FREE CIR PROCESSES

HOLGER FINK¹,a,b, HENRY PORT,*²,a,c GEORG SCHLÜCHTERMANN³,d,e

30th December 2019

ABSTRACT

For stochastic processes of non-commuting random variables we formulate a Cox-Ingersoll-Ross (CIR) stochastic differential equation in the context of free probability theory which was introduced by Voiculescu. By transforming the classical CIR equation and the Feller condition, which ensures the existence of a positive solution, into the free setting (in the sense of having a strictly positive spectrum), we show the existence of a free CIR equation. The main challenge lies in the transition from a stochastic differential equation driven by a classical Brownian motion to a stochastic differential equation driven by the free analogue to the classical Brownian motion, the so-called free Brownian motion.

AUTHORS INFO

¹ Center for Quantitative Risk Analysis CE-QURA, Ludwig-Maximilians-Universität München, Akademiestr. 1/I, 80799 Munich, Germany
² Department of Computer Science and Mathematics, Munich University of Applied Sciences, Munich, Germany
³ Chair of Financial Econometrics, Institute of Statistics, Ludwig-Maximilians-Universität München, Akademiestr. 1/I, 80799 Munich, Germany
⁴ Department of Mechanical, Automotive and Aeronautical Engineering, Munich University of Applied Sciences, Munich, Germany
⁵ Faculty of Mathematics, Computer Science and Statistics, Ludwig-Maximilians-Universität München, Theresienstrasse 39/I, 80333 Munich, Germany

holger.fink@hm.edu  henry.port@stat.uni-muenchen.de  georg.schluechtermann@hm.edu

Corresponding author

KEYWORDS

free probability, CIR, volatility, stochastic differential equations, stochastic processes

MATHEMATICS SUBJECT CLASSIFICATION

60H10, 65C30, 46L54, 97M30
1 Introduction

The Black-Scholes (BS) framework is considered as one of the benchmarks for modeling the price process \( (X(t))_{t \geq 0} \) of an underlying asset. Initially mentioned in Black and Scholes (1973), it is based on the stochastic differential equation (SDE)

\[
dX(t) = \mu X(t)dt + \sigma X(t)dB_1(t),
\]

for \( t \in [0, \infty[ \), where \( \mu \in \mathbb{R}, \sigma > 0 \) and \( (B_1(t))_{t \geq 0} \) denotes a Brownian motion.

One major drawback of this model is the fact that it does not account for certain common properties of financial data, known as stylized facts, in particular, volatility clustering and the so-called leverage effect. The first term refers to the fact that volatility exhibits a highly autocorrelated structure and the second to the negative correlation between volatility and returns frequently found in financial data (cf. Pagan (1996); Mandelbrot (1997); Cont (2001)).

A possible solution is to model the variance separately as a time-dependent stochastic process that accounts for these effects. A desired property for such a model is to ensure that the variance returns to an average value in finite time, which is referred to as mean-reversion. Therefore, the Vasicek model (cf. Vasicek (1977)), which was originally developed to model the evolution of interest rates, appears as a promising candidate. That means the variance process \( (s(t))_{t \geq 0} \) of \( (X(t))_{t \geq 0} \) is modeled as the solution to the SDE

\[
ds(t)^2 = a(b-s(t)^2)dt + \sigma dB_2(t),
\]

for \( t \in [0, \infty[ \), where \( b > 0 \) is referred to as the long term mean level, \( a > 0 \) is a measure of the time it takes \( s(t)^2 \) to return to \( b \) and \( \sigma > 0 \) signifies the influence of the random shocks and \( (B_2(t))_{t \geq 0} \) is another Brownian motion, correlating with \( (B_1(t))_{t \geq 0} \) via

\[
\rho dt = dB_1(t)dB_2(t),
\]

for \( t \in [0, \infty[ \), where \(-1 \leq \rho \leq 1\). This setup appears reasonable and sufficient at first glance but unfortunately it can not guarantee positivity of \( (s(t)^2)_{t \geq 0} \), which is of course a necessity for a variance model.
However, in 1993 Heston (1993) improved upon the BS framework by addressing the aforementioned issues, allowing the variance to be modeled by a separate SDE, namely the Cox, Ingersoll and Ross (CIR) process as developed by Cox et al. (1985), via the SDE

\[ ds(t)^2 = (a - bs(t)^2)dt + \sigma \sqrt{s(t)^2} dB_2(t), \]

for \( t \in [0, \infty[ \), which allows for mean-reversion but also addresses the issue of negativity: Feller (1951) has shown that as long as the so-called Feller condition

\[ 2a \geq \sigma^2 \]  

holds there exists a positive unique solution.

Since then, many researchers have been developing and generalizing this framework for volatility modeling. Motivated by the limitation of not being able to recreate the term structure of the analyzed asset, or interest rate, in question (cf. Hull and White (1990); Yang (2005); Keller-Ressel and Steiner (2008)), the extended CIR equation was introduced in Maghsoodi (1996) allowing time dependency for all parameters. In order to model potential long memory effects in volatility which were observed by some researchers (cf. Baillie et al. (1996); Bollerslev and Mikkelsen (1996)), Comte and Renault (1998) implemented a fractional Brownian motion as the driving process. This setup was adapted by Comte et al. (2012) who proposed a fractional Heston model to explain the mystery of the steepness present in volatility smiles of long term options. Schlüchtermann and Yang (2016) permitted a dynamic term of the form \( \sigma x^q \), where \( q \geq 0 \), in the fractional setup and showed the existence of a positive solution of these so called generalized fractional CLKS-type equations by imposing a Feller-like condition on the coefficients, for pathwise forward integrals as well as for integrals in the Wick sense. Fink and Schlüchtermann (2018) expanded the fractional CIR equation even further to the Mandelbrot-Van Ness fractional Lévy process-driven case with time-dependent coefficients. Yet another modification is the implementation of a Hurst index of \( H < \frac{1}{2} \). The resulting so called rough Heston models give a good mixture of a decent fit to historical data and implied volatility without touching upon the curse of dimensions (cf. Gatheral et al. (2018); El Euch et al. (2018); Jaber and Euch (2019); El Euch and Rosenbaum (2019); Bayer et al. (2019)).

The task of modeling prices for two or more assets on the other hand poses additional challenges. Taking covariances into account necessitates a joint model. A promising candidate is given by Wishart
autoregression processes, which were introduced in Bru (1991). Since Wishart processes do not need additional constraints to ensure positive definiteness almost surely (cf. Gouriéroux (2006)), they are suited especially well for the role of a matrix-valued CIR process.

Nevertheless, with an increasing number of assets the complexity of volatility models adequately describing these systems increases rapidly, demanding the usage of more variables and therefore the curse of dimension (cf. Gourieroux and Sufana (2010)) threatens a feasible application.

Some matrix ensembles such as the GUE (Gaussian unitary ensemble) and the Wishart random matrices (these are of the form $\frac{1}{N}XX^*$, where $X$ is $N \times M$ random matrix with independent Gaussian entries and $X^*$ denotes its adjoint) behave like so-called free random variables in (eigenvalue) distribution, when their size gets very large, which is referred to as "asymptotic freeness" (cf. Mingo and Speicher (2017)). The limiting eigenvalue distribution of the former is given by the semi-circular distribution and for the latter by the Marchenko-Pastur distribution (also known as "free Poisson distribution").

Since these matrix ensembles behave like free random variables in high dimensions we may employ the tools of free probability theory, which allow for concepts like convolution and a pendant to the central-limit theorem (cf. Nourdin and Taqqu (2014)) to adequately work with random matrices and operators in the probabilistic context. Free probability, developed by Dan Voiculescu (around 1986 while trying to solve an isomorphism problem about free groups (cf. "Background and outlook" in Voiculescu et al. (2016)), is a fitting framework to analyse the distribution of non-commutative random objects in the desired generality. It can be considered as a non-commutative analogue to classical probability theory together with the notion of freeness that allows the computation of the joint distribution of non-commutative random variables. In the context of random matrices this distribution can be understood as the spectral distribution. In particular, portfolio theory benefits immensely from properties of free random variables, since it allows for cleaning of empirical correlation matrices, which basically are Wishart matrices, to derive the distribution of a "true" noise-free correlation matrix via so-called "free deconvolution" (cf. Ryan and Debbah (2007); Ryan (2008); Bouchaud and Potters (2009)). Since free probability is a fitting framework for dealing with high- or even infinite-dimensional random-matrices we can employ this setting to adequately describe a volatility-model for high-dimensional portfolios, namely via a potential free CIR model. This idea necessitates a free stochastic calculus.

Free stochastic calculus first appeared in Speicher (1990). This theory was further developed by
Kümmerer and Speicher (1992), Biane (1997) and Biane and Speicher (1998), where among other ground laying definitions the notion of free stochastic processes and a free Brownian motion were introduced. For an in-depth study of the free Itô integral, we refer to Anshelevich (2002). This framework allows for defining the notion of free SDEs. In particular free stochastic processes form a vivid research area (cf. Biane (1998); Biane and Speicher (2001); Barndorff-Nielsen et al. (2002); Fan (2006); Gao (2006); Gao et al. (2008); An and Gao (2015)).

Some processes, such as the item of interest of this paper, the CIR process, arise naturally as the solution of SDEs. A special class of free SDEs are studied in depth in Kargin (2011), where the author provided proofs for existence and uniqueness of a local solution of free SDEs of the form

$$dX(t) = a(X(t))dt + \sum_{k=1}^{m} b_k(X(t))dW(t)c_k(X(t)),$$

for $t \in [0, \infty]$, where $(W(t))_{t \geq 0}$ is a free Brownian motion and $a, b_k, c_k$ are locally operator Lipschitz functions. To recall: We call a function $f : \mathbb{R} \to \mathbb{C}$ locally operator Lipschitz, if it is locally bounded, measurable and we also have that for all $C > 0$, there is a constant $K(C) > 0$, with the property that $\|f(X) - f(Y)\| \leq K(C)\|X - Y\|$, provided that $X$ and $Y$ both are self-adjoint operators such that $\|X\|, \|Y\| < C$, with $\|\|$ denoting the operator norm. Unfortunately due to the fact that the coefficient functions do not fulfill the necessary locally operator Lipschitz condition, the local existence theorem does not imply the existence of a solution to the classical free CIR equation of the form

$$dX(t) = (a - bX(t))dt + \sigma\sqrt{X(t)}dW(t), \quad X(0) = X_0 \in \mathcal{A}_+,$$

for $t \in [0, \infty]$ and $a, b, \sigma \in \mathcal{A}_+$. We denote by $\mathcal{A}_+$ the self-adjoint elements of the von Neumann algebra $\mathcal{A}$ with a strictly positive spectrum. Those are representable by the square of a self-adjoint operator which gives meaning to the expression $\sqrt{A}$. Therefore a solution to this equation needs to be bounded below by zero (in the sense of having a positive spectrum), just as in the scalar-valued case. We introduce and show the existence of the free CIR equation and derive an equivalent condition to the classical Feller condition. Note that for possibly time-dependent operators $a(t), b(t), \sigma(t)^2$ the Feller condition is to be understood in the sense of $2a(t) - \sigma(t)^2$ having a non-negative spectrum. When referring to the Feller condition (1) in the context of operators that is how it is meant to be interpreted.

This paper is structured as follows: In Section 2 we cover the necessary basics of free probability theory. Section 3 proves our main theorems on the existence and uniqueness of positive solutions for
the free CIR setup.

2 Free Probability and Free Stochastic Calculus

In the following we give a short introduction to the theory of free probability and free SDEs. For the theory of operator theory and in particular von Neumann algebras we refer the reader to the multivolume works "Theory of operator algebras" by Takesaki (cf. Takesaki (2002); Takesaki (2013); Takesaki (2003)) and Murphy (2014). For an introduction to free probability we refer to Voiculescu et al. (1992), Nica and Speicher (2006), Voiculescu et al. (2016) and Mingo and Speicher (2017). In the following we will draw heavily on Biane and Speicher (1998) and Kargin (2011).

2.1 Definition. Let $\mathcal{A}$ be a type I von Neumann algebra and $\varphi$ a faithful, normal and unital trace on $\mathcal{A}$. We call the tuple $(\mathcal{A}, \varphi)$ a non-commutative probability space and the self-adjoint elements of $X \in \mathcal{A}$ are called (free) random variables. In this paper we only consider self-adjoint random variables.

The trace $\varphi$ induces the $p$-norms $\|\cdot\|_p$ by

$$\|X\|_p = \varphi(|X|^p)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty,$$

where $\|\cdot\| := \|\cdot\|_\infty$ is the operator norm. For more information on non-commutative integration see e.g. Fack and Kosaki (1986) and Terp (1981).

We proceed by defining the notion of free independence or freeness, as introduced by Voiculescu.

2.2 Definition. Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be subalgebras of $\mathcal{A}$. We call $\mathcal{A}_1, \ldots, \mathcal{A}_n$ free if

$$\varphi\left(A_{i(1)} \ldots A_{i(m)}\right) = 0$$

whenever

$$\varphi\left(A_{i(s)}\right) = 0 \text{ and } i(s+1) \neq i(s) \text{ for each } s.$$

We call random variables free if the algebras they generate are free.

In order to introduce free SDEs, a "free" notion of Brownian motion is necessary.

2.3 Definition. A free Brownian motion is a stochastic process $(W(t))_{t \geq 0}$ of elements in a von Neumann algebra with the the three properties: $W(0) = 0$, the increments $W(t) - W(s)$ are free from
$W_s = \{ W(\tau) \mid \tau \leq s \}$ for $t > s$, and $W(t) - W(s)$ follows a semicircular distribution with expectation $\varphi(W(t) - W(s)) = 0$ and variance $\varphi((W(t) - W(s))^2) = t - s$.

We proceed by introducing an integral of the form

$$\mathcal{I} = \int_0^T a(X(s)) dW(s)b(X(s)),$$

where $a(X(s))$ and $b(X(s))$ are operator-valued functions of $X(s)$ and $T > 0$. Certain properties on $a$ resp. $b$ will be specified later. For a detailed construction of this free stochastic integral, we refer to Anshelevich (2002) or Biane and Speicher (1998). We will describe the general construction shortly as it is done in Kargin (2011).

Given an interval $[0,T]$ and $s \in [0,T]$, we let $a_s, b_s \in \mathcal{W}_s$ and assume that $s \mapsto a_s$ and $s \mapsto b_s$ are continuous maps w.r.t. to $\|\cdot\|$. Consider further $s_0, \ldots, s_n, \tau_1, \ldots, \tau_n \in \mathbb{R}$ with $0 = s_0 \leq s_1 \leq \cdots \leq s_n = T$ and $0 \leq \tau_k \leq s_{k-1}$. We denote the collection of all $s_i$ and $\tau_j$ by $\Delta := \{ s_i, \tau_j \mid 0 \leq i \leq n \text{, } 1 \leq j \leq n \}$. Consider the expression

$$\mathcal{I}(\Delta) = \sum_{i=1}^n a_{\tau_i} (W(s_i) - W(s_{i-1})) b_{\tau_i}.$$ 

For $d(\Delta) = \max_{1 \leq k \leq n} (s_k - \tau_k)$ we get that

$$\lim_{d(\Delta) \to 0} \mathcal{I}(\Delta) = \mathcal{I},$$

where the limit is meant w.r.t. $\|\cdot\|$ and is independent of the choice of $s_i$ and $\tau_i$. $\mathcal{I}$ is called the "free stochastic integral". The convergence relies heavily on the so called free Burkholder-Gundy inequality (see (Biane and Speicher, 1998, Theorem 3.2.1)). In the following lemma we state the free analogues of the Itô formula in an abbreviated form as we will use later on. For a detailed discussion we refer to Biane and Speicher (1998).

2.4 Lemma (Free Itô). Let $a_t, b_t, c_t, d_t$ be operator-valued functions and $(W(t))_{t \geq 0}$ as above. Then

$$a_t dt \cdot b_t dt = a_t dt \cdot b_t dW(t) c_t = a_t dW(t) b_t \cdot c_t dt = 0,$$

$$a_t dW(t) b_t \cdot c_t dW(t) d_t = \varphi (b_t c_t) a_t d_t dt.$$ 

From here on we will restrict ourselves to coefficients that do not depend on time explicitly (as it is done in the local existence results in Kargin (2011) as well) and will denote them by e.g. $a(X(t))$. 

7
3 Free CIR equations

As mentioned in the introduction it is known in the (commutative) scalar-valued case that the Feller condition \((1)\) ensures for the CIR equation

\[ dx(t) = (a - bx(t))\, dt + \sqrt{x(t)} dB(t), \quad x(0) = x_0 > 0, \]

for \( t \in [0, \infty[ \), and \( a, b, \sigma > 0 \) a global positive solution, where \((B(t))_{t \geq 0}\) is a (classical) Brownian motion.

Therefore it is natural to ask, if the above Feller condition guarantees global existence of a positive solution for a free SDE

\[ dX(t) = (a - bX(t))\, dt + \sqrt{X(t)} dW(t), \quad X(0) = X_0 \in A_+, \tag{2} \]

for \( t \in [0, \infty[ \), and \( a, b, \sigma \in A_+ \), where \((W(t))_{t \geq 0}\) is a free Brownian motion. From here on we will refer to the (classical) Brownian motion by \((B(t))_{t \geq 0}\) and to the free Brownian motion by \((W(t))_{t \geq 0}\).

Due to the non-commutativity of \(X(t)\) the free SDE, which we call (non-classical) free CIR equation,

\[ dX(t) = (a - bX(t))\, dt + \frac{\sigma}{2} \sqrt{X(t)} dW(t) + \frac{\sigma}{2} dW(t) \frac{\sqrt{X(t)}}{\sqrt{X(t)}}, \quad X(0) = X_0 \in A_+, \tag{3} \]

for \( t \in [0, \infty[ \), and \( a, b, \sigma \in A_+ \), may differ from the classical free CIR equation \((2)\). But, since all elements involved are self-adjoint, an easy argument shows that the traces of the solutions to \((2)\) and \((3)\) coincide and therefore it is enough to show the existence of the solution to the latter.

We start showing existence and uniqueness of these free SDEs by introducing a SDE with a simple additive Brownian motion term of the form

\[ dV(t) = \left( a - \frac{\sigma^2}{2} \right) \frac{1}{2} V^{-1}(t) \frac{1}{2} bV(t) dt + \frac{\sigma}{2} dB(t), \quad V(0) = V_0 > 0, \]

for \( t \in [0, \infty[ \), and \( a, b, \sigma > 0 \), which we will refer to as square-root process (the reason for that will become clear later), and by step by step transforming it into a free SDE (driven by a free Brownian motion). Since the existence of a positive solution to a scalar-valued SDE of the form above and the classical CIR equation are equivalent by the classical Itô lemma we know that such an equation
has a positive solution as long as the Feller condition is satisfied. We will transform this connection first into the setting of (commutative) function spaces (see Theorem 3.3), followed up by the general (non-commutative) von Neumann algebra-valued case (see Theorem 3.5). Note that the driving process is still a classical Brownian motion. The final and most elaborate part (Theorem 3.9) consists of changing the driving process to a free Brownian motion. We will do this by showing that the solutions to the von Neumann algebra-valued SDE driven by a classical and the one driven by a free Brownian motion are $L^2$-isometric for $t \geq 0$. Finally the free Itô lemma gives us the existence of a global solution to our free CIR equation (under Feller).

Given a positive $V_0$ we can select a special probability space to transfer the SDE into a usual vector-valued SDE. Using functional calculus resp. the spectral theorem we have an isometric homomorphism

$$T : B(\sigma(V_0)) \longrightarrow (V_0, \text{id}),$$

where $(V_0, \text{id})$ is the von Neumann algebra generated by $V_0$ (and the identity) and $B(\sigma(V_0))$ is the function space of bounded, measurable functions on $\sigma(V_0)$, the spectrum of $V_0$. If $\varphi$ is a unital, faithful trace, then consider $\mathbb{E}_P^\varphi = T^*(\varphi)$ with the identity

$$\mathbb{E}_P^\varphi(g) = \int_{\sigma(V_0)} g dP^\varphi = \varphi(T(g)) \text{ for all } g \in B(\sigma(V_0)).$$

Next, for dealing with operator-valued coefficients, we introduce the (free) conditional expectation $\mathbb{E}_B$ by the following lemma.

**3.1 Lemma.** Let $B \subset A$ be a von Neumann subalgebra. Then there exists a conditional expectation

$$\mathbb{E}_B : A \longrightarrow B$$

s.t.

$$\varphi(ab) = \varphi(\mathbb{E}_B(a)b) \text{ for all } a \in A, b \in B.$$

(cf. Biane and Speicher (2001))

We fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ and start with a vector-valued version for the existence of a local solution of a classical SDE. For this let $(B(t))_{t \geq 0}$ be a classical Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$. First, we state a result which seems folklore, but is not cited explicitly.
3.2 Theorem. Let $E$ be a Banach space. Let $a$ and $b_i$ for $i = 1, \ldots, m$ be continuous and locally Lipschitz. For $V_0 \in E$ there exists a $T > 0$ and a unique continuous solution $V : [0,T] \to E$, such that $V(0) = V_0$, $V \in C([0,T],E)$ and for all $t \in [0,T]$

$$V(t) = V_0 + \int_0^t a(V(s))ds + \sum_{i=1}^m \int_0^t b_i(V(s))dB(s).$$

Proof. The proof follows a standard argument via Picard approximations and the Banach fixed-point theorem. Applying the Hahn-Banach theorem we use a $x^* \in X^*$ in the dual of $X$ to transform our approximations into the real setting in order to make use of the (classical) Burkholder-Gundy inequality. The details are analogous to the proof of (Kargin, 2011, Theorem 3.1.).

Having established the existence of a local solution we will show that the classical Feller condition ensures a positive solution in the case of the commutative von Neumann algebra $C(K)$ of continuous functions of a compact Hausdorff space $K$. We further denote the positive cone by

$$C(K)_+ = \{ f \in C(K) \mid f(u) > 0, \text{ for all } u \in K \}.$$

We note that using the Itô formula, under the Feller condition the square-root process $V(t)$ enjoys a global solution provided the initial condition is positive. We state a vector-valued extension.

3.3 Theorem. Let $K \subset ]0,\infty[$ be compact. Let $a,b,\sigma : [0,\infty[ \to C(K)_+$ be continuous, such that (1) holds and let $\tilde{V}_0 \in C(K)_+$. Then the SDE

$$d\tilde{V}(t) = \left( a(t) - \frac{\sigma(t)^2}{2} \right) \tilde{V}^{-1}(t) \frac{1}{2} \tilde{V}(t) dt + \frac{\sigma(t)}{2} I_K dB(t), \quad \tilde{V}(0) = \tilde{V}_0,$$

for $t \in ]0,\infty[$, has a global solution $V \in C([0,\infty[, L_2(\mathbb{P}, C(K)_+))$.

Proof. Consider a classical Brownian motion $(B(t))_{t \geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then using point mass measures $\delta_k$ for $k \in K$ and the classical Feller condition prove the global existence of

$$d\tilde{V}_k(t) = \left( a_k(t) - \frac{\sigma_k(t)^2}{2} \right) \frac{1}{2} \tilde{V}_k^{-1}(t) - \frac{1}{2} b_k(t) \tilde{V}_k(t) dt + \frac{\sigma_k(t)}{2} dB(t), \quad \tilde{V}_k(0) = \tilde{V}_{0,k} > 0,$$

for $t \in ]0,\infty[$.

Using a countable dense subset $\tilde{K} \subset K$ and the point mass functional we show that the paths keep positive except at a $\mathbb{P}$-zero set $N := \bigcup_{k \in \tilde{K}} N_k$. So for all $\omega \in \Omega \setminus N$ the paths stay positive on $[0,\infty[$.
for all $k$.

### 3.4 Corollary

Let $K,a,b,\sigma$ be given as in Theorem 3.3. Then the generalized CIR equation

$$d\overline{X}(t) = \left( a(t) - b(t)\overline{X}(t) \right) dt + \sigma(t)\sqrt{\overline{X}(t)} dB(t), \quad \overline{X}(0) = \overline{X}_0 \in C(K)_+,$$

for $t \in [0, \infty[$, has a global solution.

**Proof.** The proof is immediate by Itô's lemma.

The next step is to transform the square-root process into the setting of a non-commutative von Neumann algebra $\mathcal{A}$ but still with a classical Brownian motion as driving process.

### 3.5 Theorem

Let $V_0 \in \mathcal{A}_+$ and let $a,\sigma,b : [0,\infty[ \to (V_0,\text{id})_+$ such that (1) holds for all $t \in [0,\infty[$. Then the SDE

$$d\overline{V}(t) = \left( \left( a(t) - \frac{\sigma^2(t)}{2} \right) \frac{1}{2} \overline{V}^{-1}(t) - \frac{b(t)}{2} \overline{V}(t) \right) dt + \frac{\sigma(t)}{2} dB(t), \quad \overline{V}(0) = V_0, \quad (4)$$

for $t \in [0,\infty[$, has a global solution in $V \in C([0,\infty[,L_2(\mathbb{P},\mathcal{A}_+))$. Note that $\text{id}$ is the unit in the corresponding von Neumann algebra.

**Proof.** Let $\hat{V}(t)$ be a global solution in $C(K)_+$ by Theorem 3.3. By the functional calculus we see that $T(\hat{V}(t))$ is a positive solution to (4) under the Feller condition, where $T : C(K) \to (V_0,\text{id})_+$ is the functional calculus mapping for the self-adjoint element $V_0$. In particular we have that $\overline{V}(t) \in (V_0,\text{id})_+$ for all $t \in [0,\infty[$.

Using the Itô calculus we get:

### 3.6 Corollary

Let $a(t),b(t),\sigma(t)$ be given as in Theorem 3.5. Then the generalized CIR equation

$$dX(t) = \left( a(t) - b(t)X(t) \right) dt + \sigma(t)\sqrt{X(t)} dB(t), \quad X(0) = X_0 \in \mathcal{A}_+,$$

for $t \in [0,\infty[$, has a global solution $X \in C([0,\infty[,L_2(\mathbb{P},\mathcal{A}_+))$.

In the above Theorem 3.5 we restricted the coefficient functions to the von Neumann subalgebra $(V_0,\text{id})_+$. We now follow up with our main Theorem 3.9 where we show the existence of our solution.
in the case of a free Brownian motion. For this setting we can allow $a : [0, \infty] \to \mathcal{A}_+$ but will have to restrict $b$ this time to the case of a scalar $b > 0$.

3.7 Remark. 1. The last step we need for the existence of a positive solution in the context of free probability, is to change the driving process in the general von Neumann algebra-valued SDE from a classical Brownian motion to a free Brownian motion. Having established that the corresponding SDE driven by a classical Brownian motion stays positive under Feller by Theorem 3.5 we will show that both solutions (the classical and the free Brownian motion case) are $L_2$-isometric for all $t \in [0, T]$, a common existence interval of both solutions. Finally we will prove, using the upcoming proposition, that $V(T)$, the solution to the free equation evaluated at $T$, is invertible and therefore the free SDE has a global positive solution as well.

2. For a more convenient notation we may write $\varphi$ instead of $\mathbb{E}_{\varphi}$ in the sequel for the process $(\nabla(t))_{t \geq 0}$. No confusion will arise at any point.

3. For Lipschitz-continuous and $\mathcal{A}$-valued functions $a, b, c$ we call a process

$$V(t) = V_0 + \int_0^t a(V(s))ds + \int_0^t b(V(s))dW(s)c(V(s)),$$

for $t \in [0, T]$, a free Itô process. The existence is guaranteed by the main result due to Kargin (2011).

Before stating the main result on the existence of a global solution, we formulate a result, which may be of independent interest.

3.8 Proposition. Let $T > 0$ and let $V \in C([0, T], \mathcal{A}) \cap C([0, T], L_2(\varphi))$ be a free Itô process and let $\nabla \in C([0, T], L_2(\mathbb{P}_{\varphi}, \mathcal{A}))$ be a vector-valued Itô processes. Suppose for all $t \in [0, T]$ and all orthogonal projections $p$ free of $V(t)$ that we have

$$\|pV(t)^2p\|_1 = \|p\nabla(t)^2p\|_1. \quad (5)$$

1. If $V(t) \in \mathcal{A}$ and $V(T) \in L_2(\varphi)$ then $V(T) \in \mathcal{A}$.

2. If $\nabla(t)$ is invertible for all $t \in [0, T]$, $V(t)$ is invertible for all $t \in [0, T]$ and $V(T) \in \mathcal{A}$, then $V(T) \in \mathcal{A}$ is also invertible.
Proof. 1. We suppose that $V(T) \notin \mathcal{A}$. Then, since $V(T) \in L_2(\varphi)$, we find a sequence of projections $(p_n)$, such that

$$\frac{1}{\varphi(p_n)} \|p_n V(T)^2 p_n\|_1 \geq n.$$ 

Since $\nabla \in C([0,T], L_2(\mathbb{P} \cap \mathcal{A}))$, we find a $\hat{T} < T$ such that

$$\|\nabla(T)^2 - \nabla(\hat{T})^2\|_1 < 1.$$ 

Since $\nabla(T) \in \mathcal{A}$, we get

$$\frac{1}{\varphi(p_n)} \|p_n \nabla(T)^2 p_n\|_1 \leq \|\nabla(T)^2\| < \infty.$$ 

Since $V(T) = V(\hat{T}) + (V(T) - V(\hat{T}))$, the element $V(T)$ is generated by the projections in the von Neumann algebras $\mathcal{A}_{\hat{T}}$ and $\mathcal{A}_T$, which are generated by $\{W(t) \mid t \leq \hat{T}\} \cup \{V_0, \text{id}\}$ and $V(T) - V(\hat{T})$, respectively. Therefore $\mathcal{A}_{\hat{T}}$ and $\mathcal{A}_T$ are free. Consider two cases:

a) $p_n \in \mathcal{A}_{\hat{T}}$ for infinitely many $n \in \mathbb{N}$. (For simplicity we assume for all $n \in \mathbb{N}$). In this case we consider a new starting value $V(\hat{T})$ instead of $V_0$ and instead of the interval $[0,T]$ the interval $[\hat{T},T]$. Then for all $n \in \mathbb{N}$ the projection $p_n$ is free of increments of the free Brownian motion $W(t) - W(s)$ for $\hat{T} \leq s < t \leq T$ and the proof runs as in the case that all $p_n \in \langle V_0, \text{id} \rangle_+$. 

b) Only for finitely many $n \in \mathbb{N}$ we have $p_n \in \mathcal{A}_{\hat{T}}$. Then we choose the infinitely many $p_n \in \mathcal{A}_T$, for simplicity all $n \in \mathbb{N}$. Here, all those $p_n$ are free of $\mathcal{A}_{\hat{T}}$ and thus again we have that $W(t) - W(s)$ for $0 \leq s < t \leq \hat{T}$ are free of $p_n$.

In both cases we get by (5)

$$\varphi(p_n \nabla(t)^2 p_n) = \varphi(p_n V(t)^2 p_n) \text{ for } t \in [0,\hat{T}]. \quad (6)$$
Consequently, for both cases it holds that
\[
\infty > \|V(T)^2\| \geq \frac{1}{\varphi(p_n)}\|p_nV(T)^2p_n\|_1 \\
= \frac{1}{\varphi(p_n)}\|p_n(V(T)^2 + (V(T)^2 - \overline{V}(T)^2))p_n\|_1 \\
\geq (6) \frac{1}{\varphi(p_n)}\|p_nV(T)^2p_n\|_1 - \frac{1}{\varphi(p_n)}\|p_n(V(T)^2 - \overline{V}(T)^2)p_n\|_1 \\
> n - 1,
\]
by the isometry (5). This is a contradiction.

2. Suppose \(V(T)\) and hence \(V(T)^2\) is not invertible. We find a sequence of projections \((p_n)\) in the von Neumann algebra generated by the self-adjoint element \(V(T)\) such that
\[
\alpha_n = \frac{1}{\varphi(p_n)}\|p_nV(T)^2p_n\|_1 \rightarrow 0, \ n \rightarrow \infty. \tag{7}
\]
We know that \(\overline{V}(T)\) is invertible and therefore we get by the properties of \(p_n\) that
\[
0 < \frac{1}{\|(V(T)^2)^{-1}\|} \leq \frac{1}{\varphi(p_n)}\|p_nV(T)^2p_n\|_1.
\]
Since both processes are continuous mappings
\[
\overline{V},V : [0,T] \rightarrow \mathcal{A},
\]
i.e. \(\overline{V},V \in C([0,T],\mathcal{A})\), we can find a \(\tilde{T} \in]0,T[\), s.t.
\[
\left|\frac{1}{\|(V(T)^2)^{-1}\|} - \frac{1}{\|(V(T)^2)^{-1}\|}\right| < \frac{1}{4\|(V(T)^2)^{-1}\|}
\]
and
\[
\|V(T)^2 - \overline{V}(T)^2\| < \frac{1}{4\|(V(T)^2)^{-1}\|}.
\]
Thus, for all \(n \in \mathbb{N}\):
\[
\beta_n = \frac{1}{\varphi(p_n)}\|p_n(V(T)^2 - \overline{V}(T)^2)p_n\|_1 \leq \frac{1}{\varphi(p_n)}\varphi(p_n)\|V(T)^2 - \overline{V}(T)^2\| < \frac{1}{4\|(V(T)^2)^{-1}\|}.
\]
Again, since $V(T) = V(\tilde{T}) + (V(T) - V(\tilde{T}))$, we consider as in the proof to 1. two cases for the projections $(p_n)$ and conclude by (5) that

$$\varphi(p_n V(t)^2 p_n) = \varphi(p_n V(t)^2) \text{ for } t \in [0, \tilde{T}].$$

(8)

Consequently, it holds that

$$\frac{1}{\varphi(p_n)} \|p_n V(T)^2 p_n\|_1 = \frac{1}{\varphi(p_n)} \|p_n (V(\tilde{T})^2 + (V(T)^2 - V(\tilde{T})^2)) p_n\|_1$$

$$\geq \frac{3}{4\|V(T)^2\|^{-1}} - \frac{1}{2\|V(T)^2\|^{-1}} > 0.$$  

This is a contradiction to the assumption (7) and hence $V(T)$ is invertible.

3.9 Theorem. Let $V_0 \in \mathcal{A}_+$ be given. Let $a : [0, \infty[ \to \mathcal{A}_+$, $\sigma : [0, \infty[ \to (V_0, \text{id})_+$ and $b > 0$ a constant such that (1) holds. Then the free SDE

$$dV(t) = \left( a(t) - \frac{\sigma^2(t)}{2} \right) \frac{1}{2} V^{-1}(t) - \frac{b}{2} V(t) \right) dt + \frac{\sigma(t)}{2} dW(t), \quad V(0) = V_0,$$

(9)

for $t \in [0, \infty[$, has a global solution $V \in C([0, \infty[, \mathcal{A}_+)$.

Proof. 1. In the first step we choose a maximal interval $[0, T]$, where a solution of the equation

$$dV(t) = \left( a(t) - \frac{\sigma^2(t)}{2} \right) \frac{1}{2} V^{-1}(t) - \frac{b}{2} V(t) \right) dt + \frac{\sigma(t)}{2} dW(t), \quad V(0) = V_0,$$

for $t \in [0, T]$, exists according to Kargin (2011). By Theorem 3.5 we know that the solution for

$$d\overline{V}(t) = \left( a(t) - \frac{\sigma^2(t)}{2} \right) \frac{1}{2} \overline{V}^{-1}(t) - \frac{b}{2} \overline{V}(t) \right) dt + \frac{\sigma(t)}{2} dB(t), \quad \overline{V}(0) = V_0,$$
for \( t \in [0, \infty[, \) exists globally. First we prove the following three isometries.

\[
\|V(t)\|_2 = \|\nabla(t)\|_2 \quad \text{for} \quad t \in [0,T] \\
\|pV(t)^2p\|_1 = \|p\nabla(t)^2p\|_1 \quad \text{for} \quad t \in [0,T] \quad \text{and} \quad p \in \langle V_0, \text{id} \rangle_+ \\
\|pV(t)^2p\|_1 = \|p\nabla(t)^2p\|_1 \quad \text{for} \quad t \in [0,T] \quad \text{and} \quad p \text{ free of } V(t)
\]

(10)

The third isometry directly follows from the first by freeness.

We approximate the solutions on \([t_0, T]\) and thus, select a partition \( Z \) of the interval \([t_0, T]\) namely \( 0 = t_0 < t_1 < \cdots < t_n = T \). We will omit the variable "\( t \)" in the expression of \( a \) and \( \sigma \). The proof for the isometry

\[
\|V(t)\|_2 = \|\nabla(t)\|_2 \quad \text{for} \quad t \in [0,T]
\]

is basically a complete induction. We will show it for \([t_0, t_1]\) and then the induction step from \([t_0, t_1]\) to \([t_1, t_2]\) to illustrate the procedure. We start with the interval \([t_0, t_1]\). For \( t \in [t_0, t_1]\) we have

\[
\nabla_Z(t) = V_0 + \int_{t_0}^t \left( \left( a - \frac{\sigma^2}{2} \right) \frac{1}{2} V_Z^{-1}(s) - \frac{b}{2} V_Z(s) \right) ds
\]

and

\[
V_Z(t) = V_0 + \int_{t_0}^t \left( \left( a - \frac{\sigma^2}{2} \right) \frac{1}{2} V_Z^{-1}(s) - \frac{b}{2} V_Z(s) \right) ds.
\]

Thus we have by an easy approximation by step functions that \( \nabla_Z(t), V_Z(t) \in \langle V_0, \text{id} \rangle_+ \) and

\[
\varphi(V_Z(t)^2) = \varphi(\nabla_Z(t)^2).
\]

By adding the classical Brownian motion term \( \frac{\sigma}{2} B(t_1) \) and the free Brownian motion term \( \frac{\sigma}{2} W(t_1) \), respectively we define

\[
\nabla_Z(t_1) = \nabla_Z(t_1-) + \frac{\sigma}{2} B(t_1) \\
V_Z(t_1) = V_Z(t_1-) + \frac{\sigma}{2} W(t_1),
\]

where the first one is just a discretization of the global solution of Theorem 3.5 (which serves as our "reference solution") evaluated at \( t_1 \) and \( V_Z(t_1-) \) and \( \nabla_Z(t_1-) \), respectively is the local solution for \( t \in [t_0, t_1[ \). Since \( \sigma \in \langle V_0, \text{id} \rangle_+ \) is free of \( W(t_1) \) and \( \varphi(W(t_1)) = 0 \), we conclude with (Kargin, 2011,
\[
\varphi \left( \frac{\sigma}{2} B(t_1) \frac{\sigma}{2} B(t_1) \right) = \varphi \left( W(t_1) \frac{\sigma}{2} W(t_1) \frac{\sigma}{2} \right) = \frac{1}{4} \varphi(\sigma^2) t_1
\]

and see by the independence, resp. the freeness that

\[
\varphi(\sqrt{V_Z(t_1)})^2 = \varphi(V_Z(t_1))^2.
\]

On \([t_1, t_2]\) we again consider approximations of our solutions starting in \(\sqrt{V_Z(t_1)}\) and \(V_Z(t_1)\), (not equal but \(L_2\)-isometric) respectively:

\[
\sqrt{V_Z}(t) = \sqrt{V_Z}(t_1) + \int_{t_1}^{t} \left( \left( a - \frac{\sigma^2}{2} \right) \frac{1}{2} V_Z^{-1}(s) - \frac{b}{2} V_Z(s) \right) ds
\]

Defining \(\overline{X}(t) := \sqrt{V_Z}(t)^2\), resp. \(X(t) := V_Z(t)^2\) and applying the trace we get a general ordinary differential equation (ODE) of the form

\[
\frac{dy}{dt} = \alpha - by
\]

with \(\alpha = \varphi(a - \frac{\sigma^2}{2})\) and \(y \in \{\varphi(\overline{X}(t)), \varphi(X(t))\}\) with the initial value

\[
y(t_1) = \varphi(X(t_1)) = \varphi(\overline{X}(t_1)).
\]

Since this ODE has a unique solution we get

\[
\|\sqrt{V_Z}(t)\|_2^2 = \varphi(X(t)) = \varphi(\overline{X}(t)) = \|V_Z(t)\|_2^2 \text{ for } t \in [t_1, t_2].
\]

Again we add the classical Brownian motion term \(\frac{\sigma}{2} (B(t_2) - B(t_1))\) and the free Brownian motion term \(\frac{\sigma}{2} (W(t_2) - W(t_1))\), respectively and get

\[
\sqrt{V_Z}(t_2) = \sqrt{V_Z}(t_2^-) + \frac{\sigma}{2} (B(t_2) - B(t_1))
\]

\[
V_Z(t_2) = V_Z(t_2^-) + \frac{\sigma}{2} (W(t_2) - W(t_1)).
\]
As before by the independence, resp. freeness we conclude

\[ \varphi(\nabla Z(t_2)^2) = \varphi(Z(t_2)^2). \]

We may extend \( \nabla Z \), resp. \( Z \) beyond the interval \([0,t_2]\). In the same respect as above we have for both processes extensions on \([0,t_n]\) such that

\[ \varphi(\nabla Z(t)^2) = \varphi(Z(t)^2) \text{ for all } t \in [0,t_n]. \]

Since these isometries hold by the continuity of the solutions for all partitions of \([0,T]\), we see that \( Z \) converges for \(|Z| \to 0\), i.e. if the length of the partition converges to 0:

\[ \varphi(\nabla(t)^2) = \lim_{|Z| \to 0} \varphi(\nabla Z(t)^2) = \lim_{|Z| \to 0} \varphi(Z(t)^2) = \varphi(V(t)^2) \text{ for all } t \in [0,T]. \]

We proceed by showing the second isometry of (10) and therefore introduce the equation

\[
pV p(t) = pV_0p + \int_0^t \left( p \left( a - \frac{\sigma^2}{2} \right) + pV_p(s) \right) ds \]

\[
+ \int_0^t \frac{\sigma}{2} p dW(s), \quad pV_p(0) = pV_0p \in p\mathcal{A}_+ p,
\]

with \( a \in \mathcal{A}_+ \), \( \sigma \in (V_0,\text{id})_+ \) and \( b > 0 \), all strictly positive, in \( p\mathcal{A}_+ \). Since \( p \in (V_0,\text{id})_+ \) by assumption, we have \( pVp, p\sigma p \in (pV_0p,\text{id})_+ \) and the proof for the \( L_2 \)-isometry above can be mimicked. For this we consider \( d(pVp(t))^2 = 2pVp(t) d(pVp(t)) \) to get a similar general ODE to the case above.

2. According to Kargin (2011), we know that the process \( V(t) \) exists for \( t < T \). The reference process \( \nabla(t) \) exists globally with values in \( \mathcal{A} \). Because of the isometry we can define

\[ V(T) = L_2 \lim_{t \to T} V(t) = V_0 + \int_0^T \frac{1}{2} \left( a - \frac{\sigma^2}{2} \right) V^{-1}(t) dt + \frac{\sigma}{2} W(T). \]

Using Proposition 3.8 (1.) we can deduce that \( V(T) \in \mathcal{A} \).

3. Step 2 told us that \( V(T) \in \mathcal{A} \). We want to extend the (unique) solution \( V(t) \) beyond \( T \). To apply the basic result in Kargin (2011), we need the invertibility of \( V(T) \). This would allow us an extension and the solution is global. Again by Proposition 3.8 (2.), we see that \( V(T) \) is invertible. Therefore the solution to (9) is global. \( \square \)
Having established the existence of a global solution to the square-root process we now apply the free Itô formula and get the global solution to the free CIR equation. Let \( X(t) = V(t)^2 \). Then according to the free Itô formula:

\[
\begin{align*}
    dX(t) &= (V(t) + dV(t))^2 - (V(t))^2 = (dV(t))^2 + dV(t)V(t) + V(t)dV(t) \\
    &= \left(2\left(\frac{a - \sigma^2}{2V(t)} - \frac{b}{2}V(t)\right)V(t) + \frac{\sigma^2}{2}\right)dt + \frac{\sigma}{2}V(t)dW(t) + \frac{\sigma^2}{2}dW(t)V(t) \\
    &= (a - bV^2(t))dt + \frac{\sigma}{2}V(t)dW(t) + \frac{\sigma^2}{2}dW(t)V(t) \\
    &= (a - bX(t))dt + \frac{\sigma^2}{2}\sqrt{X(t)}dW(t) + \frac{\sigma^2}{2}dW(t)\sqrt{X(t)},
\end{align*}
\]

for \( t \in [0, \infty[ \). Thus, we state

3.10 Theorem. Let \( a, b, \sigma \) such that (1) and the assumptions made in (3.9) hold. Then the free SDE

\[
dX(t) = (a - bX(t))dt + \frac{\sigma^2}{2}\sqrt{X(t)}dW(t) + \frac{\sigma^2}{2}dW(t)\sqrt{X(t)},
\]

for \( t \in [0, \infty[ \), has a global solution \( X \in C([0, \infty[, A_+) \).

3.11 Remark. As mentioned in the introduction the above theorem guarantees the existence of a unique positive solution to the classical free CIR equation (2) as well and for a constant real-valued \( \sigma \) the solutions even coincide.

4 Conclusion

We introduced the CIR equation to the world of free probability and made a contribution to the study of free SDEs. Initially developed as a tool for solving operator-theoretic problems free probability theory has been evolving into its own field of research inviting researchers from various disciplines such as finance, physics and signal processing to profit from and contribute to. As motivated in the introduction the connection between random matrix ensembles and their free operator-valued limit arouses the interest of many researchers by its attractive properties. Having introduced the free CIR equation(s) in this paper we paved the way for future applications to exploit this connection known as asymptotic freeness to process our setting for utilization in the real world. In particular the recent developments in big data and machine learning allow for the usage of very high-dimensional data as e.g. very large random matrices, such as portfolios incorporating a huge number of assets. We hope
that this paper will be used for portfolio optimization when the number of assets makes classical probabilistic approaches unfeasible.

Funding

This research received no external funding.

Conflicts of interest

The authors declare no conflict of interest.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

An, G. and Gao, M. (2015). Poisson processes in free probability. arXiv preprint. https://arxiv.org/abs/1506.03130.

Anshelevich, M. (2002). Itô formula for free stochastic integrals. Journal of Functional Analysis, 188(1):292–315. https://doi.org/10.1006/jfan.2001.3849.

Baillie, R. T., Bollerslev, T., and Mikkelsen, H. O. (1996). Fractionally integrated generalized autoregressive conditional heteroskedasticity. Journal of Econometrics, 74(1):3–30. https://doi.org/10.1016/S0304-4076(95)01749-6.

Barndorff-Nielsen, O. E., Thorbjørnsen, S., et al. (2002). Self-decomposability and Lévy processes in free probability. Bernoulli, 8(3):323–366. https://www.jstor.org/stable/3318705.

Bayer, C., Friz, P. K., Gassiat, P., Martin, J., and Stemper, B. (2019). A regularity structure for rough volatility. Mathematical Finance. https://doi.org/10.1111/mafi.12233.

Biane, P. (1997). Free Brownian motion, free stochastic calculus and random matrices, in free probability theory. Fields Institute Communications, 12:1–19. https://ci.nii.ac.jp/naid/10004590009/.

Biane, P. (1998). Processes with free increments. Mathematische Zeitschrift, 227(1):143–174. https://doi.org/10.1007/PL00004363.

Biane, P. and Speicher, R. (1998). Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. Probability Theory and Related Fields, 112(3):373–409. https://doi.org/10.1007/s00440050194.
Biane, P. and Speicher, R. (2001). Free diffusions, free entropy and free Fisher information. In *Annales de l’Institut Henri Poincare (B) Probability and Statistics*, volume 37, pages 581–606. No longer published by Elsevier. https://doi.org/10.1016/S0246-0203(00)01074-8.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654. https://www.jstor.org/stable/1831029.

Bollerslev, T. and Mikkelsen, H. O. (1996). Modeling and pricing long memory in stock market volatility. *Journal of Econometrics*, 73(1):151–184. https://doi.org/10.1016/0304-4076(95)01736-4.

Bouchaud, J.-P. and Potters, M. (2009). Financial applications of random matrix theory: a short review. *arXiv preprint*. https://arxiv.org/abs/0910.1205.

Bru, M.-F. (1991). Wishart processes. *Journal of Theoretical Probability*, 4(4):725–751. https://doi.org/10.1007/BF01259552.

Comte, F., Coutin, L., and Renault, E. (2012). Affine fractional stochastic volatility models. *Annals of Finance*, 8(2-3):337–378. https://doi.org/10.1007/s10436-010-0165-3.

Comte, F. and Renault, E. (1998). Long memory in continuous-time stochastic volatility models. *Mathematical Finance*, 8(4):291–323. https://doi.org/10.1111/1467-9965.00057.

Cont, R. (2001). Empirical properties of asset returns: Stylized facts and statistical issues. *Quantitative Finance*, 1:223–236. http://citeseer.ist.psu.edu/viewdoc/summary?doi=10.1.1.16.5992.

Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407. https://www.jstor.org/stable/1911242.

El Euch, O. and Rosenbaum, M. (2019). The characteristic function of rough Heston models. *Mathematical Finance*, 29(1):3–38. https://doi.org/10.1111/mafi.12173.

El Euch, O., Rosenbaum, M., et al. (2018). Perfect hedging in rough Heston models. *The Annals of Applied Probability*, 28(6):3813–3856. https://projecteuclid.org/euclid.aop/1538985636.

Fack, T. and Kosaki, H. (1986). Generalized s-numbers and τ-measurable operators. *Pacific Journal of Mathematics*, 123:269–300. https://projecteuclid.org/euclid.pjm/1102701004.

Fan, Z. (2006). Self-similarity of free stochastic processes. *Infinite Dimensional Analysis, Quantum Probability and Related Topics*, 9(03):451–469. https://doi.org/10.1142/S0219025706002482.

Feller, W. (1951). Two singular diffusion problems. *Annals of Mathematics*, pages 173–182. https://www.jstor.org/stable/1969318.

Fink, H. and Schlüchtermann, G. (2018). Fractional Lévy Cox–Ingersoll–Ross and Jacobi processes. *Statistics & Probability Letters*, 142(C):84–91. https://doi.org/10.1016/j.spl.2018.07.004.
Gao, M. (2006). Free Ornstein-Uhlenbeck processes. *Journal of Mathematical Analysis and Applications*, 322(1):177–192. https://doi.org/10.1016/j.jmaa.2005.09.013.

Gao, M. et al. (2008). Free Markov processes and stochastic differential equations in von Neumann algebras. *Illinois Journal of Mathematics*, 52(1):153–180. https://projecteuclid.org/euclid.ijm/1242414126.

Gatheral, J., Jaisson, T., and Rosenbaum, M. (2018). Volatility is rough. *Quantitative Finance*, 18(6):933–949. https://doi.org/10.1080/14697688.2017.1393551.

Gourieroux, C. (2006). Continuous time Wishart process for stochastic risk. *Econometric Reviews*, 25(2-3):177–217. https://doi.org/10.1080/14697688.2017.1393551.

Gourieroux, C. and Sufana, R. (2010). Derivative pricing with Wishart multivariate stochastic volatility. *Journal of Business & Economic statistics*, 28(3):438–451. https://www.jstor.org/stable/20750851.

Heston, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *The Review of Financial Studies*, 6(2):327–343. https://doi.org/10.1093/rfs/6.2.327.

Hull, J. and White, A. (1990). Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3(4):573–592. https://doi.org/10.1093/rfs/3.4.573.

Jaber, E. A. and Euch, O. E. (2019). Multi-factor approximation of rough volatility models. *SIAM Journal of Financial Mathematics*, 10(2):309–349. https://doi.org/10.1137/18M1170236.

Kargin, V. (2011). On free stochastic differential equations. *Journal of Theoretical Probability*, 24(3):821–848. https://doi.org/10.1007/s10959-011-0341-z.

Keller-Ressel, M. and Steiner, T. (2008). Yield curve shapes and the asymptotic short rate distribution in affine one-factor models. *Finance and Stochastics*, 12(2):149–172. https://doi.org/10.1007/s00780-007-0059-z.

Kümmerer, B. and Speicher, R. (1992). Stochastic integration on the Cuntz algebra $O_\infty$. *Journal of Functional Analysis*, 103(2):372–408. https://doi.org/10.1016/0022-1236(92)90126-4.

Maghsoodi, Y. (1996). Solution of the extended CIR term structure and bond option valuation. *Mathematical Finance*, 6(1):89–109. https://doi.org/10.1111/j.1467-9965.1996.tb00113.x.

Mandelbrot, B. B. (1997). The variation of certain speculative prices. In *Fractals and Scaling in Finance*, pages 371–418. Springer. https://doi.org/10.1007/978-1-4757-2763-0_14.

Mingo, J. A. and Speicher, R. (2017). *Free probability and random matrices*, volume 4. Springer. https://www.springer.com/de/book/9781493969418.

Murphy, G. J. (2014). *$C^*$-algebras and operator theory*. Academic Press. https://doi.org/10.1016/C2009-0-22289-6.

Nica, A. and Speicher, R. (2006). *Lectures on the combinatorics of free probability*, volume 13. Cambridge University Press. https://doi.org/10.1017/CBO9780511735127.
Nourdin, I. and Taqqu, M. S. (2014). Central and non-central limit theorems in a free probability setting. Journal of Theoretical Probability, 27(1):220–248. https://link.springer.com/article/10.1007/s10959-012-0443-2.

Pagan, A. (1996). The econometrics of financial markets. Journal of Empirical Finance, 3(1):15–102. https://doi.org/10.1016/0927-5398(95)00020-8.

Ryan, Ø. (2008). Free probability: Basic concepts, tools, applications, and relations to other fields. http://folk.uio.no/oyvindry/stocsem1.pdf.

Ryan, Ø. and Debbah, M. (2007). Free deconvolution for signal processing applications. In International Symposium on Information Theory, pages 1846–1850. IEEE. https://ieeexplore.ieee.org/document/4557490.

Schlüchtermann, G. and Yang, Y. (2016). Note on fractional CLKS-type stochastic differential equation–path-wise and in the Wick sense. https://www.researchgate.net/publication/299670926.

Speicher, R. (1990). A new example of ‘independence’ and ‘white noise’. Probability Theory and Related Fields, 84(2):141–159. https://doi.org/10.1007/BF01197843.

Takesaki, M. (2002). Theory of operator algebras I, Encyclopaedia of Mathematical Sciences, volume 124. Springer. https://www.springer.com/de/book/9783540422488.

Takesaki, M. (2003). Theory of operator algebras III, Encyclopaedia of Mathematical Sciences, volume 127. Springer-Verlag, Berlin. https://www.springer.com/de/book/9783540429135.

Takesaki, M. (2013). Theory of operator algebras II, volume 125. Springer Science & Business Media. https://www.springer.com/gp/book/9783540429142.

Terp, M. (1981). $L^p$-spaces associated with von Neumann algebras. Notes Copenhagen University.

Vasicek, O. (1977). An equilibrium characterization of the term structure. Journal of Financial Economics, 5(2):177–188. https://doi.org/10.1016/0304-405X(77)90016-2.

Voiculescu, D., Dykema, K. J., and Nica, A. (1992). Free random variables. Number 1. American Mathematical Society. https://books.google.de/books/about/Free_Random_Variables.html?id=r7BJAAQBAJ&redir_esc=y.

Voiculescu, D., Stammeyer, N., and Weber, M. (2016). Free probability and operator algebras. https://www.ems-ph.org/books/book.php?proj_nr=208.

Yang, H. (2005). Calibration of the extended CIR model. SIAM Journal on Applied Mathematics, 66(2):721–735. https://www.jstor.org/stable/4096135.