Abstract

Inspired by regularization in quantum field theory, we study topological and metric properties of spaces in which a cut-off is introduced. We work in the framework of noncommutative geometry, and focus on Connes distance associated to a spectral triple \((\mathcal{A}, \mathcal{H}, D)\). A high momentum (short distance) cut-off is implemented by the action of a projection \(P\) on the Dirac operator \(D\) and/or on the algebra \(\mathcal{A}\). This action induces two new distances. We first individuate conditions making them equivalent to the original distance. We then study the Gromov-Hausdorff limit of the set of truncated states, first for quantum metric spaces in the sense of Rieffel, and then for arbitrary spectral triples. In the commutative case, we show that the cut-off induces a minimal length between points, which is infinite if \(P\) has finite rank. When \(P\) is a spectral projection of \(D\), we work out an approximation of points by non-pure states that are at finite distance from each other. We apply the results to Moyal plane and to the fuzzy sphere, obtained as Berezin quantization of the plane and the sphere. Along the way, we introduce a notion of “state with finite moment of order 1” for noncommutative algebras, give a new proof that the spectral distance between coherent states of Moyal plane is the Euclidean distance between the peaks, and present some new results about Wasserstein distance on the real line and on the circle, including a sharp approximation of the distance between Fejér probability distributions. Finally we discuss discrete approximations of the derivative on the real line: the \(h\)-derivative and the \(q\)-derivative.
## Contents

1 Introduction ................................................................. 3  
2 Preliminaries ................................................................. 5  
   2.1 Metric aspect of noncommutative geometry ....................... 5  
   2.2 Compact quantum metric spaces .................................. 6  
   2.3 Hausdorff and Gromov-Hausdorff distance ....................... 7  
3 Truncations ........................................................................ 7  
   3.1 Regularization of the geometry .................................... 7  
   3.2 Truncated topologies .................................................. 8  
   3.3 Multi-ranked truncations .......................................... 11  
4 Convergence of truncations ................................................ 12  
   4.1 Compact quantum metric spaces & Gromov-Hausdorff convergence .. 12  
   4.2 Beyond compact quantum metric spaces: states with finite $1^{st}$ moment 14  
   4.3 Example: the lattice $\mathbb{Z}$ ...................................... 17  
5 Pure states and approximation of points ................................ 20  
   5.1 Bounded regularization ............................................. 21  
   5.2 Finite rank regularization ......................................... 21  
   5.3 Regularization by spectral projection & geodesic flow ........... 22  
   5.4 Gromov-Hausdorff convergence on the circle .................... 24  
   5.5 Optimal transport on $S^1$ for rectangular distributions ....... 28  
6 Wasserstein distance and Berezin quantization ....................... 30  
   6.1 Gauged Dirac operators ........................................... 31  
   6.2 Berezin quantization of the plane ................................ 31  
   6.3 Quantum discs, fuzzy spaces and other examples ............... 35  
   6.4 The fuzzy sphere as a coherent state quantization ............. 36  
7 $h$-derivative and $q$-derivative ......................................... 40  
   7.1 $h$-derivative and “fat points” ................................... 40  
   7.2 $q$-derivative and French Rail metric ............................ 42
1 Introduction

We study the topological and metric aspects of spaces in which a cut-off is implemented. The physical motivation is the presence of divergent quantities in quantum field theory, and the techniques used to tame them. Specifically, the calculations in quantum field theory are usually done in the Fourier space and there are divergences due to the high values of the momentum. To obtain finite quantities, one introduces a large scale (usually called a cut-off) which represents the maximum attainable momentum. By the momentum/position duality, this corresponds to a short distance cut-off.

Noncommutative geometry [15] is an appropriate framework to study such spaces, whose short distance behaviour is profoundly changed. The algebraization at the roots of noncommutative geometry enables the generalization of all the relevant geometrical concepts to a setting where a cut-off can be naturally implemented through the action of a projection $P_{\Lambda}$. More precisely, we aim at studying spectral triples $(\mathcal{A}, \mathcal{H}, D)$ in which the initial Dirac operator $D$ is substituted by a “truncated” Dirac operator

$$D_{\Lambda} := P_{\Lambda} D P_{\Lambda}.$$  

(1.1)

Let us be more precise on the physical motivations and their geometrical consequences. Consider for instance Euclidean field theories. Given a Riemannian spin manifold $M$, the action functional describing the dynamics of free fermions in classical field theory is $S_F(\psi) = \langle \psi, D\psi \rangle$ where $D$ is the Dirac operator of $M$, a unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$ of square integrable spinors on $M$, and $\psi \in \mathcal{H}$ is a vector in the domain of $D$. Formally, second quantization consists in replacing $\mathcal{H}$ with a (complex or real) Grassmann algebra and define the partition function $Z$ as Berezin integral of the exponential $e^{-S_F(\psi)}$. If $\mathcal{H}$ is finite dimensional (hence $M$ is a finite set), this is well-defined and yields the determinant or the Pfaffian of $D$ (depending on whether $\mathcal{H}$ is complex or real and even-dimensional). In the infinite dimensional case $Z$ is not well-defined and one needs to regularize the theory with the introduction of some dimensional quantity, like a momentum cut-off. The later might have a physical meaning as the scale at which the theory is not valid anymore, or be considered a technical device in order to obtain finite quantities. This is part of the general process of renormalization.

In this paper we are interested in the geometrical consequence of the cut-off: by momentum/position duality, cutting away high momenta means cutting away short distances. This means that the usual tools of differential geometry do not apply anymore, but the setting is ideal for the methods of noncommutative geometry [15, 27, 36, 17]. The latter provides a translation of Riemannian geometry in completely algebraic terms, using a $\ast$-algebra represented on a Hilbert space (which capture the topological aspects), and a not necessarily bounded generalized Dirac operator (which captures the metric aspects). These elements form a spectral triple and are at the basis of the construction. These ingredients are naturally present in any quantum field theory: the algebra is the one of complex-valued functions on spacetime, the Hilbert space is the one representing the matter fields of the theory, and the generalized Dirac operator contains the relevant physical information of the system.

A cut-off $\Lambda > 0$ is naturally implemented through the action of a suitably chosen projection. For instance when $M$ is closed, a natural way to define a regularized partition function is to decompose the infinite dimensional $\mathcal{H} = \bigoplus_{\lambda} V_{\lambda}$ as the (completed) direct sum of eigenspaces of the Dirac operator $D$. Let $P_{\Lambda}$ denote the projection on the direct
sum of eigenspaces with $|\lambda| \leq \Lambda$. Since $D$ has compact resolvent, the projection $P_\Lambda$, and hence $D_\Lambda$ in (1.1) are finite rank operators. The determinant of $D_\Lambda$ (thought of as an operator on $\mathcal{H}_\Lambda := P_\Lambda \mathcal{H}$) is well defined and gives a regularized partition function. This procedure is called finite mode regularization \cite{1, 25, 2}. Although it is very much in the spirit of Noncommutative Geometry, it was in fact originally developed before it.

As recalled in §2, the Dirac operator $D$ induces a distance $d_{A,D}$ on the state space of the algebra $A = C_0(M)$. This distance coincides with the geodesic distance if the states are pure (Dirac deltas), and with the Wasserstein distance of order 1 of transport theory - with cost the geodesic distance - if the states are given by arbitrary probability distributions on $M$ (see e.g. \cite{20}). Inspired by finite mode regularization, a natural question is how replacing $D$ with $D_\Lambda$ changes the metric properties of the state space. In particular, we investigate whether the regularized theory is an “approximation” of the original theory in some precise mathematical sense.

Although in finite mode regularization $P_\Lambda$ is an eigenprojection of $D$, i.e. $[D, P_\Lambda] = 0$, we work under the general hypothesis that $P_\Lambda$ is any projection on $\mathcal{H}$, non-necessarily commuting with $D$. Furthermore, our starting point is any (not necessarily commutative) unital spectral triple $(A, \mathcal{H}, D)$. Except for sections 5 and 6, we do not assume this is the canonical spectral triple of a closed Riemannian spin manifold.

The main results of that paper are the following:

- **Equivalence of the topologies induced by truncated distances:** we introduce two new distances $d_{A,D_\Lambda}$ and $d_{\mathcal{H},D_\Lambda}$ corresponding to the truncation of the Dirac operator $D$ only, and to the truncation of both the Dirac operator and the algebra. The main result is Prop. 3.5, in which we work out some conditions on $D$ and $P_\Lambda$ under which these two distances are equivalent to the initial one $d_{A,D}$. With weaker conditions, we also obtain in Prop. 3.6 some inequalities between the three distances.

- **Approximation of states and Gromov-Hausdorff convergence:** for compact quantum metric spaces in the sense of Rieffel, we show in Prop. 4.2 that any normal state of the initial algebra $A$ can be approximated in the metric topology of $d_{A,D}$ by a sequence of truncated states. We also show in Prop. 4.3 that the convergence holds true not only for individual state, but also for metric spaces, in the Gromov-Hausdorff sense. Similar results are obtained in Prop. 4.8 and Cor. 4.9 for spectral triples that are not quantum metric spaces. There, normal states are substituted by a noncommutative generalization of states with finite moment of order 1, a notion which is introduced in Def. 4.5. The difference between the weak* and the metric topologies is illustrated on a simple example: the lattice $\mathbb{Z}$. It is shown in Prop. 4.13 that the closure of the inductive limit of truncated normal states of $C_0(\mathbb{Z})$ is the space of normal states for the weak* topology, and the space of states with finite moment of order 1 for the metric topology.

- **Approximation of points:** we show in Prop. 5.1 that on a commutative spectral geometry with a cut-off, the distance $d_{A,D_\Lambda}$ between points (i.e. pure states) is never smaller than the cut-off, and is even infinite when $P_\Lambda$ has finite rank (Prop. 5.4). Coming back to the original physical motivation, namely for $P_\Lambda$ an eigenprojection of the Dirac operator, we show in Prop. 5.6 how to approximate points by non-pure states that remain at finite distance from one another. Specifically on the real line, we work out an approximation of points by non-pure states such that both distances $d_{A,D_\Lambda}, d_{\mathcal{H},D_\Lambda}$ are finite, and the latter actually coincides with $d_{A,D}$ between points (Prop. 5.7). On the circle, we approximate
points by the non-pure state given by the evaluation of the Fejér transform of \( f \). We show in Prop. 5.11 that the distances are always smaller than the geodesic one and converge to it as the rank of the Fejér transform goes to infinity. We also provide a tight lower bound. Alternative approximations of points on the circle are investigated in §5.5.

\textit{Wasserstein distance and Berezin quantization:} in Prop. 6.4 we show how Moyal plane can be seen as the complex plane with a cut-off, taking for \( P_\Lambda \) the projection on holomorphic functions (Berezin-Toeplitz quantization). We obtain in Prop. 6.9 a new proof that the distance between coherent states of Moyal plane is the Euclidean distance. Similar results are obtained for the fuzzy sphere in Prop. 6.17.

Several results of this paper have an interest independently of noncommutative geometry, as explicit computations of the Wasserstein-Monge-Kantorovich distance: on the real line (Prop. 5.7 and §7), and on the circle (Prop. 5.11 and 5.13).

\section{Preliminaries}

We recall some basics on the metric aspect of Connes noncommutative geometry, Rieffel theory of compact quantum metric spaces, and Hausdorff convergence.

\subsection{Metric aspect of noncommutative geometry}

A spectral triple \((A, \mathcal{H}, D)\) is the datum of a separable Hilbert space \( \mathcal{H} \), a \( \ast \)-subalgebra \( A \subset \mathcal{B}(\mathcal{H}) \) and a self-adjoint operator \( D \) on \( \mathcal{H} \) such that, for all \( a \in A \), \([D,a] \in \mathcal{B}(\mathcal{H})\) and \( a(D + i)^{-1} \in \mathcal{K}(\mathcal{H}) \). We say that \((A, \mathcal{H}, D)\) is \textit{unital} if \( A \) is a unital algebra. The latter condition is then equivalent to \( D \) having compact resolvent.

Although one can work with real Hilbert spaces and algebras (as for the spectral triple of the Standard Model [13]), we assume that \( \mathcal{H} \) and \( A \) are complex. With some additional assumptions, from any commutative unital spectral triple one reconstructs an underlying Riemannian manifold \( M \) [16]. That is the reason why a spectral triple over a noncommutative algebra is viewed as the noncommutative analogue of a manifold.

A state of a \( C^* \)-algebra \( A \) is a positive linear functional \( \varphi : A \to \mathbb{C} \) with norm 1. The set of states, denoted \( S(A) \), is convex, with extreme points the pure states. It is often convenient to work with a dense subalgebra \( A \) of \( A \) (like \( C_0^\infty(M) \subset C_0(M) \)). In that case, by a state of \( A \) we mean a state of its \( C^* \)-completion \( \hat{A} \).

An important class of states is given by normal states. They are usually defined for von Neumann algebras as completely additive states [32, Def. 7.1.1]. We use the following equivalent definition [9, Thm. 2.4.21], generalized to \( C^* \)-algebras [54][9, Def. 2.4.25]): a state \( \varphi \) of \( A \subset \mathcal{B}(\mathcal{H}) \) is \textit{normal} if there exists a positive trace-class operator \( R \) on \( \mathcal{H} \) with \( \text{Tr}(R) = 1 \), called \textit{density matrix}, such that

\[ \varphi(a) = \text{Tr}(Ra) \quad \forall a \in A. \quad (2.1) \]

We denote by \( \mathcal{N}(A) \) the set of all normal states of \( A \). Notice that the map (2.1) from density matrices to \( \mathcal{N}(A) \) is surjective but not always injective: as soon as \( A \) is not the whole of \( \mathcal{B}(\mathcal{H}) \), there may be different density matrices associated to the same state.
Given \((A, \mathcal{H}, D)\) with \(A\) a (pre) \(C^*\)-algebra, \(S(A)\) is an extended metric space\(^1\) with distance
\[
d_{A,D}(\varphi, \varphi') := \sup_{a \in A^{sa}} \{ |\varphi(a) - \varphi'(a)| : L_D(a) \leq 1 \}
\]
(2.2)
for all \(\varphi, \varphi' \in S(A)\), where \(L_D\) denotes the seminorm defined on \(A\) by the operator \(D\),
\[
L_D(a) := \| [D, a] \|.
\]
(2.3)

We refer to it as the spectral distance. Although in the original definition \([14]\) the supremum is over all \(a \in A\) obeying the side condition, it was noted in \([30]\) that the supremum can be equivalently searched on the set \(A^{sa}\) of self-adjoint elements of \(A\).

When \(A = C_0^\infty(M)\) for \(M\) a Riemannian (spin) manifold and \(D\) is a Dirac type operator, the spectral distance (2.2) on pure states coincides with the geodesic distance of the Riemannian metric.\(^2\) On arbitrary states, if \(M\) is complete, it coincides with the Wasserstein distance of optimal transport theory (see e.g. \([20]\)).

### 2.2 Compact quantum metric spaces

An order unit space \([32]\) is a real partially ordered vector space \(O\) with a distinguished element \(e\), called the order unit, such that: i) \(\forall a \in O \exists r \in \mathbb{R}\) such that \(a \leq re\); ii) if \(a \leq re \forall r > 0\), then \(a \leq 0\). A norm on \(O\) is given by
\[
\|a\| := \inf \{ r > 0 : -re \leq a \leq re \}.
\]
(2.4)

A state on \(O\) is a bounded linear map \(\varphi : O \to \mathbb{R}\) with norm 1, that is \([32, \text{Thm. } 4.3.2]\) \(\varphi(e) = 1\). States are automatically positive. The collection \(S(O)\) of all states of \(O\) is a compact topological space with respect to the weak* topology.

Any real vector subspace \(O\) of \(B(\mathcal{H})^{sa}\) containing the identity 1 is an order unit space for the partial ordering of operators, with order unit \(e = 1\). Actually any order unit space comes in this way \([51]\), so it makes sense to talk about normal states for order unit spaces.

A seminorm \(L\) on \(O\) defines on \(S(O)\) an extended metric
\[
\rho_L(\varphi, \varphi') := \sup_{a \in O} \{ |\varphi(a) - \varphi'(a)| : L(a) \leq 1 \}.
\]
(2.5)

Given a unital spectral triple \((A, \mathcal{H}, D)\), taking \(O = A^{sa}\) and \(L = L_D\), one recovers the spectral distance (2.2). The seminorm \(L\) is called Lipschitz \([50]\) if \(L(a) = 0\) implies \(a \in \mathbb{R}e\). This is a necessary (but not sufficient) condition in order for \(\rho_L\) to be finite.

If \(X\) is a compact metric space and \(O = C(X, \mathbb{R})\), then
\[
L(f) = \sup_{x \neq y} |f(x) - f(y)| / d(x, y)
\]
(2.6)
is a Lipschitz seminorm and the associated metric \(\rho_L\) induces on \(S(O)\) the weak* topology. This motivates the definition of a compact quantum metric space \([49, 50]\) as a pair \((O, L)\) such that \(L\) is Lipschitz and \(\rho_L\) induces on \(S(O)\) the weak* topology. These two conditions guarantee that \(\rho_L\) is finite on \(S(O)\) \([49, \text{Thm. } 2.1]\).

Locally compact quantum metric spaces have been recently introduced in \([37]\).

\(^1\)An extended metric space is a pair \((X, d)\) with \(X\) a set and \(d : X \times X \to [0, \infty]\) a symmetric map satisfying the triangle inequality and such that \(d(x, y) = 0\) iff \(x = y\). It differs from an ordinary metric only in that the value +\(\infty\) is allowed.

\(^2\)Any point \(x \in M\) is recovered as the pure state “evaluation at \(x\)”, \(\delta_x(f) := f(x)\), and any pure state of \(\mathcal{C}_0^\infty(M)\) comes from a point.
2.3 Hausdorff and Gromov-Hausdorff distance

Let $X, Y$ be subsets of a metric space $(M, d)$, $d(x, Y) := \inf_{y \in Y} d(x, y)$ the distance between $x \in X$ and the set $Y$, and $d(X, Y) := \sup_{x \in X} d(x, Y)$ the largest possible distance between a point of $X$ and the set $Y$. The Hausdorff distance between $X$ and $Y$ is (see e.g. [10, 28]):

$$d_H(X, Y) := \max \{ d(X, Y), d(Y, X) \}.$$  \hspace{1cm} (2.7)

It is a semi-metric on the set of subsets of $M$ (meaning that distinct subsets of $M$ can be at zero distance), as $d_H(X, Y) = 0$ iff $X$ and $Y$ have the same closure. It becomes an extended metric if we consider only closed subsets of $M$ [10, Prop. 7.3.3]. In particular, $d_H$ is an extended metric on the collection of compact subsets of $M$, and a metric if $M$ has finite diameter.

3 Truncations

The regularization procedure motivated by quantum field theory, consisting in cutting off the spectrum of $D$, is implemented by the action of a finite-rank projection $P_\Lambda \in \mathcal{B}(\mathcal{H})$. Substituting in (2.2) the Dirac operator with $D_\Lambda$ as in (1.1) modifies the spectral distance. In this section we study the relation between the distances associated to $D$ and $D_\Lambda$.

3.1 Regularization of the geometry

Since $D_\Lambda$ is bounded (it has finite rank) and compact operators form a two-sided ideal in $\mathcal{B}(\mathcal{H})$, if $b = a(D_\Lambda + i)^{-1}$ is compact then $a = b(D_\Lambda + i)$ is compact too. Thus $a(D_\Lambda + i)^{-1}$ cannot be compact for all $a \in \mathcal{A}$, unless $\mathcal{A} \subset \mathcal{K}(\mathcal{H})$. So $(\mathcal{A}, \mathcal{K}, D_\Lambda)$ in general is not a spectral triple. Nevertheless $[D_\Lambda, a]$ is bounded for any $a \in \mathcal{A}$ and equation (2.2) still defines an extended metric $d_{A,D_\Lambda}$ on $S(\mathcal{A})$.

Assuming that $\mathcal{A}$ is unital, let $\pi_\Lambda : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be the linear map

$$\pi_\Lambda(a) := P_\Lambda a P_\Lambda$$  \hspace{1cm} (3.1)

and $\mathcal{O}_\Lambda$ the image of $\mathcal{A}^{sa}$:

$$\mathcal{O}_\Lambda := \pi_\Lambda(\mathcal{A}^{sa})$$  \hspace{1cm} (3.2)

Proposition 3.1. $\mathcal{O}_\Lambda$ is a finite-dimensional order unit space, with order unit $P_\Lambda$. Every state of $\mathcal{O}_\Lambda$ is normal,

$$S(\mathcal{O}_\Lambda) = N(\mathcal{O}_\Lambda).$$  \hspace{1cm} (3.3)

Furthermore, if

$$L_\Lambda(\cdot) := \|[D_\Lambda, \cdot]\|$$  \hspace{1cm} (3.4)

is a Lipschitz seminorm, then $(\mathcal{O}_\Lambda, L_\Lambda)$ is a compact quantum metric space.

\footnote{What we call “extended metric” is simply called a “metric” in [10].}
Proof. Call $\mathcal{H}_A := P_\Lambda \mathcal{H}$ the range of $P_\Lambda$. Then $\mathcal{O}_\Lambda$ is a real vector subspace of $\mathcal{B}(\mathcal{H}_A)^{sa}$. Since $1 \in A$, $\mathcal{O}_\Lambda$ contains the identity operator of $\mathcal{H}_A$, that is $P_\Lambda = \pi_\Lambda(1)$. Hence $\mathcal{O}_\Lambda$ is an order unit space. By [32, Thm. 4.3.13(i)] every state of $\mathcal{O}_\Lambda$ can be extended to a state of $\mathcal{B}(\mathcal{H}_A)$, hence it is normal, being the latter a finite dimensional matrix algebra.

For the second statement, one can repeat verbatim the proof of [12, Prop. 4.2].

\textbf{Remark 3.2.} If $A$ is not unital, one may consider its minimal unitization $A^+ = A \oplus \mathbb{C}$ with $z \in \mathbb{C}$ acting on $\mathcal{H}$ as a multiple of the identity operator 1. $(A^+, \mathcal{H}, D)$ may not be a spectral triple (if $D$ has not a compact resolvent, the condition $\sigma(D + i)^{-1}$ is not satisfied). Nevertheless the spectral distance $d_{A^+, D}$ is still well-defined on $S(A^+)$, and coincides with $d_{A,D}$ on $S(A) \subset S(A^+)$ [44, Lemma V.6]. The same is true for $d_{A,D_A}$ and $d_{A,D}$.\vspace{2mm}

\textbf{Example 3.3.} The complexification $\pi_\Lambda(A)$ of $\mathcal{O}_\Lambda$ is a vector subspace of $\mathcal{B}(\mathcal{H}_A)$ but not necessarily a subalgebra. For instance take $\mathcal{H} = \mathbb{C}^4$, $A \simeq M_2(\mathbb{C})$ the subalgebra of $M_4(\mathbb{C})$ of block-diagonal matrices with identical blocks:

$$
\begin{bmatrix}
    a_{11} & a_{12} & 0 & 0 \\
    a_{21} & a_{22} & 0 & 0 \\
    0 & 0 & a_{11} & a_{12} \\
    0 & 0 & a_{21} & a_{22}
\end{bmatrix}, \quad a_{ij} \in \mathbb{C},
$$

and $P_\Lambda = \text{diag}(1, 1, 1, 0)$. Every element of $\pi_\Lambda(A)$ is a matrix with the same element in position $(1, 1)$ and $(3, 3)$. If $a \in A$ is the element with $a_{11} = a_{22} = 0$ and $a_{12} = a_{21} = 1$, clearly

$$
\pi_\Lambda(a) \cdot \pi_\Lambda(a) = 
\begin{bmatrix}
    1 & 0 & 0 & 0 \\
    0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0
\end{bmatrix}
$$

is not in $\pi_\Lambda(A)$, meaning that $\pi_\Lambda(A)$ is not a subalgebra of $\mathcal{B}(\mathcal{H})$. However $\pi_\Lambda(A)$ is an algebra as soon as $P_\Lambda$ is in $A$ or in its commutant $A'$ (that is $[P_\Lambda, a] = 0$ for all $a \in A$). \hfill \Box$

\section*{3.2 Truncated topologies}

Given a unital spectral triple $(A, \mathcal{H}, D)$ and a finite-rank projection $P_\Lambda$, we thus obtain three distinct extended metric spaces:

$$
(S(A), d_{A,D}), \quad (S(A), d_{A,D_A}), \quad (S(\mathcal{O}_\Lambda), d_{\mathcal{O}_\Lambda,D_A})
$$

where $d_{\mathcal{O}_\Lambda,D_A}$ denotes the distance defined by (2.5) on $S(\mathcal{O}_\Lambda)$ by the seminorm $L_\Lambda$. In the passage $d_{A,D} \rightarrow d_{A,D_A}$, only the metric structure incoded in $D$ changes. In the passage $d_{A,D_A} \rightarrow d_{\mathcal{O}_\Lambda,D_A}$ the state space itself is modified. We aim at answering two questions:

- \textbf{Equivalence:} Under which conditions are these distances equivalent?
- \textbf{Convergence:} Given $(A, \mathcal{H}, D)$ with infinite dimensional $A$, can the extended metric space $(S(A), d_{A,D})$ be approximated in a suitable sense by a sequence of metric spaces associated to truncated operators $D_A$?

We study below the first point. The second one is discussed in the next section.

To begin with, let us note that the last distance in (3.7) is defined on a different space than the other two. So in order to compare them one needs to clarify the relation between states of $A$ and of $\mathcal{O}_\Lambda$. This is where normal states turn out to be important.
Lemma 3.4. There is an injective (not necessarily surjective) map $\sharp : S(O_A) \rightarrow N(A)$, $\varphi \mapsto \varphi^\sharp$, given by:

$$\varphi^\sharp := \varphi \circ \pi_A .$$

\textbf{Proof.} Since $\pi_A$ in (3.1) preserves positivity, and $\varphi^\sharp(1) = \varphi(\pi_A(1)) = \varphi(\rho_A) = 1$, clearly $\varphi^\sharp$ is a state of $A$ (it is actually a state of $A^{\text{sa}}$, extended in a unique way to $A$ by $\mathbb{C}$-linearity).

Let $\varphi, \psi \in S(O_A)$ and suppose $\varphi^\sharp(a) = \psi^\sharp(a)$ for all $a \in A^{\text{sa}}$. Since $O_A = \pi_A(A^{\text{sa}})$, for any $b \in O_A$ there exists $a \in A$ such that $b = \pi_A(a)$. One has $\varphi(b) = \varphi^\sharp(a) = \psi^\sharp(a) = \psi(b)$. Hence $\varphi = \psi$, and the map $\varphi \mapsto \varphi^\sharp$ is injective.

Any $\varphi \in S(O_A)$ is normal. Let $R$ be a density matrix for $\varphi$. For any $a \in A$ one has $\varphi^\sharp(a) = \text{Tr}(RP_AaP_A) = \text{Tr}(\rho a)$ with $\rho := P_ARP_A$, meaning that $\rho$ is a density matrix for $\varphi^\sharp$, hence the latter is normal. \hfill $\blacksquare$

The map $\sharp$ being injective allows to define two extended metrics on $S(O_A)$:

$$d_{A,D}^\vartheta(\varphi, \psi) := d_{A,D}(\varphi^\sharp, \psi^\sharp) \quad \text{and} \quad d_{A,D_A}^\vartheta(\varphi, \psi) := d_{A,D_A}(\varphi^\sharp, \psi^\sharp).$$

(3.9)

In the following proposition we discuss conditions for their equivalence.

**Proposition 3.5.** If $L_A$ is a Lipschitz seminorm, then $d_{A,D}^\vartheta$ and $d_{O_A,D_A}$ are strongly equivalent on $S(O_A)$. If in addition i) $L_D$ is Lipschitz, or ii) $P_A$ is in the commutant $A'$ of $A$, or iii) $[D, P_A] = 0$, then $d_{A,D}^\vartheta$ and $d_{O_A,D_A}$ are strongly equivalent.

\textbf{Proof.} Since $P_A$ has finite rank, $O_A \subset L^1(\mathcal{H})$ and we can consider traceless operators

$$V_A = \{ b \in O_A : \text{Tr}(b) = 0 \} .$$

(3.10)

By (1.1) and (3.1) one has $[D_A, P_A] = 0$, thus adding a multiple of the order unit $P_A$ to $b$ does not change $L_A(b)$ nor $\varphi(b) - \varphi'(b)$. Therefore for any $\varphi, \varphi' \in S(O_A)$:

$$d_{O_A,D_A}(\varphi, \psi) = \sup_{b \in V_A} \{ \varphi(b) - \psi(b) : L_A(b) \leq 1 \} .$$

(3.11a)

Likewise, adding a multiple of $1$ to $a \in A$ doesn’t change $L_D(a), L_A(a)$ nor $\varphi^\sharp(a) - \varphi^\sharp'(a)$, so

$$d_{A,D}^\vartheta(\varphi, \psi) = \sup_{a \in \pi_A^{-1}(V_A), L_D(a) \leq 1} \varphi^\sharp(a) - \varphi^\sharp'(a) = \sup_{b \in V_A, L'(b) \leq 1} \varphi(b) - \psi(b) ,$$

(3.11b)

$$d_{A,D_A}^\vartheta(\varphi, \psi) = \sup_{a \in \pi_A^{-1}(V_A), L_A(a) \leq 1} \varphi^\sharp(a) - \varphi^\sharp'(a) = \sup_{b \in V_A, L''(b) \leq 1} \varphi(b) - \psi(b) ,$$

(3.11c)

where $\pi_A^{-1}(V_A)$ is the preimage of $V_A$ in $A$ and we call

$$L'(b) := \sup_{a \in \pi_A^{-1}(V_A), \pi_A(a) = b} L_D(a) , \quad L''(b) := \sup_{a \in \pi_A^{-1}(V_A), \pi_A(a) = b} L_A(a) .$$

(3.12)

The proposition is proved as soon as we show that $L_A, L'$ and $L''$ are norm on $V_A$: all norms on a finite-dimensional vector space are equivalent, so that the distances (3.11a), (3.11b) and (3.11c) dual to $L_A, L'$ and $L''$ are strongly equivalent. $L_A$ being a norm follows from the Lipschitz hypothesis: $L_A(b) = 0$ implies $b = \lambda P_A$, but since $b$ is traceless, it must be $b = 0$. Regarding $L'$ and $L''$, one easily checks the triangle inequality, so they are seminorms. That $L''$ is actually a norm comes from $[D_A, P_A] = 0$ that, for any $b = \pi_A(a)$, implies

$$L_A(a) \geq L_A(b) .$$

(3.13)
Hence $L''(b) \geq L_A(b)$, and $L''$ is a norm too on $V_A$.

For $L'$, in case $L_D$ is Lipschitz one has

$$L'(b) = 0, \ b \in V_A \implies L_D(a) = 0 \ \forall a \in \pi_A^{-1}(V_A),$$

(3.14)
hence $a = \lambda 1$, so that $b = \lambda P_A = 0$ because of the traceless condition. Thus $L'$ is a norm. Otherwise one notices that

$$[D_A, \pi_A(a)] = P_A ([D, a] + [[P_A, a], [D, P_A]] P_A,$$

(3.15)
so that $[P, D_A] = 0$ or $P_A \in \mathcal{A}'$ implies

$$L_D(a) \geq L_A(b)$$

(3.16)
for all $b = \pi_A(a)$. Hence $L'(b) \geq L_A(b)$ and $L'$ is a norm.

With some conditions on the projection $P_A$, but no Lipschitz condition on $L_D$ nor $L_A$, one gets that truncating the algebra yields less states (the map $\#$ is not surjective on $S(\mathcal{A})$) but larger distances.

**Proposition 3.6.** For all $\varphi, \psi \in S(\mathcal{O}_A)$,

$$d_{\mathcal{O}_A, D_A}(\varphi, \psi) \leq d_{\mathcal{O}_A, D_A}(\varphi, \psi),$$

(3.17)
with equality if $P_A \in \mathcal{A}$. If $[D, P_A] = 0$ or $P_A \in \mathcal{A}'$, then

$$d_{\mathcal{O}_A, D}(\varphi, \psi) \leq d_{\mathcal{O}_A, D_A}(\varphi, \psi).$$

(3.18)

If $[D, P_A] = 0$ and $P_A \in \mathcal{A}$, then

$$d_{\mathcal{O}_A, D_A}(\varphi, \psi) \leq d_{\mathcal{O}_A, D_A}(\varphi, \psi).$$

(3.19)

Furthermore, if $P_A \in \mathcal{A}'$, one also has for all $\varphi, \psi \in S(\mathcal{A})$:

$$d_{\mathcal{A}', D}(\varphi, \psi) \leq d_{\mathcal{A}', D_A}(\varphi, \psi).$$

(3.20)

**Proof.** Eq. (3.17) is the dual of (3.13):

$$d_{\mathcal{O}_A, D_A}(\varphi, \psi) = \sup_{a \in \mathcal{A}^{=a}} \{ \varphi^\flat(a) - \psi^\flat(a) : L_A(a) \leq 1 \} = \sup_{a \in \mathcal{A}^{=a}} \{ \varphi(b) - \psi(b) : L_A(a) \leq 1 \}$$

$$\leq \sup_{b \in \mathcal{O}_A} \{ \varphi(b) - \psi(b) : L_A(b) \leq 1 \} = d_{\mathcal{O}_A, D_A}(\varphi, \psi).$$

(3.21)

If $P_A \in \mathcal{A}$, then $\mathcal{O}_A \subset \mathcal{A}$ and $d_{\mathcal{A}', D_A}(\varphi^\flat, \psi^\flat) \geq d_{\mathcal{O}_A, D_A}(\varphi, \psi)$, showing the previous inequality is an equality. Eq. (3.18) is the dual of (3.16). Eq. (3.20) is the dual of

$$L_D(a) \geq L_A(a)$$

(3.22)
which follows from (3.16) and the observation that $P_A \in \mathcal{A}'$ implies $L_A(a) = L_A(b)$.

Assume $[D, P_A] = 0$ and $P_A \in \mathcal{A}$. Then $L_A(b) = L_D(b)$ for any $b \in \mathcal{O}_A$ so that

$$d_{\mathcal{O}_A, D_A}(\varphi, \psi) = \sup_{b \in \mathcal{O}_A} \{ \varphi(b) - \psi(b) : L_D(b) = 1 \}$$

(3.23a)

$$\leq \sup_{a \in \mathcal{A}} \{ \varphi^\flat(a) - \psi^\flat(a) : L_D(a) = 1 \} = d_{\mathcal{O}_A, D}(\varphi, \psi),$$

(3.23b)
where we identify $\varphi, \psi$ (defined on the subalgebra $\pi_A(\mathcal{A}) \subset \mathcal{A}$) to their extension $\varphi^\flat, \psi^\flat$.

Note that unlike proposition 3.5, proposition 3.6 does not require $P_A$ to be finite rank, not even $D_A$ to be bounded.

**Remark 3.7.** When $P_A$ is a central projection commuting with $D$, then (3.18) and (3.19) combine to give $d_{\mathcal{O}_A, D} = d_{\mathcal{O}_A, D_A}$. This is Lemma 1 of [45], that allows to compute the distance in the spectral triple of the Standard Model, by reducing $C \oplus H \oplus M_3(C)$ to $R \oplus R$. 

10
3.3 Multi-ranked truncations

We now consider a sequence of increasing finite rank projections in $\mathcal{B}(\mathcal{H})$:

$$P_N \leq P_{N+1}, \quad \forall N \in \mathbb{N}. \quad (3.24)$$

Once fixed a truncation $P_N$ for some $N \in \mathbb{N}$, instead of considering the non-truncated algebra $\mathcal{A}$ one may consider a less-truncated order unit space $\mathcal{O}_M, M > N$. All the results of the preceding section hold, substituting $\mathcal{A}, D$ with $\mathcal{O}_M, D_M$ and $\mathcal{O}_\Lambda, D_\Lambda$ with $\mathcal{O}_N, D_N$.

Lemma 3.8. There is an injective (but not necessarily surjective) map from $S(\mathcal{O}_N)$ to $S(\mathcal{O}_N)$ given by $\varphi \to \varphi^\dagger := \varphi \circ \pi_N$, where we write $\pi_N := P_N(\cdot)P_N \forall N \in \mathbb{N}$.

Proof. For any $M \geq N$, one has [32, Prop. 2.5.2]:

$$P_N P_M = P_M P_N = P_N, \quad (3.25)$$

so that $\varphi^\dagger(P_M) = \varphi(\pi_N(P_M)) = \varphi(P_N) = 1$. The proves that $\varphi^\dagger$ is a state and of injectivity are then identical to Prop. 3.4, using that any $b$ in $\mathcal{O}_N$ can be written as $\pi_N(c)$ for some $c$ in $\mathcal{O}_M$ (for $b = \pi_N(a)$ one takes $c = \pi_M(a)$).

For any $\varphi, \psi \in S(\mathcal{O}_N)$ we write

$$d_{\mathcal{O}_M, D}(\varphi, \psi) := d_{\mathcal{O}_M, D}(\varphi^\dagger, \psi^\dagger) \quad \text{and} \quad d_{\mathcal{O}_M, D_N}(\varphi, \psi) := d_{\mathcal{O}_M, D_N}(\varphi^\dagger, \psi^\dagger). \quad (3.26)$$

We collect in a single proposition the multi-ranked version of Prop. 3.5 and 3.6.

Proposition 3.9. i) For all $\varphi, \psi \in S(\mathcal{O}_N)$ and $M \geq N$,

$$d_{\mathcal{O}_M, D_N}(\varphi, \psi) \leq d_{\mathcal{O}_N, D_N}(\varphi, \psi). \quad (3.27)$$

The two distances are strongly equivalent if $L_N$ is Lipschitz on $\mathcal{O}_N$, and equal if $P_N \in \mathcal{O}_M$.

ii) If $[D_M, P_N] = 0$ or $P_N \in \mathcal{O}_M'$ then

$$d_{\mathcal{O}_M, D_M}(\varphi, \psi) \leq d_{\mathcal{O}_N, D_N}(\varphi, \psi) \quad (3.28)$$

If $[D_M, P_N] = 0$ and $P_N \in \mathcal{O}_M$ then

$$d_{\mathcal{O}_N, D_N}(\varphi, \psi) \leq d_{\mathcal{O}_M, D_M}(\varphi, \psi). \quad (3.29)$$

The two distances are strongly equivalent if $L_N$ is Lipschitz on $\mathcal{O}_N$ and either $[D_M, P_N] = 0$, or $P_N \in \mathcal{O}_M'$, or $L_M$ is Lipschitz on $\mathcal{O}_M$.

iii) If $P_N$ is in the commutant $\mathcal{O}_M$, then for all $\varphi, \psi \in S(\mathcal{O}_M)$ one has

$$d_{\mathcal{O}_M, D_M}(\varphi, \psi) \leq d_{\mathcal{O}_M, D_N}(\varphi, \psi). \quad (3.30)$$

Proof. Eq. (3.27) is dual to $L_N(c) \geq L_N(b)$ for $b = \pi_N(c), b \in \mathcal{O}_N, c \in \mathcal{O}_M$, coming from $[D_N, P_N] = 0$. When $P_N \in \mathcal{O}_M$, then $\mathcal{O}_N \subset \mathcal{O}_M$ and (3.27) is an equality.

Eq. (3.28) is dual to $L_M(c) \geq L_N(b)$, coming from the hypothesis on $P_N$ (see (3.15)).

Eq. (3.30) is dual to $L_M(c) \geq L_N(c)$, coming from (3.25).

Eq. (3.29) is proved as eq. (3.23a).
For the strong equivalences one proceeds as in Prop. 3.5: \([D_N, P_N] = 0\), so with \(V_N\) the set of traceless elements of \(O_N\), one has
\[
ed_{O,N,D_N}(\varphi, \psi) = \sup_{b \in V_N} \{\varphi(b) - \psi(b) : L_N(b) \leq 1\}. \tag{3.31}
\]
Adding a multiple of \(P_M\) doesn’t change \(L_N(c)\) (because \([D_N, P_M] = 0\) by (3.25)) nor \(L_M(c)\) (because \([D_M, P_M] = 0\)). Thus instead of (3.11b) and (3.11c) one has
\[
d_{O,M,D_M}(\varphi, \psi) = \sup_{b \in V_N, L'(b) \leq 1} \varphi(b) - \psi(b), \quad d_{O,M,D_N}(\varphi, \psi) = \sup_{b \in V_N, L''(b) \leq 1} \varphi(b) - \psi(b) \tag{3.32}
\]
with
\[
L'(b) := \sup_{c \in \pi_{\pi}^{-1}(V_N), \pi_N(c) = b} L_M(c) \quad \text{and} \quad L''(b) := \sup_{c \in \pi_{\pi}^{-1}(V_N), \pi_N(c) = b} L_N(c). \tag{3.33}
\]
\(L_N\) is a norm on \(V_N\) by the Lipschitz condition, as well for \(L''\) (because \(L_N(c) \geq L_N(b)\)) and \(L'\) (either because \(L_M\) is a norm on \(O_M\) by Lipschitz hypothesis, or because \(L_M(c) \geq L_N(b)\) by the hypothesis on \(P_N\)).

4 Convergence of truncations

In this section we discuss how elements of \(\mathcal{S}(\mathcal{A})\) can be approximated by sequences of elements in \(\mathcal{S}(O_N)\), as well as the convergence of \((\mathcal{S}(O_N), d_{O,N,D_N})\) to \((\mathcal{S}(\mathcal{A}), d_{A,D})\) in the Gromov-Hausdorff sense. In §4.1 we focus on compact quantum metric spaces, in §4.2 we extend some of the results to the non-compact case, and in §4.3 we illustrate our results by the example of the lattice \(\mathbb{Z}\).

In all this section, \(\{P_N\}_{N \in \mathbb{N}}\) is a sequence of increasing finite-rank projections, as in (3.24). We make the extra-assumption that they converge to 1 in the weak operator topology, that is
\[
\lim_{N \to \infty} \langle v, P_N w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in \mathcal{H}. \tag{4.1}
\]

4.1 Compact quantum metric spaces & Gromov-Hausdorff convergence

Let us begin with a technical lemma.

**Lemma 4.1.** For any \(a \in \mathcal{B}({\mathcal{H}})\), the sequence \(\{\pi_N(a)\}_{N \in \mathbb{N}}\) weakly converges to \(a\).

**Proof.** For any \(v, w \in \mathcal{H}\) one has
\[
\langle v, (P_N a P_N - a) w \rangle = \langle v, a (P_N - 1) w \rangle + \langle (P_N - 1) v, a P_N w \rangle. \tag{4.2}
\]
Since \(\|P_N\| \leq 1\),
\[
\langle v, (P_N a P_N - a) w \rangle \leq 2 \|a\| \cdot \|v\| \cdot \|w\|. \tag{4.3}
\]
From (4.1) it follows \(\lim_{N \to \infty} \langle v, (P_N - 1) w \rangle = 0\), that concludes the proof. ■

For any projection \(P_N\), we call “truncated states” the image in \(\mathcal{S}(\mathcal{A})\) of the map \(\#\) defined in Lemma 3.4. On an arbitrary unital spectral triple, any normal state of \(\mathcal{A}\) can be weakly approximated by a sequence of truncated states.

**Proposition 4.2.** For any \(\varphi \in \mathcal{N}(\mathcal{A})\) there is a sequence of states \(\{\varphi_N\}_{N \in \mathbb{N}}\) such that:
1) \(\varphi_N \in \mathcal{S}(O_N)\) for all \(N \geq 0\); 2) \(\varphi_N \to \varphi\) in the weak\(^*\) topology.
Proof. Let us choose a density matrix $R$ for $\varphi$ and define

$$Z_N := \text{Tr}(RP_N) = \text{Tr}(\pi_N(R)) \ . \quad (4.4)$$

Any normal states is continuous on the unit ball of $\mathcal{B}(\mathcal{H})$ for the weak operator topology [32, Thm. 7.1.12]. Since $\|\pi_N(R)\| \leq 1$, Lemma 4.1 yields

$$\lim_{N \to \infty} Z_N = 1 \ . \quad (4.5)$$

Let $N_\varphi$ denote the smallest integer such that $Z_{N_\varphi} \neq 0$. Since $\pi_{N_\varphi} \circ \pi_N = \pi_{N_\varphi}$ for all $N \geq N_\varphi$, we have $Z_N \neq 0$ for all $N \geq N_\varphi$. Thus

$$\varphi_N(b) := Z_N^{-1} \text{Tr}(Rb) \quad (4.6)$$

is a well-defined state of $\mathcal{O}_N$ for all $N \geq N_\varphi$. For $N < N_\varphi$, choosing arbitrary states $\varphi_N \in S(\mathcal{O}_N)$ does not modify the limit, and we do not lose generality assuming $N_\varphi = 0$.

We now prove the weak limit. Due to the linearity of states, it is enough to show it for $\|a\| \leq 1$. Note that in this case $\|\pi_N(a)\| \leq 1$ too. By Lemma 4.1, $\pi_N(a) \to a$ weakly. Again by [32, Thm. 7.1.12], $\text{Tr}(R\pi_N(a)) \to \text{Tr}(Ra) = \varphi(a)$. Hence by (4.5):

$$\varphi_N^N(a) = Z_N^{-1} \text{Tr}(R\pi_N(a)) \to \varphi(a) \ . \quad \blacksquare$$

When $(\mathcal{A}^{sa}, L_D)$ is a compact quantum metric space, Prop. 4.2 shows that any normal state is the limit of truncated state in the metric topology induced by $d_{A,D}$. In this case one also has convergence of metric spaces in the Gromov-Hausdorff sense.

**Proposition 4.3.** Let $(\mathcal{A}^{sa}, L_D)$ be a compact quantum metric space and $\overline{N(A)}$ the weak closure of $N(A)$. Then $(S(\mathcal{O}_N), d^0_{A,D})$ converges to $(\overline{N(A)}, d_{A,D})$ for the Gromov-Hausdorff distance.

*Proof.* Since the map $\sharp : S(\mathcal{O}_N) \to N(A)$ in Prop. 3.4 is an isometric embedding, it is enough to prove that the subspaces $X_N := \sharp(S(\mathcal{O}_N))$ of $M := S(A)$ converge to $\overline{N(A)}$ in the Hausdorff sense.

Since $(\mathcal{A}^{sa}, L_D)$ is a compact quantum metric space, the metric topology coincides with the weak* topology on $S(A)$. Hence $M$ is compact and $X_N$ are compact subspaces $\forall N$.

For a sequence of compact subspaces $\{X_N\}$ of a compact metric space $M$, such that $X_N \subset X_{N+1}$ for all $N$, the Hausdorff limit $X$ is the closure of the union $\bigcup_X X_N$ [10, pag. 253]. Since $X_N \subset N(A)$, then $X \subset \overline{N(A)}$. On the other hand, from Prop. 4.2 it follows that $N(A) \subset X$. Hence $\overline{N(A)} = X$. $\blacksquare$

One may wonder what the closure of $N(A)$ is. If the the $C^*$-completion $A$ of $\mathcal{A}$ is a von Neumann algebra, then $N(A)$ is already closed [21, Lemma 1]. As well, if $A = \mathcal{K}$ then every state is normal and $N(A)$ is closed.

Another important class of examples is given by $A = C^\infty(M)$ and $\mathcal{H} = L^2(M)$, with $M$ a compact oriented Riemannian manifold. In this case $N(A)$ is a proper subset of $S(A)$, since for example pure states are not normal. Nevertheless, it is easy to prove that $\overline{N(A)} = S(A)$. Indeed, let $\psi_{\epsilon,x}$ be the (normalized) characteristic function of the ball with radius $\epsilon$ centered at $x \in M$. Since $f \in A$ is continuous, $\langle \psi_{\epsilon,x}, f \psi_{\epsilon,x} \rangle \to f(x)$ for $\epsilon \to 0$, and the pure state $\delta_x$ is the weak* limit of normal states. Hence, $\overline{N(A)}$ contains all finite convex combinations of pure states. By Krein-Milman theorem [3], every compact convex set (in
a locally convex space) is the closure of the convex hull of its extreme points. Thus, \( \mathcal{S}(A) \) is the closure of finite convex combinations of pure states, and this means \( \mathcal{S}(A) \subset \overline{\mathcal{N}(A)} \) (the opposite inclusion is obvious). The same holds if \( \mathcal{H} = L^2(M, E) \) with \( E \) a vector bundle, since \( E \) is locally trivial and for \( \epsilon \) small enough we can define a family of sections playing the role of \( \psi_{\epsilon, x} \). Hence,

**Corollary 4.4.** For \( A = C^\infty(M), \mathcal{H} = L^2(M, E) \) as above, and \( D \) a Dirac-type operator, \((\mathcal{S}(\mathcal{O}_N), \delta_{A,D})\) converge to \((\mathcal{S}(A), d_{A,D})\) for the Gromov-Hausdorff distance.

### 4.2 Beyond compact quantum metric spaces: states with finite moment of order 1

Let us now consider a spectral triple \((A, \mathcal{H}, D)\) such that \((A^{sa}, L_D)\) is not a compact quantum metric space, and \(\{P_N\}_{N \in \mathbb{N}}\) an increasing sequence of finite rank projections convergent to 1 in the weak operator topology. We can always assume that \(A\) is unital (replacing it by its unitization if needed, as explained in remark 3.2). The order unit spaces \(\mathcal{O}_N\) are well defined, and it makes sense to study the convergence of the sequence \((\mathcal{S}(\mathcal{O}_N), \delta_{A,D})\).

In the commutative case \(A = C^\infty_0(M), M\) a non-compact Riemannian manifold, an important class of states regarding the topology induced by the Wasserstein distance are those with finite moment of order 1 (on a connected manifold, the distance between any two such states is finite). Any \(\varphi \in \mathcal{S}(C^\infty_0(M))\) is given by a unique probability measure \(\mu\) on \(\mathcal{P}(C^\infty_0(M)) \simeq M\),

\[
\varphi(f) = \int_M f(x) \, d\mu_x \quad \forall f \in C_0(M)
\]  

and its moment of order 1 with respect to \(x' \in M\) is defined as

\[
M_1(\varphi, x') := \int_M d_{\text{geo}}(x, x') \, d\mu(x)
\]

For \(M\) connected, the finiteness of \(M_1(\varphi, x')\) does not depend on the choice of \(x'\): either it is finite for all \(x'\) or infinite for all \(x'\).

In the noncommutative case, a similar notion can be defined for states \(\varphi\) that are given by probability measures \(\mu\) on the pure state space, that is such that

\[
\varphi(a) = \int_{\mathcal{P}(A)} \omega(a) \, d\mu_\omega \quad \forall a \in A.
\]

This is not always the case, but there is a large class of noncommutative algebras \(A\) for which this does happen (e.g. unital separable \(C^*\)-algebras). We then say that \(\mu\) has a **finite moment of order 1 with respect to the pure state \(\omega'\)** if the expectation of the spectral distance from \(\omega'\), viewed as a function on \(\mathcal{P}(A)\), namely

\[
M_1(\mu, \omega') := \int_{\mathcal{P}(A)} d_{A,D}(\omega, \omega') \, d\mu_\omega,
\]

is finite. Notice that unlike the commutative case, for noncommutative \(A\) there may be different measures \(\mu\) on \(\mathcal{P}(A)\) giving the same state \(\varphi\): the quantity \(M_1(\mu, \omega')\) (in particular its finiteness) may depend to the choice of \(\mu\), as illustrated in example 4.6 below.

For a normal state \(\varphi\), we use the following alternative definition. Any density matrix \(R\) for \(\varphi\) is a positive compact operator, hence it is diagonalizable. Let \(\mathfrak{B} = \{\psi_n\}_{n \in \mathbb{N}}\) be
an orthonormal basis of $\mathcal{H}$ made of eigenvectors of $R$, with eigenvalues $p_n \in \mathbb{R}^+$. Denote $\Psi_n(a) := \langle \psi_n, a\psi_n \rangle$ the corresponding vector states in $S(A)$. Then one has

$$\varphi(a) = \sum_{n \geq 0} p_n \Psi_n(a) \quad \forall a \in A. \quad (4.11)$$

**Definition 4.5.** Given an arbitrary spectral triple $(A, \mathcal{H}, D)$ and a density matrix $R$, we call moment of order 1 of $R$ with respect to an eigenbasis $\mathcal{B}$ and to a state $\Psi_n$ (induced by a vector $\psi_n \in \mathcal{B}$) the moment of the distribution $\{p_n\}_{n \in \mathbb{N}}$, viewed as a discrete probability measure on the lattice, the latter being equipped with cost function $d_{A,D}$. Explicitly, we define

$$\mathcal{M}_1(R, \mathcal{B}, \Psi_n) := \sum_{k \geq 0} p_k d_{A,D}(\Psi_k, \Psi_n). \quad (4.12)$$

It is not difficult to prove that, once fixed $R$ and an eigenbasis $\mathcal{B}$, the finiteness of $\mathcal{M}_1(R, \mathcal{B}, \Psi_n)$ does not depend on the choice of the vector state $\Psi_n$.

We stress that definition 4.5 does not necessarily coincide with (4.10), because the vector states $\Psi_k$ are not necessarily pure: they are pure if e.g. $A = \mathcal{K}(\mathcal{H})$, but they are not for $\varphi$ a normal state of $A = C^\infty(M)$.

**Example 4.6.** Let $A = M_2(\mathbb{C})$. Any pure state $\Psi$ is a vector state, that is $\Psi(a) = \langle \psi, a\psi \rangle$ for any $a \in M_2(\mathbb{C})$, where $\psi$ is a unit vector in $\mathbb{C}^2$ and the inner product is the usual one. Any two vectors $\psi$ equal up to a phase determine the same pure state, so that $\mathcal{P}(M_2(\mathbb{C}))$ is the projective space $CP^1$. The latter is in 1-to-1 correspondence with the sphere $S^2$: $\psi_+ := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is mapped to the north pole of $S^2$, $\psi_- := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to the south pole, and the set of vectors

$$\psi_\theta := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ e^{i\theta} \end{pmatrix} \quad \theta \in [0, 2\pi] \quad (4.13)$$

is mapped to the equator. The whole state space (with weak*-topology) is homeomorphic to the unit ball in $\mathbb{R}^3$. For instance the center of the ball is the state $\varphi(a) := \frac{1}{2} \text{Tr}(a)$.

We consider the spectral triple described in [30], such that the distance between any two states is finite if and only if they are at the same latitude. In particular

$$d_{A,D}(\psi_+, \psi_-) = \infty, \quad d_{A,D}(\psi_\theta, \psi_{\theta'}) < \infty \quad \forall \theta, \theta' \in [0, 2\pi]. \quad (4.14)$$

The state $\varphi$ has density matrix $R = \frac{1}{2^n} I_2$, meaning that any orthonormal basis of $\mathbb{C}^2$ is an eigenbasis of $R$. In particular the canonical basis $\mathcal{B} := \{\psi_+, \psi_-\}$ and the basis $\mathcal{B}_\theta := \{\psi_\theta, \psi_{\pi+\theta}\}$ for any value of $0 \leq \theta \leq \pi$ yields two distinct decompositions of $\varphi$ on pure states:

$$\varphi = \frac{1}{2}(\psi_+ + \psi_-) = \frac{1}{2}(\psi_\theta + \psi_{\pi+\theta}). \quad (4.15)$$

Explicitly, for any $a = \{a_{ij}\} \in M_2(\mathbb{C})$ one has

$$\Psi_+(a) = \langle \psi_+, a \psi_+ \rangle = a_{11}, \quad \Psi_-(a) = \langle \psi_-, a \psi_- \rangle = a_{22}, \quad (4.16)$$

$$\Psi_\theta(a) = \langle \psi_\theta, a \psi_\theta \rangle = \frac{1}{2}(a_{11} + a_{12} e^{i\theta} + a_{21} e^{-i\theta} + a_{22}). \quad (4.17)$$

Notice that $\Psi_+, \Psi_-$ and $\Psi_\theta, \Psi_{\pi+\theta}$ are pure, so that here (4.15) corresponds to both decompositions (4.9) and (4.11): the first term in (4.15) may be viewed as the discrete measure $\mu := \{p_+ = \frac{1}{2}, p_- = \frac{1}{2}\}$ with support on the north and south poles, and the second term
as the discrete measure $\mu_\theta := \{p_\theta = \frac{1}{2}, p_{\theta+\pi} = \frac{1}{2}\}$ with support on the equatorial points with meridian coordinates $\theta, \theta + \pi$.

From (4.14) we see that moment of order 1 of $\varphi$ depends on the choice of the eigenbasis of $R$ (or equivalently on the choice of the measure):

$$M_1(\mu, \Psi_+) = M_1(R, \mathcal{B}, \Psi_+) = \frac{1}{2} d_{A,D}(\Psi_+, \Psi_-) = \infty \quad (4.18)$$

and similarly for $M_1(\mu, \Psi_-)$, whereas for any value $\theta \in [0, \pi]$

$$M_1(\mu_\theta, \Psi_\theta) = M_1(R, \mathcal{B}_\theta, \Psi_\theta) = \frac{1}{2} d_{A,D}(\Psi_\theta, \Psi_{\theta+\varphi}) < \infty \quad (4.19)$$

and similarly for $M_1(\mu_\theta, \Psi_{\theta+\pi})$. □

Among the normal states of $\mathcal{A}$, we single out the set $\mathcal{N}_0(\mathcal{A})$ of those for which there exists at least one density matrix $R$ with an eigenbasis $\mathcal{B} = \{\psi_n\}$ such that (4.12) is finite.

**Proposition 4.7.** Let $\varphi \in \mathcal{N}(\mathcal{A})$. For any choice of $(R, \mathcal{B}, \Psi_n)$ one has

$$d_{A,D}(\varphi, \Psi_n) \leq M_1(R, \mathcal{B}, \Psi_n). \quad (4.20)$$

In particular, if $\varphi \in \mathcal{N}_0(\mathcal{A})$ then $d_{A,D}(\varphi, \Psi_n)$ is finite.

**Proof.** From (4.11) it follows:

$$\varphi(a) - \Psi_n(a) = \sum_k p_k (\Psi_k(a) - \Psi_n(a)) \leq L_D(a) \sum_k p_k d_{A,D}(\Psi_k, \Psi_n) \quad (4.21)$$

for all $a \in \mathcal{A}^{\text{sa}}$. The last sum is the definition of $M_1(R, \mathcal{B}, \Psi_n)$. □

In a similar way, one obtains that $d_{A,D}(\varphi, \Psi_n) \leq M_1(\mu, \Psi_n)$ for all choices of $\mu$. In the commutative case one has the equality $d_{A,D}(\varphi, x') = M_1(\varphi, x')$ [20, Prop. 2.2]. In the noncommutative case the equality between spectral distance and moments of order 1 defined in (4.10) and (4.12) does not hold in general, as this would imply that these moments do not depend on how one decomposes $\varphi$, in contradiction with example 4.6.

We now prove an analogue of Prop. 4.2 for spectral triples that are not necessarily compact quantum metric spaces.

**Proposition 4.8.** Let $(\mathcal{A}, \mathcal{K}, D)$ be an arbitrary spectral triple and $\{P_N\}_{N \in \mathbb{N}}$ an increasing sequence of projections in $\mathcal{B}(\mathcal{H})$ convergent weakly to 1. For any $\varphi \in \mathcal{N}_0(\mathcal{A})$ such that $M_1(R, \mathcal{B}, \Psi_n)$ is finite for an eigenbasis $\mathcal{B}$ in which the $P_N$’s are all diagonal, then there exists a sequence $\{\varphi_N\}_{N \in \mathbb{N}}$ such that: i) $\varphi_N \in S(\mathcal{O}_N)$ for all $N \geq 0$; ii) $\varphi_N^2 \to \varphi$ in the metric topology.

**Proof.** Once fixed $(R, \mathcal{B})$, the state $\varphi_N$ is defined as in (4.6), namely

$$\varphi_N^\pi(a) = Z_N^{-1} \varphi(\pi_N(a)) = Z_N^{-1} \text{Tr} (P_N R P_N a). \quad (4.22)$$

By hypothesis $[P_N, R] = 0$ for any $N \in \mathbb{N}$, hence

$$\varphi(a) - \varphi_N^\pi(a) = \text{Tr} \left( (1 - P_N) R \left( a - \varphi_N^\pi(a) \right) \right). \quad (4.23)$$

Writing $a_{ij}$ the components of $a$ in the basis $\mathcal{B}$, one has

$$\text{Tr} ((1 - P_N) R) = \sum_{n>N} p_n a_{nm}, \quad \text{Tr} ((1 - P_N) R \varphi_N(a)) = Z_N^{-1} \sum_{n>N} (p_n \sum_{k<N} p_k a_{kk})$$
so that
\[
\varphi(a) - \varphi_N^\sharp(a) = Z_N^{-1} \sum_{n>N} p_n \left( \sum_{k<n} p_k (a_{nn} - a_{kk}) \right).
\] (4.24)

For any \(a\) such that \(L_D(a) \leq 1\) one has \(|a_{nn} - a_{kk}| = |\Psi_n(a) - \Psi_k(a)| \leq d_{A,D}(\Psi_n, \Psi_k)\), therefore

\[
\sup_{a \in A, L_D(a) \leq 1} |\varphi(a) - \varphi_N^\sharp(a)| \leq Z_N^{-1} \sum_{n>N} p_n \left( \sum_{k<n} p_k d_{A,D}(\Psi_n, \Psi_k) \right)
\]

\[
\leq Z_N^{-1} \sum_{n>N} p_n \left( \sum_{k<n} p_k d_{A,D}(\Psi_n, \Psi_0) + p_k d_{A,D}(\Psi_0, \Psi_k) \right)
\]

\[
\leq Z_N^{-1} \left( \sum_{n>N} p_n d_{A,D}(\Psi_n, \Psi_0) + p_n M_1(R, B, \Psi_0) \right).
\]

Both terms in the parenthesis are the rest of series converging to \(M_1(R, B, \Psi_0)\), and so vanish as \(N \to \infty\). Since \(Z_N \to 1\), one gets \(\lim_{N \to \infty} d_{A,D}(\varphi, \varphi_N^\sharp) = 0\). □

As a corollary, one obtains that any state \(\varphi\) in \(\mathcal{N}_0(A)\) can be approximated in the metric topology by a sequence of states with finite-rank density matrices.

**Corollary 4.9.** Let \((A, \mathcal{H}, D)\) be an arbitrary spectral triple and \(\varphi \in \mathcal{N}_0(A)\). There exists a sequence \(\{\varphi_N\}_{N \in \mathbb{N}}\) of normal states with finite-rank density matrix that is convergent to \(\varphi\) in the metric topology,

\[
\lim_{N \to \infty} d_{A,D}(\varphi, \varphi_N^\sharp) = 0.
\] (4.25)

Furthermore, for any \(\varphi, \varphi' \in \mathcal{N}_0(A)\),

\[
d_{A,D}(\varphi, \varphi') = \lim_{N \to \infty} d_{A,D}(\varphi_N^\sharp, \varphi_N^\sharp).
\] (4.26)

**Proof.** Take \((R, B)\) such that \(M_1(R, B, \Psi_0)\) is finite, and \(P_N\) the projection on the first \(N\) vectors of \(B\). Then (4.25) follows from proposition 4.8. Eq. (4.26) comes from the \(N \to \infty\) limit of the two following equations (obtained by the triangle inequality)

\[
d_{A,D}(\varphi, \varphi') \leq d_{A,D}(\varphi, \varphi_N^\sharp) + d_{A,D}(\varphi_N^\sharp, \varphi_N^\sharp) + d_{A,D}(\varphi_N^\sharp, \varphi_N^\sharp),
\]

\[
d_{A,D}(\varphi_N^\sharp, \varphi_N^\sharp) \leq d_{A,D}(\varphi, \varphi_N^\sharp) + d_{A,D}(\varphi_N^\sharp, \varphi_N^\sharp) + d_{A,D}(\varphi, \varphi') \text{ .} \]

Corollary 4.9 shows that the states with a finite rank density matrix are dense in \(\mathcal{N}_0(A)\). There is an important difference with the situation in the weak* topology: once fixed the net of projection \(P_N\), any normal state can be weakly approximated by states in \(\mathcal{S}(\Omega_N)\) (in fact from proposition 4.2 one has \(\lim_{N \to \infty} \mathcal{N}(\Omega_N) = \mathcal{N}(\mathcal{A})\)). On a non-compact quantum metric space, any state with finite moment of order 1 can be approximated by truncated states, but the truncations (i.e. the \(P_N\)-s) depends on the state. We investigate below simple, a one dimensional lattice, where the \(P_N\)-s are actually the same for all states.

### 4.3 Example: the lattice \(\mathbb{Z}\)

We identify \(A = C_0(\mathbb{Z})\) with the algebra of complex diagonal matrices,

\[
a = \text{diag}(\ldots a_{-1}, a_0, a_1, \ldots, a_n, \ldots) \quad \text{with} \quad \lim_{n \to \pm \infty} a_n = 0,
\] (4.27)
It is normal, with density matrix \( B \) basis by \( |\phi\rangle \) acting on \( H \).

On the other hand, if \( |\parallel\rangle \) which proves introduced in [42] by taking as cost function \( d_S \) We denote by \( \parallel \) the set of states with finite moment of order 1.

This concludes the proof of the first equation in the proposition.

Lemma 4.10. For any \( a \in \mathcal{A} \), one has \( \|[D, a]\| = \sup_n |a_n - a_{n+1}| \), hence

\[
\|[D, a]\| \leq 1 \quad \text{iff} \quad |a_n - a_k| \leq |n - k| \quad \forall n, k.
\]

Proof. Both sides of the first equation are invariant if we add a constant to \( a \), thus we assume that \( a_0 = 0 \). Noticing that \( 1 - D^2 \) is the projection operator on \( |0\rangle_\perp \), one has \( \|D\|^2 = \|D^2\| = 1 \). Since \( a_0 = 0 \), we have also \([D, a] = D^2[D, a]\). Thus

\[
\|[D, a]\| \leq \|D\| \cdot \|[D, a]\| = \|[D, a]\|.
\]

On the other hand

\[
\|[D, a]\| \leq \|[D]\| \cdot \|[D, a]\| = \|[D, a]\|
\]

which proves \( \|[D, a]\| = \|[D, a]\| \). This norm is easy to compute since:

\[
[D, a] |n\rangle_+ = (a_n - a_{n+1}) |n\rangle_+ , \quad D[D, a] |n\rangle_+ = (a_n - a_{n+1}) |n\rangle_+ .
\]

This concludes the proof of the first equation in the proposition.

Due to the triangle inequality, \( \|[D, a]\| \leq 1 \) implies

\[
|a_n - a_k| \leq \sum_{j=\min(n,k)}^{\max(n,k)-1} |a_j - a_{j+1}| \leq 1 = |n - k| .
\]

On the other hand, if \( |a_n - a_k| \leq |n - k| \) then \( |a_n - a_{n+1}| \leq 1 \) and \( \|[D, a]\| \leq 1 \).

The condition \( |a_n - a_k| \leq |n - k| \) is the discrete analogue of the 1-Lipschitz condition. This is what makes \( d_D \) equal to \( W_D \), as in the continuous case \( \mathcal{A} = C_0(M) \) (see e.g. [20]).
Proposition 4.11. For any states \( \varphi, \varphi' \in S(C_0(\mathbb{Z})) \) one has

\[
d_{A,D}(\varphi, \varphi') = W_D(\varphi, \varphi').
\] (4.37)

In particular, the spectral distance between any state and the pure state \( \delta_n, n \in \mathbb{Z} \) is

\[
d_{A,D}(\varphi, \delta_n) = \sum_{k \in \mathbb{Z}} |k - n| p_k,
\] (4.38)

meaning that the spectral distance between pure states of the lattice is

\[
d_{A,D}(\delta_m, \delta_n) = |m - n|.
\] (4.39)

Proof. By (4.32) one has \( d_D(\delta_m, \delta_n) \leq |m-n| \). The upper bound is attained by the element \( a \) with components \( a_i = i \) for \( i \leq \text{sup}(m,n) \), zero otherwise. Hence (4.39). Eq. (4.37) follows noticing that \( L_D(a) \leq 1 \) is equivalent to \( a \in \text{Lip}_D(A) \).

To prove (4.38), we use again Lemma 4.10 which yields

\[
|\varphi(a) - \delta_n(a)| = \left| \sum_{k \in \mathbb{Z}} p_k (a_k - a_n) \right| \leq \sum_{k \in \mathbb{Z}} |k - n| p_k.
\] (4.40)

This upper bound is attained by the sequence of elements

\[
a^{(m)}_k = \begin{cases} 
    k & \text{if } k \leq m, \\
    2m - k & \text{if } m < k \leq 2m, \\
    0 & \text{if } k > 2m.
\end{cases}
\]

\[\blacksquare\]

Remark 4.12. The result for the distance between pure states for the finite case \( (A = \mathbb{C}^N) \) had been obtained [23]. Note that the “spinorial” character of the Hilbert space \( l^2(\mathbb{Z}) \otimes \mathbb{C}^2 \) plays a crucial role. Consider instead \( \mathcal{H}' = l^2(\mathbb{Z}) \), with orthonormal basis \( |n\rangle \) and the Dirac operator acting as \( D'|n\rangle = |n+1\rangle - |n-1\rangle \). This is a finite approximation of the derivative on \( \mathbb{R} \) and has been considered in [6, 4]. In this case the distance between pure states \( \delta_n \) and \( \delta_n' \) is

\[
d_{A,D'}(\delta_m, \delta_n) = |m - n| + 1 \quad \text{if } m - n \text{ is odd}, \quad \text{(4.41a)}
\]

\[
d_{A,D'}(\delta_m, \delta_n) = \sqrt{|m - n|(|m - n| + 1)} \quad \text{if } m - n \text{ is even}. \quad \text{(4.41b)}
\]

On the lattice, the approximation of a state by its truncations is always possible in the weak* topology, but only for states with finite moment of order 1 in the metric topology.

Proposition 4.13. In the metric topology induced by \( d_{A,D} \) one has

\[
\lim_{N \to \infty} S(O_N) = S_0(A).
\] (4.42)

In the weak* topology one has

\[
\lim_{N \to \infty} S(O_N) = S(A).
\] (4.43)

Proof. \( S_0(A) \subseteq \lim_{N \to \infty} S(O_N) \) follows from corollary 4.9, noticing that on the lattice there is only one eigenbasis \( \mathcal{B} \), hence only one possible choice of the \( P_N \)'s. Eq. (4.42) comes from the observation that \( S_0(A) \) can be equivalently characterized as the connected component \( \text{Con}(\delta_n) = \{ \varphi \in S(A), d_{A,D}(\varphi, \delta_n) < \infty \} \) of any pure states \( \delta_n \). As such, it is closed (and open as well) for the metric topology [20, Def. 2.1].

Eq. (4.43) follows from (5.9), remembering that \( N(A) = S(A) \) and that for any \( C^* \)-algebra \( S(A) \) is closed in the weak* topology. \[\blacksquare\]
The weak* topology is induced by the distance \([9, \text{Prop. 2.6.15}]\)
\[
d(R, R') := \|R - R'\|_{T'}.
\]
The difference between the weak* and the metric topologies can be seen computing the diameters of the space of states for the corresponding distances.

**Proposition 4.14.** \(S(A)\) has infinite diameter for the spectral distance, diameter 2 for the metric \(d\) inducing the weak* topology.

**Proof.** For all \(\varphi, \varphi' \in S(\mathcal{O}_N)\) and \(a\) with \(L_D(a) \leq 1\) we have
\[
\varphi(a) - \varphi'(a) = \sum_{n,k=0}^N (a_n - a_k)p_np'_k \leq \sum_{n,k=0}^N |n - k|p_np'_k \leq N \sum_{n,k=0}^N p_np'_k = N,
\]
so \(d_{A,D}(\varphi, \varphi') \leq N\). This upper bound is reached by \(d_{A,D}(\Psi_0, \Psi_N) = N\). Hence \(S(\mathcal{O}_N)\) has diameter \(N\) for the spectral distance, and from (4.42) \(S(A)\) has infinite diameter.

For all \(\varphi, \varphi' \in S(A)\), one has
\[
d(\varphi, \varphi') = \sum_n |p_n - p'_n| \leq \sum_n (p_n + p'_n) = 2.
\]
The upper bound is reach by \(\varphi = \Psi_m, \varphi = \Psi_n\) with \(n \neq m\). \(\blacksquare\)

## 5 Pure states and approximation of points

Having studied in the preceding sections the general topological and metric properties of the various truncated distances defined in (3.7), we now come back to the initial motivation of this work, that is understanding what happens to the short distance behaviour of a classical (i.e. commutative) space once a momentum cut-off has been implemented, through the substitution of \(D\) with \(D_A\).

Specifically, for \(A = C_0^\infty(M)\) (as usual \(M\) is an orientable, without boundary, Riemannian manifold), we study how the cut-off in the spectrum of \(D\) changes the topology of the pure state space, i.e. the points of \(M\). We first consider bounded regularization in §5.1, that is \(D_A\) is a bounded operator with norm \(\Lambda > 0\). We prove that the distance \(d_{A,D_A}\) between two distinct pure states cannot be smaller than \(\Lambda^{-1}\), meaning that the pure state space \(\mathcal{P}(A)\) with the metric topology induced by \(d_{A,D_A}\) is not homeomorphic to \(M\) (recall that \(\mathcal{P}(A) \simeq M\) in the weak* topology). We investigate the case of finite rank operator \(D_A\) in §5.2, and prove that any two distinct pure states are at infinite \(d_{A,D_A}\) distance.

It is then clear that in a spectral geometry with a cut-off, points must be replaced by states that are not pure. In §5.3 we investigate the regularization \(D_A = P_\Lambda D\) of the Dirac operator by its spectral projection \(P_\Lambda\) on the interval \([-\Lambda, \Lambda]\), \(\Lambda \in \mathbb{R}^+\). We individuate a class of states that are at finite distance, namely the orbits under the geodesic flow of \(D\) of any vector states in the range of \(P_\Lambda\). We stress that this result is valid for any spectral triple, not necessarily commutative. Applied to the real line, it allows to work out states that approximate points in the weak* topology, and whose distance \(d_{0_\Lambda,D_A}\) is exactly the Euclidean one. Applications to the circle are the object of §5.4.

To remain as general as possible, we make \(A = C_0^\infty(M)\) act by pointwise multiplication on the Hilbert space \(\mathcal{H} := L^2(M, E)\) of square integrable sections of an arbitrary smooth vector bundle \(E \to M\) (not necessarily the spinor or the cotangent bundle), so that
\[
\|f\| = \sup_{x \in M} |f(x)|.
\]
5.1 Bounded regularization

We consider regularization by a bounded operator $D_A$ on $\mathcal{H}$ (not necessarily with finite rank). $[D_A, f]$ is clearly bounded and the spectral distance $d_{A,D_A}$ is well-defined. Borrowing the terminology of [15], the line element $"ds = D_A^{-1}"$ is no longer an infinitesimal (because $f(D_A + f)^{-1}$ is no longer compact for any $f$), so it is reasonable to expect that points can no longer be taken as close as we want. A minimum length should appear. From a physical point of view, this means one cannot probe the space with a resolution better than $\Lambda^{-1}$ [24].

**Proposition 5.1.** Let $D_A$ be a bounded operator with norm $\Lambda > 0$. Then for any $x \neq y$,

$$d_{A,D_A}(\delta_x, \delta_y) \geq \Lambda^{-1},$$

i.e. the distance between two points cannot be smaller than the cut-off.

**Proof.** From (5.1) one has $\|[D, f]\| \leq \|Df\| + \|fD\| \leq 2\|f\|\Lambda$. Any $f \in A$ with maximum $f(x) = 1/2\Lambda$ and minimum $f(y) = -1/2\Lambda$ satisfies $\|[D_A, f]\| \leq 1$ and yields (5.2). $\blacksquare$

Although the inequality (5.2) could be trivial (the distance could be simply infinite for all $x \neq y$), it is still a remarkable result, for it shows that the extended metric $d_{A,D_A}$ and $d_{A,D}$ are never strongly equivalent as soon as $D_A$ is bounded.

**Example 5.2.** A finite distance can be obtained in case $E = M \times \mathbb{C}^2$ (that is $\mathcal{H} = L^2(M) \oplus L^2(M)$) by taking $D_A := \Lambda F$ proportional to the flip operator

$$F(\psi_\uparrow \oplus \psi_\downarrow) = \psi_\downarrow \oplus \psi_\uparrow \quad \forall \psi_\uparrow, \psi_\downarrow \in L^2(M),$$

and making $A$ acts as $\pi(f) = f \oplus f(x_0)$ where $x_0 \in M$ is a fixed base-point ($f$ acts by pointwise multiplication on the first factor, and through the irreducible representation $f \mapsto f(x_0)$ on the second). Then $d_{A,D_A}$ is the discrete metric

$$d_{A,D_A}(\delta_x, \delta_y) = 2\Lambda^{-1} \quad \forall x \neq y.$$  

Indeed, for any $f \in A$ one has $[F, \pi(f)] = ((f(x_0) - f) \oplus (f - f(x_0)))F$. Since $F$ is unitary, one gets $\|[F, \pi(f)]\| = \|f - f(x_0)\|_\infty$ (even though the representation is slightly more involved that the pointwise one, (5.1) remains valid). Hence

$$|f(x) - f(y)| \leq |f(x) - f(x_0)| + |f(x_0) - f(y)| \leq 2$$

for all $f$ such that $\|[F, \pi(f)]\| \leq 1$. This upper bound is attained by any $f$ with maximum $f(x) = 1$, minimum $f(y) = -1$ and $f(x_0) = 0$. $\blacksquare$

5.2 Finite rank regularization

We now consider a finite rank operator $D_A$. In case $M$ is compact, it can be obtained as the truncation $D_A = P_A D P_A$ of the Dirac operator of $M$ by the action of one of its spectral projection $P_A$ ($D$ has compact resolvent, thus $P_A$’s have finite rank). The following results however are valid for arbitrary $M$ and arbitrary finite-rank operator $D_A$.

**Lemma 5.3.** Let $P_0$ be a rank 1 projection and $\psi_0$ a unit vector in the range of $P_0$. For any $f = f^* \in A$ one has

$$\|[P_0, f]\|^2 = \langle f\psi_0, f\psi_0 \rangle - |\langle \psi_0, f\psi_0 \rangle|^2.$$
Proof. For any \( f \in A \), call \( f_0 := f - \langle \psi_0, f \psi_0 \rangle \). If \( f_0 \psi_0 \neq 0 \), we consider the unit vector \( \psi_1 := \frac{1}{\| f_0 \psi_0 \|} f_0 \psi_0 \). One easily checks that \( \langle \psi_0, \psi_1 \rangle = 0 \), so that \{ \psi_0, \psi_1 \} is an orthonormal basis of a 2-dimensional vector subspace \( V \) of \( \mathcal{H} \). For any \( \eta \in \mathcal{H} \) and \( f = f^* \),

\[
[P_0, f] \eta = [P_0, f_0] \eta = \psi_0 \langle \psi_0, f_0 \eta \rangle - f_0 \psi_0 (\psi_0, \eta) = \| f_0 \psi_0 \|_2 (\psi_0 (\psi_1, \eta) - \psi_1 (\psi_0, \eta)).
\]

(5.7)

Hence \( [P_0, f]^2 = -\| f_0 \psi_0 \|^2 \frac{1}{2} \mathrm{id}_V \) and \( \| [P_0, f] \| = \| f_0 \psi_0 \|_2 \). If \( f_0 \psi_0 = 0 \), one has \( [P_0, f] = 0 \) from (5.7), and the previous equation is trivially true. The lemma follows by linearity of the inner product. \( \blacksquare \)

\[
\|[P_0, f]\|^2 \text{ is the variance of the random variable } f \text{ with respect to the probability measure with density } |\psi_0|^2 \text{ or, in physicists language, the uncertainty } \Delta f \text{ of the observable } f \text{ relative to the vector state } \psi_0. \text{ When working with the spectral distance, it is not uncommon that the corresponding seminorm is some kind of standard deviation, as recently stressed by Rieffel [53].}
\]

**Proposition 5.4.** Let \( D_\Lambda \) be any selfadjoint finite-rank operator on \( \mathcal{H} \). For any \( x \neq y \),

\[
d_{A, D_\Lambda}(\delta_x, \delta_y) = \infty .
\]

(5.8)

**Proof.** Using the spectral decomposition of \( D_\Lambda \), i.e. \( D_\Lambda = \sum_{n=1}^r \lambda_n P_n \) where \( r := \mathrm{rk}(D_\Lambda) \) and \( P_n := \psi_n(\psi_n, \cdot) \) are the rank 1 eigenprojections of \( D \), one obtains from Lemma 5.3

\[
\|[D_\Lambda, f]\| \leq \sum_{n=1}^r |\lambda_n||[P_n, f]| \leq \sum_{n=1}^r |\lambda_n||f\psi_n|.
\]

(5.9)

We can always find an open neighbourhood \( U \) of \( x \) with \( y \notin U \), and a real smooth function \( f \) with support in \( U \) such that \( ||f\psi_n\| \) is as small as we want for any \( n \in [1, r] \) and \( f(x) \) is arbitrarily large (take \( f \) with a sufficiently narrow peak around \( x \)). Hence the result. \( \blacksquare \)

### 5.3 Regularization by spectral projection and geodesic flow

The results of the preceding sections indicate that in order to have a reasonable topological space associated with the distance \( d_{A, D_\Lambda} \), points must be replaced by states that are not pure. This is particularly true for finite rank regularizations, as shown in proposition 5.4.

In this section we consider the regularization \( D_\Lambda = P_\Lambda D P_\Lambda \) of the Dirac operator by its spectral projections \( P_\Lambda \) [48], and work out some non-pure states that i) remain at finite distance, ii) weakly approximate points in the commutative case.

Let \( (A, \mathcal{H}, D) \) be an arbitrary spectral triple. Given \( \psi_0 \in \mathcal{H} \) we write \( \psi_t := U_t \psi \) and

\[
\Psi_t(a) = \langle \psi_t, a \psi_t \rangle \quad \forall a \in A
\]

(5.10)

the orbits of the vector \( \psi_0 \) and of the state \( \Psi_0 \) under the geodesic flow of \( D \) [48]:

\[
U_t = e^{itD}, \quad t \in \mathbb{R}.
\]

(5.11)

**Lemma 5.5.** Let \( \psi_0 \in P_\Lambda \mathcal{H} \). For all \( a \in \mathcal{B}(\mathcal{H}) \), \( \Psi_t(a) \) is a differentiable function of \( t \) and

\[
\frac{d}{dt} \Psi_t(a) = -i \Psi_t([D_\Lambda, a]) .
\]

(5.12)
Proof. $P_{A}$ is the identity operator on $P_{A}\mathcal{H}$ so that $\psi_{t} = e^{itD}\psi_{0} = e^{itA}\psi_{0}$ and
\[
\frac{d}{dt}\Psi_{t}(a) = \lim_{\tau \to 0} \left< \psi_{t}, \frac{(e^{i\tau D} \Lambda a e^{i\tau D\Lambda} - 1)\psi_{t}}{\tau} \right> = \lim_{\tau \to 0} \left< \psi_{t}, \frac{(e^{i\tau D\Lambda} - 1)ae^{i\tau D\Lambda}\psi_{t}}{\tau} + a\frac{(e^{i\tau D\Lambda} - 1)\psi_{t}}{\tau} \right>
= \left< \psi_{t}, -iD_{A}a\psi_{t} \right> + \left< \psi_{t}, iaD_{A}\psi_{t} \right> = -i\Psi_{t}([D_{A}, a]) ,
\] (5.13)
where we use $\frac{e^{i\tau D\Lambda}\psi_{t} - \psi_{0}}{\tau} \to iD_{A}\psi_{t}$ and lim $\psi_{t+\tau} = \lim_{\tau \to 0} e^{i\tau D\Lambda}\psi_{t} = \psi_{t}$ \cite[Theo. VIII]{48}. ■

Proposition 5.6. For any $\psi$ in the range of $P_{A}$, the various spectral distances introduced so far are all finite on any orbit $\Psi_{t}$ of the geodesic flow of $D$:
\[
d_{A,D}(\Psi_{t_{1}}^{1}, \Psi_{t_{2}}^{1}) \leq |t_{1} - t_{2}| ,
\] (5.14)
\[
d_{A,D}^{0}(\Psi_{t_{1}}, \Psi_{t_{2}}) \leq d_{0,A,D}(\Psi_{t_{1}}, \Psi_{t_{2}}) \leq |t_{1} - t_{2}| .
\] (5.15)

Proof. From (5.12) one has $\Psi_{t_{1}}(a) - \Psi_{t_{2}}(a) = i \int_{t_{1}}^{t_{2}} \Psi_{t}([D_{A}, a]) dt$ . Since $|\Psi_{t}(A)| \leq \|A\|$ for any bounded operator $A$, from Jensen’s inequality we get
\[
|\Psi_{t_{1}}(a) - \Psi_{t_{2}}(a)| \leq \|D_{A}, a\| \|\int_{t_{1}}^{t_{2}} dt\| = \|D_{A}, a\| |t_{1} - t_{2}| .
\] (5.16)
It is valid for all $a \in \mathcal{B}(\mathcal{H})$, proving both $d_{A,D}(\Psi_{t_{1}}, \Psi_{t_{2}}) \leq |t_{1} - t_{2}|$ and $d_{0,A,D}(\Psi_{t_{1}}, \Psi_{t_{2}}) \leq |t_{1} - t_{2}|$. Since $[P_{A}, D] = 0$, eq. (3.18) yields $d_{A,D}^{0}(\Psi_{t_{1}}, \Psi_{t_{2}}) \leq d_{0,A,D}(\Psi_{t_{1}}, \Psi_{t_{2}})$.

We stress that Prop. 5.6 is true for arbitrary spectral triples, not necessarily commutative. However it is particularly relevant in the commutative case, because $\psi_{0}$ can be chosen in such a way that $\Psi_{t}$ approximates the pure state $\delta_{t}$. We begin with the real line and investigate the case of the circle in the next section. To make clear that $t$ is no longer an abstract parameter but a point of space, we switch notation $t \to x$.

Take
\[
A = C_{0}^{\infty}(\mathbb{R}), \quad \mathcal{H} = L^{2}(\mathbb{R}), \quad D = -id/ dx .
\] (5.17)
Since $\|D_{A}\| = \Lambda$, from Prop. 5.1 there is a minimum length $\Lambda^{-1}$. Since $P_{A}$ is not of finite rank, Prop. 5.4 does not apply to this particular example. Whether $d_{A,D}(\delta_{x}, \delta_{y})$ is finite or not is still an open problem.

To obtain approximation of points that are at finite distance, we thus consider the orbit \{ $\Psi_{x} = \langle \psi_{x}, \psi_{x} \rangle, x \in \mathbb{R}$ \} under the geodesic flow of $D$ of the state $\Psi_{0} = \langle \psi_{0}, \psi_{0} \rangle$, where $\psi_{0}$ is a suitably chosen vector in $P_{A}\mathcal{H}$ as explained in remark 5.8 below.

Proposition 5.7. For any $\Lambda$ and $x, y \in \mathbb{R}$ one has
\[
d_{A,D}(\Psi_{x}^{1}, \Psi_{y}^{1}) \leq d_{0,A,D}(\Psi_{x}, \Psi_{y}) = d_{A,D}^{0}(\Psi_{x}, \Psi_{y}) = |x - y| .
\] (5.18)

Proof. The last equality follows noticing that $\Psi_{x}^{1}$ is the non-pure state of $C_{0}^{\infty}(\mathbb{R})$ given by the probability density $|\psi_{x}|^{2}$, and $\Psi_{y}$ is its pull back under the translation $x \to x + y$, namely
\[
\Psi_{x}^{1}(f) = \int_{\mathbb{R}} f(t)|\psi_{x}(t)|^{2}dt , \quad \Psi_{y}^{1}(f) = \int_{\mathbb{R}} f(t - y)|\psi_{x}(t)|^{2}dt .
\] (5.19)
It is well known that the Wasserstein distance between translated states on the real line is the amplitude of translation, that is $d_{A,D}^{0}(\Psi_{x}, \Psi_{y}) = |x - y|$ \cite[see e.g. \cite{20}]. The thesis then follows from Prop. 5.6. ■
Viewing the orbit \( \mathbb{R}_A \psi_0 := \{ \Psi_x \}_{x \in \mathbb{R}} \) as a “replica” of the real line inside the state space of \( C_0(\mathbb{R}) \), one has that \( (\mathbb{R}_A \psi_0, d_{\Lambda}, D_\Lambda) \) and \( (\mathbb{R}^5_A \psi_0^\dagger, d_{\tilde{A},D}) \) (with obvious notations) are isometric to \( (\mathbb{R}, |\cdot|) \) for any \( \Lambda \) and \( \psi_0 \).

**Remark 5.8.** In order that \( \Psi_x \rightarrow \delta_x \) in the weak* topology as \( \Lambda \rightarrow \infty \), \( \psi_0 \) can be taken as the Fourier transform of the (normalized) characteristic function of the interval \([-\Lambda, \Lambda] \):

\[
\psi_0(t) = \frac{1}{2\sqrt{\Lambda \pi}} \int_{-\Lambda}^{\Lambda} e^{ipt} dp = \frac{1}{\sqrt{\Lambda \pi}} \sin \Lambda t. \tag{5.20}
\]

Indeed one then has

\[
\Psi_x(f) = \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^2 t \frac{f(x + \frac{t}{\Lambda})}{t^2} dt, \tag{5.21}
\]

and clearly \( \lim_{\Lambda \rightarrow \infty} \Psi_x(f) = f(x) \) for all \( f \in \mathcal{A} \) and \( x \in \mathbb{R} \).

### 5.4 Gromov-Hausdorff convergence on the circle

We apply some of the previous results to the circle. The spectral triple is

\[
\mathcal{A} = C^\infty(S^1), \quad \mathcal{H} = L^2(S^1, dx), \quad D = -id/dx, \tag{5.22}
\]

and we identify functions on \( S^1 \) with \( 2\pi \)-periodic functions on \( \mathbb{R} \). We use as orthonormal basis of \( \mathcal{H} \) the Fourier modes

\[
\{ e_n : x \rightarrow e^{inx}, n \in \mathbb{Z} \} \tag{5.23}
\]

in which \( D \) acts as an infinite diagonal matrix, and \( f \) as an infinite matrix with constant diagonals, that is

\[
\langle e_n, f e_m \rangle_{L^2} = f_{n-m} \tag{5.24}
\]

where \( f_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx \) are the Fourier coefficients of \( f \) (by standard Fourier analysis they of rapid decay for \( f \in C^\infty(S^1) \)).

We consider the regularization by (spectral) projection on the first \( N \) positive and negative Fourier modes. Namely for any \( N \in \mathbb{N} \) we write \( D_N = P_N D P_N \) where \( P_N \) denotes the projection on

\[
\mathcal{H}_N := \text{Span}\{ e_n : |n| \leq N \}. \tag{5.25}
\]

The geodesic flow \( U_t = e^{it \frac{D}{\Lambda}} \) acts as \( U_t e_n = e^{int} e_n \). Its adjoint action \( \alpha_t(f) := U_t f U_t^* \) implements the translation: \( \alpha_t(f)(x) = f(x + t) \). As any automorphism that preserves the Lipschitz seminorm, \( \alpha_t \) is an isometry of the space of states.

**Lemma 5.9.** Let \( (\mathcal{A}, \mathcal{H}, D) \) be an arbitrary spectral triple (not necessarily (5.22)). Any automorphism of \( \mathcal{A} \) such that \( L_D(\alpha(a)) = L_D(a) \) \( \forall a \in \mathcal{A} \) is an isometry of the extended metric space \( (\mathcal{S}(\mathcal{A}), d_{\mathcal{A},D}) \): writing \( \alpha^* \varphi := \varphi \circ \alpha \) the pull back of \( \alpha \) on states, one has

\[
d_{\mathcal{A},D}(\varphi, \varphi') = d_{\mathcal{A},D}(\alpha^* \varphi, \alpha^* \varphi') \quad \forall \varphi, \varphi' \in \mathcal{S}(\mathcal{A}). \tag{5.26}
\]

In particular, for the spectral triple (5.22), any translation \( \alpha_t, t \in \mathbb{R} \), and the reflection \( \beta(f)(x) := f(-x) \) are isometries of both \( (\mathcal{S}(\mathcal{A}), d_{\mathcal{A},D}) \) and \( (\mathcal{S}(\mathcal{A}), d_{\tilde{A},D_N}) \).

24
Proof. Eq. (5.26) follows from

$$\sup_{a \in \text{Lip}_D(A)} \alpha^* \varphi(a) - \alpha^* \varphi'(a) = \sup_{b \in \text{Lip}_D(A)} \varphi(b) - \varphi'(b) = \sup_{b \in \text{Lip}_D(A)} \varphi(b) - \varphi'(b). \quad (5.27)$$

For any $t$ the unitary operator $U_t$ commutes with $D$ and $D_N$, hence $L_D(\alpha_t(a)) = L_D(a)$ for any $a$ and similarly for $L_N$. The reflection $\beta$ is implemented by the adjoint action of the unitary operator $C e_n := e_{-n}$. Since $C$ anticommutes with $D$ and $D_N$, one has $[D, \beta(a)] = -C[D, a]C$ so that $L_D(\beta(a)) = L_D(a)$, and similarly for $L_N$.

As approximation of the point $x \in [-\pi, \pi]$, we consider the vector state $\Psi_{x,N} \in \mathcal{S}(\mathcal{O}_N)$ defined by the vector in $\mathcal{H}_N$:

$$\psi_{x,N} := \frac{1}{\sqrt{N+1}} \sum_{n=0}^{N} e^{-inx} e_n. \quad (5.28)$$

**Lemma 5.10.** For any $f \in \mathcal{C}^\infty(S^1)$, $\Psi_{x,N}^f(f)$ is the Fejé transform of $f$:

$$\Psi_{x,N}^f(f) = \sum_{n=-N}^{N} \left(1 - \frac{|n|}{N+1}\right) f_n e^{inx} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) F_{N+1}(x - t) dt \quad (5.29)$$

where $F_N(t) := \frac{1}{N} \left(\frac{\sin(Nt/2)}{\sin(t/2)}\right)^2$ is the Fejé kernel. For any $x \in S^1$, the sequence of non-pure states $\{\Psi_{x,N}^f\}_{N \in \mathbb{N}}$ converges to the pure state $\delta_x$:

$$\lim_{N \to \infty} \Psi_{x,N}^f(f) = f(x) \quad \forall f \in \mathcal{C}^\infty(S^1). \quad (5.30)$$

Proof. By (5.24) and (5.28) one has

$$\Psi_{x,N}^f(f) = \langle \psi_{x,N}, f \psi_{x,N} \rangle = \frac{1}{N+1} \sum_{n,m=0}^{N} e^{i(m-n)x} \langle e_m, f e_n \rangle = \frac{1}{N+1} \sum_{n,m=0}^{N} e^{i(m-n)x} f_{m-n}. \quad (5.29)$$

With some combinatorics, one obtains the r.h.s. of (5.29):

$$(N + 1) \Psi_{x,N}^f(f) = \sum_{k=-N}^{N} f_k e^{ikx} \sum_{n=0, \ldots, N}^{N} 1 \sum_{n=0, \ldots, N}^{N} f_k e^{ikx} \sum_{n=\max(0,-k)}^{N} 1$$

$$= \sum_{k=-N}^{N} f_k e^{ikx} (\min(N, N - k) - \max(0, -k) + 1) = \sum_{k=-N}^{N} \min(N, N - k) - \max(0, -k) + 1 = \sum_{k=-N}^{N} \sum_{n=0}^{N} \min(N, N - k) - \max(0, -k) + 1 = \sum_{n=0}^{N} (N + 1 - |k|) f_k e^{ikx}.$$  

Recall that, by induction, the Cesàro sum of a sequence $\{a_k\}_{k \in \mathbb{Z}}$ is

$$\frac{1}{N+1} \sum_{n=0}^{N} S_n(a_k) = S_N \left(1 - \frac{|k|}{N+1} \frac{k}{a_k} \right) \quad (5.31)$$

where $S_n(a_k) := \sum_{k=-n}^{n} a_k$. Therefore

$$\Psi_{x,N}^f(f) = \frac{1}{N+1} \sum_{n=0}^{N} S_n(f_k e^{ikx}). \quad (5.32)$$

This is precisely the Fejé transform of $f$, whose integral formula is given by the second term in (5.29) (see e.g. [47]). By Fejé theorem, the Fejé transforms uniformly converges to $f$ as $N \to \infty$. Thus in particular $\Psi_{x,N}^f$ converges to $\delta_x$. \qed
Since $\psi_{x,N} = U_{-x}\psi_{0,N}$, one has that

$$S^1_N = \{ \psi^x_{x,N} : x \in S^1 \} \quad (5.33)$$

is the orbit of $\Psi_{0,N}$ under the geodesic flow. On the real line, both $d_{O,D}$ and $d^\rho_{A,D}$ on similar orbits coincides with the geodesic distance (Prop. 5.7). For the circle the same is true only in the $N \to \infty$ limit.

**Proposition 5.11.** i) For all $x, y \in S^1$, one has

$$d^\rho_{A,D}(\Psi_{x,N}, \Psi_{y,N}) \leq d_{O,D}(\Psi_{x,N}, \Psi_{y,N}) \leq d_{geo}(x, y), \quad (5.34)$$

and

$$\lim_{N \to \infty} d^\rho_{A,D}(\Psi_{x,N}, \Psi_{y,N}) = \lim_{N \to \infty} d_{O,D}(\Psi_{x,N}, \Psi_{y,N}) = d_{geo}(x, y). \quad (5.35)$$

ii) A lower bound for $d^\rho_{A,D}$ and an alternative upper bound are provided by

$$\rho_N(x - y) \leq d^\rho_{A,D}(\Psi_{x,N}, \Psi_{y,N}) \leq \rho_N'(x - y) \quad (5.36)$$

where

$$\rho_N(x) := \frac{8}{\pi} \sum_{1 \leq n \leq N} \frac{(-1)^{(n-1)/2}}{n^2} \left(1 - \frac{n}{N + 1}\right) \sin \frac{nx}{2}, \quad (5.37)$$

$$\rho_N'(x) := 2\sqrt{2} \left[ \sum_{1 \leq n \leq N} \frac{1}{n^2} \left(1 - \frac{n}{N + 1}\right)^2 \left(\sin \frac{nx}{2}\right)^2 \right]. \quad (5.38)$$

**Proof.** Since $\alpha^*_t \Psi_{x,N} = \Psi_{x+t,N}$ and $\beta^*_t \Psi_{x,N} = \Psi_{-x,N}$, by Lemma 5.9 it is enough to prove the proposition for $y = 0$ and $0 < x \leq \pi$.

i) Eq. (5.34) follows from Prop. 5.6 noticing that for $x \in [0, \pi]$ the geodesic distance is $d_{geo}(x, y) = |x - y|$. Since $(A, L_D)$ is a compact quantum metric space (see e.g. the introduction of [50]), the weak* topology and metric topology of $d_{A,D}$ coincide. By Lemma 5.10 one has $\Psi_{x,N} \to \delta_x$, and so $\lim_{N \to \infty} d_{A,D}(\Psi_{x,N}, \Psi_{y,N}) = d_{geo}(x, y)$, proving (5.35).

ii) Note that $d^\rho_{A,D}(\Psi_{x,N}, \Psi_{0,N}) = d^\rho_{A,D}(\Psi_{x/2,N}, \Psi_{-x/2,N})$, that by (5.29) is the sup of

$$\Psi_{x/2,N}^*(f) - \Psi_{-x/2,N}^*(f) = -4 \sum_{1 \leq n \leq N} \left(1 - \frac{n}{N + 1}\right) \Im(f_n) \sin \left(\frac{nx}{2}\right), \quad (5.39)$$

over real 1-Lipschitz functions $f$ (we used $f_{-n} = f_n^*.$)

The periodic 1-Lipschitz function defined by $f(t) = |t|$ for $t \in [-\pi, \pi]$ (so, the geodesic distance of the circle) has Fourier coefficients $f_0 = \frac{\pi}{2}$, $f_n = f_{-n} = -\frac{\pi}{n^2}$ for $n$ odd, and $f_n = 0$ for even $n \neq 0$. Translating this function by $\pi/2$ amounts to rescaling $f_n$ by $e^{in\pi/2}$, which is equal to $i(-1)^{(n-1)/2}$ for $n$ odd and gives the lower bound (5.37).

On the other hand, from (5.39), for any 1-Lipschitz function $f$:

$$\Psi_{x/2,N}^*(f) - \Psi_{-x/2,N}^*(f) \leq 4 \sum_{1 \leq n \leq N} \left(1 - \frac{n}{N + 1}\right) |f_n \sin \left(\frac{nx}{2}\right)| \leq 2\sqrt{2} \sum_{n=1}^{N} C_n^2 \quad (5.40)$$

where

$$C_n := \frac{1}{n} \left(1 - \frac{n}{N + 1}\right) \sin \frac{nx}{2} \quad (5.41)$$

26
Proof. (Proposition 5.12.

Gromov-Hausdorff distance.

circle with its geodesic distance. However the sequence geodesic distance (see Figure 1). Unlike the real line, the orbit S^1\{ and by translation invariance:

second we used

function defined by f

exists

such that

\\{ x,N \mid \exists \pi \}

\sum_{n=1}^{N} |f_n|^2 = \sum_{n=-N}^{N} |f_n|^2 \leq \sum_{n=-\infty}^{\infty} |f_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(t)|^2 dt \leq 1. \quad (5.42)

This proves (5.38).

Eq. (5.36) shows that at fixed N, none of truncated distances actually equals the geodesic distance (see Figure 1). Unlike the real line, the orbit S^1\{ is not a replica of the circle with its geodesic distance. However the sequence \{S^1_{N}\}_{N \in \mathbb{N}} converges to it in the Gromov-Hausdorff distance.

Proposition 5.12. \( (S^1_1, d_{A,D}) \) converges to \((S^1_1, d_{geo})\) for the Gromov-Hausdorff distance.

Proof. From [10, pag. 253], the Gromov-Hausdorff limit of S^1_1 is the set X of limits of all convergent sequences \{\varphi_n \in S^1_1\}_{n \in \mathbb{N}}. But \((C^\infty(S^1_1), L_D)\) is a compact quantum metric space [49] and by Lemma 5.10 \(\Psi_{x,N} \to \delta_x\) in the weak* topology, thus \(\Psi_{x,N} \to \delta_x\) also in the metric topology of \(d_{A,D}\). Hence \(S^1 \subset X\).

Consider a convergent sequence \{\varphi_n \in S^1_1\}_{n \in \mathbb{N}}. By definition, for any \(\varphi_n \in S^1_1\) there exists \(x_n \in S^1\) such that \(\varphi_n = \Psi_{x_n,N}\). Let \(N' > N\) and consider the Lipschitz periodic function defined by \(f(t) := |t|\) for \(t \in [-\pi, \pi]\). By easy calculation one obtains

\[\Psi_{0,N'}(f) - \Psi_{x,N'}^\sharp(f) = \frac{8}{\pi} \sum_{1 \leq n \leq N'} \frac{1}{n^2} \left( 1 - \frac{n}{N'} \right) \left( \sin \frac{n}{2} \right)^2 \geq \frac{4}{\pi} \left( \sin \frac{x}{2} \right)^2 = \rho_1(x),\]

\[\Psi_{0,N}(f) - \Psi_{x,N}^\sharp(f) \geq \frac{4}{\pi} \sum_{N < n \leq N'} \frac{1}{n^2} \left( 1 - \frac{n}{N + 1} \right) \geq 0,\]

with \(\rho_1(x)\) as in (5.37) (the first inequality follows from the observation that we have a sum of positive terms - hence superior to the term \(n = 1\) - and \(\frac{N'}{N+1} \geq \frac{1}{2}\) if \(N' \geq 1\); in the second we used \(-\frac{1}{N+1} \geq -\frac{1}{N'}\)). Hence

\[d_{A,D}(\Psi_{0,N}, \Psi_{x,N'}) \geq \{\Psi_{0,N'}(f) - \Psi_{x,N'}^\sharp(f)\} + \{\Psi_{0,N}(f) - \Psi_{x,N}^\sharp(f)\} \geq \rho_1(x), \quad (5.43)\]

and by translation invariance:

\[\rho_1(x-y) \leq d_{A,D}(\Psi_{x,N}, \Psi_{y,N'}). \quad (5.44)\]
Now, since $S(C^\infty(S^1))$ is complete, $\{\varphi_n\}$ is a Cauchy sequence for the spectral distance. Hence (5.44) shows that for all $\epsilon > 0$ there exists $N \geq 1$ such that
\[
\rho_1(x_m - x_n) \leq d_{A,D}(\varphi_m, \varphi_n) < \epsilon \quad \forall \ m > n \geq N.
\] (5.45)

This means $|x_m - x_n| < 2\arcsin\sqrt{\frac{\pi}{4}\epsilon}$, proving that $\{x_n\}$ is a Cauchy sequence, thus convergent. Let $x := \lim_{n \to \infty} x_n$. From the triangle inequality and Prop. 5.11 we get
\[
d_{A,D}(\varphi_n, \delta_x) \leq \rho_1(x_m - x_n) + d_{A,D}(\Psi^\sharp_{x,n}, \delta_x).
\] (5.46)

Since $\Psi^\sharp_{x,n} \to \delta_x$ in the weak* topology, $d_{A,D}(\Psi^\sharp_{x,n}, \delta_x) \to 0$. On the other hand $x_n \to x$, hence $d_{geo}(x_n, x) \to 0$ too. Thus $\lim_{n \to \infty} d_{A,D}(\varphi_n, \delta_x) = 0$ proving that $\varphi_n \to \delta_x$ converges to a pure state, and $X \subset S^1$. This concludes the proof.  

5.5 Optimal transport on $S^1$ for rectangular distributions

For completeness, we now give a family of non-pure states approximating points of the circle for which the Wasserstein distance can be explicitly computed. These results have an interest on their own independently of noncommutative geometry since, as far as we know, there are few examples of explicit computation of the Wasserstein on the circle (discrete distribution has been recently studied in [22]).

Consider the compactly supported distribution
\[
\delta_{x,\epsilon}(f) := \int_{-\epsilon}^{\epsilon} f(t + x)d\mu_t,
\] (5.47)
where $0 < \epsilon < \pi$ and $d\mu_t$ is any distribution with support in $[-\epsilon, \epsilon]$ normalized to 1. This is more general class of states than in §5.3, for $\delta_{x,\epsilon}$ is not necessarily a vector state. It is enough to compute the distance for $y = 0$ and $0 < x \leq \pi$ since Lemma 5.10 and obvious computation yield
\[
d_{A,D}(\delta_{x,\epsilon}, \delta_{y,\epsilon}) = d_{A,D}(\delta_{|x-y|,\epsilon}, \delta_{0,\epsilon}) \leq d_{geo}(x, y).
\] (5.48)

Proposition 5.13. For $0 < x \leq \pi - 2\epsilon$ one has
\[
d_{A,D}(\delta_{0,\epsilon}, \delta_{x,\epsilon}) = x.
\] (5.49)

For $d\mu_t = \frac{1}{2\pi} \chi_{[-\epsilon,\epsilon]}(t)dt$ the rectangular distribution and $\pi - 2\epsilon \leq x \leq \pi$, one has (Fig. 2)
\[
d_{A,D}(\delta_{0,\epsilon}, \delta_{x,\epsilon}) = \frac{1}{\pi} \left( -x^2 + 2\pi x - (\pi - 2\epsilon)^2 \right).
\] (5.50)

Proof. Consider the 1-Lipschitz function:

![Diagram](https://via.placeholder.com/150)

\[
f(t) = \begin{cases} 
0 & 0 \leq t \leq x + 2\epsilon \\
\pi & x + 2\epsilon < t \leq 2(x + 2\epsilon) \\
2\pi & 2(x + 2\epsilon) < t \leq 2\pi 
\end{cases}
\]
With the replacement \( f(x) \to f(x + \epsilon) \), we get for \( x + 2\epsilon \leq \pi \):

\[
d_{A,D}(\delta_0, \delta_{x,\epsilon}) \geq \int_0^{2\epsilon} d\mu_{t-\epsilon} f(t + x) - f(t) = \int_0^{2\epsilon} d\mu_x t + x - t = x \int_{-\epsilon}^\epsilon d\mu_x = x. \tag{5.51}
\]

Hence (5.49) from Lemma (5.48).

For the rectangular distributions, the following function

\[
\xi \mapsto \pi - \xi + (x - \xi)(2\epsilon - \xi)
\]

yields

\[
2\epsilon d_{A,D}(\delta_0, \delta_{x,\epsilon}) \geq \int_0^{2\epsilon} \{ g(t + x) - g(t) \} dt \\
= \int_0^\xi \{ g(t + \pi + \xi) - g(t) \} dt + \int_\xi^{2\epsilon} \{ g(t + x - \xi) - g(t) \} dx \\
\geq \int_0^\xi \{ (\pi - t) - (\xi - t) \} dt + \int_\xi^{2\epsilon} \{ (t + x - 2\xi) - (t - \xi) \} dt \\
= \{ (\pi - \xi)\xi + (x - \xi)(2\epsilon - \xi) \} = \frac{1}{2} ( -x^2 + 2\pi x - (\pi - 2\epsilon)^2 ).
\]

where \( \xi := \frac{1}{2}(x + 2\epsilon - \pi) \). On the other hand, for any 1-Lipschitz function \( f \):

\[
f(t + \pi + \xi) - f(t) \leq d_{geo}(0, \pi + \xi) = 2\pi - (\pi + \xi) = \pi - \xi ,
\]

\[
f(t + x - \xi) - f(t) \leq d_{geo}(0, x - \xi) = x - \xi ,
\]

where in the first equation we noticed that \( \pi \leq \pi + \xi \leq 2\pi \). Therefore

\[
d_{A,D}(\delta_0, \delta_{x,\epsilon}) \leq \frac{1}{2\epsilon} \left( \int_0^\xi (\pi - \xi) dt + \int_\xi^{2\epsilon} (x - \xi) dt \right) \\
= \frac{1}{2\epsilon} \{ (\pi - \xi)\xi + (x - \xi)(2\epsilon - \xi) \} = \frac{1}{4\epsilon} ( -x^2 + 2\pi x - (\pi - 2\epsilon)^2 ) .
\]

Hence the inequality is actually an equality. \( \blacksquare \)

When \( x + 2\epsilon > \pi \) the distance is less than the geodesic one (fig. 2) because one can optimize the transport by moving part of the distribution to the left and part to the right along the circle (see fig. 3). As stressed in [11], computing the Wassertein distance on the circle amounts to cut the circle at a well chosen point, then compute the same distance on the real line. This is what we do here: we cut the circle at \( \frac{1}{2}(x - \pi) \).

The distance is also smoother than the Euclidean one (not at 0 though). Interestingly the same phenomenon appears in a totally different context (covariant Dirac operator on a \( U(n) \)-bundle on the circle [39]).
6 Wasserstein distance and Berezin quantization

We now discuss an application of the truncation procedure to quantum spaces. Given a spectral triple \((A, \mathcal{H}, D)\) and a projection \(P \in \mathcal{B}(\mathcal{H})\) such that \(P \cdot \text{dom}(D) \subset \text{dom}(D)\), we denote by \(A_P\) the algebra generated by the elements \(\pi(a) := PaP\), with \(a \in A\). Let \(D_P := PDP\) be the truncated Dirac operator. Note that unlike section 5, we do not assume that \(D_P\) is bounded.

Many well known noncommutative spectral triples \((A_P, P\mathcal{H}, D_P)\) are obtained in this way, that is by the action of a projection \(P\) on a commutative spectral triple \((A, \mathcal{H}, D)\): Moyal plane, fuzzy spaces, quantum discs, and more generally any Berezin-Toeplitz quantization of a classical space. Specifically we study the quantization of the plane in §6.2, and of the sphere in §6.4. Before that, we give in §6.1 an application to gauge theory.

Having in mind the analogy between the spectral and the Wasserstein distances, a first general result is that a quantum transport is more expensive than a classical transport.

**Lemma 6.1.** There is a map \(\sharp : S(A_P) \to S(A)\), \(\varphi \mapsto \varphi^\sharp\), given by

\[
\varphi^\sharp = \varphi \circ \pi .
\]  

(6.1)
If \([P,D] = 0\) or \([P,a] = 0\) for all \(a \in A\), then for any \(\varphi, \psi \in \mathcal{S}(A_P)\) one has

\[
d_{A_P,D_P}(\varphi, \psi) \geq d_{A,D}(\varphi^\sharp, \psi^\sharp).
\] (6.2)

**Proof.** Since \(\pi\) preserves positivity of operators, \(\varphi^\sharp\) is positive. When \(A_P\) is unital, with unit \(P\), then \(\varphi^\sharp(1) = \varphi(P) = 1\) and \(\varphi^\sharp\) is a state. Otherwise, \(\varphi\) extends in a unique way to the unitization of \(A_P\), whose unit is the identity on \(P\mathcal{K}\), that is \(P\). Hence this extension satisfies \(\varphi(P) = 1\), and as before \(\varphi\) is a state. The proof of (6.2) is the same as in Prop. 3.6. \(\blacksquare\)

**Remark 6.2.** The map \(\varphi^\sharp\) is not necessarily injective, unlike Prop. 3.4, because in general \(\pi\) is not surjective.

### 6.1 Gauged Dirac operators

The Dirac operator \(D\) of a Riemannian spin manifold \(M\) can be lifted to any vector bundle \(E \to M\) as follows. By Serre-Swan theorem, the set \(\mathcal{E}\) of smooth sections of \(E\) vanishing at infinity is a finitely generated projective \(A\)-module, with \(A = C_0^\infty(M)\): namely \(\mathcal{E} \simeq \mathcal{P}A^n\) for some \(n \geq 1\) and some projection \(P \in M_n(A)\). Let \(\mathcal{K} = L^2(M, S)\) be the space of square integrable spinors and \(\mathcal{K}^n = \mathcal{K} \otimes \mathbb{C}^n\). We denote \(D_P = P(D \otimes \mathbb{1}_N)\) the lift of \(D\) to \(E\). It acts on \(\mathcal{K}^n\) and is well defined, because \(P\) being smooth sends the domain of \(D \otimes \mathbb{1}_N\) into itself. In gauge theories, \(E\) is a \(SU(n)\) bundle describing a fermion paired with a \(su(n)\) gauge field [36], and \(D_P\) is then called “gauged Dirac operator”.

**Lemma 6.3.** One has \(A_P \simeq A\).

**Proof.** Since \(A\) is commutative and \(P \in M_n(A)\), \(\pi(a) := P(a \otimes \mathbb{1}_n)\) is a representation of \(A\) and \(A_P \simeq A/\ker \pi\). Let \(k = \text{Tr}(P)\) be the matrix trace of \(P\) (not the trace on \(\mathcal{H}\)). This should be an element of \(A\), but in fact it is an integer, since it coincides with the rank of the vector bundle \(E\). Since \(\text{Tr}(\pi(a)) = ka, \pi(a) = 0\) implies \(a = 0\) and \(\ker \pi = \{0\}\). \(\blacksquare\)

One can prove that \((A_P, P\mathcal{K}^n, D_P)\) is a spectral triple. Indeed, the construction described here is very common in index theory, because for \(M\) an even-dimensional manifold, the Fredholm index of \(D_P\) gives an integer-valued pairing between \(D\) (or more generally, a \(K\)-homology class for \(M\)) and the class of \(E\) in \(K^0(M)\) (see [15, 46]).

The spectral triple \((A, \mathcal{K}, D)\) is metrically equivalent to \((A, \mathcal{K}^n, D \otimes \mathbb{1}_n)\), where the algebra acts diagonally on \(\mathcal{K}^n\). By Lemma 6.3 we identify \(A\) with \(A_P\) and \(\varphi\) with \(\varphi^\sharp\). Lemma 6.1 then tells us that “gauging” a Dirac operator makes distances larger:

\[
d_{A_P,D_P}(\varphi, \psi) \geq d_{A,D}(\varphi, \psi) \quad \forall \varphi, \psi \in \mathcal{S}(A).
\] (6.3)

By further adding a connection on \(E\), one obtains a covariant Dirac operator. The corresponding distance then strongly depends on the holonomy of the connections, as shown in [40, 41].

### 6.2 Berezin quantization of the plane

We first recall how to quantize \(\mathbb{C} \simeq \mathbb{R}^2\) by projecting \(\mathcal{H} = L^2(\mathbb{C}, \frac{dz}{\pi})\), with inner product

\[
\langle f, g \rangle_{L^2} := \frac{1}{\pi} \int_{\mathbb{C}} \overline{f(z)} g(z) d^2 z,
\] (6.4)
on a suitable Hilbert subspace. Then we prove a result similar to Lemma 6.1.

Fix a real deformation parameter $\theta > 0$ and denote by $\mathcal{H}_\theta \subset \mathcal{H}$ the Hilbert subspace spanned by the set of orthonormal functions

$$h_n(z) := \frac{z^n}{\sqrt{\theta^{n+1}n!}} e^{-\frac{|z|^2}{2\theta}} \quad n \in \mathbb{N}. \quad (6.5)$$

Notice that $\mathcal{H}_\theta$ is isomorphic to the holomorphic Fock space $L^2_{hol}(\mathbb{C}, e^{-\frac{|z|^2}{2\theta}})$ via the module map $f \mapsto \tilde{f} := \sqrt{\theta} e^{\frac{1}{2\theta}z^2}f$. Let $P_\theta$ be the orthogonal projection $\mathcal{H} \to \mathcal{H}_\theta$, namely

$$P_\theta := \sum_{n=0}^\infty h_n \langle h_n, \cdot \rangle_{L^2}. \quad (6.6)$$

Two maps are naturally associated to it [8]: the Toeplitz quantization $\pi_\theta$ from bounded continuous functions $f$ to bounded operators on $\mathcal{H}_\theta$:

$$\pi_\theta(f) := P_\theta f P_\theta, \quad (6.7)$$

and the Berezin symbol $\sigma_\theta(T)$, defined for $T \in \mathcal{B}(\mathcal{H})$ by

$$\sigma_\theta(T)(z) = \Psi_z(T), \quad (6.8)$$

where $\Psi_z(T) = \langle \psi_z, T\psi_z \rangle_{L^2}$ is the vector state defined by the unit vector in $\mathcal{H}_\theta$:

$$\psi_z := e^{-\frac{|z|^2}{2\theta}} \sum_{n=0}^\infty \frac{z^n}{\sqrt{\theta^{n+1}n!}} h_n. \quad (6.9)$$

Their composition $B_\theta := \sigma_\theta \circ \pi_\theta$ is called Berezin transform. Both maps $\pi_\theta$ and $\sigma_\theta$ (and then $B_\theta$), are unital, positive and norm non-increasing, that is: $\|\pi_\theta(f)\| \leq \|f\|_\infty$ and $\|\sigma_\theta(T)\|_\infty \leq \|T\|$ for all $f, T$ (the latter simply follows from $\Psi_z$ being a state).

Let us now consider the canonical spectral triple of $\mathbb{C} \cong \mathbb{R}^2$, given by:

$$A = S(\mathbb{R}^2), \quad \mathcal{H} \otimes \mathbb{C}^2, \quad D = \begin{pmatrix} 0 & -\partial \\ \partial & 0 \end{pmatrix}, \quad (6.10)$$

where $S(\mathbb{R}^2)$ is the algebra of Schwartz functions on the plane. We write $z = x + iy$, so that the derivatives are $\partial = \frac{1}{2}(\partial_x - i\partial_y)$ and $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$. Let $0_\theta$ be the order unit space spanned by $\pi_\theta(f), f \in A^{sa}$, and $\pi_\theta(1) = P_\theta$. The action of $P_\theta$ on $D$, with a proper normalization factor\footnote{The factor 2 for $D_\theta$ is required so that $d_{A,D}(\delta_x, \delta_{x'}) = |z - z'| = \|((x - x', y - y'))\|$ coincides with the geodesic distance on the plane.}, yields the Dirac operator of the irreducible spectral triple of Moyal plane [44, 18].

**Proposition 6.4.** The truncated Dirac operator $D_\theta := 2(P_\theta \otimes I_2)D(P_\theta \otimes I_2)$ is given by:

$$D_\theta = \frac{2}{\sqrt{\theta}} \begin{pmatrix} 0 & a^1 \\ a & 0 \end{pmatrix} \quad (6.11)$$

where $a^1, a$ are the creation, annihilation operators: $a^1 h_n = \sqrt{n+1} h_{n+1}, ah_n = \sqrt{n} h_{n-1}$.

**Proof.** One has $\bar{\partial} e^{-\frac{|z|^2}{2\theta}} = -\frac{z}{\sqrt{\theta}} e^{-\frac{|z|^2}{2\theta}}$, so that $\bar{\partial} h_n = -\frac{a^1}{2\sqrt{\theta}} h_n$. Thus $P_\theta \bar{\partial} P_\theta = -\frac{a^1}{2\sqrt{\theta}}$ and by conjugation $P_\theta \partial P_\theta = \frac{a}{2\sqrt{\theta}}$. Hence (6.11).
Although we work with Schwartz functions, the quantization map makes sense for more general (even unbounded) functions. In particular one has $\pi_\theta(z) = \sqrt{\theta} a^\dagger$ (coming from $zh_n(z) = \sqrt{\theta}(n+1)h_{n+1}$), and by conjugation $\pi_\theta(\bar{z}) = \sqrt{\theta} a$. Hence

$$ [\pi_\theta(\bar{z}), \pi_\theta(z)] = \theta. \quad (6.12) $$

In other terms, cutting-off the Euclidean plane by $P$ yields a canonical quantization of the plane.\textsuperscript{5} The map $\mathfrak{J} : \mathcal{S}(\mathcal{O}_\theta) \to \mathcal{S}(\mathcal{A})$ in (6.1) is injective, because $\pi_\theta$ in (6.7) is surjective by construction, and maps “quantum states” into “classical states”. Even though $P_\theta$ and $D$ do not commute, we are able to obtain in Prop. 6.7 below a result similar to lemma 6.1, together with an upper bound for $d_{\mathcal{O}_\theta, D_\theta}$ given by the distance

$$ d_{\mathcal{A}, D}(\varphi, \psi) := \sup_{f = f^* \in \mathcal{A}} \{ \varphi(f) - \psi(f) : \|D, B_\theta(f)\| \leq 1 \} \quad \forall \varphi, \psi \in \mathcal{S}(\mathcal{A}). \quad (6.13) $$

We begin with two technical lemmas.

**Lemma 6.5.** For any $f \in \mathcal{A}$ and $T \in \mathcal{O}_\theta$, one has\textsuperscript{6}

$$ [D_\theta, \pi_\theta(f) \otimes \mathbb{I}_2] = (\pi_\theta \otimes \mathbb{I}_2)([D, f]), \quad [D_\theta, \sigma_\theta(T)] = (\sigma_\theta \otimes \mathbb{I}_2)([D_\theta, T]). \quad (6.14) $$

**Proof.** The rank 1 projection in direction of $\psi_z$,

$$ Q_z := e^{-|z|^2} \sum_{m,n \geq 0} \frac{z^m \bar{z}^n}{\sqrt{\theta}^{m+n} m! n!} h_m \langle h_n, \cdot \rangle_{L^2}. \quad (6.15) $$

is the density matrix of the coherent state $\Psi_z$. With some computations one verifies that

$$ \partial Q_z + \frac{1}{\sqrt{\theta}} [a, Q_z] = 0, \quad -\partial Q_z + \frac{1}{\sqrt{\theta}} [a^\dagger, Q_z] = 0. \quad (6.16) $$

Using the explicit form of the Toeplitz operator,

$$ \pi_\theta(f) = \sum_{m,n \geq 0} h_m \langle h_m, f h_n \rangle_{L^2} \langle h_n, \cdot \rangle_{L^2} \equiv \frac{1}{\pi \theta} \int_{\mathbb{C}} f(z) Q_z dz \quad (6.17) $$

one obtains, after integration by part:

$$ [a, \pi_\theta(f)] = \sqrt{\theta} \pi_\theta(\partial f), \quad [a^\dagger, \pi_\theta(f)] = -\sqrt{\theta} \pi_\theta(\partial f). \quad (6.18) $$

Hence the first equation in (6.14).

From (6.8) one has $\sigma_\theta(T)(z) = \text{Tr}(Q_z T)$. Together with (6.16) this yields

$$ \partial \sigma_\theta(T)_z = -\frac{1}{\sqrt{\theta}} \text{Tr}([a, Q_z] T) = -\frac{1}{\sqrt{\theta}} \text{Tr}(Q_z [T, a]) = \frac{1}{\sqrt{\theta}} \sigma_\theta([a, T])_z. \quad (6.19) $$

Similarly $\partial \sigma_\theta(T)_z = -\frac{1}{\sqrt{\theta}} \sigma_\theta([a^\dagger, T])(z)$. Hence the second equation in (6.14). \hfill \blacksquare

**Lemma 6.6.** $B_\theta$ is a self-adjoint automorphism of the vector space $\mathcal{S}(\mathbb{R}^2)$.

\textsuperscript{5}As usual, the commutation relation holds on a dense subspace of $\mathcal{J}_\theta$ containing the linear span of the basis elements $h_n$.

\textsuperscript{6}Inside the commutator with $D$, we identify an element $f$ of $\mathcal{A}$ with its representation $f \otimes \mathbb{I}_2$ on $\mathcal{H} \otimes \mathbb{C}^2$. 

33
Proof. The set $S(\mathbb{R}^2)$ is a pre-Hilbert space with inner product $\langle \cdot , \cdot \rangle_{L^2}$. Introducing the reproducing kernel,
\begin{equation}
K_z(\xi) = \theta^{-1} |\langle \psi_\xi, \psi_z \rangle|^2 = \theta^{-1} e^{-\frac{1}{\theta} |z - \xi|^2},
\end{equation}
on one obtains the integral form of the Berezin transform
\begin{equation}
B_\theta(f)(z) = \frac{1}{\pi} \int_\mathbb{C} K_z(\xi) f(\xi) d^2\xi = \langle K_z, f \rangle_{L^2}.
\end{equation}

Since $e^{-\frac{1}{\theta} |z - \xi|^2}$ is a Schwartz function, and the Schwartz space is closed under convolution, $B_\theta(f) \in A$ for all $f \in A$. With a simple explicit computation one verifies that $\langle f, B_\theta(g) \rangle_{L^2} = \langle B_\theta(f), g \rangle_{L^2}$ for all $f, g \in A$, that is the Berezin transform is self-adjoint.

In Fourier space, the Berezin transform becomes the pointwise multiplication of the Fourier transform $\hat{f}$ of $f$ by a Gaussian (the Fourier transform of the Gaussian kernel $K_z$). This is identically zero if and only if $\hat{f} = 0$, i.e. only iff $f = 0$. This proves injectivity.

Let $V = B_\theta(A)$ and $V^\perp$ its orthogonal complement in the Hilbert space closure of $S(\mathbb{R}^2)$. For any $f \in V^\perp$ one has $0 = \langle f, B_\theta(g) \rangle_{L^2} = \langle B_\theta(f), g \rangle_{L^2}$ for all $g$. Choosing $g = B_\theta(f)$, one proves that $B_\theta(f)$ (hence $f$ by injectivity) vanishes. Thus $V^\perp = \{0\}$ and $B_\theta$ is surjective.

Proposition 6.7. For all $\varphi, \psi \in S(\mathbb{O}_\theta)$:
\begin{equation}
d_{A,D}(\varphi^\delta, \psi^\delta) \leq d_{D,B}(\varphi, \psi) \leq d_{A,D}^{(\theta)}(\varphi^\delta, \psi^\delta).
\end{equation}

Proof. $\sigma_\theta$ and $\pi_\theta$ are norm non-increasing. Moreover, being $\pi_\theta$ surjective, there is always an $f$ such that $T = \pi_\theta(f)$, that is $\sigma_\theta(T) = B_\theta(f)$. Omitting the identity, (6.14) yields
\begin{equation}
\|[D, \pi_\theta(f)]\| \leq \|[D, f]\|, \quad \|[D, B_\theta(f)]\| \leq \|[D, T]\|.
\end{equation}
The opposite inequalities for the dual distances then follow.

Let us apply these results to the coherent states $\Psi_\sharp, z \in \mathbb{C}$, that are the (pure) states of $\mathbb{O}_\theta$ defined by the vectors $\psi_z$ in (6.9). Recall that coherent states are the “best approximation” of points in a quantum context, because they minimize the uncertainty of $\hat{z} \hat{z}^*$ (the square of the “distance operator” [24, 43, 5]), where $\hat{z} = \pi_\theta(z)$. Another way to see that coherent states are a good approximation of points is to notice that $\Psi_\sharp$ is the (non-pure) state of the Schwartz algebra $A = S(\mathbb{C})$ given by the evaluation at $z$ of the Berezin transform,
\begin{equation}
\Psi_\sharp(f) = B_\theta(f)(z).
\end{equation}

As such $\Psi_\sharp$ converges to the pure state $\delta_z$ as $\theta \to 0$, as follows from:  

Lemma 6.8. For any $f \in A$, one has $\|f - B_\theta(f)\|_\infty \leq \sqrt{\pi \theta} L_D(f)$.

Proof. We use (6.21). For any $z \in \mathbb{C}$, $\pi^{-1} K_z(\xi)$ is a Gaussian probability measure on $\mathbb{C}$. Similarly to [52, Theo. 2.3], one has ($r := |\xi|$):
\begin{align*}
|f(z) - B_\theta(f)(z)| &= \frac{1}{\pi} \left| \int_\mathbb{C} K_z(\xi) (f(z) - f(\xi)) d^2\xi \right| \leq \frac{1}{\pi} \int_\mathbb{C} K_z(\xi) |f(z) - f(\xi)| d^2\xi \\
&\leq L_D(f) \frac{1}{\pi} \int_\mathbb{C} |z - \xi| K_z(\xi) d^2\xi = L_D(f) \int_0^{\infty} \frac{2}{\theta} e^{-\frac{1}{\theta} r^2} r^2 dr = \sqrt{\pi \theta} L_D(f).
\end{align*}

Prop. 6.7 allows to compute the distance between coherent states, and retrieve a result proved in [44] from a completely different perspective.
Proposition 6.9. For any \( z, z' \in \mathbb{C} \),

\[
d_{0,\mu,\nu}(\Psi z, \Psi z') = |z - z'|.
\]  

\[(6.25)\]

**Proof.** Due to (6.22), it is enough to prove that

\[
d_{A,D}(\Psi z_x^2, \Psi z_y^2) = d_{\theta}^{(\theta)}(\Psi z, \Psi z') = |z - z'|.
\]  

\[(6.26)\]

This would be immediate if \( \mathcal{A} \) were the algebra \( C_0^\infty(\mathbb{R}^2) \): in this case \( d_{A,D} \) would be the Wasserstein distance, and it is known that the distance between two Gaussians (with the same variance) is the Euclidean distance between the peaks (see e.g. [20]).

Here \( \mathcal{A} = S(\mathbb{R}^2) \) is smaller than \( C_0^\infty(\mathbb{R}^2) \), so in principle \( d_{A,D}(\Psi z_x^2, \Psi z_y^2) \leq |z - z'| \), but one easily checks that the supremum is attained on the sequence of 1-Lipschitz Schwartz functions (of the variable \( \xi \)):

\[
f_{n,z,z'}(\xi) = \xi \cdot \frac{z - z'}{|z - z'|} e^{-\frac{1}{2}|\xi|^2}.
\]  

\[(6.27)\]

On the other hand, since \( B_\theta \) is surjective on \( \mathcal{A} \),

\[
d_{A,D}(\Psi z_x^2, \Psi z_y^2) = \sup_{f = f' \in \mathcal{A}} \left\{ B_\theta(f)(z) - B_\theta(f)(z') : \|D, B_\theta(f)\| \leq 1 \right\}
\]

\[
= \sup_{g = g' \in \mathcal{A}} \left\{ g(z) - g(z') : \|D, g\| \leq 1 \right\} \leq |z - z'|,
\]

where \( g = B_\theta(f) \) and, since \( \mathcal{A} \subset C_0^\infty(\mathbb{R}^2) \), the distance above is no greater than the geodesic distance. Using again the sequence (6.27) one proves that the supremum is attained on Schwartz functions, and last inequality is in fact an equality. \( \blacksquare \)

**Remark 6.10.** Let \( \langle A, B \rangle_{HS} = \theta \text{Tr}(A^* B) \) be the Hilbert-Schmidt inner product on \( \mathcal{L}^2(\mathcal{H}_\theta) \). By an explicit computation one checks that \( \langle \pi_\theta(f), \pi_\theta(g) \rangle_{HS} = \langle f, B_\theta(g) \rangle_{L^2} \) for all \( f, g \in \mathcal{A} \). Since \( \|B_\theta(f)\|_{\infty} \leq \|f\|_{\infty} \),

\[
\|\pi_\theta(f)\|_{HS}^2 \leq \|f\|_{L^1} \|f\|_{\infty}
\]  

\[(6.28)\]

is finite, proving that elements of \( \pi_\theta(\mathcal{A}) \) are Hilbert-Schmidt operators. If we replace \( \mathcal{O}_\theta \) by the algebra \( \mathcal{A}_\theta \) generated by \( \pi_\theta(A) \), we get a spectral triple \( (\mathcal{A}_\theta, \mathcal{H}_\theta \otimes \mathbb{C}^2, D_\theta) \) close to the irreducible spectral triple of Moyal plane (for the latter, one uses the algebra of rapid decay matrices, in the basis \( h_n \), that is dense in the algebra of Hilbert-Schmidt operators).

### 6.3 Quantum discs, fuzzy spaces and other examples

Berezin quantization applies as well to the unit disc \( \mathfrak{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). With measure

\[
d\mu = \frac{1}{(1 - |z|^2)^2} d^2z,
\]  

\[(6.29)\]

one projects on the subspace \( \mathbb{H} := L^2_{\text{hol}}(\mathfrak{D}, d\mu) \) of holomorphic functions (the Bergman space) of the Hilbert space \( \mathcal{H} = L^2(\mathfrak{D}, d\mu) \). This yields the quantum disc of [33]. Using instead the measure

\[
d\mu_\alpha = \pi^{-1}(\alpha + 1)(1 - |z|^2)^\alpha d^2z \quad -1 < \alpha < \infty,
\]  

\[(6.30)\]

the corresponding spaces \( \mathbb{H}_\alpha := L^2_{\text{hol}}(\mathfrak{D}, d\mu_\alpha) \) are the weighted Bergman space (the Hardy space if \( \alpha = 0 \)). The corresponding truncated algebra describes the quantum disc of [29].
With a more complicated measure (not absolutely continuous with respect to the Lebesgue one) one get the $q$-disc [34, Eq. (3)]. Another example, where the projection operator has finite rank, is given by the fuzzy disk [38].

A similar construction holds for the torus $T^2$ and the projective space $\mathbb{C}P^n$. For $E \to T^2$, resp. $E' \to \mathbb{C}P^n$, holomorphic line bundles with Chern number $N$, the subspaces of $L^2(T^2, E)$, resp. $L^2(\mathbb{C}P^n, E)$, of holomorphic sections is finite-dimensional with dimension $|N|$, resp. $(n+1)^{N\choose n}$. In both cases the corresponding projection has finite rank so that the truncation yield a finite-dimensional spectral triple: for instance for $n = 1$, $\mathbb{C}P^1 \simeq S^2$ and one gets the fuzzy sphere (see e.g. [55]).

A more general class of examples is given by the Berezin quantization of a compact Kähler manifold, that is always given by finite-dimensional full matrix algebras. The quantization map $\pi_N : A \to A_N$ is surjective (so $\pi_N(A)$ is already an algebra), and provides a strict deformation quantization in the sense of Rieffel [7].

Among all these examples, we study in the following the fuzzy sphere (and keep the other examples for further works). Clearly the Dirac operator of $\mathbb{C}P^n$ does not commute with the projection (for example, $P\partial/\partial \bar{z}_i P = 0$ for $z_i$ a homogeneous coordinate on $\mathbb{C}P^n$ or the complex coordinate on the covering $\mathbb{C}$ of $T^2$), so that Lemma 6.1 does not apply. However, it is possible to obtain the fuzzy $\mathbb{C}P^n$ (and more generally fuzzy homogeneous spaces) using projections that commute with the Dirac operator. For $\mathbb{C}P^n$ one projects on a finite direct sum of irreducible representations of $SU(n+1)$, the so called Weyl-Wigner formalism (see [56]). At least for the fuzzy sphere, this gives rise to the same quantized space as Berezin quantization. A third way to obtain fuzzy spaces is via coherent states quantization, that we investigate in the next section. Metric properties of the fuzzy sphere are investigated in [19], where we show that the distance between coherent state converges to the geodesic distance in the $N \to \infty$ limit.

Before that, let us point out that the regularization of the real line by spectral projection, investigated in §5.3, is an example of Berezin quantization. Indeed consider the spectral triple (5.17), with $P_\Lambda$ the spectral projection of $D$ in the interval $[-\Lambda, \Lambda]$. By Fourier transform one proves that:

$$P_\Lambda = \int_{-\infty}^{\infty} \frac{\sin \Lambda t}{\pi t} U_t ,$$

(6.31)

where $U_t f(x) = f(x + t)$. Let $K_x$ be the following kernel:

$$K_x(t) := \frac{\sin \Lambda(x - t)}{\pi(x - t)} .$$

(6.32)

For any $x$, $K_x$ is a vector in $P_\Lambda \mathcal{H}$ (with norm $\sqrt{\Lambda/\pi}$). One has $P_\Lambda(f)(x) = \langle f, K_x \rangle$ hence for any $f \in \mathcal{H}(\Lambda)$, $\langle f, K_x \rangle = f(x)$. In other words, $P_\Lambda \mathcal{H}$ is a reproducing kernel Hilbert space, and the usual cut-off procedure on the real line is yet another example of Berezin quantization.

### 6.4 The fuzzy sphere as a coherent state quantization

The standard Berezin quantization of the sphere $S^2 \simeq \mathbb{C}P^1$ consists in taking a power of the quantum line bundle (i.e. the dual of the tautological bundle) and project on the finite-dimensional space of holomorphic sections. Here we follow the alternative approach of coherent state quantization.
The canonical spectral triple for \( S^2 \) is \((C^\infty(S^2), L^2(S^2) \otimes \mathbb{C}^2, D)\) where

\[
D = \begin{pmatrix} \frac{1}{2} + \partial_H & \partial_F \\ \partial_E & \frac{1}{2} - \partial_H \end{pmatrix}
\]

(6.33)

and in spherical coordinates \( \phi \in [0, 2\pi] \) and \( \vartheta \in [0, \pi] \) the derivatives in (6.33) are:

\[
\partial_H = -i \frac{\partial}{\partial \phi}, \quad \partial_E = e^{i\phi} \left( \frac{\partial}{\partial \vartheta} + i \cot \vartheta \frac{\partial}{\partial \phi} \right), \quad \partial_F = -e^{-i\phi} \left( \frac{\partial}{\partial \vartheta} - i \cot \vartheta \frac{\partial}{\partial \phi} \right).
\]

(6.34)

We write the inner product on \( L^2(S^2) \) as

\[
\langle f, g \rangle_{L^2} = \int_{S^2} f(x) g(x) d\mu_x
\]

(6.35)

with \( d\mu_x \) the \( SU(2) \)-invariant measure normalized to 1. An orthonormal basis is given by Laplace spherical harmonics \( Y_{\ell,m} \).

The notation \( \partial_H, \partial_E, \partial_F \) comes from the fact that these operators are the image of the standard Chevalley generators \( H = H^* \), \( E \) and \( F = E^* \) of the Lie algebra \( \mathfrak{su}(2) \) under the representation \( \partial : \mathfrak{su}(2) \to \text{Der}(C^\infty(S^2)) \) as vector fields on \( S^2 \). Let us recall that, in the Chevalley basis, the defining relations of \( \mathfrak{su}(2) \) are \([E,F] = 2H\), \([H,E] = E\), \([H,F] = -F\). The irreducible representation \( \rho_\ell : \mathfrak{su}(2) \to \text{End}(V_\ell) \) with highest \( \ell \in \frac{1}{2}\mathbb{N} \) is defined as follows; the underlying vector space \( V_\ell \simeq \mathbb{C}^{2\ell+1} \) has orthonormal basis \( |\ell, m\rangle \), with \( m = -\ell, \ldots, \ell \), and

\[
\rho_\ell(H) |\ell, m\rangle = m |\ell, m\rangle, \quad \rho_\ell(E) |\ell, m\rangle = \sqrt{(\ell - m)(\ell + m + 1)} |\ell, m + 1\rangle,
\]

(6.36)

with \( \rho_\ell(F) = \rho_\ell(E)^* \). The representation \( \partial \) decomposes as direct sum of all \( \rho_\ell \) with integer \( \ell \), and the equivalence

\[
U \partial_\ell U^* = \oplus_\ell \rho_\ell(\xi) \quad \forall \xi \in \mathfrak{su}(2)
\]

(6.37)

is implemented by the unitary map

\[
U : L^2(S) \to \mathcal{K} := \bigoplus_{\ell \in \mathbb{N}} V_\ell, \quad U(Y_{\ell,m}) := |\ell, m\rangle.
\]

(6.38)

The irreducible spectral triple\(^8\) \((A_\ell, V_\ell \otimes \mathbb{C}^2, D_\ell)\) of the fuzzy sphere is obtained by the action of the orthogonal projection \( P_\ell : \mathcal{K} \to V_\ell \) on \((C^\infty(S^2), \mathcal{K}, UDU^*)\), which is unitary equivalent to the canonical spectral triple of the sphere. Namely \([19, \text{eq. (4.1)}]\]

\[
A_\ell := \text{End}(V_\ell) \simeq M_{2\ell+1}(\mathbb{C}), \quad D_\ell := P_\ell(UDU^*)P_\ell = \begin{pmatrix} \frac{1}{2} + \rho_\ell(H) & \rho_\ell(F) \\ \rho_\ell(E) & \frac{1}{2} - \rho_\ell(H) \end{pmatrix}.
\]

(6.39)

We equip \( A_\ell \) with the Hilbert-Schmidt inner product:

\[
\langle A, B \rangle_{\text{HS}} := \gamma_\ell^{-1} \text{Tr}(A^*B) \quad \text{with} \quad \gamma_\ell := 2\ell + 1.
\]

(6.40)

The covariant Berezin symbol \( \sigma_\ell : A_\ell \to A \) is defined as \( \sigma_\ell(a) = \Psi_{x,\ell}(a) \), where \( \Psi_{x,\ell} \) is the Bloch coherent state. Namely \( \Psi_{x,\ell}(a) = \text{Tr}(Q_{x,\ell} a) \) is the vector state of \( A_\ell \) defined by the rank 1 projection

\[
Q_{x,\ell} = \sum_{m,n=-\ell}^{\ell} \binom{2\ell}{\ell+m} \binom{2\ell}{\ell+n} e^{i(\ell-m)\phi} (\sin \frac{\theta}{2})^{2\ell+m+n} (\cos \frac{\theta}{2})^{2\ell-m-n} |\ell,m\rangle \langle \ell,n|.
\]

(6.41)

\(^7\)Within our normalization, one has e.g. \( Y_{0,0}(x) = 1 \) and not \( 1/\sqrt{4\pi} \) as more commonly used.

\(^8\)As for Moyal plane, we use an index notation instead of \( A,H,D \) to stress the \( \ell \)-dependence of the objects.
One easily checks that for any fixed $\ell$, the map $S^2 \to S(A_\ell)$, $x \mapsto \Psi_{x,\ell}$, is injective. We denote by $\pi_\ell$ the adjoint map:

$$\langle f, \sigma_\ell(a) \rangle_{L^2} = \langle \pi_\ell(f), a \rangle_{HS} \quad \forall f \in A, \ a \in A_\ell.$$  \hspace{1cm} (6.42)

**Remark 6.11.** The operator $\pi_\ell(f)$ is not the Toeplitz operator $\pi_\ell(f) := PuFU^*P_\ell$ given by the action of the projection, as in Moyal case. It is an easy exercise to check that:

$$\pi_\ell(f) = \gamma_\ell \int_{S^2} f(x) R_{x,\ell} \, d\mu_x,$$

(6.43)

where $R_{x,\ell}$ is the rank 1 projection in the direction of the vector $\sum_{m=-\ell}^{\ell} Y_{\ell,m}(x) |\ell,m\rangle$, while

$$\pi_\ell(f) = \gamma_\ell \int_{S^2} f(x) Q_{x,\ell} \, d\mu_x.$$  \hspace{1cm} (6.44)

That the quantization maps $\pi_\ell$ and $\bar{\pi}_\ell$ are different can be seen for instance when $x = x_0$ is the north pole (so $\vartheta = 0$): then $R_{x_0,\ell}$ projects in the direction of $|\ell,0\rangle$, whereas $Q_{x_0,\ell}$ projects in the direction of the lowest weight vector $|\ell,-\ell\rangle$.

Since $[UDU^*, P_\ell] = 0$, using the quantization map $\bar{\pi}_\ell$ one applies Lemma 6.1 and get that the distance $d_{A_\ell, D_\ell}$ on the fuzzy sphere is not smaller than the distance $d_{A,D}$ on $S(C^\infty(S^2))$ induced by the pull back of $\bar{\pi}_\ell$. In the following we will use instead the map $\pi_\ell$ so that to find – as in Moyal plane – a lower bound $d^2_{A,D}$ to $d_{A_\ell,D_\ell}$ and also an upper bound given by the distance

$$d^2_{A,D}(\varphi, \psi) := \sup_{f = f^* \in A} \{ \varphi(f) - \psi(f) : ||D,B_\ell(f)|| \leq 1 \},$$

(6.45)

where $B_\ell := \sigma_\ell \circ \pi_\ell : A \to A$ is the Berezin transform,

$$B_\ell(f)(x) = \int_{S^2} K_x(y) f(y) \, d\mu_y = \langle K_x, f \rangle_{L^2}.$$  \hspace{1cm} (6.46)

with reproducing kernel $K_x(y) = \gamma_\ell \text{Tr}(Q_{x,\ell}Q_{y,\ell})$.

We begin with technical lemmas, similar to the ones for Moyal plane.

**Lemma 6.12.** Both the maps $\sigma_\ell$ and $\pi_\ell$ are unital, positive and norm non-increasing. Moreover $\sigma_\ell : A_\ell \to A$ is injective, hence the adjoint map $\pi_\ell : A \to A_\ell$ is surjective and the pull back $\xi : S(A_\ell) \to S(A)$ of $\pi_\ell$ on the space of state is injective.

**Proof.** For $\sigma_\ell$, all properties but injectivity follow from $\Psi_{x,\ell}$ being a state: $|\sigma_\ell(a)(x)| = |\Psi_{x,\ell}(a)| \leq \|a\|$ for all $x$, hence $\|\sigma_\ell(a)\|_\infty \leq \|a\|$. The injectivity of $\sigma_\ell$ is checked by explicit computation. For $\pi_\ell$, the only non-trivial point is unitality, i.e. $\pi_\ell(1) = P_\ell$. From (6.48) one has $[\rho_\ell(\xi), \pi_\ell(1)] = 0$ for all $\xi \in su(2)$. Since the representation $V_\ell$ is irreducible, by Shur lemma $\pi_\ell(1) = \lambda P_\ell$ is proportional to the identity endomorphism of $V_\ell$. Since Tr$(\pi_\ell(1)) = \gamma_\ell \int_{S^2} \text{Tr}(Q_{x,\ell}) \, d\mu_x = 2\ell + 1$ and Tr$(\lambda P_\ell) = \lambda(2\ell + 1)$, one gets $\lambda = 1$.

**Lemma 6.13.** For all $\xi \in su(2)$ and $x \in S^2$:

$$\partial_\xi Q_{x,\ell} + [\rho_\ell(\xi), Q_{x,\ell}] = 0.$$  \hspace{1cm} (6.47)

**Proof.** From [19, Lem. 4.2], $\Psi_{x,\ell}([\rho_\ell(\xi), a]) = \partial_\xi \Psi_{x,\ell}(a) \forall \xi \in su(2), a \in A_\ell$ and $x \in S^2$. By cyclicity of the trace, $\langle a, \partial_\xi Q_{x,\ell} + [\rho_\ell(\xi), Q_{x,\ell}] \rangle_{HS} = 0 \forall a \in A_\ell$, hence the lemma.
Corollary 6.14. For all \( \xi \in \mathfrak{su}(2) \), \( f \in A \) and \( a \in A_\ell \) one has
\[
\pi_\ell(\partial_\xi f) = [\rho_\ell(\xi), \pi_\ell(f)], \quad \partial_\xi \sigma_\ell(a) = \sigma_\ell([\rho_\ell(\xi), a]) .
\] (6.48)

Hence
\[
\pi_\ell([D, f]) = [D_\ell, \pi_\ell(f)], \quad [D, \sigma_\ell(a)] = \sigma_\ell([D_\ell, a]) .
\] (6.49)

Proof. The first equation in (6.48) comes from Lemma 6.13 using integration by parts. The second comes from Lemma 6.13 and the cyclic property of the trace. Eq. (6.49) then follows from the explicit form of \( D \) and \( D_\ell \).

Following Prop. 6.7, one gets the announced upper and lower bound to the distance on the quantum sphere.

Proposition 6.15. For any \( \varphi, \psi \in S(A_\ell) \), one has
\[
d_{A,D}^f(\varphi, \psi) \leq d_{A,\ell \cdot D_\ell}(\varphi, \psi) \leq d_{A,D}^{(f)}(\varphi^\#, \psi^\#) .
\] (6.50)

Remark 6.16. As for Moyal, coherent states converge to points in the weak* topology. Indeed for all \( f \in A \) one has
\[
\| f - B_\ell(f) \|_\infty \leq L_D(f) \cdot \frac{\pi}{2\gamma + 2} \left( \frac{2\gamma}{\gamma_\ell} \right) , \quad \forall f \in A .
\] (6.51)

This is a particular case of [52, Thm. 2.3]. The coefficient multiplying \( L_D(f) \) is given by \( \int_{S^2} d_{\text{geo}}(x_0, y) K_{x_0}(y) \, d\mu_y \), and is independent on \( x_0 \). Taking for \( x_0 \) the north pole \( \vartheta = 0 \), one has \( K_{x_0}(y) = \gamma_\ell(\cos \frac{\vartheta}{\vartheta'})^4 \), and \( d_{\text{geo}}(x_0, y) = \vartheta' \) where \( \vartheta' \) is the polar angle of \( y \). An explicit computation of the integral yields (6.51). From the asymptotic behaviour \( (\frac{2n}{n}) \sim \frac{2\pi}{\sqrt{n}} \), we deduce that \( \Psi^\#_{x,\ell}(f) = B_\ell(f) \xrightarrow{\ell \to \infty} f(x) \).

To conclude, we observe that since \( \|[D_\ell, \cdot]\| \) is a Lipschitz seminorm on \( A_\ell \), and the algebra is finite-dimensional, we know that \((A_\ell, V_\ell \otimes \mathbb{C}^2, D_\ell)\) is a compact quantum metric space, so that the distance \( d_{A_\ell, \ell \cdot D_\ell} \) is finite. But \( d_{A,D}^{(f)} \) is not: as for the regularization by eigenprojections of section 5.3, the representation \( \pi_\ell \) cuts the components of \( f \) with high angular momentum and the distance between pure states is infinite.

Proposition 6.17.
\[
d_{A,D}^{(f)}(\delta_x, \delta_y) = \infty \quad \forall x \neq y ,
\] (6.52)
\[
d_{A,D}^{(f)}(\varphi^\#, \psi^\#) < \infty \quad \forall \varphi, \psi \in S(A_\ell) .
\] (6.53)

Proof. Let \( f_k(x) = e^{-ik\phi} \sin \vartheta \) with \( k > 2\ell \). Then \( \pi_\ell(f_k) = 0 \) as one can see by performing the integral in \( d\phi \) in (6.44). Let \( x = (\phi, \vartheta) \) and \( y = (\phi', \vartheta') \). For \( x \neq y \) we can always find a \( k > 2\ell \) such that \( f_k(x) \neq f_k(y) \), which proves (6.52):
- when \( \vartheta \neq \vartheta' \) or when \( \vartheta = \vartheta' \) and \( \frac{\varphi - \varphi'}{2\pi} \) is irrational, any \( k \) is fine;
- when \( \vartheta = \vartheta' \) and \( \frac{\varphi - \varphi'}{2\pi} = \frac{p}{q} \) with \( p \) and \( q \) coprime, then any \( k \) coprime to \( q \) is fine.

For the same reason above, \( \pi_\ell(Y_{km}) = 0 \) if \( k > 2\ell \) and the support of \( \pi_\ell \) is \( V = \text{Span}\{Y_{km} : k \leq 2\ell\} \). Now \( \dim(V) = (2\ell + 1)^2 \), and being \( \pi_\ell \) surjective, \( \dim(\text{Im}(\pi_\ell)) = \dim(M_{2\ell+1}(\mathbb{C})) = (2\ell + 1)^2 \) too, proving that the restriction of \( \pi_\ell \) to \( V \) is also injective.

\(^9\)When acting on \( V_\ell \otimes \mathbb{C}^2 \) and \( V_\ell \otimes \mathbb{C}^2 \), we write \( f \) and \( a \) for the operators \( f \otimes \text{id}_{\mathbb{C}^2} \) and \( a \otimes \text{id}_{\mathbb{C}^2} \). Similarly, the maps \( \pi_\ell \otimes \text{id}_{M_2(\mathbb{C})} \) and \( \sigma_\ell \otimes \text{id}_{M_2(\mathbb{C})} \) are denoted by \( \pi_\ell \) and \( \sigma_\ell \).
Since $\sigma_k$ is injective, we deduce that the map $B_\ell : V \to V$ is injective too. Therefore $[D, B_\ell(f)] = B_\ell([D, f])$ (from (6.49)) is zero only if $[D, f] = 0$, i.e. $f$ is constant. Hence $L_B := \|[D, B_\ell(\cdot)]\|$ is a Lipschitz seminorm on $V$. By construction $\varphi^\sharp$ and $\psi^\sharp$ depends only on the component of $f$ belonging to $V$, thus

$$d_{A, D}^\varphi(\varphi^\sharp, \psi^\sharp) = \sup_{f, f' \in V^{\infty}} \left\{ \varphi^\sharp(f) - \psi^\sharp(f) : \|[D, B_\ell(f)]\| \leq 1 \right\}. \quad (6.54)$$

In fact, since we can add a constant to $f$ without changing $\varphi^\sharp(f) - \psi^\sharp(f)$ nor $\|[D, B_\ell(f)]\|$, the space $V$ can be replaced by $W = \text{Span}\{Y_{km} : 0 < k \leq 2\ell\}$ (obtained from $V$ by removing the constant functions multiple of $Y_{00}$). Since $L_B$ and $L_D$ are norms on $W$, and the latter is finite-dimensional, they are equivalent. In particular, $d_{A, D}^\varphi(\varphi^\sharp, \psi^\sharp)$ is strongly equivalent to $d_{A, D}(\varphi^\sharp, \psi^\sharp)$, and the latter (the Wasserstein distance) is no greater than $2$ (the diameter of $\mathbb{S}^2$). This proves (6.53).

## 7 Approximations of the derivative on $\mathbb{R}$

Here we consider some approximations of the derivative on $\mathbb{R}$ by finite differences (opposed to the approximation by the action of projections studied in the rest of the paper): the $h$ and the $q$-derivative, where $h > 0$ and $0 < q < 1$ are (real) deformation parameters.

### 7.1 $h$-derivative and “fat points”

The $h$-derivative is defined as [31]

$$D_h \psi(x) = \frac{\psi(x + h) - \psi(x)}{h}. \quad (7.1)$$

It has norm $\Lambda = 2h^{-1}$ and converges to the derivative in the weak sense for $h \to 0$.

It is not difficult to prove that two points are at finite distance $d_{A, D_h}$ iff they are in the same orbit for the action of $\mathbb{Z}$ generated by the translation $x \mapsto x + h$.

**Lemma 7.1.** For any $f \in A$, we have

$$\|[D_h, f]\| = \|D_h f\|_{\infty} \leq \|f'\|_{\infty}. \quad (7.2)$$

**Proof.** Let $U_t f(x) := f(x + t)$. Since $D_h = h^{-1}(U_h - 1)$, one checks that $[D_h, f] = (D_h f)U_h$. $U_h$ being unitary we get $\|[D_h, f]\| = \|D_h f\|_{\infty} = h^{-1} \sup_{x \in \mathbb{R}} |f(x + h) - f(x)|$. The inequality in (7.2) then follows from $|f(x) - f(y)| \leq |x - y| \cdot \|f'\|_{\infty}$. \hfill $\blacksquare$

**Proposition 7.2.** $d_{A, D_h}(\delta_x, \delta_y) = |x - y|$ if $x - y \in h\mathbb{Z}$, and is infinite otherwise.

**Proof.** The distance being symmetric, we can assume $x > y$. Take $x = y + nh$ with $n \geq 1$, and define $y_k := y + kh$. Lemma 7.1 yields

$$|f(x) - f(y)| \leq \sum_{k=0}^{n-1} |f(y_k) - f(y_k + h)| \leq n \sup_{t \in \mathbb{R}} |f(t) - f(t + h)| = nh \|[D_h, f]\|. \quad (7.3)$$

Thus $d_{A, D_h}(\delta_x, \delta_y) \leq nh = |x - y|$. Now, $\|[D_h, f]\| \leq \|f'\|_{\infty}$ implies the opposite inequality between the dual distance, that is $d_{A, D_h}(\delta_x, \delta_y) \geq |x - y|$. This proves the first statement.

For $K > 0$ let

$$f_K(x)(t) = K \sin^2 \frac{\pi(x - t)}{h}. \quad (7.5)$$
This function is $h$-periodic, hence $||D_h, f_{K,x}|| = 0$ so that
\[ d_{A,D_h}(\delta_x, \delta_y) \geq |f_{K,x}(x) - f_{K,x}(y)| = K \sin^2 \frac{\pi(x-y)}{h} . \] (7.6)

If $x - y \notin h\mathbb{Z}$, the lower bound is not zero, and goes to infinity for $K \to \infty$. □

As in section 5.3 we wonder if there exists a family of states convergent to pure states whose distance is the Euclidean one. Natural candidates are the rectangular distributions
\[ \delta_{x,\epsilon}(f) = \frac{1}{\epsilon} \int_{x-\epsilon/2}^{x+\epsilon/2} f(t)dt \quad \epsilon > 0, \] (7.7)
since the net $\{\delta_{x,\epsilon}\}_{\epsilon>0}$ is weakly convergent to $\delta_x$. For fixed $\epsilon$, call $\mathbb{R}_\epsilon = \{\delta_{x,\epsilon}\}_{x \in \mathbb{R}}$.

**Proposition 7.3.** $(\mathbb{R}_\epsilon, d_{A,D_h})$ is a metric space isometric to the Euclidean real line iff $\epsilon \neq 0$ is a multiple of $h$:

\[ d_{A,D_h}(\delta_{x,\epsilon}, \delta_{y,\epsilon}) = |x - y| \qquad \forall x, y \in \mathbb{R}. \] (7.8)

\[ d_{A,D_h}(\delta_{x,\epsilon}, \delta_{y,\epsilon}) = \begin{cases} |x - y| & \text{if } x - y \in h\mathbb{Z}, \\ \infty & \text{otherwise.} \end{cases} \] (7.9)

**Proof.** Assume that $x > y$. From the inequality in (7.2) follows
\[ d_{A,D_h}(\delta_{x,\epsilon}, \delta_{y,\epsilon}) \geq d_{A,D}(\delta_{x,\epsilon}, \delta_{y,\epsilon}) = |x - y|, \] (7.10)
where $D$ is the usual derivative and the last equality comes e.g. from [20, Prop. 3.2].

Let $\epsilon \in h\mathbb{N}^*$ and $n$ be the integer part of $(x-y)/\epsilon$, that is $x - y - \epsilon < n\epsilon \leq x - y$. Then
\[ \epsilon \delta_{x,\epsilon}(f) = \int_{x-\epsilon/2}^{x+\epsilon/2} f(t)dt \] (7.11)
\[ \epsilon(\delta_{x,\epsilon} - \delta_{y,\epsilon})(f) = \int_{x-\epsilon/2}^{x+\epsilon/2} (f(t + y + n\epsilon) - f(t + y))dt \] (7.12)
For any $f$ with $||[D_h, f]|| \leq 1$,
\[ \epsilon(\delta_{x,\epsilon} - \delta_{y,\epsilon})(f) \leq ne \int_{x-\epsilon/2}^{x+\epsilon/2} dt + \epsilon \int_{x-\epsilon/2}^{x+\epsilon/2} dt = ne + \epsilon(x - y - n\epsilon) = \epsilon(x - y), \]
where we use $f(t + y + n\epsilon) - f(t + y) \leq n\epsilon$ that follows from (7.3). Hence $d_{A,D_h}(\delta_{x,\epsilon}, \delta_{y,\epsilon}) \leq |x - y|$. Eq. (7.8) follows from (7.10).

Similarly, for any $\epsilon$ and any $x = y + nh$ with $n \geq 1$, one has
\[ \delta_{x,\epsilon}(f) - \delta_{y,\epsilon}(f) = \frac{1}{\epsilon} \int_{x-\epsilon/2}^{x+\epsilon/2} (f(t + x) - f(t + y))dt \leq \frac{Nh}{\epsilon} \int_{x-\epsilon/2}^{x+\epsilon/2} 1dt = Nh = |x - y|. \] (7.11)

Hence the first equation in (7.9). The second is obtained considering the function $f_{K,y}$ defined in (7.5). One has
\[ \delta_{x,\epsilon}(f_{K,y}) = \frac{K}{2\epsilon} \left[ t - \frac{h}{2\pi} \sin \frac{2\pi t}{h} \right]_{x-y}^{x-y+\epsilon} \] (7.12)
and using some trigonometric identities we get
\[ \delta_{x,\epsilon}(f_{K,y}) - \delta_{y,\epsilon}(f_{K,y}) = \frac{Kh}{\pi \epsilon} \sin \frac{\pi \epsilon}{h} \sin \frac{\pi(x-y)}{h}. \] (7.13)
If $\epsilon \notin h\mathbb{N}$ and $x - y \notin h\mathbb{Z}$, the right hand side goes to infinity for $K \to \infty$. □
Let us comment on this result. Since $\|D_h\| \sim h^{-1}$, we expect a maximum resolution of order $h$. What we proved is that if we replace points by rectangular states with width $\epsilon < h$ the distance is infinite, while for $\epsilon = h$ we recover the Euclidean distance.

7.2 $q$-derivative and French Rail metric

Let us consider another well-known “approximation” of the derivative, the $q$-derivative [35]

$$D_q f(x) = \frac{f(x) - f(qx)}{(1 - q)x} \quad \forall x \neq 0. \quad (7.14)$$

$D_q$ is extended by continuity at 0 (provided the r.h.s. is well-defined): $D_q f(0) = f'(0)$.

Because of the behaviour at $x = 0$, $D_q$ is not a bounded operator, so one cannot expect a minimum length. We show that the distance between pure states is always finite, and bounded by the “French Rail metric”: $d_{\text{snrcf}}(x, y) := |x| + |y|$ for any $x \neq y$, $d(x, x) = 0$.

**Lemma 7.4.** For any $f \in A$, we have $q^{\frac{1}{2}} \|[D_q, f]\| = \|[D_q f]\|_\infty \leq \|f'\|_\infty$.

**Proof.** From the $q$-analogue of the Leibniz rule, we get $[D_q, f] \psi(x) = \psi(qx) D_q f(x)$ for all $x \in \mathbb{R}$ (including $x = 0$). For $\rho > 0$, let $T_\rho$ be the unitary operator defined by $T_\rho \psi(x) = \rho^2 \psi(\rho x)$. Then $[D_q, f] = q^{-\frac{1}{2}} (D_q f) T_\rho$, and

$$[D_q, f]^* [D_q, f] = q^{-1} T_\rho^* D_q f^2 T_\rho. \quad (7.15)$$

Hence $\|[D_q, f]\|^2 = q^{-1} \|D_q f\|^2_{\text{snrcf}}$. The result follows from $|f(x) - f(y)| \leq |x - y| \|f'\|_\infty$. ■

**Proposition 7.5.** For any $x, y$,

$$d_{A, D_q}(\delta_x, \delta_y) \leq q^{\frac{1}{2}} d_{\text{snrcf}}(x, y). \quad (7.16)$$

If further $x$ and $y$ are in the same orbit of $q^\mathbb{Z}$, then

$$d_{A, D_q}(\delta_x, \delta_y) = q^{\frac{1}{2}} |x - y| \quad (7.17)$$

**Proof.** Assume $x > y$. By Lemma 7.4 we get $d_{A, D_q}(\delta_x, \delta_y) \geq q^{\frac{1}{2}} |x - y|$. If $y = q^n x$, with $n \geq 1$, then

$$|f(x) - f(y)| \leq \sum_{k=0}^{n-1} |f(x_k) - f(x_{k+1})| \leq \|D_q f\|_\infty \sum_{k=0}^{n-1} (1-q)|x_k| = \|D_q f\|_\infty (1-q^n)|x| = q^{\frac{1}{2}} \|[D_q, f]\|_\infty |x - y|,$$

where $x_k := q^k x$. Thus $d_{A, D_q}(\delta_x, \delta_y) \leq q^{\frac{1}{2}} |x - y|$, which proves (7.17).

In the $n \to \infty$ limit, $y = q^n x \to 0$ and we get $|f(x) - f(0)| \leq q^{\frac{1}{2}} \|[D_q, f]\|_\infty |x|$ $\forall x \in \mathbb{R}$. Hence (7.16), since for any $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \leq |f(x) - f(0)| + |f(0) - f(y)| \leq q^{\frac{1}{2}} \|[D_q, f]\|_\infty d_{\text{snrcf}}(x, y). \quad (7.17)$$

Note that if $x, y$ are small, $d_{A, D_q}(x, y)$ can be as small as we want: as expected, there is no minimum length. The geometrical reason why the distance is always finite is that all the orbits of $q^\mathbb{Z}$ have a common accumulation point, given by $x = 0$. 

42
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