ON THE CESÀRO AVERAGE OF THE NUMBERS THAT CAN BE WRITTEN AS SUM OF A PRIME AND TWO SQUARES OF PRIMES

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Abstract. Let $\Lambda(n)$ be the Von Mangoldt function and $r_{SP}(n) = \sum_{m_1 + m_2^2 + m_3^2 = n} \Lambda(m_1) \Lambda(m_2) \Lambda(m_3)$ be the counting function for the numbers that can be written as sum of a prime and two squares. Let $N$ be a sufficiently large integer. We prove that
\[
\sum_{n \leq N} r_{SP}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{N^{k+2} \pi}{4 \Gamma(k+3)} + E(N,k)
\]
for $k > 3/2$, where $E(N,k)$ consists of lower order terms that are given in terms of $k$ and sum over the non-trivial zeros of the Riemann zeta function.

1. Introduction

We continue the recent work of Languasco, Zaccagnini and the author on additive problems with prime summands. In [12] and [13] Languasco and Zaccagnini study the Cesàro weighted explicit formula for the Goldbach numbers (the integers that can be written as sum of two primes) and for the Hardy-Littlewood numbers (the integers that can be written as sum of a prime and a square). Recently [2] the author wrote a paper regarding the Cesàro average of the integers that can be written as sum of a prime and two squares. In a similar manner, we will study a Cesàro weighted explicit formula for the integers that can be written as sum of a prime and two squares of primes. We will obtain an asymptotic formula with a main term and more terms depending explicitly on the zeros of the Riemann zeta function. This technique allow us to obtain a large number of terms in our asymptotic but unfortunately the bound $k > 3/2$ seems to be very difficult to improve. We recall that, for $k = 0$, the Cesàro weights vanish so a result for $k \geq 0$ would allow us to get an asymptotic for the mean of $r_{SP}(n)$.

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We let 
\[ r_{SP}(n) = \sum_{m_1 + m_2^2 + m_3^2 = n} \Lambda(m_1) \Lambda(m_2) \Lambda(m_3) \]
where \( \Lambda(n) \) is the Von Mangoldt function and

\[ M_1(N, k) = \frac{N^{k+2} \pi}{4 \Gamma(k+3)} \]

\[ M_2(N, k) = \frac{N^{k+1} \pi}{4} \sum_{\rho} \frac{N^{\rho} \Gamma(\rho)}{\Gamma(k+2+\rho)} - \frac{N^{k+3/2} \sqrt{\pi}}{2} \sum_{\rho} \frac{N^{\rho/2} \Gamma(\rho/2)}{\Gamma(k+5/2+\rho/2)} \]

\[ M_3(N, k) = \frac{N^{k+1/2}}{2} \sum_{\rho_1} \sum_{\rho_2} N^{\rho_1+\rho_2/2} \frac{\Gamma(\rho_1) \Gamma(\rho_2/2)}{\Gamma(k+3/2+\rho_1+\rho_2/2)} \]

\[ M_4(N, k) = \frac{N^k}{4} \sum_{\rho_1} \sum_{\rho_2} \sum_{\rho_3} N^{\rho_1+\rho_2+\rho_3/2} \frac{\Gamma(\rho_1) \Gamma(\rho_2/2)}{\Gamma(k+\rho_1+\rho_2/2+\rho_3/2)} \]

The main result of this paper is the following

**Theorem 1.** Let \( N \) be a sufficient large integer. We have

\[ \sum_{n \leq N} r_{SP}(n) \frac{(N-n)^k}{\Gamma(k+1)} = M_1(N, k) + M_2(N, k) + M_3(N, k) + M_4(N, k) + O(N^{k+1}) \]

for \( k > 3/2 \), where \( \rho = \beta + i \gamma \), with or without subscripts, runs over the non-trivial zeros of the Riemann zeta function \( \zeta(s) \).

Note that an upper bound for \( M_i(N, k) \), \( i = 2, \ldots, 4 \) depends closely on \( \beta \). Let us define

\[ \beta := \sup \{ \beta : \Re(\rho) = \beta \} . \]

We have that

\[ M_2(N, k) \ll_k N^{k+3/2 + \beta/2} \]

\[ M_3(N, k) \ll_k N^{k+1 + \beta} \]

\[ M_4(N, k) \ll_k N^{k+2\beta} . \]

Note also that, if the Riemann hypothesis is true, then \( M_4(N, k) \) can be incorporated in the error term. The problem of representing an integer as sum of a prime and two prime squares is classical. Let

\[ A = \{ n \in \mathbb{N} : n \equiv 1 \mod 2; n \not\equiv 1 \mod 3 \} ; \]

it is conjectured that every sufficiently large natural number \( n \in A \) is a sum of a prime and two prime squares. Many authors studied the cardinality \( E(N) \) of the set of integers \( n \leq N, n \in A \) that are not representable as a sum of prime and two squares of primes. We recall Hua [10], Schwarz [19], Leung-Liu [10], Wang [21], Wang-Meng [22], Li [17], Harman-Kumchev [9]. Zhao [24] proved that

\[ E(N) \ll N^{1/3 + \varepsilon} \]

and so every integer \( n \in [1, N] \cap A \), with at most \( O(N^{1/3+\varepsilon}) \) exceptions, is a sum of a prime and two squares of primes. Letting

\[ r(n) := \sum_{p_1 + p_2^2 + p_3^2 = n} \log(p_1) \log(p_2) \log(p_3) \]

Zhao also proved that an asymptotic formula for \( r(n) \) holds for \( n \in [1, N] \cap A \), with at most \( O(N^{1/3+\varepsilon}) \) exceptions. Similar averages of arithmetical functions are common in literature, see, e.g., Chandrasekharan - Narasimhan [3].
and Berndt [1] who built on earlier classical work. The method we will use in this additive problem is based on a formula due to Laplace [15], namely

\[
\frac{1}{2\pi i} \int_{(a)} v^{-s} e^v dv = \frac{1}{\Gamma(s)}
\]

with \( \text{Re}(s) > 0 \) and \( a > 0 \) (see, e.g., formula 5.4 (1) on page 238 of [5]), where the notation \( \int_{(a)} \) means \( \int_{a-i\infty}^{a+i\infty} \). As in [13], we combine this approach with line integrals with the classical methods dealing with infinite sum over primes and integers.

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2. Preliminary definitions and Lemmas

Let \( z = a + iy, \ a > 0, \) let

\[
\widetilde{S}_1(z) = \sum_{m \geq 1} \Lambda(m) e^{-mz}
\]

\[
\widetilde{S}_2(z) = \sum_{m \geq 1} \Lambda(m) e^{-m^2z}
\]

and let us introduce the following

**Lemma 2.** Let \( z = a + iy, \ a > 0 \) and \( y \in \mathbb{R} \). Then

\[
\widetilde{S}_1(z) = \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) + E(a, y)
\]

where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \) and

\[
E(a, y) \ll |z|^{1/2} \begin{cases} 1, & |y| \leq a \\ 1 + \log^2(|y|/a), & |y| > a. \end{cases}
\]

(For a proof see Lemma 1 of [12]. The bound for \( E(a, y) \) has been corrected in [11]). So in particular, taking \( z = \frac{1}{N} + iy \) we have

\[
\left| \sum_{\rho} z^{-\rho} \Gamma(\rho) \right| = \left| \frac{1}{z} - \widetilde{S}(z) \right| + E \left( \frac{1}{N}, y \right) \ll N + \frac{1}{|z|} + |E \left( \frac{1}{N}, y \right)|
\]

\[
\ll \begin{cases} N, & |y| \leq 1/N \\ N + |z|^{1/2} \log^2(2N|y|), & |y| > 1/N. \end{cases}
\]

We now introduce the following

**Lemma 3.** Let \( z = a + iy, \ a > 0, \ y \in \mathbb{R} \) and \( \ell \) a fixed positive integer. Then

\[
\widetilde{S}_\ell(z) = \frac{\Gamma(1/\ell)}{\ell^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma \left( \frac{\rho}{\ell} \right) + E_\ell(a, y)
\]

where \( \rho = \beta + i\gamma \) runs over the non-trivial zeros of \( \zeta(s) \) and

\[
E_\ell(a, y) \ll \ell E(a, y).
\]

**Proof.** It is well known (see for example formula 5 of [14]) that, for \( \ell \in \mathbb{N}_0, \)

\[
\widetilde{S}_\ell(z) = \sum_{m \geq 1} \Lambda(m) e^{-m^{1/\ell}z}
\]

\[
= \frac{\Gamma(1/\ell)}{\ell^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma \left( \frac{\rho}{\ell} \right) - \frac{\zeta'}{\zeta}(0) - \frac{1}{2\pi \ell} \int_{(-1/2)} \frac{\zeta'}{\zeta}(\ell w) \Gamma(w) z^{-w} dw
\]
so, taking \( w = -\frac{1}{2} + it \), following the proof of the Lemma 1 in [12] and observing that
\[
\left| \frac{\zeta'}{\zeta} (\ell w) \right| \ll \ell \log (|t| + 2)
\]
we can conclude that we may estimate the integral in (13) exactly as in [12], so the claim follows. □

Now we have to recall that the Prime Number Theorem (PNT) is equivalent, via Lemma 2, to the statement
\[
(14) \quad \mathcal{S}_1 (a) \sim a^{-1}, \quad \text{when } a \to 0^+
\]
(see Lemma 9 of [8]) and from Lemma 3 we have
\[
(15) \quad \mathcal{S}_2 (a) \sim \frac{\sqrt{\pi}}{2a^{1/2}}, \quad \text{when } a \to 0^+.
\]
For our purposes it is important to introduce the Stirling approximation (see for example §4.42 of [20])
\[
|\Gamma (x + iy)| \sim 2\pi e^{-\pi |y|/2} |y|^{-x/2}
\]
uniformly for \( x \in [x_1, x_2] \), \( x_1 \) and \( x_2 \) fixed, as well as the identity
\[
(16) \quad |z^{-w}| = |z|^{-\text{Re}(w)} \exp (\text{Im}(w) \arctan (y/a)).
\]
We now quote Lemmas 2 and 3 from [12]:

**Lemma 4.** Let \( \beta + i \gamma \) run over the non-trivial zeros of the Riemann zeta function and let \( \alpha > 1 \) be a parameter. The series
\[
\sum_{\rho, \gamma > 0} \gamma^{\beta - 1/2} \int_1^\infty \exp (-\gamma \arctan (1/u)) \frac{du}{u^\alpha + \beta}
\]
converges provided that \( \alpha > 3/2 \). For \( \alpha \leq 3/2 \) the series does not converge. The result remains true if we insert in the integral a factor \( \log^c (u) \), for any fixed \( c \geq 0 \).

**Lemma 5.** Let \( \beta + i \gamma \) run over the non-trivial zeros of the Riemann zeta function, let \( z = a + iy \), \( a \in (0, 1) \), \( y \in \mathbb{R} \) and \( \alpha > 1 \). We have
\[
\sum_\rho |\gamma|^{\beta - 1/2} \int_{\mathbb{R}_1} \exp \left( \gamma \arctan \left( \frac{y}{a} \right) - \frac{\pi}{2} |\gamma| \right) \frac{dy}{|z|^\alpha + \beta} \ll_\alpha a^{-\alpha}
\]
where \( \mathbb{R}_1 = \{ y \in \mathbb{R} : \gamma y \leq 0 \} \) and \( \mathbb{R}_2 = \{ y \in [-a, a] : y \gamma > 0 \} \). The result remains true if we insert in the integral a factor \( \log^c \left( |y/a| \right) \), for any fixed \( c \geq 0 \).

Let us introduce another lemma

**Lemma 6.** Let \( \rho = \beta + i \gamma \) run over the non-trivial zeros of the Riemann zeta function, let \( z = \frac{1}{\ell} + iy \), where \( N > 1 \) is a natural number, \( y \in \mathbb{R} \), \( \ell \geq 1 \) an integer and \( \alpha > 3/2 \). We have
\[
\sum_\rho \left| \Gamma \left( \frac{\rho}{\ell} \right) \right| \int_{(1/N)} |e^{Nz}| \left| z^{-\rho/\ell} \right| |z|^{-\alpha} |dz| \ll_\alpha N^\alpha.
\]
**Proof.** Put \( a = \frac{1}{\ell N} \). Using the identity \([17], (18)\) and
\[
(18) \quad |z|^{-1} \simeq \begin{cases} 
    a^{-1} & |y| \leq a, \\
    |y|^{-1} & |y| \geq a,
\end{cases}
\]
we get that the left hand side in the statement above is
\[
(19) \quad \sum_\rho |\gamma|^{\beta/\ell - 1/2} \int_\mathbb{R} \exp \left( \frac{\gamma}{\ell} \arctan \left( \frac{y}{a} \right) - \frac{\pi}{2} |\gamma| \right) \frac{dy}{|z|^\alpha + \beta/\ell}.
\]
The case \( \ell = 1 \) has already been discussed in Lemma 6 of [2]. For \( \ell > 1 \), observing Lemmas 2 and 3 of [12] and Lemma 6 [2], we can conclude that the presence of \( \ell \) does not alter the proofs. Hence using the same argument of Lemma 6 of [2] we have the convergence for \( \alpha > 3/2 \). □
3. **Setting**

From (6) and (7) it is not hard to see that
\[
\tilde{S}_1(z) \tilde{S}_2(z) = \sum_{m_1 \geq 1} \sum_{m_2 \geq 1} \sum_{m_3 \geq 1} \Lambda (m_1) \Lambda (m_2) \Lambda (m_3) e^{-(m_1 + m_2 + m_3)z} = \sum_{n \geq 1} r_{SP} (n) e^{-nz}
\]
so let \( z = a + iy \) and \( a > 0 \) and let us consider
\[
\frac{1}{2\pi i} \int_{(a)} e^{Nz^{-k-1}} \tilde{S}_1(z) \tilde{S}_2(z) \, dz = \frac{1}{2\pi i} \int_{(a)} e^{Nz^{-k-1}} \sum_{n \geq 1} r_{SP} (n) e^{-nz} \, dz.
\]
Now we prove that we can exchange the integral with the series. From (14) and (15) we have
\[
\sum_{n \geq 1} |r_{SP} (n) e^{-nz}| = \tilde{S}_1(a) \tilde{S}_2(a) \ll a^{-2}
\]
and hence
\[
\int_{(a)} |e^{Nz^{-k-1}}| |\tilde{S}_1(z) \tilde{S}_2(z)| \, dz \ll a^{-2} e^{Na} \left( \int_{-a}^a a^{-k-1} \, dy + 2 \int_{-\infty}^\infty y^{-k-1} \, dy \right) \ll k a^{-2-k} e^{Na}
\]
assuming \( k > 0 \), so we have that
\[
\sum_{n \leq N} r_{SP} (n) \frac{(N-n)^{k}}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(a)} e^{Nz^{-k-1}} \tilde{S}_1(z) \tilde{S}_2(z) \, dz.
\]
Now from (8), (11), (14) and (15) and observing that, for \( \ell \geq 1 \),
\[
\frac{\Gamma(1/\ell)}{\ell z^{1/\ell}} - \frac{1}{\ell} \sum_{\rho} z^{-\rho/\ell} \Gamma \left( \frac{\rho}{\ell} \right) = \tilde{S}_\ell(z) - E_\ell(a, y) \ll a^{-1/\ell} + |E_\ell(a, y)|
\]
we have
\[
\tilde{S}_1(z) \tilde{S}_2(z) = \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma (\rho) + E_1(a, y) \right) \left( \frac{\sqrt{\pi}}{2 z^{1/2}} - \frac{1}{2} \sum_{\rho} z^{-\rho/2} \Gamma \left( \frac{\rho}{2} \right) + E_2(a, y) \right)^2
\]
\[
= \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma (\rho) \right) \left( \frac{\sqrt{\pi}}{2 z^{1/2}} - \frac{1}{2} \sum_{\rho} z^{-\rho/2} \Gamma \left( \frac{\rho}{2} \right) \right)^2 + O \left( |E_1(a, y)| a^{-1} + |E_1(a, y)| |E_2(a, y)| \right)
\]
(21)
\[
|E_2(a, y)|^2 a^{-3/2} \ll |E_2(a, y)|.
\]
(22)
Now let us consider \( l, m, r, s \geq 1 \) integers. From (9) and (12) we have that
\[
\int_{(a)} |e^{Nz^{-k-1}}| \, |E_1(a, y)| |E_m(a, y)|^r \, |dz|
\]
\[
\ll_{l,m,r,s} e^{Na} a^{-k-1+l \frac{r+s}{2}} \int_0^a dy + \int_a^\infty y^{-k-1+l \frac{r+s}{2}} \log^{2r+2s} \left( \frac{y}{a} \right) \, dy
\]
\[
\ll_{l,m,r,s} e^{Na} a^{-k-1+l \frac{r+s}{2}}
\]
assuming \( k > \frac{r+s}{2} \). We now have to deal with the terms in (21) and (22): taking \( a = 1/N \) we can observe that
\[
\int_{(1/N)} |e^{Nz^{-k-1}}| |E_2(1/N, y)|^2 |E_1(1/N, y)| \, |dz| \ll k N^{k-3/2},
\]
\[
N^{1/2} \int_{(1/N)} |e^{Nz^{-k-1}}| |E_2(1/N, y)| |E_1(1/N, y)| \, |dz| \ll N^{k-1/2},
\]

From \( \frac{N}{1/N} \) and hence the Cesàro average of \( r_{SP} \) can be broken down as
\[
\sum_{n \leq N} r_{SP}(n) \frac{(N-n)^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_{(1/N)} e^{\frac{N}{z} - k-1} \left( \frac{1}{z} - \sum_{\rho} z^{-\rho} \Gamma(\rho) \right) \left( \frac{\sqrt{\pi}}{2z^{1/2}} - \frac{1}{2} \sum_{\rho} z^{-\rho/2} \Gamma \left( \frac{\rho}{2} \right) \right)^2 \, dz + O_k \left( N^{k+1} \right) 
\]
\[
= \frac{1}{8i} \int_{(1/N)} e^{\frac{N}{z} - k-3} \, dz
\]
\[
+ \frac{1}{8i} \int_{(1/N)} e^{\frac{N}{z} - k-2} \sum_{\rho} z^{-\rho} \Gamma(\rho) \, dz
\]
\[
- \frac{1}{4\sqrt{\pi}} \int_{(1/N)} e^{\frac{N}{z} - k-5/2} \sum_{\rho} z^{-\rho/2} \Gamma(\rho) \, dz
\]
\[
+ \frac{1}{4\sqrt{\pi}} \int_{(1/N)} e^{\frac{N}{z} - k-3/2} \sum_{\rho_1, \rho_2} z^{-\rho_1/2} \Gamma(\rho_1) \sum_{\rho_2} z^{-\rho_2/2} \Gamma(\rho_2) \, dz
\]
\[
+ \frac{1}{8\pi i} \int_{(1/N)} e^{\frac{N}{z} - k-2} \sum_{\rho_1} z^{-\rho_1} \Gamma(\rho_1) \sum_{\rho_2} z^{-\rho_2} \Gamma(\rho_2) \, dz
\]
\[
- \frac{1}{8\pi i} \int_{(1/N)} e^{\frac{N}{z} - k-1} \sum_{\rho_1} z^{-\rho_1} \Gamma(\rho_1) \sum_{\rho_2} z^{-\rho_2} \Gamma(\rho_2) \sum_{\rho_3} z^{-\rho_3} \Gamma(\rho_3) \, dz
\]
\[
+ O_k \left( N^{k+1} \right)
\]
\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + O_k \left( N^{k+1} \right),
\]
say. In the next sections we will prove that \( I_1 = M_1(N,k) \), \( I_2 + I_3 = M_2(N,k) \), \( I_4 + I_5 = M_3(N,k) \) and \( I_6 = M_4(N,k) \).

### 4. Evaluation of \( I_1 \)

From \( I_1 \) we will find the main term. If we put \( Nz = s \) we get
\[
I_1 = \frac{1}{8i} \int_{(1/N)} e^{Nz} - k-3 \, dz = \frac{N^{k+2}}{8i} \int_{(1)} e^{s} - k-3 \, ds = \frac{N^{k+2} \pi}{4 \Gamma(k+3)}
\]
using (23). Then \( I_1 = M_1(N,k) \).

### 5. Evaluation of \( I_2 \) and \( I_3 \)

We have
\[
I_2 = \frac{1}{8i} \int_{(1/N)} e^{Nz} - k-2 \sum_{\rho} z^{-\rho} \Gamma(\rho) \, dz
\]
and
\[
I_3 = -\frac{1}{4\sqrt{\pi}i} \int_{(1/N)} e^{Nz} - k-5/2 \sum_{\rho} z^{-\rho/2} \Gamma(\rho) \, dz.
\]
We want to exchange the integral with the series, then we will prove the absolute convergence for a suitable choice of \( k \). Hence we have to study

\[
A_2 := \left| \sum_\rho \Gamma (\rho) \int_{(1/N)} e^{Nz} \left| z^{-k-2} \right| \left| z^{-\rho} \right| |dz|
\]

and

\[
A_3 := \left| \sum_\rho \Gamma \left( \frac{\rho}{2} \right) \int_{(1/N)} e^{Nz} \left| z^{-k-5/2} \right| \left| z^{-\rho/2} \right| |dz|
\]

and from Lemma 6 we have the convergence for \( k > -1/2 \) and \( k > -1 \) respectively. So we can switch the integral and the series and get

\[
I_2 = \frac{1}{8i} \sum_\rho \Gamma (\rho) \int_{(1/N)} e^{Nz} z^{-k-2-\rho} |dz| = \frac{N^{k+1}}{4} \sum_\rho N^\rho \frac{\Gamma (\rho)}{\Gamma (k+2+\rho)}
\]

and

\[
I_3 = -\frac{1}{4\sqrt{\pi} i} \sum_\rho \Gamma \left( \frac{\rho}{2} \right) \int_{(1/N)} e^{Nz} z^{-k-5/2-\rho/2} |dz| = -\frac{N^{k+3/2}}{2} \sum_\rho N^{\rho/2} \frac{\Gamma (\rho/2)}{\Gamma (k+5/2+\rho/2)}
\]

then \( I_2 + I_3 = M_2 (N, k) \).

6. Evaluation of \( I_4 \)

We have to evaluate

\[
I_4 = \frac{1}{4\sqrt{\pi} i} \int_{(1/N)} e^{Nz} z^{-k-3} \sum_{\rho_1} z^{-\rho_1} \Gamma (\rho_1) \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) |dz|.
\]

We want to switch the integral with two series so we will prove the absolute convergence of

\[
A_{4,1} := \sum_{\rho_1} \left| \Gamma (\rho_1) \right| \int_{(1/N)} e^{Nz} \left| z^{-k-3/2} \right| \left| z^{-\rho_1} \right| \left| \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) \right| |dz|
\]

and

\[
A_{4,2} := \sum_{\rho_1} \left| \Gamma (\rho_1) \right| \sum_{\rho_2} \left| \Gamma \left( \frac{\rho_2}{2} \right) \right| \int_{(1/N)} e^{Nz} \left| z^{-k-3/2} \right| \left| z^{-\rho_1} \right| \left| z^{-\rho_2/2} \right| |dz|.
\]

Now we have to introduce some notations, which is necessary since the evaluation of the integrals depends strictly on the sign of \( y \) and the sign of the imaginary part of \( \rho \). Assume that \( A_{m,n} := \int_{(1/N)} \ldots |dz| \). Hereafter we will use the symbol

\[
A_{m,n}^+ := \int_0^{1/N+\infty} \ldots |dz|
\]

and

\[
A_{m,n}^- := \int_0^{1/N-\infty} \ldots |dz|.
\]

From (11) we can see that

\[
\left| \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) \right| = \left| \tilde{S}_2 (z) - \frac{\sqrt{\pi}}{2 z^{1/2}} - E_2 (1/N, y) \right| \ll N^{1/2} + \frac{1}{|z|^{1/2}} + |E_2 (1/N, y)|
\]

(24)

(25)

\[
\ll \begin{cases} N, & |y| \leq 1/N \\ N + |z|^{1/2} \log^2 (2N |y|), & |y| > 1/N. \end{cases}
\]

(26)
Let us consider $y \leq 0$ and, recalling the notation $\rho_j = \beta_j + i\gamma_j$, the notation (25) and assuming $\gamma_1 > 0$ for symmetry, we have to study

$$A_{4,1}^- \ll \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_0^1 \exp \left( \gamma_1 \arctan (Ny) - \frac{\pi}{2} \gamma_1 \right) |z|^{k - 3/2 + 3 \beta_1} |dz| +$$

$$N \int_{-\infty}^{1/N} \exp \left( \gamma_1 \arctan (Ny) - \frac{\pi}{2} \gamma_1 \right) \frac{dy}{|y|^{k + 3/2 + \beta_1}} + \int_{-\infty}^{1/N} \exp \left( \gamma_1 \arctan (Ny) - \frac{\pi}{2} \gamma_1 \right) \log^2 (2N |y|) \frac{dy}{|y|^{k + 1 + \beta_1}} \ll_k N^{k + 5/2}$$

from Lemma 5, assuming that $\gamma > 0$. Note that we have to split the integral since, from (15) and (20), we have different evaluations if $|y| \leq 1/N$ or $|y| > 1/N$. Now let us consider $y > 0$. Recalling (24), we have to study

$$A_{4,1}^+ \ll N \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_0^{1/N} \exp \left( \gamma_1 \arctan (Ny) - \frac{\pi}{2} \gamma_1 \right) |z|^{k + 3/2 + \beta_1} |dz|$$

$$+ \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty \exp \left( \gamma_1 \arctan (Ny) - \frac{\pi}{2} \gamma_1 \right) N + \frac{y}{2} \gamma_1 \frac{dy}{|y|^{k + 3/2 + \beta_1}} = A_1 + A_2$$

say, and we have that

$$A_1 \ll_k N^{k + 5/2}$$

from Lemma 5 and

$$A_2 \ll N^{k + 5/2} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_1^\infty \frac{\exp \left( -\gamma_1 \arctan (1/u) \right)}{u^{k + 3/2 + \beta_1}} |du|$$

$$+ \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_1^\infty \frac{\exp \left( -\gamma_1 \arctan (1/u) \right) \log^2 (2u)}{u^{k + 1/2 + \beta_1}} |du| \ll_k N^{k + 5/2}$$

from Lemma 4, assuming $k > 1/2$. Now let us consider

$$A_{4,2} = \sum_{\rho_1} |\Gamma (\rho_1)| \sum_{\rho_2} |\Gamma \left( \frac{\rho_2}{2} \right)| \int_{(1/N)} |e^{Nz}| |z^{-k - 3/2}| |z^{-\rho_1}| |z^{-\rho_2/2}| |dz| .$$

By symmetry, it suffices to consider only the cases $\gamma_1, \gamma_2 > 0$ and $\gamma_1 > 0, \gamma_2 < 0$. As in (24) and (25) we have to introduce some new notations since the evaluation depends on the sign of the product $\gamma_1 \gamma_2$ and the sign of $y$. Hereafter we will use the symbol $B_{m,n}$ when we consider $A_{m,n}$ with the assumption $\gamma_1, \gamma_2 > 0$ and the symbol $C_{m,n}$ when we consider $A_{m,n}$ with the assumption $\gamma_1 > 0, \gamma_2 < 0$. Since

$$\arctan (Ny) - \frac{\pi}{2} \leq -\frac{\pi}{2}$$

and recalling (26), we have

$$B_{4,2}^- \ll \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2 - 2/1/2} \exp \left( \frac{\pi}{4} \gamma_2 \right) \left( \int_{-\infty}^0 \frac{dy}{|z|^{k - 3/2 + \beta_1 + \beta_2/2}} \right)$$

$$\ll_k N^{k + 2} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2 - 2/1/2} \exp \left( -\frac{\pi}{4} \gamma_2 \right) \ll_k N^{k + 2}.$$
say. If \( y \in (0, 1/N] \) we obviously have \( \arctan(Ny) - \frac{\pi}{2} \leq -\frac{\pi}{4} \) and so

\[
A_3 \ll_k \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{4} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2 / 2 - 1/2} \exp \left( -\frac{\pi}{8} \gamma_2 \right) \int_0^{1/N} N^{k + 3/2 + \beta_1 + \beta_2 / 2} dy \ll_k N^{k + 2}
\]

For \( A_4 \) we can see, following the proof of the Lemma 4, that we have

\[
A_4 \ll N^{k+1/2} \sum_{\rho_1: \gamma_1 > 0} N^{\beta_1 \gamma_1} \sum_{\rho_2: \gamma_2 > 0} N^{\beta_2 \gamma_2} \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2}} \frac{1}{\gamma_1 + \gamma_2} \sum_{0 < \gamma_2 \leq \gamma_1} \log \frac{\gamma_1}{\gamma_2}
\]

and observing that

\[
\frac{\gamma_1^{\beta_2 / 2} \gamma_2^{\beta_2 / 2}}{2} \leq \left( \gamma_1 + \frac{\gamma_2}{2} \right)^{\beta_1 + \beta_2 / 2}
\]

we get

\[
A_4 \ll_k N^{k+1/2} \sum_{\rho_1: \gamma_1 > 0} \sum_{\rho_2: \gamma_2 > 0} N^{\beta_1 + \beta_2 / 2} \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2}} \frac{1}{\gamma_1 + \gamma_2} \sum_{0 < \gamma_2 \leq \gamma_1} \log \frac{\gamma_1}{\gamma_2}
\]

and so we proved the convergence if \( k > 1/2 \) using the Riemann - Von Mangoldt formula. Let us consider the case \( \gamma_1 > 0, \gamma_2 < 0 \) (and so we will use the symbol \( C_{m,n} \)) and let \( y \leq 0 \). Using again (27) we have to study

\[
C_{4,2}^{-} \ll \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2 / 2 - 1/2} \exp \left( -\frac{\pi}{8} |\gamma_2| \right) \exp \left( \frac{\pi |\gamma_2|}{4} \right) \int_{-\infty}^{0} \frac{1}{|z|^{k+3/2 + \beta_1 + \beta_2 / 2}} \left( \gamma_2 \arctan(Ny) - \pi |\gamma_2| \right) \, |dz|
\]

and using Lemma 4, Lemma 5 and the identity \( \arctan(x) + \arctan(1/x) = -\pi/2 \), \( x < 0 \) we have

\[
C_{4,2}^{-} \ll_k N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2 / 2 - 1/2} \exp \left( -\frac{\pi}{8} |\gamma_2| \right) + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2 / 2 - 1/2} \exp \left( -\frac{\pi}{8} |\gamma_2| \right) \exp \left( -\frac{|\gamma_2|}{2} \arctan \left( \frac{N}{y} \right) \right) \int_{-\infty}^{-1/N} \frac{1}{|y|^{k+3/2 + \beta_1 + \beta_2 / 2}} \exp \left( -\frac{|\gamma_2|}{2} \arctan \left( \frac{N}{y} \right) \right) \, |dy|
\]

\[
\ll_k N^{k+3} + N^{k+2} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2 / 2 - 1/2} \exp \left( -\frac{\pi}{8} |\gamma_2| \right) \exp \left( -\frac{|\gamma_2|}{2} \arctan \left( \frac{N}{y} \right) \right) \int_{1}^{\infty} \frac{1}{|y|^{k+3/2 + \beta_1 + \beta_2 / 2}} \exp \left( -\frac{|\gamma_2|}{2} \arctan \left( \frac{N}{y} \right) \right) \, |dy| \ll_k N^{k+3}
\]

for \( k > -1/2 \). If \( y > 0 \) we have essentially the same situation exchanging the role of \( \gamma_1 \) and \( \gamma_2 \). So we can switch the integral with the series and get

\[
I_4 = \frac{1}{4\sqrt{\pi t}} \sum_{\rho_1} \Gamma \left( \rho_1 \right) \sum_{\rho_2} \Gamma \left( \frac{\rho_2}{2} \right) \int_{(1/N)} e^{Nz} z^{-\gamma_1 - 3/2 - \rho_1 - \rho_2 / 2} \, |dz|
\]

\[
= \frac{N^{k+1/2} \sqrt{\pi}}{2} \sum_{\rho_1} \sum_{\rho_2} N^{\rho_1 + \rho_2 / 2} \frac{\Gamma \left( \rho_1 \right) \Gamma \left( \frac{\rho_2}{2} \right)}{\Gamma \left( k + 3/2 + \rho_1 + \rho_2 / 2 \right)}
\]
7. Evaluation of $I_5$

We have to evaluate

$$I_5 = \frac{1}{8\pi i} \int_{(1/N)} e^{Nz}z^{-k-2} \sum_{\rho_1} z^{-\rho_1/2} \Gamma \left( \frac{\rho_1}{2} \right) \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) \, dz$$

and we can see that the argument used in $I_4$ works also in this case since the presence of $\beta_1/2$ instead of $\beta_1$ does not alter the validity of the proof. So repeating the reasoning we can obtain the convergence for $k > 1/2$ and so

$$I_5 = \frac{1}{8\pi i} \sum_{\rho_1} \Gamma \left( \frac{\rho_1}{2} \right) \sum_{\rho_2} \Gamma \left( \frac{\rho_2}{2} \right) \int_{(1/N)} e^{Nz}z^{-k-2-\rho_1/2-\rho_2/2} \, dz$$

$$= \frac{N^{k+1}}{4} \sum_{\rho_1} \sum_{\rho_2} N^{\rho_1/2+\rho_2/2} \Gamma \left( \frac{\rho_1}{2} \right) \Gamma \left( \frac{\rho_2}{2} \right) \Gamma \left( \frac{\rho_1}{2} + \frac{\rho_2}{2} \right) \Gamma \left( k + 2 + \frac{\rho_1}{2} + \frac{\rho_2}{2} \right)$$

then $I_4 + I_5 = M_3(N, k)$.

8. Evaluation of $I_6$

We have to evaluate

$$I_6 = \frac{1}{8\pi i} \int_{(1/N)} e^{Nz}z^{-k-1} \sum_{\rho_1} z^{-\rho_1} \Gamma \left( \rho_1 \right) \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) \sum_{\rho_3} z^{-\rho_3/2} \Gamma \left( \frac{\rho_3}{2} \right) \, dz.$$ 

We want to switch the integral with three series, so we will prove the absolute convergence of

$$A_{6,1} := \sum_{\rho_1} \left| \Gamma \left( \rho_1 \right) \right| \int_{(1/N)} \left| e^{Nz} \right| \left| z^{-k-1} \right| \left| z^{-\rho_1} \right| \left| \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) \right| \left| \sum_{\rho_3} z^{-\rho_3/2} \Gamma \left( \frac{\rho_3}{2} \right) \right| \, dz,$$

$$A_{6,2} := \sum_{\rho_1} \left| \Gamma \left( \rho_1 \right) \right| \sum_{\rho_2} \left| \Gamma \left( \frac{\rho_2}{2} \right) \right| \int_{(1/N)} \left| e^{Nz} \right| \left| z^{-k-1} \right| \left| z^{-\rho_1} \right| \left| z^{-\rho_2/2} \right| \left| \sum_{\rho_3} z^{-\rho_3/2} \Gamma \left( \frac{\rho_3}{2} \right) \right| \, dz$$

and

$$A_{6,3} := \sum_{\rho_1} \left| \Gamma \left( \rho_1 \right) \right| \sum_{\rho_2} \left| \Gamma \left( \frac{\rho_2}{2} \right) \right| \sum_{\rho_3} \left| \Gamma \left( \frac{\rho_3}{2} \right) \right| \int_{(1/N)} \left| e^{Nz} \right| \left| z^{-k-1} \right| \left| z^{-\rho_1} \right| \left| z^{-\rho_2/2} \right| \left| z^{-\rho_3/2} \right| \, dz.$$ 

Let us consider

$$A_{6,1} = \sum_{\rho_1} \left| \Gamma \left( \rho_1 \right) \right| \int_{(1/N)} \left| e^{Nz} \right| \left| z^{-k-1} \right| \left| z^{-\rho_1} \right| \left| \sum_{\rho_2} z^{-\rho_2/2} \Gamma \left( \frac{\rho_2}{2} \right) \right| \left| \sum_{\rho_3} z^{-\rho_3/2} \Gamma \left( \frac{\rho_3}{2} \right) \right| \, dz,$$

and we assume, by symmetry, that $\gamma_1 > 0$. Let $y \leq 0$. From (27) and recalling the notation (25) we have that

$$A_{6,1} \ll N^{k+3} \sum_{\rho_1: \gamma_1 > 0} N^{\beta_1} \gamma_1^{-1/2} \exp \left( \frac{\pi}{2} \gamma_1 \right) \int_{-1/N}^{0} \exp (\gamma_1 \arctan (N |y|)) \, dy$$

$$+ \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \int_{-\infty}^{-1/N} |z|^{-k-1-\beta_1} \exp (\gamma_1 \arctan (N |y|)) \left( N + |z|^{1/2} \log^2 (2N |y|) \right)^2 \, dy$$

which is bounded by

$$A_{6,1} \ll N^{k+3} + N^2 \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \int_{-\infty}^{-1/N} |y|^{-k-1-\beta_1} \, dy$$

$$+ 2N \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \int_{-\infty}^{-1/N} |y|^{-k-1/2-\beta_1} \log^2 (2N |y|) \, dy$$

$$+ \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp \left( \frac{\pi}{2} \gamma_1 \right) \int_{-\infty}^{-1/N} |y|^{-k-\beta_1} \log^4 (2N |y|) \, dy \ll N^{k+3}.$$
for $k > 1$. Let $y > 0$. We have

$$A^+_{6,1} \ll N^2 \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_0^{1/N} \exp \left( \gamma_1 \arctan (N y) - \frac{\pi}{2} \gamma_1 \right) \frac{|dz|}{|z|^{k+1+\beta_1}}$$

$$+ \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty |z|^{-k-1-\beta_1} \exp \left( \gamma_1 \arctan (N y) - \frac{\pi}{2} \gamma_1 \right) \left( N + |z|^{1/2} \log^2 (2N |y|) \right)^2 dy.$$  

From Lemma 5 we have

$$\sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_0^{1/N} \exp \left( \gamma_1 \arctan (N y) - \frac{\pi}{2} \gamma_1 \right) \frac{|dz|}{|z|^{k+1+\beta_1}} \ll_k N^{k+1}$$

for $k > 0$ so

$$A^+_{6,1} \ll N^{k+3} + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty |z|^{-k-1-\beta_1} \exp \left( \gamma_1 \arctan (N y) - \frac{\pi}{2} \gamma_1 \right) \left( N + |z|^{1/2} \log (2N |y|) \right)^2 dy$$

$$\ll N^{k+3} + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty \exp \left( \gamma_1 \arctan (N y) - \frac{\pi}{2} \gamma_1 \right) \frac{\log^2 (2N y)}{y^{k+1+\beta_1}} dy$$

$$+ 2N \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty \exp \left( \gamma_1 \arctan (N y) - \frac{\pi}{2} \gamma_1 \right) \exp \left( \gamma_1 \arctan \left( \frac{1}{u} \right) \right) \frac{\log^4 (2N y)}{y^{k+1+\beta_1}} dy$$

and using the well known identity

$$\arctan (x) - \frac{\pi}{2} = - \arctan \left( \frac{1}{x} \right), \ x > 0$$

and placing $Ny = u$ we get

$$A^+_{6,1} \ll N^{k+3} + N^{k+2} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty \exp \left( -\gamma_1 \arctan \left( \frac{1}{u} \right) \right) u^{-k-1-\beta_1} dy$$

$$+ 2N^{k+1/2} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty \exp \left( -\gamma_1 \arctan \left( \frac{1}{u} \right) \right) \frac{\log^2 (2u)}{u^{k+1/2+\beta_1}} dy$$

$$+ N^{k-1} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_{1/N}^\infty \exp \left( -\gamma_1 \arctan \left( \frac{1}{u} \right) \right) \frac{\log^4 (2u)}{u^{k+1+\beta_1}} dy \ll_k N^{k+3}$$

from Lemma 4, assuming $k > 3/2$. Now we have to study

$$A_{6,2} = \sum_{\rho_1} |\Gamma (\rho_1)| \sum_{\rho_2} \left| \frac{\beta_2}{2} \right| \int_{(1/N)} \left| \exp (\gamma_2 z) \right| \left| z^{-k-1} \right| \left| z^{-\rho_2/2} \right| \left| \sum_{\rho_3} z^{-\rho_3/2} \Gamma \left( \frac{\beta_3}{2} \right) \right| |dz|$$

and, by symmetry, we can consider the cases $\gamma_1, \gamma_2 > 0$ or $\gamma_1 > 0, \gamma_2 < 0$. Let $\gamma_1, \gamma_2 > 0$ and $y \leq 0$. From (27) we have

$$B_{6,2}^- \ll N \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2 - 1/2} \exp \left( -\frac{\pi}{4} \gamma_2 \right) \int_{-1/N}^0 \frac{|dz|}{|z|^{k+1+\beta_1+\beta_2/2}}$$

$$+ \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2 - 1/2} \exp \left( -\frac{\pi}{4} \gamma_2 \right) \int_{-\infty}^{-1/N} \frac{N + |y|^{1/2} \log^2 (2N |y|)}{|y|^{k+1+\beta_1+\beta_2/2}} dy |y|$$

$$\ll N^{k+3} + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2 - 1/2} \exp \left( -\frac{\pi}{4} \gamma_2 \right) \int_{-\infty}^{-1/N} \frac{1}{|y|^{k+1+\beta_1+\beta_2/2}} dy |y|.$$
\[ + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{1/2} \exp \left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{1/2} \exp \left( -\frac{\pi}{2} \gamma_2 \right) \int_{-\infty}^{-1/N} \frac{\log^2 (2N |y|)}{|y|^{k+1/2 + \beta_1 + \beta_2/2}} \quad |y| \ll k \quad N^{k+3} \]

for \( k > 1/2 \). Let \( y > 0 \), and so the symbol \( B_{6.2} \). We recall again that we have to split the integral for \( y \in (0, 1/N) \) and \( y \in (1/N, \infty) \) since, by (13) and (20), we have different estimation in these two set. We have that

\[ B_{6.2}^+ \ll N \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{1/2} \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{1/2} \int_{1/N}^{1} \exp \left( \left( \gamma_1 + \frac{\gamma_2}{2} \right) \left( \arctan (Ny) - \frac{\pi}{2} \right) \right) \left( N + y^{1/2} \log^2 (2Ny) \right) \quad dy \]

which is bounded by

\[ B_{6.2}^+ \ll N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{1/2} \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{1/2} \int_{1/N}^{1} \exp \left( \left( \gamma_1 + \frac{\gamma_2}{2} \right) \left( \arctan (Ny) - \frac{\pi}{2} \right) \right) \left( N + y^{1/2} \log^2 (2Ny) \right) \quad dy \]

and again from (28) and placing \( Ny = u \) we get

\[ B_{6.2}^+ \ll N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{1/2} \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{1/2} \int_{1}^{\infty} \exp \left( -\frac{\gamma_1 + \frac{\gamma_2}{2}}{Ny} \right) \arctan \left( \frac{1}{Ny} \right) \log^2 (2u) \quad du \]

and from the proof of Lemma 4 we have

\[ B_{6.2}^+ \ll_k N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{1/2} \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{1/2} \left( \gamma_1 + \frac{\gamma_2}{2} \right)^{-k+1/2 - \beta_1 - \beta_2/2} \]

and observing that

\[ \gamma_1^{1/2} \left( \frac{\gamma_2}{2} \right)^{\beta_2/2} \leq \left( \gamma_1 + \frac{\gamma_2}{2} \right)^{\beta_1} \left( \gamma_1 + \frac{\gamma_2}{2} \right)^{\beta_2/2} = \left( \gamma_1 + \frac{\gamma_2}{2} \right)^{\beta_1 + \beta_2/2} \]

we get

\[ B_{6.2}^+ \ll_k N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{1/2} \sum_{\rho_2: \gamma_2 > 0} \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2} \left( \gamma_1 + \gamma_2 \right)^{k-1/2}} \]

\[ \ll_k N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \frac{1}{\gamma_1^{1/2}} \sum_{\rho_2: 0 < \gamma_2 \leq \gamma_1} \frac{1}{\gamma_2^{1/2}} \]

\[ \ll_k N^{k+3} \sum_{\rho_1: \gamma_1 > 0} \frac{\log \left( \gamma_1 \right)}{\gamma_1^{k-1/2}} \]

and so the convergence if \( k > 3/2 \). Let us assume that \( \gamma_1 > 0 \), \( \gamma_2 < 0 \) and \( y \leq 0 \). From (27) we have
\[ + N \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp\left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \int_{-\infty}^{-1/N} \frac{\exp\left( -\frac{|\gamma_2|}{2} \frac{\arctan(Ny + \frac{\pi}{2})}{2} \right)}{|z|^{k+1+\beta_1+\beta_2/2}} |dz| \]

\[ + \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp\left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \int_{-\infty}^{-1/N} \frac{\exp\left( -\frac{|\gamma_2|}{2} \frac{\arctan(Ny + \frac{\pi}{2})}{2} \right)}{|z|^{k+1/2+\beta_1+\beta_2/2}} |dz| \]

hence

\[ C_{6,2} \ll N^{k+3/2} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp\left( \frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left( -\frac{\pi}{8} |\gamma_2| \right) \int_{1}^{\infty} \frac{\exp\left( -\frac{|\gamma_2|}{2} \frac{\arctan\left( \frac{\rho_2}{\gamma_2} \right)}{2} \right)}{u^{k+1+\beta_1+\beta_2/2}} du \]

\[ + N^{k+5/2} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp\left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} |\gamma_2|^{\beta_2/2-1/2} \int_{1}^{\infty} \frac{\exp\left( -\frac{|\gamma_2|}{2} \frac{\arctan\left( \frac{\rho_2}{\gamma_2} \right)}{2} \right)}{u^{k+1/2+\beta_1+\beta_2/2}} du \ll N^{k+5/2} \]

for \( k > 1 \). If \( y > 0 \) we have essentially the same calculations exchanging the role of \( \gamma_1 \) and \( \gamma_2 \). So we have to consider

\[ A_{6,3} = \sum_{\gamma_1 > 0} \sum_{\rho_2 \gamma_2 > 0} |\Gamma(\rho_1)| \sum_{\gamma_3 > 0} \sum_{\rho_3} |\Gamma(\rho_3)| \int_{(1/N)} |e^{Nz}| |z^{-k-1}| |z^{-\rho_1}| |z^{-\rho_2}| |z^{-\rho_3}| |dz|. \]

It is sufficient to consider the cases \( \gamma_i > 0 \), \( i = 1, 2, 3 \), \( \gamma_1, \gamma_2 > 0 \) and \( \gamma_3 < 0 \) and lastly \( \gamma_1 > 0 \), \( \gamma_2, \gamma_3 < 0 \). We will use the symbol \( D_{6,3} \) when we consider \( A_{6,3} \) with the assumption \( \gamma_i > 0 \), \( i = 1, 2, 3 \), the symbol \( E_{6,3} \) when we consider \( A_{6,3} \) with the assumption \( \gamma_1, \gamma_2 > 0 \) and \( \gamma_3 < 0 \) and \( F_{6,3} \) when we consider \( A_{6,3} \) with the assumption \( \gamma_1 > 0 \), \( \gamma_2, \gamma_3 < 0 \). From (27) we have

\[ D_{6,3}^- \ll \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp\left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left( -\frac{\pi}{4} \gamma_2 \right) \sum_{\rho_3: \gamma_3 > 0} |\gamma_3|^{\beta_3/2-1/2} \exp\left( -\frac{\pi}{4} \gamma_3 \right) \]

\[ \cdot \int_{-1/N}^{0} N^{k+1+\beta_1+\beta_2/2+\beta_3/2} dy \]

\[ + \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \exp\left( -\frac{\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \exp\left( -\frac{\pi}{4} \gamma_2 \right) \sum_{\rho_3: \gamma_3 > 0} |\gamma_3|^{\beta_3/2-1/2} \exp\left( -\frac{\pi}{4} \gamma_3 \right) \]

\[ \cdot \int_{-1/N}^{1} |y|^{-k-1-\beta_1-\beta_2/2-\beta_3/2} dy \ll N^{k+2} \]

for \( k > 1 \). Let \( y > 0 \). We have

\[ D_{6,3}^+ \ll \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3: \gamma_3 > 0} |\gamma_3|^{\beta_3/2-1/2} \]

\[ \cdot \int_{1/N}^{1} \exp\left( \left( \gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2} \right) \frac{\arctan(Ny - \frac{\pi}{2})}{2} \right) N^{k+1+\beta_1+\beta_2/2+\beta_3/2} dy \]

\[ + \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3: \gamma_3 > 0} |\gamma_3|^{\beta_3/2-1/2} \int_{1/N}^{\infty} \exp\left( \left( \gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2} \right) \frac{\arctan(Ny - \frac{\pi}{2})}{2} \right) dy \]

\[ \ll N^{k+2} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3: \gamma_3 > 0} |\gamma_3|^{\beta_3/2-1/2} \int_{1}^{\infty} \exp\left( -\left( \gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2} \right) \arctan\left( \frac{\rho_2}{\gamma_2} \right) \right) du \]

and from the proof of the Lemma 4 we get

\[ D_{6,3}^+ \ll N^{k+2} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3: \gamma_3 > 0} \gamma_3^{\beta_3/2-1/2} \left( \gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2} \right)^{-k-\beta_1-\beta_2/2-\beta_3/2} \]

and

\[ D_{6,3}^- \ll N^{k+2} \sum_{\gamma_1 > 0} \gamma_1^{\beta_1-1/2} \sum_{\rho_2: \gamma_2 > 0} |\gamma_2|^{\beta_2/2-1/2} \sum_{\rho_3: \gamma_3 > 0} |\gamma_3|^{\beta_3/2-1/2} \left( \gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2} \right)^{-k-\beta_1-\beta_2/2-\beta_3/2} \]
and observing that
\[
\frac{\gamma_1^{\beta_1} \gamma_2^{\beta_2/2} \gamma_3^{\beta_3/2}}{2} \leq \left( \gamma_1 + \frac{\gamma_2}{2} + \frac{\gamma_3}{2} \right)^{\beta_1 + \beta_2/2 + \beta_3/2}
\]
we get
\[
D_{6,3}^+ \ll_k N^{k+2} + N^{k+2} \sum_{\rho_1: \gamma_1 > 0} \sum_{\rho_2: \gamma_2 > 0} \sum_{\rho_3: \gamma_3 > 0} \frac{1}{\gamma_1^{1/2} \gamma_2^{1/2} \gamma_3^{1/2} \left( \gamma_1 + \frac{\rho_2}{2} + \frac{\rho_3}{2} \right)^k}
\]
and from AM-GM inequality we get
\[
D_{6,3}^+ \ll N^{k+2} + N^{k+2} \sum_{\rho_1: \gamma_1 > 0} \sum_{\rho_2: \gamma_2 > 0} \sum_{\rho_3: \gamma_3 > 0} \frac{1}{\gamma_1^{k/3+1/2} \gamma_2^{k/3+1/2} \gamma_3^{1/3}} \ll N^{k+2}
\]
for \( k > 3/2 \). Let \( \gamma_1, \gamma_2 > 0, \gamma_3 < 0 \) (and so the symbol \( E_{m,n} \)) and \( y \leq 0 \). From (27) we have
\[
E_{6,3}^- \ll \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( \frac{-\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2/2 - 1/2} \exp \left( \frac{-\pi}{4} \gamma_2 \right) \sum_{\rho_3: \gamma_3 < 0} \left| \gamma_3 \right|^{\beta_3/2 - 1/2} \frac{\exp \left( -\frac{|\gamma_3|}{4} \arctan \left( \frac{Ny}{\gamma_3} \right) + \frac{\pi}{4} \right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} \bigg| dz \bigg| + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( \frac{-\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 > 0} \gamma_2^{\beta_2/2 - 1/2} \exp \left( \frac{-\pi}{4} \gamma_2 \right) \sum_{\rho_3: \gamma_3 < 0} \left| \gamma_3 \right|^{\beta_3/2 - 1/2} \frac{\exp \left( -|\gamma_3| \arctan \left( \frac{1}{\gamma_3} \right) \right)}{|u|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} \bigg| du \bigg| \ll N^{k+2}
\]
from the proof of Lemma 4, for \( k > 1/2 \). If \( y > 0 \) we have essentially the same calculations exchanging the role of \( \gamma_2 \) and \( \gamma_3 \). Let \( \gamma_2, \gamma_3 < 0 \, , \, \gamma_1 > 0 \) and \( y < 0 \). Recalling (25) we have
\[
F_{6,3}^- \ll \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( \frac{-\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} \gamma_2^{\beta_2/2 - 1/2} \sum_{\rho_3: \gamma_3 < 0} \left| \gamma_3 \right|^{\beta_3/2 - 1/2} \frac{\exp \left( -\frac{|\gamma_2|+|\gamma_3|}{2} \arctan \left( \frac{Ny}{\gamma_2} \right) + \frac{\pi}{4} \right)}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} \bigg| dz \bigg| + \sum_{\rho_1: \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left( \frac{-\pi}{2} \gamma_1 \right) \sum_{\rho_2: \gamma_2 < 0} \gamma_2^{\beta_2/2 - 1/2} \sum_{\rho_3: \gamma_3 < 0} \left| \gamma_3 \right|^{\beta_3/2 - 1/2} \frac{\exp \left( -|\gamma_2| \arctan \left( \frac{1}{\gamma_2} \right) \right)}{|u|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} \bigg| du \bigg| \ll N^{k+2}
\]
\[
\ll N^{k+2} \sum_{\rho_2: \gamma_2 < 0} \sum_{\rho_3: \gamma_3 < 0} \left| \gamma_2 \right|^{\beta_2/2 - 1/2} \left| \gamma_3 \right|^{\beta_3/2 - 1/2} \left( \left| \gamma_2 \right| + \left| \gamma_3 \right| \right)^{-k-\beta_2/2-\beta_3/2} \ll N^{k+2}
\]
using Lemma 4, for $k > 1/2$. Let $y > 0$. Observing that $-\left(\frac{\gamma_2 + \gamma_1}{2}\right) (\arctan(Ny) + \frac{\pi}{2}) \leq \left(-\frac{\gamma_2 + \gamma_1}{2}\right) \frac{\pi}{2}$ we have

\[ F_{k,3}^+ \ll \sum_{\rho_1, \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \exp \left(-\frac{\pi}{4} \gamma_1\right) \sum_{\rho_2, \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp \left(-\frac{\pi}{8} |\gamma_2|\right) \sum_{\rho_3, \gamma_3 < 0} |\gamma_3|^{\beta_3/2 - 1/2} \exp \left(-\frac{\pi}{8} |\gamma_3|\right) \]

\[ + \sum_{\rho_1, \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \sum_{\rho_2, \gamma_2 < 0} |\gamma_2|^{\beta_2/2 - 1/2} \exp \left(-\frac{\pi}{8} |\gamma_2|\right) \sum_{\rho_3, \gamma_3 < 0} |\gamma_3|^{\beta_3/2 - 1/2} \exp \left(-\frac{\pi}{8} |\gamma_3|\right) \]

\[ \int_0^{1/N} \frac{dz}{|z|^{k+1+\beta_1+\beta_2/2+\beta_3/2}} \int_0^{1/N} \exp \left(\gamma_1 (\arctan(Ny) - \frac{\pi}{4})\right) dy \]

\[ \ll N^{k+2} \sum_{\rho_1, \gamma_1 > 0} \gamma_1^{\beta_1 - 1/2} \int_1^{1/N} \frac{1}{|z|^{k+1+\beta_1}} \frac{dz}{dz} \ll N^{k+2} \]

from Lemma 4 for $k > 1/2$. Now we can exchange the integral with the series and get

\[ I_6 = \frac{1}{8\pi i} \sum_{\rho_1} \Gamma (\rho_1) \sum_{\rho_2} \Gamma \left(\frac{\rho_2}{2}\right) \sum_{\rho_3} \Gamma \left(\frac{\rho_3}{2}\right) \int_{1/N} e^{Nz^2} e^{z^{-1} - \rho_1 - \rho_2/2 - \rho_3/2} dz \]

\[ = \frac{N^k}{4} \sum_{\rho_1} \sum_{\rho_2} \sum_{\rho_3} \frac{N^{\rho_1+\rho_2+2+\rho_3/2} \Gamma (\rho_1) \Gamma \left(\frac{\rho_2}{2}\right) \Gamma \left(\frac{\rho_3}{2}\right)}{\Gamma (k + \rho_1 + \rho_2/2 + \rho_3/2)} \]

then $I_6 = M_4 (N, k)$.

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