SIGNED $q$-ANALOGS OF TORNHEIM’S DOUBLE SERIES

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Abstract. We introduce signed $q$-analogs of Tornheim’s double series, and evaluate them in terms of double $q$-Euler sums. As a consequence, we provide explicit evaluations of signed and unsigned Tornheim double series, and correct some mistakes in the literature.

1. Introduction

Let $k$ be a positive integer. Sums of the form

$$\zeta(s_1, s_2, \ldots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} n_j^{-s_j}, \quad \sum_{j=1}^{m} \Re(s_j) > m, \quad m = 1, 2, \ldots, k. \quad (1)$$

have attracted increasing attention in recent years; see eg. [2, 3, 4, 5, 6, 8, 9, 10, 11, 13, 16, 24, 26, 27, 29, 30, 31]. The survey articles [7, 18, 19, 36, 37, 38] provide an extensive list of references. In (1) the sum is over all positive integers $n_1, n_2, \ldots, n_k$ satisfying the indicated inequalities. Of course (1) reduces to the familiar Riemann zeta function when $k = 1$. When the arguments are all positive integers, we refer to (1) as a multiple zeta value and note that in this case, $s_1 > 1$ is necessary and sufficient for convergence.

The problem of evaluating $\zeta(s_1, s_2)$ with integers $s_1 > 1$ and $s_2 > 0$ seems to have been first proposed in a letter from Goldbach to Euler [21] in 1742. (See also [20, 22] and [1, p. 253].) Calculating several examples led Euler to infer a closed form evaluation in terms of values of the Riemann zeta function in the case when $s_1 + s_2$ is odd. In [4], Borwein, Bradley and Broadhurst considered the more general Euler sum

$$\zeta(s_1, s_2, \ldots, s_k; \sigma_1, \sigma_2, \ldots, \sigma_k) := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} \sigma_j^{n_j} n_j^{-s_j} \quad (2)$$

with each $\sigma_j \in \{-1, 1\}$. Among the many other results for (2) listed therein is an explicit formula for the case $k = 2$ that reduces to Euler’s evaluation when $\sigma_1 = \sigma_2 = 1$. For the case of arbitrary $k$, several infinite classes of closed form evaluations for (1) and (2) are
proved in [5, 6, 8, 9, 10]. Subsequently, Bradley [12, 13, 14, 15] found $q$-analogs of (1) and (2) and thoroughly investigated their properties.

In [32] Tornheim introduced the double series

$$T(r, s, t) := \sum_{u,v=1}^{\infty} \frac{1}{u^r v^s (u+v)^t},$$

with nonnegative integers $r$, $s$ and $t$ satisfying $r + t > 1$, $s + t > 1$ and $r + s + t > 2$. Huard, Williams and Zhang [25] evaluated Tornheim’s series in terms of sums of products of Riemann zeta values when $r + s + t$ is odd. Subbarao and Sitaramachandrarao [28] considered the alternating variants

$$R(r, s, t) := \sum_{u,v=1}^{\infty} \frac{(-1)^v}{u^r v^s (u+v)^t}, \quad S(r, s, t) := \sum_{u,v=1}^{\infty} \frac{(-1)^u+v}{u^r v^s (u+v)^t},$$

and posed the problem to evaluate $R(r, r, r)$ and $S(r, r, r)$ for any positive integer $r$. Tsumura [33, Cor. 3] and [34, Theorem 3.6] tackled this problem and later [35] evaluated $S(r, s, t)$ for any positive integers $r$, $s$, $t$ such that $r + s + t$ is odd. Tsumura’s method is elementary but complicated, and also has some mistakes. In this paper, we give a simple formula that expresses $R$, $S$, and $T$ in terms of double Euler sums. In light of the aforementioned formula of Borwein, Bradley and Broadhurst for the double Euler sums, our results yield a closed form evaluation for the series $R$, $S$, and $T$ whenever the arguments $r$, $s$, and $t$ are positive integers with $r + s + t$ odd. More generally, we consider $q$-analogs of $R$, $S$, $T$ and show how they may all be evaluated in terms of double $q$-Euler sums.

2. Q-Analogs

Henceforth assume $q$ is real and $q > 1$. The $q$-analog of a positive integer $n$ is

$$[n]_q := \sum_{j=0}^{n-1} q^j = \frac{q^n - 1}{q - 1}.$$

Let $k$ be a positive integer, let $s_1, s_2, \ldots, s_k$ be real numbers, and let $\sigma_1, \sigma_2, \ldots, \sigma_k \in \{-1, 1\}$. Define the $q$-Euler sum

$$\zeta_q[s_1, s_2, \ldots, s_k; \sigma_1, \sigma_2, \ldots, \sigma_k] := \sum_{n_1 > n_2 > \cdots > n_k > 0} \prod_{j=1}^{k} \frac{\sigma_j^{n_j} q^{(s_j-1)n_j}}{[n_j]_q^{s_j}}$$

and note that this coincides with the special case $\text{Li}_{s_1, \ldots, s_k}[\sigma_1 q^{s_1-1}, \ldots, \sigma_k q^{s_k-1}]$ of the multiple $q$-polylogarithm [12, eq. (6.2)]. If $\sigma_j = 1$ for each $j = 1, 2, \ldots, k$, then we recover the multiple $q$-zeta value $\zeta[s_1, s_2, \ldots, s_k]$ of [12, 14, 15].
Let $\sigma, \tau \in \{-1, 1\}$. Define $q$-analogues of the signed and unsigned Tornheim double series $R, S$ and $T$ by

$$T[r, s, t; \sigma, \tau] := \sum_{u, v=1}^{\infty} \frac{\sigma^u \tau^v q^{(r+t-1)u+(s+t-1)v}}{[u]_q^r [v]_q^s [u+v]_q^t}.$$  

The sum

$$\varphi[s; \sigma] := \sum_{n=1}^{\infty} \left( \frac{n-1}{[n]_q^s} \right)^{\sigma} q^{(s-1)n} = \sum_{n=1}^{\infty} \frac{n \sigma^n q^{(s-1)n}}{[n]_q^s} - \zeta_q[s; \sigma],$$

the case $\sigma = 1$ of which was defined in [15], will also be needed. As in [1], it is convenient to combine signs and exponents in [2] and [3] into a single list by writing $s_j$ if $\sigma_j = 1$ and $\overline{s}_j$ if $\sigma_j = -1$. For consistency one may do this also for $T$ and $\varphi$; thus for example, $\varphi[s; 1] = \varphi[s]$, $\varphi[s; -1] = \varphi[\overline{s}]$, $R(r, s, t) = T(r, \overline{s}, t)$ and $S(r, s, t) = T(1, \overline{s}, t)$. We also employ the notation

$$\begin{pmatrix} z \cr a, b \end{pmatrix} := \begin{pmatrix} z \cr a \end{pmatrix} \begin{pmatrix} z-a \cr b \end{pmatrix} = \begin{pmatrix} z \cr b \end{pmatrix} \begin{pmatrix} z-b \cr a \end{pmatrix}$$

for the trinomial coefficient, in which $a, b$ are nonnegative integers, and which reduces to $z!/a!b!(z-a-b)!$ if $z$ is also an integer exceeding $a + b$.

The following theorem shows how the $q$-analogues of $R$, $S$ and $T$ are related to the $q$-Euler sums.

**Theorem 1.** Let $r$ and $s$ be positive integers, and let $t$ be a real number. Then

$$T[r, s, t] = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \begin{pmatrix} a+s-1 \cr a, b \end{pmatrix} (1-q)^b \zeta_q[s+t+a, r-a-b]$$

$$+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \begin{pmatrix} a+r-1 \cr a, b \end{pmatrix} (1-q)^b \zeta_q[r+t+a, s-a-b]$$

$$- \sum_{j=1}^{\min(r,s)} \begin{pmatrix} r+s-j-1 \cr r-j, s-j \end{pmatrix} (1-q)^j \varphi[r+s+t-j],$$

$$T[r, \overline{s}, t] = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \begin{pmatrix} a+s-1 \cr a, b \end{pmatrix} (1-q)^b \zeta_q[s+t+a, r-a-b]$$

$$+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \begin{pmatrix} a+r-1 \cr a, b \end{pmatrix} (1-q)^b \zeta_q[r+t+a, s-a-b]$$

$$- \sum_{j=1}^{\min(r,s)} \begin{pmatrix} r+s-j-1 \cr r-j, s-j \end{pmatrix} (1-q)^j \varphi[r+s+t-j],$$
\[
T[r, s, t] = \sum_{a=0}^{r-1} \sum_{b=0}^{s-1} \binom{a + s - 1}{a, b} (1 - q)^b \zeta_q[s + t + a, r - a - b] \\
+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a + r - 1}{a, b} (1 - q)^b \zeta_q[r + t + a, s - a - b] \\
- \sum_{j=1}^{\min(r, s)} \binom{r + s - j - 1}{r - j, s - j} (1 - q)^j (1 + q)^{j - s - t} \zeta_q[r + s + t - j].
\]

Taking the limit as \( q \to 1+ \) in Theorem 1 and noting the restrictions on \( r, s \) and \( t \) now needed for convergence yields the following

**Corollary 1.** Let \( r \) and \( s \) be positive integers and let \( t \) be a real number. If \( r + t > 1 \) and \( s + t > 1 \), then

\[
T(r, s, t) = \sum_{a=0}^{r-1} \binom{a + s - 1}{s - 1} \zeta(s + t + a, r - a) + \sum_{a=0}^{s-1} \binom{a + r - 1}{r - 1} \zeta(r + t + a, s - a);
\]

if \( r + t > 0 \) and \( s + t > 0 \), then

\[
S(r, s, t) = \sum_{a=0}^{r-1} \binom{a + s - 1}{s - 1} \zeta(s + t + a, r - a) + \sum_{a=0}^{s-1} \binom{a + r - 1}{r - 1} \zeta(r + t + a, s - a);
\]

if \( r + t > 1 \) and \( s + t > 0 \), then

\[
R(r, s, t) = \sum_{a=0}^{r-1} \binom{a + s - 1}{s - 1} \zeta(s + t + a, r - a) + \sum_{a=0}^{s-1} \binom{a + r - 1}{r - 1} \zeta(r + t + a, s - a).
\]

Putting \( t = 0 \) in Theorem 1 yields the following decomposition formulas, the first of which was given under slightly more restrictive hypotheses in [15] Theorem 2.1.

**Corollary 2.** If \( r \) and \( s \) are positive integers, then

\[
\zeta_q[r] \zeta_q[s] = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a + s - 1}{s - 1} \binom{s - 1}{b} (1 - q)^b \zeta_q[s + a, r - a - b] \\
+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a + r - 1}{r - 1} \binom{r - 1}{b} (1 - q)^b \zeta_q[r + a, s - a - b] \\
- \sum_{j=1}^{\min(r, s)} \binom{r + s - j - 1}{r - j, s - j} (1 - q)^j \varphi[r + s - j];
\]
\[ \zeta_q[r] \zeta_q[s] = \sum_{a=0}^{r-1} \sum_{b=0}^{s-1-a} \left( \frac{a + s - 1}{s - 1} \right) \left( \frac{s - 1}{b} \right) (1 - q)^b \zeta_q[s + a, r - a - b] + \sum_{a=0}^{s-1} \sum_{b=0}^{r-1-a} \left( \frac{a + r - 1}{r - 1} \right) \left( \frac{r - 1}{b} \right) (1 - q)^b \zeta_q[r + a, s - a - b] \]

\[ - \sum_{j=1}^{\min(r,s)} \left( \frac{r + s - j - 1}{r - j, s - j} \right) (1 - q)^j \varphi[r + s - j]; \]

\[ \zeta_q[r] \zeta_q[s] = \sum_{a=0}^{r-1} \sum_{b=0}^{s-1-a} \left( \frac{a + s - 1}{s - 1} \right) \left( \frac{s - 1}{b} \right) (1 - q)^b \zeta_q[s + a, r - a - b] + \sum_{a=0}^{s-1} \sum_{b=0}^{r-1-a} \left( \frac{a + r - 1}{r - 1} \right) \left( \frac{r - 1}{b} \right) (1 - q)^b \zeta_q[r + a, s - a - b] \]

\[ - \sum_{j=1}^{\min(r,s)} \left( \frac{r + s - j - 1}{r - j, s - j} \right) (1 - q)^j (1 + q)^j - s \zeta_q^2[r + s - j]. \]

Taking the limit as \( q \to 1+ \) in Corollary 2 and noting the additional restrictions needed on \( r \) and \( s \) to guarantee convergence in this case yields the following decomposition formulas, the first of which was known to Euler.

**Corollary 3.** If \( r - 1 \) and \( s - 1 \) are positive integers, then

\[ \zeta(r) \zeta(s) = \sum_{a=0}^{r-1} \left( \frac{a + s - 1}{s - 1} \right) \zeta(s + a, r - a) + \sum_{a=0}^{s-1} \left( \frac{a + r - 1}{r - 1} \right) \zeta(r + a, s - a); \]

if \( r \) and \( s \) are positive integers, then

\[ \zeta(\overline{r}) \zeta(\overline{s}) = \sum_{a=0}^{r-1} \left( \frac{a + s - 1}{s - 1} \right) \zeta(s + a, r - a) + \sum_{a=0}^{s-1} \left( \frac{a + r - 1}{r - 1} \right) \zeta(r + a, s - a); \]

if \( r - 1 \) and \( s \) are positive integers, then

\[ \zeta(r) \zeta(\overline{s}) = \sum_{a=0}^{r-1} \left( \frac{a + s - 1}{s - 1} \right) \zeta(s + a, r - a) + \sum_{a=0}^{s-1} \left( \frac{a + r - 1}{r - 1} \right) \zeta(r + a, s - a). \]

### 3. Proof of Theorem \( \square \)

The key ingredient is the following partial fraction decomposition.
Lemma 1. If \( r, s, u, \) and \( v \) are all positive integers, then
\[
\frac{1}{[u]_q^r[v]_q^s} = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b q^{(s-1-b)u+av} [u]_q^{r-a-b}[u+v]_q^{s+a} + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b q^{au+(r-1-b)v} [v]_q^{s-a-b}[u+v]_q^{r+a} - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j q^{(s-j)u+(r-j)v} [u+v]_q^{r+s-j}.
\]

Proof. Let \( x \) and \( y \) be nonzero real numbers such that \( x + y + (q-1)xy \neq 0 \). As in [15], observe that if we apply the partial differential operator
\[
\frac{1}{(r-1)!} \left( - \frac{\partial}{\partial x} \right)^{r-1} \frac{1}{(s-1)!} \left( - \frac{\partial}{\partial y} \right)^{s-1}
\]
to both sides of the identity
\[
\frac{1}{xy} = \frac{1}{x+y+(q-1)xy} \left( \frac{1}{x} + \frac{1}{y} + q-1 \right),
\]
then we obtain the identity [15, Lemma 3.1]
\[
\frac{1}{x^r y^s} = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b (1+(q-1)y)^a (1+(q-1)x)^{s-1-b} x^{r-a-b}(x+y+(q-1)xy)^{s+a} + \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \binom{a+r-1}{a, b} (1-q)^b (1+(q-1)x)^a (1+(q-1)y)^{r-1-b} y^{s-a-b}(x+y+(q-1)xy)^{r+a} - \sum_{j=1}^{\min(r,s)} \binom{r+s-j-1}{r-j, s-j} (1-q)^j (1+(q-1)y)^{r-j}(1+(q-1)x)^{s-j} (x+y+(q-1)xy)^{r+s-j}.
\]

Now let \( x = [u]_q, y = [v]_q \) and note that then \( 1 + (q-1)x = q^u, 1 + (q-1)y = q^v \) and \( x + y + (q-1)xy = [u+v]_q \).

To prove Theorem 1 multiply both sides of Lemma 1 by
\[
\frac{\sigma^{u+v} q^{(r+t-1)u+(s+t-1)v}}{[u+v]_q^t q}
\]
and sum over all ordered pairs of positive integers \((u, v)\) to obtain
\[
T[r, s; t; \sigma, \tau] = \sum_{a=0}^{r-1} \sum_{b=0}^{r-1-a} \binom{a+s-1}{a, b} (1-q)^b \sum_{u,v=1}^{\infty} \frac{\sigma^{u+v} q^{(r-a-b-1)u+(s+t+a-1)(u+v)}}{[u]_q^{r-a-b}[u+v]_q^{s+t+a}}.
\]
It follows that

\begin{align*}
+ \sum_{a=0}^{s-1} \sum_{b=0}^{s-1-a} \left( \frac{a + r - 1}{a, b} \right) (1 - q)^b & \sum_{u,v=1}^{\infty} \frac{\sigma^u r^v q^{(s-a-b-1)v} q^{(r+t+a-1)(u+v)}}{[v]_q^{s-a-b} [u + v]_q^{r+t+a}}, \\
- \sum_{j=1}^{\min(r,s)} \left( \frac{r + s - j - 1}{r - j, s - j} \right) (1 - q)^j & \sum_{u,v=1}^{\infty} \frac{\sigma^u r^v q^{(r+s+t-j-1)(u+v)}}{[u + v]_q^{r+s+t-j}}.
\end{align*}

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\[
+ \sum_{j=1}^{\min(r,s)} \left(\frac{r + s - j - 1}{r - j, s - j}\right)(1 - q)^j \sum_{m=0}^{(r+s-t-1)/2m} \frac{q^{(r+s-t-1)/2m}}{[2m]q^{r+s-t-j}}.
\]
Since
\[
[2m]q = \frac{q^{2m} - 1}{q - 1} = \frac{q^2 - 1}{q - 1} \cdot q^m = (q + 1)[m]q^m,
\]
the proof of Theorem 1 is complete. \(\square\)

4. Examples

Again, let \(\sigma, \tau \in \{-1, 1\}\). It is known \[23\] that if \(s\) and \(t\) are positive integers such that \(s + t\) is odd and \(s > (1 + \sigma)/2\), then \(\zeta(s, t; \sigma, \tau)\) lies in the polynomial ring \(\mathbb{Q}[\{\zeta(k; \pm 1) : k \in \mathbb{Z}, 2 \leq k \leq s + t\} \cup \{\zeta(1; -1)\}]\). Since
\[
\zeta(k; -1) = \zeta(k) = \begin{cases} (2^{1-k} - 1)\zeta(k) & \text{if } k > 1, \\ -\log 2 & \text{if } k = 1, \end{cases}
\]
it is clear that in fact, \(\zeta(s, t; \sigma, \tau) \in \mathbb{Q}[\{\zeta(k) : k \in \mathbb{Z}, 2 \leq k \leq s + t\} \cup \{-\log 2\}]\). It follows that if \(r, s, t\) satisfy the conditions of Corollary 1 and if in addition \(r + s + t\) is an odd integer, then the signed and unsigned double Tornheim series \(R(r, s, t)\), \(S(r, s, t)\) and \(T(r, s, t)\) also lie in this ring. It is possible to evaluate these series explicitly if we recall the following formula from \[24\ eq. (75)\].

**Proposition 1.** Let \(\sigma, \tau \in \{-1, 1\}\), and let \(s\) and \(t\) be positive integers such that \(s + t\) is odd, \(s > (1 + \sigma)/2\), and \(t > (1 + \tau)/2\). Then
\[
\zeta(s, t; \sigma, \tau) = \frac{1}{2}(1 - (-1)^s)\zeta(s; \sigma)\zeta(t; \tau) - \frac{1}{2}\zeta(s + t; \sigma \tau) + (-1)^t \sum_{0 \leq k \leq t/2} \left(\frac{s + t - 2k - 1}{s - 1}\right)\zeta(2k; \sigma \tau)\zeta(s + t - 2k; \sigma) + (-1)^t \sum_{0 \leq k \leq s/2} \left(\frac{s + t - 2k - 1}{t - 1}\right)\zeta(2k; \sigma \tau)\zeta(s + t - 2k; \tau). \tag{4}
\]

In Proposition 1, it is understood that \(\zeta(0; \sigma \tau) = -1/2\) in accordance with the analytic continuation of \(s \mapsto \zeta(s; \sigma \tau)\). The restriction \(t > (1 + \tau)/2\) can be removed if in \[24\] we interpret \(\zeta(1; 1) = 0\) wherever it occurs. That is, if \(\sigma \in \{-1, 1\}\) and \(s\) is an even positive integer, then
\[
\zeta(s, 1; \sigma, 1) = \frac{1}{2}(s - 1)\zeta(s + 1; \sigma) + \frac{1}{2}\zeta(s + 1) - \sum_{k=1}^{(s/2) - 1} \zeta(2k; \sigma)\zeta(s + 1 - 2k). \tag{5}
\]
The case $\sigma = 1$ of (5) is subsumed by another formula [4, eq. (31)] of Euler, namely

$$\zeta(s, 1) = \frac{s}{2} \zeta(s + 1) - \frac{1}{2} \sum_{k=2}^{s-1} \zeta(k)\zeta(s + 1 - k), \quad (6)$$

which is valid for all integers $s > 1$, not just for even $s$.

In [35], Tsumura listed evaluation formulas for $S(r, s, t)$ when $r + s + t \leq 9$ is odd. From Corollary 1, Proposition 1 and equation (5), we can deduce explicit formulas for $R(r, s, t)$, $S(r, s, t)$ and $T(r, s, t)$ when $r + s + t$ is odd. In particular, we have the following new results:

$$R(1, 1, 1) = -\frac{5}{8}\pi^2\zeta(3), \quad R(1, 1, 3) = \frac{5}{16}\pi^2\zeta(3) - \frac{27}{32}\zeta(5),$$
$$R(1, 2, 2) = \frac{5}{48}\pi^2\zeta(3) - \frac{3}{2}\zeta(5), \quad R(1, 3, 1) = \frac{1}{12}\pi^2\zeta(3) - \frac{59}{32}\zeta(5),$$
$$R(2, 1, 2) = -\frac{5}{16}\pi^2\zeta(3) + \frac{107}{32}\zeta(5), \quad R(2, 2, 1) = -\frac{5}{24}\pi^2\zeta(3) + \frac{59}{32}\zeta(5),$$
$$R(3, 1, 1) = \frac{1}{8}\pi^2\zeta(3) - \frac{59}{32}\zeta(5).$$

$$S(5, 5, 5) = \frac{7}{73728}\pi^4\zeta(11) + \frac{35}{24576}\pi^2\zeta(13) + \frac{63}{8192}\zeta(15),$$
$$S(7, 7, 7) = \frac{31}{35389440}\pi^6\zeta(15) + \frac{49}{1960608}\pi^4\zeta(17) + \frac{77}{262144}\pi^2\zeta(19) + \frac{429}{262144}\zeta(21).$$

The values of $R(5, 5, 5)$, $R(7, 7, 7)$ and $R(9, 9, 9)$ listed in [34] appear to be incorrect. They should be

$$R(5, 5, 5) = \frac{16375}{147456}\pi^4\zeta(11) + \frac{573335}{49152}\pi^2\zeta(13) - \frac{2064195}{16384}\zeta(15),$$
$$R(7, 7, 7) = \frac{1048543}{70778880}\pi^6\zeta(15) + \frac{7339969}{3932160}\pi^4\zeta(17) + \frac{80740121}{524288}\pi^2\zeta(19) - \frac{899676921}{524288}\zeta(21),$$
$$R(9, 9, 9) = \frac{13421747}{7046430720}\pi^8\zeta(19) + \frac{738197141}{2113929216}\pi^6\zeta(21) + \frac{1919313253}{67108864}\pi^4\zeta(23) + \frac{143948506845}{1631416447375}\pi^2\zeta(25) - \frac{1631416447375}{67108864}\zeta(27).$$

References

[1] B. Berndt, Ramanujan’s Notebooks Part I, Springer, New York, 1985. [MR 0781125] (86c:01062)
[2] D. Borwein, J. M. Borwein, and D. M. Bradley, Parametric Euler sum identities, J. Math. Anal. Appl., 316 (2006), no. 1, 328–338. doi: 10.1016/j.jmaa.2005.04.040 [MR 2201764] (2007b:11132)
http://arxiv.org/abs/math.CA/0505058
[3] J. M. Borwein and D. M. Bradley, Thirty-two Goldbach variations, Internat. J. Number Theory, 2 (2006), no. 1, 65–103. doi: 10.1142/S1793042106000383 [MR 2217795] (2007e:11109)
[4] J. M. Borwein, D. M. Bradley and D. J. Broadhurst, Evaluations of $k$-fold Euler/Zagier sums: a compendium of results for arbitrary $k$, *Electron. J. Combin.*, 4 (1997), no. 2, #R5. Wilf Festschrift. [MR 1444152] (98b:11091)

[5] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Combinatorial aspects of multiple zeta values, *Electron. J. Combin.*, 5 (1998), no. 1, #R38. [MR 1637378] (99g:11100)

[6] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Combinatorial aspects of multiple zeta values, *Electron. J. Combin.*, 5 (1998), no. 1, #R38. [MR 1637378] (99g:11100)

http://arXiv.org/abs/math.NT/9812020

[7] D. Bowman and D. M. Bradley, Multiple polylogarithms: a brief survey, *Proceedings of a Conference on $q$-Series with Applications to Combinatorics, Number Theory and Physics* (B. C. Berndt and K. Ono eds., Urbana, IL, 2000), Contemporary Math., 291, Amer. Math. Soc., Providence, RI, 2001, pp. 71–92. [MR 1874522] (2003c:33021)

http://arXiv.org/abs/math.CA/0310062

[8] D. Bowman, D. M. Bradley, and J. Ryoo, Some multi-set inclusions associated with shuffle convolutions and multiple zeta values, *European J. Combin.*, 24 (2003), no. 1, 121–127. [MR 1957970] (2004c:11115)

[9] D. M. Bradley, Partition identities for the multiple zeta function, *Zeta Functions, Topology, and Quantum Physics*, Developments in Mathematics, 14, T. Aoki, S. Kanemitsu, M. Nakahara, Y. Ohno (eds.) Springer-Verlag, New York, 2005, pp. 19–29. ISBN: 0-387-24972-9. [MR 2179270] (2006f:11105)

http://arXiv.org/abs/math.CO/0402091

[10] D. Bowman, D. M. Bradley, and J. Ryoo, The algebra and combinatorics of shuffles and multiple zeta values, *J. Combin. Theory, Ser. A*, 97 (2002), no. 1, 43–61. [MR 1879045] (2003j:05010)

http://arXiv.org/abs/math.CO/0310082

[11] D. Bowman, D. M. Bradley, J. Li, and J. Ryoo, Resolution of some open problems concerning multiple zeta evaluations of arbitrary depth, *Compositio Math.*, 139 (2003), no. 1, 85–100. doi: 10.1023/B:COMP:0000005036.52387.da [MR 2024966] (2005f:1106)

http://arXiv.org/abs/math.CO/0310061

[12] D. Bowman, D. M. Bradley, and J. Yang, Multiple $q$-zeta values, *J. Algebra*, 283 (2005), no. 2, 752–798. doi: 10.1016/j.jalgebra.2004.09.017 [MR 2111222] (2006f:11106)

http://arXiv.org/abs/math.QA/0402093

[13] D. Bowman, D. M. Bradley, and J. Yang, Duality for finite multiple harmonic $q$-series, *Discrete Math.*, 300 (2005), no. 1–3, 44–56. doi: 10.1016/j.disc.2005.06.008 [MR 2170113] (2006m:05019)

http://arXiv.org/abs/math.CO/0402092

[14] D. Bowman, D. M. Bradley, and J. Yang, On the sum formula for multiple $q$-zeta values, *Rocky Mountain J. Math.*, to appear.

http://arXiv.org/abs/math.QA/0411274

[15] D. M. Bradley, A $q$-analogue of Euler’s decomposition formula for the double zeta function, *Internat. J. Math. Math. Sci.*, 2005 (2005), no. 21, 3453–3458. doi:10.1155/IJMMS.2005.3453 [MR 2206867] (2006k:11174)

http://arXiv.org/abs/math.NT/0502002

[16] D. J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, *Phys. Lett. B*, 393 (1997) no. 3-4, 403–412. [MR 1435933] (98g:11011)

[17] F. C. Brown, Périodes des espaces des modules $\mathcal{M}_{0,n}$ et valeurs zêtas multiples [Multiple zeta values and periods of the moduli spaces $\mathcal{M}_{0,n}$] *C. R. Math. Acad. Sci. Paris*, 342 (2006), no. 12, 949–954. [MR 2235616]

[18] P. Cartier, Fonctions polylogarithmes, nombres polyzêtas et groupes pro-unipotents, *Astérisque*, 282 (2002), viii, 137–173, Séminaire Bourbaki, 53ème année, 2000–2001, Exp. No. 885. [MR 1975178] (2004i:11101)

[19] P. Cartier, Values of the $\zeta$-function, *Surveys in Modern Mathematics*, 260–273, London Math. Soc. Lecture Note Ser., 321, Cambridge Univ. Press, Cambridge, 2005. [MR 2166932] (2006g:11182)
[20] L. Euler, *Meditationes Circa Singulare Serierum Genus*, Novi Comm. Acad. Sci. Petropol., 20 (1775), 140–186. Reprinted in “Opera Omnia,” ser. I, 15, B. G. Teubner, Berlin (1927), 217–267.

[21] ______, *Briefwechsel*, vol. 1, Birhäuser, Basel, 1975. [MR 0497632]

[22] L. Euler and C. Goldbach, *Briefwechsel* 1729–1764, Akademie-Verlag, Berlin, 1965.

[23] P. Flajolet and B. Salvy, Euler sums and contour integral representations, *Experiment. Math.*, 7 (1998), no. 1, 15–35. [MR 1618286] (99c:11110)

[24] M. E. Hoffman, Periods of mirrors and multiple zeta values, *Proc. Amer. Math. Soc.*, 130 (2002), no. 4, 971–974. [MR 1873769] (2002k:14068)

[25] J. G. Huard, K. S. Williams and N. Y. Zhang, On Tornheim’s double series, Acta Arith., 75 (1996), no. 2, 105–117. [MR 1379394] (97f:11073)

[26] K. Ihara and T. Taka- muki, The quantum $g_2$ invariant and relations of multiple zeta values, *J. Knot Theory Ramifications*, 10 (2001), no. 7, 983–997. [MR 1867104] (2002m:57016)

[27] T. Q. T. Le and J. Murakami, Kontsevich’s integral for the Homfly polynomial and relations between values of multiple zeta functions, *Topology Appl.*, 62 (1995), no. 2, 193–206. [MR 1320252] (96c:57017)

[28] M. V. Subbarao and R. Sitaramachandra Rao, On some infinite series of L. J. Mordell and their analogues, *Pacific J. Math.*, 119 (1985), no. 1, 245–255. [MR 0797027] (87c:11091)

[29] T. Terasoma, Selberg integrals and multiple zeta values, *Compositio Math.*, 133 (2002), no. 1, 1–24. [MR 1918286] (2003k:11142)

[30] ______, Mixed Tate motives and multiple zeta values, *Invent. Math.*, 149 (2002), no. 2, 339–369. [MR 1918675] (2003h:11073)

[31] ______, Period integrals and multiple zeta values, *Sūgaku*, 57 (2005), no. 3, 255–266. [MR 2163671] (2006h:11110)

[32] L. Tornheim, Harmonic double series, *Amer. J. Math.*, 72 (1950), 303–314. [MR 0034860] (11,654a)

[33] H. Tsumura, On some combinatorial relations for Tornheim’s double series, Acta Arith., 105 (2002), no. 3, 239–252. [MR 1931792] (2003k:11134)

[34] ______, On alternating analogues of Tornheim’s double series, *Proc. Amer. Math. Soc.*, 131 (2003), no. 12, 3633–3641. [MR 1998168] (2004e:11102)

[35] ______, Evaluation formulas for Tornheim’s type of alternating double series, *Math. Comp.*, 73 (2004), no. 245, 251–258. [MR 2034120] (2005d:11137)

[36] M. Waldschmidt, Valeurs zêta multiples: une introduction, *J. Théor. Nombres Bordeaux*, 12 (2000), no. 2, 581–595. [MR 1823204] (2002a:11106)

[37] ______, Multiple polylogarithms: an introduction, in *Number Theory and Discrete Mathematics*, (Chandigarh, 2000), Trends Math., Birkhäuser, Basel, 2002, pp. 1–12. [MR 1952273] (2004d:33003)

[38] V. V. Zudilin, Algebraic relations for multiple zeta values (Russian), *Uspekhi Mat. Nauk*, 58 (2003), no. 1, 3–32; translation in *Russian Math. Surveys*, 58 (2003), vol. 1, 1–29. [MR 1992130] (2004k:11150)
