Small World Graphs by the iterated “My Friends are Your Friends” Principle

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Abstract

We study graphs obtained by successive creation and destruction of edges into small neighborhoods of the vertices. Starting with a circle graph of large diameter we obtain small world graphs with logarithmic diameter, high clustering coefficients and a fat tail distribution for the degree. Only local edge formation processes are involved and no preferential attachment was used. Furthermore we found an interesting phase transition with respect to the initial conditions.

1 Introduction

In 1998 Watts and Strogatz introduced in a seminal paper Small World Networks and provided a simple model of such graphs [1, 2]. The key property of Small World Graphs is the simultaneous presence of small diameter (at most logarithmic in the vertex size) and high clustering coefficient. Extensive investigations over the last decade of real networks like the WorldWideWeb, citation networks, friendship networks and many others have shown that these graphs are really of Small World type. But there is a second remarkable property shared by most real networks namely that they have a power law like fat tail distribution (see [3] for an excellent recent survey about the whole subject).

There are many models which produce graphs with small diameter and power law distribution. If additionally local search rules are incorporated one easily obtains high clustering as well. But up to now it was common believe that for the small diameter property a certain amount of essentially independent random edge formations like in classical random graphs where necessary. To obtain a scale free distribution for the degree the basic conviction was that the random edge formation should be biased according to the “the richer you are the richer you get- principle” [4] where richness is just measured by the degree of a vertex. In the present paper we will present a model which shows that even by entirely local edge formation rules without preferential choices graphs are obtainable which admit all the three above mentioned properties. The results are entirely based on numerical simulations since an analytic treatment of the model seems to be rather difficult. Nevertheless we hope to present in a forthcoming paper a more theoretical study of the main properties.
2 The iterated “My Friends are Your Friends” Principle

Looking at the edge formation process in social networks there are essential two ways to get a new contact. First by the random event of meeting somebody for instance in a train or airplane. Humans refer to this kind of relation creation often as fate. Second there are contacts which where created by the local rule “let me introduce you to one of my friends”, certainly a very common process in real life. In the language of graphs this translates to the formation of an edge between two vertices say $x$ and $y$ which had distance 2 with respect to the underlying graph metric and we refer to this kind of edge formation as the “My Friends are your Friends” Principle (FFP). Graphs which where partially build by using the FF-Principle are already described and analyzed in [5]. In the following we want to describe a model based on the iteration of this principle - the IFF-model. A typical example of an essential FF-graph (although a directed one and with increasing number of vertices) is the citation network.

3 The model description and results

In detail, we want to discuss the following model of a time dependent random graph space $G_t(N, t_0)$ with $N$ vertices and an explicit integer parameter $t_0$ tuning the total number of edges. We start with a circular chain of $N$ vertices with the usual nearest neighbor edges, i.e. vertex 1 is connected to vertex $N$ and 2, vertex 2 is connected to vertex 1 and 3 and so on. We call this configuration a circle and refer to the corresponding edges as fixed base edges. The random graph space $G_t(N, t_0)$ will be defined through the following algorithm: In each of the first $t_0$ time-steps, a vertex say $x$ is chosen randomly. From the set of neighbors $N_1(x,t)$ of this vertex (at the given time $t$) one element say $y$ is chosen randomly and then randomly from $N_1(y,t) \setminus \{x\}$ – the neighbor set of $y$ at time $t$ with the exclusion of $x$ – another element say $z$. If there was no edge between $z$ and $x$ a new one is created between the two vertices otherwise nothing happens. In the first case we say that $x$ has created an FF-edge to $z$– respectively $z$ was chosen by $x$. After repeating $t_0$ times the above procedure one continues for $t > t_0$ the same way with the additional rule that whenever a vertex say $x$ has created a new FF-edge randomly an FF-edge containing $x$ is deleted. Note that the total number of edges stays therefore constant for $t > t_0$.

3.1 The diameter

The first surprising observation is the collapse of the diameter from $\text{const} \cdot N$ to $\text{const} \cdot \log N$ for sufficiently large $t$, $t_0$ and typical elements of $G_t(N, t_0)$. This happens despite the fact that only local edge formations where used. Fig. II shows the diameter versus $t$ for different $N$ and $t_0$. Our simulations indicate that in the case $t_0/N \gtrsim 2$ one gets an asymptotic regime in which the diameter is very small and shows no strong dependence on $N$. Tab. III shows for three choices of $t_0/N$ the mean diameter $\bar{d}$ in the asymptotic regime. The diameter $\bar{d}$ is approximately pro-
Figure 1: Diameter of $G_t(N, t_0)$; $d$ versus time, averaged over 10 realizations.

proportional to $\log N$ which is one of the characteristics of small world graphs. In the case $t_0/N = 2$ an asymptotic regime with still small diameter but large fluctuations starts approximately at $t/N \approx 10^4$. Note that this value is much bigger than the collapse time in the cases $t/N = 4, 10$. It seems that $t_0/N \approx 2$ is just the borderline where a collapse of the diameter appears. This corresponds to the case where the mean number of non-base edges equals $N$. For smaller values of $t_0$ we found the diameter to stay of order $f(t_0) \cdot N$ with an almost linear function $f$ (see Fig. 2). This indicates an interesting phase transition in the total number of edges reminiscent to the famous phase transition in the size of the largest component in classical random graph spaces like the $G(N, M)$ – the space of all graphs with $N$ vertices and $M$ edges equipped with the uniform probability measure.

Fig. 3 shows one realization of the time evolution of a typical graph from $G_t(N, t_0)$ with $t_0/N = 4$ and $N = 10^3$. It illustrates how the diameter of the graph becomes very small only by using the IFF principle. Before the asymptotic state is reached an interesting symmetry breaking can be observed: the appearance of a few components with long range edges (taken the circle as a reference system) and hence small diameter. The components itself are only connected via the circle skeleton. It takes
Table 1: Mean diameter of the circle system

| $t_0/N$ | $N$   | $M(t_0)/N$ | $\bar{d}$ | $\bar{d}/\log_{10} N$ | $\bar{d}_{\text{random}}$ |
|---------|-------|-------------|------------|------------------------|------------------------|
| 2       | $10^3$| 1.1         | 24.0       | 8.00                   | 7.7                    |
| 2       | $10^4$| 1.1         | 47.0       | 11.75                  | 10.1                   |
| 2       | $10^5$| 1.1         | 70.6       | 14.12                  | 12.3                   |
| 4       | $10^3$| 2.0         | 8.3        | 2.78                   | 6.1                    |
| 4       | $10^4$| 2.0         | 11.6       | 2.90                   | 7.7                    |
| 4       | $10^5$| 2.0         | 15.0       | 3.00                   | 9.2                    |
| 10      | $10^3$| 4.4         | 5.1        | 1.70                   | 4.2                    |
| 10      | $10^4$| 4.4         | 6.7        | 1.68                   | 5.7                    |
| 10      | $10^5$| 4.4         | 8.2        | 1.64                   | 7.0                    |

calculated by averaging the values $d(t)$ for $10^4 < t/N < 2 \cdot 10^4$ ($t_0/N = 2$) or $10^3 < t/N < 4 \cdot 10^3$ ($t_0/N = 4, t_0/N = 10$) and 10 realizations; effective non-base edges $M(t_0)$; the mean diameter $\bar{d}_{\text{random}}$ of $G_c(N, M(t_0))$ is calculated by averaging over 100 realizations.

Figure 2: Transition in the circle system; $N = 10^4$, left axis: mean diameter $d$ calculated at $t/N = 10^4$ (boxes with lines) and $t/N = 2 \cdot 10^4$ (triangles with lines), right axis: effective non-base edges $M(t_0)$ (crosses) versus $t_0/N$ calculated at $t/N = 10^4$; averaged over 10 realizations.
a relative long time till the components finally merge and the circle is filled uniformly with edges. The same phenomenon can be seen for larger values of \(N\) and different \(t_0/N\).

It is interesting to compare our model with a version of the random graph space \(G(N, M)\) where an underlying circle is added to keep the graph connected. To be precise let \(G_c(N, M)\) be the random graph space where \(M\) edges are randomly added to a circle graph with \(N\) vertices. Fig. 4 shows the diameters \(d_{\text{random}}\) of the average of a few samples of \(G_c(N, M)\) as a function of \(M/N\). Here the diameter clearly decays like \(\frac{\text{const} \cdot N}{M}\) and reaches the log \(N\) regime already for very small values of \(M/N\) (\(\sim 0.05\)). Note that this value is much smaller than the threshold value \(M/N \sim 0.5\) for the emergence of a giant component in \(G(N, M)\). Let us remark that the \(G_c(N, M)\) model is very close in spirit to the original Small World model of Watts and Strogatz. The main difference is with respect to the clustering coefficient since the underlying skeleton – the circle – has zero clustering coefficient. Replacing the standard circle with next neighbor connections by one where also next-next neighbor connections are edges would give an essentially equivalent model to the Watts-Strogatz one.

### 3.2 Fat tail of the degree distribution

Another unexpected property of our model is the fat tail of the degree distribution. Fig. 5 shows the degree distribution for \(N = 10^5\) and \(t_0/N = 4\) at various times.

Let us compare the degree distribution in the asymptotic regime with those of the random graph \(G_c(N, M(t_0))\), where \(M(t_0)\) is the expectation of the number of edges in the IFF random graph space \(G(t, N, t_0)\) (see Fig. 2 for a plot of \(M(t_0)/N\)).
Figure 5: Degree distribution using the circle system; $N = 10^5$, a $t_0/N = 4$, $t/N = 4$ (circles), $t/N = 600$ (boxes), $t/N = 4000$ (triangles), b $t_0/N = 10$, $t/N = 10$ (circles), $t/N = 600$ (boxes), $t/N = 4000$ (triangles); the big dotted line shows the degree distribution of $G_c(N, M(t_0))$, all values averaged over 10 realizations.

The resulting degree distributions are shown in Fig. 5a and Fig. 5b by big dotted lines. At the beginning (that is for $t \sim t_0$) the IFF-model has essential the degree distribution of $G_c(N, M(t_0))$ but in the asymptotic regime the distributions differ drastically since the IFF-model has gotten a fat tail. At the moment the limited numerical data still don’t allow to check whether it is really a power law.

We want to close this section with a remark about the degree preferences if a vertex $x$ was chosen in the formation of a new edge. Although never explicitly included in the model there is a strong numerical evidence (see Fig. 4) that the probability of a vertex to be chosen is proportional to its degree. That is one of the basic assumptions in the Albert–Barabási model. A heuristic explanation for this property in our model is the following argument based on two independence assumptions. Let $N_1$ and $N_2$ be the expected values for the first and second neighborhood sizes. Assume that the conditional expectation of $N_2(x)$ for $x$ having degree $k$ equals $k \cdot N_1$ and assume further that for $z \in N_2(x \mid d(x) = k)$ the expected value of $N_2(z)$ equals $N_2$. Clearly with these conditions one can easily compute the probability of a vertex $x$ with degree $k$ to be chosen within the process of an edge formation to be equal to $\frac{k \cdot N_1}{N} \cdot \frac{1}{N_2}$.

3.3 Clustering properties

A basic quantity to measure the local clustering around a vertex $x$ is the number of triangles $C_3(x)$ containing $x$ as a vertex. Note that for classical random graph spaces the expectation of this number is is of order $N^{-1}$. We are interested in the averaged number of triangles in the asymptotic regime of the IFF-model. Fig. 6 shows the
mean number of triangles $C_3$ for all vertices with a fixed degree for $t/N = 600$ and $t/N = 4000$. The mean number of triangles per vertex (independent of its degree) is shown by the big symbols on the figure frame. The plot for the two time-values coincides practically. The number of triangles increases nearly linear with the degree $k$. The reason for the fluctuations for high $k$ is the low number of vertices with such high degrees. Fig. 7 shows also the mean clustering coefficient $C_c$ which is directly connected to the mean number of triangles. Namely the clustering coefficient $C_c(x)$ of a vertex $x$ with degree $k$ is given by the normalized triangle coefficient $\frac{2C_3(x)}{k(k-1)}$.

There is another interesting quantity, defined via the second shortest distance between two vertices, which characterizes additional clustering properties. For pairs of vertices with a common edge let $d_2(x,y)$ be the distance between $x$ and $y$ after removal of the connecting edge. Define for a given graph $G$ the quantity $\varphi^{(n)}(G)$ as the fraction of pairs of vertices with distance 1 whose $d_2$-distance is larger $n$ (for a given random graph space $\mathcal{G}$ let $\varphi^{(n)}(\mathcal{G})$ the expectation of $\varphi^{(n)}$). In case when the diameter of a random graph space is small due to the presence of independently generated edges the $\varphi^{(2)}$ value is usually very large since no short second-shortest paths between two vertices connected by an independently generated edge exist.
Table 2: Mean value of $\varphi^{(n)}$ of the circle system

| $t_0/N$ | $t/N$ | $\varphi^{(2)}$ | $\varphi^{(3)}$ |
|---------|-------|----------------|----------------|
| 4       | 600   | 0.28           | 0.15           |
| 4       | 4000  | 0.29           | 0.15           |
| 10      | 600   | 0.30           | 0.16           |
| 10      | 4000  | 0.30           | 0.16           |

Calculated with $N = 10^5$ and after $t$ steps, averaged over 10 realizations

Table 3: Mean diameter of the torus system

| $t_0/N$ | $N$ | $M(t_0)/N$ | $\bar{d}$ | $\bar{d}/\log_{10} N$ | $\bar{d}_{\text{random}}$ |
|---------|-----|------------|--------|---------------------|---------------------|
| 2       | $30^2$ | 1.5 | 7.9   | 2.67               | 5.6                |
| 2       | $100^2$ | 1.5 | 14.1  | 3.53               | 7.0                |
| 2       | $300^2$ | 1.5 | 27.6  | 5.57               | 8.4                |
| 4       | $30^2$ | 2.8 | 5.1   | 1.73               | 4.7                |
| 4       | $100^2$ | 2.8 | 6.7   | 1.68               | 6.0                |
| 4       | $300^2$ | 2.8 | 8.0   | 1.61               | 7.0                |
| 10      | $30^2$ | 6.6 | 4.0   | 1.35               | 4.0                |
| 10      | $100^2$ | 6.7 | 5.0   | 1.25               | 4.8                |
| 10      | $300^2$ | 6.7 | 6.0   | 1.21               | 5.5                |

Calculated with the values of the mean diameters in the range $1000 < t/N < 4000$ averaged over 10 realizations; the value $\bar{d}_{\text{random}}$ is calculated by averaging over 100 realizations

This remains also true for the Watts-Strogatz Small World graph. In our model we get by the very construction process a small value of $\varphi^{(3)}$ similar to the situation met in real networks (see Tab. 2).

3.4 Replacing the circle by a torus

Up to now, the system of base edges forms a circle. It is natural to ask if one gets qualitatively the same results with a two-dimensional system of base edges like a torus lattice. This means that at the beginning the vertices are connected in form of a torus or a lattice with periodic boundaries. The algorithm to generate (and define) $G_t(N,t_0)$ remains the same.

For the diameter one gets again an asymptotic regime with collapse up to logarithmic size but the asymptotics starts earlier as in the “circle”-case. Even for $t_0/N = 2$ we get an asymptotic regime starting at $t/N \approx 200$. Tab. 3 shows the mean diameter $\bar{d}$ in the case of a torus base for various values of $t_0$ and $N$.

There is another difference to the one-dimensional case which can be seen in the realization shown in Fig. 8: the graph evolution is more uniformly with a two-dimensional base edge system than with a one-dimensional (compare to Fig. 3).

We obtain qualitatively the same results as in the one-dimensional case with respect to the degree distribution and the clustering properties for $N = 300^2$ and $t_0/N = 4$, $t_0/N = 10$. In the asymptotic regime one sees again a clear fat tail distri-
Figure 8: Realization using the torus system; $N = 30^2$, $t_0/N = 4$; a, b the edges at different times are plotted, the dots characterize the vertices, the color of an edge characterizes its length $l$; c the dots characterize the vertices, the color characterizes the length $l$ of the longest edge of the vertex.

bution. The distribution of the number of triangles and the clustering coefficient for $N = 300^2$ and $t_0/N = 4$ gives qualitatively the same results as in the “circle”-case, i.e. the distributions are nearly equal for $t/N = 600$ and $t/N = 4000$ and we have a linear dependence of the number of triangles on the degree $k$.

3.5 Removal of the base edge system

Finally we want to discuss the stability of the graphs in the asymptotic regime under removal of the circle edges or torus edges, i.e. at $t/N = 1000$ we remove the base edges and examine in which way the graph breaks into connected components and how these components evolve by continuing using our algorithm. Fig. 9 shows the number of connected components with different size $s$ (number of vertices) and different diameter $d$ for $t_0/N = 4$ directly after deleting the base edges at $t/N = 1000$ for the “circle” and the “torus” case. There is still one big component. The time evolution of the biggest component can be seen in Fig. 10 which shows its size $s$ and diameter $d$ versus time. An interesting result is that the diameter of the biggest component is nearly constant for $1500 < t/N < 4000$, nearly independent of $t_0/N$ and also nearly independent of using a circle or a torus as base edge system. Let us note that without the skeleton there can be no merging of different components by the very nature of the IFF principle. Therefore, as can easily be seen, the only stable asymptotic configurations are components which are totally connected (i.e. every vertex has an edge to all other vertices in the component) since no further changes in such components can happen. But it is very likely that the time scale till this phenomenon can be seen is huge compared to the time scales we where studying.

To compare with the classical random graph situation observe that the removal of the circle in $G_c (N, M)$ gives just the model $G(N, M)$. The component distribution
Figure 9: Number of components with a given number of vertices and diameter, after deleting the basic edges, \( t/N = 1000, t_0/N = 4 \), one realization

for this random graph space is well known. Especially for \( M > N/2 \) there is always a giant component of size \( \text{const} \cdot N \). But in contrast to our model, the diameter \( d \) of the biggest component depends via the constant on \( M (t_0) \).

4 Summary and outlook

We have presented a model where the iteration of the "My Friends are Your Friends (FF)" principle has produced random graphs with small world properties (logarithmic diameter, high clustering) and a fat tail distribution for the degree. The model used a fixed regular graph of large diameter as a skeleton, reminiscent to a pre-given geographical structure with pre-given neighborhood relations for vertex pairs. But in contrast to the small world graphs by Watts and Strogatz no random, global edge formation processes are involved to obtain small diameter. The high clustering is an immediate consequence of the FF-principle. The fat tail distribution is surprising since no preferential choice mechanisms are contained in the graph generation algorithm. So far our investigations are entirely numerical and clearly a more theoretical explanation of the observed phenomena is the desired next step.

There are several natural variants of the model which we would like to mention shortly.

First, in our model the total number of edges is kept fixed after the build up phase. Instead of that one could use probabilistic rules which keep the number of edges only fixed in mean. This would match better real situations where the FF-principle is of relevance.

Second, like in the Albert & Barabási network [4], a growing number of vertices could be considered. Growth should happen here in form of offsprings of already existing vertices to be able to apply the FF-principle to the "newcomers". In this situation it could well be the case that even without the iteration of the FF-principle – every vertex when entering the network forms just once a number of edges according to the FF-rule – Small World graphs plus fat tail for the degree distribution are obtainable. This model variant is actually the typical situation for the growth and formation of the network of citations or collaboration.
Figure 10: Parameters of the biggest component after deleting the basic edges; a time dependence of its diameter $d$; b time dependence of its size $s$; circle, $N = 10^5$, $t_0/N = 4$ (plus signs); circle, $N = 10^5$, $t_0/N = 10$ (crosses); torus, $N = 300^2$, $t_0/N = 4$ (circles); torus, $N = 300^2$, $t_0/N = 10$ (boxes); the symbols on the left vertical axes indicate the values of the parameters directly before deleting the basic edges; all values averaged over 10 realizations
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