Dynamical Relativistic Systems and the Generalized 
Gauge Fields of Manifestly Covariant Theories*

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Abstract: The problem of the classical non-relativistic electromagnetically kicked oscillator can be cast into the form of an iterative map on phase space. The original work of Zaslovskii et al showed that the resulting evolution contains a stochastic flow in phase space to unbounded energy. Subsequent studies have formulated the problem in terms of a relativistically charged particle in interaction with the electromagnetic field. We review the standard derivation of the covariant Lorentz force, and review the structure of the relativistic equations used to study this problem. We show that the Lorentz force equation can be derived as well from the manifestly covariant mechanics of Stueckelberg in the presence of a standard Maxwell field. We show how this agreement is achieved, and criticize some of the fundamental assumptions underlying these derivations. We argue that a more complete theory, involving “off-shell” electromagnetic fields should be utilized. We then discuss the formulation of the off-shell electromagnetism implied by the full gauge invariance of the Stueckelberg mechanics (based on its quantized form), and show that a more general class of physical phenomena can occur.

I. Introduction

In recent years, there has been considerable interest in classical relativistic dynamical systems, engendered, in part, by the results of Zaslovskii et al [1] on the electromagnetically kicked oscillator. In this system, a model for real phenomena in plasmas, it was shown that (under certain conditions) there is an Arnol’d type diffusion along a stochastic web in phase space, and that the energy goes to infinity. The equation studied by Zaslovskii et al, for a non-relativistic system in a periodically δ-function kicking electric field with transverse magnetic field is

\[
\frac{d^2 x}{dt^2} + \omega_H^2 x(t) = \frac{e}{m} E(x,t),
\]

(1.1)

where the electric field was taken to be

\[
E(x,t) = -E_0 T \sin(k_0 x - \omega_0 t) \sum_{n=-\infty}^{\infty} \delta(t - nT),
\]

(1.2)

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and $\omega_H = eB_0/mc$ is the Larmor (cyclotron) frequency. For rational ratios of the frequency $2\pi/T$ and the oscillator frequency $\omega_H$, the phase space $(x, dx/dt)$ of the system is covered by a mesh of finite thickness for which the space within the cells contains regular dynamics, and the filaments contain motion of a stochastic nature, even for arbitrarily small fields. The particles may then diffuse (in a way analogous to Arnol’d diffusion) arbitrarily far into the region of high energies.

Since the energies of such systems are apparently unbounded, some authors reformulated the problem using relativistic kinematics. Longcope and Sudan [2] start from the equations

$$\frac{dv_x}{dt} + \frac{1}{\gamma} v_x \frac{d\gamma}{dt} + \frac{\omega_H^2}{\gamma^2} x = \frac{e}{m\gamma} E(x, t)$$

$$v_y = -(\omega_H/\gamma)x.$$  

They take $B = (0, 0, B_0)$ and $E = (E(x, t), 0, 0)$, where

$$E(x, t) = E_0T \sin kx \sum_{n=-\infty}^{\infty} \delta(t - nT).$$

One can set $z = v_z = 0$, as we shall see below.

These equations can be obtained from the covariant form of the Lorentz force [3-5] (we use the metric $(-, +, +, +)$)

$$\ddot{x}^\mu = \frac{e}{mc} F^\mu_\nu \dot{x}^\nu,$$  

(1.4)

where we take, in this section, $\dot{x}^\mu = dx^\mu/ds$, and

$$ds^2 = dt^2 - \frac{dx^2}{c^2}$$  

(1.5)

is the square of the proper time interval on the particle world line. Using the relation

$$\left(\frac{dx^\mu}{ds}\right)\left(\frac{dx^\mu}{ds}\right) = -c^2,$$  

(1.6)

actually assumed in the derivation of (1.4) [4], we see that

$$\left(\frac{dt}{ds}\right)^2 = 1 - \frac{v^2}{c^2},$$  

(1.7)

from which (taking the positive square root)

$$\frac{dt}{ds} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma.$$  

(1.8)

This result is consistent with the Lorentz transformation, but appears to be stronger. The transformation laws of special relativity are valid only in inertial frames. If the source of
(1.8) were an explicit Lorentz transformation, corresponding to a method which is sometimes used in developing the consequences of (1.4) ([2], [6], [7]), the second derivative \(d^2t/ds^2\) would clearly not have a reasonable physical interpretation. The result (1.7) is, however, the consequence of an identity (Eq.(1.6)) and hence it appears that it can be differentiated with respect to \(s\) [4]. However, we shall argue in the next section that the formula (1.5) relating “proper time” to the interval is highly dynamical, and cannot be understood as a relation with actual proper time if the particle is accelerating.* We continue here to discuss the consequences of (1.5) to show how the dynamical map constructed by Longcope and Sudan [2] follows.

Longcope and Sudan [2] take, using (1.7),
\[
\frac{d\mathbf{x}}{ds} = \frac{dt}{ds} \mathbf{v} = \gamma \mathbf{v},
\]
and replacing the second derivatives with respect to proper time by \((\gamma \frac{dt}{ds})^2\), (1.4) implies the differential equation
\[
m \frac{d}{dt}(\gamma \mathbf{v}) = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}),
\]
where we have used
\[
\frac{1}{\gamma} \left( \frac{d\mathbf{x}}{ds}, \frac{dt}{ds} \right) = (\mathbf{v}, 1).
\]
The three components of (1.9) are then
\[
m \frac{d}{dt}(\gamma v_x) = eE(x,t) + \frac{v_y}{c}B_0
\]
\[
m \frac{d}{dt}(\gamma v_y) = -\frac{e}{c}v_xB_0
\]
\[
m \frac{d}{dt}(\gamma v_z) = 0.
\]
It is therefore consistent to take \(v_z = z = 0\). Integrating the second of (1.11), one then finds (1.3).

As Longcope and Sudan [2] show by direct calculation using the map derived from (1.3) \((E(x,t) = 0)\) between \(\delta\)-function kicks; one obtains a map by integrating between kicks and establishing new initial conditions after the kick by integrating over the \(\delta\)-function), one recovers the stochastic web of Zaslavskii et al [1] for small velocities, but for larger velocities, the distribution in phase space becomes stochastic.

Let us now re-examine briefly the structure of the covariant Lorentz force (1.4). In addition to the vector equation
\[
\ddot{x}^j = \frac{e}{mc}(F_0^j \dot{x}^0 + F_k^j \dot{x}^k),
\]

* There is no question that \(c^2 dt^2 - d\mathbf{x}^2\), picking out the endpoints of the interval from a particle trajectory, is a Lorentz invariant. For an accelerating particle, the Lorentz transformation cannot, however, reach an inertial frame for this particle, and therefore this invariant does not correspond to the proper time on the particle.
one has the time component

\[ \ddot{x}^0 = \frac{e}{mc} F_0^0 \dot{x}^i. \]  

(1.13)

Directly changing the independent variable from \( s \) to \( t \), one obtains

\[ \frac{d}{ds} = \frac{dt}{ds} \frac{d}{dt}, \]

\[ \frac{d^2}{ds^2} = \frac{d^2 t}{ds^2} \frac{d}{dt} + \left( \frac{d t}{ds} \right)^2 \frac{d^2}{dt^2}. \]  

(1.14)

Hence Eq. (1.12) becomes

\[ \frac{d^2 t}{ds^2} v^j + \gamma^2 \frac{d^2 x^i}{dt^2} = \gamma \frac{e}{m} (E^j + \frac{1}{c} F^{jk} v^k). \]

From (1.13), we see that

\[ \frac{d^2 t}{ds^2} = \frac{e}{mc^2} \gamma E \cdot v, \]  

(1.15)

and hence \[5, 6\]

\[ \frac{d^2 x}{dt^2} = \frac{1}{\gamma m} \left( E + \frac{1}{c} (v \times B) - \frac{1}{c^2} v (v \cdot E) \right), \]  

(1.16)

or, using the identity \[8\]

\[ v \times (v \times E) = v (v \cdot E) - v^2 E, \]

one sees that the effective magnetic field is corrected by a relativistically induced magnetic field. The repeated acceleration of an electron by an electric field eventually becomes ineffective in the direction of motion of the particle. Its velocity becomes bounded dynamically by the velocity of light. Comparing (1.16) and (1.10), the term \( (e/\gamma mc^2)v (v \cdot E) \) must coincide with \( (v/\gamma)(d\gamma/dt) \).

Landau \[5\] remarks that

\[ \frac{d E_{\text{kin}}}{dt} = mc^2 \frac{d \gamma}{dt} \]  

(1.18)

can be shown to be \( e (v \cdot E) \). This follows directly by noting that

\[ \frac{d}{dt} \left( 1 - \frac{v^2}{c^2} \right)^{-\frac{1}{2}} = \frac{\gamma^3}{c^2} v \cdot \frac{dv}{dt}; \]

the result then follows from (1.17). It is physically quite reasonable, since \( e (v \cdot E) \) can be interpreted as the rate at which the field does work on the particle. However, the interpretation of \( mc^2 \gamma \) as the particle energy is derived from the Lorentz transformation from a state at rest, and is only valid in an inertial frame. It is not clear what the derivative of such an expression, implying a change of velocity, means.
We learn, furthermore, from the form of (1.13) and (1.15), that the variable \( t \), as a function of \( s \), undergoes acceleration, involving more than the Lorentz dilatation. It must therefore be considered a dynamical variable both in classical and quantum mechanics, as is done in the framework of Stueckelberg [9,10]

II. The Covariant Stueckelberg Formulation

In order to examine more clearly the assumptions and structure of the results outlined above, we formulate the problem in the manifestly covariant framework of Stueckelberg [9,10], using the standard Maxwell electromagnetic fields. We remark that Mendonça and Oliveira e Silva [11] have studied the dynamics generated by a "super Hamiltonian", where the energy \( E \) and time \( t \) are considered as dynamical variables. This formulation is equivalent to that of refs. [9] and [10]. We shall consider later the more general pre-Maxwell fields [12].

The Stueckelberg theory constitutes a formulation of relativistic dynamics in terms of forces, or interactions, in a given Lorentz frame. The motions, including acceleration (of any order) are not associated with the motion of a frame and hence the theory is applicable directly to the many-body problem (Horwitz and Piron [10]). The theory is constructed in a manifestly covariant way, and has the same form in any Lorentz (or Poincaré) frame. It has a symplectic (Hamiltonian) structure, and constitutes a direct generalization of the standard non-relativistic dynamics, but on an \( 8N \) dimensional phase space, including the time \( t \) and energy \( E \) of each particle. Given an invariant Hamiltonian \( K(x_1, x_2, \ldots, x_N, p_1, p_2, \ldots, p_N) \), where \( x_i \equiv x_i^\mu = (ct_i, \mathbf{x}_i) \), \( p_i \equiv p_i^\mu = (E_i/c, \mathbf{p}_i) \) are four-vectors, the Hamilton equations are

\[
\dot{x}_i^\mu = \frac{\partial K}{\partial p_i^\mu}, \quad \dot{p}_i^\mu = -\frac{\partial K}{\partial x_i^\mu},
\]

where the dot now indicates differentiation with respect to an invariant (universal) parameter \( \tau \). For a single free particle, an effective choice is

\[
K_0 = \frac{p_\mu p_\mu^\mu}{2M},
\]

where \( M \) is a parameter with dimensions of mass. Then,

\[
\dot{x}_\mu = \frac{p_\mu}{M}, \quad \dot{p}_\mu = 0.
\]

From the first of these, we see that

\[
\frac{dx}{d\tau} = \frac{P}{M}, \quad c\frac{dt}{d\tau} = \frac{E}{Mc}
\]

and hence

\[
\frac{dx}{dt} = c^2 \frac{P}{E},
\]

consistent with the definition of velocity in special relativity. This definition, based on directly observable quantities, does not depend on whether the particle is accelerating or
not. The occurrence of $p^\mu p_\mu$ in the generator (2.2) of evolution, and the existence of the dynamical variables $t, E$ as independent of $x, p$, imply that the theory is in principle not on “mass shell”. The quantity $E^2/c^2 - p^2$ is a dynamical variable, and its values are determined by the equations of motion. In the case of the free particle, $p^\mu p_\mu \equiv -m^2 c^2$ is a constant of the motion, and may be chosen arbitrarily in terms of initial conditions.

Choosing $m^2 = M^2$ ("on-shell") so that *

\[
\frac{E^2}{c^2} - p^2 = M^2 c^2,
\]

it follows from Eq. (2.4) that

\[
\frac{dt}{d\tau} = \frac{E}{Mc^2} = \frac{1}{\sqrt{1 - c^2 \frac{p^2}{E^2}}} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.
\]

(2.6)

Modifying (2.2) to represent a minimal gauge invariant interacting Hamiltonian [9], we write [9]-[11]

\[
K = \frac{(p^\mu - \frac{e}{c} A^\mu)(p_\mu - \frac{e}{c} A_\mu)}{2M}.
\]

(2.7)

The equation of motion for $x^\mu$ is

\[
\dot{x}^\mu = \frac{d x^\mu}{d\tau} = \frac{\partial K}{\partial p_\mu} = \frac{p^\mu - \frac{e}{c} A^\mu}{M}
\]

(2.8)

and we see that

\[
\frac{d x^\mu}{d\tau} \frac{d x_\mu}{d\tau} = c^2 \left( \frac{d s}{d\tau} \right)^2 = \frac{(p^\mu - \frac{e}{c} A^\mu)(p_\mu - \frac{e}{c} A_\mu)}{M},
\]

(2.9)

a quantity which (since it is proportional to the $\tau$-independent Hamiltonian) is strictly conserved. In fact, this quantity is the gauge invariant mass-squared:

\[
(p^\mu - \frac{e}{c} A^\mu)(p_\mu - \frac{e}{c} A_\mu) = -m^2 c^2,
\]

(2.10)

so that

\[
c^2 \left( \frac{d s}{d\tau} \right)^2 = c^2 \left( \frac{dt}{d\tau} \right)^2 - \left( \frac{dx}{d\tau} \right)^2 = \frac{m^2 c^2}{M^2}
\]

(2.11)

and

\[
\left( \frac{dt}{d\tau} \right)^2 = \frac{m^2 /M^2}{1 - \frac{v^2}{c^2}},
\]

(2.12)

where $m^2$ is a constant of the motion.

* It is sometimes convenient to choose $m = M$ as the Galilean (non-relativistic) limit of the variable $m$. The Galilean group requires a fixed mass for any representation. In this sense, we use the terminology “on-shell”.

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We now derive the Lorentz force from the Hamilton equation (this derivation has also been carried out independently by C. Piron [13])

\[
\frac{dp^\mu}{d\tau} = -\frac{\partial K}{\partial x^\mu} = \left(\frac{p^\nu - e A^\nu}{M}\right) \frac{e}{c} \frac{\partial A^\nu}{\partial x^\mu}
\]

(2.13)

Since \( p^\mu = M \frac{dx^\mu}{d\tau} + \frac{e}{c} A^\mu \), the left hand side is \( (A^\mu \text{ is evaluated on the particle world line } x^\nu(\tau)) \)

\[
\frac{dp^\mu}{d\tau} = M \frac{d^2 x^\mu}{d\tau^2} + \frac{e}{c} \frac{\partial A^\mu}{\partial x^\mu} \frac{dx^\nu}{d\tau},
\]

and hence

\[
M \frac{d^2 x^\mu}{d\tau^2} = \frac{e}{c} \left( \frac{\partial A^\mu}{\partial x^\mu} - \frac{\partial A^\mu}{\partial x^\nu} \right) \frac{dx^\nu}{d\tau},
\]

or

\[
M \frac{d^2 x^\mu}{d\tau^2} = \frac{e}{c} F^\mu_\nu \frac{dx^\nu}{d\tau},
\]

where \( (\partial^\mu \equiv \partial/\partial x^\mu) \)

\[
F^\mu_\nu = \partial^\mu A^\nu - \partial^\nu A^\mu.
\]

The form of (2.15) is identical to that of (1.4), but the temporal derivative is not with respect to the variable \( s \), the Minkowski distance along the particle trajectory, but with respect to the universal evolution parameter \( \tau \).

One might argue that these should be equal, or at least proportional by a constant, since the proper time is equal to the time which may be read on a clock on the particle in its rest frame. For an accelerating particle, however, one cannot transform by a Lorentz transformation, other than instantaneously, to the particle rest frame. The properties of an accelerating particle in an instantaneous rest frame are not equivalent to those in an inertial frame. As pointed out by Mashoon [14] *, the usual assumed equivalence is based on a postulate of “locality” by which the state of the particle is determined by its position and velocity. He emphasized that this postulate is not, in general, physically correct. He cites the example of a particle in interaction with the radiation field of electromagnetism. Such a particle radiates if it is accelerating, but the corresponding particle in a comoving inertial frame does not (in ref. [5], the radiation formula is derived by going to an instantaneous rest frame for the particle, and computing the radiation field of a non-relativistic particle with acceleration in that frame). Due to this inequivalence, the formula (2.15) appears to have a more reliable interpretation. The parameter of evolution \( \tau \) does not require a Lorentz transformation to achieve its meaning.

However, since \( m^2 \) is absolutely conserved by the Hamiltonian model (2.7), we have the constant relation

\[
ds = \frac{m}{M} d\tau,
\]

(2.17)

* I thank J. Beckenstein for bringing this work to my attention.
assuming the positive root (as we shall also do for the root of (2.12); we do not wish to discuss the antiparticle solutions here). Eq. (2.15) can therefore be written

\[ m \frac{d^2 x^\mu}{ds^2} = e F^\mu_\nu \frac{dx^\nu}{ds}, \]  

(1.4')

as in (1.4), with the “proper time”.

This result is somewhat puzzling. We have obtained the Lorentz force in terms of “proper time”, a concept which has somewhat insecure foundations (without an equivalence principle, as in Einstein’s gravity) in an arbitrary force field. In fact \( ds \) cannot be the proper time interval for the freely falling system; the system is accelerating, and this quantity is not accessible through a Lorentz transformation. Since \( \tau \) can be understood as the time one can observe on a freely falling ideal clock, the correspondence (2.17) seems too strong.

One can check with (2.15) (or (1.4)) that the conservation law is trivially maintained by the equations of motion. Since \( F^\mu_\nu \) is antisymmetric, multiplying (1.4') by \( \dot{x}_\mu \) yields zero, but \( \ddot{x}^\mu \ddot{x}_\mu = \frac{1}{2} \frac{d}{ds} (\ddot{x}^\mu \ddot{x}_\mu) = 0 \) is just this conservation law.

As we have pointed out, Mashoon’s [14] counterexample involves the phenomenon of radiation. Very clear discussions of the derivation of radiation corrections to (1.4) are given by Rohrlich [3] and Sokolov and Ternov [4], based on the original discussion of Dirac [15]. In these derivations, the identity (1.6) is used in the analysis of the series expansion, resulting in the well-known \( \frac{d}{ds} \ddot{x}^\mu \) term. The result contains the Abraham-Lorentz radiation reaction terms [16]:

\[ m \ddot{x}^\mu = \frac{e}{c} \dot{x}_\mu F^\mu_\nu + \frac{2}{3} \frac{r_0}{c} m \left( \frac{d}{ds} \ddot{x}^\mu - \frac{1}{c^2} \ddot{x}^\mu \ddot{x}_\nu \right), \]  

(2.18)

where \( r_0 = e^2/mc^2 \), the classical electron radius, and the dots refer here (to the end of this section), as in (1.4), to derivatives with respect to \( s \). It follows from the identity (1.6) that

\[ \dot{x}_\mu \ddot{x}^\mu = 0, \quad \dot{x}_\mu \frac{d}{ds} \ddot{x}^\mu + \dddot{x}_\mu \ddot{x}^\mu = 0, \]  

(2.19)

and hence (2.18) can be written as

\[ m \ddot{x}^\mu = \frac{e}{c} \dot{x}_\nu F^\nu_\mu + \frac{2}{3} \frac{r_0}{c} m \frac{d}{ds} \ddot{x}^\nu (\delta^\mu_\nu + \frac{1}{c^2} \ddot{x}^\mu \ddot{x}_\nu). \]  

(2.20)

The last factor on the right is a projection orthogonal to \( \dot{x}^\mu \) (if \( \dddot{x}^\mu \ddot{x}_\mu = -c^2 \)), and (2.20) is therefore consistent with the conservation of \( \dot{x}^\mu \ddot{x}_\mu \). Sokolov and Ternov [4] state that this conservation law follows “automatically” from (2.18), as from (1.4'), but it is apparently only consistent.
III. The Pre-Maxwell Fields

It seems remarkable that electromagnetism appears to be consistent with the use of accelerating frames under some circumstances, preserving the relation

\[ dt = \frac{ds}{\sqrt{1 - \frac{v^2}{c^2}}} \]  

(3.1)

One may attribute this to the fact that this relation is not a direct consequence of the Lorentz transformation (although it is clearly related to it by the invariance of the definition \( c^2 ds^2 = c^2 dt^2 - dx^2 \); recall, however, that this \( ds \) is not the proper time for an accelerating system). In the more dynamical treatment in terms of Stueckelberg mechanics, we see this relation emerging due to the dynamical conservation of \( m^2 \), equivalent to the constancy of \( \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \). And yet, the result is somewhat disturbing.

As pointed out by Sokolov and Ternov [4], there is no Lagrangian or Hamiltonian starting point for deriving (2.18). It is a result of a series of assumptions and decisions dealing with the propagators of the field. They understand this obstacle as being associated with the irreversible nature of, in particular, the radiation term proportional to \( \ddot{x}, \ddot{x}^\nu \). It is therefore very difficult to verify conservation laws.

As we have seen, although (2.18) is consistent with (1.6), even this equation (in which this identity is used) does not imply, without using the relation again, as in (2.20), that (1.6) is necessarily true.

The possibility that the relation \( c^2 ds^2 = -dx^\mu dx_\mu \), for \( ds \) the actual proper time of the particle, should acquire a non-trivial metric (instead of the Euclidean Minkowski form) in acceleration may be equivalent to the requirement that the particle, as well as the radiation field, go off mass shell during interaction, and hence invalidates the argument leading to (2.12) with constant \( m \) (see, for example, ref. [17], where it is shown that an effective potential in the Stueckelberg evolution function, giving rise to an explicit local mass shift, can be expressed equivalently in terms of a conformal metric tensor). Stueckelberg [9], in attempting to illustrate pair production and annihilation in a classical framework, discovered that the Hamiltonian (2.7) presents an obstacle; the particle world line cannot turn continuously to negative \( dt/d\tau \) if the proper time does not go through zero (he added an additional vector field interaction to complete the illustration).

The quantum form of the Stueckelberg theory, in which the Hamiltonian generates evolution in \( \tau \) by means of the Stueckelberg-Schrödinger equation (we take \( \hbar = c = 1 \) in the following)

\[ i \frac{\partial \psi_\tau}{\partial \tau} = K \psi_\tau \]  

(3.2)

presents a structure suggesting such a useful generalization. The invariance of the probability density \( |\psi_\tau(x)|^2 \) in spacetime is preserved by \( \psi_\tau(x) \rightarrow e^{i\xi_0 \Lambda} \psi_\tau(x) \), where \( \Lambda \) is a pointwise function of \( \tau \) and \( x^\mu \). To compensate for the derivatives of \( \Lambda \), one must introduce gauge compensation fields which we shall call \( a^\alpha(x, \tau) \) (see ref. [12] for discussion and further references),

\[ a^\alpha(x, \tau) = \{a^\mu(x, \tau), a^5(x, \tau)\} \]  

(3.3)
The fifth gauge field is required by the derivative \( i \partial_\tau \) (as for the non-relativistic Schrödinger equation, where the fourth gauge field \( A^0(x) \) is required by the derivative \( i \partial_t \)). The gauge fields transform as

\[
a'^\alpha = a^\alpha + \partial^\alpha \Lambda. \tag{3.4}
\]

The gauge covariant minimal coupling form of (3.2) is then

\[
i \frac{\partial}{\partial \tau} \psi_\tau(x) = \left\{ \frac{1}{2M} (p_\mu - e_0 a_\mu) (p^\mu - e_0 a^\mu) - e_0 a_5 \right\} \psi_\tau(x). \tag{3.5}
\]

It follows from this equation, in a way analogous to the Schrödinger non-relativistic theory, that there is a current

\[
j^\mu_\tau = -\frac{i}{2M} \{ \psi^*_\tau (\partial^\mu - ie_0 a^\mu) \psi_\tau - \psi_\tau (\partial^\mu + ie_0 a^\mu) \psi^*_\tau \}, \tag{3.6}
\]

which, with

\[
\rho_\tau \equiv j^5_\tau = |\psi_\tau(x)|^2,
\]
satisfies

\[
\partial_\tau \rho_\tau + \partial_\mu j^\mu_\tau \equiv \partial_\alpha j^\alpha = 0. \tag{3.7}
\]

We see that for \( \rho_\tau \to 0 \) pointwise (\( \int \rho_\tau(x) d^4x = 1 \) for any \( \tau \)),

\[
J^\mu(x) = \int_{-\infty}^{\infty} j^\mu_\tau(x) d\tau \tag{3.8}
\]
satisfies

\[
\partial_\mu J^\mu(x) = 0, \tag{3.9}
\]

and can be a source for the standard Maxwell fields.

We understand the operator on the right hand side of (3.5) as the quantum form of a classical evolution function

\[
K = \frac{1}{2M} (p_\mu - e_0 a_\mu) (p^\mu - e_0 a^\mu) - e_0 a_5. \tag{3.10}
\]

It follows from the Hamilton equations that

\[
\frac{dx^\mu}{d\tau} = \frac{p^\mu - e_0 a^\mu}{M} \tag{3.11}
\]

and

\[
\frac{dp^\mu}{d\tau} = e_0 \frac{dx^\nu}{d\tau} \frac{\partial a_\nu}{\partial x_\mu} + e_0 \frac{\partial a_5}{\partial x_\mu}.
\]

Hence

\[
M \frac{d^2 x^\mu}{d\tau^2} = e_0 \frac{dx^\nu}{d\tau} f^\mu_\nu + e_0 \left( \frac{\partial a_5}{\partial x_\mu} - \frac{\partial a^\mu}{\partial \tau} \right). \tag{3.12}
\]
If we define $x^5 \equiv \tau$, the last term can be written as $\partial^\mu a_5 - \partial_5 a^\mu = f_5^\mu$, so that

$$M \frac{d^2 x^\mu}{d\tau^2} = e_0 \frac{dx^\nu}{d\tau} f_\nu^\mu + e_0 f_5^\mu. \quad (3.13)$$

Note that in this equation, the last term appears in the place of the radiation correction terms of (2.18). It plays the role of a generalized electric field. Furthermore, we see that the relation (1.6) consistent with the standard Maxwell theory no longer holds as an identity; the Stueckelberg form of this result (2.10) in the presence of standard Maxwell fields, where $m^2$ is conserved, is also not valid. We now have

$$\frac{d}{d\tau} \frac{1}{2} M \left( \frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau} \right) = e_0 \frac{dx^\mu}{d\tau} f_\mu^5, \quad (3.14)$$

with the physical interpretation that the field $f_\mu^5$ does work on the system, resulting in a mass change. We see that, in general, $\frac{dx^\mu}{d\tau} \frac{dx_\mu}{d\tau}$ is not conserved.

Writing a Lagrangian that yields, upon variation, the equations of motion (3.13), and adding the kinetic term

$$-\frac{\lambda}{4} \int d\tau d^4 x f^{\alpha\beta}(x, \tau) f_{\alpha\beta}(x, \tau), \quad (3.16)$$

one obtains the equations of motion

$$\partial_\beta f^{\alpha\beta} = \frac{e_0}{\lambda} \frac{dx^\alpha}{d\tau} \delta^4(x - x(\tau)) \equiv e J^\alpha(x, \tau), \quad (3.17)$$

where we have identified $e_0/\lambda$ with the Maxwell electric charge $e$ (see below). Note that we have raised and lowered the index $\alpha$; the signature for raising and lowering the fifth index may be $\pm$, and we leave this unspecified here. The current $j^\alpha$ is the classical analog of the quantum mechanical current appearing in (3.7). Considering the $\mu$-components of (3.17), and integrating over $\tau$ (assuming that $f_5^\mu \to 0$ for $\tau \to \pm\infty$ pointwise), we find that, with (3.8) (see, e.g., [3] p.81 for the classical case),

$$\partial_\mu F^{\mu\nu} = e J^\nu, \quad (3.18)$$

where we have identified

$$\int d\tau a^\mu(x, \tau) = A^\mu(x) \quad (3.19)$$

with the Maxwell field. We see that if $A^\mu$ has dimension $L^{-1}$ (reciprocal length), then $a^\mu$ has dimension $L^{-2}$. The charge $e_0$ therefore has dimension $L$, as has $\lambda$. It then follows that $e_0/\lambda$ is dimensionless, and may (classically) be identified with the Maxwell charge, as indicated in (3.17).
IV. Discussion

The greater generality of the pre-Maxwell-Lorentz equations appears to admit a deeper study of the radiation process; we are presently applying it to the kicked relativistic oscillator for which the perturbation may be taken to be $f^5\mu$ (whose source is local event density in spacetime) or $f^{(5)}$ (whose zero-mode is the Maxwell electric field). The $\tau$-dependence of the fields, and the spacetime dependence of $a_5$, makes it possible for the particle to move off-shell during the radiation process.

In conclusion, we remark that in the quantized version of conventional electrodynamics, one thinks of the photon, the quantum of radiation contained in the field $A_{\mu}(x)$, as being emitted or absorbed by an accelerating charge (as in Compton scattering), or during the change of state of an atom. The spacial size scales of the emitter or absorber is of order $10^{-8}$ cm. or smaller, and yet the photon is a (nonlocalizable if massless) plane wave. It seems much more reasonable to think of quantized radiation as being carried by off-shell (massive) photons [12], which can be very localized during emission and absorption, and only asymptotically acquire masses in the neighborhood of zero (in the standard theory, an off-shell photon is considered as “virtual”; this property is understood in that the photon was emitted by some charged particle source and must be eventually absorbed). The standard QED teaches us that emission and absorption vertices are not trivially pointlike [18]. We see that both Einstein’s relativity and quantum electrodynamics suggest a generalization of Maxwell’s electrodynamics and its interaction with charged particles. The Stueckelberg formulation of relativistic dynamics, along with the pre-Maxwell fields that it implies, may provide a useful model for studying these effects.

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