THE FREQUENCY SPACE OF A FREE GROUP

ILYA KAPOVICH

Abstract. We analyze the structure of the frequency space $Q(F)$ of a non-abelian free group $F = F(a_1, \ldots, a_k)$ consisting of all shift-invariant Borel probability measures on $\partial F$ and construct a natural action of $\text{Out}(F)$ on $Q(F)$. In particular we prove that for any outer automorphism $\phi$ of $F$ the conjugacy distortion spectrum of $\phi$, consisting of all numbers $||\phi(w)||/||w||$, where $w$ is a nontrivial conjugacy class, is the intersection of $Q$ and a closed subinterval of $\mathbb{R}$ with rational endpoints.

1. Introduction

Let $F = F(a_1, \ldots, a_k)$ be a free group of rank $k > 1$ with a fixed free basis $X = \{a_1, \ldots, a_k\}$. We identify the hyperbolic boundary $\partial F$ with the set of all semi-infinite freely reduced words in $F$. Let $T = T_X : \partial F \to \partial F$ be the standard shift operator, which erases the first letter of every such semi-infinite word over $X^{\pm 1}$.

The frequency space $Q(F) = Q_X(F)$ can be defined as the space of all $T$-invariant Borel probability measures on $\partial F$. Objects of this sort appear naturally in ergodic theory, but in the present paper we concentrate on the algebraic and combinatorial structure of $Q(F)$ rather than on the more traditional analytical and probabilistic questions. Moreover, there is a natural correspondence between $Q(F)$ and the space of “geodesic currents” on $F$, that is the space of scalar equivalence classes of all $F$-invariant positive Borel measures on $\partial^2 F := \{(x, y) \in \partial F \times \partial F : x \neq y\}$ (in many instances one considers “unordered” currents, that is measures which are invariant both with respect to the $F$-action and the flip map $(x, y) \mapsto (y, x)$). The space of geodesic currents on a surface group, and, more generally on a word-hyperbolic group, has been extensively studied by Bonahon [6, 7, 8] (whose work served as a substantial source of inspiration for the present paper) and other authors and it plays an important role in modern 3-manifold topology. We provide a brief discussion of geodesic currents in Section 7 below.

In this paper we study a natural action of $\text{Out}(F)$ on $Q(F)$ by homeomorphisms. The existence of such an action was well understood in the surface group case and it can in fact be established via a correspondence between $Q(F)$ and geodesic currents, where it was first studied by Reiner Martin [21]. We provide a direct construction here which, as we will see, also yields valuable new information about how this action looks like in the natural “frequency coordinates” on $Q(F)$.

We first observe (this also follows from Bonahon’s work and was proved by Martin in the geodesic currents setting [21]) that there is a natural map $\alpha : C \to Q(F)$, where $C$ is the set of nontrivial conjugacy classes in $F$, with dense image in $Q(F)$. 

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Also, \(\alpha(C) = \alpha(C_0)\) where \(C_0\) is the set of root-free nontrivial conjugacy classes and \(\alpha\) is injective on \(C_0\). It turns out that the natural action of \(\text{Out}(F)\) on \(C_0\) translates into a continuous action on \(Q(F)\).

We will briefly discuss the map \(\alpha\) here, which will also motivate the name “frequency space” for \(Q(F)\). Let \(w\) be a nontrivial conjugacy class in \(F\). We will think of \(w\) as a cyclic word that is a cyclically reduced word written on a circle in a clockwise direction without specifying a base-point. We denote the length of \(w\) by \(|w|\). For any freely reduced word \(v\) one can define in the obvious way the number \(n_w(v)\) of occurrences of \(v\) in \(w\). Namely, we count the number of positions on the circle, starting from which it is possible to read the word \(v\) going clockwise along the circle (and wrapping around more than once, if necessary). Thus \(n_w(v)\) is defined for all \(v \in F\), even for those that are longer than \(w\). We also define the frequency \(f_w(v)\) of \(v\) in \(w\) as \(f_w(v) = n_w(v)/|w|\). A cyclic word \(w\) defines a \(T\)-invariant probability measure \(\mu_w = \alpha(w)\) on \(\partial F\) as follows. If \(v \in F\) is a nontrivial element, we set

\[
\mu_w(Cyl(v)) := f_w(v),
\]

where \(\text{Cyl}(v) = \text{Cyl}_X(v)\) consists of all semi-infinite freely reduced words with initial segment \(v\). It is not hard to see that \(\mu_w\) is indeed a \(T\)-invariant probability measure on \(\partial F\). Moreover, there is another way to view the measure \(\mu_w\). Namely, for any cyclically reduced word \(v\) defining the cyclic word \(w\) the point \(v^\infty = vv\ldots \in \partial F\) is \(T\)-periodic with the period equal to the length of the deepest root \(w_0\) of \(w\). The \(T\)-orbit \(A(w) = \{T^n v^\infty | n \geq 0\}\) is independent of \(v\) and consists of \(|w_0|\) points. It is not hard to see that \(\mu_w\) is precisely the discrete measure uniformly supported on \(A(w)\):

\[
\mu_w = \frac{1}{\# A(w)} \sum_{x \in A(w)} \delta_x.
\]

It follows that that \(\alpha(C)\) is dense in \(Q(F)\), since the finite sets \(A(w)\) are exactly the \(T\)-periodic orbits in \(\partial F\). Thus for any measure \(\mu \in Q(F)\) one may call the number \(f_\mu(v) := \mu(Cyl(v))\) the frequency of \(v\) in \(\mu\). Note that the collection of frequencies \((f_\mu(v))_{v \in F}\) uniquely determines \(\mu \in Q(F)\) and thus \((f_\mu(v))_{v \in F}\) can be considered as a global “coordinate system” for \(Q(F)\). We prove (see Theorem \ref{thm:coordinates} below) that the canonical action of \(\text{Out}(F)\) on \(C_0\) extends via \(\alpha\) (uniquely since \(\alpha(C_0)\) is dense in \(Q(F)\)) to an action of \(\text{Out}(F)\) on \(Q(F)\) by homeomorphisms. Moreover, it turns out that in the frequency coordinates described above the action of each \(\phi \in \text{Out}(F)\) on \(Q(F)\) is expressible by fractional linear transformations, where each coordinate function \(f_{\phi(p)}(v)\) (with \(v \in F\) being a fixed element) depends in a fractional-linear manner on only finitely many frequency coordinates \((f_\mu(v))_{v \in F}\) of the argument. Moreover, these transformations are, in a sense, linear. Namely, for a fixed point \(p \in Q(F)\) all the fractional-linear coordinate functions \(f_{\phi(p)}(v)\), \(v \in F\), have the same denominators.

We also study in detail the structure of \(Q(F)\) and the image of the map \(\alpha\), concentrating on the description of \(Q(F)\) as the inverse limit of a sequence of affine maps between finite-dimension convex compact polyhedra \(Q_m\). Roughly speaking \(Q_m\) captures “finitary chunks” of \(T\)-invariant measures \(\mu\), namely \((f_\mu(u))_{|u|=m}\). In particular, we obtain an explicit geometric and algorithmically verifiable criterion (Theorem \ref{thm:coordinates}) characterizing those tuples of rational numbers \(q = (q_u)_{u \in F, |u|=m}\) that can be realized as frequency tuples of cyclic words.
Although the definition of $Q(F) = Q_X(F)$ is less equivariant than that of the space of geodesic currents (because of the dependence on $X$), it turns out that the action of $Out(F)$ on $Q(F)$ is particularly informative for algebraic purposes, as it contains some new interesting algebraic, geometric and algorithmic information about automorphisms of $F$. In fact, it appears that the action of $Out(F)$ on $Q(F)$ often provides data of a different kind from what one usually gets by traditional outer space and train-track methods.

Our main applications concern the metric distortion properties of automorphisms. For an automorphism $\phi \in Aut(F)$ we define the conjugacy distortion spectrum $I(\phi)$ of $\phi$ (with respect to a fixed basis $A$ of $F$) as

$$I(\phi) := \{ ||\phi(w)|| : w \in F, w \neq 1 \}.$$  

It is obvious that $I(\phi) \subseteq \mathbb{Q}$ and one can easily show that there exist numbers $0 < C_1, C_2 < \infty$ such that $I(\phi) \subseteq [C_1, C_2]$. Surprisingly, it is possible to obtain a very precise description of the possible shape of $I(\phi)$. Thus we prove:

**Theorem A.** For any $\phi \in Aut(F)$ there exists a closed interval $J \subseteq \mathbb{R}$ such that

$$I(\phi) = J \cap \mathbb{Q}.$$

Moreover, the endpoints of $J$ are positive rational numbers that are algorithmically computable in terms of $\phi$.

Theorem A implies, in particular, that the “extremal distortion factors” $\sup I(\phi)$ and $\inf I(\phi)$ actually belong to $I(\phi)$ and are therefore realized as distortion factors of some nontrivial conjugacy classes in $F$.

In a subsequent work with Kaimanovich and Schupp [13] we combine the results of this paper with a number of algebraic and probabilistic methods to obtain certain rigidity results regarding $I(\phi)$. In particular, it turns out that for any $\phi \in Out(F)$ there exists a nontrivial conjugacy class $w$ such that $||w|| = ||\phi(w)||$. Moreover, there exists an open interval $I_0 \subseteq \mathbb{R}$ containing 1 such that for any $\phi \in Out(F)$ either $\phi$ is equal in $Out(F)$ to an automorphism induced by a permutation of $X^{\pm 1}$, in which case $I(\phi) = \{1\}$, or $\mathbb{Q} \cap I_0 \subseteq I(\phi)$.

We also obtain some applications to hyperbolic automorphisms of free groups. An outer automorphism $\phi \in Out(F)$ is strictly hyperbolic if there is $\lambda > 1$ such that

$$\lambda ||w|| \leq \max\{||\phi(w)||, ||\phi^{-1}(w)||\}$$

for every cyclic word $w$.

An outer automorphism $\phi \in Out(F)$ is hyperbolic if there is $n > 0$ such that $\phi^n$ is strictly hyperbolic (this definition is equivalent to the one used in [2], as observed in [11]).

This notion is important because of its connection with the Combination Theorem of Bestvina and Feigl [2, 3] (see also [13]) and the structure of free-by-cyclic groups. Peter Brinkmann [10] proved that $\phi \in Out(F)$ is hyperbolic if and only if $\phi$ does not have nontrivial periodic conjugacy classes. Using the Combination Theorem this implies that for $\psi \in Aut(F)$ the free-by-cyclic group $G = F \rtimes_{\psi} \mathbb{Z}$ is hyperbolic if and only if the outer automorphism $\phi$ defined by $\psi$ has no periodic conjugacy classes, that is if and only if $G$ does not contain $\mathbb{Z} \times \mathbb{Z}$-subgroups.

For $\phi \in Out(F)$ let $\lambda_0(\phi)$ be the infimum over all nontrivial cyclic words $w$ of

$$\max\{||\phi(w)||, ||\phi^{-1}(w)||\}.$$
Theorem B. Let \( \phi \in \text{Out}(F) \). Then \( \lambda_0(\phi) \) is a rational number, algorithmically computable in terms of \( \phi \). The outer automorphism \( \phi \) is strictly hyperbolic if and only if \( \lambda_0(\phi) > 1 \), and hence it is algorithmically decidable whether \( \phi \) is strictly hyperbolic.

This provides an algorithm (apparently the first known one) for deciding if \( \phi \) is strictly hyperbolic: compute \( \lambda_0(\phi) \) and check if \( \lambda_0(\phi) > 1 \). Moreover, a detailed analysis of the proof produces an explicit worst-case complexity bound for the above algorithms, namely, double exponential time in terms of the length of \( \phi \) as a product of generators of \( \text{Out}(F) \). As we note later, one can also use Theorem B together with a result of Brinkmann \[10\] to provide a new algorithm for deciding if an automorphism is hyperbolic.

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After the first version of this paper has been released, the author was alerted by Mladen Bestvina of the existence of an earlier unpublished dissertation of his UCLA student Reiner Martin [21]. In his 1995 PhD thesis [21], Martin studied, in particular, the space of geodesic currents on a free group. In particular, he obtained a description of an action of \( \text{Out}(F) \) on the space of currents which, via its identification with the frequency space, can be seen to coincide with the action constructed here. Moreover, there are, inevitably, many similarities in the general treatment of the space of currents by Martin and frequency space in the present paper. However, as it turned out, our approaches also differ in many details and in most applications. Thus our treatment of \( Q_X(F) \) as being approximated by finite-dimensional polyhedra \( Q_m \) is new, as are our main applications, namely the “realization theorem” Theorem 4.5, Theorem A and Theorem B. On the other hand, Martin’s thesis contains many interesting results that are not addressed in the present paper.

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2. FREQUENCIES IN CYCLIC WORDS

Convention 2.1. Fix an integer \( k \geq 2 \) and \( F = F(a_1, \ldots, a_k) \). Denote \( X = \left\{ a_1, \ldots, a_k \right\}^{\pm 1} \). We denote by \( CR \) the set of all cyclically reduced words in \( F \).

A cyclic word is an equivalence class of cyclically reduced words, where two cyclically reduced words are equivalent if they are cyclic permutations of each other. If \( u \) is a cyclically reduced word, we denote by \( (u) \) the cyclic word defined by \( u \).

If \( u \) is a freely reduced word, we denote the length of \( u \) by \( |u| \) and the length of the cyclically reduced form of \( u \) by \( ||u|| \). If \( w = (v) \) is a cyclic word, we denote by \( ||w|| := ||v|| \) the length of \( w \).

We will denote the set of all nontrivial cyclic words by \( C \).
Let \( w \) be a nontrivial cyclic word and let \( u \) be a nontrivial freely reduced word. We denote by \( n_w(u) \) the number of occurrences of \( u \) in \( w \). This notation has obvious meaning even if \( |u| > |w| \). Namely, let \( w = (z) \) for a cyclically reduced word \( z \).

Take the smallest \( p > 0 \) such that \( |z^{p-1}| \geq 2|u| \) and count the number of those \( i, 0 \leq i < |w| \) such that \( z^p \equiv z_1 uz_2 \) where \( |z_1| = i \). This number by definition is \( n_w(u) \). If \( u = 1 \) we define \( n_w(u) := ||w|| \).

Another way to think of \( n_w(u) \) is as follows. Identify a cyclic word \( w \) with a circle subdivided in \( ||w|| \) directed edges labeled by letters from \( X \) so that starting at any position and going clockwise around the circle once we read a cyclically reduced word \( i \), \( n \) subdivided in \(||\) meaning even if \( which and going clockwise it is possible to read the word \( u \) without getting off the circle.

Also, we denote \( f_w(u) := \frac{n_w(u)}{||w||} \) and call it the frequency of \( u \) in \( w \).

Thus, for example, \( n_u(aaa) = 1 \) and \( f_w(1) = 1 \) for any \( w \).

If \( w \) is a nontrivial freely reduced word (that is not necessarily cyclically reduced), and \( v \) is a freely reduced word, we denote by \( n_w(v) \) the number of occurrences of \( v \) in \( w \). If \( |w| = n > 0 \) then by definition \( n_w(v) \) is the number of those \( i, 0 \leq i < n \) for which \( w \) decomposes as a freely reduced product \( w = w'v_{w''} \) with \( |w'| = i \). Thus if \( |v| \leq |w| \) then necessarily \( n_w(v) = 0 \) (unlike the situation if \( w \) is a cyclic word).

**Lemma 2.2.** Let \( w \) be a nontrivial cyclic word. Then:

1. For any \( m \geq 0 \) and for any freely reduced word \( u \) with \( |u| = m \) we have:
   \[
   n_w(u) = \sum_{x \in X, |ux| = |u|+1} n_w(xu) + \sum_{x \in X, |ux| = |u|+1} n_w(xu),
   \]
   and
   \[
   f_w(u) = \sum_{x \in X, |ux| = |u|+1} f_w(xu) + \sum_{x \in X, |ux| = |u|+1} f_w(xu).
   \]

2. For any \( m \geq 1 \)
   \[
   \sum_{|u|=m} n_w(u) = ||w|| \quad \text{and} \quad \sum_{|u|=m} f_w(u) = 1.
   \]

3. For any \( s > 0 \) and any \( u \in F \)
   \[
   n_w(u) = sn_w(u) \quad \text{and} \quad f_w(u) = f_w(u).
   \]

**Proof.** Parts (1) and (3) are obvious. We establish (2) by induction on \( m \). For \( m = 1 \) the statement is clear. Suppose that \( m > 1 \) and that (2) has been established for \( m - 1 \).

We have:
\[
\sum_{|u|=m} n_w(u) = \sum_{|v|=m-1, x \in X} n_w(vx) = \sum_{|v|=m-1} n_w(v) = ||w||,
\]

as required.

Note that \( 0 \leq f_w(v) \leq 1 \) and \( f_w(v) \in \mathbb{Q} \). Part (2) of Lemma 2.2 implies that if \( w \) is a cyclic word and \( m \geq 1 \) is an integer, then the set of frequencies \( \{f_w(v)\}_{v \in F, |v|=m} \) defines a point in the standard simplex of dimension \( 2k(2k-1)^{m-1} - 1 \) standardly embedded in the Euclidean space of dimension \( 2k(2k-1)^{m-1} \).
Convention 2.3. Denote $D(m) := 2k(2k-1)^{m-1}$ for $m > 0$ and $D(0) := 0$. Note that for any $m \geq 0$ $D(m)$ is exactly the number of elements of length $m$ in $F$. We also identify $\mathbb{R}^0 = \mathbb{R}^{D(0)} = \{1\}$.

In the subsequent discussion we will think of the coordinates of points $q \in \mathbb{R}^{D(m)}$ as indexed by elements $v \in F$ with $|v| = m$. Thus a point $q \in \mathbb{R}^{D(m)}$ is a $D(m)$-tuple $(q_v)_{|v|=m}$ of real numbers.

Definition 2.4. Let $m \geq 0$ be an integer. We define the set $Q_m \subseteq \mathbb{R}^{D(m)}$ as the collection of all points $q = (q_v)_{|v|=m} \in \mathbb{R}^{D(m)}$ that satisfy the following conditions:

1. We have $\sum_{|v|=m} q_v = 1$.
2. For each $v \in F$ with $|v| = m$ we have $q_v \geq 0$.
3. For each $u \in F$ with $|u| = m - 1$ we have

$$\sum_{\{x \in X : |ux| = m\}} q_{ux} = \sum_{\{y \in X : |yu| = m\}} q_{yu}.$$ 

We also define $Q_0 := \{1\}$. Sometimes, we will also denote $Q_m$ as $Q_{m,k}$ to stress the dependence of this notion on the rank $k$ of the free group $F$.

Thus $Q_m$ is a convex compact polyhedron in $\mathbb{R}^{D(m)}$ that, for $m > 0$, is contained in the standardly embedded $D(m) - 1$-dimensional simplex in $\mathbb{R}^{D(m)}$.

Note that for $m = 1$ condition (3) in the above definition is vacuously satisfied.

Definition 2.5. Let $m \geq 2$. We define maps $\hat{\pi}_m : \mathbb{R}^{D(m)} \to \mathbb{R}^{D(m-1)}$ as follows. Let $q = (q_v)_{|v|=m} \in \mathbb{R}^{D(m)}$. Then for any $u \in F$ with $|u| = m - 1$ we set the $u$ coordinate of $\hat{\pi}_m(q)$ to be:

$$[\hat{\pi}_m(q)]_u := \sum_{\{x \in X : |ux| = m\}} q_{ux}.$$ 

Lemma 2.6. For any $m \geq 1$ we have

$$\hat{\pi}_m(Q_m) \subseteq Q_{m-1}$$

Proof. Let $q \in Q_m$ and $z = \hat{\pi}_m(q)$. It is obvious that for every $u \in F$ with $|u| = m - 1$ $z_u \geq 0$.

We have

$$\sum_{|u|=m-1} z_u = \sum_{|u|=m-1} \sum_{\{x \in X : |ux| = m\}} q_{ux} = \sum_{|v|=m} q_v = 1,$$

so that condition (1) of Definition 2.4 holds for $z$.

To verify condition (3) of Definition 2.4 let $u' \in F$ be an element with $|u'| = m - 2$.

Then

$$\sum_{\{x \in X : |ux'| = m-1\}} z_{ux'} = \sum_{\{x \in X : |ux'| = m-1\}} \sum_{\{x_1 \in X : |u'xx_1| = m\}} q_{u'xx_1} = \sum_{\{x \in X : |ux'| = m-1\}} \sum_{\{y \in X : |yu'| = m\}} q_{yu'} = \sum_{\{x,y \in X : |yu'| = m\}} q_{yu'.$
Here the first equality holds by the definition of \( \tilde{\pi}_m \) and the second equality holds by condition (3) of Definition \ref{def:2.4} for \( q \).

Similarly,

\[
\sum_{\{y \in X:|y'\cdot x|=m-1\}} z_{yu} = \sum_{\{y \in X:|y'\cdot x|=m\}} \sum_{\{x \in X:|y\cdot x|=m\}} q_{yu'x} = \sum_{\{x,y \in X:|y\cdot x|=m\}} q_{yu'x}.
\]

Thus

\[
\sum_{\{x \in X:|u'\cdot x|=m-1\}} z_{u'x} = \sum_{\{y \in X:|y'\cdot x|=m-1\}} z_{yu'}
\]

and condition (3) of Definition \ref{def:2.4} holds for \( z \).

\[\square\]

**Definition 2.7.** In view of Lemma \ref{lem:2.6} for \( m \geq 2 \) we define

\[\pi_m : Q_m \to Q_{m-1}\]

to be the restriction of \( \tilde{\pi}_m \) to \( Q_m \). For \( m = 1 \) define \( \pi_1 : Q_1 \to Q_0 \) as \( \pi_1(q) = 1 \) for each \( q \in Q_1 \). Note that this agrees with the formula defining \( \hat{\pi}_m \) in Definition \ref{def:2.4}.

We will later on see that the map \( \pi_m \) is actually “onto”.

Any cyclic word \( w \) comes equipped with its set of frequencies \( \{f_w(v)\}_{|v|=m} \) which by Lemma \ref{lem:2.5} satisfy all the conditions of Definition \ref{def:2.4}.

**Definition 2.8.** Let \( m \geq 1 \). We define the map \( \alpha_m : C \to Q_m \) as follows. For a cyclic word \( w \in C \) and \( v \in F \) with \( |v| = m \) we set

\[ [\alpha_m(w)]_v := f_w(v). \]

We have seen that the sets \( Q_m \) are compact convex Euclidean polyhedra and that the maps \( \pi_m : Q_m \to Q_{m-1} \) are affine for each \( m \geq 1 \). We can now formulate the main definition of the paper.

**Definition 2.9** (The Frequency Space). Consider the systems of maps

\[ (\dagger) \quad \cdots \to Q_m \xrightarrow{\pi_m} Q_{m-1} \xrightarrow{\pi_{m-1}} \cdots \to Q_2 \xrightarrow{\pi_2} Q_1 \xrightarrow{\pi_1} Q_0 \]

We define the frequency space \( Q(F) = Q_X(F) \) to be the inverse limit of this sequence:

\[ Q(F) := \varprojlim_{\leftarrow m}(Q_m, \pi_m) \]

Thus an element of \( Q(F) \) is a tuple of numbers \( (q_v)_{v \in F} \) that are consistent with respect to the sequence \((\dagger)\). This means that:

(a) For any \( m \geq 1 \) \( (q_v)_{|v|=m} \) defines a point \( q_m \) of \( Q_m \).

(b) For any \( m \geq 2 \) \( \pi_m(q_m) = q_{m-1} \).

Note that \( Q(F) \) comes equipped with a canonical inverse limit topology induced by the inclusion \( Q(F) \subseteq \prod_{m \geq 0} Q_m \). Namely, two points \( q = (q_v)_{v \in F} \) and \( q' = (q'_v)_{v \in F} \) are close if there is large \( M \geq 1 \) such that \( q_m \) is close to \( q'_m \) in \( Q_m \) for all \( 1 \leq m \leq M \) (or, equivalently, that \( q_M \) is close to \( q'_M \) in \( Q_M \)).
Remark 2.10. It is easy to see that the natural maps \( \alpha_m : C \to Q_m \) define a map
\[ \alpha : C \to Q(F) \]

If \( w \in C \) is a cyclic word, then its image \( \alpha(w) \in Q(F) \) can be thought of as an infinite tuple \((f_w(v))_{v \in F}\).

Part (3) of Lemma 2.12 immediately implies that the maps \( \alpha_m \) and \( \alpha \) have a certain "projective" character in the sense that the image of \( w \) depends only on the semigroup generated by \( w \):

Lemma 2.11. Let \( w \in C \) be a nontrivial cyclic word. Then for any \( s \geq 1 \) we have \( \alpha(w) = \alpha(w^s) \) and \( \alpha_m(w) = \alpha_m(w^s) \), where \( m \geq 0 \).

Moreover, having common positive powers is the only reason why two cyclic words can have the same image in \( Q(F) \) under \( \alpha \):

Proposition 2.12. Let \( w, u \) be nontrivial cyclic words. Then \( \alpha(w) = \alpha(u) \) if and only if both \( w \) and \( u \) are positive powers of the same cyclic word.

Proof. The "if" implication is obvious by Lemma 2.11. Suppose now that \( \alpha(w) = \alpha(u) \). Hence \( \alpha_m(w) = \alpha_m(u) \) for every \( m \geq 0 \).

There exist some \( s, t > 0 \) such that \( ||w^s|| = ||u^t|| \). Denote \( j = ||w^s|| = ||u^t|| \). Then \( \alpha_m(w^s) = \alpha_m(u^t) \) for each \( m \geq 0 \) and, in particular \( \alpha_j(w^s) = \alpha_j(u^t) \).

Since \( j = ||w^s|| \), whenever \( f_{w^s}(v) > 0 \) for \( v \in F \) with \( |v| = j \), then \( v \) must be a cyclically reduced word with \( (v) = w^s \). The same is true for \( u^t \). Choose \( v \in F \) with \( |v| = j \) such that \( f_{w^s}(v) = f_{u^t}(v) > 0 \). Then \( w^s = (v) = u^t \), which implies the statement of Proposition 2.12. \( \square \)

3. THE FREQUENCY SPACE AS THE SPACE OF INVARIANT MEASURES

Recall that if \((\Omega, \mathcal{F}, \mu)\) is a probability space, and \( T : \Omega \to \Omega \) be a measurable map, then \( \mu \) is said to be \( T \)-invariant if for any \( A \in \mathcal{F} \) we have \( \mu(A) = \mu(T^{-1}A) \).

Convention 3.1. Suppose \( F = F(a_1, \ldots, a_k) \) if a free group of finite rank \( k > 1 \). Then the hyperbolic boundary \( \partial F \) is naturally identified with the set of semi-infinite freely reduced words in \( \{a_1, \ldots, a_k\}^{\pm 1} \) corresponding to geodesic rays from 1 in the standard Cayley graph of \( F \). We endow \( \partial F \) with the \( \sigma \)-algebra \( \mathcal{F} \) of Borel sets. If \( u \) is a freely reduced word of length \( n \) then by \( Cyl(u) \) we denote the set of all \( x \in \partial X \) that have \( u \) as initial segment. Denote by \( T : \partial F \to \partial F \) the shift operator which erases the first letter of every semi-infinite freely reduced word.

Theorem 3.2. The frequency space \( Q(F) \) is canonically identified with the space of all \( T \)-invariant Borel probability measures on \( \partial F \).

Proof. Indeed, suppose \( \mu \) is a \( T \)-invariant Borel probability measure on \( \partial F \).

For each \( v \in F \) put \( q_v := \mu(Cyl(v)) \). We claim that the tuple \((q_v)_{v \in F} \) defines a point of \( Q(F) \). We need to show that for any \( m \geq 1 \) \( q_m := (q_v)_{|v|=m} \in Q_m \) and that for \( m \geq 1 \) we have \( \pi_m(q_m) = q_{m-1} \).

For any \( m \geq 0 \)
\[ 1 = \mu(\partial F) = \sum_{|v|=m} \mu(Cyl(v)) = \sum_{|v|=m} q_v, \]
so that condition (1) of Definition 2.4 holds for \( q_m \). For any \( v \in F \) we have

\[
Cyl(v) = \bigcup_{\{x \in X : |vx| = |v| + 1\}} Cyl(vx)
\]

which implies

\[
q_v = \sum_{\{x \in X : |vx| = |v| + 1\}} q_{vx}.
\]

Moreover, \( T^{-1}(Cyl(v)) = \bigcup_{\{x \in X : |xv| = |v| + 1\}} Cyl(xv) \) and therefore by \( T \)-invariance of \( \mu \)

\[
q_v = \sum_{\{x \in X : |xv| = |v| + 1\}} q_{xv}
\]

This implies that for each \( m \geq 1 \) all the conditions of Definition 2.4 hold for \( q_m \) and so \( q_m \in Q_m \).

Moreover, by definition of \( \pi_m \) the above equation also implies that for any \( m \geq 1 \) we have \( \pi_m(q_m) = q_{m-1} \), as required. Thus indeed \( q = (q_v)_{v \in F} \in Q(F) \).

Suppose now that \( q = (q_v)_{v \in F} \in Q(F) \). Thus for any \( m \geq 1 \), \( q_m := (q_v)_{|v| = m} \in Q_m \) and \( \pi_m(q_m) = q_{m-1} \) provided \( m \geq 1 \). We will specify a Borel measure \( \mu \) on \( \partial F \) by its values on all the sets \( Cyl(v), v \in F \). For \( v \in F \) set

\[
\mu(Cyl(v)) := q_v.
\]

Then, by conditions (1)-(3) of Definition 2.4 and because \( \pi_m(q_m) = q_{m-1} \) it is easy to see that \( \mu \) is indeed a \( T \)-invariant Borel probability measure on \( \partial F \).

We have constructed two maps between \( Q(F) \) and the set of \( T \)-invariant Borel probability measures on \( \partial F \) that are mutually inverse. This maps provide the required identification asserted in Theorem 3.2. Moreover, it is easy to see that the inverse limit topology on \( Q(F) \) coincides with the weak topology on the space of Borel \( T \)-invariant measures. \( \square \)

Recall that for a cyclic word \( w \) the measure \( \alpha(w) \) on \( \partial F \) was defined by \( \alpha(w)(Cyl(u)) = f_w(u) \) for \( u \in F \). There is another simple description of \( \alpha(w) \).

If \( v \) is a cyclically reduced word, we denote by \( v^\omega \) the element of \( \partial F \) corresponding to the geodesic ray from 1 with the label \( vvvv \ldots \). For a cyclic word \( w \) denote

\[
A(w) := \{v^\omega | (v) = w\}.
\]

It is not hard to see that \( A(w) \) is a finite subset of \( \partial F \) consisting of \( |w| \) elements where \( w = w_0^s \) with maximal possible \( s \geq 1 \). In particular, if \( w \) is not a proper power then \( \#A(w) = |w| \). Recall that a point \( x \in \partial F \) is said to be \( T \)-periodic if there is \( n > 0 \) such that \( T^n x = x \).

The following is an easy corollary of the definitions:

**Lemma 3.3.** The following hold:

1. Let \( w \) be a cyclic word that is not a proper power and let \( u \in F \) be a nontrivial freely reduced word. Then \( w_u \) is equal to the number of those \( x \in A(w) \) that have initial segment \( u \), that is, to the number of those \( x \in A(w) \) that are contained in \( C(u) \).
2. A point \( x \in \partial F \) is \( T \)-periodic if and only if \( x = v^\omega \) for some cyclically reduced word \( v \).
(3) For any cyclic word $w$ and any $v \in F$ with $(v) = w$ the $T$-orbit of $v^\infty$ is precisely $A(w)$:

$$A(w) = \{ T^n v^\infty | n \geq 0 \}. $$

This in turn immediately implies:

**Lemma 3.4.** For any cyclic word $w$ the measure $\alpha(w)$ is the discrete measure uniformly supported on $A(w)$:

$$\alpha(w) = \frac{1}{\# A(w)} \sum_{x \in A(w)} \delta_x.$$ 

**Corollary 3.5.** The set $\alpha(C)$ is dense in $Q(F)$. 

**Proof.** By Theorem 3.2 the space $Q(F)$ is exactly the set of $T$-invariant Borel probability measures on $\partial F$. It follows from the basic results of symbolic dynamics [17] (cf [24, 22]) that the set $Z$ of discrete measures uniformly supported on $T$-periodic orbits is dense in $Q(F)$. By Lemma 3.3 and Lemma 3.4 $Z = \alpha(C)$ and the statement follows. 

4. Analyzing the image of the map $\alpha_m$

Our next goal is to understand the image of the map $\alpha_m$, namely $\alpha_m(C) \subseteq Q_m$. Obviously, each point in $\alpha_m(C)$ must have all rational coordinates, but it is not clear for the moment if this conditions is also sufficient.

To study this question we need to introduce the following useful notion.

**Definition 4.1** (Initial graph). Let $m \geq 2$ and let $q = (q_v)_{|v|=m}$ be a point in $Q_m$. We define the initial graph $\Gamma_q$ as follows.

The vertex set of $\Gamma_q$ is the set $\{ u \in F : |u| = m - 1 \}$.

For each $v \in F$ with $|v| = m - 1$ there is a directed edge in $\Gamma_q$ with label $q_v$ from the vertex $v_-$ to the vertex $v_+$. Here $v_-$ denotes the initial segment of $v$ of length $m - 1$ and $v_+$ denotes the terminal segment of $v$ of length $m - 1$. Thus $\Gamma_q$ is a labelled directed graph with the sum of the edge-labels equal to 1.

We also denote by $\Gamma'_q$ the graph obtained from $\Gamma_q$ by first removing all the edges labeled by 0 and then removing all isolated vertices from the result. We call $\Gamma'_q$ the improved initial graph of $q$. The edge-set of $\Gamma'_q$ is in 1-to-1 correspondence with the set of those $v \in F, |v| = m$ for which $q_v > 0$.

If $w$ is a nontrivial cyclic word and $q = \alpha_m(w)$, we denote the graph $\Gamma_q$ by $\Gamma_{w,m}$ (or just $\Gamma_w$ if the value of $m$ is fixed). Similarly, we denote $\Gamma'_q$ by $\Gamma'_{w,m}$ (or just $\Gamma'_w$).

The following lemma is an immediate corollary of Definition 4.1.

**Lemma 4.2.** If $m \geq 2$ then for any $q \in Q_m$ the graphs $\Gamma_q$ and $\Gamma'_q$ have the following properties:

1. The sum of the edge-labels in each of $\Gamma_q, \Gamma'_q$ is equal to 1.
2. For any vertex $u$ of $\Gamma_q$ the in-degree of $u$ is equal to the out-degree of $u$ in $\Gamma_q$. The same is true for $\Gamma'_q$. Moreover, for any vertex $u$ of $\Gamma_q$ its in-degree in $\Gamma'_q$ is the same as its in-degree in $\Gamma_q$. 

The frequency space of a free group

Notation 4.3. Let $\Gamma$ be a labeled directed graph and let $r \in \mathbb{R}$. We denote by $r\Gamma$ the graph obtained from $\Gamma$ by multiplying the label of every edge of $\Gamma$ by $r$.

Let $\Gamma$ be a labeled directed graph where the label of any edge is a positive integer. We denote by $[\Gamma]$ the directed unlabelled graph obtained from $\Gamma$ by replacing every directed edge $e$ from a vertex $u$ to a vertex $v$ with label $M > 0$ by $M$ directed unlabelled edges from $u$ to $v$.

If $\Gamma$ is a directed unlabelled graph, and $n \geq 1$ is an integer, we denote by $n\Gamma$ the directed unlabelled graph obtained from $\Gamma$ by multiplying the number of directed edges between any two vertices of $\Gamma$ by $n$.

By a labelled directed graph we will mean a digraph where every edge is equipped with a real number called the label of the edge. A cycle in a directed graph is a closed directed edge-path. A circuit in a directed graph is an equivalence class of cycles, where two cycles are equivalent if they are cyclic permutations of each other. Thus the notion of a circuit is similar to that of a cyclic word in the free group context. An Euler cycle in a directed graph is a cycle that passes through each edge of the graph exactly once. An Euler circuit in a directed graph is an equivalence class of an Euler cycle.

We need the following elementary fact from graph theory.

Lemma 4.4. Let $\Gamma$ be a directed graph.

(1) Suppose the underlying undirected graph of $\Gamma$ is connected. Then $\Gamma$ possesses an Euler circuit (in the digraph sense) if and only if for every vertex $u$ of $\Gamma$ the in-degree at $u$ is equal to the out-degree at $u$.

(2) Suppose that for every vertex $u$ of $\Gamma$ the in-degree at $u$ is equal to the out-degree at $u$. Then the underlying undirected graph of $\Gamma$ is connected if and only if $\Gamma$ is connected as a digraph (that is for any vertices $u_1, u_2$ of $\Gamma$ there exists a directed path from $u_1$ to $u_2$ in $\Gamma$).

We can now give a precise description of those rational points in $Q_m$ that come from some actual cyclic words in $F$.

Theorem 4.5. Let $m \geq 2$ and $q \in Q_m$. Then $q \in \alpha_m(C)$ if and only if the underlying topological graph of $\Gamma'_q$ is connected.

Proof. Suppose $q \in Q_m$ is a rational point where the underlying graph of $\Gamma'_q$ is connected.

Choose a positive integer $N > 0$ such that for every $v \in F$ with $|v| = m$ we have $Nq_v \in \mathbb{Z}$.

Let $\Delta := [N\Gamma'_q]$. Thus the vertex sets of $\Delta$ and of $\Gamma'_q$ are the same.

Then $\Delta$ is a connected digraph with $N$ directed edges where for each vertex the in-degree is equal to the out-degree. Therefore $\Delta$ possesses an Euler circuit (in the digraph sense).

Choose such an Euler-circuit and also, by specifying an initial vertex, choose a closed edge-path $e_1, \ldots, e_N$ in $\Delta$ realizing this circuit. Each $e_i$ corresponds to a freely reduced word $v_i$ of length $m$ such that the origin of $e_i$ is $(v_i)_-$ and the terminus of $e_i$ is $(v_i)_+$.

We will create a cyclically reduced word $w$ of length $N$ as follows. Put $z_1$ to be the last letter of $v_1$. If the words $z_1, \ldots, z_{i-1}$ are already constructed, then we define $z_i$ to be the word $z_i x$ where $x$ is the last letter of $v_i$ (and so of $(v_i)_+$). Put $z := z_t$. It is easy to see that $z$ is cyclically reduced. Let $w = (z)$ be the cyclic...
word defined by \( z \). Then by construction we have \( |w| = N \). Moreover, it is easy to see that for every \( v \in F \) with \( |v| = m \) we have \( n_w(v) = Nq_v, f_w(v) = q_v \). Hence \( \alpha_m(w) = q \).

Note that in the above construction the cyclic word \( w \) depends only on the Euler circuit in \( \Delta \) but not on an actual closed-edge-path realizing this circuit.

Suppose now that \( q = \alpha_m(w) \in Q_m \) for some nontrivial cyclic word \( w \) in \( F \). Put \( N = ||w|| \). We need to show that the graph \( \Gamma'_q \) is connected.

For each \( v \in F \) with \( |v| = m \) we have \( n_w(v) = Nw' \in \mathbb{Z} \). Hence with this choice of \( N \) the graph \( N\Gamma'_q \) has positive integer edge-labels whose sum is equal to \( N \). We will produce a path in \( N\Gamma'_q \) passing through each edge of \( N\Gamma'_q \). This would imply that \( \Gamma'_q \) is connected.

Recall that the edge-set of \( \Gamma'_q \) (and hence of \( N\Gamma'_q \)) is in a canonical one-to-one correspondence with the set \( \{v \in F : |v| = m, f_w(v) \neq 0\} \). Choose a cyclically reduced word \( z \) with \( (z) = w \). Construct an edge-path \( e_1, \ldots, e_N \) in \( N\Gamma'_q \) as follows.

Write \( z \) as \( z = x_1x_2 \ldots x_N \) where \( x_j \in X \). Also, for \( t \geq N \) put \( x_t = x_i \) where \( 1 \leq i \leq N \) and \( t \equiv i \mod N \). For \( i = 1, \ldots, N \) put \( e_i \) to be the edge in \( N\Gamma'_q \) corresponding to \( x_i \ldots x_{i+m} \), that is an edge from \( x_i \ldots x_{i+m} \) to \( x_{i+1} \ldots x_{i+m-1} \). It is easy to see that \( e_1, \ldots, e_N \) is a closed edge-path in \( N\Gamma'_q \), where each edge corresponding to a word \( v \in F \) with \( |v| = m \) is repeated exactly \( Nf_w(v) = n_w(v) \) times. Since this path passes through every edge of \( N\Gamma_q \), the underlying graph of \( \Gamma'_q \) is connected, as claimed.

\[ \square \]

**Example 4.6.** Let \( k = 2, F = F(a, b) \) and \( m = 3 \).

Consider a point \( q \in \mathbb{Q}_3 \) given by \( q_{aba} = 2/5, q_{bab} = q_{aba^2} = q_{ba^2} = 1/5 \) and all other \( q_v = 0, |v| = 3, v \notin \{aba, bab, a^2b, ba^2\} \).

Take \( N = 5 \). Then the graph \( 5\Gamma_q' \) has the vertices \( ab, ba, a^2 \) and four directed edges: the edge \( e_1 \) from \( ab \) to \( ba \) with label 2, the edge \( e_2 \) from \( ba \) to \( ab \) with label 1, the edge \( e_3 \) from \( ba \) to \( a^2 \) with label 1 and the edge \( e_4 \) from \( a^2 \) to \( ab \) with label 1. Then the path \( e_1, e_2, e_1, e_3, e_4 \) defines an Euler cycle at a vertex \( ab \) in the graph \( [5\Gamma_q'] \). The corresponding cyclically reduced word \( z \) is

\[ z = abaab \]

Then for \( w = (z) \) we have \( |w| = 5, n_w(aba) = 2, n_w(bab) = n_w(a^2b) = n_w(ab^2) = 1 \) and \( \alpha_3(w) = q \).

Similarly, the path

\[ e_1, e_2, e_1, e_2, e_1, e_3, e_4 \]

defines an Euler cycle at vertex \( ab \) in the graph \([10\Gamma_q'] \). The corresponding cyclically reduced word \( z_1 \) is

\[ z_1 = ababaabab \]

Then for \( w_1 = (z_1) \) we have \( |w| = 10, n_w(aba) = 4, n_w(bab) = n_w(a^2b) = n_w(ab^2) = 2 \) and \( \alpha_3(w) = q \).

**Lemma 4.7.** Let \( q \in \mathbb{Q}_1 \) be a rational point. Then \( q \in \alpha_1(C) \) if and only if there does not exist a letter \( x \in X \) such that \( q_x > 0, q_{x^{-1}} > 0 \) and \( q_x + q_{x^{-1}} = 1 \) (and hence \( q_y = 0 \) for all \( y \in X \) such that \( y \neq x, y \neq x^{-1} \)).

**Proof.** To simplify notation we denote \( A_i = a_i^{-1} \) throughout the proof. It is clear that if \( q = \alpha_1(w) \) for a nontrivial cyclic word \( w \) then there does not exist a letter \( x \in X \) such that \( q_x > 0, q_{x^{-1}} > 0 \) and \( q_x + q_{x^{-1}} = 1 \).
Therefore by Lemma 4.9, it contains a point \( q \) which is a proper compact convex sub-polyhedron. Choose an integer \( N > 0 \) such that all coordinates of the point \( Nq \) are integers, that is \( Nq = 1 \) for each letter \( a \in \{a_1, \ldots, a_k\}^{\pm 1} \).

We define the words \( \alpha_i, \beta_i \) for \( i = 1, \ldots, k \) as follows.

- If \( q_{a_i} = 0 \), we put \( \alpha_i = \beta_i = 1 \).
- If \( q_{a_i} > 0 \), \( q_{a_i} = 0 \), we put \( \alpha_i = \beta_i = a_i^{Nq_{a_i}} \).
- If \( q_{a_i} > 0 \), \( q_{a_i} = 0 \), we put \( \alpha_i = a_i^{2Nq_{a_i}}, \beta_i = A_i^{2Nq_{a_i}} \).

Now put \( v = \alpha_1 \ldots \alpha_k \beta_1 \ldots \beta_k \). It is clear that \( v \) is a cyclically reduced word of length \( 2N \). For the cyclic word \( w = (v) \) we have \( n_w(a) = 2Nq_a \) for every letter \( a \) and hence \( q = \alpha_1(w) \in c_1(C) \), as required.

\[ \square \]

**Conjecture 4.9.** For \( m \geq 1 \) let \( Q_m^+ \) be the set of all points \( q \in Q_m \) such that for every \( v \in F \) with \( |v| = m \) we have \( q_v > 0 \). Let \( QQ_m^+ \) be the set of rational points in \( Q_m^+ \).

It is clear that if \( m \geq 1 \) then for any \( q \in Q_m^+ \) the graph \( \Gamma_q^+ \) is connected and hence by Theorem 4.2 and Lemma 4.4, \( QQ_m^+ \subseteq Q_m^+ \subseteq \alpha_m(C) \).

**Lemma 4.9.** Let \( m \geq 1 \). Then the set \( QQ_m^+ \) is dense in \( Q_m \).

**Proof.** The statement is obvious when \( m = 1 \), so we will assume that \( m \geq 2 \).

Since \( Q_m \) is a compact convex finite-dimensional polyhedron defined by equations and inequalities with rational coefficients, the set of rational points \( QQ_m \) is dense in \( Q_m \). Thus it suffices to show that \( QQ_m^+ \) is dense in \( QQ_m \).

Recall that \( D(m) \) is the number of freely reduced words of length \( m \) in \( F \). Put \( z_u = 1/D(m) \) for all \( u \in F \) with \( |u| = m \) and put \( z = (z_u)_{|u|=m} \). Then it is easy to see that \( z \in Q_m \) and in fact \( z \in QQ_m^+ \).

Let \( q \in QQ_m \) be arbitrary. Let \( 1 > \epsilon > 0 \) be a rational number. By convexity of \( QQ_m \) we have \( \epsilon z + (1-\epsilon)q \in QQ_m \) and, moreover \( \epsilon z + (1-\epsilon)q \in QQ_m^+ \). Since

\[ \lim_{\epsilon \to 0} \epsilon z + (1-\epsilon)q = q, \]

and \( q \in QQ_m \) was arbitrary, it follows that \( QQ_m^+ \) is dense in \( QQ_m \) as required.

**Note.** That Lemma 4.3 and the definition of \( Q(F) \) immediately imply that \( \alpha(C) \) is dense in \( Q(F) \), yielding another proof of Corollary 3.4.

**Corollary 4.10.** If \( m \geq 1 \) then \( \pi_m(Q_m) = Q_m - 1 \).

**Proof.** The case \( m = 1 \) is obvious, so we assume \( m \geq 2 \). It follows from the definition of \( \pi_m \) that \( \pi_m(Q_m^+) \subseteq Q_m - 1 \) and \( \pi_m(QQ_m^+) \subseteq QQ_m^+ - 1 \).

Moreover, since \( Q_m \) and \( Q_m - 1 \) are compact and \( QQ_m^+ \) is dense in \( Q_m \), the image of \( \pi_m \) is equal to the closure of \( \pi_m(QQ_m^+) \). Suppose \( \pi_m \) is not onto, so that \( \pi_m(Q_m) \) is a proper compact convex sub-polyhedron is a compact convex polyhedron \( Q_m - 1 \). Therefore \( Q_m - 1 \) contains a nonempty open subset of \( Q_m - 1 \) and therefore, by Lemma 4.3, it contains a point \( q \) of \( QQ_m^+ - 1 \).

By Theorem 4.3 and Lemma 4.4, there exists a cyclic word \( w \) such that \( \alpha_m(w) = q \). Then \( q = \pi_m(\alpha_m(w)) \) and hence \( q \in \pi_m(Q_m) \), contrary to our assumption that \( q \in Q_m - 1 \), a contradiction.

\[ \square \]
Theorem 4.14. The following hold:

Proof. Note that all $p_u$ (where $u \in F, |u| = m$) are rational numbers since $p$ is an extremal point of a compact convex polyhedron $Q_m$ defined by equations and inequalities with integer coefficients.

Suppose that $\Gamma'_p$ is not connected. Then there exists two nonempty disjoint labelled subgraphs $G_1, G_2$ of $\Gamma'_p$ such that $\Gamma'_p = G_1 \sqcup G_2$. Recall that $G'_p$ consists of all the edges of $G_p$ with positive labels and of their endpoints. Let $s$ be the sum of the edge-labels in $G_1$ and let $t$ be the sum of the edge-labels in $G_2$. Then $s + t = 1, s > 0, t > 0$.

Define $r = (r_u)_{|u|=m}$ as follows. Put $r_u = 0$ if $u \in F, |u| = m$ does not correspond to an edge of $G_1$. Put $r_u = \frac{q_u}{t}$ if $u \in F, |u| = m$ does correspond to an edge of $G_1$. Then $r \in Q_m$ and $\Gamma'_r = G_1$.

Similarly, define $q = (q_u)_{|u|=m}$ as follows. Put $q_u = 0$ if $u \in F, |u| = m$ does not correspond to an edge of $G_2$. Put $q_u = \frac{p_u}{t}$ if $u \in F, |u| = m$ does correspond to an edge of $G_2$. Then $q \in Q_m$ and $\Gamma'_q = G_2$.

By construction we have $sr + tq = p$ and $r \neq p, q \neq p$. Thus $q \in Q_m$ is a convex linear combination of two points of $Q_m$ different from $p$. This contradicts our assumption that $p$ is an extremal point of $Q_m$. \hfill \Box

In order to estimate the topological dimension of $Q_m$ we need the following elementary fact:

Lemma 4.12. Let $f_1, \ldots, f_s, g_1, \ldots, g_t : \mathbb{R}^n \to \mathbb{R}$ be affine functions. Let

\[ J = \{ x \in \mathbb{R}^n : f_i(x) = 0, 1 \leq i \leq s \} \]

and

\[ Z = \{ x \in \mathbb{R}^n : f_i(x) = 0, g_j(x) > 0, 1 \leq i \leq s, 1 \leq j \leq t \}. \]

Suppose $Z$ is nonempty. Then $\dim Z = \dim J$.

Corollary 4.13. Let $m \geq 2$. Then

\[ D(m) - D(m - 1) - 1 \leq \dim Q_m \leq D(m) - 1. \]

Proof. The upper bound is obvious from condition (1) of Definition 2.4. Let $J_m$ be the subset of $\mathbb{R}^{D(m)}$ defined by equations from parts (1) and (3) in Definition 2.4. Thus $J_m$ is defined by $D(m - 1) + 1$ affine equations, of which $D(m - 1)$ equations (those coming from part (3)) are homogeneous. Let $J'_m$ be the set defined by equations from part (3) of Definition 2.4 and the homogeneous version of the equation from part (1), namely $\sum_{|v|=m} q_v = 0$. The set $J'_m$ is given by $D(m - 1) + 1$ linear homogeneous equations in $D(m)$ variables. Hence $\dim J'_m \geq D(m) - D(m - 1) - 1$. Since $Q_m$ is nonempty, we can choose $q_0 \in Q_m \subseteq J_m$. Then $J_m = q_0 + J'_m$ and hence $\dim J_m = \dim J'_m \geq D(m) - D(m - 1) - 1$. Lemma 4.12 implies that $\dim J_m = \dim Q_m$, so that $\dim Q_m \geq D(m) - D(m - 1) - 1$, as required. \hfill \Box

Note that directly from Definition 2.4 we get $\dim Q_1 = D(1) - 1 = 2k - 1$.

We summarize our results regarding $Q_m$ and $Q(F)$ accumulated so far:

Theorem 4.14. The following hold:

1. If $m \geq 1$, then $\pi_m(Q_m) = Q_{m-1}$.
2. We have $QQ^+_m \subseteq \alpha_m(C) \subseteq Q_m$ and $QQ^+_m$ is dense in $Q_m$ for $m \geq 0$. 

The space $Q(F)$ with the inverse limit topology is compact.

If $w \in C$ is a nontrivial cyclic word then for any $n \geq 1$ we have $\alpha(w) = \alpha(w^n)$ and $\alpha_m(w) = \alpha_m(w^n)$, where $m \geq 1$.

Let $w, u$ be nontrivial cyclic words. Then $\alpha(w) = \alpha(u)$ if and only if both $w$ and $u$ are positive powers of the same cyclic word.

For $q \in Q_m$ is a rational point, where $m \geq 2$, then we have $q \in \alpha_m(C)$ if and only if the underlying undirected graph of $\Gamma_q$ is connected.

If $q \in Q_1$ is a rational point then $q \in \alpha_1(C)$ if and only if there does not exist $x \in X$ such that $q_x + q_{x-1} = 1$, $q_x > 0$, $q_{x-1} > 0$.

We have

$$D(m) - D(m - 1) - 1 \leq \dim Q_m \leq D(m) - 1$$

for any $m \geq 2$. Also we have $\dim Q_1 = D(1) - 1 = 2k - 1$ and $\dim Q_0 = 0$.

If $m \geq 2$ and $p \in Q_m$ is an extremal point of $Q_m$ then $p \in \alpha_m(C)$.

5. The Action of $\text{Out}(F)$ on the Frequency Space

**Definition 5.1** (Nielsen automorphisms). A Nielsen automorphism of $F$ is an automorphism $\tau$ of one of the following types:

1. There is some $i, 1 \leq i \leq k$ such that $\tau(a_i) = a_i^{-1}$ and $\tau(a_j) = a_j$ for all $j \neq i$.
2. There are some $1 \leq i < j \leq k$ such that $\tau(a_i) = a_j$, $\tau(a_j) = a_i$ and $\tau(a_l) = a_l$ when $l \neq i, j$.
3. There are some $1 \leq i < j \leq k$ such that $\tau(a_i) = a_i a_j$ and $\tau(a_j) = a_i$ for $l \neq i$.

It is well-known (see, for example, [19]) that $\text{Aut}(F)$ is generated by the set of Nielsen automorphisms. The following two propositions are crucial for our arguments (as well as for our results in [15]).

**Proposition 5.2.** For each Nielsen automorphism $\tau$ of $F$ and for any $u \in F$ with $|u| = m$ there exist integers $c(v) = c(v, u, \tau) \geq 0$, where $v \in F, |v| = 2m + 6$, such that for any cyclic word $w$ we have

$$n_{\tau(w)}(u) = \sum_{|v|=2m+6} c(v)n_w(v).$$

**Proof.** Note that in view of part (1) of Lemma 2.2 it suffices to express $n_{\tau(w)}(u)$ as a linear function in terms of $n_w(v)$ where $|v| \leq 2m + 6$.

The statement of the proposition is obvious if $\tau$ interchanges two generators or inverts a generator since in that case $n_{\tau(w)}(u) = n_w(\tau^{-1}(u))$.

Without loss of generality we may now assume that $F = F(a, b, a_3, ..., a_k)$ with $a = a_1, b = a_2$ and that $\tau(a) = a, \tau(b) = ba, \tau(a_i) = a_i$ for $2 < i \leq k$.

Note that if $i \neq 1$ then no occurrences of $a_i^{-1}$ cancel when we reduce $\tau(w)$. The cancellation structure involving $a_i^{\pm 1}$ is also very controlled. Only the occurrences of $ba^{-1}$ and $ab^{-1}$ in $w$ produce appearances of $aa^{-1}$ in the non-reduced form of $\tau(w)$. There are no appearances of $a^{-1}a$ in the non-reduced form of $\tau(w)$. We will say that an occurrence of $a$ in the reduced form of $\tau(w)$ is old if it comes from an occurrence of $a$ in $w$ that was not followed by $b^{-1}$ in $w$ (and hence was not cancelled in the process of freely reducing $\tau(w)$.) We will say that an occurrence of $a$ in the
reduced form of \(\tau(w)\) is \textit{new} if it comes from an occurrence of \(b\) in \(w\) that was not followed by \(a^{-1}\) in \(w\) (and hence was not cancelled in the process of reducing \(\tau(w)\)). Similarly, an occurrence of \(a^{-1}\) in the reduced form of \(\tau(w)\) is \textit{old} if it comes from an occurrence of \(a^{-1}\) in \(w\) that was not preceded by \(b\) in \(w\). An occurrence of \(a^{-1}\) in the reduced form of \(\tau(w)\) is \textit{new} if it comes from an occurrence of \(b^{-1}\) in \(w\) that was not preceded by \(a\) in \(w\).

The argument involves a straightforward (but tedious) analysis of several cases for the possible configurations at the beginning and the end of \(u\). We present the details for completeness. In various summations below we will implicitly assume that \(x\) varies over the set \(\{a, b, a_3, \ldots, a_k\}\). We will also implicitly assume that the summation occurs only over those words satisfying the explicitly specified conditions that are freely reduced.

**Case 1.** Let \(u = a^s\) where \(s > 0\).

The number of those occurrences of \(a^s\) in \(\tau(w)\) that start with an old occurrence of \(a\) is equal to \(\sum_{x \neq b^{-1}} n_w(a^sx)\). The number of those occurrences of \(a^s\) in \(\tau(w)\) that start with a new occurrence of \(a\) is equal to \(\sum_{x \neq b^{-1}} n_w(ba^{s-1}x)\).

Thus

\[
 n_{\tau(w)}(u) = \sum_{x \neq b^{-1}} n_w(a^sx) + \sum_{x \neq b^{-1}} n_w(ba^{s-1}x).
\]

**Case 2.** Let \(u = a^{-s}\) where \(s > 0\).

The number of those occurrences of \(a^{-s}\) in \(\tau(w)\) that start with an old occurrence of \(a^{-1}\) is equal to \(\sum_{x \neq b} n_w(xa^{-s})\). The number of those occurrences of \(a^{-s}\) in \(\tau(w)\) that start with a new occurrence of \(a^{-1}\) is equal to \(\sum_{x \neq a} n_w(xb^{-1})\) if \(s = 1\).

Thus for \(s = 1\)

\[
 n_{\tau(w)}(a^{-1}) = \sum_{x \neq b} n_w(xa^{-1}) + \sum_{x \neq a} n_w(xb^{-1})
\]

and for \(s > 1\)

\[
 n_{\tau(w)}(a^{-s}) = \sum_{x \neq b} n_w(xa^{-s}).
\]

**Case 3.** Let \(u = a^sz^t\) where \(s \in \mathbb{Z}, t \in \mathbb{Z}\) and \(z\) is a nontrivial reduced word that neither begins nor ends with \(a^{\pm 1}\).

Let \(z'\) be obtained from the reduced form of \(\tau^{-1}(z)\) by deleting the first letter if that letter is \(a^{\pm 1}\) and by deleting the last letter if that letter is \(a^{\pm 1}\). Note that now the first letters of \(z'\) and \(z\) are the same and the last letters of \(z'\) and \(z\) are the same.

**Subcase 3.1** Suppose that \(s = t = 0\), so that \(u = z\).

It is easy to see that \(n_{\tau(w)}(u) = n_w(z')\).

**Subcase 3.2** Suppose that \(s > 0, t > 0\).

**Subcase 3.2.a** Assume that \(z\) does not end in \(b\).

If \(z\) does not start with \(b^{-1}\) then the number of occurrences of \(u\) in \(w\) that start with an old occurrence of \(a\) is equal to \(\sum_{x \neq b^{-1}} n_w(a^sz^t'a^tx)\). In this case the number of occurrences of \(u\) in \(w\) that start with a new occurrence of \(a\) is equal to \(\sum_{x \neq b^{-1}} n_w(ba^{s-1}z^t'a^tx)\). Thus if \(z\) does not start with \(b^{-1}\) then
In this case the number of occurrences of \( u \) begins with \( b^{-1} \). Then the number of occurrences of \( u \) that start with an old occurrence of \( a \) is equal to \( \sum_{x \neq b^{-1}} n_{w}(a^{s} z' a^{t} x) \). In this case the number of occurrences of \( u \) in \( w \) that start with a new occurrence of \( a \) is equal to \( \sum_{x \neq b^{-1}} n_{w}(ba^{s} z' a^{t} x) \). Thus if \( z \) starts with \( b^{-1} \) then

\[
n_{\tau(w)}(u) = \sum_{x \neq b^{-1}} n_{w}(a^{s} z' a^{t} x) + \sum_{x \neq b^{-1}} n_{w}(ba^{s} z' a^{t} x).
\]

**Subcase 3.2.B** Suppose now that \( z \) ends in \( b \).

Assume first that \( z \) does not begin with \( b^{-1} \).

Then the number of occurrences of \( u \) in \( w \) that start with an old occurrence of \( a \) is equal to \( \sum_{x \neq b^{-1}} n_{w}(a^{s} z' a^{t} x) \). In this case the number of occurrences of \( u \) in \( w \) that start with a new occurrence of \( a \) is equal to \( \sum_{x \neq b^{-1}} n_{w}(ba^{s} z' a^{t} x) \). Thus if \( z \) does not start with \( b^{-1} \) then

\[
n_{\tau(w)}(u) = \sum_{x \neq b^{-1}} n_{w}(a^{s} z' a^{t} x) + \sum_{x \neq b^{-1}} n_{w}(ba^{s} z' a^{t} x).
\]

Assume now that \( z \) does begin with \( b^{-1} \). Then the number of those occurrences of \( u \) in \( w \) that start with an old occurrence of \( a \) is equal to \( \sum_{x \neq b^{-1}} n_{w}(a^{s} z' a^{t} x) \). In this case the number of occurrences of \( u \) in \( w \) that start with a new occurrence of \( a \) is equal to \( \sum_{x \neq b^{-1}} n_{w}(ba^{s} z' a^{t} x) \). Thus if \( z \) starts with \( b^{-1} \) then

\[
n_{\tau(w)}(u) = \sum_{x \neq b^{-1}} n_{w}(a^{s} z' a^{t} x) + \sum_{x \neq b^{-1}} n_{w}(ba^{s} z' a^{t} x).
\]

**Subcase 3.3.** Suppose that \( s > 0 \) and \( t = 0 \) so that \( u = a^{s} z \).

Then, as in Subcase 3.2, if \( z \) does not begin with \( b^{-1} \) then

\[
n_{\tau(w)}(u) = n_{w}(a^{s} z') + n_{w}(ba^{s} z')
\]

Similarly, if \( z \) begins with \( b^{-1} \) then arguing as in Subcase 3.2 we get

\[
n_{\tau(w)}(u) = n_{w}(a^{s+1} z') + n_{w}(ba^{s} z').
\]

**Subcase 3.4.** Suppose that \( s = 0 \) and \( t > 0 \) so that \( u = za^{t} \).

It is easy to see that if \( z \) does not end in \( b \) then

\[
n_{\tau(w)}(u) = \sum_{x \neq b^{-1}} n_{w}(z' a^{t} x).
\]

Similarly, if \( z \) ends in \( b \) then

\[
n_{\tau(w)}(u) = \sum_{x \neq b^{-1}} n_{w}(z' a^{t-1} x).
\]

**Subcase 3.5.** We will consider in detail one more case and leave the remaining possibilities to the reader.

Suppose that \( u = a^{-t} z a^{-t} \) where \( s, t > 0 \) and \( z \) is a nontrivial reduced word that neither begins nor ends with \( a^\pm 1 \).

**Subcase 3.5.A** Suppose that \( z \) does not end in \( b \).
If $z$ does not start with $b^{-1}$ then
\[ n_{\tau(w)}(u) = \sum_{x \neq b} n_w(xa^{-s}z'a^{-t}). \]

If $z$ starts with $b^{-1}$ and $s > 1$ then
\[ n_{\tau(w)}(u) = \sum_{x \neq b} n_w(xa^{-s+1}z'a^{-t}). \]

If $z$ starts with $b^{-1}$ and $s = 1$, so that $u = a^{-1}za^{-t}$, then
\[ n_{\tau(w)}(u) = \sum_{x \neq a} n_w(xz'a^{-t}). \]

**Subcase 3.5.B** Suppose that $z$ ends in $b$.
If $z$ does not start with $b^{-1}$ then
\[ n_{\tau(w)}(u) = \sum_{x \neq b} n_w(xa^{-s}z'a^{-t-1}). \]

If $z$ starts with $b^{-1}$ and $s > 1$ then
\[ n_{\tau(w)}(u) = \sum_{x \neq b} n_w(xa^{-s+1}z'a^{-t-1}). \]

If $z$ starts with $b^{-1}$ and $s = 1$ then
\[ n_{\tau(w)}(u) = \sum_{x \neq a} n_w(xz'a^{-t-1}). \]

\[ \square \]

**Proposition 5.3.** Let $\phi \in Out(F)$ be an outer automorphism and let $t$ be such that $\phi$ can be represented, modulo $\text{Inn}(F)$, as a product of $t$ Nielsen automorphisms.

Then for any freely reduced word $v \in F$ with $|v| = m$ there exists a collection of integers $c(u, v) = c(u, v, \phi) \geq 0$, where $u \in F$, $|u| = 8^t m$, such that for any nontrivial cyclic word $w$ we have
\[ n_{\phi(w)}(v) = \sum_{|u|=8^t m} c(u, v)n_w(u). \]

**Proof.** Let $\psi = \tau_1 \ldots \tau_t \in Out(F)$ where each $\tau_i$ is a Nielsen automorphism. We prove Proposition 5.3 by induction on $t$.

For $t = 1$ Proposition 5.3 follows from Proposition 5.2 since $2m + 6 \leq 8m$ for $m \geq 1$. Suppose now that $t > 1$ and that Proposition 5.3 has been established for $t - 1$.

Put $\psi = \tau_2 \ldots \tau_t$.

By assumption there is a collection of numbers $b(v, z) \geq 0$, where $|v| = 8^{t-1}(2m+6), |z| = 2m + 6$, such that for any cyclic word $w$ we have
\[ n_{\psi(w)}(z) = \sum_{|v|=8^{t-1}(2m+6)} b(v, z)n_w(v). \]

Since $\phi(w) = \tau_1(\psi(w))$, Proposition 5.2 implies that
Therefore and Proof. By Proposition 5.3 we have

\[ n_{\phi(w)}(u) = \sum_{|z|=2m+6} c(z, u, m)n_{\psi(w)}(z) = \]

\[ = \sum_{|z|=2m+6} c(z, u, m) \sum_{|v|=8^{t-1}(2m+6)} b(v, z)n_w(v) = \]

\[ = \sum_{|v|=8^{t-1}(2m+6)} \left[ \sum_{|z|=2m+6} b(v, z)c(z, u, m) \right]n_w(v). \]

Since \(8^{t-1}(2m + 6) \leq 8^t m\), in view of part (1) of Lemma 2.2, the statement of Proposition 5.3 follows.

\[
\]

Proposition 5.4. Let \(\phi \in Out(F)\) be an outer automorphism and let \(t\) be such that \(\phi\) can be represented, modulo \(\text{Inn}(F)\), as a product of \(t\) Nielsen automorphisms.

Then there exists a collection of integers \(d(u, \phi) \geq 0\), where \(u \in F\), \(|u| = 8^t\), such that for any nontrivial cyclic word \(w\) we have

\[ f_{\phi(w)}(v) = \frac{\sum_{|u|=8^t} c(u, v)f_w(u)}{\sum_{|u|=8^t} d(u, \phi)f_w(u)} \]

and

\[ ||\phi(w)|| = \sum_{|u|=8^t} d(u, \phi)n_w(u). \]

Here \(c(u, v) = c(u, v, \phi)\) are as in Proposition 5.3.

Moreover, there is \(c_0 > 0\) such that for any \(q \in Q_{8^t}\)

\[ \sum_{|u|=8^t} d(u, \phi)q_u \geq c_0 > 0. \]

Proof. By Proposition 5.3 we have

\[ n_{\phi(w)}(v) = \sum_{|u|=8^t} c(u, v)n_w(u) \]

and

\[ ||\phi(w)|| = \sum_{|z|=1} n_{\phi(w)}(z) = \sum_{|z|=1} \sum_{|u|=8^t} c(u, z)n_w(u). \]

Therefore

\[ f_{\phi(w)}(v) = \frac{n_{\phi(w)}(v)}{||\phi(w)||} = \frac{\sum_{|u|=8^t} c(u, v)n_w(u)}{\sum_{|z|=1} \sum_{|u|=8^t} c(u, z)n_w(u)} = \frac{\sum_{|u|=8^t} c(u, v)f_w(u)}{\sum_{|z|=1} \sum_{|u|=8^t} c(u, z)f_w(u)}, \]

and the first part of the statement follows with \(d(u) = \sum_{|z|=1} c(u, z)\).

Note that by construction

\[ \frac{||\phi(w)||}{||w||} = \sum_{|u|=8^t} \left( \sum_{|z|=1} c(u, z) \right)f_w(u) = \sum_{|u|=8^t} d(u)f_w(u), \]
for any \( w \in C \). Choose an automorphism \( \psi \in \text{Aut}(F) \) representing \( \phi \in \text{Out}(F) \) (so that \( \psi^{-1} \) represents \( \phi^{-1} \)).

Obviously \( \frac{||\phi(w)||}{||w||} \leq L(\psi) \) where \( L(\psi) = \max\{|\psi(x)| : x \in X\} \) and \( 1 \leq L(\psi) < \infty \). Therefore \( \frac{||\phi(w)||}{||w||} \geq \frac{1}{L(\psi^{-1})} \) for any \( w \in C \). Hence the function \( h(q) = \sum_{|u|=8} d(u)q_u \) on \( Qst \) satisfies \( h(q) \geq \frac{1}{L(\psi^{-1})} \) for each \( q \in \alpha_{st}(C) \). Since \( \alpha_{st}(C) \) is dense in \( Qst \), this implies that \( h(q) \geq \frac{1}{L(\psi^{-1})} > 0 \) for every \( q \in Qst \), as claimed.

\[ \square \]

**Convention 5.5.** For each \( \phi \in \text{Out}(F) \) we choose and fix a shortest representation of \( \phi \) modulo \( \text{Inn}F \) as a product Nielsen automorphisms. Denote by \( t = t(\phi) \) the number of automorphisms in this product. We also fix the numbers \( \phi \) and \( \psi \) for any \( \phi \in \text{Out}(F) \).

**Theorem 5.6.** For \( \phi \in \text{Out}(F) \) and \( q \in \alpha(C) \subseteq Q(F) \) put \( \tilde{\phi}(q) := \phi(w)^{m} \), where \( w \in C \) is any cyclic word with \( w^{m} = q \).

Then \( \tilde{\phi}(q) \) is well-defined and the map \( \tilde{\phi} : \alpha(C) \to \alpha(C) \) is continuous in the induced from \( Q(F) \) topology. This map extends uniquely to a continuous homeomorphism \( \hat{\phi} : Q(F) \to Q(F) \). The map

\[ \hat{\phi} : Q(F) \to \text{Homeo}(Q(F)), \quad \phi \mapsto \hat{\phi}, \text{ for each } \phi \in \text{Out}(F), \]

defines a left action of \( \text{Out}(F) \) on \( Q(F) \) by homeomorphisms.

**Proof.** By Proposition 5.5 we know that if \( q = w_{1}^{m} = w_{2}^{m} \) then \( w_{1} \), \( w_{2} \) are positive powers of the same cyclic word and therefore the same is true for \( \phi(w_{1}) \) and \( \phi(w_{2}) \). Hence \( \phi(w_{1})^{m} = \phi(w_{2})^{m} \) and so \( \tilde{\phi} : \alpha(C) \to \alpha(C) \) is well-defined.

To see that this map is continuous and extends uniquely to a continuous map \( Q(F) \to Q(F) \), choose an arbitrary integer \( m > 0 \). Then for any \( q = w^{m} \) and any \( v \in F \) with \( |v| = m \) we have

\[ \tilde{\phi}(q)_{v} = f_{\phi(w)}(v) = \frac{\sum_{|u|=8} c(u,v,\phi)f_{w}(u)}{\sum_{|u|=8} d(u,\phi)f_{w}(u)} = \frac{\sum_{|u|=8} c(u,v,\phi)f_{w}(u)}{\sum_{|u|=8} d(u,\phi)f_{w}(u)} \]

Fix \( v \in F \) with \( |v| = m \). Then by Proposition 5.3 the fractional-linear function

\[ r_v(q) := \frac{\sum_{|u|=8} c(u,v,\phi)f_{w}(u)}{\sum_{|u|=8} d(u,\phi)f_{w}(u)} \]

has denominator satisfying

\[ \sum_{|u|=8} d(u,\phi)f_{w}(u) \geq c_{0} > 0 \]

on \( Q(F) \) for some constant \( c_{0} > 0 \) independent of \( q \in Q(F) \). Hence \( r_v(q) \) is continuous on \( Q(F) \).

Since \( \alpha(C) \) is dense in \( Q(F) \), this implies that the above map \( \tilde{\phi} : \alpha(C) \to \alpha(C) \) extends uniquely to a continuous map \( \phi : Q(F) \to Q(F) \). The map \( \tilde{\phi} : \phi \mapsto \hat{\phi} \) defines an action of \( \text{Out}(F) \) on \( Q(F) \) by homeomorphisms. Indeed, since \( \phi(\psi(w)) = \phi(\psi(w)) \) for any \( w \in C \) and \( \phi, \psi \in \text{Out}(F) \), we conclude that \( \hat{\phi} \circ \hat{\phi} = (\phi \circ \psi) \) on \( \alpha(C) \) and therefore, by continuity, on \( Q(F) \). Also, for the trivial automorphism \( 1 \in \text{Out}(F) \) the action of \( 1 \) on \( C \) is trivial, and hence \( \hat{1}_{\alpha(C)} = Id_{\alpha(C)} \). Therefore,
by continuity $\tilde{1}_{Q(F)} = Id_{Q(F)}$. Thus $\tilde{\phi} : \to \bar{\phi}$, $\phi \in Out(F)$ indeed is an action of $Out(F)$ on $Q(F)$ by homeomorphisms. \hfill \square

**Remark 5.7.** As we mentioned in the introduction, the formulas defining the action of $Out(F)$ on $Q(F)$ are, in a sense linear. Indeed, for a fixed $q \in Q(F)$ all the coordinate functions $r_v(q) = \phi(q)_v$, (where $v \in F$) are fractional-linear functions with the same denominators (independent of $v$).

### 6. Applications to the geometry of automorphisms

In this section we will prove Theorem A and Theorem B stated in the introduction.

**Definition 6.1.** We say that an outer automorphism $\phi \in Out(F)$ is **strictly hyperbolic** if there is $\lambda > 1$ such that
$$\lambda \|w\| \leq \max\{||\phi(w)||, ||\phi^{-1}||\}$$
for every cyclic word $w$.

An outer automorphism $\phi \in Out(F)$ is **hyperbolic** if there is $n > 0$ such that $\phi^n$ is strictly hyperbolic.

For an outer automorphism $\phi \in Out(F)$ set
$$\lambda_0(\phi) := \inf_{w \neq 1} \max\{\frac{||\phi(w)||}{||w||}, \frac{||\phi^{-1}||}{||w||}\}.$$  

**Theorem 6.2.** The following hold:

1. An an outer automorphism $\phi \in Out(F)$ is strictly hyperbolic if and only if $\lambda_0(\phi) > 1$.
2. For any $\phi \in Out(F)$ the number $\lambda_0(\phi)$ is rational.
3. There is a double exponential time (in terms of the number $t$ of Nielsen automorphisms in the expression of $\phi$) algorithm that computes $\lambda_0(\phi)$ and decides if an outer automorphism $\phi \in Out(F)$ is strictly hyperbolic.

**Proof.** Let $\phi$ be an outer automorphism given as a product (modulo $InnF$) of $t$ Nielsen automorphisms. For $u \in F$ with $|u| = 8^t$ let $d(u, \phi)$ and $d(u, \phi^{-1})$ be the integer constants provided by Proposition 5.4.

Then for any cyclic word $w$ by Proposition 5.4 we have
$$\frac{||\phi(w)||}{||w||} = \sum_{|u| = 8^t} d(u, \phi)f_w(u)$$
and
$$\frac{||\phi^{-1}(w)||}{||w||} = \sum_{|u| = 8^t} d(u, \phi^{-1})f_w(u).$$

Define the functions $g, h : Q_{8^t} \to \mathbb{R}$ by formulas
$$g(p) := \sum_{|u| = 8^t} d(u, \phi)p_u \quad \text{and} \quad h(p) := \sum_{|u| = 8^t} d(u, \phi^{-1})p_u.$$  

Thus both $h$ and $g$ are affine functions with integer coefficients on $Q_{8^t}$. Also, put $e := \max\{g, h\}$. By definition $\phi$ is strictly hyperbolic if and only if $\inf_{p \in C}(C) e(p) > 1$. Since $e$ is continuous and $\alpha_{8^t}(C)$ is dense in a compact space $Q_{8^t}$, this condition is equivalent to:
$$(\forall) \quad z := \min_{p \in Q_{8^t}} e(p) > 1.$$
Moreover, again by density of $\alpha_8(t)$ in $Q_8t$, if $z > 1$ then $z = \lambda_0(\phi)$. Since $h, g$ are affine functions with integer coefficients on a polyhedron $Q_8t$ given by affine equations and inequalities with integer coefficients, this implies that $z$ is a rational number. Thus we have established statements (1) and (2) of the Theorem.

The number of $v \in F$ with $|u| = 8t$ is $D(8^t) = 2k(2k - 1)^{8^t}$, which is double-exponential in $t$. Since a linear programming problem is solvable in polynomial time, both in the number of variables and the size of the entries (see, for example, [16]), the value of $z$ can be computed algorithmically is double exponential time in $t$ and part (3) of the theorem follows.

Note that Theorem 6.2 provides the following new algorithm for deciding if $\phi \in \text{Out}(F)$ is hyperbolic. By Brinkmann’s result [10] $\phi$ is hyperbolic if and only if it does not have nontrivial periodic conjugacy classes. Therefore we will run the following two procedures in parallel. For $n = 1, 2, \ldots$ we start checking, using Theorem 6.2 if $\phi^n$ is strictly hyperbolic. At the same time for each such $n$ we will start enumerating elements of $F$ and checking if $\phi^n(f)$ is ever conjugate to $f$.

Eventually we will find $n$ such that either $\phi^n$ is strictly hyperbolic or $\phi^n$ fixes a nontrivial conjugacy class.

The only other algorithm for checking hyperbolicity of $\phi$ known to the author proceeds as follows. Form the group $G = F \rtimes_\phi \mathbb{Z}$ and in parallel start checking if $G$ is word-hyperbolic (using, say, Papasoglu’s algorithm [23]) and if $\phi$ has a periodic conjugacy class. By Brinkmann’s theorem one of these procedures will eventually terminate, and $\phi$ is hyperbolic if and only if $G$ is word-hyperbolic.

For an outer automorphism $\phi \in \text{Out}(F)$ set

$$\nu_+(\phi) := \sup_{w \neq 1} \frac{||\phi(w)||}{||w||}$$

and

$$\nu_-(\phi) := \inf_{w \neq 1} \frac{||\phi(w)||}{||w||}.$$ 

Note that by definition $\nu_+(\phi^{-1}) = \nu_-(\phi)$.

The following is a restatement of Theorem A.

**Theorem 6.3.** Let $\phi \in \text{Out}(F)$. Then:

1. The numbers $\nu_+(\phi), \nu_-(\phi)$ are rational and algorithmically computable in terms of $\phi$.
2. For any rational number $\nu_-(\phi) < r < \nu_+(\phi)$ there exists a nontrivial cyclic word $w$ with

$$\frac{||\phi(w)||}{||w||} = r.$$ 

3. We have $\nu_+(\phi), \nu_-(\phi) \in I(\phi)$.

**Proof.** Let $\phi$ be an outer automorphism given as a product (modulo $\text{Inn}F$) of $t$ Nielsen automorphisms. Let for $x \in X$ and $u \in F$ with $|u| = 8^t$ let $d(u, \phi)$ be the constants provided by Proposition 5.

Then as in the proof of Theorem 6.2, for any cyclic word $w$ we have

$$\frac{||\phi(w)||}{||w||} = \sum_{|u|=8^t} d(u, \phi)f_w(u)$$

As before, define the function $g : Q_{8^t} \to \mathbb{R}$ as:

$$g(p) := \sum_{|u|=8^t} d(u, \phi)p_u$$
Then
\[ \nu_+(\phi) = \sup_{p \in \alpha_{8t}(C)} g(p) = \max_{p \in Q_{8t}} g(p), \]
where the last equality holds since \( \alpha_{8t}(C) \) is dense in \( Q_{8t} \).

Similarly, then
\[ \nu_-(\phi) = \inf_{p \in \alpha_{8t}(C)} g(p) = \min_{p \in Q_{8t}} g(p). \]

Recall that \( Q_{8t} \) is a compact convex polyhedron given by equations and inequalities with rational coefficients and that \( g(p) \) is a linear function with integer coefficients. Hence the maximum of \( g(p) \) over \( Q_{8t} \) is a rational number that is algorithmically computable. The same applies to the minimum of \( g \) over \( Q_{8t} \). This proves assertion (1) of the theorem.

To see that (2) holds, suppose now that \( r \) is a rational number satisfying \( \nu_-(\phi) < r < \nu_+(\phi) \). Let \( p_+, p_- \in Q_{8t} \) be such that \( g(p_+) = \nu_+(\phi) \) and \( g(p_-) = \nu_-(\phi) \).

Since \( Q_{8t} \) is dense in \( Q_{8t} \), there exist \( p_n, q_n \in Q_{8t} \), such that \( p_n \rightarrow p_+ \) and \( q_n \rightarrow p_- \) as \( n \rightarrow \infty \).

Hence \( \lim_{n \rightarrow \infty} g(p_n) = \nu_+(\phi) \) and \( \lim_{n \rightarrow \infty} g(q_n) = \nu_-(\phi) \). Therefore for some \( n \) we have
\[ g(q_n) < r < g(p_n). \]

Choose a rational number \( s \in (0,1) \) so that \( r = sg(q_n) + (1-s)g(p_n) \). Recall that both \( p_n, q_n \) are vectors in \( \mathbb{R}^{D(8t)} \) with all their coordinates being positive rational numbers. Therefore the same is true for their convex combination \( y_n := sq_n + (1-s)p_n \). Since \( Q_{8t} \) is convex, \( y_n \in Q_{8t} \) and hence \( y_n \in Q_{8t}^+ \). By linearity of \( g \) we have \( g(y_n) = r \). Since by Theorem 4.4, \( Q_{8t}^+ \) is contained in \( \alpha_{8t}(C) \), there is a cyclic word \( w \) with \( \alpha_{8t}(w) = y_n \). Then
\[ g(y_n) = \frac{||\phi(w)||}{||w||} = r, \]
as required.

Note that \( g : Q_{8t} \rightarrow \mathbb{R} \) is a linear function on a compact convex finite-dimensional polyhedron \( Q_{8t} \). Therefore there exists an extremal point \( z \) of \( Q_{8t} \) such that \( g(z) = \max_{Q_{8t}} g = \nu_+(\phi) \). By Lemma 4.11, the improved intial graph \( \Gamma_z^\prime \) is connected and hence by Theorem 4.3, there exists \( w \in C \) such that \( z = \alpha_{8t}(w) \). Then
\[ \frac{||\phi(w)||}{||w||} = g(z) = \nu_+(\phi), \]
so that \( \nu_+(\phi) \in I(\phi) \).

The same argument, with replacing max by min, shows that \( \nu_-(\phi) \in I(\phi) \), which completes the proof of the theorem. \( \Box \)

**Remark 6.4.** One can provide a slightly more uniform argument for Algorithmic computability of \( \nu_{\pm}(\phi) \) than that given in the above proof.

In the notation of that proof, a linear function \( g \) on the compact convex polyhedron \( Q_{8t} \) attains its maximal and minimal values at some extremal points of \( Q_{8t} \). We have seen in Lemma 4.11 that extremal points of \( Q_{8t} \) have connected improved initial graphs and are thus realizable by some cyclic words in \( F \). First, we can find and enumerate all the extremal points of \( Q_{8t} \). For each extremal point \( p \), using the Euler circuit algorithm from the proof of Theorem 4.3, we can find a cyclic word \( w_p \) realizing that point. We then compute \( ||\phi(w_p)||/||w_p|| \). The maximal of these
values as $p$ varies over the set of extremal points of $Q_{s,t}$ is $\nu_+(\phi)$ and the minimal is $\nu_-(\phi)$.

The advantage of this approach is that we do not have to compute an explicit formula for the linear function $g$ at all and the same set of words $w_p$ will work for arbitrary outer automorphisms that are products of at most $t$ Nielsen moves.

7. Geodesic Currents

We provide here a very brief and informal discussion about geodesic currents on free groups. The details can be found in the dissertation of Martin [21]. We also intend to elaborate this discussion in the future. An excellent source on geodesic currents in the context of Gromov-hyperbolic spaces is provided by a paper of Furman [12].

Let $F$ be a finitely generated free group. In this section we will think of $\partial F$ as the topological boundary of $F$ in the sense of the theory of Gromov-hyperbolic groups. Thus as a set $\partial F$ consists of all equivalence classes of sequences $(g_n)_{n \geq 1}$ of elements $g_n \in F$ that are convergent at infinity. A sequence $(g_n)$ is convergent at infinity if for some (and hence for any) free basis $X$ of $F$ the distance from 1 to the geodesic segment from $g_n$ to $g_m$ in the Cayley graph of $F$ with respect to $X$ tends to infinity as $m,n \to \infty$. Two sequences $(g_n)_{n \geq 1}$ and $(h_n)_{n \geq 1}$ are convergent at infinity if the sequence $g_1, h_1, g_2, h_2, \ldots, g_n, h_n, \ldots$ is also convergent at infinity. For each free basis $X$ of $F$ there is a canonical identification of $\partial F$ with the boundary of $F$ in the sense of the previous sections, that is with the set of geodesic rays starting at 1 in the Cayley graph of $F$ with respect to $X$. Every such identification endows $\partial F$ with a topology, that turns out to be independent of the choice of $X$.

**Definition 7.1** (Geodesic Currents). Let $F$ be a finitely generated free group. Denote

$$\partial^2 F := \{(s, t) : s, t \in \partial F, s \neq t\},$$

the set of ordered pairs of distinct elements of $\partial F$.

A **geodesic current** on $F$ is an $F$-invariant positive Borel measure on $\partial^2 F$.

Two currents $\nu_1, \nu_2$ are **equivalent**, denoted $\nu_1 \sim \nu_2$, if there is $c > 0$ such that $\nu_1 = c\nu_2$.

Put

$$\text{Curr}(F) := \{\nu : \nu \text{ is a nonzero geodesic current on } F\}/\sim$$

We will denote the $\sim$-equivalence class of a geodesic current $\nu$ by $[\nu]$. Note that it is often customary to require geodesic currents to be not just $F$-invariant, but also invariant under the flip-map $\partial^2 F \to \partial^2 F$, $(s, t) \to (t, s)$. However, the two notions are very close and most arguments work in both contexts.

The set of all geodesic currents $\nu$ on $F$ comes equipped with the natural weak*-topology. Hence $\text{Curr}(F)$ inherits the quotient topology under the $\sim$-quotient map.

Let $X$ be a free basis of $F$. Then $X$ defines a generating set of the Borel $\sigma$-algebra of $\partial^2 F$ as follows. Let $\Gamma(F, X)$ be the Cayley graph of $F$ with respect to $X$. For any two distinct points $s, t \in F \cup \partial F$ there exists a unique directed geodesic line $[s, t]_A \subseteq \Gamma(F, X)$ from $s$ to $t$. Thus we think of $[s, t]_X$ as a union of closed edges in $\Gamma(F, X)$ together with a choice of direction (from $s$ to $t$). We will say that $[s, t]_X \subseteq [s', t']_X$ if the underlying set of $[s, t]_X$ is contained in that of $[s', t']_X$ and the directions on $[s, t]_X$ and $[s', t']_X$ agree.
For any \( u_1, u_2 \in F, u_1 \neq u_2 \) put

\[
\text{Cyl}_F(u_1, u_2) := \{(s, t) \in \partial^2 F : [u_1, u_2]_X \subseteq [s, t]_X\}
\]

It is easy to see that \( B_X := \bigcup_{(u, u') \in F^2, u \neq u'} \text{Cyl}_F(u, u') \) is a generating set for the Borel \( \sigma \)-algebra of \( \partial^2 F \).

The following lemma shows that the frequency space \( Q_A(F) \) is essentially the same object as the space of currents \( \text{Curr}(F) \).

\textbf{Lemma 7.2.} Let \( A \) be a free basis of \( F \). For each \( \mu \in Q_X(F) \) define a set function \( \beta_X(\mu) : B_X \to \mathbb{R} \) as \( \beta_X(\mu)(\text{Cyl}_F(u_1, u_2)) := \mu(\text{Cyl}_F(u_1^{-1}u_2)) \).

Then for every \( \mu \in Q_X(F) \) the function \( \beta_X(\mu) \) defines a geodesic current on \( F \) and the map \( \hat{\beta}_X : Q_X(F) \to \text{Curr}(F), \mu \mapsto [\beta_X(\mu)] \) is a homeomorphism.

\textit{Proof.} The proof is a straightforward corollary of the definitions and we will omit most of the details.

We will indicate why the map \( \hat{\beta}_X \) is onto. Let \( \nu \) be a geodesic current. After multiplying \( \nu \) by an appropriate constant (which preserves the \( * \)-equivalence class of \( \nu \)), we may assume that

\[
\sum_{x \in X \cup X^{-1}} \nu(\text{Cyl}_F(1, x)) = 1.
\]

Since \( \nu \) is finitely additive, a simple inductive argument now shows that for any \( m \geq 1 \)

\[
\sum_{u \in F, |u|_X = m} \nu(\text{Cyl}_F(1, u)) = 1.
\]

For each \( u \in F, u \neq 1 \) define \( \mu(\text{Cyl}_F(u)) := \nu(\text{Cyl}_F(1, u)) \). We claim that \( \mu \in Q_X(F) \). Indeed, for any \( u \in F, u \neq 1 \) we have

\[
\text{Cyl}_F(1, u) = \bigcup_{|x| = 1, |ux| = |u| + 1} \text{Cyl}_F(1, ux).
\]

Since \( \nu \) is a measure, we have

\[
\mu(\text{Cyl}_F(u)) = \nu(\text{Cyl}_F(1, u)) = \sum_{|x| = 1, |ux| = |u| + 1} \nu(\text{Cyl}_F(1, ux)) = \sum_{|x| = 1, |ux| = |u| + 1} \mu(\text{Cyl}_F(ux)).
\]

This shows that \( \mu \) is a Borel probability measure on \( \partial F \).

Also, for any \( u \in F, u \neq 1 \) we have

\[
\text{Cyl}_F(1, u) = \bigcup_{|y| = 1, |yu| = |u| + 1} \text{Cyl}_F(y^{-1}, u).
\]

Since \( \nu \) is \( F \)-invariant, we have \( \nu(\text{Cyl}_F(y^{-1}, u)) = \nu(\text{Cyl}_F(1, yu)) \). Hence we have

\[
\mu(\text{Cyl}_F(u)) = \sum_{|y| = 1, |yu| = |u| + 1} \mu(\text{Cyl}_F(yu)).
\]

Hence \( \mu \) is shift-invariant (with respect to the shift \( T_X \) corresponding to \( X \)) and hence \( \mu \in Q_X(F) \). By construction we have \( \beta_X(\mu) = \nu \), so that \( \hat{\beta}_X \) is onto, as required. \( \square \)

We can define an action of \( \text{Aut}(F) \) on \( \text{Curr}(F) \) as follows. Let \( \nu \) be a nonzero geodesic current on \( F \) and let \( \phi \in \text{Aut}(F) \) be an automorphism. Thus \( \phi \) extends to a homeomorphism of \( \partial F \) and so to a homeomorphism of \( \partial^2 F \). Define a positive
Borel measure $\phi \nu$ on $\partial^2 F$ by $(\phi \nu)(B) := \nu(\phi^{-1} B)$ for an arbitrary Borel $B \subseteq \partial F$. Obviously, $\phi \nu$ is a positive Borel measure on $\partial^2 F$. It can also be verified directly, via a Gromov-hyperbolic argument, that $\phi \nu$ is $F$-invariant, and hence is a geodesic current.

The measure $\phi \nu$ can be understood “coordinatewise” as follows. Let $X$ be a free basis of $F$, so that $X' = \phi(X)$ is also a free basis. Then $\phi \nu$ has the same $B_X'$-coordinates as the $B_X$-coordinates of $\nu$. That is, for any $u_1, u_2 \in F, u_1 \neq u_2$ we have $(\phi \nu)(\text{Cyl}_X(\phi(u_1), \phi(u_2))) = \nu(\text{Cyl}_X(u_1, u_2))$. This defines a continuous action of $\text{Aut}(F)$ on $\text{Curr}(F)$. Moreover, if $\alpha$ is an inner automorphism of $F$ corresponding to conjugation by $h$, $g \mapsto hgh^{-1}$, then $\alpha(t) = ht$ for any $t \in \partial F$ and hence $\alpha B = hB$ for any $B \subseteq \partial^2 F$. This easily implies that $\phi \nu$ depends only on the outer automorphism class of $\phi$ in $\text{Out}(F) = \text{Aut}(F)/\text{Inn}(F)$ and that the action of $\text{Aut}(F)$ on $\text{Curr}(F)$ factors through to an action of $\text{Out}(F)$ on $\text{Curr}(F)$.

If we fix a free basis $X$ of $F$, then one can show that this action of $\text{Out}(F)$ on $\text{Curr}(F)$ is exactly the same (via the $\beta_X$-identification of $\text{Curr}(F)$ and $Q_X(F)$) as the action of $\text{Out}(F)$ on $Q_X(F)$ constructed earlier. One way to see this is to check that the actions coincide on the dense set of measures coming from the conjugacy classes in $F$, that is $\alpha_X(C) \subseteq Q_X(F)$, as well as the corresponding dense set of geodesic currents $\beta_X \alpha_X(C) \subseteq \text{Curr}(F)$.

There is also a natural topological embedding $j : \mathcal{X}(F) \to \text{Curr}(F)$ of the Culler-Vogtmann [11] outer space $\mathcal{X}(F)$ into the space of currents $\text{Curr}(F)$ (and hence into $Q_X(F)$ as well). A point $\omega$ of an outer space can be represented by a minimal free discrete isometric action of $F$ on an $\mathbb{R}$-tree $Y$ (or the “marked length spectrum” on $F$ defined by such an action). This tree $Y$ can be thought of as the universal cover of a finite metric graph $K$. The tree $Y$ is a Gromov-hyperbolic metric space and as such $\partial Y$ comes equipped with the so-called Patterson-Sullivan measure. This measure, via an explicit formula involving the Busemann function, defines an $F$-invariant measure on $\partial^2 Y$ and hence, via the orbit map $F \to Y$, on $\partial F$. The $\sim$-equivalence class of that measure is precisely $j(\omega)$. We refer the reader to an article by Alex Furman [12] for a detailed discussion about Patterson-Sullivan measures in the context of Gromov-hyperbolic spaces and the way in which they determine geodesic currents.

Reiner Martin [21] produced a different family of $\text{Out}(F_n)$-equivariant embeddings of the outer space into the space of currents, which, since $\text{Curr}(F)$ is compact, provide ways of equivariantly compactifying the outer space. Martin [21] proved, that these compactifications are different from the standard one (given by weak limits of length functions).

Yet another way of embedding the outer space into $\text{Curr}(F)$ comes from the so-called “visibility currents” defined by weighted non-backtracking random walks on the tree $Y = \tilde{K}$ associated with a point of the outer space in the above notations.

As demonstrated by the work of Lyons [20] for simplicial graphs, one should expect the Patterson-Sullivan embedding, the ”visibility current embedding” and Reiner Martin’s embeddings of the outer space in the space of currents to be rather different. The precise relationship between these embeddings and the properties of the compactifications of the outer space that they provide remain to be explored.
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Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 West Green Street, Urbana, IL 61801, USA

http://www.math.uiuc.edu/~kapovich/

E-mail address: kapovich@math.uiuc.edu