Geometric approach to fragile topological phases

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We present a framework to systematically address topological phases when finer partitionings of bands are taken into account, rather than only considering the two subspaces spanned by valence and conduction bands. Focusing on $C_2T$-symmetric systems that have gained recent attention, for example in the context of layered van-der-Waals graphene heterostructures, we relate these insights to homotopy evaluations and mathematical varieties, which in turn correspond to Wilson flow approaches. We make use of a geometric construction, the so-called Plücker embedding, to induce windings in the band structure necessary to facilitate non-trivial topology. Specifically, this directly relates to the parametrization of the Grassmannian, which describes partitioning of an arbitrary band structure and is embedded in a better manageable exterior product space. From a physical perspective, our construction encapsulates and elucidates the concepts of fragile topological phases and new kinds of band node braiding processes that arise when different band gaps are taken into account. The adopted geometric viewpoint most importantly culminates in a direct and easily implementable method to construct model Hamiltonians to study such phases, constituting a versatile theoretical tool.

I. INTRODUCTION

Whereas the conceptional discovery of topological insulators [1, 2] is nearing a fifteen-year anniversary, the research into their properties and material realizations remains increasingly active. The consideration of spatial symmetries and of gapless systems has by now resulted in a rich variety of topological phases and characterizations [3–27]. Recently, consistency equations for representations in momentum space were used to describe the possible topological band configurations [16, 18], which has provided several schemes to compare these configurations against those that have an atomic limit [19, 21]. More specifically, band representations that cannot be written as an integer sum of band structures corresponding to atomic orbitals are diagnosed as topological. However, there is a possibility that a band representation amounts to a difference of trivial (i.e. atomic) configurations, inducing the so-called fragile topology [28].

Conventionally the topology of band structures is characterized under the condition of a single spectral gap. This can be thought of as partitioning the bands into two subspaces, i.e. an “occupied” subspace spanned by states with energies below the energy gap, and the complementary “unoccupied” subspace spanned by states with energies above the energy gap. However as pointed out by general vector bundle theory [29], this is in fact the coarsest partitioning of bands that can enable nontrivial topology.

In this work, we consider a finer characterization of band topology, which is obtained by assuming multiple spectral gaps. Such a refined partitioning of energy bands has been recently applied to certain $C_2T$-symmetric and $\mathcal{PT}$-symmetric systems ($C_2$ is $\pi$-rotation, $T$ is time reversal, and $\mathcal{P}$ is space inversion). More specifically, this new perspective recently provided new insights into the fragile band topology [30], and has led to the prediction of a new kind of reciprocal braiding of band nodes inside the momentum space [31–34]. In this regard we also point out that these topological insights and their interplay with $C_2T$-symmetry further touch upon real experimentally viable systems as they constitute the key elements in the modelling of twisted layer graphene systems [35, 36] and non-Abelian braiding in ZrTe [33]. Specifically, we present a construction that involves a so-called Plücker embedding which enables us to parametrize real oriented Grassmannians that describe the band structures in question. As a next step, we can then readily address the topology by considering the homotopy classes of these systems. Such homotopy evaluations in turn intimately relate to Wilson flow arguments, which provide in many circumstances a readily implementable viewpoint to discern band topologies [3, 11, 13, 18, 30, 37–42].

The manuscript is organized as follows. We begin in Sec. II by specifying the symmetry settings and the assumptions on tight-binding models of band structures. In this context we also introduce the notion of a total Bloch bundle. In Sec. III we define several notions related to vector bundles and frame bundles, including the appropriate classifying spaces, and we argue that they provide the natural language to completely characterize the studied band topology. We also comment here on several related but distinct mathematical notions, attempting to resolve possible sources of misconception. We continue in Sec. IV by discussing the homotopy groups of...
the classifying spaces of vector subbundles, and we relate the identified topological invariants to the Euler and the Stiefel-Whitney characteristic classes. In Sec. IV we generalize the mathematical description to the presence of multiple band gaps (c.f. Fig. 1) and relate the obtained topological invariants again to the characteristic classes. This generalized “multi-gapped” context allows us to define fragile topology via repartitioning of energy bands. Indeed, we note that in our case information from the irreducible representations [16, 18, 19], c.q. elementary band representations [21], is not sufficient to diagnose the fragile criterion, rather similar to how they cannot detect Chern number in certain scenarios. We argue that an observable signature of both the Euler and the second Stiefel-Whitney class of a band subspace is given by a minimum number of stable nodal points formed within the band subspace.

After introducing this set of key mathematical notions, we use the developed machinery to generate physical models corresponding to various fragile topological phases. First, in Sec. VI we discuss our strategy in a general abstract setting. We show that that a representative of any topological class can be obtained as a pullback of the tautological total gapped bundle on the classifying space, and that this can be geometrically visualized by virtue of the Plücker embedding. We then turn our attention to specific few-band examples. Specifically, in Sec. VII we focus on the case of three bands that are partitioned into a two-band and single band subspace. We similarly perform this analysis for the four band case in Sec. VIII, where the extra band gives rise to various different partitionings in terms of single and two-band blocks. In both instances we use our general insights to address the classification aspects as well as their topological stability, c.q. fragility, that are of direct physical interest. Finally, we turn to the conclusions and discussions, Sec. IX, where we underpin the generality of the framework upon referring to directions of extension.

II. REAL BAND STRUCTURES

We model crystalline systems through a Hermitian Bloch Hamiltonian $\mathcal{H} = \sum_{\mu\nu, k} |\phi_\mu, k\rangle H_{\mu\nu}(k) |\phi_\nu, k\rangle$, where the Bloch state $|\phi_\mu, k\rangle = \sum_k e^{ik\cdot R} |w_\mu, R + r_\mu\rangle$ is the Fourier transform of the Wannier state $|w_\mu, R\rangle$ that represents the physical orbital $\mu$ at site $R + r_\mu$ (possibly with a spin), where $R$ is a Bravais lattice vector, and $r_\mu$ is the (sub-lattice) position within the $R$-th unit cell. The Bloch wave vector $k$ is a point of the Brillouin zone $B$, that is a 2-torus ($B = \mathbb{T}^2$) for two-dimensional crystals. In later sections, we sometimes replace the base space $B$ by a 2-sphere $\mathbb{S}^2$. In this work we assume that the Bloch states $|\phi_\mu, k\rangle$ are fully trivial, i.e. they carry no Berry phase and their Wannier representations $\langle r | w_\mu, R + r_\mu \rangle = w_\mu(R + r_\mu - r)$ are exponentially localized. This implies that the real-space hopping amplitudes $H_{\mu\nu}(R - R')$ have an exponential decay in $|R - R'|$, such that the Fourier transform $H_{\mu\nu}(k)$ is smooth in $k$. In practice the hopping amplitudes in the tight-binding models of materials are cut-off beyond a finite support. We remark that in this convention the states $|\phi_\mu, k\rangle$ and the Bloch Hamiltonian $H(k)$ are not necessarily periodic in reciprocal lattice vectors.

It is known that $C_2T$-symmetry with $(C_2T)^2 = +1$ implies the existence of a basis in which $H(k)$ is real [33], irrespective of the spinfulness. In the subsequent text we assume this choice of basis, i.e. $H(k)$ is an $N \times N$ real and symmetric matrix where $N \geq 2$ is the number of degrees of freedom per unit cell. This property implies that all eigenstates of $H(k)$ can be gauged to be real vectors, allowing us to drop the difference between bra-states and ket-states, $\langle u_n(k)|^\dagger = |u_n(k)\rangle \equiv u_n(k)$.

From the eigenvalue problem $H(k)u_n(k) = E_n(k)u_n(k)$, we get the spectral decomposition $H(k) = R(k)D(k)R(k)^T$, with eigenvalues $D(k) = \text{diag}[E_1(k), \ldots, E_N(k)]$, and the diagonalizing matrix $R = (u_1 \cdots u_N)$ formed by the real column eigenvectors. The eigenvectors define a rank-$N$ orthonormal frame, $R \in \mathbb{O}(N)$, and serve as a basis of the real vector space $V_k = \text{Span}[u_1, \ldots, u_N] \cong \mathbb{R}^N$ at each point $k \in B$. The collection of fibers $V_k$ at each point $k$ of the base space $B$ allows us to construct a real vector bundle.

More precisely, we define the Bloch bundle [43] as the union of the fibers, $\mathcal{E}_{N,N} = \bigcup_{k \in B} V_k$, with the continuous projection onto the base space, i.e. $\pi : \mathcal{E}_{N,N} \to B$, and so that it is locally homeomorphic to a direct product space, i.e. $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^N$ for any contractible open subset $U \subset B$. By virtue of the later property we say that $\mathcal{E}_{N,N}$ is locally trivializable. In the following we fix the ordering of the eigenvalues, $E_1 \leq \cdots \leq E_N$, and we assume the same ordering for the eigenvectors in $R$.

III. GAP CONDITION AND CLASSIFYING SPACES

A. Vector bundles and subbundles

In this work we assume that the “total” Bloch bundle $\mathcal{E}_{N,N}$ as defined in Sec. II is trivial, which corresponds to situations in which the Bloch Hamiltonian can be brought to real-symmetric form periodic in reciprocal lattice vectors. We explain this assumption in the following Sec. III C. Nontrivial topology may then arise by considering subbundles defined through a spectral gap condition. Under the condition of a single energy gap, say

\begin{equation}
E_1 \leq \cdots \leq E_p \leq E_{p+1} \leq \cdots \leq E_N \quad \text{with} \quad 1 \leq p < N,
\end{equation}

i.e. with a finite gap $\delta(k) = E_{p+1}(k) - E_p(k) > 0$ for all $k \in B$, the total frame $R = (R_I R_{II})$ splits into the subframes $R_I = (u_1 \cdots u_p)$ and $R_{II} = (u_{p+1} \cdots u_N)$. The
FIG. 1. Band partitioning with multiple gap conditions. Each block of energy bands (colored strips) is separated from all other bands by energy gaps (white regions) both from above and from below. The topology of the i-th subspace with a number \( p_i \) of bands is classified by free homotopy classes that are derived from the second homotopy group of the Grassmannian, \( \pi_2[\text{Gr}_{p,N}] \), when the bands are orientable, i.e., when the subspace does not carry \( \pi \text{-} \)Berry phase (see text), with a reduction \( \mathbb{Z} \to \mathbb{N} \) for the two-band subspaces (see Section IV). The multiple gap conditions generalize the conventional partition of the system into the occupied and unoccupied sectors by means of a single energy gap.

collection of all \( p \)-component subframes of \( \mathbb{R}^N \) is called the Stiefel manifold, labelled \( \text{P}_p(\mathbb{R}^N) \). We now define the rank-\( p \) “occupied” vector subbundle

\[
\mathcal{B}_I(p) = \bigcup_{k \in B} V_{I,k} \quad \text{with} \quad V_{I,k} = \text{Span}\{u_1(k) \ldots u_p(k)\},
\]

and the rank-(\( N-p \)) “unoccupied” subbundle \( \mathcal{B}_{II}(N-p) \) similarly via \( V_{II,k} = \text{Span}\{u_{p+1}(k) \ldots u_N(k)\} \).

By flattening the spectrum, i.e. \( \text{diag}[E_1, \ldots, E_p] \to \pm 1 \) and \( \text{diag}[E_{p+1}, \ldots, E_N] \to \pm 1 \), we get the flattened Hamiltonian \( Q = R \cdot [\mp \mathbb{1}_p \mp \mathbb{1}_{N-p}] \cdot R^T \). The constructed \( Q \) is invariant under any orthogonal (gauge) transformation \( R \to R \cdot [G_I \oplus G_{II}] \) with \( G_I \in \text{O}(p) \) and \( G_{II} \in \text{O}(N-p) \).

The classifying space of the flattened Hamiltonian is then obtained as the space of \( R \) “divided” by the group of gauge symmetries, resulting in the quotient space

\[
\text{Gr}_{p,N} = \text{O}(N)/[\text{O}(p) \times \text{O}(N-p)]
\]

called the real Grassmannian, having the property \( \text{Gr}_{p,N} = \text{Gr}_{N-p,N} \). A specific matrix \( R(k) \) defines a point of the Grassmannian through the left coset

\[
[R(k)] = [R(k) \cdot [G_I \oplus G_{II}]], \quad \text{such that} \quad G_I \in \text{O}(p) \quad \text{and} \quad G_{II} \in \text{O}(N-p).
\]

We will occasionally consider a restriction of the bundle \( \mathcal{B}_I(p) \) to a loop \( l \subset B \) in the Brillouin zone, i.e. \( \{[R(k)] \mid k \in l\} \cong \mathcal{B}_I(p)|_l \). Such a restriction defines a loop in the space \( \text{Gr}_{p,N} \). Similarly, setting \( B = S^2 \) for the base space, the collection \( \{[R(k)] \mid k \in S^2\} \) defines a sphere inside the space \( \text{Gr}_{p,N} \). The equivalence classes of loops (spheres) inside the Grassmannian can be characterized by homotopy group \( \pi_1(\text{Gr}_{p,N}) \), and are analyzed below in Sec. IV.

To any orthogonal matrix \( G \in \text{O}(N) \) we can associate an orientation through \( \det G = \pm 1 \). By definition the coset \( [R] = [R_l R_{II}] \in \text{Gr}_{p,N} \) is invariant under any orientation flipping gauge transformation of \( R_l \) and \( R_{II} \), hence \( \text{Gr}_{p,N} \) is called the real unoriented Grassmannian. Restricting to orientation preserving cosets, referred to as \( [R(k)]^+ \), we similarly define the real oriented Grassmannian \( \text{Gr}_{p,N}^+ = \text{SO}(N)/[\text{SO}(p) \times \text{SO}(N-p)] \). In analogy with the unoriented case, one can consider loops and 2-spheres inside \( \text{Gr}_{p,N}^+ \), characterized by \( \pi_1(\text{Gr}_{p,N}^+) \).

While it is customary to consider only one vector subbundle at a time, a band structure with an energy gap really consists of the ordered collection of two subbundles \( \mathcal{B}_I(p) \) and \( \mathcal{B}_{II}(N-p) \), which we write as \( \mathcal{E}_{p,N} = \mathcal{B}_I(p) \cup \mathcal{B}_{II}(N-p) \). We call this the total gapped bundle. Importantly, \( \mathcal{E}_{p,N} \neq \mathcal{B}_I(p) \oplus \mathcal{B}_{II}(N-p) \cong \mathcal{G}_{E,N} \). Indeed, the direct sum allows us to take arbitrary intra- and inter-subspace linear combinations of eigenvectors, i.e. mixing the vectors of \( \mathcal{B}_I(p) \) with those of \( \mathcal{B}_{II}(N-p) \), see Section III C, while only intra-subspace linear combinations of eigenvectors are allowed in \( \mathcal{E}_{p,N} \). In other words, the direct sum \( \mathcal{E}_{E,N} \) “forgets” about the gap condition.

We finally consider the isomorphism (i.e. equivalence) classes of the introduced bundles under continuous deformations that preserve the gap condition. Assuming a fixed choice of the base space \( B \), we write \( \mathcal{E}_I(p) \) for the isomorphism classes of rank-\( p \) vector bundles that are subbundles of \( \mathcal{E}_{E,N} \). We further write \( \mathcal{E}_{p,N} \) for the isomorphism classes of total gapped bundles that split into the vector subbundles \( \mathcal{B}_I(p) \) and \( \mathcal{B}_{II}(N-p) \). Labeling the isomorphism classes with integers, we label the trivial class by \( 0 \). We point out that by assumption the total Bloch bundle \( \mathcal{E}_{E,N} \cong 0 \) is a trivial rank-\( N \) bundle. It is important to note that for us, and contrary to what is usually done in the classification schemes based on \( K \)-theory, we keep \( N \), i.e. the rank of the underlying band structure, finite and fixed.

B. Orientability of bands and bundles

We associate to the vector subbundle \( \mathcal{B}_I(p) \) certain orthonormal frame bundle. Using \( \text{O}(R_I(k)) \subset \text{P}_p(\mathbb{R}^N) \) to indicate the orbit of subframe \( R_I(k) \) under the right transitive action of \( G_I \in \text{O}(p) \), we define the associated frame subbundle

\[
\mathcal{F}_I(p) = \bigcup_{k \in B} \text{O}[R_I(k)].
\]

Each fiber of \( \mathcal{F}_I(p) \) is isomorphic to the structure group \( \text{O}(p) \), thus making \( \mathcal{F}_I(p) \) a principal \( \text{O}(p) \)-bundle [29, 44]. An analogous construction can be carried for the unoc-
cupid sector, defining the associated frame subbundle $F_t(N - p)$.

Importantly, each fiber of $F_t(p)$ is equipped with an $SO(p)$-invariant exterior product $w_p = u_1 \wedge \cdots \wedge u_p$ resp. $\omega_{N-p} = u_{p+1} \wedge \cdots \wedge u_N$. (This form will be especially useful when discussing the Plücker embedding in Sec. VII D.)

The associated frame subbundle allows us to introduce the notion of orientability. Given a local trivialization $\phi : \pi^{-1}(U) \to U \times (\mathbb{R}^p \oplus \mathbb{R}^{N-p})$ of a total gapped bundle $\mathcal{E}_{p,N}$, the pushforwards $\phi_* \omega_p = \mathbf{o}_1 |_{U} e_1 \wedge \cdots \wedge e_p$ and $\phi_* \omega_{N-p} = \mathbf{o}_{I} |_{U} e_{p+1} \wedge \cdots \wedge e_N$, where $(e_1, \ldots, e_N)$ are orthogonal coordinate vectors on $\mathbb{R}^N$, allow us to define $\mathbf{o}_{I,J} |_{U} = \pm 1$ called the local orientations of the vector/frame subbundles. Considering a good open cover $\{U_i \to B\}$ of the base space with local trivializations $\phi_i$, every non-empty pairwise overlap $U_i \cap U_j \neq \emptyset$ is characterized by $\mathbb{Z}_2$-valued functions $t_i^I$ and $t_i^I$. More precisely, starting with an arbitrary unoriented transition functions $t_i^I = (\mathbf{o}_I |_{U_i})(\mathbf{o}_I |_{U_j}) = \pm 1$, and similarly for the unoccupied bands.

Change of a local trivialization $\phi_i$ or frames $R_{ij}^I$ may lead to a reversal of $t_i^I$. We say that a vector subbundle $B_{I,I}(N)$ is unorientable if for all trivializations $\phi_i$ (and for all choices of $R_{ij}^I$) there are some transition functions $t_i^I \neq \pm 1$. We call the total gapped bundle $\mathcal{E}_{p,N}$ unorientable if either the occupied or unoccupied vector subbundle is unorientable. The classifying space of the corresponding gapped band structure is the unoriented Grassmannian $Gr_{p,N}$. In contrast, when local trivializations can be found such that simultaneously all transition functions are equal to $\pm 1$, the vector subbundle is called orientable. The total gapped bundle is called orientable if both the occupied and unoccupied vector subbundles are orientable. In the case of a trivial total bundle $\mathcal{E}_{p,N}$, as it is assumed in this work, the subbundles $B_{I}(p)$ and $B_{I,I}(N - p)$ are either both orientable or both unorientable.

We remark that the classifying space of an unorientable gapped bundle is the unoriented Grassmannian, since there is no canonical choice of an orientation for the matrix of eigenvectors $R$. Nevertheless, the mapping $\eta : B \to Gr_{p,N}$ characterizing any orientable total gapped bundle $\mathcal{E}_{p,N}$ (i.e. the map that assigns to a point $k \in B$ with Hamiltonian $H(k)$ the coset $[R(k)] \in Gr_{p,N}$) can be lifted to a mapping $\tilde{\eta} : B \to Gr^+_{p,N}$ to the oriented Grassmannian [44]. In other words, any orientable occupied (unoccupied) vector subbundle $B_{I,I}(N)$ can be equipped with an orientation, in which case it is called oriented vector subbundle and is written $B_{I,I}(p)$. Taking them together, we obtain an oriented total gapped bundle $\mathcal{E}^+_{p,N} = B^-_I(p) \cup B^+_{I,I}(p)$.

Focusing on the orientable case, we will build in later sections explicit tight-binding models by representing them as oriented gapped bundles classified by $Gr^+_{p,N}$. Then, we show that the arbitrariness in the choice of an orientation implies that every pair of oppositely oriented bundles correspond to the same orientable bundle, i.e. there is a two-to-one reduction from the oriented bundles to the orientable bundles. We also simplify from now on the terminology, whereas a band sector characterized by an orientable (oriented) vector subbundle would be called orientable (oriented) bands [45].

C. Conceptual clarifications

We importantly remark that total Bloch bundle $\mathcal{E}_{N,N}$ as defined in Sec. II is not necessarily trivial. A simple example of such a non-trivial case is provided by the two-band 2D Mielke model discussed in Ref. [46] which exhibits total $\pi$-Berry phases in both directions of the Brillouin zone torus [34], making its total Bloch bundle non-orientable (see the definition of the first Stiefel-Whitney class in Sec. IV). This can be understood as an effect of the body-centered lattice structure (this will be elaborated elsewhere). Nevertheless, a theorem in vector bundle theory asserts that any vector subbundle $B_I(p)$ can be trivialized through the direct sum with an appropriate vector subbundle $B_I(N - p)$, i.e. $B_I(p) \oplus B_I(N - p) \cong B \times \mathbb{R}^N$. This especially also applies to a non-trivial total Bloch bundle $\mathcal{E}_{N,N}$, for which there exists a vector bundle $\overline{\mathcal{E}}(N' - N)$ such that $\mathcal{E}_{N,N} \oplus \overline{\mathcal{E}}(N' - N) \cong \mathcal{E}_{N',N} \cong B \times \mathbb{R}^N$. Then $\mathcal{E}_{N',N}$ can be interpreted as a total trivial Bloch bundle, of which the original $\mathcal{E}_{N,N}$ and the trivializing $\overline{\mathcal{E}}(N' - N)$ are two complementary subbundles. For instance, for the two-band Mielke model [46], the triviality of the total bundle is achieved for a four-band model obtained through the direct sum of two Mielke models.

We now comment on the relevance of the concept of vector bundle for band structures. We defined $F_t(p)$ in Eq. (5) by gluing together the orbits $O[R_{I}(k)]$ of the $p$-frame spanned by the occupied eigenvectors under the action of the gauge group $O(p)$. One could instead consider the finer notion of an eigenbundle [47], which corresponds to gluing together the ordered collection of eigenvectors, rather than their orbit. While local trivializability belongs to the axioms of fiber bundles, the eigenbundle may not have this property. This notably happens when the eigenvalues form a topologically stable cross-dire

Dirac strings to the orientable bundles. We also simplify from now on the terminology, whereas a band sector characterized by an orientable (oriented) vector subbundle would be called orientable (oriented) bands [45].
\( G_I \in \text{SO}(p) \) (here \([u_n]_l\) is the \(l\)-th component of the vector \(u_n\)). Clearly, the vectors \(v_\nu(k)\) need not be eigenvectors in general. Since a section of a \(p\)-frame bundle is essentially an ordered collection of \(p\) pointwise orthonormal vector bundles, the vector subbundle \(B_I(p)\) is also locally trivializable. Therefore, in contrast to eigenbundles, the higher flexibility of the vector and frame subbundles permits the local trivialization, as has been scrupulously analyzed e.g. in the Supplementary Material of Ref. [33].

It also follows that the occupied subbundle of a topological semimetal does not form a vector bundle over the whole Brillouin zone, while it does so over any closed manifold that avoids the semimetallic nodes. For this reason, we refrain from identifying the concepts of a band structure and of a vector bundle, and rather talk about the vector (sub)bundles associated to a block of energy bands separated from all other bands at each \(k\) in Brillouin zone by the gap conditions.

IV. HOMOTOPY CLASSIFICATION AND HOMOTOPY INVARIANTS

A. General description

The topological classification of gapped band structures is given by the set of all allowed isomorphism classes \([\mathcal{E}_{p,N}]\) of total gapped bundles. The later is isomorphic to the set of free homotopy classes of continuous maps from the base space (the Brillouin zone \(B = \mathbb{T}^2\)) to the classifying space of gapped band structures. We denote the set of such homotopy classes as \([T^2, \text{Gr}_{p,N}]\). These can be expressed [49–52] as

\[
[T^2, \text{Gr}_{p,N}] = \bigcup_{\alpha_x,\alpha_y} [T^2, \text{Gr}_{p,N}]^{(\alpha_x,\alpha_y)}. \tag{6}
\]

Here, the weak invariants \(\alpha_x,\alpha_y \in \pi_1[\text{Gr}_{p,N}]\) characterize the total gapped bundle \(\mathcal{E}_{p,N} = B_I(p) \cup B_{II}(N-p)\) along the two non-contractible loops \(l_x\) (resp. \(l_y\)) of \(T^2\), as discussed in Sec. IV B. Further, \([T^2, \text{Gr}_{p,N}]^{(\alpha_x,\alpha_y)}\) is the set of free homotopy classes from a square \(l_x^2\) (the inside of the BZ) to the space \(\pi_1\), which are compatible with the weak invariants \((\alpha_x,\alpha_y)\) on the BZ boundary \(\partial l_x^2\), and which we discuss in more detail in Sec. IV C. The decomposition in Eq. (6) mimics the CW-complex decomposition of \(T^2\), namely the wedge sum of the two non-contractible loops \(l_x \lor l_y\) together with a two-dimensional sheet \(l_y^2\) with its boundary glued along the loops.

When the total number of bands \(N\) is large enough, the homotopy groups for few-band models may depend on \(N\), in which case we speak of unstable topology. Note that in our definition of stability of topological invariants, and contrary to works based on \(K\)-theory, we keep the number \(p\) of occupied bands fixed. In this section we discuss the stable results, while an in-depth analysis of the unstable topology of 3-band and 4-band systems is presented in Secs. VII and VIII.

B. The first homotopy group

The stable limit for the first homotopy group is reached for \(N \geq 3\), when \(\pi_1[\text{Gr}_{p,N}] = \mathbb{Z}_2\). The element \(\alpha_l\) in the first homotopy group for a non-contractible loop \(l \in B\) coincides with the first Stiefel-Whitney (SW) class \(w_1[B_I(p)]|_l \in H^1(S^1, \mathbb{Z}_2)\) [34] (i.e. the characteristic class of the bundle that is captured by the first cohomology group of \(l \simeq S^1\) with \(\mathbb{Z}_2\) coefficients), which is known to capture the orientability of the vector subbundle \(B_I(p)\) restricted to \(l\) [44]. Considering now the occupied vector subbundle \(B_I(p)\) inside the full Brillouin zone, one can independently study the first SW class on the two non-contractible paths of the torus, which define an element \((\alpha_x,\alpha_y) \equiv w_1[B_I(p)] \in H^1(T^2, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2\). Accordingly, the vector subbundle \(B_I(p)\) and the band subspace it represents are orientable if-and-only-if we have \(\alpha_x = \alpha_y = 0\). The first SW class can be computed through the Berry phase factor \(\alpha_l = e^{i\gamma_1[l]} = \det W_l[l] \in \{+1, -1\}\) along a loop \(l\), where the \(O(p)\) Wilson loop \(W_l\) is obtained from the \(p\) eigenvectors spanning \(B_I\).

The first SW class resp. the Berry phase can also be computed for the unoccupied vector subbundle. From the assumed triviality of \(\mathcal{E}_{N,N}\) and from the Whitney sum formula for the cup product of cohomology classes, the first SW class satisfies the sum rule \(0 = w_1[B_I(p) \oplus B_{II}(N-p)] = (w_1[B_I(p)] + w_1[B_{II}(N-p)]) \mod 2\) [53], so that

\[
w_1[B_I(p)] = w_1[B_{II}(N-p)], \tag{7}
\]

and similarly for the Berry phase, i.e. \(\gamma_1[l] \mod 2\pi = \gamma_1[l] \mod 2\pi\) for both non-contractible loops of the Brillouin zone torus. This relation clarifies our statement below Eq. (6) that \(\alpha_x,\alpha_y\) characterize the topology of the total gapped bundle (rather than of just the occupied or unoccupied vector subbundle) – at least along the non-contractible loops \(l_x, l_y\). A similar relation is found in Sec. IV D also for the topological classification on the Brillouin zone square.

C. The second homotopy group

Ascending now one dimension higher, Eq. (6) suggests that the classification of vector subbundles \(B_I(p)\), and \(B_{II}(N-p)\) depends on the weak indices \((\alpha_x,\alpha_y)\). In the nonorientable case, the free homotopy set on the Brillouin zone square is

\[
[T^2, \text{Gr}_{p,N}]^{(1,0)} = [T^2, \text{Gr}_{p,N}]^{(0,1)} = [T^2, \text{Gr}_{p,N}]^{(1,1)} = \mathbb{Z}_2 \tag{8}
\]

where the \(\mathbb{Z}_2\) invariant corresponds to the second SW class [34], \(w_2[B_I(p)] \in H^2(T^2, \mathbb{Z}_2) = \mathbb{Z}_2\), which we discuss in more detail in Sec. IV D below.
In the following, we focus on the more interesting orientable case, i.e., when \( \alpha_x = \alpha_y = 0 \), such that the Berry phases are zero along both non-contractible loops of the Brillouin zone. Then the topological classification is given by the free homotopy set \( [I^2, \text{Gr}_{p,N}]^{(0,0)} \). The triviality of the weak invariants implies that the mapping to the classifying space can be deformed into a constant on the boundary \( \partial I^2 \) of the Brillouin zone square. This allows us to identifying the boundary as a single point, resulting in \( I^2/\partial I^2 \simeq S^2 \), i.e., a sphere. Therefore, \( [I^2, \cdot]^{(0,0)} = [S^2, \cdot] \), which differs from the second homotopy group \( \pi_2[\cdot] \) only by the absence of a base point.

More precisely, the unoriented Grassmannian \( \text{Gr}_{p,N} \) contains a non-contractible loop \( \ell \), and evolving the base point along this loop induces a non-trivial automorphism \( \pi_2[\text{Gr}_{p,N}] \to \pi_2[\text{Gr}_{p,N}] \) on the based homotopy group \([52, 54]\). This automorphism corresponds to orientation reversal, and it acts as \( \pi_2 : \beta \mapsto \beta^{-1} \) on elements \( \beta \in \pi_2[\text{Gr}_{p,N}] \). Knowing that second (and higher) homotopy groups are Abelian, one can always represent them as a direct sum of several \( \pi_2 \)'s, and indicate the composition with “+”, i.e., as addition, such that \( \beta^{-1} = -\beta \). The automorphism \( \pi_2 \) then reduces the second homotopy group into orbits \( \{\beta, -\beta\} \), and relaxing the condition on the base point (i.e., the reduction from based to free homotopy classes) corresponds to replacing \( \pi_2[\text{Gr}_{p,N}] \) by the set of orbits. This allows us to express the free homotopy classes concisely as

\[
[I^2, \text{Gr}_{p,N}]^{(0,0)} = \pi_2[\text{Gr}_{p,N}]/\{+1, -1\}. \tag{9}
\]

It is worth noting that the non-contractible loop \( \ell \subset \text{Gr}_{p,N} \) that appears in the construction is not homotopy equivalent to the image of any of the non-contractible loops of the Brillouin zone torus. Rather, the motion of the base point along \( \ell \) can be understood as an adiabatic and orientation-reversing deformation of the Hamiltonian. We elucidate this adiabatic transformation on explicit three-band and four-band tight-binding models below in Secs. VII and VIII.

Crucially, note that \( \pi_2[\text{Gr}_{p,N}^+] = \pi_2[\text{Gr}_{p,N}] \) because the sphere is simply connected and because \( \text{Gr}_{p,N}^+ \to \text{Gr}_{p,N} \) is a double cover. Therefore, orientable vector subbundles can be classified in terms of oriented subbundles modulo the two-to-one redundancy \( \beta \sim -\beta \) for all \( \beta \in \pi_2[\text{Gr}_{p,N}^+] \). Since maps to the oriented Grassmannians are easier to analyze, in the remainder of the manuscript we very often explicitly focus on oriented vector subbundles, nonetheless, one should always keep in mind that the actual topological classification would be obtained by the “modulo sign” reduction of the presented results. The same comments also apply to our strategy for the construction of explicit tight-binding models for all homotopy classes of three-band and four-band orientable phases.

In the same spirit, it is worth noting that whenever we are given a concrete collection of eigenvectors of a band subspace (rather than just the unoriented vector spaces they span), then the bundle has been equipped with a specific choice of orientation, and as such it can be classified by a unique element \( \beta \in \pi_2[\text{Gr}_{p,N}^+] \). However, after dropping the (arbitrary) choice of the eigenvector gauge, the bundle becomes indistinguishable from a bundle with the opposite second homotopy invariant \(-\beta\), and as such is classified by an element \( \{\beta, -\beta\} \in \{S, \text{Gr}_{p,N}\} \). Nonetheless, as long as the gauge is fixed and manipulated continuously, the oriented bundles corresponding to \( \beta \) and \(-\beta\) cannot be evolved into one another. Below, whenever we say that we deal with an explicit model, we mean an oriented bundle. In contrast, when we discuss the (free) homotopy class representative, we mean an orientable bundle, that is an equivalence class of two explicit models with opposite orientations.

D. Euler class and second Stiefel-Whitney class

The relevant second homotopy groups for oriented classifying spaces are listed in Table I [26]. The stable limit of the second homotopy group is given by \( N-p \geq 3 \), for which we have

\[
\begin{align*}
\pi_2[\text{Gr}_{1,N}^{+}] &= 0, \\
\pi_2[\text{Gr}_{2,N}^{+}] &= Z, \\
\pi_2[\text{Gr}_{p,3,N}^{+}] &= Z_2. \tag{10}
\end{align*}
\]

Notably, the second homotopy invariant characterizing an oriented two-band vector subbundle \( B^+(p=2) \) in the stable limit corresponds to the Euler class [44], \( \chi[B^+_p(p=2)] \in H^2(T^2, Z) = Z \). The Euler class is computed as the integral of the Pfaffian of the Berry curvature [33, 55, 56] over the Brillouin zone. It can also be conveniently computed as a Wilson loop winding [30, 53]. In contrast, when the oriented vector subbundle under consideration consists of three or more bands, the second homotopy invariant in the stable limit corresponds to the second SW class \( w_2[B^+_p(p \geq 3)] \in H^2(T^2, Z_2) = Z_2 \). The second SW class can be conveniently computed as the parity of the number of \( \pi \) crossings in the Wilson loop flow [34]. Single-band orientable subspaces (i.e. orientable real line bundles) are always trivial.

Because of the assumed triviality of \( \mathcal{E}_{N,N}^- \), the second SW class satisfies the sum rule 0 = \( w_2[B^+_p(p) \oplus B^+_N(N-p)] \equiv (w_2[B^+_p(p)] + w_2[B^+_N(N-p)]) \mod 2 \), where we have used the fact that the first SW class is zero for oriented vector bundles. Therefore

\[
w_2[B^+_p(p)] = w_2[B^+_N(N-p)], \tag{11}
\]

implying that the same element of \( H^2(T^2, Z) \) characterizes both the occupied and the unoccupied vector subbundle, i.e., it entirely characterizes the total oriented gapped bundle \( \mathcal{E}_{p,N}^+ \). For a rank-2 oriented vector subbundle, the second SW class is given as the parity of the Euler class [53],

\[
w_2[B^+_2(2)] = \chi[B^+_2(2)] \mod 2, \tag{12}
\]
which implies that the Euler class must also satisfy the sum rule in Eq. (11) mod 2, i.e.

\[ \chi[B^+(2)] \mod 2 = w_2[B^+(N-2)]. \tag{13} \]

Since the Euler class contains more information than the mod 2 reduction, Eq. (13) implies that it entirely characterizes the oriented total gapped bundle \( E_{2,N} \).

We finally consider the reduction, up to a sign, when dropping the explicit choice of orientation. We find that the topology of orientable gapped band structures is classified by the following stable free homotopy sets

\begin{align*}
[S^2, \text{Gr}_{1,N \geq 4}]^{(0,0)} &= 0 , \\
[S^2, \text{Gr}_{2,N \geq 5}]^{(0,0)} &= \mathbb{N} , \\
[S^2, \text{Gr}_{p \geq 3,N \geq p+3}]^{(0,0)} &= Z_2 ,
\end{align*}

where for orientable two-band subspaces we define the reduced Euler class \( \chi \), obtained through the reduction modulo sign of the Euler class of the associated oriented subbundle, i.e.

\[ \chi[B^+(2)] = |\chi[B^+(2)]|. \tag{15} \]

The orientable subspaces with more bands are characterized by the second SW class which, contrary to the Euler class, does not require a definite orientation,

\[ w_2[B(p \geq 3)] = w_2[B^+(p \geq 3)] \in Z_2 . \tag{16} \]

| \( N = p_1 + p_2 + \ldots \) | \( \text{Fl}_{p_1,p_2}^+ \) | \( \pi_2 \) |
|---|---|---|
| 2 | \( \text{Fl}_{1,1}^+ \) | \( \text{Gr}_{1,2}^+ = S^1 \) | \( 0 \) |
| 3 | \( \text{Fl}_{1,2}^+ \) | \( \text{Gr}_{2,3}^+ = S^2 \) | \( 2\mathbb{Z} \) |
| 4 | \( \text{Fl}_{2,1}^+ \) | \( \text{Gr}_{3,4}^+ = S^3 \) | \( 0 \) |
| | \( \text{Fl}_{2,2}^+ \) | \( \text{Gr}_{2,2}^+ = S^2 \times S^2 \) | \( Z \oplus Z \) |
| | \( \text{Fl}_{1,1}^+ \) | \( \text{Gr}_{3,3}^+ = S^2 \) | \( 2\mathbb{Z} \) |
| | \( \text{Fl}_{1,1,1,1}^+ \) | \( \text{Gr}_{3,3,3,3}^+ = S^2 \) | \( 0 \) |
| \( (m \geq 3) \) | \( \text{Fl}_{i}^+ \) | \( \text{Gr}_{i+1,i+1}^+ = S^m \) | \( 0 \) |
| \( 1 + m \) | \( \text{Fl}_{2,1}^+ \) | \( \text{Gr}_{2,2}^+ = S^2 \) | \( Z \) |
| \( 2 + m \) | \( \text{Fl}_{2,2}^+ \) | \( \text{Gr}_{3,3}^+ = S^2 \) | \( Z_2 \) |

TABLE I. Classification of orientable band structures, i.e. over the simply-connected base space \( B = S^2 \) representing the Brillouin zone torus in the absence of Berry phases. Table indicates the second homotopy groups, \( \pi_2 \), of oriented Grassmannian and flag varieties as discussed in the text. The factor 2 in \( 2\mathbb{Z} \) is a convention in order to match with the computed value of the Euler class, see Section VII. The topologically inequivalent orientable phases are then classified, up to a sign, by the reduction of the second homotopy group.

V. REFINED BAND PARTITIONING

A. Multiple gap conditions

The single gap condition is naturally generalized to multiple gap conditions when several blocks of bands are separated from each other by energy gaps both from above and from below everywhere in the Brillouin zone \( B \), cf. Fig. 1. We use \( \mathfrak{N} \) to indicate the total number of band subspaces, and we write the subbundle of the \( i \)-th band subspace \( (i = I, II, III \ldots , \mathfrak{N}) \) as \( B_i(p_i) \) with \( p_i \) its number of bands, and \( N = \sum_{i=1}^{\mathfrak{N}} p_i \) the total number of bands. The total gapped bundle can be expressed as

\[ E_{p_1, \ldots , p_\mathfrak{N}} = B_1(p_1) \cup \ldots \cup B_\mathfrak{N}(p_\mathfrak{N}) \tag{17} \]

where the ordering of the subspaces follows the increasing band energy. Similar to Sec. IV, in the present section we assume the stable limit, i.e. \( \forall i : N - p_i \geq 3 \).

Formally, the classifying space of a Hamiltonian with multiple gap conditions generalizes the Grassmannian to a flag variety

\[ \text{Fl}^+_{p_1,p_2,\ldots , p_\mathfrak{N}} = O(N) / [O(p_1) \times O(p_2) \times \ldots \times O(p_\mathfrak{N})] \tag{18} \]

where the quotient corresponds to the gauge structure obtained by flattening every block of bands separately. The work of Ref. [31] revealed non-Abelian band topology of nodal lines in \( PT \)-symmetric systems by considering the complete flag manifold \( O(N) / O(1)^N = \text{Fl}_{1,1,\ldots ,1} \), while ideas interpretable in terms of a partial flag \( \text{Fl}_{p-1,2,N-p-1} \) were employed by the work of Ref. [33] to analyze the topological properties of principal band nodes in \( C_2T \)-symmetric models. One can also construct an oriented flag variety \( \text{Fl}^+ \) by replacing \( O \mapsto SO \) in Eq. (18) for both the total space and the quotients.

B. Homotopy classes of flag varieties

The first homotopy group of the flag variety in Eq. (18) is easily shown [57] to be \( \pi_1[\text{Fl}^+_{p_1,\ldots , p_\mathfrak{N}}] = Z_2^{\mathfrak{N}-1} \). This result is interpretable in terms of the quantized Berry phases of each subbundle (i.e. by their first SW classes) on a closed path \( \ell \), subject to the constraint \( \sum_{i=1}^{\mathfrak{N}} \gamma_i[\ell] = 0 \mod 2\pi \) that follows from the Whitney sum formula and from the triviality of the total Bloch bundle. In analogy with the single gap case discussed in Sec. IV C, the generators of the first homotopy group are associated with certain paths \( \{ \ell_i \}_{i=1}^{\mathfrak{N}-1} \) in \( \text{Fl}^+_{p_1,\ldots , p_\mathfrak{N}} \), such that adiabatically evolving the Hamiltonian along \( \ell_i \) reverses the local orientation of subbundles \( B_i(p_i) \) and \( B_{i+1}(p_{i+1}) \) [33, 52].

We further consider the topological classification of total multi-gapped bundles in two dimensions. We explicitly consider only the case when all \( B_i(p_i) \) are orientable. For simplicity, we first assume that each subbundle is equipped with an explicit orientation, becoming \( B_i^+(p_i) \).
and we implement the effect of dropping the orientations as the second step. Under these assumptions, the discussion in Sec. IV B implies that the first homotopy groups play no role, and according to Sec. IV C the homotopy classification of each oriented subbundle is captured by the stable second homotopy group \( \pi_2[Gr_{p_i,N \geq p_i+3}] \), which depends on \( p_i \). It follows that (i) one-band oriented subspaces \( (p_i = 1) \) have a trivial topology, (ii) two-band oriented subspaces \( (p_i = 2) \) have a \( \mathbb{Z} \) topology indicated by the Euler class, and (iii) multiband subspaces \( (p_i \geq 3) \) have a \( \mathbb{Z}_2 \) topology indicated by the second SW class. We can indicate a generic equivalence class of total gapped bundles with a prescribed partitioning of bands as \( (\beta_1, \beta_{11}, \ldots, \beta_n) \) where the indicators \( \beta_i \) is a 0, \( \mathbb{Z} \) resp. \( \mathbb{Z}_2 \) number depending on the value of \( p_i \). It follows from the Whitney sum formula for orientable subbundles, from the triviality of the total Bloch bundle, and from the discussion in Sec. IV D that \( \sum_{i=1}^n n_i = 0 \mod 2 \).

Similar to Sec. IV C, dropping the orientations of the subbundles reduces the second homotopy groups into orbits under automorphism induced by the first homotopy group of the classifying space. Since adiabatic evolution of the Hamiltonian along \( \ell_i \) reverses the orientation of subbundles \( B_i(p_i) \) and \( B_{i+1}(p_{i+1}) \), the corresponding automorphism reverses \( \beta_{\ell_i} : (\beta_i, \beta_{i+1}) \to (-\beta_i, -\beta_{i+1}) \) while keeping the other indicators intact. By forming arbitrary compositions of automorphisms \( \{\beta_{\ell_i}\}_{i=1}^{n-1} \), we can flip the sign of any even number of the indicators \( \beta_i \). In other words, the orbits (i.e. the elements of the free homotopy set \( \{T^2, F_{p_1}, \ldots, p_m\}\) where \( (0,0) \) indicates the vanishing Berry phases of each subbundle along the two non-contractible cycles of the Brillouin zone) consists of collections \( (\pm \beta_1, \pm \beta_{11}, \ldots, \pm \beta_m) \) that differ from each other by an even number of sign reversals. Whenever any of the indicators is 0 or \( \mathbb{Z}_2 \) valued, but also when it is \( \mathbb{Z} \) valued but takes the zero value, its sign reversal does not correspond to any change of topology, meaning that the orbits under automorphisms \( \{\beta_{\ell_i}\}_{i=1}^{n-1} \) also admit arbitrary (including odd) number of sign reversals.

C. Repartitioning of bands and fragile topology

With the obtained understanding of the topology of the generalized flag manifold, let us consider the effect of repartitioning the bands

\[
B_i(2) \cup B_{i+1}(1) \rightarrow B'_i(3) = B_i(2) \oplus B_{i+1}(1),
\]

caused by the closing (or discarding) of the energy gap between band subspaces \( B_i \) and \( B_{i+1} \). The repartitioning induces the following reduction of topological charge

\[
\mathbb{N} \to \mathbb{Z}_2 ; \quad \chi[B_i(2)] \to w_2[B'_i(3)] = \chi[B_i(2)] \mod 2.
\]

We thus conclude that whenever a two-band subspace has an even (odd) Euler class, the effect of adding an extra trivial band trivializes (respectively reduces) the topology of the combined 3-band subspace. For this reason, Euler class is described as a fragile topology [28, 30]. Fragile topology is thus weaker than the stable topology known from Chern insulators where the nontrivial topology is robust under the addition of trivial bands. However, fragile topology must be sharply contrasted from the unstable topology of Hopf insulators that only exists in strictly two-level systems \( (\pi_2[Gr_{p}(\mathbb{C}^2)] = \mathbb{Z}) \) [58–61]. Indeed, in Hopf insulators the embedding of the two-level Hamiltonian into three-(or more)-band Hamiltonian destroys the whole topology, while the nontrivial fragile topology of a few-band subspace is conserved as long as the energy gaps separating it from the other bands are maintained.

D. Nodal points

The principal observable linked to the reduced Euler class of an orientable two-band subspace \( B_i(2) \) is the number of stable nodal points formed between the two bands, i.e. there is a minimal number of nodal points \( \#_{NP} = 2\chi[B_i(2)] = 2|\chi[B'_i(2)]| \) that cannot be annihilated as long as the gaps with the adjacent bands, \( B_{i-1} \) and \( B_{i+1} \), remain open [33, 34]. We emphasize that this result is only valid in the orientable case [34], otherwise the Euler class cannot be defined [46].

We emphasize that stable nodal points here indicates those that cannot be removed within a two-band subspace as long as the adjacent gaps remain open. Band structures may host additional pairs of nodal points within a two-band subspace that can be annihilated when the nodes are collapsed onto each other. This may however require a large deformation of the band structure (similarly to generic Weyl points), i.e. unstable nodal points in this context are topologically robust relatively to small local perturbations of the band structure. In that sense, the “stability” of unstable nodal points can be measured, crudely, by the shortest distance that separates them in the Brillouin zone. A more detailed analysis would be needed to obtain the measure of stability in terms of the deformation of the tight-binding parameters, which will be studied in an upcoming work.

By allowing band inversions of the two principal bands with a third band, additional nodal points can be generated or annihilated in pairs within the two adjacent (below and above) energy gaps [33, 34]. This facilitates the braiding of principal and adjacent band nodes which is accompanied by non-Abelian phase factors [31]. The \( \mathbb{Z}_2 \) second SW class of the three-band subspace then indicates the stable parity of the minimal number of pairs of nodal points, i.e. \( w_2[B_i(p_i)] = 1 \) indicates that at least one pair of nodal points cannot be annihilated within the \( p_i \)-band subspace.
VI. GEOMETRIC CONSTRUCTION

A. Strategy

We now embark on employing the notions developed in the previous sections to construct a general geometrical framework. Before we turn to the topic, we emphasize that out strategy is again to first develop explicit models equipped with a specific orientation of each subbundle. We subsequently drop the orientation and arrive at homotopy class representatives of orientable bands.

Accordingly, we note that all real vector bundles over a sphere are orientable since all base loops can be contracted to a point, i.e. $e^{2\pi i l} = +1$ for all $l \in B = S^2$, which further implies that $\pi_2[Gr_{p,N}] = \pi_2[Gr_{p,N}^+]$. Inversely, all orientable topological phases can be effectively modeled over the sphere since $[T^2, Gr_{p,N}] = \pi_2[Gr_{p,N}^+]$. This motivates the strategy [26] to generate representative tight-binding models for all the homotopy classes.

The general framework is then presented as follows. After setting up some definitions and identifying the appropriate bundle structure, which concretely amounts to introducing the tautological and according gapped bundle, we then describe how this structure can be pulled back to the torus to obtain specific coordinates to facilitate the desired maps. To achieved this we make use of the so-called Plücker embedding into more manageable exterior product spaces that allows us to parametrize the map into the Grassmannians in a tractable manner. After having discussed this embedding, we close with the homotopy aspects of our construction.

B. The tautological bundle

The tautological bundle of the oriented Grassmannian, $F_{p,N} \to Gr_{p,N}^+$, is defined as the vector bundle obtained by taking the oriented $p$-dimensional hyperplane $V_I = \Span\{u_1, \ldots, u_p\}$ at every point $[R]^+$ of the oriented Grassmannian, where $R = (R_I R_{II}) = (u_1 \cdots u_p u_{p+1} \cdots u_N) \in SO(N)$ [62]. As mentioned in Sec. III.B, and as more carefully elaborated in Sec. VID below, the oriented $p$-plane can also be expressed in an $SO(p)$-invariant fashion as the wedge product $u_1 \wedge \cdots \wedge u_p$. The tautological bundle is canonical, in the sense that its structure follows directly (without extra assumptions) from the construction of the Grassmannian.

We note that by fixing an oriented $p$-dimensional hyperplane in $\mathbb{R}^N$, we implicitly but purposely also define the complementary oriented $(N-p)$-dimensional hyperplane $V_{II} = \Span\{u_{p+1}, \ldots, u_N\}$ such that $\mathbb{R}^N = V_I \oplus V_{II}$. This can be also seen as the Hodge dual of $u_1 \wedge \cdots \wedge u^p$. However, in general for the equivalence classes we have $[F_{p,N}^+] \neq [F_{p,N}^+\cap N-P,N]$ since they do not need to have equal ranks, while $Gr_{p,N} = Gr_{N-p,N}^+$. For this reason we introduce the notion of oriented tautological total gapped bundle, in analogy with our definition of $E_{p,N}^+$ in Sec. III.B, as $T_{p,N}^+ = F_{p,N}^+ \cup F_{N-p,N}^-$. We now define a reference total gapped bundle from which all the phases can be generated. This is achieved through a map $f_1 : \mathbb{S}^2 \to Gr_{p,N}^+$ such that $f_1(\mathbb{S}^2)$ belongs to the homotopy class that generates $\pi_2[Gr_{p,N}]$ (by abuse of language, we will say that $f_1(\mathbb{S}^2)$ is the generator of the second homotopy group). The reference bundle is then defined as the pullback $R_{p,N}^+ = f_1^* T_{p,N}^+$, i.e. the restriction of the tautological total gapped bundle $T_{p,N}^+$ induced by $f_1$. Now, in analogy with the way every orientable vector (sub)bundle can be obtained as a pullback of the tautological bundle, i.e. $B^+(q) = f_B^* F_{p,N}$, an explicitly constructed $f_1$ allows us to express an arbitrary oriented total gapped bundle $E_{p,N}$ as a pullback of $T_{p,N}^+$.

C. Pullback to the Brillouin zone torus

In order to connect with tight-binding models, we define a continuous function $t_q : T^2 \to \mathbb{S}^2$ that maps the Brillouin zone torus onto the sphere with $\deg t_q = q \in \mathbb{Z}$, i.e. $t_q$ wraps $|q|$ times over the sphere with the orientation $\text{sgn} q = \pm 1$. Parametrizing the sphere $S^2$ with the usual spherical coordinates, a simple choice of $t_q$ is obtained by taking inside the first Brillouin zone $t^2 \simeq |k_x, y| \leq \pi$ the mapping

$$
t_q : k \mapsto (\theta_q(k), \phi_q(k)) ,
$$

(21a)

with

$$
\theta_q(k) = \max(|k_x|, |k_y|) ,
\phi_q(k) = q \arg (k_x + i k_y) ,
$$

(21b)

where we set $\phi_q(0,0) = 0$. Note that $\theta_q$ has a branch cut on $|k_x| \leq 0, k_y = 0$ and that is discontinuous at $k = (0, 0)$. However, these discontinuities disappear in the Cartesian coordinates of a point of the sphere $e_r = (\cos \phi_q \sin \theta_q, \sin \phi_q \sin \theta_q, \cos \theta_q)$. Furthermore, although $\theta_q$ is not differentiable at $|k_x| = |k_y|$, the map is continuous. (The differentiability of the resulting Bloch Hamiltonians representatives of each topological phase will be easily restored in Secs. VII and VII.). The composition map $\eta_q = f_1 \circ t_q$ thus sends each point $k \in T^2$ of the Brillouin zone to a point $\eta_q(k) = f_1(\theta_q(k), \phi_q(k))$ of a sphere inside the Grassmannian, cf. Fig. 2.

A generic oriented total gapped bundle over the Brillouin zone torus is obtained as a pullback by the composition map $\eta_q$,

$$
E_{p,N}^+ = t_q^* R_{p,N}^+ = t_q^*(f_1^* F_{p,N}^+) \equiv \eta_q^* F_{p,N}^+ ,
$$

(22)

and according to the diagram

$$
\begin{array}{ccc}
E_{p,N}^+ & \xrightarrow{h} & R_{p,N}^+ \\
\downarrow \pi_2 & & \downarrow \pi_2 \\
T_{p,N}^+ & \xrightarrow{f_1} & \text{Gr}_{p,N}^+ \\
\end{array}
\begin{array}{c}
\text{Sym}^{-1} \\
\mathbb{S}^2 \\
\mathbb{S}^2 \\
\end{array}
\xrightarrow{f_1} \mathbb{R}^p |\mathbb{R}^N| .
$$

(23)
where $h$, and $h'$, are bundle maps obtained as the inverse of the pullbacks $f_1$, and $f_2$, respectively, and the mapping $\iota$ on the bottom-right is the Plücker embedding of the Grassmannian explained in Sec. V1D. It follows from the described construction, that the integer $q$ determines the isomorphism class of the oriented total gapped bundle, such that $[\mathcal{E}^{q+}_{p,N}] = \beta_q \in \pi_2(\text{Gr}^+_{p,N})$. This defines the homomorphism of groups

$$\beta : Z \rightarrow \pi_2(\text{Gr}^+_{p,N})$$

with $\beta^{-1} = \beta_q - \beta_{q' + q} + \beta_q$, and $\beta_0 = 0$.

Finally, writing the oriented total gapped bundle as $\mathcal{E}^{q+}_{p,N} = B^+_f(p) \cup B^+_R(N-p)$, the corresponding diagram for the vector subbundle $B^+_f(p)$ is

$$
\begin{array}{ccc}
\mathbb{T}^2 & B^+_f(p) & \mathcal{F}^+_p \\
\downarrow^{t_q} & \downarrow^{h} & \downarrow^{\pi_{f_1}} \\
\mathbb{S}^2 & \text{Gr}^+_{p,N} & \text{Gr}^+_{p,N} \\
\end{array}
$$

and similarly for $B^+_R(N-p)$, where $B^+_S$ is the vector subbundle of the bundle $\mathcal{R}^+_p = f^+_1 \mathcal{F}^+_p$ defined over $\mathbb{S}^2$. We emphasize that since the associated total gapped bundle $\mathcal{E}^{q+}_{p,N}$ is generated from the pullback by a map to the oriented Grassmannian ($\eta_q : \mathbb{T}^2 \rightarrow \text{Gr}^+_{p,N}$) it has no nontrivial weak invariants. In other words, the Berry phases over either the occupied and the unoccupied band subspaces are all trivial, even though the base space ($\mathbb{T}^2$) contains non-contractible loops.

**D. Plücker embedding**

We now motivate how to represent the image $f_1(\mathbb{S}^2) \subset \text{Gr}^+_{p,N}$ via a general procedure that would be discussed more explicitly for three-band and four-band models in Secs. VII and VIII. To achieve this, we employ the Plücker embedding $\iota : \text{Gr}^+_{p,N} \rightarrow \Lambda^N(\mathbb{R}^N)$ which represents the oriented Grassmannian as a $p(N-p)$-dimensional submanifold of the $p$-th exterior power of $\mathbb{R}^N$ (i.e. the $\Lambda^N(\mathbb{R}^N)$-dimensional Euclidean vector space spanned by $N$-component and fully anti-symmetric $p$-vectors). As a result, the image of the Plücker embedding can be generally represented by $[63]$

$$\iota(\text{Gr}^+_{p,N}) = \mathbb{K}_p \cap \mathbb{S}^{N(p)-1}.$$  \hspace{1cm} (26)

Here, $\mathbb{K}_p$ is the cone of simple (or decomposable) $p$-vectors, i.e. those of the form $\bigwedge_{i=1}^p v_i$ for some collection of vectors $\{v_i\}_{i=1}^p$ in $\mathbb{R}^N$ (not necessarily pairwise orthogonal), and $\mathbb{S}^{N(p)-1}$ is the unit sphere in $\Lambda^p(\mathbb{R}^N)$ with respect to the linear inner product defined on simple $p$-vectors as $[64]$

$$\left\langle \bigwedge_{i=1}^p v_i, \bigwedge_{i=1}^p v'_i \right\rangle = \sum_{\sigma \in \mathbb{S}_p} (-1)^{\text{sign } \sigma} \prod_{i=1}^p \langle v_i, v'_{\sigma(i)} \rangle.$$  \hspace{1cm} (27)

where $\mathbb{S}_p$ is the permutation group of $p$ elements.

The Plücker embedding is defined explicitly as follows. For a given point $[R]^+ \in \text{Gr}^+_{p,N}$ we take a representative $R = (R_I R_{II})$, and we construct $\iota([R]^+)$ as the wedge product of the columns of subframe $R_I$, that is

$$\iota([R]^+) = u_1 \wedge \cdots \wedge u_p \equiv \omega_p.$$  \hspace{1cm} (28)

Crucially, the $p$-vector in Eq. (28) is invariant under the $\text{SO}(p) \times \text{SO}(N-p)$ gauge transformations of the frame $R$, meaning that all choices of the representative of $[R]^+$ result in the same image $\iota([R]^+)$. Furthermore, note that $\omega_p$ is by definition a simple $p$-vector, and it is easy to check that it has a unit norm, implying $\iota(\text{Gr}^+_{p,N}) \subseteq \mathbb{K}_p \cap \mathbb{S}^{N(p)-1}$. The validity of the equality in Eq. (26) is less obvious $[65]$, but can be proved $[63]$. 

FIG. 2. Composition map $\eta_q = f_1 \circ t_q$ through which the pullback bundle $\mathcal{E}^{q+}_{p,N} = \eta_q^* \mathcal{T}^{q-}_{p,N}$ is built. We define the map $t_q$ such that the Brillouin zone center is mapped to the “blue pole” of the sphere, and the Brillouin boundary to the “red pole”. The points with the same distance from the Brillouin zone center, $\max\{|k_x|,|k_y|\}$, are mapped to the same polar angle $\theta$ on the sphere. The map $f_1$ is then constructed such that its image $f_1(\mathbb{S}^2)$ induces the generator(s) of $\pi_2(\text{Gr}^+_{p,N})$. As a result, windings producing non-trivial Euler class can be imposed.
Note that one can similarly define \( \iota(\Gr_{N-p,N}^{+}) \) as the exterior product \( \omega_{N-p} \) of the \( N - p \) eigenvectors of subframe \( R_{II} \). Although in general \( \omega_{p} \neq \omega_{N-p} \) (in a similar way as generically \( F_{p,N}^{+} \neq F_{p,N}^{+} \)), the two objects are canonically related as Hodge duals, namely \( * \omega_{p} = \omega_{N-p} \). The invariance of \( \omega_{p} \) and of \( \omega_{N-p} \) under gauge transformations \( R \rightarrow R[G_{I} \otimes G_{II}] = (R_{I}G_{I} \otimes R_{II}G_{II}) \) with \( G_{I} \in SO(p) \) and \( G_{II} \in SO(N - p) \), and the fact that the images of the two subframes are uniquely related as Hodge duals, together imply that Eq. (28) is a faithful representation of the oriented Grassmannian.

In the following Sections, we obtain an explicit parametrization of \( \Gr_{p,N}^{+} \) for \( N = 3 \) and \( N = 4 \) through the Plücker embedding.

Starting from the general parametrization of an element \( R \in SO(N) \), the task is to find the restriction to the parametrization of an element \( \omega_{p} \) of the image \( (\iota \circ f_{j})(S^{2}) \) (corresponding to the generator of the second homotopy group of the Grassmannian), as this provides the parametrization of the subframes \( R_{I} \) and \( R_{II} \), which directly encode the Hamiltonian \( H(k) \). By explicitly solving this problem for \( SO(3) \) and \( SO(4) \), we derive explicit three-band and four-band tight-binding models for all the homotopy classes.

\section*{E. Homotopy classes of total gapped bundles}

We argued in Sec. IVD that two-band subspaces are characterized by the Euler class, \( \chi[B^{2} \times S^{2}] \), while \( p \)-band subspaces with \( p \geq 3 \) are characterized by the second SW class, \( w_{2}[B^{2} \times (p)] \). We will see in Secs. VII and VIII for several concrete examples that the element of the second homotopy group for the oriented total gapped bundle determines the Euler and the second SW classes of both vector subbundles \( B_{I}^{2}(p) \) and \( B_{II}^{2}(N - p) \), depending on \( q \).

When the classifying space is a product space, i.e. \( C = \prod_{j \in A} X_{j} \) (\( A \) being some indexing set), the second homotopy group splits as a direct sum \( \pi_{2}[C] = \bigoplus_{j \in A} \pi_{2}[X_{j}] \). Thus the map \( f_{j} \) splits accordingly into the components \( \{f_{j,i} : S^{2} \rightarrow X_{j}\} \) where the indexing set \( A' \) contains the components \( X_{j} \) with a non-trivial \( \pi_{2}[X_{j}] \). This scenario notably occurs for \( \Gr_{2,4}^{+} \) that is discussed in Sec. VIII. Under these circumstances, one needs to replace the base space of the reference total gapped bundle in Eq. (25) by a product \( \mathcal{S}^2 [A'] = \bigcup_{j \in A'} S^{2}_{j} \), i.e. one copy of \( S^{2} \) for each generator of \( \pi_{2}[C] \), and the maps relate \( f_{j,i} : S^{2}_{j} \rightarrow X_{j} \).

The map from \( \mathcal{S}^2 \) to \( \mathcal{S}^2 [A'] \) is characterized by a vector of integers \( \eta = (q_{1}, \ldots, q_{A'}) \), with each element encoding the map in Eq. (21) to the respective \( S^{2}_{j} \). The composition map \( \eta_{q} = f_{j} \circ \iota_{q} \) then determines the homotopy class (and therefore also the Euler resp. the second SW class of both band subspaces) of the mapping \( \eta_{q} : \mathcal{T}^{2} \rightarrow \Gr_{p,N}^{+} \).

\section*{VII. THREE-BAND MODELS}

\subsection*{A. The three-band classifying space}

We now turn the attention to the specific case of \( N = 3 \) bands, and we deploy the machinery developed in the previous sections in a concrete context. From a physical perspective, we point out that the \( N = 3 \) topology has appeared in numerous physical settings. In particular, for nematic systems the associated topology has been extensively studied \([66–70]\), as well as non-Hermitian band topology has been related to it \([52]\). Noting that \( \Gr_{2,3}^{+} = SO(3)/[SO(2) \times \{1\}] \cong S^{2} \), we have \( \pi_{2}[\Gr_{2,3}^{+}] = 2\mathbb{Z} \). The factor “2” here is a convention made such that the value of the topological invariant agrees with the computed Euler class, as we elaborate below.

We start from an arbitrary element

\[ R = (u_{1}u_{2}u_{3}) \in SO(3) \]  

(29)

that is parametrized by three continuous angles (e.g. the Euler angles). Choosing a Cartesian frame \( (e_{1},e_{2},e_{3}) \) of \( \mathbb{R}^{3} \) to decompose the eigenvectors, i.e. \( u_{i} = u_{i1}e_{1} + u_{i2}e_{2} + u_{i3}e_{3} \) for \( i = 1,2,3 \), the Plücker embedding \( \iota(\Gr_{2,3}^{+}) \) of the two-band subspace is given through the bivectors

\[ u_{1} \wedge u_{2} = (u_{1}^{2}u_{3}^{2} - u_{1}^{3}u_{2}^{2})e_{2} \wedge e_{3} + (u_{1}^{3}u_{2}^{1} - u_{1}^{2}u_{3}^{3})e_{3} \wedge e_{1} + (u_{1}^{2}u_{3}^{2} - u_{1}^{3}u_{2}^{2})e_{1} \wedge e_{2}. \]  

(30)

Note that the expressions in the parentheses correspond to components of \( u_{3} \), by virtue of the property in Eq. (29).

By formally identifying the basis of the three-dimensional Euclidean vector space \( \mathbb{R}^{2} \times \mathbb{R}^{3} \) with the Cartesian frame of \( \mathbb{R}^{3} \) via the Hodge dual, i.e. \( \star(e_{2} \wedge e_{3} - e_{1} \wedge e_{1} \wedge e_{2}) = (e_{1},e_{2},e_{3}) \), we get \( \star(u_{1} \wedge u_{2}) = u_{3} \in S^{2} \), which is a specific instance of the duality discussed in Sec. VED.

We thus infer that spherical coordinates for three-dimensional orthonormal frames defined on \( S^{2} \) provide the desired mapping \( f_{1} \) onto the non-trivial sphere inside the Grassmannian, i.e. we use

\[ f_{1}(\theta, \phi) = [(u_{1}u_{2}u_{3})]^{+} \]  

with

\[ u_{3} = e_{r} = (\cos \phi \sin \theta, \sin \phi \sin \theta, \cos \theta) \]  

\[ u_{1} = e_{\theta} = \frac{\partial \phi}{\partial e_{r}} = (\cos \phi \cos \theta, \sin \phi \cos \theta, -\sin \theta) \]  

\[ u_{2} = e_{\phi} = \frac{\partial \theta}{\partial e_{r}} = (-\sin \phi, \cos \phi, 0) \]  

(32)

which correspond respectively to the directions “up”, “south”, and “east” at every point \( (\theta, \phi) \) of the sphere. Indeed, \( u_{1} \wedge u_{2} = e_{\theta} \wedge e_{\phi} \), which is invariant under any \( G_{I} \in SO(2) \) gauge transformation \( R_{I} \rightarrow R_{I}G_{I} \), represents the oriented plane perpendicular to \( u_{3} = e_{r} \) and, by definition, an element of \( \Gr_{2,3}^{+} \).
B. Three-band reference total gapped bundle

Note that $(e_\theta,e_\phi)$ is an oriented orthonormal frame of $T(\theta,\phi)S^2$, i.e. of the tangent space at the point $(\theta,\phi)$ of the sphere, and $e_r$ is the basis of the normal space to the point of the sphere. Therefore, these vectors span the tangent bundle $TS^2$, and the normal bundle $NS^2$, respectively. Furthermore, since $Gr^+_2S^2 \cong S^2$, the total gapped tautological bundle $\mathcal{T}^+_{2,3} \to Gr^+_2S^2$ is topologically equivalent to the couple $TS^2 \cup NS^2$, so we write

$$\mathcal{T}^+_{2,3} \sim TS^2 \cup NS^2.$$

Then, since $f_1(S^2) \cong S^2$, the same is true for the pullback bundle

$$\mathcal{R}^+_{2,3} = f_1^*\mathcal{T}^+_{2,3} = B^+(2) \cup B^+(1)$$

with $B^+(2) \sim TS^2$ and $B^+(1) \sim NS^2$.

It is well known through the hairy ball theorem that the tangent bundle of the 2-sphere is non-trivial, namely any global section, (i.e. any smooth tangent vector field) must have zeros associated with a vortex structure. We illustrate this known fact on an example in Fig. 3(c), which displays the eigenvectors of a tight-binding model presented in the next section. The hairy ball theorem is formalized by the *Poincaré-Hopf theorem*, which states

$$\sum_j \text{index}_{x_j}(v) = \chi(S^2) = 2,$$

where $x_j$ is the location of a zero of the vector field $v$, $\text{index}_{x_j}(v)$ is the winding number of $v/|v|$ around the zero $x_j$, and $\chi(S^2) = 2$ is the Euler characteristic of the sphere [71]. Thus, any tangent vector field must have two sources of vorticity 1. In the more general context of our classification scheme, the Euler characteristic of the sphere is substituted by the Euler class of the rank-2 subspace of our reference bundle, which however is still $\chi(B^+_{2,3}(2)) = 2$ by virtue of Eq. (34) [c.f. also to the diagram in Eq. (25)]. We show below that the vortices of the tangent vector field correspond to nodal points within the two-band subspace.

In this context, we emphasize the *stable triviality* of the tangent bundle of the sphere, i.e. that it becomes trivial upon the direct product with a trivial bundle. The normal bundle of the sphere is trivial, and the direct sum gives $TS^2 \oplus NS^2 \cong S^2 \times \mathbb{R}^3$. In other words, the non-trivial tangent bundle (with the non-zero characteristic Euler class $\chi(TS^2) = 2$) is trivialized by the trivial normal bundle [44, 72] (resulting in the vanishing characteristic second SW class $w_2(S^2 \times \mathbb{R}^3) = 0$). The trivialization (or reduction) of the fragile topology of two-band subspaces upon closing the adjacent energy gap as discussed in Sec. V C can thus be perceived as a fingerprint of the stable triviality of the tangent bundle of the sphere.

C. Simple model generation

Importantly, we can make use of our machinery to generate explicit models of fragile topological phases over the Brillouin zone torus for an arbitrary homotopy class $[\mathcal{B}^+_2] = [f,q] = 2q \in 2\pi$ and Euler class $\chi(\mathcal{B}^+_2) = 2q$. As elaborated previously, we first construct explicit models with a specific orientation. Then the orientation can be dropped, resulting in a 2N topological classification. The continuous deformations that relate oriented tight-binding Hamiltonians with Euler class $\pm q$ are explicitly presented in Appendix B.

For $q = 1$ the mapping $t_{q=1}$ (cf. Fig. 2) wraps the Brillouin zone torus around the sphere once, and we have $c_{2,3} \equiv \chi(S^2) \sim TS^2 \cup NS^2$. The Hamiltonian representative is then readily given by [26]

$$H(k) = R(k)[(-1)^q \pm 1]R(k)^T,$$

where $\mathbb{I}_2$ is a $2 \times 2$ identity matrix, 1 is the identity in the single-band unoccupied subspace, and the frame $R(k) = t_{q=1} f_1(\theta, \phi) = (e_\theta, e_\phi, e_r)$ with

$$e_a(t_{q=1}(k)) = e_a(\theta_{q=1}(k), \phi_{q=1}(k))$$

for $a = \theta, \phi, r$, with $f_1(\theta, \phi)$ defined in Eq. (32) and $t_q(k)$ from Eqs. (21). Sampling $H(k)$ over a grid and performing an inverse discrete Fourier transform, we obtain the hopping parameters of a tight-binding model which we truncate to a few neighbors (see Appendix A for more details) without affecting the topological features. Accordingly, we note that while the bands of the Hamiltonian $H(k)$ constructed above are flat and the two occupied bands are fully degenerate, these features are lost in our tight-binding models after performing the truncation. Indeed, imposing the perfect flatness and degeneracy results after performing the inverse Fourier transform in an infinite-range hopping amplitudes.

We show in Fig. 3(a) the band structure of the obtained tight-binding model (truncated at two nearest neighbors in both directions of the square lattice, see Appendix A) that is a representative Hamiltonian for $c_{2,3} \equiv \chi(S^2) \sim TS^2 \cup NS^2$ mentioned below Eq. (34). To achieve this, we transfer the eigenvectors $\{u_1, u_2, u_3\}$ of the band structure in Fig. 3(a) defined over the torus Brillouin zone to two tangent vectors and one normal vector over each point of the sphere. This is done through the mapping $t_{q=1}$ as a “pushforward” of vector fields, i.e. $t_{q=1}(u_i(k)) \mapsto u_i(t_1(k)) = u_i(\partial_\theta(k), \phi_i(k))$ for $i = 1, 2, 3$. We thus plot on the right Fig. 3(a) the tangent vector field given by
FIG. 3. Band structure and tangent vector field realization of fragile topology of $Gr^+_{2,3}$, together with the Wilson loop winding of occupied two-band subspace indicating the Euler class. Panel (a) shows $E^{1,1+}_{q}$ with Euler class $|\chi| = 2q = 2$. The mapping $t_{q=1}$ of Fig. 2 from the Brillouin zone covers the sphere once. We show one vector field directly given by the eigenvectors of lowest energy of the two-band subspace $B_2^2$ as a global section of the tangent bundle of the sphere, it is characterized through the Poincaré-Hopf theorem with the Euler characteristic $|\chi| = 2$, see Eq. (35). Similarly, panel (b) shows $E^{2,2+}_{q}$ with Euler class $|\chi| = 2q = 4$. The mapping $t_{q=2}$ of Fig. 2 from the Brillouin zone covers the sphere two times. We show one vector field given by the eigenvectors of lowest energy, over the halves $-\pi \leq k_x \leq 0$ (black), and $0 \leq k_x \leq \pi$ (red), of the Brillouin zone. In both cases the base sphere is shown on the side of the pole image of $\Gamma$, i.e. $t_q(0,0)$ as in Fig. 2. We thus see in both cases that the vortex structures of the tangent vector fields directly reflects the nodal points of the eigenvalues band structure, with $\#NP = 2|\chi|$ globally. Although these nodes come in two pairs that are pairwise close in momentum space, making them hard to distinguish visually, inspecting the nodes in more detail, as shown in Fig. 4 for the panel (a), confirms their presence in the anticipated number.

the eigenvectors of the lower energy band of the two-band subspace of the band structure shown on the left of Fig. 3(a). As a global section of the tangent bundle $T\mathbb{S}^2$, it can be characterized by invoking the Poincaré-Hopf theorem from Eq. (35). This indicates the equivalence between the Euler characteristic $|\chi(\mathbb{S}^2)| = 2$ and the Euler class $\chi[B_2^{1,1+}(2)] = 2$. We also observe that the nodal points of the band structure with fragile topology (Fig. 3(a)) correspond to vortices of the section of $T\mathbb{S}^2$.

We can repeat the exercise for arbitrary $q \in \mathbb{Z}$. We show an example of band structure for $E^{2,2+}_{q}$ in Fig. 3(b) with the Euler class $\chi = 2q = 4$. We find $\#NP = 2|\chi| = 2 \cdot 4 = 8$ nodal points in the two occupied bands, namely 4 on the Brillouin zone boundary and 4 around the $\Gamma$ point. Owing to the pullback construction, we can force a geometric picture of $E^{2,2+}_{2,3}$ as the tangent
and normal bundles of a generalized surface (not a manifold). Since \( t_q : \mathbb{T}^2 \rightarrow S^2 \) wraps the sphere \( q \)-times, we can divide the Brillouin zone into \( q \) cells that are each mapped onto \( S^2 \), i.e. we get a tangent vector field over the sphere for each of the \( q \) cells. Taken together we can think of it as a tangent field over a surface that wraps on itself with \( q \) sheets and with the two \( q \)-fold ramification points \( t_q(0, 0) \) and \( t_q(k) \) \( k \in \partial BZ \), where the former is the image of \( \Gamma \) (blue point in Fig. 2), and the latter is the image of the boundary of the first Brillouin zone \( \partial BZ = \{ k | \max(|k_x|, |k_y|) = \pi \} \) (red point in Fig. 2).

VIII. FOUR-BAND MODELS

In this section we study the \( Gr^+_{2,4} \) case, i.e. the classifying space of a band structure with four bands and with a single gap condition that separates an “occupied” two-band subspace from an “unoccupied” two-band subspace. We first discuss in detail the parametrization of the classifying space, and then we briefly address the stability of the topological invariants under re-partitioning of the bands.

A. Parametrization

The diagonalizing matrix, \( R = (u_1 \ u_2 \ u_3 \ u_4) \), underlying the Hamiltonian belongs to \( SO(4) \) and thus can be parametrized by six continuous angles. The exterior product of the occupied states \( \omega = u_1 \wedge u_2 \) is now a point of the image of the Plücke r embedding \( \iota(Gr^+_{2,4}) \), i.e. a simple unit bivector. Notably, since the number of unoccupied bands is the same, the Hodge dual \( +\omega = u^3 \wedge u^4 \) also is a unit bivector in the same space \( \iota(Gr^+_{2,4}) \). We show in Appendix C that their linear combinations \( v_\pm = \frac{1}{\sqrt{2}} (u_1 \wedge u_2 \pm u_3 \wedge u_4) \in V_+ \), where \( V_+ \) are two complementary three-dimensional vector spaces of bivectors which partition the second exterior power of \( \mathbb{R}^4 \) (a space of dimension \( \binom{4}{2} = 6 \)) into two halves, i.e. \( V_+ \oplus V_- \). Furthermore \( |v_+| = |v_-| = 1 \), thus \( v_+ (v_-) \) belongs to a unit sphere in \( V_+ (V_-) \). Therefore, the image of the Plücke r embedding is the four-dimensional submanifold \( S^2_+ \times S^2_- \) with points \( \omega = v_+ \wedge v_- \). The inverse embedding induces a parametrization \( Gr^+_{2,4} \cong S^2_+ \times S^2_- \), implying \( \pi_2[Gr^+_{2,4}] = \mathbb{Z} \oplus \mathbb{Z} \). We observe that \( f_1 \) splits into two generators \( \{ f_1^{(j)}(S^2_j) = S^2_j \}_{j=+, -} \) of \( \pi_2[Gr^+_{2,4}] \) parametrized by \( \{ (\theta_j, \phi_j) \}_{j=+, -} \).

Let us now consider the image \( \mathcal{M} = \iota(Gr^+_{2,4}) = S^2_+ \times S^2_- \) whose points are bivectors parametrized by a pair of angles \( (\theta_+, \phi_+, \theta_-, \phi_-) \) (see Appendix C for more details). The inverse embedding induces a parametrization \( P \) of the four-band diagonalizing matrices \( \mathcal{P} : \mathcal{M} \rightarrow Gr^+_{2,4} \rightarrow SO(4) \) through the assignment \( P : \omega(\theta_+, \phi_+, \theta_-, \phi_-) \mapsto [R]^+ \mapsto R(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \).

B. Generating the models

Once the parametrization in Eqs. (38) is found, we can readily apply our machinery to generate a tight-binding model corresponding to any homotopy class of \( \pi_2[Gr^+_{2,4}] = \mathbb{Z} \oplus \mathbb{Z} \). Adapting the discussion from Sec. VII E to the present situation, we replace the base space of the reference total gapped bundle \( T^+_{2,4} \rightarrow Gr^+_{2,4} \). The explicit parametrization \( P \) depends on the chosen encoding of elements \( R \in SO(4) \). This is done in Appendix C in terms of the Lie algebra of real and anti-symmetric matrices.
target spheres as we cover the Brillouin zone torus. The map \( f_1 \) splits into \( (f_{1+, f_{1-}}) \) such that \( \tau \circ f_{1+} \) wraps \( S^2_1 \) around \( S^2_2 \) (and \( \tau \circ f_{1-} \) wraps \( S^2_2 \) around \( S^2_1 \)) exactly once. Then we generate all the topological phases through the pullback of the tautological total gapped bundle, \( \mathcal{E}^+_2 \times_t \mathcal{T}^+_4 \), where \( \eta_q = f_1 \circ t_q \). The pair \( (q_+, q_-) \) is \( Z \oplus Z \) determined uniquely by the homotopy invariant \( \{ \beta_1(q_+, q_-), \beta_2(q_+, q_-) \} \in \pi_2[Gr^+_{2,4}] = Z \oplus Z \), and we can thus directly take \( \{ \beta_1, \beta_2 \} = \{ q_+, q_- \} \).

We show in the left column of Fig. 5 band structures of truncated tight-binding models (for details see Appendix A) associated to \( \mathcal{E}^{(q_+, q_-)} \) for different combinations of \( (q_+, q_-) \), which were generated in a similar way as the three-band models discussed in Sec. VII. By construction, each band structure is composed of two two-band subspaces, \( B_i^I(2) \) and \( B_i^{II}(2) \). The topology of each oriented subspace is characterized by the Euler class, i.e., \( \chi_i = \chi[B_i^I(2)] \in Z \) and \( \chi_{II} = \chi[B_i^{II}(2)] \in Z \), which we compute here as a two-band Wilson loop winding \( [30] \) (middle column of Fig. 5) [73].

Both Euler classes are directly determined by the numbers \( (q_+, q_-) \). While \( (q_+, q_-) \) takes value in a free group, the pair of Euler classes \( (\chi_I, \chi_{II}) \) must satisfy the sum rule \( (\chi_I + \chi_{II}) \mod 2 = 0 [26, 34] [74] \). From the data presented in Fig. 5, we conclude that there is the following homomorphism of groups from the homotopy invariants to the cohomology invariants,

\[
m : (q_+, q_-) \rightarrow \{ \begin{align*}
\chi_I &= q_+ - q_- \\
\chi_{II} &= q_+ + q_-.
\end{align*}
\]

We explain in Appendix C how the homomorphism \( m \) follows from the chosen parametrization of \( SO(4) \). We emphasize that while the parity of the Euler classes \( (\chi_I, \chi_{II}) \) must be equal (by virtue of the Whitney sum formula), their sum does not need to vanish contrary to Chern numbers.

As anticipated, we again observe that the number of noble nodal points within each two-band subspace follows \#NP\( [B_i^I(2)] = 2|\chi_i|, i = I, II \), see the right column of Fig. 5. This does not prevent accidental pairs of nodal points as is found in Fig. 5(e) which exhibits 8 nodal points in the unoccupied two-band subspace while the minimum of only 6 stable nodal points is expected.

Beyond the phases that are represented in Fig. 5, all the other topologically nontrivial phases within \( 0 \leq |\chi_I|, |\chi_{II}| \leq 3 \) can readily be obtained through the transformations

\[
\begin{align*}
(i) \quad (q'_+, q'_-) &= (q_+, -q_-) \quad \text{for} \quad (\chi'_I, \chi'_{II}) = (\chi_{II}, \chi_I), \\
(ii) \quad (q'_+, q'_-) &= (-q_+, q_-) \quad \text{for} \quad (\chi'_I, \chi'_{II}) = (-\chi_I, \chi_{II}), \\
(iii) \quad (q'_+, q'_-) &= (q_+, q_-) \quad \text{for} \quad (\chi'_I, \chi'_{II}) = (-\chi_{II}, \chi_I).
\end{align*}
\]

(The topologically trivial case with \( \chi_{I,II} = 0 \) is easily obtained as a constant Hamiltonian, and therefore not listed in Fig. 5.)

### C. Dropping of orientation

The topology of orientable band structures is classified by the free homotopy classes (see Section IV C) for which there is no canonical definition of an orientation. Therefore, the orientation assumed so far must be dropped. This has the effect of the following reduction of the homotopy classification of band structures (see Section V B)

\[
Z \oplus Z \rightarrow [Z \oplus Z] / \tau, \quad (42)
\]

where the quotient set is defined through the equivalence relation given by the automorphism \( \tau(\beta_+, \beta_-) = (-\beta_+, \beta_-) \) that reverses the orientations of both subbundles \( B_i^I(2) \) and \( B_i^{II}(2) \).

This implies that any two phases with, on the one hand, \( (\chi_I, \chi_{II}) \) and, on the other hand, \( (-\chi_I, -\chi_{II}) \), belong to the same homotopy class, which we write

\[
(\chi_I, \chi_{II}) \sim (-\chi_I, -\chi_{II}). \quad (43)
\]

On the contrary,

\[
(\chi_I, \chi_{II}) \sim (\chi_I, -\chi_{II}) \sim (-\chi_I, \chi_{II}). \quad (44)
\]

Given the sum rule of Euler classes, and given the above reduction, we readily obtain the following list of all equivalence classes of topologically nonequivalent phases that are bounded by \( 0 \leq |\chi_I|, |\chi_{II}| \leq 3 \), and written in terms of Euler class, i.e.,

\[
(\chi_I, \chi_{II}) : \begin{cases}
(0, 0) \\
(1, 1) \sim (1, -1) \\
(2, 0) \sim (2, 0) \\
(0, 2) \sim (0, 2) \\
(1, 3) \sim (1, -3) \\
(3, 1) \sim (3, -1) \\
(-1, 3) \sim (1, -3) \\
(3, 1) \sim (-3, 3) \\
(-3, 3) \sim (3, -3)
\end{cases}, \quad (45)
\]

of which Fig. 5 only presents a subset.

### D. Fragile topology of four-band models

We conclude this section by commenting on the fragility through a repartitioning of the bands. Let us start from the band structure with \( (\chi_I, \chi_{II}) = (2, 0) \) [Fig. 5(d)]. As indicated by its Euler class, the two higher bands have no nodal points and they can be separated by an energy gap [as readily visible in Fig. 5(d)] thus enabling a finer partitioning \( \{B_i^I(2), B_{II}(1), B_{III}(1)\} \).

Then we can lower the band \( IIa \) in energy and close the gap with the block \( I \). The (oriented) classifying space corresponding to the repartitioned bands is \( Gr^+_{3,4} = S^3 \).

Then \( \pi_2[S^3] = 0 \) tells us that the nontrivial fragile topology of the occupied two-band subspace \( (\chi_I = 2) \) has
been trivialized, in agreement with $w_2[B_I(2) \oplus B_{IIa}(1)] = (w_2[B_I(2)] + w_2[B_{IIa}(1)]) \mod 2 = \chi_I \mod 2 + 0 = 0$. This can alternatively be obtained in terms of the tangent bundle to the 3-sphere which is associated to the

FIG. 5. Band structures (left column) associated to the real oriented vector bundles $E^{(q_+, q_-)+}$ based on the Grassmannian $Gr_{2,4}$. Wilson loop flow (middle column) of the lower two-band subspaces (blue) and, when different, of the higher two-band subspaces (dashed red). The Wilson loop winding gives the (unsigned) Euler class $|\chi_I, \chi_{II}|$. The correspondence between the geometric construction and the topology follows the bijection $(\chi_I, \chi_{II}) = (q_+ - q_-, q_+ + q_-)$. Location of the $\#NP = 2\chi_i$ nodal points (right column, black spots) of the $i$-th two-band subspace.
classifying space $S^3$. It is a well established result that $TS^3$ is fully parallelizable, i.e. it is topologically trivial. Then, since $F^+_3 \cong TS^1$, any two-dimensional band structure associated to the pullback bundle $E^+_3 \cong f^*_1 F^+_3$ is also trivial.

IX. CONCLUSION AND DISCUSSION

In this work we have provided a unified perspective on fragile topological phases by addressing structures that emerge when a refined partitioning of bands is taken into account. This topological analysis also underlies the novel braiding properties found in \cite{31,33} when nodes of different bands are considered. The framework rests on direct homotopy evaluations of the relevant Grassmannians using a geometrical construction, which involves the so-called Plücker embedding into the more manageable projective exterior product spaces. This construction does not only provide descriptive power, enumerating the possible topologies on a generic footing, but in reverse also allows for a direct construction of rather tractable models displaying the desired topological features.

We close with pointing out some possible future directions. Firstly, it would be interesting to explore how Grassmannians naturally fit within the more general framework of flag varieties, which naturally describe band structures with multiple band gaps. In fact, this approach allowed us to discuss the trivialization of the two occupied band in Sec. VIII.D. Moreover such generalizations have also been briefly foreshadowed in Secs. VA and VB. For example, we could think here the of concrete case of $Fl_{2,2}^+$, for which we predict a $Z^3$ topology. In this case the total gapped bundle is composed of three vector subbundles, $E_{2,2,6} = B_1(2) \cup B_{II}(2) \cup B_{III}(2)$. The two-band subspaces are characterized by an Euler class, $\chi_i = \chi[B_i(2)] \in Z$, $i = 1, 2, 3$, which must satisfy the sum rule $[\chi_1 + \chi_2 + \chi_3] \mod 2 = 0$ which follows directly from the Whitney sum formula, i.e. the total second SW class must be zero. However, the flag manifold $Fl_{2,2}^+$ has a dimension of 12 which makes explicit parametrization more challenging.

Secondly, our approach suggests a natural generalization to systems in higher (e.g. three) spatial dimensions (by considering higher homotopy groups), and within certain other symmetry classes, corresponding to e.g. complex varieties. Related to this observation, when the third momentum is played by the time dimension, it appears that the language of flag varieties may provide a natural language to describe other classes of topological systems, especially in the case of periodically driven Floquet systems \cite{75–77}. Here, the periodicity of the quasienergy implies that there is no canonical choice of chemical potential, and one often assigns the same importance to all spectral gaps \cite{78}. Another possible generalization arises in the context of non-Hermitian models \cite{79–81}, where non-standard gap conditions were recently investigated using homotopy theory \cite{54,82}. Indeed, as noted above, the topology of a generic two-band non-Hermitian Hamiltonian has been shown by Ref. \cite{52} to be essentially equivalent to the fragile topology of 3-band real-symmetric Hamiltonians discussed here.

The final extension, which is in particular important for the study of materials, concerns the interplay with crystalline symmetries. We have shown in Ref. \cite{30} that a point group of crystalline symmetries can lead to an obstruction on the Wilson loop winding (Euler class) of two-band subspaces. This was proved to be directly rooted in the representation theory of the Wilson loop. All the observations we made here should have a similar natural explanation from the exhaustive topological classification of band structures and their explicit realization. We will report on this in due time at another occasion.

X. ACKNOWLEDGEMENTS

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In contrast, computing the homotopy classes $[T^d, F_{\mathbb{L}(B(1))}]$ is a nontrivial problem. Nevertheless, by restricting to two-dimensional and orientable systems, the topologies of any band structure can be inferred from the second homotopy groups of Grassmannians that are discussed in Sec. IV.

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Alternatively, the tautological vector bundle can be defined in the following way. First, the band vector subspaces can be defined as the range (i.e. image) $V_i = \text{ran} P_i$, with the projectors $P_i = R_i R_i^T$, for $i = I, II$. Then, $V_{II} = \text{ran} P_{II} = \text{ran} Q_I$ with $Q_I = I_N - \mathbb{P}_r$. Thus, $\mathcal{F}_{I,P,N}^+ = \bigcup_{[R] \in \mathbb{C}(I,P,N)} \text{ran} P_I$ and $\mathcal{F}_{II,N-P,N}^+ = \bigcup_{[R] \in \mathbb{C}(I,N-P,N)} \text{ran} (I_N - \mathbb{P}_r)$. 

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Equivalently, the right-hand side of Eq. (27) is the determinant of a matrix $A$ with elements $A_{ij}(v_i, v_j')$.

The inverse map, i.e., the one that assigns to any simple $p$-vector with unit norm a unique oriented $p$-plane, can be obtained by considering the QR decomposition of the matrix $(v_1, \ldots, v_p)$ representing the $p$-vector.

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The Wilson loop winding gives the unsigned Euler class $|\chi|$. The sign invariant can be obtained from the winding of the Pfaffian of the Wilsomian Hamiltonian Ref. [33].

This follows from the sum rule for the second Stiefel-Whitney class of the total bundle, namely $w_2(B_1(2)) \oplus B_1(2)) = (w_2(B_1(2)) + w_2(B_1(2))) \mod 2 = (\chi(B_1(2)) + \chi(B_1(2))) \mod 2 = 0$.

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Appendix A: Tight-binding models

In order to get an explicit tight-binding model, we first sample $H(k)$ over a grid $\Lambda^*$ in the Brillouin zone and perform an inverse discrete Fourier transform (FT). This gives us the hopping matrix elements $t_{\mu\nu}(R_j - 0) = FT[\{H_{\mu\nu}(k_m)\}_{m\in\Lambda^*}]\{R_j - 0\}$. Typically, the hopping elements decay exponentially and we truncate them beyond a finite support including a few neighbors $R_j$ around the center 0. The three-band example with Euler class 2, and all the four-band examples shown in Section VIII are truncated beyond the second neighbors in both directions, i.e. $t_{\mu\nu}(R_j - 0) = 0$ for all $R_j \in \{n_1a_1 + n_2a_2\}_{n_1, n_2 \neq 0, 1, 2, 3}$, while the three-band example with Euler class 4 has been truncated beyond the third neighbors in both directions, i.e. $t_{\mu\nu}(R_j - 0) = 0$ for all $R_j \in \{n_1a_1 + n_2a_2\}_{n_1, n_2 \neq 0, 1, 2, 3}$.

Appendix B: Euler class reversal in $\mathbb{RP}^2$

For completeness, in this Appendix we reproduce from Ref. [52] the continuous and adiabatic transformation that reverses the Euler class of the two-band oriented subbundle of an orientable gapped three-band model classified by $\mathbb{RP}^2$, hence realizing the automorphism $\triangleright_\pi : \chi \rightarrow -\chi$ of Section IV C.

Our representative Hamiltonians of orientable gapped three-band systems [cf. Eqs. (21) and (36)] can conveniently be parametrized as [26]

$$H[n](\theta, \phi) = 2n(\theta, \phi) \cdot n(\theta, \phi)^T - I_3,$$  \hspace{1cm} (B1)

with $n(\theta, \phi) = u_3 \in S^2$ the unit eigenvector of the single unoccupied band, and $n = n_1e_1 + n_2e_2 + n_3e_3$.

Since $H[n]$ is explicitly invariant under the inversion $n \rightarrow -n$ there is not canonical signed Euler class associated with the Hamiltonian. The indeterminacy can be lifted though by assigning a smooth structure to the vector field $\{n(\theta, \phi)\}_\theta \in S^2$, which is allowed by virtue of the triviality of any real line bundle defined over the sphere (see Section IV D).

As in Section VII this is achieved by setting [26] $n(\theta, \phi) = e_r = (\cos \phi_q \sin \theta_q, \sin \phi_q \sin \theta_q, \cos \theta_q) \in S^2$, where $q \in \mathbb{Z}$ defines the number of times $n$ wraps around the sphere [cf. Eq. (32)]. The Euler class of the oriented occupied two-band subbundle is then $\chi_0 = 2q \in 2\mathbb{Z}$. We have thereby promoted the Hamiltonian Eq. (B1) to an oriented total gapped bundle with a well defined Euler class.

Since we are interested in an automorphism of the based homotopy group $\pi_2(\mathbb{RP}^2)$ it is crucial to specify a chosen base point that will serve as a reference for comparing any two elements of the group. Let us fix $n(\theta = 0, \phi = 0) = e_3$ at the blue pole of the sphere [Fig. 2].

Defining the rotation

$$S(s) = \begin{pmatrix} \cos s & 0 & -\sin s \\ 0 & 1 & 0 \\ \sin s & 0 & \cos s \end{pmatrix}.$$  \hspace{1cm} (B2)

for $s \in [0, \pi]$, we obtain a smooth deformation of the Hamiltonian through

$$H[S(s) \cdot n] = 2(S(s) \cdot n) \cdot (S(s) \cdot n)^T - I_3,$$  \hspace{1cm} (B3)

which is adiabatic (i.e. it preserves the gap between the eigenvalues) since it can be rewritten as the change of basis, i.e.

$$H[S(s) \cdot n] = S(s) \cdot H[n] \cdot S(s)^T.$$  \hspace{1cm} (B4)

Exploiting the gauge freedom of the Hamiltonian (i.e. $H[n] = H[\bar{n}]$), we eventually find

$$H[-S(\pi) \cdot n] = 2(-S(\pi) \cdot n) \cdot (-S(\pi) \cdot n)^T - I_3,$$  \hspace{1cm} (B5)

which at $s = \pi$ preserves the base point of the original Hamiltonian, i.e. $S(\pi) \cdot n(0, 0) = n(0, 0) = e_3$. Furthermore, we find

$$-S(\pi) \cdot n = n_1e_1 - n_2e_2 + n_3e_3,$$  \hspace{1cm} (B6)

such that the Euler class of $H[-S(\pi) \cdot n]$ is $\chi_\pi = -\chi_0 = -2q$. Therefore, at $s = \pi$ the transformation realizes the automorphism $\triangleright_\pi : \chi_0 \rightarrow -\chi_0$.

As a conclusion, the above construction defines the continuous deformation of Hamiltonian

$$\triangleright(s) : \mathbb{RP}^2 \rightarrow \mathbb{RP}^2 : H[n] \rightarrow H[-S(s) \cdot n].$$  \hspace{1cm} (B7)

with $\triangleright(0) = \text{id}$ and $\triangleright(\pi) = \triangleright_\pi$. Then, keeping track of $H[n]$ at the base point $(\theta, \phi) = (0, 0)$ through the deformation, i.e. $\{\triangleright(s)H[n](0, 0)\}_{s \in [0, \pi]}$, this defines a non-contractible loop within $\mathbb{RP}^2$, i.e. a generator of $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ [52].

Appendix C: Plücker embedding for $\text{Gr}_{2,4}^+$

In this Appendix we derive the explicit Plücker embedding for $\text{Gr}_{2,4}^+ = \text{SO}(4)/[\text{SO}(2) \times \text{SO}(2)] \cong S^2 \times S^2$. We do it starting from the parametrizations of $\text{SO}(4)$ in terms of the Lie algebra of real and anti-symmetric matrices.
1. Parametrization of $SO(4)$

A matrix $R \in SO(4)$ can be decomposed as $[83]$ $R = Q_c U_r$ with

$$Q_c = \exp\left( \begin{array}{cc} C & D \\ -C & D \end{array} \right), \quad U_r = \begin{pmatrix} \text{Re} U & -\text{Im} U \\ \text{Im} U & \text{Re} U \end{pmatrix},$$

(C1)

where $C$ and $D$ are arbitrary real antisymmetric matrices, and $U_r$ is a generic matrix in $U(2)$. $Q_c$ can be parametrized as $[83]$

$$Q_c = \begin{pmatrix} \cos \rho & \sin \rho & 0 & \sin \rho \cos \xi \\ -\sin \rho \sin \xi & \cos \rho & -\sin \rho \cos \xi & 0 \\ -\sin \rho \cos \xi & 0 & \cos \rho & -\sin \rho \sin \xi \\ \sin \rho \sin \xi & \cos \rho & 0 & \sin \rho \cos \xi \end{pmatrix},$$

(C2)

with the angle $\rho = \sqrt{c^2 + d^2}$, where $c = \text{Pf}[C]$ and $d = \text{Pf}[D]$, and an other angle defined through $\cos \xi = c/\rho$ and $\sin \xi = d/\rho$. The range of these angles are $\rho, \xi \in [0, 2\pi)$. A generic matrix $U_r \in U(2)$ can be decomposed as

$$U_r = e^{i\varphi/2} \begin{pmatrix} e^{i\psi_1} \cos \psi & e^{i\psi_2} \sin \psi \\ -e^{-i\psi_2} \sin \psi & e^{-i\psi_1} \cos \psi \end{pmatrix},$$

(C3)

with the angles $\varphi, \phi_1, \phi_2 \in [0, 2\pi)$ and $\psi \in [0, \pi)$. This results in

$$R(\rho, \xi, \phi, \phi_1, \phi_2) = Q_c(\rho, \xi) U_r(\varphi, \psi, \phi_1, \phi_2) \in SO(4).$$

(C4)

We now need the constraints among the six angles, $\{\rho, \xi, \phi, \psi, \phi_1, \phi_2\}$, as to only cover the quotient space $SO(4)/[SO(2) \times SO(2)] \cong S^2 \times S^2$. Before doing so, we first review the diffeomorphism of spaces $SO(4)/[SO(2) \times SO(2)] \cong S^2 \times S^2$. The readers familiar with the Plücker embedding may jump to the solution Eq. (C15).

2. $SO(4)/[SO(2) \times SO(2)] \cong S^2 \times S^2$

We review here the standard result $SO(4)/[SO(2) \times SO(2)] \cong S^2 \times S^2$ obtained through the Plücker embedding $[84]$. This section follows the argument of $[84]$ with a few more steps.

The Plücker embedding $\iota : \mathbb{Gr}_2^+(R^4) \rightarrow \Lambda^2 R^4$ represents the points of the oriented Grassmannian as elements of the second exterior power of $R^4$, $\Lambda^2 R^4$, that is a vector space of dimension $(\frac{1}{2})$ is 6 spanned by bivectors, i.e. the exterior product $\langle \cdot, \cdot \rangle$. More specifically, for $x \in \Lambda^2 R^4$ the image of the Plücker embedding is defined by the solutions to the system

$$x \wedge x = 0, \quad |x|^2 = 2,$$

(C5)

where the norm $|\cdot, \cdot| = \sqrt{\langle \cdot, \cdot \rangle}$ is defined in terms of a inner product in $\Lambda^2 R^4$, see below.

Let us take $(u_1, u_2, u_3, u_4)$ an oriented orthonormal frame of $R^4$. There is a bijection between any oriented plane $V \subset R^4$ and an element $u_1 \wedge u_2 \in \Lambda^2 R^4$, given that $V$ is spanned by the orthonormal frame $(u_1, u_2)$. The orthogonal complement $V^c = \{u \in R^4| \langle u, v \rangle = 0, \forall v \in V\}$ is then represented by the Hodge dual $\star(u_1 \wedge u_2) = u_3 \wedge u_4$.

We have $\star(\alpha u_1 \wedge u_2 \pm \beta u_3 \wedge u_4) = \pm(\beta u_1 \wedge u_2 \pm \alpha u_3 \wedge u_4) \in \Lambda^2 R^4$, $\alpha, \beta \in R$. Thus, the $\pm 1$-eigenspaces of the Hodge star $\star$, which we note $\Lambda_+^2$ and $\Lambda_-^2$, are composed of elements of the form $v_+ = \alpha (u_1 \wedge u_2 \pm u_3 \wedge u_4)$. These are perpendicular with respect to the exterior and the inner products, i.e. $v_+ \wedge v_- = \langle v_+, v_- \rangle = 0$ for $v_+ \in \Lambda_+^2$ and $v_- \in \Lambda_-^2$. Let $V_+ = \langle a, b, c, d \rangle = \langle a, c \rangle \langle b, d \rangle - \langle a, d \rangle \langle b, c \rangle$ with $a, b, c, d \in R^4$.

Choosing a Cartesian frame for $R^4$, each column vector of $R = (u_1 u_2 u_3 u_4) \in SO(4)$ reads

$$u_i = u_1^i \hat{e}_1 + u_2^i \hat{e}_2 + u_3^i \hat{e}_3 + u_4^i \hat{e}_4, \quad \text{for } i = 1, 2, 3, 4, \quad \text{and with } \hat{e}_i = \delta_{ij} \

(C6)

We then choose a reference basis for $\Lambda^2 R^4$,

$$\{\hat{e}_1, \hat{e}_2, \hat{e}_3, \hat{e}_4, \hat{e}_5, \hat{e}_6\} = \{\hat{e}_4 \wedge \hat{e}_2, \hat{e}_3 \wedge \hat{e}_1, \hat{e}_1 \wedge \hat{e}_2, \hat{e}_4 \wedge \hat{e}_1, \hat{e}_2 \wedge \hat{e}_4, \hat{e}_3 \wedge \hat{e}_4\},$$

(C7)

and compute the elements $v_+ = u_1 \wedge u_2 + u_3 \wedge u_4 \in \Lambda_+^2$ and $v_- = u_1 \wedge u_2 - u_3 \wedge u_4 \in \Lambda_-^2$.

3. From $SO(4)$ to $SO(4)/[SO(2) \times SO(2)]$

The previous section provides the guidelines for the derivation of the constraints Eq. (C15) that map the elements of $SO(4)$ to the elements of $SO(4)/[SO(2) \times SO(2)]$.

Choosing a Cartesian frame for $R^4$, each column vector of $R = (u_1 u_2 u_3 u_4) \in SO(4)$ reads
For the parametrization $R(\rho, \xi, \varphi, \psi, \phi_1, \phi_2)$ derived in Eq. (C4), we find
\begin{align}
v_+ &= (\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6) \cdot (a \ b \ c \ a \ b \ c)^T, \\
v_- &= (\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6) \cdot (d \ e \ f - d - e - f)^T,
\end{align}
with
\begin{align*}
a &= \cos(\psi)^2 \sin(2\phi_1) + \sin(\psi)^2 \sin(2\phi_2), & d &= \cos(\rho)^2 \sin(\varphi) + \sin(\rho)^2 \sin(\varphi + 2\xi), \\
b &= \sin(2\psi) \sin(\phi_1 - \phi_2), & e &= -\cos(\varphi + \xi) \sin(2\rho), \\
c &= \cos(\psi)^2 \cos(2\phi_1) + \sin(\psi)^2 \cos(2\phi_2), & f &= \cos(\rho)^2 \cos(\varphi) - \sin(\rho)^2 \cos(\varphi + 2\xi),
\end{align*}
Note that $\langle v_+, v_- \rangle = ad + be + cf - ad - be - cf = 0$.

Let us make the following change of basis for $\Lambda^2 \mathbb{R}^4$,
\begin{align}
\hat{e}'_1 &= \hat{e}_1 + \hat{e}_4, & \hat{e}'_2 &= \hat{e}_2 + \hat{e}_5, & \hat{e}'_3 &= \hat{e}_3 + \hat{e}_6, \\
\hat{e}'_4 &= \hat{e}_1 - \hat{e}_4, & \hat{e}'_5 &= \hat{e}_2 - \hat{e}_5, & \hat{e}'_6 &= \hat{e}_3 - \hat{e}_6,
\end{align}
which we rewrite as
\begin{equation}
(\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_4 \hat{e}_5 \hat{e}_6) = (\hat{e}'_1 \hat{e}'_2 \hat{e}'_3 \hat{e}'_4 \hat{e}'_5 \hat{e}'_6) \cdot S, \quad \text{with } S = \frac{1}{2} \begin{pmatrix} 1_{3 \times 3} & 1_{3 \times 3} \\ -1_{3 \times 3} & 1_{3 \times 3} \end{pmatrix}.
\end{equation}
In the new basis, we then get
\begin{align}
v_+ &= (\hat{e}'_1 \hat{e}'_2 \hat{e}'_3 \hat{e}'_4 \hat{e}'_5 \hat{e}'_6) \cdot S \cdot (a \ b \ c \ a \ b \ c)^T, \\
&= (\hat{e}'_1 \hat{e}'_2 \hat{e}'_3 \hat{e}'_4 \hat{e}'_5 \hat{e}'_6) \cdot (a \ b \ c \ 0 \ 0 \ 0)^T = (a, b, c, 0, 0, 0), \\
v_- &= (\hat{e}'_1 \hat{e}'_2 \hat{e}'_3 \hat{e}'_4 \hat{e}'_5 \hat{e}'_6) \cdot S \cdot (d \ e \ f - d - e - f)^T, \\
&= (\hat{e}'_1 \hat{e}'_2 \hat{e}'_3 \hat{e}'_4 \hat{e}'_5 \hat{e}'_6) \cdot (0 \ 0 \ 0 \ d \ e \ f)^T = (0, 0, 0, d, e, f),
\end{align}
i.e. this basis emphasizes that $v_+$ and $v_-$ live in three-dimensional orthogonal complements $V_\pm$ of the six-dimensional vector space $\Lambda^2 \mathbb{R}^4 = V_+ \oplus V_-$. The restriction of $v_+ = (a, b, c)$ and $v_- = (d, e, f)$ to unit spheres is then obtained through
\begin{equation}
\phi_1 = \phi_2 = \theta_+/2, \quad \psi = \phi_+/2, \quad \rho = \theta_-/2, \quad \varphi = -\xi = \phi_-,
\end{equation}
with the spherical angles $(\phi_+(-), \theta_+(-)) \in [0, 2\pi) \times [0, \pi] \cong S^2_+(-)$. Indeed, substituting Eq. (C15) we find in the basis $\{\hat{e}'_i\}_{i=1,\ldots,6}$,
\begin{align*}
v_+ &= (a, b, c, 0, 0, 0) = (\cos \phi_+ \sin \theta_+, \sin \phi_+ \sin \theta_+, \cos \theta_+, 0, 0, 0) \\
&\in S^2_+ \subset \text{Span}\{\hat{e}'_1, \hat{e}'_2, \hat{e}'_3\}, \\
v_- &= (0, 0, 0, d, e, f) = (0, 0, 0, \sin \phi_- \cos \theta_-, -\sin \theta_-, \cos \phi_- \cos \theta_-) \\
&\in S^2_- \subset \text{Span}\{\hat{e}'_4, \hat{e}'_5, \hat{e}'_6\}.
\end{align*}
The inverse Plücker embedding then gives a bijection,
\begin{equation}
\iota^{-1}: S^2_+ \times S^2_- \to \text{SO}(4)/[\text{SO}(2) \times \text{SO}(2)] : (\phi_+, \theta_+, \phi_-, \theta_-) \mapsto [R(\phi_+, \theta_+, \phi_-, \theta_-)],
\end{equation}
that we use in Section VIII for building explicit tight-binding models for all homotopy classes.