Scotogenic $A_4$ Neutrino Model for Nonzero $\theta_{13}$ and Large $\delta_{CP}$

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Abstract

Assuming that neutrinos acquire radiative seesaw Majorana masses through their interactions with dark matter, i.e. scotogenic from the Greek 'scotos' meaning darkness, and using the non-Abelian discrete symmetry $A_4$, we propose a model of neutrino masses and mixing with nonzero $\theta_{13}$ and necessarily large leptonic $CP$ violation, allowing both the normal and inverted hierarchies of neutrino masses, as well as quasi-degenerate solutions.
In 2006, a one-loop mechanism was introduced linking neutrino mass with dark matter. The idea is very simple. The standard model of particle interactions is extended to include a second scalar doublet \((\eta^+, \eta^-)\) which is odd under an exactly conserved \(Z_2\) symmetry \(^2\), as well as three neutral fermion singlets \(N_i\) which are also odd under \(Z_2\). This requirement immediately allows the possibility of having the real (or imaginary) part of \(\eta^0\) as a dark-matter candidate, which was first pointed out also in Ref. \(^1\). As shown in Fig. 1, this results in the radiative generation of seesaw Majorana neutrino masses from dark matter, i.e. scotogenic from the Greek ‘scotos’ meaning darkness.

The non-Abelian discrete symmetry \(A_4\) was introduced \(^3\) to achieve the seemingly impossible, i.e. the existence of a lepton family symmetry consistent with the three very different charged-lepton masses \(m_e, m_\mu, m_\tau\). It was subsequently shown \(^6\) to be a natural theoretical framework for neutrino tribimaximal mixing, i.e. \(\sin^2 \theta_{23} = 1, \tan^2 \theta_{12} = 0.5, \) and \(\theta_{13} = 0\). This pattern was consistent with experimental data until recently, when the Daya Bay Collaboration reported \(^7\) the first precise measurement of \(\theta_{13}\), i.e.

\[
\sin^2 2\theta_{13} = 0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst}),
\]

followed shortly \(^8\) by the RENO Collaboration, i.e.

\[
\sin^2 2\theta_{13} = 0.113 \pm 0.013(\text{stat}) \pm 0.019(\text{syst}).
\]
This means that tribimaximal mixing is not a good description, and more importantly, leptonic CP violation is now possible because $\theta_{13} \neq 0$, just as hadronic CP violation in the quark sector is possible because $V_{ub} \neq 0$.

Recently, it was shown [9] that $A_4$ is still a good symmetry for understanding this pattern, using a new simple variation of the original idea [6]. In that proposal, neutrinos acquire Majorana masses through their direct interactions with Higgs triplets. We study here instead the corresponding scenario with the radiative mechanism of Fig. 1.

The symmetry $A_4$ is that of the even permutation of four objects. It has twelve elements and is the smallest group which admits an irreducible three-dimensional representation. Its character table is given below. The basic multiplication rule of $A_4$ is

$$3 \times 3 = 1 + 1' + 1'' + 3 + 3'.$$ (3)

As first shown in Ref. [3], for $(\nu_i, l_i) \sim 3$, $l'_i \sim 1, 1', 1''$, and $\Phi_i = (\phi_i^0, \phi_i^-) \sim 2$, the charged-lepton mass matrix is given by

$$M_l = \begin{pmatrix} v_1 & 0 & 0 \\ 0 & v_2 & 0 \\ 0 & 0 & v_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix},$$ (4)

where $v_i = \langle \phi_i^0 \rangle$ and $\omega = e^{2\pi i/3} = -1/2 + i\sqrt{3}/2$. For $v_1 = v_2 = v_3 = v/\sqrt{3}$, we then obtain

$$M_l = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix},$$ (5)

Table 1: Character table of $A_4$. 

| class | $n$ | $h$ | $\chi_1$ | $\chi_1'$ | $\chi_1''$ | $\chi_3$ |
|-------|----|----|---------|----------|-----------|---------|
| $C_1$ | 1  | 1  | 1       | 1        | 1         | 3       |
| $C_2$ | 4  | 3  | $\omega$ | $\omega^2$ | 0         |         |
| $C_3$ | 4  | 3  | $\omega^2$ | $\omega$ | 0         |         |
| $C_4$ | 3  | 2  | 0       | 0        | -1        |         |
where $m_e = f_1 v$, $m_\mu = f_2 v$, $m_\tau = f_3 v$. The original $A_4$ symmetry is now broken to the residual symmetry $Z_3$, i.e. lepton flavor triality [10], with $e \sim 1$, $\mu \sim \omega^2$, $\tau \sim \omega$. This is a good symmetry of the Lagrangian as long as neutrino masses are zero. Exotic scalar decays are predicted and may be observable at the Large Hadron Collider (LHC) in some regions of parameter space [11, 12].

To obtain nonzero neutrino masses, we assign $\eta \sim 1$ and $N_i \sim 3$ under $A_4$. We also add the scalar singlets $\sigma_i \sim 3$ with nonzero $\langle \sigma_i \rangle$. The resulting $3 \times 3$ Majorana mass matrix for $N_i$ is then

$$M_N = \begin{pmatrix} A & F & E \\ F & A & D \\ E & D & A \end{pmatrix},$$

which is the analog of

$$M_\nu = \begin{pmatrix} a & f & e \\ f & a & d \\ e & d & a \end{pmatrix},$$

considered in Ref. [9]. (A better way to enforce Eq. (6) is to postulate gauged $B - L$ and assume complex neutral scalars which transform as $1, 3$ under $A_4$, in complete analogy to the scalar triplets of Ref. [9].) Instead of enforcing $E = F = 0$ which is required for tribimaximal mixing, we assume here that $F = -E$ which may be maintained by an interchange symmetry [6, 13].

Consider now the tribimaximal basis, i.e.

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}. $$

Since $\nu_{1,2,3}$ are connected to $N_{1,2,3}$ through the identity matrix, we find

$$M_N^{(1,2,3)} = \begin{pmatrix} A + D & 0 & 0 \\ 0 & A & C \\ 0 & C & A - D \end{pmatrix},$$

where $C = (E - F)/\sqrt{2} = \sqrt{2}E$. 

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The diagram of Fig. 1 is exactly calculable from the exchange of Re($\eta^0$) and Im($\eta^0$) and is given by \[1\]

\[
(M\nu)_{ij} = \sum_{k} h_{ik} h_{jk} M_k \left[ \frac{m_R^2}{m_R^2 - M_k^2} \ln \frac{m_R^2}{M_k^2} - \frac{m_I^2}{m_I^2 - M_k^2} \ln \frac{m_I^2}{M_k^2} \right],
\]

(10)

where $\sum_k h_{ik}(h_{jk})^* = |h|^2 \delta_{ij}$, and $m_{R,I}$ are the masses of $\sqrt{2}$Re($\eta^0$) and $\sqrt{2}$Im($\eta^0$), respectively. In the limit $m_R^2 - m_I^2 = 2\lambda_5 v^2$ is small compared to $m_0^2 = (m_R^2 + m_I^2)/2$, and $m_0^2 << M_k^2$, Eq. (10) reduces to

\[
(M\nu)_{ij} = \frac{\lambda_5 v^2}{8\pi^2} \sum_k \frac{h_{ik} h_{jk}}{M_k} \left[ \ln \frac{M_k^2}{m_0^2} - 1 \right].
\]

(11)

In the tribimaximal basis of Eq. (9), we then have

\[
h_{ik} = h \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta e^{i\phi} \\ 0 & \sin \theta e^{-i\phi} & \cos \theta \end{pmatrix} \begin{pmatrix} e^{i\alpha_1/2} & 0 & 0 \\ 0 & e^{i\alpha_2/2} & 0 \\ 0 & 0 & e^{i\alpha_3/2} \end{pmatrix},
\]

(12)

with

\[
\begin{pmatrix} \cos \theta & \sin \theta e^{i\phi} \\ -\sin \theta e^{-i\phi} & \cos \theta \end{pmatrix} \begin{pmatrix} A & C \\ C & A - D \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta e^{-i\phi} \\ \sin \theta e^{i\phi} & \cos \theta \end{pmatrix} = \begin{pmatrix} e^{i\alpha_2} M_2 & 0 \\ 0 & e^{i\alpha_3} M_3 \end{pmatrix}.
\]

(13)

The neutrino mixing matrix $U$ has 4 parameters: $s_{12}, s_{23}, s_{13}$ and $\delta_{CP}$ \[14\]. We choose the convention $U_{\tau 1}, U_{\tau 2}, U_{e3}, U_{\mu 3} \rightarrow -U_{\tau 1}, -U_{\tau 2}, -U_{e3}, -U_{\mu 3}$ to conform with that of the tribimaximal mixing matrix of Eq. (8), then

\[
(M\nu)^{1,2,3}_\nu = U_{TB}^T U \begin{pmatrix} e^{i\alpha_1} m'_1 & 0 & 0 \\ 0 & e^{i\alpha_2} m'_2 & 0 \\ 0 & 0 & m'_3 \end{pmatrix} U^T U_{TB},
\]

(14)

where $m'_{1,2,3}$ are the physical neutrino masses, with

\[
m'_2 = \sqrt{m_1^2 + \Delta m_{21}^2},
\]

(15)

\[
m'_3 = \sqrt{m_1^2 + \Delta m_{21}^2/2 + \Delta m_{32}^2} \quad \text{(normal hierarchy),}
\]

(16)

\[
m'_3 = \sqrt{m_1^2 + \Delta m_{21}^2/2 - \Delta m_{32}^2} \quad \text{(inverted hierarchy).}
\]

(17)
We now diagonalize $M^{(1,2,3)}$ using
\[
U_e M^{(1,2,3)} U_e^T = \begin{pmatrix}
  e^{i\alpha_1 m_1'} & 0 & 0 \\
  0 & e^{i\alpha_2 m_2'} & 0 \\
  0 & 0 & e^{i\alpha_3 m_3'}
\end{pmatrix},
\]
from which we obtain $U' = U_T U_e^T$. To obtain $U$ with the usual convention, we rotate the phases of the $\mu$ and $\tau$ rows so that $U'_{\mu 3} e^{-i\alpha_3/2}$ is real and negative, and $U'_{\tau 3} e^{-i\alpha_3/2}$ is real and positive. These phases are absorbed by the $\mu$ and $\tau$ leptons and are unobservable. We then rotate the $\nu_{1,2}$ columns so that $U'_{e1} e^{-i\alpha_1/2} = U_{e1} e^{i\alpha_1''/2}$ and $U'_{e2} e^{-i\alpha_2/2} = U_{e2} e^{i\alpha_2''/2}$, where $U_{e1}$ and $U_{e2}$ are real and positive. The physical relative Majorana phases of $\nu_{1,2}$ are then $\alpha_{1,2} = \alpha_{1,2}' + \alpha_{1,2}''$. The three angles and the Dirac phase are extracted according to
\[
\tan^2 \theta_{12} = |U'_{e2}/U'_{e1}|^2,
\tan^2 \theta_{23} = |U'_{\mu 3}/U'_{\tau 3}|^2,
\sin \theta_{13} e^{-i\delta_{CP}} = U'_{e3} e^{-i\alpha_3/2}.
\]
The effective Majorana neutrino mass in neutrinoless double beta decay is then given by
\[
m_{ee} = |U'_{e1} e^{i\alpha_1 m_1'} + U'_{e2} e^{i\alpha_2 m_2'} + U'_{e3} m_3'|. 
\]

In Eq. (9), let $A$ be real and positive by convention, then both $C$ and $D$ may be complex, i.e. $C = C_R + i C_I$ and $D = D_R + i D_I$. The $2 \times 2$ matrix of Eq. (13) can be solved exactly to yield
\[
\tan \phi = \frac{C_R D_I - C_I D_R}{C_R (2A - D_R) - C_I D_I},
\]
\[
\tan 2\theta = \frac{2[4A^2 C_R^2 - 4AC_R (C_R D_R + C_I D_I) + (C_R^2 + C_I^2)(D_R^2 + D_I^2)]^{1/2}}{2AD_R - (D_R^2 + D_I^2)},
\]
with
\[
e^{i\alpha_2} M_2 = \cos^2 \theta A + 2 \sin \theta \cos \theta e^{i\phi} C + \sin^2 \theta e^{2i\phi} (A - D),
\]
\[
e^{i\alpha_3} M_3 = \cos^2 \theta (A - D) - 2 \sin \theta \cos \theta e^{-i\phi} C + \sin^2 \theta e^{-2i\phi} A.
\]
The corresponding $U'$ elements are
\[
U'_{e1} = \sqrt{\frac{2}{3}}, \quad U'_{e2} = \frac{\cos \theta}{\sqrt{3}}, \quad U'_{e3} = -\frac{\sin \theta}{\sqrt{3}} e^{-i\phi},
\]
\[
U'_{\mu 3} = -\frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{3}} e^{-i\phi}, \quad U'_{\tau 3} = \frac{\cos \theta}{\sqrt{2}} - \frac{\sin \theta}{\sqrt{3}} e^{-i\phi}.
\]
If we absorb the scale factor $\lambda \equiv h^2 v^2 / 8\pi^2$ into the parameters $A, C, D$ as well as $m_0$, then the mass eigenvalues of Eq. (11) are given by

$$m'_k = \frac{1}{M_k} \left[ \ln \frac{M_k^2}{m_0^2} - 1 \right],$$

(27)

which are the ones used in Eqs. (14) and (18). Since $m_0$ is an unknown, having to do with the dark-matter scalar mass, we fix it by requiring $M_1/m_0 = 10$, where $M_1 = |A + D|$. If we input the five parameters $A, C_R, C_I, D_R, D_I$, we will obtain $m'_{1,2,3}$ as well as the three mixing angles and the three $CP$ phases. For our numerical analysis, we set

$$\Delta m^2_{21} = 7.59 \times 10^{-5} \text{ eV}^2, \quad \Delta m^2_{32} = 2.45 \times 10^{-3} \text{ eV}^2,$$

(28)

and vary $\theta_{13}$ in the range

$$\sin^2 2\theta_{13} = 0.05 \text{ to } 0.15.$$  

(29)

Following Ref. [9], we look for solutions with $\sin^2 2\theta_{23} = 0.92$ and 0.96. Whereas only normal hierarchy is allowed in the model of Ref. [9], we find solutions for both normal and inverted hierarchies, as well as quasi-degenerate solutions, as detailed below.

The predictions of this model regarding mixing angles are basically the same as in Ref. [9] for the special case of $b = 0$ there. Using Eqs. (19), (25), and (26), we find

$$\tan^2 \theta_{12} = \frac{1 - 3 \sin^2 \theta_{13}}{2},$$

(30)

$$\tan^2 \theta_{23} = \frac{1 - \sqrt{2} \sin \theta_{13} \cos \phi}{1 - 3 \sin^2 \theta_{13}} + \frac{2 \sin^2 \theta_{13} \sin^2 \phi}{1 - 3 \sin^2 \theta_{13}} + \frac{2 \sin^2 \theta_{13} \sin^2 \phi}{1 - 3 \sin^2 \theta_{13}},$$

(31)

The conventionally defined Dirac $CP$ phase is given by $\delta_{CP} = \phi + \alpha'_3 / 2$, where $\alpha'_3$ is defined in Eq. (18) and depends on the specific values of Eq. (9). For $\sin \theta_{13} = 0.16$, corresponding to $\sin^2 2\theta_{13} = 0.1$, this predicts $\tan^2 \theta_{12} = 0.46$. If $\text{Im}(C) = 0$, then $\delta_{CP} = \alpha'_3 = 0$, so this would predict $\sin^2 2\theta_{23} = 0.80$ which is of course ruled out. Using $\sin^2 2\theta_{23} > 0.92$, we find in this case $|\tan \phi| > 1.2$. 

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For each of the two values $\sin^2 2\theta_{23} = 0.92$ and 0.96, we obtain 5 representative solutions, all as functions of $\sin^2 2\theta_{13}$. Using Eq. (30), we plot $\sin^2 2\theta_{12}$ versus $\sin^2 2\theta_{13}$ in Fig. 2. The characteristic features of the 5 solutions are listed in Table 2. For $\text{Im}(D) = 0$, we find one solution for inverted ordering of neutrino masses, and two solutions for normal ordering (one of which is quasi-degenerate). For $\text{Im}(D) = \text{Re}(D)$, we again find one solution for inverted ordering, but the only solution for normal ordering is quasi-degenerate.

In Fig. 3 we show the physical neutrino masses $m'_{1,2,3}$ and the effective mass in neutrinoless double beta decay $m_{ee}$ (in eV) as well as the model parameters (in eV$^{-1}$) for solution (I) in the case $\sin^2 2\theta_{23} = 0.96$. In Figs. 4-7 we show the same quantities for solutions (II),(III),(IV),(V),

![Figure 2: $\sin^2 2\theta_{12}$ versus $\sin^2 2\theta_{13}$.](image)

Table 2: Five representative solutions. Three have $\text{Im}(D) = 0$, and two have $\text{Im}(D) = \text{Re}(D)$. NH denotes normal hierarchy of neutrino masses, IH inverted, and QD quasi-degenerate. The values of $|\tan \delta_{CP}|$ and $m_{ee}$ (in eV) are for $\sin^2 2\theta_{23} = 0.96$ and $\sin^2 2\theta_{13} = 0.10$. 

| solution | $\text{Im}(D)$ | $\text{class}$ | $|\tan \delta_{CP}|$ | $m_{ee}$ |
|----------|----------------|----------------|------------------|--------|
| I        | 0              | IH             | 2.05             | 0.020  |
| II       | $\text{Re}(D)$| IH             | 4.64             | 0.022  |
| III      | 0              | NH             | 3.59             | 0.002  |
| IV       | 0              | QD             | 2.20             | 0.046  |
| V        | $\text{Re}(D)$| QD             | 1.84             | 0.051  |
in the cases of $\sin^2\theta_{23} = 0.92, 0.96, 0.92, 0.96$ respectively. Finally we show in Fig. 8 the values of $|\tan\delta_{CP}|$ for all 5 solutions in the case of $\sin^2\theta_{23} = 0.92$. It is clear that at $\sin^2\theta_{13} = 0.10$, large $|\tan\delta_{CP}|$ is predicted.

Acknowledgments: This work is supported in part by the U. S. Department of Energy under Grant No. DE-AC02-06CH11357. The work of A.R. is supported in part by the National Science Foundation under Grant No. NSF PHY-1068052 and in part by the Graduate Student Council Research Grant Award 2012-2013. A.R. thanks the University of California, Riverside for hospitality.

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Figure 3: $A_4$ parameters and the physical neutrino masses and effective neutrino mass $m_{ee}$ in neutrinoless double beta decay for the inverted hierarchy with $\text{Im}(D)=0$ and $\sin^2 2\theta_{23} = 0.96$.

Figure 4: $A_4$ parameters and the physical neutrino masses and effective neutrino mass $m_{ee}$ in neutrinoless double beta decay for the inverted hierarchy with $\text{Im}(D)=\text{Re}(D)$ and $\sin^2 2\theta_{23} = 0.92$. 

Figure 5: $A_4$ parameters and the physical neutrino masses and effective neutrino mass $m_{ee}$ in neutrinoless double beta decay for the normal hierarchy with $\text{Im}(D)=0$ and $\sin^2 2\theta_{23} = 0.96$.

Figure 6: $A_4$ parameters and the physical neutrino masses and effective neutrino mass $m_{ee}$ in neutrinoless double beta decay for quasi-degenerate neutrino masses with $\text{Im}(D)=0$ and $\sin^2 2\theta_{23} = 0.96$. 
Figure 7: $A_4$ parameters and the physical neutrino masses and effective neutrino mass $m_{ee}$ in neutrinoless double beta decay for quasi-degenerate neutrino masses with $\text{Im}(D)=\text{Re}(D)$ and $\sin^2 2\theta_{23} = 0.96$.

Figure 8: $|\tan \delta_{CP}|$ versus $\sin^2 2\theta_{13}$ for $\sin^2 2\theta_{23} = 0.92$. 