A short proof of Ornstein’s non-inequality in $\mathbb{R}^{2 \times 2}$

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Abstract

We give a very short proof of Ornstein’s $L^1$ non-inequality for first- and second-order operators in two dimensions.

Given two linear constant-coefficient homogeneous $k$-th order differential operators $P_1, P_2$ and a number $1 \leq p \leq \infty$, consider the inequality

$$
\|P_1 \varphi\|_{L^p(\mathbb{R}^n)} \leq C_p \|P_2 \varphi\|_{L^p(\mathbb{R}^n)},
$$

for all $\varphi \in C_\infty^c(\mathbb{R}^n, \mathbb{R}^m)$, where $0 < C_p$ is some constant. When does such an estimate hold?

The case $1 < p < \infty$ is very classical: if we take $P_1 = D^k$ to be the $k$-th order gradient then (1) holds if and only if $P_2$ is an elliptic operator (in the sense that it has injective symbol); this is a classical result which goes back to the work of Calderón–Zygmund [4]. At the end-points $p = 1$ or $p = \infty$, (1) never holds, except in trivial situations: this was proved, respectively, by Ornstein [22] and Mityagin [19], but see also [9] for the $p = \infty$ case. In some circumstances one can deduce the result for $p = \infty$ from the one for $p = 1$, see for instance [3, 18] for the case $P_2 = \text{div}$, and in fact the result for $p = 1$ is much more difficult. Similar results also hold in the anisotropic setting, see [13, 23].

The purpose of this note is to give a simple proof of Ornstein’s result when $n = m = 2$ and $k = 1$. These assumptions encompass the case where $P_1 = D$ and $P_2$ is any of the operators

$$(\text{div, curl}), \quad \mathcal{E} u \equiv \frac{1}{2} (Du + (Du)^T), \quad \mathcal{D} u \equiv \mathcal{E} u - \frac{\text{div} u}{2} \text{Id}_2,$$

which appear respectively in electromagnetism, linearized elasticity and complex analysis. The proof establishes a connection between the failure of singular integral estimates and rigidity properties of a particular class of quasiconvex functions. Our strategy is similar to the one

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of Kirchheim–Kristensen [14, 15] and in fact we prove a particular case of their convexity theorem, but our approach is somewhat simpler since we rely on laminates. Besides providing a concise proof of Ornstein’s result, their theorem also has applications to the regularity of hessians of rank-one convex functions and to the characterization of gradient Young measures [16, 17]. Both the failure of (1) when \( p = 1 \), \( P_1 = D, P_2 = \mathcal{E} \), as well as the existence of rank-one convex functions on 3-dimensional spaces with irregular hessians, were proved by the first named author and collaborators in [5, 6] through constructions with with unbounded laminates (the so-called staircase laminates) introduced in [10, 11]. See [1] and [12, 21] for related problems for \( p > 1 \).

For a bounded open set \( \Omega \subset \mathbb{R}^n \) and a function \( f: \mathbb{R}^{m \times n} \to \mathbb{R} \), we say that

(a) \( f \) is quasiconvex at \( A \in \mathbb{R}^{m \times n} \) if \( 0 \leq \int_{\Omega} f(A + D\varphi) - f(A) \, dx \) for all \( \varphi \in C^\infty_c(\Omega, \mathbb{R}^m) \);

(b) \( f \) is rank-one convex if \( t \mapsto f(A + tX) \) is convex for all \( A, X \in \mathbb{R}^{m \times n} \) with rank \( X = 1 \);

(c) \( f \) is positively 1-homogeneous if \( f(tA) = tf(A) \) for all \( t > 0 \) and all \( A \in \mathbb{R}^{m \times n} \);

(d) \( f \) is 1-homogeneous if \( f(tA) = |t|f(A) \) for all \( t \in \mathbb{R} \) and all \( A \in \mathbb{R}^{m \times n} \).

It is well-known that the definition in (a) is independent of \( \Omega \) and that (a) \( \Rightarrow \) (b), see [7], although in general the converse is not true [26]. In [15], the following theorem was proved:

**Theorem 2** (Kirchheim–Kristensen). Let \( f: \mathbb{R}^{m \times n} \to \mathbb{R} \) be positively 1-homogeneous and rank-one convex. Then \( f \) is convex at all matrices \( X \) with rank \( X \leq 1 \).

The reader may find other results concerning positively 1-homogeneous rank-one convex functions in [8, 20, 24]. In the planar case there is a particularly simple proof of Theorem 2 for 1-homogeneous functions:

**Lemma 3.** Let \( f: \mathbb{R}^{2 \times 2} \to \mathbb{R} \) be 1-homogeneous and rank-one convex. Then \( f \geq 0 \) and moreover, as \( f(0) = 0 \), \( f \) is convex at zero.

**Proof:** For \( A \in \mathbb{R}^{2 \times 2} \) the singular value decomposition yields \( Q, R \in O(2) \) and \( \Lambda \in \mathbb{R}_{\text{diag}}^{2 \times 2} \) such that \( A = QAR \); moreover, the entries of \( \Lambda \) are non-negative. Let us write \( (x, y) \equiv \text{diag}(x, y) \). If \( 0 \neq A \) then, by homogeneity, we can assume that \( \Lambda = (1, y) \), where \( y \geq 0 \). The measure

\[
\nu = \frac{1}{2} \delta_{Q(2,-2y)R} + \frac{1}{3} \delta_{Q(1,y)R} + \frac{1}{6} \delta_{Q(-2,-2y)R}
\]

is a laminate with barycentre \( Q(1,-y)R \). Indeed, we have the splittings

\[
(1, -y) \to \frac{1}{3} (1, y) + \frac{2}{3} (1, -2y) \to \frac{1}{3} (1, y) + \frac{1}{6} (-2, -2y) + \frac{1}{2} (2, -2y)
\]

and the map \( A \mapsto QAR \) is rank-preserving. Since \( f \) is rank-one convex and 1-homogeneous,

\[
f(Q(1,-y)R) \leq \frac{1}{2} f(Q(2,-2y)R) + \frac{1}{3} f(Q(1,y)R) + \frac{1}{6} f(Q(-2,-2y)R)
\]

\[
= f(Q(1,-y)R) + \frac{1}{3} f(A) + \frac{1}{3} f(-A)
\]

Hence \( 0 \leq f(A) + f(-A) = 2f(A) \) and the proof is finished. \( \square \)
**Remark 4.** An identical proof gives the same conclusion if \( f: \mathbb{R}_{sym}^{2 \times 2} \to \mathbb{R} \); in this case one takes \( R = Q^T \), since symmetric matrices are diagonalisable by orthogonal matrices.

From Lemma 3 we get a two-dimensional version of Ornstein’s non-inequality:

**Theorem 5.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set and let \( P_i \) be first-order differential operators, \( i = 1, 2 \), acting on \( \varphi \in C_c^\infty(\Omega, \mathbb{R}^2) \) by \( P_i \varphi = P_i(D\varphi) \), where \( P_i \in \text{Lin}(\mathbb{R}^{2 \times 2}, \mathbb{R}^{d_i}) \).

Suppose that there is a constant \( C \) such that

\[
\|P_1 \varphi\|_{L^1} \leq C \|P_2 \varphi\|_{L^1} \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^2).
\]

Then there is \( T \in \text{Lin}(\mathbb{R}^{d_2}, \mathbb{R}^{d_1}) \) such that \( P_1 = T \circ P_2 \). Moreover, the same conclusion is true if we require that (6) holds only for those \( \varphi \) of the form \( \varphi = \nabla \phi \) for some \( \phi \in C_c^\infty(\Omega, \mathbb{R}) \).

**Proof:** Consider the function \( f: \mathbb{R}^{2 \times 2} \to \mathbb{R} \) defined by \( f(A) = C \|P_2 A\| - \|P_1 A\| \). Its quasiconvex envelope \( f^{qc}: \mathbb{R}^{2 \times 2} \to [-\infty, \infty] \) is given by the Dacorogna formula

\[
f^{qc}(A) = \inf_{\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)} \int_{\mathbb{R}^2} f(A + D\varphi) \, dx;
\]

it is easily checked that \( f^{qc} \) is 1-homogeneous, since the same holds for \( f \). Note that (6) is equivalent to \( f^{qc}(0) \geq 0 \); thus \( f^{qc} > -\infty \) everywhere and hence \( f^{qc} \) is rank-one convex. Applying Lemma 3 we see that \( 0 \leq f^{qc} \leq f \) and so we must have \( \ker P_2 \subseteq \ker P_1 \). Take \( T = P_1 P_2^\dagger \), where \( P_2^\dagger \) is the Moore–Penrose inverse, defined by

\[
P_2^\dagger \equiv \left( P_2|_{(\ker P_2)}^\perp \right)^{-1} \text{Proj}_{\ker P_2}.
\]

Since \( P_2^\dagger P_2 \) is the orthogonal projection onto \( (\ker P_2)^\perp \), the conclusion follows.

The last part is identical, except that we replace Lemma 3 with Remark 4: if (6) holds for all potential vector fields then \( (f|_{\mathbb{R}^{2 \times 2}})^{qc}(0) \geq 0 \), see [2, 25] for quasiconvexity on \( \mathbb{R}^{n \times n}_{sym} \). \( \square \)

In particular, from the second part of Theorem 5 we recover [22, Part 1]:

**Corollary 7.** Given a bounded open set \( \Omega \subset \mathbb{R}^2 \), there is no constant \( C \) such that

\[
\int_{\Omega} |\partial_{x_1 x_2} \phi(x)| \, dx \leq C \int_{\Omega} |\partial_{x_1} \phi(x)| + |\partial_{x_2} \phi(x)| \, dx \quad \text{for all } \phi \in C_c^\infty(\Omega).
\]

**References**

[1] Astala, K., Faraco, D., and Székelyhidi Jr., L. Convex integration and the \( L^p \) theory of elliptic equations. *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* 7, 5 (2008), 1–50.

[2] Ball, J., Currie, J., and Oliver, P. Null Lagrangians, weak continuity, and variational problems of arbitrary order. *Journal of Functional Analysis* 41, 2 (apr 1981), 135–174.

[3] Bourgain, J., and Brezis, H. On the equation \( \text{div} Y = f \) and application to control of phases. *Journal of the American Mathematical Society* 16, 02 (nov 2002), 393–427.

[4] Calderon, A. P., and Zygmund, A. On the existence of certain singular integrals. *Acta Mathematica* 88, 1 (1952), 85–139.

[5] Conti, S., Faraco, D., and Maggi, F. A New Approach to Counterexamples to \( L^1 \) Estimates: Korn’s Inequality, Geometric Rigidity, and Regularity for Gradients of Separately Convex Functions. *Archive for Rational Mechanics and Analysis* 175, 2 (feb 2005), 287–300.
[6] Conti, S., Faraco, D., Maggi, F., and Müller, S. Rank-one convex functions on $2 \times 2$ symmetric matrices and laminates on rank-three lines. *Calculus of Variations and Partial Differential Equations* 24, 4 (dec 2005), 479–493.

[7] Dacorogna, B. *Direct Methods in the Calculus of Variations*, vol. 78 of *Applied Mathematical Sciences*. Springer New York, New York, NY, 2007.

[8] Dacorogna, B., and Marechal, P. The role of perspective functions in convexity, polyconvexity, rank-one convexity and separate convexity. *Journal of Convex Analysis* 15, 2 (2008), 271–284.

[9] de Leeuw, K., and Mirksl, H. Majorations dans $L^\infty$ des opérateurs différentiels à coefficients constants. *Comptes Rendus de l’Académie des Sciences - Series I - Mathematics* 254 (1962), 2286–2288.

[10] Faraco, D. Milton’s conjecture on the regularity of solutions to isotropic equations. *Annales de l’Institut Henri Poincare (C) Analyse Non Lineaire* 20, 5 (2003), 889–909.

[11] Faraco, D. Tartar conjecture and Beltrami operators. *The Michigan Mathematical Journal* 52, 1 (apr 2004), 83–104.

[12] Faraco, D., Mora-Corral, C., and Oliva, M. Sobolev homeomorphisms with gradients of low rank via laminates. *Advances in Calculus of Variations* 11, 2 (apr 2018), 111–138.

[13] Kazaniecki, K., Stolyarov, D. M., and Wójcichowski, M. Anisotropic Ornstein noninequalities. *Analysis & PDE* 10, 2 (feb 2017), 351–366.

[14] Kirchheim, B., and Kristensen, J. Automatic convexity of rank-1 convex functions. *Comptes Rendus Mathematique* 349, 7-8 (apr 2011), 407–409.

[15] Kirchheim, B., and Kristensen, J. On Rank One Convex Functions that are Homogeneous of Degree One. *Archive for Rational Mechanics and Analysis* 221, 1 (jul 2016), 527–558.

[16] Kristensen, J., and Raiť, B. Oscillation and concentration in sequences of PDE constrained measures. http://arxiv.org/abs/1912.09190 (dec 2019), 1–20.

[17] McMullen, C. T. Lipschitz maps and nets in Euclidean space. *Geometric and Functional Analysis* 8, 2 (1998), 304–314.

[18] Mityagin, B. S. On second mixed derivative. In *Doklady Akademii Nauk*, vol. 123. Russian Academy of Sciences, 1958, pp. 606–609.

[19] Müller, S. On quasiconvex functions which are homogeneous of degree. *Indiana University Mathematics Journal* 41, 1 (1992), 295–301.

[20] Oliva, M. Bi-Sobolev homeomorphisms $f$ with $Df$ and $Df^{-1}$ of low rank using laminates. *Calculus of Variations and Partial Differential Equations* 55, 6 (dec 2016), 1–38.

[21] Ornstein, D. A non-inequality for differential operators in the $L^1$ norm. *Archive for Rational Mechanics and Analysis* 11, 1 (1962), 40–49.

[22] Prosinski, A. Existence of minimisers of variational problems posed in spaces of mixed smoothness. *In preparation* (2020).

[23] Šverák, V. Quasiconvex functions with subquadratic growth. *Proceedings: Mathematical and Physical Sciences* (1991), 723–725.

[24] Šverák, V. New examples of quasiconvex functions. *Archive for Rational Mechanics and Analysis* 119, 4 (nov 1992), 293–300.

[25] Šverák, V. Rank-one convexity does not imply quasiconvexity. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* 120, 1-2 (nov 1992), 185–189.