Implications of a viscosity bound on black hole accretion

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Motivated by the viscosity bound in gauge/gravity duality, we consider the ratio of shear viscosity ($\eta$) to entropy density ($s$) in black hole accretion flows. We use both an ideal gas equation of state and the QCD equation of state obtained from lattice for the fluid accreting onto a Kerr black hole. The QCD equation of state is considered since the temperature of accreting matter is expected to approach $10^{12}$K in certain hot flows. We find that in both the cases $\eta/s$ is small only for primordial black holes and several orders of magnitude larger than any known fluid for stellar and supermassive black holes. We show that a lower bound on the mass of primordial black holes leads to a lower bound on $\eta/s$ and vice versa. Finally we speculate that the Shakura-Sunyaev viscosity parameter should decrease with increasing density and/or temperatures.

Accretion flows around black holes and neutron stars are very hot. In the geometrically thin (and optically thick, i.e., strong interaction between dense matter and radiation) regime, the temperature ($T$) of the flow could be $\sim 10^7$K [1]. The flows in certain temporal classes of micro-quasar GRS 1915+105 [2] are of this type. In the geometrically thick (and optically thin, i.e., weak interaction between tenuous matter and radiation) hotter regime, we expect a two-temperature flow with the ion temperature ($T_i$) as high as $7 \times 10^{11}$K and the electron temperature being much lower.

While originally $\mu$ was thought to be unity [8], we know now that $\mu < 1$ [9] and quite likely non-zero [10] (for recent reviews, see [11]). For example, in the gravity model [12], $\mu \approx 0.414$. For the purpose of this work we assume that there is a bound and that $\mu \sim 1$. One of the main questions that we wish to answer is: How small is $\eta/s$ in black hole accretion? We can anticipate the answer to this question before getting into details. First, $\eta$ is a combination of a molecular/physical viscosity and a turbulent viscosity. If the molecular viscosity dominates then for temperatures around or above $T_c$, we expect $\eta/s$ to be close to $0.1\hbar/k_B$. More precisely, we need to use the QCD EoS. If the turbulent viscosity dominates, then the ratio is much higher. However, as we will argue, close to the horizon where gravity effects on the flow are strong, or in the early universe when density was high, we expect a small ratio.

There is a huge literature discussing accretion of optically thin hot matter around a Kerr black hole in full general relativistic as well as pseudo-Newtonian frameworks (e.g. [13–15]). For the present purpose, we will consider the one temperature advection-dominated accretion flow (ADAF) model around a Kerr black hole [14,15]. The units are $G = M = c = 1$, with $M$ being the mass of the black hole. In what follows we will work in terms of normalized temperatures where the normalization is done through $m_p c^2/k_B$ with $m_p$ being the mass of a proton.

The equations of continuity and the stress energy conserva-
tion satisfy
\[ 4\pi r^2 \rho H_\theta \theta \left( \frac{D}{1 - \vartheta^2} \right)^{1/2} = -\dot{M}, \]
\[ \vartheta \left( \frac{D}{1 - \vartheta^2} \right)^{1/2} \left[ \partial_t \epsilon - (\epsilon + p) \frac{\partial \rho}{\rho} \right] = f \Phi, \]
\[ \vartheta \left( \frac{D}{1 - \vartheta^2} \right)^{1/2} \frac{\partial \vartheta}{\partial t} = f_r - \frac{1}{\rho \kappa} \partial_r \rho, \]
\[ \dot{M} \lambda \epsilon - 4\pi \rho H_\theta r^2 W_r = \dot{M} j. \] (2)

Here \( r \) is the Boyer-Lindquist radius, \( \rho \) the rest-mass density, \( H_\theta = H/r \) with \( H \) the half thickness of the flow, \( \vartheta \) the radial velocity measured in the co-rotating frame, \( \lambda \) the specific angular momentum of the flow, \( D = 1 - 2/r + a^2/r^2 \) with \( a \) being the rotation parameter and \( \dot{M} \) the rest-mass accretion rate. The total energy density \( \epsilon = \rho + \rho T g(T) \) with \( g(T) = (4/(\gamma_0 - 1) + 15T)/4(1 + 5T) \) and \( \kappa = (\epsilon + p)/\rho \) is the relativistic enthalpy. The function \( f_r \) is given by
\[ f_r = -\frac{1}{r^2} \frac{A^2}{D} \left( 1 - \frac{\Omega}{\Omega_+} \right) \left( 1 - \frac{\Omega}{\Omega_-} \right), \]
with \( A = 1 + a^2/r^2 + 2a^2/r^3, \gamma_0^2 = 1 + \lambda^2(1 - \vartheta^2)/(r^2 A) \) and \( \Omega_\pm = \pm (r^3/2 \pm a)^{-1} \) with \( \Omega = 2a/(Ar^3) + \lambda(1 - \vartheta^2)^{1/2} D^{1/2} / (\gamma_0^2 A^{3/2}) \) being the angular velocity. When complemented with the gas EoS \( p = \rho T \), the equations (2) enable us to solve for \( \vartheta, T, \rho, \lambda \) as functions of \( r \). In (2), \( j \) is a constant and should be treated as an eigenvalue for transonic flows. The form of the equations enables us to rescale \( \rho \rightarrow \dot{M} / \dot{M} \) to set \( \dot{M} = 1 \) in the numerics. To reinstate physical units, for radial velocity we use \( \vartheta c \), for specific angular momentum \( \lambda G M / c \), for density \( \rho M c^3 / G^2 M^2 \) and for temperature \( T m_p c^2 / k_B \). The viscous stress tensor components entering the equations are \( W_\theta \) and \( \Phi \), given by
\[ W_\theta = FS, \]
\[ S = \frac{\rho \tau_r u_r^2 \left( \frac{d \lambda}{dr} \right)}{1 - \tau_r u_r^2 (2/r + d \ln F/dr)} + \frac{2 \kappa \eta \sigma}{1 - \tau_r u_r^2 (2/r + d \ln F/dr)} \]
\[ \Phi = -2 S \sigma, \] (3)
with \( u_r = \vartheta D^{1/2} / (1 - \vartheta^2)^{1/2} \). The expression for \( \sigma \) is very lengthy and can be found in the appendix of [14]. Parametrizing the turbulent viscosity based on the famous Shakura-Sunyaev prescription [11] and accounting for the relativistic enthalpy factor as in [14, 15] we have
\[ \eta = \alpha \kappa \rho c_s H_\theta r. \] (4)

Here \( \alpha \) is a constant typically taking values \( 0.01 - 0.1 \). The contribution from the physical (molecular) viscosity is small and the viscosity is turbulence induced. In the present work, we take \( \alpha = 0.01 \) and \( \gamma_0 = 1.4444 \). The quantity \( \tau_r \) is a relaxation time and is given by \( \tau_r = \eta / (\kappa \rho c_s^2) \). At this stage it is a phenomenological input. The presence of the relativistic enthalpy makes \( \eta \) proportional to \( \epsilon + p \). Using the fact that the entropy density \( s = (\epsilon + p)/T \approx \rho (c^2 + c_s^2)/T \), we have
\[ \frac{\eta}{s} \approx \kappa \frac{\alpha \rho c_s H_\theta r}{c^2 + c_s^2} T. \] (5)

Following previous work [14, 15], we reproduce the results using a shooting method shown in FIG.1. We estimate a rough order of magnitude for \( \eta/s \) for \( \alpha = 0.999 \), when the minimum value of \( H_\theta \) is of the order of 0.3 near the horizon where \( \kappa \sim O(1), c_s \sim 0.4 \) and \( T \sim 0.3 \) for \( \alpha = 0.01 \). This gives us
\[ \frac{\eta}{s} \approx 4 \times 10^{-4} G M m_p c^2 / k_B \approx 10^{15} \frac{M}{M_\odot} \frac{h}{k_B}, \] (6)
so that for stellar mass black holes, this ratio is of the order of \( 10^{16} \) in units of \( h/k_B \), while for a supermassive black hole this is \( > 10^{22} \) compared to the QGP value of 0.1! Only for primordial black holes with \( M/M_\odot \sim 10^{-10} \) will this ratio be comparable to the QGP value. In fact in order to be consistent with the viscosity bound, it appears that
\[ M \gtrsim 10^{-16} M_\odot. \] (7)

Now, note the following curious fact. Had we been using a larger \( \alpha \sim 1 \) we would have got \( \eta/s \sim 10^{17} M/M_\odot h/k_B \). If we use the fact that in the present universe the surviving primordial black holes must have \( M > 10^{15} \)gm we obtain \( \eta/s > 0.1 h/k_B \).

FIG.1 shows that close to the black hole event horizon \( T \approx 0.25 \), corresponding to \( \sim 2.5 \times 10^{12} K \), which is above \( T_c \). For QGP matter at this temperature we expect a small \( \eta/s \). Hence, the remarkably high value of \( \eta/s \) for astrophysical black holes is quite puzzling. Even away from the event horizon, while \( T \) decreases (only an order of magnitude) below the crossover temperature, it is large enough to question the very large \( \eta/s \) we find. One possible resolution to this puzzle is that at these temperatures the EoS being used is problematic and it is more appropriate to use the QCD EoS [5, 6].
The QCD EoS from lattice calculations is specified by the energy density $\epsilon$ and pressure $p$. It is useful to express the result in terms of $I(T) = \epsilon - 3p$ which is proportional to the trace anomaly. If we had conformal matter, this quantity would be zero. At high temperatures ($T \gtrsim 1.97 \times 10^{12} K$), a useful way of fitting the lattice data is given by [6]

$$I(T) = \frac{d_2}{T^2} + \frac{d_4}{T^4} + c_1 + \frac{c_2}{T^{18}}, \quad (8)$$

where $d_2 = 0.2405 GeV^2$, $d_4 = 0.01355 GeV^4$, $c_1 = -0.0003237 GeV^5$, $c_2 = 1.439 \times 10^{-14} GeV^{18}$. A somewhat different fitting function is given in [5]. The crossover temperature in [5] works out to be around $150 MeV$ in comparison to what is used in [6], which is closer to $190 MeV$. At low temperatures ($\lesssim 1.97 \times 10^{12} K$),

$$I(T) = a_1 T + a_2 T^3 + a_3 T^4 + a_4 T^{10}, \quad (9)$$

where $a_1 = 4.654 GeV^{-1}$, $a_2 = -879 GeV^{-3}$, $a_3 = 8081 GeV^{-4}$, $a_4 = -7039000 GeV^{-10}$. The low temperature approximation is found using a matching procedure with the Hadron Resonance Gas model. From these we find

$$p(T) = \int_0^T \frac{dT}{T^5} I(T), \quad (10)$$

shown in FIG. 2. In order to go from $1 MeV$ to the normalized units we have to multiply by $\approx 0.0011$.

![FIG. 2: Variations of (a) $\epsilon/T^4$, (b) $p/T^4$, as functions of temperature.](image)

In order to explain the QGP observables for heavy ion flows at RHIC/LHC, it has become apparent that a non-zero shear viscosity, $\eta$, is needed. One way of characterizing $\eta$ which is a dimensionful number, is to take its ratio with the entropy density. It turns out that the RHIC/LHC plasma has [16]

$$\frac{\eta}{s} \sim 0.1 \frac{\hbar}{k_B}. \quad (11)$$

Although $\eta$ itself is large (in cgs units $\sim 10^{12}$ gm/cm s), its ratio with $s$ is small. To put things into perspective, this ratio for water is around 2 orders of magnitude greater. Since for optically thin accretion flows (e.g. ADAF [18], GAAF [17]) the temperature near the horizon can be of the order of $10^{12} K$, it is an interesting question to ask if $\eta/s$ becomes as small as in equation (11). The QCD EoS (see FIG. 2) cannot be valid everywhere since the low temperature behaviours of QCD matter and ordinary matter are quite different, and far away from the black hole when flow is cooler, the accreting matter will presumably follow the ideal gas EoS. Hence, an accurate analysis would involve a knowledge of EoS that is valid everywhere, but this appears to be a very hard problem. However, we know that close to the horizon $r_h$ ($r < 10 r_h$) flow must be transonic. Therefore, we start our computations for the QCD EoS with temperatures at an outer boundary that are similar to what we found using the ideal gas EoS.

Before we start discussing the solutions, note that equations (2)-(8) exhibit the following scaling symmetry: (a) Rescaling $\rho$ to $\rho \rightarrow M \rho$ and $j \rightarrow j/M$ leaves the equations invariant. (b) Rescaling $\epsilon, p$ and $j$ to $\epsilon \rightarrow \beta \epsilon, p \rightarrow \beta p$ and $j \rightarrow \beta j$ leaves the equations invariant ($\beta$ being any constant). Thus we do not need to carry the actual units of $\epsilon, p, j$. The relaxation time $\tau_r$ is taken to be $\tau_r = b \eta/\langle s T \rangle$ [19]. Here $b$ parametrizes our ignorance of the QCD coupling constant, influence of gravity etc. We choose $b = 10^4$. The choice of $b$ does not alter our conclusions.

FIG. 3 shows the variations of Mach number and temperature as functions of radius. The variations [20] of $\eta/s$ for various values of $a$ are shown in FIG. 4. This seems to suggest the same inequality as in equation (7).

![FIG. 3: Variations of (a) Mach number, (b) temperature, as functions of flow radius. The solid dashed and dotted lines are for different fitting function is given in [5]. The crossover temperature in [5] works out to be around $150 MeV$ in comparison to what is used in [6], which is closer to $190 MeV$. At low temperatures ($\lesssim 1.97 \times 10^{12} K$),

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where $a_1 = 4.654 GeV^{-1}$, $a_2 = -879 GeV^{-3}$, $a_3 = 8081 GeV^{-4}$, $a_4 = -7039000 GeV^{-10}$. The low temperature approximation is found using a matching procedure with the Hadron Resonance Gas model. From these we find

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So what have we learnt? In using the QCD EoS we have found the following:
The temperature close to $r_h$ never reaches as high as what the ideal gas EoS dictates.

The value of $\eta/s$ is small only for primordial black holes and gives a result that is quite similar to what was found using the ideal gas EoS, for all black holes.

Using the fact that the mass of primordial black holes in the present universe must be greater than $10^{13}$g, we find that $\eta/s > 0.1\alpha\hbar/k_B$. Although for $\alpha < 1$, $\eta/s < \frac{\pi}{10}\hbar/k_B$, this is perfectly consistent with the fact that the original KSS bound can in fact be violated in holography [9].

Let us conclude with some speculations. In order to apply the viscosity bound for black holes of mass $0.1M_\odot$ in the early universe around $T_c$ (which is in the radiation dominated era), one must consider an optically thick flow model [1, 21], unlike the one we considered previously. In this quark-hadron phase transition era, the black holes must be accreting QGP matter. However, here we face the following conundrum. First, for transition era, the black holes must be accreting QGP matter. In this quark-hadron phase one must consider an optically thick flow model [1, 21], unlike the one we considered previously. Second, the temperature in the present universe must be greater than $10^{13}$K for $M = \xi M_{Edd}$ [22], the value of $\alpha$ should be very small ($< 10^{-15}$) which in turn leads to a small $\eta/s < \xi^2\hbar/k_B$ with $Re \sim 10$. Such a low $\alpha$ also leads to a large density which is consistent with the fact that the universe was very dense in this era. This seems to suggest that $\alpha$ should decrease with increasing density. Now, does it make sense to have $\eta/s > O(10^{15})\hbar/k_B$ near $T_c$, which appears to be the case for stellar and supermassive black holes discussed above? Near the black hole, we expect gravity to make the flow sub-Keplerian and quasi-spherical, which is more "ordered", diminishing turbulence. This is supported by the fact that $Re$ for QGP and the optically thin disk flows are comparable and as such if matter was in the form of QGP, turbulent viscosity would not dominate. A more natural solution is to demand that $\alpha$ becomes small near $T_c$. Thus our conjecture is the following:

The Shakura-Sunyaev viscosity parameter $\alpha$ should be a function of temperature and density such that it decreases with increasing temperature and/or density. However, low $\alpha$ would only support sub-Keplerian optically thin flows. Therefore, the inner accretion is necessarily sub-Keplerian.

Finally let us point out a somewhat worrying aspect to having a large $\eta/s$. First, we note that for the hydrodynamic approach to be valid, it had better be true that the stress tensor is treated as an expansion in gradients of the velocity. In particular $T_{\mu\nu} = pg_{\mu\nu} + p\kappa u^\mu u^\nu + \theta_{\mu\nu}$, where $t_{\mu\nu}$ is proportional to $\eta$ and involves gradients of the velocity. We have to ensure that the first order terms, $t_{\mu\nu}$, are much smaller than the zeroth order terms, $pg_{\mu\nu}, p\kappa u^\mu u^\nu$. When we compute the ratio of these two terms along the flow, it turns out that the contribution from $t_{\mu\nu}$ is around $20 - 25\%$ of that of the leading term. Hence, the numerics appear to be suspect since the subleading terms in the gradient expansion will be important. It may be worth investigating effects such as cavitation [23] in this context. It seems that lowering the viscosity parameter along the lines we have hinted at above may ameliorate this problem.

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