Universal Diamagnetism of Charged Scalar Fields

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We show that charged scalar fields are always diamagnetic, even in the presence of interactions and at finite temperatures. This generalises earlier work on the diamagnetism of charged spinless bosons to the case of infinite degrees of freedom.

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I. INTRODUCTION

A classical gas of charged point particles is non-magnetic, by Van Leeuwen’s Theorem [1]. But quantum mechanically, free spinless Bose particles in a uniform magnetic field show diamagnetism [2]. One finds that the energy of the system in a magnetic field is higher than in the absence of the field. Simon [3] proved quite generally that the ground state of non-relativistic spinless Bosons interacting through an arbitrary potential always increases in a magnetic field. He went on to extend this result by showing that the free energy in the presence of a magnetic field is always greater than the free energy in the absence of a magnetic field at all temperatures [3]. An alternative proof of this result is given in [4]. All this work which deals with systems with a finite number of degrees of freedom suggests that diamagnetism is a universal property of Spinless Bosons. In field theory (which describes systems with an infinite number of degrees of freedom) charged spinless Bosons are described by complex scalar fields. One might therefore expect that charged scalar fields would also show diamagnetic behaviour. With this motivation we study the magnetic behavior of scalar field theories.

The paper is organised as follows. The first part deals with finite temperature free scalar field theory in the presence of an external homogeneous magnetic field. Here we explicitly calculate the partition function and the free energy as a function of the applied magnetic field. This expression is formally divergent. Using a suitable regularization scheme we compute the difference in the free energy (with and without the magnetic field) and obtain a finite answer. This difference is also shown to be positive, thus establishing the diamagnetic behaviour of free charged scalar fields.

We then move on to interacting scalar field theory in the second part. Here, we cannot evaluate the partition function explicitly. Nevertheless we prove the universal diamagnetism of scalar fields by assuming a finite momentum cutoff in the theory. If the theory is renormalizable, then one can take this cutoff to infinity while maintaining finiteness of all physical quantities. In both cases the results obtained are exact.
II. FREE CASE

In this section we calculate the free energy of free scalar fields in the presence of an external uniform magnetic field. For ease of presentation we work in two spatial dimensions. The interesting physics takes place in the plane normal to the applied field. Generalization to higher dimensions is straightforward.

Let $\Phi$ be a complex scalar field which describes charged spinless Bosons. The Lagrangian density of a free charged scalar field in the presence of a constant homogeneous external magnetic field is given by

$$\mathcal{L} = (D_\mu \Phi)^* (D^\mu \Phi) - m^2 (\Phi^* \Phi)$$

where $\mu = 0, 1, 2$,

$$D_\mu = \partial_\mu - ieA_\mu$$

and $m$ and $e$ are the mass and charge respectively. (We set $\hbar = 1$ and $c = 1$). Now, we write the complex field in term of two real fields $\Phi_1$ and $\Phi_2$.

$$\Phi = \frac{\Phi_1 + i\Phi_2}{\sqrt{2}}, \quad \Phi^* = \frac{\Phi_1 - i\Phi_2}{\sqrt{2}}$$

This theory has a global U(1) symmetry and therefore a conserved Noether charge $Q$, given by

$$Q = \int d^2 x (\Pi_1 \Phi_2 - \Phi_1 \Pi_2)$$

where

$$\Pi_i = \partial_0 \Phi_i$$

The Hamiltonian density of the system is given by

$$\mathcal{H} = \frac{1}{2} (\Pi_1^2 + \Pi_2^2) + \frac{1}{2} (\nabla \Phi_1)^2 + \frac{1}{2} (\nabla \Phi_2)^2 + \frac{1}{2} (m^2 + e^2 A^2) (\Phi_1^2 + \Phi_2^2) - j \cdot A$$
where the current density $j$ is given by

$$ j = -e(\Phi_1 \nabla \Phi_2 - \Phi_2 \nabla \Phi_1). $$

(7)

We now suppose that the external magnetic field is uniform in the $x - y$ plane.

We choose the temporal gauge ($A_0 = 0$). The constant magnetic field $B$ is

$$ B = \partial_x A_y - \partial_y A_x $$

(8)

where $A(x)$ is independent of $t$.

The action of this theory is

$$ S = \int_0^\beta \int d^2x d\tau L(\Phi, \Phi^*, A), $$

(9)

where $\tau$ is the imaginary time variable which runs from 0 to $\beta (=1/(k_B T))$, the inverse temperature. The action defined above is quadratic and so the partition function can be evaluated exactly. As is usual in finite temperature field theory [5], we impose periodic boundary conditions for Bosonic fields

$$ \Phi(x, 0) = \Phi(x, \beta). $$

(10)

Now, the partition function of this theory can be written as

$$ Z(B) = \int \mathcal{D}[\Pi_1] \mathcal{D}[\Pi_2] \int \mathcal{D}[\Phi_1] \mathcal{D}[\Phi_2] \exp \left[ \int d\tau d^2x \left( i\Pi_1 \frac{\partial \Phi_1}{\partial \tau} + i\Pi_2 \frac{\partial \Phi_2}{\partial \tau} - H(\Phi_1, \Phi_2, \Pi_1, \Pi_2) + \mu(\Phi_2 \Pi_1 - \Phi_1 \Pi_2) \right) \right] $$

(11)

Here $\mu$ is the chemical potential associated with the conserved charge. We pick the gauge in which the vector potential $A$ is (-By, 0), and expand the complex scalar field in terms of modes adapted to the present situation. These modes solve the Klein-Gordon equation in an external magnetic field. The eigenfunctions are labelled by one discrete ($l$) and one continuous $p_x$ quantum number and the spectrum depends on $l$ only. In the gauge we choose, the modes are plane waves in the $x$ direction and harmonic oscillator (i.e. gaussian) wavefunctions in the $y$ direction.
The spectrum is given by

$$\omega_l^2 = m^2 + (2l + 1)eB, \quad l = 0, 1, 2, \ldots \infty$$  \hfill (12)

The degeneracy of these states for fixed $l$ is $eAB/2\pi$, where $A$ is the area of the system. So, these modes can be thought of as quantized harmonic oscillators. Expanding the fields $\Phi_1$ and $\Phi_2$ in these modes the system reduces to a collection of harmonic oscillators with frequency $\omega_l$.

By standard manipulations we get the free energy as

$$F(B) = -\ln Z(B)/\beta = 2\pi eAB \sum_{l=0}^{\infty} \left[ \omega_l + \frac{1}{\beta} \ln(1 - \exp(-\beta(\omega_l - \mu))) + \frac{1}{\beta} \ln(1 - \exp(-\beta(\omega_l + \mu))) \right]$$ \hfill (13)

The first term in the square brackets corresponds to the zero point fluctuation of the vacuum and the other two terms are finite temperature contributions. We will first compare the free energy of the system with and without the magnetic field at zero temperature.

The free energy of the system in presence of the magnetic field at zero temperature is given by

$$F_0(B) = 2\pi A eB \sum_{l=0}^{\infty} \omega_l,$$ \hfill (14)

where $\omega_l^2 = m^2 + (2l + 1)eB$. Obviously, this sum diverges. In order to obtain a finite answer we need to impose a cutoff $L$ in the sum (14). Then the free energy becomes

$$F_0(B, L) = 2\pi A eB \sum_{l=0}^{L} \omega_l.$$ \hfill (15)

The free energy in the absence of the magnetic field is also given by a divergent expression

$$F_0(0) = 2\pi A \int_0^{\infty} dp \sqrt{(p_x^2 + p_y^2) + m^2}$$ \hfill (16)

We regularize this expression by imposing a cutoff $\Lambda$. Thus the free energy (16) becomes

$$F_0(0, \Lambda) = 2\pi A \int_0^{\Lambda} dp \sqrt{(p_x^2 + p_y^2) + m^2}$$ \hfill (17)
In order to compare the free energies in equations (15) and (17), we choose the cutoffs $L$ and $\Lambda$ in such a way that both systems have the same number of modes. Such a procedure can be justified on physical grounds if one imagines turning on the magnetic field adiabatically.

Counting the modes up to the $L$-th Landau level we find

$$2\pi A eB \sum_{l=0}^{L} l = 2\pi A eB(L + 1)$$

(18)

Similarly, for the momentum cutoff up to $\Lambda$ we get the modes without the magnetic field as

$$2\pi A \int_{0}^{\Lambda} p dp = \pi A\Lambda^2$$

(19)

Equating these gives us

$$\Lambda^2 = 2 eB (L + 1)$$

(20)

Now, the free energy in absence of the magnetic field depends on magnetic field through the momentum cutoff and is given by

$$F_0(0, L) = 2\pi A \int_{0}^{\Lambda(B)} p dp \sqrt{p^2 + m^2}$$

(21)

The difference between the two free energies is given by

$$\Delta F(B, L) = F_0(B, L) - F_0(0, L)$$

(22)

We define $G(B) = F_0(B, L)/2\pi A$, $H(B) = F_0(0, L)/2\pi A$ and $K(B) = G(B) - H(B)$.

Numerically evaluating these sums for large but finite $L$ and plotting them in figure 1 and figure 2 shows that $K(B)$ is the difference between two large quantities. As the cutoff $L$ goes to infinity then $K(B)$ becomes the difference between two infinities. In this limit we find that $K(B)$ tends to a finite value. Thus, the susceptibility at zero temperature in the relativistic case is non-zero. This vacuum susceptibility can be interpreted as due to virtual currents.

We now show analytically that $\Delta F(B)$ is positive i.e. the vacuum is diamagnetic. Note that
\[ \Delta F(B) = F(B) - F(0) = \sum_{l=0}^{\infty} a_l(B, m) \] (23)

where \( a_l(B, m) \) is given by

\[ a_l(B, m) = eB \left[ \sqrt{m^2 + (2l+1)eB} - \int_0^1 d\alpha \sqrt{m^2 + 2(l+\alpha)eB} \right] \] (24)

Introducing a dimensionless quantity \( \rho = \frac{eB}{m^2} \) the above equation becomes

\[ a_l(\rho) = \rho \left[ \sqrt{1 + (2l+1)\rho} - \int_0^1 d\alpha \sqrt{1 + 2(l+\alpha)\rho} \right] \] (25)

The positivity of \( a_l(\rho) \) for each \( l \) can be proved geometrically. Defining \( z_l = \frac{1 + 2l\rho}{2\rho} \) and \( f(\alpha) = \sqrt{z_l + \alpha} \), the coefficient \( a_l(\rho) \) can be rewritten in terms of \( c_l(\rho) \) as

\[ c_l(\rho) = \frac{a_l(\rho)}{\sqrt{2} \rho^{3/2}} = f(1/2) - \int_0^1 d\alpha f(\alpha). \] (26)

Since, the function \( f(\alpha) \) is convex, the area under the tangent drawn at \( \alpha = 1/2 \) is greater than the area under the curve (see figure 3). This shows that \( c_l(\rho) \) is positive. To show the convergence of the sum (23) we note that

\[ \int_0^1 d\alpha f(\alpha) \leq \left[ f(1/2) - \frac{f(0) + f(1)}{2} \right] \] (27)

Now, applying mean value theorem twice one can easily show that

\[ \int_0^1 d\alpha f(\alpha) \leq - \frac{1}{16(z_l + \alpha)^{3/2}} \] (28)

Thus the coefficient \( c_l(\rho) \) is positive for each \( l \) and the sum converges, hence the diamagnetic inequality is established.

**Massless Limit:** The magnetisation at zero temperature in the zero mass limit is given by

\[ M(B) \sim - \sqrt{B} \] (29)

So, the susceptibility in this zero mass limit is given by

\[ \chi(B) \sim - \frac{1}{\sqrt{B}} \] (30)
which diverges \[8,9\] as \( B \) goes to zero. This feature of the susceptibility has already been noticed in the magnetised pair Bose gas \[8\]. So, in this case the external field will be totally expelled. This happens because of large number of virtual particle and antiparticle produced in the ground state so that the overall diamagnetism of the system is high enough to totally expel the external field.

Another interesting point is that there exists a critical magnetic field below which magnetic field will be totally expelled. This critical field can be estimated as follows. The effective magnetic field can be defined as \( B_{\text{eff}} = B + 2\pi M \) Here, the magnetisation \( M \) varies as \(- const\sqrt{B}\). Therefore, there exists a critical magnetic field \( B_c \) where the effective field \( B_{\text{eff}} \) vanishes.

Now, for the finite temperature case one can regulate the free energy through the same mode matching regularisation method. Finally, one can write down the free energy difference in dimensionless form as before

\[
\Delta F(B) = F(B) - F(0) = \sum_{l=0}^{\infty} b_l(\rho, \delta, \zeta)
\]

where,

\[
b_l(\rho, \delta, \zeta) = \frac{\rho}{\delta} \left[ g(\rho, l, 1/2) - \int_0^1 d\alpha \ g(\rho, l, \alpha) \right].
\]

The dimensionless variables are defined as \( \delta = \beta m \) and \( \zeta = \beta \mu \).

The coefficient \( g(\rho, l, \alpha) \) is given by

\[
g(\rho, l, \alpha) = \log \left( 1 - \exp(-\delta(\sqrt{1 + 2(l + \alpha)}\rho - \zeta)) \right) + \\
\log \left( 1 - \exp(-\delta(\sqrt{1 + 2(l + \alpha)}\rho + \zeta)) \right).
\]

Now, defining \( z_l = \frac{\delta^2(1+2\rho)}{2\rho} \) we can write the equation \[33\]

\[
g(\rho, l, \alpha) = \log \left( 1 - \exp(-(\sqrt{z_l + \alpha} - \zeta)) \right) + \\
\log \left( 1 - \exp(-(\sqrt{z_l + \alpha + \zeta}) \right).
\]

The function \( g(\rho, l, \alpha) \) is convex and so the zero temperature argument applies unchanged. It follows that the free energy satisfies the following inequality

\[\Box\]
Thus the response of the system to the magnetic field will be diamagnetic.

III. INTERACTING CASE

In this section we want to extend the diamagnetic inequality to the self-interacting field theory case including the dynamical interaction between scalar fields. The partition function of this charged self-interacting field theory in the presence of the magnetic field can be written as

$$Z(B) = \int \int \mathcal{D}[\Phi] \mathcal{D}[\Phi^*] \exp(-S(\Phi, \Phi^*, A)),$$

where the action $S$ is defined as

$$S = \int \int d^2x \, d\tau \left[ (D_\mu \Phi)(D^\mu \Phi)^* + m^2(\Phi^* \Phi) + V(\Phi, \Phi^*) \right].$$

The action is not quadratic and $Z(B)$ cannot be evaluated in closed form. Nevertheless, we show that the response of the system to an external magnetic field is diamagnetic. Since the formal expression for the partition function may not exist (the integrals may not exist) we impose a cut off in momentum space. The functional integral in (36) signifies that one only integrates over those field configurations whose Fourier transforms have support within a sphere of radius $\Lambda$ in momentum space. The partition function then explicitly depends on $\Lambda$. We do not explicitly indicate the $\Lambda$ and $\mu$ dependence of $Z(B, \lambda, \mu)$ below.

We divide the action into two parts $S_0$ and $S_{int}$, where $S_0$ is the action in the absence of the external field.

$$S = S_0 + S_{int},$$

where

$$S_0 = \int \int d^2x d\tau \left[ (\partial_\mu \Phi)(\partial^\mu \Phi)^* + m^2(\Phi^* \Phi) + V(\Phi^*, \Phi) \right].$$
\[ S_{\text{int}} = \int \int d^2xd\tau \left[ -ie(\partial_\mu \Phi)(A^\mu \Phi^*) + ie(A_\mu \Phi)(\partial^\mu \Phi^*) + e^2(\mathbf{A} \cdot \mathbf{A})(\Phi \Phi^*) \right]. \quad (40) \]

Notice that \( \exp(-S_0) \) is a positive measure on the space of field configurations. The ratio \( Z(B)/Z(0) \) can therefore be regarded as the expectation value of \( \exp(-S_{\text{int}}) \). Since \( \exp(-S_{\text{int}}) \) is an oscillatory function whose modulus is less than or equal to 1, we conclude that

\[
\frac{Z(B)}{Z(0)} = \ll \exp-S_{\text{int}} \gg \leq 1 \quad (41)
\]

This implies that

\[
F(B) \geq F(0) \quad (42)
\]

In this derivation, we have not assumed any form for the vector potential. So, the result derived above is true for both homogeneous or inhomogeneous magnetic fields of any strength. Since \( \beta \) is arbitrary, the result holds at all temperatures. The argument presented here works for any arbitrary interaction \( V(\Phi^*\Phi) \). (Generally, it is assumed that \( V(\Phi^*\Phi) \) is a smooth function, for instance, a polynomial \([9]\).

Upto now we have considered the cases of charged scalar fields interacting through a potential. It is also possible to consider interaction mediated by a dynamical electromagnetic field \( A_\mu \). The fields in the system are now \( \Phi \) (charged scalar fields) and \( A_\mu \). If one applies an external magnetic field \( A_{\text{ext}} \) then the full Lagrangian is given by

\[
\mathcal{L} = -\frac{1}{4}F^2 + (D_\mu \Phi)^*(D^\mu \Phi) - m^2(\Phi^*\Phi) - V(\Phi^*, \Phi) \quad (43)
\]

where \( D_\mu = \partial_\mu - i e A_{\text{ext}}^\mu - i e A_\mu \) and \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

The argument given above can be modified as follows. The definition of \( S_0 \) changes slightly while \( S_{\text{int}} \) remains the same.

\[
S_0 = \int \int d^2xd\tau \left[ -\frac{1}{4}F^2 + (\partial_\mu - ieA_\mu \Phi)^*(\partial^\mu + ieA^\mu \Phi) + m^2(\Phi^*\Phi) + V(\Phi^*\Phi) \right], \quad (44)
\]

and
\[ S_{\text{int}} = \int \int d^2x d\tau \left[ -ie(\partial_\mu \Phi)(A^\mu \Phi^*) + ie(A_\mu \Phi)(\partial^\mu \Phi^*) + e^2(A \cdot A)(\Phi \Phi^*) \right]. \] (45)

Again one can repeat the same argument by noting that \( \exp(-S_0) \) is a positive measure and the ratio \( Z(B)/Z(0) \) as an expectation value of \( \exp(-S_{\text{int}}) \) to establish the diamagnetic inequality.

**IV. CONCLUSIONS AND PERSPECTIVES**

The response of a system to an electric field is completely different from its response to a magnetic field. The basic difference between the responses of a system on application of an electric field or a magnetic field lies in the Hamiltonian of the system.

The Lagrangian of a system in the presence of an electric field can be written as

\[ \mathcal{L} = (D_0 \Phi)^*(D_0 \Phi) - (\nabla \Phi)^*(\nabla \Phi) - m^2(\Phi^* \Phi) - V(\Phi^* \Phi) \] (46)

where

\[ D_0 = \partial_0 - ieA_0 \] (47)

For statistical mechanics to make sense, the Hamiltonian \( H \) must be independent of time. Then it follows that

\[ \mathcal{H} = (\Pi^*)(\Pi) + (\nabla \Phi)^*(\nabla \Phi) + m^2(\Phi^* \Phi) + V(\Phi^* \Phi) - ie [(\Pi^*)(A_0 \Phi) - (\Pi)(A_0 \Phi^*)] \] (48)

The electric field appears in the Hamiltonian through the linear vector potential \( A_0 \) term. Now, from finite temperature second order perturbation theory, one can show easily that the free energy of the system always decreases with the electric field. Hence, the dielectric susceptibility is always positive in thermal equilibrium.

But in the case of a magnetic field the Hamiltonian contains both linear and quadratic terms in \( A \). The net effect of an applied magnetic field is not \textit{a priori} clear. However, as our analysis makes clear, for charged scalar field theories the net effect is always diamagnetic.
In case of Spinless Bosons, there is no Zeeman term \cite{12} coupled with a magnetic field and hence the system consisting of Spinless Bosons always has higher energy in a magnetic field than without the magnetic field. It has been already pointed \cite{3, 4, 13, 14} out that there is no corresponding theorem for Fermions.

Let us consider some illustrative examples of Spinless Bose systems. One obvious example in the laboratory is the Cooper pair formation in superconductors which shows perfect diamagnetism (known as the Meissner \cite{15} effect) below the critical temperature. Cooper pairs also exist in Neutron Stars \cite{16} where the magnetic field is very high compared to any laboratory field. Of course, the operators which create and destroy Cooper pairs are not strictly Bose operators, so this is only an analogy. It has been shown explicitly by J. Daicic et. al. \cite{8} that the magnetised pair Bose systems are relativistic superconductors. These systems are not covered by previous analyses \cite{1–4} which apply to non-relativistic quantum mechanical systems. Pions would be a suitable candidate for application of this theory with \( \pi^+ \) and \( \pi^- \) regarded as the particles and antiparticles. The choice is also motivated by the fact that these pions are massive \( (mc^2 = 139.5673 \text{ Mev}) \), obey Bose-Einstein statistics and that they possess no spin. It is also well known that hard core Bosons \cite{13} in any dimension on any lattice show a preference for zero flux.

Throughout this paper we have considered two spatial dimensions both for the free and interacting cases, but the formalism easily generalizes to higher dimensions. In summary, we have shown exactly, that charged scalar fields at all temperatures are diamagnetic.
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[7] This divergence of the susceptibility is reminiscent of the fact that the free energy $F(B) \sim B^{3/2}$. This variation of the free energy in this massless limit can also be understood from dimensional argument. Since we are working in natural units $\hbar = 1$ and $c = 1$, then $[m] \sim [L]^{-1}$. Then the free energy density (i.e.per unit area ) varies as $[L]^{-3}$. However, the dimension of $B$ is $[L]^{-2}$. So, the massless limit restricts the free energy density variation with magnetic field $B$ as $B^{3/2}$ only.

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FIGURES

FIG. 1. The functions $G(B)$ and $H(B)$ have been plotted against $B$ for $L = 3$ and $m = .1$. For simplicity we have drawn the figure for $B$ instead of $eB$.

FIG. 2. The function $K(B) = G(B) - H(B)$ has been plotted against $B$. The values of $L$ and $m$ are the same as figure 1. Note that the scale in this figure in $y$ direction is much expanded compared to figure 1.

FIG. 3. The full line is the curve $f(\alpha) = \sqrt{1 + \alpha}$, the dashed line is the tangent to the above curve at $\alpha = .5$. It is straightforward to see that the area under the curve is less than the area under the tangent. Also the area under the tangent is same as the area of the rectangle defined by the dotted line. The area of this rectangle is given by $f(1/2) \times 1 = f(1/2)$. Hence, the positivity of $a(.1)$ is proved. This can be generalised to any positive value of $z_l$. 

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