DENSITY FUNCTION ANALYSIS FOR A STOCHASTIC SEIS EPIDEMIC MODEL WITH NON-DEGENERATE DIFFUSION

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Abstract. In this paper, we construct a stochastic SEIS epidemic model that incorporates constant recruitment, non-degenerate diffusion and infectious force in the latent period and infected period. By solving the corresponding Fokker-Planck equation, we obtain the exact expression of the density function around the endemic equilibrium of the deterministic system provided that the basic reproduction number is greater than one. Our work greatly improves the result of Chen [A new idea on density function and covariance matrix analysis of a stochastic SEIS epidemic model with degenerate diffusion, Appl. Math. Lett., 2020, 106200].

1. Introduction. Mathematical modelling is of considerable significance in the investigation of epidemiology because it can provide understanding of the underlying mechanisms which affect the transmission of disease and may suggest control strategies [16]. The first known mathematical model of epidemiology is formulated and solved by Daniel Bernoulli in 1760. From then on, various epidemic models have been developed and analyzed [13, 14, 9, 6, 21, 10, 20, 12]. The SEIS (susceptible-exposed-infectious) model is one of the most important models among them which supposes that after the initial infection, a host stays in a latent period before becoming infectious. An infectious host may die from disease or recover with no acquired immunity to the disease and again become susceptible [6]. Motivated by the models in Fan et al. [6] and Li and Jin [12], we first propose an SEIS epidemic model that incorporates constant recruitment, and infectious force in the latent period and

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infected period which can be described by the following system

\[
\begin{align*}
\frac{dS}{dt} &= A - \lambda_1 SI - \lambda_2 SE - dS + \gamma I, \\
\frac{dE}{dt} &= \lambda_1 SI + \lambda_2 SE - (d + \epsilon + \alpha_1) E, \\
\frac{dI}{dt} &= \epsilon E - (d + \gamma + \alpha_2) I,
\end{align*}
\]

where the parameter \(A\) denotes the influx of individuals into the susceptible; \(d\) represents the natural death rate of \(S, E\) and \(I\) compartments; \(\lambda_1, \lambda_2\) are the rates of the efficient contact in the infected period and latent period; \(\alpha_1\) and \(\alpha_2\) denote disease-caused death rates; \(\epsilon\) and \(\gamma\) are the transfer rates among the corresponding compartments. Therefore, \(\frac{1}{\epsilon}\) represents the mean latent period and \(\frac{1}{\gamma}\) is the mean infectious period. The parameters \(A, \lambda_1, \lambda_2, d, \gamma\) and \(\epsilon\) are positive constants, \(\alpha_1\) and \(\alpha_2\) denote nonnegative constants. According to the theory of Kermack and McKendrick [11], the basic reproduction number for system (1) is given by \(R_0 = \frac{\lambda_1 A}{d + \epsilon + \alpha_1(\gamma + d)} + \frac{\lambda_2 A}{d + \gamma + \alpha_2}\). If \(R_0 \leq 1\), there exists a disease-free equilibrium \(P_0 = (\frac{\lambda_1 A}{d + \gamma + \alpha_2}, 0, 0)\) which is locally asymptotically stable in the invariant set \(\Gamma\) while if \(R_0 > 1\), there is a unique endemic equilibrium \(P^* = (S^*, E^*, I^*)\) in the interior of \(\Gamma\) with \(S^* > 0, E^* > 0, I^* > 0\), where \(\Gamma = \{(S, E, I)|S + E + I \leq \frac{d}{\gamma}\}\).

On the other hand, recent research findings show that the transmission of infectious diseases, travel of populations and the design of control strategies may be disturbed by environmental fluctuations [18, 4]. Therefore, it is important and interesting to study the effects of stochastic perturbations on epidemic models. Recently, many scholars have paid much attention to stochastic epidemic models in biological research [17, 2, 1, 15, 7, 3, 5]. For example, Liu et al. [15] studied a stochastic SIR epidemic model with distributed delay and degenerate diffusion. They obtained the existence and uniqueness of a stable stationary distribution by using the Markov semigroup theory. They also proved the densities of the distributions of the solutions can converge in \(L^1\) to an invariant density. Feng et al. [7] analyzed the stochastic dynamics of a stochastic HIV-1 infection model with degenerate diffusion. Chen [5] considered a stochastic SEIS epidemic model with degenerate diffusion. By solving the corresponding Fokker-Planck equation, she obtained the exact expression of the density function around the endemic equilibrium of the deterministic system provided that the basic reproduction number is greater than one. However, as far as we know, there is little study on the density function analysis for a stochastic SEIS epidemic model with non-degenerate diffusion in the existing literature. In this paper, we will tend to do some work in this area to fill the gap.

Following the idea of [3], in this paper, we assume that stochastic perturbations are of the white noise type which are directly proportional to the variables \(S, E, I\), respectively. Then we obtain the following stochastic SEIS epidemic model with non-degenerate diffusion in Stratonovich form

\[
\begin{align*}
\frac{dS}{dt} &= [A - \lambda_1 SI - \lambda_2 SE - dS + \gamma I]dt + \sigma_1 S \circ dW_1(t), \\
\frac{dE}{dt} &= [\lambda_1 SI + \lambda_2 SE - (d + \epsilon + \alpha_1) E]dt + \sigma_2 E \circ dW_2(t), \\
\frac{dI}{dt} &= [\epsilon E - (d + \gamma + \alpha_2) I]dt + \sigma_3 I \circ dW_3(t),
\end{align*}
\]

where \(\{W_i(t)\}\) are mutually independent standard Brownian motions defined on a complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) with a filtration \(\mathcal{F}_{t \geq 0}\) satisfying the usual conditions (i.e., it is increasing and right continuous and \(\mathcal{F}_0\) contains all...
P-null sets) [19], $i = 1, 2, 3$, $\sigma_i > 0$ and $\sigma_i^2$ denote the intensities of the white noise, $i = 1, 2, 3$. Other parameters are the same as in system (1).

To proceed, we should first give some conditions under which system (2) has a unique global positive solution. Since the proof is similar to the statement of Lemma 1.1 in Chen [5], we only state the result without proof.

**Lemma 1.1.** For any initial value $(S(0), E(0), I(0))^T \in \mathbb{R}_+^3$, there is a unique solution $(S(t), E(t), I(t))^T$ to system (2) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one, namely, the solution $(S(t), E(t), I(t))^T \in \mathbb{R}_+^3$ for all $t \geq 0$ almost surely (a.s), where $\mathbb{R}_+^3 = \{y = (y_1, y_2, y_3)^T \in \mathbb{R}^3 : y_i > 0, i = 1, 2, 3\}$.

2. **Density function analysis for system (2).** In this section, we aim to obtain the local probability density function of system (2) around the endemic equilibrium $P^*$. To this end, let $x_1 = \ln S$, $x_2 = \ln E$ and $x_3 = \ln I$, then in view of system (2), we obtain

$$
\begin{align*}
    dx_1 &= [Ae^{-x_1} - \lambda_1 e^{x_3} - \lambda_2 e^{x_2} - d + \gamma e^{x_3-x_1}] dt + \sigma_1 dW_1(t), \\
    dx_2 &= [\lambda_1 e^{x_1} + x_3 - x_2 - (d + \epsilon + \alpha_1)] dt + \sigma_2 dW_2(t), \\
    dx_3 &= [\epsilon e^{x_2-x_3} - (d + \gamma + \alpha_2)] dt + \sigma_3 dW_3(t).
\end{align*}
$$

Let $U = (u_1, u_2, u_3)^T$, where $u_i = x_i - x_i^*$, $i = 1, 2, 3$ and $x_1^* = \ln S^*$, $x_2^* = \ln E^*$, $x_3^* = \ln I^*$. The linearized system of (3) is as follows

$$
\begin{align*}
    du_1 &= [-a_{11} u_1 - a_{12} u_2 - a_{13} u_3] dt + \sigma_1 dW_1(t), \\
    du_2 &= [a_{21} u_1 - a_{22} u_2 + a_{23} u_3] dt + \sigma_2 dW_2(t), \\
    du_3 &= [a_{32} u_2 - a_{33} u_3] dt + \sigma_3 dW_3(t),
\end{align*}
$$

where

$$
\begin{align*}
    a_{11} &= Ae^{-x_1^*} + \gamma e^{x_3^* - x_1^*},
    a_{12} &= \lambda_2 e^{x_2^*},
    a_{13} &= \lambda_1 e^{x_3^*} - \gamma e^{x_3^* - x_1^*}, \\
    a_{21} &= \lambda_1 e^{x_1^*} + x_3^* - x_2^* + \lambda_2 e^{x_1^*},
    a_{22} &= \lambda_1 e^{x_1^*} + x_3^* - x_2^*,
    a_{32} &= \epsilon e^{x_2^* - x_3^*}.
\end{align*}
$$

It is easy to see that $a_{11} > 0$, $a_{12} > 0$, $a_{21} > 0$, $a_{22} > 0$, $a_{32} > 0$. Next, we will obtain the explicit expression of density function around the endemic equilibrium $P^*$ by solving the corresponding Fokker-Planck equation.

**Theorem 2.1.** Let $U = (u_1, u_2, u_3)^T$ be a solution to system (4) with the initial value $(u_1(0), u_2(0), u_3(0))^T \in \mathbb{R}^3$. If $R_0 > 1$, then there is a unique approximate normal density function $\Phi(U)$ around the endemic equilibrium $P^*$ which satisfies the following form

$$
\Phi(U) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(u_1, u_2, u_3)^T \Sigma^{-1} (u_1, u_2, u_3)^T},
$$

where the covariance matrix $\Sigma$ is positive definite, and it satisfies the following form

(i) If $\Delta_1 \neq 0$ and $\Delta_2 \neq 0$, then

$$
\Sigma = \rho_1^2 M_{\Delta_1}^{-1} \Sigma_0 (M_{\Delta_1}^{-1})^T + \rho_1^2 (M_2 P_2 J_2)^{-1} \Sigma_0 [(M_2 P_2 J_2)^{-1}]^T + \rho_3^2 (M_3 P_3 J_3)^{-1} \Sigma_0 [(M_3 P_3 J_3)^{-1}]^T.
$$

(ii) If $\Delta_1 \neq 0$ and $\Delta_2 = 0$, then

$$
\Sigma = \rho_1^2 M_{\Delta_1}^{-1} \Sigma_0 (M_{\Delta_1}^{-1})^T + \rho_1^2 (M_2 P_2 J_2)^{-1} \Sigma_0 [(M_2 P_2 J_2)^{-1}]^T + \rho_3^2 w_2 (M_3 w_2 P_3 J_3)^{-1} \Sigma_0 [(M_3 w_2 P_3 J_3)^{-1}]^T.
$$
(iii) If $\Delta_1 = 0$ and $\Delta_2 \neq 0$, then
\[
\Sigma = \rho_1^2 M_1^{-1} \Sigma_0 (M_1^{-1})^T + \rho_2^2 w_1 (M_{2w_1} P_{2w_2} J_2) (M_{2w_1} P_{2w_2} J_2)^{-1} \theta_0 [(M_{2w_1} P_{2w_2} J_2)^{-1}]^T
\]
\[+ \rho_3^2 (M_3 P_3 J_3)^{-1} \Sigma_0 [(M_3 P_3 J_3)^{-1}]^T,
\]
where
\[
\Delta_1 = \lambda_2 A - (d + \epsilon + a_1)(d + \gamma + a_2), \quad \Delta_2 = \epsilon + a_1, \quad w_1 = \frac{a_{11} a_{12}}{a_{32}} - (a_{12} + a_{13}),
\]
\[
w_2 = \frac{a_{13}(a_{13} a_{21} - a_{11} a_{22}) + a_{22}^2 (a_{12} + a_{13})}{a_{13}^2},
\]
Here $\Delta_1 = 0$, $\Delta_2 = 0$ are equivalent to $w_1 = 0$ and $w_2 = 0$, respectively.

\[
\Sigma_0 = \begin{pmatrix}
\frac{2(a_1 a_2 - a_3)}{2(a_1 a_2 - a_3)} & 0 & -\frac{1}{2(a_1 a_2 - a_3)} \\
0 & 1 & 0 \\
-\frac{1}{2(a_1 a_2 - a_3)} & 0 & \frac{a_1}{2a_3 (a_1 a_2 - a_3)}
\end{pmatrix}, \quad \theta_0 = \begin{pmatrix}
\frac{1}{2b_1} & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{2b_1 b_2}{2b_1} & 0
\end{pmatrix},
\]
\[
\tilde{\theta}_0 = \begin{pmatrix}
\frac{1}{2b_1} & 0 & 0 \\
0 & \frac{1}{2b_1 b_2} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad a_1 = a_{11} + a_{22} + a_{32}, \quad a_2 = a_{11} (a_{22} + a_{32}) + a_{12} a_{21},
\]
\[
a_3 = a_{21} a_{32} (a_{12} + a_{13}), \quad b_1 = a_{22} + a_{32}, \quad b_2 = a_{12} a_{21}, \quad \tilde{b}_1 = a_{22} + a_{32} + \frac{a_{13} a_{21}}{a_{22}},
\]
\[
b_2 = \frac{a_{13} a_{21} a_{32}}{a_{22}}, \quad J_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & a_{12} & 1
\end{pmatrix},
\]
\[
P_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \frac{a_{22}}{a_{13}} & 1
\end{pmatrix}, \quad M_1 = \begin{pmatrix}
a_{21} a_{32} & -a_{32} (a_{22} + a_{32}) & a_{32} (a_{22} + a_{32}) \\
0 & a_{32} & -a_{32} \\
0 & 0 & 1
\end{pmatrix},
\]
\[
M_2 = \begin{pmatrix}
a_{32} w_1 & w_1 (a_{11} + a_{32}) & a_{11} \\
0 & w_1 & -a_{11} \\
0 & 0 & 1
\end{pmatrix}, \quad M_{2w_1} = \begin{pmatrix}
a_{32} & -a_{32} & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
\[
M_3 = \begin{pmatrix}
-a_{13} w_2 & -w_2 (a_{11} + a_{22}) & -a_{12} w_2 + \frac{a_{12}^2 (a_{12} + a_{13})^2}{a_{13}^2} \\
0 & w_2 & -a_{22} (a_{12} + a_{13}) \\
0 & 0 & a_{13}
\end{pmatrix},
\]
\[
M_{3w_2} = \begin{pmatrix}
-a_{13} & -a_{11} & a_{12} a_{22} \\
0 & 1 & a_{13} \\
0 & 0 & 1
\end{pmatrix},
\]
\[
\rho_1 = a_{21} a_{32} \sigma_1, \quad \rho_2 = a_{32} w_1 \sigma_2, \quad \rho_{2w_1} = a_{32} \sigma_2, \quad \rho_3 = -a_{13} w_2 \sigma_3, \quad \rho_{3w_2} = -a_{13} \sigma_3.
\]
Before giving the proof of Theorem 2.1, we should first introduce two lemmas which are associated with the corresponding standard matrix form.

**Lemma 2.2.** For the algebraic equation

\[ G_0^2 + A_0 \Sigma_0 + \Sigma_0 A_0^T = 0, \]

where \( G_0 = \text{diag}(1, 0, 0), \)

\[ A_0 = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \] (5)

If \( a_1 > 0, a_3 > 0 \) and \( a_1a_2 > a_3, \) then the matrix \( \Sigma_0 \) is positive definite. (Here \( A_0 \) in this form is called the standard \( R_1 \) matrix, we can easily obtain that \( a_1, a_2 \) and \( a_3 \) are the coefficients of the characteristic polynomial \( \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 \) of \( A_0. \)

The form of \( \Sigma_0 \) refers to (6).

**Proof.** Consider the equation

\[ G_0^2 + A_0 \Sigma_0 + \Sigma_0 A_0^T = 0, \]

where \( \Sigma_0 = (\sigma_{ij})_{3 \times 3} \) is a real symmetric matrix, then we have

\[ \Sigma_0 = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} \\ 0 & \sigma_{22} & 0 \\ \sigma_{13} & 0 & \sigma_{33} \end{pmatrix}, \] (6)

where

\[ \sigma_{12} = \sigma_{23} = 0, \quad \sigma_{22} = \frac{1}{2(a_1a_2 - a_3)}, \quad \sigma_{13} = -\sigma_{22}, \quad \sigma_{33} = \frac{a_1}{a_3}\sigma_{22}, \quad \sigma_{11} = a_2\sigma_{22}. \]

If \( a_1 > 0, a_3 > 0, a_1a_2 > a_3, \)

then

\[ \sigma_{11} > 0, \quad \sigma_{11}\sigma_{22} > 0, \quad \sigma_{22}(\sigma_{11}\sigma_{33} - \sigma_{13}^2) > 0, \]

which implies that all the leading principal minors of \( \Sigma_0 \) are positive. Hence \( \Sigma_0 \) is positive definite. This completes the proof. \( \square \)

**Lemma 2.3.** For the algebraic equation

\[ G_0^2 + B_0 \theta_0 + \theta_0 B_0^T = 0, \]

where \( G_0 = \text{diag}(1, 0, 0), \)

\[ B_0 = \begin{pmatrix} -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 \\ 0 & 0 & b_{33} \end{pmatrix}. \] (7)

If \( b_1 > 0 \) and \( b_2 > 0, \) then the matrix \( \theta_0 \) is semipositive definite. (Here \( B_0 \) in this form is called the standard \( R_2 \) matrix, we can easily obtain that \( b_1 - b_{33}, b_2 - b_1b_{33} \) and \( -b_2b_{33} \) are the coefficients of the characteristic polynomial \( \lambda^3 + (b_1 - b_{33})\lambda^2 + (b_2 - b_1b_{33})\lambda - b_2b_{33} \) of \( B_0. \)

The form of \( \Sigma_0 \) refers to (8).

**Proof.** Consider the equation

\[ G_0^2 + B_0 \theta_0 + \theta_0 B_0^T = 0, \]
where \( \theta_0 = (\theta_{ij})_{3 \times 3} \) is a real symmetric matrix, then we get
\[
\theta_0 = \begin{pmatrix}
\theta_{11} & 0 & 0 \\
0 & \theta_{22} & 0 \\
0 & 0 & 0
\end{pmatrix},
\]
(8)
where
\[
\theta_{12} = \theta_{13} = \theta_{23} = \theta_{33} = 0, \quad \theta_{11} = \frac{1}{2b_1}, \quad \theta_{22} = \frac{1}{2b_1b_2}.
\]
If
\[
b_1 > 0, \quad b_2 > 0,
\]
then
\[
\theta_{11} > 0, \quad \theta_{11} \theta_{22} > 0,
\]
which shows that \( \theta_0 \) is semipositive definite. This completes the proof.

Now we are in the position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. In view of system (2), we can rewrite system (2) in the following form
\[
dU = AU dt + G \circ dW(t),
\]
where \( G = \text{diag}(\sigma_1, \sigma_2, \sigma_3), \quad W(t) = (W_1(t), W_2(t), W_3(t))^T \) and
\[
A = \begin{pmatrix}
-a_{11} & -a_{12} & -a_{13} \\
a_{21} & -a_{22} & a_{23} \\
0 & a_{32} & -a_{32}
\end{pmatrix}.
\]
From the result of Grasman [8], it satisfies the following Fokker-Planck equation at the endemic equilibrium \( P^* \),
\[
-3 \sum_{i=1}^{3} \frac{\partial^2 \Phi}{\partial u_i^2} + \frac{\partial}{\partial u_1} [(a_{11}u_1 - a_{12}u_2 - a_{13}u_3)\Phi] + \frac{\partial}{\partial u_2} [(a_{21}u_1 - a_{22}u_2 + a_{23}u_3)\Phi] + \frac{\partial}{\partial u_3} [(a_{32}u_2 - a_{33}u_3)\Phi] = 0,
\]
which is approximated by a Gaussian distribution
\[
\Phi(U) = C \exp \left\{ -\frac{1}{2} (U - U^*)^T Q (U - U^*) \right\},
\]
where \( U^* = (0, 0, 0) \) and \( Q \) is a real symmetric matrix which satisfies the equation
\[
QG^2 + A^T Q + QA = 0.
\]
If \( Q \) is positive definite, we can adjust the value of \( C \) to make \( \Phi(U) \) normal, that is, \( C = (2\pi)^{-\frac{3}{2}} |Q|^\frac{1}{2} \) and let \( Q^{-1} = \Sigma \), then
\[
G^2 + A\Sigma + \Sigma A^T = 0.
\]
In view of the finite independent superposition principle, (9) is equivalent to the sum of the following three equations
\[
G_i^2 + A\Sigma_i + \Sigma_i A^T = 0, \quad i = 1, 2, 3,
\]
where \( G_1 = \text{diag}(\sigma_1, 0, 0), \quad G_2 = \text{diag}(0, \sigma_2, 0), \quad G_3 = \text{diag}(0, 0, \sigma_3), \quad \Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3, \quad G^2 = G_1^2 + G_2^2 + G_3^2.
\]
Before showing the positive definiteness of \( \Sigma \), we first define the corresponding characteristic polynomial of \( A \) as follows
\[
\phi_A(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3,
\]
where $a_1$, $a_2$ and $a_3$ are positive constants satisfying

(a) $a_1 = a_{11} + a_{22} + a_{32} > 0$;

(b) $a_2 = a_{11}(a_{22} + a_{32}) + a_{12}a_{21} > 0$;

(c) $a_3 = a_{21}a_{32}(a_{12} + a_{13}) > 0$.

From the theory of matrix’s similar transformation it follows that $\phi_A(\lambda)$ is the similarity invariant of the matrix $A$. It is easy to see that $a_1$, $a_2$ and $a_3$ are also similarity invariants. Thus we can realize the standard $R_1$ matrix of $A$ (i.e., $A_0$) is unique.

Now we are in the position to solve Equation (9) and prove the positive definiteness of $\Sigma$ through three steps. More precisely, we will verify that $\Sigma_1$ is positive definite and $\Sigma_2$, $\Sigma_3$ are at least semipositive definite.

Step 1. Consider the algebraic equation

$$G_1^2 + A\Sigma_1 + \Sigma_1 A^T = 0.$$  \hspace{1cm} (10)

Let $A_1 = M_1AM_1^{-1}$, where the standardized matrix

$$M_1 = \begin{pmatrix}
    a_{21}a_{32} & -a_{32}(a_{22} + a_{32}) & a_{32}(a_{22} + a_{32}) \\
    0 & a_{32} & -a_{32} \\
    0 & 0 & 1
\end{pmatrix},$$

which can be derived in the subsection (I) of Appendix, then $A_1$ is a standard $R_1$ matrix, that is to say,

$$A_1 = \begin{pmatrix}
    -a_1 & -a_2 & -a_3 \\
    1 & 0 & 0 \\
    0 & 1 & 0
\end{pmatrix}.$$

Moreover, note that

(i') $A - dS^* = \lambda_1 S^*I^* + \lambda_2 S^*E^* - \gamma I^*$;

(ii') $a_{11}a_{22} - a_{13}a_{21} = \lambda_1 e^{x_1^2 + x_2^2 - \gamma x_1^2} - \lambda_2 e^{x_1^2 + x_2^2 - \gamma x_1^2} + 2\gamma \lambda_1 e^{x_1^2 + x_2^2 - \gamma x_1^2} > 0$.

Then

$$a_1a_2 - a_3 = (a_{11} + a_{22} + a_{32})[a_{11}(a_{22} + a_{32}) + a_{12}a_{21}] - a_{21}a_{32}(a_{12} + a_{13})
= (a_{11} + a_{22})(a_{12}a_{21} + a_{11}a_{22} + a_{11}a_{32}) + a_{11}a_{32}^2 + a_{32}(a_{11}a_{22} - a_{13}a_{21})
> (a_{11} + a_{22})(a_{12}a_{21} + a_{11}a_{22} + a_{11}a_{32}) + a_{11}a_{32}^2
> 0.$$

According to Lemma 2.2, we obtain that $\Sigma_1$ is positive definite, and (10) is transformed into the following form

$$M_1G_1^2M_1^T + (M_1AM_1^{-1})(M_1\Sigma_1M_1^T) + (M_1\Sigma_1M_1^T)(M_1AM_1^{-1})^T = 0,$$

i.e.,

$$G_0^2 + A_1\Sigma_0 + \Sigma_0 A_1^T = 0.$$
where $\Sigma_0 = \frac{1}{\rho_1^2} M_1 \Sigma_1 M_1^T$, $\rho_1 = a_{21} a_{32} \sigma_1$ and the positive definiteness and form of $\Sigma_0$ is introduced in Lemma 2.2,

\[
\Sigma_0 = \begin{pmatrix}
\frac{a_2}{2(a_1 a_2 - a_3)} & 0 & -\frac{1}{2(a_1 a_2 - a_3)} \\
0 & \frac{1}{2(a_1 a_2 - a_3)} & 0 \\
-\frac{1}{2(a_1 a_2 - a_3)} & 0 & \frac{a_1}{2(a_1 a_2 - a_3)}
\end{pmatrix}.
\]

Therefore, the matrix $\Sigma_1$ is positive definite and $\Sigma_1 = \rho_1^2 M_1^{-1} \Sigma_0 (M_1^{-1})^T$.

**Step 2.** Consider the algebraic equation

\[
G_2^2 + A \Sigma_2 + \Sigma_2 A^T = 0.
\] (11)

Let $A_2 = J_2 A J_2^{-1}$, where the matrix

\[
J_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\]

then

\[
A_2 = \begin{pmatrix}
a_{22} & a_{22} & a_{21} \\
a_{32} & -a_{32} & 0 \\
-a_{12} & -a_{13} & -a_{11}
\end{pmatrix}.
\]

Define $B_2 = P_2 A_2 P_2^{-1}$, where the matrix

\[
P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & a_{12} & 1
\end{pmatrix},
\]

we obtain

\[
B_2 = \begin{pmatrix}
-a_{22} & -a_{22} - a_{12} a_{21} & a_{21} \\
-a_{32} & -a_{32} & 0 \\
0 & a_{32} & -a_{11}
\end{pmatrix}.
\]

In view of the value of $w_1$, where $w_1 = \frac{a_{11} a_{12} + a_{21} a_{32}}{a_{12} + a_{13}} - (a_{12} + a_{13})$, we discuss it in two cases.

**Case 1.** If $w_1 \neq 0$, by the method in Step 1, we can choose $C_2 = M_2 B_2 M_2^{-1}$, where the standardized matrix

\[
M_2 = \begin{pmatrix}
a_{32} w_1 & -w_1 (a_{11} + a_{32}) & a_{11}^2 \\
0 & w_1 & -a_{11} \\
0 & 0 & 1
\end{pmatrix}.
\]

In view of the uniqueness of standard $R_1$ matrix, we can easily get $C_2 = A_1$. Thus, (11) can be transformed into the following form

\[
(M_2 P_2 J_2) G_2^2 (M_2 P_2 J_2)^T + [(M_2 P_2 J_2) A (M_2 P_2 J_2)^{-1}][((M_2 P_2 J_2) \Sigma_2 (M_2 P_2 J_2)^T] + [((M_2 P_2 J_2) \Sigma_2 (M_2 P_2 J_2)^T][((M_2 P_2 J_2) A (M_2 P_2 J_2)^{-1}]^T = 0,
\]

i.e.,

\[
G_2^2 + C_2 \Sigma_0 + \Sigma_0 C_2^T = 0,
\]

where \( \Sigma_0 = \frac{1}{\rho_1^2} M_1 \Sigma_1 M_1^T \), \( \rho_1 = a_{21} a_{32} \sigma_1 \) and the positive definiteness and form of \( \Sigma_0 \) is introduced in Lemma 2.2,
with $\Sigma_0 = \frac{1}{\rho_2^2}(M_2P_2J_2)\Sigma_2(M_2P_2J_2)^T$, $\rho_2 = a_{32}w_1\sigma_2$. So the matrix $\Sigma_2$ is positive definite and $\Sigma_2 = \rho_2^2(M_2P_2J_2)^{-1}\Sigma_0[(M_2P_2J_2)^{-1}]^T$.

**Case 2.** If $w_1 = 0$, let $C_{2w_1} = M_{2w_1}B_2M_{2w_1}^{-1}$, where the standardized matrix

$$M_{2w_1} = \begin{pmatrix} a_{32} & -a_{32} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which can be obtained by the method (II) in Appendix, then

$$C_{2w_1} = \begin{pmatrix} -b_1 & -b_2 & -b_3 \\ 1 & 0 & 0 \\ 0 & 0 & -a_{11} \end{pmatrix}.$$ 

By the similarity invariant of the characteristic polynomial of $A$, we have

$$\phi_A(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = \phi_{C_{2w_1}}(\lambda) = \lambda^3 + (b_1 + a_{11})\lambda^2 + (b_2 + a_{11}b_1)\lambda + a_{11}b_2.$$ 

Hence

(i) $b_1 = a_1 - a_{11} = a_{22} + a_{32} > 0$;

(ii) $b_2 = a_2 - a_{11}b_1 = a_{11}(a_{22} + a_{32}) + a_{12}a_{21} - a_{11}(a_{22} + a_{32}) = a_{12}a_{21} > 0$.

Analogously, we can transform (11) into the following form

$$(M_{2w_1}P_2J_2)G_0^2(M_{2w_1}P_2J_2)^T + [(M_{2w_1}P_2J_2)A(M_{2w_1}P_2J_2)^{-1}][(M_{2w_1}P_2J_2)\Sigma_2(M_{2w_1}P_2J_2)^T]$$

$$+ [(M_{2w_1}P_2J_2)\Sigma_2(M_{2w_1}P_2J_2)^T][(M_{2w_1}P_2J_2)A(M_{2w_1}P_2J_2)^{-1}]^T = 0,$$

that is

$$G_0^2 + C_{2w_1}\theta_0 + \theta_0 C_{2w_1}^T = 0,$$

with $\theta_0 = \frac{1}{\rho_{2w_1}^2}(M_{2w_1}P_2J_2)\Sigma_2(M_{2w_1}P_2J_2)^T$, $\rho_{2w_1} = a_{32}\sigma_2$. According to Lemma 2.3, we obtain that $\theta_0$ is semipositive definite, and

$$\theta_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2b_1} & 0 \\ 0 & 0 & \frac{1}{2b_1b_2} \end{pmatrix}.$$ 

Therefore, $\Sigma_2 = \rho_{2w_1}^2(M_{2w_1}P_2J_2)^{-1}\theta_0[(M_{2w_1}P_2J_2)^{-1}]^T$, and $\Sigma_2$ is semipositive definite.

**Step 3.** Consider the algebraic equation

$$G_3^2 + A\Sigma_3 + \Sigma_3A^T = 0.$$ 

Let $A_3 = J_3AJ_3^{-1}$, where the matrix

$$J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

then

$$A_3 = \begin{pmatrix} -a_{32} & 0 & a_{32} \\ -a_{13} & -a_{11} & -a_{12} \\ a_{22} & a_{21} & -a_{22} \end{pmatrix}. $$
Let $B_3 = P_3 A_3 P_3^{-1}$, where the matrix

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{22} & a_{13} \end{pmatrix},$$

we get

$$B_3 = \begin{pmatrix} -a_{32} & -a_{22}a_{32} & a_{32} \\ -a_{13} & a_{13}a_{12}a_{22} & -a_{12} \\ 0 & w_2 & -a_{22}(a_{12} + a_{13})/a_{13} \end{pmatrix}.$$

Analogously, according to the value of $w_2$, where $w_2 = \frac{a_{13}(a_{13}a_{21} - a_{11}a_{22}) + a_{22}^2(a_{12} + a_{13})}{a_{13}^2}$, we study the following two cases.

**Case a.** If $w_2 \neq 0$, similarly, let $C_3 = M_3 B_3 M_3^{-1}$, where the standardized matrix

$$M_3 = \begin{pmatrix} -a_{13}w_2 & -a_{12}w_2 + \frac{a_{22}^2(a_{12} + a_{13})^2}{a_{13}^2} \\ -a_{11} + a_{22} & 0 \\ 0 & 0 \end{pmatrix},$$

Similarly, we have $C_3 = A_1$, And so (12) can be transformed into the following form

$$G_2^2 + C_3 \Sigma_0 + \Sigma_0 C_3^T = 0,$$

in which $\Sigma_0 = \frac{1}{\rho_3^2}(M_3 P_3 J_3) \Sigma_3 (M_3 P_3 J_3)^T$, $\rho_3 = -a_{13}w_2 \sigma_3$. Hence $\Sigma_3$ is positive definite and

$$\Sigma_3 = \rho_3^2 (M_3 P_3 J_3)^{-1} \Sigma_0 [(M_3 P_3 J_3)^{-1}]^T.$$

**Case b.** If $w_2 = 0$, that is, $a_{11}a_{13} - a_{12}a_{22} = a_{13}a_{22} + \frac{a_{13}^2a_{21}}{a_{22}}$. Let $C_{3w_2} = M_{3w_2} B_3 M_{3w_2}^{-1}$, where the standardized matrix

$$M_{3w_2} = \begin{pmatrix} -a_{13} & -a_{11} + \frac{a_{12}a_{22}}{a_{13}} & -a_{12} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which can be obtained by the method (II) in Appendix, so

$$C_{3w_2} = \begin{pmatrix} -\tilde{b}_1 & -\tilde{b}_2 & -\tilde{b}_3 \\ 1 & 0 & 0 \\ 0 & a_{22}(a_{12} + a_{13})/a_{13} \end{pmatrix}.$$

By the method in Case 2 of Step 2, we have

$$\phi_A(\lambda) = \phi_{C_{3w_2}}(\lambda) = \lambda^3 + \left[\tilde{b}_1 + \frac{a_{22}(a_{12} + a_{13})}{a_{13}}\right] \lambda^2 + \left[\tilde{b}_2 + \frac{a_{22}\tilde{b}_1(a_{12} + a_{13})}{a_{13}}\right] \lambda + \frac{a_{22}\tilde{b}_2(a_{12} + a_{13})}{a_{13}}.$$
So

\[
(i) \quad \hat{b}_1 = a_1 - \frac{a_{22}(a_{12} + a_{13})}{a_{13}} = a_{11} + a_{32} - \frac{a_{12}a_{22}}{a_{13}} = a_{32} + \frac{a_{11}a_{13} - a_{12}a_{22}}{a_{13}} = a_{32} + a_{22} + \frac{a_{13}a_{21}}{a_{22}} > 0;
\]

\[
(ii) \quad \hat{b}_2 = \frac{a_{3}a_{13}}{a_{22}(a_{12} + a_{13})} = \frac{a_{13}a_{21}a_{32}}{a_{22}} > 0.
\]

Likewise, (12) can be transformed into the following form

\[
G_0^2 + C_{3w_2} \theta_0 + \hat{b}_0 C_{3w_2}^T = 0,
\]

where \( \tilde{\theta}_0 = \frac{1}{\rho_{3w_2}} \Sigma_3 (M_{3w_2} P_3 J_3) \Sigma_3 (M_{3w_2} P_3 J_3)^T \), \( \rho_{3w_2} = -a_{13}\sigma_3 \). In view of Lemma 2.3, we obtain that \( \tilde{\theta}_0 \) is semipositive definite and

\[
\tilde{\theta}_0 = \begin{pmatrix}
\frac{1}{2b_1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

Therefore, \( \Sigma_3 = \rho_{3w_2}^2 (M_{3w_2} P_3 J_3)^{-1} \tilde{\theta}_0 [(M_{3w_2} P_3 J_3)^{-1}]^T \) and \( \Sigma_3 \) is semipositive definite. This completes the proof. \( \Box \)

**Remark 1.** Note that if \( \lambda_2 = \alpha_1 = \sigma_2 = \sigma_3 = 0 \), then system (2) becomes the model in Chen [5], while the model in Chen [5] is with degenerate diffusion, in other words, our work greatly improves the result of Chen [5].

3. **Conclusion.** This paper is concerned with a stochastic SEIS epidemic model that incorporates constant recruitment, non-degenerate diffusion and infectious force in the latent period and infected period. By solving the corresponding Fokker-Planck equation, we obtain the exact expression of the density function around the endemic equilibrium \( P^* \) of the deterministic system (1) provided that the basic reproduction number is greater than one. More precisely, we have obtained the following result:

- Let \( U = (u_1, u_2, u_3)^T \) be a solution to system (4) with the initial value \( (u_1(0), u_2(0), u_3(0))^T \in \mathbb{R}^3 \). If \( R_0 > 1 \), then there is a unique approximate normal density function \( \Phi(U) \) around the endemic equilibrium \( P^* \) which satisfies the following form

\[
\Phi(U) = (2\pi)^{-\frac{3}{2}} |\Sigma|^{-\frac{1}{2}} e^{-\frac{1}{2}(u_1, u_2, u_3) \Sigma^{-1} (u_1, u_2, u_3)^T},
\]

where the covariance matrix \( \Sigma \) is positive definite, and it satisfies the following form

(i) If \( \Delta_1 \neq 0 \) and \( \Delta_2 \neq 0 \), then

\[
\Sigma = \rho_1^2 M_1^{-1} \Sigma_0 (M_1^{-1})^T + \rho_2^2 (M_2 P_2 J_2)^{-1} \Sigma_0 [(M_2 P_2 J_2)^{-1}]^T + \rho_3^2 (M_3 P_3 J_3)^{-1} \Sigma_0 [(M_3 P_3 J_3)^{-1}]^T.
\]

(ii) If \( \Delta_1 \neq 0 \) and \( \Delta_2 = 0 \), then

\[
\Sigma = \rho_1^2 M_1^{-1} \Sigma_0 (M_1^{-1})^T + \rho_2^2 (M_2 P_2 J_2)^{-1} \Sigma_0 [(M_2 P_2 J_2)^{-1}]^T + \rho_3^2 (M_{3w_2} P_3 J_3)^{-1} \tilde{\theta}_0 [(M_{3w_2} P_3 J_3)^{-1}]^T.
\]
(iii) If $\Delta_1 = 0$ and $\Delta_2 \neq 0$, then
\[
\Sigma = \rho_1^2 M_1^{-1} \Sigma_0 (M_1^{-1})^T + \rho_2^2 w_1 (M_{2w_1} P_{2w_1} J_2)^{-1} \theta_0 [(M_{2w_1} P_{2w_1} J_2)^{-1}]^T \\
+ \rho_3^2 (M_3 P_3 J_3)^{-1} \Sigma_0 [(M_3 P_3 J_3)^{-1}]^T.
\]
(iv) If $\Delta_1 = 0$ and $\Delta_2 = 0$, then
\[
\Sigma = \rho_1^2 M_1^{-1} \Sigma_0 (M_1^{-1})^T + \rho_2^2 w_1 (M_{2w_1} P_{2w_1} J_2)^{-1} \theta_0 [(M_{2w_1} P_{2w_1} J_2)^{-1}]^T \\
+ \rho_3^2 (M_{3w_2} P_{3w_2} J_3)^{-1} \theta_0 [(M_{3w_2} P_{3w_2} J_3)^{-1}]^T,
\]
where
\[
\Delta_1 = \lambda_2 A - (d + \epsilon + a_1)(d + \gamma + a_2), \quad \Delta_2 = \epsilon + a_1, \quad w_1 = \frac{a_{11} a_{12}}{a_{32}} - (a_{12} + a_{13}),
\]
\[
w_2 = \frac{a_{13}(a_{13} a_{21} - a_{11} a_{22}) + a_{22}^2 (a_{12} + a_{13})}{a_{13}^2}.
\]
Here $\Delta_1 = 0$, $\Delta_2 = 0$ are equivalent to $w_1 = 0$ and $w_2 = 0$, respectively.

$$
\Sigma_0 = \begin{pmatrix}
\frac{1}{2(a_1 a_2 - a_3)} & 0 & -1 \\
0 & \frac{1}{2(a_1 a_2 - a_3)} & 0 \\
-\frac{1}{2(a_1 a_2 - a_3)} & 0 & \frac{a_1}{2a_3(a_1 a_2 - a_3)}
\end{pmatrix},
\theta_0 = \begin{pmatrix}
\frac{1}{2b_1} & 0 & 0 \\
0 & \frac{1}{2b_1 b_2} & 0 \\
0 & 0 & 0
\end{pmatrix},$

\[
\lambda_1 = \frac{1}{a_{22}}, \quad J_2 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}, \quad J_3 = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}, \quad P_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & a_{32}
\end{pmatrix},
\]

\[
\rho_1 = \frac{a_{21} a_{32} (a_{12} + a_{13})}{a_{22}}, \quad \rho_2 = a_{32} w_1 \sigma_2, \quad \rho_{2w_1} = a_{32} \sigma_2, \quad \rho_3 = -a_{13} w_2 \sigma_3, \quad \rho_{3w_2} = -a_{13} \sigma_3.
\]
Moreover, we need to point out that the method used in this paper can be also applied to study other multi-dimensional epidemic models with non-degenerate diffusion, such as malaria transmission model, rabies transmission model and syphilis transmission model, multiple infected cholera model, etc.

Appendix. We will find the corresponding standardized transformation matrix in terms of invertible linear transformations.

(I) The method of transforming standard $R_1$ matrix: For the algebraic equation

$$G^2 + A\Sigma + \Sigma A^T = 0,$$

where $G = \text{diag}(\sigma, 0, 0)$,

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix},$$

We assume that $a_{21} \neq 0$, $a_{32} \neq 0$. From the linear transformation of ordinary differential equations, we construct

$$dX = AX dt, \quad X = (x_1, x_2, x_3)^T.$$

Let $U = (u_1, u_2, u_3)^T$, $u_3 = x_3$, $u_2 = a_{32}x_2 + a_{33}x_3$, $u_1 = u'_2 = a_{32}dx_2 + a_{33}dx_3 = a_{21}a_{32}x_1 + a_{32}(a_{22} + a_{32})x_2 + a_{33}(a_{32} + a_{33})x_3$, then

$$M = \begin{pmatrix} a_{21}a_{32} & a_{32}(a_{22} + a_{32}) & a_{32}(a_{32} + a_{23}) \\ 0 & a_{32} & a_{32} \\ 0 & 0 & 1 \end{pmatrix},$$

which satisfies

$$U = MX,$$

i.e.,

$$dU = M dX = MAX dt = MA^{-1}U dt.$$

By the relationship of components of the vector $U$, we have

$$dU = d\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} dt.$$

Hence we derive the standard transform matrix $M$ and $MA^{-1} = A_0$, which refers to (5).

Let $\rho_1 = a_{21}a_{32}\sigma$, $\Sigma_0 = \frac{1}{\rho_1}M \Sigma M^T$ and the above equation can be transformed into the following form

$$G^2_0 + A_0\Sigma_0 + \Sigma_0 A_0^T = 0.$$

(II) The method of transforming standard $R_2$ matrix: For the algebraic equation

$$G^2 + \Theta \theta + \theta \Theta^T = 0,$$

where $G = \text{diag}(\sigma, 0, 0)$,

$$\Theta = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{22} \\ 0 & 0 & b_{33} \end{pmatrix}. $$

We assume that $b_{21} \neq 0$, then define $X = (x_1, x_2, x_3)^T$, $U = (u_1, u_2, u_3)^T$, let

$$dX = \Theta X dt,$$
and \( u_3 = x_3, u_2 = x_2, u_1 = u'_2 = b_{21}x_1 + b_{22}x_2 + b_{23}x_3, \) then
\[
M = \begin{pmatrix}
b_{21} & b_{22} & b_{23} \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

It satisfies
\[
U = MX,
\]
i.e.,
\[
dU = MdX = (M \mathbb{B} M^{-1})U dt,
\]
In view of the relationship of components of the vector \( U \), we get
\[
dU = d \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = \begin{pmatrix}
-b_1 & -b_2 & -b_3 \\
1 & 0 & 0 \\
0 & 0 & b_{33}
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} dt.
\]
So we get the standard transform matrix \( M \) and \( \mathbb{B} M^{-1} = B_0 \), which refers to (7).

Let \( \rho_2 = b_{21} \sigma, \theta_0 = \frac{1}{\rho_2} M \theta M^T \) and the above equation can be transformed into the following form
\[
G_0^2 + B_0 \theta_0 + \theta_0 B_0^T = 0.
\]

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