CLASSIFICATION OF NAHM POLE SOLUTIONS OF THE
KAPUSTIN-WITTEN EQUATIONS ON $S^1 \times \Sigma \times \mathbb{R}^+$

SIQI HE AND RAFE MAZZEO

Abstract. In this note, we classify all solutions to the SU(n) Kapustin-Witten equations on $S^1 \times \Sigma \times \mathbb{R}^+$, where $\Sigma$ is a compact Riemann surface, with Nahm pole singularity at $S^1 \times \Sigma \times \{0\}$. We provide a similar classification of solutions with generalized Nahm pole singularities along a simple divisor (a “knot”) in $S^1 \times \Sigma \times \{0\}$.

1. Introduction

An important conjecture by Witten [20] posits a relationship between the Jones polynomial of a knot and a count of solutions to the Kapustin-Witten equations. More specifically, let $K$ be a knot in $X = \mathbb{R}^3$ or $S^3$, and fix an SU(n) bundle $P$ over $X \times \mathbb{R}^+$ with associated adjoint bundle $\mathfrak{g}_P$. The Kapustin-Witten (KW) equations [11] are equations for a pair $(A, \Phi)$, where $A$ is a connection on $P$ and $\Phi$ is a $\mathfrak{g}_P$-valued 1-form. We augment these with the singular Nahm pole boundary conditions at $y = 0$ (where $y$ is a linear variable on the $\mathbb{R}^+$ factor), and with an additional singularity imposed along $K \times \{0\}$. The conjecture states that an appropriate count of solutions to the KW equations with these boundary conditions computes the Jones polynomial. One can define these equations when $X$ is a more general Riemannian 3-manifold, and in that case this gauge-theoretic enumeration may lead to new 3-manifold invariants when $K = \emptyset$, or to a generalization of the Jones polynomial for $K$ lying in a general 3-manifold, see [21, 3].

The core of all of this is to investigate the properties of the moduli space of solutions. Significant partial progress has been made, see [12, 13, 4, 15, 16], as well as Taubes’ recent advance [18] regarding compactness properties.

As usual in gauge theory, it is reasonable to seek to understand a dimensionally reduced version of this problem. Thus suppose that $X = S^1 \times \Sigma$, where $\Sigma$ is a compact Riemann surface of genus $g$. Solutions which are invariant in the $S^1$ direction are solutions of the so-called extended Bogomolny equations. General existence theorems for solutions of these dimensionally reduced equations were proved in [6, 7]. In the present paper, we adapt arguments from [13] and prove that every solution to the KW equation on $S^1 \times \Sigma \times \mathbb{R}^+$ satisfying Nahm pole boundary conditions is necessarily invariant in the $S^1$ direction. This leads to a complete classification of solutions in this special case.

Theorem 1.1. Consider the Kapustin-Witten equations on $S^1 \times \Sigma \times \mathbb{R}^+_y$ for fields satisfying the Nahm pole boundary condition at $y = 0$ (with no knot singularity) and which converge to a flat SL(n, $\mathbb{C}$) connection as $y \to \infty$.

i) There are no solutions if $g = 0$;

ii) There is a unique solution (up to unitary gauge equivalence) if $g = 1$;

iii) If $g > 1$, there exists a solution if and only if the limiting flat connection as $y \to \infty$ lies in the Hitchin section in the SL(n, $\mathbb{C}$) Hitchin moduli space, and in that case, this solution is unique up to unitary gauge.

Part ii) here largely comes from the uniqueness theorem in [12] for solutions on $\mathbb{R}^3 \times \mathbb{R}^+$. The Hitchin section in part iii) is also known as the Hitchin component of the SL(n, $\mathbb{R}$)
representation variety, cf. [10]. We recall that there are in fact \( n^{2g} \) equivalent Hitchin components, depending on the different choices of spin structure.

Next suppose that the knot \( K \subset S^1 \times \Sigma \) is a union of ‘parallel’ copies of \( S^1 \), \( K = \bigsqcup (S^1 \times \{ p_i \}) \). The Nahm boundary conditions at a knot require that we specify a weight \( k^i \), i.e., an \((n-1)\)-tuple of positive integers \((k^i_1, \ldots, k^i_{n-1}) \in \mathbb{N}^{n-1} \) for each component \( K_i \).

**Theorem 1.2.** Consider the Kapustin-Witten equations on \( S^1 \times \Sigma \times \mathbb{R}^+ \) for fields which satisfy the Nahm pole boundary condition with knot singularities with weights \( k^i \), as described above, along \( K \times \{0\} \), where \( K = \bigsqcup (S^1 \times \{ p_i \}) \), and which converge to a flat \( SL(n, \mathbb{C}) \) connection, corresponding to a stable Higgs pair \((E, \varphi)\), as \( y \to \infty \).

i) There are no solutions when \( g = 0 \);

ii) If \( g > 1 \) and \( \rho \) is irreducible, there exists a solutions with these boundary conditions at \( K \) if and only if there exists a holomorphic line subbundle \( L \) of \( E \) such that the data set \( d(E, \varphi, L) = \{(p_i, k^i)\} \).

The definition of data sets \( d(E, \varphi, L) \) is recalled in Section 3.3.

**Remark.** We do not discuss the case \( g = 1 \) here. Indeed, it is not clear what the correct existence theory for solutions with knot singularities should be in this case.

**Corollary 1.3.** There exists, up to unitary gauge, at most \( n^{2g} \) solutions to the KW equations which converge to the given flat connection associated to \((E, \varphi)\) and with Nahm singularity along \( K = \bigsqcup S^1 \times \{ p_i \} \).

The knot points \( p_i \) and the weights \( k^i \) determine the divisor \( D = \sum_i p_i \sum_j k^i_j \).

**Theorem 1.4.** If \( \text{deg} \, D \) is not divisible by \( n \), there exist no Nahm pole solutions to the KW equations with knot singularity along \( K \). In particular, there are no solutions to the \( SU(2) \) extended Bogomolny equations with only a single knot singularity of weight 1.

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### 2. The Kapustin-Witten Equations and the Nahm Pole Boundary Conditions

We begin with some background materials on the Kapustin-Witten equations [11] and Nahm pole boundary conditions:

#### 2.1. The Kapustin-Witten Equations

Let \((M, g)\) be a Riemannian 4-manifold, and \( P \) an \( SU(n) \) bundle over \( M \) with the adjoint bundle \( g_P \). The Kapustin-Witten equations for a connection \( A \) and a \( g_P \)-valued 1-form \( \Phi \) are

\[
F_A - \Phi \wedge \Phi + \star d_A \Phi = 0, \quad d^*_A \Phi = 0.
\]

When \( M \) is closed, all solutions to the KW equations are flat \( SL(n, \mathbb{C}) \) connections [11]. Indeed, in this setting, a Weitzenböck formula shows that solutions must satisfy the decoupled equations

\[
F_A - \Phi \wedge \Phi = 0, \quad d_A \Phi = 0, \quad d_A \star \Phi = 0,
\]

or equivalently, \( F_A = 0 \) where \( A := A + i\Phi \) and \( d_A \star \Phi = 0 \).

Following [20, 21], the main case of interest here is when \( M = X \times \mathbb{R}^+ \), where \( X \) is a closed 3-manifold and \( \mathbb{R}^+ : = (0, \infty) \) with linear coordinate \( y \). From now on, we fix a Riemannian metric on \( X \) with volume 1, and endow \( X \times \mathbb{R}^+ \) with the product metric.
2.1.1. The Nahm Pole Boundary Condition. Let $G := \text{SU}(n)$, with Lie algebra $\mathfrak{g}$ and choose a principal embedding $\varrho : \mathfrak{su}(2) \rightarrow \mathfrak{g}$ as well as a global orthonormal coframe $\{e_a^x, a = 1, 2, 3\}$ of $T^*X$, which is possible since $X$ is parallelizable. Next, choose a section $e$ of $T^*X \otimes \mathfrak{g}_P$, $e = \sum t_a e_a^x$ for some everywhere nonvanishing sections $t_a$, $a = 1, 2, 3$, of the adjoint bundle $\mathfrak{g}_P$ which satisfy the commutation relations $[k_a, t_b] = \epsilon_{abc} t_c$, and which lie in the conjugacy class of the image of $\varrho$. This choice of $e$ is called a dreibein form.

Definition 2.1. With all notation as above, the pair $(A, \Phi)$ satisfies the Nahm pole boundary condition at $y = 0$ if, in some gauge, $A = A_0 + \mathcal{O}(y^\epsilon)$ and $\Phi = \frac{x}{y} + \mathcal{O}(y^{-1+\epsilon})$ for some $\epsilon > 0$.

The rationale for this name is that the dimensional reduction of the KW equations to $\mathbb{R}^+$ are the Nahm equations, and in this case $(0, \frac{x}{y})$ is a ‘standard’ solution of the Nahm equations with a so-called pole at $y = 0$. We remark also that as proved in [12], it is sufficient to assume that $A = \mathcal{O}(y^{-1+\epsilon})$, since the regularity theory for solutions shows that there is automatically a leading coefficient $A_0$.

2.1.2. The Nahm Pole Boundary Condition with Knot Singularities. A generalization of this boundary condition incorporates certain ‘knot’ singularities at $y = 0$. Before describing this, recall from [20] the model solution when $G = \text{SU}(2)$ and $X = \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$ with coordinate $(x_1, z = x_2 + ix_3)$. Introduce spherical coordinate $(R, s, \theta)$ in the $(z, y)$ half-space: $z = re^{i\theta}$, $R = \sqrt{r^2 + y^2}$, $y = R \sin s$, $r = |z| = R \cos s$. The model knot is the line $(x_1, 0, 0) \subset \mathbb{R}^3 \times \{0\}$. Writing $\Phi = \phi_z dz + \phi_y dy + \phi_1 dx_1 + \phi_2 + \phi_3 + \phi_4 + \phi_5$, the model solution of weight $k$ takes the form

$$
A = -(k + 1) \cos^2 s \frac{(1 + \sin s)^k - (1 - \sin s)^k}{(1 + \sin s)^{k+1} - (1 - \sin s)^{k+1}} \frac{dz}{d\theta} \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix},
$$

$$
\phi_z = \frac{2(k + 1)e^{ik\theta} \cos^2 s}{R(1 + \sin s)^{k+1} - R(1 - \sin s)^{k+1}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
$$

$$
\phi_1 = \frac{k + 1 (1 + \sin s)^{k+1} + (1 - \sin s)^{k+1}}{R (1 + \sin s)^{k+1} - (1 - \sin s)^{k+1}} \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & \frac{i}{2} \end{pmatrix}, \quad \phi_y = 0.
$$

There is a less explicit model solution when $G = \text{SU}(n)$, due to Mikhaylov [14]. The weight in that case is an $(n - 1)$-tuple $k = (k_1, \ldots, k_{n-1})$, and the corresponding solution is denoted $(A_k^{\text{mod}}, \Phi_k^{\text{mod}})$. As in the case $n = 2$, $|A_k^{\text{mod}}| \sim R^{-1}s^0$ and $|\Phi_k^{\text{mod}}| \sim R^{-1}s^{-1}$ near $z = 0, y = 0$.

In general, given a knot $K \subset X \times \{0\}$, introduce local coordinates $(x_1, z = x_2 + ix_3, y)$ near $K$, where $K = \{z = y = 0\}$ and $t$ is a coordinate along $K$. We can use cylindrical coordinates $(R, s, \theta, x_1)$ near $K$, where $y = R \sin s$, $z = R \cos s e^{i\theta}$. Then, as in [20, 13], we make the

Definition 2.2. With $P$ and $G$ as above, and $K \subset X$ a knot, then $(A, \Phi)$ satisfies Nahm pole boundary condition with knot $K$ and weight $k$ if in some gauge

i) $(A, \Phi)$ satisfies the Nahm pole boundary condition away from knots $K$,

ii) near $K$, $A = A_k^{\text{mod}} + \mathcal{O}(R^{-1+\epsilon}s^{-1+\epsilon})$, $\Phi = \Phi_k^{\text{mod}} + \mathcal{O}(R^{-1+\epsilon}s^{-1+\epsilon})$.

2.1.3. The Boundary Condition at $y = \infty$. We must also impose an asymptotic boundary condition at the cylindrical end, as $y \rightarrow \infty$. We change to a temporal gauge, i.e., so that $A_y = 0$. Then writing $\Phi = \phi + \phi_y dy$ (so $\phi$ includes the $\phi_1$ part), the KW equations become
flow equations
\[\begin{align*}
\partial_y A &= * d_A \phi + [\phi_y, \phi], \\
\partial_y \phi &= d_A \phi_y + * (F_A - \phi \wedge \phi), \\
\partial_y \phi_y &= d_A^* \phi.
\end{align*}\]

We shall assume that \((A, \Phi)\) converges to a "steady-state" \((y\)-independent) solution as \(y \to \infty\), which is then necessarily a flat \(SL(n, \mathbb{C})\) connection. The \(y\)-independence, together with the equations (2) yield that \([\phi, \phi_y] = d_A \phi_y = 0\); this shows that if \(\phi_y \neq 0\), then \(A\) is reducible.

**Proposition 2.3.** If \((A, \Phi)\) satisfies the KW equations together with Nahm pole boundary conditions (possibly with knots), and converges to an irreducible flat \(SL(n, \mathbb{C})\) connection as \(y \to \infty\), then \(\phi_y \equiv 0\).

Indeed, the hypothesis and the remark above shows that \(\lim_{y \to \infty} \phi_y = 0\). A well-known vanishing theorem then implies that \(\phi_y \equiv 0\), see [17, Page 36] or [4, Corollary 4.7] for a proof. We assume henceforth, as in [18], that \(\phi_y \equiv 0\).

We now define the moduli spaces
\[\mathcal{M}_{NP}^\text{KW} := \{(A, \Phi) : \text{KW}(A, \Phi) = 0, \ (A, \Phi) \text{ converges to a flat SL}(n, \mathbb{C}) \text{ connection as } y \to \infty \} / \mathcal{G}_0,\]
and
\[\mathcal{M}_{NPK}^\text{KW} := \{(A, \Phi) : \text{KW}(A, \Phi) = 0, \ (A, \phi, \phi_1) \text{ satisfies the Nahm pole boundary condition with knot } K \text{ and converges to a flat } SL(n, \mathbb{C}) \text{ connection as } y \to \infty \} / \mathcal{G}_0,\]
where \(\mathcal{G}_0\) is the space of gauge transformations preserving the boundary conditions.

2.2. The Regularity theorems of Nahm pole Solutions. We next recall the regularity theory for this singular boundary condition at \(y = 0\), as developed in [12, 13]. Still working on \(X \times \mathbb{R}^+\), fix a smooth background connection \(\nabla\), and write \(\nabla_x, \nabla_y\) for the covariant derivatives in the \(x \in X\) and \(y\) directions.

**Theorem 2.4.** [12, 13] Let \((A, \Phi)\) satisfy the KW equations with Nahm pole boundary condition, and write \(A = A_0 + a, \ \Phi = \varphi + b\) near \(y = 0\) where \(a = \mathcal{O}(y^s), b = \mathcal{O}(y^{-1+s})\). Then \(a\) and \(b\) are polyhomogeneous. Furthermore, the leading term \(A_0\) of \(A\) must correspond, under the intertwining provided by the dreibein \(e\), with the Levi-Civita connection on \(X\).

If \((A, \Phi)\) satisfies the Nahm pole boundary condition with a knot singularity along \(K\) of weight \(k\) at \(y = 0\), then writing \(A = A_{0, k} + a, \ \Phi = \Phi_{k} + b, \) where \((A_{0, k}, \Phi_{k})\) is the model solution and \(a, b = \mathcal{O}(R^{-1+s} s^{-1+e})\), then \(a, b\) are polyhomogeneous, i.e., have expansions in positive powers of \(R\) and \(s\), and nonnegative integer powers of \(\log R\) and \(\log s\), with coefficients smooth in the tangential variables. These expansions are of product type at the corner \(R = s = 0\).

**Remark.** We recall that a function (or section of some bundle) \(u\) is polyhomogeneous on \(X \times \mathbb{R}^+\) at \(X \times \{0\}\) if, near any boundary point,
\[u \sim \sum_{j} \sum_{\ell=0}^{N_j} u_{j, \ell}(x) y^{\gamma_j} (\log y)^\ell \text{ as } y \to 0.\]

Here \(x\) is a local coordinate on \(X\) and each coefficient \(u_{j, \ell}(x)\) is \(C^\infty\), while \(\gamma_j\) is a sequence of complex numbers with real parts tending to infinity. In our setting, the \(\gamma_j\) are explicit real numbers calculated in [12].
The second polyhomogeneity statement, near \( K \), may be phrased similarly once we introduce the blowup \([X \times \mathbb{R}^+; K \times \{0\}]\). This is a new manifold with corners of codimension two obtained by replacing the knot \( K \) at \( y = 0 \) with its inward-pointing spherical normal bundle. The cylindrical coordinates \((x_1, R, s, \theta)\) are nonsingular on this space, and the two boundaries are defined by \( \{R = 0\} \) and \( \{s = 0\} \). A function or section \( u \) is polyhomogeneous on this space if it admits a classical expansion as described above near each point in the interior of the codimension one boundaries, while near the corner \( \{R = s = 0\} \) it admits a product type expansion

\[
u \sim \sum_{j,k} \sum_{\ell=0}^{N_j} \sum_{m=0}^{M_j} u_{j,\ell,m}(x_1, \theta) s^{\gamma_j} R^{\mu_k} (\log s)^\ell (\log R)^m,
\]

where as before, each coefficient function is smooth in the variables \( t, \theta \) along the corner. In our setting the \( \gamma_j \) and \( N_j \) are the same numbers as in the previous expansion, while the \( \mu_k \) are real numbers calculated (somewhat less explicitly, i.e., only in terms of spectral data of some auxiliary operator) in \([13]\).

The paper \([5]\) considers various refined aspects of the higher terms in the expansion in \( y \).

We have described this precise regularity for the sake of completeness, but in fact, we do not use the full power of these expansions here, but only the estimates

\[
|\nabla_x^{\ell} \nabla_y^m u|_C^0 \leq C_{\ell,m} y^{2-m+\epsilon}, \quad |\nabla_x^{\ell} \nabla_y^m b|_C^0 \leq C_{\ell,m} y^{1-m+\epsilon},
\]

\[
|\nabla_{x_1}^{\ell} \nabla_R^{m} \nabla_s^n u|_C^0 \leq C_{\ell,m,n} R^{-\epsilon-m} s^{2-\epsilon-n}, \quad |\nabla_{x_1}^{\ell} \nabla_R^{m} \nabla_s^n b|_C^0 \leq C_{\ell,m,n} R^{-\epsilon-m} s^{1-\epsilon-n}
\]

for any \( \epsilon > 0 \) and any \( \ell, m, n \in \mathbb{N} \).

3. The Extended Bogomolny Equations

We next recall the dimensional reduction of the Kapustin-Witten equations from \( S^1 \times \Sigma \times \mathbb{R}^+ \) to \( \Sigma \times \mathbb{R}^+ \), obtained by considering fields invariant in the \( S^1 \) direction. This was previously studied in \([6, 7]\), and is closely related to the Atiyah-Floer approach to counting Kapustin-Witten solutions \([3]\).

Assume on the one hand that the bundle \( P \) on \( S^1 \times \Sigma \times \mathbb{R}^+ \) is pulled back from \( \Sigma \times \mathbb{R}^+ \). Changing notation slightly, given a solution \((\hat{A}, \hat{\Phi})\) of the KW equations on \( S^1 \times \Sigma \times \mathbb{R}^+ \), choose a gauge for which the \( S^1 \) component of \( A \) vanishes and \( A_y \equiv 0 \) as well. By virtue of the Nahm pole boundary conditions at \( y = 0 \) and the asymptotic condition as \( y \to \infty \), Proposition 2.3 gives that \( \phi_y = 0 \), but we cannot gauge away the \( S^1 \) component \( \phi_1 \). Thus the remaining fields are \((\hat{A}_\Sigma, \hat{\Phi}_1, \hat{\Phi}_\Sigma)\). We regard \( \hat{A}_\Sigma \) as a connection \( A \) on \( \Sigma \), and write \( \hat{\Phi}_1 = \phi_1 \), \( \hat{\Phi}_\Sigma = \phi \). These remaining fields satisfy the extended Bogomolny equations

\[
F_A - \phi \wedge \phi - *d_A \phi_1 = 0
\]

\[
d_A \phi + *[\phi, \phi_1] = 0,
\]

\[
d_A^* \phi = 0.
\]

On the other hand, given a solution \((A, \phi, \phi_1)\) of the extended Bogomolny equations on \( \Sigma \times \mathbb{R}^+ \), then denoting by \( \pi : S^1 \times \Sigma \times \mathbb{R}^+ \to \Sigma \times \mathbb{R}^+ \) the natural projection, we define the connection \( \hat{A} = \pi^* A \) and Higgs field \( \hat{\Phi} = \pi^* \phi + \pi^* \phi_1 dx_1 \). It is straightforward to check that \((\hat{A}, \hat{\Phi})\) satisfies the KW equations.

Let \( D = \{ (p_i, k_i = (k^i_1, \cdots, k^i_{n-1})) \} \) where for each \( i \), \( k^i_j \) are non-negative integers with at least one of them nonzero.

**Definition 3.1.** Let \((A, \phi, \phi_1)\) be a solution to the extended Bogomolny equations on \( \Sigma \times \mathbb{R}^+ \).

i) The fields \((A, \phi, \phi_1)\) satisfy the **Nahm pole boundary condition** if the corresponding fields \((\hat{A}, \hat{\Phi})\) satisfy the Nahm pole boundary condition on \( S^1 \times \Sigma \times \mathbb{R}^+ \).
ii) Similarly, \((A, \phi, \phi_1)\) satisfies the **Nahm pole boundary condition with knot data** \(D\) if the corresponding pull back fields \((\hat{A}, \hat{\Phi})\) satisfy the Nahm pole boundary condition with knots at \(K_1 := S^1 \times \{p_1\}\) with weight \(k_1\).

The moduli space we shall consider are:

\[
\mathcal{M}^{\text{EBE}}_{\text{NP}} := \{(A, \phi, \phi_1) : \text{EBE}(A, \phi, \phi_1) = 0, (A, \phi, \phi_1) \text{ converges to a flat } SL(n, \mathbb{C}) \text{ connection as } y \to \infty \text{ and satisfies the Nahm Pole boundary condition at } y = 0\}/\mathcal{G}_0,
\]

and

\[
\mathcal{M}^{\text{EBE}}_{\text{NPK}} := \{(A, \phi, \phi_1) : \text{EBE}(A, \phi, \phi_1) = 0, (A, \phi, \phi_1) \text{ satisfies the Nahm pole boundary condition with knot and converges to a flat } SL(n, \mathbb{C}) \text{ connection as } y \to \infty\}/\mathcal{G}_0,
\]

where \(\mathcal{G}_0\) is the gauge transformations that preserve the boundary condition.

3.1. **Hermitian-Yang-Mills Structure.** In [3, 20], it is observed that the extended Bogomolny equations have a Hermitian-Yang-Mills structure. By this we mean the following. Let \(E\) be complex vector bundle of rank \(n\) over \(\Sigma \times \mathbb{R}^+\) with \(\det E = 0\). A choice of Hermitian metric \(H\) on \(E\) induces an \(SU(n)\) structure on this bundle, and we denote by \(\mathfrak{g}_E\) the associated adjoint bundle. Writing

\[
d_A = \nabla_2 dx_2 + \nabla_3 dx_3 + \nabla_y dy, \quad \phi = \phi_2 dx_2 + \phi_3 dx_3 = \frac{1}{2}(\varphi_z dz + \varphi_\bar{z} d\bar{z}),
\]

we define the operators

\[
\begin{align*}
\mathcal{D}_1 &= (\nabla_2 + i\nabla_3) d\bar{z} = (2\partial_\bar{z} + A_1 + iA_2) d\bar{z}, \\
\mathcal{D}_2 &= \text{ad} \varphi = [\varphi, \cdot] = [(\phi_2 - i\phi_3) dz, \cdot], \\
\mathcal{D}_3 &= \nabla_y - i\phi_1 = \partial_y + A_y - i\phi_1.
\end{align*}
\]

Their adjoints with respect to \(H\) are denoted \(\mathcal{D}^\dagger_i\). The extended Bogomolny equations can then be written in the elegant form

\[
[D_i, D_j] = 0, \quad i, j = 1, 2, 3,
\]

\[
\frac{i}{2} \Lambda \left([D_1, D_2]\right) + \left([D_2, D_2]\right) + \left([D_3, D_3]\right) = 0,
\]

where \(\Lambda : \Omega^{1,1} \to \Omega^n\) is the inner product with the Kähler form (normalized as \((i/2)dz \wedge d\bar{z}\) when the metric on \(\Sigma\) is flat).

The action \(D_i \to g^{-1} D_i g\) of the gauge group \(\mathcal{G}\) preserves the Hermitian metric; the complex gauge group is denoted \(\mathcal{G}_C\). The smaller system \([D_i, D_j] = 0\) is invariant under \(\mathcal{G}_C\), while the full set of equations (11) is invariant only under \(\mathcal{G}\). The final equation is a real moment map condition. Following Donaldson [2] and Uhlenbeck-Yau [19], geometric data from the \(\mathcal{G}_C\)-invariant equations play an important role in understanding the moment map equation.

3.2. **Higgs Bundles and Flat Connections.** The appearance of Higgs bundles over \(\Sigma\) in this story is motivated by the fact that the \(y\)-independent versions of the equations of (7), when in addition \(\phi_1 = 0\), are simply the Hitchin equations.

Recall that a Higgs bundle over \(\Sigma\) is a pair \((\mathcal{E}, \varphi)\) where \(\mathcal{E}\) is a holomorphic bundle of rank \(n\) with \(\det \mathcal{E} = 0\) and \(\varphi \in H^0(\text{End}(\mathcal{E}) \otimes K)\). A Higgs pair (which is an alternate phrase for Higgs bundles) \((\mathcal{E}, \varphi)\) is called stable if for any holomorphic subbundle \(V\) with \(\varphi(V) \subset V \otimes K\), we have \(\text{deg}(V) < 0\), and polystable if it is a direct sum of stable Higgs pairs.
Setting $D_3 = 0$ in the extended Bogomolny equations (or alternately, considering only the equations for $D_1$ and $D_2$ on each slice $\Sigma_y := \Sigma \times \{y\}$), we obtain the Hitchin equations:

$$F_H + [\varphi, \varphi^*{}] = 0, \quad \bar{\partial}\varphi = 0. \quad (12)$$

The initial term $F_H$ is the curvature of the Chern connection $\nabla_H$ associated to $H$ and the holomorphic structure, and $\varphi^*{}$ is the adjoint with respect to $H$. Irreducibility of the fields $(A, \varphi + \varphi^*{})$ is defined in the obvious way. One may regard (12) as an equation for the fields $(A, \varphi)$ or else for the Hermitian metric $H$; we consider $H$ as the variable here.

**Theorem 3.2.** [8] For any Higgs pair $(\mathcal{E}, \varphi)$ on $\Sigma$, there exists an irreducible solution $H$ to the Hitchin equations if and only if this pair is stable, and a reducible solution if and only if it is polystable.

To any solution $H$ of (12) we associate the flat $\text{SL}(n, \mathbb{C})$ connection $D = \nabla_H + \varphi + \varphi^*{}$. This determines, in turn, a representation $\rho : \pi_1(\Sigma) \to \text{SL}(n, \mathbb{C})$ which is well-defined up to conjugation. Irreducibility of the solution is the same as irreducibility of the representation, while complete reducibility corresponds to the fact that $\rho$ is reductive. The map from flat connections back to solutions of the Hitchin system is defined as follows: first find a harmonic metric, cf. [1], which determines a decomposition $D = D_{\text{skew}} + D_{\text{Herm}}$ into skew-Hermitian and Hermitian parts. After that, the further decomposition $D_{\text{Herm}} = \varphi + \varphi^*{}$ determines $\varphi$, and hence the Hitchin bundle $(D_{\text{skew}})^{0,1}, \varphi$.

Denoting by $\mathcal{M}_{\text{Higgs}} := \{(\mathcal{E}, \varphi)\}^{\text{stable}}/\mathcal{G}_\mathbb{C}$ the moduli space of stable $\text{SL}(n, \mathbb{C})$ Higgs bundle, we are then led to define

$$P^\infty_{\text{NP}} : \mathcal{M}_{\text{NP}}^\text{EBE} \to \mathcal{M}_{\text{Higgs}}, \quad P^\infty_{\text{NP}} : \mathcal{M}_{\text{NP}}^\text{EBE} \to \mathcal{M}_{\text{Higgs}};$$

this is the map which assigns to a solution $(A, \phi, \phi_1)$ of the extended Bogomolny equations its limiting flat connection, and then, under Theorem 3.2, the corresponding Higgs bundle.

The Hitchin fibration is the map

$$\pi : \mathcal{M}_{\text{Higgs}} \to \bigoplus_{i=2}^n H^0(\Sigma, K^i)$$

$$\pi(\varphi) = (p_2(\varphi), \ldots, p_n(\varphi)), \quad (13)$$

where $\det(\lambda - \varphi) = \sum \lambda^{n-i}(-1)^j p_j(\varphi)$. By [9], this is a proper map.

We next introduce the Hitchin component (also called the Hitchin section). Choose a spin structure $K^\frac{2}{2}$ and set $B_i = i(n - i)$. Now define the Higgs bundle $(\mathcal{E}, \varphi)$, where

$$\mathcal{E} := S^{n-1}(K^{-\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2} + 1} \oplus \cdots \oplus K^{-\frac{n-1}{2}})$$

$$(14)$$

$$\varphi = \begin{pmatrix}
0 & \sqrt{B_1} & 0 & \cdots & 0 \\
0 & 0 & \sqrt{B_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \vdots & \cdots & 0 & \sqrt{B_{n-1}} \\
q_n & q_{n-1} & \cdots & q_2 & 0
\end{pmatrix}. \quad (15)$$

The constant $\sqrt{B_i}$ in the $(i, i + 1)$ entry represents this multiple of the natural isomorphism $K^{-\frac{n-1}{2} + i} \to K^{-\frac{n-1}{2} + i - 1} \otimes K$, and similarly, $H^0(\Sigma, K^{n-i}) \ni q_{n-i} : K^{-\frac{n-1}{2} + i} \to K^{-\frac{n-1}{2}} \otimes K$. The Hitchin component $\mathcal{M}_{\text{Hitch}}$ is the complex gauge orbit of this family of Higgs bundle,

$$\mathcal{M}_{\text{Hitch}} := \{(\mathcal{E} := S^{n-1}(K^{-\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2}}), \varphi \text{ as in (15)})\}/\mathcal{G}_\mathbb{C}. \quad (16)$$

The following theorem explains its importance.
Theorem 3.3. [10] Every element in $\mathcal{M}_{\text{Hit}}$ is a stable Higgs pair. Furthermore, the map assigning to each element of $\oplus_{i=3}^n H^0(\Sigma, K^i)$ the unique solution of the Hitchin equations corresponding to the associated Higgs pair is a diffeomorphism to one of the $n^{2g}$ choices for the Hitchin component; thus its inverse, the restriction of the Hitchin fibration $\pi|_{\mathcal{M}_{\text{Hit}}}$, is also a diffeomorphism.

Note that the image of this map is only one component of the space of all irreducible flat $\text{SL}(n, \mathbb{R})$ connections, which explains the name ‘Hitchin component.’

3.3. The Kobayashi-Hitchin Correspondence. We now recall the Kobayashi-Hitchin correspondence for the extended Bogomolny equations moduli space $[3, 6, 7]$.

As noted earlier, from the Hermitian structure in (11) and the commutation relationship $[D_1, D_2] = 0$, we obtain a Higgs bundle $(\mathcal{E}_y, \varphi_y)$ on each slice $\Sigma \times \{y\}$. The commutation relationship $[D_3, D_1] = [D_3, D_2] = 0$ means that parallel transport by $D_3$ identifies these Higgs bundles for different values of $y$.

Suppose first that the solution of the extended Bogomolny equations satisfies the Nahm pole boundary condition without knots. As explained in more detail in [7, Section 4], there is a holomorphic line subbundle $L \subset E$ determined by the property that the parallel transports (under $D_3$ parallel transport) of its sections vanish at the fastest possible rate as $y \to 0$, measured with respect to the Hermitian metric $H$. In other words, a solution of the extended Bogomolny equations satisfying these boundary conditions determines a triple $(\mathcal{E}, \varphi, L)$, consisting of a Higgs bundle and a line subbundle.

More generally, consider any triple $(\mathcal{E}, \varphi, L)$ where $L$ is any holomorphic line subbundle of $\mathcal{E}$. Define holomorphic maps

$$f_i := 1 \wedge \varphi \cdots \wedge \varphi^{i-1} = H^0(\Sigma; L^{-i} \otimes \wedge^i E \otimes K^{i(\cdot - 1)}), \quad 1 \leq i \leq n.$$  

Note that $Z(f_j) - Z(f_{j-1}) = \sum_i k_i^i p_i$ for some $k_i^i \in \mathbb{N}$. Setting $k_i := (k_1^i, \ldots, k_n^i) \in \mathbb{N}^{n-1}$, then we define the knot data set to be $\mathfrak{d}(\mathcal{E}, \varphi, L) := \{(p_i, k_i)\}$. Note the important special case (which holds by noting that $f_n \neq 0$ everywhere):

Proposition 3.4. [7, Section 4] If $\mathfrak{d}(\mathcal{E}, \varphi, L) = \emptyset$, then $(\mathcal{E}, \varphi) \in \mathcal{M}_{\text{Hit}}$ and $L \cong K^{\frac{np_h}{p_h-1}}$.

We then state the main equivalences between the extended Bogomolny equations moduli spaces and the spaces of triples $(\mathcal{E}, \varphi, L)$, first for data in the Hitchin component and then for general data.

Theorem 3.5. [6, 7] There is a diffeomorphism of moduli spaces

$$\mathcal{M}_{EFE}^{\text{NP}} \cong \mathcal{M}_{\text{Hit}}.$$  

More specifically, recall the map $P_\infty^{\text{NP}}$ from (13).

i) For any $(\mathcal{E}, \varphi) \in \mathcal{M}_{\text{Hit}}$, there exists a unique Nahm pole solution $(A, \phi, \phi_1) \in \mathcal{M}_{EFE}^{\text{NP}}$ such that $P_\infty^{\text{NP}}(A, \phi, \phi_1) = (\mathcal{E}, \varphi)$;

ii) Given any Higgs bundle $(\mathcal{E}, \varphi) \notin \mathcal{M}_{\text{Hit}}$, there is no solution to the extended Bogomolny equations which converges to the flat connection determined by $(\mathcal{E}, \varphi)$. In other words, $(P_\infty^{\text{NP}})^{-1}(\mathcal{E}, \varphi) = \emptyset$.

Theorem 3.6. [6, 7] Fix a data set $D = \{(p_i, k_i = (k_1^i, \ldots, k_n^i))\}$. If $(\mathcal{E}, \varphi)$ is any stable Higgs bundle over $\Sigma$ with genus $g(\Sigma) > 1$, there exists a solution to the extended Bogomolny equations satisfying the general Nahm pole boundary condition with knot singularities at $p_i$ with weight $k_i$, if and only if there exists a line bundle $L \subset \mathcal{E}$ such that $\mathfrak{d}(\mathcal{E}, \varphi, L) = D$. In other words, there is a bijection

$$\mathcal{M}_{EFE}^{\text{NP}} \cong \{(\mathcal{E}, \varphi, L)\}/\mathcal{G}_C,$$

where the pairs $(\mathcal{E}, \varphi)$ on the right are stable Higgs bundles and $L \subset \mathcal{E}$ is a line subbundle.
Remark. Notice that in the second result, when knot singularities are allowed, we do not claim that this bijection of moduli spaces is a diffeomorphism. Indeed, while the space of triples \((\mathcal{E}, \varphi, L)\) maps onto the space of all stable Higgs pairs, i.e., onto the entire Hitchin moduli space, it is not clear that this space of triples is even a manifold.

As a second remark, if \((\mathcal{E}, \varphi)\) is polystable, it seems likely that there are no solutions to the extended Bogomolny equations which satisfy Nahm pole boundary conditions with knot singularities which converge to \((\mathcal{E}, \varphi)\). However, we do not prove this.

4. A Weitzenböck Identity for the Kapustin-Witten Equations

In this section, we establish a Weitzenböck identity analogous to the one in [13], and use this to show that all solutions to the KW equations over \(M := S^1 \times \Sigma \times \mathbb{R}^+\) are invariant in the \(S^1\) direction, hence determine solutions to the extended Bogomolny equations. In all the following, we use coordinates \(x_1 \in S^1, z \in \Sigma\) and \(y \in \mathbb{R}^+\).

4.1. Weitzenböck Identity. As before, let \(P\) be an \(\text{SU}(n)\) bundle over \(M := S^1 \times \Sigma \times \mathbb{R}^+\), and fix a connection \(\tilde{A}\) and a \(\mathfrak{g}_P\)-valued 1-form \(\tilde{\Phi}\) on \(M\); assume that \(\tilde{A}_1 = \tilde{A}_y = \tilde{\Phi}_y \equiv 0\).

Write \(d_A = d_A + dx_1 \wedge \nabla_1\) and \(\tilde{\Phi} = \phi + \phi_1 dx_1\). We also fix a product metric on \(S^1 \times \Sigma \times \mathbb{R}^+\) with orientation \(dx_1 \wedge d\Sigma \wedge dy\).

Now write \(F_A = F_A + B_A \wedge dx_1\); the Bianchi identity \(d_A F_A = 0\) is equivalent to

\[
\nabla_1 F_A + d_A B_A = 0
\]

In the following, we write \(\ast_4\) and \(\ast\) for the Hodge star operators on \(M\) and \(S^1 \times \Sigma\), respectively.

We first compute

\[
F_A - \tilde{\Phi} \wedge \tilde{\Phi} + \ast_4 d_A \tilde{\Phi} = (F_A - \phi \wedge \phi + \ast(\nabla_1 \phi - d_A \phi_1)) + (B_A - [\phi, \phi_1] - \ast d_A \phi) \wedge dx_1,
\]

\[
d_A \ast \Phi = d_A \ast \phi - \nabla_1 \phi_1.
\]

Next, for any \(\epsilon \in (0, 1)\), write \(M_\epsilon := S^1 \times \Sigma \times [\epsilon, \epsilon^{-1}]\). Then

\[
\int_{M_\epsilon} |KW|^2 = \int_{M_\epsilon} |F_A - \phi \wedge \phi + \ast(\nabla_1 \phi - d_A \phi_1)|^2 + |B_A - [\phi, \phi_1] - \ast d_A \phi|^2 + |d_A \phi - \nabla_1 \phi_1|^2
\]

\[
= \int_{M_\epsilon} |F_A - \phi \wedge \phi - \ast d_A \phi_1|^2 + |\nabla_1 \phi|^2 + |B_A|^2 + |[[\phi, \phi_1] + \ast d_A \phi]|^2 + |d_A \phi|^2 + |\nabla_1 \phi_1|^2 + \int_{M_\epsilon} \chi,
\]

where

\[
\chi := 2 \langle F_A - \phi \wedge \phi - \ast d_A \phi_1, \ast \nabla_1 \phi \rangle - 2 \langle B_A, \ast d_A \phi + [\phi, \phi_1] \rangle - 2 \langle d_A \phi, \nabla_1 \phi_1 \rangle.
\]

The inner product here is \(\langle A, B \rangle := -\text{Tr}(A \wedge \ast B)\).

Lemma 4.1. We have the following identities:

i) \(\langle F_A, \ast \nabla_1 \phi \rangle - \langle B_A, \ast d_A \phi \rangle = \nabla_1 \text{Tr}(F_A \wedge \phi) \wedge dx_1 + d\text{Tr}(B_A \wedge \phi) \wedge dx_1\),

ii) \(\langle \ast d_A \phi_1, \ast \nabla_1 \phi \rangle + \langle B_A, [\phi, \phi_1] \rangle + \langle \ast d_A \phi_1, \ast \nabla_1 \phi \rangle = \nabla_1 \text{Tr}(\phi \wedge \ast d_A \phi_1) \wedge dx_1 - \nabla_1 \text{Tr}(\phi_1 \wedge d_A \ast \phi \wedge dx_1)\),

iii) \(\langle \phi \wedge \phi, \ast \nabla_1 \phi \rangle = \frac{1}{2} \nabla_1 \text{Tr}(\phi \wedge \phi \wedge dx_1)\)
Lemma 4.3. Let $A^\rho + \phi^\rho$ be a flat $SL(n, \mathbb{C})$ connection over $S^1 \times \Sigma$, and write $A^\rho = A_1^\rho + A_2^\rho$, $\phi^\rho = \phi_1^\rho + \phi_2^\rho$.

i) If we write $F_{A^\rho} = B_{A^\rho} \wedge dx_1 + E_{A^\rho}$, then $B_{A^\rho} = 0$;

ii) Up to a unitary gauge transformation, we can assume $A_1^\rho$ and $\phi_1^\rho$ are invariant in the $\Sigma$ directions and $A_2^\rho$, $\phi_2^\rho$ are invariant in the $S^1$ directions.

iii) Up to a unitary gauge transformation, $\phi_1^\rho = 0$.

Proof. Items i) and ii) follow from the fact that $\pi_1(\Sigma \times S^1) = \pi_1(\Sigma) \times \pi_1(S^1)$.

For iii), observe that $A_1^\rho$ and $\phi_1^\rho$ come from the contribution of $\pi_1(S^1) \to SL(n, \mathbb{C})$. Since $\pi_1(S^1)$ is abelian, and $A_1^\rho + i\phi_1^\rho$ is an unitary connection, we obtain that $\phi_1^\rho = 0$.

We now prove vanishing of the second part of the boundary contribution:

Lemma 4.4. Suppose that $(A, \phi, \phi_1)$ is a solution to the extended Bogomolny equations.

i) If $(A, \phi)$ satisfies the Nahm pole boundary conditions at $y = 0$, with or without knot singularities, then

\[ \lim_{\epsilon \to 0} \int_{S^1 \times \Sigma \times \{\epsilon\}} \text{Tr}(B_A \wedge \phi) = 0; \]

ii) If $(A, \phi)$ converges to a flat $SL(n, \mathbb{C})$ connection as $y \to \infty$, then

\[ \lim_{\epsilon \to 0} \int_{S^1 \times \Sigma \times (1/\epsilon)} \text{Tr}(B_A \wedge \phi) = 0. \]
Proof. First consider i). Away from knots, Theorem 2.4 gives that $A \sim A_{LC} + \mathcal{O}(y^{2-\epsilon})$ for any $\epsilon > 0$, which implies that $B_A = B_{A_{LC}} + \mathcal{O}(y^{2-\epsilon}) + dy \wedge (B_A)_y$. (The ‘LC’ subscript denotes Levi-Civita.) The $dy$ component vanishes in the integration so we may disregard it. In addition, since we are using the product metric, $B_{A_{LC}} = 0$. Finally, since $\phi \sim e^{-y}$, we conclude that $B_A \wedge \phi \sim \mathcal{O}(y^{1-\epsilon})$, so there are no boundary contributions in this region.

Near a knot $K$, we use spherical coordinates $(R, s, x_1)$ as before, and consider the boundary term as $R \to 0$. By Theorem 2.4, $B_A \sim B_{A_{mod}} + \mathcal{O}(1) \sim \mathcal{O}(1)$ because $B_{A_{mod}}$. In addition, $\phi \sim R^{-1}$, so $B_A \wedge \phi \sim R^{-1}$. Since the volume form is $R^2dRdsdx_1$, this boundary contribution vanishes too.

Part ii) follows directly from the previous lemma.

The other terms in $\chi$ are derivatives with respect to $x_1$, and hence vanish once we integrate over $S^1$.

**Corollary 4.5.** Under the previous assumptions, $\int_M \chi = 0$.

### 4.2. $S^1$-invariance.

In summary, we may now conclude the

**Theorem 4.6.** Any solution to the KW equations over $S^1 \times \Sigma \times \mathbb{R}^+$ satisfying Nahm pole boundary condition at $y = 0$ (possibly with knot singularities at $K = S^1 \times D \times \{0\}$), and which converges to a flat SL($n, \mathbb{C}$) connection as $y \to \infty$, is $S^1$ invariant and reduces to a solution of the extended Bogomolny equations. In addition, $A_1 \equiv 0$.

**Proof.** By Corollary 4.5, any solution to the KW equations with these boundary and asymptotic conditions must satisfy

$$\text{EBE}(A, \phi, \phi_1) = 0, \nabla_1 \phi = 0, B_A = 0, \nabla_1 \phi_1 = 0,$$

where EBE is the extended Bogomolny equation operator.

By Lemma 4.3, up to gauge we can assume that $(A, \phi, \phi_1)$ converges to $(A^0, \phi^0, 0)$ as $y \to \infty$, where $A^0, \phi^0$ is $S^1$ invariant. Since $(A, \phi, \phi_1)$ is a solution to the extended Bogomolny equations, Theorem 3.5 and Theorem 3.6 imply that $(A, \phi, \phi_1)$ is $S^1$ invariant. From $B_A = \nabla_1 \phi = 0$, we obtain $d_A A_1 = 0$ and $[\phi, A_1] = 0$. Irreducibility of solutions to the extended Bogomolny equations with Nahm pole boundary conditions give finally that $A_1 = 0$. □

The projection map $\pi : S^1 \times \Sigma \times \mathbb{R}^+ \to \Sigma \times \mathbb{R}^+$ naturally induces morphisms

$$\pi^* : \mathcal{M}_{\text{EBE}}^\text{NP} \to \mathcal{M}_{\text{KW}}^\text{NP},$$
$$\pi^* : \mathcal{M}_{\text{EBE}}^\text{NPK} \to \mathcal{M}_{\text{KW}}^\text{NPK}$$

We obtain from this the

**Corollary 4.7.** $\pi^* : \mathcal{M}_{\text{EBE}}^\text{NP} \to \mathcal{M}_{\text{NP}}^\text{NP}$ and $\pi^* : \mathcal{M}_{\text{EBE}}^\text{NPK} \to \mathcal{M}_{\text{NPK}}^\text{NP}$ are bijections.

### 5. Classification

We are now able to complete our main theorem.

#### 5.1. Case 1: $\Sigma = S^2$. 

**Proposition 5.1.** There is no Nahm pole solution to the KW equations on $S^1 \times S^2 \times \mathbb{R}^+$. 

**Proof.** By Theorem 4.6, all such solutions must be $S^1$-invariant and reduce to solutions of the Extended Bogomolny equations. Hence any such solution would lead to a stable Higgs bundle over $S^2$ with nonvanishing Higgs field. However, these do not exist [8]. □
5.2. **Case 2:** $\Sigma = T^2$. We next classify Nahm pole solutions over $T^3 \times \mathbb{R}^+$. Let $M = T^3 \times \mathbb{R}^+$ with flat metric $g$. If $A$ is a connection, then $d_A = \nabla_A^\perp + \nabla_y$, where $\nabla_A^\perp$ is the covariant derivative on $T^3$.

We quote the following identity for solutions of the KW equations from [15, 12]:

\[
\int_{M_\epsilon} |KW(A, \Phi)|^2 = \int_{M_\epsilon} (|F_A|^2 + |\nabla_A^\perp|^2 + |\nabla_y \phi + \phi \phi^\perp|^2 + \langle \text{Ric}(\phi), \phi \rangle) + 2 \int_{\partial M_\epsilon} \phi^\perp F_A,
\]

where $M_\epsilon := T^3 \times (\epsilon, \frac{1}{\epsilon})$ and $\star$ is the Hodge star operator on $T^3$.

**Proposition 5.2.** If $(A, \Phi)$ is a solution to the KW equations over $T^3 \times \mathbb{R}^+$ satisfying the Nahm pole boundary conditions, then

\[
F_A = 0, \quad \nabla_A^\perp = 0, \quad \nabla_y \phi + \phi^\perp \phi = 0.
\]

**Proof.** From [5], $F_A \sim F_{A_{LC}} + O(y^2) = O(y^2)$, where $A_{LC}$ is the Levi-Civita connection on $T^3$, but since the metric on $T^3$ is flat, $F_{A_{LC}} = 0$. This shows that $\text{lim}_{\epsilon \to 0} \int_{T^3 \times \{\epsilon\}} \phi^\perp F_A = 0$. Furthermore, since $(A, \Phi)$ converges to a flat connection on $T^3$, Lemma 4.3 implies that $\text{lim}_{\epsilon \to 0} \int_{T^3 \times \{\epsilon\}} \phi \wedge F_A = 0$. \(\square\)

**Proposition 5.3.** Let $e$ be a dreibein which is parallel along $T^3$. Then $(0, \frac{\hat{e}}{y})$ is the only solution to (22).

**Proof.** Use the temporal gauge in the $y$-direction, so $\nabla_y \phi = \partial_y$. Then $\nabla_y \phi + \star \phi \wedge \phi = 0$ is just the Nahm equations. Uniqueness of solutions to the Nahm equations with these boundary conditions implies that $\phi \equiv \frac{\hat{e}}{y}$ for some dreibein $e$. Up to a unitary gauge transformation, we can write $e = \sum_{i=1}^3 dx_i t_i$ where $de = 0$, $dx_i$ is an orthogonal basis of $T^\perp T^3$ and the triplet $t_i \in g_\rho$ satisfies $[t_i, t_j] = \epsilon_{ijk} t_k$. Finally, $\nabla_{A^\perp}^\perp \Phi = 0$ together with $de = 0$ implies that $A^\perp = 0$. \(\square\)

5.3. **Case 3:** $g(\Sigma) > 1$.

**Proposition 5.4.** Let $(A, \Phi)$ be a solution to the KW equations on $S^1 \times \Sigma \times \mathbb{R}^+_\epsilon$ satisfying Nahm pole boundary conditions and which converges to a flat $\text{SL}(n, \mathbb{C})$ connection $(A^\rho, \phi^\rho)$ as $y \to \infty$. If $g(\Sigma) > 1$, then there exists a unique solution if and only if $\rho$ is $S^1$ independent and lies in the Hitchin component.

**Proof.** By Theorem 4.6, all Nahm pole solutions are $S^1$ invariant and thus satisfy the extended Bogomolny equations and the statement then follows from Theorem 3.5. \(\square\)

5.4. **Case 4:** **Knots.** Suppose now that the Nahm pole boundary condition has an additional singularity along the knot $K = \cup_i K_i$ where $K_i = S^1 \times \{p_i\} \subset S^1 \times \Sigma$ with weight $k_i = (k_i^1, \cdots, k_i^{n-1})$.

**Theorem 5.5.** There is no solution $(A, \Phi)$ to the KW equations over $S^1 \times S^2 \times \mathbb{R}^+_y$ satisfying the Nahm pole boundary conditions with knots $K_i$ and weight $k_i$, and which converges to a flat $\text{SL}(n, \mathbb{C})$ connection as $y \to \infty$.

On the other hand, solutions to these equations with these boundary and asymptotic conditions on $S^1 \times \Sigma \times \mathbb{R}^+_\epsilon$ exist when $g(\Sigma) > 1$ if and only if there exists a line subbundle $L \subset \mathcal{E}$, where $(\mathcal{E}, \varphi)$ is the Higgs data corresponding to the flat bundle at infinity, such that $\mathcal{E}(\mathcal{E}, \varphi, L) = \{(p_i, k_i)\}$. 
Proof. By Proposition 4.6, solutions in either case are necessarily $S^1$-invariant. As there are no Higgs bundles with non-vanishing Higgs field over $S^2$, there is no solution over $S^1 \times S^2 \times \mathbb{R}^+$. The rest of the statement is just Theorem 3.6.

Corollary 5.6. Let $\rho$ be an irreducible flat $SL(n, \mathbb{C})$ connection. Then there exists at most $n^{2g}$ solutions to the KW equations satisfying Nahm pole boundary condition with a knot singularity along $K$ at $y = 0$ and which converges to $\rho$ in the cylindrical end.

Proof. Denote by $(\mathcal{E}, \varphi)$ the Higgs bundle corresponding to $\rho$. By Theorem 5.5, existence of a solution is equivalent to the existence of a line bundle $L \subset \mathcal{E}$ for which $\mathcal{D}(\mathcal{E}, \varphi, L) = \{ p_i, k_i = (k_{i1}, \ldots, k_{in-1}) \}$. The knot data determines the divisor $D = \sum_i p_i(\sum_{j=1}^{n-1} k_{ij})$, and we have $Z(f_n) = D$ where $f_n := 1 \wedge \varphi \wedge \cdots \wedge \varphi^{n-1}$. If $L_D$ is the line bundle associated to $D$, then $L^n = L_D^{-1} \otimes K^{\frac{n(n-1)}{2}}$. However, this determines $L$ only up an $n^{th}$ root of unity: if $N$ is any line bundle with $N^n = \mathcal{O}$, then $(L \otimes N)^n = L^n$. There are $n^{2g}$ choice of $N$, hence $n^{2g}$ possible solutions. However, it is not necessarily the case that each $(L \otimes N)^n$ is a subbundle of $\mathcal{E}$, so there may not be $n^{2g}$ actual solutions.

Theorem 5.7. Let $D = \sum_i p_i(\sum_{j=1}^{n-1} k_{ij})$ be the divisor determined by the given knot data. If $\deg D$ is not divisible by $n$, then there exists no solution.

Proof. Let $L_D$ be the line bundle associated to $D$ and $(\mathcal{E}, \varphi)$ the Higgs bundle determined by $\rho$. Suppose there exists a solution; then there exists a subbundle $L \subset \mathcal{E}$ such that $L^n = L_D^{-1} \otimes K^{\frac{n(n-1)}{2}}$. Therefore, $\deg D = -n \deg(L) + n(n-1)(g-1)$, so $n$ divides $\deg D$.

Corollary 5.8. Let $K = S^1 \times \{ p \} \subset S^1 \times \Sigma$ with weight 1 and suppose $g(\Sigma) > 1$. Then there is no SU(2) solution to the KW equations with Nahm pole singularity and knot $K$.

We now focus on the special case where $\rho$ lies in one of the “non-Hitchin” components of $SL(2, \mathbb{R})$ Higgs bundles. These components are described as follows. Let $\ell$ be a line bundle with $0 < \deg \ell < g-1$ and consider the stable Higgs bundle

$$\mathcal{E} = \ell^{-1} \oplus \ell, \quad \varphi = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

where $\alpha \in H^0(\ell^{-2} \otimes K)$ and $\beta \in H^0(\ell^2 \otimes K)$ are nontrivial sections. Then the zeroes of $f_2 := 1 \wedge \varphi : \ell^2 \to K$ coincide with those of $\alpha$, and the number of zeroes counted with multiplicity equals $2g - 2 - 2\deg \ell$.

Proposition 5.9. With all notation as above, fix the knot data $D = \sum_i p_i k_i$.

(i) If $\deg D = 2g - 2 - 2\deg \ell$, then there exists a unique Nahm pole solution if and only if $D = \alpha$ and no solution otherwise;

(ii) if $2g - 2 > \deg D > 2g - 2 - 2\deg \ell$, there is no solution.

Proof. With $L_D$ the line bundle for $D$, by Theorem 5.5 the necessary condition for existence of a Nahm pole solution is that there exists $L \subset \mathcal{E}$ such that $L^2 = L_D^{-1} \otimes K$. For (i), if $\deg(D) = 2g - 2 - 2\deg \ell$, then $\deg L = \deg \ell$. However, since $\mathcal{E}$ has rank 2 and deg $\mathcal{E} = 0$, there is a unique subbundle of positive degree, so $L = \ell$. By the form of the Higgs bundle, we conclude that $D = \alpha$. For (ii), if $2g - 2 > \deg D > 2g - 2 - 2\deg \ell$, if there is solution with line bundle $L$, then $0 < \deg L < \deg \ell$, which is impossible.

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**Simons Center for Geometry and Physics, StonyBrook University, Stonybrook, NY 11794**

*E-mail address: she@scgp.stonybrook.edu*

**Department of Mathematics, Stanford University, Stanford, CA 94305**

*E-mail address: rmazzeo@stanford.edu*