A note on one-dimensional symmetry for Hamilton–Jacobi equations with extremal Pucci operators and application to Bernstein type estimate

Rodrigo Fuentes and Alexander Quaas

Abstract. We prove a Liouville-type theorem that is one-dimensional symmetry and classification results for non-negative $L^q$-viscosity solutions of the equation

$$-\mathcal{M}_{\lambda,\Lambda}^\pm(D^2u) \pm |Du|^p = 0, \quad x \in \mathbb{R}^n_+,$$

with boundary condition $u(\tilde{x}, 0) = M \geq 0, \tilde{x} \in \mathbb{R}^{n-1}$, where $\mathcal{M}_{\lambda,\Lambda}^\pm$ are the Pucci’s operators with parameters $\lambda, \Lambda \in \mathbb{R}_+ 0 < \lambda \leq \Lambda$ and $p > 1$. The results are an extension of the results by Porretta and Véron (Adv Nonlinear Stud 6:351–378, 2006) for the case $p \in (1, 2]$ and by Filippucci et al. (Commun Partial Differ Equ 45(2):321–349, 2019) for the case $p > 2$, both for the Laplacian case (i.e. $\lambda = \Lambda = 1$). As an application in the case $p > 2$, we prove a sharp Bernstein estimation for $L^q$-viscosity solutions of the fully nonlinear equation

$$-\mathcal{M}_{\lambda,\Lambda}^\pm(D^2u) = |Du|^p + f(x), \quad x \in \Omega,$$

with boundary condition $u = 0$ on $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$.

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1. Introduction

In this paper we consider, for $n \geq 2$, the second order elliptic equation

$$\begin{cases}
-\mathcal{M}_{\lambda,\Lambda}^\pm(D^2u) \pm |Du|^p = 0, & x \in \mathbb{R}^n_+, \\
u(\tilde{x}, 0) = M, & \tilde{x} \in \mathbb{R}^{n-1},
\end{cases}$$

(1)
where \( \mathbb{R}^n_+ = \{ (\tilde{x}, x_n) = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0 \} \), \( p > 1, M \geq 0 \) and \( \mathcal{M}^{\pm}_{\lambda, \Lambda} \) are the extremal Pucci’s operators with parameters \( 0 \leq \lambda \leq \Lambda \) defined by

\[
\mathcal{M}^+_{\lambda, \Lambda}(N) = \Lambda \sum_{e_i > 0} e_i + \lambda \sum_{e_i < 0} e_i,
\]

\[
\mathcal{M}^-_{\lambda, \Lambda}(N) = \lambda \sum_{e_i > 0} e_i + \Lambda \sum_{e_i < 0} e_i,
\]

with \( N \) any symmetric \( n \times n \) matrix, and \( e_i \) the eigenvalues of \( N \). One first aim of this paper is to establish one-dimensional symmetry for any \( L^q \)-viscosity solution of Eq. (1). We split the results in two cases depending on \( p \).

**Theorem 1.1.** Let \( p \in (1, 2] \) and \( u \in C(\mathbb{R}^n_+) \) an \( L^q \)-viscosity solution of

\[
\begin{cases}
-\mathcal{M}^\pm_{\lambda, \Lambda}(D^2 u) + |Du|^p = 0, & x \in \mathbb{R}^n_+,

u(\tilde{x}, 0) = M, & \tilde{x} \in \mathbb{R}^{n-1},

u \geq 0, & \text{in } \mathbb{R}^n_+,
\end{cases}
\]

with \( q \geq n \) and \( M \geq 0 \). Then \( u \) depends only on the variable \( x_n \).

and for \( p > 2 \) we have

**Theorem 1.2.** Let \( p > 2 \) and \( u \in C(\mathbb{R}^n_+) \) an \( L^q \)-viscosity solution of

\[
\begin{cases}
-\mathcal{M}^\pm_{\lambda, \Lambda}(D^2 u) \pm |Du|^p = 0, & x \in \mathbb{R}^n_+,

u(\tilde{x}, 0) = 0, & \tilde{x} \in \mathbb{R}^{n-1},
\end{cases}
\]

with \( q \geq n \). Then \( u \) depends only on the variable \( x_n \).

From these theorems, classification results can be established by solving the ODE, see the beginning of Sect. 3. Notice that the only work we know of one-dimensional symmetry or rigidity results in the half space for fully non-linear operator is in [7], for explosive boundary condition. Other symmetry type results in a bounded domain can be found in [19].

In the case of the Laplacian, rigidity results in the half-space is widely studied for \( -\Delta u = f(u) \) in different setting see for example [4,5,16], other references can be found in these works.

In the context of our problem and the Laplacian et al. [28] studied the case \( p \in (1, 2] \) and Filippucci et al. the case \( p > 2 \) in [22], that are connected with our setting.

One useful tool in establishing the above theorems is a Bernstein type estimate by Birindelli et al. [8], specifically Proposition 2.3. with \( \alpha = 0 \), which is an extension of the result from Capuzzo et al. [13].

These Bernstein-type results are for second order elliptic equations of the form

\[
-\mathcal{M}^\pm_{\lambda, \Lambda}(D^2 u) + g(Du) = f(x), \quad x \in \Omega,
\]

where \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \).
Using a regularity argument and the Bernstein estimate, we prove first that any $L^q$-viscosity solution of problem (1) has to be a classical solution.

Now we come back to the proof of our one-dimensional symmetry results. For $p \in (1, 2]$ the proof is based on the fact that the solution of the problem has a finite limit when the variable $x_n$ goes to infinity. In [28] this fact is guaranteed by the Bernstein estimation from [27], and in this paper, the result is obtained thanks to Theorem 4.1 in Sect. 4. We note that for $p \in (1, 2]$ the assumption $u \geq 0$ is necessary for the technique used, that is why we only consider the Eq.

$$-\mathcal{M}^\pm_{\lambda, \Lambda}(D^2 u) + |Du|^p = 0.$$ 

For $p > 2$, the solution does not have a finite limit, so is not possible to use the same argument. In this case, we use a moving planes argument and the maximum principle to prove that any $L^q$-viscosity solution of (1) is one dimensional. The technique used in this case also can not be replicated for $p \in (1, 2]$, since we can not guarantee uniform convergence in compacts from a sequence to a solution of the problem. The moving planes technique was used in [22] to prove the same result for classical solutions for $p > 2$ of the equation

\[
\begin{cases}
-\Delta u = |Du|^p, & x \in \mathbb{R}_+^n, \\
u(\tilde{x}, 0) = 0, & \tilde{x} \in \mathbb{R}^{n-1},
\end{cases}
\]

using the Bernstein estimation in [27], and a regularity argument.

We present here for completeness in Sect. 4 a Bernstein estimate that we will use throughout the paper. This result is a simplification of the work done in [8], and we present a shorter proof, that is based on the same ideas of [13] as the proof in [8], that works also for the case $\lambda = 0$.

The Bernstein technique was implemented for the first time by Bernstein himself in [6], and it has been replicated for equations with different elliptic operators. For example, in 1985, Lions [27] proved for the equation

$$-\Delta u + \alpha |Du|^p = f(x), \quad \text{in } \Omega,$$

with $\alpha > 0$, the estimation

$$|Du| \leq C \left\{ d(x, \partial \Omega)^{-\frac{1}{p-1}} + C_1 \right\},$$

where $C_1$ depends on $f$, for $p > 1$. Lions used that $f \in C^1(\Omega)$, and the proof was based on the traditional Bernstein technique, using the equation for the function $|Du|^2$. Many other Bernstein type results have been proved for other elliptic operators. In [2], Barles extended this idea to a weak Bernstein method to prove global Lipschitz continuity, using only continuous viscosity solutions and bases his work not on the differentiation of the equation, but only in properties of the operator $F$. More recently in [3], Barles uses this weak Bernstein method to prove local Lipschitz continuity for a continuous solution of

$$F(x, u, Du, D^2 u) = 0 \quad \text{in } \Omega,$$
using locally Lipschitz continuity and ellipticity, with other properties of \( F \). Other regularity results for viscosity solutions were proved by Kawohl and Kutev [25], and by Armstrong and Tran [1] using different assumptions. In [25] they used a boundary Lipschitz condition (and more recently in [24] using \( C^2 \)-boundary condition) to prove Lipschitz regularity in all the domain \( \Omega \), and in [1] they require the diffusion matrix \( A \) from the equation

\[-\text{tr}(A(x)D^2 u) + H(x, Du) = 0, \quad \text{in } B_2\]

to be regular enough in view of proving that any continuous solution is Lipschitz continuous in \( B_1 \). Hölder regularity results for extensions of equation (6) have recently been published in [9,18]. Da Silva and Nornberg use in [18] \( L^q \)-viscosity solutions for elliptic equations with Hamiltonian with coefficients not necessarily regular, as in our Eq. (5), to prove local Hölder continuity.

As an application to their Liouville result, Filipucci et al. proved an improvement of the Bernstein estimation in [27], now with the boundary value, for the inhomogeneous Dirichlet problem (specifically Theorem 1.6 in [22]). The result establishes a more precise constant in the Bernstein estimate, which is also valid for an equation of the form

\[
\begin{aligned}
-\mathcal{M}^\pm_{\lambda, \Lambda}(D^2 u) &= |Du|^p + f(x), \quad \text{in } \Omega, \\
 u &= 0, \quad \text{on } \partial \Omega,
\end{aligned}
\]

(7)

this by using our Theorem 1.2 and similar ideas as in [22].

Specifically, we have

**Theorem 1.3.** Let \( p > 2 \) and \( f \) a Lipschitz function with \( \|f\|_{W^{1,\infty}} \leq M \). If \( u \in C(\Omega) \) is an \( L^q \)-viscosity solution of (7), then for every \( \epsilon > 0 \) there exists a constant \( C = C(\epsilon, M) > 0 \) such that

\[
|Du(x)| \leq (1 + \epsilon) \left( \frac{\lambda}{p - 1} \right)^{\frac{1}{p - 1}} \text{d}_\partial \Omega(x)^{-\frac{1}{p - 1}} + C \quad \text{in } \Omega,
\]

if the operator in (7) is \( \mathcal{M}^+ \). If the operator is \( \mathcal{M}^- \), the constant must be \( \Lambda \).

The paper is organized as follows: In Sect. 2 we give some preliminaries; in Sect. 3 we prove our main results; in Sect. 4 we give and prove our first Bernstein type estimate.

## 2. Preliminaries

Next we are going to give the different definitions of solutions that we will use throughout the paper. For the definitions, with \( \Omega \subset \mathbb{R}^n \), we consider a measurable function \( F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times S(n) \to \mathbb{R} \), where \( S(n) \) denotes the set of \( n \times n \) real symmetric matrices, and the equation

\[
F(x, u(x), Du, D^2 u) = f(x), \quad \text{in } \Omega.
\]

(8)
Definition 2.1. Let \( n \leq q \) and \( f \in L^q_{\text{loc}}(\Omega) \). A function \( u \in C(\Omega) \) is an \( L^q \)-viscosity subsolution (supersolution) of (8) if for all \( \varphi \in W^{2,q}_{\text{loc}}(\Omega) \), and \( \hat{x} \in \Omega \) at which \( u - \varphi \) has a local maximum (respectively minimum) we have

\[
\begin{align*}
\text{ess lim inf}_{x \to \hat{x}} (F(x, u(x), D\varphi(x), D^2\varphi(x)) - f(x)) &\leq 0 \\
(\text{ess lim sup}_{x \to \hat{x}} (F(x, u(x), D\varphi(x), D^2\varphi(x)) - f(x)) &\geq 0).
\end{align*}
\]

In addition, \( u \) is an \( L^q \)-viscosity solution of (8) if it is both an \( L^q \)-viscosity subsolution and an \( L^q \)-viscosity supersolution.

This notion of solutions was studied by Caffarelli et al. [12], and rightly related with the notion of \( L^q \)-strong solutions. The use of \( L^q \)-viscosity solutions falls in that no differentiability is required for the test functions, neither the continuity of \( F \) and \( f \) at \( x \), therefore, pointwise relations are not required. If the functions \( F \) and \( f \) are continuous, and the test function \( \varphi \) is twice continuously differentiable we employ the notion of classical viscosity solution that we define next.

Definition 2.2. For \( f \) and \( F \) continuous in \( \Omega \), a function \( u \in C(\Omega) \) is a viscosity subsolution (supersolution) of (8) if, for any function \( \varphi \in C(\Omega) \cap C^2(\Omega) \), if \( \hat{x} \in \Omega \) is a local maximum (minimum) point of \( u - \varphi \), then

\[
\begin{align*}
F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) &\leq f(\hat{x}) \\
(F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) &\geq f(\hat{x})).
\end{align*}
\]

In addition, \( u \) is a viscosity solution of (8) if it is both a viscosity subsolution and a viscosity supersolution.

The use of these types of solutions allows establishing results for uniqueness, existence and stability for a wide class of equations. The notion of viscosity solutions was introduced by Crandall and Lions [15] for first-order equations of Hamilton–Jacobi type. A second-order extension was introduced by Lions [26] for Hamilton-Jacobi-Bellman equations, where he showed uniqueness results for viscosity solutions of these types of problems.

We use that the Pucci operators are extremal, and can be written by

\[
\begin{align*}
\mathcal{M}^+_{\lambda,\Lambda}(N) &= \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \text{tr}(AN), \\
\mathcal{M}^-_{\lambda,\Lambda}(N) &= \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \text{tr}(AN),
\end{align*}
\]

where \( \mathcal{A}_{\lambda,\Lambda} \) denotes the set of all symmetric matrix whose eigenvalues lie in the interval \([\lambda, \Lambda]\). With these definitions, it is easy to see that the Pucci operators \( \mathcal{M}^\pm_{\lambda,\Lambda} \) have the following properties: for any \( \alpha \geq 0 \) and for any \( n \times n \) symmetric matrices \( M, N \) we have

\[
\begin{align*}
\mathcal{M}^\pm_{\lambda,\Lambda}(\alpha M) &= \alpha \mathcal{M}^\pm_{\lambda,\Lambda}(M), \\
\mathcal{M}^-_{\lambda,\Lambda}(M) &\leq \mathcal{M}^\pm_{\lambda,\Lambda}(M + N) - \mathcal{M}^\pm_{\lambda,\Lambda}(N) \leq \mathcal{M}^+_{\lambda,\Lambda}(M).
\end{align*}
\]

(For these and other properties of Pucci operators see [11]).
3. Proof of main results and classification of solutions

In this section, we prove our main results. We first start this section with the classification results that can be deduced from our main theorems.

As a consequence of Theorem 1.1 we have that any solution of (3) is given by

(i) For \( p \in (1, 2) \), \( u \equiv 0 \) or there exists a constant \( l \in [0, M) \) such that
\[
(\tilde{x}, y) = l + c_{p,\lambda,\Lambda}(y + C_{M,l})^{\frac{p-2}{p-1}},
\]
where
\[
c_{p,\lambda,\Lambda} = \left( \frac{\Lambda}{p-1} \right)^{\frac{1}{p-1}} \left( \frac{p-1}{2-p} \right) \text{ and } C_{M,l} \text{ is determined by the relation }
\]
\[
\int_0^\infty \left[ \left( \frac{p-1}{\Lambda} \right) (t + C_{M,l}) \right]^{-\frac{1}{p-1}} dt = M - l,
\]
in the case of \( M_{\lambda,\Lambda}^+ \). If the operator is \( M_{\lambda,\Lambda}^- \), the constant \( \Lambda \) in the solution must be replaced by \( \lambda \).

(ii) For \( p = 2 \) necessarily \( u \equiv M \).

As a consequence of Theorem 1.2, in this case we have that any solution of (4) is given by
\[
u \equiv 0 \text{ or } u = \hat{u}_c(x_n) = u_0(x_n + \hat{c}) - u_0(\hat{c}) \text{ for some } \hat{c} \geq 0, \text{ with }
\]
\[
u_0(t) = c_{p,\lambda,\Lambda} t^{\frac{p-2}{p-1}}, \quad t \geq 0,
\]
where
\[
c_{p,\lambda,\Lambda} = \left( \frac{\lambda}{p-1} \right)^{\frac{1}{p-1}} \left( \frac{p-1}{p-2} \right) \text{ if the operator is } M_{\lambda,\Lambda}^+ \text{ and the equation is } -M_{\lambda,\Lambda}^+(D^2u) - |Du|^p = 0. \text{ In the case of } M_{\lambda,\Lambda}^-; \text{ the constant } \lambda \text{ in } c_{p,\lambda,\Lambda} \text{ must be replaced by } \Lambda \text{ (in the same equation). If the equation is } -M_{\lambda,\Lambda}^\pm(D^2u) + |Du|^p = 0, \text{ the solution is the negative of the one given above.}
\]

We note that in all cases, the functions are classical solutions, this will be the first thing we will show next.

**Proof of Theorem 1.1.** First, we will show that \( u \) is a classical solution of (3).

By Theorem 4.1 below, with \( g(Du) = |Du|^p \), we have
\[
|Du(\tilde{x}, y)| \leq C(n, p) y^{\frac{1}{p-1}}, \quad \text{a.e. } (\tilde{x}, y) \in \mathbb{R}^{n-1} \times (0, \infty).
\]

Considering \( f(\tilde{x}, y) = |Du(\tilde{x}, y)|^p \) in Corollary 3.10. in [12], recalling that \( \tilde{B} = B((\tilde{x}, y), \frac{y}{2}) \) satisfies the uniform exterior cone condition and \( u \) is continuous in \( \mathbb{R}^n_+ \), by (12) we have that \( f \in L^q(\tilde{B}) \) for all \( n \leq q \leq \infty \), therefore \( u \in W^{2,q}_{loc}(\mathbb{R}^n_+) \) (using Lemma 2.5 in [12] and the fact that the unique \( L^q \)-strong solution of (3) coincides with our \( L^q \)-viscosity solution). This implies that \( Du \in W^{1,q}_{loc}(\mathbb{R}^n_+) \), and by Morrey’s inequality (Theorem 4 in 5.6.2, [21]), we have that \( Du \in C^{\alpha}_{loc}(\mathbb{R}^n_+) \), for some \( \alpha \in (0, 1) \). Therefore, by Theorem 1.2 in [10], the function \( u \) is in \( C^{2,\alpha}_{loc}(\mathbb{R}^n_+) \), and this implies that \( u \) is a classical solution of (3).

Now we will prove the symmetry result.
First we take $p \in (1, 2)$, and we follow the ideas in the proof in [28]. By Theorem 4.1 and recalling that $u$ is a classical solution of (3) we have that

$$|Du(\tilde{x}, y)| \leq C(n, p)y^{-\frac{1}{p-1}}, \quad \forall (\tilde{x}, y) \in \mathbb{R}^{n-1} \times (0, \infty). \quad (13)$$

and therefore

$$|u(\tilde{x}, \eta) - u(\tilde{x}, y)| \leq C \int_{\eta}^{y} t^{-\frac{1}{p-1}} dt. \quad (14)$$

Since $p \in (1, 2)$, then $1 - p > 1$ and therefore $u(\tilde{x}, y)$ has a finite limit as $y \to \infty$. By (13) and using the mean value theorem, the limit does not depend on $\tilde{x}$, and then we can define $l := \lim_{y \to \infty} u(\tilde{x}, y)$, and by (14) we have that

$$l - Cy^{\frac{2-p}{p-1}} \leq u(\tilde{x}, y) \leq l + Cy^{\frac{2-p}{p-1}}, \quad \forall (\tilde{x}, y) \in \mathbb{R}^{n}_{+}. \quad (15)$$

Now we want to prove that $u(\tilde{x}, y) = v_{l}(y)$, where $v_{l}$ is the unique solution of the one-dimensional problem

$$
\begin{cases}
\Lambda v''_{l} &= |v'_{l}|^{p} \text{ in } (0, \infty), \\
v_{l}(0) &= M, \\
\lim_{y \to \infty} v_{l}(y) &= l.
\end{cases}
(16)
$$

We use here the constant $\Lambda$. If the operator is $\mathcal{M}_{\Lambda, \Lambda}$, this constant is replaced by $\lambda$. To prove that $u \leq v_{l}$, for $t \in (0, 1)$ and $C_{R} \in \mathbb{R}$, we consider the problem

$$
\begin{cases}
-M_{\lambda}^{+}(D^{2} \psi_{t,R}) + \sqrt{1-t^{2}}|D\psi_{t,R}|^{p} + C_{R} = 0 & \text{in } B^{n-1}_{R} \subset \mathbb{R}^{n-1}, \\
\psi_{t,R}(0) = 0, & \lim_{|\tilde{x}| \to 0^{+}} \psi_{t,R}(\tilde{x}) = \infty.
\end{cases}
(17)
$$

We know by [20] that there exists a unique constant $C_{R}$ such that the problem (17) has a solution $\psi_{t,R}$, and by [9], this solution is also unique. Also by [20] we know that $C_{R} > 0$ and that $\psi_{t,R}$ achieves its minimum in zero, and writing $C_{1}$ and $\psi_{t,1}$ the solutions of (17) in $B^{n-1}_{1}$, we have

$$
C_{R} = R^{-\frac{p}{p-1}}C_{1}, \quad \psi_{t,R} = R^{-\frac{2-p}{p-1}}\psi_{t,1} \left( \frac{|\tilde{x}|}{R} \right). \quad (18)
$$

Consider now, for $L \in \mathbb{R}_{+}$, the unique solution $\varphi_{t,L,R}$ of the problem

$$
\begin{cases}
-\Lambda \varphi'' + t|\varphi'|^{p} = \frac{\sqrt{1-t^{2}}}{t}C_{R}, & \text{in } (0, L), \\
\varphi(0) = \frac{M}{t}, \quad \varphi(L) = \frac{1}{t} \left( l + CL^{-\frac{2-p}{p-1}} \right).
\end{cases}
(19)
$$

Define now the function $\overline{u}(\tilde{x}, y) = t\varphi_{t,L,R}(y) + \sqrt{1-t^{2}}\psi_{t,R}(\tilde{x})$, we want to see that $\overline{u}$ is a supersolution of (3) in the cylinder $B^{n-1}_{R} \times (0, L)$. Indeed, using
Eqs. (17) and (19) we have
\[-\mathcal{M}_{\Lambda,\lambda}(D^2\pi) + |D\pi|^p\]
\[= t \left( \sqrt{1 - t^2} C_R - t|\varphi'_{t,L,R}|^p \right) + \sqrt{1 - t^2} \left( -\sqrt{1 - t^2}|D\psi_{t,R}|^p - C_R \right) + \left( t^2|\varphi'_{t,L,R}|^2 + (1 - t^2)|D\psi_{t,R}|^2 \right)^{\frac{p}{2}},\]
and using the concavity of the function $s^\frac{p}{2}$, since $1 < p < 2$, we have
\[\left( t^2|\varphi'_{t,L,R}|^2 + (1 - t^2)|D\psi_{t,R}|^2 \right)^{\frac{p}{2}} \geq t^2|\varphi'_{t,L,R}|^p + (1 - t^2)|D\psi_{t,R}|^p,\]
therefore
\[-\mathcal{M}_{\Lambda,\lambda}(D^2\pi) + |D\pi|^p \geq 0.\]
By (15), and since the function $\psi_{t,R}$ blows up at the boundary, we have that
\[u(\bar{x}, y) \leq u(\bar{x}, y) \text{ at the boundary of the cylinder, and using comparison as in section V.1. in [23], we have that}\]
\[u(\bar{x}, y) \leq t\varphi_{t,L,R}(y) + \sqrt{1 - t^2}\psi_{t,R}(\bar{x}), \quad \forall(\bar{x}, y) \in B_{R}^{n-1} \times (0, L).\]
Therefore, at the origin we have $u(0, y) \leq t\varphi_{t,L,R}(y)$, and translating the origin in the $x-$axis we have that
\[u(\bar{x}, y) \leq t\varphi_{t,L}(y), \quad \forall(\bar{x}, y) \in \mathbb{R}^n_{+}.\]
Letting $R \to \infty$, by (18) we see that $C_R$ tends to 0, therefore
\[u(\bar{x}, y) \leq t\varphi_{t,L}(y), \quad \forall(\bar{x}, y) \in \mathbb{R}^n_{+}.\]
(20)
Now, letting $L$ goes to infinity, the function $\varphi_{t,L}$ converges to the solution of the problem
\[
\begin{align*}
-\Lambda \varphi''_t + t|\varphi'|^p &= 0, \quad \text{in } (0, \infty), \\
\varphi_t(0) &= \frac{M}{t}, \quad \lim_{y \to \infty} \varphi_t(y) = \frac{l}{t}.
\end{align*}
\]
Using (20), and $L$ tending to infinity, we have that $u(\bar{x}, y) \leq t\varphi_t(y)$ for any $t \in (0, 1)$, and letting $t$ tends to 1, the function $\varphi_t$ converges to the function $v_l(y)$ defined in (16). Then we have the inequality
\[u(\bar{x}, y) \leq v_l(y).\]
(21)
Now we want to prove the inequality $u(\bar{x}, y) \geq v_l(y)$. Let $a \geq 0$, consider the functions $\nu = v_{a,R,S}(\rho)$, radial solutions of
\[
\begin{align*}
-\mathcal{M}_{\Lambda,\lambda}(D^2\nu) + |D\nu|^p &= 0, \quad \text{in } B_{R+S}(0) \setminus B_R(0), \\
v(R) &= M, \quad v(R + S) = a,
\end{align*}
\]
and for $x = (\bar{x}, y)$ fixed with $y \in (0, S)$, the sequence $\{v_{a,R,S}(x - x_R)\}_R = \{v_{a,R,S}(0, y + R)\}_R$, with $x_R = (\bar{x}, -R)$. Letting $R$ tends to infinity, we can
see using Lemma 3.1 in [17] that this sequence of radial solutions converges to the unique one-dimensional solution \( v_{a,S}(y) \) of the problem

\[
\begin{aligned}
\Lambda v''_{a,S} &= |v'_{a,S}|^p \text{ in } (0,S), \\
v_{a,S}(0) &= M, \quad v_{a,S}(S) = a.
\end{aligned}
\]

Now as \( S \) goes to infinity we can see that \( v_{a,S} \) converges to the function \( v_a(y) \), which is the unique solution of

\[
\begin{aligned}
\Lambda v''_a &= |v'_a|^p \text{ in } (0,\infty), \\
v_a(0) &= M, \quad \lim_{y \to \infty} v_a(y) = a.
\end{aligned}
\]

Since \( u(\tilde{x},\cdot) \geq 0 \) for all \( \tilde{x} \in \mathbb{R}^{n-1} \) and \( \lim_{y \to \infty} v_0(y) = 0 \), using comparison again as in [23], we have that \( u \geq v_0(y) \), and by (15):

\[
u(\tilde{x}, y) \geq a_1 := \min \left\{ \max \left\{ v_0(y), l - Cy^{\frac{2-p}{p-1}} \right\} \right\}.
\]

Now, if we know that \( u(\tilde{x}, y) \geq a \) for all \( (\tilde{x}, y) \in \mathbb{R}^n_+ \), as above, using comparison we deduce that \( u(\tilde{x}, y) \geq v_{a,R,S}(x-x_R) \), and letting \( R \) and \( S \) tend to infinity, we have that \( u(\tilde{x}, y) \geq v_a(y) \). Applying this to \( a_1 \) we deduce that \( u(\tilde{x}, y) \geq v_{a_1}(y) \) and then

\[
u(\tilde{x}, y) \geq a_2 := \min \left\{ \max \left\{ v_{a_1}(y), l - Cy^{\frac{2-p}{p-1}} \right\} \right\}.
\]

Then, iterating the process, we get the positive real sequence \( \{a_n\} \) and the functions sequence \( \{v_{a_n}(y)\} \) such that

\[
u \geq v_{a_n}(y), \quad \text{with } a_n = \min \left\{ \max \left\{ v_{a_{n-1}}(y), l - Cy^{\frac{2-p}{p-1}} \right\} \right\}.
\]

Now we letting \( n \) goes to infinity, and we have that \( a_n \to l \) and \( v_{a_n} \) converges to \( v_l(y) \), so we have the inequality

\[
u(\tilde{x}, y) \geq v_l(y).
\]

With this and (21) we conclude that \( u(\tilde{x}, y) = v_l(y) \).

For the case \( p = 2 \), with a simple ODE analysis we have that the only non-negative solution of the equation \( \Lambda v'' = |v'|^2 \) is \( v = M \). We also note that, for \( p \in (1,2) \), the solution of Eq. (16) (that is \( u \)) is given by

\[
u(\tilde{x}, y) = l + \int_y^{\infty} \left( \left( \frac{p-1}{\Lambda} \right) (t+C_{M,l}) \right)^{-\frac{1}{p-1}} dt,
\]

where \( C_{M,l} \) is determined by the relation

\[
\int_0^{\infty} \left( \left( \frac{p-1}{\Lambda} \right) (t+C_{M,l}) \right)^{-\frac{1}{p-1}} dt = M - l.
\]

\( \square \)

**Proof of Theorem 1.2.** As in Theorem 1.1, we have that, using Theorem 4.1 applied to the function \(-u\) in order to have (12), \( u \) is a classical solution of equation (4). The proof will be given for equation \(-M_{\pm,\Lambda}(D^2 u) - |Du|^p = 0\). For \(-M_{\pm,\Lambda}(D^2 u) + |Du|^p = 0\) we just change \( u \) for \(-u\).
Let now \( x = (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty) \) and \( h \in \mathbb{R}^{n-1} \setminus \{0\} \) fixed. We define the function
\[
v(\tilde{x}, y) = u(\tilde{x} + h, y) - u(\tilde{x}, y), \quad (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty).
\]
It suffices to prove that \( v \equiv 0 \). We assume for contradiction that \( \sigma := \sup_{\mathbb{R}^{n-1}_+} v > 0 \) (the case with \( \inf \) is analogous). Using now that \( u \) is a classical solution, we have the same Bernstein estimation as in (13), therefore
\[
|v(\tilde{x}, y)| \leq C(n, p) |h| y^{-\frac{1}{p-1}}, \quad (\tilde{x}, y) \in \mathbb{R}^{n-1} \times (0, \infty).
\]
It follows that \( |v| \leq \frac{\sigma}{2} \) for \( y \geq A \) large. Then
\[
\sigma = \sup_{\mathbb{R}^{n-1} \times (0, A)} v.
\]
On the other hand, we can write \( u_h = u(\tilde{x} + h, y) \) and if we define
\[
F(s) = |sDu_h + (1 - s)Du|^p,
\]
with \( s \in \mathbb{R} \), we have
\[
F'(s) = p|sDu_h + (1 - s)Du|^{p-2}(sDu_h + (1 - s)Du) \cdot Dv.
\]
Writing \( G(\xi) = p|\xi|^{p-2}\xi \), it follows that
\[
|Du_h|^p - |Du|^p = F(1) - F(0) = \int_0^1 F'(s) ds,
\]
and then
\[
-\mathcal{M}^{\pm}_{\lambda, \Lambda}(D^2 u_h) + \mathcal{M}^{\pm}_{\lambda, \Lambda}(D^2 u) = a(\tilde{x}, y) \cdot Dv,
\]
with \( a(\tilde{x}, y) = \int_0^1 G(sDu_h + (1 - s)Du) ds \). Using now (10) we have that
\[
-\mathcal{M}^{\pm}_{\lambda, \Lambda}(D^2 u_h) + \mathcal{M}^{\pm}_{\lambda, \Lambda}(D^2 u) \geq -\mathcal{M}^{\pm}_{\lambda, \Lambda}(D^2 v),
\]
and then \( v \) satisfies
\[
-\mathcal{M}^{\pm}_{\lambda, \Lambda}(D^2 v) \leq a(\tilde{x}, y) \cdot Dv, \quad \text{in} \ \mathbb{R}^{n}_+.
\]
By Theorem 4.1, the function \( a \) is bounded in compact subsets of \( \mathbb{R}^{n}_+ \), recalling that now \( u \) is a classical solution of (4). Therefore, by the strong maximum principle, the solution \( v \) of (25) cannot achieve any local maximum in \( \mathbb{R}^{n}_+ \), and the function \( v \) is not constant either by the fact that \( v(\tilde{x}, 0) = 0 \) and (23). Therefore, \( \sigma \) in (24) is not attained and there exists a sequence \( (\tilde{x}_j, y_j) \in \mathbb{R}^{n-1} \times (0, A) \) with \( |x_j| \to \infty \) such that
\[
v(\tilde{x}_j, y_j) \to \sigma.
\]
We define the sequence
\[
u_j(\tilde{x}, y) = u(\tilde{x}_j + \tilde{x}, y), \quad (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty),
\]
and it follows that
\[
\sup_{(\tilde{x}, y) \in \mathbb{R}_+^n} (u_j(\tilde{x} + h, y) - u_j(\tilde{x}, y)) = \sup_{(\tilde{x}, y) \in \mathbb{R}_+^n} (u(\tilde{x}_j + \tilde{x} + h, y) - u(\tilde{x}_j + \tilde{x}, y)) = \sup_{\mathbb{R}_+^n} v = \sigma, \tag{26}
\]
and
\[
u_j(h, y_j) - u_j(0, y_j) = v(\tilde{x}_j, y_j) \to \sigma. \tag{27}
\]

Since \(u_j\) is a solution of (1) \(\forall j\), by Theorem 4.1 we have
\[
|Du_j(\tilde{x}, y)| \leq C(n, p) y^{\frac{p-2}{p-1}}, \quad \forall (\tilde{x}, y) \in \mathbb{R}_+^{n-1} \times (0, \infty), \tag{28}
\]
and since \(u_j(\tilde{x}, 0) = 0\), integrating in the \(y\) direction, it follows that for all \(j\):
\[
|u_j(\tilde{x}, y)| \leq C(n, p) y^{\frac{p-2}{p-1}}, \quad \forall (\tilde{x}, y) \in \mathbb{R}_+^{n-1} \times [0, \infty). \tag{29}
\]

As before, by Theorem 1.2 in [10], recalling that the functions \(u_j\) are classical solutions of (4), we have together with (28) that \(u_j \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}_+^n)\) for some \(\alpha \in (0, 1)\), then as a consequence of Arzela-Ascoli theorem we have that \(\{u_j\}\) has a subsequence which converges uniformly in compacts sets to a classical solution \(U\) of (4). Also, as a consequence of (29) we have \(U \in C(\mathbb{R}_+^n)\) and \(U(\tilde{x}, 0) = 0\), and by (24) we also have \(y_j \to y_\infty \in [0, A]\), then, by (27) we get
\[
U(h, y_\infty) - U(0, y_\infty) = \sigma, \tag{30}
\]
which implies \(y_\infty > 0\).

Let now
\[
V(\tilde{x}, y) = U(\tilde{x} + h, y) - U(\tilde{x}, y), \quad (\tilde{x}, y) \in \mathbb{R}_+^{n-1} \times [0, \infty).
\]

By (26) and (30) we have \(\sigma = \sup_{\mathbb{R}_+^n} V = V(0, y_\infty)\), but \(V\) satisfies
\[
-\mathcal{M}^\pm_{\bar{\lambda}, \bar{\Lambda}}(D^2 V) \leq A(\tilde{x}, y) \cdot DV,
\]
where \(A(\tilde{x}, y) = \int_0^1 G(sDU(\tilde{x} + h, y) + (1 - s)DU(\tilde{x}, y)) ds\) is bounded on compact subsets of \(\mathbb{R}_+^n\). Finally, this contradicts the strong maximum principle. \(\square\)

As we present in Sect. 1, as an application to their result, Filipucci et al. [22] proved an improvement of the Bernstein estimation in [27]. The result establishes a more precise Bernstein constant when the solution \(u\) is zero at the boundary of \(\Omega\), which is also valid for an \(L^p\)-viscosity solution of the Pucci equation (7) in the case \(p > 2\). As in the beginning of the proof of Theorem 1.1, considering \(\bar{f}(x) = |Du|^p + f(x)\), it can be proved that the solution \(u\) is classical.

Theorem 1.3 is an application to the Bernstein estimation in Theorem 4.1 and the Liouville-type result from Theorem 1.2. In the proof we follow the ideas from [22].
Proof of the Theorem 1.3. We will use, as mentioned above, that \( u \) is a classical solution. Let \( c_p = \left( \frac{\lambda}{p-1} \right)^{1/p} \), and assume for contradiction that there exist \( c > c_p \) and sequences \( \{f_j\}, \{u_j\} \) and \( \{x_j\} \), \( f_j \) Lipschitz in \( \Omega \) with \( \|f_j\|_{W^{1,\infty}} \leq M \), such that

\[
\begin{align*}
-M_{\lambda,\Lambda}^\pm(D^2u_j) &= |Du_j|^p + f_j(x), & \text{in } \Omega, \\
d_{\partial\Omega}(x_j) &\to 0, \\
-d_{\partial\Omega}^{-1}(x_j)|Du_j(x_j)| &\geq c.
\end{align*}
\]  

(31)

Let \( P(x) \) the projection of \( x \) onto \( \partial\Omega \), and

\[
z_j := P(x_j), \quad \alpha_j := d_{\partial\Omega}(x_j) = |x_j - z_j|.
\]

By extracting a subsequence if necessary, we assume that \( z_j \to a \in \partial\Omega \), and without loss of generality we may assume that \( a = 0 \), therefore \( x_j \to 0 \) and we have that

\[
\nu_j := \nu(z_j) \to e_n.
\]  

(32)

Let now \( v_j(y) = \alpha_j^{\frac{2-p}{p-1}}u_j(z_j + \alpha_jy) \), we have that

\[
\begin{align*}
Du_j(y) &= \alpha_j^{\frac{p}{p-1}}Du_j(z_j + \alpha_jy), \\
M_{\lambda,\Lambda}^\pm(D^2v_j(y)) &= \alpha_j^{\frac{p}{p-1}}M_{\lambda,\Lambda}^\pm(D^2u_j(z_j + \alpha_jy)).
\end{align*}
\]

If we define \( \Omega_j := \alpha_j^{-1}(\Omega - z_j) \), we have that the function \( v_j \) satisfies the equation

\[
M_{\lambda,\Lambda}^\pm(D^2v_j) + |Dv_j|^p = \alpha_j^{\frac{p}{p-1}}[M_{\lambda,\Lambda}^\pm(D^2u_j) + |Du_j|^p](z_j + \alpha_jy), \quad \text{in } \Omega_j,
\]

therefore

\[
-M_{\lambda,\Lambda}^\pm(D^2v_j) = |Dv_j|^p + \tilde{f}_j(y), \quad \text{in } \Omega_j,
\]

where \( \tilde{f}_j(y) = \alpha_j^{\frac{p}{p-1}}f_j(z_j + \alpha_jy) \). We note that \( \Omega_j \) converges to the half-space \( \mathbb{R}_+^n \) as \( j \to \infty \).

Now, using Bernstein estimation from Theorem 4.1 it follows that

\[
|Du_j(x)| \leq Cd_{\partial\Omega}^{-\frac{1}{p-1}}(x) \quad \text{in } \Omega,
\]

and

\[
|u_j(x)| \leq Cd_{\partial\Omega}^{\frac{p-2}{p-1}}(x) \quad \text{in } \Omega,
\]

with \( C = C(n, p, M) > 0 \), independent of \( j \). Now we need to find a uniform bound for \( v_j \) and \( Du_j \). Letting

\[
Q_{R,\epsilon} = B_R \cap \{y_n > \epsilon\},
\]

as in the proof of Proposition 3.1 in [22], it can be proved that \( Q_{R,\epsilon} \subset \Omega_j \) for all \( j \geq j_0 \), where \( j_0 \) depends on \( R \) and \( \epsilon \), and

\[
\frac{1}{2}y_n \leq d_{\partial\Omega_j}(y) \leq 3y_n, \quad \forall y \in Q_{R,\epsilon}, \forall j \geq j_0.
\]  

(33)
Using now the Bernstein estimation from Theorem 4.1, we have that, for all \( j \geq j_0 \) and \( y \in Q_{R,\epsilon} \),
\[
|Dv_j(y)| = \alpha_j^{\frac{1}{p-1}} |Du_j(z_j + \alpha_j y)| \leq C \alpha_j^{\frac{1}{p-1}} d_{\partial \Omega}^{-\frac{1}{p-1}} (z_j + \alpha_j y),
\]
and since \( d_{\partial \Omega}(z_j + \alpha_j y) = \alpha_j d_{\partial \Omega}(y) \), using (33), we have
\[
|Dv_j(y)| \leq C 2^{\frac{1}{p-1}} y_n^{\frac{1}{p-1}}. \tag{34}
\]
Similarly, we have the estimation
\[
|v_j(y)| \leq C 3^{\frac{p}{p-2}} y_n^{\frac{p-2}{p-1}}. \tag{35}
\]
Using now interior elliptic \( L^q \) estimates, we have that the sequence \( \{v_j\}_j \) is precompact in \( W^{2,q}(Q_{R,\epsilon}) \), and using a diagonal procedure, we deduce that some subsequence of \( (v_j)_j \) converges in \( W^{2,q}(Q_{R,\epsilon}) \) for each \( q, R, \epsilon > 0 \), to a strong solution \( V(y) \in C^2(\mathbb{R}_+^n) \cap C(\mathbb{R}_+^n) \) of
\[
\begin{cases}
-\mathcal{L}_{\lambda, \Lambda}^+(D^2V) = |DV|^p, & y \in \mathbb{R}_+^n, \\
V(y) = 0, & y \in \partial \mathbb{R}_+^n.
\end{cases}
\]
It follows from Theorem 1.2 that either \( V = 0 \) or \( V(y) = U_{\hat{c}}(y_n) \) for some \( \hat{c} \geq 0 \), and by (11) we have
\[
|DV(y)| = c_p (y_n + \hat{c})^{-\frac{1}{p-1}} \leq c_p y_n^{-\frac{1}{p-1}}.
\]
Finally, with this fact and using (32):
\[
\lim_{j \to \infty} d_{\partial \Omega}^{-\frac{1}{p-1}} |Du_j(x_j)| = \lim_{j \to \infty} |Dv_j \left( \frac{x_j - z_j}{\alpha_j} \right)| = \lim_{j \to \infty} |Dv_j(v_j)| = |DV(e_n)| \leq c_p,
\]
which is a contradiction with (31), being \( c > c_p \). \( \square \)

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4. Appendix

In this section we prove Lipschitz regularity for any continuous \( L^q \)-viscosity solution of the equation (5). Here, we assume that there exist positive constants \( \gamma_0 \) and \( \gamma_1 \) such that, for all \( \xi, \eta \in \mathbb{R}^n \),
\[
\begin{align*}
g(\xi) & \geq \gamma_0 |\xi|^p, & p > 1, \\
|g(\xi) - g(\eta)| & \leq \gamma_1 (|\xi|^{p-1} + |\eta|^{p-1}) |\xi - \eta|.
\end{align*}
\]
Theorem 4.1. Let $\Omega \subset \mathbb{R}^n$ open and bounded with Lipschitz boundary and satisfy the uniform interior sphere condition. Let $p > 1$ and $u \in C(\Omega)$ a bounded $L^q$-viscosity solution of (5) in $\Omega$, with $n \leq q$. Assume that (36) hold true, and that the function $f$ is Lipschitz and bounded in $\Omega$ (hence $d_{\partial \Omega}^p f$ is also bounded in $\Omega$, where $d_{\partial \Omega}(x) = \text{dist}(x, \partial \Omega)$). Then $u$ is locally Lipschitz continuous and there exists a constant $C > 0$, depending on $p$ and $\|d_{\partial \Omega}^p f\|_{L^\infty(\Omega)}$, such that

$$|Du(x)| \leq \frac{C}{d_{\partial \Omega}(x)^{p-1}} \text{ for a.e. } x \in \Omega.$$ 

For the proof of Theorem 4.1, we present a simplification of the work done in [8], following the ideas from [13]. The work of Capuzzo Dolcetta et al. [13] (specifically, Sect. 3) is for the equation

$$-\text{tr}(A(x)D^2u) + \alpha u + H(x, Du) = 0, \quad \text{in } \Omega,$$

where $A$ is a bounded and Lipschitz continuous map, and $\alpha \geq 0$. As in [13], the assumption of non-degeneracy is not required for Theorem 4.1, considering that the proof is valid in the case $\lambda = 0$. The technique consists of a variable bending, using a $C^2$-test function depending on the distance to the boundary, with the calculation of the Pucci operators from derivatives of the test function, which are non-linear operators.

In the first case we are going to prove the result in $B(0, 1)$ with exponent $\frac{1}{p-1}$ instead of $\frac{1}{p-1}$ as in the theorem. Then, when rescaling to a domain $\Omega$, we will improve the Bernstein bound.

Proof of Theorem 4.1. First case: $\Omega = B(0, 1) = B$. We choose a smooth monotone radial function $d \in C^2(\overline{B})$ satisfying the properties:

$$d(x) = d_{\partial B}(x) \text{ if } d_{\partial B} \leq \frac{1}{2},$$

$$\frac{d_{\partial B}(x)}{2} \leq d(x) \leq d_{\partial B}(x) \quad \forall x \in \overline{B},$$

$$|Dd(x)| \leq 1, D^2d(x) \leq 0 \quad \forall x \in \overline{B}.$$

Now consider the function

$$\Phi(x, y) = k|x - y|\varphi(x, y),$$

where

$$\varphi(x, y) = \frac{1}{d(y)^\gamma} \left[ L + \left( \frac{|x - y|}{d(x)} \right)^\beta \right]. \quad (37)$$

The constants $\gamma, \beta$ and $L$ are positive and will be fixed later. We want to prove that, choosing $k >> 1$ we have

$$u(x) - u(y) - \Phi(x, y) \leq 0 \quad \forall x, y \in B.$$

Suppose by contradiction, that $w(x, y) = u(x) - u(y) - \Phi(x, y) > 0$ at some point $(x, y) \in B \times B$, this says that $w$ has a maximum point at $(x, y) \in B \times B$, with $x \neq y$. Observing that the function $\Phi$ is $C^2(B)$, from [14], specifically
**Theorem 3.2**, we have that for every $\epsilon > 0$ there exist matrices $X = X(\epsilon)$, $Y(\epsilon) \in S_n$ such that

$$
(D_x \Phi(x,y), X) \in \mathcal{T}^{2,+} u(x)_B, (-D_y \Phi(x,y), Y) \in \mathcal{T}^{2,-}_B u(y),
$$

$$
- \left( \frac{1}{\epsilon} + \|D^2 \Phi(x,y)\| \right) I_{2n} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \epsilon (D^2 \Phi(x,y))(38)
$$

where $\mathcal{T}^{2,\pm}$ denotes the closure of the second order super(sub)-jet. Using the definition from sub and super-jet in [14] we have:

$$
- M_{\alpha \lambda}^\pm (X) + g(D_x \Phi) \leq f(x),
$$

$$
- M_{\alpha \lambda}^\pm (Y) + g(-D_y \Phi) \geq f(y),
$$

and multiplying the first equation by $(1+t)$, for fixed $t > 0$:

$$
M_{\alpha \lambda}^\pm (Y) - (1 + t)M_{\alpha \lambda}^\pm (X)
+ (1 + t)g(D_x \Phi) - g(-D_y \Phi) \leq (1 + t)f(x) - f(y).
$$

Using (10) and the fact that $M_{\alpha \lambda}^\pm (M) = -M_{\alpha \lambda}^\pm (-M)$ we have

$$
- M_{\alpha \lambda}^\pm ((1 + t)X - Y) + (1 + t)g(D_x \Phi) - g(-D_y \Phi) \leq (1 + t)f(x) - f(y),
$$

and using (36):

$$
\gamma_0 t \left| D_x \Phi \right|^p \leq M_{\alpha \lambda}^\pm ((1 + t)X - Y)
+ g(-D_y \Phi) - g(D_x \Phi) + (1 + t)f(x) - f(y).
$$

In the calculation of derivatives of $\Phi$, we have:

$$
D_x \Phi = \frac{k}{d(y)^\gamma} \left[ \left( L + (1 + \beta) \left( \frac{|x - y|}{d(x)} \right)^\beta \right) \left( \frac{x - y}{d(x)} \right) - \beta \left( \frac{|x - y|}{d(x)} \right)^{\beta+1} Dd(x) \right].
$$

Denoting $\xi = \frac{|x - y|}{d(x)}$, and using the notation $\hat{v} = \frac{v}{|v|}$ we have the estimation

$$
\left| D_x \Phi \right|^2 \geq \frac{k^2}{d(y)^{2\gamma}} \left[ \frac{1}{2} (L + (1 + \beta)\xi^\beta)^2 + \beta^2 |Dd(x)|^2 \xi^{2(\beta+1)} - 2\beta^2 \xi^{2\beta} \right].
$$

Since $|Dd(x)| = 1$ if $d(x) \leq \frac{1}{2}$ and $\xi = \frac{|x - y|}{d(x)} \leq 4$ if $d(x) \geq \frac{1}{2}$, in both cases there exists $c > 0$, depending only on $\beta$, such that:

$$
\left| D_x \Phi \right|^2 \geq \frac{k^2}{d(y)^{2\gamma}} \left[ \frac{1}{2} (L + (1 + \beta)\xi^\beta)^2 + \frac{\beta^2}{2} \xi^{2(\beta+1)} - c \right].
$$

We choose $L$ sufficiently large such that

$$
\left| D_x \Phi \right|^2 \geq c \frac{k^2}{d(y)^{2\gamma}} \left[ (L + (1 + \beta)\xi^\beta)^2 + \beta^2 \xi^{2(\beta+1)} \right],
$$

and therefore

$$
\left| D_x \Phi \right| \geq c \frac{k}{d(y)^\gamma} (L + \xi^\beta)(1 + \xi) = ck\varphi(1 + \xi).
$$

\[(44)\]
Now, let $A \in A_{\lambda, \Lambda}$ and the non negative matrix

$$A_t = \begin{pmatrix} (1 + t)A & \sqrt{1 + t}A \\ \sqrt{1 + t}A & A \end{pmatrix}.$$  \hfill (45)

Multiplying the right inequality in (38) by $A_t$, taking traces and then taking supremum over $A \in A_{\lambda, \Lambda}$:

$$\mathcal{M}^{+}_{\lambda, \Lambda}((1 + t)X - Y) \leq \sup_{A \in A_{\lambda, \Lambda}} \text{tr}(A_t D^2 \Phi) + \epsilon \sup_{A \in A_{\lambda, \Lambda}} \text{tr}(A_t (D^2 \Phi)^2),$$

and from (42) we have:

$$t|D_x \Phi|^p \leq \sup_{A \in A_{\lambda, \Lambda}} \text{tr}(A_t D^2 \Phi) + \epsilon \sup_{A \in A_{\lambda, \Lambda}} \text{tr}(A_t (D^2 \Phi)^2) + |D_y \Phi|^p - |D_x \Phi|^p.$$ \hfill (46)

Now letting $\epsilon$ tend to 0:

$$t|D_x \Phi|^p \leq \sup_{A \in A_{\lambda, \Lambda}} \text{tr}(A_t D^2 \Phi) + |D_y \Phi|^p - |D_x \Phi|^p.$$ \hfill (47)

In calculating $D^2 \Phi$, multiplying by $A_t$ and taking traces we get the expression

$$\text{tr}(A_t D^2 \Phi(x, y)) = \frac{k}{d(y)^\gamma} \left\{ \left( L + (1 + \beta)\xi^\beta \right) (2 - 2\sqrt{1 + t} + t) \mathcal{M}^{+}_{\lambda, \Lambda}(B) \\
+ \frac{\beta(1 + \beta)\xi^\beta}{|x - y|} (2 - 2\sqrt{1 + t} + t) \mathcal{M}^{+}_{\lambda, \Lambda}(T) \\
+ \frac{\beta(1 + \beta)\xi^\beta}{d(x)} \sqrt{1 + t} \mathcal{M}^{+}_{\lambda, \Lambda}(-\sqrt{1 + t}C - (\sqrt{1 + t} - 2)C^T) \\
+ \frac{\gamma(L + (1 + \beta)\xi^\beta)}{d(y)} (\sqrt{1 + t} - 1) \mathcal{M}^{+}_{\lambda, \Lambda}(-D - D^T) \\
+ \frac{\beta(1 + \beta)\xi^\beta + 1}{d(x)} (1 + t) \mathcal{M}^{+}_{\lambda, \Lambda}(Dd(x) \otimes Dd(x)) \\
+ \frac{\gamma(\gamma + 1)|x - y|}{d(y)^2} (L + \xi^\beta) \mathcal{M}^{+}_{\lambda, \Lambda}(Dd(y) \otimes Dd(y)) \\
+ \frac{\beta\gamma\xi^\beta + 1}{d(y)} \sqrt{1 + t} \left[ \mathcal{M}^{+}_{\lambda, \Lambda}(Dd(x) \otimes Dd(y)) + \mathcal{M}^{+}_{\lambda, \Lambda}(Dd(y) \otimes Dd(x)) \right] \\
+ \beta\xi^\beta + 1 (1 + t) \mathcal{M}^{+}_{\lambda, \Lambda}(-D^2 d(x)) + \frac{\gamma|x - y|}{d(y)} (L + \xi^\beta) \mathcal{M}^{+}_{\lambda, \Lambda}(D^2 d(y)) \right\},$$

where $B, T, C$ and $D$ are the matrices defined by $B = I - \frac{x - y \otimes x - y}{d(y)}$, $T = \frac{x - y \otimes x - y}{d(x)}$, $C = \frac{x - y \otimes Dd(x)}{d(y)}$ and $D = \frac{x - y \otimes Dd(y)}{d(y)}$, with $\otimes$ the Kronecker product between two vectors. Using the fact that for any vectors $v_1, v_2 \in \mathbb{R}^n$ the only non trivial eigenvalue from the matrix $v_1 \otimes v_2$ is $v_1 \cdot v_2$ and property in (10), the next estimation is valid:

$$\mathcal{M}^{+}_{\lambda, \Lambda}(-\sqrt{1 + t}C - (\sqrt{1 + t} - 2)C^T) = -2(\sqrt{1 + t} - 1)\mathcal{M}^{+}_{\lambda, \Lambda}(C),$$
and since $\mathcal{M}_{\lambda,\Lambda}(C) = \mathcal{M}_{\lambda,\Lambda}(x - y \otimes Dd(x)) = C_{\lambda,\Lambda}(x - y) \cdot Dd(x)$, we have

$$\mathcal{M}_{\lambda,\Lambda}^-(-\sqrt{1 + t}C - (\sqrt{1 + t} - 1)C^T) \leq -2C_{\lambda,\Lambda}(\sqrt{1 + t} - 1)(x - y) \cdot Dd(x).$$

On the other hand:

$$\mathcal{M}_{\lambda,\Lambda}^+(D - D^T) \leq -\left[\mathcal{M}_{\lambda,\Lambda}^-(D) + \mathcal{M}_{\lambda,\Lambda}^-(D^T)\right] = -2\mathcal{M}_{\lambda,\Lambda}(D),$$

and since $\mathcal{M}_{\lambda,\Lambda}^-(D) = \mathcal{M}_{\lambda,\Lambda}^-(x - y \otimes Dd(y)) = C_{\lambda,\Lambda}(x - y) \cdot Dd(y)$ we have

$$\mathcal{M}_{\lambda,\Lambda}^+(D - D^T) \leq -2C_{\lambda,\Lambda}(x - y) \cdot Dd(y).$$

Now observe that for all $t > 0$ we have

$$\begin{align*}
(2 - 2\sqrt{1 + t} + t) \leq t^2 \leq t^2 + |x - y|^2, \\
2\sqrt{1 + t}(\sqrt{1 + t} - 1) \leq 2t \leq 2t + |x - y|,
\end{align*}$$

and we obtain then,

$$\text{tr}(A_tD^2\Phi) \leq \frac{ck}{d(y)^\gamma} \left\{ \left(\frac{L + \xi^\beta}{|x - y|}\right)(t^2 + |x - y|^2) + \left(\frac{\xi^\beta}{d(x)} + \frac{1 + \xi^\beta}{d(y)}\right)(t + |x - y|) \\
+ \xi^{\beta+1}\left(\frac{1}{d(x)} + \frac{1}{d(y)}\right)t + \left(\frac{\xi^{\beta+1}}{d(x)} + \frac{\xi^{\beta+1}}{d(y)} + \frac{(1 + \xi^\beta)|x - y|}{d(y)^2}\right) \right\}. \quad (48)
$$

We also have the estimation:

$$\frac{\xi^\beta}{d(x)} \leq 2\frac{(1 + \xi^{\beta+1})}{d(y)},$$

and with this, we obtain

$$\frac{\xi^{\beta+1}}{d(x)} + \frac{\xi^{\beta+1}}{d(y)} + \frac{(1 + \xi^\beta)|x - y|}{d(y)^2} \leq c|x - y|(1 + \xi^{\beta+2}) \frac{d(y)^2}{d(y)},$$

and

$$\left(\frac{\xi^\beta}{d(x)} + \frac{1 + \xi^\beta}{d(y)}\right)(t + |x - y|) + \xi^{\beta+1}\left(\frac{1}{d(x)} + \frac{1}{d(y)}\right)t \leq c\frac{1 + \xi^{\beta+1}}{d(y)}|x - y| + ct\frac{1 + \xi^{\beta+2}}{d(y)}.$$

Using this in (48) and by (47) we have:

$$\begin{align*}
\gamma_0 t |D_x\Phi|^p \\
\leq g(-D_y\Phi) - g(D_x\Phi) + (1 + t)f(x) - f(y) \\
+ c\frac{k}{d(y)^\gamma} \left\{ \left(\frac{L + \xi^\beta}{|x - y|}\right)(t^2 + |x - y|^2) \\
+ t\left(\frac{1 + \xi^{\beta+2}}{d(y)} + \frac{1 + \xi^{\beta+1}}{d(y)}\right)|x - y| + \frac{(1 + \xi^{\beta+2})}{d(y)^2}|x - y| \right\}.
\end{align*}$$
Considering now $\gamma \geq \frac{1}{p - 1}$ and $\beta \geq \frac{2 - p}{p - 1}$ we have $\gamma p \geq \gamma + 1$ and $(\beta + 1)p \geq \beta + 2$, and by estimation of $|D_x \Phi|^p$ in (44): \[
|D_x \Phi|^p \geq ck^p \frac{(1 + \xi^{\beta + 1})}{d(y)^\gamma p} \geq ck^p \frac{(1 + \xi^{\beta + 2})}{d(y)^{\gamma + 1}}.
\]
Since $p > 1$, taking $k^{p - 1} > 1$ we have: \[
ctk \frac{(1 + \xi^{\beta + 2})}{d(y)^{\gamma + 1}} \leq ct \frac{k^p (1 + \xi^{\beta + 2})}{d(y)^{\gamma + 1}} \leq \frac{t}{2} |D_x \Phi|^p,
\]
and therefore: \[
\frac{t}{2} |D_x \Phi|^p - c t k^2 \frac{(L + \xi^\beta)}{|x - y|d(y)^\gamma} \leq t f(x) + g(-D_y \Phi) - g(D_x \Phi) + f(x) - f(y)
+ c \frac{k |x - y| (1 + \xi^\beta) + 1 + \xi^{\beta + 1}}{d(y)^{\gamma + 1}} + 1 + \xi^{\beta + 2}.
\]
Now we choose optimal $t$ to maximize the left side: \[
t = c \frac{|D_x \Phi|^p}{k c (L + \xi^\beta) |x - y|d(y)^\gamma} = c \frac{|D_x \Phi|^p}{k} \cdot \frac{|x - y|}{\varphi}. \tag{49}
\]
Then \[
|D_x \Phi|^{2p} \frac{|x - y|}{k \varphi} \leq c \left\{ \frac{|D_x \Phi|^p}{k} \cdot \frac{|x - y|}{\varphi} \varphi + g(-D_y \Phi) - g(D_x \Phi) + f(x) - f(y)
+ k |x - y| \left[ \varphi + \frac{1 + \xi^\beta + 1}{d(y)^{\gamma + 1}} + 1 + \xi^{\beta + 2} \right] \right\},
\]
and therefore: \[
|D_x \Phi|^{2p} \leq c \left\{ \frac{|D_x \Phi|^p \varphi + k^p \varphi + k^{p + 1} \varphi (1 + \xi^{\beta + 1})}{d(y)^{\gamma p + p}} + k^2 \varphi \left[ \varphi + \frac{1 + \xi^{\beta + 2}}{d(y)^{\gamma + 2}} \right] \right\}. \tag{50}
\]
On the estimation of the term $g(-D_y \Phi) - g(D_x \Phi)$, using hypothesis in (36) we can get \[
g(-D_y \Phi) - g(D_x \Phi) \leq ck^p \frac{(1 + \xi^{\beta + 1})}{d(y)^{\gamma p + p}} |x - y|.
\]
Using this in (50), with the fact that $f$ is Lipschitz we have: \[
|D_x \Phi|^{2p} \leq c \left\{ |D_x \Phi|^{p \varphi} + k^p \varphi + k^{p + 1} \varphi (1 + \xi^{\beta + 1}){d(y)^{\gamma p + p}} + k^2 \varphi \left( \varphi + \frac{1 + \xi^{\beta + 2}}{d(y)^{\gamma + 2}} \right) \right\},
\]
and choosing now $\gamma = \frac{p}{p - 1}$, using (44), we can find the estimation: \[
k^{p + 1} \varphi \frac{(1 + \xi^{\beta + 1})}{d(y)^{\gamma p + p}} \leq \frac{c}{k^{p - 1}} |D_x \Phi|^{2p}.
\]
Also, since $2(\gamma + 1) \leq 2\gamma p$ and $\beta + 2 \leq (\beta + 1)p$ we have:

$$k^2 \varphi \frac{(1 + \xi^{\beta + 2})}{d(y)^{\gamma + 2}} \leq c \frac{|D_x \Phi|^{2p}}{k^{2(p-1)}}.$$ 

Therefore:

$$|D_x \Phi|^{2p} \leq c \left\{ |D_x \Phi|^p \|f\| + k \varphi + k^2 \varphi^2 + \frac{|D_x \Phi|^{2p}}{k^{p-1}} + \frac{|D_x \Phi|^{2p}}{k^{2(p-1)}} \right\} \leq c \left\{ |D_x \Phi|^p + |D_x \Phi| + |D_x \Phi|^2 + \frac{|D_x \Phi|^{2p}}{k^{p-1}} + \frac{|D_x \Phi|^{2p}}{k^{2(p-1)}} \right\},$$

which is a contradiction taking $k >> 1$, by (44). Therefore there is a constant $M > 0$, depending on $p$ and $f$, such that

$$u(x) - u(y) \leq M \frac{|x - y|}{d_{\partial B}(y)} \left[ 1 + \left( \frac{|x - y|}{d_{\partial B}(x)} \right)^\beta \right] \forall x, y \in B.$$ 

Changing the roles of $x$ and $y$:

$$|u(x) - u(y)| \leq M \frac{|x - y|}{(d_{\partial B}(y) \wedge d_{\partial B}(x))^{\gamma}} \left[ 1 + \left( \frac{|x - y|}{d_{\partial B}(x) \wedge d_{\partial B}(y)} \right)^\beta \right],$$

for all $x, y \in B$, so that $u$ is locally Lipschitz in $B$ and satisfies

$$|Du(x)| \leq \frac{M}{d_{\partial B}(x)^\gamma}, \text{ for a.e. } x \in B,$$

with $\gamma = \frac{p}{p-1}$.

**Second case** In order to have a Bernstein estimation of the solution in $\Omega$, we do the following rescaling. Let $x_0 \in \Omega$ a differentiability point for $u$, and let $r = \frac{d_{\partial \Omega}(x_0)}{2}$, we define

$$v(x) = r^{\frac{2-p}{p-1}} u(x_0 + rx), \ x \in B(0,1).$$

For this function we have

$$Du(x) = r^{\frac{1}{p-1}} Du(x_0 + rx),$$

and

$$-\mathcal{M}_{\lambda,\lambda}^+(D^2 v(x)) = -r^{\frac{p}{p-1}} \mathcal{M}_{\lambda,\lambda}^+(D^2 u(x_0 + rx)),$$

therefore, $v$ is a viscosity solution of the equation

$$-\mathcal{M}_{\lambda,\lambda}^+(D^2 v) + g_r(Dv) = r^{\frac{p}{p-1}} f(x_0 + rx), \ x \in B,$$

with $g_r(\xi) = r^{\frac{p}{p-1}} g \left( r^{-\frac{p}{p-1}} \xi \right)$. We note that $g_r$ satisfies (36) with the same constants. By the choice of $r$ we have $r \leq d_{\partial \Omega}(x_0 + rx) \leq 3r$, hence

$$\|r^{\frac{p}{p-1}} f\|_{L^\infty(B)} \leq \frac{\|d_{\partial \Omega}^\frac{p}{p-1} f(x_0 + rx)\|_{L^\infty(B)}}{2} \leq \frac{\|d_{\partial \Omega}^\frac{p}{p-1} f(x_0 + rx)\|_{L^\infty(B)}}{2} \leq \|d_{\partial \Omega}^\frac{p}{p-1} f\|_{L^\infty(\Omega)},$$
therefore

\[ |Dv(x)| \leq \frac{C}{d_{\partial B}(x)^{\frac{p}{p-1}}}, \]

and evaluating at \( x = 0 \):

\[ |Du(x_0)| \leq \frac{C}{r^{\frac{p}{p-1}}} = \frac{C}{d_{\partial \Omega}(x_0)^{\frac{p}{p-1}}}, \tag{51} \]

where \( C \) depends on \( p \) and \( \|d_{\partial \Omega}^{\frac{p}{p-1}} f\|_{L^\infty(\Omega)} \).

**Remark 1.** The above rescaling also applies to unbounded domain \( \mathbb{R}^n_+ \), taking \( r = \frac{x_{0,n}}{2} \), where \( x_{0,n} \) is the \( n \)-th coordinate from a point \( x_0 \).

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Rodrigo Fuentes and Alexander Quaas
Departamento de Matemática
Universidad Técnica Federico Santa María
Casilla 110-V
Valparaíso
Chile
e-mail: rfuentes.montecinos@gmail.com

Alexander Quaas
e-mail: alexander.quaas@usm.cl

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