Landau Levels in QED in Time-Dependent Magnetic Fields

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The Landau levels of charged fermions in QED make continuous transitions under a homogeneous, time-dependent magnetic field. We analytically formulate the quantum motion as the Cauchy initial value problem in the two-component first order formalism. We put forth a measure that characterizes and classifies the quantum motions of fermion into the adiabatic change, the nonadiabatic change and the sudden change. We find the exact quantum motion and calculate the pair-production rate when the magnetic field suddenly changes as a step function.

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I. INTRODUCTION

The interaction of charged particles with background electromagnetic fields has been intensively studied since the seminal works by Heisenberg and Euler, and Weisskopf on the effective action in a constant electromagnetic field in spinor QED [1] and scalar QED [2]. Schwinger’s proper-time formalism has laid a cornerstone for probing vacuum polarization in the constant electromagnetic field [3]. The quantum states of charged particles are an essential ingredient in understanding the electromagnetic interaction and vacuum polarization, in particular, in strongly magnetized neutron stars [4] and in extremely high-intensity lasers [3].

In a constant magnetic field charged spin-1/2 fermions or spinless scalars have Landau levels [6–9] and the energy spectrum leads to the QED effective action via the zeta-function regularization [10]. However, the quantum motion becomes non-trivial when the background electromagnetic field is localized in space or time. Recently the effective actions in spinor or scalar QED have been found for the Sauter-type electric field localized in time or space, in which the quantum states of charged spin-1/2 fermion or spinless scalar are used in the in-out formalism [11, 12], and the effective action for the Sauter-type magnetic field has been found in Ref. [13].

The resolvent method has also been used to compute the effective action in the Sauter-type electric field [14] and the magnetic field [15]. The fermionic determinants have been investigated in static inhomogeneous magnetic fields [16, 17]. For numerical purposes, the worldline formalism has also been applied to scalar QED in a static magnetic field of step function [18] and the real space split operator has been introduced for scalar QED in arbitrary electromagnetic potential [19]. Hence analytical studies of quantum states of charged particles beyond the Sauter-type electric or magnetic field or in time-dependent magnetic fields will be of not only theoretical interest but also practical applicability.

In this paper we study the quantum states of charged spin-1/2 fermions in homogeneous, time-dependent magnetic fields. For that purpose we observe that the spin-diagonal component of the Dirac equation in a time-dependent magnetic field and the Wheeler-DeWitt equation for the Friedman-Robertson-Walker universe with a minimal massive scalar field have the same mathematical structure: a relativistic wave equation with transverse motion of time-dependent harmonic oscillator. The Cauchy initial value problem for the Wheeler-DeWitt equation with a general scalar field has been studied in the two-component first order formalism [20, 22] and in the third quantization [23]. Feshbach and Villars have introduced another two-component first order formalism [24], which was applied to the Klein-Gordon equations in Refs. [25, 26].

The instantaneous Landau levels in time-dependent magnetic field do not decouple the Dirac equation and cause continuous transitions among themselves during the quantum evolution. We formulate the Cauchy initial value problem, which expresses the squared Dirac equation as the two-component first order equation and incorporates the changing rate of Landau levels in addition to the instantaneous energy spectrum. The ratio of the changing rate of each Landau level to the corresponding dynamical phase during any time interval may provide a measure that characterizes and classifies quantum motions of charged particles into (i) the adiabatic change, (ii) the sudden change, and (iii) the nonadiabatic change.

In the adiabatic change of magnetic field the Landau levels change so slowly that the charged fermion remains in the same time-dependent Landau level and the adiabatic theorem holds. In the sudden change the Landau levels
change so rapidly that the charged fermion cannot follow the time-dependent Landau level and is thus frozen to the initial Landau level. The magnetic field that suddenly changes as a step function is an exactly solvable model, in which the dynamical phase is extremely small compared to a finite change of Landau levels during the infinitesimal interval and the measure becomes arbitrary large. In the last case of nonadiabatic change the charged fermion makes continuously transitions among time-dependent Landau levels, keeping the parity of the initial Landau level.

The organization of the paper is as follows. In Sec. II we introduce the two-component first order formalism for the squared Dirac equation in time-dependent magnetic fields. Using the instantaneous Landau levels, we formulate the Cauchy initial value problem that evolves an initial Landau level. The two-component propagator is entirely determined by the changing rate of the instantaneous Landau levels and the energy spectrum. In Sec. III we further propose a dimensionless measure which depends on the relative ratio of the changing rate of Landau levels to the dynamical phase over the time interval under study and which classifies the quantum motions into the adiabatic change, the sudden change and the nonadiabatic change. In Sec. IV we find the quantum states of charged fermions and calculate the pair-production rate from the Dirac sea when the magnetic field changes abruptly as a step-function. In Sec. V we discuss the physical implications of the quantum states in time-dependent magnetic fields.

II. QUANTUM MOTION OF FERMION

We study quantum states of charged spin-1/2 fermions in spinor QED under the background of time-dependent magnetic fields. The vector potential

\[ \mathbf{A}(t, \mathbf{x}) = \frac{1}{2} \mathbf{B}(t) \times \mathbf{x} \]  

leads to both the time-dependent magnetic field \( \mathbf{B}(t) \) and the electric field \( \mathbf{E}(t, \mathbf{x}) = -\partial \mathbf{A}/\partial t \). The magnetic field is assumed to be along the \( z \)-direction and to have positive magnitude \( |\mathbf{B}(t)| \) to guarantee the Landau levels. In the second order formalism the Dirac equation for a charge \( q \) and mass \( m \) in the vector potential takes the form (in units of \( \hbar = c = 1 \))

\[ \left[ \eta^{\mu\nu} (i \partial_\mu - q A_\mu) (i \partial_\nu - q A_\nu) - m^2 + 2q B(t) \sigma^{12} \right] \Psi(t, \mathbf{x}) = 0. \]  

The eigenstates of the spin tensor \( \sigma^{12} = \text{diag}(\sigma_z, \sigma_z)/2 \) thus leads to the equation

\[ \left[ \frac{d^2}{dt^2} + \hat{p}_\perp^2 + \left( \frac{q B(t)}{2} \right)^2 \hat{x}_\perp^2 - q B(t) \hat{L}_z + 2q B(t) \sigma_z + k_z^2 + m^2 \right] \Psi_{\sigma_\perp}(t, \mathbf{x}_\perp) = 0, \]  

where \( \hat{p}_\perp = -i \nabla \perp \), \( \hat{L}_z = \hat{x}_\perp \times \hat{p}_\perp \), \( \sigma_z = \pm 1/2 \), and \( \Psi_{\sigma_\perp}(t, \mathbf{x}_\perp) \) is the longitudinal Fourier-mode.

Note that the transverse Hamiltonian

\[ \hat{H}_\perp(t) = \hat{p}_\perp^2 + \left( \frac{q B(t)}{2} \right)^2 \hat{x}_\perp^2 - q B(t) \hat{L}_z \]  

describes a two-dimensional time-dependent oscillator coupled to the angular momentum \( \hat{L}_z \). In the oscillator representation

\[ \hat{a}_x(t) = \sqrt{\frac{q B(t)}{2}} \hat{x} + \frac{i}{\sqrt{2q B(t)}} \hat{p}_x, \quad \hat{a}_x^\dagger(t) = \text{H.C.}, \]

\[ \hat{a}_y(t) = \sqrt{\frac{q B(t)}{2}} \hat{y} + \frac{i}{\sqrt{2q B(t)}} \hat{p}_y, \quad \hat{a}_y^\dagger(t) = \text{H.C.}, \]

the transverse Hamiltonian is given by

\[ \hat{H}_\perp(t) = q B(t) [\hat{a}_x^\dagger(t) \hat{a}_x(t) + \hat{a}_y^\dagger(t) \hat{a}_y(t) + 1] + iq B(t) [\hat{a}_x(t) \hat{a}_y^\dagger(t) - \hat{a}_x^\dagger(t) \hat{a}_y(t)]. \]  

Further, in the new basis

\[ c_\pm(t) = \frac{1}{\sqrt{2}} (\hat{a}_x(t) \pm i \hat{a}_y(t)) \]  

is valid.
with equal-time commutators
\[ [\hat{c}_\pm(t), \hat{c}^\dagger\pm(t)] = 1, \quad [\hat{c}_\pm(t), \hat{c}_\mp(t)] = [\hat{c}_\pm(t), \hat{c}^\dagger_\mp(t)] = 0, \] (8)

Eq. (6) can be written in the diagonal form
\[ \hat{H}_\perp(t) = qB(t) [\hat{c}^\dagger_-(t)\hat{c}_-(t) + \hat{c}_-(t)\hat{c}^\dagger_-(t)]. \] (9)

Hence the Landau levels for Eq. (9) are the number states of \( \hat{c}^\dagger_-(t)\hat{c}_-(t) \):
\[ \hat{c}_-(t)|0, t\rangle = 0, \quad |n, t\rangle = \frac{\hat{c}^\dagger_-(t))^n}{\sqrt{n!}} |0, t\rangle. \] (10)

However, note that the Landau levels (10) do not separate Eq. (3) into a diagonal one since they explicitly depend on time and make continuous transitions among themselves.

The Landau levels, arranged into a column vector in increasing quantum numbers,
\[ \vec{\Phi}(t) = \begin{pmatrix} |0, t\rangle \\ |1, t\rangle \\ \vdots \end{pmatrix}, \] (11)
change as
\[ \frac{d}{dt} \vec{\Phi}(t) = \Omega(t)\vec{\Phi}(t). \] (12)

Here the changing rate of the basis has the oscillator representation \[21, 22\]
\[ \Omega(t) = \frac{\dot{B}(t)}{4B(t)} (\hat{c}^2(t) - \hat{c}^\dagger_2(t)), \] (13)
and the matrix representation
\[ \langle m, t|\Omega(t)|n, t\rangle = \frac{\dot{B}(t)}{4B(t)} (\sqrt{n(n+1)}\delta_{m,n-2} - \sqrt{(n+1)(n+2)}\delta_{m,n+2}). \] (14)

Thus the Landau levels unitarily transform as
\[ \vec{\Phi}(t) = \mathbf{S}(t, t_0)\vec{\Phi}(t_0), \] (15)
where the transition matrix yields a one-mode squeezed operator \[29\]
\[ \mathbf{S}(t, t_0) = \mathcal{T} \exp \left[ \int_{t_0}^t \Omega(t')dt' \right] = \exp \left[ \frac{1}{4} \ln \left( \frac{B(t)}{B(t_0)} \right) (\hat{c}^2(t_0) - \hat{c}^\dagger_2(t_0)) \right]. \] (16)

Following Refs. \[21, 22\], we may expand the field as
\[ \Psi_{\sigma\perp}(t, x_\perp) = \vec{\Phi}^T(t, x_\perp)\mathbf{S}(t, t_0)\vec{\Phi}_{\sigma\perp}(t) \] (17)
with \( \vec{\Phi}^T(t, x_\perp) \) denoting the transpose of the coordinate representation of Eq. (14), and then write Eq. (3) in the two-component first order formalism
\[ \frac{d}{dt} \begin{pmatrix} \Psi_{\sigma\perp}(t) \\ \frac{d\Psi_{\sigma\perp}(t)}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\mathbf{S}^{-1}(t, t_0)\omega^2(t)\mathbf{S}(t, t_0) & 0 \end{pmatrix} \begin{pmatrix} \Psi_{\sigma\perp}(t) \\ \frac{d\Psi_{\sigma\perp}(t)}{dt} \end{pmatrix}, \] (18)
where
\[ \omega^2(t) = qB(t)(2\hat{c}^\dagger_-(t)\hat{c}(t) + 1 + 2\sigma) + m^2 + k_z^2. \] (19)
Remarkably, the Cauchy data are given by\[21, 22\]
\[
\left( \frac{\partial \Psi(t, x)}{\partial t} \right) = \begin{pmatrix} \Phi^T(t, x) & 0 \\ 0 & \Phi^T(t, x) \end{pmatrix} U(t, t_0) \left( \frac{\partial \Psi(t_0)}{\partial t_0} \right),
\]
(20)
where \( S \) from Eq. (18) cancels \( S \) in Eq. (17) and the two-component propagator can be given by the time-ordered integral
\[
U(t, t_0) = T \exp \left[ \int_{t_0}^{t} \begin{pmatrix} \Omega(t') & I \\ -\omega^2(t') & \Omega(t') \end{pmatrix} dt' \right].
\]
(21)
The Cauchy data (20) guarantees the inner product between the positive and negative frequency solutions for the squared Dirac equation
\[
i \int d^3 x \left( \Psi^-(t, x) \frac{\partial}{\partial t} \Psi^+(t, x) - \Psi^+(t, x) \frac{\partial}{\partial t} \Psi^-(t, x) \right) = I.
\]
(22)

III. CLASSIFICATION OF QUANTUM MOTIONS

As the evolution (21) is carried by \( \omega^2 \) and \( \Omega \), we may put forth a dimensionless measure that characterizes the quantum motion of the \( n \)th Landau level during any time interval \((t_i, t_f)\) as
\[
\mathcal{R}_n = \frac{n}{4} \left| \frac{\ln \left( \frac{B(t_f)}{B(t_i)} \right)}{\int_{t_i}^{t_f} \omega(t', n) dt'} \right|.
\]
(23)
Note further that the fermion energy (19) in the \( n \)th Landau level is dominated by the mass \( m \) when the magnetic field has an under-critical strength, \( nB(t)/m^2 \ll 1 \), while it is dominated by \( nB(t) \) when the field has an over-critical strength, \( nB(t)/m^2 \gg 1 \). Since the lower bound for the dynamical phase is \( m\Delta t \) for \( \Delta t = t_f - t_i \), the upper bound for the measure \( \mathcal{R}_n \) is
\[
\mathcal{R}_n \leq \frac{n}{4} \left| \frac{\ln \left( \frac{B(t_f)}{B(t_i)} \right)}{m\Delta t} \right|.
\]
(24)
Hence we may classify the quantum motions into three categories: (i) the adiabatic change when \( \mathcal{R}_n \ll 1 \), (ii) the sudden change when \( \mathcal{R}_n \gg 1 \), and (iii) the nonadiabatic change, otherwise. An interesting model is provided by a modulated magnetic field in a constant background
\[
B(t) = B_0 + B_1 \cos \left( \frac{t}{T} \right), \quad (B_0 > B_1).
\]
(25)
The background field should be larger than the modulated field in order to exclude the tachyonic states for the over-critical strength, but not necessary otherwise. For under-critical strengths \( B_0, B_1 \ll m \), the measure is given by
\[
\mathcal{R}_n \approx \frac{n}{4} \left| \frac{\ln \left( \frac{B_0 + B_1}{B_0 - B_1} \right)}{m\Delta T} \right|.
\]
(26)
The measure can be made large by choosing a very small \( T \) and \( \Delta B = B_0 - B_1 \) in the Compton scale for \( m \). For instance, \( \mathcal{R}_n = 1 \) requires \( \ln(B_0/\Delta B)/T = 10^{20} \) for electrons and positrons.

In the first case (i) of the adiabatic change, \( \Omega \) can be neglected and the two-component propagator is approximately given by
\[
U(t, t_0) \approx \mathcal{P}(t, t_0),
\]
(27)
where
\[
\mathcal{P}(t, t_0) = T \exp \left[ \int_{t_0}^{t} \begin{pmatrix} 0 & I \\ -\omega^2(t') & 0 \end{pmatrix} dt' \right].
\]
(28)
Note that Eq. (28) can be written as

$$\mathcal{P}(t, t_0) = \mathcal{P}(t)\mathcal{P}^{-1}(t_0), \quad \mathcal{P}(t) = \begin{pmatrix} P_1(t) & P_2(t) \\ \dot{P}_1(t) & \dot{P}_2(t) \end{pmatrix}$$

(29)

where $P_1(t)$ and $P_2(t)$ are two independent solutions to the diagonal matrix equation

$$\frac{d^2 P(t)}{dt^2} + \omega^2(t)P(t) = 0.$$  

(30)

The fermion remains in the same Landau level which adiabatically changes when the magnetic field slowly changes, and thus the adiabatic theorem holds.

In the second case (ii) of the sudden change, $\Omega$ dominates over $\omega$ and $I$, so the two-component propagator is approximately given by

$$\mathcal{U}(t, t_0) \approx \begin{pmatrix} S(t, t_0) & 0 \\ 0 & S(t, t_0) \end{pmatrix}.$$  

(31)

The one-mode squeezed operator $S$ in Eq. (31) cancels another $S^\dagger$ from $\tilde{\Phi}^T(t, \mathbf{x}_\perp) = \tilde{\Phi}^T(t_0, \mathbf{x}_\perp)S^\dagger$ in Eq. (16), so the Cauchy data are approximately given by

$$\begin{pmatrix} \Psi_\sigma(t, \mathbf{x}_\perp) \\ \partial_0 \Psi_\sigma(t, \mathbf{x}_\perp) \end{pmatrix} \approx \begin{pmatrix} \tilde{\Phi}^T(t_0, \mathbf{x}_\perp) \cdot \tilde{\Psi}_\sigma(t_0) \\ \tilde{\Phi}^T(t_0, \mathbf{x}_\perp) \cdot \partial_0 \tilde{\Psi}_\sigma(t_0) \end{pmatrix}.$$  

(32)

The fermion does not follow the time-dependent Landau level and is frozen to the initial one when the magnetic field suffers a step-function change and $R_n \gg 1$. In Sec. IV we shall consider the most typical model of this category, in which the magnetic field suffers a step-function change and $R_n = \infty$.

In the third case (iii) of the nonadiabatic change, $\Omega$ is comparable to $\omega$, so we may write the propagator using the similarity formula

$$\mathcal{U}(t, t_0) = \mathcal{P}(t)T \exp \left[ \int_{t_0}^t P^{-1}(t') \begin{pmatrix} \Omega(t') & 0 \\ 0 & \Omega(t') \end{pmatrix} P(t')dt' \right] \mathcal{P}^{-1}(t_0).$$  

(33)

The equivalence between Eqs. (21) and (33) can be shown by taking a derivative with respect to time. The fermion makes continuous transitions among Landau levels due to the transition matrix from the time-ordered integral in Eq. (33), for instance, the first two terms

$$\mathcal{U}(t, t_0) = \mathcal{P}(t)T \left[ I + \int_{t_0}^t P^{-1}(t') \begin{pmatrix} \Omega(t') & 0 \\ 0 & \Omega(t') \end{pmatrix} P(t')dt' + \cdots \right] \mathcal{P}^{-1}(t_0).$$  

(34)

The first term is the adiabatic evolution and the second term comes from continuous transitions of Landau levels. The nonperturbative form may be found using the Magnus expansion.

IV. SUDDEN CHANGE MODEL

As a solvable model, we consider a sudden change in which the magnetic field jumps from $B_0$ to $B_1$ with a step function

$$B(t) = (B_1 - B_0)\theta(t) + B_0.$$  

(35)

The magnetic field (35) is a mathematical model in that the induced electric field has a delta function profile. We denote $c_{\text{in}}$ when $t < 0$ and $c_{\text{out}}$ when $t > 0$ for the basis (7) for the Landau levels. Since the changing rate $\Omega$ in Eq. (15) is proportional to $\delta(t)$, the transition matrix (16) is the one-mode squeezed operator

$$S = \exp \left[ \frac{1}{4} \ln \left( \frac{B_1}{B_0} \right) (c_{\text{in}}^2 - c_{\text{in}}^2) \right].$$  

(36)
Alternatively, the Bogoliubov transformation
\[ \hat{c}_{\text{out}} = \frac{1}{2} \left( \sqrt{\frac{B_1}{B_0}} - \sqrt{\frac{B_0}{B_1}} \right) \hat{c}_{\text{in}} + \frac{1}{2} \left( \sqrt{\frac{B_1}{B_0}} + \sqrt{\frac{B_0}{B_1}} \right) \hat{c}^\dagger_{\text{in}} \] (37)
leads to the unitary transformation \[ 29 \]
\[ \hat{c}_{\text{out}} = S \hat{c}_{\text{in}} S^\dagger, \quad \hat{c}^\dagger_{\text{in}} = S^\dagger \hat{c}^\dagger_{\text{out}} S. \] (38)
Hence each Landau level transforms into a squeezed one, \([n, \text{out}] = S[n, \text{in}],\) and similarly the column vector \([11]\) of Landau levels transforms as \(\Phi_{\text{out}} = S\Phi_{\text{in}}\) after the sudden change of magnetic field.

We decompose the two-component propagator \([21]\) into three parts
\[ \mathcal{U}(t, t_0) = \mathcal{P}(t, \epsilon; \hat{c}_{\text{out}}, \hat{c}^\dagger_{\text{out}}) \mathcal{T} \exp \left[ \int_{t_0}^t \left( \frac{\Omega(t')}{\omega^2(t')} \mathcal{I} \Omega(t') \right) dt' \right] \mathcal{P}(-\epsilon, t_0; \hat{c}_{\text{in}}, \hat{c}^\dagger_{\text{in}}). \] (39)
The post-factor in Eq. \([39]\) is the propagator from the initial time \(t_0\) to an infinitesimal time \(-\epsilon\) in the basis of \(\hat{c}_{\text{in}}\) and \(\hat{c}^\dagger_{\text{in}}\), the mid-factor is the propagator from \(-\epsilon\) to \(\epsilon\), and the pre-factor is the propagator from \(\epsilon\) to the final time \(t\) in the basis of \(\hat{c}_{\text{out}}\) and \(\hat{c}^\dagger_{\text{out}}\). In the limit of \(\epsilon = 0\), evaluating the propagator by the transition matrix \([30]\) as
\[ \mathcal{T} \exp \left[ \int_{t_0}^t \left( \frac{\Omega(t')}{\omega^2(t')} \mathcal{I} \Omega(t') \right) dt' \right] = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \] (40)
and using \([35]\), we arrive at the Cauchy data
\[ \begin{pmatrix} \Psi_{\sigma \perp}(t_0, x_\perp) \\ \frac{\partial \Psi_{\sigma \perp}(t, x_\perp)}{\partial t} \end{pmatrix} = \begin{pmatrix} \Phi^T(t_0, x_\perp) & 0 \\ 0 & \Phi^T(t_0, x_\perp) \end{pmatrix} \mathcal{P}(t, 0; \hat{c}_{\text{in}}, \hat{c}^\dagger_{\text{in}}) \mathcal{P}(0, t_0; \hat{c}_{\text{in}}, \hat{c}^\dagger_{\text{in}}) \begin{pmatrix} \Psi_{\sigma \perp}(t_0) \\ \frac{\partial \Psi_{\sigma \perp}(t_0)}{\partial t} \end{pmatrix}. \] (41)
The Magnus expansion \([32]\) shows that Eq. \([10]\) is the correct limit since the leading correction of \(O(\epsilon^2)\) vanishes and the higher terms are order of \(O(\epsilon^3)\).

The two-component propagator
\[ \mathcal{P}(t) = \frac{1}{\sqrt{2\omega}} \begin{pmatrix} e^{-i\omega t} & e^{i\omega t} \\ -i\omega e^{-i\omega t} & i\omega e^{i\omega t} \end{pmatrix}, \] (42)
consists of the positive and negative frequency solutions column-wise, respectively. Hence the charged fermion has the quantum state
\[ \Psi_{\sigma \perp}(t, x_\perp) = \Phi^T(t_0, x_\perp) \mathbf{F}(t) \Psi_{\sigma \perp}(t_0), \] (43)
with the amplitude matrix
\[ \mathbf{F}(t) = \frac{\omega_{\text{out}} + \omega_{\text{in}}}{2\omega_{\text{out}}} e^{-i\omega_{\text{out}} t + i\omega_{\text{in}} t_0} + \frac{\omega_{\text{out}} - \omega_{\text{in}}}{2\omega_{\text{out}}} e^{i\omega_{\text{out}} t + i\omega_{\text{in}} t_0}, \] (44)
where \(\omega_{\text{in}}\) and \(\omega_{\text{out}}\) are the diagonal matrix \([19]\) with \(B_0\) for \(t < 0\) and \(B_1\) for \(t > 0\), respectively. As explained in Sec. \([13]\) the fermion initially in the Landau level \(\Phi_{\sigma}(t_0, x_\perp)\) is frozen to that level with the time-dependent amplitude \(\mathbf{F}_\sigma(t)\). The coefficients in Eq. \([41]\) can also be obtained from quantum scattering of a wave \(e^{-i\omega_{\text{in}} t}\) into \(e^{-i\omega_{\text{out}} t}\) and \(e^{i\omega_{\text{out}} t}\) by a potential step. Thus the temporal oscillation of \(|\mathbf{F}_\sigma(t)|^2\), as shown in Fig. 1, is a consequence of partial scattering of the positive frequency into the negative one after the sudden change of the energy. The probability interpretation of \(|\mathbf{F}_\sigma(t)|^2\) is not the correct prescription for the Dirac equation. Instead, the inner product \([22]\) with respect to the in-state and the out-state
\[ |\text{in}, t\rangle = \frac{e^{-i\omega_{\text{in}} t}}{\sqrt{2\omega_{\text{in}}}} \Phi(t_0), \quad |\text{out}, t\rangle = \frac{e^{-i\omega_{\text{out}} t}}{\sqrt{2\omega_{\text{out}}}} \Phi(t_0), \] (45)
leads to the Bogoliubov coefficients
\[ \alpha = \frac{1}{2} \left( \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} + \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} \right), \quad \beta = \frac{1}{2} \left( \sqrt{\frac{\omega_{\text{out}}}{\omega_{\text{in}}}} - \sqrt{\frac{\omega_{\text{in}}}{\omega_{\text{out}}}} \right). \] (46)
Considering the quantum motion of virtual fermion pairs in the Dirac sea, the pair-production rate is $|\beta(n)|^2$ for each Landau level. The induced electric field reinforces the argument of pair production from a time-dependent magnetic field. As far as pair production is concerned, time-dependent magnetic fields strongly contrast constant magnetic fields, in which the Dirac sea is stable at one loop and forbids charged pairs from being emitted. Though the direction of the magnetic field is fixed in this paper, it would be interesting to compare pair production from a rotating magnetic field in Ref. [33].

V. CONCLUSION

We have studied the quantum evolution of charged spin-1/2 fermions in a homogeneous, time-dependent magnetic field. Contrary to a constant magnetic field, the Landau levels that instantaneously diagonalize the squared Dirac equation are a unitary transformation of initial ones via a one-mode squeezed operator when the magnetic field changes from a constant value. The two-component first order formalism has been employed to solve the Cauchy initial problem (20) in the basis of time-dependent Landau levels. We have introduced a dimensionless measure (23) that classifies the quantum motions into three categories: (i) the adiabatic change, (ii) the sudden change, and (iii) the nonadiabatic change. The measure is the ratio of the changing rate of each Landau level to the corresponding dynamical phase for the time interval of study. When the magnetic field changes so slowly that the ratio is very small, the time-dependent Landau level adiabatically changes and the fermion remains in the same Landau level, while when the magnetic field changes so rapidly that the ratio is very large, the fermion cannot follow the rapidly changing Landau level and is frozen to the initial Landau level. On the other hand, when the ratio is order of unity and the changing rate of the Landau level is comparable to the corresponding energy, the fermion makes continuous transitions among time-dependent Landau levels, keeping the parity of the initial state, during the quantum evolution.

We have explicitly analyzed the quantum states for the charged fermion when the magnetic field changes from one constant value to another as a step function. The two-component propagator (20) and (21) for an infinitesimal interval for the change of the magnetic field reduces to the one-mode squeezed operator for the change of Landau levels, and the resulting quantum state remains the same initial Landau level. This implies that the fermion does not follow the instantaneously changing Landau level and is frozen in the initial Landau level. We have found the Bogoliubov coefficients when the out-state is the frozen Landau levels with the energy spectrum in the changed magnetic field. This implies that a time-dependent magnetic field may produce pairs of charged particles from the Dirac sea, which is supported from the induced electric field.

The use of the quantum states for vacuum polarization and pair production and possible applications to astrophysics and extremely high-intensity lasers in general time-dependent magnetic fields are beyond the scope of this paper and will be addressed in the future publication.
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