Symmetries and Invariants in Higher-Spin Theory

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Abstract

General aspects of higher-spin gauge theory and unfolded formulation are briefly recalled with some emphasize on the recent results on the breaking of $sp(8)$ symmetry by current interactions and construction of invariant functionals relevant to the higher-spin holography.
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1 Introduction

Higher-spin (HS) gauge theory is based on higher symmetries associated with HS massless fields. HS gauge symmetries are expected to become manifest at ultra high energies possibly beyond the Planck energy. Since such energies are unreachable by modern accelerator devices the conjecture that a fundamental theory exhibits the HS symmetries at ultra high energies provides a unique chance to explore properties of this regime. HS symmetries severely restrict the structure of HS theory.

The study of HS fields has long history starting from seminal papers of Dirac [1], Fierz and Pauli [2], and others including the Tamm group [3, 4, 5]. The role of HS gauge symmetries for massless fields in four dimensions was originally appreciated at the linearized level for spin 3/2 by Rarita and Schwinger [6] and for any spin by Fronsdal [7].

Extension to the interacting level was not simple encountering difficulties of combining nonAbelian symmetries of different types. First positive results were obtained in eighties of the last century in the papers by A. Bengtsson, I. Bengtsson, Brink [8, 9] and Berends, Burgers, van Dam [10, 11] who found that the action consistent with HS gauge symmetries in the cubic order contains higher derivatives in interactions

\[ S = S^2 + S^3 + \ldots , \quad S^3 = \sum_{p,q,r} (D^p \phi)(D^q \phi)(D^r \phi) \rho^{p+q+r+\frac{1}{2}d-3} , \]

where the order of higher derivatives increases with spins of the fields \( \phi \) in the vertex. Since full HS theory necessarily involves infinite towers of HS fields, such a theory is somewhat nonlocal (note however that no higher derivatives appear at the quadratic level within the expansion around AdS background). Of course some kind of nonlocality beyond Planck scale should be expected of the theory anticipated to capture the quantum gravity regime.

Appearance of higher derivatives in interactions requires a dimensionful coupling constant \( \rho \) in the action, whose origin in a theory involving only massless fields was obscure. Resolution of this puzzle was proposed in [12, 13] via identification of the parameter \( \rho \) with the radius of the background (anti-)de Sitter space. In this setup, the higher derivative vertices do not allow a meaningful flat limit. This is consistent with numerous no-go statements ruling out consistent interactions of massless HS fields in Minkowski space [14, 15].

In eighties, the fact that consistent HS interactions require non-zero cosmological constant looked like a peculiarity. It acquired a much deeper interpretation after discovery of the AdS/CFT correspondence [16, 17, 18]. The fact that HS theories are most naturally formulated in the AdS background was conjectured to play a role in the context of the HS holography in [19, 20, 21]. The precise conjecture on the AdS\(_4\)/CFT\(_3\) correspondence was put forward by Klebanov and Polyakov [22] while its first explicit check was performed by Giombi and Yin in [23]. This research triggered a great increase of interest in HS theories and HS holographic duality (see e.g. [24]-[38] and references therein). CFT\(_2\) duals of 3d HS symmetries were studied in [39, 40]. The conjecture on AdS\(_3\)/CFT\(_2\) HS holography proposed by Gaberdiel and Gopakumar [41] also formed an active research direction in the recent years.
One of the driving forces of the study of interacting HS theories is the hope that HS holography may help to uncover the origin of AdS/CFT. However, the subtlety is that despite significant progress in the construction of actions during last thirty years [8]-[13], [42]-[51] the full nonlinear generalization of the Fronsdal action is still unavailable. This complicates the standard construction of the generating functional. One of the aims of this talk is to discuss a recently proposed construction for invariant on-shell functionals [52] conjectured to give rise to both the generating functional for boundary correlators and black hole charges.

2 Unfolded Dynamics

2.1 General setup

Modern formulation of nonlinear HS theory [53] is based on the so-called unfolded approach [54] which reformulates dynamics in question in terms of differential forms. The unfolded form of dynamical equations provides a covariant generalization of the first-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)),$$

which is convenient in many respects. In particular, since initial data can be given in terms of the values of variables $q^i(t_0)$ at any given point $t_0$, the number of degrees of freedom in the first-order formulation equals to the number of dynamical variables.

Unfolded formulation is a multidimensional generalization achieved via replacement of the time derivative by the de Rham derivative

$$\frac{\partial}{\partial t} \to d = dx^\nu \partial_\nu$$

and the dynamical variables $q^i$ by a set of differential forms

$$q^i(t) \to W^\Omega(dx, x) = dx^{\nu_1} \cdots dx^{\nu_p} W^\Omega_{\nu_{1} \cdots \nu_{p}}(x)$$

to reformulate a system of partial differential equations in the first-order covariant form

$$dW^\Omega(dx, x) = G^\Omega(W(dx, x)). \quad (2.1)$$

Here $dx^\nu$ are anticommuting differentials ($dx^\nu dx^\mu = -dx^\mu dx^\nu$; the wedge product symbol is implicit) and $G^\Omega(W)$ are some functions of the “supercoordinates” $W^\Omega$

$$G^\Omega(W) = \sum_n f^\Omega_{\Lambda_1 \cdots \Lambda_n} W^{\Lambda_1} \cdots W^{\Lambda_n}. $$

Since $d^2 = 0$ at $d > 1$ the functions $G^\Lambda(W)$ have to obey the compatibility conditions

$$G^\Lambda(W) \frac{\partial G^\Omega(W)}{\partial W^\Lambda} \equiv 0. \quad (2.2)$$
(Not that all products of the differential forms $W(dx,x)$ are the wedge products due to anticommutativity of $dx^\nu$.) Let us stress that these are conditions on the functions $G^\Lambda(W)$ rather than on $W$.

As a consequence of (2.2) system (2.1) is invariant under the gauge transformation

$$\delta W^\Omega = d\varepsilon^\Omega + \varepsilon^\Lambda \frac{\partial G^\Omega(W)}{\partial W^\Lambda},$$

(2.3)

where the gauge parameter $\varepsilon^\Omega(x)$ is a $(p_\Omega - 1)$-form for a $p_\Omega$-form $W^\Omega$. Strictly speaking, this is true for the class of universal unfolded systems in which the compatibility conditions (2.2) hold independently of the dimension $d$ of space-time, i.e., (2.2) should be true disregarding the fact that any $(d+1)$-form is zero. Let us stress that all unfolded systems appearing in HS theories are universal.

The unfolded formulation can be applied to description of invariant functionals. As shown in [55], the variety of gauge invariant functionals associated with the unfolded equations (2.1) is described by the cohomology of the operator

$$Q = G^\Omega(W) \frac{\partial}{\partial W^\Omega},$$

(2.4)

which obeys

$$Q^2 = 0$$

as a consequence of (2.2). By virtue of (2.1), $Q$-closed $p$-form functions $L_p(W)$ are $d$-closed, giving rise to the gauge invariant functionals

$$S = \int_{\Sigma^p} L_p.$$

In the off-shell case they can be used to construct invariant actions while in the on-shell case they describe conserved charges. (For more detail and examples see [55].)

### 2.2 Properties

The unfolded formulation of partial differential equations has a number of remarkable properties.

- First of all, it has general applicability: every system of partial differential equations can be reformulated in the unfolded form.

- Due to using the exterior algebra formalism, the system is invariant under diffeomorphisms, being coordinate independent.

- Interactions can be understood as nonlinear deformations of $G^\Omega(W)$. 


• Unfolded formulation gives clear group-theoretical interpretation of fields and equations in terms of modules and Chevalley-Eilenberg cohomology of a symmetry algebra $h$. In particular, background fields are described by a flat connection of $h$. Dynamical fields are described in terms of $h$-modules while equations of motion have a form of covariant constancy conditions.

• Local degrees of freedom are carried by 0-forms $C^i(x_0)$ at any $x = x_0$ (as $q(t_0)$), valued in the infinite-dimensional module dual to the space of single-particle states: $C^i(x_0)$ are moduli of solutions replacing initial data in the conventional Cauchy problem. It is worth to mention that this property of unfolded dynamics gives a tool to control unitarity in presence of higher derivatives via the requirement on the space of zero-forms like $C^i(x_0)$ to admit a positive-definite norm preserved by the unfolded equations in question.

• The most striking feature of the unfolded formulation is that it makes it possible to describe one and the same dynamical system in space-times of different dimensions. Unfolded dynamics exhibits independence of the “world-volume” space-time with coordinates $x$. Instead, geometry is encoded by the functions $G^\Omega(W)$ in the “target space” of fields $W^\Omega$. Indeed, the universal unfolded equations make sense in any space-time independently of a particular realization of the de Rham derivative $d$. For instance one can extend space time by adding additional coordinates $z$

$$dW^\Omega(x) = G^\Omega(W(x)), \quad x \rightarrow X = (x, z), \quad d_x \rightarrow d_X = d_x + d_z, \quad d_z = dz^u \frac{\partial}{\partial z^u}. $$

The unfolded equations reconstruct the $X$-dependence in terms of values of the fields $W^\Omega(X_0) = W^\Omega(x_0, z_0)$ at any $X_0$. Clearly, to take $W^\Omega(x_0, z_0)$ in space $M_X$ with coordinates $X_0$ is the same as to take $W^\Omega(x_0)$ in the space $M_x \subset M_X$ with coordinates $x$.

Such uplifting becomes most interesting provided that there is a non trivial vacuum connection along the additional coordinates $z$. This is in particular the case for $AdS/CFT$ correspondence where the conformal flat connection at the boundary is extended to the flat $AdS$ connection in the bulk with $z$ being a radial Poincaré coordinate.

3 Field-current-field correspondence

The $AdS_4/CFT_3$ HS holography [22] relates the HS gauge theory in $AdS_4$ to the quantum theory of conformal currents in three dimensions. To see how it works from the unfolded dynamics perspective, let us first discuss the unfolded equations for free massless fields and currents on the 3d boundary.

The unfolded equations of 3d conformal massless fields is formulated in terms of 0-forms $C(y|x)$ [56, 57] which depend on the coordinates $x^{\alpha\beta} = x^{\beta\alpha}$ of 3d space-time and auxiliary
commuting spinorial variables $y^\alpha$ ($\alpha, \beta = 1, 2$ are 3d spinorial indices). Unfolded equations for conformal massless fields are

$$
\left( \frac{\partial}{\partial x^{\alpha\beta}} \pm i \frac{\partial^2}{\partial y^{\alpha} \partial y^{\beta}} \right) C_{j}^{\pm}(y|x) = 0, \quad \alpha, \beta = 1, 2, \quad (3.1)
$$

where $j = 1, \ldots N$ is a color index.

The meaning of these equations is twofold. Firstly, they express all terms of degree two or higher in the $y$ variables via $x$-derivatives of the fields $C^{\pm}(x) := C^{\pm}(0|x)$ and $C_{\alpha}^{\pm}(x) := \frac{\partial}{\partial y^{\alpha}} C^{\pm}(y|x)\big|_{y=0}$. The latter are the usual scalar and spinor fields which obey, respectively, Klein-Gordon and Dirac equations by virtue of (3.1). More precisely, the $\pm$ components should be identified with positive- and negative-frequency parts of the solutions of free field equations. Note that the fields $C^{\pm}(x)$ and $C_{\alpha}^{\pm}(x)$ are primaries of the conformal modules underlying equations (3.1).

The unfolded equations for 3d conformal conserved currents have the rank-two form

$$
\left\{ \frac{\partial}{\partial x^{\alpha\beta}} \right\} J(u, y|x) = 0.
$$

(3.2)

$J(u, y|x)$ contains all 3d HS currents along with their derivatives.

Elementary 3d conformal currents, which are conformal primaries, contain conserved currents of all spins

$$
J(u, 0|x) = \sum_{2s=0}^{\infty} u^{\alpha_1} \ldots u^{\alpha_{2s}} J_{\alpha_1 \ldots \alpha_{2s}}(x), \quad \tilde{J}(0, y|x) = \sum_{2s=0}^{\infty} y^{\alpha_1} \ldots y^{\alpha_{2s}} \tilde{J}_{\alpha_1 \ldots \alpha_{2s}}(x)
$$

along with the additional scalar current

$$
J^{asym}(u, y|x) = u_{\alpha} y^{\alpha} J^{asym}(x).
$$

Their conformal dimensions are

$$
\Delta J_{\alpha_1 \ldots \alpha_{2s}}(x) = \Delta \tilde{J}_{\alpha_1 \ldots \alpha_{2s}}(x) = s + 1 \quad \Delta J^{asym}(x) = 2.
$$

The unfolded equations express all other components of $J(u, y|x)$ in terms of derivatives of the primaries, also imposing the differential equations on the latter, which are just the conservation conditions

$$
\frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial u_{\alpha} \partial u_{\beta}} J(u, 0|x) = 0, \quad \frac{\partial}{\partial x^{\alpha\beta}} \frac{\partial^2}{\partial y_{\alpha} \partial y_{\beta}} \tilde{J}(0, y|x) = 0
$$

for all currents except for the scalar ones that obey no differential equations.

The rank-two equation is obeyed by

$$
J(u, y|x) = \sum_{i=1}^{N} C_{i}^{-}(u + y|x) C_{i}^{+}(y - u|x).
$$
This simple formula gives the explicit realization of the HS conformal conserved currents in terms of bilinear combinations of derivatives of free massless fields in three dimensions.

Generally, the rank-two fields and hence conserved currents can be interpreted as bi-local fields in the twistor space. In this respect they are somewhat analogous to space-time bi-local fields also used for the description of currents (see e.g. [29] and references therein).

To relate 3\text{d} currents to 4\text{d} massless fields we extend the 3\text{d} current equation to the 4\text{d} massless equations. This is easy to achieve in the unfolded dynamics via the extension of the 3\text{d} coordinates $x^{\alpha\beta}$ to the 4\text{d} coordinates $x^{\alpha\dot{\alpha}}$, extending 3\text{d} equations (3.2) to

$$
\left( \frac{\partial}{\partial x^{\alpha\dot{\alpha}}} + \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}} \right) C(y, \bar{y}|x) = 0 .
$$

These are just the free unfolded equations [54] for 4\text{d} massless fields of all spins in Minkowski space, i.e., at $\Lambda = 0$.

The analysis in $AdS_4$, which is also simple, is performed analogously. In this case, $x^{\alpha\beta} = \frac{1}{2}(x^{\alpha\beta} + x^{\beta\alpha})$ are boundary coordinates, while $z^{-1} = x^{\alpha\dot{\alpha}} \epsilon_{\alpha\dot{\alpha}}$ is the radial coordinate. (For more detail see [25].) At the non-linear level, the full HS theory in $AdS_4$ turns out to be equivalent to the theory of 3\text{d} currents of all spins interacting through conformal HS gauge fields [25].

A rank-two field (current) in $AdS_3$ is equivalent to a rank-one field in a larger space with ten coordinates $X^{AB} = X^{BA}$

$$
\left( \frac{\partial}{\partial X^{AB}} + \frac{\partial^2}{\partial y^A \partial y^B} \right) J^3(y|x) = 0 , \quad A, B = 1, \ldots, 4 , \quad X^{AB} = X^{BA} ,
$$

$$
X^{AB} = (x^{\alpha\dot{\alpha}}, x^{\alpha\beta}, \bar{x}^{\dot{\alpha}\dot{\beta}}) , \quad x^{\alpha\dot{\alpha}} = (x^{\alpha\dot{\alpha}}, \epsilon^{\alpha\dot{\alpha}}) z .
$$

Reduction to Minkowski coordinates $x^{\alpha\dot{\alpha}}$ gives 4\text{d} massless equations for all spins with $J^3 \rightarrow C^4$. Mathematically this is the manifestation of the Flato-Fronsdal theorem [59] stating that the tensor product of unitary modules associated with 3\text{d} massless fields gives the unitary module associated with all 4\text{d} massless fields:

$$(3d, m = 0) \otimes (3d, m = 0) = \sum_{s=0}^{\infty} (4d, m = 0) . \quad (3.4)$$

The full system of 4\text{d} massless fields of all spins exhibits $sp(8)$ symmetry [60, 61, 62, 57]. A rank-two field in four dimensions describes 4\text{d} conserved currents equivalent to a rank-one field in six dimensions [62, 63]

$$
C^4 C^4 \sim J^4 \sim C^6 .
$$

Dualities of this type can be called field-current-field correspondence.
4 From free massless equations to current interactions and holography

4.1 Central on-shell theorem

The infinite set of 4d massless fields of all spins \( s = 0, 1, 2 \ldots \) is conveniently described by a 1-form \( \omega(y, \bar{y} \mid x) \), and 0-form \( C(y, \bar{y} \mid x) \)

\[
A(y, \bar{y} \mid x) = i \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{\alpha_1} \ldots y_{\alpha_n} \bar{y}_{\beta_1} \ldots \bar{y}_{\beta_m} A^{\alpha_1 \ldots \alpha_n, \beta_1 \ldots \beta_m}(x).
\]

The central fact of the analysis of free massless fields in four dimensions known as Central on-shell theorem is that unfolded system for free massless fields has the form [54]

\[
R_1(y, \bar{y} \mid x) = \bar{H}^{\alpha\dot{\beta}} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}} \bar{C}(0, \bar{y} \mid x) + H^{\alpha\dot{\beta}} \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}} C(y, 0 \mid x),
\]

(4.1)

\[
\tilde{D}_0 C(y, \bar{y} \mid x) = 0,
\]

(4.2)

where

\[
R_1(y, \bar{y}|x) = D^a \omega(y, \bar{y}|x) = D^L \omega(y, \bar{y}|x) - \lambda e^{\alpha \dot{\beta}} \left( y_\alpha \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} + \frac{\partial}{\partial y^\alpha} \bar{y}_\dot{\beta} \right) \omega(y, \bar{y}|x),
\]

(4.3)

\[
\tilde{D}C(y, \bar{y}|x) = D^L C(y, \bar{y}|x) + \frac{i}{2} \lambda e^{\alpha \dot{\beta}} \left( y_\alpha \bar{y}_\dot{\beta} - \frac{\partial^2}{\partial y^\alpha \partial \bar{y}^{\dot{\beta}}} \right) C(y, \bar{y}|x),
\]

(4.4)

\[
D^L A(y, \bar{y}|x) = dA(y, \bar{y}|x) - \left( \omega^{\alpha \dot{\beta}} y_\alpha \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} + \bar{e}^{\dot{\beta} \dot{\gamma}} y_\dot{\gamma} \frac{\partial}{\partial \bar{y}^{\dot{\beta}}} \right) A(y, \bar{y}|x).
\]

(4.5)

Here the background \( AdS_4 \) Lorentz connection \( \omega_{\alpha \dot{\beta}}, \bar{\omega}_{\dot{\alpha} \dot{\beta}} \) and vierbein \( e_{\alpha \dot{\beta}} \) obey the \( AdS_4 \) equations

\[
R_{\alpha \dot{\beta}} = 0, \quad \bar{R}_{\dot{\alpha} \dot{\beta}} = 0, \quad R_{\alpha \dot{\alpha}} = 0,
\]

(4.6)

where \( \lambda^{-1} \) is the \( AdS_4 \) radius and

\[
R_{\alpha \dot{\beta}} = d\omega_{\alpha \dot{\beta}} + \omega_\alpha \gamma \wedge \omega_{\beta \gamma} + \lambda^2 e_\alpha \dot{\dot{\beta}} \wedge e_{\dot{\beta}},
\]

(4.7)

\[
\bar{R}_{\dot{\alpha} \dot{\beta}} = d\bar{\omega}_{\dot{\alpha} \dot{\beta}} + \bar{\omega}_{\dot{\alpha} \dot{\gamma}} \wedge \bar{\omega}_{\dot{\beta} \dot{\gamma}} + \lambda^2 \bar{e}_{\dot{\gamma}} \dot{\dot{\gamma}} \wedge e_{\dot{\beta}},
\]

(4.8)

\( (\text{Two-component indices are raised and lowered by } \varepsilon_{\alpha \dot{\beta}} \text{ or } \varepsilon_{\dot{\alpha} \dot{\beta}}. \) \( H^{\alpha \dot{\beta}} = H^{\alpha \dot{\beta}} \text{ and } \bar{H}^{\dot{\alpha} \dot{\beta}} = \bar{H}^{\dot{\alpha} \dot{\beta}} \)

are the basis 2-forms

\[
H^{\alpha \dot{\beta}} := e^{\alpha \dot{\beta}} e^{\dot{\beta}}, \quad \bar{H}^{\dot{\alpha} \dot{\beta}} := e_{\dot{\alpha}} \dot{\dot{\beta}} e^{\dot{\beta}}.
\]

(4.9)

The 0-forms \( C(Y|x) \) form a Weyl module equivalent to the boundary current module. 1-form HS connections \( \omega(y, \bar{y} \mid x) \) contain HS gauge fields. For spins \( s \geq 1 \), equation (4.11) expresses the Weyl 0-forms \( C(Y|x) \) via gauge invariant combinations of derivatives of the HS gauge connections. From this perspective the Weyl 0-forms \( C(Y|x) \) generalize the spin-two Weyl tensor along with all its derivatives to any spin.
### 4.2 Current deformation

Schematically, for the flat connection $D = d + w$ the current deformation of the free equations (4.1), (4.2) has the form

\[
\begin{cases}
D\omega^4 + L(C^4, w) = 0 \\
\bar{D}C^4 = 0 \\
D_2 J^4 = 0
\end{cases}
\implies
\begin{cases}
D\omega^4 + L(C^4, w) + G(w, J^4) = 0 \\
\bar{D}C^4 + F(w, J^4) = 0 \\
D_2 J^4 = 0
\end{cases}
\]

The sector of 0-forms of this system was analyzed in detail in [64, 65]. Here $J^4$ can be interpreted either as a 4d current or as a 6d massless field. As a result, 4d current interactions can be interpreted as a mixed linear system of 4d and 6d fields [64]. Algebraically this is the semidirect sum of a rank-one and rank-two systems.

An interesting question is what symmetry is preserved by the deformed system? When unmixed, both rank-one and rank-two system are $sp(8)$-invariant. The question whether $sp(8)$ is preserved by the deformation is equivalent to that whether formal consistency of the deformation takes place with any connection $w \in sp(8)$. The analysis of this question [65] shows that current interactions break $sp(8)$ down to the conformal algebra $su(2, 2)$.

### 4.3 Kinematics of $AdS_4/CFT_3$ HS holography

To make boundary conformal invariance manifest it is convenient to use the following basis

\[
y_+^\alpha = \frac{1}{2}(y_\alpha - iy_\bar{\alpha}), \quad y_-^\alpha = \frac{1}{2}(y_\bar{\alpha} - iy_\alpha), \quad \left[y_-^\alpha, y_+^{+, \beta}\right]_w = \delta_\alpha^\beta.
\]

$AdS_4$ can be foliated as $x^n = (x^a, z)$, where $x^a$ are coordinates of leaves ($a = 0, 1, 2$.) and the Poincaré coordinate $z$ is the foliation parameter. $AdS_4$ infinity is at $z = 0$. In these coordinates the background connection at $\lambda = 1$ is

\[
W = \frac{i}{z} dx^\alpha y^\alpha y^\beta - \frac{dz}{2z} y^\beta y^\alpha,
\]

\[
e^{a\bar{\alpha}} = \frac{1}{2i} dx^{a\bar{\alpha}}, \quad \omega^{\alpha\beta} = -\frac{i}{4z} dx^{\alpha\beta}, \quad \omega^{\alpha\bar{\beta}} = \frac{i}{4z} dx^{\alpha\bar{\beta}}.
\]

Using insensitiveness of unfolded equations to the extension to a larger space, the vacuum connection can be analytically extended to the complex plane of $z$ with all components containing $dz$ being zero. In these terms the generating functional for the boundary correlators takes the form

\[
S = \frac{1}{2\pi i} \oint_{z=0} \mathcal{L}(\omega(C), C)
\]

if $\mathcal{L}(\omega(C), C)$ is an on-shell closed $(d + 1)$-form for a $d$-dimensional boundary

\[
d\mathcal{L}(\omega(C), C) = 0, \quad \mathcal{L} \neq dM.
\]
The resulting functional is the residue at $z = 0$ giving the boundary functional of the structure analogous to $\phi_{n_1 \ldots n_s}$

$$S_{M^3}(\omega) = \int_{M^3} \mathcal{L}, \quad \mathcal{L} = \frac{1}{2} \omega_{\alpha_1 \ldots \alpha_{2^{s-1}}} e^\beta e^{\alpha_2 \ldots \alpha_{2^s}} (a C_{\alpha_1 \ldots \alpha_{2^s}} (\omega) + \bar{a} \bar{C}_{\alpha_1 \ldots \alpha_{2^s}} (\omega)).$$

Here $C_{\alpha_1 \ldots \alpha_{2^s}} (\omega)$, which have conformal properties of currents $J$, are expressed via the HS connections $\omega$ by Eq. (4.1). On the other hand $\omega_{\alpha_1 \ldots \alpha_{2^{s-1}}} (s-1)$ have conformal dimensions of the shadow sources $\phi$ to the currents. Being related to $C$ via unfolded equations it does not describe new degrees of freedom however.

The $C$-dependent terms can be represented in the form

$$a C_{\alpha_1 \ldots \alpha_{2^s}} (\omega) + \bar{a} \bar{C}_{\alpha_1 \ldots \alpha_{2^s}} (\omega) = a_- T_{\alpha_1 \ldots \alpha_{2^s}} (\omega) + a_+ T_{\alpha_1 \ldots \alpha_{2^s}} (\omega),$$

where $T_-$ describes local boundary terms while $T_+$ describes nontrivial correlators via the variation of $S_{M^3}$ over the HS gauge fields $\omega_{\alpha_1 \ldots \alpha_{2^{s-1}}} (s-1)$

$$\langle J(x_1) J(x_2) \ldots \rangle = \frac{\delta^n \exp [-S_{M^3} (\omega, C(\omega))] |_{\omega=0}}{\delta \omega(x_1) \delta \omega(x_2) \ldots}.$$

The main problem is to find an appropriate nonlinear invariant functional $\mathcal{L}$.

## 5 Nonlinear HS equations in $AdS_4$

To explain the construction of invariant functionals we first recall the form of nonlinear massless field equations of [53]. The key element is the doubling of auxiliary Majorana spinor variables $Y_A$ in the HS 1-forms and 0-forms

$$\omega(Y; \mathcal{K}|x) \rightarrow W(Z; Y; \mathcal{K}|x), \quad C(Y; \mathcal{K}|x) \rightarrow B(Z; Y; \mathcal{K}|x) \quad (5.1)$$

supplemented with equations which determine dependence on the additional variables $Z_A$ in terms of “initial data”

$$\omega(Y; \mathcal{K}|x) = W(0; Y; \mathcal{K}|x), \quad C(Y; \mathcal{K}|x) = B(0; Y; \mathcal{K}|x). \quad (5.2)$$

An additional spinor field $S_A(Z; Y; \mathcal{K}|x)$, that carries only pure gauge degrees of freedom, plays a role of connection in $Z^A$ directions. It is convenient to introduce anticommuting $Z-$differentials $dZ^A dZ^B = -dZ^B dZ^A$ to interpret $S_A(Z; Y; \mathcal{K}|x)$ as a $Z$–1-form,

$$S = dZ^A S_A(Z; Y; \mathcal{K}|x). \quad (5.3)$$

The variables $\mathcal{K} = (k, \bar{k})$ are Klein operators that satisfy

$$k w^\alpha = - w^\alpha k, \quad k \bar{w}^\alpha = \bar{w}^\alpha k, \quad \bar{k} w^\alpha = w^\alpha \bar{k}, \quad \bar{k} \bar{w}^\alpha = - \bar{w}^\alpha \bar{k}, \quad k^2 = \bar{k}^2 = 1, \quad k \bar{k} = \bar{k} k \quad (5.4)$$

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with \( w^\alpha = (y^\alpha, z^\alpha, dz^\alpha), \bar{w}^{\bar{\alpha}} = (\bar{y}^{\bar{\alpha}}, \bar{z}^{\bar{\alpha}}, d\bar{z}^{\bar{\alpha}}) \).

The nonlinear HS equations are 53

\[
\begin{align*}
  dW + W \ast W &= 0, \\
  dB + W \ast B - B \ast W &= 0, \\
  dS + W \ast S - S \ast W &= 0, \\
  S \ast B &= B \ast S, \\
  S \ast S &= -i(dZ^A dZ_A + dz^\alpha dz_\alpha F_*(B)k\nu + d\bar{z}^{\bar{\alpha}} d\bar{z}_{\bar{\alpha}} \tilde{F}_*(B)\tilde{k}\tilde{\kappa}),
\end{align*}
\]

where \( F_*(B) \) is some star-product function of the field \( B \).

Setting \( W = d + W + S \) brings equations (5.5)-5.9 to the concise form

\[
\begin{align*}
  W \ast W &= -i[dZ_A dZ_A^* + \eta \delta^2(dz)B \ast k \ast \kappa + \bar{\eta} \delta^2(d\bar{z})B \ast \bar{k} \ast \bar{\kappa}], \\
  W \ast B &= B \ast W.
\end{align*}
\]

The simplest choice of linear functions

\[
F_*(B) = \eta B, \quad \tilde{F}_*(B) = \bar{\eta} B,
\]

where \( \eta \) is some phase factor (its absolute value can be absorbed into redefinition of \( B \)) leads to a class of pairwise nonequivalent nonlinear HS theories. The particular cases of \( \eta = 1 \) and \( \eta = \exp \frac{\pi \nu}{2} \) are especially interesting, corresponding to so called \( A \) and \( B \) HS models. These two cases are distinguished by the property that they respect parity 60.

The associative star product \( \ast \) acts on functions of two spinor variables

\[
(f \ast g)(Z; Y) = \frac{1}{(2\pi)^4} \int d^4U \, d^4V \exp [iU^A V^B C_{AB}] f(Z + U; Y + U)g(Z - V; Y + V),
\]

where \( C_{AB} = (\varepsilon_{\alpha\beta}, \varepsilon_{\bar{\alpha}\bar{\beta}}) \) is the 4d charge conjugation matrix and \( U^A, V^B \) are real integration variables. It is normalized so that 1 is a unit element of the star-product algebra, i.e., \( f \ast 1 = 1 \ast f = f \). Star product (5.13) provides a particular realization of the Weyl algebra

\[
[Y_A, Y_B] = -[Z_A, Z_B] = 2iC_{AB}, \quad [Y_A, Z_B] = 0
\]

([a, b]* = a \ast b - b \ast a).

The left and right inner Klein operators

\[
\kappa = \exp iz_\alpha y^\alpha, \quad \bar{\kappa} = \exp iz_{\bar{\alpha}} \bar{y}^{\bar{\alpha}},
\]

which enter Eq. (5.9), change a sign of undotted and dotted spinors, respectively

\[
\begin{align*}
  (\kappa \ast f)(z, \bar{z}; y, \bar{y}) &= \exp iz_\alpha y^\alpha f(y, \bar{z}; z, \bar{y}), \quad (\bar{\kappa} \ast f)(z, \bar{z}; y, \bar{y}) = \exp iz_{\bar{\alpha}} \bar{y}^{\bar{\alpha}} f(z, \bar{y}; y, \bar{z}), \\
  \kappa \ast f(z, \bar{z}; y, \bar{y}) &= f(-z, \bar{z}; -y, \bar{y}) \ast \kappa, \quad \bar{\kappa} \ast f(z, \bar{z}; y, \bar{y}) = f(z, -\bar{z}; y, -\bar{y}) \ast \bar{\kappa}.
\end{align*}
\]
κ * κ = ̄κ * ̄κ = 1, \quad κ * ̄κ = ̄κ * κ. \quad (5.18)

To analyze Eqs. (5.5)-(5.9) perturbatively, one has to linearize them around some vacuum solution. The simplest choice is

\[ W_0(Z; Y|x) = W_0(Y|x), \quad S_0(Z; Y|x) = dZ^A Z_A, \quad B_0 = 0, \quad (5.19) \]

where \( W_0(Y|x) \) is some solution of the flatness condition

\[ dW_0(Y|x) + W_0(Y|x) * W_0(Y|x) = 0. \quad (5.20) \]

\( W_0(Y|x) \) bilinear in \( Y^A \) describes \( AdS_4 \).

Propagating massless fields are described by the fields \( W(Z; Y; \kappa|x) \) even in \( \kappa \) and fields \( B(Z; Y; \kappa|x) \) odd in \( \kappa \)

\[ W(Z; Y; -\kappa|x) = W(Z; Y; \kappa|x), \quad B(Z; Y; -\kappa|x) = -B(Z; Y; \kappa|x). \quad (5.21) \]

In this sector, linearization of system (5.5)-(5.9) around vacuum (5.19) just reproduces free field equations (4.1), (4.2).

The fields of opposite parity in the Klein operators

\[ W(Z; Y; -\kappa|x) = -W(Z; Y; \kappa|x), \quad B(Z; Y; -\kappa|x) = B(Z; Y; \kappa|x) \quad (5.22) \]

are topological in the sense that irreducible fields describe at most a finite number of degrees of freedom. (For more detail see [53, 67, 68]). As such they can be treated as describing infinite sets of the coupling constants in HS theory.

6 \hspace{1em} \textbf{Invariants of the} \hspace{1em} AdS_4 \hspace{1em} \textbf{HS theory}

To explain the idea of our construction let us first consider an example of a contractible unfolded system of the form

\[ d\omega = \mathcal{L}, \quad d\mathcal{L} = 0. \quad (6.1) \]

It is obviously consistent and hence is invariant under gauge transformations (2.3)

\[ \delta \omega(x) = \epsilon(x), \quad \delta \mathcal{L}(x) = d\epsilon(x). \quad (6.2) \]

As such it is dynamically empty since the gauge transformation allows one to gauge fix \( \omega = 0 \). By virtue of (6.1) it follows then that \( \mathcal{L} = 0 \).

A more interesting system is

\[ d\omega + L(W) = \mathcal{L}, \quad d\mathcal{L} = 0, \quad (6.3) \]

where \( L(W) \) is some closed function of other fields \( W \) that obey some unfolded equations (2.1). In the canonical gauge \( \omega = 0 \) it takes the form

\[ \mathcal{L} = L(W), \quad dL(W) = 0. \]
The singlet field \( L \) becomes a Lagrangian giving rise to an invariant action

\[
S = \int_{\Sigma} L(W) .
\]  

(6.4)

So defined functional is independent of local variations of the integration cycle and gauge invariant. Indeed, being formally consistent, the system is invariant under gauge transformations \((2.3)\) with respect to the gauge parameter \( \epsilon \) associated with \( w \) and the gauge parameters \( \epsilon^\Omega \) associated with \( W^\Omega \). In the gauge \( w = 0 \), the parameter \( \epsilon \) is expressed by the condition \( \delta w = 0 \) via the gauge parameters \( \epsilon^\Omega \) and the gauge fields \( W^\Omega \)

\[
\epsilon = \epsilon(\epsilon, W) .
\]  

(6.5)

Though \( L \) is not gauge invariant under the gauge transformations of the system, it transforms by a total derivative of a function of fields \( W^\Omega \) and gauge parameters \( \epsilon^\Omega \). As a result, the action \( S \) is gauge invariant.

Note that though the system \( dw + L(W) = 0 \) is formally consistent it is not guaranteed that it admits a solution with regular \( w \). In fact, the Lagrangian \( L \) defined by \((6.3)\) is nontrivial for non-exact \( L(W) \).

The proposal of \([52]\) is to consider invariants resulting from the following extension of the HS unfolded equations

\[
W \ast W = F(B) + L Id , \quad W \ast B = B \ast W , \quad dL = 0 ,
\]  

where \( W = d + W \) and \( B \) are differential forms of all odd and even degrees, respectively (both in \( dx \) and \( dZ \)). An appropriate choice is

\[
iF(B) = dZ_A dZ^A + \eta \delta^2(dz) B \ast k \ast \kappa + \bar{\eta} \delta^2(d\bar{z}) \bar{B} \ast \bar{k} \ast \bar{\kappa} + G(B) \delta^4(dZ) k \ast \bar{k} \ast \kappa \ast \bar{\kappa} .
\]

\( G = g + O(B) \), where \( g \) is the coupling constant. \( L(x) \) are \( x \)-dependent space-time differential forms of positive even degrees since the left-hand side of \((6.6)\) contains a product of forms of even degrees. That it enters as a coefficient in front of the unit element \( Id \) of the star-product algebra means that \( L(x) \) is independent of \( Y^A \) and \( Z^A \). As a result, application of the covariant derivative to the right-hand side of \((6.6)\) gives \( dL(x) = 0 \).

It should be stressed that the modification of the system by the “Lagrangians” \( L \) does not affect the form of all equations except for the single \( Z, Y \)-independent equation proportional to \( Id \), which just acquires the form \((6.3)\). The form of the Lagrangian \( L(W) \) \((6.3)\) now results from the perturbative solution of the other equations, \( i.e., \) nonlinear HS equations.

The density relevant to the generating functional of correlators in \( AdS_4/CFT_3 \) HS holography is a 4-form \( \mathcal{L}^4 \). The density relevant to BH entropy is a 2-form \( \mathcal{L}^2 \) (for recent progress in this direction see \([69]\).)

7 Conclusions

A very general property illustrated by the analysis of HS theory is that the unfolding machinery makes holographic duality manifest at the level of the unfolded formulation of HS
equations. Following [52], the duality extends to the level of generating functionals. The latter can be identified with integrals of differential forms of positive even degrees valued in the center of the star-product algebra. So defined functionals are gauge invariant, coordinate independent and can be evaluated for any boundaries and bulk solutions.

In 4d HS theory the 4-form $L^4$ is conjectured to give rise to the generating functional for boundary correlators while the 2-form $L^2$ gives black-hole charges opening new perspectives for the understanding of black-hole physics including the informational paradox [69].

As shown in [52] a similar construction applies to the HS theory in $AdS_3$. In this case the only Lagrangian density is a 2-form $L^2$. An exciting peculiarity of this construction is that the boundary functional results from the integration over a one-dimensional cycle at the boundary (times a cycle over the complexified Poincaré coordinate $z$). It is tempting to speculate that this property expresses holomorphicity of the 2d boundary conformal theory.

By virtue of unfolded dynamics usual field-current correspondence can be extended via interpretation of further nonlinear combinations of fields with linear fields in higher dimensions. An interesting subtlety here is that the mixing of fields in different dimensions representing nonlinear interactions in the original system can decrease the symmetries of unmixed fields. This is illustrated by current interactions of massless fields of all spins in $d = 4$ which break the $sp(8)$ symmetry of free fields down to the conformal symmetry $su(2, 2) \subset sp(8)$ [65].

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