Remarks on attractors for affine schemes

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Abstract. Let $S$ be a scheme. We remark that for a diagonalizable group scheme $D(M)_{S}$ acting on an $S$-affine scheme $X$ and for any submonoid $N$ of $M$, we have a natural closed subscheme $X^{N}$ of $X$.

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1. Introduction

For a diagonalizable group scheme $D(M)_{S}$ acting on an $S$-affine scheme $X$ over a scheme $S$, we introduce for any submonoid $N$ of $M$ an attractor scheme $X^{N}$. If $M = Z$ and $N = 0$ (resp. $N = N$ and $N = -N$), our definition coincides with [Ri16] and gives the fixed points (resp. algebraic attractors and repellers). Recall that the definition given in [Ri16] generalizes [Dr13] in the sense that [Dr13] considers only bases that are spectra of fields. In [JS18], over fields, the authors generalize the theory to actions of much more general groups than $G_{m}$ and prove the
existence of Bialynicki-Birula [Bi73] decompositions in this context. Our construction goes in a different generality than [JS18] for two main reasons:

(i) in the case of $\mathbb{G}_m$ (i.e $M = \mathbb{Z}$) over fields, monoids appearing in the construction of [JS18] are the same than [Dr13] or [Ri16], whereas we can also consider monoids like $0 \cup \mathbb{N}_{\geq k}$, $a\mathbb{N}$, $a\mathbb{N} + b\mathbb{N}$ or $\mathbb{N} \setminus \{1, 2, 4, 5, 7, 10, 13\}$.

(ii) we allow bases that are not fields.

To our knowledge, in the $\mathbb{G}_m$-case, the definition of attractors as functors using equivariant morphisms appeared first in the works of Hesselink [He80, §4] and Jurciewick [Ju85, §1.2.10].

At the end of the document, we explain that our formalism allows to see roots groups of reductive groups as attractors.

Section 2 introduces notations and definitions about schemes associated to monoids. Section 3 gives the definition of attractors as functors. Section 4 introduces attractors with prescribed limits. Section 5 shows that attractors are representable in the affine case. Section 6 is about non trivial cartesian squares. Section 7 is about the interpretation of root groups as attractors for split reductive groups over fields. We continue our work in [Ma22] and extend it to algebraic spaces.

2. Rings associated to monoids and their spectra

We refer to [Og] for a detailed introduction to monoids. We recall in this section some basic definitions and facts that will turn out to be important in our work. In this article, monoids are always abelian monoids.

**Definition 2.1.** A monoid is a set endowed with an associative and commutative operation, with a neutral element.

**Definition 2.2.** Let $R$ be a commutative ring and let $N$ be a monoid, then we denote by $R[N]$ the associated $R$-algebra whose underlying $R$-module is $\oplus_{n \in N} R.X^n$ and multiplication is induced by the operation of the monoid: $X^n \times X^{n'} = X^{n+n'}$. By convention we have $X^0 = 1$.

**Example 2.3.** Let $R$ be a ring.

(i) $R[\mathbb{N}] \simeq R[X]$ is the $R$-algebra of polynomials.

(ii) $R[\mathbb{Z}] \simeq R[X, X^{-1}]$ is the $R$-algebra of Laurent polynomials.

(iii) $R[0 \cup \mathbb{N}_{\geq k}]$ is the algebra of polynomials with monomials of degree 0 or bigger than $k$.

(iv) $R[\mathbb{N}^n] \simeq R[X_1, \ldots, X_n]$ is the algebra of polynomials with $n$ variables.

(v) $R[\mathbb{Z}/n\mathbb{Z}] \simeq R[X]/X^n - 1$.

(vi) If $N' \subset N$, then we have a canonical injective morphism of rings $R[N'] \to R[N]$.

**Definition 2.4.** Let $N$ be a monoid, then we define $A(N)$ to be the scheme $\text{Spec}(\mathbb{Z}[N])$. If $S$ is a scheme, then we put $A(N) \times S = A(N) \times_{\text{Spec}(\mathbb{Z})} S$. This is a monoid scheme over $S$. If $M$ is an abelian group, then $A(M)$ is denoted $D(M)$ and is a group scheme called the diagonalizable group scheme associated to $M$ [SGA3, Exp 1, §4.4].

If $N' \subset N$, then we obtain morphisms of $S$-schemes $A(N) \times S \to A(N') \times S$, for every scheme $S$. 

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Let $M$ be an abelian group and $N$ be a submonoid of $M$. Then we have an algebraic action of $D(M)$ on $A(N)$ over $\mathbb{Z}$ given by:
\[
Z[N] \to Z[N] \otimes Z[M] \\
X^n \to X^n \otimes X^n.
\]

By base change, we obtain an action of $D(M)_S$ on $A(N)_S$ for every scheme $S$. Remark that the action of $D(M)_S$ on $A(N)_S$ comes from the morphism of monoid schemes $D(M)_S \to A(N)_S$.

**Definition 2.5.** Let $N$ and $L$ be submonoids such that $N \subset L$. We say that $N$ is a face of $L$ if the projection map
\[
Z[L] \to Z[N], \quad X^l \mapsto 0 \text{ if } l \in L \setminus N \text{ and } X^l \mapsto X^l \text{ if } l \in N
\]
is a morphism of rings.

Remark that if $N$ is a face of $L$, then the associated morphism of schemes $A(N)_S \to A(L)_S$ is $D(M)_S$-equivariant for any scheme $S$.

**Proposition 2.6.** Let $N \subset L$ be monoids. Then $N$ is a face of $L$ if and only if for all $x, y \in L$
\[
x + y \in N \iff x \in N \text{ and } y \in N.
\]

**Proof.** Let $\phi$ denote the projection and assume it is a morphism of rings. Let $x, y \in L$. Then
\[
x + y \in N \iff \phi(X^{x+y}) = X^{x+y} = \phi(X^x)\phi(X^y)
\]
is not zero $\iff$ both $x$ and $y$ are in $N$. Reciprocally assume that for all $x, y \in L$, $x + y \in N \iff x \in N$ and $y \in N$, then we have
\[
\phi(X^xX^y) = \phi(X^x)\phi(X^y).
\]

**Proposition 2.7.** Let $N$ be a monoid in $M$. Let $N^* = \{x \in N | \exists y \in N \text{ such that } x + y = 0\}$, then $N^*$ is a submonoid of $N$ and a group, moreover $N^*$ is a face of $N$.

**Proof.** Take $x, y \in N$. Assume $x + y \in N^*$, then there exists $z \in N$ such that $x + y + z = 0$, this shows that $x$ and $y$ are in $N^\times$. Now apply Proposition 2.6.

**Remark 2.8.** Let $M$ be a monoid and $N \subset M$ and $L \subset M$ be two submonoids. Then $N \cap L$ is a submonoid. Let $N + L$ be $\{n + l \in M | n \in N, l \in L\}$. Then $N + L$ is a monoid.

**Definition 2.9.** Let $M$ be a monoid and let $\Sigma$ be a subset of $M$, then we denote by $N_\Sigma$ or by $[\Sigma]$ the smallest monoid of $M$ containing $\Sigma$, this is the intersection of all monoids of $M$ containing $\Sigma$.

3. Definition of attractors associated to monoids and elementary properties

Let $X$ be an $S$-functor over a base scheme $S$. Let $M$ be an abelian group and let $D(M)_S$ be the associated diagonalizable $S$-group scheme. Assume that $D(M)_S$ acts on $X$. Let $N$ be a submonoid of $M$ and consider $A(N)_S$ with the canonical action of $D(M)_S$ as in the previous section. We now introduce the attractor $X^N$ associated to the monoid $N$ under the action of $D(M)_S$ on $X$.

**Definition 3.1.** Let $X^N$ be the functor
\[
(Sch/S) \to Sets, (T \to S) \mapsto \text{Hom}^{D(M)_T}(A(N)_T, X_T)
\]
where $\text{Hom}^{D(M)_T}(A(N)_T, X_T)$ is the set of $D(M)_T$-equivariant $T$-morphisms from $A(N)_T$ to $X_T = X \times_S T$. 

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In most part of this article $X$ will be $S$-affine. In [Ma22], we study attractors in the context of $S$-algebraic spaces, the $S$-affine case is a prerequisite in [Ma22].

**Remark 3.2.** Taking attractors commutes with base change: if $S' \to S$ is a morphism of schemes, then $X^N \times_S S' = (X \times_S S')^N$. Moreover if $X$ and $Y$ are two $S$-schemes with actions of $D(M)_S$, then $D(M)_S$ acts on $X \times_S Y$ componentwise and we have $X^N \times_S Y^N \simeq (X \times_S Y)^N$.

**Remark 3.3.** Let 0 be the trivial monoid, then $X^0$ is the well-known space of fixed points $X^{D(M)_S}$.

**Remark 3.4.** We have an identification $X \simeq X^M$. For all scheme $T \to S$ and $x \in X^M(T)$, we associate the morphism $T \xrightarrow{\varepsilon} D(M)_T \xrightarrow{\varepsilon} X_T \in X(T)$ where $\varepsilon$ is the neutral element.

**Remark 3.5.** Let $N \subset L$ be submonoids of $M$. For $T \to S$, we have a morphism

$$\text{Hom}_{T}^{D(M)_T}(A(N)_T, X_T) \to \text{Hom}_{T}^{D(M)_T}(A(L)_T, X_T)$$

obtained using the morphism $A(L)_T \to A(N)_T$ corresponding to the morphisms of rings $\mathbb{Z}[N] \subset \mathbb{Z}[L]$. So we get a morphism of functors $t_{N,L} : X^N \to X^L$.

**Remark 3.6.** We have an action of $D(M)_S$ on $X^N$ given as follows. For any $S$-scheme $T$, we have an action of $D(M)(T)$ on $X_T$ (for any $t \in D(M)(T)$, we have an arrow $X_T \xrightarrow{t} X_T$). Now let $f \in X^N(T) = \text{Hom}_{T}^{D(M)_T}(A(N)_T, X_T)$ and $t \in D(M)(T)$. We define $t \cdot f$ to be $A(N)_T \xrightarrow{f} X_T \xrightarrow{t} X_T$. The morphism $X^N \to X$ is $D(M)_S$-equivariant.

**Proposition 3.7.** If $X$ is a group scheme over $S$ and the action of $D(M)_S$ is by group automorphisms, then $X^N$ is a group functor.

**Proof.** Let $T$ be a scheme over $S$. Let $A(N)_T \xrightarrow{g} X_T$ and $A(N)_T \xrightarrow{h} X_T$ be two elements in $X^N(T)$. Then we define $gh$ as the composition

$$A(N)_T \to A(N)_T \times_X A(N)_T \to X_T \times_X X_T \to X_T$$

where the first morphism is the diagonal morphism, the second is $g \times h$, and the third is the multiplication morphism coming from the group structure on $X$. The two first are equivariant by definitions and the third is equivariant because $D(M)_S$ acts on $X_S$ by group automorphisms. This defines a group law on $X^N$.

**Definition 3.8.** Let $N$ and $L$ be submonoids of an abelian group $M$ and assume that $N$ is a face of $L$. Let $X$ be an $S$-scheme with an action of $D(M)_S$. Then for all $T \to S$ the morphism $A(N)_T \to A(L)_T$ induces a morphism

$$\text{Hom}_{T}^{D(M)_T}(A(L)_T, X_T) \to \text{Hom}_{T}^{D(M)_T}(A(N)_T, X_T).$$

So we obtain a morphism of functors $X^L \to X^N$, that we denote $p_{N,L}$.

When $N$ is a face of $L$, the morphism $p$ satisfies $p \circ t = \text{Id}$.

**4. Attractors with prescribed limits**

We introduce in this section an other functor. Let $X$ be an $S$-scheme over a base scheme $S$. Let $M$ be an abelian group and let $D(M)_S$ be the associated diagonalizable $S$-group scheme. Assume that $D(M)_S$ acts on $X$. Let $F$ be a face of $N$. Let $Z$ be another $S$-scheme with a monomorphism $Z \to X^F$. 

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We now introduce the attractor $X^N_{F,Z}$ associated to the monoid $N$ under the action of $D(M)_S$ on $X$ with prescribed limit in $Z$ relatively to the face $F$. Since being a monomorphism is stable by base change, $Z(T) \subset X^F(T)$ for any $S$-scheme $T$, so the following definition makes sense.

**Definition 4.1.** Let $X^N_{F,Z}$ be the functor

$$(\text{Sch}/S) \to \text{Sets}, (T \to S) \mapsto \{ f \in X^N(T) \mid \text{the image of } f \text{ in } X^F(T) \text{ belongs to } Z(T) \}.$$ 

If $F = N^*$, we omit $F$ from the notation.

**Proposition 4.2.** We have a canonical isomorphism $X^N_{F,Z} \simeq X^N \times_X^F Z$.

**Proof.** Clear since

$$X^N_{F,Z}(T) = \{ f \in X^N(T) \mid \text{the image of } f \text{ in } X^F(T) \text{ belongs in } Z(T) \}$$

$$= \{(f, g) \in X^N(T) \times Z(T) \mid f = g \text{ in } X^F(T) \}$$

$$= X^N(T) \times_{X^F(T)} Z(T).$$

\[ \square \]

5. Representability and properties of attractors in the affine case

Let us prove that attractors are representable in the affine case and state some results about them. Let $M$ be an abelian group and $N$ be a submonoid of $M$. Let $S$ be a scheme and $X$ an $S$-affine scheme endowed with an action of $D(M)_S$. Let $\mathcal{A}$ be the $O_S$-quasi-coherent algebra of $X$. By [SGA3, SGA3.1 Corollaire 4.7.3.1], $\mathcal{A}$ is $M$-graded. So we have a decomposition $\mathcal{A} = \oplus_{m \in M} A_m$ with each $A_m$ quasi-coherent.

**Theorem 5.1.** The functor $X^N$ is representable by a closed subscheme of $X$ whose quasi-coherent ideal sheaf $\mathcal{J}$ is the ideal sheaf generated locally by homogeneous element in $A_m$ for $m \in M \setminus N$.

**Proof.** The quasi-coherent ideal $\mathcal{J}$ is the image of the morphism $\mathcal{A} \times \oplus_{m \in M \setminus N} A_m \to \mathcal{A}$, so it is quasi-coherent. We claim that $X^N = \text{Spec}_S \mathcal{A}/\mathcal{J}$. Let $p : T \to S$ be an $S$-scheme. Let $\mathcal{A}_T = p^* \mathcal{A}$, this is an $O_T$ quasi-coherent algebra satisfying $\text{Spec}_{O_T} A_T = X \times_S T$ by [StP, 01LQ]. Let $O_T[N]$ be the $O_T$ quasi-coherent algebra of the $T$-affine scheme $A(N)_T$. On a first hand, we have

$$X^N(T) = \text{Hom}^{D(M)_T}_{O_T}(A(N)_T, X_T)$$

$$= \left\{ f \in \text{Hom}_{O_T}(A_T, O_T[N]) \mid \begin{array}{ccc}
A_T & \to & A_T \otimes_{O_T} O_T[M] \\
\downarrow & & \downarrow \\
O_T[N] & \to & O_T[N] \otimes_{O_T} O_T[M]
\end{array} \right. \text{ commutes} \right\}$$

$$= \left\{ f \in \text{Hom}_{O_S}(A, p_* O_T[N]) \mid \begin{array}{ccc}
A & \to & A \otimes_{O_S} O_S[M] \\
p_* O_T[N] & \to & p_* O_T[N] \otimes_{O_S} O_S[M]
\end{array} \right. \text{ commutes} \right\}. \text{ On the other hand, we have }$$

$$(\text{Spec}_S \mathcal{A}/\mathcal{J})(T) = \text{Hom}_{O_T}(T, \text{Spec}_S \mathcal{A}/\mathcal{J}) = \text{Hom}_{O_S}(A/\mathcal{J}, p_* O_T).$$

Now take an open $U \subset S$ and put

$$O_S(U) = B, \mathcal{A}(U) = A, \mathcal{J}(U) = J, p_* O_T(U) = B'.$$
It is enough to define functorial maps $\Theta$ and $\Psi$

$$\begin{array}{c}
\text{Hom}_B(A/J, B') \xrightarrow{\Theta} \begin{cases}
f \in \text{Hom}_B(A, B'[N]) & \\
A \xrightarrow{\Phi} A \otimes_B B'[M] & \\
A \downarrow & A \downarrow \\
B'[N] \xrightarrow{\Phi} B'[N] \otimes_B B'[M] & B'[N] \downarrow \\
\end{cases}
\end{array}$$

such that $\Theta \circ \Psi = \text{Id}$ and $\Psi \circ \Theta = \text{Id}$.

**Lemma 5.2.** Let $a_m \in A_m$ for some $m \in M$ and let $f$ in the right-hand-side set above. Then $f(a_m) = 0$ if $m \in M \setminus N$ and $f(a_m) = \lambda m X^m$ for some $\lambda m \in B'$ if $m \in N$. In other words, a $D(M)$-equivariant morphism preserves $M$-gradings.

**Proof.** Easy and well-known. \[ \Box \]

Take $A/J \xrightarrow{\ell} B'$ on the left-hand-side and define a map $f = \Theta(F)$ on the right-hand-side as

$$A = \oplus_{m \in M} A_m \xrightarrow{\ell} B'[N] \quad a_m \in A_m \mapsto F([a_m]) X^m$$

Let us check that this map $\Theta$ is well-defined. Since $F([a_m]) = 0$ if $m \in M \setminus N$, the element $F([a_m]) X^m$ belongs to $B'[N]$. Now we have to explain that $f$ is a morphism of $B$-algebras. This is a consequence of the identity

$$f(a_m a_{m'}) = F([a_m][a_{m'}]) X^{m+m'} = F([a_m][a_{m'}]) X^m X^{m'} = f(a_m)f(a_{m'}).$$

Now we check that the diagram of the right-hand-side condition commutes. Let $a_m \in A_m$ for some $m \in M$, if $m \in M \setminus N$, $f(a_m) = 0$ and there is nothing to prove. Otherwise $m \in N$ and $a_m$ is sent to $F([a_m]) X^m \otimes X^m$ by the upper way or the lower way in the diagram. So $\Theta$ is well-defined. Now take $A \xrightarrow{\ell} B'[N]$ on the right-hand-side. Then $f(a_m) = 0$ for all $a_m \in A_m$ for all $m \in M \setminus N$ by Lemma 5.2, so $f$ vanishes on $J$, i.e $f$ factors through $A \xrightarrow{\ell} A/J \xrightarrow{\ell} B'[N]$.

Now we define $F = \Psi(f)$ as the composition $A/J \xrightarrow{\ell} B'[N] \xrightarrow{X^n \mapsto 1} B'$, this is a morphism of $B$-algebras. Now let us prove that $\Theta \circ \Psi = \text{Id}$. Let $f$ be a morphism on the right-hand-side. Let $a_n \in A_n$ for $n \in N$, we have $f(a_n) = \lambda_n X^n$ by Lemma 5.2. Then

$$((\Theta \circ \Psi)(f))(a_n) = (\Theta(\Psi(f)))(a_n) = (\Psi(f))(a_n) \cdot X^n = (f(a_n))|_{X^n} \cdot X^n = \lambda_n X^n = f(a_n).$$

Now let $a_m \in N \setminus M$, then

$$((\Theta \circ \Psi)(f))(a_m) = (\Theta(\Psi(f)))(a_m) = (\Psi(f))(a_m) \cdot X^n = 0 = f(a_m).$$

This proves that $\Theta \circ \Psi = \text{Id}$. Now let us prove that $\Psi \circ \Theta = \text{Id}$. Let $F$ be a morphism on the left-hand-side, and let us look at the image of $a_n$ for some $a_n \in A_n$ with $n \in N$ under $(\Psi \circ \Theta)(F) = \Psi(\Theta(F))$:

$$\begin{array}{c}
A/J \xrightarrow{\Theta(F)} B'[N] \xrightarrow{\Psi} B' \\
[a_n] \mapsto F([a_n]) X^n \mapsto F([a_n]).
\end{array}$$

This finishes the proof of the Proposition. \[ \Box \]

**Proposition 5.3.** If $N \subset L$ are submonoids in $M$, then $\iota : X^N \to X^L$ is a closed immersion.

**Proof.** With the notation of the previous proof, we have $\mathcal{J}_L \subset \mathcal{J}_N$. \[ \Box \]
**Proposition 5.4.** Let \( N, L \) be two submonoids of \( M \). Then \( X^{N \cap L} = X^N \cap X^L \) and \( (X^N)^L = X^{N \cap L} \).

**Proof.** Let \( J_{N \cap L} \) be the quasi-coherent ideal of the quasi-coherent algebra of \( X \) defining \( X^{N \cap L} \). It is generated by homogeneous elements of degree belonging in \( M \setminus (N \cap L) \). Then \( J_{N \cap L} = J_N + J_L \) where \( J_N \) and \( J_L \) are the quasi-coherent ideals defining \( X^N \) and \( X^L \). Moreover we know that the quasi-coherent ideal of \( X^N \cap X^L \) equals \( J_N + J_L \). The second assertion also follows from the description in terms of ideals and the relation \((M \setminus N) \cup (M \setminus L) = M \setminus (N \cap L)\). This finishes the proof.

**Lemma 5.5.** Let \( S \) be a scheme and let \( f : X \to Y \) be a \( D(M)_S \)-equivariant closed immersion of \( S \)-affine \( D(M)_S \)-schemes. Let \( N \) be a submonoid of \( M \). Then the morphism \( X^N \to Y^N \) is a closed immersion, and more precisely \( X^N = X \times_Y Y^N \).

**Proof.** This is local on \( S \) so we assume \( S = \text{Spec}(R) \) is affine, moreover we identify \( X \) with a closed subscheme of \( Y \). Let \( A \) be the \( R \)-algebra of \( Y \) and \( I \) be the ideal of \( A \) defining \( X \). Let \( J \) be the ideal of \( A \) defining \( Y^N \), cf. Proof of Theorem 5.1. Then the ideal of \( A/I \) defining \( X^N \) is \( I + J/I \) using Proposition 5.1. So the ideal of \( A \) defining \( X^N \) is \( I + J \). Now the isomorphism \( A/I \otimes_A A/J \cong A/(I + J) \) finishes the proof.

**Lemma 5.6.** Let \( f : M \to Z \) be a morphism of abelian groups, and let \( D(Z)_S \to D(M)_S \) be the corresponding morphism of group schemes. Assume that \( X = \text{Spec}(A) \) is an \( S \)-affine scheme with a \( D(M)_S \)-action, then we can see it as a \( D(Z)_S \)-scheme. Let \( Y \) be a submonoid of \( Z \). Let \( N \) be \( f^{-1}(Y) \), this is a submonoid in \( M \). Then we have an isomorphism of \( S \)-schemes \( X^N \cong X^Y \), where on the left-hand-side \( X \) is seen as a \( D(M)_S \)-scheme and on the right-hand-side as a \( D(Z)_S \)-scheme.

**Proof.** We reduce to the case where \( S \) and \( X = \text{Spec}(A) \) are affine. We have two compatible gradings on \( A \), one given by \( Z \) and one given by \( M \). For any \( y \in Y \), we have \( A_y = \bigoplus_{n \in f^{-1}(y)} A_n \). So \( \bigoplus_{z \in Z \setminus Y} A_z = \bigoplus_{n \in M \setminus N} A_n \). Then the ideal defining \( X^N \) equals the ideal defining \( X^Y \), cf. Proposition 5.1.

**Proposition 5.7.** Let \( E \) be a quasi-coherent \( \mathcal{O}_S \)-module and let \( \mathbb{V}(E) = \text{Spec}_S(\text{Sym} \ E) \) be the associated quasi-coherent bundle defined by \( E \). Assume that \( D(M)_S \) acts linearly on \( \mathbb{V}(E) \). Let \( N \) be a submonoid of \( M \). Then the attractor \( (\mathbb{V}(E))^N \) associated to \( N \) is canonically isomorphic to \( \mathbb{V}(E^N) \) where \( E^N \) is the \( N \)-graded component of \( E \) relatively to the \( D(M)_S \)-action on \( E \).

**Proof.** By [SGA3, Proposition 4.7.3], \( E \) is \( M \)-graded so that the \( E^N \) is well-defined. Let \( p : T \to S \) be a scheme over \( S \). We have \( \mathbb{V}(E) \times_S T = \mathbb{V}(p^*E) \). The quasi-coherent \( \mathcal{O}_T \)-module inherits a \( M \)-grading and we have \( (p^*E)^N = p^*(E^N) \). The following identifications finish the proof

\[
\text{Hom}_{\mathcal{O}_T}^{D(M)_T}(A(N)_T, \mathbb{V}(p^*E)) = \text{Hom}_{\mathcal{O}_T}^{M\text{-graded}}(\text{Sym} \ p^*E, \mathcal{O}_T[N])
= \text{Hom}_{\mathcal{O}_T}^{M\text{-graded}}(p^*E, \mathcal{O}_T[N])
= \text{Hom}_{\mathcal{O}_T}^{M\text{-graded}}(p^*E^N, \mathcal{O}_T[N])
= \text{Hom}_{\mathcal{O}_T}^{M\text{-graded}}(p^*E^N, \mathcal{O}_T)
= \text{Hom}_{\mathcal{O}_T}^{\mathcal{O}_T\text{-alg}}(\text{Sym} \ p^*E^N, \mathcal{O}_T)
= \mathbb{V}(E^N)(T).
\]
6. Cartesian squares

In this section, we show that we can obtain non trivial cartesian squares using the formalism of attractors. We thank Simon Riche for a question concerning the possibility to interpret the cartesian square appearing in Remark 7.4 in our attractor formalism. Riche’s question leads to the following general statement.

**Proposition 6.1.** Let $S$ be a scheme. Let $M$ be an abelian group and let $X/S$ be a $D(M)_{S}$-scheme. Let $N, N', L$ and $L'$ be submonoids in $M$ such that $L \subset L'$, $N' \subset L'$, $N = L \cap N'$ and $L' = L + N'$. We assume that $X^{E}$ is representable by a scheme for all $E \in \{N, N', L, L'\}$. We assume that $L$ is a face of $L'$. Then $N$ is a face of $N'$. Assume that one of the following conditions holds

(i) we have an equality $N' = L' \setminus (L \setminus N)$
(ii) $S = \text{Spec}(R)$ and $X = \text{Spec}(A)$ are affine, $A_{l}A_{n'} = A_{l+n'}$ for all $l \in L \setminus N$ and $n' \in N'$ (as usual $A_{m}$ denote the $m$-graded part of $A$)

then the following diagram is a cartesian square in the category of schemes

\[
\begin{array}{ccc}
X^{N'} & \xrightarrow{\iota_{N',L'}} & X^{N} \\
\downarrow{p_{N,N'}} & & \downarrow{p_{L,N'}} \\
X^{L'} & \xrightarrow{\iota_{L,L'}} & X^{L} \\
\end{array}
\]

**Proof.** The monoid $(N' \cap L)$ is a face of $N'$ because for any ring $R$, the projection $R[N'] \to R[N' \cap L]$ is the restriction of the projection $R[L'] \to R[L]$ to $R[N']$, now $R[L'] \to R[L]$ is a morphism of rings because $L$ is a face of $L'$, and so $R[N'] \to R[N' \cap L]$ is a morphism of rings.

(i) Assume (i) is satisfied. Let $R$ be a ring. We have $R[L'] \times_{R[L]} R[N] \simeq \{(x, y) \in R[L'] \times R[N] | f(x) = g(y)\}$, indeed an element $x$ in $R[L']$ maps to an element in $R[N]$ under the projection $R[L'] \to R[L]$ if and only if $x \in R[\langle L' \setminus L \rangle \cup N]$. The map $R[N'] \to R[N]$ is the projection morphism associated to the face inclusion $N \subset N'$. The map $R[N'] \to R[L']$ is the morphism associated to the inclusion $N' \subset L'$. So by [StP, 0ET0] the scheme $A(N')_{R}$ is the push-out, in the category of schemes, of the diagram

\[
\begin{array}{ccc}
A(L)_{R} & & A(N)_{R} \\
\downarrow & & \downarrow \\
A(L')_{R} & & A(N')_{R} \\
\end{array}
\]

We need the following Lemma.

**Lemma 6.2.** For any scheme $S$, $A(N')_{S}$ is the push-out, in the category of schemes, of

\[
\begin{array}{ccc}
A(L)_{S} & & A(N)_{S} \\
\downarrow & & \downarrow \\
A(L')_{S} & & A(N')_{S} \\
\end{array}
\]

**Proof.** Let $S = \cup_{i \in I} U_{i}$ be an affine open covering and write $U_{i} = \text{Spec}(R_{i})$. Let $Y$ be a scheme and let $A(L')_{S} \to Y$ and $A(N)_{S} \to Y$ be two morphisms such that the following
We then obtain, for any \( i \in I \), a commutative diagram

![Diagram](https://example.com/diagram.png)

Now since \( U_i \) is affine, we obtain a unique morphism \( f_i : A(N'_U) \to Y \) such that the following diagram commutes

![Diagram](https://example.com/diagram.png)

For \( i, j \in I \), we have \( U_i \times_S U_j = U_i \cap U_j \). Let \( U_i \cap U_j = \bigcup_{q \in Q} V_q \) be an affine open covering. We have \( f_i \big|_{A(N'_U)_{V_q}} = f_j \big|_{A(N'_U)_{V_q}} \) for all \( q \in Q \) by the affine case done before the statement of Lemma 6.2. So we have \( f_i \big|_{A(N')_{U_i \cap U_j}} = f_j \big|_{A(N')_{U_i \cap U_j}} \) by [GW, Prop. 3.5]. Thus using [GW, Prop. 3.5] again, we obtain a unique morphism \( f : A(N'_S) \to Y \) such that the following diagram commutes

![Diagram](https://example.com/diagram.png)

Now let \( T \) be a scheme and let \( T \to X', T \to X \) be two morphisms of schemes such that
the following diagram commutes

\[
\begin{array}{ccc}
T & \overset{}{\longrightarrow} & X_N \\
\downarrow & & \downarrow \\
X_L' & \overset{\nu_{L,L'}}{\longrightarrow} & X_L \\
\end{array}
\]

This corresponds to a diagram

\[
\begin{array}{ccc}
A(L)_T & \overset{}{\longrightarrow} & A(N)_T \\
\downarrow & & \downarrow \\
A(L')_T & \overset{}{\longrightarrow} & A(N)_T \\
\downarrow & & \downarrow \\
X_T & \overset{}{\longrightarrow} & A(N')_T \\
\end{array}
\]

where all arrows are $D(M)_T$-equivariant. By Lemma 6.2, we obtain a unique arrow $A(N')_T \to X_T$ such that the following diagram commutes

\[
\begin{array}{ccc}
A(L)_T & \overset{}{\longrightarrow} & A(N)_T \\
\downarrow & & \downarrow \\
A(L')_T & \overset{}{\longrightarrow} & A(N)_T \\
\downarrow & & \downarrow \\
X_T & \overset{}{\longrightarrow} & A(N')_T \\
\end{array}
\]

It is enough to show that the arrow $A(N')_T \to X_T$ is $D(M)_T$-equivariant. Consider the diagram

\[
\begin{array}{ccc}
A(L)_T & \overset{}{\longrightarrow} & A(N)_T \\
\downarrow & & \downarrow \\
A(L')_T & \overset{}{\longrightarrow} & A(N)_T \\
\downarrow & & \downarrow \\
X_T & \overset{}{\longrightarrow} & A(N')_T \\
\end{array}
\]

obtained by fiber product with $D(M)_T$. We have

\[
A(E)_T \times_T D(M)_T = (A(E) \times_{\text{Spec}(\mathbb{Z})} T) \times_T D(M)_T = A(E)_{D(M)_T}
\]

for $E \in \{N, N', L, L'\}$, so the left diamond is a push-out by Lemma 6.2. Now we want to show that the lower rectangle is commutative. Consider the upper right composition in this rectangle and precompose it with the right part of the left diamond, denote this arrow by $a_1$. Consider the lower left composition in the rectangle and precompose it with the left part
of the left diamond, denote this arrow by \( a_2 \). Now using the commutative diagrams coming from the \( D(M)_T \)-equivariant morphisms on the right, we see that \( a_1 \) and \( a_2 \) are both equal to the arrow

\[ A(L)_T \times_T D(M)_T \to A(L)_T \to A(N')_T \to X_T. \]

Using the left push-out diamond, this now implies that the lower rectangle is commutative. So the arrow \( A(N')_T \to X_T \) is \( D(M)_T \)-equivariant. This finishes the proof.

(ii) Let \( x \in X^N(R) \). Then \( x \) is a morphism \( A \to R[N'] \). Now we have

\[ (A \to R[N'] \to R[L] \to R[L]) = (A \to R[N'] \to R[N' \cap L] \to R[L]). \]

This shows that the diagram is commutative. Now let us prove that it is cartesian. Let \( Y = \text{Spec}(B) \) be an affine \( R \)-scheme and let \( f : Y \to X^{L'} \) and \( g : Y \to X^N \) be two morphisms such that \( p_{L,L'} \circ f = \iota_{L,N} \circ g \). So \( f \) is a morphism of graded algebras \( A \to B[L'] \) and \( g \) is a morphism of graded algebras \( A \to B[N] \). Let \( m \in L' = L + N' \) and let \( A_m \) be the \( m \)-graded part of \( A \). Let \( x \in A_m \) and let \( \lambda_m \) such that \( f(x) = \lambda_m x^m \). Then since \( p_{L,L'} \circ f = \iota_{L,N} \circ g \), we obtain that \( \lambda_m = 0 \) for all \( l \in L \setminus (N' \cap L) \). So we get \( f(x) = 0 \) for all \( x \in A_m \) for all \( m \in L' \setminus N' \) (we use that \( A_l A_{m'} = A_{l+n'} \) for all \( l \in L \setminus (N' \cap L) \) and \( n' \in N' \)). So we obtain a unique morphism \( h \) from \( Y \) to \( X^{N'} \) with the cartesian property.

\[ \square \]

**Remark 6.3.** Let \( L' \) be a monoid and \( N \subset L \subset L' \) be submonoids. Assume \( L \) is a face of \( L' \). Then \( N' := L' \setminus (L \setminus N) \) is a monoid, moreover \( N' \cap L = N \) and \( N' \cup L = N' + L = L' \).

### 7. Root groups, parabolic and Levi subgroups in reductive groups

Let \( G \) be a split connected reductive group scheme over a field \( R \). Let \( T \) be a maximal split torus and choose a Borel \( B \) containing \( T \). Let \( \Phi = \Phi(G, T) \subset X^*(T) \) denote the set of roots associated to \( (G, T) \) and \( \Phi^+ = \Phi(B, T) \) the roots in \( B \). Let \( B \) be the basis of \( \Phi \) determined by \( \Phi^+ \). For \( \alpha \in \Phi \), let \( U_\alpha \) be the associated unipotent root group and \( u_\alpha \) be the root group associated to \( \alpha \) in the Lie algebra of \( G \). We refer to \([\text{SGA} 3, \text{Exp. XXII}]\) for the definition of \( U_\alpha \) and \( u_\alpha \). Let \( U \) be the unipotent radical of \( B \) and let \( u \subset b \subset g \) be the Lie algebras of \( U, B \) and \( G \). Consider the adjoint action of \( T \) on \( G, U \) and \( u \).

**Proposition 7.1.** There exists a \( T \)-equivariant isomorphism of \( R \)-schemes \( u \simeq U \).

**Proof.** This is a direct consequence of \([\text{SGA} 3, \text{Exp. XXII Th. 1.1, Exp. XXVI Prop. 1.11}]\), indeed these results imply the following assertions. For each root \( \alpha \in \Phi^+ \), we have a \( T \)-equivariant isomorphism \( U_\alpha \simeq u_\alpha \). We have \( T \)-equivariant isomorphisms of schemes \( u = \Pi_{\alpha \in \Phi^+} u_\alpha \) and \( U = \Pi_{\alpha \in \Phi^+} U_\alpha \). This finishes the proof.

Let us now fix \( \alpha \in \Phi^+ \). Recall that \( [\alpha] \subset X^*(T) \) is the submonoid generated by \( \alpha \).

**Proposition 7.2.** We have a canonical isomorphism \( u_\alpha \simeq u^{[\alpha]} \) (resp. \( U_\alpha \simeq U^{[\alpha]} \)), between root group and \( [\alpha] \)-attractor for the action of \( T = \text{Spec}(R[X^*(T)]) \) on \( u \) (resp. \( U \)).

**Proof.** By Proposition 7.1, we have a \( T \)-equivariant isomorphism \( u \simeq U \), so it is enough to prove the statement for \( u \). Since \( G \) is split, \( \Phi \) is reduced and Proposition 5.7 finishes the proof.

Recall that we have a bijection between parabolic subgroups of \( G \) containing \( B \) and subsets of \( B \), cf. e.g. \([\text{Co} 14, \text{Page 35, lines 4-5}]\) or \([\text{CPG} 10, 2.2.8]\). Recall that if \( \Sigma \) is a subset of \( X^*(T) \), we define \( N_\Sigma \) to be the monoid generated by \( \Sigma \) in \( X^*(T) \).
Proposition 7.3. Let \( \zeta \subset B \). Let \( \Theta \) be \( \zeta \cup -\zeta \). Let \( \Sigma \) be \( \mathcal{B} \cup -\zeta \).

(i) The attractor \( G^N_{\Theta} \) is the Levi subgroup \( L_{\Theta} \) such that \( \Phi(L_{\Theta}, T) = N_{\Theta} \cap \Phi \).

(ii) The attractor \( G^N_{\Sigma} \) is the associated parabolic subgroup, moreover \( L_{\Theta} \) is a Levi component of \( P \).

(iii) Let \( \xi \subset \zeta \). Let \( \Gamma = \mathcal{B} \cup -\xi \). Let \( N \) be a submonoid of \( N_{\Sigma} \) such that \( N \cap \Sigma = \Gamma \). Then \( G^N = G^{N_{\Gamma}} \).

Proof. Let \( P_{\Sigma} \) be the parabolic corresponding to \( \zeta \). By [CGP10, 2.2.8, 2.2.9], there exists a \( \lambda \in X^*_m(T) \) such that \( P_{\Sigma} \) is the associative submonoid of \( \mathbb{N} \) relatively to the action of \( G_m = D(Z)_S \) on \( G \) via \( x.g = \text{ad}_{\lambda(x)}g \) and such that \( \lambda(\beta) \geq 0 \) for all \( \beta \in \Sigma \) and \( \lambda(\beta) = 0 \) for all \( \beta \in \Theta \). The Levi subgroup \( L_{\Theta} \) corresponding to \( \Theta \) is the fixed point in \( G \) of the action of \( \lambda \) by conjugation, i.e. \( L_{\Theta} = G^0 \). Now we prove the Proposition.

(i) Assume first that \( \zeta = B \). Then \( L_{\Theta} = G \) and \( N_{\Theta} = N_{\Phi(G, T)} \). Using Prop. 7.2, we deduce that the big cell \( \Omega = \Pi_{\alpha \in \Phi} U_{\alpha} \times T \times \Pi_{\alpha \in \Phi^*} U_{\alpha} \) is in \( G^N_{\Theta} \). Now we have inclusions \( \Omega \subset G^N_{\Theta} \subset G \) with \( \Omega \) dense in \( G \) and \( G^N_{\Theta} \) closed in \( G \). This implies \( G^N_{\Theta} = G \). Let us now prove the general case, let \( \zeta \subset B \). By Lemma 5.6, we have \( G^{f^{-1}(0)}_\Theta = G^0 = L_{\Theta} \). We have \( \Theta \subset f^{-1}(0) \) and so \( N_{\Theta} \subset f^{-1}(\mathbb{N}) \). Consequently, \( G^{N_{\Theta}} \subset G^{f^{-1}(\mathbb{N})} \). So we have proved that \( G^{N_{\Theta}} \subset L \) and let us now prove that this is an equality. We remark that \( N_{\Theta} = N_{\Phi(L_{\Theta}, T)} \). Now since \( L \subset G \) and using the first case done before, we have \( L = L^{N_{\Phi(L_{\Theta}, T)}} \subset G^N_{\Theta} \). This finishes the proof.

(ii) Recall that \( \lambda : G_m \to T \) corresponds to the morphism of abelian groups \( f : X^*(T) \to \mathbb{Z}, \chi \mapsto (\lambda, \chi) \). Now we see \( G \) as a \( G_m \)-scheme and as a \( T \)-scheme. By Lemma 5.6, we have \( G^N = G^{f^{-1}(\mathbb{N})} \). Since, for all \( \beta \in \Sigma \), \( f(\beta) = \lambda(\beta) \geq 0 \), we have \( f(\beta) \in \mathbb{N} \) and so \( \Sigma \subset f^{-1}(\mathbb{N}) \). Consequently, \( P_{\Sigma} = G^N = G^{f^{-1}(\mathbb{N})} \subset G^{N_{\Sigma}} \). Let us prove that \( P_{\Sigma} \subset G^{N_{\Sigma}} \). We have \( P_{\Sigma} = L_{\Theta} \times R_u(P_{\Sigma}) \) where \( R_u(P_{\Sigma}) \) is the unipotent radical of \( P_{\Sigma} \). Using (i), we have \( L_{\Theta} = L^{N_{\Theta}}_{\Theta} \subset L^{N_{\Sigma}}_{\Theta} \subset L_{\Theta} \), and so \( L^{N_{\Sigma}}_{\Theta} = L_{\Theta} \). Using Proposition 7.2, one has \( (R_u(P_{\Sigma}))^{N_{\Sigma}} = R_u(P_{\Sigma}) \). So \( P^{N_{\Sigma}}_{\Sigma} = P_{\Sigma} \), and so \( P_{\Sigma} \subset G^{N_{\Sigma}} \).

(iii) We have a canonical closed immersion \( G^{N_{\Gamma}} \subset G^N \). We have \( N \cap N_{\Theta} = N_{\mathcal{B} \cup -\xi} \) and \( N \cap N_{\Theta} \) it is thus included in \( N_{\Gamma} \). By Proposition 5.4 we have \( G^N = (G^{N_{\Sigma}})^N \). By the previous assertions we have \( G^{N_{\Sigma}} = G^{N_{\Theta}} \times R_u(G^{N_{\Sigma}}) \). Now by 5.4 we have

\[
\left(G^{N_{\Theta}} \times R_u(G^{N_{\Sigma}})\right)^N = (G^{N_{\Theta}})^N \times (R_u(G^{N_{\Sigma}}))^N = G^{N_{\mathcal{B} \cup -\xi}} \times R_u(G^{N_{\Sigma}}).
\]

This implies that \( G^{N_{\Gamma}} \supset G^N \) and finishes the proof.

Remark 7.4. Proposition 7.3 implies that any parabolic or Levi subgroup of \( G \) containing \( T \) can be obtained as an attractor under the conjugation action of \( T \) on \( G \). Moreover, assume that \( B \) is a parabolic subgroup in a parabolic \( P \) and \( L \) is a Levi component of \( P \). We assume that \( B, P \) and \( L \) contain \( T \). Let \( M \) be \( B \cap L \), this is a parabolic subgroup in \( L \). Then one has a cartesian
This square can be obtained using Proposition 6.1. Indeed let $L'$ be the submonoid generated by $\Phi(P, T)$, let $L$ be the submonoid generated by $\Phi(L, T)$ and $N$ be the submonoid generated by $\Phi(M, T)$. Using Proposition 2.7, we deduce that $L$ is a face of $L'$. Now let $N'$ be $L' \setminus (L \setminus N)$. We have $G^{L'} = P, G^N = M$ and $G^L = L$. Using Proposition 7.3 we have $G^{N'} = B$.

For any root $\alpha \in \Phi^+$ we denote by $H_\alpha \subset G$ the semidirect product $T \rtimes U_\alpha$, this is a group scheme whose unipotent radical equals $U_\alpha$.

**Proposition 7.5.** We have a canonical isomorphism $H_\alpha \simeq G^{(\alpha)}$.

**Proof.** Since $[\alpha] \subset \Sigma_{\Phi^+}$ and by Proposition 7.3, we have a closed immersion $G^{(\alpha)} \subset B = G^{\Sigma_{\Phi^+}}$ where $B$ is the Borel subgroup. So by Lemma 5.5, we get a closed immersion $G^{(\alpha)} \subset B^{(\alpha)}$ and thus an equality $G^{(\alpha)} = B^{(\alpha)}$. Now we have a $T$-equivariant isomorphism of schemes $B \simeq T \times U$. Using Proposition 7.2, we get

$$U_\alpha = U^{(\alpha)}.$$ \hfill (7.1)

It is obvious that

$$T = T^{(\alpha)}.$$ \hfill (7.2)

Now equations 7.1 and 7.2 and Remark 3.2 imply that $B^{(\alpha)} = H_\alpha$. This finishes the proof. \hfill \square

**Corollary 7.6.** (i) Let $e_G^G$ be the closed subscheme of $G$ corresponding to the unit section.

Then the attractors $G^{(\alpha)}_{e_G^G}$ with prescribed limit $e_G$ equals $U_\alpha$.

(ii) We have a canonical isomorphism $U_\alpha = R_u (G^{(\alpha)})$ where $R_u$ means the unipotent radical.

**Proof.** Clear from Proposition 7.5 and Section 4. \hfill \square

**Remark 7.7.** In [Ma22], we generalize this Corollary to general bases.

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