Bilinear maps and their classifying tensor products are well-known in the theory of linear algebra, and their generalization to algebras of commutative monads is a classical result of monad theory. Motivated by constructions needed in categorical approaches to finite model theory, we generalize the notion of bimorphism much further. To illustrate these maps are mathematically natural notions, we show that many common axioms in category theory can be phrased as certain morphisms being bimorphisms. We also show that much of the established theory of bimorphisms goes through in much greater generality. Our results carefully identify which assumptions are needed for the different components of the theory, including when good properties hold globally, or can at least be established locally. We include a brief string diagrammatic account of the bimorphism axiom, and conclude by recovering a simple proof of a classical theorem, emphasizing the efficacy of the bimorphism perspective.
1 Introduction

In linear algebra, the operation of tensor product $\otimes$ is characterised by the
universal property that bilinear maps of type $U \times V \to W$, involving vector
spaces $U, V$ and $W$, are in bijective correspondence with linear maps of type
$U \otimes V \to W$. (See for example [Alu21] for an expository account in categorical
language.)

The tensor product in vector spaces can be seen as a lifting of the cartesian
product over $\text{Set}$ to algebras over the free vector space monad. Under mild
assumptions, this situation generalizes to any commutative monad $T$ by defining
a lifting of a monoidal product on a category $C$ to a monoidal product on the
category of algebras $C^T$. The definition of commutative monad involves a lot of
assumptions, but the notion of bilinear map, and the corresponding classifying
object construction make no essential use of many of them.

In this paper, we identify a minimal notion of bimorphism, generalizing the
classical theory of bilinear maps in the setting of monads [Koc71] (see also
[Jac94, Sea13]). At this greater level of generality, bimorphisms subsume ordi-
nary algebra morphisms. Furthermore, many axioms of classical monad theory,
such as those for being a monad, or an Eilenberg-Moore algebra can be phrased
as certain maps being bimorphisms. We establish that classifying objects for
bimorphisms exist under very mild assumptions, and carefully identify the ax-
ioms required for good properties of the classifying object construction, such as
functoriality or commuting with the free algebra functors.

Though the lifting of monoidal products to the category of algebras has been
studied extensively, one of our main technical contribution is in extracting the
minimal conditions under which various liftings of functors to endofunctor al-
gebras exist. Another contribution is demonstrating, that in practice, a global
assumption that all diagrams of a certain type commute can be weakened to a
local assumption, that the diagrams at a particular index commute. For exam-
ple, existence of a natural transformation can be weakened to commutativity
of specific naturality squares. We also provide a string diagrammatic proof of
a well-known adjoint lifting theorem [Joh75, Kei75] which utilises the theory of
bimorphisms. Underlying this entire presentation, is the key unifying idea of a
$\lambda$-bimorphism between endofunctor algebras.
Our main motivation for investigating bimorphisms is to study Feferman-Vaught-Mostowski (FVM) theorems [Mos52, FV67]. Such theorems characterize how logical equivalence behaves under composition and transformation of models. They have applications ranging from model theory [Gur85] to algorithmic meta-theorems [Con90, CMR00, CE12, OS06, Mak04].

An FVM theorem typically states that, for a logic $L$, and $n$-ary operation $H$ on models, the $L$-theory of $H(A_1,\ldots,A_n)$ can be computed from the $L$-theories of $A_1,\ldots,A_n$. In our forthcoming paper [JMS22] we make use of the technology built here, and demonstrate that in the setting of game comonads [ADW17, AS21] the core of Feferman-Vaught-Mostowski theorems is lifting $H$ to the Eilenberg-Moore category of a comonad capturing $L$, and analysing properties of bimorphisms. Our emphasis on identifying the minimal technical considerations is motivated by practical considerations in that work. We focus on monads in the present work as the classical motivating example is algebraic in nature.

2 Preliminaries

We assume familiarity of the standard category-theoretic notions of functor, natural transformations, adjunctions, and (co)monads. These notions can be found in any introductory writing on the subject (see e.g. [ML13]) For an endofunctor $T : C \to C$, the category of $T$-algebras is denoted $\text{Alg}(T)$ and the category of $T$-coalgebras is denoted $\text{CoAlg}(T)$. For a monad $T = (T, \mu_T, \eta_T)$ over $C$, the Eilenberg-Moore category of $T$ is denoted $C^T$. The Kleisli category of $T$ is denoted $C_T$. Recall that $C_T$ and $C^T$ are the initial and terminal object in the category of adjunctions which yield $T$ as monad over $C$. In particular, there is a comparison functor $K : C_T \to C^T$ which maps the Kleisli adjunction $U_T \dashv F_T$ to the Eilenberg-Moore adjunction $F^T \dashv U^T$ associated to $T$ as described in figure 1.

In an effort to declutter notation, with natural transformations $\lambda : T \Rightarrow S$, we will often write $\lambda$ for the individual components $\lambda_A : T(A) \to S(A)$ when it does not lead to confusion. We will also write $\mu$ for $\mu^T$ when it does not lead to confusion and similarly for $\eta$. In section 8, we will work with comonads $D = (D, \delta^D, \epsilon^D)$ where will employ the same terminological and notational conventions.
3 Bilinear maps

For real vector spaces $U, V, W$, a standard algebraic notion is that of a bilinear map. We say that a function $h : U \times V \to W$ is \textit{bilinear} if:

1. For all $u \in U$, $h(u, -)$ is a linear map $V \to W$.
2. For all $v \in V$, $h(-, v)$ is a linear map $U \to W$.

Furthermore, there is a vector space $U \otimes V$, the \textit{tensor product} of $U$ and $V$, such that there is a bijective correspondence between:

1. Bilinear maps $U \times V \to W$.
2. Linear maps $U \otimes V \to W$.

This universal property of the tensor product is very convenient, as it allows us to move freely between the perspective of bilinear maps, and that of linear maps, for which there is a great deal of linear algebraic machinery.

It is a classical fact of monad theory that this situation generalizes to many other settings [Koc71], particularly to that of commutative monads (See also [Jac94, Sea13]). In this paper, we explore this underappreciated fact in some detail. We establish connections with so-called Kleisli laws, and use this to show that the notion of bilinear map can be generalized yet further, with a corresponding universal object. We show that the abstract formulation of bilinear maps captures a wide range of familiar notions from mathematics and computer science.

4 Generalizing Bilinear maps to commutative monads

This section discusses some classical categorical results relating to bilinear maps. We must first introduce the notions of strong and commutative monads. The material in this section is standard, but provides motivation for later constructions, and gives us a convenient opportunity to fix some terminology and notation.

4.1 Strong and Commutative Monads

Let $(\mathcal{V}, \otimes, I)$ be a symmetric monoidal category, with coherence isomorphisms:

\[
\begin{align*}
I &: I \otimes A \to A \\
r &: A \otimes I \to A \\
a &: (A \otimes B) \otimes C \to A \otimes (B \otimes C) \\
s &: A \otimes B \to B \otimes A
\end{align*}
\]
A strength for an endofunctor $F : \mathcal{V} \to \mathcal{V}$ is a natural transformation:

$$\text{st} : A \otimes F(B) \to F(A \otimes B)$$

such that the following two diagrams commute:

$$I \otimes F(A) \xrightarrow{\text{st}} F(I \otimes A) \xleftarrow{\text{I}} F(A)$$

$$\begin{array}{ccc}
(A \otimes B) \otimes F(C) & \xrightarrow{\text{st}} & F((A \otimes B) \otimes C) \\
\downarrow{s} & & \downarrow{F(a)} \\
A \otimes (B \otimes F(C)) & \xrightarrow{A \otimes \text{st}} & A \otimes F(B \otimes C) \xrightarrow{\text{st}} F(A \otimes (B \otimes C))
\end{array}$$

Dually, a costrength for $F$ is a natural transformation:

$$\text{st}' : F(A) \otimes B \to F(A \otimes B)$$

satisfying the dual of the conditions above. If $\mathcal{V}$ is a symmetric monoidal category, we can define a costrength from a strength using the symmetry $s$ as the composite:

$$F(A) \otimes B \xrightarrow{s} B \otimes F(A) \xrightarrow{\text{st}} F(B \otimes A) \xrightarrow{F(s)} F(A \otimes B)$$

A strong functor is a functor with a specified strength.

A monad $(T, \eta, \mu)$ is strong [Koc70] if $T$ is a strong functor, such that the strength is compatible with the monad structure, in that the following diagrams commute:

$$\begin{array}{ccc}
A \otimes T(B) & \xrightarrow{\text{st}} & T(A \otimes B) \\
\downarrow{A \otimes \eta} & & \downarrow{\eta} \\
A \otimes B & \xrightarrow{\eta} & A \otimes B
\end{array}$$

$$\begin{array}{ccc}
A \otimes T^2(B) & \xrightarrow{A \otimes \text{mu}} & A \otimes T(B) \\
\downarrow{\text{st}} & & \downarrow{\text{st}} \\
T(A \otimes T(B)) & \xrightarrow{\text{Tst}} & T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B)
\end{array}$$

Remark 4.1. On a monoidal closed category, a strong monad is the same thing as an enriched monad [Koc72]. Every $\text{Set}$ monad is $\text{Set}$-enriched, and hence canonically strong, with $\text{st}(a, t) := T(\lambda b. (a, b))(t)$. 

5
For a strong monad on a symmetric monoidal category, we define

\[ \text{dst} : T(A) \otimes T(B) \to T(A \otimes B), \]

using the induced costrength, as the composite:

\[ T(A) \otimes T(B) \xrightarrow{\text{st}} T(T(A) \otimes B) \xrightarrow{T(\text{st})} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B) \]

We can also define natural transformation \( \text{dst}' \):

\[ T(A) \otimes T(B) \to T(A \otimes B) \]

as the composite:

\[ T(A) \otimes T(B) \xrightarrow{\text{st}'} T(T(A) \otimes B) \xrightarrow{T(\text{st})} T^2(A \otimes B) \xrightarrow{\mu} T(A \otimes B) \]

The morphisms \( \text{dst} \) and \( \text{dst}' \) are referred to as double strengths. A strong monad is said to be commutative [Koc70] if \( \text{dst} = \text{dst}' \).

**Remark 4.2.** On a symmetric monoidal closed category, a commutative monad is the same thing as a symmetric monoidal monad [Koc72].

**Example 4.3.** For the list monad \( L \), we have:

\[
\text{st}(a, [b_1, \ldots, b_n]) = [(a, b_1), \ldots, (a, b_n)] \\
\text{st}'(a_1, \ldots, a_n, b) = [(a_1, b), \ldots, (a_n, b)] \\
\text{dst}([a_1, \ldots, a_m], [b_1, \ldots, b_m]) = [(a_1, b_1), \ldots, (a_m, b_1), \ldots, (a_1, b_n), \ldots, (a_m, b_n)] \\
\text{dst}'([a_1, \ldots, a_m], [b_1, \ldots, b_m]) = [(a_1, b_1), \ldots, (a_1, b_n), \ldots, (a_m, b_1), \ldots, (a_m, b_n)]
\]

In particular, the list monad is not commutative.

**Example 4.4.** If \( T \) is a monad induced by the presentation of some algebraic theory \( (\Sigma, E) \), \( \text{st}(a, t) \) is defined on the representative term \( t \) by replacing the occurrence of each variable \( b \) with \( (a, b) \). Dually, \( \text{st}'(t, b) \) replaces the occurrence of each variable \( a \) in \( t \) with \( (a, b) \).

For example, if \( \Sigma = \{\times, 1\} \), and the equations \( E \) are those for the theory of monoids, then we have:

\[
\text{st}(a, (b_1 \times b_2) \times b_3) = ((a, b_1) \times (a, b_2)) \times (a, b_3) \\
\text{st}'((a_1 \times a_2) \times a_3, b) = ((a_1, b) \times (a_2, b)) \times (a_3, b)
\]

As we would expect, this is a syntactic rephrasing of the actions for the list monad.

**Example 4.5.** For a semiring \( S \), we can define a monad \( M_S : \text{Set} \to \text{Set} \):

**Endofunction** : \( M_S(A) \) is the set of finite formal sums of the form:

\[
\sum_i s_i a_i
\]

Here each \( s_i \) is an element of the semiring, and each \( a_i \) is in \( A \).
**Unit** : The unit maps \( a \) to the trivial sum.

**Multiplication** : The multiplication flattens a sum of sums to a sum:

\[
\sum_i s_i (\sum_j s_{i,j} a_{i,j}) \mapsto \sum_{i,j} (s_i \times s_{i,j}) a_{i,j}
\]

The strength and costrength are given by:

\[
\text{st}(a, \sum_j s_j b_j) = \sum_j s_j (a, b_j)
\]

\[
\text{st}'(\sum_i s_i a_i, b) = \sum_i s_i (a_i, b)
\]

Therefore we have

\[
\text{dst}(\sum_i s_i a_i, \sum_j r_j b_j) = \sum_{i,j} (r_j \times s_i)(a_i, b_j)
\]

\[
\text{dst}'(\sum_i s_i a_i, \sum_j r_j b_j) = \sum_{i,j} (s_i \times r_j)(a_i, b_j)
\]

\( M_S \) will be a commutative monad if and only if the semiring multiplication is commutative. The algebras for this monad are known as semimodules, and modules if \( S \) is in fact a ring. We note a few special cases of interest, in each case, the corresponding monad is commutative:

- If \( S = \mathbb{N} \), the algebras are abelian monoids.
- If \( S = \mathbb{Z} \), the algebras are abelian groups.
- If \( S \) is a field \( F \), the algebras are the vector spaces over \( F \).

### 4.2 Abstracting Bilinear Maps

For a commutative monad \( T \), and algebras \((A, \alpha), (B, \beta)\) and \((C, \gamma)\), following [Koc71], we say that a morphism \( h : A \otimes B \to C \) is **bilinear**, or a **bimorphism**, if the following diagram commutes:

\[
\begin{array}{c}
\mathbb{T}(A) \otimes \mathbb{T}(B) \xrightarrow{\text{dst}} \mathbb{T}(A \otimes B) \xrightarrow{\mathbb{T}(h)} \mathbb{T}(C) \\
\downarrow_{\alpha \otimes_{\otimes} \beta} \quad \quad \quad \quad \downarrow_{\gamma} \\
A \otimes B \xrightarrow{h} C
\end{array}
\]

**Example 4.6.** If \( T \) is the abelian monoid monad, a function \( h : A \times B \to C \) is bilinear if it satisfies:

\[
h(\sum_i a_i, \sum_j b_j) = \sum_i \sum_j h(a_i, b_j)
\]
This is equivalent to it being a homomorphism component-wise. That is, satisfying the equations
\[ h(\sum_i a_i, b) = \sum_i h(a_i, b) \quad \text{and} \quad h(a, \sum_j b_j) = \sum_j h(a, b_j). \]

Bilinearity can also be phrased in terms of the left and right components separately, using the monad strength and costrength.

\[ A \otimes T(B) \xrightarrow{\text{st}} T(A \otimes B) \xrightarrow{T(h)} T(C) \]
\[ A \otimes B \xrightarrow{\alpha \otimes B} T(A) \otimes T(B) \xrightarrow{\text{st}'} T(A \otimes B) \xrightarrow{T(h)} T(C) \]

A morphism \( h : A \otimes B \rightarrow C \) is bilinear in the sense that diagram (1) commutes if and only if both diagrams (2) and (3) commute.

**Example 4.7.** If \( T \) is the monoid monad, that diagrams (2) and (3) commute is equivalent to equations:
\[ h(\prod_i a_i, b) = \prod_i h(a_i, b) \quad \text{and} \quad h(a, \prod_j b_j) = \prod_j h(a, b_j). \]

Here, crucially, the notation intends that we preserve the order of the elements in the products.

As we know the monoid monad is not commutative, we would expect that the conditions implied by \( \text{dst} \) and \( \text{dst}' \) are different. This is the case, with the two properties being
\[ h(\prod_i a_i, \prod_j b_j) = \prod_j \prod_i h(a_i, b_j) \quad \text{and} \quad h(\prod_i a_i, \prod_j b_j) = \prod_i \prod_j h(a_i, b_j). \]

Again, it is critical we preserve the order of the elements in the products. It is perhaps easier to see the difference on smaller examples of each condition:
\[ h(a \times a', b \times b') = h(b, a) \times h(b, a') \times h(b', a) \times h(a', b') \]
\[ h(a \times a', b \times b') = h(b, a) \times h(b', a) \times h(b, a') \times h(a', b') \]

The two conditions are not equivalent, as the monoid multiplication is not in general commutative.
4.3 Lifting tensor products of commutative monads

The following is a well-known and very useful result, see for example [Jac94].

**Theorem 4.8.** If $T$ is a commutative monad on a symmetric monoidal category $(\mathcal{V}, \otimes, I)$ then:

1. The symmetric monoidal structure lifts to a symmetric monoidal structure $(\otimes_T, I_T)$ on $\mathcal{V}_T$.

2. If $\mathcal{V}_T$ has coequalizers of reflexive pairs, then the symmetric monoidal structure lifts to a symmetric monoidal structure $(\otimes_T, I_T)$ on $\mathcal{V}_T$, and the full and faithful functor $K: \mathcal{V}_T \to \mathcal{V}_T$ preserves the symmetric monoidal structure (in sense of [Jac94]).

3. Furthermore, $\otimes_T$ is universal, in the sense that for algebras $(A, \alpha)$ and $(B, \beta)$ there is a bimorphism $u: A \otimes B \to (\alpha \otimes_T \beta)$ such that for every coalgebra $(C, \gamma)$, every bimorphism $h: A \otimes B \to C$ factors through a unique $\hat{h}$ as:

\[
\begin{array}{c}
A \otimes B \xrightarrow{u} (\alpha \otimes_T \beta) \\
\downarrow h \\
C
\end{array}
\]

Here we see the connection between bilinear maps and universal tensors extended to the setting of commutative monads on some monoidal category. We shall move beyond this setting in later sections.

5 Beyond Commutative Monads

As far as we are aware, the existing literature on bimorphisms, and their corresponding classifying objects in Eilenberg-Moore categories, restricts attention to commutative monads on monoidal categories. We now pursue a wide generalization of these notions. Throughout, we will be careful to identify the roles of the various assumptions play in the theory, disentangling the large number of assumptions that are typically combined in the commutative monad setting.

5.1 Bimorphisms

We observe that diagram (1) makes no essential use of the following features:

1. That $T$ is a monad rather than a mere endofunctor.
2. That the algebras on the left and right of the diagram are both $T$-algebras.
3. The monoidal structure.
4. That $\otimes$ is a bifunctor, i.e. that it is 2-ary.
5. That $dst$ is natural and satisfies various coherence axioms.

Abstracting away from these inessential details, we arrive at the following.

**Definition 5.1** (Left $\lambda$-morphism for endofunctor-algebras). Let $S : C \to C$ and $T : D \to D$ be endofunctors, and $H : C \to D$ a functor. For a morphism $\lambda : H(S(A)) \to T(H(A))$, $S$-algebra $(A, \alpha)$ and $T$-algebra $(B, \beta)$, a morphism $h : H(A) \to B$ is a (left) $\lambda$-morphism from $\alpha$ to $\beta$ if the following diagram commutes:

$$
\begin{array}{ccc}
H(S(A)) & \xrightarrow{\lambda} & T(H(A)) \\
\downarrow_{H(\alpha)} & & \downarrow_{\beta} \\
H(A) & \xrightarrow{h} & B
\end{array}
$$

In which case we shall write $h : \alpha \to _{\lambda} \beta$. We shall also say $h$ is a left bimorphism when we do not wish to specify $\lambda$, or it is clear from the context.

We shall also have need of a related condition.

**Definition 5.2** (Right $\rho$-morphism for endofunctor-algebras). Let $S : C \to C$ and $T : D \to D$ be endofunctors, and $G : D \to C$ a functor. For a morphism $\rho : S(G(B)) \to G(T(B))$, $S$-algebra $(A, \alpha)$ and $T$-algebra $(B, \beta)$, a morphism $h : A \to G(B)$ is a (right) $\rho$-morphism from $\alpha$ to $\beta$ if the following diagram commutes:

$$
\begin{array}{ccc}
S(A) & \xrightarrow{S(h)} & S(G(B)) \\
\downarrow_{\alpha} & & \downarrow_{G(\beta)} \\
A & \xrightarrow{h} & G(B)
\end{array}
$$

In which case we shall write $h : \alpha \to _{\rho} \beta$. We shall also say $h$ is a right bimorphism when we do not wish to specify $\rho$, or it is clear from the context.

**Remark 5.3.** The terms left and right correspond to the position of $\lambda$ or $\rho$ in diagrams (4) and (5).

We shall refer to morphisms satisfying either equation (4) or (5) as bimorphisms. This terminology follows [Jac94], and the extension of bilinearity terminology from the original [Koc71], although we push the notion even further. Admittedly, there is nothing intrinsically binary about these morphisms, but the terminology is concise, and maintains the connection with historical conventions.

**Remark 5.4.** Note that conditions (4) and (5) are not dual, although both capture being an algebra morphism “up-to a mediating morphism”. As we shall see later the theory develops rather differently for the two, with condition (4) perhaps being the more interesting concept.
Although we will generally start imposing stronger conditions on \( \lambda \) in the notion of bimorphism, for example naturality or compatibility with other structures, it is useful to identify this level of abstraction. This will allow us to correctly isolate which conditions are essential for later results.

**Example 5.5** (Endofunctor algebra morphisms). \( T \)-algebra morphisms are a special case of bimorphisms, both in the sense of definition 5.1 and 5.2. For \( T \)-algebras \((A, \alpha)\) and \((B, \beta)\), a morphism \( h : A \to B \) is an algebra morphism if and only if:

\[
\begin{array}{c}
\text{Id}(T(A)) \xrightarrow{\text{id}} T(\text{Id}(A)) \xrightarrow{T(h)} T(B) \\
\text{Id}(A) \xrightarrow{\text{Id}(\alpha)} \xrightarrow{h} B
\end{array}
\]

That is, if it is a left \( \text{id} \)-morphism. We also have that \( h \) is an algebra morphism if and only if it is a right \( \text{id} \)-morphism.

The following example shows that by generalizing to endofunctor algebras, we can state well-known axioms as requiring certain morphisms be bimorphisms.

**Example 5.6** (Eilenberg-Moore axioms as bimorphisms). For a monad \( T \), let \( \alpha : T(A) \to A \) be an endofunctor algebra for \( T \). That \( \text{id} \) is a bimorphism of type \( \text{id} \to \eta \alpha \) is equivalent to requiring the following diagram to commute:

\[
\begin{array}{c}
\text{Id}(\text{Id}(A)) \xrightarrow{\eta} T(\text{Id}(A)) \xrightarrow{T(\text{id})} T(A) \\
\text{Id}(A) \xrightarrow{\text{id}} \xrightarrow{\alpha} A
\end{array}
\]

Some trivial simplifications show this is the same as \( \alpha \) satisfying the unit axiom for an Eilenberg-Moore algebra.

Similarly, that \( \alpha \) is a bimorphism of type \( \alpha \to \mu \text{id} \) is equivalent to requiring the following diagram to commute:

\[
\begin{array}{c}
T^2(A) \xrightarrow{\mu} \text{Id}(T(A)) \xrightarrow{T(\text{id})} T(A) \\
T(A) \xrightarrow{\alpha} \xrightarrow{T(\text{id})} A
\end{array}
\]

Simplifying, this is exactly that \( \alpha \) satisfies the multiplication axiom for an Eilenberg-Moore algebra.

As a second example of the ubiquity of bimorphism conditions, we note that the monad axioms themselves can be phrased as bimorphism conditions.

**Example 5.7** (Monad axioms). The unit axioms for a monad require that the following diagram commutes for all \( A \), both in the case where \( \lambda_A = \eta_{T(A)} \).
and \( \lambda_A = T(\eta_A) \):

\[
\begin{array}{ccc}
T(id(A)) & \xrightarrow{\lambda} & T(T(A)) \\
\downarrow_{T(id)} & & \downarrow_{\mu} \\
A & \xrightarrow{id} & A
\end{array}
\]

That is \( id_A \) is both:

1. A left \( \eta_{T(A)} \)-morphism of type \( id \rightarrow \eta_{T(A)} \mu \).
2. A left \( T(\eta_A) \)-morphism of type \( id \rightarrow T(\eta_A) \mu \).

Similarly, the associativity axiom requires that the following commutes for all \( A \):

\[
\begin{array}{ccc}
Id(T^3(A)) & \xrightarrow{\mu} & T(T(A)) \\
\downarrow_{Id(\mu)} & & \downarrow_{\mu} \\
T^2(A) & \xrightarrow{\mu} & T(A)
\end{array}
\]

That is, the components of \( \mu \) are left \( \mu \)-morphisms of type \( \mu \rightarrow \mu \mu \).

5.2 Kleisli and Eilenberg-Moore Laws

In later sections, we will require more structure on the morphism \( \lambda \) in diagrams (4) and (5). Of particular interest will be those that interact well with monad structure.

Let \( S : C \rightarrow C \) and \( T : D \rightarrow D \) be monads, and \( H : C \rightarrow D \) a functor. A Kleisli law is a natural transformation

\[
\lambda : H \circ S \Rightarrow T \circ H
\]

such that the following diagrams commute:

\[
\begin{array}{ccc}
HS(A) & \xrightarrow{\lambda} & TH(A) \\
\downarrow_{H\eta} & & \downarrow_{\eta} \\
H(A) & \xrightarrow{\eta} & T(A)
\end{array}
\]

\[
\begin{array}{ccc}
HS^2(A) & \xrightarrow{\lambda} & THS(A) \\
\downarrow_{H\mu} & & \downarrow_{\mu} \\
HS(A) & \xrightarrow{\lambda} & TH(A)
\end{array}
\]

The significance of Kleisli laws is captured in the following theorem, which is probably folklore. See for example [MM07].
Theorem 5.8 (Kleisli laws Classify Liftings). Let \( S : C \to C \) and \( T : D \to D \) be monads, and \( H : C \to D \) a functor. There is a bijective correspondence between:

1. Kleisli laws of type \( H \circ S \Rightarrow T \circ H \).
2. Kleisli liftings \( C_S \to D_T \), i.e. functors \( H : C_S \to D_T \) such that the following diagram commutes:

\[
\begin{array}{ccc}
C_S & \xrightarrow{H} & D_T \\
\uparrow F_S & & \uparrow F_T \\
C & \xrightarrow{H} & D
\end{array}
\]

For a Kleisli law \( \lambda : H \circ S \Rightarrow T \circ H \), the action on morphisms of its corresponding Kleisli lifting \( H_\lambda : C_S \to D_T \) is:

\[
H_\lambda : \ x \xrightarrow{f} y \mapsto H(A) \xrightarrow{H(f)} H(S(B)) \xrightarrow{\lambda} T(H(B))
\]

Example 5.9. If \( T \) is a strong monad, for each \( A \), the morphisms \( A \otimes T(B) \Rightarrow T(A \otimes B) \) form a Kleisli law.

We will also have need of a dual notion to Kleisli laws. Let \( S : C \to C \) and \( T : D \to D \) be monads, and \( G : D \to C \) a functor. An Eilenberg-Moore law is a natural transformation \( \rho : S \circ G \Rightarrow G \circ T \) such that the following diagrams commute:

\[
\begin{array}{ccc}
S(G(A)) & \xrightarrow{\rho} & G(T(A)) \\
\downarrow \eta & & \downarrow \eta \\
G(A) & & G(T(A))
\end{array}
\]

\[
\begin{array}{ccc}
S^2(G(A)) & \xrightarrow{S(\rho)} & S(G(T(A))) \\
\downarrow \mu & & \downarrow \mu \\
S(G(A)) & \xrightarrow{\rho} & G(T^2(A))
\end{array}
\]

Similarly to Kleisli laws, the significance of Eilenberg-Moore laws is captured by the following well-known result, credited to [App65] in [Joh75].

Theorem 5.10 (Eilenberg-Moore laws Classify Liftings). Let \( S : C \to C \) and \( T : D \to D \) be monads, and \( K : D \to C \) a functor. There is a bijective correspondence between:

1. Eilenberg-Moore laws of type \( S \circ G \Rightarrow G \circ T \).
2. Eilenberg-Moore liftings \( D^T \to C^S \), i.e. functors \( G : D^T \to C^S \) such that the following diagram commutes:

\[
\begin{array}{ccc}
D^T & \xrightarrow{G} & C^S \\
\downarrow U^T & & \downarrow U^S \\
D & \xrightarrow{G} & C
\end{array}
\]

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For a given Eilenberg-Moore law $\rho : S \circ G \Rightarrow G \circ T$, the action on objects of the corresponding Eilenberg-Moore lifting $G^{\rho} : D^{T} \rightarrow C^{S}$ is:

$$G^{\rho} : \ T(A) \xrightarrow{\alpha} A \mapsto \ S(G(A)) \xrightarrow{\rho} G(T(A)) \xrightarrow{G(\alpha)} G(A) \quad (8)$$

The following well-known facts about Kleisli and Eilenberg-Moore laws will be useful.

**Proposition 5.11.** If $L : C \rightarrow D$ is a functor, then:

1. If $\rho : S \circ L \Rightarrow L \circ T$ is an invertible Eilenberg-Moore law, then the inverse $\rho^{-1} : L \circ T \Rightarrow S \circ L$ is a Kleisli law.

2. If we have adjunction $L \dashv R$, with unit $\eta : \text{Id} \Rightarrow R \circ L$ and counit $\epsilon : L \circ R \Rightarrow \text{Id}$, then natural transformation $\rho : S \circ R \Rightarrow R \circ T$ is an Eilenberg-Moore law if and only if its transpose:

$$L \circ S \xrightarrow{L S(\eta)} L \circ S \circ R \circ L \xrightarrow{L(\rho L)} L \circ R \circ T \circ L \xrightarrow{\epsilon L} T \circ L$$

is a Kleisli law.

**5.3 General Arities**

It is natural to consider functors, and corresponding bimorphisms, of more general arities. To do so, we consider categories $C_{1}, \ldots, C_{n}$ and $D$, with:

1. A functor $H : C_{1} \times \ldots \times C_{n} \rightarrow D$.

2. Endofunctors

$$S_{1} : C_{1} \rightarrow C_{1}, \ldots, S_{n} : C_{n} \rightarrow C_{n},$$

and algebras

$$S_{1}(A_{1}) \xrightarrow{\alpha_{1}} A_{1}, \ldots, S_{n}(A_{n}) \xrightarrow{\alpha_{n}} A_{n}.$$  

3. An endofunctor $T : D \rightarrow D$ and algebra $T(B) \xrightarrow{\beta} B$.

We can then generalize condition (4) to

$$H(S_{1}(A_{1}), \ldots, S_{n}(A_{n})) \xrightarrow{H(\alpha_{1}, \ldots, \alpha_{n})} T(H(A_{1}, \ldots, A_{n})) \xrightarrow{T(\beta)} T(B)$$

Assuming further that $S_{1}, \ldots, S_{n}$ are monads, we could then define an $n$-ary Kleisli law as a natural transformation of type

$$H(S_{1}(A_{1}), \ldots, S_{n}(A_{n})) \Rightarrow T(H(A_{1}, \ldots, A_{n}))$$
such that the following two diagrams commute:

\[
\begin{array}{ccc}
H(S_1(A_1), \ldots, S_n(A_n)) & \xrightarrow{\lambda} & \mathcal{T}H(A_1, \ldots, A_n) \\
\downarrow_{H(\eta, \ldots, \eta)} & & \downarrow_{\eta} \\
H(A_1, \ldots, A_n) & \xrightarrow{\eta} & \mathcal{T}H(A_1, \ldots, A_n)
\end{array}
\] (10)

\[
\begin{array}{ccc}
H(S_1^2(A_1), \ldots, S_n^2(A_n)) & \xrightarrow{\lambda} & \mathcal{T}H(S_1(A_1), \ldots, S_n(A_n)) \\
\downarrow_{H(\mu, \ldots, \mu)} & & \downarrow_{\mu} \\
H(S_1(A_1), \ldots, S_n(A_n)) & \xrightarrow{\lambda} & \mathcal{T}H(A_1, \ldots, A_n)
\end{array}
\] (11)

We note that for a family of endofunctors \(T_i : C_i \to C_i\), there is a product endofunctor \(\prod_i T_i : \prod_i C_i \to \prod_i C_i\). In the case that the \(T_i\) are monads, \(\prod_i T_i\) also carries a monad structure pointwise.

**Proposition 5.12.** Assume \(T_i : C_i \to C_i\) is a family of monads. There is an induced monad with:

**Functor:** The endofunctor \(\prod_i T_i : \prod_i C_i \to C_i\) acts pointwise:

\[
\prod_i T_i(A_1, \ldots, A_n) := (T_1(A_1), \ldots, T_n(A_n))
\]

**Unit:** The unit has components:

\[
\eta(A_1, \ldots, A_n) := (\eta A_1, \ldots, \eta A_n)
\]

**Multiplication:** The multiplication has components:

\[
\mu(A_1, \ldots, A_n) := (\mu A_1, \ldots, \mu A_n)
\]

Unravelling definitions, for endofunctors \(T_1, \ldots, T_n\), a \(\prod_i T_i\)-algebra is simply a product of \(T_i\)-algebras. In the case of monads \(\mathcal{T}_1, \ldots, \mathcal{T}_n\) a \(\prod_i \mathcal{T}_i\)-algebra \(((A_1, \ldots, A_n), (\alpha_1, \ldots, \alpha_n))\) is exactly a tuple with each \(\alpha_i\) a \(\mathcal{T}_i\)-algebra structure map. Furthermore, a natural transformation \(\lambda : H \circ \prod_i S_i \Rightarrow \mathcal{T} \circ H\) is a natural family

\[
\lambda : H(\mathcal{T}_1(A_1), \ldots, \mathcal{T}_n(A_n)) \to \mathcal{T}(H(A_1, \ldots, A_n))
\]

\(\lambda\) is a Kleisli law exactly when it satisfies diagrams (10) and (11).

The fact that an \(n\)-ary Kleisli law is the same thing as a Kleisli law for the induced monad on the product category provides two useful perspectives:

1. The product monad perspective means we can prove theoretical results in full generality using the simpler formulation, using condition (4).
2. The explicit condition (9) given above can clarify concrete situations in applications by exposing the separate components.

A similar strategy can be applied to condition (5) for right bimorphisms.

**Example 5.13.** For any monad, dst and dst' are Kleisli laws of type

\[ T(A) \otimes T(B) \to T(A \otimes B) \]

In the terminology of Kleisli laws, for a commutative monad, \( h : A \otimes B \to C \) is a bimorphism exactly if it is a left dst-morphism.

### 5.4 Examples of bimorphisms for Kleisli laws

**Example 5.14.** For any Kleisli law \( \lambda : H \circ S \Rightarrow T \circ H \), the axiom in diagram (7) can be interpreted as saying the components of \( \lambda \) are \( \lambda \)-morphisms \( \mu \to \lambda \mu \).

**Example 5.15.** Let \( \lambda : S \circ T \Rightarrow T \circ S \) be a distributive law in the sense of Beck [Bec69]. An algebra for the induced composite monad \( T \circ S \) consists of

1. A \( S \)-algebra \( \alpha^S : S(A) \to A \).
2. A \( T \)-algebra \( \alpha^T : T(A) \to A \).

These must satisfy the following compatibility condition:

\[
\begin{array}{ccc}
S\lambda & \xrightarrow{\lambda} & T\lambda S \xrightarrow{\lambda T} T \xrightarrow{\alpha^T} A \\
S(\alpha^T) & \downarrow & S(\alpha^T) & \downarrow & A \\
S(A) & \xrightarrow{\alpha^S} & A \\
\end{array}
\]

That is, \( \alpha^S : \alpha^T \to \lambda \alpha^T \).

**Example 5.16.** Every monad morphism \( \sigma : S \Rightarrow T \) induces:

1. A functor \( \text{Id}_\sigma : C_S \to D_T \), identity on objects, with action on morphisms:

\[ f \mapsto \sigma \circ f \]

2. A functor \( \text{Id}_\sigma : D_T \to C_S \), identity on objects, with action on objects:

\[ (A, \alpha) \mapsto (A, \alpha \circ \sigma) \]

Such a \( \sigma \) is a special case of a Kleisli law, of type \( \text{Id} \circ S \Rightarrow T \circ \text{Id} \). A left \( \sigma \)-morphism from \( \alpha \) to \( \beta \) satisfies:

\[
\begin{array}{ccc}
S(A) & \xrightarrow{\sigma} & T(A) & \xrightarrow{T_h} & T(B) \\
\alpha & \downarrow & & \downarrow & \beta \\
A & \xrightarrow{h} & B \\
\end{array}
\]

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By the naturality of $\sigma$, this is equivalent to:

$$
S(A) \xrightarrow{S(h)} S(B) \xrightarrow{\sigma} T(B)
$$

Which is exactly the condition that $h$ is an $S$-algebra morphism of type $\alpha \to \text{Id}^\sigma(\beta)$.

6 Bimorphisms Graphically

We now briefly explore bimorphisms in terms of string diagrams. For background on the notation, see for example [HM16b, HM16a]. For $T$-algebras $(A, \alpha)$ and $(B, \beta)$, $h$ is an algebra morphism of type $\alpha \to \beta$ if the following equation holds.

Intuitively, we see this as the ability to slide “the” algebra past the homomorphism.

Now let $(A, \alpha)$ be an $S$-algebra, and $(B, \beta)$ an $T$-algebra, and $\lambda : H \circ S \Rightarrow T \circ H$ a natural transformation. Then, $h$ is a left $\lambda$-morphism if the following equation holds:

$$
H S A \xrightarrow{\alpha} H S A
$$

Our bimorphism $h : H(A) \to B$ now has a structured input, and visually the mediating natural transformation $\lambda$ allows us to “cross wires” so that we can slide the algebra over the bimorphism.

If $(A, \alpha)$ is a $S$-algebra, and $(B, \beta)$ an $T$-algebra, and $\rho : S \circ G \Rightarrow G \circ T$,
then $h : A \to G(B)$ is a right $\rho$-morphism if the following equation holds:

$$h \circ A \to G(B) \alpha\circ S \beta = h \circ G \circ A \circ S \circ \beta \circ \rho (13)$$

Now our bimorphism $h : A \to G(B)$ has a structured output, and the mediating natural transformation $\rho$ allows us to cross wires, so we can slide the algebras over the bimorphism.

The visual nature of the string diagram renditions of the bimorphism conditions (4) and (5) as (12) and (13) gives a more intuitive sense for how the natural transformation is needed for the bimorphism to commute with the algebra structures.

Drawing the identity functor explicitly, as a dashed edge, we see that the ordinary homomorphism condition is a special case of both of the above. Specializing equation (12):

$$H \circ S \circ A \alpha \circ h \circ B = H \circ S \circ A \circ h \circ B \circ h \circ B (13)$$

Similarly, specializing condition (12):

$$S \circ A \alpha \circ k \circ K \circ B = S \circ A \circ k \circ B \circ k \circ B$$

Diagrammatically, it is easy to see that if $h : \alpha \to \lambda \beta$ and $g : \beta \to \lambda' \gamma$, then
As a special case of the above observation, we note that if \( m \) is a left \( \lambda \)-morphism of type \( \alpha \to \lambda \beta \), and \( h : \alpha' \to \alpha \) and \( k : \beta \to \beta' \) are algebra morphisms, then \( k \circ m \circ H(h) : \alpha' \to \lambda \beta' \).

7 Universal Constructions

The results of this section generalize the lifting of monoidal structure in the case of commutative monads discussed in section 4.3. Probably the most explicit proofs for the analogous commutative monads constructions appear in [Sea13]. We wish to acknowledge this paper had great impact on our own approach.

7.1 Classifying Objects for \( H_\lambda \)-morphisms

Let \( S : C \to C \) and \( T : D \to D \) be monads, \( H : C \to D \) a functor and \( \lambda : H(S(A)) \to T(H(A)) \) a morphism in \( D \). An important feature of bimorphisms is that under sufficient cocompleteness conditions, they are in bijection with ordinary algebra homomorphisms of a suitable type. Specifically, given an \( S \)-algebra \( (A, \alpha) \), if the following coequalizer exists in \( D \):

\[
\begin{array}{c}
F^T(H(S(A))) \\
\mu^T \circ F^T(\lambda) \\
\mu^T(\lambda)
\end{array} \xrightarrow{\eta} F^T(H(A)) \xrightarrow{q_\alpha} (W_\alpha, \omega_\alpha)
\]

then bimorphisms \( \alpha \to \lambda \beta \) are in bijection with \( T \)-algebra morphisms \( \omega_\alpha \to \beta \), with \((W_\alpha, \omega_\alpha)\) as defined in diagram (14).

This statement is made precise in the following theorem.

**Theorem 7.1 (Universal Bimorphisms).** Assuming the coequalizer in (14) exists, the composite morphism

\[
u := H(A) \xrightarrow{\eta} F^T(H(A)) \xrightarrow{q_\alpha} W_\alpha
\]

is a left \( \lambda \)-morphism \( \nu : \alpha \to \lambda \omega_\alpha \). Furthermore, it is universal in the sense that every \( T \)-algebra \( (B, \beta) \) and \( h : \alpha \to \lambda \beta \) there exists a unique \( T \)-algebra
morphism \( \hat{h} : \omega_\alpha \to \beta \) such that the following commutes:

\[
\begin{array}{ccc}
H(A) & \xrightarrow{u} & W_\alpha \\
\downarrow h & & \downarrow \hat{h} \\
\downarrow & & \downarrow B
\end{array}
\]

**Proof.** Firstly, we must show \( u : H(A) \to W_\alpha \) is a bimorphism. We calculate:

\[
\begin{align*}
\omega_\alpha & \circ T(q_\alpha) \circ \eta \circ \lambda \\
= & \{ \text{\( q_\alpha \) is a \( T \)-algebra homomorphism} \} \\
= & \{ \text{\( q_\alpha \) is a \( T \)-algebra homomorphism} \} \\
= & \{ \text{\( q_\alpha \) \( \mu \) \( \eta \) \( \lambda \)} \} \\
= & \{ \text{\( q_\alpha \) \( \mu \) \( \eta \) \( \lambda \)} \} \\
= & \{ \text{\( q_\alpha \) \( \mu \) \( \eta \) \( \lambda \)} \} \\
= & \{ \text{\( q_\alpha \) \( \mu \) \( \eta \) \( \lambda \)} \} \\
= & \{ \text{\( q_\alpha \) \( \mu \) \( \eta \) \( \lambda \)} \} \\
= & \{ \text{\( q_\alpha \) \( \eta \)-naturality} \} \\
= & \{ \text{\( q_\alpha \) \( \eta \)-naturality} \} \\
= & \{ \text{\( q_\alpha \) \( \eta \)-naturality} \} \\
\end{align*}
\]

We then note that if \( h : \alpha \to \lambda \beta \) then \( \beta \circ F^\Sigma(h) \) equalizes the parallel pair in diagram (14). Reasoning in the base category:

\[
\begin{align*}
\beta & \circ T(h) \circ \mu \circ T(\lambda) \\
= & \{ \text{\( \mu \)-naturality} \} \\
\beta & \circ T^2(h) \circ T(\lambda) \\
= & \{ \text{\( \mu \)-naturality} \} \\
\beta & \circ T(\beta) \circ T^2(h) \circ T(\lambda) \\
= & \{ \text{\( \beta \)-naturality plus functoriality of \( T \)} \} \\
\beta & \circ T(h) \circ T(H(\alpha))
\end{align*}
\]

Therefore by the coequalizer universal property, there is an induced algebra morphism \( \hat{h} : \omega_\alpha \to \beta \) such that \( \hat{h} \circ q_\alpha = \beta \circ F^\Sigma(h) \). Conversely, given any algebra morphism \( k : \omega_\alpha \to \beta \), \( k \circ u \) is a bimorphism, as bimorphisms compose and algebra morphisms are bimorphisms too (cf. the end of section 6).

We aim to show the mappings \( h \mapsto \hat{h} \) and \( k \mapsto k \circ u \) are mutually inverse. In one direction:

\[
\hat{h} \circ u
\]
\[
\begin{align*}
\hat{h} \circ q_\alpha \circ \eta &= \{ \text{definition} \} \\
\beta \circ \eta \\
\beta \circ \eta \circ \hat{h} &= \{ \text{Eilenberg-Moore algebra unit axiom} \}
\end{align*}
\]

In the other direction, as:

\[
\begin{align*}
\hat{k} \circ u \circ q_\alpha &= \{ \text{definition of } u \text{ and the coequalizer} \} \\
\beta \circ T(k) \circ T(q_\alpha) \circ T(\eta) &= \{ \text{algebra homomorphism} \} \\
k \circ \omega_\alpha \circ T(q_\alpha) \circ T(\eta) &= \{ \text{algebra homomorphism} \} \\
k \circ q_\alpha \circ \mu \circ T(\eta) &= \{ \text{monad unit axiom} \} \\
k \circ q_\alpha
\end{align*}
\]

the coequalizer universal property completes the proof.

Note that, in view of the discussion at the end of section 6, the diagram in theorem 7.1 expresses that the bimorphisms \( h : \alpha \to \lambda \beta \) decompose as the composition of bimorphisms \( u : \alpha \to \omega_\alpha \) and \( \hat{h} : \omega_\alpha \to \id \beta \).

**Theorem 7.2** (Functorial Universal Bimorphisms). If \( \lambda \) is in fact a natural transformation \( H \circ S \Rightarrow T \circ H \), and the coequalizer (14) exists at \( \alpha \), for every \( S \)-algebra \((A, \alpha)\), then \( \hat{H}_\lambda \) extends to a functor \( C^S \to D^T \).

This theorem follows from the following localised version of this statement.

**Proposition 7.3.** If \((A, \alpha)\) and \((A', \alpha')\) are \( S \)-algebras, the coequalizer (14) exists at \( \alpha \) and \( \alpha' \), and there exist \( \lambda_A : H(S(A)) \to T(H(A)) \) and \( \lambda_{A'} : H(S(A')) \to T(H(A')) \), every \( C^S \)-morphism \( f : \alpha \to \alpha' \) such that \( \lambda \) is natural with respect to \( f \), in that the following commutes:

\[
\begin{array}{ccc}
H(S(A)) & \xrightarrow{\lambda_A} & T(H(A)) \\
\downarrow_{H(S(f))} & & \downarrow_{T(H(f))} \\
H(S(A')) & \xrightarrow{\lambda_{A'}} & T(H(A'))
\end{array}
\]
induces a $\mathcal{D}^T$-morphism $\widehat{H}_\lambda(f) : \widehat{H}_\lambda(\alpha) \to \widehat{H}_\lambda(\alpha)$. Moreover, $\widehat{H}_\lambda(\text{id}) = \text{id}$ and $\widehat{H}_\lambda(f \circ g) = \widehat{H}_\lambda(f) \circ \widehat{H}_\lambda(g)$ whenever $\widehat{H}_\lambda(f)$ and $\widehat{H}_\lambda(g)$ are induced this way.

Proof. In diagram (15) below, the two rows are coequalizer diagrams.

\[ \begin{array}{ccc}
F^T(H(T(A))) & \xrightarrow{\mu \circ F^T(\lambda_A)} & F^T(H(A)) & \xrightarrow{q_\alpha} & (W_\alpha, \omega_\alpha) \\
F^T(\lambda_A) & \downarrow & F^T(H(\alpha)) \downarrow \mu & & \\
F^T(H(T(A'))) & \xrightarrow{\mu \circ F^T(\lambda_{A'})} & F^T(H(A')) & \xrightarrow{q_{\alpha'}} & (W_{\alpha'}, \omega_{\alpha'})
\end{array} \]

(15)

We wish to show that $q_{\alpha'} \circ F^T(H(f))$ equalizes the parallel pair in the top of the diagram.

\[
q_{\alpha'} \circ T(H(f)) \circ \mu \circ T(\lambda_A) = \{ \mu\text{-naturality} \} \\
= \{ q_{\alpha'} \circ \mu \circ T^2(H(f)) \circ T(\lambda_A) \} \\
= \{ T\text{-functoriality} \} \\
q_{\alpha'} \circ \mu \circ T(T(H(f)) \circ \lambda_A)) = \{ \lambda\text{-naturality with respect to } f \} \\
q_{\alpha'} \circ \mu \circ T(\lambda_{A'} \circ H(S(f))) = \{ T\text{-functoriality} \} \\
q_{\alpha'} \circ \mu \circ T(\lambda_{A'} \circ T(H(S(f)))) = \{ \text{coequalizer} \} \\
q_{\alpha'} \circ T(H(\alpha')) \circ T(H(S(f))) = \{ \text{functoriality twice} \} \\
q_{\alpha'} \circ T(H(\alpha')) \circ T(H(S(f))) = \{ \text{algebra homomorphism} \} \\
q_{\alpha'} \circ T(H(f \circ \alpha)) = \{ \text{functoriality twice} \} \\
q_{\alpha'} \circ T(H(f)) \circ T(H(\alpha)) = \{ \text{universal property of coequalizers} \}
\]

This therefore induces a morphism $\widehat{H}_\lambda(f) : (W_\alpha, \omega_\alpha) \to (W_{\alpha'}, \omega_{\alpha'})$. By the universal property of coequalizers, we clearly have $\widehat{H}_\lambda(\text{id}) = \text{id}$. If we consider two algebra homomorphism $f : \alpha \to \alpha'$ and $g : \alpha' \to \alpha''$, and morphisms $\lambda_A, \lambda_{A'}$ and $\lambda_{A''}$ which are natural with respect to $f$ and $g$, then consider the diagram...
below:

\[
\begin{aligned}
F^T(H(T(A))) & \xrightarrow{\mu \circ F^T(\lambda_A)} F^T(H(A)) \xrightarrow{\eta_A} (W_\alpha, \omega_\alpha) \\
& \downarrow F^T(H(f)) \\
F^T(H(T(A'))) & \xrightarrow{\mu \circ F^T(\lambda_{A'})} F^T(H(A')) \xrightarrow{\eta_{A'}} (W_{\alpha'}, \omega_{\alpha'}) \\
& \downarrow F^T(H(g)) \\
F^T(H(T(A''))) & \xrightarrow{\mu \circ F^T(\lambda_{{A''}})} F^T(H(A'')) \xrightarrow{\eta_{A''}} (W_{\alpha''}, \omega_{\alpha''})
\end{aligned}
\]

Then we must have \( \hat{H}_\lambda(g \circ f) = \hat{H}_\lambda(g) \circ \hat{H}_\lambda(f) \), as both are suitable fill-in morphisms satisfying the universal property of coequalizers from the top row to the bottom.

**Theorem 7.4** (Universal Bimorphisms and Free Constructions). If \( \lambda : H \circ S \Rightarrow T \circ H \) is a Kleisli law and satisfies the assumptions of theorem 7.2, then there is a natural isomorphism:

\[
F^T(H(A)) \cong \hat{H}_\lambda(F^S(A))
\]

**Proof.** Beyond what is proved in 7.5, we must establish that the isomorphism is natural. So we require the following diagram to commute:

\[
\begin{array}{c}
\hat{H}(F^S(A)) \xrightarrow{\iota_A} F^T(H(A)) \\
\downarrow F^T(H(h)) \\
\hat{H}(F^S(B)) \xrightarrow{\iota_B} F^T(H(B))
\end{array}
\]

Here the morphisms \( \iota_A \) are the induced isomorphisms from proposition 7.5. The following diagram commutes:

\[
\begin{array}{c}
\hat{H}(F^S(A)) \leftarrow \hat{H}(F^S(h)) \xrightarrow{q_{\nu A}} T(H(S(A))) \xrightarrow{q_{\eta A}} \hat{H}(F^S(A)) \\
\downarrow \hat{H}(F^S(h)) \\
\hat{H}(F^S(B)) \xrightarrow{q_{\nu B}} T(H(S(A))) \xrightarrow{T(H(\lambda_A))} \hat{H}(F^S(A)) \\
\downarrow \hat{H}(F^S(h)) \\
F^T(H(B)) \xrightarrow{\mu_{H(B)}} T(H(B)) \xrightarrow{T^2(H(A))} F^T(H(A)) \\
\downarrow \mu_{H(B)} \\
F^T(H(B)) \xrightarrow{\mu_{H(B)}} T(H(B)) \xrightarrow{T^2(H(h))} F^T(H(h))
\end{array}
\]

The diamond and the bottom right trapezium commute by naturality, and the remaining parts by the definitions of the coequalizer constructions involved. As the paths from the centre top of the diagram to the bottom corners are equal, both the left and right hand side are candidate coequalizer fill-in morphisms, induced by the same morphism. By the universal property of coequalizers,
both the sides of the diagram, and hence both paths around diagram (16), are equal.

Again, we recover the construction from a local version.

**Proposition 7.5.** If

\[ \lambda_A : H(S(A)) \to T(H(A)) \quad \text{and} \quad \lambda_{S(A)} : H(S^2(A)) \to T(H(S(A))) \]

are \( \mathcal{D} \)-morphisms satisfying Kleisli axioms (6) and (7) at \( A \) and natural with respect to \( \eta^A \) : \( A \to S(A) \), then the coequalizer defining \( H_\lambda(F^\gamma(T(A))) \) exists, and is isomorphic to \( F^\gamma(T(H(A))) \).

**Proof.** We aim to show that (17) is a coequalizer diagram:

\[
\begin{align*}
F^\gamma(T(H(S^2(A)))) & \xrightarrow{\mu \circ F^\gamma(\lambda_{S(A)})} F^\gamma(T(H(S(A)))) \xrightarrow{F^\gamma(\lambda_A)} F^\gamma(T(H(A))) \xrightarrow{\mu} F^\gamma(T(H(A)))
\end{align*}
\]

(17)

Firstly, to see \( \mu \circ F^\gamma(\lambda_A) \) coequalizes the parallel pair:

\[
\begin{align*}
\mu \circ T(\lambda_A) & \circ T(\lambda_{S(A)}) \\
& = \{ \mu \text{-naturality} \} \\
& \mu \circ \mu \circ T^2(\lambda_A) \circ T(\lambda_{S(A)}) \\
& = \{ \text{monad associativity axiom} \} \\
& \mu \circ T(\mu) \circ T^2(\lambda_A) \circ T(\lambda_{S(A)}) \\
& = \{ T\text{-functoriality} \} \\
& \mu \circ T(\mu \circ T(\lambda_A) \circ \lambda_{S(A)}) \\
& = \{ \text{Kleisli axiom (7) at } A \} \\
& \mu \circ T(\lambda_A \circ H(\mu)) \\
& = \{ T\text{-functoriality} \} \\
& \mu \circ T(\lambda_A) \circ T(H(\mu))
\end{align*}
\]

If algebra morphism \( f : F^\gamma(T(H(S(A))) \to (B, \beta) \) coequalizes the parallel pair in diagram (17) then \( f = f \circ F^\gamma(T(H(\eta))) \circ \mu \circ F^\gamma(\lambda_A) \) as:

\[
\begin{align*}
f \circ T(H(\eta)) & \circ \mu \circ T(\lambda_A) \\
& = \{ \mu \text{-naturality} \} \\
f \circ \mu \circ T^2(H(\eta)) \circ T(\lambda_A) \\
& = \{ T\text{-functoriality} \} \\
f \circ \mu \circ T(T(H(\eta)) \circ \lambda_A)
\end{align*}
\]

\(^1\)That is, \( \eta^A_{H(A)} = \lambda_A \circ H(\eta^A) \) and \( \mu^A_{H(A)} \circ T(\lambda_A) \circ \lambda_{S(A)} = \lambda_A \circ H(\mu^A_A) \).
\[
\begin{align*}
&= \{ \text{naturality of } \lambda \text{ wrt } \eta \} \\
&= \{ \text{T-functoriality } \} \\
&= \{ f \text{ coequalizes } \mu \circ \mathbb{T}(\lambda(A)) \text{ and } \mathbb{T}(H(\mu)) \} \\
&= \{ \text{functoriality and monad unit axiom } \}
\end{align*}
\]

We then note that \( \mu \circ \mathbb{T}(\lambda_A) \) is a split epimorphism, as:

\[
\begin{align*}
\mu \circ \mathbb{T}(\lambda_A) \circ \mathbb{T}(H(\eta)) &= \{ \text{T-functoriality } \} \\
\mu \circ \mathbb{T}(\lambda_A \circ H(\eta)) &= \{ \text{Kleisli axiom (6) at } A \} \\
\mu \circ \mathbb{T}(\eta) &= \{ \text{monad unit axiom } \} \\
\text{id}
\end{align*}
\]

Therefore, diagram (17) satisfies the universal property of a coequalizer diagram.

By theorem 7.4, we may view \( \overline{H}_\lambda \) as the lifting of \( H_\lambda \) to the category of algebras, as depicted in the following commutative diagram (up to isomorphism).

This is gives us a generalised version of theorem 4.8.

**Example 7.6** (Lifting Binary Coproducts). Let \( \mathbb{T} \) a monad on a category \( \mathcal{C} \) with binary coproduct. For every pair \( A, B \) of \( \mathcal{C} \) objects, there is a canonical morphism:

\[
\mathbb{T}(\kappa_1), \mathbb{T}(\kappa_2) : T(A) + T(B) \to T(A + B),
\]

Observe that under the assumptions that \( \lambda \) is a Kleisli law, the coequalizer in (14) is in fact a coequalizers of reflexive pairs. Indeed, \( \mathbb{T}(H(\eta)) \) is always a section of \( \mathbb{T}(H(\alpha)) \) and by (6) the same holds for \( \mu^\mathbb{T} \circ F^\mathbb{T}(\lambda) \). Although the existences of coequalizers in Eilenberg-Moore categories is by no means automatic, standard conditions under which they do exist are known. See for example the accounts in [BW00], [Bor94, Chapter 4], and [PT04, Chapter 5].
where the $\kappa_i$ are the coproduct injections, and $[f, g]$ is the morphism induced by $f$ and $g$ by the coproduct universal property. This is clearly natural in $A$ and $B$. In fact it is a Kleisli morphism. For the unit axiom we calculate:

$$
[T(\kappa_1), T(\kappa_2)] \circ \eta + \eta
= \{ \text{coproducts } \}
[T(\kappa_1) \circ \eta, T(\kappa_2) \circ \eta]
= \{ \text{naturality } \}
[\eta \circ \kappa_1, \eta \circ \kappa_2]
= \{ \text{coproducts } \}
\eta \circ [\kappa_1, \kappa_2]
= \{ \text{coproducts } \}
\eta
$$

For the multiplication axiom:

$$
[T(\kappa_1), T(\kappa_2)] \circ \mu + \mu
= \{ \text{coproducts } \}
[T(\kappa_1) \circ \mu, T(\kappa_2) \circ \mu]
= \{ \text{naturality } \}
[\mu \circ T^2(\kappa_1), \mu \circ T^2(\kappa_2)]
= \{ \text{coproducts } \}
\mu \circ [T^2(\kappa_1), T^2(\kappa_2)]
= \{ \text{coproducts } \}
\mu \circ [T[T(\kappa_1), T(\kappa_2)] \circ T(\kappa_1), T[T(\kappa_1), T(\kappa_2)] \circ T(\kappa_2)]
= \{ \text{coproducts } \}
\mu \circ T[T(\kappa_1), T(\kappa_2)] \circ [T(\kappa_1), T(\kappa_2)]
$$

The bimorphism condition with respect to the canonical morphism is:

$$
T(A) + T(B) \xrightarrow{T(\kappa_1) + T(\kappa_2)} T(A + B) \xrightarrow{T(h)} T(C)
\downarrow \alpha + \beta
\downarrow h
A + B \xrightarrow{h} C
\downarrow \gamma
$$

We can define $h_1 = h \circ \kappa_1$ and $h_2 = h \circ \kappa_2$, and by the universal property of
coproducts, \( h = [h_1, h_2] \), so we can rewrite our diagram as:

\[
T(A) \oplus T(B) \xrightarrow{[T(\kappa_1) + T(\kappa_2)]} T(A + B) \xrightarrow{T([h_1, h_2])} T(C)
\]

which commutes if and only if \( h_1 : \alpha \to \gamma \) and \( h_2 : \beta \to \gamma \). If \( C^T \) has coequalizers of reflexive pairs, we have bijective correspondences:

\[
\begin{align*}
&h_1 : \alpha \to \gamma \\
&h_2 : \beta \to \gamma \\
&h : (\alpha, \beta) \to_{[T(\kappa_1), T(\kappa_2)]} \gamma
\end{align*}
\]

This establishes that \( \hat{\oplus} \) is the coproduct in \( C^T \).

### 7.2 Classifying Objects for \( H^\rho \)-morphisms

Given the straightforward functor action induced by Eilenberg-Moore laws (8), relating right bimorphisms with respect to an Eilenberg-Moore law

\[
\rho : S \circ G \Rightarrow G \circ T
\]

to ordinary algebra morphisms is relatively trivial. We observe that for algebras \((A, \alpha)\) and \((B, \beta)\), the following are equivalent for a morphism \( h : A \to G(B) \):

1. \( h \) is an algebra morphism \( \alpha \to G^\rho(\beta) \).
2. \( h \) a right \( \rho \)-morphism \( \alpha \to^\rho \beta \).

Observe that the identity \( \text{id} : G(B) \to G(B) \) is a bimorphism \( G^\rho(\beta) \to^\rho \beta \). Consequently, any bimorphism \( h : \alpha \to^\rho \beta \) is equal to the composition of bimorphisms:

\[
\alpha \xrightarrow{h} \text{id} G^\rho(\beta) \xrightarrow{\text{id}^\rho} \beta
\]

Note that this is true even in the more general setting where \( T \) and \( S \) are only required to be endofunctors and \( \rho \) is just a natural transformation inducing a functor \( G^\rho : \text{Alg}(T) \to \text{Alg}(S) \) between categories of endofunctor algebras.

### 8 Dualizing to Comonads

Naturally, as with every categorical concept, the notions introduced in the previous sections dualize. Although dualizing is routine, we provide explicit description of the main results for concreteness. The results for Kleisli laws of comonads have applications in the emerging theory of *game comonads* [AS21], which motivated our original investigations.
First, we dualize the notion bimorphism suitable for endofunctor coalgebras. Because of dualizing, the Kleisli laws are stated in terms of right instead of left bimorphisms. Namely, given functors \( G: \mathcal{C} \to \mathcal{D} \), \( C: \mathcal{C} \to \mathcal{C} \) and \( D: \mathcal{D} \to \mathcal{D} \) and a morphism \( \rho: D(G(B)) \to G(C(B)) \) in \( \mathcal{D} \), we say that \( h: A \to G(B) \) is a (coalgebraic right) \( \rho \)-morphisms, or just bimorphism, from a \( D \)-coalgebra \((A, \alpha)\) to a \( C \)-coalgebra \((B, \beta)\), if it makes the following diagram commute:

\[
\begin{array}{ccc}
A & \xrightarrow{h} & G(B) \\
\downarrow{\alpha} & & \downarrow{G\beta} \\
D(A) & \xrightarrow{D(h)} & D(G(B)) & \xrightarrow{\rho} & G(C(B))
\end{array}
\] (18)

Further, if \( C, D \) are comonads then a natural transformation \( \rho: D \circ G \Rightarrow G \circ C \) is a Kleisli law if the following commute:

\[
\begin{array}{ccc}
\mathbb{D}(G(A)) & \xrightarrow{\rho} & G(C(A)) \\
\downarrow{\epsilon} & & \downarrow{K\epsilon} \\
G(A) & \xrightarrow{=} & G(C(A))
\end{array}
\]

\[
\begin{array}{ccc}
\mathbb{D}(G(A)) & \xrightarrow{\rho} & G(C(A)) \\
\downarrow{\delta} & & \downarrow{G\delta} \\
\mathbb{D}^2(G(A)) & \xrightarrow{\mathbb{D}(\rho)} & \mathbb{D}(G(C(A))) & \xrightarrow{\rho} & G(C^2(A))
\end{array}
\]

Then, the dual situation of comonads behaves dually to that of monads. In particular, theorem 7.1 dualises as:

**Theorem 8.1.** For comonads \( C, D \) and a functor \( G \) as above, a \( C \)-coalgebra \((A, \alpha)\) and a morphism \( \rho: D(G(A)) \to G(C(A)) \), assume that the following diagram is an equalizer in \( \mathcal{D}^2 \):

\[
(W_{\alpha}, \omega_{\alpha}) \xrightarrow{} F^D(G(A)) \xrightarrow{F^D(\lambda) \circ \delta^D} F^D(G(C(A)))
\] (19)

Then, there exists a \( \rho \)-morphism \( u: W_{\alpha} \to G(A) \) from \( \omega_{\alpha} \) to \( \alpha \) such that, for every \( \rho \)-morphism \( h: B \to G(A) \) from a \( D \)-coalgebra \( \beta \) to \( \alpha \), there exists a coalgebra morphism \( \hat{h}: \beta \to \omega_{\alpha} \) such that \( h = u \circ \hat{h} \).

And formally dualizing of theorems 7.2 and 7.4 into a single theorem for conciseness, gives us:

**Theorem 8.2.** If \( \rho: D(G(A)) \Rightarrow G(C(A)) \) is a natural transformation, and all coequalizers of the form (19) exist, then there exists a functor \( \widehat{G}_\rho: C^C \to D^D \).

If, furthermore, \( \lambda \) is a Kleisli law, then there is a natural isomorphism:

\[
F^D(G(A)) \cong \widehat{G}_\rho(F^C(A))
\] (20)
9 Lifting Adjunctions via Bimorphisms

We now prove a well-known adjoint lifting theorem [Joh75, Theorem 2] also given in dual form in [Kei75]. Our aim is to make explicit the application of the theory of bimorphisms in the proof.

Firstly, note that by proposition 5.11, item 2, if we have an adjunction $L \dashv R$, then the transpose operation:

\[
\rho : S \circ R \Rightarrow R \circ T
\]

yields a bijection between natural transformations of type $S \circ R \Rightarrow R \circ T$ and those of type $L \circ S \Rightarrow T \circ L$. Furthermore, it is easy to check this restricts to a bijection between Eilenberg-Moore laws and Kleisli laws of the corresponding types.

Assuming an $L \dashv R$, and an Eilenberg-Moore law $\rho : S \circ R \Rightarrow R \circ T$, let $\lambda : L \circ S \Rightarrow T \circ L$ be defined by the mapping (21). We then observe, writing $\eta$ and $\epsilon$ for the unit and counit of the adjunction, that:

\[
\alpha : A \to R(B) \text{ is a right } \rho\text{-morphism if and only if its transpose } L(h) \circ \epsilon \text{ of type } L(A) \to B \text{ is a left } \lambda\text{-morphism.}
\]

By the snake equation for the unit an counit of the adjunction. This establishes that $h : A \to R(B)$ is a right $\rho$-morphism if and only if its transpose $L(h) \circ \epsilon$ of type $L(A) \to B$ is a left $\lambda$-morphism. Therefore we have a bijection:

\[
\frac{\alpha \to R\rho(\beta)}{\alpha \to \lambda \beta}
\]

where $R\rho$ denotes the functor induced by the Eilenberg-Moore law $\rho$.

If $D^T$ has coequalizers of reflexive pairs, then the universal property of the classifying objects gives a bijection:

\[
\frac{\alpha \to \lambda \beta}{\hat{L}_\lambda(\alpha) \to \beta}
\]

Combining these two bijections, we have show that $\hat{L}_\lambda \dashv R\rho$.
Example 9.1 (Adjoints to functors induced by monad morphisms). It is well-known that a monad morphism $\sigma : S \Rightarrow T$ can be viewed as an Eilenberg-Moore law $S \circ \text{Id} \Rightarrow \text{Id} \circ T$, and trivially $\text{Id}_C \dashv \text{Id}_C$. Therefore, we can apply the previous result if $C^T$ has coequalizers of reflexive pairs. The Eilenberg-Moore law induces a functor $\text{Id}^\sigma : C^T \rightarrow C^S$, which has a left adjoint $\text{Id}^\sigma : C^S \rightarrow C^T$.

We also note that if we have an invertible Eilenberg-Moore $\lambda^{-1} : T \circ L \Rightarrow L \circ S$, then by proposition 5.11, item 1, $\lambda : L \circ S \Rightarrow T \circ L$ is a Kleisli law, and by item 2 its transpose $\rho : S \circ R \Rightarrow R \circ T$ is an Eilenberg-Moore law. We can then apply the previous result to lift this adjunction to the Eilenberg-Moore categories. This is [Joh75, Theorem 4].

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