STANLEY CONJECTURE IN SMALL EMBEDDING DIMENSION

IMRAN ANWAR, DORIN POPESCU

Abstract. We show that Stanley’s conjecture holds for a polynomial ring over a field in four variables. In the case of polynomial ring in five variables, we prove that the monomial ideals with all associated primes of height two, are Stanley ideals.

Key words: Monomial Ideals, Prime Filtrations, Pretty Clean Filtrations, Stanley Ideals.

2000 Mathematics Subject Classification: Primary 13P10, Secondary 13H10, 13F20, 13C14.

1. Introduction

Let $S = K[x_1, x_2, ..., x_n]$ be a polynomial ring in $n$ variables over a field $K$ and $I \subset S$ a monomial ideal. In this paper a prime filtration of $I$ is assumed to be a monomial prime filtration, that is a monomial filtration

$$\mathcal{F}: \quad I = I_0 \subset I_1 \subset \ldots \subset I_r = S$$

with $I_j/I_{j-1} \cong S/P_j(-a_j)$ for some monomial prime ideals $P_j$ of $S$, $a_j \in \mathbb{N}^n$ and $j = 1, 2, ..., r$. Set $\text{Supp}(\mathcal{F}) = \{P_1, \ldots, P_r\}$. After [4], the prime filtration $\mathcal{F}$ is called pretty clean, if for all $i < j$ for which $P_i \subseteq P_j$ it follows that $P_i = P_j$. The monomial ideal $I$ is called pretty clean, if it has a pretty clean filtration.

Let $I \subset S$ be a monomial ideal, any decomposition of $S/I$ as a direct sum of $K$-vector spaces of the form $uK[Z]$ where $u$ is a monomial in $S$, and $Z \subseteq \{x_1, x_2, ..., x_n\}$ is called Stanley decomposition. Stanley [9] conjectured that there always exists a Stanley decomposition

$$S/I = \bigoplus_{i=1}^{r} u_i K[Z_i]$$

such that $|Z_i| \geq \text{depth}(S/I)$ for all $i$, $1 \leq i \leq r$. If this is the case, we call $I$ a Stanley ideal. Sometimes Stanley decompositions of $S/I$ arise from prime filtrations. In fact, if $\mathcal{F}$ is a prime filtration of $S/I$ with factors $(S/P_i)(-a_i)$ for $i = 1, 2, ..., r$ then set

The authors are highly grateful to the School of Mathematical Sciences, GC University, Lahore, Pakistan in supporting and facilitating this research. The second author was supported by CNCSIS and the Contract 2-CEX06-11-20/2006 of the Romanian Ministry of Education and Research and the Higher Education Commission of Pakistan.
\[ u_i = \prod_{j=1}^{n} x_j \text{ and } Z_i = \{x_j^{a_{ij}} : x_j \not\in P_i\} \]
and we have
\[ S/I = \bigoplus_{i=1}^{r} u_i K[Z_i] \]

If \( F \) is a pretty clean filtration of \( S/I \), then by [4, Corollary 3.4]
\[ \text{Ass}(S/I) = \text{Supp}(F) \]
The converse is not always true (see [8, Example 4.4]). However not all Stanley decompositions arise from the prime filtrations (see [3, Example 3.8]). A prime filtration \( F \) is a Stanley filtration if the Stanley decomposition arising from \( F \) satisfies the Stanley conjecture. In [8, Proposition 2.2] it is shown that all prime filtrations \( F \) for which \( \text{Ass}(S/I) = \text{Supp}(F) \) are Stanley filtrations, in particular all monomial ideals \( I \subset S \) for which \( S/I \) is pretty clean, are Stanley (see [4, Theorem 6.5]). In case \( n = 3 \), for any monomial ideal \( I \subset S \) we have \( S/I \) pretty clean by [8, Theorem 1.10] and so \( I \) is Stanley. This result was first obtained by different methods in [2]. Recently, Herzog, Soleyman Jahan, Yassemi [3, Proposition 1.4] showed that if \( I \) is a monomial ideal of \( S \) (for any \( n \)) such that \( S/I \) is Cohen-Macaulay of codimension two then \( I \) is a Stanley ideal.

It is the purpose of our note to describe the Stanley ideals of the polynomial ring \( S = K[x_1, x_2, \ldots, x_n] \), \( n \leq 5 \). If \( n = 4 \) we show that all monomial ideals of \( S \) are Stanley (see Theorem 2.4). This extends [1, Corollary 1.4], which says that a sequentially Cohen-Macaulay monomial ideal \( I \subset K[x_1, \ldots, x_4] \) is Stanley. If \( n = 5 \) we show that all monomial ideals \( I \subset S \) having all the associated prime ideals of height 2 are Stanley ideals (see Corollary 2.3).

2. Stanley’s Conjecture in small embedding dimension

We start with a very elementary Lemma.

**Lemma 2.1.** Let \( S = K[x_1, x_2, \ldots, x_n] \), \( T = K[x_1, x_2, \ldots, x_r] \) for some \( 1 \leq r \leq n \) and \( J \subset T \) a monomial ideal. Then \( T/J \) is pretty clean if and only if \( S/J S \) is pretty clean.

The following lemma is a key result in this note.

**Lemma 2.2.** Let \( S = K[x_1, x_2, \ldots, x_n] \), \( n \leq 5 \) be a polynomial ring and \( I \subset S \) a monomial ideal having all the associated primes of height 2. Then there exists a prime filtration \( F \) of \( S/I \) such that \( \text{ht}(P) \leq 3 \) for all \( P \in \text{Supp}(F) \).

**Proof.** We use induction on \( s(I) \), where \( s(I) \) denotes the number of irreducible monomial ideals appearing in the unique decomposition of \( I \) as an intersection of irreducible monomial ideals (see [10, Theorem 5.1.17]), let us say
\[ I = \bigcap_{i=1}^{s(I)} Q_i; \]
where $Q_i$’s are irreducible monomial ideals of codimension 2. If $s(I) = 1$, then the result follows because $S/I$ is clean. If $s(I) \geq 1$, then set

$$J = \bigcap_{i=2}^{s(I)} Q_i.$$ 

Therefore $I = J \cap Q_1$. We may suppose $Q_1 = (x_1^{d_1}, x_2^{d_2})$ after renumbering of variables, with $d_1$ the largest power of $x_1$ in $\cup_{i=1}^{s(I)} G(Q_i)$, where $G(Q_i)$ is the set of minimal monomial generators of $I$. We claim that

$$F_0 = I \subset F_1 = (I, x_2^{d_2}) \subset F_2 = (x_1^{d_1}, x_2^{d_2}) \subset F_3 = S$$

is a filtration of $S/I$, which will give the desired filtration by refining. Note that $F_3/F_2 = S/(x_1^{d_1}, x_2^{d_2})$ is a clean module, so $F_3/F_2$ has a prime filtration involving only the prime $(x_1, x_2)$.

Now for $F_2/F_1 = (x_1^{d_1}, x_2^{d_2})/(I, x_2^{d_2}) \cong S/((I, x_2^{d_2}) : x_1^{d_1})$ we have

$$E := ((I, x_2^{d_2}) : x_1^{d_1}) = (I : x_1^{d_1}, x_2^{d_2}) = (J : x_1^{d_1}, x_2^{d_2})$$

and we get

$$E = \bigcap_{i=2}^{s(I)} ((Q_i : x_1^{d_1}), x_2^{d_2}).$$

Set $T = K[x_2, \ldots, x_n]$. Since $U_i := ((Q_i : x_1^{d_1}), x_2^{d_2})$ is either $S$, or an irreducible ideal of height 2 or 3 in the variables $x_2, \ldots, x_n$, we note that $E = WS$ for a monomial ideal $W \subset T$ with all associated prime ideals of dimension $n-2$ or $n-3$. If $n = 4$ then dim $T = 3$ and $T/W$ is pretty clean by [5, Theorem 1.10] and so $S/E$ is pretty clean by Lemma 2.1. If $n = 5$ then set $G := \bigcap_{i=2, \ ht(U_i) = 2} U_i$ and consider the filtration $W \subset G \subset T$ (this is the dimension filtration of [4]). As $s(G) < s(I)$ we get by induction hypothesis a prime filtration of $S/G$ involving just prime of height $\leq 3$. Since Ass($G/W$) contains just prime ideals of height 3 we get $G/W$ clean by [5, Corollary 2.2]. So we get a prime filtration of $T/W$ and by extension of $S/E$, involving only prime ideals of height $\leq 3$. Therefore in both cases, $F_2/F_1 \cong S/E$ has a prime filtration involving only prime ideals of height at most 3.

Finally, $F_1/F_0 = (I, x_2^{d_2})/I \cong S/(I : x_2^{d_2})$, and we have

$$(I : x_2^{d_2}) = (J : x_2^{d_2}) = \bigcap_{i=2}^{s(I)} (Q_i : x_2^{d_2}),$$

where $(Q_i : x_2^{d_2})$ is either $S$, or irreducible of height 2. Since $s(I : x_2^{d_2}) \leq s(I) - 1 < s(I)$, we get by induction hypothesis that $S/(I : x_2^{d_2})$ (and so $F_1/F_0$) has a prime filtration with prime ideals of height at most 3. Then by gluing together all these prime filtrations we get the desired one.

\[\square\]

**Corollary 2.3.** Let $S = K[x_1, x_2, \ldots, x_n]$, $n \leq 5$ be a polynomial ring and $I \subset S$ a monomial ideal having all the associated prime ideals of height 2. Then $I$ is a Stanley ideal.
Proof. If $S/I$ is Cohen-Macaulay then $I$ is a Stanley ideal by [3 Proposition 1.4]. Now if $S/I$ is not Cohen-Macaulay then $\text{depth}(S/I) \leq n - 3$ because $\text{dim}(S/I) = n - 2$. Let $\mathcal{F}$ be the filtration given by Lemma 2.2. Then all the associated primes $P$ of $\text{Supp}(\mathcal{F})$ satisfy the condition,

$$\text{dim}(S/P) \geq n - 3 \geq \text{depth}(S/I).$$

Thus $I$ is a Stanley ideal. □

Theorem 2.4. Any monomial ideal $I$ of $S = K[x_1, x_2, x_3, x_4]$ is a Stanley ideal.

Proof. Let $s(I)$ be the number of irreducible monomial ideals appearing in the unique decomposition of $I$ as an intersection of irreducible monomial ideals, let us say $I = \bigcap_{i=1}^{4} Q_{i_1} \cap Q_{i_2} \cap Q_{i_3} \subset \mathcal{F}_2 = Q_1 \cap Q_2 \cap \mathcal{F}_3 = Q_1 \subset \mathcal{F}_4 = S$

where $Q_{i_j}$’s are irreducible monomial ideals of height $i$. If $ht(I) = t$, $1 \leq t \leq 4$ then $Q_k = S$ and $s(Q_k) = 0$ for all $1 < k < t$. After Schenzel [7], the dimension filtration of $I$ will be

$$\mathcal{F}_0 = I \subset \mathcal{F}_1 = Q_1 \cap Q_2 \cap Q_3 \subset \mathcal{F}_2 = Q_1 \cap Q_2 \subset \mathcal{F}_3 = Q_1 \subset \mathcal{F}_4 = S$$

Now consider $\mathcal{F}_4/\mathcal{F}_3 \cong S/\mathcal{Q}_1$, where $Q_1 = \bigcap_{i_j=1}^{s(Q_1)} Q_{i_j}$ with $Q_{i_j}$’s principal ideals. Therefore $Q_1 = (u)$ for a monomial $u$ in $S$ (it is factorial ring) and so $S/Q_1$ is pretty clean (see e.g. the proof of [8, Lemma 1.9]). Hence $\mathcal{F}_3$ is a Stanley ideal.

Now take $\mathcal{F}_3/\mathcal{F}_2 \cong Q_1/(Q_1 \cap Q_2) \cong S/(Q_2 : u)$, where

$$(Q_2 : u) = \bigcap_{j=1}^{s(Q_2)} (Q_{2j} : u)$$

Also $(Q_{2j} : u)$ is either $S$ or of height 2 for all $j$. By Corollary 2.3 $(Q_2 : u)$ is a Stanley ideal and so is $\mathcal{F}_2$ because the clean filtrations of $S/(u)$ involve only prime ideals of depth 3. If $s(Q_3) = s(Q_4) = 0$ then we are done. If $s(Q_3) \neq 0$ then $\text{depth}(S/I) \leq 1$ and $\text{Ass}(\mathcal{F}_2/\mathcal{F}_1)$ contains only prime ideals of height 3. Hence $\mathcal{F}_2/\mathcal{F}_1$ is pretty clean by [6 Corollary 2.2] and so $\mathcal{F}_2/\mathcal{F}_1$ has a prime filtration involving only prime ideals of height 3. But as above $S/\mathcal{F}_2$ has a prime filtration with prime ideals of height $\leq 3$. Gluing together these two prime filtrations we get a prime filtration with prime ideals $P$ such that $\text{dim}(S/P) \geq 1 \geq \text{depth}(S/I)$. So $\mathcal{F}_1$ is a Stanley ideal. If $s(Q_4) \neq 0$ then $\text{depth}(S/I) = 0$ and every prime filtration gives a Stanley filtration. □

References

[1] S. Ahmad, D. Popescu, Sequentially Cohen-Macaulay monomial ideals of embedding dimension four, Preprint Lahore 2007, Arxiv:math.AC/0702569.
[2] J. Apel, On a conjecture of R.P. Stanley, in Part I - Monomial ideals. J. Algebra. Comb., 17, 36-59(2003).
[3] J. Herzog, A. Soleyman Jahan, S. Yassemi, Stanley decompositions and partitionable simplicial complexes, Preprint 2007, [Arxiv:math.AC/0612848v2].
[4] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicomplexes, Manuscripta Math. 121, (2006), 385-410.
[5] D. Maclagan, G. Smith, Uniform bounds on multigraded regularity J.Alg.Geom. 14(2005), 137-164.
[6] D. Popescu, Criterions for shellable multicomplexes, An. St. Univ. Ovidius, Constanta 14(2), (2006),73-84. [Arxiv:math.AC/0505655]
[7] P. Schenzel, On the dimension filtrations and Cohen-Macaulay filtered modules, In: Commutative algebra and algebraic geometry (Ferrara), Lecture Notes in Pure and Appl. Math. 206, Dekker, New York, 1999, 245-264.
[8] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations, to appear in J. Alg, [Arxiv:math.AC/0605119]
[9] R. P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math.68, (1982), 175-193.
[10] R. H. Villarreal, Monomial algebras, Dekker, New York, 2001.

Imran Anwar, School of Mathematical Sciences, 68-B New Muslim Town, Lahore, Pakistan.
E-mail address: iimrananwar@gmail.com

Dorin Popescu, Institute of Mathematics "Simion Stoilow", University of Bucharest, P.O.Box 1-764, Bucharest 014700, Romania
E-mail address: dorin.popescu@imar.ro