ON THE GENERALIZED SCARF COMPLEX
OF LATTICE IDEALS

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Abstract. Let \( k \) be a field, \( \mathcal{L} \subset \mathbb{Z}^n \) be a lattice such that \( \mathcal{L} \cap \mathbb{N}^n = \{0\} \), and \( I_{\mathcal{L}} \subset k[x_1, \ldots, x_n] \) the corresponding lattice ideal. We present the generalized Scarf complex of \( I_{\mathcal{L}} \) and show that it is indispensable in the sense that it is contained in every minimal free resolution of \( R/I_{\mathcal{L}} \).

1. Introduction

Let \( k \) be a field, let \( \mathcal{L} \subset \mathbb{Z}^n \) be a lattice and \( I_{\mathcal{L}} \) the corresponding lattice ideal in \( k[x_1, \ldots, x_n] \). In a series of recent papers, see [1], [5], [9], [10], [13], [16], [17] and [25], the problem of getting different minimal generating systems for lattice ideals \( I_{\mathcal{L}} \) was investigated. One of the motivations was a related question from Algebraic Statistics that asked when a prime lattice ideal possesses a unique minimal system of binomial generators. As a result the notion of indispensable binomials was introduced by Ohsugi and Hibi [16] to describe the binomials that up to constant multiples are part of all minimal systems of generators of \( I_{\mathcal{L}} \). The corresponding notion for higher syzygies is the motivating question for the present article. Three landmark papers dealing with free resolutions of lattice ideals by Bayer, Peeva and Sturmfels provide crucial leads: see [4], [20], and [21]. The algebraic Scarf complex is defined in [20] and it is shown to be a minimal free resolution when \( I_{\mathcal{L}} \) is generic. The Taylor complex defined in [4] is always a resolution, alas hardly ever a minimal one. The complexity of computing minimal resolutions becomes even more apparent in [21], and the importance of the topological structure of the fibers is strongly hinted. Apart from the above, the problem of computing syzygies of lattice ideals has also been addressed by many recent articles, see for example [6], [8], [18], [19].

The structure of this paper is as follows: in section 2 we introduce the basic notions and terminology. In section 3 we introduce simple minimal resolutions of \( I_{\mathcal{L}} \); these are complexes such that the syzygies determined by \( \mathbb{Z}^n/\mathcal{L} \)-homogeneous bases have minimal support. In section 4 we generalize the Scarf simplicial complex and basic fibers of [20]. In Section 5 we introduce the generalized algebraic Scarf complex and show that it is an indispensable complex. The last section contains examples and open problems.

2. Notation

Let \( \mathcal{L} \subset \mathbb{Z}^n \) be a lattice such that \( \mathcal{L} \cap \mathbb{N}^n = \{0\} \). The polynomial ring \( R = k[x_1, \ldots, x_n] \) is positively multigraded by the group \( \mathbb{Z}^n/\mathcal{L} \), see [15]. Let \( \mathbf{a}_i = e_i + \mathcal{L} \) where \( \{e_i : 1 \leq i \leq n\} \) is the canonical basis of \( \mathbb{Z}^n \). By \( \mathcal{A} \) we denote the

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subsemigroup of the group \( \mathbb{Z}^n / \mathcal{L} \) generated by \( \{ a_i : 1 \leq i \leq n \} \). Since \( \mathcal{L} \cap \mathbb{N}^n = \{ 0 \} \), the semigroup \( \mathcal{A} \) is pointed meaning that \( \{ x : x \in \mathcal{A} \text{ and } -x \in \mathcal{A} \} = \{ 0 \} \). Equivalently if \( \mathcal{A} \) is a pointed semigroup generated by \( a_1, \ldots, a_n \) then \( \mathcal{L} \subset \mathbb{Z}^n \) is the lattice of relations of \( a_1, \ldots, a_n \) and \( \mathbb{L} \cap \mathbb{N}^n = \{ 0 \} \). We set \( \deg_{\mathcal{A}}(x_i) = a_i \). The \( \mathcal{A} \)-degree of the monomial \( x^v = x_1^{v_1} \cdots x_n^{v_n} \) is

\[
\deg_{\mathcal{A}}(x^v) := v_1 a_1 + \cdots + v_n a_n \in \mathcal{A}.
\]

When we want to put the emphasis on \( \mathcal{L} \) we occasionally write \( \deg_{\mathcal{L}}(x^v) := \deg_{\mathcal{A}}(x^v) \). The lattice ideal \( \mathcal{I}_{\mathcal{L}} \) (or \( \mathcal{I}_A \)), associated to \( \mathcal{L} \) is the ideal generated by all the binomials \( x^u_+ - x^u_- \) where \( u_+, u_- \in \mathbb{N}^n \) and \( u = u_+ - u_- \in \mathcal{L} \). Prime lattice ideals are called toric ideals [24] and are the defining ideals of toric varieties. For binomials in \( \mathcal{I}_{\mathcal{L}} \), we define \( \deg_{\mathcal{A}}(x^u_+ - x^u_-) := \deg_{\mathcal{A}} x^u_+ \). Lattice ideals are \( \mathcal{A} \)-homogeneous. For \( b \in \mathcal{A} \) we let \( R[-b] \) be the \( \mathcal{A} \)-graded free \( R \)-module of rank 1 whose generator has \( \mathcal{A} \)-degree \( b \). Let

\[
(F_{\mathcal{L}}, \phi) : 0 \to F_p \overset{\phi_p}{\to} \cdots \overset{\phi_1}{\to} F_0 \overset{\phi_0}{\to} R / \mathcal{I}_{\mathcal{L}} \to 0
\]

be a minimal \( \mathcal{A} \)-graded free resolution of \( R / \mathcal{I}_{\mathcal{L}} \). The \( i \)-Betti number of \( \mathcal{I}_{\mathcal{L}} \) of \( \mathcal{A} \)-degree \( b \) is equal to the rank of the \( R \)-summand of \( F_i \) of \( \mathcal{A} \)-degree \( b \):

\[
\beta_{i,b}(R / \mathcal{I}_{\mathcal{L}}) = \dim_k \text{Tor}_i(R / \mathcal{I}_{\mathcal{L}}, k)_b
\]

and is denoted by \( \beta_{i,b}(R / \mathcal{I}_{\mathcal{L}}) \). This is an invariant of \( \mathcal{I}_{\mathcal{L}} \), see [15]. For \( b \in \mathcal{A} \), we let \( C_b \) equal the fiber

\[
C_b := \text{deg}_{\mathcal{A}}^{-1}(b) = \text{deg}_{\mathcal{L}}^{-1}(b) := \{ x^u : \deg_{\mathcal{A}}(x^u) = b \}
\]

and

\[
\Delta_b := \{ F \subset \{ 1, \ldots, n \} : \exists x^a \in C_b, \ F \subset \text{supp} x^a \}
\]

It is well known that

\[
\beta_{i,b}(R / \mathcal{I}_{\mathcal{L}}) = \dim_k \tilde{H}_i(\Delta_b)
\]

see [2, 6, 7, 23]. The degrees \( b \) for which \( \tilde{H}_i(\Delta_b) \neq 0 \) are called \( i \)-Betti degrees.

The semigroup \( \mathcal{A} \) is pointed, so we can partially order \( \mathcal{A} \) with the relation

\[
c \geq d \iff \text{there is } e \in \mathcal{A} \text{ such that } c = d + e.
\]

The minimal elements of the set \( \{ b : \beta_{i,b}(R / \mathcal{I}_{\mathcal{L}}) \neq 0 \} \) with respect to \( \geq \) are called minimal \( i \)-Betti degrees.

3. Simple syzygies

In this section we present the theory of simple syzygies of \( \mathcal{A} \)-homogeneous ideals for a positive grading \( \mathcal{A} \). We will apply these results for lattice ideals.

Let \( I \) be an \( \mathcal{A} \)-homogeneous ideal and \( (F, \phi) \) be a minimal \( \mathcal{A} \)-graded free resolution of \( R / I \), where

\[
F_i = \bigoplus_{1 \leq i \leq \deg} R \cdot E_{it}.
\]

In particular \( E_{10} \) is the basis element of \( F_0 \cong R \) of \( \mathcal{A} \)-degree 0. The elements of \( \text{Im} \phi_{i+1} = \ker \phi_i \) are called \( i \)-syzygies. In the sequel all syzygies are \( \mathcal{A} \)-homogeneous. We note that the zero syzygies of \( R / I \) are the elements of \( I \). Let \( h \) be an \( \mathcal{A} \)-homogeneous element of \( F_i \). We write \( h \) as a combination of the basis elements \( E_{it} \) with nonzero coefficients:

\[
h = \sum_{1 \leq i \leq \deg} \left( \sum_{c_{ni} \neq 0} c_{ni} x^a \right) E_{it}.
\]
Since \( h \) is \( \mathcal{A} \)-homogeneous, \( \deg_A x^{a_1} + \deg_A (E_{i_1}) = \deg_A (h) \). When \( h \in \ker \phi_i \) we define \( S(h) \), the syzygy support of \( h \) to be the set
\[
S(h) = \{ x^{a_1} E_{i_1} : c_{a_1} \neq 0 \},
\]
and partially order the \( \mathcal{A} \)-homogeneous \( i \)-syzygies by \( h' \leq h \) if and only if \( S(h') \subseteq S(h) \). We note that \( E_{i_1} \) may appear in \( S(h) \) more than once with different monomial coefficients.

**Definition 3.1.** We say that a nonzero \( \mathcal{A} \)-homogeneous \( h \in \ker \phi_i \) is simple if there is no nonzero \( \mathcal{A} \)-homogeneous \( h' \in \ker \phi_i \) such that \( h' < h \).

We note that the syzygy support of \( h \in \ker \phi_i \) and thus also the simplicity of \( h \) depends on the basis \( \{ E_{i_1} \} \) of \( F_i \). We also note that when \( I_L \) is a lattice ideal then \( h \in I_L \) is simple (as a zero syzygy of \( R/I_L \)) if and only if \( h \) is a binomial. Next we define \( m\text{-supp}(h) \), the monomial support of \( h \in F_i \). Let \( C_1, C_2 \) two fibers, \( T_1 \subset C_1, T_2 \subset C_2 \). We let \( T_1 \cdot T_2 = \{ m_1 m_2 : m_i \in T_i \} \). Let \( h \) be a sum as in (1). We recursively define \( m\text{-supp}(h) \) by setting \( m\text{-supp}(E_{i_1}) = \{ 1 \} \) and
\[
m\text{-supp}(h) = \bigcup_{x^{a_1} E_{i_1} \in S(h)} \{ x^{a_1} \} \cdot m\text{-supp}(\phi_i(E_{i_1})).
\]
We note that if the \( \mathcal{A} \)-degree of \( h \) is \( b \) then \( m\text{-supp}(h) \subset C_b \). We also note that \( m\text{-supp}(E_{i_1}) = m\text{-supp}(\phi_i(E_{i_1})) \).

If \( T \) is a subset of monomials then we set
\[
\gcd(T) := \gcd(m : m \in T).
\]

**Definition 3.2.** For a vector \( b \in \mathcal{A} \) we define the gcd-complex \( \Delta_{\gcd}(b) \) to be the simplicial complex with vertices the elements of the fiber \( \deg^{-1}_{\mathcal{L}}(b) \) and faces all subsets \( T \subset \deg^{-1}_{\mathcal{L}}(b) \) such that \( \gcd(T) \neq 1 \).

Let \( b \in \mathcal{A} \). We remark that the gcd complex \( \Delta_{\gcd}(b) \) and the complex \( \Delta b \) have the same homology, see [11].

**Lemma 3.3.** Let \( I \) be an \( \mathcal{A} \)-homogeneous ideal, \( (F, \phi) \) a minimal \( \mathcal{A} \)-graded free resolution of \( R/I \) and \( h \) a simple \( i \)-syzygy of \( \mathcal{A} \)-degree \( b \), \( i \geq 1 \). Then \( m\text{-supp}(h) \) is a connected subset of \( \Delta_{\gcd}(b) \).

**Proof.** Let \( h \) be given as in equation (1). Since \( F_L \) is minimal it follows that \( x^{a_1} \neq 1 \), for all \( x^{a_1} \) such that \( x^{a_1} E_{i_1} \in S(h) \), see [23]. Therefore for any fixed \( x^{a_1} \), the set \( \{ x^{a_1} \} \cdot m\text{-supp}(\phi_i(E_{i_1})) \) is a face of \( \Delta_{\gcd}(b) \). Moreover, since \( i \geq 1 \) and \( F_L \) is minimal, \( 1 \notin m\text{-supp}(\phi_i(E_{i_1})) \). Therefore for each \( E_{i_1} \), the monomials in the sets \( \{ x^{a_1} \} \cdot m\text{-supp}(\phi_i(E_{i_1})) \) where \( x^{a_1} E_{i_1} \in S(h) \) are in the same connected component of \( \Delta_{\gcd}(b) \). Suppose now that \( m\text{-supp}(h) \) is disconnected and has \( l \geq 2 \) components. This means that there are disjoint index sets \( J_r, r = 1, \ldots, l \) such that for each \( r \)
\[
\bigcup_{j \in J_r} \bigcup_{x^{a_1} E_{i_1} \in S(h)} \{ x^{a_1} \} \cdot m\text{-supp}(\phi_i(E_{j_{i_1}}))
\]
is a connected component of \( m\text{-supp}(h) \). Note that the different components of \( m\text{-supp}(h) \) in \( \Delta_{\gcd}(b) \) have no variable in common. It follows that for a fixed \( r \)
\[
h' = \sum_{j \in J_r} (\sum_{c_{a_1} \neq 0} c_{a_1} x^{a_1}) E_{ji}
\]
is a syzygy and \( h' < h \), a contradiction. \( \square \)
We also remark that if \((F, \phi)\) is a minimal \(\mathcal{A}\)-graded free resolution of \(R/I\) and \(\{H_t : t = 1, \ldots, b_i\}\) is an \(\mathcal{A}\)-homogeneous basis for \(F_i\) then by induction it can be shown that gcd(m-supp\((H_t)\)) = 1. Next we will consider minimal free \(\mathcal{A}\)-graded resolutions of \(R/I\) with a special property:

**Theorem 3.4.** There exists a minimal \(\mathcal{A}\)-homogeneous generating set of \(\ker \phi_i\) consisting of simple \(i\)-syzygies.

**Proof.** Since \((F, \phi)\) is a minimal \(\mathcal{A}\)-graded free resolution of \(S/I\) and \(\mathcal{A}\) is pointed \(\ker \phi_i\) can be generated by \(\mathcal{A}\)-homogeneous elements. It is enough to show that any \(\mathcal{A}\)-homogeneous \(i\)-syzygy can be written as a sum of simple \(\mathcal{A}\)-homogeneous \(i\)-syzygies. Let \(h \in \ker \phi_i\) not simple. We will use induction on \(|S(h)|\). By hypothesis there exists a simple \(h'\) such that \(h' < h\). Since \(h' \neq 0\) there is an \(a\) such that

\[
h' = c_a x^a E_a + \sum_{b \neq a} c_b x^b E_b, \quad h = c_a x^a E_a + \sum_{b \neq a} c_b x^b E_b, \quad c_a c_a' \neq 0.
\]

Note that if \(c_b' \neq 0\) then \(c_b \neq 0\). It follows that

\[
h = (h - \frac{c_a}{c_a'} h') + \frac{c_a}{c_a'} h'
\]

while \(|S(h - \frac{c_a}{c_a'} h)| < |S(h)|\). Induction finishes the proof. \(\square\)

**Definition 3.5.** Let \(I\) be an \(\mathcal{A}\)-homogeneous ideal, \((F, \phi)\) be an \(\mathcal{A}\)-graded free resolution of \(R/I\) and for each \(F_i\) we let \(B_i = \{E_i\}\) be an \(\mathcal{A}\)-homogeneous basis for \(F_i\). We say that \(F\) is simple (with respect to the bases \(B_i\)) if \(\phi_i(E_i)\) is simple for each \(i\) and \(t\).

The corollary follows immediately from Theorem 3.4.

**Corollary 3.6.** Let \(I\) be an \(\mathcal{A}\)-homogeneous ideal. There exists a minimal simple \(\mathcal{A}\)-graded free resolution of \(R/I\).

We note that there might be more than one minimal simple free resolution of \(R/I\).

**Example 3.7.** Let \(I = \langle f_1, \ldots, f_s \rangle\) be a complete intersection lattice ideal where \(f_i : i = 1, \ldots, s\) is an \(R\)-sequence of binomials and let \((K_s, \theta)\) be the Koszul complex on the \(f_i\). The \(\mathcal{A}\)-homogeneous standard basis of \(K_s\) consists of elements \(E_J\) where \(J\) ranges over all subsets of \([s]\); if \(J = \{k_1, \ldots, k_t\}\), where \(k_1 < \ldots < k_t\) then

\[
E_J = e_{k_1} \wedge \cdots \wedge e_{k_t}, \quad \text{and} \quad \theta(E_J) = \sum (-1)^{t+1} f_i E_{J \setminus \{k_i\}}.
\]

\((K_s, \theta)\) is a simple resolution of \(R/I\) (with respect to the basis \(\{E_J\}\)). Indeed let \(h \neq 0\) be a syzygy such that \(h < \theta(E_J)\). Clearly this can only happen if \(|J| > 1\). Suppose that \(m_i E_{J \setminus \{k_i\}} \notin S(h)\) while \(m_j E_{J \setminus \{k_j\}} \in S(h)\); where \(f_i = m_i - n_i, f_j = m_j - n_j\). Note that since \(R\) has no zero divisors and the coefficient of \(E_{J \setminus \{k_i, k_j\}}\) in \(\theta(h)\) has to be zero, it follows that \(n_i E_{J \setminus \{k_i\}} \in S(h)\). Therefore the coefficient of \(E_{J \setminus \{k_i, k_j\}}\) in \(\theta(h)\) is \((c_i n_i f_k - (c_j m_j - d_j n_j) f_k)\) for some \(c_i, c_j, d_j \in k\) which cannot be zero since \(f_k, f_k\) is an \(R\)-sequence, a contradiction.

We note that a simple syzygy is determined by its syzygy support:

**Theorem 3.8.** If \(h\) and \(h'\) are two simple \(i\)-syzygies and \(S(h) = S(h')\), then there exists a \(c \neq 0\) in \(k\) such that \(h = ch'\).
Proof. Since $h' \neq 0$ there exists $c_a \neq 0$ such that $h'_t = c_a x^a E_a + \sum_{b \neq a} c_b x^b E_b$. Therefore $c_a \neq 0$ and

$$h'' = h - \frac{c_a}{c_a} h' \in \ker \phi_i,$$

while $S(h'') \subseteq S(h)$. Therefore $h'' = 0$ and $h = c_a/c_a h'$.

Next we give the definition of an indispensable complex. We note the related notion of rigidity, see [14].

**Definition 3.9.** Let $I$ be an $\mathcal{A}$-homogeneous ideal. We say that an $\mathcal{A}$-graded complex $(G, \theta)$ is an indispensable complex for $R/I$ if for any simple minimal $\mathcal{A}$-graded free resolution $(F, \phi)$ of $R/I$ (with respect to bases $B_j$ of $F_j$), there is an inclusion map $i : (G, \theta) \rightarrow (F, \phi)$ so that the image of $G$ is a subcomplex of $F$. In particular for each $j$ there is a subset $B_j'$ of $B_j$ so that the set $i^{-1}(B_j')$ is an $\mathcal{A}$-homogeneous basis of $G_j$. If $H_{ij} \in i^{-1}(B_j')$ we call $\theta_j(H_{ij})$ an indispensable $j-1$ syzygy of $R/I$.

We say that $G$ is a strongly indispensable complex for $R/I$ if the above holds without the requirement for $(F, \phi)$ to be simple.

Let $I_L$ be a toric ideal. A polynomial of $I_L$ is simple, as a zero syzygy, if and only if it is a binomial. In this case the indispensable zero-syzygies of $R/I_L$ are also strongly indispensable. More precisely in [10] it was shown that the binomial $f$ of degree $b$ is contained in every minimal set of binomial generators of $I_L$ if and only if $b$ is a minimal 1-Betti degree and the fiber $(b)$ consists of just two monomials with no common divisor: the difference of these two monomials is $f$ up to a constant multiple. Such a binomial is not only contained in every minimal set of binomial generators of $I_L$; it is necessarily contained in every minimal system of generators of $I_L$ and thus is strongly indispensable, see [20]. In [20], Peeva and Sturmfels introduced the algebraic Scarf complex for $R/I_L$. They showed that this complex is contained in every minimal resolution of $R/I_L$. It follows that the algebraic Scarf complex is a strongly indispensable complex. We will generalize this construction and show that the generalized algebraic Scarf complex is also indispensable.

4. THE GENERALIZED SCARF COMPLEX

Let $\max (\emptyset) = (-\infty, \ldots, -\infty)$. For $J \subset L$, $0 < |J| < \infty$, we let

$$\max (J) = (\max \{a_i : a \in J\}, \ldots, \max \{a_n : a \in J\}) \in \mathbb{Z}^n.$$

We have $\max (J') \leq \max (J)$ if and only if $\max (J')_i \leq \max (J)_i$, while $\max (\emptyset)$ is the smallest element. For $J \subset K \subset L$ we define $v\text{-supp}_K(J)$, the variable support of $J$ in $K$: if $J = \emptyset$ we set $v\text{-supp}_K(\emptyset) = \emptyset$ and for all other $J$ we set

$$v\text{-supp}_K(J) := \{i : \exists a \in J, \text{ such that } \max (K)_i - a_i > 0\}.$$

We note that

$$v\text{-supp}_K(J) = \bigcup_{a \in J} \text{supp}(x^{\max (K)-a}),$$

where $\text{supp}(x^a) = \{i : x_i | x^a\}$.

In [20] the complex $\Delta_L$ was defined to be the collection of all finite subsets $J$ of $L$ with unique $\max (J)$. We extend this complex to $\Delta_L$ as follows:

**Definition 4.1.** Let $\overline{\Delta}_L$ be the collection of all finite subsets $J$ of $L$ that satisfy the following conditions:

1) if $J' \subset J$ then $\max (J') < \max (J)$
(2) if \( a \notin J \) and \( |J| \leq 2 \) then \( a \notin \text{max} (J) \)
(3) if \( a \notin J \), \( |J| > 2 \) and \( a \leq \text{max} (J) \) then

\[
\text{supp}(x^{\text{max} (J)-a}) \cap \text{v-supp}_J(J) = \emptyset .
\]

We note that \( \text{max} (J) \) is determined by at most \( n \) elements of \( \Delta_L \). If \( J \notin \tilde{\Delta}_L \) then the first condition implies that \( |J| \leq n \). On the other hand if \( |J| > 2 \) then the first two conditions imply that \( \text{max} (J) \) is unique. Finally all sets \( J \) with unique \( \text{max} (J) \) are in \( \Delta_L \) and thus \( \Delta_L \subset \tilde{\Delta}_L \).

**Proposition 4.2.** \( \tilde{\Delta}_L \) is a simplicial complex.

**Proof.** We have \( \{a\} \in \tilde{\Delta}_L \) for every \( a \in L \): \( b \leq a \implies a - b \geq 0 \in L \), a contradiction. The case \( |J| = 2 \) is trivial. We now examine the case \( |J| > 2 \), \( J \in \Delta_L \).

Let \( a \in J \); we will show that \( J_1 = J \setminus \{a\} \in \tilde{\Delta}_L \). Let \( J_2 \subseteq J_1 \) such that \( \text{max} (J_2) = \text{max} (J_1) \). It follows that \( \text{max} (J_2 \cup \{a\}) = \text{max} (J) \), a contradiction.

Suppose now that \( c \notin J_1 \) and \( c \leq \text{max} (J_1) \). Therefore \( c \neq a, c < \text{max} (J) \) and \( \text{supp}(x^{\text{max} (J)-c}) \cap \text{v-supp}_J(J) = \emptyset \). On the other hand since \( \text{max} (J_1) < \text{max} (J) \), it follows that for some \( i \), \( \text{max} (J_1)_i < \text{max} (J)_i \). Therefore \( a_i = \text{max} (J)_1 \) while for all \( b \in J_1, b_i < \text{max} (J)_i \). In particular \( i \in \text{v-supp}_J(J) \). This implies that \( i \notin \text{supp}(x^{\text{max} (J)-c}) \), and \( c_i = \text{max} (J)_i \). Thus \( c \notin \text{max} (J)_1 \), a contradiction. It follows that \( \text{max} (J_1) \) is unique and \( J_1 \in \tilde{\Delta}_L \). \( \square \)

There is a natural action of the lattice \( L \) in \( \tilde{\Delta}_L \), since \( J \in \tilde{\Delta}_L \) if and only if \( J + a \in \tilde{\Delta}_L \) for any \( a \in L \). We identify \( \Delta_L \) with its poset of nonempty faces, and we form the quotient poset \( \tilde{\Delta}_L / L \). This poset is called the generalized Scarf complex of \( L \).

**Proposition 4.3.** The generalized Scarf complex \( \tilde{\Delta}_L / L \) is a finite poset.

**Proof.** Let \( \tilde{\Delta}_L^0 \) be the link to zero: \( \tilde{\Delta}_L^0 = \{J \subseteq \Delta_L \setminus \{0\} : J \cup \{0\} \in \Delta_L \} \). As in [20], since \( L \) acts transitively on the vertices of \( \Delta_L \) it is enough to show that \( \tilde{\Delta}_L^0 \) has finitely many vertices. The vertices \( a \) of \( \tilde{\Delta}_L^0 \) are such that \( \{a, 0\} \in \tilde{\Delta}_L \), therefore \( \text{max}((a, 0)) = a^+ \) is unique and we are exactly in the case of [20, Proposition 2.2]: there are finitely many primitive elements of \( L \) and \( a \) is one of them, see also [3]. \( \square \)

For all \( J \subseteq L \) we define

\[
C_J := \{x^{\text{max} (J)-a} : a \in J \} .
\]

We note that \( |C_J| = |J| \). Moreover if \( x^u \in C_J \) then \( \deg_L x^u = \text{max} (J) + L \). It follows that \( C_J \) is a subset of the fiber \( \deg_L^{-1} \text{max} (J) + L \). We also note that \( C_J \) is not necessarily an entire fiber and there are fibers or part of fibers that cannot be expressed in the form \( C_J \). The following Lemma determines exactly the cases when this can happen.

**Lemma 4.4.** Let \( G \) be a subset of a fiber \( C_J \). Then \( G = C_J \) for some \( J \subseteq L \) if and only if \( \gcd(G) = 1 \).

**Proof.** Suppose that \( G = C_J \) for some \( J \subseteq L \). For each \( 1 \leq i \leq n \) there exists \( a_i \in J \) such that \( \text{max} (J)_i = a_i \). It follows that \( i \notin \text{supp}(x^{\text{max} (J)-a}) \). Therefore \( \gcd(G) = 1 \).
Suppose that \( \gcd(G) = 1 \) and \( x^e \in G \). Let \( J = \{ a \in \mathbb{Z}^n : x^{e-a} \in G \} \). Since \( \deg_{C}(x^e) = \deg_{C}(x^{e-a}) = b \) it follows that \( J \) is a subset of \( L \). We claim that \( \max(J) = e \). Indeed, since \( x^{e-a} \in G \) it is clear that \( e \geq a \) and therefore \( \max(J) \geq e \). Moreover \( \gcd(G) = 1 \) implies that for each \( 1 \leq i \leq n \) there exists an \( a \in J \) such that \( a_i = e_i \). Therefore \( \max(J) = e \) and \( G = C_J \).

We isolate a slight variation of a useful remark of [20].

**Lemma 4.5.** If \( b \) is an \( i \)-Betti degree then \( C_b = C_J \) for some subset \( J \) of \( L \).

*Proof.* Since \( \tilde{H}_i(\Delta_b) \neq 0 \) it is enough to show that \( \gcd(C_b) = 1 \), see Lemma 4.4. Indeed if \( \gcd(C_b) \neq 1 \) then \( \Delta_b \) would be a cone with apex any variable in the support of \( \gcd(C_b) \) and \( \Delta_b \) would have no homology, a contradiction.

We will also make use of the following lemma:

**Lemma 4.6.** \( C_J = C_{J'} \) if and only if \( J' = u + J \), for some \( u \in L \).

*Proof.* One direction is direct. For the opposite let \( a \in J \) and assume that \( C_J = C_{J'} \). Then there exists \( a' \in J' \) such that \( x^{\max(J)-a} = x^{\max(J')-a'} \) and thus \( \max(J)-a = \max(J')-a' \). We will show that \( J = J'-a'+a \). Let \( b \in J'-a'+a \). Then \( b + a' - a \in J' \). We note that \( C_{J'-a'+a} = C_{J'} = C_J \). Therefore
\[
\max(J')-b-a'+a = \max(J')-a' + a = \max(J)-b \in C_J
\]
and \( b \in J \).

When a fiber \( C_b \) is not of the form \( C_J \) some of its subsets may be expressed as such.

**Definition 4.7.** Let \( C_b \) be a fiber and \( G \subset C_b \). We say that \( G \) is a basic component of \( C_b \) if the following are satisfied:

- \( G = C_J \) for some \( J \subset L \)
- \( \gcd(G \setminus \{ m \}) \neq 1 \) for all \( m \in G \) and
- \( G \) is a connected component of \( \Delta_{\gcd(b)} \) if \( |C_b| > 2 \).

If \( C_b \) satisfies the above properties then we call \( C_b \) a basic fiber.

We note that not all fibers contain subsets that are basic fiber components. As a matter of fact we will show, see Theorem 4.12, that the set of basic fiber components is finite. The definition of a basic fiber first appeared in [20]. It follows directly from definition 4.7 that if \( C_b \) is a basic fiber and \( |C_b| > 2 \) then \( C_b \) has only one connected component. Moreover if \( |C_b| = 2 \) then \( C_b \) is a basic fiber if and only if \( C_b \) is disconnected. We also note that if \( G \) is a basic component of \( C_b \) and \( |C_b| = 2 \) then \( G = C_b \) is a basic fiber. This is the only way \( C \) can be a basic fiber component when \( |C| = 2 \) as the following Lemma shows:

**Lemma 4.8.** If \( |J| = 2 \) and \( C_J \) is a basic component of \( C_b \) then \( C_J \) is a basic fiber.

*Proof.* We have \( |C_J| = 2 \). By Lemma 4.4 \( C_J \) is disconnected in \( \Delta_{\gcd(b)} \). The third condition of definition 4.7 implies that \( C_J = C_b \).

In [20, Theorem 3.2, Lemma 3.3] the following was shown:

**Lemma 4.9.** If \( C_b \) is a basic fiber with \( i + 1 \) elements then \( \dim \tilde{H}_i(\Delta_b) = 1 \) and \( b \) is a minimal \( i \)-Betti degree.
Let $T$ be a subset of a fiber $C_b$. We denote by $|T|$ the set of monomials in $T$ divided by gcd($T$). Let $I' = I \cup \{m\} \subset T$, $m \notin I$. It is clear that
\[ |T \setminus I'| = \left| \left( |T \setminus I| \setminus \{ \frac{m}{\gcd(T \setminus I)} \} \right) \right|. \]

**Lemma 4.10.** If $C_J$ is a basic component of $C_b$ and $\emptyset \neq I \subset C_J$ then $|C_J \setminus I|$ is a basic fiber.

**Proof.** By the preceding comment it suffices to prove the statement when $|I| = 1$. Let $d = \deg_A(gcd(C_J \setminus I))$ and $b' = b - d$. Since gcd($C_J \setminus I$) $\neq 1$ it follows that $b' \neq b$. We will show that $|C_J \setminus I| = C_{b'}$. Indeed if $m \in C_{b'}$ then $mx^d \in C_b$ and clearly $mx^d$ is in same connected component of $C_b$ as $C_J \setminus I$, thus $mx^d \in C_J$ and $|C_J \setminus I| = C_{b'}$. It is immediate that gcd($C_J \setminus I$) = 1. The remaining condition follows as in [20, Lemma 2.4].

The next lemma helps in computing gcd($C_J \setminus \{m\}$).

**Lemma 4.11.** Let $C_J$ be a subset of $C_b$, $a \in J$ and $m = x_{\max(J) - a}$. Then
\[ \gcd(C_J \setminus \{m\}) = x_{\max(J) - \max(J \setminus \{a\})}. \]

**Proof.** An arbitrary element of $C_J \setminus \{m\}$ is of the form $x_{\max(J) - b}$ where $b \in J \setminus \{a\}$. We note that $b \leq \max(J \setminus \{a\})$. It follows that
\[ x_{\max(J) - b} = x_{\max(J) - \max(J \setminus \{a\})}^\ast x_{\max(J \setminus \{a\}) - b}. \]

For each $1 \leq i \leq n$ there exist a $b \in J \setminus \{a\}$ such that $b_i = (\max(J \setminus \{a\})_i$.
Therefore gcd($C_J \setminus \{m\}$) = $x_{\max(J) - \max(J \setminus \{a\})}$.

The set of basic fiber components forms a poset by setting $C_J' \leq C_J$ if and only if there exists a monomial $x^f$ such that $x^f C_J' \subset C_J$. We state and prove the analogue of [20, Theorem 2.5].

**Theorem 4.12.** The poset of basic fiber components is isomorphic to the generalized Scarf complex $\Delta_{\mathcal{L}}/\mathcal{L}$ and is finite.

**Proof.** Let $F$ be an element in $\Delta_{\mathcal{L}}/\mathcal{L}$. Choose a representative $J$ of $F$. Let $b = \max(J) + \mathcal{L}$. For any other representative $J'$ of $F$, $C_J = C_{J'} \subset C_b$. We will show that $C_J$ is a basic component of $C_b$.

Let $m \in C_J$. We will show that gcd($C_J \setminus \{m\}$) $\neq 1$. Suppose that $m = x_{\max(J) - a}$, for $a \in J$. Since $J \in \Delta_{\mathcal{L}}$, we have that $\max(J \setminus \{a\}) < \max(J)$ and $x_{\max(J) - \max(J \setminus \{a\})} \neq 1$. The desired inequality now follows by Lemma 4.11.

Let $|J| = 2$. Then $J \in \Delta_{\mathcal{L}}$ implies that $\max(J)$ is unique and $C_J = C_b$. Suppose now that $|C_J| = |J| > 2$. Since gcd($C_J \setminus \{m\}$) $\neq 1$, it follows that $\gcd(C_J \setminus \{m\})$ is a face of $\Delta_{\mathcal{L}}(b)$ and thus $C_J$ is connected. To show that $C_J$ is a connected component of $C_b$ we consider an element $x^u \in C_b \setminus C_J$. Then $u + \mathcal{L} = b$ and therefore $u = \max(J) - a$ where $a \in \mathcal{L}$ and $a \notin J$. Since $a \leq \max(J)$ and
\[ \text{supp}(x_{\max(J) - a}) \bigcap v \text{-supp}_J(J) = \emptyset \]
we have that gcd($x^u, m$) = 1, $\forall m \in C_J$. It follows that $C_J$ is a connected component of $\Delta_{\mathcal{L}}(b)$. Set $C_F = C_J$. It is immediate that $\psi : F \mapsto C_F$ is order preserving.

We will show that $\psi$ is bijective. The injectivity follows immediately from Lemma 4.6. To show that $\psi$ is surjective we let $C_J$ be a basic component of $C_b$. We need
to show that \( J \in \Delta \). Let \( J' \subseteq J \). For \( I = \{ x_{\max(J)-a} : a \notin J' \} \) we have that \( \gcd(C_J \setminus I) = \gcd(\{ x_{\max(J)-a} : a \notin J' \}) \neq 1 \). Therefore there is an \( i \) such that \( \max(I) > a \) for all \( a \in J' \). It follows that \( a < \max(J) \), \( \forall a \in J' \) and \( \max(J') < \max(J) \). Suppose now that \( a \in \mathcal{L} \) is such that \( a \notin J \) and \( a \leq \max(J) \). It follows that \( \max(J \cup \{ a \}) = \max(J) \) and \( C_J \subseteq C_{J \cup \{ a \}} \subset C_J \). If \( |J| = 2 \) then Lemma 4.8 gives a contradiction, so in this case \( \max(J) \) is unique and \( J \in \Delta \). Suppose now that \( |J| > 2 \). Since \( m = x_{\max(J)-a} \in C_J \) and \( C_J \) is a connected component of \( \Delta_{\gcd(b)} \), it follows that no variable in the support of \( m \) is in the support of any monomial in \( C_J \) and thus
\[
\text{supp}(x_{\max(J)-a}) \cap \text{v-sup} \subseteq J = \emptyset
\]
as required.

5. The generalized algebraic Scarf complex

We generalize the notion of the algebraic Scarf complex introduced in [20].

**Definition 5.1.** The generalized algebraic Scarf complex is the complex of free \( R \)-modules
\[
(\mathbf{G}_\mathcal{L}, \theta) := \bigoplus_{C \in \Delta_{\mathcal{L}} / \mathcal{L}} R \cdot E_C
\]
where \( E_C \) denotes a basis vector in homological degree \(|C| - 1\) and the sum runs over all basic fiber components \( C \), identified as elements of \( \Delta_{\mathcal{L}} / \mathcal{L} \). We let
\[
\theta(E_C) = \sum_{m \in C} \text{sign}(m, C) \gcd(C \setminus \{ m \}) E_{C \setminus \{ m \}},
\]
where \( \text{sign}(m, C) \) is \((-1)^{l+1}\) if \( m \) is in the \( l \)th position in the lexicographic ordering of \( C \).

Our first remark is that \((\mathbf{G}_\mathcal{L}, \theta)\) is a subcomplex of the Taylor resolution, see [4, Proposition 3.10]. Indeed the canonical basis of the Taylor complex in homological degree \( i \) consists of vectors \( E_C \) where \( C \) is a subset of a fiber such that \( \gcd(C) = 1 \) and \(|C| = i+1 \). The differential of \((\mathbf{G}_\mathcal{L}, \theta)\) is the restriction of the differential of the Taylor complex on the elements of \((\mathbf{G}_\mathcal{L}, \theta)\). Moreover we note that the algebraic Scarf complex of [20] is a subcomplex of \((\mathbf{G}_\mathcal{L}, \theta)\). Indeed the canonical basis of the Scarf complex in homological degree \( i \) consists of vectors \( E_C \) where \( C \) is a basic fiber and the differential coincides for these elements. We also note that for \( i \leq 1 \) the algebraic Scarf complex is identical to the generalized algebraic Scarf complex.

**Theorem 5.2.** The complex \( \mathbf{G}_\mathcal{L} \) is an indispensable complex for \( R/I_\mathcal{L} \).

**Proof.** Let \((\mathbf{F}_\mathcal{L}, \phi)\) be a simple \( \mathcal{A} \)-graded minimal resolution of \( R/I_\mathcal{L} \) with respect to an \( \mathcal{A} \)-homogeneous basis \( \{ H_{ij}, t = 1, \ldots, b_j \} \) of \( F_j \). We will use induction on the homological degree \( i \), the case \( i = 0 \) being trivial. For \( i = 1 \), let \( C = \{ m_1, m_2 \} \) be a basic fiber. Then \( \theta_1(E_C) = m_2 - m_1 \) is an indispensable binomial of \( I_\mathcal{L} \) meaning that up to a constant multiple, \( \theta_1(E_C) \) is part of any minimal system of generators of the \( I_\mathcal{L} \), see [10], [20]. Note that binomials in a lattice ideal are always simple. Thus for \( i \leq 1 \), \( \theta_i(E_C) \) are indispensable.

Suppose now that \( \mathbf{G}_\mathcal{L} \) is indispensable for homological degrees less than \( i \) and thus \( \phi_j |_{G_i} = \theta_i \), \( \forall j < i \). Let \( C \) be a basic component of \( C_h \) with cardinality \( i + 1 \). It is clear that \( \theta_i(E_C) \in \ker \phi_{i-1} \). We will show that \( \theta_i(E_C) \) is a simple
syzygy. The proof is essentially the same as in Example 3.7. Indeed suppose that $0 \neq h \in \ker \phi_{t-1}$ and $h < \theta_i(E_C)$. Since $S(h) \subset S(\theta_i(E_C))$ it follows that $h \in \mathcal{G}_{t-1}$ and $\phi_{t-1}(h) = \theta_{t-1}(h)$. Moreover since $S(h) \subset S(\theta_i(E_C))$ and $S(\theta_i(E_C)) = \{ \gcd(C \setminus \{ m \})_{|E_{C}(\{m\}) : m \in C} \}$, it follows that for some monomial $m_1 \in C$,
\[ \gcd(C \setminus \{ m_1 \})_{|E_{C}(\{m_1\})} \notin S(h). \]

On the other hand since $h \neq 0$ we can find $m_2 \in C$ such that
\[ \gcd(C \setminus \{ m_2 \})_{|E_{C}(\{m_2\})} \in S(h). \]

It follows that $\gcd(C \setminus \{ m_1, m_2 \})_{|E_{C}(\{m_1, m_2\})} \in S(\phi_{t-1}(h))$, a contradiction since $\phi_{t-1}(h) = 0$.

Next suppose that $\theta_i(E_C)$ is an $R$-linear combination of $i$-syzygies of strictly smaller $A$-degree. It follows that $F_C$ in homological degree $i$ has a basis generator $h$ of $A$-degree $b_1$, where $b_1 < b$ and
\[ m \setminus \text{supp}(\theta_i(E_C)) \cap m \setminus \text{supp}(mh) \neq \emptyset, \text{ for some } m \in C_{b-b_1}. \]

By Lemma 4.5 we have that $\gcd(C_{b_1}) = 1$. Therefore $mC_{b_1} \cap C \neq \emptyset$. Since $C$ is a connected component it follows that $mC_{b_1} \subset C$ and therefore $C_{b_1} = |C \setminus I|$ for some $I \subset C$. Lemma 4.10 implies that $C_{b_1}$ is basic while Lemma 4.9 implies that $C_{b_1}$ has at least $i + 1$ elements. Recall that by construction $|C| = i + 1$. Since $\gcd(C) = 1$, $mC_{b_1} \subset C$ and $|mC_{b_1}| = i + 1$ it follows that $|C| > i + 1$, a contradiction.

We still need to show that we can identify $E_C$ with a constant multiple of a basis element $H_{ti}$ for some $t$. Suppose that $\theta_i(E_C) = \sum c_i \phi_i(H_{ti})$ where $\deg_A(H_{ti}) = b$ for at least one $t$. Then for some $t$ with $\deg_A(H_{ti}) = b$, we have that
\[ \gcd(C \setminus \{ m \})_{|E_{C}(\{m\})} \in S(\theta_i(E_C)) \cap S(\phi_i(H_{ti})), \]

for a monomial $m \in C$. Therefore $m \setminus \text{supp}(\theta_i(E_C)) \cap m \setminus \text{supp}(\phi_i(H_{ti})) \neq \emptyset$. Since $m \setminus \text{supp}(\theta_i(E_C)) = C$, $m \setminus \text{supp}(\phi_i(H_{ti})) = m \setminus \text{supp}(\phi_i(H_{ti}))$ and by Lemma 3.3 $m \setminus \text{supp}(\phi_i(H_{ti}))$ is connected, it follows that $m \setminus \text{supp}(H_{ti}) \subset C$. Suppose that
\[ \phi_i(H_{ti}) = \sum s p_s H_{s,i-1} \]

where $H_{s,i-1}$ are basis generators of $F_{i-1}$ of $A$-degree $b_s$ and $p_s \in R$ is $A$-homogeneous of $A$-degree $b - b_s$. It is clear that $b_s < b$ and that $C_{b_s} < C$. Moreover $\bar{H}_{i-1}(d_{b_s}) \neq 0$ and therefore $C_{b_s} = |C \setminus I|$ for some subset $I$ of $C$. We note that the set $I$ need not be unique. By Lemma 4.10 it follows that $C_{b_s}$ is a basic fiber. By induction $E_{C_{b_s}}$ is indispensable and is the unique basis element of $F_{i-1}$ of $A$-degree $b_s$, up to a constant multiple. By Lemma 4.9 $|C_{b_s}| \geq i$. Since $C_{b_s} = |C \setminus I|$, it follows that $|C_{b_s}| \leq i$. Therefore $|C_{b_s}| = i$ and $C_{b_s} = |C \setminus \{ m_i \}|$ for a monomial $m_i$. Let $c x^\gamma$ be a monomial term of $p_s$ where $c \in k - \{ 0 \}$. Since $m \setminus \text{supp}(\phi_i(H_{ti})) \subset C$, $\gcd(C) = 1$ and $\gcd(C_{b_s}) = 1$ it follows that $x^\gamma = \gcd(C \setminus \{ m_i \})$, so actually $p_s = c x^\gamma$ for a $c \neq 0$. Thus $\phi_i(H_{ti}) \leq \theta_i(E_C)$ and since both are simple we get that $S(\phi_i(H_{ti})) = S(\theta_i(E_C))$. We apply Theorem 3.8 to obtain the desired conclusion. \(\square\)

Next we consider a complex that sits between the algebraic Scarf complex and the generalized algebraic Scarf complex.
Definition 5.3. Let $B_i = \{ E_C : C \text{ basic component of } C_b, |C| = i+1, b \text{ minimal i-Betti degree}, \dim_k \tilde{H}_i(\Delta_b) = 1 \}$. The generalized strongly algebraic Scarf complex $S_L$ is the subcomplex of $G_L$ with basis in homological degree $i$ the set $B_i$.

We note that if $C$ is a basic fiber then $C$ satisfies the conditions of Definition 5.3 and thus $S_L$ contains the algebraic Scarf complex of \cite{20}. This containment can be strict as Example 6.5 shows. We point out that if $C$ is a basic component of $C_b$ then for $i > 0$ the reduced $i$-homology group of the corresponding component of $\Delta_b$ has dimension 1.

Theorem 5.4. The complex $S_L$ is a strongly indispensable complex for $R/I_L$.

Proof. Let $C_b$ be a fiber with a component satisfying the conditions of Definition 5.3. These conditions imply the following for any $A$-graded free resolution of $R/I_L$, $(F_L, \phi)$, and $\{ H_{ti} : t = 1, \ldots, b_i \}$ an $A$-homogeneous basis of $F_i$: there is a unique basis element $H_{ti}$ of $A$-degree $b$ and there is no generator $H_{t_0}$ of smaller $A$-degree. Following the proof of Theorem 5.2 this automatically implies that $\theta_i(E_C) = \phi_i(H_{ti})$. 

In \cite{20} it was shown that whenever the ideal $I_L$ is generic, meaning that the support of each minimal binomial generator of $I_L$ is $[n]$, then the algebraic Scarf complex is a minimal resolution of $R/I_L$. The converse is not necessarily true as example 6.1 shows: there are ideals which are not generic but the algebraic Scarf complex is a minimal resolution of $R/I_L$. We note the following:

Theorem 5.5. If $I_L$ is generated by indispensable binomials then the generalized Scarf complex equals the Scarf complex.

Proof. Suppose that $C_b$ is not a basic fiber so that $C \neq C_b$ is a basic fiber component. It is immediate that $C_b$ has more than one connected components. By \cite{10} any binomial which is the difference of two monomials belonging to different connected components of $C_b$ is a minimal binomial generator of $I_L$. Since $|C_b| > 2$ we obtain a contradiction.

The following is now immediate:

Corollary 5.6. If the generalized algebraic Scarf complex $(G_L, \theta)$ is a free resolution of $R/I_L$ then $(G_L, \theta)$ is the algebraic Scarf complex and all $C$ that are basic components of fibers are basic fibers themselves.

6. Examples

In this section we compute the generalized algebraic Scarf complex and compare it with the algebraic Scarf complex and the generalized strongly algebraic Scarf complex for some examples. We start with an example where the three complexes coincide.

Example 6.1. Let $A$ be the semigroup generated by the elements of the set $\{(4,0), (3,1), (1,3), (0,4)\}$, and let $R = \mathbb{k}[a, b, c, d]$. The ideal $I_A$ is minimally generated by $bc - ad, ac^2 - b^2d, b^3 - a^2c, c^3 - bd^2$. The Scarf complex is a minimal resolution of $R/I_A$. We note that $I_A$ is not a generic ideal in any subring of $R$.

In Example 3.7 we showed that when $I = \langle f_1, \ldots, f_s \rangle$ is a complete intersection lattice ideal where $f_i : i = 1, \ldots, s$ is an $R$-sequence of binomials then the Koszul
complex on the \( f_i \), \((K_\bullet, \theta)\), is a simple minimal resolution of \( R/I_L \). In the following example we compare \((K_\bullet, \theta)\) with the Scarf complex. We also discuss the largest index for which there is an indispensable syzygy. First we give the following definition:

**Definition 6.2.** We define \( \text{ideg}(I_L) \), the indispensability degree of \( I_L \), to be the largest \( t \) for which there is an indispensable complex for \( R/I_L \) of length \( t \).

**Example 6.3.** Let the \( f_i \) be as the above preceding remarks. If the \( f_i \) are indispensable generators for \( I = \langle f_1, \ldots, f_s \rangle \) then it is not hard to show that \( K_\bullet \) is a maximum indispensable complex \( R/I_L \). In this case \( \text{ideg}(I_L) \) is equal to the projective dimension of \( R/I_L \). However the generalized algebraic Scarf complex may have length just one, as is the case for the toric ideal \( I = \langle ae - fg, bd - cg \rangle \). If the \( f_i \) are not indispensable then \( \text{ideg}(I_L) \) can be 0, for example in the case of the toric ideal \( \langle x_1 - x_2, x_2 - x_3 \rangle \).

The generalized algebraic Scarf complex is also computed in the following two examples.

**Example 6.4.** Let \( \mathcal{A} \) be the semigroup generated by the elements of the set \{\( (6, 0), (4, 2), (2, 4), (0, 6), (5, 4) \)\}, and let \( R = \mathbb{k}[a, b, c, d, e] \). Then \( I_\mathcal{A} \) is minimally generated by \(-bc + ad, -b^2 + ac, -c^2 + bd, abd - e^2\). The corresponding lattice \( L \) is given by the rows of the following matrix:

\[
\begin{pmatrix}
1 & -2 & 1 & 0 & 0 \\
0 & 1 & -2 & 1 & 0 \\
0 & 2 & 1 & 0 & -2
\end{pmatrix}
\]

A minimal resolution of \( R/I_\mathcal{A} \), by [12], is of the form:

\[
0 \rightarrow R^2 \rightarrow R^5 \rightarrow R^4 \rightarrow R \rightarrow R/I_\mathcal{A} \rightarrow 0
\]

and the \( i \)-Betti degrees are as follows:

- for \( i = 1 \): \( (6, 6), (8, 4), (4, 8), (10, 8) \)
- for \( i = 2 \): \( (8, 10), (10, 8), (14, 16), (16, 14), (18, 12) \)
- for \( i = 3 \): \( (18, 18), (20, 16) \).

For \( b = (6, 6) \), the fiber \( C_b \) contains exactly 2 monomials: \( bc, ad \). Their greatest common divisor is 1 and thus the corresponding binomial is an indispensable generator of \( I_\mathcal{A} \). This is also the case for \( b = (8, 4) \), and for \( b = (4, 8) \). For \( b = (10, 8) \) the fiber \( C_b \) consists of 4 monomials, \( ac^2, abd, b^2c, e^2 \). Thus \( \Delta_{gcd}(b) \) consists of a triangle and a point:

\[
\begin{array}{c}
\text{abd} \\
\text{ac}^2 \\
\text{b}^2c \\
\end{array}
\]

\[
\begin{array}{c}
\text{e}^2 \\
\end{array}
\]

The basic component of \( \Delta_{gcd}(b) \), the triangle, equals \( C_{J_1} \), where \( J_1 = \{(0, 0, 0, 0, 0), (0, 1, -2, 1, 0), (1, -1, -1, 0, 0)\} \). We note that \( \max (J_1) = \max (J) \) where \( J = J_1 \cup \{(1, 1, 0, 1, -2)\} \) and that \( C_b = C_J \). It follows that \( I_\mathcal{A} \) has a generator of
\(A\)-degree \(b\): any binomial formed by taking the difference of \(e^2\) from any element of \(C_{J_1}\) will do. Clearly this generator is not indispensable. It is also immediate that \(C_{J_1}\) is a basic component of \(C_b\). Since \((10,8)\) is a minimal 1-Betti degree it follows that the image of \(E_{C_{J_1}}\) gives a strongly indispensable syzygy.

Finally the fiber for \((8,10)\) is a basic fiber of cardinality 3 and equals \(C_{J_2}\) for \(J_2 = \{(0,0,0,0,0), (1,-1,-1,1,0), (1,-2,1,0,0)\}\). The generalized algebraic Scarf complex equals

\[
0 \to RE_{C_{J_1}} \oplus RE_{C_{J_2}} \to R^3 \to R
\]

and is strongly indispensable. We note that the generalized algebraic Scarf complex equals the generalized strongly algebraic Scarf complex and differs from the algebraic Scarf complex.

In the final example we give an ideal \(I_L\) for which the generalized algebraic Scarf complex is not strongly indispensable. In fact we give a fiber which consists of two basic components.

**Example 6.5.** Let \(A\) be the semigroup of \(Z\) generated by the 6 elements \(3 \cdot 13, 4 \cdot 13, 5 \cdot 13, 3 \cdot 14, 4 \cdot 14, 5 \cdot 14\). In the ring \(R = k[a, \ldots, f]\), the ideal \(I_A\) is generated by the binomials \(-b^2 + ac, e^2 - df, -a^3 + bc, d^3 - ef, -a^2b + c^2, d^2e - f^2,\) and \(bc^2 - f^2d\). The first 6 of these generators are indispensable binomials. The last generator has \(A\)-degree \(b = 13 \cdot 14\). In this case \(\Delta_{gcd}(b)\) has two basic components each of cardinality 3:

Thus any binomial formed by choosing one monomial from each component is part of some minimal binomial generating set of \(I_A\). At the same time there are two indispensable 2-syzygies of \(A\)-degree \(13 \cdot 14\). Therefore these syzygies are not strongly indispensable. Moreover we point out that there are exactly two basic fibers with 3 elements: \(C_{13 \cdot 13}\) and \(C_{14 \cdot 14}\). This means that no fiber has a basic component of cardinality 4, since any such basic component would imply the existence of 4 smaller basic fibers. A minimal resolution of \(R/I_A\), by [12], is of the form:

\[
0 \to R^4 \to R^{16} \to R^{25} \to R^{19} \to R^7 \to R \to R/I_A \to 0 ,
\]

while the generalized algebraic Scarf complex is of the form

\[
0 \to R^4 \to R^6 \to R .
\]

We remark that the generalized strongly algebraic Scarf complex is equal to the algebraic Scarf complex and differs from the generalized algebraic Scarf complex.

As a final note we remark that it is easy to produce examples where the three complexes are different, by combining the two previous examples and using the
technique of gluing semigroups, see [22].

We finish with a list of related open questions.

- Determine all lattice ideals $I_L$ so that the algebraic Scarf complex is a minimal resolution of $R/I_L$.
- Determine the maximum (strongly) indispensable complex for $R/I_L$.
- Determine all lattice ideals $I_L$ so that the maximum (strongly) indispensable complex is a minimal resolution of $R/I_L$.
- Does there exist a $t$ such that if the monomial support of all $i$-syzygies of $R/I_L$ for $i \leq t$ form basic fibers, then the Scarf complex is a free resolution for $I_L$?
- Does there exist a $t$ such that if ideg$(I_L) \geq t$ then ideg$(I_L)$ is equal to the projective dimension of $R/I_L$ and the maximum indispensable complex is a free resolution of $R/I_L$?

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