1. Introduction

Let $K$ be a closed subset in $\mathbb{R}^d$ with the Euclidean metric, and let $\mu$ be an $\alpha$–regular measure on $K$, that is, there exists $\alpha > 0$ such that for any ball $B(x, r)$ with $0 < r < \text{diam}(K)$,

$$\mu(B(x, r)) \approx r^\alpha. \quad (1.1)$$

(Here $f \asymp g$ means there exists constant $C > 0$ such that $C^{-1} g \leq f \leq C g$.) Fix $\sigma > 0$, for $u \in L^2(K, \mu)$, let

$$[u]_{B_{2,\infty}}^\sigma := \sup_{0 < r < 1} r^{\sigma - 2\sigma} \int_K \int_{B(x, r)} |u(x) - u(y)|^2 d\mu(y) d\mu(x), \quad (1.2)$$

and define $B_{2,\infty}^\sigma := \{ u \in L^2(K, \mu) : ||u||_{B_{2,\infty}^\sigma} < \infty \}$ with norm $||u||_{B_{2,\infty}^\sigma} := ||u||_2 + [u]_{B_{2,\infty}^\sigma}$. The space is a Banach space and belongs to the class of Besov spaces (cf., for example [9], [17], [19]; note that this space is also denoted by Lip($\sigma, 2, \infty$).)

Obviously, $B_{2,\infty}^\sigma \subseteq B_{2,\infty}^{\sigma'}$ if $0 < \sigma' < \sigma$. The space $B_{2,\infty}^\sigma$ can be dense in $C(K)$, or dense in $L^2(K, \mu)$; it can also become trivial as $\sigma$ increases, depending on the geometry of $K$ and $\mu$. Let us define the critical exponents on $(K, \mu)$ by

$$\sigma^* := \sup \{ \sigma : B_{2,\infty}^\sigma \cap C(K) \text{ is dense in } C(K) \}.$$

For many self-similar sets $K$, $B_{2,\infty}^{\sigma^*}$ are the domains of some local regular Dirichlet forms (if exist), and they are essential in the study of the Laplacians, Brownian motions, and the associated heat kernels [1, 9, 16, 17, 20, 27, 29]. The value $\beta^* = 2\sigma^*$ is called the walk dimension of $(K, \mu)$. It is an important parameter in the study of heat kernels, which corresponding to the speed of diffusion on the underlying sets. Heuristically, the larger

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the value $\beta^*$, the harder is for the diffusion process (Brownian motion) to drift away from the initial position. It is well-known that if $K$ is a domain in $\mathbb{R}^d$, then $\sigma^* = 1$; if $K$ is the $d$-dimensional Sierpinski gasket, then $\sigma^* = \log(d+3)/(2 \log 2)$ [17]. There are extensions on the nested fractals [27], and approximate value of the Sierpinski carpet [2]; also for Cantor-type sets, $\sigma^* = \infty$ [21]. More generally, the notion of Besov space and critical exponents have been extended to metric measure spaces $(K, d, \mu)$, where $(K, d)$ is a locally compact, separable metric space, and $\mu$ is $\alpha$-regular as before. It is known that with a heat kernel assumption and a chain condition on $(X, d)$, we have $1 \leq \sigma^* \leq \frac{1}{2}(\alpha + 1)$ [9].

In the previous studies of Dirichlet forms on $B_{2,\infty}^\sigma$, one often assumes that the space admits a Brownian motion with a Gaussian or a sub-Gaussian heat kernel. In such cases, it is known that if $\sigma > \sigma^*$, then $B_{2,\infty}^\sigma$ consists of constant functions only. For this we define another critical exponent

$$\sigma^\# := \sup \{ \sigma : \ B_{2,\infty}^\sigma \text{ contains non-constant functions} \},$$

Clearly, $\sigma^\leq \leq \sigma^\#$, and for the standard examples, we always have $\sigma^* = \sigma^\#$. In this paper, we will bring up the different situations through two asymmetric self-similar sets, and give a detail study of the two critical exponents as well as the functional behaviors of the associated Besov spaces. The investigation is intended to get a better understanding of the local regular Dirichlet forms, of which the existence is still not clear on the more general fractal sets.

Our consideration is on the post critically finite (p.c.f.) self-similar sets [20]. Let $\{F_i\}_{i=1}^N$ be an iterated function system (IFS) of the form $F_i(x) = \rho(x-b_i) + b_i$ with $0 < \rho < 1$, $b_i \in \mathbb{R}^d$, and let $K$ be the self-similar set. Assume that the IFS has the p.c.f. property [20], let $V_0$ be the boundary of $K$, $V_n = \bigcup_{i=1}^N F_i(V_{n-1}) = \bigcup_{|\omega|=n} F_\omega(V_0)$, and $V_\ast = \bigcup_{n=1}^\infty V_n$. We write $V_\omega = F_\omega(V_0)$, and for a function $u$ on $V_n$, we define

$$E_n[u] = \sum_{x,y \in V_n; \ |\omega|=n} |u(x) - u(y)|^2,$$

and call it a primal energy on $V_n$. According to Jonsson [17] and Bodin [3] (see also [7]), we have the following discrete expression of the Besov semi-norm.

**Proposition 1.1.** For a p.c.f. self-similar set defined by the IFS as above, and for $2\sigma > \alpha$ we have

$$[u]_{B_{2,\infty}^\sigma} \leq \sup \left\{ \rho^{-2(\sigma-\alpha)} E_j[u] \right\},$$

(1.4)

In the notion of electrical network, the primal energy form $E_n$ corresponds to a network $G_n$ on $V_n$ with unit resistance on each pair of vertices in $V_\omega (|\omega| = n)$ in $V_n$. We denote by $R_n(p,q)$, $p, q \in V_0$, the trace (or induced resistance) of $G_n$ on $V_0$. For example, on the Sierpinski gasket, we have $R_n(p,q) = r^{-n} = \left(\frac{1}{2}\right)^n$ for all $p, q \in V_0$, and for the Vicsek cross, $R_n(p,q) = r^{-n} = 3^n$; in these cases, for any $n \geq 1$ and $u$ on $V_0$, there is a minimal energy extension of $u$ on $V_r$ (and hence on $K$) such that $r^{-n} E_n[u] = E_0[u]$, and for $u$ on $V_\ast$, the sequence $\{r^{-n} E_n[u]\}_n$ is increasing, so that $\mathcal{E}[u] := \lim_{n \to \infty} r^{-n} E_n[u]$ exists, and $\mathcal{E}$ is the classical local regular Dirichlet form on $K$. This $r$ is called a renormalizing factor, and $r = \rho^{(2\sigma-\alpha)}$ in (1.4).
In this paper, we will study the case that the traces $R_n(p, q), p, q \in V_0$ have different growth rates, which give rise to the two critical exponents of the Besov spaces on $K$. Our first task is to develop the needed theoretical background for the two critical exponents. Roughly speaking, $\sigma^*$ is determined by the minimum growth rate of the sequences $(R_n(p, q))_n$, for $p, q \in V_0$ (Theorem 4.1), and $\sigma^#$ is determined by the maximum of such growth rates (Theorem 4.3). We also provide some criteria of the density of $B_{2,\infty}^{\sigma^*}$ and $B_{2,\infty}^{\sigma^#}$ in $C(K)$ and $L^2(K, \mu)$ respectively (Theorem 4.1)-(ii), Propositions 4.2 and 4.4).

One of the major techniques in this study is to define an equivalent relation to partition the $V_n$'s according to the growth rates of $R_n(p, q), p, q \in V_0$, and consider the equivalent classes and the quotient network. From the electrical network point of view, taking quotient means shorting the circuit (putting zero resistance) at the vertices in the equivalent classes. The quotient network can simplify calculations, and give interesting properties and different geometries, in particular, the growth rate of the traces of the original network can be modified to the need on the quotient. We give a detail study of the structure of the quotient network in Section 3 (Theorem 3.2): we show that the equivalent classes are related to the attractors of a graph directed system [6, 24], and give a sufficient condition for $R_n(p, q)$ to be a renormalization factor localized on certain equivalent classes (Lemma 3.4).

We remark that the device of quotient network was first considered by Sabot [28] with a different definition ($G$-relation, $G$ for group), which is used to study the existence and uniqueness of Dirichlet forms on symmetric ramified self-similar fractals. We will draw some comparison of the two in Section 3.

We will present two asymmetric p.c.f. sets that the traces $R_n(p, q), p, q \in V_0$, have different growth rates. We make special emphasis on the constructive aspects to illustrate the new situations. The first example is modified from the Vicsek cross by adding two eyebolts on the cross to produce the irregularity (see Figure 5), we call it the eyebolted Vicsek cross. It consists of 21 maps with contraction ratio $1/9$, and has four boundary points $V_0$. By equipping the $V_n$ with the primal energy, and using a generalized $\Delta$-Y transform from the electrical network theory, we show that the traces $R_n(p, q)$ have different growth rates, but the same power of growth $9^n$ (Proposition 5.2). By using this we conclude that (Theorems 5.4 and 6.1).

**Theorem 1.2.** For the eyebolted Vicsek cross $K$ in Figure 5 the critical exponents are

$$\sigma^* = \sigma^# = \frac{1}{2} \left(1 + \frac{\log 21}{\log 9}\right).$$

Moreover,

(i) $B_{2,\infty}^{\sigma^*} (\subset C(K))$ is dense in $L^2(K, \mu)$, but not dense in $C(K)$;

(ii) there are two kinds of (non-primal) local regular Dirichlet forms that can be constructed on $K$, one satisfies the energy self-similar identity; the other follows from a “reverse recursive construction”. Their domains are different from $B_{2,\infty}^{\sigma^*}$.

From (i) and Proposition 1.1 we see that we can not have a regular (sufficiently many continuous functions) Dirichlet form from the renormalized limit of the primal energy and has $B_{2,\infty}^{\sigma^*}$ as its domain. On the other hand, in (ii), we can use different conductances to obtain energy forms that yield local regular Dirichlet forms on $K$. One construction gives
an energy form that satisfies the energy self-similar identity \([20]\); it provides a concrete constructive proof to implement the abstract proof (fixed point theory) for the existence of such energy form on asymmetric p.c.f. sets \([23, 25, 28, 13, 26]\). The other construction, we call it reverse recursive method, is to fix an initial data at \(V_0\), and iterate this to \(V_n\) to obtain a sequence of compatible networks. This method first appeared in a probabilistic study by Hattori, Hattori and Watanabe \([15]\) on the Sierpinski gasket \(K\) (abc-gasket), they showed that there is an asymptotically one-dimensional diffusion process on \(K\). Some further development and extensions can be found in \([10, 11, 12, 14]\) by Hambly et al, and in \([8]\) by the authors.

We call the second example a Sierpinski sickle. It is a connected p.c.f. set \(K\) generated by an IFS of 17 similitudes of contraction ration \(1/7\) (see Figure 7); the boundary \(V_0\) has three points. By using the \(\Delta-Y\) transform, we show that the traces \(R_n(p, q), p, q \in V_0\) are comparable to \(7^n\) and \((17/2)^n\) (Proposition 5.7). By using this, we conclude that (Theorems 5.8, and 6.2).

**Theorem 1.3.** For the Sierpinski sickle (Figure 7), we have

\[
\sigma^* = \frac{1}{2} \left(1 + \frac{\log 17}{\log 7}\right), \quad \sigma^# = \frac{1}{2} \left(\frac{2 \log 17 - \log 2}{\log 7}\right).
\]

Moreover,

(i) \(B^*_{2,\infty} \subseteq C(K)\) is dense in \(C(K)\), and \(B^#_{2,\infty}\) is dense in \(L^2(K, \mu)\).

(ii) there are (non-primal) Dirichlet forms on \(K\) that satisfy the energy self-similar identity; but the reverse recursive method does not yield a Dirichlet form on \(K\).

We remark that not all asymmetric p.c.f. set \(K\) will give inhomogeneous rate on the \(R_n(p, q)\)'s. In fact, in the above two examples, the construction is quite delicate; if we make small variances on the IFS, then the growth rate of the \(R_n(p, q)\)'s on \(K\) will have the same power (as in the Theorem 1.2), and there are Dirichlet forms with energy self-similar identities, and have \(B^*_{2,\infty}\) as domain. We will discuss this in the remark section.

For the organization of the paper, in Section 2, we recall some basic definitions, and make some comments of Proposition 1.1 on the discretization of the Besov norm. We also introduce the notion of the trace (induced resistance) and the \(\Delta-Y\) transform. In Section 3, we introduce the compatible equivalent relations on the network induced by the primal energy, and study the structure of the quotient network. In Section 4, we make use of this to prove some theoretical results for the two critical exponents. We present the two examples in Section 5, and prove the first part of Theorems 1.2 and 1.3. The last part of the two theorems on the construction of Dirichlet forms are given in Section 6. In Section 7, we give some remarks of the two examples, and discuss briefly on another related Besov space \(B^*_{2,2}\) of the non-local Dirichlet forms. We also give an Appendix of the directed graph self-similar sets that is associated with the quotient networks in Section 3.
2. Preliminaries

We first recall the definition of a Dirichlet form. Let \((M,d)\) be a locally compact, separable metric space, and let \(\nu\) be a Radon measure on \(M\) with \(\text{supp}(\nu) = M\); the triple \((M,d,\nu)\) is called a metric measure space. Let \(C_0(M)\) denote the space of continuous functions with compact support.

**Definition 2.1.** On \((M,d,\nu)\), a Dirichlet form \(E\) with domain \(\mathcal{F}\) is a symmetric bilinear form which is non-negative definite, closed, densely defined on \(L^2(M,\nu)\), and satisfies the Markovian property: \(u \in \mathcal{F} \Rightarrow \tilde{u} := (u \vee 0) \wedge 1 \in \mathcal{F}\) and \(E[\tilde{u}] \leq E[u]\). (Here \(E[u] := E(u,u)\) denote the energy of \(u\).)

A Dirichlet form is called regular if \(\mathcal{F} \cap C_0(M)\) is dense in \(C_0(M)\) with the supremum norm, and dense in \(\mathcal{F}\) with the \(E_{1/2}\)-norm. It is called local if \(E(u,v) = 0\) for \(u,v \in \mathcal{F}\) having disjoint compact supports.

The importance of a local regular Dirichlet form is that it induces a Laplacian on \(M\). However it is a non-trivial matter to construct or to prove the existence of such form. In fact there are only limited class of self-similar sets on which the existence of Laplacians is known. Throughout we will consider the specific class of p.c.f. self-similar sets \([20]\), which is defined in the following. Unless otherwise specify, we will assume \(M = K\) as a compact subset in \(\mathbb{R}^d\) and \(d\) is the Euclidean metric.

Let \(\{F_i\}_{i=1}^N\) be an IFS on \(\mathbb{R}^d\) such that
\[
F_i(x) = \rho(x - b_i) + b_i, \quad 1 \leq i \leq N,
\]
where \(0 < \rho < 1\) and \(b_i \in \mathbb{R}^d\). Let \(K = \bigcup_{i=1}^N F_i(K)\) be the self-similar set, and let \(\mu\) be the self-similar measure defined by \(\mu = \frac{1}{N} \sum_{i=1}^N \mu \circ F_i^{-1}\). If the IFS satisfies the open set condition (OSC), i.e., there is a nonempty bounded open set \(O\) such that \(F_i(O) \subset O\) and \(F_i(O) \cap F_j(O) = \emptyset\) for \(i \neq j\), then the Hausdorff dimension of \(K\) is \(\text{dim}_H(K) = \alpha = \frac{\log N}{\log \rho}\), and \(\mu\) is the \(\alpha\)-Hausdorff measure normalized on \(K\), it is \(\alpha\)-regular in the sense of [1.1]. Without loss of generality, we always assume that \(K\) is connected.

We define the symbolic space of \(K\) as usual. Let \(\Sigma = \{1, \cdots, N\}\) be the alphabets, \(\Sigma^n\) the set of words of length \(n\), and \(\Sigma^\infty\) the set of infinite words \(\omega = \omega_1 \omega_2 \cdots\); let \(\tau : \Sigma^\infty \to K\) be defined by \(\{x\} = \{\pi(\omega)\} = \bigcap_{n \geq 1} K_{\omega_1 \cdots \omega_n}\), a symbolic representation of \(x \in K\) by \(\omega\).

Following Kigami [20], we define the critical set \(C\) and the post-critical set \(\mathcal{P}\) for \(K\) by
\[
C = \pi^{-1}\left(\bigcup_{1 \leq i < j \leq N} (K_i \cap K_j)\right), \quad \mathcal{P} = \bigcup_{m \geq 1} \tau^m(C),
\]
where \(K_i = F_i(K)\), \(\tau : \Sigma^\infty \to \Sigma^\infty\) is the left shift by one index. If \(\mathcal{P}\) is a finite set, we call \(\{F_i\}_{i=1}^N\) a post-critically finite (p.c.f.) IFS, and \(K\) is a p.c.f. self-similar set. The boundary of \(K\) is defined to be \(V_0 = \pi(\mathcal{P})\). (We always assume \(#(V_0) \geq 2\) to avoid triviality.) We also define
\[
V_n = \bigcup_{i \in \{1, \cdots, N\}} F_i(V_{n-1}), \quad V_\omega = \bigcup_{n \geq 1} V_n.
\]
It is clear that \(\{V_\omega\}_{\omega \in \Sigma^n}\) is an increasing sequence of sets, and \(K\) is the closure of \(V_\omega\). We call \(V_\omega := F_\omega(V_0)\) a cell of \(V_n\) for any \(\omega \in \Sigma^n\), where \(F_\omega = F_{\omega_1} \circ \cdots \circ F_{\omega_n}\).
It is known that a p.c.f. IFS in (2.1) satisfies the open set condition [5] (More generally, this is true if the associate similar matrices $A_i$ of $F_i$ (instead of the $\rho$ in (2.1)) are commensurable i.e., there exists $A$ such that $A_i = A^n$; but it is not true without this assumption [30].) Hence the p.c.f. self-similar set $K$ has dimension $\alpha$, and is associated with a self-similar measure $\mu$ that is $\alpha$-regular.

For a Besov space $B^\sigma_{2,\infty}$ on a compact set $K$ with an $\alpha$-regular measure, we recall a continuity property of its functions ([9], over there the following proposition is put under the assumption that a heat kernel exists, but it was not used in the proof).

**Proposition 2.2.** For $2\sigma > \alpha$, then the identity map $\iota: B^\sigma_{2,\infty} \rightarrow C^{(2\sigma-\alpha)/2}(K)$ is a continuous embedding. (Here $C^\alpha(K)$ denotes the class of Lipschitz functions on $K$.)

The discretized version of a Besov space in Proposition [1.1] was first established by Jonsson [17] on the Sierpinski gasket, and he showed that the critical exponent $\sigma^* = \log 5 \over \log 2$. He also introduced the notion of regular triangular system (RTS) on the $d$-sets ($d$ is $\alpha$ here) to study the piecewise linear bases of the Besov space generated by this system [18]. In [3], Bodin extended Jonsson’s discretization theorem to the Besov spaces $B^\sigma_{p,q}$, $1 \leq p, q \leq \infty$, for a $d$-set that admits a RTS. He stated without proof that similar to the RTS case, the discretization is also true for p.c.f. sets. Actually there are technical steps that need to be justified, and they are provided in [7]. For our purpose, we will need Proposition [1.1] in a slightly more general form.

**Corollary 2.3.** With the p.c.f. self-similar set defined as above, then for $2\sigma > \alpha$ and for any integer $\ell > 0$,

$$\left[u\right]^2_{B^\sigma_{2,\infty}} = \sup_{j \geq 0} \left\{ \rho^{-2(\sigma-\alpha)\ell} \sum_{x, y \in F_\ell(V_0); |\omega| = \ell} |u(x) - u(y)|^2 \right\}. \quad (2.2)$$

We will make frequently use of the following proposition to construct functions in $B^\sigma_{2,\infty}$ [7].

**Proposition 2.4.** Assume $2\sigma > \alpha$, then for any function $u$ on $V_\ast$, if $u$ satisfies

$$\sup_{j \geq 0} \left\{ \rho^{-2(\sigma-\alpha)j} \sum_{x, y \in V_\ast; |\omega| = j} |u(x) - u(y)|^2 \right\} < \infty,$$

$u$ can be extended continuously to $\bar{u}$ on $K$, and $\bar{u} \in B^\sigma_{2,\infty}$.

Let $\ell(V_\ast)$ denote the class of real-valued functions on $V_\ast$. For $u \in \ell(V_\ast)$, we define an energy form $\mathcal{E}_n[u]$ on $V_\ast$, $n \geq 0$, by

$$\mathcal{E}_n[u] = \sum_{x, y \in F_\omega(V_0); |\omega| = n} c_n(x, y)|u(x) - u(y)|^2, \quad (2.3)$$

where $c_n(x, y)$ is the conductance of the nodes $x, y$. In literature, the most studied approach to construct a Dirichlet form on a p.c.f. set is to consider the sequence $\mathcal{E}_{n+1}[u] = \sum_{i=1}^{N} \tau_i^{-1} \mathcal{E}_n[u \circ F_i]$, where $0 < \tau_i < 1$ are the renormalization factors. If $\mathcal{E}[u] = \lim_{n \rightarrow \infty} \mathcal{E}_n[u]$ exists for all $u \in \ell(V_\ast)$, then $\mathcal{E}$ satisfies the energy self-similar identity

$$\mathcal{E}[u] = \sum_{i=1}^{N} \tau_i^{-1} \mathcal{E}[u \circ F_i], \quad u \in \mathcal{F}, \quad (2.4)$$
and defines a local regular Dirichlet form on $L^2(K, \mu)$ for a given Radon measure $\mu$ fully supported on $K$ \cite{20, 29}. If all the $\tau_i$ are equal, then the Dirichlet form $\mathcal{E}$ on the metric measure space $(K, |\cdot|, \mu)$ has domain $\mathcal{F} = B^\infty_{\sigma, \infty}$ ($\mu$ is the normalized $\alpha$-Hausdorff measure on $K$). If the $\tau_i$’s are not all equal, then we can consider the metric measure space $(K, d_r, \nu)$, where $d_r$ is the resistance metric on $K$, and $\nu$ is the self-similar measure with weights $\{\tau_i\}_{i=1}^N$ where $\sum_{i=1}^N \tau_i = 1$, and the domain $\mathcal{F}$ is a modified Besov space with respect to $(K, d_r, \nu)$ \cite{20, 27, 9, 16}.

Let $G_n := (V_n, r_n)$ denote the corresponding electrical network of \cite{25} with resistance $r_n(x, y) = c_n(x, y)^{-1}$, $x, y \in V_n$ as resistance. It is known that \cite[Theorem 2.1.6]{20} for any $m < n$, there is an induced network of $G_n$ on $V_m$ with resistance $R_{n,m}(x, y)$ such that for $u \in \ell(V_m)$,

$$\min \{ \mathcal{E}_n[u] : u \in \ell(V_n), u|_{V_m} = u \} = \sum_{x, y \in V_m} \frac{1}{R_{n,m}(x, y)} |u(x) - u(y)|^2. \quad (2.5)$$

Let $\{R_i\}_{i=0}^\infty$ be an increasing sequence of positive real numbers, suppose there exists $R > 1$ such that for any $\varepsilon > 0$, there exists $N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$,

$$R^{\varepsilon n} \leq R_n \leq R^{(1+\varepsilon)n},$$

then we call $R$ the asymptotic geometric growth rate of $R_n$.

\textbf{Definition 2.5.} We call $R_{n,m}(x, y)$, $x, y \in V_m$ the trace (or the induced resistance) of $G_n$ on $V_m$. In particular, for $m = 0$, we will use the notation $R_n(p, q)$, $p, q \in V_0$ for simplicity. We also use $R(p, q)$ to denote the asymptotic geometric growth rate of $R_n(p, q)$ if it exists.

A function $h$ on $V_n$ is called harmonic on a subset $E \subset V_n$ if $h(x) = \sum_{x-y} c_n(x, y)h(y)$, $x \in E$. In the above, the function $v \in \ell(V_n)$ that attains the minimum (always exists) is a harmonic function on $V_n \setminus V_m$; we call it a harmonic extension of $u$ on $V_m$ to $V_n$. As $v$ is harmonic on the “interior” of each subcell of $V_m$, we see that $v$ is a “piecewise harmonic” function on $V_n$. These functions will be used to construct continuous functions in $B^\infty_{\sigma, \infty}$ as in the following.

\textbf{Proposition 2.6.} For the primal energy $E_n[u], n \geq 0$ as defined in \cite{1, 3}, suppose $\sigma$ satisfies $2\sigma > \alpha$, and there exists an integer $N \geq 1$ such that

$$\rho^{-(2\sigma-\alpha)N} \leq R_n(p, q), \quad \forall \ p, q \in V_0, \ p \neq q, \quad (2.6)$$

then $u \in \ell(V_0)$ has an extension to $K$, and consequently, $B^\infty_{\sigma, \infty}$ is dense in $C(K)$.

\textbf{Proof.} For any $u \in \ell(V_0)$, from the trace of $G_N$ on $V_0$, we have

$$\min_{v \in \ell(V_N), v|_{V_0} = u} E_N[v] = \sum_{p,q \in V_0} \frac{1}{R_n(p, q)} |u(p) - u(p)|^2,$$

where $E_N[v]$ is defined as in \cite{1, 3}. Multiplying $\rho^{-(2\sigma-\alpha)N}$ to both sides, and by (2.6), we obtain

$$\min_{v \in \ell(V_N), v|_{V_0} = u} \left\{ \rho^{-(2\sigma-\alpha)N} E_N[v] \right\} \leq \sum_{p,q \in V_0} |u(p) - u(q)|^2.$$
With no confusion, we use $u$ again to denote the unique function in $\ell(V_N)$ that attains the minimum. By using this $u$ as initial data on each $F_\omega(V_0)\setminus\{\omega\} = N$, and continue this extension procedure to $V_{2N}, V_{3N}, \cdots$, there is $u$ on $V = \bigcup_{n \geq 0} V_n$ such that for all $k \geq 1$,

$$\rho^{-(2\sigma-\alpha)kn}E_{kn}[u] \leq \sum_{p,q \in V_0} |u(p) - u(q)|^2.$$ 

By Corollary 2.3 and Proposition 2.4, $u$ can be extended continuously to $K$ and $u \in B_{2,\infty}^\sigma$.

It follows that for any $v \in C(K)$, if we let $v_n$ be the restriction of $v$ on $V_n$, we can extend $v_n$ on each cell $K_\omega, |\omega| = n$ so that $v_n \in B_{2,\infty}^\sigma$ (this $v_n$ is a piecewise harmonic function). The sequence $\{v_n\}_{n=1}^\infty$ converges to $v$ uniformly. This shows that $B_{2,\infty}^\sigma$ is dense in $C(K)$.

To evaluate the trace $R_n(p,q)$ and estimate the energy functional on a network, we will use some elementary techniques like the series law and parallel law of resistance and the $\Delta$-Y transform. Recall the $\Delta$-Y transform [20, 29] states that the $\Delta$-shaped resistors $(R_{12}, R_{23}, R_{31})$ and the $Y$-shaped resistors $(a, b, c)$ in Figure 1 in any network are equivalent by the following relation

$$a = R_{12}R_{31}R, \quad b = R_{12}R_{23}R, \quad c = R_{31}R_{23}R,$$ 

with $R = R_{12} + R_{23} + R_{31}$, and conversely,

$$R_{12} = \frac{r}{c}, \quad R_{23} = \frac{r}{b}, \quad R_{31} = \frac{r}{a},$$

where $r = ab + bc + ca$.

![Figure 1. $\Delta$-Y transform](image)

In the example of eyebolted Vicsek cross in Section 5, we need to use an electrical network with four terminals. We give a version of equivalent electrical networks similar to the $\Delta$-Y transform, and call it the $\boxplus$-X transform.

**Lemma 2.7.** For the two electrical networks as shown in Figure 2 and assume that $yz = x^2$, then they are equivalent and the resistances satisfy

$$a = \frac{xy}{2(x+y)}, \quad \text{and} \quad b = \frac{xz}{2(x+z)} \left( = \frac{x^2}{2(x+y)} \right);$$

equivalently,

$$x = 2(a+b), \quad y = \frac{2a}{b}(a+b), \quad \text{and} \quad z = \frac{2b}{a}(a+b).$$
Figure 2. equivalent networks with four vertices

Proof. We only outline the proof of the identity for $a$. By using the $\Delta$-Y transform on the square together with $p_2p_4$, it is easy to calculate the effective resistance of $p_1$ and $p_3$ is $x$ (this can also be obtained by observing that no current should pass through $p_2p_4$). Then take this in parallel with the resistance $y$ on $p_1p_3$, we have the desired expression. □

3. Quotient network

In this section, We will set up the equivalent relations and quotients on $V_n, n \geq 0$ which was first considered by Sabot [28]. It will be used to study the other critical exponent $\sigma^#$ in Theorem 4.3.

Definition 3.1. Let $\sim$ be an equivalence relation on $V_0$ that contains at least two equivalent classes. We define the induced equivalent relation $\sim_n$ to be the smallest equivalent relation on $V_n$ generated by

(i) (embedding) for $x \sim_{n-1} y$ in $V_{n-1} (\subset V_n)$, then $x \sim_n y$ in $V_n$;

(ii) (self-similar) for $x \sim_{n-1} y$ in $V_{n-1}$, then $F_i(x) \sim_n F_i(y)$ for $1 \leq i \leq N$.

We say that $\sim$ is a compatible (equivalence) relation if for any $n \geq 0$ and any $x, y \in V_n$, $x \sim_n y$ in $V_n$ if and only if $x \sim_{n+1} y$ in $V_{n+1}$.

We will omit the subscript $n$ when there is no confusion, and we write $V_0 = \bigcup J_i$ where the $J_i$’s are equivalent classes of $V_0$, and. Note that (ii) implies

$$F_\omega(p) \sim F_\omega(q), \quad \text{if} \quad p \sim q, \ p, q \in V_0;$$

furthermore if there are $q' \in V_0, |\omega'| = |\omega|$ such that $F_\omega(q) = F_{\omega'}(q')$, then for $p' \in V_0$ and $p' \sim q'$, then $F_\omega(p) \sim F_{\omega'}(p')$. We will use $V_n^-, n \geq 0$ to denote the quotient spaces, i.e.,

$$V^-_n = \{ [F_\omega(J)] : J \in V_0^-, |\omega| = n \}.$$

Here $[F_\omega(J)]$ is the union of the $F_\omega(J'), J' \in V_0^-, |\omega'| \leq n$ where $F_\omega(J_i) \cap F_{\omega_{i+1}}(J_{i+1}) \neq \emptyset$ for a finite sequence of cells in $V_m, m \leq n$ with $F_\omega(J) = F_{\omega_1}(J_1), \cdots, F_{\omega_k}(J_k) = F_\omega(J')$.

In view of (i), the compatible condition is only imposed on the sufficiency. It follows that for $m \leq n$, $V_m^-$ can be identified as a subset of $V_n^-$. The compatible relation therefore induces an equivalence relation on $V_n^-$ (also denote by $\sim$) with $V_n^- = \bigcup V_n^-$ such that
each equivalent class \( J \in V_s \) is the union of an increasing sequence of equivalent classes \( J^{(n)} \in V_n \). It is easy to show inductively that if \( J^{(n)}_1, J^{(n)}_2 \) are distinct in \( V_n^{(n)} \), then \( J^{(m)}_1, J^{(m)}_2 \) are distinct in \( V_m^{(m)} \) for \( m \geq n \), so that \( J^{(1)}_1, J^{(1)}_2 \) are distinct in \( V_s^{(s)} \).

We call an equivalent class \( J \) of \( V_n \) (or \( V_s \)) a boundary class if \( J \cap V_0 \neq \emptyset \), and a non-boundary class otherwise.

**Examples.** For the Sierpinski gasket with \( V_0 = \{ p_1, p_2, p_3 \} \), the partition \( J_1 = \{ p_1, p_3 \} \), \( J_2 = \{ p_2 \} \) defines a compatible relation. It is easy to see that an element of \( V_n^{(n)} \), \( n \geq 2 \), is either a single vertex or is consisted of consecutive vertices on a line segment parallel to \( p_1p_3 \) (see Figure 4). There are two boundary classes in \( V_s \), \( \{ p_2 \} \) and the set of dyadic points on the line segment \( p_1p_3 \).

Consider the pentagasket with \( V_0 = \{ p_i \}_{i=1}^5 \) arranged in the counterclockwise direction. \( J_i = \{ p_i \}, i = 1, 2, 3 \), \( J_4 = \{ p_4, p_5 \} \), then it defines a compatible relation. There are four boundary classes in \( V_s \): the three singletons \( J_i, i = 1, 2, 3 \), and \( J_4^* \), which is a Cantor-set on the line segment \( p_4p_5 \).

On the pentagasket, if we let \( J_1 = \{ p_1, p_2 \} \), \( J_2 = \{ p_3, p_4, p_5 \} \), then it is again a compatible relation, but the two boundary classes is more complicated (see Figure 3), its structure follows from Theorem 3.2.

**Remark.** In [28], Sabot first made use of the equivalent relation to study the Dirichlet form on a ramified self-similar set with a symmetric group \( G \) acting on \( V_0 \). He defined a \( G \)-relation on \( V_0 \) by \( x \sim y \Rightarrow gx \sim gy \) where \( g \in G \), and extended this to \( V_n \) by rule (i) and required it to be compatible (it is called preserved \( G \)-relation [28], Section 4.2.1, Definition 4.19)). This induce equivalent relation on \( V_n \) is different from ours, which is generated by both rules (i) and (ii). The former definition is more limited, as it is easy to check that on the pentagasket, under rule (ii) only, then all non-trivial relation cannot be compatible.

In the following we will prove a theorem on the structure of the equivalent classes. For convenience, we call a set equivalent set if it is consisted of equivalent elements. For \( A, B \) equivalent sets, we write \( A \sim B \) if there are \( x \in A \), \( y \in B \) such that \( x \sim y \).

**Theorem 3.2.** Let \( V_0 = \bigcup_{i=1}^s J_i \) be the union of the equivalent classes of a compatible relation, and let \( \{ J^{(s)}_i \}_{i=1}^s \) be the family of boundary classes in \( V_s \). Then \( \{ J^{(s)}_i \}_{i=1}^s \) are attractor of a graph directed system, and \( \dim_H(J^{(s)}_i) = \dim_B(J^{(s)}_i) \).

For the non-boundary classes, they are finite unions of contractive similitude images of \( \{ J^{(s)}_i \}_{i=1}^s \).

**Proof.** Let \( V_0 = \bigcup_{i=1}^s J_i \), and let \( J^{(n)}_i \) be the boundary classes in \( V_n \), \( n \geq 0 \) (\( J^{(0)}_i = J_i \)). Note that by (ii), \( F_k(J^{(n)}_i) \) is an equivalent set. It is easy to see that \( J^{(1)}_i \) is generated by the vertices in \( \mathcal{S}_i := \{ F_k(J^{(0)}_i) : F_k \) has fixed point in \( J^{(0)}_i \}\) (by (i)), together with those in \( \{ F_k(J^{(0)}_j) : F_k(J^{(0)}_j) \sim J, J \in \mathcal{S}_i, j \neq i \}\). We can hence express \( J^{(1)}_i \) as

\[
J^{(1)}_i = \bigcup_{j=1}^s \bigcup_{k \in \Gamma_{ij}} F_k(J^{(0)}_j), \quad 1 \leq i \leq s.
\]
where $\Gamma_{i,j} = \{k : F_k(J_j^{(0)}) \sim J, J \in \mathcal{S}_i\}$ (see Figure 3b). We denote these boundary classes of \{J_i^{(1)}\}_{i=1}^t by $\mathcal{B}^{(1)}$.

To determine the non-boundary classes \{I_i^{(1)}\}_{i=1}^t in $V_1$ (it may be empty), we replace the above $\mathcal{S}_i$ by $\mathcal{T}_i := \{F_k(J_j^{(0)}) : F_k(J_j^{(0)}) \sim J, \forall J \in \mathcal{S}_j, 1 \leq j \leq s\}$, and by the same way, we obtain

$$I_i^{(1)} = \bigcup_{j=1}^{s} \bigcup_{k \in \Lambda_{ij}} F_k(J_j^{(0)}), \quad 1 \leq i \leq t$$

(see Figure 3b). Note that $I_1^{(1)} \subset V_1 \setminus V_0$, and the union of all the boundary and non-boundary classes is $V_1$. We denote this class $\{I_i^{(1)}\}_{i=1}^t$ by $\mathcal{N}_1^{(1)}$.

Next, using the above procedure in $V_2$, we obtain,

$$J_i^{(2)} = \bigcup_{j=1}^{s} \bigcup_{k \in \Gamma_{i,j}} F_k(J_j^{(1)}), \quad I_i^{(2)} = \bigcup_{j=1}^{s} \bigcup_{k \in \Lambda_{ij}} F_k(J_j^{(1)}),$$

denote the two classes by $\mathcal{B}^{(2)}$ and $\mathcal{N}_1^{(2)}$. There are other non-boundary classes appeared, namely, those $F_k(\mathcal{N}_1^{(1)})$, $1 \leq k \leq N$ (see Figure 3c), note that for $I_i^{(1)} \in \mathcal{N}_1^{(1)}$, we have $I_i^{(1)} \subset V_1 \setminus V_0$; hence $F_k(I_i^{(1)}) \subset F_k(V_1) \setminus F_k(V_0)$, so that $F_k(I_i^{(1)})$ remains an equivalent class in $V_2$). We denote this family by $\mathcal{N}_2^{(2)}$. Hence the family of non-boundary classes is $\mathcal{N}_{1}^{(2)} \cup \mathcal{N}_{2}^{(2)}$.

Inductively, we obtain $\mathcal{B}^{(n)}$ and $\mathcal{N}^{(n)} := \mathcal{N}_1^{(n)} \cup \mathcal{N}_2^{(n)} \cdots \cup \mathcal{N}_n^{(n)}$ where $\mathcal{N}_\ell^{(n)} = \{F_k(\mathcal{N}_{\ell-1}^{(n)}) : 1 \leq k \leq N\} = \{F_\omega(\mathcal{N}_{\ell-1}^{(n-1)}) : |\omega| = \ell - 1, 2 \leq \ell \leq n\}$. Therefore, for $J_i^{*} \in \mathcal{B}^{\star}$, i.e., $J_i^{*} \subset V_\star$, we have

$$J_i^{*} = \bigcup_{j=1}^{s} \bigcup_{k \in \Gamma_{i,j}} F_k(J_j^{(n-1)}), \quad 1 \leq i \leq s,$$

and $\Gamma_{i,j} = \{k : F_k(J_j^{(n-1)}) \cap J_i^{(n-1)} \neq \emptyset\}$. We can set up the graph directed system $(\mathcal{V}, \Gamma)$ with $\mathcal{V} = \{1, \cdots, s\}$ as the index of the $J_i^{*}$, and $\Gamma = \bigcup_{1 \leq i,j \leq s} \Gamma_{i,j}$ as the edge set. Therefore, $J_i^{*}, 1 \leq s \leq j$ are the attractors of $(\mathcal{V}, \Gamma)$, and hence graph directed sets [24]. For $\mathcal{N}^{\star}$, we have for $I_i^{*} \in \mathcal{N}_1^{\star}$,

$$I_i^{*} = \bigcup_{j=1}^{s} \bigcup_{k \in \Lambda_{ij}} F_k(J_j^{*}),$$

and for $\ell \geq 2, \mathcal{N}_\ell^{*} = \{F_\omega(\mathcal{N}_1^{*}) : |\omega| = \ell - 1\}$. Hence for $I_i^{*} \in \mathcal{N}_\ell^{*}, \bar{I}_\ell^{*}$ is a finite union of graph directed sets.

That $\dim_B(\bar{J}_i^{*}) = \dim_B(\bar{J}_i^{*})$ is in [6, p. 42] where the graph directed system is assumed to be irreducible and the IFS is strongly separated. In the present case, the IFS satisfies the OSC; the proof can easily be adjusted by observing that every ball of radius $r$ can intersect at most a number $\ell$ of cells of comparable size for some $\ell$. For the non-irreducible case, Mauldin and Williams in [24, Theorems 4 and 5] studied the Hausdorff measures of the attractors (they can be infinite), but left out the box dimension. This can easily be supplemented, and for the sake of completeness and for the need of the box count in Lemma 3.4, we will sketch the main idea of this in the Appendix. \hfill $\Box$

To obtain some separation property of the boundary classes, we introduce a property on the compatible relation:

(B) Any 1-cell can intersect at most one boundary class in $V_1$. 

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Hence (i) and (ii) in Definition 3.1), i.e., \( J \) and \( F \).

Under property (B), any \( n \)-cell \( V \) in \( \mathbb{Z}^n \) satisfies the following properties:

**Lemma 3.3.** Under property (B), any \( n \)-cell \( V \), \( n \geq 1 \), intersects at most one boundary class in \( V \).

**Proof.** We use \( \omega^- \) to denote the parent of \( \omega \). Let \( V_\omega \) be an \( n \)-cell, \( n \geq 2 \). For any boundary class \( J \) in \( V_\omega \), \( J \cap V_\omega^- \) is a finite union of \( F_\omega^- \)-images of boundary classes in \( V_\omega \) (by rules (i) and (ii) in Definition [3.1]), i.e., \( F_\omega^- (J \cap V_\omega^-) \) is a finite union of equivalent classes in \( V_\omega \). As \( F_\omega^- (V_\omega) \) is a 1-cell in \( V_\omega \), it intersects at most one \( F_\omega^- (J \cap V_\omega^-) \) (by property (B)). Hence \( V_\omega \) intersects at most one boundary class in \( V_\omega^- \) \( \square \).

Let \( E_n[u] = \sum_{x,y \in V_n, |u| = n} |u(x) - u(y)|^2 \) be the primal energy of \( u \). We will extend the consideration to the quotient network. For \( u \in \ell(V_n^-) \), \( u \) can be considered as a function in \( \ell(V_n^-) \) that takes constant value on each equivalent class \( J \in V_n^- \). We have

\[
E_n[u] = \sum_{J,J' \in V_n^-} n_{J,J'} |u(J) - u(J')|^2 = E_n[u], \quad u \in \ell(V_n^-), \tag{3.1}
\]

where \( n_{J,J'} \) is the number of the edges connecting \( p \in J \) and \( q \in J' \). (Note that \( n_{J,J'} \) is well defined as the set of equivalent classes gives a partition of \( V_n \), hence for \( J \) as a subset in \( V_n \), each edge going out \( J \) will meet with another equivalent class.) Let \( R_n(J,J') \) denote the corresponding trace of \( V_n^- \) on \( V_n^- \), then for \( u \in \ell(V_n^-) \), as in (2.5), we have

\[
\min_\{v \in \ell(V_n^-), v|_{V_n^-} = u\} E_n[v] = \sum_{i \neq j} \frac{1}{R_n(J_i,J_j)} |u(J_i) - u(J_j)|^2, \tag{3.2}
\]

We denote by \( R^-(J_i,J_j) \) the asymptotic geometric growth rate of \( \{R_n(J_i,J_j)\}_n \) if exists (see Definition [2.5]).

For \( u \in \ell(V_n^-) \), and for a non-trivial equivalent class \( J \) (i.e., contain more than one point) of \( V_n^- \), we denote by \( E_{J,n}(u) \) the energy of \( u \) on \( J \): the summation of all the terms \( (u(x) - u(y))^2 \) in \( E_n(u) \) with \( x,y \in J \cap V_n, |\omega| = n \). Similarly, we can define \( E_{J,J',n}(u) \) with
Let \( J \) be a boundary class in \( V_\gamma \), and \( \bar{J} \) has Hausdorff dimension \( \gamma \). Suppose there exists \( p, q \in V_0, p \neq q \) such that \( R := R(p, q) < \rho^{-(2-\gamma)} \), then for any Lipschitz function \( u \) on \( J \), we have

\[
\sup_{n \geq 0} R^n E_{J, n}(u) < \infty \tag{3.3}
\]

Furthermore if \( \mathcal{H}^\gamma(\bar{J}) < \infty \), then the condition can be relax to \( R \leq \rho^{-(2-\gamma)} \) and the same result holds.

Proof. Let \( N(\rho^n) \) be the count of the \( n \)-cells \( K_\omega \) that intersects \( J \). It follows from Theorem 3.2 that the Hausdorff dimension of \( \bar{J} \) equals its box dimension, and \( \gamma = \lim_{n \to \infty} \frac{\log N(\rho^n)}{\log \rho^n} \).

Hence for \( \varepsilon > 0 \) satisfies \( R < \rho^{-(2-(\gamma+\varepsilon))} \), there exists \( C > 0 \) such that \( N(\rho^n) < C \rho^{-n(\gamma+\varepsilon)} \). This implies that for all \( n \),

\[
R^n E_{J, n}(u) \leq R^n \sum_{x,y \in J} |u(x) - u(y)|^2 \leq C' R^n \rho^{-n(\gamma+\varepsilon)} \rho^{2n} < C',
\]

and (3.3) follows. If \( \mathcal{H}^\gamma(\bar{J}) < \infty \), then we can actually have \( N(\rho^n) < C \rho^{-\varepsilon n} \) (see Appendix), and the above inequality holds for \( R \leq \rho^{-(2-\gamma)} \). \( \square \)

4. Critical exponents of Besov spaces

In this section, we prove some general results for the critical exponents \( \sigma^* \) and \( \sigma^\# \) of the Besov spaces on the p.c.f. sets with respect to the primal energy, then apply them to the two concrete cases in the next section.

Theorem 4.1. Let \( K \) be a p.c.f. self-similar set with an IFS satisfying (2.1). Assume \( R(p, q) (> 1) \), \( p, q \in V_0 \) exist, and let

\[
R^* = \min \{ R(p, q) : p \neq q, \ p, q \in V_0 \},
\]

then for the Besov spaces \( B^\sigma_{2,\infty} \) defined on \( K \), the critical exponent \( \sigma^* = \frac{1}{2} \left( \frac{\log R^*}{\log \rho} + \sigma \right) \).

Furthermore at \( \sigma^* \),

(i) if \( R^n \leq R_n(p, q) \) for all \( n \geq 0 \) and \( p, q \in V_0 \), then \( B^\sigma_{2,\infty} \) is dense in \( C(K) \);

(ii) if \( R^* \) satisfies \( \lim_{n \to \infty} \frac{R_n(p, q)}{R_{n+1}(p, q)} = \infty \) for some \( p, q \in V_0 \), then \( B^\sigma_{2,\infty} \) is not dense in \( C(K) \).

Proof. Let \( \sigma = \frac{1}{2} \left( \frac{\log R^*}{\log \rho} + \sigma \right) + 2\varepsilon \), \( \varepsilon > 0 \), then \( 2\sigma - \sigma > 0 \). We will prove that there exist \( p, q \in V_0, p \neq q \), such that \( u(p) = u(q) \) for any \( u \in B^\sigma_{2,\infty} \). This implies \( B^\sigma_{2,\infty} \) is not dense in \( C(K) \), and by definition, we have \( \sigma^* \leq \frac{1}{2} \left( \frac{\log R^*}{\log \rho} + \sigma \right) \).

To this end, for any \( u \in B^\sigma_{2,\infty} \), we restrict \( u \in \ell(V_0) \) with values \( u(p_i), \ p_i \in V_0 \). From the trace of \( G_n \) on \( V_0 \), we obtain

\[
\min_{v \in \ell(V_0), v|_{V_0} = u} E_n[v] = \sum_{i \neq j} R_n(p_i, p_j) |u(p_i) - u(p_j)|^2. \tag{4.1}
\]
Let $R^*_e = \rho^{-2\varepsilon} R^e$. Multiplying $R^*_e$ to both sides of (4.1), it reduces to

$$\min_{v \in \ell(V_0)} \left\{ \rho^{\left(\frac{\log R^e}{-\log \rho}\right) + 2\varepsilon} E^*_n[v] \right\} = \sum_{i \neq j} \frac{R^*_e}{R_n(p_i, p_j)} |u(p_i) - u(p_j)|^2. \quad (4.2)$$

By the definition of $R^*$, we have $R^* = R(p, q)$ for some $p, q \in V_0$, and

$$\frac{R^*_e}{R_n(p, q)} \geq \rho^{-\varepsilon n} \to \infty \quad \text{as} \quad n \to \infty. \quad (4.3)$$

As $u \in B_{2,\infty}^{\sigma}$, the left hand side of (4.2) is uniformly bounded for all $n > 0$ (by Proposition 1.1). Hence by (4.3), $u(p) = u(q)$ on the right hand side of (4.2). This completes the proof of $\sigma^* \leq \frac{1}{2} \left( \frac{\log R^e}{-\log \rho} + \alpha \right)$.

Next we consider $\sigma < \frac{1}{2} \left( \frac{\log R^e}{-\log \rho} + \alpha \right)$ such that $2\sigma - \alpha > 0$. By the definition of $R^*$, we can find an $N = N(\sigma)$ such that

$$\rho^{-(2\sigma - \alpha)N} \leq R_N(p, q), \quad \forall \ p, q \in V_0, \ p \neq q. \quad (4.4)$$

Then by Proposition 2.6, $B_{2,\infty}^{\sigma}$ is dense in $C(K)$. Therefore $\frac{1}{2} \left( \frac{\log R^e}{-\log \rho} + \alpha \right) \leq \sigma^*$.

For (i) in the last part, we have

$$\rho^{-(2\sigma^* - \alpha)n} = R^n \leq R_n(p, q) \quad \forall \ p, q \in V_0,$n

and the assertion follows from Proposition 2.6. For (ii), if we replace (4.3) there by the assumption, then the same argument applies and we conclude that $u(p) = u(q)$, so that $B_{2,\infty}^{\sigma*}$ is not dense in $C(K)$. \hfill \Box

It is important to know the density of $B_{2,\infty}^{\sigma}$ in $C(K)$. Part (i) covers the standard cases that renormalization factors exist [20]; there is example in Section 4 (eyebolted Vicsek cross) that satisfies (ii). In addition, we have a situation there (Sierpinski sickle) that $R^n \sim R_n(p, q)$, and the density question is not covered in the above two situations. We make use the technique of quotient network developed in the last section to handle this case.

**Proposition 4.2.** With the assumptions in Theorem 4.1, suppose there is a compatible relation with property (B), and satisfies

(i) For each non-trivial boundary class $J$ of $V_e$ and for any $u$ on $J \cap V_0$, there is an extension of $u$ on $J$ such that $\sup_{g \geq 0} R^e J(u) < \infty$.

(ii) for any two distinct equivalence classes $J_i, J_j$ of $V_0$, $R^e(J_i, J_j)$ exists and satisfies $R(J_i, J_j) > R^*$.

Then $B_{2,\infty}^{\sigma}$ is dense in $C(K)$.

**Remark.** A sufficient condition for the existence of the $u$ in condition (i) is provided in Lemma 3.4. The main idea behind condition (ii) is to use the equivalent class (i.e., shorting in the sense of electrical network) to get rid of the small $R_n(p, q)$, and reduces to an expression analogous to Theorem 4.1(i). An example of this is the Sierpinski sickle in Section 4.
Proof. We let $\mathcal{B}_n$ ($\mathcal{B}_0$) denote the family of boundary classes in $V_n$ ($V_0$ respectively), and let $B_n$ ($B_0$ respectively) be the union of the boundary classes. For $u \in \ell(V_0)$, we show that $u$ can be extended to $V_0$, and has bounded $B_{2,\infty}$-norm. This will imply $B_{2,\infty}$ is dense in $C(K)$ by a similar argument (piecewise extension) as in the last paragraph of Proposition 2.6. We will prove the assumption in three steps. (See Figure 4, note that the SG there is just for simplicity for illustration, it does not satisfies condition (ii))

Step 1. We use (i) to extend $u$ to the $J \in \mathcal{B}$, such that $\sup_{x \in B_{i,j}} R_n^\beta E_{J,n}(u) \leq M$ for some $M > 0$. Hence $u$ is defined on $B_n \subseteq V_n$. We also define $u$ on $V_1 \setminus (V_0 \cup B_1)$ to be $\min_{z \in V_0} u(z)$.

Our main task is to extend $u$ to the rest of $V_0$ and has bounded energy. We call $V_0$ a boundary cell if $V_0 \cap B_n \neq \emptyset$, and a non-boundary cell otherwise.

Step 2. We use induction on $|\omega| \geq 2$ to define $u$ on the boundary cells $V_0$ as well as $V_{0,i}$, $1 \leq i \leq N$, so that $u$ is constant on the equivalent classes. For this, suppose we have defined such $u$ on boundary cells $V_{0,i}$, $|\omega| = n - 1$. We carry out the induction in two steps. For a non-boundary cell $V_{0,i}$ that has an equivalent class $I$ intersects $V_{0,i}$, we assign $u$ on $I$ to take the value $\min_{z \in V_{0,i}} u(z)$ on the rest of the vertices in $V_{0,i}$, $1 \leq i \leq N$ that are not yet defined.

Step 3. Let $\mathcal{N} = \{\omega : V_0$ is non-boundary cell, $V_{0,\omega}$ is boundary cell$\}$. (See Figure 4 for the dotted cells.) In Step 2, we have already defined the values of $u$ on these cells. To complete the construction, we continue to define $u$ on $V_n \cap K_\omega$. By using assumption (ii) and (3.2) on $V_{0,\omega}$, for small enough $\varepsilon > 0$ such that $R^\varepsilon < \min_{\omega \in \mathcal{N}} R(J, J') - \varepsilon$, we can choose $N$ big enough so that the harmonic extension of $u$ from $V_{0,i}$ to $V_{n+1} \cap K_\omega$ satisfies

\[
\left(\min_{i \neq j} R(J, J') - \varepsilon\right)^N E_{\varepsilon}(u \circ F_\omega) \leq E_0(u \circ F_\omega),
\]

and $u$ is constant on the equivalent classes in $V_{0,N}$. By repeating the harmonic extension of $u$ on $V_{n+2} \cap K_\omega, V_{n+3} \cap K_\omega, \ldots$, we have $\sup_{0 \leq k \leq N} \left(\min_{i \neq j} R(J, J') - \varepsilon\right)^k E_{\varepsilon}(u \circ F_\omega) \leq E_0(u \circ F_\omega)$, and consequently by Corollary 2.3.

\[
\sup_{0 \leq k \leq N} \left(\min_{i \neq j} R(J, J') - \varepsilon\right)^k E_{\varepsilon}(u \circ F_\omega) \leq C E_0(u \circ F_\omega).
\]

This completes the construction of $u$ on $V_n$.

Finally we estimate the energy $E_{n}(u)$. Note that $\sum_{J \in \mathcal{B}_n} E_{J,n} = 0$ as $u$ is constant on such $J$ by construction; also for $n \geq 1$, $\sum_{J' \in \mathcal{B}_n, J \neq J'} E_{J',n} = 0$ because elements in $J$ and $J'$ have zero conductance (by Lemma 3.3). Hence we can write $E_n(u)$ as

\[
E_n(u) = \sum_{J \in \mathcal{B}_n} E_{J,n}(u) + \sum_{J' \in \mathcal{B}_n, J \neq J'} E_{J',n}(u) + \sum_{J \in \mathcal{B}_n, J \neq J'} E_{J',n}(u).
\]

For the second sum, we have

\[
\sum_{J \in \mathcal{B}_n, J \neq J'} E_{J',n}(u) = \sum_{J \in \mathcal{B}_n} \sum_{J' \in \mathcal{B}_n, J \neq J'} \left(\sum_{x \in V_0 \cap J} (u(x) - u(y))^2\right).
\]

\[
\leq C \sum_{J \in \mathcal{B}_n} \sum_{J' \in \mathcal{B}_n, J \neq J'} (u(x) - u(x'))^2.
\]

\[
\leq C' \sum_{J \in \mathcal{B}_n, J \neq J'} \left(\sum_{x \in V_0 \cap J} (u(x) - u(z))^2 + \sum_{x \in V_0 \cap J} (u(x) - u(z))^2\right).
\]
Figure 4. Let \( J_1 = \{ p_1, p_3 \}, J_2 = \{ p_2 \} \) on \( V_0 \). For \( n = 1, 2, 3 \), the vertices in heavy line segments are the equivalent classes, and the grey cells are the boundary cells. Step 2 determines \( u \) on the new vertices of the grey cells and the lined cells; Step 3 determines \( u \) in the dotted cells.

\[
\sum_{J \in B_n} E_{J, n}(u) + E_{J, n-1}(u),
\]

where \( u(y) \) takes value \( \min_{z \in V_{\omega} \cap J} u(z) = u(x') \) for some \( x' \in V_{\omega} \cap J \); the second inequality follows by choosing a path on \( J \cap V_n \cap K_{\omega} \) from \( x \) to some point \( z \) in \( J \cap V_{\omega} \), this is controlled by the first term in (4.7), and then connecting \( z \) and \( x' \) in \( J \cap V_{\omega} \) which is controlled by the second term in (4.7).

For the last sum in (4.6), as \( J, J' \notin B_n \), by the construction, \( u \) is constant on \( J \) and on \( J' \). It follows from (4.5) that

\[
\sum_{J \notin B_n, J' \notin B_n} E_{J, J', n}(u) = \sum_{J \notin B_n, J' \notin B_n} E_{J, J', n}(u) = \sum_{\omega \in N, |\omega| = k \leq n} E_{n-k}(u \circ F_{\omega})
\]

\[
\leq C \sum_{\omega \in N, |\omega| = k \leq n} \left( \frac{1}{\min_{i \neq j} R^+(J_i, J_j) - \varepsilon} \right)^{n-k} E_0(u \circ F_{\omega}). \tag{4.8}
\]

For the \( \omega \) in the sum, \( V_{\omega} \cap J \neq \emptyset \) for some boundary class \( J \), then by the construction, the values of \( u \) on \( V_k \cap K_{\omega} \) are defined from the values of \( u \) on \( J \cap K_{\omega} \), and hence

\[
E_0(u \circ F_{\omega}) \leq E_1(u \circ F_{\omega}) \leq C E_{k, J \cap K_{\omega} \cap \omega}(u).
\]

Thus (4.8) is smaller than

\[
C' \sum_{J \in B^*} \sum_{k=0}^{n} \left( \frac{1}{\min_{i \neq j} R^+(J_i, J_j) - \varepsilon} \right)^{n-k} E_{k, J}(u).
\]

To conclude, the above two estimates and (ii) imply that (4.6) satisfies

\[
R^n E_n(u) \leq C \sup_{0 \leq k \leq n} R^k E_{J, k}(u) + C' \sum_{J \in B^*} \sum_{k=0}^{n} R^k E_{k, J}(u) + E_0(u) < \infty.
\]

Therefore \( R^n E_n(u) \) is uniformly bounded on \( n \). This implies that \( u \in B^{n \sigma} \) and completes the proof.

\( \square \)

In the following, we will consider the second critical exponent \( \sigma^\# \).
Theorem 4.3. Let $K$ be a p.c.f. self-similar set with an IFS satisfying (2.7). Assume $R(p, q) > 1$, $p, q \in V_0$ exist. Let

$$R^\# = \min \{ s : \forall p \neq q \text{ in } V_0, \exists a \text{ chain } p = p_1, p_2, \cdots, p_m = q \text{ in } V_0$$

$$\exists R(p_i, p_{i+1}) \leq s, 1 \leq i \leq m - 1 \}. $$

Suppose there is a compatible relation on $V_0$ such that for any small $\varepsilon > 0$,

$$\lim_{n \to \infty} \frac{(R^\#)^{(1-\varepsilon)n}}{R_n(J_i, J_j)} = 0, \quad \forall J_i, J_j \in V_0^\sim, J_i \neq J_j. \quad (4.9)$$

Then $\sigma^\# = \frac{1}{2} \left( \frac{\log R^\#}{\log \rho} + \alpha \right)$.

Proof. To prove $\sigma^\# \leq \frac{1}{2} \left( \frac{\log R^\#}{\log \rho} + \alpha \right)$, we consider $\sigma = \frac{1}{2} \left( \frac{\log R^\#}{\log \rho} + \alpha \right) + 2\varepsilon$, we claim that for $u \in B^\sigma_{2,\infty}, u(p) = u(q)$ for all $p, q \in V_0$. Then the same argument apply to any $u \in \ell(V_n)$, and hence $u$ is a constant function.

For $\varepsilon > 0$ and for $R^\# = R^\# \cdot \rho^{-2\varepsilon}$, the definition of $R^\#$ implies that there is a chain $p = p_1, p_2, \cdots, p_m = q$ such that $R(p_i, p_{i+1}) \leq R^\#$ for $1 \leq i \leq m - 1$. Hence

$$\frac{R^\#}{R_n(p_i, p_{i+1})} \geq \rho^{-\varepsilon n} \to \infty \text{ as } n \to \infty. \quad \forall 1 \leq i \leq m - 1.$$ 

By the same argument as in Theorem 4.1 we conclude that $u(p_i) = u(p_{i+1})$ for all $1 \leq i \leq m - 1$, so that $u(p) = u(q)$, and the claim follows.

Next we consider $\sigma < \frac{1}{2} \left( \frac{\log R^\#}{\log \rho} + \alpha \right)$ such that $2\sigma - \alpha > 0$. We show that each $u \in \ell(V_0)$ (equivalently, $u \in \ell(V_0)$ and is constant on each $J_i$) can be extended to be in $B^\sigma_{2,\infty}$. Then $B^\sigma_{2,\infty}$ contains non-constant function, and by definition, $\frac{1}{2} \left( \frac{\log R^\#}{\log \rho} + \alpha \right) \leq \sigma^\#$.

The proof of the statement is the same as the corresponding part in Theorem 4.1. By (4.9), we can find an $N = N(\sigma)$ such that for all distinct $i, j$,

$$\rho^{-(2\sigma - \alpha)N} \leq R_N(J_i, J_j), \quad \forall J_i, J_j \in V_0^\sim, J_i \neq J_j, \quad (4.10)$$

then by (3.2) and Proposition 2.6 we obtain $u \in B^\sigma_{2,\infty}$. \square

Remark. It follows from the above argument that if $u \in \ell(V_n^-)$, then $u$ can be extended to be in $B^\sigma_{2,\infty}, \sigma < \sigma^\#$ and $u$ is constant on each equivalent class $\mathcal{J} \in V_0^\sim$.

Proposition 4.4. With the same assumption as in Theorem 4.3, Suppose that

$$\max \{ \dim_H J : J \text{ a boundary class} \} < \dim_H K. \quad (4.11)$$

Then for $\sigma < \sigma^\#$, $B^\sigma_{2,\infty}$ is dense in $L^2(K, \mu)$. If further, for any $u$ on $\ell(V_0)$, there is an extension of $u$ on $V_0^\sim$ such that

$$\sup_{n \geq 0} R^{\#n} E_n(u) < \infty. \quad (4.12)$$

Then $B^\sigma_{2,\infty}$ is dense in $L^2(K, \mu)$. 

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Proof. For any $\varepsilon > 0$ and for any $f \in L^2(K, \mu)$, let $g \in C(K)$ satisfying $\|g - f\|_2 \leq \varepsilon$. Let $\delta > 0$ be such that for $x, y \in K$ with $|x - y| \leq \delta$, $|g(x) - g(y)| \leq \varepsilon$. For $\delta > 0$, we let

$$Q_{\delta,n} = \bigcup_{\omega \in n} \{K_\omega: \epsilon_\omega \cap J \neq \emptyset, \text{diam}(J) \geq \delta, J \in V^-\},$$

and let $Q_\delta = \bigcap_{n=1}^{\infty} Q_{\delta,n}$. Then $Q_\delta$ is union of the closures of finitely many equivalent classes. In view of Theorem 3.1, $\dim_H(Q_\delta)$ is the maximum of the Hausdorff dimensions of the boundary classes. By (4.11), $\mu(Q_\delta) = 0$, we can find $N := N(\varepsilon, \delta)$ satisfies $\mu(Q_{\delta,n}) < \varepsilon$. Define $g_N$ on $V_N$ such that

$$g_N(x) = \sup_{y \in V_N \cap J} g(y), \quad \text{for } x \in V_N \cap J, \ J \in V^-.$$

Then $g_N \in \ell(V^-)$. In view of the above Remark, we can extend $g_N$ to $B_{2,\infty}^r$, which is continuous and is constant on each equivalent class of $V^-$, still denote this extension by $g_N$. Note that $g_N$ takes maximum and minimum values on $V_N$. It follows that

$$\|g_N - g\|_2^2 = \int_{Q_{\delta,n}} |g_N - g|^2 d\mu + \int_{K \setminus Q_{\delta,n}} |g_N - g|^2 d\mu$$

$$\leq (2\|g\|_\infty)^2 \cdot \mu(Q_{\delta,n}) + (2\varepsilon)^2 \cdot \mu(K) \leq c\varepsilon^2$$

for some $c > 0$ (independent of $g$). Hence $\|g_N - f\|_2 \leq \|g_N - g\|_2 + \|g - f\|_2 \leq (c + 1)\varepsilon$. The denseness of $B_{2,\infty}^r$ in $L^2(K, \mu)$ follows.

For the second part, the assumption (4.12) on $V^-$ implies that we can extend any $u \in \ell(V^-_0)$ to be in $\ell(V^-)$ and in $B_{2,\infty}^{\mu}$, so that the same argument shows that $B_{2,\infty}^{\mu}$ is dense in $L^2(K, \mu)$.

5. P.c.f. sets with inhomogeneous traces

In this section, we will construct and study the two asymmetric p.c.f. sets as announced. We will assume the primal energy on $V_n$, i.e., $E_n[u] = \sum_{x,y \in V_n, \|x\| = n} |u(x) - u(y)|^2$.

1. Eyebolted Vicsek cross. In $\mathbb{R}^2$, let $\{p_1, p_2, p_3, p_4\}$ be the four vertices of the unit square $S$, and let $p_0$ be the center of $S$, that is, $p_0 = (0, 0)$ and $p_1 = (-1/2, -1/2)$, $p_2 = (1/2, -1/2)$, $p_3 = (1/2, 1/2)$, $p_4 = (-1/2, 1/2)$. Divide $S$ into a mesh of sub-squares of size $1/9$, and pick 21 sub-squares as shown in Figure 5.

![Figure 5. The eyebolted Vicsek cross K](image-url)
Let \( \{a_i\}_{i=1}^{21} \) be the center of these sub-squares. Let \( \{F_i\}_{i=1}^{21} \) be the IFS on \( \mathbb{R}^2 \) with
\[
F_i(x) = \frac{1}{9}(x - a_i) + a_i, \quad 1 \leq i \leq 21,
\]
where \( F_i, 1 \leq i \leq 9 \) correspond to the 9 sub-squares along \( \overline{p_1p_3} \), and the other \( F_i \) correspond to the other 12 sub-squares; let \( K \) be the unique nonempty compact set such that \( K = \bigcup_{i=1}^{21} F_i(K) \). Then \( (K, \{F_i\}_{i=1}^{21}) \) is a p.c.f. self-similar set with boundary \( V_0 = \{p_1, p_2, p_3, p_4\} \). We call this modified Vicsek cross an eyebolted Vicsek cross. The Hausdorff dimension of \( K \) is \( \alpha = \log 21/ \log 9 \), and the self-similar measure with the natural weight is the normalized \( \alpha \)-dimensional Hausdorff measure \( \mu \) on \( K \).

In the \( \mathbb{O} \times X \) transform in Lemma 2.7, we define the vertex set on the X-side by \( V_0' = V_0 \cup \{p_0\} \) where \( p_0 \) is an added point in the center (see Figure 2), and let \( V_n' = \bigcup_{|\omega|=n} F_\omega(V_0') \).

**Lemma 5.1.** Suppose \( (V'_1, \xi) \) is a network with resistance \( \xi = (a, b, c, d) \) on the four edges of each subcell. Let \( \Phi(\xi) = (a', b', c', d') \) be the trace of \( (V'_1, \xi) \) on \( V'_0 \), then
\[
\Phi(\xi) = (5a + 4c, 4b + 3d + \varphi(\xi), 4a + 5c, 3b + 4d + \varphi(\xi)). \tag{5.1}
\]
where \( \varphi(\xi) = \frac{(b+d)(2a+2c+b+d)}{2(a+b+c+d)} \).

**Proof.** The expression of \( \varphi(\xi) \) is obtained by applying the parallel law of the resistances to the two eyebolts on \( V'_1 \) (see Figure 6). That \( a' = 5a + 4c \) is by applying the series law to the branch of \( p_1 \) to the center, and the same for \( b', c', d' \) on the other three branches. □

![Figure 6. Network on V_1' and its trace on V_0']

For \( n \geq 0 \), we let \( G_n \) denote the network on \( V_n \) with unit resistance on any two vertices of each subcell \( V_{\omega} \) (i.e., there are 6 edges on \( V_{\omega} \) as in the left picture in Figure 2).

**Proposition 5.2.** Let \( R_n(p, q), \ p, q \in V_0 \) be the trace of \( G_n \) on \( V_0 \), then we have

(i) \( R_n(p_i, p_{i+1}) = \frac{a_n + b_n}{2} = a_n \) for \( 1 \leq i \leq 4, \ p_5 = p_1 \);

(ii) \( R_n(p_1, p_3) = \frac{1}{2}(a_n + \frac{a_n^2}{b_n}) \times \frac{a_n^2}{b_n}, \) and \( R_n(p_2, p_4) = \frac{1}{2}(b_n + \frac{b_n^2}{a_n}) \times b_n \),

where \( a_n = 9^n \) and \( b_n = 9b_{n-1} - \frac{b_{n-1}^2}{9b_{n-1}} \) for \( n \geq 1 \) (\( b_0 = 1 \)). Moreover, \( \lim_{n \to \infty} b_n/9^n = 0 \), and for any \( 0 < \varepsilon < 9 \), \( \lim_{n \to \infty} b_n/(9-\varepsilon)^n = \infty \).
Proof. First we use the $\mathfrak{S}$-X transform to convert the resistances from $G_n$ to $G'_n$; it follows that each cell in $G'_n$ has resistance $\xi = \frac{1}{4}(1, 1, 1, 1)$ (Lemma 2.7). By applying Lemma 5.1 we obtain the trace $\Phi(\xi)$ on each cell of $V'_{n-1}$. By induction and some simple calculation, we see that the trace of $G'_n$ on $V_0'$ is given by

$$\Phi^n(\xi) = \frac{1}{4}(a_n, b_n, a_n, b_n),$$

where $a_n = 9^n$ and $b_n = 9b_{n-1} - \frac{b^2_{n-1}}{9^n+b_{n-1}}$, ($b_0 = 1$). Then applying the inverse $\mathfrak{S}$-X transform (Lemma 2.7), we obtain the expressions of $R_n(p, q)$ for $p, q \in V_0$ as stated.

To prove the asymptotic values of $b_n$, we let $x_n = \frac{b_n}{9^n}$, then we have $x_n = x_{n-1} - \frac{x^2_n}{9(1+x_{n-1})}$, it follows that $\{x_n\}$ is non-increasing and has a limit $x$ satisfies $x = x - \frac{x^2}{9(1+x)}$. This implies $\lim_{n \to \infty} b_n/9^n = 0$. The limit also implies that for any $0 < \varepsilon < 9$, $b_n \geq (9 - \frac{\varepsilon}{2})b_{n-1}$ for large $n$. and $\lim_{n \to \infty} b_n/(9 - \varepsilon)^n = \infty$ follows. The asymptotic values in (i), (ii) also follows. \qed

Now we apply the results in Section 3 to conclude the critical exponents of the eye-bolted Vicsek cross $K$, and the density of the Besov space in $C(K)$ and $L^2(K, \mu)$. First we prove a lemma on the quotient network.

**Lemma 5.3.** We define a compatible equivalent relation on $V_0$ by identifying $p_2, p_4$, i.e., $V_0 = \{p_1\} \cup \{p_2, p_4\} \cup \{p_3\} := J_1 \cup J_2 \cup J_3$. Then

(i) the relation satisfies (4.11) in Proposition 4.4

(ii) the trace of $G'_n$ on $V_0'$ is given by

$$R_n(J_1, J_3) = \frac{1}{2}(9^n + 9^{2n}), \quad R_n(J_1, J_2) = R_n(J_3, J_2) = \frac{1}{4}(1 + 9^n), \quad (5.2)$$

Proof. (i) It is easy to see that $J_1^*$ and $J_3^*$ are singletons, $\tilde{J}_2^*$ is the line segment $\overline{p_2p_4}$. Hence $\max\{\dim_H(J_i^*) : i = 1, 2, 3\} = 1 < \alpha$, so that (4.11) holds.

(ii) Observe that by identifying $p_2, p_4$, then $E_0^*[u] = E_0[u]$ for $u \in \ell(V_n')$. By comparing the conductances, we obtain the resistances on $V_0'$ as $R_0(J_1, J_3) = 1, R_0^*(J_1, J_2) = R_0^*(J_3, J_2) = \frac{1}{2}$. We make use of the $\Delta$-$Y$ transform in the calculation. Let $x_0, y_0, x_0$ be the corresponding resistances in the $Y$-form. Then

$$x_0 = \frac{1}{4} \quad \text{and} \quad y_0 = \frac{1}{8}.$$ 

Following the same type of proof in Lemma 5.1 and Proposition 5.2 in connection with the quotient (modify Figure 6 to a more simple graph), it is easy to show that the trace satisfies

$$x_n = 9x_{n-1} \quad \text{and} \quad y_n = \frac{1}{8}.$$ 

We then transform this back to the $\Delta$-form to get $R_n(J_1, J_2)$ inductively, which yields (5.2). \qed
Theorem 5.4. For the Besov spaces $B^r_{2,\infty}$ defined on the eyebolted Vicsek cross $K$, the critical exponents are

$$\sigma^* = \sigma^# = \frac{1}{2} \left(1 + \frac{\log 21}{\log 9}\right).$$

Moreover, (i) $B^r_{2,\infty}$ is dense in $L^2(K,\mu)$; (ii) $u \in B^r_{2,\infty}$ takes constant values on each line segment parallel to $\overline{p_2p_4}$, and $B^r_{2,\infty}$ is not dense in $C(K)$.

Proof. The cross $K$ has Hausdorff dimension $\alpha = \frac{\log 21}{\log 9}$, also $R^* = 9$, and $\lim_{n \to \infty} \frac{R^n}{R_*(p_2,p_4)} = \infty$ (Proposition 5.2). By Theorem 4.1, $\sigma^*$ has the expression as asserted, and $B^r_{2,\infty}$ is not dense in $C(K)$, and $u \in B^r_{2,\infty}$ has the property as stated.

For $\sigma^#$, we have $R^# = 9$ by Proposition 5.2. Take the compatible equivalent relation in Lemma 5.3, then the conditions in Theorem 4.3 and Proposition 4.4 are satisfied. We conclude that $\sigma^#$ has the same expression as $\sigma^*$, and $B^r_{2,\infty}$ is dense in $L^2(K,\mu)$.

2. Sierpinski sickle. Let $V_0 = \{p_1, p_2, p_3\}$ with $p_1 = (0,0), p_2 = (1,0), p_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let $\{F_i\}_{i=1}^{17}$ be the IFS of contractive similitudes on $\mathbb{R}^2$ such that

$$F_i(x) = \frac{1}{7}x + a_i, \quad 1 \leq i \leq 17,$$

where the $a_i$'s are the 17 points lie on the triangle determined by $V_0$ as indicated in Figure 7. Let $K$ be the unique nonempty compact set such that $K = \bigcup_{i=1}^{17} K_i$, and call it a Sierpinski sickle. Then $K_i \cap K_j$ contains at most one point, and satisfies the p.c.f condition. The Hausdorff dimension of $K$ is $\alpha = \log 17/\log 7$, and the self-similar measure with the natural weight is the normalized $\alpha$-dimensional Hausdorff measure $\mu$ on $K$.

![Figure 7: The Sierpinski sickle $K$](image)

On $V_0$, we arrange the three edges clockwise in the order of $\overline{p_1p_2}, \overline{p_2p_3}, \overline{p_3p_1}$, and the same way for the sub-triangles in $V_n$. Similar to the last example, we define the vertex sets $V_0^\prime$ an $V_n^\prime$ on the $Y$-side of the $\Delta$-$Y$ transform. We show that the traces $R_n(p,q), p,q \in V_0$ of the primal energy have different asymptotic geometric rate.

Lemma 5.5. Suppose $(V_1^\prime, \xi)$ is a network with resistance $\xi = (a,b,c)$ on the three edges of each subcell (as indicated in Figure 8). Then the trace of $(V_1^\prime, \xi)$ on $V_0^\prime$ is:

$$\Phi(\xi) = (a', b', c') = \left(6a + 5c + \varphi_a, 6a + 8b + 5c + \varphi_b, c + \varphi_c\right). \quad (5.3)$$

where $\varphi_a = \frac{(a+b)(a+c)}{2(a+b+c)}$, and $\varphi_b, \varphi_c$ are defined symmetrically.
Proof. We apply the $\Delta$-Y transform and obtain (5.3) through a direct calculation (see Figure 8).

$\blacksquare$

Figure 8. The trace of $G'_1$ on $G'_0$

Lemma 5.6. Let $\Phi(\xi) = (a', b', c')$ be as in (5.3). Then

(i) if $c \leq a \leq b$, then $\frac{7}{2}c' \leq a' \leq b'$;

(ii) if $\frac{7}{2}c \leq a \leq b$, then there exists $0 < \lambda_0 < 1$ such that $\frac{b}{a} \leq \lambda_0 \frac{b'}{a'}$.

Proof. It is direct to check that $a \leq b$ implies $a' \leq b'$, and $c \leq b$ implies $\frac{a + b}{a + b + c} \geq \frac{1}{2}$. Hence

$$\frac{a'}{c'} \geq \frac{6a + 5c + \frac{1}{4}(a + c)}{c + \frac{1}{2}(a + c)} \geq \frac{7}{2},$$

and (i) follows. By using (5.3) and a simple estimate, we have

$$\frac{b'}{a'} \geq \frac{6a + 8b + 5c + \frac{1}{4}(a + b)}{6a + 5c + \frac{1}{2}(a + c)} = \frac{25 + 33 \left(\frac{a}{a} + 20 \left(\frac{c}{a}\right)\right)}{22 \left(\frac{c}{a}\right) + 26} \geq \frac{33 \left(\frac{2}{a}\right)}{22 \left(\frac{2}{a}\right) + 26} = \frac{231}{226} \frac{b}{a}.$$

The assertion follows by letting $\lambda_0 = \frac{231}{226}$.

$\blacksquare$

We let $G_n$ denote the electrical network of the primal energy $E_n$ on $V_n$, and let $G'_n$ be the corresponding network in the $Y$-form. As each edge in $G_n$ has resistance 1, then each edge in $G'_n$ has resistance $(a_0, b_0, c_0) = \frac{1}{3}(1, 1, 1)$. Let $\xi = (a_n, b_n, c_n) = \Phi(\xi)$ be the traces of $G'_n$ on $V'_n$. Also let $R_n(p_1, p_2), R_n(p_2, p_3), R_n(p_3, p_1)$ be the equivalent traces in the $\Delta$-expression. Then using the $\Delta$-Y transform (2.8), we have,

$$R_n(p_1, p_2) = a_n + b_n + \frac{a_n b_n}{c_n}, \quad R_n(p_2, p_3) = b_n + c_n + \frac{b_n c_n}{a_n}, \quad R_n(p_3, p_1) = c_n + a_n + \frac{c_n a_n}{b_n}.$$

Proposition 5.7. With the above expressions, we have

$$R_n(p_1, p_2) \approx R_n(p_2, p_3) \approx \left(\frac{17}{2}\right)^n, \quad \text{and} \quad R_n(p_3, p_1) \approx 7^n.$$
Proof. We will show that \( b_n \asymp \left( \frac{17}{2} \right)^n \), and \( c_n, a_n \asymp 7^n \), and the proposition will follow. By Lemma 5.6 we have

\[
\frac{c_n}{b_n} \leq \frac{a_n}{b_n} \leq \lambda^n_0 \ (< 1).
\]

(5.5)

Then making use of this together with

\[
\frac{b_n}{b_{n-1}} = 5 \cdot \frac{c_{n-1}}{b_{n-1}} + 6 \cdot \frac{a_{n-1}}{b_{n-1}} + 8 + \left( \frac{c_{n-1}}{b_{n-1}} \right) + \left( \frac{a_{n-1}}{b_{n-1}} \right).
\]

(by (5.3)), we obtain

\[
\frac{17}{2} - \lambda_0^{n-1} \leq \frac{b_n}{b_{n-1}} \leq \frac{23}{2} - \lambda_0^{n-1},
\]

which implies \( b_n \asymp \left( \frac{17}{2} \right)^n \).

Next, by using (5.3) and a direct calculation, we have

\[
\frac{a_{n+1}}{c_{n+1}} - 11 = \frac{2 + 13a_n + c_n}{2} \left( \frac{a_n}{c_n} - 11 \right) + \frac{13a_n}{c_n} - 12a_n + \frac{13a_n}{b_n} \left( \frac{1 + \frac{a_n}{c_n}}{1 + \frac{a_n}{c_n}} \right).
\]

(5.6)

Letting \( \alpha_n = \left| \frac{a_n}{c_n} - 11 \right| \) and making use of (5.5), (5.6) implies

\[
\alpha_{n+1} \leq \delta \alpha_n + \gamma \lambda_n^n
\]

for some \( 0 < \delta < 1, \gamma > 0 \), and for \( n \) sufficiently large. An inductive argument shows that there exist \( n_0 \) and \( 0 < \delta_1 < 1 \) such that for \( n > n_0 \), we have \( \alpha_{n+1} \leq \delta_1 \alpha_n \). It follows that there is \( C_1 \) such that

\[
\alpha_n \leq C_1 \delta_1^n \quad \forall \ n \geq 0.
\]

(5.7)

Now by (5.3) again and simplify the terms, we have

\[
\frac{c_{n+1}}{c_n} = 7 + \left( \frac{a_n}{c_n} - 11 \right) + \frac{13a_n}{c_n} + \frac{13a_n}{b_n} + \frac{13a_n}{c_n}.
\]

(5.8)

By (5.5), (5.7) and (5.8), we conclude that \( c_n \asymp 7^n \). The same estimate also holds for \( a_n \), and completes the proof. \( \square \)

Theorem 5.8. For the Besov spaces \( B^\sigma_{2,\infty} \) defined on the Sierpinski sickle \( K \), the critical exponents are

\[
\sigma^* = \frac{1}{2} \left( 1 + \frac{\log 17}{\log 7} \right), \quad \sigma^# = \frac{1}{2} \left( \frac{2 \log 17 - \log 2}{\log 7} \right).
\]

Moreover, we have

(i) \( B^\sigma_{2,\infty} \) is dense in \( C(K) \).

(ii) For \( \sigma^* < \sigma \leq \sigma^# \), \( B^\sigma_{2,\infty} \) is dense in \( L^2(K, \mu) \).

Proof. The Sierpinski sickle \( K \) has Hausdorff dimension \( \alpha = \frac{\log 17}{\log 2} \), and \( R^* = 7 \) (Proposition 5.7). By Theorem 4.1, the expression of \( \sigma^* \) follows. To consider \( \sigma^# \), we know that \( R^* = \frac{1}{2} \) (Proposition 5.7), and we need to check that condition (4.9) in Theorem 4.3 is satisfied.

For this we take the compatible equivalent relation to be \( V_0 = \{ p_1, p_3 \} \cup \{ p_2 \} := J_1 \cup J_2 \). Then \( R_0(J_1, J_2) = 1/2 \), and by a simple inductive argument, we have

\[
R^*_n(J_1, J_2) = \left( \frac{1}{2} + 8 \right) \cdot R^*_{n-1}(J_1, J_2) = \frac{1}{2} \left( \frac{17}{2} \right)^n.
\]

(5.9)
Hence condition (4.9) is satisfied, and \( \sigma^q \) is as asserted.

To prove (i), we use the same equivalence relation as the above. We check the conditions in Proposition 4.2. First property (B) is satisfied trivially. Next, the closure of the boundary class \( J^* \) is the line segment \( \overline{p_1 p_3} \), therefore \( \dim_H(J^*_1) = 1 \) and \( \mathcal{H}^1(J_1^*) < \infty \). We have by Lemma 3.4, for a given \( u \) on \( J_1 \), there is extension on \( J^*_1 \) with \( 7^nE_{J^*_1,n}(u) = E_{J^*_1,0}(u) \); also \( R^*(J_1, J_2) = \frac{1}{2} > R^* = 7 \). Thus all the conditions in Proposition 4.2 are satisfied and \( B^*_2 \) is dense in \( C(K) \).

For (ii), that \( B^*_{2,\infty} \), \( \sigma > \sigma^* \), is not dense in \( C(K) \) is in the definition of \( \sigma^* \). That \( B^*_{2,\infty} \) is dense in \( L^2(K, \mu) \) follows from Proposition 4.4, as condition (4.11) holds by the same argument as in Lemma 5.3 and the condition (4.12) holds. By the decreasing property of \( B^*_{2,\infty} \) on \( \sigma \), it follows that of \( B^*_{2,\infty} \) is dense in \( L^2(K, \mu) \) for \( \sigma < \sigma^q \).

It is well-known that if a p.c.f. set admits a self-similar energy form \( E \), then the renormalized energy \( E_n[u] \) increases to \( E[u] \), which defines an (equivalent) Besov norm. This is not the case for the Sierpinski sickle.

**Corollary 5.9.** On the Sierpinski sickle, there is non-constant \( u \in B^*_{2,\infty} \) such that
\[
\lim_{n \to \infty} \rho^{-2(\sigma^q - \alpha)n} E_n[u] = \lim_{n \to \infty} 7^n E_n[u] = 0.
\]

**Proof.** Let \( u \) be defined on \( V_0 \) with values \( u(p_1) = u(p_3) = 0 \) and \( u(p_2) = 1 \). We extend \( u \) on \( V_1 \) as follows: take \( u \) to be constant 0 on the 7 sub-triangles of \( F_1(V_0) \) on the left side; the values of the \( F_1(V_0) \) in the 10 sub-triangles on the right as in Figure 9. Note that there are 8 sub-triangles that \( u \) takes non-constant values. Next we define \( u \) on \( V_2 \) by \( u_i = u(F_i(p_1)) + (u(F_i(p_2)) - u(F_i(p_1))) \cdot u \circ F_i^{-1}, 1 \leq i \leq 17 \) on \( F_i(V_1) \). We continue the same process to define \( u \) on each \( V_n \), and eventually on \( V \). A direct calculation shows that
\[
7^n \sum_{x, y \in F_n(V_0): |x-y|=n} |u(x) - u(y)|^2 \leq 7^n \cdot 8^n \cdot 2 \cdot \left( \frac{1}{8} \right)^{2n}.
\]
This implies \( \lim_{n \to \infty} 7^n E_n[u] = 0 \). Also we have
\[
\sup_{n \geq 0} \left\{ \rho^{-2q(n - \alpha)E_n[u]} \right\} = \sup_{n \geq 0} \{7^n E_n[u]\} < \infty,
\]
by Proposition 2.4, \( u \) can be extended to be in \( B^*_{2,\infty} \), and proves the statement. \( \square \)
6. Constructions of Dirichlet forms

6.1. Eyebolted Vicsek cross. Since \( B_{2,\infty}^C \) is not dense in \( C(K) \), \( B_{2,\infty}^C \) cannot be the domain of a local regular Dirichlet form on \( K \). Nevertheless we will give two constructions of such Dirichlet forms on \( K \), but the domains are different from \( B_{2,\infty}^C \).

**Theorem 6.1.** On the eyebolted Vicsek cross, there are two kinds of (non-primal) local regular Dirichlet forms that can be constructed, one satisfies the energy self-similar identity (2.4), the other one is from a reverse recursive construction and does not satisfy (2.4).

**Proof.** First construction: We assign two different renormalization factors \( \tau', \tau'' \) (to be determined) on the cells of \( K \) as follows: let \( \tau_1 = \tau_2 = \cdots = \tau_9 = \tau' \) on the 9 sub-cells \( F_1(K), \cdots, F_9(K) \) along the line \( p_1p_3 \), and let \( \tau_{10} = \tau_{11} = \cdots = \tau_{21} = \tau'' \) on the remaining 12 sub-cells \( F_{10}(K), \cdots, F_{21}(K) \); then similar to Lemma [5.1] we obtain a new trace map \( \Phi_{\tau',\tau''} \) for \( \xi = (a, b, c, d) \):

\[
\Phi_{\tau',\tau''}(\xi) = \left( \tau'(5a + 4c), \tau''(3b + 3d + \varphi(\xi)) + \tau'b, \tau'(4a + 5c), \tau''(3b + 3d + \varphi(\xi)) + \tau'd \right).
\]

where \( \varphi(\xi) = \frac{(a+b)(2a+2c+b+d)}{2(a+b+c+d)} \). Consider the equation

\[
\Phi_{\tau',\tau''}(a, b, c, d) = (a, b, c, d),
\]

i.e., the trace of \( G_0' \) coincides with the resistances on \( G_0' \). If we apply this to \( G_0' \) inductively, then we obtain a sequence of networks \( \{G'_k\}_{k=0}^n \) that is compatible in the sense of Definition 2.5, and given the energy self-similar identity. Specifically, let us take \( a = b = c = d \) in (6.1) then it reduces to be two simple linear equations, and the solution is

\[
\tau' = \frac{1}{9}, \quad \tau'' = \frac{16}{135}.
\]

Let \( E_0(u) = \sum_{p,q \in Y_0} (u(p) - u(q))^2 \), define

\[
\mathcal{E}[u] = \lim_{n \to \infty} \sum_{|\omega|=n} \tau_\omega^{-1} E_0(u \circ F_\omega),
\]

and \( \mathcal{E}[u] < \infty \) implies that \( u \in C(K) \) [20, Theorem 2.2.6(1)], thus we can let \( \mathcal{F} = \{ u \in C(K) : \mathcal{E}[u] < \infty \} \). Then (\( \mathcal{E}, \mathcal{F} \)) satisfies the self-similar identity

\[
\mathcal{E}[u] = \sum_{j=1}^{17} \tau_j^{-1} \mathcal{E}[u \circ F_j], \quad u \in \mathcal{F}.
\]

It is known that this defines a local regular Dirichlet form on the metric measure space \((K, d_r, \nu)\), where \( d_r \) is the resistance metric on \( K \), and \( \nu \) is the self-similar measure with weights \( \{\tau_i\}_{i=1}^N \) where \( \sum_{i=1}^N \tau_i = 1 \) ([13], [16]).

Second construction: The main idea is to use \( \Phi^{-a} \) (where \( \Phi \) is defined in (5.1)) to construct a sequence of conductances \( \{c_n(x, y)\}_n \) in (2.3) such that \( \mathcal{E}_n[u] \) converges for \( u \in C(K) \).

Consider the network \( G'_n \), let \( y_n \) be the resistance on each cell of \( G'_n \). We are looking for \( y_n = (s_n, t_n, s_n, t_n) \) such that the trace is \( y_0 = (1, 1, 1, 1) \), i.e., \( \Phi^{-a}(y_n) = y_0 \). As \( \Phi(s, t, s, t) = (9s, 9t - \frac{r}{s+t}, 9s, 9t - \frac{r}{s+t}) \), it follows that

\[
y_n = \Phi^{-a}(y_0) = (s_n, t_n, s_n, t_n)
\]
where \( s_n = 9^{-n} \) and \( t_{n-1} = 9t_n - \frac{t_n^2}{9} > 0 \) for \( n \geq 1 \). Hence by the compatibility of \( G'_n \) and \( G'_0 \) with resistance \( y_n \) and \( y_0 \) respectively, we have

\[
\min \left\{ \sum_{|\omega|=n} \sum_{i=1,3} |v \circ F_\omega(p_i) - v \circ F_\omega(p_0)|^2 + t_n^{-1} \sum_{j=2,4} |v \circ F_\omega(p_j) - v \circ F_\omega(p_0)|^2 \right\}
\]

\[
= \sum_{p \in V_0} |u(p) - u(p_0)|^2.
\]

(6.2)

where the minimum is taken over all \( v \in \ell(V'_n) \) such that \( v|_{V_0} = u \). Then by applying the inverse of \( \mathcal{E} \)-X transform (Lemma 2.7) to each cell in \( G'_n \), we obtain an equivalent network \( G_n \).

\[
\min \left\{ \sum_{x,y \in \ell(V_0),|\omega|=n} c_n(x,y) |v(x) - v(y)|^2 \right\} = \sum_{p,q \in V_0} \frac{1}{4} |u(p) - u(q)|^2.
\]

(6.3)

where the resistances \( c_n(x,y)^{-1} \) on \( V_n \) are given by

\[
c_n(F_\omega(p_i), F_\omega(p_{i+1}))^{-1} = 2(s_n + t_n), \quad i = 1, 2, 3, 4, \quad (p_5 = p_1)
\]

\[
c_n(F_\omega(p_1), F_\omega(p_5))^{-1} = 2(s_n + \frac{t_n^2}{s_n})
\]

\[
c_n(F_\omega(p_2), F_\omega(p_4))^{-1} = 2(t_n + \frac{t_n^2}{s_n})
\]

For \( u \in C(K) \) and \( n \geq 0 \), let

\[
\mathcal{E}_n[u] = \sum_{x,y \in \ell(V_0),|\omega|=n} c_n(x,y) |u(x) - u(y)|^2.
\]

By the compatibility of \( G_n \) and \( G_{n-1} \) through the \( y_n \) and \( y_{n-1} \), we see that \( \mathcal{E}_n[u] \) is an increasing sequence on \( n \), define

\[
\mathcal{E}[u] = \lim_{n \to \infty} \mathcal{E}_n[u], \quad \mathcal{F} = \{ u \in C(K) : \mathcal{E}(u) < \infty \}.
\]

Note that \( \mathcal{F} \) is dense in \( C(K) \) by approximating \( u \in C(K) \) through the piecewise harmonic functions constructed from (6.3) applied to the subcells. Hence it is not hard to see that \( (\mathcal{E}, \mathcal{F}) \) is a local regular Dirichlet form on the metric measure space \((K, | \cdot |, \mu)\).

Finally we show by contradiction that the above Dirichlet form does not satisfy the self-similar identity. Assume that there exist positive numbers \( \tau_1, \tau_2, \ldots, \tau_{21} \) such that for any \( u \in \mathcal{F} \),

\[
\mathcal{E}[u] = \sum_{i=1}^{21} \tau_i^{-1} \mathcal{E}[u \circ F_i].
\]

(6.4)

Recall that in our construction, the weight we put on each cell is the same, then we have \( \tau_1 = \tau_2 = \ldots = \tau_{21} = \tau \).

Let \( u_1 \) be the function that is linear on the line segment \( \overline{p_1p_3} \) with boundary values \( u_1(p_1) = 1, u_1(p_3) = 0 \), and \( u_1 \) is constant on all the eyebolted branches issued at some point on \( \overline{p_1p_3} \). Then the energy of \( u_1 \) is supported on \( \overline{p_1p_3} \). We can easily show that \( \mathcal{E}_n[u_1] = \frac{1}{2^n} \) for all \( n \geq 0 \), and thus \( \mathcal{E}[u_1] = \frac{1}{2} \). Similarly we have \( \mathcal{E}[u_1 \circ F_i] = \frac{1}{2} \cdot \frac{1}{9} \) for each \( i = 1, 2, \ldots, 9 \) along the line \( \overline{p_1p_3} \), and \( \mathcal{E}[u \circ F_i] = 0 \) for the rest twelve maps. By using (6.4), we obtain \( \tau = \frac{1}{9} \).
Let \( u_2 \) be the harmonic function with boundary values \( \left( u_2(p_1), u_2(p_2), u_2(p_3), u_2(p_4) \right) = (0, 1, 0, 0) \). By using (6.4) \( n \) times with \( \tau_1 = \tau_2 = \cdots = \tau_{21} = 1/9 \), and \( u = u_2 \), we obtain
\[
E[u_2] = \sum_{|\omega|=n} \tau_{\omega}^{-1} E[u_2 \circ F_{\omega}] = 9^n \sum_{|\omega|=n} E[u_2 \circ F_{\omega}], \quad n > 0. \tag{6.5}
\]

Since for any \( u \in \mathcal{F} \), \( E[u] \) is the limit of the increasing sequence \( E_n[u] \), the trace estimate yields
\[
9^n \sum_{|\omega|=n} E[u_2 \circ F_{\omega}] \geq 9^n \sum_{|\omega|=n} E_0[u_2 \circ F_{\omega}] = 9^n \sum_{p,q \in V_n, |\omega|=n} \frac{1}{4} |u_2(p) - u_2(q)|^2. \tag{6.6}
\]

On the other hand, by (4.1) and Proposition 5.2, we have
\[
9^n \sum_{p,q \in V_n, |\omega|=n} \frac{1}{4} |u_2(p) - u_2(q)|^2 \geq \frac{1}{4} \min_{u \in \mathcal{B}_2} \left\{ 9^n \sum_{x,y \in V_n, |\omega|=n} |u(x) - u(y)|^2 \right\}
\geq C^{-1} \frac{a_n}{b_n} |u_2(p_2) - u_2(p_4)|^2
= C^{-1} \frac{a_n}{b_n} \to \infty \text{ as } n \to \infty. \tag{6.7}
\]

Hence we see from (6.5), (6.6) and (6.7) that \( E(u_2) = \infty \), contradicting \( u_2 \in \mathcal{F} \). Therefore the Dirichlet form does not satisfy the energy self-similar identity. \( \Box \)

### 6.2. Sierpinski sickle.

Let \( K \) be the Sierpinski sickle. Despite \( B_{2,\infty}^{\nu} \) is dense in \( C(K) \), the primal energy does not give a local regular Dirichlet form in view of Corollary 5.9. We do not know if \( B_{2,\infty}^{\nu} \) can be domain of some other local regular Dirichlet form \( (\mathcal{E}, \mathcal{F}) \) on \( L^2(K, \mu) \) with \( B_{2,\infty}^{\nu} \) as domain. On the other hand, we have the following conclusion.

**Theorem 6.2.** The Sierpinski sickle admits a local regular Dirichlet form that satisfies the energy self-similar identity.

**Proof.** We will determine three renormalization factors \( \tau_L, \tau_R, \tau_T \) on the cells of \( K \) as follows: let
\[
\tau_1 = \tau_2 = \cdots = \tau_5 = \tau_L \text{ on the left 5 sub-triangles } F_1(K), F_2(K), \cdots, F_5(K);
\tau_6 = \tau_7 = \tau_8 = \tau_T \text{ on the 3 top sub-triangles } F_6(K), F_7(K), F_8(K);
\tau_9 = \tau_{10} = \cdots = \tau_{17} = \tau_R \text{ on the right 9 sub-triangles } F_9(K), F_{10}(K), \cdots, F_{17}(K).
\]

Then similar to Lemma 5.5 we obtain the trace map:
\[
\Phi_{\tau_L, \tau_R, \tau_T}(a, b, c) = (a', b', c')
= (\tau_L(5a + 5c) + \tau_T(a + \varphi_a), \tau_R(6a + 7b + 5c) + \tau_T(b + \varphi_b), \tau_T(c + \varphi_c)).
\]

where \( \varphi_a = \frac{(a+b)(a+c)}{2(a+b+c)} \), and define \( \varphi_b, \varphi_c \) symmetrically. Let us take \( a = b = kc \) with \( k > 1 \) and solve the equation
\[
\Phi_{\tau_L, \tau_R, \tau_T}(a, b, c) = (a, b, c), \tag{6.8}
\]
we obtain
\[
\tau_L = \frac{k(k-1)}{5(k^2 + 6k + 3)}, \quad \tau_R = \frac{k^2 - 1}{(13k + 5)(k^2 + 6k + 3)}, \quad \tau_T = \frac{2(k+1)}{k^2 + 6k + 3}.
\]
Let $E_0(u) = (u(p_1) - u(p_2))^2 + k(u(p_2) - u(p_3))^2 + k(u(p_3) - u(p_1))^2$ on $V_0$, define
\[ \mathcal{E}[u] = \lim_{n \to \infty} \sum_{|\omega|=n} \tau_\omega^{-1} E_0(u \circ F_\omega), \]
and let $\mathcal{F} = \{ u \in C(K) : \mathcal{E}(u) < \infty \}$. Then $(\mathcal{E}, \mathcal{F})$ is a regular local Dirichlet form on $L^2(K, \mu)$, and satisfies the self-similar identity
\[ \mathcal{E}[u] = \sum_{i=1}^{17} \tau_i^{-1} \mathcal{E}[u \circ F_i], \quad u \in \mathcal{F}. \]
\[ \square \]

**Remark.** Unlike the eyebolted Vicsek cross, we cannot get the other Dirichlet form on the Sierpinski sickle through the reverse recursive construction. Indeed, for any nonnegative initial value $(a_0, b_0, c_0)$ with $a_0 + b_0 + c_0 = 1$ for the sub-triangles in $V_n$ (we can assume this because $\Phi(\lambda(a_0, b_0, c_0)) = \lambda \Phi(a_0, b_0, c_0), \lambda > 0$), let $(a_n, b_n, c_n) = \Phi^c(a_0, b_0, c_0)$ be the trace. We claim that, $\frac{b_n}{a_n + c_n}$ goes to infinity very fast, that is
\[ \frac{1}{a_n + b_n + c_n} (a_n, b_n, c_n) \to (0, 1, 0) \quad \text{uniformly as } n \to \infty. \]
Indeed, if $c_0 \leq a_0 \leq b_0$, then by Lemma 5.6, we see that there exists $0 < \lambda_0 < 1$ such that for all $n \geq 0$,
\[ \frac{b_n}{a_n + c_n} \geq \frac{1}{2 \lambda_0^n}. \]
If $a_0 \leq c_0 \leq b_0$, then by a direct calculation, $b_1 \geq a_1 \geq c_1$ and reduces to the previous case. Finally if $b_0 \leq a_0$ (or $b_0 \leq c_0$), then $\frac{b_1}{a_1} \geq \frac{6a_0 + 5c_0 + 8b_0}{6a_0 + 5c_0 + (a_0 + b_0)} \geq \frac{12}{13}$, hence
\[ \frac{b_2}{a_2} \geq \frac{6a_1 + 5c_1 + 8b_1}{6a_1 + 5c_1 + \frac{a_1 + b_1}{2}} \geq 1. \]
Also we have $c_2 \leq a_2$ by a similar calculation, and hence reduce back to the first case. We checked all the cases and the claim follows.

Now if we adopt the same method as in the second construction in Theorem 6.1, on the one hand, $y_0 = (0, 1, 0)$ is not an interesting choice (as $y_n = \Phi^{-n}(y_0) = (\frac{2}{17})^n y_0$ by (5.3)); on the other hand, for any initial value $y_0 \neq (0, 1, 0)$, we can not expect to have a non-negative sequence $(y_n)_n$ such that $\Phi(y_n) = y_{n-1}$ for all $n > 0$.

### 7. Other variances and remarks

For the eyebolted Vicsek cross, if we lift the lower right eyebolt to the upper right position, then the abnormality of the density in Theorem 5.4 will not appear. We can show that for this new $K$ and with the primal energy, then $\sigma^0 = \sigma^# = \log \frac{21 + \log(35/4)}{9 \log 9}$. As in the first construction in Theorem 6.1, we can obtain a self-similar energy form with the renormalization factor $r = \frac{4}{35}$ by simply solving equations. By some further computation, we can obtain a local regular Dirichlet form that the domain is the associate Besov space $B^r_{2,\infty}$. We omit the detail.
Furthermore, we claim that it is clear that the normalization factor \( D \) follows by using a sub-additive argument.

Consider the self-similar set \( K_1 \) generated by the IFS with 15 maps and contraction ratio \( \rho = 1/7 \) as shown in Figure 10. Then the relationship of the resistance of the cells on any two levels is given by (as in Lemma 5.5):

\[
\Phi(a, b, c) = (6a + 5c + \varphi_a, 5a + 6b + 4c + \varphi_b, c + \varphi_c).
\]

If we let \((a_n, b_n, c_n) = \Phi^m(1, 1, 1)\) (the \((1, 1, 1)\) is from the primal energy form on \( V_n \)), then there exists \( \lambda > 1 \) such that

\[
a_n \approx b_n \approx c_n \approx \lambda^n. \tag{7.1}
\]

There exists a Dirichlet form on \( L^2(K, \mu) \) satisfying the energy self-similar identity with renormalization factor \( \lambda^{-1} \), and the Dirichlet form has \( B^2_{2, \infty} \) as the domain (similar to the Sierpinski gasket).

**Proof.** It is clear that \( c_n \leq a_n \leq b_n \), and using this, it is not hard to show that \( a_n \leq 34c_n \). Furthermore, we claim that \( b_n \leq 60a_n \), then \( a_n \approx b_n \approx c_n \).

To prove the claim, by using the fact that \( a_n + 2b_n + c_n \geq \frac{4}{3}(a_n + b_n + c_n) \), we have

\[
a_{n+1} + c_{n+1} = 6a_n + 6c_n + \frac{(a_n + c_n)(a_n + 2b_n + c_n)}{2(a_n + b_n + c_n)} \geq \frac{20}{3}(a_n + c_n). \tag{7.2}
\]

On the other hand,

\[
b_{n+1} = 5a_n + 6b_n + 4c_n + \frac{(a_n + b_n)(b_n + c_n)}{2(a_n + b_n + c_n)} \\
\leq 5a_n + \frac{13}{2}b_n + \frac{9}{2}c_n \leq \frac{13}{2}b_n + 5(a_n + c_n). \tag{7.3}
\]

Combining \(7.2\) and \(7.3\), and using induction, we obtain

\[
b_n \leq 30(a_n + c_n) \leq 60a_n,
\]

and the claim follows.

Note that \( \Phi : \mathbb{R}_+^3 \to \mathbb{R}_+^3 \) satisfies \( \Phi(x) \leq \Phi(y) \) for any \( x \leq y \) (coordinate-wise defined), and \( \Phi(cx) = c\Phi(x) \) for any \( c > 0 \). For \( n, m \geq 1 \), we have

\[
(a_{m+n}, b_{m+n}, c_{m+n}) = \Phi^{(n+m)}(1, 1, 1) = \Phi^m(a_n, b_n, c_n) = b_n \cdot \Phi^m(\frac{a_n}{b_n}, 1, \frac{c_n}{b_n}) \\
\approx b_n \cdot \Phi^m(1, 1, 1) = b_n \cdot (a_m, b_m, c_m).
\]

Then \(7.1\) follows by using a sub-additive argument.

We show that the above \( \Phi \) defines a self-similar energy by using a fixed point theorem argument. Let \( D = \{(a, b, c) : a + b + c = 1, a, b, c \geq 0\} \) be the simplex, and let \( \Phi : D \to D \) be the normalization of \( \Phi \), i.e.,

\[
\Phi(a, b, c) = \frac{1}{11a + 6b + 10c + \varphi_a + \varphi_b + \varphi_c} \Phi(a, b, c).
\]

For \( \varepsilon > 0 \), let \( D_\varepsilon = \{(a, b, c) \in D : a + c \geq \varepsilon\} \), then we can show by computation that we can choose \( \varepsilon \) small enough so that \( \Phi \) maps \( D_\varepsilon \) to \( D_\varepsilon \). By applying the
Brouwer’s fixed point theorem to $\Phi$ on $D_ε$, we can find a fixed point $p$ of $\Phi$ on $D_ε$. Obviously $p$ cannot be $(1, 0, 0)$ or $(0, 0, 1)$. Assume that $p = (a_0, b_0, c_0)$ and $\Phi(a_0, b_0, c_0) = (a_0, b_0, c_0)$, consequently $\Phi(a_0, b_0, c_0) = \lambda(a_0, b_0, c_0)$, where $\lambda$ is the same parameter as in (7.1). Thus by transforming $(a_0, b_0, c_0)$ back to the $\Delta$-form, we get the conductance $(c_0(p_1, p_2), c_0(p_2, p_3), c_0(p_3, p_1))$. We can set

$$E_0(u) = c_0(p_1, p_2)(u(p_1) - u(p_2))^2 + c_0(p_2, p_3)(u(p_2) - u(p_3))^2 + c_0(p_3, p_1)(u(p_3) - u(p_1))^2.$$  

Set $E(u) = \lim_{n \to \infty} \sum_{|ω| = n} E_0(u \circ F_ω)$ and $\mathcal{F} = \{u \in C(Ω) : E(u) < \infty\}$. $(E, \mathcal{F})$ gives the self-similar energy with renormalization factor $\lambda^{-1}$. Hence $B_{2,∞}^{ε^∗}$ is the domain of this Dirichlet form.

We remark that the using $D_ε$ instead of $D$ is to avoid the fixed point $(0, 1, 0)$. This fixed point $(0, 1, 0)$ turns out to be repulsive [25, 28]. As in Proposition 7.1, the same situation happens if we consider $ρ = 1/6$ or $ρ = 1/5$ (Figure 11 and Figure 12). For $K_2$, the resistances relationship is (see (5.3))

$$\Phi(a, b, c) = (5a + 4c + ϕ_a, 4a + 5b + 4c + ϕ_b, c + ϕ_c).$$

and for $K_3$,

$$\Phi(a, b, c) = (4a + 3c + ϕ_a, 2a + 4b + 3c + ϕ_b, c + ϕ_c).$$

We can proceed similar to Proposition 7.1.

The construction of the energy form by the reverse recursive method was implicitly used by Hattori, Hattori and Watanabe on the Sierpinski gasket [15] through a probability consideration, and they call the limit an asymptotically one dimensional diffusion. This diffusion was investigated further by Hambly and Kumagai [12] on some other nested fractals (see also [10, 14]). Recently, in [8], the authors gave a detail study of this method on the Sierpinski gasket from an analytic point of view; they showed a dichotomy result that for any initial data, the Dirichlet forms obtained are either the standard or the one in [15]. They also obtained sharp estimate of the eigenvalue counting functions of the associated Laplacian with respect to the $α$-Hausdorff measure. The construction seems to be quite intuitive, but it has limitation (as it fails on the Sierpinski sickle). It will be interesting to find out the validity of this method on the more general class of fractals, and to investigate problems related to the associated Laplacian.
Besides the spaces $B^{\sigma}_{2, \infty}$, there is another important class of Besov spaces that is associated with the Dirichlet forms. Let
\[
[u]^{2}_{B^{\sigma}_{2, 2}} := \int_{K} \int_{K} \frac{|u(x) - u(y)|^2}{|x - y|^{\alpha + 2\sigma}} d\mu(y) d\mu(x),
\]
and define $B^{\sigma}_{2, 2} := \{u \in L^2(K, \mu) : ||u||_{B^{\sigma}_{2, 2}} < \infty\}$, with norm $||u||_{B^{\sigma}_{2, 2}} := ||u||_{2} + [u]_{B^{\sigma}_{2, 2}}$. This family of spaces is the domain of some non-local Dirichlet forms, and is associated with the class fractional Laplacians, and the stable jump processes [4]. In a recent study of boundary theory of random walks [21], it was shown that for a class of random walks, the Martin boundary can be identified with the self-similar set $K$ (not necessary p.c.f. set), and the induced Dirichlet form on the boundary has the expression in (7.4). In such setting the critical exponents of the $B^{\sigma}_{2, 2}$ in connection with the random walk has also been studied in [22].

To put (7.4) into the previous framework, it is not hard to see that the semi-norm $[u]^{2}_{B^{\sigma}_{2, 2}}$ is equivalent to
\[
\int_{0}^{1} \frac{dr}{r} \frac{1}{r^{\alpha + 2\sigma}} \int_{K} \int_{B(x,r)} (u(x) - u(y))^2 d\mu(y) d\mu(x),
\]
which can also be expressed as $\sum_{n=0}^{\infty} \rho^{-n(\alpha + 2\sigma)} \int_{K} \int_{B(x,\rho^n)} (u(x) - u(y))^2 d\mu(y) d\mu(x)$. Similar to Proposition 1.1 (see also [3]) , we have the following discretization of $[u]^{2}_{B^{\sigma}_{2, 2}}$.

**Proposition 7.2.** Suppose the IFS \(\{F_i\}_{i=1}^{N}\) is as in (2.1) and has the p.c.f. property. Then for $2\sigma > \alpha$,
\[
[u]^{2}_{B^{\sigma}_{2, 2}} \asymp \sum_{j=0}^{\infty} \rho^{-(2\sigma - \alpha)j} \sum_{x,y \in V_{\alpha}} |u(x) - u(y)|^2.
\]

The spaces $B^{\sigma}_{2, 2}$ satisfy the following inclusion relation: for $0 < \epsilon < \sigma$, $B^{\sigma}_{2, 2} \subseteq B^{\sigma}_{2, \infty} \subseteq B^{\sigma - \epsilon}_{2, 2}$. Hence they share the same critical exponents as the class $B^{\sigma}_{2, \infty}$, $\sigma > 0$. In view of Theorems 5.4 and 5.8 we have

**Corollary 7.3.** For the Sierpinski sickle, $B^{\sigma}_{2, 2}$ is not dense in $C(K)$, and $B^{\sigma}_{2, 2}$ contains only constant functions. For the eyebolted Vicsek cross, $B^{\sigma}_{2, 2}$ contains only constant functions.

**Proof.** For any $u \in B^{\sigma}_{2, 2}$, by discretizing $[u]^{2}_{B^{\sigma}_{2, 2}}$ as (7.6), we have
\[
\sum_{m=0}^{\infty} 7^{m} \sum_{x,y \in V_{\alpha}|\alpha|=m} (u(x) - u(y))^2 \asymp [u]^{2}_{B^{\sigma}_{2, 2}} < \infty.
\]
Then
\[
\lim_{m \to \infty} 7^{m} \sum_{x,y \in V_{\alpha}|\alpha|=m} (u(x) - u(y))^2 = 0.
\]
From this, we claim that $u$ is constant in the direction of $\overline{p_1p_3}$. For if otherwise, there must be some finite word $\tau$ such that $u \circ F_\tau(p_1) \neq u \circ F_\tau(p_3)$, then for any $n \geq 0$,

$$7^{n+\ell} \sum_{\lvert \omega \rvert = \lvert \tau \rvert + n} (u \circ F_\omega(p_1) - u \circ F_\omega(p_3))^2 \geq 7^n (u \circ F_\tau(p_1) - u \circ F_\tau(p_3))^2 > 0,$$

a contradiction to (7.7), and the claim follows. This implies that $B^r_{2,2}$ is not dense in $C(K)$. We can also show that the rest statements are true by using similar arguments as before, the detail is omitted. □

**Appendix: Graph directed systems**

In here we give a brief supplement to the graph directed systems of Mauldin and Williams [24] on the box count and the box dimension to suit our purpose in Theorem 3.2 and Lemma 3.4.

Let $(\mathcal{V}, \Gamma)$ be a graph directed system with $\mathcal{V} = \{1, \cdots, N\}$ the set of vertices, and $\Gamma$ the set of edges on $\mathcal{V} \times \mathcal{V}$; let $\Gamma_{i,j} \subset \Gamma$ denote the set of edges from $i$ to $j$. For each $(i, j)$, let $\{F_e : e \in \Gamma_{i,j}\}$ be the associated contractive similitudes, then there exists non-empty compact sets $K_i$ such that

$$K_i = \bigcup_j \bigcup_{e \in \Gamma_{i,j}} F_e(K_j). \quad (7.8)$$

We assume that the family $\{F_e : e \in \Gamma\}$ satisfies the open set condition (OSC), i.e., there exists open sets $U_1, \cdots, U_N$ such that $\bigcup_j \bigcup_{e \in \Gamma_{i,j}} F_e(U_j) \subset U_i$, and the sets in the union are non-overlapping. We also assume that all the $F_e$ has contraction ratio $\rho$ for simplicity, and that is what we use throughout the paper.

We first consider the graph is strongly connected, i.e., for every $i, j \in \mathcal{V}$, there exists a path from $i$ to $j$. Corresponding to the graph directed system, there is an associated matrix $N \times N$ matrix $T = [n_{i,j}]$ where $n(i, j) = \#(\Gamma_{i,j})$. For $\mathbf{1}$ a column vector of 1’s, $T \mathbf{1}$ counts the number of subcells of each $K_i$ (see (7.8)). Let $N_i(n)$ be the number of subcells of the $K_i$ in the $n$-th iteration, then $N_i(n) = T^n \mathbf{1}$. As $T$ is irreducible, by the Perron-Frobenius Theorem, the maximal eigenvalue $\lambda$ of $T$ is positive, and the eigenvector $\mathbf{v} > 0$. By using $c^{-1} \mathbf{1} \leq \mathbf{v} \leq c \mathbf{1}$ for some $c > 0$, we can show that

$$N_i(n) \approx \lambda^n.$$ 

As all the $n$-level cells are of the same size $\rho^n$, and each one of them intersects at most $\ell$ of the $n$-level cells for some $\ell > 0$ (by OSC), we see that

$$\alpha = \lim_{n \to \infty} \frac{\log N_i(n)}{\log \rho^n} = \frac{\log \lambda}{\log \rho}$$

is the box dimension of the $K_i$, $1 \leq i \leq N$, which is also the Hausdorff dimension, and $0 < \mathcal{H}^\alpha(K_i) < \infty$ [6, 24].

If the directed graph is not strongly connected, we assume for simplicity that it has two strongly connected components $\mathcal{V}_1$ and $\mathcal{V}_2$, and we can write $T$ as

$$T = \begin{bmatrix} T_1 & Q \\ 0 & T_2 \end{bmatrix}$$
where $T_1$ and $T_2$ are irreducible. Let $\lambda_1, \lambda_2$ be the eigenvalues of $T_1$ and $T_2$ respectively. It is clear that $K_i, i \in \mathcal{V}_2$ behaves the same as the above irreducible case. To consider the $K_i, i \in \mathcal{V}_1$, we observe that
\[
T^n = \begin{bmatrix} T_1^n & Q^n \\ 0 & T_2^n \end{bmatrix}
\]
with $Q^n = \sum_{k=0}^{n-1} T_1^k Q T_2^{(n-1)-k}$. Similar to the above, it is easy to estimate $1' Q_n 1 \leq C \sum_{k=0}^{n-1} \lambda_1^k \lambda_2^{(n-1)-k}$, where $1'$ and $1$ are column vector of 1's, and have coordinates equals to $\#(\mathcal{V}_1), \#(\mathcal{V}_2)$ respectively.

Let $\lambda = \max\{\lambda_1, \lambda_2\}$. If $\lambda_1 \neq \lambda_2$, then $N_i(n) \approx \lambda^n$ for $i \in \mathcal{V}_1$, and $K_i$ has box dimension $\alpha = \log \lambda / \log n$. On the other hand, if $\lambda_1 = \lambda_2$, then $N_i(n) \approx n \lambda^n$ for $i \in \mathcal{V}_1$, and the box dimension is the same. The difference is that in the first case $0 < \mathcal{H}^\alpha(E_i) < \infty$, but in the second case $\mathcal{H}^\alpha(E_i) = \infty$ (which is also the only case that the $\alpha$-Hausdorff measure is infinite) [24, Theorems 4 and 5].

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References

[1] M. Barlow, Diffusions on fractals, Lect. Notes Math., vol. 1690, Springer, 1998, pp. 1–121.
[2] M. Barlow and R. Bass, Transition densities for Brownian motion on the Sierpiński carpet, Probab. Theory Related Fields, 91 (1992), pp. 307–330.
[3] M. Bodn, Discrete characterisations of Lipschitz spaces on fractals, Math. Nachr., 282 (2009), pp. 26–43.
[4] Z. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d-sets, Stochastic Process. Appl., 108 (2003), pp. 27–62.
[5] Q.-R. Deng and K.-S. Lau, Open set condition and post-critically finite self-similar sets, Nonlinearity, 21 (2008), pp. 1227–1232.
[6] K. Falconer, Techniques in fractal geometry, Wiley, 1998.
[7] Q. Gu and K.S. Lau, On a theorem of Jonsson of discretizing Besov spaces, preprint.
[8] Q. Gu, K.-S. Lau, and H. Qu, On a recursive construction of Dirichlet form on the Sierpiński gasket, arXiv:1707.01426.
[9] A. Grigor’yan, J. Hu, and K.-S. Lau, Heat kernels on metric-measure spaces and an application to semilinear elliptic equations, Trans. Amer. Math. Soc., 355 (2003), pp. 2065–2095.
[10] B. Hambly and O. Jones, Asymptotically one-dimensional diffusion on the Sierpiński gasket and multifractal branching processes with varying environment, J. Theoret. Probab., 15 (2002), pp. 285–322.
[11] B. Hambly and T. Kumagai, Transition density estimates for diffusion processes on post critically finite self-similar fractals, Proc. London Math. Soc. (3), 78 (1999), pp. 431–458.
[12] B. Hambly and T. Kumagai, Heat kernel estimates and homogenization for asymptotically lower-dimensional processes on some nested fractals, Potential Anal., 8 (1998), pp. 359–397.
[13] B. Hambly, V. Metz, and A. Teplýaev, Self-similar energies on post-critically finite self-similar fractals, J. London Math. Soc. (2), 74 (2006), pp. 93–112.
[14] B. Hambly and W. Yang, Degenerate limits for one-parameter families of non-fixed-point on fractals, arXiv: 1612.02342.
[15] K. Hattori, T. Hattori, and H. Watanabe, Asymptotically one-dimensional diffusions on the Sierpiński gasket and the abc-gaskets, Probab. Theory Related Fields, 100 (1994), pp. 85–116.
[16] J. Hu and X.-S. Wang, Domains of Dirichlet forms and effective resistance estimates on p.c.f. fractals, Studia Math., 177 (2006), pp. 153–172.
[17] A. Jonsson, Brownian motion on fractals and function spaces, Math. Zeit., 222 (1996), pp. 495–504.
[18] A. Jonsson, triangulation of close sets and bases in function spaces, Ann. Acad. Sci. Fenn. Math., 29 (2004), pp. 43-58
[19] A. Jonsson and H. Wallin, Function spaces on subsets of $\mathbb{R}^n$, Math. Reports Vol. 2, Acad. Publ., Harwood, 1984.
[20] J. Kigami, Analysis on Fractals, Cambridge Univ. Press, 2001.
[21] S.-L. Kong, K.-S. Lau and T.-K. Wong, Random walks and induced Dirichlet forms on self-similar sets, Adv. Math., 320 (2017), 1099-1134.
[22] S.-L. Kong and K.-S. Lau, Critical exponents of induced Dirichlet forms on self-similar sets, arXiv:1612.01708.
[23] T. Lindström, Brownian motion on nested fractals, Mem. Amer. Math. Soc., 83 (1990), no. 420.
[24] R. Mauldin and Williams, Hausdorff dimension in graph directed constructions, Tran. Amer. Math. Soc. 309 (1988), pp. 811-829.
[25] V. Metz, Hilbert’s projective metric on cones of Dirichlet forms, J. Funct. Anal., 127 (1995), pp. 438–455.
[26] R. Peirone, Existence of self-similar energies on finitely ramified fractals, J. Anal. Math., 123 (2014), pp. 35–94.
[27] K. Pietruska-Pałuba, Some function spaces related to the Brownian motion on simple nested fractals, Stochastics and Stochastics Reports, 67 (1999), pp. 267–285.
[28] C. Sabot, Existence and uniqueness of diffusions on finitely ramified self-similar fractals, Ann. Sci. École Norm. Sup. (4), 30 (1997), pp. 605–673.
[29] R. Strichartz, Differential equations on fractals: a tutorial, Princeton University Press, 2006.
[30] A. Tetenov, K. Kamalutdinov, D. Vaulin, Self-similar Jordan arcs which do not satisfy OSC, arXiv:1512.00290.

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