Fast and Feasible Estimation of Generalized Linear Models with High-Dimensional \( k \)-way Fixed Effects\(^*\)

Amrei Stammann\(^†\)

Heinrich-Heine University Düsseldorf

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We present a fast and memory efficient algorithm for the estimation of generalized linear models with an additive separable \( k \)-way error component. The brute force approach uses dummy variables to account for the unobserved heterogeneity, but quickly faces computational limits. Thus, we show how a weighted version of the Frisch-Waugh-Lovell theorem combined with the method of alternating projections can be incorporated into an iteratively reweighted least squares algorithm to dramatically reduce the computational costs. The algorithm is especially useful in situations, where generalized linear models with \( k \)-way fixed effects based on dummy variables are computationally demanding or even infeasible. In a simulation study and an empirical application we demonstrate the outstanding performance of our algorithm.

**Key Words:** High-dimensional fixed effects, Generalized Linear Models, Alternating Projections, Frisch-Waugh-Lovell Theorem, Logit, Probit, Poisson, Panel Data

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\(^†\)Email: Amrei.Stammann@hhu.de
1 Introduction

Fixed effects models are popular specifications to account for unobserved heterogeneity; for example in labor economics often worker, firm and time fixed effects are used, and in trade economics importer-time, exporter-time and importer-exporter fixed effects are applied. Especially in large micro-level panels like the German IAB panel or the U.S. PSID panel such model specifications can lead to high-dimensional fixed effects. In classical one-way linear regression models, it is possible to use a computational trick known as demeaning or with-in transformation to avoid having to include dummy variables to the model. Even if only one fixed effects dimension is large, it is straightforward to add the smaller dimensions as dummy variables to the regressor matrix and to demean over the larger fixed effects dimension. If all or many fixed effects dimensions are large the aforementioned approach would require the generation and inversion of a potentially large regressor matrix. To tackle this computational burden various algorithms have been proposed (see among others Ouazad et al. (2008), Guimaraes, Portugal et al. (2010), Gaure (2013b), Correia (2016), Somaini and Wolak (2016)). These algorithms rely on the Frisch-Waugh-Lovell (FWL) theorem (Frisch and Waugh (1933), Lovell (1963)) and have been so far developed in particular for linear models.

In the case of generalized linear models like probit and logit models no general efficient k-way fixed effects algorithms have been designed yet. Like in the linear case it is possible to include the lower fixed effects dimensions as additional regressors and to apply special algorithms to pseudo-demean over the larger dimension. For example the partitioned inverse approach proposed by Chamberlain (1980) or the algorithm proposed by Stammann et al. (2016) are suited. However, the inclusion of many dummies can easily lead to computational limits. In their paper Guimaraes et al. (2010) propose an Gauss-Seidel algorithm to efficiently estimate linear models with high-dimensional fixed effects and they also pioneered the way to generalized linear models by incorporating their algorithm into an iteratively reweighted least squares (IRLS) routine to estimate poisson and negative binomial models. However, these types of models have a closed form solution for the fixed effects which does not exist for all generalized linear models.

Guimaraes et al. (2016) also provide a Stata routine poi2hdfe to estimate two-way fixed effects poisson models. They did not implement the previously mentioned algorithm based on the Gauss-Seidel algorithm (Guimaraes et al. 2010), but rather use a weighted version of the method of alternating projections. Similar to the algorithm we propose in this paper, Guimaraes et al. (2016) use the fact that each iteration step of IRLS is equivalent to a weighted regression.

We contribute to the missing theoretical foundation and derive an alternative weighted version of the method of alternating projections. This results in a straightforward and memory efficient estimation routine which we call pseudo-demeaning and can be applied to all generalized linear models with a k-way error component. Our starting point is an algorithm proposed by Gaure (2013b) for linear regression models with high-dimensional fixed effects which uses an iterative demeaning procedure based on alternating projections. We extend it to generalized linear models by using a result previously shown by Stammann et al. (2016), which allows the application of the FWL theorem in each step of

\footnote{We define pseudo-demeaning as a kind of weighted demeaning (see Stammann, Heiß and McFadden (2016)).}

\footnote{Guimaraes et al. (2010) note that for models with no closed-form solution of the fixed effects the Gauss-Seidel algorithm would need to be combined with numerical optimization routines to solve for the fixed effects, which probably becomes demanding. However, they do not specify the approach further. Our approach closes this gap by proposing an efficient procedure for all kind of generalized linear models. Contrary to Guimaraes et al. (2010)’s note, within our algorithm it is not necessary to use demanding routines to update the fixed effects for IRLS directly.}

\footnote{Guimaraes et al. (2016) use the Stata routine hdfe by Correia (2016).}
the iteratively reweighted least-squares (IRLS) optimization routine.

Unlike in linear models, where the structural parameters can be estimated separably from the fixed effects by application of the FWL theorem, for generalized linear models this only holds for each iteration of the optimization routine. To be more precise, the fixed effects have to be updated in each iteration of the optimization routine to update the gradient and Hessian of the subsequent iteration. Fortunately, it turns out to be much less computational challenging to update the contribution of the fixed effects to the linear predictor in each iteration. However, we also present an efficient post estimation routine to recover the fixed effects, if these are of interest.

The paper is organized as follows. First we introduce the $k$-way fixed effects generalized linear model, and show how the IRLS step can be rewritten to make it suitable for applying the FWL theorem. Next we combine the result of the FWL theorem with the method of alternating projections to arrive at the straightforward pseudo-demeaning algorithm. Afterwards, a simulation study demonstrates the performance of the pseudo-demeaning algorithm and an empirical example in trade economics shows possible areas of applications.

2 The Model

A generalized linear model consists of three parts: a stochastic component $\mu$, a systematic component $\eta$, and a link $g(\cdot)$ between both components (McCullagh and Nelder 1989).

In a $k$-way fixed effects generalized linear model the linear predictor takes the following specific form:

$$\eta = Z\gamma = D\alpha + X\beta = \sum_{k=1}^{K} D_k\alpha_k + X\beta,$$

(2.1)

where the regressor matrix $Z$ can be split into a sparse part $D$ and a remaining part $X$. More specifically, the matrices $D_k$ arise from dummy encoding of $K$ categorical variables and capture the unobserved heterogeneity. Each dummy matrix is of dimension $(n \times l_k)$, where $n$ is the number of observations and $l_k$ is the number of levels of the $k$-th categorical variable. The corresponding parameters $\alpha = [\alpha_1, \ldots, \alpha_K]'$ are called fixed effects. The remaining part $X$ is a $(n \times p)$ matrix of variables of interest and the corresponding parameters $\beta$ are the structural parameters.

The further components of the model can be expressed as follows:

$$E(y) = \mu = g^{-1}(\eta),$$

where the link function $g(\cdot)$ is a monotonic differentiable function and $y$ is a realization of an independently distributed random variable from the exponential family $Y$. The distribution is given by:

$$f_Y(y, \theta, \phi) = \exp \left( \frac{(y\theta - b(\theta))/a(\phi) + c(y, \phi)}{} \right),$$

(2.2)

where $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ are specific functions. We consider the cases where $\phi$ is known and thus $\theta$ is a canonical parameter. Table 2 summarizes the corresponding functions and parameters for the logit and poisson model. For other generalized linear models please consult e.g. McCullagh and Nelder (1989).

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4A first version of our pseudo-demeaning algorithm is available as an R-package alpaca which can be downloaded here: [https://github.com/amrei-stammann/alpaca](https://github.com/amrei-stammann/alpaca) The underlying algorithm is implemented in Rcpp (Eddelbuettel and François (2011), Eddelbuettel (2013)).
The unknown parameters $\gamma = [\alpha, \beta]'$ are estimated using maximum likelihood. The log-likelihood is

$$L = \sum_{i=1}^{n} \left( \frac{y_i \theta_i - b(\theta_i)}{a(\phi)} + c(y_i, \phi) \right),$$

which can be maximized iteratively

$$\gamma^r - \gamma^{r-1} = -(H^r)^{-1}g^r,$$

where $g^r$ and $H^r$ are the gradient and Hessian at iteration $r$.

Since $\theta(\mu)$ is the canonical link we can apply the chain rule which leads to the following expression of the gradient:

$$\frac{\partial L}{\partial \gamma^r} = g^r = Z'W^r \left( (y - \mu^r) \otimes \frac{\partial \eta^r}{\partial \mu^r} \right),$$

where $W^r$ is a positive definite diagonal weighting matrix with its $i$-th entry equal to $(\frac{\partial \mu^r_i}{\partial \eta^r_i})^2 V^r_i = 1/(\left( \frac{\partial \eta^r_i}{\partial \mu^r_i} \right)^2 V^r_i)$. Note that $\frac{\partial \eta^r_i}{\partial \mu^r_i} = \frac{\partial g(\mu^r)}{\partial \mu^r_i}$. The Hessian can be derived in the same manner:

$$\frac{\partial^2 L}{\partial \gamma^r \partial \gamma'^r} = H^r = -Z'W^rZ.$$

We assume $Z$ to have full rank and $\text{dim}(Z) = n \times (p + l)$, where $l \leq \sum_{k=1}^{K} l_k$ denotes the columns of the sparse part of $Z$.

Brute-force implementations would require the computation and inversion of a potentially large Hessian of dimension $(p + l) \times (p + l)$ which quickly becomes computationally demanding or even infeasible.

In the next section we present a new pseudo-demeaning algorithm based on the Frisch-Waugh-Lovell (FWL) theorem in combination with the method of alternating projections. This approach substantially decreases the computational costs of the brute-force implementation.

| Table 1: Logit and Poisson | Log | Poisson |
|---------------------------|-----|---------|
| Dispersion parameter $\phi$ | 1   | 1       |
| Cumulant function $b(\theta)$ | $\log(1 + \exp(\theta))$ | $\exp(\theta)$ |
| $c(y, \phi)$ | 0   | $\log(y!)$ |
| $\mu(\theta)$ | $\exp(\theta)/(1 + \exp(\theta))$ | $\exp(\theta)$ |
| Canonical link $\theta(\mu)$ | $\log(\mu/(1 - \mu))$ | $\log(\mu)$ |
| Variance function $V(\mu)$ | $\mu(1 - \mu)$ | $\mu$ |

Note: Following McCullagh and Nelder (1989) (table 2.1).

5 Usually the sparse part of $Z$ has no full rank, such that some columns are removed for the estimation. For example in the classical two-way fixed effects model with individual and time fixed effects, one column of the dummy matrix has to be removed.
3 The Pseudo-Demeaning Algorithm

3.1 The Frisch-Waugh-Lovell theorem applied to GLM’s

In the classical fixed effects linear model the FWL theorem is applied to separate the estimation of the fixed effects from the structural parameters. Recently Stammann et al. (2016) showed how the FWL theorem can be adapted to separate the updates of the structural parameters from the fixed effects updates in a one-way fixed effects logit model. The same logic can be applied to $k$-way fixed effects generalized linear models. The idea is to rewrite the iteration step into a regression model:

$$
\gamma^r - \gamma^{r-1} = (Z^rW^rZ)^{-1}Z^rW^r(y - \mu^{r-1}) \odot \frac{\partial \eta^{r-1}}{\partial \mu^{r-1}} \tag{3.1}
$$

where $W^r = (W^r)^{1/2}$. This transformation is possible since $W^r$ is a positive definite diagonal matrix. (2.6) reduces to the regression model:

$$
\tilde{y}^{r-1} = \sum_{k=1}^{K} \left( \tilde{D}_k^{-1}(\alpha_k^r - \alpha_k^{r-1}) \right) + \tilde{X}^{r-1}(\beta^r - \beta^{r-1}) \tag{3.2}
$$

where $\tilde{y}^r = \tilde{W}^r(y - \mu^r) \odot \frac{\partial \eta^r}{\partial \mu^r}$, $\tilde{D}_k = \tilde{W}^rD_k$, and $\tilde{X}^r = \tilde{W}^rX$.

Rewrite the model

$$
\tilde{y}^{r-1} = S^{r-1}(\alpha^r - \alpha^{r-1}) + \tilde{X}^{r-1}(\beta^r - \beta^{r-1}) \tag{3.3}
$$

where $(\alpha^r - \alpha^{r-1}) = [\alpha_1^r - \alpha_1^{r-1}, \ldots, \alpha_K^r - \alpha_K^{r-1}]$ and $S^r = [\tilde{D}_1^r, \ldots, \tilde{D}_K^r]$, where $\text{dim}(S^r) = (n \times l)$. This transformation of the update formula in a regression-like problem allows to eliminate $S^{r-1}(\alpha^r - \alpha^{r-1})$ via the FWL theorem:

$$
M^{r-1}\tilde{y}^{r-1} = M^{r-1}S^{r-1}(\alpha^r - \alpha^{r-1}) + M^{r-1}\tilde{X}^{r-1}(\beta^r - \beta^{r-1})
= M^{r-1}\tilde{X}^{r-1}(\beta^r - \beta^{r-1})
$$

where $M^r = I - S^r(S^rS^r)^{-1}S^r$ is the projection on the orthogonal column space of $S^r$.

We call $M^{r-1}\tilde{y}^{r-1}$ and $M^{r-1}\tilde{X}^{r-1}$ the pseudo-demeaned variables that can be used to compute the update $(\beta^r - \beta^{r-1})$ separately from the high-dimensional fixed effects updates at very low computational costs. However, the brute-force pseudo-demeaning ends up in a computational challenge itself since the annihilator matrix $M^r$ has dimension $(n \times n)$ and is typically non-sparse. One exception is the case $K = 1$ where the block-diagonal structure of $(S^rS^r)^{-1}$ allows to derive a straightforward formula to compute the pseudo-demeaned variables without costly matrix operations (Stammann et al. 2016).

For $K > 1$ this is not possible since $(S^rS^r)^{-1}$ looses its sparse structure. Thus if $K$ becomes large, the brute-force computation of the pseudo-demeaning becomes expensive and even possibly infeasible.

Fortunately, we can use a combination of the one-way pseudo-demeaning along with the method of alternating projections to derive an algorithm to obtain the pseudo-demeaned variables directly without having to compute the annihilator matrix $M^r$.

\[\text{Note, } M \text{ is idempotent; } M = M^2 = M^r.\]
3.2 The Method of Alternating Projections

An approach to compute the pseudo-demeaned variables is a method called alternating projections (AP) tracing back to Von Neumann (1949) and Halperin (1962). Gaur (2013b) introduced AP in the context of classical linear models with many fixed effects categories.

We show how the AP approach can be adapted to generalized linear models. The idea is to subsequently subtract weighted group means (instead of simple group means as in the linear case) from the dependent variable \( \tilde{y}^r \) and the regressor matrix \( \tilde{X}^r \) respectively until they are approximately equal to the pseudo-demeaned variables \( M^r \tilde{y}^r \) and \( M^r \tilde{X}^r \).

Let \( v \) be an arbitrary vector. With help of Halperin’s theorem the non-sparse projection \( M^r v \) can be decomposed into an iterative procedure based on only sparse projections:

\[
M^r v = \lim_{N \to \infty} (M^r_{D^1} \cdots M^r_{D^K})^N v,
\]

where \( M^r_{D^1} = I - \tilde{D}^1_k \tilde{D}^1_k \) \( \tilde{D}^1_k \) The projections \( M^r_{D^1} v \) are sparse and translate in one-way pseudo-demeaning over category \( k \). Using the result shown by Stammann et al. (2016), the projections \( M^r_{D^1} v \) can be efficiently computed as follows:

\[
M^r_{D^1} v = v - \tilde{w}^r \otimes \left( \left( \sum_{i=1}^{e_k} (\tilde{w}_{i(k=1)})^r \cdot v_{i(k=1)} \right) / \left( \sum_{i=1}^{e_k} w_{i(k=1)}^r \right) \right) \otimes i_{\tilde{e}_k(k=1)}, \tag{3.4}
\]

where \( \tilde{w}^r = \text{diag}(\tilde{W}^r) \), \( w^r = \text{diag}(W^r) \), \( e_{\kappa} \) denotes the frequency a level \( \kappa \) of categorical variable \( k \) appears, and where \( i_{\tilde{e}_k} \) is a vector of \( \tilde{e}_k \) ones.

The following algorithm is subsequently applied to \( \tilde{y}^r \) and column wise to \( \tilde{X}^r \), such that it delivers the approximations \( M^r \tilde{y}^r \) and \( M^r \tilde{X}^r \).

Algorithm 1 Pseudo-Demeaning

1. Let \( v \in \{\tilde{y}^r, \tilde{X}^r_j\} \), \( j = 1, \ldots, p \).
2. Set \( i = 1, z_i = v, z_{i-1} = z_i - 1_n \), and tolerance level \( \epsilon \).
3. while \( \|z_i - z_{i-1}\|_2 \geq \epsilon \) do
4. Set \( z_{i\cdot z_i} = z_i \).
5. for \( k = 1, \ldots, K \) do
6. Compute \( z_{ik} \) by subtracting the weighted group mean from \( z_{i(k-1)} \) as given in formula (3.4).
7. Set \( i = i + 1, z_i = z_{ik} \).
8. Set \( M^r v = z_i \).

The pseudo-demeaned variables are then used to compute the updates of the structural parameters:

\[
(\beta^r - \beta^{r-1}) = (\tilde{X}^{r-1'} \tilde{X}^{r-1})^{-1} \tilde{X}^{r-1'} \tilde{y}^{r-1}, \tag{3.5}
\]

where \( \tilde{X}^r = M^r \tilde{X}^r \) and \( \tilde{y}^r = M^r \tilde{y}^r \).

Since the gradient and Hessian at iteration \( r \) are functions of the linear predictor \( \eta^r = D \alpha^{r-1} + X \beta^{r-1} \), we additionally need to recover the fixed effects in each iteration of the IRLS routine. In the next subsection a memory efficient method is presented to update \( \eta^r \).

\footnote{In other words, the vector \( v \) is orthogonally projected on \( \tilde{D}_1 \), from there orthogonally projected on \( \tilde{D}_2 \) and so, until it is projected on \( \tilde{D}_K \). This procedure is repeated until convergence.}
3.3 Updating the Weights for IRLS

Although it would be possible to recover the fixed effects $\alpha^r_{k-1}$ in each iteration of the IRLS routine it is substantially less costly to directly recover the part of the linear predictor that corresponds to the fixed effects $D\alpha^r_{k-1}$.

Reconsider the reformulation of the IRLS update into the regression model

$$\tilde{y}^{r-1} = S^{r-1}(\alpha^r - \alpha^{r-1}) + \tilde{X}^{r-1}(\beta^r - \beta^{r-1}).$$

(3.6)

The first normal equation of system (3.6) is

$$S^{r-1}S^{r-1}(\alpha^r - \alpha^{r-1}) + S^{r-1}\tilde{X}^{r-1}(\beta^r - \beta^{r-1}) = S^{r-1}\tilde{y}^{r-1}.$$  

(3.7)

We naturally would like to compute

$$(\alpha^r - \alpha^{r-1}) = (S^{r-1}S^{r-1})^{-1}S^{r-1} (\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})).$$  

(3.8)

In general, $(S^rS^r)$ is not sparse, such that no straightforward formula could be derived that abstains from matrix calculations. Therefore, one could use numerical routines to solve the following system of equations:

$$S^{r-1}(\alpha^r - \alpha^{r-1}) = S^{r-1}(S^{r-1}S^{r-1})^{-1}S^{r-1} (\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})).$$

(3.9)

However, in this representation $b^{r-1}$ involves the costly computation of $S^{r-1}(S^{r-1}S^{r-1})^{-1}S^{r-1}$. Fortunately, we can apply a trick to reduce the computational cost dramatically. Noticing that $S^{r-1}(S^{r-1}S^{r-1})^{-1}S^{r-1} = I - M^{r-1}$, (3.9) can be rewritten

$$S^{r-1}(\alpha^r - \alpha^{r-1}) = (I - M^{r-1}) (\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})).$$

(3.10)

$$= (\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})) - (\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})), $$

where $b^{r-1}$ can be easily computed with already generated variables. Now one could use (3.10) to compute the fixed effects updates. Since we only need $D\alpha^r$ to compute $\eta^{r+1}$ in the next iteration, it is more efficient to compute $D\alpha^r$ instead of $\alpha^r$.

Noticing that $S^r = \tilde{W}^rD$, (3.10) can be rewritten into:

$$D(\alpha^r - \alpha^{r-1}) = (\tilde{W}^{r-1}\tilde{W}^{r-1})^{-1}\tilde{W}^{r-1}b^{r-1}$$

(3.11)

and finally

$$D\alpha^r = (\tilde{W}^{r-1})^{-1}b^{r-1} + D\alpha^{r-1}$$

(3.12)

Another benefit of this approach is that (3.12) does not require $D$ to have full column rank.

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8For example one could use a sparse solver like the Kaczmarz method (Kaczmarz 1937) (see section 4.2).

9It is possible to implement our pseudo-demeaning algorithm (algorithm 2) with or without removing collinear columns.
4 The General Framework of the Optimization Algorithm

4.1 Estimation of the Structural Parameters and the Standard Errors

Bringing together all previously mentioned components the IRLS $k$-way pseudo-demeaning algorithm can be summarized by the following pseudo-code:

**Algorithm 2 IRLS**

1: Initialize $\beta^1$ and $D\alpha^1$.
2: Set $r = 1$, $\beta^{r-1} = \beta^r - 1_p$, and a tolerance level $\epsilon$.
3: **while** $||\beta^r - \beta^{r-1}||_2 \geq \epsilon$ **do**
   4: Compute the required quantities for the gradient and Hessian (see table 2).
   5: Compute $\tilde{y}^{r-1}$ and $\tilde{X}^{r-1}$ (see formula (3.2)).
   6: Compute $\tilde{y}^{r-1}$ and $\tilde{X}^{r-1}$ using algorithm 1.
   7: Compute $(\beta^r - \beta^{r-1})$ (see formula (3.5)).
   8: Compute $b^{r-1}$ (see formula (3.10)).
   9: Compute $D\alpha^r$ (see formula (3.12)).
10: Set $\beta^r = \beta^{r-1} + (\beta^r - \beta^{r-1})$, $D\alpha^r = D\alpha^{r-1} + (D\alpha^r - D\alpha^{r-1})$, and $r = r + 1$.

Let $r^*$ denote all quantities after convergences of algorithm 2. In order to compute the standard-errors of the structural parameters $\beta^r$, we fortunately do not need the full Hessian (2.6). The estimated variance-covariance matrix corresponding to the structural parameters $\beta^r$ can be easily computed using the concentrated Hessian at convergence:

$$\hat{V} = \left(\tilde{X}^{r^*}\tilde{X}^{r^*}\right)^{-1}. \quad (4.1)$$

4.2 Recovering the Fixed Effects Ex-Post

In some rare cases, the researcher might not only require estimates of the structural parameters but also of the fixed effects. We propose a numerical routine, called Kaczmarz method (Kaczmarz 1937), which can be applied to all kinds of generalized linear models.

System (3.10) becomes

$$S^{r-1}(\alpha^r - \alpha^{r-1}) = \left(\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})\right) - \left(\tilde{y}^{r-1} - \tilde{X}^{r-1}(\beta^r - \beta^{r-1})\right). \quad (4.2)$$

Since we want to solve the system (1.2) for $\alpha^r$, we rearrange and arrive at

$$S^{r-1}\alpha^r = b^{r-1} + S^{r-1}\alpha^{r-1}$$

$$= b^{r-1} + \tilde{w}^{r-1}D\alpha^{r-1},$$

where $b$ can be computed at minimum computational cost from already generated variables.

An efficient way to solve the system (1.2) is the Kaczmarz method (Kaczmarz 1937) (a special case of AP). We successively project each equation of (1.2) on the subsequent equation and repeat the whole process until convergence. Each projection of the $i$-th equation on the $i + 1$-th equation can be summarized as follows:

$$\psi_{i+1} = \psi_i + \frac{(b_i - \langle s_i, \psi_i \rangle)}{||s_i||^2} s_i, \quad (4.4)$$

from $D$. Both would deliver identical estimates of the structural parameters.
where $\psi$ is a vector of length $l$, $s_i$ and $b_i$ denote the $i$-th row of $S_r^*-1$ and $b$ respectively, and $|| \cdot ||_2^2$ is the squared euclidean norm. The beauty of Kaczmarz is that we can solve the system with the rank deficient $S_r^*-1$. Thus, each row of $S_r^*-1$ contains $K$ times the identical weight $\tilde{w}_i$, such that the denominator can be simplified as follows

$$
\psi_{i+1} = \psi_{i} + \frac{(b_i - \langle s_i, \psi_{i} \rangle)}{K \tilde{w}_i^2} s_i.
$$

(4.5)

Since $S_r^*-1$ is sparse, the Kaczmarz updates can be computed at minimum memory.

The following algorithm summarizes the procedure:

**Algorithm 3 Kaczmarz**

1. Set $j = 1$, $\psi_j = 0_K$, $\psi_{j-1} = \psi_j - 1_K$, and tolerance level $\epsilon$.
2. while $||\psi_j - \psi_{j-1}||_2 \geq \epsilon$ do
3. Set $\psi_{j0} = \psi_j$.
4. for $i = 1, \ldots, n$ do
5. Compute $\psi_{ji}$ as given in formula (4.5).
6. Set $j = j + 1$, $\psi_j = \psi_{jn}$.
7. Set $\alpha^* = \psi_j$.

Note, that it is possible to solve (4.3) by using the rank deficiant matrix $S_r^*-1$. However, in order to get meaningful estimates for the fixed effects it is necessary to apply an estimable function to the solution (Gaure 2013b).

5 Simulation

To demonstrate the performance of our algorithm we consider two different simulation designs for which we measure the computation times: a two-way fixed effects logit model and a three-way (pseudo-) poisson model. All simulations were done with our R-package `alpaca` on a Linux workstation (Ubuntu 14.04, Intel Xeon CPU 8 cores, 2.8 GHz, 24 GB RAM) using R version 3.4.2 (R Core Team 2016).

Please note that the package is intended for applied work and thus involves parts of data cleaning and data-transformation. These steps would not be necessary for simulated data. Thus, the reported computation times do not purely measure the performance of the pseudo-demeaning algorithm itself.

At first we analyse a two-way fixed effects logit model and generate data according to:

$$
y_{it} = \mathbf{1}[x_{it} \mathbf{b} + \alpha_i + \gamma_t + \epsilon_{it} > 0],
$$

(5.1)

where $i = 1, \ldots, N$, $t = 1, \ldots, T$, $x_{itp}$ is generated as iid. standard normal with $p = 1, \ldots, 3$, and $\epsilon_{it}$ is an iid. logistic error term with location zero and scale one, $\alpha_i \sim$ iid. $\mathcal{N}(\sum_{p=1}^3 \bar{x}_{ip}, 1)$ and $\gamma_t \sim$ iid. $\mathcal{N}(\sum_{p=1}^3 \bar{x}_{ip}, 1)$ and $\mathbf{b} = [1, -1, 1]'$.

We consider different combinations of $N$ and $T$ and measure the average computation time for each combination over 10 replications, as well as the average computation time per iteration. Table 5 summarizes the computation times in seconds for different combinations of $N$ and $T$. Even in a dataset consisting of 10 million observations and including 11,000 fixed effects our routine is able to estimate the model in roughly 2 minutes.\(^{11}\)

\(^{10}\)The application of a poisson estimator to a model with a continuous dependent model is popular in trade economics and called pseudo-poisson (see section 6).

\(^{11}\)In the first version of this paper we derived a two-way algorithm based on the partitioned-inverse formula and clever
Table 2: Average Computation Time in Seconds

| N  | T  | n.obs | n.fe | time | time p.i. |
|----|----|-------|------|------|-----------|
| 250| 50 | 12,500| 300  | 0.10 | 0.02      |
| 250| 100| 25,000| 350  | 0.15 | 0.03      |
| 500| 50 | 25,000| 550  | 0.15 | 0.02      |
| 500| 100| 50,000| 600  | 0.31 | 0.05      |
| 500| 250| 125,000| 750 | 0.84 | 0.14      |
| 1,000| 50 | 50,000| 1,050| 0.30 | 0.05      |
| 1,000| 100| 100,000| 1,100| 0.64 | 0.11      |
| 1,000| 250| 250,000| 1,250| 1.68 | 0.28      |
| 1,000| 500| 500,000| 1,500| 3.92 | 0.65      |
| 5,000| 50 | 250,000| 5,050| 1.79 | 0.30      |
| 5,000| 100| 500,000| 5,100| 4.04 | 0.67      |
| 5,000| 250| 1,250,000| 5,250| 11.49| 1.91      |
| 5,000| 500| 2,500,000| 5,500| 26.72| 4.45      |
| 5,000| 1000| 5,000,000| 6,000| 55.15| 9.19      |
| 10,000| 50 | 500,000| 10,050| 3.91 | 0.65      |
| 10,000| 100| 1,000,000| 10,100| 8.93 | 1.49      |
| 10,000| 250| 2,500,000| 10,250| 25.99| 4.33      |
| 10,000| 500| 5,000,000| 10,500| 58.91| 9.82      |
| 10,000| 1,000| 10,000,000| 11,000| 122.92| 20.49     |

Note: time denotes the average computation time in seconds based on 10 datasets according to the DGP \[5.1]\]. time p.i. denotes the average computation time per iteration in seconds based on 10 datasets according to the DGP \[5.1]\]. n.obs denotes the number of observations, n.fe denotes the number of fixed effects.

For the second experiment we consider a three-way fixed effects pseudo-poisson model \[12\]. The data are generated according to

\[
Y_{ijt} = \exp(\alpha_{it} + \gamma_{jt} + \delta_{ij} + x_{ijt}\beta_1 + d_{ijt}\beta_2) \cdot \epsilon_{ijt}, \tag{5.2}
\]

where \(i = 1, \ldots, n, j = 1, \ldots, n, t = 1, \ldots, T\), \(x_{ijt}\) is generated as iid. standard normal, \(d_{ijt} = 1[\psi_{ijt} > 0]\), with \(\psi_{ijt}\) is iid. standard normal, and \(\epsilon_{ijt}\) is an iid. log-normal error-term with mean zero and variance one, \(\alpha_{it} \sim \text{iid. } \mathcal{N}(\bar{x}_{it}, 1)\), \(\gamma_{jt} \sim \text{iid. } \mathcal{N}(\bar{x}_{jt}, 1)\), \(\delta_{ij} \sim \text{iid. } \mathcal{N}(\bar{x}_{ij}, 1)\), and \(\beta_1 = \beta_2 = 1\). \(\bar{x}\) denote the corresponding group means. Note that a dataset with \(N\) and \(T\) generated according to the \[5.2\] has \(N \cdot (N - 1) \cdot T\) observations and \((2 \cdot N \cdot T) + (N \cdot (N - 1))\) fixed effects whereof the first category has \(N \cdot (N - 1)\) levels and the second and third each have \(N \cdot T\) levels.

Table 7 summarizes the total average computation times and the average computation time per iterations over 10 replications in seconds for different combinations of \(N\) and \(T\). The enormous model matrix-arrangements (https://arxiv.org/abs/1707.01815v1). This algorithm is a direct extension of an algorithm by Somaini and Wolak (2016) for linear two-way fixed models to generalized linear models. We found that this algorithm has a negligibly small speed advantage compared to our algorithm based on alternating projections if the two category variables have rather few levels. However, when increasing the levels our algorithm based on alternating projections outperformed this approach substantially. Another disadvantage on using the partitioned-inverse approach is, that the sub-matrices need to be invertible and thus, collinearities have to be accounted for. In contrast, our algorithm is able the fit models, where collinearities between the fixed effects exist. Further, we found that both algorithms as well as the naive dummy variable approach, deliver identical parameter estimates and standard-errors.

\[12\]The term pseudo-poisson is explained in more detail in the empirical application (see section \[6\]).
with 1.99 million observations and 59,800 fixed effects takes roughly 18 minutes.

In the next section we demonstrate a possible area of application for our pseudo-dememeaning algorithm.

| N  | T  | n.obs | n.fe | time | time p.i. |
|----|----|-------|------|------|-----------|
| 10 | 5  | 450   | 190  | 0.16 | 0.01      |
| 10 | 10 | 900   | 290  | 0.15 | 0.01      |
| 10 | 25 | 2,250 | 590  | 0.77 | 0.05      |
| 10 | 50 | 4,500 | 1,090| 1.73 | 0.08      |
| 25 | 5  | 3,000 | 850  | 2.37 | 0.12      |
| 25 | 10 | 6,000 | 1,100| 2.29 | 0.11      |
| 25 | 25 | 15,000| 1,850| 4.48 | 0.24      |
| 25 | 50 | 30,000| 3,100| 10.55| 0.59      |
| 50 | 5  | 12,250| 2,950| 2.80 | 0.18      |
| 50 | 10 | 24,500| 3,450| 5.52 | 0.29      |
| 50 | 25 | 61,250| 4,950| 22.78| 1.42      |
| 50 | 50 | 122,500| 7,450| 28.10| 1.76      |
| 100| 5  | 49,500| 10,900| 39.99| 1.90      |
| 100| 10 | 99,000| 11,900| 21.78| 1.36      |
| 100| 25 | 247,500| 14,900| 69.00| 3.83      |
| 100| 50 | 495,000| 19,900| 197.97| 13.20    |
| 200| 5  | 199,000| 41,800| 135.72| 7.54     |
| 200| 10 | 398,000| 43,800| 194.62| 11.45    |
| 200| 25 | 995,000| 49,800| 481.62| 28.33    |
| 200| 50 | 1,990,000| 59,800| 1071.13| 66.95   |

Note: time denotes the average computation time in seconds based on 10 datasets according to the DGP \(5.2\). time p.i. denotes the average computation time per iteration in seconds based on 10 datasets according to the DGP \(5.2\). n.obs denotes the number of observations, n.fe denotes the number of fixed effects.

6 Empirical Example

In trade economics many empirical gravity models are estimated with the pseudo-poisson maximum likelihood estimator.\(^{13}\) A standard gravity model takes the following form

\[
Y_{ijt} = \exp(\alpha_{it} + \alpha_{jt} + \alpha_{ij} + x_{ijt}' \beta + \epsilon_{ijt},
\]

(6.1)

where \(Y_{ijt}\) denotes the trade flows from exporter \(i\) to importer \(j\) at time \(t\), \(\alpha_{it}\) is an exporter-time fixed effect, \(\alpha_{jt}\) is an importer-time fixed effect and \(\alpha_{ij}\) is an exporter-importer (dyadic) fixed effect, \(x_{ijt}\) is a vector of further regressors and \(\beta\) the corresponding parameter vector. For model specifications, where the three fixed effects are high-dimensional, researchers face computational limits. For example Glick and Rose (2016) used a panel dataset with over 200 countries trading for 65 years. In this case, the dataset consists of 879,794 observations and a three-way fixed effects specification would require roughly 11,000 exporter-time and importer-time fixed effects, respectively, as well as roughly 34,000

\(^{13}\)The estimator actually is a poisson-maximum likelihood estimator, but applied to a non-poisson distributed dependent variable (trade flows are positive and continuous). See Gouriouex, Monfort and Trognon (1984), why this application is valid.
dyadic fixed effects. Due to computational limits Glick and Rose (2016) estimated a three-way fixed effects log-linear, instead of the PPML counterpart. However, Glick and Rose (2016) suspected that a log-linear model with three-way fixed effects should give similar results as the PPML counterpart.

For the estimation of the log-linear three-way model they used the Stata command `hdfe` by Correia (2016). An estimation routine to estimate the PPML three-way model did not exist at that time. Recently, Larch, Wanner, Yotov and Zyklin (2017) proposed an iterative PPML estimator based on the algorithm of Guimaraes et al. (2010), which can handle high-dimensional three-way fixed effects. With this tool at hand, they are able to estimate the model of Glick and Rose (2016) with the full set of fixed effects. We use the same dataset to show, that our algorithm is an alternative to the one of Larch et al. (2017).

Glick and Rose (2016) estimate the following theory-consistent gravity model

$$\log X_{ijt} = \gamma CU_{ijt} + Z_{ijt}' \beta + \lambda_{it} + \psi_{jt} + \epsilon_{ijt}, \quad (6.2)$$

where $X_{ijt}$ denotes the nominal value of bilateral exports from exporter $i$ to importer $j$ at year $t$, $CU_{ijt}$ is a dummy variable, specifying whether $i$ and $j$ use the same currency at time $t$, $Z_{ijt}$ are further control variables, $\lambda_{it}$ denotes a time-varying exporter fixed effect, and $\psi_{jt}$ a time-varying importer fixed effect.

They also estimate a specification, where some of the control variables are replaced by dyadic fixed effects $\delta_{ij}$:

$$\log X_{ijt} = \gamma CU_{ijt} + Z_{ijt}' \beta + \lambda_{it} + \psi_{jt} + \delta_{ij} + \epsilon_{ijt}. \quad (6.3)$$

Using our k-way pseudo-demeaning algorithm we can check whether the three-way log-linear model (6.3) and a three-way PPML model deliver similar results as suspected by Glick and Rose (2016). We use the following PPML model specification

$$X_{ijt} = \exp(\gamma CU_{ijt} + Z_{ijt}' \beta + \lambda_{it} + \psi_{jt} + \delta_{ij}) + u_{ijt}. \quad (6.4)$$

Table 7 reproduces parts of table 5 from Glick and Rose (2016) and opposes the results with the estimates of the pseudo-poisson estimator. The left panel covers the models with importer-time and exporter time fixed effects and the right panel shows the models with additional dyadic fixed effects. The results point out, that a log-linear model and a PPML model, both with three-way fixed effects, deliver different results and thus contradict Glick and Rose (2016). For a detailed discussion of this finding please consult Larch et al. (2017). Depending on the model specification, we are able to estimate the three-way PPML in 12 to 30 minutes. However, the choice of better starting values

\[\text{In a working paper version of 2015, Glick and Rose (2016) have been able to estimate a PPML model with exporter-time and importer-time fixed effects for the dataset on a smaller horizon (1960 – 2013). However, they don’t take the resulting estimates as serious, since they probably suffer from endogeneity of the CU’s, which could be captured by dyadic fixed effects.}\]

\[\text{Basically, they extended the two-way Poisson algorithm based on Gauss-Seidel (Guimaraes et al. 2010) to three-way and put the closed form of the fixed effects inside of the IRLS routine, whereas Guimaraes et al. (2010) use it outside. Both routines are only approximations of the theoretical IRLS steps, since both do not pseudo-demean the variables. Therefore, both routines need substantially more outer iterations until convergence compared to our pseudo-demeaning algorithm. However, it is unclear in which cases which routine has a lower computation time. An advantage of our pseudo-demeaning algorithm is that our IRLS optimization directly delivers the variance-covariance of the structural parameters, whereas the routines based on Gauss-Seidel need to do some post estimations. Besides our algorithm is more flexible since it is not limited to poisson models and can handle k-way fixed effects.}\]

\[\text{We follow the suggestion of practitioners and estimate the model with a scaled dependent } X_{ijt}/\bar{X} \text{ variable, where } \bar{X} \text{ denotes the average.}\]

\[\text{For the estimation of the log-linear models we applied the R-Package lfe developed by Gaure (2013a).}\]
probably will decrease the number of required iterations and thus the computation time.

7 Conclusion

We present a new algorithm for the estimation of generalized linear models with high-dimensional 
$k$-way fixed effects models.

The algorithm incorporates a special pseudo-demeaning procedure into the iteratively reweighted
least squares estimation routine such that the updates of the structural parameters are separated from
the high-dimensional fixed effects updates.

The pseudo-demeaning algorithm delivers identical estimates as the naive maximum likelihood esti-
mation routine which includes dummy variables to control for the unobserved heterogeneity.

Whereas the dummy variable approach quickly runs into memory limits, which slows down the
computation time considerably or even makes the computation infeasible, our algorithm offers new
possibilities and reliefs to researchers. Especially in the application of gravity model estimation we
consider our algorithm to be beneficial.
Table 4: Empirical Results

|                                   | Exporter × year, importer × year | Exporter × year, importer × year, dyadic |
|-----------------------------------|----------------------------------|-----------------------------------------|
|                                   | All CUs  | Disagg. EMU | Disagg. CUs | All CUs  | Disagg. EMU | Disagg. CUs |
| All Currency Unions               |         |             |             |         |             |             |
| OLS                               | 0.51    | 0.34        |             |         |             |             |
| PPML                              | -0.13   | 0.30        |             |         |             |             |
| All Non-EMU Currency Unions       |         |             |             |         |             |             |
| OLS                               | 0.76    | -0.65       | -0.63       | 0.70    | -0.65       | -0.63       |
| PPML                              | 0.22    | 0.19        | 0.19        | 0.70    | 0.22        | 0.19        |
| EMU                               |         |             |             |         |             |             |
| OLS                               | -0.65   | 0.43        | 0.43        | -0.63   | 0.43        | 0.43        |
| PPML                              | 0.20    | 0.03        | 0.03        | 0.19    | 0.03        | 0.03        |
| CFA Franc Zone                    |         |             |             |         |             |             |
| OLS                               | 0.74    |             |             |         |             |             |
| PPML                              | 0.53    |             |             |         |             |             |
| East Caribbean Currency Union     |         |             |             |         |             |             |
| OLS                               | 1.83    | 0.43        |             |         |             |             |
| PPML                              | 0.72    | 0.43        |             |         |             |             |
| Aussie                            |         |             |             |         |             |             |
| OLS                               | 1.27    |             | 0.39        |         |             |             |
| PPML                              | 1.19    |             | 0.39        |         |             |             |
| British                           |         |             |             |         |             |             |
| OLS                               | 0.26    | 0.55        |             |         |             |             |
| PPML                              | 1.02    | 0.55        |             |         |             |             |
| French Franc                      |         |             |             |         |             |             |
| OLS                               | 1.84    | 0.87        |             |         |             |             |
| PPML                              | 2.41    | 2.10        |             |         |             |             |
| Indian Ruppee                     |         |             |             |         |             |             |
| OLS                               | 0.03    | 0.52        |             |         |             |             |
| PPML                              | -0.37   | 0.52        |             |         |             |             |
| US $                              |         |             |             |         |             |             |
| OLS                               | 0.02    |             | -0.05       |         |             |             |
| PPML                              | -0.35   |             | -0.05       |         |             |             |
| Other CUs                         |         |             |             |         |             |             |
| OLS                               | 1.37    |             | -0.10       |         |             |             |
| PPML                              | 0.12    |             | -0.10       |         |             |             |
| Importer-time fixed effects       | 11,277  | 11,277      | 11,277      | 11,277  | 11,277      | 11,277      |
| Exporter-time fixed effects       | 11,227  | 11,227      | 11,227      | 11,227  | 11,227      | 11,227      |
| Dyadic fixed effects              |         |             |             |         |             |             |
|                                   | 34,104  | 34,104      | 34,104      |         |             |             |
| PPML Information                  |         |             |             |         |             |             |
| time (in sec.)                    | 48      | 56          | 99          | 748     | 1804        | 1148        |
| time per iteration (in sec.)      | 7       | 8           | 11          | 75      | 139         | 96          |
| iterations                        | 7       | 7           | 9           | 10      | 13          | 12          |

Note: The left panel shows the estimates using exporter-time and importer-time fixed effects, the right panel shows the estimates using importer-time, exporter-time and dyadic fixed effects. OLS denotes the log-linear specifications following (Glick and Rose 2016). PPML denotes the PPML counterparts.
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