ON A GARSIDE GROUP EXTENDING THE BRAID GROUP

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Dedicated to the memory of Patrick Dehornoy.

Abstract. The submonoid of the 3-strand braid group $B_3$ generated by $\sigma_1$ and $\sigma_1\sigma_2$ is known to yield an exotic Garside structure on $B_3$. We introduce and study an infinite family $(M_n)_{n \geq 1}$ of Garside monoids generalizing this exotic Garside structure, i.e., such that $M_2$ is isomorphic to the above monoid. The corresponding Garside group $G_n$ is an extension of the $(n + 1)$-strand braid group, isomorphic to $B_3$ when $n = 2$ and to the complex braid group of the complex reflection group $G_{12}$ when $n = 3$. In general, the Garside monoid $M_n$ sujects onto the submonoid $\Sigma_n$ of $B_{n+1}$ generated by $\sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_n$, which is not a Garside monoid when $n > 2$. Using a new presentation of $B_{n+1}$ similar to the presentation of $G_n$, we nevertheless check that $\Sigma_n$ is an Ore monoid with group of fractions isomorphic to $B_{n+1}$, and give a conjectural presentation of it. We also show that the groups $G_n$ are isomorphic to groups defined by a cyclic relation; such groups were shown to be Garside groups by Dehornoy and Paris. As a byproduct the monoid $M_n$ yields a new Garside structure for these groups.

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1. INTRODUCTION

The braid group on $n$ strands is one of the most basic example of a Garside group. Garside groups, originally introduced by Dehornoy and Paris [18] following an original idea of Garside [22], are defined as groups of fractions of certain monoids, called Garside
monoids, which have enough properties to ensure that every element of the group can be written uniquely as an irreducible fraction in two elements of the monoid. Computable normal forms for elements of these monoids can be defined, allowing one to effectively compute such fractions, which in particular yields a solution to the word problem in these groups. Garside groups also have many other properties. For example, they are torsion-free, and have a solvable conjugacy problem: see Section 2 below for basic definitions and properties of Garside monoids and groups, and [17] for more on the topic.

While the word problem in the \( n \)-strand braid group has been known to be solvable since Artin’s original paper [1] and several other approaches have been shown to be fruitful in between (see [6, Section 5] for a survey), Garside’s approach allowed him to get the first solution to the conjugacy problem in this group, and his results could be generalized to get a uniform solution to these questions in Artin-Tits groups of spherical type [9] [19], i.e., Artin-Tits groups attached to finite Coxeter groups (see [24, Section 6.6] for an introduction to the topic). It also provides new proofs that Artin-Tits groups of spherical type are torsion-free, and allows one to determine their center. One can also note that Garside normal forms can be used to show faithfulness of (linear, and more recently categorical) representations of Garside groups [25, 8, 23, 26]. Roughly speaking, Garside groups are groups satisfying a set of axioms that ensures that generalizations of the techniques of Garside can be applied to solve the above-mentioned problems.

In general, the Garside group does not determine an associated Garside monoid, i.e., several non-isomorphic Garside monoids may have isomorphic group of fractions. Up to now, it seems that very few classification results of Garside monoids for a given Garside group are known. In the case of the \( n \)-strand braid group, Garside’s original paper yields a so-called classical Garside monoid, which is nothing but the positive braid monoid, while Birman, Ko and Lee [7] discovered a second Garside monoid, which strictly contains the first one, and is generated by a copy of the set of transpositions of the symmetric group. Bessis, Digne and Michel [5] generalized this monoid to Artin-Tits groups of Coxeter type \( B_n \), and then Bessis gave a generalization of these constructions, called dual braid monoid, which is valid for every Artin-Tits group attached to a finite Coxeter system [2], and even to complex braid groups of well-generated complex reflection groups [4]. Following Bessis’ approach, some Artin-Tits groups of non-spherical type were also shown to be (quasi-)Garside groups [20, 21, 3].

Birman and Brendle asked if there exist other Garside monoids for the \( n \)-strand braid group (see [6, Open Problem 10], where it is also claimed that it is very likely that the classical and dual presentations of \( B_{n+1} \) are the only presentations yielding a Garside monoid). At the time of writing of this paper, it seems that the only known Garside monoids which can be defined for the \( n \)-strand braid group for all \( n \geq 1 \) are still the classical and the dual braid monoids.

Nevertheless, for \( n = 3 \), several exotic Garside monoids for the 3-strand braid group \( B_3 \) were discovered (see [17, Section IX.2.4] for a survey). It is natural to wonder whether these monoids admit analogues in higher rank or if they should be considered as some sort of sporadic monoids only arising in low rank. The aim of this paper is to give an answer to this question for one such exotic Garside structure on \( B_3 \), given by the submonoid generated by \( \sigma_1, \sigma_1 \sigma_2 \). This monoid admits the presentation with generators \( a = \sigma_1, b = \sigma_1 \sigma_2 \) subject to a single relation \( aba = b^2 \). It was the first example of a Garside monoid where the lcm of the atoms is not equal to the Garside element (see [13, Exemple 1.5]). Indeed, in this Garside monoid, the left-lcm of \( a \) and \( b \) is \( b^2 \), while the Garside element \( \Delta \) is \( b^3 \) (the lattice of divisors of \( \Delta \) under left-divisibility is given in Figure 1). In fact, in the original paper [18], it was a requirement for the
Garside element \( \Delta \) to be the lcm of the atoms, but this condition was slightly relaxed in [14], and is not required anymore in the definition of Garside monoid which is used nowadays.

The submonoid of \( B_3 \) mentioned above admits a natural generalization to \( B_{n+1} \), \( n \geq 2 \), given by the submonoid \( \Sigma_n \subseteq B_{n+1} \) generated by \( \sigma_1, \sigma_1\sigma_2, \ldots, \sigma_1\sigma_2\cdots\sigma_n \).

In [17, Chapter IX, Question 29], the following question is asked:

**Question 1.1.** Does the submonoid \( \Sigma_n \) admit a finite presentation? Is it a Garside monoid?

A positive answer to the last question would in particular yield a new Garside structure on \( B_{n+1} \), generalizing the exotic Garside structure given by \( \Sigma_2 \) on \( B_3 \). Unfortunately, the submonoid \( \Sigma_n \) is not a Garside monoid when \( n > 3 \): in fact, as already noticed by Dehornoy before Question 1.1 was asked, this monoid does not have lcm’s (as follows easily from [15, Example 3.7]). But we shall show that there is a suitable extension \( \mathcal{G}_n \) of \( B_{n+1} \), isomorphic to it when \( n = 1 \) and \( n = 2 \), which is a Garside group with Garside monoid generalizing this exotic Garside structure.

More precisely, let \( n \geq 1 \) and let \( M_n \) be the monoid with generators \( \rho_1, \rho_2, \ldots, \rho_n \) and relations \( \rho_1\rho_n\rho_i = \rho_{i+1}\rho_n \) for all \( 1 \leq i \leq n - 1 \). Then our main results can be summarized as follows (see Theorem 3.18, Propositions 3.3, 3.21, and 4.2, and Corollaries 3.17, 3.20, and 4.3 below)

**Theorem 1.2.** We have

1. The monoid \( M_n \) is a Garside monoid, with Garside element \( \Delta = \rho_n^{n+1} \), and (left- or right-) lcm of the atoms \( \rho_n^m \).
2. The Garside group \( \mathcal{G}_n \) obtained as group of fractions of \( M_n \) is an extension of the braid group \( B_{n+1} \) on \( n+1 \) strands. For \( n = 1 \) and \( n = 2 \) we have \( \mathcal{G}_n \cong B_{n+1} \), while for \( n > 2 \) the surjection is proper.
3. The image of \( M_n \) in \( B_{n+1} \) under the above surjection is the submonoid \( \Sigma_n \). In particular \( M_2 = \Sigma_2 \).
4. The center of \( \mathcal{G}_n \) is infinite cyclic, generated by \( \Delta = \rho_n^{n+1} \) for \( n \geq 2 \) and by \( \rho_1 \) for \( n = 1 \).
5. The group \( \mathcal{G}_n \) is isomorphic to the group \( \mathcal{C}_n \) defined by the presentation with generators \( x_1, x_2, \ldots, x_n \) and relations
   \[
   x_1x_2\cdots x_n x_1 = x_2x_3\cdots x_n x_1 x_2 = x_3x_4\cdots x_n x_1 x_2 x_3 = \cdots = x_n x_1 x_2 \cdots x_n.
   \]
As a corollary, the group $G_3$ is isomorphic to the braid group of the complex reflection group $G_{12}$.

Note that the complex reflection group $G_{12}$ is not well-generated, hence its braid group does not admit a dual braid monoid; nevertheless this group was already known to be a Garside group as the groups with the presentation given in point (4) of the above Theorem were shown to be Garside groups by Dehornoy and Paris [18, Proposition 5.2], using a monoid which is different from ours. Hence as a byproduct our construction gives a new Garside structure on these groups.

Coming back to Question 1.1, one can define a presentation of the braid group $B_{n+1}$ which is closely related to that of $M_n$. Namely, let $H_n$ (respectively $H_n^+$) be the group (respectively the monoid) defined by the presentation with generators $\rho_1, \rho_2, \ldots, \rho_n$ and relations $\rho_1\rho_j \rho_i = \rho_{i+1}\rho_j$ for all $1 \leq i < j \leq n$. Then we show (see Propositions 5.2 and 5.3)

**Proposition 1.3.** We have

1. The submonoid $\Sigma_n$ of $B_{n+1}$ is an Ore monoid with group of fractions isomorphic to $B_{n+1}$.
2. We have $H_n \cong B_{n+1}$. The image of $H_n^+$ inside $B_{n+1}$ is $\Sigma_n$.

We then conjecture the following (see Conjecture 5.4 below for a more precise statement)

**Conjecture 1.4.** The monoid $H_n^+$ is cancellative. As a corollary, we have $H_n^+ \cong \Sigma_n$, and $\Sigma_n$ admits a finite presentation.

This would positively answer the first part of Question 1.1. Note that in the particular case $n = 3$, Dehornoy asked whether $H_3^+$ is (right-)cancellative and embeds into its group of fractions (see [15, Question 3.8]).

The paper is organized as follows: Section 2 is devoted to recalling definitions and properties of Garside monoids and groups, and collecting a few general results which are used later on. In Section 3 we introduce the monoids $M_n$, give several presentations of them, and show that they are Garside monoids using the so-called reversing approach. We also determine the center of the corresponding Garside group $G_n$. In Section 4 we establish isomorphisms between $G_n$ and groups defined by a cyclic relation, and derive a few applications. In Section 5 we explore the link between $G_n$ and $B_{n+1}$ and give a few properties as well as a conjectural presentation of the submonoid $\Sigma_n$ of $B_{n+1}$. The last Section 6 is devoted to showing that Artin groups of odd dihedral type can be endowed with a Garside structure that is analogous to the one given by $M_n$.

**Acknowledgements.** The author thanks Ivan Marin, Jean Michel, Matthieu Picantin, and Baptiste Rognerud for useful discussions.

2. Garside monoids and groups

The aim of this section is to recall a few basic results on Garside monoids and Garside groups for later use. We mostly adopt the definitions and conventions from [17]. Note that, while *loc. cit.* introduces most of the results used in this paper in the general framework of Garside categories, we will only need them in the case of presented monoids, and therefore reproduce them here in this less general context for the comfort of the reader. We also include proofs of a few basic results.
2.1. Definitions and properties. Every monoid considered in this paper has a unit element 1. Let \( M \) be such a monoid.

**Definition 2.1** (Divisors and multiples). Let \( a, b, c \in M \). If \( ab = c \), we say that \( a \) is a left-divisor (respectively, that \( b \) is a right-divisor of \( c \)) of \( c \) and that \( c \) is a right-multiple of \( a \) (respectively a left-multiple of \( b \)).

**Definition 2.2** (Cancellativity). We say that \( M \) is left-cancellative (respectively right-cancellative) if for all \( a, b, c \in M \), the equality \( ab = ac \) (resp. \( ba = ca \)) implies \( b = c \). If \( M \) is both left- and right-cancellative then we simply say that \( M \) is cancellative.

**Theorem 2.3** (Ore’s Theorem). If \( M \) is cancellative, and if any two elements \( a, b \in M \) admit a common left-multiple, that is, if there is \( c \in M \) such that \( d'a = c = b'b \) for some \( a', b' \in M \), then \( M \) admits a group of fractions \( G(M) \) in which it embeds. If \( \langle S, R \rangle \) is a presentation of the monoid \( M \), then \( \langle S, R \rangle \) is a presentation of \( G(M) \).

A proof of this Theorem can be found for instance in [11, Section 1.10].

**Definition 2.4** (Ore monoid). A monoid satisfying the assumptions of Theorem 2.3 is an *Ore monoid*.

**Lemma 2.5.** If \( M \) is left-cancellative (respectively right-cancellative) and 1 is the only invertible element in \( M \), then the left-divisibility (resp. right-divisibility) relation on \( M \) is a partial order.

*Proof.* Reflexivity is clear as \( M \) has a unit 1 and transitivity is also clear (and both hold without the cancellativity assumption and without the assumption on invertible elements). Let \( a, b \in M \) such that \( a \) left-divides \( b \) and \( b \) left-divides \( a \). Then there are \( c, c' \in M \) such that \( ac = b \) and \( bc' = a \). Hence \( b = ac = bc'c \). By left-cancellativity this implies that \( c'c = 1 \), hence \( c = 1 = c' \) as 1 is the only invertible element in \( M \). Hence \( a = b \), and the left-divisibility relation is reflexive. The proof for the right counterparts is similar. \(\square\)

**Definition 2.6** (Noetherian divisibility). We say that the divisibility in \( M \) is *Noetherian* if there exists a function \( \lambda : M \to \mathbb{Z}_{\geq 0} \) such that \( \forall a, b \in M, \lambda(ab) \geq \lambda(a) + \lambda(b) \) and \( a \neq 1 \Rightarrow \lambda(a) \neq 0 \). We say that \( M \) is *right-Noetherian* (respectively *left-Noetherian*) if every strictly increasing sequence of divisors with respect to left-divisibility (resp. right-divisibility) is finite. Note that if the divisibility in \( M \) is Noetherian, then \( M \) is both left- and right-Noetherian.

Note that it implies that the only invertible element in \( M \) is 1 and that \( M \) is infinite if \( M \neq \{1\} \). In particular, by Lemma 2.5 in a cancellative monoid \( M \) with Noetherian divisibility, both left-divisibility and right-divisibility induce a partial order on \( M \).

**Definition 2.7** (Garside monoid). A *Garside monoid* is a pair \((M, \Delta)\) where \( M \) is a monoid with 1 and \( \Delta \) is an element of \( M \), satisfying the following five conditions

1. \( M \) is left- and right-cancellative,
2. The divisibility in \( M \) is Noetherian,
3. Any two elements in \( M \) admit a left- and right-lcm, and a left- and right-gcd,
4. the left- and right-divisors of the element \( \Delta \) coincide and generate \( M \),
5. The set of (left- or right-)divisors of \( \Delta \) is finite.

Note that under these assumptions, the restrictions of left- and right-divisibility to the set of divisors of \( \Delta \) yield two lattice structures on this set.
In general, checking the above five conditions is a nontrivial task, especially for the left- and right-cancellativity. But these conditions have strong implications. We list some of them below, and refer the reader to [17] for complete proofs.

Let $M$ be a Garside monoid. Firstly, by Ore’s Theorem, we have that $M$ embeds into its group of fractions $G(M)$.

**Definition 2.8 (Garside group).** A group $G$ is a **Garside group** if $G = G(M)$ for some Garside monoid $M$.

Secondly, one can define normal forms for elements of $M$ as products of divisors of the Garside element: let $a \in M$. As $M$ has gcd’s, let $x_1 = \gcd(a, \Delta)$ (we consider left-gcd’s here). Hence $a = x_1y_1$, and $x_1$ is the greatest divisor of $\Delta$ which also left-divides $a$. By cancellativity, the element $y_1$ is uniquely determined, and one can go on, considering the greatest left-divisor $x_2$ of $a_1$ which also divides $\Delta$. We then write $a = x_1x_2y_2$. In this way, we get a uniquely defined sequence of divisors of $\Delta$, and as the divisibility is Noetherian in $M$, this sequence is finite. At the end we get a uniquely defined expression $a = x_1x_2 \cdots x_k$ as product of divisors of $\Delta$. This normal form is called the (left-)**Garside normal form** of $a$. It can be effectively calculated provided that left-gcd’s of the form $\gcd(xy, \Delta)$, where $x$ and $y$ are divisors of $\Delta$, can be calculated. Indeed, one can show that for $x, y \in M$, one has $\gcd(xy, \Delta) = \gcd(x(\gcd(y, \Delta)), \Delta)$; this allows to calculate the normal form of $a$ starting from any expression of $a$ as a product of divisors of $\Delta$ (which generate $M$). Namely if $a = a_1a_2 \cdots a_k$ with $a_i$ dividing $\Delta$ for all $i$, then an iterated application of the above formula reduces the computation of the first factor of the Garside normal form to an iterated calculation of gcd’s of the above form. Similarly, one can define a right-Garside normal form.

Thirdly, the important point about (left) normal forms in $M$ is that they can be used, in the case where $M$ and $G(M)$ are defined by generators and relations, to give a solution to the word problem in $G(M)$. We say that the word problem in a (finitely generated) group $G$ is **solvable** if there is an algorithm which allows one to determine in finite time whether a word in the generating set represents the identity or not. If $G(M)$ is a Garside monoid, then it can be checked that every element of $G(M)$ can be written uniquely as an irreducible fraction $x^{-1}y$ with $x, y \in M$, which can be calculated using the left-normal form in $M$. The normal form can also be used to give a solution to the conjugacy problem in Garside groups.

Finally, it can also be shown that every Garside group $G(M)$ is torsion-free, and that a power of $\Delta$ is central in $G(M)$—hence in particular, that the center of $G(M)$ is not trivial.

In Sections 2.2 and 2.3 we will recall a few existing tools for checking some of the conditions of Definition 2.7 in the case of presented monoids.

**Example 2.9.** The seminal example is given by braid groups, or more generally Artin-Tits groups of spherical type (i.e., attached to a finite Coxeter system $(W, S)$). Let $n \geq 1$. Recall that the $(n+1)$-strand braid group $B_{n+1}$ has a presentation by generators and relations with $n$ generators $\sigma_1, \sigma_2, \ldots, \sigma_n$ and relations
\[
\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \quad \forall i = 1, \ldots, n-1, \\
\sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{whenever } |i - j| > 1.
\]
A possible Garside monoid $M$ such that $G(M) \cong B_{n+1}$ is given by the positive braid monoid $B^+_{n+1}$ defined by the same presentation (but as monoid) as the one given above. The element $\Delta$ is given by the half-twist. This is the classical Garside structure on $B_{n+1}$. An alternative Garside monoid $M'$ such that $G(M') \cong B_{n+1}$ is given by the Birman-Ko-Lee braid monoid [7] (or *dual* braid monoid [2]). In this case, the monoid
The two Garside structures (classical and dual) given in Example 2.9 are the only known Garside structures on $B_{n+1}$ which can be defined for all $n \geq 1$. Whether there exist other Garside structures that can be defined for all $n \geq 1$ or not is an open problem. For $n = 3$, there are a few known exotic Garside structures (see [17] Section X.2.4). In this case, the classical braid monoid $B_3^+$ has generators $\sigma_1, \sigma_2$ and element $\Delta$ given by $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 = \Delta$ (the half-twist). The dual braid monoid $B_3^-$ has generators $\sigma_1, \sigma_2, \sigma_1 \sigma_2 \sigma_1^{-1}$ and element $\Delta$ given by $\sigma_1 \sigma_2$. An alternative Garside monoid is given by the submonoid $\Sigma_2$ with generators $\rho_1 := \sigma_1, \rho_2 := \sigma_1 \sigma_2$ and element $\Delta$ given by $(\sigma_1 \sigma_2 \sigma_1)^2 = \rho_2^3$. A presentation of $\Sigma_2$ is given by the single relation $\rho_1 \rho_2 \rho_1 = \rho_2^3$.

2.2. Cancellativity criterions for presented monoids. This section is devoted on recalling some known cancellativity criterions for presented monoids which will be used in Section 3. We recall them from [17] Section II.4 (extending approaches from [14]; see also [16] for more recent results). Most of the definitions given in this section are also borrowed from [17].

Recall that all the monoids that we consider in this paper have a unit element $1$. Assume that $M$ is such a monoid, defined by a presentation $\langle S, R \rangle$, where $S$ is a finite set of generators and $R$ a set of relations between words in $S^*$, i.e., words with letters in the generating set $S$.

Definition 2.11 (Right-complemented presentation). The presentation $\langle S, R \rangle$ is right-complemented if $R$ contains no relation where one side is equal to the empty word, no relation of the form $s \cdots = s \cdots$ with $s \in S$, and if for $s \neq t \in S$, there is at most one relation of the form $s \cdots = t \cdots$ in $R$.

Example 2.12. The classical presentation of the $(n + 1)$-strand braid group that we recalled in Example 2.9 is right-complemented. In fact every defining presentation of an Artin-Tits group is right-complemented.

Given a right-complemented presentation $\langle S, R \rangle$ of a monoid $M$, there is a uniquely determined partial map $\theta : S \times S \rightarrow S^*$ such that $\theta(s, s) = 1$ for all $s \in S$ and for $s \neq t \in S$, the words $\theta(s, t)$ and $\theta(t, s)$ are defined whenever there is a relation $s \cdots = t \cdots$ in $R$, and are such that this relation is given by $s \theta(s, t) = t \theta(t, s)$. The map $\theta$ is the syntactic right-complement of the right-complemented presentation $\langle S, R \rangle$.

If $\langle S, R \rangle$ is right-complemented, then by [17] Lemma 4.6, the map $\theta$ admits a unique minimal extension to a partial map from $S^* \times S^*$ to $S^*$ which we still denote $\theta$, and satisfying

\begin{align}
\theta(s, s) &= 1, \quad \forall s \in S, \\
\theta(ab, c) &= \theta(b, \theta(a, c)), \quad \forall a, b, c \in S^*, \\
\theta(a, bc) &= \theta(a, b) \theta(b, c), \quad \forall a, b, c \in S^*, \\
\theta(1, a) &= a \quad \text{and} \quad \theta(a, 1) = 1, \quad \forall a \in S^*.
\end{align}

Definition 2.13 (Cube condition). Given a right-complemented presentation $\langle S, R \rangle$ of a monoid $M$ with syntactic right-complement $\theta$, we say that the $\theta$-cube condition holds (respectively that the sharp $\theta$-cube condition holds) for a triple $(a, b, c) \in (S^*)^3$ if either both $\theta(\theta(a, b), \theta(a, c))$ and $\theta(\theta(b, a), \theta(b, c))$ are defined and represent words in
\( S^* \) that are equivalent under the set of relations \( \mathcal{R} \) (resp. that are equal as words), or neither of them is defined.

**Definition 2.14** (Conditional lcm). We say that a left-cancellative (respectively right-cancellative) monoid \( M \) with no nontrivial invertible elements **admits conditional right-lcms** (resp. **admits conditional left-lcms**) if any two elements of \( M \) that admit a common right-multiple (resp. left-multiple) admit a common right-lcm (resp. left-lcm).

**Proposition 2.15** (see [17, Proposition II.4.16]). If \( \langle S, \mathcal{R} \rangle \) is a right-complemented presentation of a monoid \( M \) with syntactic right-complement \( \theta \), and if \( M \) is right-Noetherian and the \( \theta \)-cube condition holds for every triple of pairwise distinct elements of \( S \), then \( M \) is left-cancellative, and admits conditional right-lcms. More precisely, \( u \) and \( v \) admit a common right-multiple if and only if \( \theta(u, v) \) exists and, then, \( u\theta(u, v) = v\theta(v, u) \) represents the right-lcm of these elements.

The presentation of the braid group (see Example 2.9) again satisfies the assumptions of the above Proposition: for more details and an explicit check of the \( \theta \)-cube condition, we refer the reader to [17, Example II.4.20].

For later use we also state the following result

**Lemma 2.16** (see [17, Lemma II.2.22]). If \( M \) is cancellative and admits conditional right-lcms (respectively left-lcms), then any two elements \( u, v \) of \( M \) admit a common right-multiple (resp. right-lcm).

### 2.3. Garside elements and induced lattices.

Most of the content of this section is folkloric. We include proofs for the sake of completeness.

**Lemma 2.17.** Let \( M \) be a cancellative monoid with no nontrivial invertible element (so that left- and right-divisibility relations are partial orders). Assume that \( M \) has conditional (left- and right-) lcms, and that \( M \) has an element \( \Delta \) satisfying the following assumptions

- The sets of left and right-divisors of \( M \) coincide, and form a finite set,
- The set of divisors of \( \Delta \) generate \( M \).

Then any two elements \( x, y \in M \) admit a left-lcm and a right-lcm.

**Proof.** As \( M \) has conditional lcms, it suffices to show that any two elements \( x, y \in M \) have a (left- or right-)common multiple. We show that \( x, y \) have a common right-multiple (the proof for left-multiples is similar). Note that under our assumptions, if \( z \) is any divisor of \( \Delta \), then \( z\Delta = \Delta z' \) for some divisor \( z' \) of \( \Delta \).

Let \( x = x_1x_2 \cdots x_k, y = y_1y_2 \cdots y_l \) where the \( x_i \)'s and \( y_i \)'s are divisors of \( \Delta \). Without loss of generality we can assume that \( l \leq k \). Then we claim that \( \Delta^k \) is a common right-multiple of \( x \) and \( y \). To this end, it suffices to show that if \( a = a_1a_2 \cdots a_m \) is an element of \( M \) which is a product of \( m \) divisors \( a_i \) of \( \Delta \), then \( a \) is a left-divisor of \( \Delta^m \).

As \( a_1 \) is a left-divisor of \( \Delta \) and left and right-divisors of \( \Delta \) coincide, we can write \( \Delta^m = a_1\Delta^{m-1}b_1 \), where \( b_1 \) is a divisor of \( \Delta \). Iterating, we eventually end up with a decomposition \( \Delta^m = a_1a_2 \cdots a_mb_m \cdots b_2b_1 \), which concludes the proof. \( \square \)

**Definition 2.18** (Garside element). If \( M \) and \( \Delta \) satisfy the assumptions of the above Lemma, we say that \( \Delta \) is a **Garside element** in \( M \). In this case we denote by \( \text{Div}(\Delta) \) the set of left-divisors of \( \Delta \) (which is equal to the set of right-divisors of \( \Delta \)). We call its elements the **simples** of \( M \).

Note that if the conditions in Lemma 2.17 are satisfied, then the set of divisors of the Garside element \( \Delta \), endowed with the restriction of the left-divisibility (which is a
Figure 2. The lattice of simples (for left-divisibility) in three different Garside monoids for \( B_3 \), expressed in terms of the classical Artin generators of \( B_3 \). The lattice for the classical Garside structure is on the left, the one for the dual Garside structure in the middle, and the one for the exotic Garside structure discussed in Example 2.10 on the right.

The same holds for the restriction of right-divisibility. In general these two lattices are not isomorphic. We shall see an example of this phenomenon in Remark 3.19 below (note that in the three examples depicted in Figure 2, they are isomorphic).

Nevertheless, we have:

**Lemma 2.19.** Let \( M \) and \( \Delta \) satisfying the assumptions of Lemma 2.17. Let \( \leq_L \) (respectively \( \leq_R \)) be the partial order induced by left-divisibility on \( \text{Div}(\Delta) \) (respectively by right-divisibility). Then the map \( x \mapsto \Delta x^{-1} \) is an isomorphism of lattices \((\text{Div}(\Delta), \leq_L) \cong (\text{Div}(\Delta), \leq_R)^{\text{op}}\). In other words, the lattice \((\text{Div}(\Delta), \leq_L)\) is isomorphic to the dual of the lattice \((\text{Div}(\Delta), \leq_R)\).

**Proof.** The fact that \( x \mapsto \Delta x^{-1} \) is well-defined is clear, as left- and right-divisors of \( \Delta \) coincide and \( M \) is cancellative. It is invertible, with inverse given by \( y \mapsto y^{-1} \Delta \). It remains to show that both \( x \mapsto \Delta x^{-1} \) and its inverse are order-preserving. Let \( x, y \in \text{Div}(\Delta) \) such that \( x \leq_L y \). Then there is \( a \in \text{Div}(\Delta) \) such that \( xa = y \). As \( a \in \text{Div}(\Delta) \), there is \( b \in \text{Div}(\Delta) \) such that \( ba = \Delta \). Similarly, as \( b \in \text{Div}(\Delta) \), there is \( c \in \text{Div}(\Delta) \) such that \( cb = \Delta \). We then have \( c \Delta y^{-1} = c \Delta a^{-1} x^{-1} = cbx^{-1} = \Delta x^{-1} \), which shows that \( \Delta y^{-1} \leq_R \Delta x^{-1} \), hence that \( x \mapsto \Delta x^{-1} \) is order-preserving.

The proof that the inverse map \( y \mapsto y^{-1} \Delta \) is also order-preserving is similar. \( \square \)

**Corollary 2.20.** Let \( \Delta \) be a Garside element in \( M \). Then
\[(\text{Div}(\Delta), \leq_L) \text{ is self-dual } \iff (\text{Div}(\Delta), \leq_L) \cong (\text{Div}(\Delta), \leq_R) \]
\[(\text{Div}(\Delta), \leq_R) \text{ is self-dual.} \]

### 3. A Garside group extending the braid group

In this section, we now define our main object of study.

#### 3.1. Definition and several presentations.
Definition 3.1. Let \( n \geq 1 \). We denote by \( G_n \) the group defined by the presentation with generators \( \rho_1, \rho_2, \ldots, \rho_n \) and relations
\[
\rho_i \rho_n \rho_i = \rho_{i+1} \rho_n, \quad \forall 1 \leq i \leq n - 1.
\]
We denote by \( M_n \) the monoid (with 1) defined by the same presentation. We will denote the set of generators of both the group and the monoid by \( S \), and the above set of relations by \( R \), omitting the dependency on \( n \).

Example 3.2. The group \( G_1 \) is isomorphic to \( \mathbb{Z} \). The group \( G_2 \) has generators \( \rho_1, \rho_2 \) and a single relation \( \rho_1 \rho_2 \rho_1 = \rho_2^2 \). It is isomorphic to the braid group \( B_3 \) on 3-strands via \( \rho_1 \mapsto \sigma_1, \rho_2 \mapsto \sigma_1 \sigma_2 \).

In fact, the group \( G_n \) is an extension of \( B_{n+1} \):

Proposition 3.3. The assignment \( \rho_i \mapsto \sigma_1 \sigma_2 \cdots \sigma_i, \forall i = 1, \ldots, n \) extends to a surjective group homomorphism \( \varphi_n : G_n \rightarrow B_{n+1} \).

This map is not an isomorphism for \( n \geq 3 \), as we shall prove in Corollary 3.20 below.

Proof of Proposition 3.3. It suffices to show that the elements \( S_i := \sigma_1 \sigma_2 \cdots \sigma_i \in B_{n+1} \) \( (i = 1, \ldots, n) \) satisfy the defining relations of \( G_n \). We show that \( S_1 S_n S_i = S_{i+1} S_n \) for all \( i = 1, \ldots, n - 1 \) by induction on \( 1 \leq i \leq n - 1 \). We have
\[
S_1 S_n S_i = \sigma_1 (\sigma_1 \sigma_2 \cdots \sigma_i) \sigma_1 = \sigma_1 \sigma_1 \sigma_2 \sigma_1 (\sigma_3 \sigma_4 \cdots \sigma_n)
\]
\[
= \sigma_1 \sigma_2 (\sigma_1 \sigma_2 \sigma_3 \cdots \sigma_n) = S_2 S_n,
\]
hence the result holds for \( i = 1 \). Now let \( 1 < i \leq n - 1 \). By induction we have
\[
S_1 S_n S_i = S_1 S_n S_{i-1} \sigma_i = S_i S_n \sigma_i = S_i (\sigma_1 \sigma_2 \cdots \sigma_n) \sigma_i
\]
\[
= S_i (\sigma_1 \sigma_2 \cdots \sigma_{i+1}) \sigma_i \sigma_{i+2} \cdots \sigma_n
\]
\[
= S_i (\sigma_1 \sigma_2 \cdots \sigma_{i-1}) \sigma_{i+1} (\sigma_i \sigma_{i+1} \cdots \sigma_n) = S_i S_{i+1} S_n = S_{i+1} S_n,
\]
which concludes the proof.

Lemma 3.4. The map \( \lambda : \{\rho_1, \rho_2, \ldots, \rho_n\} \rightarrow \mathbb{Z}_{\geq 0}, \rho_i \mapsto i \) extends to a uniquely defined length function \( \lambda \) on \( M_n \) such that \( \lambda(ab) = \lambda(a) + \lambda(b) \), for all \( a, b \in M_n \). In particular, the divisibility in \( M_n \) is Noetherian, and \( M_n \) is both left- and right-Noetherian.

Proof. It suffices to show that the extension of \( \lambda \) to \( S^* \) takes the same value on each side of any given relation in \( R \), in other words, that the relations in \( R \) are homogeneous with respect to \( \lambda \). This is clear, as \( \lambda(\rho_1 \rho_n \rho_i) = n+i+1 = \lambda(\rho_{i+1} \rho_n) \) for all \( 1 \leq i \leq n-1 \).

Unfortunately, the cancellativity criterions that we recalled in Subsection 2.2 do not work with the presentation \( (S, R) \) of \( M_n \). We need to enlarge the set \( R \) of relations, thereby making it redundant, to be able to apply such criterions. We will need two distinct enlarged sets of relations, one to show left-cancellativity, the other one to show right-cancellativity. We introduce them in the following two Lemmas.

Lemma 3.5. Let \( 1 \leq i < j \leq n \). In \( M_n \) (and hence in \( G_n \)), we have
\[
\rho_i \rho_n \rho_j = \rho_j \rho_n \rho_i.
\]

Proof. Note that when \( i = 1 \), the relations are just the defining relations of \( M_n \). Assume that \( i > 1 \). As \( \rho_k \rho_n = \rho_1 \rho_n \rho_{k-1} \) for all \( 2 \leq k \leq n \), we have
\[
\rho_i \rho^j_n = \rho_1 \rho_n \rho_{i-1} \rho^j_n = \rho_1 \rho_n \rho_1 \rho_n \rho_{i-2} \rho^j_n = \cdots = (\rho_1 \rho_n)^{i-j}.
\]
Hence we get
\[(3.2) \quad \rho_i \rho_n^i \rho_{j-i} = (\rho_1 \rho_n)^i \rho_{j-i}.\]
Similarly, as \(j > i\), we have that
\[(3.3) \quad \rho_j \rho_n^j \rho_{i-j} = \rho_1 \rho_n \rho_1 \rho_n^2 \rho_n^3 = \ldots = (\rho_1 \rho_n)^i \rho_{j-i}.\]
Putting (3.2) and (3.3) together we get \(\rho_i \rho_n^i \rho_{j-i} = (\rho_1 \rho_n)^i \rho_{j-i} = \rho_j \rho_n^j\). This concludes the proof. \(\square\)

**Lemma 3.6.** Let \(1 \leq i < j \leq n\). In \(M_n\) (and hence in \(G_n\)), we have
\[(\rho_1 \rho_n)^{n-j+1} \rho_i = \rho_{n-j+i+1}(\rho_1 \rho_n)^{n-j} \rho_j.\]

**Proof.** Note that when \(j = n\), the claimed relations are just the defining relations of \(M_n\). Assume that \(j < n\). As \(\rho_1 \rho_n \rho_k = \rho_{k+1} \rho_n\) for all \(1 \leq k < n\), we have
\[(3.4) \quad (\rho_1 \rho_n)^{n-j+1} \rho_i = (\rho_1 \rho_n)^{n-j} \rho_{i+1} \rho_n = (\rho_1 \rho_n)^{n-j-1} \rho_{i+1} \rho_n^2 = \ldots = \rho_{n-j+i+1} \rho_n^{n-j+1}.\]
Applying the same relation, we have on the other hand that
\[(3.5) \quad (\rho_1 \rho_n)^{n-j} \rho_j = (\rho_1 \rho_n)^{n-j-1} \rho_{j+1} \rho_n = \ldots = \rho_{n-j}.\]
Putting (3.4) and (3.5) together we get
\[\rho_{n-j+i+1}(\rho_1 \rho_n)^{n-j} \rho_j = \rho_{n-j+i+1} \rho_n^{n-j+1} = (\rho_1 \rho_n)^{n-j+1} \rho_i,\]
which concludes the proof. \(\square\)

**Corollary 3.7.** The monoid \(M_n\) has two presentations \(\langle S, R' \rangle\) and \(\langle S, R'' \rangle\), where \(S\) is as before the set \(\{\rho_1, \rho_2, \ldots, \rho_n\}\) and \(R'\) (respectively \(R''\)) is the set of relations given in the statement of Lemma 3.6 (respectively Lemma 3.8).

**Proof.** We have seen in Lemmas 3.6, 3.7 that all the relations in \(R', R''\) follow from the relations in \(R\), and as they contain all the relations in \(R\) and \(M_n = \langle S, R \rangle\), the claim is immediate. \(\square\)

### 3.2. Cancellativity.

#### 3.2.1. Left-cancellativity.

In order to show that the monoid \(M_n\) is left-cancellative, we will apply Proposition 2.15 using the presentation \(\langle S, R' \rangle\) defined above, which is right-complemented. The presentation \(\langle S, R \rangle\) is also right-complemented, but it is easy to see that the \(\theta\)-cube condition fails for this presentation.

Note that in the presentation \(\langle S, R' \rangle\), we have precisely one relation for each pair of indices \(i, j \in \{1, 2, \ldots, n\}, i < j\), namely \(\rho_i \rho_n^i \rho_{j-i} = \rho_j \rho_n^j\). Hence \(\theta\) is defined over all \(S \times S\), and for \(i < j\) we have
\[\theta(\rho_i, \rho_j) = \rho_n^i \rho_{j-i}, \quad \theta(\rho_j, \rho_i) = \rho_n^i.\]

**Lemma 3.8.** The presentation \(\langle S, R' \rangle\) satisfies the sharp \(\theta\)-cube condition for every triple \((\rho_i, \rho_j, \rho_k)\) of pairwise distinct generators in \(S\).

**Proof.** We need to check that either both \(\theta(\theta(\rho_i, \rho_j), \theta(\rho_i, \rho_k))\) and \(\theta(\theta(\rho_j, \rho_i), \theta(\rho_j, \rho_k))\) are defined and equal as words in \(S^*\), or neither is defined.

It is sufficient to distinguish three cases: the case where \(i < j < k\), the case where \(i < k < j\), and the case where \(k < j < i\). The three remaining cases are indeed obtained for free by swapping the roles of \(i\) and \(j\).
Hence let $i < j < k$. On one hand we have
\[ \theta(\theta(\rho_i, \rho_j), \theta(\rho_i, \rho_k)) = \theta(\rho_i^i \rho_j, \rho_n \rho_{k-1}) = \theta(\rho_j - i, \rho_{k-1}) = \rho_n^j i - \rho_{k-j}, \]
where for the middle equality we used the fact that for all $a, b, c \in S^*$, we have \( \theta(ab, ac) = \theta(b, c) \) (which is an easy consequence of the relations (2.1) - (2.4). On the other hand we have
\[ \theta(\theta(\rho_j, \rho_i), \theta(\rho_j, \rho_k)) = \theta(\rho_n^j \rho_{k-1}, \rho_n \rho_{k-1}) = \theta(\rho_{k-1}^{-i} \rho_{k-j}) = \rho_n^{j-i} \rho_{k-j}. \]
Hence we have equality of words in that case.

Now assume that $i < k < j$. One on hand we have
\[ \theta(\theta(\rho_i, \rho_j), \theta(\rho_i, \rho_k)) = \theta(\rho_n^i \rho_{k-1}, \rho_n \rho_{k-1}) = \theta(\rho_{j-1} - i, \rho_{k-1}) = \rho_n^{j-i}. \]
On the other hand we have
\[ \theta(\theta(\rho_j, \rho_i), \theta(\rho_j, \rho_k)) = \theta(\rho_n^j \rho_{k-1}, \rho_n \rho_{k-1}) = \theta(\rho_{j-1} - i, \rho_{k-1}) = \rho_n^{j-i}. \]
Finally, let $k < j < i$. On one hand we have
\[ \theta(\theta(\rho_i, \rho_j), \theta(\rho_i, \rho_k)) = \theta(\rho_n^i \rho_j, \rho_n \rho_k) = \theta(\rho_n^{j-k}, 1) = 1. \]
On the other hand we have
\[ \theta(\theta(\rho_j, \rho_i), \theta(\rho_j, \rho_k)) = \theta(\rho_n^j \rho_{k-1}, \rho_n \rho_{k-1}) = \theta(\rho_{j-1} - i, \rho_{k-1}) = 1. \]
Hence in all cases we have $\theta(\theta(\rho_i, \rho_j), \theta(\rho_i, \rho_k)) = \theta(\theta(\rho_j, \rho_i), \theta(\rho_j, \rho_k))$, which concludes the proof. \( \square \)

**Proposition 3.9.** The monoid $M_n$ is left-cancellative and admits conditional right-lcms. When it exists, the right-lcm of $u$ and $v \in M_n$ is given by $uv \theta(u, v) = v \theta(v, u)$.

**Proof.** Since $M_n$ is right-Noetherian (Lemma 3.3) and satisfies the (sharp) $\theta$-cube condition for every triple of pairwise distinct generators in $S$ (Lemma 3.5), Proposition 2.15 ensures that $M_n$ is left-cancellative and admits conditional right-lcms. \( \square \)

The following corollary will be helpful to determine the center of $G_n$ in Section 3.4.

**Corollary 3.10.** Let $1 \leq i < j \leq n$. The right-lcm of $\rho_i$ and $\rho_j$ is given by \( \rho_i^i \rho_j \rho_{j-i} \).

**Proof.** It follows immediately from the proposition above, as $\theta(\rho_i, \rho_j)$ is defined and equal to $\rho_n^i \rho_{j-i}$. \( \square \)

3.2.2. *Right-cancellativity.* Unlike many classical examples of Garside monoids (like the positive braid monoid, or more generally Artin-Tits monoids of spherical type), the defining presentation $\langle S, R \rangle$ of the monoid $M_n$ is not symmetric for $n \geq 3$. We therefore cannot deduce right-cancellativity from left-cancellativity. To show that $M_n$ is right-cancellative, we will show the equivalent statement that the opposite monoid $M_n^{op}$ is left-cancellative. This monoid has the same set of generators as $M_n$ but we will denote them $T = \{ \tau_i \}_{i=1, \ldots, n}$ to distinguish them (with $\tau_i$ corresponding to $\rho_i$ for all $i$), and relations $R^{op}$ which are obtained from $R$ by reversing all the words.

Recall the presentations $\langle S, R \rangle$, $\langle S, R' \rangle$ and $\langle S, R'' \rangle$ of $M_n$ (see Section 3.1). As for left-cancellativity, it is not hard to see that the $\theta$-cube condition fails with the right-complemented presentation $\langle T, R^{op} \rangle$ of $M_n^{op}$, hence one cannot apply Proposition 2.15 with this choice of presentation. Moreover, the presentation $\langle T, (R')^{op} \rangle$ which is the opposite of the presentation $\langle S, R' \rangle$ that we used to show left-cancellativity is not right-complemented as in general there is more than one relation of the form $\tau_i \cdots = \tau_j \cdots$.
for \( i \neq j \) (for instance \( \tau_1 \tau_3 \tau_1 = \tau_3 \tau_2 \) and \( \tau_3^3 = \tau_1 \tau_3^2 \tau_2 \) for \( n = 3 \)), hence again Proposition \[\text{2.15}\] cannot be applied with this choice of presentation. But the presentation \( \langle T, (R^n)^{op} \rangle \) of \( M_n^{op} \) is right-complemented. The set of relations \((R^n)^{op}\) is indeed given by
\[
\tau_i (\tau_n \tau_1)^n \tau_{n-j} = \tau_j (\tau_n \tau_1)^{n-1} \tau_{n-j+i+1}, \quad \forall 1 \leq i < j \leq n.
\]

The syntactic right-complement \( \theta \) attached to the right-complemented presentation \( \langle T, (R^n)^{op} \rangle \) is then given by
\[
\theta((\tau_i, \tau_j, \tau_k)) = ((\tau_n \tau_1)^{n-j+1}, (\tau_n \tau_1)^{n-k+1}) = (\tau_i, \tau_j, \tau_k) = 1.
\]

Hence equality holds in that case.

Now assume that \( i < k < j \). One one hand we have
\[
\theta((\tau_i, \tau_j, \tau_k)) = ((\tau_n \tau_1)^{n-j+1}, (\tau_n \tau_1)^{n-k+1}) = (\tau_i, \tau_j, \tau_k) = 1.
\]

Finally, let \( k < j < i \). On one hand we have
\[
\theta((\tau_j, \tau_i, \tau_k)) = ((\tau_n \tau_1)^{n-j+1}, (\tau_n \tau_1)^{n-i+1}) = (\tau_j, \tau_i, \tau_k) = 1.
\]

\[\Box\]

**Proposition 3.12.** The monoid \( M_n^{op} \) is left-cancellative and admits conditional right-lcms. Equivalently, the monoid \( M_n \) is right-cancellative and admits conditional left-lcms.

**Proof.** Since \( M_n^{op} \) is right-Noetherian (as \( M_n \) is left-Noetherian by Lemma \[\text{3.4}\]) and satisfies the (sharp) \( \theta \)-cube condition for every triple of pairwise distinct generators in \( T \) (Lemma \[\text{3.11}\]), Proposition \[\text{2.15}\] ensures that \( M_n^{op} \) is left-cancellative and admits conditional right-lcms. \[\Box\]

We also note:

**Corollary 3.13.** Let \( 1 \leq i < j \leq n \). The left-lcm of \( \rho_i \) and \( \rho_j \) is given by
\[
(\rho_1 \rho_n)^{n-j+1} \rho_i = \rho_{n-j+i+1}(\rho_1 \rho_n)^{n-j} \rho_j.
\]
second claim, as the property holds for the set
which shows the second claim.

| Corollary 3.17. Both the left and the right-lcm of the generators \( \rho_1, \rho_2, \ldots, \rho_n \) of \( M_n \) are given by \( \rho_n^k = \rho_1 (\rho_1 \rho_2 \cdots \rho_n)^{n-1} \). In particular, using also Theorem 3.18 below, the family \((M_n)_{n \geq 2}\) yields an example of a family of Garside monoids where \( \Delta \) is not the lcm of the atoms.

3.3. Garside structure. In this Section, we establish the existence of a Garside ele-
moment in \( M_n \), and deduce from it and from previously shown properties that \( M_n \) is a Garside monoid.

Notation 3.14. Let \( M_n \) be the monoid with the presentation \((S, R)\) as defined in
Section [3.2]. We set \( \Delta := \rho_n^{n+1} \), omitting the dependency on \( n \).

Proposition 3.15. The following holds in \( M_n \):

1. We have \( \rho_1 (\rho_n \rho_1)^{n-1} = \rho_n^n \). Hence \( \Delta = (\rho_1 \rho_n)^n = (\rho_n \rho_1)^n \).
2. Let \( 1 \leq i \leq n \). Set \( a_i := \rho_n (\rho_1 \rho_n)^{n-1} \). Then \( \rho_i a_i = a_i \rho_i = \Delta \). In particular,
every element in \( S \) is both a left- and a right-divisor of \( \Delta \) (and the left- and
right-complements coincide), and \( \Delta \) is central in \( M_n \).
3. Let \( a, b \in M_n \) such that \( ab = \Delta \). Then \( ba = \Delta \).

Proof. The first claim follows from the fact that for all \( 1 \leq k \leq n-1 \), we have
\[(3.6) \rho_1 (\rho_n \rho_1)^k = \rho_{k+1} \rho_n^k.\]
Indeed, for \( k = 1 \) this is just a relation in \( R \), while the general case is obtained by
induction on \( k \): \( \rho_1 (\rho_n \rho_1)^k = \rho_1 \rho_n \rho_1 (\rho_n \rho_1)^{k-1} = \rho_1 \rho_n \rho_1 \rho_n^k = \rho_{k+1} \rho_n^k \).

For the second claim, using the first claim and (3.6) we have
\[
\Delta = (\rho_1 \rho_n)^n = (\rho_1 \rho_n)^{n-1} \rho_n (\rho_1 \rho_n)^{n-i} = \rho_i \rho_n^i (\rho_1 \rho_n)^{n-i} = \rho_i a_i.
\]
Arguing as for (3.6), for all \( k \leq n-i \), we see that \( (\rho_1 \rho_n)^k \rho_i = \rho_{i+k} \rho_n^k \). Applying this
with \( k = n-i \) we get
\[
\Delta = \rho_n^{n-1} = \rho_n^i \rho_n^{n-i} = \rho_n^i (\rho_1 \rho_n)^{n-i} \rho_i = a_i \rho_i,
\]
which shows the second claim.

The last claim is an immediate consequence of the cancellativity of \( M_n \) and the second
claim, as the property holds for the set \( S \) which generates \( M_n \).

Corollary 3.16. The left and right-divisors of \( \Delta \) coincide, and form a finite set.

Proof. The fact that the left and right-divisors of \( \Delta \) coincide follows immediately from
point (3) of Proposition 3.15. The fact that this set is finite is clear by Lemma 3.4
since \( S \) is finite.

Corollary 3.17. Both the left and the right-lcm of the generators \( \rho_1, \rho_2, \ldots, \rho_n \) of \( M_n \)
generate \( \rho_n^k = \rho_1 (\rho_1 \rho_2 \cdots \rho_n)^{n-1} \). In particular, using also Theorem 3.18 below, the family
\((M_n)_{n \geq 2}\) yields an example of a family of Garside monoids where \( \Delta \) is not the lcm
of the atoms.

Proof. By Corollary 3.10 we have that the right-lcm of \( \rho_n \) and \( \rho_{n-1} \) is given by \( \rho_n^k \).
Hence to conclude it suffices to show that \( \rho_i \) left-divides \( \rho_n^i \), for all \( 1 \leq i \leq n-2 \).
This is the case, as by point (1) of the above proposition together with relation (3.1), we have
\[
\rho_n = (\rho_1 \rho_n) (\rho_1 \rho_n)^{n-1-i} \rho_1 = \rho_i \rho_n^i (\rho_1 \rho_n)^{n-1-i} \rho_1.
\]
The proof that \( \rho_n^k \) is also the left-lcm of the elements in \( S \) is similar. This time,
consider the left-lcm of \( \rho_1 \) and \( \rho_2 \). By Corollary 3.13 it is equal to \( (\rho_1 \rho_n)^{n-1} \rho_1 \) which,
by the first point of Proposition 3.15 is equal to \( \rho_n^k \). Hence to conclude the proof, it
Theorem 3.18. The pair \((M_n, \Delta)\) is a Garside monoid. The corresponding Garside group is isomorphic to \(G_n\).

Proof. The monoid \(M_n\) is cancellative and admits conditional lcm’s by Propositions 3.9 and 3.12. It has Noetherian divisibility by Lemma 3.4. Now by Proposition 3.15 and Corollary 3.16, the element \(\Delta\) satisfies the last two conditions of Definition 2.7. We then get the existence of lcm’s from the existence of conditional lcm’s, applying Lemma 2.17.

By Theorem 2.3, we get that \(G(M_n)\) and \(G_n\) are isomorphic.

Remark 3.19. The lattice of simples of \(M_3\) (for left-divisibility) is given in Figure 3.

Recall that the lattice of simples of \(M_2\) was given in Figure 1. Note that the lattice in Figure 3 is not self-dual; in particular, by Corollary 2.20 the lattice of simples for left-divisibility is not isomorphic to the lattice of simples for right-divisibility.

Corollary 3.20. The surjection \(\varphi_n : G_n \rightarrow B_{n+1}\) from Proposition 3.3 is proper for \(n \geq 3\).

Proof. Let \(x = \rho_1 \rho_2 \rho_1 \rho_2^{-2} \in G_n, n \geq 3\). We have

\[
\varphi_n(\rho_1 \rho_2 \rho_1 \rho_2^{-2}) = \sigma_1 \sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_2^{-1} = 1,
\]

hence \(x \in \ker(\varphi_n)\). But \(x \neq 1\): indeed, for otherwise, we would get \(\rho_1 \rho_2 \rho_1 = \rho_2^2\) in \(G_n\).

But as \(M_n\) embeds into \(G_n\), the same equality would hold in \(M_n\), a contradiction, as no relation can be applied in \(M_n\) (\(n \geq 3\)) to the word \(\rho_1 \rho_2 \rho_1\).

An alternative proof can also be given by observing that the image of \(M_n\) under \(\varphi_n\) is the submonoid \(\Sigma_n\) of \(B_{n+1}\) generated by \(\sigma_1, \sigma_1 \sigma_2, \ldots, \sigma_1 \sigma_2 \cdots \sigma_n\). By Remark 3.6 below, the monoid \(\Sigma_n\) does not have lcm’s, while \(M_n\) does. Hence the restriction of \(\varphi_n\) to \(M_n \subseteq G_n\) cannot be injective.

It would be interesting to have an explicit description of \(\ker(\varphi_n)\) for \(n \geq 3\).

3.4. Determination of the center. One knows that every Garside group \(G\) has a nontrivial center. Indeed, as the set \(\text{Div}(\Delta)\) of divisors of \(\Delta\) is finite and stable by conjugation by \(\Delta\), there is a power of \(\Delta\) which acts by conjugation as the identity on \(\text{Div}(\Delta)\), and hence on \(G\), as \(\text{Div}(\Delta)\) generates \(G\).

We already know by Proposition 3.15 that \(\Delta\) is central in \(M_n\) (hence in \(G_n\)). For \(n = 1\) we have \(G_1 \cong \mathbb{Z}\), hence \(Z(G_1) = G_1\). We show:

Proposition 3.21. Let \(n \geq 2\). Then \(Z(G_n)\) is infinite cyclic, generated by \(\Delta = \rho_n^{n+1}\).

Proof. Let \(x \in Z(G_n)\). As \(G_n\) is a Garside group, there are \(p \in \mathbb{Z}\) and \(y \in M_n\) such that \(x = \Delta^p y\) and \(\Delta \not\leq y\) (in this proof \(\leq\) denotes left-divisibility). Note that, as \(\Delta\) and \(x\) are central in \(G_n\), so is \(y\). Assume for contradiction that \(y \neq 1\). We will show that with this assumption, we have \(\Delta \leq y\), which yields the desired contradiction. This forces \(x\) to be a power of \(\Delta\) and, as \(\Delta\) itself is central by point (2) of Proposition 3.15, the result follows.

Claim 1: \(\rho_n \leq y\).
Indeed, as $y \neq 1$, there is $1 \leq i \leq n$ such that $\rho_i \leq y$. We can assume that $i \neq n$. Then, as $y$ is central, we have that both $\rho_i$ and $\rho_{i+1}$ left-divide $\rho_{i+1}y$. Hence the lcm of $\rho_i$ and $\rho_{i+1}$, which is given by $\rho_i \rho_{i+1}^n = \rho_{i+1} \rho_{n}^n$ (see Corollary 3.10), also left-divides $\rho_{i+1}y$. Canceling $\rho_{i+1}$ we get that $\rho_i^n \leq y$, which shows the claim.

**Claim 2:** $\rho_n^{n-1} \rho_1 \leq y$.

This follows from the fact that, using Claim 1, both $\rho_n$ and $\rho_{n-1}$ left-divide $\rho_{n-1}y$. Hence their (left-)lcm, which is given by $\rho_n^n = \rho_{n-1} \rho_n^{n-1} \rho_1$, also left divides $\rho_{n-1}y$. Canceling $\rho_{n-1}$, we get that $\rho_n^{n-1} \rho_1 \leq y$, which shows the claim.

Now by Claim 1, we have that both $\rho_1$ and $\rho_n$ left-divide $\rho_1y$. Hence their lcm, which is equal to $\rho_n^2 = \rho_1 \rho_n \rho_{n-1}$, also left-divides $\rho_1y$. Hence $\rho_n \rho_{n-1}$ left-divides $y$. But we also know from Claim 2 that $\rho_n^{n-1} \rho_1 \leq y$.

Assume that $n \neq 2$. Canceling $\rho_n$, we get that both $\rho_n, \rho_{n-1}$ left-divide $\rho_n^{n-1}y$. Hence their lcm, given by $\rho_n^n$, also left divides $\rho_n^{n-1}y$. This implies that $\Delta = \rho_n^{n+1} \leq y$, a contradiction. Hence $y = 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{The lattice of divisors of the Garside element $\Delta$ in $M_3$ for left-divisibility.}
\end{figure}
If \( n = 2 \), then by Claim 2 we have \( \rho_2 \rho_1 \leq y \). Hence, as \( y \) is central, we have that both \( \rho_1 \) and \( \rho_2 \) left-divide \( \rho_1 \rho_2 y \). Hence their lcm \( \rho_1 \rho_2 \rho_1 = \rho_2 \) also left-divides \( \rho_1 \rho_2 y \). Canceling \( \rho_1 \rho_2 \), we get that \( \rho_1 \leq y \). As, by Claim 2, \( \rho_2 \rho_1 \leq y \), we get that the lcm of \( \rho_1 \) and \( \rho_2 \rho_1 \) left-divides \( y \). But this lcm is equal to \( \Delta = \rho_2^3 \), a contradiction. Hence \( y = 1 \) in this case as well.

\[ \square \]

4. Garside groups defined by a cyclic relation and the special cases \( n = 2,3 \)

The aim of this section is to establish an isomorphism between the group \( \mathcal{G}_n \) defined in Section 3 and the group \( \mathcal{C}_n \) from Theorem 1.2 and to derive a few applications.

4.1. Groups defined by a cyclic relation.

**Definition 4.1.** Recall that \( \mathcal{C}_n \) is the group defined by the presentation with generators \( x_1, x_2, \ldots, x_n \) and relations:

\[
x_1 x_2 \cdots x_i x_1 = x_2 x_3 \cdots x_n x_1 x_2 = x_3 x_4 \cdots x_n x_1 x_2 x_3 = \cdots = x_n x_1 x_2 \cdots x_n.
\]

We call such a (family of) relation(s) a **cyclic relation**.

**Proposition 4.2.** The map

\[
\rho_1 \mapsto x_1, \rho_2 \mapsto x_n x_1, \rho_3 \mapsto x_{n-1} x_n x_1, \ldots, \rho_n \mapsto x_2 x_3 \cdots x_n x_1
\]

extends to a group isomorphism \( \phi : \mathcal{G}_n \rightarrow \mathcal{C}_n \), with inverse \( \psi \) given by

\[
x_1 \mapsto \rho_1, x_n \mapsto \rho_2 \rho_1^{-1}, x_{n-1} \mapsto \rho_3 \rho_2^{-1}, \ldots, x_2 \mapsto \rho_n \rho_{n-1}^{-1}.
\]

**Proof.** The fact that the two defined maps are inverse to each other is immediate, hence we only need to show that they extend to group homomorphisms. To this end, we first show that the \( \phi(\rho_i) \)'s satisfy the defining relations of \( \mathcal{G}_n \), which is enough to conclude that \( \phi \) is a homomorphism. We have

\[
\phi(\rho_1) \phi(\rho_n) \phi(\rho_1) = x_1 x_2 \cdots x_n x_1 x_1 = \phi(\rho_2) \phi(\rho_n).
\]

Now let \( 1 < i < n \). We have

\[
\phi(\rho_1) \phi(\rho_n) \phi(\rho_i) = x_1 x_2 x_3 \cdots x_n x_1 x_{n-i+2} \cdots x_n x_1
\]

and

\[
\phi(\rho_{i+1}) \phi(\rho_n) = x_{n-i+1} \cdots x_n x_1 x_2 x_3 \cdots x_n x_1
\]

\[
= \frac{x_{n-i+1} \cdots x_n x_1 x_2 x_3 \cdots x_n x_{n-i+1}}{x_1 x_2 \cdots x_n x_1} x_{n-i+2} \cdots x_n x_1
\]

\[
= \phi(\rho_1) \phi(\rho_n) \phi(\rho_i),
\]

hence \( \phi \) is a homomorphism. Similarly, we have to show that the \( \psi(x_i) \) satisfy the defining relations of \( \mathcal{C}_n \). We have \( \psi(x_n) \psi(x_1) \psi(x_2) \cdots \psi(x_n) = \rho_2 \rho_n \rho_1 = \rho_1 \rho_n = \psi(x_1) \psi(x_2) \cdots \psi(x_n) \psi(x_1) \). Now let \( 1 < i < n \). We have

\[
\psi(x_i) \psi(x_{i+1}) \cdots \psi(x_n) \psi(x_1) \cdots \psi(x_i)
\]

\[
= (\rho_{n+2-i} \rho_{n+1-i}^{-1}) (\rho_{n+1-i} \rho_{n-i}^{-1}) \cdots (\rho_2 \rho_1^{-1}) (\rho_\rho_n \rho_{n-1}^{-1}) (\rho_{n-1} \rho_{n-2}^{-1}) \cdots (\rho_{n+2-i} \rho_{n+1-i}^{-1})
\]

\[
= \rho_{n+2-i} \rho_{n+1-i} \rho_{n+1-i} = \rho_1 \rho_n^{-1} \rho_{n+1-i} = \rho_1 \rho_n = \psi(x_n) \psi(x_1) \psi(x_2) \cdots \psi(x_n),
\]

hence \( \psi \) is also a homomorphism. This concludes the proof. \( \square \)
Note that the groups $C_n$, $n \geq 1$, admit a Garside structure as a consequence of [18 Proposition 5.2], which is distinct from the one introduced in this paper.

4.2. Link with complex braid groups. The exceptional complex reflection group $G_{12}$ has three generators $s, t, u$ and relations $s^2 = t^2 = u^2 = 1$, $stus = tust = ustu$. Its braid group $B(G_{12})$ has generators $\sigma, \tau, \nu$ subject to the same relations as $s, t, u$ except the quadratic ones (see [10]). We can deduce the following from the Proposition 4.2.

Corollary 4.3. We have

1. The group $G_3$ is isomorphic to the complex braid group $B(G_{12})$ of the exceptional complex reflection group $G_{12}$.

2. The complex reflection group $G_{12}$ has a presentation with generators $r_1, r_2, r_3$ and relations

$$r_1r_3r_1 = r_2r_3, \quad r_1r_3r_2 = r_3^2, \quad r_3^2 = 1.$$

In Corollary 5.3 below, we give analogous presentations for the symmetric groups $S_n$.

Proof. The first statement follows immediately from Proposition 4.2, as $G_3$ and $B(G_{12})$ have the same presentation if we set $\sigma = x_1$, $\tau = x_2$, $\nu = x_3$.

For the second statement, note that the given relations are exactly those of $G_3$ (except that the generators are denoted by $r_i$ instead of $\rho_i$), with the additional relation $r_1^2 = 1$.

By the first point $G_3$ is isomorphic to the complex braid group of $G_{12}$. Now $G_{12}$ is obtained from $B(G_{12})$ by adding the relations $\sigma^2 = \tau^2 = \nu^2 = 1$, but since $\sigma, \tau$, and $\nu$ are all conjugate in $B(G_{12})$, it suffices to add the relation $\sigma^2 = 1$ to get a presentation of $G_{12}$; this translates in $G_3$ to the relation $\rho_1^2 = 1$. \qed

Remark 4.4. Corollary 4.3 yields a new Garside structure on $B(G_{12})$. Note that the complex reflection group $G_{12}$ is not well-generated. By work of Bessis [4], every well-generated irreducible complex reflection group admits a dual braid monoid, in particular, the corresponding complex braid group is a Garside group. Almost all braid groups attached to irreducible complex reflection groups which are not well-generated have been shown to be Garside groups: see Dehornoy-Paris [18, Proposition 5.2 and Example 5] (for $G_{15}, G_7, G_{11}, G_{19}, G(2d,2e,2f)$ for $d > 1$, which all have isomorphic braid group, $G_{12}$, and $G_{21}$), Picantin [27] Examples 11, 13] (for $G_{13}$, whose braid group is isomorphic to the Artin group of type $I_3(6) = G_2$), and Corran-Lee-Lee [12] (for the remaining imprimitive groups). See also [17, Example IX.3.25]. It seems that the only irreducible complex reflection group for which it remains open to determine whether the corresponding braid group is a Garside group or not is $G_{31}$.

Remark 4.5. In view of the previous remark, it is natural to wonder if $G_n$ is the complex braid group of a complex reflection group in a natural way. For $n = 2$ we know that $G_2$ is isomorphic to the 3-strand braid group, which is the complex braid group of several irreducible complex reflection groups (obtained by adding the relation $\rho_1^2 = 1$ to the presentation of $G_2$). For $i = 2$ we get the symmetric group $S_3$, and for $i = 3, 4, 5$ the exceptional groups $G_4$, $G_8$ and $G_{16}$ respectively—note that these presentations already occur in Coxeter’s paper [13] from 1959. It is easy to check that the Garside monoid $M_2$ can be obtained from the finite group $G_4$ as an interval group (another way of producing Garside monoids, which we did not recall here; see [2, Section 0.5] or [17, Chapter VII]). For $n \geq 4$, adding the relation $\rho_1^2 = 1$ to the presentation of $G_n$ seems to yield an infinite group, and the same can be expected for $i > 2$. This suggests the question below.
Question 4.6. Let \( n \geq 4 \). Consider the quotient \( \overline{G_n} \) of \( G_n \) by the relation \( \rho_i^2 = 1 \). Does this quotient admit a natural realization as an infinite complex reflection group?

Note that the same question can be asked if we replace \( \rho_i^2 = 1 \) by \( \rho_i^2 = 1, \ i \geq 3 \) (even for \( n = 2 \) and \( n = 3 \) in the cases which are not covered by the above remark or Corollary).

5. Link with the braid group on \( n \) strands

In this section, we give a new presentation of the braid group \( B_{n+1} \), obtained by adding suitable relations to the presentation \( \langle S, R \rangle \) of \( G_n \). Using it we show that the submonoid \( \Sigma_n \) of \( B_{n+1} \) generated by \( \sigma_1, \sigma_1 \sigma_2, \ldots, \sigma_1 \sigma_2 \cdots \sigma_n \) is an Ore monoid with group of fractions isomorphic to \( B_{n+1} \), and conjecture that this monoid admits a finite presentation.

Definition 5.1. Let \( H_n \) be the group defined by the presentation with generators \( \rho_1, \rho_2, \ldots, \rho_n \), and relations

\[ \rho_i \rho_j \rho_i = \rho_{i+1} \rho_j, \forall \ 1 \leq i < j \leq n. \]

As the presentation is positive, let \( H_n^+ \) be the monoid defined by the same presentation.

Proposition 5.2. There is an isomorphism \( H_n \cong B_{n+1} \) given by \( \rho_i \mapsto \sigma_1 \sigma_2 \cdots \sigma_i, \forall 1 \leq i \leq n \). Hence the presentation of Definition 5.1 is a presentation of the braid group on \( n+1 \) strands.

Proof. We show that the assignment \( \rho_i \mapsto \sigma_1 \sigma_2 \cdots \sigma_i \), \( 1 \leq i \leq n \), extends to a group isomorphism \( f : H_n \rightarrow B_{n+1} \). To this end, it suffices to show that \( f \) extends to a group homomorphism, and that the assignment \( \sigma_i \mapsto \rho_{i-1}^{-1} \rho_i \) (with the convention \( \rho_0 = 1 \)) extends to a group homomorphism \( g : B_{n+1} \rightarrow H_n \), as both induced maps are clearly inverse to each other.

Showing that the \( f(\rho_i) \)'s satisfy the claimed relations can be checked by exactly the same computation as the one given in the proof of Proposition where it is done in the case where \( j = n \) (or just derived from it by invoking the embeddings \( B_k \subseteq B_{k+1} \)). Hence \( f \) is a group homomorphism.

Conversely, let us check that the \( g(\sigma_i) \)'s satisfy the braid relations. Let \( 1 \leq i \leq n-1 \). Using the relations \( \rho_i \rho_{i+1} \rho_i = \rho_i^2 \) and \( \rho_{i+1} \rho_i \rho_{i+1} = \rho_{i+1}^2 \), we get

\[ \rho_{i-1}^{-1} \rho_{i+1}^{-1} \rho_{i} \rho_{i+1} \rho_i = \rho_{i-1}^{-1} \rho_{i+1}^{-1} \rho_{i+1}^2. \]

Replacing \( \rho_{i-1}^{-1} \rho_i \rho_{i-1} \) by \( \rho_{i-1}^{-1} \rho_{i+1} \rho_i \) in each side (using the relation \( \rho_i \rho_{i+1} \rho_i = \rho_{i+1} \rho_i \rho_{i+1} \)) we get the equality

\[ \rho_{i-1}^{-1} \rho_i \rho_{i+1} \rho_i = \rho_{i-1}^{-1} \rho_{i+1} \rho_i \rho_{i-1}^{-1} \rho_{i+1}. \]

The left hand side of the above equality is equal to \( g(\sigma_i) g(\sigma_{i+1}) g(\sigma_i) \), while the right hand side is equal to \( g(\sigma_{i+1}) g(\sigma_i) g(\sigma_{i+1}) \), thus establishing the braid relation

\[ g(\sigma_i) g(\sigma_{i+1}) g(\sigma_i) = g(\sigma_{i+1}) g(\sigma_i) g(\sigma_{i+1}). \]

It remains to check that \( g(\sigma_i) g(\sigma_j) = g(\sigma_j) g(\sigma_i) \) whenever \( 1 \leq i < j \leq n-1 \). Using the relations \( \rho_i \rho_j \rho_i = \rho_{i+1} \rho_j \) and \( \rho_i \rho_{j-1} \rho_i = \rho_{i+1} \rho_j \), we can write

\[ \rho_{i-1}^{-1} \rho_{j-1} \rho_i \rho_{j-1} \rho_i = \rho_{j-1}^{-1} \rho_i \rho_{j-1} \rho_i \rho_j. \]

Replacing \( \rho_{j-1}^{-1} \rho_i \rho_{j-1} \) by \( \rho_i \rho_{j-1}^{-1} \rho_{j-1} \) in the left hand side (using the relation \( \rho_i \rho_{j-1} \rho_i = \rho_{j-1} \rho_i \rho_{j-1} \)) and \( \rho_{j-1}^{-1} \rho_i \rho_{j-1} \) by \( \rho_{j-1} \rho_i \rho_{j-1} \) in the right hand side (using the relation \( \rho_i \rho_{j-1} \rho_i = \rho_i \rho_{j-1} \rho_i \)), we get the equality

\[ \rho_{i-1}^{-1} \rho_i \rho_{j-1} \rho_j = \rho_{j-1}^{-1} \rho_j \rho_{i-1} \rho_i. \]
This equality is nothing but the equality \(g(\sigma_i)g(\sigma_j) = g(\sigma_j)g(\sigma_i)\). This shows that \(g\) is a group homomorphism, and concludes the proof. \(\square\)

**Corollary 5.3.** The symmetric group \(S_{n+1}\) admits a presentation with generators \(r_1, r_2, \ldots, r_n\), and relations

\[ r_i^2 = 1, \quad r_1r_j r_i = r_{i+1} r_j, \quad \forall 1 \leq i < j \leq n. \]

**Proof.** The claimed set of relations is given by the relations in the statement of Proposition 5.2 where we added the relation stating that the square of the first generator is equal to one. As all the \(\sigma_i\)'s are conjugate in \(B_{n+1}\), it suffices to add to the braid relations the relation \(\sigma_1^2 = 1\) to get a presentation of the symmetric group \(S_{n+1}\). This is equivalent to adding the relation \(\rho_1^2 = 1\) to the set of relations given in Proposition 5.2. \(\square\)

Investigating the properties of the monoid \(H_n^+\) appears as a natural question.

**Lemma 5.4.** In the monoids \(\Sigma_n\) and \(H_n^+\), every two elements \(x, y\) admit both a common right-multiple and a common left-multiple.

**Proof.** This follows immediately from the fact that both \(\Sigma_n\) and \(H_n^+\) are quotients of the Garside monoid \(M_n\). Indeed, the presentation of \(H_n^+\) is obtained from the presentation \(\langle S, R \rangle\) of \(M_n\) by adding relations, and under the isomorphism \(B_{n+1} \cong H_n\), the submonoid \(\Sigma_n\) is precisely the submonoid of \(H_n\) generated by \(\rho_1, \rho_2, \ldots, \rho_n\), which is a quotient of \(H_n^+\). \(\square\)

As a corollary we get

**Proposition 5.5.** The submonoid \(\Sigma_n\) of \(B_{n+1}\) is an Ore monoid, with group of fractions isomorphic to \(B_{n+1}\).

**Proof.** For the first statement, we need cancellativity and the existence of left-multiples. The last condition is given by Lemma 5.4 while cancellativity immediately follows from the fact that \(\Sigma_n\) is a submonoid of a group. The second statement follows, as \(\Sigma_n\) embeds into \(B_{n+1}\), with image generating \(B_{n+1}\) as a group: this ensures that the induced map \(G(\Sigma_n) \to B_{n+1}\) is an isomorphism. \(\square\)

**Remark 5.6.** It was noticed by Dehornoy [15, Example 3.7] that the monoid \(H_3^+\) does not have lcm’s (and the same holds for \(n > 3\)). Indeed, both \(\rho_1 \rho_2 \rho_1 = \rho_2^2\) and \(\rho_1 \rho_3 \rho_1 = \rho_2 \rho_3\) are common right-multiples of \(\rho_1\) and \(\rho_2\), and it is straightforward to check that none of these two elements left-divides the other one. Similarly, in \(\Sigma_n\), both \(\sigma_1 \sigma_1 \sigma_2 \sigma_1\) and \(\sigma_1 \sigma_1 \sigma_2 \sigma_3 \sigma_1\) are common right-multiples of \(\sigma_1\) and \(\sigma_2\), and it is clear that none of them left-divides the other one in \(\Sigma_n\). This implies that neither \(\Sigma_n\) nor \(H_n^+\) are Garside monoids. The answer to the second part of Question 1.1 from the Introduction is therefore negative.

Dehornoy also asked whether \(H_3^+\) is cancellative or not (see [15, Question 3.8]) and conjectured that this is the case. More precisely he conjectured that \(H_3^+ \cong \Sigma_3\). We conjecture the following more general statement, which would also imply that \(\Sigma_n\) admits a finite presentation (answering the first part of Question 1.1).

**Conjecture 5.7.** Let \(n \geq 1\), let \(H_n^+\) be the monoid with generators \(\rho_1, \rho_2, \ldots, \rho_n\), and relations \(\rho_1 \rho_j \rho_i = \rho_{i+1} \rho_j\) for all \(1 \leq i < j \leq n\). Then

1. The monoid \(H_n^+\) is cancellative,
(2) The monoid $\mathcal{H}^+_n$ is isomorphic to $\Sigma_n$ via $\rho_i \mapsto \sigma_1 \sigma_2 \cdots \sigma_i$. In particular, it embeds into $B_{n+1}$, which is therefore isomorphic to its group of fractions.

Note that both items of the above conjecture are actually equivalent: clearly (2) ⇒ (1) as $\Sigma_n$ is cancellative. Conversely, assume that $\mathcal{H}^+_n$ is cancellative. Then, using Lemma 3.1 it is an Ore monoid, embedding into its group of fractions $G(\mathcal{H}^+_n)$, and by Ore’s Theorem 2.3 we have $G(\mathcal{H}^+_n) \cong \mathcal{H}_n$. But as the group $\mathcal{H}_n$ with the same presentation as $\mathcal{H}^+_n$ is isomorphic to the $(n+1)$-strand braid group, this yields

$$G(\mathcal{H}^+_n) \cong \mathcal{H}_n \cong B_{n+1},$$

and the submonoid $\mathcal{H}^+_n$ of $B_{n+1}$ then precisely corresponds under this isomorphism to the submonoid of $B_{n+1}$ generated by $\sigma_1, \sigma_1 \sigma_2, \ldots, \sigma_1 \sigma_2 \cdots \sigma_n$, that is, to $\Sigma_n$.

**Remark 5.8.** In terms of the generators $x_1, \ldots, x_n$ of the group $G_n$ from Section 4 the surjection map from Proposition 3.3 from $G_n \cong C_n$ to $B_{n+1}$ sends $x_1$ to $\sigma_1$ and $x_{n-i+1}$ to $\sigma_1 \sigma_2 \cdots \sigma_i \sigma_{i+1} \cdots \sigma_n^{-1}$ for all $1 \leq i \leq n-1$. These elements are generators of the Birman-Ko-Lee braid monoid [7] (or dual braid monoid [2] with choice of Coxeter element $\sigma_1 \sigma_2 \cdots \sigma_n$). The cyclic relation satisfied by the $x_i$’s can then also be checked using the dual braid relations.

6. Related Garside structures on dihedral Artin groups of odd type

While the exotic Garside structure on $B_3$ given in Example 2.10 which was generalized in the previous sections to the groups $(G_n)_{n \geq 1}$, does not seem to generalize to Artin groups of type $A_n$ where $n \geq 2$ (see the previous section), it is natural to wonder which Artin groups (or more generally complex braid groups, in view of Section 4.2) admit a Garside structure analogous to the one introduced for $G_n$.

The case of dihedral Artin groups appears to us as the first family to consider, as they are the Artin groups with the most elementary structure, and $B_3$ is an Artin group of dihedral type. The aim of this section is to show that dihedral Artin groups of odd type admit a Garside structure similar to the one obtained for $B_3$ by $G_2$. These Garside structures are presumably new.

Let $m \geq 3$ be odd. Recall that the dihedral group $I_2(m)$ is generated by two simple reflections $s, t$ subject to the relations $s^2 = t^2$ and the braid relation $st \cdots = ts \cdots$. The corresponding Artin group $B(I_2(m))$ is generated by $\sigma, \tau$, only $m$ factors $m$ factors subject to the braid relation of $I_2(m)$.

For $m$ an integer as above, we denote by $B(m)$ the monoid generated by two elements $\rho_1, \rho_2$, and subject to the relation $\rho_1 \rho_2^{(m-1)/2} \rho_1 = \rho_2^{(m+1)/2}$. We denote by $B(m)$ the group defined by the same presentation. Note that $M(3) = M_2$.

**Lemma 6.1.** The group $B(m)$ is isomorphic to the dihedral Artin group $B(I_2(m))$.

**Proof.** It is a easy calculation to check that an isomorphism is given by $\rho_1 \mapsto \sigma$, $\rho_2 \mapsto \sigma \tau$. $\square$

Note that $M(m)$ is cancellative, as divisibility is Noetherian (since the defining relation is homogeneous with $\lambda(\rho_1) = 1$ and $\lambda(\rho_2) = 2$) and $M(m)$ is generated by two elements $\rho_1, \rho_2$ with a single relation of the form $\rho_1 \cdots = \rho_2 \cdots$, hence the defining presentation is right-complemented and the cube condition (Definition 2.13) is vacuously true for triples of distinct generators.

Setting $\Delta := \rho_2^m$, the following Lemma is the analogue for $M(m)$ of Proposition 3.15 established in the case of $M_n$:
Lemma 6.2. The following holds in $M(m)$:

1. We have $(\rho_1 \rho_2^{(m-1)/2})^2 = (\rho_2^{(m-1)/2} \rho_1)^2 = \Delta$.
2. Let $a_1 := \rho_2^{(m-1)/2} \rho_1 \rho_2^{(m-1)/2}$. Then $\rho_1 a_1 = a_1 \rho_1 = \Delta$. In particular, both generators $\rho_1$ and $\rho_2$ are left- and right-divisors of $\Delta$ (and the left- and right-complements of a given generator coincide).
3. Let $a, b \in M(m)$ such that $ab = \Delta$. Then $ba = \Delta$.

Proof. The first claim is an immediate consequence of the defining relation of $M(m)$. The second claim follows immediately from the first one. The last claim is a consequence of the cancellativity of $M(m)$ and the second claim, as the claimed property holds for $\rho_1$ and $\rho_2$ (recall that $\Delta$ is a power of $\rho_2$), which generate $M(m)$.

Proposition 6.3. The pair $(M(m), \Delta)$ is a Garside monoid. The corresponding Garside group is $B(m)$.

Proof. The proof is exactly the same as for $G_n$ (Theorem 3.18): as noted above, the divisibility in $M(m)$ is Noetherian and the $\theta$-cube condition is vacuously true, hence we have cancellativity and the existence of conditional lcm’s in $M(m)$. By Lemma 6.2 above, the element $\Delta$ is a Garside element in $M(m)$, and we then conclude the proof by applying the same results as in the case of $G_n$.

Of course, adding the relation $\rho_2^2 = 1$ to the presentation of $B(m)$ yields a presentation of the dihedral group $I_2(m)$, as there is only one conjugacy class of reflections in $I_2(m)$.

Remark 6.4. The dihedral Artin groups of even type do not seem to admit a similar description. Indeed, let $B = B(I_2(4)) = B(B_2)$ be the Artin group of type $B_2$, with standard generators $\sigma_1, \sigma_2$ and braid relation $\sigma_1 \sigma_2 \sigma_1 \sigma_2 = \sigma_2 \sigma_1 \sigma_2 \sigma_1$. Then setting $\rho_1 = \sigma_1, \rho_2 = \sigma_1 \sigma_2$, we get a presentation for $B$ by taking this new set of generators and the relation $\rho_1 \rho_2^2 = \rho_2^2 \rho_1$. This appears to us as the natural analogue of the presentations considered in the odd case but in this case, the monoid generated by $\rho_1$ and $\rho_2$ subject to the above relation is not a Garside monoid: indeed, if it was, then the Garside element $\Delta$ would have a power which is central. Since the center of $B$ is infinite cyclic generated by $(\sigma_1 \sigma_2)^2 = \rho_2^2$, it is clear from the above defining relation that $\Delta$ itself would have to be a power of $\rho_2$ as $\rho_1$’s cannot be eliminated using the unique defining relation, say $\Delta = \rho_2^m$. But then $\rho_1$ could not divide $\Delta$ as no relation can be applied to the word $\rho_2^m$, a contradiction.

As a concluding remark, let us note the following. We introduced several monoids in this paper, which either are Garside monoids (like $M_n$ and $M(m)$), or closely related to a Garside monoid (like $H_n^+ \sigma$). All of them are defined by the same kind of presentations. The corresponding groups of fractions are braid groups of real or complex reflection groups in several cases, and presentations for these reflection groups can be naturally derived from those of the corresponding monoids (as done in Corollaries 4.3, 5.3, and Remark 4.5). This covers the following cases: $G_4, G_8, G_{16}, G_{12}, G_n$ for all $n$, and $I_2(m)$ for odd $m$. All these groups have a single conjugacy class of reflections, while the dihedral groups of even type like $I_2(4)$, for which the above remark shows that there does not seem to exist a Garside monoid similar to the ones introduced in this paper, have two conjugacy classes of reflections. While we do not have any general statement at the moment, it would be interesting to investigate whether reflection groups with a single conjugacy class of reflections, and their braid groups, admit presentations and monoids similar to those introduced in this work.
REFERENCES

[1] E. Artin, Theorie der Zöpfe, Abh. Math. Sem. Univ. Hamburg 4 (1925), no. 1, 47-72.
[2] D. Bessis, The dual braid monoid, Ann. Sci. École Norm. Sup. 36 (2003), 647-683.
[3] D. Bessis, A dual braid monoid for the free group, J. Algebra 302 (2006), 275-309.
[4] D. Bessis, Finite complex reflection arrangements are \( K(\pi, 1) \), Annals of Math. 181 (2015), Issue 1, 55-69.
[5] D. Bessis, F. Digne, and J. Michel, Springer theory in braid groups and the Birman-Ko-Lee monoid, Pacific J. Math. 205 (2002), 287–309.
[6] J. Birman and T. Brendle, Braids: a survey, Handbook of Knot Theory, 19-103, Elsevier B.V., Amsterdam, 2005.
[7] J. Birman, K.H. Ko, and S.J. Lee, A New Approach to the Word and Conjugacy Problems in the Braid Groups, Adv. in Math. 139 (1998), 322–353.
[8] C. Brav and H. Thomas, Braid groups and Kleinian singularities, Math. Ann. 351 (2011), no. 4, 1005-1017.
[9] E. Brieskorn and K. Saito, Artin-Gruppen und Coxeter-Gruppen, Invent. Math. 17 (1972), 245-271.
[10] M. Brouë, G. Malle, and R. Rouquier, Complex reflection groups, braid groups, Hecke algebras, J. Reine Angew. Math. 500 (1998), 127-190.
[11] A.H. Clifford and G.B. Preston, The algebraic theory of semigroups, Vol. II, Mathematical Surveys, No. 7, American Mathematical Society, Providence, R.I., 1967.
[12] R. Corran, E.-K. Lee, and S.-J. Lee, Braid groups of imprimitive complex reflection groups, J. Algebra 427 (2015), 387-425.
[13] H.S.M. Coxeter, Factor groups of the braid group, Proc. 4th Canad. Math. Cong. (1959) 95-122.
[14] P. Dehornoy, Groupes de Garside, Ann. Sci. École Norm. Sup. (4) 35 (2002), no. 2, 267-306.
[15] P. Dehornoy, The subword reversing method, Internat. J. Algebra Comput. 21 (2011), no. 1-2, 71-118.
[16] P. Dehornoy, A cancellativity criterion for presented monoids, Semigroup Forum 99 (2019), no. 2, 368-390.
[17] P. Dehornoy, F. Digne, D. Krammer, E. Godelle, and J. Michel. Foundations of Garside theory, Tracts in Mathematics 22, Europ. Math. Soc. (2015).
[18] P. Dehornoy and L. Paris, Gaussian groups and Garside groups, two generalisations of Artin groups, Proc. London Math. Soc. (3) 79 (1999), no. 3, 569-604.
[19] P. Deligne, Les immeubles des groupes de tresses généralisés, Invent. Math. 17 (1972), 273-302.
[20] F. Digne, Présentations duales des groupes de tresses de type affine \( A \), Comment. Math. Helv. 81 (2006), no. 1-2, 23-47.
[21] F. Digne, A Garside presentation for Artin groups of type \( C_n \), Ann. Inst. Fourier 62 (2012), no. 2, 641-666.
[22] F.A. Garside, The braid group and other groups, Quart. J. Math. Oxford Ser. 20 (1969), no. 2, 235â€“254.
[23] L.T. Jensen, The 2-braid group and Garside normal form, Math. Z. 286 (2017), no. 1-2, 491-520.
[24] C. Kassel and V. Turaev, Braid groups, Graduate Texts in Mathematics, 247. Springer, New York, 2008.
[25] D. Krammer, Braid groups are linear, Ann. of Math. (2) 155 (2002), no. 1, 131-156.
[26] T. Licata and H. Queffelec, Braid groups of type ADE, Garside structures, and the categorified root lattice, preprint (2017).
[27] M. Picantin, Petits groupes gaussiens, PhD Thesis, Université de Caen, 2000.

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