FRACTIONAL OSTROWSKI-SUGENO FUZZY UNIVARIATE INEQUALITIES

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Abstract. Here we present fractional univariate Ostrowski-Sugeno Fuzzy type inequalities. These are of Ostrowski-like inequalities in the setting of Sugeno fuzzy integral and its special-particular properties. In a fractional environment, they give tight upper bounds to the deviation of a function from its Sugeno-fuzzy averages. The fractional derivatives we use are of Canavati and Caputo types. This work is greatly inspired by [8], [1] and [2].

1. Introduction. The famous Ostrowski ([8]) inequality motivates this work and has as follows:

\[ \left| \frac{1}{b-a} \int_a^b f(y) \, dy - f(x) \right| \leq \left( \frac{1}{4} + \frac{(x-a+b)^2}{(b-a)^2} \right) (b-a) \| f' \|_\infty, \]

where \( f \in C'([a,b]), \ x \in [a,b], \) and it is a sharp inequality.

Another motivation is author’s next fractional result, see [2], p. 44:

Let \([a,b] \subset \mathbb{R}, \ \alpha > 0, \ m = \lceil \alpha \rceil \) (i.e. \( f^{(m-1)} \) is absolutely continuous), and \( \| D_{x_0}^\alpha f \|_\infty,[a,x_0] \cdot \| D_{*x_0}^\alpha f \|_\infty,[x_0,b] < \infty \)
(where \( D_{x_0}^\alpha f, D_{*x_0}^\alpha f \) are the right and left Caputo fractional derivatives of \( f \) of order \( \alpha \), respectively), \( x_0 \in [a,b] \). Assume \( f^{(k)}(x_0) = 0, \ k = 1, \ldots, m-1 \). Then

\[ \left| \frac{1}{b-a} \int_a^b f(x) \, dx - f(x_0) \right| \leq \frac{1}{(b-a) \Gamma (\alpha+2)}. \]

Another great source for inspiration is [4].

Here first we give a complete survey about Sugeno fuzzy integral and its basic properties. Then we derive a series of Ostrowski-Sugeno type fractional inequalities in the univariate case to all directions in the context of Sugeno integral and its basic important particular properties. We employ right and left fractional derivatives of Canavati and Caputo types. At the end we present a reverse fractional Polya-Sugeno inequality.

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2. Background - I. In this section, some definitions and basic important properties of the Sugeno integral which will be used in the next section are presented. Also a preparation for the main results in Section 5 is given.

**Definition 2.1.** (Fuzzy measure [10, 12]) Let \( \Sigma \) be a \( \sigma \)-algebra of subsets of \( X \), and let \( \mu : \Sigma \to [0, +\infty] \) be a non-negative extended real-valued set function. We say that \( \mu \) is a fuzzy measure iff:

1. \( \mu(\emptyset) = 0 \),
2. \( E, F \in \Sigma : E \subseteq F \) imply \( \mu(E) \leq \mu(F) \) (monotonicity),
3. \( E_n \in \Sigma \ (n \in \mathbb{N}), E_1 \subset E_2 \subset \ldots \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \) (continuity from below);\( \)
4. \( E_n \in \Sigma \ (n \in \mathbb{N}), E_1 \supset E_2 \supset \ldots, \mu(E_1) < \infty \), imply \( \lim_{n \to \infty} \mu(E_n) = \mu(\bigcap_{n=1}^{\infty} E_n) \) (continuity from above).

Let \( (X, \Sigma, \mu) \) be a fuzzy measure space and \( f \) be a non-negative real-valued function on \( X \). We denote by \( \mathcal{F}_+ \) the set of all non-negative real valued measurable functions, and by \( L_\alpha f \) the set: \( L_\alpha f := \{ x \in X : f(x) \geq \alpha \} \), the \( \alpha \)-level of \( f \) for \( \alpha \geq 0 \).

**Definition 2.2.** Let \( (X, \Sigma, \mu) \) be a fuzzy measure space. If \( f \in \mathcal{F}_+ \) and \( A \in \Sigma \), then the Sugeno integral (fuzzy integral) \([11]\) of \( f \) on \( A \) with respect to the fuzzy measure \( \mu \) is defined by

\[
(S) \int_A f \, d\mu := \vee_{\alpha \geq 0} (\alpha \wedge (A \cap L_\alpha f)),
\]

where \( \vee \) and \( \wedge \) denote the sup and inf on \([0, \infty]\), respectively.

The basic properties of Sugeno integral follow:

**Theorem 2.3.** ([9, 12]) Let \( (X, \Sigma, \mu) \) be a fuzzy measure space with \( A, B \in \Sigma \) and \( f, g \in \mathcal{F}_+ \). Then

1) \( (S) \int_A f \, d\mu \leq \mu(A) \);
2) \( (S) \int_A k \, d\mu = k \wedge \mu(A) \) for a non-negative constant \( k \);
3) if \( f \leq g \) on \( A \), then \( (S) \int_A f \, d\mu \leq (S) \int_A g \, d\mu \);
4) if \( A \subset B \), then \( (S) \int_A f \, d\mu \leq (S) \int_B f \, d\mu \);
5) \( \mu(A \cap L_\alpha f) \leq \alpha \Rightarrow (S) \int_A f \, d\mu \leq \alpha \).

Theorem 2.4. ([12], p. 135) Here \( f \in \mathcal{F}_+ \), the class of all finite nonnegative measurable functions on \((X, \Sigma, \mu)\). Then

1) if \( \mu(A) = 0 \), then \((S) \int_A f \, d\mu = 0\), for any \( f \in \mathcal{F}_+ \);
2) if \((S) \int_A f \, d\mu = 0\), then \( \mu(A \cap \{x|f(x) > 0\}) = 0 \);
3) \((S) \int_A f \, d\mu = (S) \int_X f \cdot \chi_A \, d\mu \), where \( \chi_A \) is the characteristic function of \( A \);
4) \((S) \int_A (f + a) \, d\mu \leq (S) \int_A f \, d\mu + (S) \int_A a \, d\mu \), for any constant \( a \in [0, \infty) \).

**Corollary 1.** ([12], p. 136) Here \( f, f_1, f_2 \in \mathcal{F}_+ \). Then

1) \((S) \int_A (f_1 \vee f_2) \, d\mu \geq (S) \int_A f_1 \, d\mu \vee (S) \int_A f_2 \, d\mu \);
2) \((S) \int_A (f_1 \wedge f_2) \, d\mu \leq (S) \int_A f_1 \, d\mu \wedge (S) \int_A f_2 \, d\mu \);
3) \((S) \int_{A \cup B} f \, d\mu \geq (S) \int_A f \, d\mu \vee (S) \int_B f \, d\mu \);
4) \((S) \int_{A \cap B} f \, d\mu \leq (S) \int_A f \, d\mu \wedge (S) \int_B f \, d\mu \).
In general we have
\[(S) \int_A (f_1 + f_2) d\mu \neq (S) \int_A f_1 d\mu + (S) \int_A f_2 d\mu,\]
and
\[(S) \int_A af d\mu \neq a (S) \int_A f d\mu, \text{ where } a \in \mathbb{R},\]
see [12], p. 137.

**Lemma 2.5.** ([12], p. 138) \((S) \int_A f d\mu = \infty \iff \mu(A \cap L_\alpha f) = \infty \) for any \(\alpha \in [0, \infty)\).

We need

**Definition 2.6.** ([5]) A fuzzy measure \(\mu\) is subadditive iff \(\mu(A \cup B) \leq \mu(A) + \mu(B)\), for all \(A, B \in \Sigma\).

We mention

**Theorem 2.7.** ([5]) If \(\mu\) is subadditive, then
\[(S) \int_X (f + g) d\mu \leq (S) \int_X f d\mu + (S) \int_X g d\mu, \tag{2}\]
for all measurable functions \(f, g : X \rightarrow [0, \infty)\).

Moreover, if (2) holds for all measurable functions \(f, g : X \rightarrow [0, \infty)\) and \(\mu(X) < \infty\), then \(\mu\) is subadditive.

Notice here in (1) we have that \(\alpha \in [0, \infty)\).

We have

**Corollary 2.** If \(\mu\) is subadditive, \(n \in \mathbb{N}\), and \(f : X \rightarrow [0, \infty)\) is a measurable function, then
\[(S) \int_X nf d\mu \leq n (S) \int_X f d\mu, \tag{3}\]
in particular it holds
\[(S) \int_A nf d\mu \leq n (S) \int_A f d\mu, \tag{4}\]
for any \(A \in \Sigma\).

**Proof.** By (2).

A very important property of Sugeno integral follows.

**Theorem 2.8.** If \(\mu\) is subadditive measure, and \(f : X \rightarrow [0, \infty)\) is a measurable function, and \(c > 0\), then
\[(S) \int_A cf d\mu \leq (c + 1) (S) \int_A f d\mu, \tag{5}\]
for any \(A \in \Sigma\).

**Proof.** Let the ceiling \(\lceil c \rceil = m \in \mathbb{N}\), then by Theorem 2.3 (3) and (4) we get
\[(S) \int_A cf d\mu \leq (S) \int_A mf d\mu \leq m (S) \int_A f d\mu \leq (c + 1) (S) \int_A f d\mu,
\]
proving (5).
Remark 1. Let \( f \in C ([a, b], \mathbb{R}_+) \), and \( \mu \) is a subadditive fuzzy measure such that \( \mu ([a, b]) > 0 \), \( x \in [a, b] \). We will estimate

\[
E(x) := \left| (S) \int_{[a, b]} f(x) d\mu(t) - \mu([a, b]) \wedge f(x) \right| \tag{6}
\]

(by Theorem 2.3 (2))

\[
= \left| (S) \int_{[a, b]} f(t) d\mu(t) - (S) \int_{[a, b]} f(x) d\mu(t) \right| .
\]

We notice that

\[
f(t) = f(t) - f(x) + f(x) \leq |f(t) - f(x)| + f(x) ,
\]

then (by Theorem 2.3 (3) and Theorem 2.4 (4))

\[
(S) \int_{[a, b]} f(t) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) + (S) \int_{[a, b]} f(x) d\mu(t) ,
\]

that is

\[
(S) \int_{[a, b]} f(t) d\mu(t) - (S) \int_{[a, b]} f(x) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) .
\]

Similarly, we have

\[
f(x) = f(x) - f(t) + f(t) \leq |f(t) - f(x)| + f(t) ,
\]

then (by Theorem 2.3 (3) and Theorem 2.7)

\[
(S) \int_{[a, b]} f(x) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) + (S) \int_{[a, b]} f(t) d\mu(t) ,
\]

that is

\[
(S) \int_{[a, b]} f(x) d\mu(t) - (S) \int_{[a, b]} f(t) d\mu(t) \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) .
\]

By (8) and (9) we derive that

\[
\left| (S) \int_{[a, b]} f(t) d\mu(t) - (S) \int_{[a, b]} f(x) d\mu(t) \right| \leq (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) .
\]

Consequently it holds

\[
E(x) \overset{\text{(by (6), (10))}}{\leq} (S) \int_{[a, b]} |f(t) - f(x)| d\mu(t) .
\]

We will later use (11).
3. Background - II.

We need

**Remark 2.** Here $[\cdot]$ denotes the integral part of the number. Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C ([a, b])$, $[a, b] \subset \mathbb{R}$, $x \in [a, b]$. The gamma function $\Gamma$ is given by $\Gamma (\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left( J^{\alpha}_{a^-} f \right) (x) = \frac{1}{\Gamma (\alpha)} \int_a^x (x-t)^{\alpha-1} f (t) \, dt,$$

(12)

$a \leq x \leq b$. We define the subspace $C^{\alpha}_{a^+} ([a, b])$ of $C^m ([a, b])$:

$$C^{\alpha}_{a^+} ([a, b]) = \{ f \in C^m ([a, b]) : J^{\alpha}_{1-\beta} f^{(m)} \in C^1 ([a, b]) \}.$$  

(13)

For $f \in C^{\alpha}_{a^+} ([a, b])$, we define the left generalized $\alpha$-fractional derivative of $f$ over $[a, b]$ as

$$D^\alpha_{a^+} f := \left( J^{\alpha}_{1-\beta} f^{(m)} \right)' ,$$

(14)

see [1], p. 24. Canavati first in [6] introduced the above over $[0, 1]$. Notice that $D^\alpha_{a^+} f \in C ([a, b])$.

We need the following left fractional Taylor’s formula, see [1], pp. 8-10, and in [6] the same over $[0, 1]$ that appeared first.

Let $f \in C^{\alpha}_{a^+} ([a, b])$:

(i) If $\alpha \geq 1$, then

$$f (x) = f (a) + f' (a) (x - a) + f'' (a) \frac{(x-a)^2}{2} + \ldots + f^{(m-1)} (a) \frac{(x-a)^{m-1}}{(m-1)!} +$$

$$+ \frac{1}{\Gamma (\alpha)} \int_a^x (x-t)^{\alpha-1} (D^{\alpha}_{a^+} f) (t) \, dt ,$$

(15)

all $x \in [a, b]$.

(ii) If $0 < \alpha < 1$, we have

$$f (x) = \frac{1}{\Gamma (\alpha)} \int_a^x (x-t)^{\alpha-1} (D^{\alpha}_{a^+} f) (t) \, dt ,$$

(16)

all $x \in [a, b]$.

Notice that

$$\int_a^x (x-t)^{\alpha-1} (D^{\alpha}_{a^+} f) (t) \, dt = \int_a^x (D^{\alpha}_{a^+} f) (t) \, d\left( \frac{(x-t)^{\alpha}}{-\alpha} \right)$$

(17)

$$= \left( D^{\alpha}_{a^+} f \right) (\xi_x) \frac{(x-a)^{\alpha}}{\alpha}, \text{ where } \xi_x \in [a, x],$$

by first integral mean value theorem.

Hence, when $\alpha \geq 1$ and $f^{(i)} (a) = 0$, $i = 0, 1, \ldots, m-1$ or when $0 < \alpha < 1$, we get

$$f (x) = \left( D^{\alpha}_{a^+} f \right) (\xi_x) \frac{(x-a)^{\alpha}}{\Gamma (\alpha + 1)} , \text{ all } x \in [a, b].$$

(18)

Furthermore we need:

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C ([a, b])$, call the right Riemann-Liouville fractional integral operator by

$$\left( J^{\alpha}_{b^-} f \right) (x) := \frac{1}{\Gamma (\alpha)} \int_x^b (t-x)^{\alpha-1} f (t) \, dt ,$$

(19)

$x \in [a, b]$, see also [3, pp. 333, 345].
Define the subspace of functions
\[ C^m_{b^-} ([a, b]) = \{ f \in C^m ([a, b]) : J^{1-\beta}_{b^-} f^{(m)} \in C^1 ([a, b]) \}. \] \tag{20}

Define the right generalized \(\alpha\)-fractional derivative of \(f\) over \([a, b]\) as
\[ D^\alpha_{b^-} f = (-1)^{m-1} \left( J^{1-\beta}_{b^-} f^{(m)} \right)', \] \tag{21}
see [3, p. 345]. We set \(D^0_{b^-} f = f\).

Notice that \(D^\alpha_{b^-} f \in C ([a, b])\).

From [3, p. 348], we need the following right Taylor fractional formula:
Let \(f \in C^\alpha_{b^-} ([a, b]), \alpha > 0, m = [\alpha]\). Then
(i) If \(\alpha \geq 1\), we get
\[ f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x - b)^k + (J^\alpha_{b^-} D^\alpha_{b^-} f)(x), \] \tag{22}
all \(x \in [a, b]\).

(ii) If \(0 < \alpha < 1\), we get
\[ f(x) = J^\alpha_{b^-} D^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} (D^\alpha_{b^-} f)(t) \, dt, \] \tag{23}
all \(x \in [a, b]\).

Notice that
\[ \int_x^b (t - x)^{\alpha-1} (D^\alpha_{b^-} f)(t) \, dt = \int_x^b (D^\alpha_{b^-} f)(t) \, d \left( \frac{(t - x)^\alpha}{\alpha} \right) = (D^\alpha_{b^-} f)(\eta_x) \frac{(b - x)^\alpha}{\alpha}, \text{ where } \eta_x \in [x, b], \] \tag{24}
by first integral mean value theorem.

Hence, when \(\alpha \geq 1\) and \(f^{(k)}(b) = 0, k = 0, 1, \ldots, m - 1\) or \(0 < \alpha < 1\), we obtain
\[ f(x) = \left( D^\alpha_{b^-} f \right)(\eta_x) \frac{(b - x)^\alpha}{\Gamma(\alpha + 1)}, \text{ all } x \in [a, b]. \] \tag{25}

Let \(f \in C^\alpha_{a^+} ([a, b]), \alpha \geq 1, \) and \(f^{(i)}(a) = 0, i = 1, \ldots, m - 1, \) then
\[ |f(x) - f(a)| \leq \| D^\alpha_{a^+} f \|_{L_{\alpha+1}([a, b])} \] \tag{26}
all \(x \in [a, b]\), by (15).

Again let \(f \in C^\alpha_{a^+} ([a, b]), \alpha \geq 1, \) and \(f^{(i)}(a) = 0, i = 1, \ldots, m - 1, \) then by (15) we have
\[ f(x) - f(a) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} (D^\alpha_{a^+} f)(t) \, dt, \] \tag{27}

hence
\[ |f(x) - f(a)| \leq \frac{(x - a)^{\alpha-1}}{\Gamma(\alpha)} \| D^\alpha_{a^+} f \|_{L_{\alpha+1}([a, b])}, \] \tag{28}
all \(x \in [a, b]\).

Let \(p, q > 1: \frac{1}{p} + \frac{1}{q} = 1\), continuing from (27), \(\alpha \geq 1\), we get
\[ |f(x) - f(a)| \leq \frac{1}{\Gamma(\alpha)} \left( \int_a^x (x - t)^{p(\alpha-1)} \, dt \right)^{\frac{1}{p}} \| D^\alpha_{a^+} f \|_{L_q([a, b])} = \]
\[
\frac{(x-a)^{\frac{(\nu-1)}{p}}}{\Gamma (\alpha) (p (\alpha -1) +1)^\frac{1}{p}} \| D^\alpha_{a+} f \|_{L_q([a,b])},
\]
\(\forall x \in [a,b].\)

Let \( f \in C^\alpha_{b-} ([a,b]), \alpha \geq 1, m = [\alpha], f^{(k)} (b) = 0, k = 1, \ldots, m - 1, \) then by (22) we get:
\[
f (x) - f (b) = \frac{1}{\Gamma (\alpha)} \int_x^b (t-x)^{\alpha-1} (D^\alpha_{b-} f) (t) \, dt.
\]

We derive the following estimates:

1) \[
|f (x) - f (b)| \leq \frac{(b-x)^{\alpha-1}}{\Gamma (\alpha +1)} \| D^\alpha_{b-} f \|_{\infty},
\]

2) \[
|f (x) - f (b)| \leq \frac{(b-x)^{\alpha-1}}{\Gamma (\alpha)} \| D^\alpha_{b-} f \|_{L_1([a,b])},
\]

3) let \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \) then
\[
|f (x) - f (b)| \leq \frac{(b-x)^{\frac{(p\alpha-1)}{p}+1}}{\Gamma (\alpha) (p (\alpha -1) +1)^\frac{1}{p}} \| D^\alpha_{b-} f \|_{L_q([a,b])},
\]
\(\forall x \in [a,b].\)

4. Background - III. We need

Remark 3. Let \( \nu > 0, n := [\nu], [\cdot] \) is the ceiling of the number, \( f \in AC^n ([a,b]) \)
(i.e. \( f^{(n-1)} \) is absolutely continuous on \([a,b])\). We call the left Caputo fractional
derivative ([7])
\[
D^\nu_{a} f (x) := \frac{1}{\Gamma (n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)} (t) \, dt,
\]
\(\forall x \in [a,b].\)

The above function \(D^\nu_{a} f (x)\) exists almost everywhere for \( x \in [a,b].\)

If \( \nu \in \mathbb{N}, \) then \(D^\nu_{a} f = f^{(\nu)}\) the ordinary derivative, it is also \(D^0_{a} f = f.\)

We have the left fractional Taylor formula for left Caputo fractional derivatives
[7, p. 40].

Assume \( \nu > 0, n = [\nu], \) and \( f \in AC^n ([a,b]). \) Then
\[
f (x) = \sum_{k=0}^{n-1} \frac{f^{(k)} (a)}{k!} (x-a)^k + \frac{1}{\Gamma (\nu)} \int_a^x (x-t)^{\nu-1} D^\nu_{a} f (t) \, dt,
\]
\(\forall x \in [a,b].\)

Additionally assume that
\[
f^{(k)} (a) = 0, \quad k = 1, \ldots, n - 1;
\]
then
\[
f (x) - f (a) = \frac{1}{\Gamma (\nu)} \int_a^x (x-t)^{\nu-1} D^\nu_{a} f (t) \, dt,
\]
\(\forall x \in [a,b].\)

We get the following estimates:
1) if \(D_{cb}^\nu f \in L_\infty ([a, b])\), then
\[
|f (x) - f (a)| \leq \frac{\|D_{cb}^\nu f\|_\infty}{\Gamma (\nu + 1)} (x - a)^\nu ,
\]
(37)
\(\forall x \in [a, b]\), see \([1]\), p. 619;
2) if \(\nu \geq 1\), and \(D_{cb}^\nu f \in L_1 ([a, b])\), then
\[
|f (x) - f (a)| \leq \frac{\|D_{cb}^\nu f\|_{L_1([a, b])}}{\Gamma (\nu)} (x - a)^{\nu - 1} ,
\]
(38)
\(\forall x \in [a, b]\), see \([1]\), p. 620;
3) let \(p, q > 1\) : \(\frac{1}{p} + \frac{1}{q} = 1\), and \(\nu > \frac{1}{q}\), and \(D_{cb}^\nu f \in L_q ([a, b])\), then
\[
|f (x) - f (a)| \leq \frac{\|D_{cb}^\nu f\|_{L_q([a, b])}}{\Gamma (\nu) (p (\nu - 1) + 1)^{\frac{1}{p}} } (x - a)^{\nu - \frac{1}{q}} ,
\]
(39)
\(\forall x \in [a, b]\), see \([1]\), p. 621.

Furthermore we need:
Let \(f \in AC^m ([a, b])\) \((f^{(m-1)}\) is absolutely continuous on \([a, b])\), \(m \in \mathbb{N}\), \(m = \lceil \alpha \rceil\), \(\alpha > 0\). We define the right Caputo fractional derivative of order \(\alpha > 0\) by
\[
D_{cb}^\alpha f (x) = \frac{(-1)^m}{\Gamma (m - \alpha)} \int_x^b (J - x)^{m - \alpha - 1} f^{(m)} (J) dJ ,
\]
(40)
\(\forall x \in [a, b]\), see \([3]\), p. 336).
If \(\alpha = m \in \mathbb{N}\), then
\[
D_{cb}^\alpha f (x) = (-1)^m f^{(m)} (x), \ \forall x \in [a, b].
\]
(41)

If \(x > b\) we define \(D_{cb}^\alpha f (x) = 0\).

We also need:
Let \(f \in AC^m ([a, b])\), \(\alpha > 0\), \(m = \lceil \alpha \rceil\). Then
\[
f (x) = \sum_{k=0}^{m-1} \frac{f^{(k)} (b)}{k!} (x - b)^k + \frac{1}{\Gamma (\alpha)} \int_x^b (J - x)^{\alpha - 1} D_{cb}^\alpha f (J) dJ ,
\]
(42)
\(\forall x \in [a, b]\), the right Caputo fractional Taylor formula with integral remainder, see \([3]\), p. 338.

Additionally assume that
\[
f^{(k)} (b) = 0, \ k = 1, \ldots, m - 1,
\]
then
\[
f (x) - f (b) = \frac{1}{\Gamma (\alpha)} \int_x^b (J - x)^{\alpha - 1} D_{cb}^\alpha f (J) dJ ,
\]
(43)
\(\forall x \in [a, b]\).

Following (43) we get the following estimates:
1) if \(D_{cb}^\alpha f \in L_\infty ([a, b])\), then
\[
|f (x) - f (b)| \leq \frac{(b - x)^\alpha}{\Gamma (\alpha + 1)} \|D_{cb}^\alpha f\|_\infty ,
\]
(44)
\(\forall x \in [a, b]\), see \([2]\), p. 23;
2) if \(D_{cb}^\alpha f \in L_1 ([a, b])\), \(\alpha \geq 1\), then
\[
|f (x) - f (b)| \leq \frac{\|D_{cb}^\alpha f\|_{L_1([a, b])}}{\Gamma (\alpha) } (b - x)^{\alpha - 1} ,
\]
(45)
∀ x ∈ [a, b], see [2], p. 24.

3) let p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, and \( \alpha > \frac{1}{q} \), \( m = [\alpha] \), \( D_{cb}^{\alpha} f \in L_q ([a, b]) \), then

\[
|f (x) - f (b)| \leq \frac{\| D_{cb}^{\alpha} f \|_{L_q ([a, b])}}{\Gamma (\alpha) (p (\alpha - 1) + 1)^{\frac{1}{p}}} (b - x)^{\alpha - \frac{1}{q}}, \tag{46}
\]

∀ x ∈ [a, b], see [2], p. 25.

5. Main results. We make

Remark 4. Let \( \alpha \geq 1, f \in C_{\alpha}^m ([a, b]) \) and \( f^{(i)} (a) = 0, i = 1, ..., m - 1; m = [\alpha] \).

By (11), for \( x = a \), we get that

\[
E (a) = \left| (S) \int_{[a, b]} f (t) d \mu (t) - \mu ([a, b]) \wedge f (a) \right| \leq \tag{47}
\]

\[
(S) \int_{[a, b]} |f (t) - f (a)| d \mu (t) =: \Delta (a). \]

By (26) we get

\[
\Delta (a) \leq (S) \int_{[a, b]} \frac{\| D_{\alpha}^{a} f \|_\infty}{\Gamma (\alpha + 1)} (t - a)^{\alpha} d \mu (t) \tag{5}
\]

\[
\left( \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\alpha - 1} d \mu (t). \tag{48}
\]

By (28) we obtain

\[
\Delta (a) \leq (S) \int_{[a, b]} \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha)} (t - a)^{\alpha - 1} d \mu (t) \tag{5}
\]

\[
\left( \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\alpha - 1} d \mu (t). \tag{49}
\]

And by (29) \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1 \) we derive

\[
\Delta (a) \leq (S) \int_{[a, b]} \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha) (p (\alpha - 1) + 1)} (t - a)^{\alpha - \frac{1}{p} + 1} d \mu (t) \tag{5}
\]

\[
\left( \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha) (p (\alpha - 1) + 1)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\alpha - \frac{1}{p} + 1} d \mu (t). \tag{50}
\]

We have proved that

\[
\left| (S) \int_{[a, b]} f (t) d \mu (t) - \mu ([a, b]) \wedge f (a) \right| \leq
\]

\[
\min \left\{ \left( \frac{\| D_{\alpha}^{a} f \|_\infty}{\Gamma (\alpha + 1)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\alpha} d \mu (t), \right. \right.
\]

\[
\left. \left( \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\alpha - 1} d \mu (t), \right. \right.
\]

\[
\left. \left. \left( \frac{\| D_{\alpha}^{a} f \|_{L_1 ([a, b])}}{\Gamma (\alpha) (p (\alpha - 1) + 1)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\alpha - \frac{1}{p} + 1} d \mu (t) \right\}. \tag{51}
\]

\[
\]
We have established the following left generalized fractional Ostrowski-Sugeno inequality.

**Theorem 5.1.** Here $\mu$ is a fuzzy subadditive measure with $\mu([a, b]) > 0$.
Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; \alpha \geq 1$. Let $f \in C_{\alpha+}^\infty([a, b])$ with $f^{(i)}(a) = 0, i = 1, \ldots, m - 1; m = [\alpha];$ and $f$ is $\mathbb{R}_+$-valued. Then

$$
\frac{1}{\mu([a, b])} \bigg( \int_{[a, b]} f(t) d\mu(t) - \left( 1 \wedge \frac{f(b)}{\mu([a, b])} \right) \bigg) \leq
\frac{1}{\mu([a, b])} \min \left\{ \left( \frac{\|D_{a+}^\alpha f\|_\infty}{\Gamma(\alpha + 1)} + 1 \right) \left( S \int_{[a, b]} (t-a)^\alpha d\mu(t) \right),
\left( \frac{\|D_{a+}^\alpha f\|_{L_1([a, b])}}{\Gamma(\alpha)} + 1 \right) \left( S \int_{[a, b]} (t-a)^{\alpha-1} d\mu(t) \right),
\left( \frac{\|D_{a+}^\alpha f\|_{L_\infty([a, b])}}{\Gamma(\alpha) (\nu (\alpha - 1) + 1)^{\frac{1}{\nu}}} + 1 \right) \left( S \int_{[a, b]} (t-a)^{\alpha-\frac{1}{\nu}} d\mu(t) \right) \right\}.
$$

(52)

Similarly (as in Remark 4), we get the right generalized fractional Ostrowski-Sugeno inequality (use of (11) for $x = b$, and (31), (32), (33)).

**Theorem 5.2.** Here $\mu$ is a fuzzy subadditive measure with $\mu([a, b]) > 0$.
Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1; \alpha \geq 1$. Let $f \in C_{\alpha-}^\infty([a, b])$ with $f^{(k)}(b) = 0, k = 1, \ldots, m - 1; [\alpha] = m;$ and $f$ is $\mathbb{R}_+$-valued. Then

$$
\frac{1}{\mu([a, b])} \bigg( \int_{[a, b]} f(t) d\mu(t) - \left( 1 \wedge \frac{f(a)}{\mu([a, b])} \right) \bigg) \leq
\frac{1}{\mu([a, b])} \min \left\{ \left( \frac{\|D_{b-}^\alpha f\|_\infty}{\Gamma(\alpha + 1)} + 1 \right) \left( S \int_{[a, b]} (b-t)^\alpha d\mu(t) \right),
\left( \frac{\|D_{b-}^\alpha f\|_{L_1([a, b])}}{\Gamma(\alpha)} + 1 \right) \left( S \int_{[a, b]} (b-t)^{\alpha-1} d\mu(t) \right),
\left( \frac{\|D_{b-}^\alpha f\|_{L_\infty([a, b])}}{\Gamma(\alpha) (\nu (\alpha - 1) + 1)^{\frac{1}{\nu}}} + 1 \right) \left( S \int_{[a, b]} (b-t)^{\alpha-\frac{1}{\nu}} d\mu(t) \right) \right\}.
$$

(53)

We present the following left Caputo fractional Ostrowski-Sugeno inequalities:

**Theorem 5.3.** Here $\mu$ is a fuzzy subadditive measure with $\mu([a, b]) > 0$. Let $f : [a, b] \to \mathbb{R}_+$ such that $f \in AC^\infty([a, b])$, where $n = [\nu], \nu > 0$. Assume $f^{(k)}(a) = 0, k = 1, \ldots, n - 1$. We have

1) if $D_{a+}^\nu f \in L_\infty([a, b]),$ then

$$
\frac{1}{\mu([a, b])} \left( \int_{[a, b]} f(t) d\mu(t) - \left( 1 \wedge \frac{f(a)}{\mu([a, b])} \right) \right) \leq
\frac{1}{\mu([a, b])} \left( \frac{\|D_{a+}^\nu f\|_\infty}{\Gamma(\nu + 1)} + 1 \right) \left( S \int_{[a, b]} (t-a)^\nu d\mu(t) \right),
$$

(54)

2) if $\nu \geq 1,$ and $D_{a+}^\nu f \in L_1([a, b])$, then

$$
\frac{1}{\mu([a, b])} \left( \int_{[a, b]} f(t) d\mu(t) - \left( 1 \wedge \frac{f(a)}{\mu([a, b])} \right) \right) \leq
$$
Theorem 5.4. Let \( \mu \) be a fuzzy subadditive measure with \( \mu ([a, b]) > 0 \). Let \( f : [a, b] \rightarrow \mathbb{R}_+ \) such that \( f \in AC^m ([a, b]), \) \( m \in \mathbb{N}, \) \( m = [\alpha], \) \( \alpha > 0. \) Assume \( f^{(k)} (b) = 0, \) \( k = 1, \ldots, m - 1. \) We have

1) if \( D^\alpha_{cb} f \in L_\infty ([a, b]), \) then

\[
\frac{1}{\mu ([a, b])} \left( \frac{\| D^\nu f \|_{L_\infty ([a, b])}}{\Gamma (\nu)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\nu - 1} d\mu (t),
\]

and

2) if \( \alpha \geq 1, \) and \( D^\alpha_{cb} f \in L_1 ([a, b]), \) then

\[
\frac{1}{\mu ([a, b])} \left( \frac{\| D^\alpha_{cb} f \|_{L_1 ([a, b])}}{\Gamma (\alpha + 1)} + 1 \right) (S) \int_{[a, b]} (b - t)^{\alpha - 1} d\mu (t),
\]

3) if \( p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \) and \( \alpha > \frac{1}{q}, \) and \( D^\alpha_{cb} f \in L_q ([a, b]), \) then

\[
\frac{1}{\mu ([a, b])} \left( \frac{\| D^\alpha_{cb} f \|_{L_q ([a, b])}}{\Gamma (\alpha (p - 1) + 1)} + 1 \right) (S) \int_{[a, b]} (b - t)^{\alpha - \frac{1}{q}} d\mu (t).
\]

Proof. By the use of (37), (38) and (39), acting as in the proof of Theorem 5.1. \( \square \)

Next, we give the following right Caputo fractional Ostrowski-Sugeno inequalities:

Theorem 5.4. Here \( \mu \) is a fuzzy subadditive measure with \( \mu ([a, b]) > 0. \) Let \( f : [a, b] \rightarrow \mathbb{R}_+ \) such that \( f \in AC^m ([a, b]), \) \( m \in \mathbb{N}, \) \( m = [\alpha], \) \( \alpha > 0. \) Assume \( f^{(k)} (b) = 0, \) \( k = 1, \ldots, m - 1. \) We have

1) if \( D^\alpha_{cb} f \in L_\infty ([a, b]), \) then

\[
\frac{1}{\mu ([a, b])} (S) \int_{[a, b]} f (t) d\mu (t) - \left( 1 \wedge \frac{f (b)}{\mu ([a, b])} \right) \leq \frac{1}{\mu ([a, b])} \left( \frac{\| D^\nu f \|_{L_\infty ([a, b])}}{\Gamma (\nu)} + 1 \right) (S) \int_{[a, b]} (t - a)^{\nu - 1} d\mu (t),
\]

Proof. Use of (44), (45) and (46), acting again as in the proof of Theorem 5.1. \( \square \)

Remark 5. Let \( x_0 \in [a, b]. \) Of interest will be to estimate the quantity

\[
\left| (S) \int_{[a, x_0]} f (t) d\mu (t) + (S) \int_{[x_0, b]} f (t) d\mu (t) - \{ \mu ([a, x_0]) \wedge f (x_0) + \mu ([x_0, b]) \wedge f (x_0) \} \right| \leq \left| (S) \int_{[a, x_0]} f (t) d\mu (t) - \mu ([a, x_0]) \wedge f (x_0) \right| + \left| (S) \int_{[x_0, b]} f (t) d\mu (t) - \mu ([x_0, b]) \wedge f (x_0) \right|.
\]
We have proved a reverse fractional Polya-Sugeno type inequality:

\[ \left( S \right) \int_{[x_0, b]} f(t) \, d\mu(t) - \mu([x_0, b]) \wedge f(x_0) \leq 0. \]  

(60)

An important special case is when \( x_0 = \frac{a+b}{2} \).

The above can be done with the use of our earlier results.

**Remark 6.** Here \( \mu \) is just a finite fuzzy measure on \([a, b]\) which is positive on non-empty closed subsets of \([a, b]\).

Let \( f: [a, b] \to \mathbb{R}_+ \) be continuous, such that \( f \in \mathcal{C}_{a+}^\alpha \left( [a, \frac{a+b}{2}] \right), 0 < \alpha < 1. \)

By (18) we get that

\[ f(x) \geq \left( \inf_{x \in \left[ a, \frac{a+b}{2} \right]} |D_{a+}^\alpha f| \right) \frac{(x-a)^\alpha}{\Gamma(\alpha+1)}, \text{ all } x \in \left[ a, \frac{a+b}{2} \right]. \]  

(61)

Assume also \( f \in \mathcal{C}_{b-}^\alpha \left( [\frac{a+b}{2}, b] \right), 0 < \alpha < 1. \)

By (25) we find that

\[ f(x) \geq \left( \inf_{x \in [\frac{a+b}{2}, b]} |D_{b-}^\alpha f| \right) \frac{(b-x)^\alpha}{\Gamma(\alpha+1)}, \text{ all } x \in \left[ \frac{a+b}{2}, b \right]. \]  

(62)

We notice that

\[ \left( S \right) \int_{[a, b]} f(t) \, d\mu(t) = \left( S \right) \int_{[a, \frac{a+b}{2}]} \int_{[a, \frac{a+b}{2}]} f(t) \, d\mu(t) \geq \left( S \right) \int_{[a, \frac{a+b}{2}]} f(t) \, d\mu(t) \vee \left( S \right) \int_{[\frac{a+b}{2}, b]} f(t) \, d\mu(t) \geq \right. \]

(63)

\[ \frac{1}{\Gamma(\alpha+1)} \left[ \left( S \right) \int_{[a, \frac{a+b}{2}]} \left( \inf_{t \in [a, \frac{a+b}{2}]} |D_{a+}^\alpha f| \right) (t-a)^\alpha \, d\mu(t) \vee \left( S \right) \int_{[\frac{a+b}{2}, b]} \left( \inf_{t \in [\frac{a+b}{2}, b]} |D_{b-}^\alpha f| \right) (b-t)^\alpha \, d\mu(t) \right]. \]

We have proved a reverse fractional Polya-Sugeno type inequality(11,10),(979,992):

\[ \left( S \right) \int_{[a, b]} f(t) \, d\mu(t) \geq \frac{1}{\Gamma(\alpha+1)} \left[ \left( S \right) \int_{[a, \frac{a+b}{2}]} \left( \inf_{t \in [a, \frac{a+b}{2}]} |D_{a+}^\alpha f| \right) (t-a)^\alpha \, d\mu(t) \vee \left( S \right) \int_{[\frac{a+b}{2}, b]} \left( \inf_{t \in [\frac{a+b}{2}, b]} |D_{b-}^\alpha f| \right) (b-t)^\alpha \, d\mu(t) \right]. \]  

(64)

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