On the extendibility of partially and Markov exchangeable binary sequences

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Abstract

In [Fortini et al., Stoch. Proc. Appl. 100 (2002), 147–165] it is demonstrated that a recurrent Markov exchangeable process in the sense of Diaconis and Freedman is essentially a partially exchangeable process in the sense of de Finetti. In case of finite sequences there is not such an equivalence. We analyze both finite partially exchangeable and finite Markov exchangeable binary sequences and formulate necessary and sufficient conditions for extendibility in both cases.

Keywords: Markov exchangeability; Partial exchangeability; Extendibility

2000 Mathematics Subject Classification. 60G09.

1 Introduction

A finite sequence of r.v.s \( (X_1, \ldots, X_n) \) defined on a common probability space is said exchangeable (sometimes \( n \)-exchangeable) if its joint distribution is invariant under permutations of its components. The sequence may or may not be the initial segment of a longer exchangeable sequence, i.e., as is said, it may or may not be “extendible”, and is said \( \infty \)-extendible, if it is the initial segment of an infinite exchangeable sequence. de Finetti characterized all the \( n \)-exchangeable sequences of r.v.s taking values in a finite space \( I \), disregarding their extendibility, as unique mixtures of certain \( n \)-exchangeable, not extendible distributions, namely the hypergeometric processes. From this result, he has been able to demonstrate his representation theorem for exchangeable infinite sequences by a passage to the limit, and in [5] derived necessary and sufficient conditions for extendibility of \( \{0, 1\} \)-valued finite sequences in a geometric approach (see also [7], [10], [2] and [23]).

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Under partial exchangeability, introduced in \cite{3} (pp. 193–205 of \cite{17}), \((X_1, \ldots, X_n)\) is divided into groups or subsequences (e.g. women and men) accordingly to a characteristic we consider relevant (e.g. each unit’s sex), and we retain exchangeability to hold just for variables within the same subsequence. Again, we can represent every finite partially exchangeable sequence as a mixture of not extendible, partially exchangeable sequences, and an analogous representation theorem holds if all the exchangeable subsequences forming it are \(\infty\)-extendible. de Finetti in \cite{4} (pp. 147–227 of \cite{6}), suggested to consider in a sequence of observations the last observation preceding the present one as a relevant characteristic to define an interesting case of partial exchangeability. Consider a finite state space \(I\); call the variables immediately subsequent any occurrence of \(i \in I\) the successors of \(i\). Then the subsequences forming the partially exchangeable sequence are those constituted of the successors of each state in \(I\). He apparently suggested the possibility to characterize, by the usual passage to the limit, the mixtures of Markov Chains processes as partially exchangeable processes of that kind.

Diaconis and Freedman in \cite{9} demonstrated that the limit argument does not hold for mixtures of transient Markov Chains. They dropped the intuitive idea of “relevant characteristics”, introduce a different notion of partial exchangeability in terms of sufficient statistics (we will call this case Markov exchangeability) and characterized the mixtures of Markov Chains under the additional assumption of recurrence of the process. In \cite{13} it is demonstrated that the two definitions (that in terms of subsequences and that in terms of sufficient statistics) coincide in case of recurrent processes. But they differ in case of finite sequences.

We will focus on partial exchangeability in the sense of de Finetti and Markov exchangeability for finite sequences of \(\{0, 1\}\)-valued variables, and on the respective notions of extendibility. Some necessary conditions for the extendibility of a partially exchangeable finite sequence have been studied in \cite{21} and in \cite{20}. Finite Markov exchangeable sequences have been analyzed in \cite{24, 25} but, as far as I know, no criterion for extendibility in the Markov exchangeable case has been given. In Section 3 we define a general framework in order to analyze this topic. In Sections 3 we analyze the partially exchangeable case. In particular we present two bijective transformations of the probabilities defining a binary partially exchangeable distribution (i.e. two alternative parameterizations). The first, introduced by de Finetti, allows us to establish necessary and sufficient conditions for extendibility developing the geometric approach presented in \cite{5}, \cite{7} and \cite{2} for the simply exchangeable case. The second parameterization, in
terms of generalized covariances, allows us to derive some simpler necessary conditions related to the central moments of the mixing distributions. In Section 4 we formulate analogous results for Markov exchangeable distributions.

2 A general setting

A sequence partially exchangeable in the sense of de Finetti is essentially a set of distinct exchangeable subsequences. The concept of partial exchangeability has been extended in various ways, relating to ergodic theory and extreme points of a convex set, (see [14], [12], [11], [1, chap. 12]). In our case of discrete time processes taking values in a finite state space, we will refer to a simple formalization in terms of sufficient statistics borrowed from [8], (see also [24]). With this formalization, we can represent also simple exchangeability and partial exchangeability in the sense of de Finetti.

Let \((Ω, F, P)\) be the probability space on which all the r.v.s in the sequel will be defined. Consider a sequence of \(n\) r.v.s \((X_1, \ldots, X_n)\) each taking values in a finite set \(I\). Consider a statistic \(T\) from \(I^n\) into a finite set \(\{t_1, \ldots, t_z\}\). We call the sequence, as well as its joint distribution, \(n\)-partially exchangeable with respect to \(T\) if:

\[
T(x_1) = T(x_2) \Rightarrow P(x_1) = P(x_2) \quad \forall x_1, x_2 \in I^n
\]  

(1)

That is, \(T\) induces a partition of \(I^n\) into \(z\) equivalence classes and \(P\) attributes the same probability to the elements within the same class. So we can say that \(T\) is a minimal sufficient statistic for \((X_1, \ldots, X_n)\) under \(P\). Denote with \([t_i]\) the set \(\{x \in I^n : T(x) = t_i\}\); denote \(P(x \in [t_i])\) as \(w_{t_i}\), and the probability of any specified sequence in \([t_i]\) as \(p_{t_i}\). We have \(w_{t_i} = |[t_i]| \cdot p_{t_i}\) where \(|[t_i]|\) denotes the cardinality of the set \([t_i]\), and the distribution of \((X_1, \ldots, X_n)\) is completely defined by the \(z\) probabilities \((w_{t_1}, \ldots, w_{t_z})\) subjected to \(\sum_{i=1}^{z} w_{t_i} = 1\). On the converse, any set of nonnegative values \((w_{t_1}, \ldots, w_{t_z})\) having sum 1, defines a sequence \(n\)-partially exchangeable w.r.t. \(T\). Consequently the space

\[
\Diamond_z = \left\{(w_{t_1}, \ldots, w_{t_z}) : w_{t_i} \geq 0, \; i = 1, \ldots, z, \; \sum_{i=0}^{z} w_{t_i} = 1\right\}
\]  

(2)

which is the \((z - 1)\)-dimensional unitary simplex embedded in \(\mathbb{R}^z\), represents all the distributions \(n\)-partially exchangeable w.r.t. \(T\). Let \(h_{[t_i]}(x) = P(x \mid T(x) = t_i)\) be
the conditional probability distribution on $I^n$ given $T$, assessing equal masses to all the sequences in the equivalence class $[t_i]$ and mass 0 to the other sequences:

$$
    h_{[t_i]}(x) = \begin{cases} 
    1/|t_i| & \text{if } x \in [t_i] \\
    0 & \text{otherwise} 
    \end{cases}
$$

Then the following stated in [8] is plain:

**Theorem 2.1 ([8]).** The set of all the distributions over $I^n$ partially exchangeable w.r.t. $T$ is a simplex whose vertices are the extremal distributions $h_{[t_i]}$, $i = 1, \ldots, z$, and each partially exchangeable distribution is a unique mixture of those extremal distributions with mixing weights $w_{t_1}, \ldots, w_{t_z}$.

The extremal distributions can be conceived as urn processes without replacement, and, depending on the properties of $T$, de Finetti's style theorems may be deduced by the convergence of the hypergeometric processes to the i.i.d. processes.

### 3 Partially exchangeable binary sequences in the sense of de Finetti

We say that $(X_1, \ldots, X_n)$ is partially exchangeable in the sense of de Finetti of order $(n_1, \ldots, n_g)$, and we will denote it $(n_1, \ldots, n_g)$–DFPE, if it can be divided into $g$ exchangeable subsequences $(X_{i,1}, \ldots, X_{i,n_i})$, $i = 1, \ldots, g$, $\sum_i n_i = n$. Denote $\sum_{j=1}^{n_i} X_{i,j}$ as $S_i$. If the variables are $\{0, 1\}$–valued, $(S_1, \ldots, S_g)$ is a sufficient statistic in the sense of (1). Denote $P(x \in I^n : S_1 = k_1, \ldots, S_g = k_g)$ as $w_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)}$ and the probability of any sequence consistent with $(S_1 = k_1, \ldots, S_g = k_g)$ as $p_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)}$. Then we have:

$$
    w_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)} = \binom{n_1}{k_1} \cdots \binom{n_g}{k_g} p_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)} \tag{3}
$$

An $(n_1, \ldots, n_g)$–DFPE distribution is defined by the $(n_1 + 1) \cdots (n_g + 1)$ probabilities $w_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)}$ defined for every $g$–tuple of nonnegative integers $(k_1, \ldots, k_g)$ such that $0 \leq k_i \leq n_i$ for $i = 1, \ldots, g$, subject to

$$
    \sum_{k_1=0}^{n_1} \cdots \sum_{k_g=0}^{n_g} w_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)} = 1
$$

For what we have said in (2), the $\left\{ w_{k_1, \ldots, k_g}^{(n_1, \ldots, n_g)} \right\}_{k_i \leq n_i}$ range in the unitary simplex $\Diamond_{(n_1+1) \cdots (n_g+1)}$. 

4
The exchangeability of each subsequence \((X_{i,1}, \ldots, X_{i, n_i})\) implies the exchangeability of all its subsets, and we can obtain all the probabilities of the kind \(\left\{ w^{(m_1, \ldots, m_g)}_{i_1, \ldots, i_g} \right\}_{i_1, \ldots, i_g} \leq m_i, m_i < n_i, \) from the \(\left\{ w^{(n_1, \ldots, n_g)}_{i_1, \ldots, i_g} \right\}_{i_1, \ldots, i_g \leq n_i} \) through the following easily proved formula:

\[
w^{(m_1, \ldots, m_g)}_{i_1, \ldots, i_g} = \sum_{k_1 = l_1}^{n_1 - m_1 + l_1} \cdots \sum_{k_g = l_g}^{n_g - m_g + l_g} \frac{(k_2)_{m_2 - l_1}}{(n_1)_{m_1 - l_1}} \cdots \frac{(k_g)_{n_g - k_1}}{(n_g)_{m_g - l_g}} w^{(n_1, \ldots, n_g)}_{k_1, \ldots, k_g} \tag{4}
\]

Denote in particular the probabilities \(w^{(k_1, \ldots, k_g)}_{k_1, \ldots, k_g}\) as \(w_{k_1, \ldots, k_g}\). We have

\[
P \left( \bigcap_{i=1}^{g} \{X_{i,s_i} = 1, \ldots, X_{i,s_i} = 1\} \right) = E \left[ \prod_{i=1}^{g} X_{i,s_1} \cdots X_{i,s_i} \right] = w_{k_1, \ldots, k_g} \tag{5}
\]

for every subset \((s_1, \ldots, s_{k_i})\) of \(k_i\) labels in \(\{1, \ldots, n_i\}, i = 1, \ldots, g\). By (3) we have

\[
w_{k_1, \ldots, k_g} = \sum_{i_1 = k_1}^{n_1} \cdots \sum_{i_g = k_g}^{n_g} \frac{(i_1)_{k_1}}{(n_1)_{k_1}} \cdots \frac{(i_g)_{k_g}}{(n_g)_{k_g}} w^{(n_1, \ldots, n_g)}_{i_1, \ldots, i_g} \tag{6}
\]

where, from now on, \((i)_k = i(i-1) \cdots (i-k+1)\) for \(k \leq i\) and \((i)_0 = 1\).

To define the inverse map of (6) introduce the difference operator \(\Delta_i\) w.r.t. the \(i\)-th group: \(\Delta_i \left( w_{k_1, \ldots, k_g} \right) = w_{k_1, \ldots, k_i+1, \ldots, k_g} - w_{k_1, \ldots, k_i, \ldots, k_g} \). Then we have (see [3])

\[
w^{(n_1, \ldots, n_g)}_{k_1, \ldots, k_g} = \binom{n_1}{k_1} \cdots \binom{n_g}{k_g} (-1)^{n_1-k_1 + \cdots + n_g-k_g} \Delta_{k_1}^{n_1-k_1} \cdots \Delta_{k_g}^{n_g-k_g} \left( w_{k_1, \ldots, k_g} \right) \tag{7}
\]

Where \(w_{0, \ldots, 0} = 1\). So the \(\left\{ w_{k_1, \ldots, k_g} \right\}_{k_i \leq n_i, \ i = 1, \ldots, g} \) suffice to completely define any \((n_1, \ldots, n_g)\)-DFPE binary sequence, i.e. they constitute a parameterization of an \((n_1, \ldots, n_g)\)-DFPE binary distribution.

By (7), in an \((n_1, \ldots, n_g)\)-DFPE sequence each probability \(w_{k_1, \ldots, k_g}\) should satisfy

\[
(-1)^{n_1-k_1 + \cdots + n_g-k_g} \Delta_{k_1}^{n_1-k_1} \cdots \Delta_{k_g}^{n_g-k_g} \left( w_{k_1, \ldots, k_g} \right) \geq 0 \tag{8}
\]

Moreover, since by (3) it is \(\sum_{k_1 = 0}^{n_1} \cdots \sum_{k_g = 0}^{n_g} w^{(n_1, \ldots, n_g)}_{k_1, \ldots, k_g} = w_{0, \ldots, 0} = 1\), the (8) constitute necessary and sufficient conditions for a set \(\left\{ w_{k_1, \ldots, k_g} \right\}_{k_i \leq n_i, \ i = 1, \ldots, g}\) with \(w_{0, \ldots, 0} = 1\) to define
an \((n_1, \ldots, n_g)\)-DFPE sequence. Then the \(\{w_{k_1, \ldots, k_g}\}_{k_i \leq n_i, i=1, \ldots, g}\) range in the space
\[
\Lambda_{n_1, \ldots, n_g} = \left\{ \left(w_{k_1, \ldots, k_g}\right)_{k_i \leq n_i, \sum_i k_i > 0} : \text{satisfy } \mathbb{E}\right\}
\]

### 3.1 Generalized covariances

We introduce a generalization of the usual concept of covariance defined as follows: the covariance of order \(k\) among the variables \(X_1, \ldots, X_k\) is
\[
\text{Cov}[X_1, \ldots, X_k] = E[(X_1 - E[X_1]) \cdots (X_k - E[X_k])]
\]
(9)

Under DFPE, these covariances depend only on the number of variables involved for each exchangeable subsequence. To simplify the notation, denote the value \(w_{k_1, \ldots, k_g}\) when \(k_i = 1\) and all other subscripts are zero as \(w(i)\), i.e. \(E[X_{i,1}] = w(i)\). Then under DFPE any generalized covariance involving \(k_i\) r.v.s of the \(i\)-th subsequence, \(i = 1, \ldots, g\) is equal to
\[
\text{Cov}_{k_1, \ldots, k_g} = E\left[ (X_{1,1} - w(1)) \cdots (X_{1,k_1} - w(1)) \cdots (X_{g,1} - w(g)) \cdots (X_{g,k_g} - w(g)) \right]
\]

and the relation with the previous parameterization is
\[
\text{Cov}_{k_1, \ldots, k_g} = \sum_{i_1=0}^{k_1} \cdots \sum_{i_g=0}^{k_g} \binom{k_1}{i_1} \cdots \binom{k_g}{i_g} (-1)^{i_1+\cdots+i_g} (w(1))^{i_1} \cdots (w(g))^{i_g} w_{k_1-i_1, \ldots, k_g-i_g}
\]
(10)

**Proof of (10):** For the sake of simplicity, but without loss of generality, set \(g=2\). By expanding the product,
\[
(X_{1,1} - w(1)) \cdots (X_{1,k_1} - w(1)) (X_{2,1} - w(2)) \cdots (X_{2,k_2} - w(2))
\]
results as the sum of \((k_1 + 1)(k_2 + 1)\) terms of the kind
\[
\sum_{h_1 < \ldots < h_i} \sum_{s_1 < \ldots < s_j} (-w(1))^{k_1-i} (-w(2))^{k_2-j} X_{1,h_1} \cdots X_{1,h_i} \cdot X_{1,s_1} \cdots X_{1,s_j}
\]
(11)

where the first sum ranges over all the possible \(i\)-tuples \((h_1, \ldots, h_i)\) of distinct labels in \(\{1, \ldots, n_i\}\) and consists of \(\binom{k_1}{i}\) terms, the second of \(\binom{k_2}{j}\) terms. Passing to the expectation, by (5), the term (11) results as \(\binom{k_1}{i} \binom{k_2}{j} (-w(1))^{k_1-i} (-w(2))^{k_2-j} w_{i,j}\), so
One can prove that the inverse map, which is somewhat similar to the inverse of a binomial transform, is

$$w_{k_1,\ldots,k_g} = \sum_{i_1=0}^{k_1} \cdots \sum_{i_g=0}^{k_g} \binom{k_1}{i_1} \cdots \binom{k_g}{i_g} w^{(1)}^{i_1} \cdots w^{(g)}^{i_g} Cov_{k_1-i_1,\ldots,k_g-i_g}$$

(12)

where $Cov_{0,\ldots,0} = 1$ and all the covariances having a single 1 and all zeros in the subscript are zero. So, a $(n_1,\ldots,n_g)$–DFPE binary sequence is completely defined by the $g$ probabilities $w^{(1)},\ldots,w^{(g)}$ together with the generalized covariances $\{Cov_{k_1,\ldots,k_g}\}$ defined for every $g$–tuple $(k_1,\ldots,k_g)$ with $k_i \leq n_i$ and such that $\sum_{i=1}^{g} k_i \geq 2$. The space of the $Cov_{k_1,\ldots,k_g}$ is implicitly defined by $\Lambda_{n_1,\ldots,n_g}$ and (10) and is not easily described. We can say that all the $Cov_{k_1,\ldots,k_g}$ can be both positive or negative, and by (12) are all null if, and only if, $X_1,\ldots,X_n$ are i.i.d.

3.2 Extendibility

For the sake of simplicity in this section we set $g = 2$, but all the results hold for a general $g$.

For what we have said, we can represent any $(n_1,n_2)$–DFPE distribution as a point in the linear spaces $\hat{\Lambda}_{(n_1+1)(n_2+1)}$ and $\Lambda_{n_1,n_2}$. Formulas (11) and (9) define the linear maps between the two spaces. Clearly these maps are one–one and onto and establish affine congruence of the two sets. The $(n_1+1)(n_2+1)$ vertices of $\hat{\Lambda}_{(n_1+1)(n_2+1)}$ are the points having one coordinate equal to one and the others equal to zero and represent the extremal distributions of Theorem 2.1 (6) maps this vertices onto the vertices of $\Lambda_{n_1,n_2}$. In particular, the extremal distribution having $w_{k_1,k_2}^{(n_1,n_2)} = 1$ is represented in
The points \( \{ \lambda_{k_1,k_2; n_1,n_2} \} \) are at least (appears in the right hand side of exactly one equation of the kind (14)), having coordinates

\[
w_{l_1,l_2} = \begin{cases} 
0 & \text{whenever } l_i > k_i \text{ for any } i = 1, 2 \\
(k_1)_{l_1}(k_2)_{l_2} \quad (n_1)_{l_1} \quad (n_2)_{l_2} & \text{elsewhere}
\end{cases}
\]

The points \( \{ \lambda_{k_1,k_2; n_1,n_2} \} \) are affinely independent, then \( \Lambda_{n_1,n_2} \), which is their convex hull, is a \((n_1 + 1)(n_2 + 1) - 1\) dimensional convex polytope with \((n_1 + 1)(n_2 + 1)\) vertices, i.e. a non-standard simplex.

We say that a \((n_1,n_2)\)–DFPE sequence is (at least) \((r_1,r_2)\)–extendible, \( r_i \geq n_i \), if it is the initial segment of a \((r_1,r_2)\)–DFPE sequence. So the sequence, represented by the point \( w \equiv (w_{l_1,l_2})_{l_1 \leq n_1} \) in \( \Lambda_{n_1,n_2} \), is \((r_1,r_2)\)–extendible if, and only if, there exist a point \( w^* \equiv (w_{k_1,k_2})_{k_1 \leq r_1} \) in \( \Lambda_{r_1,r_2} \) such that its orthogonal projection over the coordinates of \( \Lambda_{n_1,n_2} \) coincide with \( w \). That is, denote as \( \Lambda_{r_1,r_2} \) the projection of \( \Lambda_{n_1,n_2} \) over the coordinates of \( \Lambda_{n_1,n_2} \), and as \( \Lambda_{k_1,k_2; r_1,r_2} \) the analogous projection of \( \lambda_{k_1,k_2; r_1,r_2} \). Then \( \Lambda_{r_1,r_2} \) is exactly the subspace of \( \Lambda_{n_1,n_2} \) representing the \((n_1,n_2)\)–DFPE distributions which are at least \((r_1,r_2)\)–extendible and it results as the convex hull of the \( \{ \lambda_{k_1,k_2; r_1,r_2} \} \) \( k_1 \leq r_1 \). Moreover, we are going to see that none of this point is redundant with respect to the convex hull problem, that is they are exactly the vertices of \( \Lambda_{r_1,r_2} \).

**Theorem 3.1.**

\[
\lambda^{(n_1,n_2)}_{k_1,k_2; r_1,r_2} = \frac{r_1 - k_1}{r_1} \lambda^{(n_1,n_2)}_{k_1,k_2; r_1-1,r_2} + \frac{k_1}{r_1} \lambda^{(n_1,n_2)}_{r_1-1,k_2; r_1-1,r_2} \\
= \frac{r_2 - k_2}{r_2} \lambda^{(n_1,n_2)}_{k_1,k_2; r_1,r_2-1} + \frac{k_2}{r_2} \lambda^{(n_1,n_2)}_{k_1,k_2-1; r_1,r_2-1}
\]

**Proof.** We have

\[
p^{(n_1,n_2)}_{k_1,k_2} = p^{(n_1+1,n_2)}_{k_1,k_2} + p^{(n_1+1,n_2)}_{k_1+1,k_2} \quad (14)
\]

\[
p^{(n_1,n_2+1)}_{k_1,k_2} = p^{(n_1,n_2+1)}_{k_1,k_2+1} + p^{(n_1,n_2+1)}_{k_1+1,k_2+1} \quad (15)
\]

The point \( \lambda_{k_1,k_2; r_1,r_2} \) represents the distribution having \( w^{(r_1,r_2)} = 1 \). Any term \( p^{(r_1,r_2)}_{k_1,k_2} \) appears in the right hand side of exactly one equation of the kind (13) and one of the...
kind (13). Then, by (3) it is easily seen that if \( w_{k_1,k_2}^{(r_1,r_2)} = 1 \), it is
\[
\begin{align*}
  w_{k_1,k_2}^{(r_1-1,r_2)} &= \frac{r_1 - k_1}{r_1}, & w_{k_1-1,k_2}^{(r_1-1,r_2)} &= \frac{k_1}{r_1}, & w_{k_1,k_2}^{(r_1,r_2-1)} &= \frac{r_2 - k_2}{r_2}, & w_{k_1,k_2}^{(r_1,r_2-1)} &= \frac{k_2}{r_2},
\end{align*}
\]
then the statement follows by (3).

\[\square\]

**Proposition 3.1.** Consider a polytope \( A \) and a set of points lying on distinct edges of \( A \). Call \( A' \) their convex hull. Then those points are the vertices of \( A' \).

**Proof.** Say one of those point \( v \) lies on the edge \( e \) of \( A \). Then it can only be represented as convex combinations of points in \( e \), and no other points in \( A \). But \( v \) is the only point of \( A' \) lying on \( e \) and obviously \( A' \subset A \), then \( v \) cannot be represented as convex combinations of any other points in \( A' \), and hence is a vertex.

\[\square\]

**Theorem 3.2.** a) The \( \{\Lambda_{k_1,r_1}^{(n_1,n_2)}\}_{k_1 \leq r_1} \) are the vertices of \( \Lambda_{r_1,r_2}^{(n_1,n_2)} \).

b) Each pair of points in the right hand side of (13) constitute the vertices of an edge of their own space.

**Proof.** \( \Lambda_{n_1,n_2} \) is a simplex, so each couple of its vertices identifies an edge. By (13), the points \( \Lambda_{k_1,k_2}^{(n_1,n_2)} \) of \( \Lambda_{n_1,n_2} \) lie on distinct edges of \( \Lambda_{n_1,n_2} \) and by Proposition 3.1 are all vertices of \( \Lambda_{n_1,n_2}^{(n_1,n_2)} \). Moreover, each couple of vertices of \( \Lambda_{n_1,n_2}^{(n_1,n_2)} \) of the kind \( \Lambda_{k_1,k_2}^{(n_1,n_2)} \) lie on two adjacent edges of \( \Lambda_{n_1,n_2} \) having the vertex \( \Lambda_{k_1,k_2}^{(n_1,n_2)} \) in common, and no other vertex of \( \Lambda_{n_1,n_2}^{(n_1,n_2)} \) has \( \Lambda_{k_1,k_2}^{(n_1,n_2)} \) in its representation (13). So they identify an edge of \( \Lambda_{n_1,n_2}^{(n_1,n_2)} \). To be precise, all the points \( \Lambda_{k_1,k_2}^{(n_1,n_2)} \) having \( k_1 = 0 \) or \( k_1 = n_1 + 1 \) coincide with vertices of \( \Lambda_{n_1,n_2} \). However, as we have said, there are not three points having a common vertex of \( \Lambda_{n_1,n_2} \) in their representation (13), so they are vertices of \( \Lambda_{n_1,n_2}^{(n_1,n_2)} \) as well. In conclusion, a) and b) are valid for \( r_1 = n_1 + 1 \), and obviously also for \( r_2 = n_2 + 1 \). It is easily seen that, if we suppose a) and b) hold for \( \Lambda_{r_1,r_2}^{(n_1,n_2)} \), then they also hold for \( \Lambda_{r_1+1,r_2}^{(n_1,n_2)} \) and \( \Lambda_{r_1,r_2+1}^{(n_1,n_2)} \), so the theorem is proved by induction.

\[\square\]

In conclusion, an \( (n_1,n_2) \)-DFPE distribution, represented by a point \( w \) in \( \Lambda_{n_1,n_2} \), is at least \( (r_1,r_2) \)-extendible if, and only if, \( w \) is contained in \( \Lambda_{r_1,r_2}^{(n_1,n_2)} \), and is exactly \( (r_1,r_2) \)-extendible if \( \Lambda_{r_1,r_2}^{(n_1,n_2)} \) and \( \Lambda_{r_1+1,r_2}^{(n_1,n_2)} \) do not contain \( w \).

Note that, by virtue of (4) we can map the extremal points of \( \hat{\Phi}_{(r_1+1)(r_2+1)} \) and find the subspace of \( \hat{\Phi}_{(n_1+1)(n_2+1)} \) representing the \( (n_1,n_2) \)-DFPE distribution that are at least \( (r_1,r_2) \)-extendible. But the probabilities \( w_{k_1,k_2}^{(n_1,n_2)} \) depend on \( n_1 \) and \( n_2 \), so we
should obtain the vertices of the subspaces for each couple \((n_1, n_2)\). On the converse, the probabilities \(w_{k_1, k_2}\) do not depend on the sequence size, and once we know the vertices of \(\Lambda_{r_1, r_2}\) we can obtain the vertices of \(\Lambda_{r_1, r_2}^{(n_1, n_2)}\) for every \(n_1 < r_1, n_2 < r_2\) simply excluding certain coordinates.

The points in \(\Lambda_{r_0, 0}^{(n, 0)}\) represents the \(n\)-exchangeable distributions that are at least \(r\)-extendible. The \((n-1)\)-dimensional faces of an \(n\)-dimensional polytope are said facets. A polytope is said simplicial if all its facets are simplexes. Crisma in [2] demonstrated that the \(\Lambda_{r_0, 0}^{(n, 0)}\) are simplicial and their vertices satisfy Gale Evenness Condition (a combinatorial property characterizing the facets). As a consequence, we can easily determine if a point lies inside any \(\Lambda_{r_0, 0}^{(n, 0)}\). Moreover, Crisma has been able to compute their volumes, determining in some sense the proportion of \(n\)-exchangeable sequences that are \(r\)-extendible.

Unfortunately, the \(\Lambda_{r_1, r_2}^{(n_1, n_2)}\) are not simplicial polytopes, and we have not found an analytical way to determine their facets. Then, to determine if a point \(w\) lies inside a certain polytope \(\Lambda_{r_1, r_2}^{(n_1, n_2)}\) we can use the following linear program:

\[
\begin{align*}
\text{maximize} & \quad z^T w - z_0 = f \\
\text{subject to} & \quad z^T \lambda - z_0 \leq 0 \quad \forall \ \lambda \in \left\{ \lambda_{k_1, k_2; r_1, r_2}^{(n_1, n_2)} \right\}_{k_1 \leq r_1, k_2 \leq r_2} \\
& \quad z^T w - z_0 \leq 1
\end{align*}
\]

where \(z \in \mathbb{R}^{(n_1+1)(n_2+1)}\) and \(z_0 \in \mathbb{R}\). The last inequality is artificially added so that the linear program has a bounded solution. The optimal value \(f\) is positive if and only if there exists an hyperplane \(\{x \in \mathbb{R}^{(n_1+1)(n_2+1)} : z^T x = z_0\}\) separating the polytope \(\Lambda_{r_1, r_2}^{(n_1, n_2)}\) and \(w\), i.e. if and only if \(w\) lies outside of \(\Lambda_{r_1, r_2}^{(n_1, n_2)}\).

### 3.2.1 \(\infty\)-extendible case

If all the \(g\) subsequences of a DFPE sequence are \(\infty\)-extendible, there exists a probability measure \(\nu\) over the \(g\)-dimensional hypercube \([0, 1]^g\) and a r.v. \(\Theta = (\theta(1), \ldots, \theta(g))\) distributed accordingly such that

\[
\begin{align*}
\nu^{(n_1, \ldots, n_g)}_{k_1, \ldots, k_g} = \binom{n_1}{k_1} \cdots \binom{n_g}{k_g} \int_0^1 \cdots \int_0^1 \prod_{i=1}^g \theta(i)^{k_i} (1 - \theta(i))^{n_i - k_i} \, d\nu(\Theta)
\end{align*}
\]
So, the probabilities \( w_{k_1,\ldots,k_g} \) are the ordinary mixed moments of the mixing measure \( \nu \). Let \( \mathcal{M}^{(n_1,\ldots,n_g)} \) be the space of the mixed moments up to order \((n_1,\ldots,n_g)\) of all the probability measures over \([0,1]^g\). For what we have said, \( \{\Lambda^{(n_1,\ldots,n_g)}_{r_1,\ldots,r_g}\}_{r_1,\ldots,r_g} \) is a decreasing multisequence of polytopes and we have

\[
\bigcap_{r_1=n_1}^{\infty} \cdots \bigcap_{r_g=n_g}^{\infty} \Lambda^{(n_1,\ldots,n_g)}_{r_1,\ldots,r_g} = \mathcal{M}^{(n_1,\ldots,n_g)}
\]

As far as I know there is no practical criterion to establish if a point of \( \mathbb{R}^{(n_1+1)\cdots(n_g+1)} \) lies inside \( \mathcal{M}^{(n_1,\ldots,n_g)} \). Then we can check some simple necessary conditions for \( \infty \)-extendibility using moments’ inequalities.

Formulas (10) and (12) link the ordinary mixed moments and the central mixed moments of a multivariate distribution (see e.g. [19], equations (34.28) (34.29)), consequently we have:

\[
Cov_{k_1,\ldots,k_g} = E_{\nu} \left[ \left( \theta(1) - E_{\nu}[\theta(1)] \right)^{k_1} \cdots \left( \theta(g) - E_{\nu}[\theta(g)] \right)^{k_g} \right]
\]

So, a simple necessary condition for a representation of the kind (17) to hold is

\[
Cov_{2k_1,\ldots,2k_g} \geq 0 \quad \forall \; k_i = 1, \ldots, \lfloor n_i/2 \rfloor, \; i = 1, \ldots, g \quad (18)
\]

To simplify the notation, denote as \( Cov(i,j) \) the covariance between a r.v. of the \( i \)-th group and one of the \( j \)-th group, i.e. the value \( Cov_{k_1,\ldots,k_g} \) when \( k_i = k_j = 1 \) and all other subscripts are 0. Another simple necessary condition for (17) to hold is that, in that case, for what we have said, \( \{Cov(i,j)\}_{1\leq i,j\leq g} \) is the Variance–Covariance matrix of \( \Theta \) and hence must be nonnegative definite.

**Example.** The \((2,2)\)-DFPE distribution defined by the following values of \( w_{k_1,k_2}^{(2,2)} \):

\[
\begin{array}{c|ccc}
   & 0 & 1 & 2 \\
\hline
   0 & \frac{1}{16} & \frac{1}{16} & 0 \\
   1 & \frac{1}{16} & \frac{3}{16} & 0 \\
   2 & \frac{1}{16} & 0 & \frac{5}{16} \\
\end{array}
\]

by (6) leads to the following values of \( w_{k_1,k_2} \):
and by (10) we have

\[
\begin{array}{c|ccc}
  k_1, k_2 & 0 & 1 & 2 \\
  \hline
  0 & 1 & \frac{7}{16} & \frac{1}{16} \\
  1 & \frac{1}{2} & \frac{23}{64} & \frac{5}{16} \\
  2 & \frac{3}{4} & \frac{7}{16} & \frac{7}{16} \\
\end{array}
\]

\[
Cov_{2,0} = 1/8, \quad Cov_{0,2} = 1/16, \quad Cov_{2,2} = 1/32
\]

so (18) is satisfied. But \(Cov_{2,0} Cov_{0,2} - Cov_{1,1}^2 = -\frac{17}{4096} < 0\), so \((Cov_{2,0} Cov_{1,1}, Cov_{0,2})\) is not nonnegative definite and the distribution is not \((\infty, \infty)\)-extendible. The linear program (10) reveals that the point of \(\Lambda_{2,2}\) representing the distribution lies in \(\Lambda_{4,2}^{(2,2)}\), but not in \(\Lambda_{5,2}^{(2,2)}\) nor in \(\Lambda_{2,3}^{(2,2)}\). Hence the distribution is exactly \((4, 2)\)-extendible. Note that both the \(2\)-exchangeable subsequences identified respectively by \((w_{1,0}, w_{2,0})\) and by \((w_{0,1}, w_{0,2})\), are \(\infty\)-extendible.

4 Markov exchangeability

Consider an \(I\)-valued sequence \((x_1, \ldots, x_n)\). Define its transition counts \(n_{i,j}\) for all \(i, j\) in \(I\) as

\[
n_{i,j} = \sum_{k=1}^{n-1} I_{(i,j)}(x_k, x_{k+1})
\]

and arrange them in a matrix \(N = \{n_{i,j}\}_{i,j}\). Then, the distribution of \((X_1, \ldots, X_n)\) is Markov exchangeable (hereafter ME or \(n\)-ME if we need to highlight the number of variables) when the sufficient statistic \(T\) in (1) is the value of the first step \(x_1\), together with the transition count matrix \(N\). Introduce the number of transitions exiting from \(i\): \(n_i^+ = \sum_{j \in I} n_{i,j}\) and the number of transitions entering in \(i\): \(n_i^- = \sum_{j \in I} n_{j,i}\).

**Proposition 4.1.** Consider an \(I\)-valued sequence \((x_1, \ldots, x_n)\). Then, it is \(x_1 = x_n\) if, and only if

\[
n_i^+ = n_i^- \quad \forall i \in I
\]

while it is \(x_1 \neq x_n\) if, and only if

\[
\begin{cases}
  n_{x_1}^+ = n_{x_1}^- + 1 \\
  n_{x_n}^- = n_{x_n}^+ + 1 \\
  n_i^+ = n_i^- & \text{for } i \neq x_1 \neq x_n
\end{cases}
\]
Moreover, an integer valued matrix \( N = \{ n_{i,j} \}_{i,j} \) is a consistent transition count matrix if, and only if, it is irreducible and one between (19) and (20) is valid.

**Proof.** Consider \( H = \{(x_1, x_2), \ldots, (x_{n-1}, x_n)\} \) and let \( J \) be the set of the distinct states \((J \subseteq I)\) visited by \((x_1, \ldots, x_n)\). We can think to \( N \) as the adjacency matrix of the directed graph \( G = (J, H) \). But \( G \) is Eulerian by construction, and the result follows immediately.

Denote as \([x_1, N]\) the set of all the \( I \)–valued \( n \)–tuples starting in \( x_1 \) and having a transition count \( N \). Denote \( P(x \in [x_1, N]) \) as \( w_{x_1,N} \), and the probability of having any specified element of \([x_1, N]\) as \( p_{x_1,N} \). Denote the set of all the distinct transition count matrices of all the \( I \)–valued \( n \)–tuples starting in \( x_1 \) as \( \Phi(x_1,n) \). For what we have said, an \( I \)–valued \( n \)–ME distribution is completely defined by the probabilities \( w_{x_1,N} \) for \( N \) ranging in \( \Phi(x_1,n) \) and \( x_1 \) ranging in \( I \) subjected to

\[
\sum_{x_1 \in I} \sum_{N \in \Phi(x_1,n)} w_{x_1,N} = 1 \quad \text{and} \quad \sum_{N \in \Phi(x_1,n)} w_{x_1,N} = P(X_1 = x_1) \quad \forall x_1 \in I
\]

The cardinality of \([x_1, N]\) was first found by Whittle in [22]. Define the matrix \( B = \{b_{i,j}\}_{i,j \in I} \) as

\[
b_{i,j} = \begin{cases} 
-n_{i,j}/n_i^+ & \text{for } i \neq j \\
1 - n_{i,i}/n_i^+ & \text{for } i = j
\end{cases}
\]

By (19) and (20), if we know the starting state and the transition counts of a sequence, we also know its ending state.

**Theorem 4.1** ([22]). The number of sequences in \([x_1, N]\) is

\[
\det(B_{x_0,x_0}) \frac{\prod_i n_i^{+!}}{\prod_{i,j} n_{i,j}^{+!}}
\]

where \( x_n \) is uniquely determined by \( x_1 \) and \( N \), and where \( B_{x_0,x_0} \) is the matrix obtained by \( B \) removing the \( x_n \)-th row and the \( x_n \)-th column.

Then it is \( w_{x_1,N} = \det(B_{x_0,x_0}) \frac{\prod_i n_i^{+!}}{\prod_{i,j} n_{i,j}^{+!}} p_{x_1,N} \).

We say that an \( I \)–valued process \( X = \{X_n\}_{n \in \mathbb{N}} \) is ME if \((X_1, \ldots, X_n)\) is ME for every \( n \). In [9] it is demonstrated that a recurrent process \((X_1 = X_n \ i.o.)\) is ME if, and only if, its law is a mixture of Markov Chains. That is, let \( \mathcal{P} \) be the space of all the
stochastic matrices $\Theta = \{\theta_{i,j}\}_{i,j}$ on $I \times I$. Then there exists, and is unique, a mixing measure $\nu$ on the Borel sets of $I \times \mathcal{P}$ such that

$$P(X_1 = x_1, \ldots, X_n = x_n) = \int_{\mathcal{P}} \prod_{i=1}^{n-1} \theta_{x_i, x_{i+1}} \nu(x_1, d\Theta)$$

Let $\Gamma_i(k)$ be the step of the process $X$ at which the state $i$ occurs for the $k$–th time. Let $V_i(k)$ be the $k$–th successor of the state $i$, i.e. the variable immediately subsequent the $k$–th occurrence of $i$ ($V_i(k) = X_{\Gamma_i(k)+1}$). The hypothesis of de Finetti was that, if all the subsequences $\{V_i(k)\}_{k=1,\ldots,n_i^+}$, for $i \in I$, are exchangeable and $\infty$–extendible, then $X$ is a mixture of Markov Chains. This actually occurs if all the states in $I$ are recurrent, but Lemma 5 in [13] assures that a recurrent ME process is strongly recurrent, then all the states are recurrent and the two characterizations coincide.

Zaman in [24, 25] demonstrated that finite Markov exchangeability does not coincide with finite exchangeability of the $\{V_i(k)\}_{k=1,\ldots,n_i^+}$, $i \in I$. In fact, given $x_1$ and $N$, some of the transitions in $(X_1, \ldots, X_n)$ should necessarily occur as last. Then, the subsequences $\{V_i(k)\}_k$ are invariant only under permutations that do not alter those forced transitions. Zaman described the extremal $n$–ME distributions as particular urn processes without replacement where some balls should necessarily be drawn as last, but the characterization of the mixture of Markov Chains cannot be derived through a passage to the limit without adding some restrictions.

### 4.1 Markov exchangeable binary sequences

The proofs of the theorems (4.2), (4.4) and (4.3) of this section are in appendix.

If $I = \{0, 1\}$ we deal with $2 \times 2$ transition count matrices of the kind:

$$N = \begin{pmatrix} n_{0,0} & n_{0,1} \\ n_{1,0} & n_{1,1} \end{pmatrix}$$

and it is $n_{0,0} + n_{0,1} = n_0^+$ and $n_{1,0} + n_{1,1} = n_1^+$. The term $\det(B_{x_n,x_n})$ in Theorem 4.1 simply is $n_{1,0}/n_1^+$ if $x_n = 0$, and $n_{0,1}/n_0^+$ if $x_n = 1$. So we have

$$w_{x_1,N} = \begin{cases} \left( \frac{n_0^+}{n_{0,0}} \right) \left( \frac{n_1^+}{n_{1,0}} \right) p_{x_1,N} & \text{if } (x_1, N) \text{ imply } x_n = 0 \\ \left( \frac{n_0^-}{n_{0,0}} \right) \left( \frac{n_1^-}{n_{1,0}} \right) p_{x_1,N} & \text{if } (x_1, N) \text{ imply } x_n = 1 \end{cases}$$

\(21\)
We can consider separately the sequences depending on the initial state. From now on, we fix \( P(X_1 = 0) = 1 \) and hence we will consider only the sequences starting with 0 and the probabilities \( \{w_{0,N}\}_{N \in \Phi(0,n)} \) and \( \{p_{0,N}\}_{N \in \Phi(0,n)} \). We will also use the self-explaining notation \( p_0^{n_{0,0} n_{0,1}} \) and \( w_0^{n_{0,0} n_{0,1}} \) when we need to display the number of transitions.

Unlike the DFPE case, the number of probabilities defining an \( n \)-ME distribution is not so evident. We have to count the possible different transition count matrices for each fixed starting state. From (19) and (20) two cases are possible when \( X_1 = 0 \), and we define

\[
\Phi_1(0,n) = \{ N \in \Phi(0,n) : n_{0,1} = n_{1,0} \} \\
\Phi_2(0,n) = \{ N \in \Phi(0,n) : n_{0,1} = n_{1,0} + 1 \}
\]

such that \( \Phi_1(0,n) \cup \Phi_2(0,n) = \Phi(0,n) \). Call the transition count matrices of \( \Phi_1(0,n) \) matrices of the first kind, and those of \( \Phi_2(0,n) \) of the second kind. The following theorem corrects the assertion \( |\Phi(0,n)| = 1 + \binom{n-1}{2} \) stated in a different form in [8, page 239] and reported in [18].

**Theorem 4.2.**

\[ |\Phi(0,n)| = 1 + \binom{n}{2} \]

For symmetry reasons the same result is valid for the sequences starting in 1.

Now we state a couple of equations we will use in the following. For any \( n \) and \( N \) we have

\[
p_{0,N} = p_0^{n_{0,0} n_{0,1}} = p_0^{n_{0,0}+1 n_{0,1}} + p_0^{n_{0,0} n_{0,1}+1} \quad \text{if } N \in \Phi_1(0,n) \quad (22)
\]

\[
p_{0,N} = p_0^{n_{0,0} n_{0,1}} = p_0^{n_{0,0} n_{0,1}+1} + p_0^{n_{0,0}+1 n_{0,1}} \quad \text{if } N \in \Phi_2(0,n) \quad (23)
\]

The first \( k \) steps \((X_1, \ldots, X_k), k < n\), of an \( n \)-ME sequence are \( k \)-ME, and we can obtain all the probabilities \( \{p_{0,K}\}_{K \in \Phi(0,k)} \) from the \( \{p_{0,N}\}_{N \in \Phi(0,n)} \). Let \( K = \binom{k_{0,0} k_{0,1}}{k_{1,0} k_{1,1}} \) be the transition count matrix up to step \( k \) of a sequence starting in 0, and let \( k_{0,0} + k_{0,1} = k_0^+ \) and \( k_{1,0} + k_{1,1} = k_1^+ \). Then
Theorem 4.3.

\[ p_{0,K} = \sum_{N \in \Phi_1(0,n)} \frac{(n_{0,0})_{k_{0,0}}(n_{0,1})_{k_{0,1}}(n_{1,1})_{k_{1,1}}(n_{1,0} - 1)_{k_{1,0}}}{(n_{0}^+)^{k_0^+}(n_{1}^+ - 1)^{k_1^+}} w_{0,N} + \]

\[ + \sum_{N \in \Phi_2(0,n)} \frac{(n_{0,0})_{k_{0,0}}(n_{0,1} - 1)_{k_{0,1}}(n_{1,1})_{k_{1,1}}(n_{1,0})_{k_{1,0}}}{(n_{0}^+ - 1)^{k_0^+}(n_{1}^+)^{k_1^+}} w_{0,N} \]

where the sums should be restricted over those matrices \( N \) in \( \Phi(0,n) \) having \( n_{i,j} \geq k_{i,j} \), for all \( i, j \) in \( \{0, 1\} \). Consider the probability \( p_{0}(\begin{smallmatrix} a & 1 \\ 0 & b \end{smallmatrix}) \) of having the sequence of \( a+b+2 \) steps starting in 0 with \( a \) transitions \((0,0)\), a single transition \((0,1)\) and ending with \( b \) transitions \((1,1)\), and denote it \( w_{0,a,b} \). By the above theorem we have

\[ w_{0,a,b} = p_{0}(\begin{smallmatrix} a & 1 \\ 0 & b \end{smallmatrix}) = \sum_{N \in \Phi_1(0,n)} \frac{(n_{0,0})_{a} n_{0,1} n_{1,1} (n_{1,0} - 1)_{b}}{(n_{0}^+)^{a+1}(n_{1}^+ - 1)^{b}} w_{0,N} + \]

\[ + \sum_{N \in \Phi_2(0,n)} \frac{(n_{0,0})_{a} (n_{0,1} - 1) n_{1,1} (n_{1,0})_{b}}{(n_{0}^+ - 1)^{a+1}(n_{1}^+)^{b}} w_{0,N} \] (24)

We set \( w_{0,n-1,0} = p_{0}(\begin{smallmatrix} n-1 & 0 \\ 0 & 0 \end{smallmatrix}) \).

Define the operators \( \Delta_0 \) and \( \Delta_1 \) such that:

\[ \Delta_0 (w_{0,a,b}) = w_{0,a+1,b} - w_{0,a,b} \quad \text{and} \quad \Delta_1 (w_{0,a,b}) = w_{0,a,b+1} - w_{0,a,b} \]

Then we have

Theorem 4.4.

\[ p_{0,N} = (-1)^{n_{0,1} - 1 + n_{1,0}} \Delta_0^{n_{0,1} - 1} \Delta_1^{n_{1,0}} (w_{0,n_{0,0},n_{1,1}}) \]

In an \( n \)–ME sequence the probabilities \( \{w_{0,a,b}\} \) are well defined for every couple of nonnegative integers \( (a,b) \) having sum not greater than \( n - 2 \), together with the case \( w_{0,n-1,0} \). Denote as \( \mathcal{L}_n \) the set of couples \( (a,b) \) such defined together with the couple \( (n-1,0) \). Theorem 4.4 assures that the probabilities \( \{w_{0,a,b}\}_{\mathcal{L}_n} \) suffice to completely define an \( n \)–ME sequence starting in 0. It is easily seen that \(|\mathcal{L}_n| = \binom{n}{2} + 1\) as we would expect.
4.2 Extendibility

Unlike the DFPE case, in a ME sequence it is meaningless to consider separately the extendibility of the two subsequences \( \{V_0(k)\}_k \) and \( \{V_1(k)\}_k \). Then we say that an \( n\)–ME sequence \((X_1, \ldots, X_n)\) is \( r\)–extendible if there exist \((X_{n+1}, \ldots, X_r)\) such that \((X_1, \ldots, X_r)\) is \( r\)–ME.

The probabilities \( w_{0,a,b} \) allow us to study the extendibility of a ME sequence in a geometric approach analogous to that of Section 3.2.

The space of the probabilities \( \{w_{0,a,b}\}_{\mathcal{L}_n} \) of all the \( n\)–ME sequences starting in 0 (call it \( \Gamma_n \)) is implicitly defined by Theorem 4.4. That is, we have that every \( w_{0,a,b} \) should satisfy

\[
(-1)^{c+d-1} \Delta_0^{c-1} \Delta_1^d (w_{0,a,b}) \geq 0 \quad \forall (c,d) : \left( \begin{array}{c} c \\ d \end{array} \right) \in \Phi(0,n) \quad (25)
\]

and we can write

\[\Gamma_n = \left\{ (w_{0,a,b})_{\mathcal{L}_n} : w_{0,a,b} \geq 0, (25) \text{ is satisfied} \right\}\]

Theorem 4.4 and (24) establish affine congruence between the unitary \((\frac{n}{2})\)–dimensional simplex \( \hat{\Delta}_{\frac{n}{2}+1} \), which is the space of the probabilities \( \{w_{0,N}\}_{N \in \Phi(0,n)} \), and \( \Gamma_n \), which consequently is a \((\frac{n}{2})\)–dimensional (non standard) simplex. The vertices of \( \hat{\Delta}_{\frac{n}{2}+1} \) represent the extremal distributions of Theorem 2.1. Equation (24) maps them to the vertices of \( \Gamma_n \). We will denote as \( \gamma_N \) the vertex of \( \Gamma_n \) corresponding to the extremal distribution having \( w_{0,N} = 1 \).

An \( n\)–ME sequence starting in 0 represented in \( \Gamma_n \) by the point \((w_{0,a,b})_{\mathcal{L}_n}\) is \( r\)–extendible if, and only if, there exist probabilities \( \{w_{0,a,b}\} \) with \((n-2) < (a+b) \leq (r-2) \) together with \( w_{0,r-1,0} \), such that \((w_{0,a,b})_{\mathcal{L}_r}\) lies in \( \Gamma_r \). Let \( \Gamma_r^{(n)} \) be the orthogonal projection of \( \Gamma_r \) over the coordinates of \( \Gamma_n \), and let \( \gamma_R^{(n)} \) be the analogous projection of \( \gamma_R \). Then \( \Gamma_r^{(n)} \) represents the \( n\)–ME sequences that are (at least) \( r\)–extendible and is the convex hull of the \( \{\gamma_R^{(n)}\}_{R \in \Phi(0,r)} \).

By (21), (22), (23), (24), and with passages similar to those of the proof of Theorem
one can prove the following is valid for any \( r > n \):

\[
\gamma^{(n)}_R = \gamma^{(n)}(\frac{r_{0,0}}{r_{1,1}} r_{1,1}) = \frac{r_{0,1}}{r_0} \gamma^{(n)}(\frac{r_{0,0}}{r_{1,1}} \frac{r_{0,1}}{r_{1,1}}) + \frac{r_{0,0}}{r_0} \gamma^{(n)}(\frac{r_{0,0}}{r_{1,1}} \frac{r_{0,1}}{r_{1,1}}) \quad \text{if} \quad R \in \Phi_1(0, r)
\]

\[
\gamma^{(n)}_R = \gamma^{(n)}(\frac{r_{1,0}}{r_{1,1}} r_{1,1}) = \frac{r_{1,0}}{r_1} \gamma^{(n)}(\frac{r_{0,0}}{r_{1,1}} \frac{r_{0,1}}{r_{1,1}}) + \frac{r_{1,1}}{r_1} \gamma^{(n)}(\frac{r_{0,0}}{r_{1,1}} \frac{r_{0,1}}{r_{1,1}}) \quad \text{if} \quad R \in \Phi_2(0, r)
\]

(26)

As a consequence, \( \Gamma^{(n)}_{r+1} \) is embedded in \( \Gamma^{(n)}_r \) and \( \{ \Gamma^{(n)}_r \} \) is a nested sequence of convex polytopes. To verify whether a point representing a distribution lies inside a certain polytope, and establish its extendibility, we can use a linear program analogous to (10).

We have computationally calculated the volume of some of the polytopes \( \Gamma^{(n)}_r \). We consider the ratio of the volume of \( \Gamma^{(n)}_r \) to the volume of \( \Gamma_n \) as an index of the proportion of \( n \)-ME distribution that are \( r \)-extendible, as has been done in [2] and [23] for the exchangeable case, and we report some values in Table 1. By (26) one can see that, unlike the DFPE case, not all the points \( \gamma^{(n)}_R \) are vertices of \( \Gamma^{(n)}_r \) as some of them are redundant. A strange consequence is that \( \Gamma^{(3)}_r = \Gamma_3 \) for any \( r \), so in Table 1 we start with \( n = 4 \).

| \( n \) \( r \) | 5  | 6  | 7  | 8  | 9  | 10 |
|--------------|----|----|----|----|----|----|
| 4            | 0.75 | 0.6667 | 0.6024 | 0.5504 | 0.5105 | 0.4778 |
| 5            | 0.4445 | 0.2860 | 0.2018 | 0.1454 | 0.1091 |
| 6            | 0.1929 | 0.0738 | 0.0336 |    |    |
| 7            | 0.0625 | 0.0111 |          |    |
| 8            | 0.0146 |          |    |
| 9            |          |          |    |    |    | 0.0025 |

Table 1: Values of \( \text{Vol}(\Gamma^{(n)}_r) / \text{Vol}(\Gamma_n) \) for different values of \( n \) and \( r \). The entries relative to \( n = 6, 7, 8 \) with \( r = 10 \) are missing since it seems computationally intractable to find the relative volume of \( \Gamma^{(n)}_r \).

### 4.2.1 \( \infty \)-extendible case

An \( \infty \)-extendible \( n \)-ME sequence is not necessarily the initial segment of a mixture of Markov Chains. As pointed out in [9], an infinite ME sequence starting in 0 is a mixture of two kinds of processes: recurrent Markov Chains and processes that deterministically begin with a streak of zeros, make a single \((0, 1)\) transition and end with all ones. But if, as \( n \to \infty \), both \( n_0^+ \) and \( n_1^+ \) go to infinity, there exists a unique mixing measure \( \nu \)
over \([0,1]^2\) and a couple \((\theta_{0,0},\theta_{1,1})\) such that, conditionally on \(X_1 = 0\)
\[
p_0 \left( \begin{array}{cc} n_{0,0} & n_{0,1} \\ n_{1,0} & n_{1,1} \end{array} \right) = \int_0^1 \int_0^1 \theta_{0,0}^{n_{0,0}} (1 - \theta_{0,0})^{n_{0,1}} \theta_{1,1}^{n_{1,1}} (1 - \theta_{1,1})^{n_{1,0}} d\nu \left( \theta_{0,0},\theta_{1,1} \right)
\]

Denote the indicator function of the event \(\{\text{the } k\text{-th successor of } i \text{ is } j\}\) as \(Y_{i,j}(k)\): \(\mathbb{1}_j(Y_i(k)) = Y_{i,j}(k)\) for all \(i, j\) in \(\{0,1\}\). Then we can write
\[
w_{0,a,b} = E \left[ (1 - X_1) \cdot Y_{0,0}(1) \cdots Y_{0,0}(a) \cdot (1 - Y_{0,0}(a + 1)) \cdot Y_{1,1}(1) \cdots Y_{1,1}(b) \right]
\]
When we consider sequences starting both in 0 and 1, we introduce the probabilities \(w_{1,a,b}\), defined as the probabilities of having the sequence starting in 1 with \(b\) transitions to \((1,1)\) a single transition \((1,0)\), and ending with \(a\) transitions \((0,0)\). Then we have:
\[
w_{1,a,b} = E \left[ X_1 \cdot Y_{1,1}(1) \cdots Y_{1,1}(b) \cdot (1 - Y_{1,1}(b + 1)) \cdot Y_{0,0}(1) \cdots Y_{0,0}(a) \right]
\]
In a mixture of Markov Chains it is
\[
w_{0,a,b} = E_{\nu} \left[ (1 - X_1) (\theta_{0,0})^a (\theta_{1,1})^b \right] - E_{\nu} \left[ (1 - X_1) (\theta_{0,0})^{a+1} (\theta_{1,1})^b \right]
w_{1,a,b} = E_{\nu} \left[ X_1 (\theta_{0,0})^a (\theta_{1,1})^b \right] - E_{\nu} \left[ X_1 (\theta_{0,0})^a (\theta_{1,1})^{b+1} \right]
\]
So, unlike the DFPE case, we do not have the mixed moments of the mixing distribution, but those involved differences. It is easily seen that it is not possible to single out the mixed moments from the probabilities \(w_{0,a,b}\) and \(w_{1,a,b}\). However, let \(N\) be the transition count matrix of \((X_1, \ldots, X_n)\) intended as a r.v. Then, if the ME distribution is such that \(X_1\) and \(N\) are independent, we can obtain them. Define
\[
m_{a,b} = E \left[ Y_{0,0}(1) \cdots Y_{0,0}(a) \cdot Y_{1,1}(1) \cdots Y_{1,1}(b) \right]
\]
and let \(P(X_1 = i) = q_i\). Under independence of \(X_1\) and \(N\) we have
\[
\frac{w_{0,a,b}}{q_0} = m_{a,b} - m_{a+1,b} \quad \text{and} \quad \frac{w_{1,a,b}}{q_0} = m_{0,b}
\]
Then we have \(m_{1,b} = m_{0,b} - (w_{0,0,b}/q_0)\), and in general, by recurrence
\[
m_{a,b} = m_{a-1,b} - \frac{w_{0,a-1,b}}{q_0}
\]
So an $n$–ME distribution such that $X_1$ and $N$ are independent is defined by the quantities $m_{a,b}$, for every couple $(a, b)$ such that $a + b \leq n - 1$. In a mixture of Markov Chains it is $m_{a,b} = E_{\nu} [\theta_{0,0}^a \theta_{1,1}^b]$ and we can formulate generalized covariances as in [9] and [10] and state simple necessary conditions for $\infty$–extendibility as in the DFPE case.

5 Concluding remarks

For what we have said, either for exchangeable, DFPE and ME cases, the $\infty$–extendible sequences are a particular subset of all the sequences of a fixed length. Then, in the inferential analysis of binary data, one can look for distributions which do not need the assumption of $\infty$–extendibility as an alternative to the mixtures of i.i.d and mixtures of Markov Chains processes. So, a preliminary analysis of the extendibility of the data at hand (i.e. of their empirical distribution) can give some evidences against a mixture model, and the present paper give the tools for this purpose.

Gupta in [15, 16] looked for an extension of the Hausdorff’s moment problem for distributions over the simplex, and implicitly found the necessary and sufficient conditions for the extendibility of an exchangeable finite sequence taking values in a finite state space, with the same geometric interpretation we have given. Combining his results with those of Section 3 one can easily find the conditions for the extendibility of DFPE sequences when the variables assume more than two values. It seems hard to find an analogous extension for the ME case.

Appendix A. Proof of Theorem 4.2

We first find $|\Phi_1(0, n)|$. In a sequence of length $n$ we have $n - 1$ transitions. For every fixed value for $n_{1,0} = n_{0,1}$ equal to $k$, say, the couple $(n_{1,1}, n_{0,0})$ can assume all the possible values such that $(n_{1,1} + n_{0,0}) = n - 1 - 2k$, whose number is $(n - 2k)$. The possible values for $k = n_{1,0} = n_{0,1}$ range in $0, 1, \ldots, \lfloor (n-1)/2 \rfloor$, where $\lfloor (n-1)/2 \rfloor$ is the integer part of $(n-1)/2$. In the special case $n_{1,0} = n_{0,1} = 0$ we have only one matrix $\begin{pmatrix} n_{1,1} & 0 \\ 0 & 0 \end{pmatrix}$. So we have

$$|\Phi_1(0, n)| = 1 + \sum_{k=1}^{\lfloor (n-1)/2 \rfloor} (n - 2k)$$

Now consider the following two arguments:
• All the sequences consistent with a matrix in $\Phi_2(0, n)$ start in 0 and end in 1. If we add a transition $(1, 0)$ at the end of any such sequence, its transition count matrix belong to $\Phi_1(0, n + 1)$.

• If we reduce of one the number of transitions $(1, 0)$ in a matrix of $\Phi_1(0, n + 1)$, we obtain a matrix of $\Phi_2(0, n)$.

Consequently, each matrix of the second kind is constructible by one of the first kind of a step longer, as long as $n_{1,0}$ is not null. Then we have to exclude the matrix having $n_{1,0} = n_{0,1} = 0$ and it is:

$$|\Phi_2(0, n)| = |\Phi_1(0, n + 1)| - 1 = \sum_{k=1}^{\lfloor n/2 \rfloor} (n + 1 - 2k)$$

Clearly it is $|\Phi(0, n)| = |\Phi_1(0, n)| + |\Phi_2(0, n)|$, that is

$$|\Phi(0, n)| = 1 + \sum_{k=1}^{(n-1)/2} (n - 2k) + \sum_{k=1}^{\lfloor n/2 \rfloor} (n - 2k + 1) = 1 + \sum_{k=1}^{n-1} (n - k) = 1 + \binom{n}{2}$$

**Appendix B. Proof of Theorem 4.4**

$w_{0,a,b}$ is the probability $p_0\left(\begin{smallmatrix} a & 1 \\ 0 & b \end{smallmatrix}\right)$ of having a sequence starting in 0 and ending in 1. Then by (23) we have

$$w_{0,a,b} = p_0\left(\begin{smallmatrix} a & 1 \\ 0 & b \end{smallmatrix}\right) = p_0\left(\begin{smallmatrix} a & 1 \\ 0 & b+1 \end{smallmatrix}\right)$$

$$= p_0\left(\begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}\right) + w_{0,a,b+1}$$

Then it follows that

$$p_0\left(\begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}\right) = -\Delta_1(w_{0,a,b}) \quad (27)$$

so, we can derive the probability of having any sequence starting in 0 and consistent with the transition count matrix $\left(\begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}\right)$. These sequences end in 0, so by (22) we have

$$p_0\left(\begin{smallmatrix} a & 2 \\ 1 & b \end{smallmatrix}\right) = p_0\left(\begin{smallmatrix} a & 1 \\ 1 & b \end{smallmatrix}\right) - p_0\left(\begin{smallmatrix} a+1 & 1 \\ 1 & b \end{smallmatrix}\right) \quad (28)$$

We have just demonstrated that all the terms on the right hand side of (28) can be derived from (27), and it is $p_0\left(\begin{smallmatrix} a & 2 \\ 1 & b \end{smallmatrix}\right) = \Delta_0\left(\Delta_1(w_{0,a,b})\right)$. So, we can derive the probability of any sequence starting in 0 and consistent with the transition count matrix $\left(\begin{smallmatrix} a & 2 \\ 1 & b \end{smallmatrix}\right)$. 21
For an \( n \)-ME sequence starting in 0, it is always \( n_{0,1} = n_{1,0} \) or \( n_{0,1} = n_{1,0} + 1 \). So, repeating the previous passages, by recurrence, we obtain:

\[
p_{0,N} = p_0 \left( \begin{array}{c} n_{0,0} \\ n_{1,0} \\ n_{1,1} \end{array} \right) = \begin{cases} -\Delta_1 \left( \Delta_0 \circ \Delta_1 \right)^{n_{1,0}-1} (w_{0,n_{0,0},n_{1,1}}) & \text{if } N \in \Phi_1(0,n) \\ \left( \Delta_0 \circ \Delta_1 \right)^{n_{1,0}} (w_{0,n_{0,0},n_{1,1}}) & \text{if } N \in \Phi_2(0,n) \end{cases}
\]

which is equivalent to Theorem 4.4.

Appendix C. Proof of Theorem 4.3

Let \( K \) be the transition count matrix of the first \( k \) steps of the sequence. The number of sequences \( (x_1, \ldots, x_n) \in \{0, 1\}^n \), with \( x_1 = 0 \), such that \( K = \left( \begin{array}{cc} k_{0,0} & k_{0,1} \\ k_{1,0} & k_{1,1} \end{array} \right) \) and \( N = \left( \begin{array}{cc} n_{0,0} & n_{0,1} \\ n_{1,0} & n_{1,1} \end{array} \right) \) is equal to the number of sequences consistent with the transition count matrix \( \left( \begin{array}{cc} n_{0,0}-k_{0,0} & n_{0,1}-k_{0,1} \\ n_{1,0}-k_{1,0} & n_{1,1}-k_{1,1} \end{array} \right) \) that is

\[
\begin{cases} \left( \frac{n_{0,0}^+ - k_{0,0}^+}{n_{0,0}^+ - k_{0,0}} \right) \left( \frac{n_{0,1}^+ - k_{0,1}^+}{n_{0,1}^+ - k_{0,1}} \right) & \text{if } x_n = 0 \\ \left( \frac{n_{0,0}^- - k_{0,0}^-}{n_{0,0}^- - k_{0,0}} \right) \left( \frac{n_{0,1}^- - k_{0,1}^-}{n_{0,1}^- - k_{0,1}} \right) & \text{if } x_n = 1 \end{cases}
\]

But, as we have said, since we have fixed \( x_1 = 0 \), it is \( x_n = 0 \) if \( N \) is of the first kind, and \( x_n = 1 \) if \( N \) is of the second kind. Then it is

\[
p_{0,K} = \sum_{N \in \Phi_1(0,n)} \left( \frac{n_{0,0}^+ - k_{0,0}}{n_{0,0} - k_{0,0}} \right) \left( \frac{n_{0,1}^+ - k_{0,1}}{n_{0,1} - k_{1,1}} \right) p_{0,N^+} + \sum_{N \in \Phi_2(0,n)} \left( \frac{n_{0,0}^- - k_{0,0}}{n_{0,0} - k_{0,0}} \right) \left( \frac{n_{0,1}^- - k_{0,1}}{n_{0,1} - k_{1,1}} \right) p_{0,N}
\]

Finally the theorem follows by (21) and the fact that

\[
\left( \frac{n_{0,0}^+ - k_{0,0}}{n_{0,0}^+} \right) = \left( n_{0,0} - k_{0,0} \right) \left( n_{0,0}^+ - n_{0,0} \right) = \frac{(n_{0,0})_{k_{0,0}}(n_{0,0}^+ - n_{0,0})_{k_{0,0}}}{(n_{0,0})_{k_{0}^+}}
\]

References

[1] D. J. Aldous. Exchangeability and related topics. In École d’été de probabilités de Saint-Flour, XIII—1983, volume 1117 of Lecture Notes in Math., pages 1–198. Springer, Berlin, 1985.
[2] L. Crisma. Quantitative analysis of exchangeability in alternative processes. In *Exchangeability in probability and statistics (Rome, 1981)*, pages 207–216. North-Holland, Amsterdam, 1982.

[3] B. de Finetti. *Sur la condition d’equivalence partielle. Actualités Scientifiques et Industrielles*. Hermann, Paris, 1938. Vol.739 pp. 5–18.

[4] B. de Finetti. La probabilità e la statistica nei rapporti con l’induzione, secondo i diversi punti di vista. C.I.M.E., Induzione e Statistica (No.1), pp. 1–115, 1959.

[5] B. de Finetti. Sulla proseguibilità di processi aleatori scambiabili. *Rend. Ist. Mat. Univ. Trieste*, 1:53–67, 1969.

[6] B. de Finetti. *Probability, Induction and Statistics*. Wiley, New York, 1972.

[7] P. Diaconis. Finite forms of de Finetti’s theorem on exchangeability. *Synthese*, 36(2):271–281, 1977. Foundations of probability and statistics, II.

[8] P. Diaconis and D. Freedman. de Finetti’s generalizations of exchangeability. In *Studies in inductive logic and probability, Vol. II*, pages 233–249. Univ. California Press, Berkeley, Calif., 1980.

[9] P. Diaconis and D. Freedman. de Finetti’s theorem for Markov chains. *Ann. Probab.*, 8(1):115–130, 1980.

[10] P. Diaconis and D. Freedman. Finite exchangeable sequences. *Ann. Probab.*, 8(4):745–764, 1980.

[11] P. Diaconis and D. Freedman. Partial exchangeability and sufficiency. In *Statistics: applications and new directions (Calcutta, 1981)*, pages 205–236. Indian Statist. Inst., Calcutta, 1984.

[12] E. B. Dynkin. Sufficient statistics and extreme points. *Ann. Probab.*, 6(5):705–730, 1978.

[13] S. Fortini, L. Ladelli, G. Petris, and E. Regazzini. On mixtures of distributions of Markov chains. *Stochastic Process. Appl.*, 100:147–165, 2002.

[14] D. Freedman. Invariants under mixing which generalize de Finetti’s theorem. *Ann. Math. Statist*, 33:916–923, 1962.

[15] J. C. Gupta. The moment problem for the standard $k$-dimensional simplex. *Sankhyā Ser. A*, 61(2):286–291, 1999.

[16] J. C. Gupta. Completely monotone multisequences, symmetric probabilities and a normal limit theorem. *Proc. Indian Acad. Sci. Math. Sci.*, 110(4):415–430, 2000.

[17] R. Jeffrey, editor. *Studies in inductive logic and probability. Vol. II*. University of California Press, Berkeley, Calif., 1980.

[18] R. Jeffrey. *Subjective probability. The real thing*. Cambridge University Press, Cambridge, 2004.

[19] N. L. Johnson, S. Kotz, and N. Balakrishnan. *Discrete multivariate distributions*. Wiley Series in Probability and Statistics: Applied Probability and Statistics. John Wiley & Sons Inc., New York, 1997. A Wiley-Interscience Publication.
[20] M. Scarsini and L. Verdicchio. On the extendibility of partially exchangeable random vectors. Statist. Probab. Lett., 16(1):43–46, 1993.

[21] J. von Plato. Finite partial exchangeability. Statist. Probab. Lett., 11(2):99–102, 1991.

[22] P. Whittle. Some distribution and moment formulae for the Markov chain. J. Roy. Statist. Soc. Ser. B., 17:235–242, 1955.

[23] G. R. Wood. Binomial mixtures and finite exchangeability. Ann. Probab., 20(3):1167–1173, 1992.

[24] A. Zaman. Urn models for Markov exchangeability. Ann. Probab., 12(1):223–229, 1984.

[25] A. Zaman. A finite form of de Finetti’s theorem for stationary Markov exchangeability. Ann. Probab., 14(4):1418–1427, 1986.