L2 NORM PRESERVING FLOW IN MATRIX GEOMETRY

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Abstract. In this paper, we study L2 norm preserving heat flow in matrix geometry. We show that this flow preserves the operator convex property and enjoys the entropy stability in any finite time. Interesting properties of this flow like conserved trace free property are also derived.

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1. Introduction

In this paper we continue our study of evolution equation in the matrix geometry [13]. We introduce the L2 norm preserving flow such that starting from any initial data with unit L2 norm, we can produce a family of matrices of unit L2 norm such that their limit is a eigen-matrix of the Laplacian operator introduced in [13] and [9]. In [9], the author introduces the Ricci flow which exists globally when the initial data is a positive definite. The Ricci flow preserves the trace of the initial matrix and the flow converges the scalar matrix with the same trace as the initial matrix. In [13], we have introduced the heat equation, which also preserves the trace of the initial matrix. The advantage of the heat equation is that the corresponding flow not only preserves the trace but also can be with any initial matrix. In [2] and [14], The authors introduce the norm preserving flows which are global flow and converge to eigenfunctions. In quantum information theory, we can see the interesting works [3] [4] [5] and [6], but we need evolution flows which preserve the some norms like the L2 norm just like the normalized Ricci flow [11] preserves the volume and the curve flows [14] [16] preserve length or area. This is the main topic of the study of norm preserving flow in matrix geometry.

More precisely, we study again the matrix geometry model as in [13] [9] [17]. Let $X, Y$ be two Hermitian matrices on $C^n$. Define $U = e^{2\pi i X}$, $V = e^{2\pi i Y}$. We use $M_n$ to denote the algebra of all $n \times n$ complex matrices generated by $U$ and $V$ with the bracket $\{u, v\} = uv - vu$. Then $CI$, which is the scalar multiples of the identity matrices $I$, is the commutant of the operation $\{u, v\}$. Sometimes we simply use 1 to denote the $n \times n$ identity matrix.

We define two derivations $\delta_1$ and $\delta_2$ on the algebra $M_n$ by the commutators

$$\delta_1 := [y, \cdot] \quad \delta_2 := -[x, \cdot]$$

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Define the Laplacian operator on $M_n$ by

$$\hat{\Delta} = \delta_1^* \delta_1 + \delta_2^* \delta_2 = -\delta_1^2 - \delta_2^2 = -\delta_\mu \delta_\mu,$$

where we have used the Einstein sum convention. We use the Hilbert-Schmidt norm $| \cdot |$ defined by the inner product

$$< a, b >= (a, b) := \tau(a^* b)$$

on the algebra $M_n$ and let $\sigma(a) = < 1, a >$. Here $a^*$ is the complex conjugate of the matrix $a$, $\tau$ denotes the usual trace function on $M_n$. We now state basic properties of $\delta_1, \delta_2$ and $\hat{\Delta}$ (see also [9]).

For any $c \in M_n$, we define the Dirichlet energy

$$D(c) = \sum < \delta_\mu c, \delta_\mu c >$$

and the L2 mass

$$M(c) = < c, c >.$$

Let, for $c \neq 0$,

$$\lambda(c) = \frac{D(c)}{M(c)}.$$ 

Then the eigenvalues of the Laplacian operator $\hat{\Delta}$ correspond to the critical values of the Dirichlet energy $D(c)$ on the constrain subset

$$\Sigma = \{ c \in M_n; M(c) = 1 \}.$$ 

Hence to preserve the $L2$ norm, it is nature to study the evolution equation

$$c_t = -\hat{\Delta} c + \lambda(c) c$$ (1.1)

with its initial matrix $c|_{t=0} = c_0 \in M_n$. Assume $c = c(t)$ is the solution to the flow above. Then

$$\frac{1}{2} \frac{d}{dt} M(c) = < c, c_t >= -< c, \hat{\Delta} c > + D(c).$$

Since $< c, \hat{\Delta} c >= D(c)$, We know that $\frac{d}{dt} M(c) = 0$. Then

$$M(c(t)) = M(c_0).$$

The aim of this paper is to show that we always has a global flow to (1.1) and the flow has many very nice properties like entropy stability and operator convex preserving. Our main results, Theorem 2.1, Theorem 4.1, Theorem 5.1, and Theorem 6.1 are contained in sections 2, 3, 4, and 5.
2. EXISTENCE OF THE GLOBAL FLOW

We first consider the local existence of the flow (1.1). We prefer to follow the standard notation and we let \( \Delta = -\hat{\Delta} \). Let \( a = a(t) \in M_n/CI \) be such that

\[
a_t = \Delta a + \lambda(t)a,
\]

(2.1)

with the initial data \( a|_{t=0} = a_0 \). Here \( a_0 \in M_n/CI \) such that \( |a_0|^2 = 1 \) and \( \tau(a_0) := \bar{a}_0 = 0 \). Then for \( a = a(t) \), we let

\[
\lambda(t) = \frac{|\delta_{\mu} a|^2}{|a|^2} = -\frac{(\Delta a, a)}{(a, a)}.
\]

(2.2)

Formally, if the flow (2.1) exists, we then compute that

\[
\frac{d}{dt} |a|^2 = 2(a, a_t) = 2(a, \Delta a) + 2\lambda(t)(a, a) = 0.
\]

Then \( |a|^2(t) = |a|^2(0) = 1, \forall t > 0 \).

Our main goal in this section is to show that it exists a global solution to equation (2.1) for any \( a_0 \in M_n/CI \) with \( |a_0|^2 = 1 \).

Assume at first that \( \lambda(t) \geq 0 \) is any given continuous function and \( a = a(t) \) is the corresponding solution of (2.1). Define \( b = e^{-\int \lambda(t) dt} a \). Then \( b(0) = a(0) \) and we get

\[
b_t = e^{-\int \lambda(t) dt} (-\lambda)a + e^{-\int \lambda(t) dt} a_t
\]

(2.3)

\[
= -\lambda a e^{-\int \lambda(t) dt} + e^{-\int \lambda(t) dt} (\Delta a + \lambda a)
\]

(2.4)

\[
= e^{-\int \lambda(t) dt} \Delta a
\]

(2.5)

\[
= \Delta (e^{-\int \lambda(t) dt} a)
\]

(2.6)

\[
= \Delta b.
\]

(2.7)

The equation (2.7) can be solved by standard way. For completeness we recall it here. Assume \( \varphi_i \) and \( \lambda_i \) are eigen-matrices and eigenvalues of \( \Delta \), as we introduced in [13], such that

\[-\Delta \varphi_i = \lambda_i \varphi_i, \quad < \varphi_i, \varphi_j > = \delta_{ij}.
\]

Note that \( \lambda_i \geq 0 \).

Assume that \( b = b(t) \) is the solution to (2.7). Set

\[b = \sum < b, \varphi_i > \varphi_i := \sum u_i \varphi_i, \quad u_i \in R, \quad u_i = u_i(t).
\]

Then by (2.7), we obtain

\[(u_i)_t \varphi_i = \Delta (u_i \varphi_i) = -u_i \lambda_i \varphi_i.
\]

Then \( (u_i)_t = -u_i \lambda_i \), and \( u_i = u_i(0) e^{-\lambda_i t} \).

Hence

\[b = \sum u_i(0) e^{-\lambda_i t} \varphi_i,
\]
and
\[ a = \sum u_i(0)e^{-\lambda_it} + \int_{t_0}^t \lambda dg \varphi_i \]  \hspace{1cm} (2.8)

solves (2.1) with the given \( \lambda(t) \).

We now define a iteration relation to solve (2.1) for the unknown \( \lambda(t) \) given by (2.2).

Define \( a_1 \) such that
\[ a_1 = \Delta a + \lambda_0 a \]  \hspace{1cm} (2.9)

with
\[ \lambda_k(t) = -\frac{(\Delta a_k, a_k)}{(a_k, a_k)} = \frac{|\delta_{\mu}a_k|^2}{(a_k, a_k)}. \]  \hspace{1cm} (2.10)

Then using the formula (2.8), we get a sequence \((a_k)\). We claim that \((a_k) \subset M_n/CI\) is a bounded sequence and \((\lambda_k(t))\) is also a bounded sequence. It is clear that \((a_k) \subset M_n/CI\). If this claim is true, we may assume \( a_k \to a, \lambda_k(t) \to \tilde{\lambda}(t) \).

Then by (2.9) and (2.10), we obtain
\[ a_t = \Delta a + \tilde{\lambda}(t)a \]
and
\[ \tilde{\lambda}(t) = -\frac{(\Delta a, a)}{(a, a)}, \]
which is the same as (2.1). That is to say, \( a = a(t) \) obtained above is the desired solution to (2.1).

We first prove the Claim in a small interval \([0, T]\). Assume \(|a_k| \leq A = 1.5\) and \(|\lambda_k| \leq B = \log(4/3)\) on \([0, T]\), \( T = 1/2 \). Then, by (2.9),
\[ \frac{1}{2}|a_{k+1}|^2_t = (a_{k+1}, (a_{k+1})_t) \]
\[ = (a_{k+1}, \Delta a_{k+1} + \lambda_k a_{k+1}) \]
\[ = -|\delta_{\mu}a_{k+1}|^2 + \lambda_k |a_{k+1}|^2. \]  \hspace{1cm} (2.11)

By (2.10), we obtain \( \lambda_{k+1} = \frac{|\delta_{\mu}a_{k+1}|^2}{|a_{k+1}|^2} \). Then
\[ |\delta_{\mu}a_{k+1}|^2 = \lambda_{k+1} |a_{k+1}|^2. \]

By (2.13), we get
\[ \frac{1}{2}|a_{k+1}|^2_t = -\lambda_{k+1} |a_{k+1}|^2 + \lambda_k |a_{k+1}|^2 \]
\[ = (\lambda_k - \lambda_{k+1}) |a_{k+1}|^2. \]

Then \(|a_{k+1}|^2 = e^{2f/(\lambda_k-\lambda_{k+1})dt}|a_0|^2 = e^{2Bt} \leq A.\)

Recall that \( \bar{a}_0 = 0. \) Then
\[ \partial_t(\bar{a}_{k+1}) = (\Delta a_{k+1}, 1) + \lambda_k (a_{k+1}, 1) = \lambda_k \bar{a}_{k+1}, \]
\[
\tilde{a}_{k+1} = e^{\int \lambda_k dt} \tilde{a}_{k+1}(0) = e^{\int \lambda_k dt} a_0 = 0.
\]

Note that since \(\tilde{a}_{k+1} = 0\) and norm equivalence relation,
\[
\lambda_{k+1}|a_{k+1}|^2 = |\delta_\mu a_{k+1}|^2 \\
\leq C|a_{k+1} - \tilde{a}_{k+1}|^2 \\
= C|a_{k+1}|^2.
\]

Then \(\lambda_{k+1} \leq C\). Hence the Claim is true in \([0, T]\).

Therefore, (2.1) has a solution in \([0, T]\). By iteration we can get a solution in \([T, 2T]\) with \(u(T)\) as the initial data. We can iterate this step on and on and we get a global solution to (2.1) with initial data \(a_0\).

In conclusion we have the below.

**Theorem 2.1.** For any given initial matrix \(a_0 \in M_n/CI\) with \(|a_0|^2 = 1\), The equation (2.1) has a global solution with \(a_0\) as its initial data and \(|a(t)|^2 = 1\) for all \(t > 0\).

3. Basic Properties Preserved by the Flow

In this section we show that there are two properties of the initial matrix are preserved except the conservation of the \(L^2\) norm.

We show that if the initial matrix is positive definite, then along the flow (2.1), the evolving matrix is also positive definite.

**Theorem 3.1.** Assume \(a_0 > 0\), that is \(a_0\) is a Hermitian positive definite. Then \(a(t) > 0, \forall t > 0\) along the flow equation
\[
a_t = \Delta a + \lambda(t) a
\]
with \(a(0) = a_0\), where \(\lambda(t)\) is given by (2.2).

**Proof.** By continuity, we know that \(a(t) > 0\) for small \(t > 0\). Compute
\[
\frac{d}{dt} \log \det a = (a^{-1}, a_t) = (a^{-1}, \Delta a) + N \lambda(t)
\]
where \(N = \sigma(I)\).

Since
\[
(a^{-1}, \Delta a) = -(\delta_\mu a^{-1}, \delta_\mu a) = (a^{-1}\delta_\mu a \cdot a^{-1}, \delta_\mu a) = |a^{-1}\delta_\mu a|^2.
\]

We know that
\[
\frac{d}{dt} \log \det a = |a^{-1}\delta_\mu a|^2 + N \lambda(t) \geq N \lambda(t) \geq 0.
\]

Hence, we have \(a(t) > 0, \forall t > 0\).

Remark that by continuity, we can show that if \(a_0 \geq 0\), then \(a(t) \geq 0\) along the flow (2.1).
Proposition 3.2. Assume $\sigma(a_0) = \bar{a}_0 = 0$. Then $\bar{a}_t = 0$, $\forall t > 0$.

Proof.

$$\frac{d}{dt}\bar{a} = \sigma(a_t)$$

$$= \sigma(\Delta a + \lambda(t)a)$$

$$= \sigma(\Delta a) + \lambda(t)\sigma(a)$$

$$= \lambda(t)\bar{a},$$

so $\bar{a}(t) = \bar{a}(0)e^{\int_0^t \lambda(t)dt} = 0$. \hfill \Box$

4. Convergence of the flow $a = a(t)$ at $\infty$

We prove the the global flow converges to some eigen-matrix.

Theorem 4.1. For any given initial matrix $a_0 \in M_n/CI$ with $|a_0|^2 = 1$, the global solution to the equation (2.1) with $a_0$ as its initial data converges to some eigen-matrix with its eigenvalue $\lambda \geq \lambda_1$.

Recall that we have the global flow $a = a(t)$ such that

$$a_t = \Delta a + \lambda(t)a, \bar{a} = 0$$

with $\lambda(t) = |\delta_\mu a|^2 \geq C^{-1}|a|^2 = C^{-1}, a(t)|_{t=0} = a_0, |a_0|^2 = 1$ and $|a(t)|^2 = 1$.

Set

$$\Delta a = (\Delta a, a) + (\Delta a)^\perp,$$

where $((\Delta a)^\perp, a) = 0$.

Note that $\lambda(t) = - (\Delta a, a)$, so by the Schwartz inequality, $\lambda(t) \leq |\Delta a|$.

Compute

$$\frac{d}{dt}\lambda(t) = 2(\nabla a, \nabla a_t)$$

$$= -2(\Delta a, a_t)$$

$$= -2(\Delta a, \Delta a + \lambda(t)a)$$

$$= -2(\Delta a)^2 - 2\lambda(t)(\Delta a, a)$$

$$= -2(\Delta a)^2 + 2(\Delta a, a)^2$$

$$\leq 0.$$

Then we may assume that

$$a(t) \to a_\infty$$

and $|a_\infty|^2 = 1, \bar{a}_\infty = 0, \lambda_\infty = \lim_{t \to \infty} \lambda(t)$ and $(\Delta a_\infty)^\perp = 0$.

The latter condition implies that

$$((\Delta a_\infty)^\perp, a_\infty) = 0,$$
which is $-\Delta a_\infty = (\Delta a_\infty, a_\infty)a_\infty = \lambda_\infty a_\infty$.

Since $\lambda_\infty \geq C^{-1}$, we know that $\lambda_\infty$ is the non-zero eigenvalue of $-\Delta$ and then $\lambda_\infty \geq \lambda_1 > 0$.

This completes the proof of Theorem 4.1.

5. Entropy stability of the flow $a = a(t)$

We first prove the HS norm stability of the flow (2.1). Recall that there is a uniform constant $C > 0$ such that for any $u, v \in M_n$ and with $|u|^2 = 1 = |v|^2$,

$$|\lambda(u) - \lambda(v)| \leq C|u - v|.$$ 

Given two initial matrix $u_0, v_0$. Let $u, v$ be the corresponding solutions to (2.1) with initial datum $u_0, v_0$.

Note that

$$\frac{d}{dt}|u - v|^2 = 2 < u - v, \Delta u - \Delta v > = 2 < u - v, \lambda(u)u - \lambda(v)v >$$

$$\leq 2 < u - v, \Delta(u - v) > + 2|\lambda(u) - \lambda(v)||u - v| + |\lambda(v)||u - v|^2$$

$$\leq C_1|u - v|^2,$$

where $C_1$ is a uniform constant.

$$|u - v|^2 \leq e^{C_1 t}|u - v|^2(0)$$

which implies the HS norm stability of the flow in (2.1) in the finite time interval $[0, T]$.

Remark: Similarly, we have the Trace norm stability of solutions, where the Trace norm is denoted by $T(u, v)$ [18]. Recall that by the spectral theorem we may let $u - v = Q - P$, where $Q$ and $P$ are positive definite operators with compact support. Then

$$T(u, v) = \tau(P) + \tau(Q)$$

In below, we assume $u_0 > 0$ and define the von Neumann entropy [18] by

$$S(u) = -\tau(u \log u)$$

for the positive definite solution $u = u(t)$ with $u(0) = u_0$.

Recall the Fannes inequality [13] for $\forall a, b \in M_n$ and $a > 0 b > 0$, we have

$$|S(a) - S(b)| \leq \Omega \log d + \eta(\Omega),$$

where $\eta(s) = -s \log s$, $d = \text{dim} M_n$ and $\Omega = \sum |r_i - s_i| \leq T(a, b) \leq \frac{1}{e}$.

Then we can use the Fannes inequality to get the entropy stability of the solution of (2.1).

**Theorem 5.1.** If $T(u_0, v_0) \leq \frac{1}{e}$, $u_0 > 0 v_0 > 0$ in $M_n$, then the solution $u(t), v(t)$ to (2.1) satisfies

$$|S(u_t) - S(v_t)| \leq C_1 T(u, v)(0) \log d + \eta(C_1 T(u, v))(0).$$
Proof. By the result above we have
\[ T(u, v)(t) \leq C_1 T(u, v)(0), \]
by the Fannes inequality, we have
\[ |S(u_t) - S(v_t)| \leq T(u(t), v(t)) \log d + \eta(T(u(t), v(t))) \]
\[ \leq C_1 T(u(0), v(0)) \log d + C_1 \eta(T(u(0), v(0))), \]
where we have used the monotonicity of the function \( \eta \) in \([0, \frac{1}{e}] \).
\[ \square \]

6. OPERATOR CONVEXITY OF THE HEAT EQUATION

In this section we prove SOME nonlinear convexity properties of the initial matrix are preserved along the heat flow (6.1) below
\[ a_t = \Delta a. \] (6.1)

Recall that we say \( f : M_n \to M_n \) is operator convex if for any Hermitian symmetric matrices \( A \) and \( B \), we have
\[ \mu f(A) + (1 - \mu) f(B) - f(\mu A + (1 - \mu) B) \geq 0, \quad \mu \in (0, 1). \]
Let \( f(x) = x^2 \). Then we compute,
\[ \Delta a^2 = \delta_{\mu}(\delta_{\mu} a^2) \]
\[ = \delta_{\mu}(\delta_{\mu} a \cdot a + a\delta_{\mu} a) \]
\[ = \delta_{\mu}^2 a \cdot a + 2(\delta_{\mu} a)^2 + a\delta_{\mu}^2 a \]
\[ = \Delta a \cdot a + 2(\delta_{\mu} a)^2 + a\Delta a \]
and
\[ \partial_t(a^2) = a a_t + a_t a. \]

By \((\partial_t - \Delta)a = 0\), we obtain \((\partial_t - \Delta)a^2 = -2(\delta_{\mu} a)^2\).

Hence, we have
\[
\frac{d}{dt} \log \det(a^2) = \tau(a^{-2}(a^2)_t) \\
= \tau(a^{-2}\Delta a^2) - 2\tau(a^{-2}(\delta_{\mu} a)^2) \\
= \tau(a^{-2}\Delta a^2) + 2\tau(a^{-1}\delta_{\mu} a \cdot (\delta_{\mu} a)^{-1} a^{-1}) \\
= \tau(a^{-2}\Delta a^2) + 2\tau(a^{-1}\delta_{\mu} a \cdot (a^{-1}\delta_{\mu} a)^{-1}) \\
\geq 0.
\]
We then deduce that \( a^2 > 0 \) provided \( a_0^2 > 0 \).

Assume that \( \lambda > 0 \) is any positive constant. Recall that \((\lambda + a)^{-1} > 0 \) if and only if \( \lambda + a > 0 \).

Note that, by \( \lambda + a_0 > 0 \), we have \( \lambda + a > 0 \). Then \((\lambda + a)^{-1} > 0, \forall t > 0. \)

Theorem 6.1. Assume \( f \) is operator convex, if \( f(a_0) > 0 \), then \( f(a) > 0, \forall t > 0. \).
Proof. According to [12], we have that any continuous operator convex real function on $[0, \infty]$ can be expressed as

$$f(x) = f(0) + ax + bx^2 + \int_0^\infty \left( \frac{x}{1+\lambda} - 1 + \frac{\lambda}{x+\lambda} \right) d_\mu(\lambda),$$

where $a, b \geq 0$, $d_\mu$ is a nonnegative measure.

Since for $f(x) = x, x^2, \frac{x}{x+\lambda}$, we already know that if $f(a_0) > 0$, then $f(a) > 0$, $\forall t > 0$.

Hence, for any Hermitian positive definite matrix $a_0 > 0$ with $f(a_0) > 0$, then

$$f(a) > 0, \forall t > 0.$$

This completes the proof of Theorem 6.1.

□

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