Higher-order superintegrable systems separating in polar coordinates

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Abstract. Classical and Quantum mechanical superintegrable systems that are separating in polar coordinates are analyzed. The motion is restricted to a Euclidean plane $E_2$ and the additional integral is assumed to be a polynomial of degree $N \geq 3$ in momenta. Cases $N = 3, 4$ and 5 are investigated in detail. This leads to a general and unified description of higher-order superintegrability in the case of potentials allowing separation in polar coordinates.

1. Introduction
In classical mechanics, if a Hamiltonian system with $n$ degrees of freedom allows $n$ functionally independent integrals of motion $\{H, X_1, \ldots, X_{n-1}\}$ in involution, then it is called integrable. On the other hand existence of more integrals $\{Y_1, \ldots, Y_k\}$ makes the system superintegrable. Of course, the total set of integrals (at most $2n - 1$) must be functionally independent, however, the additional ones are not necessarily in involution among themselves, nor with the already existing $n$ integrals (except $H$). All these concepts are also introduced in quantum mechanics through well-defined linear operators which are supposed to be algebraically independent [2, 12, 13, 14].

A systematic search for the properties of superintegrable systems was started quite some time ago [13, 14]. Originally the approach concentrated on the natural Hamiltonians of type $H = -\frac{1}{2}\Delta + V(r)$, with integrals of motion that are second-order polynomials in the momenta and directly related with the multiseparability in 2- and 3-dimensional Euclidean spaces [13]. This relationship between integrability and separability breaks down in other cases. For example, for natural Hamiltonians, the existence of third-order integrals does not lead to the separation of variables [7, 15]. Furthermore, if we consider velocity dependent potentials $H = -\frac{1}{2}\Delta + V(r) + (A, p)$, then quadratic integrability no longer implies the separation of variables [3, 8]. Recently extended reviews have been published describing the current status of the subject [18, 21].

After the publication of the seminal paper “An infinite family of solvable and integrable quantum systems on a plane” [25], the direction of the research has been shifted to higher-order integrability/superintegrability [19, 22, 23, 24, 26, 27]. In these works it has been shown that two types of potentials, namely the Tremblay-Turbiner-Winternitz (TTW) and Post-Winternitz (PW)

$$V_{\text{TTW}} = br^2 + \frac{1}{r^2} \left[ \frac{\alpha}{\cos^2(k\theta)} + \frac{\beta}{\sin^2(k\theta)} \right], \quad V_{\text{PW}} = \frac{a}{r} + \frac{1}{r^2} \left[ \frac{\mu}{\cos^2(\frac{2}{3}\theta)} + \frac{\nu}{\sin^2(\frac{2}{3}\theta)} \right] , \quad (1)$$
exist and admit polynomial integral of arbitrary order $N$. Here $k = m/n$ is rational, where $m$ and $n$ are two integers with no common divisors.

More recently, exotic and standard potentials appearing in classical and quantum superintegrable systems has been studied for $N = 4$ both in Cartesian [20] and polar [9, 11] coordinates. Also, doubly exotic potentials for $N = 5$ in Cartesian coordinates has been studied in [1].

In this article we concentrate on systems that are separable in polar coordinates and posses an additional polynomial integral of higher order $N > 2$ in momenta. The systems are second-order integrable because in addition to the Hamiltonian

$$ H = -\frac{\hbar^2}{2} \left( \partial_\ell^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) + V(r, \theta), \quad V(r, \theta) = R(r) + \frac{S(\theta)}{r^2}, \quad (2) $$

they allow a second-order integral $X = L_z^2 + 2 S(\theta)$, with $L_z = x p_y - y p_x$ and $x = r \cos \theta, y = r \sin \theta$. The existence of the higher-order integral $Y$ makes the system superintegrable.

2. Higher-order integral of motion: superintegrability

In the classical case, a higher-order polynomial integral has the form [16, 23]

$$ Y = Y^{(N)} + \frac{N-2\ell}{2} \sum_{\ell=1}^{N-2} \sum_{j=0}^{N-j-2\ell} F_{j,2\ell} p_x^j p_y^{N-j-2\ell}, \quad Y^{(N)} = \sum_{0 \leq m+n \leq N} A_{N-m-n,m,n} L_z^{N-m-n} p_x^m p_y^n, \quad (3) $$

where the contributions of the highest-order terms are collected in a single term $Y^{(N)}$. Here the functions $F_{j,2\ell} = F_{j,2\ell}(x, y)$ depend on the potential $V$ figuring in the Hamiltonian and $A_{N-m-n,m,n}$ are $(N+1)(N+2)$ constants. The leading term $Y^{(N)}$ is fundamental since it defines the existence of the highest-order integral.

Introducing $P = \sqrt{p_x^2 + p_y^2}$ and $\tan \Phi = p_y/p_x$, where the basis functions generating the irreducible representations of $SO(2)$ are given by $(p_x \pm ip_y)^s = P^s (\cos s \Phi \pm i \sin s \Phi), \quad (s = 0, 1, 2 \ldots)$, $Y^{(N)}$ can be more conveniently written as

$$ Y^{(N)} = \sum_{0 \leq s+2k \leq N} L_z^{N-s-2k} P^{s+2k} \left[ B_{N-s-2k,s,k}^{(1)} \cos s \Phi + B_{N-s-2k,s,k}^{(2)} \sin s \Phi \right], \quad (4) $$

where $B_{N-s-2k,s,k}^{(\ell)} \quad (l = 1, 2)$ are $(N+1)(N+2)/2$ constants. Each of these constants $B_{N-s-2k,s,k}^{(\ell)}$ can be expressed uniquely as a linear combination of the original parameters $A_{N-m-n,m,n}$. Of course, to obtain the corresponding expression in polar coordinates we put $p_x = \cos \theta p_r - \frac{\sin \theta}{r} L_z$, $p_y = \sin \theta p_r + \frac{\cos \theta}{r} L_z$.

Similarly, in the quantum case [23] the Hermitian highest-order operator $Y^{(N)}$ is expressed as

$$ Y^{(N)} = \sum_{0 \leq s+2k \leq N} \left\{ L_z^{N-s-2k}, (p_x^2 + p_y^2)^k \left[ B_{N-s-2k,s,k}^{(1)} [c]_{Re} + B_{N-s-2k,s,k}^{(2)} [c]_{Im} \right] \right\}, \quad (5) $$

where $[c]_{Re}$ refers to the real part of $c = (p_x + ip_y)^s$ in which we formally treat the operators $p_x$ and $p_y$ as variables. The $B_{N-s-2k,s,k}^{(\ell)} \quad (l = 1, 2)$ are constants, and $\{, \}$ denotes an anticommutator.
Vanishing of the Poisson or Lie bracket \([H, Y] = 0\) give the whole set of determining equations and it has been shown \([16, 23]\) that the next to highest-order of them can be expressed as the following PDE for existence of the potential \(V\)

\[
\sum_{j=0}^{N-1} (-1)^j \partial_x^{N-1-j} \partial_y^j [(j + 1) f_{j+1,0} \partial_x V + (N - j) f_{j,0} \partial_y V] = 0 ,
\]

and called a linear compatibility condition (LCC). It is a necessary (but not sufficient) condition for the existence of the integral \(Y\). The functions \(f_{j,0}\) do not depend on the potential \(f_{j,0} = \sum_{N-m} \sum_{m=0} (N-m-n) A_{N-m-n,n,n} x^{N-j-n} (-y)^{i-m}\), and are completely determined by the coefficients \(A_{N-m-n,n,n}\). The LCC does not contain \(\hbar\) and thus it is the same for both classical and quantum systems.

For \(R(r) = 0\) some of the constants involved in \(Y^{(N)}\) do not appear in the LCC and depending on this fact we separate the \(Y^{(N)}\) into two parts; \(Y^{(N)} = Y^{(N)}_f + Y^{(N)}_l\)

\[
Y^{(N)}_f = \sum_{N-1 \leq s \leq N} \sum_{N-1 \leq s \leq N} L_z^{N-s-2k} p_s + 2k \left[ B^{(1)}_{N-s-2k,s,k} \cos s \Phi + B^{(2)}_{N-s-2k,s,k} \sin s \Phi \right] ,
\]

\[
Y^{(N)}_l = \sum_{0 \leq s \leq N-2} L_z^{N-s-2k} p_s + 2k \left[ B^{(1)}_{N-s-2k,s,k} \cos s \Phi + B^{(2)}_{N-s-2k,s,k} \sin s \Phi \right] .
\]

Of course, for the quantum case the analogs of (7) and (8) can easily be written. Thus, for both classical and quantum systems we distinguish two cases for \(R(r) = 0\):

- \(Y^{(N)}_f \neq 0\): this case corresponds to the standard potentials for which the angular component \(S(\theta)\) satisfies the LCC (6).
- \(Y^{(N)}_l = 0\): this situation corresponds to the exotic potentials. All coefficients in (6) satisfy \(f_{a,0} = 0\) and the function \(S(\theta)\) is not constrained by this linear equation.

This splitting of \(Y^{(N)}\) considerably simplifies the analysis of the determining equations since we can choose many of the constants as zero on our demand. Moreover, bearing in mind that for the specific cases \(N = 3, 4\) and \(5\) it has already been explicitly shown that \(R(r)\) can only be \(0, \frac{2}{r}, \text{ and } br^2\) \([9, 10, 11, 27]\) with \(a\) and \(b\) constants, it only remains to solve the compatibility conditions for \(S(\theta)\) in order to have a polynomial integral of order \(N > 2\). For convenience we define \(S(\theta) \equiv T^s(\theta)\) and analyze each of the specific cases in detail.

### 3. Cases: \(N = 3, 4\) and \(5\)

#### 3.1. \(N = 3\)

\(Y^{(3)}\) is composed of the following terms:

\[
A_1 (p_x^2 - 3p_x p_y^2) , \quad A_3 p_x (p_x^2 + p_y^2) , \quad B_1 \{ L_z, (p_x^2 - p_y^2) \} , \quad C_1 \{ L_z, p_x \} , \quad D_0 L_z^3 ,
\]

\[
A_2 (3p_y^2 p_y - p_y^2) , \quad A_4 p_y (p_x^2 + p_y^2) , \quad B_2 \{ L_z, 2p_x p_y \} , \quad C_2 \{ L_z, p_y \} , \quad B_0 L_z (p_x^2 + p_y^2) .
\]

In literature there appear many ways to expand the 3rd-order integral of motion and therefore it might be helpful to give the correspondence between these different notations

\[
B^{(1)}_{3,0,0} = B^{(2)}_{3,0,0} \equiv D_0 \equiv A_{3,0,0} , \quad (s = 0, k = 0)
\]

\[
B^{(1)}_{2,1,0} \equiv C_1 \equiv A_{2,1,0} , \quad B^{(2)}_{2,1,0} \equiv C_2 \equiv A_{2,0,1} , \quad (s = 1, k = 0)
\]
\[ B^{(1)}_{1,0,1} = B^{(2)}_{1,0,1} = B_0 \equiv \frac{A_{1,2,0} + A_{1,0,2}}{2}, \]  
\[ B^{(1)}_{0,1,1} = B^{(2)}_{0,1,1} \equiv A_4 = \frac{3A_{0,3,0} + A_{0,1,2}}{4}, \]  
\[ B^{(1)}_{1,2,0} = B^{(2)}_{1,2,0} \equiv B_2 = \frac{A_{1,2,0} - A_{1,0,2}}{2}, \]  
\[ B^{(1)}_{1,2,0} = B^{(2)}_{1,2,0} \equiv B_2 = \frac{A_{1,2,0} - A_{1,0,2}}{2}, \]  
\[ B^{(1)}_{0,3,0} = A_1 \equiv \frac{A_{0,3,0} - A_{0,1,2}}{4}, \]  
\[ B^{(2)}_{0,3,0} = A_2 \equiv \frac{A_{0,2,1} + A_{0,0,2}}{4}. \] 

In general, \([H,Y^2]\) is a 4\(^{th}\)-order operator, however, here the relevant information is coming from 2\(^{nd}\)- and 0\(^{th}\)-order terms. \(i.e.\) there exist 4 determining equations. This is consistent with the fact that for odd \(N\) there exist \(\frac{N(N+1)}{2}\) determining equations. We have 10 arbitrary constants \((\frac{N(N+1)(N+2)}{2})\) and 2 arbitrary functions. Depending on the fact that which constants appear in the LCC we obtain and classify various potentials. It is important to note that most of the doublets do not mix in the determining equations. We find the following standard potentials

\[ T_1(\theta) = \frac{s_1 \sin(\theta) + s_2 \cos(\theta) + s_3 \sin(3\theta) + s_4 \cos(3\theta)}{A_4 \cos(\theta) - A_3 \sin(\theta) + 3(A_2 \cos(3\theta) - A_1 \sin(3\theta))}, \]  
\[ T_2(\theta) = \frac{s_1 + s_2 \cos(2\theta) + s_3 \sin(2\theta)}{B_1 \cos(2\theta) + B_2 \sin(2\theta)} + s_4, \]  
where \(s_i\) are integration constants to be determined from the last nonlinear determining equation. In many cases the TTW potentials are recovered but for some of the solutions we obtain pure quantum potentials \([17]\) (proportional to \(\hbar^2\)) which cannot be reduced or transformed to TTW potentials.

For the exotic singlet \(D_0\) we obtain a potential expressible in terms of Weierstrass elliptic function, however, there is the following syzygy between the integrals of motion \([27]\]

\[
\left(\frac{Y}{2}\right)^2 = 8 \left(\frac{X}{2}\right)^3 - c_1 \frac{\hbar^4}{4} X + c_2 \frac{\hbar^6}{4},
\]  
and hence the obtained potential is not a new one.

For the exotic doublet \((C_1, C_2)\) we have a 4\(^{th}\)-order nonlinear equation. In this equation, after making the change of variables \(z = \tan(\theta)\) and the transformation \((z, T(z)) \rightarrow (x, W(x))\), where \(z = \frac{4\sqrt{\gamma_1 \gamma_4}}{1 - \gamma_1 \gamma_4}\), we arrive at the derivative of the first canonical subspace of the Cosgrove’s master Painlevé equation \([5, 6]\)

\[ T(x) = \hbar^2 \left[ \frac{W(x)}{\sqrt{\gamma_1 \gamma_4}} \right] + \gamma \left[ \frac{1 - 2x}{4 \sqrt{\gamma_1 \gamma_4}} \right] x \equiv \left\{ \cos^2 \left[\frac{\theta}{2}\right] \right\} \sin^2 \left[\frac{\theta}{2}\right], \]  
with \(\gamma = (\gamma_2 + \gamma_4) - (\gamma_1 + \gamma_3) + \sqrt{\gamma_2 \gamma_4 - \frac{1}{4}}\) and \((\gamma_2 + \gamma_3)(\gamma_1 + \gamma_4 - \sqrt{\gamma_2 \gamma_4}) = 0\).

In the above potential, \(W(x)\) is expressed in terms of Painlevé transcendent \(P_6\) as

\[ W(x; \gamma_1, \gamma_2, \gamma_3, \gamma_4) = \frac{x^2(x-1)^2}{4\gamma_6(\gamma_6-1)(\gamma_6-x)} \left[ P_6' - \frac{P_6(\gamma_6-1)}{x(x-1)} \right]^2 \]

\[ + \frac{1}{8}(1 - 2) \right) (1 - 2P_6) - \frac{1}{4} \gamma_2 \left[ 1 - \frac{x}{\gamma_6} \right] \left[ 1 - \frac{2(x-1)}{\gamma_6} \right] + \left[ \frac{1}{8} - \frac{\gamma_4}{\gamma_6} \right], \]

with \(\gamma_1, \gamma_2, \gamma_3\) and \(\gamma_4\) are the parameters that define the \(P_6\) which satisfies the well known second order differential equation \([4]\):

\[ P''_6 = \frac{1}{2} \left[ \frac{1}{\gamma_6} + \frac{1}{\gamma_6 - 1} + \frac{1}{\gamma_6 - x} \right] (P'_6)^2 - \left[ \frac{1}{2} + \frac{1}{x-1} + \frac{1}{\gamma_6 - x} \right] P'_6 \]

\[ + \frac{P_6(\gamma_6-1)(\gamma_6-x)}{x^2(x-1)^2} \left[ \gamma_1 + \frac{\gamma_2 x}{\gamma_6} + \frac{\gamma_3 (x-1)}{(\gamma_6-1)^2} + \frac{\gamma_4 x(x-1)}{(\gamma_6-x)^2} \right], \]  

\[ (17) \]
3.2. $N = 4$

$Y^{(4)}$ is composed of the following terms:

$$\begin{align*}
A_1 \{ L_1^2, p_3 \} & , A_3 \{ L_2, p_3(p_3^2 + p_3^2) \} & , B_3 \{ L_2^2, (p_2^2 - p_2^2) \} & , B_4 \{ L_2^2, (p_2^2 - p_2^2) \} \\
A_2 \{ L_2^2, p_3 \} & , A_4 \{ L_2, p_3(p_3^2 + p_3^2) \} & , B_2 \{ 2p_3 p_2(p_2^2 + p_2^2) \} & , B_4 \{ L_2^2, 2p_3 p_2 \} \\
C_1 \{ L_2, (3p_2^2 p_2 - p_2^2) \} & , D_1 \{ p_2^2 + p_2^2 - 6p_2^2 p_2 \} & , L_4 & , L_5 \{ p_2^2 + p_2^2 \}, (p_2^2 + p_2^2)^2. \tag{18}
\end{align*}$$

In general $[H, Y^{(4)}]$ is a $5^{th}$-order operator, however, here the relevant information is coming from $3^{rd}$ and $1^{st}$-order terms. (i.e. there exist 6 determining equations). This is consistent with the fact that for even $N$ there exist $\frac{1}{4}N(N + 2)$ determining equations. We have 15 arbitrary constants and 4 arbitrary functions. Depending on the fact that which constants appear in the LCC we obtain and classify various potentials. As in the previous case most of the doublets do not mixed in the determining equations. We find the following standard potentials

$$T_1(\theta) = \frac{s_1 + s_2 \cos(2\theta) + s_3 \sin(2\theta) + s_4 \cos(4\theta) + s_5 \sin(4\theta)}{B_2 \cos(2\theta) - B_1 \sin(2\theta) + 2[D_2 \cos(4\theta) - D_1 \sin(4\theta)],}$$

$$T_2(\theta) = \frac{s_1 + s_2 \cos(\theta) + s_3 \sin(\theta) + s_4 \cos(3\theta) + s_5 \sin(3\theta)}{A_3 \cos(\theta) + A_4 \sin(\theta) + C_2 \cos(3\theta) + C_1 \sin(3\theta)},$$

where $s_i$ are integration constants to be determined from the last nonlinear determining equation. In many cases the TTW potentials are recovered but for some of the solutions we obtain pure quantum potentials [17] (proportional to $\hbar^2$) which cannot be reduced or transformed to TTW potentials.

As an example let us investigate $T_1(\theta)$ in detail. Introducing it into the nonlinear determining equation we obtain the following solutions

$$s_1^{(\ell)} = \frac{q_\ell \hbar^2}{4 D_1^2 (B_2^2 - 8D_1^2)^2 D_2^2} \left[ B_2 D_2 (8D_1^2 - B_2^2) (B_2^2 D_1 + D_2^2) + 8D_1^2 (4D_1^2 + 3D_2^2) \right]$$

$$+ (8B_2^2 D_1^2 - 6D_1^2 D_2^2 - B_2^2 D_1^2 + 3D_2^2 D_1^2) + 64 (2D_1^2 + D_2^2) D_1^2 q_\ell$$

$$B_2 D_2^2 D_2 (3B_2^2 + 40D_1^2) q_\ell^2 - D_1^2 (B_2^2 + 8D_1^2) q_\ell^3,$$

$$s_2^{(\ell)} = \frac{q_\ell \hbar^2}{D_1 D_2^2 (B_2^2 - 8D_1^2)^2} \left[ D_2 (8D_1^2 - B_2^2) (3B_2^2 D_1^2 + D_2^2) + 8D_1^2 (2D_1^2 + D_2^2) \right]$$

$$+ (2B_2^2 D_1^2 + 4D_2^2 D_1^2 - 32B_2 D_1^2) q_\ell + D_1^2 D_2 (7B_2^2 + 8D_1^2) q_\ell^2 + 2B_2 D_2^2 q_\ell^3,$$

$$s_3^{(\ell)} = \frac{q_\ell \hbar^2}{D_2 (B_2^2 - 8D_1^2)^2} \left[ 4B_2 D_4 (D_1^2 + 2D_2^2) (B_2^2 - 8D_1^2) \right]$$

$$+ 2D_1 (B_2^2 (D_1^2 + 6D_2^2) - 16D_1^2 (D_1^2 + D_2^2)) q_\ell + 8B_2 D_1^2 D_2^2 q_\ell^2 + 2D_1^3 q_\ell^3,$$

$$s_5 = 0,$$

where $q_\ell, \ell = 1, 2, 3, 4,$ are the four roots of the quartic equation

$$\hbar^8 [D_1^2 q_\ell^4 + 4B_2 D_2 D_1^2 q_\ell^3 + (B_2^2 (D_1^2 + 6D_2^2 D_2^2) - 16D_1^2 (D_1^2 + D_2^2)) q_\ell^2$$

$$- 2B_2 D_1 (8D_1^2 - B_2^2) D_2 (D_1^2 + 2D_2^2) q_\ell + (B_2^2 - 8D_1^2)^2 D_2^2 (D_1^2 + D_2^2) ] = 0,$$

whose discriminant is

$$\Gamma = -256 \hbar^8 D_1^4 D_2^4 (B_2^2 - 8D_1^2)^2 \left[ B_2^2 (60D_1^2 - 48D_1^2) + 768B_2^2 (D_1^2 + D_2^2)^2 + B_2^4 - 4096 (D_1^2 + D_2^2)^3 \right].$$
The above solutions are obtained for $\Gamma \neq 0$. Such discriminant is zero if and only if at least two roots are equal. If the discriminant is negative there are two real roots and two complex conjugate roots. If it is positive the roots are either all real or all non-real. From a physical point of view we consider only real solutions. In general, we obtain an angular component $S_1(\theta)$ proportional to $\hbar^2$ with no classical analog, it cannot be transformed or reduced to that of the TTW model.

In particular, the discriminant vanishes for $\hbar = 0$. The highest order terms in the nonlinear determining equation are proportional to $\hbar^2$, therefore the limit $\hbar \to 0$ is singular and the above solutions are no longer valid for $\hbar = 0$.

Now, let us analyze the zeros of the discriminant

**Case I:** For $D_1 = 0$, non-trivial solutions exist only for $B_2 = 0$. The corresponding coefficients take the values $s_1 = s_2 = 0$, $s_3 = 0$, $s_4 = s_5$, which yields the potential

$$S_1(\theta) = \frac{4(D_1 \cos 4\theta + D_2 \sin 4\theta)s_1 + 4(D_3 s_5 + D_2 s_4)}{(D_1^2 - D_2^2) \cos 8\theta + 2 D_1 D_2 \sin 8\theta - (D_1^2 + D_2^2)}.$$  \hspace{1cm} (23)

It is easy to show that if we make the following transformations $\theta \to \theta + \frac{1}{2} \arctan(-D_2/D_1)$, $\alpha = -\frac{\sqrt{D_1^2 + D_2^2} + D_1 s_4 + D_2 s_5}{s(D_1^2 + D_2^2)}$, $\beta = -\frac{D_1^2 s_1 + \sqrt{D_1^2 + D_2^2} D_1 s_4 + D_2 (D_2 s_1 + \sqrt{D_1^2 + D_2^2} s_5)}{s(D_1^2 + D_2^2)}$, $k = 2$ in the angular component of the well-known TTW potential $S_{TTW}(\theta) = \frac{4k^2 (\alpha - \beta) \cos 2k \theta - 4k^2 (\alpha + \beta)}{\cos 4k \theta - 1}$, we recover $S_1(\theta)$. Therefore, $S_1(\theta)$ corresponds to a rotated TTW model (with no radial component $R(r) = 0$) which is a superintegrable system both in the classical and quantum cases.

**Case II:** For $D_1 = 0$, the corresponding coefficients vanish, $s_1 = s_2 = s_3 = s_4 = s_5 = 0$, which gives the trivial solution $S_1(\theta) = 0$.

**Case III:** For $D_2 = 0$, the corresponding coefficients are given by $s_1 = s_2 = 0$, $s_3 = B_2^2 s_4 + s D_1^2 (s_1 + s_5)$, $s_4 = s_5 = 0$, thus

$$S_1(\theta) = -\frac{2(B_2 s_4 + 2 D_1 s_1 \sin 2\theta + 2 D_1 s_4 \sin 2\theta)}{2 B_2 D_1 (1 + \cos 4\theta)}.$$  \hspace{1cm} (24)

This solution corresponds to the angular component of the TTW model with $k = 1$.

**Case IV:** For $B_2^2 - 8 D_1^2 = 0$, we put $D_1 = D_2 = 1$ thus $B_2 = \sqrt{8}$ for simplicity and find that the coefficients $s_1 = 0$, $s_2 = -2\sqrt{2} \hbar^2$, $s_3 = 2\sqrt{2} \hbar^2$, $s_4 = -4 \hbar^2$, $s_5 = 0$, lead to

$$S_1(\theta) = 2 \hbar^2 \frac{\sqrt{2} \cos 6\theta - \sqrt{2} \sin 6\theta + 2}{[\cos 4\theta - \sin 4\theta + \sqrt{2} \cos 2\theta]^2},$$  \hspace{1cm} (25)

which is a pure quantum potential. It cannot be reduced to that of the TTW model.

**Exotic Potentials**

For the exotic doublets $(A_1, A_2)$ and $(B_3, B_4)$ we have 2 separate $5^{th}$-order nonlinear equations. Each of which can be integrated once and after making the change of variables $z = \tan(\theta)$ and $z = \tan(2\theta)$ each can be integrated once more. Making one more transformation $(z, T(z)) \rightarrow (x, W(x))$ where $z = \frac{2 \sqrt{\sqrt{T^2 - 1}}}{1 - 2 z}$, we arrive the derivative of the first canonical subcase of the Cosgrove’s master Painlevé equation [5, 6].

We have the following exotic potentials

$$T_1(x) = \hbar^2 \left[ \frac{W(x)}{\sqrt{x \sqrt{1 - x}}} + \gamma \frac{(1 - 2x)}{4 \sqrt{x \sqrt{1 - x}}} \right] x \equiv \begin{cases} \cos^2 \theta \bigg[ \frac{2}{3} \bigg] \\ \sin^2 \theta \bigg[ \frac{2}{3} \bigg], \end{cases}$$

$$T_2(x) = \hbar^2 2 \left[ \frac{W(x)}{\sqrt{x \sqrt{1 - x}}} + \gamma \frac{(1 - 2x)}{4 \sqrt{x \sqrt{1 - x}}} \right] x \equiv \begin{cases} \cos^2 \theta \bigg[ \frac{2}{3} \bigg] \\ \sin^2 \theta \bigg[ \frac{2}{3} \bigg], \end{cases}$$

with $\gamma = (\gamma_2 + \gamma_4) - (\gamma_1 + \gamma_3) + \sqrt{2} \gamma_1 - \frac{2}{3}$.
3.3. $N = 5$

$Y^{(5)}$ is composed of the following terms:

\begin{align*}
A_1 \{ L^2_1, p_4 \} & \, B_1 \{ L^2_1, (p^2_x - p^2_y) \} \quad C_1 \{ L^2_3, (p^2_x - 3p_y^2, p^2_y) \} \quad F_1 \{ L^2_5, p_y (p^2_x + p^2_y) \} \quad A_0 L^2_5, \\
A_2 \{ L^2_1, p_y \} & \, B_2 \{ L^2_1, 3p_x, p_y \} \quad C_2 \{ L^2_5, (3p^2_x p_y - p^2_y) \} \quad F_2 \{ L^2_5, p_y (p^2_x + p^2_y) \} \quad M_1 \{ L^2_3, (p^2_x + p^2_y) \},
\end{align*}

\begin{align*}
K_1 p_x (p^2_x + p^2_y)^2 & \quad G_1 \{ L_2, (p^2_x - p^2_y)(p^2_x + p^2_y) \} \quad H_1 (p^2_x - 3p_y^2)(p^2_x + p^2_y) \quad D_1 \{ L_2, (p^2_x + p^4_y - 6p_x^2p_y^2) \} \\
K_2 p_y (p^2_x + p^2_y)^2 & \quad G_2 \{ L_2, 2p_x p_y (p^2_x + p^2_y) \} \quad H_2 (3p^2_x p_y - p^2_y)(p^2_x + p^2_y) \quad D_2 \{ L_2, 4p_x p_y (p^2_x - p^2_y) \},
\end{align*}

\begin{align*}
E_1 (p^2_x - 10p^2_y p_y + 5p_x^2 p_y^2) E_2 (5p^2_x p_y - 10p^2_y p_y + p^2_y) & \quad M_2 \{ L_2, (p^2_x + p^2_y)^2 \}. \tag{28}
\end{align*}

In general $[H, Y^{(5)}]$ is a 6th-order operator, however, the relevant information is coming from 4th-, 2nd-, and 0th-order terms (i.e. there exist 9 determining equations). We have 21 arbitrary constants and 6 arbitrary functions. Depending on the fact that which constants appear in the LCC we obtain and classify various potentials. Again most of the doublets do not mix in the determined equations. Although we could not give the full set of determining equations due to the lack of space we would like to show more details as opposed to the previous cases. It would be better to use the following notation:

\begin{align*}
A & = A_1 \cos(\theta) + A_2 \sin(\theta), \quad B = B_1 \cos(2\theta) + B_2 \sin(2\theta), \quad C = C_1 \cos(3\theta) + C_2 \sin(3\theta), \\
D & = D_1 \cos(4\theta) + D_2 \sin(4\theta), \quad E = E_1 \cos(5\theta) + E_2 \sin(5\theta), \quad F = F_1 \cos(\theta) + F_2 \sin(\theta), \\
G & = G_1 \cos(2\theta) + G_2 \sin(2\theta), \quad H = H_1 \cos(3\theta) + H_2 \sin(3\theta), \quad K = K_1 \cos(\theta) + K_2 \sin(\theta),
\end{align*}

and $A_0, B_0, C_0, \ldots$ for the derivatives of $A, B, C, \ldots$ with respect to $\theta$.

For the analysis of standard potentials we obtain two linear differential equations from the LCC:

\begin{align*}
96 (G_\theta - 4D_\theta) T''(\theta) + 4 \left( (44G - 496D) T''''(\theta) + 60D_\theta T^{(3)}(\theta) + 5(11D + 2G) T^{(4)}(\theta) \right) \\
-6(D_\theta + G_\theta) T^{(5)}(\theta) - (D + G) T^{(6)}(\theta) = 0, \tag{29}
\end{align*}

\begin{align*}
384(50E - 6H + K) T''(\theta) - 32(137E_\theta - 13H_\theta + 2K_\theta) T''''(\theta) \\
+20(6K - 3H - 75E) T^{(3)}(\theta) + (17E_\theta + 5H_\theta - K_\theta) T^{(4)}(\theta) \\
+6(25E + 9H + K) T^{(5)}(\theta) - (E_\theta + H_\theta + K_\theta) T^{(6)}(\theta) = 0, \tag{30}
\end{align*}

which have the following general solutions, respectively

\begin{align*}
T_1(\theta) & = \frac{s_1 + s_2 \cos(2\theta) + s_3 \sin(2\theta) + s_4 \cos(4\theta) + s_5 \sin(4\theta)}{G + D} + s_6, \tag{31} \\
T_2(\theta) & = \frac{s_1 \sin(\theta) + s_2 \cos(\theta) + s_3 \sin(3\theta) + s_4 \cos(3\theta) + s_5 \sin(5\theta) + s_6 \cos(5\theta)}{K_\theta + H_\theta + E_\theta}, \tag{32}
\end{align*}

where $s_1, \ldots, s_6$ are integration constants. These constants are specified after introducing the solutions $T_1(\theta)$ and $T_2(\theta)$ into the lower order nonlinear determining equations.

Now, let us concentrate on the singlets only and show that the potentials obtained for them are already the known ones. If we consider only the exotic singlet $M_1$ we need to solve the following two nonlinear differential equations

\begin{align*}
18T''''(\theta) + 6T''(4T'' + T^{(4)}) - \frac{h^2}{4} (4T^{(4)} + T^{(6)}) = 0, \tag{33} \\
384(T'')^2 T'' + 8 (h^4 T^{(6)} + h^2 (2h^2 - 15T^{(3)}) T^{(4)} - h^2 (36T^{(3)} + 7T^{(5)}) T'' - 6(T'')^3) \\
-4T'' (12(4h^2 + 4T^{(3)}) T'' + h^2 (32T^{(4)} + 3T^{(6)})) + h^4 T^{(8)} = 0. \tag{34}
\end{align*}

From these two nonlinear differential equations we can get many compatibility conditions. For example, taking a certain linear combination of (33), (34) and second $\theta$ derivative of (33), we obtain a fifth-order
Showed that the exotic singlet leads to a potential that we already know. It can easily be verified that the Weierstrass doublets do not mix in these equations. For example, we have two equations involving only the construction we obtain an exotic potential involving a Painlevé transcendent which is not new.

The leading part of the integral \( Y^C \) is the integral of fifth-order involving only the doublets \( (4) \), \( (2) \), and \( (1) \). The reason behind the fact that the singlets lead to only known potentials is; the integral involving these singlets are algebraically dependent to the lower-order and/or the trivial integrals \( (i.e. \) the leading order parts satisfy \( (Y^{(N)})^\delta = (Y^{(N-i)})^\alpha X^\beta H^\gamma \), where \( Y^{(N-i)} \) is the \( N^{th} \)-order integral involving singlets only, \( Y^{(N-i)} \) is the \( (N-i)^{th} \)-order integral involving singlets only and \( \delta, \alpha, \beta, \gamma \) are positive integers and the lower-order terms can always be absorbed into the arbitrary functions). Specifically, for \( N = 5 \), the leading order terms satisfy the following algebraic relations \( L_5 = L_3 X \), \( (L_2 (p_x^2 + p_y^2))^2 = X H^4 \), and \( (L_2 (p_x^2 + p_y^2))^2 = X^3 H^2 \), where \( A_0 \) is the coefficient of \( L_5 \), \( M_1 \) is the coefficient of \( \{L_1, (2p_x^2 + p_y^2)\} \) and \( M_2 \) is the coefficient of \( \{2L_2, (p_x^2 + p_y^2)\} \). Hence, we set \( A_0 = M_1 = M_2 = 0 \).

**Exotic Potentials**

We obtain three sets of nonlinear differential equations. It is important to note here that certain doublets do not mix in these equations. For example, we have two equations involving only the doublets \( (A_1, A_2) \), two equations involving only the doublets \( (B_1, B_2) \) and two equations mixing the doublets \( (C_1, C_2) \) and \( (F_1, F_2) \).

Concentrating on the doublets \( (A_1, A_2) \), which are the coefficients of \( \{L_2, p_x\} \), \( \{L_4, p_y\} \), we see that they can be expressed as the leading part of the integral \( Y^{(3)} X \), where \( Y^{(3)} \) is the integral of third-order. Remember that for such a third-order integral the potential \( V \) involves a Painlevé transcendent \( P_6 \) \([27]\). Hence, by construction we obtain an exotic potential involving a Painlevé transcendent to support our conjecture, however, it is not new.

Similarly, the doublets \( (B_1, B_2) \), which are the coefficients of \( \{(L_3, p_x (p_x^2 - p_y^2)), (L_2, 2p_x p_y)\} \), are the leading part of an expression involving the lower order integral and the trivial ones. More precisely, we have the following algebraic equation \( (Y^{(3)})^2 = X (Y^{(4)})^2 \) satisfied for the leading order terms and the lower order terms can be absorbed into the arbitrary functions. Here, \( Y^{(4)} \) is the integral of fourth-order and \( Y^{(5)} \) is the integral of fifth-order involving only the doublets \( (B_1, B_2) \). Again we want to remind that for such a fourth-order integral the potential \( V \) involves a Painlevé transcendent \( P_6 \) \([9]\) and once more by construction we obtain an exotic potential involving a Painlevé transcendent which is not new.

Finally, the doublets \( (F_1, F_2) \), which are the coefficients of \( \{(L_2, p_x (p_x^2 + p_y^2)), (L_2, 2p_x p_y (p_x^2 + p_y^2))\} \), are the leading part of the integral \( Y^{(3)} H \).

Since all of the above fifth-order integrals are algebraically dependent with the known lower order ones and the trivial integrals we consider them as obvious integrals of motion and dropped from general fifth-order integral \( i.e. \) we set \( A_1, A_2, B_1, B_2, F_1, F_2 \) to zero. This means that we are left with only one
set of nonlinear differential equations for the exotic doublets \((C_1, C_2)\), which can be written as

\[
48 \left(9C h^2 + 2(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T)\right) T''
\]

\[-4 \left(45C h^2 + 22(c_{14} \cos(3\theta) - c_{13} \sin(3\theta)) - 66CT + 40C h T''\right) T'''
\]

\[-\left(24(t_1C h^2 + c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T + 4CT') - 6C h T''\right) T^{(3)}
\]

\[+ \left(\frac{67}{3}C h^2 + 2(c_{14} \cos(3\theta) - c_{13} \sin(3\theta) - 3CT + C_5 T')\right) T^{(4)}
\]

\[+408C(T')^2 - 90C(T'')^2 + 9C h^2 T^{(5)} - \frac{1}{6}C h^2 T^{(6)} = 0,
\]

\[
48CT^{(3)} - 12 \left(95C h^2 - 2(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T)\right)(T')^2
\]

\[-\left(96h^2 \left(9C h^2 + 2(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T)\right) + 72C(T'')^2\right) T'
\]

\[+ 2C h(T''')^3 + 9 \left(115C h^2 - 2(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T)\right)(T'')^2
\]

\[+ \left(64h^2 (9C h^2 + 5(c_{14} \cos(3\theta) - c_{13} \sin(3\theta) - 3CT))
\]

\[+20(43C h^2 - 2(c_{14} \cos(3\theta) + c_{14} \sin(3\theta) - 3CT)) T' - 40C h (T')^2 \right) T''' - 96C h^2 (T^{(3)})^2
\]

\[+ \left(6h^2 (241C h^2 + 34(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T) + 224CT')
\]

\[-\left(-261C h^2 - 2(c_{14} \cos(3\theta) - c_{13} \sin(3\theta) - 3CT + C_5 T')\right) T^{(3)}
\]

\[-\left(\frac{1}{3}h^2 \left(186(c_{14} \cos(3\theta) - c_{13} \sin(3\theta)) - 9C \left(62T - 41T''\right) + 5C \left(133h^2 + 64T^2 - 3T^{(3)}\right)\right)(T^{(4)}
\]

\[-\frac{1}{6}h^2 \left(27(t_1C h^2 + 2(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T + 4CT')) - 14C h T''\right) T^{(5)}
\]

\[+ \frac{1}{12}h^2 \left(125C h^2 + 6(c_{14} \cos(3\theta) - c_{13} \sin(3\theta) - 3CT + C_5 T')\right) T^{(6)}
\]

\[+3C h^4 T^{(7)} - \frac{1}{24}C h^2 T^{(8)} = 0,
\]

where \(c_{ij}\) are integration constants. Of course, we can choose them as zero, but even keeping them as arbitrary we will arrive potentials, expressed in terms of the Painlevé transcendent order equation. The fifth-order terms can be eliminated by the help of these two compatibility that option to be ruled out. On the other hand it can be shown that \((37)\) can be expressed in terms of \(\Phi(\theta)\) (i.e. \((\text{Eq}37) = -\frac{1}{2}(\Phi''(\theta) + \Phi(\theta))\)) and thus the solution of \(\Phi(\theta) = 0\) satisfies both of the equations

\[
\Phi(\theta) = \frac{h^2}{3} \left(C_5 T^{(4)} - 36CT^{(3)}\right) - 2 \left(9C h^2 + 2(c_{14} \cos(3\theta) - c_{13} \sin(3\theta) - 3CT + C_5 T')\right) T''
\]

\[+ 12 \left(9C h^2 + 2(c_{13} \cos(3\theta) + c_{14} \sin(3\theta) + C_5 T + 2CT')\right) T'.
\]
If we make the following change of variables $z = \tan(3\theta)$ in the equation $\Phi(\theta) = 0$ and divide the result by the common factor $81(1 + z^2)^2$, we see that after applying a similar procedure as in the previous cases we obtain exactly the derivative of the first canonical subcase of the Cosgrove’s master Painlevé equation \cite{5,6}. Hence, similarly as in (15), (26) and (27), the potential is expressed in terms of $P_6$, where the highest order $N$ is reflected as an overall factor (as $(N-2)$) in the potential and as $\frac{(N-2)\theta}{2}$ in the argument of the trigonometric functions appeared in the potential.

4. Conclusions

In this article we establish some general properties of superintegrable systems separating in polar coordinates and allowing a higher-order integral by observing specific examples $N = 3, 4$ and 5. In particular, superintegrable Hamiltonians in classical and quantum mechanics differ \cite{16,17}. Terms depending on $\hbar$ appear in the quantum case. The classical limit $\hbar \to 0$ is singular and must be taken in the determining equations, not in the solutions. Two types of potentials occur which we call standard and exotic. Standard ones are solutions of a linear compatibility condition for the determining equations and it has been observed that the TTW (PW) potentials are fully contained in the angular component $S(\theta) \equiv T'(\theta)$ both in classical and quantum cases. Moreover, in the quantum case there exist more potentials, proportional to $\hbar^2$, which cannot be reduced to TTW (PW). For exotic potentials the linear compatibility condition is satisfied trivially so the potentials satisfy nonlinear equations. In quantum mechanics the nonlinear equations pass the Painlevé test and we conjecture that a new infinite family of superintegrable potentials in terms of the sixth Painlevé transcendent $P_6$ exists.

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