Degree powers in graphs with forbidden subgraphs

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Abstract
For every real $p > 0$ and simple graph $G$, set

$$f(p, G) = \sum_{u \in V(G)} d^p(u),$$

and let $\phi(p, n, r)$ be the maximum of $f(p, G)$ taken over all $K_{r+1}$-free graphs $G$ of order $n$. We prove that, if $0 < p < r$, then

$$\phi(p, n, r) = f(p, T_r(n)),$$

where $T_r(n)$ is the $r$-partite Turan graph of order $n$. For every $p \geq r + \lceil \sqrt{2r} \rceil$ and $n$ large, we show that

$$\phi(p, n, r) > (1 + \varepsilon) f(p, T_r(n))$$

for some $\varepsilon = \varepsilon(r) > 0$.

Our results settle two conjectures of Caro and Yuster.

1 Introduction

Our notation and terminology are standard (see, e.g. [1]).

Caro and Yuster [3] introduced and investigated the function

$$f(p, G) = \sum_{u \in V(G)} d^p(u),$$

where $p \geq 1$ is integer and $G$ is a graph. Writing $\phi(r, p, n)$ for the maximum value of $f(p, G)$ taken over all $K_{r+1}$-free graphs $G$ of order $n$, Caro and Yuster stated that, for every $p \geq 1$,

$$\phi(r, p, n) = f(p, T_r(n)), \quad (1)$$

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where $T_r(n)$ is the $r$-partite Turán graph of order $n$. However, simple examples show that (1) fails for every fixed $r \geq 2$ and all sufficiently large $p$ and $n$; this was observed by Schelp [4]. A natural problem arises: given $r \geq 2$, determine those real values $p > 0$, for which equality (1) holds. Furthermore, determine the asymptotic value of $\phi(r,p,n)$ for large $n$.

In this note we essentially answer these questions. In Section 2 we prove that (1) holds whenever $0 < p < r$ and $n$ is large. Next, in Section 3, we describe the asymptotic structure of $K_{r+1}$-free graphs $G$ of order $n$ such that $f(p,G) = \phi(r,p,n)$. We deduce that, if $p \geq r + \sqrt{2r}$ and $n$ is large, then

$$\phi(r,p,n) > (1 + \varepsilon)f(p,T_r(n))$$

for some $\varepsilon = \varepsilon(r) > 0$. This disproves Conjecture 6.2 in [3]. In particular,

$$\frac{r}{pe} \geq \frac{\phi(r,p,n)}{n^{r+1}} \geq \frac{r - 1}{(p + 1)e}$$

holds for large $n$, and therefore, for any fixed $r \geq 2$,

$$\lim_{n \to \infty} \frac{\phi(r,p,n)}{f(p,T_r(n))}$$

grows exponentially in $p$.

The case $r = 2$ is considered in detail in Section 4 we show that, if $r = 2$, equality (1) holds for $0 < p \leq 3$, and is false for every $p > 3$ and $n$ large.

In Section 5 we extend the above setup. For a fixed $(r + 1)$-chromatic graph $H$, $(r \geq 2)$, let $\phi(H,p,n)$ be the maximum value of $f(p,G)$ taken over all $H$-free graphs $G$ of order $n$. It turns out that, for every $r$ and $p$,

$$\phi(H,p,n) = \phi(r,p,n) + o(n^{p+1}).$$

(2)

This result completely settles, with the proper changes, Conjecture 6.1 of [3]. In fact, Pikhurko [5] proved this for $p \geq 1$, although he incorrectly assumed that (1) holds for all sufficiently large $n$.

## 2 The function $\phi(r,p,n)$ for $p < r$

In this section we shall prove the following theorem.

**Theorem 1** For every $r \geq 2$, $0 < p < r$, and sufficiently large $n$,

$$\phi(r,p,n) = f(p,T_r(n)).$$

**Proof** Erdős [2] proved that, for every $K_{r+1}$-free graph $G$, there exists an $r$-partite graph $H$ with $V(H) = V(G)$ such that $d_G(u) \leq d_H(u)$ for every $u \in V(G)$. As Caro and Yuster noticed, this implies that, for $K_{r+1}$-free graphs $G$ of order $n$, if $f(p,G)$ attains a maximum then $G$ is a complete $r$-partite graph. Every complete $r$-partite graph is defined uniquely by the size of its vertex
classes, that is, by a vector \((n_i)_{i=1}^r\) of positive integers satisfying \(n_1 + \ldots + n_r = n\); note that the Turán graph \(T_r(n)\) is uniquely characterized by the condition \(|n_i - n_j| \leq 1\) for every \(i, j \in [r]\). Thus we have

\[
\phi(r, p, n) = \max \left\{ \sum_{i=1}^{r} n_i (n - n_i)^p : n_1 + \ldots + n_r = n, \ 1 \leq n_1 \leq \ldots \leq n_r \right\}.
\]

(3)

Let \((n_i)_{i=1}^r\) be a vector on which the value of \(\phi(r, p, n)\) is attained. Routine calculations show that the function \(x (n - x)^p\) increases for \(0 \leq x \leq \frac{n - 1}{p + 1}\), decreases for \(\frac{n}{p + 1} \leq x \leq n\), and is concave for \(\frac{2n}{p + 1} \leq x \leq n\). If \(n_r \leq \frac{2n}{p + 1}\), the concavity of \(x (n - x)^p\) implies that \(n_1 - n_r \leq 1\), and the proof is completed, so we shall assume \(n_r > \frac{2n}{p + 1}\). Hence we deduce

\[
n_1 (r - 1) + \left\lfloor \frac{2n}{p + 1} \right\rfloor < n_1 + \ldots + n_r = n.
\]

(4)

We shall also assume

\[
n_1 \geq \left\lfloor \frac{n}{p + 1} \right\rfloor,
\]

(5)

since otherwise, adding 1 to \(n_r\) and subtracting 1 from \(n_1\), the value \(\sum_{i=1}^{r} n_i (n - n_i)^p\) will increase, contradicting the choice of \((n_i)_{i=1}^r\). Notice that, as \(n_1 \leq n/r\), inequality (5) is enough to prove the assertion for \(p \leq r - 1\) and every \(n\). From (4) and (5), we obtain that

\[
(r - 1) \left\lfloor \frac{n}{p + 1} \right\rfloor + \left\lfloor \frac{2n}{p + 1} \right\rfloor < n.
\]

Letting \(n \to \infty\), we see that \(p \geq r\), contradicting the assumption and completing the proof. \(\square\)

Maximizing independently each summand in (3), we see that, for every \(r \geq 2\) and \(p > 0\),

\[
\phi(r, p, n) \leq \frac{r}{p + 1} \left(\frac{p}{p + 1}\right)^p n^{p+1}.
\]

(6)

3 The asymptotics of \(\phi(r, p, n)\)

In this section we find the asymptotic structure of \(K_{r+1}\)-free graphs \(G\) of order \(n\) satisfying \(f(p, G) = \phi(r, p, n)\), and deduce asymptotic bounds on \(\phi(r, p, n)\).

**Theorem 2** For all \(r \geq 2\) and \(p > 0\), there exists \(c = c(p, r)\) such that the following assertion holds.

If \(f(p, G) = \phi(r, p, n)\) for some \(K_{r+1}\)-free graph \(G\) of order \(n\), then \(G\) is a complete \(r\)-partite graph having \(r - 1\) vertex classes of size \(cn + o(n)\).
Proof We already know that $G$ is a complete $r$-partite graph; let $n_1 \leq \ldots \leq n_r$ be the sizes of its vertex classes and, for every $i \in [r]$, set $y_i = n_i/n$. It is easy to see that
\[
\phi ( r, p, n ) = \psi ( r, p ) n^{p+1} + o ( n^{p+1} ),
\]
where the function $\psi ( r, p )$ is defined as
\[
\psi ( r, p ) = \max \left\{ \sum_{i=1}^{r} x_i (1-x_i)^p : x_1 + \ldots + x_r = 1, \ 0 \leq x_1 \leq \ldots \leq x_r \right\}.
\]

We shall show that if the above maximum is attained at $(x_i)_1^r$, then $x_1 = \ldots = x_{r-1}$. Indeed, the function $x (1-x)^p$ is concave for $0 \leq x \leq 2/(p+1)$, and convex for $2/(p+1) \leq x \leq 1$. Hence, there is at most one $x_i$ in the interval $(2/(p+1) \leq x \leq 1)$, which can only be $x_r$. Thus $x_1, \ldots, x_{r-1}$ are all in the interval $[0, 2/(p+1)]$, and so, by the concavity of $x (1-x)^p$, they are equal. We conclude that, if
\[
0 \leq x_1 \leq \ldots \leq x_r, \ x_1 + \ldots + x_r = 1,
\]
and $x_j > x_i$ for some $1 \leq i < j \leq r-1$, then $\sum_{i=1}^{r} x_i (1-x_i)^p$ is below its maximum value. Applying this conclusion to the numbers $(y_i)_1^r$, we deduce the assertion of the theorem. □

Set
\[
g ( r, p, x ) = (r-1) x (1-x)^p + (1-(r-1)) (rx)^p.
\]
From the previous theorem it follows that
\[
\psi ( r, p ) = \max_{0 \leq x \leq 1/(r-1)} g ( r, p, x ).
\]
Finding $\psi ( r, p )$ is not easy when $p > r$. In fact, for some $p > r$, there exist $0 < x < y < 1$ such that
\[
\psi ( r, p ) = g ( r, p, x ) = g ( r, p, y ).
\]

In view of the original claim concerning (1), it is somewhat surprising, that for $p > 2r - 1$, the point $x = 1/r$, corresponding to the Turán graph, not only fails to be a maximum of $g ( r, p, x )$, but, in fact, is a local minimum.

Observe that
\[
f ( p, T_r ( n ) ) = \left( \frac{r-1}{r} \right)^p + o (1),
\]
so, to find for which $p$ the function $\phi ( r, p, n )$ is significantly greater than $f ( p, T_r ( n ) )$, we shall compare $\psi ( r, p )$ to $\left( \frac{r-1}{r} \right)^p$.

**Theorem 3** Let $r \geq 2, p \geq r + \lceil \sqrt{2r} \rceil$. Then
\[
\psi ( r, p ) > (1+\varepsilon) \left( \frac{r-1}{r} \right)^p
\]
for some $\varepsilon = \varepsilon ( r ) > 0$. 
Proof We have
\[ \psi (r, p) \geq g \left( r, p, \frac{1}{p} \right) = \frac{r - 1}{p} \left( \frac{p - 1}{p} \right)^p + \left( 1 - \frac{r - 1}{p} \right) \left( \frac{r - 1}{p} \right)^p \]
\[ > \frac{r - 1}{p} \left( \frac{p - 1}{p} \right)^p. \]

To prove the theorem, it suffices to show that
\[ \frac{r - 1}{p} \left( \frac{p - 1}{p} \right)^p > 1 + \varepsilon \]
for some \( \varepsilon = \varepsilon (r) > 0 \). Routine calculations show that
\[ \frac{r - 1}{p} \left( 1 + \frac{p - r}{p (r - 1)} \right)^p \]
increases with \( p \). Thus, setting \( q = \left\lceil \sqrt{2r} \right\rceil \), we find that
\[ \frac{r - 1}{p} \left( 1 + \frac{p - r}{p (r - 1)} \right)^p \]
geq \[ \frac{r - 1}{r + q} \left( 1 + \frac{r + q}{1} \right) \frac{q}{(r + q) (r - 1)} + \left( \frac{r + q}{2} \right) \frac{q^2}{(r + q)^2 (r - 1)^2} \]
\[ = \frac{r - 1}{r + q} \frac{q}{r + q} + \frac{q^2 (r + q - 1)}{2 (r + q)^2 (r - 1)} \geq 1 - \frac{1}{r + q} + \frac{r (r + q - 1)}{(r + q)^2 (r - 1)} \]
\[ = 1 + \frac{r (r + q - 1) - (r + q) (r - 1)}{(r + q)^2 (r - 1)} = 1 + \frac{q}{(r + q)^2 (r - 1)}. \]

Hence, (7) holds with
\[ \varepsilon = \frac{\left\lceil \sqrt{2r} \right\rceil}{(r + \left\lceil \sqrt{2r} \right\rceil)^2 (r - 1)}, \]
completing the proof. \( \square \)

We have, for \( n \) sufficiently large,
\[ \frac{\phi (r, p, n)}{n^{p+1}} = \psi (r, p) + o(1) \geq g \left( r, p, \frac{1}{p} \right) + o(1) \]
\[ = \frac{r - 1}{p + 1} \left( \frac{p}{p + 1} \right)^p + \left( 1 - \frac{r - 1}{p + 1} \right) \left( \frac{r - 1}{p + 1} \right)^p + o(1) \]
\[ > \frac{r - 1}{p + 1} \left( \frac{p}{p + 1} \right)^p. \]

Hence, in view of (6), we find that, for \( n \) large,
\[ \frac{r}{p e} \geq \frac{r}{p} \left( \frac{p}{p + 1} \right)^{p+1} \geq \frac{\phi (r, p, n)}{n^{p+1}} \geq \frac{r - 1}{p + 1} \left( \frac{p}{p + 1} \right)^p \geq \frac{(r - 1)}{(p + 1) e}. \]
In particular, we deduce that, for any fixed \( r \geq 2 \),
\[
\lim_{n \to \infty} \frac{\phi(r, p, n)}{f(p, T_r(n))}
\]
grows exponentially in \( p \).

4 Triangle-free graphs

For triangle-free graphs, i.e., \( r = 2 \), we are able to pinpoint the value of \( p \) for which (1) fails, as stated in the following theorem.

**Theorem 4** If \( 0 < p \leq 3 \) then
\[
\phi(3, p, n) = f(p, T_2(n)). \tag{8}
\]

For every \( \varepsilon > 0 \), there exists \( \delta \) such that if \( p > 3 + \delta \) then
\[
\phi(3, p, n) > (1 + \varepsilon) f(p, T_2(n)) \tag{9}
\]
for \( n \) sufficiently large.

**Proof** We start by proving (8). From the proof of Theorem 1 we know that
\[
\phi(p, n, 3) = \max_{k \in \lceil n/2 \rceil} \left\{ k(n - k)^p + (n - k)k^p \right\}.
\]

Our goal is to prove that the above maximum is attained at \( k = \lceil n/2 \rceil \).

If \( 0 < p \leq 2 \), the function \( x(1 - x)^p \) is concave, and (8) follows immediately.

Next, assume that \( 2 < p \leq 3 \); we claim that the function
\[
g(x) = (1 + x)(1 - x)^p + (1 - x)(1 + x)^p
\]
is concave for \( |x| \leq 1 \). Indeed, we have
\[
g(x) = (1 - x^2) \left( (1 - x)^{p-1} + (1 + x)^{p-1} \right) = 2 \left( 1 - x^2 \right) \sum_{n=0}^{\infty} \left( \frac{p-1}{2n} \right) x^{2n}
\]
\[
= 2 + 2 \sum_{n=1}^{\infty} \left( \frac{p-1}{2n} - \frac{p-1}{2n-2} \right) x^{2n}
\]
\[
= 2 + 2 \sum_{n=1}^{\infty} \left( \frac{p-1}{2n-2} \right) \left( \frac{p-2n-1}{2n-1} \frac{p-2n-2}{2n} - 1 \right) x^{2n}.
\]

Since, for every \( n \), the coefficient of \( x^{2n} \) is nonpositive, the function \( g(x) \) is concave, as claimed.

Therefore, the function \( h(x) = x(n - x)^p + (n - x)x^p \) is concave for \( 1 \leq x \leq n \). Hence, for every integer \( k \in [n] \), we have
\[
h\left( \left\lfloor \frac{n}{2} \right\rfloor \right) + h\left( \left\lceil \frac{n}{2} \right\rceil \right) \geq h(k) + h(n - k) = 2h(k)
\]
\[
= 2 \left( k(n - k)^p + (n - k)k^p \right),
\]
proving (8).

Inequality (9) follows easily, since, in fact, for every $p > 3$, the function $g(x)$ has a local minimum at $0$. □

5 $H$-free graphs

In this section we are going to prove the following theorem.

Theorem 5 For every $r \geq 2$, and $p > 0$,

$$\phi(H, p, n) = \phi(r, p, n) + o(n^{p+1}).$$

A few words about this theorem seem in place. As already noted, Pikhurko \[5\] proved the assertion for $p \geq 1$; although he incorrectly assumed that (1) holds for all $p$ and sufficiently large $n$, his proof is valid, since it is independent of the exact value of $\phi(r, p, n)$. Our proof is close to Pikhurko’s, and is given only for the sake of completeness.

We shall need the following theorem (for a proof see, e.g., \[1\], Theorem 33, p. 132).

Theorem 6 Suppose $H$ is an $(r+1)$-chromatic graph. Every $H$-free graph $G$ of sufficiently large order $n$ can be made $K_{r+1}$-free by removing $o(n^2)$ edges.

Proof of Theorem 5 Select a $K_{r+1}$-free graph $G$ of order $n$ such that $f(p, G) = \phi(H, p, n)$. Since $G$ is $r$-partite, it is $H$-free, so we have $\phi(H, p, n) \geq \phi(r, p, n)$. Let now $G$ be a $H$-free graph of order $n$ such that

$$f(p, G) = \phi(H, p, n).$$

Theorem 6 implies that there exists a $K_{r+1}$-free graph $F$ that may be obtained from $G$ by removing at most $o(n^2)$ edges. Obviously, we have

$$e(G) = e(F) + o(n^2) \leq \frac{r-1}{2r}n^2 + o(n^2).$$

For $0 < p \leq 1$, by Jensen’s inequality, we have

$$\left(\frac{1}{n}f(p, G)\right)^{1/p} \leq \frac{1}{n}f(1, G) = \frac{1}{n}2e(G) \leq \frac{r-1}{r}n + o(n).$$

Hence, we find that

$$f(p, G) \leq \left(\frac{r-1}{r}\right)^p n^{p+1} + o(n^{p+1}) = \phi(r, p, n) + o(n^{p+1}),$$

completing the proof.
Next, assume that \( p > 1 \). Since the function \( xn^{p-1} - x^p \) is decreasing for \( 0 \leq x \leq n \), we find that
\[
d_G^p(u) - d_F^p(u) \leq (d_G(u) - d_F(u)) n^{p-1}
\]
for every \( u \in V(G) \). Summing this inequality for all \( u \in V(G) \), we obtain
\[
f(p, G) \leq f(p, F) + (d_G(u) - d_F(u)) n^{p-1} = f(p, F) + o(n^{p+1})
\]
\[
\leq \phi(r, p, n) + o(n^{p+1}),
\]
completing the proof.

\[\square\]

6 Concluding remarks

It seems interesting to find, for each \( r \geq 3 \), the minimum \( p \) for which the equality is essentially false for \( n \) large. Computer calculations show that this value is roughly 4.9 for \( r = 3 \), and 6.2 for \( r = 4 \), suggesting that the answer might not be easy.

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