Universal, low temperature, $T$-linear resistivity in two-dimensional quantum-critical metals from spatially random interactions

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We consider two-dimensional metals of fermions coupled to critical scalars, the latter representing order parameters at zero crystal momentum or emergent gauge fields. We show that at low temperatures ($T$), such metals generically exhibit a $T$-linear resistivity arising from spatially random fluctuations in the fermion-scaler Yukawa couplings about a non-zero spatial average. We also find a $T \ln(1/T)$ specific heat, and a rationale for the Planckian bound on the transport scattering time. These results are obtained in the large $N$ expansion of an ensemble of critical metals.

A major theme in the study of correlated metals has been their ‘Planckian behavior’ at low temperatures i.e. a linear-in-temperature resistivity which appears to be controlled by a dissipative relaxation time of order $h/(k_B T)$ (where $T$ is the absolute temperature)$^{[1-8]}$. Moreover, this anomalous resistivity is invariably accompanied by a logarithmic enhancement of the Sommerfeld metallic specific heat to $T \ln(1/T)$ $^{[1]}$. 

A popular approach towards describing such metals in two spatial dimensions is the theory of a Fermi surface coupled to a gapless scalar field $^{[9]}$. We will limit our considerations here to scalar fields near zero crystal momentum, in which case the scalar field can either represent order parameters breaking point group, time reversal, or spin rotation symmetry, or represent the transverse component of an emergent gauge field. Fortunately, all these cases are described by essentially the same low energy theory $^{[10]}$. An important early result was that the self energy of the fermion near the Fermi surface has the frequency dependence $\text{Im} \Sigma (\omega) \sim \omega^{2/3}$ $^{[11]}$, and so there are no quasiparticle excitations on the Fermi surface (although the Fermi surface remains sharp in momentum space). Conservation of momentum in the low energy theory implies that the d.c. conductivity in not affected by the anomalous self energy $^{[12-17]}$, and we will discuss related effects which apply also to the proposed singular behavior of the optical conductivity $^{[18]}$. In other words, the strong coupling between the Fermi surface and the scalar field places the system in the limit of strong ‘scalar drag’: this is in stark contrast to the electron-phonon system, where the weak electron-phonon coupling makes phonon drag a factor only in ultrapure samples $^{[19]}$.

In clean systems, umklapp scattering can lead to non-zero resistance, and its influence in quantum-critical metals has been investigated in other works $^{[16, 20]}$. Umklapp is suppressed at low $T$, its predictions for transport are not universal and depend upon specific Fermi surface details, and there is no corresponding $T \ln(1/T)$ specific heat.

This paper will focus on the effects of disorder, and finds a universal phenomenology that matches several aspects of the observations, including the $T$-linear resistivity and the $T \ln(1/T)$ specific heat. Earlier works $^{[13, 16, 21]}$ examined random potential scattering of the fermions, a central ingredient in the theory of disordered, interacting metals $^{[22]}$. The propagator of the scalar field has a form similar to those of diffusive density fluctuations, with dynamic critical exponent $z = 2$ $^{[21]}$, but there are no singular corrections to transport coefficients from random potential scattering, as we shall confirm below. Here, we examine spatially random fluctuations in the ‘Yukawa coupling’ between the fermions and the scalar, upon a background of a coupling with non-zero spatial average. While the fermion inelastic self energy corrections are dominated by the spatially uniform coupling, we show that the transport is nevertheless dominated by the spatially random coupling, and this leads to our main results. Our work follows other recent works with random Yukawa interactions $^{[23-32]}$ inspired by the Sachdev-Ye-Kitaev (SYK) model $^{[33, 34]}$, along with studies which found linear-in-$T$ resistivity with random interactions, but with vanishing spatial average $^{[30, 32, 35, 36]}$.

Spatially uniform quantum-critical metal. We recall the SYK-inspired large $N$ theory of the two-dimensional quantum-critical metal $^{[30, 32]}$. The imaginary time ($\tau$) action for the fermion field $\psi_i$ and scalar field $\phi_i$ (with $i = 1 \ldots N$ a flavor index) is $^{[32]}

\begin{align*}
S_g &= \int d\tau \sum_{k} \sum_{i=1}^{N} \psi_{ik}(\tau) \left[ \partial_\tau + \epsilon(\mathbf{k}) \right] \psi_{ik}(\tau) \\
&\quad + \frac{1}{2} \int d\tau \sum_{q} \sum_{i=1}^{N} \phi_{iq}(\tau) \left[ -\partial_\tau^2 + K q^2 + m_0^2 \right] \phi_{i,-q}(\tau) \\
&\quad + \frac{g_{ij}}{N} \int d\tau \sum_{i,j,l=1}^{N} \psi_{i}(\mathbf{r}, \tau) \psi_{j}(\mathbf{r}, \tau) \phi_{l}(\mathbf{r}, \tau),
\end{align*}

(1)

where the fermion dispersion $\epsilon(\mathbf{k})$ determines the Fermi
surface, the scalar $m_b$ is needed for infrared regularization but does not appear in final results, and $g_{ijl}$ is space independent but random in flavor space with

$$g_{ijl} = 0, \quad g_{ijl} g_{abc} = g^2 \delta_{ia} \delta_j \delta k_c,$$

(2)

where the overline represents average over flavor space. The hypothesis is that a large domain of flavor couplings all flow to the same universal low energy theory (as in the SYK model), so we can safely examine the average of an ensemble of theories. Momentum is conserved in each member of the ensemble, and the flavor-space randomness does not lead to any essential difference from non-random theories. This is in contrast to position-space randomness which we consider later, which does relax momentum and modify physical properties.

The large $N$ saddle point of (1) has singular fermion ($\Sigma$) and boson ($\Pi$) self energies at $T = 0$ [32]

$$\Pi(i\omega, q) = -c_i \frac{\omega}{q}, \quad \Sigma(i\omega, k) = -ic_f \text{sgn}(\omega) \omega^{2/3}$$

$$c_i = \frac{g^2}{2\pi \kappa v_F}, \quad c_f = \frac{g^2}{2\pi \sqrt{3} (\frac{2\pi v_F k}{K^2 g^2})^{1/3}}.$$  

(3)

These results are obtained on a circular Fermi surface with curvature $\kappa = 1/m$ where $m$ is the effective mass of the fermions. The result is consistent with the theory of two antipodal patches around $\pm k_0$ on the Fermi surface to which $q$ is tangent, with axes chosen so that $q = (0, q)$ and fermionic dispersion $\varepsilon(\pm k_0 + k) = \pm v_F k_0 + k k_0^2/2$.

The large $N$ computation of the optical conductivity yields only the clean Drude result $\text{Re}[\sigma(\omega)]/N = \pi N v_F^2 \delta(\omega)/2$, where $N = m/(2\pi)$ is the fermion density of states at the Fermi level. This is obtained for a circular Fermi surface when only states on the Fermi surface are considered. By coincidence, this result agrees with the patch theory, but we show in the supplement that the patch theory fails to fully capture transport properties. The absence of a $\omega \neq 0$ contribution is tied to an exact cancellation between self-energy and vertex diagrams arising from momentum conservation. Furthermore, this cancellation can be recast into a kinematical constraint for all odd harmonic modes relax slowly even for a general Fermi surface, and the leading order contribution to relaxation is due to states not exactly on the Fermi surface. Instead of the $\omega^{-2/3}$ correction [18] to the optical conductivity, we expect $\sigma(\omega) \sim 1/(-i\omega + \omega^2) \sim 1/(\omega) + |\omega|^0$ (Supplementary Information).

Potential disorder. We now add a spatially random fermion potential

$$S_c = \frac{1}{\sqrt{N}} \int d^3r d\tau v_{ij}(r) \psi_i^\dagger(r, \tau) \psi_j(r, \tau)$$

$$v_{ij}(r) = 0, \quad v_{ij}(r) v_{lm}(r') = \gamma^2 \delta(r-r') \delta_{ij} \delta_{lm},$$

(4)

and here the overline is an now average over spatial coordinates and flavor space. The present large $N$ limit is described in the supplement, and yields results similar to earlier studies [13, 16, 21]. The low frequency boson propagator is now characterized by $z = 2$, while the fermion self energy has an elastic scattering term, along with a marginal Fermi liquid [39] inelastic term at low frequencies

$$\Pi(i\omega, q) = -\frac{N g^2 |\omega|}{\Gamma}, \quad \Gamma = 2\pi v_F^2 N,$$

(5)

$$\Sigma(i\omega, k = k_F \hat{k}) = -\frac{\omega}{2} \text{sgn}(\omega) - \frac{g^2 \omega}{2\pi^2 \Gamma} \ln \left(\frac{e \Gamma^3}{N g^2 v_F^2 |\omega|}\right),$$

at $T = 0$. However, the marginal Fermi liquid self energy, while leading to a $T \ln(1/T)$ specific heat, does not lead to a corresponding [39] linear-$T$ term in the DC resistivity, as it arises from forward scattering of electrons off the $q \sim 0$ bosons. These forward scattering processes are unable to relax either current or momentum due to the small wavevector of the bosons involved and the momentum conservation of the $g$ interactions. As a result, even a perturbative computation of the conductivity at $O(g^2)$ (Fig. 1) shows a cancellation between the interaction-induced self energy contributions and the interaction-induced vertex correction, leading to a DC conductivity that is just a constant, set by the elastic potential disorder scattering rate $\Gamma$. A full summation of all diagrams at large $N$ shows that the $g$ interactions only renormalize the frequency term in the Drude formula (Supplementary Information):

$$\frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \frac{1}{2} \frac{N v_F^2 \Gamma}{\omega^2 + \Gamma^2},$$

(6)

where

$$\bar{\omega} = \omega \left(1 - \frac{\gamma^2}{2\pi^2 N^2 v_F^4} \left[\frac{v_F^2}{4} + \frac{\Gamma}{4\pi} \ln \left(\frac{\Gamma}{e \Lambda v_F}\right)\right]\right),$$

(7)

and $\Lambda \sim N v_F$ is a UV momentum cutoff. In the limit of large Fermi energy (and hence large $N v_F$), this renormalization is negligible and $\bar{\omega} \approx \omega$. The leading frequency dependence of the optical conductivity at frequencies $\omega \ll \Gamma$ is therefore just a constant, and there is no linear in frequency correction. Correspondingly, in the DC limit, there is no linear in $T$ correction, and a conventional $T^2$ correction is expected.

Interaction disorder. Our main results are obtained with additional spatially random interactions. In principle, such terms will be generated under renormalization from $S_c$. However, this does not happen in our large $N$ limit, and so we have to account for the renormalization by adding an explicit term:

$$S_{g'} = \frac{1}{\sqrt{N}} \int d^3r d\tau g'_{ij}(r) \psi_i^\dagger(r, \tau) \psi_j(r, \tau) \phi_l(r, \tau)$$

$$g'_{ij}(r) = 0, \quad g'_{ij}(r) g'_{abc}(r') = g^2 \delta(r-r') \delta_{ia} \delta_{jb} \delta_{lc},$$

(8)

Note that $v, g,$ and $g'$ are all independent flavor-random variables. Earlier work has considered the limiting case
\( g = 0, v = 0, g' \neq 0 \) [30, 32]. We will instead describe the more physically relevant regime where spatial disorder is a weaker perturbation to a clean quantum-critical system, with \( g \) the largest interaction coupling. We therefore now have all of \( g, v, g' \) nonzero. As in the discussion above on potential disorder, we find that the low frequency boson propagator is characterized by \( z = 2 \), and the low frequency fermion self energy again has an elastic scattering term, along with a marginal Fermi liquid inelastic term [40] (Supplementary Information)

\[
\Pi(i\omega, q) = -\frac{N\gamma^2|\omega|}{\Gamma} - \frac{\pi}{2} \lambda' g^2 |\omega| = -c_d|\omega|,
\]

\[
\Sigma(i\omega, k = k_F k) = -\frac{\Gamma}{2} \text{sgn}(\omega) - \frac{i\gamma \omega}{2\pi^2} \ln \left( \frac{e\Gamma^2}{v_F c_d |\omega|} \right)
\]

\[
- \frac{i\gamma g^2 |\omega|}{4\pi} \ln \left( \frac{e\Lambda_d^2}{c_d |\omega|} \right) \quad (T = 0),
\]

where \( \Lambda_d \sim \Gamma/v_F \). This self-energy leads to a \( T \ln(1/T) \) specific heat, as for the large \( g' \) case [32]. However, there is now an important difference with respect to the previous case where \( g' = 0 \), which leads to markedly different charge transport properties: the marginal Fermi liquid self energy now contains a term (last line of (9)), that does not arise solely from forward scattering of electrons. This term is produced by the disordered part of the interactions in (8). Therefore, this part of the self energy represents scattering that relaxes both current and momentum carried by the electron fluid, and therefore its imaginary part on the real frequency axis determines the inelastic transport scattering rate.

We can show this as follows by computing the conductivity using the Kubo formula. If we work perturbatively in \( g \) and \( g' \), then the conductivity at \( \mathcal{O}(g^2) \) and \( \mathcal{O}(g'^2) \) in the large \( N \) limit is given by the sum of self energy contributions and vertex corrections (Fig. 1). However, due to the isotropy of the scattering processes arising from the \( g' \) interactions, only the vertex correction due to the \( g' \) interactions survives. The conductivity up to the first sub-leading frequency dependent correction is then given by (see Supplementary Information)

\[
\frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \sigma_v + \sigma_{\Sigma, g} + \sigma_{V, g} + \sigma_{\Sigma, g'};
\]

\[
\sigma_v(\omega) = \frac{Nv_F^2}{2\Gamma}, \quad \sigma_{\Sigma, g}(\omega) = -\frac{Nv_F^2 g^2 |\omega|}{8\pi\Gamma^3},
\]

\[
\sigma_{V, g}(\omega) = \frac{Nv_F^2 g^2 |\omega|}{8\pi\Gamma^3}, \quad \sigma_{\Sigma, g'}(\omega) = -\frac{Nv_F^2 g^2 |\omega|}{16\pi\Gamma^2}.
\]

Eq. (10) can then be suitably re-expressed as

\[
\frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \frac{Nv_F^2}{2\Gamma} \left( 1 - \frac{Nv_F^2 |\omega|}{8\pi\Gamma^3} \right)
\]

\[
\approx \frac{Nv_F^2}{2\Gamma \left( 1 + \frac{Nv_F^2 |\omega|}{8\pi\Gamma^3} \right)} = \frac{Nv_F^2}{2\Gamma + \frac{Nv_F^2 |\omega|}{8\pi\Gamma^3}}.
\]

The incomplete cancellation of the self-energy contributions \( \sigma_{\Sigma, g}(\omega) + \sigma_{\Sigma, g'}(\omega) \) by the vertex correction \( \sigma_{V, g}(\omega) \) now leads to a linear in frequency correction to the constant transport scattering rate \( \Gamma \). In the opposite limit \( |\omega| \ll T \), this translates into a \( T \)-linear correction to the resistivity in the DC limit; computing the co-efficient of the linear-\( T \) resistivity requires a self-consistent numerical analysis, which has been carried out in the large \( g' \) limit [30, 32]. Remarkably, the slope of this scattering rate with respect to \( |\omega| \) (and therefore \( T \)) does not depend on \( \Gamma \) and hence on the residual \( \omega = T = 0 \) resistivity. In the Supplementary Information we show that the perturbative result described here continues to be valid under a full resummation of all diagrams at large \( N \) in the Kubo formula, as all surviving higher order contributions are merely repetitions of the interaction insertions in Fig. 1b,c.

We can also consider the case where \( v = 0 \) but \( g \neq 0 \) and \( g' \neq 0 \). In this case we have (at \( T = 0 \)) (Supplementary Information)

\[
\Pi(i\omega, q) = -c_d |\omega| - \frac{\pi}{2} N\gamma^2 g^2 |\omega|,
\]

\[
\Sigma(i\omega, k = k_F k) = -i\gamma |\omega| |\omega|^{2/3} - \frac{i\gamma g^2 |\omega|}{4\pi} \ln \left( \frac{e\Lambda^3 d}{c_d |\omega|} \right),
\]

where \( \Lambda \sim g^2/(g^2 v_F N) \) is a UV momentum cutoff. Interestingly, the disordered interactions induce a marginal Fermi liquid term in \( \Sigma \), which manifests as the first higher order correction to the translationally invariant result (3) [41].

It is sufficient in this \( v = 0 \) but \( g \neq 0 \) and \( g' \neq 0 \) case to compute the conductivity using the theory of modes in the vicinity of antipodal points on the Fermi surface [42]. We then find, as before, that \( \sigma_{\Sigma, g} \) and \( \sigma_{V, g} \) cancel, and (Supplementary Information)

\[
\frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \frac{Nv_F^2}{2\omega} \left( \frac{Nv_F^2 |\omega|}{2\pi\omega} \right) \ln \left( \frac{e\Lambda^6}{c_d |\omega|^2} \right)
\]

\[
\approx \frac{Nv_F^2}{2\omega + \frac{Nv_F^2 |\omega|}{6\pi} \ln \left( \frac{e\Lambda^6}{c_d |\omega|^2} \right)};
\]

\[
\frac{1}{N} \text{Re}[\sigma(\omega \gg T)] = \frac{6|\omega| \left( 2 + \frac{Nv_F^2}{6\pi} \ln \left( \frac{e\Lambda^6}{c_d |\omega|^2} \right) \right)^2}{N\gamma g^2 |\omega|^4 36}.
\]

The transport scattering rate is therefore still linear in \( |\omega| \) (and hence \( T \)), up to logarithms, and there is no residual resistivity when \( v = 0 \) despite the presence of disorder in \( g' \). This also turns out to be valid to all orders in perturbation theory in the large \( N \) limit (Supplementary Information).
Fermi liquid, and so deduced a ‘scattering time’ $\tau$ renormalization of the effective mass in a proximate appearing in a Drude formula for the resistivity. In our (Supplementary Information). However then, as shown above for $v = 0$, the $|\omega|$ or $T$ dependence of the transport scattering rate continues to arise from $g'$ and remains linear (up to logarithms), but with a slope that is a factor of $\sim 2/3$ times the slope in the $E < E_{c,1}$ theory.

For energy scales larger than $E_{c,1} \sim \Gamma^2/(v_F e_d)$, but smaller than $E_{c,2} \sim g^4/(g^6 v_F^2 N^4)$ ($E_{c,1} < E_{c,2}$; because $N g^2 < g^2 / \Gamma$ as disorder is a correction to the clean system), the leading frequency dependence of the inelastic part of the fermion self energy induced by $g$ changes from $\omega \ln(1/|\omega|)$ to $\text{sgn}(\omega)|\omega|^{2/3}$, as in the theory with $v = 0$ described above (Supplementary Information). However then, as shown above for $v = 0$, the $|\omega|$ or $T$ dependence of the transport scattering rate continues to arise from $g'$ and remains linear (up to logarithms), but with a slope that is a factor of $\sim 2/3$ times the slope in the $E < E_{c,1}$ theory.

For energy scales larger than $E_{c,2}$, there is an additional crossover to the theory with $g = 0$ considered in Refs. [30, 32], which also has a linear $|\omega|$ or $T$ dependence (up to logarithms) of the transport scattering rate, but now with the same slope as in the $E < E_{c,1}$ theory (Supplementary Information).

Planckian behavior. Experimental analyses [6, 7] have compared the slope of the linear-$T$ resistivity to the renormalization of the effective mass in a proximate Fermi liquid, and so deduced a ‘scattering time’ $\tau_{tr}$ appearing in a Drude formula for the resistivity. In our theory, we obtain

$$\frac{1}{\tau_{tr}} = \alpha \frac{k_B T}{h}.$$  \hfill (14)

The dimensionless number $\alpha$ has been computed previously [30, 32] in the limit $g' \gg g$ to be $\alpha \approx (\pi/2) \times$ (ratio of logarithms of $T$). For smaller $g'$ we find (at $v \neq 0$) (Supplementary Information)

$$\alpha \approx \frac{\pi}{2} \frac{g'^2}{g^2 L_1(T) + \frac{g'^2}{\Gamma} L_2(T)}, \quad L_1,2(T) \sim -\ln T. \quad (15)$$

Therefore, ‘Planckian behavior’ ($\alpha \sim \mathcal{O}(1)$ and depending only slowly on $T$ and non-universal parameters) only occurs in the regime of large $g'$ considered in Refs. [30, 32]. Otherwise, $\alpha \ll 1$ when $g$ is the largest interaction coupling. Our theory therefore provides a concrete realization of the often conjectured “Planckian bound” of $\alpha \lesssim 1$ on the transport scattering times of quantum-critical metals [1, 5, 8]. It is worth noting that quantum-critical $T$-linear resistivity with $\alpha \ll 1$ has been recently observed in experiments on heavy fermion materials [7]. Finally, for $v = 0$ but $g \neq 0$, $\alpha \ll 1$ and has a power-law dependence on $T$; therefore there is manifestly no Planckian behavior in this case.

Scalar mass disorder. Finally, we propose a route to accounting for disorder in the scalar ‘mass’ $m_b$. Such a term is not allowed for emergent gauge fields, but it can appear as a fluctuation in the position of the quantum-critical point for the cases where $\phi$ is a symmetry breaking order parameter.

$$S_w = \int d\tau \frac{1}{2 \sqrt N} \int d^2 r \sum_{ij=1}^N w_{ij}(r) \phi_i(r, \tau) \phi_j(r, \tau)$$  \hfill (16)

with

$$w_{ij}(r) w_{lm}(r') = \frac{w_0^2}{2} \delta(r - r') (\delta il \delta jm + \delta im \delta jl).$$  \hfill (17)

The large $N$ analysis shows that $S_w$ is strongly relevant. Consequently it is appropriate to account for $S_w$ first, by transforming to the bases of eigenmodes of $\phi$ which are eigenstates of the harmonic terms for $\phi$ in a given disorder realization. In this new basis, we will obtain a theory which has the same form as $S_g + S_v + S_{g'}$ with additional spatial disorder in the couplings, including in $K$. However, it is not difficult to show that spatial disorder in $K$ is unimportant. So we conclude that $S_w$ can be effectively absorbed in a renormalization of the values of $v$ and $g'$, and we can continue to use our results for $S_g + S_v + S_{g'}$.

Discussion. (i) A Phenomenologically attractive feature of our theory is that the residual resistivity and the slope of the linear-$T$ resistivity are determined by different types of disorder: respectively, the potential disorder
v (which determines the elastic scattering rate \( \Gamma \)) and the interaction disorder \( g' \) (which determines the inelastic self energy in the last term of (9)).

(ii) Our theory yields a marginal Fermi liquid electron self energy [39], as is often observed in quantum-critical metals [43].

(iii) All the computed diagrams, and their associated cancellations, apply also for \( N = 1 \); the large \( N \) method serves to systematically select a consistent set of diagrams to resum, from the saddle point of an effective action.

(iv) Our theory of the influence of spatial disorder is non-perturbative in the disorder strength, unlike the perturbative disorder analysis of earlier memory function treatments [14, 16].

(v) Unlike earlier approaches (see Ref. 2) to constructing controlled theories of strongly correlated metals with low-temperature T-linear resistivity, there is no local criticality in our new theory. The quantum-critical scalar fluctuations live in two, and not zero spatial dimensions.

(vi) When the values of the interaction couplings and \( T \) are large enough to make the fermion self energies \( \Sigma \) comparable to the Fermi energy, we expect the theories described here to cross over into a locally critical ‘bad metal’ regime [44]; it would be interesting to examine the transport properties of such a regime.

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Supplementary Information for
Universal $T$-linear resistivity in two-dimensional quantum-critical metals from spatially random interactions

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SI. SPATIALLY UNIFORM QUANTUM-CRITICAL METAL (DERIVATION OF EQ. (4))

A. The model and notations

In this section, we review some properties of the clean model studied in the previous work [1], and recapitulate some useful notations. The main result of this section is to derive Eq. (4) in the main text. For simplicity, we will work with $K = 1$ and restore it in the main text by dimensional analysis.

1. Lagrangian and $G-\Sigma$ action

The Lagrangian of the clean model is given by

$$\mathcal{L} = \sum_i \bar{\psi}_i (\partial_\tau + \varepsilon_k - \mu) \psi_i + \frac{1}{2} \sum_i \phi_i (-\partial_\tau^2 + \omega_q^2 + m_b^2) \phi_i + \sum_{ijkl} \frac{g_{ijkl}}{N} \bar{\psi}_i \psi_j \phi_k \phi_l.$$  \hspace{1cm} (S1)

Here $\varepsilon_k$ and $\omega_q^2$ which physically describe the dispersions of fermions and bosons respectively, should be understood as differential operators that act on the fields. The Yukawa couplings $g_{ijkl} = g_{jkil}^*$ are Gaussian random variables with zero mean and variance $|g_{ijkl}|^2 = g^2$. 
Assuming the system self averages, we perform disorder average over $g_{ijl}$ with a simple replica, and next we introduce bilocal variables

$$G(x_1, x_2) = -\frac{1}{N} \sum_i \psi_i(x_1)\psi_i^\dagger(x_2),$$

$$D(x_1, x_2) = \frac{1}{N} \sum_i \phi_i(x_1)\phi_i(x_2),$$

(S2)

as well $\Sigma(x_1, x_2)$ and $\Pi(x_1, x_2)$ as Lagrangian multipliers to enforce the above definitions, to obtain the $G$-$\Sigma$ action

$$\frac{1}{N} S[G, \Sigma, D, \Pi] = -\ln \det \left( (\partial_x + \varepsilon_k - \mu) \delta(x - x') + \Sigma \right) + \frac{1}{2} \ln \det \left( (-\partial_x^2 + \omega_q^2 + m_k^2) \delta(x - x') - \Pi \right)$$

$$- \text{Tr} \left( \Sigma \cdot G \right) + \frac{1}{2} \text{Tr} \left( \Pi \cdot D \right) + \frac{g^2}{2} \text{Tr} \left( (GD) \cdot G \right).$$

(S3)

Here $\delta(x - x')$ denotes a spacetime delta function.

We pause briefly the explain our notation, which is the same as in Ref. [2]. For two bilocal functions $f, g$, we define their inner product as

$$\text{Tr} (f \cdot g) \equiv f^T g \equiv \int dx_1 dx_2 f(x_2, x_1) g(x_1, x_2).$$

(S4)

The action of a linear functional $A$ is defined as:

$$A[f](x_1, x_2) \equiv \int dx_3 dx_4 A(x_1, x_2; x_3, x_4) f(x_3, x_4).$$

(S5)

The transpose acts both on functions and on functionals:

$$f^T(x_1, x_2) \equiv f(x_2, x_1),$$

(S6)

$$A^T(x_1, x_2; x_3, x_4) \equiv A(x_4, x_3; x_2, x_1).$$

(S7)

2. Saddle point

Going back to the action (S3) and differentiating it, we obtain

$$\frac{\delta S}{\delta N} = \text{Tr} \left( \delta \Sigma \cdot (G_\ast[\Sigma] - G) + \delta G \cdot (\Sigma_\ast[G] - \Sigma) + \frac{1}{2} \delta \Pi \cdot (D - D_\ast[\Pi]) + \frac{1}{2} \delta D \cdot (\Pi - \Pi_\ast[D]) \right),$$

(S8)

where

$$G_\ast[\Sigma](x_1, x_2) = (-\partial_x + \mu - \varepsilon_k - \Sigma)^{-1}(x_1, x_2),$$

(S9)

$$\Sigma_\ast[G](x_1, x_2) = \frac{g^2}{2} G(x_1, x_2) (D(x_1, x_2) + D(x_2, x_1)), $$

(S10)

$$D_\ast[\Pi](x_1, x_2) = (-\partial_x^2 + \omega_q^2 + m_k^2)^{-1}(x_1, x_2),$$

(S11)

$$\Pi_\ast[D](x_1, x_2) = -g^2 G(x_1, x_2) G(x_2, x_1).$$

(S12)

In the first and the third line the inverse is in the functional sense. Therefore the saddle point equations are simply

$$G = G_\ast[\Sigma], \quad \Sigma = \Sigma_\ast[G], \quad D = D_\ast[\Pi], \quad \Pi = \Pi_\ast[D].$$

(S13)
3. Fluctuations about the saddle point

We can further expand (S3) to second order around the saddle point to obtain the fluctuations around the saddle point. Define the collective notation $G_a = (D, G)$ and $\Xi_a = (\Pi, \Sigma)$, where $a = b, f$ denotes boson/fermion. The gaussian fluctuations around the saddle point is described by

$$\frac{1}{N} \delta^2 S = \frac{1}{2} \begin{pmatrix} \delta \Xi^T & \delta G^T \end{pmatrix} \Lambda \begin{pmatrix} W_\Sigma & -1 \\ -1 & W_G \end{pmatrix} \begin{pmatrix} \delta \Xi \\ \delta G \end{pmatrix},$$

where $\Lambda = \text{diag}(-1/2, 1)$ acts on the $b, f$ indices, and $W_\Sigma$ and $W_G$ are defined by

$$W_\Sigma(x_1, x_2; x_3, x_4)_{a\bar{a}} = \frac{\delta G_a[\Xi_\alpha(x_1, x_2)]}{\delta \Xi_{a\bar{a}}(x_3, x_4)}, \quad W_G(x_1, x_2; x_3, x_4)_{a\bar{a}} = \frac{\delta G_a[\Xi_\alpha(x_1, x_2)]}{\delta G_{a\bar{a}}(x_3, x_4)}.$$  \hspace{1cm} (S15)

Later for the evaluation of the conductivity, we will be using fluctuation of self energies, which is given by

$$\langle \delta \Xi_a(x_1, x_2) \delta \Xi_{a\bar{a}}(x_4, x_3) \rangle = \left[ W_G \frac{1}{W_\Sigma W_G - 1} \Lambda^{-1} \right]_{a\bar{a}} (x_1, x_2; x_3, x_4).$$

For the $G$-$\Sigma$ action (S3), $W_\Sigma$ and $W_G$ are given by Feynman diagrams

$$W_\Sigma(x_1, x_2; x_3, x_4) = \begin{pmatrix} 1 & \cdots & \cdots & 3 \\ 2 & \cdots & \cdots & 4 \\ 0 & \cdots & \cdots & 3 \\ 0 & \cdots & \cdots & 4 \end{pmatrix},$$

$$W_G(x_1, x_2; x_3, x_4) = \begin{pmatrix} 0 & -g^2 \left( 1 & \cdots & \cdots & 3 \\ 2 & \cdots & \cdots & 4 \\ 1 & \cdots & \cdots & 3 \\ 2 & \cdots & \cdots & 4 \right) \end{pmatrix},$$

where a black arrowed line denotes fermion propagator, a dashed arrowed line denotes boson propagator, and an unarrowed dashed line denotes spacetime $\delta$-function. The first entry is boson and the second entry is fermion. Recalling $\Lambda = \text{diag}(-1/2, 1)$, we see that $\Lambda W_\Sigma$ and $\Lambda W_G$ are explicitly symmetric as required by quadratic expansion.

In momentum space, we can explicitly write down the action of $W_\Sigma$ and $W_G$:

$$W_\Sigma \begin{pmatrix} B(k, p) \\ F(k, p) \end{pmatrix} = \begin{pmatrix} G(k + p/2)G(k - p/2) & 0 \\ 0 & D(k + p/2)D(k - p/2) \end{pmatrix} \begin{pmatrix} B(k, p) \\ F(k, p) \end{pmatrix},$$

$$W_G \begin{pmatrix} B(k, p) \\ F(k, p) \end{pmatrix} = \begin{pmatrix} \tilde{B}(k, p) \\ \tilde{F}(k, p) \end{pmatrix},$$

where

$$\tilde{B}(k_1, p) = -g^2 \int \frac{d^3k_2}{(2\pi)^3} \left[ G(k_2 - k_1)F(k_2, p) + G(k_1 - k_2)F(-k_2, p) \right],$$

$$\tilde{F}(k_1, p) = g^2 \int \frac{d^3k_2}{(2\pi)^3} \left[ \frac{1}{2} G(k_1 - k_2) (B(k_2, p) + B(-k_2, p)) + D(k_1 - k_2)F(k_2, p) \right].$$

Here $p$ denotes the CoM momentum and $k$ denotes the relative momentum. Unless stated explicitly, we will be using $\int d\omega/(2\pi)$ and $T \sum_{\omega_n}$ interchangeably.
4. Relation to patch theories

In the previous paper [1], we have studied the same theory within patch approximations \( \varepsilon_k = \pm k_x + k_y^2 \). In this paper, we will take a different route by working with the full Fermi surface and taking a patch-like approximation at a later stage. While patch theories produce the correct solution for the saddle point equations, they are inadequate for transport computations. In particular, within the patch theory the vector nature of the current operator is neglected, and it behaves very similar to the density operator. For example, in the single patch theory they are exactly proportional and in the two patch theory with two antipodal patches, they differ by a \( \mp \) sign on the left/right patch. Due to this similarity, current-current correlation function can be inferred from the density-density correlation function, and this results in zero conductivity at non-zero frequency.

As we will see later in the theory of the full Fermi surface, the current operator as a vector, is susceptible to additional scattering events than the density operator, which is a scalar. These scattering events are due to bosons carrying momentum tangential to the Fermi surface. Because the current operator contains \( l = 1 \) angular harmonics, there is a phase shift \( e^{-i\theta k \cdot k'} \) associated with the scattering event \( k \rightarrow k' \), which is absent for scalar operators. This effect has the same origin as the \( (1 - \cos \theta) \) factor in the transport scattering rate of Boltzmann equations, and this factor is set to zero in the patch theory.

B. Expression for Conductivity

1. Polarization Bubble

In this section we derive an expression for the conductivity from the \( G-\Sigma \) action. To define the electric current, we use the minimal coupling scheme \( \partial_\mu \rightarrow \partial_\mu + iA_\mu \), i.e. \( k_\mu \rightarrow k_\mu + A_\mu \), so the only relevant term is the fermion determinant term as the following:

\[
S[G, \Sigma, D, \Pi; A] = -\ln \det((\partial_\tau + \varepsilon_{k+A} - \mu)\delta(x-x') + \Sigma) - \text{Tr}(\Sigma \cdot G + S_b[D, \Pi] + S_{int}[G, D]),
\]  

(S23)

where \( S_b[D, \Pi] \) denotes the kinetic terms for the boson and \( S_{int}[G, D] \) describes the interactions.

The conductivity is given by Kubo formula

\[
\sigma^{\mu\nu}(\omega) = i\frac{\Pi^{\mu\nu}(i\omega_n \rightarrow \omega + i0, k = 0)}{\omega},
\]

(S24)

and here the polarization \( \Pi^{\mu\nu} \) is defined in real space by

\[
\Pi^{\mu\nu}_A(x,x') = -\frac{\delta^2 \ln Z[A]}{\delta A_\mu(x) \delta A_\nu(x')} \bigg|_{A=0},
\]

(S25)

where \( Z[A] = \int DGD\Sigma DD\Pi e^{-S} \) is the partition function. We can alternatively write the above expression as

\[
\Pi^{\mu\nu}_A(x,x') = \left\langle \frac{\delta^2 S}{\delta A_\mu(x) \delta A_\nu(x')} - \frac{\delta S}{\delta A_\mu(x)} \frac{\delta S}{\delta A_\nu(x')} \right\rangle_{A=0},
\]

(S26)

where the average only includes connected diagrams, and it is performed over bilocal fields. In the leading large-\( N \) order, we can take \( S \) to be the saddle-point action. The expression in fourier space is given by

\[
\Pi^{\mu\nu}_A(p) = -\frac{(2\pi)^3}{\delta(0)} \frac{\delta^2 \ln Z[A]}{\delta A_\mu(-p) \delta A_\nu(p)} \bigg|_{A=0},
\]

(S27)
where \( A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} A_\mu(p) e^{ip\cdot \hat{x} - ip_0x_0} \).

Let’s now compute the functional derivatives in (S26). Expanding (S23) in \( A \) by

\[
S[A] = S_0 + \delta_A S + \delta^2_A S ,
\]

where for the first order term we have

\[
\delta_A S = N \int_{x,x'} G_\ast[\Sigma](x,x') \delta_A \varepsilon_{k+A}(x',x).
\]

Here \( G_\ast[\Sigma] \) is a functional of \( \Sigma \) which defines the RHS of SD equations:

\[
G_\ast[\Sigma] = \frac{1}{-\partial_x + \mu - \varepsilon_{k+A} - \Sigma} ,
\]

and \( \partial_x, \mu, \varepsilon_{k+A}, \Sigma \) should be understood as bilocal fields or functionals on local fields.

We can proceed to second order in the expansion, which yields

\[
\delta^2_A S = \frac{N}{2} \left[ \int_{x,x',y,x'} G_\ast[\Sigma](x,y) \delta_A \varepsilon_{k+A}(y,y') G_\ast[\Sigma](y',x') \delta_A \varepsilon_{k+A}(x',x) + 2 \int_{x,x'} G_\ast[\Sigma](x,x') \delta^2_A \varepsilon_{k+A}(x',x) \right] .
\]

We can see that the first term of (S26) comes from (S31), which can be evaluated directly at the saddle point. The first term in (S31) is a current-current correlator and the second term is a contact term. The second term of (S26), however, is zero at the saddle point (since they are disconnected) and must be evaluated using fluctuations of the bilocal fields, i.e. summing the ladder diagrams.

2. Vertex functions

To write down explicit expressions for the functional derivatives, we need to calculate the vertex functions \( \delta_A \varepsilon_{k+A} \). For simplicity, we shall assume that we only turn on gauge field in the \( x \)-direction, and it is independent of \( y \): \( A_x(\tau,x,y) = A_x(\tau,x) \). Under this assumption, the kinetic term \( \varepsilon_{k+A} \) is

\[
\varepsilon_{k+A} = \varepsilon_k(k_x + A_x, k_y) ,
\]

where \( \varepsilon_k \) is a (smooth) function describing the dispersion, but the arguments \( k_x + A_x \) and \( k_y \) are operators. Our above assumptions of \( A_x \) means that \( A_x \) commutes with \( k_y \), and therefore we can unambiguously write down a Taylor expansion for \( \varepsilon_k \):

\[
\varepsilon_k(k_x + A_x, k_y) = \sum_{n=0}^{\infty} \frac{1}{n!} f_x^{(n)}(0)(k_x + A_x)^n ,
\]

where \( f_x(k_x) = \varepsilon_k(k_x, k_y) \).

Let’s first calculate \( \delta_{A_x} \varepsilon_{k+A} \), we can expand \( \varepsilon_{k+A} \) to first order in \( A_x \):

\[
\delta_{A_x} \varepsilon_{k+A} = \sum_{n=0}^{\infty} \frac{1}{n!} f_x^{(n)}(0) \left( k_x^{n-1} A_x + k_x^{n-2} A_x A_x + \cdots + I_{A_x A_x A_x}^{n-1} \right) .
\]

This is an operator equation, where the matrix elements are

\[
k_x(x,x') = -i \partial_x \delta(x - x') , \quad A_x(x,x') = A_x(x) \delta(x - x') .
\]

(S35)
Insert these matrix elements into $\delta A_x \varepsilon_{k+A}$, and we obtain

$$\frac{\delta \varepsilon_{k+A}}{\delta A_x(x_0)}(x_1, x_2) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \sum_{m=0}^{n-1} (k_x^{n-1-m})(x_1, x_0)(k_x^{m})(x_0, x_2)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) \sum_{m=0}^{n-1} (-i \partial_{x_1})^n (i \partial_{x_2})^m \delta(x_1 - x_0) \delta(x_2 - x_0)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) (\frac{-i \partial_{x_1}}{-i \partial_{x_1}} - (i \partial_{x_2})) \delta(x_1 - x_0) \delta(x_2 - x_0)$$

$$= f_x (-i \partial_{x_1}) - f_x (i \partial_{x_2}) \delta(x_1 - x_0) \delta(x_2 - x_0).$$

(S36)

Here $\partial_x$ only acts on the $x$-component, but the delta functions are over the spacetime. We can also write it in momentum space as

$$\frac{\delta \varepsilon_{k+A}}{\delta A_x(p)}(p, q) = \Gamma^x(p, q) \delta(r + q - p), \quad \Gamma^x(p, q) = \frac{f_x(p_x) - f_x(q_x)}{p_x - q_x}. \quad \text{(S37)}$$

We remind the reader that here the external momentum $r$ has no $y$ component $r = (r_0, r_x, 0)$.

To obtain the second derivative, we write

$$\frac{\delta^2 \varepsilon_{k+A}}{\delta A_x(-r) \delta A_x(r)}(p, r + p) = \frac{\delta(0)}{(2\pi)^3} \Delta^x(p, r), \quad \Delta^x(p, r) = 2 \left( \frac{d}{dp_x} \frac{f_x(p_x) - f_x(q_x)}{p_x - q_x} \right) \bigg|_{q=r+p}. \quad \text{(S39)}$$

The expression for the functional derivative is complicated for general external momentum, but we only need it for the case where the two $A_x$’s carry opposite momenta, and the functional derivative simplifies to

$$\frac{\delta^2 \varepsilon_{k+A}}{\delta A_x(-r) \delta A_x(r)}(p, r + p) = \frac{\delta(0)}{(2\pi)^3} \Delta^x(p, r), \quad \Delta^x(p, 0) = \frac{\partial^2 \varepsilon_{k}(p)}{\partial p_x^2}. \quad \text{(S40)}$$

Therefore we can write down the contribution to $\Pi_A$ from the first term of (S26), which originates from (S31):

$$\Pi_{A1}^{\varepsilon}(r) = N \int \frac{d^3p}{(2\pi)^3} G_s[\Sigma](p) \Gamma^x(p, r + p) G_s[\Sigma](r + p) \Gamma^x(r + p, p). \quad \text{(S41)}$$

$$\Pi_{A2}^{\varepsilon}(r) = N \int \frac{d^3p}{(2\pi)^3} G_s[\Sigma](p) \Delta_x(p, r). \quad \text{(S42)}$$

These two terms are the same as the conventional current-current correlator term and the diamagnetic term. This can be seen from the example

$$\varepsilon_k = \frac{k_x^2 + k_y^2}{2m} \quad \Gamma^x(p, p) = \frac{p_x}{m}, \quad \Delta_x(p, 0) = \frac{1}{m}, \quad \text{which agrees with well-known results.} \quad \text{(S43)}$$
Finally, we look at the second term of (S26). At leading \( N \) order we can expand \( G_\Sigma \) and obtain

\[
\Pi_{xx}^{A_3}(r) = - \frac{N^2}{(2\pi)^3\delta(0)} \int \frac{d^3p d^3q}{(2\pi)^6} G_\Sigma(p) G_\Sigma(q) G_\Sigma(p+r) G_\Sigma(q+r) \Gamma^x(p+r,p) \Gamma^x(q,q+r) \times \langle \delta \Sigma(p,p+r) \delta \Sigma(q,q+r) \rangle.
\]

(S44)

Here the \( \delta \Sigma(p,q) \) is the Fourier transform of the fluctuating bilocal field

\[
\delta \Sigma(p,q) = \int d^3x d^3y \delta \Sigma(x,y) e^{-i(\mathbf{p} \cdot \mathbf{x} - p_0 x_0)} e^{i(\mathbf{q} \cdot \mathbf{y} - q_0 y_0)}.
\]

(S45)

The correlator \( \langle \delta \Sigma(p,p+r) \delta \Sigma(q,q+r) \rangle \) is calculated in the previous paper [1], where we have derived the expression:

\[
\langle \delta \Sigma(p,p+r) \delta \Sigma(q,q+r) \rangle = N^{-1} \left( 2\pi \right)^3 \delta^3(0) \left[ W_G \frac{1}{K_G - 1} \Lambda^{-1} \right] (p + r/2, q + r/2; r),
\]

(S46)

where the delta-function comes from energy-momentum conservation. The first two arguments on the RHS label the relative momenta and the third argument denotes the CoM momentum. Since we are looking at fermionic components, the matrix \( \Lambda \) can be replaced by identity.

Also notice that the \( \Gamma_{xx} \) factors in (S41) and (S44) are nothing but \( \Sigma \), we can therefore write \( \Pi_{xx} \) as

\[
\Pi_{xx} = \Pi_{A_1}^{xx} + \Pi_{A_3}^{xx} \equiv \Pi_{A_1}^{xx}(r) + \Pi_{A_3}^{xx}(r) = N(\Gamma^{xx})^T \frac{1}{W^{xx}_{\Sigma^{-1}} - W_G} \Gamma^{xx},
\]

(S47)

where the vertex function \( \Gamma^{xx} \) is viewed as a two-point function by ignoring the leg with external momentum \( r \), and thus can be acted by \( W^{xx}_{\Sigma} \).

The total polarization is therefore

\[
\Pi_{xx} = \Pi_{A_1}^{xx} + \Pi_{A_2}^{xx}.
\]

(S48)

The above formalism can also be used to derive the charge-charge polarization function. Using the minimal coupling scheme \( \partial_\tau \rightarrow \partial_\tau + iA_\tau \), we obtain the vertex function

\[
\Gamma^\tau(p,q) = i.
\]

(S49)

There is no diamagnetic term for charge, so the charge-charge (density-density) polarization function is

\[
\Pi_{xx}^\tau = N(\Gamma^{\tau})^T \frac{1}{W^{\tau}_{\Sigma^{-1}} - W_G} \Gamma^{\tau}.
\]

(S50)

3. Polarization bubble at the DC limit

In this section we show that at the DC limit \( p_x = 0, p_0 \rightarrow 0 \), the polarization bubble vanishes:

\[
\Pi_{xx}^A(p_x = 0, p_0 \rightarrow 0) = 0.
\]

(S51)

Here, we use \( p_n \) to denote the discrete Matsubara frequency and \( p_0 \) to denote the frequency continued to real time, i.e. \( ip_n \rightarrow p_0 + i\eta \).

We introduce a renormalized vertex function \( V^\mu \):

\[
V^\mu = W^{\mu}_{\Sigma^{-1}} \frac{1}{W^{\tau}_{\Sigma^{-1}} - W_G} \Gamma^\mu.
\]

(S52)
Therefore the current-current (paramagnetic) contribution to the polarization is
\[ \Pi^{xx}_{A13}(p_n, \vec{p} = 0) = NT \sum_{q_n} \int \frac{d^2 \vec{q}}{(2\pi)^2} \Gamma^x(q, q) G(q + p) G(q) V^x(p + q, q). \] (S53)

Here we have used the fact that the bare vertex \( \Gamma^x(q, q + p) = \Gamma^x(q, q) \) because \( \vec{p} = 0 \).

The diamagnetic term is
\[ \Pi^{xx}_{A2}(p_n, \vec{p} = 0) = NT \sum_{q_n} \int \frac{d^2 \vec{q}}{(2\pi)^2} \Delta_x(q, 0) G(q). \] (S54)

Using (S40), we can integrate by parts in \( q_x \) to obtain
\[ \Pi^{xx}_{A2}(p_n, \vec{q} = 0) = -NT \sum_{q_n} \int \frac{d^2 \vec{q}}{(2\pi)^2} \Gamma^x(q, q) G(q)^2 \left( \Gamma^x(q, q) + \frac{\partial \Sigma(q)}{\partial q_x} \right). \] (S55)

We therefore needs to show that the renormalized vertex \( V^x(q, q + p) \) cancels the terms in the parenthesis in (S55) when \( p_0 \to 0 \).

Using the Ward identity (S84) in the next section, we have
\[ \eta \eta V^\mu(p + q, q) = G^{-1}(q) - G^{-1}(q + p). \] (S56)

Plugging in \( p_\mu = (-p_n, p_x, 0) \) and expanding the Green’s functions, we get
\[ -p_n V^\tau(p + q, q) + p_x V_x(p + q, q) = -i p_n + (\varepsilon_{p+q} - \varepsilon_q) + (\Sigma(p + q) - \Sigma(q)). \] (S57)

Taking the limit \( p_x \to 0 \) on both sides, and matching to linear order in \( p_x \), we obtain
\[ V^x = \Gamma^x + p_n \frac{\partial V^\tau}{\partial p_x} + \frac{\partial \Sigma(p_n + q_n, \vec{p})}{\partial q_x}. \] (S58)

Here both \( V^x \) and \( \Gamma^x \) are evaluated at \( (p + q, q) \) with \( \vec{p} = 0 \), and the derivative of \( V^\tau \) is
\[ \frac{\partial V^\tau}{\partial p_x} = \frac{\partial V^\tau(k, q)}{\partial k_x} \bigg|_{k=(p_n+q_n, \vec{q})}. \] (S59)

Now, the function \( V^x \) given by (S58), viewed as a function of \( p_n \), can be analytically continued to the complex \( p_n \) plane and it has a branch cut at \( p_n = -q_n \). There is no ambiguity in taking the limit \( p_n \to \eta \), and because \( \partial V^\tau / \partial p_x \) is finite, we have
\[ V^x = \Gamma^x + \frac{\partial \Sigma(q)}{\partial q_x}, \] (S60)
and therefore \( \Pi^{xx}(p_n \to 0, \vec{p} = 0) = 0. \)

C. Ward Identities

For the clean model, Ward identities are an important tool that makes the evaluation of conductivities possible. The main idea is the following: We will apply arguments similar to Prange and Kadanoff [3] to integrate out momentum dependence in electron Green’s functions and reduce the kernel \( W^{-1}_\Sigma - W_G \) in (S47) to act only in frequency and angular harmonic space. Under this reduction, the Ward identities become a statement of eigenvector of \( W^{-1}_\Sigma - W_G \), and it turns out that the vertex function \( \Gamma^x \) is very close to this eigenvector, which makes a resummed perturbation theory possible.
1. Master Ward identity

We first present a master Ward identity which includes both U(1) symmetry and diffeomorphism invariance. We write the \( G-\Sigma \) action in the form

\[
\frac{S}{N} = -\ln \det (\sigma_f + \Sigma) + \frac{1}{2} \ln \det (-\sigma_b - \Pi) - \text{Tr} \left( \Sigma \cdot G \right) + \frac{1}{2} \text{Tr} \left( \Pi \cdot D \right) + \frac{g^2}{2} \text{Tr} \left( (GD) \cdot G \right),
\]

where

\[
\sigma_f(x,x') = (\partial_\tau + \varepsilon_k - \mu)\delta(x-x'),
\]

and

\[
\sigma_b(x,x') = (\partial_\tau^2 - \omega^2_q)\delta(x-x')
\]

are the UV sources.

Consider the following change of variables \((G, \Sigma, D, \Pi, \sigma_f, \sigma_b) \rightarrow (\tilde{G}, \tilde{\Sigma}, \tilde{D}, \tilde{\Pi}, \tilde{\sigma}_f, \tilde{\sigma}_b)\) which makes the action invariant:

\[
G(x_1, x_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{\Delta} \left| \frac{\partial y_2}{\partial x_2} \right|^{\Delta} \tilde{G}(y_1, y_2) e^{i(\lambda(y_1) - \lambda(y_2))},
\]

\[
\Sigma(x_1, x_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{1-\Delta} \left| \frac{\partial y_2}{\partial x_2} \right|^{1-\Delta} \tilde{\Sigma}(y_1, y_2) e^{i(\lambda(y_1) - \lambda(y_2))},
\]

\[
D(x_1, x_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{1-2\Delta} \left| \frac{\partial y_2}{\partial x_2} \right|^{1-2\Delta} \tilde{D}(y_1, y_2),
\]

\[
\Pi(x_1, x_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{2\Delta} \left| \frac{\partial y_2}{\partial x_2} \right|^{2\Delta} \tilde{\Pi}(y_1, y_2),
\]

\[
\sigma_f(x_1, x_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{1-\Delta} \left| \frac{\partial y_2}{\partial x_2} \right|^{1-\Delta} \tilde{\sigma}_f(y_1, y_2) e^{i(\lambda(y_1) - \lambda(y_2))},
\]

\[
\sigma_b(x_1, x_2) = \left| \frac{\partial y_1}{\partial x_1} \right|^{2\Delta} \left| \frac{\partial y_2}{\partial x_2} \right|^{2\Delta} \tilde{\sigma}_b(y_1, y_2).
\]

Here \( |\partial y/\partial x| \) is the Jacobian of \( y = y(x) \), and \( \Delta \) is an arbitrary real number.

Define \( \delta_{\lambda,y} G = \tilde{G}(x_1, x_2) - G(x_1, x_2) \) and similarly for other variables, we can write down a master Ward identity

\[
\text{Tr} \left( \frac{\delta S}{\delta G} \delta_{\lambda,y} G + \frac{\delta S}{\delta \Sigma} \delta_{\lambda,y} \Sigma + \frac{\delta S}{\delta D} \delta_{\lambda,y} D + \frac{\delta S}{\delta \Pi} \delta_{\lambda,y} \Pi \right) = -\text{Tr} \left( \frac{\delta S}{\delta \sigma_f} \delta_{\lambda,y} \sigma_f + \frac{\delta S}{\delta \sigma_b} \delta_{\lambda,y} \sigma_b \right).
\]

Taking functional derivatives of the Master Ward identity (S70) at the saddle point, we obtain

\[
\int dx_1 dx_2 \left( \frac{\delta \Sigma(x_2, x_1)}{\delta G(x_3, x_4)} \delta_{g,y} G(x_1, x_2) - \frac{1}{2} \frac{\delta \Pi(x_2, x_1)}{\delta G(x_3, x_4)} \delta_{g,y} D(x_1, x_2) \right) = \delta_{g,y} \Sigma(x_4, x_3),
\]
\[
\int dx_1 dx_2 \left( \frac{\delta \Sigma^* (x_2, x_1)}{\delta D(x_3, x_4)} \delta y, \lambda G(x_1, x_2) - \frac{1}{2} \frac{\delta \Pi^* (x_2, x_1)}{\delta D(x_3, x_4)} \delta y, \lambda D(x_1, x_2) \right) = -\frac{1}{2} \delta y, \lambda \Pi(x_4, x_3), \quad (S72)
\]

\[
- \delta y, \lambda G(x_4, x_3) + \int dx_1 dx_2 \frac{\delta G^* (x_2, x_1)}{\delta \Sigma(x_3, x_4)} \delta y, \lambda \Sigma(x_1, x_2) = -\int dx_1 dx_2 \frac{\delta G^* (x_2, x_1)}{\delta \Sigma(x_3, x_4)} \delta y, \lambda \sigma_f (x_1, x_2), \quad (S73)
\]

\[
\frac{1}{2} \delta y, \lambda D(x_4, x_3) - \frac{1}{2} \int dx_1 dx_2 \frac{\delta D^* (x_2, x_1)}{\delta \Pi(x_3, x_4)} \delta y, \lambda \Pi(x_1, x_2) = \frac{1}{2} \int dx_1 dx_2 \frac{\delta D^* (x_2, x_1)}{\delta \Pi(x_3, x_4)} \delta y, \lambda \sigma_b (x_1, x_2). \quad (S74)
\]

Matching the above functional derivatives with the definitions of \( W_\Sigma \) and \( W_G \), and using the property that \( \Lambda W_\Sigma \) and \( \Lambda W_G \) are symmetric, we can bring the above four equations into a compact form

\[
(\delta y, \lambda \Pi, \delta y, \lambda \Sigma)^T = W_G (\delta y, \lambda D, \delta y, \lambda G)^T, \quad (S75)
\]

\[
(W^{-1}_\Sigma - W_G) (\delta y, \lambda D, \delta y, \lambda G)^T = (\delta y, \lambda \sigma_b, \delta y, \lambda \sigma_f)^T, \quad (S76)
\]

and here the transpose only acts on \( b, f \) indices and doesn’t act on functions.

2. \( U(1) \) Ward identity

Setting \( y(x) = x \), we obtain the \( U(1) \) Ward identity:

\[
\delta \chi \Sigma = W_G \delta \chi G, \quad (S77)
\]

\[
(W^{-1}_\Sigma - W_G) \delta \chi G = \delta \chi \sigma_f. \quad (S78)
\]

Here the bosons are not charged under \( U(1) \) and therefore dropped.

Using the transformations \((S64), (S65)\) and \((S68)\), we can explicitly write down \( \delta \chi \Sigma \) and \( \delta \chi G \) in momentum space:

\[
\delta \chi G(k, p) = i \left[ G \left( k - \frac{p}{2} \right) - G \left( k + \frac{p}{2} \right) \right] \lambda(p), \quad (S79)
\]

\[
\delta \chi \Sigma(k, p) = i \left[ \Sigma \left( k - \frac{p}{2} \right) - \Sigma \left( k + \frac{p}{2} \right) \right] \lambda(p), \quad (S80)
\]

\[
\delta \chi \sigma_f(k, p) = i \left[ \sigma_f \left( k - \frac{p}{2} \right) - \sigma_f \left( k + \frac{p}{2} \right) \right] \lambda(p). \quad (S81)
\]

Here \( \lambda(p) = \int d^3x \lambda(x) e^{-ip \cdot x} \), and \( p \cdot x = \vec{p} \cdot \vec{x} - p_0 x_0 \). Using \( \sigma_f(k) = -i k_0 + \varepsilon_k - \mu \), and the vertex functions, we can rewrite \( \delta \chi \sigma_f \) as

\[
\delta \chi \sigma_f(k, p) = -i \lambda(p) p_\mu \Gamma^\mu(k + p/2, k - p/2), \quad (S82)
\]

where \( p_\mu = (-p_n, \vec{p}) \).

Factoring out \( i \lambda(p) \), the \( U(1) \) Ward identity then reduces to the statements

\[
\Sigma \left( k - \frac{p}{2} \right) - \Sigma \left( k + \frac{p}{2} \right) = W_G \left[ G \left( k - \frac{p}{2} \right) - G \left( k + \frac{p}{2} \right) \right], \quad (S83)
\]
and
\[ G \left( k - \frac{p}{2} \right) - G \left( k + \frac{p}{2} \right) = \frac{1}{W^{-1}_\Sigma - W_G} \left[ -p_\mu \Gamma^\mu \right] (k, p) . \] (S84)

The above two Ward identities are easy to check using the saddle point equations. The first identity (S83) follows from the fact that \( W_G = \delta \Sigma / \delta G \) and that \( \Sigma \) is linear in \( G \). By using explicit forms of \( W_\Sigma \) and \( W_G \), the second identity (S84) is equivalent to
\[ [\Sigma(k - p/2) + G^{-1}(k - p/2) - \Sigma(k + p/2) - G^{-1}(k + p/2)] = p_\mu \Gamma^\mu(k + p/2, k - p/2) , \] (S85)
which is trivially satisfied by the vertex functions.

3. Density-Density Correlation Function

We can use the Ward identity to compute the density-density correlation function at the limit \( \vec{p} = 0 \). Setting \( p_\mu = (\Omega_n, 0) \), and using \( \Gamma^\tau = i \), the Ward identity (S84) yields
\[ \frac{1}{W^{-1}_\Sigma - W_G} \left[ 1 \right](r, p) = \frac{1}{i\Omega_n} \left[ G(ir_n - i\Omega_n/2, \vec{r}) - G(ir_n + i\Omega_n/2, \vec{r}) \right] , \] (S86)
therefore
\[ \Pi^{00}(i\Omega_n, \vec{p} = 0) = T \sum_{r_n} \int \frac{d^2 \vec{r}}{(2\pi)^2} \frac{1}{i\Omega_n} \left[ G(ir_n - i\Omega_n/2, \vec{r}) - G(ir_n + i\Omega_n/2, \vec{r}) \right] = 0 , \] (S87)
while agrees with [4].

4. Diffeomorphism Ward identity

Now we want to derive the Ward identity for translation symmetry, by setting \( \lambda = 0 \). Let \( y^\mu = x^\mu + \epsilon^\mu \), we can compute:
\[ \delta y, \lambda = 0 G = - \left( \Delta \partial_\mu \epsilon^\mu(x_1) + \Delta \partial_\mu \epsilon^\mu(x_2) + \epsilon^\mu(x_1) \partial_{x_1}^\mu + \epsilon^\mu(x_2) \partial_{x_2}^\mu \right) G(x_1, x_2) , \] (S88)
\[ \delta y, \lambda = 0 \Sigma = - \left( (1 - \Delta) \partial_\mu \epsilon^\mu(x_1) + (1 - \Delta) \partial_\mu \epsilon^\mu(x_2) + \epsilon^\mu(x_1) \partial_{x_1}^\mu + \epsilon^\mu(x_2) \partial_{x_2}^\mu \right) \Sigma(x_1, x_2) , \] (S89)
\[ \delta y, \lambda = 0 D = - \left( (1 - 2\Delta) \partial_\mu \epsilon^\mu(x_1) + (1 - 2\Delta) \partial_\mu \epsilon^\mu(x_2) + \epsilon^\mu(x_1) \partial_{x_1}^\mu + \epsilon^\mu(x_2) \partial_{x_2}^\mu \right) D(x_1, x_2) , \] (S90)
\[ \delta y, \lambda = 0 \Pi = - \left( 2\Delta \partial_\mu \epsilon^\mu(x_1) + 2\Delta \partial_\mu \epsilon^\mu(x_2) + \epsilon^\mu(x_1) \partial_{x_1}^\mu + \epsilon^\mu(x_2) \partial_{x_2}^\mu \right) \Pi(x_1, x_2) , \] (S91)
\[ \delta y, \lambda = 0 \sigma_f = - \left( (1 - \Delta) \partial_\mu \epsilon^\mu(x_1) + (1 - \Delta) \partial_\mu \epsilon^\mu(x_2) + \epsilon^\mu(x_1) \partial_{x_1}^\mu + \epsilon^\mu(x_2) \partial_{x_2}^\mu \right) \sigma_f(x_1, x_2) , \] (S92)
\[ \delta y, \lambda = 0 \sigma_b = - \left( 2\Delta \partial_\mu \epsilon^\mu(x_1) + 2\Delta \partial_\mu \epsilon^\mu(x_2) + \epsilon^\mu(x_1) \partial_{x_1}^\mu + \epsilon^\mu(x_2) \partial_{x_2}^\mu \right) \sigma_b(x_1, x_2) . \] (S93)
Since the choice of $\Delta$ is arbitrary, we expect all terms proportional to $\Delta$ to cancel identically in the master Ward identity (S70). This cancellation involves an extra ingredient, which is the UV regularization of the determinant terms [2]: $\det(\sigma_f + \Sigma) \rightarrow \det(\sigma_f + \Sigma)/\det(\sigma_f)$, $\det(\sigma_b + \Pi) \rightarrow \det(\sigma_b + \Pi)/\det(\sigma_b)$. After using this regularization, the cancellation of $\Delta$ terms becomes manifest. This regularization term is unimportant for the derived Ward identities (S75) and (S76) because they are obtained from functional derivatives of (S70) with respect to bi-local fields, but the regularization term is independent of the fields.

We can rewrite the above infinitesimal transformations in fourier space as

$$\delta_{y,\lambda=0} A(k,p) = -ip_\mu \epsilon^\mu(p) \left( \Delta_A - \frac{1}{2} \right) \left[ A \left( k - \frac{p}{2} \right) + A \left( k + \frac{p}{2} \right) \right] - ik_\mu \epsilon^\mu(p) \left[ A \left( k - \frac{p}{2} \right) - A \left( k + \frac{p}{2} \right) \right]$$

(S94)

$$= -ip_\mu \epsilon^\mu \Delta_A \left[ A \left( k - \frac{p}{2} \right) + A \left( k + \frac{p}{2} \right) \right] - ik_\mu \epsilon^\mu(p) \left[ A \left( k - \frac{p}{2} \right) A \left( k + \frac{p}{2} \right) A \left( k + \frac{p}{2} \right) \right]$$

Here $p_\mu = \eta_{\mu\nu} p^\nu$, with $\eta_{\mu\nu} = (-, +, +)$. $k$ denotes relative momentum and $p$ denotes CoM momentum. $A = G, \Sigma, D, \Pi, \sigma_f, \sigma_b$ and $\Delta_G$ denotes the corresponding value of $\Delta$ appeared above.

The two Ward identities (S75) and (S76) with diffeomorphism can also be verified by using the saddle point equations.

The Noether theorem states that

$$\delta_y S = -\int d^3x T^{\mu\nu} \partial_\mu \varepsilon_\nu(x),$$

(S95)

where $T^{\mu\nu}$ is the stress tensor. We are interested in the consequences of momentum conservation at the transport limit, therefore we set $\varepsilon^0 = 0$ and $p^\mu = (p_n, 0)$ in (S94). Applying this to $\delta_y \sigma_f$ and $\delta_y \sigma_b$, we can read out the momentum vertices:

$$\delta_{y,\lambda=0} \sigma_f(k, p) = i\Gamma^\mu(k + p/2, k - p/2)k_\nu p_\mu \varepsilon_\nu,$$

(S96)

$$\delta_{y,\lambda=0} \sigma_b(k, p) = i\bar{\Gamma}^\mu(k + p/2, k - p/2)k_\nu p_\mu \varepsilon_\nu,$$

(S97)

where $\Gamma^\mu$ is the electron current vertex and $\bar{\Gamma}^\mu = (k_n, -\vec{k})$. The momentum vertices are therefore read out to be $\Gamma^0 k_i$ and $\bar{\Gamma}^0 k_i$.

**D. Solving the saddle point**

We now solve the saddle point equations on the whole FS. We work in the units where the boson velocity $c = 1$.

The boson self energy is

$$\Pi(i\Omega_n, \vec{q}) = -g^2 T \sum_{\omega_n} \int \frac{d^2 \vec{k}}{(2\pi)^2} \frac{1}{i\omega_n - \xi_k - \Sigma(i\omega_n)} \frac{1}{i\omega_n - \xi_{k+q} - \Sigma(i\omega_n + i\Omega_n)}.$$

(S98)

We expand the dispersion with $\xi_{k+q} = \xi_k + v_F q \cos \theta_{kq}$ and then we can perform the integral over $\theta_{kq}$ and $\xi_k$ to obtain

$$\Pi(i\Omega_n, \vec{q}) = \pi \mathcal{N} g^2 T \sum_{\omega_n} \frac{\text{sgn} \omega_n (\text{sgn} (\omega_n + \Omega_n) - \text{sgn} \omega_n)}{\sqrt{v_F^2 q^2 - (i\Omega_n - \Sigma(i\omega_n + i\Omega_n) + \Sigma(i\omega_n))^2}},$$

(S99)

where $\mathcal{N} = \frac{m}{2\pi}$ is the fermion DoS. In the denominator, only the $v_F q$ term is relevant, and we get

$$\Pi(i\Omega_n, q) = -\gamma \frac{|\Omega_n|}{q}, \quad \gamma = \frac{\mathcal{N} g^2}{v_F}.$$

(S100)
As a sanity check, we compare with patch theory where \( m = 1/2 \) and \( v_F = 1 \), we get \( \gamma = g^2/(4\pi) \) which agrees with two-patch theory. At zero Matsubara frequency, we also need to include a thermal mass term in the boson propagator

\[
D(0, q) = \frac{1}{q^2 + \Delta(T)^2},
\]

where \( \Delta(T)^2 \sim T \ln(1/T) \) [5, 6].

The electron self energy \( \Sigma = \Sigma_Q + \Sigma_T \) can be decomposed into a quantum part \( \Sigma_Q \propto |\omega|^{2/3} \) and a thermal part \( \Sigma_T \propto T^{1/2} \). The quantum part is

\[
\Sigma_Q(i\omega_n, k) = g^2 \int \frac{d^2 q}{(2\pi)^2} T \sum_{\Omega_n \neq 0} \frac{1}{q^2 + \gamma |\Omega_n|/q} i\omega_n - i\Omega_n - \frac{1}{2} \varepsilon_{k-q} - \Sigma(i\omega_n - i\Omega_n). \tag{S102}
\]

We expand \( \varepsilon_{k-q} = \varepsilon_k - qv_F \cos \theta_q \), and then integrate over \( \theta_q \) to get

\[
\Sigma_Q(i\omega_n, k) = g^2 T \int_0^\infty \frac{dq}{2\pi} \sum_{\Omega_n \neq 0} \frac{1}{q^2 + \gamma |\Omega_n|/q} \frac{\text{sgn}(\Omega_n - \omega_n)}{\sqrt{(qv_F)^2 + A(\omega_n)^2}}, \tag{S103}
\]

where \( A(\omega_n) = \omega_n + i\Sigma(\omega_n) \). We now evaluate the \( q \) integral. Due to the boson propagator, the typical value of \( q \) is of order \( |\Omega_n|^{1/3} \), which is larger than \( A(\omega_n) \) in the scaling sense. Therefore we can drop \( A(\omega_n) \) in the second factor and obtain

\[
\Sigma_Q(i\omega_n, k) = \frac{ig^2}{v_F} T \sum_{\Omega_n} \int_0^\infty dq \frac{\text{sgn}(\Omega_n - \omega_n)}{2\pi} \frac{1}{q^2 + \gamma |\Omega_n|/q} = \frac{ig^2}{3\sqrt{3}v_F\gamma^{1/3} T \sum_{\Omega_n} \text{sgn}(\Omega_n - \omega_n)} |\Omega_n|^{1/3} = -\frac{i2^{2/3} g^2 T^{2/3} \text{sgn}(\omega_n)}{3\sqrt{3} \pi^{1/3} \gamma^{1/3} v_F} H_{1/3} \left( \frac{|\omega_n|}{2\pi T} - \frac{1}{2} \right) = -\frac{ig^2}{2\sqrt{3} \pi v_F \gamma^{1/3} \text{sgn}(\omega_n)|\omega_n|^{2/3}}, \quad (T = 0). \tag{S104}
\]

The above result also agrees with two-patch theory in [1] when \( \gamma = g^2/(4\pi) \), \( v_F = 1 \).

The thermal part of the self-energy is

\[
\Sigma_T(i\omega_n, k) = g^2 T \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q^2 + \Delta(T)^2} i\omega_n - \frac{1}{2} \varepsilon_{k-q} - \Sigma_Q(i\omega_n) - \Sigma_T(i\omega_n). \tag{S105}
\]

Evaluating the \( q \) integral, we obtain

\[
\Sigma_T(i\omega_n) = -\text{sgn} \omega_n \frac{g^2 T}{2\pi} \frac{\text{sec}^{-1} \left( \frac{v_F \Delta(T)}{|A(\omega_n)|} \right)}{\sqrt{v_F^2 \Delta(T)^2 - A(\omega_n)^2}}, \tag{S106}
\]

where \( A(\omega_n) = \omega_n + i\Sigma_Q(i\omega_n) + i\Sigma_T(i\omega_n) \).

At the low-frequency limit \( |\omega_n + i\Sigma_Q(\omega_n)| \ll \Delta(T) \), we obtain

\[
\Sigma_T(i\omega_n) = -\text{sgn} \omega_n h(T), \tag{S107}
\]

where \( h(T) \) satisfies

\[
h(T) = \frac{g^2 T}{2\pi} \frac{\text{cos}^{-1} \left( \frac{h(T)}{v_F \Delta(T)} \right)}{\sqrt{v_F^2 \Delta(T)^2 - h(T)^2}}. \tag{S108}
\]
Since $\Delta(T)^2/T \to \infty$ as $T \to 0$, the asymptotic behavior of $h(T)$ is

$$h(T) \to \frac{g^2T}{4v_F\Delta(T)} \left(1 + \frac{g^2T}{2\pi v_F^2\Delta(T)^2}\right)^{\frac{1}{2}}. \quad (S109)$$

## E. Conductivity Computation

### 1. Prange-Kadanoff Reduction

The saddle point computation above is consistent with a reduction method proposed by Prange and Kadanoff [3]. It assumes that the fermionic spectral function $A(\omega, \vec{k})$ has a sharp peak at the Fermi surface, and doesn’t require a well-defined quasiparticle peak in $\omega$. Therefore as an approximation, we could restrict all fermionic momenta to be exactly on the FS, and work with the angular variables. For application to our problem, there is an additional validity requirement [7]: the typical peak in the boson propagator (as a function of momentum $q$) should be much wider than the peak in the propagator (as a function of $\xi_k = v_F q$), i.e.

$$v_F |\text{Im} \Pi(\omega)|^{1/2} \gg |\text{Im} \Sigma_R(\omega)|. \quad (S110)$$

The exponent $1/2$ on the LHS is due to the fact that boson momentum appears in the propagator as $(q^2 + \Pi)^{-1}$, and therefore the typical boson momentum is order $|\text{Im} \Pi(\omega)|^{1/2}$.

The condition (S110) implies a description using only fermions on the Fermi surface: For any boson carrying momentum $q_\parallel$ normal to the fermi surface, it will excite a fermion with energy $\xi_k \sim v_F q_\parallel$. This energy is much larger than the width determined by fermion self energy and that process has a much smaller amplitude due to small fermionic spectral weight. As a consequence, we only consider bosons that connect fermions on the Fermi surface. When the two fermion momenta are close, it also implies that the boson momentum is tangent to the fermi surface — a feature also seen in patch theories.

The condition (S110) is indeed satisfied by the clean model we are considering: the fermion self energy is of order $\text{Im} \Sigma \sim \max(|\omega|^{2/3}, T^{1/2}/\ln(1/T))$, and the boson self energy is $\text{Im} \Pi(\omega \neq 0) \sim |\omega|/q \sim |\omega|^{2/3}$ and $\text{Im} \Pi(\omega = 0) \sim \Delta(T)^2 \sim T \ln(1/T)$. However, this condition is violated when we add disorder potential to the fermions, and therefore the method only applies to the translational invariant model.

We now apply the reduction idea to conductivity computation. We are interested in optical conductivity and we work at $T = 0$. We compute the paramagnetic term (S47) of the polarization function:

$$\Pi^{x\bar{x}}(i\Omega_n, \vec{p} = 0) = (\Gamma^x)^T \frac{1}{W^x_{\Sigma} - W_G} \Gamma^x, \quad (S111)$$

where we have assumed zero CoM momentum and a finite CoM frequency $\Omega_n > 0$. The diamagnetic term exactly cancels the contribution of the paramagnetic term at zero frequency, so the conductivity is

$$\sigma_{xx}(\omega) = \left. \frac{\Pi^{x\bar{x}}(i\Omega_n) - \Pi^{x\bar{x}}(0)}{\Omega_n} \right|_{\Omega_n \to \omega + i0}. \quad (S112)$$

Near the Fermi surface, we can approximate the vertex function to be $\Gamma^x(k, k) = v_F \cos \theta_k$, which only contains first harmonics of $\theta_k$. We can write $\Pi^{x\bar{x}}$ as an inner product of the form

$$\Pi^{x\bar{x}}(i\Omega_n) = v_F^2 \langle \cos \theta_k | \frac{1}{W^x_{\Sigma} - W_G} | \cos \theta_k \rangle. \quad (S113)$$
Here the inner product is defined as
\[
\langle f | g \rangle = \int \frac{d\omega}{2\pi} \frac{d^2\vec{k}}{(2\pi)^2} f(\vec{k}, i\omega) g(\vec{k}, i\omega), \quad (S114)
\]
and |cos θ_k⟩ denotes the constant function cos θ_k.

Notice that W_Σ and W_G are block operators as given in (S17) and (S18), and we are only interested in the fermionic sector, we can perform a block inversion to obtain
\[
\left( \frac{1}{W^{-1}_{\Sigma} - W_G} \right)_{EF} = \frac{1}{W^{-1}_{\Sigma,EF} - \underbrace{W_{G,FF} W_{\Sigma,FB} W_{G,BF}}_{W_{MT}}}. \quad (S115)
\]
Here the additional subscripts refer to blocks of W_Σ, W_G. The two terms that emerge from the block inversion can be interpreted as Maki-Thomson (MT) diagrams and Aslamazov-Larkin (AL) diagrams.

2. Maki-Thomson Diagrams

We apply the Prange-Kadanoff reduction to the MT diagram kernel W_MT, which is given by
\[
W_{MT}[F](\omega, \vec{k}) = g^2 \int \frac{d\omega'}{2\pi} \frac{d^2\vec{k}'}{(2\pi)^2} D(k - k') F(\omega', \vec{k}'). \quad (S116)
\]
We factorize the momentum integral as
\[
\int \frac{d^2\vec{k}'}{2\pi^2} = N' \int d\theta' \int \frac{d\xi_{k'}}{2\pi}, \quad (S117)
\]
where the density of state is \( N = k_F/(2\pi v_F) \). Assuming the function F is sharply peaked on the FS \( \xi_{k'} = 0 \), we perform the integral over \( \xi_{k'} \) first, obtaining
\[
\tilde{F}(\omega', \theta') = \int \frac{d\xi_{k'}}{2\pi} F(\omega', \vec{k}'), \quad (S118)
\]
and other factors in (S116) are assumed to have a smooth dependence on \( \xi_{k'} \), and are evaluated at \( \xi_{k'} = 0 \). In later steps we will also integrate over \( \xi_k \), and therefore we can assume \( \vec{k} \) is also on the Fermi surface, we get
\[
W_{MT}[\tilde{F}](\omega, \theta, \xi_k = 0) = N' g^2 \int \frac{d\omega' d\theta'}{2\pi} \frac{1}{|q|^2 + 2|\omega' - \omega|/|q|} \tilde{F}(\omega', \theta'), \quad (S119)
\]
where the boson momentum \( \vec{q} = k_F(\hat{\theta} - \hat{\theta}') \) and \( \hat{\theta}, \hat{\theta}' \) are unit vectors corresponding to angles \( \theta, \theta' \) respectively. To carry out the \( \theta' \) integral, we use a gradient expansion. Let \( \theta' = \theta + \delta\theta \), and expand \( \tilde{F}(\omega', \theta') = \tilde{F}(\omega', \theta) + \delta\theta \partial_{\omega'} \tilde{F}(\omega', \theta) + \frac{1}{2} \delta\theta^2 \partial_{\omega'}^2 \tilde{F}(\omega', \theta) + \ldots \). The momentum \( \vec{q} \) is parameterized as \( |q| = 2k_F \sin(\delta\theta/2) \). The result is
\[
W_{MT}[\tilde{F}](\omega, \theta, \xi_k = 0) = \frac{g^2}{v_F} \frac{2}{3\sqrt{3}} \int \frac{d\omega'}{2\pi} \frac{1}{\gamma^{1/3}|\omega' - \omega|^{1/3}} \tilde{F}(\omega', \theta) - \frac{\gamma^{1/3}|\omega - \omega'|^{1/3}}{2k_F^2} \partial_{\omega'} \tilde{F}(\omega', \theta), \quad (S120)
\]
Here we have only kept the leading order term in \( 1/k_F \) for each order of derivative in \( \theta \). As we will see later, the first term in the bracket cancels the self energies. In obtaining (S120), we used dimensional regularization by analytically continuing the following integral
\[
\int_0^{\infty} dq \frac{q^n}{q^2 + \frac{a}{q}} = \frac{\pi}{3} a^{\eta+1} \sec \left( \frac{\pi}{6}(2\eta + 1) \right) \quad (a > 0), \quad (S121)
\]
which is only convergent for \(-2 < \eta < 1\) but continued to all \( \eta \).
Next we consider the
\[ W_{\text{AL}}[F](k_1) = -\frac{g^4}{2} \int \frac{d^3q d^3k_2}{(2\pi)^6} \left( G(k_1 - q) + G(k_1 + q) \right) \left( G(k_2 - q) + G(k_2 + q) \right) \]
\[ \times D(q + p/2)D(q - p/2)F(k_2) \]
\[ = -g^4 \int \frac{d^3q d^3k_2}{(2\pi)^6} G(k_1 - q)(G(k_2 - q) + G(k_2 + q)) D(q + p/2)D(q - p/2) \]
\[ \times F(k_2), \]
where \( p = (\Omega_n, 0) \) denotes the CoM frequency. We first perform the Prange-Kadanoff reduction. We rewrite (S122) as (\( \nu \) is the frequency component of \( q \))
\[ W_{\text{AL}}[F](\omega_1, \bar{k}_1) = -g^4 \int \frac{d^3q d^3k_2 d^2\bar{k} d^2\bar{k}'}{(2\pi)^6} G(\omega_1 - \nu, \bar{k}')\delta(\bar{q} = \bar{k}_1 - \bar{k}') \]
\[ \times \left( G(\omega_2 - \nu, \bar{k}'')\delta(\bar{q} = \bar{k}_2 - \bar{k}'') + G(\omega_2 + \nu, \bar{k}'')\delta(\bar{q} = \bar{k}'' - \bar{k}_2) \right) \]
\[ \times D(q + p/2)D(q - p/2)F(\omega_2, \bar{k}_2). \]

Next, we perform integrals over \( \xi_k, \xi_\nu \) and \( \xi_{\nu'} \) assuming other terms in the integrand are slow varying, and we effectively restrict all fermionic momenta to be on the FS, parameterized by angles \( \theta_1, \theta_2, \theta', \theta'' \). The momentum delta functions then impose the following conditions on the angles:
\[ (\theta_2, \theta'') = (\theta_1, \theta') \text{ or } (\theta' + \pi, \theta_1 + \pi) \text{ if } \bar{q} = \bar{k}_1 - \bar{k}' = \bar{k}_2 - \bar{k}''; \]
\[ (\theta'', \theta_2) = (\theta_1, \theta') \text{ or } (\theta' + \pi, \theta_1 + \pi) \text{ if } \bar{q} = \bar{k}_1 - \bar{k}' = \bar{k}'' - \bar{k}_2. \]

We can therefore integrate out \( \theta_2 \) and \( \theta'' \), yielding
\[ W_{\text{AL}}[F](\omega_1, \theta_1, \xi_{k_1} = 0) = \pi^2 g^4 N^3 \int \frac{d\nu d\omega_2}{2\pi} \frac{d\theta'}{q k_F} \frac{1}{D(q + p/2)D(q - p/2)\text{sgn}(\omega_1 - \nu)} \]
\[ \times \left[ \text{sgn}(\omega_2 - \nu) \left( \hat{F}(\omega_2, \theta_1) + \hat{F}(\omega_2, \theta' + \pi) \right) \right] + \text{sgn}(\omega_2 + \nu) \left( \hat{F}(\omega_2, \theta') + \hat{F}(\omega_2, \theta_1 + \pi) \right). \]

Here the momentum \( \bar{q} = k_F(\bar{\theta}_1 - \bar{\theta}'). \) To proceed, we should assume that the function \( \hat{F} \) has a definite parity \( P = \pm 1 \) under inversion: \( \hat{F}(\theta + \pi) = P\hat{F}(\theta) \). We obtain
\[ W_{\text{AL}}[F](\omega_1, \theta_1, \xi_{k_1} = 0) = \pi^2 g^4 N^3 \int \frac{d\nu d\omega_2}{2\pi} \frac{d\theta'}{q k_F} \frac{1}{D(q + p/2)D(q - p/2)} \]
\[ \times \left( \text{sgn}(\omega_1 - \nu) + P\text{sgn}(\omega_1 + \nu) \right) \left( \text{sgn}(\omega_2 - \nu) + P\text{sgn}(\omega_2 + \nu) \right) \left( \hat{F}(\omega_2, \theta_1) + P\hat{F}(\omega_2, \theta') \right). \]

For computation of conductivity, we are interested in odd parity modes and we set \( P = -1 \) from now on. Performing gradient expansion in \( \theta \), we get
\[ W_{\text{AL}}^{\mu = -1}[F](\omega_1, \theta_1, \xi_{k_1} = 0) = -\frac{g^4}{6\sqrt{3}k_F v_F^2 \gamma^2 2^3} \int |\nu| > |\nu_1|, |\nu| > |\nu_2| \frac{d\nu d\omega_2}{(2\pi)^2} \frac{|\nu + \Omega/2|^{1/3} - |\nu - \Omega/2|^{1/3}}{|\nu + \Omega/2| - |\nu - \Omega/2|} \partial^2 \hat{F}(\omega_2, \theta_1). \]

4. Resummation

In this part we include the effects of \( W_\Sigma \) in (S113) and (S115). We expand the geometric series to write
\[ \frac{1}{W^{-1}_\Sigma - W_{\text{MT+AL}}} = W_\Sigma + W_\Sigma W_{\text{MT+AL}} W_\Sigma + \ldots. \]

(S128)
In analyzing $W_{MT}$ and $W_{AL}$ in previous sections, we have assumed that they act on functions of $\xi_k$ which are sharply peaked on the Fermi surface. This assumption is justified by noting that the $W_\Sigma$ factor as a product of two fermion Green’s functions is indeed peaked on the Fermi surface. Therefore, we have

$$\int \frac{d\xi_k}{2\pi} W_\Sigma[F](\xi, \omega, \theta) \simeq L(\omega)\theta(\Omega/2 - |\omega|)F(\omega, \theta, \xi_k = 0),$$

where $\theta(\Omega/2 - |\omega|)$ is the Heaviside theta function and

$$L(\omega) = \int \frac{d\xi_k}{2\pi} W_\Sigma(\xi_k, \omega) = \int \frac{d\xi_k}{2\pi} G(i(\omega + \Omega/2), \xi_k)G(i(\omega - \Omega/2), \xi_k) = \frac{i}{2} \frac{\text{sgn} (\omega + \Omega/2) - \text{sgn} (\omega - \Omega/2)}{\Omega - i\Sigma(i(\omega - \Omega/2)) + i\Sigma(i(\omega + \Omega/2))}.$$

We see that the effect of $W_\Sigma$ is to restrict the functional space to be supported only on $[-\Omega/2, \Omega/2]$. We arrive at the following new inner product formula for $\Pi_{xx}$, which is over functions of angle $\theta$ and frequency $\omega$ ($|\omega| \leq \Omega/2$):

$$\Pi_{xx}(\omega) = v_F^2 (\cos \theta) \frac{1}{\mathcal{L}^{-1} - W_{MT} - W_{AL}} \cos \theta,$$

where

$$\langle f || g \rangle = \mathcal{N} \int_0^{2\pi} d\theta \int_{-\Omega/2}^{\Omega/2} d\omega f(i\omega, \theta)g(i\omega, \theta).$$

The operator $L$ is defined by Eq. (S130), and $W_{MT}$ and $W_{AL}$ are given by Eqs. (S120) and (S127) respectively, understood as functionals acting on $\hat{F}$ instead of $F$.

The vertex function $f(\theta_k) = \cos \theta_k$ appearing in (S131) is frequency independent, allowing us to compute its image under $W_{MT}$ and $W_{AL}$ explicitly:

$$W_{MT}[f](\omega, \theta) = \frac{g^2}{v_F} \int_{-\Omega/2}^{\Omega/2} \frac{d\omega'}{2\pi} \left[ \frac{1}{\gamma^{1/3}|\omega - \omega'|^{1/3}} f(\theta) - \frac{\gamma^{1/3}|\omega - \omega'|^{1/3}}{2k_F^2} \partial_\theta f(\theta) \right]$$

$$= \frac{g^2 v_F^{1/3}}{8\pi\sqrt{3} v_F k_F} \left[ \text{sgn} (\omega + \Omega/2)\omega + \Omega/2 |\omega - \Omega/2|^{4/3} - \text{sgn} (\omega - \Omega/2)\omega - \Omega/2 |\omega - \Omega/2|^{4/3} \right] \partial_\theta^2 f(\theta).$$

$$W_{AL}^{-1}[f](\omega, \theta) = \frac{-g^4}{6v_F^2 \sqrt{3} k_F^{2/3}} \int_{|\nu| > |\omega|, |\nu| > |\omega'|, |\omega'| < \Omega/2} \frac{d\nu d\nu'}{(2\pi)^2} \left[ \frac{|\nu + \Omega/2|^{1/3} - |\nu - \Omega/2|^{1/3}}{\nu + \Omega/2 - |\nu - \Omega/2|} \partial_\theta^2 f(\theta) \right].$$

In obtaining (S134), we again used dimensional regularization on the exponents of $|\nu \pm \Omega/2|$ to drop the divergent parts at $\nu \to \pm \infty$.

Using the relation $\gamma = \frac{N^2 g^2}{v_F} = \frac{k_F g^2}{2\pi v_F}$, we see that the last line of (S133) exactly cancels (S134), and therefore

$$\left( L^{-1} - W_{MT} - W_{AL}^{-1} \right) [f] = \Omega f.$$

That is, any odd-parity frequency-independent function $f(\theta)$ is an eigenvector of $L^{-1} - W_{MT} - W_{AL}$ with eigenvalue $\Omega$.

This implies that the conductivity of the model is exactly Drude like

$$\sigma_{xx}(\omega) = \frac{N v_F^2}{2} \frac{1}{-i\omega}. $$
5. Discussion

a. Change of Integration Order: In obtaining the above results, we have exchanged the order of integration between frequency and momentum, which can potentially modify the value of the integral. However, the difference between two integration orders is due to UV divergence at large frequency and momentum. There is exactly one diagram that has this behavior, which is the one-loop bubble of fermions. By examining this diagram, it can be shown that changing the integration order just cancels the diamagnetic term (S42).

b. Cancellation and Ward Identity The Drude-like result (S136) is due to two cancellation related symmetries: First, the cancellation between self energies and Maki-Thomson diagrams due to U(1) symmetry and charge conservation. Second, the cancellation between the Aslamazov-Larkin diagram and the Maki-Thomson diagram is due to diffeomorphism symmetry and momentum conservation. These cancellations can be relate to the Ward identities derived in Sec. SIC by Prange-Kadanoff reduction. The almost cancellation between the self energy and the MT diagram is a consequence of the U(1) Ward identity. This can be seen by integrating both sides of Eq. (S86) over $\xi_r$ [8]. The cancellation between the rest of MT diagram and the AL diagram can be seen as the following: Within the Prange-Kadanoff formalism, we only consider momenta exactly on the Fermi surface. Therefore the current vertex function $\nu F \cos \theta$ is proportional to the momentum vertex function $k_F \cos \theta$. Because the boson self-interaction is irrelevant at the critical point, there is no boson-boson entry in the kernel $W_G$, and from the Ward identity (S75), we have

$$\delta_y \Pi = W_{G,BF}[\delta_y G].$$

(S137)

Here $\delta_y$ denotes small diffeomorphism transformation as defined in Eqs.(S88)-(S93). Substitute the above into (S76) and we obtain

$$(W_{\Sigma}^{-1} - W_{MT} - W_{AL})[\delta_y G] = \delta_y \sigma_f - W_{G,FB} W_{\Sigma} \delta_y \sigma_b.$$  

(S138)

At the critical point, the bare boson momentum term $\sigma_b$ is also irrelevant compared to the boson self energy $\Pi$, and therefore the second term on the RHS (S138) can be dropped. Multiplying $(W_{\Sigma}^{-1} - W_{MT} - W_{AL})^{-1}$ on both sides and then perform Prange-Kadanoff reduction by integrating over $\xi$, we see that the momentum vertex is exactly an eigenvector of $W_{\Sigma}^{-1} - W_{MT} - W_{AL}$ with eigenvalue $\Omega$.

c. A Would-be $|\omega|^{-2/3}$ Optical Conductivity If we ignore the cancellation between MT and AL diagrams and consider the MT diagram only, our calculation would reproduce the conventional $|\omega|^{-2/3}$ conductivity in the literature [4, 9]. This can be seen by noting that at lowest order of angular expansion, the MT diagram exactly cancels self-energy contribution (see the first line of (S133)). This is related to the U(1) Ward identity, and can be physically interpreted as forward scattering doesn’t contribute to current dissipation. We have obtained an eigenvalue statement $(L^{-1} - W_{MT}^{(0)})[f] = \Omega f$ which is valid only at zeroth order of gradient expansion. Effect of small angle scattering is included as a first order gradient expansion (the second line of (S133)), which perturbs the eigenvalue equation above by a term of order $\Omega^{1/3}$, whose leading order effect is to shift the eigenvalue by an amount of order $\Omega^{1/3}$. As a result, we would obtain a Drude formula with scattering rate $\sim \Omega^{4/3}/k_F^2$, and in the $k_F \to \infty$ limit, this turns into a $|\omega|^{-2/3}$. The cancellation between two integration orders is due to UV divergence at large frequency and momentum. There is exactly one diagram that has this behavior, which is the one-loop bubble of fermions. By examining this diagram, it can be shown that changing the integration order just cancels the diamagnetic term (S42).

d. Slow Relaxation of Odd-Parity Modes We now argue that within the Prange-Kadanoff approximation, every odd harmonic $\cos m\theta$ satisfies the eigenvalue equation (S135) at any order of gradient expansion. As a corollary,
(S136) is valid at the critical point regardless of Fermi surface shape, as long as it has inversion symmetry. This conclusion is in disagreement with Ref. [9] which assumed that MT and AL diagrams would not cancel.

Eq. (S135) has already been shown at second order in the gradient expansion. What happens at higher order? It can be seen that both in $W_{MT}$ and $W_{AL}$ associated with each $\partial \theta$ there is a factor of $\delta \theta \simeq q/k_F \sim \gamma^{1/3}|\Omega|^{1/3}/k_F$. Therefore the gradient expansion is at the same time a $1/k_F$ expansion (i.e. the series is actually in $(1/k_F)\partial \theta$). Momentum conservation implies that the series vanishes identically for first harmonics to all orders in $1/k_F$, and therefore it must also vanish to all orders in $\partial \theta$, given $P = -1$.

When the Fermi surface is not exactly circular but still inversion symmetric, we can decompose the current vertex into angular harmonics of the momentum angle $\theta_k$, and by inversion symmetry it only contains odd harmonics. Since all odd-harmonics satisfy (S135), the result (S136) continues to hold.

There is a more intuitive way to understand the statement in terms kinematic constraint for fermion collision. What happens in our model is a non-Fermi liquid generalization of a Fermi liquid story [10, 11]. Within the Prange-Kadanoff approximation, we only consider momenta on the Fermi surface scattering onto Fermi surface. Because of momentum conservation and Pauli’s exclusion principle, when two initial momenta $(\vec{k}_1, \vec{k}_2)$ are not head-on $(\vec{k}_1 + \vec{k}_2 \neq 0)$, the only kinematically allowed process is forward scattering or particle exchange. This process doesn’t cause any relaxation. When the two initial momenta are head-on, they are allowed to scatter to any head-on pairs. However, this process only relaxes even harmonics of the Fermi surface, because a pair of head-on particles have zero overlap with odd harmonics. This intuitive picture holds for any inversion symmetric Fermi surface.

e. Beyond Prange-Kadanoff According to the Fermi liquid story [10–12], the first correction to the eigenvalue equation (S135) is a superdiffusion term $\partial^4 \theta$ in the angular coordinate. The superdiffusion term can be understood as a two-particle correlated random walk on the angular coordinate which conserves center of mass coordinate due to momentum conservation. Furthermore, the superdiffusion also intertwines angular and radial relaxation, and it is therefore beyond the Prange-Kadanoff approximation. Following the analysis there, we can estimate the diffusion coefficient to be

$$D \sim \text{Im} \Sigma_R(\delta \theta)^4 \sim \frac{g^2 |\omega|^2}{k_F^4 v_F^2} \sim \frac{g^4 |\omega|^2}{(v_F^2 k_F)^3}.$$  

(S140)

This result is accurate up to logarithmic corrections of order $\log \delta \theta$ [10]. The optical conductivity is therefore

$$\sigma^{xx}(\omega) \sim \frac{1}{\omega} \langle \cos \theta | \frac{1}{i \omega - D \partial^2 \theta} | \cos \theta \rangle \sim N v_F^2 \frac{1}{-i \omega - D}.$$  

(S141)

This requires a non-circular Fermi surface since for first harmonics the correction term still vanishes by momentum conservation.

At finite temperature, the quantum-critical scaling is violated by thermal fluctuations. However, we expect the angular superdiffusion picture to still hold, but with the angular step $\delta \theta \sim \Delta(T)/k_F$ where $\Delta(T)$ is the thermal mass. Therefore we have

$$D \sim \text{Im} \Sigma_R(\delta \theta)^4 \sim T^{5/2} \ln^{3/2}(1/T).$$  

(S142)

**III. POTENTIAL DISORDER (DERIVATION OF EQ. (7))**

In this part we investigate the spatially disordered theory with potential ($v$) disorder, and compute its conductivity.
A. Lagrangian

The model we consider is

\[ L = \sum_i \psi_i^\dagger (\partial \tau + \varepsilon_k - \mu) \psi_i + \frac{1}{2} \sum_i \phi_i (-\partial_x^2 + \omega_q^2 + m_b^2) \phi_i + \sum_{ijl} \frac{g_{ijl}}{N} \psi_i^\dagger \psi_j \psi_l + \sum_{ij} \frac{v_{ij}(x)}{\sqrt{N}} \psi_i^\dagger \psi_j. \]  

(S143)

Here \( \varepsilon_k \) and \( \omega_q^2 \) should be understood as differential operators. \( g_{ijl} \) is the random interaction and \( v_{ij} \) is disorder. The boson mass term \( m_b^2 \) might be replaced by a fixed length constraint. We will assume that in the low-energy limit the disorder scattering rate \( \Gamma = 2\pi N v^2 \) (\( N \) is DOS) is the largest scale.

1. Scaling Analysis

Assuming dynamical exponent \( z = 2 \) for the bosons, we have \([\tau] = -2, [x] = [y] = -1\). At the fixed point, we assume the disorder self energy of the fermions and the boson kinetic term are invariant under scaling. We can then determine \([\psi] = 2 \) and \([\phi] = 1 \). Therefore the Yukawa coupling and the fermion-disorder coupling are irrelevant. There is also the boson mass term \( \phi^2 \) which is relevant and the boson self interaction \( \phi^4 \) which is marginal, but we assume that they have been tuned to criticality.

2. \( G-\Sigma \) action

After averaging out \( g_{ijl} \) and \( v_{ij} \), we obtain the \( G-\Sigma \) action

\[ S_N = -\ln \det \left( (\partial \tau + \varepsilon_k - \mu) \delta(x-x') + \Sigma \right) + \frac{1}{2} \ln \det \left( (-\partial_x^2 + \omega_q^2 + m_b^2) \delta(x-x') - \Pi \right) \]

\[ - \text{Tr} \left( \Sigma \cdot G \right) + \frac{1}{2} \text{Tr} \left( \Pi \cdot D \right) + \frac{g^2}{2} \text{Tr} \left( (GD) \cdot G \right) + \frac{v^2}{2} \text{Tr} \left( (G\bar{\delta}) \cdot G \right), \]  

(S144)

where \( \delta \) is a space-time delta function and \( \bar{\delta} \) is a spatial delta function.

The saddle point equations are

\[ G(i\omega_n, k) = \frac{1}{i\omega_n + \mu - \varepsilon_k - \Sigma(i\omega_n, k)}, \]

\[ D(i\Omega_n, q) = \frac{1}{\Omega_n^2 + \omega_q^2 + m_b^2 - \Pi(i\Omega_n, k)}, \]  

(S145)

\[ \Sigma(x) = g^2 G(x) D(x) + v^2 G(x) \bar{\delta}(x), \]

\[ \Pi(x) = -g^2 G(x) G(-x). \]

B. Solving the saddle point

In this disordered model, the Prange-Kadanoff method does not apply. In the presence of disorder, the disorder contribution to electron self energy \( \Sigma_{dis} = -i(\Gamma/2) \text{sgn} \omega_n \) dominates at low energy. As a consequence, the peak in the electron Green’s function is now wider than the peak in the boson Green’s function (as we will see the boson self energy scale linearly with frequency). Therefore the Prange-Kadanoff method does not apply, and it is not legitimate in the scaling sense to neglect momentum dependence in the electron self energy. However, the
momentum dependence only introduces non-dissipative corrections, and for the real part of optical conductivity we are interested in the dissipative part, so to simplify the calculation we can still set fermionic momenta to be on the Fermi surface.

1. Boson self energy

Let us compute the boson self energy first, which in momentum space reads

\[
\Pi(i\Omega_n,\vec{q}) = -g^2T \sum_{\omega_n} \frac{1}{(2\pi)^2} \int \frac{d^2\vec{k}}{i\omega_n - \xi_{\vec{k}} - \Sigma(i\omega_n)} \frac{1}{i\omega_n + \xi_{\vec{k}+\vec{q}} - \Sigma(i\omega_n + i\Omega_n)},
\]

where we have assumed that the electron self energy takes value on the Fermi surface, and \(\xi_{\vec{k}} = \varepsilon_{\vec{k}} - \mu\). Expanding in small \(\vec{q}\) and around a circular Fermi surface, we have

\[
\Pi(i\Omega_n,\vec{q}) = -g^2T \sum_{\omega_n} \frac{1}{(2\pi)^2} \int \frac{d\theta}{2\pi} \int \frac{d\xi_{\vec{k}}}{\omega_n - \xi_{\vec{k}} - \Sigma(i\omega_n)} \frac{1}{i\omega_n + \xi_{\vec{k}} - \Sigma(i\omega_n + i\Omega_n) - v_F q \cos \theta}.
\]

Taking the \(\xi_{\vec{k}}\) integral to be over the real line, we obtain

\[
\Pi(i\Omega_n,\vec{q}) = -\pi N g^2 T \sum_{\omega_n} \frac{\text{sgn} \omega_n (\text{sgn} (\omega_n + \Omega_n) - \text{sgn} \omega_n)}{\sqrt{v_F^2 q^2 - (i\Omega_n - \Sigma(i\omega_n + i\Omega_n) + \Sigma(i\omega_n))^2}} \approx -\frac{\pi g^2 |\Omega_n|}{\sqrt{v_F^2 q^2 + \Gamma^2}} = -\frac{\pi g^2 |\Omega_n|}{\Gamma}.
\]

Here \(N\) is density of states. Here we have assumed that at low frequencies the electron self energy is dominated by disorder scattering \(\Sigma \approx -i\frac{\Gamma}{2} \text{sgn} (\omega_n)\).

The thermal mass of this boson self energy has been calculated in a previous paper [1], which is

\[
\Delta(T)^2 = -\pi \gamma T W_0\left( -\frac{1}{\pi} \ln\left( \frac{2\pi T}{\gamma e^{\gamma e}}\right) \right), \quad \gamma = \frac{N g^2}{\Gamma}.
\]

2. Electron self energy

The electron self energy is given by

\[
\Sigma(i\omega_n,\vec{k}) = g^2 \int \frac{d^2\vec{q}}{(2\pi)^2} T \sum_{\Omega_n} D(i\Omega_n,\vec{q}) G(i\omega_n - i\Omega_n,\vec{k} - \vec{q}) + v^2 \int \frac{d^2\vec{q}}{(2\pi)^2} G(i\omega_n,\vec{q}).
\]

The second term gives rise to the disorder contribution

\[
\Sigma_{\text{dis}}(i\omega_n,\vec{k}) = -\frac{\Gamma}{2} \text{sgn} (\omega_n), \quad \Gamma = 2\pi v^2 N.
\]

The first term can be split into thermal part and quantum part

\[
\Sigma_T(i\omega_n,\vec{k}) = g^2 T \int \frac{d^2\vec{q}}{(2\pi)^2} D(0,\vec{q}) G(i\omega_n,\vec{k} - \vec{q}).
\]
Taking $\tilde{k}$ to be on the Fermi surface, we can expand $\xi_{\tilde{k}-\tilde{q}} = v_F q \cos \theta$, we obtain

$$\Sigma_T(i\omega_n, \tilde{k}) = -\frac{g^2 T}{2\pi} \frac{\text{sgn} (\omega_n)}{A(\omega_n) - \xi_{\tilde{k}-\tilde{q}}} \text{sgn} \left( \frac{v_F \Delta(T)}{A(\omega_n)} \right),$$

where $A(\omega_n) = \omega_n + i\Sigma(\omega_n)$ and $\Delta(T)$ is the thermal mass. Taking the large $\Gamma$ limit, we obtain

$$\Sigma_T(i\omega_n) = -\frac{ig^2 T\text{sgn} \omega_n}{2\pi|A(\omega_n)|} \ln \left| \frac{2A(\omega_n)}{v_F \Delta(T)} \right| = -\frac{ig^2 T\text{sgn} \omega_n}{\pi \Gamma} \ln \left| \frac{\Gamma}{v_F \Delta(T)} \right|$$

(S154)

The quantum part is

$$\Sigma_Q(i\omega_n) = g^2 \int \frac{d^2 q}{(2\pi)^2} T \sum_{\Omega_n \neq 0} \frac{1}{\gamma|\Omega_n| + q^2} \frac{1}{\sqrt{v_F^2 q^2 + A(\omega_n) - \xi_{\tilde{k}-\tilde{q}}}}$$

(S155)

Replace $\xi_{\tilde{k}-\tilde{q}} = v_F q \cos \theta_q$ and perform the angular integral, we obtain

$$\Sigma_Q(i\omega_n) = g^2 T \sum_{\Omega_n \neq 0} \int \frac{qdq}{(2\pi)^2} \frac{1}{\gamma|\Omega_n| + q^2} \frac{-i\text{sgn} (\omega_n - \Omega_n)}{\sqrt{v_F^2 q^2 + A(\omega_n) - \Omega_n}^2}$$

(S156)

Using low frequency and low energy approximations, we perform the frequency sum first get

$$\Sigma_Q(i\omega_n) = -\frac{ig^2 \text{sgn} \omega_n}{2\pi^2 \gamma} \int_0^\infty \frac{qdq}{\sqrt{v_F^2 q^2 + \Gamma^2 / 4}} \left[ \psi \left( \frac{|\omega_n|}{2\pi T} + \frac{q^2}{2\pi T \gamma} \right) - \psi \left( 1 + \frac{q^2}{2\pi T \gamma} \right) \right].$$

(S157)

At zero temperature, the above reduces to

$$\Sigma_Q(i\omega_n) = -\frac{ig^2 \text{sgn} \omega}{2\pi^2 \gamma} \int_0^\infty \frac{qdq}{\sqrt{v_F^2 q^2 + \Gamma^2 / 4}} \ln \left( 1 + \frac{|\omega|\gamma}{q^2} \right)$$

$$= -\frac{g^2 \text{sgn} \omega}{2\pi^2 \gamma} \left( \frac{-\Gamma}{v_F^2} \right) \left( 2 \sqrt{1 - \frac{4|\omega|\gamma v_F^2}{\Gamma^2}} \sinh^{-1} \left( \sqrt{\frac{\Gamma^2}{4|\omega|\gamma v_F^2}} - 1 \right) + \ln \left( \frac{|\omega|\gamma v_F^2}{\Gamma^2} \right) \right)$$

(S158)

This logarithmic behavior signatures the break down of Prange-Kadanoff reduction.

**Alternative calculation of $\Sigma_Q$:** In (S156), we perform the momentum integral over $q$ first:

$$\Sigma_Q(i\omega_n) = -\frac{ig^2 T}{2\pi} \sum_{\Omega_n \neq 0} \frac{\text{sgn} (\omega_n - \Omega_n)}{\sqrt{A^2 (\omega_n - \Omega_n) - v_F^2 \gamma |\Omega_n|}} \cosh^{-1} \left( \frac{|\Omega_n - \omega_n|}{v_F \sqrt{\gamma |\Omega_n|}} \right).$$

(S159)

To evaluate the sum to leading order in $\Gamma$, we replace $A$ by $\Gamma/2$, and we obtain

$$\Sigma_Q(i\omega_n) = -\frac{ig^2 T}{\pi} \frac{\text{sgn} \omega_n}{2\pi} \sum_{0<\Omega_n<|\omega_n|} \frac{2}{|\Omega_n|} \ln \left( \frac{\Gamma}{v_F \sqrt{\gamma |\Omega_n|}} \right)$$

$$= -i\text{sgn} \omega_n \frac{2g^2 T}{\pi \Gamma} \left[ \left( \frac{|\omega_n|}{2\pi T} - \frac{1}{2} \right) \ln \frac{\Gamma}{v_F \sqrt{2\pi T \gamma}} - \frac{1}{2} \ln \Gamma_F \left( \frac{|\omega_n|}{2\pi T} + \frac{1}{2} \right) \right],$$

(S160)

where the $\Gamma_F$ denotes the gamma function. Taking the $T \to 0$ limit, we recover (S158).

Combining (S154) and (S160), we obtain

$$\Sigma_Q(i\omega_n) + \Sigma_T(i\omega_n) = -i\text{sgn} \omega_n \frac{2g^2 T}{\pi \Gamma} \left[ \left( \frac{|\omega_n|}{2\pi T} \right) \ln \frac{\Gamma}{v_F \sqrt{2\pi T \gamma}} - \frac{1}{2} \ln \frac{\Delta(T)}{\sqrt{2\pi T \gamma}} - \frac{1}{2} \ln \Gamma_F \left( \frac{|\omega_n|}{2\pi T} + \frac{1}{2} \right) \right].$$

(S161)

Including momentum dependence will shift $\Gamma$ to $\Gamma + i\xi_k \text{sgn} \omega_n$, whose primary effect is to introduce a real part to the self energy, which we will ignore.
C. Conductivity in the disordered model

Now we calculate the conductivity in the disordered model. The conductivity is also given by (S47) and (S112).

We will have to invert the operator $W^{-1}_\Sigma - W_G$. Since the Prange-Kadanoff method doesn’t apply, we will treat disorder scattering exactly and treat fermion-boson scattering perturbatively in $g$.

Let’s set up the formalism. Similar to (S115), we integrate out the bosons to write

$$\left( \frac{1}{W^{-1}_\Sigma - W_G} \right)_{FF} \equiv \frac{1}{W^{-1}_{\Sigma,0} + W^{-1}_{\Sigma,1}} \frac{1}{W_{\Sigma,FF} - W_{G,FF} - W_{G,FB}W_{\Sigma,BB}W_{G,BF}W_{AL}^{-1}} .$$

(S162)

Here $W_{\Sigma,0}$ is a diagonal operator in $k$-space whose expression is

$$W_{\Sigma,0}(k, p) = G_0(k + p/2)G_0(k - p/2) ,$$

(S163)

where $G_0$ is the Green’s function which only includes disorder:

$$G_0(i\omega, \vec{k}) = \frac{1}{i\omega - \xi_k - \Sigma_{dis}(i\omega)} , \quad \Sigma_{dis}(i\omega) = -\frac{i\Gamma}{2} \text{sgn} \omega .$$

(S164)

Here $p$ denotes CoM 3-momentum and $k$ denotes relative 3-momentum.

$W^{-1}_{\Sigma,1}$ is obtained from $W^{-1}_{\Sigma,0}$ by doing first-order expansion in $g^2$:

$$W^{-1}_{\Sigma,1}(k, p) = - (\Sigma_T(k + p/2) + \Sigma_Q(k + p/2))G^{-1}_0(k - p/2)$$

$$- (\Sigma_T(k - p/2) + \Sigma_Q(k - p/2))G^{-1}_0(k + p/2) .$$

(S165)

$W_{\text{dis}}$ describes disorder scattering:

$$W_{\text{dis}}[F](i\omega, \vec{k}) = v^2 \int \frac{d^2q}{(2\pi)^2} F(i\omega, \vec{q}) ,$$

(S166)

and in $l$-th angular harmonics, it takes the form

$$W^{(l)}_{\text{dis}}[F](i\omega, \xi_k) = \Gamma \delta_{l,0} \int \frac{d\xi}{2\pi} F(i\omega, \xi_q) .$$

(S167)

Here and after the superscript $(l)$ denotes Fourier transform in the angular harmonics. The simplicity of $W_{\text{dis}}$ allows us to treat it exactly.

$W_{\text{MT}}$ describes scattering in Maki-Thomson diagrams:

$$W_{\text{MT}}[F](i\omega, \vec{k}) = g^2 \int \frac{d^2\vec{q}d\omega'}{(2\pi)^3} D(\omega - \omega', \vec{k} - \vec{q})F(i\omega', \vec{q})$$

(S168)

$W_{\text{AL}}$ describes scattering in Azlamasov-Larkin diagrams:

$$W_{\text{AL}}[F](k_1) = -\frac{g^4}{2} \int \frac{d^3qd^3k_2}{(2\pi)^6} (G(k_1 - q) + G(k_1 + q))(G(k_2 - q) + G(k_2 + q))$$

$$\times D(q + p/2)D(q - p/2)F(k_2)$$

$$= -g^4 \int \frac{d^3qd^3k_2}{(2\pi)^6} (G(k_1 - q) + G(k_2 - q) + G(k_2 + q))D(q + p/2)D(q - p/2)F(k_2) .$$

(S169)
1. Zeroth order

The zeroth order polarization is

$$
\Pi^{xx}_{0}(i\Omega) = N(\Gamma^x)T \frac{1}{W^{-1}_{\Sigma,0} - W_{\text{dis}}} \Gamma^x, \quad (S170)
$$

where the bare vertex function is approximated by

$$
\Gamma^x(k) = v_F \cos \theta_k. \quad (S171)
$$

Since $\Gamma^x(k)$ only contains first harmonics, $W_{\text{dis}}$ vanishes, and we obtain a Drude-like result

$$
\Pi^{xx}_{0}(i\Omega)/N = N \frac{v_F^2}{\Omega} \Omega + \Gamma, \quad (S172)
$$

$$
\sigma_{xx,0}(i\omega)/N = N \frac{v_F^2}{2} - i\omega + \Gamma. \quad (S173)
$$

2. First order: Self-energy and Maki-Thomson diagrams

To first order, the polarization is

$$
\Pi^{xx}_{1} = -N(\Gamma^x)^T \frac{1}{W^{-1}_{\Sigma,0} - W_{\text{dis}}} \left(W^{-1}_{\Sigma,1} - W_{MT} - W_{AL}\right) \frac{1}{W^{-1}_{\Sigma,1} - W_{\text{dis}}} \Gamma^x. \quad (S174)
$$

Using the fact that $\Gamma^x$ only contains first harmonics, we have

$$
\Pi^{xx}_{1} = N(\Gamma^x)^T W_{\Sigma,0} \left(W^{(1)}_{MT} + W^{(1)}_{AL} - W^{-1}_{\Sigma,1}\right) W_{\Sigma,0} \Gamma^x
$$

$$
= N(\Gamma^x)^T W_{\Sigma,0} \left(\tilde{\Gamma}^x_{MT} + \tilde{\Gamma}^x_{AL} - \tilde{\Gamma}^x_\Sigma\right). \quad (S175)
$$

In the transport limit $p = (\Omega_n, 0)$, the kernel $W_{\Sigma,1}$ is rotational invariant so we have dropped the superscript.

In (S175) we have defined three types of renormalized vertices $\tilde{\Gamma}^x_\Sigma$, $\tilde{\Gamma}^x_{MT}$ and $\tilde{\Gamma}^x_{AL}$.

a. $\tilde{\Gamma}^x_\Sigma$ The first type $\tilde{\Gamma}^x_\Sigma$ describes the contribution due to self-energies:

$$
\tilde{\Gamma}^x_\Sigma(i\omega_n, \vec{k}) = W^{-1}_{\Sigma,1} W_{\Sigma,0} \Gamma^x = -v_F \cos \theta_k \left(\Sigma_+ G_+ + \Sigma_- G_-\right), \quad (S176)
$$

where we have used a shorthand notation

$$
\Sigma_\pm = \Sigma_Q(i\omega_n \pm i\Omega_n/2) + \Sigma_T(i\omega_n \pm i\Omega_n/2), \quad G_\pm = G_0(i\omega_n \pm i\Omega_n/2, \vec{k}). \quad (S177)
$$

b. $\tilde{\Gamma}^x_{MT}$ Next we calculate $\tilde{\Gamma}^x_{MT}$:

$$
\tilde{\Gamma}^x_{MT}(i\omega_n, \vec{k}) = v_F \cos \theta_k g^2 T \sum_{\nu_n} \int \frac{q dq}{2\pi} \frac{d\theta_q}{2\pi} D(\nu_n, q) e^{-i\theta_{kk'}} \frac{1}{iA'_+ - \xi_k + v_F q \cos \theta_q} \frac{1}{iA'_- - \xi_k + v_F q \cos \theta_q}, \quad (S178)
$$

where

$$
A(\omega_n) = \omega_n + \frac{\Gamma}{2} \text{sgn} \omega_n, \quad A'_\pm = A(\omega_n - \nu_n \pm \Omega_n/2), \quad (S179)
$$
and $\theta_q$ is the angle between $\vec{k}$ and $\vec{q}$. The $\theta_{kk'}$ above is the angle between $\vec{k}$ and $\vec{k}' = \vec{k} - \vec{q}$, and because $\vec{q}$ is small compared to $k_F$, we approximate

$$e^{-i\theta_{kk'}} = 1 - \frac{q^2}{2k_F^2} \sin^2 \theta_q.$$  \hfill (S180)

The boson propagator is given by

$$D(\nu_n, q) = \frac{1}{q^2 + M^2(T, \nu_n)}, \quad M^2(T, \nu_n) = \begin{cases} \gamma |\nu_n|, & \nu_n \neq 0; \\ \Delta(T)^2, & \nu_n = 0. \end{cases}$$  \hfill (S181)

We can now perform the angular integrals in (S178), which yields

$$\tilde{\Gamma}_{MT}^\pi = v_F \cos \theta_k g^2 T \sum_{\nu_n} \int_0^\infty \frac{q dq}{2\pi} D(\nu_n, q) \frac{1}{A'_+ - A'_-} \left\{ \left( \frac{\text{sgn} A'_+}{\sqrt{A'_+^2 + v_F^2 q^2}} - \frac{\text{sgn} A'_-}{\sqrt{A'_-^2 + v_F^2 q^2}} \right) \right\}$$

$$+ \frac{1}{v_F^2 k_F^2} \left[ \text{sgn} A'_+ \left( \sqrt{A'_+^2 + v_F^2 q^2} - |A'_+| \right) - \text{sgn} A'_- \left( \sqrt{A'_-^2 + v_F^2 q^2} - |A'_-| \right) \right],$$

where we have assumed $\vec{k}$ to be lying on the FS and set $\xi_k = 0$.

The $1/(A'_+ - A'_-)$ factor is a piecewise constant function ($\Omega_n > 0$):

$$\frac{1}{A'_+ - A'_-} = \begin{cases} \frac{1}{\Omega_n}, & |\nu_n - \omega_n| < \Omega_n/2; \\ \frac{1}{\Omega_n + \Gamma}, & |\nu_n - \omega_n| > \Omega_n/2. \end{cases}$$  \hfill (S183)

Plugging the above into the first line of (S182), we can separate out a part which yields the self energy and a correction term:

$$\tilde{\Gamma}_{MT,a}^\pi = v_F \cos \theta_k g^2 T \sum_{\nu_n} \int_0^\infty \frac{q dq}{2\pi} D(q, \nu_n) \frac{1}{\Omega_n} \left( \frac{\text{sgn} A'_+}{\sqrt{A'_+^2 + v_F^2 q^2}} - \frac{\text{sgn} A'_-}{\sqrt{A'_-^2 + v_F^2 q^2}} \right)$$

$$= \frac{iv_F \cos \theta_k}{\Omega_n} (\Sigma_+ - \Sigma_-),$$

$$\tilde{\Gamma}_{MT,b}^\pi = v_F \cos \theta_k g^2 T \sum_{|\nu_n - \omega_n| < \Omega_n/2} \int_0^\infty \frac{q dq}{2\pi} \frac{-\Gamma}{\Omega_n(\Omega_n + \Gamma)} D(q, \nu_n) \left( \frac{\text{sgn} A'_+}{\sqrt{A'_+^2 + v_F^2 q^2}} - \frac{\text{sgn} A'_-}{\sqrt{A'_-^2 + v_F^2 q^2}} \right)$$

$$= v_F \cos \theta_k \left( \frac{-\Gamma}{\Omega_n(\Omega_n + \Gamma)} \right) \frac{g^2 T}{2\pi} \sum_{|\nu_n - \omega_n| < \Omega_n/2} \left( \cosh^{-1} \frac{A'_+}{\sqrt{A'_+^2 - v_F^2 M^2(T, \nu_n)}} \right) + (+ \rightarrow -)$$

$$= iv_F \cos \theta_k \left( \frac{-\Gamma}{\Omega_n(\Omega_n + \Gamma)} \right) (\Sigma_+ - \Sigma_-).$$

$$\tilde{\Gamma}_{MT,a}^\pi + \tilde{\Gamma}_{MT,b}^\pi = \frac{iv_F \cos \theta_k}{\Omega_n + \Gamma} (\Sigma_+ - \Sigma_-)$$  \hfill (S186)

To obtain the above results, we evaluated the $q$ integral first and next the $\nu_n$ sum with large $\Gamma$ approximation, and found the result agrees with (S161).

Finally we compute the second line of (S182), we again split it into two parts:
\[ \tilde{\Gamma}_x^{\text{MT},c} = \frac{v_F \cos \theta_k g^2 T}{v_F^2 k_F^2} \sum_{\nu_n} \int_0^\infty dq \frac{1}{2\pi} D(q, \nu_n) \left[ \text{sgn} A' \left( \sqrt{A'^2 + v_F^2 q^2} - |A'| \right) \right. \\
\left. - \text{sgn} A' \left( \sqrt{A'^2 + v_F^2 q^2} - |A'| \right) \right] \]  

(S187)

\[ \tilde{\Gamma}_x^{\text{MT},d} = \frac{v_F \cos \theta_k g^2 T}{v_F^2 k_F^2} \sum_{|\nu_n - \omega_n| < \Omega_n/2} \int_0^\infty dq \frac{-\Gamma}{2\pi} \frac{1}{\Omega_n (\Omega_n + \Gamma)} D(q, \nu_n) \left[ \text{sgn} A' \left( \sqrt{A'^2 + v_F^2 q^2} - |A'| \right) \right. \\
\left. - \text{sgn} A' \left( \sqrt{A'^2 + v_F^2 q^2} - |A'| \right) \right] \]  

(S188)

The \( q \)-integral is UV divergent and we cut it off by a Pauli-Vilas regulator \( \Lambda \sim k_F \)

\[ \int_0^\infty dq \frac{1}{2\pi} \left( \frac{1}{q^2 + M^2} - \frac{1}{q^2 + A^2} \right) \left( \sqrt{A^2 + v_F^2 q^2} - |A| \right) \]

\[ = \frac{v_F^2 A}{4} + \frac{1}{2\pi} \left[ \sqrt{|A|^2 - M^2 v_F^2} \cosh^{-1} \left( \frac{|A|}{M v_F} \right) - |A| \ln \left( \frac{A e}{M} \right) \right] \]

\[ \approx \frac{v_F^2 A}{4} + \frac{1}{2\pi} |A| \ln \left( \frac{2|M|}{e M v_F} \right) - \frac{1}{4\pi |A|} M^2 v_F^2 \ln \left( \frac{2\sqrt{e} |A|}{M v_F} \right) . \]  

(S189)

Computing the frequency sum, we obtain

\[ \tilde{\Gamma}_x^{\text{MT},c} + \tilde{\Gamma}_x^{\text{MT},d} = \frac{v_F \cos \theta_k g^2}{\pi v_F^2 k_F^2} \left[ \frac{\Omega_n}{\Omega_n + \Gamma} \right. \left[ \frac{v_F A}{4} + \frac{\Gamma}{4\pi} \ln \left( \frac{e e v_F}{\Gamma} \right) \right] \]  

(S190)

c. \text{MT+ self energy} It’s easy to check that

\[ (\Gamma_x^{S})^T W_{S,0} \left( \tilde{\Gamma}_x^{\text{MT},a+b} - \tilde{\Gamma}_S^x \right) = 0, \]  

(S191)

which can be seen after computing the \( \xi_k \) integral.

The rest from the MT diagrams contribute as

\[ (\Gamma_x^{S})^T W_{S,0} \tilde{\Gamma}_x^{\text{MT},c+d} = \frac{N v_F^2}{2} \left( \frac{\Omega_n}{\Omega_n + \Gamma} \right)^2 \frac{2g^2}{(v_F k_F)^2} \left[ \frac{v_F A}{4} + \frac{\Gamma}{4\pi} \ln \left( \frac{e e v_F}{\Gamma} \right) \right] , \]  

(S192)

and its contribution to conductivity is

\[ \sigma_{xx,1,MT}(i\omega)/N = \frac{N v_F^2}{2} \left( -i\omega - i\omega + \Gamma \right)^2 \frac{2g^2}{(v_F k_F)^2} \left[ \frac{v_F A}{4} + \frac{\Gamma}{4\pi} \ln \left( \frac{e e v_F}{\Gamma} \right) \right] . \]  

(S193)

This result can be interpreted as an additional scattering rate in the Drude formula

\[ \sigma_{xx} = N \frac{N v_F^2}{2} \left( -i\omega + \Gamma + \frac{1}{\tau_{MT}(\omega)} \right) , \]  

(S194)

where

\[ \frac{1}{\tau_{MT}(\omega)} = \frac{e e v_F}{2} \left( v_F k_F^2 \right)^2 \left[ \frac{v_F A}{4} + \frac{\Gamma}{4\pi} \ln \left( \frac{e e v_F}{\Gamma} \right) \right] . \]  

(S195)

There is no linear in \( T \) resistivity. Higher order corrections in \( 1/T \) will start at order \( |\omega_n|^2 \) or \( T^2 \), which is Fermi-liquid like.
Now we show that the contributions from AL diagrams are also subdominant. The expression to evaluate is

\[ \Pi_{i,\text{AL}}^\tau(i\Omega)/N = -\frac{g^4}{2} \int \frac{d^3q}{(2\pi)^3} D(q + \Omega/2)D(q - \Omega/2)X(q,\Omega)^2 \]  

\[ (S196) \]

where

\[ X(q,\Omega) = \int \frac{d^3k}{(2\pi)^3} v_k \cos \theta_k G_0(k + \Omega/2)G_0(k - \Omega/2) |G_0(k + q) + G_0(k - q)|. \]

\[ (S197) \]

Here \( q = (\nu,\tilde{q}) \) and \( k = (\omega,\tilde{k}) \). The notation \( q + \Omega/2 \) means adding \( \Omega/2 \) to the Matsubara component. For conductivity computation we assume \( \Omega > 0 \).

We first evaluate \( X(q,\Omega) \), plugging in the expression for \( G_0 \) we have

\[ X(q,\Omega) = 2\pi N \int \frac{d\omega}{2\pi} \frac{d\xi_k}{2\pi} \frac{d\theta_k}{2\pi} v_k \cos \theta_k \frac{1}{iA(\omega + \Omega/2) - \xi_k} \frac{1}{iA(\omega - \Omega/2) - \xi_k} \times \left[ \frac{1}{iA(\omega + \nu) - \xi_k} - v_k q \cos \theta_k - \kappa q^2 + (\nu \rightarrow -\nu, \theta_q \rightarrow \pi + \theta_q) \right]. \]

\[ (S198) \]

Here \( \kappa \) measures the angle between \( \tilde{k} \) and \( \tilde{q} \), and \( A(\omega) = \omega + (\Gamma/2)\text{sgn} \omega \). We have included the Fermi surface curvature \( \kappa = 1/(2m) \). Noticing that \( A(\omega) \) is an odd function of \( \omega \), under standard approximations \( v_k = v_F \) and \( \kappa \rightarrow 0 \), the integrand is odd under \( (\omega,\xi_k) \rightarrow -(\omega,\xi_k) \) and we get \( X(q,\omega) = 0 \). Therefore, we need to keep terms that break the \( \xi_k \rightarrow -\xi_k \) symmetry. There are two sources: Fermi surface curvature and dependence of \( v_k \) on \( \xi_k \).

To set up the expansion, we write \( v_k = \sqrt{1 + 2\xi_k/(v_F k_F^2)} \) and \( \kappa = v_F/(2k_F) \), and expand Eq. (S198) to first order in \( 1/k_F \), the first nonzero term is

\[ X(q,\Omega) = \frac{N}{2k_F} \int \frac{d\omega}{2\pi} \frac{d\xi_k}{2\pi} \frac{d\theta_k}{2\pi} \cos \theta_k \frac{1}{iA(\omega + \Omega/2) - \xi_k} \frac{1}{iA(\omega - \Omega/2) - \xi_k} \times \left[ \frac{1}{q^2 v_F^2 - 2\xi_k^2 + 2i\xi_k A(\nu + \omega)} \frac{1}{(iA(\nu + \omega) - \xi_k - q^2 v_F \cos \theta_k)^2 + (\nu \rightarrow -\nu, \theta_q \rightarrow \theta_q + \pi)} \right]. \]

\[ (S199) \]

The integral over \( \xi_k \) can be taken to be one the real line, since the finite band width only corrects the result by \( O(1/k_F^2) \). As a result the \( \xi_k \) integral can be evaluated by residue method. The angular integral is performed using the formula

\[ \int \frac{d\theta_k}{2\pi} \frac{\cos \theta_k}{(ia - b \cos \theta_k)^2} = \frac{ib \cos \theta_q \text{sgn} a}{(a^2 + b^2)^{3/2}}, \quad a \in \mathbb{R}, b > 0. \]

The final result for \( X \) contains two analytic branches depending on whether \(|\nu| < \Omega/2 \) or \(|\nu| > \Omega/2 \). The branch with \(|\nu| < \Omega/2 \) will connect to \( D_R D_A \) when Eq. (S196) is continued to real frequency, while the branch with \(|\nu| > \Omega/2 \) will connect to \( D_R D_R \) or \( D_A D_A \). It is shown in [13] that only the first branch contributes at the low frequency limit (\(|\Omega| < T \)). In this limit, we are allowed to expand in small \( |\nu| \) and small \( |\Omega| \) (both are of the same order), yielding

\[ X(|\nu| < \Omega/2, q, \Omega) = \frac{N}{k_F} \frac{2i q^4 v_F^2 \nu \cos \theta_q}{(\Omega + \Gamma)(q^2 v_F^2 + \Gamma^2)^{3/2}} + O(\nu^2, \Omega^2). \]

\[ (S200) \]

The numerator of the result has the same scaling as [13], but the denominator is different because in our large \( N \) limit we have dropped vertex correction of Yukawa interaction due to disorders. In obtaining Eq. (S200), the frequency
summation is over a piecewise constant function, and therefore Eq. (S200) should be valid at finite temperature as well.

Finally, we evaluate the integral (S196) using (S200) with \( \nu \in [-\Omega/2, \Omega/2] \). To lowest order in \( g \) we can set \( \gamma = 0 \) in the boson propagators, and we obtain

\[
\Pi_{1,AL}^{xx}(i\Omega)/N = \frac{N v_F^2}{2} \frac{g^4(\Omega^3 + 8\pi T^2 \Omega)}{96\pi^2 T^2 (\Gamma + \Omega)^2 k_F v_F}.
\]

(S201)

Here we have used \( N = k_F/(2\pi v_F) \).

Analytically continuing to real frequency \( \Omega \rightarrow -i\omega + 0^+ \), we obtain a effective scattering rate

\[
\frac{1}{\tau_{AL}(\omega)} = \frac{g^4(\omega^2 - 8\pi^2 T^2)}{96\pi^2 T^2 k_F v_F}.
\]

(S202)

Note that this is a correction to the elastic scattering rate \( \Gamma \), as in (S194). Therefore, the contribution of AL diagrams is less singular than MT + self energy diagrams.

**SIII. INTERACTION DISORDER**

This section will consider the most general model, including the translationally invariant Yukawa coupling described in Section SI, the potential disorder described in Section SII, and an additional spatial randomness in the Yukawa coupling.

**A. Action**

We recall the action for all terms, and describe the associated large \( N \) saddle point. We start with the Lagrangian for the critical Fermi surface without disorder

\[
\mathcal{L} = \sum_i \bar{\psi}_i (\partial_\tau + \varepsilon_k - \mu) \psi_i + \frac{1}{2} \sum_i \phi_i (-\partial_\tau^2 + \omega_q^2 + m_b^2) \phi_i + \sum_{ijl} g_{ijl} \bar{\psi}_i \psi_j \phi_l,
\]

(S203)

where \( g_{ijl} \) is spatially independent,

\[
|g_{ijl}|^2 = g^2.
\]

(S204)

To this we add a random potential coupling to the fermions

\[
S_v = \int d\tau \frac{1}{\sqrt{N}} \sum_{r} \sum_{ij} v_{ij}(r) \psi_i^\dagger(r,\tau) \psi_j(r,\tau),
\]

(S205)

where \( r \) labels lattice sites. The potential \( v_{ij}(r) \) is random both in position and flavor space

\[
\overline{v_{ij}(r)v_{lm}(r')} = v^2 \delta(r-r')\delta_{il}\delta_{jm}.
\]

(S206)

We can also add a similar random potential coupling to the bosons

\[
S_w = \int d\tau \frac{1}{2\sqrt{N}} \sum_{r} \sum_{ij} w_{ij}(r) \phi_i(r,\tau) \phi_j(r,\tau),
\]

(S207)
with
\[
\overline{w_{ij}(r) w_{lm}(r')} = \frac{w^2}{2} \delta(r - r') (\delta_{il} \delta_{jm} + \delta_{im} \delta_{jl}) .
\] (S208)

The large $N$ saddle point equations for $S + S_v + S_w$ are now
\[
\Sigma(\tau, r) = g^2 D(\tau, r) G(\tau, r) + v^2 G(\tau, r = 0) \delta^2(r),
\]
\[
\Pi(\tau, r) = -G(-\tau, -r) G(\tau, r) + w^2 D(\tau, r = 0) \delta^2(r),
\]
\[
G(\omega, k) = \frac{1}{i \omega - \varepsilon_k + \mu - \Sigma(i \omega, k)},
\]
\[
D(i \Omega, q) = \frac{1}{\Omega^2 + \omega_q^2 + m_b^2 - \Pi(i \Omega, q)} .
\] (S209)

We choose the conventional dispersions $\varepsilon_k = |k|^2/(2m)$ and $\omega_q^2 = |q|^2 = q^2$, and tune the system to the quantum-critical point with $m_b^2 - \Pi(0,0) = 0$ at $T = 0$. The $v^2$ term leads to an impurity scattering lifetime $\sim i v^2 \text{sgn}(\omega)$ in the fermion self energy. The $w^2$ term leads to an impurity scattering term $\sim w^2 \ln(|\omega|)$ in the boson self energy. While this term is strongly relevant, we argue in the main text that we can neglect it following a rotation of the boson basis, after which it only ends up generating the $g'$ term.

We then add disorder to the interaction term:
\[
S_{g'} = \int d\tau \frac{1}{N} \sum_r \sum_{ij=1}^N g'_{ij}(r) \psi_i^\dagger(r, \tau) \psi_j(r, \tau) \phi_i(r, \tau),
\] (S210)
where $g'_{ij}(r)$ is spatially random,
\[
g'_{ij}(r) g'_{abc}(r') = g'^2 \delta(r - r') \delta_{ia} \delta_{jb} \delta_{ic} .
\] (S211)

For $S + S_v + S_w + S_{g'}$, the most general saddle point equations are therefore
\[
\Sigma(\tau, r) = g'^2 D(\tau, r) G(\tau, r) + v^2 G(\tau, r = 0) \delta^2(r) + g'^2 G(\tau, r = 0) D(\tau, r = 0) \delta^2(r),
\]
\[
\Pi(\tau, r) = -g'^2 G(-\tau, -r) G(\tau, r) + w^2 D(\tau, r = 0) \delta^2(r) + g'^2 G(-\tau, r = 0) G(\tau, r = 0) \delta^2(r),
\]
\[
G(\omega, k) = \frac{1}{i \omega - \varepsilon_k + \mu - \Sigma(i \omega, k)},
\]
\[
D(i \Omega, q) = \frac{1}{\Omega^2 + \omega_q^2 + m_b^2 - \Pi(i \Omega, q)} .
\] (S212)

**B. Self energies**

We take $w = 0$ and focus on the critical point. We then compute the self-consistent solutions for the fermion and boson Green’s functions. We first consider the case of nonzero $v$, at low frequencies and $T = 0$. For the internal lines in the self-energy diagrams, because the interaction contribution to the fermion self energy and the bare $i \omega$ term are both much smaller than the impurity scattering rate at low frequencies, we can approximate $G(i \omega, k) \simeq G_0(i \omega, k) = 1/(i \Gamma \text{sgn}(\omega)/2 - v_F k)$, where $v_F$ is the Fermi velocity, $k = |k| - k_F = |k| - \sqrt{2m \mu}$, and $\Gamma = v^2 k_F/v_F$. Then, we obtain, for the boson self energy at criticality:
\[
\Pi(i \Omega, q) - \Pi(0,0) = \Pi_g(i \Omega, q) + \Pi_{g'}(i \Omega)
\]
\[
\Pi_g(i \Omega, q) = \frac{1}{i v_F k} \int_{-\infty}^{\infty} \frac{d\omega}{2 \pi} \int_{-\pi}^{\pi} \frac{d\theta}{2 \pi} \int_{-\infty}^{\infty} \frac{dk}{2 \pi} \frac{1}{i \frac{1}{2} \text{sgn}(\omega) - v_F k} \frac{1}{i \frac{1}{2} \text{sgn}(\omega + \Omega) - v_F (k + q \cos \theta)}
\]
\[ \Pi_{g'}(i\Omega) = -g'^2k^2_F \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{i\Gamma} \left( \frac{1}{i\frac{\Gamma}{2} \text{sgn}(\omega + \Omega) - v_F k} - \frac{1}{i\frac{\Gamma}{2} \text{sgn}(\omega) - v_F k'} \right) \]

\[ \Pi_{g'}(i\Omega) = -g'^2k^2_F [\Omega] \frac{\pi}{8\pi v_F} = \frac{\pi}{2} N g^2 [\Omega], \]

(S213)

Here, we have assumed that the Fermi surface is circular, \( k_F \) is large so that \( k, q \ll k_F \) and the Fermi energy is essentially infinite. The limit of large Fermi energy is therefore translated into a limit of large \( k_F \), while keeping \( N \) fixed. This is completely equivalent to taking the limit of large Fermi energy in other ways, such as by fixing \( k_F \) and making \( N \) small, which would be appropriate for a lattice. In such cases, one would have to adjust the values of \( v \) and \( g' \) so that \( \Gamma = 2\pi N v^2 \) and \( Ng^2 \) do not become arbitrarily large or arbitrarily small, which is not a problem.

At \( T \neq 0 \), we would expect a boson thermal mass \( m_b^2(T) = m_b^2 - \Pi(0,0) \sim T/\ln(1/T) \) due to the marginal scaling dimension of the boson self interaction, since the dynamical critical exponent of the boson is \( z = 2 \) and its spatial dimensionality is \( d = 2 \) [1, 14].

We now consider the fermion self energy due to interactions \( \Sigma(i\omega, \mathbf{k}) \), which we also split into clean and momentum-independent disordered contributions \( \Sigma(i\omega, \mathbf{k}) = \Sigma_g(i\omega, \mathbf{k}) + \Sigma_{g'}(i\omega) \). The disordered contribution \( \Sigma_{g'}(i\omega) \) is straightforward to compute:

\[ \Sigma_{g'}(i\omega) = g'^2 k_F \int_{-2\pi}^{2\pi} \frac{dk}{2\pi} \int_0^\infty \frac{qdq}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \frac{1}{\frac{\Gamma}{2} \text{sgn}(\omega + \Omega) - v_F k} \frac{1}{\text{sgn}(\omega) - v_F k'} \]

\[ \approx -ig'^2 k_F \omega \ln \left( \frac{e\Lambda_d^2}{c_d|\omega|} \right) = -i\frac{Ng^2}{4\pi} \frac{\omega}{c_d|\omega|}, \]

(S214)

where \( \Lambda_d \sim \Gamma/v_F \) is a UV cutoff on \( q \). This part of the self energy corresponds to current and momentum relaxing scattering induced by the spatially random interactions, and has a marginal Fermi liquid form. These expressions also imply a frequency cutoff of \( \Gamma^2/(v_F^2 c_d) \) on the low energy theory.

The computation of \( \Sigma_g(i\omega, \mathbf{k}) \) is a little more involved, and unlike \( \Sigma_{g'} \), it has some momentum dependence. It is given by

\[ \Sigma_g(i\omega, \mathbf{k}) = g^2 \int \frac{d^2k'}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} G_0(i\omega', \mathbf{k}') D(i\omega - i\omega', \mathbf{k} - \mathbf{k}'). \]

(S215)

For the computation of the scaling of the real part of the conductivity, the momentum dependence of \( \Sigma_g(i\omega, \mathbf{k}) \) does not matter, and one can use its value for \( \mathbf{k} \) exactly on the Fermi surface, which is given by

\[ \Sigma_g(i\omega, \mathbf{k} = k_F \mathbf{k}) \approx -i g^2 \omega \frac{\Gamma}{2\pi^2} \ln \left( \frac{e\Gamma^2}{v_F^2 c_d|\omega|} \right). \]

Note that this also has a marginal Fermi liquid form, but doesn’t correspond to current and momentum relaxing scattering, and therefore will not contribute to transport, as we will show later. For completeness, we provide the derivation of momentum dependent expression as well. Eq. (S215) may be expressed as

\[ \Sigma_g(i\omega, \mathbf{k}) = g^2 k_F \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1}{\frac{\Gamma}{2} \text{sgn}(\omega') - v_F k} \left( (k_F + k)^2 + (k_F + k')^2 \right) \]

\[ - 2(k_F + k)(k_F + k') \cos \theta + \sin \theta |\omega - \omega'| \right)^{-1}. \]

(S216)
\[
\sum_{\omega, \mathbf{k}} \simeq g^2 k_F \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1}{i^2 \text{sgn}(\omega') - v_F k'} \left(2k_F^2 + k^2 + k'^2 - 2(k_F^2 + k k') \cos \theta \right)^{-1} \\
\simeq \frac{g^2}{2} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{1}{i^2 \text{sgn}(\omega') - v_F k'} \frac{1}{\sqrt{(k - k')^2 + c_d |\omega - \omega'|}}.
\]  
(S217)

In the above, we neglected non-singular (as \( k, \omega \to k', \omega' \)) terms in the angle-integrated boson propagator, which are also additionally suppressed by additional powers of \( 1/k_F \). These will only contribute to \( \Sigma_{g}(i\omega, \mathbf{k}) \) at higher orders in \( \omega \) beyond the marginal Fermi liquid form and are therefore not of interest to us. We can split \( 1/(i\Gamma \text{sgn}(\omega')/2 - v_F k') \) into real and imaginary parts. Then,

\[
-v_F^2 \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{k'}{\sqrt{(k - k')^2 + c_d |\omega - \omega'|}}
\]

\[
= -v_F^2 \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{k'}{\sqrt{(k - k')^2 + c_d |\omega - \omega'|}} \left( \frac{1}{\sqrt{(k - k')^2 + c_d |\omega - \omega'|}} - \frac{1}{c_d |\omega - \omega'|} \right)
\]

\[
= \frac{v_F^2 \ln \left(1 + \frac{4v_F^2 \Lambda_d}{\Gamma} \right)}{2\pi^2 v_F c_d}, \quad \Lambda_d \sim \frac{\Gamma}{v_F}.
\]  
(S218)

This term is frequency independent, and therefore cannot lead to any dissipation relevant for transport; we therefore drop it as it is only a small (\( O(1/k_F) \)) renormalization of the Fermi velocity. Then we have

\[
\Sigma_{g}(i\omega, \mathbf{k}) \simeq \frac{ig^2}{4} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\Gamma \text{sgn}(\omega')}{\sqrt{(k - k')^2 + c_d |\omega - \omega'|}}
\]

\[
= -\frac{ig^2}{2\pi c_d} \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma \text{sgn}(\omega)}{\sqrt{(k - k')^2 + c_d |\omega| - |k - k'|}}
\]

\[
= \frac{g^2 \text{sgn}(\omega_n)}{8\pi^2 c_d v_F} \left( 2\Gamma \ln \left( \frac{c_d v_F^2 |\omega_n|}{\Gamma^2 + 4k^2 v_F^2} \right) + \sqrt{c_d v_F^2 |\omega_n| + \left( k v_F + \frac{\Gamma}{2} \right)^2} \left( \ln \left( c_d v_F^2 |\omega_n| + \left( k v_F + \frac{\Gamma}{2} \right)^2 \right) + \frac{\Gamma}{2} + k v_F \right) - \ln \left( \sqrt{c_d v_F^2 |\omega_n| + \left( k v_F + i \frac{\Gamma}{2} \right)^2} - i \Gamma - k v_F \right) + 2 \tanh^{-1} \left( \frac{k v_F + i \frac{\Gamma}{2}}{\sqrt{c_d v_F^2 |\omega_n| + (k v_F + i \frac{\Gamma}{2})^2}} \right) \right)
\]

\[
+ \sqrt{c_d v_F^2 |\omega_n| + \left( k v_F - i \frac{\Gamma}{2} \right)^2} \left( \ln \left( c_d v_F^2 |\omega_n| + \left( k v_F - i \frac{\Gamma}{2} \right)^2 \right) + \frac{\Gamma}{2} - k v_F \right)
\]

\[
+ \ln \left( \sqrt{c_d v_F^2 |\omega_n| + \left( k v_F - i \frac{\Gamma}{2} \right)^2} - i \Gamma + k v_F \right) + 2 \tanh^{-1} \left( \frac{-k v_F + i \frac{\Gamma}{2}}{\sqrt{c_d v_F^2 |\omega_n| + (k v_F - i \frac{\Gamma}{2})^2}} \right)
\]

\[
+ 2k v_F \left( \ln \left( -k v_F + i \frac{\Gamma}{2} \right) - \ln \left( k v_F + i \frac{\Gamma}{2} \right) + 2i \tan^{-1} \left( \frac{2kvF}{\Gamma} \right) \right) .
\]  
(S219)

The momentum dependent corrections to (S216) induced by expanding (S219) in \( k \) will not produce any dissipative contributions to the conductivity as we shall show in the next subsection, and therefore may be ignored. Furthermore, because \( |\Sigma_{g}(i\omega, \mathbf{k})| \ll \Gamma \) at small frequencies, the above fermion and boson self-energies lead to a self-consistent solution of (S212) low frequencies.
When \( v = 0, \Gamma = 0 \), and therefore \( \Pi_g(i\Omega, \mathbf{q}) = -c_b|\omega|/|\mathbf{q}| \) from (S213) [1]. \( \Pi_g'(i\Omega) \), on the other hand, stays the same as in (S213), as the value of the fermion Green’s function integrated over momentum does not depend on \( \Gamma \), due to the fermion bandwidth being large. Since \( \Pi_{g'} \ll \Pi_g \) then at small (\( \omega, \mathbf{q} \)), we can neglect \( \Pi_g \) when computing the leading contributions to \( \Sigma_g \) and \( \Sigma_{g'} \). We then have \( \Sigma_g(i\omega, \mathbf{k}) \simeq -ic_f \text{sgn}(\omega)|\omega|^{2/3} [1] \), and

\[
\Sigma_g'(i\omega) \simeq g^2 k_F \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_0^{\infty} \frac{dq}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \frac{G(i(\omega + \Omega), k)}{q^2 + c_b|\Omega|/q}
\]

\[
\simeq -i\pi Ng'^2 \int_0^{\infty} \frac{dq}{2\pi} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \frac{\text{sgn}(\omega + \Omega)}{q^2 + c_b|\Omega|/q}
\]

\[
\simeq -\frac{iNg'^2\omega}{6\pi} \ln \left( \frac{e\tilde{\Lambda}^3}{c_b|\omega|} \right),
\]

where \( \tilde{\Lambda} \) is a UV cutoff on \( q \). This part of the self energy corresponds to current and momentum relaxing scattering induced by the spatially random interactions, and has a marginal Fermi liquid form. The corrections coming from \( \Pi_{g'} \) that we neglected here can be shown to be a non-dissipative \( \mathcal{O}(i\omega) \) term to \( \Sigma_g \) and a \( \mathcal{O}(\text{sgn}(\omega)|\omega|^{4/3}) \) term to \( \Sigma_{g'} \) respectively, both of which can be neglected when compared to the above results for \( \Sigma_g \) and \( \Sigma_{g'} \). These results are also self-consistent at low energies, since \( |\Sigma_{g'}| \ll |\Sigma_g| \), and \( \Sigma_g \) is essentially the same as its self-consistent value in the theory with \( g' = 0 \) [1].

As discussed earlier in this subsection, the low energy solutions for \( v \neq 0 \) derived above are valid below a frequency scale \( E_{c,1} \sim \Gamma^2/(v_F^2 c_d) \) and a corresponding momentum scale \( \Lambda_d \sim \Gamma/v_F \). Above these scales, the \( v_F^2 q^2 \) term in the denominator of the expression for \( \Pi_g(i\Omega, \mathbf{q}) \) (S213) is no longer negligible compared to the \( \Gamma^2 \) term, and there is a crossover to \( z = 3 \) boson dynamics, with \( \Pi_g(i\Omega, \mathbf{q}) = c_b|\omega|/|\mathbf{q}| \). \( \Pi_g'(i\Omega) \), on the other hand, stays the same as given by (S213), as the value of the fermion Green’s function integrated over momentum does not change under this crossover, due to the fermion bandwidth being large. Similar to the case of \( v = 0 \), we then have \( \Sigma_g(i\omega, \mathbf{k}) \simeq -ic_f \text{sgn}(\omega)|\omega|^{2/3} \), and \( \Sigma_{g'} \) is given by (S220), with the fermion Green’s function \( G(i\omega, \mathbf{k}) \simeq 1/(i\Gamma \text{sgn}(\omega)/2 - v_F k - \Sigma_g(i\omega, \mathbf{k}) - \Sigma_{g'}(i\omega)) \). For frequency scales larger than \( E_{c,2} \sim g^4/(g^6 v_F^2 N^4) > E_{c,1} \), we can see that \( \Sigma_{g'} \) dominates over \( \Sigma_g \) (and \( \Pi_{g'} \) also dominates over \( \Pi_g \), after applying the momentum scaling \( q^2 \sim c_b|\omega| \)), which produces another crossover to an effective theory having \( v \neq 0, g = 0 \), and \( g' \neq 0 \). Such a theory has \( \Sigma = \Sigma_{g'} \) given by (S214) [1, 14]. The momentum scale \( \tilde{\Lambda} \) (which serves as the momentum cutoff in (S220)) corresponding to \( E_{c,2} \) is therefore \( \tilde{\Lambda} \sim c_b^{1/3} E_{c,2}^{1/3} \sim g^2/(g^2 v_F N) \).

C. Transport properties

We now proceed to compute the conductivity at \( T = 0 \) as a function of frequency \( \omega \) for the models with disordered interactions. We first focus on the case with \( v, g, g' \) all nonzero. Because the thermal mass of the \( z = 2 \) bosons (that is induced by boson self-interactions) scales as \( T \) up to logarithms [1, 14, 15], this leads to \( \omega/T \) scaling in \( \text{Im}[\Sigma_R(\omega, \mathbf{k})] \) up to logarithms [1, 14], in turn leading to \( T \)-linear resistivity. The last fact was explicitly demonstrated earlier in models with \( g = 0 \) [1, 14], and we will show that it continues to be valid here when \( g \neq 0 \).

To demonstrate the existence of const.\( +T \)-linear resistivity, or a const.\( +\omega \)-linear scattering rate at \( T = 0 \) in the real part of the conductivity, it is sufficient to look at the computation of the conductivity perturbatively in the interactions \( g \) and \( g' \), while keeping diagrams that are order \( N \). The full ladder computation performed in Secs. SI and SII incorporates essentially the same principles and cancellations; for the sake of simplicity and to highlight the essential features of the calculation, we will describe the generalization to the ladder summation after the perturbative computation.
The non-interacting contribution to the real part of the $T = 0$ conductivity is trivially given by computing the current-current correlation function (Fig. 1a of the main text)

\[
\frac{1}{N} \text{Re}[\sigma_v(\omega)] = \frac{\text{Im}[(f_v(\omega) - f_v(0))_{\omega \rightarrow \omega + i0^+}]}{\omega},
\]

\[
f_v(\omega) = -v_F^2 k_F \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\pi}^{\pi} d\theta \cos^2 \theta \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega + \omega') - v_F k} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega) - v_F k'},
\]

\[
\frac{1}{N} \text{Re}[\sigma_v(\omega)] = \frac{N v_F^2}{2\Gamma}.
\]

(S221)

Here, we have noted that $k_F$ is large, and therefore approximated $v_k = v_F k$.

The perturbative contribution to the conductivity from the momentum-independent $\Sigma_{g'}(\omega)$ in Fig. 1b of the main text is also straightforwardly computed at small frequencies:

\[
\frac{1}{N} \text{Re}[\sigma_{\Sigma_{g'}}(\Omega)] = \frac{\text{Im}[(f_{\Sigma_{g'}}(\omega) - f_{\Sigma_{g'}}(0))_{\omega \rightarrow \omega + i0^+}]}{\omega},
\]

\[
f_{\Sigma_{g'}}(\omega) = -2v_F^2 k_F \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \int_{-\pi}^{\pi} d\theta \cos^2 \theta \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega + \omega') - v_F k} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega') - v_F k} \Sigma_{g'}(\omega)
\]

\[
\frac{1}{N} \text{Re}[\sigma_{\Sigma_{g'}}(\Omega)] = -\frac{N^2 v_F^2 g'^2 |\omega|}{16\pi \Gamma^2}.
\]

(S222)

The perturbative correction to the real part of the conductivity in Fig. 1b of the main text from $\Sigma_g(\omega,k)$ may be computed using (S216). The computation then parallels (S222), and we obtain

\[
\frac{1}{N} \text{Re}[\sigma_{\Sigma_g}(\omega)] = -\frac{N v_F^2 g^2 |\omega|}{8\pi \Gamma^3}.
\]

(S223)

If we use the momentum dependent form of $\Sigma_g(\omega,k)$ instead, i.e., (S219) and then numerically perform the $k, \omega'$ integrals in the Kubo formula, we only obtain an analytic $\propto \omega^2$ correction to the current correlation function $f_{\Sigma_g}(\omega) - f_{\Sigma_g}(0)$, over the result utilizing (S216) for the same. Therefore, $\text{Im}[(f_{\Sigma_g}(\omega) - f_{\Sigma_g}(0))_{\omega \rightarrow \omega + i0^+}]$ is not corrected, and the dissipative part of the conductivity correction remains the same as given by (S223).

We now proceed to compute the vertex corrections to the conductivity. For the vertex corrections to the conductivity, we note that the $g'$ and $v$ vertices never contribute in the large $N$ limit due to the insertion of their non-momentum conserving lines decoupling the momentum integrals in the two current vertices of the current correlation function. This causes such diagrams to vanish as the velocity factors in the current vertices are odd under inversion of $k \rightarrow -k$ whereas the fermion propagators are even under the same. Therefore, only the vertex corrections that solely involve $g$ vertices matter in the large $N$ limit. At order $g^2$, there is only one, the Maki-Thomson diagram (Fig. 1c of the main text), given by

\[
\frac{1}{N} \text{Re}[\sigma_{V_g}(\omega)] = \frac{\text{Im}[(f_{V_g}(\omega) - f_{V_g}(0))_{\omega \rightarrow \omega + i0^+}]}{\omega},
\]

\[
f_{V_g}(\omega) = -g^2 v_F^2 k_F \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\pi}^{\pi} d\theta_1 \cos \theta_1 \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega_1) - v_F k_1} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega_2) - v_F k_2}
\]

\[
\times \frac{1}{v_F k_1 i\frac{v_F}{2} \text{sgn}(\omega_1 + \omega) - v_F k_1} \frac{1}{i\frac{v_F}{2} \text{sgn}(\omega_2 + \omega) - v_F k_2}
\]

\[
\times \frac{1}{(k_F + k_1)^2 + (k_F + k_2)^2 + c_d |\omega_1 - \omega_2| - 2(k_F + k_1)(k_F + k_2) \cos(\theta_1 - \theta_2)}. \tag{S224}
\]
The angular integrals can be computed first
\[
\int_{-\pi}^{\pi} \frac{d\theta_1 \cos \theta_1}{2\pi} \int_{-\pi}^{\pi} \frac{d\theta_2 \cos \theta_2}{2\pi} \left( \frac{1}{(k_F + k_1)^2 + (k_F + k_2)^2 + c_d|\omega_1 - \omega_2| - 2(k_F + k_1)(k_F + k_2)\cos(\theta_1 - \theta_2)} \right.
\]
\[
= \frac{1}{2} \int_{-\pi}^{\pi} \frac{d\theta_1}{2\pi} \left( \frac{1}{(k_F + k_1)^2 + (k_F + k_2)^2 + c_d|\omega_1 - \omega_2| - 2(k_F + k_1)(k_F + k_2)\cos \theta_1} \right.
\]
\[
\simeq \frac{1}{4k_F \sqrt{(k_1 - k_2)^2 + c_d|\omega_1 - \omega_2|}} + \ldots, \quad (S225)
\]
where the ... corresponds to terms that are not singular as \( k_1, \omega_1 \to k_2, \omega_2, \) and are also \( \mathcal{O}(1/k_F^2) \) and smaller. These terms may therefore be ignored just like they were in the computation of \( \Sigma_\phi(\omega, \mathbf{k}) \). Since only the singular (as \( k, \omega \to 0 \)) boson fluctuations contribute to the marginal Fermi liquid form of \( \Sigma_\phi(\omega, \mathbf{k}) \), these higher order non-singular corrections will not give rise to any \( \omega \) or \( T \)-linear transport scattering rates (instead contributing higher powers of \( \omega, T \)), and are therefore of no interest to us. Because of the additional suppression of these non-singular corrections by powers of \( 1/k_F \) both here and in \( \Sigma_\phi(\omega, \mathbf{k}) \), the conductivity corrections that they induce will also not be extensive in the number of electrons in the system. Substituting \( (S225) \) into \( (S224) \), we can then write
\[
f_{V,g}(i\omega) = -\frac{g^2 v_F^2 k_F}{4} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \frac{1}{i\frac{\Gamma}{2} \operatorname{sgn}(|\omega_1 + \omega_2|) - v_F(k_1 + k_2)}
\]
\[
\times \frac{1}{i\frac{k_F}{2} \operatorname{sgn}(|\omega_1 + \omega + \omega|) - v_F(k_1 + k_2)} \frac{1}{i\frac{k_F}{2} \operatorname{sgn}(|\omega_2 + \omega|) - v_F(k_2)}
\]
\[
= \Theta(|\omega_1| - |\omega|) \frac{i\Gamma \operatorname{sgn}(\omega_1)(2|\omega_1| - 2|\omega|)}{2\pi v_F \Gamma(v_F k - i\operatorname{sgn}(\omega) \Gamma)^3} \Theta(|\omega| - |\omega_1|) \frac{i\Gamma \operatorname{sgn}(\omega_1)(2|\omega_1| - |\omega|)}{2\pi v_F \Gamma(v_F k + i\operatorname{sgn}(\omega) \Gamma)^3}.
\quad (S227)
\]
We now integrate over \( k_2 \): this results in a complicated function of \( \omega, \omega_1, \omega_2 \) that switches discontinuously between different values depending on the signs of various linear combinations of \( \omega, \omega_1, \omega_2 \). Nevertheless, it is then possible to proceed by an additional step and and also integrate over \( \omega_2 \). This gives
\[
\int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \frac{1}{i\frac{\Gamma}{2} \operatorname{sgn}(|\omega_1 + \omega_2|) - v_F(k_1 + k_2)} \frac{1}{i\frac{k_F}{2} \operatorname{sgn}(|\omega_1 + \omega + \omega|) - v_F(k_1 + k_2)}
\]
\[
= \Theta(|\omega_1| - |\omega|) \frac{i\Gamma \operatorname{sgn}(\omega_1)(|\omega_1| - 2|\omega|)}{2\pi v_F \Gamma(v_F k - i\operatorname{sgn}(\omega) \Gamma)^3} + \Theta(|\omega| - |\omega_1|) \frac{i\Gamma \operatorname{sgn}(\omega_1)(2|\omega_1| - |\omega|)}{2\pi v_F \Gamma(v_F k + i\operatorname{sgn}(\omega) \Gamma)^3}.
\quad (S228)
\]
where \( \Theta(\ldots) \) is the Heaviside step function. The first (high frequency) term of the last line of \( (S227) \) is not interesting. Inserting it into \( (S226) \), performing the \( k_1 \) integral, and then the \( \omega_1 \) integral with the physical UV frequency cutoff of the \( z = 2 \) theory, \( \omega_{UV} \sim \Gamma^2/(v_F^2 c_d) \), yields a contribution
\[
\frac{1}{N} \operatorname{Re}[\sigma_{V,g}(\omega)] = -\frac{N g^2 c_1}{c_d \Gamma},
\quad (S228)
\]
where \( c_1 \sim \ln(\omega_{UV} v_F c_d / \Gamma^2) \) is a positive \( \mathcal{O}(1) \) number, whose precise value depends on the numerical constant of proportionality between the physical UV frequency cutoff \( \omega_{UV} \) and \( \Gamma^2/(v_F^2 c_d) \). This contribution is not extensive in the number of fermions in the system (i.e. not proportional to \( k_F^2 \) or \( v_F^2 \)), and is thus only a small (suppressed by \( 1/k_F^2 \)) \( \omega \) and \( T \)-dependent correction to the static transport scattering rate \( \Gamma \). We therefore ignore it.

The second term of the last line of \( (S227) \) is important. Inserting it into \( (S226) \) and performing the \( k_1 \) and \( \omega_1 \) integrals yields the contribution
\[
f_{V,g}(i\omega) - f_{V,g}(0) = -\frac{g^2 v_F k_F}{2\pi \Gamma} \int_{0}^{\omega} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \left( \frac{\Gamma^2 (|\omega_1| - 2\omega_1) + v_F^2 k_1^2 |\omega|}{\Gamma^2 + v_F^2 k_1^2} \right) \frac{2\Gamma^2 \omega_1}{(\Gamma^2 + v_F^2 k_1^2)^2} \frac{1}{\sqrt{k_1^2 + c_d|\omega_1|}}
\]
\[
\begin{align*}
\frac{1}{N} \text{Re}[\sigma_{V,g}(\omega)] &= \frac{N v_F^2 g^2 |\omega|}{8\pi^2 \Gamma^3}, \\
\frac{1}{N} \text{Re}[\sigma(\omega)] &= \frac{1}{N} \text{Re}[\sigma_i(\omega)] + \frac{1}{N} \text{Re}[\sigma_{\Sigma,F}(\omega)] \\
&= \frac{N v_F^2}{2\Gamma} \left( 1 - \frac{N g^2 |\omega|}{8\Gamma} \right) \rightarrow \frac{N v_F^2}{2\Gamma} (1 + \frac{N g^2 |\omega|}{8\Gamma}) = \frac{N v_F^2 / 2}{\Gamma + N g^2 |\omega|/8},
\end{align*}
\]

\[\text{(S230)}\]

This contribution exactly cancels (S223). This is expected since the singular low-momentum boson fluctuations just lead to forward scattering that does not relax current.

Therefore, perturbatively, we have

\[
\frac{1}{N} \text{Re}[\sigma(\omega)] = \frac{1}{N} \text{Re}[\sigma_i(\omega)] + \frac{1}{N} \text{Re}[\sigma_{\Sigma,F}(\omega)]
\]

\[
= \frac{N v_F^2}{2\Gamma} \left( 1 - \frac{N g^2 |\omega|}{8\Gamma} \right) \rightarrow \frac{N v_F^2}{2\Gamma} (1 + \frac{N g^2 |\omega|}{8\Gamma}) = \frac{N v_F^2 / 2}{\Gamma + N g^2 |\omega|/8},
\]

\[\text{(S230)}\]

giving the much sought after linear-in-energy correction to the static impurity scattering rate \(\Gamma\).

When \(T > 0\), the \(z = 2\) boson gains a thermal mass \(m_b^2(T) \sim c_d T \ln(\Gamma^2/(v_F^3 c_d T))/\ln(\Gamma^2/(v_F^3 c_d T))\) \([1, 14, 15]\), with \(D(\omega, q) = 1/(q^2 + c_d |\omega| + m_b^2(T))\). Because \(m_b^2(T)\) is not \(\gg |\omega| \sim q^2/c_d\) at low \(T\), the low-frequency and low-momentum boson fluctuations do not become any less singular when \(T > 0\). Thus, the singular boson fluctuations continue to induce only forward scattering of the fermions, whose contributions to the conductivity continue to cancel between \(\sigma_{\Sigma,F}\) and \(\sigma_{V,g}\), just like as demonstrated above. The finite temperature conductivity and transport scattering rate are then also simply computed using \(\Sigma_{g'}\), which was also done in Refs. \([1, 14]\). This gives a \(\sim N g^2 T \ln(\Gamma^2/(v_F^3 c_d T))\) correction to the static impurity transport scattering rate \(\Gamma\) \([1, 14]\).

While the \(T\)-linear correction to the transport scattering rate arises only from \(\Sigma_{g'}\), both \(\Sigma_g\) and \(\Sigma_{g'}\) contribute to the effective mass renormalization of the fermions. Since these are both of marginal Fermi liquid form, the effective mass renormalization is \(m^*/m \sim (a_1 g^2 / \Gamma + a_2 N g^2) \ln(\Gamma^2/(v_F^3 c_d T))\), where the numbers \(a_{1,2} \sim \mathcal{O}(1)\) \([1]\). Then, following the steps in Sec. VIII D of Ref. \([1]\), we obtain the result for the constant of proportionality \(a\) between \(1/\tau_t\) and \(k_B T/\hbar\) given in the main text (Eq. (15)).

Considering diagrams with four interaction vertices, we find the two Aslamazov-Larkin diagrams (Fig. 1d,e of the main text) in the large \(N\) limit (which are also encountered in the kernel of the full ladder resummation described in Sec. SII). We will show below that the sum of these two diagrams does not correct the conductivity when \(k_F\) is large. The other diagrams are the two rung \(g\) ladder, and additional insertions of the \(g\) and \(g'\) self energies and ladders. From the computation of Sec. SII, it can be established that the resummation of such non-Aslamazov-Larkin terms essentially just renormalizes the current-relaxing electron scattering rate, while canceling the current-conserving forward scattering at all orders. In particular, since \(g'\) does not generate new vertex corrections, its only net effect in (S162) is to add a term analogous to (S165), but involving \(\Sigma_{g'}\) instead of \(\Sigma_g\), to \(W_{\Sigma,F,F}^{-1}\) \([16]\). Therefore, the renormalization of the transport scattering rate described by (S230) actually holds to all orders in perturbation theory in the large \(N\) limit.

We now demonstrate the nullification of the sum of the two Aslamazov-Larkin diagrams in Fig. 1d,e of the main text in the large Fermi energy or large \(k_F\) limit. We can express the sum of the two order \(g^4\) Aslamazov-Larkin
diagrams in this limit as
\[
f_{\text{AL},g}(i\omega) = g^4 v_F^2 k_F^2 \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_3}{2\pi} \int_{-\infty}^{\infty} \frac{d\theta_1}{2\pi} \cos \theta_1 \int_{-\infty}^{\infty} \frac{d\theta_2}{2\pi} \cos \theta_2 \int_{-\infty}^{\infty} \frac{d\theta_3}{2\pi} \int_{-\infty}^{\infty} \frac{dk_1}{2\pi} \int_{-\infty}^{\infty} \frac{dk_2}{2\pi} \int_{-\infty}^{\infty} \frac{dk_3}{2\pi}
\]
\[
= \frac{1}{k_F^2 + c_4|\omega_2 + \frac{\omega}{2}| \frac{1}{k_F^2 + c_4|\omega_2 - \frac{\omega}{2}|} \frac{1}{i\frac{\omega}{2} \text{sgn} (\omega_1 + \frac{\omega}{2}) - v_F k_1 \frac{1}{i\frac{\omega}{2} \text{sgn} (\omega_1 - \frac{\omega}{2}) - v_F k_1}} \times \frac{1}{k_F^2 + c_4|\omega_2 + \frac{\omega}{2}| - v_F k_2 \frac{1}{i\frac{\omega}{2} \text{sgn} (\omega_1 - \omega_2) - v_F k_1 + v_F k_2 \cos (\theta_1 - \theta_2)}} \times \frac{1}{k_F^2 + c_4|\omega_3 + \omega_2| - v_F k_3 - v_F k_2 \cos (\theta_3 - \theta_2) + i\frac{\omega}{2} \text{sgn} (\omega_3 - \omega_2) - v_F k_3 + v_F k_2 \cos (\theta_1 - \theta_2)}.
\]
(S231)

We can then see that the quantity in brackets on the last line is odd under \(\omega_3 \rightarrow -\omega_3\), whereas all the other terms multiplying it are even under the same, which renders the whole integrand odd under \(\omega_3 \rightarrow -\omega_3\). Therefore, the integral over \(\omega_3\) (and hence \(f_{\text{AL},g}(i\omega)\)) vanishes identically. In fact, this continues to occur when the self-energies \(\Sigma_g, \Sigma_{g'}\) and the bare \(i\omega\) term are included in the fermion propagators, as these are all odd in the Matsubara frequency. The cancellation of the Aslamazov-Larkin diagrams is therefore completely self-consistent in the large \(k_F\) limit. The only non-zero contributions to the sum of the Aslamazov-Larkin diagrams come from going away from the large \(k_F\) limit, by including corrections to the fermion current vertex factors and fermion dispersions.

The resulting corrections to the conductivity are therefore not extensive in the number of fermions in the system, and are therefore not important to us. A computation of these non-extensive contributions is nevertheless carried out in Sec. SIIIC3, where it is demonstrated that they indeed lead to a small correction to the transport scattering rate that is \(\mathcal{O}(N'g^2\omega^2/(\Gamma^2 k_F^2))\).

Finally, we consider the case of \(v = 0\) but \(g, g' \neq 0\). In this case, the non-interacting conductivity (Fig. 1a of the main text) is trivially given by
\[
\frac{1}{N} \sigma_0(i\omega) = \frac{N v_F^2}{2\omega}.
\]
(S232)

The contributions of the momentum-independent \(\Sigma_g(i\omega, k) \simeq -i c_f |\text{sgn}(\omega)| |\omega|^{2/3}\) and \(\Sigma_{g'}(i\omega)\) (S220) to the conductivity within perturbation theory (i.e. Fig. 1b of the main text) are also straightforwardly computed as in (S222), and are given by
\[
\frac{1}{N} \sigma_{\Sigma,g'}(i\omega) \simeq \frac{3N v_F^2 c_F}{5\omega |\omega|^{1/3}} = -\frac{3N v_F g^2}{10\pi \sqrt{3} e_b^{1/3} \omega |\omega|^{1/3}}.
\]

As before, only vertex corrections with \(g\) vertices contribute in the large \(N\) limit. The contributions to \(\sigma_{V,g}(i\omega)\) from the most singular boson fluctuations can be computed using the theory of antipodal patches described in Ref. [1]. We then have
\[
\frac{1}{N} \sigma_{V,g}(i\omega) = \frac{v_F^2 g^2}{\omega} \sum_{s = \pm} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{1x}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{2x}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{1y}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{2y}}{2\pi} \int_{-\infty}^{\infty} \frac{|k_{1y} - k_{2y}|}{|k_{1y} - k_{2y}|^3 + c_6 |\omega_1 - \omega_2|}
\]
\[
\times \frac{1}{\omega_1 - s v_F k_{1x} - \kappa k_{1y}^2 \pm i(\omega_1 + \omega) - s v_F k_{1x} - \kappa k_{1y}^2 \pm i\omega_2 - s v_F k_{2x} - \kappa k_{2y}^2 \pm i(\omega_2 + \omega) - s v_F k_{2x} - \kappa k_{2y}^2}^{1/2}
\]
\[
= \frac{2g^2}{\omega} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \left( \frac{\text{sgn}(\omega_1 + \omega) - \text{sgn}(\omega_1)}{2\omega} \right) \left( \frac{\text{sgn}(\omega_2 + \omega) - \text{sgn}(\omega_2)}{2\omega} \right) \int_{-\infty}^{\infty} \frac{dk_{1y}}{2\pi} \int_{-\infty}^{\infty} \frac{dk_{2y}}{2\pi}
\]
\[ \times \frac{|k_1y - k_2y|}{|k_1y - k_2y|^3 + c_b|\omega_1 - \omega_2|} = 2g^2A_y \frac{3}{\omega} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} \left( \frac{\text{sgn}(\omega_1 + \omega) - \text{sgn}(\omega_1)}{2\omega} \right) \left( \frac{\text{sgn}(\omega_2 + \omega) - \text{sgn}(\omega_2)}{2\omega} \right) \int_{-\infty}^{\infty} \frac{dk_1y}{2\pi} \frac{|k_1y|}{|k_1y|^3 + c_b|\omega_1 - \omega_2|} \]

where we shifted \( k_1y \rightarrow k_1y + k_2y \), and fixed the cutoff on the patch size \( \Lambda_y \) so that the correct non-interacting conductivity \((S232)\) is obtained from the theory of antipodal patches when \( v = g = g' = 0 \). Therefore, we find that the most singular contribution in \( \sigma_{V,g}(i\omega) \) cancels with \( \sigma_{\Sigma,g}(i\omega) \). Further, less singular corrections to \( \sigma_{V,g}(i\omega) \) coming from going beyond the theory of antipodal patches are \( O(1/\omega^{2/3}) \) \([4]\), and are therefore negligible in comparison to \( \sigma_{\Sigma,g}(i\omega) \). Therefore, we obtain the results in Eq. \((13)\) of the main text.

The above conclusions are however even stronger, because \((S110)\) is still valid when \( v = 0, g \neq 0, g' \neq 0 \). It then follows from the analysis in Sec. SIE that the less singular contributions in \( \sigma_{V,g}(i\omega) \) arising from going beyond the theory of antipodal patches cancel with the Aslamazov-Larkin diagrams exactly like they did when \( v = 0, g \neq 0, g' = 0 \) in Sec. SIE \([17]\). Additionally, it can also be shown that within the theory of antipodal patches, where \( \sigma_{V,g}(i\omega) \) is restricted to its most singular contribution, the sum of the two Aslamazov-Larkin diagrams vanishes due to an odd/even cancellation of integrands like in \((S231)\). Therefore, the antipodal patch theory essentially reproduces the results from the full theory going beyond antipodal patches for the low-frequency behavior of the optical conductivity in the \( v = 0 \) case, even though it doesn’t capture all the physics of the system correctly. As was the case for \( v \neq 0, g \neq 0, g' \neq 0 \), the perturbative results for the conductivity here are also valid to all orders in perturbation theory in the large \( N \) limit, for the same reasons as before.

\[ \text{References} \]

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[17] Since the $|\omega|$ damping term induced by $g'$ in the boson propagator is sub-leading to the $|\omega|/|q|$ damping term induced by $g$, it can be neglected as we have always done, and the computations analogous to Sec. SIE are then actually identical to those in Sec. SIE even in this case of $v = 0, g \neq 0, g' \neq 0$. 