Galaxy rotation curves. The theory.

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Abstract

The non-gauge vector field with as simple as possible Lagrangian (1) turned out an adequate tool for macroscopic description of the main properties of dark matter. The dependence of the velocity of a star on the radius of the orbit $V(r)$ – galaxy rotation curve – is derived analytically from the first principles completely within the Einstein’s general relativity. The Milgrom’s empirical modification of Newtonian dynamics in nonrelativistic limit (MOND) gets justified and specified in detail. In particular, the transition to a plateau is accompanied by damping oscillations. In the scale of a galaxy, and in the scale of the whole universe, the dark matter is described by a vector field with the same energy-momentum tensor. It is the evidence of the common physical nature. Now, when we have the general expression (9) for the energy-momentum tensor of dark matter, it is possible to analyze its influence on the structure and evolution of super heavy stars and black holes.
I. INTRODUCTION

The “galaxy rotation curves” problem appeared after J. H. Oort discovered the galactic halo, a group of stars orbiting the Milky Way outside the main disk [1]. In 1933, F. Zwicky [2] postulated ”missing mass” to account for the orbital velocities of galaxies in clusters. Persistent investigations by Vera Rubin [3] in seventies finally dispelled the skepticism about the existence of dark matter on the periphery of the galaxies.

Among numerous attempts to solve the problem of galaxy rotation curves the most discussed one is an empirical explanation named MOND (Modified Newtonian Dynamics), proposed by Milgrom back in 1983 [4]. For a relativistic justification of MOND Bekenstein [5], Sanders [6], Brownstein and Moffat [7],[8] introduce additional scalar, vector, or tensor fields. Though these (and many others) relativistic improvements of MOND are able to fit a large number of samples for about a hundred galaxies, the concern still remains. So far we had neither self-consistent description of the dark sector as a whole, nor direct derivation of MOND from the first principles within the Einstein’s general relativity. The survey [9] by Benoît Famaey and Stacy McGaugh reflects the current state of the research and contains the most comprehensive list of references.

From my point of view the approach to the theory of dark sector based on the utilization of vector fields in general relativity is very promising, and its abilities are not yet exhausted. Vector fields with the simplest Lagrangian

\[ L = a \left( (\phi^K_K)^2 - m^2\phi^K\phi_K \right) - V_0 \]  

(1)

allowed me to describe macroscopically the main features of evolution of the Universe completely within the frames of the Einstein’s theory of general relativity [10]. The longitudinal non-gauge massive vector field displays the repulsive elasticity. As a result the Big Bang turns into a regular inflation-like state of maximum compression with the further accelerated expansion at late times. The parametric freedom of the theory allows to forget the fine tuning troubles. At the scales much larger than the distances between the galaxies the Universe is homogeneous and isotropic. Its temporal evolution depends on time only. Currently the characteristic rate of its expansion is determined by the Hubble parameter which is of the order of the inverse time from the big bang. In the much smaller galactic scales the situation is just the opposite. The space structure is essentially nonhomogeneous, while the influence of expansion is negligible.
In what follows I present the macroscopic theory of dark matter, including the derivation of galaxy rotation curves, directly from the first principles within the minimal Einstein’s general relativity. In the galactic scale the longitudinal non-gauge vector field with the same Lagrangian (1) not only fits the observed rotation curves, but also opens a promising approach to understand the origin of the substance that we name dark matter. In the non-relativistic limit the expression (26) derived analytically, justifies and specifies the empirical modification of Newtonian dynamics by M. Milgrom [4].

II. VECTOR FIELD IN GENERAL RELATIVITY

In general relativity, the Lagrangian of a vector field \( \phi_I \) consists of the scalar bilinear combinations of its covariant derivatives and a scalar potential \( V(\phi^K \phi_K) \). A bilinear combination of the covariant derivatives is a 4-index tensor \( S_{IKLM} = \phi_I;K \phi_L;M \). The most general form of the scalar \( S \), formed via contractions of \( S_{IKLM} \), is

\[
S = (ag^{IK}g^{LM} + bg^{IL}g^{KM} + cg^{IM}g^{KL})S_{IKLM},
\]

where \( a, b, \) and \( c \) are arbitrary constants. The general form of the Lagrangian of a vector field \( \phi_I \) is

\[
L = a(\phi^M_i \phi^M_i)^2 + b\phi^L_i \phi^M_i \phi^L_i \phi^M_i + c\phi^L_i \phi^M_i \phi^L_i \phi^M_i - V(\phi^M_i \phi^M_i).
\] (2)

The classification of vector fields \( \phi_I \) is most convenient in terms of the symmetric \( G_{IK} = \frac{1}{2}(\phi_{I;K} + \phi_{K;I}) \) and antisymmetric \( F_{IK} = \frac{1}{2}(\phi_{I;K} - \phi_{K;I}) \) parts of the covariant derivatives. The Lagrangian (2) gets the form

\[
L = a(G^M_i)2 + (b + c)G^L_i G^M_i + (b - c)F^L_i F^M_i - V(\phi^M_i \phi^M_i).
\]

The bilinear combination of antisymmetric derivatives \( F^L_i F^M_i \) is the same as in electrodynamics. It becomes clear in the common notations \( A_I = \phi_I/2, F_{IK} = A_{I,K} - A_{K,I} \).

The terms with symmetric covariant derivatives deserve special attention. In applications of the vector fields to elementary particles in flat space-time the divergence \( \frac{\partial \phi^K}{\partial x^K} \) is artificially set to zero [11]:

\[
\frac{\partial \phi^K}{\partial x^K} = 0.
\] (3)

This restriction allows to avoid the difficulty of negative contribution to the energy. In the electromagnetic theory it is referred to as Lorentz gauge. The negative energy problem in application to the galaxy rotation curves in view of a precaution against instability of the
vacuum was discussed by J. Bekenstein [5]. However in general relativity (in curved space-time) the energy is not a scalar, and its sign is not invariant against the arbitrary coordinate transformations. From my point of view, considering vector fields in general relativity, it is worth getting rid of the restriction (3), using instead a more weak condition of regularity.

The covariant field equations

$$a \phi^K_{;K;I} + b \phi^{;IK}_{;I;K} + c \phi^K_{;I;K} = -V' \phi_I$$

(4)

and the energy-momentum tensor

$$T_{IK} = -g_{IK} L + 2V' \phi_I \phi_K + 2a g_{IK} (\phi^M_M \phi^L_L) + 2(b + c) [(G_{IK} \phi^L)_{;L} - G^L_K F_{IL} - G^L_I F_{KL}] + 2(b - c) (2 F^L_I F_{KL} - F^L_K \phi_I - F^L_{I;L} \phi_K)$$

(5)

describe the behavior of a vector fields in the background of any arbitrary given metric $g_{IK}$ [12]. Here $V' \equiv \frac{dV}{d(\phi_M \phi^M)}$.

If the back reaction of the field on the curvature of space-time is essential, then the metric obeys the Einstein equations

$$R_{IK} - \frac{1}{2} g_{IK} R + \Lambda g_{IK} = \kappa T_{IK}$$

(6)

with (5) added to $T_{IK}$. Here $\Lambda$ and $\kappa$ are the cosmological and gravitational constants, respectively. With account of back reaction the field equations (4) are not independent. They follow from the Einstein equations (6) with $T_{IK}$ (5) due to the Bianchi identities. The field equations (4) are linear with respect to $\phi$ if the vector field is small, and the terms with the second and higher derivatives of the potential $V (\phi_M \phi^M)$ can be omitted.

III. DARK MATTER DESCRIBED BY A VECTOR FIELD

In curved space-time there is no invariance against the order of covariant differentiation:

$$\phi^K_{;K;L} - \phi^K_{;L;K} = \phi^M R_{ML}.$$
only. The opposite case \( a = 0 \), and \( b \neq 0, c \neq 0 \) corresponds to either electromagnetic field \((c = -b)\), or to vector particles \((b \neq 0, c = 0)\). This way the dark matter and the ordinary matter are separated from one another, so that the ordinary matter is not taken into account twice. The dark matter is described by the Lagrangian

\[
L_{\text{dm}} = a(\phi_{iM}^2)^2 - V(\phi_{M}\phi^M). \tag{7}
\]

Thereafter the field equation (4) and the energy-momentum tensor of the vector field (5) reduce to

\[
a \frac{\partial \phi_{iM}^M}{\partial x^i} = -V'\phi_I, \tag{8}
\]

\[
T_{\text{dm}IK} = g_{IK} \left[ (\phi_{iM}^M)^2/a + V \right] + 2V' \left( \phi_I \phi_K - g_{IK} \phi_M^M \phi^M \right). \tag{9}
\]

Though the dark matter displays itself by curving the space-time, its physical nature remains unclear so far. We don’t know the dependence \( V(\phi_{M}\phi^M) \). If the vector \( \phi_I \) remains small enough to neglect the second and higher derivatives of \( V(\phi_{M}\phi^M) \), then the parameter

\[
m^2 = \left| \frac{V'(0)}{a} \right|
\]

characterizes the field. As usual it is designated as the square of mass. In accordance with (8) the dimension of \( m \) is \( cm^{-1} \). The covariant divergence \( \phi_{iM}^M \) is a scalar, and in accordance with the equation (5) the massive \((m \neq 0)\) field has a potential: it is a gradient of a scalar.

So far there is no evidence of any direct interaction between dark and ordinary matter other than via gravitation. The gravitational interaction is described by Einstein equations (6) with

\[
T_{IK} = T_{\text{dm}IK} + T_{\text{om}IK}, \tag{10}
\]

where

\[
T_{\text{om}IK} = (\varepsilon + p) u_I u_K - pg_{IK} \tag{11}
\]

is the well known energy-momentum tensor of macroscopic objects. The energy \( \varepsilon \), pressure \( p \), and temperature \( T \) of the ordinary matter obey the equation of state. If \( T \ll \varepsilon \) the Einstein equations (3) with \( T_{IK} \) (10) together with the equation of state with \( T = 0 \) form a complete set. The field equation (5) is not independent. It is a consequence of the Einstein equations due to Bianchi identities.
IV. GALAXY ROTATION CURVES

Applying general relativity to the galaxy rotation problem it is reasonable to consider a static centrally symmetric metric

\[ ds^2 = g_{IK}dx^I dx^K = e^{\nu(r)} \left( dx^0 \right)^2 - e^{\lambda(r)} dr^2 - r^2 d\Omega^2 \]  \[ (12) \]

with two functions \( \nu (r) \) and \( \lambda (r) \) depending on only one coordinate - circular radius \( r \). Real distribution of the stars and planets in a galaxy is neither static, nor centrally symmetric. However this simplification facilitates analyzing the problem and allows to display the main results analytically. If a galaxy is concentrated around a supermassive black hole, the deviation from the central symmetry caused by the peripheral stars is small.

In the background of the centrally symmetric metric (12) the vector \( \phi^I \) is longitudinal. In accordance with the field equation (8) its only non-zero component \( \phi^r \) depends on \( r \). In view of

\[ g = \det g_{IK} = -e^{\lambda + \nu} r^4 \sin^2 \theta, \quad \frac{1}{\sqrt{-g}} \frac{\partial \sqrt{-g}}{\partial r} = \frac{2}{r} + \frac{\lambda' + \nu'}{2}. \]

the covariant divergence

\[ \phi^r_M = \frac{1}{\sqrt{-g}} \frac{\partial \left( \sqrt{-g} \phi^M \right)}{\partial x^M} = \frac{\partial \phi^r}{\partial r} + \left( \frac{2}{r} + \frac{\lambda' + \nu'}{2} \right) \phi^r. \]  \[ (13) \]

In the “dust matter” approximation \( p = 0 \), and the only nonzero component of the energy-momentum tensor (11) is \( T_{00} = \varepsilon g_{00} \). Whatever the distribution of the ordinary matter \( \varepsilon (r) \) is, the covariant divergence \( T^K_{00} L K \) is automatically zero. In the dust matter approximation the curving of space-time by ordinary matter is taken into account, but the back reaction of the gravitational field on the distribution of matter is ignored. If \( p = 0 \) the energy \( \varepsilon (r) \) is considered as a given function.

In the power series

\[ V(\phi_M \phi^M) = V_0 + V' \phi_M \phi^M + O \left( \left( \phi_M \phi^M \right)^2 \right) \]

\( V_0 = V(0) \) together with the cosmological constant \( \Lambda \) determines the expansion of the Universe. In the scale of galaxies the role of expansion of the Universe as a whole is negligible, and one can set \( \tilde{\Lambda} = \Lambda - \kappa V_0 = 0 \) in the Einstein equations. Omitting the second and higher derivatives of the potential \( V(\phi_M \phi^M) \), we have the Einstein equations as follows (see [13],
Here prime stands for \( \frac{d}{dr} \), except \( V' = \frac{\partial V(\phi_M, \phi_M)}{\partial \phi_M} \). Among the four equations (8), (14-16) for the unknowns \( \phi^r, \lambda, \) and \( \nu \) any three are independent.

Extracting (15) from (14) we get a relation

\[ \nu' + \lambda' = \kappa r e^\lambda \left[ 2 e^\lambda (\phi^r)^2 V' + \epsilon + p \right]. \]  

(17)

With account of (13) and (17) the vector field equation (8) takes the form

\[ \left[ (\phi^r)' + \left( \frac{2}{r} + \frac{\kappa r e^{2\lambda} (\phi^r)^2 V' + \frac{1}{2} \kappa r e^\lambda (\epsilon + p)}{2} \right) \phi^r \right]' = -m^2 e^\lambda \phi^r \]  

(18)

where \( m^2 = -\frac{V'(0)}{a} \). The sign in the r.h.s. of (18) corresponds to the case \( V'(0) > 0, a < 0 \).

Negative \( a \) is taken in accordance with the requirements of regularity in application of the same Lagrangian (7) to the analysis of the role of dark matter in the evolution of the Universe [10]. It is convenient to set \( a = -1 \) in what follows. Hence \( V'(0) = m^2 \). The equations (17) and (18) are derived with no assumptions concerning the strength of the gravitational field.

Excluding \( \lambda' \) from equations (14) and (17), we get the following expression for \( \nu' \):

\[ \nu' = \kappa r e^\lambda \left[ m^2 e^\lambda (\phi^r)^2 + (\phi^r_{t;M})^2 \right] + \frac{e^\lambda - 1}{r} \]

In case of the dust matter approximation \( (p = 0) \):

\[ \nu' = \kappa r e^\lambda \left[ m^2 e^\lambda (\phi^r)^2 + (\phi^r_{t;M})^2 \right] + \frac{e^\lambda - 1}{r} \]

(19)

In a static centrally symmetric gravitational field \( \nu' \) determines the centripetal acceleration of a particle (13), page 323). Without dark matter \( \phi^r = 0 \) (19) gives the Newton’s attractive potential far from the center:

\[ \varphi_N (r) = \frac{1}{2} e^2 \nu (r) \sim -r^{-1}, \quad r \to \infty. \]
The first term in the r.h.s. of (19) appears due to the dark matter. Both terms have the same sign, and the presence of dark matter increases the attraction to the center.

The curvature of space-time caused by a galaxy is small. In the linear approximation the influence of dark and ordinary matter can be separated from one another. For $\lambda \ll 1$ (19) reduces to

$$\nu' = \kappa r \left[ m^2 (\phi^r)^2 + (\phi^M_M)^2 \right] + \frac{\lambda}{r},$$

(20)

where the first term does not contain $\varepsilon$. However, the contribution of dark matter comes from both additives. The vector field equation (18) and the Einstein equation (14) at $\lambda \ll 1$ are simplified:

$$\left(\phi^r\right)'' + \left(\frac{2}{r} + \kappa m^2 r (\phi^r)^2 + \frac{1}{2} \kappa r (\varepsilon + p)\right)\phi^r)' = -m^2 \phi^r$$

(21)

and

$$\lambda' + \frac{\lambda}{r} = \kappa r \left[-(\phi^M_M)^2 + m^2 (\phi^r)^2 + \varepsilon\right].$$

(22)

The boundary conditions for these equations,

$$\phi^r = \frac{1}{3} \phi^r_0 r, \quad \lambda = \frac{1}{3} \kappa (\varepsilon_0 - \phi^2_0) r^2, \quad r \to 0,$$

(23)

are determined by the requirement of regularity in the center. Here $\varepsilon_0 = \varepsilon (0)$.

The term $\frac{1}{2} \kappa r (\varepsilon + p)$ in (21) reflects the interaction of dark and ordinary matter via gravitation. If the curvature of space-time caused by the ordinary matter is small, this term is negligible compared to $2/r$. The nonlinear term $\kappa m^2 r (\phi^r)^2$ is small compared to $2/r$ at $r \to 0$, but at $r \to \infty$, despite of being small, it decreases only a little bit quicker than $2/r$ [24]. Neglecting both nonlinear terms in square brackets, the field equation (21) reduces to

$$\left((\phi^r)' + \frac{2}{r} \phi^r\right)' = -m^2 \phi^r.$$

Its regular solution is

$$\phi^r = \frac{\phi^r_0}{m^3 r^2} (\sin mr - mr \cos mr), \quad \phi^M_M = \phi^r_0 \frac{\sin mr}{mr},$$

(24)

where $\phi^r_0 = \phi^M_M (0)$. Substitution of (24) into (20) results in

$$\nu' (r) = \frac{\kappa (\phi^r)'^2}{m^2 r} f (mr) + \frac{\lambda}{r}, \quad \lambda \ll 1.$$

Function $f (x)$,

$$f (x) = \left(1 - \frac{\sin 2x}{x} + \frac{\sin^2 x}{x^2}\right) = \begin{cases} 
  x^2 - \frac{2}{5} x^4 + \ldots, & x \to 0 \\
  1, & x \to \infty
\end{cases},$$

(25)
is presented in Figure 1 (blue curve).

![Figure 1. Function $f(x)$ (25) - blue curve, and $\Psi(x)$ (29) - green curve.](image1)

![Figure 2. Function $\sqrt{f(x) + \Psi(x)}$ in (30) found analytically coincides with the found numerically.](image2)

The balance of the centripetal $\frac{c^2 \nu'}{2}$ and centrifugal $\frac{V^2}{r}$ accelerations determines the velocity $V$ of a rotating object as a function of the radius $r$ of its orbit:

$$V(r) = \sqrt{\frac{V_{\text{pl}}^2 f(mr)}{c^2} + \frac{c^2}{2} \lambda(r)}, \quad (26)$$

$$V_{\text{pl}} = \sqrt{\frac{c \phi'_{0}}{2 m}} \quad (27)$$

Far from the center $\lambda(r)$ decreases as $1/r$, while $f(mr) \to 1$. The dependence $V(r)$ (26) turns at $r \gtrsim m^{-1}$ from a linear to a plateau with damping oscillations. The plateau appears entirely due to the vector field. At the same time the vector field contributes to $\lambda(r)$ as well. Regular at $r \to 0$ solution of the equation (22) is

$$\lambda(r) = 2 \left( \frac{V_{\text{pl}}}{c} \right)^2 \Psi(mr) + \frac{x}{r} \int_{0}^{r} \varepsilon(r) r^2 dr, \quad \lambda \ll 1. \quad (28)$$

The last term in (28) gives the Newton’s potential. The function

$$\Psi(x) = \frac{1}{x} \int_{0}^{x} \left( \frac{\sin^2 y}{y^2} - \frac{\sin 2y}{y} + \cos 2y \right) dy \quad (29)$$

is shown in Figure 1 (green curve). The radial dependence $V(r)/V_{\text{pl}}$ at \( \varepsilon \to 0 \),

$$V(r)/V_{\text{pl}} = \sqrt{f(mr) + \Psi(mr)}, \quad (30)$$

is shown in Figure 2. In the limit $\lambda \ll 1$ the transition to a plateau due to the dark matter only is a universal function (30).
The plateau value $V_{\text{pl}}$ [27] is connected with the parameter $\phi_0/m$, and the period of oscillations is $2\pi/m$. The form of a plateau allows to restore the value of the parameter $\phi_0 = \phi_{\text{M}}^M (0)$ at $r \to 0$ in the boundary conditions [23]. As far as there is no evidence of any direct interaction of dark and ordinary matter, the origin of specific values $\phi_0$ and $m$ of a particular galaxy depends on what happens in the center. As long as the gravitation is weak, in the linear approximation $\phi_0$, $\varepsilon_0$, and $m$ are free parameters. The values $V_{\text{pl}}$ and $m$ can differ from one galaxy to another. It looks like for each galaxy the values of $V_{\text{pl}}$ and $m$ are driven by some heavy object (may be a black hole, may be a neutron star) located in the center (by the way, supporting the central symmetry of the gravitational field).

Interaction with dark matter via gravitation should affect the equilibrium structure of heavy stars and can shift the collapse boundary. The Einstein equations are not linear. If the gravitation is not weak, there are restrictions on the parameters $\phi_0$ and $\varepsilon_0$ in [23]. If $p \neq 0$ the radial distribution of the ordinary matter and gravitational field are interdependent. In the approximation of cold degenerate relativistic gas it is more convenient to use the chemical potential $\mu_0$ in the boundary conditions instead of $\varepsilon_0$. It is worth reconsidering the equilibrium [14] and collapse [15] of supermassive bodies taking the dark matter into account. However, it is a different story.

Dark matter, described by a vector field with the Lagrangian [1], actually justifies the empirical Milgrom’s hypothesis of MOND - the modified Newton’s dynamics [4]. Naturally, basing only on the intuitive arguments, it was scarcely possible to guess that the transition to a plateau is accompanied by damping oscillations.

V. FITTING

The field itself is zero in the center, $\phi^r (0) = 0$, and the contributions of the dark matter and of the ordinary one are introduced to the boundary conditions [23] by the values $(\phi_0')^2$ and $\varepsilon_0$, respectively.

When there is a plateau, the speed of rotation on the plateau $V_{\text{pl}}$ [27], which is determined from the Doppler shift of spectral lines, provides us with information about the input parameter $\phi_0' = \phi_{\text{M}}^M (0)$ in the boundary conditions. The parameter $m$ is determined by scaling the radial coordinate so that the period of oscillations of $f (x)$ [25] fits the observations. While the distribution of dark matter is characterized unambiguously by the two
parameters $\phi_0'$ and $m$, the situation with the density of the ordinary matter $\varepsilon(r)$ in galaxies is not that clear. Radiation coming from the galaxies does not carry information about cooled non-emitting stars and planets. Just the opposite: the strict fitting could provide us with the distribution of the ordinary matter in galaxies.

In the dust matter approximation and weak gravitational field $\varepsilon(r)$ is an arbitrary given function. To demonstrate the relative role of dark and ordinary matter I use the Gauss distribution for the density of dust matter

$$
\varepsilon(r) = \varepsilon_0 \exp \left( -r^2 / r_0^2 \right).
$$

(31)

Qualitatively a particular form of a monotonically decreasing function $\varepsilon(r)$ is not essential. (Possible existence of a hard core in the center is a special case.) $\varepsilon_0$ is the maximum density in the center, and $r_0$ is the mean radius of a galaxy. Total mass of a galaxy $M \sim \varepsilon_0 r_0^3$. Though the dark and ordinary matter are inputted into the boundary conditions (23) via $\phi_0'$ and $\varepsilon_0 = \varepsilon(0)$, it looks more clearly to demonstrate their relative role using $\varepsilon_0 r_0^3$ (proportional to the total rest energy of a galaxy) instead of $\varepsilon_0$.

In Figures 3 and 4 a blue dashed curve is the rotation curve (30) without ordinary matter. It is the same curve as in Figure 2. In each case the radial scales are specifically chosen to clarify the difference better. Red lines in figures 3 a,b,c are rotation curves with $m^2 r_0^2 = 1, 10, 0.1$, respectively (the three cases where the radius $r_0$ of a galaxy is equal, $\sqrt{10}$ times larger, and $\sqrt{10}$ times smaller then the period $\sim m^{-1}$ of oscillations). The ratio $\frac{\varepsilon_0 r_0^3}{(\phi_0')^2} = 1$. The smaller is $\frac{\varepsilon_0 r_0^3}{(\phi_0')^2}$, the less is the difference between red and blue curves.

**Figure 3a.** Red curve - (26) with $\frac{\varepsilon_0 r_0^3}{(\phi_0')} = 1$, $m^2 r_0^2 = 1$. Blue dashed curve - (30).

**Figure 3b.** Red curve - (26) with $\frac{\varepsilon_0 r_0^3}{(\phi_0')} = 1$, $m^2 r_0^2 = 10$. Blue dashed curve - (30).

**Figure 3c.** Red curve - (26) with $\frac{\varepsilon_0 r_0^3}{(\phi_0')} = 1$, $m^2 r_0^2 = 0.1$. Blue - (30).

Red curves in figures 4 a,b,c are rotation curves for a fixed $\frac{\varepsilon_0 r_0^3}{(\phi_0')} = 10$, and $m^2 r_0^2 = 1$. Red -
1, 10, and 0.1, respectively. As $m^2 r_0^2$ grows, the oscillations are smoothed out, and when it decreases the difference between the curves moves to the center.

Figure 4a. Red curve - (26) with $\frac{\varepsilon_0 r_0^3}{(\phi_0')^2} = 10$, $m^2 r_0^2 = 1$. Blue dashed curve - (30).

Figure 4b. Red curve - (26) with $\frac{\varepsilon_0 r_0^3}{(\phi_0')^2} = 10$, $m^2 r_0^2 = 10$. Blue dashed curve - (30).

Figure 4c. Red curve - (26) with $\frac{\varepsilon_0 r_0^3}{(\phi_0')^2} = 10$, $m^2 r_0^2 = 0.1$. Blue dashed curve - (31).

One can find over a hundred graphs of galaxy rotation curves in the literature, including those displaying the transition to a plateau. One of the often referred to, marked UMa: NGC 3726, is shown in Figure 5a and 5b. Both graphs are taken from different places within the same list [7].

Figure 5a. Fitting by MSTG, practically coinciding with MOND.

Figure 5b. Another fitting by MSTG, slightly different from MOND.

Figure 5c. Points are observations, fitted by (26) together with (28) and (31).

The red points with error bars are the observations. The solid lines are the rotation curves determined from so called MSTG (metric-skew-tensor-gravity ([7], [8])). The solid line in Figure 5a practically coincides with MOND (modified Newton’s dynamics). In Figure 5b it is slightly different from MOND. Other dashed and dotted lines correspond to the ordinary Newton’s dynamics.
The blue curve in Figure 5c shows how (26) together with (28) and (31) fits the observations. The input parameters are $V_{pl} = 158 \frac{\text{Km}}{\text{sec}}, \frac{\kappa \epsilon_0}{m^2} = 0.00000005, mr_0 = 3.78$.

Figure 6 is another example of comparison of fitting by MSTG - MOND (a) and in accordance with (26)-(28)-(31) (b). The input parameters are $V_{pl} = 130 \frac{\text{Km}}{\text{sec}}, \frac{\kappa \epsilon_0}{m^2} = 0.00000005, mr_0 = 2.24$.

![Figure 6a. Solid line is fitting via MSTG and MOND. Dashed – the Newtons’s dynamics.](image)

![Figure 6b. Points are observations, Solid curve is fitting by (26) together with (28) and (31).](image)

Frankly speaking, it is a surprise for me. I did not expect such a coincidence. Deviations at small radii can be related to the presence of an additional strongly gravitating compact object located at the center. As shown in Figures 3c and 4c in case $r_0 \ll m^{-1}$ the initial growing part of the curve $V(r)$ shifts to the center. At the same time, if a heavy object in the center really exists, it supports the central symmetry, and the gravitational field becomes only slightly distorted by other stars and planets of the galaxy.

VI. SUMMARY

The non-gauge vector field with as simple as possible Lagrangian (11) provides the macroscopic description of all major observed properties of the dark sector within the Einstein’s theory of general relativity.

In the galaxy scale the field with the energy-momentum tensor (9) allows to describe analytically the galaxy rotation curves in detail. The formulae (26-29) are derived completely.
within the Einstein’s theory. Thus, there is no need in any modifications of the general relativity to explain the observable plateau in rotation curves.

As I have shown previously [10], the vector fields with the same Lagrangian (1) are adequate tools for macroscopic description of the main features of evolution of the universe. In the scale of the whole universe the zero-mass field corresponds to the dark energy, and the massive one - to dark matter. Price issue is the rejection of the prejudice (widely spread, unfortunately) that the energy should not be negative. Instead I utilized a weaker condition of regularity. In general relativity the energy is not a scalar, and its sign is not invariant against the arbitrary coordinate transformations. Described by the vector field with the same Lagrangian (1), the dark matter is of the same physical nature in both applications: to cosmology [10], and to galaxy rotation curves.

As a matter of fact, I agree with the Sanders’ statement that “.the correct theory may well be one in which MOND reflects the influence of cosmology on local particle dynamics and arises only in a cosmological setting” [6]. However, it is evident that I don’t share the Sanders’ conclusion: “ It goes without saying that this theory is not General Relativity, because in the context of General Relativity local particle dynamics is immune to the influence of cosmology” [6]. I have presented here the complete derivation from the Einstein equations (14-16) to the galaxy rotation curve (26).

There are attempts of applying the scalar, vector, and tensor fields in order “to explain the flat rotation curves of galaxies and cluster lensing without postulating exotic dark matter” [8]. In quantum physics each elementary particle is a quantum of some field, and vice versa, each field corresponds to its own quantum particle [16]. From my point of view, the various fields are just convenient mathematical instruments that we utilize for description of physical phenomena, no matter how we name them.

According to the observations the period of oscillations \( \frac{2\pi}{m} \) (see Figures 6 and 7) is some 15 kpc. If in quantum mechanics it is the de Broglie wavelength \( \lambda = \frac{\hbar}{mc} \), then the rest energy of a quantum particle, corresponding to the vector field, should be \( mc^2 \sim 2.5 \times 10^{-27} \text{ eV} \).

A few words about fine tuning. I have come across this situation for three times. The first one has been the widely used in the fifties “Bennet pinch” [17] – a fine tuned solution of equilibrium of a high current channel where the magnetic attraction is balanced by the gas pressure of electric charges. In reality it came out to be a boundary between the expansion and compression when the balance is broken [18]. For the second time it has been the
conclusion of the existence of the limiting mass of an ultra relativistic star by Chandrasekhar [19] and Landau [20]. The fine tuned solution of equilibrium with ultra relativistic equation of state turned out to be the boundary of the gravitational collapse [15]. The third time it has been the fine tuned singular cosmological solution by Friedman [21], Robertson [22], and Walker [23]. In the thirties, dark matter had not been taken seriously. With account of dark matter the FRW singular solution turned out to be a lower boundary of the regular oscillating cosmological solutions [10]. In all the cases the requirement of regularity rules out the problem of fine tuning.

From my point of view it is time to reconsider the equilibrium [14] and collapse [15] of supermassive bodies taking into account the dark matter.

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[24] This nonlinear term at $r \to \infty$ decreases as $(r \ln r)^{-1}$:

$$\kappa m^2 r (\phi r)^2 = \frac{2 \sin^2 m r}{3 r \ln \frac{r^*}{r}} \approx \frac{1}{3 r \ln \frac{r^*}{r}}, \quad r^* \sim \frac{1}{m}.$$ 

[25] MSTG is an attempt to “generalize” the general relativity on the basis of a pseudo-Riemannian metric tensor and a skew symmetric rank-3 tensor field in a hope to explain the flat rotation curves of galaxies. 

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