Simple crystallizations of 4-manifolds

Biplab Basak\textsuperscript{1}, Jonathan Spreer\textsuperscript{2}

\textsuperscript{1}Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India. biplab10@math.iisc.ernet.in.
\textsuperscript{2}School of Mathematics and Physics, The University of Queensland, Brisbane QLD 4072, Australia. j.spreer@uq.edu.au.

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Abstract

Minimal crystallizations of simply connected PL 4-manifolds are very natural objects. Many of their topological features are reflected in their combinatorial structure which, in addition, is preserved under the connected sum operation.

We present a minimal crystallization of the standard PL K3 surface. In combination with known results this yields minimal crystallizations of all simply connected PL 4-manifolds of “standard” type, that is, all connected sums of $\mathbb{C}P^2$, $S^2 \times S^2$, and the K3 surface. In particular, we obtain minimal crystallizations of a pair of homeomorphic but non-PL-homeomorphic 4-manifolds.

In addition, we give an elementary proof that the minimal 8-vertex crystallization of $\mathbb{C}P^2$ is unique and its associated pseudotriangulation is related to the 9-vertex combinatorial triangulation of $\mathbb{C}P^2$ by the minimum of four edge contractions.

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1 Introduction

In this article, we will consider the following straightforward generalization of simplicial complexes: A $d$-dimensional simplicial cell complex, or $\Delta$-complex in the terminology of \cite{K}, will be called $k$-simple, $k \leq d$, if its set of $k$-dimensional faces forms a simplicial complex. In particular, we are interested in 1-simple simplicial cell complexes where the underlying space is a (simply connected) 4-manifold and the 1-skeleton equals the 1-skeleton of a single 4-simplex. Such a simple contracted pseudotriangulation of a 4-manifold can be described in terms of a 5-colored graph, which will be called a simple crystallization of the manifold. Converting between these two representations is straightforward. Thus, statements about simple crystallizations will naturally transform to statements about simple contracted pseudotriangulation and vice versa.

Simple contracted pseudomanifolds (and hence simple crystallizations) have a number of convenient properties, namely:

(i) they are simply connected by construction;

(ii) as a consequence, using Freedman’s classification, the homeomorphism problem for simple contracted pseudomanifolds is decidable and can be determined in a polynomial time procedure;
(iii) for a given 4-manifold \( M \) they have the minimum number of faces amongst all pseudotriangulations of \( M \). In particular, their topology is presented in a very compact form;

(iv) they always have five vertices and ten edges. Hence, their Euler characteristic, homology and the rank for their intersection form are determined by their number of facets. In particular, two simple pseudotriangulations of two 4-manifolds coincide in homology if and only if they have identical \( f \)-vectors;

(v) the connected sum of two simple crystallizations is again a simple crystallization. Thus, modulo the 11/8-conjecture, simple crystallizations of all topological types of simply connected PL 4-manifolds can be constructed from simple crystallizations of \( S^4 \), \( \mathbb{C}P^2 \), \( S^2 \times S^2 \) and the K3 surface.

For these reasons, simple crystallizations, or simple contracted pseudotriangulations, are excellent objects to work with in the class of simply connected PL 4-manifolds. In many ways they correspond to the famous class of 3-neighborly combinatorial 4-manifolds in the world of simplicial complexes \([13, 32, 34]\). For instance, both simple crystallizations and 3-neighborly combinatorial 4-manifolds are simply connected by construction, and for both settings the homology groups of the underlying manifold are determined by their number of pieces.

However, not all simply connected PL 4-manifolds of “standard type”, i.e., connected sums of \( \mathbb{C}P^2 \), \( S^2 \times S^2 \), the K3 surface and their copies with opposite orientation, can be described by 3-neighborly simplicial complexes (for instance, \( S^2 \times S^2 \) does not admit such a triangulation, see \([33]\)). Moreover, the essential 3-neighborliness property is not preserved under taking the connected sum (cf. property (v) in the above list).

Simple crystallizations, and hence simple contracted pseudotriangulations, of simply connected 4-manifolds have been continuously studied over the last decades. The 8-vertex crystallization of \( \mathbb{C}P^2 \) is a well-known object in the literature \([15, 27]\). A simple crystallization of \( S^2 \times S^2 \) is presented in \([15]\) and all simple crystallizations with intersection form of rank up to three have been recently classified due to work by Casali and Gagliardi \([12]\) and Casali \([10, 11]\). Here, we complement these results by giving an elementary proof that there is a unique simple crystallization of the complex projective plane (this fact, of course, can also be followed from the classification).

Furthermore, we present the first simple crystallization of the K3 surface. This completes the list of simple crystallizations of all known simply connected prime PL 4-manifolds. As a consequence, by the connected sum property (v), we thus obtain simple crystallizations of simply connected manifolds of the form \( k\mathbb{C}P^2 \# l\overline{\mathbb{C}P^2} \) and \( mK3 \# rS^2 \times S^2 \), for all \( k, l, m, r \geq 0 \). This is a complete list of all topological types of simply connected 4-manifolds known to admit a PL structure.

Simple contracted pseudotriangulations have yet another important property. They are all isolated global minima in the \textit{Pachner graph}\(^1\) of pseudotriangulations (i.e., they don’t allow \( k \)-moves, \( k > 0 \)). By Pachner’s theorem \([40]\) (see for example \([9]\) for a version for pseudotriangulations), the connected components of the Pachner graph precisely denote PL-homeomorphism classes. That is, two simplicial cell complexes are PL-homeomorphic if

\(^1\) The Pachner graph is the graph whose vertices are complexes and two complexes are connected by an edge if and only if there is a Pachner move turning one complex into the other. The Pachner graph is sometimes also referred to as the \textit{flip graph}.
and only if one can be turned into the other by a sequence of Pachner moves (in other words, if they are connected by a path in the Pachner graph). In practice, this result is a very useful tool to establish PL-equivalence between pairs of simplicial cell complexes (see [2, 18] for more about Pachner moves for simplicial complexes, and [5, 7, 8] for the generalized triangulations setting). Typically, this is done by repeatedly applying Pachner moves to both of the complexes. Each complex generated this way is known to be PL-homeomorphic to the complex it has been constructed from. This way, we build two sets of PL-equivalent complexes, both representing connected subgraphs of the Pachner graph. Now the two complexes are PL-homeomorphic if and only if these two subgraphs eventually overlap in a joint vertex in the Pachner graph (that is, if the two sets of PL-equivalent complexes overlap in a joint complex). However, finding such a joint complex is extremely difficult due to the often very large number of complexes. Thus having well-defined, small regions in the Pachner graph where both subgraphs have a greater chance to meet is essential. Local minima are excellent candidates for such meeting points.

In Section 5, we present a heuristic routine to produce simple crystallizations from pseudotriangulations of 4-manifolds. Because of their special property, we believe that this heuristic will be useful in a number of further applications, such as an ongoing project about PL-homeomorphisms for triangulated 4-manifolds [8, 9].

2 Preliminaries

2.1 Contracted pseudotriangulations

A $d$-dimensional CW-complex $K$ is said to be regular if the attaching maps which define the incidence structure of $K$ are homeomorphisms and the maximum dimension over all cells of $K$ is $d$. Given a regular CW-complex $K$, let $\mathcal{K}$ be the set of all closed cells of $K$ together with the empty set. Then $\mathcal{K}$ is a poset, where the partial ordering is the set inclusion. This poset $\mathcal{K}$ is said to be the face poset of $K$. Clearly, if $K$ and $L$ are two finite regular CW-complexes with isomorphic face posets then $K$ and $L$ are homeomorphic. Now, let $K$ be a regular CW-complex with partial ordering $\leq$. If $\beta \leq \alpha \in K$ then we say $\beta$ is a face of $\alpha$. For $\alpha \in X$, the set $\partial \alpha := \{\gamma \in K : \alpha \neq \gamma \leq \alpha\}$ is a subcomplex of $K$ with induced partial ordering and is called the boundary of $\alpha$. If all the maximal cells of a $d$-dimensional regular CW-complex $K$ are $d$-cells then $K$ is said to be pure. Maximal cells in a pure CW-complex $K$ are called facets, 0-dimensional cells are called vertices, and 1-dimensional cells are called edges of $K$. More generally, the set of $i$-dimensional faces of $K$ will be called the $i$-skeleton of $K$, denoted by $\text{skel}_i(K)$ and the vector $f(K) = (f_0(K), f_1(K), \ldots, f_d(K))$ where $f_i(K) = |\text{skel}_i(K)|$ will be called the $f$-vector of $K$. The underlying topological space of $K$ is referred to as the geometric carrier of $K$ which will be denoted by $|K|$.

If all faces of a regular CW-complex $K$ are simplices then $K$ is often called a generalized triangulation (see for example [30], where they are referred to as triangulations). Generalized triangulations are predominantly used in 3-manifold topology and hyperbolic geometry where they are usually introduced as a set of tetrahedra together with face-pairings along their triangular faces. In particular, generalized triangulations allow self-identifications of cells and often contain no more than a single 0-dimensional cell (hence, they are sometimes also called 1-vertex triangulations.

Here, we want to focus on a slightly less general type of CW-complex. A simplicial cell complex $K$ of dimension $d$ is a regular CW-complex such that the boundary of each face
in \( K \) is isomorphic (as a poset) to the boundary of a simplex of same dimension. Note that every simplicial cell complex is a generalized triangulation but the converse is not true. More precisely, in a simplicial cell complex no self-identifications of faces are allowed. As a consequence, each simplicial cell complex of dimension \( d \) must have at least \( d + 1 \) vertices.

If a \( d \)-dimensional simplicial cell complex \( K \) has exactly \( d + 1 \) vertices then \( K \) is called contracted.

If for a \( d \)-dimensional simplicial cell complex \( K \) each \((d - 1)\)-face is contained in exactly two facets of \( K \), we say that \( K \) is a weak pseudomanifold. For \( \alpha \in K \), the set \( \{ \sigma \setminus \alpha \in K : \alpha \leq \sigma \} \) is also a simplicial cell complex and is said to be the link of \( \alpha \) in \( K \), denoted by \( \text{lk}_K(\alpha) \). Here \( \sigma \setminus \alpha \) denotes the set of all faces in \( \sigma \) which are disjoint of \( \alpha \). Furthermore, for any vertex \( v \) of a \( d \)-dimensional simplicial cell complex \( K \) the \((d - 1)\)-dimensional simplicial cell complex given by the boundary of a small neighborhood of \( v \) in \( K \) (inside the interior of the subcomplex of all faces of \( K \) containing \( v \)) is called the vertex figure of \( v \) in \( K \) (note that in a simplicial complex \( K \) a vertex figure of \( v \) in \( K \) is isomorphic to the link of \( v \) in \( K \)). If all vertex figures of \( K \) are simplicial cell decompositions with their geometric carrier being PL-homeomorphic to the standard PL \((d - 1)\)-sphere then \( K \) is said to be a pseudotriangulation of a \( d \)-manifold. By construction, all pseudotriangulations of manifolds are weak pseudomanifolds but the converse is not true. Given a PL-manifold \( M \) we say that a pseudotriangulation \( M \) is (PL-)homeomorphic to \( M \) when \( \vert M \vert \cong_{(PL)} M \).

Now let \( M \) be a \( d \)-dimensional weak pseudomanifold. Consider the graph \( \Lambda(M) \) whose vertices are the facets of \( M \) and the edges are pairs \( \{ \sigma_1, \sigma_2, \gamma \} \), where \( \sigma_1 \) and \( \sigma_2 \) are facets, and \( \gamma \) is a common \((d - 1)\)-cell (i.e., \( \gamma \) is a face of both \( \sigma_1 \) and \( \sigma_2 \)). The graph \( \Lambda(M) \) is said to be the dual graph or sometimes also the face pairing graph of \( M \). Observe that \( \Lambda(M) \) of a weak pseudomanifold \( M \) is a multi graph without loops.

Pseudotriangulations of PL-manifolds are a straightforward generalization of combinatorial manifolds where the underlying CW-complex must be a simplicial complex (see [35] [44] for an introduction into combinatorial manifolds). All together, we have three types of cell-decompositions, generalized triangulations, simplicial cell complexes, and simplicial complexes which all are closely connected: All simplicial cell complexes are generalized triangulations and the barycentric subdivision of any generalized triangulation is a simplicial cell complex. All simplicial complexes in turn are simplicial cell complexes and the barycentric subdivision of any simplicial cell complex is a simplicial complex [9].

An even richer set of classes of regular and simplicial CW-complexes of decreasing generality between simplicial cell complexes and simplicial complexes is given by the following.

**Definition 2.1.** Let \( K \) be a \( d \)-dimensional simplicial cell complex. For \( 1 \leq k \leq d \), \( K \) is said to be \( k \)-simple if any set of \( k + 1 \) vertices is in at most one \( k \)-cell.

A \( d \)-dimensional simplicial cell complex \( K \) is a simplicial complex if and only if \( K \) is \( d \)-simple. If \( K \) is contracted then \( K \) is \( k \)-simple if any set of \( k + 1 \) vertices is in a unique \( k \)-cell. If \( K \) is 1-simple we will call \( K \) simple. From the definition we get the following.

**Lemma 2.2.** Let \( M \) be a contracted \( k \)-simple pseudotriangulation of a closed connected \( d \)-manifold \( \mathbb{M} \). Then \( k \leq d - 1 \), and \( k = d - 1 \) if and only if \( M \) is a 2-facet contracted pseudotriangulation of \( S^d \).

**Lemma 2.3.** Let \( M \) be a contracted pseudotriangulation of a closed connected \( d \)-manifold \( \mathbb{M} \). If \( M \) is \( k \)-simple then \( \vert M \vert \) is \( k \)-connected.
Proof. For $k = 0$ the statement directly follows since every contracted pseudotriangulation is connected.

Let $k > 0$. $M$ is $d$-dimensional, $k$-simple ($k \leq d - 1$), and contracted. Hence the full $k$-skeleton $\text{skel}_k(M)$ of $M$ is contained in every $d$-simplex $\sigma \in M$. Thus $\text{skel}_k(M) = \text{skel}_k(\sigma) \subseteq \sigma$ but on the other hand $\{\sigma\} \subseteq M$. Hence, we have $\pi_1(M, x) \leq \pi_1(|\sigma|, x) = \{0\}$ for $x \in |\sigma|$ and $H_i(M) \leq H_i(|\sigma|) = \{0\}$ for $1 \leq i \leq k$. It follows that $|M|$ is $k$-connected.

\[ \square \]

2.2 Colored Graphs

In the following we will use the standard terminology for graphs as introduced in [3].

All graphs considered in this article are finite multi graphs without loops. Let $\Gamma = (V, E)$ be a graph and $U \subseteq V$ a subset of its vertices. Then the \textit{induced subgraph} $\Gamma[U]$ is the subgraph of $\Gamma$ with vertex set $U$ containing all edges of $\Gamma$ with both endpoints lying in $U$. For $n \geq 2$, an $n$-cycle is a closed path with $n$ distinct vertices and $n$ edges. If vertices $a_i$ and $a_{i+1}$ are adjacent in an $n$-cycle for $1 \leq i \leq n$ (addition is modulo $n$) then the $n$-cycle is denoted by $C_n(a_1, a_2, \ldots, a_n)$. A graph $\Gamma$ is called $h$-regular or $h$-valent if the number of edges adjacent to each vertex is $h$.

An edge coloring of a graph $\Gamma = (V, E)$ is a surjective map $\gamma: E \rightarrow C$ such that $\gamma(e) \neq \gamma(f)$ whenever $e$ and $f$ are adjacent (i.e., $e$ and $f$ share a common vertex). The elements of the set $C$ are called the colors. If $C$ has $h$ elements then $(\Gamma, \gamma)$ is said to be an $h$-colored graph.

Let $(\Gamma, \gamma)$ be an $h$-colored graph with color set $C$. If $B \subseteq C$ with $k$ elements then the graph $(V(\Gamma), \gamma^{-1}(B))$ is a $k$-colored graph with coloring $\gamma|_{\gamma^{-1}(B)}$. This colored graph is denoted by $\Gamma_B$. Let $(\Gamma, \gamma)$ be an $h$-colored connected graph with color set $C$. If $\Gamma_{C \setminus \{c\}}$ is connected for all $c \in C$ then $(\Gamma, \gamma)$ is called \textit{contracted}.

Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two disjoint $h$-regular $h$-colored graphs with same color set $\{1, \ldots, h\}$. Furthermore, let $v_i \in V_i$ and let $u_{j,i} \in V_i$ be their neighbors in $\Gamma_i$ such that the edge going from $v_i$ to $u_{j,i}$ is colored with color $j$, $1 \leq i \leq 2$, $1 \leq j \leq h$. Consider the graph $\Gamma$ obtained from $(\Gamma_1 \setminus \{v_1\}) \sqcup (\Gamma_2 \setminus \{v_2\})$ (here $\Gamma_i \setminus \{v_i\} = \Gamma_i[V_i \setminus \{v_i\}]$) by adding $h$ new edges $e_j$ with colors $j$ respectively, $1 \leq j \leq h$, such that the end points of edge $e_j$ are $u_{j,1}$ and $u_{j,2}$. The colored graph $\Gamma$ is called the \textit{connected sum} of $\Gamma_1$ and $\Gamma_2$ and is denoted by $\Gamma_1 \#_{v_1 v_2} \Gamma_2$. Note that permuting the colors of $\Gamma_1$ gives rise to $h!$ ways to perform the connected sum with $\Gamma_2$ along $v_1$ and $v_2$.

2.3 Crystallizations

Crystallizations are colored graphs defining contracted pseudotriangulations. Hence, they provide a way to visualize the essential properties of high dimensional manifolds in a low-dimensional setting.

Let $(\Gamma, \gamma)$ be a $(d + 1)$-colored graph with color set $C = \{0, \ldots, d\}$, $d \geq 1$. Then a $d$-dimensional simplicial cell complex $M(\Gamma)$ can be defined as follows. For each $v \in V(\Gamma)$ we take a $d$-simplex $\sigma_v$ and label its vertices by $0, \ldots, d$. If $u, v \in V(\Gamma)$ are joined by an edge $e$ and $\gamma(e) = i$, then we identify the $(d - 1)$-faces of $\sigma_u$ and $\sigma_v$ opposite to vertex $i$, such that equally labeled vertices are identified. Since there is no identification within a $d$-simplex, $M(\Gamma)$ is a simplicial cell complex. We say that $(\Gamma, \gamma)$ represents the simplicial cell complex $M(\Gamma)$. Since, in addition, the number of $i$-labeled vertices of $M(\Gamma)$ is equal to the number of components of $\Gamma_{C \setminus \{i\}}$ for each $i \in C$, the simplicial cell complex $M(\Gamma)$ is contracted if and only if $\Gamma$ is contracted [21].
Hence, for a manifold $M$ we will call a $(d+1)$-colored contracted graph $(\Gamma, \gamma)$ a crystallization of $M$ if the simplicial cell complex $M(\Gamma)$ is a pseudotriangulation of $M$. Furthermore, the crystallization $(\Gamma, \gamma)$ of some closed $d$-manifold $M$ either has two vertices (connected by $d+1$ edges, in which case $M$ is $S^d$) or the number of edges between two vertices is at most $d-1$. We will call $(\Gamma, \gamma)$ $k$-simple if $M(\Gamma)$ is $k$-simple. In [41], Pezzana showed the following.

**Proposition 2.4 (Pezzana).** Every connected closed PL-manifold admits a crystallization.

Note that the analogous statement about simply connected manifolds and simple crystallizations would imply the Smooth Poincaré conjecture (cf. Section 4.1). However, since there exist simply connected topological 4-manifolds with finite dimensional homology which admit an infinite number of PL structures this can not be true in general.

Crystallizations of manifolds admit a number of very useful combinatorial criteria which translate into topological properties of the manifolds they describe. In the following we will list some of these criteria.

**Proposition 2.5 (Cavicchioli-Grasselli-Pezzana [14]).** Let $(\Gamma, \gamma)$ be a crystallization of a $d$-manifold $M$. Then $M$ is orientable if and only if $\Gamma$ is bipartite.

Let $(\Gamma, \gamma)$ be a $(d+1)$-colored graph with color set $C = \{0, \ldots, d\}$. For any $k$-color set $D = \{i_1, i_2, \ldots, i_k\} \subset C$, the number of components of the sub graph $\Gamma_D$ will be denoted by $g_D$ or sometimes just by $g_{i_1i_2\cdots i_k}$. With this setup in mind we can state

**Proposition 2.6 (Gagliardi [25]).** Let $(\Gamma, \gamma)$ be a contracted $4$-colored graph with $n$ vertices and color set $\{0, 1, 2, 3\}$. Then $(\Gamma, \gamma)$ is a crystallization of a connected closed $3$-manifold if and only if

(i) $g_{ij} = g_{kJ}$ for $\{i,j,k,l\} = \{0,1,2,3\}$, and

(ii) $g_{01} + g_{02} + g_{03} = 2 + n/2$.

Let $(\Gamma, \gamma)$ be a crystallization (with color set $C$) of a connected closed $d$-manifold $M$. Choose two colors $i,j \in C$, let $\{G_1, \ldots, G_{s+1}\}$ be the set of all connected components of $\Gamma_{C\setminus\{i,j\}}$, and $\{H_1, \ldots, H_{t+1}\}$ be the set of all connected components of $\Gamma_{\{i,j\}}$. Since $\Gamma$ is regular, $H_k$ is an even cycle for $1 \leq k \leq t+1$ (note that in this case each $H_k$ is regular of degree two and 2-colorable). Note that, if $d = 2$, then $\Gamma_{\{i,j\}}$ is connected and hence $H_1 = \Gamma_{\{i,j\}}$. Take a set $S = \{x_1, \ldots, x_s, x_{s+1}\}$ of $s+1$ elements such that $x_m \in G_m$, $1 \leq m \leq s+1$. Choose a vertex $v_1$ in $H_k$ and let

$$H_k = C_{2l}(v_1, v_2, \ldots, v_{2l}),$$

where without loss of generality the edge between $v_1$ and $v_2$ has color $i$ and the edge between $v_2$ and $v_3$ has color $j$. Define

$$\tilde{r}_k := x_{k_2}^{x_{k_3}^{-1}x_{k_4}^{x_{k_5}^{-1}}\cdots x_{k_{2l}}^{x_{k_1}^{-1}}}, \text{ for } 1 \leq k \leq t+1 ,$$

where $G_{k_h}$ is the component of $\Gamma_{C\setminus\{i,j\}}$ containing $v_h$. For $1 \leq k \leq t+1$, let $r_k$ be the word obtained from $\tilde{r}_k$ by deleting $x_{k_{s+1}}^{\pm 1}$'s in $\tilde{r}_k$. Then we have
Example 2.9. Intersection forms of some well-known simply connected \(\mathbb{Q}\)-manifolds and \(\mathbb{Q}\)-manifolds with negative eigenvalues is referred to as its signature.

Proposition 2.8. we can deduce the following.

Note that if Proposition 2.7 (Gagliardi [26]) and then equivalently define \(\pm\) determinants \(H\) that \(\pi_1(M, x)\) is the fundamental group of \(M\) at a point \(x\), then

\[
\pi_1(M, x) \cong \{ \langle x_1, x_2, \ldots, x_s \mid r_1 \rangle \quad \text{if} \quad d = 2, \quad \langle x_1, x_2, \ldots, x_s \mid r_1, \ldots, r_t \rangle \quad \text{if} \quad d \geq 3.\
\]

For more about presentations of fundamental groups of crystallizations see [1].

### 2.4 4-manifolds and the intersection form

Given a closed topological \(d\)-manifold \(M\) we know that any smooth structure on \(M\) determines a PL-structure on \(M\), and the converse holds for dimension \(d \leq 6\), thus, for the remainder of this article we will regard PL-structures as equivalent to smooth structures and, since we are in the setting of triangulations, only refer to PL structures.

Given a closed oriented 4-manifold \(M\), its intersection form is the symmetric 2-form defined by

\[
\mathcal{Q}_M : H^2(M; \mathbb{Z}) \times H^2(M; \mathbb{Z}) \to \mathbb{Z}, \quad \mathcal{Q}_M(\alpha, \beta) = (\alpha \sim \beta)[M]
\]

where \(\sim\) denotes the cup-product.

\(\mathcal{Q}_M\) is bilinear, symmetric and is presented by a quadratic matrix of size \(\text{rk} H^2(M; \mathbb{Z})\) of determinant \(\pm 1\). The size is called the rank of \(\mathcal{Q}_M\) and the difference between positive and negative eigenvalues is referred to as its signature. If, for all \(\alpha \in H^2(M; \mathbb{Z})\) we have that \(\mathcal{Q}_M(\alpha, \alpha)\) is an even number, then \(\mathcal{Q}_M\) is called even. Otherwise, it is called odd. In order to define \(\mathcal{Q}_M\) more geometrically, one can present classes \(\alpha, \beta \in H^2(M; \mathbb{Z})\) by embedded surfaces \(S_\alpha\) and \(S_\beta\) of their Poincaré duals (this is always possible, see [28, Proposition 1.2.3]) and then equivalently define \(\mathcal{Q}_M(\alpha, \beta)\) as the intersection number of \(S_\alpha\) and \(S_\beta\):

\[
\mathcal{Q}_M(\alpha, \beta) = S_\alpha \cdot S_\beta.
\]

Note that if \(M\) is simply-connected, then \(H_2(M; \mathbb{Z})\) is a free \(\mathbb{Z}\)-module and there are isomorphisms \(H_2(M; \mathbb{Z}) \cong \mathbb{Z}^m\) where \(m = b_2(M)\) (see [23, 28, 43] for more). From the definition we can deduce the following.

Proposition 2.8. Let \(M\) and \(N\) be oriented closed 4-manifolds with intersection forms \(\mathcal{Q}_M\) and \(\mathcal{Q}_N\). Then their connected sum \(M \# N\) has intersection form \(\mathcal{Q}_M \oplus \mathcal{Q}_N\).

Example 2.9. Intersection forms of some well-known simply connected 4-manifolds

(i) The most common 4-manifold \(S^4\) does not have any 2-homology. Therefore we can take \(\emptyset\) as its intersection form.

(ii) The complex projective plane \(\mathbb{C}P^2\) has intersection form \(\mathcal{Q}_{\mathbb{C}P^2} = [+1]\) and the oppositely-oriented manifold \(\overline{\mathbb{C}P^2}\) has intersection form \(\mathcal{Q}_{\overline{\mathbb{C}P^2}} = [-1]\). Since \(\mathbb{C}P^2 \cong \overline{\mathbb{C}P^2}\), reversing orientation does not give a new manifold.

(iii) The manifold \(S^2 \times S^2\) has intersection form

\[
\mathcal{Q}_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

This matrix is often denoted by \(H\) (from “hyperbolic plane”). The oppositely-oriented manifold \(\overline{S^2 \times S^2}\) has intersection form \(\mathcal{Q}_{\overline{S^2 \times S^2}} = -H\). Since \(S^2 \times S^2 \cong \overline{S^2 \times S^2}\), reversing orientation does not give a new manifold.
(iv) The twisted product $S^2 \times S^2$ has intersection form

$$Q_{S^2 \times S^2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$ 

By a change of basis, we get $Q_{S^2 \times S^2} = [1] \oplus [−1]$. We will see that this proves $S^2 \times S^2 \cong \mathbb{CP}^2 \# \mathbb{CP}^2$.

(v) The $E_8$-manifold $M_{E_8}$ is a topological 4-manifold with (even) intersection form

$$E_8 = Q_{M_{E_8}} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix}.$$ 

By Rohlin’s theorem $M_{E_8}$ does not admit any PL structures, and thus is not of further interest when talking about simply connected PL 4-manifolds. However, its intersection form $E_8$ will re-appear as a direct summand of the intersection form of the K3 surface.

(vi) Recall that the K3 surface is a closed oriented connected and simply connected 4-manifold. Its intersection form is even of rank 22 and signature 16. In a suitable basis it is represented by the unimodular matrix $(-E_8) \oplus (-E_8) \oplus 3H$. As a PL-manifold, it is prime (i.e., it can not be expressed as connected sum of PL-manifolds).

Freedman used the intersection form for his celebrated classification of simply-connected topological 4-manifolds. More precisely, he proved the following statement.

**Theorem 2.10** (Freedman [22]). For every unimodular symmetric bilinear form $Q$ there exists a simply connected, closed, topological 4-manifold $M$ such that $Q_M = Q$. If $Q$ is even, this manifold is unique (up to homeomorphism). If $Q$ is odd, there are exactly two different homeomorphism types of manifolds with the given intersection form. At most one of these homeomorphism types carries a PL structure. Consequently, simply connected, PL 4-manifolds are determined up to homeomorphism by their intersection forms.

Using the classification theorem we now can state the converse of Proposition 2.8 for topological 4-manifolds.

**Corollary 2.11.** If $M$ is simply-connected and $Q_M$ splits as a direct sum $Q_M = Q' \oplus Q''$, then there exists topological 4-manifolds $N'$ and $N''$ with intersection form $Q'$ and $Q''$ such that $M \cong N' # N''$.

Thus, the K3 surface (which is prime as PL-manifold) can be expressed as a connected sum of the form $2M_{E_8} # 3S^2 \times S^2$.

The following result about simply connected PL 4-manifolds is a combination of Theorems 1.2.21, 1.2.30 and 1.2.31 in [28] due to work by Rohlin [42], Milnor and Husemoller [38], Donaldson [16], and Furuta [24].
Proposition 2.12. Suppose that \( Q \) is the intersection form of a simply connected PL 4-manifold. Then if \( Q \) is odd, it is isomorphic to \( k[1] \oplus l[-1] \), and if \( Q \) is even it is isomorphic to \( 2mE_8 \oplus rH \), for some integers \( k, l, r \geq 0, m \in \mathbb{Z}, r \geq 2|m| + 1 \).

In particular, if \( Q \) is even, it can not be definite. Furthermore, we have the following conjecture

Conjecture 2.13 (11/8-conjecture \([36]\)). If \( Q \), even, is the intersection form of a simply connected PL 4-manifold. Then \( Q \cong 2mE_8 \oplus rH \), \( r \geq 0, m \in \mathbb{Z}, r \geq 3|m| \).

In other words, Conjecture 2.13 states that the rank of any even intersection form admitting a PL structure is at least 11/8 times as large as its signature, hence the name. Assuming Conjecture 2.13 is true it follows that all simply connected PL 4-manifolds are homeomorphic to either \( k \mathbb{C}P^2 \# l \mathbb{C}P^2 \) or \( \tilde{m} K3 \# \tilde{r} S^2 \times S^2 \).

For complex hypersurfaces \( S_d = \{[z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 | \sum z_i^d = 0 \} \subset \mathbb{C}P^3 \), where \( d \) is a positive integer, we have a similar result.

Proposition 2.14 (Theorem 1.3.8 \([28]\), McDuff and Salamon \([37]\)). The hypersurface \( S_d \) is a PL, simply connected, complex surface. If \( d \) is odd, then \( Q_{S_d} \) is equivalent to \( \lambda_d[1] \oplus \mu_d(-[1]) \), where \( \lambda_d = \frac{1}{4}(d^3 - 6d^2 + 11d - 3) \) and \( \mu_d = \frac{1}{2}(d - 1)(2d^2 - 4d + 3) \); if \( d \) is even, then \( Q_{S_d} \) is equivalent to \( \lambda_d(-E_8) \oplus m_dH \), where \( \lambda_d = \frac{1}{24}d(d^2 - 4) \) and \( m_d = \frac{1}{8}(d^3 - 6d^2 + 11d - 3) \).

3 Simple crystallizations of simply connected PL 4-manifolds

Let \( M \) be a contracted pseudotriangulation of a (simply connected) 4-manifold \( M \), then \( M \) has at least \( \binom{5}{2} = 10 \) edges with equality if and only if \( M \) is simple. In addition, since simple crystallizations by construction always describe simply connected 4-manifolds, and simply connected 4-manifolds are always orientable, it follows from Proposition 2.5 that all simple crystallizations are bipartite. Alternatively, this can also be followed from the fact that for any crystallization of a 4-manifold \( (\Gamma, \gamma) \) with color-set \( C \), we have that \( \Gamma_{C \setminus \{c\}} \) is a crystallization of \( S^4 \) for all \( c \in C \) if and only if \( (\Gamma, \gamma) \) is simple. Thus, \( \Gamma_{C \setminus \{c\}} \) is bipartite for all \( c \in C \) in the simple case. This property will be very useful in later sections of this article and we will think of all simple crystallizations as bipartite graphs. More generally, for arbitrary contracted simplicial cell complexes we have.

Lemma 3.1. Let \( X \) be a \( d \)-dimensional contracted simplicial cell complex and let \( (\Gamma, \gamma) \) be the \((d + 1)\)-colored graph corresponding to \( X \) with color set \( C \). Then \( X \) is \( k \)-simple if and only if for all subsets \( D \subset C \) of size \( |d - k| \) the subgraph \( \Gamma_D \) is connected.

Proof. Let \( X \) be \( k \)-simple, i.e., any set of \( k + 1 \) vertices is in a unique \( k \)-cell. Let \( D = \{i_1, \ldots, i_{d-k}\} \subset C \) be of size \( (d-k) \). Let \( C \setminus D = \{j_1, \ldots, j_{k+1}\} \) and let \( v_q \) be the vertex of \( X \) corresponding to the color \( j_q \) for \( 1 \leq q \leq k + 1 \). Now, by construction, for each connected component of \( \Gamma_D \) there is a distinct \( k \)-face through \( v_1, \ldots, v_{k+1} \) in \( X \), and since \( X \) is \( k \)-simple it follows that \( \Gamma_D \) has exactly one component for every subset \( D \subset C \) of \( d-k \) colors.

Conversely, let \( \Gamma_D \) be connected for \( D \subset C \) of size \( d-k \), and let \( v_1, \ldots, v_{k+1} \) be vertices of \( X \) defined as above. Then, there is a unique \( k \)-face through \( v_1, \ldots, v_{k+1} \) in \( X \) and \( X \) is \( k \)-simple. □

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Lemma 3.2. Let $M$ and $M'$ be closed connected $d$-manifolds each admitting a $k$-simple crystallization, then $M \# M'$ admits a $k$-simple crystallization as well.

Proof. Let $(\Gamma, \gamma)$ and $(\Gamma', \gamma')$ be $k$-simple crystallizations of $M$ and $M'$ respectively with the same color set $C$.

Then by Lemma 3.1 both $\Gamma_D$ and $\Gamma'_D$ are connected for all $D \subset C$, $\# D = d - k$. Hence, for any pair of vertices $(v_1, v_2) \in (V(\Gamma), V(\Gamma'))$, $(\Gamma_{\# v_1 v_2} \Gamma')_D$ is connected and by applying Lemma 3.1 again it follows that $\Gamma_{\# v_1 v_2} \Gamma'$ is $k$-simple.

In this article we are interested in simple contracted pseudotriangulations of closed 4-manifolds (which, then, are simply connected by construction, cf. Lemma 2.3). In this special case we have the following.

Lemma 3.3. Let $M$ be a simple contracted pseudotriangulation of a closed 4-manifold, and let $(\Gamma, \gamma)$ be its corresponding crystallization with color set $C = \{0, 1, 2, 3, 4\}$. Then $g_{ij} = m$ for any 2-color set $\{i, j\} \subset C$, where $6m - 4 = f_4(M)$.

Proof. Let $n = f_4(M)$. Then $\Gamma$ has $n$ vertices. Since $M$ is simple, $\Gamma_D$ is connected for any 3-color set $D \subset C$ by Lemma 3.1. Thus, $\Gamma_D$ is a crystallization of $S^3$ for any 4-color set $E \subset C$. Now, by Proposition 2.6 (ii), we have $m = g_{ij} = g_{kl}$ for any 4-color set $E = \{i, j, k, l\} \subset C$ and hence $g_{ij} = m$ for any pair of colors $i, j \in C$. In particular, by Proposition 2.6 (ii), we have $3m = g_{01} + g_{02} + g_{03} = n/2 + 2$ since $\Gamma_{\{0,1,2,3\}}$ is a crystallization of $S^3$. This implies $3m = n/2 + 2$, i.e., $n = 6m - 4$.

Let $M$ be a simple contracted pseudotriangulation of a closed 4-manifold $M$ with $6m - 4$ facets. Let $(\Gamma, \gamma)$ be the corresponding crystallization with color set $C$. Since $\pi_1(M) \cong \{0\}$, by Hurewicz’s theorem $\pi_2(M) \cong H_2(M; \mathbb{Z})$. Hence all homology classes of $M$ can be presented as images of maps $f : S^2 \to M$. Let $\{i, j\} \subset C$ be a 2-color set. For $\{l, q, r\} = C \setminus \{i, j\}$, let $v_l, v_q$ and $v_r$ be the vertices of $M$ corresponding to the color $l, q$ and $r$ respectively. Since $\Gamma_{\{i,j\}}$ has $m$ components, there are exactly $m$ distinct triangles through the vertices $v_l, v_q$ and $v_r$ in $M$. Therefore, there is an isomorphism $H_2(M; \mathbb{Z}) \cong \mathbb{Z}^{m-1}$. Now $M$ has $6m - 4$ facets if and only if the intersection form $Q_M$ of $M$ has rank $(m - 1)$. In particular, we have for the number of vertices (facets) of simple crystallizations (simple contracted pseudotriangulations) of the following manifolds:

| Manifold | $m$ | # vertices / facets |
|----------|-----|---------------------|
| $S^4$    | 1   | 2                   |
| $\mathbb{C}P^2$ | 2   | 8                   |
| $S^2 \times S^2$ | 3   | 14                  |
| $K3$     | 23  | 134                 |

4 The unique simple crystallizations of $S^4$ and $\mathbb{C}P^2$

4.1 The standard crystallization of $S^4$

The standard 2-facet contracted pseudotriangulation of $S^4$ is given by gluing two 4-dimensional simplices together along their boundaries. The resulting complex is clearly a pseudotriangulation of $S^4$. Moreover, it has five vertices, ten edges, and is thus simple. It’s crystallization is a 2-vertex graph $\Gamma^0$ with five edges between two vertices and thus $\Gamma^0$ has a natural 5-coloring (cf. Figure 1).
Note that the standard contracted pseudotriangulation of $S^4$ is the union of two standard 4-balls glued together along the boundaries of two 4-simplexes. Hence, the standard contracted pseudotriangulation of $S^4$ is PL-homeomorphic to $S^4$ with standard PL structure. Moreover, it is unique since there is no other 5-colorable 5-valent multi graph with only two vertices. Hence, a version of Proposition 2.4 for simple crystallizations of the 4-sphere would proof the Smooth Poincaré conjecture.

### 4.2 The standard crystallization of $\mathbb{CP}^2$

The following example of a simple crystallization of $\mathbb{CP}^2$ first appeared in [27].

Let $(\Gamma^1, \gamma^1)$ be the contracted 5-colored graph with color set $C = \{0, 1, 2, 3, 4\}$ given in Figure 2 and let $M_1$ be its corresponding contracted pseudotriangulation. Since for every 2-color subgraph $\Gamma^1_{\{i_1, i_2\}}$, $\{i_1, i_2\} \subset C$, we have $g_{i_1i_2} = 2$ it follows from Proposition 2.6 that $(\Gamma^1|_{C\setminus\{c\}}, \gamma^1|_{(\gamma^1)^{-1}(C\setminus\{c\})})$ is a crystallization of a 3-manifold $M^{(c)}_1$ for all $c \in C$. By Proposition 2.7 it is easy to calculate that $\pi_1(M^{(c)}_1, x) = \{0\}$, $x \in M^{(c)}_1$, for all $c \in C$. Hence, due to Perelman’s theorem [39] $M^{(c)}_1 \cong S^3$ for all $c \in C$. Thus $(\Gamma^1, \gamma^1)$ is a crystallization and $M_1$ is a pseudotriangulation of a 4-manifold $M_1$. Since, in addition, any 3-color subgraph of $(\Gamma^1, \gamma^1)$ is connected we get by Lemma 3.1 that $M_1$ is a simple contracted pseudotriangulation and hence $M_1$ is simply connected. Furthermore, since $g_{i_1i_2} = m = 2$ the intersection form of $M_1$ has rank one, and by Freedman’s classification theorem (cf. Theorem 2.10) we know that $M_1$ must be homeomorphic to $\mathbb{CP}^2$.
number of elementary moves necessary to pass from a 9-vertex combinatorial manifold to a 5-vertex simple contracted pseudotriangulation. In particular, this shows that \((\Gamma^1, \gamma^1)\) is of standard PL type. The facet lists of the five pseudotriangulations are available from the authors upon request.

4.3 Uniqueness of the simple crystallization of \(\mathbb{C}P^2\)

From the classification of crystallizations of 4-manifolds \([10, 11, 12]\) we get that there is exactly one simple crystallization with eight vertices. In particular, it follows that there is exactly one simple crystallization of \(\mathbb{C}P^2\). In this section we give an elementary proof of the uniqueness of \((\Gamma^1, \gamma^1)\) independent of the classification of crystallizations of 3- and 4-manifolds.

**Theorem 4.2.** Up to isomorphy, \((\Gamma^1, \gamma^1)\) is the only simple crystallization of \(\mathbb{C}P^2\).

**Proof.** Let \((\Gamma, \gamma)\) be a simple crystallization of \(\mathbb{C}P^2\) with color set \(C = \{0, 1, 2, 3, 4\}\). Then \(\Gamma\) must have eight vertices, \(g_{ij} = 2\) for \(\{i, j\} \subset C\), and \((\Gamma|_{\gamma_{\{\epsilon\}_1}}, \gamma_{\{\epsilon\}_1}^{-1}(\{\gamma\}_0))\) is a crystallization of \(S^3\) for all \(c \in C\). Since \(\Gamma\) is simple we have \(g_{ijk} = 1\) for \(\{i, j, k\} \subset C\) by Lemma 3.1 and hence \(\Gamma\) cannot contain a triple edge (Otherwise, let the three edges between two vertices be colored by \(i, j\) and \(k\). Then the triple edge on its own must be a connected component of \(\Gamma_{\{i,j,k\}}\) and we have \(g_{ijk} > 1\) whenever \(\Gamma\) has more than two vertices). Furthermore, since \(\mathbb{C}P^2\) is orientable, \(\Gamma\) must be bipartite by Proposition 2.5. In particular, \(\Gamma\) can not have any cycles of odd length. It follows that \(\Gamma_{\{i,j\}}\), \(\{i, j\} \subset C\), must be of the form \(C_2 \sqcup C_6\) or \(C_4 \sqcup C_4\).

In the following we will use this fact to proof that there exist precisely three crystallizations of \(S^3\) which can occur as a 4-color subgraph of a simple crystallization of \(\mathbb{C}P^2\). The theorem then follows from the fact that there is a unique 5-colored graph such that all of its 4-color subgraphs are isomorphic to one of these three crystallizations.

Let \(\Gamma_{\{i,j\}}\) be of the form \(C_4 \sqcup C_4\) for all \(\{i, j\} \subset C\). That is, in particular, \(\Gamma_{\{0,1\}} = C_4(a_1, a_2, a_3, a_4) \sqcup C_4(b_1, b_2, b_3, b_4)\) as in Figure 3 (a). Since \(\Gamma_{\{0,1\}_1}\) must be connected, without loss of generality let \(a_1b_1 \in \gamma^{-1}(2)\). Since by assumption \(\Gamma_{\{0,2\}_2}\) and \(\Gamma_{\{1,2\}}\) are of the form \(C_2 \sqcup C_4\) it follows (up to isomorphism) that \(a_2b_2, a_3b_3, a_4b_4 \in \gamma^{-1}(2)\). Now, since \(\Gamma_{\{0,1,3\}}\) is connected, bipartite and \(\Gamma\) does not contain a 2-cycle, \(a_1b_1, a_2b_2, a_3b_3, a_4b_4 \in \gamma^{-1}(2)\) (see Figure 3 (a)). By applying Proposition 2.7 we get that \(\pi_1(|M(\Gamma_{\{0,1,2,3\}})|, x) \cong \mathbb{Z}_2\) and hence \(\Gamma_{\{0,1,2,3\}}\) can not be a crystallization of \(S^3\). Hence there exist \(\{i, j\} \subset C\) such that \(\Gamma_{\{i,j\}}\) is of the form \(C_2 \sqcup C_6\). Without loss of generality, let \(\Gamma_{\{0,1\}} = C_2(v_1, v_2) \sqcup C_6(v_3, v_4, v_5, v_6, v_7, v_8)\) as in Figure 3 (b), and since \(\Gamma_{\{0,1,2\}}\) is connected, let \(v_1v_5 \in \gamma^{-1}(2)\). Then either \(v_2v_4 \in \gamma^{-1}(2)\) or \(v_2v_8 \in \gamma^{-1}(2)\) (up to isomorphism).

**Case** \(v_2v_4 \in \gamma^{-1}(2)\): In this case we have \(v_3v_6, v_7v_8 \in \gamma^{-1}(2)\) since \(\Gamma_{\{1,2\}}\) has two connected components. We will denote this graph by \(\mathcal{F}_1\) (see Figure 3 (b)). Since \(\Gamma\) is bipartite and does not contain a triple edge, there are three possibilities to add an edges of color 3 to \(\mathcal{F}_1\) at vertex \(v_1\).

(i) Let \(v_1v_3 \in \gamma^{-1}(3)\). Then \(v_2v_6 \not\in \gamma^{-1}(3)\), since otherwise a triple edge has to be inserted at \(v_7v_8\) to ensure that \(\Gamma_{\{0,3\}}\) has two connected components. Hence either \(v_2v_4 \in \gamma^{-1}(3)\) or \(v_2v_8 \in \gamma^{-1}(3)\). If \(v_2v_4 \in \gamma^{-1}(3)\) then \(v_3v_7, v_5v_8 \in \gamma^{-1}(3)\). We will denote this graph by \(\mathcal{G}_1\) (see Figure 3 (d)). If \(v_2v_8 \in \gamma^{-1}(3)\) then \(v_5v_6, v_4v_7 \in \gamma^{-1}(3)\). We will denote this graph by \(\mathcal{G}_2\) (see Figure 3 (e)).
(ii) Let \( v_1v_5 \in \gamma^{-1}(3) \). First note that \( v_2v_4 \not\in \gamma^{-1}(3) \) since otherwise \( g_{23} > 2 \). If \( v_2v_6 \in \gamma^{-1}(3) \), then \( v_3v_8, v_4v_7 \in \gamma^{-1}(3) \) and we get a graph isomorphic to \( G_1 \). If \( v_2v_8 \in \gamma^{-1}(3) \), then \( v_3v_4, v_6v_7 \in \gamma^{-1}(3) \) since otherwise \( g_{13} < 2 \). We will denote this graph by \( G_3 \) (see Figure 3(f)).

(iii) Let \( v_1v_7 \in \gamma^{-1}(3) \). If \( v_2v_4 \) (resp., \( v_2v_6 \)) \( \in \gamma^{-1}(3) \), then we get graphs isomorphic to \( G_3 \) (resp., \( G_2 \)). Thus, assume \( v_2v_8 \in \gamma^{-1}(3) \) and hence \( v_4v_5, v_3v_6 \in \gamma^{-1}(3) \) (otherwise \( g_{13} > 2 \)). In this case we have \( \pi_1([M(\Gamma_{\{0,1,2,3\}})], x) \cong \mathbb{Z} \) by Proposition 2.7 and thus \( \Gamma_{\{0,1,2,3\}} \) is not a crystallization of \( S^3 \).

Case \( v_2v_8 \in \gamma^{-1}(2) \): It follows that \( v_3v_4, v_6v_7 \in \gamma^{-1}(2) \) since \( \Gamma_{\{1,2\}} \) has two connected components. We will denote this graph by \( J_2 \) (see Figure 3(c)).

Since \( \Gamma \) is bipartite and does not contain a triple edge, there are two possibilities (up to isomorphy) to add an edge of color 3 to \( J_2 \) at vertex \( v_1 \).

(i) Let \( v_1v_3 \in \gamma^{-1}(3) \). First note that \( v_2v_4 \not\in \gamma^{-1}(3) \) since otherwise \( \Gamma_{\{0,1,2,3\}} \) would have a triple edge or \( g_{13} > 2 \). If \( v_2v_6 \in \gamma^{-1}(3) \) then \( v_3v_4, v_7v_8 \in \gamma^{-1}(3) \) since otherwise \( g_{03} < 2 \) and \( \Gamma_{\{0,1,2,3\}} \) is isomorphic to \( G_3 \). If \( v_2v_8 \in \gamma^{-1}(3) \) then \( v_1v_7, v_5v_6 \in \gamma^{-1}(3) \) to avoid a triple edge at \( v_6v_7 \) and, again \( \Gamma_{\{0,1,2,3\}} \) is isomorphic to \( G_3 \).

(ii) Let \( v_1v_5 \in \gamma^{-1}(3) \). We have \( v_2v_8 \not\in \gamma^{-1}(3) \) since otherwise \( g_{23} > 2 \). Then, up to isomorphy we must have \( v_2v_4 \in \gamma^{-1}(3) \) and thus \( v_3v_6, v_7v_8 \in \gamma^{-1}(3) \) to avoid a triple edge and we get a graph isomorphic to \( G_3 \).
Hence, there are exactly three 8-vertex crystallizations \( G_1, G_2 \) and \( G_3 \) of \( S^3 \) such that \( g_{ij} = 2 \) for all \( \{i, j\} \subset \{1, 2, 3, 4\} \). This part of the proof can be independently checked using the classification of generalized triangulations \[4\]: Precisely ten of the 9,787,509 closed eight tetrahedra generalized triangulations of 3-manifolds are contracted pseudotriangulations, seven of which are crystallizations of the 3-sphere, three of which satisfy \( g_{ij} = 2 \) for all 2-color subsets \( \{i, j\} \subset \{0, 1, 2, 3\} \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The three 5-colored graphs obtained from \( G_2 \) and \( G_3 \).}
\end{figure}

A crystallization of \( \mathbb{CP}^2 \) can now be found by disjointly adding four more edges to \( G_1, G_2, \) and \( G_3 \) to get a 5-colored bipartite graph. A priori, there are \( 4 \cdot 3 \cdot 2 = 24 \) ways to do this per graph but we will see that many of them are invalid and the remaining ones are isomorphic to the simple crystallization of \( \mathbb{CP}^2 \) presented in Section 4.2.

**Crystallization \( G_1 \):** If \( v_5v_i \in \gamma^{-1}(4), i \in \{1, 4, 6, 8\} \), then the completed 5-colored graph \( \Gamma \) will contain a 4-color subgraph \( \Gamma_D \) with \( \pi_1(|M(\Gamma_D)|, x) \cong \mathbb{Z} \) and thus \( \Gamma \) can not be the crystallization of a 4-manifold.

**Crystallization \( G_2 \):** Analogously, we have \( v_4v_i \not\in \gamma^{-1}(4), i \in \{2, 5, 7\} \), since otherwise \( \Gamma \) can not be the crystallization of a 4-manifold. Thus \( v_3v_4 \in \gamma^{-1}(4) \) and since \( g_{24} = 2 \) it follows that we can obtain \( \Gamma \) by either adding \( v_1v_5, v_6v_7, v_2v_8 \in \gamma^{-1}(4) \) (see Figure 4 (a)) or adding \( v_1v_7, v_2v_6, v_5v_8 \in \gamma^{-1}(4) \) (see Figure 4 (b)). In the former case, \( \Gamma \) is isomorphic to the crystallization given in Section 4.2. In the latter case we once again apply Proposition 2.7 to see that \( \pi_1(|M(\Gamma_{\{0,2,3,4\}})|, x) \cong \mathbb{Z}_2 \).

**Crystallization \( G_3 \):** Again, we have \( v_3v_6, v_3v_8, v_4v_5, v_2v_4 \not\in \gamma^{-1}(4) \) to enforce valid 4-color subgraphs. It follows that \( v_1v_3, v_4v_7 \in \gamma^{-1}(4) \) and since \( g_{04} = 2 \) we have \( v_5v_6, v_2v_8 \in \gamma^{-1}(4) \) (see Figure 4 (c)). This graph is again isomorphic to the crystallization given in Section 4.2.

\( \square \)

**Remark 4.3.** Given a simple crystallization \( (\Gamma, \gamma) \) of a simply connected 4-manifold \( \mathbb{M} \), Theorem 4.2 provides a way to detect connected summands of type \( \mathbb{CP}^2 \) or \( \overline{\mathbb{CP}^2} \) in \( \mathbb{M} \) by solving a subgraph problem: Enumerate all seven vertex subgraphs of type \( (\Gamma^1, \gamma^1) \) with one vertex removed and check for each if it is connected to the rest of \( (\Gamma, \gamma) \) by five edges, one for each color. Note, that not all connected summands can be detected that way as can be followed from Theorem 8.1 (cf. \[46\]).
A heuristic to produce simple crystallizations of 4-manifolds

Using the 4-manifold branch of the computational topology software *regina* we use a simulated annealing type heuristic simplification strategy to turn combinatorial manifolds into simple contracted pseudotriangulations. The strategy uses bistellar moves and so-called edge contractions (which respect the PL-homeomorphism type of the triangulation).

In more detail, recall that a bistellar $i$-move in a simplicial complex $C$ takes $i+1$ facets joined around a common $(d-i)$-dimensional face $\delta \in C$ and replaces them with $d-i+1$ facets joined around an $i$-face $\gamma$ with the precondition that $\gamma$ is not a face of $C$. More precisely we have

$$\Phi_i(C, \delta) = (C \setminus (\delta \star \partial \gamma)) \cup (\partial \delta \star \gamma),$$

see Figure 5 for all bistellar moves in dimension four. If $C$ is a pseudotriangulation we can weaken the precondition on $\gamma$ as $C$ now can have multiple faces with equal vertex set. In fact, the only precondition we have to check in the pseudotriangular setting is that no two vertices of any facet become identified by a bistellar move. Note that in dimension four this is automatically satisfied for all 0-, 2-, 3- and 4-moves. In the case of a 1-move we have to check that the edge we are inserting is not a loop (i.e., that $u \neq v$ in Figure 5). Edge contractions have more sophisticated preconditions. However, they are also rare. This is why we let *regina* take care of the topological correctness of the modification (i.e., the check that the edge contraction does not change the PL-topological type) and explicitly check if the complex after an edge contraction is still a pseudotriangulation (that is, check that no vertex identifications have been introduced). If the resulting complex is not a pseudotriangulation we undo the edge contraction and proceed as before.

The procedure essentially performs bistellar moves and edge contractions at random where moves reducing the complexity of the triangulation (3-moves, 4-moves and edge contractions) are performed with a higher probability. Using this strategy we were able to obtain 40,651 simple contracted pseudotriangulations from the combinatorial manifold PL-homeomorphic to the K3 surface due to Kühnel and the second author and 19,129 simple contracted pseudotriangulations PL-homeomorphic to the minimum 16-vertex combinatorial manifold homeomorphic to the K3 surface due to Casella and Kühnel. We believe that the number of simple contracted pseudotriangulations of the K3 surface is orders of magnitude larger than the numbers provided above. Note that both versions of the K3 surface are conjectured to be PL-homeomorphic. This conjecture could be settled by finding a simple crystallization which occurs in both the list of simple contracted pseudotriangulations. However, as of today both lists are disjoint (see another attempt to settle this conjecture). This is work in progress.

The code, as well as all data generated using the heuristic is available from the authors upon request.

Simple crystallizations of $S^2 \times S^2$

Simple crystallizations homeomorphic to $S^2 \times S^2$ have been completely classified by [10, 11, 12], and all of them must be of standard PL type by [15]. Here we present a particularly symmetric example which was obtained from the standard combinatorial triangulation of $S^2 \times S^2$ using the heuristic described in Section 5.
Figure 5: Bistellar moves in dimension four.
Let \((\Gamma^2, \gamma^2)\) be the contracted 5-colored graph with color set \(C = \{0, 1, 2, 3, 4\}\) given in Figure 6 and let \(M_2\) be the corresponding contracted pseudotriangulation. To verify its topological type we can proceed as in the previous example. We follow that \(M_2\) is a simple contracted pseudotriangulation of a (simply connected) 4-manifold with intersection form of rank two. Taking the barycentric subdivision of \(M_2\) yields a combinatorial 4-manifold and computing its intersection form using \textit{simpcomp} \cite{18, 19, 20} yields that \(|M_2|\) is homeomorphic to \(S^2 \times S^2\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure6.png}
\caption{A simple crystallization of \(S^2 \times S^2\).}
\end{figure}

\section{Simple crystallizations of the K3 surface}

The following simple crystallization \((\Gamma^3, \gamma^3)\) of the K3 surface was obtained from a simple contracted pseudotriangulation \(M_3\) which was constructed using the 17-vertex combinatorial manifold \(K3\) with standard PL structure from \cite{15} together with the heuristic described in Section 5.

To give an independent prove that \((\Gamma^3, \gamma^3)\) is homeomorphic to the K3 surface we first have to take a look at the 2-colored graphs \(\Gamma^3_{\{i,j\}}, \{i,j\} \subset C\). For instance, we have

\[
\Gamma^3_{\{0,1\}} = 5C_2 \sqcup 7C_4 \sqcup 6C_6 \sqcup C_8 \sqcup 2C_{10} \sqcup 2C_{16},
\]

where \(kC_i\) denotes \(k\) disjoint copies of an \(i\)-cycle. Note that \(\Gamma^3_{\{0,1\}}\) has exactly \(m = 23\) connected components. More explicitly, using the notation for cycles as given in Section 2.8 we have
At this point we proceed by feeding the corresponding triangulation connected 4-manifolds of type \( k \) face respectively. Now using Lemma 3.2, we can construct simple crystallizations of simply connected. Once again, this can be done by applying Proposition 2.7. It follows that \((\Gamma^3, \tau^3)\) and the K3 surface is homeomorphic to the K3 surface.

See the first and second column of Figure 7 to verify the above list of cycles. The other nine 2-color subgraphs \( \Gamma^3_{(i,j)} \), \( \{i, j\} \subseteq C \), can be obtained by combining column \( i \) and column \( j \) in Figure 7. They all have precisely 23 connected components.

Furthermore, note that each \((\Gamma^3_{|C \setminus \{c\}}, \tau^3_{|C \setminus \{c\}})\) has 134 vertices and we have \( 3 \cdot 23 = 134/2 + 2 \). Thus, all \((\Gamma^3_{|C \setminus \{c\}}, \tau^3_{|C \setminus \{c\}})\) are crystallizations of 3-manifolds \( M^3_c \) for all \( c \in C \). It remains to show that for all \( c \in C \) the 3-manifolds \( M^3_c \) are simply connected. Once again, this can be done by applying Proposition 2.7. It follows that \((\Gamma^3, \tau^3)\) is the simple crystallization of a simply connected 4-manifold with intersection form of rank 22. At this point we proceed by feeding the corresponding triangulation \( M_3 \) into simpcomp to verify that \( |M_3| \) is homeomorphic to the K3 surface.

8 Simple crystallizations of pairs of homeomorphic but not PL-homeomorphic 4-manifolds

In Sections 4, 6, and 7 we constructed simple crystallizations of \( \mathbb{C}P^2 \), \( S^2 \times S^2 \) and the K3 surface respectively. Now using Lemma 3.2 we can construct simple crystallizations of simply connected 4-manifolds of type \( k \mathbb{C}P^2 \# l \mathbb{C}P^2 \) and \( m K3 \# r S^2 \times S^2 \), for all \( k, l, m, r \geq 0 \). As of today, these are all known topological types of simply connected 4-manifolds which allow at least one PL structure.

The following result about connected sums of simply connected 4-manifolds is due to Wall.

**Theorem 8.1** (Wall [16]). Let \( M \) and \( N \) be two simply connected closed PL 4-manifolds with isomorphic intersection forms. Then there exist a \( k \geq 0 \) such that \( M \# k (S^2 \times S^2) \) and \( N \# k (S^2 \times S^2) \) are PL-homeomorphic.

Furthermore, it is known that \( k \) in the above theorem is not always equal to zero.

**Theorem 8.2** (Kronheimer and Mrowka [31]).

\[
K3 \# \overline{\mathbb{C}P^2} \not\simeq_{PL} 3 \mathbb{C}P^2 \# 20 \overline{\mathbb{C}P^2}
\]

Note that \( K3 \# \overline{\mathbb{C}P^2} \) and \( 3 \mathbb{C}P^2 \# 20 \overline{\mathbb{C}P^2} \) have isomorphic odd intersection forms and are thus homeomorphic by Theorem 2.10.

Theorem 8.2 was proven by computing the Donaldson polynomial [17] for both manifolds. The Donaldson polynomial is a powerful PL-homeomorphism invariant to tell homeomorphic but not PL-homeomorphic 4-manifolds apart. However, it is usually very hard to compute for a pair of given manifolds. Now, using the connected-sum property of simple contracted pseudotriangulations (cf. Lemma 3.2) it follows that
Figure 7: A simple crystallization of the K3 surface.
Corollary 8.3. There is a pair of simple contracted pseudotriangulations of homeomorphic but non-PL-homeomorphic simply connected 4-manifolds.

Since simple contracted pseudotriangulations can be regarded as (strongly) minimal pseudotriangulations this also addresses a number of problems posed in [45] in a pseudotriangular setting.

Remark 8.4. Proposition 2.12 together with Freedman’s classification theorem (cf. Theorem 2.11) gives us many homeomorphic pairs of simply connected 4-manifolds with distinct connected sum decompositions: For each simply connected PL 4-manifold with even intersection form build the connected sum with $\mathbb{CP}^2$ or $\overline{\mathbb{CP}^2}$ and compare it to the suitable connected sum of the form $k\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$. Hence, we can assume that more pairs of simply connected PL 4-manifolds exist which (i) can be build from our simple crystallizations of $\mathbb{CP}^2$, $S^2 \times S^2$, and $K3$ (i.e., which are simply connected PL 4-manifolds of “standard type”) and which (ii) require $k > 0$ in Theorem 8.1.

The code for producing these pairs of homeomorphic but not PL-homeomorphic connected 4-manifolds is available from the authors upon request.

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