On systems of nonlinear equations

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Abstract

In this paper, we introduce an iterative numerical method to solve systems of nonlinear equations. The third-order convergence of this method is analyzed. Several examples are given to illustrate the efficiency of the proposed method.

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Keywords: Systems of nonlinear equations; Newton’s method; Third-order convergence.

1 Introduction

Let us consider the problem of finding a real zero of the nonlinear system \( F(x) = 0 \) which \( F : \mathbb{R}^n \rightarrow \mathbb{R}^n \). As notation throughout this paper, \( \alpha \in \mathbb{R}^n \) will denote the true solution of the nonlinear system \( F(x) = 0 \). More precisely Newton’s method may has used as the approximation of the following indefinite integral, arising from Newton’s theorem \([1]\),

\[
f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt,
\]

for nonlinear equation \( f(x) = 0 \). Noor\([2]\) by using the combination of midpoint quadrature rule and Trapezoidal rule for integral \((1)\) has introduced following iterative process for solving \( f(x) = 0 \),

\[
x_{n+1} = x_n - \frac{4f(x_n)}{f'(x_n) + 2f'(\frac{x_n + y_n}{2}) + f'(y_n)},
\]

where

\[
y_n = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

Now, corresponding to \((1)\), for nonlinear system \( F(x) = 0 \) is written, Ortega\([3, 4]\):

\[
F(x) = F(x_n) + \int_{x_n}^{x} F'(t)dt,
\]
then we can extend the discussion to solve system of nonlinear equations $F(x) = 0$, so similar to (2), the following iterative process for solving $F(x) = 0$ is obtain as,

$$x_{n+1} = x_n - 4 \left[ F'(x_n) + 2F'\left(\frac{x_n + y_n}{2}\right) + F'(y_n) \right]^{-1} F(x_n), \quad n = 0, 1, \ldots,$$

where

$$y_n = x_n - F'(x_n)^{-1} F(x_n).$$

where $F'(x_n)^{-1}$ is the Jacobian Matrix of the function $F$ evaluated in $x_k$. we call this iterative process Midpoint-Trapezoidal Newton’s method (MTN). In this paper, we analyze (MTN) in details and prove its third-order convergent theorem. Also, we have comparisons with some other variants of Newton’s method by numerical examples.

2 Description of the methods

Let $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be sufficiently differentiable function and $\alpha$ be a zero of the system of nonlinear equations $F(x) = 0$. From (2) as following

$$F(x) = F(x_n) + \int_{x_n}^{x} F'(t)dt,$$

we saw in the previous section that by using rectangular rule for above integral, classical Newton’s method (CN) is obtained as following

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n),$$

where $F'(x)^{-1}$ is the Jacobian matrix of the function $F$ evaluated in $x$. If an estimation of (3) is made by means of the trapezoidal rule and $x = \alpha$ is taken, then

$$0 \approx F(x_n) + \frac{1}{2}[F(x_n) + F(\alpha)](\alpha - x_n),$$

is obtained and a new approximation $x_{n+1}$ of $\alpha$ is given by

$$x_{n+1} = x_n - 2[F'(x_n) + F'(x_{n+1})]^{-1} F(x_n).$$

For solving of the implicit form problem that this equation involve, we use the $(n + 1)$th approximation of Newton method in right side,

$$x_{n+1} = x_n - 2[F'(x_n) + F'(y_n)]^{-1} F(x_n), \quad n = 0, 1, \ldots,$$

where

$$y_n = x_n - F'(x_n)^{-1} F(x_n).$$
This iterative method will be called \textit{Trapezoidal Newton’s method} (TN). By using harmonic mean in (3) and \( x = \alpha \) is taken, we have

\[ 0 \approx F(x_n) + \frac{2F(\alpha)F(x_n)}{F(x_n) + F(\alpha)}(\alpha - x_n), \]

and the following iterative approximation is obtained

\[ x_{n+1} = x_n - \frac{1}{2}F(y_n)^{-1}F(x_n)^{-1}[F(x_n) + F(y_n)]F(x_n), \quad n = 0, 1, \ldots, \]

where

\[ y_n = x_n - F'(x_n)^{-1}F(x_n), \]

this variant of Newton’s method is called \textit{Harmonic Newton’s method} (HN). If the midpoint rule is used to estimate integral (3) and \( x = \alpha \) is taken, it is obtained one

\[ 0 \approx F(x_n) + F\left(\frac{x_n + \alpha}{2}\right)(\alpha - x_n), \]

then, by a approximation \( x_{n+1} \) of \( \alpha \),

\[ x_{n+1} = x_n - F'\left(\frac{x_n + y_n}{2}\right)^{-1}F(x_n), \]

so, an alternative of Newton’s method is obtained as following

\[ x_{n+1} = x_n - F'\left(\frac{x_n + y_n}{2}\right)^{-1}F(x_n), \quad n = 0, 1, \ldots, \]

where

\[ y_n = x_n - F'(x_n)^{-1}F(x_n), \]

this variant of Newton’s method is called \textit{Midpoint Newton’s method} (MN).

Now, if the integral (3) is estimated using the combination of midpoint quadrature rule and Trapezoidal rule and by considering \( x = \alpha \), we have

\[ 0 \approx F(x_n) + \frac{1}{4}\left[F'(x_n) + 2F\left(\frac{x_n + \alpha}{2}\right) + F'(\alpha)\right](\alpha - x_n), \]

so, a new approximation \( x_{n+1} \) of \( \alpha \) is concluded as following:

\[ x_{n+1} = x_n - 4\left[F'(x_n) + 2F\left(\frac{x_n + \alpha}{2}\right) + F'(\alpha)\right]^{-1}F(x_n), \]

by using again the \((n+1)\)th iteration of Newton’s method in the right side of this equation, the implicit problem is avoided. Then

\[ x_{n+1} = x_n - 4\left[F'(x_n) + 2F\left(\frac{x_n + y_n}{2}\right) + F'(y_n)\right]^{-1}F(x_n), \quad n = 0, 1, \ldots, \quad (4) \]

is deduced, where

\[ y_n = x_n - F'(x_n)^{-1}F(x_n). \]

This iterative process is called \textit{Midpoint-Trapezoidal Newton’s method} (MTN).

In the next section we prove that, (MTN) has third-order convergence. The convergence of the other variants of Newton’s methods can be proved analogously.
3 Main result

In this section the third-order convergence of Midpoint-Trapezoidal Newton’s method (MTN) is proven by following theorem.

**Theorem 3.1** Let $F : \Omega \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n$, is $k$-times Fréchet differentiable in convex $\Omega$ containing the root $\alpha$ of $F(x) = 0$. The Midpoint-Trapezoidal Newton’s method has third-order convergence.

**Proof:** The Taylor’s expansion for any $x, x_n \in \Omega$, \cite{1}:

$$F(x) = F(x_n) + F'(x_n)(x - x_n) + \frac{1}{2!} F''(x_n)(x - x_n)^2 + \frac{1}{3!} F^{(3)}(x_n)(x - x_n)^3 + \cdots + \frac{1}{k!} F^{(k)}(x_n)(x - x_n)^k + \cdots,$$

with $x = \alpha$ and defining $e_n = x_n - \alpha$ we have:

$$F(\alpha) = F(x_n) + F'(x_n)(\alpha - x_n) + \frac{1}{2!} F''(x_n)(\alpha - x_n)^2 + \frac{1}{3!} F^{(3)}(\alpha - x_n)^3 + \cdots + \frac{1}{k!} F^{(k)}(\alpha - x_n)^k + \cdots,$$

$$= F(x_n) - F'(x_n)e_n + \frac{1}{2!} F''(x_n)(\alpha - x_n)^2 - \frac{1}{3!} F^{(3)}(\alpha - x_n)^3 + \cdots + (-1)^k \frac{1}{k!} F^{(k)}(x_n)e_n^k + \cdots.$$

For $k = 3$ and from $F(\alpha) = 0$ we have:

$$F(x_n) = F'(x_n)e_n - \frac{1}{2!} F''(x_n)e_n^2 + \frac{1}{3!} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4). \quad (5)$$

From (5) we can write the product $F'(x_n)^{-1}F(x_n)$ as following:

$$F'(x_n)^{-1}F(x_n) = F'(x_n)^{-1} \left( F'(x_n)e_n - \frac{1}{2} F''(x_n)e_n^2 + O(\|e_n\|^3) \right),$$

or

$$F'(x_n)^{-1}F(x_n) = e_n - \frac{1}{2} F'(x_n)^{-1} F''(x_n)e_n^2 + O(\|e_n\|^3) \quad (6)$$

From iterative process of (MTN) \cite{1} we have:

$$\left[ F'(x_n) + 2F'(\frac{x_n + y_n}{2}) + F'(y_n) \right] e_{n+1} = \left[ F'(x_n) + 2F'(\frac{x_n + y_n}{2}) + F'(y_n) \right] e_n - 4F(x_n). \quad (7)$$

To continue we need the Taylor’s expansion of $F'(x_n - \theta F'(x_n)^{-1}F(x_n))e_n$ as following:

$$F'(x_n - \theta F'(x_n)^{-1}F(x_n))e_n = F'(x_n)e_n - \theta F''(x_n)F'(x_n)^{-1}F(x_n)e_n + \frac{1}{2}\theta^2 F^{(3)}(x_n)(F'(x_n)^{-1}F(x_n))^2e_n + O(\|e_n\|^4)$$

by using (6) in above equation, we can write:

$$F'(x_n - \theta F'(x_n)^{-1}F(x_n))e_n = F'(x_n)e_n - \theta F''(x_n) \left( e_n - \frac{1}{2} F'(x_n)^{-1} F''(x_n)e_n^2 + O(\|e_n\|^3) \right) e_n$$

$$+ \frac{1}{2}\theta^2 F^{(3)}(x_n) \left( e_n - \frac{1}{2} F'(x_n)^{-1} F''(x_n)e_n^2 + O(\|e_n\|^3) \right)^2 e_n + O(\|e_n\|^4)$$

$$= F'(x_n)e_n - \theta F''(x_n)e_n + \frac{1}{2}\theta^2 F^{(3)}(x_n)e_n + O(\|e_n\|^4)$$

$$= F'(x_n)e_n - \theta F''(x_n)e_n + \frac{1}{2}\theta^2 F^{(3)}(x_n)e_n + O(\|e_n\|^4)$$
and for this prove that the order of convergence is three, then the proof is complete.

\[
F'(x_n - \theta F'(x_n)^{-1} F(x_n))e_n
= F'(x_n)^{-1}e_n - \theta F''(x_n)e_n^2 + \frac{\theta^2}{2} F''(x_n)F'(x_n)^{-1} F''(x_n)e_n^3
+ \frac{\theta^2}{2} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4),
\]

(8)

using (8) for \(\theta = \frac{1}{2}\) and \(\theta = 1\), it is obtained, respectively:

\[
F'(\frac{x_n+y_n}{2}) = F'(x_n - \frac{1}{2} F'(x_n)^{-1} F(x_n))e_n
= F'(x_n)e_n - \frac{1}{2} F''(x_n)e_n^2 + \frac{1}{4} F''(x_n)F'(x_n)^{-1} F''(x_n)e_n^3
+ \frac{1}{8} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4),
\]

(9)

and (for \(\theta = 1\))

\[
F'(y_n) = F'(x_n - F'(x_n) F(x_n))e_n = F'(x_n)e_n - F''(x_n)e_n^2
+ \frac{1}{2} F''(x_n)F'(x_n)^{-1} F''(x_n)e_n^3 + \frac{1}{2} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4).
\]

(10)

by using (5), (9) and (10), we can write right hand of Eq. (7) as following:

\[
\left( F'(x_n) + 2F'(\frac{x_n+y_n}{2}) + F'(y_n) \right)e_n - 4F(x_n)
= \left\{ F'(x_n)e_n + 2 \left( F'(x_n)e_n - \frac{1}{2} F''(x_n)e_n^2 
\quad + \frac{1}{4} F''(x_n)F'(x_n)^{-1} F''(x_n)e_n^3 
\quad + \frac{1}{8} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4) \right) \right\}
+ \left\{ F'(x_n)e_n - F''(x_n)e_n^2 + \frac{1}{2} F''(x_n)F'(x_n)^{-1} F''(x_n)e_n^3 
\quad + \frac{1}{2} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4) \right\} - 4 \left\{ F'(x_n)e_n - \frac{1}{2} F''(x_n)e_n^2 + \frac{1}{2} F^{(3)}(x_n)e_n^3 \right\} 
= F''(x_n)F'(x_n)^{-1} F''(x_n)e_n^3 - \frac{1}{24} F^{(3)}(x_n)e_n^3 + O(\|e_n\|^4)
= \left( F''(x_n)F'(x_n)^{-1} F''(x_n) - \frac{1}{24} F^{(3)}(x_n) \right)e_n^3 + O(\|e_n\|^4)
\]

so using deduced result, from Eq. (7) we obtain:

\[
\left( F'(x_n) + 2F'(\frac{x_n+y_n}{2}) + F'(y_n) \right)e_{n+1}
= \left( F''(x_n)F'(x_n)^{-1} F''(x_n) - \frac{1}{24} F^{(3)}(x_n) \right)e_n^3 + O(\|e_n\|^4).
\]

this prove that the order of convergence is three, then the proof is complete. \(\square\)

## 4 Numerical examples

In this section we will check the effectiveness of MTN (1) and other iterative methods in section 2. All computations are done by using mathematica, stopping criteria
\[ |x_{n+1} - x_n| + |F(x_n)| \leq \epsilon \] is used for computer programs. We use \( \epsilon \leq 10^{-14} \).

\[
(a) \begin{cases} \quad 4 - x_2 + x_1 \cos(x_2) = 0 \\ x_1 + x_2 - 1 = 0 \end{cases}
\]

\[
(b) \begin{cases} \quad 2x_1^2 - x_1x_2 - 5x_1 + 1 = 0 \\ x_1\tan(x_1^2 + x_2) = 0 \end{cases}
\]

\[
(c) \begin{cases} \quad x_1 + 2x_2 - 3 = 0 \\ 2x_1^2 + x_2^2 - 5 = 0 \end{cases}
\]

\[
(d) \begin{cases} \quad \ln(x_1^2) - 2\ln(x_1) - 2 = 0 \\ x_1\tan(x_1^2 + x_2) = 0 \end{cases}
\]

\[
(e) \begin{cases} \quad x_1 + x_2 - \cos(x_2) = 0 \\ 3x_1 - x_2 - \sin(x_2) = 0 \end{cases}
\]

\[
(f) \begin{cases} \quad x_1^2 + x_2^2 + x_3^2 = 9 \\ x_1x_2x_3 - 1 = 0 \\ x_1 + x_2 - x_3^2 = 0 \end{cases}
\]

\[
(g) \begin{cases} \quad \cos(x_2) - \sin(x_1) = 0 \\ (x_3)^{x_1} - \frac{1}{x_2} = 0 \\ e^{x_1} - x_2^2 = 0 \end{cases}
\]

\[
(h) \begin{cases} \quad x_2x_3 + x_4(x_2 + x_3) = 0 \\ x_1x_3 + x_4(x_1 + x_3) = 0 \\ x_1x_2 + x_4(x_1 + x_2) = 0 \\ x_1x_2 + x_1x_3 + x_2x_3 = 1 \end{cases}
\]

Approximations of \( x_i \)'s for examples (a)-(e).

| \( F(x) \) | \( x_0 \) | Method | Approximated solution | Iteration | Error estimation |
|------------|----------|--------|-----------------------|-----------|-----------------|
| \( (a) \) | \( (1, 2) \) | \( CN \) | \((-1.00000000, 1.00000000)\) | 9 | \( 9.33 \times 10^{-15} \) |
| \( \quad \) | \( \) | \( TN \) | \((-3.00000000, 1.00000000)\) | 5 | \( 9.10 \times 10^{-14} \) |
| \( \quad \) | \( \) | \( MN \) | \((-5.00000000, 1.00000000)\) | 6 | \( 3.55 \times 10^{-15} \) |
| \( \quad \) | \( \) | \( HN \) | \((-4.00000000, 1.00000000)\) | 5 | \( 1.64 \times 10^{-14} \) |
| \( \quad \) | \( \) | \( TMN \) | \((-6.00000000, 1.00000000)\) | 5 | \( 1.64 \times 10^{-14} \) |
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Approximations of $x_i$s for examples (f)-(h).

### Table 2

| $F(x)$ | $x_0$       | Method | Approximated solution                  | Iteration | Error estimation |
|--------|-------------|--------|----------------------------------------|-----------|-----------------|
| (f)    | (2, 2, 0.5) | $CN$   | $(-2.090295, 2.140258, -0.223525)$     | 8         | $8.88 \times 10^{-16}$ |
|        |             | $TN$   | $(-2.090295, 2.140258, -0.223525)$     | 5         | $8.88 \times 10^{-16}$ |
|        |             | $MN$   | $(-2.090295, 2.140258, -0.223525)$     | 5         | $9.02 \times 10^{-16}$ |
|        |             | $HN$   | $(-2.090295, 2.140258, -0.223525)$     | 6         | $1.78 \times 10^{-15}$ |
|        |             | $MTN$  | $(-2.090295, 2.140258, -0.223525)$     | 5         | $9.02 \times 10^{-16}$ |
| (g)    | (−2.5, 1, 1)| $CN$   | $(0.909569, 0.661227, 1.575834)$       | 10        | $6.82 \times 10^{-14}$ |
|        |             | $TN$   | No convergence                        |           |                 |
|        |             | $MN$   | $(0.909569, 0.661227, 1.575834)$       | 5         | $8.48 \times 10^{-14}$ |
|        |             | $HN$   | No convergence                        |           |                 |
|        |             | $MTN$  | No convergence                        |           |                 |
| (h)    | (0.5, 0.5, 0.5, 0.2)| $CN$   | $(0.5773, 0.5773, 0.5773, -0.2886)$ | 5         | $2.22 \times 10^{-16}$ |
|        |             | $TN$   | $(0.5773, 0.5773, 0.5773, -0.2886)$ | 4         | $1.11 \times 10^{-16}$ |
|        |             | $MN$   | $(0.5773, 0.5773, 0.5773, -0.2886)$ | 4         | $1.11 \times 10^{-16}$ |
|        |             | $HN$   | $(0.5773, 0.5773, 0.5773, -0.2886)$ | 6         | $1.31 \times 10^{-13}$ |
|        |             | $MTN$  | $(0.5773, 0.5773, 0.5773, -0.2886)$ | 4         | $1.11 \times 10^{-16}$ |

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