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Tensionless Strings and Supersymmetric Sigma Models

Aspects of the target space geometry
List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I  A. Bredthauer, U. Lindström and J. Persson. SL(2,\mathbb{Z}) tensionless string backgrounds in IIB string theory. *Classical and Quantum Gravity* 20 (2003) 3081. hep-th/0303225.

II A. Bredthauer, U. Lindström, J. Persson and L. Wulff. Type IIB tensionless superstrings in a pp-wave background. *Journal of High Energy Physics* 0402 (2004) 051. hep-th/0401159.

III A. Bredthauer, U. Lindström and J. Persson. First-order supersymmetric sigma models and target space geometry. *Journal of High Energy Physics* 0601 (2006) 144. hep-th/0508228.

IV A. Bredthauer, U. Lindström, J. Persson and M. Zabzine. Generalized Kähler geometry from supersymmetric sigma models. *Letters in Mathematical Physics* (2006), in press. hep-th/0603130.

V A. Bredthauer. Generalized hyperkähler geometry and supersymmetry. *Manuscript* (2006). hep-th/0608114.
## Contents

1. Introduction .................................................. 1

2. String theory basics .......................................... 7
   2.1 Non-linear sigma model .................................. 9
   2.2 Worldsheet supersymmetry ............................... 13
   2.3 Spacetime supersymmetry ............................... 17
   2.4 Low energy effective theory ............................ 19

3. Tensionless strings in plane waves .......................... 21
   3.1 Tensionless strings in flat space ....................... 22
   3.2 Plane wave geometry from $AdS_5 \times S^5$ ............. 25
   3.3 String theory in plane wave geometry .................. 27
   3.4 The tensionless superstring in the plane wave .......... 30
   3.5 Tensionless strings in homogeneous plane waves ...... 32

4. Macroscopic Tensionless Strings ............................... 35
   4.1 The gravitational shock wave ............................ 35
   4.2 The macroscopic IIB string ............................. 36
   4.3 The Tensionless $SL(2,\mathbb{Z})$ String ............... 39

5. From Complex Geometry to Generalized Complex Geometry .... 41
   5.1 Complex Geometry ........................................ 41
   5.2 Generalized Geometry .................................... 43
   5.3 Generalized Complex Structures ........................ 45
   5.4 Generalized Kähler Geometry ........................... 46
   5.5 Generalized Hyperkähler Structure ...................... 48

6. Supersymmetric Sigma Models ................................ 49
   6.1 Preliminaries ............................................. 50
   6.2 Manifest $N = (2, 2)$ supersymmetry .................... 51
   6.3 Enhanced supersymmetry in $N = 1$ phase space .......... 53
   6.4 The Poisson sigma model - A first application ......... 56
   6.5 The sigma model Hamiltonian ............................ 57
   6.6 Topological twists ........................................ 59
   6.7 $N = (4, 4)$ supersymmetric Hamiltonian ............... 60
   6.8 Twistor space for generalized complex structures ...... 62
   6.9 Generalized Supersymmetric Sigma Models .............. 65

Deutsche Zusammenfassung ........................................ 71
Svensk Sammanfattning ........................................... 75
Bibliography ...................................................... 81
1. Introduction

String theory is one of the most fascinating subjects modern theoretical physics ever developed. It unifies two fundamental concepts that at first sight do not fit together: gravity and quantum mechanics. This makes it ‘the’ candidate for a theory of nature. While electromagnetic, weak and strong interactions can be described by quantum field theories to reasonable accuracy, they fail in giving a proper description of gravity. On the other hand, we can describe gravity at large distances by Einstein’s general relativity. String theory crosses the barrier between these two different theories with a seemingly simple and naive idea: Why not consider one-dimensional objects, strings, as the basic constituents of nature instead of point-like particles?

But let us start with a short overview of the history of string theory. String theory in the way we view it today was not invented but rather discovered. At the end of the 1960’s people were analyzing scattering amplitudes of hadronic matter. String theory was proposed as a model for these interactions. Scattering of relativistic strings seemed to match with the experimental data. Unfortunately, this turned out to be wrong. String theory just was not able to describe, for example, effects in deep inelastic hadron scattering. The correct description was instead an ordinary quantum field theory. Quantum chromodynamics was born after the discovery of asymptotic freedom in 1973: The strength of the strong interaction between two quarks, the constituents of hadronic matter, decreases as they approach each other.

At around the same time, the discovery of an excitation that had gravitation-like interactions in the string spectrum triggered a new view of string theory that is still valid today. Suddenly, it became a candidate for a theory unifying the four fundamental forces. That strings have not been observed in nature was explained by their size. The typical size of a string is at the order of the Planck length, such that probing string theory directly would need much higher energy than provided by experiments. String theory is finite at high energies, in contrast to ordinary quantum field theories that all have the problem of infinities. A lot of interesting properties were discovered after 1975. At low energies corresponding to large distances, the gravitational interactions resemble exactly Einstein gravity, while they obtain corrections at short distances. This fits with the picture that general relativity breaks down below the Planck scale where quantum fluctuations are supposed to take over. Also supersymmetry, a symmetry that mixes bosons with fermions, was found to be
naturally included in string theory. Unfortunately, at those times, there were too many string theories, and there did not seem to be any principle for which one to choose.

Things changed after what is now called the first string revolution in 1985. Since then, we know that there are only five consistent theories at quantum level. All of them live in ten spacetime dimensions, they are called type I, type IIA and IIB, and the two heterotic string theories with gauge groups $SO(32)$ and $E_8 \times E_8$. The problem with the extra dimensions was solved by compactification. If the six extra dimensions are small enough, say at the order of the Planck scale or below, we would never be able to detect them with our experimental equipment. Supersymmetry was supposed to be unbroken at the compactification scale, at the size of the internal space so to speak. The four-dimensional space should be the flat space we see and this puts very strong constraints on the geometry of the internal six-dimensional space.

The second superstring revolution in 1995 revealed two things. First, the five string theories are dual to each other, related by certain duality transformations. In fact, they are perturbative expansions of one and the same theory around different vacua. It is here, the famous M-theory enters the game. However, despite the fact that we know it is there, not too many things are known about it. Also, the second revolution introduced D-branes. These solitonic objects had been known for some time but their importance to modern string theory was first realized then. Not only is their worldvolume dynamics governed by open strings attached to them, their existence allows for the idea that our world might be bounded to such a brane, explaining, for example, why gravity couples so weakly to matter.

Today, string theory is such a broad field of research that it is very hard to give a complete picture of the current research. Certainly, this is not the right place to give an introduction to string theory either. There are great books that cover this subject [GSW87, Pol98, Joh03, Zwi04]. Also, there are some useful lecture notes available [Sza02, Moh03], just to mention some.

The aim of this thesis is to give an introduction to the subjects that are covered in the publications [I] to [V], tensionless strings and supersymmetric sigma models. This serves also as a motivation for our work. In the rest of the thesis, we mainly focus on going through parts of our work in detail and providing some background information for a better understanding of our results. The list of references is not exhaustive. For a more complete list, we refer to the papers [I-V].

In particle physics massless particles play an important role. Not only is the photon, the carrier of the electromagnetic force, massless but particles at very high kinetic energies can be considered as approximately massless. The equivalent of the mass of a particle in string theory is the tension $T$ of the string, its mass per unit length. The tensionless string first appeared when discussing
strings moving at the speed of light and is still very poorly understood. Similar
to massless particles, tensionless strings are believed to have their place in the
study of the high energy behavior of string theory. For example, we can con-
consider a string that rotates with higher and higher angular momentum. As the
angular momentum increases the energy gets localized around the endpoints
of the string while its core becomes tensionless. The fact that the tension is
zero turns the string basically into a collection of freely moving particles —
it falls into pieces. However, these pieces are still connected to each other
since the string is a continuous object even in the tensionless case. Tension-
less strings have been studied for a long time, classically and quantized, with
and without supersymmetry.

Tensionless string theory exhibits a much larger spacetime symmetry than
the tensionfull theory. The quantum theory differs drastically. In flat space the
spectrum collapses to a common zero-mass level. Especially tachyonic states
that are usually unstable and have to be banned from the physical spectrum
due to their negative mass squared, become massless and thus stable for the
tensionless string. The quantum theory has either a topological spectrum or
for the case of $D = 2$ spacetime dimensions, the spacetime symmetry is re-
tained. There is no critical spacetime dimension for the tensionless string and
the spectrum has a huge symmetry involving higher spin gauge fields. The
tensionless string is supposed to be the unbroken phase of string theory where
all states are still considered on an equal footing and that breaks as the energy
decreasing giving rise to the different mass levels.

The tensionless string appears in various situations. The ordinary string is
approximately tensionless in a highly curved background and it appears in the
context of intersecting branes. In general quantization does not commute with
taking the tension to zero. In flat space, the common mass level has its origin
in the fact that string theory only has a single energy scale, the tension In the
tensionless limit there is no scale left. We show that tensionless strings have
a natural place in the context of supergravity. We find a background for type
IIB string theory that we are able to interpret as the geometry sourced by a
tensionless string.

The relation between higher spin gauge theory and tensionless strings can
probably be easiest understood in the context of the AdS/CFT correspondence.
If one looks at a hologram one sees a three dimensional picture that is stored
in a two dimensional area. In string theory this holographic principle in its
most famous version states that string theory in an Anti-de Sitter space has a
dual description in terms of a supersymmetric conformal field theory on the
boundary of the space. This correspondence has been tested ever since it has
been conjectured back in 1997 and lead to such amazing results as that cer-
tain sectors of the string theory are integrable models that can be treated with
solid state physics methods, but at least to my knowledge, no rigorous proof is
known. It relates the string tension to the coupling constant of the gauge the-
ory. Thus, the tensionless string corresponds to a vanishing gauge theory coupling where higher spin gauge fields appear. Five dimensional anti-de Sitter is part of a larger space where type IIB string theory is consistent, \( AdS_5 \times S^5 \). Unfortunately, string theory on this background is rather difficult and not much is known about the quantum theory. There are three known backgrounds for type IIB supergravity that are maximally supersymmetric. That means they preserve 32 supersymmetries. These are flat space, \( AdS_5 \times S^5 \) and a very recently discovered so-called plane wave background. This latter shares a lot of properties with \( AdS_5 \times S^5 \) but is considerably simpler. In fact, it can be derived as a certain limit of \( AdS_5 \times S^5 \). It turns out that closed string theory is a solvable model on this spacetime, at least in light-cone gauge gauge, where only the physical degrees of freedom are taken into account it has been solved and quantized. We analyze the closed tensionless type IIB string in this plane wave background and compare it to the tensile case with two main results. For the first, as opposed to flat space, the quantum theory is well-behaved and can actually be derived as a limit of the tensile theory. This can be traced back to a scale provided by the background itself that survives the tensionless limit. Secondly, the tension enters the solution only in combination with this scale parameter, which is actually related to the curvature of the space. Therefore, our result has a dual description in terms of a tensile string in a highly curved plane wave background.

The way string theory determines its own target space geometry is rather intriguing. It was already mentioned in the context of compactification, that for consistency, the internal six-dimensional manifold has to be of a certain type. This type is determined by the fact, that we want to consider four-dimensional space with \( N = 1 \) supersymmetry. If the internal space is to be Kähler, then the only choice is Calabi-Yau. Even tough people were aware, that there are solutions that are not Kähler, these possibilities were not considered for a long time. For a sigma model with supersymmetry on the worldsheet of a string, that is the area the string sweeps out in the target manifold called spacetime, the geometry of its target space is determined by the dimension of the worldsheet and the amount of supersymmetry. For example the manifest \( N = (1, 1) \) supersymmetric sigma model admits twice this amount of supersymmetry if the target space is bi-hermitian. Although classified, again the cases that are not Kähler were not considered to be of major importance. Lately, a new mathematical concept, generalized complex geometry, was founded that unifies complex and symplectic geometry. In fact, it smoothly interpolates between them. It turned out to be the right framework to discuss this interesting relation between worldsheet supersymmetry and target space geometry in. It was found that a subset of these new geometries called generalized Kähler geometry is equal to the bi-hermitian geometry and moreover that it can be completely described in terms of manifest \( N = (2, 2) \) supersymmetry. Generalized Calabi-Yau is another subset and is considered in compactifications.
with fluxes. Finally, generalized complex geometry might give a mathematical explanation for mirror symmetry. It unifies the topological A- and B-model into a single model.

Based on the fact that generalized complex geometry is related to the discussion of supersymmetry in the sigma model phase space, we show how generalized Kähler geometry arises very naturally in the Hamiltonian treatment of the supersymmetric sigma model. We argue that from the physics point of view, the relation between bi-hermitian and generalized Kähler geometry is established by the equivalence of the Hamiltonian and the Lagrangian treatment of the sigma model. We then go a step further and show how another subset, called generalized Hyperkähler geometry is related to $N = (4,4)$ supersymmetry on the worldsheet in the same way. The sigma model can be generalized by introducing auxiliary fields. We argue how supersymmetry in such a case favors a target geometry that is beyond generalized complex geometry. The lack of a proper understanding of these geometries manifests itself in the absence of a proper mathematical notion. This leaves us bound to a very simple toy-model. However, we are able to identify the relevant geometrical objects and show how generalized complex geometry is included in this new type of geometries.

We conclude with a summary of the publications included in this thesis.

**Paper I**

In the first paper, we describe how tensionless strings give rise to background solutions in IIB supergravity. Our starting point the geometry that is sourced by a macroscopic string which we then accelerate to the speed of light. In this limit, the string tension vanishes and the geometry becomes similar to a gravitational shock wave.

**Paper II**

We study the closed, tensionless IIB string in a maximally supersymmetric plane wave background. The solution is similar to the case of non-vanishing tension. Quantization of the tensionless string turns out to be unproblematic, as opposed to flat space. This can be traced back to the existence of a parameter related to the curvature of the background. We show that the tensionless string can be derived as a certain limit of the tensile string in this background and conclude that the limit commutes with quantization.

**Paper III**

In the third paper, we discuss the condition for which a generalized $N = (1, 1)$ supersymmetric sigma model admits additional supersymmetries. We find that the involved tensors naturally group together into objects that suggest an in-
interpretation beyond generalized complex geometry. Since we lack a proper understanding of this type of geometry, we are bound to a simple toy-model, such that we only can identify the relevant geometric objects and show how generalized complex geometry is embedded in this description.

**Paper IV**

We clarify the relation between generalized Kähler geometry and bi-hermitian geometry from a sigma model of view. We show that generalized Kähler geometry is the condition for $N = (2, 2)$ supersymmetry in a phase space formulation of the sigma model. The relation between generalized Kähler geometry and bi-hermitian geometry follows thus from the equivalence of the Hamiltonian and Lagrangian formulation of the sigma model. As an application of our results, we even discuss topological twists.

**Paper V**

In this paper, we study the condition for $N = (4, 4)$ supersymmetry in the Hamilton formulation of the sigma model. We find the definition of generalized hyperkähler geometry and define the twistor space of the generalized complex structures.
2. String theory basics

This chapter provides an elementary overview of those aspects of string theory that are needed to understand this thesis. We also use the opportunity to introduce our conventions and notations. For a broader introduction to string theory, we again refer to a number of good textbooks [GSW87, Pol98, Zwi04].

The motion of a relativistic point particle with mass \( m \) in spacetime is governed by the action

\[
S_{\text{part}} = m \int dt \sqrt{-\dot{X}^2}. \tag{2.1}
\]

Here, \( X(t) \) is the position of the particle at time \( t \). The action is thus equal to the length of the particle’s worldline. The action principle tells us that classically, the particle chooses the shortest path between two points. A string is a one dimensional object moving in spacetime. We can regard its motion as a two dimensional worldsheet \( \Sigma \) embedded in the spacetime \( M \) by maps \( X : \Sigma \to M \). The worldsheet has Minkowski signature with a time direction \( \tau \) and a spacial direction \( \sigma \), which we conveniently combine into a single coordinate \( \xi^a, a = 0, 1 \). We use both notations on an equal footing. Strings can be open or closed making the worldsheet either a strip or a cylinder. In this thesis, we mainly consider closed strings. Therefore, \( \Sigma = \mathbb{R} \times S^1 \). The compact direction is the spacial one, such that \( \sigma \simeq \sigma + \pi \). In analogy to the particle, the string moves classically in such a way that it minimizes the area it sweeps out in spacetime. The action is equal to the world volume of the string

\[
S_{\text{NG}} = T \int_{\Sigma} d^2 \xi \sqrt{\eta}, \tag{2.2}
\]

This action is called the Nambu-Goto action of the bosonic string. The factor \( T \) is the string tension and \( g \) is the determinant of \( g_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \). This is the pullback of the spacetime metric onto \( \Sigma \). For the moment, we consider a string in \( D \)-dimensional Minkowski space. The determinant is equal to

\[
g = -\dot{X}^2 \dot{X}'^2 + (\dot{X} \cdot \dot{X}')^2. \tag{2.3}
\]

We denote a derivative with respect to \( \tau \) by a \textit{dot} and a \( \sigma \)-derivative by a \textit{prime}. The conjugate momenta \( P_\mu = T \sqrt{-g} g^{ab} \partial_a X_\mu \) derived from the action are constrained:

\[
P_\mu X^\mu = 0, \quad P_\mu P^\mu + T^2 g^{00} = 0. \tag{2.4}
\]
These are the Virasoro constraints. Here, $g^{ab}$ is the inverse of $g_{ab}$. There is an equivalent way to write the string action that avoids taking the square root of the fields and incorporates the Virasoro constraints. It makes use of the worldsheet metric $h_{ab}$ and is given by

$$S_{\text{Poly}} = -\frac{T}{2} \int d^2\xi \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \quad (2.5)$$

This action was found by Brink, Deser, di Vecchia, Howe and Zumino [BDVH76, DZ76] but is usually known as the Polyakov action [Pol81a, Pol81b]. This action is a special case of a sigma model that maps one space into another, in this case the worldsheet $\Sigma$ into spacetime. The way the worldsheet is embedded in spacetime does not depend on how we choose to parametrize it, the action is invariant under reparametrizations of the worldsheet

$$\delta(a)X^\mu = a^a \partial_a X^\mu, \quad \delta(a)h^{ab} = a^c \partial_c h^{ab} - \partial_c a^a h^{cb}. \quad (2.6)$$

Here, $A^{(ab)} = A^{ab} + A^{ba}$ denotes symmetrization in the indices $a$ and $b$. We define symmetrization and antisymmetrization ($A^{[ab]} = A^{ab} - A^{ba}$) without a factor. Local Weyl transformations generate an additional symmetry of the worldsheet. They are parametrized by scalar functions on the worldsheet $\Lambda(\sigma, \tau)$ and multiply the worldsheet metric by a factor while leaving $X^\mu$ invariant

$$\delta(a)h^{ab} = \Lambda(\sigma, \tau)h^{ab}. \quad (2.7)$$

The field equation for $h^{ab}$ requires the two-dimensional energy momentum tensor to vanish

$$T_{ab} = (\partial_a X^\mu \partial_b X^\nu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X^\nu) \eta_{\mu\nu} \overset{!}{=} 0. \quad (2.8)$$

This is a consequence of the reparametrization invariance and it can be used to integrate out the worldsheet metric and obtain back the Nambu-Goto action, since it tells us that the determinant of $g_{ab}$ is given by

$$g = \frac{1}{4} h(h^{ab} g_{ab})^2. \quad (2.9)$$

We can use reparametrization invariance and Weyl symmetry to choose a conformally flat worldsheet metric, $h_{ab} = \eta_{ab}$. This choice is called the conformal gauge. Worldsheet light-cone coordinates $\xi^{\pm} = \tau \pm \sigma$ correspond to left and right moving modes on the string. We denote the worldsheet indices by $+-$ and $-$ in order to distinguish them from fermionic worldsheet indices $+$ and $-$ which we introduce in the discussion of supersymmetry. In these coordinates, the string action becomes

$$S = \frac{T}{2} \int d^2\xi \partial_{+} X^\mu \partial_{-} X^\nu \eta_{\mu\nu}. \quad (2.10)$$
This must be supplemented by requiring the energy momentum tensor $T_{ab}$ to vanish. This is now a constraint. $T_{ab}$ is traceless and in coordinates $\xi^{\pm}$, the constraints are given by $T_{++} = T_{--} = 0$. Since the conjugate momenta are $P_{\mu} = T_{\mu\nu}\dot{X}^{\nu}$, we recover exactly the Virasoro constraints (2.4).

After choosing a conformally flat worldsheet metric there is still some gauge freedom left. We may choose light-cone coordinates $X^{\pm} = \frac{1}{\sqrt{2}}(X^{0} \pm X^{D-1})$, $X^{I}$, $I = 1 \ldots D - 2$ on the target space. The equation of motion for $X^{\mu}$ are the wave equations

$$\partial_{+\pm} \partial_{-} X^{\mu} = (\partial_{\sigma}^{2} - \partial_{\tau}^{2})X^{\mu} = 0.$$ (2.11)

The remaining symmetry is given by reparametrizations of the worldsheet of the form

$$\tau \to f^{(+)}(\tau + \sigma) + f^{(-)}(\tau - \sigma),$$
$$\sigma \to f^{(+)}(\tau + \sigma) - f^{(-)}(\tau - \sigma),$$
$$h^{ab} \to (\partial_{+\pm} f^{(+)} \partial_{-} f^{(-)})^{-1} h^{ab}.$$ (2.12)

Herein, $f^{(+)}$ and $f^{(-)}$ are arbitrary functions that leave the form of the metric $h^{ab} = \eta^{ab}$ invariant. After such a transformation, the new time coordinate satisfies the one-dimensional wave equation $(\partial_{\sigma}^{2} - \partial_{\tau}^{2})\tau^{\text{new}} = 0$. Since $\tau$ and $X^{\pm}$ both satisfy the wave equation, we can use the remaining gauge freedom to relate them to each other by fixing

$$X^{+}(\sigma, \tau) = \frac{p^{+}}{T} \tau.$$ (2.13)

The constant $p^{+}$ is the conjugate momentum for $X^{+}$. This gauge is called the light-cone gauge and we see that $X^{+}$ and $X^{-}$ completely decouple from the action. $X^{-}$ can be determined by the Virasoro constraints which in light-cone gauge read

$$p^{+}X^{-} + T \dot{X}^{I}X_{I}' = 0, \quad 2p^{+}X^{-} + T(\dot{X}^{I}\dot{X}_{I} + X''X_{I}') = 0.$$ (2.14)

One concludes that there are only $D - 2$ physical bosonic degrees of freedom of the string given by the transverse components $X^{I}$.

### 2.1 Non-linear sigma model

String theory is a special case of a non-linear sigma model. In general, such a model embeds one space into another. It consists of a base manifold $\Sigma$ and a target manifold $M$ and a map

$$X^{\mu} : \Sigma \to M$$ (2.15)
that stands for the embedding. The case where \( \Sigma \) is a two dimensional worldsheet is very special, since it allows for conformal invariance of the worldsheet. Of course, there is no need for the target manifold to be flat. It can be a curved spacetime with metric \( G_{\mu\nu}(X) \), but it can also be supported by a two-form \( B_{\mu\nu}(X) \) and a scalar field \( \phi \) called the dilaton. Putting everything together, we obtain the most general action for a bosonic string

\[
S = -\frac{1}{2} \int d^2 \xi \left( T(\sqrt{-h}h^{ab}G_{\mu\nu} + \varepsilon^{ab}B_{\mu\nu})\partial_a X^\mu \partial_b X^\nu + 8\pi \sqrt{-h}R\phi \right), \tag{2.16}
\]

where \( R \) is the two-dimensional Ricci scalar for \( h \). We see that we can obtain (2.5) as a special case of it with a worldsheet periodic in the spacial direction \( \Sigma = S^1 \times \mathbb{R} \). The part involving the dilaton arises as a one loop effect, while the first two terms form the celebrated non-linear sigma model. In conformal gauge when the worldsheet metric is chosen to be conformally flat, the non-linear sigma model action reads

\[
S_{\text{NLSM}} = \frac{1}{2} \int d^2 \xi \partial_+ X^\mu \partial_- X^\nu (G_{\mu\nu}(X) + B_{\mu\nu}(X)). \tag{2.17}
\]

Metric and \( B \)-field can be conveniently combined into a single tensor \( e_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \). The field strength for \( B, H = dB \) is explicitly given by

\[
H_{\mu\nu\rho} = \frac{1}{2} (B_{\mu\nu,\rho} + B_{\nu\rho,\mu} + B_{\rho\mu,\nu}). \tag{2.18}
\]

Indices separated by a comma denote partial spacetime derivatives \( B_{\mu\nu,\rho} = \partial_\rho B_{\mu\nu} \). It is important to stress that the action (2.17) does not depend on \( B \) but on its field strength \( H \) only. This is seen easiest by invoking Stokes theorem. If we assume that \( \Sigma \) is the boundary of some three-dimensional worldsheet \( \Sigma^3, \Sigma = \partial \Sigma^3 \) and denote the pullback of \( B \) onto the worldsheet \( \Sigma \) by \( \varphi^*(B) \), we find

\[
\int_\Sigma \varphi^*(B) = \int_{\Sigma^3} \varphi^*(H). \tag{2.19}
\]

The term involving \( B \) respective \( H \) is called a Wess-Zumino term. It is indeed possible to consider the more general case when \( H \) is closed but not exact.

The study of sigma models in general differs somewhat from the discussion of string theory. We regard (2.17) as a field theory for \( X^\mu \). If we want to discuss string theory, we have to make use of the Virasoro constraint (2.8) as well. From the field theory point of view, the Lagrangian formulation and the action principle is just one way to study the sigma model. Equivalently, we can change to a phase space formulation and describe the worldsheet dynamics in terms of a Hamiltonian.

In the phase space formulation, the base manifold has one less dimension as compared to the Lagrangian formulation. The phase space of a worldsheet of
the two dimensional sigma model with spacial periodic boundary conditions on the worldsheet can be identified with the cotangent bundle $T^*LM$ of the loop space $LM = \{X : S^1 \to M\}$ [AS05]. The loop space consists of vector fields $X(σ)$ embedding the spacial direction of the worldsheet into the manifold. $X$ is periodic in $σ$: $X(σ + π) = X(σ)$. With this, points in $T^*LM$ are given by pairs $(X(σ), π(σ))$ where $π(σ)$ is a section of the cotangent bundle at $X$. When considering a string moving in spacetime, we can parametrize its current position and conjugate momentum by a such a pair $(X(σ), P(σ))$.

Momentum and fields are conjugated by means of a two form, the canonical symplectic structure

$$\omega = \int_{S^1} dσ (δX^\mu ∧ δP_\mu).$$

(2.20)

It yields the Poisson bracket

$$\{F, G\} = \int_{S^1} dσ F \left( \frac{δ}{δP_\mu} \frac{δ}{δX^\mu} - \frac{δ}{δX^\mu} \frac{δ}{δP_\mu} \right) G.$$

(2.21)

In phase space, we can consider generators for the symmetries of the worldsheet. The generator of $σ$-translations is given by

$$P(a) = -\int dσ P_\mu ∂X^\mu,$$

(2.22)

where $∂ ≡ ∂_σ$. It acts on the field via the Poisson bracket

$$δ(a)X^\mu = \{X^\mu, P(a)\} = a∂X^\mu, \quad δ(a)P_\mu = \{P_\mu, P(a)\} = a∂P_\mu.$$

(2.23)

In the presence of a closed three form $H \in \Omega_3(M)_{cl}$, the symplectic structure is twisted in the following way:

$$ω_H = \int_{S^1} dσ (δX^\mu ∧ δP_\mu + H_{μνρ}∂X^\mu δX^ν ∧ δX^ρ).$$

(2.24)

This is the case when the Wess-Zumino term (2.19) is present in the action of the sigma model. It yields a twisted version of the Poisson bracket, denoted by $\{F, G\}_H$. Also, $P(a)$ gets twisted appropriately. The details are part of the appendix of [IV]. If not otherwise stated, we always assume that $H$ is the field strength for $B$. The symplectic structure is invariant under transformations of the kind

$$X^\mu \to X^\mu, \quad P_\mu \to P_\mu + B_\mu ν ∂X^ν.$$

(2.25)

This is a symmetry of the symplectic structure if $B$ is a closed two-form, $B \in \Omega_2(M)_{cl}$. If $B$ is not closed, such a transformation twists the symplectic structure by $dB$. This will be an important fact in the discussion of supersymmetric sigma models and generalized complex geometry in chapter 6.
describe dynamics, the phase space is accompanied by a (canonical) Hamiltonian. It is the generator of time evolution. The Hamiltonian corresponding to \((2.17)\) with \(B = 0\) is derived in by a Legendre transformation with respect to the worldsheet coordinate \(\tau = \xi^0\). With \(P_\mu = G_{\mu \nu} \dot{X}^\nu\) we can rewrite the action \((2.17)\) in phase space

\[
S_g = \int dt d\sigma \left( P_\mu \dot{X}^\mu - \frac{1}{2} (P_\mu P_\nu G^{\mu \nu} + \partial X^\mu \partial X^\nu G_{\mu \nu}) \right).
\]

(2.26)

The first part yields a presymplectic form, the so-called Liouville form

\[
\Theta = \int d\sigma P_\mu \delta X^\mu,
\]

(2.27)

whose differential is the symplectic form \(\omega = \delta \Theta\) \((2.20)\). The second part is the Hamiltonian

\[
H(P,X) = \frac{1}{2} \int d\sigma \left( P_\mu P_\nu G^{\mu \nu} + \partial X^\mu \partial X^\nu G_{\mu \nu} \right).
\]

(2.28)

The \(B\)-field can be included using the \(B\)-transformation \((2.25)\). The second term in \((2.17)\) can be obtained in two different ways. One can perform the transformation on the presymplectic form \((2.27)\), such that

\[
\Theta_B = \int d\sigma (P_\mu + B_{\mu \nu} \partial X^\nu) \delta X^\mu.
\]

(2.29)

This results in a twisting of the symplectic structure with \(\omega_H = \delta \Theta_B\). Acting with the inverse transformation on the Hamiltonian generates the same term

\[
H_B = \frac{1}{2} \int d\sigma \left( (P_\mu - B_{\mu \rho} \partial X^\rho) G^{\mu \nu} (P_\nu - B_{\nu \sigma} \partial X^\sigma) - \partial X^\mu \partial X^\mu \right).
\]

(2.30)

The difference is that in the first way, \(P_\mu\) denotes the physical momentum, while for the second, it is the canonical momentum for \(X^\mu\). The physics described by the Hamiltonian is the same as for the action \((2.17)\). Consequently, also here, only \(H\) is important and not \(B\). Assigning the contribution from the \(B\)-field to the symplectic structure is thus the preferred choice. This makes it possible to also discuss twists with closed but not exact three forms. We will see later, that this is a crucial point in the \(N = (1,1)\) supersymmetric version of the sigma model. There, the twisted Hamiltonian contains an additional, purely fermionic piece proportional to the flux \(H = dB\) that cannot be removed by a \(B\)-transformation of the form \((2.25)\).

Let us consider a vector field \(u^\mu(X)\) and a one-form field \(\xi_\mu(X)\) on the target manifold \(M\). We can associate the following current to it:

\[
J_{\mu + \xi}(\sigma) = u^\mu(X(\sigma)) P_\mu(\sigma) + \xi_\mu(X(\sigma)) \partial X^\mu(\sigma).
\]

(2.31)
These types of currents play an important role in the discussion of symmetries for a wide class of two dimensional sigma models and have been studied in [AS05]. We already saw that the current

\[ J_P(\sigma) = P_\mu \partial X^\mu \]  

(2.32)
yields the generator of \( \sigma \)-translations (2.22). The Poisson bracket of two currents of the form (2.31) has two parts

\[ \{ J_{u+\xi}(\sigma), J_{v+\eta}(\sigma') \} = J_{[u+\xi,v+\eta]}(\sigma) \delta(\sigma - \sigma') + \frac{1}{2} (u^\mu \eta_\mu + v^\mu \xi_\mu) \delta'(\sigma - \sigma'). \]  

(2.33)
The first part is this kind of current associated to the Courant bracket of \( u + \xi \) and \( v + \eta \)

\[ [u + \xi, v + \eta]_c = [u, v] + L_u \eta - L_v \xi - \frac{1}{2} d (i_u \eta - i_v \xi). \]  

(2.34)
Here, \( L_u \cdot = d(i_u \cdot) + i_u d \cdot \) is the Lie derivative and \( i_u \xi = u^\mu \xi_\mu \) is the contraction of a vector field and a one-form. The Courant bracket reduces to the ordinary Lie bracket when restricted to vector fields \( u \) on \( TM \).

### 2.2 Worldsheet supersymmetry

If we quantize string theory with the action (2.5), or even in the more general background with (2.16), the physical spectrum only contains bosons. Since nature contains also fermions and string theory is supposed to eventually describe fundamental physics, we must include fermions. A way for doing that is to consider supersymmetry. Supersymmetry is the only possible non-trivial extension of the Poincaré algebra. If \( P_\mu \) is the generator for spacetime translations and \( M_{\mu\nu} \) generates Lorentz rotations then the spacetime symmetries consistent with a relativistic quantum field theory are generated by

\[ [P_\mu, P_\nu] = 0, \quad [M_{\mu\nu}, P_\rho] = \frac{1}{2} \eta_{\rho\mu} P_\nu, \]

\[ [M_{\mu\nu}, M_{\rho\sigma}] = \frac{1}{2} \eta_{\rho\mu} M_{\nu\sigma} - (\rho \leftrightarrow \sigma). \]  

(2.35)

For example, the commutator of a translation and a rotation is a translation. To consider supersymmetry, we introduce a generator \( Q_\alpha \) that satisfies

\[ \{ Q_\alpha, Q_\beta \} = \Gamma^\mu_{\alpha\beta} P_\mu, \]  

(2.36)
where \( \{ , \} \) is the anticommutator and \( \Gamma^\mu \) are matrices satisfying the Clifford algebra

\[ \Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = -2 \eta^{\mu\nu} 1. \]  

(2.37)
Supersymmetry can be introduced in various ways into string theory. We can think of supersymmetry on the worldsheet, on the target manifold, or both and we can vary the amount of supersymmetry. To make things clear, we consider a sigma model in flat Minkowski space and worldsheet supersymmetry. Supersymmetry is a symmetry that relates bosons and fermions. In (2.36) we see that the anticommutator of two objects with half integer statistics gives a bosonic object which has integer spin. For worldsheet supersymmetry, we introduce fields \( \psi_+^\mu = (\psi_+^\mu, \psi_-^\mu) \) that behave as real, anticommuting two-dimensional spinors on the worldsheet and transform as a vector under the Lorentz group of the target manifold:

\[
\psi_+^\mu \psi_-^\nu = - \psi_-^\nu \psi_+^\mu.
\]  

(2.38)

In our notation, worldsheet spinor indices are denoted by \( \alpha, \beta, \ldots = +, - \). We introduce two-dimensional Dirac matrices that satisfy the Clifford algebra \( \{ \gamma^a, \gamma^b \} = -2 \eta^{ab} \mathbf{1} \). With these preliminaries, we can write down the action

\[
S = -\frac{1}{2} \int d^2 \sigma \left( \partial_a X^\mu \partial^a X^\nu - \frac{1}{2} \bar{\psi}^\mu \gamma^a \partial_a \psi^\nu \right) \eta_{\mu \nu},
\]  

(2.39)

where \( \bar{\psi} = \psi' \gamma^0 \). This action is a supersymmetric extension of (2.10). The supersymmetry transformations are parametrized by a constant anticommuting spinor \( \varepsilon \)

\[
\delta(\varepsilon) X^\mu = \bar{\varepsilon} \psi^\mu, \quad \delta(\varepsilon) \psi^\mu = -\frac{1}{2} \beta \gamma^a \partial_a X^\mu \varepsilon,
\]  

(2.40)

where the contraction of spinor indices is implicit. The expression \( \bar{\varepsilon} \psi^\mu \) is a shorthand notation for \( \varepsilon_\alpha (\gamma^0)_{\alpha \beta} \psi^\mu_\beta \). Indeed, this transformation relates the bosonic field \( X^\mu \) to the spinor \( \psi^\mu \). The equations of motion for the spinors \( \gamma^a \partial_a \psi^\mu = 0 \) show that \( \psi_\pm^\mu \) are left and right moving components

\[
\partial_+ \psi_-^\mu = 0, \quad \partial_- \psi_+^\mu = 0.
\]  

(2.41)

For our purposes, it is useful to go to a Dirac matrix free notation. To this end, we define contraction of spinor indices according to the ‘up-left-down-right’ rule and raise and lower them with the antisymmetric tensor

\[
C_{+-} = -C_{-+} = \beta, \quad \psi_\alpha^\mu = (\psi^\mu)^\beta C_{\beta \alpha}, \quad (\psi^\mu)^\alpha = C^{\alpha \beta} \psi_\beta^\mu.
\]  

(2.42)

With the Dirac matrices explicitly given by

\[
\gamma^0 = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix},
\]  

(2.43)

we write out the second term in the supersymmetric action to find

\[
S = \frac{1}{2} \int d^2 \xi \left( \partial_+ X^\mu \partial_- X^\nu + \beta (\psi_-^\mu \partial_+ \psi_-^\nu + \psi_+^\mu \partial_- \psi_+^\nu) \right) \eta_{\mu \nu}.
\]  

(2.44)
The supersymmetry transformations leaving this action invariant are
\[
\delta(\varepsilon)X^\mu = (\varepsilon^- \psi^-_\mu + \varepsilon^+ \psi^+_\mu), \\
\delta(\varepsilon)\psi^-_\mu = -\beta \varepsilon^- \partial^- X^\mu, \\
\delta(\varepsilon)\psi^+_\mu = -\beta \varepsilon^+ \partial^+ X^\mu. \\
\]
(2.45)

Infinitesimal translations of the worldsheet $\xi^a \rightarrow \xi^a + \alpha^a$ act on the fields as $\delta X^\mu = a^b \partial_b X^\mu$. According to (2.36) the commutator of two supersymmetry transformations gives a translation.
\[
[\delta(\varepsilon_1), \delta(\varepsilon_2)]X^\mu = 2(\varepsilon_1^+ \varepsilon_2^+ \partial^+ + \varepsilon_2^- \varepsilon_1^- \partial^-)X^\mu.
\]
(2.46)

Concerning the spinor fields, the corresponding relation is only satisfied on-shell, i.e. by imposing the equations of motions (2.41). This can be amended by introducing an auxiliary field. A particularly useful way to implement supersymmetry is via superspace [GGRS83]. It incorporates the auxiliary field and makes supersymmetry manifest. To this end, one introduces additional directions on the worldsheet. The number of these directions depends on the amount of supersymmetry. In the present case, the worldsheet is extended by two such directions $\theta^\alpha$, $\alpha = +, -$. They are anticommuting
\[
\{\theta^\alpha, \theta^\beta\} = 0
\]
(2.47)
and usually called Grassmann coordinates. A superfield $\Phi^\mu$ is a map from this extended (super-)worldsheet $\hat{\Sigma}$ into the target manifold,
\[
\Phi(\sigma, \tau, \theta^+, \theta^-) : \hat{\Sigma} \rightarrow M.
\]
(2.48)

For each Grassmann direction, there is a generator of supersymmetry. These are odd differential operators
\[
Q_\pm = \beta \partial_\theta^\pm + \theta^\pm \partial^\pm.
\]
(2.49)

$Q_\pm$ generates a supersymmetry transformation since $Q^2_\pm = -\partial^\pm$. There are two more independent odd differential operators that one can define:
\[
D_\pm = \partial_\theta^\pm + \beta \theta^\pm \partial^\pm.
\]
(2.50)

They act like covariant derivatives for $\theta^\pm$ and satisfy the following algebraic relations together with $Q_\pm$:
\[
Q^2_\pm = -\beta \partial^\pm \quad D^2_\pm = \beta \partial^\pm \quad \{D_\pm, Q_\pm\} = 0.
\]
(2.51)

Geometrically, this means that “flat” superspace has torsion. 

15
This formulation makes supersymmetry manifest, since the whole supermultiplet is described by a single superfield $\Phi^\mu$ and supersymmetry transformations are generated by $Q_\pm$ acting simply on $\Phi^\mu$. The worldsheet coordinates transform as

$$
\delta(\varepsilon)\xi^\pm = -\beta(\varepsilon^+ Q_+ + \varepsilon^- Q_-)\xi^\pm = -\beta\varepsilon^\pm\theta, \\
\delta(\varepsilon)\xi^\mp = -\beta\varepsilon^-\theta, \\
\delta(\varepsilon)\theta^\pm = \varepsilon^\pm.
$$

The transformation of the superfield $\Phi^\mu$ is given by

$$
\delta(\varepsilon)\Phi^\mu = -\beta(\varepsilon^+ Q_+ + \varepsilon^- Q_-)\Phi^\mu
$$

To write down an action which incorporates the manifest supersymmetry, we notice that the transformation of any function of the form $L(\Phi, D_+ \Phi, D_- \Phi)$ under (2.52) is a total derivative. Therefore, the action

$$
S = \frac{1}{2} \int d^2\xi d^2\theta \Phi^\mu \eta_{\mu\nu} (D_+ \Phi^\mu (D_- \Phi^\nu) \eta_{\mu\nu})
$$

is manifestly supersymmetric. The variation of $S$ under (2.52) is a total derivative and vanishes for a topologically trivial worldsheet. The action is a straightforward generalization of (2.10). The $d\theta$ integrals are Berezin integrals and can be evaluated as

$$
S = \frac{1}{2} \int d^2\xi \left(D_+ D_-(D_+ \Phi^\mu D_- \Phi^\nu) \eta_{\mu\nu}\right) |_{\theta^\pm = 0}.
$$

We define the components of $\Phi^\mu$ with the help of the covariant derivatives $D_\pm$:

$$
X^\mu = \Phi^\mu |, \\
\psi^\mu_\pm = (D_\pm \Phi^\mu) |, \\
F^\mu = (D_+ D_- \Phi^\mu) |.
$$

The $\bar{\ }$ denotes that we set $\theta^+ = \theta^- = 0$ in the expression. $X^\mu$ and $F^\mu$ are bosonic, while $\psi^\mu_\pm$ are a worldsheet spinor. Integrating out the Grassmann directions in the action yields its component form

$$
S = \frac{1}{2} \int d^2\xi \left(\partial_+ X^\mu \partial_- X^\nu + \beta \psi^\mu_+ \partial_- \psi^\nu_+ + \beta \psi^\mu_- \partial_+ \psi^\nu_- - F^\mu F^\nu \right) \eta_{\mu\nu}.
$$

$F^\mu$ is an auxiliary field. It has algebraic equations of motion, $F^\mu = 0$, and substituting them in the action recovers (2.39).

If one solves the equations of motion and tries to write down a consistent quantized theory, then one finds that the spectrum has to be truncated in a certain way. Interestingly enough, this truncation yields spacetime supersymmetry and therefore even spacetime fermions. However, we do not persue in this direction. Instead, we turn directly to a discussion of supersymmetry in spacetime.
2.3 Spacetime supersymmetry

We introduce spacetime supersymmetry in the same way as worldsheet supersymmetry by extending the target space to superspace. To this end, we introduce a number of Grassmann coordinates $\theta^A$ where $A = 1 \ldots N$ counts the number of supersymmetries and $\alpha$ is the (spacetime) spinor index. We are only interested in the case where the target manifold is ten dimensional Minkowski space. A spinor of the ten dimensional Lorenz group $SO(1,9)$ has 32 complex components. The $32 \times 32$ dimensional Dirac matrices $\Gamma^\mu$ satisfy

$$\{ \Gamma^\mu, \Gamma^\nu \} = 2\eta^{\mu\nu}1. \quad (2.58)$$

Under supersymmetry, the coordinates $x^\mu$ and $\theta^A$ are transformed into each other similar to the case of worldsheet supersymmetry (2.52)

$$\delta(\epsilon)x^\mu = \xi_\epsilon^A \Gamma^\mu \theta^A, \quad \delta(\epsilon)\theta^A = \epsilon^A, \quad \delta(\epsilon)\bar{\theta}^A = \bar{\epsilon}^A, \quad (2.59)$$

where $\epsilon^A$ is a constant spinor. One may check that these transformations satisfy a supersymmetry algebra of the form (2.36). The simplest supersymmetric extension of the action (2.5) is given by

$$S = -\frac{1}{2} \int d^2\xi \left( \sqrt{-h} h^{ab} \hat{T}_a \hat{T}_b \eta_{\mu\nu} + 2\epsilon^{ab} \eta_{\mu\nu} \partial_a X^\mu (\bar{\theta}^1 \Gamma^\nu \partial_b \theta^1 - \bar{\theta}^2 \Gamma^\nu \partial_b \theta^2) - 2\epsilon^{ab} \eta_{\mu\nu} \bar{\theta}^1 \Gamma^\mu \partial_a \theta^1 \bar{\theta}^2 \Gamma^\nu \partial_b \theta^2 \right). \quad (2.60)$$

Here, $\hat{T}_a = \partial_a X^\mu - \beta \theta^A \Gamma^\mu \partial_a \theta^A$. As in the discussion of worldsheet supersymmetry, the contraction of spinor indices is implicit. Besides being supersymmetric, the action has a local fermionic symmetry called $\kappa$-symmetry

$$\delta \theta^A = 2\beta \Gamma^\mu \hat{T}_a \eta_{\mu\nu} \kappa^{Aa}, \quad \delta X^\mu = \beta \bar{\theta}^A \Gamma^\mu \delta \theta^A, \quad (2.61)$$

where $\kappa$ satisfies

$$\kappa^{1a} = P_{-} \kappa^{1b}, \quad \kappa^{2a} = P_{+} \kappa^{2b}, \quad P_{\pm} = \frac{1}{2}(\delta_{ab} \pm \epsilon_{ab}/\sqrt{h}). \quad (2.62)$$

In addition to (2.61), the metric transforms

$$\delta(\sqrt{-h} h^{ab}) = -16 \sqrt{-h}(P^{ac} \bar{k}^{1b} \partial_c \theta^1 + P^{ac} \bar{k}^{2b} \partial_c \theta^2). \quad (2.63)$$

The $\kappa$-symmetry allows us to make the following gauge choice for the fermions

$$\Gamma^+ \theta^A = 0, \quad (2.64)$$
where $\Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^9)$. This is sometimes also called fermionic light-cone gauge. We are only interested in type IIB string theory which has two real spacetime supersymmetries. We implement this by choosing Majorana-Weyl spinors. The Majorana condition reduces the 32 complex components to 32 real ones. The Weyl condition for the spinors is given with the help of $\Gamma^{11} = \Gamma^0 \cdots \Gamma^9$:

$$\Gamma^{11} \theta^A = \pm \theta^A, \quad (2.65)$$

For type IIB theories, both spinors have the same chirality, i.e. $\Gamma^{11} \theta^A = \theta^A$. The Dirac matrices decompose into chiral and anti-chiral representations $\gamma^\mu$ and $\bar{\gamma}^\mu$

$$\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \bar{\gamma}^\mu & 0 \end{pmatrix}. \quad (2.66)$$

The components are given by

$$\gamma^\mu = (1, \gamma', \gamma^9), \quad \bar{\gamma}^\mu = (-1, \gamma', \gamma^9) \quad (2.67)$$

with $\gamma^\mu = (\gamma^\mu)_{\alpha\beta}$ and $\bar{\gamma}^\mu = (\gamma^\mu)_{\alpha\beta}$. We assume that

$$\Gamma^{11} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.68)$$

and that $\gamma^\mu$ and $\bar{\gamma}^\mu$ are real and symmetric. The positive chirality condition reduces the number of components of $\theta^A$ to 16, given by

$$\theta^A = \begin{pmatrix} \theta^{A\alpha} \\ 0 \end{pmatrix}, \quad A = 1, 2, \quad \alpha = 1 \ldots 16. \quad (2.69)$$

In this notation, the conditions for the fermionic light-cone gauge become

$$\bar{\gamma}^+ \theta^A = 0. \quad (2.70)$$

Imposing fermionic lightcone gauge leaves us with 16 components in total. The connection to worldsheet supersymmetry can be seen in the following way: After going to lightcone gauge and fixing $\kappa$-symmetry, the equations of motion for the remaining degrees of freedom are given by

$$\partial^+ \partial^- X^I = 0, \quad \partial^+ \theta^1 = 0, \quad \partial^- \theta^2 = 0. \quad (2.71)$$

These are exactly the same as those for $X^I$, $\psi^I_\pm$ from the action (2.44) in the previous section. However, we should mention that the exact relation between the two different pictures is not just established by relabeling $\theta^{A\alpha}$ into $\psi^I_\pm$. It is a bit more involved since the $\theta^A$ transform as spacetime spinors while $\psi^I$ is a spacetime vector and a worldsheet spinor.
2.4 Low energy effective theory

When choosing a conformally flat worldsheet metric, we made use of the Weyl symmetry of the worldsheet and had to impose the Virasoro constraints by hand. In left and right moving worldsheet coordinates, $T_{++} = T_{--}$ vanishes due to the tracelessness of the energy momentum tensor. For a curved spacetime, this is only true in $D = 26$. If we go beyond the classical level and consider a quantum theory then the two-dimensional energy momentum tensor acquires an anomaly except for the case when the so-called $\beta$-functions of the background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and $\phi$ vanish. In $D = 26$ dimensions and to lowest order in the string scale $\alpha' = 4\pi T^{-1}$. The conditions for this are given by

\begin{align*}
\beta^G_{\mu\nu} &= \alpha' (R_{\mu\nu} + 2\nabla_\mu \nabla_\nu \phi - H_{\mu\rho\sigma}H^\rho\sigma_{\nu}) = 0, \\
\beta^B_{\mu\nu} &= \alpha' (-\nabla^\rho H_{\rho\mu\nu} + 2\nabla^\rho \phi H_{\rho\mu\nu}) = 0, \\
\beta^\phi &= \alpha' (-\frac{1}{2} \nabla^2 \phi + (\nabla \phi)^2 - \frac{1}{6} H^2) = 0. \ \ (2.72)
\end{align*}

All solutions to these equations yield consistent string backgrounds. The most remarkable feature of this set of equations is that they can be derived as the equations of motion for the background fields $G, B$ and $\phi$ from the spacetime action (in $D = 26$ dimensions)

\begin{equation}
S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{-G} e^{-2\phi} \left[ R + 4\nabla_\mu \phi \nabla^\mu \phi - \frac{1}{3} H^2 \right]. \ \ (2.73)
\end{equation}

This action describes the interaction of massless modes of the bosonic closed string in the long-wavelength limit, hence it is the corresponding low-energy effective theory. Here, $\kappa$ is the $D$-dimensional gravitational Newton’s constant. For supersymmetric theories, this result gets modified, the analysis however goes through in the same way. All supergravity theories share (2.73) as part of the bosonic part of the action. Supersymmetric string theory, however, requires a $D = 10$ dimensional target space. Finding consistent supergravity backgrounds was a major activity in the 1990s that lead for example to the discovery of D-branes. In 1990, Dabholkar et al. [DGHRR90] found a solution that was identified as the geometry of a heterotic superstring

\begin{align*}
\mathrm{d}s^2 &= A^{-3/4} [-\mathrm{d}t^2 + (\mathrm{d}x^1)^2] + A^{1/4} (\mathrm{d}x')^2, \\
B_{01} &= e^{2\phi} = A^{-1}, \quad A = 1 + \frac{Q}{3r^6}, \ \ (2.74)
\end{align*}

where $Q$ is the $B$-charge carried by the string and $x' = (x^2, \ldots, x^9)$ are the directions transverse to the string with $r^2 = x'^2$. The solution becomes singular at $r = 0$ and does not satisfy the equations of motion at these points. It is precisely this singularity that was interpreted as a macroscopic heterotic string. Later, after the discovery of S-duality, this solution was also identified as the geometry of a type I string [Dab95, Hul95]. S-duality relates the weakly coupled sector of one string theory to the strongly coupled sector of another, in
this particular case, it relates the heterotic string to the type I string. In chapter 4 we will see that the fundamental string and the D1-brane of IIB theory, which is also known as the D-string, yield similar solutions.
In this chapter we study the tensionless closed string on the maximally supersymmetric plane wave. This background to type IIB supergravity was found by Blau et.al. \[\text{[BFOHP02a]}\] as the ten dimensional equivalent to a family of 11d supergravity solutions called Kowalski-Glikman spaces \[\text{[KG84]}\]. It is supported by a constant selfdual five form that is directly related to the curvature of the spacetime It has parallel and planar wave fronts. Therefore, this background is sometimes also called a pp-wave. It is one of the three known maximally supersymmetric background for type IIB supergravity and is related to the other two. It is a Penrose limit of \(\text{AdS}_5 \times S^5\) on one side \[\text{[BFOHP02b, BFOP02]}\] and becomes flat space in the limit when the flux vanishes.

The AdS/CFT correspondence originally conjectured by Maldacena \[\text{[Mal98]}\] and later clarified in \[\text{[Wit98, GKP98]}\] underlies the desire to understand string theory in \(\text{AdS}_5 \times S^5\). The plane wave is a step in this direction. Metsaev and Tseytlin showed that closed string theory in light-cone gauge in this background is an integrable model and provided its solution classically and at the quantum level \[\text{[Met02, MT02]}\]. The AdS/CFT correspondence reduces to the BMN correspondence which relates certain parts of the string spectrum to planar diagrams on the gauge theory side \[\text{[BMN02, Ple04]}\]. This correspondence is not as strict as the AdS/CFT correspondence but it holds at least to first order in the expansion of \(\text{AdS}_5 \times S^5\) over the plane wave \[\text{[PR02, C*M03]}\]. In \(\text{AdS}_5 \times S^5\) the tensionless string is supposed to be related to higher spin gauge theory \[\text{[Vas99, HMS00, Sun01, SS02, Bon03, LZ04, Sav04, ES05]}\]. Part of this relation should survive the limit to the plane wave. In [II] we study the tensionless closed string in light-cone gauge on the plane wave background and find that it can be obtained as a well-behaved limit of the results of \[\text{[MT02]}\]. This behavior is traced back to the existence of a background scale which is related to the flux and allows for a reinterpretation of our results as the ordinary, tensile string moving in an infinitely curved plane wave background in accordance to \[\text{[dVGN95]}\].

This chapter proceeds in the following way. It starts out with a short introduction to the tensionless string and issues in flat space. We then present how the plane wave is obtained from \(\text{AdS}_5 \times S^5\) and review the solution of closed
string theory in this background before turning to the tensionless limit of this theory. We conclude this chapter with some remarks on the more general situation of a homogenous plane wave background.

3.1 Tensionless strings in flat space

The classical tensionless string was first mentioned in [Sch77] when strings that move with the speed of light turned out to have zero tension. This makes it a candidate for the description of the high-energy behavior of string theory [GM87]. Here, we follow the lines of [KL86, ILST94] where the classical and quantized bosonic tensionless string in flat space were discussed. The tensionless superstring has been studied in [BNRA89, LST91].

The action for a point particle is given by (2.1). By introducing an auxiliary field $e$, an einbein, the action can be brought into the form

$$S_{\text{part},P} = \int dt \left( e \dot{X}^2 + e^{-1} m^2 \right).$$

(3.1)

As long as $m \neq 0$, it is possible to gauge away $e$ using its (algebraic) field equations and rewrite the action in the first form. On the other hand, the massless particle action is obtained by taking $m \to 0$. The equivalent of (3.1) in string theory is the Polyakov action (2.5). To understand how to take the limit $T \to 0$, we have to understand how the Nambu-Goto and the Polyakov action are related to each other. The Nambu-Goto action was given in (2.2):

$$S = T \int d^2 \xi \sqrt{-g},$$

(3.2)

where $g$ was the determinant of $g_{ab} = \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}$. The conjugate momenta to $X^\mu$ are $P_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = T \sqrt{-g} g^{00} \dot{X}_\mu$ where $g^{ab}$ is the inverse of $g_{ab}$. The momenta are constrained by the Virasoro constraints (2.4)

$$P^2 + T^2 g^{00} = P \cdot X' = 0.$$

(3.3)

The Hamiltonian is given by these constraints, since the canonical Hamiltonian vanishes due to the diffeomorphism invariance of the worldsheet. If we introduce $\lambda$ and $\rho$ as Lagrange multiplies for the constraints then we can write down the phase space action corresponding to the Nambu-Goto action

$$S_{PS} = \frac{1}{2} \int d^2 \xi \left( P_\mu \dot{X}^\mu - \lambda (P_\mu P^\mu + T^2 g^{00}) - \rho P_\mu X' \mu \right).$$

(3.4)

The momenta can be integrated out using their (algebraic) field equations. This yields the configuration space action

$$S_{CS} = \frac{1}{4\lambda} \int d^2 \xi \left( (\dot{X}^\mu \dot{X}^\nu - 2\rho \dot{X}^{\mu} X'^\nu + \rho^2 X'^\mu X'^\nu) \eta_{\mu\nu} - 4\lambda^2 T^2 g^{00} \right).$$

(3.5)
This is the Polyakov action (2.5) with $h_{ab} = \left( \frac{-1}{\rho} \begin{pmatrix} \rho & 4\lambda^2 T^2 - \rho^2 \end{pmatrix} \right)$.

$$S_{\text{Poly}} = -\frac{T}{2} \int d^2 \xi \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \quad (3.6)$$

The constraints (3.3) are, of course, the Virasoro constraints. On the other hand, we can take the limit $T \rightarrow 0$ in the configuration space action. This limit is not covered by the Polyakov action since $h_{ab}$ became degenerate. Instead we can introduce a contravariant vector density $V^a = \frac{1}{\sqrt{2\lambda}} (1, \rho)$ and obtain the action for the tensionless string:

$$S_{T=0} = -\frac{1}{2} \int d^2 \xi V^a V^b \partial_a X^\mu \partial_b X^\mu \eta_{\mu\nu}. \quad (3.7)$$

This action has a reparametrization symmetry

$$\delta(a) X^\mu = a^a \partial_a X^\mu, \quad \delta(a) V^a = -V^b \partial_b a^a + a^b \partial_b V^a + \frac{1}{2} \partial_b a^b V^a \quad (3.8)$$

for a small parameter $a$. It allows to gauge away one of the components of $V^a$. A particularly useful gauge is the transverse gauge $V^a = (v^I, 0)$ in which the action takes the form

$$S_{T=0,tg} = -\frac{v^2}{2} \int d^2 \xi \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}. \quad (3.9)$$

Apart from the $d\sigma$ integral, this action looks like the action of a massless particle. As in the tensile case, the action (3.9) is still not completely gauge fixed. The residual symmetry that is left is

$$\delta \tau = f'(\sigma) \tau + g(\sigma), \quad \delta \sigma = f(\sigma). \quad (3.10)$$

Here, $f$ and $g$ are arbitrary functions of $\sigma$. Again, this allows us to go to light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1})$, $X^I$, $I = 1 \ldots D - 2$ and fix light-cone gauge by choosing $X^+ = \frac{v^+}{\nu} \tau$. The light-cone action of the tensionless string in flat space is given by

$$S_{\text{LC}} = \frac{v^2}{2} \int d^2 \xi \dot{X}^I \dot{X}^I. \quad (3.11)$$

We may compare this action to (2.10). Taking the tensionless limit amounts to replacing $T$ by $v^2$ and putting all $\sigma$-derivatives to zero. This rule of thumb can be stated more exactly. In order to take the limit $T \rightarrow 0$, we split the tension according to $T = \lambda v^2$, where $\lambda$ is a dimensionless parameter to be taken to zero and $v$ has the dimension of energy. Introducing a new worldsheet time $t = \tau/\lambda$, the action (2.10) becomes

$$S_{\text{LC}} = \frac{v^2}{2} \int dt d\sigma (\dot{X}^I \dot{X}^I - \lambda^2 X'^I X'^I). \quad (3.12)$$
Clearly, $\lambda \to 0$ amounts in (3.12) becoming (3.11). The original worldsheet parametrized by $\sigma$ and $\tau$ is now a null surface. The classical equations of motion obtained from the gauge fixed action (3.11) are

$$\ddot{X}^I = 0.$$  \hspace{1cm} (3.13)

By fixing the transverse gauge, the equations of motion for $V^a$ become the constraint equations

$$\ddot{X}^I \dot{X}^I - 2\frac{P^+}{\nu^2} \dot{X}^- = 0, \quad \ddot{X}^I \dot{X}^I - \frac{P^+}{\nu^2} X'^- = 0.$$  \hspace{1cm} (3.14)

These are the equivalent of the Virasoro constraints (2.14). Also for the tensionless string, the physical degrees of freedom are the transverse components $X^I$. At each value of $\sigma$, $X^I$ is a solution to (3.13). The string literally splits into infinitely many massless particles whose motion is restricted to be transverse to the string.

The action (3.7) has a global conformal spacetime symmetry. Dilatations are given by the scale transformation

$$\delta(\lambda)X^\mu = \lambda X^\mu, \quad \delta(\lambda)V^a = -\lambda V^a,$$  \hspace{1cm} (3.15)

and the conformal boost, or special conformal transformation, has the form

$$\delta(b)X^\mu = (b_\nu X^\nu)X^\mu - \frac{1}{2} X^2 b^\mu, \quad \delta(b)V^a = -(b_\nu X^\nu)V^a.$$  \hspace{1cm} (3.16)

There is no critical dimension for a consistent quantum theory in flat space [LRSS86]. However, the conformal symmetry survives quantization only in $D = 2$ spacetime dimensions. In any other dimension, the conformal algebra acquires an anomalous term which provides a selection rule for the physical states: The spectrum is hugely restricted and becomes topological [ILS92, GLS+95, Sal95]. This strengthens the view of the tensionless string as the unbroken, topological phase of string theory.

The vacuum state of the tensionless theory differs from the tensile case. It has more the form of a particle vacuum than a string vacuum. To obtain the quantum theory, we can proceed and introduce canonical commutation relations

$$[X^I(\sigma_1), P^J(\sigma_2)] = \beta \delta^{IJ} \delta(\sigma_1 - \sigma_2), \quad [X^-, p^+] = -\beta.$$  \hspace{1cm} (3.17)

We saw that $X^I(\sigma)$ is a collection of infinitely many degrees of freedom parametrized by $\sigma$. Therefore, the quantum theory has to be modified [ILST94] by regularizing the $\delta$-function. As long as there is little tension left, we would introduce left and right movers

$$\alpha_n^I = \frac{p_n^I}{\sqrt{T}} - \beta n \sqrt{T} x_n^I, \quad \tilde{\alpha}_n^I = \frac{p_n^I}{\sqrt{T}} + \beta n \sqrt{T} x_n^I, \quad n \neq 0,$$  \hspace{1cm} (3.18)
where $x^I_n$ and $p^I_n$ are the Fourier modes of $X^I$ and their conjugate momenta $P^I$. We would then define the vacuum state by the requirement that is annihilated by the positive frequency modes

$$\alpha_n^{1I} |0\rangle_0 = \alpha_n^{2I} |0\rangle_0 = 0, \quad n = 1, 2, \ldots \quad (3.19)$$

In the limit $T \to 0$, this implies

$$p^I_n |0\rangle_0 = p^I_{-n} |0\rangle_0 = 0. \quad (3.20)$$

From (3.19) we read off that also the $x^I_n$ annihilate the vacuum state $|0\rangle_0$ for all values $n \neq 0$. This is inconsistent with the commutation relations (3.17). The most natural possibility is therefore to choose a translation invariant vacuum state for tensionless string

$$P^I |0\rangle_0 = 0, \quad (3.21)$$

while keeping $X^I |0\rangle_0$ unspecified.

### 3.2 Plane wave geometry from $AdS_5 \times S^5$

Here, we show how the plane wave geometry arises as a Penrose limit of $AdS_5 \times S^5$. In any neighborhood of a null geodesic it is possible to choose coordinates in which the line element takes the special form

$$ds^2 = dx^+ dx^- + a(dx^+)^2 + k_I dx^+ dx^I + f_{IJ} dx^I dx^J, \quad (3.22)$$

This observation goes back to Penrose [Pen72] and is true as long as the neighborhood does not contain intersections of neighboring geodesics. The coordinates $x^+$ while $x^-$ parametrize a particle traveling along the geodesic while $x^I$ are coordinates transverse to it. Recently, this limit was extended to include the supergravity fields in ten and 11 dimensions [Gue00]. For the type IIB supergravity background $AdS_5 \times S^5$, this is the (constant) dilaton $\phi$ and the self-dual five form field strength $F_5$. The line element of $AdS_5 \times S^5$ is a combination of the part coming from $AdS$ and from the five sphere $ds^2 = ds^2_{AdS} + ds^2_{S^5}$. The radii of both subspaces are equal. Anti-de Sitter space is embedded in $\mathbb{R}^{2,4}$ as the hypersurface

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_5^2 = R^2. \quad (3.23)$$

There are a number of appropriate coordinates to parametrize $AdS$ space. We use so-called global coordinates

$$x_0 = R\cosh(\rho) \sin(t), \quad x_5 = R\cosh(\rho) \cos(t),$$

$$x_I = R\sinh(\rho) \omega_I, \quad I = 1, 2, 3, 4. \quad (3.24)$$
The coordinates $\omega_I$ parametrize the unit three sphere $\omega_I^2 = 1$. In these coordinates, the line element of $AdS$ space is given by

$$ds^2_{AdS} = R^2 \left[ -dt^2 \cosh^2(\rho) + d\rho^2 + \sinh^2(\rho) d\Omega_3^2 \right].$$

(3.25)

It is obtained by substituting (3.24) into the line element of $\mathbb{R}^{2,4}$

$$ds^2_{2,4} = -dx_0^2 - dx_1^2 + dx_I dx_I.$$

(3.26)

Analogously, we embed the five-sphere into flat six-dimensional space $\mathbb{R}^6$ by

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = R^2$$

(3.27)

and choose coordinates

$$x_0 = R \cos(\theta) \sin(\psi), \quad x_5 = R \cos(\theta) \cos(\psi),$$

$$x_I = R \sin(\theta) \omega_I^2, \quad I = 1, 2, 3, 4.$$ (3.28)

Again, $\omega_I'$ parametrize the remaining unit three sphere. The metric for $S^5$ is

$$ds^2_{S^5} = R^2 \left[ d\psi^2 \cos^2(\theta) + d\theta^2 + \sin^2(\theta) d\Omega_3^2 \right].$$

(3.29)

The five form field strength $F_5$ is given by

$$F_5 = \frac{2}{R} (d\text{Vol}(AdS_5) + d\text{Vol}(S^5)).$$

(3.30)

The plane wave geometry is obtained by considering a particle that moves along the $\psi$ direction of $S^5$ and is located at the origin in the $\theta$ and $\rho$ directions $\rho = \theta = 0$. The Penrose limit zooms into the region near the particle’s trajectory [BFOP02]. To this end, we introduce new coordinates

$$x^+ = \frac{1}{2}(t + \psi), \quad x^- = -R^2(t - \psi), \quad \rho = \frac{r}{R}, \quad \theta = \frac{y}{R},$$

(3.31)

and blow up the radius of the $S^5$, $R \to \infty$. In this limit, $\omega_I$ together with $r$ parametrize points $r$ in $\mathbb{R}^4$. The same is true for $y = (y, \omega_I')$. With the identification $x \equiv (r, y)$ the metric becomes

$$ds^2 = 2dx^+ dx^- - x^2 dx^+ dx^+ + dx^I dx^I.$$ (3.32)

The index $I$ runs over the transverse coordinates $1...8$ and the five form becomes proportional to a constant

$$F_{5;+1234} = F_{5;+5678} = \frac{f}{2}.$$ (3.33)

All other components vanish. The rescaling $x^- \to x^-/f$ and $x^+ \to fx^+$ brings the plane wave metric to the form

$$ds^2 = 2dx^+ dx^- - f^2 x^2 dx^+ dx^+ + dx^I dx^I.$$ (3.34)

This particular combination of the metric and $F_5$ is a maximally supersymmetric type IIB background [BFOHP02a].
3.3 String theory in plane wave geometry

The action for type IIB string theory in the plane wave background was found in [Met02]. Metsaev and Tseytlin studied and quantized the closed string solution in the Green-Schwarz formulation [MT02]. The action for the superstring in fermionic light-cone gauge $\tilde{\gamma}^+ \theta^A = 0$ is given by

$$S = -\frac{1}{2} \int d^2 \xi \left( T \sqrt{-h} \eta^{ab} \left( 2 \partial_a X^+ \partial_b X^- - f^2 X^I X_I \partial_a X^+ \partial_b X^+ + \partial_a X^I \partial_b X^I \right) 
+ 2 \beta \partial_a X^+ \left( \theta^1 \tilde{\gamma}^{-} \partial_b \theta^1 + \theta^2 \tilde{\gamma}^{-} \partial_b \theta^2 - 2 f \partial_b X^+ \theta^1 \tilde{\gamma}^{-} \Pi \theta^2 \right) 
- 2 \beta \epsilon^{ab} \partial_a X^+ \left( \theta^1 \tilde{\gamma}^{-} \partial_b \theta^1 - \theta^2 \tilde{\gamma}^{-} \partial_b \theta^2 \right) \right)$$

(3.35)

$\theta^A$ are Majorana-Weyl spinors as in section 2.3. The action is the equivalent of (2.60) for the plane wave background. The term with $\Pi$ is a reminiscent of terms that involve $F_5$. $\Pi$ and $\Pi'$ satisfy $\Pi^2 = \Pi'^2 = 1$ and are given by

$$\Pi^\alpha_{\beta} = (\gamma^1 \gamma^2 \gamma^3 \tilde{\gamma}^4)^{\alpha}_{\beta}, \quad \Pi'^\alpha_{\beta} = (\gamma^5 \tilde{\gamma}^5 \gamma^7 \tilde{\gamma}^8)^{\alpha}_{\beta}. \quad (3.36)$$

It is useful to choose a conformally flat worldsheet metric and use the remaining reparametrization invariance on the worldsheet to fix light-cone gauge in spacetime with $X^+ = p^+ \tau / T$. In this gauge, the action reduces drastically

$$S_{LCG} = \frac{T}{2} \int d^2 \xi \left( \partial_+ X^I \partial_- X^I - m^2 X^I X^I \right) 
+ 2 \beta \frac{p^+}{T} \left( \theta^1 \tilde{\gamma}^{-} \partial_+ \theta^1 - \theta^2 \tilde{\gamma}^{-} \partial_+ \theta^2 \right) - 4 \beta m \frac{p^+}{T} \theta^1 \tilde{\gamma}^{-} \Pi \theta^2 \right). \quad (3.37)$$

For convenience, we introduce the dimensionless parameter $m = p^+ f / T$. After choosing the conformally flat worldsheet metric, the Virasoro constraints have to be imposed by hand. In light-cone gauge, they read

$$p^+ X^- + \beta p^+ \left( \theta^1 \tilde{\gamma}^{-} \partial_\sigma \theta^1 + \theta^2 \tilde{\gamma}^{-} \partial_\sigma \theta^2 \right) + TX^I X'' = 0,$n
$$2p^+ X^- + 2ip^+ \left( \theta^1 \tilde{\gamma}^{-} \dot{\theta}^1 + \theta^2 \tilde{\gamma}^{-} \dot{\theta}^2 \right) - 4 \beta m p^+ \theta^1 \tilde{\gamma}^{-} \Pi \theta^2 
- m^2 TX^I X^I + TX^I X^I + TX'' X'' = 0. \quad (3.38)$$

They can be used to derive $X^-$, which does not enter the action any more. The equations of motion for the transverse coordinates are

$$\partial_+ \partial_- X^I + m^2 X^I = 0,$n
$$\partial_+ \theta^1 - m \Pi \theta^2 = 0, \quad \partial_- \theta^2 + m \Pi \theta^1 = 0. \quad (3.39)$$

The solution with closed string boundary conditions $X^I(\sigma + \pi) = X^I(\sigma)$ is

$$X^I(\sigma, \tau) = \cos(m \tau) x^I_0 + \frac{1}{m \tau} \sin(m \tau) p^I_0 
+ \beta \sum_{n \neq 0} \frac{1}{\omega_n} \left\{ \alpha^I_n e^{-\beta(\omega_n \tau - 2n \sigma)} + \alpha^I_n e^{-\beta(\omega_n \tau + 2n \sigma)} \right\}.$$
\[ \theta^1(\sigma, \tau) = \cos(m \tau) \theta^1_0 + \sin(m \tau) \Pi \theta^2_0 \]
\[ + \sum_{n \neq 0} c_n \left\{ \theta^1_n e^{-\beta \left( \omega_n \tau - 2n \sigma \right)} + \beta \Pi \theta^2_n \frac{\omega_n - 2n}{m} e^{-\beta \left( \omega_n \tau + 2n \sigma \right)} \right\}, \quad (3.40) \]

and similar for \( \theta^2 \). The frequencies \( \omega_n \) and the coefficients \( c_n \) are given by
\[ \omega_n = \text{sign}(n) \sqrt{m^2 + 4n^2} \quad \text{and} \quad c_n = \frac{1}{\sqrt{1 + \left( \frac{\omega_n}{m} \right)^2}}. \quad (3.41) \]

The canonical momenta for \( X^I \) and \( \theta^A \) are
\[ P^I = T \dot{X}^I, \quad \pi^A_{\alpha} = -\beta p^+ (\theta^A \gamma^-)_{\alpha}. \quad (3.42) \]

The equal time Poisson brackets for \( X^I, P^I, \theta^A \) and \( \pi^A_{\alpha} \) yield the brackets for the oscillators
\[ \{ p^I_0, \xi^J_0 \} = \delta^{IJ}, \quad \{ \alpha^A_m, \alpha^B_n \} = \frac{\beta \omega_n}{2T} \delta_{m+n} \delta^{AB} \delta^{IJ}, \]
\[ \{ \theta^A_{\alpha}^+, \theta^B_{\beta}^- \}_D = \frac{\beta}{4p^+} \gamma^{+ \alpha \beta} \delta_{m+n} \delta^{AB}, \quad A, B = 1, 2. \quad (3.43) \]

The \( D \) indicates the Poisson-Dirac bracket which has to be used for the fermionic oscillators \([\text{Dir}50]\). The light-cone Hamiltonian for the closed string, written in terms of the oscillators, is
\[ H_{LC} = \frac{1}{2T} p^2_0 + \frac{1}{2} m^2 T x^2_0 + 2\beta m p^+ \theta^1 \gamma^- \Pi \theta^2 
+ \sum_{n \neq 0, A=1,2} \left( T \alpha^A_{-n} \alpha^A_n + p^+ \omega_n \theta^A_{-n} \gamma^- \theta^A_n \right). \quad (3.44) \]

Upon integration, the first of the Virasoro constraints \((3.38)\) can be written in terms of number operators
\[ N^1 = N^2, \quad N^A = \sum_{n \neq 0} n \left( \frac{T}{\omega_n} \alpha^A_{-n} \alpha^A_n + p^+ \theta^A_{-n} \gamma^- \theta^A_n \right). \quad (3.45) \]

To quantize the theory, we follow the canonical quantization procedure and replace the Poisson brackets by equal time commutation relations promoting the Fourier modes in the expansion of the fields \((3.40)\) to operators. For our purpose, it is useful to introduce new, dimensionless creation and annihilation operators for \( n = 1, 2, \ldots \)
\[ a_0' = \sqrt{\frac{T}{2m}} (\frac{p^I_0}{T} - \beta m x^I_0), \quad \bar{a}_0' = \sqrt{\frac{T}{2m}} (\frac{p^I_0}{T} + \beta m x^I_0), \]
\[ a_n'^A = \sqrt{\frac{2T}{\omega_n}} \alpha^A_n, \quad \bar{a}_n'^A = \sqrt{\frac{2T}{\omega_n}} \alpha^A_{-n}, \]
\[ \eta_0 = \sqrt{\frac{p^+}{2}} (\theta^1_0 - \beta \theta^2_0), \quad \bar{\eta}_0 = \sqrt{\frac{p^+}{2}} (\theta^1_0 + \beta \theta^2_0), \]
\[ \eta_n^A = \sqrt{2p^+} \theta^A_n, \quad \bar{\eta}_n^A = \sqrt{2p^+} \theta^A_{-n}. \quad (3.46) \]
The Poisson brackets (3.43) yield the commutation and anti-commutation relations
\[ [a^I_0, \bar{a}^0] = \delta^{IJ}, \quad [a^I_m, \bar{a}^J_n] = \delta_{mn} \delta^{IJ} \delta^{AB}, \]
\[ \{\eta^\alpha_0, \bar{\eta}^\beta_0\} = \frac{1}{4} \gamma^{\alpha\beta}, \quad \{\eta^{A\alpha}_m, \eta^{B\beta}_n\} = \frac{1}{2} \gamma^{\alpha\beta} \delta_{mn} \delta^{AB}. \] (3.47)

The (normal ordered) light-cone Hamiltonian of the quantum theory in these new oscillators reads
\[ H_{LC} = m(4 + e_0 + \bar{a}^I_0 a^I_0 + 2\bar{\eta}^I_0 \gamma^I \eta_0) \]
\[ + \sum_{n=1}^{\infty} \omega_n (\bar{a}^{I^I}_n a^{I^I}_n + \bar{a}^{I^2}_n a^{I^2}_n + \bar{\eta}^{I^I}_n \gamma^I \eta^{I^I}_n + \bar{\eta}^{I^2}_n \gamma^I \eta^{I^2}_n). \] (3.48)

The term \(4 + e_0\) comes from the normal ordering and \(e_0\) depends on the choice of the fermionic vacuum. The Virasoro constraint (3.45) becomes a level matching for the physical states
\[ (N^1 - N^2)|\text{phys}\rangle = 0, \quad N^A = \sum_{n=1}^{\infty} n(\bar{a}^{AI}_n a^{AI}_n + \bar{\eta}^{A^I}_n \gamma^I \eta^{A^I}_n). \] (3.49)

The question is which are the physical states. The light-cone Hamiltonian can be rewritten making the ground state energy term explicit
\[ H_{LC} = E_0 + \sum_{A=1,2} \sum_{n>0} \omega_n (\bar{a}^{AI}_n a^{AI}_n + \bar{\eta}^{A^I}_n \gamma^I \eta^{A^I}_n), \]
\[ E_0 = m(\bar{a}^I_0 a^I_0 + 2\bar{\eta}^I_0 \gamma^I \eta_0 + e_0). \] (3.50)

Since the vacuum state is a direct product of the bosonic and the fermionic vacuum, it obeys
\[ \bar{a}^I_0 |0\rangle = 0, \quad \bar{a}^{AI}_n |0\rangle = 0, \quad \bar{\eta}^{A^I}_n |0\rangle = 0, \quad n = 1, 2, \ldots \] (3.51)
for the bosonic part and the higher order fermionic modes. The way to choose the fermionic zero-mode vacuum can be found by introducing projected fermionic zero modes
\[ \eta_\pm = \frac{1}{\sqrt{2}} (1 \pm \Pi) \eta_0. \] (3.52)

It turns out that there are exactly four different possible choices:
\[ \bar{\eta}_\pm |0\rangle = 0, \quad E_0 = 4, \]
\[ \eta_\pm |0\rangle = 0, \quad E_0 = 4, \]
\[ \bar{\eta}_+ |0\rangle = \eta_- |0\rangle = 0, \quad E_0 = 8, \]
\[ \eta_+ |0\rangle = \bar{\eta}_- |0\rangle = 0, \quad E_0 = 0. \] (3.53)
While the first two choices preserve the $SO(8)$ symmetry, they break the supersymmetry of the light cone Hamiltonian. The situation is vice-versa for the latter two: They break $SO(8) \rightarrow SO(4) \times SO'(4)$, but preserve supersymmetry. Metsaev and Tseytlin also showed how the spectrum of states built out of these vacua by acting with the raising and lowering operators (3.46) can be interpreted in terms of supergravity fields in the plane wave background.

3.4 The tensionless superstring in a maximally supersymmetric plane wave background

The action for the tensionless string can be derived from (3.35) in the way discussed in section [3.1]. The rigorous derivation is presented in [II]. Here, we start directly from the action in light-cone gauge (3.37) and use the shortcut. To this end, we split the tension into $T = \lambda v^2$ with $\lambda$ being a dimensionless parameter to be taken to zero and introduce the new worldsheet time $t = \tau \lambda$. In addition, we keep the combination $\mu = \lambda m = p^+ f / v^2$ fixed. The action becomes

$$S_{LCG} = \frac{v^2}{2} \int d\sigma dt \left( \partial_i X^I \partial_i X^I - \lambda^2 \partial_\sigma X^I \partial_\sigma X^I - \mu^2 X^I X^I + 2\beta \frac{p^+}{v^2} \theta^1 \bar{\gamma} \left( \partial_\tau - \lambda^2 \partial_\sigma \right) \theta^1 - \theta^2 \bar{\gamma} - 4\beta \mu \frac{p^+}{v^2} \theta^1 \bar{\gamma} \Pi \theta^2 \right).$$

(3.54)

The tensionless limit corresponding to $\lambda \rightarrow 0$ does not present any difficulty and results in discarding the $\sigma$-derivatives. The result is the light-cone action for the tensionless string

$$S^0_{LCG} = \frac{v^2}{2} \int d\sigma dt \left( \dot{X}^I \dot{X}^I - \mu^2 X^I X^I \right) + 2\beta \frac{p^+}{v^2} \left( \theta^1 \bar{\gamma} \dot{\theta}^1 - \theta^2 \bar{\gamma} \dot{\theta}^2 \right) - 4\beta \mu \frac{p^+}{v^2} \theta^1 \bar{\gamma} \Pi \theta^2.$$  

(3.55)

the dot indicating the derivative with respect to $t$. Since

$$X^+ = \frac{p^+}{T} \tau = \frac{p^+}{v^2} t,$$

(3.56)

$p^+$ is still the conjugate momentum for $X^+$. The action is accompanied by the Virasoro constraints

$$p^+ X^- + \beta p^+ (\theta^1 \bar{\gamma} \dot{\theta}^1 + \theta^2 \bar{\gamma} \dot{\theta}^2) = 0,$$

$$2p^+ \dot{X}^- + 2ip^+ (\theta^1 \bar{\gamma} \dot{\theta}^1 + \theta^2 \bar{\gamma} \dot{\theta}^2) - 4\beta m p^+ \theta^1 \bar{\gamma} \Pi \theta^2 - \mu^2 v^2 X^I X^I + v^2 \dot{X}^I \dot{X}^I = 0.$$  

(3.57)
The equations of motion for $X^I$ and $\theta^A$ are
\begin{align*}
\ddot{X}^I + \mu^2 X^I &= 0, \\
\dot{\theta}^1 - \mu \Pi \theta^2 &= 0, \\
\dot{\theta}^2 + \mu \Pi \theta^1 &= 0.
\end{align*}
(3.58)

We see that — as expected — $X^\mu$ behaves as a collection of particles with mass $\mu$ enumerated by $\sigma$. For closed string boundary conditions, the equations of motion are solved by
\begin{align*}
X^I_0(\sigma, t) &= \cos(\mu t)x^I_0 + \frac{1}{\mu \nu^2} \sin(\mu t)p^I_0 \\
&\quad + \frac{\beta}{\mu} \sum_{n \neq 0} \text{sign}(n) \left\{ \bar{\alpha}_n^I e^{-\beta(\text{sign}(n) \mu t - 2n \sigma)} + \alpha_n^I e^{-\beta(\text{sign}(n) \mu t + 2n \sigma)} \right\}, \\
\theta^1_0(\sigma, t) &= \cos(\mu t)\theta^1_0 + \sin(\mu t)\Pi \theta^2_0 \\
&\quad + \frac{1}{\sqrt{2}} \sum_{n \neq 0} \left\{ \theta_n^1 e^{-\beta(\text{sign}(n) \mu t - 2n \sigma)} + \beta \Pi \theta^2_n \text{sign}(n) e^{-\beta(\text{sign}(n) \mu t + 2n \sigma)} \right\},
\end{align*}
(3.59)
and similar for $\theta^2$. The Poisson brackets and the light-cone Hamiltonian follow in the same way as in the tensile theory. In order to quantize the tensionless string, we can make use of the fact that the solution looks very similar to the tensile case (3.40). It can be derived as a limit of the latter as opposed to the case in flat space. We then show that this limit survives quantization. In fact, the quantized tensionless string is the very same limit of the tensile quantum string. To this end, we make use of the definitions we used to obtain the tensionless action
\begin{align*}
T = \lambda \nu^2, \\
\mu = \lambda m, \\
\tau = \lambda t,
\end{align*}
(3.60)
and accompany them with $w_n = \lambda \omega_n = \text{sign}(n) \sqrt{\mu^2 + 4\lambda^2 n^2}$, where $\omega_n$ are the frequencies entering the tensile solution (3.41). Plugging these definitions into (3.40) yields
\begin{align*}
X^I_\lambda(\sigma, t) &= \cos(\mu t)x^I_0 + \frac{1}{\mu \nu^2} \sin(\mu t)p^I_0 \\
&\quad + \frac{\lambda}{w_n} \sum_{n \neq 0} \left\{ \tilde{\alpha}_n^I e^{-\beta(w_n t - 2n \sigma)} + \alpha_n^I e^{-\beta(w_n t + 2n \sigma)} \right\}. \\
\end{align*}
(3.61)

Here, we focus on the bosonic fields $X^I$ only. The fermionic fields are treated similarly. If in addition, the oscillators are rescaled as $\tilde{\alpha}_n^A = \lambda \alpha_n^A$ the tensile solution looks almost like the tensionless one except for the frequencies $w_n$. However, this is the only place where $\lambda$ enters the solution. For $\lambda \to 0$ the spectrum becomes degenerate $w_n \to \text{sign}(n) \mu$ (and $c_n \to 1/\sqrt{2}$ correspondingly). In this limit, the tensile solution matches the tensionless
\begin{align*}
X^I_\lambda \to X^I_0.
\end{align*}
(3.62)
A closer look at the dimensionless bosonic creation and annihilation operators
\[ a_0^I = \sqrt{\frac{T}{2m}} (\frac{p_0^I}{T} - \beta m x_0^I) = \sqrt{\frac{v^2}{2\mu}} (\frac{p_0^I}{v^2} - \beta \mu x_0^I) \]
\[ a_n^A = \sqrt{\frac{T}{\omega_n}} \alpha_n^A = \sqrt{\frac{2v^2}{\omega_n}} \alpha_n^A \rightarrow \sqrt{\frac{2v^2}{\mu}} \alpha_n^A. \] (3.63)

The fermionic modes do not depend on \( m \) and \( T \). As the tension goes to zero, the operators remain almost unchanged up to the fact that the frequencies entering the \( a_n^A \)'s degenerate. However, they do not enter the commutation relations (3.47). That is why the new, dimensionless modes were introduced in the beginning. A direct quantization of the tensionless string leads to the same result. Thus, we conclude that the limit \( T \rightarrow 0 \) survives and commutes with quantization. In flat space, the only scale \( T \) is lost when the string becomes tensionless. Here, the background provides a second scale with \( \mu \). We should mention that the light-cone Hamiltonian for the tensionless theory is
\[ H_{LC}^0 = \mu (4 + e_0 + \bar{a}_0^I a_0^I + 2 \bar{\eta}_0 \bar{\gamma} \Pi_0) + \mu \sum_{n=1}^{\infty} (\bar{a}_n^I a_n^I \bar{a}_n^J a_n^J + \bar{\eta}_n^I \bar{\gamma} \eta_n^I + \bar{\eta}_n^J \eta_n^J). \] (3.64)

The level matching condition for the physical states is unchanged compared to the tensile case
\[ (N^1 - N^2)|_{phys} = 0, \quad N^A = \sum_{n=1}^{\infty} n \left( \bar{a}_n^I a_n^I \bar{a}_n^J a_n^J + \bar{\eta}_n^I \bar{\gamma} \eta_n^I \right). \] (3.65)

The spectrum of the theory gets highly degenerated, since all \( \omega_n \) collapse to a single value for the tensionless string. We make the following nice observation. The tension \( T \) and the background scale \( m \) enter the theory in such a way that \( mT = \mu v^2 = p^+ f \) is kept constant when taking the tension to zero. This allows for a different interpretation of the results. \( m \) is the origin of the curvature of the plane wave. Therefore, instead of considering tensionless strings in a plane wave with finite curvature, we may change the perspective and view the solution as a string with tension \( v^2 \) moving in an infinitely curved background with \( m \rightarrow \infty \) where the contribution from the background to the energy is much higher in comparison to the splitting for the different oscillators as shown in figure 3.1 [dVGN95].

3.5 Tensionless strings in homogeneous plane waves
A question at hand is, whether the obtained results are a peculiarity of the plane wave or if there is a generalization to more complicated situations. To
determine this we look at other types of backgrounds. Homogeneous plane waves that are also known as Hpp-waves are slight generalizations of the plane wave \[\text{BOPT03}\]. These backgrounds are parametrized by two matrices \(k_{IJ}\) and \(f_{IJ}\). The line element is given by

\[
\text{d}s^2 = 2\text{d}x^+ \text{d}x^- + k_{IJ}x^I \text{d}x^J + 2f_{IJK} \text{d}x^I \text{d}x^J + \text{d}x^I \text{d}x^J.
\] (3.66)

Such a background is not maximally supersymmetric in general. It is supported by a \(\mathcal{B}\)-field given which has the \(D-2\) components \(B_{I+} = h_{IJ}x^J\). By a rotation of the transverse coordinates, \(k\) can be chosen to be diagonal:

\[k_{IJ} = \delta_{IJ}.\]

The type IIB string in this background is an integrable model and was solved by Blau et. al. \[BOPT03\] via a so-called frequency base ansatz:

\[
X^I(\sigma, \tau) = \sum_{n=\infty} \infty X^I_n(\tau) e^{2\beta n \sigma}, \quad X^I_n(\tau) = \sum_{\ell=1}^d \xi_{n\ell} a^I_{n\ell} e^{\beta \omega_{n\ell} \tau}.
\] (3.67)

In the quantized theory, \(\xi_{n\ell}\) become the raising and lowering operators. The coefficients \(a^I_{n\ell}\) are eigenvectors of the matrix

\[
M_{IJ}(\omega, n) = (\omega^2 + k_I - 4T^2 n^2) \delta_{IJ} + 2\beta \omega f_{IJK} + 4\beta T n h_{IJ}.
\] (3.68)

The allowed frequencies \(\omega_{n\ell}\) are determined by \(\text{det} M(\omega_{n\ell}, n) = 0\) and the eigenvectors are given by \(M_{IJ}(\omega_{n\ell}, n)a^I_{n\ell} = 0\). For the special choice \(h_{IJ} = f_{IJK} = 0\) and constant \(k_I = -m^2\) and for \(d = 2\), we get back the plane wave solutions \(\omega_{n\pm} = \pm \sqrt{m^2 + 4T^2 n^2}\). Previously, the frequencies in the tensionless case were obtained by letting \(T \to 0\) directly in the corresponding expression. This works here as well. The frequencies become degenerate and equal to the frequencies for \(n = 0\):

\[
\omega_{n\ell} \to \omega_\ell, \quad \text{such that } \det M(\omega_\ell) \equiv \det M(\omega_\ell, 0) = 0.
\] (3.69)

It seems plausible that this result still holds in the corresponding quantized theory. We leave this chapter with the open question how our results can lead to insights in the context of tensionless strings on \(AdS_5 \times S^5\).
4. Macroscopic Tensionless Strings

Tensionless strings appear at various places in string theory. In [I] we show how they fit into the context of supergravity backgrounds. Generalizing the results of Dabholkar and Hull which we presented in section 2.4, Schwarz found a family of backgrounds to IIB supergravity which have a macroscopic string as their source. This family is connected by $SL(2,\mathbb{Z})$ transformations, the group under which type IIB string theory is believed to be selfdual [Sch95]. Today, the macroscopic string is interpreted as a bound state of (fundamental) F-strings and D-strings, one dimensional D-branes [Wit96, dAS96]. We derive the background sourced by a tensionless string by accelerating Schwarz’ solution to the speed of light in a certain way. This limit resembles the gravitational shock wave of a massless particle which was obtained in [AS71] in the same way.

We start with a review of the shock wave geometry of a massless particle. Then we present the solution of Schwarz and show how the tensionless limit is obtained.

4.1 The gravitational shock wave

The geometry of a pointlike particle moving at the speed of light is a gravitational shock wave. In [AS71] it was obtained by considering a Lorentz transformation of a massive particle. We consider the geometry of a string traveling with the speed of light. A short introduction to the original discussion is hence appropriate. The gravitational field of a particle is derived from the Einstein-Hilbert action

$$S_{\text{EH}} = \int d^4x \sqrt{-g} R.$$  \hspace{1cm} (4.1)

A pointlike object of mass $m$ favors a spherical symmetric solution in its rest frame, the Schwarzschild metric

$$ds^2 = \frac{(1-A)^2}{(1+A)^2} dt^2 - (1+A)^4 (dx^2 + dy^2 + dz^2),$$

$$A = \frac{m}{2r}, \quad r^2 = x^2 + y^2 + z^2.$$  \hspace{1cm} (4.2)

As particles moving at the speed of light tend to be massless, one might try to send $m \to 0$, but that would recover empty Minkowski space, except for the
singularity at \( r = 0 \). Moreover, the expectation of the gravitational field of a particle traveling at the speed of light is rather a shock-wave front traveling alongside the particle. The right way to approach the question is via a Lorentz transformation and to see how the gravitational field and hence the metric behaves in the limit of an infinite transformation. We choose to act on the \( x \) and \( t \) directions,

\[
t \to t' = \gamma(t + vx), \quad x \to x' = \gamma(x + vt), \quad \gamma = \left(1 - v^2\right)^{-1/2}.
\] (4.3)

After this transformation, the metric becomes

\[
ds^2 = (1 + A)^2(dt^2 - dx^2 - dy^2 - dz^2) - \gamma^2 \left[ (1 + A)^4 - \frac{(1 - A)^2}{(1 + A)^2} \right] (dt - vx)^2,
\]

\[A = \frac{\gamma^{-1}p}{2\sqrt{\gamma^2x'^2 + (y'^2 + z'^2)}}\] (4.4)

Here, \( p = \gamma m \). In order to compare a massive particle with mass \( m \) at rest with a massless particle traveling at the speed of light, we keep the energy fixed. Especially, \( p \) becomes the momentum of the massless particle. It is tricky to take the limit \( v \to 1 \) because the transformation becomes divergent, since \( \gamma \to \infty \). This problem can be avoided by yet another change of coordinates:

\[
x'' - vt'' = x' - vt',
\]

\[
x'' + vt'' = x' + vt' - 4p \ln \left( \sqrt{(x' - vt')^2 + \gamma^{-2} - (x' - t')} \right).
\] (4.5)

In these new coordinates it is easy to ‘accelerate’ the particle to the speed of light. After going to light cone coordinates \( x^- = t'' - x'' \), \( x^+ = t'' + x'' \), the line element becomes

\[
ds''^2 = dx^+ dx^- - dy^2 - dz^2 + 8p \ln \sqrt{y^2 + z^2} \delta(x^-)(dx^-)^2.
\] (4.6)

In the eyes of a spectator looking in the boosted direction, this is indeed a gravitational shock wave front.

4.2 The macroscopic IIB string

Type IIB supergravity contains two two-form fields \( B^{(1)} \) and \( B^{(2)} \), belonging to the NS-NS and R-R sector, respectively, two real scalar fields, the dilaton \( \phi \) and the axion \( \chi \), the graviton and a four-form field \( C_4 \) with self-dual field strength \( F_5 = \ast F_5 \). The bosonic part of the supergravity action in the string
The part coming from (2.73) is obvious. Type IIB theory is self-dual under S-duality. Basically, this duality exchanges the strong and weak coupling limit of the theory. The string coupling is given by $g_s = e^{\phi_0}$ and S-duality replaces $\phi$ with $-\phi$. $\phi_0$ is the vacuum expectation value of the dilaton. In fact, S-duality is supposed to be part of a much bigger symmetry of IIB string theory, namely $SL(2, \mathbb{Z})$. However, to understand this symmetry, we would need a non-perturbative picture of the string theory. Schwarz [Sch95] slightly generalized the results of [DGHRR90] and found a type IIB background whose source is a macroscopic string that is charged under both $B$-fields. The four-form field charge is carried by a self-dual three-brane. Here, we are only interested in charges carried by strings. Therefore, $C_4$ and its field strength $F_5$ is consistently set to zero in the following. We start with rewriting the action in the Einstein frame.

$$S_{\text{IIB}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left[ e^{-2\phi} \left( R + 4\nabla \phi \cdot \nabla \phi - \frac{1}{12} (H^{(1)})^2 \right) 
- \frac{1}{12} \left( e^{-\phi} (H^{(1)})^2 + e^{\phi} (H^{(2)} + \chi H^{(1)})^2 \right) 
- \frac{1}{2} (d\chi)^2 - \frac{1}{480} (F_5 + H^{(1)} \wedge B^{(2)})^2 \right] 
+ \frac{1}{8\kappa^2} \int \left( C_4 + \frac{1}{2} B^{(1)} \wedge B^{(2)} \right) \wedge H^{(2)} \wedge H^{(1)}. \quad (4.7)$$

The two-forms and their three-form field strengths $H^{(i)}$ can conveniently be combined into two-component vectors $\mathbf{B} = (B^{(1)}, B^{(2)})$ and $\mathbf{H}$. For the sake of a better comparison to [III] we define the field strengths with an additional factor of 2 as compared to the introductory chapter 2, $H^{(i)} = 2dB^{(i)}$. The two real scalar fields, on the other hand, combine to one single complex scalar $\lambda = \chi + \beta e^{-\phi}$ and we define the matrix

$$M = e^{\phi} \begin{pmatrix} |\lambda|^2 & \chi \\ \chi & 1 \end{pmatrix}. \quad (4.9)$$

With these ingredients, the target space action for $D = 10$ IIB supergravity can be written in the form

$$S_{\text{IIB}}^{10} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} \left[ R + \frac{1}{4} \text{tr}(\partial M \partial M^{-1}) - \frac{1}{12} H^T M H \right]. \quad (4.10)$$
The action has a global $SL(2, \mathbb{R})$ symmetry that acts on $M$, $B$ and $\lambda$ as

$$M \rightarrow \Lambda M \Lambda^T, \quad B \rightarrow (\Lambda^T)^{-1} B, \quad \lambda \rightarrow \frac{a\lambda + b}{c\lambda + d},$$

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}. \quad (4.11)$$

Schwarz found the following $SL(2, \mathbb{Z})$ set of backgrounds [Sch95].

$$ds^2 = \Delta_2^{-3/4} \left( -dr^2 + (dx^1)^2 \right) + A_q^{1/4} dx^i dx_j, \quad A_q = 1 + \frac{\Delta_2^{1/2} Q}{3r^6},$$

$$\Delta_2^{1/2} = q^T M_0^{-1} q = e^{\phi_0} (q_1 - q_2 \chi_0)^2 + e^{-\phi_0} q_2^2. \quad (4.12)$$

Here, $r^2 = x^i x_j$ is the spacial distance from the string and $x^1$ parametrizes the longitudinal direction of the string. $\phi_0$ and $\chi_0$ are the vacuum expectation values of $\phi$ and $\chi$ and $M$ is built out of them in the obvious way. At first sight, this metric has an $SL(2, \mathbb{R})$ symmetry, but the restriction to $SL(2, \mathbb{Z})$ follows from the Dirac quantization condition and that $q_1$ and $q_2$ are relative prime integers, when measured in terms of the fundamental $B_{\mu\nu}$ charge $Q$. If they are not relative prime, the solution can be decomposed and interpreted as the geometry of multiple strings. Schwarz noticed that the symmetry naturally prefers both $B$ charges $q = (q_1, q_2)$ to be present. The solution is completed by the fields

$$B_{01} = M^{-1} q \Delta_2^{-1/2} A_q^{-1}, \quad \lambda = \frac{q_1 \chi_0 - q_2 |\lambda_0|^2 + \beta q_1 e^{-\phi_0} A_q^{1/2}}{q_1 - q_2 \chi_0 + \beta q_2 e^{-\phi_0} A_q^{1/2}}. \quad (4.13)$$

Here, $\lambda_0$ is the vacuum expectation value of $\lambda$. The singularity at $r = 0$ is interpreted as an infinitely long source string with a slightly modified sigma model action [Sch95, dAS96, CT97]

$$S = -\frac{T_q}{2} \int d^2 \xi \left( \partial^a X^\mu \partial_a X^\nu G_{\mu\nu} + \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}^T q + \ldots \right), \quad (4.14)$$

where the string tension is given by

$$T_q = \Delta_2^{1/2} Q = Q \sqrt{e^{\phi_0} (q_1 - q_2 \chi_0)^2 + e^{-\phi_0} q_2^2}. \quad (4.15)$$

The action is a generalization of the sigma model action (2.17) in the same way (4.10) generalized (2.73). The background fields in (4.14) are actually string condensates that arise as string loop effects [dAS96].

This background is interpreted in terms of bound states of open, F(undamental) and D-strings [Wit96] in the following way. The elementary, or fundamental string is a source for the NS-NS two-form $B^{(1)}$ but not for the R-R form $B^{(2)}$. We can say it has charge $q = (1, 0)$. R-R charges are on
the other hand carried by D-branes. These are hyperplanes on which open strings can end, but much more important is the fact that they have their own worldvolume dynamics [CvGNW97]. A one-brane in a conventional background has the same worldsheet structure as the elementary string and it is therefore natural to call it a D-string. It carries R-R charge only, and thus it is reasonable to interpret it as the $q = (0,1)$ partner of the fundamental string. Considering higher charges, we can look at combined objects of F- and D-strings with charge $q = (q_1, q_2)$. If $T$ is the tension of the fundamental string and $T_D$ that of a D-string, they are related by

$$T_D = g_s^{-1} T,$$  
(4.16)

where $g_s = e^{\phi_0}$ is the string coupling constant. In the absence of the R-R-field, the tension of a $(q_1, q_2)$-string is given by

$$T_{q_1, q_2} = T \sqrt{q_1^2 + g_s^{-2} q_2^2}.$$  
(4.17)

This can be compared to (4.15), taking into account the relation between the string tension in the Einstein and the string frame, $T_{\text{Einst}} = g_s^{1/2} T_{\text{string}}$. S-duality exchanges $g_s$ with $g_s^{-1}$ and is part of the $SL(2,\mathbb{Z})$ symmetry. Effectively, it interchanges the two types of strings. At weak coupling $g_s \rightarrow 0$, the D-strings is much heavier than the F-strings. This can thus be interpreted as a theory of weakly coupled F-strings. At strong coupling however, the situation is vice-versa and the D-strings might now be seen as the weakly coupled objects. A F-string carries the fundamental charge of the NS-NS two-form, while the D-strings carry fundamental charge under the R-R two-form. Thus, a fundamental string has B-charge $(1,0)$, while a D-string has B-charge $(0,1)$. Now, one might assume a bound state of $p$ F-strings and $q$ D-strings. In the weak coupling regime, this can be interpreted in the following way: The F-string may end on the D-string with one of its endpoints. Such a state is allowed, but not supersymmetric until this point drifts away to infinity. The D-string remains but now it does not only carry its own R-R-charge, but the NS-NS-charge of the F-string as well.

### 4.3 The Tensionless $SL(2,\mathbb{Z})$ String

From Schwarz’ solution we derive the geometry sourced by a tensionless string by considering an infinite Lorentz transformation. In comparison to the particle the situation in (4.12) is somewhat different. This starts with the fact that the string is an extended object. However, we only have to consider velocities orthogonal to the string. Without loss of generality, we perform a Lorentz transformation in the $z = x^9$ direction

$$t' = \gamma(t + vz), \quad z' = \gamma(z + vt), \quad \gamma = \left(1 - v^2\right)^{-1/2}.  \tag{4.18}$$
There are certain subtleties in taking this limit. The exact derivation is found in [I]. As in the particle case, we want to take \( v \to 1 \) while keeping the energy constant. This is achieved by introducing a rescaled fundamental charge \( Q_0 = \gamma Q \) which is kept constant. The scalars \( \phi \) and \( \chi \) tend to their (constant) vacuum expectation values, while the tension

\[
T_q = \Delta_q^{1/2} Q = \frac{\gamma}{\Delta_q^{1/2} Q_0}
\]

vanishes. After going to light-cone coordinates \( x^- = z' - t' \), \( x^+ = z' + t' \), the metric becomes

\[
(ds')^2 = dx^+ dx^- + (dx^1)^2 + dr^2 + \frac{\pi \Delta_q^{1/2} Q_0}{8r^5} \delta(x^-) (dx^-)^2. 
\]

Here, \( r^2 = (x^2)^2 + \ldots (x^8)^2 \). The metric is still invariant under the \( SL(2,\mathbb{Z}) \) transformations (4.11). It is the ten dimensional analogue of (4.6) and has the structure of a plane wave metric

\[
ds^2 = dx^+ dx^- + K(x^2, \ldots, x^8, x^-)(dx^-)^2 + \sum_{i=1}^{8} (dx^i)^2. \]

Concerning the \( B \) field, the Lorentz transformation generates four non-zero components that diverge in the limit \( v \to 1 \). This problem is overcome by considering the gauge transformation

\[
B_{01} \to B_{01} - M^{-1} q \Delta_q^{-1/2} = M^{-1} q \Delta_q^{1/2} (A_q^{-1} - 1). \]

This gauge fixed \( B \) vanishes as \( v \to 1 \), and hence does \( H = dB \). The energy momentum tensor becomes localized at the position of the string

\[
T_{--} = \frac{1}{24} \pi^3 \Delta_{q,0}^{1/2} Q_0 \delta(r) \delta(x^-). 
\]

All other components vanish. This is the energy momentum tensor for a tensionless string localized along the \( X^1 \)-direction. It can be directly derived from the action (3.7) of the tensionless string.

\[
T_{\mu\nu}(x^I) = \partial_{\tau} X_{\mu} \partial_{\tau} X_{\nu} \delta^{(8)}(x^I - x^I), 
\]

where the eight-dimensional delta function covers the space transverse to the direction of the boost, cf. [GG75]. Here, we already integrated out the world-sheet directions. This implies that \( X^- \) and \( X^1 \) are fixed to the values of \( x^- \) and \( x^1 \). Since the string is located at \( x^I = x^- = 0 \), the only non-vanishing contribution arises from

\[
X_- = G_{++} X^+ = X^+ \propto \tau. 
\]

From this, we obtain

\[
T_{--} = \partial_{\tau} X_- \partial_{\tau} X_- \delta(r) \delta(x^-) \propto \delta(r) \delta(x^-), 
\]

We conclude that (4.20) is the background geometry generated by a tensionless string.
Generalized complex geometry is a relatively new concept for the description of the geometry of a manifold. It originates in the context of generalizing the notion of Calabi-Yau manifolds to include $B$-field fluxes. These generalized Calabi-Yau manifolds play an important role in the context of compactification with fluxes \[\text{[Gra06]}\]. It was introduced by Hitchin \[\text{[Hit03]}\] and then studied in great detail by his student Gualtieri \[\text{[Gua03]}\].

Generalized complex geometry combines the tangent bundle and the cotangent bundle of a manifold and considers the complex geometry on the direct sum $E = TM \oplus T^*M$. In this way it unifies complex and symplectic geometry into a single framework. This makes it very interesting from the physics point of view. Phase space is a prominent example of a symplectic geometry — we saw in chapter 2 that the symplectic structure gives rise to the Poisson bracket in the context of sigma models. To continue in this direction, generalized complex geometry puts the metric and the $B$-field on an equal footing with the (ordinary) complex structures. This makes it an elegant notion to describe the relation between worldsheet supersymmetry of sigma models and the geometry of their target spaces.

This chapter provides the basic notions of generalized complex geometry that are needed to understand the relation to supersymmetric sigma models. It is not intended and it does not claim to be a full introduction to the topic. For this purpose, we refer to Gualtieri’s thesis \[\text{[Gua03]}\].

### 5.1 Complex Geometry

Before introducing generalized complex geometry, we start with a review of some facts of complex geometry. For a more detailed introduction, we refer to \[\text{[Nak90]}\]. A manifold $M$ is almost complex, if it can be equipped with an endomorphism on its complexified tangent bundle $J \in \text{End}(TM \otimes \mathbb{C})$ satisfying $J^2 = -1$. We denote the $\pm\beta$ eigenbundles of $J$ by $\mathcal{L}$ and $\mathcal{L}$. $\mathcal{L}$ is called integrable if it is involutive in the sense

\[
X, Y \in \mathcal{L} \Rightarrow [X, Y] \in \mathcal{L}, \tag{5.1}
\]
where \([X, Y]\) is the Lie bracket. In this case, \(J\) is called a complex structure and \(M\) is a complex manifold. On a complex manifold, there exists a chart of local holomorphic and antiholomorphic coordinate frames \(\partial_\mu = (\partial_m, \partial_{\bar{m}})\) with holomorphic and anti-holomorphic transition functions such that \(J\) is diagonal in these coordinates,

\[
J^\mu_\nu = \begin{pmatrix} \beta \delta^m_n & 0 \\ 0 & -\beta \delta^\bar{m}_{\bar{n}} \end{pmatrix}.
\] (5.2)

Every complex manifold has an even number of real dimensions, say \(2D\). The integrability condition (5.1) can also be expressed using the Nijenhuis torsion for \(J\):

\[
N(J)[X, Y] = [JX, JY] - J[JX, Y] - J[X, JY] + J^2[X, Y], \quad X, Y \in T M.
\] (5.3)

In local coordinates, the Nijenhuis tensor reads

\[
N(J)^\mu_\nu_\sigma = J^\rho_\nu J^\mu_\rho_\sigma - (\nu \leftrightarrow \sigma).
\]

The integrability condition (5.1) is equivalent to \(N(J) = 0\). For further convenience, we also introduce the Nijenhuis concomitant of two complex structures \(I\) and \(J\)

\[
N(I, J) = \frac{1}{2} ([IX, JY] - J[IX, Y] - I[X, JY] + IJ[X, Y] + (I \leftrightarrow J)).
\] (5.4)

In particular, \(N(J, J) = 2N(J)\). The product \(IJ\) of two complex structures \(I\) and \(J\) is integrable if \(N(I, J) = 0\) [AY68, MM84].

There are various types of complex manifolds. A few of them shall be presented here. A complex manifold \(M\) is hermitian if it admits a metric \(G^\mu_\nu\) that is hermitian with respect to the complex structure \(J\):

\[
J^\rho_\mu G^\rho_\sigma J^\sigma_\nu = G^\mu_\nu.
\] (5.5)

This implies that \(\omega^\mu_\nu = G^\mu_\rho J^\rho_\nu\) is a two-form of type \((1, 1)\) with respect to the complex structure. It is the Kähler form for \(J\). For closed \(\omega\) the manifold is Kähler.

\(M\) is called bi-hermitian if it admits two complex structures \(J_+\) and \(J_-\) that is hermitian with respect to both in the sense (5.5), such that the complex structures are covariantly constant with respect to a connection involving the torsion three-form \(H = dB\) of the manifold

\[
\nabla^{(\pm)} J^{\pm}_\mu = 0, \quad \Gamma^{(\pm)} = \Gamma^{(0)} \pm T,
\] (5.6)

where \(\Gamma^{(0)}\) is the Levi-Civita connection and \(T^\rho_\mu_\nu = G^\rho_\sigma H_{\sigma\mu_\nu}\) is the Bismut connection for \(H\). This implies that the Nijenhuis concomitant (5.4) of \(J_+\) and \(J_-\) vanishes. The Kähler forms \(\omega^{\pm} = G J^{\pm}_\mu\) are related to \(H\) via

\[
H^\mu_\nu_\rho = \pm J^\mu_\pm_\rho J^{\pm}_\nu J^{\sigma}_\pm (d\omega^{\pm})_{\kappa\lambda\sigma}.
\] (5.7)
An implication of (5.7) is
\[ H_{\kappa\lambda\sigma} = \pm j^\mu_{\pm|\kappa} j^\nu_{\pm|\lambda} H_{\mu\nu|\sigma}. \] (5.8)

A hermitian manifold that admits two anticommuting complex structures \( I \) and \( J \) is called hyperhermitian. Their product \( K = IJ \) is another complex structure and \( I, J \) and \( K \) satisfy the quaternion algebra \( \mathbb{H} = Cl_{0,2}(\mathbb{R}) \)
\[ I^2 = J^2 = K^2 = -1, \quad K = IJ. \] (5.9)

If the two-forms \( \omega_I, \omega_J, \omega_K \) satisfy relation (5.7) with the same sign, then \( M \) is HKT, which originally stands for ‘hyperKähler with torsion’. If \( H = 0 \), the two-forms are closed and the manifold is called hyperKähler. Then,
\[ \Omega = \omega_J + \beta \omega_K \] (5.10)
defines a (2,0)-form for \( I \) and \( \Omega^{D/2} \) is a top-holomorphic form for it.

### 5.2 Generalized Geometry

Let \( M \) be a 2\( D \)-real dimensional manifold. An element of the bundle \( E = TM \oplus T^*M \) is the sum of a vector field and a one-form: \( u + \xi \in \Gamma(E) \), where \( \Gamma(E) \) is the space of sections of \( E \). \( E \) is the direct sum of the tangent and the cotangent bundle of the manifold. There is a canonical way to define a symmetric inner product on \( \Gamma(E) \):
\[ \langle u + \xi, v + \eta \rangle = \frac{1}{2} (i_u \eta + i_v \xi), \quad u + \xi, v + \eta \in \Gamma(E). \] (5.11)

In a local coordinate frame \( (\partial_\mu, dx^\mu) \), the inner product reads
\[ \langle u + \xi, v + \eta \rangle = u^\mu \eta_\mu + v^\mu \xi_\mu \] (5.12)
and is represented by the \( 4D \times 4D \) matrix
\[ I = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (5.13)

It can be regarded as a generalized metric on \( E \). We will always assume this local coordinate frame, if we give a coordinate representation. The generalized metric has signature \( (2D, 2D) \) and defines the non-compact orthogonal group \( O(2D, 2D) \) by the symmetries of \( I \). The special orthogonal group \( SO(2D, 2D) \) preserves the natural inner product and the orientation on \( E \).

A subbundle \( \mathcal{L} \subset E \) is isotropic with respect to the inner product if for all of its sections \( u + \xi \in \Gamma(\mathcal{L}) \) the following holds
\[ v + \eta \in \Gamma(\mathcal{L}) \Rightarrow \langle u + \xi, v + \eta \rangle = 0. \] (5.14)
It is a maximally isotropic subbundle if in addition
\[\langle u + \xi, v + \eta \rangle = 0 \text{ for all } u + \xi \in \Gamma(\mathcal{L}) \Leftrightarrow v + \eta \in \Gamma(\mathcal{L}). \quad (5.15)\]
The tangent bundle of the manifold is an example of a maximally isotropic subspace of \(TM \subset E\). For a non-vanishing section \(u\) of \(TM\), we have
\[\langle u, v + \eta \rangle = i_u \eta. \quad (5.16)\]
This equation holds for all \(u \in \Gamma(TM)\) if and only if \(\eta = 0\). But \(v\) is a section of \(TM\). If \(\mathcal{L}\) is maximally isotropic, then its complement \(\mathcal{L}^*\) in \(E\) with \(\mathcal{L} \oplus \mathcal{L}^* = E\) is maximally isotropic as well. It follows that every maximally isotropic subbundle of \(E\) is 2D-dimensional.

Maximal isotropic subspaces can be identified with null spaces of pure spinors on \(M\). Based on the fact that the inner product allows us to regard \(SO(2D, 2D)\) as the structure group for \(E\), Gualtieri proved that it always admits a \(Spin(2D, 2D)\) structure. The spin bundle is isomorphic to the exterior algebra \(\wedge T^*M\). A spinor can be regarded as a formal sum of forms of different rank. A spinor \(\phi\) defines a subbundle \(\mathcal{L}_\phi \subset E\) via
\[\mathcal{L}_\phi = \{(u + \xi) \cdot \phi = i_x \phi + \xi \wedge \phi = 0\}. \quad (5.17)\]
This is the annihilator of \(\phi\) in \(E\), the spinor’s null space. By definition, \(\mathcal{L}_\phi\) is isotropic. If \(\mathcal{L}_\phi\) is maximally isotropic, then \(\phi\) is called a pure spinor. In general, the pure spinor can only be defined locally. Therefore, a maximal isotropic \(\mathcal{L}\) is identified with a pure spinor line.

In complex geometry, integrability of the complex structures is defined with the help of the Lie bracket. There is no Lie bracket action on \(TM \oplus T^*M\). However, the Courant bracket \((2.34)\) is a natural extension of the Lie bracket,
\[[u + \xi, v + \eta]_c = [u, v] + L_u \eta - L_v \xi - \frac{1}{2}d(i_u \eta - i_v \xi). \quad (5.18)\]
We drop the index \(c\) indicating the Courant bracket from now on, when there is no risk to confuse it with the Lie bracket. The Courant bracket does not satisfy the Jacobi identity but it shares a lot of properties with the Lie bracket, e.g. diffeomorphism invariance. It has an additional family of automorphisms, parametrized by closed two-forms \(B \in \Omega^2(M)_{cl}\). This \(B\)-field transformation acts on the sections of \(E\) as a shearing transformation on \(T^*M:\)
\[e^B(u + \xi) = u + (\xi + i_u B). \quad (5.19)\]
It is a symmetry of the inner product \((5.11)\) and it defines an automorphism of the Courant bracket, since
\[[e^B(u + \xi), e^B(v + \eta)] = e^B[u + \xi, v + \eta] + i_u i_v dB. \quad (5.20)\]
The last term vanishes since $B$ is closed. Integrability in generalized complex geometry is defined in the same way as in complex geometry, except that the Lie bracket is replaced by the Courant bracket. A maximally isotropic $\mathcal{L}$ that is closed under the Courant bracket

$$u + \xi, v + \eta \in \Gamma(\mathcal{L}) \rightarrow [u + \xi, v + \eta] \in \Gamma(\mathcal{L}) \quad (5.21)$$

is said to be involutive or integrable. In that case, $\mathcal{L}$ is called a Dirac structure. By (5.20), integrability of $\mathcal{L}$ is equivalent to integrability of the $B$-transformed bundle

$$\mathcal{L}_B = e^B \mathcal{L}. \quad (5.22)$$

There exists a twisted version of the Courant bracket. Let $H$ be a closed three form, then the twisted Courant bracket is defined by

$$[u + \xi, v + \eta]_H = [u + \xi, v + \eta] + i_u iv H. \quad (5.23)$$

Besides generalized geometry, one can also define twisted generalized geometry, where the Courant bracket is replaced by its twisted version. Such a twist can be achieved by a transformation with a $B$-field that is not closed. This observation provides a convenient technical trick for deriving results in twisted generalized geometry. Computations are much easier in the untwisted case and performing such a $B$-field transformation gives the corresponding results in the twisted geometry.

### 5.3 Generalized Complex Structures

An almost generalized complex structure is a maximally isotropic complex subbundle $\mathcal{L} \subset E \otimes \mathbb{C}$ such that $\mathcal{L} \oplus \bar{\mathcal{L}} = E \otimes \mathbb{C}$. To make a connection to the notion of complex geometry, an almost generalized complex structure can equally well be defined as an endomorphism $J \in \text{End}(E \otimes \mathbb{C})$ that is both complex and symplectic:

$$J^2 = -1, \quad J^t J = I. \quad (5.24)$$

We call $J$ an almost generalized complex structure, and its $+\beta$ eigenbundle is the maximally isotropic $\mathcal{L}$. $J$ is integrable and called a generalized complex structure, if $\mathcal{L}$ is integrable. Twisted generalized complex structures are defined analogously but with the Courant bracket replaced by its twisted version. In local coordinates, such a generalized complex structure can be written as a $4D \times 4D$ matrix

$$J = \begin{pmatrix} -J & P \\ L & J^t \end{pmatrix}, \quad (5.25)$$
where the components of the matrix are regarded as maps between the four possible combinations of the tangent and the cotangent bundle $J : TM \to TM$, $P : T^*M \to TM$, $L : TM \to T^*M$. Since $J$ is symplectic (5.24), $P$ and $L$ are skew-symmetric. This allows us to view $L$ as a two-form and $P$ as a bi-vector. $L$ is not to be confused with the Lie derivative of a vector field $L_u$.

The $B$-transformation acts on $J$ as

$$J_B = U_B J U_B^{-1}, \quad U_B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}$$  \hspace{1cm} (5.26)

and is a symmetry of the inner product $I$. The two basic examples of generalized complex structures are provided by the embeddings of an ordinary complex structure $J$ and a symplectic structure $\omega$ on $M$ in the notion of generalized complex geometry. They correspond to diagonal and off-diagonal generalized complex structures, respectively:

$$J_J = \begin{pmatrix} -J & 0 \\ 0 & J' \end{pmatrix}, \quad J_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}.$$  \hspace{1cm} (5.27)

The pure spinor line bundles for these two examples are given by

$$\varphi_J = e^B \Omega, \quad \varphi_\omega = e^{B+\beta\omega},$$  \hspace{1cm} (5.28)

where $B \in \Omega^2(M)_{cl}$ and $\Omega$ is the top holomorphic form corresponding to the complex structure $J$.

Locally, a generalized complex manifold can always be decomposed into a complex and a symplectic part, amounting to choosing local coordinates in which the generalized complex structure $J$ splits into complex and symplectic parts. This is a generalization of the Newlander-Nierenberg theorem for complex manifolds and the Darboux theorem for symplectic ones.

### 5.4 Generalized Kähler Geometry

Generalized Kähler Geometry is defined by two commuting generalized complex structures $J_1, J_2$. Vanishing of the commutator implies, that the product of these two generalized complex structures is a generalized product structure:

$$G = -J_1 J_2, \quad G^2 = 1.$$  \hspace{1cm} (5.29)

$G$ is the equivalent of the product structure $\hat{G}$ in complex geometry. It is called the generalized metric but is not to be mistaken for $I$. $G$ commutes with the two complex structures by construction

$$[G, J_{1,2}] = 0.$$  \hspace{1cm} (5.30)
It has signature \((2D, 2D)\) and splits \(E\) into positive and negative definite eigen-bundles. Ordinary Kähler geometry is included in generalized Kähler geometry. Let the metric \(G_{\mu\nu}\) be Kähler with respect to the complex structure \(J^\mu\) and \(\omega^\mu\) be the corresponding Kähler form. If \(J_J, \ J_\omega\) are the two generalized complex structures of the example in the previous section then \([J_J, J_\omega]\) commute by construction. The generalized metric is given by

\[
G = -J_J J_\omega = \begin{pmatrix} 0 & G^{-1} \\ G & 0 \end{pmatrix}. \tag{5.31}
\]

Equation (5.30) expressed in terms of ordinary complex geometry translates into hermiticity of \(G_{\mu\nu}\) with respect to the complex structure \(J^\mu\).

Generalized Kähler is equal to bi-hermitian geometry. The map between these two different formulations is given by

\[
J_{1,2} = \frac{1}{2} \begin{pmatrix} -(J_+ \pm J_-) & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & J_+^\prime \pm J_-^\prime \end{pmatrix}. \tag{5.32}
\]

The components \(J_\pm\) are complex structures that can be understood as follows. Since \(G\) commutes with the generalized complex structures, the projection of \(J_{1,2}\) onto the positive and negative eigenspaces of \(G\) define complex structure \(J_\pm\) and there are exactly two ways for choosing the relative sign between them, \(J_+ \pm J_-\). The generalized metric in the case of a non-vanishing two-form is obtained from (5.31) via a \(B\)-transformation

\[
G_B = U_B^t G U_B = \begin{pmatrix} G^{-1}B & G^{-1} \\ G + BG^{-1}B & BG^{-1} \end{pmatrix}. \tag{5.33}
\]

Of course, the \(B\)-transformation twists the Courant bracket accordingly. The notion of generalized Kähler geometry has major advantages in comparison to the much longer known bi-hermitian formulation. The geometric objects such as metric and the two complex structures are treated in a unified way and the \(B\)-field enters through the \(B\)-transformations. We call the triple \(\{J_1, G_B, J_2\}\) a generalized Kähler structure.

Instead of considering the generalized metric (5.33) to incorporate the contribution of the \(B\)-field, we can also twist the Courant bracket by \(H = dB\) only and define the generalized metric as in (5.31). In this way we arrive at twisted generalized Kähler geometry.

Due to the relation to bi-hermitian geometry, generalized Kähler geometry is an important tool for the study of enhanced supersymmetry in the context of supersymmetric non-linear sigma models. This relation is elaborated on in detail in the next chapter.
5.5 Generalized Hyperkähler Structure

Hyperkähler geometry is included in generalized Kähler geometry. Let $I, J, K$ be the three complex structures and $\omega_I, \omega_J$ and $\omega_K$ their Kähler forms. The relation can be seen by choosing $J_+ = I$ and $J_- = J$ in (5.32).

We define generalized hyperkähler geometry in a different way. Provided three anticommuting generalized Kähler structures $J_i$, $i = 1, 2, 3$ and a generalized metric $G$, we define generalized hyperkähler geometry by imposing the relations

\[
\{G, J_i\} = 0, \quad \{J_i, J_j\} = -\delta_{ij}. \tag{5.34}
\]

This implies that $\tilde{J}_i = GJ_i$ are three additional generalized complex structures. Each of the triples $\{J_i, G, \tilde{J}_i\}$ define a generalized Kähler structure and the six generalized complex structures and the generalized metric form a representation of the algebra of bi-quaternions $Cl_{2,1}(\mathbb{R})$:

\[
\{J_i, J_j\} = -2\delta_{ij}, \quad \{J_i, \tilde{J}_j\} = -2\delta_{ij}G, \quad [J_i, G] = 0. \tag{5.35}
\]

This definition coincides with the ones in [Huy05, Got05]. If we decompose the generalized complex structures according to (5.32) we find two sets of complex structure $J_+i$ and $J_-i$. They anticommute among themselves and the metric $G_{\mu\nu}$ is HKT with respect to both of the triples

\[
\{J_+i, J_+j\} = \{J_-i, J_-j\} = -\delta_{ij}, \quad J_{\pm i}^GJ_{\pm i} = G, \nabla^{(\pm)}J_{\pm i} = 0. \tag{5.36}
\]

This is a special case of a bi-hypercomplex geometry.
6. Supersymmetric Sigma Models

The relation between supersymmetry and geometry is very intriguing. In chapter 2 we saw how superspace is non-trivial even in the “flat” case. In the context of sigma models, the geometry of the target space is determined by the amount of supersymmetry on the sigma model worldsheet. Gates, Hull and Roček showed that a sigma model with manifest $N = (1, 1)$ supersymmetry has its supersymmetry enhanced to $N = (2, 2)$ if the target space is bi-hermitean [GHR84]. The different target space geometries have been studied and classified for around twenty years by now [HKLR87, HP88, Lin06].

Even though the possible target space geometries were known it was first the introduction of generalized complex geometry that provided a clean mathematical concept to deal with these geometries. We already discussed that bi-hermitian geometry is generalized Kähler, but the map (5.32) between these two descriptions is involved. This triggered the question of how this map can be understood in the context of sigma models. Much work has been done in this direction and by now the picture is rather clear [Lin06, Zab06b]: A phase space description favors the notion of generalized complex geometry [Zab06a]. In [IV] we show that $N = (2, 2)$ supersymmetry of the Hamiltonian of the Gates-Hull-Roček sigma model leads us directly to generalized Kähler geometry. From the physics point of view, the map between generalized Kähler geometry and bi-hermitian geometry can be derived from the equivalence of the Hamiltonian and the Lagrangian treatment of the sigma model. In [V] we elaborate this point of view and show the relation between $N = (4, 4)$ supersymmetry and generalized hyperkähler geometry.

In the Lagrangian formulation the additional supersymmetry closes only on-shell while it is off-shell in the Hamiltonian formulation. Off-shell supersymmetry for the action can be established by introducing auxiliary fields or by directly considering certain manifest $N = (2, 2)$ formulations. Recently, it has been shown that generalized Kähler geometry is in one-to-one correspondence with manifest $N = (2, 2)$ superysymmetric sigma models, where the Lagrangian serves as the generalized Kähler potential that encodes the generalized Kähler geometry [LRvUZ05b, MS06]. For a generalized sigma model including auxiliary fields we are not lead directly to generalized Kähler geometry [LMTZ05]. In [III] we elaborate this and show that supersymmetry favors geometrical objects beyond generalized complex geometry.
The chapter starts out with a review on the possible target space geometries and their relation to the supersymmetry of the sigma model. We continue with a description of manifest $N = (2,2)$ supersymmetry before turning to the phase space formulation of the sigma model and the results of [IV, V]. We conclude this chapter with a discussion of the generalized supersymmetric sigma model.

6.1 Preliminaries

Throughout this chapter, we use the notion of supersymmetry in terms of superfields introduced in chapter 2. The action for the $N = (1,1)$ supersymmetric sigma model is a straightforward generalization of the ordinary sigma model action

$$S = \int d^2 \xi d^2 \theta D_+ \Phi^\mu D_- \Phi^\nu \left( G_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi) \right). \quad (6.1)$$

By construction, the action is invariant under the supersymmetry transformation

$$\delta_0(\epsilon) \Phi^\mu = -\beta (\epsilon^+ Q_+ + \epsilon^- Q_-) \Phi^\mu. \quad (6.2)$$

Under certain circumstances, (6.1) has additional, non-manifest supersymmetries [GHR84]. By dimensional arguments, such transformations have to be of the form

$$\delta_1(\epsilon) \Phi^\mu = \beta \epsilon^+ D_+ \Phi^\nu J^\mu_{+\nu}(\Phi) + \beta \epsilon^- D_- \Phi^\nu J^\mu_{-\nu}(\Phi). \quad (6.3)$$

Otherwise, the transformation would involve a dimensionful parameter. If this is a supersymmetry for (6.1), the action is invariant under the transformation. Being a supersymmetry the transformation satisfies the algebra

$$[\delta_0(\epsilon_1), \delta_1(\epsilon_2)] \Phi^\mu = 0,$$

$$[\delta_1(\epsilon_1), \delta_1(\epsilon_2)] \Phi^\mu = 2\epsilon_1^+ \epsilon_2^+ \partial_+ \Phi^\mu + 2\epsilon_1^- \epsilon_2^- \partial_- \Phi^\mu. \quad (6.4)$$

For physical (on-shell) solutions, (6.4) may be fulfilled up to the equations of motions for $\Phi^\mu$. This is exactly the case, when $J_\pm$ are two complex structures and the target space geometry is bi-hermitian

$$J^\rho_{\pm\mu} G^\rho_{\sigma} J^\sigma_{\pm\nu} = G_{\mu\nu}, \quad \nabla^\rho_{\pm} J^\mu_{\pm\nu} = 0, \quad (6.5)$$

where the connections are given by $\Gamma^{(\pm)} = \Gamma^{(0)} \pm G^{-1} H$. Off-shell supersymmetry is achieved if the two complex structures commute and $\hat{G} = -J_+ J_-$ is an integrable product structure with $\hat{G}^2 = 1$. There is an alternative possibility that is similar to the component formulation of worldsheet supersymmetry in section 2.2. We can add auxiliary superfields $S_{\pm\mu}$. These fields anticommute
and transform as a worldsheet spinor. Under supersymmetry, they mix with \( \Phi^\mu \).

There are a number of other possibilities for extended supersymmetry. The following table summarizes some of the corresponding geometries.

| \( N \) | target space geometry                        |
|--------|---------------------------------------------|
| (0, 0), (1, 0), (1, 1) | Riemannian                                  |
| (2, 0), (2, 1) | hermitian, without \( H \): Kähler          |
| (2, 2) | bi-hermitian, without \( H \): bi-Kähler or Kähler |
| (4, 0), (4, 1) | HKT, QKT, without \( H \): hyperkähler      |
| (4, 4) | bi-hypercomplex                             |

*Table 6.1: The amount of supersymmetry restricts the target space geometry.*

### 6.2 Manifest \( N = (2, 2) \) supersymmetry

The focus in this chapter is on extended off-shell \( N = (2, 2) \) supersymmetry of the \( N = (1, 1) \) supersymmetric sigma model. One way to achieve this is to start from a manifest formulation and rewrite it in terms of \( N = (1, 1) \) superfields. In this way, two of the supersymmetries become non-manifest.

Manifest \( N = (2, 2) \) supersymmetry is introduced by extending the world-sheet with four Grassmann directions instead of two. We denote these directions by \( \theta^\pm \) and \( \bar{\theta}^\pm \). The corresponding spinorial derivatives are \( D_\pm \) and \( \bar{D}_\pm \). A general superfield depends on all four Grassmann directions. However, \( N = (2, 2) \) supersymmetry is implemented by constraints on the superfields. There are three types of \( N = (2, 2) \) superfields \cite{LRvUZ05b, MS06}. A chiral superfield \( \lambda \) is constrained by \( \bar{D}_\pm \lambda = 0 \) and a twisted chiral field \( \chi \) by \( D_+ \chi = \bar{D}_- \chi = 0 \). There is a doublet of semichiral superfields \( X \) and \( Y \) \cite{BLR88, ST97} and there are the corresponding antichiral fields. We collect the different fields and their constraints in table 6.2.

| Type             | Constraint                                      |
|------------------|-------------------------------------------------|
| chiral           | \( \bar{D}_\pm \lambda = 0 \)                 |
| antichiral       | \( D_\pm \bar{\lambda} = 0 \)                 |
| twisted chiral   | \( D_+ \chi = \bar{D}_- \chi = 0 \)           |
| twisted antichiral| \( D_- \bar{\chi} = \bar{D}_+ \bar{\chi} = 0 \) |
| semichiral       | \( D_+ \bar{X} = \bar{D}_- Y = 0 \)           |
| semi-antichiral  | \( \bar{D}_+ \bar{X} = D_- \bar{Y} = 0 \)     |

*Table 6.2: The different types of \( N = (2, 2) \) superfields.*

\(^1\)The semichiral superfields are sometimes also called left-/right-chiral.
To connect to the previous section, it is useful to write the $N = (2,2)$ superfields in their $N = (1,1)$ supersymmetric components. We define the $N = (1,1)$ covariant derivatives $D_{\pm}$ and the corresponding supercharges $\hat{Q}_{\pm}$ by

$$D_{\pm} = \mathbb{D}_{\pm} + \mathbb{D}_{\pm}, \quad \hat{Q}_{\pm} = \beta(\mathbb{D}_{\pm} - \mathbb{D}_{\pm}).$$

(6.6)

The component fields are then

$$\lambda \equiv \lambda|, \quad X \equiv \chi|, \quad Y \equiv \gamma|,$$

$$\Psi_- \equiv \hat{Q}_- \chi|, \quad \Psi_+ \equiv \hat{Q}_+ \gamma|.$$  

(6.7)

When reducing to $N = (1,1)$ superfields, two of the supersymmetries become non-manifest. The non-manifest supersymmetry transformations for these fields are found by writing the constraints for the $N = (2,2)$ superfields in terms of the component fields. For the chiral superfields, this reads

$$\hat{Q}_{\pm} \lambda = \beta D_{\pm} \lambda, \quad \hat{Q}_{\pm} \bar{\lambda} = -\beta D_{\pm} \bar{\lambda}.$$  

(6.8)

For $p$ chiral fields $\lambda^a$ and $\bar{p}$ antichiral fields $\lambda^{\bar{a}}$, it is convenient to introduce notation $A = (a, \bar{a})$ and the complex structure

$$J^A_B = \begin{pmatrix} \beta \delta^a_b & 0 \\ 0 & -\beta \delta^{\bar{a}}_{\bar{b}} \end{pmatrix}.$$  

(6.9)

The transformation for $\lambda^A$ is simply $\hat{Q}_{\pm} \lambda^A = J^A_B D_{\pm} \lambda^B$. Similarly, we introduce coordinates $A' = (a', \bar{a}')$ for twisted chiral fields $\chi^{\bar{a}}'$ and twisted antichiral fields $\chi^{a'}$ as well as $M = (m, \bar{m})$ and $M' = (m', \bar{m}')$ for the semichiral fields $\chi^M$ and $\chi^{M'}$. With the complex structures defined in the obvious way, the non-manifest supersymmetry transformations are

$$\hat{Q}_{\pm} \lambda^A = J^A_B D_{\pm} \lambda^B, \quad \hat{Q}_{\pm} \chi^{A'} = \mp J^{A'}_{B'} D_{\pm} \chi^{B'},$$

$$\hat{Q}_+ X^M = J^M_N D_+ X^N, \quad \hat{Q}_- \Psi^-_M = J^M_N D_- \Psi^-_N,$$

$$\hat{Q}_+ \chi^{M'} = \chi^{M'}, \quad \hat{Q}_- \Psi^+_M = -\beta \partial_+ \chi^{M'},$$

$$\hat{Q}_- X^M = \Psi^+_M, \quad \hat{Q}_+ \Psi^-_M = -\beta \partial_- X^M,$$

$$\hat{Q}_- \chi^{M'} = -J^{M'}_{N'} D_- \chi^{N'}, \quad \hat{Q}_+ \chi^{M'} = -J^{M'}_{N'} D_+ \chi^{N'}.$$  

(6.10)

A general $N = (2,2)$ action is a functional of the constrained superfields. It has the form

$$S = \int d^2 \xi d^2 \theta d^2 \bar{\theta} K(\lambda^A, \chi^{A'}, \chi^M, \chi^{M'}).$$  

(6.11)

The corresponding component action is obtained by the relation

$$S = \int d^2 \sigma d^2 \theta d^2 \bar{\theta} K = \int d^2 \sigma \mathbb{D}_+^2 \mathbb{D}_-^2 K$$

$$= \int d^2 \sigma D_+ D_- Q^\dagger Q | K = \int d^2 \sigma d^2 \theta \bar{K},$$  

(6.12)

(6.13)
with $\tilde{K}(\chi, X, Y, \Psi, Y) = Q_+ Q_- K(\chi, X, \Psi)$]. Comparing to the previous section, we can conclude that the chiral and twisted chiral superfields describe situations where the two complex structures commute. Semichiral superfields, on the other hand, describe situations, where the complex structures do not commute [LRvUZ05a]. All of these situations lead to generalized Kähler geometry where $K$ is the generalized Kähler potential [LRvUZ05b]. Recently, it has been shown that a manifest $N = (2, 2)$ supersymmetric sigma model cannot contain any other type of manifest $N = (2, 2)$ superfields [MS06]. Generalized Kähler geometry can be fully described in terms of chiral, twisted chiral and semichiral superfields. To get a feeling for the reduction to $N = (1, 1)$, consider the topological model

$$S = \int d^2 \xi d^2 \theta d^2 \bar{\theta} K(X, \bar{X}).$$

(6.14)

With $S_{-M} = \omega_{MN} \Psi^N_-$ and

$$\omega_{MN} = \begin{pmatrix} \beta \partial_m \partial_n K \\ -\beta \partial_m \partial_n K \end{pmatrix},$$

(6.15)

the action reduces to

$$S = -\frac{\beta}{4} \int d^2 \xi d^2 \theta D_+ X^M S_{-M}.$$  

(6.16)

If the fields are collected into $\Phi^\mu = (\lambda^A, \chi^A', X^M, Y^M')$ and $S_\pm$ are defined similar as above, the action (6.11) can be brought into the form

$$S = \int d^2 \xi d^2 \theta \left( D_+ \Phi^\mu D_- \Phi^\nu e_{\mu\nu} + S_{-M} S_{+N} \epsilon^{MN'} \right).$$

(6.17)

The tensors $e_{\mu\nu}$ and $\epsilon^{MN'}$ are determined by the generalized Kähler potential $K$ [LRvUZ05a].

### 6.3 Enhanced supersymmetry in $N = 1$ phase space

We will now turn to the Hamiltonian treatment of the $N = (2, 2)$ supersymmetric sigma model. The supersymmetric version of phase space corresponds to the cotangent bundle of the superloop space $L^* M = \{ \phi^\mu : S^{1,1} \to M \}$. Here, $S^{1,1}$ is a supercircle with coordinates $\sigma, \theta$ where $\theta$ is the Grassmann-valued direction. We have to reverse the parity on the fibers in order to get the right statistics and denote the cotangent bundle by $\Pi T^* L^* M$. The conjugate momenta are worldsheet fermions. In a local coordinate frame, we have a superfield $\phi^\mu(\sigma, \theta)$ and its conjugate momentum $S_\mu(\sigma, \theta)$. Their expansion in $\theta$ is

$$\phi^\mu(\sigma) = X^\mu(\sigma) + \theta \lambda^\mu(\sigma), \quad S_\mu(\sigma) = \psi_\mu(\sigma) + \beta \theta P_\mu(\sigma),$$

(6.18)
such that $P_\mu$ is the momentum conjugate to $X^\mu$. Here, we follow the notation of [Zab06a]. The symplectic structure on $\Pi T^* L M$ is defined such that the restriction to the bosonic part coincides with (2.20)

$$\omega = \beta \int_{S^1} d\sigma d\theta (\delta S_\mu \wedge \delta \phi^\mu). \quad (6.19)$$

If we perform the Berezin integral, then

$$\omega = \int d\sigma (\delta X^\mu \wedge \delta P_\mu - \beta \delta \psi_\mu \wedge \delta \lambda^\mu). \quad (6.20)$$

The part that involves the bosonic fields is indeed equal to (2.20). The symplectic structure yields the (super-)Poisson bracket

$$\{F, G\} = \beta \int d\sigma d\theta F(\delta S_\mu \wedge \delta \phi^\mu) G. \quad (6.21)$$

This Poisson bracket satisfies the appropriate graded versions of antisymmetry, the Leibniz rule and the Jacobi identity. On this phase space, there are two operators, a spinorial derivative and a corresponding supercharge

$$D = \partial_\theta + \beta \theta \partial, \quad Q = \partial_\theta - \beta \theta \partial. \quad (6.22)$$

They satisfy the algebra

$$D^2 = \beta \partial, \quad Q^2 = -\beta \partial, \quad \{D, Q\} = 0. \quad (6.23)$$

The generator for the manifest supersymmetry is defined by

$$Q(\epsilon) = -\int d\sigma d\theta \epsilon S_\mu Q_\phi^\mu. \quad (6.24)$$

It acts on the fields through the Poisson bracket

$$\delta_1(\epsilon) \phi^\mu = \{\phi^\mu, Q_1(\epsilon)\} = -\beta \epsilon Q_\phi^\mu \quad \delta_1(\epsilon) S_\mu = \{S_\mu, Q_1(\epsilon)\} = -\beta \epsilon Q S_\mu. \quad (6.25)$$

Taking the Poisson bracket of $Q_1$ with itself yields the generator of worldsheet translations

$$\{Q(\epsilon), Q(\tilde{\epsilon})\} = P(2\epsilon \tilde{\epsilon}), \quad P(a) = \int d\sigma d\theta a S_\mu \partial \phi^\mu. \quad (6.26)$$

Any additional generator of supersymmetry transformations $Q_1$ has to be of the form [Zab06a]

$$Q_1(\epsilon) = -\frac{1}{2} \int d\sigma d\theta \epsilon (2D\phi^\mu S_\nu J^\nu_\mu(\phi) + D\phi^\mu D\phi^\nu L_{\mu\nu}(\phi) + S_\mu S_\nu P^{\mu\nu}(\phi)). \quad (6.27)$$
It has to satisfy the Poisson brackets

\[
\{Q_1(\epsilon), Q_1(\tilde{\epsilon})\} = P(2\epsilon \tilde{\epsilon}), \quad \{Q(\epsilon), Q(\tilde{\epsilon})\} = 0. \tag{6.28}
\]

Here, we assume that the supersymmetry does not have central charges. It is shown in \cite{Zab06a} that these conditions are satisfied if and only if the tensors in (6.27) group together into a generalized complex structure $J$ and the target space manifold is generalized complex. The transformations on the fields are given by

\[
\delta(\epsilon)\phi^\mu = \{\phi^\mu, Q_1(\epsilon)\} = \beta \epsilon (D\phi^\nu J^\mu_{\nu} - S_\nu P^{\nu\mu}),
\]

\[
\delta(\epsilon)S_\mu = \{S_\mu, Q_1(\epsilon)\} = \beta \epsilon (D(S_\nu J^\nu_{\mu}) - \frac{1}{2} S_\nu S_\rho P^{\nu\rho}_{\mu} + D(D\phi L_{\nu\mu})
+ S_\nu D\phi^\rho J^\nu_{\rho\mu} - \frac{1}{2} D\phi^\nu D\phi^\rho L_{\nu\rho\mu}). \tag{6.29}
\]

For later use we observe that $Q_1$ can be written in a very compact way using the symmetric inner product (5.11). It makes explicit use of $J$

\[
Q_1(\epsilon) = -\frac{1}{2} \int d\sigma d\theta \epsilon \langle \Theta, J \Theta \rangle. \tag{6.30}
\]

$\Theta$ is given by

\[
\Theta = \left( \begin{array}{c} D\phi \\ S \end{array} \right). \tag{6.31}
\]

If the geometry is twisted by a non-vanishing three-form $H$, the symplectic form gets twisted as in the bosonic case (2.24) and the generators are modified. This modification can be generated by a $B$-transformation with $H = dB$ replacing

\[
S_\mu \rightarrow S_\mu - B_{\mu\nu} D\phi^\nu. \tag{6.32}
\]

The generator of manifest supersymmetry becomes

\[
Q(\epsilon) = -\int d\sigma d\theta \epsilon (S_\mu - B_{\mu\nu} D\phi^\nu) Q\phi^\mu. \tag{6.33}
\]

Since also the Poisson bracket gets twisted, the form of the transformations on the fields (6.25) remains unchanged. We denote the twisted Poisson bracket by $\{\cdot, \cdot\}_H$. It is given by

\[
\{F, G\}_H = \beta \int d\sigma d\theta F \left( \delta \frac{\delta}{\delta S_\mu} \delta \frac{\delta}{\delta \phi^\mu} - \delta \frac{\delta}{\delta \phi^\mu} \delta \frac{\delta}{\delta S_\mu} + 2 \delta \frac{\delta}{\delta S_\nu} H_{\nu\mu\rho} \delta \frac{\delta}{\delta S_\rho} \right) G. \tag{6.34}
\]

A more detailed description for the case $H \neq 0$ is part of \cite{Zab06a,IV}.
6.4 The Poisson sigma model - A first application

In the phase space formulation, time evolution is generated by the Hamiltonian. It is thus natural to study the condition under which a Hamiltonian is invariant under the additional supersymmetry. A relatively simple example is the Wess Zumino (WZ)-Poisson sigma model that plays an important role in the context of deformation quantization [SS94, BCZ05]. Its $N = 1$ supersymmetric version is given by the action

$$S_{PSM} = \int d^2 \xi d\theta \left( S_\mu D\phi^\mu + \frac{1}{2} S_\mu S_\nu \Pi^{\mu\nu} \right).$$  (6.35)

If $\Pi$ is a Poisson structure, it satisfies the Jacobi identity

$$\Pi^{[\mu}_\sigma \Pi^{\rho]_\sigma = 0.$$  (6.36)

This relation allows for a special choice of local coordinates, in which $\Pi$ becomes block diagonal and constant

$$\Pi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ 0 & \end{pmatrix}.$$  (6.37)

These coordinates are called Casimir-Darboux coordinates. This simplifies the local analysis around regular points. $\Pi$ can be twisted by a three form $H$ making it a WZ-Poisson structure that spoils the Jacobi identity

$$\Pi^{[\mu}_\nu \Pi^\rho_{\sigma} = \Pi^{\mu}_{\kappa \nu} \Pi^\nu_{\lambda \rho} \Pi^{\rho}_{\sigma} H_{\kappa \lambda \sigma}.$$  (6.38)

For such a model, the phase space is constrained by the equations of motion

$$\mathcal{C} : D\phi^\mu + \Pi^{\mu\nu} S_\nu = 0.$$  (6.39)

This is a first class constraint for $S_\mu$. In fact, the left hand side is the supersymmetric version of a current of the form (2.31). The canonical Hamiltonian for the WZ-Poisson sigma model vanishes and thus, the Hamiltonian is given by the constraint [AS05]

$$H(S, \phi) = \int d\sigma d\theta \Lambda_\mu \left( D\phi^\mu + \Pi^{\mu\nu} S_\nu \right).$$  (6.40)

The superfields $\Lambda_\mu (\sigma, \theta)$ act as Lagrange multipliers for the constraint. The condition that $\Pi$ is a WZ-Poisson structure is equal to the physical constraint that $\mathcal{C}$ is preserved by Hamilton dynamics

$$\{D\phi^\mu + \Pi^{\mu\nu} S_\nu, \mathcal{H} \} |_{\mathcal{C}} = 0.$$  (6.41)

By construction, the Hamiltonian is invariant under the manifest supersymmetry (6.24). We saw in the previous section that the phase space admits $N = 2$
supersymmetry if the target space geometry is generalized complex. All that remains is to find the conditions under which $H$ is invariant under the transformation $Q_2$ in the constrained phase space, i.e.

$$\{H, Q_1(\varepsilon)\}|_{\varepsilon^\prime} = 0.$$  \hspace{1cm} (6.42)

These conditions were derived and studied by Calvo [Cal05]. The solution to this equation involves the Dirac structure associated to $\Pi$

$$L_\Pi = \{u + \xi \in TM \oplus T^*M, \xi|_{\varepsilon^\prime} = \Pi(X)\}. \hspace{1cm} (6.43)$$

The WZ-Poisson sigma model admits $N = 2$ supersymmetry in phase space if and only if $L_\Pi$ is involutive with respect to the generalized complex structure $J$ associated to the second supersymmetry transformation $Q_1$:

$$J(L_\Pi) \subset L_\Pi. \hspace{1cm} (6.44)$$

### 6.5 The sigma model Hamiltonian

In [IV] we study supersymmetry of the Hamiltonian that corresponds to the sigma model (6.1)

$$S = \int d^2 \xi d^2 \theta D_+ \Phi^\mu D_- \Phi^\nu (G_{\mu\nu}(\Phi) + B_{\mu\nu}(\Phi)). \hspace{1cm} (6.45)$$

To derive the Hamiltonian we reformulate the sigma model in terms of $N = 1$ components of the $N = (1, 1)$ superfields $\Phi^\mu$ by integrating out one of the fermionic directions after a proper coordinate transformation. To this extent, we define

$$\theta^{0,1} = \frac{1}{\sqrt{2}} (\theta^+ + \beta \theta^-), \quad D_{0,1} = \frac{1}{\sqrt{2}} (D_+ \pm \beta D_-), \quad Q_{0,1} = \frac{1}{\sqrt{2}} (Q_+ \pm \beta Q_-). \hspace{1cm} (6.46)$$

With these definitions the action reads

$$S = -\frac{1}{4} \int d^2 \xi d^1 \theta d^0 \left( 2D_0 \Phi^\mu D_1 \Phi^\nu G_{\mu\nu} + (D_1 \Phi^\mu D_1 \Phi^\nu - D_0 \Phi^\mu D_0 \Phi^\nu)B_{\mu\nu} \right). \hspace{1cm} (6.47)$$

We define the component $N = 1$ superfields

$$\phi^\mu = \Phi^\mu|_{\theta^0 = 0}, \quad S_\mu = G_{\mu\nu} D_0 \Phi^\nu|_{\theta^0 = 0}. \hspace{1cm} (6.48)$$

and abuse notation to write $G_{\mu\nu}(\phi) = G_{\mu\nu}(\Phi)|_{\theta^0 = 0}$ and $B_{\mu\nu}(\phi) = B_{\mu\nu}(\Phi)|_{\theta^0 = 0}$. We denote $D = D_1|_{\theta^0 = 0}$ and $\partial = \partial_\sigma$. With this prescription, $D$ is equal to the
The phase space action is obtained by performing the $d\theta^0$ integral

$$S = \int d^2\sigma d\theta (S_\mu - B_{\mu\nu} D\phi^\nu) \partial_0 \phi^\mu - \int dt H(S, \phi), \quad (6.49)$$

where $H$ is the Hamiltonian defined as

$$H(S, \phi) = \frac{1}{2} \oint \sigma d\theta \left( \beta \partial \phi^\mu D\phi^\nu G_{\mu\nu} + S_\mu D S_\nu G^{\mu\nu} + S_\mu D\phi^\nu S_\rho G^{\rho\sigma} \Gamma_{\nu\sigma}^{\mu} \right. \left. - \frac{1}{3} S_\mu S_\nu S_\rho H^{\mu\nu\rho} + D\phi^\mu D\phi^\nu S_\rho H_{\mu\nu\rho} \right). \quad (6.50)$$

$\Gamma_{\nu\rho}^{\mu}$ is the Levi-Civita connection for $G_{\mu\nu}$. The last two terms that depend on $H_{\mu\nu\rho}$ do not appear in the bosonic sigma model Hamiltonian. In fact, those two terms do not have purely bosonic components. To find the form of the supersymmetry transformation, we introduce $\epsilon^{0,1} = 1/\sqrt{2} (\epsilon^+ \mp \beta \epsilon^-)$ and write (6.3) in the form

$$\delta(\epsilon)\Phi^\mu = -\beta (\epsilon^0 Q_0 + \epsilon^1 Q_1) \Phi^\mu \quad (6.51)$$

The term with $\epsilon = \epsilon^1$ gives rise to the manifest supersymmetry of the fields $\phi^\mu$ and $S_\mu$ with $Q = Q_1|_{\theta^0=0}$. The part involving $\epsilon^0$ on the other hand is not a source for a manifest supersymmetry. It gives rise to the non-manifest supersymmetry transformations

$$\tilde{\delta}_0(\epsilon)\phi^\mu = \epsilon G_{\mu\nu} S_\nu, \quad (6.52)$$
$$\tilde{\delta}_0(\epsilon)S_\mu = \beta \epsilon G_{\mu\nu} \partial \phi^\nu + \epsilon S_\nu S_\rho G^{\nu\sigma} \Gamma_{\mu\sigma}^{\rho}. \quad (6.53)$$

In the derivation, terms corresponding to time evolution were dropped. There is no obvious way to write down a generator for this transformation, since it cannot be of the form (6.27).

The additional supersymmetry transformation $Q_1(\epsilon)$ yields a twisted generalized complex structure $J$ due to the presence of $H$. The Hamiltonian admits enhanced supersymmetry if the target space geometry is twisted generalized Kähler, since

$$\{H, Q_1(\epsilon)\}_H = 0 \quad (6.54)$$

implies that the twisted generalized complex structure $J$ commutes with the generalized metric given in (6.97)

$$G = \begin{pmatrix} 0 & G^{-1} \\ G & 0 \end{pmatrix}. \quad (6.55)$$

58
Consequently, the Hamiltonian is invariant under two extra supersymmetries with generators \( \mathcal{Q}_1(\varepsilon) \) and \( \tilde{\mathcal{Q}}_1(\varepsilon) \) of the form (6.27). These generators satisfy (6.28) and in addition

\[
\{ \mathcal{Q}_1(\varepsilon), \tilde{\mathcal{Q}}_1(\varepsilon) \}_H = 2\varepsilon \tilde{\varepsilon} H. \tag{6.56}
\]

Since the two generalized complex structures commute with the generalized metric \( G \), the two extra supersymmetries commute with the non-manifest supersymmetry (6.53). Recently, [Mal06] provided a rigid mathematical proof of the derivation of the Hamiltonian and its supersymmetries.

In conclusion, the sigma model Hamiltonian (6.50) is \( N = (2, 2) \) supersymmetric if the target space geometry is twisted generalized Kähler. If \( H = 0 \), the target space geometry is generalized Kähler. From the physics point of view the relation between generalized Kähler and bi-hermitian geometry is thus given by the equivalence of the Hamiltonian and Lagrange formulation of the sigma model. This can be seen by rewriting the supersymmetry transformation (6.3) in the \( N = 1 \) component fields

\[
\delta(\varepsilon) \phi^\mu = \left( \varepsilon^+ D_+ \Phi^v J^\mu_{\pm v} + \varepsilon^- D_- \Phi^v J^\mu_{\mp v} \right) |\theta^0 = 0
\]

\[
= \frac{i}{2} \varepsilon^1 \left( D\phi^v (J^\mu_{\pm v} + J^\mu_{\mp v}) + S_v ((\omega^{-1})^{\mu v} - (\omega^{-1})^{\mu v}) \right)
\]

\[
+ \frac{i}{2} \varepsilon^0 \left( D\phi^v (J^\mu_{\pm v} - J^\mu_{\mp v}) - S_v ((\omega^{-1})^{\mu v} + (\omega^{-1})^{\mu v}) \right). \tag{6.57}
\]

From this, we identify the tensors in (6.29) and find exactly the relation (5.32) between \( J_{\pm} \) and the generalized complex structures \( J_{1,2} \).

### 6.6 Topological twists

Our picture of the \( N = (2, 2) \) supersymmetric sigma model can be used to discuss topological twists and the corresponding topological field theories in a very natural way. The generators of supersymmetry in phase space can be associated to BRST transformations by converting them to odd generators [Zab06a]. This is formally done by setting the odd parameter \( \varepsilon \) to one in (6.24) and (6.27). This does not change the algebra that \( \mathcal{Q} \) and \( \mathcal{Q}_1 \) satisfy. The linear combination

\[
q = Q(1) + BQ_1(1) \tag{6.58}
\]

is nilpotent. A generalized complex structure can therefore be associated to an odd differential \( s \) on \( C^\infty(\Pi T^* \mathcal{L} M) \).

\[
s\phi^\mu = \{ q, \phi^\mu \} \qquad sS_\mu = \{ q, S_\mu \}. \tag{6.59}
\]
In the case of generalized Kähler geometry, we can define two such operators $s_1, s_2$ by considering the two generators $Q_2(1)$ and $\tilde{Q}_1(1)$. Due to the relation (6.56), the Hamiltonian (6.50) is BRST exact and can be written in two ways

$$H = -\frac{\beta}{2} \int d\sigma d\theta s_1 \tilde{Q}_1(1) = -\frac{\beta}{2} \int d\sigma d\theta s_2 Q_1(1). \quad (6.60)$$

Let us focus on the first version. The topological field theory is localized at the fixed points of the BRST transformations. Purely bosonic fixed points are given by the first class constraint

$$v^\mu p_\mu + \bar{\xi}_\mu \partial X^\mu = 0, \quad (6.61)$$

where $v + \bar{\xi} \in \Gamma(\mathbb{L})$ is a section of the $+\beta$-eigenbundle of $J$. This theory was originally discussed in [AS05] and later reexamined by [BZ05]. It covers the topological A- and B-model. The phase space action that corresponds to (6.60) is

$$S = \int d^2 \xi d\theta \left( (S_\mu - B_{\mu\nu} D\phi^\nu) \partial_0 \phi^\mu + \frac{1}{2} s_1 \tilde{Q}_1(1) \right). \quad (6.62)$$

This is the gauge fixed action for the theory defined by (6.61). One of the complex structures defines the topological field theory and the operator $s_1$ while the other is used for the gauge fixing. The first term in the action can be interpreted as a topological term. The two possibilities of distributing the two generalized complex structures correspond to the two non-equivalent ways of twisting the $N = (2, 2)$ sigma model. An extensive discussion of topological strings and generalized complex geometry can be found in [Pes06].

### 6.7 $N = (4,4)$ supersymmetric Hamiltonian

In this section, we focus on the results of [V] and show how to generalized hyperkähler geometry from $N = (4,4)$ supersymmetry similar to the discussion in section 6.5. We saw that for a generalized complex target manifold, the phase space admits $N = 2$ supersymmetry and that the Hamiltonian (6.50) is $N = (2,2)$ supersymmetric on a (twisted) generalized Kähler manifold. We start with discussing $N = 4$ supersymmetry in phase space and show that the necessary condition for this is a generalized hypercomplex manifold that admits three generalized complex structures satisfying the algebra of quaternions

$$\{J_1, J_2\} = 0, \quad J_3 = J_1 J_2. \quad (6.63)$$

According to the discussion of $N = 2$ supersymmetry, the additional generators of supersymmetry besides the manifest are of the form (6.27)

$$Q_i(\epsilon) = -\frac{1}{2} \int d\sigma d\theta \epsilon \left( 2D\phi^\mu S_\mu J_i^\nu(\phi) + D\phi^\mu D\phi^\nu L_{i\mu\nu}(\phi) + S_\mu S_\nu P_i^{\mu\nu}(\phi) \right). \quad (6.64)$$
These are generators of supersymmetry transformations if we can relate them to generalized complex structures $J_i$. If we denote the generator of manifest supersymmetry by $Q_0(\epsilon) \equiv Q(\epsilon)$ then we require that the $Q_i(\epsilon)$ satisfy the supersymmetry algebra

$$\{Q_i(\epsilon), Q_j(\tilde{\epsilon})\} = \delta_{ij}P(2\epsilon\tilde{\epsilon}), \quad i = 0, 1, 2, 3. \tag{6.65}$$

We do not consider a Hamiltonian at this stage. The Poisson brackets involving $Q_0(\epsilon)$ and those with $i = j$ imply that $J_i$ are (integrable) generalized complex structures as in the previous discussion. The remaining brackets translate into conditions for the generalized complex structures

$$\{J_1, J_2\} = 0, \quad J_3 = J_1J_2, \quad N(J_1, J_2) = 0. \tag{6.66}$$

This coincides with the definition of generalized hypercomplex geometry. The two generalized complex structures anticommute and their (generalized) Nijenhuis concomitant vanishes. It is defined as in (5.4) but with the Lie bracket replaced by the Courant bracket. Actually, if $J_1$ and $J_2$ are integrable, then vanishing of the Nijenhuis concomitant implies integrability for $J_3$. We conclude that the phase space admits $N = 4$ supersymmetry if and only if the manifold is generalized hypercomplex.

Next, we show that invariance of the Hamiltonian (6.50) under the three additional supersymmetries $Q_i$ requires a generalized hyperkähler manifold. For this, we combine the discussion of $N = 4$ supersymmetry in phase space with the discussion that lead to $N = (2, 2)$ supersymmetry. The Hamiltonian is $N = 4$ supersymmetric if

$$\{Q_i(\epsilon), H\} = 0, \quad i = 0, 1, 2, 3. \tag{6.67}$$

We compare this to (6.54) and find that each of the three additional supersymmetry generators gives rise to a generalized Kähler structure since the corresponding generalized complex structures commute with the generalized metric $G$

$$[J_i, G] = 0, \quad i = 1, 2, 3. \tag{6.68}$$

As a consequence, $\tilde{J}_i = GJ_i$ are three additional generalized complex structures. They correspond to supersymmetry transformations $\tilde{Q}_i$ according to the discussion of in section 6.5 such that

$$\{Q_i(\epsilon), \tilde{Q}_j(\tilde{\epsilon})\} = 2\beta\epsilon\tilde{\epsilon}\delta_{ij}H. \tag{6.69}$$

Using anticommutativity of the $J_i$, it is not difficult to show that $J_i, \tilde{J}_i$ and $G$ satisfy the relations of a generalized hyperkähler structure (5.35).

In conclusion, the Hamiltonian (6.50) is $N = (4, 4)$ supersymmetric if and only if the target manifold is generalized hyperkähler, or twisted generalized hyperkähler for the case $H \neq 0$. 
6.8 Twistor space for generalized complex structures

In this section, we define the twistor space of generalized complex structures that is associated to the $N = (4,4)$ supersymmetry of the sigma model Hamiltonian. The idea of a twistor space is to encode the geometric properties of the target manifold $M$ in the holomorphic structure of a larger manifold, the twistor space. The original idea goes back to Penrose [Pen76] and Salamon [Sal82, Sal86]. We here follow the same approach as in the definition of the twistor space for hyperkähler geometry [HKLR87]. Twistor spaces of generalized complex structures and generalized Kähler structure are also discussed in [DM06a, DM06b] in order to find examples of generalized complex and generalized Kähler structures that are not induced by complex, symplectic and Kähler structures. Before discussing the twistor space for the generalized hyperkähler geometry, we first review the results for hyperkähler geometry. Given a hypercomplex structure $J_1, J_2, J_3$ the linear combination

$$K = c^1 J_1 + c^2 J_2 + c^3 J_3$$

is a complex structure if $c$ lies on the unit sphere: $c^2 = 1$. This sphere can be identified with $\mathbb{C}P^1$. $\mathbb{C}P^1$ is usually represented as $\mathbb{C}^2$ with coordinates $(\zeta, \bar{\zeta})$ and the identification $(\zeta, \bar{\zeta}) \simeq (\lambda \zeta, \lambda \bar{\zeta})$ for $\lambda \neq 0$. Therefore, we can cover it with two sheets of coordinates $(\zeta, 1)$ and $(1, \bar{\zeta})$ such that $\bar{\zeta} = \zeta^{-1}$ in the overlapping region. In these coordinates,

$$K = \frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_1 + \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_2 + \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_3. \quad (6.71)$$

The twistor space of complex structures is the product space $M \times S^2$, such that at any point $p \in M$, $S^2$ parametrized the space of complex structures on $T_p M$. A complex structure for the whole manifold is then given by the pair

$$K_{\xi} = \left( \frac{1 - \zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_1 + \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_2 + \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}} J_3, I \right), \quad (6.72)$$

where $I$ is the ordinary complex conjugation on the sphere. This construction allows to define hyperkähler geometry in terms of an abstract parameter space.

We now define the twistor space of generalized complex structures in a completely analogous way. Given the six generalized complex structures $J_i$ and $\tilde{J}_i$ of the previous section, we find that the linear combinations that define generalized complex structures are given by the relation

$$K = \frac{1}{2}(c^i + d^i)J_i + \frac{1}{2}(c^i - d^i)\tilde{J}_i, \quad c^2 = d^2 = 1. \quad (6.73)$$

The space of generalized complex structures for a generalized hyperkähler structure is parametrized by $Z = S^2 \times S^2$. In $\mathbb{C}P^1 \times \mathbb{C}P^1$ coordinates $z, w$, the
vectors $c$ and $d$ are given by

$$
c = \left( \frac{1 - z \bar{z}}{1 + z \bar{z}}, \frac{z + \bar{z}}{1 + z \bar{z}}, \beta(z - \bar{z}) \right), \quad d = \left( \frac{1 - w \bar{w}}{1 + w \bar{w}}, \frac{w + \bar{w}}{1 + w \bar{w}}, \beta(w - \bar{w}) \right).
$$

(6.74)

Since the generalized complex structures $J_i, \tilde{J}_i$ are a realization of the bi-quaternionic algebra, it follows that $K^2 = -1$ and $\tilde{K} = G K$ where $G$ is the generalized metric. The generalized metric $G$ acts on the parameter space by letting $d \to -d$. In the $\mathbb{C}P^1$ coordinate $w$, this corresponds to the anti-podal map

$$
\tau_w : w \to \bar{w}^{-1}
$$

(6.75)

that changes the orientation of the $w$-sphere. The ordinary complex structures for the two spheres $I_z$ and $I_w$ define a complex structure $I$ for $Z$. This complex structure induces a generalized complex structure on $TZ \oplus T^*Z$ by

$$
\hat{J} = \begin{pmatrix} -I & 0 \\ 0 & I' \end{pmatrix}.
$$

(6.76)

A generalized complex structure for the combined space $M \times S^2 \times S^2$ is then given by

$$
J = (K(z, w), \hat{J}).
$$

(6.77)

It is an interesting question, if $I$ can be chosen in a more general way in this context. Generalized complex structures for $S^2 \times S^2$ were explicitly defined in [Hit06].

The triples $\{K, G, \tilde{K} = G K\}$ form generalized Kähler structures. The two spheres parametrize the space of ordinary left- and right-complex structures on $TM$. We can clarify this by introducing

$$
J_i^{(\pm)} = \frac{1}{2}(J_i \pm \tilde{J}_i) = \frac{1}{2}(1 \pm \omega)J_i.
$$

(6.78)

These are the projections of the generalized complex structures on the $\pm$ eigenspaces of $G$. Explicitly and with relation (5.32), they are given by

$$
J_i^{(\pm)} = \frac{1}{2} \begin{pmatrix} -J_{\pm i} & -\omega_{\mp i}^{-1} \\ \omega_{\mp i} & J'_{\pm i} \end{pmatrix}.
$$

(6.79)

With this, (6.73) becomes

$$
K = c^i J_i^{(+)} + d^i J_i^{(-)}.
$$

(6.80)
We indeed find that \( c \) and \( d \) parametrize the two sets of complex structures \( J_{+i} \) and \( J_{-i} \).

It remains to show that \( J \) as defined in (6.77) is indeed a generalized complex structure. In order to see this, we reformulate the previous discussion in the pure spinor language. Let \( \pi \) be the pure spinor line associated to the generalized complex structure \( J_1 \) and \( \varphi \) be a local pure spinor representative such that for the sections \( u + \xi \) of the \( +\beta \) eigenbundle,

\[
(u + \xi) \cdot \varphi = i_X \varphi + \xi \wedge \varphi = 0. \tag{6.81}
\]

Since \( J_1 \) is integrable, the spinor is pure and satisfies

\[
d\varphi = (u + \xi) \cdot \varphi \tag{6.82}
\]

for some section \( u + \xi \) of \( TM \oplus T^*M \). Given that \( \varphi \) is a pure spinor for the \( +\beta \) eigenspace of \( J_1 \), then

\[
\sigma = (1 + \frac{1}{2}zJ_3^{(+)} + \frac{1}{2}wJ_3^{(-)}) \cdot \varphi \tag{6.83}
\]

is a pure spinor for \( K \). Since \( J_1 \) and \( \tilde{J}_1 \) are integrable by assumption, \( K \) is integrable as well. This follows from the fact that the Nijenhuis concomitants vanish. Especially, for fix \( z, w \),

\[
d\sigma|_{z,w} = (u + \xi) \cdot \sigma \tag{6.84}
\]

for some \( u + \xi \in \Gamma(TM \oplus T^*M) \). The bar indicates that the derivative is taken for fixed values of \( z \) and \( w \). The generalized complex structure \( \tilde{J} \) is integrable by construction. We can associate to it a pure spinor \( \eta \) such that \( (A + b) \cdot \eta = 0 \) for sections \( A + b \) of \( TZ \oplus T^*Z \). Explicitly, \( \eta \) is the top-holomorphic form \( \eta = dz \wedge dw \). Since \( \sigma \) is holomorphic in \( z, w \), the spinor \( \rho = \sigma \wedge \eta \) satisfies

\[
d(\sigma \wedge \eta) = d\sigma|_{z,w} \wedge \eta + (-1)^{[\sigma]} \sigma \wedge d\eta + dz \wedge \nabla_\partial z \sigma \wedge \eta + dw \wedge \nabla_\partial w \sigma \wedge \eta. \tag{6.85}
\]

\( \rho \) is a spinor. It is an element of the exterior algebra \( \wedge T^*(M \times S^2 \times S^2) = (\wedge T^*M) \wedge (\wedge T^*S^2) \wedge (\wedge T^*S^2) \). By construction the last two terms in (6.85) vanish such that

\[
d\rho = (X + \xi) \cdot \sigma \wedge \eta + (-1)^{[\sigma]} \sigma \wedge (A + b) \cdot \eta
\]

\[
= (X + \xi + A + b) \cdot \rho. \tag{6.86}
\]

\( \rho \) is a pure spinor for the almost generalized complex structure \( J = (K(z, w), \tilde{J}) \) and we conclude that \( J \) is integrable.

The case for \( H \neq 0 \) follows in the same way, except that (6.82) and (6.84) are replaced by their twisted versions

\[
(d + H \wedge) \varphi = (u + \xi) \cdot \varphi. \tag{6.87}
\]
6.9 Generalized Supersymmetric Sigma Models

We already mentioned that the \( N = (2, 2) \) supersymmetric non-linear sigma model with action (6.1) yields generalized Kähler geometry and can be parametrized completely in terms of chiral, twisted chiral and semichiral \( N = (2, 2) \) superfields. For the latter case the additional supersymmetry closes on-shell unless the action is complemented by an auxiliary term taking care of the additional components of the semichiral fields. For a sigma model with additional auxiliary fields in general, one is not guided uniquely to generalized Kähler geometry. This may have various reasons. On one possibility we elaborate in Paper III, namely a possible geometry beyond the generalized complex case. Up to field redefinitions, the most general action involving auxiliary spinorial fields \( S_{\pm \mu} \) was introduced in [Lin04]

\[
S = -\frac{1}{4} \int d^2 \sigma d^2 \theta \left( S_{[+\mu} D_{-\nu]} \Phi^\mu + S_{+\mu} e^{\mu \nu} S_{-\nu} + 2 D_+ \Phi^\mu D_- \Phi^\nu (B_{\mu \nu} - b_{\mu \nu}) \right). \tag{6.88}
\]

We refer to this model as the generalized non-linear sigma model. \( b_{\mu \nu} \) is a globally defined two form on \( M \), while \( B_{\mu \nu} \) is only locally defined in general and is the origin for the WZ-term with \( H = dB \). We are not interested in the difference between \( B \) and \( b \) and set them equal to each other throughout this section. Extended supersymmetry for this action was first considered in [Lin04]. The solution makes heavy use of the field equations for \( S_{\pm \mu} \). Also, \( e^{\mu \nu} \) was supposed to be invertible, in which case one may perform a coordinate transformation similar to a \( B \)-transformation

\[
S_{\pm \mu} \rightarrow S_{\pm \mu} + e_{\mu \nu} D_{\pm} \Phi^\nu \tag{6.89}
\]

and bring the action into the form

\[
S = -\frac{1}{4} \int d^2 \sigma d^2 \theta \left( S_{+\mu} e^{\mu \nu} S_{-\nu} + 2 D_+ \Phi^\mu D_- \Phi^\nu e_{\mu \nu} \right). \tag{6.90}
\]

The field equations for \( S_{\pm \mu} \) are \( S_{\pm \mu} = 0 \) and like in the introductory discussion of supersymmetry, it is consistent to substitute them into the action and recover the original sigma model (6.1) with \( e = G + B \). The price to pay is that the extended supersymmetry only closes on-shell. If \( e^{\mu \nu} \) is a Poisson tensor, (6.88) becomes the \( N = (1, 1) \) supersymmetric version of the Poisson sigma model. It has been shown [Ber05] that the solution of [Lin04] works even in this case, despite the fact the existence of a metric \( G_{\mu \nu} \) was a crucial point in its derivation. The most general form of an additional supersymmetry is given
by [Lin04]

\[
\delta^{(\pm)}(\epsilon^{\pm})\phi^\mu = \epsilon^{\pm} \left( D_{\pm} \phi L_{\nu}^{(\pm)\mu} \right) - S_{\pm} P_{\nu}(^{(\pm)\mu\nu})
\]

\[
\delta^{(\pm)}(\epsilon^{\pm})S_{\pm\mu} = \epsilon^{\pm} \left( D_{\pm} \phi S_{\nu}^{(\pm)\mu} \right) - D_{\pm} S_{\pm} K_{\nu}^{(\pm)\mu} + S_{\pm} S_{\pm\sigma} N_{\mu}^{(\pm)\nu\sigma}
\]

\[
+ D_{\pm} \phi^\nu D_{\pm} \phi^\rho J_{\mu\nu}^{(\pm)\rho} + D_{\pm} \phi^\nu S_{\pm\sigma} Q_{\mu\nu}^{(\pm)\sigma}
\]

\[
\delta^{(\pm)}(\epsilon^{\pm})S_{\pm\mu} = \epsilon^{\pm} \left( D_{\pm} S_{\pm\nu} R_{\mu\nu}^{(\pm)} \right) + D_{\pm} S_{\pm\nu} Z_{\mu}^{(\pm)\nu} + D_{\pm} D_{\pm} \phi^\nu T_{\mu\nu}^{(\pm)}
\]

\[
+ S_{\pm\rho} D_{\pm\nu} U_{\mu\nu}^{(\pm)\rho} + D_{\pm} \phi^\nu S_{\pm\rho} V_{\mu\nu}^{(\pm)\rho}
\]

\[
+ D_{\pm} \phi^\nu D_{\pm\nu} \phi^\rho X_{\mu\nu\rho}^{(\pm)} + S_{\pm\nu} S_{\pm\rho} Y_{\mu\nu}^{(\pm)\rho}\right). \quad (6.91)
\]

With these, the transformations are given by, e.g.

\[
\delta(\epsilon)\phi^\mu = (\delta^{(\pm)}(\epsilon^{\pm}) + \delta^{(-)}(\epsilon^{-}))\phi^\mu. \quad (6.92)
\]

As in (6.27) the form of the transformation is constrained by dimensional arguments, i.e., it must not contain any dimensionful parameter. The involved tensors are subject to a number of conditions if these transformations are to satisfy the supersymmetry algebra and in order to yield a symmetry of the action. If one disregards the third transformation in (6.91), the remaining two lines recover the form of the transformations for the cases that yield generalized complex geometry. While in that case, the conditions for closure of the algebra have a geometric meaning, here, no such interpretation is known yet. If only one half of the extended supersymmetry is considered, say only the \(\delta^{(+)}\)-transformations, it has been shown [LMTZ05] that generalized complex geometry is a solution. The authors found it intriguing and rather curious that the tensors involved in the transformation (6.91) seem to rearrange themselves into a \(6D \times 6D\) matrix rather than the \(4D \times 4D\) generalized complex matrices

\[
J^{(+)} = \begin{pmatrix}
J^{(+)} & -P^{(+)} & 0 \\
-L^{(+)} & K^{(+)} & 0 \\
T^{(+)} & -Z^{(+)} & R^{(+)}
\end{pmatrix} \quad (6.93)
\]

and that some of the non-differential conditions could be rewritten in a form resembling an almost complex structure

\[
J^{(+)}2 = -1. \quad (6.94)
\]

We proceed from here in a bottom-up approach and try to mimic the concept of generalized complex geometry as best as we can to find a solution for a very simple case of (6.88). Ignoring all differential conditions for the moment and guided by the outcome of the previous discussion, we arrange the tensors in \(6D \times 6D\) matrices. It is worth noticing that while \(D_{\pm} \phi\) live on \(TM\), both auxiliary fields \(S_{\pm}\) live on the cotangent bundle \(T^*M\). It is useful to define two
copies of the cotangent bundle $T^*M\pm$, where the index indicate the copy that $S_\pm$ is living on, respectively. We define the bundle $E = TM \oplus (T^*M_+ \oplus T^*M_-)$. A look at (6.93) reveals that it is written in local coordinates for $E$. In addition to $J^{(\pm)}$, we introduce

$$J^{(-)} = \begin{pmatrix} J^{(-)} & 0 & -P^{(-)} \\ T^{(-)} & R^{(-)} & -Z^{(-)} \\ -L^{(-)} & 0 & K^{(-)} \end{pmatrix}. \quad (6.95)$$

The non-differential conditions from the supersymmetry algebra imply that $J^{(\pm)}$ are almost complex structures on $E$ that commute.

$$J^{(\pm)2} = -1, \quad [J^{(+)}, J^{(-)}] = 0. \quad (6.96)$$

The non-differential conditions that come from invariance of the action, say (6.88), can be understood with the help of the matrix

$$G = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & e \\ -1 & e^t & 0 \end{pmatrix}. \quad (6.97)$$

This matrix has to be bi-hermitian

$$J^{(\pm)\dagger} G J^{(\pm)} = G. \quad (6.98)$$

We are tempted to refer to $G$ as a generalized metric on $E$. This interpretation is supported by the fact that it encodes the action. However, in general, it does not have maximum rank and thus fails to be a suitable candidate. The following observation is worth mentioning. The upper left $4D \times 4D$ submatrix of $G$ is equal to the generalized metric $I$ that appears in generalized complex geometry. This corresponds to projecting $E$ onto $TM \oplus T^*M_+$. Under this projection, $J^{(\pm)}$ reduce to generalized complex structures on the reduced bundle. A similar argument holds for the projection onto $TM \oplus T^*M_-$. The differential conditions are a lot more involved. However, if $e^{\mu\nu} \equiv \Pi^{\mu\nu}$ is a symplectic structure then integrating out the auxiliary fields yields the action

$$S = -\frac{1}{2} \int d^2 \sigma d^2 \theta D_+ \Phi^\mu D_- \Phi^\nu B_{\mu\nu}, \quad (6.99)$$

where $B_{\mu\nu}$ is the closed two-form given by the inverse of $\Pi$. Clearly, this model is trivial. The main issue that prevents finding a geometric condition for the admission of enhanced supersymmetry is a proper language of integrability similar to the notion of the Courant bracket on $E$. Therefore, the main purpose of [III] is to look for hints that point in the right direction. For this, we make one additional assumption. We assume that $P^{(\pm)}$ are invertible Poisson
tensors. This implies that $J^{(\pm)}$, $K^{(\pm)}$ and $R^{(\pm)}$ are covariantly constant complex structures. This and the remaining differential conditions resulting from the algebra and invariance of the action are satisfied provided that the almost complex structures $J^{(\pm)}$ are ‘covariantly constant’ with respect to a certain connection matrix

$$\nabla J^{(\pm)} = \partial J^{(\pm)} + J^{(\pm)} \cdot \Gamma - \Gamma \cdot J^{(\pm)},$$

$$\Gamma = \text{diag}(\Gamma^{(J)}, \Gamma^{(K)}, \Gamma^{(R)}). \quad (6.100)$$

The components of $\Gamma$ are connections that are related to each other through $\Pi$, $P^{(\pm)}$ and their inverses. This resembles the situation in the Gates-Hull-Roˇcek case for ordinary supersymmetric sigma models. But it is still not clear how to correctly interpret this relation as an integrability condition for $J^{(\pm)}$. For this model, the connection matrices $\Gamma$ are flat and have a vanishing generalized Riemann tensor in the sense

$$R = d\Gamma - \Gamma \circ \Gamma = 0. \quad (6.101)$$

$B$-transformations are an important ingredient in generalized complex geometry and could be interpreted as gauge transformations for the geometry. These transformations have an equivalent in the geometry of the generalized sigma model. For $B \in \Omega^2_{cl}(M)$, we define an automorphism of the bundle $E$ by

$$U_B = \begin{pmatrix} 1 & 0 & 0 \\ -B & 1 & 0 \\ -B & 0 & 1 \end{pmatrix}. \quad (6.102)$$

It transforms the complex structure matrices according to

$$J^{(\pm)} \rightarrow U_B J^{(\pm)} U_B^{-1}. \quad (6.103)$$

If $\Gamma$ transforms as a connection under this ‘gauge’ transformation, then (6.100) is invariant. This strengthens its interpretation as an integrability condition for $J^{(\pm)}$. The full geometric picture remains unclear, since we lack a proper concept of integrability for these objects.

The manifest formulation of $N = (2, 2)$ sigma models involving semichiral fields provides a way to gain a better understanding of the presented geometric framework and its relation to generalized complex geometry. The action (6.17) is a special case of (6.90). Expectedly, generalized complex and especially generalized Kähler geometry fits into the above picture as a certain subclass. In [III], we elaborate this and consider the special case of a toy model that involves semichiral superfields only and has the action

$$S = - \int d^2 \xi d^2 \theta d^2 \bar{\theta} \left( \bar{X} \bar{Y} - \bar{X} Y \right) = \int d^2 \xi d^2 \theta d^2 \bar{\theta} \left( X^M B_{MN} Y^N \right). \quad (6.104)$$
If we reduce this action to $N = (1, 1)$ superfields according to section 6.2 and make a proper field redefinition, this action can be rewritten as

$$S = -\frac{1}{4} \int d^2 \xi d^2 \theta \left( S_{+\mu} B^{\mu\nu} S_{-\nu} + D_+ \phi^\nu B_{\mu\nu} D_\mu \phi^\nu \right),$$

(6.105)

where $B_{\mu\nu}$ is constant and antisymmetric. This implies that the second term vanishes, however, it is kept here for clarity. In comparison to (6.104), this action contains twice as many spinorial fields. As a consequence of the constraints for the semichiral superfields, half of them are constrained by

$$S_- M = S_{+M'} = 0,$$

(6.106)

There is a second interpretation of this constraint. It can be interpreted as the field equations for $S_\pm$: Effectively, half of the spinorial fields are integrated out by means of their equations of motion. In the complex structure matrices, it is consistent to neglect the entries corresponding to these components. Effectively, $J^{(\pm)}$ collapse to generalized complex structures:

$$J^{(+)\, 4D} = \left( \begin{array}{ccc} J^{(+)} & -P^{(+)} & 0 \\ 0 & K^{(+)} & 0 \\ 0 & -Z^{(+)} & R^{(+)} \end{array} \right) \rightarrow J^{(+)\, 4D} = \left( \begin{array}{cc} f^{(+)} & -\hat{P}^{(+)} \\ 0 & \hat{K}^{(+)} \end{array} \right),$$

(6.107)

In terms of $M, M'$ coordinates, this reads

$$J^{(+)\, 4D} = \left( \begin{array}{ccc} J^{(+)} & 0 & 0 \\ 0 & J^{(+)\prime} & -P^{(+)} \\ 0 & 0 & K^{(+)} \end{array} \right) \rightarrow J^{(+)\, 4D} = \left( \begin{array}{cc} f^{(+)} & 0 \\ 0 & \hat{K}^{(+)} \end{array} \right),$$

(6.108)

where we identified the tensors with their components that survive the collapse, e.g. $K^{(+)} = K_{\alpha}^{(B)}$. There is a similar reduction for $J^{(-\, 6D) \rightarrow J^{(-\, 4D)}$. This result coincides with the derivation in [LRvUZ05a].

In section 6.3 we gave a shorthand notation for the additional generators of supersymmetry (6.30). Especially, the $B$-transformation reduces to $\Theta \rightarrow U_B \Theta$. Here, the situation is different, there are two derivatives $D_\pm$ and two corresponding auxiliary fields $S_\pm$. A way around this problem is to promote the matrices to operators

$$J^{(+)} = \left( \begin{array}{ccc} J D_+ & -P & 0 \\ -L D_+ & K D_+ & 0 \\ T D_+ D_- & Z D_- & R D_+ \end{array} \right),$$

(6.109)
and define \( \Lambda \) by \( \Lambda = (\phi, S_+, S_-)' \). Similarly, we can define \( J^{(-)} \) and promote \( G \) to an operator. Then, for example, the transformations (6.91) can be written in a very compact way

\[
\delta^{(\pm)} \Lambda = \varepsilon^{\pm} J^{(\pm)} \Lambda,
\]

(6.110)

up to terms involving third rank tensors.

We conclude that our results strongly point towards a geometrical interpretation beyond generalized complex geometry though the lack of a proper notion for this case presents a major obstacle for elaborating further in this direction.
Stringtheorie ist eines der faszinierendsten Teilgebiete der modernen theoretischen Physik. Sie vereint zwei Konzepte, die auf den ersten Blick unvereinbar erscheinen: Gravitation und Quantenmechanik. Das macht sie zu „dem“ Kandidaten für eine Theorie, die alle Naturgesetze beschreibt. Während Quantenfeldtheorien auf der einen Seite in der Lage sind, die elektromagnetische, die schwache und die starke Wechselwirkung in hinreichender Genauigkeit zu beschreiben, haben wir durch Einsteins allgemeine Relativitätstheorie ein Verständnis der Gravitation für verhältnismäßig große Abstände. Die Stringtheorie vereint diese zwei Konzepte mittels einer auf den ersten Blick einfachen und naiven Idee: Warum sollten die fundamentalen Bausteine der Natur nicht eindimensionale Objekte, Strings, statt punktförmige Teilchen sein?

In der vorliegenden Dissertation werden zwei Aspekte der Stringtheorie näher betrachtet: Spannungslose Strings und supersymmetrische Sigmamodelle.

In der Teilchenphysik spielen masselose Teilchen eine wichtige Rolle. Das Photon beispielsweise ist der Träger der elektromagnetischen Kraft. Zudem können Teilchen bei sehr hohen kinetischen Energien als nahezu masselos angesehen werden. In der Stringtheorie ist das Äquivalent zur Teilchenmasse die Spannung $T$ des Strings, dessen Masse pro Einheitslänge. Dem spannungslosen String wird eine ähnliche Rolle zugesprochen wie den masselosen Teilchen. Er taucht erstmals in der Literatur auf im Zusammenhang mit der Diskussion von Strings, die sich wie masselose Teilchen mit Lichtgeschwindigkeit bewegen, jedoch haben wir bis heute nur ein sehr grobes Verständnis seiner eigentlichen Natur. Auch ihm wird eine entscheidende Rolle bei der Beschreibung hochenergetischer Strings zugesprochen. Beispielsweise können wir uns einen String vorstellen, der mit wachsender Winkelgeschwindigkeit rotiert. Nach und nach wird sich der Großteil der Energie des Strings um dessen Endpunkte konzentrieren, während der überwiegende Teil spannungslos wird. Der String zerfällt bildlich gesprochen in eine Ansammlung freier Teilchen, die sich jedoch nur orthogonal zum String bewegen können.

Der spannungslose String unterscheidet sich in vielfältiger Weise von einem „gewöhnlichen“ String mit nicht-verschwindender Spannung. Die verschwindende Spannung führt zu einer Erweiterung der Symmetrie der Raumzeit, des sogenannten Zielraumes, in den wir uns die Weltfläche, die der String überstreicht, eingebettet denken. In der quantentheoretischen Betrachtung wird der
Unterschied noch drastischer. So kollabiert das Spektrum des spannungslosen String zu einem einheitlichen Massenniveau: Alle Anregungen des Strings sind masselos. Insbesondere gilt dies auch für tachyonische Anregungen, die für gewöhnlich instabil sind und aus dem physikalischen Spektrum entfernt werden müssen, da sie eine imaginäre Masse besitzen. Der spannungslose String besitzt keine kritische Dimension. Eine Quantisierung ist für jede beliebige Raumzeit-Dimension möglich und nicht auf zehn bzw. 26 Dimensionen begrenzt wie im Falle nicht-verschwindender Spannung. Jedoch wird die erweiterte Raumzeitssymmetrie nur in \( D = 2 \) Dimensionen bewahrt, während eine Quantisierung in allen anderen Fällen in einem topologischen Spektrum resultiert. Man vermutet, dass der spannungslose String die noch ungebrochene Phase der Stringtheorie beschreibt, in der alle Zustände gleichberechtigt sind, und dass zu geringeren Energien hin ein Phasenübergang stattfindet, in dem sich die verschiedenen Energieniveaus ausbilden.

Das Anregungsspektrum des spannungslosen Strings enthält Zustände mit hohem Spin. Das legt die Vermutung eines Zusammenhangs zu der sogenannten Higher Spin Gauge-Theorie („Höhere-Spin-Eichtheorie“) nahe. Diese Relation lässt sich am einfachsten im Zusammenhang mit der AdS/CFT-Korrespondenz verstehen. Bei der Betrachtung eines Hologramms sieht man ein dreidimensionales Bild, dessen Information auf einer zweidimensionalen Fläche enthalten ist. Übertragen auf die Stringtheorie besagt dieses sogenannte holographische Prinzip in seiner bekanntesten Version, dass Stringtheorie in einem Anti-de Sitter Raum äquivalent ist zu einer konformen Feldtheorie auf dem Rand dieses Raumes. Diese Korrespondenz wurde seitdem die Vermutung ihrer Existenz im Jahre 1997 erstmals aufgestellt wurde, immer wieder getestet. Das hat zu so erstaunlichen Ergebnissen geführt, wie dass gewisse Sektoren der Stringtheorie mit Hilfe der dualen Beschreibung exakt lösbare Modelle sind, die sich mit Methoden der Festkörperphysik lösen lassen. Jedoch gibt es bis heute — meines Wissens nach — keinen direkten Beweis für die Korrespondenz. Sie relatiert die Kopplungskonstante auf der Feldtheorie-Seite zur Spannung \( T \) in der Stringtheorie. Daher entspricht der spannungslose String einer freien Feldtheorie, die die Existenz von Higher Spin-Feldern zuläßt.

Fünfdimensionaler Anti-de Sitter ist Teil eines zehndimensionalen Raumes, in dem der sogenannte Typ-IIB String konsistent quantisiert werden kann, \( AdS_5 \times S^5 \). Leider ist die Betrachtung von Strings in diesem Hintergrund ein schwieriges Unterfangen, und besonders zu deren Quantisierung ist nicht viel bekannt. \( AdS_5 \times S^5 \) ist einer von drei bekannten Hintergründen für Typ-IIB Strings, die maximal supersymmetrisch sind. Das bedeutet, dass sie 32 Supersymmetrien besitzen. Neben \( AdS_5 \times S^5 \) sind dies der flache, leere Raum sowie ein erst kürzlich entdeckter, sogenannter Plane-Wave Hintergrund, der eine Reihe von Eigenschaften mit \( AdS_5 \times S^5 \) gemeinsam hat, aber deutlich einfacher ist. Wie sich herausstellt, lässt sich dieser Hintergrund in einer bestimm-
ten Weise als Grenzfall von $AdS_5 \times S^5$ ableiten. Seine Einfachheit ermöglicht es, den geschlossenen Typ-IIB String zu betrachten und für den Fall der Lichtkegelgleichung, in dem nur die transversalen Freiheitsgrade in Betracht gezogen werden, zu lösen und zu quantisieren.

Die Art, wie die Stringtheorie die Geometrie des Raumes beeinflusst ist sehr verblüffend. Wir haben schon im Rahmen von Kompaktifizierungen angesprochen, dass aus Konsistenzgründen der interne, sechsdimensionale Raum von einer bestimmten Art sein muss. Die Geometrie ist dadurch bestimmt, dass wir unser vierdimensionalen Teilraum $N = 1$ supersymmetrisch sein soll. Wenn der interne Raum zudem Kähler sein soll, bleibt nur eine Möglichkeit, nämlich, dass es sich um eine Calabi-Yau-Mannigfaltigkeit handelt. Auch wenn man wusste, dass es neben Kähler auch andere Möglichkeiten gab, so wurden diese für lange Zeit nicht in Betracht gezogen. Für ein Sigmagmodell mit Supersymmetrie auf der Weltfläche, der Fläche, die ein String in der Zielmannigfaltigkeit, sprich Raumzeit, überstreicht, wird die Geometrie des Zielraums durch die Dimension der Weltfläche und der Anzahl Supersymmetrien bestimmt. Beispielsweise besitzt das $N = (1,1)$ supersymmetrische Sigma-modell die doppelte Anzahl Supersymmetrien wenn die Zielmannigfaltigkeit bi-hermitesch ist. Auch wenn die Geometrien klassifiziert sind, so wurden die Fälle, die nicht Kähler waren doch für lange Zeit als für die Stringtheorie weniger bedeutend eingestuft. Erst in neuerer Zeit wurde mit generalisierter komplexer Geometrie ein neues mathematisches Konzept entwickelt, das komplexe und symplektische Geometrie vereint und gleichmäßig zwischen ihnen interpoliert. Es bietet genau den richtigen Rahmen, um die Verbindung zwischen Weltflächensupersymmetrie und der Geometrie der Zielmannigfaltigkeit näher zu untersuchen. So stellte sich heraus, dass eine Untermenge dieser neuen Geometrien, die sogenannten generalisierten Kähler Geometrien, identisch ist mit der bi-hermiteschen Geometrie und zudem eine vollständig Beschreibung im Rahmen von manifesten $N = (2,2)$ Supersymmetrie besitzt. Generalisierte Calabi-Yau Geometrie ist eine weitere Unterkategorie, die heutzutage bei Flusskompaktifizierungen eine wichtige Rolle spielt. Letztlich kann generalisierte komplexe Geometrie in der Lage sein, eine mathematische Erklärung für Spiegelsymmetrie zu liefern. Generalisierte komplexe Geometrie vereint die topologischen A- und B-Modelle in einem einzigen Modell.

Es folgt eine Zusammenfassung der Originalarbeiten, die dieser Dissertation zugrunde liegen.

**Artikel I**

Im ersten Artikel beschreiben wir, wie spannungslose Strings zu Supergravitationslösungen, Hintergründe, auf denen Stringtheorie konsistent ist, führen. Zu diesem Zweck betrachten wir die Geometrie eines makroskopischen Typ-IIB Strings im Grenzfall, in dem der String sich mit Lichtgeschwindigkeit
bewegt. Dies führt dazu, dass die Spannung des Strings verschwindet und die Geometrie ähnlich wie im Teilchenfall als eine gravitationelle Schockwelle beschrieben werden kann.

**Artikel II**

Wir studieren den spannungslosen, geschlossenen Typ-IIB String in der maximal supersymmetrischen Plane-Wave-Geometrie. Die Lösung ist ähnlich wie im Falle nicht-verschwindender Spannung. Auch die Quantisierung des spannungslosen Strings ist im Gegensatz zum flachen Raum unproblematisch. Dies ist auf die Existenz eines Parameters zurückzuführen, der mit der Krümmung des Hintergrundes zusammenhängt. Wir können zeigen, dass sich der spannungslose String direkt aus dem spannungsbehafteten String im Grenzfall verschwindender Spannung ableiten lässt und konstatieren, dass dieser Grenzwert mit der Quantisierung kommutiert.

**Artikel III**

Im dritten Artikel diskutieren wir die Bedingung, unter der ein generalisiertes Sigmamodell mit zwei Supersymmetrien zusätzliche Supersymmetrien besitzt. Wir finden, dass sich die dabei involvierten Tensoren in natürlicher Weise zu Objekten gruppieren, die eine geometrische Interpretation jenseits der generalisierten komplexen Geometrie nahelegen. Aufgrund unseres unzulänglichen Verständnisses dieser Art von Geometrie sind wir bei unseren Betrachtungen an ein sehr einfaches Sigmamodell gebunden und können nur die wesentlichen geometrischen Objekte identifizieren, sowie zeigen, wie generalisierte komplexe Geometrie in diese Beschreibung eingebettet ist.

**Artikel IV**

Wir erklären die Beziehung zwischen generalisierter Kählergeometrie und bi-hermitescher Geometrie von einem physikalischen Standpunkt aus. Wir zeigen, dass generalisierte Kählergeometrie die Bedingung für $N = (2, 2)$ erweiterte Supersymmetrie im Phasenraum ist. Damit lässt sich die Relation zwischen generalisierter Kählergeometrie und bi-hermitescher Geometrie mit der Äquivalenz zwischen Hamilton- und Lagrangeformalismus beschreiben. Als Anwendung unserer Resultate beschreiben wir topologische Twists.

**Artikel V**

In diesem Artikel, der auf den Ergebnissen des vorherigen basiert, studieren wir die Bedingung für $N = (4, 4)$ Supersymmetrie in der Hamiltonformulierung des Sigmamodells. Wir finden eine Definition für generalisierte Hyperkählergeometrie und definieren den Twistorraum der generalisierten komplexen Strukturen.
Svensk Sammanfattning

Strängteori är en av dem mest fascinerande ämnen som utvecklats inom den moderna teoretiska fysiken. Den förenar två koncept som inte verkar passa ihop: gravitation och kvantmekanik. Detta gör strängteori till en potentiell kandidat för en teori som beskriver naturens alla lager. Medans kvantfältteori å ena sidan kan beskriva den elektromagnetiska, den svaga samt den starka växelverkan tillräckligt exakt, så förstår vi gravitationen genom Einsteins allmänna relativitetsteori som gäller på förhållandevis stora avstånd. Strängteori förenar dessa två concept genom en idé som verkar lika enkel som naiv: Varför skall inte naturens fundamentala byggestenar vara endimensionella objekt, strängar istället för punktformiga partiklar?

I denna avhandling betraktas två olika aspekter av strängteorin: spänningslösa strängar och supersymmetriska sigmadeller.

Masslösa partiklar har en viktig roll inom partikelfysiken. Fotonen till exempel transporterar den elektromagnetiska kraften. Partiklar som rör sig med väldigt höga kinetiska energier kan anses som nästan masslösa. Massens ekvivalent inom strängteori är spänningen $T$, strängens massa per enhetslängd. Den spänningslösa strängen tilldelas en roll motsvarande masslösa partiklar. För första gången diskuteras den i samband med strängar som rör sig likt masslösa partiklar med ljushastigheten, men än idag är vår förståelse av den spänningslösa strängens natur rätt så grov. Som masslösa partiklar anses spänningslösa strängar vara viktiga för förståelsen av strängteorin vid höga energier. Till exempel kan vi tänka oss en strång som roterar med en växande vinkelhastighet. Ju högre vinkelhastigheten blir, desto mer koncentreras strängens energi kring dess ändpunkter medans den stora delen av strängen blir spänningslös. Strängen sönderfaller och blir en samling partiklar som är bundna att röra sig ortogonalt mod strången.

Den spänningslösa strängen skiljer sig från den “vanliga”, spänningsfulla strängen på flera olika sätt. Den försvinnande spänningen ger upphov till en utvidgad symmetri av rumtiden, själva målrummet som vi tänker oss strängens världsytan vara inbäddad i. I den kvantteoretiska diskussionen av strängen blir skillnaden ännu mera drastiskt. Den spänningslösa strängens spektrum kollapssar till en enhetlig massnivå: alla excitationer blir masslösa. Detta gäller även för tachyoniska excitationer som brukar vara instabila och måste elimineras ur det fysikaliska spektrumet på grund av att dem har en imaginär massa. Den
spänningslösa strängen har ingen kritisk dimension. Kvantiseringen är möjligt i alla rumstidsdimensioner och inte bara i tio eller 26 dimensioner som för den spänningsfulla strängen. Den utvidgade rumtidssymmetrien bevaras dock bara för $D = 2$ dimensioner, medans kvantiseringen annars leder till ett topologiskt spektrum. Den spänningslösa strängen tros vara en obruten fas inom strängteorin då alla tillstånd är fortfarande jämställda och att det finns en fasövergång mot lägre energier som ger upphov till dem olika energinivåer.

Den spänningslösa strängens excitationsspektrum innehåller tillstånd med högre spinn. Det tyder mot ett samband med högre spinn gaugeteori. Detta går enklast att förstå i sambandet med AdS/CFT-korrespondensen. Antagligen har alla någon gång sett ett hologram: Informationen av ett tredimensionellt objekt sparas på en tvådimensionell yta. I strängteorin säger detta holografiska princip i dess mest kända version att strängteori i ett Anti-de-Sitter rum är ekvivalent med en konform fältteori som lever på detta rummets rand. Sedan dess upptäckt 1997 testades korrespondensen på flera olika sett och levererade några intressanta resultat som att vissa sektorer av strängteorin är integrabla modeller som via den duala beskrivningen kan löstras med metoder av kondenserade materiens fysiken. En fullständig bevis saknas dock än idag. Korrespondensen relaterar gaugeteorins kopplingskonstant och strängens spänning. Den spänningslösa strängen svarar mot en fri fältteori som tillåter just högre spinn gaugeteorin.

Femdimensionellt anti-de Sitter rum är del av en tiodimensionell bakgrund för typ-IIB strängteori, $AdS_5 \times S^5$. Tyvärr ligger speciellt kvantiseringen av strängteorin i denna bakgrund utanför vår nuvarande förmåga. $AdS_5 \times S^5$ är en av tre kända bakgrunder för typ-IIB strängteorin som är maximalt supersymmetrisk: den bevarar 32 supersymmetrier. De andra två bakgrunden brevid $AdS_5 \times S^5$ är det tomma, plana rummet samt en såkallad planvägsbakgrund som upptäcktes för ett par år sedan. Den delar en del egenskapar med $AdS_5 \times S^5$ men är mycket enklare än den sistnämnda. Det visar sig även att planvägsbakgrunden är en viss gräns av $AdS_5 \times S^5$. Dess enkelhet gör det möjligt att diskutera och kvantisera den slutna typ-IIB strängen i denna bakgrund.

Det sättet på det strängteorin påverkar rumtidsgeometrin är väldigt fascinerande. När vi pratade om kompaktifiering så diskuterade vi redan att det interna sexdimensionella rummet måste vara av en viss typ. Geometrin bestäms genom att vi kräver $N = 1$ supersymmetri i vårt fyrdimensionella rum. När det interna rummet är dessutom Kahler så finns det bara en enda möjlighet: Det måste vara en Calabi-Yau mångfald. Även om det var känd att det fanns flera andra alternativ så diskuterades dem inte på allvar under en lång tid. För en sigmamodell med supersymmetrin på världsytan bestäms geometrin genom världsytan dimension och antalet supersymmetrier. Den $N = (1, 1)$ supersymmetiska sigmamodellen till exempel har två ytterligare supersymmetrier om målmångfalden är bihermitsk. Även om dessa geometrier var kända så klassades de för det mesta som mindre viktigt.
i samband med strängteorin. Först för några år sedan utvecklades ett nytt matematiskt koncept som förenar komplexa och symplektiska geometrin och dessutom interpolerar mellan dem två. Det är det rätta verktyget för studier av världsytesupersymetriens relation till målmångfaldens geometri. Det visade sig att en viss delmängd av dessa nya geometrier, de så kallade generaliserade Kähler geometrier, är identiska med den bi-hermitska geometrin och att det finns en fullständig beskrivning av generaliserad Kähler geometri med hjälp av manifest $N = (2, 2)$ supersymmetri. Generaliserad Calabi-Yau geometri är en annan viktig delmängd som idag är viktig i samband med flödeskompaktieringar. Slutligen finns det potential för att generaliserad komplex geometri kan ge en matematisk förklaring av spegelsymmetrin, eftersom den förenar den topologiska A- och B-modellen i en enda modell.

Vi slutar med en sammanfattning av originalarbeten som denna avhandling grundar på.

**Artikel I**

I den första artikeln beskriver vi hur spänningslösa strängar ger upphov till bakgrundslösningar till typ-IIB supergravitation. Det gör vi genom att betrakta geometrin som härstammar från en makroskopisk sträng i den gränsen då strängen rör sig med ljushastighet. I denna gräns försvinner strängens spänning och geometrin liknar en gravitaionell chockvåg.

**Artikel II**

Vi studerar den spänningslösa, slutna typ-IIB strängen i den maximalt supersymmetiska planvågsgeometrin. Lösningen liknar det spänningsfulla fallet. Även kvantiseringen är inte problematiskt till skillnad från det plana rummet. Detta hänger ihop med existensen av en parameter som är relaterad till bakgrundens krökning. Vi visar även att den spänningslösa strängen fås i en viss gräns av det spänningsfulla fallet och konstaterar att gränsen kommer att med kvantiseringen.

**Artikel III**

I tredje artikeln diskuterar vi villkoret för en generaliserad sigmamodell med två supersymmetrier att ha ytterligare supersymmetrier. Vi ser dem involverade tensorer grupperar sig på ett naturligt sätt till geometriska objekt som tyder mod en tolkning bortom generaliserad komplex geometri. På grund av att vi inte har tillräckligt förståelse av denna typ av geometri är vi bundna till en väldigt enkel sigmamodell där vi kan bara identifiera dem väsentliga geometriska objekt samt förklara hur generaliserad komplex geometri är inbäddad i denna beskrivning.
Artikel IV

Vi förklarar relationen mellan generaliserad Kähler geometri och bi-hermitsk geometri ur en fysikalisk synvinkel. Vi visar att generaliserad Kähler geometri uppstår som villkor för $N = (2,2)$ supersymmetri i fasrummet. Relationen mellan generaliserad Kähler geometri och den bi-hermitska geometrin kan därför tolkas genom ekvivalensen av Hamilton- och Lagrangebeskrivningen av den supersymmetriska sigmamodellen. I diskussionen av topologiska twists hittar vi en första tillämpning av våra resultat.

Artikel V

I denna artikel diskuterar vi villkoret för $N = (4,4)$ supersymmetri i Hamilton-formuleringen av sigmamodellen. Byggande på den förra artikeln hittar vi en definition av generaliserad hyperkähler geometri och definierar twistorrummet för dem generaliserade komplexa strukturer.
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