PATH INTEGRAL AND PSEUDOCCLASSICAL ACTION FOR SPINNING
PARTICLE IN EXTERNAL ELECTROMAGNETIC AND TORSION FIELDS

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Abstract. Starting from the Dirac equation in external electromagnetic and torsion
fields we derive a path integral representation for the corresponding propagator. An effective
action, which appears in the representation, is interpreted as a pseudoclassical action for a
spinning particle. It is just a generalization of Berezin-Marinov action to the background un-
der consideration. Pseudoclassical equations of motion in the nonrelativistic limit reproduce
exactly the classical limit of the Pauli quantum mechanics in the same case. Quantization
of the action appears to be nontrivial due to an ordering problem, which needs to be solved
to construct operators of first-class constraints, and to select the physical sector. Finally the
quantization reproduces the Dirac equation in the given background and, thus, justifies the
interpretation of the action.
1 Introduction

Introducing of torsion is usually regarded as a most natural way to extend the description of the space-time in General Relativity. Torsion appears naturally as a compensating field for local gauge transformations [1] (see [2, 3] for a review of this approach and further references) and also in the effective low-energy gravity induced by quantum effects of (super)strings. The main advantage of theories with torsion is that the spin of matter particles (fields), along with energy and momentum, becomes the source of gravity. This is the reason why so much attention has been paid to the interaction between torsion and fermion fields [1, 2, 3, 4]. When investigating the quantum effects together with torsion, the latter may be considered as a classical background for the quantum matter fields. Alternatively we have to think about the propagating torsion which should be subject of quantization. The dynamical torsion, despite it produces an interesting phenomenological consequences [10], meets serious theoretical obstacles [11] which put very high lower bound for the torsion mass. Indeed there remains the possibility to treat torsion as purely background field (maybe not elementary) which is, by definition, not an object for the quantization. The formulation of a renormalizable theory of matter fields in an external gravitational field with torsion has been given in [12] and some physical [16] and cosmological [13] applications, has been studied.

Important information about the spin torsion interaction provides spinning particle propagation in the torsion field. There are many works devoted to the problem. For example, using the results of [12], the Pauli equation in external electromagnetic and torsion fields has been derived in [4, 5, 6] and the corresponding equations of motion for the non-relativistic particle were obtained. These equations show an unusual behavior of the spinning particle in an external torsion field [15]. In [7, 8] actions of a massless spinning particle in an external torsion field has been obtained on the basis of supersymmetry considerations.

In the present work, we are going to construct a path integral representation for a propagator of a massive spinning particle in external electromagnetic and torsion fields. To this end we follow the ideas and technics of the papers [17, 18]. In particular, it was demonstrated there that a special kind of path integral representations for propagators of relativistic particles allow one to derive, in a sense, a form for gauge invariant classical (pseudoclassical) actions for the corresponding particles. In such a way, an action for spinning particles with anomalous magnetic moments, and also actions for spinning particles in odd- dimensional

4Earlier some particular form of the Pauli equation has been used in [15]. After [9], Pauli equation with torsion has been derived in Ref.’s [10].
space-time, have been derived for the first time \cite{24, 18}. Thus, besides the path integral representation for the Dirac propagator, we get here a possibility to construct and to study a gauge invariant pseudoclassical model for spinning particle in torsion field. One ought to mention some relevant to the problem under consideration works \cite{19, 20, 21}. In the first one some formal path integral representations for massive spinning particle in the presence of the torsion field was derived using so called perturbative approach to path integrals, developed in \cite{22}. Such an approach does not take into account possible boundary conditions for trajectories of integration, which is the case in the problem under consideration. Besides of that, effective actions in such a representation \cite{19} appeared in a non-symmetric form due to special gauges selected. In two second articles an index theorem for the Dirac operator in the presence of general gravitational background (including torsion) was studied, in particular a path integral representation for the index of the Dirac operator was derived.

The paper is organized in the following way. In the next section we present some considerations \cite{12}, which allow us to justify a form of Dirac and Klein-Gordon equations in external gravitation field with torsion. In Sect. III we introduce an equation for the Dirac propagator in electromagnetic and torsion fields. Using it, a path integral representation for the propagator is derived. Analyzing this representation, in Sect. IV, we propose a pseudoclassical action to describe a spinning particle in the above-mentioned background. It is a generalization of the Berezin-Marinov action \cite{24, 27} to the background under consideration. In a similar manner a classical action for a scalar particle can be derived. We study equations of motion and consider the nonrelativistic and classical limits. In Sect. V, doing a quantization of the action, we arrived at the corresponding Dirac equation. Results of two latter Sect. confirm the interpretation of the pseudoclassical action derived by us from the path integral representation. In the last Sect. VI we draw some conclusions.

2 Dirac and Klein-Gordon equations in an external gravitational field with torsion

Let us start with the basic notions of gravity with torsion. All our notations correspond to those of the book \cite{14}. The metric $g_{\mu\nu}$ and torsion $T^{\alpha}_{\beta\gamma}$ are independent characteristics of the space-time. When torsion is present, the covariant derivative $\tilde{\nabla}$ is based on the non-symmetric connection, namely $\tilde{\Gamma}^{\alpha}_{\beta\gamma} - \tilde{\Gamma}^{\alpha}_{\gamma\beta} = T^{\alpha}_{\beta\gamma}$. The torsion field $T^{\alpha}_{\beta\gamma}$ can be expressed
through its irreducible components as

\[ T_{\alpha\beta\mu} = \frac{1}{3} \left( T_\beta g_{\alpha\mu} - T_\mu g_{\alpha\beta} \right) - \frac{1}{6} \varepsilon_{\alpha\beta\mu\nu} S^\nu q_{\alpha\beta\mu}, \]  

where \( T_\beta = T^\alpha_{\beta \alpha} \) is the vector trace of torsion, \( S^\nu = \varepsilon^{\alpha\beta\mu\nu} T_{\alpha\beta\mu} \) is axial vector and the tensor \( q^\alpha_{\beta\gamma} \) satisfies two conditions \( q^\alpha_{\beta\alpha} = 0 \) and \( \varepsilon^{\alpha\beta\mu\nu} q_{\alpha\beta\mu} = 0 \).

The actions of the matter fields in an external gravitational field with torsion must be formulated in such a way that they lead to the consistent quantum theory. One can impose the principles of locality, general covariance and require the symmetries of the given theory (like gauge invariance for the QED or SM) in flat space-time to hold for the theory in curved space-time with torsion. Then the renormalizable field theory can be achieved through the introduction of some new non-minimal parameters of the matter-torsion interaction \( \eta_{1,2} \) and \( \xi_{1,5} \) \cite{12} (see also \cite{14}). For the GUT-like gauge theory of interacting fields with spin-0, \( \frac{1}{2} \) and 1, the full action with all these non-minimal parameters can be written as

\[
S = \int d^4x \sqrt{g} \left\{ -\frac{1}{4} \left( G_{\mu\nu}^a \right)^2 + \frac{1}{2} g^{\mu\nu} D_\mu \phi D_\nu \phi + \frac{1}{2} \left( \sum \xi_i P_i + M^2 \right) \phi^2 - V(\phi) + i \bar{\psi} \left( \gamma^\alpha D_\alpha + \sum \eta_j Q_j - im + h \phi \right) \psi \right\} + S_{\text{vacuum}},
\]

where \( D \) denotes the derivatives which are covariant with respect to both gravitational and gauge field but do not contain torsion. We accept that the vector fields do not couple with torsion even in a minimal way, in order to maintain the gauge invariance. Interaction of scalar fields with torsion is purely non-minimal. For one real scalar there are five possible non-minimal structures

\[
P_1 = R, \quad P_2 = \nabla_\alpha T^\alpha, \quad P_3 = T_\alpha T^\alpha, \quad P_4 = S_\alpha S^\alpha, \quad P_5 = q_{\alpha\beta\gamma} q^{\alpha\beta\gamma},
\]

and the non-minimal parameters \( \xi_1, ..., \xi_5 \). A more complicated scalar content may require additional non-minimal terms \cite{12}. For the Dirac spinors there are two possible non-minimal structures

\[
Q_1 = i \gamma^5 \gamma^\mu S_\mu, \quad Q_2 = i \gamma^\mu T_\mu
\]

and two non-minimal parameters \( \eta_1, \eta_2 \). The action of the minimal theory for the Dirac spinor field, is given by the spinor part of the expression (GUT) with \( \eta_1 = -1/8 \) and \( \eta_2 = 0 \) \cite{2, 14}.

Among all the non-minimal parameters, the ones related to the \( S_\mu \)-field are most essential for the renormalizability, which can be achieved by including \( \eta_1 \) and \( \xi_4 \) and \( \xi_1 \)-type structures.
into the classical action \([12]\). It is remarkable that not only spinors but also scalars have to interact with torsion if we want to have a renormalizable theory. Other parameters \(\eta_2, \xi_{2,3,5}\) are purely non-minimal. For this reason in the rest of this paper we will consider the torsion as purely antisymmetric and describe it by the axial vector \(S_\mu\). It is worth to mention that the string-induced torsion is completely antisymmetric as well.

Since torsion is metric-independent quantity, one can study the theory with torsion and the flat Minkowski metric. On the other hand, since we are going to consider torsion as purely background field, it can be always normalized in such a way that the non-minimal parameter \(\eta_1\) is set to unity. Therefore, the Dirac equation in external electromagnetic and torsion fields can be presented in the form:

\[
\left[ \gamma^\mu \left( \hat{P}_\mu - i\gamma^5 S_\mu \right) - m \right] \psi(x) = 0, 
\]

(3)

where \(\hat{P}_\nu = i\partial_\nu - qA_\nu(x)\).

The Klein-Gordon equation in external electromagnetic and torsion fields has the form

\[
\left[ \hat{\mathcal{D}}^2 + m^2 + \xi_4 S^2 \right] \varphi(x) = 0, 
\]

(4)

with an arbitrary non-minimal parameter \(\xi_4\).

3 Path integral representations for the propagators

In this section we are going to write down path integral representations for the propagators of spinning and spinless particles in the background under consideration, following the techniques of Refs. \([17, 18]\). In this relation one ought to mention the works \([23]\) where the propagators were presented as path integrals over bosonic and fermionic variables.

First we start with the case of a spinning particle and consider the causal Green function \(\Delta^c(x, y)\) of the equation (3), which is the propagator of the corresponding particle,

\[
\left[ \gamma^\mu \left( \hat{P}_\mu - i\gamma^5 S_\mu \right) - m \right] \Delta^c(x, y) = -\delta^4(x - y). 
\]

(5)

To get the result in supersymmetric form one needs to work with the transformed function \(\tilde{\Delta}^c(x, y) = \Delta^c(x, y)\gamma^5\), which obeys the equation

\[
\left[ \Gamma^\mu \left( \hat{P}_\mu - i\Gamma^4 S_\mu \right) - m\Gamma^4 \right] \tilde{\Delta}^c(x, y) = \delta^4(x - y), 
\]

(6)

where \(\Gamma^\mu = \gamma^\mu \gamma^5\).
where the set of five matrices $\Gamma^n$, $n = 0, 1, \ldots, 4$, forms a representation of the Clifford algebra in 5- dimensions,

$$\Gamma^\mu = \gamma^5 \gamma^\mu, \quad \Gamma^4 = \gamma^5, \quad \mu = 0, 1, 2, 3,$$

$$[\Gamma^n, \Gamma^m]_+ = 2 \eta^{nm}, \quad \eta_{nm} = \text{diag}(1, -1, -1, -1, -1). \quad (7)$$

Similar to Schwinger \[28\] we present $\tilde{\Delta}_{\alpha\beta}(x, y)$ as a matrix element of an operator $\tilde{\Delta}_{\alpha\beta}$, however, in contrast with the cited work, we do this in the coordinate space only,

$$\tilde{\Delta}_{\alpha\beta}(x, y) = \langle x | \Delta^c_{\alpha\beta} | y \rangle . \quad (8)$$

In (8) spinor indices are written for clarity explicitly and will be omitted hereafter; $|x\rangle$ are eigenvectors for some Hermitian operators of coordinates $X^\mu$, the corresponding canonically conjugated operators of momenta are $P_\mu$, so that:

$$X^\mu |x\rangle = x^\mu |x\rangle, \quad \langle x| y \rangle = \delta^4(x - y), \quad \int |x\rangle \langle x| dx = I,$$

$$[P_\mu, X^\nu]_- = -i \delta^\nu_\mu, \quad P_\mu |p\rangle = p_\mu |p\rangle, \quad \langle p| p' \rangle = \delta^4(p - p'),$$

$$\int |p\rangle \langle p| dp = I, \quad \langle x| P_\mu |y\rangle = -i \partial_\mu \delta^4(x - y), \quad \langle x| p \rangle = \frac{1}{(2\pi)^2} e^{ipx},$$

$$[\Pi_\mu, \Pi_\nu]_- = -iq F_{\mu\nu}(X), \quad \Pi_\mu = -P_\mu - q A_\mu(X), \quad F_{\mu\nu}(X) = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

The equation (6) implies the formal solution for the operator $\tilde{\Delta}^c$:

$$\tilde{\Delta}^c = \tilde{\mathcal{F}}^{-1}, \quad \tilde{\mathcal{F}} = \Pi_\mu \Gamma^\mu - m \Gamma^4 - i \Gamma^\mu \Gamma^4 S_\mu .$$

The operator $\tilde{\mathcal{F}}$ may be written in an equivalent form,

$$\tilde{\mathcal{F}} = \Pi_\mu \Gamma^\mu - m \Gamma^4 - i \frac{1}{6} \epsilon_{\mu\nu\alpha\beta} S^\mu \Gamma^\nu \Gamma^\alpha \Gamma^\beta , \quad (9)$$

using the following formula

$$\Gamma_\mu \Gamma^4 = \frac{1}{6} \epsilon_{\mu\nu\alpha\beta} \Gamma^\nu \Gamma^\alpha \Gamma^\beta , \quad \epsilon_{0123} = 1 .$$

That allows one to regard it as a Fermi operator, if one reckons $\Gamma$-matrices as Fermi operators. In general case an inverse operator to a Fermi operator $\tilde{\mathcal{F}}$ can be presented by means of an integral over a super-proper time $(\lambda, \chi)$ of an exponential with an even exponent \[L7\],

$$\tilde{\mathcal{F}}^{-1} = \int_0^\infty d\lambda \int e^{i(\lambda \tilde{\mathcal{F}}^2 + i\chi \tilde{\mathcal{F}})} d\chi ,$$
where $\lambda$ is an even and $\chi$ is an odd (Grassmann) variable, the latter anticommutes with $\hat{F}$ by definition. Calculating $\hat{F}^2$ in the case under consideration when $\hat{F}$ is given by (9), we find

$$
\hat{F}^2 = \Pi^2 - m^2 - S^2 - \frac{iq}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu + \hat{K}_{\mu\nu} \Gamma^\mu \Gamma^\nu + \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 ,
$$

where

$$
\hat{K}_{\mu\nu} = \frac{i}{2} \left[ [\Pi^\alpha , S^\beta] + \epsilon_{\alpha\beta\mu\nu} \right] , \quad \Pi^2 = P^2 + q^2 A^2 + q \left[ P_\mu , A^\mu \right] .
$$

Thus we get:

$$
\tilde{\Delta}^c = \int_0^\infty d\lambda \int e^{-i\hat{H}(\lambda, \chi)} d\chi ,
$$

where

$$
\hat{H}(\lambda, \chi) = \lambda \left( m^2 + S^2 - \Pi^2 + \frac{iq}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu - \hat{K}_{\mu\nu} \Gamma^\mu \Gamma^\nu - \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) - \chi \left( \Pi_\mu \Gamma^\mu - m \Gamma^4 \right) \frac{i}{6} \epsilon_{\alpha\mu\nu\alpha} S^\alpha \Gamma^\mu \Gamma^\nu \Gamma^\alpha .
$$

Then, the Green function $\tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}})$ takes the form:

$$
\tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = \lim_{N \to \infty} \int_0^\infty d\lambda \int \langle x_{\text{out}} | e^{-i\hat{H}(\lambda, \chi)} | x_{\text{in}} \rangle d\chi .
$$

Now we are going to represent the matrix element entering in the expression (12) by means of a path integral following a technique of Refs. [17, 18]. First we write, as usual, $e^{-i\hat{H}} = \left( e^{-i\hat{H}/N} \right)^N$ and then insert $(N - 1)$ resolutions of identity $\int |x\rangle \langle x| dx = I$ between all the operators $e^{-i\hat{H}/N}$. Besides, we introduce $N$ additional integrations over $\lambda$ and $\chi$ to transform then the ordinary integrals over these variables into the corresponding path-integrals:

$$
\tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = \lim_{N \to \infty} \int_0^\infty d\lambda \int \prod_{k=1}^N \langle x_k | e^{-i\hat{H}(\lambda_k, \chi_k)} \Delta \tau | x_{k-1} \rangle \times \delta(\lambda_k - \lambda_{k-1}) \delta(\chi_k - \chi_{k-1}) d\lambda_0 dx_1 \ldots dx_{N-1} d\lambda_1 \ldots d\lambda_N d\chi_1 \ldots d\chi_N ,
$$

where $\Delta \tau = 1/N$, $x_0 = x_{\text{in}}$, $x_N = x_{\text{out}}$. Bearing in mind the limiting process, we can, as usual, restrict ourselves to calculate the matrix elements from (13) approximately,

$$
\langle x_k | e^{-i\hat{H}(\lambda_k, \chi_k)} \Delta \tau | x_{k-1} \rangle \approx \langle x_k | 1 - i\hat{H}(\lambda_k, \chi_k) \Delta \tau | x_{k-1} \rangle .
$$

In this connection it is important to notice that the operator $\hat{H}(\lambda_k, \chi_k)$, by construction, is symmetric with respect to the operators $X$ and $P$. Thus, one can write

$$
\hat{H}(\lambda, \chi) = \text{Sym}_{(X, P)} H(\lambda, \chi, X, P) ,
$$
where $\mathcal{H}(\lambda, \chi, x, p)$ is the Weyl symbol of the operator $\hat{\mathcal{H}}(\lambda, \chi)$ in the sector of $x, p,$

$$\mathcal{H}(\lambda, \chi, x, p) = \lambda \left( m^2 + S^2 - \mathcal{P}^2 + \frac{iq}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu - K_{\mu\nu} \Gamma^\mu \Gamma^\nu \right) - \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) - \chi \left( \mathcal{P}_\mu \Gamma^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right), \tag{15}$$

and

$$\mathcal{P}_\nu = -p_\nu - qA_\nu(x), \quad K_{\mu\nu} = -\mathcal{P}^\alpha S^\beta \epsilon_{\alpha\beta\mu\nu} \tag{16}.$$ 

The matrix elements (14) are expressed in terms of the Weyl symbols at the middle point $\vec{x}_k = (x_k + x_{k-1})/2$, see [29]. Taking all that into account, one realizes that in the limiting process the matrix elements (14) can be replaced by the expressions

$$\int \frac{dp_k}{(2\pi)^4} \exp i \left[ p_k \frac{x_k - x_{k-1}}{\Delta \tau} - \mathcal{H}(\lambda_k, \chi_k, \vec{x}_k, p_k) \right] \Delta \tau. \tag{17}$$

Such expressions with different values of $k$ do not commute due to the $\Gamma$-matrix structure and, therefore, are to be situated in (13) in such a way that the numbers $k$ increase from the right to the left. For the two $\delta$-functions, accompanying each matrix element (14) in the expression (13), we use the integral representations

$$\delta(\lambda_k - \lambda_{k-1}) \delta(\chi_k - \chi_{k-1}) = \frac{i}{2\pi} \int e^{i[\pi_k(\lambda_k - \lambda_{k-1}) + \nu_k(\chi_k - \chi_{k-1})]} d\pi_k d\nu_k ,$$

where $\nu_k$ are odd variables. Then we attribute formally to the $\Gamma$-matrices, entering into (17), also an index $k,$ and at the same time we attribute to all quantities the “time” $\tau_k$ according the index $k$ they have, $\tau_k = k \Delta \tau$, so that $\tau \in [0, 1].$ Introducing the T-product, which acts on $\Gamma$-matrices, it is possible to gather all the expressions, entering in (13), in one exponent and deal then with the $\Gamma$-matrices like with odd variables. At equal times the $\Gamma$-matrices anticommute due to their contractions to complete antisymmetric objects. Taking into account all that, we get for the right side of (13):

$$\tilde{\Delta}^c(x_{\text{out}}, x_{\text{in}}) = T \int_0^{\infty} d\lambda_0 \int d\chi_0 \int_{x_{\text{in}}}^{x_{\text{out}}} D\lambda \int d\lambda \int_{x_{\text{in}}}^{x_{\text{out}}} D\chi \int D\pi \int D\nu \times \exp \left\{ i \int_0^1 \left[ \lambda \left( \mathcal{P}^2 - m^2 - S^2 - \frac{iq}{2} F_{\mu\nu} \Gamma^\mu \Gamma^\nu + K_{\mu\nu} \Gamma^\mu \Gamma^\nu + \partial_\mu S^\mu \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) \right. \right. \left. \left. + \chi \left( \mathcal{P}_\mu \Gamma^\mu - m \Gamma^0 \Gamma^1 \Gamma^2 \Gamma^3 \right) \right] d\tau \right\}, \tag{18}$$

where $x(\tau), p(\tau), \lambda(\tau), \pi(\tau), \nu(\tau)$ are even and $\chi(\tau), \nu(\tau)$ are odd trajectories, obeying the boundary conditions $x(0) = x_{\text{in}}, \ x(1) = x_{\text{out}}, \ \lambda(0) = \lambda_0, \ \chi(0) = \chi_0.$ The operation of $T$-ordering acts on the $\Gamma$-matrices which are supposed formally to depend on time $\tau.$ The
expression (18) can be transformed then as follows:

\[
\Delta^c(x_{out}, x_{in}) = \int_0^\infty d\lambda_0 \int d\chi_0 \int_0^\lambda d\lambda \int_0^{\lambda-x_{out}} D\chi \int_0^{x_{out}} Dx \int Dp \int D\pi \int D\nu \\
\times \exp \left\{ i \int_0^1 \left[ \lambda \left( P^2 - m^2 - S^2 - \frac{i q}{2} F_{\mu\nu} \frac{\delta_l}{\delta p_\mu} \frac{\delta_l}{\delta p_\nu} + K_{\mu\nu} \frac{\delta_l}{\delta \rho_\mu} \frac{\delta_l}{\delta \rho_\nu} + \partial_\mu S^\mu \frac{\delta_l}{\delta \rho_0} \frac{\delta_l}{\delta \rho_1} \right) + \frac{1}{6} \kappa_{\mu\nu\alpha} S^\alpha \frac{\delta_l}{\delta \rho_2} \frac{\delta_l}{\delta \rho_3} \right] + \lambda \left( P_{\mu} \frac{\delta_l}{\delta p_\mu} - m \frac{\delta_l}{\delta \rho_4} - \frac{i}{6} \kappa_{\mu\nu\alpha} S^\alpha \frac{\delta_l}{\delta \rho_2} \frac{\delta_l}{\delta \rho_3} \right) p\dot{x} + \pi\lambda + \nu\chi \right] \right\} \\
\times \exp \left[ i \int_0^1 \rho_n(\tau) \Gamma^n d\tau \right]_{\rho=0},
\]

where five odd sources \( \rho_n(\tau) \) are introduced. They anticommute with the \( \Gamma \)-matrices by definition. One can represent the quantity \( \exp \int_0^1 \rho_n(\tau) \Gamma^n d\tau \) via a path integral over odd trajectories [17, 30].

\[
\exp \int_0^1 \rho_n(\tau) \Gamma^n d\tau = \exp \left( i \frac{\partial_l}{\partial \theta^n} \int_{\psi(0)+\psi(1)=0} \exp \left[ \int_0^1 \left( \psi_n \psi_n^* - 2i \rho_n(\psi_n^* \psi_n) \right) d\tau \right] \right) \\
+ \psi_n(1) \psi_n(0) \mathcal{D}\psi_{|\theta=0}, \quad \mathcal{D}\psi = \int_{\psi(0)+\psi(1)=0} \exp \left[ \int_0^1 \psi_n \psi_n^* d\tau \right]^{-1} \mathcal{D}\psi \mathcal{D}\psi_{|\theta=0}, \quad (19)
\]

Here \( \theta^n \) are odd variables, anticommuting with the \( \Gamma \)-matrices, and \( \psi_n(\tau) \) are odd trajectories of integration, obeying the boundary conditions which are pointed out below the signs of integration. Using (19) we get the Hamiltonian path integral representation for the propagator in question:

\[
\Delta^c(x_{out}, x_{in}) = \exp \left( i \frac{\partial_l}{\partial \theta^n} \int_0^\infty d\lambda_0 \int d\chi_0 \int_0^\lambda d\lambda \int_0^{\lambda-x_{out}} D\chi \int_0^{x_{out}} Dx \int Dp \int D\pi \int D\nu \\
\times \int_{\psi(0)+\psi(1)=0} \mathcal{D}\psi \exp \left\{ i \int_0^1 \left[ \lambda \left( P_{\mu} + \frac{i}{\chi} \psi_n \psi_n^* + d_{\mu} \right) \right] - \lambda \left( m^2 + S^2 \right) \\
+ 2i\lambda q F_{\mu\nu} \psi_n^* \psi_n^\nu + 16\lambda \partial_\mu S^\mu \psi_n^0 \psi_1^1 \psi_2^2 \psi_3^3 + 2i \chi \left( m \psi_n^4 + \frac{2}{3} \psi_n^\mu d_{\mu} \right) \\
- i\psi_n \psi_n^* + p\dot{x} + \pi\lambda + \nu\chi \right] d\tau + \psi_n(1) \psi_n(0) \right\} \bigg|_{\theta=0}, \quad (20)
\]

where

\[
d_{\mu} = -2i\epsilon_{\mu\nu\alpha} S^n \psi_n^\alpha \psi^n. \]

Integrating over momenta, we get Lagrangian path integral representation for the propagator,

\[
\Delta^c(x_{out}, x_{in}) = \exp \left( i \frac{\partial_l}{\partial \theta^n} \int_0^\infty d\epsilon_0 \int d\chi_0 \int_0^{\epsilon_0} d\epsilon \int \mathcal{M}(e) De \int_0^{\epsilon_0} D\epsilon \int_0^{\epsilon_0} D\epsilon \int_0^{x_{out}} Dx \int D\pi \int D\nu \\
\times \int_{\psi(0)+\psi(1)=0} \mathcal{D}\psi \exp \left\{ i \int_0^1 \left[ \frac{z^2}{2\epsilon} - \frac{e}{2} M^2 - \dot{x}_\mu (q A^\mu - d^\mu) + i e F_{\mu\nu} \psi_n^* \psi_n^\nu \\
+ i \chi \left( m \psi_n^4 + \frac{2}{3} \psi_n^\mu d_{\mu} \right) - i\psi_n \psi_n^* + \pi \dot{e} + \nu\chi \right] d\tau + \psi_n(1) \psi_n(0) \right\} \bigg|_{\theta=0}, \quad (21)
\]
where the measure $\mathcal{M}(e)$ has the form:

$$
\mathcal{M}(e) = \int Dp \exp \left[ \frac{i}{2} \int_0^1 e p^2 d\tau \right],
$$

(22)

and

$$
M^2 = m^2 + S^2 - 16 \partial_{\mu} S^\mu \psi^0 \psi^1 \psi^2 \psi^3, \quad z^\mu = \dot{x}^\mu + i\chi \psi^\mu.
$$

The discussion of the role of the measure (22) can be found in [17].

Let us now pass to the case of a spinless particle. Here we need to consider a corresponding propagator $D^c$ which obeys the non-homogeneous Klein-Gordon equation (see (4)

$$
\left(\mathcal{P}^2 + m^2 + \xi_4 S^2\right) D^c(x, y) = -\delta^4(x - y).
$$

(23)

Its path integral representation may be obtained from one (21) for the spinning particle if one drops there all odd variables and remember that in this case the torsion field $S^\mu$ can not be anymore normalized to set the parameter $\xi_4$ to unity, and $S^2$ has to be multiplied by this parameter. Thus, we get

$$
D^c(x_{\text{out}}, x_{\text{in}}) = i \int_0^\infty d\epsilon_0 \int_{x_{\text{in}}}^{x_{\text{out}}} Dx \int D\pi \int \mathcal{M}(e) De \\
\times \exp \left\{ i \int_0^1 \left[ -\frac{\dot{x}^2}{2e} - \frac{e}{2} M_{\text{sc}}^2 - q\dot{x}^\mu A^\mu + \pi \dot{e} \right] d\tau \right\},
$$

(24)

where $M^2 = m^2 + \xi_4 S^2$.

4 Pseudoclassical action for the spinning particle and classical equations of motion in the nonrelativistic limit

The exponent in the integrand (21) can be considered as an effective and non-degenerate Lagrangian action of a spinning particle in electromagnetic and torsion fields. It consists of two principal parts. The first one, which unifies two summands with the derivatives of $e$ and $\chi$, can be treated as a gauge fixing term $S_{\text{GF}}$,

$$
S_{\text{GF}} = \int_0^1 (\pi \dot{e} + \nu \dot{\chi}) d\tau,
$$
and corresponds, in fact, to gauge conditions \( \dot{e} = \dot{\chi} = 0 \). The rest part of the effective action can be treated as a gauge invariant action of a spinning particle in the field under consideration. It has the form

\[
S = \int_0^1 \left[ -\frac{z^2}{2e} - \frac{e}{2} M^2 - \dot{x}_\mu (q A^\mu - d^\mu) + i e q F_{\mu\nu} \psi^\mu \psi^\nu \\
+ i \chi \left( m \psi^4 + \frac{2}{3} \psi^\mu d_\mu \right) - i \psi_\alpha \dot{\psi}_\alpha \right] d\tau ,
\]

(25)

where

\[
z^\mu = \dot{x}^\mu + i \chi \psi^\mu, \quad M^2 = m^2 + S^2 - 16 \partial_\mu S^\mu \psi^0 \psi^1 \psi^2 \psi^3, \quad d_\mu = -2i \epsilon_{\mu\nu\alpha\beta} S^\nu \psi^\alpha \psi^\beta .
\]

The action (25) is a generalization of Berezin-Marinov action [26, 27] to the background with torsion. One can easily verify that it is reparametrization invariant. Explicit form of supersymmetry transformations, which generalize ones for the Berezin-Marinov action, is not so easily to derive. Their presence will be proved in an indirect way. Namely we are going to prove the existence of two primary first-class constraints in Hamiltonian formulation.

Let us analyze the equations of motion for theory with the action (25).

\[
\frac{\delta S}{\delta e} = \frac{1}{e^2} \left( \frac{\dot{x}^2}{2} - i \dot{x}_\alpha \psi^\alpha \chi \right) - \frac{M^2}{2} + i e q F_{\alpha\beta} \psi^\alpha \psi^\beta = 0 ,
\]

(26)

\[
\frac{\delta S}{\delta \chi} = i \left[ \left( \frac{\dot{x}_\mu}{e} - \frac{2}{3} d_\mu \right) \psi^\mu - m \psi^4 \right] = 0 ,
\]

(27)

\[
\frac{\delta S}{\delta \psi^\alpha} = 2i \dot{\psi}_\alpha - 2i e q F_{\alpha\beta} \psi^\beta - i \frac{\dot{\alpha}}{e} \dot{\chi} + \frac{2i}{3} \chi d_\alpha + 4i \epsilon_{\mu\nu\alpha\beta} \dot{x}^\mu S^\nu \psi^\beta \\
- \frac{8}{3} \chi \epsilon_{\mu\nu\alpha\beta} \psi^\mu S^\nu \psi^\beta - \frac{4e}{3} \partial_\lambda S^\lambda \epsilon_{\mu\nu\alpha\beta} \psi^\mu \psi^\nu \psi^\beta ,
\]

(28)

\[
\frac{\delta S}{\delta \psi^5} = -2i \dot{\psi}^4 + i m \chi = 0 ,
\]

(29)

\[
\frac{\delta S}{\delta x^\alpha} = \frac{d}{d\tau} \left( \frac{\dot{x}_\alpha}{e} \right) + q \dot{x}^\beta F_{\beta\alpha} + i e q F_{\mu\nu,\alpha} \psi^\mu \psi^\nu + \dot{x}_\mu A^\mu + \frac{d}{d\tau} \left( \frac{i}{e} \psi_\alpha \chi - A_\alpha + d_\alpha \right) \\
+ e S^\mu \partial_\alpha S^\mu - 8e (\partial_\alpha \partial_\mu S^\mu) \psi^0 \psi^1 \psi^2 \psi^3 - \dot{x}_\mu (\partial_\alpha d^\mu) - \frac{2i}{3} \chi \psi_\mu (\partial_\alpha d^\mu) = 0 .
\]

(30)

One can choose (it is also may be seen from the Hamiltonian analysis which follows) the gauge conditions \( \chi = 0 \) and \( e = 1/m \) to simplify the analysis of the equations (28-30). In order to perform the nonrelativistic limit we define the three dimensional spin vector \( \vec{\sigma} \) as [26],

\[
\sigma_k = 2i \epsilon_{kjl} \psi^j \psi^l , \quad \psi^j \psi^l = \frac{i}{4} \epsilon^{kjl} \sigma_k , \quad \psi^j \psi^l = \frac{i}{4} \epsilon^{kjl} \sigma_k .
\]

(31)

Then we consider

\[
\psi^0 \approx 0 , \quad \dot{x}^0 \approx 1 , \quad \dot{x}^i \approx v^i = \frac{dx^i}{dx^0} ,
\]

(32)
as a part of the nonrelativistic approximation, and also use standard relations for the components of the stress tensor:

\[ F_{0i} = -E_i = \partial_0 A_i - \partial_i A_0 \quad \text{and} \quad F_{ij} = \epsilon_{ijk} H^k. \]

Substituting these formulas into (30) and (29), disregarding the terms of higher orders in the external fields, we arrive at the equations:

\[
m \ddot{v} = qE + \frac{q}{c} [\ddot{v} \times \vec{H}] - \nabla (\vec{\sigma} \cdot \vec{S}) - \frac{1}{c} \frac{d}{dt} (\vec{\sigma} S_0) + \frac{1}{c} (\vec{v} \cdot \vec{\sigma}) \nabla S_0 + ... \]
\[
\frac{d\vec{\sigma}}{dt} = \left( \frac{q}{mc} \vec{H} + \frac{2}{\hbar} \vec{S} - \frac{2S_0}{c} \ddot{v} \right) \times \vec{\sigma}. \tag{33}
\]

They coincide perfectly with the classical equations of motion obtained in the work [9] from the Pauli equation in the same background. That confirms our interpretation of the action (26). Additional arguments in favor of the interpretation will be obtained in the next section, where we are going to quantize of the action.

A corresponding classical gauge invariant action for spinless particle, which may be extracted from (24), has the form

\[
S_{sc} = \int \left[ -\frac{\dot{x}^2}{2e} - \frac{e}{2} M_{sc}^2 - q\dot{x} A_\mu \right] d\tau. \tag{34}
\]

We can see that it exhibits interaction between the particle and torsion. The dependence on the background torsion enters the equations of motion of the particle (in a very nontrivial way for the coordinate-dependent torsion) and therefore the presence of a torsion may be detected not only for the spinor, but also for scalar particles.

## 5 Quantization of the pseudoclassical action

Going over to the Hamiltonian formalism, we introduce the canonical momenta:

\[
p_\alpha = \frac{\partial L}{\partial \dot{x}_\alpha} = -\frac{z_\alpha}{e} - qA_\alpha + d_\alpha,
\]

\[
P_e = \frac{\partial L}{\partial \dot{e}} = 0, \quad P_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0, \quad P_n = \frac{\partial L}{\partial \dot{\psi}_n} = -i\psi_n. \tag{35}
\]

From the last equations (35), follows that there exist primary constraints

\[
\Phi_A^{(1)} = 0,
\]

\[
\Phi_A^{(1)} = \begin{cases} 
\Phi_1^{(1)} = P_\chi, \\
\Phi_2^{(1)} = P_e, \\
\Phi_3^{(1)} = P_n + i\psi_n.
\end{cases} \tag{36}
\]
We construct the total Hamiltonian $H^{(1)}$, according to the standard procedure (we use the notations of the book [32]), $H^{(1)} = H + \lambda A \Phi^{(1)}_A$, where

$$H = \left( \frac{\partial L}{\partial \dot{q}} \dot{q} - L \right) \Bigg|_{\frac{\partial L}{\partial \dot{q}} = p}, \quad q = (x, e, \chi, \psi^n), \quad P = (p, P_e, P_\chi, P_n).$$

We get for $H$:

$$H = -\frac{e^2}{2} (p^2 + 2 p \mu d^\mu + 2iqF_{\mu\nu} \psi^\mu \psi^\nu - M^2) + i\chi \left( P_\mu \psi^\mu - m \psi^4 + \frac{1}{3} d_\mu \psi^\mu \right). \quad (37)$$

Using the consistency conditions: the conservation of the primary constraints $\Phi^{(1)}_{1,2}$ in time $\tau$, $\dot{\Phi}^{(1)}_{1,2} = \{ \Phi^{(1)}_{1,2}, H^{(1)} \} = 0$, we find the secondary constraints $\Phi^{(2)}_{1,2} = 0$, 

$$\Phi^{(2)}_1 = P_\mu \psi^\mu - m \psi^4 + \frac{1}{3} d_\mu \psi^\mu = 0, \quad (38)$$

$$\Phi^{(2)}_2 = p^2 + 2 p \mu d^\mu + 2iqF_{\mu\nu} \psi^\mu \psi^\nu - M^2 = 0. \quad (39)$$

and the same conditions for the constraints $\Phi^{(1)}_{3n}$ give equations for the determination of $\lambda^{3n}$. Thus, the Hamiltonian $H$ appears to be proportional to constraints, as one can expect in the case of a reparametrization invariant theory,

$$H = i\chi \Phi^{(2)}_1 - \frac{e}{2} \Phi^{(2)}_2.$$

No more secondary constraints arise from the Dirac procedure, and the Lagrange multipliers $\lambda_1$ and $\lambda_2$ remain undetermined, in perfect correspondence with the fact that the number of gauge transformations parameters equals two for the theory in question. One can go over from the initial set of constraints $(\Phi^{(1)}, \Phi^{(2)})$ to the equivalent one $(\Phi^{(1)}, T)$, where:

$$T = \Phi^{(2)} + i \frac{\partial \Phi^{(2)}}{\partial \psi^m} \Phi^{(1)}_{3n}. \quad (40)$$

The new set of constraints can be explicitly divided in a set of the first-class constraints, which is $(\Phi^{(1)}_{1,2}, T)$ and in a set of the second-class constraints, which is $\Phi^{(1)}_{3n}$.

Now we consider an operator quantization, expecting to get in this procedure the Dirac equation (5). To this end we perform only a partial gauge fixing, by imposing the supplementary gauge conditions $\Phi^{G}_{1,2} = 0$ to the primary first-class constraints $\Phi^{(1)}_{1,2}$,

$$\Phi^{G}_1 = \chi = 0, \quad \Phi^{G}_2 = e = 1/m, \quad (41)$$

which coincide with those we used in the Lagrangian analysis. One can check that the conditions of the conservation in time of the supplementary constraints (11) give equations for
determination of the multipliers \( \lambda_1 \) and \( \lambda_2 \). Thus, on this stage we reduced our Hamiltonian theory to one with the first-class constraints \( T \) and second-class ones \( \varphi = (\Phi^{(1)}, \Phi^G) \). After that we will use the so called Dirac method for systems with first-class constraints \([31]\), which, being generalized to the presence of second-class constraints, can be formulated as follow: the commutation relations between operators are calculated according to the Dirac brackets with respect to the second-class constraints only; second-class constraints operators equal zero; first-class constraints as operators are not zero, but, are considered in sense of restrictions on state vectors. All the operator equations have to be realized in some Hilbert space.

The sub-set of the second-class constraints \( (\Phi^{(1)}_1, \Phi^G) \) has a special form \([32]\), so that one can use it for eliminating of the variables \( e, \bar{e}, \chi, \bar{\chi} \), from the consideration, then, for the rest of the variables \( x, p, \psi^n \), the Dirac brackets with respect to the constraints \( \varphi \) reduce to ones with respect to the constraints \( \Phi^{(1)}_{3n} \) only and can be easy calculated,

\[
\{x^\alpha, p_\beta\}_{D(\Phi^{(1)}_{3n})} = \delta_\beta^\alpha, \quad \{\psi^n, \psi^m\}_{D(\Phi^{(1)}_{3n})} = \frac{i}{2} \eta^{nm},
\]

while others Dirac brackets vanish. Thus, the commutation relations for the operators \( \hat{x}, \hat{p}, \hat{\psi}^n \), which correspond to the variables \( x, p, \psi^n \) respectively, are

\[
[\hat{x}^\alpha, \hat{p}_\beta]_- = i \{x^\alpha, p_\beta\}_{D(\Phi^{(1)}_{3n})} = \delta_\beta^\alpha, \\
[\hat{\psi}^m, \hat{\psi}^n]_+ = i \{\psi^m, \psi^n\}_{D(\Phi^{(1)}_{3n})} = -\frac{1}{2} \eta^{mn}.
\]

Besides, the operator equations hold:

\[
\hat{\Phi}^{(1)}_{3n} = \hat{P}_n + i \hat{\psi}_n = 0.
\]

The commutation relations (42) and the equations (43) can be realized in a space of four columns \( \Psi(x) \) dependent on \( x^\alpha \). At the same time we select \( \hat{x}^\alpha \) to be operators of multiplication, and \( \hat{p}_\alpha = -i \partial_\alpha, \quad \hat{\psi}^\alpha = i \frac{1}{2} \gamma^5 \gamma^\alpha, \quad \hat{\psi}^4 = i \frac{1}{2} \gamma^5 \), where \( \gamma^n \) are the \( \gamma \)-matrices \( (\gamma^\alpha, \gamma^5) \). The first-class constraints \( \hat{T} \) as operators have to annihilate physical vectors; in virtue of (43), (40) these conditions reduce to the equations:

\[
\hat{\Phi}^{(2)}_{1,2} \Psi(x) = 0,
\]

where \( \hat{\Phi}^{(2)}_{1,2} \) are operators, which correspond to the constraints (38), (39). There is no ambiguity in the construction of the operator \( \hat{\Phi}^{(2)}_{1} \), according to the classical function \( \Phi^{(2)}_{1} \) from (38). Thus, taking into account the realizations of the commutation relations (42), one easily
can see that the first equation (44) reproduces the Dirac equation (5). As to the construction of the operator $\hat{\Phi}_2^{(2)}$, according to the classical function $\Phi_2^{(2)}$ from (39), we meet here an ordering problem since the constraint $\Phi_2^{(2)}$ contains terms with products of the momenta and functions of the coordinates. For such terms we choose the symmetrized (Weyl) form of the corresponding operators, which, in particular, provides the hermicity of the operator $\hat{\Phi}_2^{(2)}$. But the main reason is, that such a correspondence rule provides the consistency of the two equations (44). Indeed, in this case we have

$$\hat{\Phi}_2^{(2)} = \left(\hat{\Phi}_1^{(2)}\right)^2,$$

(45)

and the second equation (44) appears to be merely the consequence of the first equation (44), i.e. of the Dirac equation (5). Thus, we see that the operator quantization of the action reproduces the Dirac quantum theory of spinning particle in electromagnetic and torsion field.

6 Conclusions

We have constructed a path integral representations for propagators of spinning and spinless particles in the torsion and electromagnetic fields. These representations allow one to study and calculate the propagators in the same manner as it was done, for example, in [33]. From the path integral representations we extract (pseudo)classical actions for the particles in an external torsion and electromagnetic fields. These actions satisfy some natural conditions: they are consistent with the renormalizable theory of interacting fields on torsion background, they manifest standard gauge symmetries, and the low-energy limit fits nicely with the expressions obtained from the Pauli equation. Upon quantization, a quantum mechanics of particles is produced again. In our opinion all that justifies the form of the actions. As a somehow unexpected consequence one meets the nontrivial interaction with torsion for the scalar particle. This interaction has a non-minimal form and results from the one between scalar filed and background torsion.

Some speculations may be done in relation to cosmology problems. According to [11], the mass of the propagating torsion axial vector $S_\mu$ has to be very large: at least some orders of magnitude above the Fermi scale. Therefore torsion can not be visible in the modern Universe. However, this opens the door to consider the string-induced torsion as an origin for the cosmological perturbations during the inflationary phase. After the early stage of inflation torsion becomes non-propagating due to its mass, in this respect it is different from the
magnetic field. There are some indications that the torsion-induced density perturbations differ from the ones induced by quantum effects of the metric [34]. The difference between torsion and magnetic fields in the particle actions (25), (34) may indicate that the torsion can produce perturbation spectrum distinct from the one of external magnetic field [35]. The expressions for the particle actions obtained here can be an appropriate basis for the formulation of the cosmological source terms in an external torsion field. These application of our results will be explored elsewhere.

Acknowledgments D.M.G. and I.L.Sh. are grateful to CNPq for permanent support. D.M.G. thanks DAAD and FAPESP for financial support of his stay at Leipzig University, and thanks the latter for hospitality.

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