KODAIRA-TYPE VANISHINGS VIA NON-ABELIAN HODGE THEORY

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Abstract. In this paper, we use non-abelian Hodge Theory to study Kodaira type vanishings and its generalizations. In particular, we generalize Saito vanishing using mixed twistor $\mathcal{D}$-modules. We also generalize it to a Kawamata-Viehweg type vanishing using $\mathbb{Q}$-divisors, and we also prove a relative version for a projective morphism.

1. Introduction

Let $X$ be a complex projective manifold. Given a reduced normal crossing divisor $D = D_1 + \ldots + D_n$ on $X$, and a vector $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$, we set $dD = d_1D_1 + \ldots + d_nD_n$. We say $d \geq a$, if $d_i \geq a_i$ for all $i$, similarly defined for other inequalities. We use $1 = (1, \ldots, 1) \in \mathbb{R}^n$, similarly for $0$. We also use $aD$ to denote $aD_1 + \ldots + aD_n$, for $a \in \mathbb{R}$. A divisor denoted by a single letter, e.g. $D$, is an effective integral divisor, unless there is a coefficient in front of it, like $dD$, $aD$, as above, or explicitly stated otherwise. We mainly consider $\mathbb{Q}$-divisors, and the $\mathbb{R}$-divisor case can be reduced to such case by an argument appeared later in Remark 1.7.

The renowned Kodaira-Akizuki-Nakano Vanishing Theorem has the following generalized form, which is due to Esnault-Viehweg [EV86], [EV92], combined with the Kawamata-Viehweg type Vanishing formulation [AMPW]:

**Theorem 1.1.** Notations as above, assume that we have an ample divisor $A$. Let $L$ be an integral divisor, such that

$$L \equiv_{\text{num}} aA + dD,$$

with $a > 0$, $0 \leq d \leq 1$. Then, we have the following vanishings:

$$H^i(X, \Omega^j_X \log D)(-D) \otimes \mathcal{O}_X(L)) = 0, \text{ for } i + j > \dim(X);$$

$$H^i(X, \Omega^j_X \log D) \otimes \mathcal{O}_X(-L)) = 0, \text{ for } i + j < \dim(X).$$

There is another direction to generalize Kodaira-Akizuki-Nakano Vanishing using Saito’s mixed Hodge Module. In this paper, we say a pair $(\mathcal{M}, F_\bullet)$ admits a mixed Hodge Module, where $\mathcal{M}$ is a right coherent $\mathcal{D}$-module, and $F_\bullet$ is its increasing Hodge filtration, if we can add a weight filtration to make it an algebraic graded polarizable mixed Hodge module in the sense of [Sai90 §2]. We remark that, among other things, the existence of the weight compatible polarization on $(\mathcal{M}, F_\bullet)$ is essential while defining mixed Hodge Module. However, for the vanishing theory we studying in this paper, we only use its formal property: the strictness of certain functors with respect to $F_\bullet$. Hence we mute those information for the ease of notation.
Theorem 1.2. Let $X$ be a complex projective manifold, with a reduced divisor $D$ and a semi-ample line bundle $\mathcal{L}$. Assume further that $\mathcal{L}(dD)$ is an ample line bundle, for some $d \geq 0$. Let $(\mathcal{M}, F_\bullet)$ be a pair admitting a mixed Hodge Module as described above. Then, we have the following vanishings:

$$H^i(\text{DR}_X \text{Gr}_F^0 \mathcal{M}[\ast D] \otimes \mathcal{L}) = 0, \text{ for } i > 0;$$

$$H^i(\text{DR}_X \text{Gr}_F^0 \mathcal{M}[!]D \otimes \mathcal{L}^{-1}) = 0, \text{ for } i < 0.$$

In the case with the absence of $D$, it is just Saito’s vanishing \cite[(2.g)]{Sai90}. Please refer \cite[Theorem 20]{Wei20b} for this general form, with the proof follows Saito’s proof of Saito’s vanishing. Please note that, in \textit{loc. cit.}, we use $\text{Sp}$ to denote the de Rham functor on right modules. We use the notation $\text{DR}$ here, to follow the convention in \cite{Sab05} and \cite{Moc15}.

In 80s’, the non-abelian Hodge Theory is mainly developed by C. Simpson. As the starting point, he studies the harmonic bundles and uses that to give extra structures on cohomologies of a compact Kähler manifold, with the coefficient being a semisimple local system, instead of the constant local system as studied in the classical Hodge theory. Simpson also proposes the Meta theorem which states that what works for classical Hodge theory shall still work for non-abelian Hodge theory. Later on, following Saito’s idea of constructing mixed Hodge Modules, the theory of mixed twistor $\mathcal{D}$-modules is initiated by C. Sabbah. Then, T. Mochizuki completes this spectacular theory in the past decade, which can also be viewed as a satisfying answer towards Simpson’s Meta Theorem. In the same vein, heuristically, the vanishing theorem above can be generalized to the corresponding mixed twistor $\mathcal{D}$-modules setting. Before we state the generalization, let’s set up the notations for mixed twistor $\mathcal{D}$-modules.

For any complex manifold $X$, we denote $\mathcal{X} = X \times \mathbb{C}_\lambda$, where $\mathbb{C}_\lambda$ is the affine line with $\lambda$ as its coordinate, and $p : \mathcal{X} \to X$, $q : \mathcal{X} \to \mathbb{C}_\lambda$ the natural projections. Following the notations in \cite{Sab05} and \cite{Moc15}, in this paper, we say that a right algebraic $\mathcal{R}_X$-module $\mathcal{M}$ on $\mathcal{X}$ admits a mixed twistor $\mathcal{D}$-module, if there exists an algebraic graded polarizable mixed twistor $\mathcal{D}$-module represented by a filtered $\mathcal{R}$-triple $(T, W)$, with the triple $T = (\mathcal{M}', \mathcal{M}'', C)$, and $\mathcal{M} = \mathcal{M}''$. We also use $\Xi_{\text{Dol}} := \mathcal{M}/\lambda \mathcal{M}$ and $\Xi_{\text{DR}} := \mathcal{M}/(\lambda - 1) \mathcal{M}$, which are exact functors from strict coherent $\mathcal{R}_X$-modules to coherent $\mathcal{D}_X$-modules and coherent $\mathcal{A}_X := \text{Gr}_F(\mathcal{R}_X)$-modules, respectively \cite[1.1.a, Definition 1.2.1]{Sab05}. As we remarked in the mixed Hodge Module case, the other data while defining a graded polarizable mixed twistor $\mathcal{D}$-module is important. However, for the vanishing theorem, we only use the holomorphic picture of a mixed twistor $\mathcal{D}$-module, and its formal property like the strictness of certain natural functors, so we will only keep track the $\mathcal{A}_X$-module $\mathcal{M} = \mathcal{M}''$.

The following vanishing result is a direct non-abelian Hodge theoretic generalization of Theorem 1.2.

Theorem 1.3. Let $X$ be a complex projective manifold, with a reduced divisor $D$ and a semi-ample line bundle $\mathcal{L}$. Assume further that $\mathcal{L}(dD)$ is an ample line bundle, for some $d \geq 0$. Let $\mathcal{M}$ be a right $\mathcal{R}_X$-module admitting a mixed twistor $\mathcal{D}$-module. Then, we have the following vanishings:

$$H^i(\text{DR}_X \Xi_{\text{Dol}} \mathcal{M}[\ast D] \otimes \mathcal{L}) = 0, \text{ for } i > 0;$$

$$H^i(\text{DR}_X \Xi_{\text{Dol}} \mathcal{M}[!]D \otimes \mathcal{L}^{-1}) = 0, \text{ for } i < 0.$$
In the previous theorem, $\mathcal{M}[*D]$ and $\mathcal{M}![D]$ are prolongations of $\mathcal{M}$ as coherent $\mathbb{R}X$-modules as in [Moc15 3.1.2, 3.1.3]. We will use $\mathcal{M}(*D)$ to denote the canonical prolongation as a coherent $\mathbb{R}X(=*D)$-module [Moc15 3.1.1].

Naturally, we want to give the previous vanishing a Kawamata-Viehweg Vanishing type formulation, as in Theorem [11]. The next vanishing theorem can be viewed as a such generalization, and it seems to be new even in the setting of mixed Hodge Modules. The formulation is quite technical, and we will define the notations in §3.

At this stage, for reader's convenience, we introduce the notation

$$V^D_\alpha \mathcal{M}^{(\alpha)}(*D) = \cap V^D_i \mathcal{M}^{(\alpha)}(*D),$$

where $V^D_\alpha \mathcal{M}^{(\alpha)}(*D)$ is the KM-filtration of $\mathcal{M}(*D)$ locally around $\lambda = \alpha$. Please note that the multi-indexed filtration $V^D$ does not behave well in general. We need the $V$-compatibility on $\mathcal{M}(*D)$, Definition 3.3.

**Theorem 1.4.** Let $X$ be a smooth projective variety, with a reduced, normal crossing divisor $D$. Let $\mathcal{M}$ be a right $\mathbb{R}X$-module admitting a mixed twistor $\mathbb{D}$-module, and assume that $\mathcal{M}(*D)$ is $V$-compatible with respect to $D$. (In particular, it is the case when $\mathcal{M}$ corresponds to a tame harmonic bundle on $X \setminus D$). Separate $D$ into two groups of components: $B + C = D$, and assume that we have a semi-ample divisor $A$, such that $A + eC$ is ample for some $e \geq 0$. Let $L$ be a divisor, such that

$$L \equiv_{num} aA + bB + cC,$$

with $a > 0$. Then, we have the following vanishings:

$$H^i(\mathbb{D}R(X, D)(\Xi_{\text{Dol}} V^B_{< -1} V^C_{< -b} V^C_{< c} \mathcal{M}^{(0)}(*D)) \otimes \mathcal{O}_X(L)) = 0, \text{ for } i > 0;$$

$$H^i(\mathbb{D}R(X, D)(\Xi_{\text{Dol}} V^B_{< b} V^C_{< c} \mathcal{M}^{(0)}(*D)) \otimes \mathcal{O}_X(-L)) = 0, \text{ for } i < 0.$$

**Remark 1.5.** Here, we consider the multi-indexed KM-filtration on the canonical prolongation $\mathcal{M}(*D)$ as a coherent $\mathbb{R}X(*D)$-module. Due to the $V$-compatibility in Definition 3.3, for any $\alpha \in \mathbb{C}_{\lambda}$ and $k \in \mathbb{Z}^n$, it satisfies

$$V^D_{d + k} \mathcal{M}^{(\alpha)}(*D) = V^D_d \mathcal{M}^{(\alpha)}(*D) \otimes p^* \mathcal{O}_X(kD).$$

Hence we can always shift the indexes of the multi-indexed $\mathbb{R}X$-filtration to $\leq 0$, and use the $V$-filtration on $\mathcal{M}[*D]$ by [Moc15 Lemma 3.1.1]. In particular, we have that, when $d \leq 0$,

$$V^D_d \mathcal{M}^{(\alpha)}(*D) = V^D_d \mathcal{M}^{(\alpha)}[*D].$$

**Remark 1.6.** Actually, we only need to show the case that $e = 0$, i.e. $A$ itself is ample, since we always have

$$L \equiv_{num} aA + eC + bB + (c - e)eC,$$

noting that $aA + eC$ is ample, and $V^C_{< e} = V^C_{< e + e}$, $V^C_{< c} = V^C_{< c - e}$, for some $0 < \epsilon \ll 1$, due to the semi-continuity of the KM-filtration.

The reason that we keep the superficially more general form as stated above, is due to that it actually carries natural geometric information that will be clarified in the proof. Furthermore, following the same proof as in §5, Theorem [14] can be further generalized to the form that assume $A + e(C + E)$ is ample for some effective divisor $E$, and replace $\mathcal{M}(*D)$ in the first (resp. second) vanishing by $\mathcal{M}[*E](*D)$ (resp. $\mathcal{M}![E](*D)$), as in Theorem [13].
Remark 1.7. The above theorem also works for $\mathbb{R}$-divisors by the following reduction. Due to the previous remark, we can assume $A$ is ample. Hence, we can find $0 < b'' \ll 1$, and $0 < c'' \ll 1$, so that $aA - b''B + c''C$ is still ample, and both $b' := b + b''$ and $c' := c - c''$ are rational. Hence, we have

$$L \equiv \text{num}(a'A - b''B + c''C) + b'B + c'C,$$

with $a'A - b''B + c''C$ be a $\mathbb{Q}$-ample divisor. We also can assume $V_{\leq -b}^B V_{-c}^C = V_{\leq -b}^B V_{-c'}^C$ and $V_b^B V_{c}^C = V_b^B V_{c'}^C$, due to the semi-continuity of the KM-filtration. From now on, we only consider $\mathbb{Q}$-divisors, unless explicitly stated otherwise.

We will also establish the relative versions of the previous two vanishing theorems in §2. Actually, it is not straight-forward to get such generalization as e.g. [KMM87 Theorem 1-2-3], using a log-smooth compactification and Serre vanishing on coherent $\mathcal{O}$-modules. This is because, in our case, we are not working in the derived category of coherent sheaves. However, due to our proof of Theorem 1.3 being functorial, we can also get a functorial proof of its relative version, Theorem 2.1 which does not use a log-smooth compactification. We then show a Nadel-type vanishing with multiplier-ideals from $\mathbb{Q}$-divisors, and state an effective global generalization theorem, as an application.

In §3, we recall the notion of multi-indexed Kashiwara-Malgrange (KM) filtration, its compatibility, and using it to get the logarithmic comparison in the setting of $\mathcal{D}$-modules as in [Wei20a]. In §4, we recall the geometric construction in [EV92], which is constructing a cyclic cover from a normal crossing $\mathbb{Q}$-divisor, and study the induced local systems and their multi-indexed KM filtration. We give the proofs of our main vanishing results in §5.

It is also natural to generalize the injectivity/surjectivity results in [EV92 §5], [Wn17, Wu21], to the Non-abelian Hodge setting. However, the author finds such type of results have a different nature of approach, and they cannot directly imply those vanishing results above using the method summarized in [EV92 §1, 2. proof], with the help of Serre Vanishing, for the same reason in the relative vanishing case. Hence, the author decides to leave it for another occasion.

Let’s explain how the previous two vanishing theorems cover essentially all Kodaira-type vanishings in algebraic geometry we know so far. Nevertheless, the author has no intention to say that we have new proofs to those brilliant results. On the contrary, the vanishing theorems above shall be taken as a natural summary of currently known Kodaira-type vanishings, and the proofs are based on the proofs of those results. The proofs mainly adopt Esnault and Viehweg’s geometric construction [EV92], Saito’s proof of Saito Vanishing, and the Theory of mixed twistor $\mathcal{D}$-modules, due to Sabbah and Mochizuki. However, we still cannot recover Nadel vanishing in the analytic setting, that is proved using $L^2$ method.

We first restrict ourselves to the case that $\mathcal{M}$ being the Rees algebra associated to a mixed Hodge Module $(\mathcal{N}, F_\bullet)$, with a right $\mathcal{D}$-module $\mathcal{N}$, and an increasing Hodge filtration $F_\bullet$. We have that $\Xi_{\text{Dol}} M := M/\lambda M$ is just taking the associated graded pieces, $\text{Gr}_\lambda^F \mathcal{N}$. Then, Theorem 1.3 in this case is just Theorem 1.2.

For Theorem 1.4 if we only consider $\mathcal{M}$ admits a graded polarizable variation of mixed Hodge structures on $X$, $C = 0$, and $0 \leq b < 1$, we have $V_b^B M(\ast D) \simeq \mathcal{M}$. In particular, if we take $\mathcal{M} \simeq \omega_X$, i.e. it corresponds to the trivial variation, we get the Kawamata-Viehweg vanishing for klt pairs. Hence, via a standard argument,
e.g. Corollary 2.3, it implies the big and nef vanishing for the lowest graded piece of a mixed Hodge Module, which has been proved by Suh [Suh18] and Wu [Wu17] independently. Let’s also remark that the C part can be useful in case we want to deal with log-canonical singularities, e.g. the application on showing the zero locus of holomorphic log-one forms, [Wu20].

If we have a reduced, possibly singular Cartier divisor $D$, by taking a log resolution $f : (X', D') \to (X, D)$, with $f^*dD = d'D'$. We have the multiplier ideal

\[ \mathcal{J}(dD) \otimes \omega_X \simeq f_* \Xi_{\text{Dol}} V_{< -d}^D \omega_{X'}^{*D'} \]

(1.1)

Hence we get the Nadel vanishing with the multiplier ideal of $\mathbb{Q}$-divisors. Such relation between multiplier ideals and Hodge modules has also been studied in [BS05] and [MP20]. See also Corollary 2.3 in the next section.

In the case that $\mathcal{M}$ is tame and underlies a variation of twistor structures on $X \setminus D$, this is equivalent to that $\Xi_{\text{Dol}} \mathcal{M}$ gives us a slope parabolic polystable higgs bundle on $X \setminus D$, due to [Moc06, Theorem 1.4]. Since a parabolic semistable higgs bundle can be realized as a sequentially extension of stable higgs bundles, as argued in [AHL19 Lemma 7.1], then the two vanishings above generalize those vanishing results proved by Arapura-Hao-Li [AHL19], and by Deng-Hao [DH]. In [DH22], their vanishing is more refined in the sense that they build a relation between the range of the vanishing degrees and the number of positive eigenvalues of the curvature form of $L$. In the case that $L$ being $k$-ample as in [Som78, §1], (which is a stronger condition, see also [Tot13],) we have a projective dominate morphism $f : X \to Y$, with $L^n = f^*A$ for some ample line bundle $A$ on $Y$, and $n \in \mathbb{Z}^+$. Hence we can achieve the vanishing with the expected range as in [DH22], due to the compatibility of the direct image, DR and $\Xi_{\text{Dol}}$ functor. We leave the details to the interested readers.

In this paper, we use $\otimes$ (without sub-index) to denote the tensor product over the corresponding structure sheaf $O$, unless specified otherwise. We use $f_+$ and $f_!$ (resp. $f_*$ and $f_!$) to denote the derived direct image and direct image with proper support in the derived category of $\mathcal{R}$-modules (resp. quasi-coherent sheaves or constructable sheaves). We use $H^*$ (resp. $H^*_!$) to denote taking the hypercohomoloy (resp. hypercohomoloy with proper support), which is the same as the (derived) functor of taking derived direct image $a_*$ (resp. $a_!$), where $a$ is the canonical map to Spec($\mathbb{C}$). We consider right $\mathcal{D}$-modules or $\mathcal{R}$-modules by default, unless specified otherwise. Although we only consider the vanishing theory for algebraic mixed twistor $\mathcal{D}$-modules on a quasi-projective variety, the constructions in §3 and §4 work for the analytic setting.

## 2. Relative Vanishings and Applications

In this section, we state the relative vanishings, a generalized nef and big vanishing, and an effective global generalization theorem, as an application. Most notations shall be standard, following [Sab05] and [Moc15]. Some notations will be carefully defined in the next section.

Let’s first state a relative version of Theorem 1.3.

**Theorem 2.1.** Let $f : X \to S$ be a projective morphism between smooth quasi-projective varieties. On $X$, assume that we have a reduced divisor $D$ and an $f$-semi-ample line bundle $L$. Assume further that $L(dD)$ is an $f$-ample line bundle,
for some $d \geq 0$. Let $\mathcal{M}$ be a right $\mathcal{R}_X$-module admitting a graded polarizable mixed twistor $\mathcal{D}$-module on $X$. Then, we have the following vanishings:

\[ R^i f_* (\mathcal{D} \mathcal{R}_X \Xi_{\text{Dol}} \mathcal{M}[sD] \otimes L) = 0, \text{ for } i > 0; \]
\[ R^i f_* (\mathcal{D} \mathcal{R}_X \Xi_{\text{Dol}} \mathcal{M}[lD] \otimes L^{-1}) = 0, \text{ for } i < -d, \]

where $d = \dim S$.

We can also show a relative version of Theorem 1.4.

**Theorem 2.2.** Fix a projective morphism between smooth quasi-projective varieties $f : X \to S$. $D$ is a reduced, normal crossing divisor on $X$. Let $\mathcal{M}$ be a right $\mathcal{R}_X$-module admitting a graded polarizable mixed twistor $\mathcal{D}$-module on $X$, and assume that $\mathcal{M}(sD)$ is $V$-compatible with respect to $D$. Separate $D$ into two groups of components: $B + C = D$, and assume that we have an $f$-semi-ample divisor $A$, such that $A + eC$ is $f$-ample for some $e \geq 0$. Let $L$ be a divisor, such that

\[ L \equiv_{\text{lin}} aA + bB + eC, \]

with $a > 0$. Then, we have the following vanishings:

\[ R^i f_* (\mathcal{D} \mathcal{R}_{(X,D)} (\Xi_{\text{Dol}} V^B_{-b} V^C_{c} \mathcal{M}(0)(sD)) \otimes \mathcal{O}_X(L)) = 0, \text{ for } i > 0; \]
\[ R^i f_* (\mathcal{D} \mathcal{R}_{(X,D)} (\Xi_{\text{Dol}} V^B_{-b} V^C_{c} \mathcal{M}(0)(sD)) \otimes \mathcal{O}_X(-L)) = 0, \text{ for } i < -d, \]

where $d = \dim S$.

Recall that, due to Hitchin-Kobayashi correspondence, we have that, for a stable vector bundle $\mathcal{E}$ on a smooth projective variety $X$ with vanishing Chern classes, we can view it as a Higgs bundle with the trivial Higgs map, that can be lift as a harmonic bundle corresponding to a unitary representation. A semistable vector bundle $\mathcal{N}$ with vanishing Chern classes can be realized as extensions of stable vector bundle $\mathcal{E}$ with vanishing Chern classes, [Sim91, Theorem 2]. Hence, $\mathcal{N}$, equipped with the trivial higgs map, can be lift as a smooth variation of mixed twistor structures.

Then, we can show the following Nadel vanishing on a semistable vector bundle with vanishing Chern classes.

**Corollary 2.3.** Let $X$ be a smooth projective variety, $\mathcal{N}$ a semistable vector bundle on $X$ with vanishing Chern classes. Given a divisor $L$, such that

\[ L \equiv_{\text{num}} aA + bB, a > 0, b > 0 \]

with $A$ a nef and big divisor, and $B$ an effective divisor. Then we have

\[ H^i (\mathcal{N} \otimes \omega(L) \otimes J(bB)) = 0, \text{ for } i > 0. \]

where $J(bB)$ stands for the multiplier ideal of the $\mathbb{Q}$-divisor $bB$.

$J(bB)$ can also be replaced by $J(b|B|)$ the multiplier ideal of linear series, or $J(||bB||)$ the asymptotic multiplier ideal. Let’s refer [Laz04b] for more details.

**Proof.** Take an embedded log-resolution of $X$ and the support of $B$, getting $\pi : X' \to X$. Set $\pi^* aA = a'A'$ and $\pi^* bB = b'B'$. Since $A$ is nef and big, so is $a'A'$. Up to a further log-resolution, we have $a'A' \equiv_{\text{lin}} a^2 A^2 + eE$, with $a^2 A^2$ ample and $0 < e \ll 1$. (Since we can apply

\[ a'A' \equiv_{\text{lin}} \left( \frac{a^2}{m} A^2 + \frac{a'(m-1)}{m} A' \right) + \frac{1}{m} eE, \]
with the $\mathbb{Q}$-divisor in the parentheses being ample.) To summarize, we have

$$\pi^*L =_{num} a^0A^0 + b'B' + eE.$$  

Set $N' = \pi^*N$, and it, equipped with the trivial higgs map, still can be lifted as a variation of mixed twistor structures, and we use $\mathcal{M}'$ to denote its corresponding $\mathcal{P}$-module. In particular, it is non-characteristic with respect to any smooth divisor, hence the multi-indexed $KM$-filtration of $\mathcal{M}'(\ast (B' + E))$ only jumps at the integers, along each component of $B'$ and $E$, e.g. (3.11). In the case that $B'$ and $E$ share no common component, we have

$$V_{\leq -1}^{B'} V_{< -\epsilon}^{E} \mathcal{M}'(\ast (B' + E)) \simeq V_{\leq -1}^{B'} V_{< 0}^{E} \mathcal{M}'(\ast (B' + E)) \simeq V_{\leq -1}^{B'} \mathcal{M}'(\ast B'),$$

where the second identity can be checked directly. In the case that $B'$ and $E$ share common components, e.g. $B'_0 = E_0$, we change the first term by omitting $c_0E_0$ in $E$, and use $b'_0 + e_0$ to replace $b'_0$, both as index of $KM$-filtration and coefficient of $B'_0$. Since $c_0$ is small, it is still isomorphic to the other two terms with index $b'$ the original coefficients of $B'$. See also the remark below. Now, due to Theorem 1.4 we have

$$\mathbb{H}^i(DR(X', B' + E)(\mathcal{E}_{\text{Dol}} V_{\leq -1}^{B'} V_{< -\epsilon}^{E} \mathcal{M}'(\ast (B' + E))) \otimes \pi^*O(L)) = 0, i > 0.$$  

Since we also have $\mathcal{E}_{\text{Dol}} V_{\leq -1}^{B'} \mathcal{M}'(\ast B') \simeq \mathcal{N}' \otimes \mathcal{E}_{\text{Dol}} V_{\leq -1}^{B'} \omega_{X'}(\ast B')$, and the Higgs connection has set to be trivial, combining (2.1),

$$\mathcal{D}R(X', B' + E)(\mathcal{E}_{\text{Dol}} V_{\leq -1}^{B'} V_{< -\epsilon}^{E} \mathcal{M}'(\ast (B' + E))))$$

decomposes as \(\bigoplus (\mathcal{N}' \otimes \mathcal{N}' \mathcal{T}(X', B' + E) \otimes \mathcal{E}_{\text{Dol}} V_{\leq -1}^{B'} \omega_{X'}(\ast B'))[i]\)

Due to local vanishing, we have

$$\mathcal{R}f_* \mathcal{E}_{\text{Dol}} V_{\leq -1}^{B'} \omega_{X'}(\ast B') \simeq \mathcal{R}f_* \omega_X(-[b']B') \simeq \mathcal{R}^0 f_* \omega_X(-[b']B') \simeq \omega_X \otimes \mathcal{J}(bB),$$

See also [BS05]. (Of course, we can directly apply Theorem 2.1, but it is not necessary.) Combining the vanishing above and the projection formula, we get the vanishing we need. \qed

**Remark 2.4.** Note that, for the nef and big vanishing, we cannot add the $C$ part in Theorem 1.4 in a naive way. The reason is that we cannot identify $V_{\leq 0}^{B'}$ and $V_{\leq 0}^{E}$

as in (2.1), even when $e$ is very small.

**Remark 2.5.** The essential point of this version of Nadel vanishing is Hitchin-Kobayashi correspondence. Once we assume that, we can view the semistable vector bundle $\mathcal{N}$ as extensions of variation of Hodge structures coming from unitary representations, i.e. with trivial Hodge filtration. Hence itself is the lowest filtered piece, and we can apply the vanishing in the setting of Hodge module, e.g. [Wu17]. Another way to achieve the vanishing is to use the fact that $\mathcal{N}$ is Nakano semi-positive, and we can apply the usual $L^2$ type Nadel vanishing.

It is natural to apply the previous Nadel vanishing to get the following effective global generation result, and we follow the exposition in [Laz04b], 10.4. Please also refer to [C98a], [C98b] for a much more comprehensive study on this topic.

**Theorem 2.6** (Theorem of Angehrn and Siu). Let $X$ be a smooth projective variety of dimension $n$, with $A$ an ample divisor on it. Fix a point $x \in X$, and assume

that, for every irreducible subvariety $Z \subset X$ passing through $x$, (including $X$ itself,) setting $m$ as its dimension, and assume that 

$$(A^m \cdot Z) > (\frac{1}{2}n(n + 1))^m.$$ 

Then, for any semistable vector bundle $N$ with vanishing Chern classes, $N \otimes \omega_X(A)$ is free at $x$, i.e. $N \otimes \omega_X(A)$ has a global section that does not vanish at $x$.

In particular, if $A \equiv_{num} kL$, for some $k \geq \frac{1}{2}n(n + 1)$ and an ample divisor $L$, then $N \otimes \omega_X(A)$ is free, i.e. globally generated.

Proof. According to [Laz04a, 10.4.C], we are able to find an effective $\mathbb{Q}$-divisor $dD$ on $X$, with $lct(D; x) = d$, and $x$ being $dD$'s isolated LC locus, and $dD \equiv_{num} \lambda A$, for some $\lambda < 1$. Due to the vanishing in Corollary 2.3, we have

$$H^1(X, N \otimes \omega_X(A) \otimes J(dD)) = 0.$$ 

Locally around $x$, $J(dD)$ is just $x$'s ideal sheaf, so we have the surjection of the following natural restriction

$$H^0(X, N \otimes \omega_X(A)) \to H^0(X, N \otimes \omega_X(A) \otimes C(x)),$$

where $C(x)$ denoting the one-dimensional sky-scraper sheaf, supported at $x$, and this is what we need. \hfill $\Box$

3. Multi-indexed KM-filtration and Logarithmic comparison

In this section, we would love to generalize some results in [Wei20a] about logarithmic comparison in mixed Hodge modules setting, to the mixed twistor modules case.

Let’s first recall the definition of the Kashiwara-Malgrange (KM-)filtration on a coherent $R_X(\ast D)$-module, with respect to a smooth component $H$ of the reduced normal crossing divisor $D$ on a complex manifold $X$. Since in this paper, given a $\mathcal{B}$-module $\mathcal{M}$ on $X$, we will find ourselves only use its information on the open part $X \setminus D$, so it is more natural and easier to consider the KM-filtration on the coherent $R_X(\ast D)$-module $\mathcal{M}(\ast D) := \mathcal{M} \otimes \mathcal{O}_X(p^*D)$ than to consider the KM-filtration on the coherent $R_X$-modules like $\mathcal{M} [+ D]$ or $\mathcal{M} ([D])$, see also Remark [15]. Actually, $\mathcal{M}(\ast D)$ and $\mathcal{M} ([D])$ themselves are built from the KM-filtration on $\mathcal{M}(\ast D)$, as in [Moc15, 3.1].

There is a canonical $\mathbb{Z}$-indexed increasing filtration on $R_X$, stalk-wise defined by, for any $(x, \alpha) \in X$,

$$V^H_k R_X(\ast D, (x, \alpha)) = \{ P \in R_X(\ast D, (x, \alpha)) | P \cdot (p^* \mathcal{O}_X( iH ) ) (x, \alpha) \subset p^* ( \mathcal{O}_X ( i + k H ) ) (x, \alpha), \forall i \in \mathbb{Z} \}.$$ 

We note that $V^H_0 R_X(\ast D)$ is a coherent ring. It can be directly checked that $V^H_k R_X(\ast D) = V^H_0 R_X(\ast D) \otimes \mathcal{O}_X ( kH )$, and $V^H_0 R_X(\ast D)$ is a good filtration, [Bjo93 Appendix III], as a $V^H_0 R_X(\ast D)$-module.

Now, we use $\mathcal{N}$ to denote a strict coherent $R_X(\ast D)$-module. For $\forall \alpha \in \mathbb{C}$, we use $\mathcal{N}^{(\alpha)}$ to denote restricting $\mathcal{N}$ onto an open subset $X \times \Delta(\alpha, \epsilon) \subset X$, for some small $\epsilon > 0$, where $\Delta(\alpha, \epsilon)$ is the open disk in $\mathbb{C}_\lambda$, centered at $\lambda = \alpha$ and with radius $\epsilon$.

Definition 3.1. Let $\mathcal{N}$ be a strict coherent $R_X(\ast D)$-module. We say that it is strictly specializable along $H$, a smooth component of $D$, [Moc15, 2.1.2.2], if, for $\forall \alpha \in \mathbb{C}$, there exists a Kashiwara-Malgrange (KM-)filtration $V^H_\ast$ on $\mathcal{N}^{(\alpha)}$, (for
some $\epsilon > 0$,) which is an exhaustive $\mathbb{R}$-indexed increasing filtration by coherent $V^H_0 \mathcal{A}_X^{(*D)}$-modules, satisfying the following conditions:

1. for all $a \in \mathbb{R}$, locally around any point $P \in X^{(\alpha)}$, there exists some $\epsilon > 0$
   \[ V^H_a N^{(\alpha)} = V^H_{a+\epsilon} N^{(\alpha)}; \]
2. each $\mathcal{A}_{H(*D)}$-module $\text{Gr}_a^{H} N^{(\alpha)} := V^H_a / V^H_{a+\epsilon} N^{(\alpha)}$ is strict, where $D' = D - H$;
3. locally around any point $P \in X^{(\alpha)}$, we have
   \[ V^H_a N^{(\alpha)} \cdot t = V^H_{a-\epsilon} N^{(\alpha)}, \text{ for all } a \in \mathbb{R}, \]
   where $t$ is any local holomorphic function on $X$ that defines $H \subset X$.
4. For any $a \in \mathbb{R}$ and $P \in X^{(\alpha)}$, there exists a finite set $K(a, \alpha, P) \subset \{ u \in \mathbb{R} \times \mathbb{C} | p(\alpha, u) = a \}$, such that
   \[ \prod_{u \in K(a, \alpha, P)} (\ell \partial_t + \epsilon(\lambda, u)) \]
   is nilpotent on $\text{Gr}_a^{H} N^{(\alpha)}$, where $p(\alpha, \bullet) : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$ and $\epsilon(\alpha, \bullet) : \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ are functions defined in [Moc07a, §2.1]. (They build a relation between the KMS-spectrum at $\lambda = 0$ and $\lambda = \alpha$. See [Moc07a, Corollary 7.71] for the case on a tame harmonic bundle on the puncture disk.)

**Remark 3.2.** The KM-filtration $V^H_\bullet$ on $N^{(\alpha)}$ is actually unique if exists, [Moc11, Lemma 22.3.4].

Then, we consider the multi-indexed KM-filtration. Let $X$ be a complex manifold, with a reduced normal crossing divisor $D = D_1 + \ldots + D_n$, with irreducible components $D_i$. For any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we denote

\[ V_a^D \mathcal{A}_X^{(*D)} = \cap_i V_{a_i}^{D_i} \mathcal{A}_X^{(*D)}. \]

For $a = 0 := (0, \ldots, 0)$, $V_0^D \mathcal{A}_X^{(*D)}$ is a coherent sub-ring of $\mathcal{A}_X$, denoted by $\mathcal{A}_X^{(*D)}$.

Let $\mathcal{N}$ be a coherent right $\mathcal{A}_X^{(*D)}$-module, which is strictly specializable with respect to all $D_i$. In particular, it is the case when $\mathcal{M}$, a coherent $\mathcal{A}_X$-module, admits a graded polarizable mixed twistor $\mathcal{D}$-module, and $\mathcal{N} = \mathcal{M}^{(*D)}$.

Define a multi-indexed Kashiwara-Malgrange filtration with respect to $D$ by

\[ V_a^D N^{(\alpha)} = \cap_i V_{a_i}^{D_i} N^{(\alpha)}, \]

for any $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$. It is not hard to see that $V_a^D N^{(\alpha)}$ is a multi-indexed filtered module over the filtered ring $V_\bullet^D \mathcal{A}_X$, and their filtrations are compatible in the sense that

\[ V_a^D N^{(\alpha)} \cdot V_b^D \mathcal{A}_X^{(*D)} \subset V_{a+b}^D N^{(\alpha)}, \]

for any $a, b \in \mathbb{R}^n$.

If we separate $D$ into two groups of components $D = B + C$, we will also use

\[ V_b^D V_c^C N^{(\alpha)} := V_b^D N^{(\alpha)} \cap V_c^C N^{(\alpha)}. \]

Recall that we say $b < a$, if $b_i < a_i$ for all $i$. We denote

\[ V_{<a}^D N^{(\alpha)} := \cup_{b < a} V_b^D N^{(\alpha)}. \]
In general, the $n$ filtrations $V^D$ do not behave well between each other. This motivates us to make the following definition, which will be essentially used in the logarithmic comparison, Proposition 3.7.

**Definition 3.3.** Notations as above, we say that $V^D$, the multi-indexed Kashiwara-Malgrange filtration with respect to $D$ on $N^{(a)}$ is $V$-compatible, if we have the following strictness relation

$$V^D_nA^{(a)} \xrightarrow{i} V^D_{n-1}A^{(a)}$$

are isomorphisms, for all $a \in \mathbb{R}^n$, where $1^i := [0, \ldots, 0, 1, 0, \ldots, 0]$, with the only 1 at the $i$-th position. If locally for any $a \in \mathbb{C}$, $V^D_nA^{(a)}$ is $V$-compatible, we say that such $\mathcal{R}(*D)$-module $N$ is $V$-compatible with respect to $D$. If we further have $N = M(*D)$ for some coherent $\mathcal{R}_X$-module $M$, then we also say that $M$ is $V$-compatible with respect to $D$.

**Remark 3.4.** Let $H$ be a smooth component of $D$, and denote $D' = D - H$. If we fix the $D'$ part of the multi-indexed KM-filtration, the $H$ part induces a filtration $V^H$ on $V^D_nA^{(a)}$, for a fixed $a' \in \mathbb{R}^n$. Assuming the $V$-compatibility, we can easily check that such an induced filtration satisfies all conditions in Definition 3.1 replacing $\mathcal{R}_X(*D)$ by $V^D_n\mathcal{R}_X(*D)$ and it is unique if exists, following the same argument. We may still call it the KM-filtration on a $V^D_n\mathcal{R}_X(*D)$-module with respect to $H$.

As in the filtered $\mathcal{D}$-module case in [Wei20a Lemma 12], we have the following

**Lemma 3.5.** Let $H$ be a smooth component of $D$ and denote $D' = D - H$. For any right coherent $\mathcal{R}_{(X,D)}$-module $V$, such that it is $t$-torsion free, for any local holomorphic function $t$ on $X$ that locally defines $H$, we have

$$H^i(V \otimes_{\mathcal{R}_{(X,D')}} \mathcal{R}_{(X,D')}) = 0, \text{ for all } i \neq 0.$$

**Proof.** Working locally on $X$, we can assume $X = Y \times \mathbb{C}$, with $H = Y \times 0$, and $D' = p^*_{Y} D^Y$, for some normal crossing $D^Y \subset Y$. Then, locally we have

$$\mathcal{R}_{(X,D)} = \mathcal{R}_{(Y,D^Y)}(t, t\partial_t),$$

$$\mathcal{R}_{(X,D')} = \mathcal{R}_{(Y,D^Y)}(t, \partial_t).$$

We also consider $\mathcal{R}_{(X,D)}(\xi)$, which is, as a left free $\mathcal{R}_{(X,D)}$-module, isomorphic to the polynomial ring $\mathcal{R}_{(X,D)}[\xi]$. It also possesses a right $\mathcal{R}_{(X,D)}$-module structure with the non-commutative relations

$$[\xi, t] = \lambda, \ [\xi, t\partial_t] = \lambda \xi.$$

We have the following left $\mathcal{R}_{(X,D)}$-linear complex

$$\mathcal{R}_{(X,D)}(\xi) \xrightarrow{-(t\xi-\partial_t)} \mathcal{R}_{(X,D)}(\xi) \rightarrow \mathcal{R}_{(X,D')}.$$

Let's show that it actually is a short exact sequence. We give both $\xi, t\partial_t$ and $\partial_t$ of grading 1, which induces an increasing filtration $F_•$ on $\mathcal{R}_{(X,D)}(\xi)$ and $\mathcal{R}_{(X,D')}$. Both maps in the complex are strict with respect to $F_•$, so to show the complex is exact, we only need to consider the induced complex of their associated graded pieces:

$$\mathcal{R}_{(Y,D^Y)}[t, t\partial_t, \xi] \xrightarrow{-(t\xi-\partial_t)} \mathcal{R}_{(Y,D^Y)}[t, t\partial_t, \xi] \rightarrow \mathcal{R}_{(Y,D^Y)}[t, \partial_t], \quad \xi \mapsto \partial_t.$$
which is exact by induction on the grading. Hence, (3.3) gives a free resolution of \( \mathcal{R}(X,D') \) as a left \( \mathcal{R}(X,D) \) module.

Now, we only need to argue that \( \mathcal{V}(\xi) \xrightarrow{\xi t \partial_3} \mathcal{V}(\xi) \) is injective, which can be checked by considering the degree of \( \xi \) combined with the \( t \)-torsion freeness of \( \mathcal{V} \).

\[ \square \]

**Proposition 3.6.** Let \( \mathcal{M} \) be a strict coherent \( \mathcal{R}_X \)-module. Assume \( \mathcal{M}(\ast D) \) is \( V \)-compatible with respect to \( D \), and \( H \) is a smooth component of \( D \). Denote \( D' = D - H \). For any \( \alpha \in \mathbb{C}_\lambda \) and \( a' \in \mathbb{R}^{n-1} \), we have

\[
\begin{align*}
V_{a'}^D V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes L_{\mathcal{R}(X,D)} \mathcal{R}(X,D') &\simeq V_{a'}^D \mathcal{M}^{(\alpha)}(\ast H)(\ast D'); \\
V_{a}^{D'} V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes L_{\mathcal{R}(X,D)} \mathcal{R}(X,D') &\simeq V_{a'}^D \mathcal{M}^{(\alpha)}(\ast H)(\ast D').
\end{align*}
\]

**Proof.** Let’s first show that,

\[
\begin{align*}
V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes L_{\mathcal{R}(X,D)} \mathcal{R}(X,D') &\simeq \mathcal{M}^{(\alpha)}(\ast H)(\ast D'); \\
V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes L_{\mathcal{R}(X,D)} \mathcal{R}(X,D') &\simeq \mathcal{M}^{(\alpha)}(\ast H)(\ast D').
\end{align*}
\]

These two identities are due to [Moe15, Lemma 3.1.2, Lemma 3.1.10], combining Lemma 3.5 above. From now on, we focus on the first identity in the statement of the proposition, since the second one follows similarly. Lemma 3.5 will be used repeatedly to show various tensor functors are exact, without being mentioned explicitly.

We want to show the following naturally induced map

\[ (3.6) \quad V_{a'}^D V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes \mathcal{R}(X,D') \to \mathcal{M}^{(\alpha)}(\ast H)(\ast D') \]

is injective, for any \( a' \in \mathbb{R}^{n-1} \). Note that, using a similar resolution as (3.5), we have

\[
\begin{align*}
\mathcal{M}^{(\alpha)}(\ast H)(\ast D') &\simeq V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes L_{\mathcal{R}(X,D)} \mathcal{R}(X,D') \\
&\simeq V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes \mathcal{R}(X,D) \mathcal{R}(X,D').
\end{align*}
\]

Now the injectivity of (3.6) can be deduced from the fact that the cokernel of the natural inclusion

\[ V_{a'}^D V_0^H \mathcal{M}^{(\alpha)}(\ast D) \to V_0^H \mathcal{M}^{(\alpha)}(\ast D) \]

is \( t \)-torsion free, due to the \( V \)-compatibility of \( \mathcal{M} \) in the assumption. In particular, we have the following injection

\[ V_0^E V_0^H \mathcal{M}^{(\alpha)}(\ast D) \otimes \mathcal{R}(X,D) \mathcal{R}(X,D') \to \mathcal{M}^{(\alpha)}(\ast H)(\ast D'), \]

for any component \( E \neq H \) of \( D \). It is straightforward to check that the filtration on \( \mathcal{M}^{(\alpha)}(\ast H)(\ast D') \) induced by the image above satisfies all of those conditions in Definition 3.1 which means it gives the KM-filtration with respect to \( E \). This implies that the multi-indexed filtration induced by the images of (3.6) is indeed the multi-indexed KM-filtration of \( \mathcal{M}^{(\alpha)}(\ast H) \), with respect to \( D' \). It can also be argued directly by using Remark 3.3.

\[ \square \]

Apply the previous proposition inductively, we get the following
Proposition 3.7 (Logarithmic Comparison). With the same assumptions in the previous proposition, separate $D$ into two groups of components $D = B + C$. For any $\alpha \in \mathbb{C}_\lambda$, we have
\begin{equation}
V_B^0 V_0^C \mathcal{M}^{(\alpha)}(\ast D) \otimes L_{\mathcal{R}(X,D)} \simeq \mathcal{M}^{(\alpha)}[B + \ast C].
\end{equation}
In particular, we have
\begin{equation}
\text{DR}_{(X,D)} V_B^0 V_0^C \mathcal{M}^{(\alpha)}(\ast D) \simeq \text{DR}_X \mathcal{M}^{(\alpha)}[B + \ast C].
\end{equation}
Recall the de Rham functor on a right coherent $\mathcal{R}_X$-module $\mathcal{M}$
\[
\text{DR}_X \mathcal{M} = \mathcal{M} \otimes_{\mathcal{R}_X} \mathcal{O}_X,
\]
which can be explicitly expressed using the Spencer complex $\text{Sp}^*(\mathcal{O}_X)$, [Sab05 §0.6], as a resolution of $\mathcal{O}_X$, by locally free left $\mathcal{R}_X$-modules. In the case of right $\mathcal{R}_{(X,D)}$-modules, we have the log-de Rham functor on a right coherent $\mathcal{R}_{(X,D)}$-module $\mathcal{V}$:
\[
\text{DR}_{(X,D)} \mathcal{V} = \mathcal{V} \otimes_{\mathcal{R}_{(X,D)}} \mathcal{O}_X.
\]

We can use the logarithmic Spencer complex in [Wei20a §2], to get the explicit expression. Please note that the $\text{Sp}$ functor in loc. cit. is the same as the de Rham functor here.

Let’s also show the following compatibility of the multi-indexed KM-filtration with respect to certain pushforward functor, which has essentially been proved in [Sab05 Theorem 3.1.8]. It will be used in the proof of Theorem 1.4.

Proposition 3.8. Assume that $f : X \to Y$ is a proper map between complex manifolds, and $F = f \times \text{Id} : X \times \mathbb{C}^n \to Y \times \mathbb{C}^n$. Let $D^X$ (resp. $D^Y$) be a normal crossing divisor on $X \times \mathbb{C}^n$ (resp. $Y \times \mathbb{C}^n$), defined by those coordinates of $\mathbb{C}^n$, and $\mathcal{M}$ a strict coherent $\mathcal{R}$-module on $X \times \mathbb{C}^n$, and $\mathcal{M}(\ast D^X)$ is $V$-compatible with respect to $D^X$. We have the following compatibility of the direct image functor with multi-indexed KM-filtration:
\begin{equation}
\mathcal{H}^i F_! (\mathcal{V}_{d^X}^{D^X} \mathcal{M}^{(\alpha)}(\ast D^X)) \simeq \mathcal{V}_{d^Y}^{D^Y} \mathcal{H}^i F_! \mathcal{M}^{(\alpha)}(\ast D^X), \text{ for any } d \in \mathbb{R}^n
\end{equation}
where the $F_!$ on the left shall be read as the direct image functor on a $\mathcal{R}_{(X \times \mathbb{C}^n,D^X)}$-module, [Wei20a §2], see also [Sab05 Remark 1.4.3(2)].

Proof. Although Sabbah’s theorem only states the case that $D$ has only one component, we can apply it inductively on the number of components of $D$. To be more precise, the inductive assumption implies that
\[
\mathcal{H}^i F_! (\mathcal{V}_{d'}^{D'^X} \mathcal{M}^{(\alpha)}(\ast D^X)) \simeq \mathcal{V}_{d'}^{D'^Y} \mathcal{H}^i F_! \mathcal{M}^{(\alpha)}(\ast D^X), \text{ for any } d' \in \mathbb{R}^{n-1}
\]
where $D'^X = D^X - D^X_n$, $D'^Y = D^Y - D^Y_n$. Now, as in Remark 3.4, $\mathcal{V}_{d^n}$ gives the KM-filtration on $\mathcal{V}_{d'}^{D'^X} \mathcal{M}^{(\alpha)}$, hence we can apply Sabbah’s argument to get that $\mathcal{H}^i F_! (\mathcal{V}_{d^n} \mathcal{V}_{d'}^{D'^X} \mathcal{M})$ gives the KM-filtration on $\mathcal{V}_{d'}^{D'^Y} \mathcal{H}^i F_! \mathcal{M}^{(\alpha)}$ with respect to $D^Y_n$. \qed

Please also refer to [Wei20a Theorem 3], and the Remark after its proof. In that case, the map is more general, and we cannot reduce to the form of the map $F$ as above. The other results in [Wei20a] about the direct image and dual functors on log-representations shall still work in the setting of $\mathcal{R}$-modules admitting graded polarizable mixed twistor $\mathcal{O}$-modules, by essentially the same arguments. Since we are not using them in this paper, we do not copy them here.
Before we end this section, we consider an easy special case. Let $X$ be a complex manifold with a reduced normal crossing divisor $D$. According to [Sab05 §3.7], assume a holonomic $\mathcal{R}_X$-module $\mathcal{M}$ is strictly non-characteristic with respect to all components of $D$. Then, $\mathcal{M}(sD)$ is strictly specializable and $V$-compatible, and the multi-indexed KM-filtration $\mathcal{V}_D^\bullet$ on $\mathcal{M}(sD)$ is globally defined, i.e. does not depend on $\alpha$, and $\mathcal{V}_D^\bullet$ only jumps at $\mathbb{Z}^n$, with

$$\mathcal{V}_D^\bullet\mathcal{M}(sD) = \mathcal{M} \otimes \mathcal{O}_X((\alpha + 1)D), \text{ for any } \alpha \in \mathbb{Z}^n.$$ 

Let $H$ be one component of $D$ and $D' := D - H$. Following [Moc15 Lemma 3.1.1], let's first compute,

$$V_H^H \mathcal{M}(sD'[sH]) := V_H^H \mathcal{M}(sD) \cap \mathcal{M}(sD'[sH]).$$

We claim that

$$(3.10) \quad V_k^H \mathcal{M}(sD'[sH]) = \begin{cases} \mathcal{M}(sD') \otimes \mathcal{O}_X((k + 1)H), & \text{if } k \leq 0; \\
\mathcal{M}(sD') \otimes \mathcal{O}_X(H) + \ldots + \mathcal{M}(sD') \otimes \mathcal{O}_X((k + 1)H) \cdot \lambda^k, & \text{if } k \geq 1. \end{cases}$$

The $k \leq 0$ part is due to [Sab05 Lemma 3.7.4]. The $k \geq 1$ part is due to that

$$m t^{-k} \cdot \partial_t = (m \partial_t) t^{-k} + m(-k)t^{-k-1} \lambda, \forall m \in \mathcal{M}(sD'),$$

which implies $V_k^H \mathcal{M}(sD'[sH]) + V_k^H \mathcal{M}(sD'[sH]) \cdot \partial_t = V_{k+1}^H \mathcal{M}(sD'[sH])$, which is what we need. Set

$$\mathcal{V}_D^D \mathcal{M}[*D] := \mathcal{V}_D^D \mathcal{M}(sD) \cap \mathcal{M}[*D].$$

By induction, we can get that it only jumps at $\mathbb{Z}^n$, and for any $\alpha \in \mathbb{Z}^n$,

$$(3.11) \quad \mathcal{V}_D^D \mathcal{M}[*D] = \sum_{k \in \mathbb{Z}^n, k \leq \alpha} \mathcal{M} \otimes \mathcal{O}_X((k + 1)D) \cdot \lambda^{s_k},$$

where

$$s_k := \sum_{\{i | k_i \geq 1\}} k_i.$$

**Remark 3.9.** When $\mathcal{M}$ is algebraic and holonomic, it is non-characteristic with respect to a general hypersurface, due to Bertini’s theorem, e.g. [Moc15 §14.3.1.3]. If we further assume that it is strictly specializable, then it is strictly non-characteristic.

4. Esnault-Viehweg’s covering construction

On a smooth variety $X$, assume we have a line bundle $\mathcal{L}$, with a section $s \in H^0(X, \mathcal{L}^\otimes N)$ defining an integral divisor $N(dD)$, with $D = D_1 + ... + D_n$ reduced and normal crossing, $d \in \mathbb{Q}^n$, $0 < d_i \leq 1$, and

$$\gcd(Nd_1, ..., Nd_n, N) = 1.$$ 

Applying the construction in [EV92 §3], the section $s$ gives a cyclic covering $g : \hat{X} \to X$, that only ramifies along $D$, with $\hat{X}$ being smooth and irreducible, and the ramification number along $D_i$ being $\frac{N}{\gcd(Nd_i, N)}$. If we denote $d_i = \frac{d_i'}{d_i}$ as reduced fractional, then the ramification number is just $d_i'$. Let's use $h : \hat{X}^0 \to X^0$ to denote $g$ restricted over $X^0 = X \setminus D$, i.e. the étale part of $g$. Fix an $\mathcal{R}_X$-module $\mathcal{M}$ that admits a graded polarizable mixed twistor $\mathcal{G}$-module, assuming that $\mathcal{M}(sD)$ is $V$-compatible with respect to $D$, and denote $\mathcal{M}[*D]$, its prolongation along $D$. 

Let $\mathcal{M}^0 = \mathcal{M}|_{X^0}$, and denote $\hat{\mathcal{M}}^0 = h^*\mathcal{M}^0$. Denote $\hat{\mathcal{M}}[\hat{\mathcal{D}}]$, the prolongation of $\mathcal{M}^0$ along $\hat{\mathcal{D}}$, which also underlies a mixed twistor $\mathcal{D}$-module.

Due to the projection formula for $\mathcal{D}$-modules, [HTT08 Corollary 1.7.5] for the $\mathcal{D}$-module case, and [Moc11 Lemma 22.7.1], on $X^0$ we have

\begin{equation}
(4.1) \quad h_*\hat{\mathcal{M}}_0 = h_*h^+\mathcal{M}^0 \simeq \mathcal{M}^0 \otimes h_*\mathcal{O}_{\hat{\mathcal{X}}^0} \simeq \bigoplus_{0 \leq i < N} (\mathcal{M}^0 \otimes p^*\mathcal{L}^{-(i,d\hat{D})}|_{X^0}),
\end{equation}

where

\[ \mathcal{L}^{-(i,d\hat{D})} = (\mathcal{L}^{(i,d\hat{D})})^{-1} = (\mathcal{L}^i(-[i d\hat{D}]))^{-1}, \]

as in [EV92 3.1 Notation]. We will just use (i) to replace $(i,d\hat{D})$, if $d\hat{D}$ is clear from the context. We will also use the notation

\[ (x) = x - \lfloor x \rfloor, \]

for the fractional part of $x$.

Note that

\[ h_*\mathcal{O}_{\hat{\mathcal{X}}^0} \simeq \bigoplus_{0 \leq i < a^0} p^*\mathcal{L}^{-(i)}|_{X^0} \]

shall also be viewed as a decomposition of left $\mathcal{D}$-modules, that underlies mixed twistor $\mathcal{D}$-modules, corresponds to the cyclic decomposition of $h_*\mathcal{C}_{\hat{\mathcal{X}}^0}$. We also note that the tensor products in (4.1) are between a right $\mathcal{D}$-module and a left $\mathcal{D}$-module, and such gives us a right $\mathcal{D}$-module, as the $\mathcal{D}$-module case in [HTT08 Proposition 1.2.9. (ii)].

Fix $\delta$ such that $0 \leq \delta \leq N$. Using the notations in [Moc07a 6.1.5], locally around a general point of $D_i$, $p^*\mathcal{L}^{-(\delta)}|_{X^0}$ underlies the tame harmonic bundle $q_1^\delta L(-[\delta d_i], 0)$. See also [EV92 3.16 Lemma c)]. In particular, after we change it into its corresponding right $\mathcal{D}$-module, [Moc07a 14.1.2], we have

\[ V_{\alpha_i}^{D_i}(\mathcal{O}_\mathcal{Y} \otimes p^*\mathcal{L}^{-(\delta)}(D_i)) = \mathcal{O}_\mathcal{Y} \otimes p^*\mathcal{L}^{-(\delta)}(nD_i), \]

if $-\langle \delta d_i \rangle + n - 1 \leq a_i < -\langle \delta d_i \rangle + n$. Recall that $\mathcal{O}_\mathcal{Y} := \lambda^{a_i} \cdot p^*\omega_X$, e.g. [Moc07a Example 14.4].

Denote $i : X^0 \rightarrow X$, the natural embedding, and denote

\[ \mathcal{M}_\delta = i_+ (\mathcal{M}^0 \otimes p^*\mathcal{L}^{-(\delta)}|_{X^0}), \]

which is an $\mathcal{D}$-module that underlies a mixed twistor $\mathcal{D}$-module on $X$, satisfying $\mathcal{M}_\delta = \mathcal{M}_\delta[* \hat{\mathcal{D}}]$. Take prolongation of (4.1) along $D$, we have

\[ g_*\hat{\mathcal{M}}[* \hat{\mathcal{D}}] \simeq \bigoplus_{0 \leq i < N} \mathcal{M}_i. \]

By using the functoriality for the tensor product in [Moc07a 7.2.6], we have

\[ V_{\alpha_i}^{D_i} \mathcal{M}_\delta^{(\alpha)}(* \hat{\mathcal{D}}) \simeq V_{\langle \delta d_i \rangle \alpha_i}^{D_i} \mathcal{M}_\delta^{(\alpha)}(* \hat{\mathcal{D}}) \otimes V_{-\langle \delta d_i \rangle - 1}^{D_i} (\mathcal{O}_\mathcal{Y} \otimes p^*\mathcal{L}^{-(\delta)}(* \hat{\mathcal{D}})) \otimes \omega_{\mathcal{Y}}^{-1} \]
\[ \simeq V_{\langle \delta d_i \rangle \alpha_i}^{D_i} \mathcal{M}_\delta^{(\alpha)}(* \hat{\mathcal{D}}) \otimes p^*\mathcal{L}^{-(\delta)}(* (D - D_i)). \]

The shifting of degree 1 in the KM-filtration is due to the convention of the shifting from the parabolic structure to the KM-filtration as in [Moc07a 15.1.2]. Twisting $\omega_{\mathcal{Y}}^{-1}$ at the end is to make the tensor product works between right and left
components of $D$.

In particular, we have the following

$$V^D_\delta M^{(\alpha)}_\delta(*D) = V^D_\delta \cdot M^{(\alpha)}_\delta(*D) \otimes V^D_{-(\delta)} \cdot M^{(\alpha)}_\delta(*D) \otimes p^* L^{-(\delta)}(*D) \otimes \omega^{-1} \leq V^D_{-(\delta)} \cdot M^{(\alpha)}_\delta(*D) \otimes p^* L^{-(\delta)}.$$ 

In particular, we have the following

**Lemma 4.1.** In the above setting, $M_\delta = M_\delta[*D]$, and $M_\delta(*D)$ is $V$-compatible with respect to $D$. For any $\alpha \in \mathbb{C}_\lambda$,

$$V^D_\delta M^{(\alpha)}_\delta(*D) \otimes p^* L^{-(\delta)} \simeq V^D_\delta \cdot M^{(\alpha)}_\delta(*D) \otimes p^* L^{-(\delta)} \simeq V^D_0 \cdot M^{(\alpha)}_\delta(*D),$$

and

$$V^D_\delta \cdot M^{(\alpha)}_\delta(*D) \otimes p^* L^{-(\delta)} \simeq V^D_{\delta d} \cdot M^{(\alpha)}_\delta(*D) \otimes p^* L^{-(\delta)} \simeq V^D_{\delta 0} \cdot M^{(\alpha)}_\delta(*D),$$

as $\mathcal{R}(X,D)$-modules.

We can also generalize an intermediate result in Saito’s proof of Saito vanishing [Sai90, (2.33.3)], which will be used in the proof of Theorem 1.3.

**Lemma 4.2.** In the above setting, if we further assume that the $A_X$-module $M$ is non-characteristic with respect to all components of $D$, then we have that, for any $1 \leq \delta \leq N$, $M_\delta = M_\delta[*D] = M_\delta[|D]|$. Furthermore, we have

$$\Xi_{\text{Dol}} M_\delta \simeq \Xi_{\text{Dol}} M[*D] \otimes L^{-(\delta)} \simeq \Xi_{\text{Dol}} M[|D|] \otimes L^{-(\delta)}(D),$$

as $A_X$-modules, where $L^{-(\delta)}$ and $L^{-(\delta)}(D)$ carry trivial higgs structure, i.e. differential operators act trivially on them.

**Proof.** We have

$$V^D_0 \cdot M(*D) \otimes p^* L^{-(\delta)} = V^D_{0 + \delta} \cdot M(*D) \otimes p^* L^{-(\delta)} \simeq V^D_0 \cdot M_\delta(*D),$$

where the first identity is due to $M$ being non-characteristic with respect to $D$, and the second one is due to the previous lemma. We also have

$$V^D_0 \cdot M(*D) \otimes p^* L^{-(\delta)} = V^D_{0 + \delta} \cdot M(*D) \otimes p^* L^{-(\delta)} \simeq V^D_{0 + \delta} \cdot M_\delta(*D),$$

which implies $V^D_0 \cdot M_\delta(*D) \simeq V^D_{0 + \delta} \cdot M_\delta(*D)$. Hence, the first statement holds due to the logarithmic comparison Proposition 3.1, since we only need to compare the $V^D_0$ and $V^D_{0 + \delta}$ part.

For the second statement, according to the previous computation, we note that

$$\Xi_{\text{Dol}} V^D_0 \cdot M_\delta = \Xi_{\text{Dol}} V^D_0 \cdot M(*D) \otimes L^{-(\delta)} = \Xi_{\text{Dol}} V^D_{0 + \delta} \cdot M(*D) \otimes L^{-(\delta)}(D).$$

Due to the logarithmic comparison again and use the explicit resolution (3.3), $\Xi_{\text{Dol}} M_\delta$, $\Xi_{\text{Dol}} M[*D] \otimes L^{-(\delta)}$ and $\Xi_{\text{Dol}} M[|D|] \otimes L^{-(\delta)}(D)$, can be computed from the terms in the previous identities respectively, using a same functor. \hfill \Box

**Remark 4.3.** Actually, we have

$$M_\delta \simeq M[*D] \otimes p^* L^{-(\delta)}.$$

Note that $M[*D] \otimes p^* L^{-(\delta)}$ is a priori just a sub-$\mathcal{R}(X,D)$-module of $M_\delta(*D)$, but we will see from the computation below that it is indeed closed under the action of $\mathcal{R}(X)$. However, we cannot identify them to $M[|D|] \otimes p^* L^{-(\delta)}(D)$ naively, since it does not carry a natural $\mathcal{R}(X)$-module structure.
Let’s compute $V^D_\mathcal{M}_\delta := V^D_\mathcal{M}_\delta(\ast D) \cap \mathcal{M}_\delta$. We first note that, due to (1.2), $V^D_\mathcal{M}_\delta$ only jumps at $\mathbb{Z}^n + \langle \delta d \rangle$. Recall (3.11) is just the special case when $\delta = 0$. We claim that

$$(4.4)\ V^D_{a + (\delta d)} \mathcal{M}[\ast D] = \sum_{k \leq a} \mathcal{M} \otimes p^* \mathcal{L}^{-(\delta)} \otimes \mathcal{O}_X((k + 1)D) \cdot \lambda^a.$$ 

Similarly, we just need to compute it component by component, which is a more general version of (3.10). Let $H = D_1$ a component of $D$, $V^H_\mathcal{M}_\delta := V^H_\mathcal{M}_\delta(\ast D) \cap \mathcal{M}_\delta$. We already know that it only jumps at $\mathbb{Z} + \langle \delta d_1 \rangle$. We want to show

$$(4.5)\ V^H_{k + (\delta d_1)} \mathcal{M}_\delta = \begin{cases} \mathcal{M}(\ast D') \otimes p^* \mathcal{L}^{-(\delta)} \otimes \mathcal{O}_X((k + 1)H), \ 	ext{if } k \leq 0; \\ \mathcal{M}(\ast D') \otimes p^* \mathcal{L}^{-(\delta)} \otimes \mathcal{O}_X(H) + \vdots \\ + \mathcal{M}(\ast D') \otimes p^* \mathcal{L}^{-(\delta)} \otimes \mathcal{O}_X((k + 1)H) \cdot \lambda^k, \ 	ext{if } k \geq 1. \end{cases}$$

The $i \leq 0$ part is due to [Sab02, Lemma 3.7.4] and (1.2). The $i \geq 1$ part is due to that, for any local sections $m \in \mathcal{M}(\ast D')$, $l \in p^* \mathcal{L}^{-(\delta)}$,

$$(m \otimes l)t^{-k} \cdot \partial_l = (m \partial_l \otimes l)t^{-k} - (m \otimes (\delta_1 \partial l))t^{-k-1} + m \otimes l(-k)t^{-k-1} \lambda,$$

$$= (m \partial_l \otimes l)t^{-k} - (m \otimes (\langle \delta d_1 \rangle \lambda)t^{-k-1} + m \otimes l(-k)t^{-k-1} \lambda,$$

$$= (m \partial_l \otimes l)t^{-k - m \otimes l(\langle \delta d_1 \rangle + k)t^{-k-1} \lambda.$$ 

This implies $V^H_{k + (\delta d_1)} \mathcal{M}_\delta + V^H_{k + (\delta d_1)} \mathcal{M}_\delta \cdot \partial_l = V^H_{k+1 + (\delta d_1)} \mathcal{M}_\delta$, which is what we need.

Comparing (4.4) with (3.11), we get $V^D_{k + (\delta d)} \mathcal{M}_\delta \simeq V^D_k \mathcal{M}[\ast D] \otimes p^* \mathcal{L}^{-(\delta)}$.

5. PROOFS OF MAIN THEOREMS

Although the proof of Theorem 1.3 is mainly motivated from Saito’s proof of Saito-Kodaira vanishing, we give it a more geometric and functorial treatment and hopefully provide a clearer picture. Later, we will find that it is also helpful to prove the relative version.

Fix $X$, a smooth algebraic variety of dimension $n$, with a normal crossing reduced divisor $D$, and a line bundle $\mathcal{A}$. We also fix an algebraic coherent $\mathcal{R}_X$-module $\mathcal{M}$, admitting a mixed twistor $\mathcal{D}$-module, that is $V'$-compatible with respect to $D$. We say a sequence of smooth varieties $X = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_n$, with $Y_j$ of codimension $j$ in $X$ or $Y_j = \emptyset$, is a filtration induced by $\mathcal{A}$, if $\mathcal{O}_{Y_j} \simeq \mathcal{A}|_{Y_j}$. Note that, once $\mathcal{A}|_{Y_j}$ is a trivial line bundle, then $Y_{j+1} = \emptyset$, so are the successive terms.

Such a filtration is called normal crossing, (with respect to $D$), if $D|_{Y_j}$ is still a normal crossing reduced divisor on $Y_j$, for all $j$ such a filtration is called non-characteristic, (with respect to $\mathcal{M}_j$) if $Y_j$ is non-characteristic with respect to $\mathcal{M}|_{Y_{j-1}}$, for all $j$. Note that strict specializability is part of the assumption of mixed twistor $\mathcal{D}$-module, hence being non-characteristic automatically implies being strictly non-characteristic. If $\mathcal{A}$ is base point free, then due to Remark 3.3 we know that, for a generic filtration induced by $\mathcal{A}$, it is both normal crossing and non-characteristic (NCNC). If we fix a NCNC filtration, it induces an exact sequence of $\mathcal{R}_X$-modules

$$0 \rightarrow \mathcal{M} \rightarrow \mathcal{M}[\ast Y_1] \rightarrow i_+(\mathcal{M}|_{Y_1}[\ast Y_2]) \rightarrow \ldots \rightarrow i_+(\mathcal{M}|_{Y_{n-1}}[\ast Y_n]) \rightarrow i_+\mathcal{M}|_{Y_n} \rightarrow 0.$$ 

Once $Y_{j+1} = \emptyset$, all the successive terms are just 0 by default. This can be checked directly using [Moc15, Lemma 3.1.23], and the explicit computation (3.11).
Dually, using [Moc15, Lemma 3.1.24], we can similarly consider the exact sequence of $\mathcal{R}_X$-modules

$$0 \to i_+ M|_{Y_n} \to i_+(M|_{Y_{n-1}}[Y_n]) \to ... \to i_+(M|_{Y_1}[Y_2]) \to M|_{Y_1} \to M \to 0.$$  

**Proof for Theorem 1.3.** We only prove the second vanishing here. Saito’s proof also focus on this case, and readers can compare these two approaches. The first vanishing can be derived using the dual construction with little extra effort. Please note that we cannot directly apply Grothendieck-Serre duality here, since we are not working in the derived category of coherent sheaves.

We first set $\mathcal{A} = \mathcal{L}^m$, for some integer $m$ such that it is base point free. According to the previous construction, we get an exact sequence of $\mathcal{R}_X$-modules

$$0 \to M[D] \to M[D + \ast Y_j] \to i_+(M[D]|_{Y_1}[Y_2]) \to ... \to i_+(M[D]|_{Y_{n-1}}[Y_n]) \to i_+ M[D]|_{Y_n} \to 0. \quad (5.1)$$

Apply $\Xi_{\text{Dol}}$ on each of them, we get an exact sequence of Higgs sheaves. In particular, the Higgs complex is $\mathcal{O}_X$-linear. Hence, we can twist the complex by $L^{-1}$ and keep the exactness. If we view $L^{-1}$ as a Higgs bundle with trivial Higgs connection, then the next complex we get is actually an exact sequence of Higgs sheaves:

$$0 \to \Xi_{\text{Dol}} M[D] \otimes L^{-1} \to \mathcal{E}^0 \to ... \to \mathcal{E}^n \to 0, \quad (5.2)$$

with

$$\mathcal{E}^j = \Xi_{\text{Dol}}(i_+ M[D]|_{Y_j}[Y_{j+1}]) \otimes L^{-1} = \Xi_{\text{Dol}}(i_+(M[D]|_{Y_j}[Y_{j+1}] \otimes p^* L^{-1}|_{Y_j}).$$

Due to Lemma 7.2, we get that $\mathcal{E}^j$ is isomorphic to $\Xi_{\text{Dol}} i_+ N_{Y_j}[D|_{Y_j}]$, where $N_{Y_j}$ a $\mathcal{R}_{Y_j}$-module admitting a mixed twistor $\mathcal{P}$-module, satisfying

$$N_{Y_j}[D|_{Y_j}] = N_{Y_j}[D|_{Y_j}][Y_{j+1}] = N_{Y_j}[D|_{Y_j}][Y_{j+1}].$$

Due to the assumption that $\mathcal{L}(dD)$ is ample, hence so is $\mathcal{A}|_{Y_j}(dD|_{Y_j}) = \mathcal{O}_{Y_j}(dD|_{Y_j} + Y_{j+1})$, which implies that $Z_j := Y_j \setminus (dD|_{Y_j} + Y_{j+1})_{\text{red}}$ is affine.

By Artin’s vanishing, we have

$$\mathbb{H}^k(i_* DR_{Y_j} \Xi_{\text{Dol}} N_{Y_j}[D|_{Y_j}]) = \mathbb{H}^k(DR_Z \Xi_{\text{Dol}} N_{Y_j}[D|_{Y_j}]_{Z_j}) = 0, \text{ for } k < 0.$$  

Further, recall that $q$ is the projection $\mathcal{X} \to \mathcal{C}_\lambda$. We have that $q_* N_{Y_j}[D|_{Y_j}]$ is a free module on $\mathcal{C}_\lambda$, since it shall admit a mixed twistor structure, [Moc15 Proposition 7.2.7]. In particular, $\mathbb{H}^k(DR_{\mathcal{X}} \Xi_{\text{Dol}} N_{Y_j}[D|_{Y_j}])$ has the same dimension as of $\mathbb{H}^k(DR_{\mathcal{X}} \Xi_{\text{Dol}} N_{Y_j}[D|_{Y_j}])$. Note that this argument is used to replace the classical Hodge-de Rham complex degeneration for proving Kodaira vanishing. So we have

$$\mathbb{H}^k(DR_{\mathcal{X}} \Xi_{\text{Dol}} N_{Y_j}[D|_{Y_j}]) = \mathbb{H}^k(DR_{\mathcal{X}} \mathcal{E}^j) = 0, \text{ for } k < 0. \quad (5.3)$$

By considering (5.2), now we can conclude the proof by a standard argument of the degeneration of the Hodge to de Rham spectral sequence, e.g. [B-V92 Appendix 25].

Then, we prove the relative version.

**Proof of Theorem 2.7.** We only prove the second vanishing as in the previous proof. The first one follows using the dual construction. Since we can add the pullback of
a sufficiently ample line bundle on $S$, we can assume that $\mathcal{L}$ is semi-ample on $X$. Hence, as in the proof of Theorem 1.3 we have an NCNC sequence of varieties

$$X = Y_0 \supset Y_1 \supset \ldots \supset Y_n,$$

with respect to $\mathcal{M}([D])$, induced by $\mathcal{A} = \mathcal{L}^m$. Then, we have the following exact sequence of Higgs sheaves:

$$0 \to \Xi_{\text{Del}} \mathcal{M}([D]) \otimes \mathcal{L}^{-1} \to \mathcal{E}^0 \to \mathcal{E}^1 \to \ldots \to \mathcal{E}^n \to 0,$$

satisfying

$$\mathcal{E}^i \simeq \Xi_{\text{Del}} i_+ N_{Y_j}[D|Y_j],$$

with $N_{Y_j}$ are $\mathcal{D}_{Y_j}$-modules underlying mixed twistor $\mathcal{D}$-modules, satisfying

$$N_{Y_j}[D|Y_j] = N_{Y_j}[D|Y_j + Y_{j+1}] = N_{Y_j}([D|Y_j + \ast Y_{j+1}]).$$

Due to the assumption that $\mathcal{L}(dD)$ is $f$-ample, so is $\mathcal{A}|_{Y_j}(dD|_{Y_j}) = \mathcal{O}_{Y_j}(dD|_{Y_j} + Y_{j+1})$. Set $Z_j := Y_j \setminus (dD|_{Y_j} + Y_{j+1})_{\text{red}}$ is affine, being relative ample implies that the induced $g_j : Z_j \to S$ are affine morphisms. Using Artin Vanishing, we get

$$\mathcal{H}^k_{g_j}(\mathcal{D} R^i \Xi_{\text{DR}} (N_{Y_j}([D|Y_j]|_{Z_j})) = 0, \text{for } k < 0,$$

where $\mathcal{H}^k$ stands for the $k$-th cohomology with respect to the perverse $t$-structure on constructable sheaves. This implies that, using Riemann-Hilbert correspondence, we have

$$\mathcal{R}^k f_* i_+ (N_{Y_j}([D|Y_j])) \simeq \mathcal{R}^k f_! i_! (N_{Y_j}([D|Y_j]))$$

$$\simeq \mathcal{R}^k g_! j_! (N_{Y_j}([D|Y_j]|_{Z_j}) = 0, \text{for } k < 0,$$

in the category of coherent $\mathcal{D}$-modules. Since we work locally on $S$, we can also assume that $S$ is affine. In the case that $k \geq 0$, we can apply Artin Vanishing again, (also the discussion before (5.3),) to get

$$\Xi_{\text{Del}} a_! \mathcal{R}^l f_* i_+ (N_{Y_j}([D|Y_j])) \simeq \Xi_{\text{DR}} a_! \mathcal{R}^l f_! i_! (N_{Y_j}([D|Y_j]))$$

$$\simeq \mathbb{H}^l \mathcal{D} R^s \Xi_{\text{DR}} \mathcal{R}^k g_! j_! (N_{Y_j}([D|Y_j]|_{Z_j}) = 0, \text{for } l < 0,$$

where $a$ is the universal map $S \to \text{Spec} (\mathbb{C})$.

We have the Grothendieck-Leray spectral sequence:

$$E_2^{p,q} = \Xi_{\text{Del}} a_! \mathcal{R}^p a_! \mathcal{R}^q f_! i_! (N_{Y_j}([D|Y_j])) \simeq \mathbb{H}^p \mathcal{D} R^s \Xi_{\text{Del}} \mathcal{R}^q g_! j_! N_{Y_j}([D|Y_j]),$$

which converges to

$$E^{p+q} = \mathbb{H}^{p+q} \mathcal{D} R^s \Xi_{\text{Del}} i_+ N_{Y_j}([D|Y_j]).$$

Now, using (5.5) and (5.6), we can get

$$E^k = \mathbb{H}^k \mathcal{D} R^s \Xi_{\text{Del}} i_+ N_{Y_j}([D|Y_j]) \simeq \mathbb{H}^k \mathcal{D} R^s \mathcal{E}^j = 0, \text{for } k < 0.$$

Considering (5.4), we can use a standard argument of the degeneration of the Hodge to de Rham spectral sequence as in the proof Theorem 1.3 to get

$$\mathbb{H}^k \mathcal{D} R^s \Xi_{\text{Del}} \mathcal{M}([D]) \otimes \mathcal{L}^{-1} = 0, \text{for } k < 0.$$

Note that $a_* = a_! [d]$ is an exact functor on the category of quasi-coherent sheaves, and actually it is fully faithful from the category of $\mathcal{O}_S$-modules to the category of $a_* \mathcal{O}_S$-modules, II.Exercise 5.3. Decomposing $\mathbb{H}^k$ into $a_! f_*$, and combining (5.7), we can conclude

$$\mathcal{R}^k f_* \mathcal{D} R^s \Xi_{\text{Del}} \mathcal{M}([D]) \otimes \mathcal{L}^{-1} = 0, \text{for } k < -d.$$
Let’s prove the generalized Kawamata-Viehweg vanishing with \( \mathbb{Q} \)-divisors.

**Proof for Theorem 1.4.** According to Remark 1.3 we can always assume that \( 0 < b < 1, 0 < c < 1 \). We will focus on the second vanishing, the other one is similar, by just shifting those indexes to \( 0 < -b < 1, 0 < -c < 1 \), instead. We work locally around \( \lambda = 0 \), hence we omit the super-index \((0)\) when taking the KM-filtration.

We first try to reduce numerical equivalence to linear equivalence. Note that
\[
L \equiv_{\text{num}} aA + bB + cC,
\]
is equivalent to that (see e.g. [Laz04a, Remark 1.1.20]) there exists a topologically trivial line bundle \( P \), and \( n \in \mathbb{Z}^+ \), such that
\[
\mathcal{O}(nL) \otimes P \simeq \mathcal{O}(n(aA + bB + cC)).
\]
Apply Bloch–Gieseker coverings on \( P \) and \((aA + bB + cC)\) [Laz04a, 4.1.10]. That means there exists a covering map \( \pi : \tilde{X} \to X \), satisfying \( \pi^* P \simeq \tilde{P}^n \), for some topologically trivial line bundle \( \tilde{P} \) on \( \tilde{X} \), and \( \tilde{L} = \pi^*(aA + bB + cC) \) is an integral divisor. Actually, \( \pi \) can be constructed as a cyclic cover ramified along smooth divisors of sufficient general location. Hence, we can require that \( \pi \) is non-characteristic with respect to \( \mathcal{M}(*D) \). In particular, \( \pi \) does not ramify along \( D \), and \( \tilde{A}(= \pi^* A), \tilde{B}, \tilde{C} \) are reduced and normal crossing, so we can set
\[
\tilde{\mathcal{M}}(*\tilde{D}) := \pi^! \mathcal{M}(\ast D) \simeq \pi^* \mathcal{M}(\ast D).
\]
By a local computation, the KM-filtrations on \( \tilde{\mathcal{M}}(*\tilde{D}) \) along the components of \( D \) are also just the pull back the corresponding filtrations on \( \mathcal{M}[*D] \).

In particular, \( \tilde{\mathcal{M}}[*\tilde{D}] \) is still \( V \)-compatible with respect to \( \tilde{D} \).

Due to Proposition 3.3 we have
\[
\pi_! (V^b_b \tilde{V}^{\tilde{C}}_{<c} \tilde{\mathcal{M}}(*\tilde{D})) \simeq V^b_b \tilde{V}^{\tilde{C}}_{<c} \pi_! \tilde{\mathcal{M}}(*\tilde{D}).
\]
This implies that \( V^b_b \tilde{V}^{\tilde{C}}_{<c} \tilde{\mathcal{M}}(*\tilde{D}) \) is a direct summand of \( \pi_! (V^b_b \tilde{V}^{\tilde{C}}_{<c} \tilde{\mathcal{M}}(*\tilde{D})) \). Due to the compatibility of DR, \( \Xi_{\text{Dol}} \) and the direct image functor, and by projection formula, we get
\[
\text{DR}_{(X,D)}(\Xi_{\text{Dol}} V^B_b V^C_{<c} \mathcal{M}(\ast D)) \otimes \mathcal{O}_X(-L)
\]
is a direct summand of
\[
\pi_*(\text{DR}_{(X,D)}(\Xi_{\text{Dol}} V^B_b V^C_{<c} \tilde{\mathcal{M}}(*\tilde{D})) \otimes \tilde{P} \otimes \mathcal{O}_{\tilde{X}}(-\tilde{L})),
\]
where \( \tilde{L} = a\tilde{A} + b\tilde{B} + c\tilde{C} \), and \( \tilde{P} \otimes \mathcal{O}_{\tilde{X}}(-\tilde{L}) \simeq \pi^* \mathcal{O}_X(-L) \). Note that \( \tilde{P} \), as a topologically trivial line bundle, can be viewed as a stable higgs bundle that is associated to a rank one unitary representation, i.e. a rank one harmonic bundle with trivial higgs connection, so we can use \( \tilde{\mathcal{M}} \otimes p^* \tilde{P} \) to replace \( \mathcal{M} \), still getting a mixed twisor \( \mathcal{P} \)-module. Hence we reduce the problem to the case that we replace \( \equiv_{\text{num}} \) in the initial statement by \( \equiv_{\text{lin}} \).

In this setting, we can further apply Bloch–Gieseker coverings with respect to \( A \), and we reduce to the case that \( a\tilde{A} \) is linear equivalent to some \( \mathbb{Z} \)-divisor \( A' \), hence \( b\tilde{B} + c\tilde{C} \) is also linear equivalent to some \( \mathbb{Z} \)-divisor \( L' \).
Now, we are ready to apply Esnault-Viehweg covering construction. Denote \( A = \mathcal{O}(A') \), \( B + cC = dD \) and \( \mathcal{L} = \mathcal{O}(dD) \). Let \( N \) be the least positive integer such that \( Nd \in \mathbb{Z}^n \), following the construction in §4, and using Lemma 4.1 we get

\[
V^B_0 V^C_0 M(\ast D) \otimes p^* \mathcal{L}^{-1} \simeq V^B_0 V^C_0 M_1(\ast D)
\]

This implies

\[
\begin{align*}
\text{DR}_{(X,D)}(\Xi_{\text{Dol}} V^B V^C M(\ast D)) \otimes \mathcal{O}_X(-L) & \simeq \text{DR}_{(X,D)}(\Xi_{\text{Dol}} V^B_0 V^C_0 M_1(\ast D)) \otimes A^{-1} \\
& \simeq \text{DR}_X(\Xi_{\text{Dol}} M_1(\ast |B|) \otimes |C|) \otimes A^{-1}. 
\end{align*}
\]

(5.8)

We use the logarithmic comparison (Proposition 3.7) to get the second isomorphism. Now, we can conclude the proof using Theorem 1.3.

Proof of Theorem 2.2. Once we get the identity (5.8), we can apply Theorem 2.1 to conclude.

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