Area spectrum of slowly rotating black holes

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We investigate the area spectrum for rotating black holes which are Kerr and BTZ black holes. For slowly rotating black holes, we use the Maggiore’s idea combined with Kunstatter’s method to derive their area spectra, which are equally spaced.

I. INTRODUCTION

It is believed that black holes could provide a test bed for any proposed scheme for a quantum theory of gravity. Hod has combined the perturbations of black holes with the quantum mechanics and statistical physics in order to derive the quantum of the black hole area spectrum [1]. For highly excited Schwarzschild black hole, Hod has used the real part of quasinormal frequencies (QNFs) to derive the area quanta \( \Delta A = 4 \ln(3) l_p^2 \). However, it is not consistent with \( \Delta A = 8 \pi l_p^2 \) which was obtained by Bekenstein [2] from the fact that the black hole area is adiabatically invariant.

Kunstatter has shown that the area spectrum is equally spaced for \( d \geq 4 \) dimensional Schwarzschild black holes by using the adiabatically invariant integral [3]

\[
I = \int \frac{dE}{\omega(E)} \rightarrow \int \frac{dM}{\omega_R},
\]

where \((E, \omega)\) are (energy, vibrational frequency) and \((M, \omega_R)\) are (black hole mass, real part of QNFs). On later, Maggiore has proposed that a black hole perturbed by external field is considered as a collection of damped harmonic oscillators [4]. Accordingly, he regarded \( \omega_0 = \sqrt{\omega_R^2 + \omega_I^2} \) as a physically proper frequency and thus, \( \omega_0 = \omega_I \) was used to derive \( \Delta A = 8 \pi l_p^2 \) for highly excited QNFs of \( \omega_I \gg \omega_R \) by considering the transition \( n \rightarrow n - 1 \).

For Kerr black holes, their area spectra were not clearly determined because of their spinning: these solutions are stationary but not static [5, 6]. However, it was argued that for \( J \ll M^2 \), area spectrum of Kerr black hole could be obtained and thus, is equally spaced [7]. Kerr black holes are the most interesting black holes in astronomical point of view, while in theoretical point of view they are more complicated than Schwarzschild and Reissner-Nordström black holes. Also, the BTZ black hole is important because it is closely related to the Kerr black hole even it belongs to a toy model for studying rotating black holes [8].

In this work, we will show that area spectrum could be consistently derived only for slowly rotating black holes [9–14]. Generally, “slowly rotating (sr)” means that one may consider up to linear order of rotating parameter \( a = J/M (a \ll 1) \) in the metric functions, equations of motion, and thermodynamic quantities. Further, the slowly rotating black hole must be far away from extremality because the extremal black hole means \( J = M^2 (a/M = 1) \). Thus, surface gravity (temperature) and area of event horizon do not change to \( O(a) \). However, we consider the angular momentum \( J \) at this order and thus, angular velocity of \( \Omega + \sim J \) at the event horizon.

II. SLOWLY ROTATING KERR BLACK HOLE

The metric of a four-dimensional Kerr black hole given in Boyer-Lindquist coordinates is

\[
ds^2_{Kerr} = -(1 - \frac{2Mr}{\Sigma})dt^2 - \frac{4Mar \sin^2 \theta}{\Sigma}dt d\varphi + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + (r^2 + a^2 + 2Ma^2 \sin^2 \theta) \sin^2 \theta d\varphi^2
\]

where, as always, \( M \) is the mass of the black hole, \( J \) is the angular momentum of the black hole, \( a = J/M \) is the specific angular momentum, \( \Sigma = r^2 + a^2 \cos^2 \theta \), and \( \Delta = r^2 - 2Mr + a^2 \). The roots of \( \Delta = 0 \) from \( g^{rr} = 0 \) are given

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by
\[ r_\pm = M \pm \sqrt{M^2 - a^2} \]  
where \( r_+ \) is the radius of the event (outer) black hole horizon and \( r_- \) is the radius of the inner black hole horizon. In the case of slowly rotating case, we have approximate horizon radii
\[ r_+ \approx 2M - \frac{J^2}{2M^3}, \quad r_- \approx \frac{J^2}{2M^3} \]  
which satisfy a relation of \( r_+ + r_- \approx 2M \). The Kerr black hole is rotating with angular velocity evaluated at the event horizon
\[ \Omega_+ = \frac{J}{2M(M^2 + \sqrt{M^4 - J^2})} \approx \frac{J}{4M^3} \equiv \Omega_+^{sr}, \]  
where the last expression (≈) means angular velocity for slowly rotating Kerr black hole. Furthermore, the horizon area and the Hawking temperature of Kerr black hole (in gravitational units of \( c = G = 1 \) and \( \hbar = l^p \)) are given, respectively, by
\[ A = 8\pi \left( M^2 + \sqrt{M^4 - J^2} \right) \approx 8\pi \left( 2M^2 - \frac{a^2}{2} \right) \equiv A^{sr} \]  
and
\[ T_H = \frac{\hbar \sqrt{M^4 - J^2}}{4\pi M(M^2 + \sqrt{M^4 - J^2})} \approx \frac{\hbar}{8\pi M} \equiv T_H^{sr}. \]  

For slowly rotating black hole, we keep up \( a^2 = J^2/M^2 \)-term in the horizon area but drop off \( a^2/M^2 \)-term in the Hawking temperature because the latter is very smaller than the former. If one keeps the first term in \( A^{sr} \) only, one could not get slowly rotating black hole.

Before we proceed, we mention QNFs given by [15, 16]
\[ \omega_n = \tilde{\omega}_0 - i \left[ 4\pi T_0(a)(n + 1/2) \right] \]  
with the effective temperature \( T_0(a) \). For slowly rotating black hole, one finds that \( T_0(a) = 0 \approx -T_H^{sr}/2 \). Then, we note that the transition frequency is determined by [6]
\[ \hbar \omega_c = \hbar \left[ (\omega_I)_n - (\omega_I)_{n-1} \right] = 2\pi T_H^{sr} = \frac{\hbar}{4M}. \]  

We are in a position to derive the area spectrum of slowly rotating Kerr black hole by employing Kunstatter’s method [3]. Implementing the first law of slowly rotating black hole
\[ dM = T_H^{sr} dS^{sr} + \Omega_+^{sr} dJ, \]  
the adiabatically invariant integral [11] is given as
\[ I_{K(err)}^{sr} = \int \frac{dM - \Omega_+^{sr} dJ}{\omega_c} \]  
\[ = \int \left[ 4MdM - \frac{J dJ}{M^2} \right] \]  
which is consistent with the leading-order terms in [7]. The adiabatically invariant integral leads to
\[ I_{K(err)}^{sr} = 2M^2 - \frac{1}{2} \frac{J^2}{M^2} = \frac{A^{sr}}{8\pi}, \]  
which is obviously correct when considering the adiabatically invariant integral as
\[ I_{K(err)}^{sr} = \int \frac{T_H^{sr} dS^{sr}}{\omega_c} = \frac{1}{8\pi} \int dA^{sr}. \]
with the entropy $S^{sr} = \frac{A^{sr}}{4\hbar}$. Using the Bohr-Sommerfeld quantization condition of $I_{Kerr}^{sr} \approx n\hbar$, the quantized area spectrum is

$$A^{sr}_n = 8\pi n\hbar$$

(15)

which is the universal area spectrum. On the other hand, one has found $A^{sr} = 4\hbar n$ when using the tunneling method [17]. The entropy spectrum takes the form

$$S^{sr}_n = \frac{A^{sr}_n}{4\hbar} = 2\pi n.$$  

(16)

In the case of $J \to 0$, we immediately obtain the area spectrum for the Schwarzschild black hole. On the other hand, for Kerr black hole, the adiabatically invariant integral is given by [6, 7]

$$I_{Kerr} = \int \left[ 4MdM - \frac{2JdJ}{M^2 + \sqrt{M^4 - J^2}} \right] = \frac{A}{4\pi} - 2M^2 \ln \left[ \frac{A}{8\pi} \right]$$

(17)

where the logarithmic term becomes dominant, which gives rise to difficulty to interpret the entropy.

### III. SLOWLY ROTATING BTZ BLACK HOLE

In this section, we study the slowly rotating BTZ black hole in gravitational units of $c = 8G = 1$. In three dimensional spacetimes, the metric of the BTZ black hole is given by [8]

$$ds^2_{BTZ} = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right) dt^2 + \frac{dr^2}{(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2})} + r^2 \left( d\phi - \frac{J}{2r^2} dt \right)^2 ,$$

(18)

where the cosmological constant is given by $\Lambda = \frac{-1}{l^2}$. From the condition of $g^{rr} = 0$, we obtain the outer and inner horizons as

$$r^2_{\pm} = \frac{l^2 M}{2} \left[ 1 \pm \sqrt{1 - \frac{J^2 l^2 M}{2}} \right].$$

(19)

The mass $M$ and angular momentum $J$ of the black hole can be expressed in terms of $r_{\pm}$ as

$$M = \frac{r^2_+ + r^2_-}{l^2}, \quad J = \frac{2r_+ r_-}{l}.$$  

(20)

Using these expressions, we rewrite two horizons as

$$r_{\pm} = \frac{l}{2} \left[ \sqrt{M + \frac{J}{l}} \pm \sqrt{M - \frac{J}{l}} \right].$$

(21)

The slowly rotating BTZ black hole implies that

$$a = \frac{J}{M} \ll 1.$$  

(22)

Then, from (19) and (21), we have approximate two horizons

$$r_+ \approx l\sqrt{M} \left[ 1 - \frac{1}{8} \left( \frac{J}{lM} \right)^2 \right], \quad r_- \approx \frac{J}{2\sqrt{M}}.$$  

(23)

The BTZ black hole is rotating with angular velocity evaluated at the event horizon

$$\Omega_+ = \frac{J}{2r^2_+} = \frac{J}{l^2 \left( M^2 + \sqrt{M^4 - \frac{J^2}{l^2}} \right)} \approx \frac{J}{2l^2 M} \equiv \Omega^{sr}_+.$$  

(24)
where the last expression \((\approx)\) means angular velocity for slowly rotating BTZ black hole. Also, the horizon area and the Hawking temperature of BTZ black hole are given, respectively, by \[ 8, 18 \]

\[
A = 2\pi r_+ \approx 2\pi l\sqrt{M \left[ 1 - \frac{1}{8} \left( \frac{J}{M} \right)^2 \right]} \equiv A^{sr} \tag{25}
\]

and

\[
T_H = \frac{h(r_+^2 - r_-^2)}{2\pi l^2 r_+} \approx \frac{h\sqrt{M}}{2\pi l} \equiv T_H^{sr}. \tag{26}
\]

For slowly rotating BTZ black hole with \(l \gg 1\), we keep up \(a^2 l = j^2 M^2\) term in the horizon area but drop off \(a^3 l^3 = \frac{j^2}{r^3 M^2}\) term in the Hawking temperature.

The two types of quasinormal modes of the BTZ black hole for a massive scalar field are given by \[20–22\]

\[
\omega_R = \frac{m}{l} - i \frac{(r_+ + r_-)}{l^2} \left( 2n + 1 + \sqrt{1 + \mu} \right), \tag{27}
\]

\[
\omega_L = \frac{m}{l} - i \frac{(r_+ - r_-)}{l^2} \left( 2n + 1 + \sqrt{1 + \mu} \right), \tag{28}
\]

where \(m\) and \(n\) are the angular quantum number and the overtone quantum number respectively. \(\mu\) is the mass parameter defined by \(\mu \equiv \frac{m^2 l^2}{\hbar^2}\), where \(m\) is the mass of the scalar field. At large \(n\) for a fixed \(|m| (n \gg |m|)\), the two transition frequencies of \(\omega_{Rc}\) and \(\omega_{Lc}\) are given by \[23\]

\[
\omega_{Rc} = 2 \frac{(r_+ + r_-)}{l^2} = 2 \frac{\sqrt{M + J/l}}{l}, \tag{29}
\]

\[
\omega_{Lc} = 2 \frac{(r_+ - r_-)}{l^2} = 2 \frac{\sqrt{M - J/l}}{l}. \tag{30}
\]

For the slowly rotating BTZ black hole, however, we have one-type as

\[
\omega_{Rc} \approx \omega_{Lc} \approx 2 \frac{\sqrt{M}}{l} = \frac{4\pi T_H^{sr}}{\hbar}. \tag{31}
\]

The adiabatically invariant integral is calculated to be

\[
I_{BTZ}^{sr} = \int \frac{dM - \Omega^{sr} dJ}{\omega_{Rc}} = \frac{I\sqrt{M \left[ 1 - \frac{1}{8} \left( \frac{J}{M} \right)^2 \right]} = \frac{A^{sr}}{2\pi}}{r_+}. \tag{32}
\]

For 3D slowly sinning dilaton black hole, Fernando has obtained a similar result \[14\], but he has neglected \(J^2\)-term to derive area spectrum. Also, Kwon and Nam have used \(J_R = \int dM/\omega_{Rc}\) and \(J_L = \int dM/\omega_{Lc}\) to derive two area spectra \[23\]. In this case, however, it is questionable that \(J_R\) and \(J_L\) could represent the area (entropy) of the BTZ black hole because they did not take into account \(\Omega dJ\) seriously. Without \(\Omega^{sr} dJ\), the first law of thermodynamics is not satisfied for slowly rotating BTZ black hole.

Applying the Bohr-Sommerfeld condition of \(I_{BTZ}^{sr} \approx nh\) to the slowly rotating BTZ black hole, the quantized area spectrum is determined to be

\[
A^{sr}_n = 2\pi n \hbar \tag{33}
\]

which is the universal area spectrum. Considering the unit of \(G = 1/8\), the entropy spectrum takes the form

\[
S^{sr}_n = \frac{2A^{sr}_n}{\hbar} = 4\pi n. \tag{34}
\]
In the case of $J \to 0$, we obtain the area spectrum for the non-rotating black hole \cite{24, 25}. On the other hand, for BTZ black hole, the adiabatically invariant integral take a complicated form when using \cite{31}

\begin{equation}
I_{\text{BTZ}} = l \int \left[ \frac{dM}{2\sqrt{M} - JdJ} \right]
\end{equation}

\begin{equation}
= l\sqrt{M} + \frac{l}{2\sqrt{M}} \sqrt{\frac{M^2 - J^2}{l^2} - \frac{l\sqrt{M}}{2} \ln \left[ M + \sqrt{M^2 - J^2} \right]},
\end{equation}

where the logarithmic term becomes dominant. It implies that the presence of logarithmic term gives rise to difficulty in order to interpret $I_{\text{BTZ}}$ as the entropy.

\section{IV. DISCUSSION}

We have investigated the area spectrum for rotating black holes such as Kerr and BTZ black holes. Rotating black holes provided the ill-defined (adiabatically invariant) integral which contains a large logarithmic term without any quantum corrections. This means that Kunstatter’s method to does not work for rotating black holes well. However, for slowly rotating black holes, we have used the Maggiore’s idea combined with Kunstatter’s method to derive their area spectra, which are equally spaced. This shows clearly that slowly rotating black holes provide the quantum of area spectrum very well, like the static black hole of Schwarzschild black hole.

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