Regression in Nonstandard Spaces with Fréchet and Geodesic Approaches

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One approach to tackle regression in nonstandard spaces is Fréchet regression, where the value of the regression function at each point is estimated via a Fréchet mean calculated from an estimated objective function. A second approach is geodesic regression, which builds upon fitting geodesics to observations by a least squares method. We compare these two approaches by using them to transform three of the most important regression estimators in statistics – linear regression, local linear regression, and trigonometric projection estimator – to settings where responses live in a metric space. The resulting procedures consist of known estimators as well as new methods. We investigate their rates of convergence in general settings and compare their performance in a simulation study on the sphere.

1. Introduction

Our goal is to estimate an unknown function $[0,1] \rightarrow \mathbb{Q}, t \mapsto m_t$, where $(\mathbb{Q},d)$ is a metric space. To this end, we have access to independent data $(x_i,y_i)_{i=1,...,n}$. We assume that
the covariates are fixed, e.g., \( x_i = \frac{i}{n} \), and \( y_i \) is a random variable with values in \( \mathcal{Q} \) such that its Fréchet mean (or barycenter) is equal to \( m_{x_i} \), i.e., 
\[
m_{x_i} = \arg \min_{q \in \mathcal{Q}} \mathbb{E}[d(y_i, q)^2].
\]
We consider \( \mathcal{Q} \) to be nonstandard, i.e., a metric space that is not isometric to a convex subset of a separable Hilbert space, in particular, we exclude the euclidean spaces \( \mathbb{R}^k \) with its euclidean metric. Examples of nonstandard spaces are Riemannian manifolds, like the hypersphere \( S^k \), Hadamard spaces, like the space of phylogenetic trees [BHV01], or Wasserstein spaces [ACT11] in dimension greater than one.

The literature on statistical analysis in nonstandard spaces is vast. We refer the reader to [HE20] for an overview and only present a small glimpse here. The Fréchet mean or barycenter \( m = \arg \min_{q \in \mathcal{Q}} \mathbb{E}[d(Y, q)^2] \) of a random variable \( Y \) with values in the metric space \( \mathcal{Q} \) lies at the heart of most analysis in nonstandard spaces. In Alexandrov spaces, [GPRS19] present conditions for a parametric rates of convergence of the sample Fréchet mean. [Stu03] discusses the Fréchet mean in Hadamard spaces. [Bac14] presents algorithms for its calculation. The Fréchet mean has been studied on Riemannian manifolds, e.g., [BP03]. In this setting [EH19] show a central limit theorem. Nonparametric regression with metric target values is developed, e.g., in [DFBJ10, Hei09, PM19]. [LM19] present a regression technique with regularization by total variation. [SHS10] discuss nonparametric regression techniques between Riemannian manifolds, e.g., [BP03]. In this setting [EH19] show a central limit theorem. Nonparametric regression with metric target values is developed, e.g., in [DFBJ10, Hei09, PM19]. [LM19] present a regression technique with regularization by total variation. [SHS10] discuss nonparametric regression techniques between Riemannian manifolds. Based on the notion of geodesics, [Ele13] introduces an analog of linear regression in symmetric Riemannian manifolds. These results are generalized and extended in [CZKI17].

1.1. Settings

We will present our results in three levels of abstraction: the hypersphere \( S^k \), certain classes of metric spaces \( \mathcal{Q} \) like Hadamard spaces and metric spaces of finite diameter, and an even more general setting which is governed by what kinds of meaningful statements can be proven for abstract mathematical objects.

**Hypersphere.** Let \( k \in \mathbb{N} \). Let \( S^k = \{ x \in \mathbb{R}^{k+1} : |x| = 1 \} \) be the hypersphere with radius 1 as a subset of \( \mathbb{R}^{k+1} \). We equip \( S^k \) with its intrinsic metric \( d(q, p) = \arccos(\langle q, p \rangle) \). Let \( T\mathbb{S}^k = \bigcup_{q \in \mathbb{S}^k} (\{q\} \times T_q\mathbb{S}^k) \) be the tangent bundle, where \( T_q\mathbb{S}^k = \{ v \in \mathbb{R}^{k+1} | q^\top v = 0 \} \) is the tangent space at \( q \in \mathbb{S}^k \). The exponential map is \( \text{Exp}: T\mathbb{S}^k \to \mathbb{S}^k \), \( (q, v) \mapsto \cos(|v|)q + \sin(|v|) \frac{v}{|v|} \), where \( |v| \) denotes the euclidean norm. Geodesics can be represented by a tuple \((p, v) \in T\mathbb{S}^k \) as \( x \mapsto \text{Exp}(p, xv) \).

For \( t \in [0, 1] \), let \( Y_t \) be a \( \mathbb{S}^k \)-valued random variable. Let the regression function \( m: [0, 1] \to \mathbb{S}^k \) be a minimizer \( m_t = \arg \min_{q \in \mathbb{S}^k} \mathbb{E}[d(Y_t, q)^2] \). Let \( x_i = \frac{i}{n} \) and let \((y_i)_{i=1,...,n} \) be independent random variables with values in \( \mathbb{S}^k \) such that \( y_i \) has the same distribution as \( Y_{x_i} \).

**Metric.** Let \((\mathcal{Q}, d) \) be a metric space. For \( t \in [0, 1] \), let \( Y_t \) be a \( \mathcal{Q} \)-valued random variable with finite second moment, i.e., \( \mathbb{E}[d(Y_t, q)^2] < \infty \) for all \( t \in [0, 1] \) and \( q \in \mathcal{Q} \). Let the regression function \( m: [0, 1] \to \mathcal{Q} \) be a minimizer \( m_t = \arg \min_{q \in \mathcal{Q}} \mathbb{E}[d(Y_t, q)^2] \). Let \( x_i = \frac{i}{n} \) and let \((y_i)_{i=1,...,n} \) be independent random variables with values in \( \mathcal{Q} \) such that \( y_i \) has the same distribution as \( Y_{x_i} \).
General. This setting is based on the notion of generalized Fréchet mean introduced in [Huc11] and similarly used, e.g., in [EH19, Sch19].

Let $\mathcal{X}$ be the space of covariates, $\mathcal{Y}$ a set called data space, $\mathcal{Q}$ a set called descriptor space. Let $c: \mathcal{Y} \times \mathcal{Q} \to \mathbb{R}$ be a cost function, $l: \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}$ be a loss function. For $t \in \mathcal{X}$, let $Y_t$ be a $\mathcal{Y}$-valued random variable with finite expected cost, i.e., $\mathbb{E}[|c(Y_t, q)|] < \infty$ for all $t \in \mathcal{X}$ and $q \in \mathcal{Q}$. Let the regression function $m: \mathcal{X} \to \mathcal{Q}$ be a minimizer $m_t \in \arg\min_{q \in \mathcal{Q}} \mathbb{E}[c(Y_t, q)]$. Let $x_i \in \mathcal{X}$ be deterministic and let $(y_i)_{i=1,...,n}$ be independent random variables with values in $\mathcal{Y}$ such that $y_i$ has the same distribution as $Y_{x_i}$.

1.2. Two Approaches

To construct an estimator for $m$ in these settings, one may try to adapt a known euclidean estimator to the new scenario. Two prominent approaches to this task are Fréchet regression [PM19] and geodesic regression [Fle13].

Fréchet Regression. The regression function $m_t$ is the Fréchet mean of $Y_t$, i.e., the minimizer of $\mathbb{E}[d(Y_t, q)^2]$ over $q \in \mathcal{Q}$. In Fréchet regression, we estimate the function $t \mapsto \mathbb{E}[d(Y_t, q)^2]$ for every fixed $q \in \mathcal{Q}$ by an euclidean estimator $t \mapsto \hat{F}_t(q)$ using the data $(x_i, z_{q,i})_{i=1,...,n} \subseteq [0,1] \times \mathbb{R}$ with $z_{q,i} = d(y_i, q)^2$. In this step we may use one of the standard parametric or nonparametric regression estimators for certain classes of functions $[0,1] \to \mathbb{R}$. Then $\hat{F}_t(q)$ is minimized over $q \in \mathcal{Q}$ for a fixed $t$ to obtain the estimator $\hat{m}_t$.

Geodesic Regression. Assume our metric space $\mathcal{Q}$ is equipped with an exponential map $\text{Exp}: \Theta \to \mathcal{Q}$, where $\Theta \subseteq T\mathcal{Q} \subseteq \mathcal{Q} \times \mathbb{R}^k$ is a subset of the tangent bundle of $\mathcal{Q}$. A geodesic starting in point $p \in \mathcal{Q}$ and continuing in the direction $v \in T_p \mathcal{Q}$ of the tangent space of $\mathcal{Q}$ at $p$ can be described as a function $\mathbb{R} \to \mathcal{Q}$, $x \mapsto \text{Exp}(p, xv)$ with $(p, v) \in \Theta$. In geodesic regression with covariates $x_i \in \mathbb{R}$, we minimize the empirical squared error

$$\sum_{i=1}^n d(y_i, \text{Exp}(p, x_i v))^2$$

over $(p, v) \in \Theta$ to find the best fitting geodesic. All forms of geodesic regression built on this criterion or a modification of it. For example, we can extend it to multivariate regression

$$\sum_{i=1}^n d(y_i, \text{Exp}(p, \sum_{j=1}^k x_{i,j} v_j))^2,$$

where $x_i \in \mathbb{R}^k$ and $v_1, \ldots, v_k \in T_p \mathcal{Q}$ or more general feature regression

$$\sum_{i=1}^n d(y_i, \text{Exp}(p, \sum_{j=1}^k \psi_j(x_i) v_j))^2,$$

where $x_i \in \mathcal{X}$ for an arbitrary space of covariates $\mathcal{X}$ and features $\psi_j: \mathcal{X} \to \mathbb{R}$. Furthermore, we may introduce weights $w_{i,t}$, e.g., $w_{i,t} = K((x_i - t)/h)$ for a kernel $K$ and a
bandwidth $h > 0$ to localize the procedure, and obtain (for one-dimensional covariates)
\[
\hat{m}_t = \arg \min_{(p,v) \in \Theta} \sum_{i=1}^n w_i d(y_i, \text{Exp}(p, x_i v))^2.
\]
In this paper, we do not require the existence of an exponential map in the sense of Riemannian geometry. Instead, we introduce a link function $g: \Theta \times X \to Q$ for a set of covariates $X$ and a set $\Theta$, which we can think of as parameterizing geodesics. We then minimize $\sum_{i=1}^n d(y_i, g(\theta, x_i))^2$. For $X \subseteq \mathbb{R}$, this generalizes the setting used above via $\Theta \subseteq TQ$ and $g((p,v),x) = \text{Exp}(p, xv)$.

### 1.3. Contribution

We compare the two approaches of geodesic (Geo) and Frechet (Fre) regression on three regression estimators, namely linear regression (Lin), local linear regression (Loc), and the trigonometric orthogonal series projection estimator (Tri). This makes six estimation procedures, which we refer to as LinGeo, LinFre, LocGeo, LocFre, TriGeo, and TriFre. For the resulting estimators, which we denote as $\hat{m}_t$, our goal is to show explicit finite sample bounds of the form $\mathbb{E}[d(m_t, \hat{m}_t)^2] \leq Cn^{-\alpha}$ (in the metric setting), where $C > 0$ is a constant. We are not interested in optimal universal constants, e.g., whether $C = 2$ or $C = 2000$, but the dependence on further parameters, like a moment bound, is to be explicit.

- **LinGeo** (section [2]): For standard geodesic regression in symmetric Riemannian manifolds, [Fle13] shows existence and uniqueness of the estimator as well as equivalence of the least squares estimator and the maximum likelihood estimator with Gaussian errors. [CZKI17] prove asymptotic normality results in this setting. We show $\mathbb{E}[(1/n) \sum_{i=1}^n d(m_{x_i}, \hat{m}_{x_i})^2] \leq Cn^{-1}$ for $n \in \mathbb{N}$ and a constant $C > 0$ in Hadamard spaces Corollary 2 and general metric spaces of finite diameter Corollary 3. These results are derived from an even more general statement, Theorem 1.

- **LinFre** (section [3]): Among other Fréchet regression methods, linear (or global) Fréchet regression was developed in [PM19]. Here we show a negative result. The estimator of the objective function $t \mapsto \mathbb{E}[d(Y_t, q)^2]$ is only consistent in standard spaces, Theorem 2. Our simulations show inconsistency of the estimator on the sphere in our model. We suggest a modified estimator, LinCos, which maximizes the cosine of the distance instead of minimizing the squared distance. The simulations suggest consistency of LinCos. We also give some theoretical justification. But we do not investigate rates of convergence, as LinCos is a method specific to the sphere $S^2$ (with possible extensions to the hypersphere $S^k$ and hyperbolic spaces) and not a regression technique for more general nonstandard spaces, which is the topic of this article.

- **LocGeo** (section [4]): We apply the approach of geodesic regression to the well-known local linear estimator and arrive at a new estimator, LocGeo. We show
\[ E[d(m_t, \hat{m}_t)^2] \leq C_n^{-\frac{2\beta}{2\beta+1}} \text{ for all } t \in [0, 1], n \in \mathbb{N}, \text{ a smoothness parameter } \beta \in (1, 2], \text{ and a constant } C > 0, \] 

Theorem 3, Corollary 5, Corollary 6. For this result, we assume a smoothness condition, which generalizes the Hölder condition that is common for local linear estimators. It demands that the true function \( t \mapsto m_t \) can be locally approximated at \( t \) by a geodesic up to an error of order \( |x - t|^\beta \) for \( x \) close to \( t \).

- **LocFre (section 5):** PM19 introduce local constant (Nadaray–Watson) and local linear Fréchet regression for general metric spaces. For the local linear estimator, they show \( d(\hat{m}_t, m_t) \in O(n^{-\frac{2}{5}}) \) and a more general version of this result, see Corollary 1 in their article. We show, for a general local polynomial Fréchet estimator of order \( \ell \in \mathbb{N}_0 \), that \( E[d(m_t, \hat{m}_t)^2] \leq C n^{-\frac{2\beta}{2\beta+1}} \) for a constant \( C > 0 \) and a smoothness parameter \( \beta > \ell \), Theorem 4, Corollary 8, Corollary 9. Our results are slightly more general with conditions slightly less demanding. Furthermore, bounds in expectation for finite \( n \) are stronger than in \( O_P \). As PM19, we demand a smoothness condition not directly on \( t \mapsto m_t \), but on the change of the probability density of \( Y_t \) in \( t \).

- **TriGeo (section 6):** We apply the Fréchet regression approach to the trigonometric projection estimator and arrive at a new estimator, TriGeo. We are not able to derive results on rates of convergence. We argue, that this estimator may be suboptimal as the properties that make it appealing in euclidean spaces are lost in nonstandard spaces. Nonetheless, we include the estimator in our simulation study.

- **TriFre (section 7):** We apply the approach of Fréchet regression to the trigonometric projection estimator and arrive at a new estimator, TriFre. We show \( E\left[ \int_0^1 d(m_t, \hat{m}_t)^2 dt \right] \leq C n^{-\frac{2\beta}{2\beta+1}} \) for a smoothness parameter \( \beta \geq 1 \) and a constant \( C > 0 \), Theorem 5, Corollary 11, Corollary 12. As for LocFre the smoothness condition is a requirement on the change of the density of \( Y_t \) in \( t \).

The comparison of these estimation procedures in one article underlines the importance to have a versatile tool belt when tackling new challenges: There is not one approach alone that solves the problem of nonstandard regression in every scenario. For linear regression, only the geodesic approach leads to a consistent estimator. For trigonometric regression the results are basically reversed: We can prove rates of convergence only for the Fréchet approach. For local linear estimation both approaches seem equally well suited. This comparison of geodesic and Fréchet approach on three different euclidean estimators leads to three different outcomes. Thus, focusing on one setting alone would not reveal the complexities in the general comparison of the two approaches.

Our goal is to make all theorems as general as reasonably possible. This manifests in quite abstract statements. To get a gist of the meaning of the abstract objects we start most sections by a corollary of a general theorem on the sphere: Corollary 1, Corollary 4, Corollary 7, and Corollary 10. These corollaries illustrate our results and show that they are indeed applicable to explicit interesting nonstandard spaces. Furthermore, abstract
assumptions of the general theorems are justified by showing that they are fulfilled on the sphere.

The sphere is also the metric space used in our simulation study, section 8. To fulfill a variance inequality, which is an assumption for all our results, we introduce a new family of distributions on the sphere, the contracted uniform distributions. All estimators are implemented using the statistical programming language R. The resulting package is freely available at https://github.com/ChristofSch/spheregr.

Our experiments confirm and illustrate the theoretical findings.

The proofs (appendix A) partially built upon techniques developed in [Sch19]. Therein a so-called weak quadruple inequality is assumed to prove rates of convergence for the (generalized) Fréchet mean without requiring that the descriptor space Q is bounded. We fulfill this condition by definition of our moment conditions. Generally, the major tools to prove results in this setting are empirical process theory with chaining, e.g., [vdVW96] or [Tal14] and appendix B, and a technique called slicing or peeling, e.g., [vdG00]. The proofs for local regression techniques follow the euclidean version in [Tsy08, section 1.6] as far as possible, for trigonometric regression we make use of [Tsy08, section 1.7].

1.4. Notation and Conventions

We use a lower case c for universal constants c > 0. If the value depends on a variable, we indicate this by an index, e.g., c_κ is a constant that depends only on κ. We do not introduce or define every such constant. They are silently understood to take an appropriate value. Furthermore, the value may vary between two occurrences of such a constant. Alternatively we may use c', c'', ... for the same purpose.

A capital C indicates a constant that has further meaning, which is usually described by a three letter index, e.g., we may require a moment condition \( \mathbb{E}[d(Y_t, m_t)^2] \leq C_{\text{Mom}} \) for all t to be fulfilled. For simplicity, we assume these constants to be \( \geq 1 \), so that \( C_{\text{Abc}}^2 + C_{\text{Abc}} C_{\text{Xyz}} \leq c C_{\text{Abc}}^2 C_{\text{Xyz}} \).

Assumptions are named in small caps, e.g., Moment. Different assumptions in different sections may have the same name. Hence, assumptions always refer to the assumptions defined in the same section. Nonetheless, the names are consistent across the sections insofar as assumptions with the same name are – if not identical – expressions of the same underlying requirement.

There is a silently underlying probability space (\( \Omega, \Sigma, \mathbb{P} \)). If a random variable, say \( Y \), has values in a set, say \( Y \), that set is silently understood to be a measurable space (\( Y', \Sigma_Y \)) and the random variable is a measurable map \( Y: (\Omega, \Sigma_\Omega) \rightarrow (Y', \Sigma_Y) \).

In each section the estimator of the regression function at \( t \) is denoted as \( \hat{m}_t \). It depends on \( n \) and potentially on further parameters like a bandwidth \( h \), which will not be indicated in the notation but should be clear in the context.

Let \((Q, d)\) be a metric space. To shorten the notation, we sometimes write \( \overline{q,p} \) instead of \( d(q,p) \) for \( q,p \in Q \). Define the ball \( B(o, d, \delta) = \{ q \in Q : d(q, o) < \delta \} \) and the diameter \( \text{diam}(Q, d) = \sup_{q,p \in Q} d(q, p) \).

For a vector \( v \in \mathbb{R}^k \), we denote its euclidean norm by \( |v| \). For a matrix \( A \in \mathbb{R}^{k \times \ell} \), we denote its operator norm by \( \|A\|_{\text{op}} = \sup_{v \in \mathbb{R}^\ell, |v| = 1} |Av| \).
Appendix C introduces important concepts of metric geometry, which are used in this paper, like geodesic and Hadamard space.

2. Linear Geodesic Regression

We apply the geodesic approach to linear regression, which yields the standard geodesic regression, LinGeo, introduced in [Fle13].

2.1. Hypersphere

Before we present the general and abstract results, we illustrate them in the hypersphere setting, see section 1.1. Let \( \Lambda \in [1, \infty) \). Let \( \Theta = \{(p, v) \in TS^k \mid |v| \leq \Lambda\} \). The regression function \( m: [-1, 1] \to S^k \) is assumed to be a geodesic \( m_t = \text{Exp}(p^*, tv^*) \), \( (p^*, v^*) \in \Theta \).

We observe \((x_i, y_i)_{i=1,...,n}\) on a regular grid \((x_i)_{i=1,...,n}\) of \([-1, 1]\) (instead of \([0, 1]\)).

We estimate the starting point \( p^* \) and velocity vector \( v^* \) by the least squares method in \( S^k \), i.e.,

\[
(\hat{p}, \hat{v}) = \arg\min_{(p, v) \in \Theta} \frac{1}{n} \sum_{i=1}^{n} d(y_i, \text{Exp}(p, x_i v))^2.
\]

The estimated curve then is \( t \mapsto \hat{m}_t = \text{Exp}(\hat{p}, tv) \).

Using our general theory in the next section, we obtain following corollary.

**Corollary 1 (LinGeo Hypersphere).** Assume there is \( C_{Vlo} \geq 1 \) such that \( C_{Vlo}^{-1} d(m_t, q)^2 \leq \mathbb{E}[d(Y, q)^2] \) for all \( t \in [-1, 1] \) and \( q \in S^k \). Then

\[
\mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} d(m_{x_i}, \hat{m}_{x_i})^2 \right] \leq C \frac{1}{n},
\]

where \( C = ckC_{Vlo}\Lambda^{-2} \).

2.2. General

We now present a result in the general setting of section 1.1. Let \( \Theta \) be a space of parameters. Let \( g: X \times \Theta \to Q \) be the link function. Our model assumption is \( g(t, \theta^*) = m_t \) for the true parameter \( \theta^* \in \Theta \). The canonical M-estimator of \( \theta^* \) is

\[
\hat{\theta} \in \arg\min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^{n} c(y_i, g(x_i, \theta)).
\]

The resulting plug-in estimator for the regression function \( m_t \) is \( \hat{m}_t = g(t, \hat{\theta}) \).

We introduce some further notation. Define \( c_t(y, \theta) = c(y, g(t, \theta)) \) and \( \diamond_t(y, \bar{y}, \hat{\theta}) = c_t(y, \theta) - c_t(\bar{y}, \theta) - c_t(y, \hat{\theta}) + c_t(\bar{y}, \hat{\theta}) \). Define \( x = (x_1, \ldots, x_n) \). For \( \theta_0 \in \Theta \), define \( B_X(\theta_0, l, \delta) = \{ \theta \in \Theta : \frac{1}{n} \sum_{i=1}^{n} l(g(x_i, \theta), g(x_i, \theta_0)) \leq \delta \} \). Define \( a_t(y, z) = \sup_{\theta, \delta \in \Theta, \theta \neq \delta} \frac{\diamond_t(y, z, \theta, \delta)}{b(\theta, \delta)} \).
Assumptions.

- **Variance:**
  There is $C_{Vlo} \in [1, \infty)$ such that $C_{Vlo}^{-1} I(m_t, q) \leq \mathbb{E}[\ell(Y_t, q) - \ell(Y_t, m_t)]$ for all $t \in X$ and $q \in Q$.

- **Entropy:**
  There are $T_n \geq 0$, $C_{Ent} \in [1, \infty)$, and $\xi \in (0, 1)$ such that $\gamma_2(B_X(\theta_0, l, \delta), b) \leq B\delta^{\xi}$ for all $\delta \geq T_n$ and $\theta_0 \in \Theta$, where $\gamma_2$ is a measure of entropy defined in Definition 3 (appendix B).

- **Moment:**
  There are $\kappa \geq 2$ and $C_{Mom} \in [1, \infty)$ such that $\mathbb{E}[(a_t(Y_t, m_t))^\kappa]^{1/\kappa} \leq C_{Mom}$ for all $t \in X$.

**Theorem 1 (LinGeo General).** Assume **Variance**, **Entropy**, **Moment**. Assume $\kappa(1 - \xi) > 1$. Then

$$\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} l(m_{x,i}, \hat{m}_{x,i})\right] \leq C n^{\frac{1}{\kappa(1 - \xi)}} + C_{Vlo} T_n,$$

where $C = c_{\kappa, \xi} C_{Vlo} (C_{Ent} C_{Mom})^{1/\kappa}$.

In many settings like in euclidean linear regression with $\ell(y, q) = |y - q|^2$, $l(m, q) = |m - q|^2$, we have $\xi = \frac{1}{2}$ and retrieve the parametric rate of convergence.

**Remark 1.**

- **Variance:**
  This condition is also called variance inequality and is well-known in the context of Fréchet means in Alexandrov spaces, [Stu03, Oht12, GPRS19]. **Variance** is a condition on the noise distribution and the geometry of involved spaces. It can be viewed as a quantitative version of the condition of unique Fréchet means $m_x$ of $Y_x$. The variance inequality not only ensures uniqueness of $m_x$, it also requires the objective function $\mathbb{E}[d(Y, q)^2]$ (in the metric space setting) or $\mathbb{E}[\ell(Y, q)]$ (in the general setting) to grow quadratically in the distance of a test point $q$ to the minimizer $m_x$ (metric) or linearly in $l$ (general). Intuitively, this is fulfilled when the noise distribution is not too similar to a distribution that has nonunique Fréchet means.

In the metric setting, **Variance** is true in Hadamard spaces [Stu03, Proposition 4.4], which are geodesic metric spaces with nonpositive curvature and
include the euclidean spaces. For a variance inequality in spaces of nonnegative curvature, see [ACLGP20] Theorem 3.3.

- **Entropy:**
  Entropy restricts the size of the sets $B_\kappa(\theta^*, l, \delta)$. It can be viewed as a quantitative version of the requirement that these sets are totally bounded.
  We use Talagrand’s $\gamma_2$ [Tal14] (Definition 3 in section B) to formulate the entropy condition. Let $(Q, d)$ be a metric space and $B \subseteq Q$. It holds
  \[ \gamma_2(B, b) \leq \int_0^\infty \sqrt{\log(N(B, b, r))}dr, \]
  where the integral is called entropy integral and
  \[ N(B, b, r) = \min \left\{ k \in \mathbb{N} \mid \exists q_1, \ldots, q_k \in Q: B \subseteq \bigcup_{j=1}^k B(q_j, b, r) \right\} \]
  is the covering number. Thus, we can use bounds on the entropy integral to fulfill Entropy, which is more common in the statistics literature. In some circumstances $\gamma_2$ is strictly lower than the entropy integral [Tal14, Exercise 4.3.11]. One can further weaken the entropy condition as done in [ACLGP20] and [Sch19] at the cost of worse rates of convergence, but it is not clear whether these results are optimal.

- **Moment:**
  We partially follow [Sch19] to prove the theorem. Therein a so-called weak quadruple inequality is assumed to prove rates of convergence for the (generalized) Fréchet mean. We fulfill this condition by the definition of $a_x$, i.e., it holds
  \[ \diamondsuit_x(y, z, \theta, \tilde{\theta}) \leq b(\theta, \tilde{\theta})a_x(y, z). \]
  This inequality can be understood as a generalization of Cauchy-Schwartz inequality: If $\Theta = \mathcal{Y} = \mathbb{R}^k$ and $c_x(y, \theta) = |y - \theta|^2$, then
  \[ \diamondsuit_x(y, z, \theta, \tilde{\theta}) = 2(y - z, \tilde{\theta} - \theta) \leq a_x(y, z)b(\theta, \tilde{\theta}) \]
  if $b(\theta, \tilde{\theta}) = |\tilde{\theta} - \theta|$ and $a_x(y, z) = 2|y - z|$. Moment then is nothing but the condition that the $\kappa$-th moment of the noise distribution is finite.

### 2.3. Corollaries

Next, we apply [Theorem 1] to the metric setting of section 1.1. We will replace Entropy by two conditions that compare the distances of the metric space to the euclidean distance.

#### Assumptions.

- **MetricUp:**
There is $C_{\text{Mup}} \in [1, \infty)$ such that $d(g(x, \theta), g(x, \tilde{\theta})) \leq C_{\text{Mup}} |\theta - \tilde{\theta}|$ for all $x \in \mathcal{X}$, $\theta, \tilde{\theta} \in \Theta$.

- **MetricLo:**
  There are $C_{\text{Res}}, C_{\text{Mlo}} \in [1, \infty)$ such that
  
  \[ \frac{1}{n} \sum_{i=1}^{n} d(g(x_i, \theta), g(x_i, \tilde{\theta}))^2 \geq C_{\text{Mlo}}^{-1} |\theta - \tilde{\theta}|^2 - C_{\text{Res}} n^{-1} \]

  for all $\theta, \tilde{\theta} \in \Theta$.

If we assume that $\mathcal{Q}$ is a Hadamard space, VARIANCE holds and we can set $a_x = d$ in MOMENT.

**Corollary 2** (LinGeo Hadamard). Let $(\mathcal{Q}, d)$ be a Hadamard space. Assume METRICUP, METRICLO, and MOMENT with $a_x = d$. Then

\[ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} d(m_{x_i}, \hat{m}_{x_i})^2 \right] \leq C n^{-1}, \]

where $C = c_d d_\Theta C_{\text{Mlo}} C_{\text{Mup}} C_{\text{Mom}} C_{\text{Res}}$.

In bounded metric spaces, i.e., metric spaces $(\mathcal{Q}, d)$ with $\text{diam}(\mathcal{Q}) < \infty$, MOMENT is trivial, but VARIANCE needs to be assumed.

**Corollary 3** (LinGeo Bounded). Let $(\mathcal{Q}, d)$ be a bounded metric space. Assume METRICUP, METRICLO, and VARIANCE. Then

\[ \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} d(m_{x_i}, \hat{m}_{x_i})^2 \right] \leq C n^{-1} \]

where $C = c d_\Theta C_{\text{Vlo}} C_{\text{Mlo}} C_{\text{Mup}} C_{\text{Res}} \text{diam}(\mathcal{Q})^2$.

### 3. Linear Fréchet Regression

First, we directly apply the Fréchet approach to linear regression, which leads to an estimator, LinFre, introduced in [PM19], that may be inconsistent in nonstandard spaces. Then, with a more relaxed interpretation of the Fréchet approach applied on the sphere, we introduce cosine regression, LinCos.

#### 3.1. Model and Consistency

We use the metric model of section 1.1 with covariates in $[-1, 1]$ (instead of $[0, 1]$). The approach of Fréchet regression, as proposed in [PM19], is to estimate $F_t(q) = \mathbb{E}[d(Y_t, q)^2]$ and then minimize that estimator over $q$ to obtain an estimation $\hat{m}_t$. When applying
this idea to the concept of linear regression, we estimate the function \( t \mapsto F_t(q) \) for each \( q \in \mathcal{Q} \) using a linear regression estimator on the real-valued quantities \( d(y_i, q)^2 \).

As a first step to validate this approach, we apply it to the case \( \mathcal{Q} = \mathbb{R}^{d_y} \) with the euclidean metric. To make this work, we estimate not \( F_t(q, o) = F_t(q) - F_t(o) \) for all \( q \in \mathcal{Q} \) and a fixed element \( o \in \mathcal{Q} \). This is necessary to have a linear objective in the case of a linear model as following calculations show: Assume \( Y_t = \beta_0 + t\beta_1 + \epsilon \), where \( \beta_0, \beta_1 \in \mathbb{R}^{d_y} \), and \( \epsilon \) is a centered random variable in \( \mathbb{R}^{d_y} \) with finite second moment. Denote \( \beta = (\beta_0, \beta_1) \in \mathbb{R}^{d_y \times 2} \) and \( t' = (1, t) \in \mathbb{R}^2 \). Then, for \( o, q \in \mathbb{R}^{d_y} \),

\[
F_t(q) = \mathbb{E}[|Y_t - q|^2] = |\beta t'|^2 + \mathbb{E}[|\epsilon|^2] + |q|^2 - 2(\beta t')^\top(q - o),
\]

\[
F_t(q, o) = |q|^2 - |o|^2 - 2(\beta t')^\top(q - o).
\]

For fixed \( q, o \in \mathcal{Q} \) the function \( t \mapsto F_t(q, o) \) is linear in \( t \), whereas \( t \mapsto F_t(q) \) is quadratic. Note that \( \arg \min_q F_t(q) = \arg \min_q F_t(q, o) \).

With these considerations in mind, we define the linear Fréchet regression estimator \( \hat{m}_t \) in the metric setting as follows:

\[
X_n = \begin{pmatrix} 1 & \ldots & 1 \\ x_1 & \ldots & x_n \end{pmatrix} \in \mathbb{R}^{2 \times n}, \quad \tilde{F}_t(q, o) = \sum_{i=1}^n w(t, x_i) \left( d(y_i, q)^2 - d(y_i, o)^2 \right),
\]

\[
B_n = X_n X_n^\top, \quad \hat{m}_t = \arg \min_{q \in \mathcal{Q}} \tilde{F}_t(q, o),
\]

\[
w(t, x) = t^\top B_n^{-1} x',
\]

where \( o \in \mathcal{Q} \) is an arbitrary fixed element. The empirical objective \( \tilde{F}_t(q, o) \) is the linear regression estimator of \( F_t(q, o) \): Define \( \hat{\theta}(q, o) = \arg \min_{\theta \in \mathbb{R}^{d_y \times 2}} \frac{1}{n} \sum_{i=1}^n (z_i - \theta_0 - \theta_1 x_i)^2 \) with \( z_i = d(y_i, q)^2 - d(y_i, o)^2 \). Then \( \tilde{F}_t(q, o) = \hat{\theta}(q, o)^\top t \). [PM19] showed that, under some conditions, \( \hat{m}_t \) converges to \( \tilde{m}_t \) for \( n \to \infty \), where

\[
B = \int_{-1}^1 x \cdot x^\top dx = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}, \quad \tilde{F}_t(q, o) = \int_{-1}^1 \mathbb{E} \left[ w(t, x) \left( d(Y_x, q)^2 - d(Y_x, o)^2 \right) \right] dx,
\]

\[
w(t, x) = t^\top B_n^{-1} x = \frac{1}{2} + \frac{3}{2} xt, \quad \hat{m}_t = \arg \min_{q \in \mathcal{Q}} \tilde{F}_t(q, o).
\]

The objective \( t \mapsto \tilde{F}_t(q, o) \) is the linear approximation of \( t \mapsto F_t(q, o) \) in the least squares sense. In particular, if \( t \mapsto F_t(q, o) \) is linear for every \( q \in \mathcal{Q} \), then \( \tilde{F} = F \) and \( \hat{m}_t = m_t \).

These considerations yield two ways of extending the idea of a linear model to arbitrary metric spaces.

**Definition 1.**

(i) The distributions of \( Y_x \) for \( x \in [-1, 1] \) follow the strict linear Fréchet regression model, if \( F = F \).
(ii) The distributions of $Y_x$ for $x \in [-1, 1]$ follow the relaxed linear Fréchet regression model, if $\bar{m} = m$.

As the idea of Fréchet regression is to estimate $\bar{F}$, this function being the true objective would give rise to a meaningful model, in which the linear Fréchet regression estimator is consistent due to the aforementioned results in [PM19]. But in the end, we are only interested in the minimizers. Thus, it is sufficient (and necessary) to have $\bar{m} = m$ for a model, so that the linear Fréchet regression estimator is consistent.

To further affirm that these model assumptions are reasonable, the following proposition shows that they generalize the euclidean linear model assumptions. As calculated above $F_t(q, o)$ is linear in $t$ for euclidean spaces. As $\bar{F}$ is the linear approximation of $F$, we have $F = \bar{F}$.

**Proposition 1.** Any euclidean linear model, $Y_x = \beta_0 + \beta_1 x + \epsilon$, $x \in \mathbb{R}$, $\beta_0, \beta_1 \in \mathbb{R}^d$, $\mathbb{E}[\epsilon] = 0 \in \mathbb{R}^d$, follows the strict linear Fréchet regression model for the space $(Q, d) = (\mathbb{R}^d, |\cdot|)$.

It is not clear how to write a generalization of a model equation like $Y_x = \beta_0 + \beta_1 x + \epsilon$ in arbitrary metric spaces (except on Riemannian manifolds with an exponential map at hand, where $Y_x = \text{Exp}(m_x, \epsilon)$, $m_x = \text{Exp}(\beta_0, x \beta_1)$ seems to be meaningful). But there are some elements of the model that should reasonably be included: If the metric space is a geodesic space, we should demand that a meaningful regression estimator is consistent at least in the no-noise settings $Y_t = \gamma_t$ for global geodesics $\gamma : [-1, 1] \rightarrow Q, t \mapsto \gamma_t$, as this is the simplest distribution with a non-constant regression function. Furthermore, geodesics in euclidean spaces are linear functions, which can be estimated by linear Fréchet regression, as linear Fréchet regression is equivalent to linear regression in euclidean spaces.

Unfortunately, Hilbert spaces are essentially the only spaces where this no-noise setting fulfills the strict linear Fréchet regression model, as the following theorem shows.

**Theorem 2 (LinFre inconsistency).** Let $(Q, d)$ be a nonempty geodesic space. It is also a Hilbert space if and only if it is geodesically complete (see appendix C), and for each minimizing geodesic $\gamma_t : [-1, 1] \rightarrow Q$, $t \mapsto \gamma_t$, the strict linear Fréchet regression model holds for the no-noise setting $Y_t = \gamma_t$.

**Theorem 2** shows that it does not make sense to assume the strict linear Fréchet regression model in noneuclidean spaces. It is not clear to the author, whether a statement similar to **Theorem 2** holds for the relaxed model. But simulations in appendix 8 indicate inconsistency of linear Fréchet regression on the sphere.
3.2. Parametric Cosine Regression

One important aspect of linear regression is that the regression function has a simple parametric form. The idea behind Fréchet regression is to apply regression not directly to the regression function $t \mapsto m_t$, but to the objective function $t \mapsto F_t(q)$. Thus, for a generalization of parametric regression in the sense of Fréchet regression, we would want to target function $t \mapsto F_t(q)$ to have a simple parametric form.

On the sphere $S^2$, instead of minimizing the squared distance, we will maximize the cosine of the distance (or minimize the hyperbolic cosine in the hyperbolic plane $H^2$), which will yield a simple parametric form of the objective function. It holds

$$\cos(x) = 1 - \frac{1}{2} x^2 + O(x^4), \quad \cosh(x) = 1 + \frac{1}{2} x^2 + O(x^4).$$

Thus, minimizing $E[Y^2]$ is closely related to maximizing $E[\cos(Y,q)]$ or minimizing $E[\cosh(Y,q)]$. Furthermore, using the cosine on $S^2$ (or hyperbolic cosine in $H^2$) seems appealing as laws of cosines hold in $S^2$ and $H^2$ analogously to the euclidean space: In a triangle with side lengths $a, b, c$ and angle $\alpha$ opposing side $c$, it holds, in the respective space with intrinsic metric,

- Euclidean:
  $$c^2 = a^2 + b^2 - 2ab \cos(\alpha)$$
- Spherical:
  $$\cos(c) = \cos(a) \cos(b) + \sin(a) \sin(b) \cos(\alpha)$$
- Hyperbolic:
  $$\cosh(c) = \cosh(a) \cosh(b) - \sinh(a) \sinh(b) \cos(\alpha)$$

We will only discuss $Q = S^2$ with intrinsic metric $d$. Similar considerations are valid for the hyperbolic space. In our new model, we replace the Fréchet mean, $\arg \min_q E[Y^2(q)]$, of a random variable $Y$ in $S^2$ by the cosine mean, $\arg \max_q E[\cos(Y,q)]$. For distributions with enough symmetry, the cosine mean can be characterized easily.

**Proposition 2.** Let $Y$ be a random variable with values in $S^2 = \{(\vartheta, \phi) \in [-\frac{\pi}{2}, \frac{\pi}{2}] \times [0, 2\pi)\}$ such that its distribution is symmetric with respect to rotation around one axis, without loss of generality the axis connecting $(-\frac{\pi}{2}, 0)$ and $(\frac{\pi}{2}, 0)$. The distribution of $Y$ is given by $P(Y \in B) = \frac{1}{2\pi} \int_0^{2\pi} \mathbb{1}_B(\vartheta, \varphi) d\varphi d\nu(\varphi)$ for all measurable $B \subseteq S^2$, where $\nu$ is a probability measure on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Let $A = \int \sin(\vartheta) d\nu(\vartheta)$. Let $M = \arg \max_{q \in S^2} E[\cos(Y,q)]$ the set of cosine means. Then it holds,

- $A < 0$ if and only if $M = \{(-\frac{\pi}{2}, 0)\}$,
- $A = 0$ if and only if $M = S^2$,
- $A > 0$ if and only if $M = \{(\frac{\pi}{2}, 0)\}$.

As expected from a mean-value, the cosine mean of a symmetric distribution is its center. If the regression function is equal to a geodesic, the objective function has a simple parametric form.
Proposition 3. Let $\gamma: \mathbb{R} \to S^2$, $s \mapsto \gamma_s$ be a unit-speed geodesic. Assume that $m_t = \arg\max_{q \in S^2} \mathbb{E}[\cos(Y_t,q)]$ is unique for all $t \in [0,1]$. Assume $m_t = \gamma_{t_0 + t}$ with $t_0 \in [0,2\pi)$ and $\lambda \in (0,\infty)$. For $q \in S^2$, let $s_q \in [0,2\pi)$ be such that $\min_s q, \gamma_s = q, \gamma_{s_q}$. Then

$$\mathbb{E}[\cos(Y_t,q)] = A_q \cos(B_q + \lambda t) = a_q \cos(\lambda t) + b_q \sin(\lambda t),$$

where

$$A_q = \mathbb{E}[\cos(Y_t,m_t)] \cos(\gamma_{s_q}, q), \quad a_q = A_q \cos(B_q), \quad b_q = -A_q \sin(B_q).$$

Proposition 3 shows that following model is appropriate for estimating geodesics on the sphere. For $t \in [0,1]$, let $Y_t$ be a $S^2$-valued random variable. Let the regression function $m: [0,1] \to S^2$ be a maximizer $m_t \in \arg\max_{q \in S^2} \mathbb{E}[\cos(Y_t,q)]$. Let $\Lambda \in (0,\infty)$ and assume that $t \mapsto m_t$ is a geodesic with speed bounded by $\Lambda$. Assume that $\mathbb{E}[\cos(Y_t,m_t)]$ does not depend on $t$. Let $x_i = \frac{i}{n}$ and let $(y_i)_{i=1,...,n}$ be independent random variables with values in $S^2$ such that $y_i$ has the same distribution as $Y_{x_i}$.

Set $z_{q,i} = \cos(\gamma_{s_q},q)$ and define the least squares estimators

$$(\hat{a}_{q,\lambda}, \hat{b}_{q,\lambda}) \in \arg\min_{a,b \in [-1,1]} \frac{1}{n} \sum_{i=1}^{n} (z_{q,i} - a \cos(\lambda x_i) - b \sin(\lambda x_i))^2,$$

$$\hat{\lambda} \in \arg\min_{\lambda \in [0,\Lambda]} \int_{S^2} \frac{1}{n} \sum_{i=1}^{n} (z_{q,i} - \hat{a}_{q,\lambda} \cos(\lambda x_i) - \hat{b}_{q,\lambda} \sin(\lambda x_i))^2 \, d\mu(q),$$

where $\mu$ is the Hausdorff measure on $S^2$ (for an implementation it is enough to use a three points measure $\mu = \frac{1}{3} (\delta_{q_1} + \delta_{q_2} + \delta_{q_3})$ with $q_1, q_2, q_3 \in S^2$ not on the same geodesic).

Now set $\hat{a}_q = \hat{a}_{q,\lambda}, \hat{b}_q = \hat{b}_{q,\lambda}$ and $\hat{F}_t(q) = \hat{a}_q \cos(\lambda t) + \hat{b}_q \sin(\lambda t)$. The LinCos-estimator for $m_t$ is $\hat{m}_t \in \arg\max_{q \in \Omega} \hat{F}_t(q)$.

We do not investigate LinCos deeply, as it mainly serves to illustrate the comparison of LinFre to LinGeo. Moreover, it does not fit into the scheme of this article, as we want to compare general regression methods which are not limited to one specific metric space and LinCos is only applicable in $S^2$ (with possible extensions to hyper-spheres and hyperbolic spaces). But note that, for fixed $q$, the estimation of $a_q$, $b_q$, and $\lambda$ is well-studied in the literature on sinusoidal regression, see e.g., [NK13].

4. Local Geodesic Regression

We apply the geodesic approach to the classical local linear estimator and arrive at a new procedure, local geodesic regression or LocGeo.
4.1. Hypersphere

In the hypersphere setting of section 1.1 let \( \Theta \subseteq T\mathbb{S}^k \) be the subset of the tangent bundle with \(|v| < \pi\) for all \((q, v) \in \Theta\). This set parameterizes a set of geodesics \( x \mapsto \text{Exp}(q, xv) \).

We investigate an estimator that locally fits geodesics. Let \( h \geq \frac{2}{n} \). Let \( K : \mathbb{R} \rightarrow \mathbb{R} \) be a function, the kernel, such that \( C_{\text{Ker}} \leq K(x) \leq C_{\text{Ker}} \) for a constant \( C_{\text{Ker}} \geq 1 \) \((\text{KERNEL condition})\). For \( t \in [0,1] \), define \( w_h(t, x) = \frac{1}{h} K\left( \frac{x-t}{h} \right) \) and \( w_j = \sum_{i=1}^n w_h(t, x_j) \). Let \((\hat{m}_t, \hat{v}_t) \in \arg \min_{(p,v) \in \Theta} \sum_{i=1}^n w_i d(y_i, \text{Exp}(p, \frac{x_i-t}{h} v)) \).

To be able to estimate \( m_t \), it must fulfill a Hölder-type SMOOTHNESS condition: Assume there are \( \beta > 0 \) and \( C_{\text{Smo}} \in [1, \infty) \) such that \( d(m_{x,\text{Exp}(m_t, (x-t)\hat{m}_t)}) \leq C_{\text{Smo}} |x-t|^{\beta} \) for all \( x \in [t-h, t+h] \cap [0,1], t \in [0,1], \) where \( \hat{m}_t \in T_{m_t}\mathbb{S}^k \) is the derivative of \( m_t \).

Furthermore, we again assume a VARIANCE condition: There is \( C_{\text{Vlo}} \geq 1 \) such that \( C_{\text{Vlo}}^{-1} d(m_t, q)^2 \leq \mathbb{E}[d(Y_t, q)^2 - d(Y_t, m_t)^2] \) for all \( t \in [-1, 1] \) and \( q \in \mathbb{S}^k \).

\[ \text{Corollary 4 (LocGeo Hypersphere). Assume VARIANCE, KERNEL and SMOOTHNESS. Choose } h = n^{-\frac{1}{3+\beta}}. \text{ Then } \mathbb{E}[d(m_t, \hat{m}_t)^2] \leq Cn^{-\frac{2\beta}{3+\beta}} \]

for all \( t \in [0,1], n \geq 2, \) where \( C = ckC_{\text{Ker}}^3 C_{\text{Vlo}}^2 C_{\text{Smo}}^2 \).

4.2. General

We prove the main theorem of this section in the metric setting of section 1.1 instead of the general setting.

We investigate an estimator that locally fits (generalized) geodesics of the form \( x \mapsto g(x, \theta) \), where \( \theta \in \Theta \) parametrizes geodesics: Let \( h \geq \frac{2}{n} \). Let \( K : \mathbb{R} \rightarrow \mathbb{R} \) be a function, the kernel. For \( t \in [0,1] \), define \( w_h(t, x) = \frac{1}{h} K\left( \frac{x-t}{h} \right) \) and \( w_i = \sum_{j=1}^n w_h(t, x_j) \). Note that \( w_i \) depends on \( n, t, h, K \), which is not indicated in the notation. Let \( \Theta \) be a set. Let \( g : \mathbb{R} \times \Theta \rightarrow \mathcal{Q} \) be a function, the link function. Define \( g_i(\theta) = g\left( \frac{x_i-t}{h}, \theta \right) \). Let \( \hat{\theta}_{t,h} \in \arg \min_{\theta \in \Theta} \sum_{i=1}^n w_i d(y_i, g_i(\theta))^2 \) and \( \hat{m}_t = g(0, \hat{\theta}_{t,h}) \).

We show that this estimator attains the classical nonparametric rate of convergence. To formulate the assumptions for this theorem, we first need to define two \((\text{semi})\)-metrics on \( \mathcal{Q} \): For \( \theta, \hat{\theta} \in \Theta \), define \( b(\theta, \hat{\theta}) = \sup_{x \in [-1,1]} d(g(x, \theta), g(x, \hat{\theta})) \). For \( y, z \in \mathcal{Q} \), define \( a(y, z) = \sup_{q, p \in \mathcal{Q}, q \neq p} \frac{d(g(y, q) - g(z, p), g(y, p) - g(z, q))}{d(q, p)} \).

Assumptions.

- **Lipschitz**: There is \( C_{\text{Lip}} \in [1, \infty) \) such that \([-1,1] \rightarrow \mathbb{R}, x \mapsto d\left(g(x, \theta), g(x, \hat{\theta})\right)^2 \) is Lipschitz continuous with constant \( C_{\text{Lip}} \) for all \( \theta, \hat{\theta} \in \Theta \).
• **Kernel:** There are $C_{Kmi}, C_{Kma} \in [1, \infty)$ such that
  \[
  C_{Kmi}^{-1} 1_{[-\frac{1}{2}, \frac{1}{2}]}(x) \leq K(x) \leq C_{Kma} 1_{[-1,1]}(x)
  \]
  for all $x \in \mathbb{R}$.

• **Smoothness:** Let $\beta > 0$. There is $C_{\text{Smo}} \in [1, \infty)$ such that $t \mapsto m_t$ belongs to the generalized Hölder class with parameters $\beta$ and $C_{\text{Smo}}$, i.e., there is $\theta_{t,h} \in \Theta$ such that $d(m_x, g(\frac{x-t}{h}, \theta_{t,h})) \leq C_{\text{Smo}} |x-t|^{\beta}$ for all $x \in [t-h, t+h] \cap [0, 1]$, $t \in [0, 1]$.

• **Variance:** There is $C_{\text{Vlo}} \in [1, \infty)$ such that $C_{\text{Vlo}}^{-1} d(q, m_t)^2 \leq E[d(Y_t, q)^2 - d(Y_t, m_t)^2]$ for all $q \in \mathcal{Q}$ and $t \in [0, 1]$.

• **R–Variance:** There is $C_{\text{Vup}} \in [1, \infty)$ such that $E[d(Y_t, q)^2 - d(Y_t, m_t)^2] \leq C_{\text{Vup}} d(q, m_t)^2$ for all $q \in \mathcal{Q}$ and $t \in [0, 1]$.

• **Moment:** Let $\kappa > 2$. There is $C_{\text{Mom}} \in [1, \infty)$ such that $E[a(Y_t, m_t)^\kappa]^{\frac{1}{\kappa}} \leq C_{\text{Mom}}$ for all $t \in [0, 1]$.

• **Entropy:** For $\theta_0 \in \Theta$ and $\delta > 0$, let
  \[
  B_\delta(\theta_0) = \left\{ \theta \in \Theta : \int_{-\frac{1}{2}}^{\frac{1}{2}} d(g(x, \theta), g(x, \theta_0))^2 \, dx \leq \delta \right\}.
  \]
  There is $C_{\text{Ent}} \in [1, \infty)$ such that $\gamma_2(B_\delta(\theta_0), b) \leq C_{\text{Ent}} \delta^{\frac{1}{2}}$ for all $\delta > 0$ and all $\theta_0 \in \Theta$.

**Theorem 3 (LocGeo General).** Assume **Variance**, **Smoothness**, **R–Variance**, **Moment**, **Kernel**, **Entropy**, and **Lipschitz**. Then, for all $n \in \mathbb{N}$, $h \geq \frac{2}{n}$, and $t \in [0, 1]$, it holds
\[
E \left[ \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(g(x, \hat{\theta}_{t,h}), g(x, \theta_{t,h})\right)^2 \, dx \right] \leq C_1 (nh)^{-1} + C_2 h^{2\beta},
\]
where
\[
C_1 = c_\kappa C_{\text{Ent}}^3 C_{Kmi}^3 C_{Kma}^3 C_{\text{Lip}}^2 C_{\text{Mom}}^2 C_{\text{Vlo}}^2 C_{\text{Vup}}^2
\]
\[
C_2 = c'_\kappa C_{Kma}^2 C_{Kmi}^2 C_{\text{Lip}}^2 C_{\text{Smo}}^3 C_{\text{Vlo}} C_{\text{Vup}}.
\]

We essentially obtain the classical bound of a squared bias term $h^{2\beta}$ and a variance term $(nh)^{-1}$, which yield the usual nonparametric rate of convergence for an appropriate choice of $h$. 17
Remark 2.

- **Lipschitz:** In Euclidean spaces Lipschitz bounds the slope of linear functions for the local fit. This is not a restrictive requirement as for increasing number of data points, the fit is done on an increasingly stretched version of the function, which has a lower and lower absolute slope. Thus, for every finite slope, we eventually meet this requirement.

- **Kernel:** This is a typical condition on kernels for local kernel regression, see also [Tsy08, Lemma 1.5]. It is fulfilled e.g., by the rectangular kernel $1_{[-\frac{1}{2}, \frac{1}{2}]}(x)$ or the Epanechnikov kernel $\frac{2}{3}(1-x^2)1_{[-1,1]}(x)$. Kernel likely could be weakened to allow for a greater variety of kernels.

- **Smoothness:** Smoothness can be understood as a Hölder-smoothness condition. It bound the residual of the first order approximation of $m$ at $t$, i.e., the approximation of $x \mapsto m_x$ by a generalized geodesic $x \mapsto g((x-t)/h, \theta)$ for $x$ close to $t$.

- **R-Variance:** Together with Variance, reverse variance inequality R-Variance requires $\mathbb{E}[d(Y_t, q)^2 - d(Y_t, m_t)^2]$ to behave like $d(q, m_t)^2$ up to constants. [GPRS19, Theorem 8] introduce a variance equality, from which both inequalities may be deduced. R-Variance always holds in proper Alexandrov spaces of non-negative curvature with $C_{\text{Vup}} = 1$ [Oht12, Theorem 5.2].

For a discussion of Variance, Moment, Entropy see Remark 1 in section 2.

4.3. Corollaries

Next, we apply [Theorem 3] to the metric setting of section 1.1. We, first need to make further assumptions to be able to relate the bound on the integral of the parameters $\theta_{t,h}$ and $\hat{\theta}_{t,h}$ to the distance of the true and estimated regression function $m_t$ and $\hat{m}_t$.

Assumptions.

- **Connection:** There is $C_{\text{Con}} \in [1, \infty)$ such that

$$d\left( g(0, \theta), g(0, \hat{\theta}) \right)^2 \leq C_{\text{Con}} \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left( g(x, \theta), g(x, \hat{\theta}) \right)^2 dx$$

for all $\theta, \hat{\theta} \in \Theta$. 

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• **Convexity**: The function \( x \mapsto d\left(g(x, \theta), g(x, \tilde{\theta})\right)^2 \) is convex for all \( \theta, \tilde{\theta} \in \Theta \).

The theorem bounds \( E\left[\frac{1}{2} d(g(x, \hat{\theta}_t,h), g(x, \theta_t,h))^2\right] \). Note that \( g(0, \theta_t,h) = m_t \) due to Smoothness. To obtain a bound on \( E[d(\hat{m}_t, m_t)^2] \), we may require Connection. Connection with \( C_{\text{Con}} = 1 \) is implied by Convexity due to Jensen’s inequality. Convexity is true in Hadamard spaces (including the Euclidean spaces). [Oht12, Theorem 5.2] implies that in proper Alexandrov spaces of nonnegative curvature, \( \text{R–Variance} \) holds with \( C_{\text{Vup}} = 1 \). Of course, Moment is trivial in bounded spaces.

**Corollary 5** (LocGeo Bounded). Let \((Q,d)\) be a bounded proper Alexandrov space of nonnegative curvature. Assume Variance, Smoothness, Kernel, Entropy, Lipschitz, Connection. Choose \( h = n^{-\frac{1}{2\beta+1}} \). Then

\[
E[d(m_t, \hat{m}_t)^2] \leq Cn^{-\frac{2\beta}{2\beta+1}}
\]

for all \( t \in [0,1], n \geq 2^{\frac{2\beta}{2\beta+1}} \), where \( C = c_\alpha C_{\text{Con}} C_{\text{Ent}}^2 C_{\text{Kmi}}^3 C_{\text{Kma}}^3 C_{\text{Vlo}}^2 C_{\text{Vup}}^2 \).

In Hadamard spaces Variance and Convexity always hold, see [Stu03, Proposition 4.4, Corollary 2.5].

**Corollary 6** (LocGeo Hadamard). Let \((Q,d)\) be a Hadamard space and \( g \) such that \( x \mapsto g(x, \theta) \) is a geodesic. Assume Smoothness, R–Variance, Moment, Kernel, Entropy, Lipschitz. Choose \( h = n^{-\frac{1}{2\beta+1}} \). Then

\[
E[d(m_t, \hat{m}_t)^2] \leq Cn^{-\frac{2\beta}{2\beta+1}}
\]

for all \( t \in [0,1], n \geq 2^{\frac{2\beta}{2\beta+1}} \), where \( C = c_\alpha C_{\text{Ent}}^2 C_{\text{Kmi}}^3 C_{\text{Kma}}^3 C_{\text{Vlo}}^2 C_{\text{Vup}}^2 \).

### 5. Local Fréchet Regression

We use the principles of Fréchet regression on local polynomial regression. In particular, this yields local linear Fréchet regression, LocFre, introduced in [PM19].

#### 5.1. Hypersphere

We use the hypersphere setting of section 1.1. Let \( K: \mathbb{R} \to \mathbb{R} \) be a function, the kernel, such that \( C_{\text{Ker}}^3 \mathbb{I}_{[-\frac{1}{2}, \frac{1}{2}]}(x) \leq K(x) \leq C_{\text{Ker}} \mathbb{I}_{[-1,1]}(x) \) for a constant \( C_{\text{Ker}} \geq 1 \) (Kernel
condition). For \( h > 0 \) define \( K_h(x) = \frac{1}{h}K(x/h) \), \( a_{h,k}(t) = \sum_{j=1}^n (x_j - t)^k K_h(x_j - t) \) and
\[
\begin{align*}
    w_{h,i}(t) &= \frac{a_{h,2}(t) - (x_i - t)a_{h,1}(t)}{a_{h,0}(t)a_{h,2}(t) - a_{h,1}(t)^2} K_h(x_i - t),
\end{align*}
\]
whenever the denominator is not 0; in the other case, set \( w_i = 0 \). The local linear Fréchet regression estimator is \( \hat{m}_t \in \arg\min_{q \in \mathbb{Q}} \sum_{i=1}^n w_{h,i}(t) Y_t q^2 \), where \( q = d(q,p) \).

We need a smoothness assumption to be able to estimate \( m \): For \( a > 0 \), define \( \lfloor a \rfloor \) as the largest integer strictly smaller than \( a \). The Hölder class \( \Sigma(\beta,L) \) for \( \beta, L > 0 \) is defined as the set of \( \lfloor \beta \rfloor \)-times continuously differentiable functions \( f : [0,1] \rightarrow \mathbb{R} \) with
\[
    |f^{(\lfloor \beta \rfloor)}(t) - f^{(\lfloor \beta \rfloor)}(x)| \leq L |x - t|^\beta
\]
for all \( x, t \in [0,1] \). Let \( \mu \) be a the measure of the uniform distribution on \( S^k \). Assume that for all \( t \in [0,1] \), the random variable \( Y_t \) has a density \( y \mapsto \rho(y|t) \) with respect to \( \mu \). Let \( \beta \in (1,2] \). Assume, there is \( C_{\text{SmD}} \geq 1 \), such that for \( \mu \)-almost all \( y \in S^k \), \( t \mapsto \rho(y|t) \in \Sigma(\beta,C_{\text{SmD}}) \) (SmoothDensity). Furthermore, we assume VARIANCE: There is \( C_{\text{Vlo}} \in [1,\infty) \) such that \( C_{\text{Vlo}}^{-1} \mathbb{E}m_t^2 \leq \mathbb{E}Y_t q^2 - \mathbb{E}Y_t m_t^2 \) for all \( q \in S^k \) and \( t \in [0,1] \).

**Corollary 7 (LocFre Hypersphere).** Assume VARIANCE, SmoothDensity, Kernel. Let \( n \geq n_0 \) for a universal constant \( n_0 \) and set \( h = n^{-\frac{1}{2\beta+1}} \). Then
\[
    \mathbb{E}m_t^2 \leq Cn^{-\frac{2\beta}{2\beta+1}}
\]
for all \( t \in [0,1] \), where \( C = ck (C_{\text{Vlo}} C_{\text{Ker}} C_{\text{SmD}})^2 \).

We obtain the usual nonparametric rate of convergence.

### 5.2. General

The general theorem of this section uses the general setting of \([11]\) but with a specific loss function: Let \( d \) be a metric on \( \mathbb{Q} \). Let \( \alpha > 1 \). We use \( d^\alpha \) as loss function. Define
\[
    \Diamond(y,z,q,p) = c(y,q) - c(y,p) - c(z,q) + c(z,p)
\]
and \( a(y,z) = \sup_{q \in \mathbb{Q}, q \neq p} \Diamond(y,z,q,p) d(q,p)^{-1} \). Let \( K : \mathbb{R} \rightarrow \mathbb{R} \) be a function. For \( \ell \in \mathbb{N}_0 \), \( h = h_n > 0 \), and \( x, t \in [0,1] \) define
\[
    \begin{align*}
        \Psi(x) &= \left( \frac{x^k}{k!} \right)_{k=0,\ldots,\ell}, \\
        B_{n,t} &= \frac{1}{nh} \sum_{i=1}^n \Psi\left( \frac{x_i - t}{h} \right) \Psi\left( \frac{x_i - t}{h} \right)^\top K\left( \frac{x_i - t}{h} \right), \\
        w_i &= \Psi(0)^\top B_{n,t}^{-1} \Psi\left( \frac{x_i - t}{h} \right) K\left( \frac{x_i - t}{h} \right),
    \end{align*}
\]
whenever \( B_{n,t} \) is invertible. Note that \( w_i \) depends on \( n, t, h, K \). A local polynomial Fréchet estimator of order \( \ell \) is any element
\[
    \hat{m}_t \in \arg\min_{q \in \mathbb{Q}} \sum_{i=1}^n w_i c(y_i, q).
\]
Assumptions.

- **Moment**: There are $\kappa \geq 2$ and $C_{\text{Mom}} \in [1, \infty)$ such that $E[a(Y_t, m_t) \kappa]^{\frac{1}{\kappa}} \leq C_{\text{Mom}}$ for all $x \in [0, 1]$.

- **Smoothness**: Let $\beta > 0$. For all $q, p \in Q$ there is $L(q, p) > 0$ such that $t \mapsto E[c(Y_t, q) - c(Y_t, p)] \in \Sigma(\beta, q, p, L(q, p))$. There is $C_{\text{Smo}} \in [1, \infty)$ such that $E[L(m_t, \hat{m}_t)^\kappa]^{\frac{1}{\kappa}} \leq C_{\text{Smo}}$ for all $t \in [0, 1]$.

- **Kernel**: There is $C_{\text{Ker}} \in [1, \infty)$ such that $K(x) \leq C_{\text{Ker}} I_{[-1, 1]}(x)$ for all $x \in \mathbb{R}$. There are $n_0 \in \mathbb{N}, \lambda_0 \in (0, \infty)$ such that $\lambda_{\text{min}}(B_{n, t}) > \lambda_0$ for all $x \in [-1, 1], n \geq n_0$ and the given choice of $h = h_n$, where $\lambda_{\text{min}}(B_{n, t})$ is the smallest eigenvalue of $B_{n, t}$. The constants $n_0$ and $\lambda_0$ give rise to a constant $C_{\text{Ker}} \in [1, \infty)$, see Lemma 13.

- **Variance**: There is $C_{\text{Vlo}} \in [1, \infty)$ such that $C^{-1} Vio q, m_t \alpha \leq E[c(Y_t, q) - c(Y_t, m_t)]$ for all $q \in Q$ and $t \in [0, 1]$.

- **Entropy**: There is $C_{\text{Ent}} \in [1, \infty)$ such that $\gamma_2(B, d) \leq C_{\text{Ent}} \text{diam}(B)$ for all $B \subseteq Q$.

---

**Theorem 4 (LocFre General).** Assume Smoothness, Kernel, Variance, Entropy, Moment and $\kappa > \frac{\alpha}{\alpha - 1}$. Let $\ell = [\beta]$. Then, for $t \in [0, 1]$ and $n \geq n_0$, the local polynomial Fréchet estimator $\hat{m}_t$ of order $\ell$ fulfills,

$$E[m_t, \hat{m}_t \alpha] \leq \left(C_1 h^\beta + C_2 (nh)^{-\frac{1}{2}} \right)^{\alpha-1},$$

where $C_1 = c_{\kappa, \alpha} C_{\text{Vlo}} C_{\text{Ker}} C_{\text{Smo}}$ and $C_2 = c_{\kappa, \alpha} C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}}$.

For $\alpha = 2$, we obtain the classical error bound for local polynomial estimators with a bias term $h^\beta$ and a variance term $(nh)^{-\frac{1}{2}}$. The theorem does not necessarily give bounds for different powers $\alpha$ of the distance between estimator and true value, but possibly only for one specific $\alpha$, which is determined by Variance.

Smoothness and Kernel are classical conditions for local polynomial estimators [Tsy08, Proposition 1.13]. Variance, Entropy, Moment are conditions needed to ensure the rate of convergence for a generalized Fréchet mean, see [Sch19, Theorem 1]. For a discussion see Remark 1 in section 2.

**Remark 3.**

- **Smoothness**: In this theorem, we have to insert a loose bound $E[L(m_t, \hat{m}_t)^\kappa]^{\frac{1}{\kappa}} \leq C_{\text{Smo}} < \infty$.
independent of \( n \) and \( h \) to obtain a bound on \( E\left[ m_t, m_t^\alpha \right] \) that vanishes for \( n \to \infty \) and \( h = h_n \) chosen appropriately. In the corollaries below, we see that this is not too difficult to fulfill.

In euclidean spaces with \( c = d^2 \), where \( d \) is the euclidean metric, we have \( E[\epsilon(Y_t, q) - \epsilon(Y_t, p)] = -2\langle m(t), q - p \rangle + \|q\|^2 - \|p\|^2 \) and Smoothness is equivalent to \( m \in \Sigma(L, \beta) \) with \( L(q, p) = 2L \).

**Kernel:**

Kernel is fulfilled for \( C^{-1}_{Km} I_{[-\frac{1}{2}, \frac{1}{2}]}(x) \leq K(x) \leq C_{Kma} I_{[-\frac{1}{2}, \frac{1}{2}]}(x) \), appropriately chosen \( h_n \), and \( n \) large enough, see [Tsy08, Lemma 1.5].

5.3. Corollaries

Next, we apply Theorem 4 to the metric setting of section 1.1. Smoothness can be replaced by SmoothDensity, see Lemma 14 in the appendix, and BiasMoment. To fulfill BiasMoment we can assume BomBound.

**Assumptions.**

- **SmoothDensity:**
  Let \( \mu \) be a probability measure on \( Q \) with \( \int \overline{\rho} \overline{\rho} \mu(dy) < \infty \) for an arbitrary \( o \in Q \). Let \( \beta > 0 \) with \( \ell = \lfloor \beta \rfloor \). For \( \mu \)-almost all \( y \in Q \), there is \( L(y) \geq 0 \) such that \( t \mapsto \rho(y|t) \in \Sigma(\beta, L(y)) \). Furthermore there is a constant \( C_{SmD} > 0 \),
  \[ \int L(y)^2 d\mu(y) \leq C_{SmD}^2 \]

- **BiasMoment:**
  Define \( H(q, p) = \left( \int (\overline{\rho} + \overline{\rho})^2 \mu(dy) \right)^\frac{1}{2} \). There is \( C_{Bom} \in [1, \infty) \) such that \( E[H(\hat{m}_t, m_t)^\frac{1}{2}] \leq C_{Bom} \) for all \( t \in [0, 1] \).

- **BomBound:**
  There are \( C_{Int}, C_{Len} \in [1, \infty) \) such that
  \[ \int \overline{\rho} \overline{\rho} m_t^2 \mu(dy) \leq C_{Int}^2, \quad a(m_t, m_s) \leq C_{Len} \]
  for all \( s, t \in [0, 1] \).

**Corollary 8** (LocFre Bounded). Let \((Q, d)\) be a bounded metric space. Let \( \beta > 0 \) with \( \ell = \lfloor \beta \rfloor \). Let \( \hat{m}_t \) be the local polynomial estimator of order \( \ell \). Assume
Variance, Entropy, SmoothDensity, Kernel. Set $h = n^{-\frac{1}{2\beta+1}}$. Then

$$\mathbb{E}[\hat{m}_t, \hat{m}_t] \leq C n^{-\frac{2\beta}{2\beta+1}}$$

for all $t \in [0,1]$, where $C = c (\text{diam}(Q, d) C_{\text{Vlo}} C_{\text{Ker}} C_{\text{SmD}} C_{\text{Ent}})^2$.

Variance is always true in Hadamard spaces.

**Corollary 9 (LocFre Hadamard).** Let $(Q, d)$ be a Hadamard space. Let $\beta > 0$ with $\ell = \lfloor \beta \rfloor$. Let $\hat{m}_t$ be the local polynomial estimator of order $\ell$. Let $\kappa > 2$. Assume Moment, Entropy, SmoothDensity, BomBound, Kernel. Set $h = n^{-\frac{1}{2\beta+1}}$. Then

$$\mathbb{E}[\hat{m}_t, \hat{m}_t] \leq C n^{-\frac{2\beta}{2\beta+1}}$$

for all $t \in [0,1]$, where $C = c_\kappa (C_{\text{Ker}} C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} C_{\text{SmD}} C_{\text{Ent}})^2$.

**Remark 4.**

- **BomBound:**
  We require the length of $[0,1] \to Q, t \mapsto m_t$ to be finite, measured with respect the measure $\mu$ and with respect to the pseudo metric $a$. This is a mild condition and should be fulfilled for smooth functions $m$. See [Proposition 6](appendix A.4.3) for a result on Lipschitz continuity of the regression function.

- **BiasMoment:**
  This is a technical condition that we also use as an intermediate step to prove corollaries in metric spaces for LocFre. It is fulfilled in bounded metric spaces and can also be replaced by BomBound.

- **SmoothDensity:**
  If the noise distribution has a $\mu$-density and this density is smooth enough, SmoothDensity can be interpreted as a smoothness condition on $t \mapsto m_t$.
  In an euclidean space $Q = \mathbb{R}^k$ with a location model $\rho(y|t) = \rho((y - m(t))^2)$, we have $\partial_t \rho(y|t) = -2(y - t) \dot{m}(t) \rho'(y|t)$. Informally, the density should be as least as smooth as the regression function, to view this condition as a typical smoothness assumption on the regression function. It is likely an artifact of the proof that we require the error density to be smooth.

6. **Trigonometric Geodesic Regression**

We apply the principles of geodesic regression to transfer the euclidean trigonometric series estimator to a new method, TriGeo, for nonstandard spaces.
Let \((\psi_\ell)_{\ell \in \mathbb{N}}\) be the trigonometric basis of \(L^2[0,1]\), i.e., for \(x \in [0,1]\), \(k \in \mathbb{N}\),
\[
\psi_1(x) = 1, \quad \psi_{2k}(x) = \sqrt{2} \cos(2\pi kx), \quad \psi_{2k+1}(x) = \sqrt{2} \sin(2\pi kx).
\]
The trigonometric basis is orthonormal, i.e.,
\[
\int_0^1 \psi_k(x) \psi_\ell(x) dx = \delta_{k\ell}
\]
for all \(\ell, k \in \mathbb{N}\), where \(\delta_{k\ell}\) is the Kronecker delta.

In the metric space setting of section 1.1 with the assumption of the existence of an exponential map \(\text{Exp}(p, \cdot)\), the resulting method is \text{TriGeo}:
\[
\hat{p}, \hat{v}_1, \ldots, \hat{v}_N = \arg\min_{p \in \mathcal{Q}, \psi \in T_p \mathcal{Q}} d\left(\text{Exp}\left(p, \sum_{\ell=1}^N \psi_\ell(x) \hat{v}_\ell\right), y_i\right)^2,
\]
\[
\hat{m}(t) = \text{Exp}\left(\hat{p}, \sum_{\ell=1}^N \psi_\ell(t) \hat{v}_\ell\right).
\]

For trigonometric series estimators, one usually bounds the mean integrated squared error (MISE), as this makes it possible to utilize the orthogonality property of \((\psi_\ell)_{\ell \in \mathbb{N}}\) in \(L^2[0,1]\). To be able to use the same properties in the metric space setting, one could take the integrated mean squared euclidean error in the tangent space \(T_o \mathcal{Q}\), where, e.g., \(o = m(0)\). Then the problem reduces to the standard euclidean trigonometric estimator which is discussed, e.g., in [Tsy08, chapter 1.7]. If we assume that an inverse \(\text{Log}(o, \cdot)\) of \(\text{Exp}(o, \cdot)\) exists, a smoothness condition should be applied to \(t \mapsto \text{Log}(o,m(t))\).

The condition of centered / zero-mean noise of the euclidean model for trigonometric estimation translates to \(E[\text{Log}_o(Y_t)] = \text{Log}_o(m_t)\). Unfortunately, this seems to be far from the condition of centered noise in our metric setting, as it introduces distortions which highly depend on \(o = m(0)\). Compare this to our usual assumption, \(m_t = \arg\min_{q \in \mathcal{Q}} E[d(Y_t, q)^2]\), which implies (under mild assumptions) \(E[\text{Log}_{m_t}(Y_t)] = \text{Log}_{m_t}(m_t) = 0\), cf [Kar77, Theorem 1.2].

We were not able to show a theorem similar to Theorem 5 or Theorem 3 using our usual settings. Of course, this does not mean that the estimator above will necessarily perform badly.

The estimator was implemented for simulations (section 8). This revealed another drawback: High-dimensional non-convex optimization is required so that \text{TriGeo} is – by far – the slowest of all tested methods. The MISE values seem to be worse than for the other estimators on average. It is not clear, whether this is due to theoretical disadvantages or a worse outcome of the general purpose optimizer used for finding \((\hat{p}, \hat{v}_1, \ldots, \hat{v}_N)\).

7. Trigonometric Fréchet Regression

Using the Fréchet approach, we create a new trigonometric estimator, \text{TriFre}.
Confer section [0] for the definition of the trigonometric basis of $\mathbb{L}_2[0,1]$. In every setting, we will require a smoothness condition. The appropriate smoothness class connected to the trigonometric basis $(\psi_k)_{k \in \mathbb{N}}$ is the periodic Sobolev class $W^{\text{per}}(\beta, L)$, see [Tsy08, Definition 1.11]. A function $f(x) = \sum_{k=1}^{\infty} \theta_k \psi_k(x)$ belongs to $W^{\text{per}}(\beta, L)$ if and only if the sequence $\theta = (\theta_k)_{k \in \mathbb{N}}$, $\theta_k = \int_0^1 f(x) \psi_k(x) dx$, of the Fourier coefficients of $f$ belongs to the ellipsoid $\Theta(\beta, L)$, which is defined as

$$\Theta(\beta, L) = \left\{ \theta \in \ell^2 : \sum_{k=1}^{\infty} \theta_k^2 w_k^{-2} \leq L^2 \right\},$$

where $w_{2k+1} = w_{2k} = (2k)^{-\beta}$, see [Tsy08] Proposition 1.14.

7.1. Hypersphere

We use the hypersphere setting of section [1.1]. For $N \in \mathbb{N}$, define the vector $\Psi_N = (\psi_k)_{k=1,...,N} : [0,1] \to \mathbb{R}^N$. For $t \in [0,1]$ and $q \in S^k$ set $\hat{F}_t(q) = \Psi_N(t)^{\frac{1}{2}} \sum_{i=1}^{N} \Psi_N(x_i) y_i q^2$. The trigonometric Fréchet estimator on the hypersphere is $\hat{m}_t \in \arg \min_{q \in S^k} \hat{F}_t(q)$.

To be able to estimate $m$, we require a smoothness condition: Let $\mu$ be a the measure of the uniform distribution on $S^k$. Assume that for all $t \in [0,1]$, the random variable $Y_t$ has a density $y \to \rho(y|t)$ with respect to $\mu$. Let $\beta \geq 1$. Assume, there is $C_{\text{SmoothDensity}} \geq 1$, such that for $\mu$-almost all $q \in S^k$, $t \mapsto \rho(y|t) \in W^{\text{per}}(\beta, C_{\text{SmoothDensity}})$ (SmoothDensity). Furthermore, we again assume Variance: There is $C_{\text{Vlo}} \in [1, \infty)$ such that $C_{\text{Vlo}}^{-1}d(q, m_t)^2 \leq E[d(Y_t, q)^2 - d(Y_t, m_t)^2]$ for all $q \in S^k$ and $t \in [0,1]$.

**Corollary 10** (TriFre Hypersphere). Assume Variance and SmoothDensity.

Set $N = n^{\frac{1}{2\beta+1}}$. Then

$$\mathbb{E} \left[ \int_0^1 \frac{1}{m_t} \sigma_t^2 dt \right] \leq C n^{-\frac{2\beta}{2\beta+1}}$$

for all $n \in \mathbb{N}$, where $C = c_\beta C_{\text{Vlo}} C_{\text{SmoothDensity}}^2$.

7.2. General

We will only show a theorem in the metric setting of [1.1]. For $N \in \mathbb{N}$ with $\Psi_N = (\psi_k)_{k=1,...,N}$, define $\hat{F}_t(q) = \Psi_N(t)^{\frac{1}{2}} \sum_{i=1}^{N} \Psi_N(x_i) y_i q^2$. Let $\hat{m}_t \in \arg \min_{q \in Q} \hat{F}_t(q)$. Essentially, we estimate $t \mapsto F_t(q)$ for every fixed $q \in Q$ by a trigonometric series estimator described in [Tsy08] section 1.7. Instead of the unknown function $F_t(q)$, we then minimize $\hat{F}_t(q)$ with respect to $q$. Our goal is to bound the mean integrated squared error $\mathbb{E}[\int_0^1 \frac{1}{m_t} \sigma_t^2 dt]$. For $y, z \in Q$ define $a(y, z) = \sup_{q, p \in Q} |q-p| (\sigma_q q^2 - y \cdot y^2 - \sigma_p p^2 + \sigma_p p^2)/q \cdot p$. 

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Assumptions.

- **SmoothDensity**: Let $\mu$ be a probability measure on $\mathcal{Q}$ with $\int \frac{y^2}{\mu} \mu(dy) < \infty$ for an arbitrary $o \in \mathcal{Q}$. For all $t \in [0, 1]$, the random variable $Y_t$ has a density $y \mapsto \rho(y|t)$ with respect to $\mu$. Let $\beta \geq 1$. For $\mu$-almost all $y \in \mathcal{Y}$, there is $L(y) \geq 0$ such that $t \mapsto \rho(y|t) \in W^{\text{per}}(\beta, L(y))$. Furthermore, there is $C_{\text{SmD}} \in [1, \infty)$ such that $\int L(y)^2 d\mu(y) \leq C_{\text{SmD}}^2$.

- **Variance**: There is $C_{\text{Vlo}} \in [1, \infty)$ such that $\frac{1}{C_{\text{Vlo}}} \leq E[Y_t.q - Y_t.m_t^2]$ for all $q \in \mathcal{Q}$ and $t \in [0, 1]$.

- **Moment**: Let $\kappa > 2$. There is $C_{\text{Mom}} \in [1, \infty)$ such that $E[(Y_t.q)^\kappa]^{\frac{1}{\kappa}} \leq C_{\text{Mom}}$ for all $t \in [0, 1]$.

- **BiasMoment**: Define $H(q, p) = \{ \int (\frac{y}{q} + \frac{y}{p})^2 \mu(dy) \}^{\frac{1}{2}}$. There is $C_{\text{Bom}} \in [1, \infty)$ such that $E[H(\hat{m}_t, m_t)^\kappa]^{\frac{1}{\kappa}} \leq C_{\text{Bom}}$ for all $t \in [0, 1]$.

- **Entropy**: There is $C_{\text{Ent}} \in [1, \infty)$ such that $\gamma_2(B, d) \leq C_{\text{Ent}} \text{diam}(B)$ for all $B \subseteq \mathcal{Q}$.

**Theorem 5** (TriFre General). Assume Variance, Moment, BiasMoment, Entropy, SmoothDensity. Then

$$E\left[ \int_0^1 \frac{1}{m_t, \hat{m}_t^2} dt \right] \leq C_1 \left( N^{-2\beta} + Nn^{1-2\beta} \right) + C_2 \frac{N}{n},$$

where $C_1 = c_{\kappa, \beta} C_{\text{Vlo}}^2 C_{\text{SmD}}^2 C_{\text{Bom}}^2$ and $C_2 = c_{\kappa, \beta} C_{\text{Vlo}}^2 C_{\text{Mom}}^2 C_{\text{Ent}}^2$.

We obtain a bound with the same rates as in the euclidean setting, which lead to the classical nonparametric rate of convergence, see corollaries below.

All condition have previously been discussed, see Remark 1 and Remark 4.

**7.3. Corollaries**

In bounded spaces Moment and BiasMoment are trivial.

**Corollary 11** (TriFre Bounded). Let $(\mathcal{Q}, d)$ be a metric space with $\text{diam} \mathcal{Q} < \infty$. Assume Variance, Entropy, SmoothDensity. Set $N_n = n^{\frac{1}{2\beta + 1}}$. Then

$$E\left[ \int_0^1 \frac{1}{m_t, \hat{m}_t^2} dt \right] \leq Cn^{-\frac{2\beta}{2\beta + 1}},$$
where $C = c_\beta C_{\text{Vol}}^2 C_{\text{SmD}}^2 C_{\text{Ent}}^2 \text{diam}(Q)^2$.

In Hadamard spaces, $a(y,z) \leq 2d(y,z)$ because of the quadruple inequality [Stu03, Theorem 4.9]. Furthermore, VARIANCE is fulfilled as noted before. Lastly, we replace BIASMOMENT by BOMBOUND, which introduces an additional $\log(n)$-factor.

Assumptions.

- **BOMBOUND:**
  There are $C_{\text{Int}}, C_{\text{Len}} \in [1, \infty)$ such that
  
  \[
  \int y_m^2 \mu(dy) \leq C_{\text{Int}}^2, \quad a(m_t, m_s) \leq C_{\text{Len}}
  \]
  
  for all $s,t \in [0,1]$.

**Corollary 12** (TriFre Hadamard). Let $(Q,d)$ be a Hadamard metric space. Assume MOMENT, BOMBOUND, ENTROPY, SMOOTHDENSITY. Set $N_n = n^{\frac{1}{2\beta+1}}$. Then

\[
E \left[ \int_0^1 m_t \dot m_t^2 dt \right] \leq C n^{-\frac{2\beta}{2\beta+1} \log(n)^2},
\]

where $C = c_{\kappa, \beta} C_{\text{SmD}} C_{\text{Mom}}^2 C_{\text{Ent}}^2 C_{\text{Len}}^2 C_{\text{Int}}^2$.

8. Simulation

There is a total of 7 methods discussed in this article: LinGeo, LinFre, LinCos, LocGeo, LocFre, TriGeo, TriFre. To illustrate and compare these methods on the sphere, the R-package spheregr was developed. All code used for this paper, including all scripts which create the plots and run and evaluate the experiments shown in this section, are freely available at [https://github.com/ChristofSch/spheregr](https://github.com/ChristofSch/spheregr).

Each method requires numerical optimization. We use R’s general purpose optimizers `stats::optim(method = "L-BFGS-B")` and `stats::optimize()`, both without explicit implementation of derivatives, but with several starting points. The implementations could potentially be improved by using the algorithm presented in [EHW19]. For alternative implementation of geodesic regression, see [SO20].

The parametric methods are much faster than the nonparametric ones and Fréchet methods are faster than geodesic methods, as the optimization problem for geodesics is of higher dimension. We use leave-one-out cross-validation to estimate the hyperparameters ($h$ for LocGeo and LocFre, $N$ for TriFre). For TriGeo it did not seem feasible to do many repetitions of the experiments with cross-validation in each run. Instead we set $N = 3$ for this method, which seems to be a good choice in many runs. For LocGeo and LocFre, we use the Epanechnikov-kernel.
Let $S^2 = \{ x \in \mathbb{R}^3 : |x| = 1 \}$ be the sphere with radius 1 and intrinsic metric $d(q,p) = \arccos(q^T p)$. For $t \in [0,1]$, let $Y_t$ be a $S^2$-valued random variable. Let the regression function $m : [0,1] \rightarrow S^2$ be a minimizer $m_t \in \text{arg min}_{q \in S^2} \mathbb{E}[Y_t, q]^2$. Let $x_i = \frac{i - 1}{n - 1}$ and let $y_i \sim \text{i.i.d.}$ with values in $S^2$ such that $y_i$ has the same distribution as $Y_{x_i}$.

For the distribution of $Y_t$, we choose the contracted uniform distribution $\text{CntrUnif}(m_t, a)$ with $a \in (0,1)$, which we define next. The contracted uniform distribution is obtained from the uniform distribution on the sphere by moving all points towards a center point along the connecting geodesic by a given fraction of the total distance.

**Definition 2.** Let $a \in [0,1]$. Let $(\Theta, \Phi)$ be random angles with values in $[0, \pi] \times [0, 2\pi)$ that form a uniform distribution on the sphere, i.e., they are independent, $\Theta$ has Lebesgue density $\frac{1}{2} \sin(x) \mathbb{1}_{[0,\pi]}(x)$, and $\Phi$ is uniformly distributed on $[0, 2\pi)$.

Let

$$Z_a = \begin{pmatrix}
\sin(a\Theta) \cos(\Phi) \\
\sin(a\Theta) \sin(\Phi) \\
\cos(a\Theta)
\end{pmatrix}.$$ 

Let $m \in S^2$. Let $R_m \in O(3) \subseteq \mathbb{R}^{3\times3}$ be any orthogonal matrix that fulfills $m = R_me_3$, where $e_3 = (0,0,1)$. Then the **contracted uniform distribution** $\text{CntrUnif}(m, a)$ at $m$ with contraction parameter $a$ is defined as the distribution of $R_m Z_a$.

The matrix $R_m$ in the definition above is not unique, but the symmetry of the distribution of $Z_a$ ensures that the contracted uniform distribution is well-defined.

Two important properties are implied by the following proposition: For $a \in [0,1)$, $m \in S^2$ is the unique Fréchet mean of $\text{CntrUnif}(m, a)$. Furthermore, $\text{Variance}$ is fulfilled with $C_{V_{m}} = (1 - a)^{-1}$.

**Proposition 4 ([Oht12, section 5]).** Let $(\mathcal{Q}, d)$ be a proper Alexandrov space of nonnegative curvature. Let $Y_1$ be a random variable with values $\mathcal{Q}$ such that $\mathbb{E}[d(Y, q)^2] < \infty$ for all $q \in \mathcal{Q}$. Let $m \in \text{arg min}_{q \in \mathcal{Q}} \mathbb{E}[Y_1, q]^2$ be any Fréchet mean of $Y_1$. For $a \in [0,1)$, let $Y_a = \gamma_{m \rightarrow Y}(a)$, where, for $y \in \mathcal{Q}$, $\gamma_{m \rightarrow y}$ is a geodesic with $\gamma_{m \rightarrow y}(0) = m$, $\gamma_{m \rightarrow y}(1) = y$. Then

$$(1 - a)\gamma_{m}m^2 \leq \mathbb{E}[Y_a, q]^2 - Y_a, m^2$$

for all $a \in [0,1]$. 

Lastly, we calculate the variance of the contracted uniform distribution. Let $m \in S^2$, $a \in [0,1)$, and $Y \sim \text{CntrUnif}(m, a)$. Let $Z_a$ and $\Theta$ as in **Definition 2**. Then $\mathbb{E}[d(Y, m)^2] =$
For a true geodesic of length 3, we sample \( n \in \{10, 90\} \) observations with contracted uniform noise of standard deviation \( \sigma \in \{\frac{1}{3}, 1\} \). Then we apply LinGeo, LinFre, and LinCos.

\[
\mathbb{E}[d(Z_a, e_3)^2] \text{ because of symmetry. The distance does only depend on } \Theta \text{ and is equal to } a\Theta. \text{ Thus, } \mathbb{E}[d(Y, m)^2] = \frac{1}{2}a^2 \int_0^\pi x^2 \sin(x)dx = \frac{1}{2}(\pi^2 - 4)a^2.
\]

8.2. Parametric Regression

We draw a random geodesic \( m \) with fixed speed and sample independent \( y_i \sim \text{CntrUnif}(m_{x_i}, a) \) to obtain our data \((x_i, y_i)_{i=1,...,n}\). Then we calculate the three different parametric regression estimators LinGeo, LinFre, and LinCos.

We will describe points \( q \in S^2 \) via two angles \((\theta_q, \varphi_q) \in [0, \pi] \times [0, 2\pi]\) such that \( q = (\sin(\theta_q)\cos(\varphi_q), \sin(\theta_q)\sin(\varphi_q), \cos(\theta_q))\). We first show some illustrating plots \( \text{Figure 1} \) and \( \text{Figure 2} \). We want to depict functions of the form \([0, 1] \to [0, \pi] \times [0, 2\pi], t \mapsto (\theta_{m_t}, \varphi_{m_t})\). The graph of such a function is 3-dimensional and hard to understand on 2D-paper. Creating two plots, one for \([0, 1] \to [0, \pi], t \mapsto \theta_{m_t}\) and another for \([0, 1] \to [0, 2\pi], t \mapsto \varphi_{m_t}\), is also difficult to interpret, as one has to always take both graphs into account at the same time. Instead we show the image of the functions \( \{(\theta_{m_t}, \varphi_{m_t}) : t \in [0, 1]\} \subseteq [0, \pi] \times [0, 2\pi] \) and encode the dependence on \( t \) via color.

The rectangle of the two angles \((\theta, \varphi) \in [0, \pi] \times [0, 2\pi]\) parameterizing the sphere is the...
Figure 2: For a true geodesic of length 6, we sample $n \in \{10, 90\}$ observations with contracted uniform noise of standard deviation $sd \in \{\frac{1}{3}, 1\}$. Then we apply LinGeo, LinFre, and LinCos.
**Mercator projection.** This projection (as any projection of the sphere to the euclidean plane) distorts the surface of the sphere. This is made visible by the thin gray lines in the plots, which are geodesics and replace the usual grid lines. The plots show the image of \( t \mapsto m_t \) (line with black border) and the different estimators \( t \mapsto \hat{m}_t \) (lines with colored border). The covariate \( t \) is represented by the rainbow color inside each line. To visually compare the deviations of \( \hat{m}_t \) from \( m_t \), one has to compare the positions on the lines with the same inner color. But note that distances are distorted: Distances close to the equator \( (\vartheta = \frac{1}{2} \pi) \) are larger than they appear and smaller at the poles \( (\vartheta \in \{0, \pi\}) \). The observations \( y_i \) are also color-coded to identify which \( x_i \) they belong to. Furthermore, thin colored lines are drawn between \( y_i \) and \( m(x_i) \).

A geodesic of length 3 (Figure 1) is estimated similarly well by all estimators. This is true in different settings. Compare this with the estimation of a length 6 geodesic in Figure 2. Only LinCos and LinGeo perform well but not LinFre. This strongly suggests that LinFre is not consistent if noneuclidean properties of the descriptor space play a significant role. Note that the errors in the settings \((n = 10, \text{sd} = \frac{1}{3})\) and \((n = 90, \text{sd} = 1)\) are similar and \(\text{sd}^2/n\) is the same in both settings.

Next we repeat this experiment 1000 times for 12 different settings. The setting specifies the number of samples drawn \( n \), the noise standard deviation \( \text{sd} = \frac{1}{2} (\pi^2 - 4) a \), and the speed of the true geodesic. For each run we calculate the integrated squared error, ISE, \( \int_0^1 d(\hat{m}_t, m_t)^2 dt \). Then we take the mean of those 1000 ISE values to approximate the mean integrated squared error, MISE. Table 1 shows the results.
Table 1: Approximated MISE values for parametric regression methods. The colors give a visual indication of the MISE value of the given methods divided by the best MISE value in the row.

We can see that for geodesics with small speed, all three methods perform well. For high speed geodesics LinFre does not give meaningful results. LinGeo is by far the slowest method in our implementation, as it has the most complex optimization problem to solve.

8.3. Nonparametric Regression

Next we want to investigate the nonparametric methods LocGeo, LocFre, TriGeo, TriFre. We test two different regression functions \( t \mapsto m_t \). The first one, named simple has angles \( t \mapsto (\frac{1}{4}\pi, \frac{1}{2} + 2\pi t) \), see Figure 3. This seems to be a straight line in the Mercator projection but is a curved function on the sphere and cannot be estimated well by the parametric methods of the previous subsection. This simple curve is periodic. The second curve is described by \( t \mapsto (\frac{1}{8}\pi + \frac{3}{4}\pi t, \frac{1}{2} + 3\pi t) \). Again this curve is not geodesic. It spirals around the sphere, see Figure 4 and is not periodic. To estimate nonperiodic functions with TriGeo and TriFre, which require periodicity, we copy the data and
Figure 3: For the simple curve, we sample \( n \in \{10, 90\} \) observations with contracted uniform noise of standard deviation \( sd \in \{\frac{1}{3}, 1\} \). Then we apply LocGeo, LocFre, TriGeo, TriFre. Append it in reverse order to estimate the periodic function

\[
t \mapsto \begin{cases} 
  m_2t & \text{if } t < \frac{1}{2}, \\
  m_2 - 2t & \text{if } t \geq \frac{1}{2}.
\end{cases}
\]

This may lead to boundary effects.

On a broad scale, all estimators seem to perform similarly, except for a worse outcome for TriGeo on the spiral. In the setting \((n = 10, sd = 1)\) the estimators are not able to come close to the true curve. Compare this to the same setting in the parametric cases, where performance of estimators is still good enough to potentially be useful.

As with the parametric methods, we approximate the MISE values in different settings. The simulations are repeated 500 times. Only the two curves simple and spiral described above are used. The results are presented in Table 2.
Figure 4: For the *spiral*, we sample $n \in \{10, 90\}$ observations with contracted uniform noise of standard deviation $sd \in \{\frac{1}{3}, 1\}$. Then we apply LocGeo, LocFre, TriGeo, TriFre.
Table 2: Approximated MISE values for nonparametric regression methods. The colors give a visual indication of the MISE value of the given methods divided by the best MISE value in the row.

The more reliable analysis of the approximated MISE-values confirms that all estimators behave similar, except TriGeo, which has some bad outcomes. This may have several reasons. We were not able to show an error bound for this method and argued that it may be sub-optimal, i.e., it may be inherently worse than the other methods. We do not use cross-validation for TriGeo, as we do for the other methods, but fix $N = 3$. Thus, the comparison might be unfair, because the hyper-parameters are not tuned equally. Lastly, in TriGeo, we have to numerically solve an 8-dimensional non-convex optimization problem (2 dimensions for each of $\hat{p}, \hat{v}_1, \hat{v}_2, \hat{v}_3$). There are 4 dimensions for LocGeo and 2 for the Fréchet methods. Our program might return values farther away from the optimum in those methods with higher dimensional optimization problems.

A. Proofs

A.1. Section 2: LinGeo

A.1.1. Theorem

We prove Theorem 1. We first apply VARIANCE to relate the difference between the objective functions to the loss between their minimizers. Then chaining is used to bound the objective functions and a peeling device leads to tail bounds on the loss. Lastly, we integrate the tails.
Define the objective functions

\[
F_x(\theta) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[c(Y_{xi}, g(x_i, \theta))], \quad F_x(\theta_1, \theta_2) = F_x(\theta_1) - F_x(\theta_2),
\]

\[
\hat{F}_x(\theta) = \frac{1}{n} \sum_{i=1}^{n} c(y_i, g(x_i, \theta)), \quad \hat{F}_x(\theta_1, \theta_2) = \hat{F}_x(\theta_1) - \hat{F}_x(\theta_2).
\]

VARIANCE and the minimizing property of \( \hat{\theta} \) yield

\[
C_{\text{Vlo}} \frac{1}{n} \sum_{i=1}^{n} \mathbb{I}(m_{xi}, \hat{m}_{xi}) \leq F_x(\hat{\theta}, \theta^*) \leq F_x(\theta, \theta^*) - \hat{F}_x(\hat{\theta}, \theta^*).
\]

Define

\[
\Delta_x(\delta) = \sup_{\theta \in B_x(\theta^*, \delta)} \left( F_x(\hat{\theta}, \theta^*) - \hat{F}_x(\hat{\theta}, \theta^*) \right)
\]

and

\[
Z_i(\theta) = \frac{1}{n} \left( \mathbb{E}[c_{xi}(Y_{xi}, \theta) - c_{xi}(Y_{xi}, \theta^*)] - c_{xi}(y_i, \theta) + c_{xi}(y_i, \theta^*) \right).
\]

Then \( Z_1, \ldots, Z_n \) are independent and centered processes with \( Z_i(\theta^*) = 0 \). They are also integrable due to \textsc{Moment}. By the definition of \( a_x \), it holds

\[
n \left( Z_i(\theta_1) - Z_i(\theta_2) - Z'_i(\theta_1) + Z'_i(\theta_2) \right) \leq b(\theta_1, \theta_2) a_{xi}(y_i, y_i').
\]

Thus, the chaining result of \textbf{Theorem 6} (appendix \textbf{B}) yields

\[
\mathbb{E}[\Delta_x(\delta)^\kappa] \leq c_\kappa \left( \mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} a_{x_i}(y_i, y_i')^2 \right)^{\frac{\kappa}{2}} \right]^{\frac{1}{2}} \gamma_2(B_x(\theta^*, \delta), b)n^{-\frac{\delta}{2}} \right)^\kappa.
\]

By \text{Entropy} \( \gamma_2(B_x(\theta^*, \delta), b) \leq C_{\text{Ent}} \delta^\delta \) for \( \delta > T_n \). As \( \kappa \geq 2 \), by \textsc{Moment},

\[
\mathbb{E} \left[ \left( \frac{1}{n} \sum_{i=1}^{n} a_{x_i}(y_i, y_i')^2 \right)^{\frac{\kappa}{2}} \right]^{\frac{1}{2}} \leq \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[a_{x_i}(y_i, y_i')^\kappa] \right)^{\frac{1}{2}} \leq C_{\text{Mom}}.
\]

Thus, for \( \delta > T_n \),

\[
\mathbb{E}[\Delta_x(\delta)^\kappa] \leq c_\kappa \left( n^{-\frac{3}{2}} C_{\text{Ent}} C_{\text{Mom}} \delta^\delta \right)^\kappa.
\]

For \( T_n < a < b < \infty \), using Markov’s inequality, we obtain

\[
\mathbb{P} \left( C_{\text{Vlo}}^{-\frac{1}{n}} \sum_{i=1}^{n} \mathbb{I}(m_{xi}, \hat{m}_{xi}) \in [a, b] \right) \leq \mathbb{P}(a \leq \Delta_x(b)) \leq a^{-\kappa} \mathbb{E}[\Delta_x(b)^\kappa] \leq c_\kappa \left( n^{-\frac{3}{2}} C_{\text{Ent}} C_{\text{Mom}} b^\delta \right)^\kappa.
\]

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Hence, we set \( q > 0 \), and we use this bound in the peeling device, to obtain a tail bound for \( t \geq T_n \):

\[
\mathbb{P}\left( C_{\text{Vlo}}^{-1} \frac{1}{n} \sum_{i=1}^{n} l(m_{x_i}, \hat{m}_{x_i}) > t \right) \leq \sum_{k=0}^{\infty} \mathbb{P}\left( C_{\text{Vlo}}^{-1} \frac{1}{n} \sum_{i=1}^{n} l(m_{x_i}, \hat{m}_{x_i}) \in [2^k t, 2^{k+1} t) \right)
\]

\[
\leq 2^{\kappa_c} c_{\kappa} \sum_{k=0}^{\infty} \left( \frac{n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} t^{\frac{\kappa}{2}}}{t^{2k}} \right)^{\kappa}
\]

\[
= 2^{\kappa_c} c_{\kappa} \left( n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} \right)^{\kappa} t^{\kappa(\xi-1)} \sum_{k=0}^{\infty} \left( 2^{\kappa(\xi-1)} \right)^k
\]

\[
= c_{\kappa, \xi} \left( n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} \right)^{\kappa} t^{\kappa(\xi-1)}
\]

with \( c_{\kappa, \xi} = \frac{2^{\kappa_c} c_{\kappa}}{1 - 2^\kappa(\xi-1)} \). To obtain the desired bound on the expectation, we integrate the tail probability

\[
\mathbb{E}\left[ C_{\text{Vlo}}^{-1} \frac{1}{n} \sum_{i=1}^{n} l(m_{x_i}, \hat{m}_{x_i}) \right] \leq T_n + \int_{T_n}^{\infty} \mathbb{P}\left( C_{\text{Vlo}}^{-1} \frac{1}{n} \sum_{i=1}^{n} l(m_{x_i}, \hat{m}_{x_i}) > t \right) dt
\]

\[
\leq T_n + \int_{0}^{\infty} \min\left( 1, c_{\kappa, \xi} \left( n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} \right)^{\kappa} t^{\kappa(\xi-1)} \right) dt.
\]

It holds

\[
\int_{0}^{\infty} \min\left( 1, b t^{-\alpha} \right) dt = \frac{a}{a-1} b^{\frac{1}{\alpha}}
\]

for all \( a > 1, b > 0 \). Now set \( a = \kappa(1-\xi) \) and \( b = c_{\kappa, \xi} \left( n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} \right)^{\kappa} \). We obtain

\[
C_{\text{Vlo}}^{-1} \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} l(m_{x_i}, \hat{m}_{x_i}) \right] \leq T_n + \frac{\kappa(1-\xi)}{\kappa(1-\xi) - 1} \left( c_{\kappa, \xi} \left( n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} \right)^{\kappa} \right)^{\frac{1}{\kappa(1-\xi)}}
\]

\[
= T_n + c'_{\kappa, \xi} \left( n^{-\frac{\kappa}{2}} C_{\text{Ent}} C_{\text{Mom}} \right)^{\frac{1}{\kappa(1-\xi)}}
\]

with \( c'_{\kappa, \xi} = \frac{\kappa(1-\xi)}{\kappa(1-\xi) - 1} c_{\kappa, \xi}^{\frac{1}{\kappa(1-\xi)}} \).

### A.1.2. Corollaries

**Proof of Corollary 2.** In Hadamard spaces the variance inequality \( d(q, m)^2 \leq \mathbb{E}[d(Y, q)^2 - d(Y, m)^2] \) for all \( q \in Q \) and \( m = \arg \min_{q \in Q} \mathbb{E}[d(Y, q)^2] \) holds for all distributions of \( Y \) with \( \mathbb{E}[d(Y, q)^2] < \infty \), [Stu03, Theorem 4.9]. This shows VARIANCE. Furthermore, the quadruple inequality

\[
d(y, q)^2 - d(\tilde{y}, q)^2 - d(y, \tilde{q})^2 + d(\tilde{y}, \tilde{q})^2 \leq 2d(y, \tilde{y})d(q, \tilde{q})
\]

holds for all \( q, \tilde{q}, y, \tilde{y} \in Q \), [Stu03, Theorem 4.9]. Thus, with MUP we get

\[
\nabla_x (y, \tilde{y}, \theta, \tilde{\theta}) \leq 2d(y, \tilde{y})d(g(x, \theta), g(x, \tilde{\theta})) \leq 2C_{\text{Mup}} d(y, \tilde{y}) \mid \theta - \tilde{\theta} \mid.
\]

Hence, we set \( a_x(y, \tilde{y}) = 2d(y, \tilde{y})C_{\text{Mup}} \) and \( b = | \cdot - | \) when applying Theorem 1. Next,
to check Entropy, use MetricLo

\[
\frac{1}{n} \sum_{i=1}^{n} l(g(x_i, \theta), g(x_i, \theta^*)) = \frac{1}{n} \sum_{i=1}^{n} d(g(x_i, \theta), g(x_i, \theta^*))^2 \geq C_{\text{Mlo}}^{-1} |\theta - \theta|^2 - T_n,
\]

where \( T_n = C_{\text{Res}} n^{-1} \). Thus, for \( \delta > T_n \),

\[
B_x(\theta^*, l, \delta) \subseteq \{ \theta \in \Theta : |\theta - \theta^*| \leq (2C_{\text{Mlo}}\delta)^{\frac{1}{2}} \}.
\]

From this, together with the bound on \( \gamma_2 \) for Euclidean spaces Lemma 25 (appendix B), we obtain

\[
\gamma_2(B_x(\theta^*, l, \delta), b) \leq \gamma_2\left( \{ \theta \in \Theta : |\theta - \theta^*| \leq (2C_{\text{Mlo}}\delta)^{\frac{1}{2}} \}, b \right)
\]

\[
\leq c \left( d_{\Theta} C_{\text{Mlo}} \delta \right)^{\frac{1}{2}}
\]

\[
= C_{\text{Ent}} \delta^{\frac{1}{2}}
\]

with \( C_{\text{Ent}} = c \left( d_{\Theta} C_{\text{Mlo}} \right)^{\frac{1}{2}}. \)

\[\square\]

**Proof of Corollary 3.** We want to apply Theorem 1. Hence, we have to check its assumptions. **Variance** is a condition of the corollary. In order show **Moment**, we note

\[
d(y, g(x, \theta))^2 - d(y, g(x, \hat{\theta}))^2 \leq 2 \text{diam}(Q) d(g(x, \theta), g(x, \hat{\theta})) \leq 2 \text{diam}(Q) C_{\text{Mup}} |\theta - \hat{\theta}|
\]

using the triangle inequality and **METRICUP**. Thus,

\[
\hat{\diamond}_x(y, \hat{y}, \theta, \hat{\theta}) \leq 4 \text{diam}(Q) C_{\text{Mup}} |\theta - \hat{\theta}|.
\]

We can set \( a_x(y, \hat{y}) = 4 \text{diam}(Q) C_{\text{Mup}} \) and \( b = |\cdot - \cdot| \) when applying Theorem 1. The moment condition is trivial, as \( a_x \) is a finite constant. As before

\[
\gamma_2(B_x(\theta^*, l, \delta), b) \leq C_{\text{Ent}} \delta^{\frac{1}{2}}
\]

with \( C_{\text{Ent}} = c \left( d_{\Theta} C_{\text{Mlo}} \right)^{\frac{1}{2}}. \)

\[\square\]

Next, we want to apply Corollary 3 to show Corollary 1. To do this, we need to show **METRICUP** and **METRICLO** translated to the spherical setting:

- **METRICUP:**
  There is \( C_{\text{Mup}} \in [1, \infty) \) such that \( d(\text{Exp}(q, xv), \text{Exp}(p, xu)) \leq C_{\text{Mup}} (|p - q| + |u - v|) \)
  for all \( x \in [-1, 1], (q, u), (p, v) \in T^k \).

- **METRICLO:**
  There are \( T_n \geq 0 \) and \( C_{\text{Mlo}} \in [1, \infty) \) such that
  \[
  \frac{1}{n} \sum_{i=1}^{n} d(\text{Exp}(q, x_i v), \text{Exp}(p, x_i u))^2 \geq C_{\text{Mlo}}^{-1} \left(|p - q|^2 + |u - v|^2\right) - C_{\text{Res}} n^{-1}.
  \]
The following lemma shows MetricUp with $C_{Mup} = 4\pi$. This constant may not be sharp.

**Lemma 1.** Let $(p, u), (q, v) \in TS^k$. Then

$$d(\Exp(q, v), \Exp(p, u)) \leq \frac{\pi}{2} |q - p| + 2\pi |v - u| .$$

**Proof.** We can bound the intrinsic metric on the sphere by the extrinsic one,

$$d(\Exp(q, v), \Exp(p, u)) \leq \frac{\pi}{2} |\Exp(q, v) - \Exp(p, u)|$$
$$\leq \frac{\pi}{2} \left( |\cos(|v|)q - \cos(|u|)p| + \left| \frac{\sin(|v|)}{|v|} v - \frac{\sin(|u|)}{|u|} u \right| \right) .$$

For the cos-terms, it holds

$$|\cos(|v|)q - \cos(|u|)p| \leq |\cos(|v|)||q - p| + |p| |\cos(|v|) - \cos(|u|)|$$
$$\leq |q - p| + ||v| - |u|| .$$

For the sin-terms, let $J(x)$ be the Jacobi matrix of the function $\mathbb{R}^k \rightarrow \mathbb{R}^k$, $x \mapsto \frac{\sin(|x|)}{|x|} x$. Then

$$\left| \frac{\sin(|v|)}{|v|} v - \frac{\sin(|u|)}{|u|} u \right| \leq \sup_{x \in \mathbb{R}^k} \|J(x)\|_{\text{op}} ||u - v|| .$$

As

$$J(x) = \left( \cos(|x|) - \frac{\sin(|x|)}{|x|} \right) |x|^{-2} xx^\top + \frac{\sin(|x|)}{|x|} I_k ,$$

it holds

$$\|J(x)\|_{\text{op}} \leq \left( |\cos(|x|)| + \left| \frac{\sin(|x|)}{|x|} \right| \right) \| |x|^{-2} xx^\top \|_{\text{op}} + \left| \frac{\sin(|x|)}{|x|} \right| \|I_k\|_{\text{op}} \leq 3 .$$

Thus, $d(\Exp(q, v), \Exp(p, u)) \leq \frac{\pi}{2} (|q - p| + ||v| - |u|| + 3||u - v||) . \quad \square$

For MetricLo we prove following lemma.

**Lemma 2.** Let $(p, u), (q, v) \in TS^k$ with $|u|, |v| \leq \frac{\pi}{2}$. Then

$$\int_{-1}^{1} d_{S^k}(\Exp(p, xu), \Exp(q, xv))^2 dx \geq \frac{2}{\pi} |p - q|^2 + \frac{8}{\pi^2} |v - u|^2 .$$
Proof. First we lower bound the intrinsic distance $d_{gk}$ by the euclidean one and use the explicit representation of the $\text{Exp}$-function,

$$d_{gk}(\text{Exp}(p, xu), \text{Exp}(q, xv))^2 \geq |\cos(x |u|)p + \sin(x |u|)\frac{u}{|u|} - \cos(x |v|)q - \sin(x |v|)\frac{v}{|v|}|^2.$$ 

When integrating after calculating the squared norm, all summands with a $\cos() \sin()$-factor disappear, because of symmetry. Thus, we obtain

$$\int_{-1}^{1} d_{gk}(\text{Exp}(p, xu), \text{Exp}(q, xv))^2 dx$$

$$\geq \int_{-1}^{1} \cos(x |u|)^2 p^T p - 2 \cos(x |u|) \cos(x |v|) p^T q + \cos(x |v|)^2 q^T q dx$$

$$+ \int_{-1}^{1} \sin(x |u|)^2 \frac{u^T u}{|u|^2} - 2 \sin(x |u|) \sin(x |v|) \frac{u^T v}{|u| |v|} + \sin(x |v|)^2 \frac{v^T v}{|v|^2} dx.$$ 

As $|p| = |q| = 1$, $\cos(x)^2 + \sin(x)^2 = 1$, $2 \cos(\alpha) \cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta)$, and $2 \sin(\alpha) \sin(\beta) = \cos(\alpha - \beta) - \cos(\alpha + \beta)$, the right hand side reduces to

$$\int_{-1}^{1} 2 - (\cos(xa) + \cos(xb))^2 p^T q - (\cos(xa) - \cos(xb)) z dx,$$

where we set $a = |u| - |v|$, $b = |u| + |v|$, and $z = \frac{\frac{u}{|u|}\frac{v}{|v|}}{\frac{u}{|u|}\frac{v}{|v|}}$. Integrating yields

$$4 - 2 \left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) q^T p - 2 \left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) z.$$ 

As $\left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) \geq \frac{2}{\pi}$ for $|a|, |b| \leq \frac{\pi}{2}$ and as $\frac{1}{2} |q - p|^2 = (1 - q^T p)$, we have shown

$$\int_{-1}^{1} d_{gk}(\text{Exp}(p, xu), \text{Exp}(q, xv))^2 dx$$

$$\geq 2 \pi |p - q|^2 + \left(4 - 2 \left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) - 2 \left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) z\right).$$ 

To complete the proof, we will show $f(a, b, z) \geq 0$ for all $a \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $b \in [0, \pi]$, and $z \in [-1, 1]$, where

$$f(a, b, z) = 4 - 2 \left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) - 2 \left(\frac{\sin(a)}{a} + \frac{\sin(b)}{b}\right) z - c \left(a^2 + b^2 + (a^2 - b^2)z\right)$$

with $c = \frac{4}{\pi}$. This suffices as $a^2 + b^2 + (a^2 - b^2)z = 2 |v - u|^2$. As $f$ is linear in $z$, it is minimized either at $z = 1$ or at $z = -1$. It holds

$$f(a, b, 1) = 4 - 4 \frac{\sin(a)}{a} - ca^2; \quad f(a, b, -1) = 4 - 4 \frac{\sin(b)}{b} - cb^2.$$ 

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where $C_0 = \frac{8\pi^2}{3}$.

**Proof of Theorem 2.** We show that for each pair $y \in Q$ such that for all $q \in Q$ it holds that $d(m,y) = \frac{1}{2}d(q,y) + \frac{1}{2}d(q,z) - \frac{1}{2}d(m,z)$.

In the worst case, $x = \exp(y,A)$ and $x = \exp(q,A)$ move in opposite directions and the distance changes with a rate of $C \frac{\pi}{2} + \frac{1}{2}A$. Thus, we obtain

$$T_n \leq \frac{2(\pi + A)}{n} \leq \frac{8\pi^2}{3} C \exp^{-1}.$$

With the use of the Lemma 2 above on $(q,y) \in T_2$, $\frac{|u|}{|v|} \leq A$, we obtain

$$T_n = \frac{1}{2} \int_0^1 \left( \sum_{i=1}^n \frac{d(\exp(y,x_i),\exp(p,x_i))}{\exp(p,x_i)} \right).$$

Thus, it attains its minimum on $\pi$ at $x = \pi$. For $c = \frac{\pi}{2} = 4g(\pi)$, we hereby have shown $f(z,y)$ and thus have proven the lemma.

**Proof of Corollary 3.**

We want to apply Corollary 3 and have to check its conditions. **Variance** is the assumption stated in [Corollary 1](#), [MetricLo](#). To show **MetricLo**, let

$$g(x) = \frac{1}{x^2} - \frac{\pi^2}{3} \sin(x)$$

with derivative $g'(x) = \cos(x) - 2$. It is symmetric at 0 and decreasing for positive $x$. Thus, it attains its minimum on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ at $x = \pi$. For $c = \frac{\pi}{2} = 4g(\pi)$, we thereby have shown $f(z,y)$ and thus have proven the lemma.

Consider the function $g(x) = \frac{1}{x^2} - \frac{\pi^2}{3} \sin(x)$, with derivative $g'(x) = \cos(x) - 2$. It is symmetric at 0 and decreasing for positive $x$. Thus, it attains its minimum on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ at $x = \pi$. For $c = \frac{\pi}{2} = 4g(\pi)$, we thereby have shown $f(z,y)$ and thus have proven the lemma.
be arbitrary. The **strict linear Fréchet regression model** implies that there are \( \theta_0, \theta_1 \in \mathbb{R} \) such that
\[
\theta_0 + \theta_1 t = \mathbb{E}[d(Y_t, q)^2 - d(Y_t, m)^2] = d(\gamma_t, q)^2 - d(\gamma_t, m)^2. \tag{1}
\]
Adding this equality with \( t = +1 \) and \( t = -1 \), we obtain
\[
2\theta_0 = d(\gamma_1, q)^2 - d(\gamma_1, m)^2 + d(\gamma_{-1}, q)^2 - d(\gamma_{-1}, m)^2 = d(y, q)^2 + d(z, q)^2 - \frac{1}{2}d(y, z)^2
\]
as \( d(y, m) = d(z, m) = \frac{1}{2}d(y, z) \). Evaluating (1) at \( t = 0 \) yields \( \theta_0 = d(m, q)^2 \). Together, we arrive at the result
\[
2d(m, q)^2 = d(y, q)^2 + d(z, q)^2 - \frac{1}{2}d(y, z)^2.
\]

**Proof of Proposition 2.** Let \((\alpha, \beta) \in S^2\). For \( \varphi \in [0, 2\pi) \), let \( \angle(\varphi, \beta) \in [0, \pi] \) be the distance of the two angles on the circle. We calculate the objective function,
\[
\mathbb{E}[\cos(d(Y, (\alpha, \beta)))] = \frac{1}{2\pi} \int_0^{2\pi} \sin(\vartheta) \sin(\alpha) + \sin(\vartheta) \sin(\alpha) \cos(\angle(\varphi, \beta)) \, d\vartheta \, d\nu(\vartheta)
\]
\[
= \int \left( \sin(\vartheta) \sin(\alpha) + \frac{1}{\pi} \sin(\vartheta) \sin(\alpha) \int_0^\pi \cos(\varphi) \, d\varphi \right) \, d\nu(\vartheta)
\]
\[
= \int \sin(\vartheta) \sin(\alpha) \, d\nu(\vartheta)
\]
\[
= A \sin(\alpha).
\]
Thus, if \( A > 0 \), \( \mathbb{E}[\cos(d(Y, (\alpha, \beta)))] \) is uniquely maximized at \( \alpha = \pi/2 \), analogously for \( A < 0 \). If \( A = 0 \), \( \mathbb{E}[\cos(d(Y, (\alpha, \beta)))] = 0 \) for all \((\alpha, \beta) \in S^2\). \qed

**Proof of Proposition 3.** By the law of cosines
\[
\mathbb{E}[\cos(Y_t, q)] = \mathbb{E}[\cos(m_t, q) \cos(Y_t, m_t)] + \mathbb{E}[\sin(m_t, q) \cos(Y_t, m_t) \cos(\angle(Y_t, m_t, q))].
\]
By Lemma 3 below, \( \mathbb{E}[\sin(Y_t, m_t) \cos(\angle(Y_t, m_t, q))] = 0 \). By the Pythagorean theorem with \( \angle(m_t, \gamma_{s_1}, q) = \frac{\pi}{2} \),
\[
\cos(m_t, q) = \cos(m_t, \gamma_{s_1}) \cos(\gamma_{s_1}, q).
\]
By definition, \( \cos(m_t, \gamma_{s_1}) = \cos(\gamma_{s_0 + \lambda t, s_1}) = \cos(B_q + \lambda t) \). It holds
\[
\cos(B_q + \lambda t) = \cos(B_q) \cos(\lambda t) - \sin(B_q) \sin(\lambda t)
\]
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and, thus, 
\[ E[\cos(Y_t,q)] = A_q \cos(B_q + \lambda t) = a_q \cos(\lambda t) + b_q \sin(\lambda t). \]

**Lemma 3.** Let \((Q,d)\) be a Alexandrov space of nonpositive or nonnegative curvature \([BBI01\text{, section 4}]\). Let be a geodesic metric space. Let \(f : [0, \infty) \to \mathbb{R}\) be a continuously differentiable function with derivative \(f'\). Let \(Y\) be a random variable with values in \(Q\) such that \(E[|f(d(Y,q))|] < \infty\) and \(E[|f'(d(Y,q))|] < \infty\) for all \(q \in Q\). Let \(m \in \text{arg max}_{q \in Q} E[f(d(Y,q))]\). Then \(E[f'(Y,m) \cos(\angle(Y,m,q))] = 0\), where \(\angle(Y,m,q)\) is the angle between \(Y\), \(m\), and \(q\) at \(m\).

**Proof.** Let \((\gamma_t)_{t \in [0,T]}\) be the minimizing unit-speed geodesic between \(\gamma_0 = m\) and \(\gamma_T = q\). \([BBI01\text{, Theorem 4.5.6}]\) yields \(\partial_t d(Y,\gamma_t)_{t=0} = -\cos(\alpha)\) where \(\alpha = \angle(Y,m,q)\). Thus,
\[ 0 = \partial_t E[f(d(Y,\gamma_t))]_{t=0} = E[\partial_t f(d(Y,\gamma_t))]_{t=0} = E[f'(d(Y,\gamma_t)) \partial_t d(Y,\gamma_t)]_{t=0} = -E[f'(d(Y,m)) \cos(\alpha)]. \]

\[ \square \]

### A.3. Section 4: LocGeo

#### A.3.1. Theorem

We prove \[\text{Theorem 3}\]. To this end, we first replace the integral over \(x\) by a sum over \(x_i\) in \[\text{Lemma 5}\]. Then the comparison of the estimated parameter \(\hat{\theta}_{t,h}\) with the best local parameter \(\theta_{t,h}\) is replaced by the comparison of \(\hat{\theta}_{t,h}\) to the true function \(m\) in \[\text{Lemma 7}\]. This is necessary to apply the variance inequality, which makes it possible to translate a bound on the objective functions to a bound on their minimizers, which are elements of the metric space. For the remaining part, we bound a variance term via chaining \[\text{Lemma 8}\] and a bias term using the smoothness assumption \[\text{Lemma 9}\].

These are used in \[\text{Lemma 10}\], where a peeling device is applied to bound the tails of the error distribution (and via integration also its expectation). This is supplemented by the auxiliary lemmata \[\text{Lemma 11}\] and \[\text{Lemma 12}\]. But first we start out with another auxiliary result, \[\text{Lemma 3}\] which shows that \(a\) and \(b\) are semi-metrics.

A map \(d : Q \times Q \to [0, \infty]\) is called semi-metric on \(Q\), if \(d\) is symmetric with \(d(q,q) = 0\) for all \(q \in Q\) and obeys the triangle inequality.

**Lemma 4.** The functions \(a\) and \(b\) are semi-metrics on \(Q\) and \(\Theta\), respectively.

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Proof. Recall $q,p = d(q,p)$. All properties for $a$ are straightforward. For the triangle inequality, as
\[
\frac{y,q^2 - y,p^2 - z,q^2 + z,p^2}{q,p} = \frac{y,q^2 - y,p^2 - v,q^2 + v,p^2}{q,p} + \frac{v,q^2 - v,p^2 - z,q^2 + z,p^2}{q,p},
\]
we obtain
\[
\sup_{q \neq p} \frac{y,q^2 - y,p^2 - z,q^2 + z,p^2}{q,p} \leq \sup_{q \neq p} \frac{y,q^2 - y,p^2 - v,q^2 + v,p^2}{q,p} + \sup_{q \neq p} \frac{v,q^2 - v,p^2 - z,q^2 + z,p^2}{q,p}.
\]

For $b$ the argument is almost identical. \hfill \Box

Using the properties of the kernel, we bound the integrated squared error by a sum.

\begin{lemma}
Assume Kernel and Lipschitz. Then
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} d\left( g(x, \theta), g(x, \bar{\theta}) \right)^2 \, dx \leq 6C_{Kmi}C_{Kma} \left( \sum_{i=1}^{n} w_i d\left( g_i(\theta), g_i(\bar{\theta}) \right)^2 + \frac{2C_{\text{Lip}}}{nh} \right)
\]
for all $\theta, \bar{\theta} \in \Theta, h \geq \frac{2}{n}$.
\end{lemma}

Proof. Kernel implies
\[
\frac{C_{Kmi}^{-1}}{C_{Kma}} \mathbb{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]} \left( \frac{x_i - t}{h} \right) \leq w_i,
\]
where $I_{t,h} = \{i \in \{1, \ldots, n\} : t - h \leq x_i \leq t + h \}$. We bound the difference between the Riemann sum and its corresponding integral using Lipschitz
\[
\left| \frac{1}{\#I_{t,h}} \sum_{i \in I_{t,h}} d\left( g\left( \frac{x_i - t}{h}, \theta \right), g\left( \frac{x_i - t}{h}, \bar{\theta} \right) \right)^2 - \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left( g(x, \theta), g(x, \bar{\theta}) \right)^2 \, dx \right| \leq \frac{C_{\text{Lip}}}{\#I_{t,h}}.
\]
Thus,
\[
\sum_{i=1}^{n} w_i d\left( g_i(\theta), g_i(\bar{\theta}) \right)^2 \geq \frac{C_{Kmi}^{-1} \#I_{t,h}}{C_{Kma} \#I_{t,h}} \frac{1}{\#I_{t,h}} \sum_{i \in I_{t,h}} d\left( g\left( \frac{x_i - t}{h}, \theta \right), g\left( \frac{x_i - t}{h}, \bar{\theta} \right) \right)^2 \geq \frac{C_{Kmi}^{-1} \#I_{t,h}}{C_{Kma} \#I_{t,h}} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left( g(x, \theta), g(x, \bar{\theta}) \right)^2 \, dx - \frac{C_{\text{Lip}}}{\#I_{t,h}} \right).
\]
As $h \geq \frac{2}{n}$, we obtain

\[
\sum_{i=1}^{n} w_i d\left(g_i(\theta), g_i(\bar{\theta})\right)^2 \geq \frac{C_{Kmi}^{-1}}{6C_{Kma}} \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(g(x, \theta), g(x, \bar{\theta})\right)^2 \, dx - 2C_{\text{Lip}} \frac{nh}{n} \right).
\]

The weights $w_i$ have following properties, see [Tsy08, Proposition 1.13].

**Lemma 6.** Assume Kernel and $h \geq \frac{2}{n}$. Then

\[
w_i \geq 0, \quad \sum_{i=1}^{n} w_i = 1, \quad w_i \leq \frac{6C_{Kmi}C_{Kma}}{nh},
\]

\[
w_i = 0 \text{ if } |x_i - t| > h, \quad \sum_{i=1}^{n} w_i^2 \leq \frac{6C_{Kmi}C_{Kma}}{nh}
\]

for all $t \in [0, 1]$ and $h \geq \frac{2}{n}$.

Define $U(\theta) = \sum_{i=1}^{n} w_i d(g_i(\theta), m_{x_i})^2$. We make use of Smoothness to obtain a bound on $\sum_{i=1}^{n} w_i d(g_i(\hat{\theta}_{t,h}), g_i(\theta_{t,h}))^2$.

**Lemma 7.** Assume Kernel, Lipschitz, Smoothness. Then,

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(g(x, \hat{\theta}_{t,h}), g(x, \theta_{t,h})\right)^2 \, dx \leq 6C_{Kmi}C_{Kma} \left( U(\hat{\theta}_{t,h}) + 6C_{Kmi}C_{Kma}C_{\text{Smo}}^2h^{2\beta} + \frac{2C_{\text{Lip}}}{nh} \right).
\]

**Proof.** [Lemma 5] with Lipschitz states

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(g(x, \hat{\theta}_{t,h}), g(x, \theta_{t,h})\right)^2 \, dx \leq 6C_{Kmi}C_{Kma} \left( \sum_{i=1}^{n} w_i d\left(g_i(\hat{\theta}_{t,h}), g_i(\theta_{t,h})\right)^2 + \frac{2C_{\text{Lip}}}{nh} \right).
\]

The remaining sum can be bounded using Smoothness and Kernel by

\[
\sum_{i=1}^{n} w_i d\left(g_i(\hat{\theta}_{t,h}), g_i(\theta_{t,h})\right)^2 \leq \sum_{i=1}^{n} w_i \left( d(g_i(\hat{\theta}_{t,h}), m_{x_i})^2 + d(m_{x_i}, g_i(\theta_{t,h}))^2 \right)
\]

\[
\leq \sum_{i=1}^{n} w_i d(g_i(\hat{\theta}_{t,h}), m_{x_i})^2 + C_{\text{Ker}}C_{\text{Smo}}^2h^{2\beta}.
\]
As we set up the processes $E_i \geq 0$ for $i = 0$ with $\theta, \theta \in B \subseteq \Theta$. Assume Moment and Kernel. Then,

$$E \left[ \sup_{\theta \in B} \left| \tilde{F}_i(\theta, \theta_0) - \tilde{F}_i(\theta, \theta_0) \right| \right] \leq C_{\text{Mom}} C_{\text{Ent}} \gamma_2 \left( B, b \right) \left( n h \right)^{-\frac{1}{2}}.$$

**Proof.** Define

$$Z_i(\theta) = w_i \left( d(y_i, \tilde{g}_i(\theta))^2 - d(y_i, \tilde{g}_i(\theta_0))^2 - E \left[ d(y_i, \tilde{g}_i(\theta))^2 - d(y_i, \tilde{g}_i(\theta_0))^2 \right] \right)$$

We set $y_i$ to be independent of $Y_{x_i}$ and obtain

$$E[|Z_i(\theta)|] \leq E \left[ w_i E \left[ d(y_i, \tilde{g}_i(\theta))^2 - d(y_i, \tilde{g}_i(\theta_0))^2 - d(Y_{x_i}, \tilde{g}_i(\theta))^2 - d(Y_{x_i}, \tilde{g}_i(\theta_0))^2 \mid y_i \right] \right]$$

As $\sup_{x \in \mathcal{X}} E[a(Y_x, Y'_x)] \leq \sigma_\kappa < \infty$, the processes $Z_i$ are integrable. The stochastic processes $Z_1, \ldots, Z_n$ with index set $\Theta$ are independent and integrable. Furthermore, $E[Z_i(\theta)] = 0$ for all $\theta \in \Theta$, and $Z_i(\theta_0) = 0$. They fulfill the following quadruple property: Let $Z'_i$ be independent copies of $Z_i$ with $y_i$ replaced by the independent copy $y'_i$. Then, for $\theta, \theta' \in \Theta$,

$$|Z_i(\theta) - Z_i(\theta') - Z'_i(\theta) + Z'_i(\theta')| \leq w_i a(y_i, y'_i) d(g_i(\theta), g_i(\theta')).$$

As $w_i = 0$ for $\left| \frac{x_i - \theta_0}{\theta_0} \right| > h$, we have

$$w_i d(g_i(x, \theta), g_i(x, \theta')) \leq w_i b(\theta, \theta').$$

Thus, Theorem 6 implies

$$E \left[ \sup_{\theta \in B} \left| \sum_{i=1}^n Z_i(\theta) \right|^n \right] \leq c_\kappa \gamma_2 \left( B, b \right) \left( \sum_{i=1}^n w_i^2 a(y_i, y'_i)^2 \right)^{\frac{n}{2}}.$$
Define \( W = \sum_{i=1}^{n} w_i^2 \) and \( v_i = w_i^2/W \). We obtain, using Jensen’s inequality,

\[
\mathbb{E} \left[ \left( \sum_{i=1}^{n} w_i^2 a(y_i, y_i')^2 \right)^{p/2} \right] = \mathbb{E} \left[ \left( W \sum_{i=1}^{n} v_i a(y_i, y_i')^2 \right)^{p/2} \right] \\
\leq W^{p/2} \sum_{i=1}^{n} v_i \mathbb{E} \left[ a(y_i, y_i')^p \right].
\]

Lemma 4 shows that \( a \) and \( b \) are semi-metrics. Thus, \( \mathbb{E} [a(y_i, y_i')^p] \leq 2^p \mathbb{E} [a(y_i, m_{x_i})^p] \leq 2^p C_{Mom}^\circ \) and, by Lemma 6, \( W \leq \frac{6C_{km} C_{Kma}}{n h} \), we obtain

\[
\mathbb{E} \left[ \sup_{\theta \in B} \left| \hat{F}_t(\theta, \theta_0) - \tilde{F}_t(\theta, \theta_0) \right|^p \right] \leq C_n \left( 2 \left( 6C_{km} C_{Kma} \right)^{1/2} C_{Mom} \gamma_2 (B, b)(n h)^{-1/2} \right)^p.
\]

The bias term can be bounded because of the smoothness assumption again.

**Lemma 9.** Assume Smoothness, R–Variance, and Kernel. Then

\[
\left| \sum_{i=1}^{n} w_i \mathbb{E} [d(Y_{x_i}, g_i(\theta_{t,h}))^2 - d(Y_{x_i}, m_{x_i})^2] \right| \leq C_{Vup} C_{Smo}^2 h^{2\beta}.
\]

**Proof.** By R–Variance and Smoothness

\[
\mathbb{E} \left[ d \left( Y_x, g \left( \frac{x_i - t}{h}, \theta_{t,h} \right) \right)^2 - d(Y_x, m_x)^2 \right] \leq C_{Vup} d \left( g \left( \frac{x_i - t}{h}, \theta_{t,h} \right), m_x \right)^2 \\
\leq C_{Vup} C_{Smo}^2 |x - t|^{2\beta}.
\]

for all \( x, t \in \mathbb{R} \). Hence, Kernel implies

\[
\left| \sum_{i=1}^{n} w_i \mathbb{E} [d(Y_{x_i}, g_i(\theta_{t,h}))^2 - d(Y_{x_i}, m_{x_i})^2] \right| \leq C_{Vup} C_{Smo}^2 \sum_{i=1}^{n} w_i |x_i - t|^{2\beta} \\
\leq C_{Vup} C_{Smo}^2 h^{2\beta}.
\]

A major step for obtaining a bound on the objects of interest instead of their objective function consists in using a peeling device (also called slicing). This technique is applied in the next 3 lemmata. Recall \( U(\theta) = \sum_{i=1}^{n} w_i d(g_i(\theta), m_{x_i})^2 \).

**Lemma 10.** Assume Variance, Smoothness, R–Variance, Moment, Kernel,
Entropy, and Lipschitz. Then

$$\mathbb{E}[U(\hat{\theta}_{t,h})] \leq \frac{C_1}{nh} + C_2 h^{2\beta},$$

where

$$C_1 = c_\kappa C_{\text{Lip}}^2 C_{\text{Vlo}} C_{\text{Mom}}^2 C_{\text{Ent}}^2 C_{\text{Km}} C_{\text{Kma}},$$

$$C_2 = c'_\kappa C_{\text{Vlo}} C_{\text{Vup}} C_{\text{Lip}}^2 C_{\text{Smo}}^2.$$

Proof. Recall

$$\bar{F}_t(\theta) = \frac{1}{n} \sum_{i=1}^{n} w_i \mathbb{E}\left[d(Y_{x,i}, g_i(\theta))^2\right].$$

Assume $U(\hat{\theta}_{t,h}) \in [a, b]$. Then by Variance

$$C_{\text{Vlo}}^{-1} a \leq C_{\text{Vlo}}^{-1} U(\hat{\theta}_{t,h})$$

$$\leq \sum_{i=1}^{n} w_i \mathbb{E}[d(Y_{x,i}, g_i(\hat{\theta}_{t,h}))^2 - d(Y_{x,i}, m_{x,i})^2]$$

$$\leq F_t(\hat{\theta}_{t,h}, \theta_{t,h}) + \sum_{i=1}^{n} w_i \mathbb{E}[d(Y_{x,i}, g_i(\theta_{t,h}))^2 - d(Y_{x,i}, m_{x,i})^2].$$

By Lemma 9

$$\sum_{i=1}^{n} w_i \mathbb{E}[d(Y_{x,i}, g_i(\theta_{t,h}))^2 - d(Y_{x,i}, m_{x,i})^2] \leq C_{\text{Vup}} C_{\text{Smo}}^2 h^{2\beta}.$$

For $b > 0$, let

$$\tilde{B}_b = \left\{ \theta \in \Theta : \sum_{i=1}^{n} w_i d(g_i(\theta), m_{x,i})^2 \leq b \right\}.$$

By the minimizing property of $\hat{\theta}_{t,h}$,

$$\bar{F}_t(\hat{\theta}_{t,h}, \theta_{t,h}) \leq \bar{F}_t(\hat{\theta}_{t,h}, \theta_{t,h}) - \hat{F}_t(\hat{\theta}_{t,h}, \theta_{t,h})$$

$$\leq \sup_{\theta \in \tilde{B}_b} \left| \bar{F}_t(\theta, \theta_{t,h}) - \hat{F}_t(\theta, \theta_{t,h}) \right|.$$
Using Markov’s inequality,

\[ \mathbb{P}(U(\hat{\theta}_{t,h}) \in [a, b]) \leq \mathbb{P}\left( C_{V_{up}} C_{S_{mo}} h^{2\beta} + C_{V_{lo}} \sup_{\theta \in \tilde{B}_b} \left| \bar{F}_t(\theta, \theta_{t,h}) - \hat{F}_t(\theta, \theta_{t,h}) \right| \geq a \right) \]

\[ \leq 2^{\kappa-1} C_{V_{lo}}^{\kappa} C_{V_{up}} C_{S_{mo}} h^{2\beta \kappa} + \mathbb{E}\left[ \sup_{\theta \in \tilde{B}_b} \left| \bar{F}_t(\theta, \theta_{t,h}) - \hat{F}_t(\theta, \theta_{t,h}) \right| \right]. \]

By Lemma 8 with \( \theta_0 = \theta_{t,h} \), with Lemma 12 below and Entropy

\[ \mathbb{E}\left[ \sup_{\theta \in \tilde{B}_b} \left| \bar{F}_t(\theta, \theta_{t,h}) - \hat{F}_t(\theta, \theta_{t,h}) \right| \right] \leq \kappa \left( 2 (6 C_{K_{mi}} C_{K_{ma}})^{\frac{1}{2}} C_{Mom} C_{Ent} (12 C_{K_{mi}} C_{K_{ma}}) \right) \frac{1}{n h}. \]

for \( b \geq 5 C_{Lip}^2 h^{2\beta} + 2 C_{lip} n h \). Thus,

\[ \mathbb{P}(U(\hat{\theta}_{t,h}) \in [a, b]) \leq 2^{\kappa-1} c_1^{\kappa} h^{2\beta \kappa} + \left( c_2^{\frac{1}{2}} (n h)^{-\frac{1}{2}} \right)^{\kappa}, \]

where

\[ c_1 = C_{V_{lo}} C_{V_{up}} C_{S_{mo}}; \]
\[ c_2 = 18 \kappa \frac{1}{2} C_{V_{lo}} C_{Mom} C_{Ent} C_{K_{mi}} C_{K_{ma}}. \]

Thus, by Lemma 11 below

\[ \mathbb{E}[U(\hat{\theta}_{t,h})] \leq 5 C_{Lip}^2 h^{2\beta} + \frac{2 C_{lip}}{n h} + c_1^\kappa \left( c_1 h^{2\beta} + c_2^\frac{1}{2} \right). \]

As all constants are chosen to be in \([1, \infty)\), we obtain the desired result. \( \square \)

**Lemma 11.** Let \( V \) be a nonnegative random variable. Assume that for \( 0 \leq a_0 < a < b < \infty \), it holds

\[ \mathbb{P}(V \in [a, b]) \leq c \frac{u^\kappa + (v b^2)^\kappa}{a^\kappa}. \]

where \( c \geq 1, u, v > 0, \kappa > 2 \). Then

\[ \mathbb{E}[V] \leq a_0 + c_\kappa c^\frac{1}{2} \left( u + v^2 \right). \]
Proof. For \( s > a_0 \),
\[
\mathbb{P}(V > s) \leq \sum_{k=0}^{\infty} \mathbb{P}(V \in [s2^k, s2^{k+1}])
\leq \sum_{k=0}^{\infty} c \frac{u^\kappa + v^\kappa s^{-\frac{1}{2}\kappa} 2^{\frac{1}{2}\kappa} 2^{\frac{1}{2}\kappa} s^k}{s^k 2^{k\kappa}}
\leq c \left( u^\kappa s^{-\kappa} \sum_{k=0}^{\infty} 2^{-k\kappa} + 2^{\frac{1}{2}\kappa} v^\kappa s^{-\frac{1}{2}\kappa} \sum_{k=0}^{\infty} 2^{-\frac{1}{2}k\kappa} \right)
\leq c' c \left( u^\kappa s^{-\kappa} + v^\kappa s^{-\frac{1}{2}\kappa} \right).
\]
We integrate the tail to bound the expectation,
\[
\mathbb{E}[V] \leq a_0 + \int_{a_0}^{\infty} \mathbb{P}(V > s) ds.
\]
For \( A, B \geq 0 \), it holds
\[
\int_0^{\infty} \min(1, As^{-\kappa}) \leq \kappa \frac{A^{\frac{1}{\kappa}}}{1 - \kappa}, \quad \int_0^{\infty} \min(1, Bs^{-\frac{1}{2}\kappa}) \leq \frac{\kappa}{2 - \kappa} B^{\frac{2}{2}}.
\]
Applying these inequalities to the tail bound above, we obtain
\[
\mathbb{E}[V] \leq a_0 + c_c c^{\frac{2}{2}} \left( u + v^2 \right).
\]

**Lemma 12.** For \( b > 0 \), let
\[
\mathcal{B}_b = \left\{ \theta \in \Theta : \int_{-\frac{1}{2}}^{\frac{1}{2}} d(g(x, \theta), g(x, \theta_{t,h}))^2 dx \leq b \right\}
\]
\[
\bar{\mathcal{B}}_b = \left\{ \theta \in \Theta : \sum_{i=1}^{n} w_i d(g_i(\theta), g_i(\theta_{t,h}))^2 \leq b \right\}
\]
\[
\bar{\mathcal{B}}_b = \left\{ \theta \in \Theta : \sum_{i=1}^{n} w_i d(g_i(\theta), m_{x_i})^2 \leq b \right\}
\]
Assume Smoothness, Kernel, and Lipschitz. Then, for all \( b, s > 0 \),
\[
\mathcal{B}_b \subseteq \mathcal{B}_{b+r} \quad \text{and} \quad \bar{\mathcal{B}}_s \subseteq \mathcal{B}_{s'}
\]
where
\[ r = 2C_{\text{Lip}} h^\beta b^\frac{1}{2} + C_{\text{Lip}}^2 h^{2\beta} \]
\[ s' = 6C_{\text{Kmi}} C_{\text{Kma}} \left( s + \frac{2C_{\text{Lip}}}{nh} \right). \]

**Proof.** For $\theta \in \Theta$, we obtain using the triangle inequality
\[
d(g_i(\theta), m_{x_i})^2 - d(g_i(\theta), g_i(\theta_{t,h}))^2 \leq d(m_{x_i}, g_i(\theta_{t,h}))(2d(g_i(\theta), m_{x_i}) + d(m_{x_i}, g_i(\theta_{t,h}))) \leq 2C_{\text{Smo}} |x_i - t|^\beta d(m_{x_i}, g_i(\theta)) + C_{\text{Smo}}^2 |x_i - t|^{2\beta}
\]
because of Smoothness. Thus, Kernel implies
\[
\left| \frac{1}{n} \sum_{i=1}^{n} w_i \left( d(g_i(\theta), m_{x_i})^2 - d(g_i(\theta), g_i(\theta_{t,h}))^2 \right) \right| \leq 2C_{\text{Smo}} \frac{1}{n} \sum_{i=1}^{n} w_i |x_i - t|^\beta d(m_{x_i}, g_i(\theta)) + C_{\text{Smo}}^2 \frac{1}{n} \sum_{i=1}^{n} w_i |x_i - t|^{2\beta}
\]
\[
\leq 2C_{\text{Smo}} h^\beta \frac{1}{n} \sum_{i=1}^{n} w_i d(m_{x_i}, g_i(\theta)) + C_{\text{Smo}}^2 h^{2\beta}.
\]
Now assume $\theta \in \tilde{B}_b$. Then $\sum_{i=1}^{n} w_i d(g_i(\theta), m_{x_i})^2 \leq b$. We obtain, via Jensen’s inequality,
\[
\sum_{i=1}^{n} w_i d(m_{x_i}, g_i(\theta_{t,h})) \leq \left( \frac{\sum_{i=1}^{n} w_i d(m_{x_i}, g_i(\theta_{t,h}))}{\sum_{i=1}^{n} w_i} \right)^{\frac{1}{2}} \leq b^{\frac{1}{2}}.
\]
Together, we get
\[
\left| \frac{1}{n} \sum_{i=1}^{n} w_i \left( d(g_i(\theta), m_{x_i})^2 - d(g_i(\theta), g_i(\theta_{t,h}))^2 \right) \right| \leq 2C_{\text{Smo}} h^\beta b^{\frac{1}{2}} + C_{\text{Smo}}^2 h^{2\beta} =: r.
\]
Thus,
\[
\sum_{i=1}^{n} w_i d(g_i(\theta), g_i(\theta_{t,h}))^2 \leq \sum_{i=1}^{n} w_i d(g_i(\theta), m_{x_i})^2 + r \leq b + r
\]
which shows $\tilde{B}_b \subseteq \tilde{B}_{b+r}$. The relation $\tilde{B}_s \subseteq \tilde{B}_{s'}$ follows from Lemma 5 by
\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} d\left(g(x, \theta), g(x, \tilde{\theta})\right)^2 dx \leq 6C_{\text{Kmi}} C_{\text{Kma}} \left( \sum_{i=1}^{n} w_i d\left(g_i(\theta), g_i(\tilde{\theta})\right)^2 + \frac{2C_{\text{Lip}}}{nh} \right).
\]

Finally, we can put together the results obtained so far to finish the proof of the main theorem.
Proof of Theorem 3. By Lemma 7,
\[
\int_{-1/2}^{1/2} \left( g(x, \theta_{t,h}), g(x, \theta_{t,h}) \right)^2 \, dx \leq 6C_{Kmi}C_{Kma} \left( U(\theta_{t,h}) + 6C_{Kmi}C_{Kma}C_{Smo}^2h^{2\beta} + \frac{2C_{\text{Lip}}}{nh} \right).
\]
By Lemma 10,
\[
\mathbb{E}[U(\hat{\theta}_{t,h})] \leq \frac{C'_1}{nh} + C'_2h^{2\beta},
\]
where
\[
C'_1 = c\kappa C_{\text{Lip}} C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} \left( c\kappa C_{\text{Lip}} C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} + 2C_{\text{Lip}} \right),
\]
\[
C'_2 = c'\kappa C_{\text{Vlo}} C_{\text{Vup}} C_{\text{Lip}} C_{\text{Smo}}^2.
\]
Thus, we obtain
\[
\mathbb{E}\left[ \int_{-1/2}^{1/2} \left( g(x, \hat{\theta}_{t,h}), g(x, \theta_{t,h}) \right)^2 \, dx \right] \leq \frac{C_1}{nh} + C_2h^{2\beta},
\]
where
\[
C_1 = 6C_{Kmi}C_{Kma} \left( c\kappa C_{\text{Lip}} C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} + 2C_{\text{Lip}} \right),
\]
\[
C_2 = 6C_{Kmi}C_{Kma} \left( c'\kappa C_{\text{Vlo}} C_{\text{Vup}} C_{\text{Lip}} C_{\text{Smo}}^2 + 6C_{Kmi}C_{Kma}C_{\text{Smo}}^2 \right).
\]

A.3.2. Corollaries

Corollary 5 and Corollary 6 are direct implications of Theorem 3. We want to prove Corollary 4. It is a consequence of Corollary 5 with \( Q = S^k \) and \( g(x, (q,v)) = \text{Exp}(q, xv) \) for \((q,v) \in \Theta \subseteq TS^k\). To apply this corollary, we need to show Entropy, Lipschitz, and Connection for the sphere, as Variance, Smoothness and Kernel are assumed.

- **Connection:** As \((q,v), (p,u) \in \Theta\), it holds \(|u|, |v| \leq \pi\). Lemma 2 implies
  \[
  \int_{-1/2}^{1/2} d(\text{Exp}(q, xu), \text{Exp}(p, xv))^2 \, dx = \frac{1}{2} \int_{-1}^{1} d\left( \text{Exp}\left(q, \frac{1}{2}xu\right), \text{Exp}\left(p, \frac{1}{2}xv\right)\right)^2 \, dx \\
  \geq \frac{1}{\pi} ||p - q||^2 \\
  \geq \frac{1}{\pi} d_{S^k}(p, q)^2.
  \]
  Thus, we can choose \( C_{\text{Con}} = \pi^2 \).

- **Lipschitz:** Let \( \gamma_1(x) = \text{Exp}(q, xv) \) and \( \gamma_2(x) = \text{Exp}(p, xv) \) be two geodesics. The squared distance \( d(\gamma_1(x), \gamma_2(x))^2 \) can be bounded by \( \pi \)-times the Euclidean distance. Furthermore, it changes not more than the distance of straight lines in
\( \mathbb{R}^{k+1} \) moving in opposite directions. Without loss of generality \( d(\gamma_1(x), \gamma_2(x)) \leq d(\gamma_1(y), \gamma_2(y)) \). Then

\[
\left| d(\gamma_1(x), \gamma_2(x))^2 - d(\gamma_1(y), \gamma_2(y))^2 \right|
\leq (d(\gamma_1(x), \gamma_2(x)) + \pi |x - y| (|u| + |v|))^2 - d(\gamma_1(x), \gamma_2(x))^2
\leq |x - y| \left( \pi^2 |x - y| (|u| + |v|)^2 + \pi d(\gamma_1(x), \gamma_2(x)) (|u| + |v|) \right)
\leq C_{\text{Lip}} |x - y|
\]

with \( C_{\text{Lip}} = 8\pi^4 + 4\pi \), where we used \( |u|, |v| \leq \pi \) for \((q, v), (p, u) \in \Theta\).

- **Entropy:** Lemma 2 implies

\[
\int_{-\frac{1}{2}}^{\frac{1}{2}} d(\text{Exp}(q, xu), \text{Exp}(p, xv))^2 dx \geq \frac{1}{\pi} \left( |p - q|^2 + |u - v|^2 \right).
\]

Thus, \( B_b(\theta_0) \subseteq \{ x \in \mathbb{R}^{2k+2} : |x| \leq \sqrt{\pi b} \} \). By Lemma 1, \( b((q, v), (p, u)) \leq 2\pi (|q - p| + |v - u|) \), yielding \( \gamma_2(B_b, b) \leq c\gamma_2(B_{\sqrt{\pi b}}, |\cdot|) \leq c'\sqrt{kb} \). Thus, we can choose \( C_{\text{Ent}} = c'\sqrt{kb} \).

**A.4. Section 5: LocFre**

First we state some properties of the weights \( w_i \) to be used later.

| Lemma 13 ([Tsy08 Lemma 1.3)] | Assume Kernel. Then there is \( C_{\text{Ker}} \in [1, \infty) \) such that |
|-------------------------------|--------------------------------------------------------------------------------|
| \( \sum_{i=1}^{n} w_i = 1 \), \( w_i = 0 \) if \( |x_i - t| > h \), \( w_i \leq \frac{C_{\text{Ker}}}{nh} \), |
| \( \sum_{i=1}^{n} |w_i| \leq C_{\text{Ker}}, \sum_{i=1}^{n} w_i^2 \leq \frac{C_{\text{Ker}}}{nh} \) |
| for all \( t \in [0, 1] \), \( n \geq n_0 \). |

**A.4.1. Theorem**

We prove [Theorem 4] We first apply the variance inequality to relate a bound on the objective functions to a bound on the minimizers. The required uniform bound on the objective functions can be split into a bias and a variance part, which are bounded separately thereafter. Then, these results are put together in the application of a peeling device, which is used to bound the tail probabilities of the error. Integrating the tails leads to the required bounds in expectation.
Variance Inequality and Split. We define following notation for the objective functions
\[
\hat{F}_t(q) = \sum_{i=1}^n w_i c(y_i, q) \quad \hat{F}_t(q, p) = \hat{F}_t(q) - \hat{F}_t(p),
\]
\[
\hat{F}_t(q) = \sum_{i=1}^n w_i E[c(y_i, q)] \quad \hat{F}_t(q, p) = \hat{F}_t(q) - \hat{F}_t(p),
\]
\[
F_t(q) = E[c(Y_t, q)] \quad F_t(q, p) = F_t(q) - F_t(p).
\]
Using VARIANCE and the minimizing property of \(\hat{m}_t\) we obtain
\[
C_{\square}\left(\hat{m}_t, m_t\right)^{\alpha} \leq \hat{F}_t(m_t, m_t)
\]
\[
\leq \hat{F}_t(\hat{m}_t, m_t) - \hat{F}_t(\hat{m}_t, m_t)
\]
\[
= \left(\hat{F}_t(\hat{m}_t, m_t) - \hat{F}_t(\hat{m}_t, m_t)\right) + \left(\hat{F}_t(\hat{m}_t, m_t) - \hat{F}_t(\hat{m}_t, m_t)\right)
\]

The first parenthesis represents the bias part, the second one the variance part. We will bound the former using SMOOTHNESS, the later by an empirical process argument.

Variance. Define
\[
Z_t(q) = w_i (c(y_i, q) - c(y_i, m_t)) - E[w_i (c(y_i, q) - c(y_i, m_t))].
\]
Then \(Z_1, \ldots, Z_n\) are independent and centered processes with \(Z_t(m_t) = 0\). They are integrable due to MOMENT. By the definition of \(a\),
\[
|Z_t(q) - Z_t(p) - Z_t'(q) + Z_t'(p)| \leq |w_i| a(y_i, y'_i)d(q, p),
\]
where \(Z_t(q)'\) and \(y'_i\) are independent copies of \(Z_t(q)\) and \(y_i\), respectively. Theorem 6 yields
\[
E \left[ \sup_{q \in B(m_t, d, \delta)} \left| \hat{F}_t(q, m_t) - \hat{F}_t(q, m_t) \right| \right] = E \left[ \sup_{q \in B(m_t, d, \delta)} \left| \sum_{i=1}^n Z_t(q) \right| \right]
\]
\[
\leq c_\kappa \left( E \left( \sum_{i=1}^n w_i^2 a(y_i, y'_i)^{2} \right) \right)^{1/2} \gamma_2(B(m_t, d, \delta), d)
\]
for a constant \(c_\kappa\) depending only on \(\kappa\). Define \(W = \sum_{i=1}^n w_i^2\) and \(v_i = w_i^2/W\). We apply MOMENT,
\[
E \left( \sum_{i=1}^n w_i^2 a(y_i, y'_i)^{2} \right)^{2} = E \left( W \sum_{i=1}^n v_i a(y_i, y'_i)^{2} \right)^{2}
\]
\[
\leq E \left( W \sum_{i=1}^n v_i a(y_i, y'_i)^{2} \right)^{2}
\]
\[
= W \sum_{i=1}^n v_i E[a(y_i, y'_i)^{2}]
\]
\[
\leq W \gamma_2 c_{\text{Mom}}^\kappa.
\]
By Lemma 13 \( W \leq C_{\text{Ker}}(nh)^{-1} \). By ENTROPY, \( \gamma_2(B(m_t, d, \delta), d) \leq C_{\text{Ent}} \delta \). Thus,
\[
\mathbb{E}
\left[
\sup_{q \in B(m_t, d, \delta)} \left| F_t(q, m_t) - \tilde{F}_t(q, m_t) \right| \right] ^{\kappa} \leq c_\kappa \left(C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}} \delta (nh)^{-\frac{1}{2}} \right)^{\kappa}.
\]

**Bias.** As \( \sum_{i=1}^{n} w_i = 1 \) (Lemma 13), we have
\[
F_t(q, m_t) - \tilde{F}_t(q, m_t) = \sum_{i=1}^{n} w_i \mathbb{E}[\hat{o}(Y_i, y_i, q, m_t)].
\]
Set \( f(t) = \mathbb{E}[\epsilon(Y_t, q) - \epsilon(Y_t, p)] \). Applying SMOOTHNESS, a Taylor expansion, and the property that the weights annihilate polynomials \( T_{\text{Sy08}} \) equation (1.68), we obtain
\[
\sum_{i=1}^{n} w_i \mathbb{E}[\hat{o}(Y_i, y_i, q, p)] = \sum_{i=1}^{n} w_i \left(R_i + \sum_{k=1}^{\ell} \frac{f'(t)(x_i - t)^k}{k!} \right)
\]
\[
= \sum_{i=1}^{n} w_i R_i
\]
\[
\leq \sum_{i=1}^{n} |w_i||R_i|,
\]
for values \( R_i \in \mathbb{R} \) with \( |R_i| \leq d(q, p)L(q, p)|t - x_i|^\beta \). Thus,
\[
\sum_{i=1}^{n} w_i \mathbb{E}[\hat{o}(Y_i, y_i, q, m_t)] \leq C_{\text{Ker}} d(q, p)L(q, p)h^\beta,
\]
see Lemma 13 Finally we obtain
\[
\mathbb{E}
\left[
\left| F_t(\hat{m}_t, m_t) - \tilde{F}_t(\hat{m}_t, m_t) \right| \mathbb{1}_{[0, \delta]}(d(\hat{m}_t, m_t)) \right] ^{\frac{1}{\kappa}}
\]
\[
\leq \mathbb{E}
\left[
\left| C_{\text{Ker}} d(\hat{m}_t, m_t)L(\hat{m}_t, m_t)h^\beta \mathbb{1}_{[0, \delta]}(d(\hat{m}_t, m_t)) \right| ^{\kappa} \right] ^{\frac{1}{\kappa}}
\]
\[
\leq C_{\text{Ker}} C_{\text{Smo}} \delta h^\beta.
\]

**Peeling.** For \( \delta > 0 \) define
\[
\Delta_t(q, p) = \left( |F_t(q, p) - \tilde{F}_t(q, p)| + |F_t(q, p) - \hat{F}_t(q, p)| \right) \mathbb{1}_{[0, \delta]}(d(q, p)).
\]
Recall that the variance inequality implies
\[
C_{\text{Vlo}}^{-1} d(\hat{m}_t, m_t) \alpha \leq \left( F_t(\hat{m}_t, m_t) - \tilde{F}_t(\hat{m}_t, m_t) \right) + \left( \hat{F}_t(\hat{m}_t, m_t) - \tilde{F}_t(\hat{m}_t, m_t) \right).
\]
Let \( 0 < a < b < \infty \). The inequality above and Markov’s inequality yield
\[
\mathbb{P}(d(\hat{m}_t, m_t) \in [a, b]) \leq \mathbb{P}(a^\alpha \leq C_{\text{Vlo}} \Delta_t(\hat{m}_t, m_t)) \leq \frac{C_{\text{Vlo}}^\alpha \mathbb{E}[\Delta_b(\hat{m}_t, m_t)^\kappa]}{a^\alpha \kappa}.
\]
Our previous consideration allow us the bound the expectation by a variance and a bias term:

\[
\mathbb{E}[\Delta_\delta(\hat{m}_t, m_t)^\kappa] \leq 2^{\kappa-1} \left( \mathbb{E} \left[ \left| F_\delta(\hat{m}_t, m_t) - \hat{F}_\delta(\hat{m}_t, m_t) \right| \mathbb{I}_{[0,\delta]}(d(\hat{m}_t, m_t)) \right] + \mathbb{E} \left[ \sup_{q \in B(m_t, d, \delta)} \left| \hat{F}_\delta(q, m_t) - \hat{F}_\delta(q, m_t) \right| \right] \right)
\]

\[\leq c_\kappa \left( C_{\text{Ker}} C_{\text{Smo}} h^\beta + C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}} (nh)^{-\frac{1}{2}} \right)^\kappa \delta^\kappa .
\]

We are now prepared to apply peeling (also called slicing): Let \( s > 0 \). Set \( A = C_{\text{Vlo}} C_{\text{Ker}} C_{\text{Smo}} h^\beta + C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}} (nh)^{-\frac{1}{2}} \). It holds

\[
\mathbb{P}(d(\hat{m}_t, m_t) > s) \leq \sum_{k=0}^{\infty} \mathbb{P}(d(\hat{m}_t, m_t) \in [2^k s, 2^{k+1} s])
\]

\[\leq \sum_{k=0}^{\infty} c_\kappa A^\kappa (2^{k+1} s^\kappa) \frac{1}{(2^k s)^{\alpha \kappa}}
\]

\[\leq 2^\kappa c_\kappa A^\kappa s^{\kappa(1-\alpha)} \sum_{k=0}^{\infty} 2^{k\kappa(1-\alpha)}
\]

\[\leq \frac{2^\kappa}{1 - 2^{\kappa(1-\alpha)}} c_\kappa A^\kappa s^{\kappa(1-\alpha)} .
\]

We integrate this tail bound to bound the expectation. For this we require \( \kappa > \alpha/(\alpha-1) \).

Set \( B = \frac{2^\kappa}{1 - 2^{\kappa(1-\alpha)}} c_\kappa A^\kappa \), then

\[\mathbb{E}[d(\hat{m}_t, m_t)^\alpha] = \alpha \int_0^{\infty} s^{\alpha-1} \mathbb{P}(d(\hat{m}_t, m_t) > s) ds
\]

\[\leq \alpha \int_0^{\infty} s^{\alpha-1} \min(1, B s^{\kappa(1-\alpha)}) ds
\]

\[= \frac{1}{\alpha} \left( \frac{\alpha}{\kappa(\alpha-1)} + \frac{1}{\kappa(\alpha-1)} \right) B^{\alpha-\kappa(\alpha-1)}
\]

\[= \frac{1}{\alpha} + \frac{1}{\kappa(\alpha-1)} B^{\alpha-\kappa(\alpha-1)}
\]

\[= c_{\kappa, \alpha} A^{\frac{\alpha}{\alpha-1}} .
\]

Thus,

\[\mathbb{E}[d(\hat{m}_t, m_t)^\alpha] \leq c_{\kappa, \alpha} \left( C_{\text{Vlo}} C_{\text{Ker}} C_{\text{Smo}} h^\beta + C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}} (nh)^{-\frac{1}{2}} \right)^{\alpha/(\alpha-1)} .
\]

A.4.2. Corollaries
Lemma 14. In Theorem 4 Smoothness can be replaced by SmoothDensity and BiasMoment when we replace $C_{\text{smo}}$ by $C_{\text{Bom}}C_{\text{smD}}$.

Proof. Using the $\mu$-density $y \mapsto \rho(y|t)$ of $Y_t$, we can write $E[Y_t, q^2 - Y_t, p^2] = \int (\overline{y}, q^2 - \overline{y}, p^2) \rho(y|t)\,d\mu(y)$. By SmoothDensity, $t \mapsto \rho(y|t) \in \Sigma(\beta, L(y))$. Thus, there are $a_k(y)$ such that $\rho(y|x) = R_y(x, x_0) + \sum_{k=0}^{\ell} a_k(y)(x - x_0)^k$ with $|R_y(x, x_0)| \leq L(y)|x - x_0|^\beta$. Using that the weights annihilate polynomials of order $\ell$, we obtain

$$
\sum_{i=1}^{n} w_i E[\hat{\Theta}(Y_t, y_i, q, p)] = \int \sum_{i=1}^{n} w_i \left(\overline{y}, q^2 - \overline{y}, p^2\right) (\rho(y|t) - \rho(y|x_i)) \,d\mu(y) \\
= \int \sum_{i=1}^{n} w_i \left(\overline{y}, q^2 - \overline{y}, p^2\right) R_y(t, x_i) \,d\mu(y) \\
\leq \int \sum_{i=1}^{n} |w_i| \left|\overline{y}, q^2 - \overline{y}, p^2\right| |R_y(t, x_i)| \,d\mu(y).
$$

It holds

$$
|\overline{y}, q^2 - \overline{y}, p^2| |R_y(x, x_0)| \leq \overline{y}, p|x - x_0|^\beta (\overline{y}, q + \overline{y}, p) L(y).
$$

Together with $\sum_{i=1}^{n} |w_i| \leq C_{\text{ker}}$ from Lemma 13 we obtain

$$
\left|\sum_{i=1}^{n} w_i E[\hat{\Theta}(Y_t, y_i, q, p)]\right| \leq C_{\text{ker}} \overline{y}, p \beta \int (\overline{y}, q + \overline{y}, p) L(y) \,d\mu(y)
$$

This replaces equation (2) in the proof of Theorem 4 with $L(q, p) = \int (\overline{y}, q + \overline{y}, p) L(y) \,d\mu(y)$. To make the replacement valid we have to ensure $E[L(m_t, \hat{m}_t)^n]^{\frac{1}{2}} \leq C_{\text{smo}}$. By Cauchy–Schwartz inequality,

$$
\int (\overline{y}, q + \overline{y}, p) L(y) \,d\mu(y) \leq H(q, p) \left(\int L(y)^2 \,d\mu(y)\right)^{\frac{1}{2}} \leq H(q, p) C_{\text{smD}}.
$$

BiasMoment states $E[H(\hat{m}_t, m_t)^n]^{\frac{1}{2}} \leq C_{\text{Bom}}$. Thus, we can choose $C_{\text{smo}} = C_{\text{Bom}} C_{\text{smD}}$. □

Recall $H(q, p) = \left(\int (\overline{y}, q + \overline{y}, p)^2 \,\mu(dy)\right)^{\frac{1}{2}}$.

Proposition 5. Assume BomBound, Variance, Kernel, Moment. To fulfill $E[H(\hat{m}_t, m_t)^n]^{\frac{1}{2}} \leq C_{\text{Bom}}$ in BiasMoment, we can choose

$$
C_{\text{Bom}} = c_n C_{\text{Vlo}} C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} C_{\text{ker}}.
$$
Proof of Proposition 5. Using the triangle inequality

\[
H(q, p) = \int (\frac{q(y) - q(x)}{p(x)})^2 \mu(dy)
\leq \int (\frac{q(y) - q(x)}{p(x)})^2 \mu(dy)
\leq 2 \int q(y)^2 + \int y(p)^2 \mu(dy)
\leq 2 \int q(y)^2 + 8 \int \frac{y}{p} \mu(dy)
\]

as \( \mu \) is a probability measure.

\[
E[H(\hat{m}_t, m_t)^n]^{\frac{1}{n}} \leq E\left[\left(2\bar{m}_t, m_t^2 + 8 \int \frac{y}{m_t} \mu(dy)\right)^\frac{1}{n}\right]
\leq c_n \left(E[\bar{m}_t, m_t^n]^{\frac{1}{n}} + \left(\int \frac{y}{m_t} \mu(dy)\right)^\frac{1}{n}\right).
\]

Next, we will bound \( E[\bar{m}_t, m_t^n] \). Let \( W = \sum_{i=1}^{n} |w_i| \). First, by Variance and the minimizing property of \( \hat{m}_t \),

\[
C_{\text{VIQ}}^{-1}\bar{m}_t, m_t^2 \leq F_i(\hat{m}_t, m_t)
\leq F_i(\hat{m}_t, m_t) - \hat{F}_i(\hat{m}_t, m_t)
= \sum_{i=1}^{n} w_i E[\hat{\angle}(Y_i, y_i, m_t, \hat{m}_t) \mid y_1...n]
\leq \sum_{i=1}^{n} |w_i| \bar{m}_t, m_t^2 E[a(Y_i, y_i) \mid y_i].
\]

Thus,

\[
C_{\text{VIQ}}^{-1}\bar{m}_t, m_t \leq \sum_{i=1}^{n} |w_i| E[a(Y_i, y_i) \mid y_i]
\]

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With Jensen’s inequality

\[
C_{\text{Vlo}}^{-\kappa} \mathbb{E}[m_t \bar{m}_t] \leq \mathbb{E} \left[ \left( \sum_{i=1}^n |w_i| \mathbb{E}[a(Y_t, y_i) | y_i] \right)^\kappa \right]
\]

\[
= W^\kappa \mathbb{E} \left[ \left( \sum_{i=1}^n \frac{|w_i|}{W} \mathbb{E}[a(Y_t, y_i) | y_i] \right)^\kappa \right]
\]

\[
\leq W^\kappa \sum_{i=1}^n \frac{|w_i|}{W} \mathbb{E}[a(Y_t, y_i)]
\]

\[
\leq W^\kappa \sup_{s,t \in [0,1]} \mathbb{E}[a(Y_t, Y_s)].
\]

As \( a \) is a semi-metric,

\[
\mathbb{E}[a(Y_t, Y_s)] \leq \mathbb{E}[\{(a(Y_t, m_t) + a(m_t, m_s) + a(m_s, Y_s))^\kappa\}]
\]

\[
\leq 3^\kappa \left( 2 \sup_{t \in [0,1]} \mathbb{E}[a(Y_t, m_t)] + a(m_t, m_s)^\kappa \right)
\]

\[
\leq c_\kappa (C_{\text{Mom}} + C_{\text{Len}}).
\]

Lemma 13 shows \( W \leq C_{\text{Ker}} \). This completes the proof. \( \square \)

**Proof of Corollary 8** If \( \text{diam}(Q, d) < \infty \), then

\[
H(q, p) \leq \left( \int (2 \text{diam}(Q, d))^2 \mu(dy) \right)^{\frac{1}{2}} = 2 \text{diam}(Q, d)
\]

Thus, we can choose \( C_{\text{Bom}} = 2 \text{diam}(Q, d) \). Using the triangle inequality we get \( \sqrt{\gamma q^2 - \gamma p^2 - \gamma q^2} \leq 4\gamma q \text{diam}(Q, d) \). Thus, \( a(y, z) \leq 4 \text{diam}(Q, d) \) and we can choose \( C_{\text{Mom}} = 4 \text{diam}(Q, d) \). To summarize,

\[
C_1 = c_\kappa, a C_{\text{Vlo}} C_{\text{Ker}} C_{\text{Smo}}
\]

\[
C_2 = c_\kappa, a C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}}
\]

\[
C_{\text{Mom}} = 4 \text{diam}(Q, d)
\]

\[
C_{\text{Smo}} = C_{\text{Bom}} C_{\text{SmD}}
\]

\[
C_{\text{Bom}} = 2 \text{diam}(Q, d).
\]

\( \square \)

**Proof of Corollary 9** VARIANCE holds in Hadamard spaces with \( C_{\text{Vlo}} = 1 \). We bound
\[ \mathbb{E}[H(\hat{m}_t, m_t)]^\frac{1}{2} \leq C_{\text{Bom}} \] using

\[ C_{\text{Bom}} = c_\kappa C_{\text{Vlo}} C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} C_{\text{Ker}} \]

see Proposition 5. To summarize,

\[
\begin{align*}
C_1 &= c_\kappa,\alpha C_{\text{Vlo}} C_{\text{Ker}} C_{\text{Smo}} \\
C_2 &= c_\kappa,\alpha C_{\text{Vlo}} C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}} \\
C_{\text{Vlo}} &= 1 \\
C_{\text{Smo}} &= C_{\text{Bom}} C_{\text{SmD}} \\
C_{\text{Bom}} &= c_\kappa C_{\text{Vlo}} C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} C_{\text{Ker}} \\
(C_1 + C_2)^2 &\leq c_\kappa \left( C^2_{\text{Ker}} C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} C_{\text{Ker}} C_{\text{SmD}} + C_{\text{Mom}} C_{\text{Ent}} C_{\text{Ker}} \right)^2 \\
&\leq c''_\kappa \left( C^2_{\text{Ker}} C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} C_{\text{SmD}} C_{\text{Ent}} \right)^2.
\end{align*}
\]

\[ \square \]

**Proof of Corollary 7.** By Lemma 25 we can choose \( C_{\text{Ent}} = 2\sqrt{k} \). The diameter is \( \text{diam}(S^k, d) = 2\pi \). \( \square \)

### A.4.3. Smoothness of regression function

**Proposition 6** (Smoothness of regression function). Let \((Q, d)\) be a Hadamard space. Assume \( t \to \rho(y|t) \in \Sigma(1, L(y)) \). Assume there are \( C_{\text{Int}}, C_{\text{SmD}} \in (0, \infty) \) with \( \int p(y, m^2) d\mu(y) \leq C^2_{\text{Int}} \) and \( \int L(y)^2 d\mu(y) \leq C_{\text{SmD}} \). Then \( t \mapsto m_t \) is Lipschitz continuous with constant \( C_{\text{Int}} C_{\text{SmD}} \). In particular, we can choose \( C_{\text{Len}} = C_{\text{Int}} C_{\text{SmD}} \).

**Proof of Proposition 6.** Using the variance inequality twice, we have

\[
2m_s, m_t^2 \leq (F_s(m_t, m_s) + F_t(m_s, m_t)) \\
= \int \left( \frac{y, m^2}{y, m_s^2} \right) (p(y|t) - p(y|s)) d\mu(y) \\
\leq m_s, m_t \int \left( \frac{y, m_t}{y, m_s} + \frac{y, m_s}{y, m_t} \right) |p(y|t) - p(y|s)| d\mu(y).
\]
Thus, with the Lipschitz assumption on the density,

\[
\begin{align*}
    \mathbb{E}_{s,m_t} \leq & \frac{1}{2} \int \left( \frac{y}{m_t} + \frac{y}{m_s} \right) |p(y|t) - p(y|s)| \, d\mu(y) \\
    \leq & \frac{1}{2} |s - t| \int \left( \frac{y}{m_t} + \frac{y}{m_s} \right) L(y) \, d\mu(y) \\
    \leq & |s - t| \sup_{t \in [0,1]} \left( \int \frac{y}{m_t}^2 \, d\mu(y) \int L(y)^2 \, d\mu(y) \right)^{\frac{1}{2}} \\
    \leq & |s - t| C_{\text{smD}} C_{\text{Int}}. 
\end{align*}
\]

\[
\square
\]

A.5. Section 7: TriFre

A.5.1. Theorem

We prove Theorem 5. The difference of the objective functions is split into three parts in Lemma 15. In Lemma 16, we use a peeling device and the variance inequality to relate this difference to the distance between the minimizers \( \hat{m}_t \) and \( m_t \), which is the quantity to be bounded in the theorem. Of the three parts, two bias related quantities are bounded in Lemma 17 and Lemma 18 with an auxiliary result in Lemma 19. The third part, a variance term, is bounded in Lemma 20 via chaining. The bounds on the three parts are summarized in Lemma 21. In the end, the integral over \( t \) is applied to calculate the mean integrated squared error. Here, the auxiliary result Lemma 22 is applied.

For shorter notation define \( F_t(q,p) = F_t(q) - F_t(p) \) and \( \hat{F}_t(q,p) = \hat{F}_t(q) - \hat{F}_t(p) \). We introduce the Fourier coefficients \( \vartheta_k(q,p) \) of \( t \mapsto F_t(q,p) \) with respect to the trigonometric basis

\[
\vartheta_k(q,p) = \int_0^1 \psi_k(x) F_x(q,p) \, dx
\]

such that \( F_t(q,p) = \sum_{k=1}^{\infty} \vartheta_k(q,p) \psi_k(t) \) due to SmoothDensity. Define

\[
\begin{align*}
    r_t(q,p) &= \sum_{k=N+1}^{\infty} \vartheta_k(q,p) \psi_k(t), \\
    F^r_t(q,p) &= \Psi_N(t)^\top \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) r_{x_i}(q,p), \\
    \varepsilon_t(y,q,p) &= F_t(q,p) - \left( \frac{y}{q} q^2 - \frac{y}{p} p^2 \right), \\
    F^\varepsilon_t(q,p) &= \Psi_N(t)^\top \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) \varepsilon_{x_i}(y,q,p).
\end{align*}
\]

**Lemma 15.** If \( N < n \), then

\[
F_t(q,p) - \hat{F}_t(q,p) = r_t(q,p) + F^r_t(q,p) - F^\varepsilon_t(q,p).
\]
Proof of Lemma 15. It holds
\[
\frac{1}{n} \sum_{i=1}^{n} \psi_k(x_i) \psi_\ell(x_i) = \delta_{k\ell}
\]
for \(k, \ell \in \{1, \ldots, n-1\}\), see [Tsy08, Lemma 1.7]. Set
\[
F^N_t(q,p) = \sum_{k=1}^{N} \vartheta_k(q,p) \psi_k(t).
\]
Then \(\frac{1}{n} \sum_{i=1}^{n} \psi_k(x_i) F^N_{x_i}(q,p) = \vartheta_k(q,p)\) for \(k \leq N < n\). Thus,
\[
F^N_t(q,p) = \Psi_N(t) \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) F^N_{x_i}(q,p).
\]
As \(F_t(q,p) - r_t(q,p) = F^N_t(q,p)\), we obtain
\[
F_t(q,p) - \hat{F}_t(q,p) - r_t(q,p)
\]
\[
= \Psi_N(t) \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) F^N_{x_i}(q,p) - \Psi_N(t) \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) \left( y_{i,q}^2 - y_{i,p}^2 \right)
\]
\[
= \Psi_N(t) \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) \left( F^N_{x_i}(q,p) - F_{x_i}(q,p) + F_{x_i}(q,p) - \left( y_{i,q}^2 - y_{i,p}^2 \right) \right)
\]
\[
= \Psi_N(t) \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) (-r_{x_i}(q,p) + \varepsilon_{x_i}(y_i,q,p))
\]
\[
= F^*_t(q,p) - F^*_t(q,p).
\]

Next, we apply the peeling device.

**Lemma 16.** For \(b > 0\), define
\[
U_{t,b} = \sup_{q \in B(m_t,b,d)} F^t(q,m_t) + (r_t(\hat{m}_t,m_t) - F^t_t(\hat{m}_t,m_t)) \mathbf{1}_{[0,b]}(\bar{m}_t,m_t)).
\]

Let \(\kappa > 2\). Define
\[
h(t) = \sup_{b>0} \left( \frac{\mathbb{E}[U_{t,b}]}{b^\kappa} \right)^{1/\kappa}
\]
Assume **VARIANCE**. Then
\[
\mathbb{E}\left[ \bar{m}_t,m_t^2 \right] \leq \frac{4\kappa}{\kappa - 2} C_{\lambda b}^2 h(t)^2.
\]
Proof of Lemma 16. For a function \( h(t) > 0 \), we have
\[
\mathbb{E} \left[ \frac{\hat{m}_t, m_t}{h(t)^2} \right] = \int_0^\infty 2s \mathbb{P}(\hat{m}_t, m_t > sh(t)) ds.
\]

By VARIANCE, the minimizing property of \( \hat{m}_t \), and Lemma 15, we obtain
\[
C_{\text{Vlo}}^{-1} \frac{\hat{m}_t, m_t^2}{h(t)^2} \leq F_t(\hat{m}_t, m_t) \\
\leq F_t(\hat{m}_t, m_t) - \hat{F}_t(\hat{m}_t, m_t) \\
= r_t(\hat{m}_t, m_t) + \hat{F}_t(\hat{m}_t, m_t) - F_t(\hat{m}_t, m_t).
\]

If \( \hat{m}_t, m_t \in [a, b] \) for \( 0 < a < b \), then
\[
C_{\text{Vlo}}^{-1} a^2 \leq C_{\text{Vlo}}^{-1} \frac{\hat{m}_t, m_t^2}{h(t)^2} \\
\leq \sup_{q \in B(m_t, b, d)} F_t^* (q, m_t) + (r_t(\hat{m}_t, m_t) - F_t^* (\hat{m}_t, m_t)) 1_{[0, b]}(\hat{m}_t, m_t) \\
= U_t, b.
\]

Thus, by Markov’s inequality
\[
\mathbb{P}(\hat{m}_t, m_t \in [a, b]) \leq \mathbb{P}\left( a^2 \leq C_{\text{Vlo}} U_t, b \right) \leq \frac{C_{\text{Vlo}} \mathbb{E}[U_{t, b}^\kappa]}{a^{2\kappa}}.
\]

Let \( a_k(s) = 2^k h(t) \). As \( \mathbb{E}[U_{t, b}^\kappa] \leq b^\kappa h(t)^\kappa \), we have
\[
\mathbb{P}(\hat{m}_t, m_t > sh(t)) \leq \min \left( 1, \sum_{k=0}^\infty \mathbb{P}(\hat{m}_t, m_t \in [a_k, a_{k+1}]) \right) \\
\leq \min \left( 1, C_{\text{Vlo}} \sum_{k=0}^\infty \frac{a_{k+1}^\kappa h(t)^\kappa}{a_k^{2\kappa}} \right).
\]

We obtain
\[
\frac{a_{k+1}^\kappa h(t)^\kappa}{a_k^{2\kappa}} = \left( \frac{2^{k+1} h(t)}{2^k h(t)} \right)^\kappa = \left( \frac{2 \cdot 2^k s h(t) h(t)}{2^{2k} s^2 h(t)^2} \right)^\kappa = \left( 2 \cdot 2^{-k} s^{-1} \right)^\kappa
\]
and thus
\[
\sum_{k=0}^\infty \frac{a_{k+1}^\kappa h(t)^\kappa}{a_k^{2\kappa}} = 2^\kappa s^{-\kappa} \sum_{k=0}^\infty 2^{-k\kappa} = \frac{2^\kappa}{1 - 2^{-\kappa}} s^{-\kappa}
\]

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Putting everything together with $c_\kappa = \frac{2^\kappa}{1-2^{-\kappa}} C_{Vlo}^\kappa$ yields

$$h(t)^{-2} \mathbb{E} \left[ \left( \hat{m}_t, \hat{m}_t \right)^2 \right] = 2 \int_0^\infty s \mathbb{P} \left( \hat{m}_t, \hat{m}_t > sh(t) \right) ds$$
$$\leq 2 \int_0^\infty s \min(1, c_\kappa s^{-\kappa}) ds$$
$$= \int_0^{c_\kappa^{-\frac{1}{\kappa}}} 2 s ds + 2 c_\kappa \int_{c_\kappa^{-\frac{1}{\kappa}}}^{\infty} s^{1-\kappa} ds$$
$$= \frac{c_\kappa^{\frac{2}{\kappa}}}{\kappa} + 2 c_\kappa \frac{1}{\kappa - 2} \left( c_\kappa^{-\frac{1}{\kappa}} \right)^{2-\kappa}$$
$$= \frac{c_\kappa^{\frac{2}{\kappa}}}{\kappa - 2} \left( 1 + \frac{2}{\kappa - 2} \right)$$
$$\leq \frac{4}{\kappa - 2} C_{Vlo}^2.$$

Using the smoothness assumption, we are able to bound the $r$-term.

**Lemma 17 (Bound on $r$).** Assume SmoothDensity. Then

$$\mathbb{E} \left[ |r_t(\hat{m}_t, m_t)|^\kappa \right] \leq b_\kappa h_N(t)^\kappa C_{Bom}^\kappa,$$

where

$$h_N(t) = \left( \int \left( \sum_{\ell=N+1}^\infty \xi_\ell(y) \psi_\ell(t) \right)^2 \mu(dy) \right)^{\frac{1}{2}}$$

and

$$H(q, p) = \left( \int \left( \frac{yq - yp}{2} \right)^2 \mu(dy) \right)^{\frac{1}{2}}.$$

**Proof.** It holds

$$\vartheta_k(q, p) = \int_0^1 \psi_k(x) F_x(q, p) dx$$
$$= \int_0^1 \int \psi_k(x) \left( \frac{yq}{2} - \frac{yp}{2} \right)^2 \rho(y|x) d\mu(y) dx$$
$$= \int \left( \frac{yq}{2} - \frac{yp}{2} \right) \int_0^1 \psi_k(x) \rho(y|x) dx d\mu(y)$$
$$= \int \left( \frac{yq}{2} - \frac{yp}{2} \right) \xi(y) d\mu(y).$$

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Thus,
\[ r_t(q, p) = \int \left( \frac{\gamma q^2 - \gamma p^2}{\gamma q^2 - \gamma p^2} \right) \sum_{\ell = N+1}^{\infty} \xi_\ell(y) \psi_\ell(t) \mu(dy) \]
\[ \leq \left( \int \left( \frac{\gamma q^2 - \gamma p^2}{\gamma q^2 - \gamma p^2} \right)^2 \mu(dy) \right)^{\frac{1}{2}} \left( \int \left( \sum_{\ell = N+1}^{\infty} \xi_\ell(y) \psi_\ell(t) \right)^2 \mu(dy) \right)^{\frac{1}{2}} \]
\[ \leq \frac{q}{p} H(q, p) h_N(t) . \]

Finally, we obtain
\[ \mathbb{E}[|r_t(\hat{m}_t, m_t)|^\kappa 1_{[0,b]}(\hat{m}_t, m_t)] \leq b^\kappa h_N(t)^\kappa \mathbb{E}[H(\hat{m}_t, m_t)^\kappa] . \]

Using the previous result, we can also establish a bound on \( F^r \).

**Lemma 18** (Bound on \( F^r \)).
\[ \mathbb{E}[F^r_t(\hat{m}_t, m_t)^\kappa 1_{[0,b]}(\hat{m}_t, m_t)] \leq c_\kappa \left( N n^{1-2\beta} C_{\text{SmD}} \right)^\kappa b^n C_{\text{Bom}} \]
where \( c_\kappa \in [1, \infty) \) depends only on \( \kappa \).

**Proof.** We will show that asymptotically \( F^r_t(q, p) \lesssim r_t(q, p) \). Recall
\[ F^r_t(q, p) = \Psi_N(t)^{-1} \sum_{i=1}^{n} \Psi_N(x_i) r_{x_i}(q, p) \]
\[ r_t(q, p) = \sum_{k=N+1}^{\infty} \vartheta_k(q, p) \psi_k(t) \]
and define
\[ r_{n,t}(q, p) = \sum_{\ell=n}^{\infty} \vartheta_\ell(q, p) \psi_\ell(t) \]
It holds
\[ F^r_t(q, p) \leq \| \Psi_N(t) \| \| \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) r_{x_i}(q, p) \| \]
By **Lemma 19** below, to be shown below,
\[ \| \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) r_{x_i}(q, p) \|^2 \leq \frac{1}{n} \sum_{i=1}^{n} r_{x_i}(q, p)^2 \]
As in the proof of Lemma 17 we have

\[ |r_{n,t}(q,p)| \leq q \cdot ph_n(t)^2 H(q,p) , \]

where

\[ h_n(t)^2 = \int \left( \sum_{\ell=n}^{\infty} \xi_\ell(y) \psi_\ell(t) \right)^2 \mu(dy) \]

Thus,

\[ F_t^r(q,p)^2 \leq q^2 H(q,p)^2 \| \Psi_N(t) \|^2 \frac{1}{n} \sum_{i=1}^{n} h_n(x_i)^2 \]

\[ \| \Psi_N(t) \|^2 \leq 2N \]

As \( \xi(y) \in \Theta(\beta, L(y)) \), we have \( \sum_{k=1}^{\infty} \xi_k(y)^2 w_k^{-2} \leq L(y)^2 \) with \( w_{2k+1} = w_{2k} = (2k)^{-\beta} \).

\[ \sum_{k=n}^{\infty} w_k^2 \leq cn^{1-2\beta} \]

Thus,

\[ \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=n}^{\infty} \xi_k(y) \psi_k(x_i) \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} \sum_{k=n}^{\infty} w_k^{-2} \xi_k(y)^2 \sum_{k=n}^{\infty} w_k^2 \psi_k(x_i)^2 \]

\[ \leq 2 \sum_{k=n}^{\infty} w_k^{-2} \xi_k(y)^2 \sum_{k=n}^{\infty} w_k^2 \]

\[ \leq c_0 L(y)^2 n^{1-2\beta} . \]

We obtain

\[ \frac{1}{n} \sum_{i=1}^{n} h_n(x_i)^2 \leq \frac{1}{n} \sum_{i=1}^{n} \int \left( \sum_{\ell=n}^{\infty} \xi_\ell(y) \psi_\ell(x_i) \right)^2 \mu(dy) \]

\[ \leq c_0 n^{1-2\beta} \int L(y)^2 \mu(dy) \]

and can bound

\[ F_t^r(q,p)^2 \leq 2c_0 q^2 H(q,p)^2 N n^{1-2\beta} \int L(y)^2 \mu(dy) . \]

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Finally, the inequalities above yield
\[
E[F_t(\hat{m}_t, m_t)^\kappa \mathbf{1}_{[0,b]}(\hat{m}_t, m_t)] \leq \left(2c_0 N n^{1-2\beta} \int L(y)^2 \mu(dy) \right)^{\frac{n}{2}} \eta^n E[H(\hat{m}_t, m_t)^\kappa].
\]

We still have to prove following lemma, which was used in the previous proof.

**Lemma 19.** Let \( f : [0,1] \to \mathbb{R} \) be any function and \( N < n \). Then
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i) f(x_i) \right\|^2 \leq \frac{1}{n} \sum_{i=1}^{n} f(x_i)^2
\]

**Proof of Lemma 19** Let \( a_\ell = \frac{1}{n} \sum_{i=1}^{n} \psi_\ell(x_i) f(x_i) \) and \( s(t) = f(t) - \sum_{\ell=1}^{N} a_\ell \psi_\ell(t) \). Then
\[
\frac{1}{n} \sum_{i=1}^{n} f(x_i) \psi_k(x_i) = \frac{1}{n} \sum_{i=1}^{n} \left( f(x_i) - \sum_{\ell=1}^{N} a_\ell \psi_\ell(x_i) \right) \psi_k(x_i)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} f(x_i) \psi_k(x_i) - \sum_{\ell=1}^{N} a_\ell \frac{1}{n} \sum_{i=1}^{n} \psi_\ell(x_i) \psi_k(x_i)
\]
\[
= a_k - a_k
\]
\[
= 0
\]
and thus
\[
\frac{1}{n} \sum_{i=1}^{n} f(x_i)^2 = \frac{1}{n} \sum_{i=1}^{n} \left( s(x_i) + \sum_{\ell=1}^{N} a_\ell \psi_\ell(x_i) \right)^2
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} \left( s(x_i)^2 + s(x_i) \sum_{\ell=1}^{N} a_\ell \psi_\ell(x_i) + \sum_{\ell,k=1}^{N} a_\ell a_k \psi_\ell(x_i) \psi_k(x_i) \right)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} s(x_i)^2 + \sum_{\ell=1}^{N} a_\ell \frac{1}{n} \sum_{i=1}^{n} s(x_i) \psi_\ell(x_i) + \sum_{\ell,k=1}^{N} a_\ell a_k \frac{1}{n} \sum_{i=1}^{n} \psi_\ell(x_i) \psi_k(x_i)
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} s(x_i)^2 + \sum_{\ell} a_\ell^2.
\]
Furthermore,
\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i)f(x_i) \right\|^2 = \sum_{\ell=1}^{N} (\psi(x_i)f(x_i))^2 = \sum_{\ell=1}^{N} a^2_{\ell}
\]
As \( \frac{1}{n} \sum_{i=1}^{n} s(x_i)^2 \geq 0 \) we have proved the claim.

Next, we tackle the variance term.

**Lemma 20** (Bound on \( F^\varepsilon \)). Assume Moment, Entropy. Then there is a constant \( c > 0 \) depending only on \( \kappa \) such that
\[
E \left[ \sup_{q \in B} F^\varepsilon_t(q,p)^\kappa \right] \leq c n \gamma_{\kappa}^2 \gamma_{\kappa}^2 (B,d)^\kappa.
\]

*Proof of Lemma 20.* Recall \( F^\varepsilon_t(q,p) = \Psi_N(t)^\top \frac{1}{n} \sum_{i=1}^{n} \Psi_N(x_i)\varepsilon_{x_i}(y_i,q,p) \). Define \( \alpha_i = \frac{1}{n} \Psi_N(t)^\top \Psi_N(x_i), \varepsilon_{x_i}(q,p) = \varepsilon_{x_i}(y_i,q,p) \). Then
\[
F^\varepsilon_t(q,p) = \sum_{i=1}^{n} \alpha_i \varepsilon_{x_i}(q,p),
\]
where \( \varepsilon_1, \ldots, \varepsilon_n \) are independent and \( E[\varepsilon_{x_i}(q,p)] = 0 \). We want to apply Theorem 6 with \( Z_i(q) - Z_i(p) = \alpha_i \varepsilon_{x_i}(q,p) \) and \( A_i = \alpha_i a(y_i,y'_i) \). We need to show
\[
|Z_i(q) - Z_i(p) - Z'_i(q) + Z'_i(p)| \leq A_i q \cdot p
\]
to obtain
\[
E \left[ \sup_{q \in B} \left| \sum_{i=1}^{n} Z_i(q) \right|^\kappa \right] \leq C E[||A||^2] \gamma_2(B,d)^\kappa.
\]
Using the quadruple property, we obtain
\[
\varepsilon_i(q,p) - \varepsilon'_i(q,p) = \left( F(q,p,x_i) - (\bar{y}_i q^2 - \bar{y}_i p^2) \right) - \left( F(q,p,x_i) - (\bar{y}_i q^2 - \bar{y}_i p^2) \right)
\]
\[
\leq a(y_i,y'_i) q \cdot p.
\]
Thus, Theorem 6 yields
\[
E \left[ \sup_{q \in B} F^\varepsilon_t(q,p)^\kappa \right] \leq C \gamma_2(B,d)^\kappa E \left[ \left( \sum_{i=1}^{n} \alpha^2_i a(y_i,y'_i)^2 \right)^{\frac{\kappa}{2}} \right].
\]

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Let $a_i = \frac{\alpha_i^2}{\sum_{i=1}^{\alpha_i^2}}$.

$$
E \left[ \left( \sum_{i=1}^{n} \alpha_i^2 a(y_i, y'_i)^2 \right)^{\frac{\kappa}{2}} \right] = \left( \sum_{i=1}^{n} \alpha_i^2 \right)^{\frac{\kappa}{2}} E \left[ \left( \sum_{i=1}^{n} a_i a(y_i, y'_i)^2 \right)^{\frac{\kappa}{2}} \right] \\
\leq \left( \sum_{i=1}^{n} \alpha_i^2 \right)^{\frac{\kappa}{2}} E \left[ \sum_{i=1}^{n} a_i a(y_i, y'_i)^\kappa \right] \\
= \left( \sum_{i=1}^{n} \alpha_i^2 \right)^{\frac{\kappa}{2}} \sum_{i=1}^{n}a_i E[a(y_i, y'_i)^\kappa] \\
\leq \left( \sum_{i=1}^{n} \alpha_i^2 \right)^{\frac{\kappa}{2}} \sup_t E[a(Y_t, Y'_t)^\kappa].
$$

As $a$ is a semi metric, we have, using Moment,

$$
E[a(Y_t, Y'_t)^\kappa] \leq 2^\kappa C_{Mom}^\kappa.
$$

Furthermore, it holds

$$
\sum_{i=1}^{n} \alpha_i^2 = \frac{1}{n^2} \sum_{i=1}^{n} \Psi_N(t)^\top \Psi_N(x_i) \Psi_N(x_i)^\top \Psi_N(t) = \frac{1}{n} \Psi_N(t)^\top \Psi_N(t).
$$

Together we get

$$
E \left[ \sup_{q \in B} F_t^\kappa(q, p)^\kappa \right] \leq c_\kappa C_{Mom}^\kappa n^{-\frac{\kappa}{2}} \gamma_2(B, d)^\kappa \left( \Psi_N(t)^\top \Psi_N(t) \right)^{\frac{\kappa}{2}}.
$$

Finally, we put the previous results together to proof our main theorem of this section.

\begin{lemma}
There is a constant $c > 0$ depending only on $\kappa$ such that

$$
h(t)^\kappa \leq c_\kappa \left( h_N(t)^\kappa C_{Bom}^{\kappa} + \left( N n^{1-2\beta} C_{SmD} \right)^\kappa C_{Bom}^{\kappa} + C_{Mom}^{\kappa} n^{-\frac{\kappa}{2}} C_{Ent}^{\kappa} \| \Psi_N(t) \|^\kappa \right)
$$
\end{lemma}

\begin{proof}
\textbf{[Lemma 21]} \textbf{[Lemma 17]} \textbf{[Lemma 18]} and \textbf{[Lemma 20]}
\end{proof}

\begin{lemma}
For the function $h_N$ defined in \textbf{[Lemma 17]} it holds

$$
\int_0^1 h_N(t)^2 dt \leq c_\beta N^{-2\beta} C_{SmD}^2.
$$
\end{lemma}
Proof of Lemma 22. We use Fubini’s theorem and the weights \( w_{2k+1} = w_{2k} = (2k)^{-\beta} \) from the definition of the ellipsoid \( \Theta(\beta, L) \) and obtain

\[
\int_0^1 h_N(t)^2 \, dt = \int_0^1 \left( \sum_{\ell=N+1}^{\infty} \xi_\ell(y) \psi_\ell(t) \right)^2 \, d\mu(y)
\]

\[
= \int_0^1 \left( \sum_{\ell=N+1}^{\infty} \xi_\ell(y) \psi_\ell(t) \right)^2 \, d\mu(y) dt
\]

\[
= \int \sum_{\ell=N+1}^{\infty} \xi_\ell(y)^2 d\mu(y)
\]

\[
\leq \int w_{N+1}^2 \sum_{\ell=N+1}^{\infty} \xi_\ell(y)^2 w_\ell^{-2} d\mu(y)
\]

\[
\leq c\beta N^{-2\beta} \int L(y)^2 d\mu(y).
\]

Proof of Theorem 5. We apply Lemma 16, Lemma 21, and Lemma 22 together with

\[
\int_0^1 \| \Psi_N(t) \|^2 \, dt = \int_0^1 N \sum_{\ell=1}^{N} \psi_\ell(t)^2 dt = N
\]

to finally obtain

\[
\int_0^1 E \left[ \bar{m}_t, \bar{m}_t^2 \right] dt \leq \frac{4\kappa}{\kappa - 2} C_{\text{Vlo}}^2 \int_0^1 h(t)^2 \, dt
\]

\[
\leq c_\kappa C_{\text{Vlo}}^2 \left( C_{\text{Bom}}^2 \int_0^1 h_N(t)^2 \, dt + N n_{1-2\beta} C_{\text{SmD}}^2 C_{\text{Bom}}^2 + C_{\text{Mom}}^2 n_{-1} C_{\text{Ent}}^2 \int_0^1 \| \Psi_N(t) \|^2 \, dt \right)
\]

\[
\leq c_{\kappa, \beta} C_{\text{Vlo}}^2 \left( C_{\text{Bom}}^2 C_{\text{SmD}}^2 N^{-2\beta} + C_{\text{SmD}}^2 C_{\text{Bom}}^2 N n_{1-2\beta} + C_{\text{Mom}}^2 C_{\text{Ent}}^2 \frac{N}{n} \right).
\]

\[
A.5.2. \text{Corollaries}
\]

We first need to prove an auxiliary results before we can tackle the corollaries themselves.

Recall \( H(q, p) = \left( \int (\bar{y}, q + \bar{y}, p)^2 \mu(dy) \right)^{\frac{1}{2}} \).

**Proposition 7.** Assume BOMBON. To fulfill \( E[H(\bar{m}_t, m_t)^{\frac{1}{2}}] \leq C_{\text{Bom}}, \) we can
choose

\[ C_{\text{Bom}} = c_\kappa C_{\text{Len}} C_{\text{Mom}} C_{\text{Int}} \left( 1 + \log(N) + \frac{N^2}{n} \right) \]

where \( c_\kappa > 0 \) depends only on \( \kappa \).

This proposition is proven in two steps: Lemma 23 and Lemma 24. Let \( w_i = \frac{1}{n} |\Psi_N(t)\Psi_N(x_i)| \) and \( W = \sum_{i=1}^{n} |w_i| \).

**Lemma 23.** Assume Variance. There is a constant \( c_\kappa \in [1, \infty) \) depending only on \( \kappa \) such that

\[
E[H(\hat{m}_t, m_t)^\kappa] \leq c_\kappa \left( C_{\text{Vlo}} W (C_{\text{Len}} + C_{\text{Mom}}) + C_{\text{Int}} \right).
\]

**Proof of Lemma 23.** Using the triangle inequality

\[
H(q, p)^2 = \int (y_q + y_p)^2 \mu(dy) 
\leq \int (q_y + 2y_p)^2 \mu(dy) 
\leq 2 \int q_y^2 + 4y_p^2 \mu(dy) 
\leq 2q_y^2 + 8 \int y_p^2 \mu(dy)
\]
as \( \mu \) is a probability measure.

\[
E[H(\hat{m}_t, m_t)^\kappa] \leq E \left[ \left( 2\hat{m}_t, m_t \right)^2 + 8 \int \hat{y}_m^2 \mu(dy) \right]^\frac{\kappa}{2} 
\leq c_\kappa \left( E[\hat{m}_t, m_t]^\frac{\kappa}{2} + \left( \int \hat{y}_m^2 \mu(dy) \right)^\frac{\kappa}{2} \right)
\]

Next, we will bound \( E[\hat{m}_t, m_t]^\kappa \). First, by Variance and the minimizing property of \( \hat{m}_t \),

\[
C_{\text{Vlo}}^{-1} \hat{m}_t, m_t^2 \leq F_t(\hat{m}_t, m_t) 
\leq F_t(\hat{m}_t, m_t) - \hat{F}_t(\hat{m}_t, m_t) 
\leq \sum_{i=1}^{n} |w_i| \hat{m}_t, m_t E[a(Y_t, y_i) \mid y_i]
\]

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Thus,
\[
C^{-\alpha}_{\text{lo}} m_t, \hat{m}_t \leq \sum_{i=1}^{n} |w_i| E[\alpha(Y_t, y_i) | y_i]
\]

With Jensen’s inequality
\[
C^{-\alpha}_{\text{lo}} E[m_t, \hat{m}_t] \leq \left( \sum_{i=1}^{n} \frac{|w_i|}{W} E[\alpha(Y_t, y_i) | y_i] \right)^\alpha
\]
\[
= W^\alpha E \left[ \left( \sum_{i=1}^{n} \frac{|w_i|}{W} E[\alpha(Y_t, y_i) | y_i] \right)^\alpha \right]
\]
\[
\leq W^\alpha \sum_{i=1}^{n} \frac{|w_i|}{W} E[\alpha(Y_t, y_i)^\alpha]
\]
\[
\leq W^\alpha \sup_{s,t \in [0,1]} E[\alpha(Y_t, Y_s')^\alpha].
\]

As \(\alpha\) is a semi-metric,
\[
E[\alpha(Y_t, Y_s')^\alpha] \leq E[(\alpha(Y_t, m_t) + \alpha(m_t, m_s) + \alpha(m_s, Y_s')^\alpha]
\]
\[
\leq 3^\alpha \left( 2 \sup_{t \in [0,1]} E[\alpha(Y_t, m_t)^\alpha] + \alpha(m_t, m_s)^\alpha \right)
\]
\[
\leq c_\alpha (C_{\text{Mom}}^\alpha + C_{\text{Len}}^\alpha).
\]

\[\Box\]

**Lemma 24.** There is an universal constant \(c \in (0, \infty)\) such that
\[
W \leq c \left( 1 + \log(N) + \frac{N^2}{n} \right).
\]

**Proof of Lemma 24.** Let \(g_t(s) = \sum_{t=1}^{N} \psi(t) \psi_t(s)\). Then
\[
W = \sum_{i=1}^{n} |w_i| = \frac{1}{n} \sum_{i=1}^{n} \left| \Psi_N(t)^T \Psi_N(x_i) \right| = \frac{1}{n} \sum_{i=1}^{n} g_t(x_i).
\]

By the standard comparison between an integral of a Lipschitz–continuous function an
the corresponding Riemann sum, we obtain
\[
\left| \int_0^1 g_t(s) \, ds - \frac{1}{n} \sum_{i=1}^n g_t(x_i) \right| \leq \sup_{s \in [0,1]} \frac{|g'_t(s)|}{n} \leq 4\pi \frac{N^2}{n}.
\]
This bound is quite rough and could be improved. But we will choose \( N \leq \frac{n}{4} \) and thus \( \frac{N^2}{n} \to 0 \). For \( x \in \mathbb{R} \) denote \([x]\) the fractional part of \( x \), i.e., the number \([x] \in [0,1)\) that fulfills \([x] = x - k\) for a \( k \in \mathbb{Z} \). For \( \ell \geq 2\),
\[
\psi_\ell(t) \psi_\ell(s) = \frac{1}{2} \left( (-1)^\ell \cos(2\pi \ell[t + s]) + \cos(2\pi \ell[t - s]) \right).
\]
The function \( (s,t) \mapsto \sum_{\ell=1}^N \psi_\ell(t) \psi_\ell(s) \) only depends on \([s + t]\) and \([s - t]\). When integrating \( s \) from 0 to 1, \([s + t]\) and \([s - t]\) run through every value in \([0,1)\). Thus
\[
\sup_{t \in [0,1]} \int_0^1 \left| 1 + \sum_{\ell=2}^N \psi_\ell(t) \psi_\ell(s) \right| \, ds = \sup_{t \in [0,1]} \int_0^1 \left| 1 + \frac{1}{2} \sum_{\ell=2}^N \left( (-1)^\ell \cos(2\pi \ell[t + s]) + \cos(2\pi \ell[t - s]) \right) \right| \, ds
\leq 1 + \frac{1}{2} \sup_{t \in [0,1]} \int_0^1 \left| \sum_{\ell=2}^N (-1)^\ell \cos(2\pi \ell[t + s]) \right| \, ds + \frac{1}{2} \sup_{t \in [0,1]} \int_0^1 \left| \sum_{\ell=2}^N \cos(2\pi \ell[t - s]) \right| \, ds
= 1 + \frac{1}{2} \int_0^1 \left| \sum_{\ell=2}^N (-1)^\ell \cos(2\pi \ell s) \right| \, ds + \frac{1}{2} \int_0^1 \left| \sum_{\ell=2}^N \cos(2\pi \ell s) \right| \, ds.
\]
Lagrange’s trigonometric identities state
\[
2 \sum_{\ell=1}^L \cos(\ell x) = -1 + \frac{\sin \left( \left( L + \frac{1}{2} \right) x \right)}{\sin \left( \frac{x}{2} \right)},
\]
\[
2 \sum_{\ell=1}^L (-1)^\ell \cos(\ell x) = -1 + \frac{(-1)^{L+1} \sin \left( \left( L + \frac{1}{2} \right) x \right)}{-\sin \left( \frac{x}{2} \right)}.
\]
Thus, we have to bound the integral
\[
\int_0^1 \left| \frac{\sin((2L + 1)\pi s)}{\sin(\pi s)} \right| \, ds.
\]
It holds \(|\sin(\pi x)| \geq \frac{1}{2}\pi \min(x, 1-x)\) for \(x \in [0, 1]\). Let \(a = k\pi\) for \(k \in \mathbb{N}\). Then

\[
\int_0^1 \left| \frac{\sin(as)}{\sin(\pi s)} \right| ds = \frac{2}{\pi} \int_0^1 \frac{|\sin(as)|}{\min(s, 1-s)} ds = \frac{4}{\pi} \int_0^{\frac{1}{2}} \frac{|\sin(as)|}{s} ds = \frac{4}{\pi} \int_0^{\frac{1}{2}} \frac{|\sin(t)|}{t} dt.
\]

We bound this integral as follows,

\[
\int_0^{\frac{1}{2}k\pi} \frac{|\sin(t)|}{t} dt = \int_0^{\pi} \frac{|\sin(t)|}{t} dt + \int_\pi^{\frac{1}{2}k\pi} \frac{|\sin(t)|}{t} dt \leq \int_0^{\pi} \frac{\sin(t)}{t} dt + \int_\pi^{\frac{1}{2}k\pi} \frac{1}{t} dt \leq 2 + \log(\frac{1}{2}k\pi) - \log(\pi) = 2 + \log(\frac{1}{2}k).
\]

Thus, we obtain

\[
\int_0^1 \left| \frac{\sin(2k\pi s)}{\sin(\pi s)} \right| ds \leq \frac{8}{\pi} + \frac{4}{\pi} \log\left(\frac{1}{2}k\right),
\]

which yields

\[
\sup_{t \in [0,1]} \int_0^1 \left| 1 + \sum_{\ell=2}^N \psi(t) \psi(s) \right| ds \leq c_0 + c_1 \log(N).
\]

\[
\square
\]

**Proof of Corollary 11** If \(\text{diam}(Q, d) < \infty\), then

\[
H(q, p) \leq \left( \int (2 \text{diam}(Q, d))^2 \mu(dy) \right)^{\frac{1}{2}} = 2 \text{diam}(Q, d).
\]

Thus, we can choose \(C_{\text{Bom}} = 2 \text{diam}(Q, d)\). Using the triangle inequality we get \(\sqrt{yq^2} - \sqrt{yp^2} - \sqrt{qy^2} \leq 4 \text{diam}(Q, d)\). Thus, \(a(y, z) \leq 4 \text{diam}(Q, d)\) and we can choose \(C_{\text{Mom}} = 4 \text{diam}(Q, d)\).

**Proof of Corollary 12** Variance holds in Hadamard spaces with \(C_{\text{Vlo}} = 1\). We bound

\[
\mathbb{E}[H(\hat{m}_t, m_t)^2] \leq C_{\text{Bom}} \left( 1 + \log(N) + \frac{N^2}{n} \right),
\]

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see Proposition 7.

**B. Chaining**

**Definition 3** (Measures of Entropy \[Tal14\]).

(i) Given a set \( Q \) an *admissible sequence* is an increasing sequence \((A_k)_{k \in \mathbb{N}_0}\) of partitions of \( Q \) such that \( A_0 = Q \) and \( \text{card}(A_k) \leq 2^{2^k} \) for \( k \geq 1 \).

By an increasing sequence of partitions we mean that every set of \( A_{k+1} \) is contained in a set of \( A_k \). We denote by \( A_k(q) \) the unique element of \( A_k \) which contains \( q \in Q \).

(ii) Let \((Q, d)\) be a pseudo-metric space. Define

\[
\gamma_2(Q, d) = \inf \sup_{q \in Q} \sum_{k=0}^{\infty} 2^{\frac{k}{2}} \text{diam}(A_k(q), d),
\]

where the infimum is taken over all admissible sequences in \( Q \) and

\[
\text{diam}(A, d) = \sup_{q, p \in A} d(q, p)
\]

for \( A \subseteq Q \).

**Theorem 6** (Empirical process bound). Let \((Q, d)\) be a separable pseudo-metric space and \( B \subseteq Q \). Let \( Z_1, \ldots, Z_n \) be centered, independent, and integrable stochastic processes indexed by \( Q \) with a \( q_0 \in B \) such that \( Z_i(q_0) = 0 \) for \( i = 1, \ldots, n \). Let \((Z'_1, \ldots, Z'_n)\) be an independent copy of \((Z_1, \ldots, Z_n)\). Assume the following Lipschitz-property: There is a random vector \( A \) with values in \( \mathbb{R}^n \) such that

\[
|Z_i(q) - Z_i(p) - Z'_i(q) + Z'_i(p)| \leq A_i d(q, p)
\]

for \( i = 1, \ldots, n \) and all \( q, p \in B \). Let \( \kappa \geq 1 \). Then

\[
\mathbb{E} \left[ \sup_{q \in B} \left| \sum_{i=1}^{n} Z_i(q) \right|^\kappa \right] \leq c_\kappa \mathbb{E}[\|A\|_2^\kappa] \gamma_2(B, d)\kappa,
\]

where \( c_\kappa \in (0, \infty) \) depends only on \( \kappa \).

**Proof.** See [Sch19, Theorem 6].
Define the inner metric of \((Q,d)\) as \(d(q,p) = \inf L(\gamma)\), where the infimum is taken over all continuous maps \(\gamma: [a,b] \to Q\) with \(\gamma(a) = q\) and \(\gamma(b) = p\). A length space is a metric space \((Q,d)\) with \(d = d_i\). Now, let \((Q,d)\) be a length space. A continuous map \(\gamma: [a,b] \to Q\) is called shortest path if \(L(\gamma) \leq L(\tilde{\gamma})\) for all continuous maps \(\tilde{\gamma}: [\tilde{a},\tilde{b}] \to Q\) with \(\gamma(a) = \tilde{\gamma}(\tilde{a})\) and \(\gamma(b) = \tilde{\gamma}(\tilde{b})\). A continuous map \(\gamma: [a,b] \to Q\) is locally minimizing if for every \(t \in [a,b]\) there is \(\epsilon > 0\) such that \(\gamma|_{[t-t+\epsilon]}\) is a shortest path. A continuous map \(\gamma: [a,b] \to Q\) has constant speed if there is \(v \geq 0\) such that for every \(t \in [a,b]\) there is \(\epsilon > 0\) such that \(L(\gamma|_{[t-t+\epsilon]}) = 2vt\). A geodesic is a locally minimizing continuous map with constant speed. A minimizing geodesic between two points \(q,p \in Q\) is a geodesic \(\gamma: [a,b] \to Q\) with \(L(\gamma) = d(\gamma(a),\gamma(b))\) and \(\gamma(a) = q\), \(\gamma(b) = p\). A geodesic \(\gamma: [a,b] \to Q\) is extendible (through both ends) if there is \(\epsilon > 0\) and a geodesic \(\tilde{\gamma}: [a-\epsilon,b+\epsilon] \to Q\) such that \(\tilde{\gamma}|_{[a,b]} = \gamma\). The tuple \((Q,d)\) is a geodesic space if there is a connecting geodesic for every pair of points. A geodesic space \((Q,d)\) is geodesically complete, if it is complete and all geodesics are extendible.

A Hadamard space is a nonempty complete metric space \((Q,d)\) such that for all \(q,p \in Q\), there is \(m \in Q\) such that \(d(y,m)^2 \leq \frac{1}{2}d(y,q)^2 + \frac{1}{2}d(y,p)^2 - \frac{1}{4}d(q,p)^2\) for all \(y \in Q\). In Hadamard spaces, all geodesics are minimizing. Hilbert spaces and Riemannian manifolds of nonpositive sectional curvature are Hadamard spaces. Hadamard spaces are also called global NPC-spaces, complete \(CAT(0)\) spaces or Alexandrov spaces of nonpositive curvature.

An Alexandrov spaces of nonnegative curvature is a geodesic space \((Q,d)\) such that for all \(q,p \in Q\), there is \(m \in Q\) such that \(d(y,m)^2 \geq \frac{1}{2}d(y,q)^2 + \frac{1}{2}d(y,p)^2 - \frac{1}{4}d(q,p)^2\) for all \(y \in Q\). More generally Alexandrov spaces can be defined with an arbitrary curvature bound. They generalize Riemannian manifolds with a bound on the sectional curvature.

\[\text{Lemma 25.}\] In the Euclidean space \(\mathbb{R}^k\) with the metric induced by the Euclidean norm \(|\cdot|\), it holds \(\gamma_2(B(x,r,|\cdot|,|\cdot|)) \leq 2r\sqrt{k}\) for any point \(x \in \mathbb{R}^k\) and radius \(r > 0\).

\[\text{Proof.}\] See [Pol90, section 4] and comparison to the entropy integral as in Remark 1. Q.E.D.

\[\text{C. Geometry}\]

We introduce some terms from (metric) geometry, which are used in this article. See [BBI01] for a in depth introduction.

A metric space is called proper if every closed ball is compact. Let \((Q,d)\) be a metric space. For a continuous map \(\gamma: [a,b] \to Q\) define its length as
\[L(\gamma) = \sup\left\{ \sum_{i=1}^{n} d(\gamma(x_{i-1}),\gamma(x_i)) \mid a = x_0 < x_1 < \cdots < x_n = b, n \in \mathbb{N} \right\}.\]

Define the inner metric of \((Q,d)\) as \(d_i(q,p) = \inf L(\gamma)\), where the infimum is taken over all continuous maps \(\gamma: [a,b] \to Q\) with \(\gamma(a) = q\) and \(\gamma(b) = p\). A length space is a metric space \((Q,d)\) with \(d = d_i\). Now, let \((Q,d)\) be a length space. A continuous map \(\gamma: [a,b] \to Q\) is called shortest path if \(L(\gamma) \leq L(\tilde{\gamma})\) for all continuous maps \(\tilde{\gamma}: [\tilde{a},\tilde{b}] \to Q\) with \(\gamma(a) = \tilde{\gamma}(\tilde{a})\) and \(\gamma(b) = \tilde{\gamma}(\tilde{b})\). A continuous map \(\gamma: [a,b] \to Q\) is locally minimizing if for every \(t \in [a,b]\) there is \(\epsilon > 0\) such that \(\gamma|_{[t-t+\epsilon]}\) is a shortest path. A continuous map \(\gamma: [a,b] \to Q\) has constant speed if there is \(v \geq 0\) such that for every \(t \in [a,b]\) there is \(\epsilon > 0\) such that \(L(\gamma|_{[t-t+\epsilon]}) = 2vt\). A geodesic is a locally minimizing continuous map with constant speed. A minimizing geodesic between two points \(q,p \in Q\) is a geodesic \(\gamma: [a,b] \to Q\) with \(L(\gamma) = d(\gamma(a),\gamma(b))\) and \(\gamma(a) = q\), \(\gamma(b) = p\). A geodesic \(\gamma: [a,b] \to Q\) is extendible (through both ends) if there is \(\epsilon > 0\) and a geodesic \(\tilde{\gamma}: [a-\epsilon,b+\epsilon] \to Q\) such that \(\tilde{\gamma}|_{[a,b]} = \gamma\). The tuple \((Q,d)\) is a geodesic space if there is a connecting geodesic for every pair of points. A geodesic space \((Q,d)\) is geodesically complete, if it is complete and all geodesics are extendible.

A Hadamard space is a nonempty complete metric space \((Q,d)\) such that for all \(q,p \in Q\), there is \(m \in Q\) such that \(d(y,m)^2 \leq \frac{1}{2}d(y,q)^2 + \frac{1}{2}d(y,p)^2 - \frac{1}{4}d(q,p)^2\) for all \(y \in Q\). In Hadamard spaces, all geodesics are minimizing. Hilbert spaces and Riemannian manifolds of nonpositive sectional curvature are Hadamard spaces. Hadamard spaces are also called global NPC-spaces, complete \(CAT(0)\) spaces or Alexandrov spaces of nonpositive curvature.

An Alexandrov spaces of nonnegative curvature is a geodesic space \((Q,d)\) such that for all \(q,p \in Q\), there is \(m \in Q\) such that \(d(y,m)^2 \geq \frac{1}{2}d(y,q)^2 + \frac{1}{2}d(y,p)^2 - \frac{1}{4}d(q,p)^2\) for all \(y \in Q\). More generally Alexandrov spaces can be defined with an arbitrary curvature bound. They generalize Riemannian manifolds with a bound on the sectional curvature.
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