Local linear smoothing in additive models as data projection

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Abstract. We discuss local linear smooth backfitting for additive non-parametric models. This procedure is well known for achieving optimal convergence rates under appropriate smoothness conditions. In particular, it allows for the estimation of each component of an additive model with the same asymptotic accuracy as if the other components were known. The asymptotic discussion of local linear smooth backfitting is rather complex because typically an overwhelming notation is required for a detailed discussion. In this paper we interpret the local linear smooth backfitting estimator as a projection of the data onto a linear space with a suitably chosen semi-norm. This approach simplifies both the mathematical discussion as well as the intuitive understanding of properties of this version of smooth backfitting.

Keywords: Additive models, local linear estimation, backfitting, data projection, kernel smoothing

1 Introduction

In this paper we consider local linear smoothing in an additive model

\[ E[Y_i | X_i] = m_0 + m_1(X_{i1}) + \cdots + m_d(X_{id}), \]

where \((Y_i, X_i) (i = 1, \ldots, n)\) are iid observations with values in \(\mathbb{R} \times \mathcal{X}\) for a bounded connected open subset \(\mathcal{X} \subseteq \mathbb{R}^d\). Here, \(m_j (j = 1, \ldots, d)\) are some smooth functions which we aim to estimate and \(m_0 \in \mathbb{R}\). Below, we will add norming conditions on \(m_0, \ldots, m_d\) such that they are uniquely defined given the sum. In [Mammen et al. (1999)] a local linear smooth backfitting estimator based on smoothing kernels was proposed for the additive functions \(m_j\). There, it was shown that their version of a local linear estimator \(\hat{m}_j\) of the function \(m_j\) has the same pointwise asymptotic variance and bias as a classical local linear estimator in the oracle model, where one observes i.i.d. observations \((Y_{i^*}, X_{ij})\) with

\[ E[Y_{i^*} | X_{ij}] = m_j(X_{ij}), \quad Y_{i^*} = Y_i - \sum_{k \neq j} m_k(X_{ik}). \]
In this respect the local linear estimator differs from other smoothing methods where the asymptotic bias of the estimator of the function \( m_j \) depends on the shape of the functions \( m_k \) for \( k \neq j \). An example for an estimator with this disadvantageous bias property is the local constant smooth backfitting estimator which is based on a backfitting implementation of one-dimensional Nadaraya-Watson estimators. It is also the case for other smoothing estimators as regression splines, smoothing splines and orthogonal series estimators, where in addition also no closed form expression for the asymptotic bias is available. Asymptotic properties of local linear smoothing simplify the choice of bandwidths as well as the statistical interpretation of the estimators \( \hat{m}_j \). These aspects have made local linear smooth backfitting a preferred choice for estimation in additive models. Deriving asymptotic theory for local linear smooth backfitting is typically complicated by an overloaded notation that is required for detailed proofs. In this note we will use that the local linear smooth backfitting estimator has a nice geometric interpretation. This simplifies mathematical arguments and allows for a more intuitive derivation of asymptotic properties. In particular, we will see that the estimator can be characterized as a solution of an empirical integral equation of the second kind as is the case for local constant smooth backfitting, see Mammen and Yu (2009).

Our main point is that the local linear estimator can be seen as an orthogonal projection of the response vector \( Y = (Y_i)_{i=1,...,n} \) onto a subspace of a suitably chosen linear space. A similar point of view is taken in Mammen et al. (2001) for a related construction where it was also shown that regression splines, smoothing splines and orthogonal series estimators can be interpreted as projection of the data in an appropriately chosen Hilbert space. Whereas this interpretation is rather straightforward for these classes of estimators it is not immediately clear that it also applies for kernel smoothing and local polynomial smoothing, see Mammen et al. (2001). In this paper we will introduce a new and simple view of local linear smoothing as data projection. In the next section we will define the required spaces together with a corresponding semi-norm. We will also introduce a new algorithm motivated by our interpretation of local linear smooth backfitting. The algorithm will be discussed in Section 3. In Section 4 we will see that our geometric point of view allows for simplified arguments for the asymptotic study of properties of the local linear smooth backfitting estimator.

The additive model (1) was first introduced in Friedman and Stuetzle (1981) and enjoys great popularity for two main reasons. The first is estimation performance. While not being as restrictive as a linear model, in contrast to a fully flexible model, it is not subject to the curse of dimensionality. Assuming that \( E[Y_i | X_i = x] \) is twice continuously differentiable, the optimal rate of convergence of an estimator of \( E[Y_i | X_i = x] \) is \( n^{-2/(d+4)} \) if no further structural assumptions are made, see Stone (1982). This means the rate deteriorates exponentially in the dimension of the covariates \( d \). Under the additive model assumption (1) and assuming that each function \( m_j, j = 1, \ldots, d \) is twice continuously differentiable, the optimal rate of convergence is \( n^{-2/5} \). The second reason is interpretability. In many applications it is desirable to understand the relationship between pre-
dictors and the response. Even if the goal is prediction only, understanding this relationship may help detect systematic biases in the estimator, so that out of sample performance can be improved or adjusted for. While it is almost impossible to grasp the global structure of a multivariate function \( m \) in general, the additive structure (1) allows for visualisation of each of the univariate functions, providing a comprehensible connection between predictors and the response.

Though the setting considered in this paper is fairly simple, it can be seen as a baseline for more complicated settings. One main drawback is the additive structure which cannot account for interactions between covariates. It is assuring however that even if the true model is not additive, the smooth backfitting estimator is still defined as the closest additive approximation. This will be shown in the next section. If the true regression function is far away from an additive structure, then a more complex structure may be preferable. This could be done by adding higher-dimensional covariates, products of univariate functions or considering a generalized additive model. For testing procedures that compare such specifications, see also Härdle et al. (2001); Mammen and Sperlich (2021). Besides such structural assumptions, other directions the ideas in this paper can be extended to are the consideration of time-series data or high dimensional settings. Settings using more complicated responses like survival times, densities or other functional data may also be approached. Some of these cases have been considered, e.g., in Mammen and Nielsen (2003), Yu et al. (2008), Mammen and Yu (2009), Mammen et al. (2014), Han et al. (2018), Mammen and Sperlich (2021), Han et al. (2020), Jeon et al. (2020), Hiabu et al. (2020) and Gregory et al. (2020). We hope that a better understanding of local linear estimation in this simple setting will help advance theory and methodology for more complicated settings in the future.

2 Local linear smoothing in additive models

The local linear smooth backfitting estimator \( \hat{m} = (\hat{m}_0, \hat{m}_1, \ldots, \hat{m}_d, \hat{m}^{(1)}_1, \ldots, \hat{m}^{(1)}_d) \) is defined as the minimizer of the criterion

\[
S(f_0, \ldots, f_d, f^{(1)}_1, \ldots, f^{(1)}_d) = n^{-1} \sum_{i=1}^n \int_{\mathcal{X}} \left\{ Y_i - f_0 - \sum_{j=1}^d f_j(x_j) - \sum_{j=1}^d f^{(1)}_j(x_j)(X_{ij} - x_j) \right\}^2 \times K_h^{\mathcal{X}}(X_i - x) dx
\]

under the constraint

\[
\sum_{i=1}^n \int_{\mathcal{X}} f_j(x_j) K_h^{\mathcal{X}}(X_i - x) dx = 0
\]

for \( j = 1, \ldots, d \). The minimization runs over all values \( f_0 \in \mathbb{R} \) and all functions \( f_j, f^{(1)}_j : \mathcal{X}_j \to \mathbb{R} \) with \( \mathcal{X}_j = \{ u \in \mathbb{R} : \text{there exists an } x \in \mathcal{X} \text{ with } x_j = u \} \). Under
the constraint (2) and some conditions introduced in Section 3, the minimizer is unique. For \( j = 1, \ldots, d \) the local linear estimator of \( m_j \) is defined by \( \hat{m}_j \).

In the definition of \( S \) the function \( K_n^h(\cdot) \) is a boundary corrected product kernel, i.e.,

\[
K_n^h(u - x) = \frac{\prod_{j=1}^d K(u_j - x_j)}{\int_X \prod_{j=1}^d K(u_j - v_j) \, dv_j},
\]

Here, \( h = (h_1, \ldots, h_d) \) is a bandwidth vector with \( h_1, \ldots, h_d > 0 \) and \( \kappa : \mathbb{R} \to \mathbb{R} \) is some given univariate density function, i.e., \( \kappa(t) \geq 0 \) and \( \int \kappa(t) \, dt = 1 \). We use the variable \( u \) twice in the notation because away from the boundary of \( X \), the kernel \( K_n^h(u - x) \) only depends on \( u - x \).

It is worth emphasizing that the empirical minimization criterion \( S \) depends on a choice of a kernel \( \kappa \) and a smoothing bandwidth \( h \). While the choice of \( \kappa \) is not of great importance, see similar to e.g. (Silverman 2018, Section 3.3.2), the quality of estimation heavily depends on an appropriate choice of the smoothing parameter \( h \). We will not discuss the choice of a (data-driven) bandwidth in this paper, but we note that the asymptotic properties of the local linear smoothing estimator do simplify the choice of bandwidths compared to other estimators. The reason is that the asymptotic bias of one additive component does not depend on the shape of the other components and on the bandwidths used for the other components.

We now argue that the local linear smooth backfitting estimator can be interpreted as an empirical projection of the data onto a space of additive functions. We introduce the linear space

\[
\mathcal{H} = \{(f^{i,j})_{i=1,\ldots,n; \ j=0,\ldots,d} \mid f^{i,j} : X \to \mathbb{R}, \|f\|_n < \infty \}
\]

with inner product

\[
\langle f, g \rangle_n = n^{-1} \sum_{i=1}^n \int_X \left\{ f^{i,0}(x) + \sum_{j=1}^d f^{i,j}(x)(X_{ij} - x_j) \right\} \times \left\{ g^{i,0}(x) + \sum_{k=1}^d g^{i,k}(x)(X_{ij} - x_j) \right\} K_n(X_i - x) \, dx
\]

and norm \( \|f\|_n = \sqrt{\langle f, f \rangle_n} \).

We identify the response \( Y = (Y_i)_{i=1,\ldots,n} \) as an element of \( \mathcal{H} \) via \( Y^{i,0} = Y_i \) and \( Y^{i,j} = 0 \) for \( j \geq 1 \). We will later assume that the functions \( m_j \) are differentiable. We identify the regression function

\[
m : X \to \mathbb{R}, \ m(x) = m_0 + m_1(x_1) + \cdots + m_d(x_d)
\]

as an element of \( \mathcal{H} \) via \( m^{i,0}(x) = m_0 + \sum_j m_j(x_j) \) and \( m^{i,j} = \partial m_j(x_j) / \partial x_j \) for \( j \geq 1 \). Note that the components of \( m \in \mathcal{H} \) do not depend on \( i \). We define the
following subspaces of $\mathcal{H}$:

$$
\mathcal{H}_{\text{full}} = \{ f \in \mathcal{H} \mid \text{the components of } f \text{ do not depend on } i \}, \\
\mathcal{H}_{\text{add}} = \left\{ f \in \mathcal{H}_{\text{full}} \mid f^{i,0}(x) = f_0 + f_1(x_1) + \cdots + f_d(x_d), f^{i,j}(x) = f_j^{(1)}(x_j) \text{ for some } f_0 \in \mathbb{R} \text{ and some univariate functions } f_j, f_j^{(1)} : \mathcal{X}_j \to \mathbb{R}, j = 1, \ldots, d \right\}
$$

For a function $f \in \mathcal{H}_{\text{add}}$ we write $f_0 \in \mathbb{R}$ and $f_j, f_j^{(1)} : \mathcal{X}_j \to \mathbb{R}$ for $j = 1, \ldots, d$ for the constant and functions that define $f$. In the next section we will state conditions under which the minimization has a unique solution. Equation (4) provides a geometric interpretation of local linear backfitting. The local linear smooth backfitting estimator is an orthogonal projection of the response vector $Y$ onto the linear subspace $\mathcal{H}_{\text{add}} \subseteq \mathcal{H}$. We will make repeated use of this fact in this paper. We now introduce the following subspaces of $\mathcal{H}$:

$$
\mathcal{H}_0 = \{ f \in \mathcal{H} \mid f^{i,0}(x) \equiv c \text{ for some } c \in \mathbb{R}, f^{i,j}(x) \equiv 0 \text{ for } j \neq 0 \}, \\
\mathcal{H}_k = \{ f \in \mathcal{H} \mid f^{i,j}(x) \equiv 0 \text{ for } j \neq 0, \text{ and } f^{i,0}(x) = f_k(x_k) \text{ for some univariate function } f_k : \mathcal{X}_k \to \mathbb{R} \text{ with } \sum_{i=1}^{n} \int_{\mathcal{X}} f_k(x_k) K^X_h(X_i - x) dx = 0 \}, \\
\mathcal{H}_{k'} = \left\{ f \in \mathcal{H} \mid f^{i,j}(x) \equiv 0 \text{ for } j \neq k, f^{i,k}(x) = f_k^{(1)}(x_k) \text{ for some univariate function } f_k^{(1)} : \mathcal{X}_k \to \mathbb{R} \right\}
$$
for \( k = 1, \ldots, d \) and \( k' := k + d \). Using these definitions we have \( \mathcal{H}_{add} = \sum_{j=0}^{2d} \mathcal{H}_j \) with \( \mathcal{H}_j \cap \mathcal{H}_k = \{0\}, j \neq k \). In particular, the functions \( f_j \) in (3) are unique elements in \( \mathcal{H}_j, j = 0, \ldots, 2d \). For \( k = 0, \ldots, 2d \) we denote the orthogonal projection of \( \mathcal{H} \) onto the space \( \mathcal{H}_k \) by \( \mathcal{P}_k \). Note that for \( k = 0, \ldots, d \) the operators \( \mathcal{P}_k \) set all components of an element \( f = (f^{i,j})_{i=1,\ldots,n; j=0,\ldots,d} \in \mathcal{H} \) to zero except the components with indices \((i,0), i = 1, \ldots, n \). Furthermore, for \( k = d+1, \ldots, 2d \), only components with index \((i,k-d)\) are not set to zero. Because \( \mathcal{H}_0 \) is orthogonal to \( \mathcal{H}_k \) for \( k = 1, \ldots, d \), the orthogonal projection onto the space \( \mathcal{H}_k \) is given by \( \mathcal{P}_k = \mathcal{P}_k - \mathcal{P}_0 \) where \( \mathcal{P}_k \) is the projection onto \( \mathcal{H}_0 + \mathcal{H}_k \). In Appendix [A] we will state explicit formulas for the orthogonal projection operators.

The operators \( \mathcal{P}_k \) can be used to define an iterative algorithm for the approximation of \( \hat{m} \). For an explanation observe that \( \hat{m} \) is the projection of \( Y \) onto \( \mathcal{H}_{add} \) and \( \mathcal{H}_k \) is a linear subspace of \( \mathcal{H}_{add} \). Thus \( \mathcal{P}_k(Y) = \mathcal{P}_k(\hat{m}) \) holds for \( k = 0, \ldots, 2d \). This gives

\[
\mathcal{P}_k(Y) = \mathcal{P}_k(\hat{m}) = \mathcal{P}_k \left( \sum_{j=0}^{2d} \hat{m}_j \right) = \hat{m}_k + \mathcal{P}_k \left( \sum_{j \neq k} \hat{m}_j \right)
\]

or, equivalently,

\[
\hat{m}_k = \mathcal{P}_k(Y) - \sum_{j \neq k} \mathcal{P}_k(\hat{m}_j) = \mathcal{P}_k(Y) - \hat{Y} - \sum_{j \neq k} \mathcal{P}_k(\hat{m}_j),
\]

where \( \hat{Y} = \mathcal{P}_0(Y) = \mathcal{P}_0(\hat{m}) \) is the element of \( \mathcal{H} \) with \((\hat{Y})^i,0 \equiv 0 \), \((\hat{Y})^i,j \equiv 0 \) for \( j \geq 1 \). This equation inspires an iterative algorithm where in each step approximations \( \hat{m}^{\text{old}}_k \) of \( \hat{m}_k \) are updated by

\[
\hat{m}^{\text{new}}_k = \mathcal{P}_k(Y) - \hat{Y} - \sum_{j \neq k} \mathcal{P}_k(\hat{m}^{\text{old}}_j).
\]

Algorithm 1 Smooth Backfitting algorithm

1: Start: \( \hat{m}_k(x_k) \equiv 0, \hat{m}_k = \mathcal{P}_0(Y), \text{error} = \infty \) \( \triangleright k = 0, \ldots, 2d \)
2: while \( \text{error} > \text{tolerance} \) do
3: \( \text{error} \leftarrow 0 \)
4: for \( k = 0, \ldots, 2d \) do
5: \( \hat{m}^{\text{old}}_k \leftarrow \hat{m}_k \)
6: \( \hat{m}_k \leftarrow \hat{m}_k - \sum_{j \neq k} \mathcal{P}_k(\hat{m}_j) \)
7: \( \text{error} \leftarrow \text{error} + |\hat{m}_k - \hat{m}^{\text{old}}_k| \)
8: return \( \hat{m} = (\hat{m}_0, \hat{m}_1, \ldots, \hat{m}_{2d}) \)

Algorithm 1 provides a compact definition of our algorithm for the approximation of \( \hat{m} \). In each iteration step, either \( \hat{m}_j \) or \( \hat{m}^{(1)}_j \) is updated for some \( j = 1, \ldots, d \). This is different from the algorithm proposed in [Mammen et al.].
where in each step a function tuple \((\hat{m}_j, \hat{m}_j^{(1)})\) is updated. For the orthogonal projections of functions \(m \in \mathcal{H}_{add}\) one can use simplified formulas. They will be given in Appendix A. Note that \(\tilde{m}_k = 0, \ldots, 2^d\) only needs to be calculated once at the beginning. Also the marginals \(p_k(x_k), p_k^*(x_k), p^*_k(x_k)\) and \(p^**_k(x_k)\) which are needed in the evaluation of \(P_k\) only need to be calculated once at the beginning. Precise definitions of these marginals can be found in the following sections. In each iteration of the for-loop in line 4 of Algorithm 1, \(O(d \times n \times gs)\) calculations are performed. Hence for a full cycle, the algorithm needs \(O(d^2 \times n \times gs \times \log(1/tolerance))\) calculations. Here \(gs\) is the number of evaluation points for each coordinate \(x_k\).

Existence and uniqueness of the local linear smooth backfitting estimator will be discussed in the next section. Additionally, convergence of the proposed iterative algorithm will be shown.

### 3 Existence and uniqueness of the estimator, convergence of the algorithm

In this section we will establish conditions for existence and uniqueness of the local linear smooth backfitting estimator \(\hat{m}\). Afterwards we will discuss convergence of the iterative algorithm provided in Algorithm 1. Note that convergence is shown for arbitrary starting values, i.e., we can set \(\hat{m}_k(x_k)\) to values other than zero in step 1 of Algorithm 1. For these statements we require the following weak condition on the kernel.

(A1) The kernel \(k\) has support \([-1, 1]\). Furthermore, \(k\) is strictly positive on \((-1, 1)\) and continuous on \(\mathbb{R}\).

For \(k = 1, \ldots, d\) and \(x \in \mathbb{R}^d\) we write

\[
x_{-k} := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_d).
\]

In the following, we will show that our claims hold on the following event:

\[
\mathcal{E} = \left\{ \text{For } k = 1, \ldots, d \text{ and } x_k \in \overline{\mathcal{X}_k} \text{ there exist two observations } i_1, i_2 \in \{1, \ldots, n\} \begin{array}{l}
\text{such that } X_{i_1,k} \neq X_{i_2,k}, \\ |X_{i,k} - x_k| < h_k, (x_k, X_{i,-k}) \in \overline{\mathcal{X}} \text{ for } i = i_1, i_2.
\end{array} \right\},
\]

Furthermore, there exist no \(b_0, \ldots, b_d \in \mathbb{R}\) with \(b_0 + \sum_{j=1}^{d} b_j X_{i,j} = 0 \forall i = 1, \ldots, n\),

where \(\overline{\mathcal{X}_k}\) is the closure of \(\mathcal{X}_k\) and by a slight abuse of notation

\[
(x_k, X_{i,-k}) := (X_{i,1}, \ldots, X_{i,k-1}, x_k, X_{i,k+1}, \ldots, X_{i,d}).
\]
Throughout this paper, we require the following definitions.

\[ \hat{p}_k(x_k) = \frac{1}{n} \sum_{i=1}^{n} \int_{X_{-k}(x_k)} K_h X_i(x_k) dx_{-k}, \]

\[ \hat{p}_k^*(x_k) = \frac{1}{n} \sum_{i=1}^{n} \int_{X_{-k}(x_k)} (X_{ik} - x_k) K_h X_i(x_k) dx_{-k}, \]

\[ \hat{p}_k^{**}(x_k) = \frac{1}{n} \sum_{i=1}^{n} \int_{X_{-k}(x_k)} (X_{ik} - x_k)^2 K_h X_i(x_k) dx_{-k}, \]

where \( X_{-k}(x_k) = \{ u_{-k} \mid (x_k, u_{-k}) \in X \}. \)

**Lemma 1.** Make Assumption (A1). Then, on the event \( E \) it holds that \( \| f \|_n = 0 \) implies \( f_0 = 0 \) as well as \( f_j \equiv 0 \) almost everywhere for \( j = 1 \ldots 2d \) and all \( f \in H_{add}. \)

One can easily see that the lemma implies the following. On the event \( E \), if a minimizer \( \hat{m} = \hat{m}_0 + \cdots + \hat{m}_{2d} \) of \( \| Y - f \|_n \) over \( f \in H_{add} \) exists, the components \( \hat{m}_0, \ldots, \hat{m}_{2d} \) are uniquely determined: Suppose there exists another minimizer \( \tilde{m} \in H_{add}. \) Then it holds that \( \langle Y - \hat{m}, \tilde{m} - \hat{m} \rangle_n = 0 \) and \( \langle Y - \tilde{m}, \tilde{m} - \hat{m} \rangle_n = 0 \) which gives \( \| \hat{m} - \tilde{m} \|_n = 0. \) An application of the lemma yields uniqueness of the components \( \hat{m}_0, \ldots, \hat{m}_{2d}. \)

**Remark 1.** In Figure 1 we give an example where a set \( X \) and data points \( X_i \) do not belong to the event \( E \) and where the components of the function \( f \in H_{add} \) are not identified. The data is visualized by blue dots. The size of the parameter \( h = h_1 = h_2 \) is showcased on the right hand side. For explanatory reasons, the interval \( a \) is included.

**Fig. 1.** An example of a possible data set \( X \subseteq \mathbb{R}^2 \) including data points where the conditions of the event \( E \) are not satisfied and where the components of functions \( f \in H_{add} \) are not identified. The data is visualized by blue dots. The size of the parameter \( h = h_1 = h_2 \) is showcased on the right hand side. For explanatory reasons, the interval \( a \) is included.
\begin{equation}
i_1, i_2 \in \{1, \ldots, n\} \text{ such that } X_{i_1} \neq X_{i_2} \text{ and } |X_{i,k} - x_k| < h, \text{ for } i = i_1, i_2. \text{ However, for } x_1 \in a \text{ the condition } (x_1, X_{i_2}) \in \mathcal{X} \text{ is not fulfilled for any } i = 1, \ldots, n \text{ with } |X_{i_1} - x_1| < h. \text{ Therefore, } K_h^{X_{i_1}}(X_i - x) = 0 \text{ for all } x \in \mathcal{X} \text{ with } x_1 \in a. \text{ Thus, any function satisfying } f \in \mathcal{H}_1 \text{ with } f_1(x) = 0 \text{ for } x \in \mathcal{X}_1 \setminus \{a\} \text{ has the property } \|f\|_n = 0.
\end{equation}

Proof (of Lemma \[\text{7}].
First, for each pair \(i_1, i_2 = 1, \ldots, n\) define the set
\[M_{i_1,i_2} := \{x_1 \in \mathcal{X}_1 \mid |X_{i,1} - x_1| < h, (x_1, X_i_{i-1}) \in \mathcal{X} \text{ for } i = i_1, i_2\}\]
if \(X_{i_1,1} \neq X_{i_2,1}\) and \(M_{i_1,i_2} = \emptyset\) otherwise. It is easy to see that \(M_{i_1,i_2}\) is open as an intersection of open sets. Note that on the event \(\mathcal{E}\) we have
\[\bigcup_{i_1,i_2} M_{i_1,i_2} = \mathcal{X}_1. \quad (6)\]

Now, suppose that for some \(f \in \mathcal{H}_{add}\) we have \(\|f\|_n = 0\). We want to show that \(f_0 = 0\) and that \(f_j \equiv 0\) for \(j = 1, \ldots, 2d\). From \(\|f\|_n = 0\) we obtain
\[\left\{f_0 + \sum_{j=1}^{d} f_j(x_j) + \sum_{j=1}^{d} f_j'(x_j) (X_{ij} - x_j)\right\}^2 K_h^{X_{i}}(X_i - x) = 0\]
for \(i = 1, \ldots, n\) and almost all \(x \in \mathcal{X}\). Let \(i_1, i_2 \in \{1, \ldots, n\}\). Then
\[f_0 + f_1(x_1) + \sum_{j=2}^{d} f_j(x_{ij}) + f_{d+1}(x_1)(X_{i1} - x_1) = 0 \quad (7)\]
holds for all \(i = i_1, i_2\) and \(x_1 \in M_{i_1,i_2}\) almost surely. By subtraction of Equation \[\text{7}\] for \(i = i_1\) and \(i = i_2\) we receive
\[f_{d+1}(x_1) = v_1\]
with constant \(v_1 = -\sum_{j=2}^{d} (f_j(X_{i_1,j}) - f_j(X_{i_2,j}))(X_{i_1,1} - X_{i_2,1})\) for \(x_1 \in M_{i_1,i_2}\). Furthermore, by using \[\text{7}\] once again we obtain
\[f_1(x_1) = u_1 + v_1 x_1\]
with another constant \(u_1 \in \mathbb{R}\). Following \[\text{6}\], since \(\mathcal{X}_1\) is connected and the sets \(M_{i_1,i_2}\) are open we can conclude
\[f_{d+1}(x_1) = v_1 \text{ and } f_1(x_1) = u_1 + v_1 x_1\]
for almost all \(x_1 \in \mathcal{X}_1\) since the sets must overlap. Similarly one shows
\[f_j'(x_j) = v_j \text{ and } f_j(x_j) = u_j + v_j x_j\]
for \( j = 2, \ldots, d \) and almost all \( x_j \in X_j \). We conclude that

\[
0 = \| f \|^2_n = \frac{1}{n} \int \sum_{i=1}^{n} \left\{ f_0 + \sum_{j=1}^{d} f_j(x_j) + \sum_{j=1}^{d} f_j(x_j)(X_{ij} - x_j) \right\}^2 K_h^X_i(X_i - x) dx
\]

\[
= \frac{1}{n} \int \sum_{i=1}^{n} \left\{ f_0 + \sum_{j=1}^{d} u_j + \sum_{j=1}^{d} v_j X_{ij} \right\}^2 K_h^X_i(X_i - x) dx
\]

\[
= \left\{ f_0 + \sum_{j=1}^{d} u_j + \sum_{j=1}^{d} v_j X_{ij} \right\}^2.
\]

On the event \( \mathcal{E} \) the covariates \( X_i \) do not lie in a linear subspace of \( \mathbb{R}^d \). This shows \( v_j = 0 \) for \( 1 \leq j \leq d \). Thus \( f_j \equiv 0 \) for \( d + 1 \leq j \leq 2d \) and \( f_j = u_j \) for \( 1 \leq j \leq d \).

Now, \( \int f_j(x_j) \hat{p}_j(x_j) dx_j = 0 \) implies that \( f_j \equiv 0 \) for \( 1 \leq j \leq d \) and \( f_0 = 0 \). This concludes the proof of the lemma.

Existence and uniqueness of \( \hat{m} \) on the event \( \mathcal{E} \) under Assumption (A1) follows immediately from the following lemma.

**Lemma 2.** Make Assumption (A1). Then, on the event \( \mathcal{E} \), for every \( D \subseteq \{0, \ldots, 2d\} \) the linear space \( \sum_{k \in D} \mathcal{H}_k \) is a closed subset of \( \mathcal{H} \). In particular, \( \mathcal{H}_{\text{ada}} \) is closed.

For the proof of this lemma we make use of some propositions introduced below. In the following, we consider sums \( L = L_1 + L_2 \) of closed subspaces \( L_1 \) and \( L_2 \) of a Hilbert space with \( L_1 \cap L_2 = \{0\} \). In this setup, an element \( g \in L \) has a unique decomposition \( g = g_1 + g_2 \) with \( g_1 \in L_1 \) and \( g_2 \in L_2 \). Thus, the projection operator from \( L \) onto \( L_1 \) along \( L_2 \) given by

\[
\Pi_1(L_2) : L \to L_1, \quad \Pi_1(L_2)(g) = g_1
\]

is well defined.

**Proposition 1.** For the sum \( L = L_1 + L_2 \) of two closed subspaces \( L_1 \) and \( L_2 \) of a Hilbert space with \( L_1 \cap L_2 = \{0\} \), the following conditions are equivalent

(i) \( L \) is closed.

(ii) There exists a constant \( c > 0 \) such that for every \( g = g_1 + g_2 \in L \) with \( g_1 \in L_1 \) and \( g_2 \in L_2 \) we have

\[
\| g \| \geq c \max\{\| g_1 \|, \| g_2 \|\}.
\]

(iii) The projection operator \( \Pi_1(L_2) \) from \( L \) onto \( L_1 \) along \( L_2 \) is bounded.
(iv) The gap from $L_1$ to $L_2$ is greater than zero, i.e.,

$$\gamma(L_1, L_2) := \inf_{g_1 \in L_1} \frac{\text{dist}(g_1, L_2)}{\|g_1\|} > 0,$$

where $\text{dist}(f, V) := \inf_{h \in V} \|f - h\|$ with the convention $0/0 = 1$.

Remark 2. A version of Proposition 1 is also true if $L_1 \cap L_2 \neq \{0\}$. In this case, the quantities involved need to be identified as objects in the quotient space $L/(L_1 \cap L_2)$.

**Proposition 2.** The sum $L = L_1 + L_2$ of two closed subspaces $L_1$ and $L_2$ of a Hilbert space with $L_1 \cap L_2 = \{0\}$ is closed if the orthogonal projection of $L_2$ on $L_1$ is compact.

The proofs of Propositions 1 and 2 can be reconstructed from (Bickel et al., 1993, A.4 Proposition 2), (Kato, 2013, Chapter 4, Theorem 4.2) and Kober (1940). For completeness, we have added proofs of the propositions in Appendix B.

We now come to the proof of Lemma 2.

**Proof (of Lemma 2).** First note that the spaces $H_k$ are closed for $k = 0, \ldots, 2d$.

We show that $H_k + H_{k'}$ is closed for $1 \leq k \leq d$. Consider $R = \min M$ where

$$M := \{ r \geq 0 \mid (\hat{p}_k^*(x_k))^2 \leq r \hat{p}_k(x_k) \hat{p}_k^{**}(x_k) \text{ for all } x_k \in \mathcal{X}_k \text{ and } 1 \leq k \leq d \}$$

and

$$I_{x_k} := \left\{ i \in \{1, \ldots, n\} \mid \int_{u \in X_{-k}(x_k)} K_h^{X_i}(X_i - u) du_{-k} > 0 \right\}$$

for $x_k \in \mathcal{X}_k$. By the Cauchy-Schwarz inequality we have $(\hat{p}_k^*(x_k))^2 \leq \hat{p}_k(x_k) \hat{p}_k^{**}(x_k)$ for $x_k \in \mathcal{X}_k$ and $1 \leq k \leq d$. This implies $R \leq 1$. Now, equality in the inequality only holds if $X_{ik} - x_k$ does not depend on $i \in I_{x_k}$. On the event $\mathcal{E}$ for $x_k \in \mathcal{X}_k$ there exist $1 \leq i_1, i_2 \leq n$ with $|x_k - X_{i,k}| < h$ for $i = i_1, i_2$ and $X_{i_1,k} \neq X_{i_2,k}$. Thus, $X_{ik} - x_k$ depends on $i$ for $i \in I_{x_k}$ and the strict inequality holds for all $x_k$. Furthermore, because the kernel function $k$ is continuous, we have that $\hat{p}_k$, $\hat{p}_k^*$ and $\hat{p}_k^{**}$ are continuous. Together with the compactness of $\mathcal{X}_k$ this implies that $R < 1$ on the event $\mathcal{E}$.

Now let $f \in H_k$ and $g \in H_{k'}$ for some $1 \leq k \leq d$. We will show

$$\|f + g\|^2_n \geq (1 - R)(\|f\|^2_n + \|g\|^2_n). \quad (9)$$
By application of Proposition 1, this immediately implies that $\mathcal{H}_k + \mathcal{H}_{k'}$ is closed. For a proof of (9) note that

$$\|f + g\|_n^2 = n^{-1} \sum_{i=1}^{n} (f_k(x_k) + g_0 + (X_{ik} - x_k)g_{k'}(x_k))^2 K_n^k(X_i - x)dx$$

$$= \int (f_k(x_k) + g_0)^2 \hat{p}_k(x_k)dx_k + 2 \int (f_k(x_k) + g_0)g_{k'}(x_k)\hat{p}_k^*(x_k)dx_k$$

$$+ \int g_{k'}(x_k)^2 \hat{p}_k^*(x_k)dx_k$$

$$\geq \int (f_k(x_k) + g_0)^2 \hat{p}_k(x_k)dx_k + \int g_{k'}(x_k)^2 \hat{p}_k^*(x_k)dx_k$$

$$- 2R \int |f_k(x_k) + g_0|g_{k'}(x_k)(\hat{p}_k(x_k)\hat{p}_k^*(x_k))^{1/2}dx_k$$

$$\geq \int (f_k(x_k) + g_0)^2 \hat{p}_k(x_k)dx_k + \int g_{k'}(x_k)^2 \hat{p}_k^*(x_k)dx_k$$

$$- 2R \left( \int (f_k(x_k) + g_0)^2 \hat{p}_k(x_k)dx_k \right)^{1/2} \left( \int g_{k'}(x_k)^2 \hat{p}_k^*(x_k)dx_k \right)^{1/2}$$

$$\geq (1 - R) \int (f_k(x_k) + g_0)^2 \hat{p}_k(x_k)dx_k + (1 - R) \int g_{k'}(x_k)^2 \hat{p}_k^*(x_k)dx_k$$

$$= (1 - R) \left( \int f_k(x_k)^2 \hat{p}_k(x_k)dx_k + g_0^2 + \int g_{k'}(x_k)^2 \hat{p}_k^*(x_k)dx_k \right)$$

$$\geq (1 - R)(\|f\|_n^2 + \|g\|_n^2),$$

where in the second to last row, we used that $f \in \mathcal{H}_k$. This concludes the proof of (9).

Note that the statement of the lemma is equivalent to the following statement: For $D_1, D_2 \subseteq \{1, \ldots, d\}$ and $\delta \in \{0, 1\}$ the space $\delta \mathcal{H}_0 + \sum_{k \in D_1} \mathcal{H}_k + \sum_{k \in D_2} \mathcal{H}_{k'}$ is closed. We show this inductively over the number of elements $s = |D_2 \cap D_1|$ of $D_1 \cap D_2$.

For the case $s = 0$, note that for $D_1 \cap D_2 = \emptyset$, the space $\delta \mathcal{H}_0 + \sum_{k \in D_1} \mathcal{H}_k + \sum_{k \in D_2} \mathcal{H}_{k'}$ is closed, which can be shown with similar but simpler arguments than the ones used below. Now let $s \geq 1, \delta \in \{0, 1\}, D_1, D_2 \subseteq \{1, \ldots, d\}$ with $|D_2 \cap D_1| = s - 1$ and assume $L_2 = \delta \mathcal{H}_0 + \sum_{j \in D_1 \cup \{k\}} \mathcal{H}_j + \sum_{j \in D_2 \cup \{k\}} \mathcal{H}_{j'}$ is closed. Without loss of generality, let $k \in \{1, \ldots, d\}\setminus(D_1 \cup D_2)$. We will argue that on the event $E$ the orthogonal projection of $L_2$ on $L_1 = \mathcal{H}_k + \mathcal{H}_{k'}$ is Hilbert-Schmidt, noting that a Hilbert-Schmidt operator is compact. Using Proposition 2, since $L_1$ and $L_2$ are closed, this implies that $L = \delta \mathcal{H}_0 + \sum_{j \in D_1 \cup \{k\}} \mathcal{H}_j + \sum_{j \in D_2 \cup \{k\}} \mathcal{H}_{j'}$ is closed which completes the inductive argument.

For an element $f \in L_2$ with decomposition $f = f_0 + \sum_{j \in D_1} f_j + \sum_{j \in D_2} f_j'$ the projection onto $L_1 + \mathcal{H}_0$ is given by univariate functions $g_k, g_{k'}$ and $g_0 \in \mathbb{R}$ which
The algorithm is used to approximate $\hat{\mathcal{O}}$ operator. This concludes the proof.

We prove the algorithm for arbitrary starting values, i.e. we can set the $\hat{\mathcal{O}}$.

Lemma 3. Make Assumption (A1). Then, on the event $\mathcal{E}$, for Algorithm 1 and all choices of starting values $\hat{\mathcal{O}}[0] \in \mathcal{H}_{add}$ we have

$$\|\hat{\mathcal{O}}[r] - \hat{\mathcal{O}}\|_n \leq V^r\|\hat{\mathcal{O}}[0] - \hat{\mathcal{O}}\|_n,$$

where $0 \leq V = 1 - \prod_{k=0}^{2d-1} \gamma^2(\mathcal{H}_k, \mathcal{H}_{k+1} + \cdots + \mathcal{H}_{2d}) < 1$ is a random variable depending on the observations.

Remark 3. On the event $\mathcal{E}$, the algorithm converges with a geometric rate where in every iteration step the distance to the limiting value, $\hat{\mathcal{O}}$, is reduced by a factor smaller or equal to $V$. If the columns of the design matrix $X$ are orthogonal,
$V$ will be close to zero and if they are highly correlated, $V$ will be close to 1. The variable $V$ depends on $n$ and is random. Under additional assumptions, as stated in the next section, one can show that with probability tending to one, $V$ is bounded by a constant smaller than 1.

Proof (of Lemma 3). For a subspace $V \subseteq H_{add}$ we denote by $P_V$ the orthogonal projection onto $V$. For $k = 0, \ldots, 2d$ let $Q_k := P_{H_k^0} = 1 - P_k$ be the projection onto the orthogonal complement $H_k^\perp$ of $H_k$. The idea is to show the following statements.

(i) $Y - \hat{m}^{[r]} = (Q_{2d} \ldots Q_0)(Y - \hat{m}^{[0]})$,
(ii) $(Q_{2d} \ldots Q_0)'(Y - \hat{m}) = Y - \hat{m}$,

This then implies

$$\left\| \hat{m}^{[r]} - \hat{m} \right\|_n = \left\| \hat{m}^{[r]} - Y + Y - \hat{m} \right\|_n = \left\| (Q_{2d} \ldots Q_0)'(\hat{m}^{[0]} - \hat{m}) \right\|_n.$$ 

The proof is concluded by showing

$$\left\| Q_{2d} \ldots Q_0 g \right\|_n^2 \leq \left( 1 - \prod_{k=0}^{2d-1} \gamma^2(H_k, H_{k+1} + \cdots + H_{2d}) \right) \left\| g \right\|_n^2 \tag{10}$$

for all $g \in H_{add}$. Note that $0 \leq V := 1 - \prod_{k=0}^{2d-1} \gamma^2(H_k, H_{k+1} + \cdots + H_{2d}) < 1$ by Lemma 2 and Proposition 4.

For (i), observe that for all $r \geq 1$ and $k = 0, \ldots, 2d$ we have

$$Y - \hat{m}^{[r-1]} - \cdots - \hat{m}^{[r-1]} - \hat{m}_k^{[r]} - \cdots - \hat{m}_2^{[r]} = (1 - P_k)(Y - \hat{m}^{[r-1]} - \cdots - \hat{m}_k^{[r-1]} - \hat{m}_k^{[r]} - \cdots - \hat{m}_2^{[r]})$$

$$= (1 - P_k)(Y - \hat{m}_0^{[r-1]} - \cdots - \hat{m}_k^{[r-1]} - \hat{m}_k^{[r]} - \cdots - \hat{m}_2^{[r]})$$

$$= Q_k(Y - \hat{m}_0^{[r-1]} - \cdots - \hat{m}_k^{[r-1]} - \hat{m}_k^{[r]} - \cdots - \hat{m}_2^{[r]}).$$

The statement follows inductively by beginning with the case $r = 1, k = 0$. Secondly, (ii) follows from

$$Q_r \ldots Q_0(Y - \hat{m}) = Q_r \ldots Q_0 P_{H_0^+ \cap \cdots \cap H_{2d}^+}(Y) = P_{H_0^+ \cap \cdots \cap H_{2d}^+}(Y) = Y - \hat{m}.$$ 

It remains to show the inequality in (10).

For $0 \leq k \leq 2d$ define $N_k := H_k + \cdots + H_{2d}$. We prove $\left\| Q_{2d} \ldots Q_j g \right\|_n^2 \leq (1 - \prod_{k=j}^{2d-1} \gamma^2(H_k, H_{k+1} + \cdots + H_{2d})) \left\| g \right\|_n^2$ for all $g \in H_{add}$ and $0 \leq j \leq 2d$ using an inductive argument.

The case $j = 2d$ is trivial. For $0 \leq j < 2d$ and any $g \in H_{add}$ let $g'_j := Q_j g = g' + g''$ with $g' := P_{N_{j+1}}(g)$ and $g'' := P_{N_{j+1}}(g)$. Then, by orthogonality, we have

$$\left\| Q_{2d} \ldots Q_{j+1} g'_j \right\|_n^2 = \left\| g' + Q_{2d} \ldots Q_{j+1} g'' \right\|_n^2 = \left\| g' \right\|_n^2 + \left\| Q_{2d} \ldots Q_{j+1} g'' \right\|_n^2.$$
Induction gives
\[
\|Q_{2d} \cdots Q_{j+1}g''\|_n^2 \leq \left( 1 - \prod_{k=j+1}^{2d-1} \gamma^2(\mathcal{H}_k, \mathcal{H}_{k+1} + \cdots + \mathcal{H}_{2d}) \right) \left( \|g_j^\perp\|_n^2 - \|g''\|_n^2 \right)
\]
which implies
\[
\|Q_{2d} \cdots Q_{j+1}g_j^\perp\|_n^2 \leq \left( 1 - \prod_{k=j+1}^{2d-1} \gamma^2(\mathcal{H}_k, \mathcal{H}_{k+1} + \cdots + \mathcal{H}_{2d}) \right) \|g_j^\perp\|_n^2
\]
\[+ \prod_{k=j+1}^{2d-1} \gamma^2(\mathcal{H}_k, \mathcal{H}_{k+1} + \cdots + \mathcal{H}_{2d}) \|g''\|_n^2.
\]

By Lemma 2 and Lemma 9 we have
\[
\|g''\|_n^2 \leq \|P_{N_j^\perp}Q_1\|_n^2 = \|P_{N_j^\perp}P_{H_j}\|_n^2 = 1 - \gamma^2(\mathcal{H}_j, \mathcal{H}_{j+1} + \cdots + \mathcal{H}_{2d}).
\]
This concludes the proof by noting that \(\|g_j^\perp\|_n \leq \|g\|_n\).

4 Asymptotic properties of the estimator

In this section we will discuss asymptotic properties of the local linear smooth backfitting estimator. For simplicity we consider only the case that \(X\) is a product of intervals \(X_j = (a_j, b_j) \subset \mathbb{R}\).

We make the following additional assumptions:

(A2) The observations \((Y_i, X_i)\) are i.i.d. and the covariates \(X_i\) have one-dimensional marginal densities \(p_j\) which are strictly positive on \([a_k, b_k]\). The two-dimensional marginal densities \(p_{jk}\) of \((X_{i,j}, X_{i,k})\) are continuous on their support \([a_j, b_j] \times [a_k, b_k]\).

(A3) It holds
\[
Y_i = m_0 + m_1(X_{i1}) + \cdots + m_d(X_{id}) + \varepsilon_i,
\]
for twice continuously differentiable functions \(m_j : X_j \to \mathbb{R}\) with \(\int m_j(x_j) p_j(x_j)dx_j = 0\). The error variables \(\varepsilon_i\) satisfy \(E[\varepsilon_i|X_i] = 0\) and
\[
\sup_{x \in X} E[|\varepsilon_i|^{5/2} |X_i = x] < \infty.
\]

(A4) There exist constants \(c_1, \ldots, c_d > 0\) with \(n^{1/5}h_j \to c_j\) for \(n \to \infty\). To simplify notation we assume that \(h_1 = \cdots = h_d\). In abuse of notation we write \(h\) for \(h_j\) and \(c_h\) for \(c_j\).

From now on we will write \(\hat{m}^n = (\hat{m}_0^n, \hat{m}_1^n, \ldots, \hat{m}_d^n)\) for the estimator \(\hat{m}\) to indicate its dependence on the sample size \(n\). The following theorem states an asymptotic expansion for the components \(\hat{m}_1^n, \ldots, \hat{m}_d^n\). Later in this section we will state some lemmas which will be used to prove the result.
Theorem 1. Make assumptions (A1) – (A4). Then

\[ \left| \hat{m}^n_j(x_j) - m_j(x_j) - \left( \beta_j(x_j) - \int \beta_j(u_j)p_j(u_j)du_j \right) - v_j(x_j) \right| = o_P(h^2 + \{nh\}^{-1/2}) = o_P(n^{-2/5}), \]

holds uniformly over \( 1 \leq j \leq d \) and \( a_j \leq x_j \leq b_j \), where \( v_j \) is a stochastic variance term

\[ v_j(x_j) = \frac{1}{n} \sum_{i=1}^{n} h^{-1} k(h^{-1}(X_{ij} - x_j)\varepsilon_i) = O_P(\{nh\}^{-1/2}) \]

and \( \beta_j \) is a deterministic bias term

\[ \beta_j(x_j) = \frac{1}{2} h^2 m''_j(x_j) \frac{b_{j,2}(x_j)^2 - b_{j,1}(x_j) b_{j,3}(x_j)}{b_{j,0}(x_j)b_{j,2}(x_j) - b_{j,1}(x_j)^2} = O(h^2), \]

with \( b_{j,l}(x_j) = \int_X k(h^{-1}(u_j - x_j))(u_j - x_j)^l h^{-1} b_j(u_j)^{-1} du_j \) and \( b_j(x_j) = \int_X k(h^{-1}(x_j - w_j))h^{-1} dw_j \) for \( 0 \leq l \leq 2 \).

The expansion for \( \hat{m}^n_j \) stated in the theorem neither depends on \( d \) nor on functions \( m_k \) \( (k \neq j) \). In particular, this shows that the same expansion holds for the local linear estimator \( \hat{m}^n_j \) in the oracle model where the functions \( m_k \) \( (k \neq j) \) are known. More precisely, in the oracle model one observes i.i.d. observations \( (Y_i^*, X_{ij}) \) with

\[ Y_i^* = m_j(X_{ij}) + \varepsilon_i, \quad Y_i^* = Y_i - \sum_{k \neq j} m_k(X_{ik}), \quad (12) \]

and the local linear estimator \( \hat{m}^n_j \) is defined as the second component that minimises the criterion

\[ \bar{S}(f_0, f_j, f_j^{(1)}) = \sum_{i=1}^{n} \int_X \left( Y_i^* - f_0 - f_j(x_j) - f_j^{(1)}(X_{ij} - x_j) \right)^2 \times \kappa_h(X_{ij} - x_j)dx_j \]

with boundary corrected kernel

\[ k^n_h(u - x) = \frac{\kappa \left( \frac{u - x}{h} \right)}{\int_{X_j} \kappa \left( \frac{v - x}{h} \right)dv}. \]

We conclude that the local linear smooth backfitting estimator \( \hat{m}_j \) is asymptotically equivalent to the local linear estimator \( \hat{m}^n_j \) in the oracle model. We formulate this asymptotic equivalence as a first corollary of Theorem 1. In particular, it implies that the estimators have the same first order asymptotic properties.
Corollary 1. Make assumptions (A1) – (A4). Then it holds uniformly over $1 \leq k \leq d$ and $a_j \leq x_j \leq b_j$ that

$$\left| \hat{m}^n_j(x_j) - \hat{m}^n_j(x_j) \right| = o_P(h^2).$$

For $x_j \in (a_j + 2h, b_j - 2h)$ the bias term $\beta_j$ simplifies and we have that

$$\beta_j(x_j) = h^2 \frac{1}{2} m''_j(x_j) \int k(v)^2 dv.$$ 

This implies the following corollary of Theorem 1.

Corollary 2. Make assumptions (A1) – (A4). Then it holds uniformly over $1 \leq k \leq d$ and $a_j + 2h \leq x_j \leq b_j - 2h$ that

$$\left| \hat{m}^n_j(x_j) - m_j(x_j) - \frac{1}{2} \left( m''_j(x_j) - \int m''_j(u_j)p_j(u_j)du_j \right) h^2 \int k(v)^2 dv \right| = o_P(h^2).$$

Corollary 2 can be used to derive the asymptotic distribution of $\hat{m}^n_j(x_j)$ for an $x_j \in (a_j, b_j)$. Under the additional assumption that $\sigma^2_j(u) = \mathbb{E}[\epsilon_i^2 | X_{ij} = u]$ is continuous in $u = x_j$ we get under (A1) – (A4) that $n^{2/5}(\hat{m}^n_j(x_j) - m_j(x_j))$ has an asymptotic normal distribution with mean $c_n \frac{1}{2} \left( m''_j(x_j) - \int m''_j(u_j)p_j(u_j)du_j \right) \int k(v)^2 dv$ and variance $c_n^{-1}\sigma^2_j(x_j)p_j^{-1}(x_j) \int k(v)^2 dv$. This is equal to the asymptotic limit distribution of the classical local linear estimator in the oracle model in accordance with Corollary 1.

Now, we come to the proof of Theorem 1.

First, we define the operator $S_n = (S_{n,0}, S_{n,1}, \ldots, S_{n,2d}) : G^n \rightarrow G^n$ with

$$G^n = \{(g_0, \ldots, g_{2d}) | g_0 \in \mathbb{R}, g_l, g_{2d-l} \in L_2(p_l) \text{ with } P_0(g_l) = 0 \text{ for } l = 1, \ldots, d\},$$

where $S_{n,k}$ maps $g = (g_0, \ldots, g_{2d}) \in G^n$ to $f_k$ with

$$f_0 = P_0 \left( \sum_{1 \leq l \leq 2d} g_l \right) = P_0 \left( \sum_{d+1 \leq l \leq 2d} g_l \right) \in \mathbb{R},$$

$$f_k(x_k) = P_k \left( \sum_{0 \leq l \leq 2d, l \neq k} g_l \right)(x),$$

for $1 \leq k \leq 2d$. With this notation we can rewrite the backfitting equation (5) as

$$\hat{m}^n(Y) = \hat{m}^n + S_n \hat{m}^n,$$

(13)
where for \( z \in \mathbb{R}^n \) we define \( \bar{m}_n(0) = \bar{z} = \frac{1}{n} \sum_{i=1}^n z_i \) and for \( 1 \leq j \leq d \),

\[
\bar{m}_n^j(z)(x_j) = \bar{p}_j(x_j)^{-1} \frac{1}{n} \sum_{i=1}^n (z_i - \bar{z}) K_h^{X_i}(X_i - x) dx_j,
\]

\[
\bar{m}_n^j(x_j) = \bar{p}_j^*(x_j)^{-1} \frac{1}{n} \sum_{i=1}^n (X_{ij} - x_j) z_i \int K_h^{X_i}(X_i - x) dx_j.
\]

The following lemma shows that \( I + S_n \) is invertible on the event \( E \). Here we denote the identity operator by \( I \).

**Lemma 4.** On the event \( E \) the operator \( I + S_n : G^n \to G^n \) is invertible.

**Proof.** Suppose that for some \( g \in G^n \) it holds that \((I + S_n)(g) = 0\). We have to show that this implies \( g = 0 \).

For the proof of this claim note that \( g_k + S_{n,k}(g) \) is the orthogonal projection of \( \sum_{j=0}^{2d} g_j \) onto \( H_k \). Furthermore, we have that \( g_k \) is an element of \( H_k \). This gives that

\[
\left\langle g_k, \sum_{j=0}^{2d} g_j \right\rangle_n = 0.
\]

Summing over \( k \) gives

\[
\left\langle \sum_{j=0}^{2d} g_j, \sum_{j=0}^{2d} g_j \right\rangle_n = 0.
\]

According to Lemma 4 on the event \( E \) we have \( g_0 = 0 \) and \( g_j \equiv 0 \) for \( j = 1, \ldots, 2d \). This concludes the proof of the lemma.

One can show that under conditions (A1) – (A4) the probability of the event \( E \) converges to one. Note that we have assumed that \( \mathcal{X} = \prod_{j=1}^d \mathcal{X}_j \). We conclude that under (A1) – (A4) \( I + S_n \) is invertible with probability tending to one.

Thus we have that with probability tending to one

\[
\hat{m}^n - m - \bar{m}^n(\varepsilon) - \beta_n + \Delta_n m + \Delta_n \beta_n
\]

\[
= (I + S_n)^{-1}(I + S_n)(\hat{m}^n - m - \bar{m}^n(\varepsilon) - \beta_n + \Delta_n m + \Delta_n \beta_n),
\]

(14)

where \( m \) has components \( m_0, \ldots, m_{2d} \) with \( m_0, \ldots, m_d \) as in (11) and with \( m_j = m_j' \) for \( 1 \leq j \leq d \). Furthermore, \( \beta(x) \) has components \( \beta_0 = 0, \beta_j(x) \) and

\[
\beta_j'(x_j) = \frac{1}{2} m_j''(x_j) \frac{b_{j,0}(x_j)b_{j,3}(x_j) - b_{j,1}(x_j)b_{j,2}(x_j)}{b_{j,0}(x_j)b_{j,2}(x_j) - b_{j,1}(x_j)^2} h
\]
for \( j = 1, \ldots, d \) with \( b_j(x_j) \) defined above. Additionally, the norming constants are given by

\[
(\Delta_n\beta)_j = \int \beta_j(x_j) \hat{p}_j(x_j) \, dx_j,
\]
\[
(\Delta_n m)_j = \int m_j(x_j) \hat{p}_j(x_j) \, dx_j,
\]
\[
(\Delta_n \beta)'_j = (\Delta_n m)'_j = 0 \quad \text{for} \; j = 1, \ldots, d,
\]
\[
(\Delta_n \beta)_0 = \sum_{j=1}^{d} \int \beta_j(x_j) \hat{p}_j^*(x_j) \, dx_j,
\]
\[
(\Delta_n m)_0 = \sum_{j=1}^{d} \int m_j(x_j) \hat{p}_j^*(x_j) \, dx_j.
\]

One can verify that for \( a_j + 2h_j \leq x_j \leq b_j - 2h_j \) one has \( \beta_j(x_j) = o_P(h) \). We have already seen that

\[
\beta_j(x_j) = \frac{1}{2} m_j''(x_j) \int k(v)^2 \, dv + o_P(h^2)
\]
holds for such \( x_j \).

For the statement of Theorem 1 we have to show that for \( 1 \leq j \leq d \) the \( j \)-th component on the left hand side of equation (14) is of order \( o_P(h^2) \) uniformly for \( a_j + 2h \leq x_j \leq b_j - 2h \).

For a proof of this claim we first analyze the term

\[
D_n = (I + S_n)(\hat{m}^n - m - \bar{m}^n(\varepsilon) - \beta_n + \Delta_n m + \Delta_n \beta_n)
\]
\[
= \hat{m}^n(Y) - (I + S_n)(m + \bar{m}^n(\varepsilon) + \beta_n - \Delta_n m - \Delta_n \beta_n).
\]

For this sake we split the term \( \hat{m}^n(Y) \) into the sum of a stochastic variance term and a deterministic expectation term:

\[
\hat{m}^n(Y) = \bar{m}^n(\varepsilon) + \sum_{j=0}^{d} \bar{m}^n(\mu_{n,j}),
\]
where

\[
\varepsilon = Y - \sum_{j=0}^{d} \mu_{n,j},
\]
\[
\mu_{n,j} = (m_j(X_{ij}))_{i=1,...,n} \quad \text{for} \; j = 1, \ldots, d,
\]
\[
\mu_{n,0} = (m_0)_{i=1,...,n}.
\]

We write \( D_n = D_n^\beta + D_n^\varepsilon \), with

\[
D_n^\beta = \sum_{j=0}^{d} \bar{m}^n(\mu_{n,j}) - (I + S_n)(m + \beta_n - \Delta_n m - \Delta_n \beta_n),
\]
\[
D_n^\varepsilon = S_n(\bar{m}^n(\varepsilon)).
\]

The following lemma treats the conditional expectation term \( D_n^\beta \).
Lemma 5. Assume (A1) – (A4). It holds $D_{n,0}^3 = o_p(h^2)$ and
\[
\sup_{x_k \in X_k} |D_{n,k}^3(x_k)| = \begin{cases} o_p(h^2) & \text{for } 1 \leq k \leq d, \\ o_p(h) & \text{for } d + 1 \leq k \leq 2d. \end{cases}
\]

Proof. The lemma follows by application of lengthy calculations using second order Taylor expansions for $m_j(X_j)$ and by application of laws of large numbers.

We now turn to the variance term.

Lemma 6. Assume (A1) – (A4). It holds $D_{n,0}^\varepsilon = o_p(h^2)$ and
\[
\sup_{x_k \in X_k} |D_{n,k}^\varepsilon(x_k)| = \begin{cases} o_p(h^2) & \text{for } 1 \leq k \leq d, \\ o_p(h) & \text{for } d + 1 \leq k \leq 2d. \end{cases}
\]

Proof. One can easily check that $D_{n,k}^\varepsilon(x_k)$ consists of weighted sums of $\varepsilon_i$ where the weights are of the same order for all $1 \leq i \leq n$. For fixed $x_k$ the sums are of order $O_P(n^{-1/2})$ for $1 \leq k \leq d$ and of order $O_P(h^{-1}n^{-1/2})$ for $d + 1 \leq k \leq 2d$.

Using the conditional moment conditions on $\varepsilon_i$ in Assumption (A3) we get the uniform rates stated in the lemma.

It remains to study the behaviour of $(I + S_n)^{-1}D_n^\varepsilon$ and $(I + S_n)^{-1}D_n^3$. We will use a small transformation of $S_n$ here which is better suitable for an inversion.

Define the following $2 \times 2$ matrix $A_{n,k}(x)$ by
\[
A_{n,k}(x) = \frac{1}{\hat{p}_k \hat{p}_l^{**}} \left( \frac{\hat{p}_k^* \hat{p}_l^* \hat{p}_k \hat{p}_l^{**}}{\hat{p}_k \hat{p}_l^{**}} \right) (x_k).
\]

Furthermore, define the $2d \times 2d$ matrix $A_n(x)$ where the elements with indices $(k, k), (k, k'), (k', k), (k', k')$ are equal to the elements of $A_{n,k}(x)$ with indices $(1, 1), (1, 2), (2, 1), (2, 2)$. We now define $\hat{S}_n$ by the equation $I + \hat{S}_n = A_n(I + S_n)$.

Below we will make use of the fact that $\hat{S}_n$ is of the form
\[
\hat{S}_{n,k,m}(x) = \sum_{l \notin \{k,k'\}} \int q_{k,l}(x_k,u)m_l(u)du + \sum_{l \in \{k,k'\}} \int q_l(u)m_l(u)du, \tag{17}
\]
\[
\hat{S}_{n,k,m}(x) = \sum_{l \notin \{k,k'\}} \int q_{k',l}(x_k,u)m_l(u)du + \sum_{l \in \{k,k'\}} \int q_l(u)m_l(u)du \tag{18}
\]

for $1 \leq k \leq d$ with some random functions $q_{k,l}, q_l$ which fulfill that $\int q_{k,l}(x_k,u)^2du$ and $\int q_l(u)^2du$ are of order $O_P(1)$ uniformly over $1 \leq k, l \leq 2d$ and $x_k$.

Note that we need $\hat{S}_n$ because $S_n$ can not be written in the form of (17) and (18). The operator $\hat{S}_n$ differs from $S_n$ in the $h$-neighbourhood of the boundary by terms of order $h^2$. Otherwise the difference is of order $o_p(h^2)$. Outside of the $h$-neighbourhood of the boundary, for $n \to \infty$, the matrix $A_n(x)$ converges to the identity matrix. Thus $\hat{S}_n$ is a second order modification of $S_n$ with the advantage of having (17)-(18).
For our further discussion we now introduce the space $G^0$ of tuples $f = (f_0, f_1, \ldots, f_{2d})$ with $f_0 = 0$ and $f_k, f_k': X_k \to \mathbb{R}$ with $\int f_k(x_k)p_k(x_k)dx_k = 0$ and endow it with the norm $\|f\|^2 = \sum_{k=1}^{d} (f_k(x_k)^2 + f_k'(x_k)^2)p_k(x_k)dx_k$. The next lemma shows that the norm of $H_n(I + S_n)^{-1}D_n^x$ and $H_n(I + S_n)^{-1}D_n^\beta$ is of order $o_P(h^2)$. Here $H_n$ is a diagonal matrix where the first $d + 1$ diagonal elements equal 1. The remaining elements are equal to $h$.

**Lemma 7.** Assume (A1) – (A4). Then it holds that $\|H_n(I + S_n)^{-1}D_n^x\| = \|H_n(I + \tilde{Q}_n)^{-1}A_nD_n^x\| = o_P(h^2)$ for $D_n^x = D_n^\epsilon$ and $D_n^\beta = D_n^\beta$.

**Proof.** Define $\bar{D}_n^\epsilon$ and $\bar{D}_n^\beta$ by $\bar{D}_n^\epsilon(x_k) = D_n^\epsilon(x_k) - \int D_n^\epsilon(u_k)p_k(u_k)du_k$ and $\bar{D}_n^\beta(x_k) = D_n^\beta(x_k) - \int D_n^\beta(u_k)p_k(u_k)du_k$ for $1 \leq k \leq d$ and $\bar{D}_n^\beta = D_n^\beta$ otherwise. It can be checked that it suffices to prove the lemma with $D_n^\epsilon$ and $D_n^\beta$ replaced by $\bar{D}_n^\epsilon$ and $\bar{D}_n^\beta$. Note that $\bar{D}_n^\epsilon$ and $\bar{D}_n^\beta$ are elements of $G^0$. For the proof of this claim we compare the operator $S_n$ with the operator $S_0$ defined by $S_{0,0}(g) = 0$, $S_{0,k}(g)(x_k) = 0$ and

$$S_{0,k}(g)(x_k) = \sum_{j \neq k} \int X_j g_j(u_j)\frac{p_{j,k}(u_j, x_k)}{p_k(x_k)}du_j$$

for $1 \leq k \leq d$. By standard kernel smoothing theory one can show that

$$\sup_{g \in G^0, \|g\| \leq 1} \|S_0 - H_n\tilde{S}_nH_n^{-1}g\| = o_P(1).$$

For the proof of this claim one makes use of the fact that non-vanishing differences in the $h$-neighbourhood of the boundary are asymptotically negligible in the calculation of the norm because the size of the neighbourhood converges to zero.

In the next lemma we will show that $I + S_0$ has a bounded inverse. This implies the statement of the lemma by applying the following expansion:

$$(I + H_n\tilde{S}_nH_n^{-1})^{-1} - (I + S_0)^{-1} = (I + S_0)^{-1}((I + S_0)(I + H_n\tilde{S}_nH_n^{-1})^{-1} - I)$$

$$= (I + S_0)^{-1}(((I + H_n\tilde{S}_nH_n^{-1})(I + S_0)^{-1} - I)$$

$$= (I + S_0)^{-1}((I + H_n\tilde{S}_nH_n^{-1} - S_0)(I + S_0)^{-1} - I)$$

$$= (I + S_0)^{-1}\sum_{j=1}^{\infty} (I + (-1)^j (H_n\tilde{S}_nH_n^{-1} - S_0)(I + S_0)^{-1}).$$

This shows the lemma because of $H_n(I + \tilde{Q}_n)^{-1}A_nD_n^\epsilon = H_n(I + \tilde{Q}_n)^{-1}H_n^{-1}H_nA_nD_n^\epsilon = (I + H_n\tilde{S}_nH_n^{-1} - S_0)(I + S_0)^{-1}$.

**Lemma 8.** Assume (A1) – (A4). The operator $I + S_0 : G^0 \to G^0$ is bijective and has a bounded inverse.
Proof. For a proof of this claim it suffices to show that the operator $I + S_*$ : $G^* \to G^*$ is bijective and has a bounded inverse where $G^*$ is the space of tuples $f = (f_1, \ldots, f_d)$ where $f_k : X_j \to \mathbb{R}$ with $\int f_k(x_k)dx_k = 0$ with norm $\|f\|^2 = \sum_{k=1}^d f_k(x_k)^2 p_k(x_k)dx_k$ and
\[
S_{*,k}(g)(x_k) = \sum_{j \neq k} \int_{X_j} g_j(u_j) \frac{p_{j,k}(u_j, x_k)}{p_k(x_k)} du_j
\]
for $1 \leq k \leq d$. We will apply the bounded inverse theorem. For an application of this theorem we have to show that $I + S_*$ is bounded and bijective. It can easily be seen that the operator is bounded. It remains to show that it is surjective.

We will show that

(i) $(I + S_*)g^n \to 0$ for a sequence $g^n \in G^*$ implies that $g^n \to 0$.

(ii) $\int g_k(I + S_*)r(x_k)p_k(x_k)dx_k = 0$ for all $g \in G^*$ implies that $r = 0$.

Note that (i) implies that $G^{**} = \{(I + S_*)g : g \in G^*\}$ is a closed subset of $G^*$. To see this suppose that $(I + S_*)g^n \to g$ for $g, g^n \in G^*$. Then (i) implies that $g^n$ is a Cauchy sequence and thus $g^n$ has a limit in $G^*$ which implies that $(I + S_*)g^n$ has a limit in $G^{**}$. Thus $G^{**}$ is closed.

From (ii) we conclude that the orthogonal complement of $G^{**}$ is equal to $\{0\}$. Thus the closure of $G^{**}$ is equal to $G^*$. This shows that $G^* = G^{**}$ because $G^{**}$ is closed. We conclude that $(I + S_*)$ is surjective.

It remains to show (i) and (ii). For a proof of (i) note that $(I + S_*)g^n \to 0$ implies that
\[
\int g_k^n(x_k)(I + S_*)k g^n(x_k)p_k(x_k)dx_k \to 0
\]
which shows
\[
\sum_{k=1}^d \int g_k^n(x_k)^2 p_k(x_k)dx_k + \sum_{k \neq j} g_k^n(x_k)p_{kj}(x_k, x_j)g_j(x_j)dx_k dx_j \to 0.
\]
Thus we have
\[
E[\left\{ \sum_{k=1}^d g_k(X_{ik}) \right\}^2] \to 0.
\]
By application of Proposition 1 (ii) we get that $\max_{1 \leq k \leq d} E[|g_k(X_{ik})|^2] \to 0$, which shows (i).

Claim (ii) can be seen by a similar argument. Note that $\int g_k(I + S_*)r(x_k)dx_k = 0$ for all $g \in G^*$ implies that $\int r_k(I + S_*)r(x_k)p_k(x_k)dx_k = 0$.

We now apply the results stated in the lemma for the final proof of Theorem 1.

Proof (of Theorem 2). From 14 and Lemma 4 we know that the L2 norm of $\tilde{m}^n - m - \tilde{m}^n(\varepsilon) - \tilde{b}_n + \Delta_nm + \Delta_n\tilde{b}_n = H_n(I + \tilde{S}_n)^{-1}(D_n^\alpha + D_n^\beta) = H_n(I + \tilde{S}_n)^{-1}A_n(D_n^\alpha + D_n^\beta)$ is of order $o_P(h^2)$. Note that $H_n(I + \tilde{S}_n)^{-1} = H_n - H_n\tilde{S}_n(I + \tilde{S}_n)^{-1}$. We already know that the sup norm of all components in $H_nA_n(D_n^\alpha + D_n^\beta)$ are of order $o_P(h^2)$. Thus, it remains to check that the sup norm of the
components of $H_n \tilde{S}_n(I + \tilde{S}_n)^{-1} A_n(D_n^+ + D_n^-)$ is of order $o_P(h^2)$. But this follows by application of the just mentioned bound on the $L_2$ norm of $H_n(I + S_n)^{-1}(D_n^+ + D_n^-)$, by equations (17) – (18), and the bounds for the random functions $q_{i,k}$ and $q_l$ mentioned after the statement of the equations. One gets a bound for the sup norms by application of the Cauchy Schwarz inequality.

A Projection operators

In this section we will state expressions for the projection operators $P_0$, $P_k$, $P_{k'}$ $(1 \leq k \leq d)$ mapping elements of $\mathcal{H}$ to $\mathcal{H}_0$, $\mathcal{H}_k$, $\mathcal{H}_k + \mathcal{H}_0$ and $\mathcal{H}_{k'}$, respectively, see Section 2. For an element $f = (f_{i,j})_{i=1, \ldots, n; j=0, \ldots, d}$ the operators $P_0$, $P_k$, and $P_{k'}$ $(1 \leq k \leq d)$ set all components to zero but the components with indices $(i,0)$, $i=1, \ldots, n$. Furthermore, in the case $d < k \leq 2d$ only the components with index $(i,k-d)$, $i=1, \ldots, n$ are non-zero. Thus, for the definition of the operators it remains to set

$$ (P_0(f))^{i,0}(x) = \frac{1}{n} \sum_{i=1}^{n} \int_{X} f^{i,0}(u) + \sum_{j=1}^{d} f^{i,j}(x)(X_{ij} - u_j) \{ f_{i,0}(u) + \sum_{j=1}^{d} f_{i,j}(u)(X_{ij} - u_j) \} K_{h}^{X_i}(X_i - u) du. $$

For $1 \leq k \leq d$ it suffices to define $(P_k(f))^{i,0}(x) = (P_{k'}(f))^{i,0}(x) = (P_0(f))^{i,0}$ and

$$ (P_k(f))^{i,0}(x) = \frac{1}{\hat{p}_k^*(x_k)} \left[ \frac{1}{n} \sum_{i=1}^{n} \int_{u \in X_{i,-k}(x_k)} f^{i,0}(u) + \sum_{j=1}^{d} f^{i,j}(u)(X_{ij} - u_j) \right] \times K_{h}^{X_i}(X_i - u) du_{-k}. $$

$$ (P_{k'}(f))^{i,0}(x) = \frac{1}{\hat{p}_{k'}^*(x_k)} \left[ \frac{1}{n} \sum_{i=1}^{n} \int_{u \in X_{i,-k}(x_k)} f^{i,0}(u) + \sum_{j=1}^{d} f^{i,j}(u)(X_{ij} - u_j) \right] \times (X_{ik} - x_k) K_{h}^{X_i}(X_i - u) du_{-k}. $$

For the orthogonal projections of functions $m \in \mathcal{H}_{add}$ one can use simplified formulas. In particular, these formulas can be used in our algorithm for updating functions $m \in \mathcal{H}_{add}$. If $m \in \mathcal{H}_{add}$ has components $m_0, \ldots, m_d, m_1^{(1)}, \ldots, m_d^{(1)}$ the
operators $P_k$ and $P_{k'}$ are defined as follows

$$
(P_0(m)(x))^{i,0} = m_0 + \sum_{j=1}^{d} \int_{X_j} m_j^{(1)}(u_j) \hat{p}_j^*(u_j) \, du_j,
$$

$$
(P_k(m)(x))^{i,0} = m_0 + m_k(x_k) + m_k^{(1)}(x_k) \frac{\hat{p}_k^*(x_k)}{\tilde{p}_k(x_k)}
\quad + \sum_{1 \leq j \leq d, j \neq k} \int_{X_{-j,k}(x_k)} \left[m_j(u_j) \frac{\hat{p}_{jk}(u_j, x_k)}{\tilde{p}_k(x_k)} + m_j^{(1)}(u_j) \frac{\hat{p}_{jk}^*(u_j, x_k)}{\tilde{p}_k^*(x_k)}\right] \, du_j,
$$

$$
(P_k(m)(x))^{i,0} = m_k(x_k) + m_k^{(1)}(x_k) \frac{\hat{p}_k^*(x_k)}{\tilde{p}_k^*(x_k)} - \sum_{1 \leq j \leq d} \int_{X_j} m_j^{(1)}(u_j) \hat{p}_j^*(u_j) \, du_j,
$$

$$
(P_{k'}(m)(x))^{i,0} = m_k^{(1)}(x_k) + (m_0 + m_k(x_k)) \frac{\hat{p}_k^*(x_k)}{\tilde{p}_k^*(x_k)}
\quad + \sum_{1 \leq j \leq d, j \neq k} \int_{X_{-j,k}(x_k)} \left[m_j(u_j) \frac{\hat{p}_{jk}^*(x_k, u_j)}{\tilde{p}_k^*(x_k)} + m_j^{(1)}(u_j) \frac{\hat{p}_{jk}^{**}(u_j, x_k)}{\tilde{p}_k^{**}(x_k)}\right] \, du_j,
$$

where for $1 \leq j, k \leq d$ with $k \neq j$

$$
\hat{p}_{jk}(x_j, x_k) = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathcal{X}_{-jk}(x_j, x_k)} K_h^{X_i}(X_i - x) \, dx_{-jk},
$$

$$
\hat{p}_{jk}^*(x_j, x_k) = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathcal{X}_{-jk}(x_j, x_k)} (X_{ij} - u_j) K_h^{X_i}(X_i - x) \, dx_{-jk},
$$

$$
\hat{p}_{jk}^{**}(x_j, x_k) = \frac{1}{n} \sum_{i=1}^{n} \int_{\mathcal{X}_{-jk}(x_j, x_k)} (X_{ij} - u_j)(X_{ik} - x_k) K_h^{X_i}(X_i - x) \, dx_{-jk}
$$

with $\mathcal{X}_{-jk}(x_j, x_k) = \{ u \in \mathcal{X} : u_k = x_k, u_j = x_j \}$ and $\mathcal{X}_{-k,j}(x_k) = \{ u \in \mathcal{X} : u_k = x_k \}$; there exists $v \in \mathcal{X}$ with $v_k = x_k$ and $v_j = u$ and $u_{-jk}$ denoting the vector $(u_l : l \in \{1, \ldots, d\}\setminus\{j, k\})$.

B Proofs of Propositions 1 and 2

In this section we will give proofs for Propositions 1 and 2. They were used in Section 3 for the discussion of the existence of the smooth backfitting estimator as well as the convergence of an algorithm for its calculation.

Proof (of Proposition 1).

(ii) $\Rightarrow$ (i). Let $g^{(n)} \in L$ be a Cauchy sequence. We must show $\lim_{n \to \infty} g^{(n)} \in L$. By definition of $L$ there exist sequences $g_1^{(n)} \in L_1$ and $g_2^{(n)} \in L_2$ such that
Let \( L \) be a closed subspace of a Hilbert space. For \( \Pi_1 \Pi_2 g(n) \in L_1 \) be converging sequences with limits \( g, g_1 \), then \( \Pi_1 g = g_1 \).

Let \( g(n) \in L \) and \( \Pi_1 g(n) \in L_1 \) be sequences with limits \( g \) and \( g_1 \), respectively. Write \( g(n) = g_1(n) + g_2(n) \). Since
\[
\| g_2(n) - g_2(m) \| \leq \| g_1(n) - g_1(m) \| + \| g(n) - g(m) \|
\]
g\(_2(n)\) is a Cauchy sequence converging to a limit \( g_2 \in L_2 \). We conclude \( g = g_1 + g_2 \), meaning \( \Pi_1 g = g_1 \).

(iii) \(\Rightarrow\) (ii). If \( \Pi_1 \) is a bounded operator, then so is \( \Pi_2 \), since \( \| g_2 \| \leq \| g \| + \| g_1 \| \).

Denote the corresponding operator norms by \( C_1 \) and \( C_2 \), respectively.

Then
\[
\max\{\| g_1 \|, \| g_2 \| \} \leq \max\{C_1, C_2\} \| g \|
\]
which concludes the proof by choosing \( c = \frac{1}{\max\{C_1, C_2\}} \).

(iii) \(\Leftrightarrow\) (iv). This follows from
\[
\| \Pi_1 \| = \sup_{g \in L} \frac{\| g_1 \|}{\| g \|} = \sup_{g_1 \in L_1, g_2 \in L_2} \| g_1 \|_{g_1 + g_2} = \sup_{g_1 \in L_1} \frac{\| g_1 \|}{\text{dist}(g_1, L_2)} = \frac{1}{\gamma(L_1, L_2)}.
\]

Lemma 9. Let \( L_1, L_2 \) be closed subspaces of a Hilbert space. For \( \gamma \) defined as in Proposition 7 we have
\[
\gamma(L_1, L_2)^2 = 1 - \| \mathcal{P}_2 \mathcal{P}_1 \|^2.
\]

Proof.
\[
\gamma(L_1, L_2)^2 = \inf_{g_1 \in L_1, \| g_1 \| = 1} \| g_1 - \mathcal{P}_2 g_1 \|
\]
\[
= \inf_{g_1 \in L_1, \| g_1 \| = 1} \langle g_1 - \mathcal{P}_2 g_1, g_1 - \mathcal{P}_2 g_1 \rangle
\]
\[
= \inf_{g_1 \in L_1, \| g_1 \| = 1} \langle g_1, g_1 \rangle - \langle \mathcal{P}_2 g_1, \mathcal{P}_2 g_1 \rangle
\]
\[
= 1 - \| \mathcal{P}_2 g_1 \| \| \mathcal{P}_2 g_1 \|
\]
\[
= 1 - \| \mathcal{P}_2 \mathcal{P}_1 \|^2.
\]
Proof (of Proposition 2). Let $P_j$ be the orthogonal projection onto $L_j$. Following Lemma 9 we have

$$1 - \|P_2 P_1\|^2 = \gamma(L_1, L_2)^2.$$ 

Using Proposition 1, proving $\|P_2 P_1\| < 1$ implies that $L$ is closed. Observe that $\|P_i\| \leq 1$ because for $g \in L$

$$\|P_j g\|^2 = \langle P_j g, P_j g \rangle = \langle g, P_j g \rangle \leq \|g\| \|P_j g\|,$$

which yields $\|P_i\| \leq 1$ for $i = 1, 2$. To show the strict inequality, note that if $P_2|_{L_1}$ is compact, so is $\|P_2 P_1\|$ since the composition of two operators is compact if at least one is compact. Thus, for every $\varepsilon > 0$, $P_2 P_1$ has at most a finite number of eigenvalues greater than $\varepsilon$. Since 1 is clearly not an eigenvalue, we conclude $\|P_1 P_2\| < 1$. 
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