Frames in Pretriangulated Dg-Categories

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Abstract

Triangulated categories arising in algebra can often be described as the homotopy category of a pretriangulated dg-category, a category enriched in chain complexes with a natural notion of shifts and cones that is accessible with all the machinery of homological algebra. Dg-categories are algebraic models of ∞-categories and thus fit into a wide ecosystem of higher-categorical models and translations between them. In this paper we describe an equivalence between two methods to turn a pretriangulated dg-category into a quasicategory.

The dg-nerve of a dg-category is a quasicategory whose simplices are coherent families of maps in the mapping complexes. In contrast, the cycle category of a pretriangulated category forgets all higher-degree elements of the mapping complexes but becomes a cofibration category that encodes the homotopical structure indirectly. This cofibration category then has an associated quasicategory of frames in which simplices are Reedy-cofibrant resolutions.

For every simplex in the dg-nerve of a pretriangulated dg-category we construct such a Reedy-cofibrant resolution and then prove that this construction defines an equivalence of quasicategories which is natural up to simplicial homotopy. Our construction is explicit enough for calculations and provides an intuitive explanation of the resolutions in the quasicategory of frames as a generalisation of the mapping cylinder.

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1 Introduction

Homotopical structure arises in various subfields of mathematics, which has lead to the development of a multitude of approaches to abstract homotopy theory. These approaches make different tradeoffs and are each accessible to respective flavours of either concrete calculation or high-level constructions. By considering models of a single theory in a variety of these frameworks, manipulations can be performed in the context where they are most natural.

Differentially graded categories are categories that are enriched in chain complexes. The 0-dimensional cycles in the mapping complexes of a dg-category $\mathcal{C}$ are the maps of the underlying ordinary category $\mathcal{C}_0$, while the 1-dimensional chains function as homotopies
and the higher-degree chains as coherence data. A pretriangulated dg-category additionally satisfies some representability properties that allow the construction of cones and shifts, making the underlying category into a triangulated category. Many triangulated categories in algebra are induced in this way by a pretriangulated dg-category [BK91].

Coherent families of higher-degree maps in the mapping complexes of a dg-category \( C \) organise into a quasicategory \( N_{dg}(C) \) called the dg-nerve [Fao17 Lur17]. When \( C \) is pretriangulated, the cycle category \( C_0 \) can also be seen as a cofibration category [Bro73 Sch12] which captures the homotopical structure in form of two distinguished classes of maps called weak equivalences and cofibrations. This cofibration category \( C_0 \) can in turn be made into a quasicategory \( \mathbb{N}_f(C_0) \) by a construction called the quasicategory of frames [Szn14], based on resolutions with homotopical Reedy cofibrant diagrams.

While the transition from a pretriangulated dg-category \( C \) to its cycle category \( C_0 \) forgets all higher-degree maps, the shifts and cones guarantee that no relevant information is lost. This can already be seen via an abstract argument: The dg-nerve of the dg-category of chain complexes is the \( \infty \)-localization of the underlying ordinary category at the weak homotopy equivalences [Lur17 Prop. 1.3.4.5], which generalizes to arbitrary pretriangulated dg-categories with slight modifications. Since the quasicategory of frames also implements this \( \infty \)-localization [KSI15], both of these nerve constructions translate pretriangulated dg-categories into equivalent quasicategories.

In this paper we construct a simplicial map \( B : N_{dg}(C) \to \mathbb{N}_f(C_0) \) that directly exhibits this equivalence for any pretriangulated dg-category \( C \). Objects in the resolution diagrams produced by this map are immediate generalizations of the mapping cylinder of chain complexes that simultaneously accomodate multiple maps with non-trivial coherence structure and resolutions of the individual objects. This way we do not only establish the equivalence of the two models, but provide an intuitively accessible algebraic motivation for the simplices of the quasicategory of frames.

After reviewing the theory of cofibration categories in Section 2 and pretriangulated dg-categories in Section 3, we proceed as follows: In Section 4, we define a Reedy cofibrant diagram \( B(X, f) : D[n] \to C_0 \) for any \( n \)-simplex \((X, f)\) of the dg-nerve \( N_{dg}(C) \) of a pretriangulated dg-category \( C \). In Section 5 we see that this diagram is also homotopical and thus is an \( n \)-simplex of the quasicategory of frames \( \mathbb{N}_f(C_0) \). In Section 6, we show that \( B \) is compatible with the simplicial structure and hence defines a functor of quasicategories. We conclude in Section 7 by proving that this functor is an equivalence.

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Notation The category of finite ordinals \( \Delta \) consists of objects given by the sets \([n] = \{0, \ldots, n\}\) for all natural numbers \( n \in \mathbb{N} \), together with the order preserving maps as morphisms. Where unambiguous, we also denote a morphism \( f : [n] \to [m] \) as a sequence by \((f(0), \ldots, f(n))\) and write \( f \cdot g \) for the concatenation of two sequences \( f \) and \( g \).

We use the theory and notation for quasicategories as discussed in [Lur09]. In particular \( \text{Map}_C^R \) denotes the space of right morphisms in a quasicategory \( C \), and \( K \star L \) the join of two simplicial sets \( K \) and \( L \).

A graded map \( f : X \to Y \) between chain complexes of degree \( |f| \) is a family of maps \( f_n : X_n \to Y_{n+|f|} \) of abelian groups for every \( n \in \mathbb{N} \). For graded maps \( f : X \to X' \) and \( g : Y \to Y' \) between chain complexes and homogeneous elements \( x \in X, y \in Y \), we use the sign conventions

\[
(f \otimes g)(x \otimes y) = (-1)^{|x||y|} f(x) \otimes g(y),
\]

\[
d(x \otimes y) = dx \otimes y + (-1)^{|x|}x \otimes dy.
\]
In some categories the isomorphisms do not capture the appropriate notion of equivalence of the idealised concepts that the category was intended to model. Prominently, this is the case for the categories of topological spaces or chain complexes as models for homotopy types, which are detected by the weak homotopy equivalences and quasi-isomorphisms, respectively.

Generalising from this, abstract homotopy theory starts with a category in which some morphisms are marked as weak equivalences and localises the category such that these morphisms become isomorphisms. It then provides tools to study the localised category, which often behaves dramatically differently than the original. Homotopical categories \cite{dwyer2005} impose some additional condition on the chosen weak equivalences that makes the theory better behaved:

**Definition 2.1.** A **homotopical category** is a category $\mathcal{C}$ together with a collection of morphisms called **weak equivalences** that contains all identity morphisms and satisfies the 2-out-of-6 property: If

$$W \overset{f}{\longrightarrow} X \overset{g}{\longrightarrow} Y \overset{h}{\longrightarrow} Z$$

is a composable sequence in $\mathcal{C}$ where $h \circ g$ and $g \circ f$ are weak equivalences, then $f$, $g$, $h$, and $h \circ g \circ f$ are weak equivalences as well. A functor between homotopical categories is homotopical if it preserves weak equivalences.

The data of a homotopical category already suffices to define all usual notions of abstract homotopy theory, such as homotopy categories, derived functors, and homotopy limits and colimits \cite{riehl2014}. In practice, it is worthwhile to consider homotopical categories with additional structure that makes calculations much more feasible. Quillen’s model categories \cite{quillen1967} equip homotopical categories with two extra collections of morphisms, called the fibrations and cofibrations, that interact with each other and the weak equivalences via weak factorisation systems. Brown’s categories of fibrant objects \cite{brown1973} are a weakening of this theory whose axioms are both easier to verify and applicable to more examples, while retaining some of the power of model categories. In this paper, we will consider the dual of this notion:

**Definition 2.2.** A **cofibration category** is a homotopical category $\mathcal{C}$ equipped additionally with a class of morphisms called **cofibrations** (where an **acyclic cofibration** is a morphism that is both a cofibration and a weak equivalence) such that:

1. All isomorphisms are acyclic cofibrations. Cofibrations are stable under composition.
   The category $\mathcal{C}$ has an initial object and every morphism from the initial object is a cofibration.
2. Cofibrations and acyclic cofibrations are closed under pushouts.
3. Every morphism in $\mathcal{C}$ can be factored as the composite of a cofibration followed by a weak equivalence.

The subcategory of cofibrant objects of a model category canonically becomes a cofibration category, but conversely it is not always possible or convenient to find a model structure that gives rise to a particular cofibration category of interest; see for example \cite[Section 1.4]{szumilo2014}.

Cofibration categories are models of finitely complete homotopy theories \cite{szumilo2014} and admit a theory of Reedy cofibrant diagrams, with direct categories playing the role of Reedy categories to compensate for the lack of fibrations. We replicate a few definitions and lemmas that are used in the main part of the thesis; for a full discussion see \cite{szumilo2014} and \cite{riehl2014}.

**Definition 2.3.**

1. A category $I$ is **direct** if it admits a functor $\text{deg} : I \to \mathbb{N}$ that reflects identities, where $\mathbb{N}$ is equipped with the usual order.
Let $I$ be a direct category.

2. Let $i \in I$, then the latching category $\partial(I \downarrow i)$ is the full subcategory of the slice category $I \downarrow i$ without the object $id_i$.
3. Let $i \in I$ and $X : I \to C$, then the latching object $L_i X$ is (if it exists) a colimit of

$$
\partial(I \downarrow i) \xrightarrow{\text{source}} I \xrightarrow{X} C.
$$

The latching map is the canonical map $L_i X \to X$ induced by the inclusion $\partial(I \downarrow i) \hookrightarrow (I \downarrow i)$.

4. Let $C$ be a cofibration category. A diagram $X : I \to C$ is Reedy cofibrant if for every $i \in I$ the latching object $L_i X$ exists and the latching map is a cofibration.
5. Let $C$ be a cofibration category. A natural transformation $f : X \to Y$ between Reedy cofibrant diagrams $X \to Y$ is a Reedy cofibration if for all $i \in I$ the induced map

$$
X(i) \sqcup_{L_i X} L_i Y \longrightarrow Y(i)
$$

is a cofibration.
6. A map $I \to J$ of (small) categories is a sieve if it is injective on objects, fully faithful and closed under precomposition.

We will use the following two lemmas that let us manipulate Reedy cofibrant diagrams:

**Lemma 2.4 ([RVY14 Theorem 9.3.8(1a)])**. Let $I$ be a homotopical direct category with finite latching categories and $C$ a cofibration category. Then the category $\mathcal{C}_R^I$ of homotopical Reedy cofibrant diagrams $I \to C$ and natural transformations is a cofibration category with levelwise weak equivalences and Reedy cofibrations.

**Lemma 2.5 ([KS15 Lemma 2.18])**. Let $I \hookrightarrow J$ be a sieve of direct categories and let $X : J \to C$ be a diagram such that the restriction $X|I$ is Reedy cofibrant. Then there exists a Reedy cofibrant diagram $\tilde{X} : J \to C$ together with a weak equivalence $\tilde{X} \to X$ whose restriction to $I$ is the identity map. In particular $\tilde{X}|I = X|I$.

To any cofibration category $C$ one can associate a quasicategory by a construction called the quasicategory of frames $N_I(C)$, built on homotopical Reedy-cofibrant diagrams of the following shape:

**Definition 2.6 ([Szu14])**. Let $I$ be a homotopical category, then $DI$ is the category with objects given by functors $[m] \to I$ and morphisms between $\varphi : [m] \to I$ and $\psi : [n] \to I$ given by injective maps $\sigma : [m] \hookrightarrow [n]$ such that $\varphi = \psi \circ \sigma$. The forgetful functor $DI \to I$ sends $x : [n] \to I$ to $x(n)$ and creates weak equivalences in $DI$, making $DI$ into a homotopical category.

For any homotopical category $I$, $DI$ is straightforwardly a direct category by sending $[n] \to I$ to $n$. The latching category $\partial(DI \downarrow i)$ for some object $i : [n] \to I$ consists of restrictions of $i$ to proper, non-empty subsets of $[n]$. Hence $DI$ has finite latching categories and for any homotopical functor $f : I \to J$ the functor $DF : DI \to DJ$ induces isomorphisms on latching categories.

**Definition 2.7 ([Szu14])**. Let $C$ be a cofibration category. The quasicategory of frames $N_I(C)$ is the simplicial set where $N_I(C)_n$ is the set of homotopical, Reedy cofibrant diagrams $D[n] \to C$ and simplicial maps are induced by precomposition.

**Theorem 2.8 ([Szu14 Theorem 3.3])**. Let $C$ be a cofibration category. Then $N_I(C)$ is a finitely cocomplete quasicategory.
The category $D[0]$ is isomorphic to the category $Δ_2$ of finite ordinals and injective order-preserving maps, and every map in $D[0]$ is a weak equivalence. Hence the 0-simplices of $N_t(\mathcal{C})$ for some cofibration category $\mathcal{C}$ are precisely the homotopy constant Reedy cofibrant cospimisimplicial resolutions $Δ_2 \to \mathcal{C}$. This can be seen as a cospimisimplicial version of a frame in a model category ([Hov07], Chapter 5), which motivates the name.

For a partially ordered set $I$, define $\text{Sd } I$ to be the full subcategory of $DI$ in which objects are the injective order-preserving maps $[k] \to I$. Then we have a useful lifting property:

**Lemma 2.9.** Let $K \subseteq L$ be an inclusion of partially ordered sets, $I \hookrightarrow J$ a sieve of direct homotopical categories with finite latching categories, and $\mathcal{C}$ a cofibration category. Then any square

$$
\begin{array}{ccc}
I & \xrightarrow{X} & C^D_{DL} \\
\downarrow & & \downarrow p \\
J & \xrightarrow{Y} & C^R_{(DK \cup \text{Sd } L)}
\end{array}
$$

in which $X$ and $Y$ are Reedy cofibrant diagrams admits a Reedy cofibrant lift $J \to C^D_{DL}$.

**Proof.** By [Szu14, Lemma 3.19] the restriction map $p$ is an acyclic fibration of cofibration categories, which satisfies the required lifting property by [KS15, Lemma 2.15].

**Lemma 2.10.** Restriction of diagrams in a cofibration category along the canonical map $D([k] \times [m]) \to D[k] \times D[m]$ preserves Reedy cofibrant diagrams.

**Proof.** By [KS15, Lemma 3.11].

The quasicategory of frames $N_t(\mathcal{C})$ has more 0-simplices than the cofibration category $\mathcal{C}$ has objects, since homotopy-constant Reedy cofibrant resolutions are not necessarily unique. However, any two resolutions of the same object (or of a pair of weakly equivalent objects) are equivalent as objects of the quasicategory:

**Lemma 2.11.** Let $\mathcal{C}$ be a cofibration category, $X, Y : D[0] \to \mathcal{C}$ homotopical Reedy cofibrant diagrams such that $X(\langle 0 \rangle)$ and $Y(\langle 0 \rangle)$ are weakly equivalent. Then $X$ and $Y$ are equivalent as 0-simplices of $N_t(\mathcal{C})$.

**Proof.** Let $\widehat{[1]}$ be the homotopical category with underlying category $[1]$, in which every map is a weak equivalence. We consider the weak equivalence $X(\langle 0 \rangle) \to Y(\langle 0 \rangle)$ as a homotopical functor $\widehat{[1]} \to \mathcal{C}$ and precompose with the projection $\text{Sd } \widehat{[1]} \to \widehat{[1]}$ to obtain a homotopical functor $\text{Sd } \widehat{[1]} \to \mathcal{C}$. We then use Lemma 2.9 to modify this to be Reedy cofibrant while keeping the value at $\langle 0 \rangle$ and $\langle 1 \rangle$ the same. The resulting functor $\text{Sd } \widehat{[1]} \to \mathcal{C}$ assembles together with $X$ and $Y$ into a homotopical Reedy cofibrant functor

$$
D(\Delta\{0\} \cup \Delta\{1\}) \cup \text{Sd } \widehat{[1]} \to \mathcal{C},
$$

which by Lemma 2.9 extends to a homotopical Reedy cofibrant functor $D[1] \to \mathcal{C}$. So by [Szu14, Corollary 3.14] we have an equivalence of $X$ and $Y$ as 0-simplices.

### 3 Pretriangulated Dg-Categories and the Dg-nerve

The homotopy category of a stable $(\infty, 1)$-category contains traces of the stable structure in form of a triangulation. Triangulated categories have several technical problems, including for example the non-functoriality of cones, that can be rectified in some cases where the triangulated category admits a dg-enhancement ([BK91]), i.e. if it is the homology category of a dg-category with certain properties that naturally give rise to the triangulation. While many triangulated categories arising from an algebraic context have a dg-enhancement, including for instance the derived category of any abelian category ([CS18], the stable homotopy category does not ([Sch12], p. 14).
Definition 3.1. A **dg-category** is a category enriched in the category of chain complexes, i.e. a dg-category $\mathcal{C}$ consists of:

1. A collection of objects $\text{Ob}\mathcal{C}$,
2. a chain complex $\text{Map}_\mathcal{C}(X,Y)$ for all $X, Y \in \text{Ob}\mathcal{C}$,
3. a chain map $\circ : \text{Map}_\mathcal{C}(Y,Z) \otimes \text{Map}_\mathcal{C}(X,Y) \to \text{Map}_\mathcal{C}(X,Z)$, for all $X, Y, Z \in \text{Ob}\mathcal{C}$, and
4. an element $\text{id}_X \in \text{Map}_\mathcal{C}(X,X)_0$ for all $X \in \text{Ob}\mathcal{C}$,

such that $\circ$ is associative and the elements $\text{id}_X$ are the identity elements of $\circ$. A **dg-functor** $F$ between dg-categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $\mathcal{C} \to \mathcal{D}$ of enriched categories.

The identity maps are automatically cycles, since composition is a chain map and thus

$$d \text{id}_X = d(\text{id}_X \circ \text{id}_X) = d\text{id}_X \circ \text{id}_X + \text{id}_X \circ d\text{id}_X = 2d\text{id}_X.$$

For more details on enriched category theory, see for instance [Rie14, Chapter 3]. Dg-categories can also be defined more generally for chain complexes of $A$-modules for any commutative ring $A$. All constructions in this paper are independent of the ground ring, so we simplify by only considering chain complexes over abelian groups. We will further assume all dg-categories to be small.

The category of (small) chain complexes $\text{Ch}$ is closed symmetric monoidal with the tensor product as the monoidal product. Hence it can be enriched over itself, with mapping chain complexes given by

$$\text{Map}_{\text{Ch}}(X,Y)_n = \prod_{k \in \mathbb{Z}} \text{Ab}(X_k, Y_{k+n})$$

and the differential acting on $f \in \text{Map}_{\text{Ch}}(X,Y)_n$ by

$$df = dy \circ f - (-1)^n f \circ dX.$$ 

A dg-functor $\mathcal{C} \to \text{Ch}$ will also be called a $\mathcal{C}$-module. Such dg-functors take the place of presheaves in ordinary category theory and satisfy an enriched version of the Yoneda lemma.

Definition 3.2. Let $\mathcal{C}$ be a dg-category. The cycle category $\mathcal{C}_0$ of $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and morphisms given by 0-cycles of the mapping complexes:

$$\text{Hom}_{\mathcal{C}_0}(X,Y) = \{ f \in \text{Map}_\mathcal{C}(X,Y)_0 \mid df = 0 \}.$$ 

The homology category $H(\mathcal{C})$ of $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and morphisms given by the homology of the mapping complexes in degree 0:

$$\text{Hom}_{H(\mathcal{C})}(X,Y) = H_0(\text{Map}_\mathcal{C}(X,Y)).$$

We observe that $\mathbb{Z}(0)$, the chain complex that is $\mathbb{Z}$ in degree 0 and vanishes anywhere else, is the tensor unit in the monoidal structure of chain complexes, and chain maps $\mathbb{Z}(0) \to \mathcal{C}$ are precisely the 0-cycles of $\mathcal{C}$, the cycle category is the same as the underlying ordinary category in the sense of enriched category theory.

Definition 3.3. Let $\mathcal{C}$ be a dg-category. A map $f : X \to Y$ in the cycle category $\mathcal{C}_0$ is a weak equivalence, if it becomes an isomorphism in the homology category.

By unrolling the definitions, we can see that a 0-cycle $f : X \to Y$ is a weak equivalence if there exists a 0-cycle $g : Y \to X$ and 1-chains $h \in \text{Map}_\mathcal{C}(X,X)_1, h' \in \text{Map}_\mathcal{C}(Y,Y)_1$ such that

$$f \circ g - \text{id}_Y = dh', \quad g \circ f - \text{id}_X = dh.$$

In particular, weak equivalences in $\text{Ch}$ are precisely the chain homotopy equivalences.
Definition 3.4. Let $C$ be a dg-category. The $n$-translation of an object $X \in C$ for some $n \in \mathbb{Z}$ is an object $X[n] \in C$ representing the $n$-translated mapping complex:

$$\text{Map}_C(X[n], -) \cong \text{Map}_C(X, -)[−n].$$

The mapping cone of a 0-cycle $f : X \to Y$ is an object $\text{Cone}(f)$ representing the mapping cone of the induced map on mapping complexes:

$$\text{Map}_C(\text{Cone}(f), -) \cong \text{Cone}(f^* : \text{Map}_C(X, -) \to \text{Map}_C(Y, -)).$$

A dg-category $C$ is pretriangulated if it has a zero object, it admits translations of all objects, and it admits mapping cones for all 0-cycles.

When $C$ is a pretriangulated dg-category, its homology category $H(C)$ can be canonically triangulated: the shift functor is induced by the translations, and the distinguished triangles come from the image of mapping cone sequences. More importantly for this paper, the cycle category of a pretriangulated dg-category can be made into a cofibration category:

Definition 3.5. Let $C$ be a pretriangulated dg-category. A 0-cycle $i : X \to Y$ is a cofibration if the precomposition map $i^* : \text{Map}_C(Y, -) \to \text{Map}_C(X, -)$ is surjective as a map of $C$-modules and has a representable kernel.

Proposition 3.6 (Sch12 Prop 3.2). Let $C$ be a pretriangulated dg-category. Then the cofibrations and weak equivalences make the cycle category $C_0$ into a cofibration category.

The following lemma allows for particular convenient inductive constructions of objects in pretriangulated dg-categories via extensions of representable $C$-modules:

Lemma 3.7. Let $C$ be a pretriangulated dg-category and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

a short exact sequence of $C$-modules such that $A$ and $C$ are representable. Then $B$ is representable as well. Moreover, the map of $C_0$ represented by $B \to C$ is a cofibration.

Proof. Representability of $B$ is part of the proof of [Sch12 Prop. 3.2]. Since $B \to C$ is surjective and its kernel $A$ is representable, the map represented by $B \to C$ is a cofibration.

Lemma 3.8. Let $C$ be a pretriangulated dg-category and $i : A \to B$ an acyclic cofibration. Then there exists a retraction $p : B \to A$ of $i$ and a homotopy $h : i \circ p \simeq \text{id}$ with $h \circ i = 0$.

Proof. The retraction exists by [Sch12 Prop 3.2] since every object is fibrant. Then

$$(i \circ p - \text{id}) \circ i = i \circ p \circ i - i = 0,$$

so $(i \circ p - \text{id})$ is a cycle in the kernel of the precomposition map $i^*$. Since $i$ is a cofibration, $\ker(i^*)$ is contractible and thus there exists a $h \in \ker(i^*)_1$ such that

$$dh = i \circ p - \text{id}.$$

A dg-category can be transformed into a quasicategory by first considering the mapping complexes as simplicial sets via the Dold-Kan correspondence to obtain a simplicially enriched category and then taking the homotopy coherent nerve. Alternatively, there is a more direct construction:

Definition 3.9 (Fao17). Let $C$ be a dg-category. The $dg$-nerve $N_{dg}(C)$ is the simplicial set where an $n$-simplex consists of a family of objects $X_0, \ldots, X_n$ together with a family of maps:
1. For each sequence \( (i_0, \ldots, i_k) : [k] \to [n] \) with \( k > 0 \) there is a chosen map
\[
f((i_0, \ldots, i_k)) \in \text{Map}_C(X_{i_0}, X_{i_k})_{k-1}
\]
2. The maps satisfy a differential coherence condition:
\[
-df((i_0, \ldots, i_k)) = \sum_{j=1}^{k-1} (-1)^j f((i_0, \ldots, \hat{i_j}, \ldots, i_k)) + \sum_{j=1}^{k-1} (-1)^{j-1} f((i_j, \ldots, i_k)) \circ f((i_0, \ldots, i_j)),
\]
3. The maps satisfy strict unitality:
\[
f((i_0, i_1)) = \text{id}_{X_{i_0}} \quad \text{(if } i_0 = i_1)\]
\[
f((i_0, \ldots, i_k)) = 0 \quad \text{(if } i_j = i_{j+1} \text{ for some } j)\]

An order-preserving map \( \alpha : [m] \to [n] \) acts on such an \( m \)-simplex by reindexing the objects and by precomposition of the sequences.

Simplices of the dg-nerve of a dg-category \( C \) can be seen as homotopy-coherent diagrams, where the coherence data is given by higher-degree morphisms. Concretely, the 0-simplices of \( N_{dg}(C) \) are just the objects of \( C \), the 1-simplices are the maps of the cycle category of \( C \), and a 2-simplex is given by a triple of objects \( X_0, X_1, X_2 \) and 0-cycles
\[
f((01)) \in \text{Map}_C(X_0, X_1)_0, \quad f((12)) \in \text{Map}_C(X_1, X_2)_0, \quad f((02)) \in \text{Map}_C(X_0, X_2)_0,
\]

\[f((012)) \in \text{Map}_C(X_0, X_2)_1\]

Proposition 3.10. Let \( C \) be a dg-category, then \( N_{dg}(C) \) is a quasicategory and the map \( N(G_0) \to N_{dg}(C) \) descends to an isomorphism \( H(C) \to \text{Ho}(N_{dg}(C)) \) from the homology category to the homotopy category.

Proof. By [Fao17, Proposition 2.2.12] and [Lur17, Remark 1.3.1.11].

Remark 3.11. Consider the poset \([n]\) as a dg-category with mapping complexes either \( Z(0) \) or 0, then the dg-nerve of a dg-category \( C \) as defined above is precisely the simplicial set of strictly unital \( A_\infty \)-functors \([n] \to C\), with the simplicial maps given by precomposition [Fao17]. By strict unitality, a simplex of the dg-nerve is already determined by the maps associated to injective sequences, but including the non-injective sequences in the definition makes some constructions (in particular that of the simplicial action) more convenient. This definition is still equivalent to that in [Lur17], which differs only in the omission of the maps forced by strict unitality and in the choice of signs.

We conclude the exposition by a technically useful alternative view on simplices of the dg-nerve that simplifies the calculation in the main part of the thesis.

Definition 3.12. For \( n \geq 0 \), define \( \text{Path}(n) \) to be the dg-coalgebra that as a graded abelian group is given in degree \( k > 0 \) by the freely generated abelian group
\[
\text{Path}(n)_k = \mathbb{Z} \cdot \{ \alpha : [k] \to [n] \}
\]
and which vanishes in degrees $k \leq 0$. The differential of $\text{Path}(n)$ sends a homogeneous element $\alpha : [k] \to [n]$ to the alternating sum of its inner faces:

$$d(\alpha_0, \ldots, \alpha_k) = \sum_{j=1}^{k-1} (-1)^j (\alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_k).$$

The comultiplication $\Delta : \text{Path}(n) \to \text{Path}(n) \otimes \text{Path}(n)$ splits sequences:

$$\Delta(\alpha_0, \ldots, \alpha_k) = \sum_{j=1}^{k-1} (-1)^{j(k-j)} \langle \alpha_j, \ldots, \alpha_k \rangle \otimes \langle \alpha_0, \ldots, \alpha_j \rangle.$$

A dg-category $\mathcal{C}$ induces a dg-algebra $\mathcal{C}^{\text{alg}} := \bigoplus_{X,Y} \text{Map}_\mathcal{C}(X,Y)$ with the multiplication $m$ that composes compatible maps and is 0 otherwise. The chain complex of maps $\text{Path}(n) \to \mathcal{C}^{\text{alg}}$ then is a dg-algebra with the convolution product:

$$p * q := m \circ (p \otimes q) \circ \Delta.$$

This leads to the following characterisation of simplices of the dg-nerve:

**Lemma 3.13.** Let $\mathcal{C}$ be a dg-category. An $n$-simplex of $\mathbf{N}_d\text{g}(\mathcal{C})$ corresponds to a family of objects $X_0, \ldots, X_n$ and a graded map $f : \text{Path}(n) \to \mathcal{C}^{\text{alg}}$ of degree $-1$ satisfying the following conditions:

1. Sequences are sent to the mapping spaces between the correct objects:

$$f((i_0, \ldots, i_k)) \in \text{Map}_\mathcal{C}(X_{i_0}, X_{i_k})_{k-1}$$

2. $f$ satisfies the Maurer-Cartan condition in the convolution algebra:

$$df + f * f = 0.$$

3. $f$ satisfies strict unitality:

$$f((i_0, i_1)) = \text{id}_{X_{i_0}} \quad (\text{if } i_0 = i_1)$$

$$f((i_0, \ldots, i_k)) = 0 \quad (\text{if } i_j = i_{j+1} \text{ for some } j).$$

**Proof.** Conditions 1 and 3 correspond directly to those in Definition 3.9. The Maurer-Cartan condition is equivalent to the differential coherence condition: Let $\alpha : [k] \to [n]$ be an element of $\text{Path}(n)$, then we have:

$$f(\alpha) = \sum_{j=1}^{k-1} (-1)^j f(\langle \alpha_0, \ldots, \hat{\alpha}_j, \ldots, \alpha_k \rangle)$$

$$(f * f)(\alpha) = \sum_{j=1}^{k-1} (-1)^{j-1} f(\langle \alpha_j, \ldots, \alpha_k \rangle) \circ f(\langle \alpha_0, \ldots, \alpha_j \rangle).$$

**Remark 3.14.** Graded maps of degree $-1$ from a coalgebra into an algebra satisfying the Maurer-Cartan condition are also called twisted cochains. This fits into a more general viewpoint: When we consider an $n$-simplex of $\mathbf{N}_d\text{g}(\mathcal{C})$ as an $A_\infty$-functor $[n] \to \mathcal{C}$ like in Remark 3.11, this induces a map $[n]^{\text{alg}} \to \mathcal{C}^{\text{alg}}$ of $A_\infty$-algebras, which corresponds in the terminology of [Kel06] to a twisting cochain from the bar construction $B[n]^{\text{alg}} \to \mathcal{C}^{\text{alg}}$. We can unroll the definitions to see that $B[n]^{\text{alg}}$ is $\text{Path}(n)$. 

9
4 Constructing Resolutions

The dg-nerve $N_{dg}(C)$ of a dg-category $C$ makes use of the higher-degree functions in the mapping complexes of $C$, whereas the transition from $C$ to its underlying ordinary category $C_0$ forgets all but the 0-cycles, with only some hints about higher dimensions encoded in weak equivalences and cofibrations. When $C$ is pretriangulated, however, all of the homotopical structure of $C$ can be recovered, which we show by an explicit construction that coherently encodes an $n$-simplex of the dg-nerve into a homotopical Reedy cofibrant diagram $D[n] \to C_0$.

The core idea is to encode the maps of an simplex in the dg-nerve as the differential of a complex, and then filter this complex to obtain a resolution by inclusion maps. The most prominent example of such a construction is the mapping cylinder of a chain map. To generalize from a cylinder to a shape that can accommodate higher coherence structure, we define:

**Definition 4.1.** For an object $\alpha \in D[n]$ define $\text{Cell}(\alpha)$ to be the $\text{Path}(n)$-comodule that in degree $k \geq 0$ is the free abelian group

$$\text{Cell}(\alpha)_k = \mathbb{Z} \cdot \{(i : \beta \hookrightarrow \alpha) \in (D[n] \downarrow \alpha) | \beta : [k] \to [n]\}$$

together with the differential

$$d^* (i_0, \ldots, i_k) = \sum_{j=1}^{k} (-1)^j (i_0, \ldots, \hat{i}_j, \ldots, i_k),$$

and the comultiplication $\delta : \text{Cell}(\alpha) \to \text{Cell}(\alpha) \otimes \text{Path}(n)$

$$\delta (i_0, \ldots, i_k) = \sum_{j=1}^{k} (-1)^{(k-j)} (i_j, \ldots, i_k) \otimes (\alpha \circ (i_0, \ldots, i_j)).$$

By postcomposition this definition extends to a functor from $D[n]$ to $\text{Path}(n)$-comodules.

Consider a dg-category $C$ as a dg-algebra with multiplication $m$ that composes compatible maps and let $\alpha \in D[n]$. The chain complex of graded maps $\varphi : \text{Cell}(\alpha) \to \mathcal{O}alg$ becomes a dg-module over the convolution algebra of maps $f : \text{Path}(n) \to \mathcal{O}alg$ via the multiplication

$$\varphi \ast f := m \circ (\varphi \otimes f) \circ \delta.$$

In the category of chain complexes, we can directly construct an object with a specified differential that generalizes that of the mapping cylinder by applying the higher coherence maps. In an arbitrary pretriangulated dg-category $C$, we have to proceed indirectly. We specify our main construction as a diagram of $C$-modules, which we show to be representable shortly after:

**Definition 4.2.** Let $C$ be a dg-category, $(X, f)$ an $n$-simplex of $N_{dg}(C)$ and $\alpha \in D[n]$. Then let $C(X, f)(\alpha)$ denote the $C$-module defined by the graded abelian group

$$C(X, f)(\alpha) = \prod_{i \in D[n] \downarrow \alpha} \text{Map}_C(X_{(\alpha \circ i)(0)}, -) \subseteq \text{Map}_C(\text{Cell}(\alpha), \oplus_{X \in C} \text{Map}_C(X, -))$$

together with the differential that sends homogeneous elements $\varphi \in C(X, f)(\alpha)$ to

$$d_f \varphi := d \varphi - (-1)^{|\varphi|} \varphi \ast f,$$

where $d$ is the differential of the underlying mapping space. By functoriality of $\text{Cell}$, this definition extends to a functor from $D[n]^{\text{op}}$ to $C$-modules.
Concretely, for any object $Y$ a homogeneous element $\varphi \in C(X,f)(\alpha)(Y)$ of degree $|\varphi|$ assigns to every $i \in D[n] \downarrow \alpha$ of the form
\[
\begin{array}{c}
[b] \\
\downarrow \beta \downarrow \alpha \\
[n] \\
\uparrow \alpha \\
[a]
\end{array}
\]
a homogeneous map $X_{(\alpha \cap 0)(0)} \to Y$ of degree $b + |\varphi|$. We can expand the differential $\text{d}$ on an element $(i_0, \ldots, i_k) \in \text{Cell}(\alpha)$ as:
\[
d_f \varphi((i_0, \ldots, i_k)) = \sum_{j=1}^{k} (-1)^j (i_0, \ldots, \hat{i}_j, \ldots, i_k)
\]
\[
- \sum_{j=1}^{k} (-1)^{|\varphi| + k(j-1)} \varphi((i_j, \ldots, i_k)) \circ f((\alpha(i_0), \ldots, \alpha(i_j)))
\]
\[
- (-1)^{|\varphi|} d(\varphi((i_0, \ldots, i_k))).
\]

In Lemma 3.13 we have seen that the maps $f : \text{Path}(n) \to \text{Map}_C(X,X)$ arising from an $n$-simplex of the dg-nerve $N_{d\mathbb{g}}(C)$ satisfy the Maurer-Cartan condition $d_f + f \circ f = 0$ in the convolution algebra. Here we can use this fact to show:

**Lemma 4.3.** Let $C$ be a dg-category, $(X, f)$ an $n$-simplex of $N_{d\mathbb{g}}(C)$ and $\alpha \in D[n]$. Then $d_f^2 = 0$, so $C(X,f)(\alpha)$ is a well-defined $C$-module.

**Proof.** Let $\varphi \in C(X,f)(\alpha)$ be homogeneous. Then we calculate:
\[
d_f^2 \varphi = d_f(d\varphi - (-1)^{|\varphi|} \varphi \circ f)
\]
\[
= d^2 \varphi - (-1)^{|\varphi|} d(\varphi \circ f) + (-1)^{|\varphi|} d\varphi \circ f - (\varphi \circ f) \circ f
\]
\[
= -(-1)^{|\varphi|} d\varphi \circ f - \varphi \circ df + (-1)^{|\varphi|} d\varphi \circ f - \varphi \circ (df \circ f)
\]
\[
= -\varphi \circ (df + f \circ f).
\]

By Lemma 3.13 we have that $df + f \circ f = 0$, so this last term vanishes. $\square$

Because the generating set of $\text{Cell}(\alpha)$ is finite, we can use an inductive argument to construct representations of $C(X, f)$ in any pretriangulated dg-category:

**Proposition 4.4.** Let $C$ be a pretriangulated dg-category, $(X, f)$ an $n$-simplex of $N_{d\mathbb{g}}(C)$ and $\alpha \in D[n]$. Then the $C$-module $C(X,f)(\alpha)$ is representable by some object $B(X,f)(\alpha)$. By varying $\alpha$, this assembles into a Reedy cofibrant functor
\[
B(X,f) : D[n] \longrightarrow C.
\]

**Proof.** For a sieve $S \subseteq (D[n] \downarrow \alpha)$, denote by $C(X,f)(S)$ the $C$-module
\[
C(X,f)(S) := \prod_{\beta \in S} \text{Map}_C(X_{(\alpha \cap \beta)(0)}, -),
\]
obtained from $C(X, f)$ by restriction of the product to indices in $S$. This can equivalently be seen as the quotient of $C(X, f)$ by those components of the product not contained in $S$.

For the empty sieve $\emptyset$ we have $C(X,f)(\emptyset) = 0$, which is representable by the zero object of $C$. Suppose that $C(X,f)(S)$ is representable for some sieve $S$ and let $i \in (D[n] \downarrow \alpha) \setminus S$ be minimal, where $i : \beta \rightarrow \alpha$ for some $\beta : [k] \to [n]$. By restriction of the product we then obtain a short exact sequence of $C$-modules
\[
0 \longrightarrow \text{Map}_C(X_{\beta(0)}, -)[k] \longrightarrow C(X,f)(S \cup \{i\}) \longrightarrow C(X,f)(S) \longrightarrow 0 \tag{2}
\]
Map_C(X_{β(0)}, -)[-k] is representable by the shifted object X[k] and C(X, f)(S) is representable by the inductive assumption. Thus by Lemma 3.7 we have that S(X, f)(S ∪ {β}) is representable as well. For Reedy cofibrancy, observe that the latching map for an object α ∈ D[n] is represented by the map of C-modules

\[ C(X, f)(D[n] \downarrow α) \longrightarrow C(X, f)(∅(D[n] \downarrow α)), \]

which is a cofibration by construction. □

**Example 4.5.** Let C be a pretriangulated dg-category and (X, f) a 0-simplex in the dg-nerve of C. Then f vanishes anywhere but for f((0, 0)) = id, so for any α : [n] → [0] we have a natural isomorphism

\[ C(X, f)(α) ≃ \text{Map}_{Ch}(N_*(Δ[n]), \text{Map}_C(X_0, -)) \]

where N_*(Δ[n]) is the complex of normalized chains of Δ[n]. Thus we see that the representation B(X, f) is given by a copowering with N_*(Δ[n]).

**Example 4.6.** Let C be a pretriangulated dg-category and (X, f) an n-simplex of N_{dg}(C). For any i ∈ [n] we have

\[ C(X, f)((i)) = \text{Map}_{Ch}(\mathbb{Z}(0), \text{Map}_C(X_i, -)) \cong \text{Map}_C(X_i, -), \]

and thus B(X, f)((i)) ≃ X_i.

**Example 4.7.** Let (X, f) be an n-simplex of N_{dg}(Ch). Then for any 0 ≤ i < j ≤ n, B(X, f)((i, j)) is the mapping cylinder of the chain map f((i, j)) : X_i → X_j. The inclusions of the sequences (i) and (j) into (i, j) induce the standard inclusions of X_i and X_j into the mapping cylinder.

5 The Resolutions are Homotopical

The weak equivalences in the category D[n] are those injections of sequences that preserve the maximum element. For a pretriangulated dg-category C and an n-simplex (X, f), we will show that the functor B(X, f) : D[n] → C is homotopical. We begin by defining a homotopy retraction of C(X, f)(α) onto its restriction to the last index of α. For notational convenience, we identify the isomorphic C-modules C(X, f)((max α)) with Map_C(X_{max α}, -).

**Lemma 5.1.** Let C be a dg-category, (X, f) an n-simplex of the dg-nerve of C, and α : [k] → [n]. Consider the map of C-modules

\[ I : C(X, f)(α) \longrightarrow C(X, f)((\text{max } α)) \]

induced by the inclusion \langle \text{max } α \rangle \hookrightarrow α at the last index. Then

\[ P : \text{Map}_C(X_{\text{max } α}, -) \longrightarrow C(X, f)(α) \]

\[ ω \mapsto i \mapsto (-1)^{|i|} ω \circ f((α \circ i) : (\text{max } α)) \]

is a closed 0-cycle and a retraction of I. Moreover,

\[ H : C(X, f)(α) \longrightarrow C(X, f)(α) \]

\[ ϕ \mapsto i \mapsto (-1)^{|ϕ|+|i|} ϕ(i \cdot (k)) \quad \text{(if max } i < k) \]

\[ ϕ \mapsto i \mapsto 0 \quad \text{(if max } i = k). \]

defines a homotopy P ∘ I ≃ id.
Proof. Let $\omega \in \text{Map}_C(X_{\max \alpha}, A)$ for any object $A$ and $i : \beta \hookrightarrow \alpha$. Then we have

\[
(dP)(\omega)(i) = (-1)^{|i|} d\omega \circ f(\alpha \circ i) - (-1)^{|i|} d\omega \circ f(\alpha \circ i)
\]

(3)

$+ (-1)^{|\omega|+|i|} \omega \circ df(\alpha \circ i \cdot \langle \max \alpha \rangle) - (-1)^{|\omega|}(P(\omega) \ast f)(i)$

(4)

$- (-1)^{|\omega|+|i|} \omega \circ f(d(\alpha \circ i \cdot \langle \max \alpha \rangle)) - (-1)^{|\omega|}P(\omega)(d^i).$

(5)

The term (3) vanishes immediately. In (4) we can substitute $-(f \ast f)$ for $df$ and observe that the convolutions cancel after expansion. Analogously, the alternating face sums in (4) cancel after expansion. Thus $P$ is a chain map.

For any $\omega \in \text{Map}_C(X_{\max \alpha}^{-})$, we see that

\[
(I \circ P)(\omega) = P(\omega)((\max \alpha)) = \omega \circ f(\langle \max \alpha, \max \alpha \rangle) = \omega,
\]

so $P$ is a retraction of $I$.

It remains to show that $dH = P \circ I - \text{id}$. For $\varphi \in C(X, f)(\alpha)$, we have

\[
(dH)(\varphi) = d \circ H(\varphi) + H(d \circ \varphi)
\]

(6)

$+ (-1)^{|\varphi|} H(\varphi) \ast f - (-1)^{|\varphi|} H(\varphi \ast f)$

(7)

$+ (-1)^{|\varphi|} H(\varphi) \circ d^* - (-1)^{|\varphi|} H(\varphi \circ d^*)$.

(8)

We evaluate this on elements $i \in \text{Cell}(\alpha)$ in three cases to see that $dH = P \circ I - \text{id}$:

1. Let $i \in \text{Cell}(\alpha)$ with $\max i < k$. The terms in (6) cancel each other. Evaluated on $i$, the convolutions in (7) cancel except for one summand

\[
(-1)^{|i|} \varphi(\langle k \rangle) \circ f(\alpha \circ i \cdot \langle \max \alpha \rangle) = (P \circ I)(\varphi)(i).
\]

Similarly, the alternating face sums of (8) cancel except for one summand

$-\varphi(i) = -\text{id}(\varphi)(i)$.

2. Now consider $i \in \text{Cell}(\alpha)$ with $\max i = |\alpha|$ and $|i| > 0$ and calculate

\[
(dH)(\varphi)(i) = (-1)^{|\varphi|} (H(\varphi) \ast f)(i)
\]

(9)

$+ (-1)^{|\varphi|} (H(\varphi) \circ d^*)(i)$.

(10)

Then (10) vanishes since $H$ vanishes on any suffix of $i$, so it agrees with

\[
(P \circ I)(\varphi)(i) = (-1)^{|\varphi|} \varphi(\langle k \rangle) \circ f(\alpha \circ i \cdot \langle \max \alpha \rangle) = 0,
\]

which is zero since $\alpha \circ i \cdot \langle \max \alpha \rangle$ is not injective and has more than two elements. The only summand in $d^i$ that does not contain $k$ is the prefix of $i$ of length $|i| - 1$, which is promptly extended again in (10) to the original $i$. So (10) evaluates to $\varphi(i)$.

3. Finally observe that $dH(\varphi)(\langle k \rangle) = 0$ as well as

\[
(P \circ I - \text{id})(\varphi)(\langle k \rangle) = \varphi(\langle k \rangle) \circ f(\langle \max \alpha, \max \alpha \rangle) - \varphi(\langle k \rangle) = 0.
\]

The case distinction in the definition of the homotopy $H$ was necessary since $\text{Cell}(\alpha)$ is not closed under appending the maximum element of $\alpha$.

Proposition 5.2. Let $C$ be a dg-category and $(X, f)$ an $n$-simplex of the dg-nerve of $C$.

1. The functor $C(X, f) : D[n]^{op} \to C\text{-Mod}$ is homotopical.

2. The functor $B(X, f) : D[n] \to C_0$ is homotopical.
Proof. For the first claim, we consider order-preserving maps \( \alpha : [k] \to [n], \beta : [r] \to [n] \) and let \( i : \alpha \leftrightarrow \beta \) be a weak equivalence in \( D[n] \). Then we have a diagram

\[
\begin{array}{ccc}
\alpha & \xleftarrow{i} & \beta \\
\downarrow{p} & & \downarrow{q} \\
\langle \alpha(k) \rangle & & \langle \beta(k) \rangle
\end{array}
\]

in \( D[n] \) where \( p \) is the inclusion at the last index and \( q \) the inclusion at index \( i(k) \). Then we get an induced diagram of \( C \)-modules

\[
C(X, f)(\alpha) \xleftarrow{i^*} C(X, f)(\beta)
\]

\[
\begin{array}{ccc}
p^* & & q^* \\
\downarrow{\text{Map}_C(X_{\alpha(k)}, -)} & & \downarrow{\text{Map}_C(X_{\beta(k)}, -)} \\
\end{array}
\]

\( p^* \) is a weak equivalence by Lemma 5.1. If \( i(k) = r \), then \( q \) also is the inclusion at the last index and thus also a weak equivalence, whence \( i^* \) is a weak equivalence by 2-out-of-3. Otherwise \( i(k) < r \), but then

\[
C(X, f)(\beta) \longrightarrow \text{Map}_C(X_{\alpha(k)}, -)
\]

\[
\varphi \longmapsto (-1)^{|\varphi|} \varphi((i(k), r))
\]

is a homotopy between \( q^* \) and the map induced by the inclusion at \( r \), so \( q^* \) is a weak equivalence as well.

Now the second claim follows since a chain map of representable \( C \)-modules is a chain homotopy equivalence precisely if it represents a map in \( C_0 \) that is a weak equivalence. \( \square \)

6 Functor of Quasicategories

For any \( n \)-simplex \((X, f)\) of the dg-nerve \( \mathbb{N}_{\text{dg}}(C) \) of a pretriangulated dg-category, by Proposition 4.4 we have constructed a Reedy cofibrant functor \( B(X, f) : D[n] \to C_0 \) which is homotopical by Proposition 5.2 and thus an \( n \)-simplex of \( \mathbb{N}_f(C_0) \). For \( B \) to define a functor between these quasicategories, it has to be compatible with the simplicial structure maps.

Objects of the slice categories \( D[m] \downarrow \alpha \) for some \( \alpha : [k] \to [m] \) are restrictions of \( \alpha \) to some subset of \( S \subseteq [k] \), and morphisms are inclusions of these subsets. Hence postcomposition with some map \( \sigma : [m] \to [n] \) induces an isomorphism of the slice categories \( D[m] \downarrow \alpha \to D[m] \downarrow (\sigma \circ \alpha) \). We use this to show:

Lemma 6.1. Let \( \alpha : [k] \to [m] \) and \( \sigma : [m] \to [n] \). Consider \( \text{Cell}(\alpha) \) to be a \( \text{Path}(n) \)-comodule via base change along the map of dg-coalgebras \( \text{Path}(m) \to \text{Path}(n) \) induced by \( \sigma \). Then the map \( \sigma_* : \text{Cell}(\alpha) \to \text{Cell}(\sigma \circ \alpha) \) induced by \( \sigma \) is an isomorphism of \( \text{Path}(n) \)-comodules.

Proof. The map \( \text{Cell}(\sigma \circ \alpha) \to \text{Cell}(\alpha) \) that sends a restriction \( (\sigma \circ \alpha)|_S \) to \( \alpha|_S \) is the inverse chain map of \( \sigma_* \). Both comultiplications split subsequences of \([k]\) and then apply \( \sigma \circ \alpha \) to the suffix, which is preserved by both \( \sigma_* \) and its inverse. \( \square \)

In the definition of \( B \) we have chosen representatives in \( C \) for the representable \( C \)-modules. Assuming that we choose a single representing object for every isomorphism class of \( C \)-modules, we can show the following:

Proposition 6.2. Let \( C \) be a dg-category. Then the assignment \( (X, f) \mapsto B(X, f) \) is a map of simplicial sets \( B : \mathbb{N}_{\text{dg}}(C) \to \mathbb{N}_f(C_0) \), and thus a functor of quasicategories.
Proof. Let $(X, f) \in N_{dg}(C)_n$ and let $\sigma : [m] \to [n]$. The map $\sigma$ acts on the diagram $B(X, f) : D[m] \to C_0$ by precomposition with $D \sigma : D[m] \to D[n]$. The result is represented by the diagram $D[m] \to C\cdot\text{Mod}$ given on $\alpha \in D[m]$ by

$$\sigma^*(C(X, f))(\alpha) = \prod_{i \in D[m] \downarrow \alpha} \text{Map}_C(X_{(\sigma \circ \alpha)(0)}, -)$$

with differential that sends a homogeneous element $\varphi$ to

$$d\varphi = d\varphi - (-1)^{|\varphi|} \varphi \cdot f = d\varphi - (-1)^{|\varphi|} m \circ (\varphi \otimes f) \circ \delta.$$  \hspace{1cm} (11)

Let $\sigma$ act on the $n$-simplex $(X, f)$, then $B(\sigma^*(X, f))$ is represented on $\alpha \in D[m]$ by

$$C(\sigma^*(X, f))(\alpha) = \prod_{i \in D[m] \downarrow \alpha} \text{Map}_C(X_{(\sigma \circ \alpha)(0)}, -)$$

with the differential sending a homogeneous element $\varphi$ to

$$d\sigma \cdot \varphi = d\varphi - (-1)^{|\varphi|} \varphi \cdot \sigma^* f = d\varphi - (-1)^{|\varphi|} m \circ (\varphi \otimes f) \circ (\text{id} \otimes \text{Path}(\sigma)) \circ \delta.$$  \hspace{1cm} (12)

Observe that $(\text{id} \otimes \text{Path}(\sigma)) \circ \delta$ is the comultiplication of $\text{Cell}(\alpha)$ considered as a $\text{Path}(n)$-comodule. The isomorphism $\sigma^* : \text{Cell}(\alpha) \to \text{Cell}(\sigma \circ \alpha)$ induces an isomorphism of graded objects

$$\sigma^* (C(X, f))(\alpha) \cong C(\sigma^*(X, f))(\alpha)$$

by reindexing the product, which is compatible with the differentials (11) and (12) since $\sigma^*$ and its inverse respect the $\text{Path}(n)$-comodule structures by Lemma [6.1].

Now it follows that $\sigma^* (B(X, f)) \cong B(\sigma^*(X, f))$ since they both are the representative for the common isomorphism class of $\sigma^*(C(X, f))$ and $C(\sigma^*(X, f)).$ \hfill \Box

Both $N_{dg}(C)$ and $N_{i}(C_0)$ are functorial in $C$, in that they send $\text{dg}$-functors to functors of quasicategories. The construction of $B$ required choices of representing objects and universal constructions that in general are not respected by arbitrary $\text{dg}$-functors, so $B$ is not strictly natural. However, we can show:

\textbf{Lemma 6.3.} Let $C$, $D$ be pretriangulated $\text{dg}$-categories and $F : C \to D$ a $\text{dg}$-functor. Then

$$
\begin{array}{ccc}
N_{dg}(C) & \xrightarrow{B} & N_{i}(C_0) \\
F_* & \Downarrow \cong & F_* \\
N_{dg}(D) & \xrightarrow{B} & N_{i}(D_0)
\end{array}
$$

commutes up to simplicial homotopy.

\textbf{Proof.} Let $(X, f)$ be an $n$-simplex of $N_{dg}(C)$. We have

$$
\text{Map}_D(B(F_*(X, f))(\alpha), FB(X, f)(\alpha))
\cong C(F_*(X, f))(\alpha)(FB(X, f)(\alpha))
\cong \prod_{i \in D[m] \downarrow \alpha} \text{Map}_D(FX_{(\sigma \circ \alpha)(0)}, FB(X, f)(\alpha))
$$

\hspace{1cm} (13)

with differential $dF(f)$, natural in $\alpha \in D[n]$. From the representation of the identity map of $B(X, f)(\alpha)$ we get maps $X_{(\sigma \circ \alpha)(0)} \to B(X, f)(\alpha)$ for all $i \in D[n] \downarrow \alpha$. We apply the functor $F$ and obtain an element of (13), representing a map $\eta(\alpha) : B(F_*(X, f))(\alpha) \to FB(X, f)(\alpha)$. 

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These piece together into a natural transformation
\[ \eta : BF_\ast(X,f) \to FB(X,f). \]

For any \( \alpha \in D[n] \), we thus have a 1-simplex \( \eta(\alpha) \) of \( N_{dg}(D) \), which has a homotopical Reedy cofibrant resolution \( B(\eta(\alpha)) : D[1] \to D_0 \). Since \( B \) constructs compatible resolutions, we get a homotopical diagram \( D[1] \times D[n] \to D_0 \), which is Reedy cofibrant by an analogous argument as in the proof of Proposition 4.4. We then get a homotopical Reedy cofibrant diagram \( D[n] \to (D_0)_R^{[1]} \). Again by the compatibility of the resolutions created by \( B \), these maps piece together into a map of simplicial sets
\[ N_{dg}(C) \to N_f((D_0)_R^{[1]}). \]

By composing this map with the canonical map \( N_f((D_0)_R^{[1]}) \to N_f(D_0)\Delta[1] \), we obtain the simplicial homotopy.

The maps \( \eta(\langle i \rangle) \) for any \( i \in [n] \) that occur in the proof above are isomorphisms, and so by 2-out-of-3 we have that \( \eta \) is a natural weak equivalence. By carefully choosing compatible homotopy inverses and resolutions, it is likely to be possible to extend the simplicial homotopy to an \( E[1]-\)homotopy.

### 7 Equivalence of Quasicategories

In this last section we will conclude the construction by showing that the functor \( B \) defines an equivalence of quasicategories. Just as in ordinary category theory, we have to show that \( B \) is essentially surjective and fully faithful \([Lur09, Definition 1.2.10.1]\).

**Proposition 7.1.** Let \( C \) be a pretriangulated dg-category. Then \( B : N_{dg}(C) \to N_f(C_0) \) is essentially surjective as a functor of quasicategories.

**Proof.** Let \( X : D[0] \to C_0 \) be a homotopical Reedy cofibrant functor representing an object of the homotopy category of \( N_f(C) \). Let \( Y := B(X(\langle 0 \rangle)) \), then \( Y : D[0] \to C_0 \) is Reedy cofibrant by Proposition 4.4 and homotopical by Proposition 5.2 and we have \( Y(\langle 0 \rangle) \cong X(\langle 0 \rangle) \). Thus by Lemma 2.11, \( X \) and \( Y \) are equivalent 0-simplices, and thus isomorphic in the homotopy category. Hence \( B \) induces an essentially surjective functor on the homotopy categories, so by definition it is essentially surjective as a functor of quasicategories.

It remains to show that \( B \) is fully faithful. We will use the auxiliary lemma:

**Lemma 7.2** ([KS15 Lemma 4.3]). Let \( f : K \to L \) be a map of simplicial sets. Suppose that for each \( n \in \mathbb{N} \) and a square:
\[
\begin{array}{ccc}
\partial \Delta[n] & \xrightarrow{u} & K \\
\downarrow & & \downarrow f \\
\Delta[n] & \xrightarrow{v} & L
\end{array}
\]
there are: a map \( w : \Delta[n] \to K \) such that \( w|_{\partial \Delta[n]} = u \) and a homotopy from \( f \circ w \) to \( v \) relative to the boundary. Then \( f \) is a weak homotopy equivalence.

Given the mapping cylinder for some unknown map \( f : A \to B \) of chain complexes, \( f \) can be recovered (up to homotopy) by composing the inclusion \( A \to Cyl(f) \) with a homotopy inverse of the inclusion \( B \to Cyl(f) \). Based on this idea, we can prove:

**Proposition 7.3.** Let \( C \) be a pretriangulated dg-category. Then
\[ B : N_{dg}(C) \to N_f(C_0) \]
is fully faithful as a functor of quasicategories.
Proof. $B$ is fully faithful if it induces a fully faithful functor of homotopy categories enriched in the homotopy category of spaces. In particular, this is the case when $B$ induces a weak equivalence of simplicial sets

$$B_* : \text{Map}_{N_{	ext{ad}}(C)}^R(X,Y) \longrightarrow \text{Map}_{N_{i}(C_0)}^R(B(X), B(Y))$$

for all objects $X,Y$, which we prove using Lemma \[\ref{lem:reedy-cofibrant-functor}. Consider a diagram

$$\begin{array}{ccc}
\partial \Delta[n] & \xrightarrow{\partial} & \text{Map}_{N_{\text{ad}}(C)}^R(X,Y) \\
\downarrow & & \downarrow \partial_* \\
\Delta[n] & \xrightarrow{v} & \text{Map}_{N_{i}(C_0)}^R(B(X), B(Y)).
\end{array}$$

To deal with the partially defined objects arising in this proof, we implicitly take the pointwise Kan extension of simplicial objects to finite simplicial sets. The necessary colimits exist in $C_0$ because of Reedy cofibrancy and the closure of $C_0$ under pushouts along cofibrations.

By unpacking the definition of the mapping complex, $u$ corresponds to a map

$$\partial \Delta[n] \star \Delta\{n + 1\} \longrightarrow N_{\text{ad}}(C)$$

which sends $\partial \Delta[n]$ constantly to $X$ and $\Delta\{n + 1\}$ constantly to $Y$. By letting the additional coherence map be zero, we can trivially extend this to a map

$$U : \Delta[n] \cup (\partial \Delta[n] \star \Delta\{n + 1\}) \longrightarrow N_{\text{ad}}(C),$$

which is compatible by strict unitality with all extensions that send $\Delta[n]$ constantly to $X$ and so get a graded map

$$f : \text{Path}(\Delta[n] \cup (\partial \Delta[n] \star \Delta\{n + 1\})) \longrightarrow C$$

of degree $−1$. On the other hand, we obtain from $v$ a homotopical Reedy cofibrant functor

$$V : D([n+1]) \longrightarrow C_0.$$

By the definition of the mapping space and commutativity of the square we have

$$V|D(\Delta[n]) = B(X) \circ D([n] \rightarrow [0]),$$

$$V|D(\Delta\{n + 1\}) = B(Y) \circ D([n + 1] \rightarrow [0]),$$

$$V|D(\Delta[n] \cup \partial \Delta[n] \star \Delta\{n + 1\}) = B(U).$$

In particular, $V$ induces a commutative diagram

$$\begin{array}{ccc}
B(U)(\Delta[n]) & \xrightarrow{b_1} & V(\Delta[n + 1]) \\
\downarrow & & \downarrow i \\
B(U)(\partial \Delta[n]) & \xleftarrow{b_2} & B(U)(\partial \Delta[n] \star \Delta\{n + 1\}).
\end{array}$$

Let $p_2$ be the homotopy inverse to $i_2$ and $H_2 : i_2 \circ p_2 \simeq \text{id}$ the homotopy that were constructed in Section \[\ref{sec:homotopy-theory}. $\iota$ is a weak equivalence, because $i_1$ and $i_2$ are weak equivalences, and it is a cofibration by Reedy-cofibrancy of $V$. Thus by Lemma \[\ref{lem:reedy-cofibrant-functor} there exists a map

$$p : B \rightarrow A$$

such that $p \circ \iota = \text{id}$ and a homotopy $h : \iota \circ p \simeq \text{id}$ such that $h \circ \iota = 0$. Let

$$p_1 = p_2 \circ p, \quad H_1 = \iota \circ H_2 \circ p + h,$$

then $H_1 : i_1 \circ p_1 \simeq \text{id}$ and the following diagrams commute:

$$\begin{array}{ccc}
V(\Delta[n + 1]) & \xrightarrow{p_1} & Y \\
\downarrow & & \downarrow \\
B(U)(\partial \Delta[n] \star \Delta\{n + 1\})
\end{array} \quad \begin{array}{ccc}
V(\Delta[n + 1]) & \xrightarrow{H_1} & V(\Delta[n + 1]) \\
\downarrow & & \downarrow \\
B(U)(\partial \Delta[n] \star \Delta\{n + 1\})
\end{array} \quad \begin{array}{ccc}
B(U)(\partial \Delta[n] \star \Delta\{n + 1\}) & \xrightarrow{H_2} & B(U)(\partial \Delta[n] \star \Delta\{n + 1\})
\end{array}$$

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To fill in the interior of $u$, we seek an extension of $f$ to a map
\[ \hat{f} : \text{Path}(\Delta[n+1]) \to \mathcal{C}. \]

The only injective argument $[k] \to [n+1]$ missing from the domain of $f$ (and thus the only one whose value is not forced by strict unitality) is the identity $[n+1] \to [n+1]$, so we seek an element $f([n+1]) \in \text{Map}_{\mathcal{C}}(X,Y)_{n}$. To ensure that the extension satisfies the Maurer-Cartan condition, the map $f([n+1])$ must the chosen such that
\[ d(f([n+1])) = -f(d([n+1])) - (f \circ f)([n+1]). \tag{14} \]

Consider the diagram of $\mathcal{C}$-modules
\[
\begin{array}{ccc}
\text{Map}_\mathcal{C}(X,-) & \xleftarrow{a_1} & C(U)(\Delta[n]) & \xrightarrow{Yb_1} & \text{Map}_\mathcal{C}(V(\Delta[n+1]),-) & \xleftarrow{Yp_1} & \text{Map}_\mathcal{C}(Y,-) \\
\downarrow{da_1} & & \downarrow{Y} & & \downarrow{Y} & & \downarrow{Y} \\
\text{Map}_\mathcal{C}(X,-) & \xleftarrow{a_2} & C(U)(\partial\Delta[n]) & \xrightarrow{Yb_2} & C(U)(\partial\Delta[n] \cup \Delta\{n+1\}) & & \\
\end{array}
\]

where $Y$ is the Yoneda embedding and $a_1, a_2$ are the maps of degree $n$ and $n-1$, respectively, which are defined by the following terms:
\[
a_1(\varphi) = (-1)^{|\varphi|n}\varphi([n]),
\]
\[
a_2(\varphi) = (-1)^{|\varphi|(n-1)} \sum_{i=1}^{n} (-1)^{i}\varphi([n] - i) - (-1)^{|\varphi|(n-1)}\varphi([n] - 0).
\]

Observe that $a_1, a_2$ are chosen such that the differential of the composition of the top row is the composition of the bottom row of the diagram:
\[ d(a_1 \circ Yj_1 \circ Yp_1) = da_1 \circ Yj_1 \circ Yp_1 = a_2 \circ Yj_2 \circ Yp_2. \]

We can then evaluate the composition of the bottom row for some $\omega \in \text{Map}_\mathcal{C}(Y,A)$:
\[
(a_2 \circ Yj_2 \circ Yp_2)(\omega)
\]
\[
= (-1)^{|\omega|+n} \sum_{i=1}^{n} (-1)^{i}Yp_2([n] - i) - Yp_2([n] - 0)(\omega)
\]
\[
= (-1)^{|\omega|+n} \sum_{i=1}^{n} (-1)^{i}\omega \circ f([n+1] - i) - \omega \circ f([n+1] - 0)
\]
\[
= (-1)^{(n+1)|\omega|} \sum_{i=1}^{n} (-1)^{i}\omega \circ f([n+1] - i) + \omega \circ f([n+1] - 0) \circ f((0,1))
\]
\[
= (-1)^{n}Y(f(d([n+1])) + (f \circ f)([n+1]))(\omega).
\]

Hence the map $\hat{f}([n+1])$ represented by $-a_1 \circ Yb_1 \circ Yp_1$ satisfies (14). This induces an extension $\hat{u}$ of $u$ that fills in the interior, and thus also extends $U$ to
\[ \hat{U} : \text{Path}(\Delta[n+1]) \to \text{N}_{dg}(\mathcal{C}). \]

It remains to show that there is a simplicial homotopy from $\hat{u}$ to $v$, relative to the boundary. As the essential ingedient for this homotopy, we construct a map
\[ F([n+1]) : B(\hat{U})(\Delta[n+1]) \to V(\Delta[n+1]) \]
in $\mathcal{C}_0$. Since $B(\hat{U})(\Delta[n+1])$ represents $C(\hat{U})(\Delta[n+1])$, such a map amounts to an element
\[ \varphi \in C(\hat{U})(\Delta[n+1])(V(\Delta[n+1])) \]

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of degree zero such that \( d \varphi = 0 \). Restricted to the proper subsets of \([n + 1]\), we let \( \varphi \) be determined by the element representing the map

\[
i : C(U)(\Delta[n] \cup \partial\Delta[n] \star \Delta(n + 1)) \rightarrow V(\Delta[n + 1]),
\]

and construct the missing \( \varphi([n + 1]) \) as follows: Consider the diagram of \( C \)-modules

\[
\begin{array}{ccc}
\Map_C(X, -) & \xleftarrow{a_1} & C(U)(\Delta[n]) \xrightarrow{Yb_1} \Map_C(V(\Delta[n + 1]), -) \xrightarrow{YH_1} \Map_C(V(\Delta[n + 1]), -) \\
\Map_C(X, -) & \xrightarrow{a_2} & C(U)(\partial\Delta[n]) \xrightarrow{Y\iota_2} C(U)(\partial\Delta[n] \star \Delta(n + 1)) \xrightarrow{YH_2} C(U)(\partial\Delta[n] \star \Delta(n + 1))
\end{array}
\]

where \( a_1, a_2 \) are defined as above and calculate the differential of the top row:

\[
d(a_1 \circ Yb_1 \circ YH_1) = a_2 \circ Yb_2 \circ YH_2 \circ Y\iota
\]

\[+ (-1)^n a_1 \circ Yb_1 \circ Yp_1 \circ Y\iota_1
\]

\[= (-1)^n a_1 \circ Yb_1.
\]

We evaluate this for some \( \omega \in \Map_C(V(\Delta[n + 1]), A) \) and get

\[
d(a_1 \circ Yb_1 \circ YH_1) = (-1)^{n|\omega|} \sum_{i=1}^{n} (-1)^i Y\iota(\omega)([n + 1] - i)
\]

\[- (-1)^{n|\omega|} Y\iota(\omega)([n + 1] - 0)
\]

\[+ (-1)^{n|\omega|+n+1} Y\iota_1(\omega) \circ \hat{f}([n + 1])
\]

\[+ (-1)^{n|\omega|+n+1} Yb_1(\omega)([n]).
\]

Then (15) and (16) together form the sum

\[
\sum_{i=1}^{n+1} (-1)^i \omega \circ \varphi([n + 1] - i).
\]

The terms (16) and (17) are the only non-vanishing summands of

\[
\sum_{i=1}^{n+1} (-1)^{ni+n+1} \omega \circ \varphi([i, \ldots, n + 1]) \circ \hat{f}([0, \ldots, i]).
\]

Let \( \varphi([n + 1]) \) be represented by \( a_1 \circ Yb_1 \circ YH_1 \), then we have shown that

\[
d(\varphi([n + 1])) = \varphi(d\varphi([n + 1])) + (\varphi \circ \hat{f})([n + 1])
\]

and thus \( d \varphi = d\varphi - \varphi \circ \hat{f} = 0 \).

Now we have constructed a well-defined map \( F([n + 1]) \) that is compatible with the inclusion maps for proper subsets of \([n + 1]\). For some \( \alpha : [k] \rightarrow [n + 1] \) in \( D[n + 1] \) which factors through either \([n]\) or \( \partial[n] \star \{n + 1\} \), we have \( B(\hat{U})(\alpha) = V(\alpha) \), so we can pad \( F([n + 1]) \) with identity maps to obtain a natural transformation

\[
B(\hat{U})|S \rightarrow V|S
\]

where \( S \) is the sieve \( Sd[n + 1] \cup D(\Delta[n] \cup \partial\Delta[n] \star \Delta(n + 1)) \hookrightarrow D[n + 1] \). Consider the natural transformation as a homotopical Reedy cofibrant map

\[
[1] \rightarrow (C_d^S)_R.
\]
Precompose with the canonical map $D[1] \rightarrow [1]$ to obtain a homotopical, yet not necessarily Reedy cofibrant map

$$D[1] \rightarrow (\mathcal{C}_0^S)_R,$$

and then use Lemma 2.9 to modify this to be Reedy cofibrant while leaving the restriction to $[1] \subset D[1]$ fixed. By Lemma 2.9 there exists a lift

$$\begin{array}{ccc}
\{0,1\} & \xrightarrow{B(\hat{U}) \cup V} & (\mathcal{C}_0^{D[n+1]})_R \\
\downarrow & & \downarrow \\
D[1] & \xrightarrow{\text{dotted}} & (\mathcal{C}_0^S)_R
\end{array},$$

which is also homotopical and Reedy cofibrant. Transpose this to a diagram

$$D[1] \times D[n+1] \rightarrow C_0$$

and precompose with the canonical map $D([1] \times [n+1]) \rightarrow D[1] \times D[n+1]$ to get

$$D([1] \times [n+1]) \rightarrow C_0,$$

which is still homotopical and Reedy cofibrant by Lemma 2.10. By construction, this functor agrees with $B(\hat{U})$ when restricted to $D((0) \times [n+1])$, with $V$ when restricted to $D([1] \times [n+1])$, and is the identity on the constant diagrams $B(X)$ and $B(Y)$ on $D([1] \times [n])$ and $D([1] \times [n+1])$, respectively. Hence we obtain a simplicial homotopy between $\hat{u}$ and $v$ that leaves the boundary $u$ fixed.

We can thus conclude with our main result:

**Theorem 7.4.** Let $\mathcal{C}$ be a pretriangulated dg-category. Then

$$B : N_{\Delta}(\mathcal{C}) \rightarrow N_f(\mathcal{C}_0),$$

as defined in Section 4, is an equivalence of quasicategories.

**Proof.** $B$ sends simplices of $N_{\Delta}(\mathcal{C})$ to Reedy cofibrant diagrams by Proposition 4.4, which are homotopical by Proposition 5.2. $B$ is a functor of quasicategories by Proposition 6.2, which is essentially surjective by Proposition 7.1 and fully faithful by Proposition 7.3. Hence $B$ is an equivalence.

**Bibliography**

[BK91] A. I. Bondal and M. M. Kapranov. Enhanced triangulated categories. *Math. USSR, Sb.*, 70(1):93–107, 1991.

[Bro73] Kenneth S. Brown. Abstract homotopy theory and generalized sheaf cohomology. *Transactions of the American Mathematical Society*, 186:419–458, 1973.

[CS18] Alberto Canonaco and Paolo Stellari. Uniqueness of dg enhancements for the derived category of a Grothendieck category. *Journal of the European Mathematical Society*, 20(11):2607 – 2641, Oct 2018.

[Dwy05] W.G. Dwyer. *Homotopy Limit Functors on Model Categories and Homotopical Categories*. Mathematical surveys and monographs. American Mathematical Society, 2005.

[Fao17] Giovanni Faonte. Simplicial nerve of an A-infinity category. *Theory and Applications of Categories*, 32:32 – 52, 2017.
[Hov07] M. Hovey. *Model Categories*. Mathematical surveys and monographs. American Mathematical Society, 2007.

[Kel06] Bernhard Keller. A-infinity algebras, modules and functor categories. *Contemporary Mathematics*, 406:67–94, 2006.

[KS15] Chris Kapulkin and Karol Szumiło. Quasicategories of frames of cofibration categories. *Applied Categorical Structures*, 25, 06 2015.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009.

[Lur17] Jacob Lurie. Higher Algebra. Sep 2017.

[Qui67] Daniel G. Quillen. *Homotopical algebra*. Springer, 1967.

[Rie14] Emily Riehl. *Categorical Homotopy Theory*. New Mathematical Monographs. Cambridge University Press, 2014.

[RV14] Emily Riehl and Dominic Verity. The theory and practice of Reedy categories. *Theory and Applications of Categories*, 29:256 – 301, 2014.

[Sch12] Stefan Schwede. Topological triangulated categories. *arXiv e-prints*, page arXiv:1201.0899, Jan 2012.

[Szu14] Karol Szumiło. Two Models for the Homotopy Theory of Cocomplete Homotopy Theories. *arXiv e-prints*, page arXiv:1411.0303, Nov 2014.