Abstract—In this article, we consider a continuous-type Bayesian Nash equilibrium (BNE) seeking problem in subnetwork zero-sum games, which is a generalization of either deterministic subnetwork zero-sum games or discrete-type Bayesian zero-sum games. In this model, because the feasible strategy set is composed of infinite-dimensional functions and is not compact, it is hard to seek a BNE in a noncompact set and convey such complex strategies in network communication. To this end, we give a two-step design. One is a discretization step, where we discretize continuous types and prove that the BNE of the discretized model is an approximate BNE of the continuous model with an explicit error bound. The other is a communication step, where we adopt a novel compression scheme with a designed sparsification rule and prove that agents can obtain unbiased estimations through the compressed communication. Based on the two steps, we propose a distributed communication-efficient algorithm to practically seek an approximate BNE, and further provide the convergence analysis and explicit error bounds.

Index Terms—Bayesian game, communication compression, discretization, distributed algorithm, equilibrium approximation, subnetwork game, zero-sum game.

I. INTRODUCTION

In recent years, distributed design for decision and control has become more and more important, and distributed algorithms have been proposed for various games [1], [2], [3], [4], [5], [6], [7], [8]. With the rapid development of multiagent systems these days, subnetwork zero-sum games, as extensions of zero-sum games, have attracted the attention of researchers. For example, Gharesifard and Cortés [3] provided a continuous-time distributed equilibrium seeking algorithm for both undirected and directed graphs, while Lou et al. [9] proposed a discrete-time algorithm for equilibrium seeking, and analyzed its convergence. Moreover, Huang et al. [10] considered an online learning scheme and provided a distributed mirror descent algorithm.

Because of the uncertainties in reality, Bayesian games have attracted a large amount of attention in engineering, computer science, and social science [1], [11], [12], [13]. In Bayesian games [14], [15], players cannot obtain some characteristics (called types) of the other players, and the joint distribution of types is known to all players. Due to the broad applications, the existence and computation of the Bayesian Nash equilibrium (BNE) are fundamental problems in the study of various Bayesian games. Many works have investigated BNE with discrete types [1], [16] by fixing the types and converting the games to complete-information ones. In addition to the centralized algorithms, there are also many works on distributed Bayesian games [1], [17], where players make decisions based on their own types and local information.

Particularly, Bayesian zero-sum games also have drawn much attention recently. The zero-sum condition in two-player games describes the circumstance that the players have opposite goals and cannot improve their payoffs by cooperation. For example, the authors in [16] and [18] modeled a jamming problem in an underwater acoustic sensor network as a Bayesian zero-sum game, where one player aims to maximize the overall network capacity and the other one (jammer) needs to minimize the throughput of the network as their positions are uncertain. Moreover, Tsai et al. [19] studied adversarial domains with uncertainties in social networks with Bayesian zero-sum games, where an influencer aims to maximize the influence in a social network and a mitigator attempts to minimize the efforts of the influencer to spread its agenda across the network.

However, most of the aforementioned works concentrate on discrete-type Bayesian games. In fact, continuous-type
Bayesian games are also widespread in computer science and economics [12], [13]. The continuity of types poses challenges in seeking and verifying the BNE. Specifically, in continuous-type games, the feasible strategy sets lie in infinite-dimensional spaces and are not compact [15]. Due to the lack of the compactness, we cannot apply the fixed point theorem to guarantee the existence of BNE, let alone seek a BNE. To this end, many researchers have tried to demonstrate the existence of continuous-type BNE and to design its computation. For instance, Milgram and Weber [15] analyzed the existence of BNE in virtue of equicontinuous payoffs and absolutely continuous information, while Meirowitz [20] investigated the situation when the best responses are equicontinuous. Afterward, Guo et al. [21] provided an equivalent condition of the equicontinuity and proposed an approximation algorithm. Also, Ui [22] regarded the BNE as the solution to a variational inequality and gave a sufficient condition for the existence of BNE, while Guo et al. [23] gave two variational-inequality-based algorithms when the strategy forms are prior knowledge.

With the development of subnetwork zero-sum games and Bayesian games, it is important to construct distributed algorithms for seeking BNE in continuous-type games, which can be regarded as generalizations of both discrete-type Bayesian zero-sum games [16], [19] and deterministic subnetwork zero-sum games [3], [9], [10]. Nevertheless, the continuous-type models are more challenging to handle than the discrete-type ones in distributed subnetwork zero-sum games. Actually, the challenges come from both continuous types and network communication. On the one hand, to seek a continuous-type BNE in a distributed manner, we need an effective method to convert the infinite-dimensional BNE seeking problem into a finite-dimensional one, and the method should be friendly to distributed design. On the other hand, since players need to exchange their strategies with their neighbors, the according strategies, which are infinite-dimensional functions, are hard to convey directly under limited communication capabilities, and we need an effective method to handle this complex communication.

Therefore, we consider seeking a continuous-type BNE in distributed subnetwork zero-sum games in this article, where agents in each subnetwork cooperate against the adversarial subnetwork and both the subnetworks are engaged in a zero-sum game. Each subnetwork has its own type following a continuous joint distribution, and each agent knows the type of its own subnetwork. The challenges lie in how to seek a BNE in this continuous-type model and how to efficiently exchange information through the networks. Existing works are not good enough to solve this problem. Although the authors in [23] and [24] adopted the concept of the $\epsilon$-Bayesian Nash equilibrium ($\epsilon$-BNE) to describe the approximate solution and the authors in [12], [24], and [21] proposed computation methods for BNE, their approaches were limited in heuristic methods without quantitative analysis. Moreover, due to the interactions among players, communication compression methods for optimization [25], [26] can hardly be applied to this subnetwork game model. To this end, we introduce a discretization step and a communication step to carry forward, and correspondingly propose a distributed BNE seeking algorithm to communication-efficiently solve an approximate BNE. The contributions are summarized as follows.

1) In the discretization step, to approximate a BNE of continuous-type models, we discretize continuous types to make the algorithm implementable. By conducting a distributed-friendly discretization, we prove that the derived BNE sequence in the discretized model converges to the BNE of the continuous model. Moreover, compared with existing works on continuous-type Bayesian games, our method provides an explicit error bound by taking into account the zero-sum condition [21], [24], and serves as a practicable method different from heuristics [12], [23], [24].

2) In the communication step, we adopt compression in the distributed algorithm design to reduce the communication complexity. For this purpose, we design a novel sparsification rule to reduce communication burden to an acceptable level, since existing compression methods for optimization [25], [26] can hardly be applied due to players’ interactions here. Correspondingly, we give a communication scheme to handle the interactions of players, which is well adapted to time-varying networks in subnetwork zero-sum games. On this basis, we show that agents can get unbiased estimations.

3) Based on the above two steps, we design a distributed algorithm for seeking a BNE in subnetwork zero-sum games. With discretizing continuous types and compressing network communication, the algorithm leads to an approximate BNE with an explicit error bound. We prove the convergence of the algorithm, as well as its $O(\ln T/\sqrt{T})$ convergence rate.

The rest of this article is organized as follows. Section II summarizes preliminaries. Then Section III formulates the continuous-type BNE seeking problem in subnetwork zero-sum games. The following two sections provide the technical details for seeking an approximate BNE, involving the discretization step in Section IV and the communication step in Section V. Section VI proposes the distributed algorithm based on the above two steps, while Section VII provides the convergence analysis. Section VIII provides numerical simulations for illustration. Finally, Section IX concludes this article.

II. PRELIMINARIES

A. Notations

Denote the $n$-dimensional real Euclidean space by $\mathbb{R}^n$ and its measure by $\mu$. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ (\lceil x \rceil) is the greatest (least) integer less (greater) than or equal to $x$. $B(a, \varepsilon)$ is a ball with the center $a$ and the radius $\varepsilon > 0$. Denote $\text{col}(x_1, \ldots, x_n) = (x_1^T, \ldots, x_n^T)^T$, and $I_n = I_{n,n}^n$ as the identity matrix. For column vectors $x, y \in \mathbb{R}^n$, $(x, y)$ denotes the inner product, and $\| \cdot \|$ denotes the 2-norm. For a vector $x \in \mathbb{R}^n$, $|x|_k$ denotes the $k$th element of $x$, $k \in \{1, \ldots, n\}$. For a matrix $X \in \mathbb{R}^{m \times n}$, $[X]_{ij}$ denotes the element in the $i$th row and $j$th column of
Subnetwork games.

$V_{i,j} \subseteq \Xi \ni G$ with respect to $\ell$ for a.e. $l \in G$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is (strictly) convex if $f(\lambda x_1 + (1 - \lambda)x_2) < \lambda f(x_1) + (1 - \lambda)f(x_2)$ for all $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in (0, 1)$. For a convex function $f(x), w(x)$ is a subgradient of $f$ at point $x$ if $f(y) \geq f(x) + \langle y - x, w(x) \rangle$ for all $y \in \mathbb{R}^n$. The set of all subgradients of convex function $f$ at $x$ is denoted by $\partial f(x)$. For a convex function $f(x_1, \ldots, x_n)$, denote $\partial_i f$ as the subdifferential of $f$ with respect to $x_i$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is $\nu$-strongly convex $(\nu > 0)$ if $f(y) \geq f(x) + \langle y - x, w(x) \rangle + \frac{\nu}{2} \| y - x \|^2 \forall x, y \in \mathbb{R}^n, w(x) \in \partial f(x)$. Moreover, if $f(x) - \frac{\nu}{2} \| x \|^2$ is convex, $f(x)$ is $\nu$-strongly convex.

### Bayesian Game

Consider a Bayesian game with players $\mathcal{V} = \{1, \ldots, n\}$, denoted by $G = (\mathcal{V}, \{X_i\}_{i=1}^n, \Theta, P, \{f_\sigma(V)\}_{\sigma \in \Theta})$. Player $i$ has its action set $X_i \subseteq \mathbb{R}^{m_i}$. The incomplete information of player $i$ is referred to the type, that is, a random variable $\theta_i \in \Theta_i \subseteq \mathcal{R}$. Denote $\theta_{-i} \in \Theta_{-i}$ as the type vectors of all players except player $i$. The joint distribution of types over $\Theta = \Theta_1 \times \cdots \times \Theta_n$ and its density function are denoted by $P$ and $p$, respectively, with the marginal density $p_i(\theta_i) = \int_{\Theta_{-i}} p(\theta_i, \theta_{-i}) d\theta_{-i}$, and the conditional probability density $p_{i|\theta_{-i}}(\theta_i|\theta_{-i}) = p(\theta_i, \theta_{-i})/p(\theta_{-i})$, $\theta_i \in \Theta_i$. Throughout the article, we use $(\theta_1, \ldots, \theta_n)$ to denote a random variable mapping from a probability space $(\Omega, \mathcal{B}, P)$ to $\mathbb{R}^n$, or a deterministic element in $\mathcal{R}^n$ depending on the context.

As discussed in Bayesian games [14, 15], it is a basic assumption that each player only knows its own type but not those of its rivals, and instead, the joint probability distribution of types is public information. The cost function of player $i$ is defined as $f_i : X_1 \times \cdots \times X_n \times \Theta \to \mathbb{R}$ depending on all players’ actions and types. The strategy $\sigma_i$ of player $i$ is a measurable function mapping from $\Theta_i$ to $X_i$, and $\sigma_i(\theta_i) \in X_i$ is the action when player $i$ receives $\theta_i \in \Theta_i$. Denote player $i$’s feasible strategy set by $\Sigma_i$. Define the Hilbert space $\mathcal{H}_i$ consisting of measurable functions $\beta : \mathbb{R} \to \mathbb{R}^{m_i}$ with the inner product $(\alpha \sigma, \beta \sigma)_{\mathcal{H}_i} = \int_{\Theta_i} \alpha \sigma(\theta_i) \beta \sigma(\theta_i) d\theta_i$, $\sigma, \sigma_i \in \mathcal{H}_i$. Thus, the strategy set $\Sigma_i$ lies in $\mathcal{H}_i$. If player $i$ adopts a strategy $\sigma_i$, its conditional expectation of the cost at $\theta_i$ is $U_i(\sigma_i, \sigma_{-i}, \theta_i) = \int_{\Theta_i} f_i(\sigma_i(\theta_i), \sigma_{-i,\theta_i}) p_i(\theta_{-i} \mid \theta_i) d\theta_i$, where $\sigma_{-i}$ is the profile of all players’ strategies except for player $i$.

### Graph Theory

A digraph (or directed graph) $G = (\mathcal{V}, \mathcal{E})$ consists of a node set $\mathcal{V} = \{1, \ldots, n\}$ and an edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Node $j$ is a neighbor of $i$ if $(j, i) \in \mathcal{E}$, and take $(i, i) \notin \mathcal{E}$. A path in $G$ from $i_1$ to $i_k$ is an alternating sequence $i_1 i_2 \cdots i_k$ of nodes such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j \in \{1, \ldots, k-1\}$. (The weighted associated adjacency matrix $A = (A_{ij}) \in \mathbb{R}^{n \times n}$ is composed of nonnegative adjacency elements $A_{ij}$, which is positive if and only if $(j, i) \in \mathcal{E}$. $G$ is bipartite if $\mathcal{V}$ can be partitioned into two disjoint parts $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{E} \subseteq \bigcup_{l=1}^L \{G_1 \times G_2: G_1 \in \mathcal{V}_1, G_2 \in \mathcal{V}_2\}$. Digraph $G$ is strongly connected if there is a path in $G$ from $i_j$ to $j$ for any nodes $i, j \in \mathcal{V}$.

### III. DISTRIBUTED SUBNETWORK ZERO-SUM GAMES

In this section, we formulate a distributed BNE seeking problem in subnetwork zero-sum games.

Consider a multiagent network $\Xi$ consisting of two subnetworks $\Xi_1$ and $\Xi_2$, with agents $\mathcal{V} = \{v_1, \ldots, v_n\}$, $l \in \{1, 2\}$. $\Xi$ is described by a time-varying digraph, denoted as $G^t = (\mathcal{V}_t \cup \mathcal{V}_2, E^t)$, $E^t$ can be partitioned into four digraphs: $G^t_1 = (\mathcal{V}_1, E_1^t)$ with $E_1^t \subseteq \{(j, i) \mid j \in \mathcal{V}_1, i \in \mathcal{V}_2\}$, and two bipartite graphs $G_{3-l}^t = (\mathcal{V}_1 \cup \mathcal{V}_2, E_{3-l}^t)$ with $E_{3-l}^t \subseteq \{(j, i) \mid j \in \mathcal{V}_1, i \in \mathcal{V}_{3-l}\}$, $l \in \{1, 2\}$. The set of neighbors in subnetwork $\Xi_k (k \in \{1, 2\})$ for agent $v_j$ is denoted by $N^k_j$. Agents in $\Xi$ are engaged in a Bayesian game $G = (\{\Xi_i\}_{i=1}^n, \{X_i\}_{i=1}^n, \Theta \times \Theta, P, \{f_i(\cdot, \cdot)\}_{i \in \{1, 2\}})$. For agent $v_i$, its action set, type set, joint probability distribution, and cost function are denoted by $X_i \subseteq \mathbb{R}^{m_i}$, $\Theta_i \subseteq \mathcal{R}$, $P(\theta_1, \theta_2)$, and $f_i(x_1, x_2, \theta_1, \theta_2)$, respectively. The conditional expectation of $f_i$ with respect to $\theta_{3-l}$ is $U_{i,l}(\sigma_1, \sigma_{3-l}, \theta_i) = \int_{\Theta_{3-l}} f_i(\sigma_1(\theta_1), \sigma_2(\theta_2), \theta_1, \theta_2) p_i(\theta_{3-l} \mid \theta_i) d\theta_{3-l}$.

At time $t$, agents exchange strategies with neighbors in $\Xi_1$ via $G_1^t$, and with neighbors in $\Xi_{3-1}$ via $G_{3-1}^t$. Denote the strategy sets by $\Sigma_i \subseteq \mathcal{H}_i$. The cost function of $\Xi_1$ is $f_1 = \frac{1}{n_i} \sum_{l=1}^{n_i} f_i(\cdot)$. The subnetworks are engaged in a zero-sum game, namely, $f_1(x_1, x_2, \theta_1, \theta_2) + f_2(x_1, x_2, \theta_1, \theta_2) = 0 \forall x_i \in \mathcal{X}_i, \theta_i \in \Theta_i, l \in \{1, 2\}$. Each agent pursues its subnetwork goal with its own and neighbor’s strategies, and the goal of $\Xi_1$ is to choose a strategy to minimize the global conditional expectation $U_1(\sigma_1, \sigma_{3-1}, \theta_i) = \frac{1}{n_i} \sum_{l=1}^{n_i} U_{l,i}$ for almost every $\theta_i \in \Theta_i$.

We introduce the best response strategy and the BNE as in [14].

**Definition 1**: Considering subnetwork zero-sum game $G$, 1) for subnetwork $\Xi_1$, a strategy $\sigma_1$ is a best response with respect to the adversarial strategy $\sigma_{3-1} \in \Sigma_{3-1}$ if for a.e. $\Xi_2$. 

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\[ \theta_l \in \Theta_l \]

\[ \sigma_l(\theta_l) = \arg \min_{\sigma_l(\theta_l) \in \mathcal{X}_l} U_l(\sigma_l, \sigma_{3-l}, \theta_l) \]

where the best response set is denoted by \( BR_l(\sigma_{3-l}) \).

2) A strategy pair \( (\sigma^*_1, \sigma^*_2) \) is a BNE of \( G \) if \( \sigma^*_1 \in BR_l(\sigma^*_{3-l}) \).

In Definition 1, a best response \( \sigma_l \) with respect to a given strategy \( \sigma_{3-l} \) means that it is an optimal solution for \( \Xi_l \) when \( \Xi_{3-l} \) adopts \( \sigma_{3-l} \). Moreover, reaching a BNE profile means that there is no strategy that a subnetwork can adopt to yields a lower cost, when the adversarial subnetwork adopts its strategy.

**Example 1 ([23], [27]):** Consider a rent-seeking game, where two power firms compete and gain profits from the share of the power supply. The two firms are engaged in a zero-sum game, which means that they cannot improve their profits through cooperation. Each firm \( \Xi_l \) has \( n_l \) plants. The actions are power outputs of firms. The costs of power generation are uncertain, and are influenced by continuous random variables \( \theta_l \), such as the coal price. The joint distribution of types is obtained through historical data and is known to all the plants. The cost of each plant is obtained as

\[ f_{l,i}(x_1, x_2, \theta_l, \theta_l) = g_{l,i}(x_1, \theta_l, \theta_{3-l}) - \frac{d_{l,i}x_l}{x_1 + x_2} + h_{l,i}(x_3-l, \theta_{3-l}) \]

where \( d_{l,i} \) is a constant. Here, the first term is the cost of power generation, the second term is the payoff of the supply, and the third term is to ensure the zero-sum condition which does not influence the decision of firm \( \Xi_l \). The collective goal of plants in \( \Xi_l \) is to find a strategy to minimize the conditional expectation \( U_{l,i}(\sigma_l, \sigma_{3-l}, \theta_l) \) for a.e. \( \theta_l \in \Theta_l \).

We make the following assumptions for \( G \).

**Assumption 1:** For \( l \in \{1, 2\} \),

i) (Action set) \( \mathcal{X}_l \) is nonempty, compact, and convex;

ii) (Type set) \( \Theta_l \) is compact. Without loss of generality, take \( \Theta_l = [\theta_l', \theta_l'] \);

iii) (Cost function) For \( j \in \{1, 2\} \), \( f_{l,i,j} \) is \( L_{l,j} \)-Lipschitz continuous over \( \mathcal{X}_l \) and \( L_{l,j} \)-Lipschitz continuous over \( \Theta_j \).

Besides, \( f_{l,i,j} \) is strictly convex in \( x_l \in \mathcal{X}_l \), and \( f_{l,i} \) is continuously differentiable and \( \nu \)-strongly convex in \( x_l \in \mathcal{X}_l \);

iv) (Distribution) \( P_l \) is atomless, i.e., \( P(\theta_l = \theta_{l1}') = P(\theta_l = \theta_{l2}') = 0 \) for any \( \theta_{l1}' \in \Theta_l \) and \( \theta_{l2}' \in \Theta_l \), and the measure \( \mu(\{\theta_l' \mid p_l(\theta_l') > 0, \theta_l' \in \Theta_l\}) = \mu(\Theta_l) \);

v) (Graph) \( G_l^t \) is \( \mathcal{R}_0 \)-jointly strongly connected, i.e., \( \cup_{t=1}^{l} \mathcal{X}_{l t-1} \) is strongly connected for \( t \geq 0 \). Moreover, each agent \( v_l^t \) has at least a neighbor in \( \mathcal{X}_{l t-1} \) in a finite time interval, i.e., there exists an integer \( S_0 > 0 \) that agent \( v_l^t \)‘s neighbor set in \( G_l^t \) satisfies \( \cup_{r=t}^{l} \mathcal{X}_{l r-1}(r) \neq \emptyset \) for \( t \geq 0 \).

Assumption 1 was widely used in Bayesian games and distributed equilibrium seeking problems [10], [15], [21], [23], [24], [28]. Assumption 1(i) and (iii) ensure that \( U_{l,i,j} \) is well defined for each \( \theta_l \in \Theta_l \) and \( (\sigma_1, \sigma_2) \in \Sigma_1 \times \Sigma_2 \). The atomless property in Assumption 1(iv) is common in Bayesian games [15], [21], [24], and the measure condition can be guaranteed by removing types in \( \{\theta_l \in \Theta_l \mid 3 \varepsilon > 0, p_l(\theta_l') = 0, \theta_l' \in B(\theta_l, \varepsilon)\} \). Assumption 1(v) ensures the connectivity of the network, which was also used in [9] and [24].

In continuous-type Bayesian games, we cannot apply the fixed point theorem to ensure the existence of BNE as in the discrete cases [15]. Fortunately, the authors in [22] and [23] provided the existence condition for continuous-type BNE using variational inequalities, which is summarized as follows.

**Lemma 1:** Under Assumption 1(i)–(iii), there exists a unique BNE of \( G \).

Based on the existence, our goal is to compute the BNE of the proposed model, and we formulate our problem as follows.

**Problem:** Seek the BNE in the continuous-type Bayesian game

\[ G = \left\{ \left( \Xi_l \right)_{l=1}, \left( \mathcal{X}_l \right)_{l=1}, \Theta_l, P, \{f_{l,i,j}(.)\}_{l \in \{1, 2\}} \right\} \]

via distributed computation through time-varying graphs \( G_l^t \).

Notice that even if we assume the action sets and the type sets to be compact, the strategy sets are not compact due to the continuity of types. The strategies are functions defined over the continuous set rather than finite-dimensional vectors in the discrete cases. As Riesz’s Lemma shows [29], any infinite-dimensional normed space contains a sequence of unit vectors \( \{x_n\} \) with \( \|x_n - x_m\| > \alpha \) for any \( 0 < \alpha < 1 \) and \( n \neq m \). Thus, the strategy set \( \Sigma_l \) in the infinite-dimensional space \( \mathcal{H}_l \) is not compact, which poses challenges for seeking BNE. There are only a few attempts to seek a continuous-type BNE, and these methods are difficult to implement or use heuristic approximations. For example, Guo et al. [23] considered that the forms of strategies are prior knowledge, maybe unavailable in practice, while [21] utilized polynomial approximation to estimate a BNE without an explicit estimation error. Moreover, Huang et al. [24] adopted a heuristic approximation in a discrete-action Bayesian game, but their algorithm was NP-hard and not practical in the continuous-action cases.

Thus, since it is not easy to seek a BNE directly, following [23] and [24], we introduce the concept of the \( \epsilon \)-BNE.

**Definition 2:** For \( l \in \{1, 2\} \), denote the expectation of \( U_l \) by

\[ EU_l(\sigma_1, \sigma_2) = \int_{\Theta_l} U_l(\sigma_1, \sigma_{3-l}, \theta_l) p_l(\theta_l) d\theta_l \]

for any \( \epsilon > 0 \), a strategy pair \( (\sigma^*_1, \sigma^*_2) \) is an \( \epsilon \)-BNE of \( G \) if for any \( \sigma_1 \in \Sigma_1 \) and \( \sigma_2 \in \Sigma_2 \)

\[ EU_l(\sigma_1, \sigma_{3-l}) \geq EU_l(\sigma^*_1, \sigma^*_3-l) + \epsilon, l \in \{1, 2\} \]

When seeking a BNE in a distributed manner, agents exchange their strategies through the network. To deal with the BNE seeking problem, we discretize continuous types to approximate a BNE. Thus, the dimension of the discretized strategy depends on the number of discrete points and can be explosively large. However, since communication capabilities are usually limited, conveying such high-dimensional strategies is difficult in distributed settings. To this end, we adopt communication compression to reduce the amount of communication.

To overcome the above difficulties in seeking BNE, we propose a method in the following sections with the two steps.

1) Discretize the continuous types to convert the infinite-dimensional problem into a finite-dimensional one.

2) Compress the high-dimensional strategies to implement distributed algorithms.
IV. DISCRETIZATION STEP

In this section, we provide a discretization method for an approximate BNE, and establish a relation between the BNE of the continuous model and the BNE of the discretized model.

Discretization Scheme:
1) Select \( N_i \) points \( \theta_i^1, \ldots, \theta_i^{N_i} \) from \( \Theta_i \), which satisfy
\[
P_i(\theta_i) = i/N_i, \quad i \in \{1, \ldots, N_i\}.
\]
Define the corresponding discrete type set by \( \tilde{\Theta}_i = \{\theta_i^1, \ldots, \theta_i^{N_i}\} \) and \( \theta_i^0 = \theta_i \).
2) Regard all types lying in \( [\theta_i^{l-1}, \theta_i^l) \) as \( \theta_i^l \), then for \( \theta_i^l \in \tilde{\Theta}_1 \) and \( \theta_i^l \in \tilde{\Theta}_2 \), the discretized distribution is
\[
\tilde{P}(\theta_i^l, \theta_2^l) = \int_{\theta_i^{l-1}}^{\theta_i^l} p(\theta_1, \theta_2) d\theta_1 d\theta_2.
\]
3) For \( l \in \{1, 2\} \), extend the domain of strategies in the discretized model from \( \tilde{\Theta}_i \) to \( \Theta_i \) to approximate the strategies in \( G \). For any type \( \theta \in (\theta_i^{l-1}, \theta_i^l) \)
\[
\tilde{\sigma}_i(\theta) = \tilde{\sigma}_i(\theta_i^l).
\]

On this basis, we formulate a discretized model, denoted by \( \tilde{G} = \left(\{\Xi_i\}_{i=1}^{\infty}, \{X_i\}_{i=1}^{\infty}, \tilde{\Theta}_1 \times \tilde{\Theta}_2, \tilde{P}(\cdot), \{f_i, i(\cdot)\}_{i=1}^{\infty}\right) \).
Since we use \( \theta_i^l \) to represent \( [\theta_i^{l-1}, \theta_i^l) \), we choose the length of such intervals as small as possible, which can effectively reduce the error. Moreover, our choice of discrete points facilitates the analysis of distributed algorithms. Otherwise, agents need to additionally exchange the marginal distributions, and do more computation when updating the strategies.

In this model, strategies are restricted to \( \mathbb{N}_m \)-dimensional vectors. Denote the strategy set of \( \Xi_i \) in \( \tilde{G} \) as \( \tilde{\Xi}_i \), and the marginal distribution and conditional distribution of \( P \) by \( \tilde{P}(\theta_i) \) and \( \tilde{P}_i(\theta_i, \theta_2) \), respectively. For any \( \tilde{\sigma}_1 \in \tilde{\Xi}_1 \) and \( \tilde{\sigma}_2 \in \tilde{\Xi}_2 \), the conditional expectation of \( f_{i, l} \) at \( \theta_i \in \tilde{\Theta}_i \) is
\[
\tilde{U}_{i, l}(\tilde{\sigma}_1, \tilde{\sigma}_3, \ldots, \theta_i) = \sum_{\theta_{3, l} \in \tilde{\Theta}_3} f_{i, l}(\tilde{\sigma}_1(\theta_1), \tilde{\sigma}_2(\theta_2), \theta_1, \theta_2) \tilde{P}_i(\theta_3, \ldots, \theta_i) \tilde{P}_i(\theta_3 \ldots, \theta_i).
\]
Correspondingly, denote the conditional expectation of \( f_i \) in \( \tilde{G} \) by \( \tilde{U}_i \).

Since the best responses of \( \tilde{G} \) need to respond to strategies in \( \Xi_i \) rather than \( \tilde{\Xi}_i \), we define the following best responses and BNE.

**Definition 3:** Considering the discretized model \( \tilde{G} \)
1) for the subnetwork \( \Xi_i \), a strategy \( \sigma_i^{N_i} \) is a best response to the rivals’ strategies \( \sigma_{3, l} \in \Sigma_{3, l} \) if for any \( \theta_i^l \in \tilde{\Theta}_i \)
\[
\sigma_i^{N_i}(\theta_i^l) = \min_{\sigma_1(\theta_1) \in \Xi_1} \int_{\theta_i^{l-1}}^{\theta_i^l} f_{i, l}(\sigma_1(\theta_1), \sigma_2(\theta_2), \theta_1, \theta_2) d\theta_1 d\theta_2.
\]
Denote the best response set by \( BR^{N_i}_{i}(\sigma_{3, l}) \);
2) a strategy pair \( (\tilde{\sigma}_1^*, \tilde{\sigma}_2^*) \in \tilde{\Xi}_1 \times \tilde{\Xi}_2 \) is a BNE of \( \tilde{G} \), or a DBNE \((N_1, N_2)\) if \( \tilde{\sigma}_i^* \in BR^{N_i}_{i}(\sigma_{3, l}) \).

The existence of DBNE can be guaranteed by variational inequalities [23] or the fixed point theorem [15]. We summarize the result as follows.

**Lemma 2:** Under Assumption 1(iii)–(iii), there exists a unique DBNE \((N_1, N_2)\) of the discretized model \( \tilde{G} \).

The following result shows the relation between the best responses in \( \tilde{G} \) and in \( G \), whose proof is in Appendix A.

**Lemma 3:** Let Assumption 1(iv) hold. With \( \sigma_{3, l} \in \Sigma_{3, l} \), if all the best responses in \( BR_i(\sigma_{3, l}) \) of \( G \) are piecewise continuous, then the best responses in \( BR^{N_i}_{i}(\sigma_{3, l}) \) of \( \tilde{G} \) are almost surely the best responses of \( G \), as \( N_i \) tends to infinity. Specifically, for any \( \tilde{\sigma}_i^* \in BR^{N_i}_{i}(\sigma_{3, l}) \), there exists \( \sigma_i^* \in BR_i(\sigma_{3, l}) \) such that
\[
\lim_{N_i \to \infty} \tilde{\sigma}_i^*(\theta_i) = \sigma_i^*(\theta_i), \quad \text{for a.e. } \theta_i \in \Theta_i.
\]

**Remark 1:** The condition for the (piecewise) continuity of the best responses has been also widely investigated in studies of Bayesian games [20], [21]. With Assumption 1(ii), (iv), and the Lipschitz continuity in (iii), if there exist positive constants \( \kappa \) and \( \nu \) such that for any \( \sigma_{3, l} \in \Sigma_{3, l}, \theta_i \in \Theta_i \), and \( \sigma_i^*(\theta_i) \in X_i \)
\[
-\mathcal{U}_i(\sigma_i^*, \sigma_{3, l}, \theta_i) = \mathcal{U}_i(\sigma_i^*, \sigma_{3, l}, \theta_i) + \omega \|\sigma_i^*(\theta_i) - X_i(\sigma_{3, l}, \theta_i)\|_\kappa
\]
where \( \mathcal{U}_i(\sigma_i^*, \sigma_{3, l}, \theta_i) \) is the minimum of \( \mathcal{U}_i(\sigma_i, \sigma_{3, l}, \theta_i) \) and \( \sigma_i^* = \arg\min_{\sigma_i \in \Xi_i} \mathcal{U}_i(\sigma_i, \sigma_{3, l}, \theta_i) \), then the best responses are (piecewise) continuous.

With Lemma 3, each subnetwork believes that the best response of \( \tilde{G} \) is a near-optimal strategy of \( G \) as the response to any rival’s strategy. With such a belief, both subnetworks adopt the best responses of \( \tilde{G} \), and form a DBNE.

Denote the supremum of intervals of discrete types as
\[
\Delta = \max_{l \in \{1, 2\}, i \in \{1, \ldots, N_i\}} (\theta_i^l - \theta_i^{l-1}).
\]
The next result gives a relation between the DBNE of \( \tilde{G} \) and the BNE of \( G \), whose proof can be found in Appendix B.

**Theorem 1:** Let \( (\tilde{\sigma}_1^*, \tilde{\sigma}_2^*) \) be a DBNE \((N_1, N_2)\) and \( (\sigma_1^*, \sigma_2^*) \) the BNE of \( G \). Under Assumption 1(iii)–(iv)
a) \( (\tilde{\sigma}_1^*, \tilde{\sigma}_2^*) \) is an \( \epsilon \)-BNE of \( G \), where \( \epsilon = O(D) \); 
b) for \( l = \{1, 2\} \), \( \|\tilde{\sigma}_i^* - \sigma_i^*\|_{H_i} \leq 4\epsilon/\nu \), that is to say, \( \lim_{N_1, N_2 \to \infty} \|\tilde{\sigma}_i^* - \sigma_i^*\|_{H_i} = 0 \).

By taking full advantage of the zero-sum condition, Theorem 1 shows that the DBNE \((N_1, N_2)\) of \( G \) converges to the BNE of \( G \), and provides an explicit error bound, compared with existing heuristic approximations [21], [23].

**Remark 2:** Here, we utilize the zero-sum condition to compute the error between the DBNE and the BNE. In general cases, Theorem 1(a) still holds. However, since \( \Theta_i \) is not compact, without the zero-sum condition, we cannot obtain Theorem 1(b) that the DBNE converges to the BNE of \( G \).

**Remark 3:** According to Assumption 1(iv) that \( \mu(\Theta_i) = \mu(\{\theta_i| p(\theta_i) > 0\}) \), when \( N_1, N_2 \to \infty \), \( \Delta \to 0 \), that is, the error bound \( \epsilon \to 0 \). Specifically, for a variety of distributions over \( \Theta_i \times \Theta_2 \), such as the uniform distribution and the beta distribution, we learn \( \epsilon = O(\max\{1/N_1, 1/N_2\}) \).

In this section, we regard the subnetworks as two players for simplification. In fact, the agents in the subnetworks are the real
players. Although more discrete points lead to higher approximate accuracy, this also increases the dimension of strategies in $G$, which brings barriers to implementation when agents exchange such high-dimensional strategies. Thus, we will break through the above barriers in the next section.

V. COMMUNICATION STEP

Since the communication resource of a network is limited, the high dimension of strategies is an obstacle to implementing our discretization in a distributed manner [25], [30], [31], [32]. Therefore, we provide a designed sparsification operator and a novel communication scheme to reduce communication loads, and show that agents can get unbiased estimations through our communication scheme.

Communication compression is a practical technique for reducing communication burden. Although this will slow down the convergence and increase the computation, it can still considerably improve the performance by greatly reducing communication loads. Motivated by [25], we use sparsification to reduce the size of data in order to meet the limited communication capabilities.

**Remark 4:** Existing communication compression methods focused on how to eliminate the bias from compression [30], [31], [32]. In their works, to keep the estimations unbiased, each agent (in a static network [30], [31]) or a center server (in a centralized network [32]) had to record others’ states. Besides, such an imperfect communication situation was also investigated in [33]. However, in a time-varying network, since agents cannot receive neighbors’ updates every iteration, the above methods are not practical and we need a novel one to ensure unbiased estimations.

Following the discretization step, we take a sparsification operator $Q_t: \mathbb{R}^{N_{m_1}} \to \mathbb{R}^{N_{m_1}}$ for $t \in \{1, 2\}$. Each agent selects and sends $d_1$ out of $N_{m_1}$ entries of a vector to its neighbors, while sending empty messages in the other entries. Denote the compression ratio by $p_1 = d_1/N_{m_1}$ and the encoded data size of each element by $C$. Thus, the communication data size $W$ at $t \geq 0$ is

$$W = C((E_1^t + E_{12}^t)N_{m_1}p_1 + (E_2^t + E_{21}^t)N_{m_2}p_2)$$

where $E_l^t$ is the number of edges in $G_l^t$, $l \in \{1, 2, 12, 21\}$. As (3) indicates, taking a large number of discrete points $N_l$ leads to heavy communication burden. Hence, we take a sparsification operator with its compression ratio $p_1 < 1$ to reduce the dimension of strategies to $N_{m_1}p_1$. Then we can reduce the communication loads to an affordable level, which is necessary for designing distributed algorithms. Here is a simple example.

**Example 2:** Given $X_1 = X_2 = [0, 1]^{10}$. Take $N_1 = N_2 = 1000$ for a certain accuracy. Thus, the dimension of strategies is 10 000. We adopt two different compression ratios to solve the DBNE: (i) $p_1 = 1$, i.e., without compression; (ii) $p_1 = 0.1$. Clearly, the amount of communication in (ii) is 1/10 of (i) as (3).

Following distributed optimizations [25], [34], we introduce a surplus vector for $v_t^l$ denoted by $\hat{s}_{t,i}^l \in \mathbb{R}^{N_{m_1}}$, which records variations of the strategies over time and is used to help the strategies approach the consensus.

For convenience, we split the vector-valued communication problem each entry by entry into individual scale-valued subproblems. In each subproblem, agents regard those who send nonempty messages as their neighbors. First, $v_t^l$ compresses both strategy and surplus, and sends them to current neighbors in $\Xi_l$. For $l \in \{1, 2\}$, $k \in \{1, \ldots, 2N_{m_1}\}$, define the graph for the $k$th entry of the sparsified vectors $col(Q_l(\sigma_t^l, Q_l(\hat{s}_{t,i}^l)))$ at time $t$ as $G_{t,i}^l$ and its in-neighbor set of $v_t^l$ as $A_{t,i}^l = \{v_t^l \mid v_t^l \in N_{t}^l, \{Q_l(\sigma_t^l, Q_l(\hat{s}_{t,i}^l))) \neq \emptyset\}$. The in-neighbor adjacency is: if $v_t^l \in A_{t,i}^l$, $|A_{t,i}^l| = 1$; otherwise, $|A_{t,i}^l| = 0$. Similarly, we define the $out$-neighbor adjacency matrix $B_{t,i}^l$.

Then $v_t^l$ compresses the estimation $\hat{s}_{t,i}^l$ of $\Xi_l$ and sends it to neighbors in $\Xi_{3-i}l$. For $k \in \{1, \ldots, N_{3-3m_1-l}\}$, the graph for the $k$th entry of $Q_{3-i-l}(\hat{s}_{3-i-l}^t)$ at time $t$ is denoted by $G_{3-i-l}^t$. Define its in-neighbor set as $C_{i,i}^t = \{v_{3-i-1}^t \mid v_{3-i-1}^t \in N_{3-3-i}^t, \{Q_{3-i-l}(\hat{s}_{3-i-l}^t) \neq \emptyset\}$ and its in-neighbor adjacency matrix $C_{i,i}^t$ as: if $v_{3-i-1}^t \in C_{i,i}^t$, $|C_{i,i}^t| = 1$; otherwise, $|C_{i,i}^t| = 0$.

On this basis, we design the following communication scheme with the sparsification operator to show how agents communicate and make estimations of both subnetworks.

**Communication Scheme:** For agent $v_t^l$ at time $t$,

1) send $Q_l(\sigma_t^l)$ and $Q_l(\hat{s}_{t,i}^l)$ to neighbors in $\Xi_l$, where $Q_l$ satisfies for $x(t) \in \mathbb{R}^{N_{m_1}}$ and $t = qR_0 + 1, \ldots, (q + 1)R_0$, $q \geq 0$,

$$Q_l(x(t))_k = \begin{cases} (qd_1 + 1) \mod N_{m_1} & \text{if } k \in \{q_1d_1 + 1 \mod N_{m_1} \}, \\ \ldots, (q + 1)d_1 \mod N_{m_1} & \text{empty otherwise} \end{cases}$$

2) estimate the $k$th entry $k \in \{1, \ldots, N_{m_1}\}$ of $\Xi_l$’s strategies based on $\hat{s}_{t,i}^l = \sum_{j=1}^{n_1} [A_{t,i}^l|j] [\hat{s}_{t,i}^l]|j$;

3) send $Q_l(\hat{s}_{t,i}^l)$ to neighbors in $\Xi_{3-i}$;

4) estimate the $k$th entry $k \in \{1, \ldots, N_{m_1}\}$ of $\Xi_{3-i}$’s strategies based on

$$\hat{s}_{t,i}^l = \sum_{j=1}^{N_{m_1}} [C_{i,i}^t|j] [\hat{s}_{3-i-l}^t]|j, \quad \text{if } |C_{i,i}^t| \neq 0,$$

otherwise.

In the Communication scheme, the estimation in step 2 is unbiased since agents can get access to their own strategies directly. In the inter-subnetwork communication, since agents may not receive messages in some subproblems, we use historical information for the estimations in step 4. Moreover, according to surplus-based algorithms, agents need to additionally compress and exchange surplus $s_{t,i}^l$.

Here, we propose a novel sparsification operator $Q_l$. Different from sparsification operators [25] that randomly select entries, our sparsification can better adapt to time-varying networks, as shown in Proposition 1, whose proof is presented in Appendix C.

**Proposition 1:** Define $R = \max\{R_0|N_{m_1}/d_1|\}$ and $S = S_0R$. Under Assumption 1, for $l \in \{1, 2\}$
1) for \( k \in \{1, \ldots, 2N_1m_1\} \), the digraph \( G^r_{l,k} \) is \( R \)-jointly strongly connected, i.e., \( U^{r+R-1} G^r_{l,k}(r) \) is strongly connected for any \( t \geq 0 \);  
2) for \( k \in \{1, \ldots, N_3m_3-1\} \), each agent’s neighbor set in \( G^r_{(3-l)-t}(t) \) satisfies \( \bigcup_{r=1}^{t+R-1} C_{l,k}(r) \neq \emptyset \) for any \( t \geq 0 \).

Proposition 1 shows that \( Q_t \) ensures the connectivity of \( G^r_{l,k} \) when \( G^r_{l,k} \) holds the connectivity in Assumption 1(v). Note that our design is to satisfy the worst case. For specific networks, we can improve our design to get smaller \( R \) and \( S \).

Define \( \sigma^r_{l,i} = \sum_{k=1}^{2N_1m_1} (\sigma^r_{l,i,k} + \hat{s}^r_{l,i,k}) \). The following result reveals the estimations are unbiased, whose proof is presented in Appendix C.

**Theorem 2:** Let Assumption 1(v) hold. Then 
1) \( \|\hat{\sigma}^r_{l,i} - \sigma^r_{l,i}\| \leq N_1m_1\epsilon_0 + S\sqrt{N_1m_1}\epsilon_1 \) for \( t \geq S \),  
2) \( \|\hat{\sigma}^r_{l,i} - \sigma^r_{l,i}\| \leq N_1m_1\epsilon_0 + S\sqrt{N_1m_1}\epsilon_1 \) for \( t \geq S \),  
where \( \epsilon_0 = \max_{t \leq S} \epsilon_0 \) and \( \epsilon_1 = \max_{t \leq S} \epsilon_1 \).

Theorem 2 shows that the estimation errors are controlled by \( \|\hat{\sigma}^r_{l,i} - \sigma^r_{l,i}\| \) and \( \|\hat{\sigma}^r_{l,i} - \sigma^r_{l,i}\| \). Since we will show that the above two terms tend to 0 in Section VII, which indicates that the estimation errors tend to 0, the designed communication is effective in exchanging information.

**VI. DISTRIBUTED ALGORITHM**

In this section, we present a distributed algorithm for an approximate BNE based on the above two steps.

Due to the special update rule of surplus-based algorithms, we cannot employ widely used constraint methods, such as projections and Lagrange multipliers. Thus, we give the following penalty function \( H_l(x) : \mathbb{R}^{m_l} \to \mathbb{R} \) to ensure the results lie in the action set \( \mathcal{X}_l \). Take a constant \( K_l > L_{l,1} \)

\[
H_l(x) = K_l \|x - \Pi_{\mathcal{X}_l}(x)\|.
\]

**Proposition 2:** For \( l \in \{1, 2\} \), \( \sigma_{3-l} \in \Sigma_{3-l} \), and \( \theta_{l}^r \in \Theta_l \),

a) \( H_l(x) \) is convex and \( K_l \)-Lipschitz continuous in \( \mathbb{R}^{m_l} \);  
b) \( h_l(x) = K_l \sum_{i=1}^{N_l} \phi_l^r \left( \sum_{k=1}^{2N_1m_1} \frac{1}{N_1m_1} \phi_{l,k}^r \right) \) is a subgradient of \( H_l(x) \);  
c) for any given \( \sigma_{3-l} \) and \( \theta_{l}^r \), all the minimizers \( \sigma_l(\theta_{l}^r) \) of \( \tilde{U}_l(\sigma_l, \sigma_{3-l}, \theta_{l}^r) + H_l(\sigma_l(\theta_{l}^r)) \) are in the action set \( \mathcal{X}_l \).

The proof of Proposition 2 is presented in Appendix D. Since \( H_l(x) = 0 \) for \( x \in \mathcal{X}_l \), by Proposition 2(c), all the equilibria of \( G \) with the expectation \( \tilde{U}_l + H_l \) are DBNE. Thus, we take \( \tilde{U}_l + H_l \) instead of \( \tilde{U}_l \) to seek a DBNE.

Denote the state of \( \Xi_l \) in the \( k \)th subproblem \( (k \in \{1, \ldots, N_l\}) \) by

\[
z^r_{l,k} = \text{col}(\sigma^r_{l,1,k}, \ldots, \sigma^r_{l,n_1,k}, \hat{s}^r_{l,1,k}, \ldots, \hat{s}^r_{l,n_1,k}).
\]

With the estimation \( \hat{\sigma}^r_{l,i}, \nu^r_{l,i} \) evaluates its subgradient by

\[
g_l,i(t) = \text{col} \left( \tilde{g}_l,i(t), \ldots, N_l \tilde{g}_l,i(t) \right)
\]

where \( \tilde{g}_l,i(t) = (h^r_{l,i}(t) + w^r_{l,i}(t))/N_l \), \( h^r_{l,i}(t) = h_l(\sigma^r_{l,i}(\theta_{l}^r)) \), and \( w^r_{l,i}(t) \) in \( \partial^T \tilde{U}_l(\sigma^r_{l,i}, \hat{\sigma}^r_{l,3-i}, \theta_{l}^r, r \in \{1, \ldots, N_l\} \). We summarize the distributed algorithm as follows.

**Algorithm 1:**

**Initialization:** For \( l \in \{1, 2\} \): let 

\[
\sigma^0_{l,i} = s^0_{l,i} = \sigma^0_{l,3-l} = x^0_l \in \tilde{\Sigma}_l \text{ for each } i \in \{1, \ldots, n_l\} \text{ and } j \in \{1, \ldots, n_{l-1}\}.
\]

**Discretization:** For \( l \in \{1, 2\} \), take \( N_l \) points from \( \Theta_l \) as the Discretization scheme.

Iterate until \( t \geq T \):

**Communication:** Agent \( v^r_{l,i} \in \mathcal{V}_l \) communicates and makes estimations \( \hat{\sigma}^r_{l,i}, \hat{\sigma}^r_{l,3-i} \) of both subnetworks based on the Communication scheme, respectively.  

**Update:** Agent \( v^r_{l,i} \) evaluates the subgradient \( g_l,i(t) \) based on (4), and updates \( z^r_{l,k} \) by

\[
z^{r+1}_{l,k} = \sum_{j=1}^{2n_l} \left( M_{l,i,k} z^r_{l,j,k} \right) + 1_{\{t \mod R = R-1\}} \eta \left( F^r_{l,i,k} \right)_{\{t/R\}j} + 1_{\{t \mod R = R-1\}} \alpha \left( t/R \right) \times \text{col}(g_l,i(R/t/R), 0_{N_l})
\]

**Theorem 3:** Let \( \tilde{\sigma}^r_{l,i}, \tilde{\sigma}^r_{l,3-i} \) be the BNE of \( G \). Under Assumption 1, Algorithm 1 generates a convergent sequence, and its limit point \( (\tilde{\sigma}^r_{l,i}, \tilde{\sigma}^r_{l,3-i}) \) satisfies that, for \( l \in \{1, 2\} \)

\[
\|\tilde{\sigma}^r_{l,i} - \sigma^r_{l,i}\|_{\mathcal{H}_l} \leq O(\Delta)
\]

where \( \Delta \) is the supremum of intervals of discrete types defined in (2). As \( N_l \) tends to infinity, \( \|\tilde{\sigma}^r_{l,i} - \sigma^r_{l,i}\|_{\mathcal{H}_l} \to 0 \).
Theorem 3 shows the convergence of Algorithm 1 to an approximate BNE, i.e., the DBNE, with an explicit error bound, as well as the convergence of the DBNE to the BNE.

In Algorithm 1, we should set up the number of discrete points \( N_l \) and the compression ratio \( \rho_l \) according to the actual situation. To achieve high estimation accuracy, we need to take more discrete points. However, as shown in (3), this brings excessive communication burden. Although the compression tool can reduce the burden, too small compression ratios will significantly slow down the convergence. Therefore, we need to take appropriate \( N_l \) and \( \rho_l \) for a trade-off between the estimation accuracy and the communication burden.

Since the stepsize adopted in Algorithm 1 is \( o(1/\sqrt{T}) \), it is hard to study the convergence rate directly. Notice that a stepsize which decays faster usually brings a slower convergence rate. Following the subgradient method [25], [35], we take the limit point \( \alpha(t) = 1/\sqrt{T + 1} \) to demonstrate that Algorithm 1 converges at a rate of \( O(\ln T/\sqrt{T}) \). Define \( E\tilde{U}_{l,i}(\sigma_l, \sigma_{3-l}) = \sum_{i=1}^{N_l} \tilde{U}_{l,i}(\sigma_l, \sigma_{3-l}, \theta_i) \).

**Theorem 4:** Take the stepsize \( \alpha(t) = 1/\sqrt{T + 1} \) for \( t \geq 0 \). Let \((\tilde{\sigma}_l^*, \bar{\sigma}_2^*)\) be the DBNE of \( \tilde{G} \). Under Assumption 1, Algorithm 1 generates a sequence \( \{\sigma_{l,i}^t\} \) satisfying

\[
\sum_{l=1}^{2} \sum_{i=1}^{n_l} (E\tilde{U}_{l,i}(\sigma_{l,i}^t, \sigma_{3-l}^t) - E\tilde{U}_{l,i}(\tilde{\sigma}_l^*, \bar{\sigma}_2^*)) = O(\ln T/\sqrt{T}).
\]

Theorem 4 shows that, as \( T \) tends to infinity, the cost function \( E\tilde{U}_{l,i}(\sigma_{l,i}^t, \sigma_{3-l}^t) \) converges to the optimal value \( E\tilde{U}_{l,i}(\tilde{\sigma}_l^*, \bar{\sigma}_2^*) \) at a rate of \( O(\ln T/\sqrt{T}) \) given the stepsize \( \alpha(t) = 1/\sqrt{T + 1} \). This result is consistent with existing distributed algorithms [25], [35], that is to say, the compression does not affect the order of magnitude of the convergence rate.

**VII. CONVERGENCE ANALYSIS**

In this section, we provide the convergence of Algorithm 1 with its convergence rate, namely, the proofs of Theorems 3 and 4.

**Proof of Theorem 3:** Recalling Theorem 1 that the DBNE of \( \tilde{G} \) is an approximate BNE of \( G \) with an explicit error bound, we only need to prove that Algorithm 1 generates a sequence that converges to the DBNE of \( \tilde{G} \). To this end, we show that agents reach consensus through the compressed communication and \( \sigma_{l,i}^t \) converges to the DBNE \( \tilde{\sigma}_l^* \) of \( \tilde{G} \).

First, we show the consensus, i.e., \( \lim_{t \to \infty} \left\| \sigma_{l,i}^t - \tilde{\sigma}_l^* \right\| \to 0 \). From the update rule (5), for \( q \geq 1 \), the \( k \)-th entry \( \{k \in \{1, \ldots, N_l m_l\} \} \) of \( \sigma_{l,i}^t \) can be reconstructed as

\[
\begin{align*}
\sigma_{l,i}^t &= \sum_{j=1}^{2 n_l} [M_{l,k}(q:0)]_{ij} z_{ij}^0 - \sum_{r=0}^{q-2} \sum_{j=1}^{n_l} [M_{l,k}(q-1:k)]_{ij} \alpha(q-1) [g_{l,i}((q-1)R)]_j]_k \\
&= \sum_{j=1}^{2 n_l} [M_{l,k}(q:0)]_{ij} z_{ij}^0 - \sum_{r=0}^{q-2} \sum_{j=1}^{n_l} [M_{l,k}(q-1:k)]_{ij} \alpha(q-1) [g_{l,i}((q-1)R)]_j]_k.
\end{align*}
\]

Because the column sum in \( M_{l,k}(q:0) \) equals to 1 for any \( q > q_1 \), the average state \( \sigma_{l,i}^R \) can be represented as

\[
[\sigma_{l,i}^R]_k = \frac{1}{n_l} \left( \sum_{j=1}^{2 n_l} [z_{ij}^0]_{ij} - \sum_{j=1}^{n_l} \sum_{r=0}^{q-2} \alpha(r) [g_{l,i}((q-1)R)]_j]_k \right) - \sum_{j=1}^{n_l} \alpha(q-1) [g_{l,i}((q-1)R)]_j]_k.
\]

Claim 1 (34): For \( k = 1, \ldots, N_l m_l \), there exists \( \Gamma_l = \sqrt{2 n_l m_l} > 0 \) such that

\[
\|M_{l,k}(q:0) - \frac{1}{m_l} [T^T 0^T [T^T 1^T] \|_\infty \leq \Gamma_l \xi_l
\]

where \( \xi_l = \max \lambda_2(M_{l,k}(q + 1:q)) \) and \( \lambda_2 \) is the second largest eigenvalue of \( M_{l,k}(q + 1:q) \).

Denote \( D_l = L_{l,1} + K_l \). The subgradients satisfy \( \|g_{l,i}(t)\| \leq D_l/\sqrt{N_l} \). Since \( \sqrt{\|x\|} \geq \sum_{k=1}^{n_l} \|x_k\| \geq \|x\| \) and \( \|x\|^2 = \sum_{k=1}^{n_l} \|x_k\|^2, x \in \mathbb{R}^l \), applying Claim 1 to (6) and (7)

\[
\left\| \sigma_{l,i}^R - \sigma_{l,i}^R \right\| \leq \sum_{k=1}^{n_l} \left( |\sigma_{l,i}^R| - |\sigma_{l,i}^R| \right) \leq 2 \Gamma_l R\xi_l q + D_l \sqrt{\Gamma_l} \left( n_l \Gamma_l \sum_{r=0}^{q-2} \xi_{l}^{q-1} \alpha(r) + 2 \alpha(q-1) \right)
\]

where \( R_l = \sum_{i=1}^{2 n_l} N_l m_l \|q \xi_l \| \). Furthermore, take \( \alpha(q) = 0 \) and \( \xi_q = 0 \) for all \( q < 0 \). Then

\[
\begin{align*}
\sum_{q=0}^{\infty} \alpha(q) \left| \sigma_{l,i}^R - \sigma_{l,i}^R \right| &\leq \sum_{q=0}^{\infty} \left( \Gamma_l R\xi_l q + 2 \alpha(q-1) \right) \leq \sum_{q=0}^{\infty} \alpha^2(r) \\
&\leq \sum_{q=0}^{\infty} \alpha(q) \left| \sigma_{l,i}^R - \sigma_{l,i}^R \right| \leq \sum_{q=0}^{\infty} \alpha(q) \left| \sigma_{l,i}^R - \sigma_{l,i}^R \right| \leq \sum_{q=0}^{\infty} \alpha^2(r) \leq \infty.
\end{align*}
\]

With \( \sum_{q=0}^{\infty} \alpha(q) = \infty \) and \( \sum_{q=0}^{\infty} \alpha^2(q) < \infty \), as \( q \) tends to infinity, \( \sigma_{l,i}^R \) converges to the average state \( \sigma_{l,i}^R \). Hence, the consensus is obtained for \( t = qR_l, q \to \infty \).

Then we consider the case for all \( t \geq 0 \). For \( i \in \{1, \ldots, n_i\} \) and \( t \in [qR_l, (q + 1)R_l - 1] \), there exists a matrix \( A \) such that \( \sigma_{l,i}^t = \sum_{j=1}^{n_j} [A]_{ij} \sigma_{l,j}^t \), where \( \sum_{j=1}^{n_j} [A]_{ij} = 1 \). Since the average state \( \sigma_{l,i}^t \) remains unchanged for \( t \in [qR_l, (q + 1)R_l - 1] \), \( q \geq 0 \), \( \| \sigma_{l,i}^t - \sigma_{l,i}^t \| \leq \sum_{j=1}^{n_j} [A]_{ij} \| \sigma_{l,i}^R - \sigma_{l,i}^R \| \leq \sum_{j=1}^{n_j} [A]_{ij} \| \sigma_{l,i}^R - \sigma_{l,i}^R \| \leq \max \| \sigma_{l,i}^R - \sigma_{l,i}^R \| \). Therefore, as \( t \to \infty \), \( \| \sigma_{l,i}^t - \sigma_{l,i}^t \| \to 0 \). Similarly, the surplus \( s_{l,i}^t \) vanishes.
Second, we show that $\sigma_i^{t}$ converges to the DBNE. With (5),
$$\sigma_{i}^{(q+1)R} = \sigma_{i}^{qR} - \frac{\alpha(q)}{n_i} \sum_{i=1}^{n_i} g_{i,i}(qR).$$
Thus
$$\|\sigma_{i}^{(q+1)R} - \sigma_{i}^{qR}\|^2 = \left\| \frac{\alpha(q)}{n_i} \sum_{i=1}^{n_i} g_{i,i}(qR) \right\|^2 + \|\sigma_{i}^{qR} - \sigma_{i}^{qR}\|^2 - 2\alpha(q) \sum_{i=1}^{n_i} |\sigma_{i}^{qR} - \sigma_{i}^{qR}, g_{i,i}(qR)|.$$ 

Consider the following Lyapunov candidate function:
$$V(q) = \|\sigma_{i}^{qR} - \sigma_{i}^{qR}\|^2 + \|\sigma_{i}^{qR} - \sigma_{i}^{qR}\|^2.$$

Then
$$V(q+1) \leq V(q) + (D_{i}^2/N_i + D_{i}^2/N_2)\alpha^2(q) + \sum_{i=1}^{2\alpha(q)} \left( \alpha(q) \sum_{i=1}^{n_i} |\sigma_{i}^{qR} - \sigma_{i}^{qR}, g_{i,i}(qR)| \right).$$ (9)

**Claim 2 ([36]):** Let $\{a_i\}, \{b_i\}$, and $\{c_i\}$ be nonnegative sequences satisfying $\sum_{i=0}^{\infty} b_i < \infty$. If $a_{i+1} \leq a_i + b_i - c_i$ for any $t$, then $a_t$ converges to a finite number and $\sum_{t=0}^{\infty} c_i < \infty$.

Next, we only need to show that the terms on the right side of (9) satisfy the conditions in Claim 2. Since $\sum_{t=0}^{\infty} \alpha^2(q) < \infty$, the second term satisfies $\sum_{t=0}^{\infty} (D_{i}^2/N_i + D_{i}^2/N_2)\alpha^2(q) < \infty$.

Based on the property of subgradients, for $\theta_i$,
$$N_l(\sigma_{i}^{qR} - \sigma_{i}^{qR}, g_{i,i}(qR)) = N_l(\sigma_{i}^{qR} - \sigma_{i}^{qR}, g_{i,i}(qR))$$
$$\leq 2D_l \left( \|\sigma_{i}^{qR} - \sigma_{i}^{qR}\|^2 + \sum_{i=1}^{n_i} |\sigma_{i}^{qR} - \sigma_{i}^{qR}, g_{i,i}(qR)| \right)$$
$$\leq U_i \left( \sigma_{i}^{qR}, \bar{\sigma}_{i}^{qR}, \theta_i \right) + H_i \left( \sigma_{i}^{qR}, \theta_i \right).$$

Moreover, by the Lipschitz continuity of $f_{i,i}$, for any $\theta_i \in \bar{\Theta}_i$,
$$|U_{i-1}(\sigma_{i}^{qR}, \bar{\sigma}_{i}^{qR}, \theta_i) - U_{i-1}(\sigma_{i}^{qR}, \bar{\sigma}_{i}^{qR}, \theta_i)| \leq 2L_{i-1} \|\sigma_{i}^{qR} - \bar{\sigma}_{i}^{qR}||.$$ Thus, the last term of (9) can be rewritten as
$$\sum_{t=1}^{2} \alpha(q) \sum_{i=1}^{n_i} |\sigma_{i}^{qR} - \sigma_{i}^{qR}, g_{i,i}(qR)| \leq \sum_{t=1}^{2} \alpha(q) \sum_{i=1}^{n_i} \left( \|\sigma_{i}^{qR} - \bar{\sigma}_{i}^{qR}\|^2 \right).$$

where $EH_i(\sigma_i) = \sum_{t=1}^{n_i} H_i(\sigma_i^{(t)}), \|\sigma_{i}^{qR} - \sigma_{i}^{qR}\|^2 \leq N_{i}m_{i} \left( \|\sigma_{i}^{qR} - \sigma_{i}^{qR}\|^2 + SD_{i} \sqrt{N_{i}m_{i} \alpha(q)} \right)$.

Next, we prove Theorem 4 about the convergence rate.

**Proof of Theorem 4:** Take $T_0 = |T/R|$. Since $H_i(x) \geq 0$, according to (10) and (11),
$$\sum_{t=1}^{2} \sum_{i=1}^{T_0} \left( \|\sigma_{i}^{qR} - \bar{\sigma}_{i}^{qR}\|^2 \right) \leq \sum_{t=1}^{2} \sum_{i=1}^{T_0} \left( \|\sigma_{i}^{qR} - \bar{\sigma}_{i}^{qR}\|^2 \right) \alpha(q).$$
where $C_2' = V(0) + C_2$. Due to the convexity of $E\tilde{U}_I$

\[
\frac{\sum_{q=0}^{T_0} \alpha(q) E\tilde{U}_I \left( \sigma^R_1, \tilde{\sigma}^R_{3-1} \right)}{\sum_{q=0}^{T_0} \alpha(q)} \geq E\tilde{U}_I \left( \frac{\sum_{q=0}^{T_0} \alpha(q) \sigma^R_1}{\sum_{q=0}^{T_0} \alpha(q)}, \tilde{\sigma}^R_{3-1} \right).
\]

(13)

Since $f_{i,i}$ is Lipschitz continuous in $x_i$, $E\tilde{U}_I(\sigma_i, \tilde{\sigma}_{3-1})$ is Lipschitz continuous in $\Sigma_I$. With (12) and (13)

\[
\sum_{l=1}^{2} \sum_{i=1}^{n_1} \left( E\tilde{U}_{l,i} \left( \sigma^R_{i,i}, \tilde{\sigma}^R_{3-1} \right) - E\tilde{U}_{l,i} \left( \sigma^R_{i,i}, \tilde{\sigma}^R_{3-1} \right) \right) \leq \sum_{l=1}^{2} \sum_{i=1}^{n_1} \frac{L_{l,i} \sum_{q=0}^{T_0} \alpha(q) \left\Vert \sigma^R_1 - \sigma^R_{i,i} \right\Vert}{\sum_{q=0}^{T_0} \alpha(q)} + C_5 \sum_{q=0}^{T_0} \alpha(q)
\]

\[
\leq C_4 \sum_{q=0}^{T_0} \alpha(q) + C_5 \sum_{q=0}^{T_0} \alpha(q)
\]

(14)

where $C_4 = C_1 + \sum_{l=1}^{2} n_1 L_{l,i} D_1 \sqrt{m_l} \left( \frac{n_1 \tau_1 \ell}{1 - \ell} + 2 \right)$, $C_5 = C_2' + \sum_{l=1}^{2} n_1 L_{l,i} \left( T_0 D_1 \sqrt{m_1 + \frac{T_0 R_1(\alpha(0))}{\tau_1}} \right)$. With $\alpha(q) = \frac{1}{\sqrt{T}}$, the terms on the right hand of (14) satisfy

\[
\frac{C_4 \sum_{q=0}^{T_0} \alpha(q)}{\sum_{q=0}^{T_0} \alpha(q)} = O \left( \frac{\ln T}{\sqrt{T}} \right), \quad \frac{C_5 \sum_{q=0}^{T_0} \alpha(q)}{\sum_{q=0}^{T_0} \alpha(q)} = O \left( \frac{1}{\sqrt{T}} \right).
\]

Thus, we complete the proof. \hfill \blacksquare

**VIII. NUMERICAL SIMULATIONS**

In this section, we provide numerical simulations to illustrate the effectiveness of Algorithm 1.

As in Example 1, we consider a rent-seeking game [23, 27], where two electricity companies $\Xi_1$ compete for the power supply, and each company has five power plants. Each company earns a profit according to its share of the supply. The action $x_1 = (x^1_1, x^1_2)$ is the effort to win the market share. The type $\theta_1$ is a continuous parameter that influences the cost, such as fuel price. Take $\chi_1 = \chi_2 = [0,1]^2$ and $\Theta_1 = \Theta_2 = [0.01, 1.01]$. Also, $\theta_1$ and $\theta_2$ are independent and uniformly distributed over $\Theta_1$ and $\Theta_2$, respectively. The cost functions of plants are as follows:

\[
f_{l,i}(x_1, x_2, \theta_1, \theta_2) = \sum_{k=1}^{2} \left( c \left( x^k_1 - 1 \right)^2 - (x^k_{3-1})^2 \right) + c_i \left( x^k_1 - x^k_{3-1} \right) (\theta_1 + \theta_2) + \frac{d_i}{2},
\]

where $c_i, \ldots, c_3 = (c_2, \ldots, c_7) = (0.2, 0.6, 1, 1.4, 1.8)$, $c = 0.6$, $(d_1, \ldots, d_5) = (0.5, 0.5, 1, 1.5, 1.5)$.

The communication graph switches periodically between graphs $G^r$ and $G^s$ given in Fig. 2, where $G(2k) = G^r$ and $G(2k+1) = G^s$, $k \geq 0$. In fact, the switching graph can describe signal switching and the link failure in network communication [9, 28].

Take the number of discrete points $N_1 = N_2 = N$ and the compression ratio $\rho_1 = \rho_2 = \rho$. First, we illustrate the convergence of Algorithm 1. We present the trajectories of strategies in $\Xi_1$ for types $\theta_1 = 0.1, 0.8$ under $N = 1000$ and $\rho = 0.5$ in Fig. 3. We can see that the agents’ strategies achieve consensus and convergence, which is consistent with Theorem 3. Fig. 4 shows strategy trajectories of agent 1 in $\Xi_1$ for types $\theta_1 = 0.1, 0.8$ under $N = 10, 20, 50, 200$ discrete points and the compression ratio $\rho = 1$. (a) $\theta_1 = 0.1$. (b) $\theta_1 = 0.8$.

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Next, we verify the approximation by Algorithm 1. Since the true continuous-type equilibrium in the infinite-dimensional space is hard to compute, we compare the DBNE obtained from
is continuous and thus, \( \theta \) is piecewise continuous, for any \( \theta \to 1 \) that is \( 2 \) except for finite points and \( \theta \in (0,1) \). In this article, we proposed a distributed algorithm for seeking a continuous-type BNE in subnetwork zero-sum games. We showed that the algorithm can obtain an approximate BNE with an explicit error bound with communication-efficient computation. Our algorithm involved two main steps. In the discretization step, we established a discretized model and revealed the relation with an explicit error bound between the DBNE of the discretized model and the BNE of the original model. In the communication step, we provided a novel communication scheme with a designed sparsification rule, which can effectively reduce the amount of communication and adapt well to time-varying networks. We also proved that agents can reach unbiased estimations through such communication. Finally, we provided the convergence analysis of the algorithm with its convergence rate.

**APPENDIX A**

**Proof of Lemma 3:** For any \( \theta^i_1 \in \tilde{\Theta}_1 \) and \( \theta^3_1 \in \Theta_3 \), according to Assumption 1(iv) and L’Hospital’s rule,

\[
\lim_{N_1 \to \infty} \frac{\int_{\Theta_1} \rho(\theta_1, \theta_2) d\theta_1}{\int_{\Theta_1} p(\theta_1) d\theta_1} = \frac{p(\theta_1, \theta_2) |_{\theta_1 = \theta^i_1}}{p(\theta^i_1)} = p(\theta_3 - 1 | \theta^i_1).
\]

Thus, for any \( \tilde{\sigma}_1 \in BR(\sigma_{3-1}) \) and \( \theta \in \tilde{\Theta}_1 \), there exists \( \tilde{\sigma}_1 \in BR(\sigma_{3-1}) \) that \( \tilde{\sigma}_1(N_1) \to \sigma_1(\theta) \) as \( N_1 \to \infty \). Since \( \sigma_1 \) is piecewise continuous, for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for any \( \theta \in \Theta \) except for finite points and \( \theta \in (0,1) \), \( |\sigma_1(\theta) - \sigma_1(\theta')| < \varepsilon \). Take \( \varepsilon = \Delta \), and then \( \int_{\Theta_1} |\tilde{\sigma}_1(N_1) - \sigma_1(\theta)|^2 d\theta \leq \varepsilon^2(\tilde{\Theta}_1 - \Theta_1) \). Therefore, as \( N_1 \to \infty \), \( \tilde{\sigma}_1(N_1) \to \sigma_1(\theta) \) for a.e. \( \theta \in \tilde{\Theta}_1 \).

**APPENDIX B**

**Proof of Theorem 1:** From Assumption 1(iii), for \( k \in \{1,2\} \), \( f_k \) is \( l_p \)-Lipschitz continuous over \( \Theta_k \), i.e., for \( \theta_1 \in (\theta^{-1}_1, \theta^i_1) \) and \( \theta_2 \in (\theta^{-1}_2, \theta^i_2) \), \( |f_k(x, x_2, \theta_1, \theta_2) - f_k(x_1, x_2, \theta_1, \theta_2)| \leq 2L_\theta \Delta \). According to (1), for \( \theta_1 \in (\theta^{-1}_1, \theta^i_1) \) and \( \theta_2 \in (\theta^{-1}_2, \theta^i_2) \), \( f_1(\tilde{\sigma}_1(\theta_1), \tilde{\sigma}_2(\theta_2), \theta^i_1, \theta^i_2) \). Due to the Lipschitz continuity of \( f_k \), for any \( \tilde{\sigma}_i \in \bar{\Sigma}_i \)

\[
EU_1(\tilde{\sigma}_1, \tilde{\sigma}_3) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \int_{\Theta_1} \int_{\Theta_2} \left( p(\theta_1, \theta_2) \times (f_1(\tilde{\sigma}_1(\theta_1), \tilde{\sigma}_2(\theta_2), \theta^i_1, \theta^i_2) - f_1(\tilde{\sigma}_1(\theta_1), \tilde{\sigma}_2(\theta_2), \theta^i_1, \theta^i_2)) \right) \leq EU_1(\tilde{\sigma}_1, \tilde{\sigma}_3) + 2L_\theta \Delta.
\]

Due to Assumption 1(iv), \( P(\theta_1, \theta_2) \) is continuous and thus, \( p(\theta_1, \theta_2) \) is \( L_p \)-Lipschitz continuous over \( \Theta_1 \times \Theta_2 \). Therefore,

\[
\left| p(\theta_1, \theta_2) - \frac{1}{\theta^i_1 - \theta^{-1}_1} \int_{\theta^{-1}_1}^{\theta^i_1} p(\theta_1', \theta_2) d\theta'_1 \right| \leq \frac{1}{\theta^i_1 - \theta^{-1}_1} \int_{\theta^{-1}_1}^{\theta^i_1} |p(\theta_1, \theta_2) - p(\theta_1', \theta_2)| d\theta'_1 \leq L_p \Delta.
\]
Since \( f_1 \) is Lipschitz continuous and \( \lambda_t \) and \( \Theta_t \) are compact, there exists \( M > 0 \) such that \( |f_1| \leq M \). From (16)
\[
\sum_{j=1}^{N_2} \int_{0}^{T_j^0} f_1(x(\theta_1), \sigma_2(\theta_2), \theta_1, \theta_2) \, d\theta_2 \, p(\theta_1, \theta_2) \, d\theta_1 \, d\theta_2
\]
\[
\geq \sum_{j=1}^{N_2} \int_{0}^{T_j^0} f_1(x(\theta_1), \sigma_2(\theta_2), \theta_1, \theta_2) \, d\theta_2 \, p(\theta_1, \theta_2) \, d\theta_1 - M L_{\rho} (\theta_1 - \theta_1^{-1}) (\bar{y}_2 - \bar{y}_2) \Delta
\]
\[
\geq \bar{U}_1(\sigma_1^*, \sigma_2^*, \theta_1)/N_1 - M L_{\rho} (\theta_1 - \theta_1^{-1}) (\bar{y}_2 - \bar{y}_2) \Delta.
\]
Thus, for any \( \sigma_1 \in \Sigma_1 \), recalling the definition of the DBNE
\[
E \bar{U}_1(\sigma_1^*, \sigma_2^*) \leq M L_{\rho} (\theta_1 - \theta_1^{-1}) (\bar{y}_2 - \bar{y}_2) \Delta + E U_1(\sigma_1, \sigma_2^*).
\]
Define \( C = 2L_0 + M L_{\rho}(\theta_1 - \theta_1^{-1})(\bar{y}_2 - \bar{y}_2) \). With combining (15) and (17), for any \( \sigma_1 \in \Sigma_1 \)
\[
E U_1(\sigma_1^*, \sigma_2^*) \leq E \bar{U}_1(\sigma_1^*, \sigma_2^*) + 2L_0 \Delta \leq E U_1(\sigma_1^*, \sigma_2^*) + C \Delta.
\]
Similarly, for any \( \sigma_2 \in \Sigma_2 \), \( E U_2(\sigma_2^*, \sigma_1^*) \leq E U_2(\sigma_2^*, \sigma_1^*) + C \Delta \). Hence, the DBNE is an \( \epsilon \)-BNE with \( \epsilon = C \Delta \).

Next, we prove that the DBNE converges to the BNE. By the zero-sum condition
\[
E U(\sigma_1^*, \sigma_2^*) \geq E U(\sigma_1^*, \sigma_2^*) \geq E U(\sigma_1^*, \sigma_2^*) - \epsilon
\]
\[
\geq E U(\sigma_1^*, \sigma_2^*) - 2\epsilon \geq E U(\sigma_1^*, \sigma_2^*) - 2\epsilon.
\]
Since \( f_t \) is \( \nu \)-strongly convex, for any \( \theta_1 \in \Theta_1 \) and \( \theta_2 \in \Theta_2 \)
\[
f_t(\sigma_1(\theta_1), \sigma_2(\theta_2), \theta_1, \theta_2) - f_t(\sigma_1^*(\theta_1), \sigma_2^*(\theta_2), \theta_1, \theta_2)
\]
\[
\geq (\nabla f_t(\sigma_1^*(\theta_1), \sigma_2^*(\theta_2), \theta_1, \theta_2))^T(\bar{\sigma}_1(\theta_1) - \sigma_1^*(\theta_1)) + \frac{\nu}{2} \| \bar{\sigma}_1(\theta_1) - \sigma_1^*(\theta_1) \|^2.
\]
Compute the conditional expectation of (19) over \( \Theta_2 \), and then
\[
U_t(\sigma_1^*, \sigma_2^*, \theta_1) - U_t(\sigma_1^*, \sigma_2^*, \theta_1)
\]
\[
\geq \frac{\nu}{2} \| \bar{\sigma}_1(\theta_1) - \sigma_1^*(\theta_1) \|^2.
\]
Since \( (\sigma_1^*, \sigma_2^*) \) is the BNE, i.e., for a.e. \( \theta_1 \in \Theta_1 \),
\[
\sigma_1^*(\theta_1) = \arg \min_{\sigma_1(\theta_1) \in \Theta_1} U_t(\sigma_1(\theta_1), \sigma_2^*, \theta_1), \text{ we have } (\bar{\sigma}_1(\theta_1) - \sigma_1^*(\theta_1)) \nabla U_t(\sigma_1^*, \sigma_2^*, \theta_1) \geq 0.
\]
Thus, applying (20) to (18)
\[
2\epsilon \geq E U(\sigma_1^*, \sigma_2^*) - E U(\sigma_1^*, \sigma_2^*) \geq \frac{\nu}{2} \| \bar{\sigma}_1(\theta_1) - \sigma_1^*(\theta_1) \|^2,
\]
namely, \( \| \bar{\sigma}_1(\theta_1) - \sigma_1^*(\theta_1) \|^2 \leq \frac{4\epsilon}{\nu} \). Similarly, \( \| \bar{\sigma}_2(\theta_2) - \sigma_2^*(\theta_2) \|^2 \leq \frac{4\epsilon}{\nu} \). As \( N_1, N_2 \rightarrow \infty, \epsilon \rightarrow 0 \), and thus, \( (\bar{\sigma}_1(\theta_1), \bar{\sigma}_2(\theta_2)) \rightarrow (\sigma_1^*, \sigma_2^*) \).

APPENDIX C

Proof of Proposition 1: With Assumption 1(v), each agent \( v_j^0 \) is able to communicate with agents in \( \Xi_1 \) every \( \mathcal{R}_0 \) iterations and agents in \( \Xi_3 \) every \( S_0 \) iterations, which means that \( v_j^0 \) can receive nonempty messages at each entry from \( \Xi_1 \) in \( \mathcal{R}_0[N_1 m_1/d_1] \) iterations and from \( \Xi_3 \) in \( S_0 \) \( [N_3 m_3/d_3] \) iterations. Thus, \( G_{j,k}^t \) satisfies Proposition 1 with \( \mathcal{R} = \max\{\mathcal{R}_0[N_1 m_1/d_1]\} \) and \( S = S_0 \).
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