Byzantine-Tolerant Register in a System with Continuous Churn

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Abstract—A shared read/write register emulation provides the illusion of shared-memory on top of message-passing models. The main hurdle with such emulations is dealing with server faults in the system. Several crash-tolerant register emulations in static systems require algorithms to replicate the value of the shared register onto a majority of servers. Majority correctness is necessary for such emulations. Byzantine faults are considered to be the worst kind of faults that can happen in any distributed system. Emulating a Byzantine-tolerant register requires replicating the register value on to more than two-thirds of the servers. Emulating a register in a dynamic system where servers and clients can enter and leave the system and be faulty is harder than in static systems. There are several crash-tolerant register emulations for dynamic systems.

This paper presents the first emulation of a multi-reader multi-writer atomic register in a system that can withstand nodes continually entering and leaving, imposes no upper bound on the system size and can tolerate Byzantine servers. The algorithm works as long as the number of servers entering and leaving during a fixed time interval is at most a constant fraction $\alpha$ of the system size at the beginning of the interval, and as long as the number of Byzantine servers in the system is at most $f$. Although our algorithm requires that there be a constant known upper bound on the number of Byzantine servers, this restriction is unavoidable, as we show that it is impossible to emulate an atomic register if the system size and maximum number of servers that can be Byzantine in the system is unknown to the nodes.

I. INTRODUCTION

A long-standing vision in distributed systems is to build reliable systems from unreliable components. We are increasingly dependent on services provided by distributed systems resulting in added vulnerability when it comes to failures in computer systems. In a dependable computing system, the term “Byzantine” fault is used to represent the worst kind of failures imaginable. Malicious attacks, operator mistakes, software errors and conventional crash faults are all encompassed by the term Byzantine faults [22]. The growing reliance of industry and government on distributed systems and increased connectivity to the Internet exposes systems to malicious attacks. Operator mistakes are also a very common cause of Byzantine faults [30]. The growth in the size of software in general leads to an increased number of software errors. Naturally, over the past four decades, there has been a significant work on consensus and replication techniques that tolerate Byzantine faults [4], [14], [15], [22] as it promises dependable systems that can tolerate any type of bad behavior. The shared-memory model is a more convenient programming model than message-passing, and shared register emulations provide the illusion of shared-memory on top of message-passing models.

In this paper, we emulate the first Byzantine-tolerant atomic register on top of a dynamic message-passing system that never stops changing.

A. Related Work

Typically, crash-tolerant emulations [6], [24] of a shared read/write register replicate the value of the register in multiple servers and require readers and writers to communicate with majority of servers. For instance, the ABD emulation [6] replicates the value of the shared register in a static set of servers. It assumes that a minority of the servers may fail by crashing. This problem of emulating a shared register has been extended to static systems with servers subject to Byzantine faults and these emulations typically assume that two-thirds [7] or three-fourths [1] of the servers are non-faulty. It is shown in [29] that more than two-third correctness is necessary for Byzantine-tolerant register simulation. Byzantine quorum systems (BQS) [4], [26], [27], [29] are a well known tool for ensuring consistency and availability of a shared register. A BQS is a collection of subsets of servers, each pair of which intersect in a set containing sufficiently many correct servers to guarantee consistency of the replicated register as seen by clients. Vukolic [33] provides an extensive overview of the evolution of quorum systems for distributed storage and consensus. Dantas et. al [16] present a comparative evaluation of several Byzantine quorum based storage algorithms in the literature.

The success of this replicated approach for static systems, where the set of readers, writers, and servers is fixed, has motivated several similar emulations for dynamic systems, where nodes may enter and leave. Change in system composition due to nodes entering and leaving is called churn. Ko et. al [19] provide a detailed discussion of churn behavior in practice. Most existing emulations of atomic registers in dynamic systems deal with crash-faults and rely either on the assumption that churn eventually stops for a long enough
period (e.g., DynaStore [2] and RAMBO [25]) or on the assumption that the system size is bounded (e.g., [10], [12]). Attiya et al. [8], [9] proposed an emulation of a crash-tolerant shared register in a system that does not require churn to ever stop. Bonomi et al. [13] present an emulation of a server-based regular read/write storage in a synchronous message-passing system that is subject to “mobile Byzantine failures”. They prove that the problem is impossible to solve in an asynchronous setting. The system size, however, is fixed and mobility, in this paper refers to the Byzantine agents that can be moved from server to server. Baldoni et al. [11] provide the first emulation of a Byzantine-tolerant safe [21] register in an eventually synchronous system with churn but the size of the system is upper bounded by a known parameter. To the best of our knowledge, there is no work on implementing an atomic register in a (i) dynamic system (ii) with no upper bound on the system size where (iii) servers are subject to Byzantine faults and (iv) any number of clients can crash.

### B. Contributions

The first contribution of this paper is the first algorithm to emulate an atomic multi-reader/multi-writer register that does not require churn to ever stop, does not have an upper bound on the system size and tolerates up to a constant number of Byzantine servers in the system. It is a common practice to assume that clients cannot be Byzantine in a system [11], [28] as a Byzantine client can maliciously contact separate sets of servers and write different values which results in an inconsistent register thus violating safety. In our model, clients can only crash. Although our algorithm requires that there be a constant known upper bound on the number of Byzantine servers that can be tolerated, this restriction is unavoidable as shown in our second contribution. Our second contribution a proof that it is impossible to emulate an atomic register in a system with churn if the maximum number of Byzantine servers is unknown to the nodes.

Our system model is similar to the one in [9]. We assume that there exists a parameter $D$, an upper bound, unknown to the nodes, on the delay of any message (between correct nodes). There is no lower bound on message delays and nodes do not have real-time clocks.

Churn is modeled as follows: we assume that in any time interval of length $D$, the number of servers that enter or leave the system is at most a constant fraction, $\alpha$ (known to all nodes), of the number of servers in the system at the beginning of the interval. We also assume the messages are authenticated with digital signatures [23]. In real world systems digital signatures in messages are implemented using public-key signatures [31] and message authentication codes [32]. Intuitively, this means that Byzantine servers cannot lie about the sender of a message.

### C. Challenges with Byzantine Servers

Our algorithm is called ABCC register, for Atomic Byzantine-tolerant Continuous Churn. Unlike crash faults, data may easily be corrupted by Byzantine servers by sending old information, modified information or even different information to different sets of nodes while replying to a particular message. A Byzantine server may choose to not reply to a message at all, even if it is active or it may even choose to reply to a single message multiple times. Our algorithm uses a mechanism where at every stage of the algorithm, nodes wait for at least $f+1$ replies from distinct servers before taking any major steps, to make sure at least one reply from a non-faulty server was received. The algorithm also has to make sure that a decision is not affected by multiple replies corresponding to a single message from Byzantine servers. A Byzantine server can also lie about its joined state and propagate misinformation by sending out messages after “pretending” to leave the system.

### II. System Model and Problem Statement

We model each node $p$, whether client or server, as a state machine with a set of states. The set of states for $p$ contains two initial states, $s_{p}^{i}$ and $s_{p}^{f}$. Initial state $s_{p}^{i}$ is used if node $p$ is in the system initially, whereas $s_{p}^{f}$ is used if $p$ enters the system later.

State transitions are triggered by the occurrences of events. Possible triggering events include node $p$ entering the system ($\text{ENTER}_{p}$), leaving the system ($\text{LEAVE}_{p}$), and receiving a message $m$ ($\text{RECEIVE}_{p}(m)$). In addition, triggering events for a client $p$ include the invocation of an operation ($\text{READ}_{p}$ or $\text{WRITE}_{p}(v)$) and the client crashing ($\text{CRASH}_{p}$).

A step of a node $p$ is a 5-tuple $(s', T, m, R, s)$, where $s'$ is the old state, $T$ is the triggering event, $m$ is the message to be sent, $R$ is either a response ($\text{RETURN}_{p}(v)$, $\text{ACK}_{p}$, or $\text{JOIN}_{p}$) or $\perp$, and $s$ is the new state. The message $m$ includes an indication as to whether it should be sent to all servers or to all clients, indicated as “s-bcast” or “c-bcast”. $\text{RETURN}_{p}$ is the response to $\text{READ}_{p}$, $\text{ACK}_{p}$ is the response to $\text{WRITE}_{p}$, and $\text{JOIN}_{p}$ is the response to $\text{ENTER}_{p}$; these responses are only done by clients. If $T$ is $\text{CRASH}_{p}$, then $m$ and $R$ are both $\perp$.

If the values of $m$, $R$, and $s$ are determined by the node’s transition function applied to $s'$ and $T$, then the step is said to be valid. A step by a client is always valid. In an invalid step (taken by server $p$), the values of $m$, $R$, and $s$ can be arbitrary, with the restriction that $p$ cannot modify values containing information about node ids. There is more detail on how this assumptions applies to our algorithm in Section IV

A view of a node $p$ is a sequence of steps such that:

- the old state of the first step is an initial state;
- the new state of each step equals the old state of the next step

if the old state of the first step is $s_{p}^{i}$, then no $\text{ENTER}_{p}$ event occurs

if the old state of the first step is $s_{p}^{f}$, then the triggering event of the first step is $\text{ENTER}_{p}$ and there is no other occurrence of $\text{ENTER}_{p}$;

at most one $\text{LEAVE}_{p}$ occurs, and if it occurs there are no later steps;
at most one CRASH<p> occurs; if it occurs, then <p> is a client and there are no later steps.

A view is valid if every step in it is valid.

In our model, a node that leaves the system cannot re-enter and a client node that crashes cannot recover.

Time is represented by nonnegative real numbers. A timed view is a view whose steps are labeled with nondecreasing times (the real times when the steps occur) such that:

- for each node <p>, if the old state in <p>’s first step is <s'>, then the time of <p>’s first step is 0 (such a node is said to be in the system initially), and
- if a view is infinite, the step times must increase without bound.

Given a timed view of a node <p>, if (<s'>, <T>, <m>, <R>, <s> ) is the step with the largest time less than or equal to <t>, then <s> is the state of <p> at time <t>. A node is said to be present at time <t> if its first step has time at most <t> but has not left (i.e., LEAVE<p> does not occur at or before time <t>). The number of servers present at time <t> is denoted by NS(<t>). A node is said to be active at time <t> if it is present at <t> and CRASH<p> has not occurred before time <t>. Since servers never experience crashes, a server that is present is also active.

We define the following system parameters that are used in the upcoming definition of an execution:

- <D > 0 is the maximum message delay (the delay of a message sent at time <t> and received at time <t'> is <t'> - <t>).
- <α > 0 is the churn rate, which bounds how fast servers can enter and leave; there are no bounds for clients.
- <f > ≥ 1 is the maximum number of Byzantine-faulty servers.
- NS<sub>min</sub> > 0 is the minimum number of servers. This value is unknown to all nodes in the system.

The parameters <α> and <f> are known to the nodes, but <D> is not.

An execution is a possibly infinite set of timed views, one for each node that is ever present in the system, that satisfies the following assumptions:

A1: For all <t> ≥ 0, the number of nodes present at time <t> is finite and NS(<t>) ≥ NS<sub>min</sub>.

A2: Every message s-bcast (respectively, c-bcast) has at most one matching receipt at each server (respectively, client) and every message receipt has exactly one matching bcast.

A3: If message <m> is s-bcast (respectively, c-bcast) at time <t> and server (respectively, client) node <q> is active throughout <[t, t + D]>, then <q> receives <m>. The delay of every received message is in (0, <D>].

A4: Messages from the same sender are received in the order they are sent (i.e., if node <p> sends message <m>₁ before sending message <m>₂, then no node receives <m>₂ before it receives <m>₁).

A5: For all times <t> > 0, the number of ENTER and LEAVE events for servers in <[t, t + D]> is at most <α · NS(<t>).

A6: The timed view of every client is valid. A client <p> whose timed view does not contain CRASH<p> is a correct client. There is a set <F> of servers, with |<F>| ≤ <f>, such that the timed view of every server not in <F> is valid. The servers not in <F> are correct servers. The servers in <F> are Byzantine-faulty.

A7: If a READ<p> or WRITE<p> invocation occurs at time <t>, then <p> is an active client that has already joined (JOIN<p> occurs before <t>). (no LEAVE<p> or CRASH<p> occurs by time <t>).

A8: At each client node <p>, no READ<p> or WRITE<p> occurs until there have been responses to all previous READ<p> and WRITE<p> invocations.

Assumption A1 states that the system size is always finite and there is always some minimum number of servers in the system. Assumptions A2 through A4 model a reliable broadcast communication service that provides nodes with a mechanism to send the same message to all servers or to all clients in the system with message delays in (0, <D>]. However, Byzantine servers may choose not to s-bcast (respectively, c-bcast) the same message to every server/client in the system and just send different unicast messages to different nodes in the system. Assumption A5 bounds the server churn. Assumption A6 bounds the number of Byzantine-faulty servers and restricts the clients to experience only crash failures. Assumptions A7 and A8 ensure that operations are only invoked by joined and active clients and at, any time, at most one operation is pending at each client node.

We consider an algorithm to be correct if every execution of the algorithm satisfies the following conditions:

C1: For every client <p> that is in the system initially, JOINED<p> does not occur. Every client <p> that enters the system later and does not leave or crash eventually joins.

C2: In the view of each client <p>, ignoring message-receipt events, each READ<p> or WRITE<p> is immediately followed by either LEAVE<p>, CRASH<p>, or a matching response (RETURN<p> or ACK<p>), and each RETURN<p> or ACK<p> is immediately preceded by a matching invocation (READ<p> or WRITE<p>).

C3: The read and write operations are atomic (also called linearizable) [13], [20], [21]: there is an ordering of all completed reads and writes and some subset of the uncompleted writes such that every read returns the value of the latest preceding write (or the initial value of the register if there is no preceding write) and, if an operation op₁ finishes before another operation op₂ begins, then op₁ occurs before op₂ in the ordering.

It is the responsibility of the algorithm to complete joins, reads and writes, and choose the right values for the reads, as long as Assumptions A1–A8 are satisfied.

Although our model places an upper bound on message delays, it does not place any lower bound on the message delays or on local computation times. Moreover, nodes cannot access clocks to measure the passage of real time. Consequently, the well-known consensus problem is unsolvable in our model as proved in [9], just as it is unsolvable in a model with no upper bound on message delays [17].

III. IMPOSSIBILITY OF A UNIFORM ALGORITHM WITH BYZANTINE SERVERS
In this section we show it is impossible to emulate an atomic register in our system model if nodes have no information about the maximum number of Byzantine-faulty servers or about the total number of servers. We model the lack of such information through the notion of a “uniform” algorithm. There have been similar results proved for the impossibility of self-stabilization [5] and consensus [3] in Byzantine-tolerant systems with unknown participants.

An algorithm is called uniform if the code run by every node is independent of both the system size and the maximum number of Byzantine servers in the system. Thus in a uniform algorithm, for any particular node id \( p \), there are only two possible state machines that \( p \) can have. One is for the situation when \( p \) is in the system initially and the other is for the situation when \( p \) enters the system later. Otherwise, the state machines are completely independent of the initial size of the system (or the size when \( p \) enters) and of the maximum number of Byzantine servers.

**Theorem 1.** It is impossible to emulate an atomic\(^1\) read/write register in our model with a uniform algorithm.

**Proof.** Suppose in contradiction, there is a uniform algorithm, \( \mathcal{A} \), which simulates an atomic register and can tolerate \( f \) Byzantine server failures for any \( f \in \mathbb{N}^+ \), as long as the system size at all times is at least \( NS_{\text{min}}(f) \) for some function \( NS_{\text{min}} \). Consider the following two executions of \( \mathcal{A} \).

**Execution** \( e_1 \): The maximum number of Byzantine servers is 1. The set of servers in the system initially is \( S_1 \), where \( |S_1| = NS_{\text{min}}(1) \), and all servers are correct. All message delays are \( D \). The initial value of the simulated register is \( v \). A new client \( p \) enters the system at time \( t_e \) and joins by time \( t_j \geq t_e \). Client \( p \) invokes a read on the simulated register at time \( t_r > t_j \). No other operation on the simulated register is invoked. By assumed correctness of algorithm \( \mathcal{A} \), the read invoked by \( p \) returns \( v \) at some time \( t_r' \geq t_r \).

**Execution** \( e_2 \): Multiply the real time of every event in \( e_1 \) by \( \min\left(\frac{t_j}{t_r}, 1\right) \). As a result, all events in the time interval \([0, t_r']\) in \( e_1 \) are compressed into the interval \([0, D]\) in \( e_2 \).

**Execution** \( e_3 \): The maximum number of Byzantine servers is \( f_2 = |S_2| \). The set of servers in the system initially is \( S_2 \), where \( |S_2| = NS_{\text{min}}(f_2) \) and \( S_1 \) is a subset of \( S_2 \).

Note that the system sizes are chosen such that we have execution \( e_1 \) with \( |S_1| \) servers, all correct, and execution \( e_2 \) with exactly \( |S_1| \) Byzantine servers.

There is at least one client in the system initially. All message delays are \( D \). The initial value of the simulated register is \( v \). No churn happens in this execution. Client \( q \) that was in the system at time 0 invokes a write of \( v' \neq v \) at time \( t_w \). By the assumed correctness of algorithm \( \mathcal{A} \), the write completes at some time \( t_w' \geq t_w \). No other operation on the simulated register is invoked.

Finally we construct a prefix \( e_3 \) of a new execution from executions \( e_1' \) and \( e_2 \). First, we specify a set of timed views and then we show that this set indeed forms the prefix of an execution. Let \( e_3 \) be the set of timed views in \( e_2 \). Note that \( e_3 \) includes the write operation invoked by client \( q \). Truncate each timed view in \( e_3 \) immediately after the latest step with associated time at most \( t_w' \), i.e., just after the write by client \( q \) finishes. Then append steps that result in the immediate delivery of all messages that are in transit at \( t_w' \). Call the resulting set of timed views \( e_3 \). Construct \( e_3 \) from \( e_3 \) as follows.

**Execution prefix** \( e_3 \): Add to the set the prefix of the timed view of client \( p \) from \( e_1' \) that ends at time \( D \), but change the time associated with each step by adding \( t_w' \) to it. For each server \( s \) in \( S_1 \), append the prefix of \( s \)'s timed view in \( e_1' \) that ends at time \( D \), but change the time associated with each step by adding \( t_w' \) to it. Append nothing to the timed views for the remaining nodes (client or server).

The idea behind \( e_3 \) is to have all the nodes behave correctly through \( q \)'s write of \( v' \), and then have a new client \( p \) enter, join, and invoke a read during which time it communicates only with the servers in \( S_1 \). However, the servers in \( S_1 \) are Byzantine and start acting as if they did in \( e_1' \), causing \( p \)'s read to incorrectly return the value \( v' \) instead of \( v' \). An important technicality in the construction of \( e_3 \) is to adjust the time of steps taken from \( e_1' \). The assumed uniform nature of the algorithm is what allows us to combine timed views from \( e_1 \), in which at most one server can be Byzantine, with timed views from \( e_2 \), in which \( f_2 > 1 \) servers can be Byzantine.

In order for the existence of the incorrect read by \( p \) in \( e_3 \) to contradict the assumed correctness of \( \mathcal{A} \), we must show that \( e_3 \) is the prefix of an execution (otherwise, bad behavior by \( \mathcal{A} \) is irrelevant).

We show that \( e_3 \) is a prefix of an execution by verifying properties \( A_1 \) through \( A_8 \), \( A_1 \), \( A_5 \), and \( A_6 \)-\( A_8 \) are clear.

\( A_2-A_4 \): Every message sent by a node (client or server) has exactly one matching receipt. We show this in two parts: (i) If the message was sent before \( t_w' \), it was either delivered before \( t_w' \) (from \( e_2 \)) or it was pending at \( t_w' \) (from construction of \( e_3 \)). (ii) If the message was sent after \( t_w' \): Messages exchanged between \( p \) and \( S_1 \) after \( t_w' \) are all delivered within \( D \) time (from \( e_1' \)) and all other messages in \( e_3 \) after \( t_w' \) are delivered with delay \( D \). All message delays in \( e_1' \) are \( \leq D \) and the message delays in \( e_2 \) are \( D \). Therefore, the message delays in \( e_3 \) are at most \( D \).

In \( e_3 \), \( p \)'s read returns \( v \), whereas the latest preceding write wrote \( v' \neq v \). The value returned by node \( p \) is incorrect and as \( e_3 \) is the prefix of an execution, this violates the safety property of the register. Therefore, it is impossible to simulate a shared register in dynamic systems where new nodes entering have no information about the system size and no information on the maximum number of Byzantine servers present in the system.

IV. THE ABCC ALGORITHM

The ABCC algorithm is loosely based on the algorithm in [9] along with modifications to accommodate the new client-server model, as opposed to the peer-to-peer model, and Byzantine servers as opposed to crashes in [9]. The ABCC

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\(^1\)This theorem and proof holds for a *safe* [21] register as well.
algorithm is divided into two main parts: Algorithm 1 for servers and Algorithm 2 for clients. Algorithm 3 contains a set of common procedures used by both servers and clients. The server algorithm contains a mechanism for tracking the composition of the system with respect to servers and for assisting clients with reads and writes. The client algorithm is for newly entered clients to join the system and for joined clients to read from and write to the shared register.

Initially the system consists of a set of servers $S_0$ and a set of clients $C_0$ such that $|S_0| \geq NS_{min}$ and $|C_0| \geq 0$. A server $p \in S_0$ is joined at time 0 and it knows about all other servers $q \in S_0$. In the following code description, we use the convention that local variables of node $p$ are subscripted with $p$. Each node $p$ maintains a set of events, $Server\_Changes_p$, concerning the servers that have entered, joined and left the system. A node $p$ also maintains the set $Present_p$ that stores information about servers that have entered, but have not left, as far as $p$ knows. A server $p$ is called a member if it has joined the system but not left. Client $p$ maintains the derived variable $Members_p$ of servers that $p$ considers as members. The variables $val_p$, $num_p$ and $w_{idp}$ store the latest register value and its timestamp, consisting of the ordered pair $(num_p, w_{idp})$, known by $p$. The variable $w_{idp}$ stores the id of the writer (client) that wrote $val_p$. The set $Known\_Writes[p]$ stores an entry for all nodes $q$ that $p$ thinks has entered. An entry $Known\_Writes[q]_p$ stores all the values written to the register that server $q$ has declared to know about. So, at all times $Known\_Writes[p]_p$ stores the values of all writes that $p$ has heard of through an “update” message from a client performing a write or (ii) that occur in more than $f$ entries in $Known\_Writes[p]_p$.

Algorithm 1. When a server $p$ enters the system, it broadcasts to all the servers an enter message requesting information about prior events. When a server $q$ finds out that node $p$ has entered (or joined or left) the system, $q$ updates $Server\_Changes_q$ accordingly and sends out an echo message with information about the system (stored in $Server\_Changes_q$) and the shared register (stored in the variable $Known\_Writes[q]_q$). When node $p$ receives at least $f+1$ enter-echo messages from joined servers (to make sure at least one reply is from a correct server), it calculates the number of replies it needs in order to join as a fraction of the number of servers it knows are present, i.e., $\gamma \cdot |Present_p|$. This value is in the $join\_bound_p$ local variable. Setting $\gamma$ is a key challenge in the algorithm as setting it too small might not propagate updated information, whereas setting it too large might not guarantee termination of the join. A server $q \notin S_0$ is considered joined once it has executed line number 130 of Algorithm 3.

The algorithm sends out messages that are authenticated with digital signatures. As a result Byzantine servers can send out incorrect information about everything except for node ids. Byzantine servers can modify information about anything sent out in messages of Algorithm 1 subject to the following restrictions:

- $Server\_Changes_p$: A Byzantine server $p$ can only send out subsets of the $Server\_Changes_q$ set for some $q$ that has entered the system. Server $p$ cannot modify entries as each entry in this variable contains a server node id which was digitally signed by the sending server.
- $val_p$, $num_p$ and $w_{idp}$: A Byzantine server can modify variables $val_p$ and $num_p$ while sending them out. But it cannot modify the $w_{idp}$ variable which is the id of the client that invoked the write or $\perp$.
- $Known\_Writes[p]$: A Byzantine server can send out subsets of $Known\_Writes[p]$, but cannot add entries. For an entry $(val_q, (num_q, w_{idq})) \in Known\_Writes[q]_p$, Byzantine servers can modify the $val$ and $num$ variables of this entry for node $q$.

The $JoinProtocol_p$ procedure in Algorithm 2 is used by both newly entered servers and clients to join the system. Once joined, servers can reply to read/write queries from clients. In addition to that, for all nodes $p$, there exists an in-built procedure, $IsClient_p(q)$ that can check, based on the node id $q$, if $q$ is a client or not. This procedure prevents Byzantine servers from pretending to be clients.

Algorithm 2. Clients might be in the system from the start or may enter the system at any time. Similar to servers, a newly entered client $p$ runs the $JoinProtocol_p$ procedure in Algorithm 3 to join the system. Clients treat both read and write operations in a similar manner. Both operations start with a read phase, which requests the current value of the register, using a query message, followed by a write phase, using an update message. A write operation broadcasts to all servers the new value it wishes to write, together with a timestamp, which consists of a sequence number that is one larger than the largest sequence number it has seen and its id that is used to break ties. A read operation just broadcasts to all servers the value it is about to return, keeping its sequence number as is. As in $\ref{c}$, write-back is needed to ensure the atomicity of read operations. Both the read phase and the write phase wait to receive sufficiently many reply messages. The fraction $\beta$ calculates the number of replies it needs for the operations to terminate, as a fraction of the number of nodes it believes are members, i.e., $\beta \cdot |Members_p|$. This value is stored in the $rw\_bound$ local variable. Setting $\beta$ is also a key challenge in the algorithm as setting it too small might not return/update correct information from/to the register, whereas setting it too large might not guarantee termination of the reads and writes. The fraction $\beta$ also has to ensure that enough replies from correct servers are heard so that these replies can efficiently mask incorrect replies from Byzantine servers.

Algorithm 3. The $JoinProtocol()$ procedure helps newly entered nodes to join the system. The other procedures in this algorithm are used to deal with Byzantine servers and their arbitrary nature. The procedure $SetValueTimestamp()$ checks and updates the value-timestamp triple $((val, (num, w_{id}))_p$ to $valid\_val_p$ if the timestamp of $valid\_val_p$ is higher than the

\footnote{Note that joining is not the same as entering. Once a node (client or server) enters the system, it has to complete running the $JoinProtocol$ subroutine to become “joined”.}
Algorithm 1 ABCC—Code for server $p$.

In-built Procedure:  
IsClient(q) {returns true if $q$ is a client and false if $q$ is a server }

Local Variables:  
Server_Changes {set that stores information about entering, leaving and joining of servers known by $p$ }
    {initially $\{\text{enter}(q) \mid q \in S_0\} \cup \{\text{join}(q) \mid q \in S_0\}$, if $p$ is in the system at time 0 and 0 otherwise }
join_bound {if non-zero, the number of enter-echo messages $p$ should receive before joining; initially 0}
enter_echo_counter {number of enter-echo messages received so far; initially 0}
enter_echo_from_joined_counter {number of enter-echo messages from joined servers received so far; initially 0}
is_joined {Boolean to check if $p$ has joined the system; initially false}
val {latest register value known to $p$; initially $\bot$}
num {sequence number of latest value known to $p$; initially 0}
w_id {id of node that wrote latest value known to $p$; initially $\bot$}
Known_Writes[] {map from the set of node ids to the powerset of value-timestamp pairs. Initially each entry is $\emptyset$}

Derived Variables:  
$\text{Present} = \{q \mid \text{enter}(q) \in \text{Server_Changes} \land \text{leave}(q) \notin \text{Server_Changes}\}$
valid_val = value-timestamp pair with latest timestamp that occurs in at least $(f+1)$ elements of Known_Writes[], else $(\bot, (0, \bot))$

When Enter$_p$ occurs:
1. Server_Changes := Server_Changes $\cup \{\text{enter}(p)\}$
2. s-bcast ("enter", $p$")
3. c-bcast ("server-info", Server_Changes)

When Receive$_p$("enter", $q$) occurs:
4. if IsValidMessage ("enter", $q$) then
5. Server_Changes := Server_Changes $\cup \{\text{enter}(q)\}$
6. s-bcast ("enter-echo", Server_Changes, Known_Writes[$p$], is_joined, $q$, $p$)
7. c-bcast ("server-info", Server_Changes)
8. end if

When Receive$_p$("enter-client", $q$) occurs:
9. if IsClient($q$) then
10. c-bcast ("enter-client-echo", Server_Changes, Known_Writes[$p$], is_joined, $q$, $p$)
11. end if

When Receive$_p$("enter-echo", $C$, $K$, $j$, $q$, $r$) occurs:
12. if IsValidMessage ("enter-echo", $q$, $r$) then
13. Server_Changes := Server_Changes $\cup C$
14. if ($j$ = true) then
15. Known_Writes[$r$] := Known_Writes[$r$] $\cup K$
16. end if
17. if $\neg$is_joined $\land$ ($p$ = $q$) then
18. call JoinProtocol($j$)
19. end if
20. call SetValueTimestamp()
21. end if

When Receive$_p$("joined", $q$) occurs:
22. if IsValidMessage ("joined", $q$) then
23. Server_Changes := Server_Changes $\cup \{\text{enter}(q), \text{join}(q)\}$
24. s-bcast ("joined-echo", $q$, $p$)
25. c-bcast ("server-info", Server_Changes)
26. end if

When Receive$_p$("joined-echo", $q$, $s$) occurs:
27. if IsValidMessage ("joined-echo", $q$, $s$) then
28. Server_Changes := Server_Changes $\cup \{\text{enter}(q), \text{join}(q)\}$
29. c-bcast ("server-info", Server_Changes)
30. end if

When Leave$_p$ occurs:
31. Server_Changes := Server_Changes $\cup \{\text{leave}(p)\}$
32. s-bcast ("leave", $p$")
33. c-bcast ("server-info", Server_Changes)
34. halt

When Receive$_p$("leave", $q$) occurs:
35. if IsValidMessage ("leave", $q$) then
36. Server_Changes := Server_Changes $\cup \{\text{leave}(q)\}$
37. s-bcast ("leave-echo", $q$, $p$)
38. c-bcast ("server-info", Server_Changes)
39. end if

When Receive$_p$("leave-echo", $q$, $s$) occurs:
40. if IsValidMessage ("leave-echo", $q$, $s$) then
41. Server_Changes := Server_Changes $\cup \{\text{leave}(q)\}$
42. c-bcast ("server-info", Server_Changes)
43. end if

When Receive$_p$("query", $r$, $q$) occurs:
44. if is_joined $\land$ IsClient($q$) then
45. c-bcast ("reply", Known_Writes[$p$], $r$, $q$,$p$
46. end if

When Receive$_p$("update", $(v, s, i)$, $wt$, $q$) occurs:
47. if IsClient($q$) then
48. if ($s$, $i$) > (num, w_id) then
49. (val, num, w_id) := ($v$, $s$, $i$
50. Known_Writes[$p$] := Known_Writes[$p$] $\cup$
      {(val, num, w_id)}
51. end if
52. if $\neg$is_joined then
53. c-bcast ("ack", $wt$, $q$, $p$
54. end if
55. s-bcast ("update-echo", Known_Writes[$p$], $p$
56. end if

When Receive$_p$("update-echo", $K$, $s$) occurs:
57. Known_Writes[$s$] := Known_Writes[$s$] $\cup K$
58. call SetValueTimestamp()
Algorithm 2 ABCC—Code for client, p.

In-built Procedure:
IsClient(q) {returns true if q is a client and false if q is a server}

Local Variables:
Server_Changes {set that stores information about entering, leaving and joining of servers known by p}
{initially \{enter(q) | q ∈ S₀\} ∪ \{join(q) | q ∈ S₀\}, if p is in the system at time 0 and ∅ otherwise}
enter_echo_counter {number of enter-echo messages received so far; initially 0}
enter_echo_from_joined_counter {number of enter-echo messages from joined servers received so far; initially 0}
is_joined {Boolean to check if p has joined the system; initially false}
val {latest register value known to p; initially ⊥}
num {sequence number of latest value known to p; initially 0}
w_id {id of node that wrote latest value known to p; initially ⊥}
Known_Writes[] {map from set of node ids to the powerset of value-timestamp pairs. Initially each entry is ∅}
tag {used to uniquely identify read and write phases of an operation; initially 0}

Derived Variables:
Present = \{q | enter(q) ∈ Server_Changes ∧ leave(q) \notin Server_Changes\}
Members = \{q | join(q) ∈ Server_Changes ∧ leave(q) \notin Server_Changes\}
valid_val = value-timestamp pair with latest timestamp that occurs in at least (f + 1) elements of Known_Writes[], else (⊥, (0, ⊥))

When ENTER_p occurs:
59: s-bcast (“enter-client”, p)

When RECEIVE_p(“enter-client-echo”, C, j, q, r) occurs:
60: if IsValidMessage(“enter-client-echo”, q) ∧ (p = q) then
61: Server_Changes := Server_Changes ∪ C
62: if (j = true) then
63: Known_Writes[r] := Known_Writes[r] ∪ K
64: end if
65: if ¬is_joined ∧ (p = q) then
66: call JoinProtocol(j)
67: end if
68: end if
69: call SetValueTimestamp()

When RECEIVE_p(“server-info”, C) occurs:
70: Server_Changes := Server_Changes ∪ C

Procedure BeginReadPhase()
71: tag++
72: s-bcast (“query”, tag, p)
73: rw_counter := β ∩ Members
74: rw_counter := 0
75: rw_pending := true

When RECEIVE_p(“reply”, K, r, q, s) occurs:
76: if IsValidMessage(“reply”, q, rt, s) then
77: if rw_pending ∧ (rt = tag) ∧ (q = p) then
78: rw_counter++
79: Known_Writes[s] := Known_Writes[s] ∪ K
80: if rw_counter ≥ rw_bound then
81: call SetValueTimestamp()
82: rw_pending := false
83: end if
84: end if
85: end if
86: end if

Procedure BeginWritePhase()
87: if write_pending then
88: val := temp
89: num++
90: w_id := p
91: end if
92: if read_pending then
93: temp := val
94: end if
95: s-bcast (“update”, (temp, num, w_id), tag, p)
96: rw_bound := β ∩ Members
97: rw_counter := 0
98: wp_pending := true

When RECEIVE_p(“ack”, wt, q, s) occurs:
99: if IsValidMessage(“ack”, q, wt, s) then
100: if wp_pending ∧ (wt = tag) ∧ (q = p) then
101: rw_counter++
102: if rw_counter ≥ rw_bound then
103: wp_pending := false
104: if read_pending then
105: read_pending := false
106: generate RETURN(temp) response
107: end if
108: if write_pending then
109: write_pending := false
110: generate ACK response
111: end if
112: end if
113: end if
114: end if
115: halt

When LEAVE_p occurs:
The correctness of ABCC relies on the system parameters $\alpha$, $f$, and $NS_{min}$ satisfying the following constraints, for some choice of algorithm parameters $\beta$ and $\gamma$:

\begin{align}
\alpha &\leq 1 - 2^{-1/4} \approx 0.159 \tag{1} \\
1 &\leq (1-\alpha)^3NS_{min} - 2f \tag{2} \\
\gamma &\geq \frac{1 + 2f}{(1-\alpha)^3NS_{min}} + \frac{(1+\alpha)^3}{(1-\alpha)^3} - 1 \tag{3} \\
\gamma &\leq \frac{(1-\alpha)^3}{(1-\alpha)^3} - \frac{f}{(1-\alpha)^3NS_{min}} \tag{4} \\
\beta &\leq \frac{(1+\alpha)^2}{(1-\alpha)^2NS_{min}} - \frac{f}{NS_{min}} \tag{5} \\
\beta &> \frac{(1+\alpha)^5 - 1 + 2f/NS_{min}}{(1-\alpha)^4 - f/NS_{min}} \tag{6} \\
\beta &> \frac{(1+\alpha)^3 - (1-\alpha)^3 + 1 + (1 + 3f)/NS_{min}}{[(2 + 2\alpha + \alpha^2)(1-\alpha)^2(1+\alpha)^2 - 2f/NS_{min}] - (7)}
\end{align}

Constraint (1) is an upper bound on the churn rate to ensure that not too many servers can leave the system in an interval of length $4D$. Constraint (2) is a lower bound on the minimum system size to ensure that at least $f + 1$ correct servers are in the system throughout an interval of length $3D$ encompassing the time a node enters, thus ensuring that the newly entered node successfully terminates its joining protocol. Constraint (3) ensures that the $join\_bound$ fraction, $\gamma$, is large enough such that updated information about the system is obtained by an entered node before it joins the system. Constraint (4) ensures that $\gamma$ is small enough such that for all entered nodes, a join operation terminates if the entered node is not faulty and does not leave. Constraint (5) ensures that the $rw\_bound$ fraction, $\beta$, is small enough such that termination of reads and writes is guaranteed. Constraints (6) and (7) ensure that $\beta$ is large enough such that atomicity is not violated by read and write operations. Table I gives a few sets of values for which the above constraints are satisfied. In all consistent sets of parameter values, the churn rate $\alpha$ is never more than 0.05 and $NS_{min} > 8.5f$. The algorithm can tolerate any size of $f$ as long as $NS_{min}$ is proportionally big.

ABCC violates atomicity if Assumption A5 is violated.

| maximum failures | minimum system size ($NS_{min}$) | churn rate ($\alpha$) | $join\_bound$ fraction ($\gamma$) | $rw\_bound$ fraction ($\beta$) |
|------------------|----------------------------------|----------------------|----------------------------------|-------------------------------|
| 1                | 8                                | 0                    | N/A                              | 0.86                          |
| 1                | 10                               | 0.01                 | 0.82                             | 0.84                          |
| 1                | 13                               | 0.02                 | 0.79                             | 0.80                          |
| 1                | 190                              | 0.05                 | 0.79                             | 0.80                          |
| 2                | 19                               | 0.01                 | 0.80                             | 0.83                          |
| 2                | 24                               | 0.02                 | 0.81                             | 0.82                          |
| 2                | 347                              | 0.05                 | 0.70                             | 0.77                          |
| 5                | 44                               | 0.01                 | 0.80                             | 0.83                          |
| 5                | 57                               | 0.02                 | 0.79                             | 0.82                          |
| 5                | 826                              | 0.05                 | 0.79                             | 0.82                          |
| 10               | 85                               | 0.01                 | 0.80                             | 0.83                          |
| 10               | 113                              | 0.02                 | 0.79                             | 0.82                          |
| 100              | 838                              | 0.01                 | 0.79                             | 0.82                          |
| 100              | 1107                             | 0.02                 | 0.79                             | 0.82                          |
| 100              | 16015                            | 0.05                 | 0.79                             | 0.82                          |
| 100              | 8360                             | 0.01                 | 0.79                             | 0.82                          |
| 1000             | 11042                            | 0.02                 | 0.79                             | 0.82                          |
| 1000             | 159935                           | 0.05                 | 0.79                             | 0.82                          |

TABLE I: Values for the parameters that satisfy constraints (1)–(7).

V. CORRECTNESS PROOF OF ABCC

We will show that ABCC satisfies the properties C1 to C3 listed at the end of Section II. Lemmas 1 through 6 are used to prove Theorem 2, which states that every client and any correct server eventually joins, provided it does not crash or leave. Lemmas 7 through 9 are used to prove Theorem 3, which states that every operation invoked by a client that remains active eventually completes. Lemmas 10 through 13 are used to prove Theorem 4, which states that atomicity is satisfied.

Consider any execution. We begin by bounding the number of servers that enter during an interval of time and the number of servers that are present at the end of the interval, as compared to the number present at the beginning.

Lemma 1. For all $i \in \mathbb{N}$ and all $t \geq 0$, at most $((1 + \alpha)^3 - 1)NS(t)$ servers enter during $(t, t + Di]$ and $(1 - \alpha)^4NS(t) \leq NS(t + Di) \leq (1 + \alpha)^4NS(t)$.

Proof. The proof is by induction on $i$ and is adapted from [9]. For $i = 0$ and all $t \geq 0$, $(t, t + Di]$ is empty, and hence, $0 = ((1 + \alpha)^3 - 1)NS(t)$ servers enter during this interval and $NS(t + iD) = NS(t) = (1 + \alpha)^4NS(t) = (1 - \alpha)^4NS(t)$. Now let $i \geq 0$ and $t \geq 0$. Suppose at most $((1 + \alpha)^3 - 1)NS(t)$ servers enter during $(t, t + Di]$ and $(1 - \alpha)^4NS(t) \leq NS(t + Di) \leq (1 + \alpha)^4NS(t)$.
Let $e \geq 0$ and $\ell \geq 0$ be the number of servers that enter and leave, respectively, during $(t+D_i, t+D(i+1)]$. By Assumption A5, $e + \ell \leq \alpha NS(t + D_i)$, so $e, \ell \leq \alpha NS(t + D_i) \leq (1 + \alpha)^i NS(t)$. The number of servers that enter during $(t, t + D(i + 1)]$ is at most

\[
((1 + \alpha)^i - 1)NS(t) + e \leq ((1 + \alpha)^i - 1)NS(t) + (1 + \alpha)^i NS(t) = ((1 + \alpha)^{i+1} - 1)NS(t).
\]

Hence,

\[
NS(t + D(i + 1)) \leq NS(t) + ((1 + \alpha)^{i+1} - 1)NS(t) = (1 + \alpha)^{i+1}NS(t).
\]

Furthermore,

\[
NS(t + D(i + 1)) \geq NS(t + D_i) - \ell \\
\geq NS(t + D_i) - \alpha NS(t + D_i) \\
= (1 - \alpha)NS(t + D_i) \geq (1 - \alpha)^{i+1}NS(t).
\]

By induction, the claim is true for all $i \in \mathbb{N}$.

We are also interested in the number of servers that leave during an interval of time. The calculation of the maximum number of servers that leave during an interval is complicated by the possibility of servers entering during the interval, allowing additional servers to leave.

**Lemma 2.** For $\alpha > 0$, all nonnegative integers $i \leq -1/\log_2(1 - \alpha)$ and every time $t \geq 0$, at most $(1 - (1 - \alpha)^i)NS(t)$ servers leave during $(t, t + D_i]$.

**Proof.** The proof is by induction on $i$ and is adapted from [9]. When $i = 0$, the interval is empty, so $0 = (1 - (1 - \alpha)^0)NS(t)$ servers leave during the interval. Now let $i \geq 0$, let $t \geq 0$, and suppose at most $(1 - (1 - \alpha)^i)NS(t + D)$ servers leave during $(t + D, t + D(i + 1)]$.

Let $e \geq 0$ and $\ell \geq 0$ be the number of servers that enter and leave, respectively, during $(t, t + D)$. By Assumption A5, $e + \ell \leq \alpha NS(t + D)$, so $e, \ell \leq \alpha NS(t + D) \leq (1 + \alpha)^i NS(t)$. The number of servers that leave during $(t, t + D(i + 1)]$ is the number that leave during $(t, t + D) + \ell$ plus the number that leave during $(t + D, t + D(i + 1)]$, which is at most

\[
\ell + (1 - (1 - \alpha)^i)NS(t + D) \\
\leq \ell + (1 - (1 - \alpha)^i)((1 + \alpha)NS(t) - 2\ell) \\
= (1 - (1 - \alpha)^i)(1 + \alpha)NS(t) + 2(1 - (1 - \alpha)^i)\ell \\
\leq (1 - (1 - \alpha)^i)(1 + \alpha)NS(t) + 2(1 - (1 - \alpha)^i - 1)\alpha NS(t) \\
= (1 - (1 - \alpha)^i + 1)NS(t).
\]

Note that $2(1 - (1 - \alpha)^i - 1 \geq 0$, since $i \leq -1/\log_2(1 - \alpha)$. By induction, the claim is true for all $i \in \mathbb{N}$.

Lemma 3 proves that at least $f + 1$ correct servers are active throughout any interval of length $3D$. This lemma is necessary to ensure that at all times, an active node (client or server) that expects replies, hears back from at least $f + 1$ correct servers in order to mask the bad information sent by Byzantine servers.

**Lemma 3.** For every $t > 0$, at least $f + 1$ correct servers are active throughout $[\max\{0, t - 2D\}, t + D]$. 

**Proof.** Let $S$ be the set of servers present at time $t' = \max\{0, t - 2D\}$, so $|S| = NS(t') \geq NS_{\text{min}}$. Constraint [1]
implies that $-1/\log_2(1 - \alpha) \geq 4 \geq 3$. So, by Lemma 2, at most $(1 - (1 - \alpha)^3)|S|$ servers leave during $[t', t + D]$ and there are at least $(1 - \alpha)^3|S|$ servers present throughout time interval $[t', t + D]$. At any point in time, there are at most $f$ Byzantine servers in the system. Thus, at least

$$(1 - \alpha)^3|S| - f \geq (1 - \alpha)^3 NS_{\min} - f$$

correct servers in $S$ are active at time $t + D$. By Constraint 4, $(1 - \alpha)^3 NS_{\min} - f \geq f + 1$, so at least $f + 1$ correct servers in $S$ are still active at time $t + D$.

Below, a local variable name is superscripted with $t$ to denote the value of that variable at time $t$; e.g., $v_p^t$ is the value of node $p$'s local variable $v$ at time $t$.

In the analysis, we frequently compare the data in nodes' Server_Changes sets to the set of ENTER, JOINED, and LEAVE events that have actually occurred. To facilitate this comparison, we define a set SysInfo$^t$ that contains perfect information about correct servers for the time interval $I$. For each server $q$, let $t_q^e$ and $t_q^l$ be the times when the events ENTER$^t_q$ and LEAVE$^t_q$ occur, and let $t_q^j$ be the time when server $q$ sends out a joined message. Similarly, for each client $q$, let $t_q^c$, $t_q^q$, and $t_q^l$ be the times when the events ENTER$^t_q$, JOINED$^t_q$, and LEAVE$^t_q$ occur, respectively.

Recall that $S_0$ is the set of servers that were in the system initially. If $q \in S_0$, then we set $t_q^e = t_q^l = 0$. Then we have:

$$SysInfo^t = \{\text{enter}(q) \mid t_q^e \in I\} \cup \{\text{join}(q) \mid t_q^j \in I\} \cup \{\text{leave}(q) \mid t_q^l \in I\}.$$ 

In particular,

$$SysInfo^{[0,0]} = \{\text{enter}(q) \mid q \in S_0\} \cup \{\text{join}(q) \mid q \in S_0\}.$$ 

Since a client or correct server $p$ that is active throughout $[t_p^e, t + D]$ directly receives all enter, joined, and leave messages broadcast by active clients or correct servers during $[t_p^e, t]$, within $D$ time, we have:

**Observation 1.** For every client and any correct server $p$ and all times $t \geq t_p^e$, if $p$ is active at time $t + D$, then $SysInfo^{[p,t]} \subseteq Server\_Changes^{t + D}$.

Let $C_0$ be the set of clients that are in the system initially. By assumption, for every node $p \in S_0 \cup C_0$, $SysInfo^{[0,0]} \subseteq Server\_Changes^p$, and hence Observation 1 implies:

**Observation 2.** For every client and any correct server $p \in S_0 \cup C_0$ if $p$ is active at time $t \geq 0$, then $SysInfo^{[0,\max(0,t-D)]} \subseteq Server\_Changes^p$.

The purpose of Lemmas 4 to 6 show that information about correct servers entering, joining, and leaving is propagated to active clients and correct servers properly, via the Server_Changes sets.

**Lemma 4.** Suppose that, at time $T''$, a client or correct server $p \notin S_0 \cup C_0$ receives an enter-echo message from a correct server $q$ sent at time $T'$ in reply to an enter message from $p$. Let $T$ be any time such that $\max\{0,T'' - 2D\} \leq T \leq t_p^e$. Suppose $p$ is active at time $T + 2D$ and $q$ is active throughout $[U, T + D]$, where $U = \max\{0, T'' - 2D\}$. Then $SysInfo^{[U,T]} \subseteq Server\_Changes_{T'}^{p + 2D}$.

**Proof.** The proof is adapted from [9] to include Byzantine servers. Consider any node $r$ that enters, joins, or leaves at time $t \in (U, T]$. Note that $q$ directly receives this event's announcement, since $q$ is active throughout $[U, T + D]$, which contains $[t, t + D]$, the interval during which the announcement message is in transit. There are two cases, depending on the time $v$, at which $q$ receives this message.

Case 1: $v \leq T'$. Since $q$ receives the enter message from $p$ at $T'$, information about this change to $r$ is in $Server\_Changes^q_T$, in the enter-echo message that $q$ sends to $p$ at time $T'$. Thus, this information is in $Server\_Changes^q_{T'} \subseteq Server\_Changes_{T'}^{p + 2D}$.

Case 2: $v > T'$. Messages are not received before they are sent, so $T' \geq t_p^e$. Since $v \leq t + D$, it follows that $v + D \leq t + 2D \leq T + 2D$. Thus $[v, v + D]$ is contained in $[t_p^e, T + 2D]$. Immediately after receiving the announcement about $r$, server $q$ broadcasts an echo message in reply. Since $p$ is active throughout this interval, it directly receives this echo message.

In both cases, the information about $r$'s change reaches $p$ by time $T + 2D$. It follows that $SysInfo^{[U,T]} \subseteq Server\_Changes_{T'}^{p + 2D}$.

**Lemma 5.** For every client and any correct server $p$, if $p$ is active at time $t \geq t_p^e + 2D$, then $SysInfo^{[0,t-D]} \subseteq Server\_Changes^p$.

**Proof.** The proof is adapted from [9] to include Byzantine servers. The proof is by induction on the order in which nodes enter the system. If $p \in S_0 \cup C_0$, then $t_p^e = 0$, so $SysInfo^{[0,t-D]} \subseteq Server\_Changes^p$ follows from Observation 2.

Now consider any node $p \notin S_0 \cup C_0$ and suppose that the claim holds for all nodes that enter earlier than $p$. Suppose $p$ is active at time $t \geq t_p^e + 2D$. By Lemma 3, there is at least $f + 1$ servers (let $q$ be one of these) that are active throughout $[\max\{0, t_p^e - 2D\}, t_p^e + D]$. Server $q$ receives an enter message from $p$ at some time $t' \in [t_p^e, t_p^e + D]$ and sends an enter-echo message back to $p$. This message is received by $p$ at some time $t'' \in [t', t + D]$.

If $q \in S_0$, then $SysInfo^{[0,\max(0,t'-D)]} \subseteq Server\_Changes_q^{t'}$, by Observation 2. If $q \notin S_0$, then $0 < t'' \leq \max\{0, t_p^e - 2D\}$, so $t'' \leq t_p^e - 2D$. Therefore $t'' + 2D \leq t_p^e - t'$. Since $q$ entered earlier than $p$, it follows from the induction hypothesis that $SysInfo^{[0,t'-D]} \subseteq Server\_Changes_q^{t'}$. Thus, in both cases, $SysInfo^{[0,\max(0,t'-D)]} \subseteq Server\_Changes^p_{t''}$. At time $t'' \leq t$, $p$ receives the enter-echo message from $q$, so $SysInfo^{[0,\max(0,t'-D)]} \subseteq Server\_Changes^p_{t''} \subseteq Server\_Changes^p$. 

Applying Lemma 4 for $q$, with $U = \max\{0, t'_{p} - D\}$, $T = t'_{p}$, $T^{r} = t'$ and $T'' = t''$ implies
\[ \text{SysInfo}_{\max\{0, t'_{p} - D\}, t'_{p}} \subseteq \text{Server}_{changes}^{t'_{p} + 2D} \].

Since $t \geq t'_{p} + 2D$, $\text{Server}_{changes}^{t'_{p} + 2D}$ is a subset of $\text{Server}_{changes}^{t'_{p}}$. Observation 1 implies $\text{SysInfo}_{t'_{p}, t^{r} - D} \subseteq \text{Server}_{changes}^{t'_{p}}$. Hence, $\text{SysInfo}_{0, t'_{p} - D} \subseteq \text{Server}_{changes}^{t'_{p}}$.

**Lemma 6.** For every client and any correct server $p \notin S_{0} \cup C_{0}$ if $p$ joins at time $t'_{p}$ and is active at time $t \geq t'_{p}$, then $\text{SysInfo}_{0, \max\{0, t'_{p} - 2D\}} \subseteq \text{Server}_{changes}^{t''_{p}}$.

**Proof.** The proof is by induction on the order in which clients and correct servers join the system. Let $p \notin S_{0} \cup C_{0}$ be a client or correct server that joins at time $t'_{p}$ and suppose the claim holds for all clients and correct servers that join before $p$. If $t \geq t'_{p} + 2D$, then the claim follows by Lemma 5. So, suppose $t < t'_{p} + 2D$.

Before joining, $p$ receives $f + 1$ enter-echo message from joined servers in reply to its enter message (Line number 119). Out of these, at most $f$ can be from Byzantine servers. Thus, at least one reply is from a correct server. Suppose $p$ receives the first enter-echo message at time $t'$ sent by a correct server $q$ at time $t'$; $t'_{p} \leq t' \leq t'' \leq t'_{p}$. From Lemma 5, we know that this message from $q$ has a perfect information about the $\text{Server}_{changes}^{t' - 2D}$ set. This in turn means that it has perfect information about the derived set $\text{Present}^{t' - 2D}$. Byzantine servers can only modify the information about the $\text{Server}_{changes}$ set by sending a subset of its $\text{Server}_{changes}$ set. So, when node $p$ receives at least one reply is from a correct server, the incomplete information sent by Byzantine servers is overshadowed by this one reply from $q$ and thus $p$ has a perfect information about $\text{Present}^{t' - 2D}$.

If correct server $q \in S_{0}$, then by Observation 2, $\text{SysInfo}_{0, \max\{0, t'_{p} - D\}} \subseteq \text{Server}_{changes}^{t''_{p}}$. Otherwise, by the induction hypothesis, $\text{SysInfo}_{0, \max\{0, t' - 2D\}} \subseteq \text{Server}_{changes}^{t''_{p}}$. Since $q$ joined prior to $p$ and is active at time $t' \geq t'_{p}$, Note that $\text{Server}_{changes}^{t''_{p}} \subseteq \text{Server}_{changes}^{t''_{p}} \subseteq \text{Server}_{changes}^{t''_{p}}$. If $t \leq 2D$, then $\max\{0, t'_{p} - 2D\} = 0$ and the claim holds.

If $t > 2D$, then let $S$ be the set of servers present at time $\max\{0, t'_{p} - 2D\}$; $|S| = NS_{\max\{0, t'_{p} - 2D\}}$. By Lemma 2 and Constraint 1, at most $(1 - (1 - \alpha)^{3})|S|$ servers leave during $\max\{0, t'_{p} - 2D\}, t' + D)$. Since $t' \leq t' + D$, it follows that $|\text{Present}^{t'_{p}}| \geq |S| - (1 - (1 - \alpha)^{3})|S| = (1 - \alpha)^{3}|S|$. Hence, from lines 120 and 123 of Algorithm 3, $p$ waits until it has received at least $\text{join\_bound} = \gamma \cdot |\text{Present}^{t'_{p}}| \geq \gamma \cdot (1 - \alpha)^{3}|S|$ enter-echo messages before joining.

By Lemma 7, at most $(1 - (1 - \alpha)^{3})|S|$ servers enter during $\max\{0, t'_{p} - 2D\}, t' + D)$. Thus, at time $t' + D$, at most $(1 + \alpha)^{3}|S|$ servers are present, at most $f$ of which are Byzantine.

Hence, the number of enter-echo messages $p$ receives before joining from servers that were active throughout $\max\{0, t' - 2D\}, t' + D$ is $\text{join\_bound}$ minus the total number of server enters, leaves and faults (as Byzantine servers may not reply at all), which is at least
\[
\gamma \cdot (1 - \alpha)^{3}|S| - (1 + (1 + \alpha)^{3} - 1)|S| + (1 - (1 - \alpha)^{3})|S| + f
\]
\[
= [(1 + \gamma)(1 - \alpha)^{3} - (1 + (1 + \alpha)^{3})|S| - f
\]
\[
\geq [(1 + \gamma)(1 - \alpha)^{3} - (1 + (1 + \alpha)^{3})]NS_{\min} - f - f
\]

Rearranging Constraint 3, we get
\[
[(1 + \gamma)(1 - \alpha)^{3} - (1 + (1 + \alpha)^{3})]NS_{\min} - f \geq f + 1,
\]
so expression 9 is at least $f + 1$. Hence $p$ receives an enter-echo message at some time $t'' \leq t'_{p}$ from a correct server $q'$ that is active throughout $\max\{0, t'_{p} - 2D\}, t'_{p} + D \geq [\max\{0, t'_{p} - 2D\}, t - D]$. Let $T''$ be the time that $q'$ sent its enter-echo message in reply to the enter message from $p$. Applying Lemma 4 for $q'$, with $U = \max\{0, t'_{p} - 2D\}$, and $T = t'_{p} - 2D$ gives $\text{SysInfo}_{0, \max\{0, t'_{p} - 2D\}, t'_{p} - 2D} \subseteq \text{Server}_{changes}^{t''_{p}}$.

Thus, we get $\text{SysInfo}_{0, \max\{0, t'_{p} - 2D\}} \subseteq \text{SysInfo}_{0, \max\{0, t'_{p} - 2D\}} \subseteq \text{Server}_{changes}^{t''_{p}}$.

**Lemmas 1 through 6** are used to prove **Theorem 2** as follows:

**Theorem 2.** Every client and any correct server $p \notin S_{0} \cup C_{0}$ that is active for at least 2D time after it enters succeeds in joining.

**Proof.** The proof is by induction on the order in which clients and correct servers enter the system. Let $p \notin S_{0} \cup C_{0}$ be a client or correct server that enters at time $t'_{p}$ and is active at time $t'_{p} + 2D$. Suppose the claim holds for all client and correct servers that enter before $p$.

By Lemma 5, there are $f + 1$ correct servers that are active throughout $\max\{t'_{p} - 2D, 0\}, t'_{p} + D\}$. Let $q$ be such server. If $q \in S_{0}$, then $q$ joins at time $0$. If not, then $t'_{q} \leq t'_{p} - 2D$, so, by the induction hypothesis, $q$ joins by time $t'_{q} + 2D \leq t'_{p}$. Since $q$ is active at time $t'_{q} + D$, it receives the enter message from $p$ during $[t'_{p}, t'_{q} + D]$, and sends an enter-echo message in reply. Since $p$ is active at time $t'_{p} + 2D$, it receives the enter-echo message from $q$ by time $t'_{p} + 2D$. Hence, by time $t'_{p} + 2D$, $p$ receives at least one enter-echo message from a correct joined server in reply to its enter message.

Suppose the first enter-echo message $p$ receives from a correct joined server in reply to its enter message is sent by server $q'$ at time $t''$, and received by $p$ at time $t''$. By Lemma 6, $\text{SysInfo}_{\max\{0, t'_{p} - 2D\}} \subseteq \text{Server}_{changes}^{t''_{p}} \subseteq \text{Server}_{changes}^{t''_{p}}$.

Let $S$ be the set of servers present at time $\max\{0, t'_{p} - 2D\}$. Since $t'' \leq t' + D$, it follows from Lemma 4 that at most $(1 - (1 + \alpha)^{3} - 1)|S|$ servers enter during $\max\{0, t'_{p} - 2D\}, t''$. Thus, $|\text{Present}^{t''_{p}}| \leq |S| + ((1 + \alpha)^{3} - 1)|S| = (1 + (1 + \alpha)^{3})|S|$. From line 120 in Algorithm 3, it follows that $\text{join\_bound} \leq \gamma \cdot (1 + (1 + \alpha)^{3})|S|$.
By Lemma \ref{lem:sysinfo} and Constraint \ref{con:sysinfo}, at most \((1-(1-\alpha)^3)|S|\) servers leave during \([\max\{0,t-2D\},t+D]\). At most \(f\) servers are Byzantine at \(t' + D\). Since \(t_p^c \leq t' \leq t_p^c + D\), the servers in \(S\) that do not leave during \([\max\{0,t-2D\},t+D]\) and are not Byzantine at \(t' + D\) are active throughout \([t_p^c,t_p^c + D]\) and send enter-echo messages in reply to \(p\)'s enter message. By time \(t_p^c + 2D\), \(p\) receives all these enter-echo messages. There are at least

\[
|S| - (1-(1-\alpha)^3)|S| - f = (1-\alpha)^3|S| - f
\]

such enter-echo messages. By Constraint \ref{con:sysinfo},

\[
\frac{(1-\alpha)^3}{(1+\alpha)^3} - \frac{f}{(1+\alpha)^3|S|} \geq \gamma,
\]

so the value of \(\text{join\_bound}\) is at most

\[
\gamma \cdot (1+\alpha)^3|S| \leq \left(\frac{(1-\alpha)^3}{(1+\alpha)^3} - \frac{f}{(1+\alpha)^3|S|}\right) 
\cdot (1+\alpha)^3|S| = (1-\alpha)^3|S| - f.
\]

Thus, by time \(t_p^c + 2D\), the condition in line \ref{alg:sysinfo} holds and node \(p\) joins.

Next, we show that all read and write operations terminate. Specifically, we show that the number of replies for which an operation waits is at most the number that it is guaranteed to receive.

Since \(\text{enter}(q)\) is added to \(\text{Server\_Changes}_p\) whenever \(\text{join}(q)\) is, for server \(q\), we get the following observation.

**Observation 3.** For every time \(t \geq 0\) and every client \(p\) that is active at time \(t\), \(\text{Members}_p^t \subseteq \text{Present}_p^t\).

Lemma \ref{lem:sysinfo} relates an active node's (client or correct server) current estimate of the number of servers present to the number of servers that were present in the system \(2D\) time units earlier. Lemma \ref{lem:sysinfo} relates an active client's current estimate of the number of servers that are members to the number of servers that were present in the system \(4D\) time units earlier. The lower bounds stated in these lemmas had to take into consideration that Byzantine servers may enter the system and never send a message and yet affect the system size. This scenario is impossible in the case of crash failures.

**Lemma 7.** For every node \(p\) that is either a client or a correct server and for every time \(t \geq t_p^c\) at which \(p\) is active,

\[
(1-\alpha)^2 \cdot N(\max\{0,t-2D\}) - f \leq |\text{Present}_p^t| \leq (1+\alpha)^2 \cdot N(\max\{0,t-2D\}).
\]

**Proof.** The proof is adapted from \ref{lem:sysinfo} to include \(f\) Byzantine servers in the lower bound. By Lemma \ref{lem:sysinfo} \(\text{SysInfo}^{[0,\max\{0,t-2D\}]} \subseteq \text{Server\_Changes}_p^t\), thus \(\text{Present}_p^t\) contains all nodes that are present at time \(\max\{0,t-2D\}\), plus any nodes that enter in \(\max\{0,t-2D\},t\] which \(p\) has learned about, minus any nodes that leave in \(\max\{0,t-2D\},t\] which \(p\) has learned about. Then, by Lemma \ref{lem:sysinfo}

\[
|\text{Present}_p^t| \leq NS(\max\{0,t-2D\}) + ((1+\alpha)^2 - 1) \cdot NS(\max\{0,t-2D\})
= (1+\alpha)^2 \cdot NS(\max\{0,t-2D\}).
\]

Similarly, by Lemma \ref{lem:sysinfo} and Constraint \ref{con:sysinfo},

\[
|\text{Present}_p^t| \geq NS(\max\{0,t-2D\}) - (1-(1-\alpha)^2) \cdot NS(\max\{0,t-2D\})
= (1-\alpha)^2 \cdot NS(\max\{0,t-2D\}).
\]

**Lemma 8.** For every client \(p\) and every time \(t \geq t_p^c\) at which \(p\) is active,

\[
(1-\alpha)^4 \cdot NS(\max\{0,t-4D\}) - f \leq |\text{Members}_p^t|
\leq (1+\alpha)^4 \cdot NS(\max\{0,t-4D\}).
\]

**Proof.** The proof is adapted from \ref{lem:sysinfo} to include \(f\) Byzantine servers in the lower bound. By Lemma \ref{lem:sysinfo} \(\text{SysInfo}^{[0,\max\{0,t-2D\}]} \subseteq \text{Server\_Changes}_p^t\) and, by Theorem \ref{thm:sysinfo} every node that enters by time \(\max\{0,t-4D\}\) joins by time \(\max\{0,t-2D\}\) if it is still active. Thus \(\text{Members}_p^t\) contains all nodes that are present at time \(\max\{0,t-4D\}\) plus any nodes that enter in \(\max\{0,t-4D\},t\] which \(p\) learns have joined, minus any nodes that leave in \(\max\{0,t-4D\},t\] which \(p\) learns have left. Then, by Lemma \ref{lem:sysinfo}

\[
|\text{Members}_p^t| \leq NS(\max\{0,t-4D\}) + ((1+\alpha)^4 - 1) \cdot NS(\max\{0,t-4D\})
= (1+\alpha)^4 \cdot NS(\max\{0,t-4D\}).
\]

Similarly, by Lemma \ref{lem:sysinfo} and Constraint \ref{con:sysinfo},

\[
|\text{Members}_p^t| \geq NS(\max\{0,t-2D\}) - (1-(1-\alpha)^4) \cdot NS(\max\{0,t-4D\})
= (1-\alpha)^4 \cdot NS(\max\{0,t-4D\}).
\]

**Lemma 9.** If a client or correct server \(p\) is active at time \(t \geq t_p^c\), then the number of correct servers that are joined by time \(t\) and are still active at time \(t + D\) is at least \([\frac{(1-\alpha)^3}{(1+\alpha)^3}] \cdot |\text{Present}_p^t| - f\).
(1 - (1 - \alpha)^3) \cdot \text{NS}(\max\{0, t - 2D\})$. Thus, there are at least
\[
\text{NS}(\max\{0, t - 2D\}) - (1 - (1 - \alpha)^3) \cdot \text{NS}(\max\{0, t - 2D\}) - f
\]
correct servers that were present at time $\max\{0, t - 2D\}$ and are still active at time $t + D$. This number is bounded below by
\[
\left(\frac{(1 - \alpha)^3}{1 + \alpha^2}\right) \cdot \text{NS}(\max\{0, t - 2D\}) - f
\]
since, by Lemma 7
\[
\text{NS}(\max\{0, t - 2D\}) \geq \frac{\text{Present}_p}{(1 + \alpha)^2}. \quad \text{By Theorem 5}
\]
all of these servers are joined by time $t$. 

Lemmas 7 through 9 are used to prove the following theorem.

**Theorem 3.** Every read or write operation invoked by a client that remains active completes.

**Proof.** Each operation consists of a read phase and a write phase. We show that each phase terminates within $2D$ time, provided the client remains active (does not crash or leave).

Consider a phase of an operation by client $p$ that starts at time $t$. Every correct server that joins by time $t$ and is still active at time $t + D$ receives $p$’s query or update message and replies with a reply message or an ack message by time $t + D$. By Lemma 9, there are at least
\[
\left(\frac{(1 - \alpha)^3}{1 + \alpha^2}\right) \cdot \text{NS}(\max\{0, t - 2D\}) - f
\]
such servers.

From Constraint 6, Lemma 7 and Observation 8
\[
\left(\frac{(1 - \alpha)^3}{1 + \alpha^2}\right) \cdot \text{Present}_p - f \geq \beta \cdot \text{Present}_p
\]
\[
\geq \beta \cdot \text{Members}_p = \text{rw_bound}_p.
\]
Thus, by time $t + 2D$, $p$ receives sufficiently many replies or ack messages to complete the phase. 

Now we prove atomicity of the ABCC algorithm. Let $T$ be the set of read operations that complete and write operations that execute line 95 of Algorithm 2. For any node $p$, let $ts_p^t = (\text{num}_p^t, w_{\text{id}})$ denote the timestamp of the latest value known to node $p$ that is recorded in its $\text{Known Writes}[p]$. Note that new timestamps are created by write operations (on lines 89, 90 of Algorithm 2) and are sent via enter-echo, update, and update-echo messages. Initially, $ts_p^0 = (0, \perp)$ for all nodes $p$.

For any operation $o$ in $T$ by client $p$, the timestamp of its read phase, $ts_{\text{rp}}(o)$, is $ts_p^t$, where $t$ is the end of its read phase (i.e., when the condition on line 88 of Algorithm 2 evaluates to true). The timestamp of its write phase, $ts_{\text{wp}}(o)$, is $ts_p^t$, where $t$ is the beginning of its write phase (i.e., when it $s$-bcasts on line 93 of Algorithm 2). The timestamp of a read operation in $T$ is the timestamp of its read phase. The timestamp of a write operation in $T$ is the timestamp of its write phase.

Note that $w_{\text{id}}$ is equal to $p$ and $\text{num}$ is set to one greater than the largest sequence number occurring in at least $f + 1$ replies observed during an operation’s read phase. This implies the next observation.

**Observation 4.** Each write operation in $T$ has a unique timestamp.

The next observation follows by a simple induction, since every timestamp other than $(0, \perp)$ comes from Lines 89, 90 of Algorithm 2.

**Observation 5.** Consider any read $o_p$ in $T$. If the timestamp of a read $o_p$ is $(0, \perp)$, then $o_p$ returns $\perp$. Otherwise, there is a write $o_p^2$ in $T$ such that $ts(o_p) = ts(o_p^2)$ and the value returned by $o_p$ equals the value written by $o_p^2$.

If a read operation $o_p$ returns the value written by a write operation $o_p^2$, then we say that $o_p$ reads from $o_p^2$. 

Lemmas 10 through 13 show that information written in the write phase of an operation propagates properly through the system. It is very important that at every step, the algorithm ensures that outdated information or wrong information sent by Byzantine servers does not corrupt the state of the replicated register. The IsValidMessage() procedure helps mask two types of bad behavior (multiple replies and replies sent after announcing a leave). The variables $\text{valid_val}_p$ and $\text{Known Writes}[p]$ help mask (bad) replies from Byzantine servers. These lemmas are analogous to the Lemmas 4 to 6 regarding the propagation of information about ENTER, JOINED, and LEAVE events.

**Lemma 10.** If $o$ is an operation in $T$ whose write phase $w$ starts at $t_w$, correct server $p$ is active at time $t \geq t_w + D$, and $t'_p \leq t_w$, then $ts_p^o \geq ts_{\text{wp}}(o)$.

**Proof.** Since server $p$ is active throughout $[t_w, t_w + D]$, it directly receives the update message $s$-bcast by $w$ at time $t_w$. Hence, from lines 89, 90 of Algorithm 2 $ts_p^o \geq ts_{\text{wp}}(o)$. 

**Lemma 11.** Suppose a correct server $p \notin S_0$ receives $(f + 1)$ enter-echo messages from correct servers by time $t'$. Let the $f + 1$ enter-echo message from a correct server be received from $q$ that sends it at time $t''$ reply to an enter message from $p$. If $o$ is an operation whose write phase $w$ starts at $t_w$, $p$ is active at time $t \geq \max\{t'', t_w + 2D\}$, and the $f + 1$ correct servers that send enter-echo messages are active throughout $[t_w, t_w + D]$, then $ts_p^o \geq ts_{\text{wp}}(o)$.

**Proof.** By Lemma 3 there are at least $f + 1$ correct joined servers that are active throughout $[t_w, t_w + D]$. Since $q$ is active throughout $[t_w, t_w + D]$, it receives the update message from $w$ at some time $\hat{t} \in [t_w, t_w + D]$, so $ts_q^\hat{t} \geq ts_{\text{wp}}(o)$. At time $t'' \leq t$, $p$ receives the enter-echo sent by $q$ at time $t'$. By the above argument, all $f$ earlier enter-echo messages have timestamp $\geq ts_{\text{wp}}(o)$. So, the value of the timestamp in $\text{valid_val}_p$ and in $\text{Known Writes}[p]$ is set to $\geq ts_{\text{wp}}(o)$. So $ts_p^o \geq ts_p^t \geq ts_q^\hat{t}$. If $t' \geq t$, then $ts_q^\hat{t} \geq ts_q^t$, so $ts_p^t \geq ts_{\text{wp}}(o)$. If $\hat{t} > t'$, then $q$ sends an update-echo at time $\hat{t} \leq t_w + D$, and $p$ receives it by time $\hat{t} + D \leq t_w + 2D \leq t$. The same argument works for the other $f$ correct, active servers in the system. Thus the timestamp of the variable $\text{valid_val}_p$ and $\text{Known Writes}[p]$ is either the timestamp of $w$ or of a later write. Thus, $ts_p^o \geq ts_q^\hat{t} \geq ts_{\text{wp}}(o)$. 


Lemma 12. If \( o \) is an operation in \( T \) whose write phase \( w \) starts at \( t_w \) and correct server \( p \) is active at time \( t \geq \max\{t_p^w + 2D, t_w + D\} \), then \( ts_p^t \geq ts^{\text{wpp}}(o) \).

Proof. The proof is by induction on the order in which correct servers enter the system. Suppose the claim holds for all correct servers that enter before \( p \). If \( t_p^w \leq t_w \), then the claim follows from Lemma 10.

If \( t_w < t_p^w \), then by Lemma 3 there are at least \( f + 1 \) correct joined servers that are active throughout \([\max\{0, t_p^w - 2D\}, t_p^w + 2D] \). These servers receive an enter-echo message from \( p \) and send an enter-echo message containing \( ts_q^t \) back to \( p \). Let \( q \) be the server whose enter-echo is the \((f + 1)\)th enter-echo from a correct joined server to reach \( p \). Let server \( q \) receive the enter message from \( p \) at some time \( t' \in [t_p^w, t_p^w + D] \). The enter-echo message sent by \( q \) is received by \( p \) at some time \( t'' \leq t' + D \leq t_p^w + 2D \). The above argument is true for all the other \( f \) correct joined servers that \( p \) hears from. Hence, \( ts_p^t \geq ts^{\text{wpp}}(o) \).

The first case is when \( t_w \geq \max\{0, t_p^w - 2D\} \). Since \( t_w + D < t_p^w + D \), it follows that the \( f + 1 \) correct joined servers including \( q \) are active throughout \([t_w, t_w + D] \). Furthermore, \( t \geq t_p^w + 2D \geq \max\{t_w, t_w + 2D\} \). Hence, Lemma 11 implies that \( ts_p^t \geq ts^{\text{wpp}}(o) \).

The second case is when \( t_w < \min\{0, t_p^w - 2D\} \). Since \( t_w > 0 \), it follows that \( t_p^w - 2D > 0 \). Since \( t < \max\{0, t_p^w - 2D\} \), \( t_w < t_p^w + 2D \leq t' \leq t'' \leq t_p^w + 2D \). So \( t \geq \max\{t_p^w + 2D, t_w + D\} \). Note that \( q \) is active at time \( t'' \) and \( q \) enters before \( p \), so, by the induction hypothesis, \( ts_q^t \geq ts^{\text{wpp}}(o) \). The above argument is true for all the other \( f \) correct joined servers that \( p \) hears from. Hence, \( ts_p^t \geq ts^{\text{wpp}}(o) \). \( \square \)

Lemma 13. If \( o \) is an operation in \( T \) whose write phase starts at \( t_w \), correct server \( p \) \( \not\in \) \( S \) joins at time \( t_p^w \), and \( p \) is active at time \( t \geq \max\{t_p^w, t_w + D\} \), then \( ts_p^t \geq ts^{\text{wpp}}(o) \).

Proof. The proof is adapted from Lemma 9 to tolerate \( f \) Byzantine servers. The proof is by induction on the order in which servers enter the system. Suppose the claim holds for all servers that join before \( p \). If \( t \geq t_p^w + 2D \), then the claim follows by Lemma 12. So, suppose \( t < t_p^w + 2D \). If \( t_p^w \leq t_w \), then the claim follows by Lemma 10. So, suppose \( t_p^w < t_w \).

Before \( p \) joins, it receives an enter-echo message from a joined server in reply to its enter message. Suppose \( p \) first receives such an enter-echo message at time \( t'' \) and the enter-echo was sent by \( q \) at time \( t' \). Then \( t'' \leq t_p^w \leq t \) and \( ts_q^t \leq ts_p^t \leq ts_p^{t'} \).

Now we prove that \( p \) receives an enter-echo message from a server \( q' \) that is active throughout \([\max\{0, t'' - 2D\}, t'' + D] \). Let \( S \) be the set of servers present at time \( \max\{0, t'' - 2D\} \), so \( |S| = NS(\max\{0, t'' - 2D\}) \). By Lemma 2 and Constraint 1, at most \( (1 - (1 - \alpha)^3)|S| \) servers leave during \([\max\{0, t'' - 2D\}, t'' + D] \). Since \( t'' \leq t' \leq t + D \), it follows that \( |\text{Present}_{t'}^p| \geq |S| - (1 - (1 - \alpha)^3)|S| = (1 - \alpha)^3|S| \). Hence, from lines 120 and 123 of Algorithm 3 \( p \) waits until it has received at least \( \text{join bound} = \gamma \cdot |\text{Present}_{t'}^p| \geq \gamma \cdot (1 - \alpha)^3|S| \) enter-echo messages before joining.

Hence, the number of enter-echo messages \( p \) receives before joining from servers that were active throughout \([\max\{0, t'' - 2D\}, t'' + D] \) is \( \text{join bound} \) minus the total number of server entries, leaves and faults (as Byzantine servers may not reply at all), which is at least

\[
\gamma \cdot (1 - \alpha)^3 |S| - \left( (1 + \alpha)^3 - 1 \right) |S| = \left( (1 + \gamma)(1 - \alpha)^3 - (1 - \alpha)^3 \right) |S| - \gamma \\
\geq \left( (1 + \gamma)(1 - \alpha)^3 - (1 + \alpha)^3 \right) NS_{\min} - \gamma \quad (9)
\]

Rearranging Constraint 3, we get

\[
\left( (1 + \gamma)(1 - \alpha)^3 - (1 + \alpha)^3 \right) NS_{\min} - \gamma \geq f + 1,
\]

so expression (9) is at least \( f + 1 \). Hence \( p \) receives an enter-echo message at some time \( T'' \geq t_p^w \) from a correct server \( q' \) that is active throughout \([\max\{0, t'' - 2D\}, t'' + D] \).

Let \( T' \) be the time that \( q' \) sent its enter-echo message in reply to the enter message from \( p \). Since \( ts_q^t \geq ts^{\text{wpp}}(o) \), \( ts_p^t \geq ts^{\text{wpp}}(o) \). So, suppose \( t_w < \max\{0, t'' - 2D\} \). Since \( ts_q^t \geq ts^{\text{wpp}}(o) \), \( ts_p^t \geq ts^{\text{wpp}}(o) \), since \( q \) joins at time \( t_q' < t_p^w \). Thus, in both cases, \( ts_p^t \geq ts^{\text{wpp}}(o) \). \( \square \)

Lemmas 10 through 13 are used to prove Lemma 14, which is the key lemma for proving atomicity of ABCC. It shows that for two non-overlapping operations, the timestamp of the read phase of the latter operation is at least as large as the timestamp of the write phase of the former.

Lemma 14. For any two operations \( op_1 \) and \( op_2 \) in \( T \), if \( op_1 \) finishes before \( op_2 \) starts, then \( ts^{\text{wpp}}(op_1) \leq ts^{\text{wpp}}(op_2) \).

Proof. Let \( p_1 \) be the client that invokes \( op_1 \), let \( w \) denote the write phase of \( op_1 \), let \( t_w \) be the start time of \( w \), and let \( \tau_w = ts^{\text{wp}}(op_1) = ts_p^t \). Similarly, let \( p_2 \) be the client that invokes \( op_2 \), let \( r \) denote the read phase of \( op_2 \), let \( t_r \) be the start time of \( r \), and let \( \tau_r = ts^{\text{rp}}(op_2) = ts_p^t \).

Let \( Q_w \) be the set of servers that \( p_1 \) hears from during \( w \) (i.e., that send messages causing \( p_1 \) to increment \( rw_{\text{counter}} \) on line 101 of Algorithm 2) and \( Q_r \) be the set of servers that \( p_2 \) hears from during \( r \) (i.e., that send messages causing \( p_2 \) to increment \( rw_{\text{counter}} \) on line 78 of Algorithm 2). Let \( F_w = |\text{Present}_{t_p^w}^p| \) and \( M_w = |\text{Members}_{t_p^w}^p| \) be the sizes of the \( \text{Present} \) and \( \text{Members} \) sets belonging to \( p_1 \) at time \( t_w \), and let \( F_r = |\text{Present}_{t_p^t}^p| \) and \( M_r = |\text{Members}_{t_p^t}^p| \) be the sizes of the \( \text{Present} \) and \( \text{Members} \) sets belonging to \( p_2 \) at time \( t_r \).
Case I: $t_r > t_w + 2D$. We start by showing that there exists $f + 1$ correct servers in $Q_r$ such that $t_{q}^{r} \leq t_r - 2D$.

Each server $q \in Q_r$ receives and responds to r’s query, so $q$ is joined by time $t_r + D$. By Theorem 2, the number of servers that can join during $(t_r - 2D, t_r + D)$ is at most the number of servers that can enter in $(\max\{0, t_r - 4D\}, t_r + D)$. By Lemma 1, the number of servers that can enter during $(\max\{0, t_r - 4D\}, t_r + D)$ is at most $((1 + \alpha)^5 - 1) \cdot N_S(\max\{0, t_r - 4D\})$. By Lemma 8, $(1 - \alpha)^4 N_S(\max\{0, t_r - 4D\}) - f \leq M_r$.

From the code and Constraint 6, it follows that

$$|Q_r| \geq \beta M_r \geq \left[\frac{(1 + \alpha)^5 - 1 + 2f}{(1 - \alpha)^4 - f} N_S_{\min}\right] \cdot M_r$$

$$\geq \left[\frac{(1 + \alpha)^5 - 1 + 2f}{(1 - \alpha)^4 - f} N_S_{\min}\right] \cdot ((1 + \alpha)^4 N_S(\max\{0, t_r - 4D\}) - f)$$

$$\geq [((1 + \alpha)^5 - 1) \cdot N_S(\max\{0, t_r - 4D\}) + 2f]$$

which is $2f + 1$ more than the maximum number of servers that can enter in $(\max\{0, t_r - 4D\}, t_r + D)$. At most $f$ of these can be Byzantine. Thus, at least $f + 1$ correct servers in $Q_r$ join by time $t_r - 2D$.

Suppose correct server $q \in Q_r$ receives r’s query message at time $t' \geq t_r \geq t_w + 2D$. If $q \in S_0$, then $t_{q}^{r} = 0 \leq t_w$, so, by Lemma 10 $t_{q}^{r} \geq t_{w}^{q}(op) = t_{w}$. Otherwise, $q \notin S_0$, so $0 < t_{q}^{r} \leq t_{w} < t'$. Since $t_w + 2D < t_r \leq t'$, Lemma 13 implies that $t_{q}^{r} \geq t_{w}^{q}(op_1) = t_{w}$. In either case, $q$ responds to r’s query message with a timestamp at least as large as $t_{w}$ and, hence, $t_r \geq t_{w}$.

Case II: $t_r \leq t_w + 2D$. Let $J = \{p \mid t_{q}^{r} < t_r \text{ and } p \text{ is an active server at time } t_r \}$, which contains the set of all servers that reply to r’s query. By Theorem 2 all correct servers that are present at time $t_r - 2D$ join by time $t_r$ if they remain active. Therefore all servers in $J$ are either active at time $\max(0, t_r - 2D)$ or enter during $(\max(0, t_r - 2D), t_r + D)$.

Let $K$ be the set of all servers that are present at time $\max(0, t_r - 2D)$ and do not leave during $(\max(0, t_r - 2D), t_r + D)$. Note that $K$ contains all the servers in $Q_w$ that do not leave during $[t_w, t_r + D] \subseteq [\max(0, t_r - 2D), t_r + D]$. By Lemma 1 and Constraint 1, at most $(1 - \alpha)^3 N_S(\max\{0, t_r - 2D\})$ servers leave during $[\max(0, t_r - 2D), t_r + D]$.

From the code, $|Q_r| \geq \beta M_r$ and, by Lemma 8, $M_r \geq (1 - \alpha)^4 N_S(\max\{0, t_r - 4D\}) - f$. So,

$$|Q_r| \geq \beta \left[(1 - \alpha)^4 N_S(\max\{0, t_r - 4D\}) - f\right].$$

Similarly,

$$|Q_w| \geq \beta M_w \geq \beta \left[(1 - \alpha)^4 N_S(\max\{0, t_w - 4D\}) - f\right].$$

Therefore, the size of $K$ is at least

$$|K| \geq |Q_w| - (1 - (1 - \alpha)^3) N_S(\max\{0, t_r - 2D\}) - f$$

$$\geq \beta \left[(1 - \alpha)^4 N_S(\max\{0, t_w - 4D\}) - f\right] - (1 - (1 - \alpha)^3) N_S(\max\{0, t_r - 2D\}) - f.$$
The proof of Theorem 4 uses Lemma 14 to show that the timestamps of two non-overlapping operations respect real time ordering and completes the proof of atomicity.

**Theorem 4.** ABCC ensures atomicity.

**Proof.** This proof is taken from from Theorem 23 in [9]. We show that, for every execution, there is a total order on the set of all completed read operations and all write operations that execute Line 95 of Algorithm 2 such that every read returns the value of the latest preceding write (or the initial value if there is no preceding write) and, if an operation \( op_1 \) finishes before another operation \( op_2 \) begins, then \( op_1 \) is ordered before \( op_2 \).

We first order the write operations in order of their (unique) timestamps. Then, we go over all reads in the ordering of the start times, and place a read with timestamp \((0, \perp)\) at the beginning of the total order. Place every other read after the write operation it reads from, and after all the previous reads that read from this write operation. By the Observation 5, every read in the total order returns the value of the latest preceding write (or \( \perp \) if there is no preceding write).

We show that the total order respects the real-time order of non-overlapping operations in the execution. Let \( op_1 \) and \( op_2 \) be two operations in \( T \) such that \( op_1 \) finishes before \( op_2 \) starts. By the definition of timestamps, \( ts(op_1) \leq ts^{up}(op_1) \) and \( ts^{up}(op_2) \leq ts(op_2) \). By Lemma 14, \( ts^{up}(op_1) \leq ts^{up}(op_2) \). Therefore, if \( op_2 \) is a read, then

\[
    ts(op_1) \leq ts(op_2)
\]

If \( op_2 \) is a write, then \( ts^{up}(op_2) = ts^{up}(op_2) + 1 \), and

\[
    ts(op_1) < ts(op_2)
\]

We consider the following cases:

- Suppose \( op_1 \) and \( op_2 \) are both writes. By (11), \( ts(op_1) < ts(op_2) \) and thus the construction orders \( op_1 \) before \( op_2 \).
- Suppose \( op_1 \) is a write and \( op_2 \) is a read. By (10) and the construction, \( op_2 \) is placed after the write \( op_3 \) that \( op_2 \) reads from. If \( ts(op_1) = ts(op_2) \) then \( op_1 = op_3 \) and \( op_2 \) is placed after \( op_1 \). If \( ts(op_1) < ts(op_2) \) then \( op_3 \) is placed after \( op_1 \) as \( ts(op_1) < ts(op_3) \) and thus \( op_2 \) is placed after \( op_1 \) in the total order.
- Suppose \( op_1 \) is a read and \( op_2 \) is a write. By (11) \( ts(op_1) < ts(op_2) \). Now, either \( op_1 \) is the first write in the execution and \( op_1 \)'s timestamp is \((0, \perp)\) or there exists another write \( op_3 \) that \( op_1 \) reads from. If \( op_1 \)'s timestamp is \((0, \perp)\) then the construction orders \( op_1 \) before \( op_2 \). Otherwise, the construction orders \( op_3 \) before \( op_2 \). Since \( op_1 \) is ordered after \( op_3 \) but before any subsequent write, \( op_1 \) precedes \( op_2 \) in the total order.
- Finally, suppose that \( op_1 \) and \( op_2 \) are both reads. By (10) \( ts(op_1) \leq ts(op_2) \). If \( op_1 \) and \( op_2 \) have the same timestamp, then they are placed after the same write (or before the first write) and the construction orders them based on their starting times. Since \( op_1 \) completes before \( op_2 \) starts, the construction places \( op_1 \) before \( op_2 \). If \( op_2 \) has a timestamp greater than that of \( op_1 \), then \( ts(op_2) \) cannot be \((0, \perp)\) and so there is a write operation \( op_3 \) whose timestamp is greater than that of \( op_1 \) and equal to that of \( op_2 \). The construction places \( op_1 \) before \( op_3 \) and \( op_2 \) after \( op_3 \).

Thus, ABCC ensures atomicity. \( \square \)

**VI. Discussion**

Our paper provides an algorithm that emulates a Byzantine-tolerant atomic register in a dynamic system (i) where consensus is impossible to solve, (ii) that never stops changing and (iii) has no upper bound on the system size. We also provide an impossibility proof that in our model, a uniform algorithm cannot be implemented.

There are several directions for future work. The values of \( \alpha \), \( f \) and \( NS_{min} \) that satisfy our algorithm are quite restrictive. It will be nice to see if such restrictions are necessary or if they can be improved either with a better algorithm or a tighter correctness analysis.

Currently our model tolerates the most severe end of the fault severity spectrum and paper [9] considered the most benign end of the fault severity spectrum. In future, we would like to to see if our impossibility result extends to the other failure models?

Our current way of restricting churn relies on the unknown upper bound \( D \) on message delay. An alternative may be to allow the upper bound on the message delays to be unbounded and to define the churn rate with respect to messages in transit. For example, at all times, the number of servers that can enter/leave the system when any message is in transit at most \( \alpha \) times the system size when the message was sent. It may be possible to prove that the two models are indeed equivalent, or to show that the algorithms still work, or can be modified to work, in the new model.

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