Remarks about the microcanonical description of astrophysical systems.

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We reconsider some general aspects about the mean field thermodynamical description of the astrophysical systems based on the microcanonical ensemble. Starting from these basis, we devote a special attention to the analysis of the scaling laws of the thermodynamical variables and potentials in the thermodynamic limit. Geometrical considerations motivate a way by means of which could be carried out a well-defined generalized canonical-like description for this kind of systems, even being nonextensive. This interesting possibility allows us to extend the applicability of the Standard Thermodynamic methods, even in the cases in which the system exhibits a negative specific heat. As example of application, we reconsider the classical Antonov problem of the isothermal spheres.

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I. INTRODUCTION

Traditional thermodynamics is not able to describe the astrophysical systems because of their nonextensive nature: They are nonhomogeneous systems whose total energy does not scale asymptotically with the particles number during the increasing of the system size, as well as they exhibit energetic regions with a negative specific heat, which makes the canonical description unsuitable for describing their thermodynamical properties.

An adequate thermodynamical description for astrophysical systems can be carried out starting from microcanonical basis under certain conditions. This possibility is very attractive because the microcanonical ensemble is the only well-defined statistical ensemble: since dynamics is always well defined, and because microcanonical ensemble is a “dynamical” ensemble.

The aim of the present paper is to reconsider some general questions about the thermodynamical formalism for astrophysical systems based on the consideration of the microcanonical ensemble. The organization of the paper is the following. Sec. II is devoted to review some aspects of this microcanonical mean field formalism for the astrophysical systems. The entropy function for the selfgravitating system in the microcanonical description is obtained from the Boltzmann principle:

\[ S_B = \ln W, \]  

II. MICROCANONICAL DESCRIPTION

A direct microcanonical calculation in a selfgravitating system is an formidable task which can be performed by using Montecarlo numerical simulations (see for example in refs. [11, 12]). The statistical description of \( N \)-body selfgravitating system is usually carried out by using a mean field approximation when the particles number \( N \) is large enough. We will perform in this section a brief review about some aspects of this microcanonical mean field formalism for the astrophysical systems.

A. The microcanonical mean field approximation

Let us consider an astrophysical Hamiltonian system composed by a huge number of identical particles enclosed in a rigid container with a characteristic linear dimension. The consideration of this rigid container is a regularization procedure for the thermodynamical description for this kind of system which avoids the particles evaporation. Let us also suppose that these particles interact among them by means of the gravity and short-range interactions, as example, forces with an electrostatic origin. The Hamiltonian of this system can be discomposed into two terms as follows:

\[ H_N = h_N + V_N, \]  

where \( V_N \) is the potential energy associated to the Newtonian gravitational interaction, while \( h_N \) contains the energy contribution of the short-range interactions and the kinetic energy of the particles.

The entropy function for the selfgravitating system in the microcanonical description is obtained from the Boltzmann principle:
where the microcanonical accessible volume \( W(E, N; L) \) is given by:

\[
W(E, N; L) = S_\rho [\delta(E - H_N)] \delta\epsilon_0,
\]

being \( \delta\epsilon_0 \) a suitable energy constant which makes \( W \) dimensionless. Quantum effects or/and the consideration of a natural size for the particles will be taken into account in an implicit manner in the equation \( \delta\epsilon_0 \), which is a necessary consideration in order to avoid the short-range singularity of the Newtonian interaction, and therefore, the position of its center.

The number of particles inside the cell is very large. We denoted by \( n_\alpha \) the number of particles inside the cell \( c_\alpha \), being \( r_\alpha \) the position of its center.

The short-range interactions among the particles belonging to different cells can be neglected, and therefore, the term \( h_N \) of the Hamiltonian \( H \) can be approximated by:

\[
h_N \simeq \sum_\alpha h_{n_\alpha},
\]

where \( h_{n_\alpha} \) is the energetic contribution of the \( n_\alpha \) particles inside the cell \( c_\alpha \). Hereafter, we refer this energetic contribution as the internal energy. The potential energy \( V_N \) associated to the gravitational interaction can be approximated as follows:

\[
V_N \rightarrow V[n] = -\sum_{\alpha > \beta} \frac{Gm^2 n_\alpha n_\beta}{|r_\alpha - r_\beta|}.
\]

Taking into consideration all approximations exposed above, the microcanonical volume \( W \) can be rewritten as follows:

\[
W(E, N, L) \simeq \sum_{\{n_\alpha\}} \delta_D \left( N - \sum_\alpha n_\alpha \right) W[n; E, L],
\]

being:

\[
\delta_D (k) = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{\{n_\alpha\}} \equiv \sum_{n_1} \sum_{n_2} \cdots.
\]

The functional \( W[n; E, L] \) is given by:

\[
W[n; E, L] = \int \left( \prod_\alpha \frac{d\epsilon_\alpha}{\epsilon_\alpha} \right) \delta\epsilon_0 \delta \left( E - V[n] - \sum_\beta c_\beta \right) \times 
\]

\[
\times \exp \left[ \sum_\alpha S_\alpha (e_\alpha, n_\alpha, v_\alpha) \right],
\]

where \( S_\alpha (e_\alpha, n_\alpha, v_\alpha) = \ln [\omega_\alpha (e_\alpha, n_\alpha, v_\alpha)] \) is the Boltzmann entropy associated to the subsystem of \( n_\alpha \) particles inside the cell \( c_\alpha \), where \( e_\alpha \) is its internal energy, \( v_\alpha \) the volume of this cell. The quantity \( \omega_\alpha (e_\alpha, n_\alpha, v_\alpha) \) is the number of microstates of this subsystem with the above macroscopic configuration:

\[
\omega_\alpha (e_\alpha, n_\alpha, v_\alpha) = S_\rho [\delta (e_\alpha - h_{n_\alpha})] \delta e_\alpha,
\]

where \( \delta e_\alpha \) is a suitable energy constant. Since \( h_{n_\alpha} \) involves only short-range interactions, this subsystem will look-like as an extensive system for \( n_\alpha \) large, and therefore, the entropy per volume \( s_\alpha \) depends only on the internal energy density \( e_\alpha = e_\alpha / v_\alpha \) and the particles density \( \rho_\alpha = n_\alpha / v_\alpha \) as follows:

\[
s_\alpha = S_\alpha (e_\alpha, n_\alpha, v_\alpha) / v_\alpha = s(e_\alpha, \rho_\alpha).
\]

The microcanonical volume \( W \) can be conveniently rewritten by using the following mean field approximation:

\[
W \rightarrow W_{MF} (E, N, L) = A \int \mathcal{D}\rho (r) \mathcal{D}\epsilon (r) \exp \left( S[\epsilon, \rho] \right) \times 
\]

\[
\times \delta (N - N[\rho]) \delta (E - H[\epsilon, \rho]),
\]

where \( A \) is an unimportant constant which involves the constants \( \delta\epsilon_\alpha \)'s and \( v_\alpha \)'s. The functionals

\[
S[\epsilon, \rho] = \int_{\mathbb{R}^3} d^3 r \ s(\epsilon (r), \rho (r)),
\]

\[
H[\epsilon, \rho, \phi[\rho]] = \int_{\mathbb{R}^3} d^3 r \ \epsilon (r) + \frac{1}{2} m\rho (r) \phi (r),
\]

\[
N[\rho] = \int_{\mathbb{R}^3} d^3 r \ \rho (r),
\]

represent the total entropy, energy and particles number for a given profile with \( \epsilon (r) \) and \( \rho (r) \), being \( \phi (r) \) the Newtonian potential:

\[
\phi (r) = \mathcal{G}[\rho; r] = -\int_{\mathbb{R}^3} \frac{Gm\rho (r_1)}{|r - r_1|} d^3 r_1,
\]
where $\mathcal{G}[\rho; \mathbf{r}]$ is the Green solution of the Poisson problem:

$$\Delta \phi = 4\pi G \rho \rho.$$ (16)

The expression of the microcanonical volume in mean field approximation can be rewritten in order to avoid the $\rho$ dependence of the Newtonian potential as follows:

$$W_{MF}(E, N, L) = A \int \mathcal{D}\rho(\mathbf{r}) \mathcal{D}\epsilon(\mathbf{r}) \mathcal{D}\phi(\mathbf{r}) \exp\{S[\epsilon, \rho]\} \times$$

$$\times \delta\{\phi(\mathbf{r}) - \mathcal{G}[\rho; \mathbf{r}]\} \delta(N - N[\rho]) \delta(E - H[\epsilon, \rho, \phi]).$$ (17)

$W_{MF}(E, N, L)$ can be rewritten again by using the Fourier representation of the delta functions, yielding:

$$W_{MF}(E, N) \sim \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk d\eta}{(2\pi)^2} \int \mathcal{D}\rho \mathcal{D}\epsilon \mathcal{D}\phi \mathcal{D}h \times$$

$$\times \exp\{\mathcal{L}[\epsilon, \rho; \phi; z_1, J]\},$$ (18)

where $z = \beta + ik$ and $z_1 = \mu + i\eta$ with $\beta, \eta \in \mathbb{R}$, being the functional $\mathcal{L}[\epsilon, \rho; \phi; z_1, J]$ defined by:

$$\mathcal{L}[\epsilon, \rho; \phi; z_1, J] = S[\epsilon, \rho] + z(E - H[\epsilon, \rho, \phi]) + +z_1(N - N[\rho]) + J \cdot (\phi - \mathcal{G}[\rho]).$$ (19)

The functional term $J \cdot (\phi - \mathcal{G}[\rho]) = \int_{\mathbb{R}^3} d^3r \int \mathcal{R}(\mathbf{r}) \{\phi(\mathbf{r}) - \mathcal{G}[\rho; \mathbf{r}]\}$ appears as consequence of the Fourier representation of the delta functional $\delta\{\phi(\mathbf{r}) - \mathcal{G}[\rho; \mathbf{r}]\}$. Here, $J(\mathbf{r})$ is a complex function, $J(\mathbf{r}) = j(\mathbf{r}) + i \mathcal{H}(\mathbf{r})$, with $j(\mathbf{r}) \in \mathbb{R}$.

**B. Scaling properties of the thermodynamical variables and potentials**

The integration of expression (18) is usually carried out by using the *steepest decent method*. The application of this method is based on the asymptotic behavior of the thermodynamical variables and potentials in the many particle limit $N \gg 1$.

The presence of an additive kinetic part in the Hamiltonian of certain system leads to an exponential growing of the microcanonical volume $W$ with the $N$ increasing, and therefore, the Boltzmann entropy will grow proportional to $N$ in the many particles limit, $S_B = \ln W \propto N$.

The usual thermodynamic limit for the extensive systems:

$$N \to \infty, \text{ keeping constant } \frac{E}{N} \text{ and } \frac{N}{V},$$ (20)

where $V$ is the volume of the system, is directly related with the *extensive properties* of these systems when the thermodynamical variables of the system are scaled by some scaling parameter $\alpha$ as follows:

$$N \to N(\alpha) = \alpha N,$$

$$E \to E(\alpha) = \alpha E,$$

$$V \to V(\alpha) = \alpha V,$$

$$\Rightarrow W \to W(\alpha) = \exp(\alpha \ln W).$$ (21)

In analogy with the extensive properties of the traditional systems, we will analyze the necessary conditions for the existence of the following *power law self-similarity* scaling behavior of the microcanonical variables $E$, $N$ and $L$ for the astrophysical systems:

$$N \to N(\alpha) = \alpha N,$$

$$E \to E(\alpha) = \alpha^{\Lambda_E} E,$$

$$L \to L(\alpha) = \alpha^{\Lambda_L} L,$$

$$\Rightarrow W \to W(\alpha) = \exp(\alpha \ln W),$$ (22)

where $\Lambda_E$ and $\Lambda_L$ are certain constant scaling exponent which lead to an *extensive* character of the Boltzmann entropy. This kind of self-similarity behavior is directly related with a thermodynamic limit of the form:

$$N \to \infty, \text{ keeping constant } \frac{E}{N^{\Lambda_E}} \text{ and } \frac{L}{N^{\Lambda_L}}.$$ (23)

The existence of this kind of self-similarity condition allows a considerable simplification of the thermodynamical description: the study can be performed by setting $N = 1$ and considering the $N$-dependence in the scaling laws by taking $\alpha = N$. This scaling behavior is very useful in numerical experiments, since it allows us to extend the results of this kind of study on a finite system to much bigger systems. Contrary, the nontrivial $N$-dependent behavior of the thermodynamical variables and potentials leads to a complication of the analysis.

In order to satisfy this scaling behavior for the global variables $E$, $N$, $L$ and the Boltzmann entropy $S_B$, the local functions $\epsilon(\mathbf{r})$, $\rho(\mathbf{r})$, $\phi(\mathbf{r})$ and $s(\epsilon, \rho, \phi)$ should be scaled as follows:

$$\rho \to \rho(\alpha) = \alpha^{\Lambda_\rho} \rho \} \Rightarrow \{\phi \to \phi(\alpha) = \alpha^{\Lambda_\phi} \phi,$$

$$\epsilon \to \epsilon(\alpha) = \alpha^{\Lambda_\epsilon} \epsilon,$$

$$s \to s(\alpha) = \alpha^{\Lambda_s} s.$$ (24)

Since the characteristic particles density behaves as $\rho_c \sim N/L^3$, the scaling exponent for the particles density is $\Lambda_\rho = 1 - 3\Lambda_L$. From the expression of the Newtonian potential (18) it is derived that its characteristic unit is $\phi_c \sim \rho_c L^2$, and therefore, $\Lambda_\phi = 1 - \Lambda_L$. The energy scaling exponent is equal to the scaling exponent of the total gravitational potential energy, so that, $\Lambda_E = 2 - \Lambda_L$. The other scaling exponents are obtained by using identical reasonings. All these scaling exponents depend on the scaling exponent $\Lambda_L$ as follows:

$$\Lambda_\rho = 1 - 3\Lambda_L = \Lambda_S,$$

$$\Lambda_\phi = 1 - \Lambda_L,$$

$$\Lambda_\epsilon = 2 - 4\Lambda_L,$$

$$\Lambda_E = 2 - \Lambda_L.$$ (25)
In order to satisfy these scaling laws is also necessary that the entropy density exhibits to the following scaling behavior:

\[ s\left(\alpha^L, \alpha^\rho \right) = \alpha^L \cdot s\left(\epsilon, \rho \right), \]  

(26)

This scaling property is satisfy if \( s\left(\epsilon, \rho \right) \) obeys to the following functional form:

\[ s\left(\epsilon, \rho \right) = \rho \cdot F\left(\epsilon/\rho^n\right), \]  

(27)

where \( n = \Lambda_e/\Lambda_\rho \), being \( F\left(x\right) \) an arbitrary function. It is easy to show that the functional form (27) leads to the following relation between the pressure \( p \) and the internal energy density \( \epsilon \):

\[ p = \gamma \epsilon, \]  

(28)

where \( \gamma = n - 1 \). There are some well-known Hamiltonian systems which satisfy this kind of relation, as example, the system of nonrelativistic or ultrarelativistic noninteracting particles, without matter if they obey to the Boltzmann, Fermi-Dirac or Bose-Einstein Statistics. The scaling parameter \( \Lambda_e \) and \( \Lambda_\rho \) are obtained from the parameter \( \gamma \) as follows:

\[ \Lambda_e = \frac{\gamma - 1}{3\gamma - 1}, \quad \Lambda_\rho = \frac{5\gamma - 1}{3\gamma - 1}. \]  

(29)

This result evidences that the power laws form for the self-similarity conditions (22) can be only satisfied by a reduced group of models whose microscopic picture obeys to the relation (28). Since \( \gamma = 2 \) for the ideal gas of particles, the scaling exponents for the Antonov problem [5] and the selfgravitating fermions model [13, 14] are given by \( \Lambda_e = \frac{1}{3} \) and \( \Lambda_L = \frac{1}{3} \), and therefore, they obey to the following thermodynamic limit:

\[ N \rightarrow \infty, \quad \text{keeping constant} \quad \frac{E}{N^\frac{1}{3}} \quad \text{and} \quad LN^\frac{1}{3}. \]  

(30)

This thermodynamic limit was established in the ref. [10] for the self-gravitating nonrelativistic fermions by using other reasonings. Since the selfgravitating relativistic gas and classic hard sphere model [15, 16] do not obey to the relation (28), they do not posses a scaling behavior with a power law form (22).

C. Legendre formalism

For the sake of simplicity, let us consider firstly those models exhibiting a power law thermodynamic limit (22). In such cases, the steepest decent method leads to a thermodynamic formalism very analogue to the one used for the extensive systems. The main contribution of the functional integral [13] will come from the maxima of the functional \( \mathcal{L}[\epsilon, \rho, \phi; z, z_1, J] \). This last one can be rephrased as the Legendre functional, in which the entropy functional \( S[\epsilon, \rho] \) is maximized under the constrains of the energy and particles number, where it is also taken into account the relation between the Newtonian potential and the particles density. As elsewhere discussed [10], the existence of a multimodal functional \( \mathcal{L}[\epsilon, \rho, \phi; z, z_1, J] \) for a given value of the total energy \( E \) tells us about the existence of several quasi-equilibrium profiles, which could be related with the existence of phase transitions [10, 17]. When there is only one sharp peak, the Boltzmann entropy can be appropriately estimated as follows:

\[ S_{B}(E, N, L) \approx \max_{\epsilon, \rho, \phi} \left\{ \min_{\beta, \mu, j} \mathcal{L}[\epsilon, \rho, \phi; \beta, \mu, j] \right\}, \]  

(31)

where \( \beta \) and \( \mu \) are the canonical parameters (real numbers), while \( j \) is a real field which acts as a Lagrange multiplier for the relation between the Newtonian potential and the particles density. The stationary conditions lead to the following relations:

\[ \beta = \partial_{\rho} s\left(\epsilon, \rho \right), \quad \mu = -\frac{1}{2} \beta m \phi - G\left[j\right] + \partial_{\epsilon} s\left(\epsilon, \rho \right), \quad j = \frac{1}{2} \beta m \rho, \]  

(32)

\[ N[\rho] = N, \quad H[\epsilon, \rho, \phi] = E, \quad \phi = G\left[\rho\right]. \]  

(33)

The first relation of (32) states that the canonical parameter \( \beta \) is constant through all points of the system. Therefore, it is convenient to use the Planck density \( p_c(\beta, \rho) = \max_{\epsilon} \left[\beta \epsilon - s\left(\epsilon, \rho \right)\right] \) instead of the entropy density, which allows us to rewrite the relations (32) as follows:

\[ \epsilon = \partial_{\rho} p_c(\beta, \rho), \quad \mu = -\beta m \phi - \partial_{\rho} p_c(\beta, \rho), \]  

(34)

where the field \( j \) was eliminated and the identity

\[ \partial_{\rho} s\left(\epsilon, \rho \right) = -\partial_{\rho} p_c(\beta, \rho), \]  

(35)

was taken into account. The second relation in the equation (34) allows the obtaining of the state equation for the particles density, \( \rho = \rho(\phi; \beta, \mu) \). The consideration of the constrains (33) lead to a self-consistent integral equations system whose solution represents the most probable equilibrium configuration of the astrophysical system for a given total energy \( E \).

Fluctuations around this equilibrium profile for the particles density \( \rho \) could be estimated by using a Gaussian approximation as follows:

\[ \left( \frac{\delta \rho}{\rho} \right)^2 \leq \left[ \frac{1}{n^2} \frac{\partial^2 \mathcal{L}}{\partial \rho^2} \right]_{eq}^{-1}. \]  

(36)
Since $\mathcal{L}$ grows proportional to $N$ when $N \gg 1$, the relative fluctuations depends on the system size as $\delta \rho / \rho \sim 1/\sqrt{N}$, in analogue form that the traditional systems do obey. Although all reasonings have been made for astrophysical models with a power law thermodynamical limit \[29\], the estimation of the relative fluctuations of the particles density $\rho$ \[30\] evidences the general applicability of Legendre formalism exposed above. This conclusion is straightforward followed from the vanishing of the relative fluctuations of the particles density $\rho$ due to the general linear growing of the functional $\mathcal{L}$ with the $N$ increasing.

This theoretical fact explains why several tools of the Traditional Thermodynamics can be extended for the astrophysical objects in spite of their nonextensive character. This conclusion has been elsewhere experimentally confirmed by observations. For example, the evidence about the isothermal character of the core in globular clusters and elliptical galaxies, where these last ones display a quasi-universal luminosity profile described by de Vaucouleur’s $R^{1/4}$ law \[14\]: the presence of a dark matter halo density profile decreasing as $r^{-2}$ at large distances in the spiral galaxies, which is is also related with the isothermal distributions \[1]. However, in spite of the common applicability of several tools of the standard thermodynamic formalism, the thermodynamical properties of astrophysical systems turn to be very different from the ones exhibited by the ordinary extensive systems due to the long-range character of the gravitational interaction, which leads to the existence of some striking phenomena, i.e. the gravitational collapse and the existence of a negative heat capacity for certain energetic region \[1, 2, 4, 8, 9, 10\].

### III. GEOMETRICAL CONSIDERATIONS

Is it only microcanonically that could be performed a well-defined description for the astrophysical systems? or, will there be also a well-defined canonical-like description? In our opinion, the answer of the last question is yes, which will be proved in the following subsections.

#### A. Physics in the microcanonical ensemble is reparametrization invariant.

Let us consider certain functional $\varphi = \varphi (E, N)$ which is a bijective function of the total energy $E$ (there is a bijective map between $E$ and $\varphi$). In this case, the function $\varphi (E, N)$ is also an integral of motion for the system. Let us introduce the microcanonical state density $\Omega_{\varphi}$:

$$\Omega_{\varphi} (\varphi (E, N), N) = \int \delta [\varphi (E, N) - \varphi (H_{N} (X), N)] dX.$$  

(37)

By using the identities:

$$\delta [\varphi (E, N) - \varphi (H_{N}, N)] = \left| \frac{\partial \varphi (E, N)}{\partial E} \right|^{-1} \delta (E - H_{N}),$$  

(38)

and

$$\Omega_{\varphi} (\varphi, N) = \left| \frac{\partial \varphi (E, N)}{\partial E} \right|^{-1} \Omega_{E} (E, N),$$  

(39)

is straightforward derived the reparametrization invariance of the microcanonical probabilistic distribution function:

$$\frac{1}{\Omega_{\varphi} (\varphi, N)} \delta [\varphi - \varphi (H_{N}, N)] = \frac{1}{\Omega_{E} (E, N)} \delta (E - H_{N})$$

$$\omega_{M} (X; \varphi, N) = \omega_{M} (X; E, N).$$  

(40)

From this property immediately follows the reparametrization invariance of the microcanonical expectation values:

$$\int F (X) \omega_{M} (X; \varphi, N) dX = \int F (X) \omega_{M} (X; E, N) dX,$$

$$F_{M} (\varphi, N) = F_{M} (E, N).$$  

(41)

The previous results point out the following conclusion: the microcanonical description is equivalently performed by using the thermodynamical variables $(E, N)$ or $(\varphi, N)$. We say that $(E, N)$ and $(\varphi, N)$ are two equivalent representations for the abstract space of the microcanonical macroscopic description $\Im_{N}$. The set of all those representation changes among equivalent representations of $\Im_{N}$ constitute the group of diffeomorphisms or reparametrizations of $\Im_{N}$, which is denoted by $Diff (\Im_{N})$. Thus, Physics in microcanonical ensemble is invariant under the reparametrizations changes of $\Im_{N}$.

#### B. Extensivity of the Planck thermodynamic potential.

The analysis of the necessary conditions for the ensemble equivalence starts from the consideration of the Laplace transformation between the microcanonical and the canonical-like partition functions, $\Omega_{\varphi} (\varphi, N)$ and $Z_{\varphi} (\beta_{\varphi}, N)$:

$$Z_{\varphi} (\beta_{\varphi}, N) = \int \exp (-\beta_{\varphi}) \Omega_{\varphi} (\varphi, N) d\varphi.$$  

(42)
Let us show that it is always possible to choose a representation \((\varphi, N)\) of \(\mathbb{Z}_N\) where the Planck potential:

\[
\mathcal{P}_\varphi(\beta_\varphi, N) = -\ln Z_\varphi(\beta_\varphi, N),
\]

is extensive for a system with a power law self-similarity scaling behavior \([22]\).

Let \(W\) be the microcanonical phase-space accessible volume:

\[
W(E, N) = \Omega(E, N) \delta \epsilon,
\]

(44)

For continuous variables, \(W\) is only well-defined after a coarsened partition of phase space, which is the reason why it is considered a small energy \(\delta \epsilon\) in order to make \(W\) dimensionless. However, during the thermodynamic limit \(N \to \infty\), we can make an arbitrary selection of \(\delta \epsilon\), whenever this last be small. In this case, the microcanonical phase-space accessible volume \(W\) appears as a reparametrization invariant function:

\[
W = \Omega_E \delta \epsilon = \Omega_\varphi \delta \varphi,
\]

(45)

where \(\delta \epsilon\) and \(\delta \varphi\) are very small in order to satisfy the condition:

\[
\delta \epsilon \simeq \left| \frac{\partial \varphi}{\partial E} \right|^{-1} \delta \varphi.
\]

In this case, the Boltzmann entropy:

\[
S_B = \ln W,
\]

(46)

becomes a scalar function:

\[
S_B(\varphi, N) = S_B(E, N).
\]

(47)

Taking into account all the exposed above, the equation \([12]\) can be rewritten as follows:

\[
\exp[-\mathcal{P}_\varphi(\beta_\varphi, N)] = \int \exp[-\beta_\varphi \varphi + S_B(\varphi, N)] \frac{d\varphi}{\delta \varphi}.
\]

(48)

\(S_B(\varphi, N)\) is extensive when the thermodynamic limit \([23]\) is carried out. The extensivity of the Planck potential \(\mathcal{P}_\varphi(\beta_\varphi, N)\) when \(N\) is tended to infinity is guaranteed by choosing a functional \(\varphi(E, N)\) which must be extensive when the thermodynamic limit is performed. This is a necessary condition for the ensemble equivalence. Although in the microcanonical description any representation \((\varphi, N)\) of \(\mathbb{Z}_N\) can be considered as thermodynamic variables, in the canonical-like description are only admissible those representations satisfying the extensivity condition. This condition is arisen as the generalization of the additivity condition of the traditional Thermodynamics.

The simplest choice for \(\varphi(E, N)\), in accordance with the thermodynamic limit \([23]\), is given by the scaled energy \(E\):

\[
E = E/N^\Lambda_E^{-1}.
\]

(49)

In general, the extensivity of \(\varphi(E, N)\) is ensured by considering the following dependence:

\[
\varphi(E, N) = N \phi(E/N),
\]

(50)

where \(\phi(\epsilon)\) is a bijective function of \(\epsilon\). All those representation changes \(\varphi \in \text{Diff}(\mathbb{Z}_N)\) preserving the extensivity of \(E\) in the thermodynamic limit constitute a subgroup, which can be referred as the homogeneous subgroup \(\mathcal{M}_C \subset \text{Diff}(\mathbb{Z}_N)\). All the admissible representations for the canonical-like description are parametrized by considering all transformations of \(\mathcal{M}_C\) on the scaled energy \(E\), the pair \((E, \mathcal{M}_C)\).

\section{Ensemble equivalence}

When \(N\) is tended to infinity by using a representation \(\mathcal{R}_E \in (E, \mathcal{M}_C)\) in the equation \([18]\), the main contribution of this integral will come from the maxima of the exponential argument. The equivalence between the microcanonical and the generalized canonical ensemble will take place when there is only one sharp peak. In such cases the validity of the Legendre transformation is ensured:

\[
\mathcal{P}_{\varphi}(\beta_\varphi, N) = \min \left[ \beta_\varphi \varphi - S_B(\varphi, N) \right].
\]

(51)

The minimization leads to the conditions:

\[
\beta_\varphi = \frac{\partial}{\partial \varphi} S_B(\varphi, N), \quad K_\varphi = \frac{\partial^2}{\partial \varphi^2} S_B(\varphi, N) < 0.
\]

(52)

In this case, \(K_\varphi = \partial_\varphi^2 \beta_\varphi\) appears as a generalization of the curvature tensor of the microcanonical statistical theory of Gross \([17]\). The topology of this tensor defines the ordering information of the system by using a microcanonical description. As usual, all those points of \(\mathbb{Z}_N\) where \(K_\varphi > 0\) can not be accessed by using the canonical-like description in the \(\varphi\)-representation, \(\mathcal{R}_\varphi\). However, it can be easily shown that the sign of \(K_\varphi\) is representation dependent.

Let \(\mathcal{P}_{\varphi}^m(\varphi, N)\) be the microcanonical Planck potential:

\[
\mathcal{P}_{\varphi}^m(\varphi, N) = \varphi \frac{\partial}{\partial \varphi} S(\varphi, N) - S_B(\varphi, N).
\]

(53)

\(\mathcal{P}_{\varphi}^m(\varphi, N)\) becomes in the canonical Planck potential when the ensemble equivalence takes place in the thermodynamic limit. Let us now consider two representations
\( \mathcal{R}_{\varphi_1} \) and \( \mathcal{R}_{\varphi_2} \) of \((\mathcal{E}, \mathcal{M}_C)\), where the reparametrization change \( T_{1 \to 2} = \varphi_2 \circ \varphi_1^{-1} \in \mathcal{M}_C \):

\[
\varphi_2 = (\varphi_2 \circ \varphi_1^{-1})(\varphi_1) = \varphi_2(\varphi_1, N).
\]

(54)

Since \( T_{1 \to 2} \) preserves the extensivity of \( \varphi_1 \), hence, \( \varphi_2(\varphi_1, N) \) is a homogeneous function:

\[
\varphi_2(\alpha \varphi_1, \alpha N) = \alpha \varphi_2(\varphi_1, N),
\]

(55)

and therefore, it obeys to the identity:

\[
\varphi_2 = \varphi_1 \frac{\partial \varphi_2}{\partial \varphi_1} + N \frac{\partial \varphi_2}{\partial N}.
\]

(56)

Taking into consideration the definitions (52) and (53), as well as the identity (55), it is straightforward derived the identities:

\[
\beta_{\varphi_2} = \left( \frac{\partial \varphi_1}{\partial \varphi_2} \right) \beta_{\varphi_1},
\]

(57)

\[
K_{\varphi_2} = \left( \frac{\partial \varphi_1}{\partial \varphi_2} \right)^2 [K_{\varphi_1} + \frac{\partial}{\partial \varphi_1} \ln \left( \frac{\partial \varphi_1}{\partial \varphi_2} \right) \beta_{\varphi_1}],
\]

(58)

\[
P_{\varphi_2}^m(\varphi_2, N) = P_{\varphi_1}^m(\varphi_1, N) + N \frac{\partial \varphi_2}{\partial N} \left( \frac{\partial \varphi_1}{\partial \varphi_2} \right) \beta_{\varphi_1}.
\]

(59)

It is evident from (58) that the sign of \( K_\varphi \) is non invariant under the reparametrization changes \( \mathcal{M}_C \). Note that \( \beta_\varphi \) obeys the transformation rule for covariant vectors in a Riemannian geometry. However, \( K_\varphi \) does not obey the correct transformation rule of second-rank covariant tensors. This is the reason why the curvature tensor is not a real tensor, and the ordering information which it contains is non invariant to under the reparametrization changes of \( \mathcal{M}_C \). A question arises: is it possible to choose a representation \( \mathcal{R}_\varphi \) in which the ensemble equivalence takes place? The answer is yes, and we will prove it now.

As already said, when it is used the scaled energy representation \( \mathcal{R}_\mathcal{E} = (\mathcal{E}, N) \), there is an energetic region where the astrophysical systems exhibits a negative heat capacity: \( K_\varphi > 0 \). Let us consider another representation \( \mathcal{R}_\Phi = (\Phi, N) \) which can be parametrized by a bijective function \( \varphi(\epsilon) \) as follows:

\[
\Phi = N \varphi(\epsilon) \text{ with } \epsilon = \mathcal{E}/N.
\]

(60)

In order to disregard the \( N \)-dependence, let us work with the Boltzmann entropy per particle in the thermodynamic limit:

\[
s(\epsilon) = \lim_{N \to \infty} \frac{S_B(\mathcal{E}, N)}{N} = s(\varphi),
\]

(61)

as well the \( N \)-independent thermodynamic variables \( \epsilon \) and \( \varphi = \Phi/N \). In this case, the canonical parameter \( \beta \) and the \( N \)-independent curvature \( k \) are denoted by:

\[
\beta_\epsilon = \partial_\epsilon s, \quad k_\epsilon = \partial_\epsilon \beta_\epsilon,
\]

(62)

in the \( \mathcal{R}_\mathcal{E} \) representation, or

\[
\beta_\varphi = \partial_\varphi s, \quad k_\varphi = \partial_\varphi \beta_\varphi,
\]

(63)

in the \( \mathcal{R}_\Phi \) representation. The notation \( \partial_\epsilon s \) is equivalent to \( ds/d\epsilon \). The expression (62) can be rewritten in this case as follows:

\[
\partial_\varphi \beta_\varphi(\varphi) = (\partial_\epsilon \varphi(\epsilon))^{-2} [\partial_\epsilon \beta_\epsilon(\epsilon) - \beta_\epsilon(\epsilon) \partial_\epsilon \ln(\partial_\varphi \varphi(\epsilon))]
\]

\[
\partial_\varphi \beta_\varphi = -(\partial_\epsilon \varphi(\epsilon))^{-2} a(\epsilon),
\]

(64)

where the function \( a(\epsilon) \):

\[
\partial_\epsilon \beta_\epsilon - \beta_\epsilon \partial_\epsilon \ln(\partial_\varphi \varphi(\epsilon)) = -a(\epsilon).
\]

(65)

should be positive in order to ensure the ensemble equivalence in the \( \mathcal{R}_\Phi \) representation. Direct integration of (64) yields:

\[
\partial_\epsilon \varphi(\epsilon) = C \beta_\epsilon(\epsilon) \exp \left( \int \frac{a(\epsilon)}{\beta_\epsilon(\epsilon)} d\epsilon \right),
\]

(66)

where \( C \) is a positive constant which could be set as unity. It is easy to see that a convenient choice for the function \( a(\epsilon) \), from the theoretical viewpoint, could be given by:

\[
a(\epsilon) = \begin{cases} 
-k_\epsilon(\epsilon) & \text{if } k_\epsilon(\epsilon) < 0, \\
\beta_\epsilon^2(\epsilon) = \beta_\epsilon^2 \sim \text{const} & \text{if } k_\epsilon(\epsilon) = 0, \\
k_\epsilon(\epsilon) & \text{if } k_\epsilon(\epsilon) > 0,
\end{cases}
\]

(67)

which ensures the positivity of \( a(\epsilon) \). Thus, \( \partial_\epsilon \varphi(\epsilon) \) is given by:

\[
\partial_\epsilon \varphi(\epsilon) = \begin{cases} 
1 & \text{if } k_\epsilon(\epsilon) < 0, \\
\beta_\epsilon \exp(\beta_\epsilon \epsilon) & \text{if } k_\epsilon(\epsilon) = 0, \\
\beta_\epsilon^2(\epsilon) & \text{if } k_\epsilon(\epsilon) > 0.
\end{cases}
\]

(68)

According to (64), \( \varphi(\epsilon) \) is an piecewise monotonic function of \( \epsilon \), and therefore, there is a bijective correspondence between \( \epsilon \) and \( \varphi(\epsilon) \). Hence, \( \epsilon \) and \( \varphi \) are equivalent representations, where \( k_\epsilon = \partial_\varphi \beta_\varphi \) is negative; and therefore, the canonical-like description which uses the representation \( \mathcal{R}_\Phi = (\Phi, N) \) is equivalent to the micro-canonical description in the thermodynamic limit. It was shown in this way that it is always possible to choose a representation where the ensemble equivalence is guaranteed.
D. An example: the Antonov problem.

As example, let us consider the classical Antonov problem \[5\]: the self-gravitating gas interacting by means of the Newtonian potential, which is enclosed with a spherical rigid container of radio \(R\). Considering that \(G = M = R = \hbar = 1\), the microcanonical description of this model system is derived from the Planck density:

\[
p_c (\beta, \rho) = \frac{3}{2} \rho \ln (2\pi \beta) + \rho \ln \rho - \rho, \tag{69}
\]

by using the thermodynamic formalism presented in subsection \[II\]. The internal energy density \(\epsilon\) and particles density \(\rho\) are given by:

\[
\epsilon = \frac{3}{2} \beta \rho, \quad \rho = C \exp (\Phi), \tag{70}
\]

where \(C = (2\pi \beta)^{-\frac{3}{2}} \exp (-\mu)\) and \(\Phi = -\beta \phi\), being \(\phi\) the Newtonian potential. The numerical study is carried out by solving the Poisson-Boltzmann equation:

\[
\Delta \Psi = -4\pi \exp (\Psi), \tag{71}
\]

where \(\Psi = \Phi + \ln (\beta C) \equiv \ln (\beta \rho)\), imposing the following conditions at the origin:

\[
\Psi (0) = \psi, \quad \Psi' (0) = 0, \tag{72}
\]

being \(\psi = \ln (\beta \rho_0)\) an integration parameter related with the central density \(\rho_0\) of the system. The canonical parameters \(\beta\) and \(\mu\) are obtained through the relations:

\[
\Psi (1) = \beta + \ln (\beta C), \quad \Psi' (1) = -\beta, \tag{73}
\]

while the entropy and the total energy are obtained from the relations \[KL\] and \[KP\] respectively. It is easy to see that all the thermodynamical variables and potentials are obtained as functions of the parameter \(\psi\).

Figure \[1\] shows the well-known caloric curve of the Antonov problem. The states belonging to the interval \(AB\) exhibit a negative specific heat, and therefore, they are not accessible by using the canonical ensemble.

![FIG. 1: Caloric curve for the Antonov problem. The states belonging to the interval \(AB\) exhibit a negative specific heat, and therefore, they are not accessible by using the canonical ensemble.](image1)

Figure \[2\] shows the dependence between the thermodynamical variables \(\epsilon\) and \(\psi\). Disregarding the unstable branch, it is easy to see that there is a univocal correspondence between these variables for all equilibrium states (\(ABC\) branch).

![FIG. 2: The curve \(\psi - \epsilon\). There is a univocal correspondence between these variables for all equilibrium states (\(ABC\) branch).](image2)

According to all the exposed in subsection above, the Antonov problem obeys to the thermodynamic limit \[KL\], and therefore, we are able to choose an appropriate representation for the microcanonical variables in order to guarantee that the canonical-like ensemble associated with this new representation be equivalent to the microcanonical one.

Figure \[2\] shows the dependence between the thermodynamical variables \(\epsilon\) and \(\psi\). Disregarding the unstable branch, it is easy to see that there is a univocal correspondence between these variables for all equilibrium states. We can select an appropriate representation \(\varphi = \varphi (\epsilon)\) for the microcanonical description by rephrasing \(\psi\), or more exactly, \(\chi = \exp (\psi)\) as a canonical-like parameter. The map \(\epsilon \rightarrow \varphi (\epsilon)\) can be established by using the transfor-
mation rule \(\text{57}\) for the canonical parameters:

\[
\chi = \exp (\psi) = \exp \left( \frac{\partial \epsilon}{\partial \varphi} \beta \right) = \beta (\psi) \exp \left( -\psi \right) \frac{d \epsilon (\psi)}{d \psi}.
\]

(74)

This map is shown at figure 3. The bijective character of this representation change assures the reparametrization invariance of Physics in the microcanonical ensemble. Figures 4 and 5 show the “caloric curve” \(\chi - \varphi\) and the entropy function \(s - \varphi\) in the \(\varphi\) representation for the microcanonical variable \(\epsilon\). The first shows the equivalence between the canonical-like ensemble with the microcanonical ones, while the second one confirms the concavity of the entropy function in this representation.

Thus, the physical observables obtained from the canonical-like probabilistic distribution function:

\[
\omega_c (E ; \chi, N) = \frac{1}{Z_\varphi (\chi, N)} \exp \left[ -N \chi \cdot \varphi \left( \frac{E}{N \beta} \right) \right],
\]

(75)

are equal to those obtained by using the microcanonical description in the thermodynamic limit \(\text{51}\). The numerical solution of this model exposed above was obtained by considering fixed the parameter \(\psi = \ln \chi\), and this supposition allows us to determine all the thermodynamical observables of this model system. Therefore, we are able to affirm that this numerical solution was obtained by using the \(\chi\)-canonical-like description.

It is remarkable that this generalized ensemble allows us to access to all equilibrium states of the model, even in the cases in which these last ones exhibit a negative specific heat. This is an important difference regarding to the usual canonical ensemble. Although the similar appearance between these two descriptions, the simple representation change of the microcanonical variables (compatible with the scaling laws of the thermodynamical variables and potentials in the thermodynamic limit) allows the extension of a canonical-like description to those cases in which the usual Gibbs’ canonical one fails.

**IV. CONCLUDING REMARKS**

We have reconsidered some aspects about the microcanonical mean field description for astrophysical Hamiltonian systems. Particularly, we derived a formalism which takes into account those cases is which the par-
particles also interact by means of short-range forces. This general framework allows us to perform the analysis of the necessary conditions for the existence of a power law self-similarity behavior of the microcanonical variables of the form \( \xi \), which is an extension of extensivity of the traditional systems. This study revealed that this property could be only satisfied by a reduced set of models where the nature of the short-range interactions leads to the relation \( p = \gamma \xi \) between the pressure \( p \) and the internal energy per volume \( \xi \), being \( \gamma \) certain constant.

As already shown, the consideration of the reparametrization invariance of the microcanonical ensemble allows us to extend the applicability of a canonical-like description to those situations where the standard Thermodynamics based on the canonical ensemble is not able to describe, for instance, when the systems exhibit a negative specific heat. This possibility was illustrated in this paper by reconsidering the classical isothermal model of Antonov \[5\].

The physical significance of such generalized canonical-like ensemble relies on the reparametrization invariance of Physics in the microcanonical ensemble as well as the equivalence of these descriptions by the consideration of an appropriate thermodynamic limit. It is important to remark that such thermodynamic limit is derived from the scaling properties of the thermodynamical variables and potentials of a system with a large number of degrees of freedom, and therefore, the same one can not be arbitrarily introduced.

An interesting application field of this geometrical ideas could be in the Montecarlo methods based on the canonical weight \( \exp (-\beta E) \). It is well-known that these numerical methods are more simple than those based on the microcanonical ensemble, but the first ones diverge when the ensemble equivalence is not ensured. Therefore, the consideration of a generalized canonical-like weight of the form \( \exp \left[ -N \chi \cdot \varphi \left( E/N^{\Lambda_E} \right) \right] \) (where \( \Lambda_E \) is the scaling exponent of energy in the thermodynamic limit \( N \to \infty \)), could guarantee the convergence of such methods for all the energetic range.

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