A reanalysis of a strong-flow gyrokinetic formalism

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We reanalyse an arbitrary-wavelength gyrokinetic formalism [A. M. Dimits, Phys. Plasmas 17, 055901 (2010)], which orders only the vorticity to be small and allows strong, time-varying flows on medium and long wavelengths. We obtain a simpler gyrocentre Lagrangian up to second order. In addition, the gyrokinetic Poisson equation, derived either via variation of the system Lagrangian or explicit density calculation, is consistent with that of the weak-flow gyrokinetic formalism [T. S. Hahm, Phys. Fluids 31, 2670 (1988)] at all wavelengths in the weak flow limit. The reanalysed formalism has been numerically implemented as a particle-in-cell code. An iterative scheme is described which allows for numerical solution of this system of equations, given the implicit dependence of the Euler-Lagrange equations on the time derivative of the potential. © 2015 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution 3.0 Unported License. [http://dx.doi.org/10.1063/1.4916129]

I. INTRODUCTION

The weak-flow gyrokinetic formalism\(^1\),\(^2\) uses a gyrokinetic ordering parameter
\[
e \sim \omega / \Omega \sim v_{E \times B} / v_\text{t} \ll 1,
\] (1)
with \(\omega\) a characteristic frequency, \(\Omega\) the gyrofrequency, \(v_{E \times B}\) the \(E \times B\) drift speed, and \(v_\text{t}\) the typical thermal speed.

The ordering (1) may be poorly satisfied in the core and edge of tokamak plasmas because of either large overall rotation or relatively strong flows in the pedestal. It is also frequently broken in astrophysical plasmas. Various approaches\(^3\),\(^4\) to include stronger flows in a gyrokinetic framework have been proposed, but the most general so far\(^5\) is based on ordering the vorticity to be small,
\[
e \sim v_{E \times B}' / \Omega,
\] (2)
where \(v_{E \times B}'\) is the characteristic magnitude of the spatial derivatives of the \(E \times B\) drift velocity. This is a maximal ordering in the sense that a larger vorticity on any scale would lead to breaking of the magnetic moment invariance, as nonlinear frequencies are comparable to the vorticity. Ordering the vorticity allows for general large, time-varying flows on large length scales as well as gyroscale perturbations, and includes them within a single description, unlike schemes based on separation of scales\(^6\),\(^7\) or long-wavelength schemes.\(^4\)

However, in the weak-flow limit, the gyrokinetic Poisson equation of Ref. 5 disagrees with that of the weak-flow gyrokinetic formalism at wavelengths comparable to the gyroradius. We rederive this theory and explain some minor but important departures from the derivation of the weak-flow theory. In our reanalysis, we obtain a Poisson equation, via both a variational and direct method, that, in the weak-flow limit, agrees with the weak-flow gyrokinetic Poisson equation at all wavelengths.

II. GUIDING-CENTRE LAGRANGIAN

The particle fundamental 1-form for electrostatic perturbations in a slab uniform equilibrium magnetic field is
\[
\gamma = [A(x) + v] \cdot dx - \frac{1}{2} \left[ v^2 + \hat{\phi}(x,t) \right] dt,
\] (3)
where we use units such that \(q = T = m = v_t = 1\), \(q\) is the particle charge, \(T\) is the temperature, \(m\) is the particle mass, \(A\) is the magnetic vector potential, \(x\) is the particle position, \(v\) is the particle velocity, and \(t\) is time. We redefine \(v\) as the velocity in a frame moving with a velocity \(u(x,v,t)\) such that Eq. (3) becomes
\[
\gamma = [A(x) + v + u] \cdot dx - \frac{1}{2} \left[ (v + u)^2 + \hat{\phi} \right] dt.
\] (4)
The guiding-centre fundamental 1-form (Appendix A) is
\[
\Gamma = [A(R) + U \hat{b} + u] \cdot dR - \rho \cdot du + m d\theta
- \left( \frac{1}{2} U^2 + \mu \Omega + \frac{1}{2} u^2 + \langle \phi \rangle + \frac{1}{2} \delta_{ij} \hat{\phi} \right) dt,
\] (5)
where \(R = x - \rho\) is the guiding-centre position, \(\rho = v_j \Omega^{-1} (\cos \theta_1 - \sin \theta_2)\) is the gyroradius, \(v_j\) is the perpendicular speed, \(\theta\) is the gyroangle defined with the opposite sign to that of Ref. 5, 1 = \(\hat{b} \times \hat{b}\) is the magnetic field unit vector, \(U = \hat{b} \cdot v\) is the parallel speed, \(\hat{b}\) is the magnetic vector potential, \(\hat{b} \cdot v\) is the parallel speed, \(\mu = \frac{1}{2} v_j \Omega^{-1}\) is the magnetic moment, \(\langle \phi \rangle = \left( 2 \pi \right)^{-1} \frac{1}{2} d \phi \ldots \phi = \hat{\phi} - \langle \phi \rangle\), \(\Omega = \Omega \hat{b}\) and we have used
\[
\hat{b} \cdot u = 0
\]
and the gauge
\[
S = -\rho \cdot \left( \frac{1}{2} \rho \cdot \nabla + 1 \right) A(R) + u.
\] (6)
III. GYROCENTRE LAGRANGIAN

Using the ordering (2), magnitude of the particle position \( x \sim 1 \) and
\[
\mathbf{u} = \Omega^{-1}\dot{\mathbf{b}} \times \nabla\langle\phi\rangle,
\]
we can order the terms in the Lagrangian in terms of their variation over typical length scales as
\[
\Gamma = \Gamma_0 + \Gamma_1,
\]
with
\[
\Gamma_1 = -\rho \cdot \delta_{\dot{b}} - \delta_{\dot{\phi}} \phi dt.
\]
As in weak-flow formalisms, the lowest order Lagrangian \( \Gamma_0 \) contains terms which may be large on sufficiently long length scales. In addition to the conditions in Appendix B, \( \mathbf{u} \) must satisfy the condition
\[
(\partial_t + \mathbf{u} \cdot \nabla)\mathbf{u} \sim c^2.
\]
We use noncanonical Hamiltonian Lie-transform perturbation theory\(^8,9\) to determine a set of gyrocentre coordinates where the Lagrangian is \( \theta \)-independent. This procedure systematically removes the \( \theta \)-dependence from the Lagrangian order by order. The transformation between guiding-centre and gyrocentre space is then given in terms of a Lie transform of the form
\[
T^{\pm 1} = \exp \left( \pm \sum_{n=1}^{\infty} \epsilon^n \mathcal{L}_n \right),
\]
where \( \mathcal{L}_n = \gamma_n^{ab} \omega_{ab} dz^b \), \( \gamma_n^{ab} \) are the generators, \( a, b \in \{0, \ldots, 6\} \),
\[
\omega_{ab} = \Gamma_{b, a} - \Gamma_{a, b}
\]
are the Lagrange matrix components and \( \Gamma_{b, a} = \partial_a \Gamma_b \) (Einstein notation is used). The requirement that the first-order Lagrangian be \( \theta \)-independent, with the choice \( \gamma_0^{ab} = 0 \), yields (Appendix B) the non-zero first-order generators
\[
\gamma^{R}_1 = \Omega^{-2} \nabla\phi \times \dot{\mathbf{b}},
\gamma^{L}_1 = \Omega^{-1} \delta_{\dot{\phi}} \phi,
\gamma^{2}_1 = \rho \cdot \mathbf{u}_\mu - \Omega^{-1} \delta_{\dot{\phi}} \Phi_{\mu} = -\Omega^{-1} \Phi_{\mu} - \mathbf{u} \cdot \rho_{\mu},
\]
where \( \delta_{\dot{\phi}} \phi = \int d\theta \delta_{\dot{\phi}} \phi = \int d\theta \dot{\phi} \). Given a long wavelength flow, \( \gamma^{R}_1 \) and \( \gamma^{2}_1 \) are smaller in this strong-flow formalism than in the equivalent weak-flow formalism, reflecting the improvement in the ordering scheme for such a case. Unlike Ref. 5, we simplify the second order Lagrangian by moving the second order terms into the time component (Appendix B). The gyrocentre Lagrangian up to second order is
\[
\bar{\Gamma} = \left[ A(\dot{\mathbf{R}}) + \mathbf{U} \dot{\mathbf{b}} \right] \cdot \delta \dot{\mathbf{R}} + \mu \dot{\phi} - \left( \frac{1}{2} \dot{\mathbf{R}}^2 + \dot{\mu} \Omega + \langle \phi \rangle \right) - \frac{1}{2} \langle \gamma^{R}_1 \cdot \nabla \phi \rangle - \frac{1}{2} \Omega^{-1} \langle \dot{\phi}^2 \rangle \mu dt + \mathbf{u} \cdot (\delta \dot{\mathbf{R}} - \delta \mathbf{u} dt),
\]
where the overbar denotes a gyrocentre quantity. The last term is the only one absent from the weak-flow gyrocentre Lagrangian at this order; the main qualitative difference with the weak-flow formalism is simply the presence of the electric potential in the symplectic part of the Lagrangian.

IV. EULER-LAGRANGE EQUATIONS

Using the gyrocentre Lagrangian up to first order, the gyrocentre Euler-Lagrange equations,
\[
\ddot{\mathbf{x}}_j = \omega_{ij} \dot{L}_j = \omega_{ij},
\]
where \( i, j \in \{1, \ldots, 6\} \), yield (Appendix C)
\[
\dot{\mathbf{R}} = \mathbf{U} + \Omega^{-1} \mathbf{b} \times (\partial_t + \mathbf{u} \cdot \nabla + \mathbf{U} \nabla) \dot{\mathbf{b}} + \mathbf{U} \dot{\mathbf{b}},
\dot{\mathbf{U}} = -\langle \phi \rangle \partial_z + \Omega^{-1} \mathbf{b} \cdot \mathbf{u} \cdot (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u},
\dot{\mu} = 0,
\dot{\phi} = \Omega - \langle \phi \rangle \mu - \Omega^{-1} \mathbf{b} \cdot \mathbf{u} \cdot (\partial_t + \mathbf{u} \cdot \nabla + \mathbf{U} \nabla) \mathbf{u},
\Omega^2 = \mathbf{b} \cdot \nabla \times \mathbf{u}.
\]
Note that we recover an additional term in the \( \dot{U} \) equation which appears to be missing in Ref. 5. Physically, it is a ponderomotive term that typically results from the appearance of a \( \hat{\phi}^2 \) term in the Lagrangian;\(^10\) the analogue of this term is present in Ref. 3. The contributions to the Euler-Lagrange equations from the second order part of the Lagrangian are
\[
\dot{\mathbf{R}}_2 = -\Omega^{-1} \mathbf{b} \times \nabla H_2,
\dot{\mathbf{U}}_2 = H_2 \mu - \Omega^{-1} \mathbf{b} \cdot \mathbf{u} \cdot \mathbf{b} \times \nabla H_2,
\dot{\mu}_2 = -H_2 \mu,
\dot{\phi}_2 = (\langle \mathbf{b} \cdot \nabla \rho \rangle + \frac{1}{2} \Omega^{-1} \langle \delta_{\dot{\phi}} \phi \rangle \mu \mu + \mathbf{b} \times \langle \delta_{\dot{\phi}} \phi \rangle \mu \mu),
\]
where \( H_2 \) is the second order part of the gyrocentre Hamiltonian. The Euler-Lagrange equations that include the contributions from the second order part of the Lagrangian can be simplified by renormalising the potential.\(^11\)

V. POISSON EQUATION

Gyrokinetic Poisson and Ampère equations have previously been obtained by varying the system Lagrangian with respect to the field variables.\(^12,13\) We find it helpful to give an elementary explanation of why this should be possible.

First, consider the many-body Lagrangian for a set of point particles interacting with a field, with integral terms for the field self-interaction: this is a well posed problem at least if we restrict the fields to be sufficiently smooth, and Euler-Lagrange equations for the particles and the usual Maxwell equations are directly obtained by varying particle coordinates and fields. We now apply our guiding and gyrocentre transformations to write this many-body Lagrangian in terms of the particle gyrocentre variables. The system Lagrangian, which is the sum of the particle Lagrangians, plus the field component integrated over space, then directly leads to gyrocentre Euler-Lagrange equations, and Poisson and Ampère equations for the fields. We are usually interested in the
smooth limit of these equations (potentially with a collision operator representing short spatial scale correlations), with particles described by a distribution function $F(Z)$, in which case the time evolution of $F$ can be evaluated in terms of the Euler-Lagrange equations of the gyroparticles (a gyrokinetic Vlasov equation) and in field equations sums over particles are replaced by integrals of $F$.

We note the contrast between this approach, which is similar to that of Refs. 4 and 12, and attempts to vary a system Lagrangian written in terms of the distribution function: the Euler-Lagrange equations appear naturally, rather than being inserted by hand as a constraint. At this point, it is useful to introduce some notation: we denote a mapping from coordinate system $Z$ to $z$ as $T_{Z\to z}$ and the associated Jacobian as $J_{Z\to z} = |\partial_x T_{Z\to z}|$.

We will consider only the electrostatic, quasineutral limit where the field terms have been ignored and species sums, charges, and masses have been suppressed. The Poisson equation can be obtained from the stationary variation of the system Lagrangian in original coordinates with respect to $\phi$, and this can also be written directly in gyrocentre coordinates, based on the above consideration of interpretation as the limit of a many body theory,

$$\frac{\partial}{\partial \phi} \int d^6 z \delta(z) L_\phi(z) = \frac{\partial}{\partial \phi} \int d^6 \tilde{Z} \delta(\tilde{Z}) L_\phi(\tilde{Z});$$

the invariance of the value is also what we expect due to the covariance of the form of the integral. Note, however, that, here, $f$ must be defined so that it transforms as a scalar density; the “usual” gyrocentre distribution function is actually $F^0(Z) = f(T_{Z\to z}) = (J_{Z\to z})^{-1} F(Z)$. This Jacobian is a function of $\phi$, unlike for the transformations in the weak-flow case, and varying $\phi$ with fixed $\tilde{F}$ is not identical to varying $\phi$ with fixed $F$. Performing this variation (Appendix D) yields

$$0 = (\delta L)_\phi = \int d^3 r \delta \phi(r) \int d^6 \tilde{Z} \delta(\tilde{R} + \tilde{p} - \tilde{r}) \times [\{1 + \Omega^{-2} \tilde{V} \tilde{\phi} \tilde{\phi} + \tilde{b} \cdot \tilde{V} + \Omega^{-1} \tilde{\phi} \tilde{\phi} \} \tilde{F} + \Omega^{-1} \tilde{b} \cdot \tilde{V} \times (\tilde{F} \tilde{R} - 2\tilde{F} \tilde{u})].$$

If the distribution function $F^0$ is uniform, and we neglect terms which are of order $\epsilon^2$, this Poisson equation reduces to the usual weak-flow Poisson equation as shown in Appendix D.

For weak flows, it has been shown that the variational method for obtaining the Poisson equation is equivalent to the direct method of setting the charge-density to zero, up to the chosen order of approximation. Here, we have the quasi-neutrality equation

$$0 = \int d^6 z \delta(x - r)f(z),$$

where $f$ is the original distribution function. A change of variables can be made to guiding-centre coordinates, and the guiding-centre distribution function $F^0(Z)$ can be expressed in terms of the gyrocentre distribution function $\tilde{F}(Z)$ using the Lie transform,\cite{14} to yield

$$0 = \int d^6 \tilde{z} \delta(\tilde{R} + \tilde{p} - \tilde{r}) T \tilde{F}.'$$
system composed of the first-order Euler-Lagrange equation (14) and the linearised Poisson equation with uniform \( F_0' \).
The convergence ratio per iteration is of order \( c \). This has been used to investigate the Kelvin-Helmholtz instability of a shear layer, to demonstrate that the numerical scheme converges, is well-behaved, and reduces to the weak-flow model in the appropriate limit. We have also simulated a simplified problem that reduces the spatial dynamics to three-wave coupling, to verify that the numerical implementation is correct in certain limits.

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APPENDIX A: GUIDING-CENTRE LAGRANGIAN

Substituting \( x = R + \rho \) and \( v = Ub + v_\perp \) into Eq. (4) yields
\[
\gamma = \left[ A(R + \rho) + Ub + v_\perp + u \right] \cdot (dR + d\rho) \nonumber \\
- \left( \frac{1}{2} U^2 + \mu \Omega + \frac{1}{2} u^2 + \langle \phi \rangle + \delta_1 \phi \right) dt. \tag{A1}
\]
Using \( A(R + \rho) = A(R) + (\rho \cdot \nabla)A(R) \), the gauge (6) and \( v_\perp \cdot dR = \rho \times [\nabla \times A(R)] \cdot dR \) in Eq. (A1) yields
\[
\gamma = \left[ A(R) + Ub + u \right] \cdot dR - dR \cdot \left[ \nabla [A(R) \cdot \rho] \right] - (\rho \cdot \nabla)A(R) - \rho \times [\nabla \times A(R)] \cdot \rho + \mu d\theta \nonumber \\
- \left( \frac{1}{2} U^2 + \mu \Omega + \frac{1}{2} u^2 + \langle \phi \rangle + \delta_1 \phi \right) dt. \tag{A2}
\]
By identifying the terms in curly brackets in Eq. (A2) as \([A(R) \cdot \nabla] \rho + A(R) \times (\nabla \times \rho) = 0\), we obtain Eq. (5).

APPENDIX B: GYROCENTRE LAGRANGIAN

The requirement
\[
\delta_1 \tilde{\phi} = O(\epsilon)
\]
is equivalent to restrictions on the possible choices for the \( \theta \)-independent potential appearing in Eq. (5) and \( u \) given by
\[
\phi_g - \phi(R) \leq O(\epsilon) \tag{B1}
\]
and
\[
u - \Omega^{-1} \hat{b} \times \nabla \phi(R) \leq O(\epsilon), \tag{B2}
\]
respectively, where \( \phi_g \) is a general \( \theta \)-independent potential. Some possible choices for \( \phi_g \) and \( u \) that satisfy orderings (B1) and (B2) are \( \phi_g = \phi(R) \),
\[
\phi_g = \langle \phi \rangle, \quad k = \Omega^{-1} \hat{b} \times \nabla \phi(R), \text{ and Eq. (7)}.
\]
Using Eq. (10), we can compute the non-zero Lagrange matrix components of \( \Gamma_0 \) as
\[
\begin{align*}
\omega_{i0R,j} &= \epsilon_i / \epsilon \Omega^2, \\
\omega_{0R,j} &= -u_{j\mu}, \\
\omega_{0q} &= -\nabla(\phi) - u \times (\nabla \times u) - (u \cdot \nabla + \partial_t)u, \\
\omega_{0u\mu} &= -\langle \phi \rangle_{,\mu} - u \cdot u_{,\mu} - \Omega, \\
\omega_{0R\mu} &= -\hat{b}, \\
\omega_{0RU} &= -U, \\
\omega_{0U\mu} &= 1,
\end{align*}
\tag{B3}
\]
where \( i, j, k' \in \{1, 2, 3\} \) and
\[
\Omega^* = \Omega + \nabla \times u. \tag{B4}
\]
The first-order part of the gyrocentre Lagrangian is
\[
\Gamma_1 = \Gamma - L_1 \Gamma_0 + dS_1,
\]
where
\[
\begin{align*}
\Gamma_1 &= (-\rho \cdot \nabla u) \cdot dR - \rho \cdot u_{,\mu} d\mu - (\rho \cdot u_{,t} + \delta_1 \tilde{\phi}) d\theta, \\
- L_1 \Gamma_0 &= g_1^R \times \Omega \cdot dR + g_2^\theta d\mu - g_2^\theta d\theta + (g_1^R \cdot \nabla \phi) \\
&+ g_1^\theta \Omega^* dt + O(e^2),
\end{align*}
\]
and
\[
dS_1 = \nabla S_1 \cdot dR + S_1_{,t} dR + S_1_{,\mu} d\mu + S_1_{,\theta} d\theta + S_1_{,\phi} d\phi,
\]
Solving for \( g_1 \) in terms of \( S_1 \) such that \( \Gamma_1 \) is only composed of a first-order time component,
\[
\Gamma_1 = (-\delta_1 \tilde{\phi} + \Omega^{-1} \nabla S_1 \times \hat{b} \cdot \nabla \phi + \Omega S_1_{,\theta} + S_1_{,\phi}) dt \\
+ O(e^2),
\]
yields the non-zero \( g_1 \) components
\[
\begin{align*}
g_1^R &= \Omega^{-1} [\rho \cdot (\hat{b} \times \nabla)u + \nabla S_1 \times \hat{b}], \\
g_1^\theta &= S_1_{,\theta}, \\
g_1^\phi &= \rho \cdot u_{,\mu} - S_1_{,\mu}.
\end{align*}
\]
By using
\[
(\partial_t + u \cdot \nabla)S_1 \sim e^2,
\]
as in Ref. 5,
\[
\Gamma_1 = (-\delta_1 \tilde{\phi} + \Omega S_1_{,\theta}) dt + O(e^2).
\]
By using the freedom of \( S_1 \) to remove the first-order \( \theta \)-dependent terms in \( \Gamma_1 \), we have
\[
\Gamma_1 = O(e^2)
\]
for
\[
S_1 = \Omega^{-1} \delta_1 \tilde{\phi}.
\]
\[ \Gamma_1 \text{ yields} \]
\[
\begin{align*}
\omega_{1Rp} &= \nabla u \cdot \rho_{,p}, \\
\omega_{1R\theta} &= \rho_{,\theta} \cdot \nabla u, \\
\omega_{1Rt} &= -\nabla \delta_{1} \dot{\phi}, \\
\omega_{1\mu p} &= -u_{,p} \cdot \nabla \mu, \\
\omega_{1\mu \theta} &= -u_{,\theta} \cdot \mu_{,\theta}, \\
\omega_{1\phi \phi} &= -(\rho \cdot u_{,p} + \delta_{1} \dot{\phi}_{,p},
\end{align*}
\]

and the expression for \( \Gamma_2 \) is
\[
\Gamma_2 = \Gamma_2 - L_1 \Gamma_1 + \left( \frac{1}{2} L_1^2 - L_2 \right) \Gamma_0 + dS_2
\]
\[= \Gamma_2 - L_1 \Gamma_1 + \frac{1}{2} L_1 (L_1 \Gamma_0) - L_2 \Gamma_0 + dS_2
\]
\[= \Gamma_2 - L_1 \Gamma_1 + \frac{1}{2} L_1 (\Gamma_1 + dS_1 - \Gamma_1) - L_2 \Gamma_0 + dS_2
\]
\[= \Gamma_2 - \frac{1}{2} L_1 \Gamma_1 - L_2 \Gamma_0 + dS_2 + O(\varepsilon^3),
\]

where \( L_1 dS_1 = 0 \).

\[
\begin{align*}
\Gamma_2 &= \left[ g^R \times (\nabla \times u) - g^R_{\mu} u_{,\mu} \right] \cdot dR - g^R_{\rho,\theta} u_{,\mu} d\mu \\
&+ \left( g^R_{\rho, \mu} \cdot u_{,\mu} + g^R_{\rho, \mu} \cdot u_{,\mu} + g^R_{\rho, \mu} \cdot u_{,\mu} + \delta_{1} \dot{\phi}, \right) \\
&+ (\partial_t + u \cdot \nabla) (S_1 - \rho \cdot u) \\n&+ O(\varepsilon^3),
\end{align*}
\]

\[
\frac{-1}{2} L_1 \Gamma_1 = \frac{1}{2} \left[ g^R_{\rho} \cdot u_{,\mu} + (g^R_{R, \mu} \cdot u_{,\mu}) d\mu - g^R_{\rho, \mu} \cdot u_{,\mu} d\mu \\
- g^R_{\rho} \cdot u_{,\mu} \cdot dR - g^R_{\rho, \mu} d\mu - g^R_{\rho, \mu} d\mu \\
+ \left[ g^R_{\rho} \cdot (\partial_t \phi + g^R_{\rho} \cdot \mu + \delta_{1} \dot{\phi}), \right] \\
+ g^R_{\rho} \cdot u_{,\mu} + g^R_{\rho, \mu} \right] d\mu, \right.
\]

\[
- L_2 \Gamma_0 = g^R \times \Omega \cdot dR + g^R_{\mu} d\mu - g^R_{\mu} d\mu \\
+ \left( g^R \times (\phi + g^R_{\mu} \cdot \mu + \delta_{1} \dot{\phi}), \right) \\
+ (\partial_t + u \cdot \nabla) S_2 \right] dR + O(\varepsilon^3)
\]

and
\[
dS_2 = \nabla S_2 \cdot dR + S_{2,\mu} dU + S_{2,\mu} d\mu + S_{2,\theta} d\theta + S_{2,\phi} d\phi.
\]

Choosing \( u \) to be the \( E \times B \) drift velocity associated with the \( \theta \)-independent potential that appears in Eq. (5) facilitates several cancelations during the computation of the second-order gyrocentre Lagrangian. Solving for \( g_2 \) in terms of \( S_2 \) such that \( \Gamma_2 \) is only composed of a second-order time component,
\[
\begin{align*}
\Gamma_2 &= \left[ g^R \times (\nabla \times u) - g^R_{\mu} u_{,\mu} \right] \cdot dR - g^R_{\rho,\theta} u_{,\mu} d\mu \\
&+ \left( g^R_{\rho, \mu} \cdot u_{,\mu} + g^R_{\rho, \mu} \cdot u_{,\mu} + g^R_{\rho, \mu} \cdot u_{,\mu} + \delta_{1} \dot{\phi}, \right) \\
&+ (\partial_t + u \cdot \nabla) (S_1 - \rho \cdot u) \\n&+ O(\varepsilon^3),
\end{align*}
\]

yields the non-zero \( g_2 \) components
\[
\begin{align*}
g^R_{20} &= \Omega^{-1} \left[ g^R \times (\nabla \times u) - g^R_{\mu} u_{,\mu} \right] \\
&+ \frac{1}{2} g^R_{\mu} \cdot \nabla u + \nabla S_2 \right] \times \dot{b},
\end{align*}
\]

and
\[
\begin{align*}
g^R_{21} &= S_{2,\theta} - \frac{1}{2} g^R_{\rho, \theta} \cdot u_{,\mu}, \\
&= g^R_{21} + \frac{1}{2} \left( g^R_{\rho, \theta} \cdot u_{,\mu} - g^R_{\rho, \theta} \cdot u_{,\mu} \right) - S_{2,\mu},
\end{align*}
\]

By using
\[
(\partial_t + u \cdot \nabla) S_2 \sim \varepsilon^3,
\]
\[
\Gamma_2 = \left[ g^R \times (\phi + g^R_{\mu} \cdot \mu + \delta_{1} \dot{\phi}) \right] dR + O(\varepsilon^3),
\]

By using the freedom of \( S_2 \) to remove the second-order \( \theta \)-dependent terms in \( \Gamma_2 \), we have
\[
\begin{align*}
\Gamma_2 &= \left[ g^R \times (\phi + g^R_{\mu} \cdot \mu + \delta_{1} \dot{\phi}) \right] dR \\
&= \frac{1}{2} \left( g^R \times (\phi + g^R_{\mu} \cdot \mu + \delta_{1} \dot{\phi}) \right) dR \\
&= \frac{1}{2} \left( g^R \times (\phi + g^R_{\mu} \cdot \mu + \delta_{1} \dot{\phi}) \right) dR + O(\varepsilon^3),
\end{align*}
\]

**APPENDIX C: EULER-LAGRANGE EQUATIONS**

Using the Lagrange matrix components computed from the gyrocentre Lagrangian up to first-order, or equivalently those computed from the guiding-centre Lagrangian up to zeroth-order (B3), in the gyrocentre Euler-Lagrange equation (13) with \( i = \{ R, \dot{U}, \mu, \dot{\theta} \} \) yields
\[
\dot{R} \times \Omega^* - \dot{\Omega} \dot{b} = \omega_{iR},
\]
\[
\dot{\mu} = 0,
\]
\[
\dot{\theta} = \Omega + \langle \phi \rangle_{,\mu} - \dot{u}, \dot{\mu},
\]
\[
\dot{b} \cdot \dot{R} = U,
\]

respectively. Taking the cross product of \( \dot{b} \) and (C1), expanding the resultant triple product and using (C2) yields
\[
\dot{R} = \Omega^{-1} \left\{ \Omega \dot{U} + \dot{b} \times (\nabla \times \dot{u}) + (u \cdot \nabla + \delta_{1} \dot{\phi}) \dot{u} \right\} + \dot{U} \Omega^*.
\]

By expanding the triple product and using
\[
\Omega^* = \Omega^{-1} \dot{b} + \dot{b} \times \dot{u},
\]

\[
\dot{R} = \dot{u} + \Omega^{-1} \dot{b} \times (\partial_t + u \cdot \nabla + \dot{U} \nabla) \dot{u} + \dot{U} \dot{b}.
\]

Projecting (C1) onto \( \Omega^* \) yields
\[ \dot{U} = -\Omega_{11}^{-1} \Omega^* \cdot [\nabla \langle \phi \rangle + \vec{u} \times (\nabla \times \vec{u}) + (\vec{u} \cdot \nabla + \partial_t) \vec{u}] \]  

By using (B4) and (C3) appropriately and expanding the cross product,

\[ \dot{U} = -(\phi)_{,} + \dot{\Omega}_{11}^{-1} \vec{u}_{,} \cdot \vec{b} + (\partial_t + \vec{u} \cdot \nabla) \vec{u} \]  

**APPENDIX D: POISSON EQUATION**

The variation with respect to \( \phi \) of the gyrocentre system Lagrangian up to second order is

\[
\langle \delta L \rangle_{\phi} = -\int d^6 Z \left\{ \delta \left[ \frac{1}{2} \Omega^{-2} \nabla \cdot \vec{b} \cdot \nabla \phi \right] - \frac{1}{2} \Omega^{-2} \nabla \nabla \cdot \phi \cdot \vec{b} \cdot \nabla \phi \right\}_i = -\int d^6 Z \left\{ \delta \left[ \frac{1}{2} \Omega^{-2} \nabla \nabla \cdot \phi \right] - \frac{1}{2} \Omega^{-2} \nabla \cdot \phi \cdot \vec{b} \cdot \nabla \phi \right\}_i 
\]

\[
= -\int d^6 Z \delta [\phi, \nabla \cdot \phi, \nabla \times \phi] 
\]

from which we obtain Eq. (16).

Using an alternative form for \( \Gamma_2 \) (B5) and \( J_{2,zz} = \tilde{\Omega}_{11}^{-1} \), the Euler-Lagrange equation for \( \phi \) up to first order is

\[
0 = \Omega \int d^6 Z \delta \left( \vec{R} + \vec{\rho} - r \right) \left[ (1 + \Omega^{-2} \nabla \nabla \cdot \phi \cdot \vec{b} \cdot \nabla + \Omega^{-1} \phi \partial_t) \vec{F}' + \Omega^{-2} \nabla^2 \phi \right] 
\]

\[
= \Omega \int d^6 Z \delta \left( \vec{R} + \vec{\rho} - r \right) \left[ (1 + \Omega^{-2} \nabla \nabla \cdot \phi \cdot \vec{b} \cdot \nabla + \Omega^{-1} \phi \partial_t) \vec{F}' + \Omega^{-2} \nabla^2 \phi \right] 
\]

Using the guiding-centre Jacobian up to first order \( J_{2,zz} = \Omega_{11}^* + \vec{\rho} \cdot \nabla \times \vec{u} \) and the action of the Lie transform on scalars up to first order \( T = (1 + g \vec{b} \cdot \nabla) \), an evaluation of Eq. (18) up to first order yields Eq. (D1). In other words, we obtain equivalent Poisson equations up to first order using either a variational or direct method.

We will now consider uniform \( \vec{F}' \). Using \( \nabla \langle \phi \rangle = -\int d^3 k \langle \vec{E} \rangle \langle k, \vec{\mu} \rangle e^{i k \cdot \vec{R}} \), the last two terms in Eq. (D1) are

\[ 2\pi i \int d\vec{r} d^3 k \left\{ \left[ \delta \langle \vec{J}_1 \rangle \right]_{\vec{k} \cdot \vec{\mu}} - k \omega \Omega^{-1} J_0 (k \cdot \vec{\mu}) \right\} \langle \vec{E} \rangle e^{i k \cdot \vec{r}} F' = 0. \]

In other words, in the weak-flow limit and for uniform \( \vec{F}' \), the weak- and strong-flow Poisson equations up to first order are identical,

\[ 0 = \Omega \int d^6 Z \delta (\vec{R} + \vec{\rho} - r) (1 + \Omega^{-1} \phi \partial_t) \vec{F}' , \]

where for uniform \( \vec{F}' \), the second weak-flow polarisation density term does not appear.

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