The dynamics of zeros of the finite-gap solutions to the Schrödinger equation

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Abstract
Following the recent paper by J. F. van Diejen and H. Puschmann we investigate the behavior of zeros of the finite-gap solutions to the Schrödinger equation. As a result, a new system of particles on a punctured Riemann surface is constructed. It is shown to be Hamiltonian and integrable.

1 Introduction
It was noticed in [1] that the dynamics of zeros of n-solitonic solutions to the Schrödinger equation with the reflectionless potential is governed by the rational Ruijsenaars–Schneider system with the harmonic term ([2]). This result appears to be surprising since the aforementioned dynamics was described long ago, though in the different form. In [3] it was shown that the solution to the Schrödinger equation

\[(\partial_x^2 - u(x))\psi(x, E) = E\psi(x, E)\]

with the finite-gap potential \(u(x)\) is a well-defined function on the hyperelliptic curve

\[y^2 = R_g(E) = \prod_{i=1}^{2g+1} (E - E_i).\]

The projections \(\zeta_j\) of zeros of this function onto the \(E\)-plane satisfy the Dubrovin equations ([4]):

\[\frac{\partial \zeta_s}{\partial x} = \frac{2\sqrt{R_g(\zeta_s)}}{\prod_{j \neq s} (\zeta_s - \zeta_j)}.\]
Notice that these equations contain the parameters of the curve. The analog of the Dubrovin equations holds also for degenerate hyperelliptic curves (the latter are described by the same equation where not all $E_i$'s are distinct) and in particular for fully degenerate hyperelliptic curves which can be thought of as a Riemann sphere with $n$ couples of pair-wise identified points. As it was shown in [1] in the latter case the parameters of the curve can be excluded from the system. The modified system then is a system of the second-order differential equations written solely in terms of zeros of the corresponding function. It coincides with the Ruijsenaars–Schneider system and therefore is Hamiltonian, the expressions for the parameters of the curve being the integrals of motion.

In this paper we exploit the algebraic-geometrical approach developed in [5] to apply these ideas to the case of the potentials, coming from the hyperelliptic curves with arbitrary degree of degeneracy. The dynamics of zeros of the corresponding solutions to the Schrödinger equation is described by the system, which is shown to be Hamiltonian and completely integrable, the angle-type variables being the analogs of the components of the Abel map.

In the second section we study the simplest possible case, when a hyperelliptic curve is in fact an elliptic curve with several self-intersections. In the third section we generalize the results obtained to the case of arbitrary hyperelliptic curve.

## 2 Genus one case

We start with some basic facts from the finite-gap theory. Consider an elliptic curve $\Gamma$, given by the equation

$$y^2 = E^3 - g_2E - g_3. \quad (1)$$

It’s compactified at infinity by one point which we denote by $\infty$. The only (up to multiplication by constant) holomorphic differential on $\Gamma$ has the following form: $\omega^h = \frac{dE}{y}$.

It defines the map from $\Gamma$ to the torus $\hat{\Gamma} = \mathbb{C}/\mathbb{Z}[2\omega_1, 2\omega_2]$, where $2\omega_1$ and $2\omega_2$ are $a$- and $b$-periods of $\omega^h$, respectively. This map, given by

$$A: P \mapsto z = \int_\infty^P \omega^h$$

and known as the Abel map, allows us to identify $\Gamma$ and $\hat{\Gamma}$.

Corresponding to the torus $\hat{\Gamma}$ are the standard Weierstrass functions

$$\sigma(z|\omega_1, \omega_2), \quad \zeta(z|\omega_1, \omega_2) = \frac{\sigma'(z|\omega_1, \omega_2)}{\sigma(z|\omega_1, \omega_2)}, \quad \wp(z|\omega_1, \omega_2) = -\zeta'(z|\omega_1, \omega_2)$$

(see [7] for reference). The function $\sigma(z)$ has the following properties:

i) in the neighborhood of zero $\sigma(z) = z + O(z^5)$;

ii) $\sigma(z + 2\omega_j) = e^{2\eta_j(z + \omega_j)}\sigma(z)$, where $\eta_j = \zeta(\omega_j)$.

Notice that $\wp(z)$ is an elliptic function with the only (double) pole at $z = 0$ and $\zeta(z)$ has the simple pole at $z = 0$ and satisfies the following monodromy conditions:

$$\zeta(z + 2\omega_1) = \zeta(z) + \eta_1, \quad \zeta(z + 2\omega_2) = \zeta(z) + \eta_2.$$
The map \( z \mapsto (E = \varphi(z), y = \varphi'(z)) \) is inverse to the Abel map.

Let us fix \( n - 1 \) points \( \kappa_1, \ldots, \kappa_{n-1} \) on \( \hat{\Gamma} \).

**Proposition 1** \[6\] For generic divisor \( D = \gamma_1 + \ldots + \gamma_n \) on the curve \( \hat{\Gamma} \) there exists a unique function \( \psi(x, z|D) \) satisfying the following conditions:

1. It’s meromorphic on the curve \( \hat{\Gamma} \) outside the point \( z = 0 \) and has poles of at most first order at the points \( \gamma_i, i = 1, \ldots, n \).
2. In the neighborhood of \( z = 0 \) it has a form
   \[
   \psi(x, z) = e^{xz} \left( 1 + \sum_{s=1}^{\infty} \xi_s(x) z^s \right).
   \]
3. \( \psi(x, \kappa_i) = \psi(x, -\kappa_i) \).

**Remark.** In general the function \( \psi(x, z|D) \) is defined on the curve \( \Gamma \) itself, but here for the sake of brevity we use the identification between \( \Gamma \) and \( \hat{\Gamma} \).

**Proof.** The uniqueness of such a function follows immediately from the Riemann–Roch Theorem. To show the existence we shall consider the following function

\[
\psi(x, z|D) = e^{z(x)} \frac{\prod_{s=1}^{n} \sigma(z - \gamma_s(x))}{\prod_{s=1}^{n} \sigma(z - \gamma_s) \prod_{s=1}^{n} \sigma(z_i(x))}.
\]

The set of conditions \( \psi(x, \kappa_i) = \psi(x, -\kappa_i) \) and the constraint \( \sum_{i=1}^{n} z_i(x) = x \) (the latter means that \( \psi \) is an elliptic function) form the system of \( n \) equations on the functions \( z_i(x) \).

For generic data this system is non-degenerate. Then it has the only solution (up to the permutations) and therefore defines the function \( \psi(x, z|D) \) uniquely.

**Corollary 1** The above-constructed function \( \psi(x, z|D) \) is a solution to the Schrödinger equation

\[
(\partial_x^2 + u(x))\psi(x, z) = \varphi(z)\psi(x, z),
\]

where

\[
u(x) = -2 \sum_{i=1}^{n} \varphi(z_i(x)) z_i'(x).
\]

**Proof.** Consider a function \( \psi_0(x, z) = (\partial_x^2 + u(x) - \varphi(z))\psi(x, z) \). It’s straightforward to check that the function \( \psi + \psi_0 \) satisfy all defining properties of the function \( \psi \). The uniqueness of \( \psi \) implies that \( \psi_0 = 0 \).

**Theorem 1** The zeros of the function \( \psi(x, z|D) \) satisfy the following dynamics:

\[
\frac{d^2 z_i}{dt^2} = \sum_{k \neq i} z_i z_k \frac{\varphi'(z_i) + \varphi'(z_k)}{\varphi(z_i) - \varphi(z_k)}, \quad i = 1, \ldots, n.
\]

**Proof.** To obtain these equations one has to divide (3) by \( \psi(x, z) \) and compare the residues of the both sides of the obtained equation at the points \( z_i(x) \).

**Remark.** Theorem 1 provides us with a wide class of solutions to system (3) coming from the algebraic-geometrical data. The simple "dimensional" argument shows that in fact these
are all solutions. We could reverse the whole reasoning starting with the solution to (4) and showing that the corresponding elliptic function solves the Shr"odinger equation.

¿From now on in this section we shall study system (4). Let us introduce the variables
\[ z_i = \ln z_i', \quad i = 1, \ldots, n. \]
In the variables \( z_i, \xi_i \) system (4) has the following form:
\[ \begin{align*}
  z_i' &= e^{\xi_i}, \\
  \xi_i' &= \sum_{k \neq i} e^{\xi_k} \frac{\wp'(z_i) + \wp'(z_k)}{\wp(z_i) - \wp(z_k)}, \quad i = 1, \ldots, n.
\end{align*} \tag{5} \]

Proposition 2 System (5) is Hamiltonian with respect to the Hamiltonian
\[ H = \sum_{i=1}^{n} e^{\xi_i} \] and the 2-form
\[ \omega = \sum_{i=1}^{n} dz_i \wedge d\xi_i - \frac{1}{2} \sum_{i \neq j} \frac{\wp'(z_i) + \wp'(z_j)}{\wp(z_i) - \wp(z_j)} dz_i \wedge dz_j. \tag{6} \]

The proof is a straightforward calculation.

Note that
\[ \omega = \sum_{i=1}^{n} dz_i \wedge d\xi_i - \frac{1}{2} \sum_{i \neq j} \frac{\wp'(z_i) + \wp'(z_j)}{\wp(z_i) - \wp(z_j)} dz_i \wedge dz_j = \]
\[ = \sum_{i=1}^{n} dz_i \wedge d\xi_i + \sum_{i \neq j} dz_i \wedge \frac{\wp'(z_j) dz_j - \wp'(z_i) dz_i}{\wp(z_j) - \wp(z_i)} = \]
\[ = \sum_{i=1}^{n} dz_i \wedge d\xi_i + \sum_{i \neq j} dz_i \wedge d \left( \ln(\wp(z_j) - \wp(z_i)) \right) = \sum_{i=1}^{n} dz_i \wedge d\rho_i, \]
where
\[ \rho_i = \xi_i + \sum_{j \neq i} \ln(\wp(z_j) - \wp(z_i)). \]

The algebraic-geometrical construction from the previous section provides us with a hint on how the first integrals of system (4) should look like. The constraints \( \psi(x, \kappa_s) = \psi(x, -\kappa_s) \) imply the equations
\[ \sum_{j=1}^{n} \frac{z_j'}{\wp(\kappa_s) - \wp(z_j)} = 0, \]
which can be rewritten in the following form
\[ \sum_{i=1}^{n} z_i' \prod_{j \neq i} (\wp(z_j) - \wp(\kappa_s)) = 0. \]
These considerations motivate

Theorem 2 The coefficients \( H_k \) of the polynomial
\[ L(\lambda|z, z') = \sum_{k=0}^{n-1} H_k(z, z') \lambda^k = \sum_{i=1}^{n} z_i' \prod_{j \neq i} (\wp(z_j) - \lambda) \tag{7} \]
are the integrals of motion of system (4).
Remark. Note that the leading coefficient $H_{n-1}(z, z')$ of $L$ is equal up to the sign to the Hamiltonian $H(z, z')$ of system (4).

The statement of the theorem is clear since we know that all solutions are algebraic-geometrical. However, we would like to present an independent direct proof. It can be found in the Appendix I.

Let us notice that $L(\wp(z_j)) = e^{\rho_j}$. Using this identity we can rewrite the form $\omega$ in the following way:

$$
\omega = \sum_{i=1}^{n} dz_i \wedge d\rho_i = \sum_{i=1}^{n} dz_i \wedge d\ln L(\wp(z_i)) = \\
= \sum_{i=1}^{n} \frac{1}{L(\wp(z_i))} dz_i \wedge \left(\sum_{s=0}^{n-1} H_s(\wp(z_i))\right) = \sum_{i=1}^{n} \sum_{s=0}^{n-1} \frac{\wp^s(z_i)}{L(\wp(z_i))} dz_i \wedge dH_s = \\
= \sum_{s=0}^{n-1} d\left(\sum_{i=1}^{n} \int \frac{E^s dE}{L(E)\wp(z_i)}\right) \wedge H + \sum_{s,k=0}^{n-1} \left(\sum_{i=1}^{n} \int \frac{E^{s+k} dE}{L(E)^2\wp(z_i)}\right) dH_k \wedge dH_s = \\
= \sum_{s=0}^{n-1} d\left(\sum_{i=1}^{n} \frac{\wp(z_i)}{E^s L(E)\wp(z_i)}\right) \wedge H,$$

where the function $y(E)$ is given by (1).

Thus we have proved the following statement.

**Theorem 3** The variables

$$
\wp_s = \sum_{i=1}^{n} \int \frac{E^s dE}{L(E)\wp(z_i)}, \quad s = 0, \ldots, n - 1
$$

and $H_s$ defined by (7) are the action-angle type variables for system (4).

We would like however to rewrite the form $\omega$ once again in terms of the zeros of the polynomial $L(\lambda z, z')$ which we shall denote by $\bar{\kappa}_j$, $j = 1, \ldots, n - 1$. In order to do this we introduce the new variables

$$
\chi_j = \sum_{i=1}^{n} \int \frac{dE}{(E - \bar{\kappa}_j)\wp(z_i)}, \quad j = 1, \ldots, n - 1.
$$

Let us also introduce the variable

$$
\chi = \sum_{i=1}^{n} \int \frac{dE}{\wp(z_i)}.
$$

**Theorem 4** The above-defined form $\omega$ admits the following representation

$$
\omega = d\chi \wedge d(\ln H) + \sum_{j=1}^{n-1} d\chi_j \wedge d\bar{\kappa}_j.
$$
Remark. We want to emphasize the fact that the variables \( \{ \chi, \chi_j, j = 1, \ldots, n - 1 \} \) are the degenerate curve analogs of the components of the Abel map. So our Hamiltonian structure fits in the general scheme proposed in [8] and developed in [9].

The proof is a straightforward computation (see Appendix II).

3 General case

Consider the hyperelliptic curve \( \Gamma \) of genus \( g, g \geq 1 \), given by the following equation

\[
\Gamma = \{ Q = (y, E) \mid y^2 = R_g(E) = \prod_{i=1}^{2g+1} (E - E_i) \}.
\]

It’s compactified at infinity by one point which we denote by \( \infty \). The curve \( \Gamma \) is a 2-sheeted branched covering over the complex plane of the variable \( E \).

Corresponding to the curve \( \Gamma \) is the following system of differential equations

\[
\zeta''_j = \frac{R'_g(\zeta_j)}{2R_g(\zeta_j)} (\zeta'_j)^2 + \sum_{k \neq j} \frac{\zeta'_j \zeta'_k}{\zeta_j - \zeta_k} \left( 1 + \sqrt{\frac{R_g(\zeta_j)}{R_g(\zeta_k)}} \right), \tag{9}
\]

**Theorem 5** System (9) is Hamiltonian with respect to the 2-form

\[
\omega = \sum_{j=1}^{n} \frac{d\zeta_j}{\sqrt{R_g(\zeta_j)}} \wedge d\rho_j,
\]

where \( \rho_j = \zeta'_j + \sum_{k \neq j} \ln(\zeta_k - \zeta_j) \), and the Hamiltonian \( H = \sum_{i=1}^{n} \frac{\zeta'_i}{2\sqrt{R_g(\zeta_i)}} \). Coefficients \( H_s(\zeta, \zeta') \), \( s = 0, \ldots, n - 1 \) of the polynomial

\[
L(\lambda) = \sum_{k=0}^{n-1} H_s(\zeta, \zeta') \lambda^k = \sum_{j=1}^{n} \frac{\zeta'_j}{\sqrt{R_g(\zeta_j)}} \prod_{k \neq j} (\zeta_k - \lambda)
\]

are the first integrals of system (9). Along with the functions

\[
\varphi_s = \sum_{j=1}^{n} \int \frac{E^s dE}{L(E)g(E)}, \quad s = 0, \ldots, n - 1
\]

they form the set of action-angle type variables for system (9).

Remark. The level sets of the first integrals are not compact. Thus, there is no canonical choice of action-angle type variables for system (9).

Proof. Let a function \( \varphi_g(x) \) be a solution to the differential equation

\[
\frac{d\varphi_g}{dx} = 2\sqrt{R_g(\varphi_g)}.
\]
We define new variables $z_j, \quad j = 1, \ldots, n$ by the conditions $\zeta_j = \wp_g(z_j)$. In terms of these variables system (9) has the following form

$$
z''_i = \sum_{k \neq i} z'_i z'_k \left( \frac{\wp'_g(z_i)}{\wp'_g(z_i) - \wp'_g(z_k)} \right), \quad i = 1, \ldots, n.
$$

The rest of the proof is parallel to the genus one case considered above.

**Remark.** Since $\wp_g(z)$ is a well-defined local parameter on the curve $\Gamma$ outside infinity, it also follows that in fact system (9) describes the motion of $n$ particles on $\Gamma \setminus \infty$.

Our next goal is to show that the dynamics of zeros of solutions to the Schrödinger equation associated with the curve $\Gamma$ is described by the system (9) (the number $n$ of zeros being bigger than the genus $g$ of $\Gamma$).

The way of constructing these solutions is standard in the theory of finite-gap operators.

**Proposition 3** Let’s choose a divisor $R = \kappa_1 + \ldots + \kappa_{n-g}$ on the $E$-plane. For generic divisor $D = \gamma_1 + \ldots + \gamma_n$ on the curve $\Gamma$ there exists a unique function $\psi(x,Q|R,D)$ satisfying the following conditions:

1. It’s meromorphic on the curve $\Gamma$ outside the point $\infty$ and has poles of at most first order at the points $\gamma_i, \quad i = 1, \ldots, n$.
2. In the neighborhood of $\infty$ it has a form

$$
\psi(x,Q) = e^{x E^{1/2}} \left( 1 + \sum_{s=1}^{\infty} \xi_s(x) E^{-s/2} \right).
$$

3. $\psi(x,\kappa^+_i) = \psi(x,\kappa^-_i)$, where $\kappa^+_i$ are preimages of the point $\kappa_i$ under the projection onto the $E$-plane.

**Proof.** The uniqueness of such a function (provided it exists) follows immediately from the Riemann–Roch Theorem. To show the existence we write this function down explicitly in terms of the Riemann $\theta$-function.

The Riemann $\theta$-function, associated with an algebraic curve $\Gamma$ of genus $g$ is an entire function of $g$ complex variables $z = (z_1, \ldots, z_g)$, and is defined by its Fourier expansion

$$
\theta(z_1, \ldots, z_g) = \sum_{m \in \mathbb{Z}^g} e^{2\pi i (m,z) + \pi i (Bm,m)},
$$

where $B = B_{ij}$ is a matrix of $b$-periods, $B_{ij} = \oint_{b_i} \omega_j$, of normalized holomorphic differentials $\omega_j(P)$ on $\Gamma$: $\oint_{a_i} \omega_i = \delta_{ij}$. Here $a_i, b_i$ is a basis of cycles on $\Gamma$ with the canonical matrix of intersections: $a_i \circ a_j = b_i \circ b_j = 0, \quad a_i \circ b_j = \delta_{ij}$.

The $\theta$-function has the following monodromy properties with respect to the lattice $\mathcal{B}$, spanned by the basis vectors $e_i \in \mathbb{C}^g$ and the vectors $B_j \in \mathbb{C}^g$ with coordinates $B_{ij}$:

$$
\theta(z + l) = \theta(z), \quad \theta(z + Bl) = \exp[-i\pi(Bl,l) - 2i\pi(l,z)] \theta(z)
$$

where $l$ is an integer vector, $l \in \mathbb{Z}^g$. The complex torus $J(\Gamma) = \mathbb{C}^g/\mathcal{B}$ is called the Jacobian variety of the algebraic curve $\Gamma$. The vector $A(Q)$ with coordinates

$$
A_k(Q) = \int_{Q_0}^Q \omega_k
$$

is
defines the so-called Abel transform: $\Gamma \mapsto J(\Gamma)$.

According to the Riemann–Roch theorem, if the divisors $D = \gamma_1 + \ldots + \gamma_n$ and $R = \kappa^+_1 + \ldots + \kappa^+_{n-g+1}$, where $\kappa^+_{n-g+1} = \infty$, are in the general position then there exists a unique meromorphic function $r_\alpha(Q)$ such that $D$ is its poles’ divisor and $r_\alpha(\kappa^+_\beta) = \delta_{\alpha\beta}$. It can be written in the form:

$$r_\alpha(Q) = \frac{f_\alpha(Q)}{f_\alpha(\kappa^+_\alpha)}, \quad f_\alpha(Q) = \theta(A(Q) + Z_\alpha) \frac{\prod_{\beta \neq \alpha} \theta(A(Q) + F_\beta)}{\prod_{m=1}^{n-g+1} \theta(A(Q) + S_m)},$$

where

$$F_\beta = -K - A(\kappa^+_\beta) - \sum_{s=1}^{g-1} A(\gamma_s), \quad S_m = -K - A(\gamma_{g-1+m}) - \sum_{s=1}^{g-1} A(\gamma_s),$$

$$Z_\alpha = Z_0 - A(R_\alpha), \quad Z_0 = -K - \sum_{s=1}^{n} A(\gamma_s) + \sum_{\alpha=1}^{n-g+1} A(R_\alpha),$$

where $K$ is the vector of Riemann constants.

Let $d\Omega$ be a unique meromorphic differential on $\Gamma$, which is holomorphic outside $\infty$, where it has double pole, and is normalized by conditions

$$\oint_{a_k} d\Omega = 0.$$

It defines a vector $V$ with the coordinates

$$V_k = \frac{1}{2\pi i} \oint_{b_k} d\Omega.$$

Define functions $\psi_\alpha(x, Q|D, R)$ by

$$\psi_\alpha = r_\alpha(Q) \frac{\theta(A(Q) + xV + Z_\alpha) \theta(Z_0)}{\theta(A(Q) + Z_\alpha) \theta((xV + Z_0) \exp \left( x \int_{R_\alpha} d\Omega \right)$$

Let a vector $c_\alpha, \alpha = 1, \ldots, n-g$ be a solution to the system

$$\sum_{\alpha=1}^{n-g} c_\alpha \psi_\alpha(\kappa^-_\beta) + \psi_{n+g-1}(\kappa^-_\beta) = c_\beta, \quad \beta = 1, \ldots, n-g.$$

The function

$$\psi(x, Q|D, R) = \psi_{n-g+1}(x, Q|D, R) + \sum_{\alpha=1}^{n-g} c_\alpha \psi_\alpha(x, Q|D, R)$$

satisfies conditions 1–3.

**Corollary 2** The above-constructed function $\psi(x, Q|D, R)$ is a solution to the Schrödinger equation

$$\partial_x^2 + u(x)\psi(x, Q) = E\psi(x, Q),$$

where $u(x) = -2\partial_x \xi_1(x)$. 

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The function \( \psi(x, Q) \) has \( n \) zeros on the curve \( \Gamma \), whose projections \( \zeta_1, \ldots, \zeta_n \) satisfy the set of Dubrovin equations which in this case read as

\[
\frac{\partial \zeta_s}{\partial x} = \frac{2 \sqrt{R_g(\zeta_s)} \prod_{\alpha=1}^{n-g} (\zeta_s - \kappa_\alpha)}{\prod_{j \neq s} (\zeta_s - \zeta_j)}.
\]

These equations can be rewritten in the matrix form

\[
Mv = e,
\]

where

\[
v_i = \frac{\zeta_i'}{2 \sqrt{R_g(\zeta_i)}}, \quad e_j = \delta_{jg} \quad \text{and} \quad M_{ji} = \begin{cases} \zeta_i^{j-1}, & j \leq g, \\ \zeta_i - \kappa_{j-g}, & j > g. \end{cases}
\]

**Theorem 6** Projections \( \zeta_j(x), j = 1, \ldots, n \) of zeros of the function \( \psi(x, Q|R, D) \), constructed in the Proposition 3, onto the \( E \)-plane satisfy the restriction of system (9) on the joint level set of its \( g \) first integrals:

\[
H_{n-k} = \delta_{kg}, \quad k = 1, \ldots, g.
\]

**Proof.** Consider a function \( \psi(x, Q|R, D) \). Projections \( \zeta_j \) of its zeros onto \( E \)-plane satisfy matrix equation (10). The first \( g \) equations ensure that restrictions (11) are satisfied. Therefore the polynomial \( L(\lambda|\zeta, \zeta') \) is of degree \( n - g \) and the last \( n - g \) equations in system (10) state that the points \( \kappa_1, \ldots, \kappa_{n-g} \) are its zeros. The coefficients of \( L \) then are time-independent, i.e. \( H_j'(x) = 0, j = 1, \ldots, n \). The last set of equations is equivalent to system (9).

**Remark.** In fact using the construction from Proposition 3 we can obtain almost all solutions to the restricted system. Namely, for general solution \( \zeta_j, j = 1, \ldots, n \) to system (9) satisfying conditions (11) we can define algebraic-geometrical data choosing \( R \) to be the zero-divisor of the polynomial \( L \) and \( D \) to be \( \left( \sqrt{R_g(\zeta_1(0))}, \zeta_1(0) \right) + \ldots + \left( \sqrt{R_g(\zeta_n(0))}, \zeta_n(0) \right) \).

**Appendix I**

This appendix contains the proof of Theorem 3.

Let us consider the polynomial

\[
L(\lambda|z, z') = \sum_{j=1}^{n} z_j' \prod_{i \neq j} (\varphi(z_i) - \lambda) = \sum_{k=0}^{n-1} H_k(z, z') \lambda^k.
\]

The explicit formulae for the coefficients \( H_k \) are

\[
H_k = \sum_{|J|=n-k-1} (-1)^k \prod_{j \in J} \varphi(z_j) \left( \sum_{k \notin J} z_k' \right),
\]
where summation is taken over all subsets \( J \subset \{1, \ldots, n\} \) of cardinality \( n - k - 1 \).

We are going to show that the functions \( H_k \) are time-independent, i.e. \( dH_k/dx = 0 \). Indeed,

\[
\frac{d(1)^k H_k(z, z')}{dx} = \sum_{J} \prod_{j \in J} \phi(z_j) \sum_{k \notin J} z''_k + \sum_{J} \left( \sum_{s \in J} \frac{\phi'(z_s)}{\phi(z_s)} z'_s \right) \prod_{j \in J} \phi(z_j) \sum_{k \notin J} z''_k =
\]

\[
= \sum_{J} \prod_{j \in J} \phi(z_j) \left( \sum_{k \notin J, s \in J} \phi'(z_k) z'_k \right) + \sum_{J} \prod_{j \in J} \phi(z_j) \left( \sum_{k \notin J, s \in J} \phi'(z_s) z'_s \right) =
\]

\[
= \sum_{J} \prod_{j \in J} \phi(z_j) \left( \sum_{k \notin J, s \in J} \phi'(z_k) z'_k \right) = \sum_{J,k \notin J, s \in J} \alpha(J, k, s),
\]

where

\[
\alpha(J, k, s) = \prod_{j \in J} \phi(z_j) \left[ \frac{\phi'(z_s)}{\phi(z_s)} + \frac{\phi'(z_k) + \phi'(z_s)}{\phi(z_k) - \phi(z_s)} \right] z'_k z'_s.
\]

Let us consider the involution on the set of triples \( \{J, k \notin J, s \in J\} \) which maps \( \{J, k, s\} \) into \( \{J', s, k\} \), where \( J' = J \cup \{k\} \setminus \{s\} \).

Now note that

\[
\alpha(J, k, s) + \alpha(J', s, k) = \prod_{j \in J \cup J'} \phi(z_j) z'_k z'_s \left[ \frac{\phi'(z_s)}{\phi(z_s)} + \frac{\phi'(z_k) + \phi'(z_s)}{\phi(z_k) - \phi(z_s)} \right] = 0
\]

and therefore the whole sum \( \sum_{J,k \notin J, s \in J} \alpha(J, k, s) \) vanishes.

**Appendix II**

Here we present the proof of Theorem 4.

Consider the 2-form \( \omega = \sum_{s=0}^{n-1} d\varphi_s \wedge dH_s \). Recall that \( H_s = (-1)^s H_{\sigma_{n-s-1}(\bar{\kappa})} \) for \( s = 0, \ldots, n - 1 \), where \( \sigma_{n-s-1}(\bar{\kappa}) \) denotes the coefficient of \( \lambda^s \) in the polynomial \( \prod_{i=1}^{n-s-1}(\lambda + \bar{\kappa}_i) \). By \( \sigma_{n-s-2}(\bar{\kappa}) \) we denote the coefficient of \( \lambda^s \) in the polynomial \( \prod_{i \neq j}(\lambda + \bar{\kappa}_i) \). Then

\[
\omega = \sum_{s=0}^{n-1} d\varphi_s \wedge dH_s = \sum_{s=0}^{n-2} d\varphi_s \wedge (-1)^s d(H_{\sigma_{n-s-1}(\bar{\kappa})}) + (-1)^{n-1} d\varphi_{n-1} \wedge dH =
\]

\[
= \sum_{s=0}^{n-1} (-1)^s \sigma_{n-s-1}(\bar{\kappa}) d\varphi_s \wedge dH + \sum_{j=1}^{n-2} \sum_{s=0}^{n-1} (-1)^s H \sigma_{n-s-2}(\bar{\kappa}) d\varphi_s \wedge d\bar{\kappa}_j. \tag{12}
\]
Now let us notice that
\[
\sum_{s=0}^{n-2} (-1)^s H^{j}_{n-s-2}(\tilde{\kappa}) \, d\varphi_s = \sum_{s=0}^{n-2} \sum_{l=1}^{n} (-1)^s H^{j}_{n-s-2}(\tilde{\kappa}) \, d \left( \int \frac{E^s \, dE}{L(E)y(E)} \right) =
\]
\[
= \sum_{l=1}^{n} d \left( \sum_{s=0}^{n-2} (-1)^s H^{j}_{n-s-2}(\tilde{\kappa}) \, d\varphi_s \right) = \sum_{l=1}^{n} \int \frac{dE}{L(E)y(E)} \, d\varphi_s.
\]

In the same way one can show that
\[
\sum_{s=0}^{n-1} (-1)^s H^{j}_{n-s-1}(\tilde{\kappa}) \, d\varphi_s = \sum_{l=1}^{n} \int \frac{dE}{H y(E)} \, d\varphi_s + \sum_{l=1}^{n} \int \frac{dE}{H^2 y(E)} \, dH.
\]

Plugging these two formulae into (12) we obtain
\[
d\omega = \sum_{j=1}^{n} d \left( \sum_{l=1}^{n} \int \frac{dE}{(E - \tilde{\kappa}_j)y(E)} \right) \wedge d\tilde{\kappa}_j + d \left( \sum_{l=1}^{n} \int \frac{dE}{H y(E)} \right) \wedge dH =
\]
\[
= \sum_{j=1}^{n} d\chi_j \wedge d\tilde{\kappa}_j + d\chi \wedge d(\ln H).
\]

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