Geometric Engineering of Seiberg-Witten Theories with Massive Hypermultiplets

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Abstract

We analyze the geometric engineering of the $N = 2$ $SU(2)$ gauge theories with $1 \leq N_f \leq 3$ massive hypermultiplets in the vector representation. The set of partial differential equations satisfied by the periods of the Seiberg-Witten differential is obtained from the Picard-Fuchs equations of the local B-model. The differential equations and its solutions are consistent with the massless case. We show that the Yukawa coupling of the local A-model gives rise to the correct instanton expansion in the gauge theory, and propose the pattern of the distribution of the world-sheet instanton number from it. As a side result, we obtain the asymptotic form of the instanton amplitude in the gauge theories with massless hypermultiplets.
1 Introduction

Geometric engineering of Seiberg-Witten theories [1] is the technique for extracting the moduli of the Coulomb phase of the four-dimensional $N = 2$ gauge theories from the moduli of the mirror symmetry model. It realizes the former in an infinitesimally small neighborhood of a singularity of the latter. This technology opened a door toward the systematic derivation of exact results in arbitrary $N = 2$ gauge theories, for example see [2, 3, 4]. However, detailed analysis of the prepotentials has been restricted to only one example in the $N = 2 SU(2)$ pure Yang-Mills theory [1]. There has been no attempt to extend such study to the gauge theories with hypermultiplets in various representations.

In this article we will analyze the geometric engineering of the $N = 2 SU(2)$ gauge theories with $1 \leq N_f \leq 3$ massive hypermultiplets in the vector representation. It is necessary for a mirror model corresponding to the $SU(2)$ gauge theory [5] to contain $A_1$-root lattice in $H_2(V, \mathbb{Z})$, where $V$ is a Calabi-Yau three-fold of the A-model. The corresponding non-compact Calabi-Yau manifold was found to be the canonical bundle of the Hirzebruch surface, which has the structure of a fibration of ALE-spaces of $A_1$ type over $\mathbb{P}^1$. It has been noticed in [6, 1] that cases of the massive hypermultiplets should be obtained by $N_f$ point blow ups of Hirzebruch surface $F_2$. The corresponding manifolds again have the structure of a fibration over $\mathbb{P}^1$. We will adopt such local mirror models in [7].

The geometric engineering requires that the moduli of the Seiberg-Witten theory is identified with an infinitesimal neighborhood of a singular point within the moduli of the mirror model. This is the limit of decoupling the gravitational effects by taking the string scale $M_{\text{string}}$ to $\infty$ in the type IIA string compactification. We will call this the gauge theory limit. Precisely, the gauge theory limit is the limit $\epsilon = M_{\text{string}}^{-1} \to 0$ satisfying: (1) on the A-model side, the size of the base $\mathbb{P}^1$ must become divergent as $(4 - N_f) \log \epsilon$ and the size of the exceptional curve of the ALE space must be proportional to $\epsilon a$ where $a$ is the mass of the gauge field; (2) on the B-model side, this must be the limit where the local B-model curve degenerates to the Seiberg-Witten curve. We will show that the gauge theory limit of the local mirror model actually exists, as expected. We will analyze the mirror model around the gauge theory limit and obtain the following two results.

The one result in this article is to obtain the differential equations satisfied by the periods $(a, a_D)$ of the Seiberg-Witten differential. This completes the previous attempts in the massive cases [8, 9, 10, 11]. Taking the gauge theory limit of the Picard-Fuchs operators of the local B-model, we obtained a set of partial differential operators for each $N_f$. We confirmed that these actually annihilate $(a, a_D)$. Then we solved the differential equations: the solutions are two functions $g_1(u, m_i)$, $g_2(u, m_i)$ which are identified with $(a, a_D)$, and the bare mass parameters $m_i (1 \leq i \leq N_f)$. The appearance of the mass parameters is consistent with the fact that the
Seiberg-Witten differential has a linear combination of the mass parameters as its residue.

The other result is to find out the distribution pattern of world-sheet instanton numbers for the Calabi-Yau three-fold $V$ of the local A-model. The case of $N_f = 0$ was analyzed in [1]: there, the asymptotic distribution of the world-sheet instanton numbers is controlled by the instanton amplitude of the gauge theory. Then, it is natural to ask how to extend it to the local A-models corresponding to $N_f$ hypermultiplets ($1 \leq N_f \leq 3$), and we will answer this question affirmatively. Let us explain our results briefly. At the gauge theory limit $\epsilon \to 0$, the Kähler parameter $t_1$ of the base $\mathbb{P}^1$ behaves as $(4 - N_f) \log \epsilon$. The Kähler parameters $t_2, \ldots, t_{N_f+2}$ of the other two-cycles are proportional to $\epsilon$ and linear combinations of $a$ and $m_i$ ($1 \leq i \leq N_f$). We found out that if we denote the two-cycle whose Kähler parameter is $-2\epsilon a$ by $R_0$ and the two-cycle whose Kähler parameter is $-\epsilon(a + m_i)$ by $R_i$, the asymptotic form of the world-sheet instanton number $d_\beta$ for a homology class $\beta = n_1[\mathbb{P}^1] + \sum_{i=0}^{N_f} k_i[R_i]$ is

$$d_\beta \sim \gamma_{n_1} (-1)^{k_1+\cdots+k_{N_f}} (2k_0)^{4n_1-3} \prod_{i=1}^{N_f} n_1 C_{k_i} \quad (0 \leq k_i \leq n_1),$$

at the region $k_0 \gg n_1$ for fixed $n_1 \geq 1$. For the other values of $k_i$’s, $d_\beta$ is negligible. Here $\gamma_{n_1}$ is assumed to depend on $n_1$, and it turned out that $\gamma_{n_1}$ is related to the instanton amplitude $\mathcal{F}_n$ of the $SU(2)$ Seiberg-Witten theory without hypermultiplets as follows:

$$\frac{\gamma_{n_1}}{d_\beta[R_0]} \propto \frac{2 \cdot 4^{n_1} \mathcal{F}_{n_1}}{\Gamma(4n_1 - 2)}.$$  

This relation holds for all the local A-models because of the decoupling of a hypermultiplet in the gauge theories.

Although not directly related to the geometric engineering, we studied the asymptotic form of the instanton amplitude with large instanton number in the Seiberg-Witten theories when all mass parameters are zero. The idea is the same as [12] that such asymptotic form is governed by a singularity of the moduli space.

This article is organized as follows. In section 2, we will give a brief review on the Seiberg-Witten theories, and derive the asymptotic form of the instanton amplitude with large instanton numbers. In section 3 we will analyze mirror models that we use in the geometric engineering. In section 4 we will carry out the geometric engineering of the Seiberg-Witten theories with $N_f$ hypermultiplets. A set of partial differential equations satisfied by periods $(a, a_D)$ and its solutions will be derived in subsection 4.2. We will suggest the pattern of the distribution of the world-sheet instanton number in subsection 4.3. Section 5 includes a conclusion and an outlook. Appendices contain: A: GKZ-hypergeometric differential system, B: Table of the world-sheet instanton numbers ($N_f = 0, 1$), C: the Yukawa coupling at the gauge theory limit ($N_f = 0, 1, 2, 3$), D: the Picard-Fuchs differential operator for the periods $(a, a_D)$ ($N_f = 0, 1, 2$).
We will use the following notations: (1) $$\theta_x := x \partial_x$$ is a logarithmic derivative. (2) $$a^b$$ is a multi-index notation of $$a_1^{b_1}a_2^{b_2} \cdots a_m^{b_m}$$ for given vectors $$a = (a_1, \ldots, a_m)$$ and $$b = (b_1, \ldots, b_m)$$. (3) For a series in several variables with summation $$\sum_{n_i}$$, the summation is assumed over non-negative integers $$n_i$$ so that the arguments in all the factorials are non-negative integers.

2. N = 2 SU(2) Gauge Theories

In this section, we will summarize basic facts about the exact solution of the $$N = 2$$ SU(2) gauge theories. Then in subsection 2.2, we will derive the distribution pattern of the instanton amplitude with large instanton numbers when all mass parameters become zero. This result will be obtained by an application of the techniques in a context of mirror symmetry [12].

2.1 Seiberg-Witten Curves and Periods

The moduli space of the Coulomb branch of the $$N = 2$$ SU(2) gauge theory in four-dimensions with $$0 \leq N_f \leq 3$$ hypermultiplets is determined by a holomorphic function $$F_{gauge}(a)$$ called prepotential [5]. We define an expectation value of the $$2 \times 2$$ matrix-valued complex scalar field $$\phi$$ in the $$N = 2$$ vector multiplet and the parameter $$u$$ of the moduli space as follows

$$u := \frac{1}{2} \langle \text{Tr} \phi^2 \rangle, \quad \langle \phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}. \quad (3)$$

We denote by $$m_1, \ldots, m_{N_f}$$ the bare mass parameters of $$N_f$$ hypermultiplets. In the Seiberg-Witten theory, $$a$$ and $$a_D := \frac{\partial F_{gauge}}{\partial a}$$ are periods of the meromorphic one-form (Seiberg-Witten differential $$\lambda_{SW}$$) on an elliptic curve (Seiberg-Witten curve) parameterized by $$u, m_1, \ldots, m_{N_f}$$ and the dynamical mass parameter $$\Lambda$$.

| $$N_f$$ | $$F(x)$$ | $$H(x)$$ |
|---------|---------|--------|
| 0       | $$x^2 - u$$ | $$\Lambda^4$$ |
| 1       | $$x^2 - u$$ | $$\Lambda^3(x + m_1)$$ |
| 2       | $$x^2 - u + \frac{\Lambda^2}{4}$$ | $$\Lambda^2(x + m_1)(x + m_2)$$ |
| 3       | $$x^2 - u + \frac{\Lambda^2}{4} + \frac{\Lambda(m_1 + m_2 + m_3)}{8}$$ | $$\Lambda(x + m_1)(x + m_2)(x + m_3)$$ |

Table 1: Left: $$F(x)$$ and $$H(x)$$ in (4). Right: $$\alpha, z, C$$ in (8).

The Seiberg-Witten curve and the Seiberg-Witten differential are written as follows [13]

$$y^2 = F(x)^2 - H(x), \quad \lambda_{SW} = \frac{1}{2\pi i} \frac{xdx}{y} \left[ -F'(x) + \frac{F(x)H'(x)}{2H(x)} \right],$$

(4)

3
where \( \prime \) denotes the differentiation with respect to \( x \). The functions \( F(x), H(x) \) are shown in Table 1. The Seiberg-Witten differential \( \lambda_{SW} \) is a meromorphic one-form determined so that it satisfies \( \partial_u \lambda_{SW} \propto \frac{dx}{y} \). Then \( a \) (resp. \( a_D \)) is represented as a period integral of \( \lambda_{SW} \) along the \( \alpha \)- (resp. \( \beta \)-) cycle

\[
a = \oint_{\alpha} \lambda_{SW}, \quad a_D = \oint_{\beta} \lambda_{SW}. \tag{5}
\]

Here we specify the \( \alpha \)- and \( \beta \)-cycles. We have four branching points of the curve at \( x = e_1, e_2, e_3, e_4 \) with \( e_i \) as follows (for \( N_f = 0, e_1 = -\sqrt{u + \Lambda^4}, e_2 = -\sqrt{u - \Lambda^4}, e_3 = \sqrt{u - \Lambda^4}, e_4 = \sqrt{u + \Lambda^4} \))

\[
\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = -\sqrt{u} \pm \frac{1}{2} \left( \frac{H(\sqrt{u})}{u} \right)^{\frac{1}{2}} + O(\Lambda^4 - N_f), \quad \begin{bmatrix} e_3 \\ e_4 \end{bmatrix} = \sqrt{u} \pm \frac{1}{2} \left( \frac{H(\sqrt{u})}{u} \right)^{\frac{1}{2}} + O(\Lambda^4 - N_f), \tag{6}
\]

which give rise to the cuts to run from \( e_1 \) to \( e_2 \) and \( e_3 \) to \( e_4 \). Then \( \alpha \)- (resp. \( \beta \)-) cycle is chosen to be a loop going around a pair of the points \( e_1 \) and \( e_2 \) (resp. \( e_2 \) and \( e_3 \)) counterclockwise as shown in Figure 1.

\[
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\]

Figure 1: \( \alpha \)- and \( \beta \)- cycles.

**Massless case**

First consider the case where all mass parameters are zero. We can tell the behavior of the periods at the weak coupling region \( |u| \to \infty \) to the lowest order in \( \Lambda \),

\[
a \sim \sqrt{u}, \quad a_D \sim \frac{i(4 - N_f)}{2\pi} \sqrt{u} \left[ \log \frac{u}{\Lambda^2} - 2 + \text{const.} \right], \tag{7}
\]

by evaluating the period integrals (5). Meanwhile \( a \) and \( a_D \) are solutions of the Picard-Fuchs differential equation \[14, 15, 16\]

\[
\left( \theta_z + \alpha \right)^2 - z\theta_z(\theta_z + 2\alpha) \left( \frac{a}{a_D} \right) = 0. \tag{8}
\]

The values of the constant \( \alpha \) and the variable \( z \) are given in Table 1. This is an ordinary differential equation of second order with only regular singular points at \( z = 0, 1, \infty \). It has
therefore two solutions. Around \( z = 0 \), they are

\[
w_1(z) = w(z; 0) = z^{-\alpha} \sum_{n=0}^{\infty} \frac{(\alpha)_n(-\alpha)_n}{(n!)^2} z^n, \quad \text{with} \quad w(z; \rho) = \sum_{n=0}^{\infty} \frac{(\alpha + \rho)_n(-\alpha + \rho)_n}{(1 + \rho)_n} z^{n+\rho-\alpha},
\]

\[
w_2(z) = \partial_\rho w(z; \rho)|_{\rho=0} = w_1 \log z + z^{-\alpha} \sum_{n=1}^{\infty} \frac{(\alpha)_n(-\alpha)_n}{(n!)^2} z^n \times \left[ \psi(\alpha + n) + \psi(-\alpha + n) - 2\psi(n + 1) - \psi(\alpha) - \psi(-\alpha) + 2\psi(1) \right],
\]

Comparing the solutions \( w_1(z) \), \( w_2(z) \) and \( a \), \( a_D \) in (7), we obtain \( a = C^\alpha \Lambda w_1(z), \ a_D = \frac{i(4-N_f)}{2\pi} C^\alpha \Lambda \left[ -2\alpha w_2(z) + \text{const.} \ w_1(z) \right] \) where \( C \) is defined by \( z = C(\frac{u}{\Lambda^2})^\frac{1}{2} \).

Substituting the inverted series of \( a \) into \( a_D \) and then integrating it by \( a \), we obtain the prepotential. Since \( u \) is expanded in terms of \( (\frac{\Lambda}{a})^\frac{1}{2} \), so is the instanton correction part of the prepotential. Therefore the form of the prepotential must be

\[
F_{\text{gauge}} = \frac{i(4-N_f)}{4\pi} a^2 \left[ \log \frac{a^2}{\Lambda^2} - \sum_{n=1}^{\infty} F_n \left( \frac{\Lambda}{a} \right)^n \right].
\]

The instanton amplitude \( F_n \) is shown in Table 2.

| \( n \) | \( F_n \) | \( n \) | \( F_n \) |
|---|---|---|---|
| 1 | \( \frac{1}{16} \) | 1 | \( \frac{1}{1024} \) |
| 2 | \( \frac{5}{2048} \) | 2 | \( \frac{5}{8388608} \) |
| 3 | \( \frac{3}{8192} \) | 3 | \( \frac{3}{2147483648} \) |
| 4 | \( \frac{1469}{16777216} \) | 4 | \( \frac{1469}{449448137937} \) |
| 5 | \( \frac{40397}{29514790517935282582} \) | 5 | \( \frac{40397}{866589165} \) |
| 6 | \( \frac{1024996534121312}{5024996534121312} \) | 6 | \( \frac{1024996534121312}{5024996534121312} \) |
| \( \frac{1}{512} \) | \( \frac{1}{4194304} \) | 7 | \( \frac{1}{4194304} \) |
| 2 | \( \frac{3}{1024} \) | 3 | \( \frac{3}{2147483648} \) |
| 4 | \( \frac{1469}{16777216} \) | 4 | \( \frac{1469}{449448137937} \) |
| 5 | \( \frac{40397}{29514790517935282582} \) | 5 | \( \frac{40397}{866589165} \) |
| 6 | \( \frac{1024996534121312}{5024996534121312} \) | 6 | \( \frac{1024996534121312}{5024996534121312} \) |
| \( \frac{1}{512} \) | \( \frac{1}{4194304} \) | 7 | \( \frac{1}{4194304} \) |
| 2 | \( \frac{3}{1024} \) | 3 | \( \frac{3}{2147483648} \) |
| 4 | \( \frac{1469}{16777216} \) | 4 | \( \frac{1469}{449448137937} \) |
| 5 | \( \frac{40397}{29514790517935282582} \) | 5 | \( \frac{40397}{866589165} \) |
| 6 | \( \frac{1024996534121312}{5024996534121312} \) | 6 | \( \frac{1024996534121312}{5024996534121312} \) |

Table 2: Instanton amplitude \( F_n \) with small instanton number \( n \) (1 \( \leq \) \( n \) \( \leq \) 8).

**Massive case**
When values of the mass parameters are generic, the analysis of the instanton correction is not as simple as that in the massless case. Already there exist much effort to obtain the prepotential in the massive case [8, 9, 10]. Rather than review these results, we will only remark on two points that will be of use later; on differential equations satisfied by \((a, a_D)\) and on general structure of the prepotential.

The extension of the Picard-Fuchs equation (8) to the massive case \(p(u, m)\partial_u^2 + q(u, m)\partial_u + r(u, m)\partial_u(a, a_D) = 0\) has been known [8, 9, 10, 11]. This should be regarded as an ordinary differential equation with one variable \(u\) keeping the mass parameters \(m_i\) \((1 \leq i \leq N_f)\) constant. This equation, in principle, allows us to obtain \(a\) and \(a_D\), and actually the instanton correction to some low order in \(\Lambda\) has been calculated. However, the complicated expression of \(p(u, m_i), q(u, m_i), r(u, m_i)\) makes it difficult to solve the equation completely.

As it turns out, we could obtain exact solutions from a different approach. In section 4 we will show that the geometric engineering of the gauge theory provides a set of partial differential equations that \(a\) and \(a_D\) must satisfy. There, the mass parameters \(m_i\) as well will be treated as variables. We will solve the equations and obtain expressions of \(a\) and \(a_D\) for generic values of the mass parameters. Moreover, we will obtain an ordinary differential operator with respect to \(u\) and it is consistent with the known results and the massless Picard-Fuchs equation (8).

Next, we turn to general structure of the prepotential. The parameters in the case of \(N_f\) and that of \(N_f - 1\) are related by the following limit corresponding to the decoupling of a hypermultiplet:

\[
m_{N_f} \rightarrow \infty, \quad m_{N_f}\Lambda^{4-N_f} \rightarrow \Lambda^{4-(N_f-1)}.
\]  

In the second relation, \(\Lambda\) in the left-hand side is that of the gauge theory with \(N_f\) hypermultiplets and one in the right-hand side is that of the gauge theory with \(N_f - 1\) hypermultiplets. This relation imposes a very strong constraint on the possible form of \(u\) as a inverse series of \(a\) and on that of \(\partial_3^3\mathcal{F}_{gauge}\).

First, \(u\) must take the form

\[
u = a^2 \left[ 1 + \sum_{n=1}^{\infty} b_n \cdot \left( \frac{\Lambda}{a} \right)^{(4-N_f)n} \right],
\]

\[
b_n : \text{polynomial of degree } n \text{ in each } \frac{m_i}{a}, \text{ symmetric with respect to } \frac{m_1}{a}, \cdots, \frac{m_{N_f}}{a}.
\]

The reason for the form of \(b_n\) is clear for \(N_f = 0\) because \(\frac{m_i}{a^2}\) is a series in \((\frac{\Lambda}{a})^4\), see (10). For \(N_f \geq 1\), this form is determined so that the instanton correction comes in the power of \(\Lambda^{4-N_f}\) and the terms of \(\Lambda^{4-N_f}\) descend to those of \(\Lambda^{5-N_f}\) at the limit (11) of the decoupling of a hypermultiplet.
Second, $\partial_a^3 F_{\text{gauge}}$ must take the following form

$$
\partial_a^3 F_{\text{gauge}} \propto \frac{8}{a} + \sum_{i=1}^{N_f} \left( \frac{1}{a + m_i} + \frac{1}{a - m_i} \right) - \frac{8}{a} \sum_{n=1}^{\infty} P_n \cdot \left( \frac{\Lambda}{a} \right)^{(4-N_f)n},
$$

$$
P_n = B_n \prod_{i=1}^{N_f} \left( \frac{m_i}{a} \right)^n + \cdots \quad \text{polynomial of degree } n \text{ in each } \frac{m_i}{a}, \text{ symmetric with respect to } \frac{m_1}{a}, \ldots, \frac{m_{N_f}}{a}.
$$

Note that the terms $\frac{1}{a}, \frac{1}{a \pm m_i}$ correspond to the one-loop corrected terms while the higher order terms in $\Lambda$ correspond to contributions of the instanton correction. Here $B_n$ is a number common to all the cases with $N_f = 0, 1, 2, 3$. And it is written in terms of the instanton amplitude $F_n$ of the gauge theory without hypermultiplets ($N_f = 0$) in (10) as follows

$$
B_n = \frac{4n(4n-1)(4n-2)F_n}{4}.
$$

The reason for this form is the following. Consider first the case of $N_f = 0$. From the Picard-Fuchs equation (8), we can obtain the relation

$$
\partial_a^3 F_{\text{gauge}} \propto \frac{1}{u-\Lambda} \quad \text{(derived from the argument in next paragraph)}.
$$

Considering that $\frac{n}{a}$ is a series in $(\frac{\Lambda}{a})^4$, $\partial_a^3 F_{\text{gauge}}$ must be proportional to $\frac{1}{a} \times \left[ \text{series in } (\frac{\Lambda}{a})^4 \right]$. Thus (13) is proved for the case with $N_f = 0$. We denote by $B_n$ the coefficient of $\frac{1}{a}(\frac{\Lambda}{a})^{4n}$ divided by that of $\frac{1}{a}$. Next consider the case with $N_f \geq 1$. The one-loop correction gives rise to not only the term $a^2 \log \frac{a}{\Lambda}$ but also the terms $(a \pm m_i)^2 \log \frac{a \pm m_i}{\Lambda}$ because a massless particle also appears at $a = \pm m_i$. This comes from the terms in the Lagrangian, $m_i \tilde{Q}_i Q_i + \tilde{Q}_i \Phi Q_i$, with the $N = 1$ chiral superfield $\Phi$ in the $N = 2$ vector multiplet and two $N = 1$ chiral superfields $Q_i, \tilde{Q}_i \ (1 \leq i \leq N_f)$ in the $N = 2$ hypermultiplets. The ratio of coefficients of $\frac{1}{a}$ and $\frac{1}{a \pm m_i}$ are determined as follows: if we assume the ratio to be $1$ to $r$, then the prepotential at the massless limit is proportional to $(1 + 2r)a^2 \log a + \cdots$. Comparison with (10) gives $r = -\frac{1}{8}$. As to the instanton correction term, it must take the form above by the consistency with the decoupling limit of a hypermultiplet.

We note the identity

$$
\partial_a^3 F_{\text{gauge}} = \left( \frac{da}{du} \right)^{-3} \left( \frac{d^2 a_D}{du^2} \frac{da}{du} - \frac{d^2 a_D}{du^2} \frac{da}{du} \right).
$$

The second factor in the right hand side is the Wronskian of $\partial_a a$ and $\partial_a a_D$. If we knew the differential operator that annihilates $(a, a_D)$, we would have the differential operator of the form

$$
\mathcal{P} = P(u, m_i) \partial_a^3 + Q(u, m_i) \partial_a^2 + R(u, m_i) \partial_a,
$$

to provide the Wronskian as follows

$$
\frac{d^2 a_D}{du^2} \frac{da}{du} - \frac{d^2 a_D}{du^2} \frac{da}{du} \propto e^{-\int du \frac{\partial a}{\mathcal{P}}}.
$$

This follows from the differential equation $\partial_a a \mathcal{P} a_D - \partial_a a_D \mathcal{P} a = (P \partial_a + Q)(\partial_a a_D a_D - \partial_a a_D \partial_a^2 a) = 0$. We will actually derive the differential operator in section (1). Then we will obtain expressions
of \( u \) and \( \partial_a^3 F_{\text{gauge}} \) for generic values of the mass parameters to some low orders in \( \Lambda \), and find that they actually satisfy (12) and (13).

2.2 Asymptotic Form of Instanton Amplitude

Now let us return to the massless case. We could see from Table 2 that the instanton amplitude \( F_n \) decreases rapidly as \( n \) increases. Such distribution of the instanton amplitude turns out to be governed by a singularity of the moduli space at \( z = 1 \). We will derive the asymptotic form of the instanton amplitude \( F_n \) by the analysis of \( \partial_a^3 F_{\text{gauge}} \) around \( z = 1 \).

Before proceeding, we must remark that our analysis was inspired by and is almost the same as that of the number of rational curves of large degree in the quintic hypersurface in \( \mathbb{P}^4 \). The apparent difference that our analysis is on the instanton amplitude of the gauge theory, and the one in [12] is on the world-sheet instanton number of the quintic, does not matter here. It is because quantities used in the analysis are determined only by a generalized hypergeometric differential equation, where similarity between the two system lies.

Note that the third derivative of the prepotential can be written as follows

\[
\partial_a^3 F_{\text{gauge}} = C_2 \left( \frac{dw_1}{dz} \right)^{-3} \frac{\alpha^2}{(1-z)z^{2\alpha+3}},
\]

where \( C_1, \text{resp. } C_2 \) is the coefficient of \( w_1(z) \) (resp. \( w_2(z) \)) in \( a \) (resp. \( a_D \)) divided by \( \Lambda \). The last factor in the right-hand side has come from the Wronskian of \( \partial_\alpha a \) and \( \partial_a a_D \). Together with the behavior of \( w_1(z) \) around \( z = 1 \)

\[
w_1(z) \sim \frac{\sin \pi \alpha}{\pi \alpha} \left[ 1 + \alpha^2 z' \log z' + \cdots \right], \quad z' := \frac{z - 1}{z},
\]

we can see that (18) diverges at \( z = 1 \). Therefore the radius of convergence of the instanton expansion (10) is determined by the value of \( a \) at \( z = 1 \). The instanton expansion converges on the domain \( |a(z)| < |a(1)| \) and, by the theorem of Hadamard, the asymptotic form of \( F_n \) should satisfy

\[
\lim_{n \to \infty} \frac{\sqrt{F_n}}{n} = \left| \frac{\Lambda}{a(1)} \right|^{\frac{1}{\beta}}.
\]

To obtain a little more elaborate asymptotic form of \( F_n \) for large \( n \), let us adopt the ansatz similar to the one in [12]

\[
(\beta n - 2)(\beta n - 1)(\beta n)F_n \sim Bn^\lambda (\log n)^\mu \left| \frac{a(1)}{\Lambda} \right|^{\beta n} (n \gg 1),
\]

with three constants to be determined, \( \lambda, \mu, B \). \( \beta \) is \( \frac{1}{\alpha} = 4, 6, 4 \) for \( N_f = 0, 1, 2 \). For \( N_f = 3 \), the instanton expansion is actually an expansion by \( (\frac{\Lambda}{a})^4 \) rather than \( (\frac{\Lambda}{a})^2 \), and we should redefine \( F_{2n} \) as \( F_n \) and set \( \beta = 4 \). Substituting this into (10), differentiating by \( a \) three times and evaluating it around \( z' = 0 \), we obtain

\[
\partial_a^3 F_{\text{gauge}} \sim \frac{i(4-N_f)B\Gamma(\lambda+1)(-\log z')^\mu}{4\pi a} \left( \frac{\alpha^2 \beta}{\lambda+1}(z' \log z')^{\lambda+1} \right), \quad z' \sim 0.
\]
In the process we have replaced the summation in $n$ with an integration and changed the variable $n$ into $t := n\beta \log \frac{a(1)}{a(0)}$. On the other hand, near $z' = 0$, the expression (18) becomes

$$\frac{C_{\alpha}}{c_{\alpha}^2 \Lambda} \left( \frac{\pi \alpha}{\sin \pi \alpha} \right)^3 \frac{1}{\alpha^4} \frac{-1}{z'(\log z')^2}.$$ Comparing these, we could obtain $\lambda = 0$, $\mu = -2$ and $B = \frac{2\beta}{\alpha} \left( \frac{\pi \alpha}{\sin \pi \alpha} \right)^2$.

Therefore we conclude that the asymptotic form of the instanton amplitude is as follows

$$F_n \sim \frac{4}{\alpha \beta^2} \left( \frac{\pi \alpha}{\sin \pi \alpha} \right)^2 \left( \frac{|C_{\alpha}| \sin \pi \alpha}{\pi \alpha} \right)^{\beta n} \frac{1}{n^3 (\log n)^2} =: r_n \quad (n \gg 1).$$ (22)

Note that $r_n$ declines as $n$ becomes large because in our cases, $|\frac{a(1)}{\Lambda}| = \frac{|C_{\alpha}| \sin \pi \alpha}{\pi \alpha}$ is smaller than 1. (0.900316, 0.656385, 0.31831, 0.0397887 for $N_f = 0, 1, 2, 3$.) Note also that the convergence would be very slow because of the factor $(\log n)^{-2}$ in $r_n$.

We plot the logarithm of ratio $\log_e \frac{F_n}{r_n}$ up to $2 \leq n \leq 32$ in Figure 2. Clearly the data is not enough to show the convergence to 1, but it is natural believe that the computation up to higher value of $n$ would show the correctness of the asymptotic form (22).

![Figure 2](image.png)

Figure 2: $\log_e \frac{F_n}{r_n}$. $N_f = 0$ (top left), 1 (bottom left), 2 (top right) and 3 (bottom right).

### 3 Models of Mirror Symmetry

In this section we will construct four examples of the mirror symmetry. These will appear as the mirror models that reproduce the Seiberg-Witten theories in the previous section via...
geometric engineering in the next section. First we will construct them within the framework of local mirror symmetry [7] in subsection 3.1. Then in subsection 3.2 we will derive their Yukawa coupling using the framework of mirror symmetry [17, 18, 19]. We note that a compactification of the non-compact Calabi-Yau three-fold of the local A-model is the Calabi-Yau three-fold of the A-model. We will utilize both frameworks because the former is convenient to identify the Seiberg-Witten theory, and the latter is necessary to see a structure of the Yukawa coupling. A definition of the Yukawa coupling of the B-model is not yet clear in the framework of local mirror symmetry.

3.1 Local Mirror Construction

The local mirror symmetry [7] is the duality between two moduli spaces. The one is the complexified Kähler moduli of the canonical bundle $V$ of a two-dimensional toric variety $P_{\text{base}}$. On the Kähler moduli, there is a holomorphic function called prepotential which can be written in terms of world-sheet instanton numbers. The other is the moduli space of monomial deformations of the curve in another two-dimensional toric variety. There, period integrals of a meromorphic one-form on the curve.

A local mirror model is constructed from a two-dimensional polytope. We will study four local mirror models associated with the reflexive polytopes $\Delta_{\text{local}}$ shown in Figure 3. The word “Model $i$” ($0 \leq i \leq 3$) under each polytope means that we will call the local mirror model constructed from it by Model $i$. Integral points in each polytope are denoted by $\nu_1, \cdots, \nu_r$ ($r = k + 3$), see Table 3.

Figure 3: Polytope $\Delta_{\text{local}}$. 
Let us define the lattice $L$

\[
\tilde{\nu}_i := \begin{pmatrix} 1 \\ \nu_i \end{pmatrix} \in \mathbb{Z}^3, \quad A := \left( \tilde{\nu}_1, \ldots, \tilde{\nu}_{k+3} \right) : 3 \times (k + 3) \text{ matrix},
\]

\[
L := \left\{ l \in \mathbb{Z}^{k+3} : A \cdot l = \vec{0} \right\},
\]

$l^{(1)}, \ldots, l^{(k)}$ : basis of $L$.

Our choice of the basis $l^{(i)}$ is shown in Table 4.

\[
\begin{array}{c|ccc|ccc}
\text{Model} & l^{(i)}, & (1 \leq i \leq k) & \text{Model} & l^{(i)}, & (1 \leq i \leq k) \\
0 & \begin{bmatrix} 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & 0 & -2 \end{bmatrix} & 2 & \begin{bmatrix} 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} \\
1 & \begin{bmatrix} 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 & -1 \end{bmatrix} & 3 & \begin{bmatrix} 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}
\end{array}
\]

Table 4: The basis $l^{(1)}, \ldots, l^{(k)}$. $i$-th row of the matrix is $l^{(i)}$.

Local $A$-model

Let us regard a polytope $\Delta_{\text{local}}$ as a two-dimensional complete fan $\Sigma_{\text{local}}$ whose 1-cones have $\nu_i$ ($1 \leq i \leq k + 2$) as its generators; cones in $\Sigma_{\text{local}}$ are

\[
\begin{align*}
2\text{-cones}: & \left[ \nu_i, \nu_{i+1} \right] & (1 \leq i \leq k + 1), & \left[ \nu_1, \nu_{k+2} \right], \\
1\text{-cones}: & \left[ \nu_i \right] & (1 \leq i \leq k + 2), & 0\text{-cone}: \left\{ \right\}.
\end{align*}
\]

Here $[v_1, \ldots, v_j]$ means the cone spanned by vectors $v_1, \ldots, v_j \in \mathbb{Z}^2$, i.e. the set of points in $\mathbb{R}^2$ which are a linear combination of $v_1, \ldots, v_j$ with non-negative real coefficients. Then, we can
construct $\mathbf{P}_{\text{base}}$ of the canonical bundle $V$ as the toric variety $\mathbf{P}_{\Sigma_{\text{local}}}$: $\mathbf{P}_{\text{base}}$ is the Hirzebruch surface $\mathbf{F}_2$ for Model 0 and its $N_f$ point blow ups for Model $N_f$ ($1 \leq N_f \leq 3$).

We note that $H_2(\mathbf{P}_{\text{base}}, \mathbf{Z}) = H_2(V; \mathbf{Z})$ can be identified with the lattice $L$. Let us denote by $D_i$ a divisor corresponding to 1-cone $\nu_i$ ($1 \leq i \leq k + 2$). Between the divisors, there are two linear equivalence relations $\sum_{i=1}^{k+2} \nu_i D_i = 0$. Thus, if we define a vector $l = (C \cdot D_1, \ldots, C \cdot D_{k+2}, -C \cdot (D_1 + \cdots + D_{k+2}))$ from a two-cycle $C$ in $V$, it satisfies $A \cdot l = 0$. Thus we can identify the space $L = \ker_Z A$ with $H_2(V; \mathbf{Z})$. We will identify the basis $\{l^{(1)}, \ldots, l^{(k)}\}$ of $L$ with the basis of $H_2(V; \mathbf{Z})$, and also use $l^{(i)}$ to denote the corresponding two-cycle.

We study the complexified Kähler moduli space of $V$ which is the space of complexified Kähler classes. The complexified Kähler class is a cohomology class of the sum $\omega$ of a two-form $B$ and a Kähler form $J$ of $V$ multiplied by $\sqrt{-1}$: $\omega = B + iJ$. We define the Kähler parameter of the two-cycle $l^{(i)}$

$$t_i := \oint_{l^{(i)}} 2\pi i \omega \quad (1 \leq i \leq k).$$

Then the object of the study is the prepotential. Its general form is

$$F_{\text{mirror}} = \sum_{1 \leq i,j,p \leq k} \frac{J_i \cdot J_j \cdot J_p}{6} t_i t_j t_p + \sum_{1 \leq i \leq k} c_2 \cdot J_i t_i - i \frac{\zeta(3)}{2(2\pi)^3} c_3 + \sum_{n_1,\ldots,n_k} d_{\vec{n}} \text{Li}_3(e^{n_1 t_1 + \cdots + n_k t_k}),$$

where $d_{\vec{n}} = (n_1, \ldots, n_k)$ is the world-sheet instanton number. $J_i \in H^2(V, \mathbf{R})$ is the dual of the two-cycle $l^{(i)}$ by the pairing of $H^2(V, \mathbf{R})$ and $H_2(V, \mathbf{R})$. $\text{Li}_3(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^3}$, $c_2$ is the second Chern class of $V$ and $c_3$ is the third Chern number of $V$. The sum in the last term is over $(n_1, \ldots, n_k)$ such that $\sum_i n_i l^{(i)}$ is the homology class of curves in $V$. For Model 0, 1 and 2, the summation is over $\forall n_i \geq 0$. For model 3, the summation should be over the domain where $n_1, n_3, n_4, n_5 \geq 0$ and for given $(n_1, n_3, n_4, n_5)$, over $n_2$ that satisfies $n_2 \leq n_5$ or $n_2 \leq 2n_3$ or $n_2 \leq n_3, n_4$. This is because $l^{(2)}$ does not correspond to a curve in $V$ (see also Table 5). The Yukawa coupling is defined to be the third derivative of the prepotential by the Kähler parameters

$$F_{t_it_jt_p} := \partial_{t_i} \partial_{t_j} \partial_{t_p} F_{\text{mirror}} = J_i \cdot J_j \cdot J_p + \sum_{n_1,\ldots,n_k} d_{\vec{n}} \frac{e^{n_1 t_1 + \cdots + n_k t_k}}{1 - e^{n_1 t_1 + \cdots + n_k t_k}}.$$  

**Local B-model**

The curve of the local B-model is a hypersurface in the toric variety $\mathbf{P}_{\Delta_{\text{local}}}$

$$P_{\text{local}}(X_1, X_2) = a_1 X_1^{\nu_1} + \cdots + a_{k+2} X_1^{\nu_{k+2}} + a_{k+3} = 0,$$

with $k + 3$ parameters $a_1, \ldots, a_{k+3}$. $X_1, X_2$ are local coordinates of $\mathbf{P}_{\Delta_{\text{local}}}$. The dimensions of moduli space of the curve is $k$ because we must subtract from the number of parameters $k + 3$ the dimensions of the toric automorphism $X_i \partial{X_i}$ ($1 \leq i \leq 2$) and one degree of freedom of
multiplying $P_{\text{local}}$ by a constant. Then, we have the following local coordinates of the moduli space using the basis of $L$

$$z_i = a^{l(i)} \quad (1 \leq i \leq k).$$

(29)

| Model | $l$ |
|-------|-----|
| 0     | $l^{(1)}, l^{(2)}$ |
| 1     | $l^{(1)}, l^{(2)}, l^{(3)}, l^{(1)} + l^{(2)} + l^{(3)}$ |
| 2     | $l^{(1)}, l^{(2)}, l^{(3)}, l^{(4)}, l^{(1)} + l^{(2)} + l^{(3)} + l^{(4)}$, $l^{(2)} + l^{(3)} + l^{(4)}, l^{(1)} + 2l^{(2)} + l^{(3)}$ |
| 3     | $l^{(1)}, l^{(3)}, l^{(4)}, l^{(5)}, l^{(2)} + l^{(5)}, l^{(3)} + l^{(4)}, l^{(1)} + l^{(2)} + l^{(5)}$, $l^{(2)} + l^{(3)} + l^{(5)}, l^{(2)} + l^{(3)} + l^{(4)} + l^{(5)}$, $l^{(1)} + l^{(2)} + l^{(3)} + l^{(4)} + l^{(5)}$ |

Table 5: The minimal generators of the toric ideal. A vector $l$ corresponds to a differential operator $\Pi_{i: l_i > 0} \partial_{a_i}^{l_i} - \Pi_{i: l_i < 0} \partial_{a_i}^{-l_i}$ in the toric ideal.

On the moduli space, we consider period integrals of a meromorphic one-form

$$\int_{P_{\text{local}} = 0} \log P_{\text{local}} \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2}. \quad (30)$$

Then, the Picard-Fuchs equations that annihilate this period integrals are provided by the GKZ-hypergeometric differential system $H_A(\beta)$ with $A$ in [23] and $\beta = (0, 0, 0)$ [7]. $H_A(\beta)$ consists of two parts. The one is the part made of the three differential operators

$$\theta_{a_1} + \cdots + \theta_{a_{k+3}}, \quad \sum_{i=1}^{k+3} (\nu_i)_j \theta_{a_i} \quad (j = 1, 2). \quad (31)$$

The other is the toric ideal part generated by

$$\prod_{i: l_i > 0} \partial_{a_i}^{l_i} - \prod_{i: l_i < 0} \partial_{a_i}^{-l_i} \quad \forall l \in L. \quad (32)$$

This toric ideal is actually generated by finite number of generators (see Appendix A). We show the generators in Table 5. The first part (31) means that the period integrals depend only on $z_i$’s and a differential operator $\theta_{a_j}$ acts on the period integrals as $(l^{(i)})_j \theta_{z_i}$. Therefore, if we write differential operators in the toric ideal using $\theta_{a_j}$’s and $z_i$’s and further replace $\theta_{a_j}$ with $(l^{(i)})_j \theta_{z_i}$, we arrive at Picard-Fuchs operators. For example, in the case of Model 1, the
Picard-Fuchs operators are
\[
\mathcal{L}_1 = \theta_1(\theta_1 - \theta_2 + \theta_3) - z_1(-2\theta_1 + \theta_2)(-2\theta_1 + \theta_2 - 1),
\]
\[
\mathcal{L}_2 = (\theta_2 - \theta_3)(-2\theta_1 + \theta_2) - z_2(\theta_1 - \theta_2 + \theta_3)(-\theta_2 - \theta_3),
\]
\[
\mathcal{L}_3 = \theta_3(-\theta_1 + \theta_2 - \theta_3) - z_3(\theta_2 - \theta_3)(-\theta_2 - \theta_3),
\]
\[
\mathcal{L}_4 = (\theta_2 - \theta_3)\theta_1 - z_1z_2(-2\theta_1 + \theta_2)(-\theta_2 - \theta_3),
\]
\[
\mathcal{L}_5 = \theta_3(-2\theta_1 + \theta_2) - z_2z_3(-\theta_2 - \theta_3)(-\theta_2 - \theta_3 - 1).
\]
Now we can obtain the period integrals as solutions of the Picard-Fuchs equations. Let us define
\[
f(z; \rho) = \sum_{\vec{n} = (n_1, \ldots, n_k)} a_{\vec{n}} z^{\vec{n} + \rho}, \quad a_{\vec{n}} = \prod_{i=1}^{k+3} \frac{\Gamma(1 + \sum_{j=1}^{k} \rho_j (l^{(j)}_i))}{\Gamma(1 + \sum_{j=1}^{k} (n_j + \rho_j) (l^{(j)}_i))},
\]
where \( \rho = (\rho_1, \ldots, \rho_k) \) is a set of parameters. Then the solutions are
\[
f(z; \rho) = 1,
\]
\[
\partial_{\rho_i} f(z; \rho)|_{\rho = \vec{0}} = \log z_i + \cdots \quad (1 \leq i \leq k),
\]
\[
\sum_{1 \leq i \leq j \leq k} c_{i,j} \partial_{\rho_i} \partial_{\rho_j} f(z; \rho)|_{\rho = \vec{0}} = \sum_{1 \leq i \leq j \leq k} c_{i,j} \log z_i \log z_j + \cdots.
\]
Coefficients \((c_{i,j})\) in the last double logarithmic solution are shown in Table 6. These coefficients can be obtained from the formula in (7) or by directly substituting the solution into the Picard-Fuchs equations. There are enough equations \((\frac{k(k+1)}{2} - 1)\) of them for the \(\frac{k(k+1)}{2}\) coefficients \(c_{ij}\) \((1 \leq i \leq j \leq k)\).

| Model | \((c_{i,j})\) |
|-------|--------------|
| 0     | \((0, 1, 1)\) |
| 1     | \((0, 2, 2, 2, 4, 1)\) |
| 2     | \((0, 2, 2, 2, 4, 4, 1, 2, 0)\) |
| 3     | \((0, 4, 4, 4, 4, 4, 8, 8, 8, 3, 4, 8, 0, 8, 2)\) |

Table 6: The coefficients \((c_{i,j})\) in the double logarithmic solution. They are arranged in the lexicographic order \((c_{1,1}, c_{1,2}, \ldots, c_{2,2}, \ldots, c_{k,k})\).

The local mirror symmetry states that the single logarithmic solutions in (35) constitute the mirror map to the Kähler parameters of the local A-model as follows
\[
t_i = \partial_{\rho_i} f(z; \rho)|_{\rho = \vec{0}} = \log z_i + \cdots \quad (1 \leq i \leq k).
\]
And the double logarithmic solution in (35) is translated into the following derivative of the prepotential (26) under the mirror map (36)
\[
\{-2\partial_{t_2}, -(\partial_{t_2} + \partial_{t_3}), -(\partial_{t_2} + \partial_{t_4}), -(\partial_{t_4} + \partial_{t_5})\} \mathcal{F}_{\text{mirror}},
\]
for Model 0, 1, 2 and 3. This allows us to obtain the world-sheet instanton numbers of the canonical bundle \( V \) of the local A-model (see Table in Appendix B for the results of \( N_f = 0, 1 \)).

### 3.2 Yukawa Coupling

In this subsection, we will derive the Yukawa coupling. Since the Yukawa coupling of the B-model is not yet incorporated into the framework of local mirror symmetry, we must resort to the mirror symmetry. We will construct a mirror model corresponding to each local mirror model by reversing the process that leads to the local mirror symmetry [7]. We will only give minimal explanation. The reader may consult [17, 18, 7] for more detail.

We construct a following four-dimensional polytope \( \Delta^* \) which contain \( \Delta_{\text{local}} \) as a two-dimensional face

\[
\Delta^* = \text{conv}[\nu_1, \cdots, \nu_r, \nu_{r+1}, \nu_{r+2}],
\]

\[
\nu_i = (\nu_i, 2, 3) \quad (1 \leq i \leq r), \quad \nu_{r+1} = (0, 0, -1, 0), \quad \nu_{r+2} = (0, 0, 0, -1).
\]

\[
(38)
\]

\text{conv}[p_1, \cdots, p_s] \text{ means the convex hull of points } p_1, \cdots, p_s, \text{ i.e. the set of all points in } \mathbb{R}^4 \text{ that can be written as } c_1 p_1 + \cdots + c_s p_s \text{ with } \forall c_j \geq 0, c_1 + \cdots + c_s = 1. \text{ This is a reflexive integral polytope, and has the dual integral polytope } \Delta. \text{ We will denote by } \Sigma (\text{resp. } \Sigma^*) \text{ a fan determined by a maximal triangulation of } \Delta^* (\text{resp. } \Delta).

#### A-model

A Calabi-Yau three-fold \( \tilde{V} \) in the A-model is realized as a hypersurface in the toric variety \( P_\Sigma \) associated to the fan \( \Sigma \). This is a compactification of the canonical bundle \( V \) of \( P_{\text{base}} \) in the local A-model: \( \tilde{V} \) has a structure of a fibration of elliptic curves over \( P_{\text{base}} \). Quantities in the complexified Kähler moduli space of \( \tilde{V} \) are defined in the same way as the local A-model. Let us denote by \( t_0 \) the Kähler parameter corresponding to the elliptic fiber. This is an integral of a complexified Kähler form over the elliptic fiber. Then, the Kähler moduli space of \( V \) of the local A-model appears at the limit \( t_0 \to -\infty \).

#### B-model

The moduli space of the B-model is that of complex deformations of another Calabi-Yau three-fold \( \tilde{V}^* \). It is a hypersurface determined by \( P^* = 0 \) in the toric variety \( P_{\Sigma^*} \).

\[
P^* = 1 + z_0 X_3^2 X_4^3 P_{\text{local}} + \frac{1}{X_3} + \frac{1}{X_4}.
\]

\[
X_1, \cdots, X_4 \text{ are local coordinates of } P_{\Sigma^*} \text{ and } z_0 \text{ is a one more parameter other than } a_1, \cdots, a_{k+3}. \text{ The limit } t_0 \to -\infty \text{ in the A-model corresponds to the limit } z_0 \to 0. \text{ On this moduli space, we consider period integrals of a holomorphic three-form } \Omega \text{ over three-cycles in } \tilde{V}^*. \text{ The period }
integrals are solutions of Picard-Fuchs equations. The Picard-Fuchs operators of the B-model are provided with the Picard-Fuchs operators of the local B-model $L_i$

$$\tilde{L}_i = \mathcal{L}_i \quad \text{with } \theta_{a_{k+3}} \text{ replaced by } \theta_{z_0} + \theta_{a_{k+3}}.$$ (40)

Then, we have a mirror map by solutions of the Picard-Fuchs equations.

We are ready to introduce the Yukawa coupling of the mirror B-model. We can always fix a symplectic basis of $H_3(\tilde{V}^*, \mathbb{Z})$, with $\alpha_{a_1}^*, \beta_{I^*}^I = 1, \ldots, h^{2,1} + 1 \ (h^{2,1} = k + 1)$ satisfying $\alpha_{a_1}^* \cdot \beta_{I^*}^I = \delta_{a_1}^I$, $\alpha_{a_1}^* \cdot \alpha_{a_2}^* = \beta_{I^*}^I \cdot \beta_{J^*}^J = 0$, and define the period integrals

$$A^I := \int_{\alpha_{a_1}^I} \Omega, \quad B_I := \int_{\beta_{I^*}^I} \Omega.$$ (41)

Then, the Yukawa coupling of the B-model is defined as

$$F_{z_iz_iz_p} := \sum_{I=1}^{k+2} \left[ A^I(z) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_p} B_I(z) - B_I(z) \frac{\partial}{\partial z_i} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_p} A^I(z) \right].$$ (42)

This is a contravariant tensor of rank three because of the Griffith transversality. The Yukawa coupling is determined from the Picard-Fuchs operators (40). We note that it does not depend on a choice of the symplectic basis.

We can obtain the Yukawa coupling of the local B-model at the limit $z_0 \to 0$. The results are too long to list here. In the next section, we will transform the variables of the B-model into $(z_0, \cdots, z_k)$ to the variables $(z_0, \epsilon, u, m_1, \cdots, m_{k-2})$. Here $(u, m_1, \cdots, m_{k-2})$ are parameters of the Seiberg-Witten theory and $\epsilon$ is another parameter to be $\epsilon \to 0$. We will first transform $F_{z_iz_iz_p}$ ($0 \leq i, j, p \leq k$) into $F_{z_0z_0z_0}, \cdots, F_{uuu}, \cdots$ and then obtain the Yukawa coupling of the A-model $F_{t_it_jt_p}$ as a transform of these by the mirror map. Remarkably, we find out that we can neglect $F_{z_0**}$ and even $F_{**}$ (here $* = z_0, \epsilon, u, or \ m_i$) to obtain the Yukawa coupling of the local A-model $F_{t_it_jt_p}$ with $2 \leq i, j, p \leq k$ at the gauge theory limit $z_0 \to 0$ and $\epsilon \to 0$.

4 Geometric Engineering of Seiberg-Witten Theories

In this section, we will carry out the geometric engineering of the Seiberg-Witten theories. We will pick up the Model $N_f$ as the local mirror model corresponding to the gauge theory with $N_f$-hypermultiplets [1, 6]. In subsection 4.1, we will first identify the moduli coordinates so that the curve of the local B-model reproduces the Seiberg-Witten curve at the gauge theory limit $\epsilon \to 0$. Then we will study the behavior of the period integrals under the limit $\epsilon \to 0$. We will also obtain a set of differential equations that annihilates the period integrals $(a, a_B)$ and check the equivalence of the prepotentials at the gauge theory limit. These will be addressed in
Finally, in subsection 4.3 we will suggest that the asymptotic distribution of the world-sheet instanton numbers of the local A-model is controlled by the instanton amplitude of the gauge theory. This is an extension of the argument about Model 0 in [1].

4.1 Identification of Moduli Coordinates

We give the identification of the moduli coordinates \((z_1, z_2, \cdots, z_k)\) \((k = N_f + 2)\) of a local B-model with parameters \((\epsilon, u, m_1, \cdots, m_{N_f})\) at the gauge theory limit \(\epsilon \to 0\) in Table 7. This transforms the curve of the local B-model into the Seiberg-Witten curve (4) at the limit \(\epsilon \to 0\). This indicates that the moduli space of the Seiberg-Witten theory is realized as an infinitesimal neighborhood of a singularity in the moduli space of the local mirror model [1]. We can check that the discriminant of the local B-model reduces that of the Seiberg-Witten theory. Then, we expect that period integrals of the local B-model separate into irreducible spaces under the monodromy transformation, and one of them should be identified with the space of periods \((a, a_D)\) of the Seiberg-Witten differential. We note that the space includes the mass parameters of the gauge theory. We will make this expectation explicit in next subsection.

| \(N_f\) | \((z_i)\) \((i = 1, \ldots, N_f + 2)\) |
|--------|------------------------------------------|
| 0      | \((z_1, z_2) = \left(\frac{\epsilon^2 \Lambda^2}{4}, \frac{1 - u \epsilon^2}{4}\right)\) |
| 1      | \((z_1, z_2, z_3, z_4) = \left(\frac{\epsilon^2 \Lambda^2}{4}, \frac{1 - u \epsilon^2}{4}, \frac{1 - m_1 \epsilon}{2}\right)\) |
| 2      | \((z_1, z_2, z_3, z_4, z_5) = \left(\frac{\epsilon^2 \Lambda^2}{4}, \frac{1 - u \epsilon^2}{4}, \frac{1 - n \epsilon^2}{4}, 1 + m \epsilon, 1 + m \epsilon\right)\) |
| 3      | \((z_1, z_2, z_3, z_4, z_5, z_6) = \left(\frac{\epsilon \Lambda}{4}, \frac{1 - u \epsilon^2}{4}, \frac{1 - n \epsilon^2}{4}, 1 + m \epsilon, \frac{1 + m \epsilon}{2}\right)\) |

Table 7: The change of the moduli coordinates of the local B-model.

4.2 Set of Partial Differential Equations

In this subsection, we will relate the prepotential of the local A-model to that of the Seiberg-Witten theory. For that, we will derive the behavior of the Kähler parameters \(t_1, \cdots, t_k\) \((k = N_f + 2)\) at the gauge theory limit \(\epsilon \to 0\). This should be determined from the behavior of the period integrals of the local B-model at the limit \(\epsilon \to 0\). Thus, we will first study solutions of set of partial differential equations obtained from the Picard-Fuchs equations of the local B-model at the limit \(\epsilon \to 0\). For illustrative purposes, we will explain the results in the case \(N_f = 1\) as an example.
We first assume the form of a solution of the Picard-Fuchs equations at the limit $\epsilon \to 0$ as $e^\rho \sum_{n=0}^\infty f_n(u, m_1) e^n$ and substitute it into the equations. Then, the part of the lowest order with respect to $\epsilon$ becomes as follows ($m = m_1$)

$$(\rho - 1) \partial_u f_0(u, m) = 0,$$

$$(\rho - 1) \partial_u f_0(u, m) = 0,$$

$$\left[ \theta_m(\theta_m - 1) + \frac{2m^2}{3u} \theta_u(2\theta_u + 4\theta_m - 1) \right] f_0(u, m) = 0,$$

$$\left[ \theta_m(2\theta_u + \theta_m - 1) + \frac{3m\Lambda^3}{4u^2} \theta_u(\theta_u - 1) \right] f_0(u, m) = 0,$$

$$\left[ (2\theta_u + \theta_m - 1)(2\theta_u + 4\theta_m - 1) + (1 - \rho^2) - \frac{9\Lambda^3}{8mu} \theta_u \theta_m \right] f_0(u, m) = 0.$$

The first two equation means that $f_0(u, m)$ must be a constant or otherwise $\rho = 1$. Then, solving the last three equations with $\rho = 1$, we find solutions

$$g_1(u, m) := g(u, m; \rho)|_{\rho=0}, \quad g_2(u, m) := \partial_\rho g(u, m; \rho)|_{\rho=0}, \quad m,$$

where $g(u, m; \rho)$ is the following function with an auxiliary parameter $\rho$

$$g(u, m; \rho) = \sum_{n_1, n_2} \frac{(-\frac{1}{2} + 3\rho)_{n_1+n_2}(-1)^{n_1+n_2}}{(1+2\rho)_{n_1}(1+\rho)_{n_1-n_2}(-n_1+2n_2)!} \left( \frac{\Lambda^3}{16mu} \right)^{n_1} \left( \frac{4m^2}{u} \right)^{n_2}.$$

We remind that the Picard-Fuchs equations have five independent solutions; $1, t_1, t_2, t_3$ and one double logarithmic solution. The behavior of $t_1 = \log \frac{4t_2}{(1+\sqrt{1-4\epsilon})^2}$ at $\epsilon \to 0$ is $\log \frac{\Lambda^4}{4}$. Thus, adding 1 and $t_1$ to the three solutions of order $O(\epsilon)$ above, we obtain the following solutions around $\epsilon = 0$

$$1, \quad \log \epsilon, \quad \epsilon g_1(u, m), \quad \epsilon g_2(u, m), \quad \epsilon m.$$

It still remains to identify $t_2, t_3$ and the double logarithmic solution with these solutions. We can perform analytic continuation from the expansion around $z = 0$ to the expansion around $(z_1, z_2, z_3) = (0, \frac{1}{4}, \frac{1}{2})$ with the help of the transformation formula of the Gauss’ hypergeometric function [20]. For $t_2, t_3$, the result is

$$t_2 = -\epsilon \left( \frac{\Lambda}{2} g_1(u, m) + m \right) + O(\epsilon^2), \quad t_3 = -\epsilon \left( \frac{\Lambda}{2} g_1(u, m) - m \right) + O(\epsilon^2).$$

We should check that differential operators just appeared from the Picard-Fuchs equations annihilate the periods $(a, a_D)$ of the Seiberg-Witten differential $\lambda_{SW}$ in [4]. Let us denote the differential operators in the last three equations of (43) with $\rho = 1$ by $D_1, D_2, D_3$

$$D_1 := \theta_m(\theta_m - 1) + \frac{2m^2}{3u} \theta_u(2\theta_u + 4\theta_m - 1),$$

$$D_2 := \theta_m(2\theta_u + \theta_m - 1) + \frac{3m\Lambda^3}{4u^2} \theta_u(\theta_u - 1),$$

$$D_3 := (2\theta_u + \theta_m - 1)(2\theta_u + 4\theta_m - 1) - \frac{9\Lambda^3}{8mu} \theta_u \theta_m.$$
It is easy to see that the differential operator $\mathcal{D}_i (1 \leq i \leq 3)$ satisfies

$$\mathcal{D}_i \frac{dx}{y} = \text{non-singular function}, \quad \mathcal{D}_i' \text{ is defined by } \mathcal{D}_i' \partial_u = \partial_u \mathcal{D}_i. \quad (49)$$

Here, $y = \sqrt{(x^2 - u)^2 - \Lambda^3(x + m)}$. Thus, $\mathcal{D}_i$ annihilates $(a, a_D)$. Then, $(a, a_D)$ must be linear combinations of $g_1(u, m), g_2(u, m)$. We note that we can safely think of $(a, a_D)$ as linear combinations of $g_1, g_2$ because adding $m$ to $(a, a_D)$ does not change the massless limit. By the comparison of the terms of the lowest order in $\Lambda$ at the limit $m \to 0$, we obtain

$$a = \frac{\Lambda}{2} g_1(u, m). \quad (50)$$

After all, we may conclude that the space of period integrals of the Seiberg-Witten theory (including the mass parameter $m$) is the subspace of the period integrals of the local B-model that is closed under the monodromy transformation around $\epsilon = 0$. We note that we can state the behavior of the Kähler parameter $t_2, t_3$ in terms of $a$

$$t_2 = -\epsilon(a + m) + \mathcal{O}(\epsilon^2), \quad t_3 = -\epsilon(a - m) + \mathcal{O}(\epsilon^2). \quad (51)$$

The analysis of the cases with $N_f = 2, 3$ proceeds analogously. We obtained the behavior of the period integrals of the local B-model around $\epsilon = 0$ as follows

$$1, \quad \log \left(\frac{(\epsilon \Lambda)^{4-N_f}}{4} \right), \quad \epsilon g_1(u, m_i), \quad \epsilon g_2(u, m_i), \quad \epsilon m_i \quad (1 \leq i \leq N_f), \quad (52)$$

where $g_1(u, m_i) := g(u, m_i; \rho)|_{\rho=0}$ and $g_2(u, m_i) := \partial_\rho g(u, m_i; \rho)|_{\rho=0}$. It is straightforward to see that $g_1, g_2$ and $m_i$ are annihilated by the differential operators $\mathcal{D}_j$ derived from the Picard-Fuchs equations of the local B-model. We show independent $\mathcal{D}_j$'s in Table 11 and present $g(u, m_i; \rho)$ in Table 12. Then, we can check that these $\mathcal{D}_j$'s annihilate $(a, a_D)$. Thus, $(a, a_D)$ are linear combinations of $g_1, g_2$ and for $a$, the expression is

$$a = \left\{ \Lambda, \frac{\Lambda}{2}, \frac{i \Lambda}{2}, -i \Lambda \right\} g_1(u, m_i) \quad \text{for } N_f = 0, 1, 2, 3. \quad (53)$$

It is instructive to write down the parameter $u$ as an inverse series in $a$ in Table 10. This is consistent with (12) of the gauge theory. We can also derive the behaviors of the Kähler parameters at the limit $\epsilon \to 0$. The results are summarized in Table 11. We note that only one Kähler parameter $t_1$ diverges logarithmically and the other parameters approach to zero.

Now we equate the prepotential of the local A-model $\mathcal{F}_{\text{mirror}}$ at the limit $\epsilon \to 0$ with that of the Seiberg-Witten theory $\mathcal{F}_{\text{gauge}}$. It is of use to redefine Kähler parameters $t_2, \cdots, t_k$ as

$$s_0 = -ae + \mathcal{O}(\epsilon^2), \quad s_i = -m_i \epsilon + \mathcal{O}(\epsilon^2) \quad (1 \leq i \leq N_f). \quad (54)$$

The explicit transformation is shown in Table 12. With these parameters, the double logarithmic solution should coincide with $\partial_{s_0} \mathcal{F}_{\text{mirror}}$. Then we expect that the relation

$$\partial_{s_0} \mathcal{F}_{\text{mirror}} \sim \text{const.} + \epsilon a_D + \mathcal{O}(\epsilon^2) \quad (55)$$
| $N_f$ | differential operators |
|-------|------------------------|
| 0     | $D_1 = (\theta_u - \frac{1}{2})^2 - \frac{\Lambda^2}{8\pi} \theta_u (\theta_u - 1)$ |
| 1     | $D_1 = \theta_m (\theta_m - 1) + \frac{2m^2}{3n} \theta_u (2\theta_u + 4\theta_m - 1)$  
$D_2 = \theta_m (\theta_u + \theta_m - 1) + \frac{3m\Lambda}{4\pi} \theta_u (\theta_u - 1)$  
$D_3 = (2\theta_u + \theta_m - 1) (2\theta_u + 4\theta_m - 1) - \frac{9\Lambda^3}{8m^2} \theta_u \theta_m$ |
| 2     | $D_1 = (2\theta_v + \theta_m + \theta_n - 1) (2\theta_v + 2\theta_m + 2\theta_n - 1) + \frac{\Lambda^2}{20} \theta_v (2\theta_v + 2\theta_m + 2\theta_n - 2)$  
$D_2 = \theta_n (2\theta_v + \theta_n + \theta_m - 1) - \frac{n^2 \Lambda^2}{v} \theta_v (\theta_v - 1)$  
$D_3 = \theta_m (\theta_m - 1) - \frac{m^2}{n} \theta_n (2\theta_v + \theta_m + 2\theta_n - 2)$  
$D_4 = \theta_n (2\theta_v + \theta_m + 2\theta_n - 2) + \frac{2n^2}{v} \theta_v (2\theta_v + 2\theta_m + 2\theta_n - 1)$  
$D_5 = \theta_m \theta_n + \frac{2m^2}{v} \theta_v \theta_n + \frac{m^2 \Lambda^2}{v} \theta_v (\theta_v - 1)$  
$D_6 = \theta_m (2\theta_v + \theta_m + \theta_n - 1) + \frac{\Lambda^2}{20} \theta_m \theta_v + \frac{m^2 \Lambda^2}{v^2} \theta_v (\theta_v - 1)$ |
| 3     | $D_1 = 2\partial_n \partial_p + \Lambda n \partial_w^2$  
$D_2 = \partial_n (\partial_m + \partial_p) + 2 (m \partial_n + n \partial_m) \partial_w$  
$D_3 = \partial_n (4\theta_w + 2\theta_p + 3\theta_n + 2\theta_m - 3) - n \partial_m^2$  
$D_4 = \partial_n (\partial_m + \partial_p) + 2\partial_w (4\theta_w + 2\theta_p + 3\theta_n + 3\theta_m - 2)$  
$D_5 = \partial_p (\partial_m + \partial_p) + 2\partial_w (2\theta_w + 2\theta_p + \theta_m + \theta_n - 1)$  
$D_6 = 4\partial_p (m \partial_n + n \partial_m) - \Lambda n \partial_w (\partial_p + \partial_m)$  
$D_7 = \partial_p (4\theta_w + 2\theta_p + 3\theta_n + 3\theta_m - 2) - \partial_n (2\theta_w + 2\theta_p + \theta_n + \theta_m - 1)$  
$D_8 = \partial_n (2\theta_w + 2\theta_p + \theta_n + \theta_m - 1) - (m \partial_n + n \partial_m) \partial_p$  
$D_9 = 2 (m \partial_n + n \partial_m) (2\theta_w + 2\theta_p + \theta_n + \theta_m - 1) - \Lambda n \partial_w (3\theta_w + 2\theta_p + 2\theta_m + 2\theta_n - 2)$  
$D_{10} = 4 (2\theta_w + 2\theta_p + \theta_n + \theta_m - 1) (4\theta_w + 2\theta_p + 3\theta_n + 3\theta_m - 2)$  
$+ 2\Lambda m \partial_w (3\theta_w + 2\theta_p + 2\theta_m + 2\theta_n - 2) + \Lambda (\partial_p + \partial_m) (3\theta_w + 2\theta_p + 2\theta_n + 2\theta_m - 3)$ |

Table 8: Set of partial differential operators. Those differential operators $D_i$ annihilate the lowest order part $f_0(u; m_i)$ of the solution $\sum_{n=0}^{\infty} f_n(u; m_i) e^{n+1}$ for the Picard-Fuchs equations of the local B-model. This is also the set of differential operators annihilating the periods $(a, a_D)$ of the Seiberg-Witten differential $\lambda_{SW}$ of the $N = 2$ $SU(2)$ gauge theory.
Table 9: $g(u, m_i; \rho)$. $g(u, m_i; 0)$ and $\partial_\rho g(u, m_i; \rho)|_{\rho=0}$ and $m_i$ ($i = 1, \ldots, N_f$) are solutions of the set of the partial differential equations in Table 3. For $N_f = 3$, $p_i = m_i - \frac{A}{8}$ in the last line.

\[
N_f \quad g(u, m_i; \rho) \\
0 \quad \sum_n \left( -\frac{\pi}{2} + \rho \right) n + \frac{A}{2} n \left( m_i + \rho - \frac{1}{2} \right) \\
1 \quad \sum_{n_1,n_2} \frac{\Gamma(\rho+1)}{n_1!n_2!} \frac{\Gamma(\frac{3}{2} + \rho + n_1 + n_2)}{\Gamma(\frac{3}{2} + \rho + n_1 + n_2 - 2\rho)} \frac{\left( \frac{A}{2} \right)^2 n_1 + \rho - n_1 - \frac{1}{2} n_2 \left( m_i + \rho \right)}{2 \Gamma(n_1 + 1)} \left( 4m_i^2 u \right)^n_2 \frac{4m_i^2}{n_2} \\
2 \quad \sum_{n_1,n_2} \frac{\Gamma(\rho+1)}{n_1!n_2!} \frac{\Gamma(\frac{3}{2} + \rho + n_1 + n_2)}{\Gamma(\frac{3}{2} + \rho + n_1 + n_2 - 2\rho)} \frac{\left( \frac{A}{2} \right)^2 n_1 + \rho - n_1 - \frac{1}{2} n_2 \left( m_i + \rho \right)}{2 \Gamma(n_1 + 1)} \left( 4m_i^2 u \right)^n_2 \frac{4m_i^2}{n_2} \\
3 \quad \left( \frac{\pi}{2} \right)^{2\rho} \sum_{n_1,n_2} \frac{\Gamma(\rho+1)}{n_1!n_2!} \frac{\Gamma(\frac{3}{2} + \rho + n_1 + n_2)}{\Gamma(\frac{3}{2} + \rho + n_1 + n_2 - 2\rho)} \frac{\left( \frac{A}{2} \right)^2 n_1 + \rho - n_1 - \frac{1}{2} n_2 \left( m_i + \rho \right)}{2 \Gamma(n_1 + 1)} \left( 4m_i^2 u \right)^n_2 \frac{4m_i^2}{n_2} \\
\]

Table 10: $u$ as an infinite power series in $a$. 

| $N_f$ | $u$ |
|---|---|
| 0 | $a^2 + \frac{\Lambda^4}{8a^2} + \frac{5\Lambda^8}{4a^2} + \frac{9\Lambda^{12}}{4096a^{16}} + \frac{1469\Lambda^{16}}{2097152a^{25}} + \frac{4471\Lambda^{20}}{512a^{60}} + \cdots$ |
| 1 | $a^2 + \frac{\Lambda^4 m_1}{8a^2} + \frac{\Lambda^4(3a^2 + 5m_1^2)}{512a^6} + \frac{\Lambda^4 m_1(7a^2 + 9m_1^2)}{512a^{12}} + \frac{4096a^{10}}{2097152a^{24}} + \frac{16777216a^{18}}{16777216a^{24}} + \cdots$ |
| 2 | $a^2 + \frac{\Lambda^4 m_1 m_2}{8a^2} + \frac{\Lambda^4(5m_1^2 + 3a^2 + 4m_1^2 + m_2^2)}{512a^6} + \frac{\Lambda^4 m_1(313a^4 - 5250a^2 m_1^2 + 4471m_1^4)}{512a^{12}} + \cdots$ |
| 3 | $a^2 + \frac{\Lambda^4 m_1 m_2}{8a^2} + \frac{\Lambda^4(5m_1^2 + 3a^2 + 4m_1^2 + m_2^2)}{512a^6} + \frac{\Lambda^4(5m_1^2 + 3a^2 + 4m_1^2 + m_2^2)}{512a^{12}} + \cdots$ |
| $N_f$ | $t_1, t_2, \ldots, t_k$ ($k = N_f + 2$) |
|-------|------------------------------------------|
| 0     | $\log \frac{\Lambda^{4\epsilon}}{4}, -2\epsilon a$ |
| 1     | $\log \frac{\Lambda^{4\epsilon}}{4}, -\epsilon(m_1 + a), -\epsilon(-m_1 + a)$ |
| 2     | $\log \frac{\Lambda^{4\epsilon}}{4}, -\epsilon(m_2 + a), -\epsilon(m_1 - m_2), -\epsilon(-m_1 + a)$ |
| 3     | $\log \frac{\Lambda^{4\epsilon}}{4}, -\epsilon(m_2 + m_3), -\epsilon(m_1 - m_2), -\epsilon(a - m_1), -\epsilon(a - m_3)$ |

Table 11: Behavior of the Kähler parameters at the gauge theory limit $\epsilon \to 0$.

holds at $\epsilon \to 0$. It would be straightforward but tedious to check this relation by an analytic continuation of the double logarithmic solution to $\epsilon = 0$. Alternatively, it turned out that we will only need the following behavior of the Yukawa coupling $\partial_{s_0} F_{\text{mirror}}$ at the limit $\epsilon \to 0$ in next subsection

$$
\epsilon \partial_{s_0}^3 F_{\text{mirror}} \propto \partial_{s_0}^3 F_{\text{gauge}}.
$$

From the B-model, we obtained the Yukawa coupling to some low order in $\Lambda$ in Table 13. It is easy to see that the Yukawa coupling has the structure of $\partial_{s_0}^3 F_{\text{gauge}}$ in (56).

| Model | $(N_0, N_1, \ldots, N_{N_f})$ | $(s_0, \ldots, s_{N_f})$ |
|-------|-------------------------------|----------------------------|
| 0     | $(2n_2)$                      | $(\frac{t_2}{2})$          |
| 1     | $(n_2 + n_3, n_2 - n_3)$      | $(\frac{t_2 + t_4}{2}, \frac{t_2 - t_4}{2})$ |
| 2     | $(n_2 + n_4, n_3 - n_4, n_2 - n_3)$ | $(\frac{t_2 + t_4 + t_5}{2}, \frac{t_2 - t_4}{2}, \frac{t_2 - t_5}{2})$ |
| 3     | $(n_4 + n_5, n_3 - n_4, n_2 - n_3, n_2 - n_5)$ | $(\frac{t_2 + t_4 + t_5}{2}, \frac{t_2 - t_4}{2}, \frac{t_2 - t_5}{2})$ |

Table 12: $s_0, s_1, \ldots, s_{N_f}$ and $N_0, N_1, \ldots, N_{N_f}$.

| $N_f$ | $\epsilon \partial_{s_0}^3 F_{\text{mirror}}$ |
|-------|-----------------------------------------------|
| 0     | $-\frac{8}{a} + \frac{3\Lambda^4}{a^2} + \frac{105\Lambda^8}{64a^4} - \frac{495\Lambda^{12}}{512a^8} - \frac{154245\Lambda^{16}}{258541a^{12}} + \cdots$ |
| 1     | $-\frac{8}{a} + \frac{1}{a+m_1} + \frac{1}{a-m_1} + \frac{1}{a+m_2} + \frac{1}{a-m_2} - \frac{3\Lambda^2m_1m_2}{a^2} - \frac{3\Lambda^4(a^2+70m_1^2m_2^2-15\Lambda^2(m_1^2+m_2^2))}{a^2} - \frac{5\Lambda^6m_1m_2(14a^2+99m_1^2m_2^2-42a^2(m_1^2+m_2^2))}{128a^6} + \cdots$ |
| 2     | $-\frac{8}{a} + \frac{1}{a+m_1} + \frac{1}{a-m_1} + \frac{1}{a+m_2} + \frac{1}{a-m_2} + \frac{1}{a+m_3} + \frac{1}{a-m_3} - \frac{3\Lambda^2m_1m_2}{a^2} - \frac{3\Lambda^4(a^2+70m_1^2m_2^2-15\Lambda^2(m_1^2+m_2^2))}{a^2} - \frac{128a^6}{512a^{14}} + \cdots$ |

Table 13: The Yukawa coupling $\partial_{s_0}^3 F_{\text{mirror}}$ multiplied by $\epsilon$.

The rest of this subsection is devoted to a derivation of the behavior of Yukawa coupling $\partial_{s_0}^3 F_{\text{mirror}}$. We explain in the case of $N_f = 1$ as an example. We redefine the Kähler parameters
\[
(m_1 = m)
\]
\[
s_0 := \frac{t_2 + t_3}{2} \sim -ae, \quad s_1 := \frac{t_2 - t_3}{2} \sim -m\epsilon.
\]

The Yukawa coupling at the limit \(\epsilon \to 0\) is evaluated as
\[
\partial^3_{s_0} F_{\text{mirror}} = \left( \frac{\partial u}{\partial s_0} \right)^3 F_{uuu} + \left( \frac{\partial u}{\partial s_0} \right)^2 \left( \frac{\partial m}{\partial s_0} \right) F_{uum} + \left( \frac{\partial u}{\partial s_0} \right) \left( \frac{\partial m}{\partial s_0} \right)^2 F_{umm} + \left( \frac{\partial m}{\partial s_0} \right)^3 F_{mmm}
\]
\[
\sim -\frac{1}{\epsilon^3} \left( \frac{\partial u}{\partial a} \right)^3 F_{uuu}.
\]

The transition from the first line to the second follows because all of \(F_{uuu}, F_{uum}, F_{umm}, F_{mmm}\) have the same order in \(\epsilon (O(\epsilon^2))\). Then, we calculated the Yukawa coupling \(F_{uuu}\) as
\[
F_{uuu} = \epsilon^2 \frac{64(-3u + 4m^2)}{-256u^2(u - m^2) + 32\Lambda^3m(9u - 8m^2) - 27\Lambda^6} + O(\epsilon^3).
\]

This has been obtained from the Yukawa coupling of the B-model \(F_{z_i z_j z_p}\) \((0 \leq i, j, p \leq 3)\) by the transformation as the contravariant tensor of rank three. The form shown above is the lowest order term with respect to \(z_0\). On the other hand, we can obtain the expression \(\partial^3_a F_{\text{gauge}}\) by (3). Through the geometric engineering, we do know the differential operator \(\mathcal{P}\) in (16) from the \(\mathcal{D}_1, \mathcal{D}_2\) and \(\mathcal{D}_3\). \(P, Q, R\) in (16) turned out to be
\[
P = (4m^2 - 3u)(-256m^2u^2 + 256u^3 + \Lambda^3m(256m^2 - 288u) + 27\Lambda^6),
\]
\[
Q = -2048m^4u + 3840m^2u^2 - 1536u^3 - 384\Lambda^3m^3 + 81\Lambda^6,
\]
\[
R = -8(32m^4 - 72m^2u + 24u^2 + 9\Lambda^3m).
\]

Performing an indefinite integration \(\int du \frac{Q}{P}\), we can check the relation \(F_{uuu} \propto e^{-\int du \frac{Q}{P}}\) up to a factor independent of \(u\). Thus, we have confirmed the relation (56). It is straightforward to check (56) for all cases \(N_f = 0, 1, 2, 3\). We calculated the Yukawa coupling \(F_{uuu}\) in Table in Appendix C, and obtained the differential operator \(\mathcal{P}\) in Table in Appendix D. We note that the expression of \(\mathcal{P}\) is consistent with the Picard-Fuchs equation of the massless case (8).

### 4.3 Distribution Pattern of World-sheet Instanton Numbers

To begin with, let us recall the definition of the Yukawa coupling of the local A-model. It encodes the world-sheet instanton numbers as follows
\[
\partial^3_{s_0} F_{\text{mirror}} := \text{term of the triple intersection} \quad + \sum_{n_1, \ldots, n_f} d_{n_1, \ldots, n_f} N_f^3 \frac{e^{\sum_{i=1}^{N_f} n_i s_i}}{1 - e^{\sum_{i=1}^{N_f} n_i s_i}}.
\]

Here, we have introduced \(N_0, N_1, \ldots, N_{N_f}\) as coefficients of \(s_0, \ldots, s_{N_f}\) in \(n_2 t_2 + \cdots + n_{N_f} t_{N_f + 2}\) (see Table 12). On the other hand, we have seen in subsection 2.1 that the third derivative of
\[ F_{\text{gauge}} \] by \( a \) has the expansion

\[
\partial_a^3 F_{\text{gauge}} \propto - \frac{8}{a} + \sum_{i=1}^{N_f} \left( \frac{1}{a + m_i} + \frac{1}{a - m_i} \right) - \frac{8}{a} \sum_{n=1}^{\infty} P_n \cdot \left( \frac{\Lambda}{a} \right)^{(4 - N_f)n},
\]

\[
P_n := \left( \text{polynomial of degree } n \text{ in each } \frac{m_i}{a}, \text{ symmetric with respect to } \frac{m_1}{a}, \ldots, \frac{m_{N_f}}{a} \right),
\]

\[
B_n := \left[ \text{coefficient of } \prod_{i=1}^{N_f} \left( \frac{m_i}{a} \right)^n \text{ in } P_n \right] = \frac{4n(4n - 1)(4n - 2)F_n}{4},
\]

\[
F_n := \left[ \text{the instanton amplitude of the gauge theory } (N_f = 0) \right].
\]

We have also explained that \( \partial_s^3 F_{\text{mirror}} \)'s derived from the B-model in Table 13 actually have this structure in previous subsection. Then, combining these matters, it is natural to suspect that the world-sheet instanton numbers in (61) would be translated into the instanton amplitudes of the gauge theory. Actually, by comparing two instanton expansions in (61) and (62), we will obtain the asymptotic distribution of the world-sheet instanton number \( d_{n=(n_1,\ldots,n_{N_f+2})} \) at the limit \( \epsilon \to 0 \) controlled by the instanton amplitude \( F_n \). We note that the strategy in this section is essentially the same as one in subsection 2.2 or [12].

Before proceeding the results, we would like to give several remarks on the expression (61) at the limit \( \epsilon \to 0 \). First, the constant term of the triple intersection can be neglected since the Yukawa coupling diverges as \( O(\epsilon^{-1}) \) at the limit \( \epsilon \to 0 \). Secondly, the contributions from the terms of the world-sheet instanton numbers with \( n_1 = 0 \) and the those terms with \( n_1 \geq 1 \) are different in that we can neglect the factor of the multiple cover contribution \( 1/(1 - e^{\Sigma_{1}^{n_1} t_i}) \) when \( n_1 \geq 1 \), but we can not when \( n_1 = 0 \). Thus we will have to treat each contribution separately. Thirdly, for \( n_1 \geq 1 \), the expansion (61) contains the power series in \( \Lambda^{4-N_f} \) because of the factor \( e^{n_1 t_1} \) in the numerator. Thus we will compare the contributions of \( \Lambda^{4-N_f} n_1 \) in (61) and (62). Finally, to read off and guess the behavior of the world-sheet instanton numbers, it will be helpful to consult Tables of them of low degree in Appendix B.

We begin with the analysis of Model 0 of II in a slightly different manner so that we can continue to the case of Model 1 smoothly. Let us start from the terms with \( n_1 = 0 \). We can see from Table in Appendix B, that values of \( d_{n_1,n_2} \) is nonzero only at \( n_2 = 1 \) at least for a small value of \( n_2 \). We note that we ignore the contribution of \( d_{0,0} \) here because this could not be determined neither from the Yukawa coupling nor from the double logarithmic solution. With the value \( (n_1, n_2) = (0, 1) \), the corresponding term in (61) is \( \frac{1}{1 - e^2} \sim \frac{1}{2\epsilon a} \) at \( \epsilon \to 0 \). On the other hand, we know that the term in (62) corresponding to the contribution with \( n_1 = 0 \) is \( \frac{8}{\epsilon a} \) only. Thus, we naturally expect that \( d_{0,n_2} = 0 \) for all \( n_2 \geq 2 \). Meanwhile, we turn to the terms with \( n_1 \geq 1 \). The corresponding terms in (62) is \( e^{n_1 t_1}(\epsilon s_0)^{-4(n_1+1)} \). This expression must coincide with \( \sum_{n_2} d_{n_1,n_2} e^{n_1 t_1 + 2s_0 n_2} \) in (61) up to a constant factor. Now we consider the case where the contribution from the terms with large values of \( n_2 \) is dominant, and the summation could be
replaced with an integration. This specialization is sensible because the Kähler parameter $t_2$ goes to zero at the $\epsilon \to 0$. Then, recalling the formula $\int dx x^n e^{-bx} = \Gamma(n+1)b^{-n-1}$, it is natural to adopt the following ansatz on $d_{n_1,n_2}$ for a given value of $n_1$

$$d_{0,n_2} = c_1 \delta_{n_2,1} \quad (n_2 \neq 0), \quad d_{n_1,n_2} = \gamma_{n_1}(2n_2)^{\alpha_{n_1}} \quad (n_1 \geq 1, n_2 \gg n_1). \quad (63)$$

Here, $c_1$ is a constant and $\gamma_{n_1}, \alpha_{n_1}$ are constants depending on $n_1$. With this ansatz, we can estimate the Yukawa coupling (61) at the limit $\epsilon \to 0$ by the integration of $n_2$

$$\partial_s^3 F_{\text{mirror}} \sim \frac{4c_1}{ae} + \gamma_{n_1} \frac{\Gamma(\alpha_{n_1} + 4) \cdot (\epsilon^4 \Lambda^4)^{n_1}}{2(a\epsilon)^{\alpha_{n_1} + 4} 4^{n_1}}. \quad (64)$$

We have shown that this must be of order $O(\epsilon^{-1})$ in previous subsection. Hence, we obtain $\alpha_{n_1} = 4n_1 - 3$. Then, the Yukawa coupling becomes the series in $\frac{\Lambda^4}{a}$: $\partial_s^3 F_{\text{mirror}} \sim \frac{4c_1}{ae} [1 + \sum_{n_1=1}^{\infty} \frac{\gamma_{n_1} \Gamma(4n_1 + 1)}{8c_1 4^{n_1}} \cdot \left(\frac{\Lambda}{a}\right)^{4n_1}]$. By comparing this series with the expansion (62), we obtain the relation between $\gamma_{n_1}$ and $F_n$. In summary, we arrive at the following distribution of the world-sheet instanton numbers

$$d_{0,n_2} = c_1 \delta_{n_2,1} \quad (n_2 \neq 0), \quad d_{n_1,n_2} = \gamma_{n_1}(2n_2)^{4n_1 - 3} \quad (n_1 \geq 1, n_2 \gg n_1), \quad \frac{\gamma_{n_1}}{c_1} = \frac{2 \cdot 4^{n_1} F_{n_1}}{\Gamma(4n_1 - 2)}. \quad (65)$$

where $F_n$ is the instanton amplitude of the gauge theory with $N_f = 0$.

Next we explain how the result of Model 0 is extended to Model 1. To grasp a matter, let us consult Table of the world-sheet instanton numbers in Appendix B again. First, for the numbers $d_{n_1,n_2,n_3}$ with $n_1 = 0$, we can see that nonzero values exist only at $(n_2, n_3) = (1,0), (0,1)$ and $(1,1)$. These values of $(n_1, n_2, n_3)$ give rise to the contributions $\frac{1}{1-e^{-(a+m)}} \sim \frac{1}{e^{(a+m)}}$ and $\frac{1}{1-e^{-2a}} \sim \frac{1}{2a}$ in (61) $(m_1 = m)$. On the other hand, the terms corresponding to $n_1 = 0$ in (62) give rise to just enough contributions. Thus, we can conclude that $d_{0,n_2,n_3} = 0$ unless $(n_2, n_3) = (1,0), (0,1)$ or $(1,1)$. Then, all we have to do will be to determine the numbers $d_{0,1,0}, d_{0,0,1}$ and $d_{0,1,1}$ with $n_1 = 0$. Secondly, we proceed to the numbers $d_{n_1,n_2,n_3}$ with $n_1 \geq 1$. From the Table, it is likely that there is a rule that $d_{n_1,n_2,n_3} = 0$ unless $0 \leq n_2 - n_3 \leq n_1$. Furthermore, the distribution of $|d_{n_1,n_2,n_3}|$ looks like a binomial distribution centered around $n_2 - n_3 = \frac{n_1}{2}$ and the sign of $d_{n_1,n_2,n_3}$ seems to be $(-1)^{n_2-n_3}$. Then, we arrive at the following ansatz on $d_{n_1,n_2,n_3}$

$$d_{0,n_2,n_3} = c_{10} \delta_{n_2,1} \delta_{n_3,0} + c_{01} \delta_{n_2,0} \delta_{n_3,1} + c_{11} \delta_{n_2,1} \delta_{n_3,1} \quad ((n_2, n_3) \neq (0,0)), \quad d_{n_1,n_2,n_3} = \gamma_{n_1} (-1)^{n_2-n_3} (n_2 + n_3) \delta_{n_2,n_3} \quad (n_1 \geq 1, n_2 + n_3 \gg n_1), \quad (66)$$

where $c_{ij}$’s are constants, and $\beta_{n_1}, \gamma_{n_1}$ are numbers depending on $n_1$. Substituting this into (61) and changing the variables from $(n_2, n_3)$ to $N_0 = n_2 + n_3$ and $N_1 = n_2 - n_3$, the Yukawa
coupling is evaluated in the region \( N_0 \gg n_1 \) as follows

\[
\partial^3_{s_0} F_{\text{mirror}} \sim \frac{1}{\epsilon} \frac{4c_{11}}{a} + \frac{c_{10}}{a + m} + \frac{c_{01}}{a - m}
\]

\[
+ \frac{1}{2} \sum_{n_1=1}^{\infty} \gamma_{n_1} \left( \frac{\epsilon^3 \Lambda^3}{4} \right)^{n_1} \int d N_0 \, N_0^{\beta_{n_1} + 3} e^{-\epsilon a N_0} \sum_{N_1=0}^{n_1} (-1)^{N_1} n_1 C_{N_1} e^{-\epsilon m N_1},
\]

\[
\sim \frac{1}{\epsilon} \frac{4c_{11}}{a} + \frac{c_{10}}{a + m} + \frac{c_{01}}{a - m} + \frac{1}{2} \sum_{n_1=1}^{\infty} \gamma_{n_1} \left( \frac{\epsilon^3 \Lambda^3}{4} \right)^{n_1} \frac{\Gamma(\beta_{n_1} + 4)}{(a \epsilon)^{\beta_{n_1} + 4}} (m \epsilon)^{n_1}. \tag{67}
\]

The factor \( \frac{1}{2} \) before the summation has entered by the Jacobian of the change of variables. By imposing that the Yukawa coupling is \( O(\epsilon^{-1}) \) and the ratio of \( \frac{1}{a} \) to \( \frac{1}{a \pm m} \) is 8 to -1, we obtain

\[ c_{10} = c_{01} = -\frac{c_{11}}{2}, \quad \beta_{n_1} = 4n_1 - 3. \tag{68} \]

After that, the Yukawa coupling is expressed as

\[
\partial^3_{s_0} F_{\text{mirror}} \sim -\frac{c_{11}}{2ae} \left[ -\frac{8}{a} + \frac{1}{a + m} + \frac{1}{a - m} - \sum_{n_1=1}^{\infty} \gamma_{n_1} \frac{\Gamma(4n_1 + 1)}{4^{n_1} c_1} \left( m \epsilon \right)^{n_1} \left( \frac{\Lambda}{a} \right)^{3n_1} \right]. \tag{69}
\]

Comparing this with (62), we can relate \( \gamma_{n_1} \) to \( F_{n_1} \)

\[
\frac{\gamma_{n_1}}{c_{11}} = \frac{2 \cdot 4^{n_1} F_{n_1}}{\Gamma(4n_1 - 2)}. \tag{70}
\]

Therefore, we arrived at the asymptotic behavior of \( d_{n_1,n_2,n_3} \) (66) with (68) and (70). We note that only the terms of highest degree in \( m \) in (62) have appeared in (69). To reproduce the remaining contributions, one may be tempted to adopt alternative ansatz; \( d_{n_1,n_2,n_3} \sim \sum_{\mu=0}^{\min[N_1,n_1]} \gamma_{n_1}^\mu N_0^{\beta_{n_1}^\mu} \mu C_{N_1} \) sorted out according to each value of \( \mu \). Then one would find that \( \beta_{n_1}^\mu = 3n_1 + \mu - 3 \) and \( \gamma_{n_1}^\mu \) can be written in terms of the coefficients of \( \frac{1}{a \epsilon} (\frac{m}{a})^\mu (\frac{\Lambda}{a})^{3\mu} \). However, given that \( N_0 \gg n_1 \), it is clear that the most dominant term in the ansatz is the term with \( \mu = n_1 \). Thus, we do not extract the subleading contributions in the strategy here.

The extension to Model 2 and Model 3 is now straightforward. We propose the distribution of the world-sheet instanton numbers at the gauge theory limit \( \epsilon \to 0 \) altogether with the notation in Table 12 as follows: for \( n_1 \geq 1 \),

\[
d_{n_1,n_2,\cdots,n_{N_f}+2} \begin{cases} 
\sim \gamma_{n_1} (-1)^{N_1+\cdots+N_{N_f}} N_0^{4n_1-3} \prod_{i=1}^{N_f} n_i C_{N_i} \quad (0 \leq \forall N_i \leq n_1) \\
0 \quad \text{otherwise}
\end{cases}, \tag{71}
\]

where this is effective for the region \( N_0 \gg n_1 \); for \( n_1 = 0 \),

\[
d_{0,\vec{n}} = \epsilon \left[ \delta(\vec{n}',1) - \frac{1}{2} \sum_{\alpha \in I} \delta_{\vec{n}',\alpha} \right] \quad (\vec{n}' = (n_2, \cdots, n_{N_f}) \neq \vec{0}),
\]

\[
I = \begin{cases} 
\text{Model 0} & \text{empty set} \\
\text{Model 1} & \{(1,0),(0,1)\} \\
\text{Model 2} & \{(1,0,0),(1,1,0),(0,1,1),(0,0,1)\} \\
\text{Model 3} & \{(0,0,0,1),(0,0,1,0),(0,1,1,0),(1,0,0,1),(1,1,1,0),(1,1,0,1)\}
\end{cases}, \tag{72}
\]

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where the constants $c$ and $\gamma_{n_1}$ are correlated as

$$\frac{\gamma_{n_1}}{c} = \frac{2 \cdot 4^{n_1} F_{n_1}}{\Gamma(4n_1 - 2)}. \quad (73)$$

Here, $\delta_{q, q'} := \prod_{i=2}^{N_f+2} \delta_{q_i, q'_i}$ and $F_n$ is the instanton amplitude of the gauge theory with $N_f = 0$.

It is instructive to state the result with the basis of $H_2(V; \mathbb{Z})$ in Table 14. At the gauge theory limit $\epsilon \to 0$, $P_b^1, R_0$ and $R_i$ ($i = 1 \leq i \leq N_f$) are the curves whose Kähler parameter behave as $(4 - N_f) \log \epsilon, -2\epsilon a$ and $-\epsilon (a + m_i)$. Then, the intersection matrices among the curves $(P_b^1, R_0, R_1, \cdots, R_{N_f})$ in $V$ of Model $N_f$ ($N_f = 0, 1, 2$ and 3) are

$$\begin{bmatrix}
-2 & 1 \\
1 & 0 \\
0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
-2 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}, \quad \begin{bmatrix}
-2 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}. \quad (74)$$

These matrices make clear the structure of the $P_{\text{base}}$ of Model $N_f$ as $N_f$ point blow ups of $F_2$. Now we state the distribution of the world-sheet instanton numbers $d_\beta$ with homology class $\beta = n_1[P_b^1] + \sum_{i=0}^{N_f} k_i[R_i]$: for $n_1 \geq 1$

$$d_\beta \sim \gamma_{n_1} (-1)^{k_1 + \cdots + k_{N_f}} (2k_0)^{4n_1 - 2} \prod_{i=1}^{N_f} C_{k_i} \quad (2k_0 \gg n_1, 0 \leq k_i \leq n_1), \quad (75)$$

and for other $\beta$'s, $d_\beta$ is negligible; for $n_1 = 0$, $d_\beta \neq 0$ only at $\beta = [R_0], [R_i]$ or $[R_0 - R_i]$ with ratio between the values of $[R_0]$ and the others is always 2 to $-1$. Constants in these expressions are normalized using the instanton amplitude $F_n$

$$\frac{\gamma_{n_1}}{d_\beta = [R_0]} = \frac{2 \cdot 4^{n_1} F_{n_1}}{\Gamma(4n_1 - 2)}. \quad (76)$$

| $N_f$ | $(P_b^1; R_0, R_1, \cdots, R_{N_f})$ |
|-------|---------------------------------|
| 0     | $(D_3; D_2)$                    |
| 1     | $(D_4; D_5, D_2)$               |
| 2     | $(D_5; D_6, D_2 + D_3, D_4)$    |
| 3     | $(D_5; D_6 + D_7, D_3 + D_4, D_4, D_6)$ |

Table 14: Basis of $H_2(V; \mathbb{Z})$. $D_i$ is the divisor in $P_{\text{base}}$ corresponding to the 1-cone generated by $\nu_i$ in Table 3.
5 Conclusion

In this article, we presented the calculation for the instanton expansions in the $N=2$ $SU(2)$ gauge theories with $1 \leq N_f \leq 3$ massive hypermultiplets through the geometric engineering. We checked the equivalence of the Yukawa coupling at the gauge theory limit and $\partial^3 F_{\text{gauge}}$, and conjectured the pattern of the distribution of the world-sheet instanton numbers. This proposal matches with general expectation in the geometric engineering of $N=2$ gauge theories that the asymptotic growth of the world-sheet instanton numbers is controlled by the instanton amplitude of the gauge theory. Further, it might be the universal phenomenon for the mirror pair of Calabi-Yau manifolds that the distribution of the world-sheet instanton numbers is governed by a singularity of the moduli space where the discriminant of the B-model manifold becomes zero. We also analyzed the asymptotic form of the instanton amplitude of the gauge theory with massless hypermultiplets, making use of the singularity of the moduli space at $u = \Lambda^2$, and observed the characteristic factor. In principle, it should be possible to clarify how the factor originates from the analysis on the instanton background in the gauge theory [21, 22, 23]. There has been some developments in the direct evaluation of the instanton amplitude of Seiberg-Witten theories [24, 25, 26, 27] and so on. It would be very interesting to relate this with the localization technique of the world-sheet instanton numbers [7].

The geometric engineering of the $N=2$ $SU(2)$ gauge theory with $N_f = 4$ hypermultiplets remains to be done. It would be very interesting to understand how physics of the $N_f = 4$ theory are geometrically realized in the local mirror model. Then, we mention an extension of the $N=2$ $SU(2)$ gauge theories into five-dimensional gauge theories. The five-dimensional theories receive no instanton corrections and it is known how to construct geometrically the gauge theories with $N_f \leq 4$ hypermultiplets [28]. It would be also interesting to tailor the local mirror models to the five-dimensional gauge theories exactly. Meanwhile, there has been a surge of developments in our understanding of supersymmetric gauge theories [29]. It would be interesting to develop a tool for our instanton expansions in this context.

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Appendix A: GKZ-hypergeometric System

In this appendix we give the definition of the GKZ-hypergeometric differential system (GKZ-system for short). For details, see [30, 31, 32, 33].

Let $A$ be an $n \times (n + k)$ matrix of integers, which has the following properties: (a) the columns $A_1, \cdots, A_{n+k}$ of $A$ generate the lattice $\mathbb{Z}^n$, (b) there exist integers $(c_1, \cdots, c_{n+k})$ such that $\sum_{i=1}^{n+k} c_i \ (i \text{-th row of } A) = (1, \cdots, 1)$. Let $\beta \in \mathbb{C}^n$. Then the GKZ hypergeometric system is the left ideal in the Weyl algebra of dimensions $n+k$. We denote the variables by $a_1, \cdots, a_{n+k}$. Its generators are

$$Z_i = \sum_{i=1}^{n+k} A_{ij} \theta_{a_j} - \beta_j, \quad (1 \leq i \leq n),$$

$$I_A = \left\{ \prod_{i:l_i>0} \left( \frac{\partial}{\partial a_i} \right)^{l_i} - \prod_{i:l_i<0} \left( \frac{\partial}{\partial a_i} \right)^{-l_i} \bigg| l \in L \right\},$$

$$L = \left\{ l \in \mathbb{Z}^{n+k} \big| A \cdot l = (0, \cdots, 0) \right\}. \quad (78)$$

$I_A$ is called the toric ideal. By the property (b), the system is regular holonomic.

The finite set of generators of the toric ideal can be obtained as follows (Algorithm 4.5 of [30]). We write the column vectors $A_i \ (1 \leq i \leq n + k)$ of $A$ as $A_i^+ - A_i^-$, where $j$-th component of $A_i^+$ (resp. $A_i^-$) is the $j$-th component of $A_i$ if it is positive (resp. negative), otherwise zero. And consider the ideal $I_0$ in $\mathbb{Q} \langle x_1, \cdots, x_{n+k}, t_0, t_1, \cdots, t_n \rangle$ whose generators are $t_0 t_1 \cdots t_n - 1$ and $x_i \left( \prod_{j: (A_i)_j > 0} t_j^{(A_i)_j} \right) - \left( \prod_{j: (A_i)_j < 0} t_j^{-(A_i)_j} \right) \ (1 \leq i \leq n + k)$. Then the generators of $I_0 \cap \mathbb{Q} [x_1, \cdots, x_{n+k}]$ are the generators of $I_A$ with the identification of $x_i$ and $\partial_{a_i}$.

Consider the case where the first row of $A$ is $(1, \cdots, 1)$ and write $A$ in the following form

$$A = \begin{pmatrix} 1 & \cdots & 1 \\ \nu_1 & \cdots & \nu_{n+k} \end{pmatrix}, \quad \nu_i \in \mathbb{Z}^{n-1}, \quad (1 \leq i \leq n + k).$$

Then there is the following formal solution to the GKZ-system in the integral form

$$\int P(X)^{\beta_1} \frac{dX_1}{X_1^{\beta_2+1}} \wedge \cdots \wedge \frac{dX_n}{X_{n-1}^{\beta_n+1}}, \quad (81)$$

where $P(X) = \sum_{i=1}^{n+k} a_i \prod_{j=1}^{n-1} X_j^{(\nu_j)_i}$. The proof is found in [31]. This statement for $\beta_1 = 0$ is trivial. However, the analysis of the local B-model suggests that $\int \log P(X) \prod_{j=1}^{n} \frac{dX_j}{X_j^{\beta_{j+1}+1}}$ might be the formal solution to the GKZ-system with $\beta_1 = 0$. 

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Appendix B: World-sheet Instanton Numbers

Following tables are the world-sheet instanton numbers of low degree for Model 0 and 1 \[\text{[7]}\].

Model 0: \(d_{n_1,n_2}\). The degree \(n_1\) (resp. \(n_2\)) grows down (resp. right).

\[
\begin{bmatrix}
d_{0,0} & -2 & 0 & 0 & 0 & 0 & 0 \\
d_{1,0} & -2 & -4 & -6 & -8 & -10 & -12 \\
d_{2,0} & 0 & 0 & -6 & -32 & -110 & -288 \\
d_{3,0} & 0 & 0 & 0 & -8 & -110 & -756 \\
d_{4,0} & 0 & 0 & 0 & 0 & -10 & -288 \\
d_{5,0} & 0 & 0 & 0 & 0 & 0 & -12 \\
\end{bmatrix}
\]

Model 1: \(d_{n_1,n_2,n_3}\). The degree \(n_2\) (resp. \(n_3\)) grows down (resp. right).

| \(n_1\) | \(d_{n_1,n_2,n_3}\) | \(n_1\) | \(d_{n_1,n_2,n_3}\) |
|---|---|---|---|
| 0 | \[
\begin{bmatrix}
d_{0,0,0} & 2 & 0 & 0 & 0 & 0 & 0 \\
2 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] | 1 | \[
\begin{bmatrix}
d_{1,0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -4 & 0 & 0 & 0 & 0 \\
0 & 6 & -8 & 0 & 0 & 0 \\
0 & 0 & 10 & -12 & 0 & 0 \\
0 & 0 & 0 & 14 & -16 & 0 \\
0 & 0 & 0 & 0 & 18 & -20 \\
\end{bmatrix}
\] |
| 2 | \[
\begin{bmatrix}
d_{2,0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 10 & -12 & 0 & 0 \\
0 & 0 & -12 & 70 & -64 & 0 \\
0 & 0 & 0 & -64 & 270 & -220 \\
\end{bmatrix}
\] | 3 | \[
\begin{bmatrix}
d_{3,0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 14 & -16 & 0 \\
0 & 0 & 0 & -64 & 270 & -220 \\
\end{bmatrix}
\] |
| 4 | \[
\begin{bmatrix}
d_{4,0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 18 & -20 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] | 5 | \[
\begin{bmatrix}
d_{5,0,0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\] |

Appendix C: Yukawa Coupling at the Gauge Theory Limit

We present the form of the Yukawa coupling \(F_{uuu}\) (or \(F_{vvv}, F_{www}\) for \(N_f = 2, 3\)) in the leading order at \(\epsilon \to 0\). The parameters are chosen as \(m = m_1\) for \(N_f = 1\), and as in Table \[\text{[7]}\] for \(N_f = 2, 3\). \(\Delta_2, \Delta_3\) in the denominator of the form for \(N_f = 2, 3\) are also given below.
| $N_f$ | $F_{uu} = \frac{e^2}{8(u^2-\Lambda^4)} \left( \frac{64u^2}{256 u^2 (u-m^2) + 32 \Lambda^4 (9u^2 - m^2)^2 - 27 \Lambda^8} \right)$ |
|-------|---------------------------------------------------------------|
| 0     | $F_{uu} = \frac{e^2}{256 u^2 (u-m^2) + 32 \Lambda^4 (9u^2 - m^2)^2 - 27 \Lambda^8}$ |
| 1     | $F_{vv} = \frac{e^2}{\Delta^2} \left[ -8m^4 + 16m^2 n^2 - 8n^4 + 12n^2 v + 12n^2 v - 4v^2 + \Lambda^2 (m^2 + 2n^2 - v) \right]$ |
| 2     | $F_{ww} = \frac{e^2}{\Delta^3} \left[ -32m^4 p^2 + 64m^2 n^2 p^2 - 32n^4 p^2 + 24m^4 w - 48m^2 n^2 w + 24n^4 w + 48m^2 p^2 w + 48n^2 p^2 w - 32m^2 w^2 - 32n^2 w^2 - 16p^2 w^2 + 8w^3 + \Lambda (6m^5 - 12m^3 n^2 + 6mn^4 - 9m^4 p + 10m^2 n^2 p - n^4 p - 4m^3 p^2 + 28mn^2 p^2 + 4m^2 p^3 + 8n^2 p^3 - 4m^3 w - 28mnw^2 + 10m^2 pw) - 2n^2 pw + 4mp^2 w - 4p^3 w - 2mw^2 + p^2 w \right] + \Lambda^2 (m^4 - 2m^2 n^2 + n^4 - 2m^3 p + 2mn^2 p + m^2 p^2 + 3n^2 p^2 - m^2 w - 3n^2 w + 2mpw - p^2 w)$ |

$$\triangle_2 = -48v^2 (m^4 + (n^2 - v)^2 - 2m^2 (n^2 + v)) + 4m^2 p^3 + 8n^2 p^3 - 4m^3 w$$

$$- 28mn^2 w + 10m^2 pw - 2n^2 pw + 4mp^2 w - 4p^3 w - 2mw^2 - pw^2$$

$$+ 24 \Lambda^2 (2m^6 - 2n^6 + 4n^2 v - n^2 v^2 - v^3 - m^4 (6n^2 + 5v) + m^2 (6n^4 + n^2 v + 4v^2))$$

$$- 3 \Lambda^4 (m^4 - 8n^4 + 8n^2 v + v^2 - 2m^2 (10n^2 + v)) + 3 \Lambda^6 n^2.$$ 

$$\triangle_3 = -\frac{5}{8} \left[ 256m^4 p^2 w^2 - 512m^2 n^2 p^2 w^2 + 256n^4 p^2 w^2 - 256m^4 w^3 + 512m^2 n^2 w^3 
- 256n^4 w^3 - 512m^2 p^2 w^3 - 512n^2 p^2 w^3 + 512m^2 w^4 + 512n^2 w^4 
+ 256p^2 w^4 - 256w^5 
+ \Lambda (-256m^6 p^3 + 768m^4 n^2 p^3 - 768m^2 n^4 p^3 + 256n^6 p^3 + 288m^6 pw 
- 864m^4 n^2 pw + 864m^2 n^4 pw - 288n^6 pw + 128m^5 p^2 w - 256m^3 n^2 p^2 w 
+ 128mn^4 p^2 w + 640m^4 p^3 w - 128m^2 n^2 p^3 w - 512n^4 p^3 w - 192m^5 w^2 
+ 384m^3 n^2 w^2 - 192mn^4 w^2 - 672m^4 pw^2 + 64m^2 n^2 pw^2 + 608n^4 pw^2 
- 256m^3 p^2 w^2 - 512mn^2 p^2 w^2 - 512m^2 p^3 w^2 + 128n^2 p^3 w^2 + 384m^3 w^3 
+ 640m^2 w^3 + 480m^2 pw^3 - 224n^2 pw^3 + 128mp^2 w^3 + 128p^3 w^3 
- 192mw^4 - 96pw^4) 
+ \Lambda^2 (-27m^8 + 108m^6 n^2 - 162m^4 n^4 + 108m^2 n^6 - 27n^8 + 72n^7 p - 216m^5 n^2 p 
+ 216m^3 n^4 p - 72mn^6 p - 56m^6 p^2 + 160m^4 n^2 p^2 - 152m^2 n^4 p^2 + 48n^6 p^2 
- 32m^5 p^3 + 352m^3 n^2 p^3 - 320mn^4 p^3 + 16m^4 p^4 - 320m^2 n^2 p^4 - 128n^4 p^4 
+ 60m^6 w - 84m^4 n^2 w - 12m^2 n^4 w + 36n^6 w - 120m^5 pw - 400m^3 n^2 pw 
+ 520mn^4 pw + 136m^4 p^2 w + 192m^2 n^2 p^2 w + 56n^4 pw + 64m^3 p^3 w 
+ 32mn^3 p^3 w - 32m^2 p^4 w + 128n^2 p^4 w - 66m^4 w^2 + 324n^2 w^2 - 2n^4 w^2 
+ 24m^3 pw^2 - 152mn^2 pw^2 - 104m^2 p^2 w^2 - 160n^2 p^2 w^2 - 32mp^3 w^2 + 16p^4 w^2 
+ 60m^2 w^3 + 36n^2 w^3 + 24mpw^3 + 24p^3 w^3 - 27w^4) \right].$
+ \Lambda^3(-4m^7 + 44m^5n^2 - 76m^3n^4 + 36mn^6 + 12m^6p - 140m^4n^2p + 116m^2n^4p \\
+ 12m^6p - 12m^5p^2 + 168m^3n^2p^2 - 28mn^4p^2 + 4m^4p^3 - 40m^2n^2p^3 - 60n^4p^3 \\
- 48mn^2p^4 + 16n^2p^5 + 8m^5w - 16m^3n^2w + 8mn^4w - 24m^4pw - 16m^2n^2pw \\
+ 40n^4pw + 24m^3p^2w + 8mn^2p^2w - 8m^2p^3w + 24n^2p^3w - 4m^3w^2 + 36mn^2w^2 \\
+ 12m^2pw^2 - 36n^2pw^2 - 12mp^2w^2 + 4p^3w^2) \\
+ \Lambda^4(-4m^4n^2 - 8m^2n^4 + 4n^6 - 16m^3n^2p + 16mn^4p + 24m^2n^2p^2 - 8n^4p^2 \\
- 16mn^2p^3 + 4n^2p^4)].

Appendix D: Differential Operators for the Periods

We show the differential operators \( \mathcal{P} \) for \( N_f = 0, 1 \) and 2.

| \( N_f \) | \( \mathcal{P} = P(u, m_i)\partial_u^3 + Q(u, m_i)\partial_u^2 + R(u, m_i)\partial_u \) |
|---|---|
| 0 | \( P = u^2 - \Lambda^4 \) |
| | \( Q = 2u \) |
| | \( R = \frac{1}{4} \) |
| 1 | \( P = (4m^2 - 3u)(-256m^2u^2 + 256u^3 + \Lambda^3m(256m^2 - 288u) + 27\Lambda^6) \) |
| | \( Q = -2048m^4u + 3840m^2u^2 - 1536u^3 - 384\Lambda^3m^3 + 81\Lambda^6 \) |
| | \( R = -8(32m^4 - 72m^2u + 24u^2 + 9\Lambda^3m) \) |
| 2 | \( P = [-4(2m^4 + 2n^4 - 3n^2v + v^2 - m^2(4n^2 + 3v)) + \Lambda^2(m^2 + 2n^2 - v)] \times [16u^2(m^4 + (n^2 - u)^2 - 2m^2(n^2 + v)) \raisepoint \\
- 8\Lambda^2(2m^6 - 2n^6 + 4n^4v - n^2v^2 - v^3 - m^4(6n^2 + 5v) \raisepoint \\
+ m^2(6n^4 + n^2v + 4v^2)) \raisepoint \\
+ \Lambda^4(m^4 - 8n^4 + 8n^2v + v^2 - 2m^2(10n^2 + v)) + \Lambda^6n^2] \raisepoint \\
Q = -64u(4m^8 - m^6(16n^2 + 15v) + (n^2 - v)^2(4n^4 - 7n^2v + 2v^2) \raisepoint \\
+ m^4(24n^4 + 15m^2v + 20v^2) + m^2(-16n^6 + 15n^4v + 8n^2v^2 - 11v^3)) \raisepoint \\
- 16\Lambda^2(8m^8 - 4n^8 - 4n^6v + 27n^4v^2 - 24n^2v^3 + 5v^4 - 2m^6(10n^2 + 13v) \raisepoint \\
+ 3m^4(4n^4 + 16n^2v + 11v^2) + 2m^2(2n^6 - 9n^4v + 6n^2v^2 - 10v^3)) \raisepoint \\
+ 4\Lambda^4(7m^6 - 4n^6 + 9m^4(7n^2 - 2v) - 12n^4v + 21n^2v^2 - 4v^3 \raisepoint \\
+ 3m^2(32n^4 - 20n^2v + 5v^2)) \raisepoint \\
+ \Lambda^6(-m^4 - 28m^2n^2 - 4n^4 + 2m^2v + 12n^2v - v^2) + \Lambda^8n^2 \raisepoint \\
R = -16(2m^6 - m^6(8n^2 + 9v) + (n^2 - v)^2(2n^4 - 5n^2v + v^2) \raisepoint \\
+ m^4(12n^4 - 9n^2v + 13v^2) - m^2(8n^6 - 9n^4v + 2n^2v^2 + 7v^3)) \raisepoint \\
+ 4\Lambda^2(m^6 + 8n^6 - 22n^4v + 16n^2v^2 - 2v^3 - 2m^4(15n^2 + 2v) \raisepoint \\
+ m^2(21n^4 + 2n^2v + 5v^2)) \raisepoint \\
+ \Lambda^4(-m^4 - 13m^2n^2 - 10n^4 + 2m^2v + 13n^2v - v^2) + \Lambda^6n^2 \raisepoint |
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