Luzin’s Problem on Fourier Convergence and Homeomorphisms

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Received November 14, 2021; revised February 8, 2022; accepted March 18, 2022

Abstract—We show that for every continuous function \(f\) there exists an absolutely continuous circle homeomorphism \(\phi\) such that the Fourier series of \(f \circ \phi\) converges uniformly. This resolves a problem posed by N. N. Luzin.

DOI: 10.1134/S008154382205011X

1. INTRODUCTION

Is it possible to improve the convergence properties of the Fourier series by an appropriate change of variable (homeomorphism of the circle)? The history of the problem goes back to a result of Julius Pál [14] (a Hungarian mathematician who worked for many years in Copenhagen), improved by Harald Bohr [3].

Theorem 1.1 (Pál–Bohr). Given a real function \(f \in C(\mathbb{T})\), one can find a homeomorphism \(\phi: \mathbb{T} \to \mathbb{T}\) such that the Fourier series of the composition \(f \circ \phi\) converges uniformly on \(\mathbb{T}\).

For the convenience of the reader we include a proof of the Pál–Bohr theorem in Section 2, essentially following the original idea, a surprising use of the Riemann conformal mapping theorem.

The Pál–Bohr theorem inspired a number of problems that have been actively studied. In particular, real proofs providing a visible construction of \(\phi\) were given by Saakjan [15] and by Kahane and Katznelson [8]. Kahane and Katznelson also proved the complex version of the theorem, and further showed that for any compact family of functions \(f\) there exists a single homeomorphism \(\phi\) carrying the entire family into the space of functions with uniformly converging Fourier series. On the other hand, it is possible to construct a real function \(f \in C(\mathbb{T})\) such that no homeomorphism \(\phi\) can bring it to the Wiener algebra (answering a problem posed by N. Luzin), see [12].

In all known proofs of the Pál–Bohr theorem, the homeomorphism \(\phi\) is in general singular. This prompted Luzin to ask whether it is possible, for an arbitrary \(f \in C(\mathbb{T})\), to find an absolutely continuous homeomorphism \(\phi\) such that \(f \circ \phi\) has a uniformly converging Fourier series.\textsuperscript{1} This problem has remained open so far. In this paper we resolve this conjecture.

Theorem 1.2. Given a real function \(f \in C(\mathbb{T})\), one can find an absolutely continuous homeomorphism \(\phi: \mathbb{T} \to \mathbb{T}\) such that the Fourier series of the composition \(f \circ \phi\) converges uniformly on \(\mathbb{T}\).

Further, for any \(p < \infty\) it is possible to have in addition \(\phi' \in L^p\).

\textsuperscript{1}Our attempts to discover the history of the problem were only partially successful. It is mentioned in Bary’s book [2, Ch. IV, §12], attributed to Luzin, who passed away in 1950, before Bary’s book was published (as far as we know, Luzin did not conjecture one direction or the other for the expected answer). Another piece of information is from the MathSciNet entry for [12], which states that Luzin’s other problem on the same subject is from the 1920s. It is reasonable to assume that both problems were asked together, but we have no evidence of that. We thank Carruth McGehee for help with our historical research.

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We remark that, as shown by Kahane and Katznelson, this cannot be done if we also require that $\phi \in C^1$ (see [8, Example 5]). Thus an interesting gap remains between our results and those of [8]: is it possible to do the same with a Lipschitz homeomorphism?

We should also mention that certain kinds of improvements in the behaviour of the Fourier transform are impossible to achieve using absolutely continuous homeomorphisms. Consider, for example, the result of Saakjan [15] which states that for any continuous function $f$ there exists a homeomorphism $\phi$ such that the Fourier coefficients of $f \circ \phi$ are $o(1/n)$, and in particular are in $l^p$ for all $p > 1$. This result cannot be achieved by an absolutely continuous homeomorphism. Indeed, there exists an $f \in C(\mathbb{T})$ such that $f \circ \phi \notin l^p$ for any $1 \leq p < 2$ and any absolutely continuous homeomorphism $\phi$ (see [13]). This result also implies that it would be difficult to answer Luzin’s conjecture using a version of the original construction of Pál and Bohr (for example, replacing conformal maps with quasi-conformal ones), as that construction gives an $H^{1/2}$ function (see Section 2 below), and any $g \in H^{1/2}$ has $\tilde{g} \in l^p$ for all $p > 1$.

The paper is organised as follows. In Section 2 we prove the Pál–Bohr theorem. In Section 3 we extend to a one-to-one map of $\mathbb{D}$ onto $\Omega$, and hence $g(t) := |u(e^{it})|$ is a change of variables of $f$ (in other words, $g = f \circ \phi$). We claim that $g$ has a uniformly converging Fourier expansion. To see this, write $u(z) = \sum c_k z^k$ and get

$$\iint_{\mathbb{D}} |u'(z)|^2 \, dz = \iint_{\mathbb{D}} \left( \sum_{k=1}^{\infty} k c_k z^{k-1} \right)^2 \, dz = \int_0^1 \int_0^{2\pi} \left( \sum_{k=1}^{\infty} k c_k r^{k-1} e^{i(k-1)\theta} \right)^2 r \, d\theta \, dr \leq 2\pi \int_0^1 \sum_{k=1}^{\infty} k^2 |c_k|^2 r^{2k-1} \, dr = \pi \sum_{k=1}^{\infty} k |c_k|^2,$$

where $(\ast)$ is by Parseval’s identity. Since $\iint_{\mathbb{D}} |u'(z)|^2 \, dz$ is the area of $\Omega$ and is finite, we get $g \in H^{1/2}$, where $H^{1/2}$ is the Sobolev space of functions whose $(1/2)$-derivative is in $L^2$. The theorem is finished by the following three observations. First, $H^{1/2} \cap C$ is a Banach algebra (Lemma 2.1). Hence $|g|$ is also in $H^{1/2}$ (Lemma 2.3). Finally, a continuous $H^{1/2}$ function has a uniformly converging Fourier expansion (Lemma 2.4).

**Lemma 2.1.** The space $H^{1/2} \cap C$, that is, the space of all continuous functions $f : [0, 2\pi] \to \mathbb{C}$ with $\sum |k| \cdot |\hat{f}(k)|^2 < \infty$ equipped with the norm $\max\{ \|f\|_{\infty}, (\sum |k| \cdot |\hat{f}(k)|^2)^{1/2} \}$, is a Banach algebra with respect to pointwise multiplication.

See [10, Lemma X4].
Lemma 2.2. If \( f \in H^{1/2} \cap C \) and if \( z \) is an analytic function in a neighbourhood of the set \( \{ f(x) : x \in \mathbb{T} \} \), then \( z \circ f \in H^{1/2} \cap C \).

Proof. We follow Wiener’s proof of the same fact for the Wiener algebra. We first note that being in the algebra \( A = H^{1/2} \cap C \) is a local property; namely, if some \( f : \mathbb{T} \to \mathbb{C} \) satisfies the condition that for every \( x \) there is an \( \varepsilon > 0 \) and a function \( a_x \in A \) such that \( a_x(x-x,x+x) = f(x-x,x+x) \), then \( f \in A \). Indeed, we take a finite cover of \( \mathbb{T} \) by intervals \( (x_1-\varepsilon_1, x_1+\varepsilon_1), \ldots, (x_n-\varepsilon_n, x_n+\varepsilon_n) \) and take some corresponding partition of unity \( p_1, \ldots, p_n \) with \( p_i \in A \); then \( f = f p_1 + \ldots + f p_n = a_{x_1} p_1 + \ldots + a_{x_n} p_n \in A \) (recall that \( A \) is a Banach algebra, by Lemma 2.1). This shows locality, and it is therefore enough to show that for every \( x \) there exists some \( \varepsilon > 0 \) and \( a \in A \) such that \( a(x-x,x+x) = z \circ f(x) \). By translation invariance we may assume that \( x = 0 \), and by invariance under addition of constants we may assume that \( f(0) = 0 \).

We next find some functions \( \rho_n \in A \) such that \( \text{supp} \rho_n \subseteq [-2/n, 2/n] \), \( \rho_n \equiv 1 \) on \([-1/n, 1/n] \), \( \| \rho_n \|_A \leq C \) and

\[
\lim_{n \to \infty} \| \rho_n(1 - e^{itk}) \|_A = 0 \quad \forall k \in \mathbb{Z}
\]

(trapezoidal functions satisfy all these properties, for example). We fix some \( \delta > 0 \) and write

\[
f = f_1 + f_2 \text{ where } f_1 \text{ is a trigonometric polynomial with } |f_1(0)| < \delta \text{ and } \| f_2 \|_A < \delta \text{ (for example, we can take } f_1 \text{ to be a Cesàro partial sum of } f) \text{. Then we have }
\]

\[
\| f_2 \rho_n \|_A \leq C \| f_2 \|_A \| \rho_n \|_A \leq C \delta,
\]

\[
\| f_1 \rho_n \|_A \leq \sum_k |\hat{f}_1(k)| \| \rho_n \|_A + \sum_k |\hat{f}_1(k)| (1 - e^{itk}) \rho_n \|_A \leq C \delta + \sum_k |\hat{f}_1(k)| \cdot o(1).
\]

Hence for \( n \) sufficiently large (depending on \( \delta \), of course), we have \( \| f \rho_n \|_A \leq C \delta \). Using the Taylor expansion of \( z \) near zero, we can write

\[
z(f \rho_n) = \sum_{j=1}^{\infty} z^{(j)}(0) \frac{t^j}{j!} (f \rho_n)^j,
\]

and since \( \| (f \rho_n)^k \|_A \leq C^k \| f \rho_n \|^k \|_A \leq (C \delta)^k \), we can make the sum converge in \( A \) by taking \( \delta \) sufficiently small. This shows that locally \( z \circ f \in A \) and proves the lemma. \( \square \)

Lemma 2.3. \( |g| \in H^{1/2} \cap C \).

Proof. By Lemma 2.1, \( A = H^{1/2} \cap C \) is a Banach algebra, and since we already know that \( g \in A \), we get \( |g|^2 = (\text{Re } g)^2 + (\text{Im } g)^2 \in A \). Since \( |g|^2 > 0 \) and since \( \sqrt{\cdot} \) is analytic in the neighbourhood of any interval \( [\varepsilon, 1/\varepsilon] \), the previous lemma shows that \( |g| = \sqrt{|g|^2} \in A \). \( \square \)

Lemma 2.4. If \( f \in H^{1/2} \cap C \), then \( f \) has a uniformly converging Fourier expansion.

Proof. By a standard approximation argument, it is enough to show that

\[
\| S_n(f) \|_\infty \leq C \| f \|_A \tag{2.1}
\]

for some universal constant \( C \), where \( S_n(f) \) is the \( n \)th Fourier partial sum. Indeed, once (2.1) is established, we may write \( f = f_1 + f_2 \) with \( f_1 \) a trigonometric polynomial and \( \| f_2 \|_A < \varepsilon \) and get

\[
\lim \| f - S_n(f) \|_\infty = \lim \| f_2 - S_n(f_2) \|_\infty < \varepsilon + C \varepsilon.
\]

Since \( \varepsilon > 0 \) was arbitrary, the lemma is proved. So we only need to demonstrate (2.1).
To prove (2.1), we note that $F_n$, the $n$th Cesàro partial sum of $f$, satisfies $\|F_n\|_\infty \leq \|f\|_\infty \leq \|f\|_A$, so it is enough to prove the claim for $S_n - F_n$. We write

$$\|S_n - F_n\|_\infty = \left\| \sum_{|k| \leq n} \frac{|k|}{n} \hat{f}(k) e^{ikx} \right\|_\infty = \frac{1}{n} \max_x \left| \sum_{|k| \leq n} \sqrt{|k|} \sqrt{|k|} \hat{f}(k) e^{ikx} \right| \leq \frac{1}{n} \sqrt{\sum_{|k| \leq n} |k| \|f\|_{H^{1/2}}}$$

$$\leq C\|f\|_{H^{1/2}},$$

where $(\ast)$ is from the Cauchy–Schwarz inequality. The lemma is proved, and so is the Pál–Bohr theorem. □

3. A PROOF SKETCH

The proof of Theorem 1.2 is probabilistic. Let us compare the approach of this paper to the approach of our earlier paper [11]. There we investigated the Dubins–Freedman homeomorphism $\phi$ (see below for more details) and showed that the Fourier series of $f \circ \phi$ is better behaved than that of $f$, but not quite uniformly converging or even bounded. Here we again start with a Dubins–Freedman-like homeomorphism, but this time we do not investigate its almost sure behaviour (as in [11]) but its average behaviour. We then start reducing the randomness step by step, always keeping the same averaged behaviour, until reaching a degenerate random variable, and that is the required homeomorphism.

In this section we are going to sketch a proof of an easier result: that for every continuous function $f$ there is a H"older homeomorphism $\phi$ such that $f \circ \phi$ has a uniformly bounded Fourier expansion. This result is weaker than our main result in two aspects: the homeomorphism is only H"older rather than absolutely continuous (this is the most important difference); and the Fourier series is uniformly bounded rather than uniformly converging. After the sketch we will briefly describe what additional ideas are needed to get the main result (whose proof, of course, occupies the rest of the paper).

**Lemma 3.1.** Let $\gamma > 0$ and $M$ be two parameters. Assume $(v_{i,j}: i \in \{1, \ldots, n\}, j \in \mathbb{N})$ satisfy the following: for every $j$ there is an $l(j) \in \{1, \ldots, n\}$ and a $b(j) \in \mathbb{N}$ such that

$$|v_{i,j}| \leq \min\left\{ \frac{1}{|i - l(j)| \bmod n + 1}, \frac{1}{b(j)} \right\},$$

and assume further that $|\{j: l(j) = l, b(j) = b\}| \leq Mb^{-1}$ for all $l$ and $b$. Then there exists a choice of signs $\varepsilon_1, \ldots, \varepsilon_n \in \{\pm 1\}$ such that

$$\left| \sum_{i=1}^n \varepsilon_i v_{i,j} \right| \leq \frac{C}{b(j)^{1/50}} \quad \forall j,$$

where $C$ is a constant that depends only on $\gamma$ and $M$.

Here and below, $(\cdot) \bmod n$ means a number in $\{-[(n-1)/2], \ldots, [n/2]\}$.

This lemma and its proof are inspired by a result of Kashin [9], who proved a discrete version of Menshov’s correction theorem. Roughly speaking, the proof goes as follows. We divide the values of $i$ into blocks and choose the signs randomly in each block, ensuring good local behaviour. We then group the blocks into higher level blocks and choose signs randomly inside each second-level block, multiplying them by the previously chosen signs, and so on. This is similar to the method of renormalisation in mathematical physics.

There is a connection between Lemma 3.1 and the Komlós conjecture. Recall that Komlós conjectured that under the condition $\sum_j |v_{i,j}|^2 \leq 1$ for all $i$ one may choose $\varepsilon_i$ such that $\sum_i \varepsilon_i v_{i,j} \leq C$ for all $j$. We could have used the Komlós conjecture for Lemma 3.1 if we needed the lemma only
for $\gamma < 1$, if we did not have the factor $b(j)^{1/50}$, and if it were proved. Unfortunately, we need Lemma 3.1 for $\gamma = 91$ (see the last paragraph of the proof of Lemma 7.6) and the Komlós conjecture has not been proved.

This lemma is the only part of the proof of the H"older result which exists in the main result verbatim. The reader is invited to see the details in the proof of Lemma 4.1 (Lemma 4.1 has two additional parameters, when compared to Lemma 3.1, which are needed just because the proof is by induction).

We have opted to state this seemingly technical lemma first because, in a way, it was our starting point. Inspired by the use of Spencer’s theorem (a partial result on the Komlós conjecture, see [17]) to solve a version of the flat polynomials conjecture [1], we asked ourselves whether these techniques might be useful for Luzin’s conjecture. Even though that path did lead us to a proof, eventually our proof does not use Spencer’s theorem or any other result from that area.

Continuing, throughout we will identify the circle $\mathbb{T}$ with the interval $[0, 1)$. We will consider homeomorphisms which fix 0, so they are simply increasing functions from $[0, 1]$ into itself.

Let $I$ be a map assigning to each dyadic rational $d = k/2^n \in (0, 1)$ an interval $I(d) \subseteq [0, 1]$, possibly degenerate. We call such a function an RH-descriptor, where RH stands for “random homeomorphism.” For each $I$ we introduce a random increasing function $\phi_I$ as follows. We set $\phi_I(0) = 0$ and $\phi_I(1) = 1$ deterministically. Next, we choose $\phi_I(1/2)$ uniformly in $I(1/2)$. We then choose $\phi_I(1/4)$ uniformly in $\phi_I(1/2)I(1/4)$ and choose $\phi_I(3/4)$ uniformly in $\phi_I(1/2) + (1 - \phi_I(1/2))I(3/4)$. The process then continues. Assume that we have already chosen $\phi_I(k/2^{n-1})$ for all $k = 0, \ldots, 2^{n-1}$. Then, for each odd $1 \leq k < 2^n$, let $\phi_I(k/2^n)$ be uniform in the interval $\phi_I\left(\frac{k-1}{2^n}\right) + \left(\phi_I\left(\frac{k+1}{2^n}\right) - \phi_I\left(\frac{k-1}{2^n}\right)\right)I\left(\frac{k}{2^n}\right)$.

This allows us to define $\phi_I$ on all dyadic rationals, after which it can be continued to the entire interval $[0, 1]$ by continuity. For example, if $I \equiv [0, 1]$, then the resulting $\phi_I$ is exactly the Dubins–Freedman random homeomorphism [4]. Our focus in this paper, though, is on the case where $I(d) \subseteq J$ for some $J$ strictly contained in $[0, 1]$, in which case continuity and even the Hölder property are not only trivial but also hold deterministically, and not just with probability 1, as they do for the Dubins–Freedman homeomorphism (see [7]).

Our first two lemmas are for constant $I$. It will be convenient, therefore, to make the following definition. For a $q \in (0, 1/2)$ we let $I_q \equiv [1/2-q, 1/2+q]$ and $\psi = \psi_q = \phi_{I_q}$.

**Lemma 3.2.** Let $f$ be continuous and let $x, y \in [0, 1/2]$. Then

$$|\mathbb{E}(f(\psi(x))) - \mathbb{E}(f(\psi(y)))| \leq C\|f\|_\infty \sqrt{\frac{|x-y|}{x+y}},$$

where $C$ may depend on $q$.

This lemma is not very surprising, as it merely says that if $x$ and $y$ are close (and not too close to 0 or 1), then the densities of $\psi(x)$ and $\psi(y)$ are quite close. Unfortunately, the densities of $\psi(x)$ do not have nice closed formulas. The definition of $\psi$ can certainly be traced to give closed formulas for $x = k/2^n$ for small $n$. For example, the density of $\psi(1/2)$ is $(2q)^{-1} \cdot 1_{[1/2-q, 1/2+q]}$ and the density of $\psi(1/4)$ is

$$\frac{1}{4q^2} \log \frac{\min\{1/2+q, x/(1/2-q)\}}{\max\{1/2-q, x/(1/2+q)\}} \cdot 1_{[(1/2-q)^2, (1/2+q)^2]}(x).$$

Similar formulas can be written for $\psi(3/4)$, $\psi(1/8)$, etc., but they become unmanageable quickly and we know no general formula.

The proof of the lemma, very roughly, is as follows. We first note that $\mathbb{E}(\psi(x)) = x$ (this holds for any $\phi_I$ just under the condition that all $I(d)$ are symmetric around $1/2$). From this we get
\[ \mathbb{E}(\psi(2x) - \psi(2y)) = 2x - 2y; \] that is, the variables \( \psi(2x) \) and \( \psi(2y) \) are close in the 1-Wasserstein distance. We now note that \( \psi(x) - \psi(y) \) has the same distribution as \( U \cdot (\psi(2x) - \psi(2y)) \) where \( U \) is an independent random variable uniform on \([1/2 - q, 1/2 + q]\). This smoothens the estimate on the probabilities and makes it into an estimate on densities (this is similar to the way one uses convolution to translate an estimate in a coarser norm to an estimate in a finer norm; the convolution here is multiplicative but the idea is the same). We omit all further details of the proof. The square root in Lemma 3.2 is just an artefact of our proof; the real behaviour is probably \( |x - y|/(x + y) \). This argument is not used as such in the proof of the main result, but formula (6.6) is quite similar (although it has a different proof from the one sketched above).

**Lemma 3.3.** Let \( J_1, J_2 \subseteq [1/2 - q, 1/2 + q] \) be two intervals with \( d(y_1, y_2) < \varepsilon \) for all \( y_i \in J_i \) for some \( \varepsilon > 0 \) (in other words, the intervals are both short and close to one another). Let \( f \) and \( g \) be two continuous functions. Then

\[
\left| \int_0^1 \left( \mathbb{E}(f(\psi(x)) \mid \psi(\frac{1}{2}) \in J_1) - \mathbb{E}(f(\psi(x)) \mid \psi(\frac{1}{2}) \in J_2) \right) g(x) \, dx \right| \leq C \varepsilon \| f \|_\infty \| g \|_\infty.
\]

The proof is similar to that of Lemma 3.2, and we omit it. The corresponding lemma in the proof of the main result is Lemma 6.5.

The next lemma is the main lemma in the proof of the result. It implements one step of a “reduction of randomness” strategy, that is, of starting from \( \phi_I \) for \( I \equiv [1/2 - q, 1/2 + q] \) and reducing the randomness step by step (always keeping an estimate similar to the one given by Lemma 3.2) until reaching a single homeomorphism with good properties. Unfortunately, the formulation is a mouthful.

**Lemma 3.4.** Let \( f \) be a continuous function satisfying \( \| f \|_\infty \leq 1 \), let \( q \in (0, 1/2) \) and let \( n, j \in \mathbb{N} \). Let \( I \) be a function from the dyadic rationals into subintervals of \([1/2 - q, 1/2 + q]\) with the following properties:

(i) for all \( m < n \) and all \( k \in \{1, \ldots, 2^m - 1\} \), the interval \( I(k/2^m) \) is degenerate, i.e., is a single point;

(ii) for all \( k \in \{1, \ldots, 2^n - 1\} \) odd, \( |I(k/2^n)| = 2^{2-j} q \);

(iii) for all \( m > n \) and all \( k \in \{1, \ldots, 2^m - 1\} \) odd, \( I(k/2^m) = [1/2 - q, 1/2 + q] \).

Then there exists a function \( J \) from the dyadic rationals into subintervals of \([1/2 - q, 1/2 + q]\) that has properties (i) and (iii) and is such that for each \( k \in \{1, \ldots, 2^{n-1}\} \) odd, \( J(k/2^n) \) is either the left or right half of \( I(k/2^n) \) (in particular, property (ii) is satisfied for \( J \) with \( j + 1 \) instead of \( j \)).

Further, for every \( u \in \mathbb{N} \), every \( r \in \{2^{u-1}, \ldots, 2^u - 1\} \) and every \( \xi \in 2^{-u-2} \mathbb{Z} \cap [0, 1) \) we have the estimate

\[
\left| \int_{E_{\xi,n}} (\mathbb{E}(f(\phi_I(x))) - \mathbb{E}(f(\phi_J(x)))) D_r(x - \xi) \, dx \right| \leq C \cdot 2^{-c(j+|n-u|)}, \tag{3.1}
\]

where

\[
E_{\xi,n} := \begin{cases} 
[0, 1] \setminus \left[ (-2^{-n}, 2^{-n}) + 2^{1-n}[\xi \cdot 2^{n-1}] \right], & u > n, \\
[0, 1], & u \leq n,
\end{cases}
\]

and where \( D_r \) is the Dirichlet kernel.
Before starting the proof, let us discuss quickly the integration interval $E_{\xi,n}$. In fact, this has been added only for convenience. Lemma 3.4 could have been proved with (3.1) replaced with

$$\int_0^1 (\mathbb{E}(f(\phi_J(x))) - \mathbb{E}(f(\phi_J(x))))D_r(x - \xi) \, dx \leq C \cdot 2^{-e(j+|n-u|)}.$$ 

This would certainly simplify the formulation of the lemma, but, unfortunately, would significantly complicate its proof. This is due to various technical issues stemming from the vicinity of the peak of the Dirichlet kernel. (The complications arise in the proofs of Lemmas 3.5 and 3.6 below, and since these are not spelled out in detail, the reader would have to trust us on this point. But indeed, the complications incurred are significant.) Hence we opted for the statement above, which “delays” the handling of the peak of $D_r$ until $n \approx r$.

**Sketch of the proof of Lemma 3.4.** Choosing $J$ is equivalent to choosing $2^{n-1}$ signs, as our only freedom is in choosing, for $k \in \{1, \ldots, 2^n - 1\}$ odd, whether we take $J(k/2^n)$ to be the left or right half of $I(k/2^n)$. Denote therefore, for any $\varepsilon = (\varepsilon_1, \varepsilon_3, \ldots, \varepsilon_{2^n-1})$, $\varepsilon_k \in \{\pm 1\}$, the corresponding $J$ by $J^\varepsilon$ ($\varepsilon_k = 1$ means that we take the right half of $I(k/2^n)$ and $\varepsilon_k = -1$ means we take the left half). With this notation, our goal becomes finding some $\varepsilon$ such that $J^\varepsilon$ satisfies (3.1).

Fix one $k \in \{1, \ldots, 2^{n-1}\}$, odd. Define

$$F^\pm(x) := \begin{cases} \mathbb{E}(f(\phi_{J^\varepsilon}(x))), & x \in \left[\frac{k-1}{2^n}, \frac{k+1}{2^n}\right], \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon^\pm$ is an $\varepsilon$ with $\varepsilon_k = \pm 1$ and the rest chosen arbitrarily (clearly, for $x \in [(k-1)/2^n, (k+1)/2^n]$ only the choice of $\varepsilon_k$ matters). Let

$$\Delta_k(x) := \frac{1}{2}(F^+(x) - F^-(x)).$$

We get $\mathbb{E}(f(\phi_{J^\varepsilon}(x))) = \mathbb{E}(f(\phi_I(x))) \pm \Delta_k(x)$ for all $x \in [(k-1)/2^n, (k+1)/2^n]$. Summing over $k$ gives

$$\mathbb{E}(f \circ \phi_{J^\varepsilon}) = \mathbb{E}(f \circ \phi_I) + \sum_k \varepsilon_k \Delta_k.$$ 

Our strategy would be to find estimates for $\int \Delta_k D_r$ and feed them into Lemma 3.1. Here are two such estimates (the corresponding lemmas in the main proof are Lemmas 7.3 and 7.4).

**Lemma 3.5.** If $|r|, |s| > 2^n$ and $\xi \notin [(k-3)/2^n, (k+3)/2^n]$, then

$$\left| \int_{V} F^\pm(x) \sum_{l=1}^{s-1} e(l(x - \xi)) \, dx \right| \leq \left( \frac{2^n}{|r|} + \frac{2^n}{|s|} \right) C \frac{|C|}{2^n \xi - k}.$$ 

(As usual, $e(x) = e^{2\pi i x}$.)

**Proof sketch.** We condition on $\phi_J(k/2^n)$ and consider the intervals $[(k-1)/2^n, k/2^n]$ and $[k/2^n, (k+1)/2^n]$ independently. Let $V$ be one of these intervals. Then $\phi_J$ is deterministic at $(k \pm 1)/2^n$ by assumption and at $k/2^n$ by the conditioning, so it is deterministic at both ends of $V$. This means that $\phi_J$ restricted to $V$ is simply an appropriately linearly mapped version of the $\phi_I$ from Lemma 3.2. Applying that lemma implies that $F^\pm$ satisfies an appropriate “Hölder away from the boundaries” estimate. Integrating by parts gives a good estimate for $\int_V F^\pm \sum e(l(x - \xi))$, still conditioned on $\phi_{J^\varepsilon}(k/2^n)$ (this is the standard argument that shows that a Hölder function has power-law decaying Fourier coefficients; the fact that it is only Hölder away from the boundaries...
requires no change in the argument; and the $2^n$ factors come from the scaling. Integrating the conditioning gives Lemma 3.5.

**Lemma 3.6.** For every $r < s$ and every $\xi$,

$$\left| \int_0^1 \Delta_k(x) \sum_{l=r}^{s-1} e(l(x - \xi)) \, dx \right| \leq C \min \left\{ \frac{1}{|2^n \xi - k|}, \frac{s-r}{2^n} \right\} \cdot 2^{-cj}.$$

Lemma 3.6 follows from Lemma 3.3 using the same scaling argument that was used to derive Lemma 3.5 from Lemma 3.2, and we omit the details. (The factor $2^{-cj}$ in Lemma 3.6 comes from the factor $\varepsilon^c$ in Lemma 3.3. Unlike Lemma 3.5, no cancellation in the exponential sum is used here; this lemma uses only the fact that the maximum of $\sum e(l(x - \xi))$ on $[(k-1)/2^n, (k+1)/2^n]$ can be estimated by $\min\{|\xi - k/2^n|^{-1}, s-r\}$. The cancellation would come in by combining this lemma with Lemma 3.5, later.)

**Proof of Lemma 3.4, continued.** For every $u \in \mathbb{N}$ define $U := 2^\max\{u,n\} + 2$. For every odd $k \in \{1, \ldots, 2^n - 1\}$ and every $\xi \in U^{-1}\mathbb{Z} \cap [0, 1)$ define

$$w_{k,\xi,u}^0 := \int \Delta_k(x) D_{2^n}(x - \xi) \, dx.$$

(Formally the integral is from 0 to 1, but since $\Delta_k$ is supported on $[(k-1)/2^n, (k+1)/2^n]$, only this interval contributes.) Next, for every $u \in \mathbb{N}$, every $0 \leq s < u$ and every $t \in [2^{u-1}, 2^u]$ divisible by $2^{s+1}$, every $k \in \{1, \ldots, 2^n - 1\}$ odd and every $\xi \in U^{-1}\mathbb{Z} \cap [0, 1)$, we define

$$w_{k,\xi,u,s,t}^+ := \int \Delta_k(x) \sum_{z=t+1}^{t+2^s} e(z(x - \xi)) \, dx, \quad w_{k,\xi,u,s,t}^- := \int \Delta_k(x) \sum_{z=-t-2^s}^{t} e(z(x - \xi)) \, dx.$$

We rearrange the vectors $w^0$, $w^+$ and $w^-$ into one long list $v_{k,j}$ (we consider them to be vectors in $k$, so, for example, we could take $v_{k,1} = w_{k,0,1}^0$ if we like to start with $w^0$, or we could take $v_{k,1} = w_{k,0,0,2}^+$ if we like to start with $w^+$). We now apply Lemma 3.1 with $n_{\text{Lemma 3.1}} = 2^n$. Lemma 3.1 requires us to find for every $v_{k,j}$ some $l(j)$ and some $b(j)$ such that

$$|v_{k,j}| \leq \min \left\{ \frac{1}{|l(j)| \bmod n} + 1, \frac{1}{b(j)} \right\} \quad (3.2)$$

and to count how many $j$ exist for each choice of $l$ and $b$. We take $l(j) = \lfloor 2^n \xi \rfloor$ for the corresponding $\xi$, and this gives an estimate of the form (3.2) from Lemmas 3.5 and 3.6. Checking which $b$ is appropriate for each vector is straightforward and we save the reader all the index checking (note that occasionally we have to combine the estimates from the two lemmas using the fact that for any two numbers $x$ and $y$, $\min\{x, y\} \leq \sqrt{xy}$). Eventually we get

$$\{|j: b(j) = b, l(j) = l\}| \leq C b^9 \quad \forall b, l$$

for some absolute constant $C$. The conclusion of Lemma 3.1 is then that there exists a choice of $\varepsilon_k$ such that

$$\left| \sum_k \varepsilon_k v_{k,j} \right| \leq \frac{C}{b(j)^{1/50}} \quad \forall j. \quad (3.3)$$

Finally, to estimate $\int \Delta D_r$ for an arbitrary $r$, we write $r$ in its binary expansion

$$r = 2^{u-1} + \sum_{i} 2^{s_i}, \quad t_i := 2^{u-1} + 2^{s_1} + \ldots + 2^{s_i},$$
for some decreasing sequence \( s_j \). We get

\[
D_r = D_{2r-1} + \sum_{i} \left( \sum_{z=-t_i}^{t_i} e(zx) + \sum_{z=-t_i-1}^{t_i-1} e(zx) \right).
\]

Hence for any \( \xi \in U^{-1}\mathbb{Z} \cap [0,1) \) we have

\[
\int \sum_k \epsilon_k \Delta_k(x) D_r(x - \xi) \, dx = \sum_k \epsilon_k w^0_k,\xi,u + \sum_i \sum_k \epsilon_k w^+,\xi,u,s_i,t_i - \sum_i \sum_k \epsilon_k w^-,\xi,u,s_i,t_i - 1.
\]

Plugging in estimate (3.3) proves the lemma. The factor \( b^{1/50} \) in (3.3) plays an important role here, as it makes the estimates for \( w^{0,\pm} \) decay exponentially as \( |u - n| \) increases and as \( s \) decreases (the factor \( 2^{-c_j} \) from Lemma 3.6 multiplies all the terms equally and translates to the \( 2^{-c_j} \) factor in Lemma 3.4 directly). The fact that the integral is only over \( E_{\xi,n} \) and not over the whole of \([0,1]\) comes from the fact that Lemmas 3.5 and 3.6 do not work for \( \xi \) too close to \( k/2^n \) (Lemma 3.5 explicitly and Lemma 3.6 gives a useless estimate), so we have to drop the corresponding \( w^0 \) and \( w^\pm \) and remove the corresponding element from \( \sum \epsilon_k \Delta_k \). \( \square \)

**Theorem 3.1.** For every real continuous function \( f \) and every \( \lambda < 1 \), there is a \( \lambda \)-Hölder homeomorphism \( \psi \) such that the Fourier partial sums \( S_r(f \circ \psi) \) are uniformly bounded.

**Proof sketch.** Let us fix a sufficiently small \( q \) such that any RH-descriptor \( I \) with \( I(d) \subseteq [1/2 - q, 1/2 + q] \) gives rise to a homeomorphism \( \phi_I \) which is \( \lambda \)-Hölder surely (not just almost surely). Now construct a sequence of RH-descriptors as follows. We start with \( I_0 \equiv [1/2 - q, 1/2 + q] \). We invoke Lemma 3.4 with \( I_{\text{Lemma 3.4}} = I_0 \), \( n_{\text{Lemma 3.4}} = 1 \) and \( j_{\text{Lemma 3.4}} = 1 \). We denote the output of the lemma (i.e., \( J_{\text{Lemma 3.4}} \)) by \( I_{0,1} \). This halves the length of the interval at 1/2, so now we have \( |I_{0,1}(1/2)| = q \). We then apply Lemma 3.4 again, again with \( n_{\text{Lemma 3.4}} = 1 \) but this time with \( j_{\text{Lemma 3.4}} = 2 \) and \( I_{\text{Lemma 3.4}} = I_{0,1} \), and set \( I_{0,2} := J_{\text{Lemma 3.4}} \) (which will have \( |I_{0,2}(1/2)| = q/2 \)). We continue like this, shrinking the interval at 1/2 more and more, and finally define

\[
I_1(d) = \lim_{j \to \infty} I_{0,j}(d).
\]

We find that \( I_1(1/2) \) is a single point while \( I_1(d) = [1/2 - q, 1/2 + q] \) for all \( d \neq 1/2 \). This makes \( I_1 \) suitable as input to Lemma 3.4 with \( n_{\text{Lemma 3.4}} = 2 \) and \( j_{\text{Lemma 3.4}} = 1 \). Again we apply Lemma 3.4 infinitely many times and get in the limit an RH-descriptor \( I_2 \) with \( I_2(1/4), I_2(1/2) \) and \( I_2(3/4) \) all degenerate. We continue this process infinitely many times and get an

\[
I_\infty(d) = \lim_{n \to \infty} I_n(d),
\]

which is completely degenerate, i.e., corresponds to a single homeomorphism \( \phi \). This \( \phi \) will be \( \lambda \)-Hölder because it is in the support of \( \phi_{I_0} \) (here we are thinking about \( \phi_{I_0} \) as a measure on functions, so its support is a set of functions, which are all \( \lambda \)-Hölder).

To estimate

\[
\int_0^1 (f \circ \phi) D_r,
\]

we simply sum the differences coming from Lemma 3.4. Since the error there is \( C \cdot 2^{-c(j+|n-u|)} \), the sum over \( j \) simply gives \( C \cdot 2^{-c|n-u|} \). A crude estimate for the difference between \( E_{\xi,n} \) and
$E_{\xi,n}+1$ shows that this can also be bounded by $C \cdot 2^{-c|n-u|}$. We find that for every $u \in \mathbb{N}$, every $r \in \{2^{n-1},\ldots,2^n-1\}$ and every $\xi \in 2^{-u-1}\mathbb{Z} \cap [0,1)$,

$$\left| \int_0^1 f(\phi(x))D_r(x-\xi) \, dx \right| \leq C \sum_{n=1}^{\infty} 2^{-c|n-u|},$$

which is bounded by a constant independent of $u$. Applying Bernstein’s inequality shows that the estimate holds for all $\xi \in [0,1]$. □

What is needed to go from Theorem 3.1 to our main theorem? Clearly we can no longer allow our random homeomorphism the flexibility to change by a constant proportion (our $q$) at every step, as that could never be absolutely continuous. We have to allow more flexibility where our function $f$ fluctuates a lot, and less flexibility where $f$ is more flat. In other words, we need to make $q$ into a function of the relevant dyadic rational $d$, such that the “local $q$” depends on the behaviour of $f$ in the appropriate area. The “flatness” of $f$ is naturally encoded using its Haar decomposition, and indeed, a necessary condition for the homeomorphism to be absolutely continuous is a bound for certain sums of squares of Haar coefficients. Interestingly, these sums turn out to be dyadic BMO functions, and can thus be estimated by the (dyadic) John–Nirenberg inequality (see Lemma 5.5).

Had we applied this “local $q$” strategy naively, we would have ruined the nice local structure of $\phi$, i.e., the fact that every dyadic interval can be handled with little knowledge of the surrounding, a fact that was extremely useful in the proofs of Lemmas 3.5 and 3.6. Why would that ruin this local structure? Because the local $q$ near $d$ needs to depend on $f$ in the vicinity of $\phi(d)$, which is a global property. We solve this problem by replacing $\phi$ with $\phi^{-1}$, its inverse as a homeomorphism. In other words, instead of first choosing $\phi(1/2)$ (as we did in the proof sketch above) we first choose which $x$ will have $\phi(x) = 1/2$, and we choose it uniformly in an interval $[1/2 - q, 1/2 + q]$ where $q$ depends on the behaviour of $f$ near $1/2$. We then choose $x_2$ such that $\phi(x_2) = 1/4$, and we choose it uniformly in an interval symmetric around $x/2$ whose width depends on the behaviour of $f$ near $1/4$, and so on.

This change, however, breaks Lemma 3.1, which requires the partition into intervals “in the $x$ axis” to be uniform, but our replacement of $\phi$ with $\phi^{-1}$ made the partition in the $y$ axis uniform, while the partition in the $x$ axis becomes random. To fix this problem, we do not break all intervals at every step (i.e., whenever we move from $I_n$ to $I_{n+1}$ in the proof of the theorem). Instead we break into halves only intervals larger than a certain threshold, leaving the smaller intervals fixed. The details of this can be seen in the beginning of Section 7.

Finally, the transition from “uniformly bounded Fourier series” to “uniformly converging Fourier series” is not particularly problematic. The modulus of continuity of $f$ has to be taken into account in Lemma 7.2 (the analog of Lemma 3.4 in the proof of the main result). The proof of the theorem from Lemma 7.2 then proceeds by showing that $\int_{E_{\xi,n}} \mathbb{E}(f \circ \phi_{In})D_r$ is a good approximation of $\mathbb{E}(f(\phi_{In}(\xi))) \int_{E_{\xi,n}} D_r$. As $n \to \infty$, the first integral converges to $\int_0^1 (f \circ \phi_{\infty})D_r = S_r(f \circ \phi_{\infty}; \xi)$, the second integral converges to $\int_0^1 D_r = 1$, and of course $\mathbb{E}(f(\phi_{In}(\xi))) \to f(\phi_{\infty}(\xi))$.

The details of all this fill the next four sections.

4. A HIERARCHICAL PROBABILISTIC CONSTRUCTION

Our first lemma is proved using a hierarchical random construction, in the spirit of Kashin (and also related to the Komlós conjecture, as discussed in Section 3). Its formulation is a bit strange because we had to add two parameters ($\alpha$ and $\beta$) to make the induction work. We note that Lemma 3.1 follows from it by simply setting $\alpha = 1$ and $\beta = 1/50$ (and increasing $M$ to 2 if necessary).
Lemma 4.1. Let $\alpha \in (99/100, 1]$, $\beta \in [1/50, 1/25)$, $\gamma > 0$ and $M \geq 2$ be some parameters. Let $n \in \mathbb{N}$ and $v_{i,j} \in \mathbb{R}$ ($i \in \{0, \ldots, n-1\}$ and $j \in \mathbb{N}$) and let $l(j) \in \{0, \ldots, n-1\}$ and $b(j) \in \mathbb{N}$ satisfy
\[
|v_{i,j}| \leq \min\left\{ \frac{1}{(|(i-l(j)) \mod n| + 1)^{\alpha} \cdot b(j)} \right\}, \quad |\{j : l(j) = l, b(j) = b\}| \leq Mb^\gamma \quad \forall b, l.
\]
Then there exists a choice of $\varepsilon_0, \ldots, \varepsilon_{n-1} \in \{\pm 1\}$ such that
\[
\left| \sum_{i=0}^{n-1} \varepsilon_i v_{i,j} \right| \leq \frac{A(\gamma + 1)\log M}{(\alpha - 99/100)(1/25 - \beta)b(j)^\beta} \quad \forall j
\]
for some absolute constant $A$.

Recall that $(\cdot) \mod n$ means a number in $\{-(n-1)/2, \ldots, [n/2]\}$.

**Proof.** The constant $A$ will be chosen later. We argue by induction on $n$. The case $n = 1$ is clear (if $A > 1/\log 2$), for all values of $\alpha$, $\beta$, $\gamma$ and $M$. Assume therefore the claim has been proved for all $n' < n$ and for all values of $\alpha'$, $\beta'$, $\gamma'$ and $M'$.

Thus we are given $\alpha, \beta, \gamma, M, v, l$ and $b$. Let $\alpha' := (\alpha + 99/100)/2$ and $\beta' := (\beta + 1/25)/2$. Let $K \geq 2$ be some integer parameter to be chosen later. Divide $\{0, \ldots, n-1\}$ into blocks of size $K$ (the last block may be smaller) and let $n' := \lfloor n/K \rfloor$ be the number of blocks. We will first choose signs $\delta_1, \ldots, \delta_n \in \{\pm 1\}$ such that the sum in each block will be controlled. We denote these sums by $w'$; namely, for every $s \in \{0, \ldots, n'-1\}$ and $j \in \mathbb{N}$ we define
\[
w_{s,j} := \sum_{r=0}^{K-1} \delta_{sK+r} v_{sK+r,j}.
\]
If the last block is shorter than $K$, we truncate the sum defining $w$. We wish to choose the $\delta$ such that for every $s \in \{0, \ldots, n'-1\}$ and every $j$ such that $|(sK-l(j)) \mod n| \geq 2K$ we have
\[
|w_{s,j}| \leq \min\left\{ \frac{\sigma K^{1/2}}{|(sK-l(j)) \mod n| + 1}^{\alpha}, \frac{\sigma K^{1/2}}{b(j)^{\beta/\beta'}} \right\}, \quad \sigma := \sqrt{\lambda(\gamma + 1)\left( \frac{1}{\alpha - \alpha'} + \frac{1}{\beta' - \beta} + \log M \right)}
\]
(4.1)
where $\lambda$ is some absolute constant to be defined shortly. When $|(sK-l(j)) \mod n| < 2K$, it will be convenient to define $w_{s,j} = 0$, so let us do so. The choice of $\delta$ will be random, uniform and i.i.d. Since the estimates for each block are independent of the other blocks, let us fix the number of the block $s$. We divide the $j$ into two sets, $\mathcal{J}$ and $\mathcal{F}$, according to whether $b(j) \geq \frac{\sigma K^{1/2}}{(sK-l(j)) \mod n} + 1)\gamma \beta'/\beta' \beta'$ (this is $\mathcal{J}$) or not ($\mathcal{F}$). In the first case the second term in (4.1) is the smaller, so we need to estimate the probability that $|w_{s,j}| \leq \sigma K^{1/2}b(j)^{-\beta/\beta'}$. The estimate $|v_{sK+r,j}| \leq 1/b(j)$ and Bernstein’s inequality for sums of random variables tell us that
\[
P\left(|w_{s,j}| \geq \frac{\sigma K^{1/2}}{b(j)^{\beta/\beta'}} \right) \leq 2 \exp\left(-c\sigma^2 b(j)^2(1-\beta/\beta')\right).
\]
Now, for every $b$ there are no more than $Cb^{\beta'\beta'/\beta'} \leq Cb^2$ relevant values of $l$ (recall that we fixed the value of $s$, and note that we used the inequalities $\alpha' > 99/100$ and $\beta < \beta'$). Further, for each couple $(b, l)$ we have no more than $Mb^\gamma$ values of $j$, so all in all we have no more than $CMb^{\gamma+2}$ values of $j$ for $b$. A union bound gives
\[
P\left( \exists j \in \mathcal{J}, |w_{s,j}| \geq \frac{\sigma K^{1/2}}{b(j)^{\beta/\beta'}} \right) \leq \sum_{b=1}^{CMb^{\gamma+2}} C_1 Mb^{\gamma+2} \exp\left(-c\sigma^2 b(j)^2(1-\beta/\beta')\right).
\]
We claim that if \( \lambda \) is sufficiently large, then this sum is small. This is a simple calculation, but let us do it in detail nonetheless. We write
\[
\exp\left(-c\alpha^2 b^{2(1-\beta'/\beta)}\right) = \exp\left(-c\lambda(\gamma + 1)\left(\frac{1}{\alpha - \alpha'} + \frac{1}{\beta' - \beta} + \log M\right) b^{2(\beta' - \beta)}/\beta'\right)
\leq M^{-c\lambda} \exp\left(-\frac{c\lambda(\gamma + 1)}{\beta' - \beta} b^{50(\beta' - \beta)}\right).
\]
For any \( \varepsilon > 0 \) we have \( b^\varepsilon \geq \varepsilon \log b \), and thus the second factor can be bounded from above by \( \exp(-50c\lambda(\gamma + 1)e \log b) = b^{-50c\lambda(\gamma + 1)e} \). Pick \( \lambda \) so large such that
\[
M^{-c\lambda} \leq \frac{1}{10C_1 M}, \quad b^{-50c\lambda(\gamma + 1)e} \leq b^{-4-\gamma}
\]
(this can be done independently of \( \gamma \) or \( M \); recall that \( M \geq 2 \)) and get
\[
\mathbb{P}\left( \exists j \in \mathcal{S}, \ |w_{s,j}| \geq \frac{\sigma K^{1/2}}{b(j)^{\beta'/\beta}} \right) \leq \frac{1}{10} \sum_{b=1}^{\infty} \frac{1}{b^2} \leq \frac{1}{4}.
\]
This finishes the estimate for \( \mathcal{S} \).

In the case when \( j \in \mathcal{S} \), i.e., when \( b(j) < (|(sK - l(j)) \bmod n| + 1)^{\alpha'/\beta} \), we use the estimate
\[
|w_{s,j}| \leq (|(sK + r - l(j)) \bmod n| + 1)^{-\alpha} \leq C(|(sK - l(j)) \bmod n| + 1)^{-\alpha},
\]
where the second inequality uses our assumption that \( (|sK - l(j)) \bmod n| \geq 2K \). Again using Bernstein’s inequality, we have
\[
\mathbb{P}\left( |w_{s,j}| > \frac{\sigma K^{1/2}}{(|(sK - l(j)) \bmod n| + 1)^{\alpha}} \right) \leq 2 \exp\left(-c\alpha^2 (|(sK - l(j)) \bmod n| + 1)^{2(\alpha - \alpha')}\right).
\]
For every value \( q \) of \( (|(sK - l(j)) \bmod n|, there are at most two possible values of \( l(j) \) which give it. The restriction \( b < (|(sK - l) \bmod n| + 1)^{\alpha'/\beta} \) implies that for every \( q \) there are no more than \( q^{\alpha'/\beta} \leq q^2 \) values of \( b \), and for each couple \( (b,l) \) there are at most \( Mb^\gamma \leq M(q^{\alpha'/\beta})^\gamma \leq Mq^{2\gamma} \) possibilities for \( j \). All in all the value \( q \) corresponds to at most \( CMq^{2\gamma + 2} \) values of \( j \). Thus we can bound
\[
\mathbb{P}\left( \exists j \in \mathcal{S}, \ |w_{s,j}| \geq \frac{\sigma K^{1/2}}{(|(sK - l(j)) \bmod n| + 1)^{\alpha}} \right) \leq \sum_{q=1}^{\infty} 2 \exp\left(-c\alpha^2 q^{2(\alpha - \alpha')}CMq^{2\gamma + 2},
\]
and a calculation similar to the one done above for \( \mathcal{S} \) shows that the value of \( \sigma \) ensures that this sum is also smaller than \( 1/4 \), for \( \lambda \) sufficiently large. Fix \( \lambda \) to satisfy this requirement (uniformly in \( \alpha, \alpha', M \) and \( \gamma \)) and the previous one. Since the sum of the probabilities of all dissenting events is less than 1, we see that some choice of \( \delta \) for which (4.1) will be satisfied exists.

Before continuing, we make one modification of (4.1). Recall that we defined \( w_{s,j} = 0 \) whenever \( (|sK - l(j)) \bmod n| < 2K \). If not, i.e., if \( (|sK - l(j)) \bmod n| \geq 2K \), then we can write
\[
|(sK - l(j)) \bmod n| \geq \frac{1}{2} K \left| s - \left\lfloor \frac{l(j)}{K} \right\rfloor \right| \bmod n',
\]
which allows us to rewrite (4.1) as
\[
|w_{s,j}| \leq \min\left\{ \frac{2\sigma K^{1/2 - \alpha'}}{|(s - \left\lfloor l(j)/K \right\rfloor) \bmod n'| + 1}^{\alpha'}, \frac{\sigma K^{1/2}}{b(j)^{\beta'/\beta}} \right\}.
\]
We are now in a position to apply the lemma inductively. We need numbers $v_{i',j}'$, $i' \in \{0, \ldots, n' - 1\}$, and functions $l': \mathbb{N} \to \{0, \ldots, n' - 1\}$ and $b': \mathbb{N} \to \mathbb{N}$ satisfying (4.4) also holds in the case of $\alpha$. We first find how many $b$ we have such that

$$\frac{2b^{\beta'/\beta}}{K^{\alpha'}} \in [b', b' + 1)$$

or in $[0, 2)$ in the case of $b' = 1$. The number of such $b$ can be estimated by a simple derivative bound giving

$$\frac{1}{2}K^{\alpha'} \left( \frac{1}{2}K^{\alpha'}(b' + 1) \right)^{\beta'/\beta - 1} + 1 \leq CK^2b',$$

where we estimated $\beta'/\beta \leq 2$, $b' + 1 \leq 2b'$ and $\alpha' \leq 1$. The bound $CK^2b'$ for the number of $b$ satisfying (4.4) also holds in the case of $b' = 1$ (in which case we need to verify how many $b$ satisfy $2b^{\beta'/\beta}K^{-\alpha'} \in [0, 2)$, but only the constant is affected). Adding the facts that each value of $l'$ corresponds to at most $K$ different values of $l$ and that each couple $(b, l)$ corresponds to $Mb^\gamma$ different values of $j$, we conclude that the number of $j$ for each couple $(b', l')$ is at most

$$KM \left( \frac{1}{2}K^{\alpha'}(b' + 1) \right)^{\gamma\beta'/\beta} \leq C_2MK^2(Kb')^{2\gamma + 1},$$

where we again bounded $\beta'/\beta \leq 2$, $b' + 1 \leq 2b'$ and $\alpha' \leq 1$. We are ready to apply our induction assumption! We apply it with $\alpha', \beta', \gamma' := 2\gamma + 1$,

$$M' := C_2MK^{2\gamma + 3},$$

$n'$ and the vectors $w_{s,j}/2\sigma K^{1/2-\alpha'}$. We conclude that there exist signs $\mu_0, \ldots, \mu_{n'-1} \in \{\pm 1\}$ such that

$$\sum_{s=0}^{n'-1} \mu_s w_{s,j} \leq \frac{A(\gamma' + 1)\log M'}{(\alpha' - 99/100)(1/25 - \beta')b'(j)^{\beta'}} \quad \forall j.$$

Defining $\varepsilon_i = \mu_{i/|K|}\delta_i$, gives

$$\sum_{i=0}^{n-1} \varepsilon_i v_{i,j} \leq C \min \left\{ K^{1-\alpha} \log K, \frac{K}{b(j)} \right\} + \frac{2A(\gamma' + 1)\sigma K^{1/2-\alpha'} \log M'}{(\alpha' - 99/100)(1/25 - \beta')b'(j)^{\beta'}}.$$  \hfill (4.5)

The first term is from the fact that we zeroed out $w$ when $|j - sK| < 2K$. When returning to $v$, we need to bound these $v$ and we bound them using a naive absolute value bound (we use here the fact that $\sum_{i=1}^{K} i^{-\alpha} \leq CK^{1-\alpha} \log K$ with $C$ uniform in $\alpha \leq 1$, a fact which is easy to check).
To make the calculation manageable, we will now bound the second term in (4.5) by a sequence of quite rough bounds. For $M'$ we write

$$\log M' = C + \log M + (2\gamma + 3) \log K \leq C(\gamma + 1) \log M \log K.$$ 

For $b'$ we have

$$(b')^{-\beta'} \leq 2(b' + 1)^{-\beta'} \overset{(4.4)}{\leq} 2 \left( \frac{2b' K'}{K'^2} \right)^{-\beta'} \leq 2b^{-\beta} K'^{\beta'}.$$ 

For $\sigma$ we bound

$$\sigma \leq C \frac{\gamma + 1}{(\alpha - 99/100)(1/25 - \beta)} \log M.$$ 

Finally, $\gamma' + 1 = 2(\gamma + 1)$, $\alpha' - 99/100 = (\alpha - 99/100)/2$ and $1/25 - \beta' = (1/25 - \beta)/2$. Inserting everything into (4.5) gives

$$\left( \gamma' + 1 \right) \sigma K'^{1/2 - \alpha'} \log M' \overset{\gamma + 1}{\leq} CK^{1/2 - \alpha + \beta'} \log K \frac{(\gamma + 1)^3 \log^2 M}{(\alpha - 99/100)^2 (1/25 - \beta)^2 b'^3} \leq CK^{-1/3} \theta^3 b^{-\beta},$$

where

$$\theta := \frac{(\gamma + 1) \log M}{(\alpha - 99/100)(1/25 - \beta)}.$$ 

Together with (4.5) we get

$$\left| \sum_{i=0}^{n-1} \varepsilon_i v_{i,j} \right| \leq C \min \left\{ K^{1-\alpha} \log K, \frac{K}{b(j)} \right\} + C_3 A K^{-1/3} \theta^3 b(j)^{-\beta}. \quad (4.6)$$

Recall that we need to show $\left| \sum \varepsilon_i v_{i,j} \right| \leq A \theta b(j)^{-\beta}$. Now is the time to choose $K$. We choose $K = \lceil C_4 \theta^7 \rceil$ for some constant $C_4$ sufficiently large such that

$$K^{-1/3} \leq \frac{1}{2C_3 \theta^2}$$

(and also such that $K \geq 2$). To bound the first term in (4.6), note that for $b < \theta^{15}$ we have

$$\theta b^{-\beta} \geq \theta b^{1 - \alpha} \geq \theta^{2/5} \geq c K^{2/35} \geq c K^{1-\alpha} \log K,$$

while for $b \geq \theta^{15}$ we have

$$\theta b^{-\beta} = \frac{\theta b^{1-\beta}}{b} \geq \frac{\theta^{1 + 15 - 24/25}}{b} \geq c \frac{K}{b},$$

so for any $b$ we get $\min \{ K^{1-\alpha} \log K, K/b \} \leq C \theta b^{-\beta}$. Inserting both estimates into (4.6) gives

$$\left| \sum_{i=0}^{n-1} \varepsilon_i v_{i,j} \right| \leq C_5 \theta b(j)^{-\beta} + \frac{1}{2} A \theta b(j)^{-\beta}.$$ 

Choosing $A = 2C_5$ completes the induction and proves the lemma. \qed
5. A FAMILY OF HOMEOMORPHISMS

A dyadic rational is a number $d$ of the form $k/2^n$ for some integers $k$ and $n$. If $k$ is odd, we define $\text{rnk}(d) = n$ and let also $\text{rnk}(0) = -\infty$. Let $\theta$ be a map from the dyadic rationals in $(0,1)$ into $[1/4,3/4]$. Let $n$ be an integer. Our goal is to define for every $\theta$ and $n$ a homeomorphism $\psi_{\theta,n} : [0,1] \rightarrow [0,1]$. It turns out that it is slightly more natural to first define $\psi_{\theta,n}^{-1}$ which is a "Dubins–Freedman style" homeomorphism, so we do that, and then define $\psi_{\theta,n}$ as its inverse. But first we need some notation.

We define

$$T_d(x) := \max\{1 - 2^{\text{rnk}(d)}|x - d|, 0\}, \quad \nu_d \psi := \psi(d + 2^{-\text{rnk}(d)}) - \psi(d - 2^{-\text{rnk}(d)}).$$

In words, $T_d$ is a triangle function based on a certain dyadic interval $I$ (whose centre is $d$). We will only apply the functional $\nu_d$ to increasing functions, and then $\nu_d \psi$ is the variation of $\psi$ over $I$.

We may now define $\psi_{\theta,n}^{-1}$. The definition is by induction on $n$, with the induction base being

$$\psi_{\theta,0}^{-1}(x) = x \quad \forall x \in [0,1].$$

Assume $\psi_{\theta,n-1}$ has been defined for all $\theta$. We now define, for $n \geq 1$,

$$\psi_{\theta,n}^{-1}(x) = \psi_{\theta,n-1}^{-1}(x) + \sum_{\text{rnk}(d)=n} T_d(x) \left(\theta(d) - \frac{1}{2}\right) \nu_d \psi_{\theta,n-1}^{-1},$$

where the sum is over all $d \in (0,1)$ of rank $n$ ($2^{\text{rnk}(d)-1}$ terms in total). An equivalent definition is to write, for every $k \in \{0,\ldots,2^n\}$,

$$\psi_{\theta,n}^{-1}\left(\frac{k}{2^n}\right) = \begin{cases} \psi_{\theta,n-1}\left(\frac{k}{2^n}\right), & \text{k even,} \\
 \psi_{\theta,n-1}\left(\frac{k-1}{2^n}\right) + \theta\left(\frac{k}{2^n}\right) \left(\psi_{\theta,n-1}\left(\frac{k+1}{2^n}\right) - \psi_{\theta,n-1}\left(\frac{k-1}{2^n}\right)\right), & \text{k odd,}
\end{cases}$$

and interpolate linearly on each interval $[k/2^n,(k+1)/2^n]$ for $k \in \{0,\ldots,2^n-1\}$. In particular, we note that $\psi_{\theta,n}^{-1}(d)$ stabilises for all dyadic $d$ as $n \rightarrow \infty$ (in fact, when $n > \text{rnk}(d)$).

To understand why we call $\psi_{\theta,n}^{-1}$ a Dubins–Freedman style homeomorphism, we just note that if we were to take $\theta$ uniform in $[0,1]$ (which we do not allow here), then the result would be exactly the Dubins–Freedman homeomorphism of level $n$.

Finally, let us remark that $\psi_{\theta,n}^{-1}$ has a recursive formula (which could also have been used to define $\psi_{\theta,n}$). To state it, define

$$A := \theta\left(\frac{1}{2}\right), \quad \theta^-(d) = \theta\left(\frac{d}{2}\right), \quad \theta^+(d) = \theta\left(\frac{1}{2} + \frac{d}{2}\right).$$

With these definitions we get

$$\psi_{\theta,n}^{-1}(x) = \begin{cases} A \psi_{\theta-,n-1}^{-1}(2x), & x \leq \frac{1}{2}, \\
 A + (1 - A) \psi_{\theta-,n-1}^{-1}(2x - 1) & \text{otherwise.}
\end{cases}$$

Similarly we have for $\psi_{\theta,n}$ itself

$$\psi_{\theta,n}(x) = \begin{cases} \frac{1}{2} \psi_{\theta-,n-1}\left(\frac{x}{A}\right), & x \leq A, \\
 \frac{1}{2} + \frac{1}{2} \psi_{\theta+,n-1}\left(\frac{x - A}{1-A}\right) & \text{otherwise.}
\end{cases}$$
Both formulas (which are clearly equivalent) are a special case of Lemma 5.1 below (used for $I = [0, 1/2]$ and $I = [1/2, 1]$).

We say that $I$ is a dyadic interval if there exist some $n \in \mathbb{Z}^+$ and $k \in \{1, \ldots, 2^n\}$ such that $I = [(k - 1)/2^n, k/2^n]$. We call $n$ the rank of $I$. For every interval $I \subseteq [0, 1]$ we define $L_I$ to be the affine increasing map taking $I$ to $[0, 1]$. The following lemma (whose proof is merely playing around with the definitions) formalises the local structure of $\psi_{\theta,n}$ and $\psi_{\theta,n}^{-1}$.

**Lemma 5.1.** Let $\theta$ and $n$ be as above. Let $I$ be a dyadic interval $I = [(k - 1)/2^n, k/2^n]$ with $m \leq n$. Define $\alpha = \psi_{\theta,n}^{-1}((k - 1)/2^n)$ and $\beta = \psi_{\theta,n}^{-1}(k/2^n)$. Then for every $\theta$, $n$, $I$ and for every $x \in I$,

$$\psi_{\theta,n}^{-1}(x) = \alpha + (\beta - \alpha)\psi_{\theta \circ L_I, n-m}^{-1}(L_I^{-1}(x)).$$

A short version of (5.6) is

$$\psi_{\theta,n}^{-1}(\theta_{\theta,n}(I)) = L_I \circ \psi_{\theta \circ L_I, n-m}^{-1} \circ L_I^{-1}.$$

For $\psi_{\theta,n}$ itself this can be written as

$$\psi_{\theta,n}(\psi_{\theta,n}^{-1}(I)) = L_I \circ \psi_{\theta \circ L_I, n-m}^{-1} \circ L_I^{-1}.$$

**Proof of Lemma 5.1.** We first note that $\psi_{\theta,n}^{-1}(x)$ is linear on every dyadic interval of order $n$, as it is a sum of piecewise linear functions each of which has its jumps of the derivative on dyadic rationals of rank $\leq n$. This shows the case of $m = n$, as the left-hand side of (5.6) is linear on $I$, the right-hand side of (5.6) is linear on $I$ (as $\psi_{\theta,0}^{-1}(x) = x$ always), and they are equal at the boundaries of $I$.

We will prove the claim by induction on $n$, with the base case being the case of $n = m$ just proved. We first note that for $x \in I$ the only nonzero terms in (5.1) are those for which $d \in I$, and hence for every $x \in I$

$$\psi_{\theta,n}^{-1}(x) \stackrel{(5.1)}{=} \psi_{\theta,n-1}^{-1}(x) + \sum_{d \in I} T_d(x)\left(\theta(d) - \frac{1}{2}\right)\nu_d \psi_{\theta,n-1}^{-1}$$

$$= \psi_{\theta,n-1}^{-1}(x) + \sum_{d \in I} \nu_{L_I(d)}(\psi_{\theta,n-1}^{-1})$$

(we used here the fact that $\text{rnk}(L_I(d)) = \text{rnk}(d) + m$ for all $d \in (0, 1)$, and that the boundaries of the interval do not have rank $n$ so we may omit them). Recall that $\psi_{\theta,n}^{-1} = \psi_{\theta,n-1}^{-1}$ on each dyadic rational of rank $\leq n$ and in particular on the boundaries of $I$. Hence $\alpha$ and $\beta$ do not depend on whether we define them with $n$ or $n - 1$. This allows us to apply the induction assumption and get

$$\psi_{\theta,n}^{-1}(x) = \alpha + (\beta - \alpha)\psi_{\theta \circ L_I, n-m}^{-1}(L_I^{-1}(x)).$$

Similarly,

$$\nu_{L_I(d)}(\psi_{\theta,n-1}^{-1}) = (\beta - \alpha)\nu_d(\psi_{\theta \circ L_I, n-m}^{-1}).$$

Inserting all these into (5.8) gives

$$\nu_{L_I(d)}(\psi_{\theta,n}^{-1}(x)) = \nu_d(\psi_{\theta \circ L_I, n-m}^{-1}(L_I^{-1}(x)))$$

and by (5.1) this is exactly $\alpha + (\beta - \alpha)\psi_{\theta \circ L_I, n-m}^{-1}(L_I^{-1}(x))$, as needed. □
Lemma 5.2. If $θ(d) ∈ [ε, 1 − ε]$ for some $ε > 0$ and all dyadic $d$, then $ψ_{θ,n}$ converges uniformly as $n → ∞$ and the limit is a Hölder homeomorphism of $[0, 1]$. As $ε → 1/2$, the Hölder exponent tends to 1.

Proof. Examining (5.2), we see that $ψ_{θ,n}^{-1}$ is strictly increasing; hence $ψ_{θ,n}$ is well-defined. Further, a simple induction shows that

$$\varepsilon^n ≤ ψ_{θ,n}^{-1}\left(\frac{k}{2^n}\right) - ψ_{θ,n}^{-1}\left(\frac{k-1}{2^n}\right) ≤ (1 - \varepsilon)^n.$$ 

For dyadic intervals of order $m < n$, we may get the same estimate since $ψ_{θ,n}(k/2^m) = ψ_{θ,m}(k/2^m)$. For $m > n$ the linearity of $ψ_{θ,n}^{-1}$ on dyadic intervals of order $n$ gives

$$ψ_{θ,n}^{-1}\left(\frac{k}{2^m}\right) - ψ_{θ,n}^{-1}\left(\frac{k-1}{2^m}\right) ≤ (1 - \varepsilon)^n \cdot 2^{n-m} ≤ (1 - \varepsilon)^n \quad ∀m > n.$$ 

The lower bound is similar, and we can conclude that

$$2^{-m/δ} ≤ ψ_{θ,n}^{-1}\left(\frac{k}{2^m}\right) - ψ_{θ,n}^{-1}\left(\frac{k-1}{2^m}\right) ≤ 2^{-δm} \quad ∀m, n$$

for an appropriate $δ$ depending only on $ε$, which tends to 1 as $ε → 1/2$. Since the sequence $ψ_{θ,n}^{-1}(d)$ stabilises for all dyadic rationals $d$, we find that the limit (denote it by $ψ^{-1}_∞$) also satisfies (5.9).

For general $x < y ∈ [0, 1]$ we can find a dyadic interval contained in $(x, y)$ of size at least $(y - x)/4$. In the other direction we can find two dyadic intervals $I_1$ and $I_2$ such that $(x, y) ⊆ I_1 ∪ I_2$ and such that $|I_1 ∪ I_2| ≤ 4(y - x)$. Hence

$$\left(\frac{1}{4}(y - x)\right)^{1/δ} ≤ ψ_{θ,n}^{-1}(y) - ψ_{θ,n}^{-1}(x) ≤ (4(y - x))^δ$$

for all $n$, finite or infinite. Reversing gives a similar estimate for $ψ$, namely,

$$\frac{1}{4}|x - y|^{1/δ} ≤ |ψ_{θ,n}(x) - ψ_{θ,n}(y)| ≤ 4|x - y|^δ.$$  

Since $ψ_{θ,n}^{-1}$ stabilises on all dyadic rationals, we get a dense set of numbers where $ψ_{θ,n}$ stabilises, and on these numbers the limit satisfies (5.11). This of course shows that $ψ_{θ,n}(x)$ converges uniformly, and that the limit also satisfies (5.11), for all $x, y ∈ [0, 1]$. □

Define

$$ψ_{θ,∞} := \lim_{n → ∞} ψ_{θ,n}.$$ 

We note for future use that (5.5) extends to $n = ∞$, namely,

$$ψ_{θ,∞}(x) = \begin{cases} \frac{1}{2} ψ_{θ,∞}(x/A), & x ≤ A, \\ \frac{1}{2} + \frac{1}{2} ψ_{θ,∞}(x-A/A), & \text{otherwise} \end{cases}$$

where $A$ is again $θ(1/2)$. Let us also remark that during the proof of Lemma 5.2 we defined $ψ_{θ,∞}^{-1} = \lim_{n → ∞} ψ_{θ,n}^{-1}$, but this notation is not ambiguous since this limit is also the inverse of $ψ_{θ,∞}$, as both $ψ_n → ψ_∞$ and $ψ^{-1}_n → ψ^{-1}_∞$ are uniform (the fact that $ψ^{-1}_n → ψ^{-1}_∞$ uniformly is implicit in the proof of Lemma 5.2).

We now move to conditions on $θ$ which will guarantee that $ψ_{θ,∞}$ is absolutely continuous. This clearly requires $θ$ to be usually close to $1/2$, so let us introduce a function $q$ from the dyadic rationals into $[0, ∞)$ and ask that $|θ(d) − 1/2| ≤ q(d)$ for all dyadic $d$ (it will actually be convenient later to
add a constant and require \(|\theta - 1/2| \leq cq\), but let us ignore this for a while). Thus we need to find some condition on \(q\) that will ensure that \(\psi_{\theta, \infty}\) is absolutely continuous whenever \(|\theta - 1/2| \leq q\). To formulate the condition, we need two auxiliary functions. The first, \(Q_n: [0,1] \to [0, \infty)\), is defined by

\[
Q_n(x) = Q_{q,n}(x) = q(d)
\]

whenever \(x\) is in some dyadic interval \(I\), \(|I| = 2^{-n}\), and \(d\) is the middle of \(I\). With \(Q_n\) defined we introduce

\[
z(x) = z_q(x) = \sum_{n=0}^{\infty} Q_n(x). \tag{5.13}
\]

We can now state our condition.

**Lemma 5.3.** There exists a constant \(\nu_0\) such that for any \(q\) for which

\[
|\{x: z_q(x) > \lambda\}| \leq Ce^{-\lambda} \quad \forall \lambda > 0 \tag{5.14}
\]

and for any \(\theta\) such that \(|\theta - 1/2| \leq \nu_0 q\), the homeomorphisms \(\psi_{\theta, \infty}\) and \(\psi_{\theta, \infty}^{-1}\) are absolutely continuous.

Further, for any \(p < \infty\) there exists a constant \(\nu_1(p)\) such that \(|\theta - 1/2| \leq \nu_1(p)q\) ensures that

\[
\|\psi_{\theta, \infty}'\|_p, \|(\psi_{\theta, \infty}^{-1})'\|_p \leq C(p). \tag{5.15}
\]

Notice that (5.14) does not prohibit \(z_q\) from being \(\infty\) on a set of zero measure.

**Proof.** Assume \(|\theta - 1/2| \leq \nu q\) for some \(\nu\). Examine a finite \(n\). Since \(\psi_{\theta, n}\) is piecewise linear, we may consider the derivative. The relation between \(z\) and \(\psi\) is given by the following inequality. We claim that for every \(n\) and every \(x \in [0,1]\),

\[
\exp\left(-C\nu \sum_{i=0}^{n} Q_i(x)\right) \leq (\psi_{\theta,n}^{-1})'(x) \leq \exp\left(C\nu \sum_{i=0}^{n} Q_i(x)\right), \tag{5.15}
\]

where at dyadic \(x\) the left derivative is bounded by the left limits of the terms, and similarly for the right derivative (and these derivatives, of course, need not be equal).

We prove (5.15) by induction on \(n\). The base case \(n = 0\) is trivial, as \(\psi_{0}^{-1}(x) = x\) and so \((\psi_{0}^{-1})' = 1\) and satisfies the requirement. Assume therefore that (5.15) is proved for \(n - 1\).

Recall (5.4) and the definitions of \(\theta^\pm\) and \(A\) in (5.3). We wish to differentiate (5.4), and to this end we note that \(A\) is just a number, and further satisfies

\[
\exp(-C\nu Q_{q,0}) \leq 2A \leq \exp(C\nu Q_{q,0}), \tag{5.16}
\]

where we used \(|\theta(1/2) - 1/2| \leq \nu q(1/2) = \nu Q_{q,0}(x)\) for all \(x\).

Next define \(q^-\) and \(q^+\) as for \(\theta\), namely, \(q^-(x) = q(x/2)\) and \(q^+(x) = q(1/2 + x/2)\). A simple check shows that for \(x \in [0,1/2]\),

\[
Q_{q,n}(x) = Q_{q,-n-1}(2x). \tag{5.17}
\]

Hence, for \(x \in [0,1/2]\) non-dyadic,

\[
(\psi_{\theta,n}^{-1})'(x) \overset{(5.4)}{=} 2A(\psi_{\theta,-n-1}^{-1})'(2x) \overset{(5.16)}{\leq} \exp(C\nu Q_{q,0}(x))(\psi_{\theta,-n-1}^{-1})'(2x) \overset{(\ast)}{\leq} \exp\left(C\nu Q_{q,0}(x) + C\nu \sum_{i=0}^{n-1} Q_{q,-i}(2x)\right) \overset{(5.17)}{=} \exp\left(C\nu Q_{q,0}(x) + C\nu \sum_{i=0}^{n-1} Q_{q,i+1}(x)\right)
\]

\[
= \exp\left(C\nu \sum_{i=0}^{n} Q_{q,i}(x)\right),
\]
where (**) is our induction hypothesis. For \( x \) dyadic the calculation above holds identically for both left and right derivatives (for \( x = 1/2 \) only the left derivative), when \( Q \) is replaced by appropriate left or right limits. The lower bound for \( (\psi_{q,n}^{-1})' \) is identical. The case of \( x \in [1/2, 1] \) is identical, with \( q^- \) and \( \theta^- \) replaced by \( q^+ \) and \( \theta^+ \), respectively, and with (5.17) replaced by \( Q_{q,n} = Q_{q^+,n-1}(2x - 1) \). This finishes the proof of (5.15).

Set for brevity \( \psi_n := \psi_{q,n} \) and write (5.15) as

\[
(\psi_n^{-1})'(x) \leq \exp(C\nu z(x)).
\]

Our assumption (5.14) on \( z \) shows that if \( \nu \) is chosen sufficiently small, then \( \exp(C\nu z(x)) \) is integrable. Hence \( \psi^{-1}_n \) are uniformly absolutely continuous, and hence their limit, \( \psi^{-1}_\infty \), is absolutely continuous. We next claim that (5.15) to show that \( (\psi^{-1}_n)'/(\psi^{-1}_m)' \) is bounded by a partial sum of \( Q \)'s. The second is to note that \( (\psi^{-1}_n)' \) is a positive martingale with respect to the dyadic filtration \( \mathcal{F}_n \) and apply the martingale convergence theorem [5, Theorem 4.2.12]. By the dominated convergence theorem we get

\[
\lim_{n \to \infty} (\psi^{-1}_n)'(x) = (\psi^{-1}_\infty)'(x) \quad \text{for almost every } x \in [0, 1],
\]

and in particular \( (\psi^{-1}_\infty)' \leq \exp(C\nu z(x)) \) for almost every \( x \in [0, 1] \). This shows the “further” clause of the lemma for \( \psi^{-1} \), as for \( \nu \) sufficiently small (5.14) gives \( \|\exp(C\nu z(x))\|_p < C(p) \). The results for \( \psi \) are then concluded from Lemma 5.4 below. \( \square \)

**Lemma 5.4.** If \( \phi \) is an absolutely continuous homeomorphism of \([0, 1]\) satisfying \( 1/\phi' \in L^p \) for some \( p > 0 \), then \( \phi^{-1} \) is also absolutely continuous, with \( \int_0^1 |(\phi^{-1})'|^{p+1} = \int_0^1 |\phi'|^{-p} \).

**Proof.** Assume without loss of generality that \( \phi \) is increasing. We first show that \( \phi^{-1} \) is absolutely continuous. Let therefore \( A \subset [0, 1] \) be a countable union of intervals. Let \( B = \phi^{-1}A \), so it is also a countable union of intervals. Then for every \( \varepsilon > 0 \)

\[
|A| = \int_B \phi' \geq \varepsilon (|B| - |\{x: \phi'(x) < \varepsilon\}|) \geq \varepsilon \left(|B| - \varepsilon \left\|\frac{1}{\phi'}\right\|_p^p\right),
\]

where the last inequality follows from Markov’s inequality. Choosing \( \varepsilon = (|B|/2)^{1/p}/\|1/\phi'\|_p \) gives

\[
|\phi^{-1}A| = |B| \leq 2 \left(|A| \left\|\frac{1}{\phi'}\right\|_p^{p/(p+1)}\right),
\]

so \( \phi^{-1} \) is absolutely continuous. To show the derivative equality, we write

\[
\int_0^1 ((\phi^{-1})')^{p+1} = \int_0^1 (\phi^{p-1})' = \int_0^1 \frac{1}{(\phi')^p},
\]

where the first equality is a change of variables. \( \square \)

**Tailoring the homeomorphism family to the function.** Recall from the discussion after the proof sketch of Theorem 3.1 that we need to tailor the function \( q \), which describes a family of homeomorphisms in this section and will be used to construct a measure on it in the next section, to our function \( f \) from the statement of Theorem 1.2. Here we do that, but first we need some notation.

**Definition 5.1.** Recall the Haar functions \( 1_{[0,1]}, 1_{[0,1/2]} - 1_{[1/2,1]} \), etc. It will be convenient to index them using their support, so for a dyadic interval \( J \), let \( h_J \) be the function which is \( |J|^{-1/2} \) on
the left half and \(-|J|^{-1/2}\) on the right half (the function \(1_{[0,1]}\) will not be associated to any interval; this will not be a problem).

For a function \(f \in L^2\) we define \(q = q_f\), a function on the dyadic rationals, as follows. Let \(I\) be some dyadic interval and let \(d\) be its middle. Then we define

\[
q(d) := \frac{1}{|I|^{3/2}} \sum_{J \subseteq I} \langle f, h_J \rangle^2 |J|^{1/2}.
\]  

(5.19)

Recall the definition of \(z_q\) in (5.13). Applying it to the \(q\) above, we get

\[
z_q(x) = \sum_{x \in J \supseteq \gamma} \frac{|J|^{1/2}}{|I|^{3/2}} \langle f, h_J \rangle^2
\]

(her and below this notation means that the sum is over both \(I\) and \(J\); note that \(x\) does not need to belong to \(J\), only to \(I\).

**Lemma 5.5.** Let \(f\) be a bounded function with \(\|f\|_\infty \leq 1\). Then \(z = z_{q_f}\) satisfies the inequality \(|\{x : z(x) > \lambda\}| \leq 2e^{-c\lambda}\) for all \(\lambda > 0\), where \(c\) is some universal constant.

**Proof.** Let us first calculate the \(L^1\) norm of \(z = z_{q_f}\). By Fubini’s theorem,

\[
\int_0^1 z(x) \, dx = \sum_{J \subseteq I} \int_0^1 \frac{|J|^{1/2}}{|I|^{3/2}} \langle f, h_J \rangle^2 \cdot 1_I(x) \, dx = \sum_{J \subseteq I} \frac{|J|^{1/2}}{|I|^{1/2}} \langle f, h_J \rangle^2
\]

\[
\leq \frac{\sqrt{2}}{\sqrt{2} - 1} \sum_J \langle f, h_J \rangle^2 \leq \frac{\sqrt{2}}{\sqrt{2} - 1},
\]

(5.20)

where the last inequality follows since the \(h_J\) are orthonormal and \(\|f\|_2 \leq \|f\|_\infty \leq 1\).

The same calculation shows that \(z\) is a dyadic BMO function. Indeed, let \(U\) be some dyadic interval, and let \(L_U\) be the affine increasing map taking \([0,1]\) onto \(U\). Define

\[
g_U(x) = f(L_U(x))
\]

Then write, for \(x \in U\),

\[
z(x) = \sum_{U \subseteq J \supseteq \gamma} \frac{|J|^{1/2}}{|I|^{3/2}} \langle f, h_J \rangle^2 + \sum_{x \in I \subseteq U} \frac{|J|^{1/2}}{|I|^{3/2}} \langle f, h_J \rangle^2
\]

and note that the first term does not depend on \(x\). Hence we may call it \(c_U\) and get

\[
\int_U |z(x) - c_U| \, dx = \sum_{J \subseteq I \subseteq U} \frac{|J|^{1/2}}{|I|^{1/2}} \langle f, h_J \rangle^2 \leq \|U\| \sum_{J \subseteq I \subseteq [0,1]} \frac{|J|^{1/2}}{|I|^{1/2}} \langle g_U, h_J \rangle^2 \leq |U| \frac{\sqrt{2}}{\sqrt{2} - 1},
\]

where the equality marked by (*) comes from mapping \(J\) to \(L_U^{-1}(J), I\) to \(L_U^{-1}(I)\), and noting that the linear change of variables in the integration together with the fact that \(\|h_{L_U^{-1}(J)}\|_\infty = \sqrt{|U|} \|h_J\|_\infty\)

yields a factor of \(U\), and where the inequality marked by (†) comes from applying (5.20) to \(g_U\) (which is, of course, also bounded by 1).

Thus \(z\) is a dyadic BMO function. The lemma is then concluded by the John–Nirenberg inequality for dyadic BMO functions [16, Theorem 1d].

**Lemma 5.6.** There exists a universal constant \(\eta_0\) such that for all \(f\) with \(\|f\|_\infty \leq 1\) and all \(\theta\) with \(|\theta - 1/2| \leq \eta_0 q_f\), both \(\psi_{\theta, \infty}\) and \(\psi_{\theta, \infty}^{-1}\) are absolutely continuous.
Further, for any $p < \infty$ there is an $\eta_1(p)$ such that if $|\theta - 1/2| \leq \eta_1(p)q_f$ then
\[ \|\psi'_{\theta, \infty}\|_p, \|\psi^{-1}_{\theta, \infty}'\|_p < C(p). \]

**Proof.** This follows immediately from Lemmas 5.3 and 5.5. \(\square\)

Before moving forward, let us rearrange the formula for $q = q_f(1/2)$ in a way that will be useful in Section 6. We first write
\[ q = \sum_{J \subseteq [0,1]} \langle f, h_J \rangle |J|^{1/2} \geq \sum_{k=0}^{\infty} 2^{-k/2} \sum_{|J| \in [0,1]} \langle f, h_J \rangle^2. \]
(We remark that $q$ is also bounded above by $\sqrt{2}/(\sqrt{2} - 1)$ times the same sum.) We next recall that the partial sums of the Haar expansion have a simple form. Let $X_k$ be the sum of the first $2^k$ terms in the Haar expansion of $f$. Then for any dyadic $I$ with $|I| = 2^{-k}$ and any $x \in I$,
\[ X_k(x) = 2^k \int_I f(y) \, dy. \]

The sum $\sum_J \langle f, h_J \rangle h_J$ is not the partial Haar expansion of $f$ but rather that of $g := f - \int f$, since the very first Haar function does not correspond to any interval $J$. Hence
\[ \sum_{|J| \leq 2^{-k}} \langle f, h_J \rangle h_J(x) = 2^k \int_I g(y) \, dy. \]

Applying Parseval’s identity, we get from (5.21)
\[ q \geq \sum_{k=0}^{\infty} 2^{-k/2} \sum_{i=1}^{2^k} 2^k \left( \int_{(i-1)/2^k}^{i/2^k} g(x) \, dx \right)^2. \]
(5.23)

This is the form of $q$ we will use below.

**Remarks.** The decomposition above for $q$ gives a decomposition for the corresponding $z$ (recall its definition in (5.13)), which has a probabilistic intuition behind it. Indeed, write
\[ z \approx \sum_{k=0}^{\infty} 2^{-k/2} z_k, \]
where $z_k$ is the function given by applying the procedure to get $z$ from $q$ (which is linear) for one term. Then each $z_k$ is the increasing process of a martingale; for example, $z_1$ is the increasing process of the standard Haar martingale $X_1$ defined above in (5.22). Recall that for any martingale $X_n$, its increasing process is defined as
\[ A_n = \sum_{i=1}^{n-1} \mathbb{E}(X_{i+1} - X_i)^2 \mid X_1, \ldots, X_i \] (see, e.g., [5, Sect. 4.4]).

Let us sketch a probabilistic proof of Lemma 5.5. We claim that if $X_n$ is a bounded martingale, then its increasing process $A_n$ satisfies $\mathbb{P}(A_\infty > \lambda) \leq 2e^{-c\lambda}$. To see this, we apply the Skorokhod embedding theorem to embed the martingale into Brownian motion, and then $A_\infty$ becomes the time. The result then follows from the fact that the probability of Brownian motion to stay for time $t$ inside the interval $[-1,1]$ decays exponentially in $t$. Thus each $z_k$ has an exponentially decaying tail, and therefore so does their weighted sum $z$. We skip any further details.

Our last remark would be useful mostly to readers who have already examined the proof of Lemma 6.4 below. Indeed, the specific form of $q$ we use is mostly dictated by its use in that lemma. Its use there leaves two places for flexibility. First, Lemma 6.4 could have worked with the sequence $2^{-k/2}$ replaced with any sequence decaying faster that $1/k^2$. Second, the term $\sum 2^k (\int f)^2$, which is
just the $L^2$ norm of a partial Haar expansion of $g$, could have been replaced with others norms, for example, with the $L^1$ norm. But Lemma 5.5 does not hold for the $L^1$ norm, only for the $L^p$ norms with $p \geq 2$. Thus our choice of the $L^2$ norm comes from the need to have both Lemma 5.5 and 6.4 hold, each constraining $q_f$ from one side.

6. RANDOM HOMEOMORPHISMS

**Definition 6.1.** We say that a number $\eta > 0$ is admissible if $\eta < \eta_1(2)$ with $\eta_1$ from Lemma 5.6 (in other words, if for all $f$ with $\|f\|_\infty \leq 1$ and all $|\theta - 1/2| \leq \eta Q_f$ we have $\|f_{\psi_{q_f,\tau,\infty}}\|_2 \leq C$), and in addition if the condition $|\theta - 1/2| \leq \eta Q_f$ implies that (5.9) and (5.10) hold with $\delta = 4/5$. In other words,

$$\left(\frac{3}{8}\right)^m \leq \psi_{\theta,\infty}^{-1}\left(\frac{k}{2^m}\right) - \psi_{\theta,\infty}^{-1}\left(\frac{k - 1}{2^m}\right) \leq \left(\frac{5}{8}\right)^m,$$

(6.1)

$$\left(\frac{1}{4}(y - x)\right)^{5/4} \leq \psi_{\theta,\infty}^{-1}(y) - \psi_{\theta,\infty}^{-1}(x) \leq (4(y - x))^{4/5}$$

(6.2)

for all values of $m$.

The rest of the proof holds for any admissible $\eta$, so let us fix one such $\eta$ (possibly depending on $p$, if we are proving the “further” clause of Theorem 1.2) and remove it from the notation. Further, we allow arbitrary constants like $c$ and $C$ to depend on it. For a function $\tau$ from the dyadic rationals into $[-1, 1]$ we define

$$\psi_{f,\tau,n} := \psi_{1/2 + \eta f,\tau,n} \quad \text{and} \quad \psi_{f,\tau,\infty} := \psi_{1/2 + \eta f,\tau,\infty}. \tag{6.3}$$

For convenience, let us note at this point the locality formulas for $\psi_{f,\tau,n}$. Define $f^\pm$ and $\tau^\pm$ as in (5.3) and recall that $L_I$ is the affine increasing map taking $[0, 1]$ onto $I$. Then

$$\psi_{f,\tau,\infty}(x) = \begin{cases} 
\frac{1}{2} \psi_{f^-,\tau^-,\infty}\left(\frac{x}{A}\right), & x \leq A, \\
\frac{1}{2} + \frac{1}{2} \psi_{f^+,\tau^+,\infty}\left(\frac{x - A}{1 - A}\right), & x > A,
\end{cases} \tag{6.4}$$

and

$$\psi_{f,\tau,\infty}^{-1} L_I(\psi_{f,\tau,\infty}^{-1}(I)) = L_I \circ \psi_{f_0 L_I,\tau_0 L_I,\infty} \circ L_I^{-1} \psi_{f,\tau,\infty}^{-1}(I), \tag{6.5}$$

where, as usual, $A = \theta(1/2) = 1/2 + \eta f(1/2)\tau(1/2)$. The formulas are a direct consequence of (5.12), (5.7) and the facts that $(q_f)^- = q_f$ and $q_f \circ L_I = q_{f_0 L_I}$, both of which are easy to verify.

Most importantly, from now on we take $\tau$ random, i.i.d. and uniformly distributed in $[-1, 1]$. All $\mathbb{E}$ and $\mathbb{P}$ signs in this section will refer to this measure on $\tau$.

**Lemma 6.1.** For every $f$ and $n$ (including $n = \infty$),

$$\mathbb{E}(\psi_{f,\tau,n}^{-1}(y)) = 1$$

for every $y \in [0, 1]$ if $n$ is finite (for dyadic $y$ we mean one-sided derivatives) and for almost every $y$ if $n = \infty$.

**Proof.** For $n$ finite this follows from the definition of $\psi$ (5.1), because $\mathbb{E}(\tau(d)) = 0$ for each $d$ (so $\mathbb{E}(\theta(d)) = 1/2$), and further, because if $\text{rnk}(d) = n$ then $\tau(d)$ and $\psi_{f,\tau,n-1}^{-1}$ are independent. This last claim follows since $\psi_{n-1}^{-1}$ depends only on $\tau(d')$ for $d'$ with $\text{rnk}(d') < n$, in particular $d' \neq d$, and hence the corresponding values of $\tau$ are independent.

The case of $n = \infty$ follows by taking a limit, which is allowed since $(\psi_n^{-1})' \to (\psi_\infty^{-1})'$ for almost every $x$ (see (5.18)), for every $\tau$. By Fubini’s theorem, for almost every $x$ we have $(\psi_n^{-1})' \to (\psi_\infty^{-1})'$.
almost surely (in $\tau$). Exchanging the limit and the expectation is allowed since $(\psi_n^{-1})'(x)$ is bounded by a quantity depending only on $f$, by (5.15), and finite for almost every $x$, by Lemma 5.5, so the bounded convergence theorem applies. □

Lemma 6.2. Let $X_1$ and $X_2$ be two random variables on $[-1,1]$ with densities $\rho_1$ and $\rho_2$, respectively. Let

$$p = \int_{-1}^{1} \min\{\rho_1(x), \rho_2(x)\} \, dx.$$ 

Then there exists a variable $Q$ ("a coupling of $X_1$ and $X_2"$) taking values in $[-1,1]^2$ such that

(i) $Q_i$ has the same distribution as $X_i$;

(ii) $\mathbb{P}(Q_1 = Q_2) \geq p$.

(In fact, $\mathbb{P}(Q_1 = Q_2) = p$, but we will make no use of this fact.)

Proof. If $p = 0$, then we can take $Q$ to be two independent copies of $X_1$ and $X_2$, so assume this is not the case. Let $Z$ be a random variable with density $\min\{\rho_1, \rho_2\}/p$. If $p \neq 1$, then let $Y_i$ be a random variable with density $(\rho_i - \min\{\rho_1, \rho_2\})/(1-p)$. Let $Z, Y_1$ and $Y_2$ be all independent. Now throw an independent coin which succeeds with probability $p$. If the coin succeeds, let $Q = (Z, Z)$, and if not, let $Q = (Y_1, Y_2)$. Clause (ii) is now obvious, and clause (i) is not difficult either, because the density of $Q_i$ is $p$ times the density of $Z$ plus $1-p$ times the density of $Y_i$, i.e.,

$$p \cdot \frac{1}{p} \min\{\rho_1, \rho_2\} + (1-p) \cdot \frac{1}{1-p} (\rho_i - \min\{\rho_1, \rho_2\}) = \rho_i,$$

as needed. □

Lemma 6.3. For every $0 < x_1 < x_2 < 1$ and every $\lambda \in (0, 1/4)$ there exists a random variable $T = (T_1, T_2)$ taking values in $[-1,1]^2$ such that

(i) $T_1$ and $T_2$ are both uniform in $[-1,1]$;

(ii) if we set

$$y_i := \begin{cases} \frac{x_i}{A_i}, & x_i \leq A_i, \\ \frac{x_i - A_i}{1 - A_i}, & x_i > A_i, \end{cases} \quad A_i := \frac{1}{2} + \lambda T_i,$$

then

$$\mathbb{P}(y_1 = y_2, 1_{x_1 \leq A_1} = 1_{x_2 \leq A_2}) \geq 1 - C \frac{x_2 - x_1}{\lambda \min\{x_2, 1-x_1\}}.$$

Proof. Define auxiliary variables $z_i$ by

$$z_i := \begin{cases} \frac{x_i}{2A}, & x_i \leq A, \\ \frac{1}{2} + \frac{x_i - A}{2(1-A)}, & x_i > A, \end{cases}$$

where $A$ is a random variable uniform in $[1/2 - \lambda, 1/2 + \lambda]$. Let $\rho_i$ be the density of $z_i$. A simple calculation shows that

$$\rho_i(t) = 1_{I_i}(t) \frac{x_i}{4\lambda t^2} + 1_{J_i}(t) \frac{1-x_i}{4\lambda(1-t)^2},$$

$$I_i := \left[ \frac{x_i}{1 + 2\lambda}, \frac{x_i}{1 - 2\lambda} \right] \cap \left[ 0, \frac{1}{2} \right], \quad J_i := \left[ 1 - \frac{1-x_i}{1 + 2\lambda}, 1 - \frac{1-x_i}{1 + 2\lambda} \right] \cap \left[ \frac{1}{2}, 1 \right].$$
and hence
\[
\int \min\{\rho_1, \rho_2\} \geq 1 - C \frac{x_2 - x_1}{\lambda \min\{x_2, 1 - x_1\}} =: p.
\]

Now couple \(z_1\) and \(z_2\) using Lemma 6.2 (call the coupling \(Q\), and the coupling succeeds (i.e., \(Q_1 = Q_2\)) with probability at least \(p\). Now define
\[
T_i = \begin{cases} 
1 - \frac{x_i}{2Q_i} - \frac{1}{2}, & Q_i \leq \frac{1}{2}, \\
1 - \frac{1 - x_i}{2 - 2Q_i}, & Q_i > \frac{1}{2},
\end{cases}
\]
and the properties of \(T\) follow from the way we constructed it. □

**Lemma 6.4.** For every continuous function \(f\) with \(\|f\|_{\infty} \leq 1\), every interval \([r, s] \subseteq [0, 1]\) and every \(n\) (not necessarily integer),
\[
\int_r^s \left| \mathbb{E}(f(\psi_f(x))) - \mathbb{E}(f(\psi(x))) \right| e(nx) \, dx \leq Cn^{-1/11}.
\]

**Proof.** We divide the proof into two different cases according to whether \(q := q_f(1/2) \geq n^{-9/10}\) or not (\(q_f\) from (5.19)). Let for brevity \(\psi := \psi_f, r, \infty\).

We start with the case of \(q \geq n^{-9/10}\). In this case we claim that for every \(x_1 < x_2\)
\[
|\mathbb{E}(f(\psi(x_1))) - \mathbb{E}(f(\psi(x_2)))| \leq C \frac{x_2 - x_1}{q \min\{x_2, 1 - x_1\}}.
\]

(6.6)

To see (6.6), couple \(\psi(x_1)\) and \(\psi(x_2)\) as follows. We define two random (dependent) variables \(\tau_1\) and \(\tau_2\) each of which has the same distribution as \(\tau\), i.e., a uniform function from the dyadic rationals into \([-1, 1]\). The definition of the \(\tau_i\) is as follows. For all dyadic \(p \neq 1/2\) we take \(\tau_1(p) = \tau_2(p)\) (and of course independent for different \(p\) and uniformly distributed). For \(p = 1/2\) we use the variable given by Lemma 6.3, namely, we use Lemma 6.3 with \(\lambda_{\text{Lemma 6.3}} = \eta q\) and then let \(\tau_1(1/2) = T_i\).

We now note that, by (6.4),
\[
\psi_{f, \tau_1, \infty}(x_i) = \begin{cases} 
\frac{1}{2} \psi_{f, \tau_1^-; \infty} \left( \frac{x_i}{1 - A_1} \right), & x_i \leq A_1, \\
\frac{1}{2} + \frac{1}{2} \psi_{f, \tau_1^+; \infty} \left( \frac{x_i - A_1}{1 - A_1} \right), & \text{otherwise},
\end{cases}
\]

Note that \(\tau^\pm\) do not depend on \(i\) (this is where we used the fact that \(\tau_1(p) = \tau_2(p)\) for all \(p \neq 1/2\), and of course neither do \(f^\pm\). Hence Lemma 6.3 gives
\[
\mathbb{P}(\psi_{f, \tau_1, \infty}(x_1) = \psi_{f, \tau_2, \infty}(x_2)) \geq \mathbb{P}(y_1 = y_2, 1_{x_1 \leq A_1} = 1_{x_2 \leq A_2}) \geq 1 - C \frac{x_2 - x_1}{q \min\{x_2, 1 - x_1\}},
\]

where the \(y_i\) are from Lemma 6.3 (the first inequality is in fact an equality, but we will not need this fact). Denote the event on the left-hand side by \(G\). Then
\[
|\mathbb{E}(f(\psi(x_1))) - \mathbb{E}(f(\psi(x_2)))| = |\mathbb{E}(f(\psi_{f, \tau_1, \infty}(x_1))) - \mathbb{E}(f(\psi_{f, \tau_2, \infty}(x_2)))|
\overset{(*)}{=} \left| \mathbb{E}(1_G \cdot (f(\psi_{f, \tau_1, \infty}(x_1)) - f(\psi_{f, \tau_2, \infty}(x_2)))) \right| \leq 2 \mathbb{P}(\neg G) \leq C \frac{x_2 - x_1}{q \min\{x_2, 1 - x_1\}},
\]

where the equality marked by (*) follows because \(\mathbb{E}(1_G \cdot (\ldots)) = 0\), since under \(G\) we have \(\psi_{f, \tau_1, \infty}(x_1) = \psi_{f, \tau_2, \infty}(x_2)\). This shows (6.6).
To conclude from (6.6) an estimate for the integral, we write

\[
\int_{r}^{s} + \sum_{k=0}^{\lfloor (s-r)n \rfloor - 1} \int_{r+k/n}^{r+(k+1)/n} + \int_{r+(s-r)n/n}^{s}.
\]

The last integral is over an interval of length less than \(1/n\), so it can contribute no more than \(1/n\).

For the main term we write, for each \(k\),

\[
I_k := \int_{r+k/n}^{r+(k+1)/n} \mathbb{E}(f(\psi(x))) e(nx) \, dx = \int_{r+k/n}^{r+(k+1)/n} \left( \mathbb{E}(f(\psi(x))) - \mathbb{E}(f\left(\psi\left(x + \frac{1}{2n}\right)\right)) \right) e(nx) \, dx.
\]

Hence

\[
|I_k| \leq \int_{r+k/n}^{r+(k+1)/n} \left| \mathbb{E}(f(\psi(x))) - \mathbb{E}(f\left(\psi\left(x + \frac{1}{2n}\right)\right)) \right| \, dx \leq C \int_{r+k/n}^{r+(k+1)/n} \frac{1/(2n)}{q \min\{x + 1/(2n), 1 - x\}} \, dx,
\]

\[
\leq \frac{C}{n^2 q \min\{(r + (k + 1/2)/n, 1 - (r + (k + 1/2)/n)\}}.
\]

Summing over all \(k\) gives

\[
\sum_{k} |I_k| \leq \frac{C}{n^2 q} \sum_{i=0}^{n/2} \frac{1}{1/(2n) + i/n} \leq \frac{C \log n}{nq},
\]

and because of our assumption that \(q > n^{-9/10}\) (which we have not used so far!), we get a bound of \(Cn^{-1/10} \log n\). This finishes the case \(q > n^{-9/10}\).

For the case of \(q \leq n^{-9/10}\) our first step is to rearrange the question slightly. Recall that we wish to estimate \(\int \mathbb{E}(f \circ \psi) e(nx)\). We first let \(g = f - \int_{0}^{1} f\) and write

\[
\int_{r}^{s} \mathbb{E}(f(\psi(x))) e(nx) \, dx = \int_{r}^{s} \mathbb{E}(g(\psi(x))) e(nx) \, dx + \left( \int_{0}^{1} f \right) \int_{r}^{s} e(nx) \, dx,
\]

and the second term is smaller, in absolute value, than \(2/n\). For the first term, we apply Fubini’s theorem and a change of variables and get

\[
\int_{r}^{s} \mathbb{E}(g(\psi(x))) e(nx) \, dx = \mathbb{E} \int_{r}^{s} g(\psi(x)) e(nx) \, dx = \mathbb{E} \int_{\psi(r)}^{\psi(s)} g(y) e(n(\psi^{-1}(y))(\psi^{-1}(y))' \, dy. \quad (6.7)
\]

Recall next estimate (5.23) for \(q = q_f(1/2)\). Using it with the condition \(q \leq n^{-9/10}\) gives

\[
\sum_{i=1}^{2^k} 2^k \left( \int_{(i-1)/2^k}^{i/2^k} g(y) \, dy \right)^2 \leq 2^{k/2} q \leq 2^{k/2} n^{-9/10} \quad \forall k.
\]
Let \( k \) be the smallest integer such that \( 2^k > n^{7/5} \). Then
\[
\sum_{i=1}^{2^k} \left( \int_{(i-1)/2^k}^{i/2^k} g(y) \, dy \right)^2 \leq Cn^{-1/5}.
\] (6.8)

For each \( i \) write
\[
\int_{i/2^k}^{(i+1)/2^k} g(y) e(n\psi^{-1}(y))(\psi^{-1})'(y) \, dy = \int_{i/2^k}^{(i+1)/2^k} g(y) \left( e(n\psi^{-1}(y)) - e \left( n\psi^{-1} \left( \frac{i}{2^k} \right) \right) \right) (\psi^{-1})'(y) \, dy
\]
\[+ e \left( n\psi^{-1} \left( \frac{i}{2^k} \right) \right) \int_{i/2^k}^{(i+1)/2^k} g(y)(\psi^{-1})'(y) \, dy =: I_i + \Pi_i. \] (6.9)

We start with the estimate of \( \Pi_i \). We wish to condition on \( \psi^{-1}(\frac{i}{2^k}) \) and \( \psi^{-1}(\frac{i+1}{2^k}) \), so denote the \( \sigma \)-field of these two variables by \( \mathcal{B}_i \). Then
\[
\mathbb{E} \left( \int_{i/2^k}^{(i+1)/2^k} g(y)(\psi^{-1})'(y) \, dy \ \bigg| \ \mathcal{B}_i \right) = \int_{i/2^k}^{(i+1)/2^k} g(y) \mathbb{E}((\psi^{-1})'(y) \ | \ \mathcal{B}_i) \, dy
\] (6.10)

by Fubini’s theorem. We have reached a simple but crucial point in the argument. We note that \( \psi^{-1} \) conditioned on \( \mathcal{B}_i \) is a different version of \( \psi^{-1} \). Precisely, let \( L \) be the linear increasing map taking \([0,1]\) onto \([i/2^k, (i+1)/2^k]\). Let \( h(x) = f(L(x)) \). Let \( \sigma(d) = \tau(L(d)) \). Then by (6.5), for every \( x \in [i/2^k, (i+1)/2^k] \),
\[
\psi^{-1}_{f,\tau,\infty}(x) = \psi^{-1}_{f,\tau,\infty} \left( \frac{i}{2^k} \right) + b_i \psi^{-1}_{h,\sigma,\infty} (2^k x - i), \quad b_i := \psi^{-1}_{f,\tau,\infty} \left( \frac{i+1}{2^k} \right) - \psi^{-1}_{f,\tau,\infty} \left( \frac{i}{2^k} \right).
\]
This means that
\[
(\psi^{-1}_{f,\tau,\infty})'(x) = 2^k b_i (\psi^{-1}_{h,\sigma,\infty})'(2^k x - i).
\]
We now use Lemma 6.1, which states that \( \mathbb{E}((\psi^{-1}_{h,\sigma,\infty})'(y)) = 1 \) for almost all \( y \). This shows that
\[
\mathbb{E}((\psi^{-1}_{f,\tau,\infty})'(x) \ | \ \mathcal{B}_i) = 2^k b_i
\]
for almost all \( x \). With (6.10) we get
\[
\mathbb{E} \left( \int_{i/2^k}^{(i+1)/2^k} g(y)(\psi^{-1})'(y) \, dy \ \bigg| \ \mathcal{B}_i \right) = 2^k b_i \int_{i/2^k}^{(i+1)/2^k} g(y) \, dy,
\]
or
\[
|\mathbb{E}(\Pi_i \ | \ \mathcal{B}_i)| = 2^k b_i \int_{i/2^k}^{(i+1)/2^k} g(y) \, dy. \] (6.11)
Let $E$ be the set of indices $i$ such that $[i/2^k, (i + 1)/2^k] \subseteq [\psi(r), \psi(s)]$. Sum the right-hand side over all $i \in E$ and use the Cauchy–Schwarz inequality to get

$$
\sum_{i \in E} 2^k b_i \left( i/2^k \right) g(y) dy \leq \left( \sum_{i \in E} 2^k b_i^2 \right)^{1/2} \left( \sum_{i \in E} \left( i/2^k \right) g(y) dy \right)^{1/2},
$$

and by (6.8) the second term is bounded by $Cn^{-1/10}$. As for the first term, we may write

$$
\sum_{i \in E} 2^k b_i^2 = \sum_{i \in E} 2^k \left( \int_{i/2^k}^{(i+1)/2^k} \psi_{f_r, \tau, \infty}^{-1}(y) \right)^2 \leq \sum_{i \in E} \int_{i/2^k}^{(i+1)/2^k} \left( \psi_{f_r, \tau, \infty}^{-1}(y) \right)^2 \leq \int_0^1 \left( \psi_{f_r, \tau, \infty}^{-1}(y) \right)^2.
$$

Hence

$$
\left| \mathbb{E} \sum_{i \in E} \Pi_i \right| \leq \mathbb{E} \sum_{i \in E} \left| \mathbb{E} \left( \Pi_i \left| \mathcal{B}_i \right. \right) \right| \leq \mathbb{E} \sum_{i \in E} \left( i/2^k \right) g(y) dy \leq \left( \mathbb{E} Cn^{-1/10} \left\| \left( \psi_{f_r, \tau, \infty}^{-1} \right)^2 \right\|_2 \right) = Cn^{-1/10},
$$

where $(*)$ follows by (6.11) and $(†)$ by (6.12) and (6.13) (we need to condition on $\mathcal{B}_i$ in the first inequality because $E$ is itself random). In the last inequality we used the fact that $\eta$ is admissible (recall the definition of admissibility in the beginning of Section 6). This terminates the estimate of the second (more interesting) part of (6.9).

To estimate $I_i$, we note that from $2^k \geq n^{7/5}$ and (6.2) we have $\psi^{-1}(y) - \psi^{-1}(i/2^k) < Cn^{-28/25}$ for every $y \in [i/2^k, (i + 1)/2^k]$ and hence

$$
\left| e(n\psi^{-1}(y)) - e \left( n\psi^{-1} \left( \frac{i}{2^k} \right) \right) \right| \leq Cn^{-3/25}.
$$

We get

$$
I_i \leq Cn^{-3/25} \int_{i/2^k}^{(i+1)/2^k} \left( \psi^{-1}(y) \right)' dy = Cn^{-3/25} \left( \psi^{-1} \left( \frac{i + 1}{2^k} \right) - \psi^{-1} \left( \frac{i}{2^k} \right) \right),
$$

and hence

$$
\sum_{i \in E} I_i \leq Cn^{-3/25}.
$$

With the estimate of $\Pi_i$ above we get

$$
\left| \mathbb{E} \sum_{i \in E} \int_{i/2^k}^{(i+1)/2^k} g(y) e(n\psi^{-1}(y)) (\psi^{-1}(y))' dy \right| = \left| \mathbb{E} \sum_{i \in E} I_i + \mathbb{E} \sum_{i \in E} \Pi_i \right| \leq Cn^{-1/10}.
$$

This is almost what we need, but what we need precisely is the integral from $\psi(r)$ to $\psi(s)$. The difference between it and $\sum_{i \in E} \int_{i/2^k}^{(i+1)/2^k}$ is the integral over two short intervals, $[\psi(r), i_1/2^k]$ and
The integral over the other interval can be estimated similarly. We get
\[ \int_{\psi(r)}^{i_{1/2}^k} g(y) e(n\psi^{-1}(y)) (\psi^{-1})'(y) \, dy \leq 2 \int_{\psi(r)}^{i_{1/2}^k} (\psi^{-1})'(y) \, dy \]
\[ \leq 2 \left( \int_{\psi(r)}^{i_{1/2}^k} 1 \, dy \right)^{1/2} \left( \int_{0}^{1} (\psi^{-1})'(y)^2 \, dy \right)^{1/2} \leq \frac{C}{2^{k/2}} \leq C n^{-7/10}. \]

The integral over the other interval can be estimated similarly. We get
\[ \int_{\psi(s)}^{\psi(r)} g(y) e(n\psi^{-1}(y)) (\psi^{-1})'(y) \, dy \leq C n^{-1/10}. \]

Recalling (6.7) from the beginning of the proof of the small \( q \) case, we get
\[ \left| \int_{r}^{s} \mathbb{E}(g(\psi(x))) e(n x) \, dx \right| \leq C n^{-1/10}, \]
and, as explained just before (6.7), this gives the same result for \( f \). The lemma is thus proved.

**Lemma 6.5.** Let \( J_1, J_2 \subset [-1, 1] \) be intervals and let \( \varepsilon := \max\{|y_1 - y_2|: y_i \in J_i\} \). Let \([r, s] \subset [0, 1]\) be an interval, let \( n \in \mathbb{R} \) and define
\[ F_i := \int_{r}^{s} \mathbb{E}\left( f(\psi_{f,r,\infty}(x)) \right) \left| \tau \left( \frac{1}{2} \right) \in J_i \right) e(n x) \, dx. \]
Then
\[ |F_1 - F_2| \leq C \varepsilon^{1/22} \]
uniformly in \( n \).

**Proof.** We keep the notation \( q = q_f(1/2) \). Define
\[ \alpha := \min \left\{ \frac{1}{2} + \eta qx: x \in J_1 \cup J_2 \right\}, \quad \beta := \max \left\{ \frac{1}{2} + \eta qx: x \in J_1 \cup J_2 \right\}, \]
so that \( \beta - \alpha \leq C \varepsilon \), and then
\[ F_i^- := \int_{[r,s] \cap [0,\alpha]} \mathbb{E}\left( f(\psi_{f,r,\infty}(x)) \right) \left| \tau \left( \frac{1}{2} \right) \in J_i \right) e(n x) \, dx, \]
\[ F_i^+ := \int_{[r,s] \cap [\beta,1]} \mathbb{E}\left( f(\psi_{f,r,\infty}(x)) \right) \left| \tau \left( \frac{1}{2} \right) \in J_i \right) e(n x) \, dx, \]
\[ F_i^0 := F_i - (F_i^- + F_i^+). \]
Note that \( |F_i^0| \leq C \varepsilon \), as it can be written as an integral over an interval of length at most \( C \varepsilon \). Hence we need to estimate \( |F_i^0 - F_i^+| \). The proofs of the + and − cases are identical, so for brevity we will do the calculations for \( F^- \).
Let us first dispense with an easy case. Assume that $n \geq \varepsilon^{-1/2}$. In this case, fix some $y \in J_1 \cup J_2$ and condition on $\tau(1/2) = y$. Define

$$F^-(y) := \int_{[r,s]\cap[0,\alpha]} \mathbb{E}\left(f(\psi_{f,\tau,\alpha}(x)) \mid \tau(\frac{1}{2}) = y\right) e(nx) \, dx$$

under the conditioning, $\psi_{f,\tau,\alpha}(x) = \psi_{f,-\tau,-\alpha}(x/A)/2$, $A := 1/2 + \eta q y$, so

$$F^-(y) = \int_{[r,s]\cap[0,\alpha]} \mathbb{E}\left(f(\psi_{f,-\tau,-\alpha}(\frac{x}{A}))\right) e(nx) \, dx.$$ 

We change variables and get

$$F^-(y) = \int_{I(y)} \mathbb{E}(f(\psi_{f,-\tau,-\alpha}(x))) e(nxA) A \, dx, \quad I(y) := \left[\frac{r}{A}, \frac{s}{A}\right] \cap \left[0, \frac{\alpha}{A}\right]. \quad (6.14)$$

We use Lemma 6.4 with $f_{\text{Lemma } 6.4} = f^-$, $n_{\text{Lemma } 6.4} = nA$ and $[r, s]_{\text{Lemma } 6.4} = I(y)$ and get

$$|F^-(y)| \leq AC(nA)^{-1/11} \leq C\varepsilon^{1/22}.$$ 

Integrating over $y$ in either $J_i$, we find that both $F_i$ satisfy $|F_i| \leq C\varepsilon^{1/22}$, and we are done.

Assume therefore that $n < \varepsilon^{-1/2}$. Let $y \in J_1 \cup J_2$ and consider $\psi_{f,\tau,\alpha}$ conditioned on $\tau(1/2) = y$.

We keep the notation $A$, $F^-(y)$ and $I(y)$ above, and write $\psi = \psi_{f,-\tau,-\alpha}$ for brevity. We apply (6.14) to two different $y_1$ and subtract, getting

$$F^-(y_1) - F^-(y_2) = I + II + III,$$

where

$$I := \int_{I(y_1) \cap I(y_2)} \mathbb{E}(f(\psi(x))) (e(nxA_1)A_1 - e(nxA_2)A_2) \, dx,$$

$$II := \int_{I(y_1) \setminus I(y_2)} \mathbb{E}(f(\psi(x))) e(nxA_1)A_1 \, dx, \quad III := -\int_{I(y_2) \setminus I(y_1)} \mathbb{E}(f(\psi(x))) e(nxA_2)A_2 \, dx,$$

and again II and III are integrals over intervals of length at most $C\varepsilon$, so may be ignored. As for I, we write

$$|e(nxA_1)A_1 - e(nxA_2)A_2| \leq 2\pi nx|A_1 - A_2|A_1 \leq C\varepsilon,$$

so

$$|e(nxA_1)A_1 - e(nxA_2)A_2| \leq Cn\varepsilon A_1 + C\varepsilon \leq Cn\varepsilon,$$

which we plug into the integral in I to get $I \leq Cn\varepsilon$. We get

$$|F^-(y_1) - F^-(y_2)| \leq Cn\varepsilon + C\varepsilon \leq C\varepsilon^{1/2}$$

by our assumption that $n < \varepsilon^{-1/2}$. Integrating over $y_1$ and $y_2$ gives

$$|F^+_1 - F^+_2| \leq C\varepsilon^{1/2}.$$

Since we covered both small and large $n$, we get

$$|F^+_1 - F^+_2| \leq C\varepsilon^{1/22}.$$ 

The estimate for $|F^+_1 - F^+_2|$ is identical, and the lemma is proved. \(\square\)
7. Reducing Randomness

We have spent many pages on the construction of a random homeomorphism \( \psi \) such that the expectation of \( f \circ \psi \) has some good properties. In this last section we are going to remove the randomness step by step, always controlling the expectation, until we are finally left with a single \( \psi \) such that \( f \circ \psi \) has the same good properties.

Let \( I \) be a map giving for each dyadic rational \( d = k/2^n \in (0, 1) \) an interval \( I(d) \subseteq [−1, 1] \) (possibly degenerate). For each \( I \) we define \( \phi_I = \phi_{I,f,\eta} \) to be our homeomorphism \( \psi_{f,\tau,\infty} \) with \( \tau(d) \) taken uniformly in \( I(d) \) for all \( d \). We call such \( I \) an RH-restrictor (RH standing for “random homeomorphism”). Sometimes we will write \( \tau_I \) for \( \tau \) which has this distribution.

From now on, when we say “condition \( \psi \) on \( \psi^{-1}(1/2) = y \)” or “condition \( \psi \) on \( \psi(y) = 1/2 \),” we will actually mean \( \psi_{f,\tau,\infty} \) with \( \tau(1/2) = (y − 1/2)/(\eta f(1/2)) \) and the other \( \tau(d) \) uniform in \([−1, 1]\) (this is, of course, a version of the usual conditional probability, but it is defined for all \( y \) and not just for almost all \( y \), which is convenient). Another version of the same thinking is the following

**Lemma 7.1.** Let \( \mathcal{U} = (U_i) \) be a finite partition of \([0, 1]\) into dyadic intervals (i.e., \( \bigcup_i U_i = [0, 1] \) and different \( U_i \in \mathcal{U} \) are disjoint except possibly for their endpoints). Let \( I \) be an RH-restrictor which is degenerate for any \( d \) in the boundary of any \( U \in \mathcal{U} \). Then \( \phi_I^{-1}(U) \) is deterministic for any \( U \in \mathcal{U} \) (i.e., it is independent of \( f \) but not random).

**Proof.** We proceed by induction on the number of intervals in the partition. Assume the claim has been proved for all partitions with less than \( n \) intervals, and let \( \mathcal{U} \) be with \( |\mathcal{U}| = n \). Let \( U \in \mathcal{U} \) be an interval with smallest size. Let \( V \) be the parent of \( U \), i.e., \( U \subset V \) and \( |V| = 2|U| \). Then \( V \setminus U \) must also be in \( \mathcal{U} \). Replacing \( U \) and \( V \setminus U \) with \( V \) gives a new partition \( \mathcal{U}' \) with \( |\mathcal{U}'| < n \); hence, by the induction hypothesis, \( \phi_I(W) \) is deterministic for any \( W \in \mathcal{U}' \) and in particular for all \( W \in \mathcal{U} \) other than \( U \) and \( V \setminus U \). Next, let \( d_0, d_1 \) and \( d_2 \) be the beginning, centre and end of \( V \), respectively. Then \( \phi_I^{-1}(d_0) \) and \( \phi_I^{-1}(d_2) \) are deterministic by the induction hypothesis, and \( I(d_1) \) is degenerate, say \( \{x\} \) for some \( x \in [−1, 1] \) (the last claim is because \( d_1 \) is a point in the boundary of \( U \)). Hence, by Lemma 5.1,

\[
\phi_I^{-1}(d_1) = \phi_I^{-1}(d_0) + (\phi_I^{-1}(d_2) - \phi_I^{-1}(d_0)) \left( \frac{1}{2} + \eta f(d_1)\tau(d_1) \right),
\]

which is deterministic, proving the claim. \( \Box \)

**Definition 7.1.** For a continuous function \( f \) with \( \|f\|_\infty \leq 1 \) and a \( \delta > 0 \), we say that an RH-restrictor \( I \) is of type \( (f, \delta) \) if there exists a finite partition \( \mathcal{U} = (U_i) \) of \([0, 1]\) into dyadic intervals and an integer \( m \geq -1 \) with the following properties:

(i) \( I(d) \) is a single point for every \( d \) in the boundary of any \( U_i \);

(ii) \( |\phi_I^{-1}(U_i)| \in [\delta/4, \delta] \) for all \( i \);

(iii) if \( |\phi_I^{-1}(U_i)| > \delta/2 \), then \( I(d) \) is a dyadic interval of length \( 2^{-m} \) for \( d \) in the centre of \( U_i \);

(iv) \( I(d) = [−1, 1] \) for all other \( d \).

See Fig. 1.

Of course, not for every RH-restrictor \( I \) one may find some \( f \) and \( \delta \) such that \( I \) is of type \( (f, \delta) \); such \( I \) are quite special, and if we do not wish to specify \( f \) and \( \delta \), we will simply say that \( I \) has a type. Let us remark on the appearance of \( \phi_I^{-1}(U_i) \) in properties (ii) and (iii) (and indirectly in (iv) too, as it applies only when (iii) does not). Of course, \( \phi_I \) is random. But property (i) implies, using Lemma 7.1, that in fact \( \phi_I^{-1}(U_i) \) is deterministic and all randomness is inside the \( U_i \). Hence properties (ii) and (iii) are also a function of \( f \) and \( I \), and are not random.

We call the \( \mathcal{U} \) and the \( m \) the corresponding partition and the corresponding value (to \( I \)), respectively.
The following lemma is the main lemma of this paper. It implements the “reduction of randomness” strategy that we outlined in the beginning of this section. The reduction in randomness is implemented by moving from an RH-restrictor for a given \( m \) to an RH-restrictor with \( m + 1 \). Recall the standard definition of the modulus of continuity of a function,

\[
\omega_f(\delta) = \sup \{ |f(x) - f(y)| : x, y \in [0, 1], \text{dist}(x, y) < \delta \},
\]

where here and below \( \text{dist} \) is considered cyclically in \([0, 1]\) (for example, \( \text{dist}(9/10, 0) = 1/10 \)).

**Lemma 7.2.** Let \( f \) be a continuous function satisfying \( \|f\|_{\infty} \leq 1/2 \), let \( \eta \) be admissible (recall Definition 6.1), let \( \delta > 0 \) and let \( I \) be an RH-restrictor of type \((f, \delta)\). Denote by \( U = (U_i) \) the corresponding dyadic partition of \([0, 1]\), arranged in increasing order, and by \( m \) the corresponding value.

Further, for every \( \xi \in [0, 1] \) denote by \( B_\xi \) the union of the \( \phi^{-1}(U_i) \) that contains \( \xi \) and its two immediate neighbours. We understand “neighbours” cyclically; for example, if \( \xi \in \phi^{-1}(U_1) \), then \( B_\xi \) is the union of \( \phi^{-1}(U_1), \phi^{-1}(U_2) \) and the last \( \phi^{-1}(U_i) \).

Then there exists an RH-restrictor \( J \) of type \((f, \delta)\) with the same corresponding partition \( \mathcal{U} \), with the corresponding value being \( m + 1 \) and with the following properties:

(i) \( J(d) \subseteq I(d) \) for all \( d \);

(ii) for every \( u \in \mathbb{N} \), every \( r \in \{2^{u-1}, \ldots, 2^u - 1\} \) and every \( \xi \in 2^{u-2}\mathbb{Z} \cap [0, 1) \),

\[
\left| \int_{E_\xi} (\mathbb{E}(f(\phi_I(x))) - \mathbb{E}(f(\phi_J(x)))) \, D_r(x - \xi) \, dx \right| \leq C \left( \min \left\{ 2^u \delta, \frac{1}{2^u \delta} \right\} \right)^c e^{-cm} \omega_f(\delta^c)^c,
\]

where integration set \( E_\xi = E_\xi(\mathcal{U}, I, f, \delta) \) is defined by

\[
E_\xi := \begin{cases} 
[0, 1] \setminus B_\xi, & 2^u > \delta^{-1}, \\
[0, 1], & 2^u \leq \delta^{-1}.
\end{cases}
\]

See Fig. 2.
Here and below the constants $C$ and $c$ are allowed to depend on $\eta$, and $D_r$ is the usual Dirichlet kernel.

**Proof.** For any $\xi \in [0, 1]$ define $h(\xi) := \mathbb{E}(f(\phi_I(\xi)))$. Set $V_i := \phi_I^{-1}(U_i)$ for brevity. Examine one $U_i$ in our partition, and recall that $V_i$ is not random. The requirements from $J$ leave relatively little freedom for $J$ inside $U_i$. If $|V_i| \leq \delta/2$, then there are no options, $J(d)$ must be $[-1, 1]$ for every $d \in U_i^\circ$. If $|V_i| > \delta/2$, then there are two possibilities. For $d$ in the centre of $U_i$, $J(d)$ can be either the left or right half of $I(d)$. For other $d \in U_i^\circ$, $J(d)$ must be $[-1, 1]$. Thus choosing $J$ is equivalent to choosing a sequence of signs $\epsilon_i \in \{\pm 1\}$, one for each $i$ for which $|V_i| > \delta/2$ (say that $\epsilon_i = 1$ means that we take the right half of $I(d)$ and $\epsilon_i = -1$ the left, and set for brevity $Y := \{i : |V_i| > \delta/2\}$). Denote the $J$ that corresponds to a vector $\epsilon \in \{\pm 1\}^Y$ by $J^\epsilon$.

Fix one $i \in Y$ and one $\xi \in 2^{-u-2}Z \cap [0, 1)$, let $\epsilon^+$ and $\epsilon^-$ be two vectors with $\epsilon_i^+ = 1$ and $\epsilon_i^- = -1$ and define

$$\Delta_i(x) := \frac{1}{2}(F^+(x) - F^-(x)), \quad F^\pm(x) := \begin{cases} \mathbb{E}(f(\phi_{J^\epsilon}(x))) - h(\xi), & x \in V_i, \\ 0, & \text{otherwise} \end{cases}$$

(this is a good definition since the values of $\epsilon$ different from $\epsilon_i$ have no effect on $\phi_{V_i}$; hence they affect neither $F^\pm$ nor $\Delta$). In other words, if we take $J(d)$ to be the left half of $I(d)$, with $d$ the centre of $U_i$, then $\mathbb{E}(f \circ \phi_{J^\epsilon}) = \mathbb{E}(f \circ \phi_I) - \Delta$ on $V_i$, and otherwise it is $\mathbb{E}(f \circ \phi_I) + \Delta$. Summing over $i \in Y$ gives

$$\mathbb{E}(f \circ \phi_{J^\epsilon}) = \mathbb{E}(f \circ \phi_I) + \sum_{i \in Y} \epsilon_i \Delta_i. \quad (7.2)$$

This relation is important because it “linearises” the problem of choosing a homeomorphism. Our strategy will be to find estimates for $\int \Delta_i D_r(\cdot - \xi)$ for various $i$, $r$ and $\xi$, and then apply Lemma 4.1. These estimates will occupy the next three lemmas.

The reason for subtracting $h(\xi)$ in the definition of $F^\pm$ (which, of course, has no affect on $\Delta$) is to get the $\omega(\delta^r)^c$ factor in the statement of Lemma 7.2. Precisely, we claim that

$$\|F^\pm\|_\infty \leq \omega_f(C_1(\text{dist}(\xi, V_i) + \delta)^{4/5}). \quad (7.3)$$

To see (7.3), first note that if $x \in V_i$, then $|x - \xi| \leq \text{dist}(\xi, V_i) + \delta$. Let $x'$ be a point in the boundary of some $\phi_I^{-1}(U)$, $U \in \mathcal{U}$, closest to $x$ among such points, and let $\xi'$ be the point closest to $\xi$, again from among the points in the boundary of $\phi_I^{-1}(U)$, $U \in \mathcal{U}$. Then the $\delta$-uniformity of $\phi_I^{-1}(\mathcal{U})$ says that $|x - x'| \leq \delta/2$ and $|\xi - \xi'| \leq \delta/2$. We now use the deterministic H"older condition (6.2) and get

$$|\phi_{J^\epsilon}(x) - \phi_{J^\epsilon}(x')| \leq C\delta^{1/5}, \quad |\phi_I(\xi) - \phi_I(\xi')| \leq C\delta^{1/5},$$

$$|\phi_{J^\epsilon}(x') - \phi_I(\xi')| \leq C(\text{dist}(\xi, V_i) + 2\delta)^{1/5},$$

where the third inequality follows because $x'$ and $\xi'$ are both on points where $\phi_{J^\epsilon} = \phi_I$. This shows (still deterministically) that $|\phi_{J^\epsilon}(x) - \phi_I(\xi)| \leq C(\text{dist}(\xi, V_i) + \delta)^{4/5}$ and hence

$$|f(\phi_{J^\epsilon}(x)) - f(\phi_I(\xi))| \leq \omega_f(C_1(\text{dist}(\xi, V_i) + \delta)^{4/5}). \quad (7.4)$$
Taking expectations shows (7.3). It is also occasionally useful to integrate only the right term, which gives
\[ |f(\phi_{J^\pm}(x)) - h(\xi)| \leq \omega_f(C_1(\text{dist}(\xi, V_i) + \delta)^{4/5}) \quad \forall x \in V_i. \] (7.5)
This will be used below.

In light of (7.3) let us define
\[ H := \omega_f(C_1(\text{dist}(\xi, V_i) + \delta)^{4/5}), \] (7.6)
so that $\|F^\pm\|_\infty \leq H$. Of course, $H$ depends on $\xi$, $i$ and other parameters, but we suppress this in the notation. Both $h(\xi)$ and $H$ will not play a very important role until the very end of the proof, at Lemma 7.6.

Unlike Lemmas 7.4 and 7.5 below, which are only used as sublemmas of Lemma 7.2, the next lemma actually has one application after Lemma 7.2 is finished. For that application, $i$ is not necessarily in $Y$. Hence keep in mind, when reading this lemma, that $i$ is arbitrary.

**Lemma 7.3.** With the definitions above, for every $r < s$ and every $\xi$ we have
\[ \left| \int_{V_i} F^\pm(x) \sum_{l=r}^{s-1} e(l(x - \xi)) \, dx \right| \leq \min \left\{ \frac{C}{\text{dist}(\xi, V_i)}, s - r \right\} \delta H. \]

More importantly,
\[ \left| \int_{V_i} F^\pm(x) \sum_{l=r}^{s-1} e(l(x - \xi)) \, dx \right| \leq ((\delta|s|)^{-1/22} + (\delta|s|)^{-1/22}) C\delta \sqrt{H} \frac{1}{\text{dist}(\xi, V_i)}. \] (7.7)

Recall that dist is considered cyclically.

**Proof.** The first clause of the lemma is simple. Indeed, on the one hand,
\[ \left| \sum_{l=r}^{s-1} e(l(x - \xi)) \right| \leq s - r, \]
and on the other hand,
\[ \left| \sum_{l=r}^{s-1} e(l(x - \xi)) \right| = \left| \frac{e(s(x - \xi) - e(r(x - \xi))}{1 - e(x - \xi)} \right| \leq \frac{C}{\text{dist}(\xi, x)}, \]
so
\[ \left| \int_{V_i} F^\pm(x) \sum_{l=r}^{s-1} e(l(x - \xi)) \, dx \right| \leq \min \left\{ \frac{C}{\text{dist}(\xi, V_i)}, s - r \right\} \delta H \]
(recall that $|V_i| \leq \delta$ and that $\|F^\pm\|_\infty \leq H$), establishing the lemma in this case. For the second clause we may assume that $|r|$ and $|s|$ are both greater than $1/\delta$, as in the other case the first clause gives a better estimate. We will also assume that $\xi \notin V_i$, as otherwise (7.7) is vacuous. Finally, we may assume that $h(\xi) = 0$, as subtracting a constant from $f$ changes neither $\psi_{f,\tau,\infty}$ nor $\phi_J$, and the only loss in generality is that now we may only assume $\|f\|_\infty \leq 1$ (rather than $\|f\|_\infty \leq 1/2$, which is one of the implicit assumptions of the lemma).

We will work in each half of $U_i$ separately. Denote therefore by $U_i^\pm$ these two halves, and set $V_i^\pm := J^{-1}(U_i^\pm)$ (no relation to the $\pm$ in $F^\pm$, which plays no role here). For concreteness let us work on $U_i^-$, the case of $U_i^+$ being literally identical. By the locality of $\psi$ (see (6.5)), $\phi_{J^\pm}$ restricted to $V_i^-$ is a linearly mapped copy of $\psi$; more precisely,
\[ \phi_{J^\pm}(x) = L_{V_i^-}(\psi_{f\circ L_{U_i^-},\tau_{f\circ L_{U_i^-}}}^{-1}(L_{V_i^-}(x))) \quad \forall x \in V_i^-. \]
Set \( \tilde{f} := f \circ L_{V_i}^{-1} \). Condition on \( \varphi_{\tilde{f}}^{-1}(d) \), where \( d \) is the centre of \( U_i \). Under the conditioning, \( \tau_{\tilde{f}} \circ L_{U_i}^{-1} \) is our standard \( \tau \), i.e., uniform on \([-1,1]\) for all \( d \). Hence

\[
X := \int_{V_i^-} \mathbb{E}(f(\varphi_{\tilde{f}}(x))) \frac{e(r(x - \xi))}{1 - e(x - \xi)} \, dx = \int_{V_i^-} \mathbb{E}(\tilde{f}(\varphi_{\tilde{f},\tau,\infty}(L_{V_i}^{-1}(x)))) \frac{e(r(x - \xi))}{1 - e(x - \xi)} \, dx
\]

where the last equality is obtained by a (linear) change of variables. Let \( \xi^* \) be one of \( \xi, \xi - 1 \) or \( \xi + 1 \), whichever is closest to \( V_i^- \), and note that \( \text{dist}(\xi, V_i) = \inf\{|x - \xi^*| : x \in V_i\} \). Let \( \tilde{\xi} = L_{V_i}^{-1}(\xi^*) \) (note that we are extending \( L_{V_i}^{-1} \) to the whole of \( \mathbb{R} \) here; originally we defined it only on \( V_i^- \), but it is just an affine function and we extend it to an affine function on \( \mathbb{R} \) and \( \tilde{r} = r|V_i^-| \) and with this notation

\[
X = e(-\tilde{\xi})|V_i^-| \int_0^1 \mathbb{E}(\tilde{f}(\varphi_{\tilde{f},\tau,\infty}(x))) \frac{e(\tilde{r}x)}{1 - e(|V_i^-|(x - \xi))} \, dx.
\]

Integrate by parts and get

\[
X = e(-\tilde{\xi})|V_i^-| \left( -\int_0^1 \mathbb{E}(\tilde{f}(\varphi_{\tilde{f},\tau,\infty}(y))) e(\tilde{r}y) \, dy \frac{e(|V_i^-|(x - \xi)) \cdot 2\pi \sqrt{-1}|V_i^-|}{(1 - e(|V_i^-|(x - \xi))|^2) dx} \right)
\]

We now estimate the integrals over \( y \) by Lemma 6.4. We get

\[
\left| \int_0^x \mathbb{E}(\tilde{f}(\varphi_{\tilde{f},\tau,\infty}(y))) e(\tilde{r}y) \, dy \right| \leq C\tilde{r}^{-1/11} = C(r|V_i^-|)^{-1/11} \leq C(r\delta)^{-1/11}.
\]

On the other hand, we have the trivial bound

\[
\left| \int_0^x \mathbb{E}(\tilde{f}(\varphi_{\tilde{f},\tau,\infty}(y))) e(\tilde{r}y) \, dy \right| \leq \|\tilde{f}\|_{\infty} \leq H.
\]

Since \( \min\{a, b\} \leq \sqrt{ab} \) for any two positive numbers \( a \) and \( b \), we can combine these two estimates to get

\[
\left| \int_0^x \mathbb{E}(\tilde{f}(\varphi_{\tilde{f},\tau,\infty}(y))) e(\tilde{r}y) \, dy \right| \leq C(r\delta)^{-1/11}(\sqrt{H}).
\]

We estimate \( |V_i^-| \leq \delta, |1 - e(\theta)| \geq \epsilon \theta \) and get overall

\[
|X| \leq C\delta \left( \int_0^1 (r\delta)^{-1/11}(\sqrt{H}) \frac{\delta}{1 - \xi} \, dx + (r\delta)^{-1/11}(\sqrt{H}) \frac{1}{\delta|1 - \xi|} \right)
\]

\[
\leq C\delta(r\delta)^{-1/11}(\sqrt{H}) \left( \frac{1}{\delta|1 - \xi|} + \frac{1}{\delta|\xi|} + \frac{1}{\delta|1 - \xi|} \right)
\]
(note that we used here our assumption \(\xi \notin V_i\), which gives \(\tilde{\xi} \notin [0,1]\)). For every \(x \in V_i^-\), \(|x - \xi^*| \geq \text{dist}(\xi, V_i)\) and hence

\[
|x - \tilde{\xi}| \geq \frac{\text{dist}(\xi, V_i)}{|V_i^-|} \geq \frac{\text{dist}(\xi, V_i)}{\delta} \quad \forall x \in [0,1].
\]

Hence

\[
|X| \leq \frac{C\delta(r\delta)^{-1/22}\sqrt{H}}{\text{dist}(\xi, V_i)}.
\]

Recalling the definition of \(X\) (see (7.8)) gives

\[
\left| \int_{V_i^-} \mathbb{E}(f(\phi_{J^k}(x)) | \phi_{J^k}(d)) \frac{e(r(x - \xi))}{1 - e(x - \xi)} \, dx \right| \leq \frac{C\delta(r\delta)^{-1/22}\sqrt{H}}{\text{dist}(\xi, V_i)}.
\]

This terminates our estimate for \(V_i^-\). The estimate for \(V_i^+\), as already mentioned, is literally identical, and summing them we get

\[
\left| \int_{V_i} \mathbb{E}(f(\phi_{J^k}(x)) | \phi_{J^k}(d)) \frac{e(r(x - \xi))}{1 - e(x - \xi)} \, dx \right| \leq \frac{C\delta(r\delta)^{-1/22}\sqrt{H}}{\text{dist}(\xi, V_i)}.
\]

Integrating over \(\phi_{J^k}(d)\) gives

\[
\left| \int_{V_i} \mathbb{E}(f(\phi_{J^k}(x))) \frac{e(r(x - \xi))}{1 - e(x - \xi)} \, dx \right| \leq \frac{C\delta(r\delta)^{-1/22}\sqrt{H}}{\text{dist}(\xi, V_i)}.
\]

Adding these estimates for \(r\) and \(s\) gives the result. \(\square\)

**Lemma 7.4.** With the definitions before Lemma 7.3, for every \(r < s\) and every \(\xi\) we have

\[
\left| \int_{V_i} \Delta_i(x) \sum_{l=r}^{s-1} c(l(x - \xi)) \, dx \right| \leq C \min \left\{ \frac{\delta}{\text{dist}(\xi, V_i)}, \delta(s - r) \right\} \cdot 2^{-m/22}.
\]

**Proof.** Essentially, the lemma follows from Lemma 6.5 using the same integration by parts that was used to derive the previous lemma from Lemma 6.4. But let us do it in detail nonetheless. We start by showing

\[
\left| \int_{V_i} \Delta_i(x) \sum_{l=r}^{s-1} c(l(x - \xi)) \, dx \right| \leq C \delta \cdot 2^{-m/22},
\]

which is the more interesting case. Assume \(\xi \notin V_i\), as otherwise the claim is vacuous. Recall that \(\Delta_i = F^+ - F^-\) and that \(F^\pm = \mathbb{E}(f \circ \phi_{J^k}) - h(\xi)\). As in the previous lemma, we may assume that \(h(\xi) = 0\) and \(\|f\|_{\infty} \leq 1\). We use the locality of \(\psi\) (see (6.5)) to get

\[
\phi_{J^k}(x) = L_{U_i}(\psi_{f \circ L_{U_i}, \tau_{j^k}} \circ L_{U_i}, \alpha)(L_{V_i}^{-1}(x)) \quad \forall x \in V_i.
\]
Set \( \tilde{f} := f \circ L_{U_i} \). Note that \( \tilde{r}^\pm := \tau_{f^\pm} \circ L_{U_i} \) has the following distribution: if \( d \neq 1/2 \), then \( \tilde{r}^\pm(d) \) is uniform on \([-1,1]\), while \( \tilde{r}^\pm(1/2) \) is uniform on \( J_{\epsilon^\pm}(d) \), where \( d \) is as usual the centre of \( U_i \). Let

\[
X^\pm := \int_{V_i} P^\pm(x) \frac{e(r(x-\xi))}{1-e(x-\xi)} \, dx = \int_{V_i} \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^\pm,\infty}(L_{V_i}^{-1}(x))) \right) \frac{e(r(x-\xi))}{1-e(x-\xi)} \, dx
\]

\[
= \int_{V_i} \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^\pm,\infty}(x)) \right) \frac{e(r(L_{V_i}^{-1}(x)-\xi))}{1-e(L_{V_i}(x)-\xi)} \, dx.
\]

Define \( \bar{\xi} \) as in the previous lemma with respect to \( V_i \), i.e., \( \bar{\xi} = L_{V_i}^{-1}(\xi^*) \) etc., and \( \tilde{r} = \tau |V_i| \) (also as in the previous lemma) and get

\[
X^\pm = e(-\tilde{r}\bar{\xi}) |V_i| \frac{1}{0} \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^\pm,\infty}(x)) \right) \frac{e(\tilde{r}x)}{1-e(|V_i|(x-\xi))} \, dx.
\]

Integrate by parts and get

\[
X^\pm = e(-\tilde{r}\bar{\xi}) |V_i| \left( -\int_{0}^{1} \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^\pm,\infty}(y)) \right) e(\tilde{r}y) \, dy \frac{e(|V_i|(x-\xi)) \cdot 2\pi \sqrt{-1}|V_i|}{(1-e(|V_i|(x-\xi)))^2} \right.
\]

\[
+ \int_{0}^{1} \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^\pm,\infty}(y)) \right) e(\tilde{r}y) \, dy \frac{1}{1-e(|V_i|(1-\xi))} \right).
\]

This gives us the longest formula in this paper,

\[
\int_{V_i} \Delta_i(x) \frac{e(r(x-\xi))}{1-e(x-\xi)} \, dx = \frac{1}{2} (X^+ - X^-)
\]

\[
= \frac{e(-\tilde{r}\bar{\xi}) |V_i|}{2} \left( -\int_{0}^{1} \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^+,\infty}(y)) \right) - \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^-\infty}(y)) \right) \right) e(\tilde{r}y) \, dy
\]

\[
\times \frac{e(|V_i|(x-\xi)) \cdot 2\pi \sqrt{-1}|V_i|}{(1-e(|V_i|(x-\xi)))^2} \, dx
\]

\[
+ \int_{0}^{1} \left( \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^+,\infty}(y)) \right) - \mathbb{E} \left( \tilde{f}(\psi_{\tilde{r},\tilde{r}^-\infty}(y)) \right) \right) e(\tilde{r}y) \, dy \frac{1}{1-e(|V_i|(1-\xi))} \right).
\]

We now note that each integral over \( y \) above is exactly of the form given by Lemma 6.5, with \( \epsilon = 2^{-m} \). Hence they are bounded by \( C \cdot 2^{-m/22} \). Bounding \( |V_i| \leq \delta \) and \( |1-e(\theta)| \geq c\theta \) gives

\[
\left| \int_{V_i} \Delta_i(x) \frac{e(r(x-\xi))}{1-e(x-\xi)} \, dx \right| \leq C\delta \left( \int_{0}^{1} 2^{-m/22} \frac{\delta}{\delta^2|x-\xi|^2} \, dx + 2^{-m/22} \frac{1}{\delta|1-\xi|} \right)
\]

\[
\leq C\delta \cdot 2^{-m/22} \left( \frac{1}{\delta|1-\xi|} + \frac{1}{\delta|\xi|} \right).
\]
Again \( |x - \bar{\xi}| \geq \text{dist}(\xi, V_i) / \delta \) for all \( x \in [0, 1] \), so
\[
\left| \int_{V_i} \Delta_i(x) \frac{e(r(x - \xi))}{1 - e(x - \xi)} \, dx \right| \leq C \delta \cdot 2^{-m/22} \frac{C_1}{\text{dist}(\xi, V_i)}.
\]

Summing the \( r \) and \( s \) terms gives
\[
\left| \int_{V_i} \Delta_i(x) \sum_{l=r}^{s} e(l(x - \xi)) \, dx \right| \leq C \delta \cdot 2^{-m/22} \frac{1}{\text{dist}(\xi, V_i)}.
\]

This is the main estimate of the lemma.

We still need to show the simpler estimate \( C \delta (s - r) \cdot 2^{-m/22} \). We keep the notation \( \tilde{f}, \tilde{\tau} \) and \( \bar{\xi} \). For every \( l \in \{r, \ldots, s - 1\} \) we define \( l := l|V_i| \) and then the same locality argument that gave (7.9) gives
\[
\int_{V_i} \Delta_i(x) e(l(x - \xi)) \, dx = e(-l \bar{\xi})|V_i| \left( \mathbb{E} \left( \tilde{f}(\psi_{\tilde{\tau}+,\infty}(x)) \right) - \mathbb{E} \left( \tilde{f}(\psi_{\tilde{\tau}-,\infty}(x)) \right) \right) e(lx) \, dx.
\]

We apply Lemma 6.5 directly (without integration by parts) and get
\[
\left| \int_{V_i} \Delta_i(x) e(l(x - \xi)) \, dx \right| \leq C \delta \cdot 2^{-m/22}.
\]

We sum over \( l \) from \( r \) to \( s - 1 \). This gives the second bound, and proves the lemma.

Aggregating the last two lemmas, we get

**Lemma 7.5.** With the definitions before Lemma 7.3, for every \( r < s \) and every \( \xi \) we have
\[
\left| \int_{V_i} \Delta_i(x) \sum_{l=r}^{s-1} e(l(x - \xi)) \, dx \right| \leq C_1 \delta \min \left\{ \frac{\min\{ \lfloor s|\delta| \rfloor, \lfloor r|\delta| \rfloor \}}{\text{dist}(\xi, V_i)}, s - r \right\} \cdot 2^{-m/44} H^{1/4}.
\]

**Proof.** Define \( D = D(x) = \sum_{l=r}^{s-1} e(l(x - \xi)) \). Assume first that both \( |r| \) and \( |s| \) are greater than \( 1/\delta \). We use Lemma 7.3 for both \( F^+ \) and \( F^- \), and summing the result gives
\[
\left| \int_{V_i} \Delta_i D \right| \leq C \delta \frac{\min\{ \lfloor s|\delta| \rfloor, \lfloor r|\delta| \rfloor \}}{\text{dist}(\xi, V_i)} H^{1/2},
\]
while Lemma 7.4 gives
\[
\left| \int_{V_i} \Delta_i D \right| \leq C \frac{2^{-m/22}}{\text{dist}(\xi, V_i)}.
\]

For any positive numbers \( a \) and \( b \), \( \min\{a, b\} \leq \sqrt{ab} \), so
\[
\min\{2^{-m/22}, \lfloor |r|\delta \rfloor^{-1/22} H^{1/2} \} \leq 2^{-m/44} \lfloor |r|\delta \rfloor^{-1/44} H^{1/4},
\]
and similarly for \( s \). Applying this to whichever of \( r \) and \( s \) has a smaller absolute value gives
\[
\left| \int_{V_i} \Delta_i D \right| \leq C \delta \frac{\min\{ \lfloor s|\delta| \rfloor, \lfloor r|\delta| \rfloor \}}{\text{dist}(\xi, V_i)} \cdot 2^{-m/44} H^{1/4},
\]

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as needed. The case where either $|r| \leq 1/\delta$ or $|s| \leq 1/\delta$ is similar but simpler. Lemma 7.3 gives the estimate $|f \Delta D| \leq \delta(s - r)H$, Lemma 7.4 gives the estimate $|f \Delta D| \leq C\delta(s - r)/2^{m/22}$, and we combine them as above. The lemma is thus proved.

The last step in proving Lemma 7.2 is to choose the $\varepsilon_i$. Recall that we need to choose an $\varepsilon_i \in \{\pm 1\}$ for each $i \in Y = \{i: |V_i| > \delta/2\}$. It will be convenient to add dummy variables, so we will choose $\varepsilon_i$ for every $i$, and ignore those outside $Y$. Denote by $N$ the number of $U_i$ in our partition, and $\Delta_i \equiv 0$ for every $i \notin Y$. Define

$$\Delta(x) = \sum_{i=1}^{N} \varepsilon_i \Delta_i(x).$$

The $\varepsilon_i$ will be chosen using Lemma 3.1. Here are the details.

**Lemma 7.6.** There exists an $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)$ such that for every $u \in \mathbb{N}$, every $r \in \{2^{u-1}, \ldots, 2^u - 1\}$ and every $\xi \in 2^{-u-2} \cap [0, 1)$,

$$\left| \int_{E_\xi} \Delta(x) D_r(x - \xi) dx \right| \leq C \left( \min \left\{ 2^{u} \delta, \frac{1}{2^{u} \delta} \right\} \right)^c \cdot 2^{-cm \omega_f(\delta^c)^c}.$$

**Proof.** We need to prepare a sequence of vectors to “feed” into Lemma 3.1, which will give us our signs. The estimates of these vectors will come from Lemma 7.5.

We now define our vectors. For every $u \in \mathbb{N}$ let $U$ be the smallest power of 2 such that $U \geq \max\{2^{u+2}, N\}$. For each $i \in \{1, \ldots, N\}$ and each $\xi \in \{0, 1/U, \ldots, (U - 1)/U\}$, we define

$$w_{i,\xi,u}^0 = \int_{V_i} \Delta_i(x) D_{2^u}(x - \xi) dx$$

except if $2^u > 1/\delta$ and $V_i \subset B_{\xi}$ ($B_{\xi}$ from the statement of Lemma 7.2), in which case we define $w_{i,\xi,u}^0 = 0$. Next, for every $u \in \mathbb{N}$, every $0 \leq s < u$, every $t \in [2^{u-1}, 2^u]$ divisible by $2^{s+1}$, every $i \in \{1, \ldots, N\}$ and every $\xi \in \{0, 1/U, \ldots, (U - 1)/U\}$, we define

$$w_{i,\xi,u,s,t}^+ = \int_{V_i} \Delta_i(x) \sum_{z=t+1}^{t+2^s} e(z(x - \xi)) dx, \quad w_{i,\xi,u,s,t}^- = \int_{V_i} \Delta_i(x) \sum_{z=-t-2^s}^{-t-1} e(z(x - \xi)) dx,$$

again except if $2^u > 1/\delta$ and $V_i \subset B_{\xi}$, in which case we define $w_{i,\xi,u,s,t}^\pm = 0$. This terminates the list of vectors (we think about them as vectors in $i$) that we wish to feed into Lemma 3.1, after some scaling.

We note the following convenient fact. For $\xi \in [0, 1]$ define $l(\xi)$ by $\xi \in V_l(\xi)$. Then

$$\text{dist}(\xi, V_i) \geq c\delta \left( |l(\xi) - i| \text{ mod } N \right) + 1 \quad \text{whenever } V_i \not\subset B_{\xi}. \quad (7.10)$$

(We used here the fact that dist is defined cyclically and that $|V_i| > \delta/4$ for all $i$.) Similarly,

$$\text{dist}(\xi, V_i) \leq C\delta \left( |l(\xi) - i| \text{ mod } N \right) + 1, \quad (7.11)$$

this time without the restriction $V_i \not\subset B_{\xi}$. Another step that will simplify the analysis below relates to the term $H$ from Lemma 7.5. Recall that was defined in (7.6) by

$$H = \omega_f(C(\text{dist}(\xi, V_i) + \delta)^{4/5}) \leq \min \left\{ 1, C\omega_f(\delta^{4/5}) \left( \frac{\text{dist}(\xi, V_i) + \delta}{\delta} \right)^{4/5} \right\} \leq \min \left\{ 1, C\omega_f(\delta^{4/5}) \left( \frac{\text{dist}(\xi, V_i)}{\delta} + 1 \right)^{4/5} \right\} \leq \min \left\{ 1, C\omega_f(\delta^{4/5}) \left( |l(\xi) - i| \text{ mod } N \right) + 1 \right\}.$$
Define
\[
\lambda_l := 2^{-m/44} \left( \min \left\{ 1, \left( |l \text{ mod } N| + 1 \right) \omega_f(\delta^{4/5}) \right\} \right)^{1/4}
\]  
and get
\[
2^{-m/44} H^{1/4} \leq C\lambda_{l(\xi) - i}.
\]  
This will simplify the notation in the estimates below.

We start with \( w_{i,\xi,u}^0 \) for \( 2^u \leq 1/\delta \). By Lemma 7.5 and (7.13)
\[
|w_{i,\xi,u}^0| \leq C\delta \min \left\{ \frac{1}{\text{dist}(\xi, V_i)}, 2^{u+1} \right\} \lambda_{l(\xi) - i} < C \min \left\{ \frac{1}{|l(\xi) - i \text{ mod } N| + 1}, 2^{u+1}\delta \right\} \lambda_{l(\xi) - i},
\]
where \((\ast)\) comes from (7.10) and from the observation that if \( V_i \subset B_\xi \) then the minimum is achieved at \( 2^{u+1} \) (perhaps up to a multiplicative constant). Define \( \tilde{b}(u, \xi) = b(u) = \max\{1, \lfloor 1/(2^{u+1}\delta) \rfloor \} \) and get
\[
|w_{i,\xi,u}^0| \leq \min \left\{ \frac{1}{|l(\xi) - i \text{ mod } N| + 1}, \frac{1}{b(u)} \right\} C\lambda_{l(\xi) - i}.
\]  
We used here \( \tilde{b} \) instead of \( b \) because of the factor \( C\lambda_{l(\xi) - i} \). Since it appears in all cases equally, it will be more convenient to count how many vectors satisfy an inequality of the form (7.14), for any fixed values of \( l \) and \( \tilde{b} \), and then remove the factor \( C\lambda_{l(\xi) - i} \) at the end. To count the number of \( \xi \) with \( l(\xi) = l \) for some given \( l \), we note that \( \xi \in U^{-1}\mathbb{Z} \), and since \( U \leq \max\{2^{u+2}, 2N\} \leq \max\{4/\delta, 2N\} \leq 8/\delta \) and \( |V_i| \leq \delta \) for all \( i \), we see that each possible value of \( l(\xi) \) is repeated at most nine times. Thus
\[
\left| \left\{ \left( \xi, u \right): 2^u \leq \frac{1}{\delta}, l(\xi) = l, \tilde{b}(u) = \tilde{b} \right\} \right| \leq 18 \quad \forall l, \tilde{b},
\]
because we have at most nine different \( \xi \) which give the same \( l \) and at most two different values of \( u \) which give the same \( \tilde{b} \) (the largest allowed by the condition \( 2^u \leq 1/\delta \)).

For \( 2^u > 1/\delta \), Lemma 7.5, (7.10) and (7.13) give
\[
|w_{i,\xi,u}^0| \leq C \frac{(2^u\delta)^{-1/44} \lambda_{l(\xi) - i}}{|l(\xi) - i \text{ mod } N| + 1},
\]
so we define \( \tilde{b}(u) = \lfloor (2^u\delta)^{1/44} \rfloor \). The multiplicity in \( \tilde{b} \) is at most a constant in this case, but there is multiplicity \( \lfloor \delta U \rfloor + 1 \leq C \cdot 2^4\delta \) in \( l \), because \( \xi \) is taken in \( U^{-1}\mathbb{Z} \). Hence we get at most \( C \cdot 2^4\delta \leq C\tilde{b}(u)^{44} \) vectors satisfying (7.14).

For \( w^\pm \) and \( 2^u \leq 1/\delta \), Lemma 7.5, (7.10) and (7.13) give
\[
|w_{i,\xi,u,s,t}^\pm| \leq C \min \left\{ \frac{1}{|l(\xi) - i \text{ mod } N| + 1}, 2^s \delta \right\} \lambda_{l(\xi) - i}.
\]

Hence we define \( \tilde{b}(\xi, u, s, t) = \tilde{b}(s) = \lfloor 1/(2^s\delta) \rfloor \) and get
\[
|w_{i,\xi,u,s,t}^\pm| \leq \min \left\{ \frac{1}{|l(\xi) - i \text{ mod } N| + 1}, \frac{1}{\tilde{b}} \right\} C\lambda_{l(\xi) - i}.
\]  
The same argument as above shows that each \( l(\xi) \) is repeated at most nine times, but here we need to count over \( u \) and \( t \) as well (with \( s \) fixed). The number of possibilities for \( t \) given \( u \) and \( s \) is \( 2^{u-s-2} \), and \( u \) is bounded below by \( s+1 \) and above by the requirement \( 2^u \leq 1/\delta \). Totally we get
\[
\left| \left\{ \left( \xi, u, s, t \right): l(\xi) = l, \tilde{b}(s) = \tilde{b} \right\} \right| \leq 9 \sum_{u \geq s+1} 2^{u-2-s} < \frac{9}{2^{s+1}\delta} \leq \frac{9}{2} \tilde{b}(s)
\]
for all \( l \) and \( \tilde{b} \).
Finally, the most complicated case is that of the vectors \( w^\pm \) for \( 2^u > 1/\delta \). Lemma 7.5, (7.10) and (7.13) give

\[
|w^\pm_{i,\xi,u,s,t}| \leq C \min \left\{ \left( \frac{(2^u \delta)^{-1/44}}{|l(\xi) - i \mod N| + 1} \right)^{2u}, \lambda_{l(\xi) - i}. \right\}
\]  

(7.16)

We define \( \tilde{b}(\xi, u, s, t) = \max \{ 1/(2^u \delta), (2^u \delta)^{1/44} \} \), so (7.15) holds. We still need to add up the number of vectors that correspond to each bound in (7.16). The number of \( \xi \) which give every particular \( l \) is bounded by \( \lfloor \delta U \rfloor + 1 \leq C \cdot 2^u \delta \). As for the multiplicity in \( \tilde{b} \), each value can be achieved either if \( 2^u \delta \in (1/(\tilde{b} + 1), 1/\tilde{b}] \) and \( (2^u \delta)^{-1/44} \in (1/(\tilde{b} + 1), \infty) \) or if \( 2^u \delta \in (1/(\tilde{b} + 1), \infty) \) and \( (2^u \delta)^{-1/44} \in (1/(\tilde{b} + 1), 1/\tilde{b}] \), and we need to check, in each case, how many possibilities for \( u, s, t \) and \( \xi \) we have.

In the first case, we get \( 2^u \delta < (\tilde{b} + 1)^{44} \). This has two implications. First, the number of \( \xi \) which satisfy \( 1 < 2^u \delta \leq (\tilde{b} + 1)^{44} \) can be bounded by \( C \log \tilde{b} \). Second, by the above, each \( l(\xi) \) is repeated at most \( C(\tilde{b} + 1)^{44} \) times for each \( t \) and \( s \). Further, the restriction \( 2^u \delta \in (1/(\tilde{b} + 1), 1/\tilde{b}] \) implies that there is at most one possibility for \( s \). The number of possibilities for \( t \) given \( u \) and \( s \) is always \( 2^{u-s-2} \), and this can be bounded by

\[
\frac{2^u}{2^s} \leq \frac{(\tilde{b} + 1)^{44}/\delta}{(\tilde{b} + 1)^{-1/\delta}} = (\tilde{b} + 1)^{45}.
\]

So how many \( (s, t, u, \xi) \) are there that satisfy \( |w^\pm_{i,\xi,u,s,t}| \leq \min\{ 1/(|i-l \mod N| + 1), \tilde{b} \} \) for given \( l \) and \( \tilde{b} \)? As just explained, there are \( C \log \tilde{b} \) possibilities for \( u \), \( C\tilde{b}^{44} \) possibilities for \( \xi \) and \( C\tilde{b}^{45} \) possibilities for \( t \) and \( s \), so in total we get at most \( C\tilde{b}^{89} \log \tilde{b} \leq C\tilde{b}^{90} \) possibilities.

In the second case, \( (2^u \delta)^{-1/44} \in (1/(\tilde{b} + 1), 1/\tilde{b}] \), so the number of possibilities for \( u \) is bounded, and they all satisfy \( 2^u \delta < (\tilde{b} + 1)^{44} \). As for \( s \), it holds that \( 2^s \) ranges between \( 1/(\delta(\tilde{b} + 1)) \) and \( 2^u \leq (\tilde{b} + 1)^{44}/\delta \). For each \( s \) the number of possible \( t \) is \( 2^{u-s-2} \) so the total number of possibilities for \( (s, t) \) couples is bounded by

\[
\sum_{s = \lfloor \log_2(1/(\delta(\tilde{b} + 1))) \rfloor}^{\lfloor \log_2((\tilde{b} + 1)^{44}/\delta) \rfloor} 2^{u-s-2} < 2^{u-\lfloor \log_2((\delta(\tilde{b} + 1))) \rfloor-1} \leq \frac{C(\tilde{b} + 1)^{44}/\delta}{2/(1/\delta)} \leq C\tilde{b}^{45}.
\]

Together with the bound on the number of times each \( l(\xi) \) is repeated, we get a total of \( C\tilde{b}^{89} \) possibilities. This concludes our bounds. We rearrange all our vectors \( w^0_{i,\xi,u} \) and \( w^+_{i,\xi,u,s,t} \) into one list \( \tilde{v}_{i,j} \), and define \( l(j) = l(\xi, u) \) or \( l(\xi, u, s, t) \), as the case may be, and similarly for \( \tilde{b} \). Estimates (7.14) and (7.15) become

\[
|\tilde{v}_{i,j}| \leq C_1 \lambda_{l(j) - i} \min \left\{ \frac{1}{|l(j) - i \mod n| + 1}, \frac{1}{\tilde{b}(j)} \right\}
\]  

(7.17)

and \(|\{j : l(j) = l, \tilde{b}(j) = \tilde{b}\}| \leq C\tilde{b}^{90} \).

It is now time to handle the factors \( C_1 \lambda_{l(j) - i} \). Plugging (7.12) into the right-hand side of (7.17), we have

\[
|\tilde{v}_{i,j}| \leq C_1 \cdot 2^{-m/44} \min \left\{ \frac{1}{|l(j) - i \mod n| + 1}, \frac{1}{\tilde{b}(j)} \right\} \times \left( \min \{ 1, |(l(j) - i \mod n| + 1) \omega_f(\delta^{4/5}) \} \right)^{1/4}
\]  

\[
\leq C_1 \cdot 2^{-m/44} \min \left\{ \frac{1}{|l(j) - i \mod n| + 1}, \min \left\{ \frac{1}{\tilde{b}(j)}, \omega_f(\delta^{4/5})^{1/5} \right\} \right\}.
\]
Define \( b(j) := \max\{b(j), [\omega_f(\delta^{4/5})^{-1/5}] \} \), so that \( |v_{i,j}| \leq C_1 \cdot 2^{-m/44}/b(j) \). How many \( j \) correspond to given \( l \) and \( b \)? We can bound the number simply by \( \sum_{b \leq \tilde{b}} \tilde{C}^{b/0} \leq \tilde{C}^{b/1} \). Denote this last constant by \( M \), so we have \( Mb^{91} \) vectors that correspond to \( b \).

Thus we may use Lemma 3.1 with the same \( M \), with \( \gamma = 91 \) and with the vectors \( v_{i,j} = \tilde{v}_{i,j} \cdot 2^{m/44}/C_1 \). We get a sequence of \( \varepsilon_i \) such that their inner product with each vector we fed into the lemma is bounded by \( \tilde{C} b^{-1/50} \). We estimate

\[
\frac{1}{b} = \min \left\{ \frac{1}{b}, \left[ \omega_f(\delta^{4/5})^{-1/5} \right] \right\} \leq \sqrt{b \max \left\{ \frac{\omega_f(\delta^{4/5})^{-1/5}}{1}, 1 \right\}} \leq \frac{C \omega_f(\delta^{4/5})^{1/10}}{b^{1/2}},
\]
or \( b^{-1/50} \leq \tilde{C}^{-1/100} \omega_f(\delta^{4/5})^{1/500} \). Inserting the definition of \( \tilde{b} \), we get

\[
\sum_i \varepsilon_i w_{i,\xi,u}^0 \leq C \cdot 2^{-m/44} \omega_f(\delta^{4/5})^{1/500} \times \left\{ \begin{array}{ll}
(2^u \delta)^{1/100}, & 2^u \leq \delta^{-1}, \\
(2^u \delta)^{-1/4400}, & 2^u > \delta^{-1},
\end{array} \right.
\]

\[
\sum_i \varepsilon_i w_{i,\xi,u,s,t}^\pm \leq C \cdot 2^{-m/44} \omega_f(\delta^{4/5})^{1/500} \min\{ (2^u \delta)^{-1/44}, 2^s \delta \}^{1/100}.
\]

To finish the proof of Lemma 7.6, assume we are given some \( u \in \mathbb{N} \) and some \( r \in \{2^{u-1}, \ldots, 2^u - 1\} \). Write \( r \) in its binary expansion

\[
r = 2^{u-1} + \sum 2^{s_k}, \quad t_k := 2^{u-1} + 2^{s_1} + \ldots + 2^{s_k},
\]

for some decreasing sequence \( s_k \). We get

\[
D_r(x) = D_{2^{u-1}}(x) + \sum_k \left( \sum_{z = t_{k-1} + 1}^{t_k} e(zx) + \sum_{z = -t_k}^{-t_{k-1}-1} e(zx) \right).
\]

Hence for any \( \xi \in \{0, 1/U, \ldots, (U - 1)/U\} \) we have

\[
\int_{E_\xi} \Delta(x)D_r(x - \xi) \, dx = \sum_i \varepsilon_i w_{i,\xi,u-1}^0 + \sum_k \varepsilon_i w_{i,\xi,u,\xi,\xi,s_k,t_k-1}^+ + \sum_k \sum_i \varepsilon_i w_{i,\xi,u,\xi,\xi,s_k,t_k-1}^-
\]

(our assumption that \( V_i \nsubseteq B_\xi \) whenever \( 2^u > 1/\delta \) is satisfied on \( E_\xi \); recall its definition (7.1)).

With (7.18) we get

\[
\left| \int_{E_\xi} \Delta(x)D_r(x - \xi) \, dx \right| \leq C \cdot 2^{-m/44} \omega_f(\delta^{4/5})^{1/500} \min\{ (2^u \delta)^{1/100}, \log_2(2^u \delta)(2^u \delta)^{-1/4400} \}
\]

(the \( \log_2(2^u \delta) \) in the second estimate comes from the case when \( 2^u > 1/\delta \) in (7.18), from the counting on \( s \), because, as \( s \) decreases from \( u \) towards zero, there are \( \approx \log(2^u \delta) \) values of \( s \) for which \( \min\{ (2^u \delta)^{-1/44}, 2^s \delta \} \) is constant and equal to \( (2^u \delta)^{-1/44} \), and only then it starts to decrease exponentially). Lemma 7.6 is thus proved, and due to (7.2), so is Lemma 7.2. \( \Box \)

**Proof of Theorem 1.2.** Recall that the theorem states that for every continuous function \( f \) there is an absolutely continuous homeomorphism \( \psi \) such that \( S_t(f \circ \psi) \) converges uniformly. Assume without loss of generality that \( \| f \|_\infty \leq 1/2 \). We apply Lemma 7.2 in a doubly infinite process. We start with the map \( I(d) = [-1, 1] \) for all \( d \), which is of type \( (f, 1) \) with respect to the trivial
partition \{[0, 1]\} and to \(m = -1\). We apply Lemma 7.2 and get a map \(J_0\) which is of type \((f, 1)\) with respect to \{0, 1\} and to \(m = 0\) such that for every \(u, r\) and \(\xi\) as in Lemma 7.2

\[
\left| \int_{E_\xi} \left( \mathbb{E}(f(\phi_I(x))) - \mathbb{E}(f(\phi_{I_0}(x))) \right) D_r(x - \xi) \, dx \right| \leq C \cdot 2^{-\alpha u}.
\]

We apply Lemma 7.2 to \(J_0\) and get a \(J_1\), apply it again to get \(J_2\) and so on, and for each \(m\) we have, for every \(u, r\) and \(\xi\),

\[
\left| \int_{E_\xi} \left( \mathbb{E}(f(\phi_{I_m}(x))) - \mathbb{E}(f(\phi_{I_{m+1}}(x))) \right) D_r(x - \xi) \, dx \right| \leq C \cdot 2^{-\alpha u - \alpha m}.
\]

The limit of the \(J_m\) (by which we mean simply the limit for each dyadic \(d\) individually), denote it by \(J_\infty\), is such that \(J_\infty(1/2)\) is a single point. Summing the errors gives

\[
\left| \int_{E_\xi} \left( \mathbb{E}(f(\phi_{I}(x))) - \mathbb{E}(f(\phi_{I_\infty}(x))) \right) D_r(x - \xi) \, dx \right| \leq C \cdot 2^{-\alpha u}.
\]

Now, because \(J_\infty(1/2)\) is a single point, it follows that \(J_\infty\) is of type \((f, 5/8)\) with respect to the partition \{[0, 1/2], [1/2, 1]\} and \(m = -1\) (the value 5/8 comes from (6.1)).

Now define \(I_2 = J_\infty\) and repeat this procedure. Assume inductively we have discovered some \(I_k\) which is of type \((f, \delta_k)\) with respect to a partition \(\mathcal{U}_k\) and \(m = -1\). We apply Lemma 7.2 infinitely many times as just explained, get a sequence of \(J_{k,m}\) for \(m \in \mathbb{Z}^+\) and take the limit as \(m \to \infty\). We get, for every \(u, \xi\) and \(r\),

\[
\left| \int_{E_\xi} \left( \mathbb{E}(f(\phi_{I_k}(x))) - \mathbb{E}(f(\phi_{I_{k,\infty}}(x))) \right) D_r(x - \xi) \, dx \right| \leq C \left( \min \left\{ 2^{u} \delta_k, \frac{1}{2u} \delta_k \right\} \right)^c \omega_f(\delta_k)^c.
\]

Now \(J_{k,\infty}\) is a single point at the centre of every \(U \in \mathcal{U}_k\) which satisfies \(|\phi_{I_k}^{-1}(U)| > \delta_k/2\), and therefore \(J_{k,\infty}\) is of type \((f, \delta_{k+1})\), where \(\delta_{k+1} := 5\delta_k/8\), with respect to the partition \(\mathcal{U}_{k+1}\) that one gets by splitting each \(U \in \mathcal{U}_k\) into halves if \(|\phi_{I_k}^{-1}(U)| > \delta_k/2\) and with respect to \(m = -1\). Checking most conditions for an RH-restrictor of type \((f, \delta_{k+1})\) is straightforward, so we check only property (ii), i.e., that \(|\phi_{I_k}^{-1}(U)| \in [\delta_{k+1}/4, \delta_{k+1}]\) for all \(U \in \mathcal{U}_{k+1}\). There are two cases to check. If for some \(U \in \mathcal{U}_k\) we have \(|\phi_{I_k}^{-1}(U)| \leq \delta_k/2\), then \(U \in \mathcal{U}_{k+1}\), \(|\phi_{I_k}^{-1}(U)| = |\phi_{I_{k,\infty}}^{-1}(U)|\) and

\[
\frac{1}{4} \delta_{k+1} < \frac{1}{4} \delta_k \leq \frac{1}{2} \delta_k < \delta_{k+1}.
\]

The other case is \(|\phi_{I_k}^{-1}(U)| > \delta_k/2\). In this case each half of \(U\) belongs to \(\mathcal{U}_{k+1}\), and we denote these halves by \(U_{\pm}\). We use (6.1) and get

\[
|\phi_{J_{k,\infty}}(U_{\pm})| \leq \frac{5}{8}|\phi_{J_{k,\infty}}(U)| = \frac{5}{8}|\phi_I(U)| \leq \frac{5}{8} \delta_k = \delta_{k+1},
\]

\[
|\phi_{J_{k,\infty}}(U_{\pm})| \geq \frac{3}{8}|\phi_{J_{k,\infty}}(U)| = \frac{3}{8}|\phi_I(U)| > \frac{3}{8} \cdot \frac{1}{2} \delta_k > \frac{5}{32} \delta_k = \frac{1}{4} \delta_{k+1}.
\]

So property (ii) is proved. This allows us to define \(I_{k+1} = J_{k,\infty}\) and continue inductively.

We still have the issue of the integration interval. Indeed, we showed

\[
\left| \int_{E_\xi(\mathcal{U}_{k+1} \setminus I_{k+1}, f, \delta_k)} \left( \mathbb{E}(f(\phi_{I_k}(x))) - \mathbb{E}(f(\phi_{I_{k+1}}(x))) \right) D_r(x - \xi) \, dx \right| \leq C \left( \min \left\{ 2^{u} \delta_k, \frac{1}{2u} \delta_k \right\} \right)^c \omega_f(\delta_k)^c, \quad (7.19)
\]
but we wish to compare it to the integral over $E_\xi(U_{k+1}, I_{k+1}, f, \delta_{k+1})$. Set $E_k = E_\xi(U_{k+1}, I_k, f, \delta_k)$ and $E_{k+1} = E_\xi(U_{k+1}, I_{k+1}, f, \delta_{k+1})$ for brevity. Now, if $2^u \leq 1/\delta_k$, then $E_{k+1} = E_k = [0, 1]$ and there is nothing for us to do. Otherwise $E_k \subset E_{k+1}$ and if $1/\delta_k < 2^u \leq 1/\delta_{k+1}$, then $E_{k+1} \setminus E_k$ is a single interval (a union of three intervals of the type $\phi_{I_{k+1}}^{-1}(U)$, $U \in U_{k+1}$), while if $2^u > 1/\delta_{k+1}$, then it is a union of some number (from zero to three) of intervals of the type $\phi_{I_{k+1}}^{-1}(U)$, $U \in U_{k+1}$, depending on details of the splitting (these intervals might or might not be neighbours; this makes no difference to us). In all cases we apply Lemma 7.3 for the RH-restrictor $I_{k+1}$, for both $F^+$ and $F^-$ and for all relevant values of $i$. We get

$$\frac{1}{2}(F^+(x) + F^-(x)) = E(f(\phi_{I_{k+1}}(x))) - E(f(\phi_{I_{k+1}}(\xi))).$$

(Notice that we do not subtract them as we did to get $\Delta$, but sum them.) Thus Lemma 7.3 allows us to estimate $\int E(f \circ \phi_{I_{k+1}})$, and we get

$$\left| \int_{E_{k+1} \setminus E_k} E(f(\phi_{I_{k+1}}(x)) - f(\phi_{I_{k+1}}(\xi))) D_r(x - \xi) \, dx \right| \leq \min\left\{ (2r + 1)\delta_{k+1} H, (\delta_{k+1} r)^{-1/2} \sum_{V_i \subseteq E_{k+1} \setminus E_k} C \delta_{k+1} \sqrt{H} \right\} \sqrt{\delta_{k+1}} \sqrt{H} \leq C \left( \min\left\{ 2^u \delta_k, \frac{1}{2^u \delta_k} \right\} \right)^{c/2} \sqrt{\delta_{k+1}} \sqrt{H}, \quad (7.20)$$

where the inequality marked by $(*)$ follows because usually $\operatorname{dist}(\xi, V_i) \approx \delta_{k+1}$ for $V_i \subseteq E_{k+1} \setminus E_k$. The only exception is $1/\delta_k < 2^u \leq 1/\delta_{k+1}$, where $\operatorname{dist}(\xi, V_i)$ might be zero, in which case we use the estimate $(2r + 1)\delta H$ (we changed $\delta_{k+1}$ to $\delta_k$ for short; this only affects the constant). The inequality marked by $(\dagger)$ follows again because $\operatorname{dist}(\xi, V_i) \leq C \delta_{k+1}$ for the relevant intervals. We make one last observation, namely,

$$\int_{E_k} \left( E(f(\phi_I(\xi))) - E(f(\phi_{I_{k+1}}(\xi))) \right) D_r(x - \xi) \, dx$$

$$= \left( E(f(\phi_I(\xi))) - E(f(\phi_{I_{k+1}}(\xi))) \right) \int_{E_k} D_r(x - \xi) \, dx$$

$$= \left\{ \begin{array}{ll}
O\left( \frac{\omega_f(\delta_I)}{2^u \delta_k} \right), & 2^u > \delta_k^{-1}, \\
E(f(\phi_I(\xi))) - E(f(\phi_{I_{k+1}}(\xi))), & 2^u \leq \delta_k^{-1}
\end{array} \right. \quad (7.21)$$

(notice that in the second case this is an equality and not an estimate). Adding (7.19)–(7.21) gives

$$\int_{E_\xi(U_{k+1}, I_k, f, \delta_k)} E(f(\phi_I(x)) - f(\phi_I(\xi))) D_r(x - \xi) \, dx$$

$$- \int_{E_\xi(U_{k+1}, I_{k+1}, f, \delta_{k+1})} E(f(\phi_{I_{k+1}}(x)) - f(\phi_{I_{k+1}}(\xi))) D_r(x - \xi) \, dx$$

$$= \mathbb{1}_{2^u \leq 1/\delta_k} \left( E(f(\phi_I(\xi))) - E(f(\phi_{I_{k+1}}(\xi))) \right) + O\left( \left( \min\left\{ 2^u \delta_k, \frac{1}{2^u \delta_k} \right\} \right)^{c/2} \omega_f(\delta_k)^c \right). \quad (7.22)$$
Fix $u \in \mathbb{N}$, $r \in \{2^{u-1}, \ldots, 2^u - 1\}$ and $\xi \in \{0, 1/2^u, \ldots, (2^u - 1)/2^u\}$. As $k \to \infty$, the terms $(\min\{2^u \delta_k, 1/(2^u \delta_k)\})^c$ start small, increase exponentially (recall that $\delta_k = (5/8)^k$) until reaching a constant and then decrease exponentially. Thus, if we summed them (without the term $\omega(f(\delta_k)^c)$), the sum would be constant. Multiplying also by $\omega$, we get

$$\sum_{k=1}^{\infty} \left(\min\{2^u \delta_k, \frac{1}{2^u \delta_k}\}\right)^c \omega(f(\delta_k)^c) \to 0$$

as $u \to \infty$. For the first term on the right-hand side of (7.22), we claim that for any $\ell$

$$\left|\sum_{k=1}^{\ell} 2^{u-1}/\delta_k \left(\mathbb{E}(f(\phi_{I_k}(\xi))) - \mathbb{E}(f(\phi_{I_{k+1}}(\xi)))\right)\right| \leq \omega_f(C \cdot 2^{-4u/5}).$$

(7.23)

Indeed, this is a telescoping sum and hence it is equal either to zero or to the difference of two averages of $f$ by measures supported in an interval of length at most $C \cdot 2^{-4u/5}$ (recall the deterministic H"older estimate (6.2)). This shows (7.23). We conclude that

$$\left|\int_{E_\xi(\mathcal{H}_k, I_k, f, \delta_k)} \mathbb{E}(f(\phi_{I_k}(x)) - f(\phi_{I_{k+1}}(\xi))) D_r(x - \xi) \, dx\right| \leq \varepsilon(u),$$

where $\varepsilon(u)$ is some function with $\varepsilon(u) \to 0$ as $u \to \infty$ ($\varepsilon$ depends on $f$ but not on anything else; in particular, it does not depend on $k, r$ or $\xi$). Taking $k \to \infty$, we find that the $\phi_{I_k}$ converge weakly (as measures on $C([0,1])$ with the supremum norm) to a delta measure on a single homeomorphism $\Phi$, while $E_\xi(\mathcal{H}_k, I_k, f, \delta_k)$ stabilises on $[0,1]$ as soon as $\delta_k < 2^{-u}$. Hence weak convergence gives

$$\left|\int_0^1 (f(\Phi(x)) - f(\Phi(\xi))) D_r(x - \xi) \, dx\right| \leq \varepsilon(u),$$

or equivalently,

$$|S_r (f \circ \Phi)(\xi) - f(\Phi(\xi))| \leq \varepsilon(u).$$

(7.24)

Inequality (7.24) implies a uniform bound, namely,

$$\|S_r (f \circ \Phi) - f \circ \Phi\|_\infty \to 0.$$  

(7.25)

The passage from (7.24) to (7.25) is a well-known argument, but let us do it in detail anyway. We use Bernstein’s inequality for trigonometric polynomials. Recall that this inequality states that for any trigonometric polynomial $P$ of degree $n$, $\|P\|_\infty \leq 2\pi n\|P\|_\infty$. We apply Bernstein’s inequality to $P_r := S_r (f \circ \Phi) - F_r (f \circ \Phi)$, where $F_r$ is the Ces"aro partial sum of $f \circ \Phi$. Since $F_r (f \circ \Phi) \to f \circ \Phi$ uniformly, we get $|P_r(\xi)| \leq \varepsilon_2(u)$ for all $\xi \in 2^{-u-2}\mathbb{Z}$. Let $x$ be the point where $|P_r|$ attains its maximum, and let $\xi \in 2^{-u-2}\mathbb{Z}$ be such that $|x - \xi| \leq 2^{-u-3}$. The mean value theorem now claims that

$$|P_r(\xi)| \geq |P_r(x)| - |P_r(x)| \cdot 2^{-u-3} \geq |P_r(x)| - |P_r(x)| \cdot \frac{2\pi r}{2^{u+3}} \geq |P_r(x)| \left(1 - \frac{\pi}{4}\right),$$

where $(\ast)$ follows from Bernstein’s inequality and $(\dagger)$ follows from the inequality $r < 2^u$. Thus $\|P_r\|_\infty \leq \varepsilon_2(u)/(1 - \pi/4)$. Returning to $S_r$ (again using the fact that $F_r(f \circ \Phi)$ converges uniformly) shows (7.25).

The homeomorphism $\Phi$ is in fact $\psi_{f,r,\infty}$ with $\tau(d) = \lim_{k \to \infty} I_k(d)$, so by Lemma 5.6 it is absolutely continuous and, if $\eta$ was chosen sufficiently small, also satisfies $\Phi' \in L^p$ for any desired $p$, fixed in advance. The theorem is proved. $\Box$

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 319 2022
The work of the first author was supported by the Israel Science Foundation and by the Jesselson Foundation.

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