The Renormalization Group Limit Cycle for the $1/r^2$ Potential

Eric Braaten and Demian Phillips

Department of Physics, The Ohio State University, Columbus, OH 43210

(Dated: October 24, 2018)

Abstract

Previous work has shown that if an attractive $1/r^2$ potential is regularized at short distances by a spherical square-well potential, renormalization allows multiple solutions for the depth of the square well. The depth can be chosen to be a continuous function of the short-distance cutoff $R$, but it can also be a log-periodic function of $R$ with finite discontinuities, corresponding to a renormalization group (RG) limit cycle. We consider the regularization with a delta-shell potential. In this case, the coupling constant is uniquely determined to be a log-periodic function of $R$ with infinite discontinuities, and an RG limit cycle is unavoidable. In general, a regularization with an RG limit cycle is selected as the correct renormalization of the $1/r^2$ potential by the conditions that the cutoff radius $R$ can be made arbitrarily small and that physical observables are reproduced accurately at all energies much less than $\hbar^2/mR^2$.

PACS numbers: 11.10.Gh,03.65.-w,05.10.Cc
I. INTRODUCTION

The development of the renormalization group (RG) has had a profound impact on several subfields of physics. Many applications of the RG involve renormalization group flow towards a fixed point that is invariant under renormalization. An example is critical phenomena in condensed matter physics, which can be understood in terms of renormalization group flow to a fixed point in the infrared limit. Another example is quantum chromodynamics (QCD), the quantum field theory that describes the strong interactions of elementary particles, which flows under renormalization to a fixed point in the ultraviolet limit. A fixed point is the simplest topological feature that can be exhibited by an RG flow. As pointed out by Wilson in 1970, one of the next simplest possibilities is a limit cycle, a closed curve that is invariant under renormalization. The limit cycle is characterized by a discrete scaling symmetry: the renormalization group flow executes a complete cycle around the curve every time the cutoff changes by a multiplicative factor $\lambda$ called the discrete scaling factor. The discrete scaling symmetry is reflected in log-periodic behavior of physical observables as functions of the momentum scale. The possibility of RG limit cycles has received little attention until recently, partly because of the scarcity of compelling examples. One physical example that was identified long ago is the problem of identical bosons with large scattering length $a$. In the limit $a \to \pm \infty$, there is an accumulation of 3-body bound states near threshold with binding energies differing by multiplicative factors of $\lambda^2 \approx 515.03$. This phenomenon, which is called the Efimov effect, can be understood in terms of a renormalization group limit cycle with discrete scaling factor $\lambda \approx 22.7$. This application has been made more compelling by Bedaque, Hammer, and van Kolck, who reformulated the problem using effective field theory. Other examples of renormalization group limit cycles have recently begun to emerge. There are discrete Hamiltonian systems that exhibit RG limit cycles in appropriate continuum limits. LeClair, Roman, and Sierra have identified a two-dimensional field theory whose renormalization involves an RG limit cycle in apparent contradiction to Zamolodchikov’s C theorem. It has even been conjectured that QCD has an infrared RG limit cycle at special values of the quark masses.

These examples suggest that RG limit cycles may play a more important role in physics than previously realized. They provide motivation for studying simple examples of RG limit cycles. The simplest example is the quantum mechanics of a particle in a potential whose...
long-range behavior is $1/r^2$. This problem has been studied previously within the renormalization group framework by two different groups using a spherical square-well regularization potential \[11, 12\]. Beane et al. \[11\] showed that there are infinitely many choices for the coupling constant of the square-well potential, including a continuous function of the short-distance cutoff $R$ and a log-periodic function of $R$ with discontinuities which corresponds to an RG limit cycle. Bawin and Coon \[12\] presented a closed-form solution for the coupling constant that is log-periodic, which suggests that the choice with the RG limit cycle is in some sense natural.

In this paper, we clarify the role of RG limit cycles in the renormalization of the $1/r^2$ potential. We begin in Section II by summarizing how renormalization theory can be applied to the $1/r^2$ potential. In Section III, we reconsider the spherical square-well regularization potential and calculate the bound-state spectrum for alternative choices of the coupling constant. In Section IV, we consider a spherical delta-shell regularization potential. In this case, the coupling constant is uniquely determined and is governed by an RG limit cycle. We discuss our results in Section V and identify the criterion that selects the regularization with the RG limit cycle as the correct renormalization of the $1/r^2$ potential.

II. RENORMALIZATION OF THE $1/r^2$ POTENTIAL

We consider a particle in a spherically-symmetric potential $V(r)$ that is attractive and proportional to $1/r^2$ for $r$ greater than some radius $R_{\text{min}}$:

$$V(r) = \begin{cases} -\left(\frac{1}{4} + \nu^2\right) \frac{\hbar^2}{2m r^2} & r > R_{\text{min}}, \\ V_{\text{short}}(r) & r \leq R_{\text{min}}, \end{cases}$$

(1)

where $\nu$ is a positive parameter. The coefficient of the short-distance potential is written as $\frac{1}{4} + \nu^2$ because $\nu^2 = 0$ is the critical value above which the potential is too singular for the problem to be well-behaved in the limit $R_{\text{min}} \to 0$. We will not specify the short-distance potential $V_{\text{short}}(r)$. The potential $V(r)$ has infinitely-many arbitrarily-shallow S-wave bound states with an accumulation point at the scattering threshold $E = 0$. As the threshold is approached, the ratio of the binding energies of successive states approaches $\lambda^2 = e^{2\pi/\nu}$. The asymptotic spectrum near the threshold therefore has the form

$$E_n \to \frac{\hbar^2 \kappa^2}{m} (e^{-2\pi/\nu})^{n-n_*},$$

(2)
where \( n_\ast \) is an integer that can be chosen for convenience and \( \kappa_\ast \) is determined up to a multiplicative factor of \( e^{\pi/\nu} \) by the short-distance potential. This geometric spectrum reflects an asymptotic discrete scaling symmetry in which the distance from the origin is rescaled by the discrete scaling factor \( \lambda = e^{\pi/\nu} \). One might have expected an approximate continuous scaling symmetry because the long-distance potential is scale-invariant, but the continuous scaling symmetry is broken to a discrete subgroup by the boundary conditions provided by the short-distance potential. This is an example of a quantum mechanical anomaly.

Although renormalization theory was originally introduced to attack very different problems \([1]\), it can also be applied to nonrelativistic quantum mechanics \([13]\). A particularly convenient way of implementing renormalization theory in quantum mechanics is within an effective theory framework, which allows a systematically improvable description of the system at low energies \( E \) satisfying \( |E| \ll \hbar^2/mR_{\text{min}}^2 \) \([14]\). Renormalization can be implemented in this problem by introducing a cutoff radius \( R \) satisfying \( R > R_{\text{min}} \) and replacing the potential in the region \( 0 < r < R \) by a regularization potential \( V_{\text{reg}}(r; \lambda) \) that depends on a tuning parameter \( \lambda \):

\[
V(r) = -\left(\frac{1}{4} + \nu^2\right) \frac{\hbar^2}{2mr^2} \quad r > R, \\
= V_{\text{reg}}(r; \lambda(R)) \quad r \leq R. 
\]

Some quantity involving low energies \( |E| \ll \hbar^2/mR^2 \), such as the energy eigenvalue of a very shallow bound state, is selected as a matching variable. The parameter \( \lambda(R) \) in the regularization potential is then tuned so that the value of the matching variable in the true theory with potential \([1]\) is reproduced by the theory with the regularized potential \([3]\). Renormalization theory guarantees that other low-energy observables involving energies satisfying \( |E| \ll \hbar^2/mR^2 \) will also be reproduced correctly by the regularized theory up to corrections of order \( EmR^2/\hbar^2 \). As \( R \) is decreased, the errors decrease as \( R^2 \) until \( R \) reaches \( R_{\text{min}} \). If \( R \) is decreased below \( R_{\text{min}} \), there is no further decrease in the errors.

A particularly convenient choice for the matching quantity is the zero-energy wavefunction. The stationary Schroedinger equation for a radial wavefunction \( u(r)/r \) with zero angular momentum and energy eigenvalue \( E \) is

\[
\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r)\right) u(r) = E u(r). 
\]

The \( E = 0 \) solution in the \( 1/r^2 \) region of the true potential \( V(R) \) has the form
FIG. 1: Several branches of the coupling constant $\lambda(R)$ of the square-well regularization potential for $\nu = 2$ as a function of $\ln(R/r_0)$ for (a) continuous $\lambda(R)$ and (b) log-periodic $\lambda(R)$.

$$u(r) = Ar^{1/2} \sin [\nu \ln(r/r_0)] \quad r > R_{\text{min}}.$$ \hfill (5)

The parameter $r_0$ is the position of one of the nodes of the wavefunction. It is determined up to a multiplicative factor $e^{\pi/\nu}$ by the short-distance potential $V_{\text{short}}(r)$.

### III. SQUARE-WELL REGULARIZATION

In two previous studies of the $1/r^2$ potential using renormalization theory, the regularization potential was chosen to be a spherical square well \[11, 12\]:

$$V_{\text{reg}}(r; \lambda) = -\lambda \frac{\hbar^2}{2mR^2} \quad r < R.$$ \hfill (6)

The “coupling constant” $\lambda$ is dimensionless. The condition that the regularized potential reproduce the zero-energy wavefunction (5) at distances $r > R$ is

$$\lambda^{1/2} \cot(\lambda^{1/2}) = \frac{1}{2} + \nu \cot [\nu \ln(R/r_0)].$$ \hfill (7)

This equation applies not only for $\lambda > 0$, but also for $\lambda < 0$, in which case $\lambda^{1/2} \cot(\lambda^{1/2}) = |\lambda|^{1/2} \coth(|\lambda|^{1/2})$. For any value of $R$, the transcendental equation (7) has infinitely many roots.

In Ref. [11], Beane et al. pointed out that $\lambda(R)$ could be chosen to be continuous. For the case $\nu = 2$, three branches of continuous $\lambda(R)$ are illustrated in Fig. 1(a). The coupling
FIG. 2: The spectrum of the deepest bound states for $\nu = 2$ and the square-well regularization potential as a function of $\ln(R/r_0)$ for (a) continuous $\lambda(R)$ given by the middle branch of Fig. 1(a) and (b) log-periodic $\lambda(R)$ given by the middle branch of Fig. 1(b).

Constant $\lambda(R)$ decreases monotonically as the cutoff radius $R$ decreases. Once it becomes negative, it decreases very rapidly and reaches $-\infty$ at a finite value of $R$ given by

$$R^{(M)} = \left( e^{-\pi/\nu} \right)^M r_0,$$

where $M$ is an integer. Thus, if $\lambda(R)$ is continuous, there is a lower bound on the cutoff radius $R$. The three branches in Fig. 1(a) correspond to three consecutive values of $M$. The authors of Ref. [11] also pointed out that $\lambda(R)$ could equally well be chosen to jump discontinuously between the branches of the solutions to (7) at arbitrary values of $R$ without affecting the observables at extremely low energies. One particular choice would be to have $\lambda(R)$ jump up to the next branch every time $R$ decreases by a factor of $e^{\pi/\nu}$. This choice corresponds to an RG limit cycle.

In Ref. [12], Bawin and Coon presented a closed-form solution to (7) that depends on an integer parameter $n$. The resulting coupling constants $\lambda_n(R)$ are log-periodic functions of $R$. For the case $\nu = 2$, three branches of log-periodic $\lambda(R)$ are illustrated in Fig. 1(b). They have finite discontinuities at the specific values of $R$ given by (8), which differ by multiplicative factors of $e^{\pi/\nu}$. Such a choice of the solution to (7) corresponds to a renormalization group limit cycle with discrete scaling factor $e^{\pi/\nu}$.

We now consider the bound-state spectrum. The equation for the bound-state energy
eigenvalues $E_n = -\hbar^2 \kappa_n^2 / 2m$ is

$$\frac{1}{2} + \kappa R \frac{K'_{i\nu}(\kappa R)}{K_{i\nu}(\kappa R)} = (\lambda(R) - \kappa^2 R^2)^{1/2} \cot \left( (\lambda(R) - \kappa^2 R^2)^{1/2} \right),$$

where $K_{i\nu}(z)$ is a modified Bessel function with an imaginary index. The spectrum of very shallow bound states is almost completely independent of $R$ and has the form $\mathcal{E}_n = 1 + 2 \kappa^2 R^2 K_i(\nu)(\kappa R)$.

The parameter $\kappa_*$ in (2) is related to the parameter $r_0$ in the zero-energy wavefunction by

$$\kappa_* e^{n\pi/\nu} = 2 r_0 e^{\arg \Gamma(1+i\nu)/\nu},$$

where $n$ is an integer that depends on the choices for $\kappa_*$ and $r_0$, both of which are defined only up to multiplicative factors of $e^{\pi/\nu}$. The spectrum of deeper bound states depends on $R$ and on the choice of the branch for $\lambda(R)$. The spectrum of the deepest bound states for $\nu = 2$ and for continuous $\lambda(R)$ given by the middle branch in Fig. 1(a) is shown in Fig. 2(a). The curves cannot be extended below the value $R^{(M)}$ given by (8) because $\lambda(R)$ reaches $-\infty$ at that point. The binding energies are all continuous functions of $R$. For $R > R^{(M-1)}$, the order of magnitude of the deepest binding energy is $\hbar^2 / m R^2$. The spectrum of the deepest bound states for $\nu = 2$ and for log-periodic $\lambda(R)$ given by the middle branch in Fig. 1(b) is shown in Fig. 2(b). At each of the values of $R$ at which $\lambda(R)$ jumps discontinuously, a new deepest bound state appears in the spectrum. As $R$ decreases further, the third deepest bound state rapidly approaches its asymptotic value given by (2) and (10).

### IV. DELTA-SHELL REGULARIZATION

Renormalization theory is designed to give results for low-energy observables that are independent of the regularization potential. One regularization potential that is particularly convenient is the spherical delta-shell consisting of a delta function concentrated on a shell with radius $r = R^-$ infinitesimally close to but smaller than $R$:

$$V_{\text{reg}}(r; \lambda) = -\lambda \frac{\hbar^2}{2mR} \delta(r - R^-) \quad r \leq R.$$  

The coupling constant $\lambda$ is dimensionless. The radial wavefunction $u(r)/r$ must be continuous at $r = R$. Another boundary condition at $r = R$ is obtained by integrating the Schroedinger equation over an infinitesimal region including $r = R$:

$$\lim_{r \to R^+} r \frac{u'(r)}{u(r)} - \lim_{r \to R^-} r \frac{u'(r)}{u(r)} = -\lambda(R).$$

7
The coupling constant \( \lambda(R) \) for \( \nu = 2 \) and the delta-shell regularization potential as a function of \( \ln(R/r_0) \).

The scattering solution for energy \( E = \hbar^2 k^2 / 2m \) has the form

\[
\begin{align*}
u(r) &= r^{1/2} [A_+ J_{\nu}(kr) + A_- J_{-\nu}(kr)] & r > R, \\
&= A' \sin(kr) & r < R.
\end{align*}
\] (13)

This reduces to the zero-energy solution \([5]\) as \( k \to 0 \) if the limiting behavior of the coefficients is \( A_\mp \to \mp \frac{1}{2} i A (2/k r_0)^{\mp \nu} \). Applying the boundary condition \([12]\) to this solution and taking the limit \( k \to 0 \), we determine the coupling constant \( \lambda(R) \):

\[
\lambda(R) = \frac{1}{2} - \nu \cot [\nu \ln(R/r_0)].
\] (14)

The coupling constant \([14]\) is a single-valued function of \( R \), in contrast to the case of the square-well regularization where the coupling constant has infinitely many branches. As shown in Fig. 3, \( \lambda(R) \) is a log-periodic function of \( R \) with infinite discontinuities. It jumps discontinuously from \( +\infty \) to \( -\infty \) as \( R \) decreases through the critical values given by \([8]\).

We now consider the bound state spectrum. The radial wavefunction for a negative energy \( E = -\hbar^2 \kappa^2 / 2m \) has the form

\[
\begin{align*}
u(r) &= Br^{1/2} K_{\nu}(\kappa r) & r > R, \\
&= B' \sinh(\kappa r) & r < R.
\end{align*}
\] (15)

Using the boundary condition \([12]\), we find that the equation for the bound-state energy eigenvalues \( E_n = -\hbar^2 \kappa_n^2 / 2m \) is

\[
\frac{1}{2} + \kappa R \frac{K'_{\nu}(\kappa R)}{K_{\nu}(\kappa R)} - \kappa R \coth(\kappa R) = -\lambda(R).
\] (16)
FIG. 4: The spectrum of the deepest bound states for $\nu = 2$ and the delta-shell regularization potential as a function of $\ln(R/r_0)$.

The spectrum of very shallow bound states is almost completely independent of $R$ and has the form (2) with $\kappa_*$ given by (10). The spectrum for the deepest bound states is illustrated in Fig. 4. At the critical values of $R$ given by (8), where $\lambda(R)$ changes discontinuously from $-\infty$ to $+\infty$, a new bound state with infinitely deep binding energy $\kappa = +\infty$ emerges. As $R$ decreases further, that binding energy rapidly approaches its asymptotic value given by (2) and (10). Comparing with Fig. 2(b), we see that the deepest bound state corresponds to the third deepest bound state for the square-well regularization with an RG limit cycle.

V. DISCUSSION

We have studied the renormalization of an attractive $1/r^2$ potential using two regularization potentials: a spherical square well as in Refs. [11] and [12] and a spherical delta shell. In the case of the delta-shell potential, the coupling constant $\lambda$ is necessarily a log-periodic function of the cutoff radius $R$ with infinite discontinuities. It is governed by a renormalization group (RG) limit cycle. In the case of the square-well potential, there is much more freedom because there are infinitely many branches for the coupling constant $\lambda$. It might seem natural to choose $\lambda(R)$ to be a continuous function of $R$, but this choice has some drawbacks. Since $\lambda(R)$ diverges to $-\infty$ at a finite value of $R$, the cutoff radius cannot be decreased below this value. The choice of continuous $\lambda(R)$ also imposes an upper bound on the binding energy of the deepest bound state. Alternatively, the coupling constant can be
chosen to be a log-periodic function of $R$ with finite discontinuities, corresponding to an RG limit cycle. With this choice, the cutoff can be decreased to arbitrarily short distances and there is no upper bound on the binding energies.

If the value of the physical short-distance cutoff $R_{\text{min}}$ is fixed and known in advance, the choice between a log-periodic $\lambda(R)$ with an RG limit cycle and continuous $\lambda(R)$ is only a matter of taste. For continuous $\lambda(R)$, one can simply choose a branch of the coupling constant such that the minimal value of the cutoff radius is smaller than $R_{\text{min}}$. However, if $R_{\text{min}}$ is not known or if it can be varied, the continuous choice of $\lambda(R)$ will break down if $R_{\text{max}}$ happens to be smaller than the minimal cutoff radius. The choice of log-periodic $\lambda(R)$ with an RG limit cycle guarantees that $R$ can be made arbitrarily small and the effective potential still reproduces accurately all physics involving energies much smaller than $\hbar^2/mR^2$. This criterion selects the regularization with an RG limit cycle as the correct renormalization of the $1/r^2$ potential.

The RG limit cycle for the $1/r^2$ potential has also been studied recently using flow equations for RG transformations [15].

Acknowledgments

We thank K.G. Wilson for valuable discussions. This research was supported in part by the Department of Energy under grant DE-FG02-91-ER4069.

[1] K. G. Wilson, Rev. Mod. Phys. 55 583 (1983).
[2] K. G. Wilson, Phys. Rev. D 3, 1818 (1971).
[3] S. Albeverio, R. Hoegh-Krohn, and T.S. Wu, Phys. Lett. 83A, 105 (1981).
[4] V. Efimov, Phys. Lett. 33B, 563 (1970).
[5] P. F. Bedaque, H. W. Hammer and U. van Kolck, Phys. Rev. Lett. 82, 463 (1999) arXiv:nucl-th/9809025.
[6] S. D. Glazek and K. G. Wilson, Phys. Rev. Lett. 89, 230401 (2002) arXiv:hep-th/0203088.
[7] A. LeClair, J. M. Roman and G. Sierra, Phys. Rev. B 69, 020505 (2004) arXiv:cond-mat/0211338.
[8] A. Leclair, J. M. Roman and G. Sierra, Nucl. Phys. B 675, 584 (2003).

[9] A. B. Zamolodchikov, JETP Lett. 43, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. 43, 565 (1986)].

[10] E. Braaten and H. W. Hammer, Phys. Rev. Lett. 91, 102002 (2003) arXiv:nucl-th/0309030.

[11] S. R. Beane, P. F. Bedaque, L. Childress, A. Kryjevski, J. McGuire and U. v. Kolck, Phys. Rev. A 64, 042103 (2001) arXiv:quant-ph/0010073.

[12] M. Bawin and S. A. Coon, Phys. Rev. A 67, 042712 (2003) arXiv:quant-ph/0302199.

[13] S. K. Adhikari and A. Ghosh, J. Phys. A 30, 6553 (1997) arXiv:hep-th/9706193.

[14] G. P. Lepage, arXiv:nucl-th/9706029

[15] E. Mueller and T.-L. Ho, cond-mat/0403283