Extended Nonabelian Symmetries for Free Fermionic Model

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Abstract The higher spin symmetry for both Dirac and Majorana massless free fermionic field models are considered. An infinite Lie algebra which is a linear realization of the higher spin extension of the cross products of the Virasoro and affine Kac-Moody algebras is obtained. The corresponding current algebra is closed which is not the case of analogous current algebra in the WZNW model. The gauging procedure for the higher spin symmetry is given also.

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1. Introduction

In the paper [1], it was shown that the WZNW model admits the same symmetries $O(N) \otimes O(N)$, Virasoro and Kac-Moody on the classical level as (in the critical point) on the quantum level. A similar result was obtained in the papers [2] and [3] where the gauged fermionic model and the gauged WZNW model were considered. In the last few years, higher spin extensions of the Virasoro algebra were under intensive considerations (for a complete list of references see [4]). Recently, the higher spin extended symmetries for the nonabelian free massive Majorana fermionic model have been investigated in the papers [5] and [6]. The corresponding current algebra contains the $W_\infty$ current algebra as a subalgebra. A similar problem was considered in Ref. [7] where it was shown that the classical WZNW model admits nonlinearly realized $W_\infty$ symmetry, both linearly and nonlinearly realized $W_\infty$ symmetry as well as linearly realized extended affine Kac-Moody symmetry. However, the current algebras which correspond to the latter symmetries (except the $w_\infty$) are not closed. The later makes it impossible to gauge (see Refs. [8] and [9]) these symmetries in the case of the WZNW model. Consequently, there arises a difference between the nonabelian free fermionic model and the WZNW model on the extended symmetry level.

The main goal of the present article is to make a further clarification of the above mentioned difference between two-dimensional free fermionic model and the corresponding WZNW model. Here we investigate the higher spin extended symmetries for the nonabelian $SO(N)$, $SL(N)$ and $SU(N)$ free fermionic models in the Lagrangian approach. It is shown that all these symmetries admit gauging, because on both classical and quantum levels there is $W_\infty$ symmetry as well as extended Kac-Moody symmetry whose currents form an invariant space. The latter and the results from [7] show that the nonabelian free fermionic model and the WZNW model are not equivalent on the higher spin symmetry level (see [5], [7], [10]). The symmetries considered here are off-shell, however, the corresponding Noether currents are nonhermitean in the basis which is chosen for convenience. This nonhermitean form of the currents simplify the derivation of the Lie algebra in both the classical and quantum cases as well as the obtaining of the transformation laws for the gauge fields. Due to the fact that not all of the currents in the Majorana case are independent, there arises a symmetry of Stuckelberg type [9]. Moreover, in this nonhermitean basis the central terms are nondiagonal with respect to the conformal spin which means that the transition amplitudes between the states with different spins are nonzero, i.e. we
are dealing with nonphysical basis. We note, that in this case the Lie algebra and the corresponding current algebra have more simple forms than the corresponding algebra in the physical basis (with diagonal central terms) \[11\], \[6\]. The transition to the physical basis can be made by simple redefinition of the currents which induces trivial deformation of the Lie algebra as well as of the current algebra.

2. Classical free fermionic algebra

In order to show the difference between the WZNW model and the nonabelian free fermionic model we consider higher spin extension of the Virasoro and of the affine Kac-Moody algebras in the case of free complex fermionic realization. Higher spin extension of the \(O(N)\) Kac-Moody algebra for the massive free Majorana spinor field was obtained in Ref. \[5\] and \[6\]. In the present article we will show that this extension has a more simple form for the complex spinor fields than for the Majorana ones.

In the case of \(SU(N)\) or \(SL(N,C)\) free fermionic model, we have a complex spinor field for which the free field action reads:

\[
S = \int dzd\bar{z}(\bar{\psi}_+ \partial \bar{z} \psi_+ + \bar{\psi}_- \partial z \psi_-) \tag{2.1}
\]

It is easy to verify that this action is off-shell invariant with respect to the following extended conformal and extended global (semilocal) nonabelian transformations:

\[
\begin{align*}
\delta^m \{k\} \psi(z) &= k_m(z) \partial^{m+1} \psi(z), \\
\delta^m \{k\} \bar{\psi}(z) &= (-)^{m+2} \partial^{m+1} (k_m(z) \bar{\psi}(z)), \quad (m = 0, 1, \ldots), \\
\hat{\delta}^m [\hat{\alpha}] \psi(z) &= \alpha^a m t_a \partial^m \psi(z), \\
\hat{\delta}^m [\hat{\alpha}] \bar{\psi}(z) &= (-)^{m+1} \partial^m (\alpha^a m(z) \bar{\psi}(z) t_a), \quad (m = 0, 1, \ldots) \tag{2.2}
\end{align*}
\]

where \(k(z)\) and \(\alpha(z)\) are arbitrary holomorphic functions for the \(\psi_+\)-component and antiholomorphic for the \(\psi_-\)-component and \(t_a\) are generators of \(SU(N)\) or \(SL(N,C)\) in fundamental representation. In what follows we will consider only the \(\psi_+\)-component, keeping in mind that the \(\psi_-\)-component has quite similar behavior.

In the case when \(k\) and \(\alpha\) are arbitrary holomorphic functions the transformations \(\delta^0 \psi\) and \(\hat{\delta}^0 \psi\) form the classical (without central term) Virasoro algebra and the classical

\[\text{In the Minkowsky space-time} \ k \ \text{and} \ \alpha \ \text{are arbitrary function of the corresponding single light-cone variable only}\]
affine Kac-Moody algebra. For arbitrary $m$ (arbitrary spin) we derive the following Lie algebra:

$$\left[ \delta^m \{k\}, \delta^n \{h\} \right] \psi(z) = \sum_{p=0}^{\max(m+1,n+1)} \delta^{m+n-p+1} \{[k_m, h_n]^p\} \psi(z), \quad (2.4)$$

$$\left[ \delta^m \{k\}, \tilde{\delta}_a^n \{\alpha\} \right] \psi(z) = \sum_{p=0}^{\max(m+1,n)} \tilde{\delta}_a^{m+n-p+1} \{[k_m, \beta_{n-1}^a]^p\} \psi(z), \quad (2.5)$$

$$\left[ \tilde{\delta}_a^m \{\alpha\}, \tilde{\delta}_b^n \{\beta\} \right] \psi(z) = \frac{1}{2} \sum_{p=0}^{\max(m,n)} \left( f_{ab}^{\ c} \delta_c^{m+n-p+1} \{[\alpha_m^a, \beta_{n-1}^b]^p\} \psi(z) \right.$$  

$$+ d_{ab}^{\ c} \delta_c^{m+n-p+1} \{[\alpha_m^a, \beta_{n-1}^b]_+^p\} \psi(z) \right. \left. + 2\delta^{m+n-p+1} \{[\alpha_m^a, \beta_{n-1}^b]_+^p\} \psi(z) \right), \quad (2.6)$$

where

$$[k_m, h_n]^p = \left( n + 1 \frac{1}{p} \right) h_n \partial_m^k \pm \left( m + 1 \frac{1}{p} \right) \delta^{m+n-p+1} k_m \partial^p h_n, \quad (2.7)$$

$$f_{abc} = tr([t_a, t_b]t_c), d_{abc} = tr\{[t_a, t_b]t_c\},$$

the matrix generators $t_a$ are normalized by $tr(t_a t_b) = 2\delta_{ab}$ and is taken into account that the binomial coefficients $\binom{m}{p}$ $= 0$ for $p > m$. We note that for the derivation of (2.6) the identity

$$t_a t_b = \frac{1}{2} (f_{abc} + d_{abc}) t_c + 2\delta_{ab} I, \quad (2.8)$$

is used also. We remind, that this identity is satisfied for the $SU(N)$ and $SL(N,R)$ generators in fundamental representation but it is not satisfied for the $SO(N)$ generators in the adjoint representation. This property of the $SU(N)$ generators makes its higher spin extension more simple than the higher spin extension of the $SO(N)$ algebra (see [5] and [6]).

The Lie algebra (2.4) – (2.6) coincides with the obtained in [4] Lie algebra for $SU(N)$ WZNW model.

From (2.4) – (2.6) it follows that the extension of the $SU(N)$ ($SL(N,C)$) Kac-Moody algebra is not closed. To obtain a closed algebra we have to start with both Virasoro and Kac-Moody algebras, i.e. to consider the higher spin extension of the $U(N)$ ($GL(N,C)$) algebra.
The conserved Neother currents corresponding to (2.2) and (2.3) are given by:

\[ V^m(z) = \bar{\psi} \partial^{m+1} \psi(z), \]  

\[ J^m_a(z) = \bar{\psi} t_a \partial^m \psi(z). \]  

These currents are written in a nonsymmetric form and therefore they are nonhermitean which leads to the appearance of nondiagonal central terms in the OPE. This form, however, is more convenient for applications. A transition to more symmetric form of (2.2) and (2.3) can be carried out by the following simple redefinition

\[ \delta^m \psi \rightarrow \tilde{\delta}^m \psi = \sum_{l=0}^{m} C^m_l \partial^l \partial^{m-l} \psi. \]  

This redefinition leads to a redefinition of the corresponding current

\[ U^m \rightarrow \tilde{U}^m = \sum_{l=0}^{m} (-)^l C^m_l \partial^l U^{m-l} \psi. \]

It is clear that the redefinition of the transformations leads to a deformation of the algebra (2.4) – (2.6) also. Under suitable choice of the coefficients \( C \) in (2.11) the algebra (2.4) coincides with the \( W_\infty \) algebra (see [11]).

To demonstrate the difference with the WZNW model we will show that the classical currents (2.9) and (2.10) form an invariant space with respect to the transformations (2.2) and (2.3). Namely

\[ \delta^l \{ k \} V^m(z) = - \sum_{p=0}^{l+1} \sum_{q=0}^{l-p+1} (-)^{p+q} \binom{l+1}{p} \binom{l-p+1}{q} \partial^p k_i \partial^q V^{l+m-p-q+1} \]

\[ + i \sum_{p=0}^{m+1} \binom{m+1}{p} \partial^p k_i V^{l+m-p+1}, \]

\[ \delta^l \{ k \} J^m_a(z) = - \sum_{p=0}^{l+1} \sum_{q=0}^{l-p+1} (-)^{p+q} \binom{l+1}{p} \binom{l-p+1}{q} \partial^p k_i \partial^q J^{l+m-p-q+1} \]

\[ + i \sum_{p=0}^{m} \binom{m}{p} \partial^p k_i J^{l+m-p+1}, \]
\[
\tilde{\delta}^l_a \{ k \} V^m(z) = - \sum_{p=0}^{l} \sum_{q=0}^{l-p} (-)^{p+q} \binom{l}{p} \binom{l-p}{q} \partial^p k \partial^q J^m_{l+m-p-q+1} + i \sum_{p=0}^{m} \left( m + \frac{1}{2} \right) \partial^p k \partial^q J^m_{l+m-p+1},
\]

\[
\tilde{\delta}^l_a \{ \alpha \} J^m_{b}(z) = \frac{1}{2} (f_{abc} - d_{abc}) \sum_{p=0}^{l} \sum_{q=0}^{l-p} (-)^{p+q} \binom{l}{p} \binom{l-p}{q} \partial^p \alpha \partial^q J^m_{l+m-p-q} + \frac{1}{2} (f_{abc} + d_{abc}) \sum_{p=0}^{m} \left( m + p \right) \partial^p \alpha \partial^q J^m_{l+m-p} - \delta_{ab} \sum_{p=0}^{l} \sum_{q=0}^{l-p} (-)^{p+q} \binom{l}{p} \binom{l-p}{q} \partial^p \alpha \partial^q V^m_{l+m-p-q} + \delta_{ab} \sum_{p=0}^{m} \left( m + p \right) \partial^p \alpha \partial^q V^m_{l+m-p} - \delta \partial \psi \partial^m + \partial \psi \partial^m + \sum_{p=0}^{m} \left( m + p \right) \partial^p \psi \partial^m, \tag{2.14}
\]

In spite of the fact that the field transformation laws form the Lie algebra (2.4) – (2.6) which coincides with those in the WZNW model, the current transformation laws are different. In the WZNW model the classical higher spin bilinear currents, except of \( w_\infty \) nonlinear currents, do not form a closed current algebra \([7]\) which is not the case for the laws (2.13) and (2.14).

We note, that from (2.13) it follows

\[
\delta^0 V^m = (m + 2) \partial k_0 V^m + k_0 \partial V^m + \sum_{p=2}^{m+1} \left( m + \frac{1}{2} \right) \partial^p k_0 V^m - p + 1, \tag{2.15}
\]

\[
\delta^0 J^m_a = m \partial k_0 J^m_a + k_0 \partial J^m_a + \sum_{p=2}^{m} \left( m + p \right) \partial^p k_0 J^m_a - p + 1,
\]

which shows that if \( m > 0 \) the higher spin energy-momentum tensors and currents transform with respect to quasiprimary transformation law.

3. Free fermionic operator algebra

We find the quantum conserved currents from the classical ones (2.9) and (2.10) applying a suitable normal ordering prescription:

\[
V^m(z) = \tilde{\psi} \partial^m \psi, \tag{3.1}
\]

\[
J^m_a(z) = \tilde{\psi} t^a \partial^m \psi.
\]
Following Schoutens et al. [12] we define:

\[
\mathcal{A}(z) \mathcal{B}(z) := \frac{1}{2\pi i} \oint_{\Gamma} \frac{dx}{x - z} \mathcal{A}(x) \mathcal{B}(z),
\]

(3.2)

where \( \Gamma \) is a small contour around the point \( z \).

If we substitute \( \bar{\psi} = \psi \) in (3.1) we will obtain the Majorana fermion currents. In this case there arises isotopic symmetric traceless tensor conserved current [6]

\[
\mathcal{J}^m_{ab}(z) =: \psi_{ab} \partial^{m+1}\psi, \tag{3.3}
\]

where

\[
t_{ab} = \frac{1}{2} \{t_a, t_b\} - 2\delta_{ab} I. \tag{3.4}
\]

The Dirac and Majorana spinor cases we consider separately.

### 3.1. Dirac spinor field

Applying the ordering prescription we obtain the singular terms of the operator product expansion of two currents (3.1):

\[
\mathcal{V}^k(z) \mathcal{V}^l(w) \sim \sum_{p=0}^{k+1} \sum_{q=0}^{p} P^k_{pq}(z - w)^{p-k-2} \partial^{p-q} \mathcal{V}^{l+q}(w)
\]

\[
+ \sum_{p=0}^{l+1} \frac{(l + 1)!}{p!} (z - w)^{p-k-l-3} \mathcal{V}^{k+p+1}(w) + 2NC^D_{k,l}, \tag{3.5}
\]

where

\[
P^m_{pq} = (-)^{m+q-1} \frac{(m + 1)!}{(p - q)! q!}, \quad C^D_{mn} = 2(-)^m (m + 1)! (n + 1)!. \tag{3.6}
\]

From (3.5) and (3.6) it follows that nondiagonal central charges appear. In the same way we derive

\[
\mathcal{V}^k(z) \mathcal{J}^l_a(w) \sim \sum_{p=0}^{k+1} \sum_{q=0}^{p} P^k_{pq}(z - w)^{p-k-2} \partial^{p-q} \mathcal{J}^{l+q}_a(w)
\]

\[
+ \sum_{p=0}^{l} \frac{l!}{p!} (z - w)^{p-l-1} \mathcal{J}^{k+p+1}_a(w) \tag{3.7}
\]
and
\[
\mathcal{J}_a^k(z)\mathcal{J}_b^l(w) \sim \frac{f_{abc} + d_{abc}}{2} \sum_{p=0}^{k} \sum_{q=0}^{\frac{p}{2}} \sum_{p'=0}^{l} \sum_{q'=0}^{\frac{p'}{2}} P_{pq}^{k-1}(z-w)^{p-k-1} \partial^{p-q} \mathcal{J}_c^{l+q}(w)
\]
\[
+ \frac{f_{abc} - d_{abc}}{2} \sum_{p=0}^{l} \frac{l!}{p!} (z-w)^{p-l-1} \mathcal{J}_c^{k+p}(w)
\]
\[
+ \delta_{ab} \left( \sum_{p=0}^{k} \sum_{q=0}^{\frac{p}{2}} P_{pq}^{k-1}(z-w)^{p-k-1} \partial^{p-q} \mathcal{V}^{l+q-1}(w) \right)
\]
\[
+ \sum_{p=0}^{l} \frac{l!}{p!} (z-w)^{p-k-l-1} \mathcal{V}^{k+p-1}(w)
\]
\[-2C_{k-1,l-1}^D \delta_{ab}.
\] (3.8)

Here the identity
\[
\partial^l \bar{\psi} A \partial^m \psi = \sum_{p=0}^{l} \left( \frac{(-)^{l-p}}{p!} \right) \partial^p \left( \bar{\psi} \partial^{m+l-p} \psi \right),
\] (3.9)

where $A$ is a constant matrix, is used essentially. This identity is a consequence of the Leibniz formula. It is easy to see that in the case $k = l = 0$ we obtain from (3.5) — (3.8) the Virasoro — Kac-Moody operator algebra. For $k = 0$ and $l > 0$ the quasiprimary transformation laws for the quantum $\mathcal{V}$ and $\mathcal{J}$ are obtained.

The last terms of (3.5) and (3.8) show that the central terms are nondiagonal in the basis into consideration. Moreover, if the lowest spin is one then the central terms form a degenerate matrix.

Applying the operator product technique we derive quantum transformation laws:
\[
\delta^m \psi(z) = \frac{1}{2\pi i} \int \frac{dx}{x-z} k_m(x) \psi(z) \mathcal{V}^m(x) = k_m(z) \partial^{m+1} \psi(z),
\]
\[
\delta^m_a \bar{\psi}(z) = \frac{1}{2\pi i} \int \frac{dx}{x-z} k_m(x) \mathcal{V}^m(x) = (-)^{m+1} \partial^{m+1} \left( k_m(z) \bar{\psi}(z) \right),
\]
\[
\bar{\delta}^m \psi(z) = \frac{1}{2\pi i} \int \frac{dx}{x-z} \alpha_m(x) \psi(z) \mathcal{J}^m_a(x) = \alpha_m(z) \partial^m t_a \psi(z),
\]
\[
\bar{\delta}^m_a \bar{\psi}(z) = \frac{1}{2\pi i} \int \frac{dx}{x-z} \alpha_m(x) \mathcal{J}^m_a(x) \bar{\psi}(z) = (-)^{m} \partial^{m+1} \left( \alpha_m(z) \bar{\psi}(z) \right)
\] (3.10)

The form of these laws coincide with the form of the corresponding classical laws (2.2) and (2.3). In the same way we can obtain the quantum transformation laws for the currents (2.9) and (2.10) which coincide with the corresponding classical transformation laws (2.13) — (2.15).
3.2. Majorana spinor field

In the case of Majorana spinor field applying the ordering prescription \((3.1)\) we obtain the following singular terms in the product of two Majorana spinor currents:

\[
\hat{V}^m(z)\hat{V}^n(w) \sim \sum_{p=0}^{m+1} \sum_{q=0}^{p} \tilde{P}_{pq}^m(z-w)^{p-m-2} \partial^q \hat{V}^{n+p-q}(w) \\
- \sum_{p=0}^{n+1} \frac{(n+1)!}{p!}(z-x)^{p-m-n-3} \hat{V}^{m+p}(w) \\
+ \sum_{p=0}^{m+n+2} Q_{p}^{mn} (z-w)^{p-m-n-3} \hat{V}^{p-1}(w) \\
- (z-w)^{-1} \sum_{q=0}^{m+1} R_{q}^{m} \partial^q \hat{V}^{m+n+q+1}(w) + NC_{m,n}
\]

The comparison of this formula with \((3.5)\) allows to conclude that in the Majorana case additional terms appears. These terms are a consequence of the fact that for the Majorana fields there are extra pairings compared to the Dirak case. Further we obtain

\[
\hat{V}^m(z)\hat{J}_a^n(w) \sim \sum_{p=0}^{m+1} \sum_{q=0}^{p} \tilde{P}_{pq}^m(z-w)^{p-m-2} \partial^q \hat{J}_a^{n+p-q}(w) - \sum_{p=0}^{n} \frac{n!}{p!}(z-x)^{p-n-1} \hat{J}_a^{m+p+1}(w) \\
+ \sum_{p=0}^{m+n+1} \tilde{P}_p^{m,n-1}(z-w)^{p-m-n-2} \hat{J}_a^p(w) - (z-w)^{-1} \sum_{q=0}^{m+1} R_{q}^{m} \partial^q \hat{J}_a^{m+n-q}(w)
\]
\[ \hat{J}^m_a(z) \hat{J}^n_b(w) \sim \frac{1}{2} f^{c}_{ab} \left( \sum_{p=0}^{m} \sum_{q=0}^{p} \tilde{P}^{m-1}_{pq} (z-w)^{p-m-1} \partial^q \hat{J}^n_c (w) \right. \]
\[ + \sum_{p=0}^{n} \frac{n!}{p!} (z-x)^{p-n-1} \hat{J}^{m+p}_{c} (w) - \sum_{p=0}^{m} Q^{-1}_{m-1} (z-w)^{p-m-n-1} \hat{J}^n_c (w) \]
\[ - (z-w)^{-1} \sum_{q=0}^{m+n} R^{m-1}_{n} q^{q} \hat{J}^{m+n-q}_{c} (w) \]
\[ + \sum_{p=0}^{m} \sum_{q=0}^{p} \tilde{P}^{m-1}_{pq} (z-w)^{p-m-1} \partial^q \hat{J}^{n+p}_{ab} (w) - \sum_{p=0}^{m} \frac{n!}{p!} (z-x)^{p-n-1} \hat{J}^{m+p}_{ab} (w) \]
\[ + \sum_{p=0}^{m} \frac{n!}{p!} (z-x)^{p-n-1} \hat{J}^{m+p}_{ab} (w) - (z-w)^{-1} \sum_{q=0}^{m+n} R^{m-1}_{n} q^{q} \hat{J}^{m+n-q}_{ab} (w) \]
\[ - \frac{2\delta_{ab}}{N} \left( \sum_{p=0}^{m} \sum_{q=0}^{p} \tilde{P}^{m-1}_{p} (z-w)^{p-m-1} \partial^q \hat{V}^{n+p-q-1} (w) \right) \]
\[- \sum_{p=0}^{n} \frac{n!}{p!} (z-x)^{p-n-1} \hat{V}^{m+p-1} (w) + \sum_{p=0}^{m+n} Q^{-1}_{m-1} (z-w)^{p-m-n-1} \hat{V}^{p-1} (w) \]
\[- (z-w)^{-1} \sum_{q=0}^{m+n} R^{m-1}_{n} q^{q} \hat{V}^{m+n-q-1} (w) \] + \( C^{M}_{m-1,n-1} \delta_{ab} \),
\[ (3.13) \]

where \( t_{ab} \) are given by (3.4) and

\[ \tilde{P}^{m}_{pq} = (-)^{m+p-q+1} \frac{(m+1)!}{(p-q)! q!}, \]
\[ Q^{m}_{p} = (-)^{m+1} \frac{(m+n+2)!}{p!}, \]
\[ R^{m}_{p} = (-)^{m-p+1} \binom{m+1}{p}, \]
\[ C^{M}_{m,n} = 2N (-)^{m+n} \frac{(m+1)!(n+1)! - (m+n+2)!}{(z-w)^{m+n+4}}. \]
\[ (3.14) \]

We note that in the case of SO(3)–algebra \( t_{ab} \) have the following simple form

\[ (t_{ab})^k_j = \delta_{aj} \delta^k_b - \delta^k_a \delta_{bj} - \frac{2}{3} \delta_{ab} I. \]
\[ (3.15) \]
Next we derive

\[ \hat{V}^m(z) \hat{J}^n_{ab}(w) \sim \sum_{p=0}^{m+1} \sum_{q=0}^{p} \tilde{P}^m_{pq}(z - w)^{p-m-2} \partial^q \hat{J}^{n+p-q+1}_{ab}(w) \]

\[ - \frac{(n+1)!}{p!} (z - x)^{p-m-n-3} \hat{J}^{p-1}_{ab}(w) \]

\[ + (-)^{m+1} \sum_{p=0}^{m+n+2} Q^m_{p,n}(z - w)^{p-m-n-3} \hat{J}^{p-1}_{ab}(w) \]

\[ + (z - w)^{m+1} \sum_{q=0}^{n+1} R^m_{q,n} \partial^q \hat{J}^{n+m-q+1}_{ab}(w), \]

(3.16)

\[ \hat{J}^m_{a}(z) \hat{J}^n_{bc}(w) \sim D^d_{a,bc} \left( \sum_{p=0}^{m} \sum_{q=0}^{p} \tilde{P}^{m-1}_{pq}(z - w)^{p-m-1} \partial^q \hat{J}^{n+p-q+1}_{d}(w) \right) \]

\[ + \sum_{p=0}^{m} \frac{n!}{p!} (z - x)^{p-n-2} \hat{J}^{m+p}_{d}(w) - \sum_{p=0}^{m+n} R^m_{p-1,n}(z - w)^{p-m-n-2} \hat{J}^{p}_{d}(w) \]

\[ - (z - w)^{-1} \sum_{q=0}^{m} Q^m_{q-1} \partial^q \hat{J}^{m+n-q+1}_{d}(w) \]

\[ + F^d_{a,bc} \left( \sum_{p=0}^{m} \sum_{q=0}^{p} \tilde{P}^{m-1}_{pq}(z - w)^{p-m-1} \partial^q \hat{J}^{n+p-q}_{de}(w) \right) \]

\[ - \frac{(n+2)!}{p!} (z - x)^{p-n-2} \hat{J}^{m+p}_{de}(w) \]

\[ + (-)^{m} \sum_{p=0}^{m+n+1} R^m_{p-1,n}(z - w)^{p-m-n-2} \hat{J}^{p}_{de}(w) \]

\[ - (z - w)^{-1} \sum_{q=0}^{m} Q^m_{q-1} \partial^q \hat{J}^{m+n-q}_{de}(w) \]

(3.17)

Here the coefficients \( D^d_{a,bc} \) and \( F^d_{a,bc} \) are determined from the equation

\[ t_a t_{cd} = D^d_{a,bc} t_d + F^d_{a,bc} t_{de}, \]

(3.18)

where we take into account that \( t_a \) and \( t_{ab} \) form a complete basis in the space of real traceless \( N \times N \) matrices. We note, that \( D \) are symmetric, i.e. \( D^d_{a,bc} = D^d_{bc,a} \) while \( F \) are antisymmetric, i.e. \( F^d_{a,bc} = -F^d_{bc,a} \).
Further we obtain also

\[ \tilde{J}^m_{ab}(z) \tilde{J}^n_{cd}(w) \sim F^e_{ab,cd} \left( \sum_{p=0}^{m+1} \sum_{q=0}^p \tilde{P}^m_p (z-w)^{p-m-2} \partial^q \tilde{J}^n_{e+p+1}(w) \right. \\
+ \sum_{p=0}^{n+1} \frac{(n+1)!}{p!} (z-x)^{p-n-2} \tilde{J}^m_{e+p+1}(w) \\
- \sum_{p=0}^{m+n+2} Q^m_{p,n} (z-w)^{p-m-n-3} \tilde{J}^p_e(w) \\
- (z-w)^{-1} \sum_{q=0}^{m+n+1} R^m_q \partial^q \tilde{J}^n_{e+q+2}(w) \left. \right) + D^{eg}_{ab,cd} \left( \sum_{p=0}^{m+2} \sum_{q=0}^p \tilde{P}^m_{pq}(z-w)^{p-m-2} \partial^q \tilde{J}^n_{e+q+1}(w) \right. \\
- \sum_{p=0}^{n+1} \frac{(n+1)!}{p!} (z-x)^{p-n-2} \tilde{J}^m_{e+q+1}(w) \\
+ \sum_{p=0}^{m+n+2} Q^m_{p,n} (z-w)^{p-m-n-3} \tilde{J}^p_e(w) \\
- (z-w)^{-1} \sum_{q=0}^{m+n+1} R^m_q \partial^q \tilde{J}^n_{e+q+2}(w) \right) \]

(3.19)

Here the coefficients \( F \) and \( D \) are determined from the equation

\[ t_{ab} t_{cd} = F^e_{ab,cd} t_{e} + D^{ef}_{ab,cd} t_{ef} + \delta^{cd}_{ab} \]

\[ t_{cd} \]

(3.20)

where \( F^e_{ab,cd} = -F^e_{cd,ab}, D^{ef}_{ab,cd} = D^{ef}_{cd,ab} \) and \( \delta^{cd}_{ab} = tr(t_{ab} t_{cd}) \.).
Applying the operator product technique we derive the following quantum transformation laws:

$$\delta_m^\psi(z) = \frac{1}{2\pi i} \oint dx k_m(x) \hat{V}^m(x) \psi(z)$$

$$\delta^\psi(z) = -k_m(z) \partial^{m+1} \psi(z) + (-)^{m+1} \partial^{m+1} \left( k_m(z) \psi(z) \right) ,$$

$$\tilde{\delta}^\psi(z) = \frac{1}{2\pi i} \oint dx \alpha_m^a(x) \hat{J}^m_a(x) \psi(z)$$

$$\tilde{\delta}^\psi(z) = -\alpha_m^a(z) t_a \partial^m \psi(z) + (-)^m \partial^m \left( \alpha_m^a(z) \psi(z) \right) t_a ,$$

$$\hat{\delta}^\psi(z) = \frac{1}{2\pi i} \oint dx \alpha_m^{ab}(x) \hat{J}^m_{ab}(x) \psi(z)$$

$$\hat{\delta}^\psi(z) = -\alpha_m^{ab}(z) t_{ab} \partial^{m+1} \psi(z) + (-)^{m+1} \partial^{m+1} \left( \alpha_m^{ab}(z) \psi(z) \right) t_{ab} ,$$

It is easy to check that these transformations are off-shell symmetry of the Majorana spinor action.

4. Gauging the extended affine symmetry

We consider also the problem of gauging the classical symmetry corresponding to the transformations (2.2) and (2.3) and the corresponding quantum symmetry (3.10). First we consider the case of $SU(N)$ fermionic model. In this case the Neother coupling with currents (2.9) and (2.10) is given by

$$L_{int} = A^a_m J^m_a + H_m V^m$$

For simplicity we restrict our considerations only to the case of chiral gauge in which $A_m^a$ are gauge fields only with $m + 1$ antiholomorphic indeces $(\bar{z})$ and $H_m$ are gauge fields with $m + 2$ antiholomorphic indeces.

From the invariance of the total action with respect to local gauge transformations corresponding to (2.2) and (2.3) we obtain the following transformation laws for the gauge potentials $A_m^a$ and $H_m$:

$$\delta A^a_m = \sum_{l \geq 0} \sum_{p=0}^{l+1} \left( -1 \right)^{l+1} \binom{l+1}{p} k_l \partial^p_z A^a_{m-l+p-1} - \sum_{p=0}^l \binom{l}{p} \partial^p_z k_{m-l+p-1} A^a_l$$

$$\delta H_m = -\partial \bar{z} k_m + \sum_{l \geq 0} \sum_{p=0}^l \binom{l+1}{p} \left( -1 \right)^{l+1} k_l \partial^p_z H_{m-l+p-1} - \partial^p_z k_{m-l+p-1} H_l$$

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\[
\begin{align*}
\tilde{\delta}A_m^a = & -\partial_z \alpha_m^a + \frac{1}{2} \sum_{l \geq 0} \sum_{p=0}^l \left( \frac{l+1}{p} \right) \left( (-)^{l+1} (f_{abc} - d_{abc}) \alpha_l^b \partial_z^p A_{m-l+p-1}^c ight) \\
& + (f_{abc} + d_{abc}) \partial_z^p \alpha_{m-l+p-1}^b A_l^a \\
\tilde{\delta}H_m = & \sum_{l \geq 0} \left( \sum_{p=0}^l (-)^l \left( \frac{l}{p} \right) \alpha_l^a \partial_z^p A_{m-l+p-1}^a - \sum_{p=0}^{l+1} \left( \frac{l+1}{p} \right) \partial_z^p \alpha_{m-l+p-1}^a A_l^a \right)
\end{align*}
\] (4.3)

As we shall see below the gauge transformation laws in the case of \( SU(N) \) fermionic model also have a more simple form than in the Majorana spinor model. We note also, that all the currents (2.9) and (2.11) are independent and consequently in the case of \( SU(N) \) fermionic model the Stukelberg like symmetry [9] is absent.

In order to gauge the \( SO(N) \) nonabelian Majorana spinor theory we include a Noether coupling to the free field action (2.1):

\[
L_{int} = A_m^a J_m^a + B_m V_m + A_m^{ab} J_m^{ab},
\] (4.4)

where \( V, J_a \) and \( J_{ab} \) are the spinor field currents and \( A_a, A_{ab} \) and \( B \) are gauge fields (in chiral gauge). The transformation laws for the gauge filds we determine from the invariance of the total action with respect to local gauge transformations corresponding to (3.21). We note, that we obtain the current transformation laws from the singular terms of the OPE (3.11)- (3.13), (3.16), (3.17) and (3.19) applying the formulas:

\[
\begin{align*}
\delta^m O(z) = & \frac{1}{2\pi i} \oint dx k_m(x)V_m(x)O(z), \\
\tilde{\delta}^m O(z) = & \frac{1}{2\pi i} \oint dx \alpha_m^a(x)J_m^a(x)O(z), \\
\tilde{\tilde{\delta}}^m O(z) = & \frac{1}{2\pi i} \oint dx \alpha_m^{ab}(x)J_m^{ab}(x)O(z)
\end{align*}
\] (4.5)
As an example we give the explicit form of the most complicated law

\[
\hat{J}_b^m(z) \approx \sum_{p=0}^{m+n} \sum_{q=0}^p (-1)^{m+p+q} \frac{m!}{(p-q)!q!(m-p)!} \partial^{m+q-\alpha_m^a(z)} J^{n+p-q}(z)
\]

\[+
\sum_{p=0}^{m+n} \frac{n!}{(p-m)!(n-p)!} \partial^{n-p} \alpha_m^a(z) J^{m+p}(z)
\]

\[- \sum_{p=0}^{m+n} \frac{(m+n)!}{p!(m+n-p)!} \partial^{m+n-p} \alpha_m^a(z) J^p_c(z) - \alpha_m^a(z) \sum_{q=0}^m \left( \frac{m}{q} \right) \partial^q J^{m+n-q}(z)
\]

\[+
\sum_{p=0}^{m+n} \sum_{q=0}^p (-1)^{m+p+q} \frac{m!}{(p-q)!q!(m-p)!} \partial^{m+q-\alpha_m^a(z)} J^{m+p}(z)
\]

\[- \sum_{p=0}^{m+n} \frac{n!}{(p-m)!(n-p)!} \partial^{n-p} \alpha_m^a(z) J^{m+p}(z)
\]

\[- \frac{2}{N} \left( \sum_{p=0}^{m+n} \sum_{q=0}^p (-1)^{m+p+q} \frac{m!}{(p-q)!q!(m-p)!} \partial^{m+q-\alpha_m^b(z)} V^{n+p-q-1}(z)
\]

\[+
\sum_{p=0}^{m+n} \frac{n!}{(p-m)!(n-p)!} \partial^{n-p} \alpha_m^a(z) V^{m+p-1}(z)
\]

\[- \sum_{p=0}^{m+n} \frac{(m+n)!}{p!(m+n-p)!} \partial^{m+n-p} \alpha_m^a(z) V^{p-1}(z) - \alpha_m^a(z) \sum_{q=0}^m \left( \frac{m}{q} \right) \partial^q V^{m+n-q-1}(z)
\]

(4.6)

As follows from (4.3) the currents (3.1) form a closed space with respect to the gauge transformations (3.21). The latter allows the theory to be gauged. The same follows for the classical currents (2.9) (2.10) (see the formulas (2.13) and (2.14). In such a way we
obtain the following transformation laws for the gauge fields:

\[
\delta h_l(z) = -\partial_z k_l(z, \bar{z}) + \sum_{m=0}^{l+1} \sum_{p=0}^{m+1} \left( \frac{(m + 1)!}{(m - p + 1)!} \right) \sum_{q=0}^{p} \frac{(-)^{m+p}}{(p - q)!q!} \partial^q (h_{l-p+q} \partial^{m-p+1} k_m)
\]

\[
+ \frac{1}{p!} h_m \partial_{m-p+1} k_{l-p} - (-)^m \left( \frac{m + 1}{q} \right) \partial^q (k_m h_{l-m+q})
\]

\[
+ \sum_{m=0}^{l+1} \sum_{n \geq l-m-1} \frac{(m + n + 2)!}{(l + 1)(m + n - l + 1)!} h_n \partial^{m+n-l+1} k_m,
\]

\[
\delta A^a_l(z) = \sum_{m=0}^{l+1} \sum_{p=0}^{m+1} \left( \frac{(m + 1)!}{(m - p + 1)!} \right) \sum_{q=0}^{p} \frac{(-)^{m+p}}{(p - q)!q!} \partial^q (A^a_{l-p+q} \partial^{m-p+1} k_m)
\]

\[
+ \frac{1}{(m + 1)! p!} A^a_m \partial_{m-p+1} k_{l-p} - (-)^m \left( \frac{m + 1}{q} \right) \partial^q (k_m A^a_{l-m+q})
\]

\[
+ \sum_{m=0}^{l+1} \sum_{n \geq l-m-1} \frac{(m + n + 2)!}{(l + 1)(m + n - l + 1)!} A^a_n \partial^{m+n-l+1} k_m,
\]

\[
\delta C^{a b}_l = \delta (A^a \rightarrow C^{a b})
\]

The remaining transformation laws have a similar form. We note that the nonzero nonhomogenous terms are only the following:

\[
\tilde{\delta} A^a_l \approx -\partial_z \alpha^a_l, \quad \tilde{\delta} A^{a b}_l \approx -\partial_z \alpha^{a b}_l.
\]

In the Majorana case as a consequence of \((2.9), (2.10)\) and \((3.9)\) only the even spin energy-momentum tensors, the even spin isotopic tensor currents and the odd spin currents are independent. This statement is a consequence of the Leibniz formula \((3.9)\) from which we get

\[
U^{2k} = -\frac{1}{2} \sum_{p=1}^{2k+1} (-)^p \binom{2k}{p} \partial^p U^{2k-p},
\]

for symmetric matrix \(A\) and

\[
J^{2k+1}_a = -\frac{1}{2} \sum_{p=1}^{2k+1} (-)^p \binom{2k + 1}{p} \partial^p J^{2k-p+1}_a
\]

for antisymmetric \(A\). Here \(U^m = V^m\) or \(J^{m}_{ab}\). These formulas allow us to compute the coefficients in

\[
U^{2k+1} = \sum_{m=0}^{k} C^k_m \partial^{2m+1} U^{2(k-m)}.
\]
Then the formula (4.11) shows that the interaction action (4.4) admits a Stuckelberg type symmetry [9]. This symmetry express in:

\[ \Delta A_{2m+1}^a = x_m^a, \]
\[ \Delta A_{2m}^a = \sum_{p \geq 0} C_{p+1}^m \partial^{2p+1} x_m^a, \]
\[ \Delta B_{2m+1} = \eta_m, \]
\[ \Delta B_{2m} = \sum_{p \geq 0} C_{p+1}^m \partial^{2p+1} \eta_m, \]
\[ \Delta A_{2m+1}^{ab} = \eta_m^{ab}, \]
\[ \Delta A_{2m}^{ab} = \sum_{p \geq 0} C_{p+1}^m \partial^{2p+1} \eta_m^{ab}. \]

where \( \eta \) are arbitrary functions. This invariance allows us to choose the following (additional to the chiral gauge) gauge fixing:

\[ B_{2m+1} = 0, \quad A_{2m+1}^a = 0, \quad A_{2m+1}^{ab} = 0. \]

(4.13)

In this gauge the even spin gauge fields \( B_{2m+1}, A_{2m+1}^a \), and the odd spin isotopic vector potential \( A^a \) are canceled in the Lagrangian (4.4).

The one-loop two-particle vertex function in the Dirac spinor case is given by:

\[ \Gamma_{(2)} \approx \sum_{m,n \geq 0} \int dzd\bar{z} \left( A_m^a(z, \bar{z}) \frac{\partial^{m+n+1}_z}{\partial \bar{z}} A_n^a(z, \bar{z}) + \mathcal{H}_m(z, \bar{z}) \frac{\partial^{m+n+3}_z}{\partial \bar{z}} \mathcal{H}_n(z, \bar{z}) \right). \]

(4.14)

The nondiagonality of the two-particle vertex function is a consequence of the basis which we use. Taking into account (4.2) and (4.3) we obtain

\[ \delta \Gamma_{(2)} (\mathcal{A}, \mathcal{H}) \approx \sum_{m,n \geq 0} \int dzd\bar{z} \left( \mathcal{H}_m(z, \bar{z}) \partial^{m+n+3}_z k_l(z, \bar{z}) \mathcal{H}_n(z, \bar{z}) + \partial^{m+n+3}_z k_l(z, \bar{z}) \partial^{m+n+3}_z \mathcal{H}_n(z, \bar{z}) \right) \]

\[ \tilde{\delta} \Gamma_{(2)} (\mathcal{A}, \mathcal{H}) \approx \sum_{m,n \geq 0} \int dzd\bar{z} \left( \partial^{m+n+1}_z A_m^a(z, \bar{z}) \partial^{m+n+3}_z \alpha_n^a(z, \bar{z}) \mathcal{H}_n(z, \bar{z}) + \partial^{m+n+1}_z \alpha_m^a(z, \bar{z}) \partial^{m+n+3}_z A_n^a(z, \bar{z}) \right) \]

(4.15)

Consequently, the vertex function \( \Gamma_{(2)} (\mathcal{A}_m, \mathcal{A}_n) \) is invariant with respect to \( SL(m+n+1) \), while the vertex function \( \Gamma_{(2)} (\mathcal{H}_m, \mathcal{H}_n) \) is invariant with respect to \( SL(m+n+3) \). We obtain similar results in the Majorana spinor case.

We remind that in the case of WZNW model bilinear conserved currents exist, however, they do not form an invariant space and consequently this higher spin simmetry
cannot be gauged except the ordinary Virasoro-Kac-Moody symmetry which corresponds to \( m = n = 0 \). The latter shows that on higher spin level there is no equivalence between the nonabelian free fermionic model and the WZNW model.

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