Abstract—A strong converse bound for constant composition codes of the form $P_e(n) \geq 1 - A n^{-0.5(1-E'_sc(R,W,p))} e^{-nE_{sc}(R,W,p)}$ is established using the Berry-Esseen theorem through the concepts of Augustin information and mean, where $A$ is a constant determined by the channel $W$, the composition $p$, and the rate $R$, i.e., $A$ does not depend on the block length $n$.

I. INTRODUCTION

On discrete stationary product channels, the error probability of codes operating at rates above capacity is not only bounded away from zero but also converging to one. This property, first observed by Wolfowitz [1], is called the strong converse property. For arbitrary stationary product channels, a necessary and sufficient condition for the strong converse property was determined by Augustin [2, §10], [3, §13]. The strong converse property does not hold in general; nevertheless there does exist a universal asymptotic constant that bounds the error probability of codes operating at rates above the capacity of the stationary product channels, according to Beck and Csiszar [4]. In [5], Verdù and Han provided a necessary and sufficient condition for the strong converse property for channels that are not necessarily stationary or memoryless.

Using the concept of Rényi capacity, which was employed earlier by Gallager [6] for analyzing the error probability of codes operating at rates below the channel capacity, Arimoto established in [7] the following lower bound to the error probability of codes on discrete stationary product channels (DSPCs) operating at a rate $R$ above the capacity:

$$P_e(n) \geq 1 - e^{-nE_{sc}(R)}$$

where $E_{sc}(\cdot)$ is the strong converse exponent of the channel. Although Arimoto’s initial proof in [7] is for DSPCs, Arimoto’s lower bound can be proved as a one shot bound for more general channel models using Jensen’s inequality or Hölders inequality as noted by Augustin [3] and Sheverdyaev [8], see also [9]–[11]. Arimoto’s lower bound is used to establish strong converse on channels for which alternative derivations of the strong converse is much more tedious, e.g., quantum channels discussed in [11]–[21] and Poisson channels mentioned in [9, Appendices B-B and B-C].

Arimoto’s lower bound to the error probability has been derived for certain constrained codes on memoryless channels as well, see Dueck and Körner [22] for the constant composition codes on DSPCs, Oohama [23] for the Gaussian channel, and Cheng et al. [21] and Mosonyi and Ogawa [24] for the constant composition codes on classical-quantum channels. For codes on DSPCs Omura [25] has shown

$$\lim_{n \to \infty} \frac{1}{n} \ln(1-P_e(n)) \leq E_{sc}(R)$$

for all rates above the channel capacity and below a certain threshold. Dueck and Körner [22] established (2) for all rates above the channel capacity. Thus Arimoto’s bound is tight, in terms of the exponential decay rate of the probability of correct decoding with block length, for all rates above the channel capacity. An analogous result is derived for constant composition codes on DSPCs in [22], for the Gaussian channel in [23], for classical-quantum channels in [18], for classical data compression with quantum side information in [20], and for constant composition codes on classical-quantum channels in [24].

Although Arimoto’s bound, given in (1), is tight in terms of the exponential decay rate of the correct decoding probability with block length, the prefactor multiplying the exponentially decaying term can be improved. In particular, for the constant composition codes operating at rates larger than the mutual information of the composition. Theorems 1 and 2, in the following, establish a strong converse bound of the form

$$P_e(n) \geq 1 - A n^{-0.5(1-E'_sc(R))} e^{-nE_{sc}(R)}$$

where $E_{sc}(\cdot)$ is the strong converse exponent and $E'_sc(\cdot)$ is its derivative with respect to the rate. Since $1 \geq E'_{sc}(R) \geq 0$ for all rates $R$ and $E_{sc}(R) < 1$ for small enough rates $R$, the bound (3) improves (1) strictly. In accordance with the corresponding improvements of the sphere packing bound for rates below the channel capacity given in [26]–[30], we call the bounds of the form given in (3) refined strong converses.

Proof of Theorem 1 is analogous to the proof of refined sphere packing bound presented in [29], [30]; it relies on a tight characterization of the trade-off between type-I and type-II error probabilities in the hypothesis testing problem with (possibly non-stationary) independent samples through the concepts of Augustin information and mean. However, in [29], [30] for the regime of interest the optimal tilting parameter is between zero and one; whereas we are now...
interested in the regime where the optimal tilting parameter is larger than one. Similarly, in [29], [30] Agustin information measures for orders between zero and one are used together with the sphere packing exponent, whereas we employ Agustin information measures for orders larger than one together with the strong converse exponent in our analysis.

We conclude this section with an overview of the paper. In §II, we describe our model and notation. In §III, we employ the concept of tilted probability measure and the Berry-Esseen theorem to obtain a lower bound on the type-II error probability in hypothesis testing problem with independent—but not necessarily identically distributed—samples for the regime where the optimal tilting parameter is larger than one. In §IV, we review Agustin’s information measures and the strong converse exponent. In §V, we establish a refined strong converse for the constant composition codes on stationary memoryless channels. We conclude our presentation with a brief discussion of the results and future work in §VI.

II. MODEL AND NOTATION

We denote the set of all probability mass functions that are positive only for finitely many elements of $\mathcal{X}$ by $\mathcal{P}(X)$ and the set of all probability measures on a measurable space $(\mathcal{Y}, \mathcal{Y})$ by $\mathcal{P}((\mathcal{Y})$. Expected value and variance of a measurable function $f$ under the probability measure $\mu$ are denoted by $E_{\mu}[f]$ and $V_{\mu}[f]$. Cartesian product of sets $X_1, \ldots, X_n$ and product of $\sigma$-algebras $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$ are denoted by $X_1 \times \cdots \times X_n$ and $\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$, respectively. The symbol $\otimes$ is used to denote both products of $\sigma$-algebras and products of measures.

A channel $W$ is a function from the input set $X$ to the set of all probability measures on the output space $(\mathcal{Y}, \mathcal{Y})$: $W : X \rightarrow \mathcal{P}(\mathcal{Y})$. If $X$ and $\mathcal{Y}$ are both finite sets, then $W$ is a discrete channel. The product of $W_t : X_t \rightarrow \mathcal{P}(\mathcal{Y}_t)$ for $t \in \{1, \ldots, n\}$ is a channel of the form $W_{[1,n]} : X_{[1,n]} \rightarrow \mathcal{P}(\mathcal{Y}_{[1,n]})$ satisfying

$$W_{[1,n]}(x^n) = \bigotimes_{t=1}^{n} W_t(x_t) \quad \forall x^n \in X_{[1,n]}.$$  \(5\)

Any channel obtained by curtailling the input set of a length $n$ product channel is called a length $n$ memoryless channel. A product channel $W_{[1,n]}$ is stationary iff $W_t = W$ for all $t's$ for some $W$. On a stationary channel, we denote the composition (i.e. the empirical distribution, the type) of each $x^n_t$ by $\mathcal{T}(x^n_t)$; thus $\mathcal{T}(x^n_{[1,n]}) \in \mathcal{P}(\mathcal{X})$.

An $(M, L)$ channel code on $W_{[1,n]}$ is composed of an encoding function $\Psi$ from the message set $M \triangleq \{1, 2, \ldots, M\}$ to the input set $X_{[1,n]}$ and a $\mathcal{Y}_{[1,n]}$-measurable decoding function $\Theta$ from the output set $\mathcal{Y}_{[1,n]}$ to $M \triangleq \{L : L \subset M \text{ and } |L| \leq L\}$. For any channel code $(\Psi, \Theta)$ on $W_{[1,n]}$, the conditional error probability $P_e^m$ for $m \in M$ and the average error probability $P_e$ are given by

$$P_e^m \triangleq E_{W_{[1,n]}(\Psi(m))} \left[ \mathbb{I}_{\{m \notin \Theta(\psi_Y)\}} \right],$$

$$P_e \triangleq \frac{1}{M} \sum_{m \in M} P_e^m.$$

A channel code is a constant composition code iff all of its codewords have the same composition, i.e. $\exists \psi \in \mathcal{P}(\mathcal{X})$ such that $\mathcal{T}(\psi(m)) = \psi, \forall m \in M$.

III. HYPOTHESIS TESTING PROBLEM, TILTED PROBABILITY MEASURE, AND BERRY ESSEEN THEOREM

Our main aim in this section is to characterize the trade-off between type-I and type-II error probabilities using the Berry-Esseen theorem and the tilted probability measure. This trade-off can be studied in various regimes; in order to specify the regime of interest, let us first recall the definition of Rényi divergence and define the tilted probability measure.

Definition 1. For any $\alpha \in \mathbb{R}_+$ and $w, q \in \mathcal{P}(\mathcal{Y})$, the order $\alpha$ Rényi divergence between $w$ and $q$ is

$$D_{\alpha}(w || q) \triangleq \begin{cases} \frac{1}{\alpha - 1} \ln \int \left( \frac{d\mu_{\alpha}}{dx_{\alpha}} \right)^{1-\alpha} \nu(dy) & \alpha \neq 1 \\ \int \frac{d\mu_{\alpha}}{dx_{\alpha}} \ln \frac{d\mu_{\alpha}}{dx_{\alpha}} - \ln \frac{d\nu}{d\mu_{\alpha}} \nu(dy) & \alpha = 1 \end{cases}$$

where $\nu$ is any measure satisfying $w < \nu$ and $q < \nu$.

Definition 2. For any $w, q \in \mathcal{P}(\mathcal{Y})$, let $w_{ac}$ be the component of $w$ that is absolutely continuous in $q$. If $\|w_{ac}\| = 0$, then the order 1 tilted probability measure $w_1^\alpha$ is

$$w_1^\alpha \triangleq \int w_{ac} \frac{d\nu}{d\mu_{\alpha}}.$$
Lemma 1. For any $\alpha \in (1, \infty)$, $n \in \mathbb{Z}_+$, $w_t, q_t \in \mathcal{P}(\mathcal{Y}_t)$, let $w_{t,ac}$ be the component of $w_t$ that is absolutely continuous in $q_t$ and let $a_2, a_3,$ and $\Delta$ be

$$a_2 \triangleq \frac{1}{n} \sum_{t=1}^{n} E_{w_{t,ac}} \left[ \left( \ln \frac{dw_u}{dq_u} \right) - E_{w_{t,ac}} \left[ \ln \frac{dw_u}{dq_u} \right] \right]^2,$$

$$a_3 \triangleq \frac{1}{n} \sum_{t=1}^{n} E_{w_{t,ac}} \left[ \left( \frac{\ln \frac{dw_u}{dq_u}}{\frac{dw_u}{dq_u}} \right) - E_{w_{t,ac}} \left[ \frac{\ln \frac{dw_u}{dq_u}}{\frac{dw_u}{dq_u}} \right] \right]^3,$$

$$\Delta \triangleq \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\sqrt{a_2}} + 2 \sqrt{a_3/a_2} \right],$$

where $w = \otimes_{t=1}^{n} w_t$ and $q = \otimes_{t=1}^{n} q_t$. Then for any $E \in \mathcal{Y}_n$ and $\beta \in \mathbb{R}_+$ satisfying $q(E) \leq \beta e^{-D(w_t||q)}$, we have

$$w(Y^n \setminus E) \geq \prod_{t=1}^{n} \left[ \left| w_{t,ac} \right| - \frac{2 \alpha^n - e^{-D(w_t||w)}}{\left( \alpha - 1 \right)^{1/\alpha}} \right] \prod_{t=1}^{n} \left[ \left| w_{t,ac} \right| \right].$$

(6)

Thus the lower bound asserted in Lemma 1 is tight. We, however, skip the formal statement and the proof of this matching achievability result for Lemma 1 for brevity because it is not needed for establishing the strong converse.

One can calculate exact asymptotic value of the constant in the trade-off between error probabilities in the hypothesis testing problem in this regime, under stronger hypothesis. For the stationary case — i.e. the case when $w_t = w_1$, $q_t = q_1$ for all $t$ — assuming $w \sim q$, first Csiszar and Longo [33] and more recently Vazquez-Vilar et al. [34] have discussed this problem.

Proof of Lemma 1. Let the random variables $\xi_t$ and $\xi$ be

$$\xi_t \triangleq \ln \frac{dw_t}{dq_t},$$

$$\xi \triangleq \sum_{t=1}^{n} \xi_t.$$

Then $\xi = \ln \frac{dw_u}{dq_u}$, and hence $\xi_t = \ln \frac{dw_t}{dq_t}$, holds q-a.s., and the Radon-Nikodym derivatives $\frac{dw_t}{dq_t}$ and $\frac{dw_u}{dw_t}$ can be expressed in terms of $\xi$ as follows

$$\ln \frac{dw_u}{dq_u} = D_t(w_u||q) + a \xi - E_{w_{t,ac}}[\xi],$$

$$\ln \frac{dw_u}{dw_t} = D_t(w_u||w) + (\alpha - 1) \xi - E_{w_{t,ac}}[\xi].$$

(8)

For each constant $\kappa$, let the set $B_{\kappa}$ be

$$B_{\kappa} \triangleq \left\{ y^n \in \mathcal{Y}_n : \tau + \kappa \leq \xi - E_{w_{t,ac}}[\xi] < \tau + \kappa + 1 \right\}.$$

Then for any $E \in \mathcal{Y}_n$ and $\kappa \in \mathbb{Z}$, we can bound $w_{\kappa}^n(E \cap B_{\kappa})$ from above in terms of $q(E \cap B_{\kappa})$ using (8) and from below in terms of $w(E \cap B_{\kappa})$ using (9), as follows

$$w_{\kappa}^n(E \cap B_{\kappa}) \leq \frac{\frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\sqrt{a_2}} + 2 \sqrt{a_3/a_2} \right]}{(\alpha - 1)^{1/\alpha}} q(E \cap B_{\kappa}),$$

(10)

In order bound $w(Y^n \setminus E)$ we use $w(Y^n \setminus E) \geq w(B_{\kappa} \setminus E)$ and $w(B_{\kappa} \setminus E) = w(B_{\kappa}) - w(E \cap B_{\kappa})$ where $B_{\kappa} \triangleq U_{\kappa} \in \mathcal{Y}_n$. First note that for any family of reference measures $\{\nu_t\}$ satisfying $w_{\nu_t} < w_{\nu_t}$ for all $t$ we have

$$w(B_{\kappa}) = \left( \prod_{t=1}^{n} w_{\nu_t} \right) \left( \left\{ y^n : \frac{dw_{\nu_t}}{dq_{\nu_t}} > 0 \right\} \right).$$

(11)

In order to bound $w(E \cap B_{\kappa})$ we use (10), (11), the identity $q(E \cap B_{\kappa}) \leq q(E)$, the hypothesis $q(E) \leq \beta e^{-D(w_t||q)}$, and the formula for the sum of a geometric series:

$$w(E \cap B_{\kappa}) = \sum_{\kappa \in \mathbb{Z}_n} w(E \cap B_{\kappa}) \leq \sum_{\kappa \in \mathbb{Z}_n} \frac{\beta e^{\tau + \kappa + 1 - D(w_t||w)}}{\beta e^{\tau + \kappa + 1 - D(w_t||w)}},$$

(12)

On the other hand $\xi_t$’s are jointly independent under $w_{\kappa}$. Thus the Berry-Esseen theorem [35]–[37] implies

$$w_{\kappa}^n(\mathcal{B}_\kappa) \leq \Phi \left( \frac{\tau + \kappa + \beta - 1}{\sqrt{\alpha n}} \right) + \frac{2 \sqrt{\alpha n}}{\sqrt{\alpha n}} \Phi \left( \frac{\tau + \beta - \kappa - 1}{\sqrt{\alpha n}} \right) \leq e^{-\Delta_n^{-1/\alpha}}.$$

(13)

Thus we can bound $w(E \cap B_{\kappa})$ for using (11) and the formula for the sum of a geometric series, as well:

$$w(E \cap B_{\kappa}) = \sum_{\kappa \in \mathbb{Z}_n} w(E \cap B_{\kappa}) \leq \sum_{\kappa \in \mathbb{Z}_n} e^{-\Delta_n^{-1/\alpha} - D(w_t||w) + (\alpha - 1) \tau - (\alpha - 1) \kappa} \leq e^{-\Delta_n^{-1/\alpha} - D(w_t||w) + (\alpha - 1) \tau - (\alpha - 1) \kappa}.$$

(14)

IV. AUGUSTIN INFORMATION, AUGUSTIN MEAN AND THE STRONG CONVERSE EXPONENT

Our primary goal in this section is to define the Augustin information and mean and the strong converse exponent and review those properties of them that will be useful in our analysis. Let us start by defining the conditional Rényi divergence:
Definition 3. For any $\alpha \in \mathbb{R}^+$, $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, $q \in \mathcal{P}(\mathcal{Y})$, and $p \in \mathcal{P}(\mathcal{X})$ the order $\alpha$ conditional Rényi divergence for the input distribution $p$ is

$$D_\alpha(W \| q \| p) = \sum_{x \in \mathcal{X}} p(x) D_\alpha(W(x) \| q) .$$

Definition 4. For any $\alpha \in \mathbb{R}^+$, $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, and $p \in \mathcal{P}(\mathcal{X})$ the order $\alpha$ Augustin information for the input distribution $p$ is

$$I_\alpha (p; W) = \inf_{q \in \mathcal{P}(\mathcal{Y})} D_\alpha(W \| q \| p) .$$

The infimum in (16) is achieved by a unique probability measure $q_{\alpha, p}$, called the order $\alpha$ Augustin mean for the input distribution $p$, by [31, Lemma 13-(b,c,d)]. Furthermore,

$$D_\alpha \inf (q_{\alpha, p} \| q) \geq D_\alpha(W \| q \| p) - I_\alpha (p; W) \geq D_\alpha \inf (q_{\alpha, p} \| q) \geq D_\alpha \inf (q_{\alpha, p} \| q) \quad (17)$$

for all $q \in \mathcal{P}(\mathcal{Y})$ by [31, Lemma 13-(b,c,d)], as well. $I_\alpha (p; W)$ is continuously differentiable in $\alpha$ on $\mathbb{R}^+$ and

$$\frac{d}{d\alpha} I_\alpha (p; W) = \left\{ \begin{array}{ll}
& \frac{D_\alpha(W^{\alpha qo} \| W \| p)}{\kappa_{\alpha}} \quad \alpha \neq 1 \\
& \frac{d}{d\alpha} (W^{\alpha qo} \| W \| p) \quad \alpha = 1
\end{array} \right.$$

by [31, Lemma 17-(e)], where $W^{\alpha qo} = \alpha$ is the order $\alpha$ tilted probability measure between $W(q, p)$ and $W(a, p)$. $W^{\alpha qo} \| W$ is called the order $\alpha$ tilted channel for the channel $W$ and the output distribution $q_{\alpha, p}$. The tilted channel is also used to express $I_\alpha (p; W)$ in terms of the Kullback-Leibler divergences in [31, Lemma 13-(e)]:

$$I_\alpha (p; W) = \frac{1}{\alpha - 1} D_1(W^{\alpha qo} \| W \| p) + I_1 (p; W^{\alpha qo} \| W) .$$

(19)

Since $\sum_{x \in \mathcal{X}} p(x) W^{\alpha qo} = q_{\alpha, p}$ by [31, Lemma 13-(b,c,d)], we also have the following identity for all $\alpha \in \mathbb{R}^+$

$$I_1 (p; W^{\alpha qo} \| W) = D_1(W^{\alpha qo} \| q_{\alpha, p} \| p) .$$

(20)

A more comprehensive discussion of Augustin’s information measures can be found in [31].

Definition 5. For any $W : \mathcal{X} \to \mathcal{P}(\mathcal{Y})$, $p \in \mathcal{P}(\mathcal{X})$, and $R \in \mathbb{R}^+$, the strong converse exponent (SCE) is

$$E_{sc}(R, W, p) \triangleq \sup_{\alpha \in \{1, \infty\}} \frac{1}{\alpha - 1} (I_\alpha (p; W) - R) .$$

(21)

We can apply the derivative test to determine $E_{sc}(R, W, p)$, because $I_\alpha (p; W)$ is continuously differentiable in the order $\alpha$ by [31, Lemma 17-(e)]. Equations (18) and (19) imply

$$\frac{d}{d\alpha} \frac{1}{\alpha - 1} (I_\alpha (p; W) - R) = \frac{1}{\alpha - 1} (R - I_1 (p; W^{\alpha qo} \| W)) .$$

(22)

On the other hand, either $I_1 (p; W^{\alpha qo} \| W)$ is increasing and continuous in $\alpha$ on $\mathbb{R}^+$, or $I_1 (p; W^{\alpha qo} \| W) = I_1 (p; W)$ for all positive $\alpha$ by [31, Lemma 17-(f)]. Furthermore, $I_1 (p; W^{\alpha qo} \| W)$ is equal to $I_1 (p; W)$. Thus for any rate $R$ in $(I_1 (p; W), \lim_{\alpha \to \infty} I_1 (p; W^{\alpha qo} \| W))$, there exists an order $\alpha^*$ in $(1, \infty)$ satisfying

$$R = I_1 (p; W^{\alpha^* qo} \| W) .$$

(23)
V. THE REFINED STRONG CONVERSE

**Theorem 1.** For any \( W : \mathcal{X} \to \mathcal{P}(Y) \), \( M, L, n \in \mathbb{Z}_+ \), \( p \in \mathcal{P}(\mathcal{X}) \) satisfying \( I_1(p; W) < \frac{1}{n} \ln \frac{1}{\beta} < \lim_{n \to \infty} I_1(p; W; q_{0}^{c}) \) and \( np(x) \in \mathbb{Z}_{\geq 0} \) for all \( x \in \mathcal{X} \), the order \( \alpha^* \triangleq \frac{1}{1-1/n} \) satisfies

\[
I_1(p; W; q_{0}^{c}) = \frac{1}{n} \ln \frac{1}{\beta}.
\]  

(28)

Furthermore, any \((M, L)\) channel code of length \( n \) whose codewords all have the same composition \( p \) satisfies

\[
P_e^{(n)} \geq 1 - 2e^{\alpha^* \left( \frac{\beta}{n} - \frac{1}{n} \right)^{\frac{1}{\alpha^*}} - nE_{\alpha}(\frac{1}{n} \ln \frac{1}{\beta}, W; p)}
\]

(29)

where

\[
a_2 = E_{p \in \mathcal{P}(W)} \left[ \left( \ln \frac{dW}{dq_{0}^{c}} - E_{W} \ln \frac{dW}{dq_{0}^{c}} \right)^2 \right],
\]

(30)

\[
a_3 = E_{p \in \mathcal{P}(W)} \left[ \left( \ln \frac{dW}{dq_{0}^{c}} - E_{W} \ln \frac{dW}{dq_{0}^{c}} \right)^3 \right],
\]

(31)

\[
\Delta \triangleq \frac{1}{n} \ln \frac{1}{\beta} + 2 \frac{\lambda_{0} \lambda_{n}}{a_3}.
\]

(32)

**Theorem 2.** For any \( W: \mathcal{X} \to \mathcal{P}(Y) \), \( M, L, n \in \mathbb{Z}_+ \), \( p \in \mathcal{P}(\mathcal{X}) \) satisfying \( \frac{1}{n} \ln \frac{1}{\beta} \geq \lim_{n \to \infty} I_1(p; W; q_{0}^{c}) \) and \( np(x) \in \mathbb{Z}_{\geq 0} \) for all \( x \in \mathcal{X} \), any \((M, L)\) channel code of length \( n \) whose codewords all have the same composition \( p \) satisfies

\[
P_e^{(n)} \geq 1 - e^{-nE_{\alpha}(\frac{1}{n} \ln \frac{1}{\beta}, W; p)}
\]

(33)

Theorems 1 and 2 collectively imply for all \( n \in \mathbb{Z}_+ \) a strong converse of the form (3) for \( E_{ac}(R) = E_{ac}(\frac{1}{n} \ln \frac{1}{\beta}, W; p) \), for a constant \( A \) determined by the rate \( R \), the channel \( W \), and the composition \( p \). Following the convention used for the corresponding improvement of the sphere packing bound in [26]–[30], we call these bounds refined strong converses.

**Proof of Theorem 1.** The existence of a unique order \( \alpha^* \) satisfying (28) was proved and its value was determined in §IV, see (23), (24), and (25).

Let the probability measures \( w_m \), \( q \), and \( v_m \) in \( \mathcal{P}(Y_1^n) \) be

\[
w_m \triangleq \bigotimes_{t=1}^{n} W(\Psi_t(m)),
\]

\[
q \triangleq \bigotimes_{t=1}^{n} q_{0}^{c},
\]

\[
v_m \triangleq \bigotimes_{t=1}^{n} W_{\alpha^*}^{\alpha^*}(\Psi_t(m)).
\]

Then \( v_m \) is equal to the order \( \alpha^* \) tilted probability measure between \( w_m \) and \( q \). Furthermore,\(^4\)

\[
D_1(v_m \| q) = nD_1(W_{\alpha^*}^{\alpha^*} \| q_{0}^{c} \| p) \quad m \in \mathbb{M},
\]

\[
D_1(v_m \| w_m) = nD_1(W_{\alpha^*}^{\alpha^*} \| W \| p) \quad m \in \mathbb{M}.
\]

Note that \( D_1(W_{\alpha^*}^{\alpha^*} \| q_{0}^{c} \| p) = \frac{1}{n} \ln \frac{1}{\beta} \) by (20) and (28) and \( D_1(W_{\alpha^*}^{\alpha^*} \| W \| p) = E_{ac}(\frac{1}{n} \ln \frac{1}{\beta}, W; p) \) by (23), (24).

\(^4\)It is worth mentioning that both \( D_1(v_m \| q) \) and \( D_1(v_m \| w_m) \) can be expressed in this form for all messages because all \( \Psi(m)'s \) have the same composition \( p \).

and (28). Thus applying Lemma 1, for \( E = \{ y_1^n : m \in \Theta(y_1^n) \} \) and \( \beta = q(m \in \Theta) \frac{1}{n} \ln \frac{1}{\beta} \) we get

\[
P_e \geq 1 - \frac{2e^{\alpha^* \left( \frac{\beta}{n} - \frac{1}{n} \right)^{\frac{1}{\alpha^*}} - nE_{\alpha}(\frac{1}{n} \ln \frac{1}{\beta}, W; p)} }{e^{\frac{1}{n} \ln \frac{1}{\beta}} - nE_{\alpha}(\frac{1}{n} \ln \frac{1}{\beta}, W; p)}.
\]

(34)

On the other hand \( \sum_{m \in \mathbb{M}} q(m \in \Theta) \leq L \), as a result of the definition of the list decoding. Thus using the concavity of the function \( z^{-\frac{1}{\alpha^*}} \) in \( z \) together with the Jensen’s inequality we get

\[
\sum_{m \in \mathbb{M}} \frac{1}{M} \frac{q(m \in \Theta) M}{L} \left( \frac{1}{\alpha^*} \right) \leq \left( \sum_{m \in \mathbb{M}} \frac{1}{M} q(m \in \Theta) M \right) \left( \frac{1}{\alpha^*} \right) = 1.
\]

Then (29) follows from (34) and the definition error probability as the average of the conditional error probabilities. \( \square \)

**Proof of Theorem 2.** Let the probability measures \( w_m \), \( q_o \) be

\[
w_m \triangleq \bigotimes_{t=1}^{n} W(\Psi_t(m)), \quad \text{and} \quad q_o \triangleq \bigotimes_{t=1}^{n} W_{\alpha^*}^{\alpha^*}(\Psi_t(m)).
\]

Then \( D_1(w_m \| q_o) = nD_1(W; p; W) \) for all \( m \) because all \( \Psi(m)'s \) have the composition \( p \). On the other hand the data processing inequality of the Rényi divergence, [32, Thm 9], imply

\[
D_1(w_m \| q_o) \geq \left[ \ln \left( \frac{(W_{\alpha^*}^{\alpha^*})^n(q_o(m \in \Theta))^{1-n} + (1-W_{\alpha^*}^{\alpha^*})^n(q_o(m \in \Theta))^{1-n}}{q_o(m \in \Theta)} \right) \right]^{\frac{1}{\alpha^*}} - \frac{1}{\alpha^*}.
\]

Thus \( P_e \geq 1 - \left( q_o(m \in \Theta) \right)^{-\frac{1}{\alpha^*}} e^{\frac{1}{\alpha^*}nD_1(w_m \| W)} \). On the other hand the concavity of the function \( z^{-\frac{1}{\alpha^*}} \) in \( z \) for \( \alpha > 1 \), the Jensen’s inequality, and \( \sum_{m \in \mathbb{M}} q_o(m \in \Theta) \leq L \), imply

\[
\sum_{m \in \mathbb{M}} \frac{1}{M} \left( q_o(m \in \Theta) \right)^{-\frac{1}{\alpha^*}} \leq \left( \frac{1}{M} \right)^{\frac{1}{\alpha^*}}.
\]

Hence

\[
P_e^{(n)} \geq 1 - e^{-\frac{1}{n} \ln \frac{1}{\beta} - \frac{1}{\alpha^*} \ln \frac{1}{\beta}} \quad \forall \alpha \in (1, \infty).
\]

Then (33) follows from (21). \( \square \)

VI. DISCUSSION

Although we have confined our analysis to the constant composition codes for brevity, using the Augustin capacity and center—instead of Augustin information and mean—one can obtain analogous results for additive white Gaussian noise channels with quadratic cost functions and Rényi symmetric channels defined in [30]. For Rényi symmetric channels the refined strong converse (3), can be established with smaller, i.e., better, constant \( A \) using the saddle point approximation. Such a result has been reported in [34, 36], assuming a common support for all output distributions of the channel and a non-lattice structure for the random variables involved. Establishing refined strong converses without any symmetry hypothesis is the main technical challenge in this line of work.

We believe the refined strong converses of the form (3) are the best possible bounds for derivations of the strong converse relying on the asymptotic behavior of sums of independent random variables. Nevertheless for the singular symmetric channels considered in [42], it should be possible to improve (3) as \( P_e^{(n)} \geq 1 - \Re^{-\frac{1}{\alpha^*}nE_{ac}(R)} \).
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