Quasideterminant solutions of a non-Abelian Toda lattice and kink solutions of a matrix sine-Gordon equation

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Two families of solutions of a generalized non-Abelian Toda lattice are considered. These solutions are expressed in terms of quasideterminants, constructed by means of Darboux and binary Darboux transformations. As an example of the application of these solutions, we consider the 2-periodic reduction to a matrix sine-Gordon equation. In particular, we investigate the interaction properties of polarized kink solutions.

Keywords: non-Abelian Toda lattice; quasideterminant solutions; matrix sine-Gordon equation; kink solutions

1. Introduction

There has been great interest in non-commutative versions of some well-known soliton equations, such as the KP equation, the KdV equation and the Hirota–Miwa equation (Kupershmidt 2000; Paniak 2001; Hamanaka 2003; Hamanaka & Toda 2003; Wang & Wadati 2003a,b; Sakakibara 2004; Wang et al. 2004; Dimakis & Müller-Hoissen 2005; Nimmo 2006; Gilson & Nimmo 2007; Gilson et al. 2007). Often, these non-commutative versions are obtained simply by removing the assumption that the coefficients in the Lax pair of the commutative equation commute.

The non-Abelian Toda lattice

\[ U_{n,x} + U_n V_{n+1} - V_n U_n = 0, \]

\[ V_{n,t} + U_{n-1} - U_n = 0 \]

was first studied in Mikhailov (1979). A Darboux transformation for this system was given by Salle (1982). In Nimmo & Willox (1997), the following generalization:

\[ U_{n,x} + U_n V_{n+1} - V_n U_n = 0, \]

\[ V_{n,t} + \alpha_n U_{n-1} - U_n \alpha_{n+1} = 0, \]

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was studied and the Darboux and binary Darboux transformations were obtained. We note that, in general, $\alpha_n$ is not a scalar, but is independent of $t$. In the case that $U_n$, $V_n$ and $\alpha_n$ are scalars, it is easy to show, by setting $\alpha_{n+1} U_n = e^{-\theta_n}$ and eliminating $V_n$, that (1.3) and (1.4) become the standard two-dimensional Toda lattice equation
\[ \theta_{n,t} - e^{-\theta_{n-1}} + 2e^{-\theta_n} - e^{-\theta_{n+1}} = 0. \] (1.5)

Introducing new variables $X_n$ where
\[ U_n = X_n X_{n+1}^{-1}, \quad V_n = X_{n,x} X_n^{-1}, \] (1.6)
(1.3) and (1.4) can be rewritten as
\[ (X_{n,x} X_n^{-1})_t + \alpha_n X_{n-1} X_n^{-1} - X_n X_{n+1} \alpha_{n+1} = 0. \] (1.7)

From now on, we will refer to (1.7) as the non-Abelian Toda lattice. One type of quasideterminant solution of (1.7) was found in Etingof et al. (1997). We will show how these (quasi-wronskian) solutions arise from the Darboux transformation and consider a second type of quasideterminant, which we call quasi-grammian, solutions obtained using the binary Darboux transformation.

It is well known that the 2-periodic reduction of the standard two-dimensional Toda lattice leads to the scalar sine-Gordon equation. In the same way, the 2-periodic reduction of the non-Abelian Toda lattice (1.7) leads to a non-commutative sine-Gordon equation. This equation has been studied already in a number of papers (Andreev 1990; Etingof et al. 1997; Cabrera-Carnero & Moriconi 2003; Grisaru & Penati 2003; Grisaru et al. 2004; Zuevsky 2004; Lechtenfeld et al. 2005; Hamanaka 2006) concerning both the matrix and the Moyal product versions. Here we consider only the matrix version in detail.

Recently, a matrix KdV equation was considered in Goncharenko (2001). A multisoliton solution was found by using the inverse scattering method. In particular, the properties of one- and two-soliton solutions expressed in terms of projection matrices were investigated. We will apply some of these ideas to the matrix sine-Gordon equation to study the interaction of its kink solutions.

The paper is organized as follows. In §2, some properties of quasideterminants used in the paper are described. In §3, we present quasi-wronskian solutions to the non-Abelian Toda lattice constructed by iterating Darboux transformations and in §4, we present quasi-grammian solutions to the system using the related binary Darboux transformation. In the rest of the paper, we consider the 2-periodic reduction to a matrix sine-Gordon equation. In particular, we consider the matrix kink solutions obtained from the quasi-grammian solutions, we show that kink solutions for the matrix sine-Gordon equation emerge intact from interaction apart from change of polarization and phase.

### 2. Preliminaries

In this short section we recall some of the key elementary properties of quasideterminants. The reader is referred to the original papers (Gelfand & Retakh 1991; Etingof et al. 1997; Gelfand et al. 2005) for a more detailed and general treatment.
(a) Quasideterminants

An \( n \times n \) matrix \( A \) over a ring \( \mathcal{R} \) (non-commutative, in general) has \( n^2 \) quasideterminants written as \( |A|_{i,j} \) for \( i, j = 1, \ldots, n \), which are also elements of \( \mathcal{R} \). They are defined recursively by

\[
|A|_{i,j} = a_{i,j} - r^j_i(A^{-1})^{-1} c^i_j, \quad A^{-1} = (|A|^{-1})_{i,j=1, \ldots, n}.
\]

(2.1)

In the above equation, \( r^j_i \) represents the \( i \)th row of \( A \) with the \( j \)th element removed; \( c^i_j \) is the \( j \)th column with the \( i \)th element removed; and \( A^{-1} \) the submatrix obtained by removing the \( i \)th row and the \( j \)th column from \( A \). Quasideterminants can be also denoted as shown below by boxing the entry about which the expansion is made

\[
|A|_{i,j} = \begin{vmatrix} a_{i,j} & c^i_j \\ r^j_i & A^{-1} \end{vmatrix}.
\]

The case \( n = 1 \) is rather trivial; let \( A = (a) \), say, and then there is one quasideterminant \( |A|_{1,1} = [a] = a \). For \( n = 2 \), let

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]

then there are four quasideterminants

\[
|A|_{1,1} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = a - bd^{-1}c, \quad |A|_{1,2} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = b - ac^{-1}d,
\]

\[
|A|_{2,1} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = c - db^{-1}a, \quad |A|_{2,2} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = d - ca^{-1}b.
\]

Note that if the entries in \( A \) commute, the above becomes the familiar formula for the inverse of a \( 2 \times 2 \) matrix with entries expressed as ratios of determinants. Indeed, this is true for any size of square matrix; if the entries in \( A \) commute then

\[
|A|_{i,j} = (-1)^{i+j} \frac{\det(A)}{\det(A^{-1})}.
\]

(2.2)

In this paper, we will consider only quasideterminants that are expanded about a term in the last column, most usually the last entry. For a block matrix

\[
\begin{pmatrix} A & B \\ C & d \end{pmatrix},
\]

where \( d \in \mathcal{R} \), \( A \) is a square matrix over \( \mathcal{R} \) of arbitrary size and \( B, C \) are column and row vectors over \( \mathcal{R} \) of compatible lengths, respectively, we have

\[
\begin{vmatrix} A & B \\ C & d \end{vmatrix} = d - CA^{-1}B.
\]
(b) Non-commutative Jacobi identity

There is a quasideterminant version of Jacobi’s identity for determinants, called the non-commutative Sylvester’s theorem by Gelfand & Retakh (1991). The simplest version of this identity is given by

\[
\begin{vmatrix}
A & B & C \\
D & f & g \\
E & h & i
\end{vmatrix} = \begin{vmatrix}
A & C \\
E & h
\end{vmatrix} - \begin{vmatrix}
A & B \\
E & f
\end{vmatrix} - \begin{vmatrix}
A & B \\
D & f
\end{vmatrix}^{-1} \begin{vmatrix}
A & C \\
D & g
\end{vmatrix}.
\]

(2.3)

As a direct result, we have the homological relation

\[
\begin{vmatrix}
A & B & C \\
D & f & g \\
E & h & i
\end{vmatrix} = \begin{vmatrix}
A & B & 0 \\
D & f & 0 \\
E & h & 1
\end{vmatrix} \begin{vmatrix}
A & B & C \\
A & f & i \\
E & h & 1
\end{vmatrix}.
\]

(2.4)

(c) Quasi-Plücker coordinates

Given an \((n+k) \times n\) matrix \(A\), denote the \(i\)th row of \(A\) by \(A_i\), the submatrix of \(A\) having rows with indices in a subset \(I\) of \(\{1, 2, \ldots, n+k\}\) by \(A_I\) and \(A_{(i)}\) by \(A_i\). Given \(i, j \in \{1, 2, \ldots, n+k\}\) and \(I\) such that \(#I = n-1\) and \(j \not\in I\), one defines the (right) quasi-Plücker coordinates

\[
r_{ij}^I = r_{ij}^I(A) := \begin{vmatrix}
A_I \\
A_i
\end{vmatrix}_{ns} A_{j}^{-1}_{ns} = -\begin{vmatrix}
A_I \\
A_i
\end{vmatrix}_{ns} A_j, 
\]

(2.5)

for any column index \(s \in \{1, \ldots, n\}\). The final equality in (2.5) comes from an identity of the form (2.3) and proves that the definition is independent of the choice of \(s\).

**Remark 2.1.** A useful consequence of (2.5) is the identity

\[
\begin{vmatrix}
A_I & 0 \\
A_i & 0 \\
A_j & 1
\end{vmatrix}^{-1} = \begin{vmatrix}
A_i & 0 \\
A_i & 1 \\
A_j & 0
\end{vmatrix},
\]

(2.6)

that shows that quasideterminants of this form may be inverted very simply.

3. Solutions obtained by Darboux transformations

The non-Abelian Toda lattice (1.3) and (1.4) has Lax pair

\[
\phi_{n,x} = V_n \phi_n + \alpha_n \phi_{n-1},
\]

(3.1)

\[
\phi_{n,t} = U_n \phi_{n+1}.
\]

(3.2)
Let \( \theta_{n,i}, i = 1, ..., N \) be a particular set of eigenfunctions of the linear system and introduce the notation \( \Theta_n = (\theta_{n,1}, ..., \theta_{n,N}) \). The Darboux transformation, determined by particular solution \( \theta_n \), for the non-Abelian Toda lattice is

\[
\tilde{\phi}_n = \phi_n - \theta_n \theta_{n+1}^{-1} \phi_{n+1},
\]

\[
\tilde{V}_n = V_n + \alpha_n \theta_{n-1} \theta_1^{-1} - \theta_n \theta_{n+1}^{-1} \alpha_{n+1},
\]

\[
\tilde{U}_n = U_n - (\theta_n \theta_{n+1}^{-1})_t = \theta_n \theta_{n+1}^{-1} U_{n+1} \theta_{n+2} \theta_{n+1}^{-1},
\]

\[
\tilde{X}_n = \theta_n \theta_{n+1}^{-1} X_{n+1}.
\]

This may be iterated by defining

\[
\phi_n[k + 1] = \phi_n[k] - \theta_n[k] \theta_{n+1}[k]^{-1} \phi_{n+1}[k],
\]

\[
X_n[k + 1] = \theta_n[k] \theta_{n+1}[k]^{-1} X_{n+1}[k],
\]

where \( \phi_n[1] = \phi_n \), \( X_n[1] = X_n \) and

\[
\theta_n[k] = \phi_n[k] |_{\phi_n \rightarrow \theta_n[k]}.
\]

In particular,

\[
\phi_n[2] = \phi_n - \theta_{n,1} \theta_{n+1,1} \phi_{n+1},
\]

\[
X_n[2] = \theta_{n,1} \theta_{n+1,1} X_{n+1}.
\]

In what follows, we will show by induction that the results of \( N \)-repeated Darboux transformation \( \phi_n[N+1] \) and \( X_n[N+1] \) can be expressed as in closed form as quasideterminants

\[
\phi_n[N+1] = \begin{vmatrix}
\Theta_n & [\phi_n] \\
\Theta_{n+1} & \phi_{n+1} \\
\vdots & \vdots \\
\Theta_{n+N} & \phi_{n+N}
\end{vmatrix},
\]

\[
X_n[N+1] = (-1)^N \begin{vmatrix}
\Theta_n & [0] \\
\Theta_{n+1} & 0 \\
\vdots & \vdots \\
\Theta_{n+N} & 1
\end{vmatrix} X_{n+N}.
\]

The initial case \( N=1 \) follows directly from (3.10) and (3.11). Also using the non-commutative Jacobi identity (2.4) and the homological relation (2.4)
we have
\[
\phi_n[N + 2] = \phi_n[N + 1] - \theta_n[N + 1]\theta_{n+1}[N + 1]^{-1}\phi_{n+1}[N + 1]
\]
\[
\begin{array}{c|c|c|c|c|c}
\Theta_n & \phi_n & & & & \\
\hline
\Theta_{n+1} & \phi_{n+1} & \Theta_n & \theta_{n,N+1} & \Theta_{n+1} & \theta_{n+1,N+1} \\
\hline
& \Theta_{n+2} & \theta_{n+2,N+1} & \Theta_{n+1} & \phi_{n+2} & \Theta_{n+1} \\
& \vdots & \vdots & \vdots & \vdots & \vdots \\
\Theta_{n+N} & \phi_{n+N} & \Theta_{n+N} & \theta_{n+N,N+1} & \Theta_{n+N} & \theta_{n+N+1,N+1} \\
\hline
\end{array}
\]
\[
\begin{array}{c|c|c|c|c|c}
\Theta_n & \theta_{n,N+1} & \phi_n & & & \\
\hline
\Theta_{n+1} & \theta_{n+1,N+1} & \phi_{n+1} & \Theta_n & \theta_{n+1,N+1} & 0 \\
\hline
& \Theta_{n+2} & \theta_{n+2,N+1} & \vdots & \vdots & \vdots \\
& \Theta_{n+N} & \theta_{n+N,N+1} & \Theta_{n+N} & \theta_{n+N+1,N+1} & \Theta_{n+N} \\
\hline
\end{array}
\]
and
\[
X_n[N + 2] = \theta_n[N + 1]\theta_{n+1}[N + 1]^{-1}X_{n+1}[N + 1]
\]
\[
\begin{array}{c|c|c|c|c|c|c}
\Theta_n & \theta_{n,N+1} & & & & & \\
\hline
\Theta_{n+1} & \theta_{n+1,N+1} & \Theta_n & \theta_{n+1,N+1} & \Theta_{n+1} & \theta_{n+1,N+1} & 0 \\
\hline
& \Theta_{n+2} & \theta_{n+2,N+1} & \vdots & \vdots & \vdots & \vdots \\
& \Theta_{n+N} & \theta_{n+N,N+1} & \Theta_{n+N} & \theta_{n+N+1,N+1} & \Theta_{n+N} & 1 \\
\hline
\end{array}
\]
\[
\begin{array}{c|c|c|c|c|c|c}
\Theta_n & \theta_{n,N+1} & & & & & \\
\hline
\Theta_{n+1} & \theta_{n+1,N+1} & \Theta_n & \theta_{n+1,N+1} & \Theta_{n+1} & \theta_{n+1,N+1} & 0 \\
\hline
& \Theta_{n+2} & \theta_{n+2,N+1} & \vdots & \vdots & \vdots & \vdots \\
& \Theta_{n+N} & \theta_{n+N,N+1} & \Theta_{n+N} & \theta_{n+N+1,N+1} & \Theta_{n+N} & 1 \\
\hline
\end{array}
\]
and then using the quasi-Plücker coordinate formula (2.5), we get
\[
\begin{array}{c|c|c|c|c|c|c}
\Theta_n & \theta_{n,N+1} & 0 & & & & \\
\hline
\Theta_{n+1} & \theta_{n+1,N+1} & 0 & \Theta_n & \theta_{n+1,N+1} & 0 & \Theta_{n+1} \\
\hline
& \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& \Theta_{n+N} & \theta_{n+N,N+1} & \Theta_{n+N} & \theta_{n+N+1,N+1} & \Theta_{n+N} & 1 \\
\hline
\end{array}
\]
This proves the inductive step and the proof is complete.
4. Solutions obtained by binary Darboux transformation

The linear equations (3.1) and (3.2) have the formal adjoints

\[-\psi_{n,x} = V_n^\dagger \psi_n + \alpha_{n+1}^\dagger \psi_{n+1}, \quad (4.1)\]
\[-\psi_{n,t} = U_{n-1}^\dagger \psi_{n-1}. \quad (4.2)\]

Following the standard construction of a binary Darboux transformation, one introduces a potential \(\Omega_n = \Omega(\phi_n, \psi_n)\) satisfying the three conditions:

\[\Omega(\phi_n, \psi_n)_x = \psi_{n+1}^\dagger \alpha_{n+1} \phi_n, \quad (4.3)\]
\[\Omega(\phi_n, \psi_n)_t = -\psi_n^\dagger U_n \phi_{n+1}, \quad (4.4)\]
\[\Omega_n - \Omega_{n-1} = -\psi_n^\dagger \phi_n. \quad (4.5)\]

A binary Darboux transformation is then defined by

\[\phi_n[N+1] = \phi_n[N] - \theta_n[N] \Omega(\theta_n[N], \rho_n[N])^{-1} \Omega(\phi_n[N], \rho_n[N]), \quad (4.6)\]
\[\psi_n[N+1] = \psi_n[N] - \rho_n[N] \Omega(\theta_{n-1}[N], \rho_{n-1}[N])^{-1} \Omega(\theta_{n-1}[N], \psi_{n-1}[N])^\dagger, \quad (4.7)\]
\[X_n[N+1] = (I + \theta_n[N] \Omega(\theta_n[N], \rho_n[N])^{-1} \rho_n[N])^\dagger X_n[N], \quad (4.8)\]

where \(\phi_n[1] = \phi_n, \; \psi_n[1] = \psi_n, \; X_n[1] = X_n\) and
\[\theta_n[N] = \phi_n[N]|_{\theta_n \to \theta_n}, \quad \rho_n[N] = \psi_n[N]|_{\rho_n \to \rho_n}. \quad (4.9)\]

Using the notation \(\Theta_n = (\theta_{n,1}, \ldots, \theta_{n,N})\) and \(P_n = (\rho_{n,1}, \ldots, \rho_{n,N})\), it is easy to prove by induction that for \(N \geq 1\),

\[\phi_n[N + 1] = \begin{vmatrix} \Omega(\Theta_n, P_n) & \Omega(\phi_n, P_n) \\ \Theta_n & \phi_n \end{vmatrix}, \quad (4.10)\]
\[\psi_n[N + 1] = \begin{vmatrix} \Omega(\Theta_{n-1}, P_{n-1})^\dagger & \Omega(\Theta_{n-1}, \psi_{n-1})^\dagger \\ P_n & \psi_n \end{vmatrix}, \quad (4.11)\]

and

\[\Omega(\phi_n[N + 1], \psi_n[N + 1]) = \begin{vmatrix} \Omega(\Theta_n, P_n) & \Omega(\phi_n, P_n) \\ \Omega(\Theta_n, \psi_n) & \Omega(\phi_n, \psi_n) \end{vmatrix}. \quad (4.12)\]

We may thus after \(N\) binary Darboux transformations obtain

\[X_n[N + 1] = \begin{vmatrix} \Omega(\Theta_n, P_n) & P_n^\dagger \\ \Theta_n & -I \end{vmatrix} X_n. \quad (4.13)\]
In fact, we can prove the above results by induction.

\[ X_n[N + 2] = (I + \Theta_n[N + 1]Q(\Theta_n[N + 1], P_n[N + 1])^{-1}P_n[N + 1]^{\dagger})X_n[N + 1] \]

\[ = - \left( I + \begin{vmatrix} Q(\Theta_n, P_n) & Q(\theta_{n,N+1}, P_n) \\ \Theta_n & \theta_{n,N+1} \end{vmatrix} Q(\Theta_n, P_n) & Q(\theta_{n,N+1}, P_n) \\ Q(\Theta_{n-1}, P_n) & P_n^{\dagger} \\ Q(\Theta_{n-1}, \rho_{n-1,N+1}) & \rho_{n,N+1}^{\dagger} \end{vmatrix} \right)^{-1} \]

Noticing

\[ \left| \begin{array}{cc} Q(\Theta_n^{\dagger}, P_n) & P_n^{\dagger} \\ \Theta_n & -I \end{array} \right| \]

\[ = (-\rho_{n,N+1}^{\dagger} - Q(\Theta_n^{\dagger}, \rho_{n-1,N+1})Q(\Theta_n^{\dagger}, P_n) - P_n^{\dagger}) \left( I + \Theta_nQ(\Theta_n, P_n)^{-1}P_n^{\dagger} \right) \]

\[ = -\rho_{n,N+1}^{\dagger} + Q(\Theta_n^{\dagger}, \rho_{n-1,N+1})Q(\Theta_n^{\dagger}, P_n) - P_n^{\dagger} \]

\[ + (Q(\Theta_n, \rho_{n,N+1}) - Q(\Theta_n^{\dagger}, \rho_{n-1,N+1}))Q(\Theta_n, P_n)^{-1}P_n^{\dagger} \]

\[ + Q(\Theta_n^{\dagger}, \rho_{n-1,N+1})Q(\Theta_n^{\dagger}, P_n) - 1(Q(\Theta_n^{\dagger}, P_n) - 1P_n^{\dagger}) \]

\[ = -\rho_{n,N+1}^{\dagger} + Q(\Theta_n, \rho_{n,N+1})Q(\Theta_n, P_n)^{-1}P_n^{\dagger} \]

we have

\[ X_n[N + 2] = - \left| \begin{array}{cc} Q(\Theta_n, P_n) & Q(\theta_{n,N+1}, P_n) \\ \Theta_n & \theta_{n,N+1} \end{array} \right| \]

5. Matrix sine-Gordon equation and its kink solutions

It is well known in the commutative case that one may obtain reductions by imposing periodic conditions on the \( \theta_n \). Similarly in the non-Abelian case, one can make periodic reductions of (1.7). From now on, we only consider the case.

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that $X_n$ is a $d \times d$ matrix and $\alpha_n = I_{d \times d}$ and so (7.1) is
$$
(X_n, X_n^{-1})_t + X_n^{-1}X_{n-1} - X_nX_{n+1}^{-1} = 0. \tag{5.1}
$$

The simplest such reduction has period 2, that is, we take $X_n = X_{n+2}$ and (5.1) gives the system
$$
(X_0, X_0^{-1})_t + X_1X_0^{-1} - X_0X_1^{-1} = 0, \tag{5.2}
$$
$$
(X_1, X_1^{-1})_t + X_0X_1^{-1} - X_1X_0^{-1} = 0. \tag{5.3}
$$

We call this a non-Abelian sinh-Gordon equation since in the commutative case, it will be seen that $X_0 = X_1^{-1} = F_1/F_0$ and then $\theta = 2 \log(F_1/F_0)$ satisfies the standard sinh-Gordon equation
$$
\theta_{xt} = 4 \sinh \theta.
$$

By changing $\theta \rightarrow i \theta$, we can also obtain the sine-Gordon equation
$$
\theta_{xt} = 4 \sin \theta.
$$

In what follows, we will construct solutions to (5.2) and (5.3) by the reduction of the solutions (4.13) of the non-Abelian Toda lattice (5.1). It is clear that (5.1) has vacuum solution $X_n = I$ and (4.13) gives the quasi-grammian solutions
$$
X_n = \left| \begin{array}{cc} \Theta_n & P_n^T \\ \Theta_n^T & -I \end{array} \right|, \tag{5.4}
$$

where $\Theta_n$, $\rho_n$, and $\Theta_n^T$, $\rho_n^T$ satisfy
$$
(\Theta_n)_x = \Theta_{n-1}, \quad (\Theta_n)_t = \Theta_{n+1}, \quad (\rho_n)_x = -\rho_{n+1}, \quad (\rho_n)_t = -\rho_{n-1} \tag{5.5}
$$
and $\Theta$ is defined by (4.3)–(4.5). We choose the simplest non-trivial solutions of (5.5)
$$
\theta_{n,j} = B_j q_j^{-n} \exp \left( q_j x + \frac{1}{q_j} t \right), \quad \rho_{n,i} = A_i p_i^{n} \exp \left( -p_i x - \frac{1}{p_i} t \right),
$$
where $A_i$ and $B_j$ are $d \times d$ matrices and then we obtain
$$
\Theta(\theta_{n,j}, \rho_{n,i}) = \delta_{i,j} I + \frac{A_i^T B_j p_i}{q_j - p_i} \left( \frac{p_i}{q_j} \right)^n \exp \left( (q_j - p_i)x + \left( \frac{1}{q_j} - \frac{1}{p_i} \right) t \right).
$$
The choice of constant of integration as $\delta_{i,j} I$ is needed to effect the periodic reduction that will be made shortly. This can also be written as
$$
\Theta(\theta_{n,j}, \rho_{n,i}) = \left( \frac{p_i}{q_j} \right)^n \left( \delta_{i,j} \frac{q_j}{p_i} \right)^n + \frac{A_i^T B_j p_i}{q_j - p_i} \exp \left( (q_j - p_i)x + \left( \frac{1}{q_j} - \frac{1}{p_i} \right) t \right).
$$

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Now using the invariance of a quasideterminant to scaling of its rows and columns (Gelfand et al. 2005), we get
\[
X_n = -\begin{pmatrix}
\sum_i (\frac{q_i}{p_i})^n I + \frac{A_i^j B_i p_i}{q_j - p_i} \exp \left( (q_j - p_i)x + \left( \frac{1}{q_j} - \frac{1}{p_i} \right) t \right) \left( A_i \exp \left( -p_i x - \frac{1}{p_i} t \right) \right) \\
B_j \exp \left( q_j x + \frac{1}{q_j} t \right)
\end{pmatrix}.
\]

It is obvious from this expression for \(X_n\) that it is 2-periodic when \((q_i/p_i)^2 = \cdots = (q_N/p_N)^2 = 1\), i.e. \(p_i = -q_i = \lambda_i\) for \(i = 1, \ldots, N\). Therefore, the non-Abelian sinh-Gordon equation has the solutions
\[
X_0 = -\begin{pmatrix}
\sum_i \frac{A_i^j B_i \lambda_i}{\lambda_i + \lambda_j} \exp \left( (\lambda_i + \lambda_j)x - \left( \frac{1}{\lambda_i} + \frac{1}{\lambda_j} \right) t \right) \left( A_i \exp \left( -\lambda_i x - \frac{1}{\lambda_i} t \right) \right) \\
B_j \exp \left( -\lambda_j x - \frac{1}{\lambda_j} t \right)
\end{pmatrix},
\]
\[
X_1 = -\begin{pmatrix}
\sum_i \frac{A_i^j B_i \lambda_i}{\lambda_i + \lambda_j} \exp \left( (\lambda_i + \lambda_j)x - \left( \frac{1}{\lambda_i} + \frac{1}{\lambda_j} \right) t \right) \left( A_i \exp \left( -\lambda_i x - \frac{1}{\lambda_i} t \right) \right) \\
B_j \exp \left( -\lambda_j x - \frac{1}{\lambda_j} t \right)
\end{pmatrix},
\]
\[
(5.6)
\]
\[
(5.7)
\]
From now on, we will assume that \(A_i = I\) are real and \(B_j\) are pure imaginary and written as \(ir_j P_j\), where \(r_j\) are real scalars. In this case, it follows that \(X_0\) and \(X_1\) are complex conjugate to one another. For this reason we introduce
\[
X = X_0 = \bar{X}_1
\]
\[
(5.8)
\]
Next, we will derive matrix kink solutions for the matrix sine-Gordon equation using the method applied to study the soliton solutions of the matrix KdV equation in Goncharenko (2001). To get a visual representation of the solution we will consider the matrix \(W(x, t)\) defined by
\[
i W_x = \bar{X}_x X^{-1} - X_x X^{-1}.
\]
\[
(5.9)
\]
We choose this dependent variable so that in the scalar case $W = \theta$, the solution of the sine-Gordon equation.

For $N=1$, (5.8) gives

$$X = I + B \left( I - \frac{B}{2} \exp \left( -2\lambda x - \frac{2}{\lambda} t \right) \right)^{-1} \exp \left( -2\lambda x - \frac{2}{\lambda} t \right).$$

(5.10)

We first assume further that $P$ is a projection matrix (i.e. satisfies $P^2 = P$). This choice allows us to calculate the inverse matrices in the above expression explicitly using the formula

$$(I - aP)^{-1} = I + \frac{aP}{1-a},$$

(5.11)

where $a \neq 1$ is a scalar and $P$ is any projection matrix.

In this way, we find that

$$X = I + \frac{irP}{\exp(2\lambda x + \frac{2}{\lambda} t) - \frac{ir}{2}},$$

and hence

$$W_x = \frac{4\lambda P}{\cosh(2\lambda(x + t/\lambda^2 - \phi))},$$

where $\phi = \log(r/2)/2\lambda$. Note also that $X \bar{X} = I$.

Taking one final step, we integrate to obtain the one-kink solution to the matrix sine-Gordon equation

$$W = 4P \arctan(\exp(2\lambda(x + t/\lambda^2 - \phi))).$$

(5.12)

**Remark 5.1.** For the one-kink solution (5.12), we call the projection matrix $P$ as its polarization and $\phi$ its phase. In the scalar case, if we choose $P=1$, (5.12) is simply the one-kink solution to the standard sine-Gordon equation.

For $N=2$, expanding $X$ by the definition (2.1), we can rewrite $X$ as

$$X = I + \left( B_1 \exp \left( -\lambda_1 x - \frac{1}{\lambda_1} t \right), B_2 \exp \left( -\lambda_2 x - \frac{1}{\lambda_2} t \right) \right)$$

$$\left( \delta_{i,j} I - \frac{B_j \lambda_i}{(\lambda_i + \lambda_j)} \exp \left( -(\lambda_i + \lambda_j)x - \left( \frac{1}{\lambda_i} + \frac{1}{\lambda_j} \right) t \right) \right)^{-1} \begin{pmatrix} \exp \left( -\lambda_1 x - \frac{1}{\lambda_1} t \right) I \\ \exp \left( -\lambda_2 x - \frac{1}{\lambda_2} t \right) I \end{pmatrix}_{2 \times 2}$$

$$= I + \left( L_1 \exp \left( \lambda_1 x + \frac{1}{\lambda_1} t \right), L_2 \exp \left( \lambda_2 x + \frac{1}{\lambda_2} t \right) \right) \begin{pmatrix} \exp \left( -\lambda_1 x - \frac{1}{\lambda_1} t \right) I \\ \exp \left( -\lambda_2 x - \frac{1}{\lambda_2} t \right) I \end{pmatrix}$$

$$= I + L_1 + L_2,$$
and hence

\[
L_1 \left( \exp \left( 2\lambda_1 x + \frac{2}{\lambda_1} t \right) I - \frac{1}{2} B_1 \right) - \frac{\lambda_2}{\lambda_1 + \lambda_2} L_2 B_1 = B_1,
\]

\[
L_2 \left( \exp \left( 2\lambda_2 x + \frac{2}{\lambda_2} t \right) I - \frac{1}{2} B_2 \right) - \frac{\lambda_1}{\lambda_1 + \lambda_2} L_1 B_2 = B_2.
\]

In the expressions \( B_j = i r_j P_j \), \( j = 1, 2 \), we assume that \( P_j \) are the rank-1 projection matrices

\[
P_j = \frac{p_j \otimes q_j}{(p_j, q_j)} = \frac{p_j q_j^T}{p_j^T q_j}
\]

and the d-vectors \( p_j \) and \( q_j \) satisfy the condition \((p_j, q_j) \neq 0\), we can solve for \( L_1 \) and \( L_2 \) by using (5.11) to obtain

\[
L_1 = \frac{(\lambda_1 + \lambda_2)}{g} (\lambda_2 B_2 + (\lambda_1 + \lambda_2) g_2 I) B_1,
\]

\[
L_2 = \frac{(\lambda_1 + \lambda_2)}{g} (\lambda_1 B_1 + (\lambda_1 + \lambda_2) g_1 I) B_2,
\]

where

\[
g = (\lambda_1 + \lambda_2)^2 g_1 g_2 + \lambda_1 \lambda_2 r_1 r_2 \alpha, \quad \alpha = \frac{(p_1, q_2)(p_2, q_1)}{(p_1, q_1)(p_2, q_2)}
\]

and

\[
g_j = \exp(2\lambda_2 \theta_j) - \frac{ir_j}{2} \quad \text{for} \ j = 1, 2, \quad \theta_j = x + \frac{1}{\lambda_j^2} t.
\]

Therefore,

\[
X = I + \frac{(\lambda_1 + \lambda_2)}{g} (\lambda_1 B_1 B_2 + \lambda_2 B_2 B_1 + (\lambda_1 + \lambda_2)(g_1 B_2 + g_2 B_1)). \tag{5.13}
\]

We now investigate the behaviour of \( X \) as \( t \to \pm \infty \). We will use the fact that \( W \) is invariant under the transformation \( X \to XC \) for any constant matrix \( C \) and assume, without loss of generality, that \( 0 < \lambda_1 < \lambda_2 \). In the calculations that follow, we will demonstrate that kinks emerge from the interaction and undergo phase shifts as in the scalar case. In addition, however, we will see that there are changes of polarization, in other words, amplitudes may also change as a result of the interaction.

First we fix \( \theta_1 \). Then \( \theta_2 = \theta_1 + (1/\lambda_2^2 - 1/\lambda_1^2) t \) and hence as \( t \to -\infty \),

\[
X \sim I + \frac{B_1}{g_1} = I + \frac{ir_1 P_1}{\exp(2\lambda_1 \theta_1) - \frac{ir_1}{2}}.
\]
As $t \to +\infty$, using the invariance of $W$, we obtain
\[
X \sim I + \frac{2B_2i}{r_2} - \frac{(\lambda_1 + \lambda_2)(2\lambda_1 B_1 B_2 + 2\lambda_2 B_2 B_1 - (\lambda_1 + \lambda_2)r_2 B_1 i - 4\alpha\lambda_1\lambda_2 r_1 B_2 i)}{r_2(\lambda_1 + \lambda_2)^2 g_1 i - 2\alpha\lambda_1\lambda_2 r_1 r_2} - 4\alpha\lambda_1\lambda_2 r_1 B_2 i \left( \frac{I + 2B_2i}{r_2} \right) \\
\times \left( I + \frac{2B_2i}{r_2} \right) \sim I + \frac{i\hat{r}_1 \hat{P}_1}{e^{2\lambda_1 \theta_1 - i\frac{\hat{r}_1^2}{2}}},
\]
where
\[
\hat{r}_1 = \frac{r_1(\hat{p}_1, \hat{q}_1)}{(p_1, q_1)}, \quad \hat{P}_1 = \frac{\hat{p}_1 \otimes \hat{q}_1}{(\hat{p}_1, \hat{q}_1)}.
\]
\[
\hat{p}_1 = p_1 - \frac{2\lambda_2}{(\lambda_1 + \lambda_2)} \left( \frac{p_1, q_2}{p_2, q_2} \right) p_2, \quad \hat{q}_1 = q_1 - \frac{2\lambda_2}{(\lambda_1 + \lambda_2)} \left( \frac{p_2, q_1}{p_2, q_2} \right) q_2.
\]
This shows that
\[
W \sim 4P_1 \arctan(\exp(2\lambda_1(\theta_1 - \phi_1^-))), \quad t \to -\infty,
\]
\[
W \sim 4\hat{P}_1 \arctan(\exp(2\lambda_1(\theta_1 - \phi_1^+))), \quad t \to +\infty,
\]
where
\[
\phi_1^- = \frac{1}{2\lambda_1} \log \frac{r_1}{2}, \quad \phi_1^+ = \frac{1}{2\lambda_1} \log \frac{r_1(\hat{p}_1, \hat{q}_1)}{2(p_1, q_1)}.
\]
Similarly, fixing $\theta_2$, we have
\[
X \sim I + \frac{i\hat{r}_2 \hat{P}_2}{\exp(2\lambda_2 \theta_2) - \frac{i\hat{r}_2}{2}}, \quad t \to -\infty,
\]
\[
X \sim I + \frac{B_2}{q_2} = 1 + \frac{i\hat{r}_2 P_2}{\exp(2\lambda_2 \theta_2) - \frac{i\hat{r}_2}{2}}, \quad t \to +\infty,
\]
where
\[
\hat{r}_2 = \frac{r_2(\hat{p}_2, \hat{q}_2)}{(p_2, q_2)}, \quad \hat{P}_2 = \frac{\hat{p}_2 \otimes \hat{q}_2}{(\hat{p}_2, \hat{q}_2)},
\]
\[
\hat{p}_2 = p_2 - \frac{2\lambda_1}{(\lambda_1 + \lambda_2)} \left( \frac{p_2, q_1}{p_1, q_1} \right) p_1, \quad \hat{q}_2 = q_2 - \frac{2\lambda_1}{(\lambda_1 + \lambda_2)} \left( \frac{p_1, q_2}{p_1, q_1} \right) q_1.
\]
Then
\[
W \sim 4\hat{P}_2 \arctan(\exp(2\lambda_2(\theta_2 - \phi_2^-))), \quad t \to -\infty
\]
\[
W \sim 4P_2 \arctan(\exp(2\lambda_2(\theta_2 - \phi_2^+))), \quad t \to +\infty
\]
The above calculations show that $W(x, t)$ decomposes into the sum of two kink solutions as $t \to \pm \infty$ and the $j$th kink solution propagates with the speed $1/\lambda_j^2$. The phase shifts $\Delta_j = \phi_j^+ - \phi_j^-$ for the kink solutions are

$$\Delta_1 = \frac{1}{2\lambda_1} \log \beta, \quad \Delta_2 = -\frac{1}{2\lambda_2} \log \beta,$$

where

$$\beta = 1 - \frac{4\lambda_1 \lambda_2 \alpha}{(\lambda_1 + \lambda_2)^2}.$$

**Remark 5.2.** In a similar way to the matrix KdV equation in Goncharenko (2001), we find that the matrix amplitude of the first kink solution changes from $4P_1$ to $4P_1$ and the matrix amplitude of the other one changes from $4P_2$ to $4P_2$ as $t$ changes from $-\infty$ to $+\infty$. If $(p_1, q_1) = 0$ ($P_2 P_1 = 0$) or $(p_2, q_2) = 0$ ($P_1 P_2 = 0$), there is no phase shift; however, the amplitudes may change. In the case that both $P_1 P_2 = P_2 P_1 = 0$, there is neither phase shift nor change in amplitude and so the kink solutions have trivial interactions.

To illustrate the above, we will consider the case $d=2$, i.e. the 2×2 matrix sine-Gordon equation. We choose $\lambda_1 = 1, \lambda_2 = 2, r_1 = r_2 = 1$ and

$$P_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{3} \begin{pmatrix} 2 & -2 \\ -1 & 1 \end{pmatrix}.$$ 

The analysis above shows that $P_1^- = P_1, P_2^+ = P_2$ and

$$P_1^+ = \frac{1}{3} \begin{pmatrix} -1 & 1 \\ -4 & 4 \end{pmatrix}, \quad P_2^- = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.$$ 

For convenience, rather than plotting the kink $W$ given by (5.9), we plot the derivative $W_x$ and refer to it as a soliton. In figure 1, the asymptotic forms of the matrix soliton 1 are plotted as $t \to \pm \infty$. The first plot exhibits the amplitudes
given by $P_1^-$ and the second, those of $P_1^+$. Similarly, in figure 2, we show the same plots for soliton 2.

6. Conclusions

In this paper, we have considered a generalized non-Abelian Toda lattice and presented quasi-wronskian and quasi-grammian solutions obtained by means of Darboux transformations and binary Darboux transformations, respectively. Then we imposed a 2-periodic reduction on the non-Abelian Toda lattice to derive a non-commutative sine-Gordon equation. Using a method similar to that developed in Goncharenko (2001) for the matrix KdV equation, we obtained kink solutions for the matrix sine-Gordon equation from the quasi-grammian solutions of the non-Abelian Toda lattice. Then we investigated the interaction properties of matrix kink solutions. It is known (Veselov 2003) that the change of matrix amplitude of solitons for the matrix KdV equation gives rise to a Yang–Baxter map. It would be interesting to investigate whether there is a similar result for the matrix sine-Gordon equation.

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