Electromagnetic response of superconductors in the presence of multiple collective modes

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We revisit the importance of collective-mode fluctuations and gauge invariance in the electromagnetic response of superconducting systems. In particular, we show that order-parameter fluctuations, gapless or not, have no contribution to the Meissner effect in both s- and p-wave superconductors. More generally, we extend this result to uniform and nonuniform superfluids with no external wavevector scale. To facilitate this analysis, we formulate a path-integral-based matrix methodology for computing the electromagnetic response of fermionic fluids in the presence of concomitantly fluctuating collective modes. Closed-form expressions for the electromagnetic response in different scenarios are provided, including the case of fluctuations of electronic density and the phase and amplitude of the order parameter. All microscopic symmetries and invariances are manifestly satisfied in our formalism, and it can be straightforwardly extended to other scenarios.

I. INTRODUCTION

Collective modes in superfluids and superconductors play a pivotal role in understanding gauge invariance in a many-particle context \cite{1, 2, 3}. These modes comprise amplitude and phase fluctuations of the order parameter \cite{4, 5}, and in the context of neutral superfluids the presence of the phase mode is evinced as a longitudinal sound oscillation \cite{6, 7, 8}. Observation of the amplitude mode in a condensed-matter context, while possible, is rather challenging \cite{9}. Some particular cases where this mode was indeed observed include systems with emergent Lorentz invariance \cite{10, 11, 12} and superconductors coupled to either charge-density waves \cite{13, 14} or optical modes \cite{15, 16}. Collective modes in general provide non-trivial examples of the rich physics associated with broken symmetries and non-trivial ordering \cite{17, 18}.

In contrast, the Meissner effect is conventionally understood as a “transverse” response \cite{19, 20}, where “longitudinal” collective modes are not thought to participate. This issue was addressed, partly clarified, in Ref. \cite{21}. There it was shown that in nonuniform superfluids the “longitudinal” collective modes can possibly appear in what are termed – in the context of uniform systems – “transverse” response functions. In addition, Ref. \cite{22} provided an explicit calculation of the electromagnetic (EM) response of the Fulde-Ferrell (FF) superfluid, which consists of finite-momentum Cooper pairs, and showed that the amplitude mode gives a significant contribution to the superfluid density. These issues motivate the current work, where we investigate the superfluid response for systems with nonuniform pairing, such as p-wave superfluids \cite{23, 24} and superconductors \cite{25, 26}, and we provide a more general understanding on the type of superconductor where collective modes can contribute to the Meissner response.

We define a uniform superfluid or superconductor to be one where the order parameter two-point function is both translation and rotation invariant. A nonuniform system is one that is not uniform, and as such it violates either one or both of the conditions above. In the case of uniform s-wave superconductors, gauge invariance and the uniformity of the gap establishes that there is no collective-mode contribution to the Meissner effect \cite{27}. When isotropy is broken, however, this argument needs to be revisited \cite{21}.

Phase fluctuations of the order parameter must be included to derive a gauge-invariant EM response \cite{5}. On top of this, one can also consider amplitude fluctuations of the order parameter, and these have been shown \cite{5} to be necessary to satisfy a thermodynamic sum rule, namely the compressibility sum rule \cite{28, 29}. Of particular interest is the response in p-wave superfluids \cite{23, 24, 25, 26} and also in systems with other pairing symmetries \cite{30}. A complete calculation of the EM response for a p-wave system, in the presence of Coulomb and amplitude and phase fluctuations of the order parameter, has not been, to the best of our knowledge, presented in the literature, and the question of the Meissner response for such a system was unaddressed in Ref. \cite{21}. In this paper we show that collective modes do not contribute to the Meissner effect in either uniform s-wave nor nonuniform p-wave superconductors. More generally, our results show that collective modes do not contribute to the Meissner effect, independent of the pairing symmetry, in any superconductor that does not display an external wavevector scale (e.g. finite-momentum pairing).

In order to derive this result, we develop a method for computing the gauge-invariant EM response of an electronic system with multiple collective modes present. Our analysis is based on an extension of the path-integral formulation of Ref. \cite{5} and matrix linear-response approaches of Refs. \cite{4, 28, 31}. One of our central results is to demonstrate how these collective modes can be incorporated in comprehensive and illuminating EM response...
tensors using singular-value decompositions. For pedagogical purposes we consider several examples of application, including the Coulomb screening in a normal metal and phase fluctuations in the EM response of a superfluid. We demonstrate the power of our formulation by obtaining the manifestly gauge-invariant EM response tensor for superconductors with amplitude, phase, and Coulomb fluctuations present. More generally, our results are applicable to a variety of scenarios beyond the scope of this work. They are relevant in any situation where energy scales compete, leading to intertwined ordering [17], or where symmetries provide multi-dimensional order parameters. The study of the concomitant contribution of distinct collective modes to the EM response tensor provides a direct method to access signatures of broken symmetries and non-trivial ordering.

The paper is organized as follows: in Sec. II we outline general formulae for the electromagnetic susceptibility tensors; a careful derivation of these formulae is provided in Appendix A. Following this, Sec. III provides a set of applications of these formulae, including: Coulomb screening, phase fluctuations in a superfluid, the gapping of phase modes in a superconductor by Coulomb screening, and finally the mixing of phase modes with amplitude-Higgs modes in a charged superconductor. This section contains our algebraic approach to screening by use of singular-value decompositions. Finally, Sec. IV addresses our discussions regarding the Meissner effect and we conclude in Sec. V. Appendices B-D provide further details on several relevant calculations.

II. ELECTROMAGNETIC RESPONSE TENSOR

The starting point of our analysis is a fermionic system subject to a set of collective fluctuating degrees of freedom. The latter are described by a set of generalized coordinates, denoted by $\Delta$, which should be thought of as a vector of Hubbard-Stratonovich decoupling fields. In the presence of an external EM probe $A$, we consider the dynamics of the EM response at the mean-field level, which is defined by the following conditions for each component $\Delta_a$ of $\Delta$:

$$\frac{\delta S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(x)}|_{\Delta=\Delta_{\text{mf}}[A]} = 0. \quad (2.1)$$

Here, $S_{\text{eff}}$ is the effective action for the fluctuating degrees of freedom, in the presence of the external EM probe, obtained after integration over the fermionic degrees of freedom [32]. The solutions to the mean-field equations, $\Delta_{\text{mf}}[A]$, are no longer arbitrary fluctuating degrees of freedom to be functionally integrated over, but rather they are functions determined by the external EM probe [4, 5, 33]. As a result, the mean-field EM response tensor reads

$$K_{\text{mf}}^{\mu \nu}(x, y) = \frac{\delta^2 S_{\text{eff}}[\Delta_{\text{mf}}[A], A]}{\delta A_{\mu}(x) \delta A_{\nu}(y)} \bigg|_{A=0}. \quad (2.2)$$

Note that $K^{\mu \nu}(x, y) = K^{\nu \mu}(y, x)$. In this paper imaginary time will be used and thus $A^\mu = (A_0, \mathbf{A}) = (iA_t, \mathbf{A})$.

To evaluate these derivatives it is necessary to use a functional chain rule and differentiate all terms with dependence on the vector potential. This manipulation, together with an application of the mean-field equations in Eq. (2.1), is presented in Appendix A: the result is a matrix form for the mean-field-level EM response, namely,

$$K_{\text{mf}}^{\mu \nu}(x, y) = Q^{\mu \nu}(x, y) - \int_{z, z'} \left\{ R^{\mu a}(x, z) \times \left[ S^{-1}(z, z') \right]^{ab} R^{b \nu}(z', y) \right\}, \quad (2.3)$$

where

$$Q^{\mu \nu}(x, y) = \frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta A_{\mu}(x) \delta A_{\nu}(y)} \bigg|_{A=0, \Delta=\Delta_{\text{mf}}[0]} , \quad (2.4)$$

$$R^{\mu a}(x, y) = R^{a \mu}(y, x) = \frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta A_{\mu}(x) \delta \Delta_a(y)} \bigg|_{A=0, \Delta=\Delta_{\text{mf}}[0]} , \quad (2.5)$$

and

$$S^{ab}(x, y) = \frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(x) \delta \Delta_b(y)} \bigg|_{A=0, \Delta=\Delta_{\text{mf}}[0]} . \quad (2.6)$$

Here, the derivatives with respective to the gauge field $A$ act only on the explicit vector-potential dependence. In the second contribution of Eq. (2.3), we emphasize that the matrix $S^{ab}$ must be computed first, as in Eq. (2.6), and then inverted before being inserted into Eq. (2.3). In other words, Eq. (2.3) does not involve the inverse of each matrix element of Eq. (2.6), but rather the elements of the inverse of the matrix itself.

This expression contains several insightful properties. First, it manifestly decouples into two contributions which correspond, respectively, to the bubble and collective-mode linear responses. Second, this expression is reparameterization covariant, i.e., it does not change form under a basis transformation of $\Delta$. This means that all fluctuations are considered symmetrically, in an unbiased manner. In the context of superconductivity, for example, Eq. (2.3) can be equally used for considering fluctuations in the real and imaginary part of the superconducting pairing strength [31], or for fluctuations in the radial and phase degrees of freedom, as we shall do later in the paper. Third, by writing this expression in real space it affords greater generality and can thus be used, for example, in the presence of either impurities or defects occurring in collective-mode order parameters. For a translation-invariant system, the momentum-space representation is more tractable and reads

$$K_{\text{mf}}^{\mu \nu}(q) = Q^{\mu \nu}(q) - R^{ma}(q) \left[ S^{-1}(q) \right]^{ab} R^{b \nu}(q), \quad (2.7)$$

where, for example,
\[ Q^{\mu\nu}(x, y) = Q^{\mu\nu}(x - y) = \int_{q} e^{-iq \cdot (x - y)} Q^{\mu\nu}(q). \quad (2.8) \]

We use the short-hand notation \( \int_{q} = T L^{d} \sum_{\Omega_{m}} \int \frac{dq}{(2\pi)^{d}} \), where \( L \) is a length scale, \( d \) is the number of spatial dimensions, \( T \) is the temperature, and \( \Omega_{m} \) is a bosonic Matsubara frequency. Natural units \( c = \hbar = k_{B} = 1 \) are used throughout the paper.

### III. GENERAL APPLICATIONS

In this section we present several applications of Eq. (2.7). For the benefit of the reader, in the following subsections we take a pedagogical approach and start with a rather detailed calculation of the application of Eq. (2.3) in two familiar scenarios: IIIA–Electrostatic screening and IIIB–gauge-invariant response in superfluids due to phase fluctuations. With the mathematical procedures well established, we will then move on at a progressively faster pace: in III C we study the next simplest possible scenario – a superconductor with phase fluctuations – and here we introduce the concept of folding the effects of competing fluctuations using singular-value decompositions. The dénouement of this section is III D, where we put all this methodology together to compute the EM response tensor in the non-trivial case of concomitantly fluctuating Coulomb and superconducting phase and amplitude degrees of freedom. To clarify our terminology, a superconductor is a charged system with Coulomb interactions present and a superfluid is a neutral system.

#### A. Screening due to electrostatic interactions

Consider an interacting electronic system in \( D = d + 1 \) spacetime dimensions with an action given by

\[
S[A] = -\int d^{D}x d^{D}x' \psi_{\sigma}^{\dagger}(x) G_{0}^{-1}[A](x, x') \psi_{\sigma}(x') \\
+ \frac{e^{2}}{2} \int d^{D}x d^{D}x' \delta n(x) V(x - x') \delta n(x') \\
+ ie \int d^{D}x A_{\mu}(x) n_{0},
\]

where \( \delta n(x) = \psi_{\sigma}^{\dagger}(x) \psi_{\sigma}(x) - n_{0} \), with \( n_{0} \) the constant background density, \( \sigma = \uparrow, \downarrow \) is a spin index (summed if repeated) and the inverse Green’s function is

\[
G_{0}^{-1}[A](x, x') = -\left( \partial_{\tau} - ieA_{\tau}(x) + \hbar (\hat{p} - eA) \right) \delta(x - x').
\]

The single-particle Hamiltonian, denoted by \( \hbar (\hat{p}) \), is kept general at this stage. For concreteness, we assume instantaneous interactions: \( V(x - x') = V(x - x') \delta(\tau - \tau') \).

Throughout the paper we shall interchangeably refer to electronic density fluctuations as Coulomb fluctuations. The generating functional for electromagnetic response is then

\[
Z[A] = \int D[\psi_{\uparrow}, \psi_{\downarrow}] e^{-S[A]}. \quad (3.3)
\]

We are interested in how nonuniform charge distributions affect the EM response of this system. Thus it is natural to consider decoupling the electrostatic interaction terms via a Hubbard-Stratonovich decomposition as

\[
Z[A] \sim \int D[\phi,e] e^{-S_{\text{eff}}[\phi,A]}, \quad (3.4)
\]

Defining \( \beta = 1/T \) and \( G^{-1}[\phi,A] = G^{-1}[A_{\uparrow} + \phi, A] \), the effective action is

\[
S_{\text{eff}}[\phi,A] = \int d^{D}x d^{D-1}x' \frac{\phi(x, \tau) \phi(x', \tau)}{2V(x - x')} \\
+ ie \int d^{D}x (A_{\mu}(x) + \phi(x)) n_{0} \\
- Tr \ln \left( -\beta G_{0}^{-1}[\phi,A] \right). \quad (3.5)
\]

The capitalized trace denotes a trace over all space-time/momentum-frequency and internal (uncapitalized trace) degrees of freedom:

\[
Tr \ln \left( -\beta G_{0}^{-1}[\phi,A] \right) = \int d^{D}x tr \left[ \ln \left( -\beta G_{0}^{-1}[\phi,A] \right) \right] x. \quad (3.6)
\]

In this language, we obtain the building blocks for Eq. (2.3) (which are tantamount to undressed polarization tensors). In fact, due to translation invariance, we can focus on the expressions in momentum space used in Eq. (2.7). For instance,

\[
Q^{\mu\nu}(q) \equiv \frac{\delta^{2} S_{\text{eff}}[\phi,A]}{\delta A_{\mu}(-q) \delta A_{\nu}(q)} \bigg|_{A_{\phi}=0} = -\frac{\delta^{2} Tr \ln \left( -\beta G_{0}^{-1}[\phi,A] \right)}{\delta A_{\mu}(-q) \delta A_{\nu}(q)} \bigg|_{A_{\phi}=0}. \quad (3.7)
\]

Similarly, noticing that \( A_{0} = iA_{\tau} \) and that all terms involving \( \phi \) appear in the Green’s function as \( iA_{\tau} + i\phi \), one finds

\[
R^{\mu\tau}(q) \equiv \frac{\delta^{2} S_{\text{eff}}[\phi,A]}{\delta A_{\mu}(-q) \delta \phi(q)} \bigg|_{A_{\phi}=0} = iQ^{\mu 0}(q), \quad (3.8)
\]

\[
S^{\phi\phi}(q) = \frac{\delta^{2} S_{\text{eff}}[\Delta,A]}{\delta \phi(-q) \delta \phi(q)} \bigg|_{A_{\phi}=0} = V^{-1}(q) - Q^{00}(q). \quad (3.9)
\]

Conveniently, all building blocks can be expressed in terms of the undressed polarization tensor \( Q^{\mu\nu}(q) \). An
in-depth analysis of these expressions is provided in Appendix B.

Applying Eq. (2.7) now becomes a simple matter (we drop the \( q \)-dependence label for simplicity):

\[
K_{\text{mf}}^{\mu \nu} = Q^{\mu \nu} - (iQ^{00})(V^{-1} - Q^{00})^{-1}(iQ^{0 \nu}) \\
= Q^{\mu \nu} + \frac{Q^{00}VQ^{00}}{1 - VQ^{00}} \equiv \tilde{Q}^{\mu \nu}.
\] (3.10)

The last definition will be used throughout later sections of the paper. The above result reproduces the screening effect of Coulomb fluctuations. In particular, the RPA charge-charge susceptibility [29] is obtained:

\[
K_{\text{mf}}^{00} = \frac{Q^{00}}{1 - VQ^{00}}.
\] (3.11)

\section*{B. EM response for superfluids (with no amplitude fluctuations)}

Another simple application of Eq. (2.7) concerns the gauge-invariant EM response tensor for superfluids with phase fluctuations of the order parameter. In superfluids where the mean-field order parameter takes on a finite vacuum expectation value the global \( U(1) \) symmetry is spontaneously broken. To restore gauge-invariance, the phase fluctuations of the order parameter must be included. In this section we consider a superfluid where the amplitude of the order parameter is rigidly pinned down to its mean-field value, but allow the phase to depend on the external EM field.

It is straightforward to analyze this scenario with our present approach. Consider a set of non-relativistic spin-\( \frac{1}{2} \) particles, with free Hamiltonian \( \hat{h}(\hat{p}) = \hat{p}^2 / (2m) - \mu \), interacting instantaneously with each other via an attractive, translation-invariant, but possibly anisotropic potential \( g(x - x') \). In the presence of an external probe field \( A \), the action reads

\[
S[A] = -\int d^Dx d^Dx' \bar{\psi}_n^\dagger(x) G_0^{-1}[A](x, x') \psi_n(x') \\
- \int d^Dx d^Dx' \bar{\psi}_n^\dagger(x) \psi_n^\dagger(x') g(x - x') \bar{\psi}_n(x') \psi_n(x) \\
+ ie \int d^Dx A_i n_0.
\] (3.12)

Here, \( g(x - x') = g(x - x') \delta(\tau - \tau') \).

Preparing again for the mean-field treatment of the problem, we now perform a Hubbard-Stratonovich decomposition in the Cooper channel to arrive at the generating functional

\[
Z[A] \sim \int D[\Delta, \Delta^*] D[\psi^\dagger, \psi] e^{-S_{\text{bos}} - S_{\text{cl}}},
\] (3.13)

where the bosonic contribution to the action is

\[
S_{\text{bos}} = ie \int d^Dx A_i n_0 + \int d^Dx \frac{|\Delta(x, x', \tau)|^2}{g(x - x')} \\
\] (3.14)

and the electronic contribution is

\[
S_{\text{el}} = -\int d^Dx d^Dx' \bar{\psi}_n^\dagger(x) G_0^{-1}[A](x, x') \psi_n(x') \\
- \int d^Dx d^Dx' \bar{\psi}_n^\dagger(x, \tau) \Delta(x, x', \tau) \psi_n^\dagger(x', \tau) + \text{h.c.}.
\] (3.15)

Before integrating out the fermions, remember that the symmetry of the interaction potential \( g(x - x') \) is decisive in determining the symmetry structure of the pairing field. Due to the homogeneity of the problem (in the absence of strong driving external EM fields), it is advantageous to use relative and center-of-mass coordinates to describe the pairing field:

\[
\Delta(x, x', \tau) \rightarrow \Delta \left( x - x', \frac{x + x'}{2}, \tau \right).
\] (3.16)

We ignore spin-orbit coupling. In this case, spherical anisotropy in the pairing potential can be captured in a gradient expansion of \( \Delta \).

\[
\Delta \left( x - x', \frac{x + x'}{2}, \tau \right) \\
= \Delta_s \left( \frac{x + x'}{2}, \tau \right) e^{i\Phi_s \left( \frac{x + x'}{2}, \tau \right)} \delta(x - x') \\
+ \Delta_p \left( \frac{x + x'}{2}, \tau \right) e^{i\Phi_p \left( \frac{x + x'}{2}, \tau \right)} (\partial_x + i\partial_y) \delta(x - x') + ..., 
\] (3.17)

where we favor an amplitude-phase coordinate choice. In general, the pairing potential will select only one term in Eq. (3.17); the structure we chose for the interaction, in fact, favors opposite-spin pairing by construction. Nevertheless, we can remain fairly general and write

\[
S_{\text{bos}} = ie \int d^Dx A_i n_0 + \int d^Dx \frac{|\Delta(x, \tau)|^2}{2\bar{g}},
\] (3.18)

where \( \bar{g} \) is a renormalized value for \( g \), and

\[
S_{\text{el}} = -\int d^Dx d^Dx' \bar{\psi}_n^\dagger(x) G_0^{-1}[A](x, x') \psi_n(x') \\
- \int d^Dx \left[ \Delta(x, \tau) \psi_n^\dagger(x, \tau) \hat{D} \psi_n^\dagger(x, \tau) + \text{h.c.} \right],
\] (3.19)

where \( \Delta(x, \tau) = \rho(x) e^{i\theta(x)} \) for a general amplitude and phase and \( \hat{D} \) corresponds to a differential operator that depends on the symmetry channel. In Appendix C we consider an explicit application of this to a spinless \( p \)-wave problem.
We are now ready to integrate out the fermions; introducing a Nambu doubled spinor $\Psi = \left( \psi^\dagger, \psi_\downarrow, \psi_\uparrow^\dagger, \psi^\dagger \right)^T$, the electronic part of the action becomes

$$S_{\text{el}} = -\frac{1}{2} \int d^Dx d^Dx' \Psi^\dagger (x) \mathcal{G}^{-1} [A] (x, x') \Psi (x') \tag{3.20}$$

\[ \mathcal{G}^{-1} [A] (x, x') = - \left[ \partial_\tau - ie\tilde A_\tau + \left( \frac{\rho - e\tilde A}{2m} \right)^2 - \mu \right] \]

\[- \rho (x) i\sigma_y \tilde D^\dagger \left[ \partial_\tau + ie\tilde A_\tau - \left( \frac{\rho + e\tilde A}{2m} \right) - \mu \right] \delta (x - x'), \tag{3.21} \]

$\sigma_y$ acts on the spin degrees of freedom, and we have rotated away the superconducting phase, which is conveniently absorbed by the gauge fields as $\tilde A_\mu = A_\mu - \frac{1}{2e} \partial_\mu \theta$. The generating functional thus becomes

$$Z [A] \sim \int \mathcal{D} [\Delta, \Delta^\ast] e^{-S_{\text{eff}} [\Delta, \Delta^\ast, A]}, \tag{3.22}$$

where the effective action is (dropping the $A$-dependence label)

$$S_{\text{eff}} [\Delta, \Delta^\ast, A] = S_{\text{bos}} - \frac{1}{2} \text{Tr} \ln (-\beta \mathcal{G}^{-1}), \tag{3.23}$$

with $S_{\text{bos}}$ as in Eq. (3.18) and one should keep in mind the factor of $\frac{1}{\beta}$ due to Nambu doubling.

At this point we consider the mean-field response. In this section, we will neglect fluctuations of the superconducting amplitude, setting $\rho (x) \to \rho_0$. It is then possible to use the relationship between $\bar{A}_\mu$ and $A_\mu$ to write

$$\delta S_{\text{eff}} [\theta, A] = \int dy \delta S_{\text{eff}} [\theta, A] \frac{\delta \rho_0 \theta (y)}{\delta \theta (x)} \]

$$= -\partial_\alpha \delta S_{\text{eff}} [\theta, A] \frac{\delta \rho_0 \theta (x)}{\delta \partial_\alpha \theta (x)} \]

$$= \frac{1}{2e} \partial_\alpha \delta S_{\text{eff}} [\theta, A] \frac{\delta \rho_0 \theta (x)}{\delta \bar{A}_\alpha (x)}. \tag{3.24}$$

The factor of $2e$ can be safely absorbed as it will drop out from the correlation functions; we will omit it from now on. This allows us to once again write all the momentum-space tensors in terms of the undressed polarization tensors $Q^{\mu \nu}$, namely,

$$R^{\theta \rho} (q) = iQ^{\rho \beta} (q) q_\beta, \tag{3.25}$$

$$R^{\theta \nu} (q) = -iq_\nu Q^{\alpha \nu} (q), \tag{3.26}$$

$$S^{\theta \theta} (q) = q_\lambda Q^{\lambda \sigma} (q) q_\sigma. \tag{3.27}$$

At the mean-field level $\theta$ is a constant and drops out from the Green’s functions. Notice that the Green’s functions appearing in $Q^{\mu \nu} (q)$ in this case correspond to Eq. (3.21) with $\bar{A}_\mu = 0$ and $\rho (x) \to \rho_0$. Implementing Eq. (2.7), the EM response is then

$$K^{\mu \nu}_{\text{mf}} = \Pi^{\mu \nu} = \frac{Q^{\mu \beta} q_\beta q_\alpha Q^{\alpha \nu}}{q_\lambda Q^{\lambda \sigma} q_\sigma} \equiv \Pi^{\mu \nu}. \tag{3.28}$$

This is the general form of the EM response tensor for a neutral superfluid, independent of the pairing symmetry. The gapless fluctuating phase degree of freedom is crucial to ensure gauge invariance, which the form above manifestly obeys: $q_\alpha K_{\text{mf}}^{\mu \nu} (q) = K_{\text{mf}}^{\mu \nu} (q) q_\nu = 0$. Setting $q_\lambda Q^{\lambda \sigma} q_\sigma = 0$ recovers the well-known result of Anderson and Bogoliubov [34, 35]; the EM response has a pole corresponding to a long-wavelength sound mode (with speed $c_s = v_F / \sqrt{3}$ at $T = 0$) induced by phase fluctuations of the order parameter.

C. EM response for superconductors (with no amplitude fluctuations)

With the previous results established, for our first non-trivial application of Eq. (2.7) we consider a charged superconductor with both phase and Coulomb fluctuations present. This problem was also considered in Ref. [25], in the context of the EM response of a $p$-wave superconductor, via sequential functional integration of the Coulomb and phase degrees of freedom. It is natural to ask what the form of the EM response would be if this procedure were performed in the opposite order, and this will be addressed in what follows. In our case, the results from
the previous sections allow the response to be written as

\[
K_{\text{mf}}^{\mu\nu} = Q_{\mu\nu} - \frac{1}{(V^{-1} - Q^{00}) q_\alpha q_\sigma} \left( \begin{array}{c} \tilde{Q}^{\mu\nu} \\ \tilde{Q}^{0\sigma} \end{array} \right)^T \frac{1}{(V^{-1} - Q^{00}) q_\beta q_\sigma} \left( \begin{array}{c} \tilde{Q}^{\mu0} \\ \tilde{Q}^{0\beta} \end{array} \right).
\]

(3.29)

The Coulomb-screened EM response tensor \( \tilde{Q}^{\lambda\sigma} \) of Sec. IIIA naturally appears here in the denominator.

While Eq. (3.29) treats the Coulomb- and phase-screened responses of a charged superconductor in a symmetric fashion, the present form is not totally satisfactory. In particular, gauge invariance is not manifest, and it may be advantageous to recover similar results found in the previous section, as well as the polaritonic resonances of the EM response. To accomplish this, we have to “bias” the above expression towards either a Coulomb-screened type of object or a phase-screened type of object. An analogy from the process of Ref. [25] would be to consider integrating out first either the electrostatic Coulomb field or the phase degree of freedom.

Let us make this procedure more explicit. With a few manipulations, we may explicitly rewrite \( Q_{\mu\nu} \) in terms of its Coulomb-screened version \( \tilde{Q}_{\mu\nu} \) so that Eq. (3.29) then has the form

\[
K_{\text{mf}}^{\mu\nu} = \tilde{Q}_{\mu\nu} - \frac{q_\alpha q_\beta}{q_\lambda q_\sigma} \left( \begin{array}{c} Q^{\mu0} \\ Q^{\nu\sigma} \end{array} \right)^T \frac{1}{(V^{-1} - Q^{00}) q_\beta q_\sigma} \left( \begin{array}{c} Q^{\mu0} \\ Q^{0\beta} \end{array} \right) \left( \begin{array}{c} \tilde{Q}^{\mu\nu} \\ \tilde{Q}^{0\sigma} \end{array} \right).
\]

(3.30)

The 2 \times 2 matrix appearing in the EM response now has zero determinant: it is a singular matrix, which can be expressed using a singular-value decomposition (SVD). Consider the following matrix

\[
M = \begin{pmatrix} ab & a \\ b & 1 \end{pmatrix}.
\]

(3.31)

Define the matrices \( U, V, \) and \( D \) by

\[
U = \begin{pmatrix} a & a \\ 1 & -|a| \end{pmatrix}, \quad V = \begin{pmatrix} b^* & b^* \\ 1 & -|b| \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

(3.32)

The matrix \( M \) can then be written as \( M = UDV^T \). By matching the coefficients \( a \) and \( b \) with the coefficients in Eq. (3.30), one obtains

\[
K_{\text{mf}}^{\mu\nu} = \tilde{Q}_{\mu\nu} - \frac{q_\alpha q_\beta}{q_\lambda q_\sigma} \tilde{Q}_{\mu\nu} \equiv \Pi_{\mu\nu}.
\]

(3.33)

Here we have “biased” the matrix expression in Eq. (3.30) into the simpler equation above. It assumes the form of an EM response tensor in the presence of phase fluctuations, as in Eq. (3.28), but now the EM polarization tensors are substituted by their Coulomb-screened versions: \( Q_{\mu\nu} \rightarrow \tilde{Q}_{\mu\nu} \). This expression is manifestly gauge invariant as in Eq. (3.28). Interestingly, this biasing process can easily be done in the reverse manner. In performing similar manipulations to arrive at Eq. (3.30), if we had first exchanged \( Q_{\mu\nu} \) for \( \Pi_{\mu\nu} \), instead of \( \tilde{Q}_{\mu\nu} \), then it is a simple exercise to show that by an analogue SVD the EM response tensor obtained reads

\[
K_{\text{mf}}^{\mu\nu} = \Pi_{\mu\nu} + \frac{\Pi_{\mu0}^{\nu0} + \Pi_{0\nu}^{\mu0}}{1 - VQ^{00}}.
\]

(3.34)

This expression assumes a Coulomb-screened form, where each tensor participating has been replaced by its “phase-screened” version: \( Q_{\mu\nu} \rightarrow \tilde{Q}_{\mu\nu} \). Evidently, since each \( \Pi_{\mu\nu} \) is gauge invariant by itself, the whole expression above is gauge invariant again. Naturally, both expressions for \( K_{\text{mf}}^{\mu\nu} \) above are equivalent.

Thus, we have introduced a process of folding the effects of each fluctuating field via an SVD of the response tensors. This process clearly biases the form of \( K_{\text{mf}}^{\mu\nu} \), although it brings simplification. The denominators of the final form of these response tensors contain the polaritonic resonances of the dielectric functions [28, 31, 36]. Equating the two denominators equal to zero

\[
q_\lambda \tilde{Q}^{\lambda\sigma} q_\sigma = 0 = 1 - VQ^{00},
\]

(3.35)

one obtains the well-known Carlson-Goldman (CG) mode [31, 37, 38], where plasmons dress the phase fluctuation poles, gapping the phase modes of charged superconductors. At \( T = 0 \) this results in solely a (double) plasmon mode, whereas in the vicinity of \( T \sim T_c \) there is a soft mode (which was originally [37, 38] termed the CG mode) and a plasmon mode [31]. Note that the exact relation between the two denominators is:

\[
q_\lambda \tilde{Q}^{\lambda\sigma} (1 - VQ^{00}) = q_\lambda \tilde{Q}^{\lambda\sigma} q_\sigma (1 - VQ^{00}).
\]

D. EM response for superconductors (with amplitude fluctuations)

Returning to Eq. (3.21), we now include the fluctuations in \( \rho(x) \). Contrary to the phase and Coulomb responses, the amplitude part cannot be written solely in terms of the unscreened EM response bubble \( Q_{\mu\nu}(q) \). The additional objects which must be defined for calculating the EM response functions read as follows

\[
\frac{\delta^2 S_{\text{eff}}}{\delta \rho(q)}_{A=|0,\Delta=\Delta_{\text{mf}}[0]}^{\Delta \rho(-q) \delta |q} \equiv S^{\rho\rho}(q),
\]

(3.36)

\[
\frac{\delta^2 S_{\text{eff}}}{\delta \theta(q)}_{A=|0,\Delta=\Delta_{\text{mf}}[0]}^{\Delta \theta(-q) \delta |q} \equiv S^{\rho\rho}(q) = i q_\beta R^{\rho\beta}(q),
\]

(3.37)
and similarly

\[
\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \rho(-q) \delta A_\mu(q)} \bigg|_{A=0, \Delta=\Delta_{\text{mf}[0]}} = R^{\mu\nu}(q), \quad (3.38)
\]

\[
\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \rho(-q) \delta \varphi(q)} \bigg|_{A=0, \Delta=\Delta_{\text{mf}[0]}} = S^{\rho\varphi}(q) = iR^{\rho\varphi}(q). \quad (3.39)
\]

Finally, tensors with both bars and tildes are interpreted to mean first evaluate the tensors with respect to the outer screening symbol and then with respect to the inner screening type. To be concrete, as an example we have

\[
\overline{Q}^{\alpha\beta} = \overline{Q}^{\alpha\beta} - \frac{\overline{R}^{\alpha\beta}}{S^{\rho\rho}}. \quad (3.44)
\]

Using these expressions we can now perform the SVD process as in the previous sections, the only requirement is to choose a biasing order in which we want to take into account the influence of each type of fluctuation. For example, taking into account the inversion of the matrix \( S(q) \) and the determinant above, we obtain

\[
K^{\mu\nu}_{\text{mf}} = Q^{\mu\nu} - \frac{q_\alpha q_\beta}{\overline{Q}^{\alpha\beta}} q_\sigma \times \frac{R^{\mu\nu}}{Q^{\mu\nu}} T \left( \begin{array}{ccc} (V^{-1} - Q^{00}) Q^{\alpha\beta} & Q^{\alpha\beta} R^{0\nu} & (V^{-1} - Q^{00}) \overline{Q}^{\alpha\beta} \\ (V^{-1} - Q^{00}) S^{\rho\rho} & (V^{-1} - Q^{00}) S^{\rho\rho} & (V^{-1} - Q^{00}) S^{\rho\rho} \\ (V^{-1} - Q^{00}) R^{\rho\nu} \overline{Q}^{\alpha\beta} & (V^{-1} - Q^{00}) R^{\rho\nu} \overline{Q}^{\alpha\beta} & (V^{-1} - Q^{00}) R^{\rho\nu} \overline{Q}^{\alpha\beta} \end{array} \right) \left( \begin{array}{ccc} R^{\rho\nu} \\ Q^{0\nu} \\ Q^{\alpha\nu} \end{array} \right). \quad (3.47)
\]

Now we focus on the first term \( Q^{\mu\nu} \). Introducing the effects of amplitude fluctuations first (“bar” variables) and

Also note that just as \( V^{-1}(q) \) contributed to \( S^{\rho\varphi}(q) \) [c.f. Eq. (3.9)], the “mass” contribution for \( \rho(x) \) in the Hubbard-Stratonovich field in Eq. (3.18) implies that \( \overline{g}^{-1} \) contributes to \( S^{\rho\rho}(q) \).

The EM response tensor now becomes

\[
K^{\mu\nu}_{\text{mf}}(q) = Q^{\mu\nu} - \frac{q_\alpha q_\beta}{\overline{Q}^{\alpha\beta}} q_\sigma.
\]
subsequently the regular screening from Coulomb fluctuations ("tilde" variables), a straightforward calculation and simplification using the relations in Eq. (3.46) results in

\[
K^{\mu\nu}_{mf} = \overline{Q}^{\mu
u} - \frac{q_{\alpha}q_{\beta}}{q_{\lambda}Q_{\lambda\sigma}} q_{\sigma} \times \left( \begin{array}{c} R^{\rho\rho} \\ Q^{v0} \\ Q^{\mu\beta} \end{array} \right) \left( \begin{array}{c} \overline{R}^{\alpha\beta} \\ \overline{Q}^{\nu0} \\ \overline{Q}^{\mu\sigma} \end{array} \right)^T \left( \begin{array}{c} \overline{R}^{\alpha\beta} \\ \overline{Q}^{\nu0} \\ \overline{Q}^{\mu\sigma} \end{array} \right)
\]

This matrix is now of the form

\[
M = \begin{pmatrix}
ab & -ad & -a \\
-bc & cd & c \\
-b & d & 1
\end{pmatrix},
\]

where

\[
a = \frac{\overline{Q}^{\beta\rho}}{S^{\rho\rho}}, \quad b = \frac{\overline{R}^{\beta\rho}}{S^{\rho\rho}},
\]

\[
c = \frac{\overline{Q}^{v0}}{V^{-1} - Q^{v0}}, \quad d = \frac{\overline{Q}^{v0}}{V^{-1} - Q^{v0}}.
\]

It displays two linearly dependent rows, thus suggesting the singular-value decomposition. Performing the SVD and simplifying the result gives

\[
K^{\mu\nu}_{mf} = \overline{Q}^{\mu\nu} - \frac{Q^{\beta\rho} q_{\beta} q_{\rho} \overline{Q}}{q_{\lambda} Q^{\lambda\sigma} q_{\sigma}} = \overline{\Pi}^{\mu\nu}.
\]

Setting \(q_{\lambda} Q^{\lambda\sigma} q_{\sigma} = 0\) gives the collective mode dispersion for the polaritons induced by simultaneous Coulomb, phase, and amplitude fluctuations. Again, gauge invariance in the SVD-simplified EM response in Eq. (3.51) is manifest:

\[
q_{\mu} K^{\mu\nu}_{mf} = q_{\mu} \overline{Q}^{\mu\nu} - \frac{q_{\mu} \overline{Q}^{\beta\rho} q_{\beta} q_{\rho} \overline{Q}}{q_{\lambda} Q^{\lambda\sigma} q_{\sigma}} = 0.
\]

As in the previous section, other equivalent forms for the EM response can be obtained by reversing the order in the SVD processes. For example, \(K^{\mu\nu}_{mf} = \overline{\Pi}^{\mu\nu} = \overline{\Pi}^{\mu\nu}\).

\[\text{phase, and Coulomb fluctuations incorporated. It was shown in the previous section that the EM response for a system with all these three types of fluctuations can be compactly written as in Eq. (3.51). Here we will use this formula to study the Meissner response for both s- and p-wave systems. The Kubo formula for the superfluid density tensor is [27]}

\[
\frac{e^2}{m} n_{s} = \lim_{q_{\mu} \rightarrow 0} K^{\mu\nu}_{mf} (\Omega = 0, q_{\nu})
\]

with no implicit index summation. It is crucial that the static limit, \(\Omega = 0\), is taken before the long-wavelength limit \(q \rightarrow 0\) is considered. This particular order of limits is appropriate for a thermodynamic quantity, whereas the converse procedure is apt for the calculation of optical properties, namely the DC electrical conductivity for instance. For nonuniform systems, the limit \(q \rightarrow 0\) must also be carefully specified. To ascertain the appropriate definition, recall that in the presence of an external EM vector potential \(A_{\mu}\), the EM current is \(j_{\mu}(x) = \int_{x'} K^{\mu\nu}(x, x') A_{\nu}(x')\). The continuity equation is \(\partial_{\mu} j_{\mu} = 0\); this statement enforces conservation of global particle number (global U(1) symmetry) for a neutral superfluid, whereas for a charged system it enforces conservation of charge. In terms of the response kernel, this equation becomes \(\partial_{\mu} K^{\mu\nu} A_{\nu} = 0\). The solution to this equation, for an arbitrary \(A_{\nu}\), is to require a gauge-invariant EM response: \(\partial_{\mu} K^{\mu\nu} = 0\), which in momentum space reads \(q_{\mu} K^{\mu\nu} = 0\). As shown in the previous section, the SVD approach enables this to be manifestly satisfied.

To compute \(n_{s}\) it is convenient to work in the gauge where \(\partial_{\mu} A_{\mu} = 0\) (Lorenz gauge), which reduces to the Coulomb gauge \(\nabla \cdot A = 0\) in the static limit. The momentum-space form of the Coulomb gauge is \(q \cdot A = 0\).

In deriving the superfluid density \(n_{s}^{ii}\), only the \(i\)th component of the vector field must be non-vanishing: \(A_{i} \neq 0\). The Coulomb gauge condition then reduces to \(q_{i} A_{i} = 0\), demanding \(q_{i} = 0\). The other momentum components go to zero only in the limit. Thus, the appropriate Kubo formula for \(n_{s}^{ii}\) is

\[
\frac{e^2}{m} n_{s}^{xx} = \lim_{q_{\mu} \rightarrow 0} K^{xx}_{mf} (\Omega = 0, q_{\nu})
\]
This Kubo formula explains why the superfluid density is often termed a “transverse” response [27, 29]. In the particular case of nonuniform superfluids, however, the appellation “transverse” loses its significance. The importance of computing the superfluid density in the appropriate limiting fashion was discussed in Ref. [22], where it was shown that for the Fulde-Ferrell superfluid the amplitude collective mode contributes to the superfluid density. A general argument for why collective modes do not need to be considered in the superfluid density response of uniform superfluids is as follows [27]. In the presence of the external vector potential \( \mathbf{A} \), the order parameter can be expanded to quadratic order in \( \mathbf{A} \) as

\[
\Delta [A] = \Delta [A = 0] + \Delta^{(1)} [A] + O (A^2),
\]

(4.3)

Since the order parameter \( \Delta \) is a scalar, whereas the vector potential \( \mathbf{A} \) is a vector, \( \Delta \) can depend only on scalar-valued functions of \( \mathbf{A} \). For a uniform superfluid, the only such scalar quantity is \( \nabla \cdot \mathbf{A} \). In the Coulomb gauge, where \( \nabla \cdot \mathbf{A} = 0 \), it follows that \( \Delta^{(1)} = 0 \). Thus, collective modes do not contribute to the superfluid density in a uniform superfluid. In the case of a nonuniform superfluid, there are potentially other scalar quantities that depend on \( \mathbf{A} \) and thus \( \Delta^{(1)} \) need not be zero. The next section provides an explicit calculation of the superfluid density for \( s \)- and \( p \)-wave superfluids with amplitude, phase, and Coulomb interactions.

### B. Explicit superfluid density calculation

First consider the case of a uniform \( s \)-wave superfluid. Without loss of generality, since the system is uniform we only need to study the response in one direction, say \( \hat{x} \). Using the formalism developed in the previous sections, the superfluid density is given by

\[
\frac{e^2}{m} n_s^{xx} = \lim_{q_x = 0, q_y \to 0} \frac{-\nabla^{xx} q_x q_y q_l^{ij}}{q_k^{ij} q_l^{ij}},
\]

(4.4)

In the small-momentum limit, \( R^{ij} (0, \mathbf{q} \to 0) = 0 \); this is because in this limit the tensor structure requires \( R^{ij} (0, \mathbf{q} \to 0) \sim q^1 \to 0 \). Thus, the generalized response functions are

\[
\frac{Q^{ij}}{\mathbf{V} - 1 - \mathbf{Q}^{00}} = Q^{ij}.
\]

(4.5)

As a result, the superfluid density is

\[
\frac{e^2}{m} n_s^{xx} = \lim_{q_x = 0, q_y \to 0} Q^{xx}.
\]

(4.6)

This proves that without any particular assumptions about particle-hole symmetry, i.e., whether or not the amplitude and Coulomb mode decouple \( (R^{00} \neq 0) \) [39], the superfluid density for an \( s \)-wave system has no contributions from amplitude, phase, or Coulomb collective modes. This is an explicit proof of the argument presented in the previous section.

Now consider a spinless-\( (p + ip) \) superfluid in two spatial dimensions. The \( x \) and \( y \) responses are equivalent, thus we again need only to consider the former. The superfluid density is as given in Eq. (4.4). Again \( R^{ij} (0, \mathbf{q} \to 0) = 0 \) remains true, and thus

\[
\frac{e^2}{m} n_s^{xx} = \lim_{q_x = 0, q_y \to 0} Q^{xx}.
\]

(4.7)

This particular limit is computed as shown below. After performing the Matsubara frequency summation, the response function is [25, 28]:

\[
\frac{Q^{ij} (i \Omega_m, \mathbf{q})}{2} = \frac{e^2}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{p^i p^j}{m} \left[ 1 + \frac{\xi_p^{+} \xi_p^{-} + \Delta_p^{2} p_p \cdot p_p / p_p^2}{E_p^+ E_p^-} \right] \frac{E_p^+ - E_p^-}{(E_p^+ - E_p^-)^2 - (i\Omega_m)^2} \left[ f (E_p^+) - f (E_p^-) \right] + \frac{n e^2}{m} \delta^{ij},
\]

where \( p_{\pm} = p \pm q / 2, \xi_p^{\pm} \equiv \xi_{p \pm q / 2}, E_p^{\pm} \equiv E_{p \pm q / 2}, \) with \( \xi_p = p^2 / (2m) - \mu, E_p = \sqrt{\xi_p^2 + \Delta_p^2 p_p^2 / p_p^2}, \) and \( n \) is the total number density. Taking the appropriate frequency and momentum limits results in

\[
\frac{e^2}{m} n_s^{xx} = e^2 \left[ \frac{n}{m} + \int \frac{d^2 p}{(2\pi)^2} \left( p^2 / m \right)^2 \frac{\partial f (E_p)}{\partial E_p} \right] \]

\[
= e^2 \int \frac{d^2 p}{(2\pi)^2} \left( p^2 / m \right)^2 \frac{\Delta_p^2 p_p^2 / p_p^2}{E_p^2} \left[ 1 - 2 f (E_p) + \frac{\partial f (E_p)}{\partial E_p} \right].
\]

(4.9)
In general, for a superfluid system with only one external momentum, the momentum $q$ of the external vector potential $\mathbf{A}$, the EM response can be decomposed into terms comprised of $\delta^{ij}$ and $q_i q_j / q^2$. In the limit $q \to 0$, as defined above, it follows that the off-diagonal terms vanish and thus the superfluid density reduces to the standard undressed bubble term. Unless there are other external vectors that can couple to the vector potential, the superfluid density always reduces to the undressed bubble term. This statement is a generalization of the analysis in the previous section, which considered only uniform superfluids; here we extend the veracity of the previous proof to include all kinds of superfluids without other external vectors that couple to the vector potential.

### C. Transverse and longitudinal responses

In Ref. [21] the EM response for nonuniform superfluids without amplitude fluctuations was derived. This particular article highlighted that for such superfluids the collective modes are, in general, no longer solely “longitudinal”, and moreover these modes can be important in what are conventionally termed “transverse” response functions in the case of uniform systems. In this section we show that our generalized formula reproduces the particular case considered in Ref. [21], namely, a neutral system with only phase fluctuations of the order parameter. Using Eq. (3.51), the response function for such a system, in the static limit, is given by

$$K^m_{ij}(0, \mathbf{q}) = Q^{ij}(0, \mathbf{q}) - \frac{Q^{ia}(0, \mathbf{q}) q_a q_b Q^{bj}(0, \mathbf{q})}{q_c Q^{cd}(0, \mathbf{q}) q_d}. \quad (4.10)$$

The undressed EM response (for a spin $1/2$ system with $e = 1$) reads $[27, 28]$:

$$Q^{ij}(0, \mathbf{q}) = 2 \sum_p \left( \frac{p_i p_j}{m m} \right) [G(i \omega_n, \mathbf{p}_+ ) G(i \omega_n, \mathbf{p}_- ) + F^s(i \omega_n, \mathbf{p}_+ ) F(i \omega_n, \mathbf{p}_- )] + \frac{n}{m} \delta^{ij}. \quad (4.11)$$

The non-bold momenta are four-vectors $p^\mu = (i \omega_n, \mathbf{p})$ with $\omega_n$ a fermionic Matsubara frequency. For simplicity, let us focus on a system with a general momentum-angle-dependent gap $\Delta_p \equiv \Delta(\mathbf{p})$. The single-particle and anomalous Green’s functions are $[27, 28]$:

$$G(i \omega_n, \mathbf{p}) = -\frac{i \omega_n + x \mathbf{p}}{\omega_n^2 + x^2 \mathbf{p}^2 + |\Delta_p|^2}, \quad (4.12)$$

$$F(i \omega_n, \mathbf{p}) = \frac{\Delta_p}{\omega_n^2 + x^2 \mathbf{p}^2 + |\Delta_p|^2}. \quad (4.13)$$

A generic static correlation function for a uniform system has the form

$$K^{ij}(0, \mathbf{q}) = \chi_L \frac{q_i q_j}{q^2} + \chi_T \left( \delta^{ij} - \frac{q_i q_j}{q^2} \right). \quad (4.14)$$

Here, $\chi_T$ and $\chi_L$ denote the transverse and longitudinal part of the full response function, respectively. By taking the dot product with $q^i$ and $q^j$, the longitudinal part is

$$\chi_L = \frac{q_i^T q_j}{q^2}. \quad (4.15)$$

The longitudinal part of the total response gives zero contribution to the Meissner effect: the full response is purely transverse. In the small-momentum limit the collective-mode part of the response (the second term in Eq. (4.10)) is purely longitudinal, and thus it gives zero contribution to the superfluid density.

Let $i = j$ in Eq. (4.14) and take the trace to obtain $\sum_i K^{ii} = \chi_L + 2 \chi_T$. Therefore the transverse part is

$$\chi_T = \frac{1}{2} \left( \sum_i K^{ii} - \chi_L \right). \quad (4.16)$$

Let $(m/n) \chi_T \equiv \chi'_T$. Using Eq. (4.11), this becomes

$$\chi'_T(q) = \frac{1}{mn} \sum_p p^2 \sin^2(\theta) \left[ G(i \omega_n, \mathbf{p}_+) G(i \omega_n, \mathbf{p}_-) + F^s(i \omega_n, \mathbf{p}_+) F(i \omega_n, \mathbf{p}_-) \right] + 1. \quad (4.17)$$

We drop the $q$ dependence in the argument of $\chi_T$ from now on. To evaluate this quantity we invoke standard Fermi-liquid theory and assume a constant density of states near the Fermi-surface. Using this approximation, the transverse response then becomes $[27]$

$$\chi'_T = 1 + T \sum_{\omega_n} \int_0^\pi d\theta \sin^3(\theta) \int_{-\infty}^\infty d\xi \frac{\left( i \omega_n + \xi_+ \right) \left( i \omega_n + \xi_- \right) + |\Delta_p|^2}{\left( \omega_n^2 + \xi_+^2 + |\Delta_p|^2 \right)} \left( \omega_n^2 + \xi_-^2 + |\Delta_p|^2 \right). \quad (4.18)$$

Here, $\xi_\pm = \xi \pm \frac{1}{2} q v_F \cos(\theta)$ with $v_F = p_F / m$. We have also used $k_F^2 = \pi^2 n$. As discussed in Ref. [27], the result of performing the Matsubara frequency summation followed by the $\xi$ integration leads to the correct normal-state result. However, performing this procedure in the reverse order leads to a different answer, in contradiction to the absence of a normal-state Meissner effect. To circumvent this problem, the method employed is to add and subtract the normal-state density expression. This enables performing the integration over $\xi$ first, which results in

$$\chi'_T = 3 \sum_{\omega_n} \int_0^{1} \frac{dx}{\sqrt{\omega_n^2 + |\Delta_p|^2}} \frac{1 - x^2}{\omega_n^2 + |\Delta_p|^2 + \frac{q^2 v_F^2}{m} x^2}. \quad (4.19)$$

For comparison, the EM current given in Ref. [21] reads

$$\mathbf{J} (\mathbf{q}) = \int dS_p R (\mathbf{p}) \mathbf{q} \left[ \mathbf{p} \cdot \mathbf{A} (\mathbf{q}) - \mathbf{p} \cdot \mathbf{q} \phi (\mathbf{q}) \right], \quad (4.20)$$
with the function $R(\hat{\mathbf{p}}) \equiv R(\hat{\mathbf{p}}; 0, \hat{\mathbf{q}})$ given by

$$R(\hat{\mathbf{p}}; 0, \hat{\mathbf{q}}) = T \sum_{\omega_n} \frac{1}{\sqrt{\omega_n^2 + |\Delta_p|^2}} \frac{|\Delta_p|^2}{\omega_n^2 + |\Delta_p|^2 + \frac{1}{2} q^2 v_F^2 x^2}$$

and $\phi(q)$ given by

$$\phi(q) = \frac{\int dS_R \left( \mathbf{1} \cdot \mathbf{q} \mathbf{i} \cdot \mathbf{A}(q) \right)}{\int dS_R \left( \mathbf{k} \cdot \left( \mathbf{k} \cdot \mathbf{i} \right) \right)}.$$

V. CONCLUSIONS

The rich physics associated with superfluids and superconductors is most perceptible in the collective fluctuations of the order parameter. These modes show that superconductors are more than just gapped fluids of condensed electron-electron pairs. Rather, superconductors are systems replete with collective excitations due to coherent many-particle effects. Historically these modes were first studied in the context of restoring gauge invariance in a superconductor. More recently, however, a bevy of literature has studied these excitations in more general settings, and one particularly important problem has been understanding their role in the Meissner effect.

The antecedent literature to the present work suggested that collective modes may be ignored in $s$-wave systems, but must be accounted for if the order parameter is anisotropic ($p$-wave, $d$-wave, etc.). In this paper we have extended this analysis by developing a general method for computing the electromagnetic response in systems with multiple collective modes. We have shown that, in fact, collective modes do not contribute to the Meissner effect in neither uniform nor nonuniform superconductors. An exception to this scenario comes when external wavevector scales exist, as in Fulde-Ferrell finite-momentum paired superconductors. The by-product of our study was to show that through singular-value decompositions, the electromagnetic response in a system with multiple collective modes present can naturally be computed by folding the various response tensors into dressed constituents. With all details we provided, we anticipate that this methodology will also prove useful in other contexts such as charge-density waves and quantum magnetism.

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Appendix A: Derivation of the mean-field EM response tensor

In this excursion we derive in detail the EM response tensor in Eq. (2.3). For concreteness, whenever we write $\delta S_{\text{eff}}[\Delta, A]/\delta A_\nu(x')$ (no $A$ dependence in $\Delta$), we mean the explicit $A$ dependence is being differentiated, with the
collective-mode fields fixed. The functional chain rule produces

$$\frac{\delta S_{\text{eff}}[\Delta_{\text{mf}}[A], A]}{\delta A_\mu(x) \delta A_\nu(y)} = \left(\frac{\delta S_{\text{eff}}[\Delta, A]}{\delta A_\mu(x) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}}[A]} + \int_z \left(\frac{\delta S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z)}\right)_{\Delta_{\text{mf}}[A]} \frac{\delta \Delta_{\text{mf}}[A](z)}{\delta A_\nu(y)} \cdot \frac{\delta \Delta_{\text{mf}}[A](z')}{\delta A_\nu(y)}.$$  \hspace{1cm} (1.1)

At the end of the calculation the value of $\Delta$ is set to its mean-field value $\Delta_{\text{mf}}[A]$. Similarly, the second derivative of the above expression reads

$$\frac{\delta^2 S_{\text{eff}}[\Delta_{\text{mf}}[A], A]}{\delta A_\mu(x) \delta A_\nu(y)} = \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta A_\mu(x) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}}[A]} + \int_z \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}}[A]} \frac{\delta \Delta_{\text{mf}}[A](z)}{\delta A_\nu(y)} \cdot \frac{\delta \Delta_{\text{mf}}[A](z')}{\delta A_\nu(y)} + \int_z \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}}[A]} \frac{\delta \Delta_{\text{mf}}[A](z)}{\delta A_\nu(y)}.$$  \hspace{1cm} (1.2)

Since we are interested in the mean-field EM response, we can invoke the saddle-point condition

$$0 = \frac{\delta S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z)} \bigg|_{\Delta = \Delta_{\text{mf}}[A]};$$  \hspace{1cm} (1.3)

thus the last term in Eq. (1.2) gives zero mean-field contribution and can be dropped. If one were to consider the EM response at the Gaussian order, however, then this term would contribute. It remains to compute the derivatives of the collective-mode fields $\Delta_a$ with respect to the vector potential. This can be done by considering the saddle-point conditions. Differentiating Eq. (1.3) with respect to $A$ gives

$$0 = \frac{\delta}{\delta A_\nu(y)} \left(\frac{\delta S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z)}\right)_{\Delta_{\text{mf}}[A]} = \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta A_\nu(y) \delta \Delta_a(z)}\right)_{\Delta_{\text{mf}}[A]} + \int_{z'} \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z) \delta \Delta_b(z')}\right)_{\Delta_{\text{mf}}[A]} \frac{\delta \Delta_{\text{mf}}[A](z')}{\delta A_\nu(y)}.$$  \hspace{1cm} (1.4)

Inverting the saddle-point integral equation yields

$$\frac{\delta \Delta_{\text{mf}}[A](z')}{\delta A_\nu(y)} = -\int_{z'} \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_b(z') \delta \Delta_a(z)}\right)_{\Delta_{\text{mf}}[A]}^{-1} \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}}[A]}.$$  \hspace{1cm} (1.5)

Substituting this into Eq. (1.2) and taking $A \to 0$, we then obtain Eq. (2.3) of the main text:

$$R_{\mu\nu}^{\text{mf}}(x, y) = \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta A_\mu(x) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}[0]}} - \int_{z,z'} \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z) \delta \Delta_a(z')}\right)_{\Delta_{\text{mf}[0]}}^{-1} \left(\frac{\delta^2 S_{\text{eff}}[\Delta, A]}{\delta \Delta_a(z) \delta A_\nu(y)}\right)_{\Delta_{\text{mf}[0]}}.$$  \hspace{1cm} (1.6)

Appendix B: Polarization tensor calculations

In this appendix we provide a short discussion regarding polarization bubbles. If $\{\Phi\}$ collectively describes a set of fields upon which a fermionic system depends (external electromagnetic fields, Hubbard-Stratonovich auxiliary fields, etc), response tensors are computed as an expansion around a reference set of values $\{\overline{\Phi}\}$. 
\[ Q_{\Phi \Phi'} (x - x') = \left. \frac{\delta^2 S_{\text{eff}} [\{\Phi\}]}{\delta \Phi (x) \delta \Phi' (x')} \right|_{\{\Phi\} = \{\bar{\Phi}\}} \]

\[ = Q_{\text{bos}, \Phi \Phi'} (x - x') - \frac{1}{2} \left. \frac{\delta^2 \text{Tr} \ln \left[ -G^{-1} [\{\Phi\}] \right]}{\delta \Phi (x) \delta \Phi' (x')} \right|_{\{\Phi\} = \{\bar{\Phi}\}} \]

\[ = Q_{\text{bos}, \Phi \Phi'} (x - x') - \frac{1}{2} \int d^D y \text{Tr} \left( y \left[ \frac{\delta G [\{\Phi\}]}{\delta \Phi (x)} \frac{\delta G^{-1} [\{\Phi\}]}{\delta \Phi' (x')} + G [\{\Phi\}] \frac{\delta^2 G^{-1} [\{\Phi\}]}{\delta \Phi (x) \delta \Phi' (x')} y \right) \right|_{\{\Phi\} = \{\bar{\Phi}\}} \]

\[ = Q_{\text{bos}, \Phi \Phi'} (x - x') + \frac{1}{2} \int d^D y \text{Tr} \left( y \left[ G [\{\Phi\}] \frac{\delta G^{-1} [\{\Phi\}]}{\delta \Phi (x)} G [\{\Phi\}] \frac{\delta G^{-1} [\{\Phi\}]}{\delta \Phi' (x')} y \right) \right|_{\{\Phi\} = \{\bar{\Phi}\}} \]

\[ - \frac{1}{2} \int d^D y \text{Tr} \left( y \left[ \frac{\delta^2 G^{-1} [\{\Phi\}]}{\delta \Phi (x) \delta \Phi' (x')} y \right) \right|_{\{\Phi\} = \{\bar{\Phi}\}} . \tag{2.1} \]

Here, \( Q_{\text{bos}, \Phi \Phi'} (x - x') \) is the bosonic part of the response which arises from differentiating the bosonic contribution to the effective action. Define real-space vertices by

\[ \hat{\mathcal{V}}_{\Phi} (x, y, x') \equiv \left. \frac{\delta G^{-1} [\Phi] (x, x')}{\delta \Phi (y)} \right|_{\{\Phi\} = \{\bar{\Phi}\}} . \tag{2.2} \]

The standard Green’s function representation of the polarization bubbles then follows

\[ Q_{\Phi \Phi'} (x - x') = Q_{\text{bos}, \Phi \Phi'} (x - x') + \frac{1}{2} \int_{y,y',z,z'} \text{Tr} \left[ G (y, z) \hat{V}_{\Phi} (z, x, z') G (z', y') \hat{V}_{\Phi'} (y', x', y) \right]_{\Phi = \bar{\Phi}}, \tag{2.3} \]

where \( \Phi \) is an arbitrary field in the system. The bare EM vertices are defined by

\[ \gamma^\mu (x, y, x') = \left. \frac{\delta G_0^{-1} [A] (x, x')}{\delta A_\mu (y)} \right|_{A = 0} . \tag{2.4} \]

For the models of superconductivity with a quadratic free-particle dispersion studied in the main text, the components of the vertices are explicitly given by

\[ \gamma^0 (x, y, x') = e \tau_3 \delta (x - y) \delta (x - x') \tag{2.5} \]

\[ \gamma (x, y, x') = \frac{e i}{2m} \tau_0 \left[ \nabla \left( \delta (x - y) \delta (x - x') \right) + \delta (x - y) \nabla \delta (x - x') \right] \]

\[ + \frac{e^2}{m} \tau_3 A (x) \delta (x - y) \delta (x - x') , \tag{2.6} \]

and

\[ \frac{\delta \gamma^\nu (x, y, x')}{\delta A_\mu (y')} = - \frac{e^2}{m} \tau_3 \delta (x - y') \delta (x - y) \delta (x - x') \delta^{\mu i} \delta^{\nu j} \delta_{ij} . \tag{2.7} \]

For the electromagnetic response, the reference value for the external field is \( A = 0 \). The Fourier expansion of the response is

\[ Q^{\mu \nu} (x - y) = \int_q e^{-i q (x - y)} Q^{\mu \nu} (q) , \tag{2.8} \]

where \( \int_q = TL^d \sum_{\Omega_m} \int \frac{d^D q}{(2\pi)^D} \). Using the general expression in Eq. (2.3), the undressed polarization response is

\[ Q^{\mu \nu} (q) = \left. \frac{1}{2} \text{Tr} \left[ G (p + q) \gamma^\mu (p + q, p) G (p) \gamma^\nu (p, p + q) \right] \right|_{\lambda = 0} + \frac{ne^2}{m} \delta^{\mu i} \delta^{\nu j} \delta_{ij} . \tag{2.9} \]

By definition, the momentum-space vertex is defined by [1]:

\[ \gamma^\mu (x, y, x') = \int_{p,q} e^{i q (x - y)} e^{i p (x - x')} \gamma^\mu (p + q, k) . \tag{2.10} \]
Therefore, in the limit of zero external field, the momentum-space vertices are \[ 1, 28]:

\[
\gamma^0 (p + q, p) \big|_{A=0} = e \tau_3. \quad (2.11)
\]

\[
\gamma (p + q, p) \big|_{A=0} = \frac{e}{m} \tau_0 \left( p + \frac{q}{2} \right) = \gamma (p, p + q) \big|_{A=0}. \quad (2.12)
\]

For a three-dimensional system with \( p \)-wave pairing, the Nambu Green’s function is

\[
G(p) = \left[ i \omega_n - \tau_3 \xi_p + \Delta_0 (p_x \tau_1 - p_y \tau_2) \right]^{-1} = \frac{i \omega_n + \tau_3 \xi_p - \Delta_0 (p_x \tau_1 - p_y \tau_2)}{\left( i \omega_n \right)^2 - E_p^2}, \quad (2.13)
\]

where \( \xi_p = p^2 / (2m) - \mu \) and \( E_p^2 = \xi_p^2 + \Delta_0^2 p \cdot p / \mu^2 \). All other bubbles appearing in the main text can be computed in a similar fashion.

**Appendix C: Superconducting pairing in radial coordinates**

Here we transform the mean-field ansatz for the case of spinless \( p \)-wave pairing to center-of-mass and relative coordinate representation as a concrete example of Eqs. (3.18) and (3.19). The coordinate transformation is

\[
R = \frac{x + x'}{2}, \quad r = x - x'. \quad (3.1)
\]

The Jacobian for this transformation is unity. For a spinless fermionic system, the \( p \)-wave ansatz reads

\[
\Delta(x, x', \tau) = \left| \Delta \left( \frac{x + x'}{2}, \tau \right) \right| e^{i \Phi \left( \frac{x + x'}{2}, \tau \right)} (\partial_x + i \partial_y) \delta(x - x'). \quad (3.2)
\]

Thus,

\[
\int d^3 x d^3 x'\psi^\dagger(x, \tau) \Delta(x, x', \tau) \psi^\dagger(x', \tau) = \int d^3 R \left| \Delta(R, \tau) \right| e^{i \Phi(R, \tau)} \int d^3 r \psi^\dagger(R + r/2, \tau) \psi^\dagger(R - r/2, \tau) (\partial_x + i \partial_y) \delta(r)
\]

\[
= \int d^3 R \left| \Delta(R, \tau) \right| e^{i \Phi(R, \tau)} \left[ \psi^\dagger(R, \tau) (\partial_x + i \partial_y) \psi^\dagger(R, \tau) \right]
\]

\[
= \int d^3 x \left| \Delta(x, \tau) \right| e^{i \Phi(x, \tau)} \left[ \psi^\dagger(x, \tau) (\partial_x + i \partial_y) \psi^\dagger(x, \tau) \right], \quad (3.3)
\]

after integrations by parts, identifications of gradients of fermion fields with respect to \( R \) and \( r \) variables and relabelling of dummy variables.

In general, non \( s \)-wave pairing demands a spatially dependent interaction coefficient, say \( g(x - x') \). In this case, the \( p \)-wave ansatz simplifies the Gaussian part of the identity introduced in the Hubbard-Stratonovich decomposition:

\[
\int d^3 x d^3 x' \left| \Delta(x, x', \tau) \right|^2 \frac{g(x - x')}{g(x - x')}
\]

\[
= \int d^3 Rd^3 r \left| \Delta(R, \tau) \right| e^{-i \Phi(R, \tau)} \left[ (\partial_x - i \partial_y) \delta(r) \right] g^{-1}(r) \left| \Delta(R, \tau) \right| e^{i \Phi(R, \tau)} (\partial_x + i \partial_y) \delta(r)
\]

\[
= \int d^3 R \frac{\left| \Delta(R, \tau) \right|^2}{g}, \quad (3.4)
\]

where we define the renormalized value for the (inverse) mass scale of the amplitude field as

\[
\tilde{g}^{-1} = \int d^2 r \left[ (\partial_x - i \partial_y) \delta(r) \right] g^{-1}(r) (\partial_x + i \partial_y) \delta(r). \quad (3.5)
\]
Appendix D: 3 × 3 response matrix determinant calculation

Here we sketch the calculation and simplification of \( \det S (q) \) for the response of a charged superconductor in the presence of Coulomb, amplitude and phase fluctuations. We first consider biasing towards including the amplitude fluctuation effects. An expansion and consideration of the definition in Eq. (3.43) returns

\[
\det S (q) = \det \begin{pmatrix}
S_{pp} & iR_{p0} & iR_{q0} q_{\beta} \\
-iq_{p} R_{q0} & V^{-1} - Q_{p0} & -Q_{q0} q_{\beta} \\
-q_{p} R_{q0} & q_{p} Q_{\alpha0} & q_{p} Q_{\alpha0} q_{\beta}
\end{pmatrix}
= q_{\alpha} q_{\beta} S_{pp} \left[ (V^{-1} - Q_{00}) Q_{\alpha0} + Q_{\alpha0} Q_{00} + Q_{\alpha0} Q_{\alpha0} S_{pp} - Q_{00} R_{p0} R_{q0} S_{pp} \right]. \tag{4.1}
\]

With the singular-value decomposition structure in mind, we can rework the term in square brackets to produce

\[
\det S (q) = q_{\alpha} q_{\beta} S_{pp} \left[ (V^{-1} - Q_{00}) \overline{Q}_{\alpha0} + (R_{p0}) (Q_{00}) \left( \begin{array}{cc}
Q_{\alpha0} & -R_{p0} S_{pp} \\
R_{p0} & S_{pp}
\end{array} \right) \right]
= q_{\alpha} q_{\beta} S_{pp} \left[ (V^{-1} - \overline{Q}_{00}) \overline{Q}_{\alpha0} + (R_{p0}) (Q_{00}) \left( \begin{array}{cc}
R_{p0} & S_{pp} \\
-R_{p0} & -S_{pp}
\end{array} \right) \right]. \tag{4.2}
\]

Following with the decomposition, we fold the effects of Coulomb fluctuations into \( \overline{Q}_{\alpha0} \) to obtain

\[
\det S (q) = S_{pp} \left( V^{-1} - Q_{00} \right) q_{\alpha} \overline{Q}_{\alpha0} \overline{Q}_{\alpha0}. \tag{4.3}
\]

A reversed order of the fluctuation considerations allows writing

\[
\det S (q) = q_{\alpha} q_{\beta} \left( V^{-1} - Q_{00} \right) S_{pp} \left[ \overline{Q}_{\alpha0} - R_{p0} R_{q0} (V^{-1} - Q_{00}) + \frac{Q_{\alpha0} R_{p0} R_{q0} (V^{-1} - Q_{00})}{(V^{-1} - Q_{00})} + \frac{Q_{\alpha0} R_{p0} R_{q0} (V^{-1} - Q_{00})}{(V^{-1} - Q_{00})} \right]. \tag{4.4}
\]

Proceeding with a similar analysis, this leads to

\[
\det S (q) = S_{pp} \left( V^{-1} - Q_{00} \right) q_{\alpha} Q_{\alpha0} q_{\beta}, \tag{4.5}
\]

proving Eq. (3.41) in the main text.

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