Boundary behaviour of Loewner Chains

Alexander Kuznetsov

February 5, 2008

Abstract

In paper found conditions that guarantee that solution of Loewner-Kufarev equation maps unit disc onto domain with quasiconformal rectifiable boundary, or it has continuation on closed unit disc, or it’s inverse function has continuation on closure of domain.

1 Introduction

In recent time there was a significant interest to evolution processes of domains in complex plane. This phenomena are well described by equations of Loewner-Kufarev type.

Let $D(t)$ be a simple connected domain in the complex plane, changing in time. Without loss of generality we can assume that $D(t)$ contains origin. Let a function $f(z,t)$ ($f'(0,t) > 0, f(0,t) = 0$) for each fixed $t$ map unit disc $D = \{z : |z| < 1\}$ onto domain $D(t)$. If $D(t_1) \subset D(t_2), t_1 > t_2$, then function $f(z,t)$ satisfies the Loewner-Kufarev equation

$$\frac{\partial f}{\partial t} = -\frac{\partial f}{\partial t} z p(z,t), z \in D,$$

where $p(z,t) : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C}$ is such that for each fixed $t \in [0, +\infty)$, $\Re p(z,t) > 0, p(0,t) > 0$ and for each fixed $z \in \mathbb{D}$, $p(z,t)$ is measurable on interval $[0, +\infty)$. Making change of variable (1) we can assume, that for each fixed $t$ function $p(z,t)$ belongs to class $\mathcal{P}$, consisting of functions $p(0) = 1, \Re p(z) > 0$.

Such type equations arise in description of Hele-Show flow [6, 15, 16, 22]. In paper [5] L. Carleson and N. Makarov used this equations for the study of DLA processes. Parametric method of representations of univalent functions [1, 17] is based on this approach. The importance of this method and approach is shown by L. de Branges proof of famous Bieberbach hypothesis [4], that was the central problem of functions theory for long time.

If $p(z,t) = \frac{e^{u(t)}z}{e^{u(t)}z - z}$, where $u(t)$ is real function, then equation will be Loewner equation [11]. Recently there was a great activity in the study of this equation [9, 10, 19, 20], there $u(t)$ describes one-dimensional Brownian motion. In this case equation (1) becomes Stochastic Loewner Equation (SLE). The study
of geometrical characteristics of solutions of SLE, by various parameters of Brownian motion is of a great interest.

In paper [13] it was proved, that solution of Loewner equation maps unit disc on quasislit-disc if and only if function \( u(t) \) is Hölder continuous with exponent 1/2. Also in it was shown if equation

\[
\inf_{\varepsilon>0} \sup_{|t-s|<\varepsilon} \frac{|u(t) - u(s)|}{\sqrt{|t-s|}} < c,
\]

holds for some small \( c > 0 \), (in paper of Lind [12] was proved, that \( c = 4 \)), when solution of Loewner equation maps unit-disc on quasislit-disc.

Let \( P_s \) be some subclass of class \( \mathcal{P} \) and \( \mathcal{PT}_s \) be a class of functions \( p(z,t) : \mathbb{D} \times [0, +\infty) \rightarrow \mathbb{C} \) such that for each fixed \( t \in [0, +\infty) \), \( p(z,t) \in \mathcal{P}_s \) and, for each fixed \( z \in \mathbb{D} \), \( p(z,t) \) is measurable on interval \([0, +\infty)\). The main goal of this paper is to find conditions of class \( \mathcal{P}_s \), under that the solution of equation (1) with \( p(z,t) \in \mathcal{PT}_s \) will have certain boundary behaviour.

Similar question was studied by J. Becker. In papers [2, 3] it was proved, if

\[
|p(z,t) - 1| \leq k < 1,
\]

then solution of equation (1) has quasiconformal extension. Recently A. Vasil’ev [21] gave a description of \( p(z,t) \) under conditions that, solution \( f(z,t) \) of (1) has quasiconformal extension for each \( t \).

In the papers [7, 8] were discussed problems of a similar nature, but from the point of view of covering and distortion theorems for various classes of univalent functions, corresponding to various classes \( \mathcal{P}_s \).

### 2 Behaviour of inverse function

Let us show that the study of boundary behaviour of solution of equation (1) for various classes \( \mathcal{PT}_s \), can be reduced to the study of Loewner-Kufarev ODE.

Let us consider a function \( w^*(z,t,s) = f^{-1}(f(z,t_2),t_1) \), that satisfies the following equation

\[
\frac{dw^*}{dt_1} = w^*p(w^*,t_1), \quad w(z,s,s) = z, \quad z \in \mathbb{D}.
\]

Let \( t_1 \) change from 0 to \( t \), and \( t_2 = t \). When \( w(z,0,t) = f(z,t) \). Assuming that \( \tau = 1 - t_1 \), we have, that \( w(z,0,t) = f(z,t) \), where \( w(z,\tau) \) is solution of equation

\[
\frac{dw}{d\tau} = -wp(w,\tau), \quad w(z,0) = z, \quad z \in \mathbb{D}.
\]

We have the necessary result.

**Theorem 1** If there are constants \( 0 < \alpha < 1 \) and \( C_1, C_2 > 0 \) such that
\[ \frac{|p(z,t)|}{\Re p(z,t)} \leq \frac{C_1}{(1-r)^\alpha}, \quad r = |z|, \quad (3) \]
\[ \Re p(z,t) \geq C_2(1-r)^\alpha, \quad r = |z|. \quad (4) \]

Then inverse function to solution \( w(z,t) \) of equation \( (2) \) has a continuation on boundary of domain \( w(\mathbb{D},t) \).

Let us consider an integral curve \( s_{(z_0,t_0)} \), passing through point \((z_0,t_0)\), and find estimate on it’s part, laying in cylinder \( C_r = \{ z : 1/2 < r < |z| < 1 \} \).

Separating the real and the imaginary part in equation \( (2) \), we have

\[
\begin{align*}
\frac{dw}{dt} &= -\rho \Re p(w,t), \\
\frac{d\arg w}{dt} &= -3mp(w,t),
\end{align*}
\quad (5)
\]

where \( \rho = |w| \).

From system \( (5) \) follows that function \( \rho(t) = |w(z,t)| \) is monotonic on \( t \).

Thus, we can make a change of variable in equation \( (2) \) from \( t \) on \( \rho \). Making it, we have that length of part of integral curve \( s_{(z_0,t_0)} \), that laying in cylinder \( C_r \), equals to

\[ l_s(r) = \int_1^r \sqrt{\left( \frac{dw}{d\rho} \right)^2 + \left( \frac{dt}{d\rho} \right)^2} d\rho. \]

Using inequalities \( (3), (4) \) and \( |w(z,t)| \leq z \) we obtain

\[ l_s(r) \leq \int_1^r \sqrt{\frac{C_2^2}{(1-\rho)^{2\alpha}} + \frac{1}{C_2^2 \rho^2 (1-\rho)^{2\alpha}}} d\rho \leq C_3(1-r)^{1-\alpha}. \]

Where \( C_3 \) is some constant, that depends only on \( C_1, C_2 \). Thus,

\[ \lim_{r \to 1} l_s(r) = 0. \quad (6) \]

Let \( w(z,t) \) be a solution of equation \( (2) \), passing through \((z_0,t_0)\) and \( \tau \) be minimal number, for that \( w(z,t) \) exits on interval \((\tau, t_0)\). Using monotony of \( |w(z,t)| \) on \( t \) and \( (6) \), we have that there is \( \lim_{t \to \tau} w(z,t) = \gamma(z_0,t_0) \) and \( |\gamma(z_0,t_0)| = 1 \).

Let define a function \( s_{\tau_i}(z_0,t_0), |z| \leq 1, 0 \leq \tau_i \leq t_0 \) in a following way. If \( w(z,t) \) solution of equation \( (2) \) with initial condition \((z_0,t_0)\) exists for \( t_1 \), we assume that \( s_{\tau_i}(z_0,t_0) = w(z,t_1) \). Let \( Q_0 \) be a set of pairs \((z,t)\) for that this is true. Let, \( s_{\tau_i}(z_0,t_0) = \gamma(z_0,t_0) \), if \( z \in Q_1 = \{ (z_0,t_0) : |z_0| < 1, (z_0,t_0) \notin Q_0 \} \) and if \( z \in Q_2 = \{ (z_0,t_0) : |z_0| = 1 \} \), then \( s_{\tau_i}(z_0,t_0) = (z_0,t_0) \).

Let us show that function \( s_{\tau_i}(z_0,t_0) \) is a continuous one.

By theorem of continuous depending of solution form initial conditions for differential equations in integral from (see for e.g. [18] p.182), set \( Q_0 \) is open and function \( s_{\tau_i}(z_0,t_0) \) is continuous on it.
Let \((z_0, t_0) \in Q_2\) and point \((z', t')\) satisfy the following equation \(|z_0 - z'| + |t_0 - t'| < \varepsilon\). The length of part of curve \(w(z, t)\), passing through point \((z', t')\) and laying in cylinder \(C_\varepsilon\), is not greater then \(C_3\varepsilon^\alpha\). So we have following inequality

\[|s_{t_1}(z_0, t_0) - s_{t_1}(z', t')| \leq C_3\varepsilon^{1-\alpha} + \varepsilon.\]

This inequality means that function \(s_{t_1}(z, t)\) is continuous on set \(Q_2\).

Let \((z_0, t_0) \in Q_1, t^*\), such that \(1 - |w(z, t^*)| = 2\varepsilon\) (for enough small \(\varepsilon\) such point exists) and \(O\) be neighbourhood of point \((t^*, w(z_0, t^*))\), that is defined by the following inequality

\[|t^* - t'| + |w(z, t^*) - z'| < \varepsilon.\] (7)

Using the theorem of continuous depending of solution form initial conditions for differential equations we have, that there is \(\delta > 0\) so that, form inequality \(|t_0 - t'_0| + |z_0 - z'_0| < \delta\) follows that \(w^*(z', t') \in O\), where \(w^*(z, t)\) integral curve passing through point \((z'_0, t'_0)\). Taking into account, that the length of parts of integral curves \(w^*(z, t)\) and \(w(z, t)\), laying in cylinder \(C_{2\varepsilon}\), is not greater then \(2C_3(2\varepsilon)^{1-\alpha}\), we have

\[|s_{t_1}(z_0, t_0) - s_{t_1}(z', t')| < 2C_3(2\varepsilon)^{1-\alpha} + \varepsilon.\]

This inequality means that function \(s_{t_1}(z, t)\) is continuous on set \(Q_1\). Noting that \(f^{-1}(z, t) = s_0(z, t)\) we have the necessary result.

### 3 Boundary behaviour

**Theorem 2** Following statments are true:

1. If inequality

\[
\frac{\Re zp(z, t)}{|z|\Re p(z, t)} \leq \frac{k}{1 - r} + O\left(\frac{1}{(1 - r)^\alpha}\right), k < 1, r = |z|\] (8)

holds, then function \(w(z, t)\) is Hoelder continuous with exponent \(1 - k\) in closed unit disc.

2. If inequality

\[
\frac{\Re zp(z, t)}{|w|\Re p(z, t)} = O\left(\frac{1}{(1 - r)^\alpha}\right), r = |z|\] (9)

holds, then function \(w(z, t)\) maps unit disc on domain with rectifiable and quasiconformal boundary.

In the proof we are using ideas similar to ideas in papers [7, 8].

Differentiating equation (2) on \(z\), we obtain

\[
\frac{dw_z}{dt} = -w_z p(w, z) - w w_z p_z(w, t), |w| < 1, w_z(z, 0) = 1.
\]
Making change of variable from \( t \) to \( \rho \), we have

\[
\frac{dw_z}{d\rho} = -w_z p(w, t) + w p_z(w, t) \rho \Re p(w, t).
\]

Integrating of this equations gives us

\[
w_z(z, t) = e^{\int_{r_1}^{\rho} \frac{w p_z(w, t)}{\rho \Re p(w, t)} d\rho}.
\]

From that we have

\[
|w_z(z, t)| \leq e^{\int_{r_1}^{\rho} \frac{w p_z(w, t)}{\rho \Re p(w, t)} d\rho}.
\] (10)

Taking into account what \( r_1 = O(\tau) \) and inequality (8), we have

\[
C_1(1 - |z|)^k \leq |w_z(z, t)| \leq C_2 \frac{1}{(1 - |z|)^k}.
\]

From this follows that, function \( w(z, t) \) is Hölder continuous with exponent \( 1 - k \) in closed unit disc (see for e.g. [17] p.300). First part of the theorem is proved.

Taking into account (9) and (10), we have that, there is \( t_0 \) such that for all \( t < t_0 \) we have

\[
|w_z(z, t)| \leq 1/4.
\]

Let us consider three points on circle \( z_1, z_2, z_3 \), so that point \( z_2 \) lays between points \( z_1, z_3 \). From equality

\[
|w(z_1, t) - w(z_2, t)| = \left| \int_{z_1}^{z_2} w_z(z, t) dz \right|,
\]

we obtain

\[
\frac{3}{4} |z_1 - z_2| \leq |w(z_1, t) - w(z_2, t)| \leq \frac{5}{4} |z_1 - z_2|.
\]

So we have

\[
\frac{|w(z_1, t) - w(z_2, t)|}{|w(z_1, t) - w(z_3, t)|} \leq \frac{5}{3} \frac{|z_1 - z_2|}{|z_1 - z_3|}.
\]

Therefore, boundary of domain \( w(D, t) \) is quasiconformal curve and function \( w(z, t) \) have quasiconformal extension on hole complex plane, for \( t < t_0 \).

Let \( f_i(z) = w_i(z, t^*) \), where \( t^* = t/n, i = 1, 2, \ldots, n \) and \( w_i \) is solution of equation

\[
\frac{dw_i}{dt} = w_i p(w_i, t - it^*), w_i(z, 0) = z.
\]

When

\[
w(z, t) = f_1 \circ f_2 \circ \ldots \circ f_n.
\]

Thus choosing \( n \) so that \( t^* = t/n < t_0 \) we have that all \( f_i, i = 1, \ldots, n \) have quasiconformal extension. Therefore \( w(z, t) \) as well. Also we have \( |w_z(z, t)| \leq (3/2)^n \) so \( w(z, t) \) maps unit disc on domain with rectifiable boundary. The second part of theorem is proved.
4 Examples

In this section we consider various subclasses of $P$, that have a simple geometrical description and satisfy conditions of theorems 1 and 2. Let $\Omega$ be convex subdomain of the right half-plane and $P_\Omega$ be class of functions $p : \mathbb{D} \to \Omega$. Through $p_\Omega$ we define a map from $\mathbb{D}$ onto $\Omega$. Any function $p \in P_\Omega$ can be represented as $p_\Omega(\varphi(z))$, where $\varphi : \mathbb{D} \to \mathbb{D}$ and $\varphi(0) = 0$. From this and Schwarz lemma follows that if function $p_\Omega$ satisfies conditions of theorem 1, then all functions from class $p \in P_\Omega$ also satisfy the conditions.

Let us consider a characteristic of function $p$:

$$H(p) = \sup_{|z| < 1} \frac{(1 - |z|^2)|p'(z)|}{\Re p(z)},$$

and show that $H(p(\varphi(z))) \leq H(p(z))$ for all $\varphi : \mathbb{D} \to \mathbb{D}$.

$$H(p(\varphi(z))) = \sup_{|z| < 1} \frac{(1 - |z|^2)|\varphi'(z)p'(\varphi(z))|}{\Re p(\varphi(z))} = \frac{(1 - |z|^2)(1 - |\varphi(z)|^2)|\varphi'(z)||p'(\varphi(z))|}{(1 - |\varphi(z)|^2)\Re p(\varphi(z))}.$$ 

Tacking into account that

$$\frac{(1 - |z|^2)|\varphi'(z)|}{(1 - |\varphi(z)|^2)} \leq 1,$$

we obtain the necessary result.

We have

$$\frac{\Re z p'(z)}{|z|\Re p(z)} \leq \frac{p'(z)}{\Re p(z)} \leq \frac{H(p)}{(1 - r)(1 + r)},$$

If $\frac{1}{1 + r} \leq 1 + \varepsilon$, then

$$\frac{\Re z p'(z)}{|z|\Re p(z)} \leq \frac{H(p)(1 + \varepsilon)}{2(1 - r)},$$

for $|z| > r$. Taking into account that $\frac{p'(z)}{\Re p(z)}$ is bound in disc $|z| \leq r$, for all $\varepsilon > 0$ we have that

$$\frac{\Re z p'(z)}{|z|\Re p(z)} \leq \frac{H(p)(1 + \varepsilon)}{2(1 - r)} + O\left(\frac{1}{(1 - r)^n}\right).$$

Thus to find some result for class $P_\Omega$, it is sufficient to study only function $p_\Omega$.

**Theorem 3** If $\Re(p, t) \geq k > 0$, then function $p(z, t)$ satisfies conditions of theorem 1.
In this case $\Omega = \{ z : \Re z > k \}$ and function $p_\Omega = (1 - k) \frac{1 + e^{i\phi}}{1 - e^{i\phi}} + k$. We only need to show that equation (4) is true. Assuming $z = re^{i\phi}$, we have

$$\frac{|p_\Omega|}{\Re p_\Omega} = \frac{|1 + (1 - 2k)e^{i\phi}|}{\Re \left( \frac{1 + (1 - 2k)re^{i\phi}}{1 - re^{i\phi}} \right)}; \quad 0 < r < 1, \phi \in \mathbb{R}.$$ 

From this

$$\frac{|p_\Omega|}{\Re p_\Omega} = \sqrt{1 + 2(1 - 2k)r \cos \phi + (1 - 2k)^2 r^2} \sqrt{1 - 2r \cos \phi + r^2}.$$ 

Calculation of derivative on $\psi$ gives us that maximum of this function can be only in points $\cos \phi = \pm 1$, $\cos \phi = \frac{2kr}{1 - r^2}$. Thus

$$\frac{|p_\Omega|}{\Re p_\Omega} \leq \max \left\{ \sqrt{\frac{(1 + (2k - 1)r^2)^2}{(1 - r^2)(1 - (1 - 2k)r^2)}}, 1 \right\}.$$ 

Taking into account that $1 > k > 0$ we have that there is a constant $C_1 > 0$, such that

$$\frac{|p_\Omega|}{\Re p_\Omega} \leq C \frac{1}{(1 - r)^{1/2}}.$$ 

**Theorem 4** If $0 < a < \Re p(z, t) < b$ then the solution $w(z, t)$ of equation (7) is Hölder function in the closed unit disk and the boundary of domain the $w(D, t)$ is a Jordan curve.

In this case $\Omega = \{ z : a < \Im z < b \}, 0 < a < 1 < b < \infty$ and $p_\Omega(w) = \frac{b - a}{2} \log \frac{1 + w}{1 - w} + \frac{2 + a - b}{2}$. 

$$\mathcal{H}(p_\Omega) = \sup_{|z| < 1} \frac{(1 - |z|^2) \frac{b - a}{2\pi} \frac{1}{|1 - z|^2}}{\arg \frac{1 + z}{1 - z} + \frac{2 + a - b}{2}}.$$ 

Assuming $z = re^{i\pi}$ we have

$$\mathcal{H}(p_\Omega) = \frac{(1 - r^2)}{|1 - r^2 e^{i2\psi}|(\arg \frac{1 + re^{i\psi}}{1 - re^{i\psi}} + \delta)},$$ 

where $\delta = \pi \frac{b + a}{b - a}$. Thus

$$\mathcal{H}(p_\Omega) = \frac{(1 - r^2)}{|1 - r^2 e^{i2\psi}|(\arg \frac{1 + re^{i\psi}}{1 - re^{i\psi}} + \delta)}.$$ 

Taking into account that

$$\arg \frac{1 + re^{i\psi}}{1 - re^{i\psi}} = \arctan \frac{2r \sin \psi}{1 - r^2},$$ 

$$\mathcal{H}(p_\Omega) = \frac{(1 - r^2)}{|1 - r^2 e^{i2\psi}|(\arg \frac{1 + re^{i\psi}}{1 - re^{i\psi}} + \delta)}.$$ 

Taking into account that
and
\[ |1 - r^2 e^{i2\psi}| = \sqrt{1 - 2r^2 \cos 2\psi + r^4} = \sqrt{(1 - r^2)^2 + 4r^2 \sin^2 \psi}, \]
we obtain
\[ \mathcal{H}(p_{11}) = \frac{2}{\sqrt{1 + x^2} (\arctan x + \delta)}, \]
where \( x = \frac{2r \sin \psi}{1 - r^2} \). Using \( \sqrt{1 + x^2} \arctan x \geq 1 \) we have
\[ \mathcal{H}(p_{11}) < \frac{1}{1 + \delta}. \]

Therefore function \( w(z, t) \) can be extended to a Hölder function in the closed unit disk with exponent \( 1 - \frac{1 + \varepsilon}{\pi \beta} - \frac{1}{\pi \alpha} \) for each \( \varepsilon > 0 \). Thus boundary \( L \) of domain \( w(\mathbb{D}, t) \) is curve. From previous theorem we have that inverse function of \( w(z, t) \) is continuous on closer of domain \( w(\mathbb{D}, t) \). So each point from \( L \) has only one preimage. This means that \( L \) is Jordan curve.

**Theorem 5** If
\[ \frac{\Im p(z, t)}{\Re p(z, t)} < C, \quad (11) \]
then the solution \( w(z, t) \) of equation (4) is a Hölder function in the closed unit disk and the boundary of domain \( w(\mathbb{D}, t) \) is a Jordan curve.

In this case \( \Omega \) is a sector, symmetrical with respect to real axis, with angle \( 2 \arctan C \). Thus \( p_{11} = \left( \frac{1 + z}{1 - z} \right)^\alpha \), where \( \alpha = \frac{2}{\pi} \arctan C \).

\[ \mathcal{H}(p_{11}) = \frac{\alpha |1 + z|^{\alpha - 1}}{\Re \left( \frac{1 + z}{1 - z} \right)^\alpha}. \]

Assuming \( z = re^{i\phi} \), we have
\[ \mathcal{H}(p_{11}) = \frac{4\alpha |1 + re^{i\phi}|^{\alpha - 1}}{\Re \left( \alpha \frac{1 + re^{i\phi}}{1 - re^{i\phi}} \right)^\alpha + \left( \frac{1 + re^{-i\phi}}{1 - re^{-i\phi}} \right)^\alpha}. \]

Thus
\[ \mathcal{H}(p_{11}) = \frac{4\alpha (1 + 2r \cos \phi + r^2)^{\alpha - 1}}{\Re (1 + i2r \sin \phi - r^2)^\alpha + (1 - i2r \sin \phi - r^2)^\alpha}. \]

Note that
\[ \Re (1 \pm i2r \sin \phi - r^2)^\alpha \geq (1 - r^2)^\alpha, \]
and equality achieves only for \( \sin \phi = 0 \). Note that
and equality achieves only for \( \cos \phi = \pm 1 \). Thus \( \mathcal{H}(p_{\Omega}) = 2\alpha \). So, function \( w(z, t) \) is Hölder continuous in closed disc with exponent \( 1 - \alpha - \varepsilon \).

From inequality (11) follows that \( |p_{\Omega}(z)| \leq \sqrt{1 + C^2 \Re p_{\Omega}(z)} \) or conditions (3) in theorem 1. Taking into account previous argumentation we have

\[
\Re p_{\Omega}(z) \geq \frac{(1 - r^2)^{\alpha}}{(1 + r)^{2\alpha}} = \frac{(1 - r)^{\alpha}}{(1 + r)^{\alpha}},
\]

this means inequality (11). Repeat argumentation from proof of theorem 5 we have the necessary result.

5 Application

Often in many phenomena, that are described by Loewner-Kufarev equation, function \( p(z, t) \) is defined by integral representation. It is well known, what any function \( p(z) \), \( \Re p(z) > 0 \) can be represented as

\[
\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta),
\]

where \( \mu(\theta) \) is a non-decreasing function.

Let \( \xi(\theta) = \mu'(\theta) \). Thus if

\[
\xi(\theta) = \frac{1}{|f'(e^{i\theta})|^2},
\]

then equation (11) will describe the Hele-Shaw flow. L. Carleson and N. Makarov [5] used

\[
\xi(\theta) = \inf\{\epsilon : \text{dist}(f(e^{i\theta}(1 - \epsilon), t), \partial f(\mathbb{D}, t)) = \delta, \delta > 0 \}
\]

for the description of processes similar to DLA. In this part we connect behaviour of function \( \xi(\theta) \) with boundary behaviour of solution of Loewner-Kufarev equation.

First let us assume that \( 0 < a < \xi(\theta) < b \). Then taking into account that, \( p(0, t) > 0 \), we can make a change of time from \( t \) on \( \tau \) in equation 11 so, that \( p^*(0, \tau) = 1 \) and \( p^*(z, t) = \frac{dt}{d\tau} p(z, t) \). We obtain that \( \frac{\alpha}{\theta} < p^*(z, \tau) < \frac{\alpha}{b} \). Thus function \( p^*(z, \tau) \) satisfies conditions of theorem and we have that solution of Loewner-Kufarev equation maps unit disc onto domain with Hölder boundary that is Jordan curve.

Let us add condition on \( \xi(\theta) \), that it is Hölder continuous this exponent \( 0 < k < 1 \). Taking into account that (see for e.g. [14] p.69-79)

\[
|p^*_z(z, t)| = O\left(\frac{1}{(1 - r)^{1-k}}\right).
\]
Note that \( \Re p(z,t) > \frac{a}{b} \) we have, that the function satisfies conditions of item 2 in theorem 2. So we have that the solution \( w(z,t) \) maps unit disc onto domain with rectifiable and quasiconformal boundary.

**References**

[1] I. A. Aleksandrov, *Parametric continuations in the theory of univalent functions*, Nauka, Moscow, 1976. (in Russian)

[2] J. Becker, *Loewnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen*, J. Reine Angew. Math. **255** (1972), 23-43.

[3] J. Becker, *Loewnersche Differentialgleichung und Schlichtheitskriterien*, Math. Ann. **202** (1973), 321-335.

[4] L. de Branges, *A proof of the Bieberbach conjecture*, Acta Math. **154** (1985) no. 1-2, 137-152.

[5] L. Carleson, N. Makarov, *Aggregation in the plane and Loewner’s equation*, Comm. Math. Phys. **216** (2001), 583-607.

[6] L. A. Galin, *Unsteady filtration with a free surface*, Dokl. Akad. Nauk USSR, **47** (1945), 246-249. (in Russian)

[7] V. V. Goryainov, *Distortion theorems for some class of univalent function*. Theory of functions and mappings, Naukova Dumka, Kiev, 1979, 75-85. (in Russian)

[8] V. Ya. Gutlyanski, *On some classes of univalent functions*. Theory of functions and mappings, Naukova Dumka, Kiev, 1979, 85-97. (in Russian)

[9] G. Lawler, O. Schramm, W. Werner, *Values of Brownian intersection exponents I: Half-plane exponents*, Acta Math. **187** (2001), 237-273.

[10] G. Lawler, O. Schramm, W. Werner *The dimension of the planar Brownian frontier is 4/3*, Math. Res. Lett. **8** (2001), 401-411.

[11] Ch. Loewner, *Untersuchungen uber schlichte konforme Abbildungen des Einheitskreises*, Math. Ann. **89** (1923), 103-121.

[12] J. Lind, *A sharp condition for the Loewner equation to generate slits*, Ann. Acad. Sci. Fenn. **30** (2005), 143-158.

[13] D.E. Marshall and S. Rohde *The Loewner differential equation and slit mappings* J. Amer. Math. Soc. **18** (2005), 763-778.

[14] N. I. Muskhelishvili, *Singular Integral Equations*, Nauka, Moscow, 3ed, 1968. (in Russian)
[15] P. Ya. Polubarinova-Kochina, *On a problem of the motion of the contour of a petroleum shell*, Dokl. Akad. Nauk USSR, 47 (1945), no. 4, 254-257. (in Russian)

[16] P. Ya. Polubarinova-Kochina, *Concerning unsteady motions in the theory of filtration*, Prikl. Matem. Mech., 9 (1945), no. 1, 79-90. (in Russian)

[17] Ch. Pommerenke, *Univalent functions, with a chapter on quadratic differentials by G. Jensen*, Vandenhoeck & Ruprecht, Goettingen, 1975.

[18] L. S. Pontryagin, *Ordinary differential equations*, Nauka, Moscow, 4ed, 1974. (in Russian)

[19] S. Rohde, O. Schramm, *Basic properties of SLE*, arXiv:math.PR/0106036

[20] O. Schramm, *Scaling limits of loop-erased random walks and uniform spanning trees*, Israel Jour. Math. 118 (2000), 221-288.

[21] A. Vasil’ev, *Evolution dynamics of conformal maps with quasiconformal extension* Bull. Sci. Math. 129 (2005), no. 10, 831-859.

[22] A. Vasil’ev, *Univalent functions in two-dimensional free boundary problems*, Acta Applic. Math. 79 (2003), no. 3, 249-280

Alexander Kuznetsov: Alexander.A.Kuznetsov@gmail.com

Saratov State University, Department of Mathematics and Mechanics, Astrakhanskaya Str. 83, 410012 Saratov, Russia