Characterizing linear groups in terms of growth properties

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Abstract

Residual finiteness growth invariants indicate how well-approximated a group is by its finite quotients. We demonstrate that some of these invariants characterize linear groups in a class of groups that includes all non-elementary hyperbolic groups.

keywords: VD99 Linear groups, residual finiteness.

1 Introduction

The goal of this paper is to characterize linearity in terms of residual finiteness growth functions. Let Γ be a finitely generated group. One function, $F_Γ(n)$, is defined to be the maximum over all non-trivial words of length at most $n$ of a function, $D_Γ$, where $D_Γ(γ)$ is the order of the minimal finite quotient of $Γ$ where $γ$ has non-trivial image. A basic connection, established in [BM13], was a polynomial upper bound for $F_Γ(n)$ for finitely generated linear groups $Γ$. We refer to the asymptotic growth of the function $F_Γ(n)$ as the F–growth of $Γ$.

We focus on two related complexity functions. Our first function, instead of measuring the complexity of residual finiteness based on the cardinality of the image, measures it by the cardinality of the minimal finite simple group that contains the image. The function, $S_Γ(n)$, takes the maximum of this function over all non-trivial words of length at most $n$. We refer to the asymptotic growth rate of $S_Γ(n)$ as the S–growth of $Γ$. Our second function measures the complexity of residual finiteness based on the cardinality of $GL(n, F_q)$ where we have a homomorphism into $GL(n, F_q)$ where a specified word survives. The function $FL_Γ(n)$ takes the maximum of this function over all non-trivial words of length at most $n$. We refer to the asymptotic growth rate of $FL_Γ(n)$ as the FL–growth of $Γ$. Both functions dominate the residual finiteness function, $F_Γ$. In particular, if the S–growth or FL–growth

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has a polynomial upper bound, then so does the F–growth. The methods of [BM13], show that linear groups have a polynomial upper bound for the S–growth and FL–growth; we provide details in Section 3.

Γ is linear if Γ < GL(n, K) for a field K. We emphasize that we do not insist that K be characteristic zero. Our main result is the following linearity characterization for hyperbolic groups (see our more general result, Theorem 2.2).

**Theorem 1.1.** Let Γ be a (non-elementary) hyperbolic group.

(a) Γ is linear if and only if the S–growth of Γ has a polynomial upper bound.

(b) Γ is linear if and only if the FL–growth of Γ has a polynomial upper bound.

Our proof of Theorem 1.1 relies on upper bounds of orders of elements in GL(n, F_q) and finite simple groups of Lie type. Our proof of (a) relies on the classification of finite simple groups while our proof of (b) does not. One novel feature of our method is that it works regardless of the characteristic of the field. In fact, the characteristic of the field for the representation can be seen in the last step of our construction of the representation when we take an ultraproduct.

The first purely group theoretic characterization of linear groups over zero characteristic was established by Lubotzky. In [Lub88], Lubotzky found a characterization based on the existence of a tower of finite index normal subgroups. The conditions on the tower provide the group with a faithful p–adic representation, hence linearity, in a beautiful synthesis of ideas. Larsen [Lar01] proved a result in the same broad vein. Namely, if there is a rich supply of finite index normal subgroups, he was able to construct a, not necessarily faithful, linear representation. The dimension of the Zariski closure of the image, which by construction is infinite, is directly relatable to the growth rate of the indices of the normal subgroups. His argument sieves through the finite simple groups to find a source algebraic group (group scheme), and consequently, as our construction does, relies on the classification of finite simple groups. It is worth noting that both Larsen and Lubotzky use limiting arguments, as the reader has no doubt surmized we employ as well. The bound on the complexity of SΓ(n) allows us to find a residual family of finite quotients that reside in bounded complexity finite simple groups. Specifically, these finite simple groups are subgroups of general linear groups of a uniformly bounded dimension over finite fields. We then construct linear representations using ultraproducts. We also prove a more general theorem for groups that we call malabelian but postpone stating these results until after we introduce malabelian groups Section 2.

Cocompact lattices in the real rank one, simple, Lie groups Sp(n, 1) are super-rigid by Corlette [Cor92] and Gromov–Schoen [GS92], provided n ≥ 2. Consequently, if one randomly adds a relation to such a cocompact lattice, an idea due to Misha Kapovich, the resulting group will be hyperbolic and non-linear. These groups must have super-polynomial S–growth and FL–growth. It is presently unknown if these groups are residually finite and it seems to be a reasonably pervasive belief that some of them might not be. Indeed, this is why Kapovich first mentioned these groups. At any rate, our work can only guarantee that they have fairly complicated S and FL–growth.
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2 Background

In this preliminary section, we record, for future use, some basic results and concepts.

2.1 Notation

For two functions $f, g : \mathbb{N} \to \mathbb{N}$ we write $f \lesssim g$ (or $g \gtrsim f$) if there exists a natural number $C$ such that $f(n) \leq Cg(Cn)$ holds for all $n$, and we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$. Given a finitely generated group $\Gamma$ with finite generating set $X$, we denote the associated word metric by $||\cdot||$ and metric ball of radius $n$ about the identity by $B_{\Gamma, X}(n)$. For a set $T \subset \Gamma$, we denote by $T^*$, the set $T \setminus \{1\}$. We sometimes abuse notation and write $||\cdot||$ for $||\cdot||_X$ with the understanding that a finite generating set for $\Gamma$ is fixed.

2.2 Divisibility functions

For a finitely generated, residually finite group $\Gamma$ and $\gamma \in \Gamma^*$, we define $Q(\gamma, \Gamma)$ to be the set of finite quotients of $\Gamma$ where the image of $\gamma$ is non-trivial. We say that these quotients detect $\gamma$. Since $\Gamma$ is residually finite, this set is non-empty, and thus the natural number $D_{\Gamma}(\gamma) := \min \{|Q| : Q \in Q(\gamma, \Gamma)\}$ is defined and positive for each $\gamma \in \Gamma^*$. For a fixed finite generating set $X \subset \Gamma$, we define

$$F_{\Gamma,X}(n) := \max \{D_{\Gamma}(\gamma) : \gamma \in \Gamma, ||\gamma||_X \leq n, \gamma \neq 1\}.$$

It was shown in [Bou10] that if $X, Y$ are two finite generating sets for the residually finite group $\Gamma$, then $F_{\Gamma,X} \approx F_{\Gamma,Y}$. Since we will only be interested in asymptotic behavior, we let $F_{\Gamma}$ denote the equivalence class (with respect to $\approx$) of the functions $F_{\Gamma,X}$ for all possible finite generating sets $X$ of $\Gamma$. Sometimes, in an abuse of notation, $F_{\Gamma}$ will stand for some particular representative of this equivalence class, constructed with respect to a fixed, convenient generating set. It is helpful to restrict to finite quotients that come from simple groups in a special way. Set

$$s_{\Gamma}(\gamma) = \min \{|H_0| : Q \leq H_0, H_0 \text{ is simple}, Q \in Q(\gamma, \Gamma)\},$$
and note that \( s_\Gamma(\gamma) \) is defined and positive for each \( \gamma \in \Gamma^* \). For a fixed generating set \( X \subset \Gamma \), we define

\[
S_{\Gamma,X}(n) := \max \{ s_\Gamma(\gamma) : \gamma \in \Gamma, \|\gamma\|_X \leq n, \gamma \neq 1 \}.
\]

Similarly, we define

\[
fl_\Gamma(\gamma) = \min \{ |\text{GL}(n, F_q)| : Q < \text{GL}(n, F_q), Q \in Q(\gamma, \Gamma) \}
\]

and

\[
\text{FL}_{\Gamma,X}(n) = \max \{ fl_\Gamma(\gamma) : \gamma \in \Gamma, \|\gamma\|_X \leq n, \gamma \neq 1 \}.
\]

All the basic properties for \( S_{\Gamma,X} \) are easily shown to be true for \( S_{\Gamma,X}, \text{FL}_{\Gamma,X} \). That is, the growth of \( S_{\Gamma,X}, \text{FL}_{\Gamma,X} \) do not depend on choice of generating set. We, hence, drop \( X \) in our notation when speaking of the growth of \( S_\Gamma \) or \( \text{FL}_\Gamma \).

\( S_\Gamma, \text{FL}_\Gamma \) share many properties with \( F_\Gamma \) and we suspect that \( S_\Gamma, \text{FL}_\Gamma, F_\Gamma \) do not stray far from one another. We outline a simple relationship between \( S_\Gamma \) and \( \text{FL}_\Gamma \). Assume that we have a quotient \( Q \) of \( \Gamma \) for which \( \gamma \) survives. To begin, set \( Q < H \), where \( H \) is a simple group with \( s_\Gamma(\gamma) = |H| \). We first produce an upper bound for \( fl_\Gamma(\gamma) \). As there are only finitely many sporadic groups, we ignore this case (see the proof of Lemma 4.3, below, for more details). If \( H \) is cyclic of prime order \( p \), we have \( H < \text{GL}(2, F_p) \) by taking the subgroup

\[\left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} : \alpha \in F_p \right\}.
\]

In particular, \( fl_\Gamma(\gamma) \leq p^4 \). If \( H \) is of Lie type, then \( |H| \approx q_H^{t_H} \) for a prime power \( q_H \) and a positive integer \( t_H \). For these groups, by definition, we get a faithful representation into \( \text{GL}(t_H, F_{q_H}) \) for some integer \( t \). Hence, \( fl_\Gamma(\gamma) \leq q_H^{t_H} \). Finally, if \( H \) is alternating on a set of cardinality \( n \), we have \( H < \text{GL}(n, F_n) \) for any \( p \) and so \( fl_\Gamma(\gamma) \leq 2p^2 \). Next, we produce the opposite relationship. Here we have \( Q < \text{GL}(r, F_q) \) with \( |\text{GL}(r, F_q)| = fl_\Gamma(\gamma) \). We simply take \( \text{GL}(r, F_q) \to \text{PSL}(r+1, F_q) \). Note here that we cannot always ensure that our element \( \gamma \) will survive since this map is not injective. However, the kernel is the center of \( \text{GL}(r, F_q) \) and with some care (and conditions on \( \Gamma \)), the reader should not be surprised that often we can ensure the safe passage of \( \gamma \). Hence, we get \( s_\Gamma(\gamma) \leq |\text{PSL}(r+1, F_q)| \). What we take away from this brief discussion is that \( S_\Gamma \)–growth and \( \text{FL}_\Gamma \)–growth are not too different.

### 2.3 Least common multiples and malabelian groups

For a finitely generated group \( \Gamma \) and finite subset \( T \subset \Gamma^* \), least common multiples of \( T \) were defined in [BM11 Section 3]. We briefly review the definition and basic constructions here. To begin, let

\[ L_T = \bigcap_{\gamma \in T} \overline{\langle \gamma \rangle}, \]

where \( \overline{\langle \gamma \rangle} \) is the normal closure of the cyclic group \( \langle \gamma \rangle \) generated by \( \gamma \). We define a least common multiple to be any word in \( L_T \) of minimal word length and denote the set of such words by \( \text{LCM}(T) \).
It could be the case that \( L_T \) is the trivial group. Nevertheless, for a wide class of groups, the subgroup \( L_T \) is non-trivial for all finite sets \( T \subset \Gamma^* \). This class contains groups that we call \( K \)-malabelian. A \( K \)-malabelian group is a finitely generated group \( \Gamma \) such that for any pair of non-trivial words \( \gamma, \eta \), there exists a word \( \lambda \in \Gamma \) with \( ||\lambda|| \leq K \) such that \( [\gamma, \lambda^{-1}\eta\lambda] \neq 1 \). Non-elementary hyperbolic groups are 1-malabelian. Note an obstruction to being \( K \)-malabelian is having a non-trivial center. Consequently, solvable groups are never \( K \)-malabelian for any \( K \).

Two key features of a least common multiple in a \( K \)-malabelian group are the following. First, we have a basic upper bound for the word length of a least common multiple of a collection of words \( T \) that depends only on \( K \), \( |T| \), and the maximum word length that occurs in \( T \); this fact follows from a modest generalization of [BM11, Proposition 4.1]. Second, a least common multiple \( \mu_T \) of a finite set \( T \) has the property that if \( \rho : \Gamma \rightarrow H \) is a homomorphism, then if \( \rho(\mu_T) \neq 1 \), then \( \rho(\gamma) \neq 1 \) for each \( \gamma \in T \). One can construct common multiples in a \( K \)-malabelian group by taking nested commutators of the elements in \( T \). The \( K \)-malabelian assumption allows one the freedom to conjugate by a bounded length word so that these commutators are all non-trivial. For the basic example of two free generators \( x, y \in F_2 \), a least common multiple would be \([x, y]\), for instance.

As we make use of the estimate on the length of a least common multiple a few times here, we recall the construction in the setting of \( K \)-malabelian groups in the following lemma.

**Lemma 2.1.** Let \( \Gamma \) be a finitely generated \( K \)-malabelian group, and \( \gamma_1, \ldots, \gamma_n \) be non-trivial elements in \( \Gamma \) with \( ||\gamma_i|| \leq d \). If

\[
\gamma = \text{LCM}(\gamma_1, \ldots, \gamma_n),
\]

then \( ||\gamma|| \leq 20Kn^2d \).

**Proof.** Let \( 2^{k-1} < n \leq 2^k \) and set \( \gamma_j \) to be an element of length one for \( j = 2^k - n, \ldots, 2^k \). For \( \gamma_{2i-1}, \gamma_{2i} \), since \( \Gamma \) is \( K \)-malabelian, we select \( \mu_{i,1} \in \Gamma \) with \( ||\mu_{i,1}|| \leq K \) so that

\[
[\gamma_{2i-1}, \mu_{i,1}^{-1}\gamma_{2i}\mu_{i,1}] = \gamma_i^{(1)} \neq 1.
\]

Note that

\[
||\gamma_i^{(1)}|| \leq 4(d + K).
\]

We repeat this construction with \( \gamma_i^{(2)}, \gamma_i^{(2)} \), obtaining

\[
\gamma_i^{(2)} = [\gamma_{2i-1}, \mu_{i,2}^{-1}\gamma_{2i}\mu_{i,2}] \neq 1.
\]

Again, note that

\[
||\gamma_i^{(2)}|| \leq 4(4(d + K) + K) = 4^2d + 4^2K + 4K.
\]

Inductively, at the \( j \)th stage, we have \( \gamma_i^{(j)}, \gamma_i^{(j)} \) with

\[
||\gamma_i^{(j)}|| \leq 4^jd + K \sum_{i=1}^{j} 4^i,
\]
and we produce
\[ \gamma_i^{(j+1)} = [\gamma_{2i-1}^{(j)}, \mu_{i,j}^{-1} \gamma_{2i}^{(j)} \mu_{i,j}] \neq 1. \]

Again, we have
\[ \left\| \gamma_i^{(j+1)} \right\| \leq 4^{j+1}d + K \sum_{\ell=1}^{j+1} 4^\ell. \]

At the \( k \)th stage, we have one element \( \gamma_1^{(k)} \) and this element is a common multiple of the original set \( \gamma_1, \ldots, \gamma_n \). Moreover,
\[ \left\| \gamma_1^{(k)} \right\| \leq 4^k d + K \sum_{\ell=1}^{k} 4^\ell. \]

Now, by assumption on \( k \), we have \( 4^k \leq 4n^2 \). Thus, we have
\[ \sum_{\ell=1}^{k} K4^\ell = K \left( \frac{4^{k+1} - 1}{3} \right) - K \leq K4^{k+1} \leq 16Kn^2. \]

Combining the above gives
\[ \left\| \gamma_1^{(k)} \right\| \leq 20Kn^2d. \]

Since a least common multiple of the set \( \{ \gamma_1, \ldots, \gamma_n \} \) must be no longer than the common multiple, \( \gamma_1^{(k)} \), the proof is done.

\[ \square \]

### 2.4 The main general results

Having introduced the concept of malabelian groups, we can now state our main results.

**Theorem 2.2.** Let \( \Gamma \) be a finitely generated group and \( \Delta \) malabelian subgroup of \( \Gamma \) with at least one element of infinite order.

(a) If \( \Gamma \) has a polynomial upper bound for the S–growth, then there exists a representation of \( \Gamma \) into \( \text{GL}(N, K) \) that restricts to an injection on \( \Delta \).

(b) If \( \Gamma \) has a polynomial upper bound for the FL–growth, then there exists a representation of \( \Gamma \) into \( \text{GL}(N, K) \) that restricts to an injection on \( \Delta \).

Taking \( \Gamma = \Delta \) in Theorem 2.2 yields half of the following corollary.

**Corollary 2.3.** Let \( \Gamma \) be a finitely generated, malabelian group with at least one element of infinite order.

(a) \( \Gamma \) is linear if and only if the S–growth of \( \Gamma \) has a polynomial upper bound.

(b) \( \Gamma \) is linear if and only if the FL–growth of \( \Gamma \) has a polynomial upper bound.

Given that non-elementary hyperbolic groups are 1–malabelian, we see that Theorem 1.1 follows immediately from Corollary 2.3. Note that any infinite hyperbolic group has an element of infinite order ([BH99, p. 458]) and is finitely presentable ([BH99, p. 470-471]).
3 Proof of Corollary 2.3: First half of the proof

In this section, we prove one half of Corollary 2.3. Namely, we prove that finitely generated linear groups have a polynomial upper bound for $S_{\Gamma}(n)$ and $FL_{\Gamma}(n)$. The proof essentially follows from the polynomial upper bound for $F_{\Gamma}(n)$ provided by [BM13].

**Proof of the direct implication of Theorem 2.3.** We begin with $\Gamma < GL(n,K)$ for an infinite field $K$ and we set $R$ to be the ring generated by the coefficients of the matrix entries of the elements of $\Gamma$. The direct implication in (b) of Corollary 2.3 is immediate from the methods of [BM13]. In fact, the upper bounds we provide in [BM13] are precisely the upper bounds for $fl_{\Gamma}(\gamma)$. In what follows, we prove the direction implication of (b) in Corollary 2.3. Our discussion below makes more precise our loose connection between the $S$ and FL–growth given in Section 2.

We have an initial faithful representation $\rho_0 : \Gamma \to GL(N,K)$. We augment the representation into $SL(N+1,K)$ by $\rho' : \Gamma \to SL(N+1,K)$ defined by

$$\rho'(\gamma) = \begin{pmatrix} \rho_0(\gamma) & 0 \\ \det(\rho(\gamma))^{-1} & 0 \end{pmatrix}.$$ 

Note that $\det(\rho(\gamma))$ is a unit in $R$ and so $\rho'(\gamma)$ is still in $SL(N+1,R)$. For an ideal $p$ of $R$, we obtain the homomorphism

$$r_p : SL(N+1,R) \to SL(N+1,R/p)$$ 

by reducing the coefficients modulo the ideal $p$. We also have

$$\psi_{N+1,p} : SL(N+1,F_p) \to PSL(N+1,F_p).$$

Finally, we set

$$\overline{\rho}_p : SL(N+1,R) \to PSL(N+1,R/p),$$

where $\overline{\rho}_p = \psi_{N+1,p} \circ r_p$. Now, given a non-trivial element $\gamma \in \Gamma$ with word length $||\gamma|| \leq n$, we need a quotient $Q \in Q(\gamma, \Gamma)$ that is contained in a finite simple group $H$ with $|H| \leq ||\gamma||^D$, for a constant $D$ that is independent of $\gamma$. Depending on whether $K$ is characteristic zero or positive characteristic, we can use either Lemma 2.2 or Lemma 2.3 in [BM13] to find a prime ideal $p$ in $R$ such that $r_p(A_\gamma) \neq 0$ with $|R/p| \leq ||\gamma||^d$, where $d$ is independent of the word $\gamma$. Now to ensure that $\overline{\rho}_p(\gamma) \neq 1$, since $\Gamma$ is $K$–malabelian, we set $\gamma_0 = [\gamma, \mu^{-1}\eta\mu]$, where $\eta$ is a generator and $\mu$ is a word of length at most $K$. The word length of $\gamma_0$ has the bound

$$||\gamma_0|| \leq 2n + 4K + 2.$$ 

Then we find a prime $p$ in $R$ such that $r_p(\gamma_0) \neq 1$ using Lemma 2.2 or Lemma 2.3 of [BM13]. By selection of $\gamma_0$ and $p$, we know that $r_p(\gamma)$ is not central since the commutator

$$[r_p(\gamma), (r_p(\mu))^{-1} r_p(\eta) r_p(\mu)]$$

is non-trivial. In particular, $\overline{\rho}_p(\gamma) \neq 1$. By selection of the prime ideal $p$, we have $R/p \cong F_p$ with $p \leq Cn^d$. In tandem with the simplicity of $PSL(N+1,F_p)$, we have

$$sr(\gamma) \leq |PSL(N+1,F_p)| \approx p^{(N+1)^2-1} \leq C' n^{d((N+1)^2-1)}.$$ 

As $\gamma$ was arbitrary, we see that $S_{\Gamma}(n) \leq n^{d((N+1)^2-1)}$, as desired. \qed
4 Theorem 2.3: the other half of the proof

Broadly speaking, our proof succeeds because having many quotients that embed into small finite simple groups restricts the finite simple groups that can appear. Here, we will demonstrate this fact and then use ultraproducts to produce a linear representation.

4.1 Requisite results

For a finite group \( Q \), we define the representation dimension of \( Q \) to be the minimal \( n \) such that \( Q \leq \text{GL}(n, F_{p^k}) \). If we want to take note of the prime \( p^k \), we state that the representation dimension is over \( p^k \). Here, we show that if \( \Gamma \) has a polynomial upper bound on the \( S \)-growth, then we can find many finite quotients that are contained in simple groups of uniformly bounded representation dimension. We require, first, a lemma on the size of cyclic groups in certain finite groups.

**Lemma 4.1.** Let \( H_m \) be a finite simple group of Lie type of dimension \( s_m \) over a finite field \( F_{q^m} \). Then there exists a constant \( C \) such that if \( \gamma \) is an element of \( H_m \) of order \( o_\gamma \), then
\[
o \gamma \leq q_m \sqrt{s_m}.
\]

*Proof.* We may ignore the groups with uniformly bounded representation: \( E_6(q), E_7(q), E_8(q), \ldots \) (see Table 1 for the full list), as the lemma is true immediately for such groups. For the rest of the groups, the lemma follows from the maximal order bounds in Table 1.

Next, we state a similar result for \( \text{GL}(N, F_q) \) that is necessary for our treatment of \( FL \)-growth.

**Lemma 4.2.** If \( \gamma \in \text{GL}(N, F_q) \) with order \( o_\gamma \), then \( o_\gamma \leq q^{2N} \).

*Proof.* Since finite fields are perfect, by the Jordan–Chevalley decomposition theorem (see, for instance, [H75]), we have that \( \gamma = \gamma_s \gamma_u \) where \( \gamma_s \) is semisimple and \( \gamma_u \) is unipotent. Since \( \gamma_u \) is unipotent, it has order at most \( q^N \) (it is conjugate into the group of upper triangular unipotent matrices by Kolchin’s Theorem). Since \( \gamma_s \) is semisimple, it has order at most \( q^N \) by [Vdo99] Lemma 2.3. Since \( \gamma_u, \gamma_s \) commute, it follows that \( \gamma \) has order at most \( q^{2N} \).

The following lemma is the main technical step in proving Theorem 2.2 as it provides us the needed control on the representation dimensions of the simple groups arising in the \( S \)-growth.

**Lemma 4.3.** Let \( \Gamma \) be a finitely generated, malabelian group with \( S_{\Gamma}(n) \leq n^D \) and let \( \gamma \) be a non-trivial element in \( \Gamma \). Then there exists a constant \( N \), depending only on \( D \), and a finite simple group \( H \) (of Lie type) of representation dimension \( N \) such that \( \gamma \) is detected in a subgroup of \( H \).

The plan is to show that sufficiently high powers of \( \gamma \) must be detected in subgroups of finite simple groups of Lie type of uniformly bounded representation dimension. Using the classification of finite
simple groups, we know that a finite simple group is either sporadic, cyclic of prime order, alternating, or of Lie type. It is worth noting, we only require that there be finitely many sporadic groups in the classification. We first prove that we can detect \( r(G) \) to be the representation of \( G \) over \( \mathbb{F}_q \). The uncited bounds are classical (see, for instance, \[Che55\], Car72, p. 64, pp. 225-226), or [Hog82]).

**Proof.** We first assume that \( \gamma \) has infinite order. In this case, we start with the sequence of elements given by taking the least common multiple of the first \( n \) powers of \( \gamma \). Specifically, we let

\[
\gamma_n = \text{LCM}(\gamma, \gamma^2, \ldots, \gamma^n).
\]

According to Lemma 2.1, we have an estimate on the word length of \( \gamma_n \) given by \( ||\gamma_n|| \leq n^3 \). For each \( n \in \mathbb{N} \), set \( Q_n \) be a finite quotient of \( \Gamma \) that realizes \( s_F(\gamma_n) \) and let \( H_n \) be the finite simple group with

| Family | \( q \) | \( m \) | \( q^m \) | \( m \) | \( q^m \) | \( q^m \) |
|--------|--------|--------|--------|--------|--------|--------|
| \( A_m(q) \), \( m \geq 2 \) | \( \frac{1}{(m,q-1)}q^{m^2-1} \) | \( m \leq 9 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) |
| \( B_m(q) \), \( m \geq 2 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) |
| \( C_m(q) \), \( m \geq 3 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) | \( \frac{1}{(m,q-1)}q^{m^2+m} \) | \( m \leq 9 \) |
| \( D_m(q) \), \( m \geq 4 \) | \( \frac{1}{(4m-1)}q^{2m^2-m} \) | \( m \leq 9 \) | \( \frac{1}{(4m-1)}q^{2m^2-m} \) | \( m \leq 9 \) | \( \frac{1}{(4m-1)}q^{2m^2-m} \) | \( m \leq 9 \) |
| \( A_m(q^2) \), \( m \geq 2 \) | \( \frac{1}{(m+1,q-1)}q^{m^2+2m+1} \) | \( m \leq 9 \) | \( \frac{1}{(m+1,q-1)}q^{m^2+2m+1} \) | \( m \leq 9 \) | \( \frac{1}{(m+1,q-1)}q^{m^2+2m+1} \) | \( m \leq 9 \) |
| \( D_m(q^2) \), \( m \geq 4 \) | \( \frac{1}{(4m^2-1)}q^{2m^2-m} \) | \( m \leq 9 \) | \( \frac{1}{(4m^2-1)}q^{2m^2-m} \) | \( m \leq 9 \) | \( \frac{1}{(4m^2-1)}q^{2m^2-m} \) | \( m \leq 9 \) |

Table 1: Infinite families of finite simple groups of Lie types with order approximations. We assume \( m, j \in \mathbb{N} \) and that \( q = p^r \) is a prime power. The upper bounds given are true up to a universal multiplicative error. Note that we use the notation \( (\alpha, \beta) = \text{gcd}(\alpha, \beta) \). Further, for a group \( G \), set \( m_1(G) \) to be the maximal order of any element appearing in \( G \). Set \( r_q(G) \) to be the representation of \( G \) over \( \mathbb{F}_q \). The uncited bounds are classical (see, for instance, \[Che55\], Car72, p. 64, pp. 225-226), or [Hog82]).
$Q_n \leq H_n$ and $|H_n| = s_\Gamma(\gamma_n)$. From our assumption that the S–growth of $\Gamma$ has a polynomial upper bound, we have that

$$|H_n| = s_\Gamma(\gamma_n) \preceq ||\gamma_n||D^D \leq n^{3D}.$$  

Our next step is to gain some control on the simple groups $H_n$. It is here that we will employ the classification of finite simple groups. To that end, by the classification of finite simple groups, $H_n$ is either sporadic, cyclic of prime order, alternating, or of Lie type.

**Claim.** There is a cofinite subsequence $H_m$ of the simple groups $H_n$ such that the groups $H_m$ are of Lie type.

**Proof of Claim.** We argue by ruling out the other three possibilities.

**Case 1. Sporadic groups.**

By definition of least common multiples, we know that the image of $\gamma$ has order $t_n$ in $Q_n$ with $t_n \geq n$ since each element in the set $\{\gamma, \gamma^2, \ldots, \gamma^n\}$ has non-trivial image. In particular, the orders of the groups $Q_n$ are not bounded since we have $\mathbb{Z}/t_n\mathbb{Z} < Q_n$ with $n \leq t_n$. As there are only finitely many sporadic groups, eventually $n$ will be too large for any sporadic group to contain such a subgroup. Consequently, we know that the groups $H_n$ are sporadic only finitely many times.

**Case 2. Cyclic groups of prime order.**

To rule out the groups $H_n$ from being cyclic of prime order, we can simply add to our set $\{\gamma, \gamma^2, \ldots, \gamma^n\}$ a fixed, non-trivial commutator $[\alpha, \beta]$ in $\Gamma$. Since $\Gamma$ is malabelian, we know that there exist many relatively short, non-trivial commutators. The least common multiple of the new set

$$\{\gamma, \gamma^2, \ldots, \gamma^n, [\alpha, \beta]\},$$

for large enough values of $n$ will also have an estimate on the word length of the form $n^d$. By construction, if a least common multiple of this set has non-trivial image in a finite quotient, each of the elements of the set must also have non-trivial image as well. In particular, our commutator $[\alpha, \beta]$ has non-trivial image and hence the image group cannot be abelian. Consequently, we can assume that none of the groups $H_n$ are cyclic of prime order.

**Case 3. Alternating groups.**

Suppose, for the sake of a contradiction, that there are infinitely many $n$ where $H_n = \text{Alt}(k_n)$. Our goal will be to show that the order $|H_n|$ cannot have a polynomial upper bound. In order to derive a contradiction, recall the Landau function $g(n)$ is defined to be the largest order of an element in $\text{Alt}(n)$. Landau [Lan03] proved that

$$e^{n/e} > g(n).$$

By definition of least common multiples, we know that the image of $\gamma$ has order at least $n$ in $\text{Alt}(k_n)$ since each of the elements in the set $\{\gamma, \gamma^2, \ldots, \gamma^n\}$ have non-trivial image. Consequently, we obtain $\mathbb{Z}/t_n\mathbb{Z} < \text{Alt}(k_n)$, where $t_n \geq n$. In particular, we have that $g(k_n)$ must be at least $n$. Using this fact with Landau’s inequality, we have $n \leq e^{k_n/e}$ or, equivalently,

$$e \log(n) \leq k_n.$$  \hfill (1)
As \( H_n = \text{Alt}(k_n) \), we know that the order of \( H_n \) is given by \( (k_n)!/2 \). Combining this equality with (1), we obtain
\[
|H_n| \geq \frac{(\lceil e \log(n) \rceil)!}{2} \geq \frac{(e \log n)^{e \log n} \sqrt{2 \pi e \log n}}{2^{e \log n}} =\frac{1}{2}(\log n)^{e \log n} \sqrt{2 \pi e \log n},
\]
which is super-polynomial in \( n \). This lower bound contradicts that \( |H_n| \) grows polynomially in \( n \). Consequently, alternating groups occur finitely many times.

Since each of the above three classes of finite simple groups occur finitely many times, we conclude that there exists a cofinite subsequence \( m \) of \( n \) where all the groups \( H_m \) are finite of Lie type. This concludes the proof of the claim.

With the claim in hand, we return to the proof of Lemma 4.3. By inspecting Table 1, we see that each of the simple groups of Lie type has a representation in \( \text{GL}(r_m, \mathbb{F}_{q_m}) \), where
\[
|H_m| \approx q_m^{s_m} \quad \text{(†)}
\]
and
\[
r_m \leq \ell s_m \quad \text{(‡)}
\]
for some integer \( \ell \) that is independent of \( m \). By Lemma 4.1 and the fact that \( \mathbb{Z}/t_m\mathbb{Z} < H_m \) with \( m \leq t_m \), we must have
\[
m \leq q_m^{C \sqrt{s_m}}. \quad \text{(2)}
\]
Combining (†) and (2), we have \( m^{1/\ell} \leq q_m^{s_m} \leq m^D \). The last inequality comes from our assumption that the S–growth of \( \Gamma \) has a polynomial upper bound. In total, we see from this string of inequalities that \( s_m \) is uniformly bounded by \( C^2D^2 \) for \( m \) sufficiently large. By ‡, we see that \( r_m \) is also uniformly bounded for \( m \) sufficiently large. Specifically, we have \( r_m \leq \ell C^2D^2 \), which only depends on \( D \) (\( \ell \) and \( C \) are independent of \( \Gamma \)), the degree of the polynomial bound for the S–growth. Infinitely many distinct \( q_m \) must appear as \( \gamma \) has infinite order.

If \( \gamma \in \Gamma \) has finite order, we proceed as follows. We select \( \gamma_0 \) be an element of infinite order and consider, instead,
\[
\gamma_n = \text{LCM}(\gamma, \gamma_0, \gamma_0^2, \ldots, \gamma_0^{n-1}).
\]
As before, we have, from Lemma 2.1 that \( ||\gamma|| \leq n^3 \). We proceed as in the torsion-free case to conclude the proof.

We require a similar result for the function \( \text{FL}_\Gamma(n) \).

**Lemma 4.4.** Let \( \Gamma \) be a finitely generated, malabelian group with \( \text{FL}_\Gamma(n) \leq n^D \) and let \( \gamma \) be a nontrivial element in \( \Gamma \). Then there exists a constant \( N \), depending only on \( D \), such that \( \gamma \) is detected in a subgroup of \( \text{GL}(N, \mathbb{F}_q) \).
Proof. By the last paragraph of the proof of Lemma 4.3 we may assume, without loss of generality, that \( \gamma \) is nontorsion. Let

\[
\gamma_n = \text{LCM}(\gamma, \gamma^2, \ldots, \gamma^n).
\]

For each \( n \in \mathbb{N} \), set \( Q_n \) be a finite quotient of \( \Gamma \) detecting \( \gamma_n \) such that \( Q_n \leq \text{GL}(r_n, F_{q_n}) \) with

\[
|\text{GL}(r_n, F_{q_n})| = f\Gamma(\gamma_n) \leq \|\gamma_n\|^D \leq n^{3D}.
\]

Lemma 4.2 gives \( n \leq q_n^{2n} \) and a dimension counting argument gives \( |\text{GL}(r_n, F_{q_n})| \approx q_n^{r_n} \). Thus,

\[
n^{r_n} \leq q_n^{2n} \leq n^{6D}.
\]

For sufficiently large \( n \), we see that \( r_n \) is bounded above by \( 6D \).

Part (a) of Theorem 2.2 has been reduced to (b) of Theorem 2.2. Indeed, the point of Lemma 4.3 is to ensure that the simple quotients reside in general linear groups of uniformly bounded dimension.

4.2 Proof of Theorem 2.2

Proof of Theorem 2.2. We begin by enumerating nontrivial elements in \( \Delta \) by \( \{\delta_0, \delta_1, \delta_2, \ldots, \delta_j, \ldots\} \) and set \( \gamma_n \in \text{LCM}(\delta_1, \ldots, \delta_n) \). In order to apply Lemma 4.3, note that we only use the malabelian condition in two places of the proof. First, we require that there exist a non-trivial least common multiple and that we have an estimate on the word length. As \( \Delta \) is malabelian, we can form the least common multiple in \( \Delta \) and obtain an estimate on the length of such words. Second, we use malabelian to rule out finite simple groups that are cyclic of prime order. We did this by adding a non-trivial commutator to our finite set, a move that can be done within \( \Delta \). The remainder of the proof utilizes the polynomial upper bound on the S–growth and, as \( \Gamma \) is assumed to have a polynomial upper bound for \( S\Gamma(n) \), we can apply Lemma 4.3 in our present setting. As the conclusion for Lemma 4.3 is identical to the conclusion of Lemma 4.4, we will complete the proof in the second case. For each \( n \in \mathbb{N} \), by Lemma 4.4, there exists a quotient \( Q_n \) of \( \Gamma \) that is a subgroup of \( \text{GL}(r_n, F_{q_n}) \) with \( r_n \leq N \). Now, set \( K_\omega \) to be the ultraproduct of all of finite fields \( F_{q_n} \) with respect to a non-principal ultrafilter \( \omega \) on the natural numbers. If there exists a cofinite set of fields \( F_{q_n} \) that all have characteristic \( p \), for some prime \( p \), then by Łoś’s Theorem (see [BS69, p. 90]), \( K_\omega \) will have characteristic \( p \). Otherwise, by Łoś’s Theorem (see [BS69, p. 90]), \( K_\omega \) has characteristic zero. In particular, the field \( K_\omega \) either embeds in \( F_p(x) \) or \( \mathbb{C} \). Now, as we have homomorphisms to each \( \text{GL}(N, F_{q_n}) \), we obtain a homomorphism

\[
\rho : \Gamma \longrightarrow \text{GL}(N, K_\omega).
\]

By construction, the image of each \( \gamma_n \) is non-trivial and thus \( \rho \) restricts to \( \Delta \) to yield an injective homomorphism, as needed for the verification of the theorem.

As we noted in the introduction, the characteristic of the field, which we cannot control, is decided entirely on whether the characteristics of the finite fields are constant on a cofinite set or not. It is
possible that for a fixed group $\Gamma$, if we implement our selection process multiple times, we could produce representations with characteristic zero for some and positive for others. Further, for each element, $\gamma$, our process does not produce canonical choices for the quotient.

5 Final remarks

Formanek and Procesi in [FP92] proved that $\text{Aut}(F_n)$ is not linear for $n > 2$. In tandem with Theorem 2.3, we obtain the following dichotomy for automorphism groups of free groups.

**Corollary 5.1.** The group $\text{Aut}(F_n)$ has polynomial $S$–growth or $FL$–growth if and only if $n = 2$.

In contrast, we know that braid groups are linear by Bigelow [Big01] and Krammer [Kra02]. As these groups are $K$–malabelian, they must have a polynomial upper bound for their $S$–growth. Theorem 2.3 is a possible tool for addressing the linearity of mapping class groups. One challenge here lies in forming a complete understanding of characteristic, finite index subgroups of free or surface groups.

References

[BS69] J. L. Bell and A. B. Slomson, *Models and Ultraproducts: An Introduction*, Dover Publications, 1969.

[Big01] S. J. Bigelow, *Braid groups are linear*, J. Amer. Math. Soc. 14 (2001), 471–486.

[Bou10] K. Bou-Rabee, *Quantifying residual finiteness*, J. Algebra 323 (2010), 729–737.

[BM11] K. Bou-Rabee and D. B. McReynolds, *Asymptotic growth and least common multiples in groups*, Bull. Lond. Math. Soc. 43 (2011), 1059–1068.

[BM13] K. Bou-Rabee and D. B. McReynolds, *Extremal behavior of divisibility functions*, to appear in Geom. Dedicata.

[BH99] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, 1999.

[Car72] R. W. Carter, *Simple groups of Lie type*, Wiley and Son, 1972.

[Che55] C. Chevalley, *Sur certains groupes simples*, Tôhoku Math. J. (2), 7, 1955.

[Cor92] K. Corlette, *Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. (2) 135 (1992), 165–182.

[FP92] E. Formanek and C. Procesi, *The Automorphism group of a free group is not linear*, J. Algebra 149, 494–499 (1992).
[GS92] M. Gromov and R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Inst. Hautes Études Sci. Publ. Math. (1992), 165–246.

[Hog82] G. M. D. Hogeweij, *Almost-classical Lie algebras. I, II*, Nederl. Akad. Wetensch. Indag. Math., 44, 441–452, 453–460, (1982).

[H75] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975.

[KS09] W. M. Kantor and A. Seress, *Large element orders and the characteristic of Lie-type simple groups*, J. Algebra 322 (2009), 802–832.

[Kra02] D. Krammer, *Braid groups are linear*, Ann. of Math. 155 (2002), 131–156.

[Lan03] E. Landau, *Über die Maximalordnung der Permutationen gegebenen Grades*, Arch. Math. Phys. Ser. 3 5 (1903), 92–103.

[Lar01] M. Larsen, *How often is \(84(g - 1)\) achieved?*, Israel J. of Math., 126 (2001), 1–16.

[Lub88] A. Lubotzky, *A group theoretic characterization of linear groups*, J. Algebra 113 (1988), 207–214.

[Suz60] M. Suzuki, *A new type of simple groups of finite order*. Proc. Nat. Acad. Sci. U.S.A. 46 (1960) 868–870.

[Vdo99] E. P. Vdovin, *Maximal orders of abelian subgroups in finite simple groups*, Algebra and Logic, 38 (1999), 67–83.

[Wil10a] R. A. Wilson, *A simple construction of the Ree groups of type \(2^F_4\)*. J. Algebra 323 (2010), 1468–1481.

[Wil10b] R. A. Wilson, *Another new approach to the small Ree groups*. Arch. Math. (Basel) 94 (2010), 501–510.

[Wil12] R. A. Wilson, *The finite simple groups*, Graduate Texts in Mathematics, 251. Springer-Verlag, 2009.