GLSM’s, gerbes, and Kuznetsov’s homological projective duality

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Abstract. In this short note we give an overview of recent work on string propagation on stacks and applications to gauged linear sigma models. We begin by outlining noneffective orbifolds (orbifolds in which a subgroup acts trivially) and related phenomena in two-dimensional gauge theories, which realize string propagation on gerbes. We then discuss the ‘decomposition conjecture,’ equating conformal field theories of strings on gerbes and strings on disjoint unions of spaces. Finally, we apply these ideas to gauged linear sigma models for complete intersections of quadrics, and use the decomposition conjecture to show that the Landau-Ginzburg points of those models have a geometric interpretation in terms of a (sometimes noncommutative resolution of) a branched double cover, realized via nonperturbative effects rather than as the vanishing locus of a superpotential. These examples violate old unproven lore on GLSM’s (namely, that geometric phases must be related by birational transformations), and we conclude by observing that in these examples (and conjecturing more generally in GLSM’s), the phases are instead related by Kuznetsov’s ‘homological projective duality.’

1. Introduction

In this short note we will outline a few of the results in [1, 2, 3, 4, 5]. In general terms, these papers outline the definition and some applications of string compactifications on (smooth, Deligne-Mumford) stacks and, in particular, gerbes.

The original motivation for understanding string compactifications on stacks was to investigate what new string vacua, new conformal field theories, arise from the near-geometries provided by stacks, and to better understand some existing theories that were already special cases of string compactifications on stacks.

This note will focus on the special case of stacks that are gerbes. A string on a gerbe is described by a noneffective orbifold (an orbifold in which a subgroup acts trivially on the space) or noneffective
gauge theory. Despite the fact that the group action is trivial, physics nevertheless knows about that (trivially-acting) subgroup via nonperturbative effects, as we shall outline in section 2.

Massless spectra resulting from such noneffective group actions appear to violate cluster decomposition, a foundational property of most quantum field theories, but these theories are nevertheless nontrivial because they are equivalent to sigma models on disjoint unions of ordinary spaces, a result known as the ‘decomposition conjecture,’ and which we review in section 3.

The decomposition conjecture has several applications, including examples in Gromov-Witten theory and the geometric Langlands program. In section 4, we focus on a different application of the decomposition conjecture, namely to understand the Landau-Ginzburg points of certain puzzling gauged linear sigma models. We shall see that the Landau-Ginzburg points in question have, for the most part, a $\mathbb{Z}_2$ gerbe structure to which the decomposition conjecture can be applied, and find a mathematically sensible description of the Landau-Ginzburg points in terms of branched double covers (the double cover arising from the application of the decomposition conjecture to a $\mathbb{Z}_2$ gerbe). The resulting theories realize geometry from GLSM’s in a novel fashion, and also violate the lore that all geometric phases in GLSM’s are birational to one another. In examples, we observe that geometric phases are related by Kuznetsov’s ‘homological projective duality’ (instead of ‘birationality’), and conjecture that this is true more generally.

2. Noneffective orbifolds and gauge theories

This note will concern applications of noneffective orbifolds to physics and geometry, so let us beginning by defining a noneffective orbifold. It is a quotient $[X/G]$, where a subgroup of $G$, call it $K$, acts trivially on $X$. Such quotients are also examples of $K$-gerbes.

To understand why such a quotient is physically different from the corresponding effective orbifold $[X/(G/K)]$, let us work through an example. Consider the orbifold $[X/D_4]$, where the $\mathbb{Z}_2$ center of $D_4$ acts trivially on $X$. Now, $D_4$ can be described as the central extension

$$1 \to \mathbb{Z}_2 \to D_4 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

so that only the $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts nontrivially. We shall see that the one-loop partition function of $[X/D_4]$ is very different from the one-loop partition function of $[X/\mathbb{Z}_2 \times \mathbb{Z}_2]$, thus making it clear that there is a difference.

We can enumerate the elements of $D_4$ as

$$D_4 = \{1, z, a, b, az, bz, ab, ba = abz\}$$

where $z$ generates the $\mathbb{Z}_2$ center, and the elements of $D_4/\mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ are

$$\mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \overline{a}, \overline{b}, \overline{ab}\}$$

where the overline indicates the coset formed through multiplication by $z$. Now, the one-loop partition function of $[X/D_4]$ has the form

$$Z(D_4) = \frac{1}{|D_4|} \sum_{g,h \in D_4 : gh = hg} Z_{g,h}$$
Each of the $Z_{g,h}$ twisted sectors that appears, is the same as a $Z_2 \times Z_2$ twisted sector (since the $Z_2$ center of $D_4$ acts trivially), appearing with multiplicity $|Z_2|^2 = 4$, except for the sectors

$$\begin{array}{ccc}
\pi & & \pi \\
| b & & | ab \\
\pi & & \pi \\
\end{array}$$

which are omitted, because the lifts of the group elements to $D_4$ do not commute in $D_4$.

Thus, we see that

$$Z(D_4) = \frac{|Z_2 \times Z_2|}{|D_4|} |Z_2|^2 (Z(Z_2 \times Z_2) - \text{some twisted sectors})$$

$$= 2 (Z(Z_2 \times Z_2) - \text{some twisted sectors})$$

In particular, the $D_4$ orbifold is significantly different from the $Z_2 \times Z_2$ orbifold, so we see here explicitly that string theory does indeed know about even trivially-acting groups.

With a little more work, we can find a cleaner interpretation of this result. Discrete torsion acts as a sign on precisely the omitted twisted sectors, so we see that

$$Z([X/D_4]) = Z \left( [X/Z_2 \times Z_2] \prod [X/Z_2 \times Z_2] \right)$$

with discrete torsion in one component.

Although we have focused so far on orbifolds, the same issue exists in two-dimensional gauge theories, where it manifests as a question of whether e.g. an abelian gauge theory with matter of charge 2 is the same as if matter is charge 1. Briefly, the perturbative physics is identical, but nonperturbatively these two theories can differ.

For example, consider the supersymmetric $\mathbb{P}^{N-1}$ model, in which the chiral superfields are taken to have charge $k$ rather than charge 1. In the ordinary $\mathbb{P}^{N-1}$ model, the axial $U(1)_A$ is broken to $Z_{2N}$ by instantons, whereas here it is broken to $Z_{2kn}$. Similarly, A model correlation functions are different. In the ordinary $\mathbb{P}^{N-1}$ model, it is straightforward to show that

$$\langle X^{N(d+1)-1} \rangle_d = q^d$$

where $X$ corresponds to the generator of the classical cohomology ring. Here, by contrast,

$$\langle X^{N(kd+1)-1} \rangle_d = q^d$$

As a result, quantum cohomology rings differ. In the ordinary $\mathbb{P}^{N-1}$ model, the quantum cohomology ring is

$$\mathbb{C}[x]/(x^N - q)$$

whereas in the present case it is

$$\mathbb{C}[x]/(x^{kn} - q)$$

In each case, we see meaningfully different physics.

On a compact worldsheet, the distinction above follows from the fact [6] that to uniquely specify a Higgs field, one must specify to which bundle it couples. For example, if a $U(1)$ gauge field couples
to a line bundle \( L \), then to say a scalar \( \phi \) has charge \( Q \) means \( \phi \in \Gamma(L^{\otimes Q}) \). Different bundles implies different zero modes, which implies different anomalies, and hence different physics. For noncompact worldsheets, there is an analogous argument using periodicity of the theta angle [6].

This phenomenon is not specific to two dimensions, but also has analogues in four dimensions. For example, the same effect is at the heart of the (nonperturbative) distinction between \( SU(n) \) and \( SU(n)/\mathbb{Z}_n \) gauge theories, and the distinction between \( \text{Spin}(n) \) and \( \text{SO}(n) \) gauge theories. For \( \mathcal{N} = 1 \) supersymmetry in four dimensions, there are results [7, 8] describing Seiberg duality between \( \text{Spin}(n) \) gauge theory with massive spinors and \( \text{SO}(n) \) gauge theories with \( \mathbb{Z}_2 \) monopoles. In \( \mathcal{N} = 4 \) supersymmetry in four dimensions, this phenomenon is crucial for the physical understanding of the geometric Langlands program [9].

3. Decomposition conjecture

Consider a quotient \([X/H]\), where

\[
1 \rightarrow G \rightarrow H \rightarrow K \rightarrow 1
\]

and \( G \) acts trivially, and is a finite group. We claim that

\[
\text{CFT}([X/H]) = \text{CFT}\left(\left[\left(\mathbb{C} \times \hat{G}\right)/K\right]\right)
\]

(together with some \( B \) field, as specified in [4]), where \( \hat{G} \) is the set of irreducible representations of \( G \). As the right-hand side is a disjoint union, this relates strings on gerbes to strings on disjoint unions of spaces (or non-gerbe stacks). We call this the decomposition conjecture.

When \( K \) acts trivially on \( \hat{G} \), the decomposition conjecture above reduces to the statement that

\[
\text{CFT}([X/H]) = \text{CFT}\left(\prod_{\hat{G}}(X, B)\right)
\]

where the \( B \) field is determined by the image of the characteristic class of the gerbe under the map defined by an element of \( \hat{G} \):

\[
H^2(X, Z(G)) \xrightarrow{Z(G) \rightarrow U(1)} H^2(X, U(1))
\]

The decomposition conjecture satisfies a number of checks, for example:

- For global quotients by finite groups, one can check explicitly that partition functions match at arbitrary worldsheet genus.
- This implies (by virtue of D-branes) a statement about K-theory, namely that

\[
K_H(X) = \text{twisted} \ K_K(X \times \hat{G})
\]

which can be checked independently.
• It is consistent with standard results on sheaves on gerbes, namely that sheaves on gerbes decompose (in the same way), and that Ext group elements between sheaves on different components vanish.
• It implies results in Gromov-Witten theory, which are being checked in papers including [10, 11, 12, 13, 14, 15].
• One can compute the Toda mirrors to Fano toric stacks, and and compare quantum cohomology computations on either side of the duality.

The example of the previous section, \([X/D_4]\), fits into this framework: the decomposition conjecture predicts that the CFT of \([X/D_4]\) should look like two copies of \([X/\mathbb{Z}_2 \times \mathbb{Z}_2]\), one copy with a flat B field corresponding to nonzero discrete torsion, which is what we found explicitly.

The decomposition conjecture also appears implicitly in the work of [9] on the physical realization of geometric Langlands; there, after dimensionally-reducing along a Riemann surface to two dimensions, one often finds a sigma model on a disjoint union of moduli spaces. This can be understood from the fact that the moduli space of the four-dimensional theory in such cases is a gerbe, as described in [4].

4. Application to GLSM’s

Let us now apply these ideas to study gauged linear sigma models (GLSM’s) in two dimensions [16]. In particular, let us consider the GLSM describing the complete intersection of four quadric (degree two) hypersurfaces in \(\mathbb{P}^7\). This GLSM has

- Eight chiral superfields \(\phi_i\), each of charge 1, corresponding to homogeneous coordinates on \(\mathbb{P}^7\)
- Four chiral superfields \(p_a\), each of charge -2, one for each hypersurface in the complete intersection

and a superpotential

\[ W = \sum_a p_a G_a(\phi) \]

where the \(G_a\)’s are degree two homogeneous polynomials.

Let us analyze the space of supersymmetric vacua in this theory, in semiclassical regimes. The D-term constraint is

\[ \sum_i |\phi_i|^2 - 2 \sum_a |p_a|^2 - r = 0 \]

When \(r \gg 0\), the \(\phi_i\) cannot all vanish (from the D-terms above), hence vanishing of F terms requires that \(p_a = G_a = 0\), and the theory appears to flow in the IR to a nonlinear sigma model on the complete intersection \(\mathbb{P}^7[2,2,2,2]\).

The other limit of \(r\) is more interesting. It is helpful to rewrite the superpotential as

\[ W = \sum_a p_a G_a(\phi) = \sum_{ij} \phi_i A^{ij}(p) \phi_j \]
where $A^{ij}$ is a symmetric matrix with entries linear in $p$'s, that encodes the $G_a$'s. Written in this form, it is clear that the superpotential is really a mass matrix for the $\phi_i$. In the limit that $r \ll 0$, the D terms specify that the $p_a$ cannot all be zero, and the superpotential above appears to imply that the $\phi_i$ are all massive, suggesting that this limit flows to a nonlinear sigma model on $P^3$. However, since the $r \gg 0$ limit is Calabi-Yau, this limit should also be Calabi-Yau, a contradiction.

The correct analysis of this limit relies on two subtleties. The first subtlety is that the $\phi_i$ are not massive everywhere, some of their masses vanish along the locus $\{\det A = 0\} \subset P^3$. The second subtlety is the fact that the $p$'s have nonminimal charge, so over the part of $P^3$ where the $\phi_i$ are all massive, we have a nonminimally-charged abelian gauge theory, which (as outlined previously) describes a local noneffective $Z_2$ orbifold, and hence a $Z_2$ gerbe.

As a result of these subtleties and the decomposition conjecture, physics sees the $Z_2$ gerbe as a double cover of $P^3$ away from the locus $\{\det A = 0\}$. Moreover, it can be shown that there is a Berry phase around the locus $\{\det A = 0\}$, which has the effect of interchanging the two sheets of the cover. Thus, physics seems to be seeing a branched double cover of $P^3$, namely Clemens’ octic double solid [17]. As a consistency check, the locus $\{\det A = 0\}$ has the right degree for this branched double cover to be Calabi-Yau, as one would expect of a GLSM describing a Calabi-Yau at another limit in Kähler moduli space.

The result we have presented so far is noteworthy for at least two reasons:

- This is a novel realization of geometry in a GLSM, considering that GLSM’s ordinarily build Calabi-Yau’s as vanishing loci of potentials.
- This branched double cover is not birational to the geometry appearing at the other limit in Kähler moduli space, namely $P^7[2,2,2,2]$. This result contradicts standard lore in the GLSM community, namely that geometries appearing at limit points of the same GLSM are all birational to one another.

Analogous results for nonabelian GLSM’s have also been obtained in [18].

This example, the GLSM for $P^7[2,2,2,2]$, is just one of a number of examples which can be analyzed in the same form. In a subset of those cases, including $P^7[2,2,2,2]$, there is additional structure to uncover, deriving from the fact that the branched double cover is singular, but the GLSM physics behaves as if it is smooth at those singularities. In such cases, when the branched double cover is singular (a subset of all examples of the form above), we believe the Landau-Ginzburg model is actually describing a ‘noncommutative resolution’ of the branched double cover worked out by Kuznetsov.

This particular notion of ‘noncommutative space’ is one described by e.g. [19, 20, 21, 22, 23, 24, 25, 26, 27], and is distinct from other notions of noncommutative space appearing previously in the physics literature such as [28, 29]. Specifically, a ‘noncommutative space’ in this sense is defined via its sheaves.

In the present case, the noncommutative resolution in question is defined by sheaves of $B$-modules over $P^3$, where $B$ is the sheaf of even parts of Clifford algebras associated with the universal quadric over $P^3$ defined by the GLSM superpotential.
Physically, those sheaves of $B$-modules are precisely the same as matrix factorizations at the Landau-Ginzburg point. Intuitively, we can understand this result as follows. First, let us recall matrix factorizations for a quadratic superpotential: even though the bulk theory is massive, one still has $D0$-branes with a Clifford algebra structure. In the present case, we have a Landau-Ginzburg model (obtained via a finite amount of renormalization group flow) fibered over $P^3$, so a Born-Oppenheimer analysis gives sheaves of Clifford algebras (determined by the superpotential) and modules thereof. This is ultimately why the D-branes precisely duplicate the definition of the noncommutative resolution.

Thus, in addition to a novel realization of geometry, and an example in which the two geometric limits of a GLSM are not birational to one another, we see in addition that the GLSM is physically realizing a noncommutative resolution, in the sense discussed earlier.

As an aside, one way to study this noncommutative resolution is via D-brane probes. It can be shown that the moduli space of D-branes propagating on this noncommutative resolution is a necessarily non-Kähler small resolution of the singular space. The non-Kähler structure makes it essentially impossible for the D-brane moduli space to be the target of the closed string theory, as it would break worldsheet supersymmetry. It is allowed here because it is the open string moduli space, not where the closed strings propagate. (Another example where the closed string target space is different from the D-brane moduli space is orbifolds [2][section 8.2]. There, closed strings see an orbifold – a quotient stack – whereas D-branes see a resolution [34].)

Although the phases of the GLSM above are not birational, they nevertheless do have a precise mathematical relationship: they are related by Kuznetsov’s “homological projective duality” [19, 20, 21]. We conjecture that phases in all GLSM’s are always related by homological projective duality.

5. Summary

In this note we have outlined some recent results and applications of string compactifications on near-geometries provided by gerbes.

- We began by describing how physics sees noneffective group actions, which is the physics at the heart of string propagation on gerbes.
- We outlined the decomposition conjecture for strings propagating on gerbes, which states that the CFT of a nonlinear sigma model on a gerbe matches the CFT of a nonlinear sigma model on a disjoint union of spaces or non-gerbe stacks.
- We described the application of the decomposition conjecture to GLSM’s, resulting in novel realizations of geometry, some violations of old lore on GLSM’s, a physical realization of Kuznetsov’s homological projective duality, and a concrete realization of strings on noncommutative resolutions.

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[1] T. Pantev, E. Sharpe, “Notes on gauging noneffective group actions,” hep-th/0502027.
[2] T. Pantev, E. Sharpe, “String compactifications on Calabi-Yau stacks,” Nucl. Phys. B733 (2006) 233-296, hep-th/0502044.
[3] T. Pantev, E. Sharpe, “GLSM’s for gerbes (and other toric stacks),” Adv. Theor. Math. Phys. 10 (2006) 77-121, hep-th/0502053.
[4] S. Hellerman, A. Henriques, T. Pantev, E. Sharpe, M. Ando, “Cluster decomposition, T-duality, and gerby CFT’s,” Adv. Theor. Math. Phys. 11 (2007) 751-818, hep-th/0506034.
[5] T. Pantev, E. Sharpe, “Non-birational twisted derived equivalences in abelian GLSMs,” arXiv: 0709.3855.
[6] J. Distler, R. Plesser, private communication.
[7] P. Pouliot, “Chiral duals of non-chiral susy gauge theories,” Phys. Lett. B359 (1995) 108-113, hep-th/9507018.
[8] M. Strassler, “Duality, phases, spinors and monopoles in SO(n) and Spin(n) gauge theories,” JHEP 9809 (1998) 017, hep-th/9709081.
[9] A. Kapustin, E. Witten, “Electric-magnetic duality and the geometric Langlands program,” Comm. Number Theory Phys. 1 (2007) 1-236, hep-th/0604151.
[10] E. Andreini, Y. Jiang, H.-H. Tseng, “On Gromov-Witten theory of root gerbes,” arXiv: 0912.3580.
[11] E. Andreini, Y. Jiang, H.-H. Tseng, “Gromov-Witten theory of etale gerbes,” arXiv: 0907.2087.
[12] A. Kuznetsov, private communication.