SPECTRAL BOUNDS FOR PERCOLATION ON DIRECTED AND UNDIRECTED GRAPHS

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Summary
We suggest several algebraic bounds for percolation on directed and undirected graphs: proliferation of strongly-connected clusters, proliferation of in- and out-clusters, and the transition associated with the number of giant components.

Introduction
Percolation on random graphs has been successfully used in network theory as a way to understand connectivity of real-life networks\cite{1, 3, 13}. On a tree graph, mean field theory is useful to determine the location of a percolation transition \cite{12}. However the existence of cycles reduce the applicability of this method on general graphs. We seek rigorous mean field bounds on the percolation transition which would be applicable to any graph.

Spectra of associated matrices have been linked to percolation and epidemic thresholds for a variety of graph models \cite{10, 11, 4, 2, 9}. Recently, we constructed a lower bound on the percolation transition for an infinite quasi-transitive graph $G_0$, a graph-theoretical analog of a translationally-invariant system\cite{5}. The bound,

$$p_c \geq 1/\rho(H),$$

is defined by the inverse spectral radius of the Hashimoto matrix\cite{6} $H$ which is used to enumerate non-backtracking walks on $G_0$. The formal proof involved a sequential application of cycle unwrapping maps which results in a tree graph locally equivalent to the original graph. In the case of a degree-regular graph, Eq. (1) gives the well-known bound in terms of the maximum degree. In the case of random graphs with few short cycles, the same expression (1) gives a numerically exact result for the percolation transition\cite{7}.

In this work we analyze the applicability of the expression (1) to more general graphs and in particular, finite graphs. We note that the Hashimoto matrix $H$ can be viewed as the adjacency matrix of the oriented line (di)graph (OLG) associated with the original graph\cite{8}. Thus, Eq. (1) requires a mapping of the original undirected percolation problem to a directed one. Several transitions are associated with percolation on a digraph $D$, in particular, formation of a giant strongly connected component, and formation of a giant in- or out-component\cite{11}. We show that on a general digraph, Eq. (1) is most directly associated with another transition, the formation of a strongly connected component with a large number of distinct self-avoiding cycles (SACs), and argue that this property is related to uniqueness of the percolating cluster.

Main results
Consider an order-$n$ strongly-connected digraph $D$ with vertex and edge sets $V(D), E(D)$. Associated with $D$ is an adjacency matrix $A \equiv A(D)$, and Hashimoto matrix\cite{6} $H \equiv H(D)$, with elements

$$H_{u,v} = \delta_{j'j} \delta_{i'i} u \equiv i \rightarrow j, v \equiv j' \rightarrow l,$$

where $u, v$ are directed edges in $E(D)$. The digraph $D$ may have some symmetric edges (length-two cycles), whereas the OLG has none. The spectral radius of $H$ satisfies $\rho(H) \leq \rho(A)$; the equality is reached iff $D$ has no symmetric edges. Also, for any induced $q$-norm, $\rho(H) \leq \|H\|_q$.

In site percolation on $D$, each vertex is open with probability $p$ and closed with probability $1 - p$; we consider a subgraph $D'$ of $D$ induced by the open vertices. We are interested in the likelihood of forming in-, out-, or strongly connected components of a given size $m$ on $D'$ [e.g., in the case of the out-component, there exists a vertex $i_0 \in \mathcal{V}(D')$ such that $m - 1$ or more vertices can be reached by directed walks on $D'$ starting with $i_0$]. In a strongly connected component, each site can be reached from any other by directed walks; it is automatically both an in- and out-cluster. We prove the following:

**Theorem 1** For any finite $m > 0$, the probability that a site $v$ is the root of an out-component of size $m$ or greater is bounded by $m p^{\text{out}}_m(v) \leq (1 - p \|H\|_1)^{-1}, p \|H\|_1 < 1$.

This implies that out-cluster percolation threshold in large (di)graphs satisfies $p^{\text{out}}(m) > \|H\|_1^{-1}$. A similar statement can be written for in-cluster percolation, $p^{\text{in}}(m) \geq$...
Additionally, on these digraphs, the strongly-connected probability reads

\[ \rho(H) < 1 \]  \quad \text{for such a digraph, an improved bound for out-cluster percolation threshold is strictly higher than} \quad p_c^{(\text{out})} \quad \text{or} \quad p_c^{(\text{in})} \text{; they give a counterexample for the central conjecture of Ref. [11].}

We notice that OLG of a strongly connected digraph \( D \) is also strongly connected, provided that \( D \) remains strongly connected when any edge in any pair \( u = i \to j \), \( u = j \to i \) of mutually opposite edges (if any) is removed. For such a digraph, an improved bound for out-cluster probability reads

\[ mP_m^{(\text{out})}(v) \leq \gamma_L \{ 1 - pp(H) \}^{-1} \quad (\text{cf. Theorem 1}) \]

where \( \gamma_L \) is the principal ratio, \( \gamma_L = \max_{ij} (\xi_i / \xi_j) \), for the left Perron-Frobenius vector of \( H \), \( \xi \rho(H) / \xi H \). This inequality allows to extend the bound (1) from Ref. [5] to in- and out-cluster percolation on any strongly-connected quasi-transitive infinite digraph.

The trace \( s^{-1} \text{Tr}\, H^s \) counts non-backtracking directed cycles of length \( s \) on \( D \). We constructed a corresponding bound for the total number of SACs on \( D' \), \( n_E \equiv |\mathcal{E}(D)|:

\[ N \leq \sum_{s>0} s^{-1}p^s \text{Tr}\, H^s \leq n_E \left[ \ln(1 - pp(H)) \right] \]  \quad \text{(3)}

Notice that a strongly-connected cluster with \( m \) cycles supports \( m \) distinct SACs if it has an articulation point separating any two cycles; such a cluster can be separated into two by removing a single site. A two-site-connected cluster (which can be cut into two pieces by removing two sites, but not one) will support at least \( m(m - 1)/2 \) distinct SACs. A stable giant cluster which can not be easily separated into two large pieces will necessarily have large connectivity and support up to the maximum \( O(2^m) \) SACs. For a cluster occupying a finite fraction of directed edges in \( D' \), such a possibility is excluded by Eq. (3). Therefore, we expect any large clusters formed at \( pp(H) < 1 \) to be unstable; i.e., fracture easily into two or more large clusters. Thus, we give

**Conjecture 2** For \( pp(H) < 1 \), if there is a percolating cluster, it is not unique with probability approaching one at large \( n \).

**Conclusions**

We used algebraic techniques based on non-backtracking (Hashimoto) matrix to establish several new bounds for percolation on general directed (and undirected) graphs. Our results are formulated as bounds for the probability that a given site connects to a cluster of size \( m \) or greater, and are applicable for finite and infinite (di)graphs. We also discuss the stability of giant strongly-connected clusters, which is related to the uniqueness transition.

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