1. Introduction

The problem of taking the continuum limit remains to be the most important issue in the lattice calculation of $B_K$ using the quenched Kogut-Susskind quark action. We have been making effort to elucidate the effect of scaling violation over two years since 1995. Our early study carried out at $\beta = 5.85 - 6.2$ showed a linear decrease of $B_K$ in $a$ in contrast to $O(a^2)$ scaling violation theoretically predicted. The run was then extended to $\beta = 6.4$, at which a departure from a linear behavior was observed, indicating the onset of $O(a^2)$ behavior. This implied a cancellation between an $a^2$ and higher order terms that leads to an apparent $O(a)$ behavior for lower values of $\beta$. We have now extended the run to $\beta = 6.65$ employing a $56^3 \times 96$ lattice to settle the issue of the continuum extrapolation. We also briefly address the problem of $O(a^2)$ effect, on which we gained insight after the present conference.

2. Perturbative matching

We have slightly revised the method of analysis in obtaining $B_K$ in the continuum since Lattice 96. Lattice values of $B_K$ are converted to the continuum value in the $\overline{MS}$ scheme with the naive dimensional regularization (NDR) by applying one-loop renormalization at the matching scale $q^* = 1/a$.

The one-loop renormalization factor is evaluated with the 3-loop running coupling constant $\alpha_{\overline{MS}}(q^*)$ with $\Lambda_{\overline{MS}} = 0.23$ GeV, estimated in the continuum limit from our results for the $\rho$ meson mass.

The continuum value of $B_K$ at the physical scale $\mu = 2$ GeV is calculated from $q^*$ via the 2-loop running of the continuum renormalization group:

$$B_K(\text{NDR}, \mu) = \left[ 1 - \frac{\alpha_{\overline{MS}}(q^*)}{4\pi} \gamma_1 \beta_0 - \gamma_0 \beta_1 \right]^{\frac{1}{2\beta_0}} B_K(\text{NDR}, q^*)$$

$$\times \left[ \frac{\alpha_{\overline{MS}}(q^*)}{\alpha_{\overline{MS}}(\mu)} \right]^{-\gamma_0/2\beta_0} \text{ with } \beta_0 = 11, \beta_1 = 102, \gamma_0 = 4 \text{ and } \gamma_1 = -7.$$

Results for Quenched $B_K$ from JLQCD*

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A report is presented on our continued effort to elucidate the continuum limit of $B_K$ using the quenched Kogut-Susskind quark action. By adding to our previous simulations one more point at $\beta = 6.65$ employing a $56^3 \times 96$ lattice, we now confirm the expected $O(a)$ behavior. This implied a cancellation $O(a^2)$ behavior for lower values of $\beta$. We have now extended the run to $\beta = 6.65$ employing a $56^3 \times 96$ lattice to settle the issue of the continuum extrapolation quadratic in $a$ leads to $B_K$ (NDR, 2 GeV) = 0.598(5). As our final value of $B_K$ in the continuum we present $B_K$ (NDR, 2 GeV) = 0.628(42), as obtained by a fit including an $\alpha_{\overline{MS}}(1/a)^2$ term arising from the lattice-continuum matching with the one-loop renormalization.

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The new data points strongly support an onset of which was noted in Lattice behavior, the onset of which however should vanish logarithmically in the continuum limit[9,10].

4. Continuum extrapolation

In Fig. 1 we present \( B_K(NDR, 2\, \text{GeV}) \) as a function of \( m_{\rho}a \). The leftmost points are the new data taken at \( \beta = 6.65 \) on a \( 56^3 \times 96 \) lattice. The new data points strongly support an \( O(a^2) \) behavior, the onset of which was noted in Lattice 96 at \( \beta = 6.4 \). The effect of higher order terms is not discernible for \( \beta \geq 5.93 \). Therefore, we fit the five points above \( \beta = 5.93 \) with the form \( B_K = c_0 + c_1 (m_{\rho}a)^2 \), which is shown by dashed lines in Fig. 1. The continuum extrapolation gives \( B_K(NDR, 2\, \text{GeV}) = 0.616(5) \) for the gauge non-invariant operator, and 0.580(5) for the invariant operator.

Figure 1. Gauge non-invariant(circles) and invariant(diamonds) \( B_K(NDR, 2\, \text{GeV}) \) as a function of \( m_{\rho}a \), together with a simultaneous fit for the two operators including \( \alpha^2 \) term (solid lines) and separate fits quadratic in \( \alpha \) (dashed lines) to the five pairs of data points for \( \beta \geq 5.93 \), the average of the two being 0.598(5).

5. Operator dependence

It has been noted that the two operators yield different values in the continuum[3] (see Fig. 1). After Lattice 97 a further analysis was made on this point, which we report in this write-up. The difference between gauge non-invariant and invariant operators should receive not only \( O(a^2) \) scaling violation but also \( \alpha MS(q^*)^2 \) errors from the matching procedure. Figure 2 plots this difference as a function of \( m_{\rho}a \). Small errors resulting from a correlation between the two operators allow us to fit the five data points with the form \( b_1 (m_{\rho}a)^2 + b_2 \alpha MS(q^*)^2 \), giving \( b_1 = -0.23(2) \) and \( b_2 = 1.73(5) \) with \chi \^2/d.o.f. = 2.2. The solid line indicates the fit, and others show the breakdown into the \( \alpha^2 \) (dotted line) and \( \alpha^2 \) (dashed line) contributions. Allowing for a non-zero constant \( b_0 \), similar fitting gives \( b_0 = -0.032(16) \), \( b_1 = -0.44(11) \) and \( b_2 = 3.4(8) \).

Encouraged by this analysis we attempt to fit the five points at \( \beta \geq 5.93 \) simultaneously for both operators including their correlations, with
6. Conclusions

As our final value of $B_K$ in the continuum limit we adopt the result from the fit including the $\alpha^2$ term, $B_K(\text{NDR, 2GeV}) = 0.628 \pm 0.042$, which includes a systematic error due to the 2-loop uncertainty. The size of the quoted error is 6.6%, which roughly equals $3 \times \alpha_{\overline{\text{MS}}}^{\text{inv}}(q^* = 1/a)^2$ at our smallest lattice spacing $1/a = 4.87$ GeV at $\beta = 6.65$ where $\alpha_{\overline{\text{MS}}}(4.87 \text{GeV}) = 0.147$. This magnitude of error is unavoidable unless a two-loop calculation is carried out for the lattice renormalization.

Our $B_K$ is consistent with the JLQCD value obtained using the Wilson quark action, $B_K(\text{NDR, 2GeV}) = 0.562 \pm 0.064$, in which the operator mixing problem is solved non-perturbatively with the aid of chiral Ward identities.[12]

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