On the effect of the material’s anisotropy: a numerical investigation for the modified Helmholtz problems of homogeneous media

A Galsan¹, M I Azis¹,∗, S Aswad², H Halide² and R Syam³

¹Department of Mathematics, Hasanuddin University, Makassar, Indonesia
²Department of Geophysics, Hasanuddin University, Makassar, Indonesia
³Department of Mechanical Engineering, Hasanuddin University, Makassar, Indonesia

E-mail: mohivanazis@yahoo.co.id (∗Corresponding author)

Abstract. This paper focuses on the study of the effect of the anisotropy of a medium in question to the values of the dependent variable. Boundary value problems governed by the two-dimensional modified Helmholtz equation for anisotropic media are considered. Specifically, it is the diffusion or conduction process which is assumed to be anisotropic. The standard boundary element method is used to find numerical solutions to the problems. The solutions show that the anisotropy of the medium causes certain effects on the solution. Therefore the anisotropy of the medium should be taken into account in the modeling and computation for the solution, specifically for experimental studies.

1. Introduction

Various kinds of problems have been solved numerically using boundary element methods (BEMs). Advantages of BEM merely lies on its ease, accuracy and efficiency. Not only for homogeneous and isotropic media, BEM also is used for solving problems of inhomogeneous and anisotropic media. Some previous papers considering the use BEM for various problems of homogeneous/inhomogeneous and isotropic/anisotropic media include [1], [2], [3], [4], [5], [6], [7], [8], [9] and [10].

The modified Helmholtz equation has been used for modeling infiltration problems. Some previous works on the equation especially for isotropic media have been done. See for examples [11], [12], [13] and [14].

Not so many works on the modified Helmholtz equation for anisotropic media have been done. This paper focuses on finding numerical solutions to the modified Helmholtz equation for anisotropic media by using the BEM.

2. The boundary value problem

The 2D scalar anisotropic modified Helmholtz equation may be written as

\[ \beta_{ij} \frac{\partial^2 A}{\partial x_i \partial x_j} - k^2 A = 0, \quad \text{for } i, j = 1, 2 \]

where the summation convention for repeated indices is assumed to apply. The coefficient matrix \([\beta_{ij}]\) is a real symmetrical positive definite matrix, that is \(\beta_{12} = \beta_{21}\) and \(\beta_{11} \beta_{22} - \beta_{12}^2 > 0\). Thus
explicitly equation (1) may be written as
\[ \beta_{11} \frac{\partial^2 A}{\partial x_1^2} + 2\beta_{12} \frac{\partial^2 A}{\partial x_1 \partial x_2} + \beta_{22} \frac{\partial^2 A}{\partial x_2^2} - k^2 A = 0 \]

Equation (1) applies for isotropic case as a special case, that is when \( \beta_{11} = \beta_{22} = 1 \) dan \( \beta_{12} = 0 \).

Referred to a Cartesian coordinate \( Ox_1x_2 \) we seek a solution to (1) in a domain \( \Omega \) with boundary \( \Gamma \) which consists of a finite number of piecewise smooth closed surfaces. On \( \Gamma' \) \( A(x, t) \) is known and on \( \Gamma'' \) \( P(x, t) \) is specified, where \( \Gamma = \Gamma' \cup \Gamma'' \), \( P(x) = \beta_{ij} \frac{\partial A(x)}{\partial x_i} n_j(x) \) and \( n_i \) denotes the component of the outward pointing vector \( n \) normal to \( \Gamma \).

### 3. Integral equation

Multiplying both sides of (1) by function \( A^*(x, \chi) \) and then integrating it over domain gives
\[
\int_\Omega \beta_{ij} \frac{\partial^2 A}{\partial x_i \partial x_j} A^* d\Omega - \int_\Omega k^2 A A^* d\Omega = 0
\]

Using Gauss divergence theorem twice in (2) we obtain
\[
\int_\Omega \left( \beta_{ij} \frac{\partial^2 A^*}{\partial x_i \partial x_j} - k^2 A^* \right) d\Omega = -\int_{\Gamma'} (PA^* - P^* A) d\Gamma
\]

where
\[
P^* = \beta_{ij} \frac{\partial A^*}{\partial x_i} n_j
\]

If the function \( A^* \) satisfies
\[
\beta_{ij} \frac{\partial^2 A^*}{\partial x_i \partial x_j} - k^2 A^* = \delta(x - \chi)
\]
where \( \chi = (\chi_1, \chi_2) \in \Omega \) and \( \delta \) is the delta function. The function \( A^* \) satisfying (4) is (see for example Azis [15] for the derivation)
\[
A^*(x, \chi) = \frac{-K}{2\pi} K_0(\omega R)
\]
where \( K_0 \) is the modified Bessel function and the bar sign (\( \bar{\cdot} \)) denotes the conjugate of complex number.

\[
K = \dot{\tau} / D \\
D = [\beta_{11} + 2\beta_{12}(\tau + \dot{\tau}) + \beta_{22}(\tau)^2] / 2 \\
\omega = |k| / \sqrt{D} \\
R = \sqrt{(x_1 + \dot{\tau} x_2 - \chi_1 - \dot{\tau} \chi_2)^2 + (\dot{\tau} x_2 - \dot{\tau} \chi_2)^2} \\
\frac{\partial R}{\partial x_1} = \frac{1}{R} (x_1 + \dot{\tau} x_2 - \chi_1 - \dot{\tau} \chi_2) \\
\frac{\partial R}{\partial x_2} = \frac{1}{R} [\dot{\tau} (x_1 + \dot{\tau} x_2 - \chi_1 - \dot{\tau} \chi_2) + \dot{\tau} (\dot{\tau} x_2 - \dot{\tau} \chi_2)]
\]

\( \dot{\tau} \) and \( \ddot{\tau} \) represent respectively the real and the positive imaginary parts of complex root of quadratic equation
\[
\beta_{11} + 2\beta_{12} \tau + \beta_{22}(\tau)^2 = 0
\]

Substitution of (4) into (3) gives
\[
\lambda(\chi) A(\chi) = \int_{\Gamma'} [P^*(x, \chi) A(x) - P(x) A^*(x, \chi)] d\Gamma(x)
\]
where \( \lambda = \frac{1}{2} \) if \( \chi \in \Gamma \), \( \lambda = 1 \) if \( \chi \in \Omega \) and \( \lambda = 0 \) if \( \chi \notin \Omega \).
4. Numerical results
Some examples of boundary value problems governed by equation (1) will be considered. The boundary integral equation (5) is used to find the solution. The integrals in equation (5) are evaluated numerically using the Gaussian quadrature rule (see Abramowitz and Stegun [16]).

4.1. Example 1: A problem with analytical solution
Consider the boundary value problem governed by (1) with coefficients

\[
[\beta_{ij}] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \quad k^2 = -1
\]

The analytical solution to (1) is

\[
A = \exp(\mu x^2) \quad \mu = \sqrt{k^2/\beta_{22}}
\]

and the domain \(\Omega\) is chosen to be a unit square with corner points \(A = (0, 0), B = (1, 0), C = (1, 1),\) and \(D = (0, 1)\). The boundary conditions are that \(P\) is given on \(AB, BC, CD\) and \(A\) is given on \(AD\).

Table 1 shows a comparison between the boundary element method (BEM) solution and the analytical solution. The results show that the BEM solution converges to the analytical solution as the number of segments of the same length increases from 80, 160 to 320. The results are as expected.

| \((x_1, x_2)\)   | \(A\) | \(\partial A/\partial x_1\) | \(\partial A/\partial x_2\) | \(A\) | \(\partial A/\partial x_1\) | \(\partial A/\partial x_2\) |
|------------------|-----|-----------------|-----------------|-----|-----------------|-----------------|
| \(BEM\) 80 segs |     |                 |                 | \(BEM\) 160 segs |                 |                 |
| (0.5,0.1)       | 1.0731 | 0.0008          | 0.7580          | 1.0732 | 0.0003          | 0.7585          |
| (0.5,0.3)       | 1.2359 | 0.0001          | 0.8733          | 1.2361 | 0.0001          | 0.8738          |
| (0.5,0.5)       | 1.4236 | -0.0005         | 1.0062          | 1.4239 | -0.0001         | 1.0066          |
| (0.5,0.7)       | 1.6398 | -0.0007         | 1.1594          | 1.6401 | -0.0002         | 1.1597          |
| (0.5,0.9)       | 1.8889 | -0.0006         | 1.3356          | 1.8893 | -0.0002         | 1.3359          |
| (0.1,0.5)       | 1.4237 | -0.0005         | 1.0066          | 1.4239 | -0.0002         | 1.0068          |
| (0.3,0.5)       | 1.4237 | -0.0004         | 1.0063          | 1.4239 | -0.0002         | 1.0067          |
| (0.7,0.5)       | 1.4236 | 0.0008          | 1.0053          | 1.4239 | 0.0003          | 1.0063          |
| (0.9,0.5)       | 1.4239 | 0.0018          | 1.0052          | 1.4239 | 0.0007          | 1.0063          |
| \(BEM\) 320 segs |     |                 |                 | \(Analytical\) |                 |                 |
| (0.5,0.1)       | 1.0732 | 0.0001          | 0.7587          | 1.0733 | 0.0000          | 0.7589          |
| (0.5,0.3)       | 1.2362 | 0.0000          | 0.8740          | 1.2363 | 0.0000          | 0.8742          |
| (0.5,0.5)       | 1.4240 | -0.0000         | 1.0068          | 1.4241 | 0.0000          | 1.0070          |
| (0.5,0.7)       | 1.6403 | -0.0001         | 1.1598          | 1.6405 | 0.0000          | 1.1600          |
| (0.5,0.9)       | 1.8895 | -0.0001         | 1.3360          | 1.8897 | 0.0000          | 1.3362          |
| (0.1,0.5)       | 1.4240 | -0.0001         | 1.0069          | 1.4241 | 0.0000          | 1.0070          |
| (0.3,0.5)       | 1.4240 | -0.0001         | 1.0069          | 1.4241 | 0.0000          | 1.0070          |
| (0.7,0.5)       | 1.4240 | 0.0001          | 1.0067          | 1.4241 | 0.0000          | 1.0070          |
| (0.9,0.5)       | 1.4240 | 0.0003          | 1.0067          | 1.4241 | 0.0000          | 1.0070          |
4.2. Example 2: A problem without analytical solution
Now consider the problem shown in Figure 1. The coefficients $\beta_{ij}$ and $k^2$ are chosen to be

\[
\beta_{11} = 0.5, \quad \beta_{12} = 0.2, \quad \beta_{22} = 0.3, \quad k^2 = 0.1.
\]

Table 2 shows the BEM solutions converge to particular values as the number of segments increases from 160 to 1000 segments.

Table 2. BEM solutions for Example 2

| $(x_1, x_2)$ | $A$ | $\partial A/\partial x_1$ | $\partial A/\partial x_2$ | $A$ | $\partial A/\partial x_1$ | $\partial A/\partial x_2$ |
|-------------|-----|-----------------|-----------------|-----|-----------------|-----------------|
| (0.5,0.1)   | 0.2721 | -0.3242 | 2.9327 | 0.2725 | -0.3247 | 2.9347 |
| (0.5,0.3)   | 0.9291 | -0.9435 | 3.5734 | 0.9299 | -0.9445 | 3.5756 |
| (0.5,0.5)   | 1.6776 | -1.3233 | 3.8755 | 1.6788 | -1.3244 | 3.8774 |
| (0.5,0.7)   | 2.4737 | -1.4989 | 4.0833 | 2.4752 | -1.4998 | 4.0850 |
| (0.5,0.9)   | 3.3112 | -1.5737 | 4.2918 | 3.3131 | -1.5747 | 4.2930 |

Figure 1. Geometry of Example 2 and Example 3

4.3. Example 3: Comparison between isotropic and anisotropic solutions
Now consider a problem with three cases, two of them are anisotropic cases with $\beta_{11} = 1, \beta_{12} = 0, \beta_{22} = 2$ and $\beta_{11} = 1, \beta_{12} = 1, \beta_{22} = 2$, and the other one is isotropic case with
\( \beta_{11} = 1, \beta_{12} = 0, \beta_{22} = 1. \) For all cases we take \( k^2 = 0.1 \) and use 1000 segments of equal length, and the domain and boundary conditions are as depicted in Figure 1.

Figures 2–4 show a certain difference between the solutions of anisotropic and isotropic cases.

**Figure 2.** Flow vector \((\partial \Lambda/\partial x_1, \partial \Lambda/\partial x_2)\) and scattering of \( \Lambda \) values for anisotropic case with \( \beta_{11} = 1, \beta_{12} = 0, \beta_{22} = 2 \)

**Figure 3.** Flow vector \((\partial \Lambda/\partial x_1, \partial \Lambda/\partial x_2)\) and scattering of \( \Lambda \) values for anisotropic case with \( \beta_{11} = 1, \beta_{12} = 1, \beta_{22} = 2 \)

**Figure 4.** Flow vector \((\partial \Lambda/\partial x_1, \partial \Lambda/\partial x_2)\) and scattering of \( \Lambda \) values for isotropic case with \( \beta_{11} = 1, \beta_{12} = 0, \beta_{22} = 1 \)

5. Conclusion
BEM for the modified Helmholtz boundary value problem has been derived. The method are generally easy to implement to obtain numerical values for particular problems. The numerical
solutions obtained using BEM indicate that it produces accurate solutions. The results for Example 1 and Example 2 of Section 4 show the convergence of the solution. And the results for Example 3 show a consistency of the flow and the scattering solutions. They also show that the anisotropy of the medium under consideration will certainly result in an effect on the solutions. Therefore for application, the anisotropy of the medium should be considered to take account in the modeling and computation for the solution.

Acknowledgements
The authors acknowledge the research grants provided by The Ministry of Higher Education of Indonesia (KEMRISTEKDIKTI) under the contract numbered as 007/SP2H/PTNBI/DRPM/2019 and by The Hasanuddin University under the Hasanuddin University’s Rector decrees numbered as 2006/UN4.1/KEP/2019 and 641/UN4.1/KEP/2019.

References
[1] Haddade A, Salam N, Khaeruddin and Azis M I 2017 A boundary element method for 2D diffusion-convection problems in anisotropic media Far East Journal of Mathematical Sciences 102(8) 1593
[2] Azis M I, Kasbawati, Haddade A and Thamrin S A 2018 On some examples of pollutant transport problems solved numerically using the boundary element method Journal of Physics: Conference Series 979 012075
[3] Clements D L and Azis M I 2000 A note on a boundary element method for the numerical solution of boundary value problems in isotropic inhomogeneous elasticity Journal of the Chinese Institute of Engineers, Transactions of the Chinese Institute of Engineers, Series A/Chung-kuo Kung Ch'eng Hsuch K'an. 23(3) 261
[4] Azis M I and Clements D L 2001 A Boundary Element Method for Anisotropic Inhomogeneous Elasticity International Journal of Solids and Structures 38(32-33) 5747
[5] Azis M I, Toaha S, Bahri M and Ilyas N 2018 A boundary element method with analytical integration for deformation of inhomogeneous elastic materials Journal of Physics: Conference Series 979 012072
[6] Azis M I, Clements D L and Budhi W S 2003 A boundary element method for the numerical solution of a class of elliptic boundary value problems for anisotropic inhomogeneous media ANZIAM J. 44(E) C79 doi:10.21914/anziamj.v44i0.673
[7] Azis M I and Clements D L 2008 Nonlinear transient heat conduction problems for a class of inhomogeneous anisotropic materials by BEM Engineering Analysis with Boundary Elements 32(12) 1054
[8] Azis M I and Clements D L 2014 On some problems concerning deformations of functionally graded anisotropic elastic materials Far East Journal of Mathematical Sciences 87(2) 173
[9] Azis M I and Clements D L 2014 A Boundary Element Method for Transient Heat Conduction Problem of Nonhomogeneous Anisotropic Materials Far East Journal of Mathematical Sciences 89(1) 51
[10] Salam N, Haddade A, Clements D L and Azis M I 2017 A boundary element method for a class of elliptic boundary value problems of functionally graded media Engineering Analysis with Boundary Elements 84(3) 186
[11] Lobo M, Clements D L and Widana N 2005 Infiltration from irrigation channels into soil with impermeable inclusions ANZIAM J. 46(E) C1055
[12] Clements D L and Lobo M 2010 A BEM for time dependent infiltration from an irrigation channel Engineering Analysis with Boundary Elements 34 1100
[13] Solekhudin I and Ang K-C 2012 A DRBEM with a predictor-corrector scheme for steady infiltration from periodic channels with root-water uptake Engineering Analysis with Boundary Elements 36 1199
[14] Chen W, Zhang J-Y and Fu Z-J 2014 Singular boundary method for modified Helmholtz equations Engineering Analysis with Boundary Elements 44 112
[15] Azis M I 2017 Fundamental solutions to two types of 2D boundary value problems of anisotropic materials Far East Journal of Mathematical Sciences 101(11) 2405
[16] Abramowitz M and Stegun I A 1972 Handbook of mathematical functions: with formulas, graphs and mathematical tables, Dover Publications, Washington