Local Well-Posedness of the Cauchy Problem for a $p$-Adic Nagumo-Type Equation

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Abstract—We introduce a new family of $p$-adic nonlinear evolution equations. We establish the local well-posedness of the Cauchy problem for these equations in Sobolev-type spaces. For a certain subfamily, we show that the blow-up phenomenon occurs and provide numerical simulations showing this phenomenon.

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1. INTRODUCTION

Nowadays, the theory of linear partial pseudo-differential equations for complex-valued functions over $p$-adic fields is a well-established branch of mathematical analysis, see e.g. [1–6, 12–16, 22–25, 27–33], and references therein. Meanwhile very little is known about nonlinear $p$-adic equations. We can mention some semilinear evolution equations solved using $p$-adic wavelets [1, 24], a kind of equations of reaction-diffusion type and Turing patterns studied in [31, 33], a $p$-adic analog of one of the porous medium equation [17, 22], the blow-up phenomenon studied in [4], and non-linear integro-differential equations connected with $p$-adic cellular networks [30].

In this article we introduce a new family of nonlinear evolution equations that we have named as $p$-adic Nagumo-type equations:

$$u_t = -\gamma D_x^\alpha u - u^3 + (\beta + 1) u^2 - \beta u + P(D_x)(u^m), x \in \mathbb{Q}_p^n, t \in [0, T],$$

where $\gamma > 0$, $\beta \geq 0$, $D_x^\alpha$, $\alpha > 0$, is the Taibleson operator, $m$ is a positive integer and $P(D_x)$ is an operator of degree $\delta$ of the form $P(D) = \sum_{j=0}^{k} C_j D^{\delta_j}$, where the $C_j \in \mathbb{R}$ and $\delta_k = \delta$. We establish the local well-posedness of the Cauchy problem for these equations in Sobolev-type spaces, see Theorem 4.1. For a certain subfamily, we show that the blow-up phenomenon occurs, see Theorem 5.1, and we also provide numerical simulations showing this phenomenon.

The theory of Sobolev-type spaces use here was developed in [34], see also [18, 25]. This theory is based in the theory of countably Hilbert spaces of Gel’fand–Vilenkin [8]. Some generalizations are
presented in [9, 10]. We use classical techniques of operator semigroups, see e.g. [3, 20]. The family of evolution equations studied here contains as a particular case, equations of the form
\[ u_t = -\gamma D_x^\alpha u - u^3 + (\beta + 1) u^2 - \beta u, \tag{1.1} \]
where \( x \in \mathbb{Q}_p^n \), \( t \in [0, T] \), \( D_x^\alpha \) is the Taibleson operator, that resemble the classical Nagumo-type equations, see e.g. [21].

In [7], the authors study the equations
\[ u_t = D_{xx} - u (u - \kappa) (u - 1) - \varepsilon u_m^\alpha, \tag{1.2} \]
where \( D > 0 \), \( \kappa \in (0, \frac{1}{2}) \), \( \varepsilon > 0 \), \( x \in \mathbb{R}, t > 0 \). They establish the local well-posedness of the Cauchy problem for these equations in standard Sobolev spaces. There are several crucial differences between (1.1) and (1.2). The operators \( u_{xx}, u_m^\alpha \) are local while the operators \( D_x^\alpha, P(D_x)^m \) are non-local. The \( p \)-adic heat equation \( u_t = -\gamma D_x^\alpha u \) has an arbitrary order of pseudo-differentiability \( \alpha > 0 \) in the spatial variable, while in the classical fractional heat equation \( u_t = D_{xx} - u \), the degree of pseudo-differentiability \( \mu \in (0, 2) \). This implies that the Markov processes attached to \( u_t = -\gamma D_x^\alpha u \) are completely different to the ones attached to \( u_t = D_{xx} \). In other words, the diffusion mechanisms in (1.1) and (1.2) are completely different. Notice that our nonlinear term involves pseudo-derivatives of arbitrary order \( P(D_x)^m \), while in [7] only of first order \( u_m^\alpha \). Of course, the \( p \)-adic Sobolev spaces behave completely different from their real counterparts, but the semigroup techniques are the same in both cases, since time is a non-negative real variable.

The article is organized as follows. In section 2, we review some basic aspects of the \( p \)-adic analysis and fix the notation. In section 3, we present some technical results about Sobolev-type spaces and \( p \)-adic pseudo-differential operators. In section 4, we show the local well-posedness of the \( p \)-adic Nagumo-type equations, see Theorem 4.1. In section 5, we show a subfamily of \( p \)-adic Nagumo-type equations whose solutions blow-up in infinite time, see Theorem 5.1. In section 6, we present a numerical simulation showing the blow-up phenomenon.

2. \( p \)-ADIC ANALYSIS: ESSENTIAL IDEAS

In this section, we collect some basic results on \( p \)-adic analysis that we use through the article. For a detailed exposition the reader may consult [1, 14, 26, 29].

2.1. The Field of \( p \)-Adic Numbers

Along this article \( p \) will denote a prime number. The field of \( p \)-adic numbers \( \mathbb{Q}_p \) is defined as the completion of the field of rational numbers \( \mathbb{Q} \) with respect to the \( p \)-adic norm \( | \cdot |_p \), which is defined as
\[ |x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma a \end{cases}, \]
where \( a \) and \( b \) are integers coprime with \( p \). The integer \( \gamma := ord(x) \), with \( ord(0) := +\infty \), is called the \( p \)-adic order of \( x \).

Any \( p \)-adic number \( x \neq 0 \) has a unique expansion of the form
\[ x = p^{ord(x)} \sum_{j=0}^{\infty} x_j p^j, \]
where \( x_j \in \{0, \ldots, p-1\} \) and \( x_0 \neq 0 \). By using this expansion, we define the fractional part of \( x \in \mathbb{Q}_p \), denoted \( \{x\}_p \), as the rational number
\[ \{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } ord(x) \geq 0 \\ p^{ord(x)} \sum_{j=0}^{-ord(x)-1} x_j p^j & \text{if } ord(x) < 0. \end{cases} \]
2.2. Topology of $\mathbb{Q}_p^n$

For $r \in \mathbb{Z}$, denote by $B^n_r(a) = \{x \in \mathbb{Q}_p^n; ||x - a||_p \leq p^r\}$ the ball of radius $p^r$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $B^n_r(0) := B^n_r$. Note that $B^n_r(a) = B_r(a_1) \times \cdots \times B_r(a_n)$, where $B_r(a_i) := \{x_i \in \mathbb{Q}_p; ||x_i - a_i||_p \leq p^r\}$ is the one-dimensional ball of radius $p^r$ with center at $a_i \in \mathbb{Q}_p$. The ball $B^n_r$ equals the product of $n$ copies of $B_0 = \mathbb{Z}_p$, the ring of $p$-adic integers. We also denote by $S^n_r(a) = \{x \in \mathbb{Q}_p^n; ||x - a||_p = p^r\}$ the sphere of radius $p^r$ with center at $a = (a_1, \ldots, a_n) \in \mathbb{Q}_p^n$, and take $S^n_r(0) := S^n_r$. We notice that $S^n_1 = \mathbb{Z}_p^\times$ (the group of units of $\mathbb{Z}_p$), but $(\mathbb{Z}_p^\times)^n \subsetneq S^n_0$. The balls and spheres are both open and closed subsets in $\mathbb{Q}_p^n$. In addition, two balls in $\mathbb{Q}_p^n$ are either disjoint or one is contained in the other.

As a topological space $(\mathbb{Q}_p^n, || \cdot ||_p)$ is totally disconnected, i.e. the only connected subsets of $\mathbb{Q}_p^n$ are the empty set and the points. A subset of $\mathbb{Q}_p^n$ is compact if and only if it is closed and bounded in $\mathbb{Q}_p^n$, see e.g. [29, Section 1.3], or [1, Section 1.8]. The balls and spheres are compact subsets. Thus $(\mathbb{Q}_p^n, || \cdot ||_p)$ is a locally compact topological space.

Since $(\mathbb{Q}_p^n, +)$ is a locally compact topological group, there exists a Haar measure $d^nx$, which is invariant under translations, i.e. $d^nx(x + a) = d^nx$. If we normalize this measure by the condition $\int_{\mathbb{Z}_p} dx = 1$, then $d^nx$ is unique.

**Notation 1.** We will use $\Omega(p^{-r}||x - a||_p)$ to denote the characteristic function of the ball $B^n_r(a)$. For more general sets, we will use the notation $1_A$ for the characteristic function of a set $A$.

2.3. The Bruhat-Schwartz Space

A complex-valued function $\varphi$ defined on $\mathbb{Q}_p^n$ is called locally constant if for any $x \in \mathbb{Q}_p^n$ there exist an integer $l(x) \in \mathbb{Z}$ such that

$$\varphi(x + x') = \varphi(x) \quad \text{for any } x' \in B^n_{l(x)}. \tag{2.1}$$

A function $\varphi : \mathbb{Q}_p^n \to \mathbb{C}$ is called a Bruhat-Schwartz function (or a test function) if it is locally constant with compact support. Any test function can be represented as a linear combination, with complex coefficients, of characteristic functions of balls. The $\mathbb{C}$-vector space of Bruhat-Schwartz functions is denoted by $\mathcal{D}(\mathbb{Q}_p^n) := \mathcal{D}$. We denote by $\mathcal{D}_R(\mathbb{Q}_p^n) := \mathcal{D}_R$ the $\mathbb{R}$-vector space of Bruhat-Schwartz functions. For $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$, the largest number $l = l(\varphi)$ satisfying (2.1) is called the exponent of local constancy (or the parameter of constancy) of $\varphi$.

We denote by $\mathcal{D}_R^m(\mathbb{Q}_p^n)$ the finite-dimensional space of test functions from $\mathcal{D}(\mathbb{Q}_p^n)$ having supports in the ball $B^m_n$ and with parameters of constancy $\geq l$. We now define a topology on $\mathcal{D}$ as follows. We say that a sequence $\{\varphi_j\}_{j \in \mathbb{N}}$ of functions in $\mathcal{D}$ converges to zero, if the two following conditions hold:

1. There are two fixed integers $k_0$ and $m_0$ such that each $\varphi_j \in \mathcal{D}_{k_0}^{m_0}$;
2. $\varphi_j \to 0$ uniformly.

$\mathcal{D}$ endowed with the above topology becomes a topological vector space.

2.4. $L^p$ Spaces

Given $p \in [1, \infty)$, we denote by $L^p := L^p(\mathbb{Q}_p^n) := L^p(\mathbb{Q}_p^n, d^nx)$, the $\mathbb{C}$-vector space of all the complex-valued functions $g$ satisfying

$$\int_{\mathbb{Q}_p^n} |g(x)|^p d^nx < \infty.$$

The corresponding $\mathbb{R}$-vector spaces are denoted as $L^p_\mathbb{R} := L^p(\mathbb{Q}_p^n) = L^p(\mathbb{Q}_p^n, d^nx)$, $1 \leq p < \infty$. 

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If $U$ is an open subset of $\mathbb{Q}_p^n$, $\mathcal{D}(U)$ denotes the space of test functions with supports contained in $U$, then $\mathcal{D}(U)$ is dense in

$$L^\rho(U) = \left\{ \varphi : U \to \mathbb{C}; \| \varphi \|_\rho = \left\{ \int_U |\varphi(x)|^\rho d^n x \right\}^{\frac{1}{\rho}} < \infty \right\},$$

where $d^n x$ is the normalized Haar measure on $(\mathbb{Q}_p^n, +)$, for $1 \leq \rho < \infty$, see e.g. [1, Section 4.3]. We denote by $L_\mathbb{R}^\rho(U)$ the real counterpart of $L^\rho(U)$. 

### 2.5. The Fourier Transform

Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on $\mathbb{Q}_p$, i.e. a continuous map from $(\mathbb{Q}_p, +)$ into $S$ (the unit circle considered as multiplicative group) satisfying $\chi_p(x_0 + x_1) = \chi_p(x_0)\chi_p(x_1)$, $x_0, x_1 \in \mathbb{Q}_p$. The additive characters of $\mathbb{Q}_p$ form an Abelian group which is isomorphic to $(\mathbb{Q}_p, +)$. The isomorphism is given by $\kappa \to \chi_p(\kappa x)$, see e.g. [1, Section 2.3].

Given $\xi = (\xi_1, \ldots, \xi_n)$ and $y = (x_1, \ldots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \chi_p(\xi \cdot x)\varphi(x)d^n x \quad \text{for} \quad \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the normalized Haar measure on $\mathbb{Q}_p^n$. The Fourier transform is a linear isomorphism from $\mathcal{D}(\mathbb{Q}_p^n)$ onto itself satisfying

$$(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi),
(2.2)$$

see e.g. [1, Section 4.8]. We will also use the notation $\mathcal{F}_{\xi} \varphi$ and $\hat{\varphi}$ for the Fourier transform of $\varphi$.

The Fourier transform extends to $L^2$. If $f \in L^2$, its Fourier transform is defined as

$$(\mathcal{F}f)(\xi) = \lim_{k \to \infty} \int_{|x|_p \leq p^k} \chi_p(\xi \cdot x)f(x)d^n x, \quad \text{for} \quad \xi \in \mathbb{Q}_p^n$$

where the limit is taken in $L^2$. We recall that the Fourier transform is unitary on $L^2$, i.e. $\|f\|_2 = \|\mathcal{F}f\|_2$ for $f \in L^2$ and that (2.2) is also valid in $L^2$, see e.g. [26, Chapter III, Section 2].

### 2.6. Distributions

The $\mathbb{C}$-vector space $\mathcal{D}'(\mathbb{Q}_p^n) := \mathcal{D}'$ of all continuous linear functionals on $\mathcal{D}(\mathbb{Q}_p^n)$ is called the Bruhat–Schwartz space of distributions. Every linear functional on $\mathcal{D}$ is continuous, i.e. $\mathcal{D}'$ agrees with the algebraic dual of $\mathcal{D}$, see e.g. [29, Chapter 1, VI.3, Lemma]. We denote by $\mathcal{D}'_\mathbb{R}(\mathbb{Q}_p^n) := \mathcal{D}_\mathbb{R}$ the dual space of $\mathcal{D}_\mathbb{R}$.

We endow $\mathcal{D}'$ with the weak topology, i.e. a sequence $\{T_j\}_{j \in \kappa}$ in $\mathcal{D}'$ converges to $T$ if $\lim_{j \to \infty} T_j(\varphi) = T(\varphi)$ for any $\varphi \in \mathcal{D}$. The map

$$\mathcal{D}' \times \mathcal{D} \to \mathbb{C}
(T, \varphi) \to T(\varphi)$$

is a bilinear form which is continuous in $T$ and $\varphi$ separately. We call this map the pairing between $\mathcal{D}'$ and $\mathcal{D}$. From now on we will use $(T, \varphi)$ instead of $T(\varphi)$.

Every $f$ in $L^1_{\text{loc}}$ defines a distribution $f \in \mathcal{D}'(\mathbb{Q}_p^n)$ by the formula

$$(f, \varphi) = \int_{\mathbb{Q}_p^n} f(x)\varphi(x)d^n x.$$

Such distributions are called regular distributions. Notice that for $f \in L^2_{\mathbb{R}}, (f, \varphi) = \langle f, \varphi \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2_{\mathbb{R}}$. 

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2.7. The Fourier Transform of a Distribution

The Fourier transform \( \mathcal{F}[T] \) of a distribution \( T \in \mathcal{D}'(\mathbb{Q}_p^n) \) is defined by

\[
(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{Q}_p^n).
\]

The Fourier transform \( T \rightarrow \mathcal{F}[T] \) is a linear (and continuous) isomorphism from \( \mathcal{D}'(\mathbb{Q}_p^n) \) onto \( \mathcal{D}'(\mathbb{Q}_p^n) \).
Furthermore, \( T = \mathcal{F}[\mathcal{F}[T](-\xi)] \).

3. SOBOLEV-TYPE SPACES

The Sobolev-type spaces used here were introduce in [25, 34]. We follow here closely the presentation given in [18, Sections 10.1, 10.2].

We set \( [\xi]_p := \max \left\{ 1, \|\xi\|_p \right\} \) for \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}_p^n \). Given \( \varphi, \varrho \in \mathcal{D}(\mathbb{Q}_p^n) \) and \( s \in \mathbb{R} \), we define the scalar product:

\[
\langle \varphi, \varrho \rangle_s = \int_{\mathbb{Q}_p^n} [\xi]_p^s \overline{\hat{\varphi}(\xi)} \varrho(\xi) d^n\xi,
\]

where the bar denotes the complex conjugate. We also set \( \|\varphi\|_s^2 = \langle \varphi, \varphi \rangle_s \), and denote by \( \mathcal{H}_s := \mathcal{H}_s(\mathbb{Q}_p^n, \mathbb{C}) = \mathcal{H}_s(\mathbb{C}) \) the completion of \( \mathcal{D}(\mathbb{Q}_p^n) \) with respect to \( \langle \cdot, \cdot \rangle_s \). Notice that if \( r, s \in \mathbb{R} \), with \( r \leq s \), then \( \|\cdot\|_r \leq \|\cdot\|_s \) and \( \mathcal{H}_s \hookrightarrow \mathcal{H}_r \) (continuous embedding). In particular,

\[
\cdots \supset \mathcal{H}_{-1} \supset \mathcal{H}_0 \supset \mathcal{H}_1 \supset \mathcal{H}_2 \cdots,
\]

where \( \mathcal{H}_0 = L^2 \). We set

\[
\mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C}) = \mathcal{H}_\infty := \bigcap_{s \in \mathbb{N}} \mathcal{H}_s.
\]

Since \( \mathcal{H}_{[s]+1} \subseteq \mathcal{H}_s \subseteq \mathcal{H}_{[s]} \) for \( s \in \mathbb{R}_+ \), where \([\cdot]\) is the integer part function, then \( \mathcal{H}_\infty = \bigcap_{s \in \mathbb{R}_+} \mathcal{H}_s \). With the topology induced by the family of seminorms \( \{\|\cdot\|_t\}_{t \in \mathbb{N}} \), \( \mathcal{H}_\infty \) becomes a locally convex space, which is metrizable. Indeed,

\[
d(f, g) := \max_{i \in \mathbb{N}} \left\{ 2^{-i} \| f - g \|_i \right\}, \text{ with } f, g \in \mathcal{H}_\infty,
\]

is a metric for the topology of \( \mathcal{H}_\infty \) considered as a convex topological space. The metric space \( (\mathcal{H}_\infty, d) \) is the completion of the metric space \( (\mathcal{D}(\mathbb{Q}_p^n), \| \cdot \|) \), cf. [18, Lemma 10.4]. Furthermore, \( \mathcal{H}_\infty \subset L^\infty \cap C_{unif} \cap L^1 \cap L^2 \), and \( \mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C}) \) is continuously embedded in \( C_0(\mathbb{Q}_p^n, \mathbb{C}) \). This is the non-Archimedean analog of the Sobolev embedding theorem, cf. [18, Theorem 10.15].

Lemma 3.1. If \( s_1 \leq s \leq s_2 \), with \( s = \theta s_1 + (1 - \theta) s_2 \), \( 0 \leq \theta \leq 1 \), then \( \| f \|_s \leq \| f \|_{s_1}^{\theta} \| f \|_{s_2}^{(1-\theta)} \).

Proof. Take \( f \in \mathcal{H}_s \), then by using the Hölder inequality for the exponents \( \frac{1}{q} = \theta, \frac{1}{q'} = 1 - \theta \),

\[
\| f \|_s^2 = \int_{\mathbb{Q}_p^n} [\xi]_p^s |\hat{f}(\xi)|^2 d^n\xi = \int_{\mathbb{Q}_p^n} [\xi]_p^{\theta s_1 + (1-\theta) s_2} |\hat{f}(\xi)|^{2(\theta + (1-\theta))} d^n\xi
\]

\[
= \int_{\mathbb{Q}_p^n} \left( [\xi]_p^{s_1} |\hat{f}(\xi)|^2 \right)^\theta \left( [\xi]_p^{s_2} |\hat{f}(\xi)|^2 \right)^{1-\theta} d^n\xi
\]

\[
\leq \left( \int_{\mathbb{Q}_p^n} [\xi]_p^{s_1} |\hat{f}(\xi)|^2 d^n\xi \right)^\theta \left( \int_{\mathbb{Q}_p^n} [\xi]_p^{s_2} |\hat{f}(\xi)|^2 d^n\xi \right)^{1-\theta} d^n\xi.
\]

}\]
The following characterization of the spaces $\mathcal{H}_s$ and $\mathcal{H}_\infty$ is useful:

**Lemma 3.2** ([18, Lemma 10.8]). (i) $\mathcal{H}_s = \{ f \in L^2; \| f \|_s < \infty \} = \{ T' \in \mathcal{D}; \| T' \|_s < \infty \}$, (ii) $\mathcal{H}_\infty = \{ f \in L^2; \| f \|_\infty < \infty \text{ for any } s \in \mathbb{R}_+ \} = \{ T' \in \mathcal{D}; \| T' \|_s < \infty \text{ for any } s \in \mathbb{R}_+ \}$. The equalities in (i)-(ii) are in the sense of vector spaces.

**Proposition 3.3.** If $s > n/2$, then $\mathcal{H}_s$ is a Banach algebra with respect to the product of functions. That is, if $f, g \in \mathcal{H}_s$, then $fg \in \mathcal{H}_s$ and $\| fg \|_s \leq C(n,s) \| f \|_s \| g \|_s$, where $C(n,s)$ is a positive constant.

**Proof.** By the ultrametric property of $\| \cdot \|_p$, $\| \xi \|_p, \| \eta \|_p \leq \max \left\{ \| \xi - \eta \|_p, \| \eta \|_p \right\}$ for $\xi, \eta \in \mathbb{Q}_p^n$, we have

$$\max \left\{ 1, \| \xi \|_p \right\} \leq \max \left\{ 1, \| \xi - \eta \|_p, \| \eta \|_p \right\},$$

which implies that

$$\left[ \max \left\{ 1, \| \xi \|_p \right\} \right]^s \leq \max \left\{ 1, \| \xi - \eta \|_p^s, \| \eta \|_p^s \right\} = \max \left\{ 1, \| \xi - \eta \|_p, \| \eta \|_p \right\}^s$$

for $s > 0$. Therefore

$$\| \xi \|^s \leq \| \xi - \eta \|^s + \| \eta \|^s.$$  \hspace{1cm} (3.1)

Now, for $f, g \in L^2$, by using (3.1),

$$\| \xi \|_p^d \| g (\xi) \| = \| \xi \|_p^d \int_{\mathbb{Q}_p^2} \| \xi - \eta \|_p^d \| g (\eta) \| d^n \eta \leq \int_{\mathbb{Q}_p^2} \| \xi - \eta \|_p^d \| g (\eta) \| d^n \eta + \int_{\mathbb{Q}_p^2} \| \eta \|_p^d \| g (\eta) \| d^n \eta \leq \| \xi \|_p^d \| g (\xi) \| + \| \xi \|_p^d \| g (\xi) \|.$$  \hspace{1cm} (3.2)

Then

$$\| fg \|_s \leq \left\| \| \xi \|_p^d \| \xi \|_p^d \| g (\xi) \| + \| g (\xi) \| \| \xi \|_p^d \| \xi \|_p^d \| \right\|_2 \leq \left\| \| \xi \|_p^d \| \xi \|_p^d \| g (\xi) \| \right\|_2 + \left\| \| g (\xi) \| \| \xi \|_p^d \| \right\|_2.$$

Since $\| \xi \|_p^d \| \xi \|_p^d \| g (\xi) \|, \| \xi \|_p^d \| g (\xi) \| \in L^2$, by using the Cauchy-Schwarz inequality with $s > n/2$, we have

$$\| g (\xi) \|_1 \leq A(n,s) \| g \|_s \| f \|_s, \| g (\xi) \|_1 \leq A(n,s) \| f \|_s, \| f (\xi) \|_1 \leq A(n,s) \| f \|_s \| g \|_s.$$  \hspace{1cm} (3.3)

Now, by a Young-type inequality, see [26, Chap. II, Theorem 1.7], we obtain that

$$\| fg \|_s \leq \| f \|_s \| g \|_1 + \| g \|_s \| f \|_1 \leq 2A(n,s) \| f \|_s \| g \|_s.$$  \hspace{1cm} (3.4)

\[ \square \]

### 3.1. The Taibleson Operator

Let $\alpha > 0$, the Taibleson operator is defined as

$$(D^\alpha \varphi)(x) = \mathcal{F}_{\xi \rightarrow x}^{-1} (\| \xi \|_p^\alpha (\mathcal{F}_{x \rightarrow \xi} \varphi)),$$

for $\varphi \in \mathcal{D}(\mathbb{Q}_p^n)$. This operator admits the extension

$$(D^\alpha f)(x) = \frac{1 - p^{-\alpha n}}{1 - p^{-\alpha n}} \int_{\mathbb{Q}_p^n} \| y \|_p^{-\alpha n} \{ f(x - y) - f(x) \} d^n y$$
to locally constant functions satisfying
\[
\int_{\|x\|_p > 1} \|x\|^{-\alpha - n} \|f(x)\| d^n x < \infty.
\]

The Taibleson operator \( D^\alpha \) is the \( p \)-adic analog of the fractional derivative. If \( n = 1 \), \( D^\alpha \) agrees with the Vladimirov operator. The operator \( D^\alpha \) does not satisfy the chain rule neither Leibniz formula. We also use the notation \( D^\alpha_x \), when the Taibleson operator acts on functions depending on the variables \( x \in \mathbb{Q}_p^n \) and \( t \geq 0 \).

Given \( 0 = \delta_0 < \delta_1 < \cdots < \delta_{k-1} < \delta_k = \delta \), we define
\[
P(D) = \sum_{j=0}^{k} C_j D^{\delta_j}, \text{ where the } C_j \in \mathbb{R}.
\]

**Lemma 3.4** ([18, Lemma 10.13 and Theorem 10.15]). For \( s \in \mathbb{R}_+ \), the mapping \( P(D) : \mathcal{H}_{s+2\delta} \rightarrow \mathcal{H}_s \) is a well-defined continuous mapping between Banach spaces.

**Lemma 3.5.** Take \( s - 2\delta > n/2 \) and \( f, g \in \mathcal{H}_{s+2\delta} \). Then
\[
\|P(D)(fg)\|_s \leq C(n, s, \delta) \|f\|_{s+2\delta} \|g\|_{s+2\delta},
\]
where \( C(n, s, \delta) \) is a positive constant that depends on \( n, s \) and \( \delta \).

**Proof.** Since \( s > n/2 \) and \( f, g \in \mathcal{H}_{s+2\delta} \), by Proposition 3.3, \( fg \in \mathcal{H}_{s+2\delta} \), and by Lemma 3.4, \( P(D)(fg) \in \mathcal{H}_s \). Now by using Proposition 3.3,
\[
\|P(D)(fg)\|_s \leq \sum_{j=0}^{k} |C_j| \left\| D^{\delta_j} (fg) \right\|_s
\]
\[
= \sum_{j=0}^{k} |C_j| \left( \int_{\mathbb{Q}_p^n} \left| \xi \right|^s \left\| D^{\delta_j} (\tilde{fg}(\xi)) \right\|^2 d^n \xi \right)^{\frac{1}{2}} \leq \sum_{j=0}^{k} |C_j| \left( \int_{\mathbb{Q}_p^n} \left| \xi \right|^{s+2\delta_j} \left\| \tilde{fg}(\xi) \right\|^2 d^n \xi \right)^{\frac{1}{2}}
\]
\[
= \sum_{j=0}^{k} |C_j| \left\| fg \right\|_{s+2\delta_j} \leq \sum_{j=0}^{k} |C_j| C(n, s, \delta_j) \left\| f \right\|_{s+2\delta_j} \left\| g \right\|_{s+2\delta_j}
\]
\[
\leq \left( \sum_{j=0}^{k} |C_j| C(n, s, \delta_j) \right) \|f\|_{s+2\delta} \|g\|_{s+2\delta}.
\]

\( \square \)

4. LOCAL WELL-POSEDNESS OF THE \( p \)-ADIC NAGUMO-TYPE EQUATIONS

4.1. Some Technical Remarks

Let \( X, Y \) Banach spaces, \( T_0 \in (0, \infty) \) and let \( F : [0, T_0] \times Y \rightarrow X \) a continuous function. The Cauchy problem
\[
\begin{cases}
\partial_t u(t) = F(t, u(t)) \\
u(0) = \phi \in Y
\end{cases}
\]
(4.1)
is locally well-posed in \( Y \), if the following conditions are satisfied.
(i) There is \( T \in (0, T_0] \) and a function \( u \in C([0, T]; Y) \), with \( u(0) = \phi \), satisfying the differential equation in the following sense:

\[
\lim_{h \to 0} \left\| \frac{u(t + h) - u(t)}{h} - F(t, u(t)) \right\|_X = 0,
\]

where the derivatives at \( t = 0 \) and \( t = T \) are calculated from the right and left, respectively.

(ii) The initial value problem (4.1) has at most one solution in \( C([0, T]; Y) \).

(iii) The function \( \phi \to u \) is continuous. That is, let \( \{ \phi_n \} \) be a sequence in \( Y \) such that \( \phi_n \to \phi_\infty \) in \( Y \) and let \( u_n \in C \left( [0, T_n]; Y \right) \), resp. \( u_\infty \in C \left( [0, T_\infty]; Y \right) \), be the corresponding solutions. Let \( T \in (0, T_\infty) \), then the solutions \( u_n \) are defined in \([0, T]\) for all \( n \) big enough and

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| u_n(t) - u_\infty(t) \|_Y = 0.
\]

### 4.2. Main Result

Consider the following Cauchy problem:

\[
\begin{cases}
  u \in C \left( [0, T], \mathcal{H}_\alpha \right) \cap C^1 \left( [0, T], \mathcal{H}_\alpha \right); \\
  u_t = -\gamma D^\alpha u - u^3 + (\beta + 1) u^2 - \beta u + P(D_x) (u^m), \; x \in \mathbb{Q}_p^n, \; t \in [0, T]; \\
  u(0) = f_0 \in \mathcal{H}_\alpha,
\end{cases}
\]

where \( T, \gamma, \alpha, \beta > 0 \), and \( m \) is a positive integer. The main result of this work is the following:

**Theorem 4.1.** For \( s > n/2 + 2\delta \), the Cauchy problem (4.2) is locally well-posed in \( \mathcal{H}_\alpha \).

### 4.3. Preliminary Results

We denote by \( V(t) = e^{-(\gamma D^\alpha + \beta I)t}, \; t \geq 0 \), the semigroup in \( L^2 \) generated by the operator \( A = -\gamma D^\alpha - \beta I \), that is,

\[
V(t)f(x) = \mathcal{F}_{\xi \to x}^{-1} \left( e^{-(\gamma \| \xi \|_2^2 + \beta) t} \mathcal{F}_{x \to \xi} f \right), \; \text{for } f \in L^2, \; t \geq 0.
\]

**Lemma 4.2.** \( \{ V(t) \}_{t \geq 0} \) is a \( C^0 \)-semigroup of contractions in \( \mathcal{H}_\alpha, \; s \in \mathbb{R}, \) satisfying \( \| V(t) \|_s \leq e^{-\beta t} \) for \( t \geq 0 \). Moreover, \( u(x, t) = V(t)f_0(x) \) is the unique solution to the following Cauchy problem:

\[
\begin{cases}
  u \in C \left( [0, T], \mathcal{H}_\alpha \right) \cap C^1 \left( [0, T], \mathcal{H}_\alpha \right); \\
  u_t = -\gamma D^\alpha u - \beta u, \; t \in [0, T]; \\
  u(x, 0) = f_0(x) \in \mathcal{H}_\alpha,
\end{cases}
\]

where \( T \) is an arbitrary positive number.

**Proof.** We just verify the strongly continuity of the semigroup. Since

\[
\left\| \mathcal{F}_{\xi \to x}^{-1} \left( e^{-(\gamma \| \xi \|_2^2 + \beta) t} \mathcal{F}_{x \to \xi} f \right) - f(x) \right\|_s^2
\]
it follows from the dominated convergence theorem that
\[
\lim_{t \to 0^+} \| V(t) f - f \|_s = 0.
\]

The existence and uniqueness of a solution for the Cauchy problem (4.3) follows from a well-known result, see e.g. \[20, \text{Theorem 4.3.1}\]. \(\square\)

**Lemma 4.3.** Let \( f_0 \in \mathcal{H}_s, s \in \mathbb{R}, \lambda \geq 0 \). Then, there exists a positive constant \( C(\lambda, \alpha) \) that depends of \( \lambda \) and \( \alpha \) such that
\[
\| V(t) f_0 \|_{s + \lambda} \leq e^{-\beta t} \left( 1 + C(\lambda, \alpha) \left( \frac{\lambda}{2\alpha \gamma} \right)^{\frac{1}{\alpha}} \right) \| f_0 \|_s \quad \text{for } t > 0.
\]

**Proof.** We first notice that
\[
\| V(t) f_0 \|_{s + \lambda}^2 = \int_{Q^d_p} \left| e^{x + \lambda \xi^2} e^{-2\gamma \| \xi \|^p_\alpha} t \right| f_0(\xi) \| f_0 \|_s^2 \leq e^{-2\beta t} \left( 1 + \sup_{\xi \in Q^d_p, \xi^2 \neq 0} \| \xi \|_p^\lambda e^{-2\gamma \| \xi \|^p_\alpha} t \right) \| f_0 \|_s^2.
\]

We now set \( y = \| \xi \|_p \) and \( h(y) = y^\lambda e^{-2\gamma y^\alpha t} \). By using the fact that \( h(y) \) reaches its maximum at \( y_{\text{max}} = \left( \frac{\lambda}{2\alpha \gamma t} \right)^{\frac{1}{\alpha}} \), we conclude that
\[
\sup_{\xi \in Q^d_p} \| \xi \|_p^\lambda e^{-2\gamma \| \xi \|^p_\alpha} t \leq \left( \frac{\lambda}{2\alpha \gamma t} \right)^{\frac{1}{\alpha}} e^{-\frac{\lambda}{\alpha}} \leq C(\lambda, \alpha) \left( \frac{\lambda}{2\alpha \gamma t} \right)^{\frac{1}{\alpha}}.
\]

**Proposition 4.4.** Let \( s > n/2 + 2\delta \) and \( F(u) = (\beta + 1)u^2 - u^3 + P(D)(u^m) \). Then \( F : \mathcal{H}_s \rightarrow \mathcal{H}_{s - 2\delta} \) is a continuous function satisfying
\[
\| F(u) - F(w) \|_{s - 2\delta} \leq L(\| u \|_s, \| w \|_s) \| u - w \|_s,
\]
for \( u, w \in \mathcal{H}_s \), here \( L(\cdot, \cdot) \) is a continuous function, which is not decreasing with respect to each of their arguments. In particular,
\[
\| F(u) \|_{s - 2\delta} \leq L(\| u \|_s, 0) \| u \|_s.
\]

**Proof.** We first notice that
\[
F(u) - F(w) = (\beta + 1)(u^2 - w^2) - (u^3 - w^3) + P(D)(u^m - w^m)
\]
\[
= (\beta + 1)(u - w)(u + w) - (u - w)(u^2 + uw + w^2) + P(D)((u - w)q(u, w)),
\]
where \( q(u, w) = \sum_{k=0}^{m-1} u^k w^{m-1-k} \). By using Proposition 3.3 and Lemma 3.5, the condition \( s > n/2 \) implies that if \( u, w \in \mathcal{H}_s \), then any polynomial function in \( u, w \) belongs to \( \mathcal{H}_s \), and
\[
\| F(u) - F(w) \|_{s - 2\delta} \leq C \left\{ (\beta + 1)\| u - w \|_{s - 2\delta} \| u + w \|_{s - 2\delta} + \| u - w \|_{s - 2\delta}^2 + uw + w^2 \|_{s - 2\delta} + \| u - w \|_s \| q(u, w) \|_s \right\},
\]
where $C = C(n, s, \delta)$. Then
\[
\|F(u) - F(w)\|_{s-2\delta} \leq A(\|u\|_s, \|w\|_s)\|u - w\|_s,
\]
where
\[
A(\|u\|_s, \|w\|_s) = C\left\{ (\beta + 1)\|u + w\|_s + \|u^2 + uw + w^2\|_s + \|q(u, w)\|_s \right\}
\leq C \left\{ (\beta + 1)\|u\|_s + (\beta + 1)\|w\|_s + \|u^2\|_s + \|uw\|_s + \|w^2\|_s + \sum_{k=0}^{m-1} \|u^k w^{m-1-k}\|_s \right\}
\leq C(\beta + 1)\|u\|_s + C(\beta + 1)\|w\|_s + C^2\|u\|_s^2 + C^2\|u\|_s\|w\|_s + C^2\|w\|_s^2 + 
C^{m+1} \sum_{k=0}^{m-1} \|u\|_s^k \|w\|_s^{m-1-k} =: L(\|u\|_s, \|w\|_s).
\]

\[\square\]

For $M, T > 0$ and $f_0 \in \mathcal{H}_s$, we set
\[
\mathcal{X}(M, T, f_0) := \left\{ u(t) \in C([0, T]; \mathcal{H}_s) ; \sup_{t \in [0, T]} \|u(t) - V(t)f_0\|_s \leq M \right\}.
\]
We endow $\mathcal{X}(M, T, f_0)$ with the metric $d(u(t), v(t)) = \sup_{t \in [0, T]} \|u(t) - v(t)\|_s$. The resulting metric space is complete.

**Proposition 4.5.** Take $f_0 \in \mathcal{H}_s$ with $s > n/2 + 2\delta, \delta > 0$. Then, there exists $T = T(\|f_0\|_s, M) > 0$ and a unique function $u \in C([0, T]; \mathcal{H}_s)$ satisfying the integral equation
\[
u(t) = V(t)f_0 + \int_0^t V(t - \tau)F(u(\tau))d\tau, \tag{4.7}
\]
such that $u(0) = f_0$. Here $F(u) = (\beta + 1)u^2 - u^3 + P(D)(u^m)$ as before.

**Remark 4.6.** Since $F(u)$ is not a locally Lipschitz function because inequality (4.6) involves two different norms, the existence of mild solutions of type (4.7) does not follow directly from standard results in semigroup theory, see e.g. [20, Theorem 5.2.2].

**Proof.** Given $u \in \mathcal{X}(M, T, f_0)$, we set
\[
\mathbf{N}u(t) = V(t)f_0 + \int_0^t V(t - \tau)F(u(\tau))d\tau.
\]

**Claim 1.** $\mathbf{N} : \mathcal{X}(M, T, f_0) \rightarrow C([0, T]; \mathcal{H}_s)$.

Take $u \in \mathcal{X}(M, T, f_0)$, then
\[
\|\mathbf{N}u(t_1) - \mathbf{N}u(t_2)\|_s \leq \left\| (V(t_1) - V(t_2))f_0 \right\|_s + \left\| \int_0^{t_1} V(t_1 - \tau)F(u(\tau))d\tau - \int_0^{t_2} V(t_2 - \tau)F(u(\tau))d\tau \right\|_s. \tag{4.8}
\]

Since $\{V(t)\}_{t \geq 0}$ is a $C_0$-semigroup in $\mathcal{H}_s$, cf. Lemma 4.2, the first term on the right-hand side of the inequality (4.8) tends to zero when $t_2 \rightarrow t_1$. To study the second term, we assume without loss of generality that $0 < t_1 < t_2 < T$. Then
\[
\left\| \int_0^{t_1} V(t_1 - \tau)F(u(\tau))d\tau - \int_0^{t_2} V(t_2 - \tau)F(u(\tau))d\tau \right\|_s 
\leq \int_0^{t_1} \|\{V(t_1 - \tau) - V(t_2 - \tau)\}F(u(\tau))\|_s d\tau + \int_{t_1}^{t_2} \|V(t_2 - \tau)F(u(\tau))\|_s d\tau.
\]
By using Lemma 4.3 with \( \lambda = \alpha \) and Proposition 4.4,
\[
\| (V(t_1 - \tau) - V(t_2 - \tau)) F(u(\tau)) \|_s \\
\leq \| V(t_1 - \tau) F(u(\tau)) \|_s + \| V(t_2 - \tau) F(u(\tau)) \|_s \\
\leq \left\{ 2 + C_0 \left( \frac{1}{2\gamma(t_1 - \tau)} \right)^\frac{1}{2} \right\} \| F(u(\tau)) \|_{s-\alpha} \\
\leq 2 \left\{ 1 + C_0 \left( \frac{1}{2\gamma(t_1 - \tau)} \right)^\frac{1}{2} \right\} \sup_{\tau \in [0,T]} \| F(u(\tau)) \|_{s-\alpha} \\
= A(T, s, \alpha) \left\{ 1 + C_0 \left( \frac{1}{2\gamma(t_1 - \tau)} \right)^\frac{1}{2} \right\} \in L^1([0, t_1]).
\]

Now, by applying the dominated convergence theorem,
\[
\lim_{t_2 \to t_1} \int_{t_0}^{t_1} \| (V(t_1 - \tau) - V(t_2 - \tau)) F(u(\tau)) \|_s d\tau = 0.
\]

By a similar argument, one shows that
\[
\| V(t_2 - \tau) F(u(\tau)) \|_{s-2d} \leq 1 + C_0 \left( \frac{1}{2\gamma(t_2 - \tau)} \right)^\frac{1}{2} L(\| u(\tau) \|_s, 0) \| u(\tau) \|_s,
\]
and since
\[
\| u(\tau) \|_s \leq \| u(\tau) - V(\tau)f_0 \|_s + \| V(\tau)f_0 \|_s \leq M + \| f_0 \|_s, \text{ for all } \tau \in [0, T],
\]
we have
\[
\int_{t_0}^{t_2} \| V(t_2 - \tau) F(u(\tau)) \|_s d\tau
\]
\[
\leq L(M + \| f_0 \|_s, 0)(M + \| f_0 \|_s) \left( \int_{t_1}^{t_2} \left( 1 + C_0 \left( \frac{1}{2\gamma(t_2 - \tau)} \right)^\frac{1}{2} \right) d\tau \right)
\]
\[
= L(M + \| f_0 \|_s, 0)(M + \| f_0(\cdot) \|_s)(t_2 - t_1) + C_0 \left( \frac{2(t_2 - t_1)}{\gamma} \right),
\]
and consequently, by applying the dominated convergence theorem,
\[
\lim_{t_2 \to t_1} \int_{t_1}^{t_2} \| V(t_2 - \tau) F(u(\tau)) \|_s d\tau = 0.
\]

**Claim 2.** There exists \( T_0 \) such that \( N(\mathcal{X}(M, T_0, f_0)) \subseteq \mathcal{X}(M, T_0, f_0) \).

By using a reasoning similar to the one used to established inequality (4.10), one gets
\[
\| (Nu)(t) - V(t)f_0 \|_s \leq \int_{0}^{t} \| V(t - \tau) F(u(\tau)) \|_s d\tau
\]
\[
\leq L(M + \| f_0 \|_s, 0)(M + \| f_0 \|_s) \left( \int_{0}^{t} \left( 1 + C_0 \left( \frac{1}{2\gamma(t - \tau)} \right)^\frac{1}{2} \right) d\tau \right)
\]
\[
\leq L(M + \| f_0 \|_s, 0)(M + \| f_0 \|_s) \left( T + C_0 \left( \frac{2T}{\gamma} \right) \right).
\]

Now taking \( T_0 \) such that
\[
L(M + \| f_0 \|_s, 0)(M + \| f_0 \|_s) \left( T_0 + C_0 \left( \frac{2T_0}{\gamma} \right) \right) \leq M,
\]
(4.11)
we conclude that \( N u \in \mathcal{X}(M, T_0, f_0) \), for all \( u(t) \in \mathcal{X}(M, T_0, f_0) \).

**Claim 3.** There exists \( T_0' \) such that \( N \) is a contraction on \( \mathcal{X}(M, T_0', f_0) \).

Given \( u(t), v(t) \in \mathcal{X}(M, T_0, f_0) \), by using Proposition 4.4, with
\[
C_0' = L (M + \|f_0\|_s, M + \|f_0\|_s),
\]
see (4.9), we have
\[
\|N u(t) - N v(t)\|_s \leq \int_0^t \|V(t - \tau)[F(u(\tau)) - F(v(\tau))]\|_s d\tau
\]
\[
\leq \int_0^t \left( 1 + C_0 \left( \frac{1}{2\gamma(t - \tau)} \right)^\frac{1}{2} \right) \|F(u(\tau)) - F(v(\tau))\|_s d\tau
\]
\[
\leq C_0' \int_0^t \left( 1 + C_0 \left( \frac{1}{2\gamma(t - \tau)} \right)^\frac{1}{2} \right) \|u(\tau) - v(\tau)\|_s d\tau
\]
\[
\leq C_0' \left( \sup_{\tau \in [0, T_0]} \|u(\tau) - v(\tau)\|_s \right) \int_0^t \left( 1 + C_0 \left( \frac{1}{2\gamma(t - \tau)} \right)^\frac{1}{2} \right) d\tau
\]
\[
\leq C_0' \left( T_0 + C_0 \left( \frac{\sqrt{2T_0}}{\gamma} \right) \right) d(u(t), v(t)).
\]

Thus, taking \( T_0' \) such that
\[
C := C_0' \left( T_0' + C_0 \left( \frac{\sqrt{2T_0}}{\gamma} \right) \right) < 1,
\]
we obtain that \( d(N u(t), N v(t)) \leq C d(u(t), v(t)) \), that is, \( N \) is a strict contraction in \( \mathcal{X}(M, T_0', f_0) \). We pick \( T \) such that the inequalities (4.11) and (4.11) hold true, and apply the Banach Fixed Point Theorem to get \( u(t) \in \mathcal{X}(M, T, f_0) \) a unique fixed point of \( N \), which satisfies the integral equation (4.7), where
\[
T = T(\|f_0\|_s, M) > 0.
\]

**Remark 4.7.** Let \( \mathcal{X} \) be a Banach space and let \( A : Dom(A) \to \mathcal{X} \) be an operator with dense domain such that \( A \) is the infinitesimal generator of a contraction semigroup \( (S_t)_{t \geq 0} \). Fix \( T > 0 \) and let \( f : [0, T] \to \mathcal{X} \) be a continuous function. Consider the Cauchy problem:
\[
\begin{align*}
&\begin{cases}
  u \in C([0, T], Dom(A)) \cap C^1([0, T], \mathcal{X}); \\
  u_t = A u + f(t), \ t \in [0, T]; \\
  u(0) = u_0 \in \mathcal{X}.
\end{cases}
\end{align*}
\]

Then
\[
u(t) = S(t)u_0 + \int_0^t S(t - \tau)f(\tau)d\tau,
\]
for \( t \in [0, T] \), see e.g. [3, Lemma 4.1.1]. Conversely, if \( u_0 \in Dom(A) \), \( f \in C([0, T], \mathcal{X}) \),
\[
\int_{(0,T)} \|f(\tau)\|_\mathcal{X} d\tau < \infty,
\]
then a solution of (4.14) is a solution of the Cauchy problem (4.13), see e.g. [3, Proposition 4.1.6].
Proposition 4.8. The problem (4.2) is equivalent to the integral equation (4.7). More precisely, if $s > n/2 + 2\delta$, and $u(t) \in C([0, T]; \mathcal{H}_s) \cap C^1((0, T]; \mathcal{H}_{s-2\delta})$ is a solution of (4.2), then $u(t)$ satisfies the integral equation (4.7). Conversely, if $s > n/2 + 2\delta$, and $u(t) \in C([0, T]; \mathcal{H}_s)$ is a solution of (4.7), then $u(t) \in C^1([0, T]; \mathcal{H}_{s-2\delta})$ and it satisfies (4.2).

Proof. It follows from Remark 4.7, Propositions 4.5, 4.4, by taking $A = -\gamma D_x^\alpha - \beta I$, $\text{Dom}(A) = \mathcal{H}_s$, $X = \mathcal{H}_{s-2\delta}$, $f(t) = F(u(t))$. We first recall that $\mathcal{D} \hookrightarrow \mathcal{H}_s \hookrightarrow \mathcal{H}_{s-2\delta}$, where $\hookrightarrow$ means continuous embedding, and that $\mathcal{D}$ is dense in $\mathcal{H}_{s-2\delta}$. If $u(t)$ is a solution of (4.2), then, since $F(u(t)) \in C([0, T]; \mathcal{H}_{s-2\delta})$, by Proposition 4.4, $u(t)$ is a solution of (4.7). Conversely, if $u(t)$ is a solution of (4.7), since

$$
\int_{(0, T)} \|F(u(\tau))\|_{s-2\delta} d\tau < \infty,
$$

by Proposition 4.4, $u(t)$ is a solution of (4.2).

Lemma 4.9 ([20, Theorem 5.1.1]). If $h \in L^1(0, T)$, with $T > 0$, is real-valued function such that. If

$$
h(t) \leq a + b \int_0^t h(s)ds,
$$

for $t \in (0, T)$ a.e., where $a \in \mathbb{R}$ and $b \in [0, \infty)$ then $h(t) \leq ae^{bt}$ for almost all $t$ in $(0, T)$.

Proposition 4.10. Let $f_0, f_1 \in \mathcal{H}_s$ and $u(t), v(t) \in C([0, T]; \mathcal{H}_s)$ be the corresponding solutions of equation (4.7) with initial conditions $u(0) = f_0$ and $v(0) = f_1$, respectively. If $s > n/2 + 2\delta$, then

$$
\|u(t) - v(t)\|_s \leq e^{L(W, W)}\|f_0 - f_1\|_s,
$$

where $L$ is given in Proposition 3.3 and

$$
W := \max \left\{ \sup_{t \in [0, T]} \|u(t)\|_s, \sup_{t \in [0, T]} \|v(t)\|_s \right\}.
$$

Proof. By using (4.7), we have

$$
u(t) - v(t) = V(t)(f_0 - f_1) + \int_0^t V(t - \tau)\{F(u(\tau)) - F(v(\tau))\}d\tau.
$$

By using Proposition 3.3, we get

$$
\|u(t) - v(t)\|_s \leq \|f_0 - f_1\|_s + \int_0^t \|V(t - \tau)\{F(u(\tau)) - F(v(\tau))\}\|_{s-\alpha} d\tau
$$

$$
\leq \|f_0 - f_1\|_s + \int_0^t \|F(u(\tau)) - F(v(\tau))\|_{s-\alpha} d\tau
$$

$$
\leq \|f_0 - f_1\|_s + L(W, W)\int_0^t \|u(\tau) - v(\tau)\|_{s} d\tau.
$$

Now the result follow from Lemma 4.9, by taking $h(t) = \|u(t) - v(t)\|_s$, $a = \|f_0 - f_1\|_s$, $b = L(W, W)$.

Proposition 4.11. Let $s > n/2 + 2\delta$ and $\delta \geq 0$. Then, the map $f \mapsto u(t)$ is continuous in the following sense: if $f_0^{(n)} \to f_0$ in $\mathcal{H}_s$ and $u_n(t) \in C([0, T_n]; \mathcal{H}_s)$, with $T_n = T \left( \left\| f_0^{(n)} \right\|_s, \sqrt{M} \right) > 0$, are the corresponding solutions to the Cauchy problem (4.2) with $u_n(0) = f_0^{(n)}$. Then, there exist $T > 0$ and a positive integer $N = N(\gamma, f_0, T)$ such that $T_n \geq T$ for all $n \geq N$ and

$$
\lim_{n \to \infty} \sup_{t \in [0, T]} \|u_n(t) - u(t)\|_s = 0. \tag{4.15}
$$
Given \( T \) and by applying Lemma 4.9, which in turns implies (4.15).

Therefore, all the \( u_n(t) \) are defined on \([0, T]\), furthermore, \( u \in \mathcal{X}(M, T, f_0^{(n)}) \) for all \( n \), and

\[
\|u_n(t)\|_s \leq \|f_0^{(n)}\|_s + M \leq \delta + M,
\]

where \( \delta = \sup_{n \in \mathbb{N}} \|f_0^{(n)}\|_s \). Now

\[
\sup_{t \in [0, T]} \|u_n(t)\|_s \leq \delta + M \quad \text{for all} \ n, \ \text{and} \ \sup_{t \in [0, T]} \|u(t)\|_s \leq \delta + M.
\]

On the other hand, by reasoning as in the proof of Proposition 4.10, we have

\[
\|u_n(t) - u(t)\|_s \leq \|f_0^{(n)} - f_0\|_s + L(\delta + M, \delta + M) \int_0^t \|u_n(\tau) - u(\tau)\|_s d\tau,
\]

and by applying Lemma 4.9

\[
\|u_n(t) - u(t)\|_s \leq e^{TL(\delta + M, \delta + M)} \|f_0^{(n)} - f_0\|_s,
\]

which in turns implies (4.15).

\[ \square \]

4.4. Proof of the Main Result

The local well-posedness of the Cauchy problem (4.2) in \( \mathcal{H}_s, s > n/2 + 2\delta \), follows from Propositions 4.5, 4.10, 4.11.

5. THE BLOW-UP PHENOMENON

In this section, we study the blow-up phenomenon for the solution of the equation

\[
\begin{cases}
  u_t = -\gamma D_x^\alpha u + F(u) + D_x^{\gammab} u^3, \ x \in \mathbb{Q}_p^n, \ t \in [0, T] ; \\
  u(0) = f_0 \in \mathcal{H}_\infty,
\end{cases}
\]

(5.1)

where \( F(u) = -u^3 + (\beta + 1) u^2 - \beta u \). We will say that a non-negative solution \( u(x, t) \geq 0 \) of (5.1) blow-up in a finite time \( T > 0 \), if \( \lim_{t \to T^-} \sup_{x \in \mathbb{Q}_p^n} u(x, t) = +\infty \). This limit makes sense since \( \mathcal{H}_\infty(\mathbb{Q}_p^n, \mathbb{C}) \) is continuously embedded in \( C_0(\mathbb{Q}_p^n, \mathbb{C}) \), [18, Theorem 10.15].

5.1. \( p \)-Adic Wavelets and Pseudo-Differential Operators

We denote by \( C(\mathbb{Q}_p, \mathbb{C}) \) the \( \mathbb{C} \)-vector space of continuous \( \mathbb{C} \)-valued functions defined on \( \mathbb{Q}_p \).

We fix a function \( a : \mathbb{R}_+ \to \mathbb{R}_+ \) and define the pseudo-differential operator

\[
\mathcal{D} \to C(\mathbb{Q}_p, \mathbb{C}) \cap L^2
\]

\[
\varphi \to A\varphi,
\]

where \( (A\varphi)(x) = \mathcal{F}_x^{-1} \left\{ a \left( |\xi|_p \right) \mathcal{F}_x \right\} \).
The set of functions $\{\Psi_{rnj}\}$ defined as
\[
\Psi_{rnj}(x) = p^{-r}x_\alpha \left[p^{-1}j \left(p^r x - n\right)\right] \Omega \left[|p^r x - n|_p\right] ,
\]
(5.2)
where $r \in \mathbb{Z}$, $j \in \{1, \cdots, p-1\}$, and $n$ runs through a fixed set of representatives of $\mathbb{Q}_p/\mathbb{Z}_p$, is an orthonormal basis of $L^2(\mathbb{Q}_p)$ consisting of eigenvectors of operator $A$: 
\[
 A \Psi_{rnj} = a(p^{1-r}) \Psi_{rnj} \text{ for any } r, n, j ,
\]
(5.3)
see e.g. [18, Theorem 3.29], [1, Theorem 9.4.2]. Notice that
\[
\hat{\Psi}_{rnj}(\xi) = p^{-r}x_\alpha \left[p^{-r}n\xi\right] \Omega \left[|p^{-r}\xi + p^{-1}j|_p\right] ,
\]
and then 
\[
a \left(|\xi|_p\right) \hat{\Psi}_{rnj}(\xi) = a(p^{1-r}) \hat{\Psi}_{rnj}(\xi) .
\]
In particular, $D^\alpha_x \Psi_{rnj} = p^{(1-r)\alpha} \Psi_{rnj}$, for any $r, n, j$ and $\alpha > 0$, and since $p^{(1-r)\alpha}$,
\[
 D^\alpha_x \text{Re} \left(\Psi_{rnj}\right) = p^{(1-r)\alpha} \text{Re} \left(\Psi_{rnj}\right) ,
\]
\[
 D^\alpha_x \text{Im} \left(\Psi_{rnj}\right) = p^{(1-r)\alpha} \text{Im} \left(\Psi_{rnj}\right) .
\]
And,
\[
\{\Psi_{rn1}(x)\}^2 = p^{-r}x_\alpha \left[2p^{-1}(p^r x - n)\right] \Omega \left[|p^r x - n|_p\right] = p^{-r}x_\alpha \Psi_{rn2}(x) ,
\]
then 
\[
 D^\alpha_x \text{Re} \left(\{\Psi_{rn1}(x)\}^2\right) = p^{-r}x_\alpha p^{(1-r)\alpha} \text{Re} \left(\Psi_{rn2}(x)\right) = p^{(1-r)\alpha} \text{Re} \left(\{\Psi_{rn1}(x)\}^2\right) .
\]

5.2. The Blow-up

In this section, we assume that $u(x,t)$ is real-valued non-negative solution of the Cauchy problem (4.2) in $\mathcal{H}_\infty$. We set $w(x) := \text{Re} \left(\{\Psi_{rn1}(x)\}^2\right)$, so $D^\alpha_x w(x) = p^{(1-r)\alpha} w(x)$. Thus $w(x)dx$ defines a (positive) measure. We also set $G(t) := \int_{\mathbb{Q}_p} u(x,t) w(x)dx$, where $u(x,t)$ is a positive solution of (5.1), then
\[
 G'(t) = \int_{\mathbb{Q}_p} u(t,x)w(x)dx = -\gamma \int_{\mathbb{Q}_p} (D^\alpha_x u)(x,t)w(x)dx
\]
\[
+ \int_{\mathbb{Q}_p} F(u(x,t))w(x)dx + \int_{\mathbb{Q}_p} (D^\alpha_x u^3)(x,t)w(x)dx.
\]
(5.4)
Now, by using that $D^\alpha_x u(\cdot, t)$, $w \in L^2$, and $F(u(\cdot, t))$, $D^\alpha_x u^3(\cdot, t) \in L^2$ since for $s > n/2$, $\mathcal{H}_s$ is a Banach algebra contained in $L^2$ cf. Proposition 3.3, and applying the Parseval-Steklov theorem, we get (5.4) can be rewritten as
\[
 G'(t) = \int_{\mathbb{Q}_p} \left(-\gamma p^{(1-r)\alpha} u(x,t) + F(u(x,t)) + p^{(1-r)\alpha} u^3(t,x)\right)w(x)dx.
\]
Since the function $H(y) = -\gamma p^{(1-r)\alpha} y + F(y) + p^{(1-r)\alpha} y^3$ is convex because
\[
 H''(y) = -6y + 2(\beta + 1) + p^{(1-r)\alpha} 6y = 6y \left[p^{(1-r)\alpha} 6 \right] + 2(\beta + 1) \geq 0,
\]
for $y \geq 0$, and $r \leq 0$, we can use the Jensen’s inequality to get $G'(t) \geq H(G(t))$, then the function $G(t)$ can not remain finite for every $t \in [0, \infty)$. Then there exists $T \in (0, \infty)$ such that $\lim_{t \to T^-} G(t) = +\infty$, hence $u(x,t)$ blow up at the time $T$. Then we have established the following result:

Theorem 5.1. Let $u(x,t)$ be a positive solution of (5.1). Then there $T \in (0, +\infty)$ depending on the initial datum such that $\lim_{t \to T^-} \sup_{x \in \mathbb{Q}_p} u(x,t) = +\infty$. 


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6. NUMERICAL SIMULATIONS

In this section, we present two numerical simulations for the solution of problem (5.1) (in dimension one) for a suitable initial datum. We solve and visualize (using a heat map) the radial profiles of the solution of (5.1). We consider equation (5.1) for radial functions $u(x, \cdot)$. In [15], Kochubei obtained a formula for $D_2^\alpha u(x, t)$ as an absolutely convergent real series, we truncate this series and then we apply the classic Euler Forward Method (see e.g. [23]) to find the values of $u(p^{-\text{ord}}(x), t)$, when $-20 \leq \text{ord}(x) \leq 20$ (vertical axis) and when $t = \{t_k : t_k = 1/k, k = 1, \ldots, 300\}$ (horizontal axis). In Figure 1, on the left side of the Figure 1, we observe that the solution $u$ grows rapidly towards infinity ($\text{ord}(x)$), the reactive term $-u^3(x, t)$, the heat map of the numerical solution of the homogeneous equation $u_t(x, t) = -D_2^\alpha u(x, t)$ with initial data $u(x, 0) = 4e^{-p^{\text{ord}}(x)/100}$ (Gaussian bell type), and parameters $p = 3, \alpha = 0.2, \gamma = 1$. On the right side, we have the numerical solution of the equation $u_t(x, t) = -D_2^\alpha u(x, t) - u^3(x, t) + (\beta + 1)u^2(x, t) - \beta u(x, t) + D_2^{\alpha_1}u^3(x, t)$, with $p = 3, \alpha = 0.2, \alpha_1 = 0.1$, and $\beta = 0.7$.

![Fig. 1. A numerical simulation of the blowup phenomena.](image)

On the left side of the Figure 1, we observe that the solution $u$ is uniformly decreasing with respect to the variable $t$. This behavior is typical for solutions of diffusion equations. These equations have been extensively studied, see e.g. [18, 35] and the references therein.

On the right side of Figure 1, we see that the evolution of $u(x, t)$ is controlled by the diffusion term $-D_2^\alpha u(x, t)$, up to a time $T$ (blow-up time), this behavior is similar to that described above. When $t > T$, the reactive term $-u^3(x, t) + (\beta + 1)u^2(x, t) - \beta u(x, t) + D_2^{\alpha_1}u^3(x, t)$ takes over and $u(x, t)$ grows rapidly towards infinity.

The method converges quite fast, but still lacks a mathematical formalism to support it, for this reason we refer to it as a numerical simulation of the solution.

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