A novel method for constructing a dynamics model with the flexible characteristics

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Abstract. An accurate dynamics model is an essential part of ensuring the effectiveness of constructing flexible dynamical systems in the simulation. Finite element method is an effective method to build a flexible structural dynamics model, but this method will cause the model to have too many orders to increase the difficulty in the simulation. The reduction method based on the finite element principle is an effective method to build a streamlined model. This paper proposes a method that combines the critical modes of the structure and the reduction method based on the finite element model to construct an effective flexible dynamics model. Firstly, the determination of primary and secondary degrees of freedom is provided by the judgment of critical modes. Then, an adequate simulation model and an experimental setup are established respectively based on a flexible structure as an example. Finally, through the simulation comparison, the reduction model and the full-order model keep the same simulation results, but the simulation time consumed by the latter is 13 times of the former. In terms of experimental and simulation comparison, the simulation and the experimental results have the same transient oscillation frequency, and the steady-state angle deviation is 2 mrad.

1. Introduction

Virtual prototyping technology[1] is a digital design method based on computer simulation models, significantly shortening the product development cycle and reduce product development costs. This technology is widely used in the field of transportation [2], robotics [3], machine tools [4], and high-precision motion [5]. Whether the constructed virtual prototype is instructive for the system depends on how well the prototype model matches the actual model. The dynamics model is an essential part of the virtual prototype model. Typically, the dynamics model is usually constructed using the lumped parameter method [6]. This method transforms the control object into inertial, damping and elastic elements through a series of equivalences and simplifications [2], but this method ignores the flexible properties of the structure. In the precision motion system[7], the flexible vibration has affected the design of the motion system controller and the final system positioning accuracy. Therefore, it is no longer desirable to construct a simplified dynamics model.

Common dynamics modeling methods mainly include experimental modeling and theoretical modeling. The experimental modeling method mainly uses the system identification method in the time domain or frequency domain[9]to obtain the model. The prerequisite is that the control object must be available. In addition to the lumped parameter method shown above, the theoretical modeling also
includes the finite element method that can reflect the flexibility of the structure. Its principle is to approximate the mechanical structure by a discrete method [10]. This method has been applied in the simulation of systems such as flexible robotic arms [11] and flexible components [12]. In this method, the small meshes can approximate the accurate model [13], but this implies the tremendous computational effort, complicated design of controllers and difficulty in ensuring real-time operations.

The model reduction method is an effective method to build a simplified model and has been widely used in precision machinery [14]. The basic idea of model reduction is to find a reduced-order basis, in a sense that the reduced-order system can optimally approximate the original large-scale system and maintain the main characteristics of the original-scale system. This method can be divided into three categories according to the reduction principle. The first type is based on the finite element model. This method obtains the reduced-order basis and reduced-order model on the basis of determining the primary and secondary degrees of freedom of the structure and removing the secondary degrees of freedom. Common methods include such as Guyan [15], dynamic condensation [16], improved static condensation system method [17], improved dynamic condensation system method [18], system equivalent expansion condensation method (SEREP) [19]. The second and third types are respectively based on the principles of projective and non-projective methods [20] to find a reduced-order basis and then obtain the reduced-order object. The former usually includes the moment matching method and the balanced truncation method based on Krylov subspace, and the latter includes methods such as singular perturbation and Hankel optimal reduction. The first type realizes the construction of the dynamics model based on the reduction of the finite element model. These two latter types are mainly from the perspective of control performance, and the ultimate goal of reduction is to obtain a low-dimensional controller to control the original high-order system [21]. Their mathematical meaning is clear, but the physical meaning is not. The first method is mainly used in the structural field to realize the reduction of the model from the primary and secondary degrees of freedom of the structure, and its physical meaning is evident [22]. This paper aims to construct an effective flexible dynamics model using the finite element method. In the condensation methods mentioned above, the SEREP method mainly uses modal information to construct the reduced-order basis. This method not only can capture the dynamic properties of the structure better with modal information and predict the system with the high-frequency dynamic properties with pre-set constraints better than the static and dynamic condensation methods but also can be suitable for condensing the large structures [23].

The prerequisite for realizing the FEM-based reduced order is to specify the primary and secondary degrees of freedom of the structure. The eigenvalues and eigenvectors of the structure are important factors for constructing a dynamics model. Therefore, the contribution of eigenvalues to the displacement of the structure can be used to determine the order of the important eigenvalues, and then use it to judge the primary and secondary degrees of freedom of the structure. Common eigenvalue ranking methods are these based on the DC gain or peak gain at the resonance peak of the model transfer function and these based on the observability and controllability of the system [24]. The former is generally limited to eigenvalue ranking for single-input, single-output systems, while the latter can be used for single-input, single-output and multiple-input, multiple-output systems.

In this paper, it proposes a method to establish a flexible dynamics model by combining the critical modes of the structure and the reduction method based on the finite element model, and then conduct a simulation and experimental comparison analysis of this modeling method by using a single-degree-of-freedom electromechanical system with a flexible structure. The rest of the paper structure is as follows: Section 2 describes the SEREP method and the method of DC gain and peak gain for determining the critical modes, and on this basis, a comprehensive analysis method for constructing a flexible dynamics model is presented; In section 3, the method proposed in the previous section is used to construct a dynamics model of a flexible structure; Section 4 is a comparative analysis of motion system simulation and experimental results; Section 5 is a summary of the paper.

2. Construction of a reduced-order dynamics model
The motion of the equation for a linear flexible time-invariant structure with n DOFs is as follows:
where $M$, $C$, $K$, $F$ denote the mass matrix, damping matrix, stiffness matrix and excitation force, respectively, and $\ddot{x}$, $\dot{x}$, $x$ the acceleration, velocity and displacement vectors, respectively.

A coordinate transformation matrix is introduced to describe the transformation relationship between the existing coordinates and the reduced-order coordinates of the model.

\[
\begin{align*}
M_R \ddot{x}_R(t) + C_R \dot{x}_R(t) + K_R x_R(t) &= F_R(t) \\
\end{align*}
\]  

Combining Eqs (1) and (2), the obtained kinetic equations of the model after the reduced order are:

\[
M_R \ddot{x}_R(t) + C_R \dot{x}_R(t) + K_R x_R(t) = F_R(t)
\]

where the superscript $R$ is denoted as the reduced order; $T_R$ is the time-invariant reduced-order basis.

The model-based order reduction method in this paper is mainly based on the principle of primary and secondary degrees of freedom. In order not to lose generality, the effect of damping is neglected, and the Eq. (1) can be transformed as:

\[
\begin{align*}
m \ddot{q}_m(t) + K_m q_m(t) &= F_m(t) \\
\end{align*}
\]

Among them, $m$ represents the number of primary degrees of freedom of the system, $s$ represents the number of secondary degrees of freedom of the system, $m \cup s = n$, $m \cap s = \emptyset$.

Based on the reduced-order model, the dynamics model is constructed for the final simulation analysis. The mathematical model is as follows:

\[
\begin{align*}
\begin{bmatrix}
\Phi_{mn} & \Phi_{ms} \\
\Phi_{mn} & \Phi_{ss}
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_m(t) \\
\ddot{q}_s(t)
\end{bmatrix} +
\begin{bmatrix}
K_{mn} & K_{ms} \\
K_{mn} & K_{ss}
\end{bmatrix}
\begin{bmatrix}
q_m(t) \\
q_s(t)
\end{bmatrix} =
\begin{bmatrix}
F_m(t) \\
F_s(t)
\end{bmatrix}
\end{align*}
\]

The reduced-order basis of the SEREP method is constructed based on the modal information of the structure. The variables are converted from being in the physical coordinate system to being in the modal coordinate system through transformation. Compared to the static and dynamic condensation, this method effectively predicts the system with high-frequency kinematic properties with predefined constraints [19].

First, the physical coordinate system is transformed into a modal coordinate system.

\[
\begin{align*}
\Phi
\end{align*}
\]

Let $q_m(t) = 0$, then we obtain:

\[
\begin{align*}
q_m = \Phi_{mn} x_m
\end{align*}
\]

Substituting Eq. (6) into the undamped dynamic equation (4), we can obtain,

\[
\begin{align*}
x_m = \Phi_{mn} \Phi_{mn}^\dagger x_m = T_{SEREP} x_m
\end{align*}
\]
where the superscript $^{-1}$ represents the inverse of the matrix; $T_{SEREP}$ represents the reduced-order basis.

In general, the number of primary degrees of freedom is greater than the number of critical modes, that is, $\Phi_{mm}$ is not a square matrix, then

$$\Phi_{mm}^{-1} = (\Phi_{mm}^T \Phi_{mm})^{-1} \Phi_{mm}^T$$

(9)

Among them, the superscript $^+$ represents the pseudo inverse of the matrix.

The mass and stiffness matrix after the condensation can be obtained by Eq. (8):

$$\begin{align*}
M_R &= T_{SEREP}^T M T_{SEREP} \\
K_R &= T_{SEREP}^T K T_{SEREP}
\end{align*}$$

(10)

2.2. Determination of the critical modes

The determination of the primary degrees of freedom is a prerequisite for obtaining the dynamics model. This paper takes the critical modes as the goal and then finds the primary degrees of freedom that can express the critical modes. The critical modes are determined by using the $DC$ gain by judging the contribution of each mode to the system response. The premise of this application is to express the dynamic equation with the eigenvalues and eigenvectors as the uncoupled motion equation in the modal coordinate system. When the damping is included in the system, the gain is called peak gain, which has the same meaning as $DC$ gain.

$$\begin{align*}
DC: \frac{x_i}{F_{ij} \phi_{nji} \phi_{nki}} (s = j\omega = j0 = 0) \\
Peak: \frac{x_i}{F_{ij} \phi_{nji} \phi_{nki}} = \frac{-j}{2\xi_i \omega_i^2} (DC) (s = j\omega_i)
\end{align*}$$

(11)

where $\phi_{nji}\phi_{nki}$ is the product of the $j$th and $k$th row eigenvectors of the $i$th mode; $\omega_i$ is the $i$th eigenvalue; $\xi_i$ is the $i$th damping ratio.

2.3. A comprehensive method for building the dynamics model

Based on the existing theory, this section proposes the following application procedures:

1) Determine the primary and secondary degrees of freedom and the reduction thresholds according to the critical modes and control indicators of the system;
2) Obtain the critical modal set according to Eq. (11).
3) Determine the primary and secondary degrees of freedom of the structure, and then organize the stiffness and mass matrices.
4) Determine the principal eigenvalues and the eigenvectors corresponding to the primary and secondary degrees of freedom based on the sorted stiffness and mass matrix;
5) Calculate the reduced-order basis of the model using Eq. (8);
6) Obtain the reduced-order stiffness and mass matrix, and their corresponding eigenvalues and eigenvectors;
7) The effect of the condensation is judged by whether the eigenvalues obtained after the condensation are equal to the original eigenvalues ($\{f_R\} = \{f_{orig}\}$) and whether the modal confidence calculated from the eigenvectors after condensation and the original eigenvectors is equal to 1 ($MAC=1$).

$$MAC = \frac{|\Phi_R^T \Phi_R|}{\sqrt{|\Phi_{orig}^T \Phi_{orig}| \cdot |\Phi_R^T \Phi_R|}}$$

(12)

where $\Phi_{orig}$ represents the vector obtained under the original stiffness and mass matrix of the model; $\Phi_R$ represents the eigenvector obtained under the reduced-order stiffness and mass matrix.
8) The dynamics model is obtained after the condensation. The specific application flowchart of the method is shown in Fig. 1.
3. Dynamics model

In this section, a flexible beam is used as the analyzed object, and its dynamics model is constructed based on the method mentioned above.

3.1. Structural parameters of the flexible beam

The material of the flexible beam is aluminum-magnesium alloy, and its specific parameters are as follows.

| Dimension              | 302.5 mm x 1 mm x 28 mm |
|------------------------|-------------------------|
| Density                | 2463 Kg/m³              |
| Young's modulus        | 4.9149e10 Pa            |

3.2. Dynamics model of the flexible beam

As a continuum, the flexible beam is discretized to obtain 121 nodes, 20 elements, and each node has six degrees of freedom (x, y, z, Rx, Ry, Rz). Firstly, the eigenvalues, eigenvectors, stiffness matrix and mass matrix of the structure are obtained using the finite element software; secondly, the DC values corresponding to the eigenvalues of each order are calculated by Eq. (10), and the accumulated DC values are then obtained. As can be seen from Fig. 2, there are 41 orders of eigenvalues corresponding to DC values above 10e-5, and the sum of the cumulative values is equal to 0.9999, that is, these 41-order modes can accurately express the structural characteristics of the model.

Since the servo cycle of the control system used is 50 μs, the sampling frequency is 20 kHz. According to the Nyquist sampling theorem and considering the practical engineering application, the effective analysis signal is 1/2.56 ~ 1/4 times the sampling frequency, so 5 kHz is selected as the analysis frequency. Through the DC sorting, it can be seen that there is a total of 12 frequencies within 5 kHz, which account for 95% of the accumulated DC. The eigenvalue results are shown in Table 2.
Table 2 Order of the critical modes and the corresponding eigenvalues

| Mode order | 1   | 2   | 3   | 6   | 7   |
|------------|-----|-----|-----|-----|-----|
| Eigenvalues /Hz | 0  | 34.23 | 112 | 237.25 | 414.33 |
| Mode order | 9   | 11  | 14  | 16  | 19  |
| Eigenvalues /Hz | 649.6 | 951.84 | 1332.8 | 1807.99 | 2397.75 |
| Mode order | 22  | 27  | 28  | 29  | 30  |
| Eigenvalues /Hz | 3128.64 | 4035.34 |

In the electromechanical system, the movement of the flexible beam is set for the rotational movement around \( R_z \) only. Based on the obtained eigenvalues corresponding to the vibration shape and the primary degrees of freedom that have a greater impact on the system motion, the rotational degree of freedom \( R_z \) of each node is extracted as the primary degree of freedom, and the rest are as the secondary degrees of freedom. Then the degrees of freedom of the flexible beam are reduced from 121*6 to 21. The extracted mass and stiffness matrices are shown in Fig. 3.

According to the SEREP method mentioned above, the reduced-order basis is generated as shown in Fig. 4. The corresponding reduced-order stiffness and mass matrices are generated from the reduced-order basis, and the eigenvalues and eigenvectors are derived, as shown in Table 3 and Fig. 5. Combining with the original eigenvectors, the MAC calculated according to Eq. (12) is constantly equal to 1. Also, combining with the eigenvalues in Tables 2 and 3, it can be seen that the accuracy of the reduced model can be guaranteed.
Substituting the eigenvalues obtained from the above analysis and their corresponding eigenvectors into Eq. (4), the dynamics model required for the simulation with the flexible structures can be established.

4. Simulations and experiments

An electromechanical system with a flexible structure is used as an experimental setup for verifying the effectiveness of the mentioned technology based on the finite element model in the motion simulation system. This test device comprises a host computer, a control board, a driver and a motion system. Among them, the motion system mainly consists of a flexible beam and a drive motor. The flexible beam is driven by the motor to rotate. The detailed experimental setup is shown in Fig. 6.

To verify the accuracy of the above-mentioned flexible structure dynamics model, the prerequisite is to ensure that the parameter settings of the drive motor and its driver are correct. After parameter query and identification, the motor torque constant is $0.4157 \text{ N.m/Arms}$. The effective value current corresponding to the $1V$ voltage of the driver is $0.3119 \text{ Arms}$. In addition, an equivalent damping factor of $0.019 \text{ N*m/(rad/s)}$ was identified at the motor spindle using the uniform speed section of the motor operation by using the equivalent method.

The simulation and experiment are divided into two parts. The first part is to determine the correctness of the parameters of the system except for the flexible dynamics model. The second part is to verify the correctness of the flexible dynamics model based on the first part.

All simulation analyses in this chapter are carried out in the MATLAB-Simulink environment.
4.1. System parameter verification

That whether the gravitational torque of a rigid beam and the torque output by the motor are equal is applied to analyze the correction of the drive motor and its drive parameters used in the system.

The dimensions of the rigid beam used are 100 mm × 10 mm × 28 mm with a density of 5600 Kg/m³. In the simulation analysis, its model is built as a mass block, ignoring the influence of flexibility.

The position of the rigid beam with the motor is perpendicular to the position shown in Fig. 6. Under the gravity, the simulation output torque to overcome the gravity of the rigid beam is 0.09415 N.m. Compared with the measured value of 0.0946 N.m, the difference between the two is 0.00045 N.m and the error is 0.46%. This can prove the credibility of the simulated system parameters.

4.2. Simulation and experiment of the flexible beam

The important parts of the simulation model of the electromechanical system with the flexible beam include the dynamics model of the flexible beam and the driving moment of the flexible beam. The former has been introduced in Section 2. For the second part, according to the actual electromechanical system analysis, the motor torque output is equal to the sum of the driving torque of the flexible beam, the moment of inertia introduced by the motor spindle and the connection between the flexible beam and the motor, and the equivalent damping torque of the system, so the input torque of the flexible beam is as follows:

$$T_{\text{flexible}} = T_{\text{Motor}} - T_{\text{axis}} - T_{\text{base}} - T_{\text{Damp}} = T_{\text{Motor}} - J_{\text{axis}}\ddot{\theta} - J_{\text{base}}\dot{\theta} - D\dot{\theta}$$

(13)

where $T_{\text{Motor}}$, $T_{\text{axis}}$, $T_{\text{base}}$, $T_{\text{Damp}}$, $T_{\text{flexible}}$ denote the total motor torque, motor spindle torque, connection drive torque, equivalent damping torque of the motor spindle and flexible beam drive torque, respectively; $\theta, \dot{\theta}, \ddot{\theta}$ is the rotation angle, angular velocity, angular acceleration, respectively.

This part is implemented with the help of the MATLAB-function in Simulink. The displacement and velocity outputted from the dynamics model and the acceleration obtained by differentiation of the velocity are used as the input for the MATLAB-function inputs. The specific Simulink simulation block diagram is shown in Fig. 7.

The simulation and experimental comparison results are shown in Fig. 8. In the same simulation environment, using the PID controller with $K_p=10$, the simulation results of the full-order model and the reduced-order model are consistent, but the simulation time of the full-order model is 948 s while the simulation time of the reduced-order model is 71 s. By comparison between the experiment and the simulation, the transient oscillation frequency of the simulation results and the experimental results are the same, and the steady-state angle deviation of the simulation is about 2 mrad.

After analysis, the error between simulation and experiment is mainly caused by the following two factors. Firstly, the flexible beam has the obvious flexible characteristics, resulting in flexible vibrations at its end being transmitted to the motor spindle, further affecting the motor encoder feedback output. Secondly, also due to the flexible characteristics, the simple controller cannot guarantee to set a high servo stiffness for the system, which leads to the non-linear friction existing between the motor’s spindle and bearings to highlight. The above two parts are not reflected in the simulation model, which leads to a lower match between simulation and experiment.

![Fig. 7 Simulink simulation block diagram of the flexible beam](image-url)
5. Conclusions

With the purpose of establishing an effective structure with flexible characteristics to serve the simulation analysis of electromechanical systems, this paper proposes a method to establish a flexible dynamics model by combining the critical modes of the structure and the reduction method based on the finite element model. This method firstly determines the primary and secondary degrees of freedom of the structure based on the critical modes. Secondly, the dynamics model is constructed using the reduced-order basis based on the modal information. Unlike the previous model building methods, this method can determine which nodes' degrees of freedom can accurately express the flexible characteristics of the structure, laying the foundation for the optimal layout of sensors that may appear in the future. The validation is divided into two parts: simulations and experiments. In terms of the simulations, under the same simulation conditions, the simulation time of the full-order model is 13 times longer than that of the reduced-order model, while the simulation results of the reduced-order model are consistent with those of the full-order model. In terms of the experiments, through analysis and comparison, the simulation results and experimental results of the electromechanical system show that the oscillation frequency in the transient state remains the same while the angular deviation in the steady state is not more than 2 mrad. So, the above analysis results demonstrate the reasonableness and practicality of the flexible dynamics model obtained by the above method.

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