An estimate of the bulk viscosity of the hadronic medium

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Abstract

The bulk viscosity (\(\zeta\)) of the hadronic medium has been estimated within the ambit of the Hadron Resonance Gas (HRG) model including the Hagedorn density of states. The HRG thermodynamics within a grand canonical ensemble provides the mean hadron number as well as its fluctuation. The fluctuation in the chemical composition of the hadronic medium in the grand canonical ensemble can result in non-zero divergence of the hadronic fluid flow velocity, allowing us to estimate the \(\zeta\) of the hadronic matter up to a relaxation time. We study the influence of the hadronic spectrum on \(\zeta\) and find its correlation with the conformal symmetry breaking (CSB) measure, \(\epsilon - 3P\). We estimate \(\zeta\) along the contours with constant, \(S/N_B\) (total entropy/net baryon number) in the \(T - \mu\) plane (temperature-baryonic chemical potential) for \(S/N_B = 30, 45\) and \(300\). We also assess the value of \(\zeta\) on the chemical freezeout curve for various centre of mass energy (\(\sqrt{s_{NN}}\)) and find that the bulk viscosity to entropy density ratio, \(\zeta/s\) is larger at FAIR than LHC energies.

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I. INTRODUCTION

Hydrodynamics is the study of the slowly varying degrees of freedom of the system involving continuity equations of the conserved charges of the underlying microscopic interactions. Viscous relativistic hydrodynamics has been a very successful framework to describe the evolution of the fireball created in heavy ion collisions at relativistic energies (HICRE) with a few free parameters which are extracted by fits to data \[1, 2\]. The following equations along with the initial conditions and equation of state (EoS) govern the progression of the matter produced in HICRE,

\[
\partial_\mu T^{\mu \nu} = 0 \tag{1}
\]
\[
\partial_\mu J_\kappa^{\mu} = 0 \tag{2}
\]

where \(T^{\mu \nu}(\mu, \nu = 0, 1, 2, 3)\) is the energy momentum tensor and \(\{J_\kappa^{\mu}\}\) refers to the current density corresponding to the conserved charges, (\(e.g., \kappa \equiv \text{net baryon number, net electric charge, etc.}\) \(T^{ij}(i, j = 1, 2, 3)\) for a non-viscous hydrodynamical system in the local rest frame is given by,

\[
T_{ij} = P \delta_{ij} \tag{3}
\]

where \(P\) is the isotropic thermodynamic pressure of the system. In case of Navier Stokes viscous hydrodynamics, the system’s response to the gradients of the fluid flow four velocity \(u^\mu\). Upto first order in derivatives of \(u^\mu\) two such transport coefficients, namely shear (\(\eta\)) and bulk (\(\zeta\)) viscosities appear as,

\[
T_{ij} = P \delta_{ij} + \eta \left( \frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3} \delta_{ij} \nabla \cdot u \right) + \zeta \delta_{ij} \nabla \cdot u \tag{4}
\]

That is the leading order correction (\(\delta T_{ij}\)) to the energy momentum tensor due to dissipation is introduced through the shear (\(\eta\)) and bulk (\(\zeta\)) viscous coefficients such that \(T_{ij} = P \delta_{ij} + \delta T_{ij}\). The effects of these transport coefficients on the hydrodynamic evolution of the strongly interacting fireball and on various observables in HICRE, \(e.g.,\) the elliptic flow, spectra of hadronic and electromagnetically interacting particles, etc have been studied extensively \[3–8\]. The general consensus reached is that the matter created in heavy ion collisions behaves almost like a perfect liquid \[9–13\] with \(\eta/s\) close to the KSS (Kovtun, Son and Starinets) bound \[14\]. On the other hand, the issue of bulk viscosity is far from settled. In the earlier works the contribution of \(\zeta\) was neglected. Inspired by the AdS/CFT correspondence, the \(\mathcal{N} = 4\) supersymmetric Yang-Mills theory has been used by several authors to
estimate the shear viscosity of the QGP and other strongly correlated system \([14]\). The \(\zeta/s\) is always zero in this approach. However, it is shown in \([15]\) that \(\zeta/s \sim 0.1\) for certain classes of black hole solutions. Moreover, the lattice QCD (LQCD) based studies \([16, 17]\) has indicated that the bulk viscosity could be as large as shear viscosity at the vicinity of the QCD phase transition. Similar conclusions have also been drawn from calculations done by using QCD inspired effective models \([18–20]\). Thus, there have been studies where \(\zeta\) was included into hydrodynamic simulations. It was found that \(\zeta\) affects the low momentum hadron spectra as well as the elliptic flow significantly \([21]\).

The phenomenological relevance of \(\zeta\) has fuelled efforts to estimate it by using various models. In this work we provide an estimate of \(\zeta\) within the ambit of HRG model which has been quite successful in describing the low temperature QCD thermodynamics. Lately, bulk viscosity of the hadronic medium has been computed in various schemes \([22–29]\). In this work we intend to study the role played by the phase space in deciding the bulk viscosity of the hadronic medium.

The paper is organized as follows. In the next section we outline the formalism used to estimate the bulk viscosity of hadronic system. We discuss the results of the present work in section III and devote section IV for summary and discussions.

II. FORMALISM

A fluid in equilibrium can fluctuate to a non-equilibrium state in many ways. Depending on the mode of fluctuation there is an onset of the corresponding dissipative process to counter this fluctuation for maintaining the equilibrium. Within the ambit of linear response theory, the medium response allows us to compute the transport coefficients like \(\eta\), \(\zeta\) etc. As seen in Eq. 4, \(\eta\) is connected to the traceless part of \(T_{ij}\) given by Eq. 4 and \(\zeta\) is related to the trace of \(T_{ij}\) for a compressible fluid warned by the presence of \(\nabla \cdot u\) which is related to the rate of change of volume \((V)\) associated with the uniform expansion or compression through the continuity equation:

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n\vec{u}) = 0
\]

substituting the density \(n = N/V\) in Eq. 5 (\(N\) is the total number and \(V\) is the volume), one obtains \(\nabla \cdot u = V^{-1} dV/dt\) \([22]\). This also indicates that the bulk viscosity will vanish for an incompressible fluid. The deviation in the pressure due to change in the bulk not followed from the equation of state
(EoS) can be connected to the bulk viscosity as follows:

$$\delta P = \zeta \nabla \cdot u$$  \hspace{1cm} (6)

In this work we will consider the situation where such flow field arises due to change in hadron yield from the equilibrium number. For a single component hadron gas assuming adiabaticity, it has been shown in [30] that:

$$\delta P = \left( \frac{\partial P}{\partial n_i} \right) \frac{\partial n_i}{\partial s_i} \zeta \tau_R$$  \hspace{1cm} (7)

where $\tau_R$ is the relaxation time scale of the system, which is the inverse of the rate of number changing process responsible to maintain chemical equilibrium. Now within the HRG formalism, the total pressure $P$ is given by sum over the partial pressure, $P_i$ due to each hadron species, $i$,

$$P = \sum P_i$$  \hspace{1cm} (8)

Thus the fluctuation in the total pressure $P$ can be expressed as

$$\delta P = \sum_i \delta P_i = \left[ \sum_i \left( \frac{\partial P_i}{\partial n_i} \right) \frac{\partial n_i}{\partial s_i} \zeta \tau_R \right] \nabla \cdot u$$  \hspace{1cm} (9)

$\tau_R^i$ is the relaxation time of the species, $i$. The fluid flow velocity field, $u^i$ being a hydrodynamic variable is same for all the hadron species. On comparing Eqs. 6 and 9, we find the expression for $\zeta$

$$\zeta = \sum_i \left( \frac{\partial P_i}{\partial n_i} \right) \frac{\partial n_i}{\partial s_i} \zeta \tau_R$$  \hspace{1cm} (10)

This relation can be used to derive the expression for bulk viscosity (see appendix for details) as,

$$\zeta = \sum_i \left[ \left( \frac{\partial P_i}{\partial T} \right) \frac{\partial n_i}{\partial T} \right] \left[ \left( \frac{\partial n_i}{\partial \mu_i} \right) \frac{\partial \mu_i}{\partial \mu_i} \right] \left( \frac{\partial n_i}{\partial s_i} \right) \left( \frac{\partial T}{\partial s_i} \right) + \left( \frac{\partial n_i}{\partial \mu_i} \right) \left( \frac{\partial n_i}{\partial s_i} \right) \frac{\partial \mu_i}{\partial s_i} \tau_R$$  \hspace{1cm} (11)

In order to estimate $\tau_R^i$ we need to know the cross sections of all the possible processes through which hadron, $i$ interacts with all the hadrons and resonances. As all these required cross sections are not known presently and we are interested in studying the effects of phase space on $\zeta$, a constant cross sections for all the hadronic processes is assumed in the spirit of Ref. [31] and treat the relaxation time as a constant to write down the ratio $\zeta/\tau_R$ as:

$$\frac{\zeta}{\tau_R} = \sum_i \left[ \left( \frac{\partial P_i}{\partial T} \right) \frac{\partial n_i}{\partial T} \right] \left[ \left( \frac{\partial n_i}{\partial \mu_i} \right) \frac{\partial \mu_i}{\partial \mu_i} \right] \left( \frac{\partial n_i}{\partial s_i} \right) \left( \frac{\partial T}{\partial s_i} \right) + \left( \frac{\partial n_i}{\partial \mu_i} \right) \left( \frac{\partial n_i}{\partial s_i} \right) \frac{\partial \mu_i}{\partial s_i} \right] s_i \tau_R$$  \hspace{1cm} (12)
Eq. 12 has been used to estimate the ζ for HRG in this work. Each term in Eq. 12 can be computed from $P_i$ and its derivatives, where $P_i$ is given by,

$$P_i = \frac{T}{V} \ln Z_i (T, V, \mu_i)$$

$$= \sum_i \frac{a_i T^4}{2\pi^2} \int_0^\infty dx x^2 \ln \left[ 1 + a \exp \left( -\sqrt{x^2 + \left( \frac{m_i}{T} \right)^2} - \mu_i \right) \right]$$

(13)

where $a = -1$ for mesons (Bosons) and +1 for baryons (Fermions). Consequently the corresponding entropy density ($s_i$), number density ($n_i$) and energy density ($\epsilon_i$) are given by,

$$s_i = \frac{\partial P_i}{\partial T}, \quad n_i = \frac{\partial P_i}{\partial \mu_i}, \quad \epsilon_i = T \frac{\partial P_i}{\partial T} - P_i + \mu_i \frac{\partial P_i}{\partial \mu_i}$$

(14)

It can be easily checked from Eqs. 12, 13 and 14 that when the hadron, $i$ is massless with $\epsilon_i = 3P_i$ then $\zeta_i$ vanishes as $(\frac{\partial P_i}{\partial T}) \left( \frac{\partial \epsilon_i}{\partial \mu_i} \right) - \left( \frac{\partial \epsilon_i}{\partial \mu_i} \right) = 0$. Eqs. 13 and 14 can be used to reproduce the known thermodynamic expressions for pressure, entropy density, number density, energy density, etc., both for relativistic and non-relativistic limits. In turn these quantities can be used in Eq. 12 to estimate the bulk viscosity to relaxation time ratio.

III. RESULTS

The bulk viscosity, ζ for a HRG system can be calculated by using Eq. 12 where all the particles as listed in the Particle Data Book [32] of mass upto 2.5 GeV are included. In order to understand the results for the full HRG, we first investigate a system with single species of mass, $m_l$ and then the other with two different species of masses $m_l$ and $m_h$ with $m_h > m_l$. We study the interplay of the two different mass scales on the temperature dependence of ζ and its correlation with the CSB (conformal symmetry breaking) measure, $\Delta = \epsilon - 3P$ [33].

We have plotted $\Delta/T^4$ as function of $T$ in Fig. 1 for systems with different composition and masses to elucidate the role of hadronic masses in $\Delta$ and subsequently in bulk viscosity (Fig.3). The curves in Fig.1 stand for different values of the ratio, $m_h/m_l$. The qualitative features of the plots remain same when we replace the bosons by fermions. Results displayed in Fig. 1 indicate that $\Delta/T^4 \to 0$ both for the non-relativistic $m/T >> 1$ and massless limits $m/T \to 0$. To understand the variation of $\Delta/T^4$ with $T/m$, first consider a system at temperature $T$ with single species of mass $m_l$. In the high temperature limit the pressure-energy density relation becomes $P = \epsilon/3$ giving rise to $\Delta = 0$.  

5
FIG. 1: Variation of $(\epsilon - 3P)/T^4$ with $T/m_l$ for different values of the ratio of the masses of the heavy to light particle, $m_h/m_l$ for vanishing chemical potential.

In the limit of large $m/T$ (small $T/m$) the CSB measure varies as: $\Delta \sim e^{-m/T}$, becomes vanishingly small. That is for both small and large $T$, $\Delta \to 0$, with an intermediary peak at $T/m_l \sim 0.5$.

Now we consider a two-particle system with masses $m_l$ and $m_h$ ($m_h > m_l$). First consider the case with $m_h/m_l = \infty$. The heavier particle does not contribute to the thermodynamics. Hence this is essentially a single particle system. We find a single peak around $T \sim 0.5m_l$. Next, we plot for the case with $m_h/m_l = 10$. The large separation in the masses of the two particles results in distinct two peaks at $T/m_l \sim 0.5$ and $T/m_l \sim 5$ (i.e. at $T/m_h \sim 0.5$). For $m_h/m_l = 5$, similar structure is found with closer peaks and reduced dip between the two peaks. We observe that

FIG. 2: Variation of the square of the speed of sound with $T$ for zero and non-zero net baryon density.
FIG. 3: Variation of $\zeta/s$ with $T/m_l$ for different values of the ratio of the masses of the heavy to light particle, $m_h/m_l$ for vanishing chemical potential.

The peak associated with the lighter particle has converted to a shoulder-like structure. Finally for $m_h/m_l = 2$, the peaks have partially merged and we are left with only a single (broader) peak at $T \sim 0.5 (0.5m_l + 0.5m_h) = 0.75m_l$.

The presence of the massive hadrons does not allow the system to satisfy the relation $\epsilon = 3P$, i.e. the conformal symmetry is broken for the entire range of $T$ both for zero and non-zero $\mu_B$. This is evident from the estimation of the speed of sound ($c_s^2$) which remains below $1/\sqrt{3}$ (Fig. 2) for the entire $T$ range considered.

It is expected that the temperature and mass dependences of CSB discussed above will be reflected on $\zeta/s$ as these quantities are correlated [34]. For demonstrating the phase space dependence of $\zeta$ we assume $\tau_R \sim 1$ fm/c. The results are depicted in Fig. 3. The peaks corresponding to the single and two particles systems (with $m_h/m_l = 2$) get blurred. For $m_h/m_l = 5$ and 10 the peaks at lower $T/m_l$ get smeared, however, for higher $T/m_l$ the peaks become broader but distinctly visible. In summary, the $\zeta$ and $\Delta$ have similar $T$-variation with broader peaks in the later quantity. For a system with many particles the $\zeta$ will be a superposition of results obtained for each of the different hadrons with their respective masses. We use Eqs. 12, 13 and 14 to estimate the ratio $\zeta/\tau_R$ for a system of single particles with mass $m$ in the limits of $m/T \rightarrow 0$ and $m/T \rightarrow \infty$. We find that $\zeta/\tau_R \sim (m/T)^2$ for $m/T << 1$ and $\zeta/\tau_R \sim e^{-(m-\mu)/T}$ for $m/T >> 1$, i.e. the bulk viscosity vanishes both in the relativistic and non-relativistic limits - a well known result in the literature.

Now we turn our attention to the HRG system. The study of HRG is important because lattice
QCD results indicate that at lower temperatures, the HRG is a good proxy for the effective degrees of freedom of the strongly interacting matter. Therefore, it will be very useful to study the properties of HRG if it is away from equilibrium. We estimate the bulk viscosity of the HRG when it is slightly away from equilibrium - a situation may be confronted during the evolution of matter formed in nuclear collisions at relativistic energies.

The lightest hadron is the pion with $m_\pi \sim 140$ MeV and the next hadron (kaon) is heavier by about 350 MeV. The hadronic degrees of freedom are expected to survive up to $T \sim 150 - 160$ MeV. Thus, in this temperature domain, $m/T \gg 1$ for all hadrons except pion. This implies that for the full HRG system, we should expect to see features qualitatively similar to the non-relativistic end of the plots in Fig. 3 i.e. we should see an increasing trend of $\zeta$ with $T$ for constant $\tau_R$.

In Figs. 4 and 5 we have displayed the temperature variation of $\zeta/\zeta_0$ and $R_\zeta = (\zeta/s)/(\zeta_0/s_0)$ [$\zeta_0 = \zeta(T = 150$ MeV) $s_0 = s(T = 150$ MeV)] respectively for different values of baryonic chemical potential, $\mu$ within the framework of HRG model. Please note that in the ratios of $\zeta$ the effects of the constant relaxation time get cancelled. We find that the bulk viscosity increases with both temperature and baryonic density as expected from the discussions above.

The results displayed so far may be improved by the following two considerations: (i) by making the $\tau_R$ a $T$ and $\mu$ dependent quantity. We have taken constant $\tau_R$ so far, however, the relaxation time $\tau_R$ should depend on the thermodynamic state of the matter, i.e. it should vary with $T$ and $\mu$. To get the $T$ and $\mu$ dependence of $\tau_R$ we can use the relation,

$$\tau_R^j = \frac{1}{\sum_i \sigma_{ij} n_i \langle p_i \rangle \langle E_i \rangle}$$

(15)

However, as mentioned above we will assume constant cross section i.e. $\sigma_{ij} = \sigma$ and a single relaxation time for the system, $\tau_R^j = \tau_R$. We find that the relaxation time, $\tau_R$ reduces with the $T$ and $\mu$ i.e. the hotter and denser systems relax faster.

(ii) By including the Hagedorn density of states (HDS) [35, 36] for counting the resonances at higher temperatures in estimating the bulk viscosity [37]. For this purpose we have used the following mass spectrum in evaluating relevant thermodynamic quantities:

$$\rho(m) = \sum_i d_i \delta(m - m_i) + \frac{a_0}{(m^2 + m_0^2)^{5/2}} e^{m/T_H}$$

(16)

where the first part is the standard discrete contribution from all the PDG resonances while the second part is the additional contribution from the continuous HDS. $d_i = 2S_i + 1$ is the degeneracy
due to the spin of the $i$th hadron with mass $m_i$. $a_0$, $m_0$ and $T_H$ are parameters extracted from fits of the HDS to the observed spectrum. Here we have used $a_0 = 0.744$ GeV$^{3/2}$, $m_0 = 0.529$ GeV and $T_H = 180$ MeV as in [36]. The thermodynamic quantities like the energy density ($\epsilon$) may be calculated by using the formula: $\epsilon = \int dm \rho(m) \int \frac{dp}{(2\pi)^3} \sqrt{p^2 + m^2} f(p)$, where $f(p)$ is the appropriate thermal distribution for Bosons or Fermions.

With the inclusion of these two effects as described above the temperature dependence of $R_\zeta$, the normalized bulk viscosity has been evaluated and the result is displayed in Fig. 6. The $T$ and $\mu$ dependence of $\tau_R$ has changed the results both quantitatively and qualitatively. In sharp contrast to the results displayed in Fig. 5 the ratio decreases with temperature as observed also in Refs. [22, 23]. However, we recall that the results depicted in Fig 6 contains temperature dependent $\tau_R$ which decreases with $T$ as: $\tau_R^{-1} \propto T^2 e^{-m/T}$ (due to $T$ dependent density and average velocity) for constant cross sections as assumed here. This temperature variation seems to be stronger than the $T$ dependent growth of the right hand side (rhs) of Eq.12. As a result the ratio, $R_\zeta$ which is a product of these two factors - rhs of Eq.12 and $\tau_R$ decreases with $T$. The outcome of the present work has particularly been compared with the results obtained by solving Boltzmann equation - in (a) relaxation time approximation with excluded volume effects in HRG [25] and (b) Chapman-Enskog approximations for interacting pion gas [23]. At higher temperature all the results converge. At lower temperature the results from different models tend to differ. However, the outcome of the present work with the inclusion of HDS (solid line) agrees well with the results of Ref. [23].

In Fig.7 the variation of CSB measure with temperature is displayed for HRG system with (solid line) and without (dashed line) HDS. A significant enhancement of CSB is observed for temperature above pion mass, more so for the case where HDS are used in addition to the standard PDG hadrons. This is expected as the inclusion of additional Hagedorn resonances result in a stronger breaking of the conformal symmetry. We have compared the CSB measure obtained in the HRG model with that of LQCD data and find good agreement upto about $T \sim 1.1m_\pi$ which is the region of interest here. It is expected that the observed variation of CSB with $T$ will also reflect in the $T$ dependence of $\zeta$.

In HICRE, for a given $\sqrt{s_{NN}}$ a hot and dense medium will be created with entropy, $S$ and net baryon number, $N_B$. For an isentropic expansion of the system, the $S/N_B$ ratio will be constant throughout the evolution. Therefore, the system will evolve along a trajectory in the $T - \mu$ plane corresponding to a constant $S/N_B$ contour. We use this evidence to evaluate $\zeta$ along the constant
FIG. 4: The temperature variation of $\zeta(T)/\zeta_0$ is shown here with $\zeta_0 = \zeta(T = 150) \text{ MeV}$. 

$S/N_B$ contours in the $T - \mu$ plane for $S/N_B = 30, 45$ and 300. These values of $S/N_B$ may correspond to AGS (FAIR), SPS and RHIC collision conditions [38] (see also [39]). The variation of $R_\zeta$ with $T$ along these contours are depicted in Fig. 8. The relaxation time is estimated by using Eq. 15 with the value of cross section, $\sigma = \pi \text{ fm}^2$ as in [31]. It is observed that the magnitude of $\zeta/s$ at low temperature domain is higher for $S/N_B = 30$ (corresponds to higher $\mu$) compared to $S/N_B = 300$. At higher temperatures the values of bulk viscosity seems to converge for all the values of $S/N_B$.

In HICRE the centre of mass energy ($\sqrt{s_{NN}}$) can be connected to the values of $T$ and $\mu$ at the chemical freeze-out curve by analyzing the hadronic yields [40, 41] as follows. In Ref. [41] the chemical freeze-out curve has been parametrized as: $T(\mu) = 0.166 - 0.139\mu^2 - 0.053\mu^4$ and the $\sqrt{s_{NN}}$ dependence of $\mu$ has been fitted with $\mu = 1.308(1+0.273\sqrt{s_{NN}})^{-1}$. Using these parameterizations the normalized bulk viscosity has been estimated and the results have been displayed as a function of $\sqrt{s_{NN}}$ in Fig 9 with (dashed line) and without (solid) incorporating the HDS. At high $\sqrt{s_{NN}}$ the inclusion of Hagedorn spectra does not make any difference, however, at lower values of $\sqrt{s_{NN}}$ Hagedorn spectra enhances the bulk viscosity. The important point to be noted here is that at lower beam energy the bulk viscosity is larger (for larger $\mu$). Therefore, $\zeta$ will play a more important role at FAIR than LHC experiments.
FIG. 5: Depicts the temperature variation of the bulk viscosity to entropy density ratio normalized to the value of the ratio at $T = 150$ MeV (see text) with constant relaxation time.

FIG. 6: Variation of $R_\zeta$ (see text) as a function of $T$ for hadronic resonances upto mass 2.5 GeV with (red line) and without (blue dashed line) HDS including $T$ dependent relaxation time estimated by using Eq. 15. We have also displayed the same quantity as obtained in other works [23, 25].

IV. SUMMARY AND DISCUSSIONS

Using grand canonical ensemble the bulk viscosity of the hadronic medium has been estimated within the ambit of the HRG model approach. The grand canonical ensemble of HRG provides the mean hadron number as well as the fluctuations in the chemical composition of the hadronic medium. These fluctuations grant a non-zero divergence for the hadronic fluid flow velocity, offering
FIG. 7: The variation \((\epsilon - 3P)/T^4\) with \(T/m_\pi\) with (solid line) and without (dashed line) HDS. A comparison has been made with continuum extrapolated (2+1) lattice QCD data with physical quark masses \([42]\).

an opportunity to evaluate the hadronic bulk viscosity \(\zeta\) upto a relaxation time. First we have considered both single and two hadronic systems with different masses to exemplify the role of hadronic masses on the CSB and bulk viscosity. Then we proceed to evaluate the \(\zeta\) for HRG model and eventually include the HDS in the calculations. We find that the inclusion of HDS enhances the bulk viscosity of the system at lower \(\sqrt{s_{NN}}\). We would like to note here that recently it has been shown that a considerable improvement in the HRG framework in describing LQCD data is obtained by simultaneous inclusion of the HDS as well as a hard core repulsion between the hadrons within an excluded volume approach \([43]\). In the low temperature domain (\(T \simeq 150\) MeV) it is found that the pressure and energy density estimated without excluded volume effect remain within error bars of the lattice QCD results \([43]\). However, the agreement with the energy density turns to be better at higher \(T\) with excluded volume effect. We leave this interesting exercise about the simultaneous role of the finite size of the hadrons as well as the HDS on the bulk viscosity of the hadronic medium for the future. We also estimate \(\zeta\) along the constant \(S/N_B\) contours and find that \(\zeta/s\) is enhanced for lower \(S/N_B\). \(\zeta/s\) has also been evaluated along the chemical freeze-out curve obtained from the parameterization of hadronic yields \([41]\) and found that the \(\zeta/s\) is larger at FAIR than LHC energy region. This indicates that the bulk viscosity will play more crucial role in nuclear collision at FAIR than LHC energies.

A few words on the \(T\) and \(\mu\) dependence of the bulk viscosity arising from the phase space factors are in order here. In the present work we have assumed a constant \(\sigma\) for all the hadronic processes,
FIG. 8: The variation of bulk viscosity to entropy density ratio (normalized at $T = 150$ MeV) with temperature along constant $S/N_B$ contours for $S/N_B = 30, 45$ and $300$.

however, in reality the situation could be more complex with $T$ and $\mu$ dependent cross sections to be considered for all types of possible reactions undergoing in the medium. As mentioned earlier the cross sections for the hadronic reactions involving all the resonances and Hagedorn states are not known presently. Therefore, we have assumed a constant cross section $[31]$ and demonstrated the $T$ and $\mu$ dependence of the bulk viscosity originating from the phase space factors only. The lack of these cross sections does not allow us to estimate the relaxation time from microscopic interactions. Therefore, we assume that the relaxation time of a hadron $h$ is $\tau_h = \tau_R \pm \delta_h$, where $\tau_R$ is the average relaxation time scale of the full system and $\delta_h$ is the deviation of $\tau_h$ from it. In this work, it is assumed that $\tau_R >> \delta_h$ for all $h$.

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V. APPENDIX

In this appendix we will show in details the computation of the bulk viscosity. For simplicity we assume that the state of the hadronic matter under consideration can be described by two independent thermodynamic variables: i.e. temperature ($T$) and baryonic chemical potential ($\mu$). When we take partial derivative w.r.t. $T$ it is understood that $\mu$ is constant and vice-versa and hence we do not
FIG. 9: The variation of $R_\zeta$ as a function of $\sqrt{s_{NN}}$ with (red line) and without (blue line) HDS.

mention this explicitly. The differential of $P(T,\mu)$ can be written as

$$\frac{dP}{dT} = (\frac{\partial P}{\partial T})_T dT + (\frac{\partial P}{\partial \mu})_T d\mu$$

$$\frac{\partial P}{\partial n} = (\frac{\partial P}{\partial T})_T \frac{\partial T}{\partial n} + (\frac{\partial P}{\partial \mu})_T \frac{\partial \mu}{\partial n}$$

$$(\frac{\partial P}{\partial n})_\epsilon = (\frac{\partial P}{\partial T})_T (\frac{\partial T}{\partial n})_\epsilon + (\frac{\partial P}{\partial \mu})_T (\frac{\partial \mu}{\partial n})_\epsilon$$

where $(\frac{\partial n}{\partial T})_\epsilon = (\frac{\partial n}{\partial T}) + (\frac{\partial n}{\partial \mu}) (\frac{\partial \mu}{\partial T})$, $(\frac{\partial n}{\partial \mu})_\epsilon = (\frac{\partial n}{\partial T}) + (\frac{\partial n}{\partial \mu}) (\frac{\partial T}{\partial \mu})$. We have $(\frac{\partial \mu}{\partial n})_\epsilon = -\frac{(\frac{\partial n}{\partial \mu})}{(\frac{\partial n}{\partial T})}$ along the constant $\epsilon$ trajectory and finally $\frac{\partial n}{\partial s}$ can be written as $(\frac{\partial n}{\partial s}) = (\frac{\partial n}{\partial T}) (\frac{\partial T}{\partial s}) + (\frac{\partial n}{\partial \mu}) (\frac{\partial \mu}{\partial s})$. Thus, bulk viscosity to entropy density ratio in units of the relaxation time scale can be expressed as:

$$\frac{\zeta}{s\tau_R} = -\left(\frac{\partial P}{\partial n}\right)_\epsilon (\frac{\partial n}{\partial s})$$

$$= -\left(\frac{\partial P}{\partial T} - \frac{\partial P}{\partial \mu} \frac{\partial \mu}{\partial T}\right) + \left(\frac{\partial P}{\partial \mu} - \frac{\partial P}{\partial T} \frac{\partial T}{\partial \mu}\right) \left(\frac{\partial n}{\partial T} (\frac{\partial T}{\partial s}) + \frac{\partial n}{\partial \mu} (\frac{\partial \mu}{\partial s})\right)$$

(17)

Now all the derivatives in Eq. (17) can be evaluated starting from the expression of $\ln Z$ within the HRG model. To begin with, the partition function $Z(T,V,\mu)$ is given by

$$\ln Z^{GC}(T,V,\{\mu\}) = \sum_i \frac{g_i}{2\pi^2} V T^3 \sum_{n=1}^{\infty} \frac{\mp 1}{n^4} x_i^2 K_2(x_i) e^{\mp \mu K_2(x_i)}$$

The pressure $P$ is obtained by operating $T \frac{\partial}{\partial V}$ on $\ln Z^{GC}$

$$P^{GC}(T,V,\{\mu\}) = \sum_i \frac{g_i}{2\pi^2} T^4 \sum_{n=1}^{\infty} \frac{\mp 1}{n^4} x_i^2 K_2(x_i) e^{\mp \mu K_2(x_i)}$$
where \( x_i = n m_i / T \) and \( y_i = n \mu_i / T \) introduced for brevity in notation. Further the derivatives of \( P \) are obtained as

\[
\left( \frac{\partial P_{GC}}{\partial T} \right) = \frac{1}{V} \left\{ \ln Z_{GC} + \frac{1}{T} \left( E_{GC} - \sum_i \mu_i N_{i,GC} \right) \right\}
\]

\[
\left( \frac{\partial P_{GC}}{\partial \mu_i} \right) = \frac{1}{V} \sum_i N_{i,GC}
\]

The particle number and its derivatives are given by

\[
N_{i,GC}(T, V, \mu_i) = T \frac{\partial \ln Z_{GC}}{\partial \mu_i}
\]

\[
N_{i,GC}(T, V, \mu_i) = \frac{g_i}{2 \pi^2} V T^3 \sum_{n=1}^{\infty} \frac{(\pm 1)^{(n+1)}}{n^3} x_i^2 K_2(x_i) e^{y_i}
\]

\[
\left( \frac{\partial N_{i,GC}}{\partial T} \right) = \frac{g_i V m_i^2}{2 \pi^2} \sum_{n=1}^{\infty} \frac{(\pm 1)^{(n+1)}}{n} e^{y_i} \left[ \frac{x_i}{2} K_1(x_i) + (1 - y_i) K_2(x_i) + \frac{x_i}{2} K_3(x_i) \right]
\]

\[
\left( \frac{\partial N_{i,GC}}{\partial \mu_i} \right)_T = \frac{B_i g_i}{2 \pi^2} V T^2 \sum_{n=1}^{\infty} \frac{(\pm 1)^{(n+1)}}{n^2} x_i^2 K_2(x_i) e^{y_i}
\]

The energy and its derivatives are given by

\[
E_{GC}(T, V, \{\mu_i\}) = T^2 \frac{\partial \ln Z_{GC}}{\partial T} + \sum_i \mu_i N_{i,GC}
\]

\[
E_{GC}(T, V, \{\mu_i\}) = \sum_i C_i T \sum_{n=1}^{\infty} \frac{(\pm 1)^{(n+1)}}{n^2} e^{y_i} \left[ \frac{x_i}{2} K_1(x_i) + (1 - y_i) K_2(x_i) + \frac{x_i}{2} K_3(x_i) \right] + \sum_i \mu_i N_{i,GC}
\]

\[
\left( \frac{\partial E_{GC}}{\partial T} \right)_\mu = \sum_i C_i \sum_{n=1}^{\infty} \frac{(\pm 1)^{(n+1)}}{n^2} e^{y_i} \left[ \frac{x_i^2}{4} K_0(x_i) + (-x_i y_i + x_i) K_1(x_i) \right. \\
\left. + (y_i^2 + \frac{x_i^2}{2}) + 2(1 - y_i) K_2(x_i) + (-x_i y_i + x_i) K_3(x_i) + \frac{x_i^2}{4} K_4(x_i) \right] + \sum_i \mu_i \frac{\partial N_i}{\partial T}
\]

\[
\left( \frac{\partial E_{GC}}{\partial \mu} \right)_T = \sum_i C_i \sum_{n=1}^{\infty} \frac{(\pm 1)^{(n+1)}}{n} e^{y_i} \left[ \frac{x_i}{2} K_1(x_i) - y_i K_2(x_i) + \frac{x_i}{2} K_3(x_i) \right] + \sum_i \left(N_i + \mu_i \frac{\partial N_i}{\partial \mu} \right)
\]
where
\[ C_i = g_i \frac{VT}{2\pi^2 m_i^2} \]  

Finally the entropy and its derivatives are given by
\[ S^{GC}(T, V, \{\mu_i\}) = \frac{1}{T} \left\{ E^{GC}(T, V, \{\mu_i\}) + P^{GC}(T, V, \{\mu_i\})V - \sum_i \mu_i N_i^{GC}(T, V, \mu_i) \right\} \]
\[ \frac{\partial S^{GC}}{\partial T} = -\frac{S}{T} + \frac{1}{T} \left\{ \left( \frac{\partial E^{GC}}{\partial T} \right) + V \frac{\partial P}{\partial T} - \sum_i \mu_i \left( \frac{\partial N_i^{GC}}{\partial T} \right) \right\} \]
\[ \frac{\partial S^{GC}}{\partial \mu_i} = \frac{1}{T} \left\{ \left( \frac{\partial E^{GC}}{\partial \mu_i} \right) + V \frac{\partial P}{\partial \mu_i} - \sum_i N_i - \sum_i \mu_i \left( \frac{\partial N_i^{GC}}{\partial \mu_i} \right) \right\} \]

where we have used
\[ \frac{\partial}{\partial \mu_i} \sum \frac{\partial \mu_i}{\partial \mu} \frac{\partial}{\partial \mu_i} \]

The speed of sound, \( c_s \) can be used as a regulator for CSB. A system with \( c_s^2 \to 1/3 \) will indicate the restoration of CSB. Therefore, we the expression to estimate, \( c_s \) within the ambit of present model is recalled below.
\[ c_s^2 = \left( \frac{\partial P}{\partial \epsilon} \right)_{s/n_B} \]
\[ = \left( \frac{\partial P}{\partial T} \right)_{s/n_B} \left( \frac{\partial \epsilon}{\partial T} \right)_{s/n_B} + \left( \frac{\partial P}{\partial \mu} \right)_{s/n_B} \left( \frac{\partial \mu}{\partial \epsilon} \right)_{s/n_B} \]

Now for constant \( s/n_B, d(s/n_B) = \frac{\partial(s/n_B)}{\partial T} dT + \frac{\partial(s/n_B)}{\partial \mu} d\mu = 0 \). This implies that \( \frac{\partial T}{\partial \mu} = -\frac{\partial(s/n_B)}{\partial \epsilon} \). Thus
\[ \left( \frac{\partial \epsilon}{\partial T} \right)_{s/n_B} = \frac{\partial \epsilon}{\partial T} - \frac{\partial \epsilon}{\partial \mu} \left( \frac{\partial(s/n_B)}{\partial T} \right)_{s/n_B} \]
\[ \left( \frac{\partial \epsilon}{\partial \mu} \right)_{s/n_B} = \frac{\partial \epsilon}{\partial \mu} - \frac{\partial \epsilon}{\partial T} \left( \frac{\partial(s/n_B)}{\partial \mu} \right)_{s/n_B} \]

Thus finally the expression for \( c_s \) turns out to be
\[ c_s^2 = \left( \frac{\partial P}{\partial \mu} \right)_{T} \left( \frac{\partial(s/n_B)}{\partial \mu} \right)_{T} + \left( \frac{\partial P}{\partial \epsilon} \right)_{T} + \left( \frac{\partial P}{\partial \epsilon} \right)_{T} \left( \frac{\partial(s/n_B)}{\partial \epsilon} \right)_{T} \left( \frac{\partial(s/n_B)}{\partial \mu} \right)_{T} \]

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