Causality and charged spin-2 fields in an electromagnetic background

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We show that, contrary to common belief, the propagation of a spin-2 field in an electromagnetic background is causal. The proof will be given in the Fierz formalism which, as we shall see, is free of the ambiguity present in the more usual Einstein representation.

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I. INTRODUCTION

The interaction of fields with spin greater than 1 (particularly $s = 2$ fields) with a fixed gravitational or electromagnetic background has attracted a lot of attention in the last three decades. There are at least two reasons for this interest. First, on the theoretical side, interacting particles with $s > 1$ present features that are absent in Electromagnetism. In this regard, two items are specially important: the consistency of the equations of motion (EOM), and the causality of the propagation. A consistent set of EOM (and of the constraints derived from them) has been obtained quite a long time ago for free fields (see for instance [1]), but the consistency of the system is usually broken when interactions are introduced. Interactions can also excite new degrees of freedom, absent in the free-field case. This may lead to violation of causality. In fact, it has been stated that causality can be violated even when the higher spin fields have the correct number of degrees of freedom, and that in order to ensure consistency, some restrictions must be made on the kind of interaction [2].

Second, on a more phenomenological vein, particles with spin 2 are known to exist as resonances, and it is desirable to have a theory to describe their interaction with background fields. Yet another reason is furnished by string theory, in which a tower of massive states of Kaluza-Klein type with various spins interact with each other. A particular instance of this scenario is furnished by the so-called bigravity models [3, 4], from which a theory with a massless and a very light graviton can be obtained. It would be very interesting then to have a consistent theory of fields with $s > 1$ (and specifically of $s = 2$ fields) interacting with given backgrounds.

A lot of work has been devoted to the case of a gravitational background. The properties of an $s = 2$ field in this kind of background were studied for instance in [2, 5, 6]. Several interesting new results in this area were obtained in [5]. We shall analyze here instead a spin 2 field in an electromagnetic (EM) background [6]. It is common lore in this situation that massive spin 2 particles propagate acausally. This result was obtained by Kobayashi and Shamaly in the late 70’s [7], and a more recent demonstration has been given by Deser and Waldron [8], both proofs being based on the method of characteristics. The main result we shall present here is that, contrary to the aforementioned claims, a more careful application of the method of characteristics reveals that the propagation of $s = 2$ fields in an EM background is actually causal.

Let us remind the reader that a spin-2 field can be described in two equivalent ways, which we shall call the Einstein representation (ER) and the Fierz representation (FR). The former is a second order representation that uses a symmetric second-order tensor $\varphi_{\mu\nu}$ to represent the field. In the FR [9], this role is played by a third order tensor $F_{\alpha\mu\nu}$, which is antisymmetric in the first pair of indices, and obeys the cyclic identity and a further condition (which will be given in Sect.III) in order to represent a single spin-2 field. The FR is first order. In flat spacetime and in the absence of interactions both representations are equivalent. Nevertheless, in the case an EM background (or in curved spacetime) this is no longer true. As we shall see, in the ER there is an ambiguity which originates in the ordering of the non-commuting covariant derivatives. We shall show that the use of the FR yields instead a non-ambiguous description with the minimal coupling procedure.

Our proof of the causal propagation will be given in the FR. We shall begin by giving in Sect.II a review of the Fierz variables to describe a spin-2 field in Minkowski spacetime (some properties of these variables are discussed in Appendix 1). In Sect.III it will be shown how the minimal coupling in the FR avoids the ambiguity present in the ER. The causality of the propagation will be discussed in Sect.IV. We close with a discussion of the results.

II. SPIN-2 FIELD DESCRIPTION IN THE FIERZ REPRESENTATION

In this section we present a short review of the FR in a Minkowskian background and in the absence of interactions [10]. Let us start by defining a three-index tensor $F_{\alpha\beta\mu}$ which is anti-symmetric in the first pair of indices and obeys the cyclic identity:

$$F_{\alpha\mu\nu} + F_{\mu\alpha\nu} = 0,$$

(1)
\[ F_{\alpha\mu} + F_{\mu\alpha} + F_{\nu\alpha\mu} = 0. \] (2)

The former expression implies that the dual of \( F_{\alpha\mu} \) is trace-free:

\[ \tilde{F}^\alpha_{\mu} = 0, \] (3)

where the asterisk represents the dual operator, defined in terms of \( \eta_{\alpha\beta\mu\nu} \) by

\[ \tilde{F}^\mu_{\alpha} \equiv \frac{1}{2} \eta^{\mu\nu\alpha\beta} {F}^{\nu}_{\beta}. \] (4)

The tensor \( F_{\alpha\mu} \) has 20 independent components. The necessary and sufficient condition for \( F_{\alpha\mu} \) to represent an unique spin-2 field (described by 10 components) is [17]

\[ \tilde{F}^\alpha_{\mu} \equiv 0, \] (5)

which can be rewritten as

\[ F_{\alpha\beta}^{\lambda} ,_{\mu} + F_{\beta\mu}^{\lambda} ,_{\alpha} + F_{\mu\alpha}^{\lambda} ,_{\beta} - \frac{1}{2} \delta^\lambda_\alpha ( F_{\mu\beta} - F_{\beta\mu} ) + \frac{1}{2} \delta^\lambda_\mu ( F_{\alpha\beta} - F_{\beta\alpha} ) = 0. \] (6)

A direct consequence of the above equation is the identity:

\[ F_{\alpha\beta}^{\lambda} ,_{\mu} = 0. \] (7)

We will call a tensor that satisfies the conditions given in the Eqs. (1), (2) and (3) a Fierz tensor.

If \( F_{\alpha\mu} \) is a Fierz tensor, it represents a unique spin-2 field. Condition (4) yields a connection between the ER and the FR: it implies that there exists a symmetric second-order tensor \( \varphi_{\mu\nu} \) such that

\[ 2 F_{\alpha\mu\nu} = \varphi_{\nu\alpha\mu} + ( \varphi_{\alpha\mu} - \varphi^\lambda_{\alpha\lambda} ) \eta_{\mu\nu} - ( \varphi_{\mu\nu} - \varphi^\lambda_{\nu\lambda} ) \eta_{\alpha\mu}. \] (8)

where \( \eta_{\mu\nu} \) is the flat spacetime metric tensor, and the factor 2 in the l.h.s. is introduced for convenience.

The following identity can be proved using the properties of the Fierz tensor:

\[ F_{\alpha\mu} = \varphi_{\alpha\mu} - \varphi^\lambda_{\alpha\lambda} \lambda, \]

where \( F_{\alpha\mu} \equiv F_{\alpha\mu} \eta^{\mu\nu} \). Thus we can write

\[ 2 F_{\alpha\mu\nu} = \varphi_{\nu\alpha\mu} + F_{\alpha\mu\nu}. \] (9)

The divergence of \( F_{\alpha\mu\nu} \) yields the identity:

\[ F_{\alpha\mu\nu} \eta^{\mu\nu} = 0. \] (10)

Indeed,

\[ F_{\alpha\mu\nu} \eta^{\mu\nu} = F_{\alpha\mu\nu}. \] (11)

The first term vanishes identically due to the symmetry properties of the field and the second term vanishes due to equation (12). Using Eqn. (13) the identity which states that the linearized Einstein tensor \( G^{(L)}_{\mu\nu} \) is divergence-free is recovered.

We shall build now dynamical equations for the free Fierz tensor in flat spacetime. Our considerations will be restricted here to linear dynamics [15]. The most general theory can be constructed from a combination of the three invariants involving the field. These are represented by \( A, B \) and \( W \):

\[ A \equiv F_{\alpha\mu\nu} F^{\alpha\mu\nu}, \quad B \equiv F_{\mu\nu} F^{\mu\nu}, \]

\[ W \equiv F_{\alpha\beta\mu\nu} F^{\alpha\beta\mu\nu} = \frac{1}{2} F_{\alpha\beta\mu\nu} F^{\mu\nu\lambda} \eta_{\alpha\beta}^{\mu\nu}. \]

\( W \) is a topological invariant in the linear regime, so we shall use in what follows only the invariants \( A \) and \( B \).

The EOM for the massless spin-2 field in the ER is given by

\[ G^{(L)}_{\mu\nu} = 0. \] (14)

As we have seen above, in terms of the field \( F^{\lambda\mu\nu} \) this equation can be written as

\[ F^{\lambda\mu\nu},_{\lambda} = 0. \] (15)

The corresponding action takes the form

\[ S = \frac{1}{k} \int d^4x (A - B). \] (16)

Note that the Fierz tensor has dimensionality \((\text{length})^{-1}\), which is compatible with the fact that Einstein constant \( k \) has dimensionality \((\text{energy})^{-1} \cdot (\text{length})^{-1}\). From now on we set \( k = 1 \). Then,

\[ \delta S = \int 2 F^{\alpha\mu\nu},_{\alpha} \delta \varphi_{\mu\nu} d^4x. \] (17)

Using the identity

\[ F^{\alpha\mu\nu},_{\alpha} = \frac{1}{2} F^{\alpha\mu\nu},_{\alpha} - \frac{1}{2} G^{(L)}_{\mu\nu}, \]

we obtain

\[ \delta S = \int G^{(L)}_{\mu\nu} \delta \varphi_{\mu\nu} d^4x. \] (18)
where \( G^{(L)}_{\mu\nu} \) is given in Eqn. (10). Thus, the action in Eqn. (14) when written in the ER reads
\[
S = - \int G^{(L)}_{\mu\nu} \phi^{\mu\nu} d^4 x. \tag{19}
\]

Let us consider now the massive case. If we include a mass for the spin 2 field in the FR, the Lagrangian takes the form
\[
\mathcal{L} = A - B - \frac{m^2}{2} (\phi_{\mu\nu} \phi^{\mu\nu} - \phi^2), \tag{20}
\]
and the EOM that follow are
\[
F^\alpha_{(\mu\nu),\alpha} - m^2 (\phi_{\mu\nu} - \eta_{\mu\nu}) = 0, \tag{21}
\]
or equivalently,
\[
G^{(L)}_{\mu\nu} + m^2 (\phi_{\mu\nu} - \eta_{\mu\nu}) = 0. \tag{22}
\]
The trace of this equation gives
\[
F^\alpha_{\alpha,\varphi} + \frac{3}{2} m^2 \varphi = 0, \tag{23}
\]
while the divergence of Eqn. (21) yields
\[
F_{\mu} = 0. \tag{24}
\]
This result together with the trace equation gives \( \varphi = 0 \).

In terms of the potential, Eqn. (28) is equivalent to
\[
\varphi_{,\mu} - \varphi_{,\mu,\varphi} = 0. \tag{29}
\]

It follows that we must have
\[
\phi^{\mu\nu}_{\varphi,\nu} = 0. \tag{30}
\]

Thus we have shown that the original ten degrees of freedom (DOF) of \( F_{\alpha\beta\mu} \) have been reduced to five (which is the correct number for a massive spin-2 field) by means of the five constraints
\[
\phi^{\mu\nu}_{\varphi,\nu} = 0, \quad \varphi = 0. \tag{31}
\]

III. INTERACTION OF THE SPIN-2 FIELD WITH AN ELECTROMAGNETIC FIELD

As discussed for instance in [11], the minimal coupling prescription \( \partial_{\mu} \to \partial_{\mu} - ieA_{\mu} \) is ambiguous in the case of a spin 2 field interacting with an EM field. The origin of this ambiguity is rooted, as in the case of a curved background, in the non-commutativity of the derivative operator, which is manifest from
\[
\varphi_{\alpha\beta\mu\nu} - \varphi_{\alpha\beta\nu\mu} = ieA_{\nu\mu}, \tag{32}
\]
where \( A_{\nu\mu} \) is the EM field, and the semicolon is the covariant derivative \( \partial_{\mu} - ieA_{\mu} \). Let us review the argument in [11], which starts from the free Lagrangian for a charged spin 2 field in the FR:
\[
\mathcal{L} = \frac{1}{2} \phi^{\mu\nu} G^{(L)}_{\mu\nu} + \frac{1}{2} m^2 (\phi^{\mu\nu} \phi_{\mu\nu} - \phi^2). \tag{33}
\]
The EOM that follow from this Lagrangian are
\[
\Box (\phi_{\mu\nu} - \eta_{\mu\nu}\varphi) + \phi_{,\mu\nu} + \eta_{\mu\nu} \varphi_{,\alpha\beta} - \varphi_{,\varphi}\alpha = \frac{m^2}{2} \varphi_{\mu\nu} - \eta_{\mu\nu}\varphi = 0. \tag{34}
\]
It is the term before the mass term of this equation that leads to an ambiguity when minimal coupling is adopted. In [11], a one-parameter family of couplings was introduced, such that
\[
\varphi_{(\nu;\mu)\alpha} \to g \varphi_{(\nu;\mu)\alpha} + (1 - g) \varphi_{(\mu;\nu)\alpha}. \tag{35}
\]
By studying the constraints of the one-parameter theory, it was shown in [11] that the only value of the gyromagnetic factor \( g \) that maintains the correct number of DOF is \( g = 1/2 \). The resulting EOM is
\[
\Box (\phi_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu}\varphi) + \phi_{,\mu\nu} + \eta_{\mu\nu} \varphi_{,\alpha\beta} - \varphi_{,\varphi}\alpha = \frac{1}{2} \varphi_{(\nu;\mu)\alpha} \tag{36}
\]
\[
\phi_{,\nu} \alpha\beta + m^2 (\phi_{\mu\nu} - \eta_{\mu\nu}\varphi) = 0. \tag{37}
\]
Let us see how the minimal coupling procedure affects the equations for the free field in the FR, given in Sect. I First, in the presence of an EM field Eqn. (41) transforms to
\[
\tilde{F}_{\alpha\beta\lambda} = - \frac{1}{2} \eta_{\alpha\beta\lambda}, \tag{38}
\]
From this equation, the tensor \( F_{\alpha\beta\mu} \) can be written as
\[
2F_{\alpha\beta\mu} = \varphi_{(\nu;\mu)\beta} + F_{[\alpha;\eta_{\beta\nu]}, \nu}, \tag{39}
\]
with
\[
F_{\alpha} = \varphi_{\alpha} - \varphi_{,\alpha} \lambda. \tag{40}
\]
If we start with the EOM for the charged spin-2 field in the absence of interactions in the FR (Eqn. 24), and apply the minimal coupling procedure, we get
\[
\Box (\phi_{\mu\nu} - \eta_{\mu\nu}\varphi) + \phi_{,\mu\nu} + \eta_{\mu\nu} \varphi_{,\alpha\beta} - \varphi_{,\varphi}\alpha = 0. \tag{41}
\]
There is no ambiguity then in the minimal substitution. In fact, using Eqns. (38) and (40) in Eqn. (39), we get the equation derived in [11] with \( g = 1/2 \) (i.e. Eqn. (24)). In other words, the Fierz representation automatically gives a theory with the correct number of degrees of freedom when the minimal coupling scheme is used.

Let us now give two constraints that follow from Eqn. (39). If we take the divergence on the index \( \mu \) in Eqn. (39), we get
\[
- \frac{3}{2} \eta_{\alpha\beta\mu} F_{\alpha\beta} + \frac{1}{2} \eta_{\alpha\beta\mu} \varphi_{\mu} + m^2 F_{\nu} = 0, \tag{42}
\]
for a sourceless EM field \( A_{\mu\nu} \). Notice that in this constraint only first derivatives of \( \varphi_{\mu\nu} \) appear (in \( F_{\mu} \)). Taking the divergence of Eqn. (42) we obtain
\[
\eta_{\alpha\beta\mu} F_{\alpha\beta} = 3 \left( m^4 + \frac{1}{2} e^2 A^2 \right) \varphi_{\mu} + \frac{3}{2} e^2 A_{\mu} A^{\beta} \varphi_{\alpha\beta} = 0, \tag{43}
\]
where \( A^2 = A_{\alpha \beta} A^{\alpha \beta} \). Eqs. (33) and (34) correspond to the free-case equations (25). They reduce the number of DOF to five, and are necessary for the compatibility of the system. Note that a remarkable cancellation has happened: no second derivatives of \( \varphi_{\mu \nu} \) are present in this second constraint. It is precisely the absence in the constraints of second derivatives w.r.t. time that guarantees that only physical degrees of freedom propagate. Armed with the EOM (32), we shall study in the next section the causal properties of massive spin 2 particles in an EM background.

**IV. CAUSALITY IN SPIN 2 FIELDS INTERACTING WITH AN EM BACKGROUND**

In this section it will be shown, using the FR, that the propagation of a massive spin 2 field in an EM background is causal. We shall recourse to the well-known method of the characteristics, which is in fact equivalent to the infinite-momentum limit of the eikonal approximation [15]. To set the stage for the calculation, let us put together the equations we shall use. They are the EOM (32), and the two constraints

\[
F^\alpha_{(\mu \nu);\alpha} - m^2 (\varphi_{\mu \nu} - \varphi \eta_{\mu \nu}) = 0, \quad (35)
\]

its trace,

\[
F^\alpha_{\alpha} - \frac{3}{2} m^2 \varphi = 0, \quad (36)
\]

and the two constraints

\[
- \frac{3}{2} i e A^\alpha_{\mu \nu} F_{\alpha \mu \nu} + \frac{1}{2} i e A_{\nu, \alpha} \varphi^\alpha + m^2 F_\nu = 0, \quad (37)
\]

and

\[
 i e A^{\alpha \beta}_{\alpha \beta} F_{\alpha \beta} - \frac{3}{2} \left( m^4 - \frac{1}{2} c^2 A^2 \right) + \frac{3}{2} c^2 A^\mu A^\beta \varphi_{\alpha \beta} = 0. \quad (38)
\]

To these, we must add some properties of the Fierz tensor:

\[
F_{\alpha \mu \nu} + F_{\mu \alpha \nu} = 0, \quad (39)
\]

\[
F_{\alpha \mu \nu} + F_{\mu \nu \alpha} + F_{\nu \alpha \mu} = 0, \quad (40)
\]

and

\[
F^{\alpha (\beta \lambda)}_{\alpha} = - \frac{1}{2} i e A^{\mu (\beta \lambda) \nu}. \quad (41)
\]

Let \( \Sigma \) be the surface of discontinuity defined by the equation

\[
\Sigma(x^\mu) = \text{constant}.
\]

The discontinuity of a function \( J \) through \( \Sigma \) will be represented by \([J]_\Sigma\), and its definition is

\[
[J]_\Sigma = \lim_{\delta \to 0^+} \left( [J]_{\Sigma + \delta} - [J]_{\Sigma - \delta} \right).
\]

We shall assume that \( F_{\alpha \mu \nu} \) is continuous through the surface \( \Sigma \) but its first derivative is not:

\[
[F_{\alpha \mu \nu}]_\Sigma = 0, \quad (42)
\]

From the discontinuity of the EOM (35) we learn that

\[
f^\mu (\alpha \beta) k_\mu = 0. \quad (43)
\]

Taking the derivative of Eqn. (40) results in

\[
k^\alpha f_{\alpha \mu \nu} + k^\beta f_{\nu \alpha \mu} + k^\gamma f_{\mu \nu \alpha} = 0. \quad (44)
\]

This equation, together with Eqs. (42) and (43) tells us that the contraction of \( k \) with \( f \) is zero on any index of \( f \). The trace equation (41) gives

\[
f^\mu k_\mu = 0. \quad (45)
\]

Eqn. (29) can be written as

\[
F_{\alpha \mu \nu} + F_{\beta \mu \nu} + F_{\mu \alpha \beta} + F_{\mu \nu \alpha} - \frac{1}{2} \delta^\lambda_{\alpha \beta} F_{[\mu, \beta]} + \frac{1}{2} \delta^\lambda_{\beta, \alpha} F_{[\alpha, \mu]} = \frac{1}{2} i e A^\nu (\varphi^\lambda \varphi)_{\nu}. \quad (46)
\]

Notice that the r.h.s. is continuous. Taking the discontinuity of this equation, multiplying by \( k^\mu \) and \( f_\lambda \), and using Eqn. (48), we get that

\[
f_{\alpha \beta \lambda} f^\lambda k^2 = 0. \quad (47)
\]

We shall assume for the time being that \( k^2 \neq 0 \). The discontinuity of the derivatives of the constraints Eqs. (27) and (35) give

\[
A_{\alpha \beta, \mu} f^\alpha = 0, \quad (48)
\]

\[
\frac{3}{2} i e A^{\alpha \beta} f_{\alpha \beta \mu} - m^2 f_\mu = 0. \quad (49)
\]

From Eqs. (47) and (49) we deduce that

\[
f_\mu f^\mu = 0. \quad (50)
\]

Note that all the equations that resulted from taking the discontinuity (i.e., Eqs. (43) - (45) and (47) - (50)) depend only on \( f_{\mu \nu \alpha} \) and its trace. Taking the discontinuity of the derivative of \( F_{\mu \nu \alpha} \) we get that

\[
[F_{\alpha \mu \nu, \lambda}]_\Sigma = f_{\alpha \mu \nu} k_\lambda
\]

where

\[
2 f_{\alpha \mu \nu} = \epsilon_{\nu \alpha} k_\mu - \epsilon_{\mu \alpha} k_\nu + f_\lambda \eta_{\mu \nu} - f_\mu \eta_{\nu \alpha}, \quad (51)
\]

and

\[
f_\alpha = \epsilon k_\alpha - \epsilon_\alpha^\beta k_\beta. \quad (52)
\]

Consequently the equations that follow from taking the discontinuity are invariant under the transformation

\[
\epsilon_\mu^\nu = \epsilon_\mu + \Lambda k_\mu k_\nu, \quad (53)
\]
where $\Lambda$ is an arbitrary function of the coordinates. This equation implies that

$$\epsilon' = \epsilon + \Lambda k^2. \quad (53)$$

Now, this symmetry implies that observable quantities depend on the gauge choice unless $k^2 = 0$. Let us take for instance $X_\mu = \epsilon_{\mu\nu}k^\nu$. From Eqn. (15),

$$X'_\mu k^\mu = X_\mu k^\mu + (k^2)^2. \quad (54)$$

It follows that the projection of the polarization in the direction of $k_\mu$ is not a gauge-invariant quantity. Another quantity that is not gauge invariant is the norm $X_\mu X^\mu$, which transforms as

$$X'^2 = X^2 + (2\epsilon + \Lambda)(k^2)^2. \quad (55)$$

We see that a spacelike (actually, a non-null) $k_\mu$ entails an unacceptable dependence of observable quantities with the gauge choice. This dependence disappears only when $k^2 = 0$. Summing up, the propagation of spin two fields in an electromagnetic background is causal, with the characteristics that used in Electromagnetism, and then we can profit from work already done in this area for instance in construction nonlinear theories for the spin 2 field. More importantly, the use of the Fierz representation has paved the way to a clean proof of the causality in the propagation of spin 2 fields in the presence of an electromagnetic field, thus showing that previous claims about noncausal propagation were mistaken.

To close, we would like to point out that in the issue of causality, the use of the Fierz representation is not mandatory. A closer look to the relevant equations in the Einstein representation (for instance Eqs. (64) and (66) in [12], without choosing a timelike $k_\mu$) shows that the gauge invariance given by Eqn. (59) is present there too. However, it is important to remark that the gauge invariance of the equations for the discontinuity (which went unnoticed before) is clearly displayed in the Fierz representation.

## Appendix 1

We shall be concerned here with the gauge invariance of Eqn. (59) under the map

$$\varphi_\mu \rightarrow \tilde{\varphi}_\mu = \varphi_\mu + \Lambda_{\mu\nu} + \Lambda_{\nu\mu}. \quad (56)$$

Although the field $F_{\alpha\beta\mu}$ is invariant under this map only if the vector $\Lambda_\mu$ is a gradient, it is important to realize that the dynamics is invariant even when $\Lambda$ is not a gradient. Indeed, we have

$$\delta F_{\alpha\beta\mu} \equiv \tilde{F}_{\alpha\beta\mu} - F_{\alpha\beta\mu} = \frac{1}{2} X_{\alpha\beta} \cdot \lambda, \quad (57)$$

where

$$X_{\alpha\beta} \cdot \lambda \equiv (\Lambda_{\alpha\beta} - \Lambda_{\beta\alpha})\delta^2 + [\Lambda_{\sigma\alpha}\delta_{\beta}^{\lambda} - \Lambda_{\alpha}^{\lambda}]\eta_{\beta\mu} - [\Lambda_{\sigma\beta}\delta_{\alpha}^{\lambda} - \Lambda_{\beta}^{\lambda}]\eta_{\alpha\mu} \quad (58)$$

Then it follows that

$$2\delta F_\alpha = X_{\alpha} \cdot \lambda, \quad (59)$$

with

$$X_{\alpha} \cdot \lambda \equiv X_{\alpha\beta} \cdot \lambda. \quad (60)$$

As a consequence of this transformation, the invariants $A$ and $B$ change in the following way:

$$\delta A = F^{\alpha\beta\mu} X_{\alpha\beta} \cdot \lambda, \quad \delta B = F^{\alpha} X_{\alpha} \cdot \lambda. \quad (61)$$

Note that $X_{\alpha\beta} \cdot \lambda$ is not cyclic in the indices $(\alpha\beta\mu)$, but the quantity $X_{\alpha\beta} \cdot \lambda$ has such cyclic property:

$$X_{\alpha\beta} \cdot \lambda + X_{\beta\alpha} \cdot \lambda + X_{\alpha\beta} \cdot \lambda = 0. \quad (62)$$

It is straightforward to show the associated identities:

$$X_{\alpha\beta} \cdot \lambda = 0 \quad (63)$$

Thus,

$$\delta A = [\varphi^{\mu\alpha\beta} + F^{\alpha} \eta_{\mu\beta}] X_{\alpha\beta} \cdot \lambda, \quad (64)$$

or, equivalently,

$$\delta A = \varphi^{\mu\alpha\beta} X_{\alpha\beta} \cdot \lambda + F^{\alpha} X_{\alpha} \cdot \lambda. \quad (65)$$

Then,

$$\delta(A - B) = \varphi_{\mu\alpha\beta} X^{\alpha\beta} \lambda \cdot \lambda, \quad (66)$$

and

$$\int \varphi_{\mu\alpha\beta} X^{\alpha\beta} \lambda \cdot \lambda = \int d\nu - \int \varphi_{\mu\alpha} X^{\alpha\beta} \lambda \cdot \beta, \quad (67)$$

so that, because of (61),

$$\int \delta(A - B) = 0. \quad (68)$$

This shows that the transformation

$$F_{\alpha\beta\mu} \rightarrow F_{\alpha\beta\mu} + X_{\alpha\beta} \cdot \lambda, \quad (69)$$

for $X_{\alpha\beta} \cdot \lambda$ given in equation (59), leaves the dynamics invariant.
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[1] See L. P. S. Singh and C. R. Hagen, Phys. Rev D9, 898 (1974) for the bosonic case.
[2] See for instance I. L. Buchbinder, D. M. Gitman, and V. D. Pershin, Nucl. Phys.B584, 615 (2000).
[3] T. Damour and I. Kogan, Phys. Rev. D 66, 104024 (2002).
[4] T. Damour, I. Kogan, and A. Papazoglou, Phys. Rev. D 66, 104025 (2002).
[5] C. Aragone and S. Deser, Il Nuovo Cimento vol. 3, n. 4, 709 (1971).
[6] I. L. Buchbinder, D. M. Gitman, V. D. Pershin, Phys. Lett. B492, 161 (2000).
[7] S. Deser and A. Waldron, Nucl. Phys. B631, 369 (2002).
[8] M. Novello and R. P. Neves, Class. Quant. Grav. 19, 5335 (2002). Apparent mass of the graviton in a DeSitter Background, to be published in Class. Quantum Grav., gr-qc/0210058.
[9] The coupling of the spin 2 field with an EM background was analyzed previously by W. Tait, Phys. Rev D5, 3272 (1972), and C. R. Hagen, Phys. Rev D6, 984 (1972).
[10] M. Kobayashi and A. Shamaly, Phys. Rev D17, 126 (1978), and Prog. Theor. Phys.61, 656 (1979).
[11] S. Deser and A. Waldron, Nucl. Phys.B631, 369 (2002).
[12] M. Fierz and W. Pauli, Proc. Roy. Soc. (Lond.) A 175, 211 (1939), Helvetica Physica Acta 12, 297 (1939).
[13] A particular case of non-linear dynamics was studied in Propagation of perturbations in non-linear spin-2 theories, S. E. Perez Bergliaffa, contribution to M. Novello’s Festschrift, gr-qc 0302031.
[14] C. Aragone and S. Deser, Phys. Lett. 86B, 161 (1979); Il Nuovo Cimento 57B, 33 (1980).
[15] Leçons sur la propagation des ondes et les equations de l’hydrodynamique, J. Hadamard, J. Hadamard Ed. Dunod, Paris, 1958.
[16] Throughout this article we will use the signature (+−−−), and the notation $A(\alpha B_\beta) = A_\alpha B_\beta + A_\beta B_\alpha$, $A(\alpha B_\beta) = A_\alpha B_\beta - A_\beta B_\alpha$.
[17] Note that this condition is analogous to that necessary for the existence of a potential $A_\mu$ for the EM field, given by $A_\mu \gamma_\mu \gamma_\alpha = 0$.
[18] Notice that this, as a direct calculation shows, is not a symmetry of the massless theory in the presence of a nonzero background.