DUALITY, CORRESPONDENCES AND THE LEFSCHETZ MAP
IN EQUIVARIANT KK-THEORY: A SURVEY

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ABSTRACT. We survey work by the author and Ralf Meyer on equivariant KK-theory. Duality plays a key role in our approach. We have organized this survey around the objective of computing a certain homotopy-invariant of a space equipped with a (generally proper) action of a groupoid. This invariant is called the Lefschetz map. The Lefschetz map associates an equivariant K-homology class to an equivariant Kasparov self-morphism of a space X. We want to describe it explicitly in the setting of bundles of smooth manifolds over the base space of a groupoid, in which groupoid elements act by diffeomorphisms between fibres. To get the required description we develop a topological model of equivariant KK-theory by way of a theory of correspondences, building on ideas of Paul Baum, Alain Connes and Georges Skandalis in the 1980’s. This model agrees with the analytic model for bundles of smooth manifolds under some technical conditions related to the existence of equivariant vector bundles. Subject to these conditions we obtain the desired computation of the Lefschetz map in purely topological terms. Finally, we describe a generalization of the classical Lefschetz fixed-point formula to apply to correspondences, instead of just maps.

The papers [13], [11], [12], [16] present a study of the equivariant Kasparov groups $\text{KK}^G(C_0(X), C_0(Y))$ where $G$ is a locally compact Hausdorff groupoid with Haar system and $X$ and $Y$ are $G$-spaces, usually with $X$ a proper $G$-space. This program builds on work of Kasparov, Connes and Skandalis done mainly in the 1980’s. At that point, the main interest was the index theorem of Atiyah and Singer and its generalisations, and later, the Dirac dual-Dirac method and the Novikov conjecture. For us, the goal is to develop Euler characteristics and Lefschetz formulas in equivariant KK-theory. Via the Baum-Connes isomorphism – when it applies – this contributes to noncommutative topology and index theory. Our program started in [13] where we found the Lefschetz map in connection with a K-theory problem. We will give the definition of the Lefschetz map in the first section, but for now record that it has the form

\[ \text{Lef}: \text{KK}^G_{\times \mathbb{Z}}(C_0(X \times Z X), C_0(X)) \rightarrow \text{KK}^G_C(C_0(X), C_0(Z)), \]

where we always denote by $Z$ the base space of the groupoid. This map is defined under certain somewhat technical circumstances, but, again, these normally involve proper $G$-spaces $X$. The domain of the Lefschetz map is very closely related to the simpler-looking group $\text{KK}^G_C(C_0(X), C_0(X))$: the latter group maps in a natural way to the domain in (0.1) and this map is an isomorphism when the anchor map $X \rightarrow Z$ is a proper map. This means that the Lefschetz map can be used to assign an invariant, which is an equivariant K-homology class, to an equivariant Kasparov self-morphism of $X$. We call this class the Lefschetz invariant of the map. It bears consideration even when $G$ is the trivial groupoid, and the reader can do worse than

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to consider this case to begin with, although by doing so one misses the applications to noncommutative topology.

The definition of \ref{absdual} uses the notion of an abstract dual for \( X \). Abstract duals for a given \( G \)-space \( X \) are not unique but the Lefschetz map does not depend on the choice of a dual, only on the existence of one. Abstract duals do not always exist either: a Cantor set \( X \) doesn’t have an abstract dual even if \( G \) is trivial. But if \( X \) is a smooth \( G \)-manifold, with \( G \) acting smoothly and properly on \( X \), then \( X \) has an abstract dual, and if \( G \) is a group, then any \( G \)-simplicial complex has an abstract dual due to \ref{absdual}.

The Lefschetz map is functorial for \( G \)-maps \( X \to X' \) in a way made explicit in \ref{absdual} (see Theorem \ref{lefschetz 않을 수 있는 합성 단순성}). In brief, it is a homotopy invariant of the \( G \)-space \( X \). In particular, since Lef doesn’t depend on the dual used to to compute it, one can try to compute the Lefschetz invariant of a given morphism using two different duals and thereby get an identity in equivariant K-homology (see \ref{absdual}). Such examples (worked out in \ref{absdual} and \ref{absdual}), seemed to us interesting enough to support making a systematic study of the Lefschetz map. However, to get started on this question one obviously has to first describe the morphisms \( f \) themselves in some kind of satisfactory way. To this end, we have extended the theory of correspondences initiated by Baum, Connes, Skandalis and others, to the equivariant situation, in the paper \ref{absdual}. This extension presents some new features, and we will devote a part of this survey to explaining them. Of course the theory of correspondences is useful and important in its own right. But it is designed for intersection theory because of the way correspondences are composed using coincidence spaces and transversality.

The theory of correspondences requires the groupoid \( G \) to be proper. The Baum-Connes conjecture allows us to reduce to this case subject to a weaker assumption that we explain below. Let \( G \) be proper and \( X \) and \( Y \) be \( G \)-spaces. A \( G \)-equivariant correspondence from \( X \) to \( Y \) is a quadruple \((M,b,f,\xi)\) where \( M \) is a \( G \)-space, \( b: M \to X \) is a \( G \)-map (not necessarily proper), \( \xi \) is an equivariant K-theory class with compact vertical support along the fibres of \( b \), and \( f \) is a K-oriented normally non-singular map from \( M \) to \( Y \) (see Definition \ref{absdual}). For example, if \( G \) is a compact group, \( Y \) is a point and \( M \) is compact, then an normally non-singular map \( M \to Y \) is the specification of an orthogonal representation of \( G \) on some \( \mathbb{R}^n \), an equivariant vector bundle \( V \) over \( M \), and an open equivariant embedding \( \hat{f}: V \to \mathbb{R}^n \). To construct an example of such a triple, assume that \( M \) has been given the structure of a smooth manifold, and that \( G \) acts smoothly. In this case we may appeal to a theorem of Mostow to embed \( M \) in a finite-dimensional linear representation of \( G \), then take \( V \) to be the normal bundle to the embedding.

There is a topologically defined equivalence relation on correspondences that makes the set of equivalence classes of \( G \)-equivariant correspondences from \( X \) to \( Y \) the morphism set \( KK_{G}(X,Y) \) in a \( \mathbb{Z} \)-graded category \( KK_G \) which maps naturally to \( KK_{G} \). For example, let \( G \) be compact group and let both \( X \) and \( Y \) be the one-point space. Let \( M \) be a smooth, compact, equivariantly K-oriented, even-dimensional \( G \)-manifold, \( \xi \in K^0_M(M) \) be an equivariant K-theory class for \( M \) represented by an equivariant vector bundle \( V \) on \( M \). By embedding \( M \) in a finite-dimensional representation of \( G \) as in the previous paragraph, we can endow the map from \( M \) to \( Y := \ast \) with the structure of a smooth, K-oriented, normal map, and we obtain a \( G \)-equivariant correspondence \((M,\ast,\ast,\xi)\) from a point to itself. This yields a class in \( KK^0_G(\ast,\ast) \). Applying the natural map \( \tilde{KK}^0_G(\ast,\ast) \to KK^0_G(\mathbb{C},\mathbb{C}) \cong \text{Rep}(G) \) maps this correspondence to the \( G \)-equivariant topological index of \( DV \) in the sense of \( \ref{absdual} \), where \( DV \) is the Dirac operator on \( M \) twisted by the equivariant vector bundle \( V \). By the Atiyah-Singer Index theorem, this agrees with the \( G \)-equivariant analytic
index of $D_Y$ in $\text{Rep}(\mathcal{G})$, obtained by considering the difference of finite-dimensional $\mathcal{G}$-representations on the kernel and cokernel of $D_Y$.

The combination of the Atiyah-Singer index theorem and the theory of equivariant correspondences represents a powerful tool, because while the index theorem allows us to translate analytic problems into topological ones, the theory of correspondences allows us to manipulate this topological data in interesting ways. For an example of this process in connection with the representation theory of complex semisimple Lie groups, see \cite{17}.

In terms of the Lefschetz map, the fact that correspondences can be composed in an essentially topological fashion has the consequence that the Lefschetz invariants of self-correspondences of a smooth $\mathcal{G}$-manifold $X$, or, or more precisely, of their images in $\text{KK}^G$, can be computed in terms of considerations of transversality. We explain the outcome of this computation in Section \[3\], the gist is that the Lefschetz invariant of a smooth equivariant self-correspondence $\Psi$ of a smooth $\mathcal{G}$-manifold $X$ in general position, can be described in terms of a $\mathcal{G}$-space called the coincidence space $F'_{\Psi}$ of the correspondence. The coincidence space inherits from the smooth structure and $K$-orientation on $\Psi$ the structure of a smooth and equivariantly $K$-oriented $\mathcal{G}$-manifold which maps to $X$ and represents an equivariant correspondence from $X$ to $Z$, thus a cycle for $\text{KK}^G_0(X, Z)$ and then a class in $\text{KK}^G_0(\mathcal{C}_0(X), \mathcal{C}_0(Z))$. It represents the Lefschetz invariant of the morphism represented by $\Psi$. See Theorem \[4,\text{\ref{12}}\] for the exact statement. The ‘general position’ caveat is non-trivial: in the equivariant setting, it may not be possible to perturb a pair of equivariant maps to make them transverse. The framework of equivariant correspondences allows us to treat this difficulty using Bott Periodicity, but we do not discuss this much here.

For example, if $\mathcal{G}$ is a compact group, $X$ a smooth and compact manifold with a smooth action of $\mathcal{G}$, then the Lefschetz invariant of a smooth equivariant map $f: X \to X$ in general position is the fixed-point set of the map, which is a finite set of points permuted by $\mathcal{G}$, oriented by an equivariant line bundle over this finite set. This bundle depends on orientation data from the original map $f$ in a manner which reduces to the classical choice of signs at each fixed-point when $\mathcal{G}$ is trivial. Thus, the topological model of the Lefschetz map provided by the theory of correspondences yields an interpretation of $\text{Lef}$ in terms of a a fixed-point theory for correspondences.

One naturally asks when the map $\text{KK}^G_0(X, Y) \to \text{KK}^G_0(\mathcal{C}_0(X), \mathcal{C}_0(Y))$ is an isomorphism. We explain our results on this in \[4,\text{\ref{11}}\] once again, they rely on duality in a crucial way. When they apply, the topological and analytic Lefschetz maps are equivalent. As mentioned above, the Baum-Connes conjecture can be used to reduce the non-proper situation to the proper one under some weaker assumptions on the $\mathcal{G}$ action on $X$, namely that it be topologically amenable. This is explained in \[4,\text{\ref{11}}\]. Putting everything together gives a computation of the Lefschetz invariant for quite a wide spectrum of smooth $\mathcal{G}$-spaces $X$.

What is duality? It is central to our whole framework, and accordingly we begin the article with a discussion of it. It is well-known from the work of Kasparov and Connes and Skandalis (see \cite{20} and \cite{8}) that if if $X$ is a smooth manifold, then there is a natural family of isomorphisms

$$\text{KK}(\mathcal{C}_0(TX), \mathbb{C}) \cong \text{RK}^*(X) := \text{KK}^X(\mathcal{C}_0(X), \mathcal{C}_0(X))$$

where the groupoid equivariant KK group on the right is equivariant representible K-theory, or K-theory with locally finite support, denoted $\text{RK}^*(X)$ by Kasparov. There is a generalisation of this duality to the equivariant situation if $\mathcal{G}$ is a groupoid acting smoothly and properly on a bundle $X \to Z$ of smooth manifolds over the base $Z$ of $\mathcal{G}$, and furthermore, the roles of $X$ and $TX$ can be in a sense reversed, so that
one can establish a pair of natural (in a technical sense) families of isomorphisms
\begin{equation}
\text{KK}^*_G(C_0(TX) \otimes A, B) \cong \text{KK}^*_{G \ltimes X}(C_0(X) \otimes A, C_0(X) \otimes B)
\end{equation}
and
\begin{equation}
\text{KK}^*_G(C_0(X) \otimes A, B) \cong \text{KK}^*_{G \ltimes X}(C_0(X), C_0(TX) \otimes B)
\end{equation}
for all $G$-C*-algebras $A$ and $B$. (The tensor products are all in the category of $G$-C*-algebras.) These results are proved in [11]. It is the first kind of duality (0.2) which is relevant for the Lefschetz map, and the second (0.3) that is used to prove that the map $\hat{\text{KK}}_G \to \text{KK}^*_G$ is an isomorphism in certain cases. The basic idea is that since the duality isomorphisms (0.3) are themselves induced by equivariant correspondences, duality can be used simultaneously in both the analytic and topological categories to reduce the question to a problem about monovariant KK-theory, that is, equivariant K-theory with support conditions.

What is needed to make this work is then a topological model of duality. The main new issue that appears is that our equivariant correspondences require a good supply of equivariant vector bundles and this forces conditions on the groupoid $G$. These considerations have in fact already appeared in the literature in connection with the (proper) groupoids $G \ltimes E_G$ in work by Wolfgang Lück and Bob Oliver in [23] (see §3.5 for the details) where $G$ is a discrete group and $E_G$ is its classifying space for proper actions. We explain exactly what the conditions are and how they are related to embedding theorems generalizing the embedding theorem of Mostow alluded to above.

The classical Lefschetz fixed-point theorem relates fixed-points of a map and the homological invariant of the map obtained by taking the graded trace of the induced map on homology, called the Lefschetz number. We finish this survey by introducing some global, homological invariants of correspondences which generalize the Lefschetz number, at least in the case when the groupoid $G$ is trivial. Roughly speaking, a Kasparov self-morphism, and in particular a self-correspondence should be considered as determining a linear operator on homology instead of just a number. We call it the Lefschetz operator. We will explain how to extend the classical Lefschetz fixed-point theorem to correspondences by identifying the Lefschetz operator with the operator of pairing with the Lefschetz invariant. In situations where a local index formula is available, this results in a description of the Lefschetz operator in local, geometric terms.

It remains to describe the Lefschetz map in global, homological terms – as in the classical formula, in which fixed-points are related to traces on homology. This problem seems quite delicate, however. One way of proceeding is to replace K-theory groups by $G$-equivariant K-theory modules over $\text{Rep}(G)$ and replace the ordinary trace by the Hattori-Stallings trace. However, this is only defined under quite stringent conditions: such modules are only finitely presented in general only for groups for which the representation rings have finite cohomological dimension, and this requires additional hypotheses on $G$. This work is still in progress.

1. Abstract duality and the Lefschetz map

Throughout this paper, groupoid shall mean locally compact Hausdorff groupoid with Haar system. All topological spaces will be assumed paracompact, locally compact and Hausdorff. For the material in this section, see [11]. For source material on equivariant KK-theory for groupoids, see [22]. One seems to be forced to consider groupoids, as opposed to groups, in equivariant Kasparov theory, even if one is ultimately only interested in groups. This will be explained later.

Therefore we will work more or less uniformly with groupoids when discussing general theory. When we discuss topological equivariant Kasparov theory, we will
further assume that all groupoids are proper. This restriction is needed for various geometric constructions. The additional assumption of properness involves no serious loss of generality for our purposes because the Baum-Connes isomorphism, when it applies, gives a method of replacing non-proper groupoids by proper ones.

1.1. Equivariant Kasparov theory for groupoids. Let $\mathcal{G}$ be a groupoid. We let $Z$ denote the base space. A $\mathcal{G}$-$C^*$-algebra is in particular a $C^*$-algebra over $Z$. This means that there is given a non-degenerate equivariant $*$-homomorphism from $C_0(Z)$ to central multipliers of $A$. This identifies $A$ with the section algebra of a continuous bundle of $C^*$-algebras over $Z$. For a groupoid action we require in addition an isomorphism $r^*(A) \to s^*(A)$ which is compatible with the structure of $r^*(A)$ and $s^*(A)$ as $C^*$-algebras over $\mathcal{G}$. Here $r: \mathcal{G} \to Z$ and $s: \mathcal{G} \to Z$ are the range and source map of the groupoid, and $r^*$ (and similarly $s^*$) denotes the usual pullback operation of bundles. From the bundle point of view, all of this means that groupoid elements $g$ with $s(g) = x$ and $r(g) = y$ induce $*$-homomorphisms $A_x \to A_y$ between the fibres of $A$ at $x$ and $y$.

In particular, if $A$ is commutative, then $A$ is the $C^*$-algebra of continuous functions on a locally compact $\mathcal{G}$-space $X$, equipped with a map $\varrho_X: X \to Z$ called the anchor map for $X$, and a homeomorphism

$$\varrho \times_{Z,s} X \to \varrho \times_{Z,r} X, \quad (g,x) \mapsto (g, gx)$$

where the domain and range of this homeomorphism (by abuse of notation) are respectively

$$\varrho \times_{Z,s} X := \{(g,x) \in \varrho \times X \mid s(g) = \varrho_X(x)\},$$

and similarly for $\varrho \times_{Z,r} X$ using $r$ instead of $s$.

1.2. Tensor products. The category of $\mathcal{G}$-$C^*$-algebras has a symmetric monoidal structure given by tensor products. We describe this very briefly (see [11] Section 2 for details).

Let $A$ and $B$ be two $\mathcal{G}$-$C^*$-algebras. Since they are each $C^*$-algebras over $Z$, their external tensor product $A \otimes B$ is a $C^*$-algebra over $Z \times Z$. We restrict this to a $C^*$-algebra over the diagonal $Z \subset Z \times Z$. The result is called the tensor product of $A$ and $B$ over $Z$. The tensor product of $A$ and $B$ over $Z$ carries a diagonal action of $\mathcal{G}$. We leave it to the reader to check that we obtain a $\mathcal{G}$-$C^*$-algebra in this way.

In order not to complicate notation, we write just $A \otimes B$ for the tensor product of $A$ and $B$ in the category of $\mathcal{G}$-$C^*$-algebras. We emphasize that the tensor product is over $Z$; this is not the same as the tensor product in the category of $C^*$-algebras.

For commutative $C^*$-algebras, i.e. for $\mathcal{G}$-spaces, say $X$ and $Y$, with anchor maps as usual denoted $\varrho_X: X \to Z$ and $\varrho_Y: Y \to Z$, the tensor product is Gelfand dual to the operation which forms from $X$ and $Y$ the fibre product

$$X \times_Z Y := \{(x,y) \in X \times Y \mid \varrho_X(x) = \varrho_Y(y)\}.$$

The required anchor map $\varrho_{X \times_Z Y}: X \times_Z Y \to Z$ is of course the composition of the first coordinate projection and the anchor map for $X$ (or the analogue using $Y$; they are equal). Of course groupoid elements act diagonally in the obvious way. Such coincidence spaces as the one just described will appear again and again in the theory of correspondences.

Finally, for the record, we supply the following important definition.

**Definition 1.1.** Let $\mathcal{G}$ be a groupoid. A $\mathcal{G}$-space $X$ is proper if the map

$$\varrho \times_{Z} X \to X, \quad (g,x) \mapsto (gx, x)$$

is a proper map, where $\varrho \times_{Z} X := \{(g,x) \in \varrho \times X \mid s(g) = \varrho_X(x)\}$.
A groupoid is itself called \textit{proper} if it acts properly on its base space \(Z\). Explicitly, the map
\[
\mathcal{G} \to X \times_Z X, \quad g \mapsto (r(g), s(g))
\]
is required to be proper.

1.3. \textbf{Equivariant Kasparov theory.} Le Gall has defined \(\mathcal{G}\)-equivariant KK-theory in \cite{22}. We briefly sketch the definitions. Let \(A\) and \(B\) be (possibly \(\mathbb{Z}/2\)-graded) \(\mathcal{G}\)-\(\mathrm{C}^*\)-algebras.

Then a cycle for \(\mathrm{KK}^G(A, B)\) is given by a \(\mathbb{Z}/2\)-graded \(\mathcal{G}\)-equivariant Hilbert \(B\)-module \(E\), together with a \(\mathcal{G}\)-equivariant grading-preserving \(*\)-homomorphism from \(A\) to the \(\mathrm{C}^*\)-algebra of bounded, adjointable operators on \(E\), and an essentially \(\mathcal{G}\)-equivariant self-adjoint operator \(F\) on \(E\) which is graded odd and satisfies \([a, F]\) and \(a(F^2 - 1)\) are compact operators (essentially zero operators) for all \(a \in A\).

modulo an appropriate equivalence relation, the set of equivalence classes of cycles can be identified with the morphism set \(\mathrm{KK}^G(A, B)\) in an additive, symmetric monoidal category. Higher KK-groups are defined using Clifford algebras, and since these are 2-periodic, there are only two up to isomorphism. We denote by \(\mathrm{RKK}^G(A, B)\) the sum of these two groups.

If \(A\) and \(B\) are \(\mathcal{G}\)-\(\mathrm{C}^*\)-algebras, then the group \(\mathrm{RKK}^G(X; A, B)\) is by definition the groupoid-equivariant Kasparov group \(\mathrm{KK}^G\times_X (C_0(X) \otimes A, C_0(X) \otimes B)\).

The tensor products are in the category of \(\mathcal{G}\)-\(\mathrm{C}^*\)-algebras. This group differs from \(\mathrm{KK}^G(A, C_0(X) \otimes B)\) only in the support condition on cycles. For example if \(\mathcal{G}\) is trivial and \(A = B = \mathbb{C}\) then \(\mathrm{KK}^G(A, C_0(X) \otimes B)\) is the ordinary K-theory of \(X\) and \(\mathrm{RKK}^G(X; A, B)\) is the representable K-theory of \(X\) (a non-compactly supported theory.) We discuss these groups in more detail in the next section. Of course similar remarks hold for higher \(\mathrm{RKK}^G\)-groups.

1.4. \textbf{Equivariant K-theory.} In this section, we present an exceedingly brief overview of equivariant K-theory, roughly sufficient for the theory of equivariant correspondences. For more details see \cite{12}.

Let \(X\) be a proper \(\mathcal{G}\)-space. Recall that a \(\mathcal{G}\times X\)-space consists of a \(\mathcal{G}\)-space \(Y\) together with a \(\mathcal{G}\)-equivariant map \(\varrho_Y : Y \to X\) serving as the anchor map for the \(\mathcal{G}\times X\)-action.

\textbf{Definition 1.2.} Let \(Y\) be a \(\mathcal{G}\times X\)-space. The \(\mathcal{G}\)-equivariant representable K-theory of \(Y\) with \(X\)-compact supports is the group
\[
\mathrm{RKK}^G_{\mathcal{G}\times X}(Y) := \mathrm{KK}^G_{\mathcal{G}\times X}(C_0(X), C_0(Y)).
\]

The \(\mathcal{G}\)-equivariant representable K-theory of \(Y\) is
\[
\mathrm{RK}^G_{\mathcal{G}\times X}(Y) := \mathrm{RK}^G_{\mathcal{G}\times X}(Y).
\]

Cycles for \(\mathrm{KK}^G_{\mathcal{G}\times X}(C_0(X), C_0(Y))\) consist of pairs \((\mathcal{H}, F)\) where \(\mathcal{E}\) is a countably generated \(\mathbb{Z}/2\)-graded \(\mathcal{G}\times X\)-equivariant right Hilbert \(C_0(X)\)-module equipped with a \(\mathcal{G}\times X\)-equivariant non-degenerate \(*\)-homomorphism from \(C_0(X)\) to the \(\mathrm{C}^*\)-algebra of bounded, adjointable operators on \(\mathcal{E}\), and \(F\) is a bounded, odd, self-adjoint, essentially \(\mathcal{G}\)-equivariant adjointable operator on \(\mathcal{H}\) such that \(f(F^2 - 1)\) is a compact operator, for all \(f \in C_0(X)\). The properness of \(\mathcal{G}\) implies that \(F\) may be averaged to be actually \(\mathcal{G}\)-equivariant, so we assume this in the following.

The Hilbert \(C_0(Y)\)-module \(\mathcal{E}\) is the space of continuous sections of a continuous field of \(\mathbb{Z}/2\)-graded Hilbert spaces \(\{\mathcal{H}_y \mid y \in Y\}\) over \(Y\). Since \(F\) must be \(C_0(Y)\)-linear, it consists of a continuous family \(\{F_y \mid y \in Y\}\) of odd operators on these graded Hilbert spaces such that \(F_y^2 - 1\) is a compact operator on \(\mathcal{H}_y\) for all \(y \in Y\).

By \(\mathcal{G}\times X\)-equivariance, the representation of \(C_0(X)\) on \(\mathcal{E}\) must factor through the \(*\)-homomorphism \(C_0(X) \to C_0(Y)\) Gelfand dual to the anchor map \(\varrho_Y : Y \to X\)...
X. Therefore F commutes with the action of $C_0(X)$ as well; in fact the induced representation of $C_0(X)$ on each Hilbert space $H_\nu$ sends a continuous function $f \in C_0(X)$ to the operator of multiplication by the complex number $f(\nu(y))$.

In particular, the only role of the representation of $C_0(X)$ is to relax the support condition on the compact-operator valued-function $F^2 - 1$ from requiring it to vanish at $\infty$ of $Y$ to only requiring it to vanish at infinity along the fibres of $\nu_Y : Y \to X$.

If $\nu_Y : Y \to X$ is a proper map then $RK^0_{G,Y}(X) = RK^0_G(X) := RK^0_{G,X}(X)$; these two groups have exactly the same cycles.

**Example 1.3.** Any $G$-equivariant complex vector bundle $V$ on $Y$ yields a cycle for $RK^0_G(Y)$ by choosing a $G$-invariant Hermitian metric on $V$ and forming the corresponding $G \times Y$-equivariant $\mathbb{Z}/2$-graded right Hilbert $C_0(Y)$ module of sections, where the grading is the trivial one. We set the operator equal to zero.

**Example 1.4.** Let $X$ be a $G$-space and let $V$ be a $G$-equivariantly $K$-oriented vector bundle over $X$ of (real) dimension $n$. The $G$-equivariant vector bundle projection $\pi_V : V \to X$ gives the structure of a space over $X$, so that $V$ becomes a $G \times X$-space. Then the *Thom isomorphism* provides an invertible Thom class

$$t_v \in RK^0_{G,Y}(V) := KK^G_{dim V}(C_0(X), C_0(V)).$$

In the case $G = \text{Spin}^c(\mathbb{R}^n)$ and $X = \ast$ and $V := \mathbb{R}^n$ with the representation $\text{Spin}^c(\mathbb{R}^n) \to \text{Spin}(\mathbb{R}^n) \to O(n, \mathbb{R})$ the class $t_{\mathbb{R}^n}$ is the ‘Bott’ class figuring in equivariant Bott Periodicity.

Certain further normalizations can be made in order to describe the groups $RK^0_{G,Y}(X)$. A standard one is to replace the $\mathbb{Z}/2$-grading on $E$ by the standard even grading, so that $E$ consists of the sum of two copies of the same Hilbert module.

This means that $F$ can be taken to be of the form

$$\begin{pmatrix}
0 & F_1 \\
F_1 & 0
\end{pmatrix}$$

and the conditions involving $F$ are replaced by ones involving $F_1$ and $F_1^*$. We may as well replace $F$ by $F_1$. With this convention, the Fredholm conditions are that $f(FF^* - 1)$ and $f(F^*F - 1)$ are compact for all $f \in C_0(X)$. In other words, $y \mapsto F_y$ takes essentially unitary values in $\mathbb{B}(H_\nu)$ for all $y \in Y$ and the compact-operator-valued functions $FF^* - 1$ and $F^*F - 1$ vanish at infinity along the fibres of $\nu_Y : Y \to X$.

The equivariant stabilization theorem for Hilbert modules implies that we may take $H$ to have the special form $L^2(G) \otimes_{C_0(Z)} C_0(Y)$, where $L^2(G)$ is the $G$-equivariant right Hilbert $C_0(Z)$-Hilbert module defined using the Haar system of $G$, and the superscript indicates the sum of countably many copies of $L^2(G)$. The corresponding field of Hilbert space has value $L^2(G^y) \otimes_{C_0(Y)} C_0(Y)$ at $y \in Y$ where $G^y$ denotes all $G \in G$ ending in $y$, on which we have a given measure specified by the Haar system of $G$.

This leads to a description of $RK^0_{G,Y}(X)$ as the group of homotopy-classes of $G$-equivariant continuous maps from $Y$ to the space $F_G$ of Fredholm operators on the Hilbert spaces $L^2(G^y) \otimes_{C_0(Y)} C_0(Y)$, but topologizing the space $F_G$ is somewhat delicate. Similarly, the relative groups $RK^0_{G,Y}(X)$ are maps to Fredholm operators with compact vertical support with respect to the map $Y \to X$, where the support of a map to Fredholm operators is by definition the complement of the set where the map takes invertible values.

**Remark 1.5.** If $G$ acts properly and co-compactly on $X$, $A$ is a trivial $G$-$C^*$-algebra and $B$ is a $G \times X$-$C^*$-algebra, then there is a canonical isomorphism

$$\text{KK}^G_{X,Y}(C_0(X) \otimes A, B) \cong \text{KK}(A, G \times B).$$

In particular, the $G$-equivariant representable $K$-theory of $X$ agrees with the $K$-theory of the corresponding cross-product. Under this identification, classes in
RK\(_G^n(X)\) which are represented by equivariant vector bundles on \(X\) correspond to classes in \(K_0(\mathcal{G} \ltimes \mathcal{C}_0(X))\) which are represented by projections in the stabilisation of \(\mathcal{G} \ltimes \mathcal{C}_0(X)\). See [12] for more information.

Thus, even if the reader is only interested in groups, or the trivial group, it is convenient to introduce groupoids to some extent in order to describe cohomology theories with different support conditions.

1.5. Tensor and forgetful functors. The following simple functor will play an important role. If \(P\) is a \(\mathcal{G} \ltimes X\)-algebra, we denote by \(T_P\) the map

\[
RKK^\mathcal{G}\left(\mathcal{G} \ltimes X; A, B\right) := \text{KK}^\mathcal{G} \left(\mathcal{C}_0(X) \otimes A, \mathbf{1}_X \otimes B\right) \to \text{KK}^\mathcal{G} \left(P \otimes A, P \otimes B\right)
\]

which sends a \(\mathcal{G} \ltimes X\)-equivariant right Hilbert \(\mathcal{C}_0(X) \otimes B\)-Hilbert module \(E\) to \(E \otimes_X P\), the tensor product being in the category of \(\mathcal{G} \ltimes X\)-algebras (we accordingly use a subscript for emphasis) and sends \(F \in \mathcal{B}(E)\) to the operator \(F \otimes_X \text{id}_P\). This definition makes sense since \(F\) commutes with the \(\mathcal{C}_0(X)\)-structure on \(E\).

The functor \(T_P\) is the composition of external product

\[
\boxtimes \otimes 1_P : \text{KK}^\mathcal{G} \left(\mathcal{C}_0(X) \otimes A, \mathcal{C}_0(X) \otimes B\right) \to \text{KK}^\mathcal{G} \left(A \otimes_X P, B \otimes_X P\right)
\]

(where the \(X\)-structure on \(A \otimes P\) etc. is on the \(P\) factor), and the forgetful map

\[
\text{KK}^\mathcal{G} \left(A \otimes_X P, B \otimes_X P\right) \to \text{KK}^\mathcal{G} \left(A \otimes X P, B \otimes X P\right)
\]

which maps a \(\mathcal{G} \ltimes X\)-algebra or Hilbert module to the underlying \(\mathcal{G}\)-algebra, or Hilbert module, thus forgetting the \(X\)-structure.

1.6. Kasparov duals. We begin our discussion of duality by by formalizing some duality calculations of Kasparov, c.f. [20] Theorem 4.9]. Explicit examples will be discussed later.

For convenience of notation we will often write \(\mathbf{1} := \mathcal{C}_0(Z)\). This notation expresses the fact that \(\mathcal{C}_0(Z)\) is the tensor unit in the tensor category of \(\mathcal{G}\)-\(C^\ast\)-algebras. Similarly, if \(\mathcal{G}\) acts on a space \(X\) then we sometimes denote by \(\mathbf{1}_X\) the \(\mathcal{G}\)-\(C^\ast\)-algebra \(\mathcal{C}_0(X)\); thus \(\mathbf{1}_X\) is the tensor unit in the category of \(\mathcal{G} \ltimes X\) \(C^\ast\)-algebras, \(X\) being the base of \(\mathcal{G} \ltimes X\). This notation is consistent with the source of this material (see [11].)

**Definition 1.6.** Let \(n \in \mathbb{Z}\). An \(n\)-dimensional \(\mathcal{G}\)-equivariant Kasparov dual for the \(\mathcal{G}\)-space \(X\) is a triple \((P, D, \Theta)\), where

- \(P\) is a (possibly \(\mathbb{Z}/2\)-graded) \(\mathcal{G} \ltimes C^\ast\)-algebra,
- \(D \in \text{KK}^\mathcal{G}_{\ast,n}(P, \mathbf{1})\), and
- \(\Theta \in \text{RK}_G^\mathcal{G}(X; \mathbf{1}, P)\),

subject to the following conditions:

1. \(\Theta \otimes_P D = \text{id}_\mathbf{1}\) in \(\text{RK}_G^\mathcal{G}(X; \mathbf{1}, \mathbf{1})\);
2. \(\Theta \otimes f = \Theta \otimes_P T_P(f)\) in \(\text{KK}^\mathcal{G}_{\ast,n}(X; A, B \otimes P)\) for all \(\mathcal{G}\)-\(C^\ast\)-algebras \(A\) and \(B\) and all \(f \in \text{KK}^\mathcal{G}_n(X; A, B)\);
3. \(T_P(\Theta) \otimes_P \Phi_P = (-1)^n T_P(\Theta)\) in \(\text{KK}^\mathcal{G}_n(P, P \otimes P)\), where \(\Phi_P\) is the flip automorphism on \(P \otimes P\).

The following theorem is proved in [11].

**Theorem 1.7.** Let \(n \in \mathbb{Z}\), let \(P\) be a \(\mathcal{G} \ltimes X\)-\(C^\ast\)-algebra, \(D \in \text{KK}^\mathcal{G}_{\ast,n}(P, \mathbf{1})\), and \(\Theta \in \text{RK}_G^\mathcal{G}(X; \mathbf{1}, P)\). Define two natural transformations

\[
\text{PD} : \text{KK}^\mathcal{G}_{\ast-n}(P \otimes A, B) \to \text{KK}^\mathcal{G}_{\ast-n}(X; A, B), \quad f \mapsto \Theta \otimes_P f,
\]

\[
\text{PD}^* : \text{RK}_G^\mathcal{G}(X; A, B) \to \text{RK}_G^\mathcal{G}_{\ast-n}(P \otimes A, B), \quad g \mapsto (-1)^n T_P(g) \otimes_P D,
\]

\[
\Theta \otimes_P D = \text{id}_\mathbf{1}\]
These two are inverse to each other if and only if \((P, D, \Theta)\) is an \(n\)-dimensional \(G\)-equivariant Kasparov dual for \(X\).

1.7. Abstract duals. The reader may have noticed that the only place the \(C_0(X)\)-structure on \(P\) comes into play in the conditions listed in Definition 1.6 and in the statement of Theorem 1.7 is via the functor \(T_P\). In particular, if one has a Kasparov dual \((P, D, \Theta)\) and if one changes the \(C_0(X)\)-structure on \(P\), for example by composing it with a \(G\)-equivariant homeomorphism of \(X\), then the map \(PD\) of Theorem 1.7 does not change; since by the theorem \(PD^*\) is its inverse map, it would not change either, strangely, since its definition uses \(T_P\). In fact it turns out that the functor \(T_P\) can be reconstructed from \(PD\) if one knows that \(PD\) is an isomorphism. This is an important idea in connection with the Lefschetz map and suggests the following useful definition.

**Definition 1.8.** An \(n\)-dimensional abstract dual for \(X\) is a pair \((P, \Theta)\), where \(P\) is a \(G\)-C*-algebra and \(\Theta \in RKK^G_n(X; \mathbb{1}, P)\), such that the map \(PD\) defined as in Theorem 1.7 is an isomorphism for all \(G\)-C*-algebras \(A\) and \(B\).

This definition is shorter, and, as mentioned, is useful for theoretical reasons, but it seems like it should be difficult to check in practise.

In any case, it is clear from Theorem 1.7 that a pair \((P, \Theta)\) is an abstract dual if it is part of a Kasparov dual \((P, D, \Theta)\).

**Proposition 1.9.** An abstract dual for a space \(X\) is unique up to a canonical \(KK^G\)-equivalence if it exists, and even covariantly functorial in the following sense. Let \(X\) and \(Y\) be two \(G\)-spaces and let \(f : X \to Y\) be a \(G\)-equivariant continuous map. Let \((P_X, \Theta_X)\) and \((P_Y, \Theta_Y)\) be abstract duals for \(X\) and \(Y\) of dimensions \(n_X\) and \(n_Y\), respectively. Then there is a unique \(P_f \in KK^G_{n_Y-n_X}(P_X, P_Y)\) with \(\Theta_X \otimes P_X P_f = f^*(\Theta_Y)\). Given two composable maps between three spaces with duals, we have \(P_{f \circ g} = P_f \circ P_g\). If \(X = Y\), \(f = \text{id}_X\), and \((P_X, \Theta_X) = (P_Y, \Theta_Y)\), then \(P_f = \text{id}_{P_X}\). If only \(X = Y\), \(f = \text{id}_X\), then \(P_f\) is a \(KK^G\)-equivalence between the two duals of \(X\).

Although the map \(f : X \to Y\) appearing in Proposition 1.9 does not have to be proper, it nonetheless yields a morphism \(P_f\) in \(KK^G\).

1.8. Duality co-algebra. Let \((P, \Theta)\) be an \(n\)-dimensional abstract dual for a \(G\)-space \(X\). By the Yoneda Lemma, another abstract dual \((P', \Theta')\) also for \(X\) and say of dimension \(n'\) is related to \((P, \Theta)\) by an invertible element

\[
(1.10) \quad \psi \in KK^G_{n'\cdot \cdot}(P, P') \quad \text{such that} \quad \Theta \otimes_P \psi = \Theta'.
\]

We repeat for emphasis that since \((P, \Theta)\) is only an abstract dual, we are not assuming that there is a \(G \times X\)-structure on \(P\). However, we are going to attempt to reconstruct what we might consider to be a \(G \times X\)-structure on \(P\) at the level of KK-theory. Along the way we will keep track of how the change in dual from \((P, \Theta)\) to \((P', \Theta')\) affects our constructions.

Define \(D \in KK^G_n(P, \mathbb{1})\) by the requirement

\[
(1.11) \quad PD(D) := \Theta \otimes_P D = 1_\mathbb{1} \quad \text{in } RKK^G_n(X; \mathbb{1}, \mathbb{1}),
\]

as in the first condition in Definition 1.6. The class \(D\) should thus play the role of the class named \(D\) in a Kasparov dual. It is routine to check that when we change the dual, as above, \(D\) is replaced by \(\psi^{-1} \otimes_P D\).

We call \(D\) counit of the duality because it plays the algebraic role of a counit in the theory of adjoint functors (see 11 and also Remark 3.3 below).

Define \(\nabla \in KK^G_n(P, P \otimes P)\) by the requirement that

\[
PD(\nabla) := \Theta \otimes_P \nabla = \Theta \otimes_X \Theta \quad \text{in } RKK^G_{2n}(X; \mathbb{1}, P \otimes P).
\]
We call \( \nabla \) the *comultiplication of the duality*. When we change the dual, \( \nabla \) is replaced by

\[
(-1)^{n(n'-n)} \psi^{-1} \otimes_P \nabla \otimes_{P \otimes P} (\psi \otimes \psi) \in \text{KK}^G_n(P', P' \otimes P').
\]

**Remark 1.12.** If \( n = 0 \) then the object \( P \) of \( \text{KK}^G \) with counit \( D \) and comultiplication \( \nabla \) is a cocommutative, counital coalgebra object in the tensor category \( \text{KK}^G \):

\[
\begin{align*}
\nabla \otimes_{P \otimes P} (\nabla \otimes 1_P) &= \nabla \otimes_{P \otimes P} (1_P \otimes \nabla), \\
\nabla \otimes_{P \otimes P} \Phi &= \nabla, \\
\nabla \otimes_{P \otimes P} (D \otimes 1_P) &= 1_P = \nabla \otimes_{P \otimes P} (1_P \otimes D).
\end{align*}
\]

Equation (1.13) holds in \( \text{KK}^G_{2n}(P, P \otimes P) \), equation (1.14) holds in \( \text{KK}^G_n(P, P \otimes P) \), and (1.15) holds in \( \text{KK}^G_n(P, P) \).

Now, for \( G \)-C*-algebras \( A \) and \( B \), we define

\[
T'_p : \text{RKK}^G_t(X; A, B) \to \text{RKK}^G_t(P \otimes A, P \otimes B), \quad f \mapsto \nabla \otimes_P \text{PD}^{-1}(f),
\]

where \( \text{PD} \) is the duality isomorphism, \( \nabla \) is the comultiplication of the duality, and \( \otimes_P \) operates on the *second* copy of \( P \) in the target \( P \otimes P \) of \( \nabla \). A computation yields that

\[
\text{PD}(T'_p(f)) = \Theta \otimes_X f \quad \text{in} \quad \text{RKK}^G_t(X; A, P \otimes B)
\]

for all \( f \in \text{RKK}^G_t(X; A, B) \). It follows that

\[
T'_p(f) = T_p(f)
\]

if \( (P, \Theta) \) is part of a Kasparov dual, and thus \( T_p \) is in fact independent of the \( G \times X \)-structure on \( P \), verifying our guess above.

When we change the dual, we replace \( T'_p \) by the map

\[
\text{RKK}^G_t(X; A, B) \ni f \mapsto (-1)^{i(n-n')} \psi^{-1} \otimes_P T_p(f) \otimes_P \psi \in \text{KK}^G_t(P' \otimes A, P' \otimes B).
\]

In fact, one can check that the maps \( T'_p \) above define a functor

\[
T'_p : \text{RKK}^G(X) \to \text{RKK}^G.
\]

This is a \( \text{RKK}^G \)-functor in the sense that it is compatible with the tensor products \( \otimes \), and it is left adjoint to the functor \( p^*_X : \text{RKK}^G \to \text{RKK}^G \) induced from the groupoid homomorphism \( G \times X \to G \).

It follows that we can write the inverse duality map involved in an abstract dual \( (P, \Theta) \) as:

\[
\text{PD}^{-1}(f) = (-1)^{i(n)} T'_p(f) \otimes_P D \quad \text{in} \quad \text{KK}^G_{-n}(P \otimes A, B)
\]

for \( f \in \text{RKK}^G_t(X; A, B) \). By the above discussion this formula agrees with the map \( \text{PD}^* \) when we have a Kasparov dual.

**1.9. The Lefschetz map.** The formal computations summarized in the previous section allows us to single out an interesting invariant of a \( G \)-space \( X \), at least under the hypothesis that \( X \) has *some* abstract dual.

For any \( G \)-space \( X \) the diagonal embedding \( X \to X \times_Z X \) is a proper map and hence induces a \(^*\)-homomorphism

\[
\mathbb{1}_X \otimes \mathbb{1}_X \cong C_0(X \times_Z X) \to C_0(X) = \mathbb{1}_X.
\]

This map is \( G \times X \)-equivariant and hence yields

\[
\Delta_X \in \text{RKK}^G_t(X; \mathbb{1}_X, \mathbb{1}) \cong \text{RKK}^G_{* \times X}(C_0(X \times_Z X), C_0(X)).
\]
We call this the diagonal restriction class. It yields a canonical map
\[(\cup \otimes_{\Delta X} \Delta X): KK^G_{\ast}(\mathbb{I}_X \otimes A, \mathbb{I}_X \otimes B) \to RKK^G_{\ast}(X; \mathbb{I}_X \otimes A, B).\]
In particular, this contains a map
\[KK^G_{\ast}(\mathbb{I}_X, \mathbb{I}_X) \to RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I}).\]

**Example 1.20.** If \(f: X \to X\) is a proper, continuous, \(G\)-equivariant map, then
\[[f] \otimes_{\Delta X} \Delta X \in RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I})\]
is the class of the \(^\ast\)-homomorphism induced by \((id_X, f): X \to X \times_Z X\).

Now drop the assumption that \(f\) be proper. Then \((id_X, f)\) is still a proper, continuous, \(G\)-equivariant map. The class of the \(^\ast\)-homomorphism it induces is equal to \(f^*(\Delta X)\), where we use the maps
\[f^*: RKK^G_{\ast}(X; A, B) \to RKK^G_{\ast}(X; A, B)\]
for \(A = \mathbb{I}_X, B = \mathbb{I}\) induced by \(f: X \to X\) (the functor \(X \mapsto RKK^G_{\ast}(X; A, B)\) is functorial with respect to arbitrary \(G\)-maps, not just proper ones.) This suggests that we can think of \(RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I})\) as generalized, possibly non-proper self-maps of \(X\).

In fact if the anchor map \(X \to Z\) is a proper map, so that \(X\) is a bundle of compact spaces over \(Z\), then \(\cup \otimes_{\Delta X} \Delta X\) is an isomorphism (an easy exercise in the definitions.)

Now let \(T'_P\) be the tensor functor and \(\Delta X\) the diagonal restriction class of an abstract dual. We define the multiplication class of \(P\) by
\[(m) := T'_P(\Delta X) \in KK^G_{\ast}(P \otimes \mathbb{I}_X, P).\]
A change of dual as in \([1.10]\) replaces \([m]\) by \(\psi^{-1} \otimes_P [m] \otimes_P \psi\).

**Lemma 1.22.** Let \((P, D, \Theta)\) be a Kasparov dual. Then \([m]\) is the class in \(KK^G_{\ast}\) of the multiplication homomorphism \(\Theta_0(X) \otimes_Z P \to P\) that describes the \(X\)-structure on \(P\) (up to commuting the tensor factors).

We now have enough theoretical development to define the Lefschetz map and sketch the proof of its homotopy invariance.

Let \(X\) be a \(G\)-space and \((P, \Theta)\) an \(n\)-dimensional abstract dual for \(X\), \(PD\) and \(PD^{-1}\) the duality isomorphisms. As before, we write \(\mathbb{I} := \Theta_0(Z), \mathbb{I}_X := \Theta_0(X)\) and \(\Delta X \in RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I}) = KK^G_{\ast}(\mathbb{I}_X \otimes \mathbb{I}_X, \mathbb{I}_X)\) the diagonal restriction class and
\[\hat{\Theta} := \text{Forget}_X(\Theta) \in KK^G_{\ast}(\mathbb{I}_X, P \otimes \mathbb{I}_X).\]

**Definition 1.23.** The equivariant Lefschetz map
\[\text{Lef}: RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I}) \to KK^G_{\ast}(\mathbb{I}_X, \mathbb{I})\]
for a \(G\)-space \(X\) is defined as the composite map
\[RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I}) \xrightarrow{PD^{-1}} KK^G_{\ast-n}(P \otimes \mathbb{I}_X, \mathbb{I}) \xrightarrow{\hat{\Theta} \otimes_{P \otimes \mathbb{I}_X} \psi} KK^G_{\ast}(\mathbb{I}_X, \mathbb{I}).\]
The equivariant Euler characteristic of \(X\) is
\[\text{Eul}_X := \text{Lef}(\Delta X) \in KK^G_{\ast}(\mathbb{I}_X, \mathbb{I}) = KK^G_{\ast}(\Theta_0(X), \Theta_0(Z)).\]

Let \(f \in RKK^G_{\ast}(X; \mathbb{I}_X, \mathbb{I})\). Equations \([1.18]\) and \([1.21]\) yield
\[\text{Lef}(f) = (-1)^{\ast n} \hat{\Theta} \otimes_{P \otimes \mathbb{I}_X} T'_P(f) \otimes_P D,\]
\[\text{Eul}_X = (-1)^{\ast n} \hat{\Theta} \otimes_{P \otimes \mathbb{I}_X} [m] \otimes_P D.\]

We have already established that if \((P, \Theta)\) is part of a Kasparov dual, then \(T'_P = T_P\) and \([m]\) is the KK-class of the multiplication \(^\ast\)-homomorphism \(\Theta_0(X, P) \to P\), so that \([1.24]\) yields explicit formulas for \(\text{Lef}(f)\) and \(\text{Eul}_X\). This is extremely important because otherwise it would not be possible to compute these invariants.
Let $X$ and $X'$ be $\mathcal{G}$-spaces, and let $f : X \to X'$ be a $\mathcal{G}$-homotopy equivalence. Then $f$ induces an equivalence of categories $\text{KK}^G(X') \cong \text{KK}^G(X)$, that is, we get invertible maps

$$f^* : \text{KK}^G(X'; A, B) \to \text{KK}^G(X; A, B)$$

for all $\mathcal{G}$-$C^*$-algebras $A$ and $B$. Now assume, in addition, that $f$ is proper; we do not need the inverse map or the homotopies to be proper. Then $f$ induces a $\ast$-homomorphism $f^* : \mathcal{C}_0(X') \to \mathcal{C}_0(X)$, which yields $[f^*] \in \text{KK}^G(\mathcal{C}_0(X'), \mathcal{C}_0(X))$. We write $[f^*]$ instead of $[f^*]$ to better distinguish this from the map $f^*$ above. Unless $f$ is a proper $\mathcal{G}$-homotopy equivalence, $[f^*]$ need not be invertible.

**Theorem 1.26.** Let $X$ and $X'$ be $\mathcal{G}$-spaces with abstract duals, and let $f : X \to X'$ be both a proper map and a $\mathcal{G}$-homotopy equivalence. Then

$$[f^*] \otimes_{\mathcal{C}_0(X)} \text{Eul}_X = \text{Eul}_{X'}$$

in $\text{KK}^G(\mathcal{C}_0(X'), \mathcal{1})$ and the Lefschetz maps for $X$ and $X'$ are related by a commuting diagram

$$
\begin{array}{ccc}
\text{KK}^G(\mathcal{C}_0(X), \mathcal{1}) & \xrightarrow{\text{Lef}_X} & \text{KK}^G(\mathcal{C}_0(X'), \mathcal{1}) \\
\text{RKK}^G_0(X; \mathcal{C}_0(X), \mathcal{1}) \downarrow & & \downarrow \text{Lef}_{X'} \ \\
\text{RKK}^G_0(X'; \mathcal{C}_0(X'), \mathcal{1}) & \xrightarrow{[f^*]^*} & \text{RKK}^G_0(X'; \mathcal{C}_0(X'), \mathcal{1})
\end{array}
$$

where $[f^*]^*$ denotes composition with $[f^*]$.

In particular, $\text{Eul}_X$ and the map $\text{Lef}_X$ do not depend on the chosen dual.

The proof relies on the discussion preceding the theorem.

Theorem 1.26 implies that the Lefschetz maps for properly $\mathcal{G}$-homotopy equivalent spaces are equivalent because then $[f^*]$ is invertible, so that all horizontal maps in the diagram in Theorem 1.26 are invertible. In this sense, the Lefschetz map and the Euler class are invariants of the proper $\mathcal{G}$-homotopy type of $X$.

The construction in Example 1.20 associates a class $[\Delta_f] \in \text{KK}^G_0(X; \mathcal{C}_0(X), \mathcal{1})$ to any continuous, $\mathcal{G}$-equivariant map $f : X \to X$; it does not matter whether $f$ is proper. We abbreviate

$$\text{Lef}(f) := \text{Lef}([\Delta_f])$$

and call this the Lefschetz invariant of $f$. Of course, equivariantly homotopic self-maps induce the same class in $\text{RKK}^G_0(X; \mathcal{C}_0(X), \mathcal{1})$ and therefore have the same Lefschetz invariant. We have $\text{Lef}(\text{id}_X) = \text{Eul}_X$.

More generally, specializing (1.19) gives a map

$$\cup \otimes_{\mathcal{L}_X} \Delta_X : \text{KK}^G_0(\mathcal{C}_0(X), \mathcal{C}_0(X)) \to \text{KK}^G_0(X; \mathcal{C}_0(X), \mathcal{1}),$$

which we compose with the Lefschetz map; abusing notation, we still denote this composition by

$$\text{Lef} : \text{KK}^G_0(\mathcal{C}_0(X), \mathcal{C}_0(X)) \to \text{KK}^G_0(\mathcal{C}_0(X), \mathcal{1})$$

Finally, we record that Lefschetz invariants for elements of $\text{RKK}^G_0(X; \mathcal{C}_0(X), \mathcal{1})$ can be arbitrarily complicated; the Lefschetz map is rather easily seen to be split surjective. The splitting is given by specializing the inflation map

$$p_X : \text{KK}^G_0(A, B) \to \text{KK}^{\mathcal{G} \times X}(\mathcal{1}_X \otimes A, \mathcal{1}_X \otimes B)$$

(1.27) to $A := \mathcal{1}_X$ and $B := \mathcal{1}$. The fundamental example of a Kasparov dual is provided by the vertical tangent space to a bundle of smooth manifolds over the base $Z$ of a groupoid, in which morphisms act smoothly. We come back to this in 

[41]
2. Examples of computations of the Lefschetz map

In this section we will give some examples of computations of the Lefschetz map for various instances of spaces with duals and for equivariant self-morphisms coming from actual maps. The problem of computing the Lefschetz invariants of more general Kasparov self-morphisms for the next section is a central problem for us, and will be treated in §4 once we have available the theory of equivariant correspondences.

Most of the examples are quite close to proper actions, but they do not quite have to be proper. The point is that an abstract (or Kasparov) dual for a $G$-space $X$ also yields one for $X$ regarded as a $G'$-space where $G'$ is a (not necessarily closed) subgroupoid of $G$. Even if $X$ is proper as a $G$-space, it need not be proper as a $G'$-space since $G'$ may not be closed.

Known examples of duals are of two types; they are both Kasparov duals. If $G$ is a locally compact group acting as simplicial automorphisms of a finite-dimensional simplicial complex then $X$ has a Kasparov dual by [13]. This does not quite imply that the action of $G$ is proper, since the action of $G$ could be trivial. Neither does it quite imply that $G$ must be discrete, though since the connected component of the identity of $G$ must act trivially, the only interesting examples must involve disconnected groups. For instance, the non-discrete group $\text{SL}(2, \mathbb{Q}_p)$ acts simplicially on a tree.

If $X$ is a complete complete Riemann manifold then $X$ is a proper $G'$-space where $G'$ is the Lie group of isometries of $X$, and either the Clifford algebra of $X$ or the $C^*$-algebra of $\mathcal{C}_o$-functions on the tangent bundle $TX$ of $X$ is part of a Kasparov dual for $X$ (see e.g. [13] or [11]), and also §4. Hence if $G$ is any group of isometries of a Riemannian manifold $X$ with finitely many components, then the $G$-space $X$ also has a Kasparov dual. If the tangent bundle $TX$ admits an equivariant K-orientation, then $\mathcal{C}_o(TX)$ can be replaced by $\mathcal{C}_o(X)$. For instance, the circle with the group $\mathbb{Z}$ acting by an irrational rotation has a Kasparov dual of of dimension $1$, given by $\langle (\mathcal{C}(\mathbb{T}), D, \Theta) \rangle$ where $D$ is the class of the Dirac operator on the circle.

The Lefschetz invariants of equivariant self-maps of $X$ will in both these situations turn out to be in some sense zero-dimensional in the sense that they are built out of point-evaluation classes. As we will see later, the Lefschetz invariants of more general Kasparov self-morphisms are more complicated, higher-dimensional objects.

2.1. The combinatorial Lefschetz map. Let $X$ be a finite-dimensional simplicial complex and let $G$ be a locally compact group acting smoothly and simplicially on $X$ (that is, stabilisers of points are open). We follow [13] and [15]. Assume that $X$ admits a colouring (that is, $X$ is typed) and that $G$ preserves the colouring. This ensures that if $g \in G$ maps a simplex to itself, then it fixes that simplex pointwise.

Let $SX$ be the set of (non-degenerate) simplices of $X$ and let $S_dX \subseteq SX$ be the subset of $d$-dimensional simplices. The group $G$ acts on the discrete set $SX$ preserving the decomposition $SX = \bigsqcup S_dX$. Decompose $SX$ into $G$-orbits. For each orbit $\hat{\sigma} \subseteq SX$, choose a representative $\sigma \in SX$ and let $\xi_{\sigma} \in X$ be its barycentre and $\text{Stab}(\sigma) \subseteq G$ its stabiliser. Restriction to the orbit $\hat{\sigma}$ defines a $G$-equivariant *-homomorphism

$$\xi_\sigma : \mathcal{C}_o(X) \rightarrow \mathcal{C}_o(G/\text{Stab}(\sigma)) \rightarrow \mathbb{K}(\ell^2(G/\text{Stab}\sigma)),$$

where the second map is the representation by pointwise multiplication operators. We let $[\xi_\sigma]$ be its class in $\text{KK}_0^G(\mathcal{C}_o(X), \mathbb{I})$. 


Let $\varphi: X \to X$ be a $G$-equivariant self-map of $X$. Since $\varphi$ is $G$-equivariantly homotopic to a $G$-equivariant cellular map, we may assume without loss of generality that $\varphi$ is itself cellular. Hence it induces a $G$-equivariant chain map

$$\varphi: C_*(X) \to C_*(X),$$

where $C_*(X)$ is the chain complex of oriented simplices of $X$. A basis for $C_*(X)$ is given by the set of (un-)oriented simplices, by arbitrarily choosing an orientation on each simplex. We may describe the chain map $\varphi$ by its matrix coefficients $\varphi_{\sigma\tau} \in \mathbb{Z}$ with respect to this basis; thus the subscripts are unoriented simplices. For example, if $\varphi$ maps a simplex to itself, and reverses orientation, then $\varphi_{\sigma,\sigma} = -1$. Since $\varphi$ is $G$-equivariant, $\varphi_{g(\sigma), g(\sigma)} = \varphi_{\sigma\sigma}$. So the following makes sense.

**Notation 2.2.** For $\hat{\sigma} \in G\backslash S_0 X$, let $n(\varphi, \hat{\sigma}) := (-1)^d \varphi_{\sigma\sigma} \in \mathbb{Z}$ for any choice of representative $\sigma \in \hat{\sigma}$.

The following theorem is proved in [15] using the simplicial dual developed in [13], and inspired by ideas of Kasparov and Skandalis in [21].

**Theorem 2.3.** Let $X$ be a finite-dimensional coloured simplicial complex and let $G$ be a locally compact group that acts smoothly and simplicially on $X$, preserving the colouring. Let $\varphi: X \to X$ be a $G$-equivariant self-map. Define $n(\varphi, \hat{\sigma}) \in \mathbb{Z}$ and $[\xi_\sigma] \in KK^G(C_0(X), \mathbb{C})$ for $\hat{\sigma} \in G\backslash S_0 X$ as above. Then

$$\text{Lef}(\varphi) = \sum_{\hat{\sigma} \in G\backslash S_0 X} n(\varphi, \hat{\sigma})[\xi_\sigma].$$

In the non-equivariant situation, if $X$ is connected and has only a finite number of simplices, then the formula just given reduces to the ordinary Lefschetz number of $G$ by the (standard) argument that proves that the Euler characteristic of a finite simplicial complex can be computed either by counting ranks of simplicial homology groups, or by directly counting the number of simplices in the complex. In the noncompact case this doesn’t make sense anymore since the number of orbits of simplices may be infinite, but our definition of the Lefschetz invariant still makes sense. This illustrates the advantage of considering $K$-homology classes instead of numbers as fixed-point data.

### 2.2. The smooth Lefschetz map

Let $X$ be a smooth manifold and $G$ a group acting by isometric diffeomorphisms of $X$. Then we can build a Kasparov dual for $X$ using the Clifford algebra. The real (resp. complex) Clifford algebra of a Euclidean vector space $V$ is generated as a real (resp. complex) unital $*$-algebra by an orthonormal basis $\{e_i\}$ for $V$ with the relations that $e_i$ is self-adjoint and $e_i e_j + e_j e_i = 2 \delta_{ij}$. In the complex case this produces a finite-dimensional $\mathbb{Z}/2$-graded $C^*$-algebra. More generally if $V$ is a Euclidean vector bundle, this construction applies and produces a locally trivial bundle of finite-dimensional $\mathbb{Z}/2$-graded $C^*$-algebras over $X$. If $V := TX$ for a Riemannian manifold $X$ then the Clifford algebra of $X$ is the corresponding $C^*$-algebra of sections vanishing at infinity. It is denoted $\mathcal{C}_r(X)$. This $C^*$-algebra carries a canonical action of the group of isometries of $X$ and hence likewise for any subgroup.

We discuss the specific mechanics of the Clifford Kasparov dual to the following extent. Let $d$ be the de Rham differential on $X$, acting on $L^2$-forms on $X$. This Hilbert space carries a unitary action of $G$ and both $d$ and its adjoint $d^*$ are $G$-equivariant, and $d + d^*$ is an elliptic operator called the Euler (or de Rham) operator on $X$. If $\omega$ is a differential form on $X$ vanishing at infinity then the operator $\lambda_\omega$ of exterior product with $\omega$ defines an operator on $L^2$-forms which is bounded and the assignment $\omega \mapsto \lambda_\omega + \lambda_\omega^*$ determines a representation of $\mathcal{C}_r(X)$ which
graded commutes modulo bounded operators with \( d + d^* \). Hence we get a cycle for \( \text{KK}^G(\mathcal{C}_r(X), \mathbb{C}) \). It represents the class \( D \) appearing in the Kasparov dual.

We start by recording one of the easiest computations of the Lefschetz map. The interested reader can easily prove it for his or herself using after looking briefly at the definition of the class \( \Theta \) (see [13]) and reviewing the definition of the Lefschetz map. Recall that the Euler class of \( X \) is the Lefschetz invariant of the identity map of \( X \).

**Proposition 2.4.** The Euler class of \( X \) is the class of the Euler operator on \( X \).

The functoriality result [1.26] combines with Proposition 2.4 to imply the homotopy-invariance of the class of the Euler operator:

\[
f_*(\text{Eul}_X) = \text{Eul}_{X'}
\]

for a proper \( G \)-equivariant homotopy-equivalence \( f: X \to X' \) between smooth and proper \( G \)-manifolds.

We now describe the Lefschetz invariant of a more general smooth \( G \)-equivariant self-map of \( X \). This requires a preliminary discussion.

Let \( Y \) be a locally compact space and \( G \) be a locally compact group acting continuously on \( Y \), and let \( \pi: E \to Y \) be a \( G \)-equivariant Euclidean \( \mathbb{R} \)-vector bundle over \( E \). Let \( A: E \to E \) be a \( G \)-equivariant vector bundle automorphism, that is, an automorphism \( A \) preserves orientation and satisfies \( A^* A = 1 \). We are going to define a \( G \)-equivariant bilinear form on the fibres of \( E \). We are going to define a \( G \)-equivariant \( \mathbb{Z}/2 \)-graded real line bundle sign(A) over \( Y \). (Since we work with complex K-theory we will only use its complexification.)

If \( Y \) is a point, then \( G \)-equivariant real vector bundles over \( Y \) correspond to real orthogonal representations of \( G \). The endomorphism \( A \) becomes in this case an invertible linear map \( A: \mathbb{R}^n \to \mathbb{R}^n \) commuting with \( G \). The sign is a virtual 1-dimensional representation of \( G \) and hence corresponds to a pair \((\chi, n)\), where \( n \in \{0, 1\} \) is the parity (we are referring to the grading, either even or odd) of the line bundle and \( \chi: G \to \{-1, +1\} \) is a real-valued character. The overall parity will turn out to be 0 if \( A \) preserves orientation and 1 if \( A \) reverses orientation (see Example 2.6). In this sense, our invariant will refine the orientation of \( A \).

As above, let \( \text{Cliff}(E) \) be the bundle of real Clifford algebras associated to \( E \). We can also define in an analogous way \( \text{Cliff}(E) \) if \( E \) carries an indefinite bilinear form and it is a well-known fact from algebra that if the index of the bilinear form on \( E \) is divisible by 8, then the fibres of \( \text{Cliff}(E) \) are isomorphic to matrix algebras. In this case, a \( G \)-equivariant spinor bundle for \( E \) is a \( \mathbb{Z}/2 \)-graded real vector bundle \( S_E \) together with a grading preserving, \( G \)-equivariant \( * \)-algebra isomorphism \( c: \text{Cliff}(E) \to \text{End}(S_E) \). This representation is determined uniquely by its restriction to \( E \subseteq \text{Cliff}(E) \), which is a \( G \)-equivariant map \( c: E \to \text{End}(S_E) \) such that \( c(x) \) is odd and symmetric and satisfies \( c(x)^2 = \|x\|^2 \) for all \( x \in E \).

The spinor bundle is unique up to tensoring with a \( G \)-equivariant real line bundle \( L \): if \( c_1: E \to S_1 \) for \( t = 1, 2 \) are two \( G \)-equivariant spinor bundles for \( E \), then we define a \( G \)-equivariant real line bundle \( L \) over \( Y \) by

\[
L := \text{Hom}_{\text{Cliff}(E)}(S_1, S_2),
\]

and the evaluation isomorphism \( S_1 \otimes L \overset{\cong}{\to} S_2 \) intertwines the representations \( c_1 \) and \( c_2 \) of \( \text{Cliff}(E) \).

**Definition 2.5.** Let \( A: E \to E \) be a real \( G \)-equivariant vector bundle automorphism and let \( A = T \circ (A^* A)\frac{1}{2} \) be its polar decomposition with an orthogonal vector bundle automorphism \( T: E \to E \). Let \( F \) be another \( G \)-equivariant vector bundle over \( Y \) with a non-degenerate bilinear form, such that the signature of \( E \oplus F \) is divisible by 8, so that \( \text{Cliff}(E \oplus F) \)
is a bundle of matrix algebras over \( \mathbb{R} \). We assume that \( E \oplus F \) has a \( G \)-equivariant spinor bundle, that is, there exists a \( G \)-equivariant linear map \( c : E \oplus F \to \text{End}(S) \) that induces an isomorphism of graded \( \ast \)-algebras

\[
\text{Cliff}(E \oplus F) \cong \text{End}(S).
\]

Then

\[
c' : E \oplus F \to \text{End}(S), \quad (\xi, \eta) \mapsto c(T(\xi), \eta)
\]
yields another \( G \)-equivariant spinor bundle for \( E \oplus F \). We let

\[
\text{sign}(A) := \text{Hom}_{\text{Cliff}(E \oplus F)}((S, c'), (S, c)).
\]

This is a \( G \)-equivariant \( \mathbb{Z}/2 \)-graded real line bundle over \( Y \).

It is not hard to check that \( \text{sign}(A) \) is well-defined and a homotopy invariant. Furthermore, \( \text{sign}(A_1 \circ A_2) \cong \text{sign}(A_1) \otimes \text{sign}(A_2) \) for two equivariant automorphisms \( A_1, A_2 : E \to E \) of the same bundle, and \( \text{sign}(A_1 \oplus A_2) \cong \text{sign}(A_1) \otimes \text{sign}(A_2) \) for two equivariant vector bundle automorphisms \( A_1 : E_1 \to E_1 \) and \( A_2 : E_2 \to E_2 \).

If \( Y \) is a point and \( G \) is trivial, then \( \text{sign}(A) = \mathbb{R} \) for orientation-preserving \( A \) and \( \text{sign}(A) = \mathbb{R}^{\text{op}} \) for orientation-reversing \( A \), as claimed above.

**Example 2.6.** Consider \( G = \mathbb{Z}/2 \). Let \( \tau : G \to \{1\} \) be the trivial character and let \( \chi : G \to \{+1, -1\} \) be the non-trivial character. Let \( \mathbb{R}_\chi \) denote the real representation of \( G \) on \( \mathbb{R} \) with character \( \chi \). This can be considered trivially graded; let \( \mathbb{R}_\chi^{\text{op}} \) denote the same representation but with the opposite grading (the whole vector space is considered odd.)

Consider \( A : \mathbb{R}_\chi \to \mathbb{R}_\chi, \ t \mapsto -t \), so \( A \) commutes with \( G \). Then \( \text{sign}(A) \cong \mathbb{R}_\chi^{\text{op}} \) carries a non-trivial representation.

To see this, let \( F \) be \( \mathbb{R}_\chi \) with negative definite metric. Thus the Clifford algebra of \( \mathbb{R}_\chi \otimes \mathbb{R}_\chi \) is \( \text{Cliff}_{1,1} \cong \mathbb{M}_{2 \times 2}(\mathbb{R}) \). Explicitly, the map

\[
c(x, y) = \begin{pmatrix}
0 & x - y \\
x + y & 0 \\
\end{pmatrix}
\]

induces the isomorphism. We equip \( \mathbb{R}^2 \) with the representation \( \tau \oplus \chi \), so that \( c \) is equivariant.

Twisting by \( A \) yields another representation

\[
c'(x, y) := c(-x, y) = Sc(x, y)S^{-1} \quad \text{with} \quad S = S^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Since \( S \) reverses the grading and exchanges the representations \( \tau \) and \( \chi \), it induces an isomorphism \( (\mathbb{R}_\tau \oplus \mathbb{R}_\chi^{\text{op}}) \otimes \mathbb{R}_\chi^{\text{op}} \cong (\mathbb{R}_\tau \oplus \mathbb{R}_\chi^{\text{op}}) \). Hence \( \text{sign}(A) = \mathbb{R}_\chi^{\text{op}} \).

Now let \( X \) be a smooth Riemannian manifold and assume that \( G \) acts on \( X \) isometrically and continuously.

Let \( \varphi : X \to X \) be a \( G \)-equivariant self-map of \( X \). In order to write down an explicit local formula for \( \text{Lef}(\varphi) \), we impose the following restrictions on \( \varphi \):

\begin{itemize}
  \item \( \varphi \) is smooth;
  \item the fixed point subset \( \text{Fix}(\varphi) \) of \( \varphi \) is a submanifold of \( X \);
  \item if \( (p, \xi) \in TX \) is fixed by the derivative \( D\varphi \), then \( \xi \) is tangent to \( \text{Fix}(\varphi) \).
\end{itemize}

The last two conditions are automatic if \( \varphi \) is isometric with respect to some Riemannian metric (not necessarily the given one) and this of course applies by averaging the given metric if \( \varphi \) has finite order.

In the simplest case, \( \varphi \) and \( \text{id}_X \) are transverse, that is, \( \text{id} - D\varphi \) is invertible at each fixed point of \( \varphi \); this implies that \( \varphi \) has isolated fixed points.

To describe the Lefschetz invariant, we abbreviate \( Y := \text{Fix}(\varphi) \). This is a closed submanifold of \( X \) by assumption. Let \( \nu \) be the normal bundle of \( Y \) in \( X \). Since
Theorem 2.7. Let $X$ be a complete smooth Riemannian manifold, let $G$ be a locally compact group that acts on $X$ smoothly and by isometries, and let $\varphi: X \to X$ be a self-map satisfying the three conditions enumerated above. Let $\nu$ be the normal bundle of $Y$ in $X$ and let $D_{\nu}\varphi: \nu \to \nu$ be induced by the derivative of $\varphi$ as above. Let $\nu_Y: C_0(X) \to C_0(Y)$ be the restriction mapping and let $\text{Eul}_Y \in KK^G_0(C_0(Y), 1)$ be the equivariant Euler characteristic of $Y$. Then

\[ \text{Lef}(\varphi) = \nu_Y \otimes c_0(Y) \cdot \text{sign}(\text{id}_G - D_{\nu}\varphi) \otimes c_0(Y) \cdot \text{Eul}_Y. \]

Furthermore, $\text{Eul}_Y$ is the equivariant $K$-homology class of the de Rham operator on $Y$.

In brief, the Lefschetz invariant of $G$ is the Euler characteristic of the fixed-point set, twisted by an appropriate equivariant line bundle depending on orientation data.

If $\varphi$ and $\text{id}_X$ are transverse then the fixed point subset $Y$ is discrete. A discrete set is a manifold and its Euler characteristic -- a degenerate case of Proposition 2.4. It is represented by the Kasparov cycle in which the Hilbert space is $L^2(\Lambda^*_G(T^*Y)) := \ell^2(Y)$ equipped with the representation $C_0(Y) \to K(\ell^2(Y))$ by pointwise multiplication operators, and the zero operator. The group permutes the points of $Y$ and so acts by unitaries on $\ell^2(Y)$.

The normal bundle $\nu$ to $Y$ in $X$ in this case is the restriction of the vector bundle $TX$ to the subset $Y$. For $p \in X$, let $n_p$ be $+1$ if $\text{id}_p - D_{\nu}\varphi$ preserves orientation, and $-1$ otherwise. The graded equivariant line bundle sign$(\text{id}_G - D_{\nu}\varphi)$ in Theorem 2.7 is determined by pairs $(n_p, \chi_p)$ for $p \in X$, where $n_p$ is the parity of the representation at $p$ and $\chi_p$ is a certain real-valued character $\chi_p: \text{Stab}(p) \to \{-1, +1\}$ that depends on $\text{id}_p - D_{\nu}\varphi$ and the representation of the stabiliser $\text{Stab}(p) \subseteq G$ on $T_pX$. Equivariance implies that $n_p$ is constant along $G$-orbits, whereas $\chi_p$ behaves like $\chi_{g \cdot p} = \chi_p \circ \text{Ad}(g^{-1})$. Let $\ell^2_G(G)$ be the representation of the cross-product $G \times C_0(G/\text{Stab}(p))$ obtained by inducing the representation $\chi_p$ from $\text{Stab}(p)$, and let $C_0(X)$ act on $\ell^2_G(G)$ by restriction to $G/\text{Stab}(p)$. This defines a $G$-equivariant $*$-homomorphism

\[ \xi_{G,\varphi}: C_0(X) \to K(\ell^2_G). \]

Theorem 2.7 asserts the following:

Corollary 2.8. If the graph of $\varphi$ is transverse to the diagonal in $X \times X$ then,

\[ \text{Lef}(\varphi) = \sum_{G \in G/\text{Fix}(\varphi)} n_p[\xi_{G,\varphi}] \]

where $[\xi_{G,\varphi}] \in KK^G_0(C_0(X), 1)$ and the multiplicities $n_p$ are explained above.

Furthermore, the character $\chi: \text{Stab}_G(p) \to \{-1, +1\}$ at a fixed point $p$ has the explicit formula

\[ \chi(g) = \text{sign det}(\text{id} - D_{p}\varphi) \cdot \text{sign det}(\text{id} - D_{p}\varphi_{\text{Fix}(g)}). \]

If, in addition, $G$ is trivial and $X$ is connected, then $\xi_{G,\varphi} = \text{ev}_p$ for all $p \in Y$; moreover, all point evaluations have the same K-homology class because they are homotopic. Hence we get the classical Lefschetz data multiplied by the K-homology class of a point

\[ \text{Lef}(\varphi) = \left( \sum_{p \in \text{Fix}(\varphi)} \text{sign}(\text{id}_p - D_{\varphi_p}) \right) \cdot [\text{ev}] \]

as asserted above. This sum is finite if $X$ is compact.
We include the following for the benefit of the reader; it enables her or he to verify our computations by direct inspection.

**Lemma 2.9.** Let $H \subseteq G$ be compact and open, let $p, q \in X^H$ belong to the same path component of the fixed point subspace $X^H$, and let $\chi \in \text{Rep}(H)$. Then

$$[\xi_{Gp, \text{Ind}_H^{\text{stab}(Gp)}}^\text{stab}(Gp)(\chi)] = [\xi_{Gq, \text{Ind}_H^{\text{stab}(Gp)}}^\text{stab}(Gp)(\chi)] \quad \text{in } \text{KK}_0(\text{C}_0(X), \mathbb{I}).$$

**Remark 2.10.** If the identity map $\text{id}: X \to X$ can be equivariantly perturbed to be in general position in the sense explained above, then combining Proposition 2.4 and Corollary 2.8 proves that the class of the de Rham operator in $\text{KK}_0^G(\text{C}_0(X), \mathbb{C})$ is a sum of point-evaluation classes. Lück and Rosenberg show in [24] that this can always be achieved when $G$ is discrete and acts properly on $X$.

The next two explicit examples involve isolated fixed points.

**Example 2.11.** Let $G \cong \mathbb{Z} \ltimes \mathbb{Z}/2\mathbb{Z}$ be the infinite dihedral group, identified with the group of affine transformations of $\mathbb{R}$ generated by $u(x) = -x$ and $v(x) = x + 1$. Then $G$ has exactly two conjugacy classes of finite subgroups, each isomorphic to $\mathbb{Z}/2$. Its action on $\mathbb{R}$ is proper, and the closed interval $[0, 1/2]$ is a fundamental domain. There are two orbits of fixed point in $\mathbb{R}$, those of 0 and 1/2, and their stabilisers represent the two conjugacy classes of finite subgroups.

Now we use some notation from Example 2.6. Each copy of $\mathbb{Z}/2$ acting on the tangent space at the fixed point acts by multiplication by $-1$ on tangent vectors. Therefore, the computations in Example 2.6 show that for any nonzero real number $A$, viewed as a linear transformation of the tangent space that commutes with $G$, we have

$$\text{sign}(A) = \begin{cases} \mathbb{R}^\text{op} & \text{if } A < 0, \\ \mathbb{R} & \text{if } A > 0. \end{cases}$$

Let $\varphi$ be a small $G$-equivariant perturbation of the identity map $\mathbb{R} \to \mathbb{R}$ with the following properties. First, $\varphi$ maps the interval $[0, 1/2]$ to itself. Secondly, its fixed points in $[0, 1/2]$ are the endpoints 0, 1/2, and 1/4; thirdly, its derivative is bigger than 1 at both endpoints and between 0 and 1 at 1/4. Such a map clearly exists. Furthermore, it is homotopic to the identity map, so that $\text{Lef}(\varphi) = \text{Eul}_G$.

By construction, there are three fixed points modulo $G$, namely, the orbits of 0, 1/4 and 1/2. The isotropy groups of the first and third orbit are non-conjugate subgroups isomorphic to $\mathbb{Z}/2$; from Example 2.6 each of them contributes $\mathbb{R}^\text{op}$. The point 1/4 contributes the trivial character of the trivial subgroup. Hence

$$\text{Lef}(\varphi) = -[\xi_{\mathbb{Z}, \chi}] - [\xi_{\mathbb{Z}+1/2, \chi}] + [\xi_{\mathbb{Z}+1/4}].$$

On the other hand, suppose we change $\varphi$ to fix the same points but to have zero derivative at 0 and 1/2 and large derivative at 1/4. This is obviously possible. Then we get contributions of $\mathbb{R}_+$ at 0 and 1/2 and a contribution of $-\xi_{1/4}$ at 1/4. Hence

$$\text{Lef}(\varphi) = [\xi_{\mathbb{Z}, r}] + [\xi_{\mathbb{Z}+1/2, r}] - [\xi_{\mathbb{Z}+1/4}].$$

Combining both formulas yields the identity

$$(2.12) \quad [\xi_{\mathbb{Z}, r}] + [\xi_{\mathbb{Z}+1/2, r}] - [\xi_{\mathbb{Z}+1/4}] = -[\xi_{\mathbb{Z}, \chi}] - [\xi_{\mathbb{Z}+1/2, \chi}] + [\xi_{\mathbb{Z}+1/4}].$$

By the way, the left-hand side is the description of $\text{Eul}_G$ we get from the combinatorial dual with the obvious $G$-invariant triangulation of $\mathbb{R}$ with vertex set $\mathbb{Z} + 1/2\mathbb{Z} \subseteq \mathbb{R}$.

Using Lemma 2.9 one can check (2.12) by direct computation.
In the previous chapter we explained our computation of the Lefschetz map for self-maps of a $G$-space $X$ in several relatively simple situations. In these cases, $G$ was in each case a group (a groupoid with trivial base). Obviously, not all equivariant Kasparov self-morphisms $\text{KK}^G_\ast(C_0(X), C_0(X))$ are represented by maps. We have organized this survey around the problem of computing Lefschetz invariants of more general equivariant Kasparov self-morphisms. This requires describing the morphisms themselves in some geometric way. The theory of correspondences of Baum, Connes and Skandalis (see [4] [8]) would seem ideal for this purpose. Since we are working in the equivariant setting, to use it would necessitate checking that the pseudodifferential calculus which plays such a prominent role in [8] works equivariantly with respect to an action of a group (or groupoid), as well as proving the main functoriality result for $K$-oriented maps. Although it seems plausible that such an extension could be carried out, a major problem arises in connection with composing correspondences using transversality in the equivariant situation (we explain this below.) A trick of Baum and Block (see [4] [8] and [43]) is useful in this connection, but in order for it to work, some hypotheses on vector bundles are necessary. Our approach is to build in the vector bundle requirements into the definitions. This is only reasonable if $G$ is proper; we now show how to reduce to this case using the Baum-Connes conjecture.

3.1. **Using Baum-Connes to reduce to proper groupoids.** Let $G$ be a locally compact group (or groupoid). The classifying space $\mathcal{E}G$ for proper actions of $G$ is the proper $G$-space with the universal property that if $X$ is any proper $G$-space, then there is a $G$-equivariant classifying map $\chi: X \to \mathcal{E}G$ which is unique up to $G$-homotopy. If $G$ is a proper groupoid to begin with, then $\mathcal{E}G = Z$ gives a simple model for $\mathcal{E}G$, for it is proper as a $G$-space and has the required universal property. In particular, if $G$ is a compact group, then $\mathcal{E}G$ is a point.

For $G$ and $G$-spaces $X$ and $Y$, the inflation map $\text{In} [127]$

\[
p^\mathcal{E}G: \text{KK}^G_\ast(C_0(X), C_0(Y)) \to \text{RKK}^G_\ast(\mathcal{E}G; C_0(X), C_0(Y))
\]

\[
:= \text{KK}^G_{\ast \times \mathcal{E}G}(C_0(X \times \mathcal{E}G), C_0(Y \times \mathcal{E}G))
\]

is an isomorphism as soon as the $G$ action on $X$ is topologically amenable, and in particular as soon as it is proper. An abstract (respectively Kasparov) dual for the $G$-space $X$ pulls back to one for $X \times \mathcal{E}G$ as a $G \times \mathcal{E}G$-space, and the diagram

\[
\begin{array}{ccc}
\text{KK}^G_\ast(C_0(X), C_0(X)) & \xrightarrow{\text{Lef}} & \text{KK}^G_\ast(C_0(X), C) \\
\cong & & \cong \\
\text{KK}^G_{\ast \times \mathcal{E}G}(C_0(X \times \mathcal{E}G), C_0(X \times \mathcal{E}G)) & \xrightarrow{\text{Lef}} & \text{KK}^G_{\ast \times \mathcal{E}G}(C_0(X \times \mathcal{E}G), C_0(\mathcal{E}G))
\end{array}
\]

commutes. Hence the Lefschetz map for $G$ acting on $X$ is isomorphic to the Lefschetz map of $G \times \mathcal{E}G$ acting on $X \times \mathcal{E}G$.

This replaces the non-proper groupoid $G$ by the proper groupoid $\mathcal{G} := G \times \mathcal{E}G$ at no loss of information.

In terms of this situation, our definitions are going to yield a a theory of $G \times \mathcal{E}G$-equivariant correspondences based on $K$-oriented $G \times \mathcal{E}G$-equivariant vector bundles (equivalently, $\mathcal{G}$-equivariant vector bundles on $X \times \mathcal{E}G$) and $G \times \mathcal{E}G$-equivariant open embeddings. Such correspondences will yield analytic Kasparov morphisms since open embeddings do, while zero sections and projections of $G \times \mathcal{E}G$-equivariantly...
K-oriented vector bundles yield analytic Kasparov morphisms for the Thom isomorphism for K-oriented vector bundles over a space with an action of a proper groupoid, which is proved in [22].

In order to compose correspondences we need a sufficient supply of ‘trivial’ vector bundles, but for this too, the fact that we have a proper groupoid makes a big difference. If \( \mathcal{G} \) is a groupoid acting on a space, then a \( \mathcal{G} \)-equivariant vector bundle (see §3.2) over that space is trivial if it is pulled back from the unit space of \( \mathcal{G} \) using the anchor map for the space. In the case we are discussing, where \( \mathcal{G} := G \times \mathcal{E}G \), a \( G \times \mathcal{E}G \)-vector bundle on \( X \times \mathcal{E}G \) is the same as a \( G \)-vector bundle on \( X \times \mathcal{E}G \), and a trivial \( G \times \mathcal{E}G \)-vector bundle is a \( G \)-vector bundle on \( X \times \mathcal{E}G \) which is pulled back from \( \mathcal{E}G \) under the coordinate projection.

In general, if a groupoid is not proper, it may have no equivariant vector bundles on its base, e.g. if our groupoid is a group \( G \), then its base is a point, so that a trivial \( G \)-vector bundle over \( X \) is equivalent to a finite-dimensional representation of \( G \).

### 3.2. Equivariant vector bundles

As per above, \( \mathcal{G} \) shall be a proper groupoid until further notice. If \( \mathcal{G} \) is a group, this means that it must be compact. We will frequently consider this case in examples.

If \( X \) is a \( \mathcal{G} \)-space, we remind the reader that the anchor map for the action is denoted \( \alpha_X : X \to Z \). A \( \mathcal{G} \)-equivariant vector bundle on \( X \) is a vector bundle on \( X \) which is also a \( \mathcal{G} \)-space such that elements of \( \mathcal{G} \) map fibres to fibres linearly. There is an obvious notion of isomorphic \( \mathcal{G} \)-equivariant vector bundles.

If \( \mathcal{G} \) is a compact group, then a \( \mathcal{G} \)-equivariant vector bundle on a point is a finite-dimensional linear representation of \( \mathcal{G} \).

**Notation 3.3.** If \( V \) is an equivariant vector bundle over \( X \) then we denote by \( \pi_V : V \to X \) the vector bundle projection and \( \zeta_V : X \to V \) the zero section. We frequently denote by \( |V| \) the total space of \( V \), and denote by \( \text{VK}_{\mathcal{G}}(X) \) the Grothendieck group of the monoid of isomorphism classes of \( \mathcal{G} \)-equivariant vector bundles over \( X \).

Given that \( \mathcal{G} \) is assumed proper, equivariant vector bundles behave in some ways just like ordinary vector bundles. For example, if \( Y \subset X \) is a closed, \( \mathcal{G} \)-invariant subset of a \( \mathcal{G} \)-space \( X \), and if \( V \) is a \( \mathcal{G} \)-equivariant vector bundle on \( X \), then any equivariant section of \( V \) defined on \( Y \) can be extended to an equivariant section defined on an open \( \mathcal{G} \)-invariant neighbourhood of \( Y \). This involves an averaging procedure (see [16]). As a consequence, if \( f_i : X \to Y \), \( i = 0,1 \) are two \( \mathcal{G} \)-equivariantly homotopic maps, and if \( V \) is a \( \mathcal{G} \)-equivariant vector bundle on \( Y \), then \( f_0^*(V) \) and \( f_1^*(V) \) are \( \mathcal{G} \)-equivariantly isomorphic.

On the other hand, some new problems appear in connection with equivariant vector bundles. We first formalize our notion of triviality and the corresponding notion of subtriviality in a definition.

**Definition 3.4.** Let \( X \) be a \( \mathcal{G} \)-space. A \( \mathcal{G} \)-vector bundle over \( X \) is trivial if it is pulled back from \( Z \) under the anchor map \( \alpha_X : X \to Z \). We denote the pull-back of a \( \mathcal{G} \)-vector bundle \( E \) over \( Z \) to the \( \mathcal{G} \)-space \( X \) by \( E^X \). A \( \mathcal{G} \)-equivariant vector bundle is subtrivial if it is a direct summand of a trivial \( \mathcal{G} \)-vector bundle.

**Example 3.5.** If \( \mathcal{G} \) is a compact group, then a trivial \( \mathcal{G} \)-equivariant vector bundle over \( X \) has the form \( X \times \mathbb{R}^n \) where \( \mathbb{R}^n \) carries a linear representation of \( \mathcal{G} \), and where \( \mathcal{G} \) acts on \( X \times \mathbb{R}^n \) diagonally.

We will make repeated use of the following basic fact about representations of compact groups (see [27]).
Lemma 3.6. Let $G$ be a compact group and $G' \subset G$ be a subgroup. Then any finite-dimensional representation of $G'$ is contained in the restriction to $G'$ of a finite-dimensional representation of $G$.

Even if $G$ is a compact group, due to the notion of ‘trivial’ vector bundle we are using, not every $G$-vector bundle is locally trivial in the category of $G$-vector bundles, i.e. locally isomorphic to a trivial $G$-vector bundle. But it is not hard to prove the following and it is a good exercise for understanding equivariant vector bundles. We leave the proof to the reader, but see §3.3 for some important ingredients in the argument.

Lemma 3.7. Every $G$-equivariant vector bundle on $X$ is locally subtrivial in the sense that for every $x \in X$ there is a $G$-equivariant vector bundle on $Z$, a $G$-equivariant neighbourhood $U$ of $x$, and an embedding $\varphi: V_U \to E^U$.

Improving this local result to a global one is not possible, however, without an appropriate compactness assumption.

Example 3.8. Let $X := \mathbb{Z}$ with the trivial action of the compact group $G := \mathbb{T}$. Then the 1-dimensional complex vector bundle $\mathbb{Z} \times \mathbb{C}$ with the action of $z \in G$ in the fibre over $n$ by the character $z \mapsto z^n$ is not subtrivial, since it contains infinitely many distinct irreducible representations of $G$.

Definition 3.9. Let $X$ be a $G$-space.

- The space $X$ has enough $G$-equivariant vector bundles if whenever we are given $x \in X$ and a finite-dimensional representation of the compact isotropy group $G_x$ of $x$, there is a $G$-equivariant vector bundle over $X$ whose restriction to $x$ contains the given representation of $G_x$.

- The space $X$ has a full vector bundle if there is a vector bundle $V$ over $X$ such that any irreducible representation of $G_x$ in the representation of $G_x$ on $V_x$ (and we say that such a vector bundle $V$ is full.)

It is the content of Lemma 3.6 that a compact group acting on a point has enough vector bundles. It does not have a full vector bundle unless it is finite, because a compact group with a finite dual has finite-dimensional $L^2(G)$ by the Peter-Weyl theorem and must then be finite.

It is easy to check that if $f: X \to Y$ is a $G$-equivariant map then $X$ has enough equivariant vector bundles if $Y$ does, and $f^*(V)$ is a full vector bundle on $X$ if $V$ is a full vector bundle on $Y$. Both of these assertions use the basic fact Lemma 3.6.

Example 3.10. The $G$-space described in Example 3.8 does not have a full vector bundle, although it obviously has enough vector bundles.

The following example is more subtle. It is due to Julianne Sauer (see [30]).

Example 3.11. Let $X = \mathbb{R}$ and $K$ be the compact group $K := \prod_{n \in \mathbb{Z}/2} \mathbb{Z}/2$ acting trivially on $\mathbb{R}$. Since $X$ is $K$-equivariantly contractible, and by the homotopy-invariance of equivariant vector bundles (mentioned above at the beginning of §3.2), any $K$-equivariant vector bundle $V$ on $X$ is trivial, and hence all the representations of $K$ on the fibres $V_x$ are equivalent.

Now let $\sigma: K \to K$ be the shift automorphism and consider the group $G := K \rtimes_{\sigma} \mathbb{Z}$; it acts on $X$ by letting $\sigma(t) := t + 1$. This is a proper action. Of course as a groupoid $G$ is not proper, but we repair this below.

We claim that the only trivial $G$-vector bundles on $X$ yield trivial $K$-representations in their fibres. This will show that $X$ does not have enough $G$-vector bundles.

The proof is as follows: any $G$-vector bundle $V$ on $X$ must be trivial as a $K$-vector bundle, as above. On the other hand, the covariance rule for the semi-direct
product implies that representations of $K$ on $V_x$ and $V_{x+1}$ are mapped to each other (up to equivalence) by the action of $\sigma$, and therefore $\hat{\sigma} : \text{Rep}(K) \rightarrow \text{Rep}(K)$ fixes the point $[V_x]$. But $\text{Rep}(K)$ is the direct sum of $\mathbb{Z}/2\mathbb{Z}$’s and $\hat{\sigma}$ acts as the shift. The only fixed point then is the zero sequence. This corresponds to a trivial representation. This means that at every point $x \in X$ the representation of $K$ we get on $V_x$ is trivial.

To repair the non-properness of $G$, replace it by $\mathcal{G} := G \ltimes \mathcal{EG}$ and replace $X$ by $X \times \mathcal{EG}$ as explained at the beginning of this section, then we get an example of a proper groupoid $\mathcal{G}$ and a $\mathcal{G}$-space $X \times \mathcal{EG}$ such which does not have enough equivariant vector bundles. This is because $X \times \mathcal{EG}$ and $X$ are $G$-equivariantly homotopy-equivalent anyway, because the action of $G$ on $X$ is proper. Hence these spaces have canonically isomorphic monoids of isomorphism classes of equivariant vector bundles.

A Morita-equivalent approach is via a mapping cylinderr construction and produces a compact groupoid acting on a compact space without enough vector bundles. Take $[0, 1] \times K$ modulo the relation $(1, k) \sim (0, \sigma(k))$. This results in a bundle of compact groups over the circle which can shown to be locally compact groupoid with Haar system.

Let $\mathcal{G}$ be this groupoid: it is proper. Its base $Z$ is the circle. By a holonomy argument similar to the one just given, any $\mathcal{G}$-equivariant vector bundle over $Z$ must restrict in each fibre to a trivial representation of $K$. Thus, there are not enough $\mathcal{G}$-equivariant vector bundles on $Z$.

Example 3.12. If $G$ is a discrete group with a $G$-compact model for $\mathcal{EG}$, then Lück and Oliver have shown in [23] that there is a full $\mathcal{G}$-equivariant vector bundle on $Z$, where $\mathcal{G} := G \ltimes \mathcal{EG}$, (so that $Z = \mathcal{EG}$.)

3.3. The topological index of Atiyah-Singer. We now indicate why the condition of having enough vector bundles, or having a full vector bundle, is important for describing analytic equivariant $\text{KK}$-groups topologically.

We start with the problem, famously treated by Atiyah and Singer, of describing the equivariant (analytic) index of a $\mathcal{G}$-equivariant elliptic operator in topological terms, where $\mathcal{G}$ is a compact group, keeping in mind that an equivariant elliptic operator is an important example of a cycle for equivariant $\text{KK}$-theory.

Let $X$ be a smooth manifold with a smooth action of the compact group $\mathcal{G}$. The symbol of an equivariant elliptic operator on $X$ is an equivariant $K$-theory class for $TX$. The idea of Atiyah and Singer for defining the topological index of the operator is to smoothly embed $X$ in a finite-dimensional (linear) representation of $\mathcal{G}$ on $\mathbb{R}^n$. The derivative of this embedding gives a smooth embedding of $TX$ in $\mathbb{R}^{2n}$, where $TX$ has the induced action of $\mathcal{G}$. Since $TX$ is an equivariantly $K$-oriented manifold, the normal bundle $\nu$ to the embedding is a $\mathcal{G}$-equivariantly $K$-oriented vector bundle on $TX$. The tubular neighbourhood embedding identities it with an open, $\mathcal{G}$-equivariant neighbourhood of the image of $TX$ in $\mathbb{R}^{2n}$. We now obtain a composition

$$\text{K}_{\mathcal{G}}^0(TX) \rightarrow \text{K}_{\mathcal{G}}^0(N) \rightarrow \text{K}_{\mathcal{G}}^0(\mathbb{R}^{2n}) \rightarrow \text{K}_{\mathcal{G}}^0(\nu) \cong \text{Rep}(\mathcal{G}),$$

where the first map is the Thom isomorphism for the equivariantly $K$-oriented $\mathcal{G}$-vector bundle $N$, the second is the map on equivariant $K$-theory induced by the open inclusion $N \hookrightarrow \mathbb{R}^{2n}$ and the third is equivariant Bott Periodicity ($\mathbb{R}^{2n} \cong \mathbb{C}^n$) with the given action of $\mathcal{G}$ has an equivariant complex structure, so a $\mathcal{G}$-equivariant spin$^c$-structure. The spinor bundle is the trivial $\mathcal{G}$-vector bundle $\Lambda^\mathcal{G}_{2n}(\mathbb{C}^n)$ over $\mathbb{C}^n$.

The content of the index theorem is that this composition agrees with the map $\text{K}_{\mathcal{G}}^0(TX) \rightarrow \text{Rep}(\mathcal{G})$ obtained by first interpreting cycles for $\text{K}_{\mathcal{G}}^0(TX)$ as symbols of
equivariant elliptic operators on $X$, making these elliptic operators into Fredholm operators, and taking their equivariant indices.

But how do we get a smooth, equivariant embedding of $X$ in a finite-dimensional linear representation of $G$ in the first place? Since it involves important ideas for us, we will sketch the proof. The result seems due to Mostow (see [27]). Very similar arguments also prove Lemma 3.7.

First of all, we may assume (or average using the Haar system on $G$) that $X$ has an invariant Riemannian metric. Now the orbit $Gx$ is a smooth embedded submanifold of $X$ isomorphic to $G/G_x$. The tangent space of $X$ at $x$ splits into the orthogonal sum of the tangent space to the orbit and its orthogonal complements $N_x := T_x(Gx)^\perp$. The latter is a finite-dimensional representation of $G_x$, and inducing it results in a $G$-equivariant vector bundle

$$N := G \times_{G_x} N_x := G \times N_x / \sim (gh, h^{-1}n) \text{ for } h \in G_x$$
onumber

on the orbit which is precisely the normal bundle to the embedded submanifold $Gx$. By exponentiating we obtain an equivariant diffeomorphism between the total space of $N$ and an invariant open neighbourhood $U$ of the orbit. We embed this neighbourhood as follows.

By Lemma 3.6 the representation of $G_x$ on $N_x$ is contained in the restriction of some representation of $G$ on some finite-dimensional vector space $\bar{N}_x$. The naturality of induction implies that we have an inclusion of vector bundles $N \subset \bar{N} := G \times_{G_x} \bar{N}_x$. But since $\bar{N}_x$ is the restriction of a $G$-representation, $\bar{N}$ is a product bundle, i.e. a trivial $G$ vector bundle on the orbit. This provides a $G$-equivariant map $U \to \bar{N}_x$, explicitly, by mapping $[(g, n)] \in U \cong N \subset \bar{N}$ to the point $gn \in \bar{N}_x$. It is of course not necessarily an embedding; to improve it to an embedding, fix a vector $v \in \bar{N}_x$ whose isotropy in $G$ is exactly $G_x$ (for this see also [27]) and set

$$\varphi: U \cong G \times_{G_x} N_x \to \bar{N}_x \oplus \bar{N}_x, \quad \varphi([(g, n)]) := (gn, gv).$$

The map $\varphi$ is an equivariant embedding as required.

As mentioned, if $X$ is compact, then we can then we can (carefully) paste together the local embeddings to get an embedding of $X$; see [16], or the source [27]. The reader should notice that the assumption that $X$ has enough vector bundles is used implicitly to show that the representation of $G_x$ on $N_x$ can be extended to a $G$-equivariant vector bundle on the orbit of $x$ (the vector bundle $\bar{N}$ induced from $\bar{N}_x$). This was the statement of Lemma 3.6 and is just an explicit way of saying that $G$ has enough vector bundles on its one-point base space.

3.4. Embedding theorems from [16]. More generally, in [16] the following is proved. Let $X$ be a $G$-space, where $G$ is a proper groupoid.

We say that $X$ is a smooth $G$-manifold if we can cover $X$ by charts of the form $U \times \mathbb{R}^n$ where $U \subset Z$ is open, so that with respect to this product structure the anchor map $g_X: X \to Z$ identifies with the first coordinate projection, and such that groupoid elements and change of coordinates are smooth in the vertical direction.

An smooth open embedding between $G$-manifolds is a smooth equivariant map which is a diffeomorphism onto an open subset of its codomain.

**Theorem 3.13.** Let $G$ be a (proper) groupoid and $X$ and $Y$ be smooth $G$-manifolds. Suppose that either

- **A.** The $G$-space $Z$ has enough vector bundles and $G \backslash X$ is compact,
- **B.** $Z$ has a full vector bundle and $G \backslash X$ has finite covering dimension.

Then, given a smooth, $G$-equivariant map $f: X \to Y$, there exists

- A smooth $G$-equivariant vector bundle $V$ over $X$,
• A smooth $\mathcal{G}$-equivariant vector bundle $E$ over $Z$,
• An smooth, equivariant open embedding $\varphi: V \to E^Y$,

such that

\[ f = \pi_{E^Y} \circ \varphi \circ \zeta_V. \tag{3.14} \]

Furthermore, under the union of hypotheses $A, B$, any $\mathcal{G}$-equivariant vector bundle over $X$ (or $Y$) is subtrivial.

Recall that the notation $E^Y$ means the pullback of $E$ to $Y$ using the anchor map $g_V : Y \to Z$.

We call a factorisation of a map $f : X \to Y$ of the form (3.14) a normal factorisation.

3.5. Normally non-singular maps. As in the previous section, $\mathcal{G}$ is a proper groupoid. The constructions of the previous section motivate the following definition.

**Definition 3.15.** A $\mathcal{G}$-equivariant normally non-singular map $\Phi$ from $X$ to $Y$ is a triple $(V, E, f)$ where $V$ is a $\mathcal{G}$-equivariant subtrivial vector bundle over $X$, $E$ is a $\mathcal{G}$-equivariant vector bundle over $Z$ and $f : V \to E^Y$ is a $\mathcal{G}$-equivariant open embedding.

- The _trace_ of $\Phi$ is the composition $\pi_{E^Y} \circ f \circ \zeta_V$.
- The _stable normal bundle_ of $\Phi$ is the class $[V] - [E^X] \in \mathcal{V}K_{\mathcal{G}}(X)$.
- The _degree_ of $\Phi$ is $\dim(V) - \dim(E)$.
- The normally non-singular map $\Phi$ is _K-oriented_ if the $V$ and $E$ are equivariantly K-oriented.
- The normally non-singular map $\Phi$ is _smooth_ if $X$ and $Y$ are smooth $\mathcal{G}$-manifolds and $V$ and $E$ are smooth equivariant vector bundles on which $\mathcal{G}$ acts smoothly, and if $f$ is a smooth embedding.

Of course the trace of a $\mathcal{G}$-equivariant smooth normally non-singular map is itself a smooth equivariant map, and the content of Theorem 3.13 is that, conversely, any smooth equivariant map between smooth $\mathcal{G}$-manifolds is the trace of some smooth normally non-singular map, under some hypotheses about the availability of equivariant vector bundles. (This statement is improved in Theorem 3.19.)

**Example 3.16.** The simplest example of a normally non-singular map is the zero section and bundle projection

$$
\zeta_V : X \to V, \quad \pi_V : [V] \to X,
$$

of a $\mathcal{G}$-equivariant vector bundle $V$ over a compact $\mathcal{G}$-space $X$, where $\mathcal{G}$ is a compact group. The zero section is the trace of the normally non-singular map $(V, 0_{[V]}, \text{id})$. The stable normal bundle is the class $[V] \in \mathcal{V}K_{\mathcal{G}}(X) \in$ of the vector bundle itself. Since $X$ is compact, $V$ is subtrivial. If $X$ is not compact, this can fail, c.f. Example 3.18.

With the same (compact) $X, V$ etc., the bundle projection $\pi_V : [V] \to X$ is the trace of a normally non-singular map $(\pi^*_V(V'), E, \varphi)$, where $V' \in \text{Vect}_\mathcal{G}(X)$ is a choice of $\mathcal{G}$-vector bundle on $X$ such that $V \oplus V'$ is a trivial bundle $E^X$, and $\varphi : [V \oplus V'] \cong [\pi^*_V(V')] \xrightarrow{\cong} E^X$ is a trivialisation. The stable normal bundle is $\pi^*_V([V']) - [E^X] \in \mathcal{V}K_{\mathcal{G}}([V])$ respectively.

The normally non-singular map just described seems to depend on the choice of trivialisation of $V$, but it can be checked that any two choices yield equivalent normally non-singular maps in the sense explained below.
Example 3.17. Let \( G \) be a compact group acting smoothly on manifolds \( X, Y \) with \( X \) compact. By the discussion in \([41, 33]\) we can fix a smooth, equivariant embedding \( i: X \to E \) in a linear representation of \( G \). Define \( V \) to be the normal bundle to the embedding \( x \mapsto (f(x), i(x)) \) of \( X \) in \( E^Y := Y \times E \). Let \( \varphi: V \to E^Y \) be the corresponding tubular neighbourhood embedding. Then the trace of the composition \( \pi_{E^Y} \circ \varphi \circ \zeta_Y \) of the normally non-singular map \((V, E, \tilde{f})\) is \( f \). Since \( TX \oplus V \cong f^*(TY) \oplus E^X \), the stable normal bundle is \( f^*([TY]) - [TX] \in \text{VK}_G(X) \).

Example 3.18. If \( G \) is a discrete group with a \( G \)-compact model for \( \mathcal{E}G \), then Lück and Oliver have shown in \([23]\) that there is a full \( G \)-equivariant vector bundle on \( \mathcal{E}G \), where \( \mathcal{G} := G \times \mathcal{E}G \), (so that the base of \( \mathcal{G} \) is \( Z := \mathcal{E}G \).) Let \( X \) and \( Y \) be smooth manifolds equipped with smooth actions of \( G \) and \( f: X \to Y \) be a smooth, \( G \)-equivariant map. As above let \( \mathcal{G} \) be the proper groupoid \( G \times \mathcal{E}G \). Applying the Baum-Connes procedure of \([5, 13]\) to this situation we get smooth \( G \)-manifolds \( X \times \mathcal{E}G \) and \( Y \times \mathcal{E}G \) and a smooth \( \mathcal{G} \)-map \( f \times \text{id}_{\mathcal{E}G}: X \times \mathcal{E}G \to Y \times \mathcal{E}G \). It is the trace of a normally non-singular map because of Theorem 3.19 and the result of Lück and Oliver.

As we will see in the next section, if \( f \) is also \( K \)-orientable in an appropriate sense, then it will give rise to a morphism in \( \text{KK}^G(\mathcal{C}_0(\mathcal{E}G \times X), \mathcal{C}_0(\mathcal{E}G \times Y)) \). If \( X \) is a topologically amenable \( G \)-space, this gives an element of \( \text{KK}^G(\mathcal{C}_0(X), \mathcal{C}_0(Y)) \).

Two normally non-singular maps \( s \) are isomorphic if there are vector bundle isomorphisms \( V_0 \cong V_1 \) and \( E_0 \cong E_1 \) that intertwine the open embeddings \( f_0 \) and \( f_1 \). The lifting of a normally non-singular map \( \Phi = \Psi = (V, \varphi, E) \) along an equivariant vector bundle \( E^+ \) over \( Z \) is the normally non-singular map \( \Phi \oplus E^+ := (V \oplus (E^+)^X, E \oplus E^+, f \times_Z \text{id}_{E^+}) \). Two normally non-singular maps \( s \) are stably isomorphic if there are \( G \)-equivariant vector bundles \( E^+_0 \) and \( E^+_1 \) such that \( \Phi_0 \oplus E^+_0 \cong \Phi_1 \oplus E^+_1 \). Finally, two normally non-singular maps \( \Phi_0 \) and \( \Phi_1 \) are isotopic if there is a continuous 1-parameter family of normally non-singular maps \( s \) whose values at the endpoints are stably isomorphic to \( \Phi_0 \) and \( \Phi_1 \) respectively (see \([10]\) for the exact definition), and are equivalent if they have isotopic liftings. There is an obvious notion of smooth equivalence of smooth normally non-singular maps \( s \).

There are obvious \( K \)-oriented analogues of the above relations. For example, lifting must only use \( K \)-oriented trivial bundles, and isomorphism must preserve the given \( K \)-orientations. Referring to this kind of equivalence we will speak of \( K \)-oriented equivalence of \( K \)-oriented normally non-singular maps.

3.6. Manifolds with smooth normally non-singular maps to \( Z \). A useful hypothesis covering a number of geometric situations is that a given smooth \( G \)-manifold \( X \) admits a smooth normally non-singular map to the object space \( Z \) of \( G \). By the theorem above this is the case if \( A \) or \( B \) hold. It means explicitly that we have a triple \((N_X, \tilde{g}, E)\) where \( N_X \) is a smooth subtrivial vector bundle over \( X \), \( E \) is an equivariant vector bundle over \( Z \) and \( \tilde{g} \) is a smooth open equivariant embedding \( N_X \to E \). Note that \( N_X \oplus TX \cong E^X \). Such a normally non-singular map is (smoothly) stably isomorphic to a \( K \)-oriented normally non-singular map because we can replace if needed \( E \) by \( E \oplus E \), which is canonically equivariantly \( K \)-oriented using the \( G \)-equivariant complex structure, and replacing \( N_X \) by \( N_X \oplus E^X \).

If \((N_X, \tilde{g}, E)\) is a smooth normal map to \( Z \) such that \( E \) is equivariantly \( K \)-oriented, then \( K \)-orientations on \( N_X \) are in 1-1 correspondence with \( K \)-orientations on \( TX \) because of the 2-out-of-3 property. One can prove the following.

Theorem 3.19. Let \( X \) and \( Y \) be smooth \( G \)-manifolds, and assume that \( X \) admits a smooth, normal \( G \)-map to \( Z \) and that \( f^*(TY) \) is subtrivial.
Then any smooth $G$-map from $X$ to $Y$ is the trace of a smooth normal $G$-map, and two smooth normally non-singular maps from $X$ to $Y$ are smoothly equivalent if and only if their traces are smoothly homotopic.

Furthermore, smooth equivalence classes of smooth $K$-oriented normally non-singular maps from $X$ to $Y$ are in 1-1 correspondence with pairs $(f, \tau)$ where $f$ is a smooth homotopy class of equivariant smooth map $X \to Y$ and $\tau$ is an equivariant $K$-orientation on $N_X \oplus f^*(TX)$.

We sketch the existence part of this proof. Fix a smooth equivariant normally non-singular map $(N_X, \hat{g}, E)$ from $X$ to $Z$. We can assume by replacing $E$ by $E \oplus E$ and $N_X$ by $N_X \oplus E_X$ if needed that $E$ is equivariantly $K$-oriented. Let $Y$ be another smooth $G$-manifold and $f: X \to Y$ be a smooth map. Let $g: X \xrightarrow{\hat{g}} N_X \xrightarrow{\hat{\pi}} E$ the composite smooth embedding.

One obtains a smooth embedding $X \to Y \times_Z E = E^Y$, $x \mapsto (f(x), g(x))$. It has a (smooth) normal bundle $V$ with a smooth open embedding in $E^Y$. Since $V \cong N_X \oplus f^*(TY)$, $V$ is subtrivial and $(V, E, f)$ is a smooth normally non-singular map with trace $f$ and stable normal bundle $[V] - [E^X] \in VK_G(X)$. Note that $V \oplus TX \cong f^*(TY) \oplus E^X$.

The stable normal bundle is $[V] - [E^X] = f^*(|TY|) - [TX] \in VK_G(X)$. Equivariant $K$-orientations on $V$ are in 1-1 correspondence with equivariant $K$-orientations on $N_X \oplus f^*(TY)$.

3.7. Correspondences. We are now in a position to define what correspondences are. Let $G$ continue to denote a proper groupoid.

**Definition 3.20.** Let $X$ and $Y$ be $G$-spaces. A $G$-equivariant correspondence from $X$ to $Y$ is a quadruple $(M, b, f, \xi)$ where $M$ is a $G$-space, $f: M \to Y$ is a $G$-equivariantly $K$-oriented normally non-singular map, $b: M \to X$ is an equivariant map, and $\xi \in RK_G^*(M)$ is a $G$-equivariant $K$-theory class with $X$-compact support (see [14]) where the $G \times X$-structure on $M$ is that determined by the $G$-equivariant map $b: M \to X$.

The degree of the correspondence $(M, b, f, \xi)$ is the sum of the degrees of $f$ and $\xi$.

**Remark 3.21.** Thus a significant difference from the set-up of Connes and Skandalis in [3] is that the map $b: M \to X$ is not required to be proper; we have replaced this by a support condition on $\xi$.

Several equivalence relations on correspondences are imposed. The first is to consider two correspondences $(M, b_0, f_0, \xi_0)$ and $(M, b_1, f_1, \xi_1)$ to be equivalent if their normally non-singular maps are equivalent. The second is to consider bordant correspondences equivalent (we will not discuss this at all in this survey.) The third is most interesting, and is called Thom modification. The Thom modification of a correspondence $(M, b, f, \xi)$ using a subtrivial $K$-oriented vector bundle $V$ over $M$ is the correspondence

$$(V, b \circ \pi_V, f \circ \pi_V, \tau_V(\xi)),$$

where $\tau_V: RK^*_G(M) \xrightarrow{\cong} RK^*_{G, X}(|V|)$ is the Thom isomorphism, i.e. $\tau_V(\xi) := \pi_V^*(-)(\xi) \cdot \xi_V$, where $\xi_V \in RK^*_{G, X}(|V|)$ is the Thom class. We declare a correspondence and its Thom modification to be Thom equivalent. Note that applying Thom modification to a correspondence does not change its degree.

The equivalence relation on correspondences is that generated by equivalence of normally non-singular maps, bordism and Thom equivalence.
Definition 3.22. Let $\mathcal{G}$ be a proper groupoid and $X$ and $Y$ be $\mathcal{G}$-spaces. We let $K\mathcal{K}^G_\mathbb{Z}(X,Y)$ denote the $\mathbb{Z}/2$-graded set of equivalence classes of $\mathcal{G}$-equivariant correspondences from $X$ to $Y$, graded by degree.

A correspondence $(M, b, f, \xi)$ from $X$ to $Y$ is smooth if $X$, $Y$, and $M$ are smooth manifolds, $f$ is a smooth normally non-singular map (see §3.3), and $b$ is a smooth map. There is a rather obvious notion of smooth equivalence of smooth correspondences. This gives rise to a parallel theory using only smooth equivalence classes of smooth correspondences; we do not use notation for this.

3.8. $kk^G$ as a category. Classes of correspondences form a category with analogous properties to Kaspars’ equivariant KK (that is, to analytic Kasparov theory). The composition of correspondences is called the intersection product. For composition we use similar notation to Kaspars’: if $\Psi \in K\mathcal{K}^G_\mathbb{Z}(X,Y)$ is a topological morphism, i.e. an equivalence class of equivariant correspondence from $X$ to $Y$, and if $\Phi \in K\mathcal{K}^G_\mathbb{Z}(Y,W)$ is another, then we write $\Psi \otimes_Y \Phi \in K\mathcal{K}^G_\mathbb{Z}(X,W)$ for their composition.

We do not describe the general intersection product here, but will focus instead on the transversality method of [8].

Recall that two smooth $\mathcal{G}$-maps $f_1: M_1 \to Y$ and $f_2: M_2 \to Y$ are transverse if for every $(p_1, p_2) \in M_1 \times M_2$ such that $f_1(p_1) = f_2(p_2)$, we have $D_{p_1}f_1(T_{p_1}M_1) + D_{p_2}f_2(T_{p_2}M_2) = T_{f_1(p_1)}(X)$. Transversality ensures that the space $M_1 \times_X M_2 := \{(p_1, p_2) \in M_1 \times M_2 | f_1(p_1) = f_2(p_2)\}$ has the structure of a smooth $\mathcal{G}$-manifold.

Theorem 3.23. Let $\Phi_1 = (M_1, b_1, f_1, \xi_1)$ and $\Phi_2 = (M_2, b_2, f_2, \xi_2)$ be smooth correspondences from $X$ to $Y$ and from $Y$ to $U$, respectively. Assume that both $M_1$ and $M_2$ admit smooth normally non-singular maps $s$ to $Z$ (see §3.2), so that we lose nothing if we view $f_1$ and $f_2$ as K-oriented smooth maps (see Theorem 3.19).

Assume also that $f_1$ and $f_2$ are transverse, so that $M_1 \times_Y M_2$ is a smooth $\mathcal{G}$-manifold; it has a smooth normally non-singular map to $Z$ as well, and the intersection product of $\Phi_1$ and $\Phi_2$ is the class of the correspondence

$$(M_1 \times_Y M_2, b_1 \circ \pi_1, f_2 \circ \pi_2, \pi_1^*\xi_1 \cdot \pi_2^*\xi_2),$$

where $\pi_j: M_1 \times_Y M_2 \to M_j$ for $j = 1, 2$ are the canonical projections.

In the non-equivariant situation, any two smooth maps can be perturbed to be transverse, and in [8] this is shown to give rise to a bordism of correspondences. As a result, one can compose bordism classes of correspondences by the recipe described in Theorem 3.23.

However, this fails in the equivariant situation because pairs of smooth maps cannot in general be perturbed equivariantly to be transverse; this happen in even some of the simplest situations.

Example 3.24. Let $\mu$ be the non-trivial character of $\mathbb{Z}/2$. The corresponding one-dimensional representation is denoted $C_\mu$. We regard this as an equivariant vector bundle $C_\mu$ over a point. Its total space is $|C_\mu|$. The equivariant vector bundle $C_\mu$ is equivariantly K-oriented, since the $\mathbb{Z}/2$-action preserves the complex structure on $C_\mu$. We therefore obtain a smooth normal equivariant map $\cdot \to |C_\mu|$ of degree 2. But since the origin is the only fixed-point of the $\mathbb{Z}/2$-action, this map cannot be perturbed to be transverse to itself.

This means that we cannot compose, for example, the topological morphism $x \in K\mathcal{K}^{\mathbb{Z}/2}(\cdot, |C_\mu|)$ represented by the correspondence $\cdot \to \cdot \to |C_\mu|$, and the topological morphism $y \in K\mathcal{K}^{\mathbb{Z}/2}(|C_\mu|, \cdot)$ represented by $|C_\mu| \to \cdot \to \cdot$ in the order $x \otimes_{|C_\mu|} y$, using the transversality recipe of Theorem 3.22.
However, we may apply Thom modification to the correspondence \( \star \leftarrow \star \rightarrow |C_\mu| \), using \( C_\chi \). Modification replaces the middle space \( \star \) of \( x \) by \( |C_\chi| \) and replaces the normally non-singular map \( \star \rightarrow \mathcal{C} \) by the composition \( |C_\chi| \rightarrow \star \rightarrow |C_\mu| \). This latter is obviously isotopic to the identity map on \( |C_\mu| \). Finally, one adds the Thom class \( \hat{\xi}_C \in K^2_{\mathbb{Z}/2}(|C_\mu|) \). Therefore the morphism \( x \) is equivalent to the morphism represented by \( \star \leftarrow (|C_\mu|, \hat{\xi}_C) \). The identity map is transverse to any other map, since it is a submersion. Composing the modified correspondence and the original representative of \( y \) using the transversality recipe yields the class of the degree 2 correspondence

\[
\star \leftarrow (\star, (\hat{\xi}_C)) \rightarrow \star,
\]

where \( (\hat{\xi}_C)_x \) denotes the restriction of the Thom class to the point. This equals the difference \( [x] - [\xi] \in \text{Rep}(\mathbb{Z}/2) \) of the trivial and the non-trivial representation of \( \mathbb{Z}/2 \). It is the Euler class of the K-oriented vector bundle \( C_\chi \), e.g. the restriction of the Thom class to the zero section.

With the given architecture of equivariant correspondences, a similar process can be carried out when composing two arbitrary (smooth) correspondences. Let \( X \xrightarrow{b} (M,\xi) \xrightarrow{f} Y \) be such. Let \( f = (V,E,f) \). The equivariant vector bundles \( V \) and \( E \) are \( K \)-oriented by assumption. Thom modification using \( V \) results in the correspondence \( X \xrightarrow{b} ([V],\xi_V \cdot \xi) \xrightarrow{f \circ \pi_V} Y \). An obvious smooth isotopy of normally non-singular map \( s \) replaces this by \( X \xrightarrow{b} ([V],\xi_V \cdot \xi) \xrightarrow{\pi_{E^V} \circ f} Y \). Since \( \pi_{E^V} \circ f \) is a submersion, it is transverse to any other smooth map to \( Y \). Hence this correspondence can be composed using the transversality recipe of Theorem 3.23 with any other one (on the right).

An analogous procedure can be used to define a composition rule for arbitrary (not necessarily smooth) correspondences. This rule is quite topological in flavour, of course, but is only defined up to isotopy and is less satisfying than the sharp formulas one gets in the presence of traversality, which of course only apply in the presence of smooth structures. We will only compute compositions in this setting in this survey.

3.9. Further properties of topological KK-theory. We have said that \( \widehat{\text{KGR}} \) is a category. It is also additive, with the sum operation on correspondences defined by a disjoint union procedure. The other important property is the existence of external products. This means that there exists an external product map

\[
\widehat{\text{KGR}}(X,Y) \times \widehat{\text{KGR}}(U,V) \rightarrow \widehat{\text{KGR}}(X \times Z, U \times Z V).
\]

It leads to the structure on \( \widehat{\text{KGR}} \) of a symmetric monoidal category.

Finally, there is a natural map \( \widehat{\text{KGR}} \rightarrow \text{KK}^\mathcal{G} \). This is defined not using the pseudodifferential calculus, as in [5], but by purely topological considerations. Indeed, by definition, a normal \( K \)-oriented \( \mathcal{G} \)-map from \( X \) to \( Y \) factors, by definition, as a composite of a zero section of an equivariantly \( K \)-oriented vector bundle, an equivariant open embedding, and the projection map for another equivariantly \( K \)-oriented vector bundle. Zero sections and bundle projections yield elements of \( \text{KK}^\mathcal{G} \) because of the Thom isomorphism of [22]. Open embeddings clearly determine elements morphisms in \( \text{KK}^\mathcal{G} \) because they determine equivariant \( * \)-homomorphisms.

The various naturality properties of the Thom isomorphism imply corresponding facts about the map \( \widehat{\text{KGR}} \rightarrow \text{KK}^\mathcal{G} \). Other functorial properties of \( \widehat{\text{KGR}} \), e.g. with respect to homomorphisms \( \mathcal{G} \rightarrow \mathcal{G} \) of groupoids, are explained in detail in [10].
4. Topological duality and the topological Lefschetz map

We have organized this survey around the goal of computing the Lefschetz map for smooth $G$-manifolds. This problem is intertwined with that of computing equivariant KK-groups topologically and we will solve both problems at once in this section. The first step is to describe a class of $G$-spaces to which the general theory of duality described in §1.6 applies. Subject to the resulting constraints on $X$ we will obtain a topological model of the Lefschetz map and simultaneously a proof that the $KK^G \to KK^G$ to be is an isomorphism on both the domain and range of the Lefschetz map. This will complete our computation of $\text{Lef}$ for a fairly wide spectrum of $G$-spaces $X$.

4.1. Normally non-singular $G$-spaces.

**Definition 4.1.** A normally non-singular $G$-space $X$ is a $G$-space equipped with a $G$-equivariant normally non-singular map $(N_X, E, \hat{g})$ from $X$ to $Z$.

We also require of a normally non-singular $G$-space $X$ that every $G$-equivariant vector bundle on $X$ is subtrivial.

The vector bundle $N_X$ is called the stable normal bundle of $X$.

We may assume without loss of generality that $E$ is equivariantly $K$-oriented. Since the zero bundle is always uniquely $K$-oriented, we obtain an equivariant $K$-oriented normally non-singular map $(0, E, \hat{g})$ from $N_X$ to $Z$. Let $D \in \hat{KK}^G(N_X, Z)$ be the corresponding class.

Since $\hat{KK}^{G \times X}$ has external products, we can define a map

$$\hat{KK}^{G \times X}(X \times_Z U, Y \times_Z V) \to \hat{KK}^{G \times X}(N_X \times_Z U, N_X \times_Z V)$$

If every $G$-equivariant vector bundle over $X$ is subtrivial, then there is a forgetful functor $\hat{KK}^{G \times X} \to \hat{KK}^G$ and this results in a map

$$\hat{KK}^{G \times X}(N_X \times_Z U, N_X \times_Z V) \to \hat{KK}^G((N_X \times_Z U, N_X \times_Z V).$$

Composing with the previous one yields a topological analogue

$$\hat{KK}^{G \times X}(X \times_Z U, X \times_Z V) \to \hat{KK}^G(N_X \times_Z U, N_X \times_Z V)$$

of the functor denoted $T_P$ in the discussion of duality in §1.6 and, composing further with the morphism $D$ gives a topological analogue of the Kasparov duality map

$$\text{PD}^*: \hat{KK}^{G \times X}(X \times_Z U, X \times_Z V) \to \hat{KK}^G(N_X \times_Z U)$$

of Theorem 1.7. However, it is of course not the case in general that every equivariant vector bundle over $X$ is subtrivial, which is why we have added this as a hypothesis.

**Example 4.2.** Let $X$ be the integers with the trivial action of the circle group $G := \mathbb{T}$. Then there are (c.f. Example 3.8) equivariant vector bundles on $X$ which are not subtrivial. This is despite the fact that $X$ admits a normally non-singular map to a point, since it smoothly embeds in the trivial representation of $G$ on $\mathbb{R}$, with trivial normal bundle.

In any case, $X$ is not normally non-singular.

Any smooth $G$-manifold satisfying one of the hypotheses of Theorem 6.13 is normally non-singular.

To define a topological analogue of the map denoted PD we need a class $\Theta \in \hat{KK}^{G \times X}(X, X \times_Z N_X)$. Combining composition with this class and the map

$$\text{KK}^G(N_X \times_Z U, V) \to \hat{KK}^{G \times X}(X \times_Z N_X \times_Z U, X \times_Z V),$$
which is defined in the topological category in the same way as in the analytic one, we will obtain the required topological map
\[ \text{PD}: \text{KK}^G(X \times Z U, V) \to \text{KK}^G_{X \times Z}(X \times Z U, X \times Z V). \]
We just describe \( \Theta \) in a heuristic fashion. Assume for simplicity that the base \( Z \) of \( G \) is a point. So \( E \) is just a Euclidean space. Choose a point \( x \in X \). Using the zero section of \( N_X \) and the map \( \hat{\gamma} \), we see \( x \) as a point in the open subset \( [N_X] \) of \( E \), and hence by rescaling \( E \) (fibrewise) into a sufficiently small open ball around \( x \) we obtain an open embedding of \( E \) into \( N_X \). Explicitly, we use an open embedding of the form
\[ \hat{\gamma}: [E] \xrightarrow{\delta} B_*(\hat{\gamma}(\zeta_{N_X}(x))) \subset |N_X| \]
where the first map is the rescaling. This yields an obvious normally non-singular map \( (E, \emptyset|_{|N_X|}, \hat{\gamma}) \) from a point to \( |N_X| \) It is easily checked that this can be carried out continuously in the parameter \( x \in X \), and we obtain an \( X \)-equivariant normal K-oriented map \( \delta \) from \( X \) to \( X \times Z |N_X| \) with trace the graph of the zero-section \( \Xi: X \to X \times Z |N_X| \), \( \Xi(x) := (x, (x, 0)) \). This yields an element of \( \text{KK}^G_{X \times Z}(X, X \times Z |N_X|) \). The same can be checked to work equivariantly, and a fibrewise version works for groupoids with nontrivial base. We let \( \Theta \in \text{KK}^G_{X \times Z}(X, X \times Z |N_X|) \) be the corresponding class.

Theorem 4.3. In the notation above, let \( X \) be a normal \( G \)-space of finite type, let \( N_X \) be the stable normal bundle and \( D \in \text{KK}^G(N_X, Z) \) and \( \Theta \in \text{KK}^G_{X \times Z}(X, X \times Z N_X) \) the classes constructed above.

Then \( (N_X, D, \Theta) \) is a Kasparov dual for \( X \) in \( \text{KK}^G \), and the maps \( \text{PD} \) and \( \text{PD}^* \) are isomorphisms.

The same formal computations as in §1.6 then imply that the maps \( \text{PD} \) and \( \text{PD}^* \) are isomorphisms.

Again, the stronger result is proved in [16] that one gets a symmetric Kasparov dual; this, remember this is designed to give, as well, an isomorphism of the form
\[ \text{KK}^G_*(X \times Z U, V) \cong \text{KK}^G_{X \times Z}(X \times Z U, N_X \times Z V) \]
for any pair of \( G \)-spaces \( U \) and \( V \). We do not give the details. As a consequence one deduces the following theorem by using duality to reduce from the bivariant to the monovariant case.

Theorem 4.4. Let \( X \) be a normal \( G \)-space of finite type, and \( Y \) be an arbitrary \( G \)-space. Then the natural transformation \( \text{KK}^G_*(X, Y) \to \text{KK}^G_*(C_0(X), C_0(Y)) \) is invertible.

Example 4.5. Consider the space \( X := \mathbb{Z} \) with the trivial action of \( G := \mathbb{T} \). This space is not normally non-singular, though it is a smooth \( G \) manifold admitting a normally non-singular map to a point. The map
\[ \text{KK}^G(X, \ast) \to \text{KK}^G(C_0(X), \mathbb{C}) \]
is not an isomorphism in this case. By duality, the elements of \( \text{KK}^G(C_0(X), \mathbb{C}) \) are parameterised by \( G \)-equivariant complex vector bundles on \( X \). One can check that the elements of \( \text{KK}^G_*(X, \ast) \) are by contrast parameterised by \( G \)-equivariant complex vector bundles which only involve a finite number of representations of \( G \). In other words, [16] is equivalent to the embedding
\[ \oplus_{n \in \mathbb{Z}} \text{Rep}(\mathbb{T}) \to \prod_{n \in \mathbb{Z}} \text{Rep}(\mathbb{T}) \]
of the direct sum into the direct product of representation rings.
If we dropped the subtriviality requirement on vector bundles that we imposed on cycles for $\mathcal{K} \mathcal{K}$ then Problem 4.7 would be an isomorphism, but then we would not be able to define the intersection product of correspondences in general.

Remark 4.7. Restricting to smooth correspondences and smooth equivalence classes of correspondences yields a parallel 'smooth' theory. If $X$ is a smooth normally non-singular $\mathcal{G}$-space and $Y$ is smooth, then it follows that the smooth and non-smooth versions of $\mathcal{K} \mathcal{K}^*_{\mathcal{G}}(X,Y)$ agree.

We are now in a position to solve the problem we have been working towards: a topological computation of the Lefschetz map for a normal $\mathcal{G}$-space of finite type. Firstly, we define the topological Lefschetz map

$$\text{Lef}: \mathcal{K} \mathcal{K}^*_{\mathcal{G}}(X \times_Z X, X) \to \mathcal{K} \mathcal{K}^*_{\mathcal{G}}(X, Z)$$

for any normal $\mathcal{G}$-space of finite type as in Definition 4.23 using the topological Kasparov dual constructed above out of the stable normal bundle. Since a topological dual maps to an analytic dual, the diagram that the diagram parov dual constructed above out of the stable normal bundle. Since a topological computation of the Lefschetz map for a normal $\mathcal{G}$-space is smooth, we are able to define the intersection product of correspondences in general.

Explicit computation of $\text{Lef}$.

Let $X$ be a smooth $\mathcal{G}$-manifold satisfying one of the hypotheses $A$ or $B$ of Theorem 3.13. From that Theorem, and by Remark 4.7, the map $X \to Z$ is the trace of an essentially unique smooth normally non-singular $\mathcal{G}$-map. The normal bundle $N_X$ is a smooth $\mathcal{G}$-equivariant vector bundle, and $\tilde{g}: N_X \to E$ is a smooth $\mathcal{G}$-equivariant embedding.

Moreover, the general morphism in $\mathcal{K} \mathcal{K}^*_{\mathcal{G}}(X \times_Z X, X)$ is represented by a smooth, $\mathcal{G} \ltimes X$-equivariant correspondence

$$\Psi := (M, b, f, \xi),$$

or in diagram form

$$X \times_Z X \xrightarrow{b} (M, \xi) \xrightarrow{f} X$$

from $X \times_Z X$ to $X$. Let $\Psi$ denote its class. Recall that the $X$-structure on $X \times_Z X$ is on the first coordinate.

Remark 4.9. The map $f: M \to X$ embedded in the correspondence $\Psi$ is assumed a smooth $\mathcal{G} \ltimes X$-equivariant normally non-singular map. This presupposes the structure on $M$ of a smooth $\mathcal{G} \ltimes X$-manifold. Note that this is a stronger condition than being a smooth $\mathcal{G}$-manifold: it entails a bundle structure on $M$ with smooth fibres and is equivalent to requiring that the smooth normally non-singular $f$ is a submersion.

Following the definition of the Lefschetz map in 4.7 we next apply the functor $T^*_{N_X}: \mathcal{K} \mathcal{K}^\mathcal{G} \to \mathcal{K} \mathcal{K}^\mathcal{G}$ which sends a $\mathcal{G} \ltimes X$-space, i.e. a $\mathcal{G}$-space $W$ over $X$, to the $\mathcal{G}$-space $W \times X N_X$. The latter is the same as the pullback to $W$ of the vector bundle $N_X$ using the map $W \to X$; recall that this functor is well-defined provided that every $\mathcal{G}$-equivariant vector bundle over $X$ is subtrivial.

The functor $T^*_{N_X}$ maps a $\mathcal{G} \ltimes X$-equivariant map from $W$ to $V$ to the $\mathcal{G}$-equivariant map $W \times X N_X \to V \times X N_X$ given by the obvious formula. Since $X \times_Z X \times_X N_X \cong$
Let \( \Psi \in \hat{\mathcal{K}}_{\psi}(X \times X, X) \); let

\[
\text{Lef} : \hat{\mathcal{K}}_G(X \times Z, X, X) \to \hat{\mathcal{K}}_G(X, Z)
\]

be the Lefschetz map in topological equivariant KK-theory. Then the topological Lefschetz invariant of the class of an equivalence \( \Psi \) in general position in the sense explained above, is the class of the coincidence cycle of \( \Psi \),

\[
\text{Lef}([([M, b, f, \xi])]) = ([\Psi, (\rho_{\mathcal{M}})^{G \times X}]_{\Psi}, \rho_{\mathcal{F}_\psi}, \xi) \in \hat{\mathcal{K}}_G(X, Z).
\]

Similar statements follow in analytic KK.
Namely, the Lefschetz invariant of the $\text{KK}^G\times_X(X\times_ZX,X)$-morphism $\text{KK}^G\times_X(\Psi)$ determined by $\Psi$ is the class $\text{KK}^G(\text{Lef}(\Psi))$. Furthermore, this class is the pushforward under the map $\left(\partial_M\right)_{|p_q^*X}: F_\Psi \to X$ of the class of the Dirac operator on the $K$-oriented coincidence manifold $F_\Psi$, twisted by $\xi$ (by an appeal to the Index Theorem.)

We leave it to the reader to compute the Lefschetz invariant of $\Psi$ in the situation where the transversality condition (4.11) fails; in this case it becomes necessary to modify $\Theta_{\text{top}}$ using Thom modification and an isotopy as in Example 3.24.

Our computation of the Lefschetz map for smooth normal $G$-manifolds of finite type is now complete, in view of (4.8) and Theorem 4.4 in combination with Remark 4.17.

4.3. Lefschetz invariants of self-morphisms of $X$. Let $X$ be a smooth normal $G$-manifold of finite type. We now consider the composition

$$\text{KR}^G(X,X) \to \text{KK}^G\times_X(X\times_ZX,X) \xrightarrow{\text{Lef}} \text{KK}^G(X,Z),$$

where the first map is the composition of the canonical inflation map

$$p_X^*: \text{KR}^G(X,X) \to \text{KK}^G\times_X(X\times_ZX,X \times X X),$$

and the map

$$\text{KR}^G\times_X(X\times_ZX,X \times X X) \to \text{KK}^G\times_X(X \times Z X,X)$$

of composition with the diagonal restriction class $\Delta_X \in \text{KK}^G\times_X(X \times_ZX,X, X)$. Recall that the latter is the class of the $G \times X$-equivariant correspondence $X \times_Z X \xrightarrow{\delta_X} X$ where $\delta_X$ is the diagonal embedding. Let $\Psi = (M, b, f, \xi)$ be a smooth, $G$-equivariant correspondence from $X$ to $X$. Of course this implies that there is a smooth normally non-singular map $M \to Z$, by composing $f: M \to X$ and $\varphi_X: X \to Z$. The inflation map replaces $\Psi$ by the $G \times X$-equivariant correspondence

$$X \times_Z X \xrightarrow{id_X \times_Z b} (X \times_Z M, \xi) \xrightarrow{id_X \times_Z f} X \times_Z X.$$ The $X$-structures are all on the first variable. In order to compose (on the right) with the diagonal restriction class using transversality, we first easily check that Theorem 3.23 applies, and deduce that we require that the smooth maps $id_X \times_Z f: X \times_X M \to X \times_X X$ and $\Delta_X: X \to X \times_Z X$ are transverse in the sense of Theorem 3.23 in the category of $G \times X$-equivariant smooth maps. They are transverse if and only the smooth $G$-map $f: M \to X$ is a submersion. If this condition is met, then $f: M \to X$ gives $M$ not just the structure of a smooth $G$-manifold, but the structure of a smooth $G \times X$-manifold, i.e. a bundle of smooth manifolds over $X$ with morphisms in $G$ acting by diffeomorphisms between the fibres. Composing with $\Delta_X$ using transversality then yields the $G \times X$-equivariant correspondence $X \times_Z X \xrightarrow{(b,f)} (M,\xi) \xrightarrow{f} X$ where $(b,f)(m) := (b(m), f(m))$. Finally, we apply Theorem 4.12 to obtain the following.

Theorem 4.13. Let $X$ be a smooth normally non-singular $G$-manifold, let $\Psi = (M, b, f, \xi)$ be a smooth, $G$-equivariant correspondence from $X$ to $X$ such that $f$ is a fibrewise submersion and such that for every $m \in M$ such that $b(m) = f(m)$ the linear map $T_mM \to T_{b(m)}X$,

$$\zeta \mapsto D_m b(\zeta) - D_m f(\zeta)$$

is non-singular. Then the Lefschetz invariant of $\Psi$ is the class of the smooth, $G$-equivariant correspondence

$$X \xleftarrow{b} (F_\Psi', \xi|_{p_{\Psi}^*}) \to Z$$
from $X$ to $Z$, where $F'_b := \{ m \in M \mid b(m) = f(m) \}$ with its induced $K$-orientation.

Example 4.14. If $b: X \to X$ is a smooth $G$-equivariant map, then $X \xrightarrow{b} (X, \xi) \xrightarrow{id} X$ is a smooth, zero-dimensional correspondence from $X$ to $X$. Since $\text{id}: X \to X$ is obviously a submersion, the transversality condition amounts to saying that for every $x \in X$ which is fixed by $b$, the linear map $\text{id} - D_x b: T_x X \to T_x X$ is non-singular. This is the classical condition. The coincidence space is of course the fixed-point set of $b$, suitably $K$-oriented. This means a sign is attached to each point, which can be checked to agree with the usual assignment. Thus the Lefschetz invariant is the algebraic fixed-point set.

4.4. Homological invariants for correspondences. In this section we describe, in the non-equivariant case, the pairing between the index of the Lefschetz invariant of a Kasparov self-morphism of $X$, and a $K$-theory class, in terms of graded traces on $K$-theory. In combination with the theory of correspondences this yields a generalisation of the classical Lefschetz fixed-point formula.

Recall that the classical Lefschetz fixed-point theorem describes the algebraic invariants of a map satisfying a transversality condition, to the graded trace of the induced map on homology, a 'global' invariant. The latter of course only makes sense when the homology groups of $X$ have finite rank. Duality and the Universal Coefficient Theorem taken together imply this, so in particular it is the case if $X$ is a compact manifold. All the same remarks hold of course for $K$-theory as well.

The reader may take all $K$-theory groups to be tensored by $\mathbb{Q}$ in the following. Recall that the $K$-theory $K^*(X)$ is a graded ring. This ring structure is important for what follows. Let $L_z: K^*(X) \to K^*(X)$ denote the additive group homomorphism of $K^*(X)$ by multiplication by $x \in K^*(X)$. For $f \in \text{KK}_*(\mathcal{C}(X), \mathcal{C}(X))$, let $f_*$ denote the endomorphism of $K$-theory induced by $f$. We are interested in the linear transformation

$$L(f): K^*(X) \to \mathbb{Z}, \quad L(f) x := \text{trace}_x(f_* \circ L_z).$$

We call $L(f)$ the Lefschetz operator of $f$. It is a globally defined object, generalizing the classical Lefschetz number $l(f) := \text{trace}_x(f_*)$ of $f$ in the sense that evaluating $L(f)$ at the unit $[1] \in K^0(X)$ recovers the Lefschetz number:

$$L(f)([1]) = l(f).$$

The Lefschetz operator contains more information; if for example if $f$ is an odd morphism then $l(f) = 0$ but $L(f) \neq 0$ except in special cases.

Theorem 4.15. Let $X$ be a compact space admitting an abstract dual. Let $f \in \text{KK}_*(\mathcal{C}(X), \mathcal{C}(X))$ be a Kasparov morphism. Then

$$L(f) \xi = \langle \xi, \text{Lef}(f) \rangle$$

holds for every $\xi \in K^*(X)$.

By the characteristic-class formulation \cite{2} of the Atiyah-Singer Index theorem, for each compact, smooth $K$-oriented manifold $M$ there exists a cohomology class $\mathcal{J}(M)$ on $M$ which we call the orientation character of $M$ such that

$$\text{Ind}(D \cdot \xi) = \int_M \mathcal{J}(M) \cdot \text{ch}(\xi),$$

where $D \cdot \xi$ denotes the class of the Dirac operator on $M$ twisted by $\xi$. Of course we choose representative differential forms for $\mathcal{J}(M)$ and $\text{ch}(\xi)$ both can be computed more or less explicitly from Chern Weil theory. In combination with Theorem 4.15 we obtain a local formula for the Lefschetz operator of a morphism represented by a correspondence, as follows.
Corollary 4.17. Let \( X \xrightarrow{b} (M, \xi) \xrightarrow{f} X \) be a smooth self-correspondence of \( X \), assume it is of degree \( d := \dim(M) - \dim(X) + \dim(\xi) \) and denote its class by \( \Psi \).

Assume that the transversality assumptions in Theorem 4.13 are met, so that the coincidence space

\[
F'_\Psi := \{ m \in M \mid b(m) = f(m) \}
\]

has the structure of a smooth, \( K \)-oriented manifold of dimension \( d \). Then

\[
L(f)\eta = \int_{F'_\Psi} \operatorname{ch}(\xi|_{F'_\Psi}) \cdot (b^*\eta|_{F'_\Psi}) \mathcal{I}(D_{F'_\Psi}),
\]

holds, where \( \mathcal{I}(D_{F'_\Psi}) \) is the index character of \( F'_\Psi \), and \( L(f)\eta := \operatorname{trace}_G(f_\ast \circ L_\Psi) \) is the Lefschetz operator applied to \( \eta \).

In the case where \( M = X, f = \text{id} \) and \( b: X \to X \) is a smooth equivariant map in general position, the coincidence manifold \( F'_\Psi \) is a finite set of points, and (4.18) reduces to the traditional Lefschetz fixed-point theorem:

\[
\operatorname{trace}_G(b_\ast) = \sum_{x \in \operatorname{Fix}(b)} \text{sign det}(\text{id} - D_x f).
\]

Ralf Meyer and I are currently aiming at an equivariant version of the above, but the work is not yet complete. Let \( \mathcal{G} \) be compact group. Let \( X \) be a compact \( \mathcal{G} \)-space. Then the \( G \)-equivariant \( K \)-theory \( K^G_\mathcal{G}(X) \) of \( X \) is a module over the representation ring \( \text{Rep}(G) \). The Hattori-Stallings trace

\[
\operatorname{trace}_{\text{Rep}(G)}: K^G_\mathcal{G}(X) \to \text{Rep}(G)
\]

is then defined under suitable conditions. Now assume that \( X \) admits an abstract dual, so that the Lefschetz map is defined. If \( f \in KK^G(\mathcal{C}_0(X), \mathcal{C}_0(X)) \) is a morphism, then call the Lefschetz index \( \text{ind}_G \circ \text{Lef}(f) \in \text{Rep}(G) \) the pairing of the unit class \([1] \in K^G(\mathcal{C}, \mathcal{C}_0(X)) \) with \( \text{Lef}(f) \in KK^G(\mathcal{C}_0(X), \mathcal{C}) \). Our expectation is that the following result holds – we state it as a conjecture since the proof is not complete at the time of writing.

Conjecture 4.20. If \( f \in KK^G(\mathcal{C}_0(X), \mathcal{C}_0(X)) \), then \( \text{ind}_G \circ \text{Lef}(f) = \operatorname{trace}_{\text{Rep}(G)}(f_\ast) \), where \( f_\ast \) denotes the action of \( f \) on \( K^G_\mathcal{G}(X) \) and \( \operatorname{trace}_{\text{Rep}(G)} \) denotes the Hattori-Stallings trace.

This theorem can be of course combined with correspondences to achieve interesting local-global equalities of what seem to be rather subtle invariants.

The Hattori-Stallings trace is defined for modules \( M \) over rings \( R \) which are finitely presented, i.e. have finite-length resolutions by finitely generated projective \( R \)-modules. Therefore we require this hypothesis on the equivariant \( K \)-theory of \( X \) as a module over \( \text{Rep}(G) \).

This a reasonable assumption only for compact, connected Lie groups with torsion-free fundamental group \( G \). For disconnected compact Lie groups, i.e. finite groups, the homological dimension of \( \text{Rep}(G) \) typically has infinite homological dimension and we do not know at the moment how to formulate an equivariant Lefschetz theorem for finite groups along the lines of [4.20].

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