POSITIVE SOLUTIONS TO SUBLINEAR ELLIPTIC PROBLEMS

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Abstract. Let $L$ be a second order elliptic operator $L$ with smooth coefficients defined on a domain $\Omega$ in $\mathbb{R}^d$, $d \geq 3$, such that $L1 \leq 0$. We study existence and properties of continuous solutions to the following problem

\[ (0.1) \quad Lu = \varphi(\cdot, u), \]

in $\Omega$, where $\Omega$ is a Greenian domain for $L$ (possibly unbounded) in $\mathbb{R}^d$ and $\varphi$ is a nonnegative function on $\Omega \times [0, +\infty[$ increasing with respect to the second variable. By means of thinness, we obtain a characterization of $\varphi$ for which $(0.1)$ has a nonnegative nontrivial bounded solution.

Keywords. Nonlinear elliptic problems; Regular domain; Greenian domain; Thickness and thinness at $\infty$.

1. Introduction

Let $L$ be a second order elliptic operator with smooth coefficients defined on a domain $\Omega$. We assume that $L1 \leq 0$ and $\Omega$ is Greenian for $L$. Let $\varphi : \Omega \times ] - \infty, \infty[ \rightarrow [0, \infty[ \text{ be a measurable function that satisfies the following hypotheses:}$

$(H_1)$: $x \mapsto \varphi(x, c) \in K^\text{loc}_d(\Omega)$ for every $c \in [0, +\infty[$.

$(H_2)$: $t \mapsto \varphi(x_0, t)$ continuous increasing for every given $x_0 \in \Omega$.

$(H_3)$: $\varphi(x, t) = 0$ for every $x \in \Omega$ and $t \leq 0$.

We study positive continuous functions satisfying

\[ (1.1) \quad Lu = \varphi(\cdot, u), \text{ in } \Omega; \text{ in the sense of distributions.} \]

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1See Preliminaries for the definition of a Greenian domain and the Kato class $K_d$. 
Our aim is to characterize solutions to (1.1) in general domains as thoroughly as possible, preferably to obtain a one to one correspondence to $L$ harmonic functions, see Theorems 12 and 16. The only assumption we make on $\Omega$ is being Greenian, so possibly unbounded, and the general context is essential. For bounded regular domains the approach is rather standard.

Our work has been inspired by the results of El Mabrouk [4] and El Mabrouk and Hansen [5] who considered the equation

$$ \Delta u = p(x)u^{\alpha}, \quad 0 < \alpha < 1 \quad \text{and} \quad p \in L^\infty_{\text{loc}}(\mathbb{R}^d).$$

The main goal of the present paper is to show how methods of potential theory, applied so effectively in [4, 5], can be used to obtain results in much bigger generality. The latter means not only the operator but, what more important, also the semi-linear part. In particular, we improve considerably the results known for the Laplace operator in $\mathbb{R}^d$ [11, 12].

Equation (1.1) with $L$ being the Laplace operator on $\mathbb{R}^d$ has recently attracted a lot of attention [1, 3, 4, 5, 8, 10, 11, 13, 14, 17] with the semilinear part $\varphi(x, u)$ being of the form $\varphi(x, u) = p(x)u^\alpha + q(x)u^\beta$, $p, q$ positive functions, $0 < \alpha \leq \beta$ or more generally, $\varphi(x, u) = p(x)f(u) + q(x)g(u)$. Here we do not assume that the variables in $\varphi$ are separated. Neither we assume any more regularity of $\varphi$ than provided by $(H_1)$ and $(H_2)$. Positivity and monotonicity of $\varphi$, however, are crucial.

Solution to (1.1) is $L$-subharmonic. As such it may be dominated by a $L$-superharmonic function or not i.e. it may be very large. In this paper we concentrate mainly on solutions of the first kind (dominated by an $L$-superharmonic function) leaving the “so called” large solutions to a subsequent paper. Let $G_\Omega$ be the Green function for $\Omega$. We prove in Section 4 that all the positive solutions $u$ in a Greenian domain $\Omega$ dominated by a positive continuous $L$-superharmonic function $s$ are of the form

$$ u(x) + \int_\Omega G_\Omega(x, y)\varphi(y, u(y)) \, dy = h_u(x) \leq s(x), \quad (1.3) $$

where $h_u$ is the minimal $L$-harmonic majorant of $u$ in $\Omega$. In addition, the map $u \mapsto h_u$ defined for $u \leq s$ is injective. We also give sufficient conditions implying surjectivity of it, see Theorem 12. In particular, adding one more and a very natural hypothesis:

$(H_4)$ For every $c \geq 0$, the Green potential

$$ G_\Omega(\varphi(\cdot, c)) = \int_\Omega G_\Omega(\cdot, y)\varphi(y, c) \, dy $$

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$(H_4)$ For every $c \geq 0$, the Green potential

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we are able to establish one-to-one correspondence between bounded positive $L$-harmonic functions and bounded positive solutions of (1.1) (Corollary 13).

In Section 6, we assume $(H_1) - (H_3)$, $L1 = 0$ and we formulate a necessary and sufficient conditions for the existence of a nontrivial bounded solution to (1.1), Theorem 22. It is boundedness of

\begin{equation}
\int_{\Omega \setminus A} G_{\Omega}(x, y) \varphi(y, c_0) \, dy
\end{equation}

where the set $A$ is thin at $\infty$. (1.4) is essentially weaker than $(H_4)$ and it generalizes considerably Theorem 3 in [4].

At last, the author is grateful to her advisors Ewa Damek and Mohamed Sifi for their work, constant encouragement and precious feedback. She also want to express their gratitude to Krzysztof Bogdan, Konrad Kolesko, and Mohamed Selmi for their helpful and kindly suggestions.

2. Preliminaries

For every open set $\Omega$ of $\mathbb{R}^d$ with $(d \geq 3)$ let $\mathcal{B}(\Omega)$(resp. $\mathcal{C}(\Omega)$) be the set of all real valued Borel measurable (resp. continuous) functions on $\Omega$. We also consider $C^1(\Omega)$ - the space of continuously differentiable functions on $\Omega$, $C^\infty(\Omega)$ - the space of infinitely differentiable functions on $\Omega$, $C^\infty_c(\Omega)$ - the space of functions in $C^\infty(\Omega)$ with compact support, $C^{2,\alpha}(\Omega)$ - the space of functions with the second derivative being $\alpha$-Hölder continuous, $L^1(\Omega)$ (resp. $L^1_{loc}(\Omega)$)- the set of integrable (resp. locally integrable) functions in $\Omega$, $L^\infty(\Omega)$ (resp. $L^\infty_{loc}(\Omega)$) the set of bounded (resp. locally bounded) functions in $\Omega$. Also, for every set $\mathcal{F}$ of numerical functions, we denote by $\mathcal{F}^+$ the set of all functions in $\mathcal{F}$ which are nonnegative.

Let $\Omega$ be a domain in $\mathbb{R}^d$, $d \geq 3$. We assume that $L$ defined in $\Omega$ is a second order elliptic operator with smooth coefficients i.e.

\[ L = \sum_{1 \leq i,j \leq d} a_{i,j}(x) \partial_i \partial_j + \sum_{1 \leq i \leq d} b_i(x) \partial_i + c(x), \]

where $a_{i,j}, b_i, c \in C^{\infty}(\Omega)$, $a_{i,j}(x) = a_{j,i}(x)$, $1 \leq i, j \leq d$ and for every $x \in \Omega$ the quadratic form

\[ \sum_{1 \leq i,j \leq d} a_{i,j}(x) \xi_i \xi_j \]

2Then it is continuous on $\Omega$, see Preliminaries.
is strictly positive definite. The latter means that \( \sum_{1 \leq i, j \leq d} a_{i,j}(x)\xi_i\xi_j > 0 \)
for every \( x \in \Omega \) and \( \xi \in \mathbb{R}^d \setminus \{0\} \). Notice that \( L \) is locally uniformly elliptic in \( \Omega \).

Suppose that \( h \in L^1_{\text{loc}}(\Omega) \) and \( Lh = 0 \) in the sense of distributions. Then \( h \in C^\infty(\Omega) \) and \( Lh = 0 \) holds in the strong sense. Such functions will be called \( L \)-harmonic. We denote \( \mathcal{H}(\Omega) \) the set of \( L \)-harmonic functions in \( \Omega \). Let \( v \in L^1_{\text{loc}}(\Omega) \). We say that \( v \) is \( L \)-subharmonic if \( Lv \geq 0 \) in the distributional sense. Then \( v \) is equal a.e. to an upper semi-continuous function satisfying “so called” sub-mean value property. A function \( s \) such that \( -s \) is \( L \)-subharmonic on \( \Omega \) will be called \( L \)-superharmonic on \( \Omega \). We denote \( S(\Omega) \) the set of \( L \)-superharmonic functions in \( \Omega \).

Let \( u \in C^+(\Omega) \). We say that \( u \) is a solution to equation (1.1) if \( \varphi(\cdot, u) \) is locally integrable on \( \Omega \) and for all \( \psi \in C^\infty_\text{c}(\Omega) \) we have

\[
\int_{\Omega} uL^*(\psi) - \int_{\Omega} \varphi(\cdot, u)\psi = 0,
\]

where \( L^* \) is the adjoint operator \( L \) in \( \Omega \). Notice that if \((H_1), (H_2)\) are satisfied, then \( \varphi(\cdot, u) \) is always locally integrable on \( \Omega \).

We say that \( \Omega \) is Greenian if it has the Green function. \( G_\Omega : \Omega \times \Omega \to \mathbb{R} \) is called the Green function in \( \Omega \) corresponding to \( L \) if \( G_\Omega \) is \( C^\infty \) outside \( \{(x, x) : x \in \Omega \} \),

\[
(2.1) \quad LG_\Omega(\cdot, y) = -\delta_y, \quad \text{where } \delta_y \text{ denotes the Dirac measure at } y,
\]

and if \( 0 \leq h(x) \leq G(x, y) \), \( Lh = 0 \) then \( h = 0 \).

**Definition 1.** A Borel measurable function \( \psi \) on \( \Omega \) belongs locally to the Kato class (i.e. \( \psi \in K^d_{\text{loc}}(\Omega) \)) if \( \psi \) satisfies

\[
(2.2) \quad \lim_{\alpha \to 0} \sup_{x \in K} \int_{\Omega \cap (|x-y| \leq \alpha)} \frac{|\psi(y)|}{|x-y|^{d-2}} dy = 0
\]

for every compact set \( K \subset \Omega \). If (2.2) holds for every \( x \in \Omega \) instead of \( K \), we say that \( \psi \) is in Kato class and we write \( \psi \in K_d(\Omega) \).

Clearly, if \( \psi \in L^1_{\text{loc}}(\Omega) \) then \( \psi \in K^d_{\text{loc}}(\Omega) \).

**Proposition 2.** (see e.g. [15]) Let \( \psi \in K_d(\Omega) \). Then for every \( M > 0 \), we have

\[
\int_{\Omega \cap (|y| \leq M)} |\psi(y)| dy < \infty.
\]

In particular, if \( \Omega \) is a bounded domain, then \( \psi \in L^1(\Omega) \).

\[
^3 \text{i.e. } h \text{ is equal a.e. to a smooth function}
\]
Green potentials of functions belonging to the Kato class will play the main part in what follows. We are going to recall their basic properties. For a more complete overview of potential theory we refer the reader to the Appendix of [6] and Section 1 of [7].

A bounded domain \( D \) contained with its closure in \( \Omega \) is called regular if each \( f \in C(\partial D) \) admits a continuous extension \( H_D f \) on \( \overline{D} \) such that \( H_D f \) is \( L \)-harmonic in \( D \). Let \( G_D \) be the Green function for \( L \) in \( D \). Then by [16], paragraph 8, there is \( C > 0 \) such that
\[
G_D(x, y) \leq C|x - y|^{-d+2}, \quad D \times D.
\]
and so proceeding as in [15] we obtain

**Proposition 3.** (see e.g. [15] and [9])
Let \( D \) be a bounded regular domain in \( \mathbb{R}^d \) \((d \geq 3)\) and \( \psi \in K_d(D) \), then
\[
G_D \psi \in C_0(D).
\]

**Proposition 4.** (See [6]) Let \( \Omega \) be a Greenian domain, \( \varphi \in K^{loc}_d(\Omega) \) and there exists \( x_0 \in \Omega \) such that \( G_\Omega |\varphi|(x_0) \) is finite then \( G_\Omega \varphi \in C(\Omega) \).

3. Solution to \( Lu = \varphi(\cdot, u) \) in a regular domain

 Unless otherwise mentioned, throughout the paper we will suppose that \( L \) is a second order elliptic operator with smooth coefficients satisfying \( L 1 \leq 0 \) defined in the domain \( \Omega \) that is Greenian for \( L \) and \( \varphi \) satisfies \((H_1) - (H_3)\).

Let \( D \) be a regular bounded domain such that \( \overline{D} \subset \Omega \). Given \( f \in C^+(\partial D) \) there is \( u \in C(\overline{D}) \) such that
\[
\begin{align*}
Lu - \varphi(\cdot, u) &= 0, & \text{in } D; \\
u &\geq 0, & \text{in } D; \\
u &= f, & \text{on } \partial D.
\end{align*}
\]
Moreover, \( u \) is related to the solution \( H_D f \) of the classical Dirichlet problem with the boundary data \( f \) in the following way:

**Theorem 5** (Solution of (3.1) in a regular domain). Let \( f \in C^+(\partial D) \). Then there exists a unique solution \( u \) to (3.1). Furthermore, we have:
\[
(3.2) \quad u = H_D f(x) - \int_D G_D(x, y)\varphi(y, u(y)) \, dy, \quad \text{for every } x \in D.
\]

**Proof.** The statement was proved for more general \( \varphi \) in [2] in the context of balayage spaces. We may also refer the reader to [6] where the equation \( Lu + \varphi(\cdot, u) = 0 \) was considered and the proof is completely analogous. However, for the readers convenience we recall the definition of an operator, to which the Schauder theorem is applied.
Given \( f \in C^+(\partial D) \) let
\[
\beta = \sup_{x \in \overline{D}} H_D f(x) \quad \text{and} \quad \alpha = \inf_{x \in \overline{D}} [H_D f(x) - G_D(\varphi(\cdot, \beta))(x)].
\]

We consider the set \( C = \{ u \in C(\overline{D}), \alpha \leq u \leq \beta \} \) with the topology of uniform convergence. So \( C \) is a bounded, closed and convex. Let \( T : C(\overline{D}) \to C(\overline{D}) \) be the map defined by
\[
u \mapsto H_D f - G_D(\varphi(\cdot, \nu)).
\]

Since \( u \) is bounded on \( D \), \( \varphi(\cdot, u) \in K_\varphi(D) \) and so \( G_D(\varphi(\cdot, u)) \in C_0(D) \).

Therefore, \( T \) is well defined and \( T(C) \subset C \). Indeed,
\[
\alpha \leq H_D f(x) - G_D(\varphi(\cdot, \beta))(x) \leq T u(x) \leq H_D f(x) \leq \beta,
\]
for every \( u \in C \) and every \( x \in \overline{D} \).

Following the same lines of the proof of Theorem 8 in [6] and using the Schauder Theorem we conclude that \( T \) has a fixed point in \( C \) i.e.
\[
u = H_D f - G_D(\varphi(\cdot, \nu)).
\]

Moreover, \( L(G_D(\varphi(\cdot, u))) = -\varphi(\cdot, u) \), and \( u = f \) on \( \partial D \). By Lemma 6 below, using the fact that zero is a trivial solution, we get that \( u \) is positive in \( D \) and unique. □

We complete the section with a Lemma that gives comparison between sub-solutions and super-solutions to \((1.1)\) in \( D \). Suppose that \( u \) is a continuous function. We say that \( u \) is a subsolution if \( Lu - \varphi(\cdot, u) \geq 0 \) or a supersolution if \( Lu - \varphi(\cdot, u) \leq 0 \) in the sense of distributions. The Lemma holds in a considerable generality: we require only that the function \( \varphi : D \times \mathbb{R} \to \mathbb{R} \) is increasing with respect to the second variable.

**Lemma 6** (Comparison with values on the boundary). Let \( u, v \in C(D) \), \( Lu, Lv \in L^1_{\text{loc}}(D) \) and let \( \varphi : D \times \mathbb{R} \to \mathbb{R} \) be an increasing function with respect to the second variable. If
\[
\begin{cases}
Lu - \varphi(\cdot, u) \leq Lv - \varphi(\cdot, v), \\
\liminf_{\substack{x \to y \\
y \in \partial D}} (u - v)(x) \geq 0.
\end{cases}
\]

Then:
\[
u - v \geq 0 \text{ in } D
\]

In particular, there is a unique solution to \((3.1)\) 4

4 By the same argument, the result remains true for \( D \) being unbounded domain under assumption \( \liminf_{\substack{x \to y \\
y \in \partial D}} (u - v)(x) \geq 0 \) and \( \liminf_{|x| \to \infty} (u - v)(x) \geq 0 \).
Proof. Let $V = \{ x \in D, \; u(x) < v(x) \}$, $V$ is open in $D$ because $u,v$ are continuous. As in \cite{6} we prove that
\[
\begin{aligned}
L(u - v) &\leq 0 \quad \text{in } V, \\
\liminf_{x \to z} (u - v)(x) &\geq 0 \quad \text{on } \partial V.
\end{aligned}
\]
It follows that $u - v$ is a lower semi-continuous function satisfying super-mean value property. In the sense of the classical potential theory such functions are called $L$-superharmonic and they satisfy a minimum principle that implies
\[ u - v \geq 0 \quad \text{in } V, \]
and so $V$ is empty. For the details we refer the reader to the Appendix of \cite{6}, Proposition 42, and Section 1 of \cite{7}.

Proceeding as in \cite{6,7}, we obtain the following statement about regularity of solutions.

**Theorem 7.** Suppose that the assumptions of Theorem 5 are satisfied and additionally that for every $c > 0$, $\varphi(\cdot, c) \in \mathcal{L}^\infty_{\text{loc}}(D)$, then the unique solution $u$ of problem (3.1) belongs to $\mathcal{C}^+(\overline{D}) \cap \mathcal{C}^1(D)$. Furthermore, if $\varphi \in \mathcal{C}^{\alpha}_{\text{loc}}(D \times [0, \infty[)$ then $u \in \mathcal{C}^{2,\alpha}_{\text{loc}}(D) \cap \mathcal{C}(\overline{D})$.

4. **ONE-TO-ONE CORRESPONDENCE IN A GREENIAN DOMAIN**

Let $\Omega$ be a Greenian domain for $L$. We would like to obtain in $\Omega$ something in the spirit of (3.2). Clearly, in this case we cannot talk about boundary values but we may write
\[(4.1) \quad h = u + G_\Omega(\varphi(\cdot, u)) \]
and ask whether it gives a one-to-one correspondence between positive solutions $u$ and $h \in \mathcal{H}^+(\Omega)$. In general, it is not the case and there may exist solutions such that $G_\Omega(\varphi(\cdot, u))$ is not finite (see \cite{5} for the Laplace operator $\Delta$). However, under some more hypotheses we may get (1.1) for $h$ from a suitable subset of $\mathcal{H}^+(\Omega)$, see Theorems 12, 16 and Corollary 13. The statements we obtain are new even for $\Delta$.

First we need to improve the comparison principle:

**Proposition 8.** Let $u_1, u_2 \in \mathcal{C}^+(\Omega)$, $h_1, h_2 \in \mathcal{C}^+(\Omega)$ such that:
\[ h_i = u_i + G_\Omega(\varphi(\cdot, u_i)), \; 1 \leq i \leq 2. \]
where $\varphi : \Omega \times [0, +\infty[ \to [0, +\infty[$ satisfies (H$_1$) – (H$_2$). If $h_1 - h_2$ is positive $L$-superharmonic then
\[ u_1 - u_2 \geq 0 \; \text{in } \Omega. \]
In particular, for every $h \in \mathcal{H}^+(\Omega)$ there exists at most one function $u \in C^+(\Omega)$ such that
\[ u + G_\Omega \varphi(\cdot, u) = h. \]

Proof. We proceed as in the proof and the proof of Lemma 15 in [6] applying the domination principle (see e.g Proposition 44 in [6] or Proposition 35 in [7]). Let
\[ K = \{ x \in \Omega, (u_1 - u_2)(x) \geq 0 \}. \]
By assumptions $K$ is closed and non empty. Let
\[ v = \varphi(\cdot, u_2) - \varphi(\cdot, u_1). \]
Then
\[ u_1 - u_2 + G_\Omega(v^-) = h_1 - h_2 + G_\Omega(v^+), \]
with $t^+ = \max\{t, 0\}$ and $t^- = \max\{-t, 0\}$. As in [6]
\[ L(G_\Omega(v^+)) = -v^+, \text{ in } \Omega, \]
and
\[ L(G_\Omega(v^-)) = -v^-, \text{ in } \Omega. \]
Therefore, $h_1 - h_2 + G_\Omega(v^+)$ is a $L$-superharmonic positive function in $\Omega$ so it is lower semi-continuous on $\Omega - K$. In addition, $G_\Omega(v^-)$ is a potential $L$-harmonic in $\Omega \setminus K$, because $v^-$ is supported in $K$. Furthermore, it is clear that $G_\Omega(v^-)$ is continuous. Finally,
\[ h_1 - h_2 + G_\Omega(v^+) \geq G_\Omega(v^-), \]
on the boundary of $K$.
We can conclude
\[ h_1 - h_2 + G_\Omega(v^+) \geq G_\Omega(v^-), \]
holds everywhere which implies that $u_1 - u_2 \geq 0$ in $\Omega$. 

We will also need the following Lemma about convergence of a sequence of solutions:

**Lemma 9.** Let $\Omega$ be an open subset on $\mathbb{R}^d$, $\varphi$ satisfies $(H_1) - (H_2)$ and let $(u_n)$ be a sequence of nonnegative solutions of (1.1) in $\Omega$ that is uniformly bounded on compact sets and it converges pointwise to a function $u$. Then $u$ is a solution of (1.1) in $\Omega$.

**Proof.** Let $D$ a bounded regular domain such that $\bar{D} \subset \Omega$ and let $h_n$ be the $L$-harmonic function in $D$ defined by $h_n = u_n + G_D(\varphi(\cdot, u_n))$. Since the functions $(u_n)$ restricted to $D$ are uniformly bounded on $D$, say by a constant $C_D$, we conclude by dominated convergence theorem that $G_D(\varphi(\cdot, u_n))$ converges to $G_D(\varphi(\cdot, u))$. This implies that the sequence
(h_n) is bounded above by $C_D + G_D(\phi(\cdot, C_D))$ and it converges to a $L$-harmonic function $h$ such that $h = u + G_D(\phi(\cdot, u))$. Hence, $u$ is continuous and satisfies (1.1) in $D$ and so in $\Omega$. \hfill\square

Finally, let $f \in C^+(\Omega)$ and let $D$ be a regular bounded domain satisfying $\overline{D} \subset \Omega$. For $x \in D$, we define

$$U_D^\phi f(x)$$

to be the unique solution to the problem (3.1) and

$$U_D^\phi f(x) = f(x), \text{ if } x \notin D.$$

Then we have

**Lemma 10.**

(a) Let $f, g \in C^+(\Omega)$. Then: $U_D^\phi f$ is monotone nondecreasing i.e. :

$$U_D^\phi f \leq U_D^\phi g, \text{ if } f \leq g \text{ in } \Omega.$$

(b) Let $u \in C^+(\Omega)$ a $L$-supersolution and $v \in C^+(\Omega)$ a $L$-subsolution of (1.1) in $\Omega$. Suppose further that there is a subdomain $D'$ regular bounded such that $D' \subset D$. Then we have:

\begin{enumerate}
  \item[(b_1)] $U_D^\phi u \leq u$ and $U_D^\phi v \geq v.$
  \item[(b_2)] $U_D^\phi u \geq U_D^\phi u$ and $U_D^\phi v \leq U_D^\phi v.$
\end{enumerate}

The proof of Lemma 10 is the same as in [4].

Now, let $(D_n)$ be a sequence of bounded regular domains such that for every $n \in \mathbb{N}$, $\overline{D_n} \subset D_{n+1} \subset \Omega$ and $\bigcup_{n=1}^{\infty} D_n = \Omega$. Such a sequence will be called a *regular exhaustion* of $\Omega$ and it is needed to construct solutions to (1.1) in $\Omega$. The following proposition summarizes the basic properties of this construction.

**Proposition 11.** Let $s \in C^+(\Omega)$ be a $L$-superharmonic function. Then:

(i) The sequence $(U_{D_n}^\phi s)$ is decreasing to a solution $u \in C^+(\Omega)$ of (1.1) satisfying $u \leq s$ \footnote{Note here that $u$ can be zero.}
(ii) Every solution \( w \in C^+(\Omega) \) of (1.1) which is majorized by \( s \) satisfies

\[
w + G_{\Omega} \varphi(\cdot, w) = h_w, \text{ in } \Omega,
\]

where \( h_w \) is an L-harmonic minorant of \( s \). Additionally, \( w \leq u \) so \( u \) is the maximal one.

(iii) Let \( h \in H^+(\Omega) \) and \( w \in C^+(\Omega) \) be such that (4.2) holds. Then \( h_w \) is the smallest L-harmonic majorant of \( w \) and \( w \) is the maximal solution to (1.1) which is majorized by \( h_w \).

Proof.

(i) In view of Lemma 10

\[
0 \leq U^\varphi_{D_{n+1}} s \leq U^\varphi_{D_n} s \leq s.
\]

So the limit \( u = \lim_{n \to +\infty} U^\varphi_{D_n} s \) exists by Lemma 9 and it gives a solution of (1.1) in \( \Omega \).

(ii) Let \( w \) be a positive continuous solution of (1.1) in \( \Omega \) bounded by \( s \). Then by Lemma 10 and Theorem 5

\[
w = U^\varphi_{D_n} w \leq U^\varphi_{D_n} s.
\]

When \( n \) tends to \( \infty \), we get \( w \leq u \). Also, by Theorem 5

\[
w + G_{D_n} \varphi(\cdot, w) = H_{D_n} w, \text{ in } D_n.
\]

Further, by the maximum principle

\[
0 \leq H_{D_n} w \leq H_{D_n} s \leq s, \text{ in } D_n.
\]

Hence, the sequence \( (H_{D_n} w) \) is increasing to a L-harmonic function \( h_w \leq s \) and by the monotone convergence theorem,

\[
w + G_{\Omega} \varphi(\cdot, w) = h_w, \text{ in } \Omega.
\]

(iii) Let \( h \) be an L-harmonic majorant of \( w \). Then \( H_{D_n} w \leq h \).

When \( n \) tends to infinity, we deduce \( h_w \leq h \). If \( w_1 \) is a solution to (1.1) majorized by \( h_w \) and so by what has been said above

\[
w_1 + G_{\Omega} \varphi(\cdot, w_1) = h_{w_1} \leq h_w.
\]

Hence by Proposition 8 \( w_1 \leq w \).

□

Now we are ready to state a theorem about one-to-one correspondence between positive L-harmonic functions and positive continuous
solutions of (1.1). Some further assumptions are needed to guarantee that

\[ u = \lim_{n \to \infty} U_{D_n}^n \]

constructed in Proposition 11 is not trivial.

**Theorem 12.** Let \( s \in C^+(\Omega) \) be an \( L \)-superharmonic function. Assume that \( G_{\Omega}(\varphi(\cdot, s)) \) is finite at least on one point. Then

\[ h = u + G_{\Omega}(\varphi(\cdot, u)) \text{ in } \Omega. \]

gives one-to-one correspondence between \( h \in H^+_\Omega \) bounded by \( s \) and positive continuous solutions of (1.1) bounded by \( s \).

**Proof.** Let \( u \in C^+(\Omega) \) be a solution to (1.1) in \( \Omega \) bounded by \( s \). We denote \( h_n = H_{D_n}u \). By Theorem 5

\[ (4.4) \quad H_{D_n}u = h_n = u + G_{D_n}(\varphi(\cdot, u)), \text{ in } D_n. \]

This implies that for every \( n \in \mathbb{N} \),

\[ (4.5) \quad h_{n+1} \geq h_n \text{ in } D_n. \]

Indeed, \( h_{n+1} \geq u = h_n \) on \( \partial D_n \), hence (4.5) follows by the maximum principle.

Also,

\[ h_n(x) = u(x) \leq s(x), \quad x \in \partial D \]

and so \( h_n \leq s \) in \( D_n \).

We deduce that \( (h_n) \) converges to a \( L \)-harmonic function \( h \leq s \). Moreover, by the monotone convergence theorem applied to (4.4), \( h \) satisfies

\[ h = u + G_{\Omega}(\varphi(\cdot, u)) \text{ in } \Omega. \]

Finally, Proposition 8 implies that different solutions give rise to different \( L \)-harmonic functions in (1.1).

Let now \( h \leq s \) be a positive \( L \)-harmonic function in \( \Omega \). By Proposition 11, \( (u_n) = (U_{D_n}^n, h) \) is decreasing to a solution \( u \in C^+(\Omega) \) of (1.1) satisfying \( u \leq h \).

In addition

\[ (4.6) \quad h = u_n + G_{D_n} (\varphi(\cdot, u_n)), \text{ in } D_n. \]

and

\[ 0 \leq G_{D_n}(x,y) \varphi(y,u_n(y)) \leq G_{\Omega}(x,y) \varphi(y,s(y)). \]

\(^6\text{By Proposition 4 for every } x \in \Omega, G_{\Omega}(x,y) \varphi(y,s(y)) \text{ is integrable as a function of } y.\)
But the right part is integrable in \( \Omega \) as a function of \( y \), so by dominated convergence theorem

\[
u + G_\Omega(\varphi(\cdot, u)) = h, \text{ in } \Omega,
\]

which implies that \( u \) is not trivial, if \( h \) is not trivial too. Also by \((H_1)\) and Proposition 4, \( u \) is continuous in \( \Omega \).

\[\square\]

Notice that domination by \( s \) is essential in the above argument. Given a \( L \)-harmonic function \( h \) in \( \Omega \), we can establish corresponding solutions in bounded subdomains \( D_n \) but passing to the limit we may end up with a trivial one unless \((4.6)\) is preserved. The latter is guaranteed by the fact that \( G_\Omega(\varphi(\cdot, s)) \) is finite. Finally, for bounded harmonic functions and bounded solutions we have the following corollary.

**Corollary 13.** Assume that \((H_4)\) is satisfied. Then

\[
(4.7) \quad h = u + G_\Omega(\varphi(\cdot, u)), \text{ in } \Omega.
\]
gives one-to-one correspondence between bounded functions in \( \mathcal{H}^+(\Omega) \) and positive continuous bounded solutions of \((1.1)\).

**Remark 14.** Notice that if bounded \( L \)-harmonic functions are known then \((4.7)\) gives a description of bounded solutions to \((1.1)\). See Section 7 for an example.

**Remark 15.** \((4.7)\) was proved by El Mabrouk for the operator \( \Delta \) and \( \varphi(x, u) = \xi(x)u^\gamma \), \( \xi \in L^\infty_{\text{loc}}, \, 0 \leq \gamma \leq 1 \) and, up to our knowledge, there was no further development even for \( \Delta \). \((4.7)\) is also essentially stronger than Theorem 1 in [12] where the equation \( \Delta u - \xi(x)f(u) \) was considered for \( \xi \in C(\mathbb{R}^d) \) and \( f \) satisfying a so called “Keller-Osserman” condition.

A superharmonic function \( s \in C^+(\Omega) \) in Theorem 12 may be replaced by a continuous solution to \((1.1)\) such that \( G_\Omega(\varphi(\cdot, v)) \) is finite. The statement is as follows:

**Theorem 16.** Suppose that there is a continuous positive function \( v \) and a positive \( L \)-harmonic function \( h_v \) in \( \Omega \) satisfying

\[
h_v = v + G_\Omega(\varphi(\cdot, v)).
\]

Then there is one-to-one correspondence between positive \( L \)-harmonic functions \( h \) bounded by \( h_v \) and positive continuous solutions of \((1.1)\) \( u \) bounded by \( v \), given by

\[
(4.8) \quad h = u + G_\Omega(\varphi(\cdot, u)) \text{ in } \Omega.
\]

The above Theorem follows from Lemmas 17 and 18 below.
Lemma 17. Suppose that there is a continuous positive function \( v \) and a positive \( L \)-harmonic function \( h_v \) in \( \Omega \) satisfying 

\[
h_v = v + G_\Omega(\varphi(\cdot, v)).
\]

Then:

- \( h_v = \lim_{n \to +\infty} H_{D_n} v \) i.e. \( h_v \) is the minimal \( L \)-harmonic function bounded below by \( v \).
- \( v = \lim_{n \to +\infty} U_{D_n} h_v \) i.e. \( v \) is the maximal solution of \((1.1)\) bounded above by \( h_v \).

Proof. First of all, clearly \( v \) is a solution of \((1.1)\) in \( \Omega \) bounded by \( h_v \).

Second, the sequence \((h_n) = (H_{D_n} v)\) is increasing to a positive \( L \)-harmonic function \( \tilde{h} \) such that \( h_n \leq h_v \) and 

\[
v + G_{D_n} \varphi(\cdot, v) = h_n, \text{ in } D_n.
\]

We conclude by monotone convergence theorem 

\[
v + G_\Omega(\varphi(\cdot, v)) = \tilde{h}, \text{ in } \Omega.
\]

Consequently: \( h_v = \tilde{h} \) in \( \Omega \).

Third, it remains to prove that \( v \) is the maximal solution in the sense of Proposition 11. Indeed, for the maximal one \( u \) we have 

\[
h_v = v + G_\Omega(\varphi(\cdot, v)) \leq u + G_\Omega(\varphi(\cdot, u)) \leq h_v.
\]

Therefore, we have equalities above and so \( v = u \) in \( \Omega \). \( \square \)

Lemma 18. Let \((f_n)\) and \((g_n)\) be sequences of positive measurable functions on \( \mathbb{R}^d \) such that

- \( 0 \leq f_n \leq g_n \) in \( \mathbb{R}^d \).
- \((f_n)\), \((g_n)\) converge pointwise respectively to \( f \) and \( g \) on \( \mathbb{R}^d \).
- For every \( n \in \mathbb{N} \), \( g_n \) and \( g \) are integrable.
- \((\int_{\mathbb{R}^d} g_n)\) converge to \( \int_{\mathbb{R}^d} g \).

Then

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f.
\]

Proof. Applying Fatou’s Lemma to the sequences \((f_n)\) and \((g_n - f_n)\) we obtain 

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f. \text{ In more details:}
\]

\[
\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} \liminf_{n \to +\infty} f_n \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^d} f_n.
\]
Similar to:
\[ \int_{\mathbb{R}^d} (g - f) = \int_{\mathbb{R}^d} \liminf_{n \to +\infty} (g_n - f_n) \leq \liminf_{n \to +\infty} \int_{\mathbb{R}^d} (g_n - f_n). \]

But
\[ \liminf_{n \to +\infty} \int_{\mathbb{R}^d} (g_n - f_n) = \int_{\mathbb{R}^d} g + \liminf_{n \to +\infty} (- \int_{\mathbb{R}^d} f_n) \]
\[ = \int_{\mathbb{R}^d} g - \limsup_{n \to +\infty} \int_{\mathbb{R}^d} f_n. \]

Hence
\[ \int_{\mathbb{R}^d} (g - f) \leq \int_{\mathbb{R}^d} g - \limsup_{n \to +\infty} \int_{\mathbb{R}^d} f_n. \]

and so
\[ (4.11) \int_{\mathbb{R}^d} f \geq \limsup_{n \to +\infty} \int_{\mathbb{R}^d} f_n. \]

(4.9) follows now from (4.10) and (4.11).

\[ \square \]

Proof of Theorem 16. First of all, by Lemma 17, for \( v_n = U_{D_n}^\varphi h_v \), we have
\[ v = \lim_{n \to +\infty} v_n, v_n \leq u \]
and so
\[ \lim_{n \to +\infty} G_{D_n} (\cdot, v_n) = G_{\Omega} (\cdot, u). \]

Let \( u \) be a solution of (1.1) bounded by \( v_n \). Then \( h_n = (D_n u) \) is an increasing sequence of \( L \)-harmonic functions bounded by \( h_v \), satisfying
\[ u + G_{D_n} (\cdot, u) = h_n \]
in \( D_n \), such that \( \lim_{n \to +\infty} h_n = h \). When \( n \) tends to infinity, we get
\[ h = u + G_{\Omega} (\cdot, u), \]
in \( \Omega \).

On the other hand, let \( h \) be a \( L \)-harmonic function bounded by \( h_v \). Then \( (u_n) = (D_n h) \) is a decreasing sequence of solutions satisfying
\[ u_n + G_{D_n} (\cdot, u_n) = h \]
in \( D_n \), such that \( \lim_{n \to +\infty} u_n = u \). Since \( u_n |_{\partial D_n} = h \leq h_v = v_n |_{\partial D_n} \), by Lemma 14, we have
\[ u_n \leq v_n \]
in \( D_n \). Now applying lemma 18 with
\[ f_n(y) = G_{D_n} (x, y) \varphi (y, u_n(y)) 1_{D_n} \]
\[ g_n(y) = G_{D_n} (x, y) \varphi (y, v_n(y)) 1_{D_n} \]
\[ f(y) = G_{\Omega} (x, y) \varphi (y, u(y)) 1_{\Omega} \]
we conclude that \( \lim_{n \to +\infty} G_{D_n} (\cdot, u_n) = G_{\Omega} (\cdot, u) \) and so
\[ h = u + G_{\Omega} (\cdot, u). \]
5. Operations on the nonlinear term of (1.1)

In this section we consider two functions \( \varphi_1, \varphi_2 : \Omega \times [0, \infty] \to [0, \infty] \) satisfying \((H_1) - (H_3)\) and equation (1.1) related to \( \varphi_1, \varphi_2 \) and \( \varphi_1 + \varphi_2 \).

**Proposition 19.** Assume that for every \( c > 0 \), \( \varphi_1(\cdot, c) \leq \varphi_2(\cdot, c) \) and the equation (1.1) has a nonnegative nontrivial solution for \( \varphi = \varphi_2 \) bounded by an \( L \)-superharmonic function \( s \). Then the same holds also for \( \varphi = \varphi_1 \).

**Proof.** Let \( w \) denotes a nontrivial solution of (1.1) for \( \varphi = \varphi_2 \) bounded by \( s \). Then by Proposition 11

\[
\lim_{n \to +\infty} H_{D_n} w = h_w, \text{ in } \Omega,
\]

where \( h_w = \lim_{n \to +\infty} H_{D_n} w \) and \( (D_n) \) is the regular exhaustion of \( \Omega \). Let

\[
u_n = U_{D_n}^{\varphi_2} h_w \text{ in } D_n.
\]

It follows that \( u = \lim_{n \to +\infty} u_n \) is a solution of (1.1) for \( \varphi = \varphi_1 \). Let us prove that it is not trivial. By the fact \( \varphi_1(\cdot, c) \leq \varphi_2(\cdot, c) \) for every \( c > 0 \)

\[
Lu_n - \varphi_2(\cdot, u_n) \leq Lu_n - \varphi_1(\cdot, u_n) = Lw - \varphi_1(\cdot, w) = 0, \text{ in } D_n.
\]

Also \( u_n = h_w \geq w \) on \( \partial D_n \). Then by Lemma 6 we get

\[
u_n \geq w, \text{ in } D_n.
\]

Hence, \( u = \lim_{n \to +\infty} u_n \) is a nonnegative nontrivial solution of equation (1.1) with \( \varphi = \varphi_1 \) bounded by \( s \).

**Proposition 20.** Suppose, that there exists \( h, h_1 \) both \( L \)-harmonic functions such that \( h_1 \leq h \) and

\[
h = u + G_\Omega(\varphi_1(\cdot, u)), \text{ in } \Omega,
\]

where \( u \) is a solution of the equation (1.1) for \( \varphi = \varphi_1 \). Assume further that \( s \leq \lim_{n \to +\infty} U_{D_n}^{\varphi_2} h_1 \) is a nontrivial solution of (1.1) for \( \varphi = \varphi_2 \). Then

\[
u = \lim_{n \to +\infty} U_{D_n}^{\varphi_1 + \varphi_2} h_1 \text{ is a nontrivial solution of (1.1) for } \varphi = \varphi_1 + \varphi_2.
\]

**Proof.** At first, by Proposition 11 \( u = \lim_{n \to +\infty} U_{D_n}^{\varphi_1} h \). Secondly,

\[
h_1 \leq h \Rightarrow U_{D_n}^{\varphi_2} h_1 \leq U_{D_n}^{\varphi_2} h,
\]

then \( v = \lim_{n \to +\infty} U_{D_n}^{\varphi_2} h \) is a nontrivial solution of (1.1) for \( \varphi = \varphi_2 \) bounded above by \( h \) satisfying

\[
v + G_\Omega(\varphi(\cdot, v)) = h_v, \text{ in } \Omega,
\]
where $h_v$ is the minimal $L$-harmonic function bounded below by $v$.

Thirdly, we denote $u_n = U_{D_n}^{\varphi_1}h, v_n = U_{D_n}^{\varphi_2}h$ and $w_n = U_{D_n}^{\varphi_1+\varphi_2}h$.

As in the proof of Proposition 19, we get $u_n \geq w_n$ and $v_n \geq w_n$ in $D_n$. Let us prove that $w = \lim_{n \to +\infty} w_n$ is non-trivial.

$$\begin{align*}
  \begin{cases}
    L(h + w_n - u_n - v_n) \leq 0, & \text{in } D_n, \\
    h + w_n - u_n - v_n = 0, & \text{on } \partial D_n.
  \end{cases}
\end{align*}$$

Hence in $D_n$

$$h + w_n - u_n - v_n \geq 0.$$ 

Then

$$\begin{align*}
  (h - u_n) + (h_v - v_n) + w_n \geq h_v.
\end{align*}$$

Tending $n$ to infinity, we get

$$G_\Omega(\varphi_1(\cdot, u)) + G_\Omega(\varphi_2(\cdot, v)) + w = (h - u) + (h_v - v) + w \geq h_v, \text{ in } \Omega.$$ 

Suppose now that $w = 0$ then

$$G_\Omega(\varphi_1(\cdot, u)) + G_\Omega(\varphi_2(\cdot, v)) \geq h_v, \text{ in } \Omega.$$ 

Since $G_\Omega(\varphi_1(\cdot, u)) + G_\Omega(\varphi_2(\cdot, v))$ is a potential, we have $h_v = 0$ implying $v = 0$. The contradiction proves $w \neq 0$.

\[\Box\]

6. Existence of bounded solutions in Greenian domains

Throughout this section, we assume that $L1 = 0$. Then $(H_4)$ is not necessary for existence of bounded solutions in a Greenian domain $\Omega$.

A sufficient and necessary condition for that is the integrability of

$$G_\Omega(\cdot, y)\varphi(y, c_0)$$

on $\Omega \setminus A$, where $A$ is a “so called” thin set at $\infty$, see Theorem 22 below.

The latter is proved below under fairly weak assumptions on $\varphi$ and it generalizes the results known for $\Delta$ as well. A subset $A \subset \Omega$ is called thin if there is a nonnegative continuous $L$-superharmonic function $s$ on $\Omega$ such that

$$s \geq 1 \text{ on } A \text{ but } \inf_{x \in \Omega} s(x) < 1.$$ 

Given a particular operator it is often not so complicated to produce such a function and to obtain existence of bounded solutions (see the example in the next section).

We start with the following theorem.

**Theorem 21.** If there exists $c > 0$ such that $\{\varphi(\cdot, c) > 0\}$ is thin at $\infty$, then $u_c = \lim_{n \to +\infty} U_{D_n}^{\varphi}c$ is a nonnegative nontrivial bounded solution of equation (1.1).
Proof. We have to prove that \( u_c \) is nontrivial. Let \( B = \{ \phi(\cdot, c) > 0 \} \) and let \( \tilde{s}_0 \) be a continuous nonnegative \( L \)-superharmonic function \( \tilde{s}_0 \) on \( \Omega \) such that \( \tilde{s}_0 \geq 1 \) on \( B \) and there is \( x_0 \in \Omega \setminus B \) such that \( \tilde{s}_0(x_0) < 1 \). We want to extend \( B \) in order to obtain a closed thin set at \( \infty \). We denote
\[
A = \{ x \in \Omega, \tilde{s}_0(x) \geq 1 \}.
\]
It is clear that \( A \) is a closed subset of \( \Omega \) thin at \( \infty \) i.e. \( B \subset A \). Let \( s = \inf\{c, cs_0\} \). \( s \) is a continuous \( L \)-superharmonic function such that \( s(x_0) = cs_0(x_0) < c \). \( cs_0 \geq c \) on \( A \), which implies \( s = c \) on \( A \).

Let \( D \) a bounded regular subdomain of \( \Omega \) such that \( \overline{D} \subset \Omega \) and \( D \cap A \neq \emptyset \). We denote \( u = U^\phi_c \) in \( \Omega \). First we prove that
\[
(6.1) \quad s \geq G_D(\phi(\cdot, u)), \text{ on } D.
\]
\[6.1\] holds on \( A \cap \overline{D} \) because \( s \equiv c \) on \( A \) and \( G_D(\phi(\cdot, u)) \leq c \). Now we are going to use the domination principle (Proposition 44 in [6] or Proposition 35 in [7]) to prove that \[6.1\] holds in the rest of \( D \) as well. Indeed, \( s \) is a nonnegative continuous \( L \)-superharmonic function in \( \Omega \). In addition, \( p = G_D(\phi(\cdot, u)) \) is a continuous potential in \( D \) satisfying \( Lp = -\phi(\cdot, u) \). Also, \( \Omega \setminus A \subset \Omega \setminus B \), and \( 0 \leq u \leq c \). Hence
\[
\begin{align*}
0 \leq \phi(\cdot, u) &\leq \phi(\cdot, c), \quad \text{in } D; \\
\phi(\cdot, c) &\equiv 0, \quad \text{in } \Omega \setminus B.
\end{align*}
\]
meaning \( p \) being \( L \)-harmonic in \( D \setminus D \cap A \). We are allowed then to apply the domination principle and \[6.1\] follows.

Now let \((D_n)\) be a regular exhaustion of \( \Omega \), such that \( D_0 \cap A \neq \emptyset \). Then, since constants are \( L \)-harmonic, by Theorem 5 for any \( n \geq 1 \) there is a positive continuous function \( u_n \) in \( D_n \) such that
\[
c = u_n + G_{D_n}(\phi(\cdot, u_n)), \text{ in } D_n,
\]
where \( u_n = U^\phi_{D_n}c \). As in \[6.1\], we have
\[
s \geq G_{D_n}(\phi(\cdot, u_n)) = c - u_n, \text{ on } D_n.
\]
When \( n \) tends to \( \infty \), we get \( s \geq c - u_c \). In particular, \( u(x_0) \geq c - s(x_0) > 0 \) which implies that \( u \) is nontrivial on \( \Omega \).

\[\Box\]

Theorem 22.
Suppose that there exists a Borel set $A \subset \Omega$ which is thin at $\infty$ and $c_0 > 0$ such that

\begin{equation}
\int_{\Omega \setminus A} G_\Omega(\cdot, y) \varphi(y, c_0) \, dy \neq \infty.
\end{equation}

Then equation (1.1) has a nonnegative nontrivial bounded solution in $\Omega$. On the other hand, existence of a bounded solution to (1.1) implies that

\begin{equation}
\int_{\Omega \setminus A} G_\Omega(\cdot, y) \varphi(y, c_0) \, dy
\end{equation}

is bounded for a set $A$ which is thin at $\infty$.

**Remark 23.** The above statement was proved by El Mabrouk for $\Delta$ and $\varphi(x, u) = \xi(x) u^\gamma$, $\xi \in L^\infty_{\text{loc}}$, $0 \leq \gamma \leq 1$.

**Proof.** Suppose first that (6.2) is satisfied. For every $c > 0$, we have

$$
\varphi(\cdot, c) = 1_A \varphi(\cdot, c) + 1_{\Omega \setminus A} \varphi(\cdot, c) = \varphi_1(\cdot, c) + \varphi_2(\cdot, c).
$$

Observe that $\{ \varphi_1(\cdot, c) > 0 \} \subset A$ is thin at $\infty$, so in view of Theorem 21, $u_{c_0} = \lim_{n \to +\infty} U_{D_n}^{c_1} c_0$ is a nonnegative nontrivial bounded solution to (1.1) with $\varphi = \varphi_1$. In addition

$$
\int_{\Omega} G_\Omega(\cdot, y) \varphi_2(y, c_0) \, dy = \int_{\Omega \setminus A} G_\Omega(\cdot, y) \varphi(y, c_0) \, dy \neq \infty.
$$

Hence in view of Proposition 11, there exists a solution $v$ to (1.1) with $\varphi = \varphi_2$ such that

$$
c_0 = v + G_\Omega(\varphi_2(\cdot, v)), \quad \text{in } \Omega.
$$

Consequently, by Proposition 20, $w = \lim_{n \to +\infty} U_{D_n}^{c_1} c_0 = \lim_{n \to +\infty} U_{D_n}^{c_1 + \varphi_2} c_0$ is a nonnegative nontrivial bounded solution to (1.1).

Now, let $w$ be a nontrivial solution of (1.1) bounded by $c$. Then $v = \lim_{n \to +\infty} U_{D_n}^c c$ is a nonnegative nontrivial solution of equation (1.1) as well. Hence, there exist $0 < c_0 < c$ and $x_0, x_1 \in \Omega$ such that $0 < v(x_0) \leq c_0 < v(x_1) \leq c$. Let $A = \{ v \leq c_0 \}$ and $s = \frac{c - c_0}{c_0}$. Then $s$ is a nonnegative continuous $L$-superharmonic function in $\Omega$, $s \geq 1$ on $A$ and $s(x_1) < 1$. So $A$ is thin at $\infty$ and

$$
\int_{\Omega \setminus A} G_\Omega(\cdot, y) \varphi(y, c_0) \, dy \leq \int_{\Omega} G_\Omega(\cdot, y) \varphi(y, v(y)) \, dy \leq c.
$$

Hence we get not only (6.2) but also boundedness of $\int_{\Omega \setminus A} G_\Omega(\cdot, y) \varphi(y, c_0) \, dy$. \qed
7. Example

Let \( \Omega = H_n = \{(x, y), x \in \mathbb{R}^n, y > 0\} \) be the upper half space of \( \mathbb{R}^{n+1} \) and

\[
\Delta = \sum_{k=1}^{n} \partial_k^2 + \partial_y^2
\]

the Laplace operator in \( \mathbb{R}^{n+1} \). The corresponding Poisson kernel is given by (see [18])

\[
P(x, y) = c_n \frac{y}{(|x|^2 + y^2)^{\frac{n+1}{2}}},
\]

where \( y \in \mathbb{R}_+, x \in \mathbb{R}^n \) and \( c_n^{-1} = \int_{\mathbb{R}^n} \frac{1}{(|x|+1)^{\frac{n+1}{2}}} \, dx \). Then every bounded \( \Delta \)-harmonic function in \( H_n \) is the Poisson integral of a bounded function in \( \mathbb{R}^n \). Then there is one-to-one correspondence between bounded positive \( \Delta \)-harmonic function in \( H_n \) and bounded positive functions in \( \mathbb{R}^n \) given by

\[
h(x, y) = c_n \int_{\mathbb{R}^n} f(x - z) \frac{y}{(|z|^2 + y^2)^{\frac{n+1}{2}}} \, dz.
\]

In addition,

\[
\lim_{y \to 0} h(\cdot, y) = f, \quad \text{weakly}
\]

and

\[
\sup_{x \in \mathbb{R}^n} h(x, y) \leq ||f||_{\infty}, \quad \text{for every } y > 0.
\]

Let \( \mathcal{L}^{\infty,+}(\mathbb{R}^n) \) be the set of bounded positive functions in \( \mathbb{R}^n \). It follows that if \( \varphi \) satisfies \( (H_1) - (H_4) \) then there is one-to-one correspondence between \( \mathcal{L}^{\infty,+}(\mathbb{R}^n) \) and continuous positive bounded solutions of (1.1) in \( H_n \) given by

\[
f \to h \to u = h - G_{\Omega \varphi}(\cdot, u).
\]

If \( G_{\Omega \varphi}(\cdot, u) \) vanished at the boundary \( \mathbb{R}^n \times \{0\} \) of \( H_n \) then the boundary values of \( u \) and \( h \) are the same.

It is easy to construct in this case thin sets and so to be able to apply criterion (6.2) for existence of bounded solutions. Let \( s(x, y) = cy^\gamma \), for \( 0 < \gamma < 1 \). Then \( \Delta s < 0 \), \( s > 1 \) for \( y > c^{-1/\gamma} \) and \( s < 1 \) for \( y < c^{-1/\gamma} \). This shows that any set \( A = \{(x, y) : y > c^{-\frac{1}{\gamma}}\} \) is thin at \( \infty \) and so for any function \( \varphi \) such that \( \varphi(\cdot, c_0) \) is integrable against \( G_{H_n}(0, 1, \cdot) \) on \( \{(x, y) : y < c^{-\frac{1}{\gamma}}\} \) we have one-to-one correspondence of bounded solutions to (1.1) and bounded \( L \)-harmonic functions.
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