Semi-polytope decomposition of a Generalized permutohedron

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Abstract

In this short note we show explicitly how to decompose a generalized permutohedron into semi-polytopes.

1 Introduction

Given a polytope, assume we have disjoint open cells whose closures sum up to be the entire polytope. A question of naturally assigning each of the remaining points (possibly in multiple closures) to a cell has appeared in [1] for studying regular matroids and zonotopes and in [2] for studying h-vectors and Q-polynomials. In other words, we are trying to determine ownership of lattice points on boundaries of multiple polytopes. In this note, we study a more general case of doing the same for a Generalized permutohedron, a polytope that can be obtained by deforming the usual permutohedron. We will show explicitly how to construct a semi-polytope decomposition of a trimmed generalized permutohedron.

2 Generalized permutohedron $P_G$ and its fine mixed subdivision

Let $\Delta_{[n]} = \text{ConvexHull}(e_1, \ldots, e_n)$ be the standard coordinate simplex in $\mathbb{R}^n$. For a subset $I \subset [n]$, let $\Delta_I = \text{ConvexHull}(e_i | i \in I)$ denote the face of $\Delta_{[n]}$. Let $G \subseteq K_{m,n}$ be a bipartite graph with no isolated vertices. Label the vertices of $G$ by $1, \ldots, m, \bar{1}, \ldots, \bar{n}$ and call $1, \ldots, m$ the left vertices and $\bar{1}, \ldots, \bar{n}$ the right vertices. We identify the barred indices with usual non-barred cases when it is clear we are dealing with the right vertices. For example when we write $\Delta_{\{\bar{1}, \bar{3}\}}$ we think of it as $\Delta_{\{1, 3\}}$. We associate this graph with the collection $I_G$ of subsets $I_1, \ldots, I_m \subseteq [n]$ such that $j \in I_i$ if and only if $(i, \bar{j})$ is an edge of $G$. Let us define the polytope $P_G(y_1, \ldots, y_m)$ as:

$$P_G(y_1, \ldots, y_m) := y_1 \Delta_{I_1} + \cdots + y_m \Delta_{I_m},$$

where $y_i$ are nonnegative integers. This lies on a hyperplane $\sum_{i \in [n]} x_i = \sum_{j \in [m]} y_j$. An example of a coordinate simplex $\Delta_{[3]}$, a bipartite graph $G$ and a generalized permutohedron $P_G(1, 2, 3)$ is given in Figure 1.

Definition 2.1 ([3], Definition 14.1). Let $d$ be the dimension of the Minkowski sum $P_1 + \cdots + P_m$. A Minkowski cell in this sum is a polytope $B_1 + \cdots + B_m$ of dimension $d$ where $B_i$ is the convex hull of some subset of vertices of $P_i$. A mixed subdivision of the sum is the decomposition into union of Minkowski cells such that intersection of any two cells is their common face. A mixed subdivision is fine if for all cells $B_1 + \cdots + B_m$, all $B_i$ are simplices and $\sum \dim B_i = d$.

All mixed subdivisions in our note, unless otherwise stated, will be referring to fine mixed subdivisions. We will use the word cell to denote the Minkowski cells. Beware that our cells are all closed polytopes.

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Figure 1: Example of a generalized permutohedron $P_G(1, 2, 3)$

Fine Minkowski cells can be described by spanning trees of $G$. When we are looking at a fixed generalized permutohedron $P_G(y_1, \ldots, y_m)$, we will use $\prod_T$ to denote $y_1 \Delta J_1 + \cdots + y_m \Delta J_m$ where $J = (J_1, \ldots, J_m)$. We say that $J$ is a tree if the associated bipartite graph is a tree.

**Lemma 2.2** ([3], Lemma 14.7). Each fine mixed cell in a mixed subdivision of $P_G(y_1, \cdots, y_m)$ has the form $\prod_T$ such that $T$ is a spanning tree of $G$.

An example of a fine mixed subdivision of the polytope considered in Figure 1 is given in Figure 2.

Figure 2: A fine mixed subdivision of $P_G(1, 2, 3)$.

We can say a bit more about the lattice points in each $\prod_T$:

**Proposition 2.3** ([3], Proposition 14.12). Any lattice point of a fine Minkowski cell $\prod_T$ in $P_G(y_1, \cdots, y_m)$ is uniquely expressed (within $\prod_T$) as $p_1 + \cdots + p_m$ where $p_i$ is a lattice point in $y_i \Delta T_i$. 

2
3 Semi-polytope decomposition

A mixed subdivision of $P_G$ divides the polytope into cells. In this section, we show that from a mixed subdivision of $P_G$, one can obtain a way to decompose the set of lattice points of $P_G$.

**Definition 3.1** ([3], Definition 11.2). The *trimmed generalized permutohedron* $P_G^-$ is defined as:

$$P_G^-(y_1, \ldots, y_m) := \{ x \in \mathbb{R}^n | x + \Delta_{[n]} \subseteq P_G \}.$$  

This is a more general class of polytopes than generalized permutohedra $P_G(y_1, \ldots, y_m)$. With a slight abuse of notation, we will let $I \setminus j$ stand for $I \setminus \{j\}$.

**Definition 3.2** ([3], Theorem 11.3). The coordinate *semi-simplices* are defined as

$$\Delta^*_i,j = \Delta_i \setminus \Delta_{i \setminus j}$$

for $j \in I \subseteq [n]$.

For each cell $\prod_T$, we are going to turn it into a *semi-polytope* of the form $y_1 \Delta^*_{i,j_1} + \cdots + y_m \Delta^*_{i,m,j_m}$. This will involve deciding which cell takes ownership of the lattice points on several cells at the same time.

We denote the point $(\sum y_i - c, \ldots, -c)$ for $c$ sufficiently large as $\infty$. For a facet of a polytope, we say that it is negative if the defining hyperplane of the facet (inside the space $\sum x_i = \sum y_i$ which the polytope lies in) separates the point $\infty$ and the interior of the polytope. Otherwise, we say that it is positive. We will say that a point of a polytope is good if it is not on any of the positive facets of the polytope.

**Lemma 3.3.** Fix $T$, a spanning tree of $G \subseteq K_{m,n}$. Let $T_i$ be the set of neighbors of $i$. For each $i$ such that $1 \notin T_i$, there exists a unique element $t_i$ in $T_i$ such that there exists a path to 1 not passing through $i$.

**Proof.** There exists such an element since $T$ is a spanning tree of $T$. There cannot be more than one such element since otherwise, we get a cycle in $T$. \hfill $\Box$

In cases where $T_i$ does contain 1, we set $t_i$ to be 1.

**Lemma 3.4.** Let $\prod_T$ be a fine mixed cell. Removing the positive facets gives us $\sum_i \Delta^*_{i,t_i}$.

**Proof.** To prove this, we first introduce a tool that will be useful for identifying which hyperplanes the facets lie on. Let $\prod_T$ be a fine mixed cell so $T$ is a spanning tree. For any edge $e$ of $T$ that is not connected to a leaf on the left side, $T \setminus e$ has two components. Let $I_e$ denote the set of right vertices of a component that contains 1. Let $c_e$ be the sum of $y_i$’s for left vertices contained in that component. Notice that $I_e$ cannot be $[n]$ since otherwise $e$ would have a leaf as its left endpoint.

**Lemma 3.5.** Let $\prod_T$ be a fine mixed cell. For any edge $e$ of $T$ that is not connected to a leaf on the left side, $\prod_{T \setminus e}$ is a facet of $\prod_T$ that lies on $\sum x_j = c_e$. If the right endpoint of $e$ is in $I_e$, then $\prod_T$ lies in half-space $\sum x_j \geq c_e$. Otherwise it lies in $\sum x_j \leq c_e$.

**Proof.** The dimension difference between $\prod_T$ and $\prod_{T \setminus e}$ is at most one, and all endpoints of $\prod_{T \setminus e}$ lie on $\sum x_j = c_e$. If the right endpoint of $e$ is in $I_e$, that means we can find a point $x$ using $e$ so that $\sum x_j > c_e$. If not, that means we can find a point $x$ using $e$ so that $\sum x_j < c_e$.

**Proof of Lemma 3.4.** If $\prod_{T \setminus e}$ is a positive facet of $\prod_T$, then Lemma 3.5 tells us that the right endpoint of $e = (i,j)$ is in $I_e$. From definition of $t_i$, we have $j = t_i$. In other words we are removing sets of form $\Delta_T + \cdots + \Delta_T \setminus t_i + \cdots + \Delta_T$. At the end we end up with $\sum_i (\Delta_T \setminus t_i)$.
4 Application to Erhart theory

In this section we show how the semi-polytope decomposition can be used in Erhart theory as guided in \[3\]. Given any subgraph \( T \) in \( G \), define the left degree vector \( ld(T) = (d_1 - ... \]

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1, \cdots, d_n - 1) and the right degree vector \( rd(T) = (d'_1 - 1, \cdots, d'_m - 1) \) where \( d_i \) and \( d'_j \) are the degree of the vertex \( i \) and \( j \) respectively. The raising powers are defined as \( (y)^a := y(y+1) \cdots (y+a-1) \) for \( a \geq 1 \) and \( (y)^0 := 1 \).

**Corollary 4.1.** Fix a fine mixed subdivision of \( P_G(y_1, \ldots, y_m) \) where \( y_i \)'s are nonnegative integers. The number of lattice points in the trimmed generalized permutohedron \( P_G^-(y_1, \ldots, y_m) \) equals \( \sum(a_1, \ldots, a_m) \prod_i \frac{(y_i)^{a_i}}{a_i!} \) where the sum is over all left-degree vectors of fine mixed cells inside the subdivision.

**Proof.** Obtain a semi-polytope decomposition as in Theorem 3.7. Then each lattice point of \( P_G^-(y_1, \ldots, y_m) \) is in exactly one semi-polytope. The claim follows since the number of lattice points of a semi-polytope \( y_1 \Delta_{T_1,t_1} + \cdots + y_m \Delta_{T_m,t_m} \) (thanks to Proposition 2.3 different sum gives a different point) is given by \( \prod_i \frac{(y_i)^{a_i}}{|T_i|!} \).

The expression in Corollary 4.1 is called the **Generalized Erhart polynomial** of \( P_G^-(y_1, \ldots, y_m) \) by [3]. As foretold in [3], Theorem 3.7 gives us a pure counting proof of Theorem 11.3 of [3].

**Definition 4.2** ([3], Definition 9.2). Let us say that a sequence of nonnegative integers \( (a_1, \cdots, a_m) \) is a \( G \)-draconian sequence if \( \sum a_i = n - 1 \) and, for any subset \( \{i_1 < \cdots < i_k \} \subseteq [m] \), we have \( |I_{i_1} \cup \cdots \cup I_{i_k}| \geq a_{i_1} + \cdots + a_{i_k} + 1 \).

**Theorem 4.3** ([3], Theorem 11.3). For nonnegative integers \( y_1, \ldots, y_m \), the number of lattice points in the trimmed generalized permutohedron \( P_G^-(y_1, \ldots, y_m) \) equals \( \sum(a_1, \ldots, a_m) \prod_i \frac{(y_i)^{a_i}}{a_i!} \), where the sum is over all \( G \)-draconian sequences \( (a_1, \ldots, a_m) \).

**Proof.** Thanks to Corollary 4.1 all we need to do is show that the set of \( G \)-draconian sequences is exactly the set of left-degree vectors of the cells inside a fine mixed subdivision of \( P_G \). Lemma 14.9 of [3] tells us that the right degree vectors of the fine cells is exactly the set of lattice points of \( P_G^-(1, \ldots, 1) \) where \( G^* \) is obtained from \( G \) by switching left and right vertices. Then Lemma 11.7 of [3] tells us that the set of \( G \)-draconian sequences is exactly the set of lattice points of \( P_G^-(1, \ldots, 1) \). \( \square \)

This approach has an advantage that the generalized Erhart polynomial is obtained from a direct counting method, without using any comparison of formulas.

**References**

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