Existence of algebraic decay in non-Abelian ferromagnets

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The low temperature regime of non-Abelian two dimensional ferromagnets is investigated. The method involves mapping such models into certain site-bond percolation processes and using ergodicity in a novel fashion. It is concluded that all ferromagnets possessing a continuous symmetry (Abelian or not) exhibit algebraic decay of correlations at sufficiently low temperatures.

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In a recent letter [1] Seiler and I proposed studying the phase structure of the 2D $O(N)$ models by mapping them into a correlated site-bond percolation problem. This approach was applied to certain discrete spin modes and to the $O(2)$ model, for which we rederived the Froehlich and Spencer [2] result regarding the existence of a massless phase at sufficiently low temperatures $1/\beta$. In this paper I report an extension of the percolation approach to $O(N) \, N \geq 3$. It leads to the conclusion that a massless phase exists in all $O(N)$ models.

For completeness I will repeat the main points of Ref. [1] (see also Ref. [3] for a more complete discussion). With any $O(N)$ spin configuration one can associate an Ising spin configuration by dividing the sphere $S(N - 1)$ into two hemispheres and introducing an Ising variable $\sigma = \pm 1$, which specifies in which hemisphere the spin points. In this manner the standard nearest neighbor action (s.n.n.a.) for the $O(N)$ model allows rewriting the partition function as

$$Z = \sum_{\{\sigma\}} \left( \prod_{i \in \Lambda} \int ds_\parallel d\vec{s}_p_i \right) \cdot \exp \left[ \beta \sum_{\langle i,j \rangle} (s_\parallel i s_\parallel j \sigma_i \sigma_j + \vec{s}_p_i \cdot \vec{s}_p_j) \right]$$  \hspace{1cm} (1)

Here $\vec{u}$ is the unit vector chosen for specifying the hemispherical decomposition, $s_\parallel = |\vec{s} \cdot \vec{u}|$ and $\vec{s}_p \cdot \vec{u} = 0$. With respect to the Ising variables the action is ferromagnetic, hence amenable to the Fortuin-Kasteleyn transformation [4]. This procedure associates to the Ising problem a correlated site-bond percolation process defined as follows:

**FK1**-identify clusters of like-$\sigma$ spins (H-clusters)

**FK2**-within each H-cluster occupy bonds randomly with probability $1 - \exp(-2\beta_{ij})$

(obtain FK-cluster)

**FK3**-assign to every site within a given FK-cluster the same $\sigma$ value, obtained by choosing randomly + or - with probability 1/2.

Here $\beta_{ij}$ is the space dependent inverse temperature, which for the s.n.n.a. would be $\beta s_\parallel i s_\parallel j$. Fortuin and Kasteleyn proved that the mean FK-cluster size (expected size of the cluster attached to the origin) equals the magnetic susceptibility of the Ising variable

$$\chi_{Is} \equiv \frac{1}{|\Lambda|} \sum_{x,y \in \Lambda} \langle \sigma_x \sigma_y \rangle.$$  \hspace{1cm} (2)

In particular the latter diverges when the mean FK-cluster size diverges.
To apply the F-K procedure to the $O(N)$ models, Seiler and I considered a modified model called ‘cut’ action: the Gibbs factor is s.n.n.a. only if $|\vec{s}_i - \vec{s}_j| < \epsilon$, $0 < \epsilon < 2$ and 0 otherwise. We then formulated the following three conjectures:

C1: The Mermin-Wagner theorem applies to the ‘cut’ model.

C2: The $O(N)$ models (‘cut’ or not) are ergodic.

C3: On a triangular lattice $T$ a percolation process produced by a measure enjoying the symmetries of the lattice can contain at most one percolating cluster.

I refer the reader to Refs. [1] and [3] for a thorough discussion of the motivations behind these three conjectures and of the comparison of the ‘cut’ and the s.n.n.a. models. I will elaborate only on C2, which is central to the arguments presented in this paper. Imagine a very large lattice on which one has used the Monte Carlo procedure to simulate the $O(N)$ model. If one has achieved thermalization, then this configuration is ‘typical.’ In the infinite volume limit a typical configuration has two important properties:

P1: spatial averages equal ensemble averages (Birkoff’s theorem)

P2: the configuration is (statistically) invariant under additional Monte Carlo steps.

I will briefly sketch the argument used in Ref. [1] to prove that the ‘cut’ $O(2)$ model must exhibit algebraic decay of its correlation functions for $\epsilon$ sufficiently small. In Eq. (2) let $\langle \cdot \rangle$ stand for expectation value measured with the full Gibbs measure. By P1 and P2, $\chi_{is}$ can be computed as a quenched expectation value provided the spins $s_{i||}$ are assigned the values of a typical configuration. Since the Gibbs measure is invariant under lattice translations and (discrete) rotations, by C1 and C3 a typical configuration cannot contain a percolating H-cluster. An interesting theorem by Russo [5] states that if a translational invariant percolation process on a $T$ lattice is such that neither clusters of the set $E$ nor of its complement $\bar{E}$ percolate, then the mean cluster size of both $E$ and $\bar{E}$ must diverge. Taking $E$ to stand for $\sigma = +1$ and $\bar{E}$ for $\sigma = -1$ shows that the mean size of the H-clusters must diverge. (This statement is not surprising since at $\beta = 0$ and $\epsilon = 2$ the $O(N)$ model is equivalent to the Bernoulli site-percolation process with $p = 1/2$ and for the latter the critical density on a $T$ lattice is indeed $1/2$.) The FK-clusters are subclusters of the H-clusters obtained via rule FK2. In the ‘cut’ model, this rule must be amended. Indeed because of rule FK3, the constraint could be violated unless bonds are occupied at all sites having $s_{||} > d \equiv \epsilon/2$. Therefore, in a ‘cut’ $O(N)$ model, the FK-clusters must contain D-clusters defined by the condition $s_{||} > d$. In Ref. [1] we showed that for the ‘cut’ $O(2)$
model simple applications of C1 and C3 required that neither $D$-clusters nor $\bar{D}$-clusters ($s_{\parallel} < d$) can percolate and then, by Russo’s theorem, both must have divergent mean size, QED.

From the discussion presented thus far it follows that in any ‘cut’ $O(N) \ N > 1$ model on a $T$ lattice, if neither clusters of $D$ nor of $\bar{D}$ percolate, the mean FK-cluster size must diverge and hence correlations must decay algebraically. In fact in the ‘cut’ model $D$-clusters can never percolate. Indeed the set $D$ consists of two disconnected pieces, both of which are contained in H-clusters and I have already argued that H-clusters cannot percolate. Therefore, the only question is whether $\bar{D}$-clusters could percolate for $\epsilon$ sufficiently small? The reason for which a topological answer to this question exists in $O(2)$ is that in that case the set $\bar{D}$ consists also of two disconnected pieces, which, for $\epsilon < \sqrt{2}$, cannot communicate. Obviously in $O(N)N \geq 3$, $\bar{D}$ is a connected set and a new strategy must be employed. In the sequel I will state three independent arguments, that in the ‘cut’ $O(N)$ model $\bar{D}$-clusters cannot percolate for $\epsilon$ sufficiently small or $\beta$ sufficiently large. Each argument requires a new conjecture and I will address their merits too.
Argument 1

This is a proof by contradiction. For simplicity I will discuss the s.n.n.a. \(O(3)\) model \((\epsilon = 2)\) at \(\beta\) large and choose \(\vec{u} = \hat{z}\). I will take \(d\) small but independent of \(\beta\) so that by FK2, when \(\beta\) is large, the bond occupation probability for sites in \(D\) goes to 1. I will assume that a cluster of \(\bar{D}\) percolates and show that that assumption suggests that a certain magnetic susceptibility (Eq. (4)) diverges. To that end I introduce spherical coordinates and rewrite the partition function as

\[
Z = \left(\prod_{i \in \Lambda} \int_0^\pi d\theta_i \int_0^{2\pi} d\varphi_i\right) \cdot \exp \left\{ \beta \sum_{\langle i,j \rangle} [\cos \theta_i \cos \theta_j + \sin \theta_i \sin \theta_j \cos (\varphi_i - \varphi_j)] \right\}
\]  

(3)

Consider the following susceptibility

\[
\chi_\varphi \equiv \frac{1}{|\Lambda|} \sum_{x,y \in \Lambda} \langle \cos(\varphi_x - \varphi_y) \rangle
\]

(4)

By P1 and P2 \(\chi_\varphi\) could be measured by quenching the \(\theta\) variables to the values \(\bar{\theta}\) they would take in a typical configuration. That is

\[
\chi_\varphi = \frac{1}{|\Lambda|} \sum_{x,y \in \Lambda} \langle \cos(\varphi_x - \varphi_y) \rangle_q
\]

(5)

where \(\langle \cdot \rangle_q\) means expectation value computed with the measure

\[
\left(\prod_{i \in \Lambda} \int_0^{2\pi} d\varphi_i\right) \exp \left[ \beta \sum_{\langle i,j \rangle} \sin \bar{\theta}_i \sin \bar{\theta}_j \cdot \cos (\varphi_i - \varphi_j) \right].
\]

(6)

Since the quenched model is an \(O(2)\) model (albeit with space dependent couplings), one can employ Ginibre’s inequality [6] to bound \(\chi_\varphi\) from below by the value it would take if in the measure (6) one replaced \(\beta \sin \theta_i \sin \theta_j\) by 0 at all sites where \(\sqrt{\beta \sin \theta_i} < c\) for some \(c > 0\). Under the assumption that \(\bar{D}\) percolates, by C3, these sites could not possibly percolate, but would form islands. The average size of these islands relative to the average distance between them would decrease with beta. Indeed by the Mermin-Wagner theorem, the probability of finding the spin at a site taking values in some subset of the sphere \(A\) of volume \(V(A)\) is equal to \(V(A)/4\pi\). (For \(\beta\) large, one can use perturbation theory to estimate the average size of these islands, which becomes actually independent of \(\beta\).) Thus the assumption that the equatorial strip \(\bar{D}\) percolates implies that \(\chi_\varphi\) is bounded
from below by the susceptibility of an $O(2)$ model at large inverse temperature, but on a
lattice having some small, randomly distributed holes. Although I am not aware of any
rigorous result proving that, the following conjecture seems eminently reasonable.

C4: Consider a $T$ lattice and dilute bonds randomly with a probability smaller than the
percolation probability for unoccupied bonds. Then there exists a $\beta_{kt} < \infty$ such that for
any $\beta > \beta_{kt}$ the susceptibility diverges.

Before motivating this conjecture, let me say that there is no reason to expect
that if in the $O(3)$ model $\bar{D}$ percolated, the polar caps would be distributed as the holes
produced by a Bernoulli process. Their actual distribution would be controlled by the full
$O(3)$ measure. However, if $\bar{D}$ percolated and especially if the model had a mass gap, by
some central limit theorem, one would expect the polar caps to form islands and their
distribution to be random at distances much larger than the correlation length.

The intuition for C4 comes from the following rigorous results:

a) Georgii [7] proved that if one randomly dilutes sites or bonds on a regular lattice
with $D \geq 2$, then provided a remaining cluster percolates, there exists an inverse
temperature $\beta_c < \infty$ such that for $\beta > \beta_c$ there exists long range order (l.r.o.).

b) De Massi et al. [8], proved that under the same conditions as above, the Laplacian
retains its continuous spectrum.

In the language of the Coulomb gas, my conjecture is that if one introduces in the gas
perfect conductors, randomly distributed, if the perfectly conducting regions do not per-
colate, at sufficiently low temperatures, the Coulomb gas does not exhibit Debye screening
(the introduction of the perfect conductors will only affect the dielectric constant).

To conclude this argument, the contradiction is this: if one assumes that for the
$O(3)$ model $\bar{D}$ percolates and $\chi_{Is}$ is finite, then clearly so is the $s_z$-susceptibility (since
$s_z \leq 1$). On the other hand C4 strongly suggests that the $s_x - s_y$ susceptibility would
diverge when $\beta$ is large. This is a clear violation of $O(3)$ invariance, hence the assumption
that $\bar{D}$ percolates must be false. Although not transparent, the topology of $O(3)$ is crucial
for this argument. Indeed one may wonder if a similar reasoning could not be used to relate
the $O(2)$ model to the Ising model and thus prove that the latter must exhibit l.r.o. at
large $\beta$, in violation of the Mermin-Wagner theorem? The answer is no, precisely because
$\bar{D}$ is no longer a connected set and thus it could not possibly percolate.
Argument 2

This is again a proof by contradiction. For simplicity I consider the ‘cut’ $O(3)$ model and choose $\vec{u} = \hat{z}$. I would like to argue that if the equatorial strip $\bar{D}$ percolated, then $O(3)$ invariance would be broken.

Next let me consider the realistic case of a $T$ lattice and an $\epsilon$ small, yet $\epsilon > 0$. Suppose that in fact the equatorial strip $\bar{D}$ does percolate and hence its complement $D$ forms islands. In the ‘cut’ model, the lines $s_z = c > d$ will have to form closed loops, nested inside these islands. Consider now a $c$-tilted equator, namely the great circle passing thru $s_z = c$ and $s_x = 0$. Since neither the hemisphere $s_x > 0$ nor $s_x < 0$ can percolate, any site of the lattice must be surrounded by an infinite sequence $X(k) k \in Z$ of concentric closed loops $s_x = 0$. (By the line $s_x = 0$ I mean a line on the dual lattice such that $s_{x_i} \cdot s_{x_j} \leq 0$; same type of qualifications apply to all other lines appearing in this discussion.) $O(3)$ invariance requires that the average number of intersections of the $X$ lines with the $c$-tilted equators is independent of $c$. However if $\bar{D}$ percolates, then in any typical configuration there exists a $k_0 < \infty$ such that any $X(k)$ line with $k > k_0$ intersects the percolating cluster. That means that infinitely many $X$ lines cross the $c = 0$ tilted equator, while they may or may not cross the $c$-tilted equators with $c > 0$. In other words if $\bar{D}$ percolates, then one would expect the average number of crossings of the $X$ lines with the $c$-tilted equators to decrease with $c$, in violation of $O(3)$ invariance. If on the other hand $\bar{D}$ does not percolate, then both $D$ and $\bar{D}$ form rings and no a priori asymmetry in the average number of crossings of the $X$ lines with the $c$-tilted equators exists. (An example where $\bar{D}$ percolates is the Richard model [9], which is a modified $O(3)$ model in which $|s_z| < 1 - b$ for some $b > 0$, hence this model is only $O(2)$ invariant. The percolation approach used in Ref. [1] can be employed to prove rigorously that this model has to be massless for $\epsilon$ sufficiently small - see Ref. [3]; $\chi_{\phi}$ diverges, yet $\chi_{\phi s} < \infty$.)

In the discussion above I used the word ‘expect’ because one could say that even though if $\bar{D}$ percolates the regions with $s_z > c$ are hidden inside regions of smaller $s_z$ values, they are larger and thus restore $O(3)$ invariance. However $O(3)$ invariance requires that any typical configuration has the following two properties:

T1: The area is preserved.

T2: The gradient is preserved.
Property T1 means that the density of sites where the spin points in some region A is proportional to the volume $V(A)$. Property T2 says that if one selects two points on the sphere $p_1$ and $p_2$, separated by a distance $L$, the average distance between sites where the spin points in the neighborhood of $p_1$ respectively $p_2$ depends only on $L$ (it is independent of which $p_1$ and $p_2$ are chosen, provided they are at distance $L$). Obviously both properties are required by C1.

C5: If in the ‘cut’ $O(3)$ model $\tilde{D}$ percolated, then the typical configuration would violate $T_1$ or $T_2$ (or both) and, hence, $O(3)$ invariance.

The motivation for C5 is this: if $\tilde{D}$ percolated, then, as already argued, $D$ would form islands - as opposed to rings, which are formed when neither $\tilde{D}$ nor $D$ percolates on a $T$ lattice. The basic difference between a system forming islands and one forming rings is that islands are basically of finite size - the probability to find an island of diameter $L$ decreases exponentially with $L$; on the contrary, if the system forms rings, there exists an infinite sequence of clusters surrounding each other and hence no exponential suppression of large clusters. Thus if the system forms islands the typical configuration will contain mostly mappings of a hemisphere over some finite region of $T$. It is easy to check that such maps cannot preserve both T1 and T2. No such difficulty exists if one considers rings - arbitrarily large regions of $T$.

**Argument 3**

As I have already noted, if $\tilde{D}$ percolates, then $D$ forms islands. Moreover, $D$ consists of two disconnected pieces $D_u$ and $D_l$. When $\epsilon$ is sufficiently small, the volume of $D_u$, $V(D_u)$ is much larger than that of $\tilde{D}$, $V(\tilde{D})$. On the other hand the area of the boundary of $D_u$, $S(D_u)$ is half $S(\tilde{D})$. Is it reasonable to expect that under these circumstances, the mean cluster size of $D_u$ is finite while that of $\tilde{D}$ infinite? The answer is provided by the following conjecture:

C6: In the ‘cut’ $O(N)$ model, if two sets $A$ and $B$ have $V(A) = V(B)$ and $S(A) < S(B)$, then there exists $\epsilon_0(A, B) > 0$ such that for any $\epsilon < \epsilon_0$, $\langle A \rangle \geq \langle B \rangle$, where $\langle \cdot \rangle$ represents the mean cluster size.

The conjecture says that at given volume, the larger the surface of a set, the smaller its average cluster size. The reason for adding the qualifier that $\epsilon < \epsilon_0$ is that for $\epsilon > 0$ the surface of the clusters of a set $A$ need not consist of points on the surface of $A$. I believe that
this conjecture is intuitively clear. It can be proved in 1D. In 2D it was verified numerically
for $O(3)$ as follows: A was the Northern polar cap of area $4\pi/3$, B the equatorial strip
of the same area and $\epsilon$ was such that the Northern and Southern polar caps could barely
communicate. The data indicated that the mean cluster size of both A and B increased
as $L^{2-\eta}$ ($L$-linear size of the lattice) and that $\eta_A < \eta_B$.

If C6 is true, it cannot be true that in the ‘cut’ $O(N)$ models the equatorial strip $D$
percolates. Indeed if $D$ percolates, its mean cluster size is divergent. By C6, for $\epsilon$
sufficiently small, so is the mean cluster size of $\bar{D}$. By Russo’s theorem, on a $T$
lattice, that can occur only if neither $D$ nor $\bar{D}$ percolates. QED.
Discussion

The arguments presented above indicate that all 2D $O(N)$ models possess a massless phase. (This situation contradicts common wisdom. Evidence in favor of the latter is analyzed separately [10] and found wanting.) The arguments moreover suggest that although at large $\beta$ extended topological defects - instantons - may exist in non-Abelian models, they are suppressed entropically with respect to spin waves. This situation, already conjectured by the author in 1986 [11], suggests that for

$$N \geq 3$$

the 2-point function may behave as

$$\langle \vec{s}_0 \cdot \vec{s}_x \rangle \sim a(\beta) \frac{e^{-m(\beta)x}}{\sqrt{x}} + b(\beta) \frac{1}{x^{\eta(\beta)}}.$$  

(7)

I have no basis at the present time to estimate $a(\beta)$ and $b(\beta)$, nor whether $\eta$ depends on $\beta$ in any given model. However, it could be that $a$ and $b$ are such that at intermediate distances the decay is exponential to a very good approximation (a similar effect governs the time evolution of a metastable state in nonrelativistic quantum mechanics [12]).

Finally a word about perturbation theory. The fact that the 2D $O(N)$ models possess a massless phase for $\beta$ sufficiently large does not imply that in 2D perturbation theory fails to produce the correct asymptotic expansion at fixed distances (as it does in 1D for $N \geq 3$). However if one defines the Callan-Symanzik $\beta$-function by requiring that say $\langle \vec{s}(0) \cdot \vec{s}(x) \rangle/\langle \vec{s}(0) \cdot \vec{s}(y) \rangle$ is a renormalization group invariant for $x, y \gg 1$, then clearly an algebraic decay for $\beta > \beta_{kt}(N)$ implies that the Callan-Symanzik $\beta$-function could be chosen to be vanishing. If my conjecture about Eq. (7) proved to be correct, one could also define the $\beta$-function as $d\beta/d\ln(m)$, in which case one may find the famous asymptotic freedom answer. However I find it hard to believe that if that were the case, the continuum limit constructed by letting $\beta \rightarrow \infty$ would not contain (coupled) massless excitations (of course a continuum limit could also be constructed for any $\infty > \beta > \beta_{kt}(N)$ - that field theory would be a massless theory).

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References

[1] A. Patrascioiu and E. Seiler, Phys. Rev. Lett. 68, 1395 (1992).
[2] J. Froehlich and T. Spencer, Comm. Math. Phys. 81, 455 (1981).
[3] A. Patrascioiu and E. Seiler, J. Stat. Phys. 69, 55 (1992).
[4] C. M. Fortuin and P. W. Kasteleyn, J. Phys. Soc. JPN (suppl.) 24, 86 (1969).
[5] L. Russo, Z. Wahrsch. Verw. Gebiete 42, 39 (1978).
[6] J. Ginibre, Comm. Math. Phys. 16, 310 (1970).
[7] H. O. Georgii, Comm. Math. Phys. 81, 527 (1981).
[8] A. DeMassi, P. A. Ferrari, S. Goldstein and W. D. Wick, J. Stat. Phys. 55, 787 (1989).
[9] J.-L. Richard, Phys. Lett. B 134, 75 (1987).
[10] A. Patrascioiu and E. Seiler, The Difference between Abelian and Non-Abelian Models: Facts and Fancy, MPI preprint, 1991, math-ph/9903038.
[11] A. Patrascioiu, Phys. Rev. Lett. 58, 2285 (1987).
[12] A. Patrascioiu, Phys. Rev. D 24, 496 (1981).