A PERIODIC SOLUTION WITH NON-SIMPLE OSCILLATION
FOR AN EQUATION WITH STATE-DEPENDENT DELAY AND
STRICTLY MONOTONIC NEGATIVE FEEDBACK

BENJAMIN B. KENNEDY
Department of Mathematics
Gettysburg College
300 N. Washington St.
Gettysburg, PA 17325, USA

Abstract. We consider a real-valued differential equation
\[ x'(t) = f(x(t - d(x_t))), \]
with strictly monotonic negative feedback and state-dependent delay, that has
a nontrivial periodic solution \( q \) for which the planar map \( q_t \mapsto (q(t), q(t-d(q_t))) \)
is not injective on the orbit of \( q \) in phase space. This solution demonstrates
that Mallet-Paret and Sell’s version of the Poincaré-Bendixson theorem for
delay equations with constant delay and monotonic feedback does not carry
over entirely to the state-dependent delay case.

1. Notation. We write \( C = C([-1,0]; \mathbb{R}) \) for the set of real-valued continuous
functions on \([-1,0]\), equipped with the sup norm. If \( x \) is any continuous function
whose domain includes the interval \([t-1,t]\), we write \( x_t \) for the member of \( C \) given by
\[ x_t(s) = x(t + s), \quad s \in [-1,0]. \]
If \( \rho \) is any Lipschitz map, \( \ell(\rho) \) shall denote a Lipschitz constant for \( \rho \).

2. Introduction. In this paper we study equations of the form
\[ x'(t) = f(x(t - d(x_t))). \]
We will take as the phase space for Equation (1) an appropriate subset \( X \subseteq C \)
on which existence and uniqueness of solutions holds, and on which a continuous
solution semiflow is defined. In particular, the delay functional \( d \) will be defined on
\( X \) and will assume values between 0 and 1.

Consider the "corresponding" equation with constant delay
\[ x'(t) = f(x(t - 1)). \]
Suppose that \( x : \mathbb{R} \rightarrow \mathbb{R} \) is a globally defined and bounded solution of Equation (2),
and write \( \omega(x_0) \subseteq C \) for the \( \omega \)-limit set of \( x \). Mallet-Paret and Sell [4] have proven a
version of the Poincaré-Bendixson Theorem for a (considerably more general) class
of equations encompassing Equation (2); this theorem describes the structure of
\( \omega(x_0) \) in the case that \( f \) is smooth and strictly monotonic. In the first place, the
theorem states that \( \omega(x_0) \) must be either a single non-constant periodic orbit or

2010 Mathematics Subject Classification. Primary: 34K13.
Key words and phrases. Periodic solution, state-dependent delay, Poincaré-Bendixson
Theorem.
a collection of equilibria and connections between them. In the second place, the theorem states that the map $\Pi : \omega(x_0) \to \mathbb{R}^2$ given by the formula

$$\Pi(\varphi) = (\varphi(0), \varphi(-1)) \in \mathbb{R}^2$$

is injective. In particular, if $p : \mathbb{R} \to \mathbb{R}$ is a nontrivial periodic solution of Equation (2) with orbit $\{p_\tau : \tau \in \mathbb{R}\} \subseteq C$, then the map $\Pi : \{p_\tau : \tau \in \mathbb{R}\} \to \mathbb{R}^2$ given by the formula

$$\Pi(p_\tau) = (p(t), p(t-1))$$

is injective. This injectivity has the consequence that, if $z_0 < z_1 < z_2$ are three successive zeros of $p$, then $p$ has minimal period $z_2 - z_0$ — loosely speaking, $p$ has “one oscillation about zero per minimal period.” Accordingly, we shall express the conclusion that $\Pi$ is injective on $\{p_\tau : \tau \in \mathbb{R}\}$ by saying that $p$ oscillates simply.

In a companion paper [1], we describe a restricted class of state-dependent delay equations (1) in which, roughly speaking, the delay $d(x_t)$ is determined implicitly by $x(t)$ and $x(t-d(x_t))$. Building on the papers [2], [3], and [4], in [1] we show that Mallet-Paret and Sell’s Poincaré-Bendixson theorem applies to equations in this restricted class when $f$ is $C^\infty$ and strictly monotonic. In particular, if $p$ is any nontrivial periodic solution for such an equation with orbit $\{p_\tau : \tau \in \mathbb{R}\}$, then the map $\Pi : \{p_\tau : \tau \in \mathbb{R}\} \to \mathbb{R}^2$ given by

$$\Pi(p_\tau) = (p(t), p(t-d(p_\tau)))$$

is injective. We express this conclusion, too, by saying that $p$ oscillates simply.

Our objective in the current paper is, conversely, to illustrate that the conclusion just described does not hold for all Equations (1) with $f$ smooth and strictly decreasing; we shall exhibit a particular case of Equation (1), with $f$ smooth and strictly decreasing, that has a periodic solution that does not oscillate simply. Put somewhat more broadly, our example illustrates that the part of Mallet-Paret and Sell’s Poincaré-Bendixson theorem concerning the injectivity of the map

$$\omega(x_0) \ni \varphi \mapsto (\varphi(0), \varphi(-d(\varphi))) \in \mathbb{R}^2$$

does not extend to all cases of Equation (1).

As interest in equations with state-dependent delay has grown, several authors have given examples of phenomena that are possible for equations of the form (1) and various of its generalizations, but not for appropriately related equations with constant delay. (See, for example, [6], where an equation $x'(t) = -\alpha x(t-d(x_t))$ is devised with a solution that is homoclinic to the equilibrium at zero.) As the state-dependent delays in equations like (1) become more exotic, such results become, if not less interesting, at least less well-motivated by the comparison of Equations (1) and (2). Accordingly, in exploring whether various parts of the Poincaré-Bendixson theorem extend to state-dependent equations, we would like to restrict ourselves to equations (1) that are in some (admittedly subjective) sense not too wildly different from Equation (2). In particular, we will be interested in cases where the delay $d(x_t)$ is strictly positive but bounded, and where the delayed time $t-d(x_t)$ is strictly increasing for any solution $x$. This “delay time increase” condition (or close analogs) has been used by many authors in the study of state-dependent delay equations. For a detailed discussion of the monotonicity of $t - d(x_t)$ for a class of state-dependent delay equations, see [5].

Here, then, is the main theorem for the paper.

**Theorem 2.1.** There is an instance of Equation (1) for which the following hold.
There is a subset $X \subseteq C$ such that every $x_0 \in X$ has a unique continuation $x : [-1, \infty) \to \mathbb{R}$ as a solution of Equation (1), and $x_t \in X$ for all $t \geq 0$;

- $f$ is continuously differentiable and strictly decreasing, with $f(0) = 0$;
- $d : X \to (0, 1]$ is Lipschitz continuous and, for every solution $x : [-1, \infty) \to \mathbb{R}$ with $x_t \in X$ for all $t \geq 0$, $t - d(x_t)$ is strictly increasing;
- There is a nontrivial periodic solution $q : \mathbb{R} \to \mathbb{R}$, with $q_t \in X$ for all $t \in \mathbb{R}$, such that the map $\Pi : \{q_\tau : \tau \in \mathbb{R}\} \to \mathbb{R}^2$ given by the formula
  \[ \Pi(q) = (q(t), q(t - d(q_t))) \]
is not injective.

In Section 3 we establish some preliminary results on Equation (1). In Section 4 we introduce a prototype equation with discontinuous feedback for which we can explicitly compute a periodic solution $p$ that does not oscillate simply. In the remaining sections, we construct an equation with smooth strictly monotonic negative feedback that has a periodic solution $q$ that is close to $p$ in an appropriate sense; $q$ does not oscillate simply. (In particular, $q$ has the property that if $\zeta_0 < \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4$ are four successive zeros of $q$, then the minimal period of $q$ is $\zeta_4 - \zeta_0$—that is, $q$ has “two oscillations about 0 per minimal period.”)

3. Preliminary results on equation (1).

Recall Equation (1): \[ x'(t) = f(x(t - d(x_t))). \]

We shall impose the following hypotheses initially.

\[
\begin{aligned}
f : \mathbb{R} &\to \mathbb{R} \text{ is Lipschitz with } \ell(f) \leq L; \\
&\text{there is some } \mu > 0 \text{ such that } |f(u)| \leq \mu \text{ for all } u \in \mathbb{R}; \\
uf(u) &< 0 \text{ for all } u \neq 0.
\end{aligned}
\]  

(H1)

The last condition above is the so-called negative feedback condition.

We write $X^\mu$ for the compact convex subset of $C$ consisting of Lipschitz functions $\varphi$ that satisfy $\|\varphi\| \leq \mu$ and $\ell(\varphi) \leq \mu$. We shall assume that the delay functional $d$ is defined on $X^\mu$, and satisfies the following hypotheses.

\[
\begin{aligned}
0 &< d_{\min} \leq d(\varphi) \leq d_{\max} \leq 1 \text{ for all } \varphi \in X^\mu; \\
d &\text{is Lipschitz on } X^\mu, \text{ with Lipschitz constant } \ell(d).
\end{aligned}
\]  

(H2)

As usual, by a solution of Equation (1) we mean either a continuous function $x : [-1, \infty) \to \mathbb{R}$ such that (1) holds for all $t > 0$ or a function $x : \mathbb{R} \to \mathbb{R}$ such that (1) holds for all $t$. In either case we refer to $x$ as the continuation of $x_0$ as a solution of Equation (1).

Here is the basic existence and uniqueness result (see also [1], where a similar result is given). Since the idea is by now standard (express solutions on $[-1, \gamma]$ for small $\gamma > 0$ as fixed points of a contractive map), we omit the proof. The negative feedback condition and the bounds on $f$, together with the definition of $X^\mu$, serve to keep solutions in $X^\mu$ for all positive time.

**Proposition 3.1.** Assume that (H1) and (H2) hold. Given any $\varphi \in X^\mu$, there exists a unique function $x^\varphi = x : [-1, \infty) \to \mathbb{R}$ such that $x_0 = \varphi$, $x_t \in X^\mu$ for all $t \geq 0$, and $x$ satisfies (1) for all $t > 0$.

Furthermore, the solution semiflow $F : \mathbb{R}_+ \times X^\mu \to X^\mu$ defined by $F(t, \varphi) = x_t^\varphi$ is continuous in the following sense: given any $T > 0$ and $\epsilon > 0$, there is some
δ > 0 such that if \( x_0 \in X^\mu \), \( y_0 \in X^\mu \), and \( \|x_0 - y_0\| < \delta \), then
\[
\|F(t, x_0) - F(t, y_0)\| = \|x_1 - y_1\| < \epsilon \quad \text{for all } t \in [0, T]. \]

Below we shall need the following lemma describing, roughly speaking, how the solution semiflow changes under small uniform perturbations in the feedback function \( f \).

**Lemma 3.2.** Suppose that \( f \) and \( \overline{f} \) are two functions that both satisfy (H1), and suppose that \( |f(u) - \overline{f}(u)| \leq \beta \) for all \( u \in \mathbb{R} \).

Assume that \( d : X^\mu \to [\ell_{\min}, \ell_{\max}] \) satisfies (H2).

Consider solutions \( x \) and \( \overline{x} \), respectively, of the two equations
\[
x'(t) = f(x(t - d(x_t))) \quad \text{and} \quad \overline{x}'(t) = \overline{f}(\overline{x}(t - d(x_t))).
\]

Assume that \( x_0 = \overline{x}_0 = \varphi \in X^\mu \) (and hence, by Proposition 3.1, that \( x_t \in X^\mu \) and \( \overline{x}_t \in X^\mu \) for all \( t \geq 0 \)). Then, given any \( T > 0 \), there is some constant \( K = K(\ell(d), \mu, L, T) \) such that
\[
\|x_t - \overline{x}_t\| \leq K \beta
\]
for all \( t \in [0, T] \).

**Proof.** Since \( x_0 = \overline{x}_0 \), for any \( t \geq 0 \) there is some \( t_0 \in [\max(0, t-1), t] \) such that
\[
\|x_t - \overline{x}_t\| = |x(t_0) - \overline{x}(t_0)| = \left| \int_{0}^{t_0} x'(s) - \overline{x}'(s) \, ds \right| \leq \int_{0}^{t_0} |x'(s) - \overline{x}'(s)| \, ds.
\]
Therefore
\[
\|x_t - \overline{x}_t\| \leq \int_{0}^{t} |x'(s) - \overline{x}'(s)| \, ds
\]
\[
\leq \int_{0}^{t} |f(x(s - d(x_s))) - \overline{f}(\overline{x}(s - \overline{d}(x_s)))| \, ds
\]
\[
\leq \int_{0}^{t} |f(x(s - d(x_s))) - \overline{f}(x(s - d(x_s)))| + |\overline{f}(x(s - d(x_s))) - \overline{f}(\overline{x}(s - \overline{d}(x_s)))| \, ds
\]
\[
\leq \int_{0}^{t} \beta \, ds + L \int_{0}^{t} |x(s - d(x_s)) - \overline{x}(s - d(x_s))| \, ds \quad \text{[recall that } \ell(\overline{f}) \leq L]\]
\[
\leq \beta t + L \int_{0}^{t} |x(s - d(x_s)) - \overline{x}(s - x_s)| + |\overline{x}(s - d(x_s)) - \overline{x}(s - d(x_s))| \, ds
\]
\[
\leq \beta t + L \int_{0}^{t} |x(s - \overline{x}_s)| \, ds + L \mu \int_{0}^{t} |d(x_s) - d(\overline{x}_s)| \, ds \quad \text{[recall that } \ell(\overline{x}) \leq \mu]\]
\[
\leq \beta t + L \int_{0}^{t} |x(s - \overline{x}_s)| \, ds + L \mu \ell(d) \int_{0}^{t} |x(s - \overline{x}_s)| \, ds
\]
\[
=: \beta t + A \int_{0}^{t} \|x(s - \overline{x}_s)\| \, ds.
\]
Gronwall’s inequality now yields
\[
\|x_t - \overline{x}_t\| \leq \beta t e^{\lambda t}.
\]
Therefore, if we assume that \( t \in [0, T] \), there is a constant \( K > 0 \) such that
\[
\|x_t - \overline{x}_t\| \leq K \beta.
\]
We now give a simple criterion for the quantity \( t - d(x_t) \) to be strictly increasing along solutions.

**Lemma 3.3.** Suppose that \( x : [-1, \infty) \to \mathbb{R} \) is a solution of Equation (1) with \( x_t \in X^u \) for all \( t \geq 0 \). Then if \( d : X^u \to [d_{\min}, d_{\max}] \) has Lipschitz constant \( \ell(d) < 1/\mu \), the quantity \( t - d(x_t) \) is strictly increasing with respect to \( t \in [0, \infty) \).

**Proof.** Take \( 0 \leq t < s \). Since \( \mu \) is a Lipschitz constant for \( x \), we have

\[
d(x_s) - d(x_t) \leq |d(x_s) - d(x_t)| \leq \ell(d)\|x_s - x_t\| \leq \ell(d)\mu|s - t| < s - t,
\]

whence

\[
s - d(x_s) - (t - d(x_t)) > 0.
\]

\[ \square \]

4. **The example, part one.** We henceforth write \( X = X^1 \) — so \( X \) is the compact convex subset of \( C \) consisting of functions \( \varphi \) with \( \\| \varphi \| \leq 1 \) and \( \ell(\varphi) \leq 1 \). Consider the functional \( d_0 : X \to \mathbb{R} \) given by

\[
d_0(\varphi) = \frac{3}{16} + \frac{1}{4} \left( \varphi\left( -\frac{1}{2} \right) - \varphi(-1) \right).
\]

Observe that, on \( X \), \( d_0 \) has Lipschitz constant \( \ell(d_0) = \frac{1}{2} \) and has values in the interval \( \left[ \frac{1}{16}, \frac{3}{16} \right] \).

Now consider the equation

\[
y'(t) = -\text{sign}(y(t - d_0(y_t))). \tag{3}
\]

By a solution of this equation we mean either a continuous function \( y : [-1, \infty) \to \mathbb{R} \) or a continuous function \( y : \mathbb{R} \to \mathbb{R} \) satisfying the integral equation

\[
y(t) = y(0) - \int_0^t \text{sign}(y(s - d_0(y_s))) \, ds \tag{4}
\]

for all \( t \geq 0 \) or all \( t \in \mathbb{R} \), respectively.

**Lemma 4.1.** Given any \( \varphi \in X \), there is a unique solution \( y : [-1, \infty) \to \mathbb{R} \) of Equation (3) such that \( y_0 = \varphi \). For this solution \( \varphi \), \( y_1 \in X \) for all \( t \geq 0 \) and the quantity \( t - d_0(y_t) \) is strictly increasing with respect to \( t \in [0, \infty) \).

**Proof.** Suppose that \( \varphi \in X \). Let \( w \) be any extension of \( \varphi \) to the interval \([-1, 1/16]\) such that \( w_t \in X \) for all \( t \in [0, 1/16] \). Let \( t \in [0, 1/16] \). Since \( d_0(w_t) \geq 1/16 \) and \( d_0(w_t) \) depends only on the restriction of \( w_t \) to \([-1, -1/2]\), we see that the function

\[
[0, 1/16] \ni t \mapsto w(t - d_0(w_t))
\]

does not depend on our particular choice of \( w \); moreover, this function is continuous. The function

\[
[0, 1/16] \ni t \mapsto -\text{sign}(w(t - d_0(w_t))) =: h(t)
\]

(being the composition of a measurable and a continuous function) is Lebesgue measurable, and depends only on \( \varphi \); thus the function \( y : [-1, 1/16] \to \mathbb{R} \) given by

\[
y(t) = \begin{cases} 
\varphi(t), & t \in [-1, 0] \\
\varphi(0) + \int_0^t h(s) \, ds, & t \in [0, 1/16]
\end{cases}
\]

is well-defined, completely determined by \( \varphi \), and differentiable almost everywhere. \( y \) satisfies (4) for all \( t \in [0, 1/16] \) and satisfies (3) for almost every \( t \in [0, 1/16] \), and has Lipschitz constant 1 on \([-1, 1/16]\). We claim, furthermore, that \( |y(t)| \leq 1 \) for all \( t \in [-1, 1/16] \). Imagine, for example, that \( y(t_0) = 1 + \epsilon_0 \) for some \( \epsilon_0 > 0 \).
and some $t_0$ ($t_0$ is necessarily in $(0,1/16]$, since $y_0 \in X$ by assumption). Then, given any $\epsilon \in (0,\epsilon_0)$, $y$ attains the value $1 + \epsilon$ at some time $0 < \tilde{t} < t_0$; but since $\ell(y) \leq 1$ we must have that $y(\tilde{t} + s) > 0$ for all $s \in [-1,0]$ and accordingly that $y'(\tilde{t}) = -\text{sign}(y(\tilde{t} - d_0(y_0))) = -1$. It is therefore impossible that $y(t_0) = 1 + \epsilon_0$, and it follows that $y(t) \leq 1$ for all $t \in [0,1/16]$, and hence that $y_t \in X$ for all $t \in [0,1/16]$.

Continuing forward by steps shows that any $\varphi \in X$ has a unique continuation $y : [-1,\infty) \to \mathbb{R}$ as a solution of (3), and that $y_t \in X$ for all $t \geq 0$.

Finally, since $d_0$ has Lipschitz constant $1/2$ on $X$, the delay time $t - d_0(y_t)$ is strictly increasing (the proof is the same as for Lemma 3.3).

Direct inspection shows that the function $p : \mathbb{R} \to \mathbb{R}$ whose graph is illustrated in Figure 1 is a periodic solution of Equation (3). The grid squares in Figure 1 are of size $1/8$. The graph of $p$ is the thick line; the thinner line shows $d_0(p_t)$ for $t \geq 0$.

- slope 1 on $[0,1/8]$, $[3/8,3/4]$, and $[5/4,3/2]$;
- slope $-1$ on $[1/8,3/8]$ and $[3/4,5/4]$.

Figure 2 shows the planar image of the orbit of this periodic solution under the map $p_t \mapsto (p(t), p(t - d_0(p_t)))$. Observe that this map is not injective on the orbit of $p$: the image curve has a nontrivial self-intersection at the point $(1/16, -3/32)$, corresponding to the times $t = 1/16$ and $t = 9/16$. Thus $p$ does not oscillate simply.

We construct our desired example periodic solution (furnishing a proof of Theorem 2.1) in several stages. First we modify the delay functional $d_0$ to make it locally constant around the points $p_c$, where $c$ is a critical point of $p$. Then we replace the feedback function in Equation (3) with a continuous function $q$ with negative feedback. We show that the resulting equation has a periodic solution $w$ that is similar to $p$ and has very strong stability properties. Finally, we perturb the feedback function $q$ to make it strictly monotonic. Our desired periodic solution $q$ will be a periodic solution for this final equation.

5. The example, part two. Given $\alpha \in [0,1/16)$, we write $h_\alpha : \mathbb{R} \to \mathbb{R}$ for the continuous piecewise linear function that is equal to $k/8$ on every interval of the form

\[
\left[\frac{k}{8} - \alpha, \frac{k}{8} + \alpha\right], \quad k \in \mathbb{Z}
\]
and has constant slope on every interval of the form
\[ \left[ \frac{k}{8} + \alpha, \frac{k + 1}{8} - \alpha \right], \quad k \in \mathbb{Z}. \]

Observe that this function \( h_\alpha \) has Lipschitz constant given by the slope of the non-constant portions of its graph:
\[ \ell(h_\alpha) = \frac{1}{8} \frac{1}{2} \frac{1}{1 - 16\alpha}. \]

Given \( \alpha \), we now write \( d : X \to C \) for the function given by
\[ d(\varphi) = \frac{3}{16} + \frac{1}{4} h_\alpha((\varphi(-1/2) - \varphi(-1))). \]

Observe that, on \( X \), \( d \) has Lipschitz constant
\[ \frac{1}{2(1 - 16\alpha)}. \]

Let us now choose and fix \( \alpha \in (0, 1/32) \), so that \( d \) has Lipschitz constant less than 1 on \( X \).

The periodic solution \( p \) of Equation (3), shown above, is also a periodic solution of the equation
\[ y'(t) = -\text{sign}(y(t - d(y_t))). \quad (5) \]

Observe that the delay \( d(p_t) \) is constant on intervals of radius \( \alpha/2 \) around the critical points of \( p \); this is, roughly speaking, the fact that makes possible the explicit results we are about to obtain. See Figure 3, which shows \( p(t) \) (thick line) and \( d(p_t) \) (thinner line) over the course of a single period.

For the rest of this paper, we shall write \( z_i \) for the nonnegative zeros of \( p \):
\[ z_0 = 0, \quad z_1 = \frac{1}{4}, \quad z_2 = \frac{1}{2}, \quad z_3 = 1, \quad z_4 = \frac{3}{2}, \ldots. \]
We shall also write $c_i$ for the positive critical points of $p$ (i.e., the points where $p$ is not differentiable):

$$c_1 = \frac{1}{8}, \quad c_2 = \frac{3}{8}, \quad c_3 = \frac{3}{4}, \quad c_4 = \frac{5}{4}, \ldots$$

Note in particular that there is a unique critical point $c_i$ between the two successive zeros $z_{i-1}$ and $z_i$.

We record the following observations about $p$ (as a solution of Equation (5)).

(a) About each critical point $c_i$ of $p$, there is a closed interval $I_i$ of radius $\alpha/2$ on which $d(p(t))$ is constant, and equal to $|p(c_i)|$ (and so, in particular, equal to either $1/8$ or $1/4$).

(b) For all $i$, $|p(t)| \geq 1/8 - \alpha/2$ for each $t \in I_i$.

(c) As $t$ traverses each interval $I_i$, $t - d(p(t))$ traverses the interval $[z_{i-1} - \alpha/2, z_{i-1} + \alpha/2]$ with slope 1, and $p(t - d(p(t)))$ traverses the interval $[-\alpha/2, \alpha/2]$ with slope $\pm 1$. The intervals $I_i$ are precisely the intervals of $t$ for which $|p(t - d(p(t)))| \leq \alpha/2$.

(d) About each zero $z_i$ of $p$, there is a closed interval $J_i$ of radius $\alpha$. As $t$ traverses each $J_i$, $p(t)$ traverses the interval $[-\alpha, \alpha]$ with slope $\pm 1$. These intervals $J_i$ are precisely the intervals of $t$ for which $|p(t)| \leq \alpha$.

(e) $t - d(p(t))$ is strictly increasing with respect to $t$.

Now, given $\eta > 0$, let us define $g : \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x) = \begin{cases} \text{sign}(x), & |x| \geq \eta; \\ -x/\eta, & |x| \leq \eta. \end{cases}$$

$g$ is odd and nonincreasing with negative feedback, and has Lipschitz constant $1/\eta$.

Consider the following equation:

$$x'(t) = g(x(t - d(x_t))). \quad (6)$$

Note that Proposition 3.1 applies to this equation (with $\mu = 1$ and $L = 1/\eta$), and so existence, uniqueness, and continuity of the solution semiflow for Equation (6) hold on $X$. Also, since $d$ has Lipschitz constant less than 1 on $X$, Lemma 3.3 applies.
and we see that $t - d(x_t)$ is strictly increasing for any solution $x$ with segments in $X$.

Write $X_0 = \{ \varphi \in X : \varphi(0) = 0 \}$. $X_0$ is a compact and convex subset of $X$, and of $C$.

We now choose and fix $\eta > 0$ and $\epsilon > 0$ such that

\[ (*) \quad \epsilon < \alpha / 6 \quad \text{and} \quad \eta < \epsilon \quad \text{and} \quad \eta < \alpha / 2 - 3\epsilon. \]

For each $i \in \{0, 1, 2, 3, 4\}$, let us write $\Sigma_\epsilon(i)$ for the following set:

\[ \Sigma_\epsilon(i) = \left\{ x_0 \in X_0 : \|x_0 - p_{z_i}\| \leq \epsilon, \quad x(s) = p_{z_i}(s) \text{ for } s = [-\alpha / 2, 0] \right\}. \]

(Note that $\Sigma_\epsilon(0) = \Sigma_\epsilon(4)$.)

$p$ is defined on all of $\mathbb{R}$. In what follows, if $\xi$ and $t$ are any two real numbers, by $p_\xi(t)$ we shall mean $p(\xi + t)$, even if $t$ is outside the interval $[-1, 0]$.

**Lemma 5.1.** Let $\alpha \in (0, 1/32)$ be given and fixed, and let $p$ be the solution of Equation (5) described above.

Choose and fix $\eta$ and $\epsilon$ satisfying $(*).$

Choose $i \in \{0, 1, 2, 3\}$. Write $z = z_{i+1} - z_i$ and $c = z / 2$. Choose $x_0 \in \Sigma_\epsilon(i)$ with continuation $x$ as a solution of Equation (6). Then the following hold.

(i) $x(t) = p_{z_i}(t)$ for all $t \in [0, c - \alpha / 2 + \epsilon] \cup [c + \alpha / 2 - \epsilon, z]$, and $c - \alpha / 2 + \epsilon \leq |x(t)| \leq |p(t)| \leq c$ for all $t \in [c - \alpha / 2 + \epsilon, c + \alpha / 2 - \epsilon]$.

(ii) $|x(t) - p(t)| \leq \eta$ for all $t \in [0, z]$.

(iii) $x_z \in \Sigma_\epsilon(i + 1)$, and the restriction of $x$ to $[0, z]$ is independent of $x_0 \in \Sigma_\epsilon(i)$.

(iv) If $t \in [0, z]$, $|x(t - d(x_t))| \leq \alpha / 2 - 2\epsilon$ is only possible if $t \in [c - \alpha / 2, c + \alpha / 2]$.

**Proof.** Observe that $z \leq 1 / 2$. For definiteness, we shall give the proof for $i \in \{0, 2\}$ — that is, in the case that $p_{z_1}(t) > 0$ on $(0, z)$ (the proof in the other case is the same); so $p_{z_1}(t)$ has slope 1 on $[0, c]$ and slope $-1$ on $[c, z]$. By observations (a) and (c) above, $d(p_{z_1 + i} = c$ for all $t \in [c - \alpha / 2, c + \alpha / 2]$, and $[c - \alpha / 2, c + \alpha / 2]$ is precisely the subinterval of $[0, z]$ where $|p_{z_1}(t - d(p_{z_1 + i}))| \leq \alpha / 2$.

By the definition of $\Sigma_\epsilon(i)$, for any $t \in [0, 1/2]$ and any $s \in [1/2, 1]$ we have

\[ |p_{z_1}(t - s) - x(t - s)| \leq \epsilon. \]

$d$ has Lipschitz constant less than 1, and $d(\varphi)$ depends only on the restriction of $\varphi$ to $[-1, -1/2]$; thus $|d(x_t) - d(p_{z_1 + t})| < \epsilon$ for all $t \in [0, 1/2]$ and, since $x$ has Lipschitz constant 1, for $t \in [0, 1/2]$ we have

\[ (** \quad |x(t - d(x_t)) - d(p_{z_1 + t})| \leq |x(t - d(x_t)) - d(p_{z_1 + t})| + |d(x_t) - d(p_{z_1 + t})| - \epsilon + |x(t - d(p_{z_1 + t})) - p_{z_1}(t - d(p_{z_1 + t}))| \leq \epsilon. \]

Now, $c$ is the first positive critical point of $p$, and $t - d(p_{z_1 + t})$ is strictly increasing; thus for $t \in [0, c]$ we have $t - d(p_{z_1 + t}) \leq 0$ (refer to Figure 3). Applying $(**)$ and the definition of $\Sigma_\epsilon(i)$ we obtain that, for $t \in [0, c]$,

\[ |x(t - d(x_t)) - p_{z_1}(t - d(p_{z_1 + t}))| \leq 2\epsilon. \]

For all $t \in [0, c - \alpha / 2 + \epsilon]$, we have that $p_{z_1}(t - d(p_{z_1 + t})) \leq -(\alpha / 2 - \epsilon)$ (this is a consequence of point (c) above, and our choice of $i \in \{0, 2\}$. — see Figure 3). Thus, for these same $t$, we have that $x(t - d(x_t)) \leq -(\alpha / 2 - \epsilon) + 2\epsilon = -(\alpha / 2 - 3\epsilon) \leq -\eta$ (recall $(*))$. Thus $x(t - d(x_t))$ and $p_{z_1}(t - d(p_{z_1 + t}))$ are both strictly negative for
all \( t \in [0, c - \alpha/2 + \epsilon] \), and since \( |x(t - d(x_t))| \leq -\eta \) for all such \( t \) we have \( x'(t) = 1 \) and \( x(t) = p_{z_t}(t) \) for all \( t \in [0, c - \alpha/2 + \epsilon] \).

Now let us consider the behavior of \( x \) on the interval \([c - \alpha/2 + \epsilon, c + \alpha/2 - \epsilon]\).

We know that \( d(p_{z_t} + t) = c \) for all \( t \in [c - \alpha/2 + \epsilon, c + \alpha/2 - \epsilon] \), by observation (a) above. We claim that \( d(x_t) = d(p_{z_t} + t) = c \) for all \( t \in [c - \alpha/2 + \epsilon, c + \alpha/2 - \epsilon] \). Note that as \( t \) runs over this range, \( p_{z_t}(t - 1/2) \) and \( p_{z_t}(t - 1) \) are within distance \( \alpha/2 - \epsilon \) of \( \mathbb{Z}/8 \) (refer to Figure 3). Since \( t - 1/2 \leq 0 \) for all \( t \) in this range and \( \|x_0 - p_{z_t}\| \leq \epsilon \), both \( x(t - 1/2) \) and \( x(t - 1) \) are within distance \( \alpha/2 \) of \( \mathbb{Z}/8 \). Thus \( h_{\alpha}(x(t - 1/2) - x(t - 1)) \) is constant for \( t \) in this range, and is equal to \( h_{\alpha}(p_{z_t}(t - 1/2) - p_{z_t}(t - 1)) \). This proves the claim.

We know that \( x(t) = p_{z_t}(t) = t \) for all \( t \in [-\alpha/2, 0] \) (by the definition of \( \Sigma_\epsilon(i) \)) and for all \( t \in [0, c - \alpha/2 + \epsilon] \) (by the work we have already done). Note also (by (s), and since \( c \geq 1/8 \) and \( \alpha < 1/32 \)) that \( c - \alpha/2 + \epsilon > \alpha/2 - \epsilon > \eta \). Thus we see that, in particular, as \( t \) traverses the interval \([c - \alpha/2 + \epsilon, c + \alpha/2 - \epsilon] \), \( x(t - d(x_t)) \) traverses the interval \([-\alpha/2 + \epsilon, \alpha/2 - \epsilon]\) with slope 1. For \( s \in [0, \alpha - 2\epsilon] \), then, we have

\[
x'(c - \alpha/2 + \epsilon + s) = g(-\alpha/2 + \epsilon + s) \quad \text{and} \quad x(c - \alpha/2 + \epsilon + s) = c - \alpha/2 + \epsilon + \int_0^s g(-\alpha/2 + \epsilon + u) \, du.
\]

Since \( g \) is odd and bounded in absolute value by 1, we see that \( x(t) \) describes a symmetric arc over the interval \( t \in [c - \alpha/2 + \epsilon, c + \alpha/2 - \epsilon] \), with a maximum at \( t = c \), departing from the graph of \( p \) only on \([c - \eta, c + \eta]\) and satisfying \( c - \eta \leq x(t) \leq p(t) \leq c \) for \( t \in [c - \eta, c + \eta] \). In particular, we have that \( |x(t) - p(t)| \leq \eta < \epsilon \) for \( t \in [c - \alpha/2 + \epsilon, c + \alpha/2 - \epsilon] \) and

\[
x(c - \alpha/2 + \epsilon) = x(c + \alpha/2 - \epsilon) = c - \alpha/2 + \epsilon.
\]

Note again that \( c - \alpha/2 + \epsilon > \alpha/2 - \epsilon > \eta \); we shall use this fact just below.

See Figure 4, which shows the graphs of \( p_{z_t} \) (thick line) and \( x \) (thinner line) on the interval \([c - \alpha/2, c + \alpha/2]\).

Now, the work we have done so far shows that

\[
c + \alpha/2 - \epsilon - d(x_{c + \alpha/2 - \epsilon}) = \alpha/2 - \epsilon,
\]
and that $x$ is greater than or equal to $\alpha/2 - \epsilon > \eta$ on the entire interval $[\alpha/2 - \epsilon, c + \alpha/2 - \epsilon]$.

Since $t - d(x_t)$ is strictly increasing, we see that $x'(t)$ will have constant slope $-1$ at least until the first time $\tau > c + \alpha/2 - \epsilon$ for which $x(\tau - d(x_\tau)) = \eta$. Imagine that $\tau < z$. Then $x'(t) = p'_z(t)$ and $x(t) = p_z(t)$ for all $t \in [c + \alpha/2 - \epsilon, \tau]$. Then, since $x_t \in X$ for all $t \geq 0$ and $d$ has minimum value $1/16$ on $X$, we have

$$d(x_\tau) \geq \frac{1}{16}.$$ 

Thus $\tau$ occurs at least $1/16$ units after $x$ attains the value $\eta$; but since the slope of $x$ is $-1$ during these $1/16$ units and $\eta < 1/16$ (by ($*$)), and the fact that $\alpha < 1/32$, we must have $x(\tau) < 0$ and we get a contradiction with the fact that $p_z(t) > 0$ for all $t \in (0, z)$. We conclude that $x'(t) = p'_z(t) = -1$ and $x(t) = p_z(t)$ for all $t \in [c + \alpha/2 - \epsilon, \tau]$. We have proven part (i) of the lemma.

Now, since we have established that $x' = p_z$ on $[c + \alpha/2 - \epsilon, \tau]$ and $\tau > c$, we have that $x$ is a constant map. Write $w(t) = x(t)$, and that $x$ has the following additional features, which follow immediately from the above lemma. We will not need anything more refined than (iv).)

Thus (recall observation (c) above) $|x(t) - p_z(t)| < \epsilon$ for all $t \in [-1, z]$ and that $z \leq 1/2$; the estimate ($**$) above yields that, for all $t \in [0, z]$,

$$|x(t) - p_z(t)| \leq 2\epsilon.$$ 

Thus (recall observation (c) above) $|x(t) - d(x_t)|$ can be less than $\alpha/2 - 2\epsilon$ only on the interval $[c - \alpha/2, c + \alpha/2]$. This is part (iv) of the lemma. (We know from our earlier work that $|x(t) - d(x_t)| \leq \alpha/2 - 2\epsilon$ on $[c - \alpha/2, c + \alpha/2 - 2\epsilon]$, but have not ruled out that $|x(t) - d(x_t)| \leq \alpha/2 - 2\epsilon$ at some other point of the interval $[c/2 - \alpha, c/2 + \alpha]$. We will not need anything more refined than (iv).)

Now, given $x_0 \in \Sigma_c(0)$, let us write $R(x_0) = x_{3/2}$. Applying the above lemma four times tells us that $R(x_0) \in \Sigma_c(4) = \Sigma_c(0)$ and that $R : \Sigma_c(0) \rightarrow \Sigma_c(0)$ is a constant map. Write $w_0 = R(x_0)$. Since $w_0 = R(w_0) = w_{3/2}$, $w$ extends to a periodic solution $w : \mathbb{R} \rightarrow \mathbb{R}$ of Equation (6), with exactly the same zeros as $p$. $w$ has the following additional features, which follow immediately from the above lemma, and the fact that (in the notation of the above lemma) $c - \alpha/2 + \epsilon > \alpha$ and $z - (c + \alpha/2 - \epsilon) > \alpha$.

(A) $|w(t) - p(t)| \leq \eta < \epsilon$ for all $t \in \mathbb{R}$.

(B) $w(t)$ has constant slope $\pm 1$ on intervals of radius $\alpha$ about each of its zeros.

These intervals are precisely where $|w(t)| \leq \alpha$; and on each of these intervals, $|w(t - d(w_t))| \geq \alpha/2 - 2\epsilon$.

Figure 5 shows the graph of numerical approximations of $w$ (thick line) and $d(w_t)$ (thin line) over the course of a period. At this scale, the graphs of $w$ and $p$ are virtually indistinguishable; but $w$ is differentiable everywhere.
Figure 5.

Figure 6.

Figure 6 shows a numerical approximation of the image of the orbit of \( w \) under the map \( w_t \mapsto (w(t), w(t - d(\omega))) \). Note that this map is apparently not injective, and so \( w \) apparently does not oscillate simply.

Now, given \( \delta > 0 \), let us write \( B_\delta(w_0) \) for the closed ball of radius \( \delta \) about \( w_0 \) in \( X_0 \). We extend the map \( R : \Sigma_\epsilon(0) \to \Sigma_\epsilon(0) \) to a map \( R : B_\delta(w_0) \cup \Sigma_\epsilon(0) \to X_0 \) by defining \( R(x_0) = x_z \), where \( x \) is the continuation of \( x_0 \) as a solution of Equation (6) and \( z \) is the fourth positive zero of \( x \) (we will see just below that \( z \) exists and hence that \( R \) is well-defined).

We have the following lemma.

**Lemma 5.2.** There is some \( \delta > 0 \) such that, with notation as above, the following hold.

- If \( x_0 \in B_\delta(w_0) \) with continuation \( x \) as a solution of Equation (6), the fourth positive zero \( z = z(x_0) \) of \( x \) is defined and satisfies \( z \in [3/2 - \alpha/2, 3/2 + \alpha/2] \).
- The map \( R : B_\delta(w_0) \cup \Sigma_\epsilon(0) \to X_0 \) given by \( R(x_0) = x_z \) is continuous.
- \( R(B_\delta(w_0)) \subseteq \Sigma_\epsilon(0) \).

**Proof.** As above, let us write \( 0 = z_0 < z_1 < z_2 < z_3 < z_4 = 3/2 \) for the first five nonnegative zeros of \( w \) (and of \( p \)).
Recall (points (A) and (B) above) that \( w(t) \) has constant slope \( \pm 1 \) on each of the intervals \([z_i - \alpha, z_i + \alpha]\); that \(|w(t - d(w_i))| \geq \alpha/2 - 2 \epsilon \) on each of these intervals; and that these intervals are the only subintervals of \([-\alpha, z_4 + \alpha]\) where \(|w(t)| \leq \alpha\).

By Proposition 3.1, we can choose \( \delta > 0 \) small enough so that, if \( x_0 \in B_\delta(w_0) \) and \( x \) is the continuation of \( x_0 \) as a solution of Equation (6), we have

\[
\|x_t - w_t\| < \gamma := \frac{\alpha/2 - 2 \epsilon - \eta}{2} \text{ for all } t \in [0, z_4 + \alpha].
\]

(From condition (*) we know that \( \gamma \) is indeed positive.)

Since \( d \) has Lipschitz constant less than 1 it follows that \(|d(x_t) - d(w_t)| < \gamma \) for every \( t \in [0, z_4 + \alpha] \). Using that 1 is a Lipschitz constant for \( x \), we then conclude that

\[
|\|x(t - d(x_i)) - w(t - d(w_i))|| \leq |x(t - d(x_i)) - x(t - d(w_i))| + |x(t - d(w_i)) - w(t - d(w_i))|
\leq \gamma + \gamma = \alpha/2 - 2 \epsilon - \eta.
\]

We therefore have that \(|x(t - d(x_i))| \geq \eta \) and that \( \text{sign}(x(t - d(x_i))) = \text{sign}(w(t - d(w_i))) \) for all \( t \) in the set

\[
[0, \alpha] \cup [z_1 - \alpha, z_1 + \alpha] \cup \cdots \cup [z_4 - \alpha, z_4 + \alpha],
\]

whence

\[
x'(t) = w'(t) \text{ for all } t \in [0, \alpha] \cup [z_1 - \alpha, z_1 + \alpha] \cup \cdots \cup [z_4 - \alpha, z_4 + \alpha].
\]

This fact, along with the fact that \(|x(t) - w(t)| < \gamma < \alpha/2 \) for all \( t \in [0, z_4 + \alpha] \), tells us that \( x \), along each of the four positive zeros of \( [0, z_4 + \alpha] \), one on each interval \([z_i - \alpha/2, z_i + \alpha/2], 1 \leq i \leq 4 \). Thus \( z \) is defined, and lies in \([3/2 - \alpha/2, 3/2 + \alpha/2] \).

Now choose two initial conditions \( x_0 \) and \( y_0 \) in \( B_\delta(w_0) \) with continuations \( x \) and \( y \), respectively, as solutions of Equation (6). Write \( z \) and \( \zeta \) for the fourth positive zeros of \( x \) and \( y \), respectively. We know that both \( x \) and \( y \) have constant slope 1 on \([3/2 - \alpha, 3/2 + \alpha] \), and that both \( z \) and \( \zeta \) lie in \([3/2 - \alpha/2, 3/2 + \alpha/2] \). Now, given any \( \sigma < \alpha/2 \), choosing \(|x_0 - y_0|\) sufficiently small guarantees that \(|x_t - y_t| < \sigma \) for all \( t \in [0, 3/2 + \alpha] \) (Proposition 3.1). Since \( x \) has constant slope 1 on \([3/2 - \alpha, 3/2 + \alpha] \), \( x(t) \) traverses the interval \([-\sigma, \sigma]\) as \( t \) traverses the interval \([z - \sigma, z + \sigma]\); we conclude that \(|z - \zeta| < \sigma \). Since 1 is a Lipschitz constant for \( y \), we have \(|y_t - y_\zeta| \leq |z - \zeta| < \sigma \), and so

\[
|R(x_0) - R(y_0)| = ||x_0 - y_\zeta|| \leq ||x_t - y_t|| < 2 \sigma.
\]

Thus \( R \) is continuous on \( B_\delta(w_0) \). We have already observed that \( R \) is constant on \( \Sigma_s(0) \); thus \( R \) is continuous on \( B_\delta(w_0) \cup \Sigma_s(0) \).

It remains to show that \( R(x_0) \in \Sigma_s(0) \) for any \( x_0 \in B_\delta(w_0) \). We can achieve \(|R(x_0) - w_0| < \epsilon - \eta\) by shrinking \( \delta \) if necessary, since \( R \) is continuous; since \( |w_0 - p_0| \leq \eta \) by observation (A) above, we then have \(|R(x_0) - p_0| \leq \epsilon \). Since \( x'(t) = w'(t) = 1 \) on \([z_4 - \alpha, z_4 + \alpha]\) and \( z \in [z_4 - \alpha/2, z_4 + \alpha/2] \), we see that \( R(x_0)(s) = s = p_0(s) \) for \( s \in [-\alpha/2, 0] \). This completes the proof.

Note that the last two lemmas combine to show that the map \( \overline{R} : B_\delta(w_0) \cup \Sigma_s(0) \rightarrow X_0 \) that advances solutions of Equation (6) by eight zeros is constant, with constant value \( w_0 \in B_\delta(w_0) \cap \Sigma_s(0) \).

Our strategy is now to perturb \( g \) to a strictly monotonic function; to define \( \overline{R} \) to be the return map, defined near \( w_0 \), that advances solutions of the corresponding equation by four zeros; and to show that \( \overline{R} \) has a fixed point near \( w_0 \). Since \( \overline{R}^2 \)
maps the compact convex set $B_δ(w_0)$ to the constant value $w_0$, it is easy to show, if $R$ and $\mathcal{R}$ are uniformly close enough, that $\mathcal{R}^2$ has a fixed point — and indeed this would be sufficient to establish Theorem 2.1. With a little more work, however, we can obtain the sharper result that $\mathcal{R}$ has a fixed point. The following lemma is the basic tool.

**Lemma 5.3.** Let $δ$ and $R$ be as in Lemma 5.2. Then there is a number $ω > 0$ such that the following holds. Suppose that there is a continuous function $\mathcal{R}: B_δ(w_0) \to X_0$ with the feature that

$$\|R(x_0) - \mathcal{R}(x_0)\| \leq ω$$

for all $x_0 \in B_δ(w_0)$. Then $\mathcal{R}$ has a fixed point.

**Proof.** For simplicity, we write $Σ(0)$ simply as $Σ$.

Observe that $Σ$ is convex and compact.

For any $σ \in [0, ϵ]$, let us write

$$S_σ = Σ \cap B_σ(w_0).$$

$S_σ$, being an intersection of two convex compact sets, is convex and compact for every $σ \in [0, ϵ]$, and $R$ maps $S_σ$ to the single point $w_0$.

Now, for $ω > 0$, consider the set

$$S_ω^σ = \{ x_0 \in X_0 : \|x_0 - y_0\| \leq ω \text{ for some } y_0 \in S_σ \}.$$

$S_ω^σ$ is a closed subset of a compact set and so is certainly compact. We claim that $S_ω^σ$ is also convex. Choose $x_0$ and $u_0$ in $S_ω^σ$ and $y_0$ and $v_0$ in $S_σ$ such that

$$\|x_0 - y_0\| \leq ω \text{ and } \|u_0 - v_0\| \leq ω.$$ 

We have, for any $τ \in [0, 1]$,

$$\|((τx_0 + (1 - τ)u_0) - (τy_0 + (1 - τ)v_0)\|
\leq τ\|x_0 - y_0\| + (1 - τ)\|u_0 - v_0\|
\leq τω + (1 - τ)ω = ω.$$

Since $S_σ$ is convex, this proves the claim.

Now choose $σ < δ$. $R$ is continuous on $B_δ(w_0) \cup Σ$, has the constant value $w_0$ on $S_σ$, and maps $B_δ(w_0)$ into $Σ$ (Lemma 5.2); thus about every $y_0 \in S_σ$ there is an open ball $U(y_0) \subseteq B_δ(w_0)$ such that for any $x_0 \in U(y_0)$ we have $R(x_0) \in Σ$ and

$$\|R(x_0) - R(y_0)\| = \|R(x_0) - w_0\| < σ$$

— and so $R(x_0) \in S_σ$. Write $V$ for the union of these open balls $U(y_0)$; $X_0 \setminus V$ and $S_σ$ are two disjoint compact sets of $X_0$. Accordingly, taking $ω$ small enough ensures that

$$S_ω^σ \subseteq V = \bigcup_{y_0 \in S_σ} U(y_0).$$

For such $ω$, then, we have that $S_ω^σ \subseteq B_δ(w_0)$ and that $R(S_ω^σ) \subseteq S_σ$. If, then, we assume that $\|R(x_0) - R(x_0)\| \leq ω$ for all $x_0 \in B_δ(w_0)$, we conclude that $\mathcal{R}$ maps the compact convex set $S_ω^σ$ into itself and therefore has a fixed point by Schauder’s theorem.

In the following section, we shall perturb the map $R$ by, roughly speaking, perturbing the feedback function in Equation (6) to make it strictly monotonic.
6. The example, part three. Throughout this section, \( g \) will be the same feedback function as in Section 5.

**Lemma 6.1.** Given any \( \beta \in (0, 1) \), there is a continuously differentiable \( f : \mathbb{R} \to \mathbb{R} \) that is strictly decreasing with negative feedback, has Lipschitz constant \( 1/\eta \), satisfies \( |f(x)| \leq 1 \) for all \( x \in \mathbb{R} \), and satisfies \( |f(x) - g(x)| \leq \beta \) for all \( x \in \mathbb{R} \).

**Proof.** We can take, for example,

\[
f(x) = \begin{cases} 
  -x/\eta, & |x| \leq \eta(1 - \beta); \\
  -\text{sign}(x) \left[ 1 - \beta e^{-|x|/\eta} \right], & |x| \geq \eta(1 - \beta). 
\end{cases}
\]

See Figure 7, which shows the graphs of \( g \) and \( f \) for \( \eta = 0.1 \) and \( \beta = 0.3 \).

Now consider the two equations (6) and

\[
u'(t) = f(u(t - d(u_t))),
\]

where \( d \) is as described in Section 5 and \( f \) is as described in Lemma 6.1.

Suppose that \( x : [-1, \infty) \to \mathbb{R} \) and \( u : [-1, \infty) \to \mathbb{R} \) are two solutions of Equations (6) and (7), respectively, with common initial condition \( x_0 = u_0 = \varphi \in X \). By Lemma 3.2, there is some \( K > 0 \) such that

\[||x_t - u_t|| < K \beta \text{ for all } t \in [0, 2],\]

where \( \beta \) is as in Lemma 6.1. Observe that, for \( t \in (0, 2] \), since \( \ell(d) \leq 1, \ell(x) \leq 1, \) and \( \ell(u) \leq 1 \), we have

\[
|x'(t) - u'(t)| = |g(x(t - d(x_t))) - f(u(t - d(u_t)))| \\
\leq |g(x(t - d(x_t))) - g(u(t - d(u_t)))| + |g(u(t - d(u_t))) - f(u(t - d(u_t)))| \\
\leq \frac{1}{\eta} |x(t - d(x_t)) - u(t - d(u_t))| + \beta \\
\leq \frac{1}{\eta} |x(t - d(x_t)) - u(t - d(u_t))| + \frac{1}{\eta} |u(t - d(x_t)) - u(t - d(u_t))| + \beta
\]
\[
\leq \frac{1}{\eta} \beta K + \frac{1}{\eta} |d(x_t) - d(u_t)| + \beta \\
\leq \frac{1}{\eta} \beta K + \frac{1}{\eta} \beta K + \beta \leq \beta \left( 1 + \frac{2K}{\eta} \right).
\]

Thus, by choosing \( \beta < \alpha/(2K) \) and \( \beta < [2(1 + 2K/\eta)]^{-1} \), we guarantee that \( |x(t) - u(t)| < \alpha/2 \) and \( |x'(t) - u'(t)| \leq 1/2 \) for all \( t \in (0, 2] \).

Now, let us suppose that \( x_0 = u_0 \in B_\delta(w_0) \), and let us write \( z \) for the fourth positive zero of \( x \). Recall from the proof of Lemma 5.2 that \( z \in [3/2 - \alpha/2, 3/2 + \alpha/2] \), and that \( x'(t) = 1 \) for all \( t \in [3/2 - \alpha, 3/2 + \alpha] \), and hence for all \( t \in [z - \alpha/2, z + \alpha/2] \).

By the conditions we have already imposed on \( \beta, K \beta < \alpha/2 \). Thus as \( t \) traverses the interval \([z - K\beta, z + K\beta]\), \( x(t) \) traverses the interval \([-K\beta, K\beta]\). Since \( |u(t) - x(t)| \leq K\beta \) and \( |u'(t)| \geq 1/2 \) for all \( t \in [z - K\beta, z + K\beta] \), we conclude that \( u \) has a unique zero \( \tau \) on \([z - K\beta, z + K\beta]\). Similar reasoning on intervals about \( 0 \) and about the first, second and third positive zeros of \( x \) shows that \( \tau \) is actually the fourth positive zero of \( u \).

Again since \( 1 \) is a Lipschitz constant for \( x \), we have

\[
\|x_z - u_z\| \leq \|x_z - x_z\| + \|x_z - u_z\| \leq |z - \tau| + K\beta \leq 2K\beta.
\]

Thus, if we define \( \overline{R} : B_\beta(w_0) \to X_0 \) by the formula \( \overline{R}(u_0) = u_\tau \) (that is, \( \overline{R} \) is a return map for Equation (7) that advances solutions by four zeros), we have that

\[
\|\overline{R}(x_0) - R(x_0)\| \leq 2K\beta \quad \text{for all} \quad x_0 \in B_\beta(w_0).
\]

Thus, for \( \beta \) small enough, Lemma 5.3 tells us that \( \overline{R} \) has a fixed point \( q_0 \in B_\beta(w_0) \). The continuation \( q \) of \( q_0 \) as a solution of Equation (7) is a periodic solution of Equation (7).

Let us write \( 0 < \zeta_1 < \zeta_2 < \zeta_3 < \zeta_4 \) for the first four positive zeros of \( q \).

Simply because \( q \) is a special case of the solution \( u \) discussed just above, we see that, by first choosing \( \epsilon \) and \( \eta \), and then \( \delta \) and \( \beta \), small enough, we can make \( q \) uniformly as close as we wish to \( u \), and to \( p \), over the time interval \( t \in [0, 2] \). More specifically, given any \( \sigma > 0 \) we can choose \( \epsilon, \eta, \delta \) and \( \beta \) such that

\[
(\ast \ast \ast) \quad \|q_t - p_t\| \leq \sigma \quad \text{for all} \quad t \in [0, 2].
\]

This of course implies that \( |q(t) - p(t)| < \sigma \) for all \( t \in [0, 2] \), and (provided that \( \sigma < \alpha \), and using the same kinds of estimates we have used several times already) that

- \( |d(q_t) - d(p_t)| < \sigma \) and \( |q(t) - d(q_t)) - p(t - d(p_t))| < 2\sigma \) for all \( t \in [0, 2] \);
- \( |\zeta_i - z_i| < \sigma \) for all \( i \in \{1, 2, 3, 4\} \).

In particular, choosing \( \sigma \) small enough ensures that \( \zeta_4 \) (and not \( \zeta_2 \)) is the minimal period of \( q \).

To complete the proof of Theorem 2.1, it remains to show that \( q \) does not oscillate simply. That is, we wish to show that the map \( \Pi : \{q_\tau : \tau \in \mathbb{R}\} \to \mathbb{R}^2 \) given by

\[
\Pi(q_\tau) = (q(t), q(t - d(q_t)))
\]

is not injective. Since \( \zeta_4 \) is the minimal period of \( q \), it is sufficient to show that there are two times \( 0 < \tau_1 < \tau_2 < \zeta_4 \) such that

\[
\Pi(q_{\tau_1}) = (q(\tau_1), q(\tau_1 - d(q_{\tau_1}))) = (q(\tau_2), q(\tau_2 - d(q_{\tau_2}))) = \Pi(q_{\tau_2}).
\]

We will spend the rest of the paper showing that, if the number \( \sigma \) described in (\ast \ast \ast) just above is small enough, the existence of such times \( \tau_1 \) and \( \tau_2 \) is guaranteed.
We take $\sigma$ small enough that
\[ 2\sigma < \alpha/2 \quad \text{and} \quad 12\sigma < 1/8 - \alpha \]
(the first condition actually implies the second, since $\alpha < 1/32$).

Direct examination of the solution $p$ — viewed as a solution of Equation (5) — shows the following. (Here we are using the fact that $d(p_t)$ is constant on intervals of radius $\alpha/2$ about the zeros and critical points of $p$; the reader may find it helpful to refer to Figure 3.)

- At time $t_1 = \alpha/2$:
  - $p(t_1) = \alpha/2$;
  - $d(p_{t_1}) = 3/16$;
  - $p(t_1 - d(p_{t_1})) = -3/16 + \alpha/2$.
- At time $T_1 = 1/8 - \alpha/2$:
  - $p(T_1) = 1/8 - \alpha/2$;
  - $d(p_{T_1}) = 1/8$;
  - $p(T_1 - d(p_{T_1})) = -\alpha/2$.
- As $t$ moves from $t_1$ to $T_1$, $p(t)$ increases monotonically from $\alpha/2$ to $1/8 - \alpha/2$, and $p(t - d(p_t))$ increases monotonically from $-3/16 + \alpha/2$ to $-\alpha/2$.
- At time $t_2 = 1/2 + \alpha/2$:
  - $p(t_2) = \alpha/2$;
  - $d(p_{t_2}) = 3/16$;
  - $p(t_2 - d(p_{t_2})) = -1/16 - \alpha/2$.
- At time $T_2 = 5/8 - \alpha/2$:
  - $p(T_2) = 1/8 - \alpha/2$;
  - $d(p_{T_2}) = 1/4$;
  - $p(T_2 - d(p_{T_2})) = -1/8 + \alpha/2$.
- As $t$ moves from $t_2$ to $T_2$, $p(t)$ increases monotonically from $\alpha/2$ to $1/8 - \alpha/2$, and $p(t - d(p_t))$ decreases monotonically from $-1/16 - \alpha/2$ to $-1/8 + \alpha/2$.

Since $|q(t) - p(t)| \leq \sigma$ for all $t \in [0, 2]$ and $|q(t - d(q_t)) - p(t - d(p_t))| \leq 2\sigma$ for all $t \in [0, 2]$, we conclude the following (remember that $2\sigma < \alpha/2$):

1-i) $q(t_1 - d(q_{t_1})) \in [-3/16 + \alpha/2 - 2\sigma, -3/16 + \alpha/2 + 2\sigma]$, and $q(T_1 - d(q_{T_1})) \in [-\alpha/2 - 2\sigma, -\alpha/2 + 2\sigma]$.

1-ii) For all $t \in [t_1, T_1]$, $q(t - d(q_t)) \in [-3/16 + \alpha/2 - 2\sigma, -\alpha/2 + 2\sigma]$. Consequently, $q'(t) > 0$ for all $t \in [t_1, T_1]$ and $q$ is strictly increasing on $[t_1, T_1]$.

1-iii) $q(t_1) \in [\alpha/2 - \sigma, \alpha/2 + \sigma]$, and $q(T_1) \in [1/8 - \alpha/2 - \sigma, 1/8 + \alpha/2 + \sigma]$.

2-i) $q(t_2 - d(q_{t_2})) \in [-1/16 - \alpha/2 - 2\sigma, -1/16 - \alpha/2 + 2\sigma]$, and $q(T_2 - d(q_{T_2})) \in [-1/8 + \alpha/2 - 2\sigma, -1/8 + \alpha/2 + 2\sigma]$.

2-ii) For all $t \in [t_2, T_2]$, $q(t - d(q_t)) \in [-1/8 + \alpha/2 - 2\sigma, -1/16 - \alpha/2 + 2\sigma]$. Consequently, $q'(t) > 0$ for all $t \in [t_2, T_2]$ and $q$ is strictly increasing on $[t_2, T_2]$.

2-iii) $q(t_2) \in [\alpha/2 - \sigma, \alpha/2 + \sigma]$, and $q(T_2) \in [1/8 - \alpha/2 - \sigma, 1/8 + \alpha/2 + \sigma]$.

See Figure 8. The thin curve shows the image of the map $q_t \mapsto (q(t), q(t - d(q_t)))$. The two thicker curve segments show the image under this map of the orbit segments $\{q_t : t \in [t_1, T_1]\}$ and $\{q_t : t \in [t_2, T_2]\}$. (For small $\sigma$, the entire image of the map looks very much like the curve pictured in Figure 6.)

Now, observe that, as $t$ traverses $[t_1, T_1]$, $q(t)$ covers $[\alpha/2 + \sigma, 1/8 - \alpha/2 - \sigma]$. Similarly, as $t$ traverses $[t_2, T_2]$, $q(t)$ covers $[\alpha/2 + \sigma, 1/8 - \alpha/2 - \sigma]$ also. Choose $t_1 \leq \tilde{t}_1 \leq T_1$ and $t_2 \leq \tilde{t}_2 \leq T_2$ such that $q(\tilde{t}_1) = q(\tilde{t}_2) = \alpha/2 + \sigma$ and $q(T_1) = q(T_2) = 1/8 - \alpha/2 - \sigma$. Notice that these $\tilde{t}_i$ and $T_i$ are uniquely
determined, because \( q \) is strictly increasing on \([t_1, T_1]\) and on \([t_2, T_2]\). Notice also that, since \( p'(t) = 1 \) on \([t_1, T_1]\) and on \([t_2, T_2]\) and \( |q(t) - p(t)| \leq \sigma \) for all \( t \in [0, 2] \), we have

\[
|\dot{\tau}_i - \dot{\tau}| \leq 2\sigma \quad \text{and} \quad |\dot{T}_i - \dot{T}| \leq 2\sigma \quad \text{for} \quad i \in \{1, 2\}.
\]

Now, elementary estimates show that the function \( t \mapsto q(t - d(q_t)) \) has Lipschitz constant 2. Thus, for example, \( q(\dot{T}_1 - d(q_{T_1})) \) is less than or equal to \(-3/16 + \alpha/2 + 2\sigma + 4\sigma\) (the greatest possible value for \( q(t_1 - d(q_{t_1})) \), plus twice the largest possible distance between \( t_1 \) and \( \dot{t}_1 \)). Similar arguments show that

1) \( q(\dot{t}_1 - d(q_{t_1})) \leq -3/16 + \alpha/2 + 6\sigma; \)
2) \( q(\dot{T}_1 - d(q_{T_1})) \leq -\alpha/2 - 6\sigma; \)
3) \( q(\dot{t}_2 - d(q_{t_2})) \leq -1/16 + \alpha/2 - 6\sigma; \)
4) \( q(\dot{T}_2 - d(q_{T_2})) \leq -1/8 + \alpha/2 + 6\sigma. \)

Thus, in particular, since \( 12\sigma < 1/8 - \alpha \) we have

\[
q(\dot{t}_2 - d(q_{t_2})) \geq q(\dot{t}_2 - d(q_{t_2})) \quad \text{and} \quad q(\dot{T}_2 - d(q_{T_2})) \leq q(\dot{T}_2 - d(q_{T_2})).
\]

The proof of Theorem 2.1 will be complete if we show that there are times \( \tau_1 \in [\hat{t}_1, \hat{T}_1] \) and \( \tau_2 \in [\hat{t}_2, \hat{T}_2] \) such that

\[
(q(\tau_1), q(\tau_1 - d(q_{\tau_1}))) = (q(\tau_2), q(\tau_2 - d(q_{\tau_2}))).
\]

That there are such times is a consequence of the following lemma, with \( [\bar{t}_1, \bar{T}_1] \) in the role of \( [a_1, b_1] \), \( [\bar{t}_2, \bar{T}_2] \) in the role of \( [a_2, b_2] \), \( q(t) \) in the role of the functions \( x_1 \) and \( x_2 \), \( q(t - d(q_t)) \) in the role of the functions \( y_1 \) and \( y_2 \), \( \tau_1 \) in the role of \( s_1 \), and \( \tau_2 \) in the role of \( s_2 \).

**Lemma 6.2.** Suppose that \( [a_1, b_1] \) and \( [a_2, b_2] \) are two closed intervals and that there are four continuous functions

\[
x_1 : [a_1, b_1] \rightarrow \mathbb{R}, \quad y_1 : [a_1, b_1] \rightarrow \mathbb{R}, \quad x_2 : [a_2, b_2] \rightarrow \mathbb{R}, \quad y_2 : [a_2, b_2] \rightarrow \mathbb{R}
\]

such that the following hold:

- \( x_1(a_1) = x_2(a_2) \) and \( x_1(b_1) = x_2(b_2) \), and both \( x_1 \) and \( x_2 \) are differentiable with strictly positive derivative everywhere;

\[\]
Then there are numbers \( s_1 \in [a_1, b_1] \) and \( s_2 \in [a_2, b_2] \) such that
\[
(x_1(s_1), y_1(s_1)) = (x_2(s_2), y_2(s_2)).
\]

See Figure 9.

**Proof.** Since \( x_1(a_1) = x_2(a_2), \ x_1(b_1) = x_2(b_2), \) and \( x_1 \) and \( x_2 \) both have strictly positive derivative everywhere, \( x_1([a_1, b_1]) = x_2([a_2, b_2]) \) and the inverse function \( x_2^{-1} : x_1([a_1, b_1]) \to [a_2, b_2] \) is continuous. Define the function \( \rho : [a_1, b_1] \to [a_2, b_2] \) by the formula
\[
\rho(s) = x_2^{-1}(x_1(s)).
\]
\( \rho \) is continuous and satisfies
\[
x_2(\rho(s)) = x_1(s) \quad \text{for all} \quad s \in [a_1, b_1].
\]

Observe that \( \rho(a_1) = a_2 \) and \( \rho(b_1) = b_2. \)

Now define the function \( z : [a_1, b_1] \to \mathbb{R} \) as follows:
\[
z(s) = y_2(\rho(s)) - y_1(s).
\]
\( z \) is a continuous function, and satisfies
\[
z(a_1) = y_2(a_2) - y_1(a_1) \geq 0 \quad \text{and} \quad z(b_1) = y_2(b_2) - y_1(b_1) \leq 0.
\]

By the intermediate value theorem there is some \( s_1 \in [a_1, b_1] \) such that \( z(s_1) = 0 \).

This means that
\[
y_2(\rho(s_1)) = y_1(s_1).
\]

Since \( x_2(\rho(s)) = x_1(s) \) for all \( s \in [a_1, b_1] \) by the definition of \( \rho \), writing \( s_2 = \rho(s_1) \) we have
\[
(x_1(s_1), y_1(s_1)) = (x_2(s_2), y_2(s_2)),
\]
as desired. \( \square \)

**REFERENCES**

[1] B. B. Kennedy, The Poincaré-Bendixson theorem for a class of delay equations with state-dependent delay and monotonic feedback, preprint.

[2] T. Krisztin and O. Arino, The two-dimensional attractor of a differential equation with state-dependent delay, *Journal of Dynamics and Differential Equations*, 13 (2001), 453–522.

[3] J. Mallet-Paret and R. D. Nussbaum, Boundary layer phenomena for differential-delay equations with state-dependent time lags, I, *Arch. Rational Mech. Anal.*, 120 (1992), 99–146.

[4] J. Mallet-Paret and G. R. Sell, Systems of differential delay equations: The Poincaré-Bendixson theorem for monotone cyclic feedback systems with delay, *Journal of Differential Equations*, 125 (1996), 441–489.
[5] H.-O. Walther, Algebraic-delay differential systems, state-dependent delay, and temporal order of reactions, *Journal of Dynamics and Differential Equations*, 21 (2009), 195–232.

[6] H.-O. Walther, A homoclinic loop generated by variable delay, *Journal of Dynamics and Differential Equations*, 27 (2015), 1101–1139.

Received April 2017; revised July 2017.

*E-mail address: bkenne\text{dy}@gettysburg.edu*