ELECTRONS FROM MUON DECAY IN BOUND STATE

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February 12, 2009

Abstract

We present results of a study of the muon decay in orbit (DIO) contribution to the signal region of muon - electron conversion. Electrons from DIO are the dominant source of background for muon - electron conversion experiments because the endpoint of DIO electrons is the same as the energy of electrons from elastic muon - electron conversion.

The probability of DIO contribution to the signal region was considered for a tracker with Gaussian resolution function and with a realistic resolution function obtained in the application of pattern recognition and momentum reconstruction Kalman filter based procedure to GEANT simulated DIO events. It is found that the existence of non Gaussian tails in the realistic resolution function does not lead to a significant increase in DIO contribution to the signal region.

The probability of DIO contribution to the calorimeter signal was studied in dependence on the resolution, assuming a Gaussian resolution function of calorimeter. In this study the geometrical acceptance played an important role, suppressing DIO contribution of the intermediate range electrons from muon decay in orbit.

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Theoretical electron spectrum from muon decay in orbit

Electron spectrum near the endpoint

Electrons from muon decay in orbit are the dominant source of background for muon - electron conversion experiments. In the free decay of a muon at rest to an electron and two neutrinos, the electron’s energy is at most half the muon rest energy, but in the decay of a bound muon the energy approaches that of the conversion electron, \( \simeq 105 \text{ MeV} \), when the two neutrinos carry away little energy.

General formulas obtained in [1] and [2], describing the electron spectrum from muon decay in orbit (DIO), are complicated.

The results of numerical calculation of the electron spectrum for aluminum and magnesium are presented in Table 1.

Note that in Table 1 the electron spectrum for aluminum is normalized to 1 but the electron spectrum for magnesium is not normalized.

Near the endpoint the electron spectrum can be presented in an analytical form found by Shanker [3]:

\[
N(E) = 10^{-21} \cdot \left( \frac{E}{m_\mu} \right)^2 \cdot \left( \frac{\delta_1}{m_\mu} \right)^5 \cdot \left( \tilde{D} \cdot E + \tilde{E} \cdot (\delta_1 / m_\mu) + \tilde{F} \cdot (\delta / m_\mu) \right)
\]  

(1)

where \( \delta = E_{\text{max}} - E \), \( \delta_1 = E_\mu - E - E^2 / 2M_A \), and \( E \) is the electron energy. The coefficients \( \tilde{D} \), \( \tilde{E} \), \( \tilde{F} \) depending on nuclear charge \( Z \), were calculated for a wide range of elements in [3].

Eq.(1) was obtained by neglecting the variation of the weak-interaction matrix element with energy.

The maximum possible energy of the electron is:

\[
E_{\text{max}} = E_\mu - \frac{E^2}{2M_A}, \quad E_\mu = m_\mu \cdot (1 - (\alpha \cdot Z)^2 / 2)
\]

where \( M_A \) is a nuclear mass, \( m_\mu \) is a muon mass, \( \alpha \) is the fine structure constant. In particular, \( E_{\text{max}} \) is 104.963 MeV for aluminum.

In order to apply Eq.(1) to aluminum the coefficients \( \tilde{D}, \tilde{E}, \tilde{F} \) should be found for this element. This can be done, for example, by fitting numerical results presented in [3] by a polynomial of the 4th power (see Appendix A).

As a result, for aluminum these coefficients are found to be 0.3575, 0.9483, 2.2706, respectively. Note that these coefficients correspond to the normalization of the electron spectrum by the muon life time in aluminum.
$^27\text{Al}$ Watanabe $^4$ $^{\text{nat}}\text{Mg}$ Herzog $^2$

| Energy (MeV) | $N(E)$ (MeV$^{-1}$) | Energy (MeV) | $N(E)$ (MeV$^{-1}$) |
|-------------|-------------------|-------------|-------------------|
| 5.0         | $1.17 \cdot 10^{-3}$ | 5.11        | $1.249 \cdot 10^{-1}$ |
| 10.0        | $4.1 \cdot 10^{-3}$  | 10.22       | $4.382 \cdot 10^{-1}$  |
| 15.0        | $8.23 \cdot 10^{-3}$ | 15.33       | $8.814 \cdot 10^{-1}$  |
| 20.0        | $1.31 \cdot 10^{-2}$ | 20.44       | $1.41$               |
| 25.0        | $1.85 \cdot 10^{-2}$ | 25.55       | $1.981$             |
| 30.0        | $2.38 \cdot 10^{-2}$ | 30.66       | $2.549$             |
| 35.0        | $2.87 \cdot 10^{-2}$ | 35.77       | $3.068$             |
| 40.0        | $3.28 \cdot 10^{-2}$ | 40.88       | $3.487$             |
| 45.0        | $3.52 \cdot 10^{-2}$ | 45.99       | $3.706$             |
| 50.0        | $2.91 \cdot 10^{-2}$ | 51.1        | $2.683$             |
| 55.0        | $3.67 \cdot 10^{-3}$ | 56.21       | $1.751 \cdot 10^{-1}$ |
| 60.0        | $1.41 \cdot 10^{-4}$ | 61.32       | $6.056 \cdot 10^{-3}$ |
| 65.0        | $9.97 \cdot 10^{-6}$ | 66.43       | $4.44 \cdot 10^{-4}$ |
| 70.0        | $1.11 \cdot 10^{-6}$ | 71.54       | $5.008 \cdot 10^{-5}$ |
| 75.0        | $1.54 \cdot 10^{-7}$ | 76.65       | $6.846 \cdot 10^{-6}$ |
| 80.0        | $2.28 \cdot 10^{-8}$ | 81.76       | $9.704 \cdot 10^{-7}$ |
| 85.0        | $3.18 \cdot 10^{-9}$ | 86.87       | $1.237 \cdot 10^{-7}$ |
| 90.0        | $3.54 \cdot 10^{-10}$ | 91.98     | $1.158 \cdot 10^{-8}$ |
| 95.0        | $2.33 \cdot 10^{-11}$ | 97.09     | $5.112 \cdot 10^{-10}$ |
| 100.0       | $3.58 \cdot 10^{-12}$ | 102.2      | $1.632 \cdot 10^{-12}$ |

Table 1: Decay electron spectra (DIO) for $^{27}\text{Al}$ and $^{\text{nat}}\text{Mg}$

$[^5]$ $\tau_{\text{Al}} = 0.864 \mu \text{sec}$. The relation between the electron spectrum normalized by the muon life time in aluminum and the spectrum normalized to free muon decay is given by

$$N(E) = N_{\text{free}}(E) \frac{\tau_{\text{Al}}}{\tau_{\text{free}}} = N_{\text{free}}(E) \frac{\tau_{\text{Al}}}{\tau_{\text{free}}} \approx 0.4 \cdot N_{\text{free}}(E)$$

where $\tau_{\text{free}} = 2.2 \mu \text{sec}$. Below $N(E)$ will always refer to the spectrum normalized to free muon decay.

According to Eq.(1) at $E = 100$ MeV the electron spectrum for aluminum is $N(E) = 1.42 \times 10^{-13}$ MeV$^{-1}$. This number should be compared with $N(E) = 1.43 \times 10^{-13}$ MeV$^{-1}$, which is the result of numerical calculations of $[^4]$ given in Table 1 but renormalized in order to take into account a muon life time in Al. One can see that the difference in these numbers is
less than 1%. At 95 MeV this approximation underestimates the spectrum by approximately 30%. Remind that Watanabe’s spectrum in Table 1 was normalized to free muon decay and calculated for energies below 100 MeV.

Above $E = 100$ MeV the precision of Shanker’s formula, obtained in phase space approximation by neglecting the variation of the matrix element with energy. Therefore the description of the spectrum is improved as one approaches the endpoint, because near the endpoint the DIO process is defined by the available phase space. Note that the numerical calculations [4] properly take into account relativistic electron wave functions and the effect of finite nuclear size on the wave functions. Thus we conclude that above 100 MeV Shanker’s approximation (1) gives a good description of the electron spectrum with the precision better than 1%.

It is important to emphasize that neglecting of the coefficients $\tilde{E}$ and $\tilde{F}$ at about 100 MeV leads to a significant underestimation (about 40%) of the electron spectrum $N(E) = 1.0 \times 10^{-13} \text{MeV}^{-1}$.

In order to simplify analytical and numerical calculations and analysis of the results we decided to use, instead of Shanker’s formula, a simple two terms approximation to describe the electron spectrum near the endpoint:

$$N(E) = C_0 \cdot (E_{max} - E)^5 + C_1 \cdot (E_{max} - E)^6 \quad (2)$$

For aluminum the coefficients $C_0$ and $C_1$ are determined by the following conditions: 1) It is required that at $E = 100$ MeV Eq.( 2) should give the same results $N(E) = 1.43 \cdot 10^{-13}$ MeV$^{-1}$ as Watanabe’s spectrum in Table 1 renormalized in order to take into account the muon life time in Al. 2) Shanker’s formula Eq.( 1) is a good approximation above 100 MeV, therefore at specific energy (we choose this energy as 104 MeV) near the endpoint the approximate formula (2) should give the same result as Eq.( 1). These conditions give $C_0 = 0.3699 \times 10^{-16} \cdot \text{MeV}^{-6}$ and $C_1 = 0.02132 \times 10^{-16} \cdot \text{MeV}^{-7}$ for aluminum. It was found that in this case in the range 100 - 104.5 MeV the spectra differ by less than 0.7%.

At 100 MeV the difference between Shanker’s approximation (1) and two terms approximation (2) (and Watanabe’s result) is about 0.7%. At 95 MeV these approximations underestimate the spectrum by approximately 30 %. If one keeps only the leading term of Eq.( 2) below 103.5 MeV the difference with Shanker’s approximation is about 3 - 8 %

**Expected number of DIO events**

The expected number $N$ of primary DIO events produced in aluminum target during the time of the experiment can be calculated as
\[ N = I_p \cdot \varepsilon_{\mu/p} \cdot \varepsilon_{\text{gate}} \cdot \varepsilon_{\text{acc}} \cdot T \cdot P \]  

(3)

In this equation \( P \) is the probability of DIO contribution to a signal region above the threshold energy. By assuming that a proton flux \( I_p = 4 \times 10^{13}/\text{sec} \), muons stopping efficiency \( \varepsilon_{\mu/p} \) in the Al target per primary proton is 0.25\%, \( \varepsilon_{\text{acc}} = 0.2 \) is the setup overall acceptance, the time of experiment \( T = 10^7 \text{ sec} \), and that the measured efficiency \( \varepsilon_{\text{gate}} \) during the 650 nsec window, extending from 700 nsec to 1350 nsec after the pulse, is 50\%:

\[ N = 5 \cdot 10^{17} \cdot P \cdot \varepsilon_{\text{acc}}. \]  

(4)

**Approximated theoretical electron spectrum**

Since the calorimeter energy cutoff can be as low as 80 MeV and the contribution from the intermediate part of the electron spectrum can be important.

Using the numerical results from Table 1 for aluminum, the theoretical electron spectrum can be approximated by a fit.

Figure 1 shows the differential electron spectrum for muon decay in orbit in Al in linear scale. Points are the results of numerical calculations [4]. The solid line is an 8 parameter fit of the spectrum below 55 MeV in the form \( e^{f(E)} \) where \( f(E) \) is a polynomial of power 7. The parameters of the fit are presented in Appendix B.

Figure 2 shows the differential electron spectrum for muon decay in orbit in Al, a linear scale. The points are the results of numerical calculations [4]. The solid line is an 8 parameter fit of the spectrum in the range 55 - 100 MeV in the form \( e^{g(E)} \) where \( g(E) \) is a polynomial of power 7. The parameters of the fit are presented in Appendix B.

The region above 100 MeV is described by the two term approximation Eq.(2) but its contribution in the case of low threshold energies is negligible.

The probability to produce an electron of energy above \( E_{\text{min}} \) in muon decay in orbit in Al on the basis of the approximate theoretical electron spectrum is given by

\[ P = \int_{E_{\text{min}}}^{E_{\text{max}}} N(E) dE. \]  

(5)

In this expression as it was mentioned above \( E_{\text{max}} = 104.963 \text{ MeV} \), \( E \) is the true energy of electron (at this energy an electron is created in muon
Figure 1: Differential electron spectrum for muon decay in orbit in Al in a linear scale. Points are the results of numerical calculations. Solid line is a fit of the spectrum below 55 MeV. This spectrum is normalized to free muon decay.

Figure 3 shows the probability to produce an electron of energy above $E_{\text{min}}$ in Al versus $E_{\text{min}}$.

In the range 55 - 100 MeV the approximation discussed above was used, above 100 MeV the electron spectrum was taken in the form (2).

Using the probability, which is given by Eq.(5), one can calculate the number of primary DIO events during the time of the experiment. It is important to emphasize that in this section we calculate the total number of DIO electrons produced in aluminum target and we do not need to include in this calculation a resolution function and overall acceptance.

The number of DIO events, which is calculated using Eq.(3) with $\varepsilon_{\text{acc}} = 1$, is sharply increasing with decreasing $E_{\text{min}}$. In particular, for $E_{\text{min}} = 100$ MeV - $N = 5.7 \times 10^4$, for $E_{\text{min}} = 95$ MeV - $N = 6.2 \times 10^6$, and for $E_{\text{min}} = 90$ MeV - $N = 1.39 \times 10^8$.

Table 2 shows the probability $P$ and the number of DIO events $N$ with an electron energy in the given energy range.
Muon Decay in Orbit on Al$_{27}$

Figure 2: Differential electron spectrum for muon decay in orbit in Al in logarithmic scale. Points are the results of numerical calculations. Solid line is a fit of the spectrum in the range 55 - 100 MeV. This spectrum is normalized to free muon decay.

Also in Table 2 the coefficients $A$ and $B$ are given, which are necessary for Monte Carlo simulation of DIO events. For this simulation it is assumed that each 5 MeV interval from $E_1$ to $E_2$ of the electron spectrum in the range 55 - 100 MeV can be approximated by

$$N(E) = A \cdot e^{-(E-E_1)/B}$$

(6)

It is important to note that in order to simulate a process with the threshold in measured energy $E_{th}^m$, one has to take into account DIO electrons produced in target starting approximately at $E^{th} = E_{th}^m - 2\sigma$. In particular, for $E_{th}^m = 80 MeV$ and $\sigma = 5 MeV$ it would correspond to $E^{th} = 70 MeV$. According to Figure 3 and Eq. (5) in this case the number of primary DIO events would reach $N = 5.65 \times 10^{11}$ making event by event simulation unfeasible. We assume that present realistic threshold for calorimeter measured energy is about 90 MeV and the minimal true energy of DIO electrons is about 80 MeV. This corresponds to the number of DIO events $N = 1.13 \times 10^{10}$ which is about 50 times less than in the case of 80 MeV.
Figure 3: Probability to produce an electron of energy above $E_{\text{min}}$ in muon decay in orbit in Al

| Range | 80 - 85 | 85 - 90 | 90 - 95 | 95 - 100 | > 100 |
|-------|---------|---------|---------|---------|-------|
| Prob. | $1.97 \times 10^{-8}$ | $2.62 \times 10^{-9}$ | $2.64 \times 10^{-10}$ | $1.23 \times 10^{-11}$ | $1.15 \times 10^{-13}$ |
| N     | $9.85 \times 10^9$ | $1.31 \times 10^9$ | $1.32 \times 10^8$ | $6.15 \times 10^6$ | $5.7 \times 10^4$ |
| A     | $9.12 \times 10^{-9}$ | $1.272 \times 10^{-9}$ | $1.416 \times 10^{-10}$ | $9.32 \times 10^{-12}$ | – |
| B     | 2.53823 | 2.27755 | 1.83767 | 1.19741 | – |

Table 2: Expected number N of primary electrons produced in muon decay in orbit for different energy ranges (in MeV). Coefficients A and B of Eq. (6) are given.

threshold in measured energy.
DIO events in a tracker

Probability of DIO contribution

The probability to have a signal from muon decay in orbit above a threshold $E_{\text{max}} + \Delta$ is given by

\[
P = \int_{E_{\text{max}} + \Delta}^{\infty} dE_M \int_{0}^{E_{\text{max}}} N(E) \cdot f(E_M - E) dE \tag{7}
\]

where $E_M$ is the measured electron energy, $E$ is the true energy at which an electron was emitted, $\Delta$ is the threshold energy measured from the endpoint, $N(E)$ is the electron spectrum of muon decay in orbit, and $f$ is the resolution function of the tracker.

For an ideal tracker the resolution function is the delta function and in this limit the probability $P$ is given by

\[
P = \int_{E_{\text{max}} + \Delta}^{E_{\text{max}}} N(E_M) dE_M
\]

For the high energy part of the electron spectrum, $N(E)$ can be approximated by Eq.(2), leading to the probability

\[
P = C_0 \frac{\Delta^6}{6} - C_1 \frac{\Delta^7}{7} \tag{8}
\]

in the case of the ideal detector. For the threshold energy $\Delta = -0.7 \ MeV$, according to Eq.(8), the probability $P \approx 6.9 \cdot 10^{-19}$. This equation is approximately valid for the range $-5 \ MeV \leq \Delta \leq 0$. The lower limit is defined by the validity of the two term approximation (2) for the electron spectrum. For positive $\Delta$ the probability $P$ is 0 for the ideal tracker.

To deal with a realistic resolution function it is convenient to introduce a new variable $y = E_M - E$ in Eq.(7). The region of integration in $y, E_M$ variables is given in Figure 4.

By interchanging the order of integration in $E_M, y$ and substituting $E_M = y + E$ the probability to have a signal from muon decay in orbit above a threshold takes the form

\[
P = \int_{\Delta}^{E_{\text{max}} + \Delta} f(y) dy \int_{E_{\text{max}} + \Delta - y}^{E_{\text{max}}} N(E) dE \tag{9}
\]
Figure 4: Domain of integration in $y, E_M$ variables.

A second integral over the region in $y$ above $E_{max} + \Delta$ was neglected because the resolution function at such $y$ is expected to be extremely small.

It follows immediately from this representation that the probability $P$ is defined by the resolution function above the threshold energy $\Delta$.

**Gaussian detector response function**

Let’s assume that the detector response function $f(y)$ is of the Gaussian form:

$$f_G(y) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)$$

In Figure 5 the probability $P$ of muon decay in orbit as a function of threshold energy $\Delta$ is shown for different resolutions $\sigma$.

The two term approximation for the electron spectrum Eq. (2) was used for this plot. For small threshold energies $\Delta \approx 0$ the probability is sensitive to the resolution $\sigma$ and it is proportional to $\sigma^6$. At $\Delta = -1$ MeV the probabilities for $\sigma = 0.4$ MeV and $\sigma = 0.1$ MeV differ by a factor 4.

Note that Eq. (8) defines the limiting probability for Figure 5 as the resolution $\sigma$ tends to 0. For the threshold energy $\Delta = -0.7$ MeV and the
resolution $\sigma = 0.2 \text{ MeV}$ the probability $P \approx 2.1 \cdot 10^{-18}$ is about three times greater than the probability $P \approx 6.9 \cdot 10^{-19}$ in the case of the ideal detector.

If the threshold energy $\Delta > -5 \text{ MeV}$ the probability $P$, given by Eq. (9), can be calculated analytically by using the two term approximation Eq. (2) for the electron spectrum. In this case

$$P = C_0 \frac{\sigma^6}{6 \sqrt{2\pi}} I_0 + C_1 \frac{\sigma^7}{7 \sqrt{2\pi}} I_1$$  \hspace{1cm} (11)

The coefficients $I_0, I_1$ are given by

$$I_0 = \sqrt{\frac{\pi}{2}} (15 + 45u^2 + 15u^4 + u^6) \text{Erfc}(\frac{u}{\sqrt{2}}) - u(33 + 14u^2 + u^4)e^{-u^2/2},$$  \hspace{1cm} (12)

$$I_1 = -\sqrt{\frac{\pi}{2}} u(105+105u^2+21u^4+u^6) \text{Erfc}(\frac{u}{\sqrt{2}}) + (48+87u^2+20u^4+u^6)e^{-u^2/2},$$  \hspace{1cm} (13)
where $u = \Delta/\sigma$, \(Erfc(z) = 1 - \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt\)  

In the case of interest, $\Delta < -\sigma$, Eq. (11) can be approximated by a form convenient for analysis

$$P = C_0 \frac{\sigma^6}{6}(15 + 45u^2 + 15u^4 + u^6) - C_1 \frac{\sigma^7}{7}u(105 + 105u^2 + 21u^4 + u^6) \quad (14)$$

The precision of this approximation is better than 1% if $\Delta < -\sigma$. In the limit $|\Delta| >> \sigma$ for negative $\Delta$ this equation reproduces Eq. (5).

**Simulated detector response function**

The resolution function obtained as a result of simulation differs from the Gaussian form due to non-Gaussian tails, which appear due to multiple scattering, radiation processes and a non-ideal reconstruction procedure. A pattern recognition and track reconstruction procedure based on the Kalman filter technique developed in [6] was applied taking into account backgrounds, delta-rays and straw inefficiency. The distribution in the difference between the incident momentum \(P_{i\text{in}}\) reconstructed by the forward Kalman filter procedure and the generated incident momentum \(P_{i\text{in}}\) is shown in Figure 6 in linear (left) and logarithmic (right) scale. According to this distribution the intrinsic tracker resolution is $\sigma = 0.189$ MeV/c if one fits the distribution by a Gaussian in the range -0.3 to 0.7 MeV/c. It follows from Figure 6 that outside the range -0.4 to 0.4 MeV the distribution is non-Gaussian.

Using this histogram, normalized by the total area of the histogram, as the resolution function, the probability to have a signal from muon decay in orbit above a threshold can be presented in the form

$$P = \sum_i \frac{n_i}{n} \int_{E_{\text{max}} - \Delta\tilde{p}_i}^{E_{\text{max}}} N(E)dE \quad (15)$$

where $n_i$ is the number in $i^{th}$ bin of the histogram, $n = \sum_i n_i$ is the total number of events, $\tilde{p}_i = (p_{i\text{min}} + p_{i\text{max}})/2$, $p_{i\text{min}} = \Delta$.

In the two term approximation Eq. (2) for the electron spectrum this equation can be rewritten as

$$P = \sum_i \frac{n_i}{n} \left[ \frac{C_0}{6}(\tilde{p}_i - \Delta)^6 + \frac{C_1}{7}(\tilde{p}_i - \Delta)^7 \right] \quad (16)$$
Figure 6: Distribution in the difference between the input reconstructed momentum and the generated input momentum with the background rate of 550 kHz, delta-ray and straw efficiency 97%.

Figure 7 shows the probability of DIO contribution to the signal region as a function of the threshold energy $\Delta$ for a resolution function of the tracker obtained from the simulation (Figure 6).

One can see that in a logarithmic scale the probability decreases approximately linearly with an increase in threshold energy. This behavior differs from the sharp decrease of the probability in the case of Gaussian resolution function and is related to non Gaussian tails. The existence of non Gaussian tails leads to an increase in probability of DIO to contribute to the signal region but this additional contribution is not very significant. For example at the threshold energy of interest $\Delta = -0.7 MeV$ $P = 2.46 \cdot 10^{-18}$ in the case of simulated resolution function and $P = 1.93 \cdot 10^{-18}$ in the assumption of Gaussian resolution function with $\sigma = 0.189 MeV/c$.

By using the two term approximation for the electron spectrum and the simulated resolution function, the expected number of primary DIO events in the tracker during the experiment with an Al target has been calculated and is presented in Table 3.
Figure 7: Probability of DIO contribution to a signal region as a function of the threshold energy $\Delta$ for a resolution function of the tracker obtained from the simulation.

| $\Delta$, MeV | -0.9 | -0.7 | -0.5 | -0.3 | -0.1 |
|---------------|------|------|------|------|------|
| $N$           | 0.68 | 0.25 | 0.087| 0.03 | 0.014|

Table 3: Expected number $N$ of DIO events in a tracker in dependence on the threshold energy $\Delta$. Realistic resolution function and overall acceptance are included.

**DIO events in the calorimeter**

A calorimeter geometrical acceptance (Figure 8) plays an important role in suppression of low energy charged particles.

One can see that due to a rapid fall-off the geometrical acceptance suppresses the electron spectrum below 70 MeV, reducing significantly in this way the number of DIO events detected by the calorimeter above the thresh-
old energy.

Figure 9 shows a differential energy spectrum of DIO electrons multiplied by the calorimeter geometrical acceptance.

One can see that in the region below 70 MeV the rapid fall-off of the geometrical acceptance with decreasing electron energy almost compensates the rapid growth of DIO electron spectrum. In particular in the range from 70 MeV to 60 MeV an increase in the electron spectrum by 2 orders of magnitude is reduced to a factor 2 due to the fall-off in the geometrical acceptance.

The probability $P$ to have a signal in the calorimeter from muon decay in orbit above a threshold is given by Eq. (9) with the electron spectrum multiplied by the geometrical acceptance $F_{geom}$:

$$P = \frac{E_{\text{max}} + \Delta}{\Delta} \int_{E_{\text{max}} + \Delta - y}^{E_{\text{max}}} f_G(y)dy \int_{E_{\text{max}} + \Delta - y}^{E_{\text{max}}} N(E) \cdot F_{geom}(E)dE \quad (17)$$

where, as above, $y = E_M - E$, and $E_M$ is the measured electron energy, $E$ is the true energy at which an electron was emitted.

In Figure 10 the probability $P$ of muon decay in orbit contributing to the calorimeter signal as a function of threshold energy $\Delta$ is shown. The
resolution function of the calorimeter is assumed to be Gaussian.

It follows from this plot that the probability is not extremely sensitive to the resolution $\sigma$, which is the consequence of fast decreasing geometrical acceptance below 70 MeV. For example, for the measured energy above 80 MeV (the threshold energy $\Delta = -25$ MeV) even the change in $\sigma$ from 5 MeV/c to 10 MeV/c increases the probability of DIO contribution only by a factor 5. According to this plot the probability $P$ is a fast decreasing function of the threshold energy in the considered range because above 55 MeV the electron spectrum is a steep function of energy.

The expected number $N$ of primary DIO events in calorimeter during the time of the experiment can be estimated by Eq. (4) with $\varepsilon_{acc} = 1$.

By using the fit for the electron spectrum described above and the Gaussian resolution function, the expected number of primary DIO events in calorimeter during the experiment with Al target is calculated and is presented in Table 4 for $\sigma = 5$ MeV/c and 8 MeV/c.
Figure 10: Probability of DIO contribution to the calorimeter signal as a function of the threshold energy $\Delta$ for different resolutions $\sigma$.

| $\Delta$, MeV | -30 | -25 | -20 | -15 | -10 |
|---------------|-----|-----|-----|-----|-----|
| $N (\sigma = 5 \text{ MeV})$ | $1.8 \times 10^{10}$ | $5 \times 10^{9}$ | $1.2 \times 10^{9}$ | $2.5 \times 10^{8}$ | $4.3 \times 10^{7}$ |
| $N (\sigma = 8 \text{ MeV})$ | $3.5 \times 10^{10}$ | $1.4 \times 10^{10}$ | $4.5 \times 10^{9}$ | $1.3 \times 10^{9}$ | $3 \times 10^{8}$ |

Table 4: Expected number $N$ of DIO events in calorimeter in dependence on the threshold energy $\Delta$. Convolution and geometrical acceptance are included.

**Muon conversion**

Muon conversion process is characterized by the appearance of a mono-energetic electron with energy $E_{\text{max}} = 104.963$ MeV for aluminum. The rate of $\mu e$ - conversion $R_{\mu e}$ is normalized by the muon capture rate:
\[ R_{\mu e} = \frac{\Gamma_{\mu e}}{\Gamma_{\mu cap}} \]

where \( \Gamma_{\mu e} \) is a width for muon conversion on nucleus and \( \Gamma_{\mu cap} \) is a width for muon capture by nucleus.

The total width is defined by

\[ \Gamma_{\mu Total} = \Gamma_{\mu free} + \Gamma_{\mu cap} \]

or in terms of life times

\[ \frac{1}{\tau_{\mu Total}} = \frac{1}{\tau_{\mu free}} + \frac{1}{\tau_{\mu cap}} \]

where for aluminum \( \tau_{\mu Total} = 0.864 \mu sec \) and \( \tau_{\mu free} = 2.2 \mu sec \).

The probability \( P_{\mu e} \) for muon conversion is expressed as:

\[ P_{\mu e} = \frac{\Gamma_{\mu e}}{\Gamma_{\mu Total}} = R_{\mu e} \cdot \frac{\Gamma_{\mu cap}}{\Gamma_{\mu Total}} = 0.6 \cdot R_{\mu e} \]

The probability to have a signal from muon conversion on aluminum is given by:

\[ P^{\text{sig}}_{\mu e} = P_{\mu e} \cdot \int_{\Delta}^{\infty} f(y)dy = 0.6 \cdot R_{\mu e} \cdot \int_{\Delta}^{\infty} f(y)dy \quad (18) \]

The background to the signal ratio is expressed as:

\[ P^{\text{sig}}_{\text{DIO}}/P^{\text{sig}}_{\mu e} = 0.103 \cdot (10^{-16}/R_{\mu e}) \cdot \frac{\int_{\Delta}^{\infty} (y - \Delta)^6 f(y)dy}{\int_{\Delta}^{\infty} f(y)dy} \]

where it was used that \( C_0/(6 \cdot P_{\mu e}) = 0.103 \cdot 10^{-16}/R_{\mu e} \).

For a Gaussian resolution function this ratio can be approximated by

\[ P^{\text{sig}}_{\text{DIO}}/P^{\text{sig}}_{\mu e} = 0.103 \cdot (10^{-16}/R_{\mu e}) \cdot \sigma^6 \cdot (15 + 45u^2 + 15u^4 + u^6) \]

where as above \( u = \Delta/\sigma \). The precision of this formula is better than 2% for \( u < -2 \).

The expected number of registered muon conversion events can be calculated from Eqs. (3), (18). For \( R_{\mu e} = 10^{-16} \) sensitivity and setup overall acceptance = 20%, the expected number of muon conversion events is:

\[ N = 4 \cdot 10^{13} \times 2.5 \cdot 10^{-3} \times 10^7 \times 0.5 \times 0.2 \times 0.6 \times 10^{-16} = 6.0 \]
Conclusion

In this memo we studied the detection of electrons from muon decay in orbit. These electrons are the dominant source of background for muon - electron conversion experiments because the endpoint of DIO electrons is the same as the energy of electrons from elastic muon - electron conversion.

It was found that near the endpoint \((E \simeq 100\text{MeV})\) the Shanker’s formula [3], obtained by neglecting the variation of the weak-interaction matrix elements with energy in DIO process is in a good agreement with the results of numerical calculations [4] properly taking into account relativistic electron wave functions and the effect of finite nuclear size on the wave functions.

It is important to note that in order to simulate a process with the threshold in measured energy \(E_{th}^m\) one has to take into account DIO electrons produced in target starting approximately at \(E_{th}^m = E_{th}^m - 2\sigma\). In particular, for \(E_{th}^m = 80\text{MeV}\) and \(\sigma = 5\text{MeV}\) it would correspond to \(E_{th} = 70\text{ MeV}\). In this case the number of primary DIO events would reach \(N = 5.65 \times 10^{11}\) making event by event simulation unfeasible. We assume that present realistic threshold for calorimeter measured energy is about 90 MeV and the minimal true energy of DIO electrons is about 80 MeV. This corresponds to the number of DIO events \(N = 1.13 \times 10^{10}\) which is about 50 times less than in the case of 80 MeV threshold in measured energy.

The probability of DIO contribution to a signal region was considered for the tracker with Gaussian resolution function and with the realistic resolution function obtained in the application of pattern recognition and momentum reconstruction Kalman filter based procedure to GEANT simulated DIO events.

It was found that non Gaussian tails in the simulated resolution function do not lead to a significant increase in DIO contribution to the signal region. The expected number of detected DIO events during the time of the experiment for realistic resolution function was calculated to be about 0.25 if the threshold energy \(\Delta = -0.7 \text{ MeV}\) and the overall acceptance is 20%.

It was shown that the probability of DIO contribution is very sensitive to the threshold energy and proportional to \(\Delta^6\) in the limit \(|\Delta| \gg \sigma\) where \(\sigma\) is the tracker resolution.

The probability of DIO contribution to the calorimeter signal was studied in dependence on the resolution, assuming a Gaussian resolution function of the calorimeter. In this study the geometrical acceptance played an important role, suppressing DIO contribution of the intermediate range electrons from muon decay in orbit. It was found that the probability of DIO contribution is not extremely sensitive to the resolution \(\sigma\), which is the
consequence of the fast decreasing geometrical acceptance below 70 MeV. For example, for the measured energy above 80 MeV (the threshold energy $\Delta > -25$ MeV) even the change in $\sigma$ from 5 MeV/c to 10 MeV/c increases the probability of DIO contribution only by a factor 5.

The expected number $N$ of detected DIO events in calorimeter was estimated by using the approximation of the electron spectrum in the range 55 - 100 MeV. For the measured energy above 80 MeV and assuming a Gaussian resolution function of the calorimeter with $\sigma = 5$ MeV it was found that during the time of the experiment $N = 5 \times 10^9$.

We wish to thank A.Mincer and P.Nemethy for fruitful discussions and helpful remarks.

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Appendix A
Coefficients of Shanker’s expansion

The coefficient $\tilde{D}$, $\tilde{E}$, $\tilde{F}$ of Shanker’s expansion for different materials can be determined by the fitting of numerical values of $\tilde{D}$, $\tilde{E}$, $\tilde{F}$ presented in [3] by 4th power polynomial.

Figure 11 shows the results of the fit for $\tilde{D}$ coefficient.

![Figure 11: Numerical values of $\tilde{D}$ coefficient. Solid line is a result of fit by 4th power polynomial $f = p_1 + p_2 \times x + p_3 \times x^2 + p_4 \times x^3 + p_5 \times x^4$.](image)

In the same way $\tilde{E}$ and $\tilde{F}$ coefficients can be found.

Figures 12 and 13 show the results of the fit for D and E coefficients, respectively.

Table 5 represents $\tilde{D}$, $\tilde{E}$, $\tilde{F}$ coefficients for different materials.
Appendix B

Fit of electron spectrum

Below 55 MeV the electron spectrum $N_{\text{free}}(E)$ can be described by an 8 parameter fit in the form $e^{f(E)}$ where $f(E)$ is a polynomial of power 7:

$$f(E) = f_7 \cdot E^7 + f_6 \cdot E^6 + f_5 \cdot E^5 + f_4 \cdot E^4 + f_3 \cdot E^3 + f_2 \cdot E^2 + f_1 \cdot E + f_0 \quad (19)$$

For aluminum the fit, using numerical results for the electron spectrum [4], gives:

$$
\begin{align*}
    f_7 &= -4.1321585902527747 \times 10^{-10}; \\
    f_6 &= 7.26144332670667 \times 10^{-8}; \\
    f_5 &= -5.0391208057229 \times 10^{-6}; \\
    f_4 &= 1.720029756203247 \times 10^{-4}; \\
    f_3 &= -2.797753809879399 \times 10^{-3}; \\
    f_2 &= 1.003520948296598 \times 10^{-2};
\end{align*}
$$
Figure 13: Numerical values of $\tilde{F}$ coefficient. Solid line is a result of fit by 4th power polynomial $f = p_1 + p_2 \cdot x + p_3 \cdot x^2 + p_4 \cdot x^3 + p_5 \cdot x^4$.

$$f_1 = 0.35027124751754113;$$
$$f_0 = -8.495230158021982;$$

Above 55 MeV the electron spectrum $N_{free}(E)$ can be described by an 8 parameter fit in the form $e^{g(E)}$ where $g(E)$ is a polynomial of power 7:

$$f(E) = g_7 \cdot E^7 + g_6 \cdot E^6 + g_5 \cdot E^5 + g_4 \cdot E^4 + g_3 \cdot E^3 + g_2 \cdot E^2 + g_1 \cdot E + g_0 \text{ (20)}$$

For aluminum the fit, using numerical results for the electron spectrum [4], gives:

$$g_7 = 5.882860554682616 \times 10^{-10};$$
$$g_6 = -3.311494207831439 \times 10^{-7};$$
$$g_5 = 7.90482092857202 \times 10^{-5};$$
$$g_4 = -1.0375709354278761 \times 10^{-2};$$
$$g_3 = 0.808726145419064;$$
$$g_2 = -37.41798219613308;$$
Table 5: \( \tilde{D}, \tilde{E}, \tilde{F} \) coefficients for different materials

| \( Z \) | \( \tilde{D} \)       | \( \tilde{E} \)       | \( \tilde{F} \)       |
|-------|----------------|----------------|----------------|
| 10    | 0.169990093   | 0.281679779   | 1.68660045    |
| 11    | 0.221416339   | 0.487800866   | 1.8169477     |
| 12    | 0.282654375   | 0.704645038   | 1.99828386    |
| 13    | 0.357468009   | 0.948250234   | 2.27059889    |
| 14    | 0.449176699   | 1.2330687     | 2.6695714     |
| 15    | 0.560655653   | 1.57196677    | 3.22657013    |
| 16    | 0.694335938   | 1.97622538    | 3.96865273    |
| 17    | 0.852204204   | 2.45553946    | 4.91856623    |
| 18    | 1.03580284    | 3.01801825    | 6.09474707    |
| 19    | 1.24623013    | 3.67018533    | 7.5132059     |
| 20    | 1.48413992    | 4.41697836    | 9.17810249    |
| 21    | 1.74974191    | 5.26174974    | 11.1005974    |
| 22    | 2.04280138    | 6.20626593    | 13.2799997    |
| 23    | 2.36263967    | 7.25070715    | 15.713191     |
| 24    | 2.70813346    | 8.39366913    | 18.3927441    |

\( g_1 = 950.4758914390899; \)
\( g_0 = -10212.91359983421; \)