THE TAMARI LATTICE AS IT ARISES IN QUIVER REPRESENTATIONS

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ABSTRACT. In this chapter, we explain how the Tamari lattice arises in the context of the representation theory of quivers, as the poset whose elements are the torsion classes of a directed path quiver, with the order relation given by inclusion.

1. Introduction

In this chapter, we will explain how the Tamari lattice $T_n$ arises in the context of the representation theory of quivers. In the representation theory of quivers, one fixes a quiver (quiver being a synonym for “directed graph”) and then considers the category of representations of that quiver. (Terms which are not defined in this introduction will be defined shortly.) Subcategories of this category with certain natural properties are called torsion classes. We show that the Tamari lattice $T_n$ arises as the set of torsion classes, ordered by inclusion, for the quiver consisting of a directed path of length $n$. It therefore follows that, for any directed graph, we obtain a generalization of the Tamari lattice. At the end of this chapter, we will comment briefly on the lattices that arise in this way, which include the Cambrian lattices discussed in Reading’s contribution to this volume [11].

The treatment of quiver representations which we have undertaken is very elementary. In particular, we avoid all use of homological algebra. A reader familiar with quiver representations will have no trouble finding quicker proofs of the results we present here. Introductions to quiver representations from a more algebraically sophisticated point of view may be found in [3, 2].

2. Quiver representations

Let $Q$ be a quiver (i.e., a directed graph). Fix a ground field $K$. A representation of $Q$ is an assignment of a finite-dimensional vector space $V_i$ over $K$ to each vertex $i$ of $Q$, and a linear map $V_\alpha : V_i \to V_j$ to each arrow $\alpha : i \to j$ of $Q$.

For a pair of representations $V, W$ of $Q$, we define a morphism from $V$ to $W$ to consist of a collection of maps $f_i : V_i \to W_i$ for all vertices $i$, such that for any $\alpha : i \to j$, we have that $W_\alpha \circ f_i = f_j \circ V_\alpha$. We write $\text{Hom}(V, W)$ for the set of morphisms from $V$ to $W$. It as a natural $K$-vector space structure. As usual, an isomorphism is a morphism which is invertible. An injection is a morphism all of whose linear maps are injections; surjections are defined similarly.

These definitions make quiver representations into a category, which we denote $\text{rep} Q$. (The careful reader is encouraged to confirm this.)

Given two representations of $Q$, their direct sum $V \oplus W$ is defined in the obvious way: setting $(V \oplus W)_i = V_i \oplus W_i$, and $(V \oplus W)_\alpha = V_\alpha \oplus W_\alpha$. 


A representation is called \textit{indecomposable} if it is not isomorphic to the direct sum of two non-zero representations.

3. Subrepresentations, Quotient Representations, and Extensions

If \( Y \) is a representation of \( Q \), a \textit{subrepresentation} of \( Y \) is a representation \( X \) such that for each \( i \), \( X_i \) is a subspace of \( Y_i \), and for \( \alpha : i \to j \), we have that \( X_{i\alpha} \) is induced from the inclusions of \( X_i \) and \( X_j \) into \( Y_i \) and \( Y_j \) respectively. The inclusions of \( X_i \) into \( Y_i \) define an injective morphism from \( X \) to \( Y \).

If \( Y \) is a representation of \( Q \), and \( x \in Y_i \), the subrepresentation of \( Y \) generated by \( x \) is the representation \( X \) such that \( X_j \) is spanned by all images of \( x \) under linear maps corresponding to walks from \( i \) to \( j \) in \( Q \).

If \( Y \) is a representation of \( Q \), and \( X \) is a subrepresentation of \( Y \), then it is also possible to form the \textit{quotient representation} \( Y/X \). By definition \( (y/X)_i = Y_i/X_i \), and the maps of \( Y/X \) are induced from the maps of \( Y \). The quotient maps from \( Y_i \) to \( (Y/X)_i \) define a surjective morphism from \( Y \) to \( Y/X \).

Suppose \( X, Y, Z \) are representations of \( Q \). Then \( Y \) is said to be an extension of \( Z \) by \( X \) if there is a subrepresentation of \( Y \) which is isomorphic to \( X \), such that the corresponding quotient representation is isomorphic to \( Z \). The extension is called \textit{trivial} if there is a morphism \( s \) from \( Y \) to \( X \) which is the identity on \( X \). Such a morphism is said to split the inclusion of \( X \) into \( Y \).

\textbf{Lemma 3.1.} If \( Y \) is a trivial extension of \( Z \) by \( X \), then \( Y \) is isomorphic to \( X \oplus Z \).

\textit{Proof.} Let \( s \) be the map which splits the inclusion of \( X \) into \( Y \). Write \( g \) for the quotient map from \( Y \) to \( Z \). Then \( s \oplus g \) is a morphism from \( Y \) to \( X \oplus Z \), which is an isomorphism over each vertex. It follows that it is an isomorphism of representations. \(\Box\)

The following discussion is not necessary for our present considerations, but may be of interest, in that it connects our discussion to notions of homological algebra. It is possible to define a notion of equivalence on extensions, as follows: two extensions \( Y, Y' \) of \( X \) by \( Z \) are said to be equivalent if there is an isomorphism from \( Y \) to \( Y' \) which induces the identity maps on \( X \) and \( Z \). We write \( \text{Ext}(Z, X) \) for the set of extensions of \( Z \) by \( X \) up to equivalence. This turns out to have a natural \( K \)-vector space structure. \( \text{Ext}(Z, X) \) can then be identified as \( \text{Ext}^{1}(Z, X) \) in the usual sense of homological algebra. See [2, Appendix A.5].

4. Pullbacks of Extensions

\textbf{Lemma 4.1.} Let \( Y \) be an extension of \( Z \) by \( X \). Suppose we have a surjective map \( h : Z' \to Z \). Then there is a representation \( Y' \) which is an extension of \( Z' \) by \( X \), and such that \( Y' \) admits a surjection to \( Y \).

The extension which we will exhibit in order to prove this lemma is called the \textit{pullback} of the extension \( Y \) along the surjection \( h \).

\textit{Proof.} Let \( i \) be a vertex of \( Q \). We are given surjective maps \( g : Y_i \to Z_i \) and \( h : Z'_i \to Z_i \). Define \( Y'_i \) to be the pullback of these two maps, that is to say,

\[ Y'_i = \{(y, z') \mid y \in Y_i, z' \in Z'_i, \text{ and } g(y) = h(z')\}. \]

For \( \alpha : i \to j \) an arrow of \( Q \), define \( Y'_{i\alpha} = Y_\alpha \times Z'_{i\alpha} \). One verifies that this defines a map from \( Y'_i \) to \( Y'_j \). It follows that \( Y' \) is a representation of \( Q \).
Write $f_i : X_i \to Y_i$ for the given injection from $X_i$ to $Y_i$. There is an injective morphism $f'$ from $X$ to $Y'$, defined on $X_i$ by sending $x$ to $(f(x), 0)$. One checks that $(f(x), 0) \in Y'_i$, and that these maps define a morphism from $X$ to $Y'$.

It is also easy to see that there is a surjective morphism from $Y'$ to $Z'$ defined on $Y'_i$ by sending $(y, z')$ to $z' \in Z'_i$. The elements of $Y'_i$ which are sent to zero by this map are those of the form $(y, 0)$, and $(y, 0) \in Y'_i$ iff $g(y) = 0$ iff $(y, 0) \in f'_i(X_i)$. So $Z'$ is isomorphic to $Y'/X$, as desired.

Finally, one defines a map from $Y'_i$ to $Y_i$ by sending $(y, z') \to y$. One checks that this defines a morphism from $Y'$ to $Y$, which is clearly surjective. \hfill $\square$

5. **Indecomposable representations of the quiver $A_n$**

Consider the quiver which consists of an oriented path: the vertices are numbered 1 to $n$, and for $1 \leq i \leq n - 1$, there is a unique arrow $\alpha_i$ whose tail is at vertex $i$, and whose head is at vertex $i + 1$. We will refer to this quiver as $A_n$.

For $1 \leq i \leq j \leq n$, define a representation $E^{ij}$ by putting one-dimensional vector spaces at all vertices $p$ with $i \leq p \leq j$, with identity maps between successive one-dimensional vector spaces, and zero vector spaces and maps elsewhere.

**Proposition 5.1.** The representations $E^{ij}$ are indecomposable, and any indecomposable representation of $A_n$ is isomorphic to some $E^{ij}$.

**Proof.** Suppose $E^{ij} \cong X \oplus Y$. Since for $E^{ij}$ the vector spaces at each vertex are at most one-dimensional, for a given vertex $p$, at most one of $X_p$ and $Y_p$ is non-zero. If neither $X$ nor $Y$ is zero, there must be some $i \leq p < j$ such that either $X_p$ is zero and $Y_{p+1}$ is zero, or vice versa. In either case, $(X \oplus Y)_{\alpha_p} = 0$. However, $(E^{ij})_{\alpha_p} \neq 0$, and it follows that $E^{ij}$ is not isomorphic to $X \oplus Y$.

Let $V$ be an indecomposable representation of $A_n$. Write $p_j$ for $V_{\alpha_j}$. Choose $i$ minimal such that $V_i \neq 0$, and choose a non-zero $t \in V_i$. Let $T$ be the subrepresentation of $V$ generated by $t$, which admits a natural injection into $V$. We have natural inclusions of the vector space at vertex $k$ for $T$ into the vector space at vertex $k$ for $V$, and we denote this inclusion by $f_k : T_k \to V_k$.

\[
\begin{array}{cccccccc}
0 & \cdots & 0 & T_i (= Kt) & T_{i+1} & T_{i+2} & \cdots \\
V_1 & \cdots & V_{i-1} & f_j & f_{i+1} & f_{i+2} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \cdots & 0 & T_{i+1} & T_{i+2} & \cdots \\
V_1 & \cdots & V_{i-1} & f_i & f_{i+1} & f_{i+2} & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
V_i & \cdots & V_{i+1} & p_i & p_{i+1} & \cdots \\
\end{array}
\]

Let $j$ be maximal so that $p_{j-1} \cdots p_i(t) \neq 0$. Define a map $s_j$ which splits the inclusion $f_j$, that is to say, a map such that $s_j \circ f_j$ is the identity. Now inductively define $s_{j-1}, s_{j-2}, \ldots, s_i$ so that, when constructing $s_k$, we have that $s_k$ splits $f_k$, and $p_k \circ s_k = s_{k+1} \circ p_k$. For $k$ not between $i$ and $j$, define $s_k = 0$.

We claim the maps $s_k$ define a morphism from $V$ to $T$. The only conditions which we did not explicitly build into the construction of $s_k$ are the commutativity conditions $p_{i-1} \circ s_{i-1} = s_i \circ p_{i-1}$ (if $i > 1$) and $p_j \circ s_j = s_{j+1} \circ p_j$ (if $j < n$). The first is satisfied by our assumption that $i$ is minimal such that $V_i \neq 0$, and the second is satisfied by our assumption that $p_j \cdots p_i(x) = 0$, which implies that $p_j | T_i = 0$.

By Lemma 3.1 it follows that $V$ is isomorphic to the direct sum of $T$ and $V/T$. Since $V$ is indecomposable by assumption, and $T$ is non-zero, $V/T$ must be zero, so $V$ is isomorphic to $T$, which is isomorphic to $E^{ij}$, proving the proposition. $\square$. 

6. Morphisms and extensions between indecomposable representations of $A_n$

**Proposition 6.1.** The space of morphisms from $E^{ij}$ to $E^{kl}$ is either 0-dimensional or 1-dimensional. It is one-dimensional iff $k \leq i \leq l \leq j$.

*Proof.* $E^{ij}$ is generated by $(E^{ij})_1$, so a morphism $f : E^{ij} \to E^{kl}$ is determined by its restriction to $(E^{ij})_1$. The space of maps from $(E^{ij})_1$ to $(E^{kl})_1$ is one-dimensional if $k \leq i \leq l$, and zero otherwise. If $l > j$, then the commutativity condition corresponding to $\alpha_j$ cannot be satisfied for a non-zero morphism; if on the other hand $l \leq j$, then we see that non-zero morphisms do exist. □

**Proposition 6.2.** The only circumstance in which there is a non-trivial extension of $E^{ij}$ by $E^{kl}$ is if $i + 1 \leq k \leq j + 1 \leq l$. In this case, any non-trivial extension is isomorphic to $E^{kl} \oplus E^{kl}$. (If $k = j + 1$, we interpret $E^{kl}$ as zero.)

*Proof.* Let $Y$ be an extension of $E^{ij}$ by $E^{kl}$. Let $t$ be an element of $Y_1$ which maps to a non-zero element of $(E^{ij})_1$. Let $T$ be the sub-representation of $Y$ which is generated by $t$. The representation $T$ is definitely non-zero at the vertices $p$ with $i \leq p \leq j$, since the image of $t$ in $(E^{ij})_p$ is non-zero for such $p$. If $T_{j+1} = 0$, then $T$ is isomorphic to $E^{ij}$, and the projection from $Y$ to $E^{ij}$ splits the inclusion of $T$ into $Y$, so $Y$ is isomorphic to $E^{ij} \oplus E^{kl}$ by Lemma 3.1.

Therefore, in order for there to exist a non-trivial extension, we must definitely have $k \leq j + 1 \leq l$. Consider now the case that $k \leq i$, which we must also exclude. Let $v$ be the image of $t$ in $Y_{j+1}$, and suppose it is non-zero. Since $E^{ij}$ is not supported over $j + 1$, we must have that $v$ lies in the image of $E^{kl}$. By our assumption that $k \leq i$, there is an element $x$ of $(E^{kl})_1$ such that its image in $(E^{kl})_{j+1}$ coincides with $v$. Now set $t' = t - v$, and repeat the above analysis with $t'$. By construction the image of $t'$ in $Y_{j+1}$ is zero, so $E^{ij}$ is a direct summand of $Y$ and the extension is trivial.

Finally, suppose that $i + 1 \leq k \leq j + 1 \leq l$, and that we have some $t$ in $Y_1$ whose image in $Y_{j+1}$ is non-zero. It follows necessarily that the subrepresentation $T$ of $Y$ generated by $t$ must be isomorphic to $E^{kl}$. As in the proof of Proposition 5.1, we see that $T$ is a direct summand of $Y$, so $Y$ is isomorphic to $E^{kl} \oplus Z$ for some $Z$, and it is clear that we must have $Z \cong E^{kl}$ (or $Z = 0$ if $k = j + 1$). □

The only nontrivial extension of indecomposable representations of $A_2$ is an extension of $E^{11}$ by $E^{22}$, isomorphic to $E^{12}$. There are several examples of nontrivial extensions among representations of $A_3$, such as, for example, the extension of $E^{12}$ by $E^{23}$ which is isomorphic to $E^{13} \oplus E^{22}$. Note that this latter example is obviously non-trivial even though the extension is a direct sum, since the summands in the direct sum are not the same as the two indecomposables from which the extension was built.

7. Subcategories of $\text{rep } Q$

By definition, a subcategory of a category is a category whose objects and morphisms belong to those of the original category, and such that the identity maps in the subcategory and category coincide. This notion is not strong enough for our purposes. A full additive subcategory $\mathcal{B}$ of $\text{rep } Q$ is a subcategory satisfying the following conditions:

- For $X, Y$ objects of $\mathcal{B}$, we have that $\text{Hom}_{\mathcal{B}}(X, Y) = \text{Hom}(X, Y)$. 

• There is some set of indecomposable objects of \( \text{rep} Q \) such that the objects of \( \mathcal{B} \) consist of all finite direct sums of indecomposable objects from this set.

From now on, when we speak of subcategories, we always mean full additive subcategories.

In this chapter, we are particularly interested in torsion classes in \( \text{rep} Q \). A torsion class in an abelian category \( \mathcal{A} \) is a full additive subcategory closed under quotients and extensions. That is to say, if \( Y \in \mathcal{A} \), and there is a surjection from \( Y \) to \( Z \), then \( Z \in \mathcal{A} \), and if \( X, Z \) are in \( \mathcal{A} \), and \( Y \) is an extension of \( Z \) by \( X \), then \( Y \in \mathcal{A} \). Torsion classes play an important role in tilting theory, which it is beyond the scope of this chapter to review. See [3, 2] for more information on this subject.

8. Quotient-closed subcategories

As a prelude to classifying the torsion classes of \( \text{rep} A_n \), we consider the subcategories of \( \text{rep} A_n \) which are closed under quotients. This class of subcategories of \( \text{rep} Q \) was not studied classically but has received some recent attention (see [10] or, considering the equivalent dual case, [13]).

Let \( M \) be the set of \( n \)-tuples \((a_1, \ldots, a_n)\) with \( 0 \leq a_i \leq n + 1 - i \). For \( a \in M \), define

\[ \mathcal{F}_a = \{(i, j) \mid i \leq j < i + a_i\} \]

and let \( \mathcal{C}_a \) be the full subcategory consisting of direct sums of indecomposables \( E^{ij} \) with \((i, j) \in \mathcal{F}_a\).

**Proposition 8.1.** The quotient-closed subcategories of \( \text{rep} A_n \) are exactly those categories of the form \( \mathcal{C}_a \) for \( a \in M \).

In order to prove this proposition, we need a lemma:

**Lemma 8.2.** Suppose \( X \) is a representation of \( A_n \) which admits a surjection to \( E^{kl} \). Then any expression for \( X \) as a direct sum of indecomposables includes an indecomposable \( E^{kj} \) with \( j \geq l \).

(Note that the statement of the lemma avoids assuming any uniqueness of the decomposition of \( X \) as a sum of indecomposable representations. In fact, for \( X \in \text{rep} Q \), and \( Q \) any quiver, the collection of summands appearing in an expression for \( X \) as a sum of indecomposable representations is unique up to permutation. This is called the Krull-Schmidt property, and it is established, for example, in [2, Section 1.4]. However, in the interests of self-containedness, we have preferred to avoid the use of this.)

**Proof.** Consider an expression of \( X \) as a sum of indecomposables. By Proposition 6.1, \( E^{kl} \) does not admit any morphisms from \( E^{ij} \) with \( i < k \), so we may assume \( X \) contains no such summands. On the other hand, when we consider summands of \( X \) of the form \( E^{ij} \) with \( i > k \), we see that the map from such summands to \( E^{kl} \) cannot be surjective at vertex \( k \). Therefore \( X \) must have some summand of the form \( E^{kj} \) which admits a morphism to \( E^{kl} \); using Proposition 6.1 again, we see that \( l \leq j \).

Now we prove the proposition:
Proof. Clearly there are surjections:

\[ E^{ii} \leftarrow E^{i(i+1)} \leftarrow \ldots \leftarrow E^{in} \]

It follows that a quotient-closed subcategory which contains \( E^{ij} \) necessarily contains \( E^{jp} \) for all \( i \leq p \leq j \), and thus that any quotient-closed subcategory is of the form \( C_a \) for some \( a \in M \).

Next, we verify that any such subcategory is quotient-closed. Suppose \( X \in C_a \), and \( X \) admits a surjection to \( Y \). Assume for the sake of contradiction that \( Y \) is not in \( C_a \). So \( Y \) has some indecomposable summand \( E^{ij} \) which is not in \( C_a \), and \( E^{ij} \), in particular, admits a surjection from \( X \). By Lemma 8.2, it follows that \( X \) has some summand of the form \( E^{ik} \) with \( k \geq j \), so \((i,k) \in F_a\), so \((i,j) \in F_a\), contradicting the assumption that \( Y \notin C_a \). □

9. Subcategories ordered by inclusion

We consider the obvious order on \( M \), the order it inherits as a Cartesian product, and we write that \( a = (a_1, \ldots, a_n) \leq b = (b_1, \ldots, b_n) \) iff \( a_i \leq b_i \) for all \( i \).

Proposition 9.1. For \( a, b \in M \), \( C_a \subseteq C_b \) iff \( a \leq b \).

Proof. Clearly if \( a \leq b \), then \( F_a \subseteq F_b \), and therefore \( C_a \subseteq C_b \). Conversely, if \( C_a \subseteq C_b \), then in particular the indecomposable objects of \( C_a \) are contained among those of \( C_b \). Since the objects of \( C_a \) are direct sums of objects \( E^{ij} \) with \((i,j) \in F_a\), the indecomposable objects of \( C_a \) are exactly those \( E^{ij} \) with \((i,j) \in F_a\). It follows that \( F_a \subseteq F_b \), and thus \( a \leq b \). □

10. Torsion classes in \( \text{rep} A_n \)

We now define a subset of \( M \). We say that an \( n \)-tuple \((a_1, \ldots, a_n) \in M \) is a bracket vector if, for all \( 1 \leq i \leq n \) and \( j \leq a_i \), we have that \( j + a_i + j \leq a_i \). The well-formed bracket strings of length \( 2n + 2 \) correspond bijectively to bracket vectors of length \( n \): for each open-parenthesis, find the corresponding close-parenthesis, and then record the number of open-parentheses strictly between them. Reading these numbers from left to right, and skipping the last one (which is necessarily zero), we obtain a bracket vector. Thus, for example, \((())\) is encoded by the bracket vector 01, while \((())\) is encoded by the bracket vector 20. The notion of bracket vector goes back to Huang and Tamari [8]. They show in addition that the poset structure induced on bracket vectors from their inclusion into \( M \), which, as we have already remarked, is shown in [8] to be isomorphic to the Tamari lattice.

The main result of this section is the following theorem:

Theorem 10.1. The torsion classes of \( \text{rep} A_n \) are exactly the subcategories \( C_a \) for \( a \) a bracket vector.

Before we begin the proof of this theorem, we will first state and prove the corollary which is the main result of this chapter.

Corollary 10.2. The torsion classes in \( \text{rep} A_n \), ordered by inclusion, form a poset isomorphic to the Tamari lattice.

Proof. We have already observed that \( C_a \subseteq C_b \) iff \( a \leq b \). It follows that the torsion classes for \( \text{rep} A_n \), ordered by inclusion, form a poset isomorphic to the poset structure induced on bracket vectors from their inclusion into \( M \), which, as we have already remarked, is shown in [8] to be isomorphic to the Tamari lattice. □
Next, we need to establish some terminology and prove a lemma.

For \( a \) a bracket vector, let
\[
G_a = \{(i,i+a_i-1) \mid 1 \leq i \leq n, a_i \geq 1\}.
\]

Let \( D_a \) consist of the full subcategory consisting of sums of \( E^{ij} \) for \((i,j) \in G_a\). Observe that \( D_a \) is a subcategory of \( C_a \), and any object in \( C_a \) is a quotient of some object in \( D_a \).

**Lemma 10.3.** Let \( a \) be a bracket vector. If \( X \in C_a \), and \( Z \in D_a \), then any extension of \( Z \) by \( X \) is trivial.

**Proof.** Write \( Z = Z^1 \oplus \cdots \oplus Z^m \). Observe that any extension of \( Z \) by \( X \) can be realized by first forming an extension of \( Z^1 \) by \( X \), call it \( Y^1 \), then forming an extension of \( Z^2 \) by \( Y^1 \), call it \( Y^2 \), and so on. If the extension at each step is trivial, then the total extension is trivial, so it suffices to consider the case that \( Z \) is indecomposable. Suppose therefore that \( Z \cong E^{i(i+a_i-1)} \), and let \( Y \) be an extension of \( Z \) by \( X \), for some \( X \in C_a \).

Let \( t \) be an element of \( Y_i \) which maps to a nonzero generator of \( Z \). Let \( T \) be the subrepresentation of \( Y \) generated by \( t \). Let \( v \) be the image of \( t \) in \( Y_{i+a_i} \).

If \( v \) is non-zero then, since \( Z \) is not supported over \( i+a_i \), we must have that \( v \in X_{i+a_i} \). Let \( E^{kl} \) be a summand of \( X \) in which \( v \) is non-zero. So we know that \( k \leq i+a_i \leq l \). By the assumption that \((k,l) \in F_a\), it follows that \( k \leq i \). Therefore, it follows that, as in the proof of Proposition 6.2, we can find an element \( x \) of \( X_i \) whose image in \( X_{i+a_i} \) equals \( v \). Now let \( t' = t - x \). The image of \( t' \) in \( Y_{i+a_i} \) is zero, so \( T \) is isomorphic to \( Z \). Therefore, the projection from \( Y \) to \( Z \) splits the inclusion of \( T \) into \( Y \), and the extension of \( Z \) by \( X \) is trivial.

\( \square \)

**Proof of Theorem 10.7.** First, we show that if \( a = (a_1, \ldots, a_n) \in M \) is not a bracket vector, then \( C_a \) is not a torsion class. So suppose we have some \( i,j \) such that \( 1 \leq i \leq n, j \leq a_i \), and \( j + a_i + j > a_i \). We know that \( C_a \) contains \( E^{i(i+a_i-1)} \) and \( E^{i(j+j+a_i+j-1)} \), and from our assumptions, \( i + j + a_i < i + j + a_i + j \). By Proposition 6.2, it follows that \( E^{i(i+j+a_i+j-1)} \oplus E^{i(j+j+a_i+j-1)} \) is an extension of \( E^{i(i+j+a_i+j-1)} \) by \( E^{i(i+a_i-1)} \), and since \( i + j + a_i + j > i + a_i \), we know that \( E^{i(i+j+a_i+j-1)} \) is not contained in \( C_a \). Thus \( C_a \) is not closed under extensions, so it is not a torsion class.

Now, we show that if \( a \) is a bracket vector, then \( C_a \) is a torsion class. We have already shown that \( C_a \) is quotient-closed, so all that remains is to show that it is closed under extensions.

Let \( X \) and \( Z \) be representations in \( C_a \). If we could assume that \( X \) and \( Z \) were indecomposable, our lives would be much easier — an argument very similar to the converse direction would suffice. However, there is no reason that we can assume that.

Choose an object \( Z' \in D_a \) such that \( Z' \) has a surjection onto \( Z \). Let \( Y' \) be the pullback along \( Z' \to Z \) of the extension of \( Z \) by \( X \). By Lemma 10.3, this is a trivial extension, so \( Y' \in C_a \). By Lemma 4.1, \( Y' \) admits a surjective map to \( Y \). Thus \( Y \) is a quotient of an element of \( C_a \), and thus lies in \( C_a \). Therefore, \( C_a \) is closed under extensions.

\( \square \)
11. Related posets

As was already mentioned, for arbitrary $Q$, we obtain a poset of torsion classes ordered by inclusion. In fact, it is easy to check from the definition that the intersection of an arbitrary set of torsion classes is again a torsion class. Thus, this poset is closed under arbitrary meets, and it has a maximum element, so it is a lattice.

The number of indecomposable representations of $Q$ is finite if and only if $Q$ is an orientation of a simply-laced Dynkin diagram. (This is part of the celebrated theorem of Gabriel, see, for example, \[2\] Theorem VII.5.10.) For such $Q$ (and only such $Q$), the lattice of torsion classes is a (finite) Cambrian lattice, for a Coxeter element chosen based on the orientation of $Q$. See [11] for more on Cambrian lattices, and [9, 1] for this result.

The poset of torsion classes has not been classically studied in representation theory. However, a closely related poset does appear. For the remainder of this section, suppose that $Q$ is a quiver with no oriented cycles.

A representation $T$ of $Q$ is called a tilting object if the only extension of $T$ with itself is the trivial extension, and $T$ has $n$ pairwise non-isomorphic summands ($n$ being the number of vertices of $Q$). For $X \in \text{rep} \ Q$, write $\text{Gen} \ X$ for the subcategory of $\text{rep} \ Q$ consisting of all quotients of direct sums of copies of $X$. A poset was defined by Riedtmann and Schofield [12] on the tilting objects of $\text{rep} \ Q$, by $T \geq V$ iff $\text{Gen} \ T \supseteq \text{Gen} \ V$. This poset was studied further by Happel and Unger [6, 5, 7].

This poset structure is related to the one discussed in this paper, because if $T$ is a tilting object, then $\text{Gen} \ T$ is a torsion class. Further, the torsion classes arising in this way can be described: they are just the torsion classes which include all the injective representations of $Q$, see [2, Theorem VI.6.5]. Thus, the torsion classes arising in this way form an interval in the poset of all torsion classes, whose minimal element is the torsion class consisting only of injective representations, and whose maximal element is the torsion class consisting of all representations.

In the case of $\text{rep} \ A_n$, $\mathcal{C}_n$ is sincere iff $a_1 = n$, since the injective indecomposable representations are those of the form $E^{(j)}$ for $1 \leq j \leq n$. There is a bijection from bracket vectors with $a_1 = n$ to bracket vectors of length $n - 1$, by removing $a_1$. (A bracketing corresponding to a bracket vector with $a_1 = n$ has its first open parenthesis closed by the final close parenthesis of the bracketing, from which this claim follows immediately.) It therefore follows that the Riedtmann-Schofield order on tilting objects for $\text{rep} \ A_n$ is isomorphic to the Tamari lattice $T_{n-1}$. The poset structure on tilting objects in the $A_n$ case was first analyzed in [4].

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