Construction of genuinely entangled subspaces and the associated bounds on entanglement measures for mixed states

K V Antipin
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Faculty of Physics, M. V. Lomonosov Moscow State University, Leninskie gory, Moscow 119991, Russia
E-mail: kv.antipin@physics.msu.ru

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Abstract
Genuine entanglement is the strongest form of multipartite entanglement. Genuine entangled pure states contain entanglement in every bipartition and as such can be regarded as a valuable resource in the protocols of quantum information processing. A recent direction of research is the construction of genuinely entangled subspaces (GESs)—the class of subspaces consisting entirely of genuinely entangled pure states. In this paper we present methods of construction of such subspaces including those of maximal possible dimension. The approach is based on the composition of bipartite entangled subspaces and quantum channels of certain types. The examples include maximal subspaces for systems of three qubits, four qubits, three qutrits. We also provide lower bounds on two entanglement measures for mixed states, the concurrence and the convex-roof extended negativity, which are directly connected with the projection on GESs.

Keywords: genuine multipartite entanglement, entanglement measures, quantum channels, tensor networks, entangled subspaces

1. Introduction

Quantum entanglement is an incredibly rich phenomenon relevant to a wide range of fields from condensed matter physics to quantum information science to particle physics. In quantum information theory entanglement is regarded as a resource for the tasks of quantum communication and computation [1, 2].

The structure of multipartite entanglement is far more complex than that in bipartite systems. There are various inequivalent entanglement classes [3, 4]. There are also such peculiar properties as monogamy relations [5–7] exhibited by some correlations of particles.
Genuine multipartite entanglement (GME) [8] is an extreme form of the described multipartite phenomenon. Pure GME states are entangled with respect to any bipartition of a compound system, the property that has recently found applications in the protocols of quantum information processing (see, e.g. [9–11]).

An interesting direction of research is construction and characterization of genuinely entangled subspaces (GESs)—subspaces consisting entirely of genuinely entangled states. To our knowledge, this concept was first introduced in reference [12], where some approaches to construction of GESs were also proposed. GESs can be interesting in connection with generating mixed GME states—any state with support on a GES is genuinely entangled. It is believed that GESs will be useful in quantum error correction [13] and quantum cryptography [14].

The research into the new methods of constructing GESs was continued in references [15–18]. In references [15–17] the construction was based mainly on the unextendible product bases (UPB) method [19, 20] and its variations. The approach in reference [18] was aimed at construction of GESs of maximal possible dimension and relied on the characterization of some bipartite entangled subspaces.

The motivation of the present paper is to develop an alternative approach to the construction of GESs which is not based on UPB. We use the compositionality of basic quantum operations and the correspondence between quantum channels and subspaces of a tensor product Hilbert space (see section 2). In many cases such a technique allows us to obtain families of GESs with simple parameterization, including those of maximal possible dimension. Another aspect is detecting entanglement of mixed states and estimating entanglement measures for them. It is known that there is an entanglement witness associated with the projection on an entangled bipartite subspace [21]. The extension of this witness to the GME case was obtained in reference [12]. In connection with the (bipartite) witness also lower bounds on some entanglement measures for bipartite mixed states can be derived [22]. We aim to extend these bounds to the GME case and use the examples of GESs we construct to illustrate their application. It was shown that computation of the convex-roof entanglement measures is NP-complete [23], so, in general, efficient algorithms or even closed mathematical expressions for them are impossible unless $P = NP$. In this regard such bounds play an important role in the theory of entanglement.

The paper is organized as follows. In section 2 we provide the necessary definitions and theoretical background. The methods of constructing GESs and their application to concrete three- and four-partite systems are presented in section 3. In section 4 we provide estimation of entanglement of some constructed subspaces and talk about lower bounds on entanglement measures for mixed states connected with the projection on a given GES. We conclude in section 5 and discuss possible developments of the present line of research.

2. Preliminaries

2.1. Genuine entanglement

Consider $n$-partite quantum states in the finite dimensional tensor product Hilbert space $H_1 \otimes \ldots \otimes H_n$. A pure $n$-partite state $|\psi\rangle$ is called fully separable if it can be written as a tensor product of states for every subsystem, i.e.

$$|\psi\rangle = |\phi_1\rangle \otimes \ldots \otimes |\phi_n\rangle.$$  (1)

States that are not fully separable are said to be entangled.

A pure $n$-partite state $|\psi\rangle$ is called biseparable if it can be written as a tensor product

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_{\bar{A}}\rangle$$  (2)
with respect to some bipartition \( A|\bar{A} \) where \( A \) denotes a particular subset of subsystems and \( \bar{A} \) denotes its complement. A multipartite pure state is called genuinely multipartite entangled if it is not biseparable with respect to any bipartition.

There is generalization of these concepts to mixed states. A mixed multipartite state is called biseparable if it can be decomposed into a convex sum of biseparable pure states (note that different terms of the sum can be biseparable with respect to different bipartitions). Otherwise it is called genuinely multipartite entangled.

A subspace of a multipartite Hilbert space consisting entirely of entangled pure states is called completely entangled (CES). The examples of CESs are known [24–26]. The natural generalization of this notion is GESs—those composed entirely of genuinely entangled pure states.

### 2.2. Entanglement measures of states and subspaces

For the purpose of quantifying entanglement many entanglement measures were introduced, initially for bipartite systems [27–29]. Among them—the concurrence [30–32], the negativity [33, 34] and the geometric measure of entanglement [35, 36].

The concurrence of a pure bipartite state \( \psi \) with subsystems \( A \) and \( B \) is defined by

\[
C(\psi) = \sqrt{2 \left(1 - \text{Tr} \rho_A^2\right)},
\]

where \( \rho = |\psi\rangle\langle\psi| \) and \( \rho_A = \text{Tr}_B \rho \).

For mixed states \( \rho \) the concurrence is given by the convex roof construction, the minimum average concurrence taken over all ensemble decompositions of \( \rho \):

\[
C(\rho) = \min_{\{p_j, \psi_j\}} \sum_j p_j C(\psi_j).
\]

The negativity of \( \rho \) is defined as

\[
N(\rho) = \frac{1}{2} (\|\rho^{T_B}\|_1 - 1),
\]

where \( \rho^{T_B} \) is the partial transpose of \( \rho \) with respect to party \( B \), and \( \|M\|_1 = \text{Tr} \left\{ \sqrt{M^*M} \right\} \) is the trace norm of \( M \).

From the definition of \( N \) it is seen that entanglement of states with a positive partial transpose (PPT states) is not detected by this measure.

The convex-roof extended negativity (CREN) is given by

\[
N^{\text{CREN}}(\rho) = \min_{\{p_j, \psi_j\}} \sum_j p_j N(\psi_j).
\]

The two measures defined via convex roof are able to detect all entangled states.

Given the Schmidt decomposition \( |\psi\rangle = \sum_i \sqrt{\lambda_i} |i\rangle \otimes |i\rangle \) of a bipartite pure state the geometric measure of entanglement is defined by

\[
G(\psi) = 1 - \max_i \{\lambda_i\}.
\]

This measure is also extended to mixed states by the convex roof construction.
Bipartite entanglement measures can be generalized to measures of GME [37]. Given a bipartite entanglement measure $E$, the corresponding GME measure for a pure multipartite state $|\psi\rangle$ is defined as

$$E_{\text{GME}}(\psi) = \min_A E_A(\psi), \quad (8)$$

where the minimum is taken over all possible bipartitions $A|\bar{A}$ of a multipartite system and $E_A$ denotes the bipartite entanglement measure with respect to bipartition $A|\bar{A}$. The measure $E_{\text{GME}}$ is extended to mixed states by the convex roof construction

$$E_{\text{GME}}(\rho) = \min \{ (p_j, \psi_j) \} \sum_j p_j E_{\text{GME}}(\psi_j), \quad (9)$$

where, as usual, the minimization runs over all possible ensemble decompositions $\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j|$. There are several ways of quantifying entanglement of a subspace. In the present paper we will use the approach suggested in references [15, 38]: the entanglement of a subspace $W$ measured by $E$ is defined by

$$E(W) = \min_{|\psi\rangle \in W} E(\psi). \quad (10)$$

In place of $E$ here we will use the geometric measure of genuine entanglement, $G_{\text{GME}}$, which can be calculated via equations (7) and (8).

### 2.3. Quantum channels and entangled subspaces

Let $H$ be a finite dimensional Hilbert space, and $\mathcal{B}(H)$—the collection of all linear operators on $H$. Given two finite dimensional Hilbert spaces $H_A$ and $H_B$, a quantum channel is a linear, completely positive and trace-preserving map between $\mathcal{B}(H_A)$ and $\mathcal{B}(H_B)$ [1]. By definition, a quantum channel maps density operators to density operators.

Action of a quantum channel $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ on an operator $C \in \mathcal{B}(H_A)$, by Choi’s theorem [39], can be represented as

$$\Phi(C) = \sum_{i=1}^{N} K_i C K_i^\dagger, \quad (11)$$

where $N \leq \dim(H_A)\dim(H_B)$, and the operators $K_i : H_A \to H_B$ satisfy the trace-preserving property

$$\sum_{i=1}^{N} K_i^\dagger K_i = I, \quad (12)$$

where $I$—the identity operator, which in this case acts on $H_A$. Representation (11) is often referred to as the Kraus decomposition of a quantum channel with Kraus operators $\{K_i\}$.

There is a one-to-one correspondence between channels and linear subspaces of composite Hilbert spaces [40]. Given a set of three Hilbert spaces $(H_A, H_B, H_C)$, a subspace $W$ of the composite space $H_B \otimes H_C$, and an isometry $V : H_A \to H_B \otimes H_C$ whose range is $W$, the corresponding quantum channel $\Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ can be defined by

$$\Phi(\rho) = \text{Tr}_{H_C}(V \rho V^\dagger). \quad (13)$$
Conversely, due to Stinespring's dilation theorem [41], any quantum channel \( \Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B) \) can be represented by equation (13) for some subspace \( W \subset H_B \otimes H_C \).

Given some Kraus representation (11) of a quantum channel \( \Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B) \), the isometry \( V : H_A \to H_B \otimes H_C \) can be defined by the action on a vector state \( |\psi\rangle \in H_A \) in the following way:

\[
V|\psi\rangle = \sum_{i=1}^{N} K_i |\psi\rangle \otimes |i\rangle,
\]

where \( \{ |i\rangle \} \) — an orthonormal basis in \( H_C \), and \( \dim(H_C) = N \). Choosing some orthonormal basis \( \{ |\mu\rangle \} \) in \( H_A \), via equation (14) we obtain an orthonormal system of vectors \( \{ V|\mu\rangle \} \) spanning the subspace \( W \subset H_B \otimes H_C \).

Figure 1 provides the diagrammatic representation of equation (13). We adopt tensor diagram notation for quantum processes from reference [42]. References [43, 44] are also excellent introductions into diagrammatic reasoning in quantum information theory.

The entanglement of the subspace \( W \subset H_B \otimes H_C \) depends on the output characteristics of the corresponding quantum channel such as the maximal output norm [45]. Let \( \mathcal{D}(H) \) denote the set of density operators in \( \mathcal{B}(H) \). The maximal output norm of a channel \( \Phi : \mathcal{B}(H_A) \to \mathcal{B}(H_B) \) is defined by

\[
\nu_p(\Phi) = \sup_{\rho \in \mathcal{D}(H_A)} \| \Phi(\rho) \|_p, \quad p > 1,
\]

where \( \| \rho \|_p = (\text{Tr}(|\rho|^p))^{1/p} \) is the standard \( p \)-norm. Since \( p \)-norm is convex, it will take its maximum on the extremal states, and the supremum in equation (15) can be taken over pure input states. We will mostly use the Frobenius norm (\( p = 2 \)) and refer to \( (\nu_2(\Phi))^2 \) as the output purity of a channel \( \Phi \).

Obviously, a subspace \( W \subset H_B \otimes H_C \) is completely entangled if and only if the output purity of the corresponding quantum channel is strictly less than one. In this case the isometry \( V \) will transform the vector states of \( H_A \) into bipartite entangled states of the subspace \( W \subset H_B \otimes H_C \), and there will not be separable states in \( W \). Therefore, we can construct bipartite completely entangled subspaces finding examples of quantum channels with output purity less than one along with the corresponding isometries. Choosing some orthonormal basis \( \{ |\tilde{i}\rangle \} \) in \( H_A \), we obtain the orthonormal basis \( \{ V|\tilde{i}\rangle \} \) of the entangled subspace \( W \) via the isometry \( V \). As we will see, the most interesting cases, examples of entangled subspaces of maximal...
Figure 2. An isometry $V$ acting on subsystem $B$ of a pure bipartite entangled state $|\psi_{AB}\rangle$. The resulting pure state is tripartite, with subsystems $A, C, D$.

dimension, can be obtained by constructing dimension changing channels $\Phi: B(H_A) \to B(H_B)$ with $\dim(H_A) \neq \dim(H_B)$. Furthermore, such channels can be used in construction of GESs.

3. Construction of entangled subspaces

In this section we describe methods of constructing GESs with the use of quantum channels and bipartite entangled subspaces.

3.1. GES from bipartite CES

This approach was inspired by diagrammatic reasoning. Given a pure bipartite entangled state $|\psi\rangle_{AB}$, we can act on one of its two parties, $B$, with an isometry $V: H_B \to H_C \otimes H_D$ corresponding to some quantum channel $\Phi: B(H_B) \to B(H_C)$, as it is illustrated on figure 2. When will the resulting state be entangled for any bipartite cut? It is convenient to consider the initial state $|\psi\rangle_{AB}$ in the Schmidt form:

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A \otimes |i\rangle_B,$$  \hspace{1cm} (16)

where there are at least two nonzero Schmidt coefficients $\sqrt{p_i}$. The resulting state

$$|\chi\rangle_{ACD} = (I \otimes V)|\psi\rangle_{AB} = \sum_i \sqrt{p_i} |i\rangle_A \otimes V|i\rangle_B$$  \hspace{1cm} (17)

is surely entangled across bipartition $A|CD$: tracing out, for example, subsystem $CD$, we obtain a mixed state on $A$:

$$\text{Tr}_{CD} (|\chi\rangle\langle\chi|_{ACD}) = \sum_i p_i |i\rangle_A \langle i|_A.$$  \hspace{1cm} (18)

Now consider bipartition $C|AD$. Tracing out subsystem $A$, then $D$, we obtain:

$$\text{Tr}_{AD} (|\chi\rangle\langle\chi|_{ACD}) = \sum_i p_i \text{Tr}_B (V|i\rangle_B \langle i|_{B}) = \sum_i p_i \Phi(|i\rangle_B \langle i|_B),$$  \hspace{1cm} (19)

where the second equality is due to equation (13). For an arbitrary channel $\Phi$ this state may be pure—for example, we can choose a channel mapping all states to some pure state $|\phi\rangle\langle\phi|_C$. In order to guarantee mixedness, we can choose a channel with some specific property. One
such option is to pick a channel with output purity less than one—in this case each term in equation (19) will be a mixed state, and the resulting state will be a convex combination of such mixed density operators. Entanglement across bipartition $D/AC$ is analyzed similarly. Therefore, we have established that genuinely entangled tripartite states can be obtained from entangled bipartite ones by applying to one of the parties the isometry corresponding to a quantum channel with output purity less than 1.

Now, applying such an isometry to each state in a completely entangled subspace of a bipartite Hilbert space, we obtain a GES of a tripartite Hilbert space, and, since isometry preserves inner products, the orthonormal system of vectors spanning the latter is obtained by action of $V$ on the orthonormal system spanning the former.

As an illustration of this approach we construct a four-dimensional genuinely entangled $3 \otimes 3 \otimes 3$ subspace from a completely entangled $3 \otimes 3$ subspace of the same dimension. Here the bipartite $3 \otimes 3$ subspace is taken to be spanned by the following orthonormal system of vectors:

\begin{align*}
|\psi_1\rangle &= \sqrt{\lambda_1}|0\rangle \otimes |0\rangle + \sqrt{1 - \lambda_1}|2\rangle \otimes |2\rangle, \\
|\psi_2\rangle &= \sqrt{\lambda_2}|1\rangle \otimes |0\rangle + \sqrt{1 - \lambda_2}|0\rangle \otimes |1\rangle, \\
|\psi_3\rangle &= \sqrt{\lambda_3}|2\rangle \otimes |0\rangle + \sqrt{1 - \lambda_3}|1\rangle \otimes |2\rangle, \\
|\psi_4\rangle &= \sqrt{\lambda_4}|2\rangle \otimes |1\rangle + \sqrt{1 - \lambda_4}|0\rangle \otimes |2\rangle,
\end{align*}

where $0 < \lambda_i < 1$, $i = 1, 2, 3, 4$.

In fact, equation (20) determines a family of completely entangled subspaces parameterized by $\{\lambda_i\}$. They can be constructed with the use of specific dimension changing quantum channels (see appendix A).

Now, given a concrete $3 \otimes 3$ entangled subspace, we choose an isometry to act with on subsystem $B$, the second qutrit. As was shown above, the range of the isometry should be a completely entangled subspace, or, equivalently, the corresponding quantum channel should have output purity less than one. Such a role can play, for example, the isometry $V : H_B \rightarrow H_C \otimes H_D$ which maps orthonormal basis in $H_B$ to vectors spanning the antisymmetric subspace of $H_C \otimes H_D$:

\begin{align*}
V|0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle), \\
V|1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |2\rangle - |2\rangle \otimes |0\rangle), \\
V|2\rangle &= \frac{1}{\sqrt{2}}(|1\rangle \otimes |2\rangle - |2\rangle \otimes |1\rangle).
\end{align*}

Acting with $(I \otimes V)$ on the basis (20a)–(20d), we obtain the orthonormal system of vectors spanning a four-dimensional GES:

\begin{align*}
|\phi_1\rangle &= \sqrt{\frac{\lambda_1}{2}}(|0\rangle \otimes |0\rangle \otimes |1\rangle - |0\rangle \otimes |1\rangle \otimes |0\rangle) \\
&\quad + \sqrt{\frac{1 - \lambda_1}{2}}(|2\rangle \otimes |1\rangle \otimes |2\rangle - |2\rangle \otimes |2\rangle \otimes |1\rangle),
\end{align*}
\[ |\psi_2\rangle = \sqrt{\frac{\lambda_2}{2}} (|1\rangle \otimes |0\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle \otimes |0\rangle) \\
+ \sqrt{\frac{1 - \lambda_2}{2}} (|0\rangle \otimes |0\rangle \otimes |2\rangle - |0\rangle \otimes |2\rangle \otimes |0\rangle), \quad (22b) \]

\[ |\psi_3\rangle = \sqrt{\frac{\lambda_3}{2}} (|2\rangle \otimes |0\rangle \otimes |1\rangle - |2\rangle \otimes |1\rangle \otimes |0\rangle) \\
+ \sqrt{\frac{1 - \lambda_3}{2}} (|1\rangle \otimes |1\rangle \otimes |2\rangle - |1\rangle \otimes |2\rangle \otimes |1\rangle), \quad (22c) \]

\[ |\psi_4\rangle = \sqrt{\frac{\lambda_4}{2}} (|2\rangle \otimes |0\rangle \otimes |2\rangle - |2\rangle \otimes |2\rangle \otimes |0\rangle) \\
+ \sqrt{\frac{1 - \lambda_4}{2}} (|0\rangle \otimes |1\rangle \otimes |2\rangle - |0\rangle \otimes |2\rangle \otimes |1\rangle). \quad (22d) \]

Another example will be given in section 4 in the context of estimating entanglement of the constructed subspaces.

It is important to note that channels and associated isometries of the proposed type may not exist for some combinations of input/output dimensions and the number of Kraus operators, for example, when some additional requirements on the dimension of constructed GESs come into play. Indeed, suppose that one aims to construct a tripartite $d_1 \otimes d_2 \otimes d_3$ GES of maximal possible dimension, where $2 \leq d_1 \leq d_2 \leq d_3$, from some CES. According to Parthasarathy’s result [24], the maximal dimension of completely entangled subspaces of a $k_1 \otimes k_2$ Hilbert space is

\[ k_1 k_2 - (k_1 + k_2) + 1. \quad (23) \]

For each state in a $d_1 \otimes d_2 \otimes d_3$ Hilbert space there are three possible bipartitions into subsystems with dimensions $d_1$ and $d_2d_3$, $d_2$ and $d_1d_3$, $d_3$ and $d_1d_2$, respectively. One such bipartition is presented on figure 3. By definition, a tripartite GES is completely entangled across all the three bipartitions, hence, by equation (23), its dimension is upper-bounded by the minimum of three numbers: $d_1d_2d_3 - (d_1 + d_2d_3) + 1$, $d_1d_2d_3 - (d_2 + d_1d_3) + 1$ and $d_1d_2d_3 - (d_3 + d_1d_2) + 1$, which, as can be easily seen, is the first number. Consequently, as the first option, one can start with some bipartite completely entangled subspace of a tensor product $d_1 \otimes d_2d_3$ Hilbert space, with dimension equal to $d_1d_2d_3 - (d_1 + d_2d_3) + 1$, and find an appropriate isometry which can be applied to the party with dimension $d_1d_2$. This isometry acts from a Hilbert space of dimension $d_1d_2$ to a tensor product $d_2 \otimes d_3$ Hilbert space and, hence, can be regarded as the splitting operation: the subsystem of dimension $d_2d_3$ is being split into two smaller subsystems with dimensions $d_2$ and $d_3$, respectively (see figure 4). The isometry is, in fact, a unitary operation since the dimensions of initial and final Hilbert spaces are equal. A simple argument shows that such an isometry cannot correspond to a channel with output purity less than 1. Indeed, if it were the case, the isometry would have a $d_2d_3$-dimensional CES as its range, which is impossible since the whole $d_2 \otimes d_3$ Hilbert space cannot be completely entangled.

The maximal $d_1 \otimes d_2 \otimes d_3$ GES could also be obtained from $d' \otimes d_3$ or $d'' \otimes d_2$ CESs, with an isometry acting on party with dimension $d'(d'')$. The numbers $d'$ and $d''$ are to be determined. Consider, for example, the case with a $d' \otimes d_3$ CES, where the isometry acts from a
Figure 3. Gathering subsystems of a tripartite pure state $|\psi\rangle_{ABC}$ into bipartition $A|BC$. The dimensions of the corresponding Hilbert spaces are shown.

Figure 4. Splitting operation $U$ acting on subsystem $B$ of each pure state in a completely entangled subspace.

$d'$-dimensional Hilbert space to a $d_1 \otimes d_2$ tensor product Hilbert space. The maximal dimension of the CES, by equation (23), is $d'd_3 - (d' + d_3) + 1$, and, as was discussed earlier, the dimension of the maximal $d_1 \otimes d_2 \otimes d_3$ GES is equal to $d_1d_2d_3 - (d_1 + d_2d_3) + 1$. Consequently, to be able to obtain the maximal GES from this CES, one has to satisfy inequality

$$d'd_3 - (d' + d_3) + 1 \geq d_1d_2d_3 - (d_1 + d_2d_3) + 1.$$  

(24)

From this inequality we can determine the minimal possible number $d'$:

$$d'_\text{min} = 1 + d_2(d_1 - 1) + \text{ceil}\left(\frac{(d_1 - 1)(d_2 - 1)}{d_3 - 1}\right),$$  

(25)

where $\text{ceil}(x)$ maps $x$ to the least integer greater than or equal to $x$. Now, it can be easily seen that an isometry acting from a $d'_\text{min}$-dimensional (or higher) Hilbert space cannot correspond to any quantum channel with output purity strictly less than one: such isometry would have a $d'_\text{min}$-dimensional (or higher) completely entangled $d_1 \otimes d_2$ subspace as its range, which is not allowed by the Parthasarathy’s bound (23), since, obviously, $d'_\text{min} > d_1d_2 - (d_1 + d_2) + 1$.

The second case, where an isometry acts on a $d' \otimes d_2$ CES, can be considered similarly (with $d_2$ and $d_3$ interchanged)—it also gives the negative answer.

As a result, we can state that in the class of quantum channels with output purity less than one there are no channels that have certain combinations of input/output dimensions and the number of Kraus operators necessary for creating maximal GESs. Therefore, tripartite GESs of maximal possible dimensions are not generated with the described above approach.

In this regard it is useful to determine the highest dimension of a $d_1 \otimes d_2 \otimes d_3$ GES that can be achieved with the use of the present approach (as before, it is assumed that $2 \leq d_1 \leq d_2 \leq d_3$). Consider the case where a GES is obtained from a $d_1 \otimes m$ CES, where $m$ is some number. The isometry, acting on party with dimension $m$, maps $m$-dimensional Hilbert space to a completely entangled subspace of a $d_2 \otimes d_3$ Hilbert space. The highest value of $m$, denoted $m_{\text{max}}$, is hence equal to the highest possible dimension of a $d_2 \otimes d_3$ CES, which is $d_2d_3 - (d_2 + d_3) + 1$, according to equation (23). The maximal possible dimension of a $d_1 \otimes m_{\text{max}}$ CES, in
turn, equals
\[ d_1 m_{\text{max}} - (d_1 + m_{\text{max}}) + 1 = (d_1 - 1)(d_2 d_3 - d_2 - d_3), \tag{26} \]

as can be seen after substitution of \( m_{\text{max}} \) and simplifications.

The number in equation (26) hence gives the highest dimension of a \( d_1 \otimes d_2 \otimes d_3 \) GES which can be obtained from \( d_1 \otimes m \) CESs. Similar analysis, applied to the construction of a GES from \( d_2 \otimes k \) and \( d_3 \otimes l \) CESs (with isometries acting on parties with dimensions \( k \) and \( l \)), yields the highest dimensions of GESs, \( (d_2 - 1)(d_1 d_3 - d_1 - d_3) \) and \( (d_3 - 1)(d_1 d_2 - d_1 - d_2) \), respectively. Being the largest of these three numbers, the number in equation (26) gives the highest dimension of \( d_1 \otimes d_2 \otimes d_3 \) GESs achievable with the present approach:

\[ D_{\text{achieve}}^{d_1 \otimes d_2 \otimes d_3} = (d_1 - 1)(d_2 d_3 - d_2 - d_3). \tag{27} \]

Recall that the highest possible dimension of a \( d_1 \otimes d_2 \otimes d_3 \) GES is given by \( d_1 d_2 d_3 - (d_1 + d_2 d_3) + 1 \), which can also be written as

\[ D_{\text{max}}^{d_1 \otimes d_2 \otimes d_3} = (d_1 - 1)(d_2 d_3 - 1). \tag{28} \]

The ratio of these two quantities

\[ \frac{D_{\text{achieve}}^{d_1 \otimes d_2 \otimes d_3}}{D_{\text{max}}^{d_1 \otimes d_2 \otimes d_3}} = 1 - \frac{d_2 + d_3 - 1}{d_2 d_3 - 1} \tag{29} \]

does not depend on \( d_1 \) and approaches 1 with the increase of both dimensions \( d_2 \) and \( d_3 \). Therefore, when one is concerned with obtaining GESs of higher dimensions, the present approach becomes more effective on condition that \( d_2, d_3 \gg 1 \), in which case the dimensions are close (but never equal) to the maximal possible ones. Having said that, we assume that bipartite CESs of various (large) dimensions can be constructed effectively, see, for example, approaches in references [24, 26].

As an example, when one constructs \( k \otimes n \otimes n \) GESs with \( n \gg 1 \) and some \( k \), which may be fixed, the ratio in equation (29) becomes \( 1 - O(1/n) \).

### 3.2. Constructing GESs of maximal dimension

GESs of exactly highest possible dimensions can still be constructed with the use of quantum channels, albeit in a much less elegant way, namely, for each specific CES one has to pick a suitable channel to act on one of the two parties. This can be done by guessing the proper splitting operation, the method we resort to in building maximal GESs.

To be precise, when constructing a tripartite \( d_1 \otimes d_2 \otimes d_3 \) GES of maximal possible dimension, one can start with some CES of a tensor product \( d_1 \otimes d_2 d_3 \) Hilbert space. This subspace should have dimension \( d_1 d_2 d_3 - (d_1 + d_2 d_3) + 1 \), as it was discussed earlier. Next, one searches for an appropriate splitting operation to act on the subsystem \( B \) with dimension \( d_2 d_3 \) (see figure 4). The splitting operation should be chosen in such a way that the resulting tripartite states will be entangled across bipartitions \( B_1 \mid AB_2 \) and \( B_2 \mid AB_1 \) (the entanglement across \( A \mid B_1 B_2 \) is provided by the initial CES).

We emphasize that any CES of the fitting dimension, \( d_1 d_2 d_3 - (d_1 + d_2 d_3) + 1 \), can be used in the initial step of the described procedure. Reference [26], for example, provides an elegant way of constructing CESs, including those of maximal dimension, although in general it does not yield simple orthonormal basis states spanning the subspace being constructed. For convenience, we build our own examples of CESs and use them in the initial step of constructing GESs.
Our construction of CESs is based on the correspondence between entangled subspaces and quantum channels. At first we construct an isometry $V : H_E \to H_A \otimes H_B$, where $H_E$—a Hilbert space of dimension $d_1 d_2 d_3 - (d_1 + d_2 d_3) + 1$. If this isometry corresponds, via Stinespring’s dilation, to a quantum channel with output purity less than one, then its range will be a completely entangled subspace of $H_A \otimes H_B$ with maximal dimension. In order to find the isometry, we analyze the expression for the output purity of a quantum channel $\Phi : B(H_E) \to B(H_A)$ in terms of its Kraus operators. As it was mentioned above, it will suffice to consider the action of $\Phi$ only on pure states, and for an arbitrary pure state $|\psi\rangle \in H_E$ we have:

$$\text{Tr}(\Phi(|\psi\rangle\langle\psi|)^2) = \sum_{i,j} \text{Tr}\left(K_i|\psi\rangle\langle\psi|K_j^\dagger|\psi\rangle\langle\psi|K_j\right) = \sum_{i,j} \langle\psi|K_i^\dagger K_j|\psi\rangle^2 \leq \sum_{i,j} \langle\psi|K_i^\dagger K_j|\psi\rangle \langle\psi|K_j^\dagger K_i|\psi\rangle = 1,$$

(30)

where the Kraus decomposition (11) was used; the inequality follows from the Cauchy–Schwarz inequality; the last equality follows from equation (12) averaged over $|\psi\rangle$ and squared.

It is known that the Cauchy–Schwarz inequality for two vectors holds strictly, without equality if and only if they are linearly independent. Therefore, to guarantee that the trace in equation (30) is strictly less than one, we need to search for Kraus operators $\{K_i\}$ satisfying the following conditions:

(a) The eigenvalues of $K_i^\dagger K_i$, for all $i$, are strictly less than one (otherwise, the trace in equation (30) will evaluate to 1 on the corresponding eigenvector):

$$\|K_i^\dagger K_i\| < 1.$$

(31)

(b) For any $|\psi\rangle \in H_E$ there is at least one pair $(K_i, K_j)$ of distinct Kraus operators such that the vectors $K_i|\psi\rangle$ and $K_j|\psi\rangle$ are not proportional to each other.

Having found the proper Kraus operators $\{K_i\}$, we construct the corresponding isometry $V$ via equation (14).

The whole procedure of building a GES consists of two steps: constructing a CES and applying the splitting operation. These steps are shown on figure 5.

We illustrate the approach with several examples.
3.2.1. GESs in three-qubit systems. Now we consider a system of three qubits whose pure states are described by vectors in a $2 \otimes 2 \otimes 2$ Hilbert space. According to the described above procedure, we gather the first and the second subsystems into one and try to construct a completely entangled $4 \otimes 2$ subspace of maximal dimension $4 \times 2 - (4 + 2) + 1 = 3$. The corresponding isometry $V : H_E \rightarrow H_A \otimes H_B$, with $\dim(H_E) = 3$, $\dim(H_A) = 4$, $\dim(H_B) = 2$, can be expressed by equation (14) in terms of Kraus operators $K_i : H_E \rightarrow H_A$ of a quantum channel with output purity less than one.

There are two Kraus operators in this particular case, and, in search for $K_1, K_2$, we can consider the simplest situation when each $K_i^\dagger K_i$ is diagonal, with some numbers $\lambda_i^{(0)}, \lambda_i^{(1)}, \lambda_i^{(2)}$ on the main diagonal, such that

$$0 < \lambda_i^{(0)} < 1, \quad \lambda_i^{(1)} + \lambda_i^{(2)} = 1, \quad i = 1, 2; \quad j = 1, 2, 3.$$  \hspace{1cm} (32)

Conditions (12) and (31) are then satisfied. Such a choice suggests the following structure of $\{K_i\}$ in the form of singular value decomposition:

$$K_i = W_i \begin{pmatrix} \sqrt{\lambda_i^{(1)}} & 0 & 0 \\ 0 & \sqrt{\lambda_i^{(0)}} & 0 \\ 0 & 0 & \sqrt{\lambda_i^{(2)}} \end{pmatrix},$$ \hspace{1cm} (33)

where there is some freedom in the choice of unitary $4 \times 4$ matrices $W_i, i = 1, 2.$

Trying to keep everything as simple as possible, we can set $W_1 = I,$ and choose $W_2$ to be some permutation matrix, for example, the one corresponding to a permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$. 

Now let us choose an arbitrary vector state $|\phi\rangle \in H_E$ with components $(\phi_1, \phi_2, \phi_3)$ and, using equation (33) and our particular choice of $W_1, W_2$, write out the corresponding vectors $K_1|\phi\rangle$ and $K_2|\phi\rangle$ as columns of a matrix:

$$\begin{pmatrix} \sqrt{\lambda_1^{(1)}}\phi_1 \\ \sqrt{\lambda_2^{(1)}}\phi_2 \\ \sqrt{\lambda_3^{(1)}}\phi_3 \\ \sqrt{\lambda_3^{(2)}}\phi_1 \end{pmatrix}.$$

(34)

It is easily seen that these vectors are linearly independent for any nonzero $|\phi\rangle$: all $2 \times 2$ minors of the matrix evaluate to 0 simultaneously if and only if $\phi_1 = \phi_2 = \phi_3 = 0$. Therefore, we have shown that operators $K_1, K_2$ are the Kraus operators of a channel with output purity less than one. Reconstructing the associated isometry $V : H_E \rightarrow H_A \otimes H_B$ with the use of equation (14) and acting with it on an orthonormal basis in $H_E$, we obtain an orthonormal system of vectors spanning a completely entangled subspace of $H_A \otimes H_B$:

$$\sqrt{\lambda_1^{(1)}}|0\rangle \otimes |0\rangle + \sqrt{\lambda_2^{(2)}}|3\rangle \otimes |1\rangle,$$

(35a)
Figure 6. Construction of a genuinely entangled $2 \otimes 2 \otimes 2$ subspace. A properly chosen unitary $U$ creates entanglement across bipartitions $A_1 |A_2 B$ and $A_2 |A_1 B$ while preserving entanglement across $A_1 A_2 |B$. Line splitting denotes operation (36).

Next, we try to convert this subspace into a genuinely entangled tripartite one by choosing a proper way of splitting the first subsystem $A$ of dimension 4 into subsystems $A_1$ and $A_2$, each with dimension 2. A naive approach would be to choose the trivial splitting:

$$
|0\rangle \rightarrow |0\rangle \otimes |0\rangle,
|1\rangle \rightarrow |0\rangle \otimes |1\rangle,
|2\rangle \rightarrow |1\rangle \otimes |0\rangle,
|3\rangle \rightarrow |1\rangle \otimes |1\rangle, \quad (36)
$$

but, obviously, under this operation, the spanning vector (35b) itself turns into a separable one. Any permutation of scheme (36) does not help either.

Our second attempt is to act on subsystem $A$ with some simple unitary transformation $U_A$ and then split $A$ according to scheme (36). Combination of these steps gives another splitting operation. $U_A$ is a local transformation with respect to bipartition $A_1 A_2 |B$, and hence it preserves entanglement here. On the other hand, $U_A$ can change the situation across bipartitions $A_1 |A_2 B$ and $A_2 |A_1 B$ and create entanglement there (see also figure 6). We choose this transformation to be

$$
U_A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & \frac{\sqrt{2}}{1} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}, \quad (37)
$$

thus mixing $|1\rangle_A$ and $|2\rangle_A$.

Combining unitary transformation (37) and splitting (36), from the basis (35) we obtain an orthonormal system of vectors in $H_{A_1} \otimes H_{A_2} \otimes H_B$:

$$
|\psi_1\rangle = \sqrt{\lambda_1}|0\rangle \otimes |0\rangle \otimes |0\rangle + \sqrt{1 - \lambda_1}|1\rangle \otimes |1\rangle \otimes |1\rangle, \quad (38a)
$$
\[ |\psi_2\rangle = \frac{\lambda_2}{2} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \otimes |0\rangle + \sqrt{1 - \lambda_2^2} (|0\rangle \otimes |0\rangle \otimes |1\rangle), \tag{38b} \]

\[ |\psi_3\rangle = \frac{\lambda_3}{2} (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle) \otimes |0\rangle + \sqrt{1 - \lambda_3^2} (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle) \otimes |1\rangle), \tag{38c} \]

where equation (32) is taken into account and the redundant superscripts of lambdas are now omitted.

We already know that the subspace spanned by vectors (38) is completely entangled across bipartition \( A_1 | A_2 B \). The entanglement across other bipartitions, for example, \( A_1 | A_2 B \), can be certified in a standard way—vectors (38) are decomposed accordingly

\[ |\psi_\mu\rangle = \sum_{i,j,k} a_{ijk}^{(\mu)} |i\rangle_{A_1} \otimes |jk\rangle_{A_2 B}, \quad \mu = 1, 2, 3, \tag{39} \]

and it is analyzed at what complex values \( \beta_\mu \) the linear combination of matrices

\[ \sum_{\mu=1}^{3} \beta_\mu a^{(\mu)} \]

has rank less than two, i.e. all minors of order two evaluate to zero. If there is only trivial solution, \( \beta_\mu = 0 \forall \mu \), then the subspace is entangled across the cut. This can be done by hand, or, even better, with a computer algebra system supporting the Groebner basis algorithm [46].

Using the described procedure, it can be verified that the subspace determined by equation (38) is completely entangled across bipartitions \( A_1 | A_2 B \) and \( A_2 | A_1 B \) and, therefore, is genuinely entangled.

### 3.2.2. GESs in four-qubit systems

Pure states of a four-qubit system are described by vectors in a \( 2 \otimes 2 \otimes 2 \otimes 2 \) Hilbert space. Let \( A_1, A_2, B_1, B_2 \) denote the four subsystems. Joining three subsystems, say \( A_2, B_1, B_2 \), into one, we consider the bipartition \( A_1 | A_2 B_1 B_2 \) and vector states in a \( 2 \otimes 8 \) Hilbert space, where the maximal dimension of CES is equal to \( 2 \times 8 - (2 + 8) + 1 = 7 \).

Bipartitions of another type, such as \( A_1 A_2 | B_1 B_2 \), yield the maximal dimension of CES equal to \( 4 \times 4 - (4 + 4) + 1 = 9 \). Therefore, the maximal dimension of GES in a \( 2 \otimes 2 \otimes 2 \otimes 2 \) Hilbert space is upper-bounded by 7.

To construct such GES, we first construct a seven-dimensional CES in a \( 4 \otimes 4 \) Hilbert space, then split each four-dimensional subsystem into two two-dimensional ones.

To obtain the CES, we search for the isometry \( V : H_E \rightarrow H_A \otimes H_B \), with the input dimension \( \dim(H_E) = 7 \) and the output dimensions \( \dim(H_A) = 4, \dim(H_B) = 4 \). In addition, the range of \( V \) must be completely entangled.

The subsystems \( A \) and \( B \) are then split into \( A_1, A_2, B_1, B_2 \), respectively, and each splitting operation is performed with some unitary transformations \( U_1 \) and \( U_2 \) applied to the parties \( A \) and \( B \). \( U_1 \) and \( U_2 \) are aimed at creating entanglement across all bipartitions except \( A_1 A_2 | B_1 B_2 \) where entanglement is guaranteed by the structure of the isometry \( V \). The whole procedure is presented on figure 7.

The seven-dimensional CES of a \( 4 \otimes 4 \) Hilbert space can be obtained by the approach used in the previous examples. By our construction (see appendix B for the details), it is spanned

[Refer to figure 7 for visual representation of the procedure.]
by the orthonormal system.

\[
\begin{align*}
\sqrt{\lambda_1} |0\rangle \otimes |0\rangle + \sqrt{1-\lambda_1} |2\rangle \otimes |1\rangle, \\
\sqrt{\lambda_2} |1\rangle \otimes |2\rangle + \sqrt{1-\lambda_2} |3\rangle \otimes |3\rangle, \\
\sqrt{\lambda_3} |0\rangle \otimes |1\rangle + \sqrt{1-\lambda_3} |3\rangle \otimes |2\rangle, \\
\sqrt{\lambda_4} |3\rangle \otimes |0\rangle + \sqrt{1-\lambda_4} |2\rangle \otimes |2\rangle, \\
\sqrt{\lambda_5} |1\rangle \otimes |1\rangle + \sqrt{1-\lambda_5} |0\rangle \otimes |3\rangle, \\
\sqrt{\lambda_6} |1\rangle \otimes |0\rangle + \sqrt{1-\lambda_6} |2\rangle \otimes |3\rangle, \\
\sqrt{\lambda_7} |2\rangle \otimes |0\rangle + \sqrt{1-\lambda_7} |3\rangle \otimes |1\rangle,
\end{align*}
\]

where \(0 < \lambda_i < 1, i = 1, \ldots, 7\).

The trivial splitting (36) of each of the two subsystems inevitably creates separable vectors from those in equation (41), and so we need some entangling unitaries \(U_1\) and \(U_2\) to apply to \(A\) and \(B\), respectively. There are seven possible bipartitions of a four-partite system, and, choosing \(U_1\) and \(U_2\), we need to check entanglement in 6 of them. Entanglement across bipartition \(A_1A_2|B_1B_2\) is provided by the structure of the isometry \(V\). Of course, such verification can be most effectively done with the use of computer algebra and the Groebner basis algorithm.

Due to the computational complexity of the problem, we were able to calculate only with concrete values for \(\lambda_i\) from equation (41). At first we set \(\lambda_i = 1/2, i = 1, \ldots, 7\). In addition, unlike the example with three qubits, we could not come up with simple unitaries \(U_1\) and \(U_2\) mixing only 2–3 basis states. From reference [47] we took several standard unitaries, three- and four-dimensional, and on the basis of them constructed two four-dimensional unitaries for mixing basis states:

\[
Q = \frac{1}{2} \begin{pmatrix}
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
1 & 1 & -1 & 1 \\
1 & 1 & 1 & -1
\end{pmatrix},
\]

\[
T = \begin{pmatrix}
-1/3 & 2/3 & 0 & 2/3 \\
2/3 & -1/3 & 0 & 2/3 \\
0 & 0 & 1 & 0 \\
2/3 & 2/3 & 0 & -1/3
\end{pmatrix}.
\]
Setting then
\[ U_1 = QT, \quad U_2 = T \] (44)
did the job.

Combining these transformations with splitting (36), we obtain two new splitting operations. For subsystem \( A \) the operation is defined by
\begin{align*}
|0\rangle_A & \rightarrow \frac{1}{6} \left(5|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + 3|1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\right), \\
|1\rangle_A & \rightarrow \frac{1}{6} \left(-|0\rangle \otimes |0\rangle + 5|0\rangle \otimes |1\rangle + 3|1\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle\right), \\
|2\rangle_A & \rightarrow \frac{1}{2} \left(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle\right), \\
|3\rangle_A & \rightarrow \frac{1}{6} \left(-|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + 3|1\rangle \otimes |0\rangle + 5|1\rangle \otimes |1\rangle\right),
\end{align*}
(45)
for subsystem \( B \) it is given by
\begin{align*}
|0\rangle_B & \rightarrow \frac{1}{3} \left(-|0\rangle \otimes |0\rangle + 2|0\rangle \otimes |1\rangle + 2|1\rangle \otimes |1\rangle\right), \\
|1\rangle_B & \rightarrow \frac{1}{3} \left(2|0\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle + 2|1\rangle \otimes |1\rangle\right), \\
|2\rangle_B & \rightarrow |1\rangle \otimes |0\rangle, \\
|3\rangle_B & \rightarrow \frac{1}{3} \left(2|0\rangle \otimes |0\rangle + 2|0\rangle \otimes |1\rangle - |1\rangle \otimes |1\rangle\right).
\end{align*}
(46)

Now, to obtain a four-qubit GES, one should apply substitutions (45) and (46) in equation (41) for the basis states of the first and the second subsystems, respectively, and set all \( \lambda_i \) to 1/2.

Actually, we have done calculations for several different sets of values of \( \lambda_i \) and each time obtained a GES. We can conjecture that equation (41) together with equations (45) and (46) determine a GES for any set of values of \( \lambda_i \).

3.2.3. GESs in three-qutrit systems. Now we construct an example of GES of maximal dimension in a \( 3 \otimes 3 \otimes 3 \) Hilbert space. The bound on the dimension in this case is the same as the bound on the dimension of a \( 9 \otimes 3 \) CES: \( 9 \times 3 - (9 + 3) + 1 = 16 \). Consequently, we need to construct an isometry \( V : H_E \rightarrow H_A \otimes H_B \), with \( \dim(H_E) = 16, \dim(H_A) = 9, \dim(H_B) = 3 \), and with a completely entangled range.

Our first step is to construct the Kraus operators \( K_i : H_E \rightarrow H_A, i = 1, 2, 3 \), which in this case are represented by matrices with dimensions \( 9 \times 16 \). As in the previous examples, we first consider matrices corresponding to \( K_i^\dagger K_i \), which should satisfy equations (12) and (31). Any attempt to make them all be purely diagonal immediately fails: we can write out the diagonals of \( K_1^\dagger K_1, K_2^\dagger K_2, K_3^\dagger K_3 \) as columns of a matrix and, to satisfy equations (12) and (31), in each row of this matrix there must be at least two nonzero values adding up to 1. Therefore, the total number of these values in all rows must be not less than 16 \( \times 2 = 32 \). On the other hand, being the eigenvalues of \( K_i^\dagger K_i \), these numbers are the squares of singular values of \( K_i \). There can be at most nine nonzero singular values for each \( K_i \) because it is represented by a \( 9 \times 16 \) matrix. In total, for all three \( K_i \) there can be at most \( 3 \times 9 = 27 \) nonzero singular values, so we have a contradiction which shows that \( K_i^\dagger K_i \) cannot be purely diagonal.
To come around the described problem, we introduce a block-diagonal form for $K_i^\dagger K_i$. For each matrix representing $K_i^\dagger K_i$ we consider the area consisting of three diagonals: the main diagonal and the two adjacent ones below and above. This area is divided into two parts down the diagonals: the first part entirely consists of $2 \times 2$ blocks and the second one is purely diagonal—it has nonzero values only on the main diagonal. The $2 \times 2$ blocks placed on the same positions in $K_1^\dagger K_1$, $K_2^\dagger K_2$, $K_3^\dagger K_3$ should add up to a $2 \times 2$ identity matrix to satisfy equation (12). To satisfy also equation (31), they should have eigenvalues strictly less than 1. As we have shown above, for each $K_i^\dagger K_i$ we only have nine eigenvalues at our disposal to cover the diagonals, so we need to spare them. This can be achieved by choosing all $2 \times 2$ blocks to have only one nonzero eigenvalue. In this case each such block will cover two positions on the diagonal, but it will take only one eigenvalue. Having placed five such blocks down the diagonals in each $K_i^\dagger K_i$, we will have $9 - 5 = 4$ eigenvalues left at our disposal. We can then put these eigenvalues down the main diagonal, thus covering the remaining positions. If we put the described three-diagonal areas of $K_1^\dagger K_1$, $K_2^\dagger K_2$, $K_3^\dagger K_3$ in front of each other in the form of columns, the picture will look like this:

\[
\begin{align*}
K_1^\dagger K_1: & \quad \begin{bmatrix} P_1^{(1)} & P_2^{(1)} & P_3^{(1)} \\ P_4^{(2)} & P_5^{(2)} & P_6^{(2)} \\ P_7^{(3)} & P_8^{(3)} & P_9^{(3)} \end{bmatrix} \\
K_2^\dagger K_2: & \quad \begin{bmatrix} \lambda_1^{(1)} & 0 & \lambda_1^{(1)} \\ \lambda_2^{(2)} & \lambda_2^{(2)} & 0 \\ 0 & \lambda_3^{(3)} & \lambda_3^{(3)} \end{bmatrix} \\
K_3^\dagger K_3: & \quad \begin{bmatrix} \lambda_4^{(1)} & \lambda_4^{(2)} & 0 \\ 0 & \lambda_5^{(3)} & \lambda_5^{(3)} \\ 0 & 0 & \lambda_6^{(3)} \end{bmatrix}
\end{align*}
\] (47)

where lambda’s in each row are positive and add up to 1, and $P_i^{(j)}$ — $2 \times 2$ matrices having only one nonzero (positive) eigenvalue and adding up to a $2 \times 2$ identity matrix:

\[P_i^{(1)} + P_i^{(2)} + P_i^{(3)} = I, \quad i = 1, \ldots, 5.\] (48)

Any other arrangement of lambdas is acceptable as long as there are at least two of them adding up to 1 in each row. Here we have placed them in some checkerboard style.

The number of $2 \times 2$ blocks used in each $K_i^\dagger K_i$ is optimal: if we had used four blocks instead, then we would have had $9 - 4 = 5$ free lambdas to cover $16 - 4 \times 2 = 8$ positions on the diagonal, which is impossible even with the use of such a sparse checkerboard disposition.

The operators satisfying equation (48) for each fixed $i$ constitute a POVM in a two-dimensional Hilbert space. One particular choice of $P^{(k)}$ [2] can be defined by

\[
P^{(1)} = A |u\rangle \langle u|, \quad P^{(2)} = A |v\rangle \langle v|, \quad P^{(3)} = I - A (|u\rangle \langle u| + |v\rangle \langle v|),
\] (49)

where $A$ — a constant and $|u\rangle$, $|v\rangle$ — vectors parameterized by

\[
|u\rangle = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix}, \quad |v\rangle = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix}, \quad 0 < \alpha < \pi/4.
\] (50)
It can be easily verified that when
\[ A = \frac{1}{1 + \sin 2\alpha}, \]  
(51)
the operator \( P^{(3)} \) has only one nonzero eigenvalue, and thus the identity operator \( I \) is decomposed into the sum of three rank one operators \( P^{(k)} \).

By construction (47), there are five triples of operators \( \left( P^{(1)}_i, P^{(2)}_i, P^{(3)}_i \right) \), \( i = 1, \ldots, 5 \), and each is defined by its own parameter \( \alpha_i \) according to equations (49)–(51). From now on we will denote such triples as \( \left( P^{(1)}_i, P^{(2)}_i, P^{(3)}_i \right), i = 1, \ldots, 5 \), thus stressing their dependence on \( \alpha_i \).

Having found proper \( K^\dagger_i K_i \) with the use of equations (47)–(51), we can construct operators \( K_i \) themselves taking advantage of the singular value decomposition:
\[ K_i = W_i \Sigma_i V_i, \quad i = 1, 2, 3, \]  
(52)
where \( W_i \) is a \( 9 \times 9 \) unitary; \( \Sigma_i \) is a \( 9 \times 16 \) matrix consisting of \( \sqrt{\lambda_{ij}} \) and square roots of the eigenvalues of the operators \( P^{(i)}_j \); \( V_i \) is a \( 16 \times 16 \) unitary which depends on \( \alpha_j, j = 1, \ldots, 5 \) (see appendix C for details).

There is freedom in choosing the unitaries \( W_i \), which we set to some permutation matrices.

When we set particular values for \( \lambda_{ij} \) and \( \alpha_j \), the procedure of construction of a \( 9 \otimes 3 \) CES reduces to the search of appropriate permutations \( W_i \). According to the general approach, we choose an arbitrary vector state \( |\phi\rangle \in H_E \) with components \( \phi_j, j = 1, \ldots, 16 \), and write the corresponding vectors \( K_i |\phi\rangle, i = 1, 2, 3, \) as columns of a matrix. If there is only trivial solution \{\phi_j\}, for which all \( 2 \times 2 \) minors of the matrix evaluate to zero, then at least two of these vectors are linearly independent, and we have a CES. We use a computer algebra system and the Groebner basis algorithm for such an analysis. When Groebner basis reduces to \{1\}, there is only trivial solution. If it is not the case, we look at the basis, identify the minors which cause problem and adjust the permutations \( W_i \) accordingly (usually it takes one or two elementary transpositions). After that we compute the Groebner basis again and, if necessary, repeat the procedure until we obtain a CES.

We were not able to calculate in abstract variables here due to high dimensionality of the problem and the complexity of the polynomial system for minors, so we made a particular choice of \( \lambda_{ij} \) and \( \alpha_j \) for scheme (47) (see appendix C, equation (C8)). For such initial setting we came up with the proper permutations \( W_i \) (see appendix C, equation (C9)) in several iterations of the described above procedure. Calculating \( K_i \) from equation (52) and constructing the corresponding isometry with the use of equation (14), we obtained an orthonormal system of vectors spanning a \( 9 \otimes 3 \) CES (see appendix C, equation (C10)). Next, we split the nine-dimensional subsystem into two three-dimensional ones according to some particular scheme (see appendix C, equation (C11)). Finally, after the analysis of the two new bipartitions, we introduced a simple unitary transformation acting on the two subsystems and creating entanglement (see appendix C, equation (C12)). In this way we obtained a \( 3 \otimes 3 \otimes 3 \) GES.

4. Entanglement of GES and the associated bounds on entanglement measures

4.1. Controlling genuine entanglement of constructed subspaces

Let us return to the universal approach to constructing GESs described in section 3.1. Such subspaces are obtained from CESs of a bipartite Hilbert space by the action of an isometry
corresponding to a channel with output purity less than one. The entanglement of the GES can be estimated if the characteristics of both the initial CES and the channel corresponding to the isometry are known. Across one bipartition it is determined by the entanglement of the initial CES (see equation (18) for bipartition $A|CD$), across the other two—by the output norm of the quantum channel (see equation (19) for bipartition $C|AD$). The maximal output norm of the channel will be an upper bound on the norms of the reduced density operators of the states in the GES across these two bipartitions because the supremum in equation (15) is achieved on pure states.

As an illustration, we take the initial CES to be the antisymmetric subspace $W_-$ of a $3 \otimes 3$ Hilbert space, spanned by the three vectors

$$|\psi_{ij}\rangle = (|i\rangle \otimes |j\rangle - |j\rangle \otimes |i\rangle)/\sqrt{2}, \quad i, j = 0, \ldots, 2, \quad i < j.$$  \hspace{1cm} (53)

It is known [22, 48] that the maximal first Schmidt coefficient over all states in this subspace is given by

$$\sup_{\psi_{AB} \in W_-} \|\text{Tr}_B |\psi_{AB}\rangle \langle \psi_{AB}|\|^{1/2} = 1/\sqrt{2}.$$  \hspace{1cm} (54)

Consider the Holevo–Werner channel [49] acting on $d \times d$ density matrices as

$$\Phi(\rho) = \frac{1}{d-1} (I - \rho^T) = \sum_{i < j} K_{ij} \rho K_{ij}^\dagger,$$  \hspace{1cm} (55)

where

$$K_{ij} = \frac{1}{\sqrt{d-1}} (|i\rangle \langle j| - |j\rangle \langle i|), \quad i, j = 0, \ldots, d - 1, \quad i < j,$$  \hspace{1cm} (56)

and $\rho^T$ denotes the matrix transpose with respect to the computational basis.

The maximal output norm of the Holevo–Werner channel can be easily obtained from the first representation in equation (55) and is given by

$$\nu_p(\Phi) = (d - 1)^{-1 - 1/p}.$$  \hspace{1cm} (57)

In our case, $d = 3$, the channel can be represented by the three Kraus operators given by equation (56). The associated isometry $V$, obtained from equation (14), acts on the computational basis as

$$V|0\rangle = \frac{1}{\sqrt{2}} \left( -|1\rangle \otimes |0\rangle - |2\rangle \otimes |1\rangle \right),$$

$$V|1\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |0\rangle - |2\rangle \otimes |2\rangle \right),$$

$$V|2\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle \otimes |1\rangle + |1\rangle \otimes |2\rangle \right).$$  \hspace{1cm} (58)

Acting with this isometry on the second qutrit of the vectors in equation (53), we obtain an orthonormal system spanning a GES of a $3 \otimes 3 \otimes 3$ Hilbert space:

$$|\phi_{ij}\rangle = \frac{1}{2} \left( |0\rangle \otimes |0\rangle \otimes |0\rangle - |0\rangle \otimes |2\rangle \otimes |2\rangle + |1\rangle \otimes |1\rangle \otimes |0\rangle + |1\rangle \otimes |2\rangle \otimes |1\rangle \right).$$  \hspace{1cm} (59a)
\[ |\phi_2\rangle = \frac{1}{2} \left( |0\rangle \otimes |0\rangle \otimes |1\rangle + |0\rangle \otimes |1\rangle \otimes |2\rangle + |1\rangle \otimes |0\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle \otimes |1\rangle \right), \]  

(59b)

\[ |\phi_3\rangle = \frac{1}{2} \left( |1\rangle \otimes |0\rangle \otimes |1\rangle + |1\rangle \otimes |1\rangle \otimes |2\rangle - |2\rangle \otimes |0\rangle \otimes |2\rangle + |2\rangle \otimes |2\rangle \otimes |2\rangle \right). \]  

(59c)

Let \( S \) denote this subspace. Consider the geometric measure \( G_{\text{GME}} \) of genuine entanglement of \( S \). The three qutrits are referred to in the same way as on figure 2: \( A, C \) and \( D \). The maximal first Schmidt coefficient over all states in \( S \) across bipartition \( A|CD \) is determined by that of the initial antisymmetric bipartite subspace given by equation (54). The maximal first Schmidt coefficient across bipartitions \( C|AD \) and \( D|AC \) will be upper bounded by the output norm \( \nu_{\infty}(\Phi)^{1/2} \) of the Holevo–Werner channel, which, by equation (57), also equals to \( 1/\sqrt{2} \). Combining the results across all bipartitions, with the use of equations (7) and (8) we obtain the lower bound on the geometric measure of \( S \):

\[ G_{\text{GME}}(S) \geq 1/2. \]  

(60)

4.2. Lower bounds on GME concurrence and negativity of mixed states

Obtained in reference [22] lower bounds on the concurrence and the CREN of an arbitrary bipartite mixed state \( \rho \) are determined by its overlap with some completely entangled subspace \( W \) of a \( d \times d \) Hilbert space. For the concurrence the bound reads

\[ C(\rho) \geq \max \left( \sqrt{\frac{2}{d(d-1)} \frac{\text{Tr} \{ \rho \Pi_W \} - \lambda_1^W}{\lambda_1^W}}, 0 \right), \]  

(61)

where \( \Pi_W \)—an orthogonal projector onto \( W \), \( \lambda_1^W \) —the supremum of the largest Schmidt coefficient squared taken over all vector states in the subspace \( W \). The bound on the negativity is given by

\[ N_{\text{CREN}}(\rho) \geq \max \left( \frac{\text{Tr} \{ \rho \Pi_W \} - \lambda_1^W}{2\lambda_1^W}, 0 \right). \]  

(62)

The underlying separability criterion was derived earlier in reference [21] with the use of the theory of entanglement witnesses. (See also reference [12] for the generalization to the GME case.)

With the use of equations (7)–(9) these inequalities can be straightforwardly extended to the GME case:

\[ C_{\text{GME}}(\rho) \geq \sqrt{\frac{2}{d(d-1)} \frac{1}{1 - G_{\text{GME}}(W)}} \max(\text{Tr} \{ \rho \Pi_W \} + G_{\text{GME}}(W) - 1, 0), \]  

(63)

\[ N_{\text{GME}}(\rho) \geq \frac{1}{2(1 - G_{\text{GME}}(W))} \max(\text{Tr} \{ \rho \Pi_W \} + G_{\text{GME}}(W) - 1, 0), \]  

(64)

where now \( W \) is a GES. Therefore, the two GME measures can be estimated for a mixed state \( \rho \) if the value \( G_{\text{GME}}(W) \) or some good lower bound on it can be determined.

As an example, with the GES \( S \) from equation (59), for a mixed state \( \rho \) the lower bound on the negativity is given by

\[ N_{\text{GME}}(\rho) \geq \max(\text{Tr} \{ \rho \Pi_S \} - 1/2, 0). \]  

(65)
where equations (60) and (64) were used.

Equations (63) and (64) (or the underlying separability criterion) are also convenient for estimating the robustness of GME under mixing with noise. Consider a state $\rho$ with support in some GES $W$ being mixed with noise described by a density operator $N$. After mixing the state is given by the convex sum

$$
(1 - p)\rho + pN, \quad p \in [0; 1].
$$

By equations (63) and (64), this state will still be genuinely entangled if the lower bound on the entanglement measures is positive, i.e.

$$
\text{Tr}\{(1 - p)\rho \Pi_W + pN\Pi_W\} + G_{\text{GME}}(W) > 1.
$$

Taking into account that $\text{Tr}\{\rho \Pi_W\} = 1$ ($\rho$ has support in $W$), we obtain the upper bound

$$
p < \frac{G_{\text{GME}}(W)}{1 - \text{Tr}\{N\Pi_W\}}
$$

saying that under such proportions of the noise $N$ the state in equation (66) is guaranteed to be genuinely entangled.

Let $H$ denote the Hilbert space of the described states. If only the white noise is considered,

$$
N = \frac{1}{\dim(H)}I,
$$

then the bound on $p$ reads

$$
p < \frac{G_{\text{GME}}(W)}{1 - \dim(W)/\dim(H)}.
$$

The same estimate was obtained earlier in reference [15] (see equation (71)).

As an example, for the subspace $S$ given by equation (59) the robustness of entanglement under the white noise can be estimated with the use of equation (60). It is given by $p < 9/16 = 0.5625$.

Following reference [22], it is also interesting to analyze the case when the expression for $N$ is unknown and only the spectrum of the noise is given (robustness from spectrum). The immediate generalization of equation (55) in reference [22] to the GME case is given by

$$
p < \frac{G_{\text{GME}}(W)/\|N\|_{(\text{codim}(W))^*}}{1 - \text{dim}(W)/\dim(H)}.
$$

where $\text{codim}(W) = \dim(H) - \dim(W)$ and $\|N\|_{(k)}$ is the sum of $k$ largest eigenvalues of $N$ (the $k$th Ky Fan norm [50, 51]).

Equation (71) may be useful in analysis of quantum states under the influence of random noise where some results on eigenvalues of random matrices can be applied.

5. Discussions

We have presented an approach to constructing GESs based on the composition of entangled subspaces and quantum channels. In one of its variants tripartite GESs were obtained from bipartite CESs by applying isometries corresponding to quantum channels with certain properties. We have discussed that channels with the maximal output norm strictly less than 1 can generate GESs from any CESs with fitting dimensions of one of the parties. On the one hand,
this method does not yield subspaces of maximal possible dimensions. On the other hand, the
construction is pretty simple and allows us to control the measures of entanglement of the
subspaces on condition that we have control over the initial CESs and the output characteris-
tics of the channel being used. Highly entangled GESs can be generated in this manner. Quite
possibly, other characteristics, such as distillability of entanglement across every bipartition,
can be controlled in a similar way. In principle, the whole procedure can be applied to obtain
$n + 1$-partite GESs from $n$-partite ones. It is also interesting to ask whether this method can be
realized in experimental setting: the isometry, for example, can be extended in some way to a
unitary which may correspond to some physical operation. These questions will be addressed
in the forthcoming work.

As a further possible development of the proposed approach, it is interesting to note that, in
addition to the channels with output purity less than one, there are other options in choosing
an isometry acting on one of the parties of each state in CES. Consider a contractive map. Let
$M_n$ denote the space of all $n \times n$ matrices. A linear map $T : M_n \rightarrow M_m$ can be characterized
by the operator norm
$$\|T\|_{p \rightarrow p} = \sup_{A \in M_n} \frac{\|T(A)\|_p}{\|A\|_p},$$
where it is supposed that both the initial and the final matrix spaces are endowed with the $p$-
norm. $T$ is called contractive under the $p$-norm if $\|T(A)\|_p \leq\|A\|_p$ for all $A \in M_n$, which is crucial for our construction. Now let us suppose
that in the procedure of constructing a GES a quantum channel $\Phi$ is contractive under the
Frobenius norm: $\|\Phi\|_2 \leq 1$. Returning to the rightmost part of equation (19), we have:
$$\left\|\Phi \left(\sum_i p_i |i'\rangle\langle i'|_B\right)\right\|_2 \leq \left\|\sum_i p_i |i'\rangle\langle i'|_B\right\|_2 < 1,$$
where the first inequality is due to contractivity of $\Phi$, the second inequality holds because the
input of the channel is a mixed state. Hence, the channel maps mixed states to mixed states
and creates entanglement across the bipartition $C|AD$ (see figure 2 and the reasoning on page
4). As to the bipartition $D|AC$, there is a subtlety: now we trace out subsystem $D$ instead of $C$, and, as a result, from the isometry $V$ we obtain a channel $\tilde{\Phi}$ which is complementary [52] to
$\Phi$. In general, a complementary channel to a contractive channel need not be contractive, and
so there is no guarantee that $\tilde{\Phi}$ maps mixed states to mixed states and creates entanglement
across $D|AC$. Hence, for generating GES our requirement is that both $\Phi$ and its complementary
channel should be contractive. Such channels do exist: the simplest example is the completely
dehasing channel
$$N : \rho \rightarrow \sum_i |i\rangle\langle i|\rho|i\rangle\langle i|,$$
where $\{|i\rangle\}$—some orthonormal basis. The channel is complementary to itself and, in addition,
unital: $N(I) = I$. It was shown in reference [53] that unital channels are contractive under the
$p$-norm for every $1 \leq p \leq \infty$. Hence, the channel $N$ can serve for constructing GESs from
any CESs (with appropriate local dimensions). Note that the output purity of $N$ is equal to 1:
it is achieved on states $\{|i\rangle\}$.

It would be useful to find further examples of contractive channels having contractive
complementaries, especially the ones with not equal input and output dimensions.
Unfortunately, channels of the mentioned above type cannot serve for creation of maximal GESs either. Indeed, let us return to the situation where a maximal \( d_1 \otimes d_2 \otimes d_3 \) GES could potentially be generated by application of some isometry to a \( d' \otimes d_3 \) CES with dimension \( d_1 d_2 d_3 - (d_1 + d_2 d_3) + 1 \) (the other two cases, where \( d'' \otimes d_2 \) and \( d''' \otimes d_1 \) CESs are used for generating a \( d_1 \otimes d_2 \otimes d_3 \) GES, are similar). The minimal possible value of \( d', d'' \) is given by equation (25), in which case the isometry acts from a \( d'_\text{min} \)-dimensional Hilbert space to a \( d_1 \otimes d_2 \) tensor product Hilbert space, and the corresponding channel is \( \Phi : \mathcal{M}_{d'_\text{min}} \to \mathcal{M}_{d_1} \), with its complementary \( \tilde{\Phi} : \mathcal{M}_{d_2} \to \mathcal{M}_{d_3} \). As seen from equation (25), \( d'_\text{min} > \max(d_1, d_2) \). The following argument shows that a channel \( T : \mathcal{M}_n \to \mathcal{M}_r \), when \( n > r \), cannot be contractive. Consider \( T \) acting on a maximally mixed state \( \frac{1}{n} I_n \), resulting in some operator \( T(\frac{1}{n} I_n) \) in \( \mathcal{M}_r \). For its \( p \)-norm one has:

\[
\left\| T \left( \frac{1}{n} I_n \right) \right\|_p \geq \left\| \frac{1}{r} I_r \right\|_p > \left\| \frac{1}{n} I_n \right\|_p,
\]

for any \( 1 < p \leq \infty \), where the first inequality is due to Schur-convexity of the \( p \)-norm [50] and the fact that \( \frac{1}{r} I_r \) is majorized by any other density matrix in \( \mathcal{M}_r \); the second inequality follows from the definition of the \( p \)-norm combined with \( n > r \). Since the \( p \)-norm of the final state is greater than the \( p \)-norm of the initial state, the channel cannot be contractive.

It would be important to investigate whether it is in principle possible to find some class of channels which can generate maximal GESs from CESs, irrespective of what particular CES (of certain local and global dimensions) is being used. As we have seen, the described above two classes of channels cannot play this role, simply because there are no their representatives with certain combinations of input and output dimensions. In principle, a hypothetical channel with the needed input/output dimensions, which, together with its complementary channel, maps mixed states to mixed states, would do the job. Whether such channels exist or not is yet to be discovered.

For constructing maximal GESs we have used another, less elegant variant where the proper isometry (the splitting operation) was guessed for each particular CES. The method gets computationally hard with the increase of the number of parties and the dimensionalities of their systems. We were able to come up with families of maximal GESs for three-qubit systems and, to some extent, for four-qubit and three-qutrit systems where we can only conjecture that we are dealing with GESs in a wide range of the values of the parameters (we checked that they are GESs for various sets of the parameters).

One may benefit from these parameterizations in the analysis of various properties of entangled states and subspaces. As an example, starting from a maximal GES for a system of three qubits given by equation (38), one can easily find another maximal GES orthogonal to the first one:

\[
|\psi_1\rangle = \sqrt{1 - \lambda_1} |0\rangle \otimes |0\rangle \otimes |0\rangle - \sqrt{\lambda_1} |1\rangle \otimes |1\rangle \otimes |1\rangle,
\]

\[
|\psi_2\rangle = \frac{1 - \lambda_2}{2} \left( |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \right) \otimes |0\rangle - \sqrt{\lambda_2} |0\rangle \otimes |0\rangle \otimes |1\rangle,
\]

\[
|\psi_3\rangle = \frac{1 - \lambda_3}{2} \left( |0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle \right) \otimes |0\rangle - \frac{\sqrt{\lambda_3}}{2} \left( |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle \right) \otimes |1\rangle.
\]

These two GESs span in total six dimensions out of eight in a tripartite Hilbert space (but note that their direct sum cannot be genuinely entangled). Their orthogonal complement, a two-dimensional subspace, contains separable vectors, for example, \( |1\rangle \otimes |1\rangle \otimes |0\rangle \). Given an
arbitrary mixed state of three qubits, one can optimize over the parameters $\lambda_i$ and find the GES of one of the two types having maximal overlap with the state. If the state is genuinely entangled, it can be expected that the overlap will be significant. With the geometric measure of the optimal GES obtained, for example, numerically, one could then use equations (63) and (64) to detect genuine entanglement of the state or to get estimates for its entanglement measures. The same can be said in relation to the four-qubit subspaces determined by equations (41), (45) and (46), in which case the two orthogonal GESs will span together 14 dimensions out of 16 possible. The numerical analysis of the effectiveness of such an approach will be conducted elsewhere.

For each of the constructed maximal GESs it would be interesting to find which of its subspaces are optimal with respect to the robustness of entanglement under mixing with the white noise, an estimate for which is given in equation (70).

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Construction of a family of $3 \otimes 3$ CESs of maximal dimension

The maximal dimension of $3 \otimes 3$ CES, by equation (23), is equal to 4. According to the theory given in section 2, we can associate each such CES with some isometry $V : H_E \rightarrow H_A \otimes H_B$, where $\dim(H_E) = 4$, $\dim(H_A) = 3$, $\dim(H_B) = 3$.

We search for the three Kraus operators $K_i$ of the corresponding channel related to $V$ by equation (14). To make the subspace be completely entangled, we must satisfy the conditions given in section 3 on page 7. At first we consider the operators $K_i^\dagger K_i$ which in this case can be constructed in the simplest, diagonal form:

$$K_1^\dagger K_1 = \begin{pmatrix} \lambda_1^{(1)} & 0 & 0 & 0 \\ 0 & \lambda_2^{(1)} & 0 & 0 \\ 0 & 0 & \lambda_3^{(1)} & 0 \\ 0 & 0 & 0 & \lambda_4^{(1)} \end{pmatrix},$$

$$K_2^\dagger K_2 = \begin{pmatrix} \lambda_1^{(2)} & 0 & 0 & 0 \\ 0 & \lambda_2^{(2)} & 0 & 0 \\ 0 & 0 & \lambda_3^{(2)} & 0 \\ 0 & 0 & 0 & \lambda_4^{(2)} \end{pmatrix},$$

$$K_3^\dagger K_3 = \begin{pmatrix} \lambda_1^{(3)} & 0 & 0 & 0 \\ 0 & \lambda_2^{(3)} & 0 & 0 \\ 0 & 0 & \lambda_3^{(3)} & 0 \\ 0 & 0 & 0 & \lambda_4^{(3)} \end{pmatrix},$$ (A1)
where
\[ 0 < \lambda_{ij}^{(1)} < 1, \quad \lambda_{ij}^{(1)} + \lambda_{ij}^{(2)} + \lambda_{ij}^{(3)} = 1, \quad i = 1, \ldots, 3; \quad j = 1, \ldots, 4 \]  
(A2)
to satisfy the conditions of equations (12) and (31).

There can be at most three nonzero values on the main diagonal of each \(K_i^\dagger K_i\) because each \(K_i\) itself is represented by a \(3 \times 4\) matrix that can have at most three nonzero singular values. In addition, the analysis can be simplified by setting \(\lambda_1^{(2)} = 0\) without violation of the conditions in equation (A2).

Equation (A1) suggests that \(K_i\) can be written in the form of the singular value decomposition:

\[ K_1 = W_1 \begin{pmatrix} \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 \end{pmatrix}, \]
\[ K_2 = W_2 \begin{pmatrix} \sqrt{\lambda_1^{(2)}} & 0 & 0 & 0 \\ 0 & \sqrt{\lambda_2^{(2)}} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_4^{(2)}} \end{pmatrix}, \]
\[ K_3 = W_3 \begin{pmatrix} \sqrt{\lambda_1^{(3)}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\lambda_3^{(3)}} \\ 0 & 0 & \sqrt{\lambda_4^{(3)}} & 0 \end{pmatrix}, \]  
(A3)

where \(W_1, W_2\) and \(W_3\)—\(3 \times 3\) unitary matrices which we can choose appropriately to satisfy our conditions on the Kraus operators. We set \(W_1 = I\) and choose \(W_2\) and \(W_3\) to be some permutation matrices:

\[ W_2 : \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad W_3 : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}. \]  
(A4)

Now we choose an arbitrary vector state \(|\phi\rangle \in H_E\) with components \((\phi_1, \phi_2, \phi_3, \phi_4)\) and, using our particular choice of \(W_1, W_2, W_3\), write out the corresponding vectors \(K_1|\phi\rangle, K_2|\phi\rangle, K_3|\phi\rangle\) as columns of a matrix:

\[ \begin{pmatrix} \sqrt{\lambda_1^{(1)}} \phi_1 & \sqrt{\lambda_2^{(1)}} \phi_2 & \sqrt{\lambda_3^{(1)}} \phi_3 & \phi_4 \\ \sqrt{\lambda_1^{(2)}} \phi_2 & 0 & \sqrt{\lambda_3^{(2)}} \phi_3 & \sqrt{\lambda_4^{(2)}} \phi_4 \\ \sqrt{\lambda_1^{(3)}} \phi_3 & \sqrt{\lambda_2^{(3)}} \phi_4 & 0 & \sqrt{\lambda_3^{(3)}} \phi_1 \end{pmatrix}, \]  
(A5)

where \(\lambda_1^{(2)} = 0\) was taken into account.

It can be easily seen that all minors of order two evaluate to zero if and only if \(|\phi\rangle\) has all components equal to zero. Consequently, for any nonzero \(|\phi\rangle \in H_E\) there are at least two linearly independent columns. According to the theory presented in section 3 on page 7, the isometry \(V\) constructed with the Kraus operators (A3) will have a completely entangled range. Acting with \(V\) on orthonormal basis states of \(H_E\), we obtain the orthonormal system presented in equation (20).
Appendix B. Details of construction of a family of $4 \otimes 4$ CESs of dimension 7

Following along the same lines as in appendix A, we can easily come up with the operators

\[
K_1 = W_1 \left( \begin{array}{cccccc}
\sqrt{\lambda^{(1)}_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\lambda^{(1)}_5} & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\lambda^{(1)}_7} \\
0 & 0 & \sqrt{\lambda^{(1)}_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \right),
\]

\[
K_2 = W_2 \left( \begin{array}{cccccc}
\sqrt{\lambda^{(2)}_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\lambda^{(2)}_5} & 0 \\
0 & 0 & \sqrt{\lambda^{(2)}_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\lambda^{(2)}_7} \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \right),
\]

\[
K_3 = W_3 \left( \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\lambda^{(3)}_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda^{(3)}_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda^{(3)}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \right),
\]

\[
K_4 = W_4 \left( \begin{array}{cccccc}
0 & 0 & 0 & 0 & \sqrt{\lambda^{(4)}_5} & 0 \\
0 & \sqrt{\lambda^{(4)}_2} & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda^{(4)}_3} & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda^{(4)}_4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array} \right),
\]

such that all $K_i^\dagger K_i$ are diagonal. Here $W_i$ are $4 \times 4$ unitary matrices. The operators $K_i$ will satisfy equation (12) on condition that any two lambdas with the same subscript add up to 1.

Setting $W_1 = I$ and choosing $W_2, W_3, W_4$ to be permutations

\[
W_2 : \left( \begin{array}{c}
1 \\
3 \\
1 \\
\end{array} \right), \quad W_3 : \left( \begin{array}{c}
3 \\
4 \\
3 \\
\end{array} \right), \quad W_4 : \left( \begin{array}{c}
2 \\
4 \\
2 \\
\end{array} \right),
\]

we obtain proper Kraus operators (in the described above sense). Reconstructing associated isometry via equation (14), we obtain the orthonormal system of vectors spanning a CES and presented in equation (41).

Appendix C. Details of construction of a $3 \otimes 3 \otimes 3$ GES of maximal dimension

The operators $P_{\alpha}^{(i)}$, defined by equations (49)–(51), are the constituent parts of $K_i^\dagger K_i$ in scheme (47). When we write $K_i$ themselves in the form of the singular value decomposition (52), the square roots of $P_{\alpha}^{(i)}$ are involved, so it is convenient to consider the diagonalization of these operators:

\[
P_{\alpha}^{(i)} = U_{\alpha}(\alpha)D_{\alpha}(\alpha)U_{\alpha}(\alpha)^\dagger, \quad i = 1, 2, 3,
\]

(C1)
where it is easy to obtain that the matrices $U_i(\alpha)$ and $D_i(\alpha)$ are defined by

\[
U_1(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad D_1(\alpha) = \begin{pmatrix} \lambda_1^{(1)} & 0 \\ 0 & 0 \end{pmatrix}, \tag{C2a}
\]
\[
U_2(\alpha) = \begin{pmatrix} \sin \alpha & -\cos \alpha \\ \cos \alpha & \sin \alpha \end{pmatrix}, \quad D_2(\alpha) = \begin{pmatrix} \lambda_2^{(1)} & 0 \\ 0 & 0 \end{pmatrix}, \tag{C2b}
\]
\[
U_3(\alpha) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad D_3(\alpha) = \begin{pmatrix} \lambda_3^{(1)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{C2c}
\]

and the nonzero eigenvalues of $P^0_{\alpha}$ are

\[
\lambda_1^{(1)} = \lambda_2^{(2)} = \frac{1}{1 + \sin 2\alpha}, \quad \lambda_3^{(3)} = \frac{2 \sin 2\alpha}{1 + \sin 2\alpha}. \tag{C3}
\]

Now we can write out the ingredients of equation (52). The matrices $V_i$, $i = 1, 2, 3$, are block-diagonal $16 \times 16$ unitaries

\[
V_i = \begin{pmatrix} U_i(\alpha_1) & & & \\ & U_i(\alpha_2) & & \\ & & U_i(\alpha_3) & \\ & & & U_i(\alpha_4) \end{pmatrix}.	ag{C4}
\]

The matrices $\Sigma_i$ are expressed in terms of $\lambda_j^{(i)}$ and $\lambda_j^{(i)}$ as

\[
\Sigma_i = \begin{pmatrix}
\sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & 0 & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} & \sqrt{\lambda_2^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_1^{(1)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{C5}
\]
\[ \Sigma_2 = \begin{pmatrix}
\sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_2^{(1)}}
\end{pmatrix}, \]

\[ \Sigma_3 = \begin{pmatrix}
\sqrt{\lambda_3^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{\lambda_3^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_3^{(1)}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{\lambda_3^{(1)}}
\end{pmatrix}. \]

\[ (C6) \]

The structure of each matrix \( \Sigma_i^\dagger \Sigma_i \) is similar to the one in equation (47) with the only distinction that all blocks \( P_{\alpha_j}^{(i)} \) are in the diagonal forms \( D_{\alpha_j} \) given by equation (C2). Multiplication of \( \Sigma_i^\dagger \Sigma_i \) by \( V_i \) from equation (C4) recovers the exact form of \( K_i^\dagger K_i \) presented in equation (47). The unitaries \( W_i \) from equation (52) do not change this form and can be chosen arbitrarily for our purpose.

Our particular choice of values for the variables \( \lambda_{ij}^{(0)} \) and \( \alpha_j \) in scheme (47) is

\[ \alpha_i = \alpha \equiv \pi/6, \quad i = 1, \ldots, 5, \]

\[ \lambda_1^{(i)} = \frac{2}{3}, \quad \lambda_2^{(i)} = \frac{1}{3}, \quad \lambda_3^{(i)} = \frac{1}{4}, \quad \lambda_4^{(i)} = \frac{3}{4}, \quad \lambda_5^{(i)} = \frac{1}{6}. \]

\[ (C8) \]

We set all \( \alpha_i \) equal to each other for simplicity.

To satisfy the conditions on \( K_i \) described in section 3 on page 7, we search for the proper unitaries \( W_i \) in the form of permutation matrices. Using the procedure described in section 3 on page 11 and the Groebner basis algorithm, we came up with the three proper permutations. Let \( e_i \) denote a nine-dimensional vector with the \( i \)th component equal to 1 and the rest equal to
We have made such a choice trying to transform the vectors $|\psi\rangle_1, \ldots, |\psi\rangle_{10}$ of equation (C10) to entangled ones. At the same time, the vectors $|\psi\rangle_{11}$ and $|\psi\rangle_{16}$ are transformed to separable ones (the basis vectors $|2\rangle, |5\rangle, |6\rangle$ of the nine-dimensional subsystem are involved), and we could not come up with a better scheme leaving all these vectors entangled. To come around this problem, we return to the CES defined by equation (C10) and mix the vectors $|2\rangle, |5\rangle, |6\rangle$.
with each other by means of a simple $3 \times 3$ unitary transformation:

\[
\begin{align*}
|2\rangle & \rightarrow -\frac{1}{3}|2\rangle + \frac{2}{3}|5\rangle + \frac{2}{3}|6\rangle, \\
|5\rangle & \rightarrow \frac{2}{3}|2\rangle - \frac{1}{3}|5\rangle + \frac{2}{3}|6\rangle, \\
|6\rangle & \rightarrow \frac{2}{3}|2\rangle + \frac{2}{3}|5\rangle - \frac{1}{3}|6\rangle,
\end{align*}
\]

which we took from reference [47]. Being a local unitary operation, such a substitution in equation (C10) does not change entanglement of the CES. After that we proceed to the splitting operation (C11). The analysis of the newly obtained subspace with the use of the Groebner basis algorithm shows that it is a GES.

**ORCID iDs**

K V Antipin https://orcid.org/0000-0003-1597-1478

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