Distributed Computing Model: Classical vs. Quantum vs. Post-Quantum

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Computation refers to an algorithmic process of producing outputs from a given set of inputs. We consider a particular model of computation where the computing device is comprised of several components/ports placed at different spacetime locations – the inputs are distributed among different ports, whereas the output is to be assessed by some other port. Importantly, communications among the different ports are limited as the physical systems allowed to be transmitted among the different ports are constrained with information theoretic restrictions. Interestingly, this limited communication distributed computing prototype provides an efficient modus operandi to compare the computational strength of different mathematical models used to describe those physical systems. We come up with computing tasks where quantum theory outperforms its classical counterpart. More surprisingly, a broader class of operational theories with state or/and effect space structures more exotic than quantum theory turns out to be suboptimal in performing some distributed computing tasks. We characterize the computations that can be perfectly accomplished in quantum theory but not in other theories. The proposed computing model thus provides a new approach to single out quantum theory in the theory space and promises a novel perspective towards the axiomatic derivation of Hilbert space quantum mechanics.
I. INTRODUCTION

Computation is one of the most profound achievements of human scientific endeavour that shapes the modern era. A comprehensive understanding of its naive foundations demands interdisciplinary study on mathematics and logic [1], computer science [2], cognitive sciences [3], and physics [4–8]. The physics of computation provides a rudimentary view of the physical principles underlying any computing device. While functioning of present day computing machines are governed by the laws of classical physics, advent of quantum information theory identifies useful applications of several quantum features in computing [9] and welcomes new architecture of the computing devices [10–14]. Although a number of novel quantum algorithms have already been invented that are extremely efficient over the known classical algorithms [15–17], it still remains illusive from a foundational point of view: *why the nature is quantum?* There is no general consensus why our physical world should be modeled by Hilbert space quantum mechanics, which, from a mathematical standpoint, is one example of model among several other possibilities [18–23], commonly known as generalized probabilistic theories (GPTs). While classical theory is embedded within quantum as a special case and is conservative in allowing possible *space-like* and *time-like* correlations [24], there exist GPTs that allows stronger *space-like* or *time-like* correlations than quantum [25–27]. It therefore raises the question why should we model the nature with Hilbert space quantum mechanics that allows non-classical features within it but in a limited ways in comparison to other possible generalized models? During last two decades several groups of researchers have developed novel approaches to address this question [27–33]. All these approaches find seemingly impossible consequences of post-quantum models and thus reject them to be the possible theory of the physical world.

In the present work, we take a completely different approach to address the aforesaid question. We look for some operational task where quantum theory outperforms not only the classical theory but also other exotic post quantum models. Interestingly, a distributed computing model provides us such a platform. In the proposed scenario the computing device comprises of several ports placed at different spacetime locations. The inputs are distributed in distinct ports whereas the final computation is accomplished at some other port. Information theoretic constraints are imposed on the physical systems that are allowed to be communicated among the input and output ports. The allowed physical systems can be described by different GPTs satisfying the imposed restriction(s). Interestingly, we come up with distributed computing tasks that can be perfectly accomplished in quantum theory but neither in classical regime nor even in other GPTs allowing more exotic state and/or effect space structure than quantum theory. We exactly characterize a class of computing functions that can be perfectly accomplished in quantum theory but not in other generalized models. Our proposed computing model thus establishes operational utility of Hilbert space framework over the other possible classes of GPTs. It thus provides a deeper understanding regarding the mathematical modeling of the physical world and turns out to be a novel primitive in singling out Hilbert space quantum mechanics in the theory space.
Figure 1. (Color on-line) (a) Dual layer computing device receives two independent and uniformly random $n$-bit strings $x$ and $y$ from a server and outputs a bit. First, it computes the function $f$ on the bits taken pairwise from $x$ and $y$ and finally computes $F$ on the outputs of the first layer, i.e., the dual layer computation can be represented as $(F, f): \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$. (b) Corresponding distributed computing scenario: Non-communicating labs $A$ and $B$ receive two $n$-bit strings $x$ and $y$ respectively from a server. $A$ and $B$ are allowed to encode their strings’ information in the state of some systems $S_A$ and $S_B$ that can initially be prepared in some correlated state $\omega_{AB}$. The computer $C$ needs to perform a measurement on systems received from $A$ and $B$ to simulate the dual layer computing device. In oblivious version of this task, the function $F$ remains oblivious before the communications from $A$ and $B$ to $C$ are completed.

II. DISTRIBUTED COMPUTING SCENARIO

The computing scenario, we consider here, consists of three spatially separated components (see Fig. 1). Two non-communicating labs $A$ and $B$ (personified as Alice and Bob, respectively) receive $n$-bit input strings $x \equiv x_1 \cdots x_n \in \{0, 1\}^n$ and $y \equiv y_1 \cdots y_n \in \{0, 1\}^n$. The strings are independent and sampled randomly from a cloud server. Once the inputs get distributed, Alice and Bob are not allowed to communicate with each other. There is a distant computer $C$ (personified as Charlie) that has to execute a $\{0, 1\}^n \to \{0, 1\}$ bit computation $F(z_1, \cdots, z_n)$, where $\forall i \in \{1, \cdots, n\}$ and $z_i = f(x_i, y_i)$ with $f$ being a function $\{0, 1\}^2 \to \{0, 1\}$. The overall computation can be thought as a dual layer computation which we have summarized as $(F, f): \{0, 1\}^n \times \{0, 1\}^n \to \{0, 1\}$. This computing scenario is closely analogous to a particular model of communication complexity, namely the simultaneous message passing model introduced by Yao [34]. The transmitters are allowed to encode their respective bit strings in some physical systems $S_A$ and $S_B$ possessed locally, and subsequently send the encoded systems to $C$. The systems can be
prepared in some correlated state. Depending upon what kind of mathematical models are considered to describe the systems $S_A$ and $S_B$, the joint system can be prepared in exotic correlated state that does not have any classical analogy. Quantum entanglement is one such example that is non-classical in nature [35]. However, quantum theory is not the only theory that allows such non-classical states. In fact, several other mathematical models are possible that permits such states [21–23, 26, 27].

With no restriction on the information carrying capacity of $S_A$ and $S_B$, any dual layer computation $(F, f)$ can be accomplished perfectly. Interesting situations arise when only limited communications are allowed from the transmitters to the computer. Such a scenario we call distributed computing with limited communication (DCLC). For instance, not all DCLC($n$) tasks can be done perfectly if only $(n - 1)$-cbits are allowed from each of Alice and Bob to Charlie. Here and henceforward, we will use the notation DCLC($n$) to indicate that the inputs $x$ and $y$ are $n$-bit strings. For an arbitrary theory, the limitation on communication can be put through some operational means. One such way is to restrict the operational dimension (OD) of the encoding systems which corresponds to the maximum number of states that can be perfectly discriminated in a single-shot measurement (see METHODS for mathematically rigorous definition). For a DCLC($n$), we impose the restriction that OD of $S_A$ and $S_B$ can be at-most $2^{n-1}$. Note that restriction on communication can also be imposed from other information theoretic motivations. In this regard, the studies in Refs. [27, 36, 37] are quite relevant.

All the parties (Alice, Bob, and Charlie) know both the functions $F$ and $f$ prior to the communications while accomplishing a DCLC and design their encoding decoding strategy accordingly. One can, however, introduce an interesting variation of the task where part of the computing function remain oblivious to Alice and Bob prior to their communications to Charlie. More particularly, the function $f$ is known to all apriori, but Alice and Bob learn about the function $F$ only after they send the communications to Charlie. We coined the term oblivious-DCLC ($O$-DCLC) for this variant of the task. Interestingly, we will show that perfect accomplishment of some $O$-DCLC task demands specific structures for the state and effect spaces of the operational theories. Depending upon whether the systems $S_A$ and $S_B$ are taken to be classical or quantum or some post-quantum GPT, the strategies executing a DCLC/ $O$-DCLC are respectively called classical, quantum, and post-quantum strategies.

III. RESULT

We start the technical part by formally defining a trivial computation in the aforesaid distributed scenario.

**Definition 1.** A dual layer task $(F, f) \in$ DCLC($n$) is said to be trivial whenever there exists a classical strategy executing the computation exactly, otherwise it is said to be nontrivial.

For an arbitrary $n$, one can have in total $(2^n)^2 \times (2^2)^2$ numbers of different DCLC($n$) tasks - some trivial and some nontrivial. For instance, a computation $(F, f)$ is trivial whenever at least one of the functions is a constant function. Importantly, there exist trivial computations where neither $F$ nor $f$ is a constant. One such example is the computation $(F \equiv \oplus, f \equiv \oplus)$, where ‘$\oplus$’ denotes the logical exclusive disjunction (XOR) operation.
Triviality follows from the fact that $\bigoplus_{i=1}^n z_i = \bigoplus_{i=1}^n (x_i \oplus y_i) = (\bigoplus_{i=1}^n x_i) \oplus (\bigoplus_{i=1}^n y_i)$, i.e., Charlie can do the required computation if Alice and Bob inform parity of their respective strings which requires only 1-cbit communication from each of the transmitters to Charlie. 

Our next result characterizes all possible trivial (nontrivial) computations in the simplest scenario, i.e., when $x$ & $y$ are 2-bit strings.

**Proposition 1.** A dual layer computation $(\mathbb{F}, f) \in \text{DCLC}(2)$ is trivial if and only if any one of the following criteria is satisfied:

(i) atleast one of the two functions is a constant function;

(ii) atleast one of them is a single bit function;

(iii) $\mathbb{F}$ is symmetric on inputs and $f$ can be realized through $\mathbb{F}$ [and with single-bit NOT operation], i.e., $f(a_1, a_2) = \mathbb{F}(a_1, a_2)$ [f($a_1, a_2$) = $\mathbb{F}(\bar{a}_1, \bar{a}_2)$].

A function $G : \{0, 1\}^n \rightarrow \{0, 1\}$ will be called a single bit function if $\forall a \in \{0, 1\}^n$ the functional value $G(a)$ only depends on a single bit $a_i$ for some fixed $i \in \{1, \ldots, n\}$. Such a function will be called symmetric if it is of the form either $G(a) = a_1 \ast a_2 \ast \ldots \ast a_n$ or $G(a) = \bar{a}_1 \ast \bar{a}_2 \ast \ldots \ast \bar{a}_n$ for some binary operation $\ast$. Otherwise, it is called non-symmetric. We discuss the detailed proof of the proposition in Appendix section. Out of 256 computations, it turns out that 176 are trivial and the rests are nontrivial. Furthermore, among the trivial ones, 60 can be accomplished even without any communication from the transmitters, 56 require communication from only one of the transmitters, and the rests require communications from both the transmitters. The proof technique used for Proposition 1 leads us to the following generalization.

**Corollary 1.** A dual layer computation $(\mathbb{F}, f) \in \text{DCLC}(n)$, for arbitrary $n \geq 2$, is trivial if any one of the criteria in Proposition 1 is satisfied.

Note that, unlike Proposition 1, Corollary 1 characterizes only a class of trivial computations for $n > 2$. Identification of all such computations we leave here as an open question for future research. We will now explore the possibility of accomplishing a nontrivial DCLC task in broader class of GPTs [21–23]. In a GPT, a system is specified by its state space, effect space and the allowed transformations acting on the states and effects. For instance, the state space of a classical system with $d$ distinct state is $(d - 1)$ simplex, while for a $d$-label quantum system it is the set $\mathcal{D}(\mathcal{H}^d)$ of density operators acting on $d$-dimensional complex Hilbert space $\mathcal{H}^d$. Due to the convex structure of the allowed states and effects, this framework is also known as convex operational theories (see Appendix for detailed description). A GPT also specifies the description of composite systems which is given by some tensor product of the component subsystems. Composite systems can be prepared in entangled states that can not be decomposed as convex mixtures of product states of the component subsystems. Importantly, such entanglement is considered to be one of the most crucial non classical signatures. Mathematically, choice of the tensor product structure in not unique, and at this point, role of physical/information principles become crucial to single out the desired structure [23, 38]. Subsequently, we will show that performance of a nontrivial DCLC task is intriguingly connected with the structure of a GPT.

To execute a DCLC($n$) or $\mathcal{O}$-DCLC($n$) task in a GPT, Alice and Bob start their protocol with a shared bipartite state $\omega^{AB} \in \Omega^{AB}$, where $\Omega^{AB}$ is the state space for composite
system with the subsystems $S_A$ & $S_B$ satisfying the constraints imposed on their operational dimension. Depending upon the inputs $x$ and $y$, Alice and Bob will apply some local encoding operations which consists of some local reversible transformations $T_A^x$ and $T_B^y$, respectively. The encoded state will be communicated to Charlie who performs some decoding measurement $M^{AB} \equiv \{ e_k^{AB} \mid e_k^{AB} \in \mathcal{E}^{AB} \& \sum_k e_k^{AB} = u^{AB} \}$ on the received bipartite state, where $\mathcal{E}^{AB}$ is the set of all bipartite effects with $u^{AB}$ being the unit effect (see the Appendix). Post processing of the measurement outcomes completes the final computation $F(f(x_1, y_1), \cdots, f(x_n, y_n))$. Our next result specifies basic resource requirement for a nontrivial DCLC task.

**Proposition 2.** Perfect accomplishment of any nontrivial computation $(\mathcal{F}, f) \in DCLC(n)$ in any GPT necessitates presence of entanglement in bipartite state and/or effect spaces of that theory.

Proof is presented in Appendix section. Quantum theory is an example of GPT that allows entanglement in its bipartite states as well as in bipartite effects. Naturally, the question arises whether all the nontrivial DCLC tasks can be perfectly accomplished in a GPT that allows entanglement in its state and/or effect space. In the next, we will see that this is not the case in general. To this aim, we characterize the DCLC($n$) tasks that can be perfectly done in quantum theory. Recall that due to the restriction on OD, each of Alice and Bob can communicate some quantum state $\rho \in \mathcal{D}(C^d)$, where $d \leq 2^{n-1}$. Of course, they can start the protocol with some bipartite entangled state $\rho^{AB} \in \mathcal{D}(C^d \otimes C^d)$.

**Theorem 1.** A nontrivial dual layer computation $(\mathcal{F}, f) \in DCLC(2)$ is perfectly computable in quantum theory if and only if $f$ is a balanced function.

Detailed proof is presented in Appendix. It is important to note that the perfect protocol uses maximally entangled 2-qubit state (e.g. $|\phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$) shared between Alice and Bob, and Charlie performs a two outcome entangled measurement $M^{[2]} \equiv \{ P_{\phi^+}, I_4 - P_{\phi^+} \}$ on the encoded state received from Alice and Bob to decode; $P_{\chi} := |\chi\rangle\langle\chi|$. For an explicit example, $(\mathcal{F} \equiv \lor, f \equiv \oplus)$ is a nontrivial computation that can be perfectly done in quantum theory, where $\lor$ denotes logical inclusive disjunction (OR) operation. This particular computation can also be thought as equality problem that asks whether $x = y$ or not [39]. On the other hand, $(\mathcal{F} \equiv \oplus, f \equiv \lor)$ is also a nontrivial computation, but it cannot be done perfectly in quantum theory. A generalization of the above result for DCLC($n$) follows immediately.

**Corollary 2.** Any nontrivial dual layer computation $(\mathcal{F}, f) \in DCLC(n)$ is perfectly computable in quantum theory whenever $f$ is a balanced function.

Note that, unlike Theorem 1, Corollary 2 characterises only a class of nontrivial DCLC($n$) tasks for $n > 2$. Here, we are leaving an open question to identify all the quantum computable nontrivial DCLC($n$). Importantly, only $\lceil \frac{n}{2} \rceil$-qubit communication, from each of Alice and Bob to Charlie, suffices to execute all the computations in Corollary 2. This amounts to (nearly) half of the maximum allowed communication $[\lceil (n - 1) \rceil$-qubit communication from each]. It is of further interest to explore whether a lesser amount of quantum communication suffices the purpose or not. Our next result deals with nontrivial $O$-DCLC in quantum theory.
Corollary 3. All the computations of Theorem 1 and of Corollary 2 are also computable in quantum theory if we consider their oblivious version.

In this case, Alice and Bob follow the same encoding procedure(s) as of Theorem 1 (Corollary 2 for higher $n$). This time, Charlie performs a 4-outcome measurement $M_{[4]}$ instead of $M_{[2]}$ used in the non-oblivious case(s). Charlie’s decoding measurement $M_{[4]}$ is the 2-qubit Bell measurement, i.e., $M_{[4]} \equiv \{P_{\phi^+}, P_{\phi^-}, P_{\psi^+}, P_{\psi^-}\}$, where $|\phi^\pm\rangle := (|00\rangle \pm |11\rangle)/\sqrt{2}$ and $|\psi^\pm\rangle := (|01\rangle \pm |10\rangle)/\sqrt{2}$. The protocol is discussed in detail in the Appendix section.

It is also interesting to study these distributed computation tasks in GPTs other than quantum theory. In accordance with the limitation imposed on OD, we consider a class of GPTs for the DCLC(2) tasks where the state space $\Omega_k$ for a single system is described by symmetric polygons with $k$ vertices ($k \geq 4$) [40]. Their bipartite composition can be constructed in several ways that are more enriched than quantum theory in some sense. For instance, the Popescu & Rohrlich (PR)-model is one extreme composition for $k = 4$ that allows all possible product and entangled states [26]. This model exhibits stronger nonlocal behaviour than quantum theory which gets depicted in the Clauser-Horne-Shimony-Holt inequality violation [41]. On the other extreme, the hyper signaling (HS)-model allows only the product states, but incorporates all possible product and entangled effects. While this model is local by construction, it exhibits some striking feature by allowing stronger time-like correlations than that are allowed in the quantum theory [27]. In the Appendix section, we construct extreme bipartite compositions for all the polygonal models and establish the following no-go result.

Theorem 2. None of the nontrivial computations in DCLC(2) can be perfectly done in the extreme bipartite models with marginal subsystems described by symmetric polygon model.

Apart from the above two extremes, bipartite polygon models can be composed in several ways that lie in between. However, the sets of possible reversible transformations for these intermediate models are too much restricted in comparison to the extremal compositions. This restriction appears due to the consistency requirement that demands positivity of all outcome probabilities. In fact, the transformations are so limited that even all the product states (effects) can not be converted among themselves under the allowed reversible transformations, and thus it makes these intermediate compositions less interesting. We, however, conjecture that the no-go statement of Theorem 2 holds true for any such intermediate composite models. The study, made in [39], supports our conjecture where it has been shown that the intermediate compositions for the square-bit model (i.e., $\Omega_4$) cannot perfectly compute the equality problem, i.e., the computation $(F \equiv \lor, f \equiv \oplus) \in DCLC(2)$.

IV. DISCUSSION

Advent of quantum information theory initiates the second quantum revolution where non-classical aspects of the theory are well utilized to design novel information protocols. While quantum random number generator and quantum communication models are already available at the commercial level [42–46], we are also at the verge of developing
much advanced quantum internet network and quantum computer prototype [13, 47, 48]. From a mathematical point of view, however, there exist several other generalized probabilistic models that incorporate quantum like non-classical features. This motivated a huge aspiration to identify quantum theory uniquely from some physical/information theoretic prescriptive [28–33, 49–57]. Our distributed computation scenario is a novel proposal in this regard as it identifies computations in DCLC(n) scenario that can be perfectly accomplished in quantum theory only but not possible in other GPTs. For n = 2 we, in fact, have fully characterized nontrivial computations that can be perfectly done only in quantum theory.

The present study constitutes a novel framework to establish the distinctive feature of quantum theory from other GPTs. A comparative discussion with the other existing approaches is quite demanding. Firstly, the seminal Bell nonlocality scenario can be considered as multipartite computational tasks where all the parties invoke only local encoding (input) and local decoding (output) although they can share some global correlation among them [24]. On the other hand, the phenomenon of quantum nonlocality without entanglement [58–60] can be thought as a computational task invoking global encodings where local decodings are allowed. While the Bell nonlocality scenario has been well explored in the GPT framework during the last three decades, the other phenomenon has also been studied recently in this generalized framework [61]. In both the above cases computational power of quantum theory turns out to be limited compared to a class of other GPTs. The computing scenario considered in this present work allows local encoding by two non-communicating players (Alice and Bob) who subsequently send limited communication to a third party (Charlie) performing some global decoding on the received composite systems. Interestingly, we come up with computational tasks that can be perfectly done if Alice and Bob start with quantum correlation instead of stronger no-signaling correlations. In this regard, the works in Refs. [62–64] are worth mentioning. There also the authors have proposed tasks where quantum theory outperforms not only classical theory but a class of other no-signaling theories. The computational limitation of the GPTs established in those work might be explained from the impossibility of entanglement swapping for generalized nonlocal correlations [65]. However, this is not the case for the computational scenario considered here.

It is also important to recall the Quantum fingerprinting task in connecting with the present work [66]. Given two random n bit strings x and y to Alice and Bob respectively, the question of fingerprinting asks Charlie to calculate the function $e(x, y) := 1$ (if $x = y$) and $e(x, y) := 0$ (if $x \neq y$) using minimum communication from Alice and Bob. In our language, the problem is equivalent to the dual layer computation $(F \equiv \lor, f \equiv \oplus) \in \text{DCLC}(n)$. The authors in [66] have established an exponential gap between quantum and classical resource requirement for the fingerprinting problem. Similar study have been made for a class of problems in [67]. Importantly, while the study in [66, 67] is limited in comparing classical vs. quantum resources, our study makes a comparison between quantum and post quantum resources. Of course it remains open whether an exponential efficiency of quantum theory over the post quantum resources is possible or not. The problems in corollary 2 of our work indicate such a possibility. Our work also welcomes further research regarding generalization of the scenario considered here.
V. APPENDIX

A. Proof of Proposition 1

(i) If \( F \) is a constant function, Charlie can yield the required output which is independent of the bit strings received from the transmitters. On the other hand, when \( f \) is constant, a single input pair \( (z_1, z_2) \), with \( z_1 = z_2 \), will be fed into the computer \( C \), effectively. In both of these cases, the dual layer computation \( (F, f) \) can be done perfectly even without any communication from the transmitters to the computer.

(ii) If \( F \) is a single bit function, Charlie only requires information about one of \( z_1 \) and \( z_2 \) to execute the dual layer computation. The transmitters will accordingly send the corresponding bit of their strings. If \( f \) is a single bit function, the DCLC \( (F, f) \) will effectively depend on one of the input strings - \( x \) or \( y \). Now, Alice or Bob will perform the required computation accordingly and communicate the 1-bit output to Charlie.

(iii) Let \( f \) (denoted by \( \star \)) be realized by \( F \) (denoted by \( \circ \)) and single bit NOT operation. Since, we consider \( F \) as symmetric, therefore \( f(\alpha, \beta) = \alpha \star \beta = \bar{\alpha} \circ \bar{\beta} = F(\bar{\alpha}, \bar{\beta}) \), where \( \alpha, \beta \in \{0, 1\} \). In the dual-layer computation, we have,

\[
F(z_1, z_2) = z_1 \circ z_2 = f(x_1, y_1) \circ f(x_2, y_2) = (x_1 \star y_1) \circ (x_2 \star y_2) = (x_1 \circ \bar{y}_1) \circ (x_2 \circ \bar{y}_2) = (\bar{x}_1 \circ \bar{x}_2) \circ (\bar{y}_1 \circ \bar{y}_2).
\]

Alice and Bob thus compute a single bit data from their respective inputs and send it to Charlie. Same holds true if \( f(\alpha, \beta) = F(\alpha, \beta) \).

Note that (i) are the trivial computations where no communication is required. In (ii), if \( f \) is a single bit function, we need to use the communication channel partially (from a single transmitter to the computer \( C \)). For the computations in (iii) and the remaining possibilities in (ii) (i.e., where \( F \) is a single bit function), we need 1-bit communication from both the transmitters to the computer. All other remaining dual layer computations are nontrivial.

It is also noteworthy that the input strings \( \{x, y\} \) get mapped into the bit values 0 and 1 after the computation \( (F, f) \) with \( 1 : 1 \) ratio if and only if both of the functions are balanced. Then, either at least one of the computations between \( F \) and \( f \) is a single bit function, or both of them are some variant of parity checking. According to (ii) and (iii), both these cases are trivial. It is also not hard to argue that for a DCLC\((n)\) computation \( (F, f) \) is trivial if the functions satisfy the aforesaid conditions. At this point, we note down an important observation regarding the nontrivial DCLC\((2)\) tasks.

**Observation 1.** Consider a set \( G := \{x, x', y, y'\} \), where \( x \neq x' \) are the inputs at \( A \) while \( y \neq y' \) at \( B \) for a DCLC\((2)\) task. Altogether, \(^4C_2 \times ^4C_2 = 36\) different such sets are possible. Evidently, the strings in \( G \) will be mapped into the bit values 0 and 1 respectively in the ratio 4 : 0,
2 : 2, 0 : 4, 1 : 3 and 3 : 1. It turns out that at-least one \( G \), among the 36 possibilities, must have the aforesaid ratio either 1 : 3 or 3 : 1 for every nontrivial DCLC(2) task.

### B. Generalized probabilistic theory

Origin of this framework dates back to 1960’s [18–20], and it has gained renewed interest in the recent past [21–23]. A GPT is specified by a list of system types and the composition rules specifying combination of several systems. A system state \( \omega \) specifies outcome probabilities for all measurements that can be performed on it. For a given system, the set of possible normalized states forms a compact and convex set \( \Omega \) embedded in a positive convex cone \( V_+ \) of some real vector space \( V \). Convexity of \( \Omega \) assures that any statistical mixture of states is a valid state. The extremal points of \( \Omega \), that do not allow any convex decomposition in terms of other states, are called pure states or states of maximal knowledge. An effect \( e \) is a linear functional on \( \Omega \) that maps each state onto a probability, \textit{i.e.}, \( e : \Omega \mapsto [0,1] \) by a pre-defined rule \( p(e|\omega) = \text{Tr}(e^T.\omega) \).

The set of effects \( \mathcal{E} \) is embedded in the positive dual cone \( (V^*)_+ \). The normalization of \( \Omega \) is determined by \( u \) which is defined as the unit effect and a specified element of \( (V^*)_+ \), such that, \( p(u|\omega) = \text{Tr}(u^T.\omega) = 1, \forall \omega \in \Omega \). A \( d \)-outcome measurement is specified by a collection of \( d \) effects, \( M \equiv \{e_j \mid \sum_{j=1}^d e_j = u\} \), such that, \( \sum_{j=1}^d p(e_j|\omega) = 1 \), for all valid states \( \omega \). Another much needed component to complete the mathematical structure for GPT is the reversible transformation \( T \) which maps states to states, \textit{i.e.}, \( T(\Omega) = \Omega \).

They are linear in order to preserve the statistical mixtures, and they cannot increase the total probability. In a GPT one can introduce the idea of distinguishable states from an operational perspective which consequently leads to the concept of \textit{Operational dimension}.

**Definition 2.** Operational dimension of a system \((S)\) is the largest cardinality of the subset of states, \( \{\omega_i\}_{i=1}^n \subset \Omega \), that can be perfectly distinguished by a single measurement, \textit{i.e.}, there exists a measurement, \( M \equiv \{e_j \mid \sum_{j=1}^n e_j = u\} \), such that, \( p(e_j|\omega_i) = \delta_{ij} \).

Importantly, operational dimension is different from the dimension of the vector space \( V \) in which the state space \( \Omega \) is embedded. For instance, for qubit the state space, the set of density operators \( D(\mathbb{C}^2) \) acting on \( \mathbb{C}^2 \) is embedded in \( \mathbb{R}^3 \). However, the operational dimension of this system is 2, as at most two qubit state can be perfectly distinguished by a single measurement – \textit{e.g.} \( \{|0\rangle,|1\rangle\} \) states can be perfectly distinguished by Pauli measurement along \( z \)-direction.

Composite systems of a GPT must be constructed in accordance with no signaling (NS) principle that prohibits instantaneous communication between two distant locations. This, along with the assumption of \textit{tomographic locality} [38], assures that the composite state space lies in between two extremes - the maximal and the minimal tensor product [68].

**Definition 3.** The maximal tensor product, \( \Omega^A \otimes_{\text{max}} \Omega^B \), is the set of all bi-linear functionals, \( \phi : (V^A)^* \otimes (V^B)^* \mapsto \mathbb{R} \), such that, (i) \( \phi(e^A, e^B) \geq 0 \), for all \( e^A \in \mathcal{E}^A \) and \( e^B \in \mathcal{E}^B \), and (ii) \( \phi(u^A, u^B) = 1 \), where \( u^A \) and \( u^B \) are unit effects for system \( A \) and \( B \) respectively.

**Definition 4.** The minimal tensor product, \( \Omega^A \otimes_{\text{min}} \Omega^B \), is the convex hull of the product states \( \omega^{A \otimes B} (= \omega^A \otimes \omega^B) \).
States belonging in $\Omega^A \otimes_{\min} \Omega^B$ are called separable; otherwise, they are entangled. One can define effect spaces for the composite systems in a similar manner. The quantum mechanical tensor product is neither the minimal one nor the maximal; it lies strictly in between.

**Proof of Proposition 2**: Recall that a DCLC is trivial (nontrivial) if it can (cannot) be perfectly accomplished by some (any) classical strategy. In the language of GPT, the state and effect spaces of a $d$-level classical system is specified by a $(d - 1)$-simplex. A bipartite system, composed of two such classical systems, is described uniquely by the minimal tensor product. In other words, the composite system has unique state space, as in this case we have, $\Omega^A \otimes_{\min} \Omega^B = \Omega^A \otimes_{\max} \Omega^B$ [68]; hence, the composite system allows no entanglement neither in states nor in effects. Barker’s conjecture [69] concerns with the converse question, i.e., for what kind of convex sets the tensor product is unique. Recently, Aubrun et al. provide an affirmative proof to the Barker’s conjecture that the minimal and maximal tensor products of two finite-dimensional proper cones coincide if and only if one of the two cones is generated by a linearly independent set, i.e., one of the state spaces is classical [70]. The only if part of this result assures the present Proposition.

C. Proof of Theorem 1

--:if part:--
Note that there are $4^2$ balanced Boolean functions $\{0, 1\}^2 \mapsto \{0, 1\}$; out of which 4 are single bit function and hence trivial (Proposition 1). The remaining two functions are XOR and X-NOR. We first discuss the protocol for the case ($F \equiv \lor, f \equiv \oplus$). Alice and Bob start the protocol with the 2-qubit maximally entangled state $|\phi^+\rangle_{AB} := \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$. Depending on the inputs $x$ and $y$, they apply local unitary operation $\sigma_i^A$ and $\sigma_j^B$ on their respective part of the entangled state, where $i := 2x_1 + x_2 \& j := 2y_1 + y_2$, and $\sigma_0 = \mathbb{I}$, $\sigma_1 = \sigma_X$, $\sigma_2 = \sigma_X\sigma_Z$ and $\sigma_3 = \sigma_Z$. Whenever $x = y$, Charlie receives the state $|\phi^+\rangle$ else a state $\perp$ to $|\phi^+\rangle$. The required computation can be exactly done by performing the 2-outcome measurement, $M_{[2]} \equiv \{P_{\phi^+} + I_4 - P_{\phi^+}\}$ and declaring the outcome as $P_{\phi^+} \to 0$ and $\neg P_{\phi^+} \to 1$. Other nontrivial computations follow a similar protocol with suitable relabeling of encoding and decoding (see Table 1).

--:only if part:--
First we prove the following lemma.

**Lemma 1.** Any nontrivial dual layer computation $(F, f) \in DCLC(2)$ maps the set of input strings $\{x, y\}$ into the binary bit values in $1 : 3$ ratio if and only if $f$ is balanced.

**Proof.** A $\{0, 1\}^2 \to \{0, 1\}$ Boolean function produces a binary output in either of the three possible ratios – $4 : 0$ (constant), $1 : 1$ (balanced) and $1 : 3$ (unbalanced). If one of the functions between $F$ and $f$ is constant or both of them are balanced, the dual layer computation $(F, f)$ is trivial (Proposition 1). If $f$ is balanced but not a single-bit function along with an unbalanced $F$, the outputs of $(F, f)$ are in $1 : 3$ and the dual layer computation is nontrivial establishing the if part.
If $f$ is unbalanced and $F$ is balanced, the output will be either $1 : 3$ or $3 : 5$. Alternatively, it will be among $1 : 15$, $3 : 13$ and $7 : 9$ for both of $f$ and $F$ being unbalanced. Now, consider an unbalanced $f$ where the ‘output bit value 0 : output bit value 1 = $1 : 3’$. In this case, bit values of $z_1z_2$ follow the ratio $00 : 01 : 10 : 11 = 1 : 3 : 9$. If $F$ is a balanced function, such that, $\{00, 01\} \rightarrow 0/1$ or $\{00, 10\} \rightarrow 0/1$, output of $(F, f)$ is in $1 : 3$ ratio. But in this case, $F$ being a single-bit function makes $(F, f)$ trivial (Proposition 1). Therefore, no other nontrivial $(F, f)$ can be in $1 : 3$ ratio except $f$ being a balanced function.

Lemma 1 assures that to prove the only if part of Theorem 1, it is sufficient to prove that no quantum strategy (entangled states along with local unitaries and two outcome measurements) can produce two disjoint subspaces containing states other than $1 : 3$ or $1 : 1$. Consider that Alice and Bob start their protocol with a pure entangled state $|\psi\rangle = a|00\rangle + b|11\rangle$, where, $\{a, b\} \in \mathbb{R}$, s.t., $a^2 + b^2 = 1$ without loss of generality. They have some unitary encoding strategies, $\{U^A_i\}_{i=0}^3$ and $\{U^B_j\}_{j=0}^3$, respectively. Now, according to the Observation 1, for every nontrivial DCLC(2) there exists at least a group of four input strings $G := \{x, x', y, y'\}$ ($x \neq x'$ & $y \neq y'$) that follows the ratio $1 : 3$. Considering Alice’s and Bob’s encoding as $U^A_0, U^A_1$ and $U^B_0, U^B_1$, the resulting encoded states read

$$
\begin{align*}
(x, y) &\mapsto |\xi_1\rangle = a|\psi_0\rangle + b|\psi_0^+\rangle, \\
(x', y) &\mapsto |\xi_2\rangle = a|\psi_1\rangle + b|\psi_1^+\rangle, \\
(x, y') &\mapsto |\xi_3\rangle = a|\psi_0\rangle + b|\psi_0^+\rangle, \\
(x', y') &\mapsto |\xi_4\rangle = a|\psi_1\rangle + b|\psi_1^+\rangle,
\end{align*}
$$

where, $|\psi_i\rangle = U^A_i|0\rangle$ & $|\psi_i\rangle = U^B_i|0\rangle$, for $i \in \{0, 1\}$. The orthogonality conditions $\langle \xi_1|\xi_j\rangle = 0, \forall j \in \{2, 3, 4\}$, imply $|\psi_1\rangle = |\psi_1^+\rangle$ & $|\psi_1^+\rangle = \pm |\psi_0\rangle$ and $|\psi_0\rangle = |\psi_0^+\rangle, |\psi_0^+\rangle = \mp |\psi_0\rangle$. In other words, both $U^A_0 \& U^A_1 (U^B_0 \& U^B_1)$ map the states $\{|0\rangle, |1\rangle\}$ into same orthogonal pairs $\{|\psi_1\rangle, |\psi_0\rangle\} (\{|\phi_1\rangle, |\phi_0\rangle\})$. For decoding, Charlie performs the measurement, $M^A_{\{2\}} = \{P_{\xi_1}, I_4 - P_{\xi_1}\}$ and assigns the outcome as $P_{\xi_1} \rightarrow 0$ and $\neg P_{\xi_1} \rightarrow 1$. For every input string $x$, either $(x, y)$ yields outcome 0, whereas $(x, y')$ yields 1 (where

| $f \equiv \text{XNORE}$ | $f \equiv \text{XOR}$ | $\sigma^B_j$ | Outcome |
|----------------|----------------|---------|--------|
| $F(z_1, z_2) \equiv (z_1 \land z_2)$ | $F(z_1, z_2) \equiv z_1 \lor z_2$ | $j = 2y_1 + y_2$ | $P_{\phi^+} \rightarrow 0$ |
| $F(z_1, z_2) \equiv z_1 \land z_2$ | $F(z_1, z_2) \equiv (z_1 \land z_2)$ | $j = 2y_1 + y_2$ | $P_{\phi^+} \rightarrow 1$ |
| $F(z_1, z_2) \equiv z_1 \lor z_2$ | $F(z_1, z_2) \equiv (z_1 \lor z_2)$ | $j = 2y_1 + y_2$ | $P_{\phi^+} \rightarrow 1$ |
| $F(z_1, z_2) \equiv z_1 \land z_2$ | $F(z_1, z_2) \equiv z_1 \lor z_2$ | $j = 2y_1 + y_2$ | $P_{\phi^+} \rightarrow 0$ |
| $F(z_1, z_2) \equiv z_1 \lor z_2$ | $F(z_1, z_2) \equiv (z_1 \lor z_2)$ | $j = 2y_1 + y_2$ | $P_{\phi^+} \rightarrow 0$ |

Table I. Quantum strategies for nontrivial DCLC(2) tasks where $f$ is balanced. Alice’s encoding operations $\{\sigma^A_j\}$ are same as used in $(F \equiv \lor, f \equiv \oplus)$.
\( y \neq y' \) belong to Bob), or \( x \) forms a group like \( \mathcal{G} \). In both these cases, all the Unitaries \( \{U_i^A\}_{i=0}^3 \) \( \{U_j^B\}_{j=0}^3 \) map the states \( \{|0\rangle, |1\rangle\} \) into same orthogonal pairs \( \{|\psi_0\rangle, |\psi_1^+\rangle\} \) \( \{|\phi_0\rangle, |\phi_1^+\rangle\} \). Therefore, the encoding by Alice and Bob, without loss of any generality, can be chosen to be the Pauli matrices \( \{\sigma_i\}_{i=0}^3 \). Let us denote \( (\sigma_i^A \otimes \sigma_j^B ) |\psi\rangle = |\xi\rangle_{ij} \), where, \( \{i,j\} \in \{0,...,3\} \). Note that \( |\xi\rangle_{ij} \sim |\xi\rangle_{kl} \) (up-to global phase) if \( i+k = j+l = 3 \). To compute DCLC (2) perfectly, Charlie performs a measurement that divides the communicated bipartite states in two orthogonal subspace. Evidently, there exists only one such measurement which divides the above states in \( \{|00\rangle, |11\rangle\} \) and \( \{|01\rangle, |10\rangle\} \) subspace in \( 1:1 \) ratio. Therefore, the dual-layer computations for which the final outcome is balanced, i.e., only the trivial ones can be performed with non-maximally pure entangled states. Whenever \( a = b = \frac{1}{\sqrt{2}} \), the subspace can also be divided in \( 1:3 \) ratio and no other choice is possible at all.

**Remark-1** (Proof of Corollary 2): Whenever \( f \) is a balanced function, any nontrivial \( (\mathbb{F},f) \in \text{DCLC}(n) \) is quantum computable. In case of even \( n \), Alice and Bob follow the same DCLC(2) protocol for each two successive bits of their \( n \)-bit string, and thus they require \( n/2 \)-ebits. Alternatively, For odd \( n \), each of them requires \( (n-1)/2 \)-ebits and 1 unentangled qubit.

**Remark-2** (Proof of Corollary 3): Let us consider \( f \) as a balanced function in \( (\mathbb{F},f) \), while \( \mathbb{F} \) is oblivious prior to the communications from Alice & Bob to Charlie. The encoding protocol, here, is similar to that of Theorem 1. Depending upon the input strings \( x \) & \( y \), Charlie receives the bipartite state as follows:

| inputs \( x, y \) | \( \sigma_{2x_1+y_2}^A \otimes \sigma_{2y_1+x_2}^B |\phi^+\rangle_{AB} \) |
|----------------|----------------------------------|
| \( x_1 = y_1 \) & \( x_2 = y_2 \) | \( |\phi^+\rangle_{AB} \) |
| \( x_1 \neq y_1 \) & \( x_2 = y_2 \) | \( |\phi^-\rangle_{AB} := \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB}) \) |
| \( x_1 = y_1 \) & \( x_2 \neq y_2 \) | \( |\psi^+\rangle_{AB} := \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \) |
| \( x_1 \neq y_1 \) & \( x_2 \neq y_2 \) | \( |\psi^-\rangle_{AB} := \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}) \) |

For decoding, Charlie performs the 4-outcome Bell measurement, \( M_4 \equiv \{P_{\phi^+}, P_{\phi^-}, P_{\psi^+}, P_{\psi^-}\} \). He, then, calculates \( z_i = f(x_i, y_i) \) for \( f \in \{\text{XOR, XNOR}\} \) and computes the final outcome \( \mathbb{F}(z_1, z_2) \). Remark-1 ensures that the same procedure can be extended for \( \mathcal{O}\)-DCLC.

### D. Polygon theory

**Single system:** For an elementary system, the state space \( \Omega_n \) is a regular polygon with \( n \) vertices. For a fixed \( n \), \( \Omega_n \) is the convex hull of \( n \) pure states \( \{\omega_i\}_{i=0}^{n-1} \), where,

\[
\omega_i = \begin{pmatrix} r_n \cos \frac{2\pi i}{n} \\ r_n \sin \frac{2\pi i}{n} \\ 1 \end{pmatrix}, \quad \text{with} \quad r_n = \sqrt{\sec(\pi/n)}.
\]

Several interesting studies have been reported with this class of models in the recent past [40, 55, 61, 71–74]. The effect space \( \mathcal{E}_n \), collection of all the possible measurement effects, is the convex hull of the null effect \( O \), unit effect \( u \), the extremal effects \( \{e_j\}_{j=0}^{n-1} \),
and their complementary effects \( \{ \tilde{e}_j \}_{j=0}^{n-1} \), where, \( \tilde{e}_j := u - e_j \). The null and unit effects are respectively given by \( O = (0,0,0)^T \) and \( u = (0,0,1)^T \), where, \( T \) denotes the matrix transposition. The effects \( \{ e_j \}_{j=0}^{n-1} \) are given by,

\[
\begin{array}{c|c}
\text{Even-gon} & \text{Odd-gon} \\
\hline
e_j = \frac{1}{2} \begin{pmatrix} r_n \cos \left( \frac{(2j-1)\pi}{n} \right) \\ r_n \sin \left( \frac{(2j-1)\pi}{n} \right) \end{pmatrix} & e_j = \frac{1}{1+r_n} \begin{pmatrix} r_n \cos \left( \frac{2j\pi}{n} \right) \\ r_n \sin \left( \frac{2j\pi}{n} \right) \end{pmatrix}
\end{array}
\]

For even-gon, it turns out that, \( \tilde{e}_j := u - e_j = e_{(j\oplus_n \frac{n}{2})} \), where, \( \oplus_n \) denotes addition modulo \( n \). Therefore, the effects \( \{ e_j \}_{j=0}^{n-1} \) as well as their complementary effects are not the pure effects only, but they are the ray extremals of the effect cone \( (V^*)_+ \) also. In contrast, for odd-gon, only \( \{ e_j := u - e_j \}_{j=0}^{n-1} \) are the ray extremals, whereas their complementary effects \( \{ \tilde{e}_j := u - e_j \}_{j=0}^{n-1} \) are not despite being pure.

For any \( n \)-gon theory, the set of the reversible transformations (RT), \( \mathbb{T}_n \), is the dihedral group of order \( 2n \) containing \( n \) reflections and \( n \) rotations, i.e.,

\[
\mathbb{T}_n \equiv \{ T_k^p \mid k = 0, \ldots, n-1; \text{ & } p \in \{ +, - \} \},
\]

\[
T_k^p := \begin{pmatrix} \cos \frac{2\pi k}{n} & -p \sin \frac{2\pi k}{n} & 0 \\ p \sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
(2)
\]

with \( p = + \) corresponds to the rotation and \( p = - \) to the reflection.

**Bipartite system:** Any bipartite composition of \( n \)-gon systems must include \( n^2 \) factorized states,

\[
\Omega^{\text{product}} := \left\{ \omega_{ni+j}^{A \otimes B} := \omega_i^A \otimes \omega_j^B \mid i,j \in \{0, \ldots, n-1\} \right\} \subset \Omega_{n \otimes 2} := \Omega_n^{AB}.
\]

We will use the superscript \( A \otimes B \) to denote factorizability. For the bipartite system, the product effects are of the form,

\[
\mathcal{E}^{\text{product}} := \left\{ g^A \otimes g^B \right\} \subset \mathcal{E}_n^{AB},
\]

\[
(4)
\]

where, \( g^X \in \left\{ \bar{O}^X, u^X \right\} \cup \left\{ e_i^X, \tilde{e}_i^X \right\}_{i=0}^{n-1}; \ X \in \{A, B\} \).

Since, \( p(e^{A \otimes B} | \omega^{A \otimes B}) = p(e^A | \omega^A)p(e^B | \omega^B) \), therefore,

\[
0 \leq p(e^{A \otimes B} | \omega^{A \otimes B}) \leq 1; \ \forall e^{A \otimes B} \in \mathcal{E}^{\text{product}} \ \& \ \forall \omega^{A \otimes B} \in \Omega^{\text{product}}.
\]

Apart from these factorized states and effects, a bipartite system may also allow non-factorized (entangled) states and effects that we will denote as \( \omega^{AB} \) and \( e^{AB} \) respectively. Of course, they must satisfy the consistency requirements:

\[
0 \leq p(e^{A \otimes B} | \omega^{A \otimes B}) \leq 1, \ \forall e^{A \otimes B} \in \mathcal{E}^{\text{product}},
\]

\[
0 \leq p(e^{AB} | \omega^{A \otimes B}) \leq 1, \ \forall \omega^{A \otimes B} \in \Omega^{\text{product}}.
\]

(6)  

(7)
In Ref.\cite{40}, the authors have introduced an maximally entangled state for bipartite $n$-gon theories both for *odd* and *even* $n$. Applying all possible local RTs $\{T_k^p\}$ on Alice’s part and $\{T_l^q\}$ on Bob’s part, we can get all the other entangled states as follows:

**Odd $n$:**

\[
\omega_{kl}^{AB}(p, q) := \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k-l)\right) & -\sin \left(\frac{2\pi}{n}(k-l)\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k-l)\right) & \cos \left(\frac{2\pi}{n}(k-l)\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p = q,
\]

\[
\omega_{kl}^{AB}(p, q) := \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k+l)\right) & \sin \left(\frac{2\pi}{n}(k+l)\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k+l)\right) & -\cos \left(\frac{2\pi}{n}(k+l)\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p \neq q.
\]

**Even $n$:**

\[
\omega_{kl}^{AB}(p, q) := \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & -\sin \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & \cos \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p = q,
\]

\[
\omega_{kl}^{AB}(p, q) := \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & \sin \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & -\cos \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p \neq q.
\]

Similarly, all the possible maximally entangled effects are given by,

**Odd $n$:**

\[
e_{kl}^{AB}(p, q) := \frac{1}{1 + r_n^2} \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k-l)\right) & -\sin \left(\frac{2\pi}{n}(k-l)\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k-l)\right) & \cos \left(\frac{2\pi}{n}(k-l)\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p = q,
\]

\[
e_{kl}^{AB}(p, q) := \frac{1}{1 + r_n^2} \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k+l)\right) & \sin \left(\frac{2\pi}{n}(k+l)\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k+l)\right) & -\cos \left(\frac{2\pi}{n}(k+l)\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p \neq q.
\]

**Even $n$:**

\[
e_{kl}^{AB}(p, q) := \frac{1}{2} \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & -\sin \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & \cos \left(\frac{2\pi}{n}(k-l) - \frac{p\pi}{n}\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p = q,
\]

\[
e_{kl}^{AB}(p, q) := \frac{1}{2} \begin{pmatrix}
\cos \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & \sin \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & 0 \\
\sin \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & -\cos \left(\frac{2\pi}{n}(k+l) - \frac{p\pi}{n}\right) & 0 \\
0 & 0 & 1 \\
\end{pmatrix}; \text{ when } p \neq q.
\]

For an arbitrary $n$, the set of all possible RTs for bipartite system is given by

\[
T_{n^\otimes 2} := T^{AB} \equiv \{S, T_k^p \otimes T_l^q\},
\]

\[
k, l \in \{0, \ldots, n-1\}; \quad p, q \in \{+, -\}.
\]

$S$ is the SWAP map whose action is defined as,

\[
S(\omega^A \otimes \omega^B) = \omega^B \otimes \omega^A; \quad \forall \omega^A \in \Omega^A \& \omega^B \in \Omega^B.
\]
**Observation 2.** The SWAP map is a global transformation, i.e., it cannot be implemented locally. Alternatively, any local transformation $T \in T^{AB}$ never maps a product state (effect) to an entangled one and vice versa. In other words, the set of product states (effects) and the set of entangled states (effects) are two disconnected islands under the reversible transformation $T^{AB}$.

Positivity of the predicted probabilities imposes restrictions on the states, effects and transformations that can be allowed together in a composite system. Satisfying this consistency requirement, several composite models are possible. These models can be classified into three main types as discussed below.

**Type-I: Entangled states product effects model**: In this case, all possible product as well as entangled states (listed above) are allowed, i.e., $\Omega^{AB}_n$ is the convex hull of the set $\{\omega^{A\otimes B}_{ni+j},\omega^{AB}_{kl}(p,q) \mid i,j,k,l \in \{0,\cdots,n-1\}; p,q \in \{+,-\}\}$, whereas the effects are only product in nature. Due to the presence of entangled states, such model can exhibit Bell nonlocality [24]. In fact, such a model can be stronger in space-like correlation by revealing more nonlocal behaviour than quantum theory [26].

**Type-II: Product states entangled effects model**: It allows only the product states, i.e., $\Omega^{AB}_n$ is the convex hull of the set $\{\omega^{A\otimes B}_{ni+j} \mid i,j \in \{0,\cdots,n-1\}\}$. However, it allows all possible product as well as entangled effects. Such a model is local by construction. Due to the presence of all possible entangled effects, this model can also exhibit peculiar feature. For instance, the authors have shown in Ref. [27] that such a model can allow time-like correlations that are stronger than quantum theory.

**Type-III: Dynamically restricted models**: There can be some models which allow some entangled states along with some (suitably chosen) entangled effects unlike the Type-I and Type-II models. All the transformations in Eq.(12) are allowed. Further, due to the consistency requirement (i.e., positivity of the outcome probability), not all the reversible transformations can be allowed when both entangled states and entangled effects are incorporated; hence, it is named as ‘dynamically restricted models’. Such restriction prevents even all the pure states (effects) to be mapped to each other under reversible transformation which makes Type-III models quite uninteresting.

**E. Proof of Theorem 2**

Evidently, operational dimension of any polygonal model $(\Omega_n, E_n, T_n)$ is 2. Therefore, both Alice and Bob are allowed to communicate one such system while performing a DCLC(2) task. However, they can consider some composite models allowing entanglement. Now, for encoding, they will apply some reversible transformations on their respective part. Therefore, Observation 2 leads us as follows:

**Observation 3.** Encoded states, received by Charlie, are all either entangled or product.

We will now prove the Theorem 2 for Type-I and Type-II composite model. For odd-gon and even-gon, the proof will be discussed separately.

**Odd-gon theory**
Type-I Models: According to Observation 3, the encoded states, received by Charlie, are product states if Alice and Bob start their protocol with a product one. If a nontrivial DCLC(2) task can be accomplished perfectly by such a strategy, it is perfectly computable in classical theory too which is not possible at all. On the other hand, Alice and Bob may start the protocol with an entangled state. Then, Charlie receives all the encoded states as entangled, followed by Observation 3. A straightforward calculation shows \( p(e_i^A \otimes e_j^B | \omega_{kl}^{AB}(p,q)) \neq 0 \), \( \forall i, j, k, l \in \{0, ..., (n-1)\} \) \& \( \omega_{kl}^{AB}(p,q) \) in (8) which leads to an unavoidable ambiguity while decoding by Charlie.

Type-II Models: A decoding strategy with product effects ensures the equivalence with a classical strategy for a nontrivial DCLC(2), and hence perfect accomplishment is impossible. Let us then move to entangled decoding strategies, and we consider, without loss of generality, Charlie’s decoding measurement, \( \mathcal{M}^{AB} \equiv \{ e_{00}^{AB}(++) , \tilde{e}_{00}^{AB}(++) \} \). Suppose, Alice encodes her strings \( x \) and \( x' \) (\( \neq x \)) into the states \( \omega_k^A \) and \( \omega_j^B \) respectively, whereas the strings \( y \) and \( y' \) (\( \neq y \)), in Bob’s side, are encoded by \( \omega_s^B \) and \( \omega_p^B \) respectively. For a particular encoded state, any of these two effects should get clicked perfectly in case of unambiguous decoding which leads to the following restrictions. It can be easily shown that \( e_{00}^{AB}(++) \) and \( \tilde{e}_{00}^{AB}(++) \) will get clicked in 1 : 1 ratio for any combination of these conditions. However, for a nontrivial computation, there should be at least a group \( G = \{ x, x', y, y' \} \) with \( x \neq x' \) \& \( y \neq y' \) such that the string pairs are mapped into 1 : 3 ratio (Observation 1). Hence, no nontrivial computation can be performed perfectly in any Type-II odd-gon theory.

### Table II. Conditions for which either of the entangled effects click sharply.

| Input strings | \( e_{00}^{AB}(++) \) clicks | \( \tilde{e}_{00}^{AB}(++) \) clicks |
|---------------|-----------------|-----------------|
| \( x, y \)   | \( k = s \)     | \( k = s \oplus_n \frac{n+1}{2} \) |
| \( x', y' \) | \( l = t \)     | \( l = t \oplus_n \frac{n+1}{2} \) |
| \( x, y' \)  | \( k = t \)     | \( k = t \oplus_n \frac{n+1}{2} \) |
| \( x', y \)  | \( l = s \)     | \( l = s \oplus_n \frac{n+1}{2} \) |

Even-gon theory

Type-I Models: The effects, \( e_i^A \otimes e_j^B \) and \( \tilde{e}_i^A \otimes e_j^B \), can be clubbed together to get a single effect, \( E_{i\otimes j} := e_i^A \otimes e_j^B + \tilde{e}_i^A \otimes e_j^B \) since, \( p(e_i^A \otimes e_j^B | \omega_{kl}^{AB}(p,q)) = p(\tilde{e}_i^A \otimes e_j^B | \omega_{kl}^{AB}(p,q)) \), \( \forall i, j, k, l, p, q \). Similarly, we have, \( \tilde{E}_{i\otimes j} := e_i^A \otimes e_j^B + \tilde{e}_i^A \otimes e_j^B \). Clubbing the effects in a different manner will do nothing but increase the ambiguity. Consider that Alice and Bob start the protocol with \( \omega_{00}^{AB}(++) \) and Charlie performs the decoding measurement, \( \mathcal{M}^{A\otimes B} \equiv \{ E_{0\otimes 0}, \tilde{E}_{0\otimes 0} \} \) without loss of generality. Encoding of different bit-strings can be accomplished by applying the proper reversible transformations on \( \omega_{00}^{AB}(++) \). The probabilities to obtain the effect \( E_{0\otimes 0} \) on different entangled states are given by,

i) \( p( E_{0\otimes 0} | \omega_{kl}^{AB}(+++) ) = \frac{1}{2} \left[ 1 + r_n^2 \cos \left( \frac{\pi}{n} - \frac{2\pi}{n}(k-l) \right) \right] \),

ii) \( p( E_{0\otimes 0} | \omega_{kl}^{AB}(-) ) = \frac{1}{2} \left[ 1 + r_n^2 \cos \left( \frac{\pi}{n} + \frac{2\pi}{n}(k-l) \right) \right] \),
iii) \( p(E_{00} | \omega_{kl}^{AB}(+-)) = \frac{1}{2} \left[1 + r_n^2 \cos \left(\frac{\pi}{n} + \frac{2\pi}{n} (k+l)\right)\right], \)

iv) \( p(E_{00} | \omega_{kl}^{AB}(-+)) = \frac{1}{2} \left[1 + r_n^2 \cos \left(\frac{3\pi}{n} + \frac{2\pi}{n} (k+l)\right)\right], \)

where, \( \omega_{kl}^{AB}(p,q) = (\mathcal{T}_k^p \otimes \mathcal{T}_l^q) \omega_{00}^{AB}(++) \). To avoid the ambiguity, we have to choose the proper reversible transformations maintaining the restrictions listed in Table III. Suppose, Alice applies \( \mathcal{T}_k^p \) & \( \mathcal{T}_l^q \) when she receives the strings \( x \& x' (\not= x) \) respectively, and Bob applies \( \mathcal{T}_l^q \) & \( \mathcal{T}_l'^q \) for the strings \( y \& y' (\not= y) \) similarly on their shared state \( \omega_{00}^{AB}(++) \). Compared to the odd-gon theories, there are more possibilities for encoding in even-gon cases as the number of entangled states are more. We consider a particular case with \( p = +, p' = -, q = +, \) and \( q' = - \). With the help of the Table III, we arrive at the following conditions for unambiguous decoding. These conditions lead to the fact

| Input strings | Encoded State | Conditions |
|--------------|---------------|------------|
| \( x, y \)   | \( \omega_{kl}^{AB}(++) \) | \( k = l \) & \( k = l \oplus_n 1 \) or \( k = l \oplus_n \frac{n}{2} \) & \( k = l \oplus_n \left(\frac{n}{2} + 1\right) \) |
| \( x', y' \)  | \( \omega_{kl'}^{AB}(-) \) | \( k = -l' \oplus_n n \) & \( k = -l' \oplus_n (n-1) \) or \( k = -l' \oplus_n \frac{n}{2} \) & \( k = -l' \oplus_n \left(\frac{n}{2} - 1\right) \) |
| \( x, y' \)   | \( \omega_{kl'}^{AB}(+) \) | \( k = -l' \oplus_n n \) & \( k = -l' \oplus_n (n-1) \) or \( k = -l' \oplus_n \frac{n}{2} \) & \( k = -l' \oplus_n \left(\frac{n}{2} - 1\right) \) |
| \( x', y \)   | \( \omega_{kl}^{AB}(-) \) | \( k = -l \oplus_n n \) & \( k = -l \oplus_n (n-1) \) or \( k = -l \oplus_n \frac{n}{2} \) & \( k = -l \oplus_n \left(\frac{n}{2} - 1\right) \) |

Table IV. Conditions for unambiguous decoding.

that the effects \( E_{00} \) and \( E_{00} \) will get clicked either in 1:1 or in 1:0 ratio resulting a trivial computation. In a similar fashion, one can argue the same for any choice of \( p, p', q, q' \in \{+, -\} \). Hence, no nontrivial computations can be executed by this type of theories.

**Type-II Models:** In this case, the arguments go as in the case of Type-II odd-gon
models and it turns out that the decoding effects get clicked in 1 : 1 ratio. Hence, all the computations, which can be accomplished, are trivial by this kind of theories. This completes proof of the Theorem 2.

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[1] B. J. Copeland (Series Editor); The Essential Turing: Seminal Writings in Computing, Logic, Philosophy, Artificial Intelligence, and Artificial Life plus The Secrets of Enigma; Oxford University Press (2004).
[2] Donald Knuth; The Art of Computer Programming.
[3] P. McCorduck; Machines Who Think: A Personal Inquiry into the History and Prospects of Artificial Intelligence; A K Peters/CRC Press (2004).
[4] R. Landauer; Irreversibility and Heat Generation in the Computing Process; IBM J. Research and Development 5, 183 (1961).
[5] C. H. Bennett; Logical reversibility of computation; IBM J. Research and Development 17, 525 (1973).
[6] C. H. Bennett; The thermodynamics of computation—a review; Int. J. Theor. Phys. 21, 905 (1982).
[7] S. Lloyd; Ultimate physical limits to computation; Nature 406, 1047 (2000).
[8] S. Aaronson; Limits on Efficient Computation in the Physical World; UC Berkeley PhD thesis, arXiv:quant-ph/0412143 (2004).
[9] P. Kaye, R. Laflamme, and M. Mosca; An Introduction to Quantum Computing; Oxford University Press (2007).
[10] R. P. Feynman; Simulating physics with computers; Int J Theor Phys 21, 467 (1982).
[11] C. Neill et al. A blueprint for demonstrating quantum supremacy with superconducting qubits; Science 360, 195 (2018).
[12] F. Arute et al. Quantum supremacy using a programmable superconducting processor; Nature 574, 505 (2019).
[13] Quantum Computing: Progress and Prospects; National Academies of Sciences, Engineering, and Medicine (2019) https://doi.org/10.17226/25196.
[14] Han-Sen Zhong et al. Quantum computational advantage using photons, Science, 10.1126/science.abe8770 (2020).
[15] D. Deutsch and R. Jozsa; Rapid solution of problems by quantum computation; Proc. Royal
20

Soc. London A. 439, 553 (1992).

[16] P.W. Shor; Algorithms for quantum computation: discrete logarithms and factoring; Proceedings 35th Annual Symposium on Foundations of Computer Science (1994).

[17] L. K. Grover; A fast quantum mechanical algorithm for database search; Proceedings, 28th Annual ACM Symposium on the Theory of Computing (1996) [arXiv:quant-ph/9605043].

[18] G. W. Mackey; Mathematical Foundations of Quantum Mechanics; New York (1963); Dover reprint (2004).

[19] G. Ludwig, Attempt of an axiomatic foundation of quantum mechanics and more general theories II, III, Commun. Math. Phys. 4, 331-348 (1967); Commun. Math. Phys. 9, 1-12 (1968).

[20] B. Mielnik, Geometry of quantum states, Commun. Math. Phys. 9, 55-80 (1968).

[21] J. Barrett, Information processing in generalized probabilistic theories, Phys. Rev. A 75, 032304 (2007).

[22] G. Chiribella, G. M. D’Ariano, and P. Perinotti, Informational derivation of quantum theory, Phys. Rev. A 84, 012311 (2011).

[23] M. Pawłowski, T. Paterek, D. Kaszlikowski, V. Scarani, A. Winter; Information Causality as a Physical Principle, Nature 461, 1101 (2009).

[24] M. Navascués, H. Wunderlich; A glance beyond the quantum model, Proc. Roy. Soc. Lond. A 466, 881 (2009).

[25] T. Fritz, A. B. Sainz, R. Augusiak, J. B. Brask, R. Chaves, A. Leverrier, and A. Acín; Local orthogonality as a multipartite principle for quantum correlations, Nat. Comm. 4, 2263 (2013).

[26] A. C-C. Yao; Some complexity questions related to distributive computing; In Proceedings of 11th ACM STOC, pages 209–213, (1979).

[27] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki; Quantum entanglement; Rev. Mod. Phys. 81, 865 (2009).

[28] P. E. Frenkel and M. Weiner; Classical information storage in an n-level quantum system; Comm. Math. Phys. 340, 563 (2015).

[29] A. Tavakoli, E. Z. Cruzeiro, J. B. Brask, N. Gisin, and N. Brunner; Informationally restricted quantum correlations; Quantum 4, 332 (2020)

[30] L. Hardy; Reformulating and Reconstructing Quantum Theory, arXiv:1104.2066 (2013).
[39] S. Saha, S. S. Bhattacharya, T. Guha, S. Halder, and M. Banik; Advantage of Quantum Theory over Nonclassical Models of Communication; Annalen der Physik, 2000334 (2020).
[40] P. Janotta, C. Gogolin, J. Barrett, and N. Brunner; Limits on nonlocal correlations from the structure of the local state space; New J. Phys. 13, 063024 (2011).
[41] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt; Proposed experiment to test local hidden-variable theories; Phys. Rev. Lett. 23, 880 (1969).
[42] M. Herrero-Collantes and J. C. Garcia-Escartin; Quantum random number generators; Rev. Mod. Phys. 89, 015004 (2017).
[43] J. Yin et al; Satellite-based entanglement distribution over 1200 kilometers; Science 356 1140 (2017).
[44] Sheng-Kai Liao et al; Satellite-to-ground quantum key distribution; Nature 549, 43 (2017).
[45] Ji-Gang Ren et al; Ground-to-satellite quantum teleportation; Nature 549, 70 (2017).
[46] J. Yin et al; Entanglement-based secure quantum cryptography over 1,120 kilometres; Nature 582, 501 (2020).
[47] H. J. Kimble; The quantum internet; Nature 453, 1023 (2008).
[48] S. Wehner, D. Elkouss, and R. Hanson; Quantum internet: A vision for the road ahead; Science 362, 9288 (2018).
[49] S. Aaronson; Is Quantum Mechanics An Island In Theoryspace? arXiv:quant-ph/0401062.
[50] J. Oppenheim and S. Wehner; The uncertainty principle determines the non-locality of quantum mechanics, Science 330, 1072 (2010).
[51] M. P. Müller and C. Ududec; Structure of Reversible Computation Determines the Self-Duality of Quantum Theory, Phys. Rev. Lett. 108, 130401 (2012).
[52] C. Pfister and S. Wehner; An information-theoretic principle implies that any discrete physical theory is classical, Nature Communications 4, 1851 (2013).
[53] M. Banik, M. R. Gazi, S. Ghosh, and G. Kar; Degree of complementarity determines the nonlocality in quantum mechanics, Phys. Rev. A 87, 052125 (2013).
[54] M. Banik, S. S. Bhattacharya, A. Mukherjee, A. Roy, A. Ambainis, and A. Rai; Limited preparation contextuality in quantum theory and its relation to the Cirel’son bound, Phys. Rev. A 92, 030103(R) (2015).
[55] M. Banik, S. Saha, T. Guha, S. Agrawal, S. S. Bhattacharya, A. Roy, and A. S. Majumdar; Constraining the state space in any physical theory with the principle of information symmetry; Phys. Rev. A 100, 060101(R) (2019).
[56] A. Ambainis, M. Banik, A. Chaturvedi, D. Kravchenko, and A. Rai; Parity Oblivious d-Level Random Access Codes and Class of Noncontextuality Inequalities; Quantum Inf Process 18, 111 (2019).
[57] M. Krumm and M. P. Mueller; Quantum computation is the unique reversible circuit model for which bits are balls; npj Quantum Inf 5, 7 (2019).
[58] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters; Quantum nonlocality without entanglement; Phys. Rev. A 59, 1070 (1999).
[59] S. Halder, M. Banik, S. Agrawal, and S. Bandyopadhyay; Strong Quantum Nonlocality without Entanglement; Phys. Rev. Lett. 122, 040403 (2019).
[60] S. Rout, A. G. Maity, A. Mukherjee, S. Halder, and M. Banik; Genuinely nonlocal product bases: Classification and entanglement-assisted discrimination; Phys. Rev. A 100, 032321 (2019).
[61] S. S. Bhattacharya, S. Saha, T. Guha, and M. Banik; Nonlocality without entanglement: Quantum theory and beyond; Phys. Rev. Research 2, 012068(R) (2020).
[62] J. Barrett and S. Pironio; Popescu-Rohrlich Correlations as a Unit of Nonlocality, Phys. Rev. Lett. 95, 140401 (2005).
[63] R. Chao and B. W. Reichardt; Test to separate quantum theory from non-signaling theories, arXiv:1706.02008 [quant-ph].
[64] M. Weilenmann and R. Colbeck; Self-Testing of Physical Theories, or, Is Quantum Theory Optimal with Respect to Some Information-Processing Task? Phys. Rev. Lett. 125, 060406 (2020).
[65] A. J. Short, S. Popescu, and N. Gisin; Entanglement swapping for generalized nonlocal correlations, Phys. Rev. A 73, 012101 (2006).
[66] H. Buhrman, R. Cleve, J. Watrous, and R. de Wolf; Quantum fingerprinting, Phys. Rev. Lett. 87, 167902 (2001).
[67] H. Buhrman, R. Cleve and A. Wigderson; Quantum vs. classical communication and computation, STOC ’98: Proceedings of the thirtieth annual ACM symposium on Theory of computing, May 1998, Pages 63–68
[68] I. Namioka and R. R. Phelps; Tensor products of compact convex sets, Pac. J. Math. 31, 469 (1969).
[69] G. P. Barker; Monotone norms and tensor products; Linear Multilinear Algebra 4, 191 (1976); Theory of cones; Linear Algebra Appl. 39, 263, (1981).
[70] G. Aubrun, L. Lami, C. Palazuelos, and M. Plavala; Entangleability of cones; arXiv:1911.09663 [math.FA].
[71] D. A. Yopp and R. D. Hill; Extremals and exposed faces of the cone of positive maps; Linear and Multilinear Algebra 53, 167 (2007).
[72] S. Weis; Duality of non-exposed faces, Journal of Convex Analysis 19, 815 (2012); arXiv:1107.2319.
[73] S. Massar and M. K. Patra; Information and communication in polygon theories; Phys. Rev. A 89, 052124 (2014).
[74] S. W Al-Safi and J. Richens; Reversibility and the structure of the local state space, New J. Phys. 17, 123001 (2015).