ON $\gamma$- AND LOCAL $\gamma$-VECTORS OF THE INTERVAL SUBDIVISION

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Abstract. We show that the $\gamma$-vector of the interval subdivision of a simplicial complex with a nonnegative and symmetric $h$-vector is nonnegative. In particular, we prove that such $\gamma$-vector is the $f$-vector of some balanced simplicial complex. Moreover, we show that the local $\gamma$-vector of the interval subdivision of a simplex is nonnegative; answering a question by Juhnke-Kubitzke et al.

1. Introduction

The $\gamma$-vector is an important enumerative invariant of a flag homology sphere. The general question is asked about $\gamma$-vector whether it is nonnegative or not. It has been conjectured by Gal in [Gal05] that this vector is nonnegative for every such sphere.

Conjecture 1.1. [Gal05] If $\Delta$ is a flag homology sphere, then $\gamma(\Delta)$ is nonnegative.

Conjecture 1.1 is a strengthening of the well known Charney-Devis conjecture. The Gal conjecture holds for all Coxeter complexes (see [Ste08]), for the dual simplicial complexes of associahedron and cyclohedron (see [NP11]), and for barycentric subdivision of homology sphere (see [NPT11]). The authors in [NP11] conjectured further strengthening of Gal conjecture.

Conjecture 1.2. [NP11, Problem 6.4] If $\Delta$ is a flag homology sphere then $\gamma(\Delta)$ is the $f$-vector of some balanced simplicial complex.

This conjecture holds for the dual simplicial complex of all flag nestohedra, see in [Ais14]. Frohmader [Fro08, Theorem 1.1] showed that the $f$-vector of any flag simplicial complex satisfies the Frankl-Füredi-Kalai (FFK) inequalities (see [FFK88]). In [NPT11], authors showed that the $\gamma$-vector of the barycentric subdivision of a homology sphere satisfies the FFK inequalities, i.e., the $f$-vector of a balanced simplicial complex.

The first aim of this paper is the confirmation of Conjecture 1.2 in the case of the interval subdivision of a homology sphere. The main theorem is stated as:

Theorem 1.3. If $\Delta$ is a simplicial complex with a nonnegative and symmetric $h$-vector, then the $\gamma$-vector of the interval subdivision of $\Delta$ is the $f$-vector of a balanced simplicial complex.

Key words and phrases. simplicial complex, subdivision of a simplicial complex, $h$-vector, $\gamma$-vector, local $\gamma$-vector, balanced complex.

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This work is based on the study of certain refinement of Eulerian numbers of type $B$ used by the present authors in [AN18] to describe the $h$-vector of the interval subdivision $\text{Int}(\Delta)$ of a simplicial complex $\Delta$. The interval subdivision of a simplicial complex was introduced by Walker in [Wal88]. By Walker [Wal88, Theorem 6.1(a)], the simplicial complex of all chains in the partially ordered set $I(\Delta \setminus \emptyset) := \{[A,B] \mid \emptyset \neq A \subseteq B \in \Delta\}$ is a subdivision of $\Delta$. In [AN18], authors have given the combinatorial description of $f$- and $h$-vectors of a simplicial complex under the interval subdivision.

The second aim of this paper is to study the local $\gamma$-vector of the interval subdivision of a simplex. Stanley [Sta92] introduced the local $h$-vector of a topological subdivision of a simplex as a tool to study the face numbers of subdivisions of simplicial complexes. The local $h$-vector plays an important role to answer the question posed by Kalai and Stanley: whether the $h$-vector increases coordinate-wise after such subdivision of a Cohen-Macaulay complex? Since the local $h$-vector is symmetric it makes sense to define a local $\gamma$-vector, which was introduced by Athanasiadis in [Ath12]. It already follows from [AS13] that the local $\gamma$-vector of the interval subdivision of a simplex is nonnegative. Here, we give another proof of this result by answering a question asked by Juhnke-Kubitzke et al in [JKMS18].

**Question 1.4.** [JKMS18, Problem 4.9] Find classes of subdivisions $\Gamma$ such that $h(\Gamma_F, x) - h(\partial(\Gamma_F), x)$ is nonnegative, unimodal or $\gamma$-nonnegative. Moreover, for those classes try to find a combinatorial interpretation of the coefficients of $h(\Gamma_F, x) - h(\partial(\Gamma_F), x)$ respectively the coefficients of its $\gamma$-polynomial.

We show that for the interval subdivision $\Gamma$ of a simplex, $h(\Gamma_F, x) - h(\partial(\Gamma_F), x)$ is nonnegative and $\gamma$-nonnegative. Along the way, we give a combinatorial interpretation of the coefficients of $h(\Gamma_F, x) - h(\partial(\Gamma_F), x)$ and as well of the coefficients of its $\gamma$-polynomial. We give a brief description on necessary definitions and notions in Section 2. In particular, we recall some known results about the FFK-vectors and balanced complexes. In Section 3, we give the combinatorial foundation needed to prove of the main result. The proof of Theorem 1.3 is given in Section 4. In the last section, we give a proof of nonnegativity of the local $\gamma$-vector of interval subdivision of a simplex. Additionally, we give a geometrical description of the symmetric Eulerian polynomial $B_n^+(x)$, defined in Section 3.

### 2. Background and Basic Notions

A simplicial complex $\Delta$ on a finite vertex set $V$ is a collection of subsets of $V$, such that $\{v\} \in \Delta$ for all $v \in V$, and if $F \in \Delta$ and $E \subseteq F$, then $E \in \Delta$. The members of $\Delta$ are known as *faces*. The dimension of a face $F$ is $|F| - 1$. Let $d = \max\{|F| : F \in \Delta\}$ and define the dimension of $\Delta$ to be $\dim \Delta = d - 1$.

The *$f$-polynomial* of a $(d-1)$-dimensional simplicial complex $\Delta$ is defined as:

$$f_{\Delta}(t) = \sum_{F \in \Delta} t^{\dim F + 1} = \sum_{i=0}^{d} f_i t^i,$$
where \( f_i \) is the number of faces of dimension \( i - 1 \). As \( \dim \emptyset = -1 \), so \( f_0 = 1 \). The sequence \( f(\Delta) = (f_0, f_1, \ldots, f_d) \) is called the \( f \)-vector of \( \Delta \). Define the \( h \)-vector \( h(\Delta) = (h_0, h_1, \ldots, h_d) \) of \( \Delta \) by the \( h \)-polynomial:

\[
h_\Delta(t) := (1 - t)^d f_\Delta(t/(1 - t)) = \sum_{i=0}^{d} h_i t^i.
\]

If \( \Delta \) is a \((d - 1)\)-dimensional simplicial complex with the symmetric \( h \)-vector, i.e., the symmetric \( h \)-polynomial, then there exist integers \( \gamma_i \) such that

\[
h_\Delta(t) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i t^i (1 + t)^{d - 2i},
\]

the sequence \( \gamma(\Delta) = (\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor}) \) is called the \( \gamma \)-vector of \( \Delta \).

The interval subdivision \( \text{Int}(\Delta) \) of a simplicial complex \( \Delta \) is the simplicial complex on the vertex set \( I(\Delta \setminus \emptyset) \), where \( I(\Delta \setminus \emptyset) := \{ [A, B] \mid \emptyset \neq A \subseteq B \in \Delta \} \) as a partially ordered set ordered by inclusion defined as \([A, B] \subseteq [A', B'] \in I(\Delta \setminus \emptyset) \) if and only if \( A' \subseteq A \subseteq B \subseteq B' \). By Walker [Wal88], the simplicial complex of all chains in the partially ordered set \( I(\Delta \setminus \emptyset) \) is a subdivision of \( \Delta \). In [ANIS], authors have given the combinatorial description of \( f \)- and \( h \)-vectors of a simplicial complex under the interval subdivision.

**Theorem 2.1.** [ANIS, Theorem 2.2 and Theorem 3.1] Let \( \Delta \) be a \((d - 1)\)-dimensional simplicial complex. Then the transformation of \( f \)-vector of \( \Delta \) to \( f \)-vector of interval subdivision \( \text{Int}(\Delta) \) is given as

\[
f(\text{Int}(\Delta)) = [(\mathcal{F}_d)_{k,l}]_{0 \leq k,l \leq df(\Delta)},
\]

where

\[
(\mathcal{F}_d)_{0,l} = \begin{cases} 1, & l = 0; \\ 0, & l > 0. \end{cases}
\]

and for \( 1 \leq k \leq d \), we have

\[
(\mathcal{F}_d)_{k,l} = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} [(2k - 2j)^l - (2k - 2j - 1)^l] \tag{1}
\]

and the transformation of \( h \)-vector of \( \Delta \) to \( h \)-vector of \( \text{Int}(\Delta) \) is given as

\[
h(\text{Int}(\Delta)) = [B^+(d + 1, s + 1, r)]_{0 \leq s, r \leq dh(\Delta)},
\]

where \( B^+(d + 1, s + 1, r) \) is the number of \( \sigma \in B_{d+1} \) with \( \sigma_1 = s, \sigma_{d+1} > 0 \) and \( \text{des}_B(\sigma) = r \).

Our goal is to show that the \( \gamma \)-vector of the interval subdivision of \( \Delta \) is \( f \)-vector of a balanced simplicial complex.
2.1. FFK-vectors. A simplicial complex $\Delta$ on the vertex set $V$ is called $k$-colorable if there is a function $c : V \to \{1, 2, \ldots, k\}$, called a coloring of its vertices, such that every face has distinctly colored vertices. If a $(d - 1)$-dimensional complex $\Delta$ is $d$-colorable, we say it is a balanced simplicial complex.

There is an analogue of the Kruskal-Katona-Schützenberger (KKS) inequalities for $k$-colorable simplicial complexes, due to Frankl, Füredi, and Kalai [FFK88], known as FFK-inequalities. The vector satisfying FFK inequalities with respect to $k$ is called a $k$-FFK-vector. It is shown in [FFK88] that every face vector of a $k$-colorable complex is a $k$-FFK-vector and every $k$-FFK-vector is a face vector of some $k$-colorable complex. For more details, see [FFK88, NP11, NPT11]. Let us state some definitions from [NPT11].

**Definition 2.2.** [NPT11] Definition 3.6] Let \( f = (1, f_1, \ldots, f_d) \) be the $f$-vector of a $(d - 1)$-dimensional balanced complex.

1. **$(d + 1)$-good.** Let \( g = (g_1, \ldots, g_d, g_{d+1}) \) be a sum of $d$-FFK-vectors, each of which is dominated by $f$. Some, but not all, of these $d$-FFK-vectors may be shorter than $f$; that is, \( g_{d+1} \neq 0 \). Then we say that \((0, g)\) is $(d + 1)$-good for $f$. By [NPT11] Lemma 3.1, it can be noted that \( f + (0, g) \) is a $(d + 1)$-FFK-vector.

2. **$d$-good.** Let \( g = (g_1, \ldots, g_d) = f^{(1)} + \cdots + f^{(k)} \), with \( g_d \neq 0 \), be a sum of $(d - 1)$-FFK-vectors such that \( f_i \geq (i + 1) f_j^{(j)} \) for all $i$ and all $j$. Then we say that \((0, g)\) is $d$-good for $f$. By [NPT11] Lemma 3.5, it can be noted that \( f + (0, g) \) is a $d$-FFK-vector.

From the above definition, we have the following observation:

**Observation 2.3.** [NPT11] Observation 3.7] Let \( f = (1, f_1, \ldots, f_d) \) be the $f$-vector of a $(d - 1)$-dimensional balanced complex. If \((0, g)\) is $d$-good for $f$ and \((0, g')\) is $(d + 1)$-good for $f$, then \((0, g + g')\) is $(d + 1)$-good for $f$.

2.2. Coxeter Complex of type $B$: Here, we briefly discuss the simplicial complex whose $f$-vector is the $\gamma$-vector of Coxeter complex of type $B$ (details can be found in [Ste08, NP11, Pet15]). Define the set of decorated permutations, $\text{Dec}_n$, to be the set of all permutations $\sigma \in S_n$ with bars in the left peak positions. The bars can come in one of four styles: \( \{\| = |^0, |^1, |^2, |^3\} \), and thus for each $\sigma \in S_n$, we have $4^{\text{lpk}(\sigma)}$ decorated permutations. For example, here are three elements of $\text{Dec}_7$:

$$4|237|^2651, 4|^1327|^3156, 25|^3137|^1654.$$ 

Notice that the leftmost block is always increasing, even it is a singleton. Also the rightmost block may be strictly increasing.

Construct a balanced simplicial complex $\Gamma(\text{Dec}_n)$ whose faces are elements of $\text{Dec}_n$ with \( \dim \sigma = \text{lpk}(\sigma) - 1 \), where \( \text{lpk}(\sigma) := \{1 \leq i \leq n - 1 : \sigma_{i-1} < \sigma_i > \sigma_{i+1}\} \), with $\sigma_0 = 0$. A decorated permutation $\sigma$ covers $\tau$ if $\tau$ can be obtained from $\sigma$ by removing a bar from $B_i|B_{i+1}$ and reordering $B_iB_{i+1}$ as keep the decreasing part of $B_i$ as it is, and rewrite $B_{i+1}$, together with the increasing part of $B_i$, in increasing order. It is clear that vertices are elements with only one bar. $\Gamma(\text{Dec}_n)$ has dimension \( \left\lfloor \frac{n}{2} \right\rfloor - 1 \) and the color set of a face $\sigma$...
is \( \text{col}(\sigma) = \{ \lfloor i/2 \rfloor : \sigma_i > \sigma_{i+1} \} \). By [NP11 Corollary 4.5 (2)], we have
\[ f(\Gamma(\text{Dec}_n)) = \gamma(B_n). \]

**Lemma 2.4.** For all \( n \geq 1, 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \), we have
\[ (i + 1)\gamma_i(B_{n-1}) \leq \gamma_i(B_n). \]

**Proof.** Let \( \sigma = A_1 \cdots |A_{i+1} \) be any \((i - 1)\)-dimensional face of \( \Gamma(\text{Dec}_{n-1}) \). Then a face of \( \Gamma(\text{Dec}_n) \) can be obtained from \( \sigma \) by inserting \( n \) to any of the \( i + 1 \) blocks of \( \sigma \). If we insert \( n \) at the end of a block, no new left peak will be created and hence with this insertion, we get a \((i - 1)\)-dimensional face of \( \Gamma(\text{Dec}_n) \) from a \((i - 1)\)-dimensional face of \( \Gamma(\text{Dec}_{n-1}) \). Moreover, different faces in \( \Gamma(\text{Dec}_{n-1}) \) give an disjoint set of faces in \( \Gamma(\text{Dec}_n) \). Hence, the inequality follows. \( \square \)

The next lemma follows from Definition 2.2 and Lemma 2.4.

**Lemma 2.5.** Let \( n = 2d + 1 \). Then
- If \((0, f)\) is \(d\)-good for \( \gamma(B_{n-2}) \), then it is also \(d\)-good for \( \gamma(B_{n-1}) \).
- If \((0, f)\) is \((d + 1)\)-good for \( \gamma(B_{n-1}) \), then it is also \((d + 1)\)-good for \( \gamma(B_n) \).

**Proof.** The proof is similar to the proof of Lemma 5.5 [NP11]. \( \square \)

### 2.3. Subdivisions and Local \( h \)-Vectors

The notion of local \( h \)-vectors was firstly studied by Stanley [Sta92]. A **topological subdivision** of a simplicial complex \( \Delta \) is a simplicial complex \( \Delta' \) with a map \( \theta : \Delta' \to \Delta \) such that, for any face \( F \in \Delta \), the following holds:
- \( \Delta'_F := \theta^{-1}(2^F) \) is a subcomplex of \( \Delta' \) which is homeomorphic to a ball of dimension \( \dim(F) \);
- the interior of \( \Delta'_F \) is equal to \( \theta^{-1}(F) \).

The face \( \theta(G) \in \Delta \) is called the **carrier** of \( G \in \Delta' \). The subdivision \( \Delta' \) is called **quasi-geometric** if no face of \( \Delta' \) has its carriers of its vertices contained in a face of \( \Delta \) of smaller dimension. Moreover, \( \Delta' \) is called **geometric** if there exists a geometric realization of \( \Delta' \) which geometrically subdivides a geometric realization of \( \Delta \), in the way prescribed by \( \theta \). Clearly, all geometric subdivisions (such as the interval subdivisions considered in this paper) are quasi-geometric. For more detail, we refer to [Sta92] and a survey by Athanasiadis [Ath16].

Let \( V \) be a non-empty finite set. Let \( \Gamma \) be a subdivision of a \((d - 1)\)-dimensional simplex \( 2^V \). Then
\[
\ell_V(\Gamma, x) = \sum_{F \subseteq V} (-1)^{d-|F|} h(\Gamma_F, x) = \sum_{j=0}^d \ell_j x^j \tag{3}
\]
is known as the **local \( h \)-polynomial** of \( \Gamma \) (with respect to \( V \)). The sequence \( \ell(\Gamma) = (\ell_0, \ldots, \ell_d) \) is the **local \( h \)-vector** of \( \Gamma \) (with respect to \( V \)). By Stanley [Sta92], it is well-known that the local \( h \)-vector \( \ell_V(\Gamma, x) \) is symmetric. Thus it can be uniquely written in the following way:
\[
\ell_V(\Gamma, x) = \sum_{i=0}^{[d/2]} \xi_i x^i (1 + x)^{d-2i},
\]
where \( \xi_i \in \mathbb{Z} \). The sequence \( \xi(\Gamma) = (\xi_0, \ldots, \xi_{[d/2]}) \) is called the local \( \gamma \)-vector of \( \Gamma \) (w.r.t. \( V \)). It is natural to ask whether the local \( \gamma \)-vector is nonnegative or not. The local \( \gamma \)-vector
is nonnegative for special classes of subdivisions including barycentric, edgewise, cluster and interval subdivisions of a simplex \([\text{Ath12, AS12, AS13}]\). The local \(h\)-polynomial of the interval subdivision of a simplex \(2^V\) has nonnegative, symmetric and \(\gamma\)-nonnegative coefficients and has a nice combinatorial description due to Athanasiadis and Savvidou \([\text{AS13}]\).

**Theorem 2.6.** \([\text{AS13, Corollary 1.2, Theorem 1.3, Proposition 4.1}]\) Let \(\Gamma\) be the interval subdivision of a \((d - 1)\)-dimensional simplex \(2^V\). The local \(h\)-polynomial \(\ell_V(\Gamma, x)\) of \(\Gamma\) is nonnegative, symmetric and \(\gamma\)-nonnegative. More precisely,

\[
\ell_V(\Gamma, x) = \sum_{\sigma \in \mathcal{D}_d^B \cap B_d^*} x^{\text{exc}(\sigma)},
\]

where \(\text{exc}_B(\sigma)\) is the number of \(B\)-excedances of \(\sigma\); \(\mathcal{D}_d^B\) is the set of derangements of \(B_d\) and \(B_d^* := \{\sigma \in B_d : \sigma(m_\sigma) > 0\}\), \(m_\sigma\) is the minimal element of \(\{\sigma_1, \ldots, \sigma_d\}\).

Here, we give an other proof of the nonnegativity of local \(\gamma\)-vector of the interval subdivision of a simplex using the result of Juhnke-Kubitzke et al \([\text{JKMS18}]\). They gave an expression of the local \(h\)-vector which involves differences of \(h\)-vectors of restrictions of the subdivision and their boundary as well as derangement polynomials.

**Theorem 2.7.** \([\text{JKMS18, Theorem 4.4}]\) Let \(\Gamma\) be a subdivision of a simplex \(2^V\). The local \(h\)-polynomial of \(\Gamma\) can be written as

\[
\ell_V(\Gamma, x) = \sum_{F \subseteq V} [h(\Gamma_F, x) - h(\partial(\Gamma_F), x)] \varphi_{\gamma}^{A_{\gamma}}(x),
\]

where \(\varphi_{\gamma}^{A_{\gamma}}(x)\) is the usual derangement polynomial of order \(d\). In particular when \(\Gamma = \text{Int}(2^V)\), we obtain

\[
\ell_V(\text{Int}(2^V), x) = \sum_{F \subseteq V} [h(\text{Int}(2^F), x) - h(\partial(\text{Int}(2^F)), x)] \varphi_{\gamma}^{A_{\gamma}}(x)
\]

In Section 3, we answer the Question 1.4 in case of the interval subdivision by giving a nice combinatorial description on differences of \(h\)-vectors of restrictions of the subdivision and their boundary.

### 3. Symmetric Eulerian Polynomials of type \(B\)

Let \(B_n\) be the group consisting of all the bijections \(\sigma\) of the set \(\{\pm 1, \ldots, \pm n\}\) onto itself such that \(\sigma_{-i} = -\sigma_i\) for all \(i \in \{\pm 1, \ldots, \pm n\}\). For \(\sigma \in B_n\), the Descent set is defined as

\[
\text{Des}_B(\sigma) := \{i \in [0, n - 1] : \sigma_i > \sigma_{i+1}\},
\]

where \(\sigma_0 = 0\) and the type \(B\)-descent number is defined as \(\text{des}_B(\sigma) := |\text{Des}_B(\sigma)|\).

We use the notation \(\bar{s} := -s\). Let

\[B_n^+ := \{\sigma \in B_n : \sigma_n > 0\}\]

and for \(\bar{n} \leq j \leq n\),

\[B_{n,j}^+ := \{\sigma \in B_n^+ : \sigma_1 = j\}.
\]
Denote $B^+(n, j, k)$ by the number of elements in $B^+_{n,j}$ with exactly $k$ descents. Now, define the $j$-Eulerian polynomials of type $B^+$ by

$$B^+_{n,j}(t) := \sum_{\sigma \in B^+_{n,j}} t^{\text{des}}(\sigma) = \sum_{k=0}^{n-1} B^+(n, j, k) t^k,$$

(6)

Note that the usual Eulerian polynomial of type $B^+$ is the descent generating function for all of $B^+_n$:

$$B^+_n(t) = \sum_{\sigma \in B^+_n} t^{\text{des}}(\sigma) = \sum_{j=0}^{n-1} B^+_{n,j}.$$ 

Let's denote

$$B^{++}_n(x) := \sum_{j=1}^{n} B^+_{n,j} = \sum_{k=0}^{n} B^{++}(n, k) x^k,$$

where $B^{++}(n, k)$ is the number of signed permutations of $B^{++} := \{\sigma \in B^+_n : \sigma_1 > 0\}$ with exactly $k$ descents and

$$B^{-+}_n(x) := \sum_{j=1}^{n} B^-_{n,j} = \sum_{k=0}^{n} B^{-+}(n, k) x^k,$$

where $B^{-+}(n, k)$ is the number of signed permutations of $B^{-+} := \{\sigma \in B^+_n : \sigma_1 < 0\}$ with exactly $k$ descents.

Let's state the recurrence relations:

**Lemma 3.1.** [AN18, Lemma 3.4] For $1 \leq s \leq n$ and $1 \leq r \leq n-1$, we have the following relations:

1. $B^+(n, s, r) = B^+(n, n-s+1, n-r-1)$ and thus
   $$B^{n,s}_{n,r}(t) = t^{n-1} B^+_{n,n-s+1}(t).$$

2. $B^+(n, s, r) = B^+(n, n-s+1, n-r-1)$ and thus
   $$B^{n,s}_{n,r}(t) = t^{n-1} B^+_{n,n-s+1}(t).$$

3. $B^+(n, s, r) = \sum_{j=1}^{s-1} B^+(n-1, j, r-1) + \sum_{j=1}^{n-1} B^+(n-1, \tilde{j}, r) + \sum_{j=s}^{n-1} B^+(n-1, j, r).$

Thus, the recurrence relation holds:

$$B^{n,s}_{n,r}(t) = t^{s-1} \sum_{j=1}^{s-1} B^+_{n-1,j}(t) + \sum_{j=1}^{n-1} B^+_{n-1,j}(t) + \sum_{j=1}^{n-1} B^+_{n-1,j}(t),$$

with initial conditions $B^+_{1,1}(t) = 1$ and $B^+_{1,1}(t) = 0$. 
\( B^+(n, s, r) = \sum_{j=1}^{n-1} B^+(n-1, j, r-1) + \sum_{j=s}^{n-1} B^+(n-1, j, r-1) + \sum_{j=1}^{s-1} B^+(n-1, j, r) \).

Thus, the recurrence relation holds:

\[ B^+_{n,s}(t) = t \sum_{j=1}^{n-1} B^+_{n-1,j}(t) + t \sum_{j=s}^{n-1} B^+_{n-1,j}(t) + \sum_{j=1}^{s-1} B^+_{n-1,j}(t). \]

Using Lemma 3.1 and bijection between \( B^+_n \) and \( B^-_n := \{ \sigma \in B_n : \sigma_n < 0 \} \) by mapping \( \sigma = (\sigma_1, \ldots, \sigma_n) \) to \( \bar{\sigma} = (\bar{\sigma}_1, \ldots, \bar{\sigma}_n) \), we have the following relation:

**Observation 3.2.** We have \( B^+_{n,1}(t) = B^+_{n-1}(t) \) and \( B^+_{n,n}(t) = t^{n-1} B^+_{n-1}(t) = B^-_{n-1}(t) \), by [AN18 Lemma 3.4] where \( B^-_{n}(t) \) is the descent generating function for all of \( B^-_{n} \). Thus we get that \( B_n(t) = B^+_{n,1}(t) + B^+_{n,n}(t) \).

Since the \( j \)-Eulerian polynomials are not symmetric so define the symmetric \( j \)-Eulerian polynomials as:

\[ B^+_n(t) = \sum_{\sigma \in B^+_1 \cup B^+_{n-j+1}} t^{\text{des}}(\sigma) \]

and

\[ B^-_{n,j}(t) = \sum_{\sigma \in B^-_j \cup B^-_{n-j+1}} t^{\text{des}}(\sigma) \]

Observe that

\[ B^+_n(t) = \begin{cases} B^+_n(t) + B^+_{n,n-j+1}(t), & j \neq (n+1)/2; \\ B^+_n(t), & j = (n+1)/2. \end{cases} \]

and

\[ B^-_{n,j}(t) = \begin{cases} B^-_{n,j}(t) + B^-_{n,n-j+1}(t), & j \neq (n+1)/2; \\ B^-_{n,j}(t), & j = (n+1)/2. \end{cases} \]

By Lemma 3.1(1), the polynomials \( B^+_n(t) \) have symmetric coefficients, and hence a \( \gamma \)-vector exists. Clearly, \( B^+_n(t) \) has symmetry axis at degree \( \left\lfloor \frac{n-1}{2} \right\rfloor \). If

\[ B^+_n(t) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \gamma^{(n,j)}_k t^k (1 + t)^{n-1-2k}, \]

let \( \gamma^{(n,j)} = (\gamma^{(n_j)}_0, \gamma^{(n_j)}_1, \ldots, \gamma^{(n_j)}_{\left\lfloor \frac{n-1}{2} \right\rfloor}) \) denotes the corresponding gamma vector.

Lets define the following polynomials for \( 1 \leq j < (n+1)/2 \)

\[ B^+_n(t) = t B^+_n(t) + B^+_{n,n-j+1}(t) \]

and

\[ B^-_{n,j}(t) = t B^-_{n,n-j+1}(t) + B^-_{n,j}(t). \]
The above polynomials are also symmetric by Lemma 3.1 and have symmetry axis at $\lfloor n/2 \rfloor$. Let $\tilde{\gamma}^{(n,j)} = (\tilde{\gamma}^{(n,j)}_0, \tilde{\gamma}^{(n,j)}_1, \ldots, \tilde{\gamma}^{(n,j)}_{\lfloor n/2 \rfloor})$ and $\tilde{\gamma}^{(n,j)}$ denote the $\gamma$-vectors for $\mathbb{B}_{n,j}^+$ and $\mathbb{B}_{n,j}^-$ respectively.

It can be observed that $B^+_{n,j}(t)$ and $B^-_{n,j}(t)$ are symmetric polynomials by Lemma 3.1, i.e., we have

$$B^+_{n,j}(t) = B^+_{n,j}(n, n - k - 1) \quad \text{and} \quad B^-_{n,j}(t) = B^-_{n,j}(n, n - k - 1)$$

(7)

To prove Theorem 1.3, we need the following lemma:

**Lemma 3.3.** The following recurrence relations hold for $\gamma^{(n,j)}$ and $\tilde{\gamma}^{(n,j)}$:

1. If $j = (n + 1)/2$, then
   $$\gamma^{(n,(n+1)/2)} = \sum_{k=1}^{(n-1)/2} \left[ \gamma^{(n-1,k)} + \gamma^{(n-1,k)} \right].$$

2. If $j = (n+1)/2$, then
   $$\gamma^{(n,(n+1)/2)} = \sum_{k=1}^{(n-1)/2} \left[ \gamma^{(n-1,k)} + (0, \gamma^{(n-1,k)}) \right].$$

3. For $1 \leq j < (n + 1)/2$,
   $$\gamma^{(n,j)} = 2 \sum_{k=1}^{\lfloor n/2 \rfloor} \gamma^{(n-1,k)} + 2 \sum_{k=1}^{j-1} \gamma^{(n-1,k)} + \sum_{k=j}^{\lfloor n/2 \rfloor} \gamma^{(n-1,k)}.$$  

4. For $1 \leq j < (n + 1)/2$,  
   $$\gamma^{(n,j)} = 2 \sum_{k=1}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1,k)}) + 2 \sum_{k=1}^{j-1} \gamma^{(n-1,k)} + \sum_{k=j}^{\lfloor n/2 \rfloor} \gamma^{(n-1,k)}.$$  

5. For $1 \leq j < (n + 1)/2$,
   $$\tilde{\gamma}^{(n,j)} = \sum_{k=1}^{j-1} \tilde{\gamma}^{(n-1,k)} + \sum_{k=1}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1,k)}) + \sum_{k=1}^{\lfloor n/2 \rfloor} \gamma^{(n-1,k)}.$$  

6. For $1 \leq j < (n + 1)/2$,
   $$\tilde{\gamma}^{(n,j)} = \sum_{k=1}^{\lfloor n/2 \rfloor} \tilde{\gamma}^{(n-1,k)} + \sum_{k=j}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1,k)}) + \sum_{k=1}^{\lfloor n/2 \rfloor} (0, \gamma^{(n-1,k)}).$$
Proof. For $j = (n + 1)/2$, by Lemma 3.1 (3)

$$
\mathbb{B}^+_{n,(n+1)/2}(t) = B^+_{n,(n+1)/2}(t)
$$

$$
= t \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t) + \sum_{k=(n+1)/2}^{n-1} B^+_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t)
$$

$$
= t \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t) + \sum_{k=(n+1)/2}^{n-1} B^+_{n-1,k}(t)
$$

$$
= \sum_{k=1}^{(n-1)/2} (tB^+_{n-1,k}(t) + B^+_{n-1,n-k}(t)) + \sum_{k=1}^{(n-1)/2} (B^+_{n-1,k}(t) + B^+_{n-1,n-k}(t))
$$

$$
= \sum_{k=1}^{(n-1)/2} (\tilde{\mathbb{B}}^+_{n-1,k}(t) + \mathbb{B}^+_{n-1,k}(t)),
$$

For $j = (n + 1)/2$, by Lemma 3.1 (3)

$$
\mathbb{B}^+_{n,(n+1)/2}(t) = B^+_{n,(n+1)/2}(t)
$$

$$
= t \sum_{k=1}^{n-1} B^+_{n-1,k}(t) + \sum_{k=(n+1)/2}^{n-1} B^+_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t)
$$

$$
= t \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t) + t \sum_{k=(n+1)/2}^{n-1} B^+_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t) + \sum_{k=1}^{(n-1)/2} B^+_{n-1,k}(t)
$$

$$
= t \sum_{k=1}^{(n-1)/2} (B^+_{n-1,k}(t) + B^+_{n-1,n-k}(t)) + \sum_{k=1}^{(n-1)/2} (tB^+_{n-1,n-k}(t) + B^+_{n-1,k}(t))
$$

$$
= \sum_{k=1}^{(n-1)/2} (t\mathbb{B}^+_{n-1,k}(t) + \mathbb{B}^+_{n-1,k}(t)).
$$

To prove 3: if $1 \leq j < (n + 1)/2,$

$$
\mathbb{B}^+_{n,j}(t) = B^+_{n,j}(t) + B^+_{n,n-j+1}(t)
$$

$$
= t \sum_{k=1}^{j-1} B^+_{n-1,k}(t) + \sum_{k=j}^{n-1} B^+_{n-1,k}(t) + t \sum_{k=1}^{n-j} B^+_{n-1,k}(t) + \sum_{k=n-j+1}^{n-1} B^+_{n-1,k}(t) + 2 \sum_{k=1}^{n-1} B^+_{n-1,k}(t)
$$

$$
= 2t \sum_{k=1}^{j-1} B^+_{n-1,k}(t) + (1 + t) \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + 2 \sum_{k=n-j+1}^{n-1} B^+_{n-1,k}(t) + 2 \sum_{k=1}^{n-1} B^+_{n-1,k}(t)
$$
To prove 4: if \(1 \leq j < \frac{n}{2}\),

\[
\mathbb{B}^+_{n,j} = \sum_{k=1}^{n} B_{n-1,k}^+(t) + \sum_{k=1}^{n-1} B_{n-1,n-k}^+(t) + (1 + t) \sum_{k=j}^{n-1} B_{n-1,k}^+(t) + 2 \sum_{k=1}^{n-1} B_{n-1,k}^+(t)
\]

To prove 5: if \(1 \leq j < \frac{n}{2}\),

\[
\mathbb{B}^+_{n,j} = t B_{n,j}^+(t) + B_{n,n-j+1}^+(t)
\]

\[
= t^2 \sum_{k=1}^{n} B_{n-1,k}^+(t) + t \sum_{k=1}^{n} B_{n-1,n-k}^+(t) + t \sum_{k=1}^{n-1} B_{n-1,k}^+(t) + 2 \sum_{k=1}^{n-1} B_{n-1,k}^+(t) + \sum_{k=n-j+1}^{n-1} B_{n-1,k}^+(t)
\]

\[
= t^2 \sum_{k=1}^{n} B_{n-1,k}^+(t) + t \sum_{k=1}^{n} B_{n-1,n-k}^+(t) + t \sum_{k=1}^{n-1} B_{n-1,k}^+(t) + 2 \sum_{k=1}^{n-1} B_{n-1,k}^+(t)
\]
\[ + t \sum_{k=1}^{n-j} B^+_{n-1,k}(t) + \sum_{k=1}^{j-1} B^+_{n-1,n-k}(t) + (1 + t) \sum_{k=1}^{n-1} B^+_{n-1,k}(t) \]

\[ = (1 + t) \sum_{k=1}^{j-1} (t B^+_{n-1,k}(t) + B^+_{n-1,n-k}(t)) + 2t \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + (1 + t) \sum_{k=1}^{n-1} B^+_{n-1,k}(t) \]

\[ = (1 + t) \sum_{k=1}^{j-1} B^+_{n-1,k}(t) + 2t \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + (1 + t) \sum_{k=1}^{n-1} B^+_{n-1,k}(t). \]

To prove 6: if \( 1 \geq j < (n + 1)/2 \),

\[ \sum_{n,j} B^+_{n,j}(t) = t B^+_{n,n-j+1}(t) + B^+_{n,j}(t) \]

\[ = t^2 \sum_{k=1}^{n-1} B^+_{n-1,k}(t) + t^2 \sum_{k=n-j+1}^{n-1} B^+_{n-1,k}(t) + t \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + t \sum_{k=1}^{n-1} B^+_{n-1,k}(t) \]

\[ + t \sum_{k=j}^{n-j} B^+_{n-1,k}(t) + \sum_{k=1}^{j-1} B^+_{n-1,k}(t) \]

\[ = t(1 + t) \sum_{k=1}^{n-j} B^+_{n-1,k}(t) + t^2 \sum_{k=1}^{j-1} B^+_{n-1,k}(t) + \]

\[ + t \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + t \sum_{k=1}^{n-j} B^+_{n-1,n-k}(t) + \sum_{k=1}^{j-1} B^+_{n-1,k}(t) \]

\[ = t(1 + t) \sum_{k=1}^{n-j} B^+_{n-1,k}(t) + 2t \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + (1 + t) \sum_{k=1}^{j-1} (t B^+_{n-1,n-k}(t) + B^+_{n-1,k}(t)) \]

\[ = t(1 + t) \sum_{k=1}^{n-j} B^+_{n-1,k}(t) + 2t \sum_{k=j}^{n-j} B^+_{n-1,n-k}(t) + (1 + t) \sum_{k=1}^{j-1} B^+_{n-1,k}(t). \]

\[ \square \]

4. \( \gamma \)-Vector

If \( \Delta \) has symmetric \( h \)-vector then by [AN18, Theorem 3.1, Lemma 3.5], \( h(\text{Int}(\Delta)) \) is also symmetric. The following proposition holds:

**Proposition 4.1.** If \( \Delta \) is a simplicial complex of dimension \( n - 1 \) with symmetric \( h \)-vector \( h(\Delta) = (h_0, h_1, \ldots, h_n) \), then

\[ h_r(\text{Int}(\Delta)) = \sum_{j=0}^{\lfloor n/2 \rfloor} (B^+(n + 1, j + 1, r) + B^+(n + 1, n - j, r)) h_j, \]
and thus
\[ h(\text{Int}(\Delta), t) = \sum_{j=0}^{\lfloor n/2 \rfloor} h_{j}B_{n+1,j+1}(t). \]

In terms of \( \gamma \)-vectors,
\[ \gamma(\text{Int}(\Delta)) = \sum_{j=0}^{\lfloor n/2 \rfloor} h_{j} \gamma^{(n+1,j+1)}. \]

**Example 4.2.** If \( n = 5 \) and \( h(\Delta) = (h_0, h_1, h_2, h_3 = h_2, h_4 = h_1, h_5 = h_0) \), then
\[
h(\text{Int}(\Delta))^t = \begin{pmatrix}
0 \\
237h_0 + 192h_1 + 168h_2 \\
1682h_0 + 1728h_1 + 1752h_2 \\
1682h_0 + 1728h_1 + 1752h_2 \\
237h_0 + 192h_1 + 168h_2 \\
h_0
\end{pmatrix}
\]
\[=h_0 \begin{pmatrix}
1 \\
237 \\
1682 \\
1682 \\
237 \\
1
\end{pmatrix} + h_1 \begin{pmatrix}
0 \\
192 \\
1728 \\
1728 \\
192 \\
0
\end{pmatrix} + h_1 \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} = h_0 \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + h_1 \begin{pmatrix}
0 \\
192 \\
1728 \\
1728 \\
192 \\
0
\end{pmatrix} + h_1 \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Thus,
\[ h(\text{Int}\Delta, t) = h_0 \mathbb{B}_{6,1}^+(t) + h_1 \mathbb{B}_{6,2}^+(t) + h_3 \mathbb{B}_{6,3}^+(t), \]
and
\[ \gamma(\text{Int}\Delta) = h_0 \gamma^{(6,1)} + h_1 \gamma^{(6,2)} + h_2 \gamma^{(6,3)}, \]
where \( \gamma^{(6,1)} = (1, 232, 976), \gamma^{(6,2)} = (0, 192, 152) \) and \( \gamma^{(6,3)} = (0, 168, 1248) \).

4.1. **Proof of Theorem 1.3**: In this subsection, we prove the main theorem that \( \gamma(\text{Int}(\Delta)) \) is an FFK-vector, i.e., an \( f \)-vector of a balanced complex.

Since \( \mathbb{B}_{n,1}^+(t) = B_{n,1}^+(t) + B_{n,n}^+(t) = B_{n-1}(t) \), where \( B_{n-1}(t) \) is the Eulerian polynomial of type \( B \). By [NP11] Theorem 6.1(1)], the \( \gamma \)-vector \( \gamma(B_{n-1}) \) of \( B_{n-1}(t) \) is an FFK-vector, and thus \( \gamma(B_{n-1}) = \gamma^{(n,1)} \) is an FFK-vector for any \( n \). Since \( B_{n-1}(t) \) has symmetry at degree \( \lfloor \frac{n-1}{2} \rfloor \), \( \gamma^{(n,1)} = (1, f_1, \ldots, f_d) \), where \( d = \lfloor \frac{n-1}{2} \rfloor \). Since \( h_0 = 1 \) for all simplicial complexes \( \Delta \),
\[ \gamma(\text{Int}(\Delta)) = \gamma^{(n+1,1)} + h_1 \gamma^{(n+1,2)} + \cdots, \]
where \( n = \dim \Delta + 1 \). Also note that \( \gamma^{(n+1,j)} = 0 \) for all \( j \neq 1 \). We will show that \( \gamma^{(n,i)} \) is \( d \)-or \( (d+1) \)-good for \( \gamma^{(n,1)} \). More precisely, we prove the following:

**Proposition 4.3.** Let \( \gamma^{(n,1)} = (1, f_1, \ldots, f_d) \), where \( d = \lfloor \frac{n-1}{2} \rfloor \).

1. If \( n \) is odd, i.e., \( n = 2d + 1 \), then
   a: \( \gamma^{(n,j)}, 1 < j \leq (n+1)/2, \) is \( d \)-good for \( \gamma^{(n,1)}, \)
   b: \( \gamma^{(n,j)}, 1 \leq j \leq (n+1)/2, \) is \( d \)-good for \( \gamma^{(n,1)}, \)
   c: \( \gamma^{(n,j)}, 1 \leq j < (n+1)/2, \) is \( d \)-good for \( \gamma^{(n,1)} \) and
(2) If \( n \) is even, i.e., \( n = 2d + 2 \), then

\[ \gamma^{(n,j)}, 1 \leq j \leq n/2, \text{ is } d\text{-good for } \gamma^{(n,1)}. \]

\[ \text{Case 1(odd): Let } n = 2d + 1 \text{ and consider } \gamma^{(n,j)} \text{ for some } 1 < j \leq (n + 1)/2. \text{ We have to show that } \gamma^{(n,j)} \text{ is } d\text{-good for } \gamma^{(n,1)}. \text{ For special case, } j = (n + 1)/2, \text{ we have by Lemma 3.3} \]

\[ \gamma_{(n,(n+1)/2)} = \sum_{k=1}^{(n-1)/2} (\tilde{\gamma}^{(n-1,k)} + \gamma^{(n-1,k)\overline{a}}). \]

Since \( n - 1 = 2(d - 1) + 2 \) is even, by induction hypothesis, each summand is \( d\text{-good for } \gamma^{(n-1,1)}, \text{ and hence by Lemma 2.5 } \gamma_{(n,(n+1)/2)} \text{ is } d\text{-good for } \gamma^{(n,1)}. \text{ For } 1 < j < (n + 1)/2, \text{ by Lemma 3.3 we have} \]

\[ \gamma^{(n,j)} = 2 \sum_{k=1}^{[n/2]} \gamma^{(n-1,k)} + 2 \sum_{k=1}^{j-1} \gamma^{(n-1,k)} + \sum_{k=j}^{[n/2]} \gamma^{(n-1,k)}. \]

By induction hypothesis, first two summands in \( \gamma^{(n,j)} \) are \( d\text{-good for } \gamma^{(n-1,1)}, \text{ and the last summand is } (d - 1)\text{-good for } \gamma^{(n-1,1)}, \text{ followed from 3.2. Thus, the sum of } d \text{ and } (d - 1)\text{-good vectors is } d\text{-good vector, so } \gamma^{(n,j)} \text{ is } d\text{-good for } \gamma^{(n-1,1)} \text{ and hence also } d\text{-good for } \gamma^{(n,1)} \text{ by Lemma 2.5 proving part 1(a).} \]

Now, we want to show that \( \gamma^{(n,j)} \), for \( 1 \leq j \leq (n + 1)/2 \) is \( d\text{-good for } \gamma^{(n,1)}. \) By Lemma 3.3 for \( 1 \leq j < (n + 1)/2 \)

\[ \gamma^{(n,j)} = 2 \sum_{k=1}^{[n/2]} (0, \gamma^{(n-1,k)}) + 2 \sum_{k=1}^{j-1} \tilde{\gamma}^{(n-1,k)} + \sum_{k=j}^{[n/2]} \gamma^{(n-1,k)}. \]
The last two summands by induction are \( d \)-good for \( \gamma^{(n-1,1)} \), hence also \( d \)-good for \( \gamma^{(n,1)} \) by Lemma 2.5. The first sum term can be rewritten as \((0, 2\alpha)\), where

\[
\alpha = \sum_{k=1}^{[n/2]} \gamma^{(n-1,k)}
\]

which is by induction an \((d-1)\)-FFK vector and by Lemma 3.3(3)

\[
\gamma^{(n,1)} = \sum_{k=1}^{[n/2]} (\gamma^{(n-1,k)} + 2\gamma^{(n-1,\overline{k})})
\]

dominates \(\alpha\). Thus, by Definition 2.2(1), \((0, 2\alpha)\) is \( d \)-good for \( \gamma^{(n,1)} \). Thus, \( \gamma^{(n,\overline{j})} \) is \( d \)-good for \( \gamma^{(n,1)} \). For \( j = \frac{n+1}{2} \), we have

\[
\gamma^{(n,\frac{n+1}{2})} = \sum_{k=1}^{(n-1)/2} [\gamma^{(n-1,\overline{k})} + (0, \gamma^{(n-1,k)})].
\]

\( \gamma^{(n,\frac{n+1}{2})} \) is \( d \)-good for \( \gamma^{(n,1)} \) as all terms are \( d \)-good for \( \gamma^{(n,1)} \) followed by induction hypothesis and Lemma 2.5.

For part (1c), we know from Lemma 3.3

\[
\tilde{\gamma}^{(n,j)} = \sum_{k=1}^{j-1} \gamma^{(n-1,k)} + 2 \sum_{k=j}^{[n/2]} (0, \gamma^{(n-1,k)}) + \sum_{k=1}^{[n/2]} \gamma^{(n-1,\overline{k})},
\]

for \( 1 \leq j < (n+1)/2 \). In the degenerate case \( j = 1 \), this gives

\[
\tilde{\gamma}^{(n,1)} = 2 \sum_{k=1}^{[n/2]} (0, \gamma^{(n-1,k)}) + \sum_{k=1}^{[n/2]} \gamma^{(n-1,\overline{k})} = \gamma^{(n,1)}
\]

which is \( d \)-good for \( \gamma^{(n,1)} \). If \( j > 1 \), we know from Lemma 3.3(5) that:

\[
\tilde{\gamma}^{(n,j)} = \sum_{k=2}^{j-1} \gamma^{(n-1,k)} + 2 \sum_{k=j}^{[n/2]} (0, \gamma^{(n-1,k)}) + \sum_{k=1}^{[n/2]} \gamma^{(n-1,\overline{k})}
\]

that can be further rewritten as:

\[
= \sum_{k=2}^{j-1} \gamma^{(n-1,k)} + 2 \sum_{k=1}^{[n/2]} \gamma^{(n-1,\overline{k})} + 2 \sum_{k=j}^{[n/2]} (0, \gamma^{(n-1,k)}) + 2 \sum_{k=1}^{[n/2]} (0, \gamma^{(n-1,k)}) + \sum_{k=1}^{[n/2]} \gamma^{(n-1,\overline{k})}
\]

\[
= \sum_{k=2}^{j-1} \gamma^{(n-1,k)} + 2 \sum_{k=1}^{[n/2]} \gamma^{(n-1,\overline{k})} + (0, 2\beta),
\]

where

\[
\beta = \sum_{k=j}^{[n/2]} \gamma^{(n-1,k)} + \sum_{k=1}^{[n/2]} \gamma^{(n-1,k)}.
\]

The first two sums are \( d \)-good for \( \gamma^{(n-1,1)} \) by induction and hence by Lemma 2.5, are \( d \)-good for \( \gamma^{(n,1)} \). Also by induction \( \beta \) is \((d-1)\)-FFK vector and \( \gamma^{(n,1)} \) dominates \( \beta \). Thus,
by Definition 2.2(1), (0, 2β) is d-good for γ(n,1). Hence, γ(n,j) is d-good for γ(n,1).

For part (1d), we have
\[ \gamma(n,j) = \gamma(n-1,k) + 2 \sum_{k=j+1}^{n/2} (0, \gamma(n-1,k)) + \sum_{k=1}^{n/2} (0, \gamma(n-1,k)), \]
for 1 ≤ j < (n + 1)/2. In the degenerate case j = 1, this gives γ(n,1) = (0, γ(n,1)) which is clearly (d + 1)-good for γ(n,1). If j > 1, we can rewrite it by using Lemma 3.3(5):
\[ \gamma(n,j) = \sum_{k=2}^{j-1} \gamma(n-1,k) + (0, \gamma(n,1)) + (0, \eta), \]
where
\[ \eta = 2 \sum_{k=j}^{n/2} \gamma(n-1,k) + \sum_{k=1}^{n/2} \gamma(n-1,k). \]

The first sum is d-good for γ(n−1,1) by induction and hence by Lemma 2.5 is d-good for γ(n,1). Also by induction, η is (d − 1)-FFK vector and γ(n,1) dominates η. Thus, by Definition 2.2, (0, 2η) is d-good for γ(n,1). But (0, γ(n,1)) is (d + 1)-good for γ(n,1). Hence, by Observation 2.3, the sum of d-good and (d + 1)-good is (d + 1)-good. Hence, γ(n,j) is (d + 1)-good for γ(n,1).

**Case 2 (n even).** Let n = 2d + 2. For 1 < j ≤ n/2, we have
\[ \gamma(n,j) = 2 \sum_{k=1}^{n/2} \gamma(n-1,k) + 2 \sum_{k=j}^{n/2} \gamma(n-1,k). \]

Since n − 1 = 2d + 1 so by Case 1, each term in all summations is d-good for γ(n−1,1) and hence, by Lemma 2.5 we get γ(n,j) is d-good for γ(n,1).

For part (2b), by Lemma 3.3(3),
\[ \gamma(n,j) = 2 \sum_{k=1}^{n/2} (0, \gamma(n-1,k)) + 2 \sum_{k=1}^{n/2} \gamma(n-1,k) + \sum_{k=j}^{n/2} \gamma(n-1,k). \]

The first sum is d-good for γ(n,1) followed from Case 1. The second sum is (d + 1)-good for γ(n−1,1) and the last sum is d-good for γ(n−1,1) by induction. Therefore, the sum of last two summations is (d + 1)-good for γ(n−1,1) by Observation 2.3 hence, is (d + 1)-good for γ(n,1) by Lemma 2.5. Thus, γ(n,j) is (d + 1)-good for γ(n,1). The proofs of (2c) and (2d) trivially follow from (1c) and (1d).

Proposition 4.3 and the above discussion show that γ(\text{Int}(Δ)) is d-good for γ(n+1,1), i.e., d-FFK-vector, hence is f-vector of a balanced simplicial complex which completes the proof of Theorem 1.3.

Since γ(\text{Int}(Δ)) is nonnegative so here arises a natural question.

**Question 4.4.** What is the combinatorial interpretation for the γ-vectors γ(n,j)?
We conclude this section with the following remark:

**Remark 4.5.** As $\gamma(\text{Int}(\Delta))$ is $f$-vector of a simplicial complex so it would be interesting if one could explicitly construct the simplicial complex $\Gamma(\text{Int}(\Delta))$ such that $f(\Gamma(\text{Int}(\Delta))) = \gamma(\text{Int}(\Delta))$.

5. **LOCAL $\gamma$-VECTOR**

Since $B_n^{++}(x)$ and $B_n^{-+}(x)$ are symmetric polynomials and

$$B_n^+(x) = B_n^{++}(x) + B_n^{-+}(x),$$

so $B_n^+(x)$ decomposes as sum of two nonnegative, symmetric polynomials. In this section, we show that the polynomials $B_n^{++}(x)$ and $B_n^{-+}(x)$ are both $\gamma$-nonnegative by giving a combinatorial description. Moreover, a geometric interpretation of the polynomial $B_n^{++}(x)$ as the $h$-polynomial $h(\partial(\text{Int}(2^{[n]})), x)$ of the boundary of interval subdivision of a simplex is given.

5.1. **A Geometrical Interpretation:** In this subsection, we show that $B_n^{++}(x)$ is the $h$-polynomial of the boundary of interval subdivision of a simplex $2^{[n]}$. Let $\Gamma$ be the interval subdivision of a simplex $2^{[n]}$.

**Proposition 5.1.** We have $h(\partial(\Gamma), x) = B_n^{++}(x)$.

**Proof.** Viewing $\partial(\Gamma)$ as interval subdivision of $\partial(2^{[n]})$ and by Theorem 2.1, we have

$$h_r(\partial(\Gamma)) = \sum_{s=0}^{n-1} B^+(n, s+1, r)h_s(\partial(2^{[n]})).$$

Using the fact that $h(\partial(2^{[n]})) = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$, we get

$$h_r(\partial(\Gamma)) = \sum_{s=0}^{n-1} B^+(n, s+1, r) = B^{++}(n, r).$$



5.2. **The type B Expansion:** Now we are turning toward analyzing the behavior of type $B$ expansion. Let us first recall, the definition of slides. Let $\sigma = \sigma_1 \ldots \sigma_n \in B_n$ and consider $\sigma_0\sigma_1 \ldots \sigma_n\sigma_{n+1}$, where $\sigma_0 = 0$ and $\sigma_{n+1} = \infty$. Put asterisks at each end and also between $\sigma_i$ and $\sigma_{i+1}$ whenever $\sigma_i < \sigma_{i+1}$. A slide is any segment between asterisks of length at least 2. In other words, a slide of $\sigma$ is any decreasing run of $\sigma_0\sigma_1 \ldots \sigma_n\sigma_{n+1}$ of length at least 2. For example, for the permutation $357168942 \in B_9$, *0 *35 *716 *89 *4 *258* there are four slides, namely, 35, 716, 89 and 258.

Let $W_B(n, k, s) := \{\sigma \in B_n : \sigma \text{ has } k \text{ descents and } s+1 \text{ slides}\}, W_B(n, s) := W_B(n, s, s)$. It follows that every slide of an element of $W_B(n, s)$ must be of length exactly 2. Also
notice that \( k \geq s \) as each slide gives at least 1 descent except the last slide (the last slide may or may not have a descent).

Lets denote \( b^{++}(n, k, s) := |W_B(n, k, s) \cap B^{++}_n|, \ b^{-+}(n, k, s) := |W_B(n, k, s) \cap B^{-+}_n|, \ b^{++}(n, s) := b^{++}(n, s, s) \) and \( b^{-+}(n, s) := b^{-+}(n, s, s) \). It can be observed that an element \( \sigma \) in \( B^{++}_n \) has at least 1 slide and an element \( \sigma \) in \( B^{-+}_n \) has at least 2 slides.

**Proposition 5.2.** We have

\[
\begin{align*}
b^{++}(n, k, s) &= \binom{n-1-2s}{k-s} b^{++}(n, s) \\
b^{-+}(n, k, s) &= \binom{n-1-2s}{k-s} b^{-+}(n, s)
\end{align*}
\]

which give the following relations

\[
\begin{align*}
B^{++}(n, k) &= \sum_{s=0}^{k} \binom{n-1-2s}{k-s} b^{++}(n, s) \\
B^{-+}(n, k) &= \sum_{s=1}^{k} \binom{n-1-2s}{k-s} b^{-+}(n, s)
\end{align*}
\]

**Proof.** Let prove the relations for \( B^{++}_n \). Let \( \sigma = \sigma_1 \ldots \sigma_n \in B^{++}_n \cap W_B(n, s, s) \). Counting \( \sigma_{n+1} = \infty \), there are \( n+1 \) symbols and \( n+1 - 2(s+1) = n - 1 - 2s \) that are not included in the slides. Choose \( k - s \) of these \( n - 1 - 2s \) elements, move each chosen element \( \sigma_k \) to the left if \( \sigma_k < 0 \) (to right if \( \sigma_k > 0 \), respectively) into the nearest slide \( *\sigma_j\sigma_{j+1}* \) with \( \sigma_j > \sigma_k > \sigma_{j+1} \). After moving chosen elements, the resulting permutation is still in \( B^{++}_n \). Thus, the first relation holds. The second assertion follows upon summing \( b^{++}(n, k, s) \) over \( 0 \leq s \leq k \). For \( B^{-+}_n \), the proof is similar.

Figure 1 illustrates the one-one correspondence. For example, let 568129734 \( \in B^{++}_9 \), \( *0*5*68*1*29*7*3*4* \) has 3 slides and 2 descents. \( \square \)

**Theorem 5.3.** For \( n \geq 1 \),

\[
B^{++}_n(x) = \sum_{s=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} b^{++}(n, s)x^s(1 + x)^{n-1-2s}
\]

![Figure 1](image)
and

\[ B_{n}^{\pm}(x) = \sum_{s=1}^{\lfloor \frac{n-1}{2}\rfloor} b^{\pm}(n, s) x^{s} (1 + x)^{n-1-2s}. \]

**Proof.** The result follows by Proposition 5.2, the relation (7) and the fact that \( \binom{n-1-2s}{k-s} = \binom{n-1-2s}{n-1-k-s} \). \[ \square \]

**Remark 5.4.** By Theorem 2.1, we have

\[ B_{n}^{+}(x) = h(\text{int}(2^{[n]}), x), \]

so \( B_{n}^{+}(x) \) is equal to the difference of \( h(\text{int}(2^{[n]}), x) \) and \( h(\partial(\text{int}(2^{[n]})), x) \). Thus, by Theorem 5.3, \( B_{n}^{+}(x) \) is nonnegative, \( \gamma \)-nonnegative, and hence unimodal. Hence, by Theorem 2.7, the local \( h \)-vector of \( \Gamma \) can be written as

\[ \ell_{V}(\Gamma, x) = \sum_{k=0}^{n} \binom{n}{k} B_{n}^{+}(x) d_{n-k}(x). \]

Since the sum and product of \( \gamma \)-nonnegative polynomials is \( \gamma \)-nonnegative, so the local \( \gamma \)-vector of \( \Gamma \) is nonnegative, which proves Theorem 2.6.

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