Entanglement dephasing dynamics driven by a bath of spins

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Abstract
We have studied the entanglement dynamics for a two-qubit system coupled to a spin environment of different configurations by a z–x-type interaction. Quantum dynamics of the models considered in this paper is solved analytically. Moreover, we show that simple and concise results may be obtained when certain approximations are properly made. Our purpose is to find out how the entanglement of a central spin system is affected by the pre-designed factors of the system and its environment, such as their initial states and the coupling constants between the system and its environment. Clearly, how the system is coupled to its environment will inevitably change the feature of entanglement evolution of the central spin system. Our major findings include the following: (i) the entanglement of the system of interest is sensitive to the number of spins in the environment, (ii) the initial states of the environment can profoundly affect the dynamics of the entanglement of the central spin system and (iii) the entangled environment can speed up the decay and revival of the entanglement of the central spin system. Our results exhibit some interesting features that have not been found for a bosonic environment.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Entanglement of a quantum system not only has important consequences as a fundamental physical property, but also been identified as a resource in the quantum information process (QIP), quantum computing and quantum cryptography [1]. Entanglement without protection or external intervention is generally fragile when it is not isolated from the influence of its
environments\(^1\). The dynamical aspect of the entangled state in the context of the quantum open system has been investigated in many different scenarios such as disentanglement of qubit systems [3–7], continuous variable systems [8, 9], entanglement delayed creation and revival [10–13] and non-Markovian entanglement evolution [14–16], to name a few.

As is well known, environmental noises that cause quantum systems to decohere also lead to the loss of entanglement [17–19]. In its simplest form the system plus environment can be examined in different configurations of system–environment models including (i) harmonic oscillators coupled to a bath of bosons, i.e. the quantum Brownian motion model [20, 21]; (ii) two-level atoms or spin-1/2 particles (qubits) interacting with a bath of harmonic oscillators, i.e. the spin-boson model [22, 23] and (iii) qubits (or harmonic oscillators) coupled to a bath of spins-1/2, i.e. the spin bath model [24–35]. It has been shown that dynamics of quantum open systems can be significantly modified by environmental configurations.

A well-studied example on decoherence is based on the framework of system plus environment where the environment is typically described by a set of harmonic oscillators. Such bosonic models provide many insights into the decoherence caused by dissipation and phase damping. However, there are some cases where qubits are coupled to the environmental noises caused by electrons, nuclear spins or impurities that are effectively described by a set of spins. The purpose of this paper is to discuss entanglement dynamics of two central spins coupled to an environment consisting of a bath of spins. The model presented in this paper is simple to allow analytical treatment, yet sophisticated enough to reveal some non-trivial features of disentanglement processes due to the spin–spin couplings. Our strategy is to directly solve the models without invoking the approximations that may lead to the conversion of bath spins into the effective Holstein–Primakoff bosons or Schwinger bosons (e.g. see [25, 27]). Our focus will be on the important features induced from the spin environment that are not seen in the boson case [36]. We not only consider different couplings between the system and its environments, but also take into account the effect of the various initial states of the environment on the entanglement dynamics of the central qubit system. Here, we take our central qubit system as two uncoupled spin-1/2 particles (qubits), which are interacting with a set of spin-1/2 particles through the z–x-type coupling. We present several analytical results on the dynamics of the two central qubits in physically relevant situations including individual coupling and collective coupling models.

This paper is organized as follows: in section 2, we present the Hamiltonians of two-qubit dephasing models with local and global spin environments, the analytical solutions to both models are provided; the dynamics of the central qubits with various initial states is analytically investigated in section 3 for the local coupling and in section 4 for the global coupling, respectively. Specifically, for each model we have considered three different types of initial states of the environment: pure, mixed and entangled. In addition, we have extended our analytical treatments to entanglement dynamics in more generic situations in section 5, which show that the concurrence of the central qubits is sensitively dependent on the total number of environment spins and bath initial states. We conclude in section 6.

2. Models and solutions

We consider two important types of system–environment couplings: (a) the two system qubits are interacting only with their own local environments and there is no crosstalk between the two environments; (b) the two system qubits are coupled to a common environment. In each

\(^1\) An example showing robust entanglement due to the qubit–qubit interaction can be found in [2].
case, the interaction between two central spins or two bath spins is neglected. Precisely, the total Hamiltonian for these two cases is given by

$$H_{\text{tot}} = H_{\text{sys}} + H_{\text{env}} + H_{\text{int}},$$

$$H_{\text{sys}} = \omega_A \sigma_A^z + \omega_B \sigma_B^z,$$

where, for the global environment case, we have

$$H_{\text{env}} = \sum_k \omega_k \sigma_k^z,$$

$$H_{\text{int}} = (\sigma_A^z + \sigma_B^z) \sum_k c_k \sigma_k^x.$$

In the case of local environment, we have

$$H_{\text{env}} = \sum_k \omega_k \sigma_k^z + \sum_l \omega_l \sigma_l^z,$$

$$H_{\text{int}} = \sigma_A^z \sum_k c_k \sigma_k^x + \sigma_B^z \sum_l c_l \sigma_l^x.$$

\(\omega_A\) and \(\omega_B\) are the transition frequencies for the two qubits, respectively. For simplicity, but without loss of generality, we assume that \(\omega_A = \omega_B = \omega_s\). Here, \(\sigma_z\) and \(\sigma_x\) are the Pauli matrices, \(\omega_k\) is the frequency for the \(k\)th spin in the environment and \(c_k\) are dimensionless real-value coupling constants between the \(k\)th spin and the central qubits. As is shown in figure 1(a), \(k\) and \(l\) represent the two local baths for the central qubits respectively. In both cases, the dephasing coupling ensures that quantum coherence is damped, but the energy of central qubits is perfectly preserved.

For both cases, the dynamics for the central qubit systems can be solved analytically under the assumption that initially the system and environment are in a separable state, i.e. the density matrix for the system plus environment at \(t = 0\) is of the following product form:

$$\rho(0) = \rho_{\text{sys}}(0) \otimes \rho_{\text{env}}(0).$$

The environment spins may take a simple form \(\rho_{\text{env}}(0) = \prod_k \rho_k^{(0)}\). Otherwise, the environment may contain some entangled spin blocks, and its density matrix can be written as \(\rho_{\text{env}}(0) = \prod_j \rho_j^{(0)}\), where \(\rho_j^{(0)}\) stands for the density matrix of the \(j\)th block. (More precise descriptions are given in subsection 3.4.)

For our dephasing models, the time-dependent density matrix \(\rho\) for the total system may be written as

$$\rho(t) = e^{-iH_{\text{sys}}t} \rho(0) e^{iH_{\text{sys}}t} = e^{-iH_{\text{sys}}t} \left[ \prod_k e^{-i\hat{X}_kt} \right] \rho(0) \left[ \prod_k e^{i\hat{X}_kt} \right] e^{iH_{\text{sys}}t},$$

where \(H_{\text{sys}}\) is the system Hamiltonian and \(\hat{X}_k\) is the sum of the Hamiltonian of the \(k\)th environment spin and the corresponding term in the interaction Hamiltonian. For example, in case (b), \(\hat{X}_k = \omega_k \sigma_k^z + (\sigma_A^z + \sigma_B^z) c_k \sigma_k^x\). The elements of the reduced density matrix (RDM)
for the central qubit system can be obtained by tracing over the environment spin degree of freedom:

\[ \rho_{\text{sys}}^{mn}(t) = \langle m | \text{Tr}_k \rho(t) | n \rangle = \text{e}^{-i(E'_m - E'_n)t} \rho_{\text{sys}}^{mn}(0) \text{Tr}_k F_{mn}^k = \text{e}^{-i(E'_m - E'_n)t} \rho_{\text{sys}}^{mn}(0) f_{mn}(t), \]

(5)

\[ f_{mn}(t) = \prod_k \text{Tr}_k F_{mn}^k = \prod_k \text{Tr}_k \left[ \text{e}^{-i \hat{X}_k(En) t} \rho_{\text{env}}(0) \text{e}^{i \hat{X}_k(En) t} \right]. \]

(6)

\(|m\rangle, |n\rangle\) and \(E'_m, E'_n\) are the eigenvectors and eigenvalues of the system Hamiltonian \(H_{\text{sys}}\) respectively. Note that the notations \(E_m, E_n\) denote the eigenvalues of the coupling operator \(\sigma_A^z + \sigma_B^z\), thus \(E'_m, E'_n = \omega_s E_m, E_n\). The dephasing coefficient \(f_{mn}(t)\) is a time-dependent function satisfying \(|f_{mn}(t)| \leq 1\) and \(f_{nn} = 1\). The specific form of \(f_{mn}(t)\) depends on the interaction Hamiltonian and the initial state of the environment, which will be discussed in later sections.

3. Solution of local coupling model with different initial states

3.1. General solutions

When each central qubit is interacting with its own environment, as shown in figure 1(a), the RDM of the two central qubits may be obtained by using the Kraus operator technique [37]. In principle, we can always write the time-dependent RDM in such a way:

\[ \rho_{\text{sys}}(t) = \sum_i K_i \rho_{\text{sys}}(0) K_i^\dagger, \]

(7)

with \(K_i\) satisfying the condition \(\sum_i K_i^\dagger K_i = I\). It is convenient to solve the model in this way since we may just focus on the subsystem involving only one qubit plus its local environment. The RDM of such a subsystem is thus a 2 \(\times\) 2 matrix, which can be derived directly from equation (5):

\[ \rho_{\text{sys}}(t) = \rho^A(t) = \begin{pmatrix} \rho_{11}^A(0) & \rho_{12}^A(0) e^{i2\omega_s t} f(t) \\ \rho_{21}^A(0) e^{i2\omega_s t} f^*(t) & \rho_{22}^A(0) \end{pmatrix}. \]

(8)

For the dephasing model, the diagonal elements are time independent, i.e. \(\rho_{\text{sys}}^{nn}(t) = \rho_{\text{sys}}^{nn}(0)\), so we only need to consider the off-diagonal ones, for which \(E_mE_n = -1\). Since \(\rho_{12} = \rho_{21}^*\), for a 2 \(\times\) 2 density matrix, we define \(f(t) = f_{12}(t)\). The Kraus operators of the one-qubit subsystem are given by

\[ K_1^A = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\omega_s t} f^*(t) \end{pmatrix}, \quad K_2^A = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1 - |f(t)|^2} \end{pmatrix}, \]

(9)

where \(f(t)\) is the dephasing coefficient defined in equation (5). Then the RDM for the two-qubit system can be directly obtained by [6, 38]

\[ \rho_{\text{sys}}(t) = \rho^{AB}(t) = \sum_{i,j=1,2} K_{ij}^{AB} \rho^{AB}(0) K_{ij}^{AB\dagger}, \]

(10)

where the composite Kraus operators, \(K_{ij}^{AB} = K_i^A \otimes K_j^B\), are given by

\[ K_{11}^{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{i2\omega_s t} f^*(t) & 0 \\ 0 & e^{i2\omega_s t} f^*(t) & 0 & 0 \\ 0 & 0 & 0 & e^{i4\omega_s t} f^*(t) \end{pmatrix}, \]

where
\[ K_{12}^{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \sqrt{1 - |f(t)|^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha^{i2\omega t f^* (t) \sqrt{1 - |f(t)|^2}} \end{pmatrix}, \]

\[ K_{21}^{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{1 - |f(t)|^2} & 0 & 0 \\ 0 & 0 & 0 & \alpha^{i2\omega t f^* (t) \sqrt{1 - |f(t)|^2}} \end{pmatrix}, \]

\[ K_{22}^{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 - |f(t)|^2 \end{pmatrix}. \]

Then for an arbitrary initial state, the time-dependent RDM is simply given by

\[ \rho_{\text{sys}}(t) = \rho_{AB}(t) = \begin{pmatrix} \rho_{11} & \alpha^{i2\omega t f^* (t) \rho_{12}} & \alpha^{i2\omega t f^* (t) \rho_{13}} & \alpha^{i2\omega t f^* (t) \rho_{14}} \\ \alpha^{i2\omega t f^* (t) \rho_{21}} & \rho_{21} & \rho_{23} & \alpha^{i2\omega t f^* (t) \rho_{24}} \\ \alpha^{i2\omega t f^* (t) \rho_{31}} & \rho_{31} & \rho_{33} & \alpha^{i2\omega t f^* (t) \rho_{34}} \\ \alpha^{i2\omega t f^* (t) \rho_{41}} & \rho_{41} & \rho_{43} & \rho_{44} \end{pmatrix}. \]

Note that the solution (11) is obtained without any approximations. We will discuss three special cases where the function \( f(t) \) may be explicitly evaluated.

### 3.2. Mixed state environment

As our first example, we assume that the environment spins are initially in the following mixed state:

\[ \rho_{k_{\text{env}}}(0) = \begin{pmatrix} N_{k}^+ & 0 \\ 0 & N_{k}^- \end{pmatrix}, \]

(12)

where \( N_{k}^+ \), \( N_{k}^- \) are the probability of whether the \( k \)th spin is in up or down state, respectively (\( N_{k}^+ + N_{k}^- = 1 \)). Here, we may choose \( N_{k}^\pm = e^{\pm \beta \omega_k} / (e^{\beta \omega_k} + e^{-\beta \omega_k}) \), where \( \beta = 1/T \) and \( T \) is the temperature (setting the Boltzmann constant \( k = 1 \)). By using the following identity

\[ e^{i (u \sigma_z + v \sigma_x)} = \cos \sqrt{u^2 + v^2} + \frac{1}{\sqrt{u^2 + v^2}} (u \sigma_z + v \sigma_x) \]

and equation (6), it is easy to show that

\[ \text{Tr}_k \left[ F_{mn}^k \right] = \cos^2 \left( p_k t \right) + \frac{\sin^2 \left( p_k t \right)}{p_k^2} \left( \omega_k^2 + E_m E_n c_k^2 \right) \left( N_k^+ + N_k^- \right) \]

\[ = 1 - c_k^2 \left( 1 - E_m E_n \right) \frac{\sin^2 \left( p_k t \right)}{p_k^2}, \]

(13)

where \( p_k = \sqrt{\omega_k^2 + c_k^2} \).

From (13), for qubit \( A \) we obtain the matrix element:

\[ \rho_{12}^A(t) = \rho_{12}^A(0) e^{-i(E_m - E_n) t f(t)}, \]

(14)

\[ f(t) = \prod_k \left[ 1 - 2c_k^2 \frac{\sin^2 \left( p_k t \right)}{p_k^2} \right]. \]

(15)
Equation (15) is the exact solution of the model with arbitrary number of spins in the environment. From equation (13) we found that the RDM is insensitive to the distribution of $N^k_+$, $N^k_-$. However, it is worth noting that this is not a generic feature for the spin environment. Actually, as shown in the common environment case (subsection 4.2), the dephasing coefficient $f(t)$ can explicitly depend on the distributions $N^k_+, N^k_-$. Interestingly, if more information about the bath spectral density becomes available, we may obtain a rather compact form for the function $f(t)$. For example, for a bath with an infinite number of spins and the Ohmic spectral density,

$$J(\omega) = \eta \frac{\omega}{c^2(\omega)} e^{-\frac{\omega}{\omega_c}},$$

where $\omega_c$ is the cut-off frequency and $\eta$ is the phase damping coefficient [24, 39], we can apply the weak coupling approximation $c_k \ll 1$, and use equation (15) to obtain

$$f(t) = \left(1 + 4\omega^2 t^2\right)^{-\frac{1}{2}} \eta.$$  

(17)

The power law decay of quantum coherence clearly deviates from the exponential decay commonly seen in the case of Markov bosonic bath [18].

### 3.3. Pure state environment

Another interesting case is that the bath spins are initially prepared in a pure state: $|\Psi(0)\rangle = \cos \alpha_k |\uparrow\rangle + e^{i\beta_k} \sin \alpha_k |\downarrow\rangle$. Its density matrix form may be written as

$$\rho_{\text{env}}^k(0) = |\Psi(0)\rangle \langle \Psi(0)| = \begin{pmatrix} \cos^2 \alpha_k e^{-i\beta_k} \cos \alpha_k \sin \alpha_k & e^{-i\beta_k} \cos \alpha_k \sin \alpha_k \\ e^{-i\beta_k} \cos \alpha_k \sin \alpha_k & \sin^2 \alpha_k \end{pmatrix}. $$

(18)

For this type of initial states of the environment, the RDM for the central qubits at time $t$ can be determined by equation (5). Thus, we have

$$\text{Tr}_k[F_{mn}^k] = 1 - (1 - E_m E_n) \frac{c^2_k}{P^k} \sin^2(p_k t) \sin^2(\beta_k) + i(E_n - E_m) \sin 2\alpha_k \left[ \frac{c_k}{2p_k} \sin(2p_k t) \cos \beta_k - \frac{\omega_k c^2_k}{p^2_k} \sin^3 p_k t \sin \beta_k \right].$$

(19)

Equation (19) is different from equation (13) by an imaginary part, whose value is directly dependent on the choice of the angle $\alpha_k$. For simplicity we choose a special case such that $\beta_k = 0$; then, the time-dependent dephasing coefficient for the pure state environment is given by

$$f(t) = \prod_k \left[ 1 - 2 \frac{c^2_k}{p^2_k} \sin^2(p_k t) + \frac{c_k}{p_k} \sin(2p_k t) \sin 2\alpha_k \right].$$

(20)

We note that the above expression does not permit $f(t)$ to have the same compact form as equation (17) for the continuous spectral density considered before.

### 3.4. Environment with entangled blocks

In this subsection, we consider a situation where the environment spins may consist of a set of entangled blocks. This is an interesting configuration that may impose some stronger constraints on the entanglement dynamics of central qubits. More specifically, we begin with an environment consisting of $N \times M$ spins in total. We may divide the total $N \times M$ spins into $M$ blocks, containing exactly $N$ spins in each block prepared in an $N$-GHZ state [40].
\[ |\Psi_1\rangle_N = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \ldots \uparrow\rangle^N + |\downarrow \downarrow \ldots \downarrow\rangle^N). \] In this case, the matrix elements of the RDM are given by equation (5) with a modified dephasing function \( f_{mn}(t) \):

\[
f_{mn}(t) = \prod_{j=1}^{M} \text{Tr}_k^N \left[ \prod_{k=1}^{N} e^{-i\hat{x}_k t} \rho_{\text{env}}^{N,j}(0) \prod_{k=1}^{N} e^{i\hat{x}_k t} \right] = \prod_{j=1}^{M} \text{Tr}_k^N [F_{mn}^{N,j}(t)]
\]

where \( \rho_{\text{env}}^{N,j}(0) \) is the density matrix for \( N \) entangled bath spins confined in the \( j \)th block. And we have

\[
\text{Tr}_k^N [F_{mn}^{N,j}(t)] = \prod_{k=1}^{N} \left( 1 - (1 - E_m E_n) \frac{c_j^2}{p_j^2} \sin^2 p_j t \right) + \left( \frac{E_n - E_m}{2} \right)^N \prod_{k=1}^{N} \left( 1 - \cos \frac{2p_j t}{p_k} \cos \frac{\omega_k}{p_k} + i \frac{\sin 2p_j t}{p_k} \sin \frac{\omega_k}{p_k} \right).
\]

A more compact expression for the coherence function \( f(t) \) may be obtained when each block contains only two spins, which are prepared as \( |\Psi_1\rangle_j = \frac{1}{\sqrt{2}} (|\downarrow \downarrow\rangle_j + |\uparrow \uparrow\rangle_j) \). (See appendix A for more details). The expression can be simplified further if we choose \( c_{j1} = c_{j2} = c_j, p_{j1} = p_{j2} = p_j \). Then the off-diagonal elements of the RDM are given by

\[
\rho_{\text{sys}m\neq n}(t) = e^{-iE_m - E_n} \rho_{\text{sys}m\neq n}(0) f(t),
\]

\[
f(t) = \prod_{j=1}^{M} \left( 1 - \frac{c_j^2}{p_j^2} \sin^2 2p_j t \right).
\]

The decoherence here decays twice faster than that of the mixed state environment case (equation (15)). Again, with the Ohmic spectral density assumption and \( c_k \ll 1 \), the dephasing coefficient \( f(t) \) in the long time limit is simply given by

\[
f(t) = (1 + 16 \omega_c^2 t^2)^{-\frac{1}{2}} \approx (4 \omega_c t)^{-\eta}.
\]

Immediately, we can see from the above calculations that the block-entangled environment may give rise to more severe decoherence effects on the central qubits than the mixed initial state (e.g. see equation (17)). Finally, It is of interest to consider another simple case: \( c_{j1} = -c_{j2} = c_j, p_{j1} = p_{j2} = p_j \), with which we obtain

\[
f(t) = \prod_{j=1}^{M} \left( 1 - \frac{8 \omega_c^2 c_j^2}{p_j^4} \sin^4 p_j t \right).
\]

By the same assumption and approximation used above, the function \( f(t) \) in the long time limit can again be reduced to a simple function:

\[
f(t) = (1 + 16 \omega_c^2 t^2)^{-\frac{1}{2}} (1 + 4 \omega_c^2 t^2)^{-2\eta} \approx (4 \omega_c^3 t^3)^{-\eta}.
\]

4. Solution of the global coupling model with different initial states

4.1. General solution

In this subsection, we consider the case that two central qubits are coupled to the same environment, illustrated in figure 1(b). Again, this model can be solved exactly by using the
same method in the previous section. The matrix elements in the RDM can be obtained by equations (5), (6), but different conventions have to be made since now the system operator in the interaction Hamiltonian is given by $\sigma^z_A + \sigma^z_B$, and thus the eigenvalues of that are $E_1 = 2, E_2 = 0, E_3 = 0$ and $E_4 = -2$. It can be seen below that new features can arise from the backreaction on the central qubits induced by the common bath. Thus, the resultant RDM is different and more complex, which is given by

$$\rho_{AB}(t) = \begin{pmatrix}
\rho_{11} e^{-i2\omega t} g(t) & \rho_{12} e^{-i2\omega t} g(t) & \rho_{13} e^{-i2\omega t} f(t) & \rho_{14} e^{-i4\omega t} f(t) \\
\rho_{21} e^{i2\omega t} g(t)^* & \rho_{22} & \rho_{23} e^{i2\omega t} g'(t)^* & \rho_{24} e^{i4\omega t} g'(t)^* \\
\rho_{31} e^{i2\omega t} g(t)^* & \rho_{32} & \rho_{33} & \rho_{34} e^{i4\omega t} g'(t)^* \\
\rho_{41} e^{i4\omega t} f(t)^* & \rho_{42} e^{i2\omega t} g(t)^* & \rho_{43} e^{i2\omega t} g'(t)^* & \rho_{44}
\end{pmatrix},$$

(27)

where the explicit expression for $f(t), g(t), g'(t)$ can be found in appendix B. When considering the initial states of system qubits $A, B$ in Bell states $\Psi_{1AB} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle)$, the entanglement can be perfectly preserved.

4.2. Mixed state environment

For the environment spins that are initially in a mixed state given by equation (12), we have

$$f(t) = \prod_k \left[ 1 - 8 \frac{c_k^2}{q_k^2} \sin^2(q_k t) \right],$$

(28)

$$g(t) = \prod_k \left[ \cos(q_k t) \cos(\omega_k t) + \frac{\omega_k}{q_k} \sin(q_k t) \sin(\omega_k t) \right] + i \Delta_k \left[ \cos(q_k t) \sin(\omega_k t) - \frac{\omega_k}{q_k} \sin(q_k t) \cos(\omega_k t) \right],$$

(29)

$$g'(t) = g(t)^*,$$

(30)

where $q_k = \sqrt{\omega_k^2 + 4c_k^2}, \Delta_k = N_k^x - N_k^z$.

4.3. Pure state environment

If the bath is initially in a pure state (see equation (18), with the assumption that $\alpha$ is real), we may obtain an explicit form of the density matrix (equation (27)) with

$$f(t) = \prod_k \left[ 1 - 8 \frac{c_k^2}{q_k^2} \sin^2(q_k t) - i 2 \frac{c_k}{q_k} \sin(2\alpha_k t) \sin(2q_k t) \right],$$

(31)

$$g(t) = \prod_k \left[ \cos(q_k t) \cos(\omega_k t) + \frac{\omega_k}{q_k} \sin(q_k t) \sin(\omega_k t) \right] + i \cos(2\alpha_k) \left( \cos(q_k t) \sin(\omega_k t) - \frac{\omega_k}{q_k} \sin(q_k t) \cos(\omega_k t) \right) - i \frac{2c_k}{q_k} \sin(2\alpha_k t) \sin(\omega_k t) \cos(\omega_k t),$$

(32)

$$g'(t) = g(t).$$

(33)

This solution will be used to discuss entanglement dynamics later.
4.4. Environment with entangled blocs

For simplicity, we consider a case in which the environment has \( M \) blocks, each block consisting of two entangled spins (i.e. \( N = 2 \)). For this structured environment, the time-dependent coefficients in equation (27) take the following form:

\[
f(t) = \prod_j \left[ (1 - 8c_{j,1}^2S_{j,1}^2)(1 - 8c_{j,2}^2S_{j,2}^2) + 16c_{j,1}c_{j,2}(\omega_{j,1}\omega_{j,2}S_{j,1}^2S_{j,2}^2 - C_{j,1}C_{j,2}S_{j,1}S_{j,2}) \right].
\]

(34)

\[
g(t) = g'(t) = \prod_j \left[ (C_{j,10}C_{j,1} + S_{j,10}S_{j,1}^2)(C_{j,20}C_{j,2} + S_{j,20}S_{j,2}^2) \right.
\]

\[\left. - \omega_{j,1}\omega_{j,2}(C_{j,10}S_{j,1}^2 - S_{j,10}C_{j,1})(C_{j,20}S_{j,2}^2 - S_{j,20}C_{j,2}) \right] - 4c_{j,1}c_{j,2}S_{j,1}S_{j,2}\cos(\omega_{j,1} + \omega_{j,2}) \right].
\]

(35)

with the notation

\[
C_{j,i} = \cos(q_{j,i}t), \quad S_{j,i} = \frac{\sin(q_{j,i}t)}{q_{j,i}}, \quad C_{j,0} = \cos(\omega_{j,i}t), \quad S_{j,0} = \sin(\omega_{j,i}t), \quad q_{j,i} = \sqrt{\omega_{j,i}^2 + 4c_{j,i}^2}, \quad i = 1, 2.
\]

where \( \omega_{j,i} \) and \( c_{j,i} \) are the frequency and coupling strength for the \( i \)th spin in the \( j \)th block, respectively.

5. Entanglement dynamics for spin environments

In this paper, we use Wootters’ concurrence to measure the entanglement of the central qubits \([41]\), which is defined as

\[
C(\rho) = \max\{0, \sqrt{\lambda_1 - \sqrt{\lambda_2}} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},
\]

(36)

where \( \lambda_i (i = 1, 2, 3, 4) \) are the eigenvalues in decreasing order of the matrix

\[
\zeta = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y).
\]

For our model, all the off-diagonal elements in the RDM are determined by their initial value multiplied by a dephasing coefficient; thus, if any element is zero at \( t = 0 \), then it will remain zero for any \( t > 0 \). Consequently, when a Bell state is chosen as the initial state for the central qubit system, its RDM will preserve an X-form \([38]\). In the case of local environment for example (see equation (11)), the concurrence may be simplified as

\[
C(\rho) = 2\max\{0, |\rho_{14}(t)| - \sqrt{\rho_{22}(t)\rho_{33}(t)}, |\rho_{23}(t)| - \sqrt{\rho_{11}(t)\rho_{44}(t)}\} = |f^2(t)|.
\]

(37)

Since the local environments do not generate entanglement between central qubits, our focus here will be on entanglement decay. For the global environment, however, it is possible to achieve entanglement generation when initially the central qubits are in a separable state. Based on our analytical solutions to both models, we will be able to systematically study concurrence dynamics \( C \) of the central qubits in the following two subsections. In the numerical results presented below, we will make the following conventions: the frequency of system qubits is chosen as \( \omega_s = 1 \); in sections 5.2 and 5.3, for the environment spins, their frequencies and coupling constants are random uniform distributions in the intervals \( \omega_k \in (1, 2) \) and \( c_k \in (0.1, 0.2) \), respectively.
5.1. Entanglement evolution with different numbers of spins

As we mentioned before, when the initial state of the system qubits is the Bell state $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, the concurrence of the system can take a very simple form (37). Here the function $f(t)$ is the product of a set of periodic functions, each corresponding to one spin in the environment (equations (6), (15), (20), (25), etc). Hence, the entanglement of the central qubits is highly sensitive to the parameters of the environment spins. In order to study how the number of environment spins affects the system’s entanglement, we assume the same parameters for all the environment spins, i.e. $c_k = c$, $\omega_k = \omega$, $p_k = p$. Now we take the local environment case as an example; the concurrence of the system qubits is given by $C(\rho) = \frac{1}{4N} \left| \text{Tr} F_{mn}^{A_B} \right|^2$. Explicitly, we have $C_{\text{mixed}} = \left| 1 - 2c^2 \sin^2(pt) \right|^{2N}$ for system qubits interacting with a bath of spins initially prepared in the same mixed state as in equation (12), $C_{\text{pure}} = \left| 1 - 2c^2 \sin^2(pt) + ipc \sin(2pt) \right|^{2N}$ for the system with a bath of spins prepared in the pure state (equation (18)), and $C_{\text{entangled}} = \left| 1 - 2c^2 \sin^2(2pt) \right|^{2M}$ for the system with a bath of spins prepared in the block entangled state (equation (23)). Under this assumption, the concurrence is clearly a periodic function of time. With increasing number of spins the time intervals for decay and revival of entanglement become shorter. Notably, the global environment will give rise to a similar periodic function $C(t) = |f(t)|$, where $f(t)$ will have a different form for different environments accordingly (equations (30),(33),(34)). Note that for the two Bell states corresponding to the decoherence free space, $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle)$, the concurrence will be constant. To compare the rates of decay/revival processes for different numbers of bath spins, we look at a short time window (before the first revival of concurrence), and measure the time it takes for concurrence to decay to half of its initial value denoted by $T_\frac{1}{2}$. It should be noted that, for some special parameters or small number of spins, the concurrence may never reach that value in a finite time. However, this half-life measurement for the entanglement decay is generally meaningful for a large environment. For all the figures displayed in this section, the blue solid line, the red dotted line and the yellow dashed line represent the situations when the environment spins are initially in the mixed state ($\Delta_k$ is not important here for local environment), the pure state and the block-entangled environment with $N = 2$ respectively. We can see from figure 2 that as we increase the number of spins in the environment, $T_\frac{1}{2}$ decreases rapidly for all initial states. The top right corner shows the concurrence plot for $N = 6$. Within the same time period, the concurrence of the system interacting with a pure state environment is larger than that of the system with a mixed state environment. And for the block entangled environment with $N = 2$, the decay and revival rates can be two times faster than that in two other cases. These observations are consistent in the plot of the half-value decay time.

5.2. Entanglement decay in local environment model

With the system initialized in the same state as in the previous section, figures 3–5 show the concurrence dynamics with 6, 60 and 400 environment spins, respectively.

If the number of the environment spins is not too large, as illustrated in figures 3 and 4, the entanglement of central qubits is clearly a quasi-periodic function of $t$, representing a strong non-Markovian feature of the environment, since the information in the system that transferred to the environment will quickly come back to the system. It should be noted that the block-entangled environment could play a positive role in preserving the system’s concurrence compared with the cases of mixed or pure initial states. In general, the mixed environment state may cause more detrimental effect to the entanglement of central qubits.
Figure 2. The half-value decay time versus the number of spins in the environment, where we have chosen $c = 0.2, \omega = 1$.

Figure 3. Two qubits interacting locally with two 6-spin environments. The blue solid line is for the environment initially prepared in the mixed state, the red dotted line for the pure state and the yellow dashed line for the block-entangled environment.

Figure 4. Two qubits interacting locally with two 60-spin environments.
Typically, a larger environment exhibits more dramatic differences among the three initial states. For the 400-spin bath shown in figure 5, we see that the initial entanglement decays to zero more rapidly than the cases of small environments. We can simply look at the dephasing coefficient $f(t)$ in equations (6), (15), (20), (25) etc; each spin in the environment will contribute a periodic function whose absolute value is not larger than 1. The period of that function is determined by the parameters of the environment spin (i.e. $c_k$, $\omega_k$); thus, the overall effect from the environment will quickly attenuate the information initialized in the system as $t$ gets larger. In the short time limit and when the environment is sufficiently large, we can easily obtain the same decay rate $e^{-\gamma t^2}$ from equation (15) as in other literature [42]. But in a longer time scale, the decay rate strongly depends on the number of spins in the environment. Thus for this case, the information within the system (initial entanglement or coherence) will be damped by the ‘noise’ immediately. This is consistent with our general observations for the bosonic case [25]. In all the three cases, the increased degrees of freedom of the environment will cause effectively an irreversible information flow between the system and its environment. It is noted that concurrence revival is found in the case of both the block-entangled environment and the pure state environment, but not in the mixed state case. For our example, as shown in figures 3 and 4, entanglement decay and revival occur approximately at the same pace for mixed and pure initial states. But for the block-entangled bath, the structured environment may speed up both the decay and the revival processes.

### 5.3. Entanglement generation in the global environment model

When the system qubits are coupled to a global environment, it is possible to achieve entanglement generation between the two central qubits. It is clear that such generation of entanglement is partly due to the backreaction on the central qubits through system–environment couplings. For example, let us consider a separable initial state of the central qubits $A, B$ represented by $|\Psi\rangle_{AB} = \frac{1}{2}(|\uparrow\rangle_A + |\downarrow\rangle_A) \otimes (|\uparrow\rangle_B + |\downarrow\rangle_B)$. In what follows we show that the degree of entanglement generation is highly sensitive to the number of spins in the environment. And the duration of that depends on the population distribution of each spin state. The purity and entanglement in the environment spins can also effect the generation.

In order to show the different dynamics of entanglement between environment spins initialed in mixed and pure states, we compare states with the same population distribution, i.e. for equations (12), (18), we consider $N^k_+ = \cos^2 \alpha_k, N^k_- = \sin^2 \alpha_k$. We begin with the case...
of a small environment with only six spins, as shown in figure 6. Figure 6(a) shows the plot when $\alpha_k = 0$, which implies that all environment spins are in ‘up’ state. We can see from the analytic solution (equations (30), (33)) that the two results will be the same. We see again a periodic evolution as expected for a small environment. Another interesting phenomenon occurs when we increase $\alpha_k$, as shown in figure 6(b), (c), (d), which are corresponding to $\alpha = \pi/16, 2\pi/16, 3\pi/16$. The degree of entanglement generated becomes smaller. In general we can see this periodic decay and generation process, but the degree of entanglement generated within one period becomes smaller and smaller. We have found that as long as the population distribution is the same, the mixed state and the pure state will produce almost the same dynamics of entanglement. Surprisingly, when the diagonal terms of spin states are equal to each other, i.e. $\alpha = \pi/4$, there is no entanglement generation for both initial states. For the case of block-entangled environment (it is assumed that each pair is in a GHZ state), the result falls into the same conclusion.

If the environment contains a larger number of spins, in general, less entanglement is generated. Figure 7 shows the entanglement dynamics for two central qubits coupled to a global bath with 60 spins. The entanglement generated may only survive some time before decaying to zero, and stay as zero for a long period of time (at least no revival is found in our simulations). However, the entanglement generated by the mixed state environment shows a regular creation and decay pattern that is not seen for the pure initial state.

Similar oscillating patterns are also observed when the number of spins is increased to 200. In figure 8, two bumps are found at early stage, and no visible entanglement is detected later. This phenomenon occurs when the environment is larger than the size of the system, but not large enough to trap all the information leaked into the environment from the system.

We have also evaluated entanglement evolution for a very large environment consisting of 800 spins. In many ways, the model under this condition may be well represented by a Markov system if the spin–spin couplings are weak. Intuitively, for such a large environment,
the entanglement in a system, even generated due to the common environment, once lost into the environment will not come back to itself again. For our example, the time scale of this generation-decay process is $\omega_s t \approx 0.5$. The concurrence generated is smaller than 0.001 by the mixed state environment, due to the number of spins in the environment, and is about 0.004 in an early stage with the pure state environment due to the initial relaxation of the environment.
environment into its steady state. Such an initial generation of entanglement is also observed in the Holstein–Primakoff boson case [25, 27].

6. Conclusion

We have studied entanglement dynamics of the two-qubit system under the z–x interaction with different spin environments. All the models presented here are solved analytically. The power law decay of entanglement is obtained when the environment is large, the spectral density is of an ohmic type and the qubit-spin couplings are weak.

The entanglement decay and generation are highly sensitive to the number of environment spins. In general, a large environment gives rise to a fast decay, or a small degree of generation of entanglement. For the local environment model, we have found that, in the case of the small environment, the central qubits can typically maintain a high level entanglement for a long period of time. Interestingly, we have shown that the block-entangled environment provides stronger protection for the entanglement of the central qubits. In addition, the entangled spins confined in the local environments can accelerate the decay and revival of the entanglement between central qubits. For the global environment model considered in this paper, we have found that, in the case of small environments (see figure 6), the concurrence evolves in similar patterns when driven by pure state and mixed state baths. Even for a large environment (see figures 7 and 8), the entanglement generated is still seen to decay in a non-monotonic way, a feature that is commonly ascribed to the non-Markovian dynamics. For a very large environment, similar to the Markov limit, the mixed state and pure state environments considered here can only generate a very small degree of entanglement. Moreover, the pure state bath can generate a higher degree of entanglement than the mixed state environment.

Another factor that can affect the entanglement dynamics is the population distribution of the state of each environment spin. As long as the probability distributions of two possible states, up and down, are not uniform for all the environment spins, the entanglement generation can typically be observed for finite number of environment spins. Furthermore, the degree of entanglement generated is dependent on the number of environment spins and its initial state of environment.

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Appendix A. Block-entangled local environment with $N = 2$

For local environment initialed in a block entangled state, the elements in the RDM can be given by

$$
\rho_{\text{sys}}(t) = \prod_{j=1}^{M} \text{Tr}_k \left[ \prod_{k=1}^{N} e^{-i\hat{X}_kt} \rho_{\text{env}}(0) \prod_{k=1}^{N} e^{i\hat{X}_kt} \right].
$$

(A.1)
where $\rho_{\text{env}}^{N,j}(0)$ is the initial density matrix for $N$ spins in the $j$th block. We can rewrite it as
\[
\rho_{\text{env}}^{N,j}(0) = \frac{1}{2} \left( \sum_{k=1}^{N} |\uparrow\rangle_k \langle\uparrow|_k + \sum_{k=1}^{N} |\downarrow\rangle_k \langle\downarrow|_k + \sum_{k=1}^{N} |\uparrow\rangle_k \langle\downarrow|_k + \sum_{k=1}^{N} |\downarrow\rangle_k \langle\uparrow|_k \right)_{j},
\]
(A.2)

Thus, for the $N$ spins in each block, when the two time-dependent operators act on the initial density matrix of the bath spin, after tracing the above equation, the first two terms are
\[
\text{Tr} \left[ \prod_{k} e^{-i \hat{X} t} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_k e^{i \hat{X} t} \right] = \prod_{k} \left( \cos^2 \left( \frac{p_k t}{p_k} \right) + \frac{\sin^2 \left( \frac{p_k t}{p_k} \right)}{p_k^2} \left( \omega_k^2 - c_k^2 E_m E_n \right) \right),
\]
(A.3)

The last two terms will give
\[
\text{Tr} \left[ \prod_{k} e^{-i \hat{X} t} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_k e^{i \hat{X} t} \right] = \prod_{k} \left( \cos^2 \left( \frac{p_k t}{p_k} \right) + \frac{\sin^2 \left( \frac{p_k t}{p_k} \right)}{p_k^2} \left( \omega_k^2 - c_k^2 E_m E_n \right) \right).
\]
(A.4)

Combine all four terms:
\[
\text{Tr} \left[ F_{mn}^{N,j} \right] = \prod_{k} \left( 1 - \frac{c_k^2}{p_k^2} \sin^2 \left( \frac{p_k t}{p_k} \right) (1 - E_m E_n) \right) + \left( \frac{E_n - E_m}{2} \right) \prod_{k} \left( \frac{\omega_k c_k}{p_k^2} \sin^2 \left( \frac{p_k t}{p_k} \right) + \frac{\sin^2 \left( \frac{p_k t}{p_k} \right)}{p_k^2} \right) + \left( \frac{E_n - E_m}{2} \right) \prod_{k} \left( - \frac{\omega_k c_k}{p_k^2} \sin^2 \left( \frac{p_k t}{p_k} \right) + \frac{\sin^2 \left( \frac{p_k t}{p_k} \right)}{p_k^2} \right).
\]
(A.7)
Therefore, for $N = 2$ (each block contains two spins in the Bell State $|\Psi\rangle_{j} = \frac{1}{\sqrt{2}}(|\downarrow\downarrow\rangle_{j} + |\uparrow\uparrow\rangle_{j})$), the elements in the RDM are give by

$$
\rho_{\text{sys}}(t) = e^{-itE_{n} - E_{n}} \rho_{\text{sys}}(0) \prod_{j=1}^{M} \left\{ \cos^{2} p_{1j} t \cos^{2} p_{2j} t 
+ \frac{\sin^{2} p_{1j} t \sin^{2} p_{2j} t}{p_{1j}^{2} p_{2j}^{2}} \left[ (\omega_{1j} \omega_{2j} + c_{1j} c_{2j})^{2} + (\omega_{1j} c_{2j} - \omega_{2j} c_{1j})^{2} E_{m} E_{n} \right] 
+ \frac{\sin^{2} p_{1j} t \cos^{2} p_{2j} t}{p_{1j}^{2}} \left( \omega_{1j}^{2} + c_{1j}^{2} E_{m} E_{n} \right) 
+ \frac{\sin^{2} p_{2j} t \cos^{2} p_{1j} t}{p_{2j}^{2}} \left( \omega_{2j}^{2} + c_{2j}^{2} E_{m} E_{n} \right) 
- \frac{\sin 2p_{1j} t \cos 2p_{2j} t}{2p_{1j} p_{2j}} c_{1j} c_{2j} \left( 1 - E_{m} E_{n} \right) \right\}. \tag{A.8}
$$

Obviously for diagonal elements $\rho_{11}(t) = \rho_{11}(0)$, $\rho_{22}(t) = \rho_{22}(0)$, and the dephasing coefficient is

$$
f(t) = \prod_{j=1}^{M} \left( 1 - \frac{2 \sin^{2} p_{1j} t \sin^{2} p_{2j} t}{p_{1j}^{2} p_{2j}^{2}} (\omega_{1j} c_{2j} - \omega_{2j} c_{1j})^{2} - \frac{c_{1j}^{2}}{p_{1j}^{2}} (\sin^{2} p_{1j} t \cos^{2} p_{2j} t) 
- \frac{c_{2j}^{2}}{p_{2j}^{2}} (\sin^{2} p_{2j} t \cos^{2} p_{1j} t) - \frac{c_{1j} c_{2j}}{p_{1j} p_{2j}} \sin 2p_{1j} t \sin 2p_{2j} t \right). \tag{A.9}
$$

**Appendix B. General form of the reduced density matrix for global environment**

We start from equations (5), (6), and using the fact that

$$
e^{\pm it(\omega_{k} \sigma_{x} + c_{k} E_{m} \sigma_{z})} = \cos \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}} t \pm \frac{i \sin \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}} t}{\sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}}} (\omega_{k} \sigma_{z} + c_{k} E_{m} \sigma_{x}). \tag{B.1}
$$

where $E_{m}$ is the eigenvalue of $\sigma_{x}^{A} + \sigma_{y}^{B}$, $E_{1} = 2$, $E_{2} = E_{3} = 0$, $E_{4} = -2$. Using the notation $q_{k}(E_{m}) = \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}}$, we can obtain

$$
\text{Tr}_{k} F_{nn}^{k} = \text{Tr}_{k} \left[ \cos \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}} - \frac{i \sin \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}}}{\sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}}} (\omega_{k} \sigma_{z} + c_{k} E_{m} \sigma_{x}) \right] \rho_{\text{env}}^{k}(0)
\times \left[ \cos \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}} + \frac{i \sin \sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}}}{\sqrt{\omega_{k}^{2} + c_{k}^{2} E_{m}^{2}}} (\omega_{k} \sigma_{z} + c_{k} E_{m} \sigma_{x}) \right] \rho_{\text{env}}^{k}(0)
= \cos[q_{k}(E_{m}) t] \cos[q_{k}(E_{m}) t] \text{Tr}_{\hat{X}}[\rho_{\text{env}}^{k}(0)]
+ \sin[q_{k}(E_{m}) t] \sin[q_{k}(E_{m}) t] \text{Tr}_{\hat{X}}[\rho_{\text{env}}^{k}(0)]
\times \left[ \cos[q_{k}(E_{m}) t] \cos[q_{k}(E_{m}) t] \text{Tr}_{\hat{X}}[\rho_{\text{env}}^{k}(0)] \right]
+ \frac{\cos[q_{k}(E_{m}) t] \sin[q_{k}(E_{m}) t]}{q_{k}(E_{m})} \text{Tr}_{\hat{X}}[\rho_{\text{env}}^{k}(0)]
- \frac{\cos[q_{k}(E_{m}) t] \sin[q_{k}(E_{m}) t]}{q_{k}(E_{m})} \text{Tr}_{\hat{X}}[\rho_{\text{env}}^{k}(0)]. \tag{B.2}
$$
The first in the above equation after trace equals 1, while the other three give

$$\text{Tr}_k [(\omega_k \sigma_z + c_k E_m \sigma_z) \rho^k_{\text{env}} (0) (\omega_k \sigma_z + c_k E_m \sigma_z)] = \omega_k^2 + c_k^2 E_m E_n + i 2 (E_m - E_n) \omega_k c_k \text{Im} |\gamma_k|^2.$$  

(B.3)

$$\text{Tr}_k [\rho^k_{\text{env}} (0)] = \omega_k (N^k_+ - N^k_-) + 2 c_k E_n \text{Re} |\gamma_k|.$$  

(B.4)

$$\text{Tr}_k [\hat{X} (E_m) \rho^k_{\text{env}} (0)] = \omega_k (N^k_+ - N^k_-) + 2 c_k E_n \text{Re} |\gamma_k|.$$  

(B.5)

where we assume that the kth environment spin is in the general state given by

$$\rho^k_{\text{env}} (0) = \begin{pmatrix} N^k_+ & \gamma^k \\ \gamma^k & N^k_- \end{pmatrix}, \quad |\gamma^k| \leq \sqrt{N^k_+ N^k_-}.$$  

(B.6)

Thus for the global environment we have

$$\text{Tr}_k F^k_{mn} = \cos [q_k (E_m) t] \cos [q_k (E_n) t] + \frac{\sin [q_k (E_m) t] \sin [q_k (E_n) t]}{q_k (E_m) q_k (E_n)}$$

$$\times \left( \omega_k^2 + c_k^2 E_m E_n + i 2 (E_m - E_n) \omega_k c_k \text{Im} |\gamma_k|^2 \right)$$

$$+ i \omega_k (N^- - N^+) \left( \frac{\cos [q_k (E_m) t] \sin [q_k (E_n) t]}{q_k (E_n)} - \frac{\cos [q_k (E_n) t] \sin [q_k (E_m) t]}{q_k (E_m)} \right)$$

$$+ i 2 c_k \text{Re} [\gamma_k] \left( q_k \cos [q_k (E_m) t] \sin [q_k (E_n) t] - q_k \cos [q_k (E_n) t] \sin [q_k (E_m) t] \right).$$  

(B.7)

We can see from above that $F^k_{mn}$ satisfies the following properties:

$$\text{Tr} \left[ F^k_{mn} \right] = \left\{ \text{Tr} \left[ F^k_{mn} \right] \right\}^*, \quad \text{Tr} \left[ F^k_{mn} \right] = 1, \quad \text{Tr} \left[ F^k_{n2} \right] = \text{Tr} \left[ F^k_{m2} \right], \quad \text{Tr} \left[ F^k_{2n} \right] = \text{Tr} \left[ F^k_{2m} \right].$$  

(B.8)

Consequently for the elements in RDM, we have $\rho^0_{\text{env}} (t) = \rho^0_{\text{env}} (0), \rho^0_{\text{env}} (t) = \rho^0_{\text{env}} (0), \rho^0_{\text{env}} (t) = \rho^0_{\text{env}} (t)$ and $\rho^0_{\text{env}} (t) = \rho^0_{\text{env}} (t)$. Therefore, the resulting RDM takes the form as in equation (27). And the dephasing coefficient $f(t), g(t), g'(t)$ can be derived accordingly.

For $f(t)$, $E_m = -E_n = 2, q_k (E_m) = q_k (E_n) = q_k$. Thus,

$$f(t) = \prod_k \text{Tr}_k F^k_{mn} = \prod_k \left( \cos^2 [q_k t] + \frac{\sin^2 [q_k t]}{q_k^2} (\omega_k^2 - 4 c_k^2 + i 4 \omega_k c_k \text{Im} |\gamma_k|^2) \right)$$

$$- i 8 \sum_k - c_k \text{Re} [\gamma_k^2] \cos [q_k t] \sin [q_k t].$$  

(B.9)

Since $\rho_{12}^0 (t) = \rho_{21}^0 (t) = \rho_{21}^0 (t) = \rho_{21}^0 (t), E_m = 2, E_n = 0$, we have

$$g(t) = \prod_k \text{Tr}_k F^k_{13} = \prod_k \left( \cos [q_k t] \cos [q_k t] + \frac{\omega_k \sin [q_k t] \sin [q_k t]}{q_k} \left( 1 + i 4 \frac{c_k}{\omega_k} \text{Im} [\gamma_k^2] \right) \right)$$

$$+ i \omega_k (N^- - N^+) \left( \frac{\cos [q_k t] \sin [q_k t]}{q_k} - \frac{\cos [q_k t] \sin [q_k t]}{q_k} \right)$$

$$+ i 2 c_k \text{Re} [\gamma_k] \left( -2 \frac{\cos [q_k t] \sin [q_k t]}{q_k} \right)$$

$$= \prod_k \left( \cos [q_k t] \cos [q_k t] + \frac{\omega_k \sin [q_k t] \sin [q_k t]}{q_k} \right)$$

$$+ i (N^- - N^+) \left( \cos [q_k t] \sin [q_k t] - \frac{\omega_k \cos [q_k t] \sin [q_k t]}{q_k} \right)$$

$$+ i 4 \frac{c_k}{q_k} \sin [q_k t] \text{Im} [\gamma_k^2] \sin [q_k t] - \text{Re} [\gamma_k^2] \cos [q_k t].$$  

(B.10)
In a similar way, we have $\rho_{24}^{\text{sys}} = \rho_{34}^{\text{sys}} = \rho_{43}^{\text{sys}}$, here $E_m = 0$, $E_n = -2$, we have

$$g'(t) = \prod_k \text{Tr}_k F_{24}^k = \prod_k \left( \cos q_k t \cos \omega_k t + \frac{\alpha_k}{q_k} \sin q_k t \sin \omega_k t - i(N_+ - N_-) \right) \times \left( \cos q_k t \sin \omega_k t - \frac{\alpha_k}{q_k} \cos \omega_k t \sin q_k t \right) + i \frac{C_k}{q_k} \sin q_k t (\text{Im}[\gamma^k] \sin \omega_k t - \text{Re}[\gamma^k] \cos \omega_k t) \right). \tag{B.11}$$

For a specific situation, for example, mixed states where $\gamma^k = 0$, we have $g(t) = g'(t)^*$. For a special distribution of probability such as $N_i^k = N_i^2$, we have $g(t) = g'(t)$.

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