Notes on $f(T)$ Theories

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Abstract

The cosmological models based on teleparallel gravity with nonzero torsion are considered. To investigate the evolution of this theory, we consider the phase-space analysis of the $f(T)$ theory. It shows when the tension scalar can be written as an inverse function of $x$ where $x = \rho_e/(3m_{pl}^2 H^2)$ and $T = g(x)$, the system is an autonomous one. Furthermore, the $\omega_e - \omega'_e$ phase analysis is given out. We perform the dynamical analysis for the models $f(T) = \beta T \ln(T/T_0)$ and $f(T) = \alpha m_{pl}^2 (-T/m_{pl}^2)^n$ particularly. We find that the universe will settle into de-Sitter phase for both models. And we have examined the evolution behavior of the power law form in the $\omega_{ep} - \omega'_{ep}$ plane.

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I. INTRODUCTION

Recent cosmic acceleration in our universe is suggested by a combination of different cosmic probes that primarily involves Supernova data \cite{1, 2}. Besides the most well-known dark energy scenario (e.g., the scalar field quintessence model \cite{3}) as the mechanism for late time cosmic acceleration, another kind of model based on infra-red modifications to general relativity (such as the $f(R)$ theory \cite{4–7}) was also proposed. The validity of general relativity on large astrophysical or cosmological scale has never been tested but only assumed, the recent cosmic acceleration in our universe might be nothing but the signal of breakdown of General Relativity at large scale \cite{8–11}.

The Riemann-Cartan geometry was proposed with the aim of unifying gravity and electromagnetism \cite{12}. In general, a large number of connections can be defined on a manifold. The assumptions of torsion-free and metric compatibility lead to the Einstein general relativity with Levi-Civita connection. Meanwhile, the assumption of curvature-free leads to the teleparallel theory of gravity with Weitzenböck connection \cite{13 –16}. The $f(T)$ theory is proved to be equivalent to general relativity but with different origins. The $f(T)$ theory is considered as a gauge theory and general relativity is thought as a geometric theory. Recently, the $f(T)$ gravity has been proposed to explain the present cosmic accelerating expansion without dark energy \cite{17–22}. Similar to $f(R)$ gravity, based on a modification of the teleparallel equivalent of general relativity Lagrangian, the additional term related to the form of $f(T)$ could be written as an effective energy. The torsion $T$ in $f(T)$ theory will be responsible for the observed acceleration of the universe.

The $f(T)$ theory was firstly used as the source of driving inflation \cite{17}. Then it was applied to the late time acceleration \cite{18, 21}. Furthermore, the perturbation of $f(T)$ theory was discussed in Refs. \cite{23, 24}, the reconstruction of the $f(T)$ theory was presented in Refs. \cite{25, 26}, and the observational constraint was analyzed in Refs. \cite{27, 28}. In this letter, we made a dynamical analysis on two $f(T)$ models by choosing variables different from those in Refs. \cite{19, 29}.

This letter is organized as follows. In Sec. II we introduce the $f(T)$ theory, present the phase-space analysis and give out the analytical form of $\omega'$. Two particular models are discussed in Sec. III. Finally, we draw our conclusion in Sec. IV.
II. THE $f(T)$ GRAVITY

In $f(T)$ theory, the fundamental dynamical object is the vierbein field $e^A_{\mu}(x)$. In the teleparallelism, the orthonormal tetrad components $e^\mu_A(x)$ are used, where the index $A$ runs over 0, 1, 2, 3 in the tangent space at each point $x^\mu$ of the manifold, and $\mu$ is the coordinate index in the manifold and also runs over 0, 1, 2, 3. The curvature tensor and the covariant derivatives of $e^\mu_A(x)$ with respect to the connection vanish globally, therefore $e^\mu_A(x)$ are absolutely parallel vector fields, this theory is also called teleparallel gravity [12] and the geometry is the Weitzenböck space-time characterized by torsion tensor alone. The vierbein field is related with the metric $g_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{AB}e^A_{\mu}e^B_{\nu}. \quad (1)$$

The torsion $T^\rho_{\mu\nu}$ and contorsion $K^{\mu\nu\rho}$ tensors are defined as

$$T^\rho_{\mu\nu} = e^\rho_A \left( \partial_\mu e^A_\nu - \partial_\nu e^A_\mu \right), \quad (2)$$

$$K^{\mu\nu\rho} = -\frac{1}{2} \left( T^\mu_{\nu\rho} - T^\nu_{\mu\rho} - T^\mu_{\nu\rho} \right). \quad (3)$$

The action for $f(T)$ theory is

$$I_T = \frac{1}{16\pi G} \int d^4x |e| \left( T + f(T) \right), \quad (4)$$

where $|e| = \det(e^A_\mu) = \sqrt{-g}$, the torsion scalar is

$$T \equiv S^{\mu\nu\rho}T^\rho_{\mu\nu}, \quad (5)$$

and

$$S^{\mu\nu\rho} \equiv \frac{1}{2} \left( K^{\mu\nu\rho} + \delta^\mu_\rho T^{\alpha\nu}_\alpha - \delta^\nu_\rho T^{\mu\alpha}_\alpha \right). \quad (6)$$

Varying the action with respect to the vierbein yields the equation of motion,

$$-\frac{1}{4} e^A_\mu (T + f) + e^B_A T^\mu_{\nu B} S^{\nu\alpha}_\mu (1 + f_T) + e^{-1} \partial_\mu (e e^A_\rho S^{\mu\alpha}_\rho) T_T T^\alpha_T + e^\rho_A S^{\mu\alpha}_\rho f_T T^\mu_T T^\alpha_T = 4\pi G e^\rho_A T^\alpha_T \quad (7)$$

where $T^\alpha_T$ is the energy-momentum tensor and the subscript $T$ denotes the derivative with respect to the torsion scalar.

For simplicity, we assume a flat Friedmann-Robertson-Walker metric,

$$ds^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2, \quad (8)$$
where $a(t)$ is the scale factor. In the FRW space-time, $g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$ and therefore the tetrad components $e^A_{\mu} = (1, a, a, a)$ yield the torsion scalar $T = -6H^2$, where $H = \dot{a}/a$ is the Hubble parameter.

The energy conservation equations for the radiation and pressureless matter are

\begin{align}
\rho'_{\gamma} + 4\rho_{\gamma} &= 0, \quad (9) \\
\rho'_m + 3\rho_m &= 0, \quad (10)
\end{align}

where $\rho_{\gamma}$ and $\rho_m$ are the energy densities of radiation and pressureless matter respectively, and a prime means the derivative with respect to $\ln a$. From Eq. (7), the Friedmann equation for $f(T)$ theory is gotten,

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_r + \rho_e),$$

where the effective (dark) energy density due to $f(T)$ is

$$\rho_e = \frac{1}{16\pi G}(-f + 2Tf_T).$$

(12)

Obviously, when $f = 2Tf_T$, i.e., $f(T) \propto T^{1/2}$, the effective energy density vanishes. The effective dark energy density is assumed to be conserved as well

$$\rho'_e + 3(\rho_e + p_e) = 0,$$

(13)

where the effective pressure is $p_e = -\rho_e \mp 2m^2_{pl}(f_T + 2Tf_{TT})T'/6$, and the evolution of the torsion scalar $T$ could be expressed as

$$\frac{T'}{T} = \frac{-1}{3m^2_{pl}H^2} \frac{3\rho_m + 4\rho_r}{2Tf_{TT} + f_T + 1}.$$ 

(14)

A. The Phase-Space Analysis

To make the phase-space analysis, we introduce the following dimensionless variable,

$$x = \frac{\rho_e}{3m^2_{pl}H^2}, \quad y = \Omega_m = \frac{\rho_m}{3m^2_{pl}H^2}, \quad z = \Omega_r = \frac{\rho_r}{3m^2_{pl}H^2}.$$ 

(15)

In terms of these dimensionless variables, the Friedmann equation becomes the constraint equation

$$x + y + z = 1.$$ 

(16)
Combining Eqs. (9), (10), (13) and (14), we get

\[
x' = (fT - \frac{f}{T} - 2Tf_{TT}) \frac{T'}{T},
\]

(17)

\[
y' = -y(3 + \frac{T'}{T}),
\]

(18)

\[
z' = -z(4 + \frac{T'}{T}),
\]

(19)

and

\[
\frac{T'}{T} = \frac{(H^2)'}{H^2} = -\frac{3y + 4z}{2Tf_{TT} + 1 + f_T}.
\]

(20)

From the definition of \(x\) in Eq. (15), we know that once the function \(f(T)\) is specified, \(x\) can be expressed as a function of \(T\), so if the inverse function exists, then \(T\) can be expressed as a function of \(x\),

\[
T = g(x).
\]

(21)

With the help of the above relation, Eqs. (17), (18) and (19) become an autonomous system. Because of the constraint (16), there are only two independent variables, here we choose them as \(x\) and \(y\).

**B. The Equation of State**

The elucidation of the physics of models becomes difficult when the EoS parameter is close to \(-1\) [30], since the evolving slowly effective energy density may be indistinguishable from \(\Lambda\)CDM scenario. By emphasizing the dynamics, the restricted regions of the trajectories are recovered in “position” and “velocity”- the value of the equation of state ratio \(\omega\) and its time variation \(\omega'\) [31].

The EoS parameter of the effective dark energy is,

\[
w_e = \frac{p_e}{\rho_e} = -1 + \frac{T'}{T} \frac{f_T + 2Tf_{TT}}{3(-f/T + 2f_T)}.
\]

(22)

When \(f(T) = constant\), the model can be regarded as the cosmological constant theory which is attributed to the quantum zero-point energy of the particle physics vacuum.

It is currently constrained from the distance measurements of SNIa as \(\omega_0 = -1.31^{+0.22}_{-0.28}\) and \(\omega'_0 = -1.48^{+0.90}_{-0.81}\) [30]. In \(f(T)\) theories, the form of \(\omega'_e\) can be expressed as

\[
\omega'_e = \frac{f_T + 2Tf_{TT}}{3(-f/T + 2f_T)} \frac{T'}{T} + \frac{3Tf_{TT} + 2T^2f_{TTT}}{3(-f/T + 2f_T)} \frac{f_T + 2Tf_{TT}}{3(-f/T + 2f_T)^2} \frac{T'^2}{T}.
\]

(23)
It is noticed that the parameters $\omega_e$ and $\omega'_e$ are functions of $T$, $\rho_m$ and $\rho_\gamma$. But only the Friedmann equation could be used as a constraint equation. It is hard to draw the phase-space diagram since both the parameters $\omega_e$ and $\omega'_e$ are functions of two variables. Fortunately, in the pivot periods, we could get the relation between $\omega_e - \omega'_e$ once the exact form of $f(T)$ is given. In the radiation dominated phase, the energy of matter could be ignored; and once the matter is dominating, the energy density of radiation could be ignored. In the following, we will give two exact forms of $f(T)$ as examples.

III. TWO MODELS

A. The Logarithmic Form

Firstly, we consider a phenomenological model with a logarithmic form,

$$f(T) = \beta T \ln\left(\frac{T}{T_0}\right),$$

(24)

where $\beta$ and $T_0$ are constants. The effective energy density is

$$\rho_{el} = \frac{1}{16\pi G}\left(\beta T \ln\left(\frac{T}{T_0}\right) + 2\beta T\right),$$

(25)

and the dimensional variable $x$ is

$$x = \frac{2}{3} \beta \ln \frac{T}{T_0} - \frac{\beta}{3}.$$  

(26)

Therefore,

$$T = T_0 e^{\frac{\beta}{3} x + \frac{1}{2}},$$

(27)

and

$$\frac{T'}{T} = -\frac{4 - 4x - y}{3\beta - 1 - x/\beta}.$$  

(28)

The dynamical equations of the autonomous system become

$$x' = -\beta \frac{4 - 4x - y}{3\beta - 1 - x/\beta},$$

(29)

$$y' = -y(3 - \frac{4 - 4x - y}{3\beta - 1 - x/\beta}).$$

(30)

Setting $x' = y' = 0$, we get the critical point $x = 1$ and $y = 0$. The critical point corresponds to dark energy domination with $\Omega_m = \Omega_r = 0$ and $w_e = -1$. The eigenvalues are $\lambda_1 = -3$
and $\lambda_2 = 4\beta^2/(3\beta^2 - \beta - 1)$, so when $(1 - \sqrt{13})/6 < \beta < (1 + \sqrt{13})/6$ and $\beta \neq 0$, the critical point is a stable fixed point. Unlike the standard model, there is no critical points for radiation domination $z = 1$ and matter domination $y = 1$ for this model. In the view of dynamical analysis, this model is ruled out by the history of our universe. Therefore, it is not necessary to discuss its EoS parameter.

B. The Power Law Form

Now we consider the power law form,

$$f(T) = \alpha m_{pl}^2 \frac{T}{m_{pl}^2}^n,$$  \hspace{1cm} (31)

where $\alpha$ and $n$ are dimensionless parameters. When $n = 0$, $f(T)$ is a constant and the model becomes the $\Lambda$CDM model. By using the power-law form of $f(T)$, we get the effective dark energy density,

$$\rho_{ep} = \alpha m_{pl}^4 (n - \frac{1}{2}) (\frac{T}{m_{pl}^2})^n,$$  \hspace{1cm} (32)

and the effective pressure,

$$p_{ep} = -\alpha m_{pl}^4 (n - \frac{1}{2}) (\frac{T}{m_{pl}^2})^n + \frac{1}{6} \frac{(2n^2 - n)\alpha m_{pl}^2 (\frac{T}{m_{pl}^2})^{n-1} T'}{m_{pl}^2}.$$  \hspace{1cm} (33)

When $n = 1/2$, $\rho_{ep} = p_{ep} = 0$. When $n = 1$, $\rho_{ep} = -\alpha m_{pl}^2 T/2 = 3\alpha m_{pl}^2 H^2$, so the effective energy tracks the background matter. Substituting Eq. (32) into the definition of the dimensionless variable $x$, we get

$$x = \alpha (2n - 1) (\frac{T}{m_{pl}^2})^{n-1},$$  \hspace{1cm} (34)

and

$$T = -m_{pl}^2 \left( \frac{x}{\alpha (2n - 1)} \right)^{1/(n-1)}.$$  \hspace{1cm} (35)

So

$$\frac{T'}{T} = -\frac{4 - 4x - y}{1 - nx},$$  \hspace{1cm} (36)

and the dynamical equations are,

$$x' = -\frac{2(n - 1)(4 - 4x - y)}{1 - nx} x,$$  \hspace{1cm} (37)

$$y' = -y(3 - \frac{4 - 4x - y}{1 - nx}).$$  \hspace{1cm} (38)
Phases $(x, y)$ | Stability | $(\lambda_1, \lambda_2)$  
---|---|---  
$R$ | unstable | $(1, -8(n^2 - 1))$  
$M$ | unstable when $n^2 < 1$ | $(-1, 6(1-n^2))$  
$A$ | stable | $(-3, -8)$  

TABLE I: The properties of the critical points for the power law form with $n \neq 1$.

Setting $x' = y' = 0$, we obtain the critical points $(x, y) = (0, 0)$, $(x, y) = (0, 1)$ and $(x, y) = (1, 0)$. The properties of the fixed points $R$, $M$, and $A$ are summarized in Table I. These fixed points are similar to those found in Ref. [19], but we find the fixed point $M$ can be stable when $n^2 > 1$.

**R**  
The radiation dominated phase  
For the critical point $R$, $x = y = 0$ and $\Omega_r = 1$. Because $\lambda_1 = 1 > 0$, it is an unstable fixed point, so the universe could exit from the radiation dominating phase.

**M**  
The matter dominated phase  
For the critical point $M$, $x = 0$ and $y = 1$. When $n^2 > 1$, it is a stable attractor, otherwise it is an unstable fixed point. So when $n^2 < 1$, the universe could exit from the matter domination.

**A**  
The accelerating phase
FIG. 2: The trajectories of $\omega_{ep} - \omega'_ep$ in the phase space. From top to bottom, $n$ takes the values of 0.3, 0.1, -0.1, -0.3, respectively.

For the fixed point $A$, $n \neq 1$, $x = 1$ and $y = 0$ and it is a stable attractor independent of the parameters $\alpha$ and $n$. So $\Omega_m = \Omega_r = 0$, $w_e = -1$ since $T'/T = 0$, and this phase recovers the de-Sitter phase. Now we have two attractors $M$ and $A$ when $n^2 > 1$. For a given initial condition, which attractor will the universe choose in the phase space? To answer this question, we go back to the dynamical Eqs. (37) and (38). Note that there is a singularity in the system when $x = 1/n$. So if we start with $x > 1/n$, then the attractor is $A$; otherwise if we start with $x < 1/n$, the attractor is $M$. When $n^2 < 1$, we have only one attractor $A$. We plot the phase diagram for the attractor $A$ with $n = 0.1$ in Fig. 1.

For the special case $n = 1$, if $\alpha < 1$, $x = \alpha$. The constraint equation becomes $y + z = 1 - \alpha$, so there is only one independent variable. And there are only two critical points $R$ with $\Omega_r = 1 - \alpha$ and $M$ with $\Omega_m = 1 - \alpha$.

Based on the power law form of $f(T)$, the corresponding equation of state is obtained

$$\omega_{ep} = -1 - \frac{n T'}{3 T}. \quad (39)$$

In the radiation or the matter dominated phase, the evolution of the scale factor $a$ is known, then it is easily to get the effective EoS parameter which is listed in Table II. Generally,
if the effective energy density part takes the dominated part, the corresponding effective EoS parameter $\omega_{el} < 0$ which suggests $n < 1$. If the effective energy density part has the possibility to make our universe accelerate, the corresponding effective EoS parameter $\omega_{ep} < -1/3$ which suggests $n < 2/3$. If the effective EoS parameter $\omega_{ep} < -1$, $n < 0$ is suggested.

After the universe transits to the matter dominated phase, the radiation part could be ignored, and the energy density of matter could be approximately written as

$$\rho_m \simeq \frac{m_{pl}^2}{2} (-T + f - 2T f_T).$$

(40)

Putting the above equations into Eq.(22), the equation of state is expressed as

$$\omega_{ep} \simeq \frac{n - 1}{1 - \alpha n (2n - 1) (-T/m_{pl}^2)^{n-1} - 1 - \frac{n T'}{3 T}}.$$  

The derivative of the EoS parameter with respect to $\ln a$ is changed to

$$\omega_{ep}' \simeq \frac{3(n-1)^2}{n}(1 + \omega_{ep})\left(\frac{n - 1}{\omega_{ep}} - 1\right).$$

(41)

After fixing the value of $n$, the diagram of $\omega_{ep} - \omega_{ep}'$ which is Figure 2 could be drawn out. There is a fixed point $\omega_{ep} = -1, \omega_{ep}' = 0$ which is corresponding to the accelerating phase $A$.

IV. CONCLUSION

In this letter, the phase-phase analysis of the $f(T)$ theories is done and its stability of the critical points has been investigated. We can see that when the energy density can be written as an inverse function of $x$ which could be expressed as a general form $T = g(x)$, we can do the phase-analysis in the dynamical system. Furthermore, the value of the EoS parameter and its evolution are considered. Those analyses depend on the exact forms of $f(T)$ heavily.
Specially, we made the dynamical analysis for the logarithmic form and the power form of \( f(T) \). For the logarithmic form \( f(T) = \beta T \ln(T/T_0) \), only one critical point which corresponds to de-Sitter phase exists. Unlike the standard cosmology with Einstein gravity, there is no critical point for radiation and matter dominations. For the power law form \( f(T) = \alpha m^2_{\text{pl}}(-T/m^2_{\text{pl}})^n \), we find that two stable fixed points exist when \( n^2 > 1 \). One fixed point corresponds to matter domination and the other fixed point corresponds to de-Sitter phase. The reason for the existence of two stable fixed points is that the autonomous system has a singular point \( x = 1/n \). If the system starts with \( x > 1/n \), then the system will settle into de-Sitter phase \( A \), otherwise the system will settle into matter domination \( M \). When \( n = 1 \), de-Sitter phase is absent and the effective energy tracks the background matter. When \( n^2 < 1 \), only de-Sitter phase is the stable fixed point. And we have plotted the diagram of \( \omega_{ep} - \omega'_{ep} \) plane for the power law form. The fixed point \( \omega_{ep} = -1, \omega'_{ep} = 0 \) is corresponding to the accelerating phase \( A \).

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