Hidden invariance of the free classical particle

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Abstract

A formalism describing the dynamics of classical and quantum systems from a group theoretical point of view is presented. We apply it to the simple example of the classical free particle. The Galileo group $G$ is the symmetry group of the free equations of motion. Consideration of the free particle Lagrangian semi-invariance under $G$ leads to a larger symmetry group, which is a central extension of the Galileo group by the real numbers. We study the dynamics associated with this group, and characterize quantities like Noether invariants and evolution equations in terms of group geometric objects. An extension of the Galileo group by $U(1)$ leads to quantum mechanics.
1 Introduction

The importance of symmetry in physics can hardly be stressed [1]. We present a formalism that exploits symmetry in order to describe certain classical and quantum systems. The fundamental object in our approach is a Lie group that represents the physical system under consideration. All dynamical quantities (Noether invariants, evolution equations ...) are canonically defined in terms of geometric quantities.

This formalism is akin in language to Geometric Quantization [2], whose goal is to build a quantum theory from a classical one: a Hilbert space where quantum operators reproduce the classical algebra of the Poisson brackets. But there is a crucial difference in philosophy. We do not assume the existence, a priori, of any classical system: quantum or classical nature are determined, as we shall see in this work, by the geometric structure of the group [3].

We have chosen the free particle as a simple example to present our ideas. Since our aim is to write an introductory review, we do calculations in detail, using coordinates rather than more elegant and concise notation. Quantities and concepts are introduced gradually. Although sometimes this may seem very ad-hoc, we think that readers not familiar with differential geometry or group theory will be able to follow our reasonings.

First, we define the Galileo group [4]. Although the equations of motion of the free particle are Galilean invariant, it is not possible to construct a Lagrangian strictly invariant under Galilean transformations. We introduce then the concept of semi-invariance.

The equivalence of the Lagrangians \( \mathcal{L} \) and \( \mathcal{L}' = \mathcal{L} + dF/dt \), where \( F \) is an arbitrary function of coordinates and time, leads to the postulate of a new symmetry group. It is remarkable that the infinitesimal transformations of this group can be found by considering the action of the Galileo group on the arbitrary function \( F(x, t) \). In an abuse of language, we have called this symmetry hidden because it is not a symmetry involving the dynamical variables of the particle, coordinate and velocity. Nevertheless, it is related to its mass in a subtle way.

The next section is devoted to the study of the new symmetry group. We find that this group is a central extension of the Galileo group by the real numbers. Central extensions have a natural fiber bundle structure. We introduce the canonical form \( \Theta \), a connection in the bundle. Symmetries of \( \Theta \) define Noether invariants and the classical equations of motion for the particle. Finally, the Hamilton-Jacobi equation is obtained. Symplectic structure and Poisson brackets are discussed in
Historically ([5], [6]), the phase invariance of the scalar product for two quantum mechanical wave functions

\[ \langle \psi | \chi \rangle = \int dx \psi^*(x) \chi(x) \]

under a $U(1)$ transformation

\[ \psi \mapsto \zeta \psi, \quad \chi \mapsto \zeta \chi \quad / \quad \zeta \in U(1) \]

led to the consideration of ray representations of Lie groups: representations defined up to a phase. This is the motivation for the last section of this paper, where we extend the Galileo group by $U(1)$. The Schrödinger equation and the Heisenberg commutation relations are obtained. We construct the Hilbert space of wave functions. The quantum operators (position, momentum) acting on this Hilbert space are given in terms of the right invariant generators of the group. Finally, we discuss the correspondence between the classical and quantum theories.

The reader familiar with Geometric Quantization will recognize all its basic ingredients ($\Theta$ is the quantization form in a contact manifold, $d\Theta$ is a presymplectic form, etc. [3]). Nevertheless, this group quantization formalism, besides providing a canonical definition of dynamical quantities based solely on group invariance, solves one essential difficulty in Geometric Quantization: the characterization of a polarization (constraint) that reduces the representation of the symmetry group in the physical Hilbert space. Since a group structure is present, a polarization can be defined by considering a maximal subalgebra of group generators, as shown in sections 6 and 7.

There are two appendices at the end of this paper. The first one presents basic concepts of differential geometry (vector fields, forms, integral curves ...), while the second deals with the definition of the Poisson brackets (symplectic structure) and the formulation of Noether’s theorem in this group theoretical language.
2 The Galileo Group

Newton’s equations of motion for a free particle of mass \( m \)
\[
\frac{d^2 x}{dt^2} = 0
\]
(1)
can be decomposed into two first order equations
\[
\frac{dx}{dt} = v \\
m \frac{dv}{dt} = 0
\]
(2)
where the position \( x \) and the velocity \( v \) are three-dimensional real vectors. Equations (2) are invariant under the following Galilean transformations ([3], [4], [5], [6])
\[
x \rightarrow x' = Rx + x_0 + v_0 t \\
v \rightarrow v' = Rv + v_0 \\
t \rightarrow t' = t + t_0
\]
(3)
where \((x_0, v_0, t_0)\) are constants and \( R \) is a rotation matrix, \( R \in SO(3) \). Rotations complicate the notation unnecessarily without playing any role in our discussion, so in the following only Galilean transformations in one dimension, the line defined by the motion of the particle, will be considered.
\[
x \rightarrow x' = x + x_0 + v_0 t \\
v \rightarrow v' = v + v_0 \\
t \rightarrow t' = t + t_0
\]
(4)
We can write (4) in matrix form by introducing a four component column vector [7] with an extra 1 as the fourth component.
\[
\begin{pmatrix}
x' \\
v' \\
t' \\
1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & v_0 & x_0 \\
0 & 1 & 0 & v_0 \\
0 & 0 & 1 & t_0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
v \\
t \\
1
\end{pmatrix}
\]
(5)
The Galilean transformations form a group, the Galileo group \( G \): the composition law for the group parameters \((x_0, v_0, t_0)\) is obtained by multiplying two Galilean matrices. We suppress the index 0 in order to simplify notation, and distinguish different group elements by primes (’). Then
\[
\begin{pmatrix}
1 & 0 & v'' & x'' \\
0 & 1 & 0 & v'' \\
0 & 0 & 1 & t'' \\
0 & 0 & 0 & 1
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & v' & x' \\
0 & 1 & 0 & v' \\
0 & 0 & 1 & t' \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & v & x \\
0 & 1 & 0 & v \\
0 & 0 & 1 & t \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
(6)
implies the following composition law for $G$:

\[
\begin{align*}
x'' &= x' + x + v't \\
v'' &= v' + v \\
t'' &= t' + t
\end{align*}
\]

(7)

3 Lagrangian semi-invariance

Since the equations of motion (2) are invariant under $G$, we expect that the free particle will be described by a Galilean invariant Lagrangian $L(x, v, t)$. Let us check this assumption.

Under a Galilean transformation (4), the Lagrangian changes to

\[
L(x, v, t) \Rightarrow L(x + x_0 + v_0 t, v + v_0, t + t_0) \equiv L(x', v', t')
\]

(8)

It is enough to demand invariance under infinitesimal Galilean transformations. Then $(x_0, v_0, t_0)$ are very small and one can expand in a Taylor series

\[
L(x', v', t') \simeq L(x, v, t) + (x_0 X_x^R + t_0 X_t^R + v_0 X_v^R)L(x, v, t) + \ldots
\]

(9)

where

\[
X_x^R = \frac{\partial}{\partial x}, \quad X_t^R = \frac{\partial}{\partial t}, \quad X_v^R = t \frac{\partial}{\partial x} + \frac{\partial}{\partial v}
\]

(10)

are infinitesimal Galilean operators. Invariance under these transformations gives

\[
X_x^R L = 0 \quad \Rightarrow \quad L(x, v, t) = L(v, t)
\]

\[
X_t^R L = 0 \quad \Rightarrow \quad L(x, v, t) = L(v)
\]

\[
X_v^R L = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial v} = 0
\]

(11)

The most general Galilean invariant Lagrangian is a constant!.

The solution of this apparent puzzle is the following: since the equations of motion derived from a Lagrangian $L$ do not change if we add the time derivative of an arbitrary function $F(x, t)$, strict invariance is a too stringent condition. The Lagrangians $L$ and $L + dF/dt$ must be considered equivalent. We only need semi-invariant Lagrangians that change in a total derivative under an infinitesimal action of a symmetry group.

Define functions $f^{(v)}$, $f^{(t)}$ and $f^{(x)}$ of $(x, t)$, such that

\[
X_v^R L = \frac{df^{(v)}}{dt}, \quad X_t^R L = \frac{df^{(t)}}{dt}, \quad X_x^R L = \frac{df^{(x)}}{dt}
\]

(12)
Semi-invariance under the infinitesimal velocity generator, $X^R_v$, implies
\[
\frac{\partial L}{\partial x} + \frac{\partial L}{\partial v} = v \frac{\partial f^{(v)}}{\partial x} + \frac{\partial f^{(v)}}{\partial t}
\]
\[\implies L(x, v, t) = \alpha(x, t)v^2 + \beta(x, t)v + \gamma(x, t)\tag{13}\]
where $\alpha(x, t), \beta(x, t), \gamma(x, t)$ are functions to be determined. Semi-invariance under time translations further restricts $L$
\[
X^R_t L = \frac{df(t)}{dt}
\]
\[\implies \frac{\partial L}{\partial t} = v^2 \frac{\partial \alpha}{\partial t} + v \frac{\partial \beta}{\partial t} + \frac{\partial \gamma}{\partial t} = v \frac{\partial f^t}{\partial x} + \frac{\partial f^t}{\partial t}
\]
\[\implies \frac{\partial \alpha}{\partial t} = 0 \implies \alpha(x, t) = \alpha(x)\tag{14}\]
and proceeding in the same way
\[
X^R_x L = \frac{df^{(x)}}{dt}
\]
\[\implies \frac{\partial \alpha}{\partial x} = 0 \implies \alpha(x, t) = \text{constant} \equiv \frac{1}{2}m\tag{15}\]
The most general Lagrangian semi-invariant under $G$ is then
\[
L(x, v, t) = \frac{1}{2}mv^2 + \beta v + \gamma\tag{16}\]
where we have chosen the constant to be $(1/2)m$ so that the usual Lagrangian $L = (1/2)mv^2$ is obtained, plus some additional terms. According to our previous discussion on semi-invariance, we expect these terms to be a total derivative. From (16) and (12)
\[
\frac{\partial f^{(t)}}{\partial t} = \frac{\partial \gamma}{\partial t} \quad \frac{\partial f^{(t)}}{\partial t} = \frac{\partial \beta}{\partial t}
\]
\[\frac{\partial f^{(x)}}{\partial t} = \frac{\partial \gamma}{\partial x} \quad \frac{\partial f^{(x)}}{\partial x} = \frac{\partial \beta}{\partial x}\tag{17}\]
Integrability of equations (17) implies
\[\beta(x, t) = \frac{\partial F(x, t)}{\partial x} \quad \gamma = \frac{\partial F(x, t)}{\partial t}\tag{18}\]
for some arbitrary differentiable function $F(x, t)$. The semi-invariant Lagrangian $L$ is then
\[
L(x, v, t) = \frac{1}{2}mv^2 + v \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} \equiv \frac{1}{2}mv^2 + \frac{dF}{dt}\tag{19}\]
As we expected. However, there is something important to be learned from this well known result (in fact, this is our justification for our long calculations): because semi-invariant Lagrangians are defined up to a total derivative, equivalence classes of Lagrangians, not individual Lagrangians, are the relevant quantities in classical mechanics.
4 A new symmetry

We can learn more about the connection between symmetry and Lagrangian semi-invariance by studying how the arbitrary function $F(x, t)$ in (19) transforms under $G$.

It is straightforward to verify from (12), (18) and (19) that

$$
\begin{align*}
  f(t) &= \frac{\partial F}{\partial t} , \\
  f(x) &= \frac{\partial F}{\partial x} , \\
  f(v) &= t \frac{\partial F}{\partial x} + m x = t \frac{\partial F}{\partial x} + \frac{\partial F}{\partial v} + m x 
\end{align*}
$$

We can rewrite the previous expressions as (see (10))

$$
\begin{align*}
  f(t) &= X_R^t F , \\
  f(x) &= X_R^x F , \\
  f(v) &= X_R^v F + m x 
\end{align*}
$$

The semi-invariant functions $f(v), f(t)$ and $f(x)$ can be expressed as infinitesimal Galilean transformations acting on an arbitrary function $F(x, t)$, except for an extra linear piece $m x$ that appears in $f^v$.

The crucial observation is that, with the introduction of a new real variable $\theta$, (21) can be written as a set infinitesimal variations acting on $F$. Consider functions of the form (see (50))

$$
F(x, t) = F(x, t) + \theta 
$$

and define the following vector fields

$$
\begin{align*}
  \tilde{\theta} &= \frac{\partial}{\partial \theta} , \\
  \tilde{X}_t^R &= \frac{\partial}{\partial t} , \\
  \tilde{X}_x^R &= \frac{\partial}{\partial x} , \\
  \tilde{X}_v^R &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial v} + m x \tilde{\theta} 
\end{align*}
$$

Equations (20) read now

$$
\begin{align*}
  f(t) &= \tilde{X}_t^R F , \\
  f(x) &= \tilde{X}_x^R F , \\
  f(v) &= \tilde{X}_v^R F 
\end{align*}
$$

Moreover, these transformations form an algebra ([,] is the Lie bracket)

$$
\begin{align*}
[\tilde{X}_t^R, \tilde{X}_x^R] &= 0 , \\
[\tilde{X}_t^R, \tilde{X}_v^R] &= \tilde{X}_x^R , \\
[\tilde{X}_x^R, \tilde{X}_v^R] &= m \tilde{\theta} 
\end{align*}
$$

with $\tilde{\theta}$ commuting with every generator. In contrast, the Galilean commutators are

$$
\begin{align*}
[X_t^R, X_x^R] &= 0 , \\
[X_t^R, X_v^R] &= X_x^R , \\
[X_x^R, X_v^R] &= 0 
\end{align*}
$$

The relations (25) indicate the existence of a new symmetry group for the free classical particle. Since this symmetry does not involve the Galilean variables $(x, v, t)$ and is a consequence of the equivalence of the Lagrangians $L$ and $L + dF/dt$, we expect its implications to be very general, far beyond the simple system we are considering here.
5 Central Extension of the Galileo Group

The new commutators (25) are related to (26) by a mathematical structure of crucial importance in physics, the central extension of a group.

Consider two groups \( G_1 \) and \( G_2 \). One can obtain a larger group by taking the cartesian product of \( G_1 \) and \( G_2 \): if \( a, b \in G_1 \) and \( y, z \in G_2 \), one defines a new group with composition law \( \otimes \)

\[
(a, y) \otimes (b, z) = (a \star b, y \odot z)
\]

where \( \star \) and \( \odot \) are the composition laws of \( G_1 \) and \( G_2 \). This is called the direct product of \( G_1 \) and \( G_2 \). There is no new information in a direct product. In physics, one encounters direct products of groups in the study of non-interacting systems.

A more interesting way of combining the two groups is

\[
(a, y) \tilde{\otimes} (b, z) = (a \star b, y \odot z \odot \epsilon(a, b))
\]

where \( \epsilon : G_1 \mapsto G_2 \) is a function whose properties will be described later. The composition law (28) corresponds to the semi-direct product, or extension, of \( G_1 \) by \( G_2 \). Notice that \( G_1 \) is not a subgroup of \( G_1 \tilde{\otimes} G_2 \), but \( G_2 \) is always an invariant subgroup. If \( G_2 \) is abelian this is called a central extension.

The commutation relations (25) show that this is just the structure obtained after enlarging the Galileo group with the new variable \( \theta \): the generators (23) can be derived from the Galilean composition law (7) and an additional line of the form

\[
\theta'' = \theta' + \theta + \epsilon(x', v', t'; x, v, t)
\]

Because \( \tilde{\theta} \) commutes with all generators, the extension must be central. Since \( \theta \) is real and has an additive composition law, this is called a central extension of \( G \) by the real numbers \( R, G_R \).

The function \( \epsilon \) in (28) and (29) is called a cocycle. Cocycles are restricted by the group law properties: associativity and existence of an inverse [5]. Let \( a, b, c \in G_1 \). Associativity of the group law

\[
a \star (b \star c) = (a \star b) \star c
\]

implies that \( \epsilon \) must satisfy the following functional equation

\[
\epsilon(a, b) \odot \epsilon(a \star b, c) = \epsilon(a, b \star c) \odot \epsilon(b, c)
\]

Consider now an arbitrary function \( f(a) : G_1 \mapsto G_2 \), and define [11]

\[
\delta_c f(a, b) = f(a \star b) \odot (f(a))^{-1} \odot (f(b))^{-1}
\]
One can check that $\delta_c f$ fulfills (30). But $\delta_c f$ amounts to a simple change of coordinates in $G_2$

$$z \rightarrow z \circ f(a) \quad z \in G_2$$

These special cocycles, called coboundaries, do not define any true extension, but rather a direct product \[12\]. They are equivalent to the trivial cocycle $\epsilon(a, b) = 0$.

We define an extension $G_R$ by the cocycle \[5\]

$$\epsilon(x', v', t'; x, v, t) = m(v'x + \frac{1}{2}tv'^2) \quad \text{(32)}$$

The composition law for $G_R$ is then

$$x'' = x' + x + v't$$

$$v'' = v' + v$$

$$t'' = t' + t$$

$$\theta'' = \theta' + \theta + m(v'x + \frac{1}{2}tv'^2) \quad \text{(33)}$$

One obtains the expression (23) for the right-invariant generators of the group. The left-invariant generators \[13\] correspond to variation with respect to unprimed parameters in (33)

$$X_{\theta}^L = \frac{\partial}{\partial \theta}, \quad \tilde{X}_{x}^L = \frac{\partial}{\partial x} + mv\tilde{\theta}, \quad \tilde{X}_{t}^L = \frac{\partial}{\partial t} + \frac{1}{2}mv^2\tilde{\theta}, \quad \tilde{X}_{v}^L = \frac{\partial}{\partial v} \quad \text{(34)}$$

If we have a particular central extension characterized by a cocycle $\epsilon$, and we perform a change of coordinates in the group, we obtain the new cocycle $\epsilon' = \epsilon + \delta_c f$, where $\delta f$ is the coboundary (31) associated to the change of coordinates. This means that two cocycles $\epsilon$ and $\epsilon'$ must be considered equivalent if they differ in a coboundary. For instance, in his study of the Galileo group, Bargmann \[5\] considered the cocycle

$$\epsilon_B(x', v', t'; x, v, t) = -\frac{1}{2}m(x'v - v'x + vv't) \quad \text{(35)}$$

which can be obtained from (32) by adding the coboundary generated by

$$f(x, v, t) = -\frac{1}{2}m xv \quad \text{(36)}$$

This equivalence relation defines an important concept, the cohomology of a group \[5\].

Use of (35) instead of (32) in the extended group law (33) would change the expression for the $\tilde{\theta}$ dependent part of the generators. This should not affect the dynamical description of the free particle, because adding a coboundary to the group law amounts to replace a Lagrangian $L$ for $L + dF/dt$. 

9
It is important to realize that the cocycle $\bar{\epsilon} = \bar{m}(v'x + \frac{1}{2}v'^2)$, with $\bar{m} \neq m$, is not equivalent to $\epsilon$, since $\epsilon - \bar{\epsilon}$ is not a coboundary: the mass of the particle has a cohomological significance, it parametrizes the extensions of the Galileo group.

6 Geometrical structure

6.1 Fiber Bundle structure. The canonical form $\Theta$

Our central extension $G_R$ singularizes the transformations generated by $\tilde{\theta}$: two points $a, b \in G_R$ that differ in the value of the $\theta$ variable are equivalent and give the same description of the free particle, since $\theta$ is not a measurable quantity, like position or velocity. In some sense, $\theta$ is irrelevant. Nevertheless, as in the case for gauge invariance in electromagnetism, the requirement that $\theta$ must not be observable has a profound dynamical content and is responsible for a geometrical structure, a fiber bundle, which is ubiquitous in many areas of physics.

First, note that although $G$ is not a subgroup of $G_R$, $G_R/R \simeq G$ as topological spaces: any $g \in G_R$ can be decomposed in two parts, $g = (a, \theta)$, where $a \equiv (x, v, t) \in G$ and $\theta \in R$. This defines a projection $\pi : G_R \mapsto G$ / $\pi(x, v, t, \theta) = (x, v, t)$.

The triplet $(G_R, R, \pi)$ is a fiber bundle ([14], [15]): $G_R/R$ is called base manifold, $R$ is the fiber and $\pi$ is the projection from $G_R$ to the base manifold. Since the fiber $R$ has a group structure, we say that $(G_R, R, \pi)$ is a fiber bundle with structural group $R$.

One can visualize such geometric object by imagining that through each point in the base manifold there is a line (fiber) of points with different values of the fiber coordinate $\theta$. It is natural then to call vertical any transformation or quantity that involves only $\theta$, whereas quantities defined on the base manifold $G$ are called horizontal.

Since we have a group structure, it is possible to define verticality or horizontality in an invariant way. Consider [3] the dual form of $\tilde{\theta}$

$$\Theta(\tilde{X}^L) = 0 \quad \Theta(\tilde{\theta}) = 1$$

where $\tilde{X}^L$ stands for all left-invariant generators other than $\tilde{\theta}$. $\Theta$ is the vertical part of the canonical left form ([14], p. 83). We will call $\Theta$ just canonical form, for brevity.

Geometrically, $\Theta$ is a connection ([14], [15]) in the fiber bundle, and its differential $\omega = d\Theta$ is a curvature form. A quick calculation gives

$$\Theta = -mv \, dx + \frac{1}{2}mv^2 \, dt + d\theta$$

(38)
Why do we consider such an object? We give an intuitive argument: $\Theta$ is blind to the physical part (the Galileo group, the horizontal direction) of the bundle, that is, $\Theta$ is zero on any linear combination of (left) pure Galilean generators. Symmetries of $\Theta$, in a sense that will be made clear later, must be or must originate physical quantities. This is substantiated by the following: restrict $\Theta$ to the base manifold $G$

$$\Theta_{|G} = -mv \, dx + \frac{1}{2}mv^2 \, dt$$  \hspace{1cm} (39)

Rewrite the above in the form

$$\Theta_{|G} = -mv (dx - v \, dt) - \frac{1}{2}mv^2 \, dt$$  \hspace{1cm} (40)

Then, along trajectories such that $dx = v dt \implies v = dx/\dot{t}$

$$\Theta_{|G} = -\mathcal{L} dt$$  \hspace{1cm} (41)

where $\mathcal{L}$ is the free Lagrangian. Hence, the quantity

$$S = \int \Theta_{|G}$$  \hspace{1cm} (42)

is analogous to the classical action, or more precisely, to the Poincaré-Cartan form of analytical mechanics [16].

### 6.2 Symmetries

A transformation $X$ is a symmetry of $\Theta$ if $X$ leaves $\Theta$ semi-invariant, that is, the Lie derivative (see appendix I) of $\Theta$ with respect to the considered transformation $X$ is a total differential

$$L_X \Theta = df^X \implies d(i_X \Theta) + i_X (d\Theta) = df^X$$  \hspace{1cm} (43)

where $f^X$ is some function associated with $X$. From (38)

$$\omega = d\Theta = m dx \wedge dv + mvdv \wedge dt$$  \hspace{1cm} (44)

The characteristic module of $\Theta$, $C_{\Theta} = \ker \Theta \cap \ker d\Theta$ is the set of transformations that leave $\Theta$ strictly invariant.

$$d(i_X \Theta) = 0 \ , \ i_X (d\Theta) = 0 \ \ \forall X \in C_{\Theta}$$  \hspace{1cm} (45)

It is straightforward to check that the time translation generator $\dot{X}_t^L$ is the only generator of $C_{\Theta}$. Its integral curves (appendix I) give the equations of motion for the free particle.

$$X_t^L = \frac{\partial}{\partial t} + v \frac{\partial}{\partial x} + \frac{1}{2}mv^2 \dot{\theta} \implies \frac{dt}{ds} = 1 \ , \ \frac{dx}{ds} = v \ , \ \frac{dv}{ds} = 0 \ , \ \frac{d\theta}{ds} = \frac{1}{2}mv^2$$  \hspace{1cm} (46)
where \( s \) is a parameter. Then

\[
x(s) = x_0 + v_0 s, \quad v(s) = v_0 \equiv \frac{p}{m}, \quad t(s) = s, \quad \theta(s) = Es
\]

where \( E \equiv (1/2)mv_0^2 = p^2/2m \). In appendix B, we show how the characteristic module defines the classical Poisson bracket structure ([3], [16], [17]).

While the free particle Lagrangian is only semi-invariant under Galilean transformations, \( \Theta \) is invariant under the right generators (10) of \( G_R \): \( L_{X^R} \Theta = 0 \) with \( X^R \) any right generator of \( G_R \). This is the basis [17] of the group formulation of Noether’s theorem: we prove in appendix II that the functions \( h_{X^R} = \Theta(X^R) \) are constants of motion (constant on the trajectories (47)). For instance

\[
\Theta(\tilde{X}_{X^R}) = (-mv dx + \frac{1}{2} mv^2 dt + d\theta)(\frac{\partial}{\partial x}) = -mv(\frac{\partial x}{\partial x}) + \frac{1}{2} mv^2(\frac{\partial t}{\partial x}) + (\frac{\partial \theta}{\partial x}) = -mv \equiv -p
\]

Also

\[
\Theta(\tilde{X}_{v^R}) = -E \quad \Theta(\tilde{X}_{t^R}) = m(x - vt) = mx_0
\]

As it should, translation invariance implies momentum conservation and time invariance is responsible for energy conservation. We find also that velocity invariance results in the conservation of the initial position \( x_0 \).

Now, we build invariant functional spaces. Define, analogously as in (22)

\[
\mathcal{F} = \{ \Psi : G_R \rightarrow R \mid \Psi(\theta + g) = \Psi(g) + \theta \quad \forall g \in G_R \}
\]

In coordinates

\[
\mathcal{F} = \{ \Psi : G_R \rightarrow R \mid \Psi(x, v, t, \theta) = \Psi(x, v, t) + \theta \}
\]

The definition (50) reflects the triviality of the \( \theta \) dependence for physical quantities. Notice that \( \hat{\theta} \Psi = 1 \quad \forall \Psi \in \mathcal{F} \). We further constrain \( \mathcal{F} \) by imposing invariance under \( G_R \) transformations. We would like \( \mathcal{F} \) to be invariant under the generator of \( \mathcal{C}_\theta \), since its integral curves define the equations of motion. What other constraints can we impose? If one wants a maximal set of constraints on \( \Psi \) (this is called a polarization in the Geometric Quantization formalism), compatibility is the only requirement: it is not possible to have invariance under transformations (differential operators) that do not close an algebra. The generator of the characteristic module is \( \tilde{X}_L^L \), and \( \hat{\theta} \) commutes with all generators. By (27), we see that

\[
\mathcal{P} = \{ \tilde{X}_L^L, \tilde{X}_L^x \}
\]
is the only acceptable possibility: if we include ˜$X^L$, then its commutator with ˜$X^L$ will be proportional to ˜$\theta$, and ˜$\theta\Psi = 0$ is incompatible with the definition of $F$. $P$ is called the polarization subalgebra, and

$$F_P = \{ \Psi \in F \mid X\Psi = 0 \ \forall X \in P \}$$

(53)
is the space of polarized functions [18].

We have

$$\hat{X}_t^L\Psi = 0 \implies \frac{\partial \Psi}{\partial t} + v\frac{\partial \Psi}{\partial x} + \frac{1}{2}mv^2 = 0 \implies \Psi(x,v,t,\theta) = -mvx - S(v,t)$$

where $p = mv$. We recognize the Hamilton-Jacobi equation in momentum space ([17],[19])

$$\frac{\partial S}{\partial t} + H(p, \frac{\partial S}{\partial p}) = 0$$

(55)
The term in $\partial S/\partial p$ is absent due to the simple form of the Hamiltonian $H(p,x) = H(p) = p^2/2m$.

7 Quantum mechanics

Classical and quantum mechanics are beautifully related in our group invariance formalism (5, 3). First, we note that the cocycle $\epsilon$ in (29), and hence $\theta$, has dimensions of action. Let’s introduce $\hbar$ explicitly

$$\theta'' = \theta' + \theta + \frac{1}{\hbar}m(v'x + \frac{1}{2}tv'^2)$$

(56)

$\theta$ is now dimensionless. We can consider the above line as the uncompactification of a U(1) multiplicative law

$$\zeta \equiv e^{-i\theta} \in U(1)$$

(57)

and obtain an extension of $G$ by U(1), $\tilde{G}$

$$x'' = x' + x + v't$$

$$v'' = v' + v$$

$$t'' = t' + t$$

$$\zeta'' = \zeta'\zeta \exp \left( -\frac{i}{\hbar}m(v'x + \frac{1}{2}tv'^2) \right)$$

(58)
Note that \( \theta \) in (57) is not restricted to the interval \([0, 2\pi)\).

We can apply now the formalism developed in the last sections in order to study the dynamics defined by \( \tilde{G} \). The expressions for the generators and canonical form \( \Theta \) are

\[
\begin{align*}
\tilde{X}_x^L &= \frac{\partial}{\partial x} - \frac{1}{\hbar} m v \Xi \\
\tilde{X}_x^R &= \frac{\partial}{\partial x} \\
\tilde{X}_v^L &= \frac{\partial}{\partial v} - \frac{1}{2\hbar} m v^2 \Xi \\
\tilde{X}_v^R &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial v} - \frac{1}{\hbar} m x \Xi \\
\tilde{X}_\zeta^L &\equiv \Xi = i \zeta \frac{\partial}{\partial \zeta} \\
\tilde{X}_\zeta^R &\equiv \Xi = i \zeta \frac{\partial}{\partial \zeta}
\end{align*}
\]

(59)

\[
\begin{align*}
\Theta &= -mv dx + \frac{1}{2} mv^2 dt - \frac{i}{\zeta} d\zeta
\end{align*}
\]

(61)

As in the case of \( G_R \), \( C_\Theta \) is generated by the (left) time translations. The quantum equations of motion are the integral curves of \( \tilde{X}_t^L \)

\[
\begin{align*}
\frac{dt}{ds} &= 1 \\
\frac{dv}{ds} &= 0 \\
\frac{dx}{ds} &= v \\
\frac{d\zeta}{ds} &= \frac{i}{\hbar} \zeta \\
\implies t &= s \\
v &= \frac{p}{m} \\
x &= x_0 + \frac{p}{m} t \\
\zeta &= \zeta_0 \exp(\frac{i}{\hbar} Et) \\
E &= \frac{p^2}{2m}
\end{align*}
\]

(62)

(63)

The polarization subalgebra is

\[
\mathcal{P} = \{ \tilde{X}_x^L, \tilde{X}_\zeta^L \}
\]

(64)

formally the same as (52). In analogy with (50) and (53), we consider complex polarized functions

\[
\mathcal{F}_P = \{ \Psi(x,v,t,\zeta) : \tilde{G} \mapsto \mathbb{C} \mid \Psi(x,v,t,\zeta) = \zeta \Psi(x,v,t) , \ X \Psi = 0 \ \forall X \in \mathcal{P} \}
\]

(65)

Note that \( \Xi \Psi = i \Psi \)

Invariance under \( \mathcal{P} \) gives the following equations for \( \Psi \)

\[
\tilde{X}_x^L \Psi = 0 \implies \frac{\partial}{\partial x} \Psi - \frac{i}{\hbar} p \Psi = 0
\]

(66)

\[
\tilde{X}_v^L \Psi = 0 \implies \frac{\partial}{\partial v} \Psi + \frac{p}{m} \frac{\partial}{\partial x} \Psi - \frac{i}{\hbar} \frac{p^2}{2m} \Psi = 0
\]

(67)

The identification of the momentum operator

\[\hat{p} = -i\hbar \frac{\partial}{\partial x}\]

(68)

is straightforward from (66). Combining (66) and (67), we obtain

\[i\hbar \frac{\partial}{\partial t} \Psi = \frac{p^2}{2m} \Psi \implies i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi , \ \hat{H} = \frac{\hat{p}^2}{2m}
\]

(69)
which is the Schrödinger equation for the free quantum particle wave function \( \Psi \) in the momentum representation. This is to be compared with equation (54), the classical Hamilton-Jacobi equation.

A basis for the solutions of (69) are plane waves modulated by an amplitude \( \psi(p) \)

\[
\Psi = \zeta \exp \left( -\frac{i}{\hbar}(Et - px) \right) \psi(p)
\]  

(70)

Commutativity of the right and left generators makes the right invariant fields good candidates for quantum operators: because \( P \) is spanned by left invariant fields, if \( X^R \) is a right generator and \( \Psi \) is polarized, then

\[
X^P \Psi' = X^P (X^R \Psi) = X^R (X^P \Psi) = 0 \quad \forall X^P \in P, \quad \forall \Psi \in \mathcal{F}_P
\]  

(71)

so \( \Psi' \) is polarized and it has the general form (70). Hence, the action of the right invariant generators on \( \mathcal{F}_P \) is well defined. Note that the momentum operator in (68) can be written as

\[
\hat{p} = -i\hbar \tilde{X}_x^R
\]  

(72)

Consider the action of \( \hat{p} \) and

\[
\hat{x} \equiv \frac{i\hbar}{m} \tilde{X}_x^R
\]  

(73)

on the amplitudes \( \psi(p) \)

\[
\hat{p} : \psi \mapsto \psi' / \hat{p}(\Psi) = \zeta \exp \left( -\frac{i}{\hbar}(Et - px) \right) \psi'(p)
\]  

(74)

and analogously for \( \hat{x} \). After a short calculation, one obtains

\[
\hat{p} : \psi(p) \mapsto p \psi(p), \quad \hat{x} : \psi(p) \mapsto i\hbar \frac{\partial}{\partial p} \psi(p)
\]  

(75)

These are the quantum momentum and position operators in the momentum representation, with the Heisenberg commutation relations

\[
[\hat{x}, \hat{p}] = i\hbar
\]  

(76)

If the functions \( \psi \) are square integrable, one can define the scalar product in the usual way

\[
\langle \Psi_1 | \Psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle = \int dp \psi_1^*(p) \psi_2(p)
\]  

(77)

Note that the measure \( dp \) is (proportional to) the dual form of the generator not present in the polarization \( P, \tilde{X}_v^L \).
This ends our analysis of the quantum mechanics of the free particle, given by the extension of the Galileo group by $U(1)$. Since classical mechanics is described by the extension of $G$ by $R$, $G_R$, there is, as promised, a beautiful characterization for the classical limit of a quantum theory in this group formalism: the classical theory is obtained by opening the $U(1)$ (a circle) in the group extension to $R$ (a line).

8 Conclusions

We have constructed, step by step, a group theoretical formalism that can describe physical systems possessing a dynamical group with a fiber bundle structure. We found that for the particular examples of the Galileo extensions $\tilde{G}$ and $G_R$ one obtains the quantum and classical mechanics of a free particle, with dynamical quantities like Noether invariants, operators and evolution equations obtained from geometrical objects canonically defined in the group.

Here are the more relevant features of this formalism

- The physical system is described by a central extension of a Lie group, which has a natural fiber bundle structure.
- Noether invariants are given by symmetries of the canonical left-form $\Theta$, dual to the extension parameter. $\int \Theta$ is analogous to the action in classical mechanics.
- The characteristic module of $\Theta$, $C_\Theta$, determines the equations of motion. The symplectic structure (Poisson brackets) is obtained when $d\Theta$ is restricted to $C_\Theta$.
- Evolution equations and Hilbert space of states are defined by a (horizontal) maximal set of constraints, the polarization subalgebra, built from the left-invariant generators.
- The right invariant generators are well defined operators in the space of polarized functions.

Although we have checked the points above in our simple example, we have not proven them generally. Rigorous proofs can be found in [3] and [17].

Obviously, not every conceivable physical system has this geometric structure, or even if that were true, finding the corresponding dynamical group for an arbitrary system would be extremely difficult. But since symmetry plays such a crucial role in the foundations of physics, we believe that
it is of interest to see how much information can be extracted from symmetry considerations alone. And with the added bonus of aesthetic beauty.

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9 Appendix I

9.1 Vector fields

In this appendix we give concise definitions of some geometric objects, oriented towards explicit calculations so that the reader could follow the computations in the main text. More comprehensive definitions can be found in introductory books on differential geometry ([14], [15]).

Let \( M \) be a \( N \)-dimensional manifold with local coordinates \( x_i \), \( i = 1, 2 \ldots N \). A vector field \( X \) is an application that associates a first order differential operator \( X(x) \) to a point \( x \in M \). \( X(x) \) can be expressed in local coordinates as a linear combination of the base fields

\[
e_i = \frac{\partial}{\partial x_i}
\]

That is

\[
X(x) = \sum_{i=1}^{N} X_i(x) \frac{\partial}{\partial x_i}
\]

with \( X_i(x) \), \( i = 1, 2 \ldots N \), differentiable functions on \( M \). The space of the vector fields is called the tangent space of \( M \), \( T(M) \). It is cumbersome to write \( X(x) \), so we will write just \( X \).

The integral curves of \( X \) are the solution to the set of ordinary differential equations

\[
\frac{dx_i}{ds} = X_i(x(s)) \quad i = 1, 2 \ldots N
\]

where \( s \) is an integration parameter. For instance, in a \( N = 2 \) dimensional manifold, the integral curves for the vector field

\[
Y = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}
\]

are given by the differential equations

\[
\frac{dx_1}{ds} = x_2 \implies \frac{dx_2}{ds} = -x_1
\]

\[
\implies x_1(s) = a \sin(s) + b \cos(s) \quad x_2(s) = a \cos(s) - b \sin(s)
\]

with \( a \) and \( b \) constants. Note that the invariance condition \( Xf = 0 \) implies that \( f \) is constant along the integral curves of \( X \). In our case, the general solution of \( Yf = 0 \) is \( f = f(x_1^2 + x_2^2) = f(a^2 + b^2) = \text{constant} \).
9.2 \textit{forms}

A 1-form $\Gamma$ is an application that associates to every point $x \in M$ an element of the dual space of $T(M)$.

$$\Gamma : x \mapsto \Gamma(x) \quad / \quad \Gamma(x)(X(x)) = f(x)$$

(82)

with $f(x)$ a differentiable function. As with vector fields, we write $\Gamma$ for $\Gamma(x)$. The space of 1-forms is called the cotangent space of $M$ and is denoted by $T^*(M)$.

A convenient representation for the basis of $T^*(M)$ is

$$u_j \equiv dx_j \quad , \quad j = 1, 2 \ldots N$$

(83)

and its action on the basis of $T(M)$, $e_i = \partial/\partial x_i$

$$u_j(e_i) \equiv dx_j(\partial/\partial x_i) = \partial x_i / \partial x_j = \delta_{i,j}$$

(84)

2-forms, 3-forms, etc. are linear combinations of tensor products ($\otimes$) of $u_j$. A function $f(x)$ is considered a zero-form.

An important operation on forms is the differential $d$. The differential of a $n$-form is a $(n+1)$-form (or zero). For a function $f$

$$df = \frac{\partial f}{\partial x_i} dx_i$$

which is the definition of the ordinary differential, For a 1-form $\Gamma = \Gamma_i dx_i$

$$d\Gamma = \frac{\partial \Gamma_i}{\partial x_j} dx_i \wedge dx_j$$

where the wedge operator $\wedge$ is the antisymmetric combination $dx_i \wedge dx_j = dx_i \otimes dx_j - dx_j \otimes dx_i$. One can define in an analogous way differentials of higher order forms. Note that the antisymmetry of the wedge operator implies that $d(d\Gamma) \equiv d^2 \Gamma = 0$ and $d^2 f = 0$

If a $n$-form acts on $n - k$ vector fields one obtains a $k$-form. For instance the 1-form $df$ acting on a field $X$ gives a zero-form

$$df(X) = \frac{\partial f}{\partial x_i} dx_i(X_j(x) \frac{\partial}{\partial x_j}) = X_i \frac{\partial f}{\partial x_i} = X(f)$$

$d\Gamma$ acting on two fields $X$ and $Y$ also gives a zero-form

$$d\Gamma(X,Y) = \frac{\partial \Gamma_i}{\partial x_j} (X_i Y_j - X_j Y_i)$$
$d\Gamma$ acting on $X$ alone gives a 1-form

$$d\Gamma(X,.) = \frac{\partial \Gamma_i}{\partial x_j}(X_i dx_j - X_j dx_i)$$

We often use the inner product notation

$$i_X \Omega = \Omega(X)$$

to denote the action of a n-form $\Omega$ on a vector field $X$.

### 9.3 Lie Derivative

The Lie derivative is a generalization of the directional derivative for both vector fields and forms. The Lie derivative of a function $f$ with respect to a vector field $X$ is

$$L_X F = X(f) = df(X)$$

For two vector fields $X, Y$, the Lie derivative of $Y$ with respect to $X$, $L_X Y$, is a vector field defined by

$$L_X Y = [X,Y] = \left( X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j} = -L_Y X$$

That is, the ordinary Lie commutator.

The Lie derivative of a 1-form $\Gamma$ with respect to $X$, $L_X \Gamma$, is also a 1-form, and is defined to be the rate of change of $\Gamma$ along the flow lines of $X$ \cite{[4]}, p. 45. One finds, in local coordinates

$$L_X \Gamma = (L_X \Gamma)_i \ dx_i \quad / \quad (L_X \Gamma)_i = X_i \frac{\partial \Gamma_i}{\partial x_j} + \Gamma_j \frac{\partial X_i}{\partial x_j}$$

or, in a more concise notation

$$L_X \Gamma = d \left( \Gamma(X) \right) + (d\Gamma)(X,.) \equiv i_X d\Gamma + d(i_X \Gamma)$$
10 Appendix II

10.1 Poisson Brackets and Symplectic Structure.

We show explicitly that the restriction of $d\Theta$ to the characteristic module $C_\Theta$ provides a symplectic structure where Poisson brackets can be naturally defined. Notice that, since time translations generate $C_\Theta$, this quotient procedure eliminates the time evolution.

Consider the differential of the left-invariant form $\Theta$

$$\omega = d\Theta = mdx \wedge dv + mv dv \wedge dt$$

(88)

The restriction of $\omega$ to $C_\Theta$ (see (47)) is

$$\omega_{C_\Theta} = \bar{\omega} = m(x_0 + v_0 t) \wedge dv_0 + mv_0 dt \wedge dv + mv_0 dv_0 \wedge dt$$

$$\Rightarrow \bar{\omega} = dq \wedge dp$$

(89)

where we have defined $q \equiv x_0$, $p \equiv mv_0$. In matrix form,

$$\bar{\omega} = \bar{\omega}_{i,j} dx_i \otimes dx_j \quad / \quad x_1 = q, \quad x_2 = p, \quad \bar{\omega}_{i,j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(90)

The matrix above is the invariant metrics for the symplectic group (see [16] or chapter 9 of [19]).

The form $\bar{\omega}$ establishes a correspondence between functions $f = f(q,p)$ and vector fields: a differential operator $X_f$ is associated to a function $f$ such that

$$i_{X_f} \bar{\omega} = -df$$

(91)

in other words, $X_f$ is a symmetry (see (43)) of $\bar{\omega}$. One obtains

$$X_f = \frac{\partial f}{\partial q} \frac{\partial}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial}{\partial q}$$

(92)

The Poisson bracket $\{f,g\}$ of two functions $f,g$ is given by

$$\{f,g\} = \bar{\omega}(X_f,X_g)$$

(93)

In coordinates

$$\{f,g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} = X_f g = -X_g f$$

(94)

Notice that the integral curves for (92) are formally analogous to the Hamilton equations of motion. If $f \equiv H$ in (92), the integral curves are ($t$ is an integration parameter)

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \quad \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

(95)
Consider a hamiltonian \( H(p, q) \) and its associated \( X_H \). It is evident, from (94) that if the symmetries \( X_f \) of \( \bar{\omega} \) leave the hamiltonian \( H \) invariant, then

\[
L_{X_f}H = X^a H = 0 \implies \{ f, H \} = 0 \quad (96)
\]

Note that the correspondence (18) that associates to a function \( f \) the vector field defined in (92) is not an isomorphism: all constant functions are mapped to the vector field zero. Moreover, the vector fields associated to the functions \( q, p \)

\[
X_q = \frac{\partial}{\partial p}, \quad X_p = -\frac{\partial}{\partial q}
\]

commute, while its Poisson bracket is \( \{q, p\} = 1 \). If we compare the commutators (25) and (26), we see that the Lie algebra of \( G_R \) is isomorphic to the \emph{classical} Poisson brackets of the free particle. This is the key for building the quantum theory [2],[3].

### 10.2 Noether’s Theorem

Let \( \bar{G} \) be a central extended group with fiber bundle structure (section 6), and \( X^R \) a right invariant generator. We proof now that the functions (see (48) and (49))

\[
f^R = i_{X^R} \Theta \quad (97)
\]

are constants of motion. This constitutes the Noether theorem in our group theoretical formulation. An alternative proof based in the invariance of \( \Theta \) under right transformations, \( L_{X^R} \Theta = 0 \), can be found in [17]. See also [20], pp. 19-21.

The function \( f^R \) is a constant of motion if and only if

\[
L_Y f^R = Y(f^R) = 0 \quad , \quad \forall \ Y \in C_{\Theta} \quad (98)
\]

since the equations of motion are the integral curves of the generators of \( C_{\Theta} \). We have

\[
L_Y f^R = L_Y (i_{X^R} \Theta) = i_{X^R} (L_Y \Theta) + i_{[Y,X^R]} \Theta \quad (99)
\]

where the identity

\[
i_{[X,Y]} = L_X i_Y - i_X L_Y \quad (100)
\]

has been used. Since \( Y \in C_{\Theta} \), \( L_Y \Theta = 0 \). Then

\[
L_Y f^R = i_{[Y,X^R]} \Theta \quad (101)
\]
We prove now that \([Y, X^R] \in C_\Theta\), so the right hand side of the above equation is zero. Let \(Y = f_i Y_i\), where \(f_i\) are functions and \(Y_i\) are the generators of \(C_\Theta\). Consider

\[
[Y, X^R] = [f_i Y_i, X^R] = f_i [Y_i, X^R] + X^R(f_i) Y_i
\] (102)

\([Y_i, X^R] = 0\) because \(C_\Theta\) is generated by left invariant fields and left and right generators commute. Then

\[
[Y, X^R] = X^R(f_i) Y_i \in C_\Theta \quad \implies \quad L_Y f^R = 0 \quad \forall Y \in C_\Theta
\] (103)
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[6] E. Wigner and E. Inönü, Representations of the Galilei group, Novo Cimento 9, 705-718, (1952).

[7] This extra 1 is essential for writing the composition law in matrix form. Matrix product is an associative operation, and an inverse exits because the determinant of a Galilean matrix is 1. Since the product of two Galilean matrix is also a Galilean matrix (the form is the same), the composition law (7) is indeed a group law. One can check that (7) defines a group without the necessity of an intermediate matrix form, but we have chosen to do so in order to show how the affine law (7) can be written linearly. Matrix (linear) representations of Lie groups provide interesting connections between group theory and functional spaces of great importance.
in physics. A well-known example in Quantum Mechanics is $SU(2)$ (complex $2 \times 2$ matrices with determinant unity) and the Spherical Harmonics. For an extensive review, see N. Vilenkin and A.U. Klimyk, *Representations of Lie Groups and Special Functions*, (Kluwer Academic Publishers, 1991)

[8] This is because the equations of motion are second order in time. See [7] for examples of higher order equations where $F$ can also depend on $v \equiv dx/dt$.

[9] J.M. Lévy-Leblond, “Group theoretical foundations of classical mechanics: the Lagrangian gauge problem”, Comm. Math. Phys. **12**, 64-79 (1969).

[10] This is not exactly true. The quantity that must remain invariant is the action. Requirement of invariance under some groups, like the Conformal group (but not Galileo or Poincaré groups) affect the volume form in the action integral. Lagrangian semi-invariance is no longer the condition for the variation of the action to be zero.

[11] In the case of the extensions of the Galileo group by $R$, [11] reads

$$\delta_c f(a,b) = f(a \star b) - f(a) - f(b)$$

since the group law in $R$ is additive.

[12] An important case in which all cocycles are coboundaries is relativistic mechanics, where $G_1$ is the Poincaré group and $G_2$ is $U(1)$. The canonical form $\Theta$ to be defined in [17] can always be reduced to a trivial $d\Theta = d\theta$ by a change of coordinates, and apparently no dynamics is obtained. This issue and ways to circumvent this problem are considered in A. Aldaya and J.A. Azcárraga, *Group Manifold Analysis of the Structure of Relativistic Quantum Dynamics*, Annals of Physics **165**, 484-504, (1985).

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[18] One can prove that $\mathcal{F}$ is the set of coherent states associated with $G_R$. See A. M. Perelomov, *Generalized coherent states and their applications*, (Springer-Verlag, New York, 1986).

[19] H. Goldstein, *Classical Mechanics*, (second edition, Addison-Wesley, 1980). The discussion of the Hamilton-Jacobi theory in chapter 10 is done in coordinate space, hence the expression for the Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial t} + H(x, \frac{\partial S}{\partial x}) = 0$$

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