Solitons and geometrical structures
in a perfect fluid spacetime

Adara M. Blaga

Abstract

Geometrical aspects of a perfect fluid spacetime are described in terms of different curvature tensors and \( \eta \)-Ricci and \( \eta \)-Einstein solitons in a perfect fluid spacetime are determined. Conditions for the Ricci soliton to be steady, expanding or shrinking are also given. In a particular case when the potential vector field \( \xi \) of the soliton is of gradient type, \( \xi := \text{grad}(f) \), we derive from the soliton equation a Laplacian equation satisfied by \( f \).

1 Introduction

Lorentzian manifolds form a special subclass of pseudo-Riemannian manifolds of great importance in general relativity, where spacetime can be modeled as a 4-dimensional Lorentzian manifold of signature \((3,1)\) or, equivalently, \((1,3)\).

Relativistic fluid models are of great interest in different branches of astrophysics, plasma physics, nuclear physics etc. Perfect fluids are often used in general relativity to model idealized distributions of matter, such as the interior of a star or an isotropic universe. Einstein’s gravitational equation can describe the behavior of a perfect fluid inside of a spherical object and the Friedmann-Lemaître-Robertson-Walker equations are used to describe the evolution of the universe. In general relativity, the source for the gravitational field is the energy-momentum tensor. A perfect fluid can be completely characterized by its rest frame mass density and isotropic pressure. It has no shear

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stresses, viscosity, nor heat conduction and is characterized by an energy-momentum tensor of the form:

\[ T(X,Y) = pg(X,Y) + (\sigma + p)\eta(X)\eta(Y), \]

for any \( X, Y \in \chi(M) \), where \( p \) is the isotropic pressure, \( \sigma \) is the energy-density, \( g \) is the metric tensor of Minkowski spacetime, \( \xi := \sharp(\eta) \) is the velocity vector of the fluid and \( g(\xi, \xi) = -1 \). If \( \sigma = -p \), the energy-momentum tensor is Lorentz-invariant (\( T = -\sigma g \)) and in this case we talk about the vacuum. If \( \sigma = 3p \), the medium is a radiation fluid.

The field equations governing the perfect fluid motion are Einstein’s gravitational equations:

\[ kT(X,Y) = S(X,Y) + (\lambda - \text{scal})g(X,Y), \]

for any \( X, Y \in \chi(M) \), where \( \lambda \) is the cosmological constant, \( k \) is the gravitational constant (which can be taken \( 8\pi G \), with \( G \) the universal gravitational constant), \( S \) is the Ricci tensor and \( \text{scal} \) is the scalar curvature of \( g \). They are obtained from Einstein’s equations by adding a cosmological constant in order to get a static universe, according to Einstein’s idea. In modern cosmology, it is considered as a candidate for dark energy, the cause of the acceleration of the expansion of the universe.

Replacing \( T \) from (1) we obtain:

\[ S(X,Y) = -(\lambda - \frac{\text{scal}}{2} - kp)g(X,Y) + k(\sigma + p)\eta(X)\eta(Y), \]

for any \( X, Y \in \chi(M) \). Recall that a manifold having the property that the Ricci tensor \( S \) is a functional combination of \( g \) and \( \eta \otimes \eta \), for \( \eta \) a 1-form \( g \) dual to a unitary vector field, is called quasi-Einstein [4]. Quasi-Einstein manifolds arose during the study of exact solutions of Einstein field equations. For example, the Robertson-Walker spacetime are quasi-Einstein manifolds [12]. They also can be taken as a model of the perfect fluid spacetime in general relativity [10], [11].

Ricci flow and Einstein flow are intrinsic geometric flows on a pseudo-Riemannian manifold, whose fixed points are solitons. In our paper, we are interested in a generalized version of the following two types of solitons:

1. **Ricci solitons** [13], which generates self-similar solutions to the Ricci flow:

\[ \frac{\partial}{\partial t} g = -2S, \]
2. *Einstein solitons* [3], which generate self-similar solutions to the Einstein flow:

\[
\frac{\partial}{\partial t} g = -2(S - \frac{\text{scal}}{2} g).
\]

Perturbing the equations that define these types of solitons by a multiple of a certain 
\((0,2)\)-tensor field \(\eta \otimes \eta\), we obtain two slightly more general notions, namely, \(\eta\)-Ricci 
solitons and \(\eta\)-Einstein solitons, which we shall consider in a perfect fluid spacetime, i.e.
in a 4-dimensional pseudo-Riemannian manifold \(M\) with a Lorentzian metric \(g\) whose 
content is a perfect fluid.

## 2 Basic properties of a perfect fluid spacetime

Let \((M, g)\) be a general relativistic perfect fluid spacetime satisfying (3). Consider \(\{E_i\}_{1 \leq i \leq 4}\) an orthonormal frame field i.e. \(g(E_i, E_j) = \varepsilon_{ij}\delta_{ij}, i, j \in \{1, 2, 3, 4\}\) with \(\varepsilon_{11} = -1, \varepsilon_{ii} = 1, i \in \{2, 3, 4\}\), \(\varepsilon_{ij} = 0, i, j \in \{1, 2, 3, 4\}, i \neq j\). Let \(\xi = \sum_{i=1}^{4} \xi^iE_i\). Then

\[-1 = g(\xi, \xi) = \sum_{1 \leq i,j \leq 4} \xi^i\xi^j g(E_i, E_j) = \sum_{i=1}^{4} \varepsilon_{ii}(\xi^i)^2\]

and

\[\eta(E_i) = g(E_i, \xi) = \sum_{j=1}^{4} \xi^jg(E_i, E_j) = \varepsilon_{ii}\xi^i.\]

Contracting (3) and taking into account that \(g(\xi, \xi) = -1\), we get:

\[
\text{scal} = 4\lambda + k(\sigma - 3p).
\]

Therefore:

\[
S(X, Y) = (\lambda + \frac{k(\sigma - p)}{2})g(X, Y) + k(\sigma + p)\eta(X)\eta(Y),
\]

for any \(X, Y \in \chi(M)\).

**Example 2.1.** A radiation fluid has constant scalar curvature equal to 4\(\lambda\).

If we denote by \(\nabla\) the Levi-Civita connection associated to \(g\), then for any \(X, Y, Z \in \chi(M)\):

\[(\nabla_X S)(Y, Z) := X(S(Y, Z)) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z) =
\]
\[ = k(\sigma + p)\{\eta(Y)g(\nabla_X \xi, Z) + \eta(Z)g(\nabla_X \xi, Y)\} = \]

\[ = k(\sigma + p)\{\eta(Y)(\nabla_X \eta)Z + \eta(Z)(\nabla_X \eta)Y\}. \tag{8} \]

Imposing different conditions on the covariant differential of \( S \), we have that:

i) \( M \) is Ricci symmetric if \( \nabla S = 0 \);

ii) \( S \) is Codazzi tensor if \( (\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \), for any \( X, Y, Z \in \chi(M) \);

iii) \( S \) is \( \alpha \)-recurrent if \( (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) \), for any \( X, Y, Z \in \chi(M) \), for \( \alpha \) a nonzero 1-form;

iv) \( S \) is weakly pseudo Ricci symmetric if \( S \) is not identically zero and \( (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \alpha(Y)S(Z, X) + \alpha(Z)S(X, Y) \), for any \( X, Y, Z \in \chi(M) \), for \( \alpha \) a nonzero 1-form;

v) \( S \) is pseudo Ricci symmetric if \( S \) is not identically zero and \( (\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(Z, X) + \alpha(Z)S(X, Y) \), for any \( X, Y, Z \in \chi(M) \), for \( \alpha \) a nonzero 1-form.

**Proposition 2.2.** Let \((M, g)\) be a general relativistic perfect fluid spacetime satisfying \([7]\).

1. If \( M \) is Ricci symmetric or \( S \) is Codazzi tensor, then \( p = -\sigma \) or \( \nabla \xi = 0 \).

2. If \( S \) is \( \alpha \)-recurrent, then \( p = -\sigma = \frac{\lambda}{k} \) or \( \nabla \xi = 0 \).

3. If \( S \) is (weakly) pseudo Ricci symmetric, then \( p = \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3} \). In this case, \( \xi \) is torse-forming (in particular, irrotational and geodesic) vector field and \( \eta \) is closed (and Codazzi) 1-form.

**Proof.**

1. If \( \nabla S = 0 \), writing \([2]\) for \( Y = Z \) we get \( (\sigma + p)\eta(Y)g(\nabla_X \xi, Y) = 0 \), for any \( X, Y \in \chi(M) \). It follows \( \nabla \xi = 0 \) or \( p = -\sigma \).

If \( (\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z) \), for any \( X, Y, Z \in \chi(M) \), using \([2]\) for \( Y = Z := \xi \) we get \( (\sigma + p)g(\nabla_\xi \xi, X) = 0 \), for any \( X \in \chi(M) \). It follows \( \nabla_\xi \xi = 0 \) or \( p = -\sigma \). If \( \nabla_\xi \xi = 0 \), using \([2]\) for \( X := \xi \) we get \( (\sigma + p)g(\nabla_\xi \xi, Z) = 0 \), for any \( Y, Z \in \chi(M) \). It follows \( \nabla \xi = 0 \) or \( p = -\sigma \).

2. If \( (\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) \), for any \( X, Y, Z \in \chi(M) \), using \([7]\) and \([2]\) and writing the obtained relation for \( Y = Z := \xi \) we get

\[ (\lambda - \frac{k(\sigma + 3p)}{2})\alpha(X) = 0, \]
for any $X \in \chi(M)$. It follows $p = \frac{2\lambda - k\sigma}{3k}$. Writing the same relation for $Z := \xi$ we get

$$-k(\sigma + p)g(\nabla_X \xi, Y) = (\lambda - \frac{k(\sigma + 3p)}{2})\alpha(X)\eta(Y) = 0,$$

for any $X, Y \in \chi(M)$. It follows $\nabla \xi = 0$ or $p = -\sigma$ (so $p = -\sigma = \frac{\lambda}{k}$).

3. If $(\nabla_X S)(Y, Z) = \alpha(X)S(Y, Z) + \alpha(Y)S(Z, X) + \alpha(Z)S(X, Y)$, for any $X, Y, Z \in \chi(M)$, using (17) and (2) and writing the obtained relation for $Y = Z := \xi$ we get

$$(\lambda - \frac{k(\sigma + 3p)}{2})[\alpha(X) - 2\alpha(\xi)\eta(X)] = 0,$$

for any $X \in \chi(M)$. Writing the same relation for $X = Y = Z := \xi$ we get

$$(\lambda - \frac{k(\sigma + 3p)}{2})\alpha(\xi) = 0.$$ 

If $\lambda - \frac{k(\sigma + 3p)}{2} \neq 0$ follows $\alpha(\xi) = 0$ and $\alpha(X) = \alpha(\xi)\eta(X) = 0$, for any $X \in \chi(M)$, which contradicts the fact that $\alpha$ is nonzero. Therefore, $\lambda = \frac{k(\sigma + 3p)}{2} = 0$, so $p = \frac{2\lambda - k\sigma}{3k}$.

Writing now the obtained relation for $Z := \xi$ we get

$$k(\sigma + p)\{g(\nabla_X \xi, Y) + \alpha(\xi)[g(X, Y) + \eta(X)\eta(Y)]\} = 0,$$

for any $X, Y \in \chi(M)$.

If $\sigma + p = 0$ follows $p = -\sigma = \frac{\lambda}{k}$ and from (17), $S(X, Y) = 0$, for any $X, Y \in \chi(M)$, which contradicts the fact that $S$ is nonzero. Therefore:

$$g(\nabla_X \xi, Y) + \alpha(\xi)[g(X, Y) + \eta(X)\eta(Y)] = 0,$$

for any $X, Y \in \chi(M)$ which implies

$$\nabla_X \xi = -\alpha(\xi)[X + \eta(X)\xi],$$

for any $X \in \chi(M)$ i.e. $\xi$ is a torse-forming vector field. Then

$$(\text{curl}(\xi))(X, Y) := g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 0,$$

for any $X, Y \in \chi(M)$ i.e. $\xi$ is irrotational. Also

$$g(\nabla_\xi \xi, X) = g(\nabla_X \xi, \xi) = \frac{1}{2}X(g(\xi, \xi)) = 0,$$

for any $X \in \chi(M)$ i.e. $\xi$ is geodesic.
Concerning $\eta$, notice that

$$(\nabla_X \eta) Y - (\nabla_Y \eta) X := X(\eta(Y)) - \eta(\nabla_X Y) - Y(\eta(X)) + \eta(\nabla_Y X) := (d\eta)(X, Y).$$

Also

$$X(\eta(Y)) - \eta(\nabla_X Y) - Y(\eta(X)) + \eta(\nabla_Y X) = g(\nabla_X \xi, Y) - g(\nabla_Y \xi, X) = 0$$

which implies $d\eta = 0$.

If $(\nabla_X S)(Y, Z) = 2\alpha(X)S(Y, Z) + \alpha(Y)S(Z, X) + \alpha(Z)S(X, Y)$, for any $X, Y, Z \in \chi(M)$, following the steps of computations above, we get the same conclusions.

Remark the following facts:

1) If in the general relativistic perfect fluid spacetime $\xi$ is not $\nabla$-parallel and it has $\alpha$-recurrent Ricci tensor field, we have the vacuum case. The energy-momentum tensor is $T_{vac} = \frac{\lambda}{2}\eta g$.

2) $S(X, \xi) = -S(\xi, \xi)\eta(X) = [\lambda - \frac{k(\sigma + 3p)}{2}]g(X, \xi)$, for any $X \in \chi(M)$ which shows that $\lambda - \frac{k(\sigma + 3p)}{2}$ is the eigenvalue of $Q$ corresponding to the eigenvector $\xi$, where $Q$ is defined by $g(QX, Y) := S(X, Y), X, Y \in \chi(M)$. From Proposition 2.2 we deduce that if $\xi$ is not $\nabla$-parallel and $M$ is Ricci symmetric or $S$ is Codazzi tensor, then $(M, g)$ is Einstein and $S(X, \xi) = (\lambda + k\sigma)\eta(X)$, for any $X \in \chi(M)$, and if $S$ is (weakly) pseudo Ricci symmetric, then $S(X, \xi) = 0$, for any $X \in \chi(M)$, hence $\xi \in \ker Q$; also, $\text{div}(\xi) := \sum_{i=1}^{4}\varepsilon_{ii}g(\nabla_{E_i} \xi, E_i) = -3\alpha(\xi)$. Notice that in all these cases, if $\sigma > -\frac{2}{k}$, the scalar curvature $\text{scal} := \sum_{i=1}^{4}\varepsilon_{ii}S(E_i, E_i) = 4\lambda + k(\sigma - 3p) > 3(\lambda - kp)$ is positive.

3) Considering Plebanski energy conditions $\sigma \geq 0$ and $-\sigma \leq p \leq \sigma$ for perfect fluids, when $S$ is (weakly) pseudo Ricci symmetric, the energy-density is lower bounded by $\max\{-\frac{\lambda}{k}, \frac{\lambda}{2k}\}$. It was observed that a positive cosmological constant $\lambda$ acts as repulsive gravity, explaining the accelerating universe. The observations of Edwin Hubble confirmed that the universe is expanding, therefore, this seems to be the real case, so $\sigma \geq \frac{\lambda}{2k}$.

4) Let $hX := X + \eta(X)\xi$ be the projection tensor, $X \in \chi(M)$. If $S$ is (weakly) pseudo Ricci symmetric and $\alpha(\xi) \neq 0$, then $(h, \xi, \eta, g)$ is a Lorentzian concircular structure [18]. In the particular case $\alpha(\xi) = -1$ it becomes LP-Sasakian structure [14]. In the other case, if $\alpha(\xi) = 0$, then $\xi$ and $\eta$ are $\nabla$-parallel, $\xi$ is divergence free, hence harmonic vector field.
Applying the covariant derivative to (2) we obtain:

\begin{equation}
\kappa(\nabla_X T)(Y, Z) = (\nabla_X S)(Y, Z) - \frac{1}{2}X(\text{scal})g(Y, Z),
\end{equation}

for any \(X, Y, Z \in \chi(M)\).

**Proposition 2.3.** Let \((M, g)\) be a general relativistic perfect fluid spacetime satisfying (7).

1. If \(T\) is covariantly constant, then the scalar curvature is constant.
2. If \(T\) is Codazzi tensor, then the scalar curvature satisfies \(\text{grad}(\text{scal}) + (d\text{scal})(\xi)\xi + 2k(\sigma + p)\nabla_\xi \xi = 0\).
3. If \(T\) is \(\alpha\)-recurrent, then the scalar curvature satisfies \(d\text{scal} + [4\lambda - \text{scal} - k(\sigma + 3p)]\alpha = 0\).

**Proof.**

1. If \(T\) is covariantly constant i.e. \(\nabla T = 0\) follows \((\nabla_X S)(Y, Z) = \frac{1}{2}X(\text{scal})g(Y, Z)\), for any \(X, Y, Z \in \chi(M)\). For \(Y = Z := \xi\) we have:

\begin{equation}
2S(\nabla_X \xi, \xi) = \frac{1}{2}X(\text{scal}),
\end{equation}

for any \(X \in \chi(M)\).

Replacing \(S\) from (7) we obtain \(X(\text{scal}) = 0\), for any \(X \in \chi(M)\), which implies that the scalar curvature is constant. Therefore, \(\nabla S = 0\) (i.e. \(M\) is Ricci symmetric) and from Proposition 2.2 follows \(p = -\sigma\) (the vacuum case) or \(\nabla \xi = 0\). If \(\xi\) is not \(\nabla\)-parallel:

\begin{equation}
S = -(\lambda - \frac{\text{scal}}{2} - kp)g
\end{equation}

and from (6) we get:

\begin{equation}
p = \frac{4\lambda - \text{scal}}{4k},
\end{equation}

so the pressure and the energy-density are constant.

2. The condition \((\nabla_X T)(Y, Z) = (\nabla_Y T)(X, Z)\), for any \(X, Y, Z \in \chi(M)\) is equivalent to:

\[
(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = \frac{1}{2}[X(\text{scal})g(Y, Z) - Y(\text{scal})g(Z, X)],
\]

for any \(X, Y, Z \in \chi(M)\). Replacing \(\nabla S\) from (2) we get:
2k(σ + p)[η(Y)g(∇_Xξ, Z) + η(Z)g(∇_Xξ, Y) − η(X)g(∇_Yξ, Z) − η(Z)g(∇_Yξ, X)] =

(13) = X(\text{scal})g(Y, Z) − Y(\text{scal})g(Z, X),

for any \(X, Y, Z \in \chi(M)\). For \(Y = Z := \xi\) and taking into account that \(g(\nabla_X\xi, \xi) = 0\), for any \(X \in \chi(M)\), the above relation becomes:

\[2k(\sigma + p)g(\nabla_\xi\xi, X) + X(\text{scal}) + \xi(\text{scal})\eta(X) = 0,\]

for any \(X \in \chi(M)\). Also \(X(\text{scal}) = (d\text{scal})(X) = \text{grad}(\text{scal})X\) and we obtain:

\[2k(\sigma + p)\nabla_\xi\xi + \text{grad}(\text{scal}) + (d\text{scal})(\xi)\xi = 0.\]

If the scalar curvature is constant, then \(\nabla_\xi\xi = 0\) or \(p = −\sigma\). If \(\nabla_\xi\xi = 0\), taking \(X := \xi\) in (13) we get \((\sigma + p)g(\nabla_\xi\xi, Z) = 0\) which implies \(p = −\sigma\) or \(\nabla\xi = 0\).

If \(\xi\) is geodesic vector field, then \(\text{grad}(\sigma) + (d\sigma)(\xi)\xi = 3[\text{grad}(p) + (dp)(\xi)\xi]\) and the gradient of the scalar curvature is collinear with \(\xi\).

3. The condition \((\nabla_XT)(Y, Z) = \alpha(X)T(Y, Z)\), for any \(X, Y, Z \in \chi(M)\), with \(\alpha\) a nonzero 1-form, is equivalent to:

\[(\nabla_X\alpha)(Y, Z) = \frac{X(\text{scal})}{2}g(Y, Z) = \alpha(X)S(Y, Z) + \alpha(X)(\lambda − \frac{\text{scal}}{2})g(Y, Z),\]

for any \(X, Y, Z \in \chi(M)\). Replacing \(S\) from (7) and \(\nabla S\) from (2) we get:

\[k(\sigma + p)[\eta(Y)g(\nabla_X\xi, Z) + \eta(Z)g(\nabla_X\xi, Y) − \alpha(X)\eta(Y)\eta(Z)] =

(14) = \frac{1}{2}\{X(\text{scal}) + \alpha(X)[4\lambda − \text{scal} + k(\sigma − p)]\}g(Y, Z),\]

for any \(X, Y, Z \in \chi(M)\). For \(Y = Z := \xi\) and taking into account that \(g(\nabla_X\xi, \xi) = 0\), for any \(X \in \chi(M)\), the above relation becomes:

\[X(\text{scal}) = \alpha(X)[−4\lambda + \text{scal} + k(\sigma + 3p)],\]

for any \(X \in \chi(M)\).

If the scalar curvature is constant, then \(\sigma = 0\) and the pressure is constant \(p = \frac{4\lambda−\text{scal}}{3k}\).
Remark the following facts:

1) If the energy-momentum tensor $T$ of a general relativistic perfect fluid spacetime is covariantly constant, then we have the vacuum case or $\xi$ is $\nabla$-parallel (in particular, geodesic).

2) If $T$ is Codazzi and the scalar curvature is not constant but $\xi$ is a geodesic vector field, then the gradient of the scalar curvature is collinear with $\xi$. It was proved that for a Codazzi energy-momentum tensor, the Ricci tensor $S$ is conserved [2].

3) For an $\alpha$-recurrent energy-momentum tensor, if the scalar curvature is constant, then the pressure is constant ($p = \frac{4\kappa - \text{scal}}{3\kappa}$), but the energy-density vanishes.

3 Perfect fluid spacetime with torse-forming vector field $\xi$

We shall treat the special case when $\xi$ is a torse-forming vector field [21] of the form:

$$\nabla \xi = I_{\chi(M)} + \eta \otimes \xi,$$

Then $\nabla_\xi \xi = \xi + \eta(\xi)\xi = 0$, for any $X \in \chi(M)$ i.e. $\xi$ is geodesic, $g(\nabla_X \xi, \xi) = 0$ and $(d\eta)(X, Y) := X(\eta(Y)) - Y(\eta(X)) - \eta([X, Y]) = X(g(Y, \xi)) - Y(g(X, \xi)) - g(\nabla_X Y, \xi) + g(\nabla_Y X, \xi) = g(\nabla_X, Y) - g(\nabla_Y, \xi, X) = 0$. We also have:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$$

$$\eta(R(X, Y)Z) = -\eta(Y)g(X, Z) + \eta(X)g(Y, Z),$$

for any $X, Y, Z \in \chi(M)$.

In this case, we shall see which are the consequences of certain conditions imposed to different types of curvatures of this space, namely, when the curvature satisfies conditions of the type $(\xi, \cdot)_T \cdot S = 0$ and $(\xi, \cdot)_S \cdot T = 0$, where $T$ is the Riemann curvature tensor $R$, the projective curvature tensor $W$ [15], the concircular curvature tensor $P$ [20], the conformal curvature tensor $C$ [1] and the conharmonic curvature tensor $H$ [17] defined as follows:

$$W(X, Y)Z := R(X, Y)Z + \frac{1}{\dim(M) - 1}[g(Z, X)QY - g(Y, Z)QX]$$
(19) \( \mathcal{P}(X,Y)Z := R(X,Y)Z + \frac{\text{scal}}{\dim(M)(\dim(M) - 1)}[g(Z,X)Y - g(Y,Z)X] \)

\( C(X,Y)Z := R(X,Y)Z + \frac{1}{\dim(M) - 2} \left\{ - \frac{\text{scal}}{\dim(M) - 1}[g(Z,X)Y - g(Y,Z)X] + 
\right\} + 
\)

\( + g(Z,X)QY - g(Y,Z)QX + S(Z,X)Y - S(Y,Z)X \}

(20) \( \mathcal{H}(X,Y)Z := R(X,Y)Z + \frac{1}{\dim(M) - 2}[g(Z,X)QY - g(Y,Z)QX + S(Z,X)Y - S(Y,Z)X] \)

for any \( X, Y, Z \in \chi(M) \).

Remark that for an empty gravitational field characterized by vanishing Ricci tensor, the curvature tensors \( R, W \) and \( H \) coincide.

Let us denote by \( T \) a curvature tensor of type \((1,3)\) and ask for certain Ricci-semisymmetry curvature conditions, namely, \( (\xi, \cdot) \cdot T = 0 \) and \( (\xi, \cdot) \cdot S = 0 \), where by \( \cdot \) we denote the derivation of the tensor algebra at each point of the tangent space:

- \( ((\xi, X) \cdot S)(Y, Z) := ((\xi \wedge_T X) \cdot S)(Y, Z) := S((\xi \wedge_T X)Y, Z) + S(Y, (\xi \wedge_T X)Z), \) for \( (X \wedge_T Y)Z := T(X, Y)Z; \)

- \( ((\xi, X) \cdot T)(Y, Z)W := (\xi \wedge_S X)T(Y, Z)W + T((\xi \wedge_S X)Y, Z)W + T(Y, (\xi \wedge_S X)Z)W + T(Y, Z)((\xi \wedge_S X)W), \) for \( (X \wedge_S Y)Z := S(Y, Z)X - S(X, Z)Y. \)

3.1 Perfect fluid spacetime satisfying \((\xi, \cdot) \cdot T = 0\)

The condition \((\xi, \cdot) \cdot T = 0\) is equivalent to

\( S(T(\xi, X)Y, Z) + S(Y, T(\xi, X)Z) = 0, \)

for any \( X, Y, Z \in \chi(M) \) and from (17) with:

\( (\lambda + \frac{k(\sigma - p)}{2})[g(T(\xi, X)Y, Z) + g(Y, T(\xi, X)Z)] + 
\)

\( + k(\sigma + p)[\eta(T(\xi, X)Y)\eta(Z) + \eta(Y)\eta(T(\xi, X)Z)] = 0, \)

for any \( X, Y, Z \in \chi(M). \)

**Theorem 3.1.** Let \((M, g)\) be a general relativistic perfect fluid spacetime satisfying (3) with torse-forming vector field \( \xi. \)
1. If \((\xi, \cdot)_R \cdot S = 0\), then \(p = -\sigma\).
2. If \((\xi, \cdot)_W \cdot S = 0\), then \(p = -\sigma\) or \(p = \frac{2\lambda - 3}{2k}\).
3. If \((\xi, \cdot)_P \cdot S = 0\), then \(p = -\sigma\) or \(p = \frac{4\lambda + k\sigma - 12}{3k}\).
4. If \((\xi, \cdot)_C \cdot S = 0\), then \(p = -\sigma\) or \(p = \frac{2\lambda - k\sigma - 6}{3k}\).
5. If \((\xi, \cdot)_H \cdot S = 0\), then \(p = -\sigma\) or \(p = \frac{\lambda - 1}{k}\).

Proof. 1. From the symmetries of \(R\) and (16) and (17) we obtain:

\[
(23) \quad k(\sigma + p)[\eta(Z)g(X, Y) + \eta(Y)g(Z, X) + 2\eta(X)\eta(Y)\eta(Z)] = 0,
\]
for any \(X, Y, Z \in \chi(M)\). Take \(Z := \xi\) and (23) becomes:

\[
(24) \quad k(\sigma + p)[g(X, Y) + \eta(X)\eta(Y)] = 0,
\]
for any \(X, Y \in \chi(M)\) and we obtain \(p = -\sigma\).

2. From (18), from the symmetries of \(R\) and (16) and (17) we obtain:

\[
(25) \quad k(\sigma + p)(2\lambda - 2kp - 3)[\eta(Z)g(X, Y) + \eta(Y)g(Z, X) + 2\eta(X)\eta(Y)\eta(Z)] = 0,
\]
for any \(X, Y, Z \in \chi(M)\). Take \(Z := \xi\) and (25) becomes:

\[
(26) \quad k(\sigma + p)(2\lambda - 2kp - 3)[g(X, Y) + \eta(X)\eta(Y)] = 0,
\]
for any \(X, Y \in \chi(M)\) and we obtain \(p = -\sigma\) or the pressure is constant \(p = \frac{2\lambda - 3}{2k}\).

3. From (19), from the symmetries of \(R\) and (16) and (17) we obtain:

\[
(27) \quad k(\sigma + p)[4\lambda + k(\sigma - 3p) - 12][\eta(Z)g(X, Y) + \eta(Y)g(Z, X) + 2\eta(X)\eta(Y)\eta(Z)] = 0,
\]
for any \(X, Y, Z \in \chi(M)\). Take \(Z := \xi\) and (27) becomes:

\[
(28) \quad k(\sigma + p)[4\lambda + k(\sigma - 3p) - 12][g(X, Y) + \eta(X)\eta(Y)] = 0,
\]
for any \(X, Y \in \chi(M)\) and we obtain \(p = -\sigma\) or \(p = \frac{4\lambda + k\sigma - 12}{3k}\).
4. From (3), from the symmetries of $R$ and (16) and (17) we obtain:

\[ k(\sigma + p)[2\lambda - k(\sigma + 3p) - 6][\eta(Z)g(X, Y) + \eta(Y)g(Z, X) + 2\eta(X)\eta(Y)\eta(Z)] = 0, \]

for any $X, Y, Z \in \chi(M)$. Take $Z := \xi$ and (32) becomes:

\[ k(\sigma + p)[2\lambda - k(\sigma + 3p) - 6][g(X, Y) + \eta(X)\eta(Y)] = 0, \]

for any $X, Y \in \chi(M)$ and we obtain $p = -\sigma$ or $p = \frac{2\lambda - k\sigma - 6}{3k}$.

5. From (21), from the symmetries of $R$ and (16) and (17) we obtain:

\[ k(\sigma + p)(\lambda - kp - 1)[\eta(Z)g(X, Y) + \eta(Y)g(Z, X) + 2\eta(X)\eta(Y)\eta(Z)] = 0, \]

for any $X, Y, Z \in \chi(M)$. Take $Z := \xi$ and (32) becomes:

\[ k(\sigma + p)(\lambda - kp - 1)[g(X, Y) + \eta(X)\eta(Y)] = 0, \]

for any $X, Y \in \chi(M)$ and we obtain $p = -\sigma$ or $p = \frac{\lambda - 1}{k}$.

\[
\begin{align*}
3.2 \text{ Perfect fluid spacetime satisfying } (\xi, \cdot)S \cdot T &= 0 \\
\text{Denote by} & \\
A &= \lambda + \frac{k(\sigma - p)}{2}, \quad B = k(\sigma + p), \\
a &= \frac{1}{3}, \quad b = \text{scal}_{\frac{12}{12}} (= \frac{4\lambda + k(\sigma - 3p)}{12}), \quad c = -\text{scal}_{\frac{6}{6}} (= -\frac{4\lambda + k(\sigma - 3p)}{6}), \quad d = \frac{1}{2}
\end{align*}
\]

and the curvature tensors can be written:

\[ S(X, Y) = Ag(X, Y) + B\eta(X)\eta(Y) \]

\[ W(X, Y)Z = R(X, Y)Z + a[g(Z, X)QY - g(Y, Z)QX] \]

\[ P(X, Y)Z = R(X, Y)Z + b[g(Z, X)Y - g(Y, Z)X] \]

\[ C(X, Y)Z = R(X, Y)Z + c[g(Z, X)Y - g(Y, Z)X] + d[g(Z, X)QY - g(Y, Z)QX] + d[S(Z, X)Y - S(Y, Z)X] \]
(37) \( \mathcal{H}(X,Y)Z = R(X,Y)Z + d[g(Z,X)QY - g(Y,Z)QX] + d[S(Z,X)Y - S(Y,Z)X] \)
for any \( X, Y, Z \in \chi(M) \).

The condition \( (\xi, \cdot)_S \cdot \mathcal{T} = 0 \) is equivalent to

\[
S(X, \mathcal{T}(Y, Z)W)\xi - S(\xi, \mathcal{T}(Y, Z)W)X + S(X, Y)\mathcal{T}(\xi, Z)W - \\
- S(\xi, Y)\mathcal{T}(X, Z)W + S(X, Z)\mathcal{T}(Y, \xi)W - S(\xi, Z)\mathcal{T}(Y, X)W + \\
+S(X, W)\mathcal{T}(Y, Z)\xi - S(\xi, W)\mathcal{T}(Y, Z)X = 0,
\]

for any \( X, Y, Z, W \in \chi(M) \).

Taking the inner product with \( \xi \), the relation (3.2) becomes:

\[
- S(X, \mathcal{T}(Y, Z)W)\xi - S(\xi, \mathcal{T}(Y, Z)W)\eta(X) + \\
+S(X, Y)\eta(\mathcal{T}(\xi, Z)W) - S(\xi, Y)\eta(\mathcal{T}(X, Z)W) + S(X, Z)\eta(\mathcal{T}(Y, \xi)W) - \\
- S(\xi, Z)\eta(\mathcal{T}(Y, X)W) + S(X, W)\eta(\mathcal{T}(Y, Z)\xi) - S(\xi, W)\eta(\mathcal{T}(Y, Z)X) = 0,
\]

for any \( X, Y, Z, W \in \chi(M) \).

**Theorem 3.2.** Let \((M, g)\) be a general relativistic perfect fluid spacetime satisfying (3) with torse-forming vector field \( \xi \).

1. If \((\xi, \cdot)_S \cdot R = 0\), then \( p = \frac{\lambda}{k} \).

2. If \((\xi, \cdot)_S \cdot W = 0\), then \( p_{1,2} = \frac{\lambda - 6 + 3(\lambda + k\sigma) \pm \sqrt{27(\lambda + k\sigma)^2 - 3(\lambda + k\sigma) + 3}}{3k} \).

3. If \((\xi, \cdot)_S \cdot \mathcal{P} = 0\), then \( p = \frac{\lambda}{k} \) or \( p = \frac{4\lambda - k\sigma - 12}{3k} \).

4. If \((\xi, \cdot)_S \cdot \mathcal{C} = 0\), then \( p = \frac{\lambda}{k} \) or \( p = \frac{2\lambda - k\sigma - 6}{3k} \).

5. If \((\xi, \cdot)_S \cdot \mathcal{H} = 0\), then \( p = \frac{\lambda}{k} \) or \( p = \frac{\lambda - 1}{k} \).

**Proof.** 1. From (3.2) and (33) we obtain:

\[
- Ag(X, R(Y, Z)W) + A[g(X, Z)g(Y, W) - g(X, Y)g(Z, W)] + \\
\]

\[
+ S(X, W)\mathcal{T}(Y, Z)\xi - S(\xi, W)\mathcal{T}(Y, Z)X = 0,
\]

for any \( X, Y, Z, W \in \chi(M) \).
\[ +2A[\eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(Y)g(Z, W)] + \]
\[ +B[\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y)] = 0, \]
for any \( X, Y, Z, W \in \chi(M) \). Taking \( Z = W := \xi \) we get:
\[ (2A - B)[\eta(X)\eta(Y) + g(X, Y)] = 0, \]
for any \( X, Y \in \chi(M) \). It follows \( p = \frac{\lambda}{k} \).

2. From (3.2), (33) and (34) we obtain:
\[ -Ag(X, R(Y, Z)W + A(1 + aB - 2aA)[g(X, Z)g(Y, W) - g(X, Y)g(Z, W)] + \]
\[ +A(2 + aB - 2aA)[\eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(Y)g(Z, W)] + \]
\[ +B(1 + aB - aA)[\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y)] = 0, \]
for any \( X, Y, Z, W \in \chi(M) \). Taking \( Z = W := \xi \) we get:
\[ [2A - B + a(2AB - 2A^2 - B^2)][\eta(X)\eta(Y) + g(X, Y)] = 0, \]
for any \( X, Y \in \chi(M) \). It follows \( p_{1,2} = \frac{\lambda - 6 + 3(\lambda + k\sigma)\pm 2\sqrt{3(\lambda + k\sigma)^2 - 3(\lambda + k\sigma) + 3}}{k} \).

3. From (3.2), (33) and (35) we obtain:
\[ -Ag(X, R(Y, Z)W + A(1 - 2b)[g(X, Z)g(Y, W) - g(X, Y)g(Z, W)] + \]
\[ +2A(1 - b)[\eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(Y)g(Z, W)] + \]
\[ +B(1 - b)[\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y)] = 0, \]
for any \( X, Y, Z, W \in \chi(M) \). Taking \( Z = W := \xi \) we get:
\[ (2A - B)(1 - b)[\eta(X)\eta(Y) + g(X, Y)] = 0, \]
for any \( X, Y \in \chi(M) \). It follows \( p = \frac{\lambda}{k} \) or \( p = \frac{4\lambda + k\sigma - 12}{3k} \).
4. From (3.2), (33) and (3.2) we obtain:

\[-A g(X, R(Y, Z)W) + A(1 - 2c + dB - 4dA)[g(X, Z)g(Y, W) - g(X, Y)g(Z, W)] +
+ A(2 - 2c + dB - 4dA)[\eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(Y)g(Z, W)] +
+ B(1 - c + dB - 3dA)[\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y)] = 0,
\]

for any \(X, Y, Z, W \in \chi(M)\). Taking \(Z = W := \xi\) we get:

\[(2A - B)[1 - c - d(2A - B)][\eta(X)\eta(Y) + g(X, Y)] = 0,
\]

for any \(X, Y \in \chi(M)\). It follows \(p = \frac{\lambda}{k}\) or \(p = \frac{2\lambda - k\sigma - 6}{3k}\).

5. From (3.2), (33) and (37) we obtain:

\[-A g(X, R(Y, Z)W) + A(1 + dB - 4dA)[g(X, Z)g(Y, W) - g(X, Y)g(Z, W)] +
+ A(2 + dB - 4dA)[\eta(X)\eta(Z)g(Y, W) - \eta(X)\eta(Y)g(Z, W)] +
+ B(1 + dB - 3dA)[\eta(Y)\eta(W)g(X, Z) - \eta(Z)\eta(W)g(X, Y)] = 0,
\]

for any \(X, Y, Z, W \in \chi(M)\). Taking \(Z = W := \xi\) we get:

\[(2A - B)[1 - d(2A - B)][\eta(X)\eta(Y) + g(X, Y)] = 0,
\]

for any \(X, Y \in \chi(M)\). It follows \(p = \frac{\lambda}{k}\) or \(p = \frac{\lambda - 1}{k}\).

Remark the following facts:

1) If for a general relativistic perfect fluid spacetime \((M, g)\) satisfying (3) with torse-forming vector field \(\xi\) the projective curvature tensor \(\mathcal{W}\) or the conharmonic curvature tensor \(\mathcal{H}\) satisfy \((\xi, \cdot)_{\mathcal{W}} \cdot S = 0\), respectively \((\xi, \cdot)_{\mathcal{H}} \cdot S = 0\), then we have the vacuum case or the pressure is constant, but \((\xi, \cdot)_{\mathcal{R}} \cdot S = 0\) leads only to the vacuum case.

2) Under the same assumptions, the condition \((\xi, \cdot)_{S} \cdot R = 0\) or \((\xi, \cdot)_{S} \cdot \mathcal{H} = 0\) implies a constant pressure of the fluid.
4 Solitons in a perfect fluid spacetime

4.1 $\eta$-Ricci solitons

Consider the equation:

$$L_\xi g + 2S + 2ag + 2b\eta \otimes \eta = 0,$$

where $g$ is a pseudo-Riemannian metric, $S$ is the Ricci curvature, $\xi$ is a vector field, $\eta$ is a 1-form and $a$ and $b$ are real constants. The data $(g, \xi, a, b)$ which satisfy the equation (50) is said to be an $\eta$-Ricci soliton in $M$ [6]; in particular, if $b = 0$, $(g, \xi, a)$ is a Ricci soliton [13] and it is called shrinking, steady or expanding according as $a$ is negative, zero or positive, respectively [7].

Writing explicitly the Lie derivative $L_\xi g$ we get

$$(L_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi),$$

and from (50) we obtain:

$$S(X, Y) = -ag(X, Y) - b\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)],$$

for any $X, Y \in \chi(M)$.

Contracting (51) we get:

$$\text{scal} = -a \dim(M) + b - \text{div}(\xi).$$

Let $(M, g)$ be a general relativistic perfect fluid spacetime and $(g, \xi, a, b)$ be an $\eta$-Ricci soliton in $M$. From (7) and (51) we obtain:

$$[\lambda + \frac{k(\sigma - p)}{2} + a]g(X, Y) + [k(\sigma + p) + b]\eta(X)\eta(Y) + \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] = 0,$$

for any $X, Y \in \chi(M)$.

Consider $\{E_i\}_{1 \leq i \leq 4}$ an orthonormal frame field and let $\xi = \sum_{i=1}^4 \xi^i E_i$. We have shown in the previous section that $\sum_{i=1}^4 \varepsilon_{ii}(\xi^i)^2 = -1$ and $\eta(E_i) = \varepsilon_{ii}\xi^i$.

Multiplying (53) by $\varepsilon_{ii}$ and summing over $i$ for $X = Y := E_i$, we get:

$$4a - b = -4\lambda - k(\sigma - 3p) - \text{div}(\xi).$$

Writing (53) for $X = Y := \xi$, we obtain:

$$a - b = -\lambda + \frac{k(\sigma + 3p)}{2}.$$

Therefore:

$$\begin{cases}
  a = -\lambda - \frac{k(\sigma - p)}{2} - \frac{\text{div}(\xi)}{3} \\
  b = -k(\sigma + p) - \frac{\text{div}(\xi)}{3}
\end{cases}.$$
Theorem 4.1. Let \((M, g)\) be a 4-dimensional pseudo-Riemannian manifold and \(\eta\) be the \(g\)-dual 1-form of the gradient vector field \(\xi := \text{grad}(f)\) with \(g(\xi, \xi) = -1\). If (50) defines an \(\eta\)-Ricci soliton in \(M\), then the Laplacian equation satisfied by \(f\) becomes:

(57) \[\Delta(f) = -3[b + k(\sigma + p)].\]

Remark 4.2. If \(b = 0\) in (50), we obtain the Ricci soliton with \(a = -\lambda + \frac{k(\sigma + 3p)}{2}\) which is steady if \(p = \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3}\), expanding if \(p > \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3}\) and shrinking if \(p < \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3}\). In these cases, \(\text{div}(\xi) = -3k(\sigma + p)\). From Plebanski energy conditions for perfect fluids we deduce that \(\sigma \geq \max\{-\frac{1}{k}, \frac{1}{2k}\}\) for the steady case, \(\sigma > \frac{1}{2k}\) and \(\sigma > -\frac{1}{k}\) for the expanding and shrinking case, respectively.

Example 4.3. An \(\eta\)-Ricci soliton \((g, \xi, a, b)\) in a radiation fluid is given by

\[
\begin{align*}
    a &= -\lambda - kp - \frac{\text{div}(\xi)}{3} \\
    b &= -4kp - \frac{\text{div}(\xi)}{3}.
\end{align*}
\]

4.2 \(\eta\)-Einstein solitons

Consider the equation:

(58) \[\mathcal{L}_\xi g + 2S + (2a - \text{scal})g + 2b\eta \otimes \eta = 0,\]

where \(g\) is a pseudo-Riemannian metric, \(S\) is the Ricci curvature, \(\text{scal}\) is the scalar curvature, \(\xi\) is a vector field, \(\eta\) is a 1-form and \(a\) and \(b\) are real constants. The data \((g, \xi, a, b)\) which satisfy the equation (58) is said to be an \(\eta\)-Einstein soliton in \(M\); in particular, if \(b = 0\), \((g, \xi, a)\) is an Einstein soliton [3].

Writing explicitly the Lie derivative \(\mathcal{L}_\xi g\) we get \((\mathcal{L}_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)\) and from (58) we obtain:

(59) \[S(X,Y) = -(a - \frac{\text{scal}}{2})g(X,Y) - b\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)],\]

for any \(X, Y \in \chi(M)\).

Contracting (59) we get:

(60) \[\frac{2 - \dim(M)}{2} \text{scal} = -a\dim(M) + b - \text{div}(\xi).\]

Let \((M, g)\) be a general relativistic perfect fluid spacetime and \((g, \xi, a, b)\) be an \(\eta\)-Einstein soliton in \(M\). From (5), (7) and (59) we obtain:

(61) \[(\lambda - kp - a)g(X,Y) - [k(\sigma + p) + b]\eta(X)\eta(Y) - \frac{1}{2}[g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)] = 0,\]

for any \(X, Y \in \chi(M)\).
Consider \( \{E_i\}_{1 \leq i \leq 4} \) an orthonormal frame field and let \( \xi = \sum_{i=1}^{4} \xi^i E_i \). We have shown in the previous section that \( \sum_{i=1}^{4} \varepsilon_{ii}(\xi^i)^2 = -1 \) and \( \eta(E_i) = \varepsilon_{ii} \xi^i \).

Multiplying (61) by \( \varepsilon_{ii} \) and summing over \( i \) for \( X = Y := E_i \), we get:

\[
4a - b = 4\lambda + k(\sigma - 3p) - \text{div}(\xi).
\]

Writing (61) for \( X = Y := \xi \), we obtain:

\[
(63) \quad a - b = \lambda + k\sigma.
\]

Therefore:

\[
\begin{cases}
    a = \lambda - kp - \frac{\text{div}(\xi)}{3} \\
    b = -k(\sigma + p) - \frac{\text{div}(\xi)}{3}
\end{cases}
\]

**Theorem 4.4.** Let \((M, g)\) be a 4-dimensional pseudo-Riemannian manifold and \( \eta \) be the \( g \)-dual 1-form of the gradient vector field \( \xi := \text{grad}(f) \) with \( g(\xi, \xi) = -1 \). If (58) defines an \( \eta \)-Einstein soliton in \( M \), then the Laplacian equation satisfied by \( f \) becomes:

\[
\Delta(f) = -3[b + k(\sigma + p)].
\]

**Remark 4.5.** If \( b = 0 \) in (58), we obtain the Einstein soliton with \( a = \lambda + k\sigma \) which is steady if \( p = \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3} \), expanding if \( p > \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3} \) and shrinking if \( p < \frac{2}{3}(\frac{\lambda}{k}) - \frac{\sigma}{3} \). In these cases, \( \text{div}(\xi) = -3k(\sigma + p) \). From Plebanski energy conditions for perfect fluids we deduce that \( \sigma \geq \max\{-\frac{\lambda}{k}, \frac{\lambda}{2k}\} \) for the steady case, \( \sigma > \frac{\lambda}{2k} \) and \( \sigma > -\frac{\lambda}{k} \) for the expanding and shrinking case, respectively.

**Example 4.6.** An \( \eta \)-Einstein soliton \((g, \xi, a, b)\) in a radiation fluid is given by

\[
\begin{cases}
    a = \lambda - kp - \frac{\text{div}(\xi)}{3} \\
    b = -4kp - \frac{\text{div}(\xi)}{3}
\end{cases}
\]

Remark the following facts:

1) From Examples 4.3 and 4.6 we deduce that the Ricci soliton in a radiation fluid is steady if \( p = \frac{\lambda}{3k} \), expanding if \( p > \frac{\lambda}{3k} \) and shrinking if \( p < \frac{\lambda}{3k} \).

2) In a general relativistic perfect fluid spacetime, if the vector field \( \xi \) is torse-forming i.e. \( \nabla_X \xi = \alpha [X + \eta(X)\xi] \), for any \( X \in \chi(M) \) with \( \alpha \) a nonzero real number, then \( \text{div}(\xi) = 3\alpha \). In this case, the existence of a Ricci soliton given by (50) for \( b = 0 \), from Plebanski energy conditions implies \(-2k\sigma \leq \alpha < 0 \) (precisely, \( \alpha = -k(\sigma + p) \)).
3) If the vector field $\xi$ is conformally Killing i.e. $L_\xi g = rg$ with $r$ a nonzero real number, then the existence of a Ricci soliton given by (51) for $b = 0$, implies the vacuum case. Moreover, the soliton is steady if $p = \frac{1}{k} + \frac{r}{2k}$, expanding if $p > \frac{1}{k} + \frac{r}{2k}$ and shrinking if $p < \frac{1}{k} + \frac{r}{2k}$.

4) The existence of a steady Ricci soliton or a (weakly) pseudo Ricci symmetric Ricci tensor field imply the same condition on the pressure of the fluid, a surprisingly fact being that the 1-form $\alpha$, arbitrary chosen, does not appear effectively.

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*Department of Mathematics*

*West University of Timișoara*

*Bld. V. Pârvan nr. 4, 300223, Timișoara, România*

*adarablaga@yahoo.com*