Quantum vs. Classical
Read-Once Branching Programs

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Abstract. The paper presents the first nontrivial upper and lower bounds for (non-
oblivious) quantum read-once branching programs. It is shown that the computational
power of quantum and classical read-once branching programs is incomparable in the
following sense:
(i) A simple, explicit boolean function on $2n$ input bits is presented that is computable
by error-free quantum read-once branching programs of size $O(n^3)$, while each classical
randomized read-once branching program and each quantum OBDD for this function
with bounded two-sided error requires size $2^{\Omega(n)}$.
(ii) Quantum branching programs reading each input variable exactly once are shown to
require size $2^{\Omega(n)}$ for computing the set-disjointness function DISJ$_n$ from communication
complexity theory with two-sided error bounded by a constant smaller than $1/2 - 2\sqrt{3}/7$.
This function is trivially computable even by deterministic OBDDs of linear size.
The technically most involved part is the proof of the lower bound in (ii). For this, a new
model of quantum multi-partition communication protocols is introduced and a suitable
extension of the information cost technique of Jain, Radhakrishnan, and Sen (2003) to
this model is presented.

1. Introduction

This paper deals with the space complexity of sequential, nonuniform quantum algo-
rithms, modeled by quantum branching programs. It follows the general plan of devel-
oping lower bound techniques for gradually less restricted variants of the model. This
line of research is well motivated by the fact that, in the classical case, it has already
led to practically meaningful time-space tradeoff lower bounds for general randomized
branching programs solving decision problems [4,5,9].

Lower bounds and separation results generally come in two main flavors: results for
multi-output-bit functions and for single-output-bit functions or decision problems. Of
the former type are recent time-space tradeoffs for quantum circuits computing some
practically important functions, including sorting [16,1,20] and boolean matrix-vector
and matrix-matrix multiplication [20,17].

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Here we are concerned with lower bounds and separation results for decision problems, which are usually harder to obtain than for multi-output-bit problems in the same model. Such results have been proved for the uniform model of quantum finite automata (QFAs, see, e.g., [21, 25, 6]). On the nonuniform side, general quantum branching programs and quantum OBDDs (ordered binary decision diagrams) have been considered (see the next section for an introduction of these models). Extending independently obtained results by Špalek [33], it has been shown in [31] that the logarithm of the size of general quantum branching programs captures the space complexity of nonuniform quantum Turing machines. Ablayev, Moore, and Pollett [3] have proved that NC$^1$ is included in the class of functions that can be exactly computed by quantum oblivious width-2 branching programs of polynomial size, in contrast to the classical case where width 5 is necessary unless NC$^1$ = ACC. Furthermore, exponential gaps have been established between the width of quantum OBDDs and classical deterministic OBDDs (Ablayev, Gainutdinova, and Karpinski [2]) and classical randomized OBDDs, resp. (Nakanishi, Hamaguchi, and Kashiwabara [26]). Finally, it has been shown in [31] that the classes of functions with polynomial size quantum OBDDs and deterministic OBDDs are incomparable and an example of a partially defined function for which quantum OBDDs are exponentially smaller than classical randomized ones has been presented.

Proving lower bounds on the space complexity of quantum algorithms for models that are more general than QFAs or quantum OBDDs and solve explicit decision problems has been open so far. In particular, previous results in this context have been limited to models that are oblivious, i.e., are required to read their input bits in a fixed order. Here we consider the non-oblivious model of quantum read-once branching programs, which are quantum branching programs that during each computation may access each input bit at most once. The logarithm of the size of quantum read-once branching programs is a lower bound on the space-complexity of (uniform or nonuniform) quantum read-once Turing machines. This follows by an easy adaptation of the proof in [31] for general quantum branching programs. On the other hand, all upper bounds presented here in terms of quantum read-once branching programs can easily be modified to work also for (uniform or nonuniform) quantum read-once Turing machines.

We prove the first nontrivial upper and lower bounds for quantum read-once branching programs. As our first main result, we present a simple function for which quantum read-once branching programs are exponentially smaller than classical randomized ones. This result is even for a total function (compare this to the fact that analogous results for quantum OBDDs [31] and quantum one-way communication complexity [8] known so far are only for partially defined functions). We use the weighted sum function due to Savicky and Žák [32] as a building block. For a positive integer $n$ and $x = (x_1, \ldots, x_n) \in \{0, 1\}^n$, let $p(n)$ be the smallest prime larger than $n$ and let $s_n(x) = \left(\sum_{i=1}^{n} i \cdot x_i\right) \mod p(n)$. Define the weighted sum function by $WS_n(x) = x_{s_n(x)}$ if $s_n(x) \in \{1, \ldots, n\}$ and 0 otherwise. For a further input vector $y = (y_1, \ldots, y_n) \in \{0, 1\}^n$ define the mixed weighted sum function by $MWS_n(x, y) = x_i \oplus y_i$ if $i = s_n(x) = s_n(y) \in \{1, \ldots, n\}$ and 0 otherwise.

**Theorem 1.** Each randomized read-once branching program and each quantum OBDD computing $MWS_n$ with two-sided error bounded by an arbitrary constant smaller than 1/2 requires size $2^{\Omega(n)}$, while $MWS_n$ can be computed by an error-free quantum read-once branching program of size $O(n^3)$. 

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The above result shows that being able to choose different variable orders for different inputs may help a lot for quantum read-once algorithms, even compared to classical randomized read-once algorithms that are allowed the same option. On the other hand, combining the read-once property with the usual unitarity constraint for quantum algorithms (required by physics) can also turn out to be a severe restriction on the computing power. It has already been shown in [31] that quantum OBDDs for the set-disjointness function \( \text{DISJ}_n \) from communication complexity theory, defined by \( \text{DISJ}_n(x, y) = \neg(x_1 y_1 \lor \cdots \lor x_n y_n) \) for \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \{0, 1\}^n \), require size \( 2^{\Omega(n)} \). As our second main result, we prove a lower bound of the same order even for the non-oblivious case. We need the additional assumption here that the branching programs do not only read each input variable at most once, but even exactly once.

**Theorem 2.** Each quantum branching program that reads each input variable exactly once and computes \( \text{DISJ}_n \) with two-sided error bounded by a constant smaller than \( 1/2 - 2\sqrt{3}/7 \approx 0.005 \) has size \( 2^{\Omega(n)} \).

Note that \( \text{DISJ}_n \) can be trivially computed by deterministic OBDDs of linear size. With the usual “uncomputing” trick it is also easy to construct a reversible (and thus quantum) oblivious read-twice branching program of linear size for this function.

The proof of the above lower bound is considerably more involved and uses a more advanced technique than that for quantum OBDDs in [31], although both rely on arguments from information theory. We use the general information-theoretical framework that Bar-Yossef, Jayram, Kumar, and Sivakumar [7] have developed for classical randomized communication complexity and that they have applied, among other results, for an elegant new proof of a linear lower bound for the disjointness function. Furthermore, we exploit main ideas from the recent extension to the quantum case for a bounded number of rounds due to Jain, Radhakrishnan, and Sen [13, 14], who in turn relied on technical tools due to Klauck, Nayak, Ta-Shma, and Zuckerman [18, 19]. For formalizing the proof, we introduce a new model of quantum (one-way) multi-partition protocols that allows protocols to use more than one input partition and may be interesting for its own sake. (See [12] for a nondeterministic, classical variant of this model.) The core part of the proof is a lower bound of \( \Omega(1) \) on the information cost of quantum multi-partition protocols computing the AND of two bits. This complements a similar bound due to Jain, Radhakrishnan, and Sen that only works for a single input partition, but for any constant number of rounds instead of only one round here.

It remains open whether the lower bound in Theorem 2 remains true for quantum read-once branching programs that are not forced to read each variable at least once during any computation. It is easy to enforce this property for classical read-once branching programs while maintaining polynomial size, but it is not clear how to do this in the quantum case due to the required unidirectionality of the programs (see the next section).

The rest of the paper is organized in the obvious way: In the next section, we define the variants of quantum branching programs considered here. In two further sections, we present the proofs of the main results.
2. Preliminaries

We assume a general background on quantum computing and quantum information theory (as provided, e.g., by the textbook of Nielsen and Chuang [27]) and on classical branching programs (BPs) (see, e.g., the textbook of Wegener [36]). We start with the definition of general quantum branching programs.

**Definition 1.** A quantum branching program (QBP) over the variable set $X = \{x_1, \ldots, x_n\}$ is a directed multigraph $G = (V,E)$ with a start node $s \in V$ and a set $F \subseteq V$ of sinks. Each node $v \in V - F$ is labeled by a variable $x_i \in X$ and we define $\var(v) = i$. Each node $v \in F$ carries a label from $\{0,1\}$, denoted by $\text{label}(v)$. Each edge $(v,w) \in E$ is labeled by a boolean constant $b \in \{0,1\}$ and a (transition) amplitude $\delta(v,w,b) \in \mathbb{C}$. We assume that there is at most one edge carrying the same boolean label between a pair of nodes and set $\delta(v,w,b) = 0$ for all $(v,w) \notin E$ and $b \in \{0,1\}$.

The graph $G$ is required to satisfy the following two constraints. First, it has to be well-formed, meaning that for each pair of nodes $u,v \in V - F$ and all assignments $a = (a_1, \ldots, a_n)$ to the variables in $X$, $\sum_{w \in V} \delta^*(u,w,a_{\var(w)})\delta(v,w,a_{\var(v)}) = 1$ if $u = v$ and 0 otherwise. Second, $G$ has to be unidirectional, which means that for each $w \in V$, all nodes $v \in V$ such that $\delta(v,w,b) \neq 0$ for some $b \in \{0,1\}$ are labeled by the same variable.

A computational state of the QBP is a pure quantum state over the Hilbert space $\mathcal{H} = \mathbb{C}^{|V|}$ spanned by an ON-basis $\langle v \rangle_{v \in V}$. The computation for an input $a = (a_1, \ldots, a_n)$ starts with the computational state $|s\rangle$, called initial state. Let the QBP be in the computational state $|\psi\rangle = \sum_{v \in V} \alpha_v |v\rangle \in \mathcal{H}$ at the beginning of a computation step. Then the QBP first carries out a projective measurement of the output label at the nodes in $|\psi\rangle$. This yields the result $r \in \{0,1\}$ with probability $\sum_{v \in F,\text{label}(v)=r} |\alpha_v|^2$. If one of these events occurs, the respective output is produced and the computation stops. The computation carries on for the non-sink nodes with nonzero amplitude in $|\psi\rangle$. Let $|\psi'\rangle = \sum_{u \in V - F} \alpha'_u |u\rangle$ be the state obtained by projecting $|\psi\rangle$ to the subspace spanned by the non-sink nodes and renormalizing. Then the next computational state is defined as $|\psi''\rangle = \sum_{v \in V - F} \alpha'_u \sum_{u \in V} \delta(v,w,a_{\var(w)})|w\rangle$.

The probability that $G$ outputs $r \in \{0,1\}$ on input $a \in \{0,1\}^n$ is defined as the sum of the probabilities of obtaining the output $r$ after any finite number of steps. Let $G(a)$ be the random variable describing the output of $G$ on input $a$, called the output random variable of $G$ for $a$. We say that the function $f$: $\{0,1\}^n \rightarrow \{0,1\}$ defined on $X$ is computed by $G$

- with two-sided error at most $\varepsilon$, $0 \leq \varepsilon < 1/2$, if for each $a \in \{0,1\}^n$, $\Pr\{G(a) \neq f(a)\} \leq \varepsilon$; and it is computed
- exactly (or $G$ is an error-free QBP for $f$), if for each $a \in \{0,1\}^n$, $\Pr\{G(a) \neq f(a)\} = 0$.

Furthermore, by bounded two-sided error we mean two-sided error with some unspecified constant bound $\varepsilon$. (Other modes of acceptance may be defined as usual for other quantum models of computation.)

The size of a QBP $G$ is the number of its nodes and is denoted by $|G|$. Its width is the maximum number of nodes with the same distance from the start node.
The definition of QBPs is similar to that of the uniform models of quantum finite automata (QFAs) and quantum Turing machines (QTMs), whose relationships to the respective classical models have already been studied to a considerable extent (see, e.g., [21, 25, 6, 10, 33, 35]). A strong motivation why QBPs are a natural model is provided by the fact that the logarithm of their size and the space complexity for nonuniform QTMs are polynomially related [33, 31]. For the scenario of sublinear space bounds, it has turned out to be useful to work with unidirectional QTMs, i.e., QTMs whose directions of head movements depend only on the entered state of the finite control. This is the standard model in the papers of Watrous [34, 35] and also that used for the simulation between QBPs and QTMs in [33, 31]. The unidirectionality constraint for QBPs (called parental condition in [33]) turns up as a natural counterpart of that for QTMs required to make the simulations work. In order to prevent QBPs from being unreasonably powerful, it is further realistic to restrict the set of allowed amplitudes, see also [31]. This is no issue here, since the upper bounds in the paper only use amplitudes from \{0, 1, \pm 1/2\} and the lower bounds for QBPs are valid for arbitrary complex amplitudes.

For the construction of QBPs it is sometimes convenient to use unlabeled nodes with an arbitrary number of outgoing edges carrying only amplitude labels. An unlabeled node \(v\) can be regarded as an abbreviation for a node according to the standard definition labeled by a dummy variable on which the considered function does not depend. Each edge leading from the unlabeled node \(v\) to a successor \(w\) with amplitude \(\alpha\) is then regarded as a pair of edges from the node labeled by the dummy variable to \(w\) that carry the boolean labels 0 and 1, resp., and that both have amplitude \(\alpha\).

A special case of QBPs are reversible classical BPs, where each node is reachable from at most one node \(v\) by a 0-edge and from at most one node \(w\) by a 1-edge and \(v\) and \(w\) are labeled by the same variable. It has been proved by Špalek [33] that each sequence of (possibly non-reversible) classical BPs with at least linear size can be simulated by a sequence of reversible ones with at most polynomial larger size. Since randomized (general) BPs can be derandomized while maintaining polynomial size analogously to probabilistic circuits (see [29] for details), the same is true in the randomized case.

We consider the following variants of quantum BPs defined analogously to their classical counterparts.

**Definition 2.**

- A quantum BP is called **leveled** if the set of its nodes can be partitioned into disjoint sets \(V_1, \ldots, V_\ell\) such that for \(1 \leq i \leq \ell - 1\), each edge leaving a node in \(V_i\) reaches a node in \(V_{i+1}\).
- A **quantum read-once BP** is a QBP where each variable may appear at most once on each path.
- A **quantum OBDD** (quantum ordered binary decision diagram) is a quantum read-once BP with an order \(\pi\) of the variables such that for each path in the graph the order in which the variables appear is consistent with \(\pi\).
3. The Separation Result for Mixed Weighted Sum (Theorem 1)

For the whole section, let \( p = p(n) \) be the smallest prime larger than \( n \) for a fixed positive integer \( n \). We first deal with the easier upper bound. Our goal is to show that \( \text{MWS}_n \) can be computed by polynomially small error-free quantum read-once BPs.

\textit{Proof of Theorem 1 – Upper Bound.} The essence of the proof is to apply the Deutsch-Jozsa algorithm, evaluating the sums \( s_n(x) \) and \( s_n(y) \) in parallel and computing the output \( x_i \oplus y_i \) if \( i = s_n(x) = s_n(y) \). We first describe the algorithm by a quantum circuit. We use a four-part quantum register consisting of two qubits for the Deutsch-Jozsa algorithm and two further parts whose basis states are indexed by \( \{0, \ldots, p - 1\} \).

The oracle gate for the Deutsch-Jozsa algorithm unitarily extends the mapping \( S \) specified for \( a, b \in \{0, 1\} \) by \( |a\rangle|b\rangle|0\rangle|0\rangle \mapsto |a\rangle|b \oplus (1 - a)y_i \oplus ax_j\rangle|i\rangle|j\rangle \), where \( i = s_n(x) \) and \( j = s_n(y) \). This gate is applied to the initial state \((1/2)(|0\rangle + |1\rangle)(|0\rangle - |1\rangle)|0\rangle|0\rangle\), giving the final state \((1/2)((-1)^y|0\rangle + (-1)^x|1\rangle)(|0\rangle - |1\rangle)|i\rangle|j\rangle\). If a measurement of the last two parts of the quantum register yields that \( i \neq j \), the output of the circuit is 0 with probability 1. Otherwise, \( i = j \) and measuring the first two qubits in the Hadamard basis yields the output \( x_i \oplus y_i = \text{MWS}_n(x, y) \) for the first qubit with probability 1.

Next we describe the implementation of the obtained quantum circuit as a quantum read-once BP. For an easier exposition, we first use unlabeled nodes. We start with the construction of a subgraph \( G_S \) realizing the mapping \( S \). The nodes of \( G_S \) are laid out on a grid with \( 2n + 1 \) rows and \( 4p^2 \) columns, the latter labeled by \( (a, b, i, j) \) with \( a, b \in \{0, 1\} \) and \( i, j \in \{0, \ldots, p - 1\} \). Each row represents an intermediate state of the four-part quantum register used for the above algorithm. The graph \( G_S \) consists of two disjoint classical reversible OBDDs \( G_0 \) and \( G_1 \) on the subsets of nodes in the columns with \( a = 0 \) and \( a = 1 \), resp. We first describe how \( G_0 \) works. The computation starts at a node in row 1 and column \( (0, b, 0, 0) \) with \( b \in \{0, 1\} \). The variable vector \( x \) is read (the order of the variables within the vector does not matter) and the node in row \( n + 1 \) and column \( (0, b, s_n(x), 0) \) is reached. Then the variable vector \( y \) is read (again, the order of the individual variables is arbitrary) and the sink in row \( 2n + 1 \) and column \( (0, b \oplus y_n(x), s_n(x), s_n(y)) \) is reached. It is easy to see how the described computation can be implemented by a reversible OBDD with nodes on the prescribed grid. The OBDD \( G_1 \) works in the same way, but with exchanged roles of \( x \) and \( y \) and exchanged roles of the last two column indices. Altogether, we obtain a classical reversible read-once BP for \( G_S \) with at most \( (2n + 1) \cdot 4p^2 \) nodes, which is of order \( O(n^3) \) due to the prime number theorem.

We add a new, unlabeled source that for \( (a, b) \in \{0, 1\}^2 \) is connected to the node in row 1 and column \( (a, b, 0, 0) \) of \( G_S \) by an edge with amplitude \( (-1)^b(1/2) \). The sinks of \( G_S \) in row \( 2n + 1 \) and in columns \( (a, b, i, j) \) with \( i \neq j \) are replaced with 0-sinks. All other sinks of \( G_S \) are replaced with unlabeled nodes connected to a new level of sinks with boolean output labels. The outgoing edges of these unlabeled nodes are labeled by amplitudes such that, together with the sinks, a measurement in the Hadamard basis is realized. The whole graph still has size \( O(n^3) \).

Finally, we remove the unlabeled nodes. For this, we first ensure that all nodes on the first level of \( G_S \) are labeled by the same variable and the same for all nodes on the last
level of $G_S$ with variable labels. We rearrange (e.g.) the variable order of the OBDD $G_1$ and update the OBDD accordingly. W.l.o.g., let $x_1$ be the first variable read in $G_0$ and let $y_n$ be the last. We move the variable $x_1$ to the front of the variable order of $G_1$ and $y_n$ to the end. It is not hard to see that we can modify $G_1$ in such a way that it complies to the new variable order while increasing its size by at most a constant factor and maintaining reversibility. After this transformation, we merge the unlabeled nodes with their successors (in the case of the source) or with their predecessors (in the case of the nodes on the level directly above the sinks). It is obvious how the edges should be relabeled such that the resulting graph still computes the same final state as a quantum read-once BP. We observe that after the reordering process also the unidirectionality requirement for quantum BPs is satisfied. Altogether, we have obtained the desired quantum read-once BP for MWS$_n$ of size $O(n^3)$.

Next we prove the lower bound on the size of randomized read-once BPs for MWS$_n$ with bounded error. We reuse main ideas from the proof an analogous lower bound for WS$_n$ in [30]. However, the result for MWS$_n$ is no obvious consequence of that for WS$_n$. We have to carefully argue why, different from the quantum case, having two input vectors present that play the same roles does not help in the randomized case.

The proof employs a variant of the rectangle bound method from communication complexity theory (see, e.g., the textbook of Kushilevitz and Nisan [24]) suitable for read-once BPs, which we first describe. For this, we introduce some notation. We consider boolean functions defined on the union of the disjoint sets of variables $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. For a set of variables $Z \subseteq X \cup Y$, let $2^Z$ denote the set of all assignments to $Z$, i.e., mappings from $Z$ to $\{0, 1\}$ that we usually identify with vectors in $\{0, 1\}^{|Z|}$. A (combinatorial) rectangle with respect to a partition $\Pi = (\Pi_1, \Pi_2)$ of $X \cup Y$ is a set of assignments $R = A \times B$ with $A \subseteq 2^{\Pi_1}$ and $B \subseteq 2^{\Pi_2}$. For $\ell \in \{1, \ldots, n - 1\}$ call $R$ an $\ell$-rectangle if $\Pi_1$ contains exactly $\ell$ variables from $X$ and at most $\ell - 1$ variables from $Y$ or the same with exchanged roles of $X$ and $Y$. Call $R$ a one-way rectangle if $B = 2^{\Pi_2}$. Given a function $g$ on $X \cup Y$, $R$ is said to be $g$-uniform if for all $a, a' \in A$ and $b \in B$, $g(a, b) = g(a', b)$.

For the following, let a function $f$ on $X \cup Y$ and a distribution $\mathcal{D}$ on the inputs of $f$ be given. Let $0 \leq \varepsilon < 1/2$. We describe how to prove lower bounds for deterministic read-once BPs whose output is allowed to differ from $f$ on at most an $\varepsilon$-fraction of the inputs with respect to $\mathcal{D}$. By a well-known averaging argument due to Yao [37], this also gives lower bounds of the same size for randomized read-once BPs computing $f$ with the same error probability.

The essence of the proof technique is to show that, on the one hand, any small deterministic read-once BP that correctly computes $f$ on a large fraction of the inputs with respect to $\mathcal{D}$ would give a rectangle with large $\mathcal{D}$-measure on which $f$ is well approximated, while on the other hand, using the specific properties of $f$, the $\mathcal{D}$-measure of any such rectangle necessarily has to be small. We now make this more precise. Let $R = A \times B$ be a rectangle and let $0 \leq \varepsilon < 1/2$. A function $g$ on $X \cup Y$ is said to uniformly approximate $f$ on $R$ with error $\varepsilon$ with respect to $\mathcal{D}$, if for all $a \in A$, $g$ differs from $f$ for at most an $\varepsilon$-fraction of the inputs in $\{a\} \times B$ with respect to $\mathcal{D}$. The following main lemma of the proof technique is
a variant of a similar statement from \cite{30}, where the uniform distribution and functions on a single set of variables have been considered.

**Lemma 1.** Let $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$. Let $f$ be a boolean function on $X \cup Y$ and let $D$ be a distribution on the inputs of $f$. Let $\ell \in \{1, \ldots, n-1\}$ and $0 \leq \varepsilon < \varepsilon' < 1/2$. Then for every deterministic read-once BP $G$ computing a function $g$ that differs from $f$ on at most an $\varepsilon$-fraction of the inputs with respect to $D$ there is a one-way $\ell$-rectangle $R$ that is $g$-uniform, on which $g$ uniformly approximates $f$ with error at most $\varepsilon'$ with respect to $D$, and which satisfies $D(R) \geq (1 - \varepsilon/\varepsilon')/(2n|G|)$.

**Proof.** By an easy adaptation of the well-known proof technique of Borodin, Razborov, and Smolensky \cite{11} (see also \cite{30}, Section 7.6), we get a partition of the input space into at most $k \leq 2n|G|$ one-way $\ell$-rectangles $R_1 = A_1 \times B_1, \ldots, R_k = A_k \times B_k$ that are all $g$-uniform. We claim that there is an $i \in \{1, \ldots, k\}$ and a subset $A'_i \subseteq A_i$ such that for $R = A'_i \times B_i$, $D(R) \geq (1 - \varepsilon/\varepsilon')/k$ and $g$ uniformly approximates $f$ on $R$ with error $\varepsilon'$ with respect to $D$. This obviously suffices to prove the claim.

Let $A^* = A_1 \cup \cdots \cup A_k$. For each $x \in A^*$, let $(\Pi_1(x), \Pi_2(x))$ be the partition of the input variables used by the rectangle to which $x$ belongs, and let $S_x = \{x\} \times 2^{\Pi_2(x)}$. Let $A = \{x \in A^* \mid D(S_x) > 0\}$. For each $x \in A$ let $\varepsilon(x)$ be the $D$-fraction of inputs from $S_x$ for which $g$ differs from $f$. Due to the definitions, the sets $S_x$, $x \in A$, are disjoint and their union has $D$-measure 1. Hence, by the law of total probability, $\sum_{x \in A} \varepsilon(x) D(S_x) \leq \varepsilon$. Let $A' = \{x \in A \mid \varepsilon(x) \leq \varepsilon'\}$ and let $S$ be the union of all $S_x$ for $x \in A'$. By Markov’s inequality, $D(S) \geq 1 - \varepsilon/\varepsilon'$. By averaging, there is a set $A'' \subseteq A'$ such that for the union $S''$ of all $S_x$ with $x \in A''$, we have $D(S'') \geq D(S)/k$ and all inputs from $A''$ belong to the same rectangle. Let $(\Pi_1, \Pi_2)$ be the partition of input variables of this rectangle. It is now obvious that the set $R = A'' \times 2^{\Pi_2}$ with $A'' \subseteq 2^{\Pi_1}$ and $D(R) \geq (1 - \varepsilon/\varepsilon')/k$ is a one-way $\ell$-rectangle with the desired properties. \hfill $\Box$

Next we cite two technical lemmas also used in \cite{30} that build the common core of the lower bounds both for the mixed weighted sum function $\text{MWS}_n$ and the usual weighted sum function $\text{WS}_n$. The first lemma allows us to argue that partial weighted sums of enough random bits are essentially uniformly distributed over the whole range of possible values.

**Lemma 2 \cite{30}.** Let $q = q(n)$ be a sequence of primes and let $n \leq q - 1$ and $n = \Omega(q^{2/3+\delta})$ for any constant $\delta > 0$. Let $a_1, \ldots, a_n, b \in \mathbb{Z}_q^* = \mathbb{Z}_q - \{0\}$ where the numbers $a_1, \ldots, a_n$ are pairwise different. Then for $(x_1, \ldots, x_n) \in \{0, 1\}^n$ chosen uniformly at random, $|\Pr\{a_1x_1 + \cdots + a_nx_n \equiv b \text{ mod } q\} - 1/q| = 2^{-\Omega(q^{\delta})}$.

In the second lemma, we consider the index function $\text{IND}_n : \{0, 1\}^n \times \{1, \ldots, n\}$ from communication complexity theory defined for $u \in \{0, 1\}^n$ and $v \in \{1, \ldots, n\}$ by $\text{IND}_n(u, v) = u_v$. We state an upper bound on the size of one-way rectangles on which $\text{IND}_n$ is well approximated that is implicit in a couple of papers, the earliest one being probably that of Kremer, Nisan, and Ron \cite{23}. For the sake of completeness, we include the easy proof. Here and in the following, $U$ denotes the uniform distribution on the domain implied by its respective argument.
Lemma 3 \([\textsc{23}]\). Let \(\varepsilon\) be a constant with \(0 \leq \varepsilon < 1/2\). Let \(R = A \times \{1, \ldots, n\}\) with \(A \subseteq \{0,1\}^n\) be a one-way rectangle for which a function \(g\) exists such that \(R\) is \(g\)-uniform and \(g\) uniformly approximates \(\text{IND}_n\) on \(R\) with error \(\varepsilon\) with respect to \(U\). Then \(U(R) = 2^{-\Omega(n)}\).

Proof. Since \(R\) is \(g\)-uniform, there is a vector \(r \in \{0,1\}^n\) such that, for each \(a \in A\), 
\((g(a,1), \ldots, g(a,n)) = r\). Since \(g\) uniformly approximates \(\text{IND}_n\) on \(R\) with error at most \(\varepsilon\) with respect to the uniform distribution, \(r\) has Hamming distance at most \(\lfloor \varepsilon n\rfloor\) to each vector in \(A\). It follows that \(|A|\) is upper bounded by the size of Hamming balls of radius \(\lfloor \varepsilon n\rfloor\), which is known to be at most \(2^{H(\varepsilon)n}\), where 
\(H(x) = -(x \log x + (1-x) \log (1-x))\) for \(x \in [0,1]\) is the binary entropy function. Thus, 
\(U(R) = |R|/(n \cdot 2^n) = |A|/2^n \leq 2^{-(1-H(\varepsilon))n} = 2^{-\Omega(n)}\). \(\square\)

Now we describe the details that are particular to the function \(\text{MWS}_n\). For the rest of the section, let \(X = \{x_1, \ldots, x_n\}\) and \(Y = \{y_1, \ldots, y_n\}\) be the sets of variables on which \(\text{MWS}_n\) is defined. Recall that \(p = p(n)\) is the smallest prime larger than \(n\). We concentrate on the set of difficult inputs \(D = \{(x,y) : s_n(x) = s_n(y)\}\) by working with the distribution \(\mathcal{D}\) with \(\mathcal{D}(x,y) = 1/|D|\) if \((x,y) \in D\) and \(\mathcal{D}(x,y) = 0\) otherwise.

As a preparation of the proof of the lower bound for randomized read-once BPs computing \(\text{MWS}_n\), we derive some basic facts about the considered one-way rectangles. We use the following notation. For a set \(S \subseteq X\) (or \(S \subseteq Y\)) of variables and a partial assignment \(a\) that fixes at least all variables in \(S\), let \(\sigma_S(a) = (\sum_{v \in S} i(v) \cdot a(v)) \mod p\), where \(i(v) \in \{1, \ldots, n\}\) denotes the index of the variable \(v\) in \(X\) (or \(Y\), resp.), and \(a(v)\) is the value that it obtains by the assignment \(a\).

Lemma 4. Let \(\ell = n - \Theta(p^{2/3+\delta})\) for some constant \(\delta\) with \(0 < \delta < 1/3\). Let \(\Pi = (\Pi_1, \Pi_2)\) be a partition of \(X \cup Y\) with \(|\Pi_1 \cap X| = \ell\) and \(|\Pi_1 \cap Y| \leq \ell - 1\). Let \(R = A \times 2^{\Pi_2}\) with \(A \subseteq 2^{\Pi_1}\) and suppose there are \(i_x, i_y \in \{0, \ldots, p-1\}\) such that for all \(a \in A\), \(\sigma_{\Pi_1 \cap X}(a) = i_x\) and \(\sigma_{\Pi_1 \cap Y}(a) = i_y\). For each \(k \in \{0, \ldots, p-1\}\) define \(B_k\) as the set of all assignments \(b \in 2^{\Pi_2}\) with \(\sigma_{\Pi_2 \cap X}(b) \equiv (k-i_x) \mod p\) and \(\sigma_{\Pi_2 \cap Y}(b) \equiv (k-i_y) \mod p\). Then we have the following.

(i) For each \(k \in \{0, \ldots, p-1\}\) and \((a,b) \in A \times B_k\), \(\sigma_X(a,b) = \sigma_Y(a,b) = k\). Furthermore, 
\(U(B_k) = (1/p^2) \cdot (1 \pm o(1))\) and \(\mathcal{D}(A \times B_k) = (1/p) \cdot U(R) \cdot (1 \pm o(1))\).

(ii) \(\mathcal{D}(R) = U(R) \cdot (1 \pm o(1))\).

Proof. Part (i): The first part of the statement is obvious. It remains to prove the claims about \(U(B_k)\) and \(\mathcal{D}(A \times B_k)\). Let \(b\) denote an assignment from \(B_k\) chosen uniformly at random. Then, using that disjoint parts of \(b\) are independent of each other and applying Lemma 2 we get 
\[
U(B_k) = \Pr\{\sigma_{\Pi_2 \cap X}(b) \equiv k - i_x \wedge \sigma_{\Pi_2 \cap Y}(b) \equiv k - i_y\} \\
= \Pr\{\sigma_{\Pi_2 \cap X}(b) \equiv k - i_x\} \cdot \Pr\{\sigma_{\Pi_2 \cap Y}(b) \equiv k - i_y\} = \frac{1}{p^2} \cdot (1 \pm o(1)).
\]
Furthermore, also by Lemma 2 $U(D) = (1/p) \cdot (1 \pm o(1))$. Again by the independence of disjoint parts of uniformly random assignments and by observing that $A \times B_k \subseteq D$ and $U(A) = U(R)$, we obtain

$$\mathcal{D}(A \times B_k) = \frac{U((A \times B_k) \cap D)}{U(D)} = \frac{U(A) \cdot U(B_k)}{U(D)} = \frac{1}{p} \cdot U(R) \cdot (1 \pm o(1)).$$

**Part (ii):** This follows from the first part, since $R \cap D$ is the disjoint union of the sets $A \times B_k$ over all $k = 0, \ldots, p - 1$. □

Finally, we are ready to prove the desired lower bound on the size of randomized read-once BPs for $\text{MWS}_n$.

**Proof of Theorem 2 – Lower bound for randomized read-once BPs.** Following the outline above, we prove the lower bound for deterministic read-once BPs that correctly compute $\text{MWS}_n$ on a large fraction of the inputs. Let $0 \leq \varepsilon_G < 1/2$ be any constant and let $G$ be a deterministic read-once BP computing a function $g$ that differs from $\text{MWS}_n$ on at most an $\varepsilon_G$-fraction of the inputs with respect to $\mathcal{D}$. Choose $\ell = n - \Theta(p^{2/3 + \delta})$ for some constant $\delta$ with $0 < \delta < 1/3$. Let $\varepsilon$ be a constant with $\varepsilon_G < \varepsilon < 1/2$. Let $\mathcal{D}$ be a one-way $\ell$-rectangle that is $g$-uniform and on which $\text{MWS}_n$ is uniformly approximated by $g$ with error at most $\varepsilon$. We prove that $\mathcal{D}(R) = 2^{-O(n)}$. By Lemma 1 this yields the desired lower bound $|G| = 2^{\Omega(n)}$.

Let $\Pi = (\Pi_1, \Pi_2)$ be the partition of the input variables used by $R$, where w.l.o.g. $|\Pi_1 \cap X| = \ell$ and $|\Pi_1 \cap Y| \leq \ell - 1$. Let $R = A_R \times 2^{\Pi_2}$ with $A_R \subseteq 2^{\Pi_1}$. Using averaging, we fix an assignment $a \in 2^{\Pi_1 \cap Y}$ and an $i_x \in \{0, \ldots, p - 1\}$ such that for the set $A$ of all assignments $a' \in A_R$ that are consistent with $a$ and satisfy $\sigma_{\Pi_1 \cap X}(a') = i_x$, we have $\mathcal{D}(A \times 2^{\Pi_2}) \geq \mathcal{D}(R)/(p \cdot 2^{(|\Pi_1 \cap Y|)})$. Let $i_y = \sigma_{\Pi_1 \cap Y}(a)$. Let $R' = \{x \in A \times 2^{\Pi_2} \mid \mathcal{D}(x) > 0\}$. Since $g$ approximates $\text{MWS}_n$ uniformly on $R$ with error at most $\varepsilon$ with respect to $\mathcal{D}$, we know that $g$ differs from $\text{MWS}_n$ for at most an $\varepsilon$-fraction of the inputs in $R'$ with respect to $\mathcal{D}$.

Let $\Pi_1 \cap X = \{x_{j_1}, \ldots, x_{j_\ell}\}$. We observe that, due to the prime number theorem, $p \leq n + o(n)$ and thus $\ell \geq n - o(n)$ and $\ell/p \geq 1 - o(1)$. Let $B_0, \ldots, B_{p-1} \subseteq 2^{\Pi_2}$ be the sets of assignments according to Lemma 2 for $R'$ and $i_x, i_y$. Let $B = B_{j_1} \cup \cdots \cup B_{j_\ell}$. Then we have the following.

**Claim 1.** The function $g$ differs from $\text{MWS}_n$ on at most a fraction of $\varepsilon \cdot (1 + o(1))$ of the inputs in $A \times B$ with respect to the uniform distribution.

**Proof of Claim 2.** Due to part (i) of Lemma 2 $\mathcal{D}(A \times B) \geq (\ell/p) \cdot U(R') \cdot (1 - o(1)) \geq U(R') \cdot (1 - o(1))$. On the other hand, by part (ii) of Lemma 2 $\mathcal{D}(R') \leq U(R') \cdot (1 + o(1))$. Thus, the inputs in $A \times B$ cover at least a $(1 - o(1))$-fraction of the rectangle $R'$ with respect to $\mathcal{D}$. It follows that $g$ differs from $\text{MWS}_n$ on at most a fraction of $\varepsilon \cdot (1 + o(1))$ of the inputs in $A \times B$ with respect to $\mathcal{D}$. Since $A \times B \subseteq D$, the same is true for the uniform distribution. □

Next we further reduce the obtained set $A \times B$ by picking appropriate representatives of each of the subsets $B_{j_1}, \ldots, B_{j_\ell}$ of $B$. 

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Claim 2. There are \( b_1 \in B_{j_1}, \ldots, b_\ell \in B_{j_\ell} \) such that \( g \) differs from \( \text{MWS}_n \) on at most a fraction of \( \varepsilon \cdot (1 + o(1)) \) of the inputs in \( R'' = A \times \{ b_1, \ldots, b_\ell \} \) with respect to the uniform distribution.

Proof of Claim 2. We choose a collection of disjoint subsets \( \{ b_1, \ldots, b_\ell \} \) of \( B \) with \( b_1 \in B_{j_1}, \ldots, b_\ell \in B_{j_\ell} \) whose union \( B' \) is as large as possible. Since \( U(B_k) \geq (1/p^2) \cdot (1 - o(1)) \) for each \( k = 0, \ldots, \ell - 1 \) by part (i) of Lemma 4, we can ensure that \( U(B') \geq (\ell/p^2) \cdot (1 - o(1)) \geq (1/p) \cdot (1 - o(1)) \). On the other hand, also by Lemma 4 \( U(B) \leq (1/p) \cdot (1 + o(1)) \). Hence, the set \( A \times B' \) covers at least a \( (1 - o(1)) \)-fraction of the inputs in \( A \times B \). It follows that the relative error of \( g \) on \( A \times B' \) with respect to the uniform distribution is bounded by some \( \varepsilon' \) with \( \varepsilon' \leq \varepsilon \cdot (1 + o(1)) \). By averaging, there is thus at least one subset \( \{ b_1, \ldots, b_\ell \} \) in \( B' \) such that \( A \times \{ b_1, \ldots, b_\ell \} \) has relative error \( \varepsilon' \) with respect to the uniform distribution.

Let \( R'' = A \times \{ b_1, \ldots, b_\ell \} \) be a rectangle according to the above claim. Now we apply the result for the index function from Lemma 3. For simplicity, we assume that \( j_1 = 1, \ldots, j_\ell = \ell \) such that the set of all restrictions of the assignments in \( A \) to the variables in \( \Pi_1 \cap X \) can be identified in the obvious way with a subset \( A_{\text{IND}} \subseteq \{0, 1\}^\ell \) of the same size. Recall that for each assignment in \( A \), the variables in \( \Pi_1 \cap Y \) are fixed according to the assignment \( a \) chosen above. We regard \( R_{\text{IND}} = A_{\text{IND}} \times \{1, \ldots, \ell\} \) as a one-way rectangle for the index function \( \text{IND}_\ell \). Define the function \( h \) on inputs \( u \in \{0, 1\}^\ell \) and \( v \in \{1, \ldots, \ell\} \) by

\[
h(u, v) = \begin{cases} 
  g((u, a), b_v) \oplus a(y_v), & \text{if } y_v \in \Pi_1; \text{ and} \\
  g((u, a), b_v) \oplus b_v(y_v), & \text{if } y_v \in \Pi_2;
\end{cases}
\]

where we regard \( u \) as an assignment to \( \Pi_1 \cap X \) in the argument of \( g \). Since \( b_v \in B_v \) and for each \( a' \in A \), \( \sigma_X(a', b_v) = \sigma_Y(a', b_v) = v \),

\[
\text{MWS}_n((u, a), b_v) = \begin{cases} 
  u(x_v) \oplus a(y_v), & \text{if } y_v \in \Pi_1; \text{ and} \\
  u(x_v) \oplus b_v(y_v), & \text{if } y_v \in \Pi_2;
\end{cases}
\]

and \( h(u, v) = u_v = \text{IND}_\ell(u, v) \) if \( g((u, a), b_v) = \text{MWS}_n((u, a), b_v) \).

The rectangle \( R_{\text{IND}} \) is \( h \)-uniform since \( R'' \) is \( g \)-uniform and the values \( a(y_v) \) and \( b_v(y_v) \), resp., added to the output of \( g \) depend only on the second part \( v \) of the input. Since \( g \) differs from \( \text{MWS}_n \) on at most a fraction of \( \varepsilon' = \varepsilon \cdot (1 + o(1)) \) of the inputs of \( R'' \) with respect to the uniform distribution, \( h \) differs from \( \text{IND}_\ell \) on at most an \( \varepsilon' \)-fraction of \( R_{\text{IND}} \) with respect to the uniform distribution. By Lemma 3 it follows that \( U(R_{\text{IND}}) = 2^{-\Omega(\ell)} \).

Furthermore,

\[
U(R') = |A|/2^{|\Pi_1|} = 2^{-|\Pi_1 \cap Y|} \cdot |A_{\text{IND}}|/2^\ell = 2^{-|\Pi_1 \cap Y|} \cdot U(R_{\text{IND}})
\]

and, by part (ii) of Lemma 4 \( \mathcal{D}(R') \leq U(R') \cdot (1 + o(1)) \). Finally, \( \mathcal{D}(R) \leq p \cdot 2^{|\Pi_1 \cap Y|} \cdot \mathcal{D}(R') \). Putting everything together, we have shown that \( \mathcal{D}(R) = p \cdot 2^{-\Omega(\ell)} \). Since \( p \leq n + o(n) \) and \( \ell \geq n - o(n) \), this bound is of the desired size.

The lower bound for quantum OBDDs stated in Theorem 4 follows by standard communication complexity arguments and the properties of \( \text{MWS}_n \) already used above.
Proof of Theorem 1 – Lower bound for quantum OBDDs. Let $G$ be a quantum OBDD computing $\text{MWS}_n$ with error bounded by a constant $\varepsilon$, $0 \leq \varepsilon < 1/2$. Let $\ell = n - \Theta(p^{2/3+\delta})$ for some constant $\delta$ with $0 < \delta < 1/3$. Appropriately cutting the list of variables used as the variable order for $G$ in two parts gives a partition $\Pi = (\Pi_1, \Pi_2)$ of the set of variables $X \cup Y$ that, w.l.o.g., satisfies $|\Pi_1 \cap X| = \ell$ and $|\Pi_1 \cap Y| \leq \ell - 1$. Choose $a \in 2^{\Pi_1 \cap Y}$ somehow arbitrarily and let $i_y = \sigma_{\Pi_1 \cap Y}(a)$. Furthermore, again w.l.o.g., suppose that $\Pi_1 \cap X = \{1, \ldots, \ell\}$. For any $i_x \in \{0, \ldots, p - 1\}$, Lemma 2 yields the existence of assignments $b_{i_x,1}, \ldots, b_{i_x,\ell} \in 2^{\Pi_2}$ such that $\sigma_{\Pi_2 \cap X}(b_{i_x,j}) \equiv (j - i_x) \mod p$ and $\sigma_{\Pi_2 \cap Y}(b_{i_x,j}) \equiv (j - i_y) \mod p$ for $j = 1, \ldots, \ell$.

The given quantum OBDD $G$ can now be used by the two players Alice and Bob in a quantum one-way communication protocol for $\text{IND}_\ell$ as follows. Let $u \in \{0, 1\}^\ell$ and $v \in \{1, \ldots, \ell\}$ be the inputs for $\text{IND}_\ell$. Alice follows the computation in $G$ for the partial input $(u, a)$, regarding $u$ as an assignment to the variables in $\Pi_1 \cap X$, and sends the reached superposition as well as the partial weighted sum $\sigma_{\Pi_1 \cap X}(u)$ to Bob. Bob finishes the computation of $G$ using the partial input $b_{i_x,v}$ and outputs the XOR of the output bit of $G$ with $a(y_v)$, if $y_v \in \Pi_1 \cap Y$, or with $b_{i_x,v}(y_v)$, otherwise. It is easy to see that, analogously to the end of the proof of the lower bound for randomized read-once BPs, this gives a protocol for $\text{IND}_\ell$ that has the same error probability as $G$. As proved by Klauck [15], the complexity of quantum one-way communication protocols for $\text{IND}_\ell$ with bounded error is lower bounded by $\Omega(\ell)$, which together with the facts that only $O(\log p) = O(\log n)$ bits are required to communicate $i_x$ and that $\ell \geq n - o(n)$ implies $|G| = 2^{\Omega(n)}$, as claimed. \qed

4. The Lower Bound for Set-Disjointness (Theorem 2)

In this section, we prove that quantum BPs reading each variable exactly once and computing $\text{DISJ}_n$ with two-sided error bounded by a small positive constant require size $2^{\Omega(n)}$. We first present definitions and tools from information theory in the next subsection. We then introduce quantum multi-partition protocols (Subsection 4.2) and prove a lower bound on the information cost of such protocols for the AND of just two bits (Subsection 4.3). This is used as a building block for the proof of the desired main result in the last subsection.

4.1. Information Theory

We assume that the reader is familiar with classical and von Neumann entropy and refer to [27] for an introduction. We briefly review some important definitions.

Let $X$ be a classical random variable taking values in a finite set $R$ and for each $x \in R$ let $\rho(x)$ be a quantum state over a fixed Hilbert space. Then the state $\rho(X) = \sum_{x \in R} \Pr\{X = x\} \cdot \rho(x)$ is called quantum encoding of $X$ by $(\rho(x))_{x \in R}$. For the special case where $\rho(x) = |x\rangle\langle x|$ for each $x \in R$ and $(|x\rangle)_{x \in R}$ is an ON-basis, we just write $X$ instead of $\rho(X)$. For an additional random variable $Y$ and a value $y$ in the range of $Y$, let $\rho(X | Y = y) = \sum_{x \in R} \Pr\{X = x | Y = y\} \cdot \rho(x)$.
For a quantum state $\rho$, $S(\rho)$ denotes the von Neumann entropy of $\rho$. For a joint system $(A, B, C)$ with subsystems $A, B, C$, define $S(A | B) = S(A, B) - S(B)$ (conditional entropy), $I(A : B) = S(A) + S(B) - S(A, B)$ (mutual information between $A$ and $B$), and $I(A : B | C) = S(A | C) + S(B | C) - S(A, B | C)$ (conditional mutual information). For classical random variables $X, Y,$ and $Z$, a value $z$ in the range of $Z$, and quantum encodings $\rho(X), \sigma(Y)$ of $X$ and $Y$, resp., we use the notational shortcut $I(\rho(X) : \sigma(Y) | Z = z) = I(\rho(X | Z = z) : \sigma(Y | Z = z))$. We list the following standard facts for easier reference (see, e.g., [27], Sections 11.3–11.4).

**Fact 1.**

(i) Let $\rho^{AB}$ be a pure state of the joint system $(A, B)$ and let $\rho^A, \rho^B$ be the corresponding reduced states of the subsystems $A$ and $B$, resp. Then $S(\rho^A) = S(\rho^B)$.

(ii) Let $\rho(X) = \sum_{x \in R} \Pr\{X = x\} \cdot \rho(x)$ be a quantum encoding of a classical random variable $X$ taking values in the finite set $R$. Suppose that the states $\rho(x), x \in R$, have support on orthogonal subspaces. Then $S(\rho(X)) = H(X) + \sum_{x \in R} \Pr\{X = x\} \cdot S(\rho(x))$, where $H(X)$ is the classical entropy of $X$.

(iii) Let $\rho(X)$ be a quantum encoding of a classical random variable $X$ taking values in a finite set $R$. Then $S(\rho(X)) \geq \sum_{x \in R} \Pr\{X = x\} \cdot S(\rho(x))$ (concavity of the entropy).

(iv) Let $X, Y$ be classical random variables with finite range, let $R$ be the range of $Y$, and let $\rho(X)$ be a quantum encoding of $X$. Consider a bipartite system with state $(\rho(X), Y) = \sum_{y \in R} \Pr\{Y = y\} \cdot \rho(X | Y = y) \otimes |y\rangle\langle y|$. Then $S(\rho(X) | Y) = S(\rho(X), Y) - S(Y) = \sum_{y \in R} \Pr\{Y = y\} \cdot S(\rho(X | Y = y))$.

(v) Let $X, Y$ be classical random variables with finite range, let $R$ be the range of $Y$, and let $\rho(X), \sigma(X)$ be quantum encodings of $X$. Consider a tripartite system with state $(\rho(X), \sigma(X), Y) = \sum_{y \in R} \Pr\{Y = y\} \cdot \rho(X | Y = y) \otimes \sigma(X | Y = y) \otimes |y\rangle\langle y|$. Then $I(\rho(X) : \sigma(X) | Y) = \sum_{y \in R} \Pr\{Y = y\} \cdot I(\rho(X | Y = y) : \sigma(X | Y = y))$.

(vi) $I(A : B) \leq I(A : BC)$ (monotonicity of mutual information).

(vii) Let $X = (X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are independent classical random variables. Then for any quantum encoding $\rho(X)$ of $X$, $I(\rho(X) : X_1, \ldots, X_n) \geq \sum_{i=1}^n I(\rho(X) : X_i)$ (superadditivity of mutual information).

We observe the following additional property that follows from the definitions and the fact that the von Neumann entropy of pure states is zero.

**Fact 2.** Let $\rho(X)$ be a quantum encoding of a classical random variable $X$ and suppose that for each value $x$ that $X$ can attain, $\rho(x)$ is a pure state. Then $I(\rho(X) : X) = S(\rho(X))$.

Furthermore, we work with standard measures for the distance of quantum states. Let $\rho, \sigma$ be quantum states over the same Hilbert space. The trace norm of $\rho$ is defined as $\|\rho\|_1 = \text{tr} |\rho| = \text{tr} \sqrt{\rho^*\rho}$ and the trace distance of $\rho$ and $\sigma$ as $\|\rho - \sigma\|_1$. The fidelity of $\rho$ and $\sigma$ is defined as $F(\rho, \sigma) = \text{tr} \sqrt{\sqrt{\rho^*}\sigma \sqrt{\rho}}$. Note that for pure states $|\psi_1\rangle$ and $|\psi_2\rangle$, $F(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|) = |\langle\psi_1 | \psi_2\rangle|$. We will also use the following facts (see, e.g., [27], Section 9.2).
Fact 3. 
(i) Let $|\psi_1\rangle$, $|\psi_2\rangle$ denote pure quantum states. Then $\|\langle\psi_1| - |\psi_2\rangle\|^2 = 4(1 - F(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|)^2)$. 
(ii) Let $\rho_0, \rho_1$ be quantum states and suppose that there is a POV measurement with boolean results that yields the result $b \in \{0, 1\}$ on state $\rho_0$ with probability at least $1 - \varepsilon$. Then $F(\rho_0, \rho_1) \leq 2\sqrt{\varepsilon(1 - \varepsilon)}$.

Further, we note the following “weak inverse triangle inequality” for the inner product of real unit vectors.

**Proposition 1.** Let $|u\rangle, |v\rangle, |w\rangle$ be real unit vectors. Then $\langle u | w \rangle \geq 2(\langle u | v \rangle + \langle v | w \rangle) - 3$.

**Proof.** This follows from

$$\|u\rangle - |w\rangle\|^2 \leq \left(\|u\rangle - |v\rangle\| + \|v\rangle - |w\rangle\|\right)^2 \leq 2\left(\|u\rangle - |v\rangle\|^2 + \|v\rangle - |w\rangle\|^2\right)$$
on the one hand and

$$\|u\rangle - |w\rangle\|^2 = 2(1 - \langle u | w \rangle)$$

and similarly for $\|u\rangle - |v\rangle\|^2, \|v\rangle - |w\rangle\|^2$ on the other. \qed

Finally, we need one of the main technical tools from [18,19] used also in [13,14]. The strong version cited below has independently been derived in [19,14].

**Lemma 5 (Local transition lemma [19,14]).** Let $X$ describe a classical uniformly random bit. Let $\rho_0, \rho_1$ be quantum states over some finite dimensional Hilbert space $\mathcal{H}$. Let $\rho(X) = (\rho_0 + \rho_1)/2$. Let $|\psi_0\rangle$, $|\psi_1\rangle$ be purifications of $\rho_0$ and $\rho_1$, resp., in $\mathcal{H} \otimes \mathcal{K}$, where $\mathcal{K}$ is a Hilbert space of dimension at least the dimension of $\mathcal{H}$. Then there is a unitary transformation $U$ on $\mathcal{K}$ such that for $|\psi'_0\rangle = (I \otimes U)|\psi_0\rangle$, where $I$ is the identity on $\mathcal{H}$, $\|\langle\psi_1|\psi_1\rangle - |\psi'_0\rangle\langle\psi'_0\rangle\|_1 \leq 2\sqrt{2I(\rho(X) : X)}$.

### 4.2. Quantum Multi-Partition Communication Protocols

We consider the following simple quantum variant of communication protocols that may have more than one input partition. We use quantum one-way communication protocols with a single input partition as defined, e.g., in [22], as building blocks.

**Definition 3.** A quantum $k$-partition (one-way) communication protocol $P$ with respect to nontrivial partitions $\Pi_1, \ldots, \Pi_k$ of the set of input variables consists of a collection of one-way quantum protocols $P_1, \ldots, P_k$ with respect to $\Pi_1, \ldots, \Pi_k$, resp., and numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ such that $|\alpha_1|^2 + \cdots + |\alpha_k|^2 = 1$. Call $\alpha_1, \ldots, \alpha_k$ initial amplitudes of their respective subprotocols. For $i = 1, \ldots, k$ let $\mathcal{H}_i = \mathcal{H}_{i,A} \otimes \mathcal{H}_{i,C} \otimes \mathcal{H}_{i,B}$ be the state space of $P_i$. We require that $\mathcal{H}_i$ and $\mathcal{H}_j$ are orthogonal for $i \neq j$. Let $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$ be the global state space of the whole protocol.
The Hilbert space $\mathcal{H}_i$ describes the state of a register of qubits on which the subprotocol $P_i$ works. For an input $z = (x, y)$ partitioned into $x, y$ according to $\Pi_i$, the initial state of the register is $|s_i(z)\rangle = |x\rangle_{\mathcal{H}_i,x}|00\ldots0\rangle_{\mathcal{H}_i,C}|y\rangle_{\mathcal{H}_i,B}$, where the three parts of the register belong to the subspaces as indicated. The qubits belonging to $\mathcal{H}_{i,A}$ and $\mathcal{H}_{i,B}$, resp., are called the input registers of the players Alice and Bob, resp., and those belonging to $\mathcal{H}_{i,C}$ work register. The computation of $P_i$ is carried out as usual for quantum one-way protocols. Let $U_i$ be the unitary transformation on $\mathcal{H}_i$ realized by protocol $P_i$.

The global initial state of $P$ is $\sum_{i=1}^{k} \alpha_i |s_i(z)\rangle$ and the global final state is $\sum_{i=1}^{k} \alpha_i U_i |s_i(z)\rangle$.

Define $P_1(z), \ldots, P_k(z)$ and $P(z)$, the result states of the respective protocols, as the states obtained from the respective final states by a partial trace over the qubits in the input registers of the players. The output random variable of $P_i$ with values in $\{0, 1\}$ is defined as the result obtained by a POVM measurement of a designated output qubit in $P_i(z)$ owned by Bob. Let $M_{i,0}, M_{i,1}$ be the linear operators with $M_{i,0}^\dagger M_{i,0} + M_{i,1}^\dagger M_{i,1} = I$ (the identity on $\mathcal{H}_i$) that describe this measurement. Then the output random variable of $P$ is the result of the POVM measurement described by the operators $\sum_{i=1}^{k} M_{i,0}, \sum_{i=1}^{k} M_{i,1}$.

This allows to define the computation of boolean functions with different kinds of error as usual. A quantum multi-partition communication protocol is a quantum $k$-partition communication protocol for some $k$.

Remarks.

- We do not define the communication complexity of quantum multi-partition protocols here (which can be done in a straightforward way), since we measure the complexity using an appropriately defined notion of information cost (see the next subsection).

- Opposed to more generous models of quantum one-way protocols, the subprotocols of our quantum multi-partition protocols are defined such that the players do not obtain any additional, entangled qubits (EPR pairs) as part of the initial state of the protocol.

- The above definition can easily be generalized by allowing general quantum operations for initialization, more than one round, or entanglement between the players. We do not need this kind of generality for our later application, though.

- Due to the orthogonality of the subspaces of the subprotocols, for each input $z$, $P(z) = P_1(z) + \cdots + P_k(z)$. For the same reason, the measurement operators for the output random variable of $P$ defined above indeed give a POVM measurement. Finally, let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ and $\mathcal{O}$ denote the output random variables of $P_1, \ldots, P_k$ and $P$, resp. Then, for $r \in \{0, 1\}$, $\Pr\{\mathcal{O} = r\} = \sum_{i=1}^{k} |\alpha_i|^2 \Pr\{\mathcal{O}_i = r\}$.

- The initial amplitudes of a quantum multi-partition protocol may be assumed to be real and positive by pushing phase factors into the initial states of the subprotocols.

Our goal is to measure the mutual information between the result state of a protocol and a random input by the simple formula in Fact 2. Hence, it is important that the result state of the considered protocol is pure for a given input. At the first glance, this no longer seems to work if we want to run the protocols on random inputs and want to allow them to use random coins. The problem is overcome by using input conventions and a simple extension of the model as described in [18, 13]. First, we consider only protocols that are safe in the following sense.
Definition 4 (Safe protocols). A communication protocol is called *safe* if both players may access their input registers only once at the beginning to make copies of their inputs into the work register. They are not allowed to access the input registers for working, communicating, or measuring afterwards.

It is obvious that requiring protocols to be safe does not change their computational power if we restrict ourselves to classical inputs as usual. The convention prevents protocols from entangling their work qubits with the input registers during the computation, which could lead to the production of extra entropy besides that contained in the inputs by the trace-out operation at the end of the computation.

Furthermore, we want to run protocols on random inputs and allow the protocols to use public random coins, but only want to work with unitary transformations, even for the preparation of the initial state. By modifying the model as follows, this is possible.

Definition 5 (Protocols with random inputs and public random coins). There is an additional *(public) random coin register* whose number of qubits may depend on the length of the input of Alice and Bob. Furthermore, the input registers of Alice and Bob and the random coin register are each augmented by a *secret register* of the same size that are each only initialized once at the beginning and never accessed afterwards.

The protocol is run for random inputs of the two players described by random variables $X$ and $Y$ and random coins described by the random variable $Z$ as follows. At the beginning, Alice prepares the states $\sum_{x} \Pr\{X = x\}^{1/2}|x\rangle|\times\rangle$ and $\sum_{z} \Pr\{Z = z\}^{1/2}|z\rangle|\times\rangle$ in the two joint registers formed by her input register together with its secret register and by the public random coin register and its secret register (where, e.g., the first part of each state belongs to the regular register and the second part to the secret one). Analogously, Bob prepares the state $\sum_{y} \Pr\{Y = y\}^{1/2}|y\rangle|\times\rangle$ in his input register and the corresponding secret register. The *result state* of the protocol is obtained by taking the final computational state and tracing out the input registers of both players, the random coin register, and all secret registers. This is a mixed state which is equal to what we would have obtained had we started the protocol on random assignments to the input registers and the random coin register as described by $X, Y$ and $Z$, resp., in the first place. The *output random variable* of such a protocol is the result of a POV measurement of a qubit owned by Bob at the end, excluding the bits of the random coin register.

Although the result state according to the extended definition above also depends on $Z$, we stick to the notation $P(X,Y)$ for this state for convenience. We summarize the properties of the modified protocols that are crucial for the following proofs.

Fact 4. For a fixed (non-random) assignment to the input registers and the random coin register, the result state of a quantum multi-partition protocol as described in Definition 5 is pure. Furthermore, for registers initialized with pure states describing random inputs and random coins according to the convention in the definition, the computational state at the end of the protocol before tracing out the input registers, the random coin register, and the secret registers is also pure.
4.3. Information Cost of Quantum Multi-Partition Protocols for AND

Here we prove that the information cost of a quantum multi-partition protocol computing the AND of two bits is lower bounded by a positive constant. For measuring the information cost, we adapt the approach of Bar-Yossef, Jayram, Kumar, and Sivakumar [7] for classical randomized communication protocols and use the information that the result state of a protocol provides on the inputs (the result state replacing the classical transcript), rather than the weighted sum of the information in individual messages as in the paper of Jain, Radhakrishnan, and Sen [13]. This makes sense also in the quantum case since we do not use entanglement and have only a single round of communication.

Definition 6.

- Let $P$ be a quantum $k$-partition protocol. Let $D$ be any random variable and let $Z$ be a random variable describing an input for $P$. Then the information cost of $P$ with respect to $Z$ and conditioned on $D$, denoted by $IC(P; Z | D)$, is defined as $I(P(Z) : Z | D)$, where $P(Z)$ is the result state of $P$.
- For a function $f$, any random variable $D$, and a random variable $Z$ describing an input for $f$, the $\varepsilon$-error information cost of quantum $k$-partition protocols for $f$ on $Z$ conditioned on $D$, $IC_{k,\varepsilon}(f; Z | D)$, is defined as the infimum of the information cost over all quantum $k$-partition protocols computing $f$ with error at most $\varepsilon$. Furthermore, let $IC_{\varepsilon}(f; Z | D) = \min_{k \in \mathbb{N}} IC_{k,\varepsilon}(f; Z | D)$ denote the information cost of quantum multi-partition protocols for $f$ with error at most $\varepsilon$.

To explain some of the difficulties that arise if we want to extend the result of Jain, Radhakrishnan, and Sen [13] for protocols with a single partition computing AND to multi-partition protocols, we consider the situation for the XOR of two bits $z_1, z_2$. We choose the following input distribution as defined in [7, 13]: Let $D \in \{1, 2\}$ with $\Pr\{D = 1\} = \Pr\{D = 2\} = 1/2$. Let $Z = (Z_1, Z_2)$, where for $i = 1, 2$, $\Pr\{Z_i = 0 \mid D = i\} = \Pr\{Z_i = 1 \mid D = i\} = 1/2$ and $\Pr\{Z_{3-i} = 0 \mid D = i\} = 1$.

Proposition 2. There is an error-free quantum 2-partition protocol for XOR on the random input $Z$ conditioned on $D$ where the subprotocols do not communicate at all and where each subprotocol has zero information cost with respect to $Z$ and conditioned on $D$.

Proof. By an application of the Deutsch-Jozsa algorithm. We define a 2-partition protocol $P$ according to the partitions ($\{z_1\}, \{z_2\}$) and ($\{z_2\}, \{z_1\}$). The protocol uses two qubits as work space and subprotocols $P_1, P_2$ both weighted by the amplitude $1/\sqrt{2}$ (it uses no random coins). The first work qubit is used for computing, the second one only to implement a phase oracle as usual. In subprotocol $P_i$, $i = 1, 2$, the first work qubit is initialized with $|i - 1\rangle$. The only player to act in $P_i$ is Bob. The only thing he does is multiplying the phase of the first work qubit by $(-1)^{z_i}$. Then by a measurement of the first work qubit in the global result state in the Hadamard basis, the value XOR($z_1, z_2$) can be retrieved with error probability 0. It is easy to check that the mutual information between the result state $P_i(Z)$ of subprotocol $P_i$ and $Z$ is zero, since Bob only encodes his input in the phase of the work qubit. \qed
On the other hand, by examining the proof of [13] for the AND of two bits, it can be shown that for each quantum 1-partition protocol computing XOR with a bounded number of rounds and with bounded two-sided error, the communication complexity as well as the information cost in either the definition of [13] or the definition used here is lower bounded by a positive constant. In fact, the proof in [13] only exploits the fact that a protocol for AND has to be able to distinguish the inputs 01 and 10 from 11 with high probability and thus works in the same way for XOR. The example of XOR and the above proposition show that a lower bound on the information cost or communication complexity for a single partition does not simply carry over to a lower bound for multiple partitions in an obvious way.

As a preparation of the proof of our result for the AND function, we state the following concavity property of the information cost of multi-partition protocols.

**Lemma 6.** Let $P$ be a quantum $k$-partition communication protocol with subprotocols $P_1, \ldots, P_k$ and initial amplitudes $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$, where $|\alpha_1|^2 + \cdots + |\alpha_k|^2 = 1$. Let $D$ be any random variable and let $Z$ be a random variable describing a random input for $P$. Then $IC(P; Z | D) \geq \sum_{i=1}^k |\alpha_i|^2 IC(P_i; Z | D)$.

**Proof.** We regard the public random coins of $P$ as part of Alice’s input for this proof. Then by the definition of the protocols, the result state $P(z)$ for a fixed input $z$ (which in fact fixes the regular inputs of Alice and Bob, the random coins, and the values for the secret registers) is a pure state. Notice, however, that this does not mean that the proof only works for pure result states. When running $P$ on the random input $Z$ and randomly chosen random coins according to the conventions, the trace-out of the input registers and the corresponding secret registers still yields a mixed result state $P(Z)$.

By definition of the information cost, the statement in the claim is equivalent to

$$I(P(Z); Z | D) \geq \sum_{i=1}^k |\alpha_i|^2 I(P_i(Z); Z | D).$$

According to Fact 1(v), it suffices to prove this without the condition on $D$. Using that the result states of $P_1, \ldots, P_k$ and $P$ are pure states for a fixed input and Fact 2 it further suffices to prove that

$$S(P(Z)) \geq \sum_{i=1}^k |\alpha_i|^2 S(P_i(Z)).$$

For notational convenience, let $p_z = \Pr\{Z = z\}$ for any input $z$. For $i = 1, \ldots, k$ let $|P_i(z)\rangle$ denote the vector belonging to the pure result state $P_i(z)$ of the subprotocol $P_i$. Purifying the global result state $P(Z)$ of $P$, we obtain

$$|\psi\rangle = \sum_{i=1}^k \alpha_i \sum_z \sqrt{p_z} |P_i(z)\rangle \otimes |z\rangle.$$ 

Let $\rho = |\psi\rangle\langle\psi|$ and let $\rho^A$ and $\rho^B$ be the reduced states obtained from $\rho$ by a partial trace over the second and first part, resp., of the state space. Using Fact 1(ii), we get

$$S(P(Z)) = S(\rho^A) = S(\rho^B).$$
Hence, we investigate $\rho^B$. We have:

$$
\rho^B = \text{tr}_A \left( \sum_{i,j} \alpha_i \alpha_j^* \sum_{z,z'} \sqrt{p_z} \sqrt{p_{z'}} |P_i(z)\rangle \langle P_j(z')| \otimes |z\rangle \langle z'| \right)
$$

$$
= \sum_{z,z'} \sqrt{p_z} \sqrt{p_{z'}} |z\rangle \langle z'| \cdot \text{tr} \left( \sum_{i,j} \alpha_i \alpha_j^* |P_i(z)\rangle \langle P_j(z')| \right)
$$

$$
= \sum_{z,z'} \sqrt{p_z} \sqrt{p_{z'}} |z\rangle \langle z'| \cdot \sum_i |\alpha_i|^2 |P_i(z') \rangle \langle P_i(z)|.
$$

The last row follows from the fact that the state spaces of different subprotocols are mutually orthogonal. We write the result as

$$
\rho^B = \sum_{i=1}^k |\alpha_i|^2 \rho_i \quad \text{with} \quad \rho_i = \sum_{z,z'} \sqrt{p_z} \sqrt{p_{z'}} |P_i(z') \rangle \langle P_i(z)| \otimes |z\rangle \langle z'|, \ i = 1, \ldots, k.
$$

Define

$$
|\psi_i\rangle = \sum_z \sqrt{p_z} |P_i(z)\rangle \otimes |z\rangle.
$$

Then $\rho_i$ is obtained from $|\psi_i\rangle \langle \psi_i|$ by tracing over the first part of the state and tracing over the second yields $P_i(Z)$. Hence, for each $i$, $S(\rho_i) = S(P_i(Z))$, which together with the concavity of the entropy (Fact 1(iii)) proves the claim.

Next we observe that quantum multi-partition protocols with only two different partitions can be simplified to quantum 2-partition protocols. This is obviously applicable to any quantum multi-partition protocol for a function on just two variables like AND.

**Proposition 3.** Each quantum multi-partition protocol $P$ with respect to partitions from the set $\{\Pi_1, \Pi_2\}$ can be turned into a quantum 2-partition protocol $P'$ with respect to $\Pi_1$ and $\Pi_2$ that has initial amplitudes $\sqrt{q_1}, \sqrt{q_2}$ with $q_1, q_2 \geq 0$ and $q_1 + q_2 = 1$ and that for each input has the same result state as $P$.

**Proof.** Let $P$ be a quantum $(k_1 + k_2)$-partition protocol for $f$ with partitions $\Pi_{1,j} = \Pi_1$ for $j = 1, \ldots, k_1$ and $\Pi_{2,j} = \Pi_2$ for $j = 1, \ldots, k_2$. For $i = 1, 2$ let $\alpha_{i,1}, \ldots, \alpha_{i,k_i}$ be the initial amplitudes of these partitions and let $q_i = \sum_{j=1}^{k_i} |\alpha_{i,j}|^2$.

Define a quantum 2-partition protocol $P'$ with initial amplitudes $\sqrt{q_1}, \sqrt{q_2}$ and partitions $\Pi_1$ and $\Pi_2$ as follows. For $i = 1, 2$ and $j = 1, \ldots, k_i$ let $|s_{i,j}\rangle$ be the initial state of the subprotocol $P_{i,j}$ with partition $\Pi_{i,j}$ in $P$. Then for $i = 1, 2$ the initial state of the subprotocol $P_i'$ of $P'$ with partition $\Pi_i$ is defined as $\sum_j (\alpha_{i,j}/\sqrt{q_i}) |s_{i,j}\rangle$, if $q_i \neq 0$, or as an arbitrary pure state, if $q_i = 0$. In this way, we get a legal pure state that can be prepared by Alice at the beginning of the computation of the $i$th subprotocol. In $P_i'$, the players then simulate the respective subprotocols $P_{1,1}, \ldots, P_{1,k_1}$ of $P$ in parallel. By the definitions it is obvious that, for each input, the final computation state of $P'$ agrees with that of $P$. Hence, the same follows also for the result states. \qed
Furthermore, we observe that it suffices to work with real amplitudes in the protocols. For a complex vector space with basis \( b_1, \ldots, b_n \), its realification is the real vector space spanned by the basis \( b_1, \ldots, b_n, ib_1, \ldots, ib_n \) (using the operations of the complex vector space but allowing only real scalars). The realification of a complex vector is obtained by replacing each of its entries with two entries containing its real and imaginary part, resp. To get the realification of a complex matrix, replace each of its entries \( a \) with a \( 2 \times 2 \)-block \( \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \) where \( b_1 = b_4 = \text{Re}(a) \), \( b_3 = -b_2 = \text{Im}(a) \).

The problem is that, in general, having a protocol \( P \) with small overall error probability does not suffice to conclude the proof, as the example of XOR discussed above shows. But this does not suffice to conclude the proof, as the example of XOR discussed above shows.

Fact 5. The von Neumann entropy of a quantum state agrees with that of its realification. In particular, the information theoretical measures introduced at the beginning of the section are preserved if all involved states are replaced with their realifications.

Proof. Let \( \rho \) and \( \rho' \) be a quantum state and its realification, resp., as defined above. Then the realifications of the vectors \( |\psi_1\rangle, \ldots, |\psi_n\rangle \) and \( i|\psi_1\rangle, \ldots, i|\psi_n\rangle \) constitute an ON-basis of eigenvectors of \( \rho' \) with corresponding eigenvalues \( p_1, \ldots, p_n \) and the eigenvalue 0 with multiplicity \( n \). Hence, \( S(\rho') = S(\rho) \). The second part of the claim is obvious.

Now we consider the AND of two bits \( z_1, z_2 \). We consider the same input distribution for AND as described before for XOR. Recall that \( D \in \{1,2\} \) with \( \Pr\{D = 1\} = \Pr\{D = 2\} = 1/2 \) and that \( Z = (Z_1, Z_2) \), where for \( i = 1, 2 \), \( \Pr\{Z_i = 0 \mid D = i\} = \Pr\{Z_i = 1 \mid D = i\} = 1/2 \) and \( \Pr\{Z_{3-i} = 0 \mid D = i\} = 1 \). We are now ready to state and prove the main theorem of this subsection.

Theorem 3. Let \( \varepsilon \geq 0 \) and \( \delta = 2\sqrt{\varepsilon(1-\varepsilon)} \) be such that \( \delta \leq 1/7 \) (or, equivalently, \( \varepsilon \leq 1/2 - 2\sqrt{3}/7 \approx 0.005 \)). Then \( \text{IC}_\varepsilon(\text{AND}; Z \mid D) \geq 1/28 - \delta/4 \).

The plan for the proof of the theorem is as follows. We have to show that if a given protocol \( P \) computes AND with small error probability, then \( I(P(Z) : Z \mid D) \) is large. First, we can restrict ourselves to 2-partition protocols using Proposition 3. We then apply Lemma 3 to lower bound the overall information \( I(P(Z) : Z \mid D) \) by the average of that given by the subprotocols \( P_1, P_2 \) of \( P \). Due to the known results, it is clear that the information provided by an individual, single-partition subprotocol about a random input of the considered kind is large if it computes AND with small error probability. But this does not suffice to conclude the proof, as the example of XOR discussed above shows. The problem is that, in general, having a protocol \( P \) with small overall error probability for each input does not imply that there is a subprotocol which shares this property. As a way around this problem, we use the fidelity as a measure for the ability of the protocols to distinguish between the inputs 00, 01, and 10 on the one hand and the input 11 on the other. Using the properties of the fidelity, we can show that if the whole protocol can
reliably distinguish between these sets of inputs, which it has to if its error probability is to be small, then the same is true for at least one of the subprotocols. The local transitivity lemma then in turn implies that this subprotocol provides a nonnegligible amount of information about a random input as chosen above. We now make this more precise.

**Proof of Theorem 3.** Due to Proposition 3 we may assume that the given protocol for AND is a 2-partition protocol with respect to the partitions \( \Pi_1 = (\{z_1\}, \{z_2\}) \) and \( \Pi_2 = (\{z_2\}, \{z_1\}) \). Let \( P \) be such a protocol computing AND with error at most \( \varepsilon \). Let \( P_1, P_2 \) be the subprotocols of \( P \) that have initial amplitudes \( \alpha_1 = \sqrt{q_1}, \alpha_2 = \sqrt{q_2} \) with \( q_1, q_2 \geq 0 \) and \( q_1 + q_2 = 1 \). Furthermore, because of Fact 5 we may additionally assume that \( P \) uses only real numbers in its transition and measurement matrices as well as in its computational states.

Let \( Z = (Z_1, Z_2) \) be the input random variable for \( P \) as defined before and let \( Z_{1,j} = Z_j \) and \( Z_{2,j} = Z_{3-j} \) for \( j = 1, 2 \). We denote the result state of \( P \) on \( Z \) by \( P(Z) \). By Lemma 6 and Fact 1 (the latter together with Fact 1(vii) for handling the additional condition on \( D \)),

\[
I(P(Z) : Z | D) \geq q_1 I(P_1(Z) : Z | D) + q_2 I(P_2(Z) : Z | D)
\]

\[
\geq q_1 I(P_1(Z) : Z_{1,1} | D) + q_2 I(P_2(Z) : Z_{2,1} | D).
\]

Furthermore, due to the fact that \( Z_{i,1} \) conditioned on \( D = 3 - i \) is the fixed bit 0, \( I(P_i(Z) : Z_{i,1} | D) = (1/2)I(P_i(Z) : Z_{i,1} | D = i) \). For \( i = 1, 2 \) let \( \eta_i = I(P_i(Z) : Z_{i,1} | D = i) \). Altogether, we have shown that

\[
I(P(Z) : Z | D) \geq \frac{1}{2}(q_1 \eta_1 + q_2 \eta_2).
\]

Our goal is to lower bound the right hand side in terms of the error probability of the protocol \( P \).

We analyze \( \eta_1 = I(P_1(Z) : Z_{1,1} | D = 1) \) in detail. Observe that, conditioned on \( D = 1 \), \( P_1(Z) = P_1(Z_{1,1}, 0) \) and \( Z_{1,1} \) is a uniformly random bit. We also run \( P_1 \) on the fixed (non-random) input \((b_1, b_2) \in \{0, 1\}^2 \), which means that, according to our conventions, the players Alice and Bob prepare states \( |b_1\rangle |b_1\rangle \left( \sum_z \sqrt{p_z} |z\rangle |z\rangle \right) \) and \( |b_2\rangle |b_2\rangle \), resp. The first two parts of each state correspond to the regular input register and its secret register. The second two parts of Alice’s state are the contents of the public random coin register and its secret register, where \( p_z \) is the distribution of the values for the random coins. Let \( |s_1(b_1, b_2)\rangle \) be the final computational state of \( P_1 \) on input \((b_1, b_2) \in \{0, 1\}^2 \), before tracing out any register. It is obvious that this is a pure state.

Let *Alice’s extended input register* be the joint register consisting of Alice’s input register, the public random coin register, and the respective secret registers. Let \( P'_1(b_1, b_2) \) be the state obtained from \( |s_1(b_1, b_2)\rangle \) by tracing out Alice’s extended input register. In general, the obtained state is mixed due to the random coin component. We may regard the states \( |s_1(00)\rangle, |s_1(10)\rangle \) as purifications of the states \( P'_1(00), P'_1(10) \), resp., where the Hilbert space of Alice’s extended input register serves as the extension space. Since conditioned on \( D = 1 \), Bob’s part of \( Z \) is the fixed input 0, we have

\[
I(P'_1(Z) : Z_{1,1} | D = 1) = I(P_1(Z) : Z_{1,1} | D = 1).
\]
Now we apply the local transition lemma (Lemma 5) to the states \( \rho_0 = P_1^t(00) \) and \( \rho_1 = P_1^t(10) \) and their purifications \( |s_1(00)\rangle \) and \( |s_1(10)\rangle \), resp. Observe that \( P_1^t(Z|D = 1) = P_1^t(Z_{1,1},0) = (1/2)(P_1^t(00) + P_1^t(10)) \). Due to the lemma, there is a unitary correction transformation \( V \) acting nontrivially only on the Hilbert space of Alice’s extended input register such that

\[
\|V|s_1(00)\rangle - |s_1(10)\rangle\|_t \leq 2\sqrt{2}I(P_1^t(Z):Z_{1,1}|D = 1) = 2\sqrt{2}\eta_1,
\]

where for the sake of readability, pure states are only written as vectors. Let \( U(z) \) be the unitary transformation applied by Bob in the protocol \( P_1 \) if his input bit is \( z \). Then, by the unitary invariance of the trace norm and the fact that \( U(z), z \in \{0,1\} \), and \( V \) commute:

\[
\|V|s_1(01)\rangle - |s_1(11)\rangle\|_t = \|VU(1)U(0)^\dagger|s_1(00)\rangle - U(1)U(0)^\dagger|s_1(10)\rangle\|_t \\
= \|V|s_1(00)\rangle - |s_1(10)\rangle\|_t \leq 2\sqrt{2}\eta_1.
\]

As abbreviations, let \( |s'_1(00)\rangle = V|s_1(00)\rangle \) and \( |s'_1(01)\rangle = V|s_1(01)\rangle \). Observe that

\[
\langle s'_1(01) | s_1(11) \rangle = \langle s_1(01) | V^\dagger | s_1(11) \rangle \\
= \langle s_1(00) | U(0)U(1)^\dagger V^\dagger \cdot U(1)U(0)^\dagger | s_1(10) \rangle \\
= \langle s'_1(00) | s_1(10) \rangle.
\]

Using the relationship between fidelity and trace distance for pure states from Fact 3 and setting \( \gamma_1 = 2\eta_1 \) as an abbreviation, it follows that

\[
F(|s'_1(01)\rangle,|s_1(11)\rangle) = F(|s'_1(00)\rangle,|s_1(10)\rangle) \geq \sqrt{1 - 2\eta_1} \geq 1 - \gamma_1. \tag{2}
\]

We treat the subprotocol \( P_2 \) in the same way. Notice that the input of \( P_2 \) is also \( Z = (Z_1,Z_2) \), but now Alice has \( Z_2 \) and Bob has \( Z_1 \). Conditioned on \( D = 2 \), \( Z_2 \) is a random bit and \( Z_1 = 0 \). Let \( |s_2(b_1,b_2)\rangle \) be the final computational state of \( P_2 \) on input \( (b_1,b_2) \in \{0,1\}^2 \) (before tracing out any register). Again, this is a pure state. Let \( |s'_2(00)\rangle \) and \( |s'_2(01)\rangle \) be the states resulting from the application of a correction transformation according to the local transition lemma. Let \( \gamma_2 = 2\eta_2 \). Then, analogously to the above, \( \langle s'_2(10) | s_2(11) \rangle = \langle s'_2(00) | s_2(01) \rangle \) and

\[
F(|s'_2(10)\rangle,|s_2(11)\rangle) = F(|s'_2(00)\rangle,|s_2(01)\rangle) \geq 1 - \gamma_2. \tag{3}
\]

We still have to connect the local information about the subprotocols that we have just derived to the global behavior of the protocol \( P \) in order to exploit the fact that \( P \) computes AND with small error probability. For this, we first relate the distances of the states for the subprotocols to those for the whole protocol. Let \( |s(00)\rangle = \alpha_1|s'_1(00)\rangle + \alpha_2|s'_2(00)\rangle \), \( |s(01)\rangle = \alpha_1|s'_1(01)\rangle + \alpha_2|s'_2(01)\rangle \), \( |s(10)\rangle = \alpha_1|s_1(10)\rangle + \alpha_2|s_2(10)\rangle \), and \( |s(11)\rangle = \alpha_1|s_1(11)\rangle + \alpha_2|s_2(11)\rangle \). Then, using that \( P_1 \) and \( P_2 \) work on orthogonal subspaces and the definitions \( |\alpha_1|^2 = q_1 \), \( |\alpha_2|^2 = q_2 \), we get

\[
\langle s(01) | s(11) \rangle = q_1\langle s'_1(01) | s_1(11) \rangle + q_2\langle s_2(01) | s_2(11) \rangle, \tag{4}
\]
\[
\langle s(10) | s(11) \rangle = q_1\langle s_1(10) | s_1(11) \rangle + q_2\langle s'_2(10) | s_2(11) \rangle, \tag{5}
\]
\[
\langle s(00) | s(11) \rangle = q_1\langle s'_1(00) | s_1(11) \rangle + q_2\langle s'_2(00) | s_2(11) \rangle. \tag{6}
\]

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Second, we relate the absolute value of the left hand sides of equations (2)–(3), i.e., the fidelity of the respective pairs of states, to the error probability of the protocol $P$. Let $\rho$ be a quantum state over the Hilbert space of the computational states of $P$ and let $\text{tr}_{\text{non-work}}(\rho)$ denote the state obtained from $\rho$ by tracing out the input registers of both players, the random coin register, and the respective secret registers. Since the correction transformations only work on Alice’s extended input register or on Bob’s extended input register, their effect disappears after applying $\text{tr}_{\text{non-work}}$. Hence, for $(b_1, b_2) \in \{0,1\}^2$,

$$\text{tr}_{\text{non-work}}(\langle s(b_1, b_2) \rangle_s \langle s(b_1, b_2) \rangle) = P(b_1, b_2).$$

Furthermore, the qubit measured to get the output of the protocol does not belong to the bits traced out in this way. We can thus apply Fact 3(iii) to get $F(\vert s(01) \rangle, \vert s(11) \rangle) \leq \delta$ for $\delta = 2\sqrt{\varepsilon(1-\varepsilon)}$, where $\varepsilon$ is the error probability of $P$. Analogously, $F(\vert s(10) \rangle, \vert s(11) \rangle) \leq \delta$ and $F(\vert s(00) \rangle, \vert s(11) \rangle) \leq \delta$.

Third, we prove the desired relationship between error probability and information about the inputs stored in the subprotocols. Here it is crucial that the considered state vectors only have real components.

**Claim.** Let $\tau > 0$ and suppose that $\delta \leq 1 - (q_1 \gamma_1 + q_2 \gamma_2) - (3/2)(1/2 + \tau)$.

1. Suppose that there is an $i \in \{1,2\}$ such that $q_i \geq 1/2 + \tau$. Then $\delta \geq 2\tau - (1/2 + \tau)\gamma_i$.
2. Let $q_1, q_2 \in [1/2 - \tau, 1/2 + \tau]$.
   2.1. Suppose that both terms on the right hand side of equation (4) or (5), resp., have the same sign. Then $\delta \geq (1/2 - \tau)(1 - \gamma_1)$ or $\delta \geq (1/2 - \tau)(1 - \gamma_2)$, resp.
   2.2. Otherwise, $\delta \geq 1/5 - (4/5)(q_1 \gamma_1 + q_2 \gamma_2)$.

**Proof.**

**Case 1:** W.l.o.g., $q_1 \geq 1/2 + \tau$ and thus $q_2 \leq 1/2 - \tau$. By equation (4) and the lower bound on the fidelity from (2),

$$\delta \geq F(\vert s(01) \rangle, \vert s(11) \rangle) \geq q_1 |\langle s'_1(01) \vert s_1(11) \rangle| - q_2 |\langle s_2(01) \vert s_2(11) \rangle| \geq q_1 F(\vert s'_1(01) \rangle, \vert s_1(11) \rangle) - q_2 \geq q_1 (1 - \gamma_1) - q_2 \geq 2\tau - (1/2 + \tau)\gamma_1.$$

**Case 2.1:** Let, e.g., equation (4) have solely nonnegative terms on its right hand side. Then

$$\delta \geq F(\vert s(01) \rangle, \vert s(11) \rangle) = q_1 |\langle s'_1(01) \vert s_1(11) \rangle| + q_2 |\langle s_2(01) \vert s_2(11) \rangle| \geq q_1 (1 - \gamma_1) \geq (1/2 - \tau)(1 - \gamma_1).$$

**Case 2.2:** This is split into two further subcases handling the different possible signs of the inner products in equations (4) and (5).

**Case 2.2.1:** Suppose first that the first terms on the right hand sides of equation (4) and (5) have the same sign. Then due to the case distinction, the second terms have the respective opposite sign. We assume that $\langle s'_1(01) \vert s_1(11) \rangle \geq 0$ and $\langle s_1(10) \vert s_1(11) \rangle \geq 0$ (the other case, $\langle s'_1(01) \vert s_1(11) \rangle \leq 0$ and $\langle s_1(10) \vert s_1(11) \rangle \leq 0$, is handled analogously). We claim that then both inner products on the right hand side of equation (6) are nonnegative and of large absolute value. Using the weak inverse triangle inequality for inner products of real vectors (Proposition 4), we get

$$\langle s'_1(00) \vert s_1(11) \rangle \geq 2(\langle s'_1(00) \vert s_1(10) \rangle + \langle s_1(10) \vert s_1(11) \rangle) - 3. \quad (*)$$
which together with the lower bound on the fidelity from fact (3) implies

\[ \langle s_1'(00) | s_1(10) \rangle = \langle s_1'(01) | s_1(11) \rangle \geq 1 - \gamma_1. \]

Furthermore, using that \( \langle s_1(10) | s_1(11) \rangle \geq 0 \), \( \langle s_2'(10) | s_2(11) \rangle < 0 \), and equation (5), we get

\[ |\langle s_1(10) | s_1(11) \rangle| = \langle s_1(10) | s_1(11) \rangle \geq \frac{1}{q_1}(q_2|\langle s_2'(10) | s_2(11) \rangle| - \delta), \]

which together with the lower bound on the fidelity from fact (3) implies

\[ |\langle s_1(10) | s_1(11) \rangle| \geq \frac{q_2}{q_1}(1 - \gamma_2) - \frac{1}{q_1}\delta. \]

Substituting this into (4) yields

\[ \langle s_1'(00) | s_1(11) \rangle \geq 2\left(1 - \gamma_1 + \frac{q_2}{q_1}(1 - \gamma_2) - \frac{1}{q_1}\delta\right) - 3 = \frac{2}{q_1}(1 - (q_1\gamma_1 + q_2\gamma_2) - \delta) - 3. \]

Next we apply the inverse triangle inequality to the vectors \( |s_2'(00)\rangle, -|s_2(01)\rangle \), and \( |s_2(11)\rangle \). Recall that in the considered case, \( \langle s_2(01) | s_2(11) \rangle < 0 \) and \( \langle s_2'(10) | s_2(11) \rangle = \langle s_2'(00) | s_2(01) \rangle < 0 \). Analogously to the calculations above, we get

\[ \langle s_2'(00) | s_2(11) \rangle \geq 2\left(\frac{-\langle s_2'(00) | s_2(01) \rangle + \langle s_2(01) | s_2(11) \rangle}{1 - \gamma_2}\right) \geq \frac{2}{q_2}(1 - \gamma_1 + q_2\gamma_2) - \delta - 3. \]

By the derived estimates and the fact that

\[ \delta \leq 1 - (q_1\gamma_1 + q_2\gamma_2) - \frac{3}{2}\left(\frac{1}{2} + \tau\right) \leq 1 - (q_1\gamma_1 + q_2\gamma_2) - \frac{3}{2}\max\{q_1, q_2\} \]

due to the hypothesis of the claim, the inner products \( \langle s_1'(00) | s_1(11) \rangle \) and \( \langle s_2'(00) | s_2(11) \rangle \) are both nonnegative. Hence, using equation (3) we obtain

\[ \delta \geq q_1|\langle s_1'(00) | s_1(11) \rangle| + q_2|\langle s_2'(00) | s_2(11) \rangle| \geq 4\left(1 - (q_1\gamma_1 + q_2\gamma_2) - \delta\right) - 3, \]

and solving for \( \delta \),

\[ \delta \geq \frac{1}{5} - \frac{4}{5}(q_1\gamma_1 + q_2\gamma_2). \]

This completes the proof for Case 2.2.1.

Case 2.2.2: In the last remaining case, the first terms on the right hand side of equation (4) and (5) have opposite sign. W.l.o.g., let \( \langle s_1'(01) | s_1(11) \rangle \geq 0 \) and \( \langle s_1(10) | s_1(11) \rangle \leq 0 \). We claim that then both inner products on the right hand side of equations (4) and (5) are nonpositive and of large absolute value. Applying the weak inverse triangle inequality for inner products to the vectors \( |s_1'(00)\rangle, |s_1(10)\rangle \), \( -|s_1(11)\rangle \), and to the vectors \( |s_2'(00)\rangle, |s_2(01)\rangle \), \( -|s_2(11)\rangle \), resp., yields

\[ -\langle s_1'(00) | s_1(11) \rangle \geq 2\left(|\langle s_1'(00) | s_1(10) \rangle| - |\langle s_1(10) | s_1(11) \rangle|\right) - 3 \]

and

\[ -\langle s_2'(00) | s_2(11) \rangle \geq 2\left(|\langle s_2'(00) | s_2(01) \rangle| - |\langle s_2(01) | s_2(11) \rangle|\right) - 3, \]
Finally, it only remains to exploit the bounds on $\delta$ into cases as in the claim. Due to equation (1) and taking into account that $\eta_i = \gamma_i/2$ for $i = 1, 2$, we have

\[
I(P(Z):Z|D) \geq \frac{1}{2}(q_1\eta_1 + q_2\eta_2) = \frac{1}{4}(q_1\gamma_1 + q_2\gamma_2).
\]

For the following case distinction, we assume that the hypothesis of the claim, $\delta \leq 1 - (q_1\gamma_1 + q_2\gamma_2) - (3/2)(1/2 + \tau)$, is satisfied.

**Case 1:** Again, we only consider the subcase $q_1 \geq 1/2 + \tau$. Due to the claim, $\delta \geq 2\tau - (1/2 + \tau)\gamma_1$, implying $\gamma_1 \geq (1/2 + \tau)^{-1}(2\tau - \delta)$. Thus, using that $q_1 \geq 1/2 + \tau$,

\[
I(P(Z):Z|D) \geq \frac{1}{4}q_1\gamma_1 \geq \frac{1}{4}\left(\frac{1}{2} + \tau\right)^{-1}(2\tau - \delta) = \frac{1}{2}\tau - \frac{1}{4}\delta. \tag{7}
\]

**Case 2.1:** W.l.o.g., let $\delta \geq (1/2 - \tau)(1 - \gamma_1)$ by the claim, i.e., $\gamma_1 \geq 1 - (1/2 - \tau)^{-1}\delta$. Then, using that $q_1 \geq 1/2 - \tau$ in this case,

\[
I(P(Z):Z|D) \geq \frac{1}{4}q_1\gamma_1 \geq \frac{1}{4}\left(\frac{1}{2} - \tau\right)^{-1}(1 - (\frac{1}{2} - \tau)^{-1}\delta) = \frac{1}{8} - \frac{1}{4}\tau - \frac{1}{4}\delta. \tag{8}
\]

**Case 2.2:** We have $\delta \geq 1/5 - (4/5)(q_1\gamma_1 + q_2\gamma_2)$ by the claim, i.e., $q_1\gamma_1 + q_2\gamma_2 \geq 1/4 - (5/4)\delta$. Then

\[
I(P(Z):Z|D) \geq \frac{1}{4}(q_1\gamma_1 + q_2\gamma_2) \geq \frac{1}{4}\left(\frac{1}{4} - \frac{5}{4}\delta\right) = \frac{1}{16} - \frac{5}{16}\delta. \tag{9}
\]

We still have to take the upper bound on $\delta$ needed for the application of the claim into account. This requires that

\[
\delta \leq 1 - (q_1\gamma_1 + q_2\gamma_2) - \frac{3}{2}\left(\frac{1}{2} + \tau\right).
\]

Since $I(P(Z):Z|D) \geq (1/4)(q_1\gamma_1 + q_2\gamma_2)$, the above is satisfied if

\[
I(P(Z):Z|D) \leq \frac{1}{16} - \frac{3}{8}\tau - \frac{1}{4}\delta.
\]

Now either the assumption of the lemma is not satisfied and negating the last inequality gives us the lower bound

\[
I(P(Z):Z|D) \geq \frac{1}{16} - \frac{3}{8}\tau - \frac{1}{4}\delta, \tag{10}
\]

□
or we get the minimum of (7–9) as a lower bound. It remains to fix \( \tau \) such that we get a positive lower bound on the information for the largest possible \( \delta \).

We choose \( \tau = 1/14 \) and assume that \( \delta \leq 1/7 \). Then either (10) is satisfied and thus

\[
I(P(Z) : Z \| D) \geq 1/28 - (1/4)\delta,
\]

or we get the minimum of (7–9) as a lower bound. It remains to fix \( \tau \) such that we get a positive lower bound on the information for the largest possible \( \delta \).

We choose \( \tau = 1/14 \) and assume that \( \delta \leq 1/7 \). Then either (10) is satisfied and thus

\[
I(P(Z) : Z \| D) \geq 1/28 - (1/4)\delta,
\]

or we get the minimum of (7–9) as a lower bound. It remains to fix \( \tau \) such that we get a positive lower bound on the information for the largest possible \( \delta \).

The last term in the minimum is smaller than the first one only if \( \delta > 3/7 \). Since we have assumed that \( \delta \leq 1/7 \), we again get the lower bound \( I(P(Z) : Z \| D) \geq 1/28 - (1/4)\delta \).

Altogether, we have shown that for \( \delta \leq 1/7 \),

\[
I(P(Z) : Z \| D) \geq 1/28 - (1/4)\delta.
\]

The claim on the range of the error probabilities in the theorem follows by substituting \( \delta = 2\sqrt{\varepsilon(1 - \varepsilon)} \) into the bound \( \delta < 1/7 \).

\[\Box\]

4.4. Application to Quantum Read-Once BPs for the Disjointness Function

It is convenient here to work with the negation of DISJ_n, i.e., the non-disjointness function defined by \( ND_n(x, y) = x_1y_1 \lor \cdots \lor x_ny_n \). We consider the same input distribution for \( ND_n \) as in [7, 13]. Let \( D \) and \( Z \) be random variables as defined for AND in the previous subsection. Let \( \vec{D} = (D_1, \ldots, D_n) \) and \( \vec{Z} = (Z_1, \ldots, Z_n) \) consist of \( n \) independent copies of \( D \) and \( Z \), resp. For \( i = 1, \ldots, n \) let \( Z_i = (Z_{i,1}, Z_{i,2}) \). Observe that, for any value \( \vec{d} \) that \( \vec{D} \) can attain and for each \( i = 1, \ldots, n \), the random variables \( Z_{i,1}, Z_{i,2} \) are independent when conditioned on \( \vec{D} = \vec{d} \). Furthermore, if for any \( i \in \{1, \ldots, n\} \) and any \( (a, b) \in \{0, 1\}^2 \) we let \( \vec{Z}' \) be the modified input obtained from \( \vec{Z} \) by replacing \( Z_i \) with \( (a, b) \), we observe that, with probability 1, \( ND_n(\vec{Z}') = \text{AND}(a, b) \).

Call a quantum read-once BP \textit{regular} if it reads each of its variables at least once, i.e., on each of its paths each variable occurs exactly once. Observe that, in particular, this means that such a graph is leveled. For a regular quantum read-once BP \( G \) and an input \( z \) of length \( n \), let \( G(z) \) denote the final quantum state computed by \( G \) on \( z \) after \( n \) computation steps, before the measurement at the sinks occurs. Observe that \( G(z) \) is a pure state by the definition of QBPs. Using Fact [2] we get:

\[\textbf{Proposition 4.} \] Let \( G \) be a regular quantum read-once BP and let \( Z \) be a classical random variable describing an input of \( P \). Then \( I(G(Z) : Z) = S(G(Z)) \leq \log |G| \).

The next lemma describes how a regular quantum read-once BP for \( ND_n \) can be used for computing the AND of any pair of input variables \( x_i, y_i \) of \( ND_n \).

\[\textbf{Lemma 7.} \] Let \( G \) be a regular \( \varepsilon \)-error quantum read-once BP for \( ND_n \). Let \( i \in \{1, \ldots, n\} \) and let a pair of assignments \( (a, b) \) to the variable vectors \( (x_j)_{j \neq i} \) and \( (y_j)_{j \neq i} \) be given such that \( \text{AND}(a_j, b_j) = 0 \) for all \( j \neq i \). Then there is an \( \varepsilon \)-error quantum 2-partition protocol \( P_{a,b} \) for AND on \( x_i, y_i \) that does not use its public random coin register and has the property that for each input assignment \( (c, d) \) its result state \( P_{a,b}(b, c) \) agrees with the final state \( G(a, b, c, d) \) of \( G \) on \( (a, b, c, d) \).

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In the proof of the lemma, we consider quantum read-once BPs with unlabeled nodes obtained by setting variables to constants as follows. Given a quantum read-once BP $G$ and a partial assignment $a$ to some of the variables of $G$, for each node with a variable fixed by $a$ we remove the variable label, remove all outgoing edges that are inconsistent with $a$, and remove all boolean labels from the remaining edges. Then the resulting graph has the same number of nodes as $G$ and still has a well-defined semantics by the remarks in Section 2.

We prepare the proof of the lemma by describing the modifications of the given quantum BP required for the construction of the desired communication protocol in advance. Let $G'$ be the quantum read-once BP containing unlabeled nodes obtained from the given graph $G$ according to the hypothesis of the lemma by replacing variables with constants according to $(a, b)$. Observe that due to the fact that each variable is read exactly once in $G$, on each path from the source to a sink in $G'$ each of the two variables $x_i$ and $y_i$ occurs exactly once as the label of a node. Furthermore, each node labeled by a variable is either the first such node on all paths reaching it or the second. Let $S_x$ and $S_y$ be the sets of nodes labeled by $x_i$ and $y_i$, resp., where this is the first variable on each path reaching the node. Let $T_x$ and $T_y$ be the sets of immediate successors of nodes labeled by an $x_i$- or a $y_i$-variable, resp., for which this is the second variable on each path reaching it. Let $T = T_x \cup T_y$. To avoid tedious case distinctions, we assume w.l.o.g. that $S_x \neq \emptyset$ and $S_y \neq \emptyset$. In particular, this implies that the source is unlabeled. Otherwise, it is easy to use the ideas described in the following to define a quantum 1-partition communication protocol with the required properties.

Furthermore, for any node $v$ in $G'$, let $d_{\text{source}}(v)$ and $d_{\text{sinks}}(v)$ denote the number of edges on each path from the source to $v$ and the number of edges on each path from $v$ to a sink, resp. Each of this is well-defined due to the fact that in the original graph $G$ each variable occurs exactly once on each path. The graph $G'$ has the following properties:

- The sets $S = S_x \cup S_y$ and $T = T_x \cup T_y$ form cuts in $G'$, i.e., each path from the source to a sink runs through exactly one node from each set, and all nodes on paths from the source to $S$ (excluding the latter) and from $T$ to the sinks (including the former) are unlabeled nodes.
- We have $S_x \cap S_y = \emptyset$ and $T_x \cap T_y = \emptyset$ (the latter due to the unidirectionality of $G'$ inherited from $G$). Furthermore, all paths starting in $S_x$ lead to $T_y$ and all paths starting in $S_y$ lead to $T_x$ and the sets of nodes on the paths of these two types are disjoint (for the nodes not in $S \cup T$, this follows from the read-once property of $G'$ inherited from $G$).

We use these properties to partition the nodes of $G'$ into four subsets:

- The top part, including all nodes reached by paths from the source to a node in $S$, excluding the latter;
- two middle parts that consist of all nodes on paths from $S_x$ to $T_y$ and from $S_y$ to $T_x$, resp.; and
- a bottom part with all nodes on paths starting at a node in $T$ and leading to a sink, excluding the former and including the latter.
Next we further simplify the structure of $G'$, which gives us a new quantum read-once BP with unlabeled nodes. Our aim is to ensure that all first nodes in the new middle parts lie on a single level and the same for all last nodes in the new middle parts.

*Changes in the top part:* First, we replace the top part of $G'$ and extend the middle parts upwards. For each node $v \in S$ let $\alpha_v$ be the amplitude for reaching it from the source (i.e., the sum over all paths of the products of the amplitudes at the edges of these paths). Remove the top part of $G'$. For each $v \in S$, add a chain of $d_{\text{source}}(v) - 1 \geq 0$ new unlabeled dummy nodes, where a single outgoing edge without boolean label and with amplitude 1 leads to the next dummy node in the chain for the first $d_{\text{source}}(v) - 2$ nodes and to $v$ for the last dummy node in the chain. Add a new unlabeled source with outgoing edges that have no boolean label and that lead to the sources of the chains of dummy nodes. The edge leading to the chain of dummy nodes for node $v$ is labeled with amplitude $\alpha_v$.

*Changes in the bottom part:* For a node $v \in T$ and a sink $w$, let $\beta_{v,w}$ be the amplitude for reaching $w$ from $v$. Remove the bottom part except for the sinks. Also remove each node $v \in T$ and redirect all incoming edges to the first node in a chain of new unlabeled dummy nodes of length $d_{\text{sinks}}(v)+1 \geq 1$. The first $d_{\text{sinks}}(v)$ nodes of this chain have a single outgoing edge with amplitude 1 leading to the next node in the chain. Furthermore, for each sink $w$ that has been reachable from $v$ in $G'$ add an edge leading from the last node in the chain of dummy nodes to $w$ with amplitude $\beta_{v,w}$. Observe that this construction increases the length of all computation paths by 1. This ensures that the nodes at the ends of the chains of dummy nodes, which play the role of those in $T$ in the new graph, are separated from the sinks, and thus avoids unwanted case distinctions.

Call the resulting graph $G''$. We state the key properties of $G''$ in form of the following lemma.

**Lemma 8.** The graph $G''$ is a legal quantum read-once BP with unlabeled nodes. Furthermore, for any input assignment $(c, d)$ to $(x_i, y_i)$ the final state of $G''$ on the input $(c, d)$ agrees with that of $G'$ and thus also with that of $G$ on the input $(a, b, c, d)$. In particular, $G''$ computes the AND of $x_i$ and $y_i$ with the error bound $\varepsilon$ of $G$.

*Proof.* We prove that the changes that turn $G'$ into $G''$ retain well-formedness, unidirectionality, and the transformation computed by the graph as a QBP. We consider the top part and the bottom part of $G'$ separately.

*Changes in the top part:* We first introduce some notation. Number the levels of $G'$ from 0 (the level of the source) to $2n$ (the level of the sinks). Let $L_\ell$ be the set of nodes on level $\ell$. For any subset of nodes $A \subseteq L_\ell$ let $\overline{A} = L_\ell - A$. Let $\ell_1 < \cdots < \ell_k$ be the levels of $G'$ that contain nodes from $S$ (recall that these are the first nodes on paths in $G'$ that are labeled by a variable). Let $\ell_0 = 1 \leq \ell_1$.

Observe that the nodes on level $\ell$ with $0 \leq \ell \leq \ell_k$ can be classified as follows: (i) nodes in $S$, i.e., nodes that are labeled by a variable and that are reachable only by paths that solely contain unlabeled nodes; (ii) unlabeled nodes that are reached only by paths that solely contain unlabeled nodes; (iii) unlabeled nodes $v$ with the property that a node in $S$ lies on each path from the source to $v$. Let $V_S(\ell)$, $V_U(\ell)$, resp., be the sets of nodes of the
first two types on level $\ell$. For levels $\ell, \ell'$ with $\ell < \ell'$ and any sets of nodes $A \subseteq L_\ell$ and $B \subseteq L_{\ell'}$, where $B$ contains all nodes reachable from $A$ and no nodes reachable from $\overline{A}$, let $U_{A,B}$ be the transformation that acts on the basis vectors of nodes in $A$ as described by the subgraph of $G'$ consisting of all paths from $A$ to $B$ and as the identity on all basis vectors of nodes in $\overline{A}$ (due to the well-formedness of $G'$, this can be extended to a unitary transformation). For $0 \leq \ell < \ell' \leq 2n$ we use the abbreviation $U_{\ell,\ell'} = U_{L_\ell,L_{\ell'}}$.

We describe the insertion of the chains of dummy nodes, called dummy chains in the following, as an inductive process. The above definitions of the sets of nodes are meant to refer to the actual graph after the modifications carried out so far. In induction step $i$, $i = k - 1, k - 2, \ldots, 0$, we modify the levels $\ell_i - 1, \ell_i, \ldots, \ell_{i+1}$. The aim is to replace the unlabeled nodes in the top part of $G'$ between levels $\ell_i$ and $\ell_{i+1}$ by dummy chains and to modify the transformation between levels $\ell_i - 1$ and $\ell_i$ to maintain the correct overall transformation of the graph.

We choose $W'' = V_S(\ell_{i+1}) \cup V_U(\ell_{i+1})$ as the set of the end nodes of the new dummy chains. The set $V_U(\ell_{i+1})$ is empty for $i = k - 1$ and contains the start nodes of all already constructed dummy chains for $i \leq k - 2$. We observe that the nodes on level $\ell_i$ from which nodes in $W''$ are reachable are precisely those in $W'' = V_U(\ell_i)$. The immediate predecessors of $W'$ are the nodes in $W = V_U(\ell_i - 1)$. Furthermore, the set $X = W'' \cup V_S(\ell_i)$ contains all nodes reachable from $W$. Finally, note that the transformations $U_{W,X}$ and $U_{W',W''}$ are well-defined.

Now step $i$ of the inductive construction is done as follows:

- Remove all nodes and edges on paths from $W$ to $W''$, excluding the start and end nodes.
- For each node in $W''$, insert a chain of new dummy nodes from level $\ell_i$ to that node. Let $W'$ denote the set of start nodes of these chains on level $\ell_i$ and let $T_{W',W'}$ be the linear extension of the bijection that maps the basis states belonging to $W''$ to those belonging to $W'$.
- Change the edges between $W$ and $X$ such that the transformation $U_{W,X}$ realized before by these edges is replaced with the transformation $T_{W'',W'} U_{W',W''} U_{W,X}$.

Assuming that the given graph is well-formed and unidirectional, these steps can be carried out such that this is still true for the resulting graph. We claim that the modifications do not change the transformation realized by the graph if interpreted as a QBP. We only need to consider the transformations realized between the modified levels.

Originally, we have

$$U_{\ell_i-1,\ell_{i+1}} = U_{W',W''} U_{W',W''} U_{W,X} U_{W,X}.$$

Let $\mathcal{X} = W' \cup V_S(\ell_i)$ and denote the transformations in the modified graph between sets $A$ and $B$ by $U_{A,B}$. Then, by the construction, $U_{W,X} = T_{W'',W'} U_{W',W''} U_{W,X}$ and $U_{W',W''} = T_{W''}^{-1} U_{W',W''}$. Hence,

$$U_{\ell_i-1,\ell_{i+1}} = U_{W',W''} U_{W',W''} U_{W,X} U_{W,X}$$

$$= T_{W''}^{-1} U_{W',W''} U_{W',W''} U_{W,X} U_{W,X}.$$

Since $U_{W',W''}$ commutes with both $T_{W''}^{-1}$ and $U_{W',W''}$ due to the disjointness of the
respective sets of nodes, we get
\[ U_{i-1,i+1} = U_{W',W''} U_{W,W''} U_{W,X} = U_{W,X}. \]

Since the changes do not affect the transformations from \( W \) to \( X = \overline{X} \) and from \( W' = \overline{W} \to \overline{W''} \), the right hand side above is equal to the original transformation \( U_{i-1,i+1} \).

Altogether, the graph that we obtain by carrying out all inductive steps is still well-formed and unidirectional and computes the same transformation as \( G' \). It is easy to see that this is exactly the graph obtained by the modification of the top part described before the lemma.

*Changes in the bottom part:* For simplicity, we first insert a level of unlabeled dummy nodes directly above the sinks such that each sink has a corresponding dummy node in this new level which obtains its incoming edges and from which it is reached by an edge labeled with amplitude 1. This ensures that the set \( T \) of direct successors of nodes that are the second ones on each path labeled by a variable is disjoint from the sinks in the new graph. The rest of the proof for the bottom part is now analogous to that for the top part if we look at the graph turned upside down and exchange the level of the source with that of the sinks and the set \( S \) with the set \( T \).

We partition the set of nodes of \( G'' \) analogously to that of \( G \). The top part of \( G'' \) consists of the source, the bottom part consists of the sinks, and the two middle parts consist of the nodes in the middle parts of \( G' \) together with the dummy nodes on the chains added to the respective parts.

Overloading notation, we reuse \( S, S_x, S_y \) and \( T, T_x, T_y \) to denote the sets of start and end nodes, resp., in the new middle parts of \( G'' \) analogous to the respective sets in \( G' \). For any node \( v \) and any assignment \( z \in \{0,1\}^2 \) to \((x_i,y_i)\), let \( |\psi_{v,\ell}(z)| \) denote the superposition of basis states belonging to the nodes on level \( \ell \) of \( G'' \) computed by \( G'' \) on input \( z \) when starting from the basis state belonging to \( v \). For the construction of the desired quantum 2-partition protocol, we need the following property of \( G'' \), which we prove in advance.

**Lemma 9.** Let \( v_1 \in S_x, v_2 \in S_y \) and let \( z_1, z_2 \in \{0,1\}^2 \) be any assignments to \((x_i,y_i)\).
Then for any level \( \ell \in \{1, \ldots, 2n+1\} \) (where the source is on level 0 and the sinks are on level \( 2n + 1 \)), the states \( |\psi_{v_1,\ell}(z_1)| \) and \( |\psi_{v_2,\ell}(z_2)| \) are orthogonal.

**Proof.** The claim is obviously true for all levels \( \ell \in \{1, \ldots, 2n\} \), since the sets of nodes in the respective superpositions \( |\psi_{v_1,\ell}(z_1)| \) and \( |\psi_{v_2,\ell}(z_2)| \) are disjoint. We have to verify the claim for the level \( \ell = 2n + 1 \) of the sinks.

Let \( U \) be a unitary extension of the transformation realized by the edges between the last level \( 2n \) of \( G'' \) above the sinks and level \( 2n + 1 \). Since the last level above the sinks only contains unlabeled nodes, \( U \) does not depend on the input. Thus, \( |\psi_{v_i,2n+1}(z_i)| = U|\psi_{v_i,2n}(z_i)| \) for \( i = 1, 2 \) and we get
\[
\langle \psi_{v_1,2n+1}(z_1) | \psi_{v_2,2n+1}(z_2) \rangle = \langle \psi_{v_1,2n}(z_1) | U^\dagger U | \psi_{v_2,2n}(z_2) \rangle = \langle \psi_{v_1,2n}(z_1) | \psi_{v_2,2n}(z_2) \rangle = 0.
\]

\[\square\]
Proof of Lemma 7. We construct the quantum 2-partition protocol $P = P_{a,b}$ using the graph $G''$. We make sure that $P$ simulates $G''$.

For $v \in S = S_x \cup S_y$, let $\alpha_v$ be the amplitude for reaching $v$ from the source of $G''$. The protocol $P$ works on the space spanned by the basis vectors belonging to the nodes in the middle parts and in the bottom part of $G''$. It has subprotocols $P_x$ and $P_y$ with respect to the variable partitions ($\{x_i\}, \{y_i\}$) and ($\{y_i\}, \{x_i\}$), resp. Let $q_x = \sum_{v \in S_x} |\alpha_v|^2$ and $q_y = \sum_{v \in S_y} |\alpha_v|^2$. The initial amplitudes of $P_x$ and $P_y$ are defined as $\sqrt{q_x}$ and $\sqrt{q_y}$, resp. As the initial state $|s_x\rangle$ of $P_x$ we choose $\sum_{v \in S_x} (\alpha_v/\sqrt{q_x}) |v\rangle$ if $q_x \neq 0$ and some arbitrary $|v\rangle$ with $v \in S_x$ if $q_x = 0$. Define $|s_y\rangle$ analogously for $P_y$.

We only describe the computation of $P_x$ in detail, $P_y$ works in the same way. We first define further subprotocols $P_{x,v}$ belonging to each of the nodes $v \in S_x$. Let $G''_v$ be the subgraph of $G''$ with source $v \in S_x$ containing all nodes reachable from $v$. On each path starting at a node $v \in S_x$ there is exactly one $y_i$-node. There is some level $m$ where the first $y_i$-node in $G''_v$ is read. In $P_{x,v}$ the player Alice simulates the computation of $G''_v$ starting at $v$ and until level $m$. She sends the reached superposition of basis states of nodes on level $m$ to Bob. Bob continues the simulation of $G''_v$ starting with the superposition received from Alice and computing a superposition of the sinks in $G''_v$. Let $P_x$ be the protocol where the described subprotocols $P_{x,v}$, $v \in S_x$, are applied to the initial state of $P_x$.

We claim that $P_x$ designed in this way is a legal quantum one-way protocol. The state obtained after Alice has finished her computation in $P_x$ need not be reachable by any computation in $G''$. Nevertheless, it is a legal pure quantum state by the following argument. Let $S_{x,m} \subseteq S_x$ be the set of all nodes $v$ for which $m$ is the first level with $y_i$-nodes reached from $v$ in $G''_v$. Let $A_v$ be the unitary transformation applied by Alice in $P_{x,v}$. Then the state computed by Alice according to $P_x$ is $|\psi\rangle = \sum_{v \in S_x} \alpha_v A_v |v\rangle$ and we have

$$
\langle \psi | \psi \rangle = \sum_{v,v' \in S_x} \alpha_v^\ast \alpha_{v'} \langle v | A_v^\dagger A_{v'} | v' \rangle = \sum_m \sum_{v,v' \in S_{x,m}} \alpha_v^\ast \alpha_{v'} \langle v | A_v^\dagger A_{v'} | v' \rangle
= \sum_m \sum_{v \in S_{x,m}} |\alpha_v|^2 = 1.
$$

The second equality is due to the fact that the subspaces induced by the nodes on different levels of $G''$ are orthogonal. The third equality follows from the unitarity of the time evolution of $G''$. Due to the same fact, also Bob’s transformation in $P_x$ is unitary. Finally, due to Lemma 7, the state spaces of $P_x$ and $P_y$ constructed in the above way are orthogonal. Hence, putting these protocols together as described before gives a legal quantum 2-partition protocol $P$. It is obvious that $P$ simulates $G''$ and thus its result state also agrees with the final state of $G$. \(\square\)

We are now ready to prove the main theorem.

Proof of Theorem 2. Let $G$ be a regular $\varepsilon$-error quantum read-once BP for ND$_n$. We run $G$ on the random input $\vec{Z}$ conditioned on $\vec{D} = \vec{d}$. Since $\vec{Z} = (Z_1, \ldots, Z_n)$, where $Z_1, \ldots, Z_n$ are independent, Fact 17 (superadditivity of mutual information),
Fact 4 and Proposition 4 yield
\[ \sum_{i=1}^{n} I(G(\vec{Z}): Z_i | \vec{D}) \leq I(G(\vec{Z}): \vec{Z} | \vec{D}) = \sum_{\vec{d}} \Pr\{\vec{D} = \vec{d}\} \cdot I(G(\vec{Z}): Z | \vec{D} = \vec{d}) \leq \log |G|. \]

Hence, by averaging, we can fix an \( i \) such that
\[ I(G(\vec{Z}): Z_i | \vec{D}) \leq (\log |G|)/n. \]

Let \( \vec{D} = (\vec{D}_{-i}, D_i) \), where \( \vec{D}_{-i} = (D_j)_{j \neq i} \). Then again by averaging and Fact 4, there is a value \( \vec{d}_{-i} \) for \( \vec{D}_{-i} \) such that
\[ I(G(\vec{Z}): Z_i | \vec{D}_{-i} = \vec{d}_{-i}, D_i) \leq (\log |G|)/n. \]

To prove the claim, we lower bound the term on the left hand side of this inequality by the information cost of an \( \epsilon \)-error quantum 2-partition protocol for AND on input \( Z_i \) conditioned on \( D_i \). Then using the constant lower bound on the information cost from Theorem 3 and the above inequality, we get that \((\log |G|)/n = \Omega(1)\) and thus \(|G| = 2^{\Omega(n)}\), which proves the theorem.

For the following, let \( d \) be any fixed value for \( D_i \). Let \( \vec{Z}^{(d)} = (Z_1^{(d)}, \ldots, Z_n^{(d)}) \) be a random variable that is distributed as \( \vec{Z} \) conditioned on \( \vec{D}_{-i} = \vec{d}_{-i} \) and \( D_i = d \). Let \( \vec{Z}_{-i}^{(d)} = (Z_j^{(d)})_{j \neq i} \). For each fixed value \( \vec{z}_{-i} \) in the support of \( \vec{Z}_{-i}^{(d)} \), we get an \( \epsilon \)-error quantum 2-partition protocol \( P_{\vec{z}_{-i}} \) for AND on the input \( z_i = (x_i, y_i) \) by Lemma 7. This protocol does not use its public random coin register. Furthermore, the result state of \( P_{\vec{z}_{-i}} \) on \( z_i \) agrees with the final state \( G(\vec{z}_{-i}, z_i) \) of \( G \). Let \( Q \) be a quantum 2-partition protocol in which the players run \( P_{\vec{z}_{-i}} \) for \( \vec{z}_{-i} \) chosen randomly with the distribution of \( \vec{Z}_{-i} \) under the condition \( \vec{D}_{-i} = \vec{d}_{-i} \). They can do this by initializing the public random coin register and the secret part of this register appropriately according to our conventions. Then the result state of \( Q \) after trace-out of the input registers of both players, the random coin register, and the secret registers is \( Q(z_i) = P_{\vec{z}_{-i}}^{(d)}(z_i) \).

Now \( Q \) is run on the random input \( Z_i^{(d)} \) by using the secret input registers of Alice and Bob (at this point, we exploit the fact that the input bits of Alice and Bob under the condition \( D_i = d \) are independent of each other). Expanding the abbreviations and using the fact that \( P_{\vec{z}_{-i}}(z_i) = G(\vec{z}_{-i}, z_i) \) for all \( (\vec{z}_{-i}, z_i) \), we get:
\[ I(Q(Z_i^{(d)}): Z_i^{(d)}) = I(Q(Z_i): Z_i | D_i = d) = I(P_{\vec{z}_{-i}}(Z_i): Z_i | \vec{D}_{-i} = \vec{d}_{-i}, D_i = d) = I(G(\vec{Z}): Z_i | \vec{D}_{-i} = \vec{d}_{-i}, D_i = d). \]

Averaging over all values \( d \) yields
\[ \text{IC}(Q; Z_i | D_i) = I(Q(Z_i): Z_i | D_i) = I(G(\vec{Z}): Z_i | \vec{D}_{-i} = \vec{d}_{-i}, D_i). \]

Since for the vector \( \vec{Z}' \) obtained from \( \vec{Z} \) by replacing \( Z_i \) with any \( z_i \in \{0, 1\}^2 \), \( \text{ND}_n(\vec{Z}') = \text{AND}(z_i) \) with probability 1, we know that \( Q \) is an \( \epsilon \)-error quantum 2-partition protocol for AND. By the lower bound on the information cost of quantum multi-partition protocols for AND from Theorem 3 it follows that the left hand side of the above inequality is lower bounded by a positive constant. Together with our above arguments, this completes the proof. \( \square \)
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