Economics-Based Optimization of Unstable Flows

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Abstract. – As an example for the optimization of unstable flows, we present an economics-based method for deciding the optimal rates at which vehicles are allowed to enter a highway. It exploits the naturally occurring fluctuations of traffic flow and is flexible enough to adapt in real time to the transient flow characteristics of road traffic. Simulations based on realistic parameter values show that this strategy is feasible for naturally occurring traffic, and that even far from optimality, injection policies can improve traffic flow. Moreover, the same method can be applied to the optimization of flows of gases and granular media.

Unstable or even turbulent flows are a wide-spread phenomenon in the motion of gases, fluids, and granular media \cite{1,2}. Therefore, it is often desirable to have methods that allow to optimize flows and to reduce their instability. While in the new field of “econophysics” \cite{3,4,5,6,7}, physical methods are usually applied to economic systems, we propose to apply economic methods to physical problems instead. Although the same investigations could be carried out for conventional gases or granular media, in the following we will focus on the example of freeway traffic flow, where vehicles play the role of the molecular or granular particles.

The advent of powerful traffic simulators \cite{8,9,10} has led to a spate of new discoveries in the area of vehicular traffic that agree well with empirical observations \cite{11}. From moving traffic jams \cite{8} to transitions to states of coherent motion \cite{11} these new results offer insights into a complicated and socially relevant many-body problem, while also suggesting ways of designing controls that can maximize the flow of vehicles in cities and highways \cite{12,13}.

The recent discovery of a new state of congested highway traffic, called “synchronized” traffic \cite{14}, has generated a strong interest in the rich spectrum of phenomena occurring close to on-ramps \cite{15,16}. In this connection, a particularly relevant problem is that of choosing an optimal injection strategy of vehicles into the highway. Similar questions have recently been raised regarding the most efficient use of the Internet in light of its bursty congestion patterns \cite{17}. While there exist a number of heuristic approaches to optimizing vehicle injection into...
freeways by on-ramp controls, the results are still not satisfactory. What is needed is a strategy that is flexible enough to adapt in real time to the transient flow characteristics of road traffic while leading to minimal travel times for all vehicles on the highway.

Our study presents a solution to this problem that explicitly exploits the naturally occurring fluctuations of traffic flow in order to enter the freeway at optimal times. This method leads to a more homogeneous traffic flow and a reduction of inefficient stop-and-go motions. In contrast to conventional methods, the basic performance criterion behind this technique is not the traffic volume, the optimization of which usually drives the system closer to the instability point of traffic flow and, hence, reduces the reliability of travel time predictions \[18\]. Instead, we will focus on the optimization of the travel time distribution itself, which is a global measure of the overall dynamics on the whole freeway stretch. It allows the evaluation of both the expected (average) travel time of vehicles and their variance, where a high value of the variance indicates a small reliability of the expected travel time when it comes to the prediction of individual arrival times.

Both the average and the variance of travel times are influenced by the inflow of vehicles entering the freeway over an on-ramp. From these two quantities one can construct a relation between the average payoff (the negative mean value of travel times) and the risk (their variance), as is considered in many optimization problems in economics. The optimal strategy will then correspond to the point in the curve that yields the lowest risk at a high average payoff. In the following, we will show that the variance of travel times has a minimum for on-ramp flows that are different from zero, but only in the congested traffic regime (which shows that the effect is not trivial at all). This finding implies that traffic flow can be optimized by choosing the appropriate vehicle injection rate into the freeway\textsuperscript{1}.

In order to obtain the travel time distribution of vehicles on a highway, we simulated two-lane traffic flow via a discretized follow-the-leader model, which describes the empirical known features of traffic flows quite well \[9\]. In our experiments, we extended the simulation to several lanes with lane-changing maneuvers and different vehicle types (cars and trucks). We determined the travel times of all vehicles by storing the times at which they pass two successive cross sections of the road.

The model distinguishes \(I\) neighboring lanes \(i \in \{1, \ldots, I\}\) of a unidirectional freeway. All lanes are subdivided into sites \(z \in \{1, 2, \ldots, Z\}\) of equal length \(\Delta x = 2.5\) m. Each site is either empty or occupied, the latter case representing the back of a vehicle of type \(a \in \{1, \ldots, A\}\) with velocity \(v = u \Delta x / \Delta t\). Here, \(u \in \{0, 1, \ldots, u_{\max}\}\) is the number of sites that the vehicle moves per update step \(\Delta t\). We have distinguished cars \((a = 1)\) and trucks \((a = 2)\). These are characterized by different optimal velocities \(U_a(d)\) with which the vehicles would like to drive at a distance \(d\) to the vehicle in front. Their lengths \(l_a\) correspond to the maximum distances satisfying \(U_a(l_a) = 0\). The positions \(z(T)\), velocities \(u(T)\), and lanes \(i(T)\) of all vehicles are updated\textsuperscript{2} every time step \(\Delta t = 1\) s at times \(T \in \{1, 2, \ldots\}\) according to the following successive steps \[10\].

1. Determine the potential velocities \(u_j(T + 1)\) on the present and the neighboring lanes

\textsuperscript{1} Hence, in order to reach well predictable and small average travel times at high flows in the overall system, it makes sense to temporarily hold back vehicles by a suitable on-ramp control based on a traffic-dependent stop light. At intersections of freeways, this may require additional buffer lanes \[13\].

\textsuperscript{2} We applied sequential update in driving direction, which avoids conflicts of vehicles that like to change to the same lane and position from both neighboring lanes at the same time. For two-lane roads the sequential update leads to almost identical results as a parallel update, which is usually applied in cellular or lattice gas automata for flow simulations and gives realistic results.
\( i(T) + j \) with \( j \in \{-1, 0, +1\} \) according to the acceleration law

\[
u_j(T + 1) = \lfloor \lambda U_a(d_j(T)) + (1 - \lambda)u(T) \rfloor.
\]

Here, the floor function \( \lfloor x \rfloor \) is defined by the largest integer \( m \leq x \), and \( d_j(T) = z_j^+(T) - z(T) \) denotes the distance to the next vehicle ahead (+) on lane \( i(T) + j \) at position \( z_j^+(T) \). The above equation describes the typical follow-the-leader behavior of driver-vehicle units. Delayed by the reaction time \( \Delta t \), they tend to move with the distance-dependent optimal (safe) velocity \( U_a \), but the adaptation takes a certain time \( \tau = \lambda \Delta t \) because of the vehicle’s inertia. Good results, in the sense of replicating highway data, are obtained for \( \lambda = 0.77 \).

2. Change lane to the left, i.e. set

\[
i(T + 1) = i(T) + k
\]

with \( k = +1 \), if the vehicle considered can go fastest there, i.e. if

\[
u_{+1}(T + 1) > u_0(T + 1) \quad \text{and} \quad > u_{-1}(T + 1).
\]

This is usually the case if the headway on the left lane is greatest. Apart from the validity of this incentive criterion, we demand two extra safety criteria \[19\]: First, the current vehicle position should be ahead of the expected position \( z_{-1}^-(T + 1) \) of the following vehicle (−) on the left lane, i.e.

\[
z(T) > z_{-1}^-(T + 1).
\]

Second, the potential velocity on the left lane should not be considerably less than the expected velocity \( u_{-1}(T + 1) \) of the following vehicle, i.e.

\[
u_k(T + 1) \geq q u_{-1}(T + 1).
\]

Once again, realistic results are obtained for \( q = 0.7 \). A value \( q < 1 \) implies that drivers are ready to accept a braking maneuver of the follower on the destination lane at the next update step. Therefore, the values of \( q \) are a measure of how relentless drivers are in overtaking.

Assuming symmetrical (“American”) lane changing rules for simplicity, a change to the right lane \( (k = -1) \) is carried out, if the incentive criterion \( u_{-1}(T + 1) > u_0(T + 1) \) and \( \geq u_{+1}(T + 1) \) as well as the safety criteria \( z(T) > z_{+1}^-(T + 1) \) and \( u_k(T + 1) \geq q u_{-1}(T + 1) \) are fulfilled. Otherwise the vehicle stays on the same lane \( (k = 0) \).

3. If the potential velocity \( u_k(T + 1) \) on the new lane \( i(T + 1) \) is positive, diminish it by 1 with probability \( p = 0.001 \), which accounts for delayed adaptation due to reduced attention of the driver and the variation of vehicle velocities:

\[
u(T + 1) = u_k(T + 1) - \begin{cases} 1 & \text{with probability } p \text{ if } u_k(T + 1) > 0 \\ 0 & \text{otherwise.} \end{cases}
\]

The resulting value defines the updated velocity \( u(T + 1) \).

4. Update the vehicle position according to the equation of motion

\[
z(T + 1) = z(T) + u(T + 1).
\]

Our simulations were carried out for a circular two-lane road. After the overall density was selected, vehicles were homogeneously distributed over the road at the beginning, with the same densities on both lanes. The experiments started with uniform distances among the vehicles and their associated desired velocities. The vehicle type was determined randomly after specifying the percentages \( r \) of cars (90%) and \((1 - r)\) of trucks (10%). Although we did
not carry out simulations with different mixtures of cars and trucks, we expect that an increase of \( r(1 - r) \) is related with a stronger instability of traffic flow and a larger lane-changing rate, so that the effects under discussion should be more pronounced.

At a given injection rate, vehicles enter the beginning of an on-ramp lane \( \text{i.e.} \) a third lane) of 1 kilometer length with a uniform time headway and their optimal velocity related to the density in the destination lane. Our simulation stretch consists of a 1 km long 3-lane part and the \( L = 10 \) km long two-lane stretch on which the travel times are measured. No vehicle is allowed to change to the on-ramp from the main road. Injected vehicles try to change from the on-ramp to the main road as far as possible, \( \text{i.e.} \) according to the above lane-changing rules, but they do not care about the incentive criterion. The end of the on-ramp is treated like a resting vehicle, \( \text{i.e.} \) any vehicle that approaches it has to stop, but it changes to the destination lane as soon as it finds a sufficiently large gap. If the on-ramp is completely occupied by vehicles waiting to enter the main road, the injected vehicles form a queue and enter the on-ramp as soon as possible. Injected vehicles that have completed their 10 km trip on the two-lane measurement stretch are automatically removed from the freeway (which would correspond to uncongested off-ramps adjacent to the lanes). Since our evaluations started after a transient period of two hours and continued until the 1000th injected vehicle finished its trip, the results are largely independent of the initial conditions.

If we plot the average of travel times as a function of their standard deviation (fig. 1), we obtain curves parametrized by the injection rate of vehicles into the road. Several points are worth noting: 1. With growing injection rate \( Q_{\text{onramp}} = 1/(n \, s) \), the travel time increases monotonically. This is because of the increased density caused by injection of vehicles into the freeway. 2. The average travel time of injected vehicles is higher, but their standard deviation lower than for the vehicles circling on the main road. This is due to the fact that vehicle injection produces a higher density on the lane adjacent to the on-ramp, which leads to smaller velocities. The difference between the travel time distributions of injected vehicles and those on the main road decreases with the length \( L \) of the simulated road, since lane-changes tend to equilibrate densities between lanes.

In addition, the standard deviation of travel times has a minimum for finite injection rates, as entering vehicles tend to fill existing gaps and thus homogenize traffic flow. This minimum is optimal in the sense that there is no other value of the injection rate that can produce travel with smaller variance. In particular, gap-filling behavior mitigates inefficient stop-and-go traffic at medium densities. Above a density of 45 vehicles per kilometer and lane on the main road (without injection), the minimum of the travel times’ standard deviation occurs for \( n \approx 60 \).

The reduction of the average travel time by smaller injection rates is less than the increase of their standard deviation. This result suggests that, in order to generate predictable and reliable arrival times, one should operate traffic at medium injection rates. Lastly, for the case of 40 vehicles per kilometer and lane, the minimum of the standard deviation of travel times is located at \( n \approx 30 \), while for 35 vehicles per kilometer and lane, it is at \( n \approx 15 \). Below 30 vehicles per kilometer and lane, vehicle injection does not reduce the standard deviation of travel times. This is because at these densities homogeneous traffic is stable anyway, so no stop-and-go traffic exists and therefore no large gaps that can be filled.

The curves displayed in fig. 1 correspond to a given density \( \rho_{\text{main}} \) on the main road without injection of vehicles. The effective density \( \rho_{\text{eff}} \) on the freeway resulting from the injection of vehicles can be approximated by

\[
\rho_{\text{eff}} = \rho_{\text{main}} + \frac{N_{\text{inj}}}{IL},
\]

where \( I = 2 \), \( L = 10 \) km. \( N_{\text{inj}} \) is the average number of injected vehicles present on the main
road and can be written as

\[ N_{\text{inj}} = N_{\text{tot}} \frac{T_{\text{inj}}}{T_{\text{tot}} - T_{\text{inj}}}, \tag{9} \]

where \( N_{\text{tot}} = 1000 \) is the total number of injected vehicles during the simulation runs, \( T_{\text{inj}} \) is their average travel time, and \( T_{\text{tot}} \) the time interval needed by all \( N_{\text{tot}} = 1000 \) vehicles to complete their trip. We point out that, in addition to these measurements, we used two other methods of density measurement which yielded similar results.

We also investigated the dependence of the travel time characteristics on the resulting effective densities of vehicles. As figure 2 shows, vehicle injection can actually reduce the average travel times of the vehicles on the main road, while the travel times of injected cars are about the same as those of vehicles on the main road without injection. This means that, for given \( \rho_{\text{eff}} \), one can actually increase the average velocity \( V_{\text{main}} = L/T_{\text{main}} \) of vehicles by injecting vehicles at a high rate without affecting their travel times. This result follows from the increased degree of homogeneity caused by entering vehicles that fill gaps on the main road, which mitigates the less efficient stop-and-go traffic.

Finally, figure 3 shows the average of the travel times for vehicles in the main road as a function of their standard deviation. In contrast to fig. 1, the curves were computed for the resulting effective densities. This time, an increase of the injection rate (which corresponds to a smaller number of circling vehicles on the main road and a greater proportion of injected vehicles) reduces the average travel times! Once again, we observe a minimum of the standard deviation of travel times at high vehicle densities and medium injection rates.

In the limit of high injection rates, a traffic jam of maximum density \( \rho_{\text{max}} \) builds up at the end of the on-ramp, while downstream of it we find the typical density \( \rho_{\text{out}} \) related to the universal outflow \( Q_{\text{out}} \) from traffic jams [20, 9]. We conjecture that the resulting structure consists of a block of density \( \rho_{\text{max}} \) and length \( L_1 \) containing \( N_1 = \rho_{\text{max}} L_1 \) vehicles, and a block of density \( \rho_{\text{out}} \) of length \( (L - L_1) \) containing \( (N - N_1) = \rho_{\text{out}} (L - L_1) \) vehicles at a mean density of \( \rho_{\text{eff}} = N/L \). The expected travel time \( T_{\text{main}} \) would be

\[ T_{\text{main}} = \frac{L}{V_{\text{out}}} + \frac{L_1}{C} + T_{\text{acc}} + T_{\text{dec}} \]

\[ = \frac{L}{V_{\text{out}}} + \frac{\rho_{\text{eff}} - \rho_{\text{out}}}{\rho_{\text{max}} - \rho_{\text{out}}} \frac{L}{C} + T_{\text{acc}} + T_{\text{dec}}, \tag{10} \]

where \( V_{\text{out}} \) is the typical velocity emerging downstream of a traffic jam, and \( C \) is the universal dissolution velocity of traffic jams [20, 9]. Notice that, for high injection rates, the average travel time should grow linearly with the mean density \( \rho_{\text{eff}} \), which is consistent with the results displayed in fig. 2. For decreasing injection rates, travel times should increase, since the alternation of congested and free flow in the resulting stop-and-go traffic implies relevant acceleration times \( T_{\text{acc}} \) and deceleration times \( T_{\text{dec}} \) in total.

In conclusion, we have presented a strategy for optimizing traffic on highways in the sense of higher flows and more reliable predictions of individual travel times. The applied method is economics-based and resorts to the establishment of average payoff-versus-risk curves. Here, the average payoff corresponds to the negative mean value of travel times and the risk to their variance. The strategy exploits the naturally occurring fluctuations of traffic flow in order to allow the entry of new vehicles to the freeway at optimal times. Simulations based on realistic parameter values show that this strategy is feasible for naturally occurring traffic, and that even far from optimality, injection policies can improve traffic flow.

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Fig. 1. – Average and standard deviation of the travel times of vehicles on the main road and from
the ramp as a function of the injection rate $Q_{\text{rmp}} = 1/(n \, s)$ with $n = 2^k$ and $k \in \{2, 3, \ldots, 10\}$ for
various vehicle densities on the main road (measured without injection). With increasing injection
rate, the average travel times are growing due to the higher resulting vehicle density on the freeway.

Fig. 2. – The average travel times of vehicles on the main road, depicted as a function of the
resulting effective vehicle density on the freeway, decrease with growing injection rates $Q_{\text{rmp}} = 1/(n \, s)$
($n \in \{2, 3, 4, 5, 10\}$), i.e. they are reduced by an increasing proportion of injected vehicles on the
freeway. In the limit of high injection rates, one observes the predicted linear dependence of average
travel times on effective density, see eq. (10). In contrast to the vehicles on the main road, the travel
times of injected vehicles did not depend on the injection rate. However, when we checked what
happens if the vehicles on the main road try to change to the left lane along the on-ramp in order to
give way to entering vehicles, we found that both, injected vehicles and the vehicles on the main road,
profited.
Fig. 3. – As figure 1, but as a function of the resulting effective density on the freeway. We find shorter travel times at high injection rates because of the homogenization of traffic. The standard deviation of travel times is varying stronger than the average travel time, which indicates that medium injection rates are the optimal choice at high vehicle densities.

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