Attitude stability analyses for small artificial satellites

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Abstract. The objective of this paper is to analyze the stability of the rotational motion of a symmetrical spacecraft, in a circular orbit. The equilibrium points and regions of stability are established when components of the gravity gradient torque acting on the spacecraft are included in the equations of rotational motion, which are described by the Andoyer’s variables. The nonlinear stability of the equilibrium points of the rotational motion is analysed here by the Kovalev-Savchenko theorem. With the application of the Kovalev-Savchenko theorem, it is possible to verify if they remain stable under the influence of the terms of higher order of the normal Hamiltonian. In this paper, numerical simulations are made for a small hypothetical artificial satellite. Several stable equilibrium points were determined and regions around these points have been established by variations in the orbital inclination and in the spacecraft principal moment of inertia. The present analysis can directly contribute in the maintenance of the spacecraft’s attitude.

1. Introduction
This work aims at analyzing the stability of the rotational motion of artificial satellites in circular orbit with the influence of gravity gradient torque, using the Andoyer’s canonical variables. This stability analysis is very important in maintaining the attitude to ensure the success of a space mission. In this paper, Kovalev-Savchenko theorem (KST) [1] is used for the study of the stability and it ensures that the motion is Liapunov stable.

The objective of this paper is to optimize the stability analysis developed in [2,3] for the satellite in a circular orbit by applying the analytical expressions obtained in [4,5] for the coefficients of the normal 4th order Hamiltonian. The equilibrium points are established when terms associated with gravity gradient torque acting on the satellite are included in the equations of rotational motion. The stable regions around the equilibrium points are obtained by the variations in the orbital inclination and in the principal moment of inertia of the satellite.

In order to simplify the application of methods of stability in this study, the Andoyer’s variables [3,6] are used to describe the rotational motion of the satellite. These variables are represented by...
generalized moments \((L_1, L_2, L_3)\) and by generalized coordinates \((\ell_1, \ell_2, \ell_3)\) that are outlined in Figure 1. The angular variables \(\ell_1, \ell_2, \ell_3\) are angles related to the satellite system Oxyz (with axes parallel of the spacecraft’s principal axes of inertia) and equatorial system OXYZ (with axes parallel to the axis of the Earth's equatorial system). Variables metrics \(L_1, L_2, L_3\) are defined as: \(L_2\) is the magnitude of the angular momentum of rotation \(\vec{L}_2\), \(L_1\) and \(L_3\) are, respectively, the projection of \(\vec{L}_2\) z-axis’s principal axis system of inertia \((L_1 = L_2 \cos f_2)\), where \(f_2\) are the angle between the z-satellite axis and \(\vec{L}_2\) and projection \(\vec{L}_2\) on the Z-equatorial axis \((L_3 = L_2 \cos f_2)\), where \(f_2\) are the angle between Z-equatorial axis and \(\vec{L}_2\).

Figure 1. Andoyer’s canonical variables [9]

In this paper, numerical simulations are made for a small artificial satellite which has orbital data and physical characteristics similar to real satellite. The stables regions around the equilibrium points are obtained by the variations of the orbital inclination and variations in the principal moments of inertia of the satellite.

2. Equation of motion

Andoyer’s variables \((L_1, L_2, L_3, \ell_1, \ell_2, \ell_3)\), are used to characterize the rotational motion of a satellite around its center of mass [6] and the Delaunay’s variables describe the translational motion of the center of mass of the satellite around the Earth [6].

Thus, assuming that the satellites have well-defined circular orbit, the goal is to study the stability of the rotational motion of the satellite. Then the Hamiltonian of the problem is expressed in terms of the Andoyer and Delaunay variables \((L, G, H, l, g, h)\), including the gravity gradient torque, as follows [6,7]:

\[
F(L_1, L_2, L_3, \ell_2, \ell_3, L, G, H, l, g, h) = F_0(L, L_1, L_2) + F_1(L_1, L_2, L_3, \ell_2, \ell_3, L, G, H, l, g, h)
\]  

(1)

with

\[
F_0 = -\frac{\mu^2 M^3}{2l^2} + \frac{1}{2}\left(\frac{1}{C} - \frac{1}{2A} - \frac{1}{2B}\right) L_1^2 + \frac{1}{2}\left(\frac{1}{A} + \frac{1}{B}\right) L_2^2 + \frac{1}{4}\left(\frac{1}{B} - \frac{1}{A}\right)(L_2^2 - L_1^2) \cos 2\ell_1
\]  

(2)

\[
F_1 = \frac{\mu^4 M^6}{l^6}\left[\frac{2C-A-B}{2} \mathcal{A}_1(\ell_m, L_m) + \frac{A-B}{4} \mathcal{A}_2(\ell_m, L_m)\right]
\]  

(3)
where \( m = 2,3 \) and \( n = 1,2,3 \); \( A, B \) and \( C \) are the principal moments of inertia of the satellite on \( x \)-axis, \( y \)-axis and \( z \)-axis respectively; \( \ell_1 \) and \( \ell_2 \) are functions of the variables \( (\ell_m, L_n) \), where \( \ell_2 \) and \( \ell_3 \) appear in the arguments of cosines. The complete analytical expression for perturbed Hamiltonian \( F_1 \) is presented in [7] for circular orbit.

The equations of motion associated with the Hamiltonian \( F \), Eq. 1, are given by:

\[
\begin{align*}
\frac{d\ell_i}{dt} &= \frac{\partial F}{\partial L_i}, \\
\frac{dL_i}{dt} &= -\frac{\partial F}{\partial \ell_i};
\end{align*}
\]

\( i = 1,2,3 \). (4)

These equations are used to find the possible equilibrium points of the rotational motion when will be considered two of its principal moments of inertia equal, \( B = A \) (symmetrical satellite). With this relationship, the variable \( \ell_1 \) will not be present in the Hamiltonian, reducing the dynamic system to two degrees of freedom, a necessary condition for applying the stability theorem chosen for analysis of equilibrium points.

3. The Algorithm for Stability Analysis

To use the KST is necessary the normal Hamiltonian of the problem. It was discussed in [3,8] and the normal Hamiltonian \( H \) is an analytic function of generalized coordinates \( (q_v) \) and moments \( (p_v) \) to a fixed point \( P \), expressed by [3,8]:

\[
H = \sum_{v=1}^{2} \frac{\omega_v}{2} R_v + \sum_{v,u=1}^{2} \frac{\delta_{vu}}{4} R_v R_u + O_5
\]

(5)

where \( O_5 \) represents higher order terms; \( \omega_v \) is the imaginary part of eigenvalues associated with the matrix defined by the product of a 4th order matrix symplectic with the Hessian of the Hamiltonian expanded in Taylor series up to 2nd order around the equilibrium point; \( \delta_{vu} \) depend on the eigenvalues \( \omega_v \) and the coefficients of the Hamiltonian expanded in Taylor series of 3rd and 4th order around the equilibrium point, which are presented analytically in Formiga [5]; and

\[
R_m = q_m^2 + p_m^2, \quad m = 1,2
\]

(6)

The stability analysis is performed here by the theorem Kovalev and Savchenko [1] which ensures that the motion is Liapunov stable if the following conditions are satisfied:

i. The eigenvalues of the reduced linear system are pure imaginary \( \pm i\omega_1, \pm i\omega_2 \);

ii. The condition

\[
k_1\omega_1 + k_2\omega_2 \neq 0
\]

(7)

is valid for all \( k_1 \) and \( k_2 \) integer satisfying the inequality.

\[
|k_1| + |k_2| \leq 4
\]

(8)

iii. The determinant \( D \) must satisfy the inequality

\[
D = -(\delta_{11}\omega_2^2 - 2\delta_{12}\omega_1\omega_2 + \delta_{22}\omega_1^2) \neq 0
\]

(9)

where \( \delta_{uv} \) are the coefficients of the normal 4th order Hamiltonian.

4. Numerical simulations
In order to do the numerical simulations, in this paper were considered a small satellite (SS), which has similar orbital characteristics of the Second Brazilian Data Collection Satellite SCD-2 [2]. All the numerical simulations were developed using the software MATHEMATICA. The initial data for the SS satellite are (Table 1):

| Orbit sets                  | Inclination | Eccentricity | Orbital radius |
|-----------------------------|-------------|--------------|----------------|
|                             | \( I = 0.4364 \) rad | \( e = 0 \)      | \( r = 7139.61585 \) km |

**Table 1. Orbital characteristics of the Second Brazilian Data Collection Satellite SCD-2.**

| Principal moments of inertia on | \( A = 9.855 \times 10^{-6} \) kg km\(^2\) | \( B = 9.855 \times 10^{-6} \) kg km\(^2\) | \( C = 13.00 \times 10^{-6} \) kg km\(^2\) |
|--------------------------------|---------------------------------|---------------------------------|---------------------------------|
| x-axis                        | \( A = 9.855 \times 10^{-6} \) kg km\(^2\) | \( B = 9.855 \times 10^{-6} \) kg km\(^2\) | \( C = 13.00 \times 10^{-6} \) kg km\(^2\) |
| y-axis                        | \( A = 9.855 \times 10^{-6} \) kg km\(^2\) | \( B = 9.855 \times 10^{-6} \) kg km\(^2\) | \( C = 13.00 \times 10^{-6} \) kg km\(^2\) |
| z-axis                        | \( A = 9.855 \times 10^{-6} \) kg km\(^2\) | \( B = 9.855 \times 10^{-6} \) kg km\(^2\) | \( C = 13.00 \times 10^{-6} \) kg km\(^2\) |

**Delaunay Variables**

| Generalized moments | \( L = 5334653.709 \) kg km\(^2\)/s | \( G = 5334653.709 \) kg km\(^2\)/s | \( H = 4834685.585 \) kg km\(^2\)/s |
|---------------------|---------------------------------|---------------------------------|---------------------------------|
| Generalized coordinates | \( l = 0 \) rad | \( g = 4.5420 \) rad | \( h = 4.542 \) rad |

There were 50 equilibrium points and only 7 were stables, the others 43 equilibrium points had also failed in the first condition of the KST and were not linearly stable, more details are presented in [9].

4.1. **Stables region around the equilibrium points**

The behavior around the Lyapunov stable equilibrium points were obtained by the variations in the orbital inclination (\( I \)) and in the principal moments of inertia of the satellite on x-axis (\( A \)) and on z-axis (\( C \)). For this procedure, was used the algorithm presented in [9].

Figure 2 shows the linear stabilization when the principal moment of inertia on the x-axis is fixed (\( A_0 = 9.855 \times 10^{-6} \) kg km\(^2\)). The blue colour, in the region 1 and 2, represents the situation where the eigenvalues are pure imaginary. By the results it is possible to note that there are non-linear stabilization regions when \( C = A_0 \) or for low orbital inclination and there is linear stabilization regions for \( C < A_0 \) and \( C > A_0 \).

![Figure 2](image-url)
Figure 3 shows the analysis of the second condition for the KST, for the case when the principal moment of inertia on $x$–axis and the orbital inclination are fixed $A_0 = 9.855 \times 10^{-6} \text{kg km}^2$ and $C_0 = 13 \times 10^{-6} \text{kg km}^2$.

Then the second condition ($k_1 \omega_1 + k_2 \omega_2 \neq 0$) is satisfied for $I > 0.1651 \text{ rad}$, in accordance with the results presented in the Figure 2.

By Figure 4, when the $I_0$ and $A_0$ are fixed, it is possible to see that the Arnold determinant is zero for values of the principal moment of inertia on $z$-axis (C), $9.5 \times 10^{-6} \text{kg km}^2 < C < A_0$, than the third conditions are not satisfied and the rotational motion is nonlinear unstable.

**Figure 3** – Second condition for $k_1 \geq 0$ and $k_2 \geq 0$, considering $A_0 = 9.855 \times 10^{-6} \text{kg km}^2$ and $C_0 = 13 \times 10^{-6} \text{kg km}^2$.

**Figure 5.** Arnold determinant in terms of orbital inclination, $A_0 = 9.855 \times 10^{-6} \text{ kg km}^2$ and $I_0 = 0.4364 \text{ rad}$
5. Conclusions
In this paper it was presented a semi-analytical stability of the rotational motion of artificial satellites, considering the influence of gravity gradient torque for symmetric satellite in a circular orbit. Applications were made for small satellites (SS). Initially the points of equilibrium were determined using the physical, orbital and attitude characteristics of the satellite. Then the algorithm for stability analysis was applied and it was obtained 7 stable points for the SS satellite.

Several stable equilibrium points were determined and regions around these points have been established by variations in the orbital inclination and in the spacecraft principal moment of inertia. There were found 50 equilibrium points for the small size satellite (with some data similar to Second Brazilian Data Collecting Satellite) with 10 Liapunov stable points.

The results for the stable regions show that in the linear stability there is a separation between the stable and unstable region when the spacecraft principal moments of inertia are equals. It is also possible to observe that the rotational motion for the small satellite is linearly unstable in a low orbital inclination. For considered equilibrium points, the second condition is valid for all values of \( k_1 \) and \( k_2 \), for the small satellite it is necessary an orbital inclination bigger than 9.45 degrees. In the nonlinear analysis it was possible to verify that the linear stability doesn’t guaranty the non-linear stability and the stable regions are bigger for the small satellite. For the considered satellite there is nonlinear stability for orbital inclination bigger than 9.45 degrees.

Then the present analysis can directly contribute in the maintenance of the spacecraft’s attitude. Once the regions of stability are known for the rotational motion, a smaller number of maneuvers to maintain the desired attitude can be accomplished. In this case, a fuel economy can be generated to the satellite with propulsion systems control, increasing the spacecraft’s lifetime.

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