Bronnikov-like wormholes in Einstein-scalar gravity

Hyat Huang\textsuperscript{1}, H Lü\textsuperscript{2} and Jinbo Yang\textsuperscript{3,\,*}

\textsuperscript{1} College of Physics and Communication Electronics, Jiangxi Normal University, Nanchang 330022, People’s Republic of China
\textsuperscript{2} Center for Joint Quantum Studies and Department of Physics, School of Science, Tianjin University, Tianjin 300350, People’s Republic of China
\textsuperscript{3} Institute for Theoretical Physics, Kanazawa University, Kanazawa 920-1192, Japan

E-mail: hyat@mail.bnu.edu.cn, mrhonglu@gmail.com and j_yang@hep.s.kanazawa-u.ac.jp

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Abstract
In this paper, we analyse the global structure of the Bronnikov wormhole, which is the most general spherically-symmetric and static solution in Einstein gravity coupled to a free massless phantom scalar. We then introduce a scalar potential and construct a large class of exact solutions that can be viewed as generalizations of the Bronnikov wormhole. We study the global structure and classify the parameters of these new wormholes. For suitable parameters, some are regular black holes with a bouncing de Sitter spacetime inside the event horizon.

Keywords: wormholes, Einstein-scalar gravity, regular black holes

\*Author to whom any correspondence should be addressed.

1. Introduction
The bending of spacetime in general relativity (GR) provides tantalizing possibilities of spacetime geometries that inspire one’s imagination: black holes, wormholes and time machines associated with the closed time-like curves. Increasingly precise astronomical observations confirm the existence of black holes in our Universe. These include the detections of gravitational waves from two colliding black holes \cite{1, 2} and the recent direct photon picture of the black hole shadow \cite{3}. Although there is not yet any evidence of wormholes and time machines, these research subjects have continued to attract attentions in the GR community.
In addition to its traditional role as a tool for space and time travel, a wormhole may play an important role in quantum information through quantum entanglement [15–22]. In 1988, Morris and Thorne demonstrated traversable wormholes in GR must violate the null energy condition on the throat of any static and spherically symmetric [23]. It means that the traversable wormholes require exotic matter. The simplest such candidate may be the phantom scalar field. It turns out that phantom fields are one of the most promising candidates of dark energy in cosmology [24, 25]. In particle physics, a phantom field model was proposed to study the neutrino mass generation [26]. Furthermore, some studies argued that the instability problem of the phantom fields could be curable [27].

Earlier than Morris and Thorne, Ellis has constructed as an exact solution of Einstein gravity coupled to a free phantom scalar field, served as a traversable wormhole, in 1973 [28]. The Ellis wormhole is a spherically-symmetric and static wormhole, connecting two flat spacetimes. Later in the same year, Bronnikov found the same wormhole solution [29, 30]. The symmetric Bronnikov–Ellis wormhole is massless while the asymmetric one is massive. Even earlier works for spherically-symmetric and static solutions in Einstein gravity and free scalar field can be found in reference [31] for canonical scalar, and in reference [32] for phantom scalar. Recently, the Bronnikov–Ellis wormhole was generalized to the rapidly rotating case [33, 34] and also in the scalar–tensor theory [35]. Furthermore, many new wormhole solutions have been constructed in Einstein gravity or modified gravity theories, especially Martinez and Nozawa investigated phantom wormhole construction and its properties [36–56].

The purpose of this paper is to generalize the Bronnikov wormhole and construct a more general class of such solutions in Einstein-scalar gravity. To do so, we first analyse the global structure of the Bronnikov wormhole. We find that the speeds of light of the two Minkowski spacetimes connected by the wormhole are different. In other words, there is no globally defined time such that the flow of time in each asymptotic region is the same. This is very different from the Ellis wormhole, where the two asymptotic regions can be connected by the Poincare transformation without using the wormhole. Thus although the Bronnikov wormhole solution involves two integration constants, the ratio of the speeds of light must be held fixed, leaving only one adjustable parameter. Keeping this in mind, we introduce a scalar potential and obtain a large class of exact solutions that can be viewed as generalizations of the Bronnikov wormhole. These solutions connect not only the Minkowski spacetimes, but also de Sitter (dS) and/or anti-de Sitter (AdS) spacetimes.

The paper is organized as follows. In section 2, we give a brief review of the Bronnikov wormhole and then study its global structure. In section 3, we present the Lagrangian of the Einstein-scalar theory and the corresponding local solutions of the Bronnikov-type wormholes. We examine and classify the parameters of the solutions in section 4. We conclude the paper in section 5.

2. Brief review of the Bronnikov wormhole

2.1. The theory and the most general local solution

Einstein gravity minimally coupled to a massless scalar admits exact spherically-symmetric and static solutions. When the scalar is phantomlike, the solution with appropriate integration constant describes a wormhole. To be specific, the Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left( R + \frac{1}{2} (\partial \phi)^2 \right),$$

(1)
where the scalar field has negative kinetic energy and hence phantomlike. The most general spherically static wormhole solution in the theory is the Bronnikov wormhole. Following the convention in reference [30], the solution is

\[ ds^2 = -h(r)dt^2 + h(r)^{-1}dr^2 + R^2(r)\,d\Omega_2^2, \]

\[ h = e^{-\frac{\phi}{q}}, \quad R^2 = \frac{r^2 + q^2 - M^2}{h}, \]

\[ \phi = \frac{2q}{\sqrt{q^2 - M^2}} \arctan\left( \frac{r}{\sqrt{q^2 - M^2}} \right), \tag{2} \]

where \((M, q)\) are two integration constants and we should not confuse the Ricci scalar \(R\) and the coordinate function \(R(r)\). Obviously, the scalar field in the Lagrangian (1) has shift symmetry, i.e., the Lagrangian is invariant under transformation

\[ \phi \rightarrow \phi + C. \tag{3} \]

The metric part of the solution (2) seems to change under the above transformation. However, it is actually invariant under the following redefinition:

\[ t \rightarrow e^{\frac{M}{q} C} t, \quad r \rightarrow e^{\frac{M}{q} r}, \quad q \rightarrow e^{-\frac{M}{q}} q, \quad M \rightarrow e^{\frac{M}{q} C} M. \tag{4} \]

When \(M = 0\), the solution reduces to the well-known Ellis wormhole. The \(q = 0\) limit is more subtle and we find

\[ ds^2_{q=0} = -ds^2_{\text{sch}}, \tag{5} \]

where \(ds^2_{\text{sch}}\) is the Schwarzschild metric of mass \(-M\), rewritten in \(r \rightarrow r - M\) coordinate, namely

\[ ds^2_{\text{sch}} = -\frac{(r + M)}{r - M} \, dr^2 + \frac{r - M}{r + M} \, dr^2 + (r - M)^2 \, d\Omega_2^2. \tag{6} \]

Note that the reality and regularity conditions imply that there is no smooth \(q \rightarrow 0\) limit from the general wormhole solution which requires that \(q > M\).

### 2.2. Some global properties

The Bronnikov wormhole (2) connects two asymptotic Minkowski spacetimes that are not symmetric. As \(r \rightarrow \pm \infty\), we have

\[ ds^2 \rightarrow -c_+^2 \, dt^2 + dR^2 + R^2 \, d\Omega_2^2, \tag{7} \]

where

\[ c_+ = e^{\frac{-M}{2\sqrt{q^2 - M^2}}}. \tag{8} \]

In other words, the speeds of light at the two asymptotic flat regions are not the same and hence cannot be simultaneously set to 1 by rescaling the time or changing the unit. Thus the asymptotic flat worlds are intrinsically different and the wormhole is a tunnel connecting them. This should be contrasted with the Ellis wormhole where \(c_+ / c_- = 1\), in which case, the wormhole can be a shortcut connecting two locations of the same asymptotic Minkowski world. For two
given Lorentz invariant worlds, the Bronnikov wormhole connecting them contains only one free parameter, with
\[
\frac{c_+}{c_-} = e^{\pm \frac{2M}{\sqrt{q^2 - M^2}}} \text{ fixed.} \tag{9}
\]
In other words, the dimensionless ratio \( q/M > 1 \) is fixed. On the other hand, the \( c_+c_- = 1 \) is coincidental; a scaling of time coordinate can turn the product to be any constant value.

To understand the wormhole solution better, it is advantageous to write the metric in the more standard Ellis wormhole type of coordinates, namely
\[
ds^2 = -h(\rho)dt^2 + \frac{d\rho^2}{f(\rho)} + (\rho^2 + a^2)d\Omega^2, \tag{10}
\]
where \( a \) is the radius of the wormhole throat, given by
\[
a^2 = R_{\text{min}}^2 = q^2 \exp \left( -\frac{2M}{\sqrt{q^2 - M^2}} \tan^{-1} \left( \frac{M}{\sqrt{q^2 - M^2}} \right) \right), \tag{11}
\]
located at \( r = -M \) of the original radial coordinate. Note that we use three different radial coordinates \( R, r \) and \( \rho \) under appropriate circumstances. The \( R \) is the true radius of the foliating two-sphere; \( r \) is the coordinate such that we have \( g_{rr} = -1 \), and \( \rho \) is the Ellis-wormhole type of coordinate defined by \( \rho^2 = R^2 - a^2 \). The Ellis-wormhole radial coordinate \( \rho \) runs from \(-\infty \) to \( \infty \), connecting the two asymptotic Minkowski spacetimes. For our purpose, it is sufficient to obtain the metric functions \( h(\rho) \) and \( \tilde{f}(\rho) \) at large \( |\rho| \). We find for \( \rho \to \pm \infty \), the leading falloffs are
\[
h = c_\pm^2 \left( 1 + \frac{2Me^{\pm \frac{2M}{\sqrt{q^2 - M^2}}}}{\rho} \right) + O(\rho^{-3}),
\]
\[
\tilde{f} = 1 + \frac{2Me^{\pm \frac{2M}{\sqrt{q^2 - M^2}}}}{\rho} + O(\rho^{-2}). \tag{12}
\]
Note that the wormhole throat radius \( a \) first appears in \( h \) in the \( 1/\rho^3 \) falloff in the asymptotic expansion, while it is in \( 1/\rho^2 \) term in \( f \). The mass of the wormhole measured in the two asymptotic regions are not the same, given by
\[
M_\pm = \mp Me^{\mp \frac{2M}{\sqrt{q^2 - M^2}}}, \quad \frac{M_+}{M_-} = \frac{c_+}{c_-}. \tag{13}
\]
In figure 1, we plot \( h \) as a function \( \rho \), for parameters \((M, q) = (-3, 5)\), giving rise to \( M_\pm = 3e^{\pm 3\pi/8} \) and \( c_\pm = e^{\pm 3\pi/8} \).

We see from figure 1 that for the parameter \( M < 0 \), the function \( h \) is a monotonically increasing function of \( \rho \), approaching \( c_\pm^2 \) as \( \rho \to \pm \infty \). The gravitational force is thus always pointing to the negative \( \rho \) direction. In other words, the wormhole is attractive in the \( \rho > 0 \) world, whilst it is repulsive in the \( \rho < 0 \) world. This is consistent with the fact that \( M_+ > 0 \), but \( M_- < 0 \). This interesting phenomenon was also observed in late constructed wormholes [38, 57]. It is
also important to note that for static observers at the two asymptotic $\rho \to \pm \infty$ regions, the experienced time flow is different, namely

$$\Delta t^+ = \frac{c_-}{c_+} (\Delta t)^-.$$  \hfill (14)

Specifically, the time flows faster in the world where the wormhole is attractive, than the other side where the wormhole is repulsive. Thus the wormhole can be used as a time machine, provided that such two asymptotic Minkowski universes exist. It should be pointed out that this phenomenon is not universal for asymmetric wormholes. For example, the wormholes constructed in [38] are asymmetric, but the two asymptotic Minkowski regions have the same flow of time.

As we have seen, the solution appears to have two independent parameters ($M, q$); however, the mass and wormhole radius of the Bronnikov wormhole that connects them must be related so that $c_+/c_-$ is fixed. We can thus introduce a fixed dimensionless positive parameter

$$\gamma = \frac{\sqrt{q^2 - M^2}}{q} \leq 1.$$ \hfill (15)

The wormhole solution (2) becomes

$$\phi = \frac{2}{\gamma} \arctan \left( \frac{r}{\sqrt{q}} \right), \quad R^2 = e^{-\sqrt{1 - \gamma^2} \phi} (r^2 + \gamma^2 q^2), \quad h = e^{\sqrt{1 - \gamma^2} \phi}.$$ \hfill (16)

In the next section, we construct a more general class of wormholes with the fixed $\gamma$ parameter, in some Einstein-scalar theories.

3. Generalization by a scalar potential

3.1. Construction

In this section, we consider a more general phantom theory by introducing a scalar potential $V(\phi)$. The Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \left( R + \frac{1}{2} (\partial \phi)^2 - V \right).$$ \hfill (17)
The covariant equations of motion associated with the variations of $g_{\mu\nu}$ and $\phi$ are respectively given by

$$\Box \phi = - \frac{\partial V}{\partial \phi}, \quad E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - T_{\mu\nu}^{\phi} = 0, \quad (18)$$

where

$$T_{\mu\nu}^{\phi} = - \left( \frac{1}{2} \frac{\partial}{\partial \phi} \phi \frac{\partial}{\partial \phi} \phi - \frac{1}{4} g_{\mu\nu} (\partial \phi)^2 \right) - \frac{1}{2} g_{\mu\nu} V. \quad (19)$$

To construct a spherically-symmetric and static solution, we take the general ansatz,

$$\text{ds}^2 = -h \, dr^2 + h^{-1} \, dr^2 + R^2(r) d\Omega_2^2, \quad \phi = \phi(r). \quad (20)$$

Substituting this into the equation (18) gives three independent ordinary differential equations:

$$E'_{r} = 0 : -4 + 4hR^2 + R^2(2V - h\phi^{'2}) + 4R(h'R' + 2hR'') = 0,$n
$$E''_{r} = 0 : -4 + 4Rh'R' + 4hR^2 + R^2(2V + h\phi^{'2}) = 0,$n
$$E_{\text{sphere}} = 0 : R(-h\phi^{'2} + 2(V + h'')) + 4(h'R' + hR'') = 0. \quad (21)$$

It is straightforward to verify that the scalar equation of motion is automatically satisfied provided that the above are all satisfied.

We now use the reverse-engineering technique to construct new wormholes and determine the scalar potential. We assume that the scalar $\phi$ and the metric function $R$ of the new wormholes take exactly the same forms as the Bronnikov wormhole, given by (16), namely

$$\phi = \frac{2}{\gamma} \arctan \left( \frac{r}{\gamma q} \right), \quad R^2 = e^{-\sqrt{1-\gamma^2}(r^2 + \gamma^2 q^2)}, \quad (22)$$

where we take $q \geq 0$ in this paper without loss of generality. We shall comment on this ansatz further presently. Under the assumption (22), the $E'_{r} = 0$ and $E''_{r} = 0$ equations become equivalent. We can therefore choose one of them and $E_{\text{sphere}} = 0$ in (21) to obtain

$$h'' = - \frac{2}{(r^2 + \gamma^2 q^2)^2} \left( \phi(r^2 + \gamma^2 q^2) + 2qr \sqrt{1-\gamma^2} - r^2 + q^2(\gamma^2 - 2)h \right), \quad (23)$$

$$V = \frac{2}{(r^2 + \gamma^2 q^2)^2} \left( (q^2(\gamma^2 - 2) - r^2 + 2qr \sqrt{1-\gamma^2})h 
+ (r^2 + \gamma^2 q^2)(\phi + (1 \sqrt{1-\gamma^2} - r)h') \right).$$

It turns out that these two equations can be solved exactly and we obtain $h(r)$ and $V(r)$ involving two integration constants $(\alpha, \beta)$. We can then use the $\phi(r)$ expression to write $V$ in terms of the scalar. We obtain

$$V = e^{\sqrt{1-\gamma^2} \beta} \left( -3\gamma \sqrt{1-\gamma^2} \sin(\gamma \phi) + (3\gamma^2 - 2) \cos(\gamma \phi) - 2 \right)$$

$$+ e^{-\sqrt{1-\gamma^2} \alpha} \left( 3\gamma \sqrt{1-\gamma^2} \sin(\gamma \phi) + (3\gamma^2 - 2) \cos(\gamma \phi) - 2 \right). \quad (24)$$
We therefore obtain an Einstein-scalar theory (17). The corresponding analytical solution is then given by (22) together with

\[
    h = \frac{1}{2} e^{-\sqrt{1 - \gamma^2} \phi} \left( 2 e^{2 \sqrt{1 - \gamma^2} \phi} + \alpha \gamma^2 (r^2 + q^2 \gamma^2) 
    + \beta e^{2 \sqrt{1 - \gamma^2} \gamma^2} (r^2 - q^2 (7 \gamma^2 - 8) + 4qr \sqrt{1 - \gamma^2}) \right). \tag{25}
\]

Thus we see that we have a total of four parameters (\(\alpha, \beta, \gamma, q\)) in the system. The parameters (\(\alpha, \beta, \gamma\)) appear in the Lagrangian and hence they cease to be integration constants of the solutions. Only \(q\) is the integration constant. As we have emphasized earlier, this does not necessarily reduce the generality in the Bronnikov wormholes. In what follows, we shall analyse the Einstein-scalar theory and the global properties of the solutions.

Before ending this subsection, it is worth commenting further on the ansatz (22). The gauge choice associated with the reparametrization of the radial coordinate \(r\) has been fixed by the ansatz (20). A priori, having chosen the function \(R(r)\), which determines the class of solutions and their associated theory, the equations of motion will uniquely determine both \(\phi(r)\) and \(h(r)\), up to some integration constants. Thus the most general ansatz of \(\phi(r)\) for the given \(R(r)\) is

\[
    \phi = \frac{2}{\gamma} \arctan \left( \frac{r}{\gamma q} \right) X(r), \tag{26}
\]

where the function \(X\) should be determined by the equations of motion. Intriguingly, although both \(E_t^t = 0\) and \(E_r^r = 0\) equations involving the function \(h(r)\), there exists a linear combination that does not, giving

\[
    \sqrt{1 - \gamma^2} (\gamma^2 q^2 + r^2)^2 \arctan \left( \frac{r}{\gamma q} \right) X'' + \gamma (\gamma^2 q^2 + r^2)^2 \arctan \left( \frac{r}{\gamma q} \right) X''
    + 2 (\gamma^2 q^2 + r^2) \left( \sqrt{1 - \gamma^2} (\gamma q + r \arctan \left( \frac{r}{\gamma q} \right)) \right)
    + \gamma^2 q X \arctan \left( \frac{r}{\gamma q} \right) X' + \gamma^3 q^2 (X^2 - 1) = 0. \tag{27}
\]

We do not find the general analytic solution that involves two integration constants. In this paper, as we have seen earlier, we find that for a generic \(\gamma\), there exists a simple special solution, namely \(X = 1\). It should be mentioned that when \(\gamma = 1\), the term above involving \(X''\) vanishes and the second-order differential equation degenerates to the first-order one, in which case, the function \(X\) can be solved analytically, giving rise to \(\phi = 2 \arctan(r/q) + \phi_0\). The \(\gamma = 1\) solution, which can be viewed as a direct generalization of the Ellis wormhole since \(R^2 = r^2 + q^2\), was obtained in [58]. Our solution with generic \(\gamma\) is a direct generalization of the Bronnikov wormhole (2), reexpressed as (16). It is thus a very different generalization from the one given in [58], which was for the \(\gamma = 1\) only.

3.2. The scalar potential

We now analyse the scalar potential (24). We compute
The left scalar potential has $\Lambda_+ = 1, \Lambda_- = \frac{1}{2}, \gamma = \frac{\pi}{5}$, with a minimum sandwiched between two maxima; the right one has $\Lambda_+ = -1,\Lambda_- = -\frac{1}{2}, \gamma = \frac{\pi}{5}$, with a maximum sandwiched between two minima.

$$\frac{\partial V}{\partial \phi} = 2 e^{-\sqrt{1 - \gamma^2} \phi} \cos \left( \frac{1}{2} \gamma \phi \right)$$
\[\times \left( 2(\alpha - e^2 \sqrt{1 - \gamma^2} \beta) \sqrt{1 - \gamma^2} \cos \left( \frac{1}{2} \gamma \phi \right) \right.\]
\[- \left( \alpha + e^2 \sqrt{1 - \gamma^2} \beta \gamma \sin \left( \frac{1}{2} \gamma \phi \right) \right) \right]. \quad (28)

Obviously, the above vanishes when
$$\phi = \left( \frac{1}{2} + n \right) \frac{2\pi}{\gamma} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots \quad (29)$$

These give the stationary points of $V$. We take $n = 0$ and $n = -1$ and obtain two special stationary points
$$\phi_+ = \frac{\pi}{\gamma}, \quad \phi_- = \frac{-\pi}{\gamma} \quad (30)$$

As we shall discuss in the next section, these two stationary points play an important role. By convention, we define
$$V(\phi_{\pm}) = -3\gamma^2 \left( \alpha e^{\mp \sqrt{1 - \gamma^2} \phi} + \beta e^{\pm \sqrt{1 - \gamma^2} \phi} \right) \equiv 2\Lambda_{\pm}. \quad (31)$$

We shall see that $\Lambda_{\pm}$ are nothing but the effective cosmological constants in two asymptotic regions.

It is straightforward to compute
$$\frac{\partial^2 V}{\partial \phi^2} \bigg|_{\phi=\phi_{\pm}} = -\frac{2}{3} \Lambda_{\pm}, \quad \frac{\partial^2 V}{\partial \phi^2} \bigg|_{\phi=\phi_{\pm}} = -\frac{2}{3} \Lambda_{\pm}. \quad (32)$$

In the case of $\Lambda_+$ and $\Lambda_-$ are both positive, $\phi_+$ and $\phi_-$ give two local maximums, and there must exist at least one stationary point between the minima at $(\phi_-, \phi_+)$. Likewise, if $\Lambda_+$ and $\Lambda_-$ are both negative, $\phi_+$ and $\phi_-$ are two local minimums. Then there will be at least one stationary point between the two maxima. These two situations are illustrated in figure 2.
However, if $\Lambda_+$ and $\Lambda_-$ have different signs, the two stationary points then give one maximum and one minimum. In this case, there may not be further stationary point within $(\phi_-, \phi_+)$, as illustrated in figure 3.

4. Wormholes and regular black holes

4.1. General properties

For the Einstein–scalar theory (17) with the scalar potential (24), we have constructed the exact spherically-symmetric and static solution of the form (20) where the scalar and metric functions are given by (22) and (25). The radial coordinate $r$ runs from minus infinity to plus infinity, for which the scalar runs from one stationary point $\phi_-$ to $\phi_+$. In order to study the two asymptotic structure, it is instructive to use $R(r)$ as the radial coordinate and write the metric as

$$\text{d} s^2 = -h(R)\text{d} t^2 + f(R)^{-1}\text{d} R^2 + R(r)^2\text{d} \Omega^2,$$

where $f(R) = h(R)R(r)^2$. Since $R(r)$ has a non-vanishing minimum $R_{\text{min}}$, in the $R$ coordinate, $f(R)$ will have a single zero even when $h(R)$ has no zero. (As we shall see presently, there exists a possibility that $h$ also has a root, in which case, it gives either a black hole horizon or a cosmic horizon.) Thus the $R$ coordinate will not describe the full spacetime region; nevertheless, it is useful to describe the asymptotic structure and read off the mass. In the $R \to \pm \infty$ regions, the coordinate $r$ and $R$ are related by

$$r \sim e^{\sqrt{1 - \frac{\gamma^2}{2\pi}} R} - \frac{2q^2 \sqrt{1 - \gamma^2}}{2R} - q \sqrt{1 - \gamma^2} + e^{\sqrt{1 - \frac{\gamma^2}{2\pi}}} + \cdots.$$  

Thus in the $R \to \pm \infty$ asymptotic region, the solutions become the (A)dS vacua

$$\text{d} s^2 = -e^{\pm\sqrt{\frac{\gamma^2}{2\pi} R}} \left( -\frac{1}{3} \Lambda \pm R^2 + 1 \right) \text{d} r^2 + \frac{\text{d} R^2}{\left( -\frac{1}{3} \Lambda \pm R^2 + 1 \right)} + R^2\text{d} \Omega^2, \quad \phi = \phi_\pm.$$
where
\[ c_{\pm} = e^{\pm \frac{2\sqrt{1-\gamma^2}}{\gamma}}. \] (36)

Note that the speeds of light in the two vacua take the same forms as those in the Bronnikov wormhole discussed earlier. The masses of the solutions in both asymptotic regions can be obtained by examining the asymptotic falloffs and we find
\[ M_{\pm} = \pm \frac{1}{3} q \sqrt{1 - \gamma^2} (3 - 2q^2\gamma^2\beta(3\gamma^2 - 4)) e^{\pm \frac{\sqrt{1-\gamma^2}}{\gamma}}. \] (37)

As in the case of the Bronnikov wormhole, the best coordinate to describe the full wormhole geometry is the Ellis-wormhole coordinate, namely (10), where the radial coordinate \( \rho \) runs smoothly from \(-\infty \) to \(+\infty \), connecting the two asymptotic spacetimes. The radius \( a \) of the wormhole throat (\( \rho = 0 \)) is given by
\[ a^2 = R_{\min}^2 = q e^{\frac{\sqrt{1-\gamma^2}}{\gamma} \arctan\left(\frac{\sqrt{1-\gamma^2}}{\gamma}\right)}, \] (38)
located at \( r = q \sqrt{1 - \gamma^2} \) of the origin coordinate. In the next, we shall study the global structures in detail depending on the various choices of the parameters.

### 4.2. Symmetric (A)dS Wormhole

We first consider \( \gamma = 1 \), in which case, the scalar potential (24) is
\[ V = \frac{2}{3} \Lambda^*(\cos \phi - 2), \] (39)
where \( \Lambda^* = -\frac{1}{2}(\alpha + \beta) \) is a constant. The corresponding solution reduces to
\[ ds^2 = -h \, dt^2 + h^{-1} \, dr^2 + (r^2 + q^2) \, d\Omega^2, \]
\[ \phi = 2 \arctan \left( \frac{r}{q} \right), \quad h = 1 - \frac{1}{3} \Lambda^*(r^2 + q^2). \] (40)

Thus we see that the solution is already naturally in the Ellis-wormhole ansatz with the wormhole throat at \( r = 0 \) and wormhole radius \( q \). As \( r \) runs from \(-\infty \) to \(+\infty \), the scalar runs from \( \phi_- = -\pi \) to \( \phi_+ = \pi \), with \( V_{\pm} = \Lambda^* \). The metric (40) was a special case of those obtained in references [58–60]. This can be viewed as a direct (A)dS generalization of the Ellis wormhole and the mass is zero. The solution describes a symmetric AdS wormhole when \( \Lambda^* < 0 \), and it describes a symmetric dS wormhole when \( \Lambda^* > 0 \). Interestingly, for the latter case, when the wormhole radius is sufficiently large, larger than the cosmic horizon, i.e. \( q \geq \sqrt{3/\Lambda^*} \), the metric becomes purely cosmological, portraying a bouncing Universe.

### 4.3. Asymmetric (A)dS Wormhole

We now consider the more general \( 0 < \gamma < 1 \) case. We first consider that the two asymptotic spacetimes are both AdS or dS. In other words, the effective cosmological constants of the two sides satisfy
\[ \Lambda_+ \Lambda_- > 0. \] (41)
The easiest way to achieve this is to set $\alpha = 0$, which leads to

$$h = \frac{1}{2}e^{\sqrt{1-\gamma^2} \beta (\beta \gamma^2 r^2 + (4\beta q \gamma^2 \sqrt{1-\gamma^2}) r + 2 + \beta q^2 \gamma^2 (8 - 7 \gamma^2))}. \quad (42)$$

For $\beta > 0$, the effective cosmological constants

$$\Lambda_{\pm} = -\frac{3}{2} \beta \gamma^2 e^{\pm \sqrt{1-\gamma^2} \beta} \quad (43)$$

are both negative, giving rise to the two asymptotic AdS spacetime. It can be verified that there is no real root in $h = 0$ for $0 < \gamma < 1$ and $\beta > 0$. The solution describes a wormhole connecting two AdS worlds with different cosmological constants. The function $h$ in terms of the Ellis-wormhole radius $\rho$ is depicted in figure 4.

On the other hand, the parameter choice $\beta < 0$ gives rise to two asymptotic dS spacetimes. There is a cosmic horizon associated with the dS spacetime in each side, given by

$$r_{\pm} = -2 \sqrt{1-\gamma^2} q \pm \sqrt{-\frac{2}{\beta \gamma^2} - q^2 (4 - 3 \gamma^2)}. \quad (44)$$

The reality condition requires that

$$q \in [0, \frac{\sqrt{2}}{\sqrt{-\beta \gamma^2 (4 - 3 \gamma^2)}}].$$

The wormhole throat is located at $r = q \sqrt{1-\gamma^2} (R = R_{\text{min}})$. The asymmetric dS wormhole is depicted in figure 5.

Note that if we turn on the $\alpha$ parameter while maintaining $\Lambda_+ \Lambda_- > 0$, the wormhole solutions are analogous and we shall not give further discussions.

### 4.4. Connecting (A)dS to flat spacetime

We now consider the case with one the effective cosmological constant vanishes. Without loss of generality, we choose $\Lambda_+ = 0$. This can be achieved by setting $\alpha = -e^{\frac{2 \pi \sqrt{1-\gamma^2}}{\beta}}$, in which
Figure 5. The asymptotic dS spacetime: the parameters are set in $\alpha = 0$, $\beta = -\frac{225}{752}$, $q = \frac{8}{10}$, $\gamma = \frac{5}{6}$. The two cosmological horizons locate at $r_- = -4$ and $r_+ = 2$ respectively. The effective cosmological constant in asymptotic region of $\rho < 0$ is $\Lambda_\rho = 0.034$ while in the $\rho > 0$ side is $\Lambda_\rho = 3.788$.

Figure 6. This is the function $h(\rho)$ of the wormhole connecting AdS to flat spacetimes, with $\alpha = -e^{\frac{2\sqrt{1-\gamma^2}}{\gamma}}$, $\beta = -11.1318$, $\beta = -0.1$, $q = 1$, $\gamma = 0.8$. The effective cosmological constant of the two asymptotic regions are $\Lambda_- = -140.927$ and $\Lambda_+ = 0$ respectively.

In this case, we have

$$\Lambda_- = \frac{3}{2} \beta \gamma e^{-\frac{2\sqrt{1-\gamma^2}}{\gamma}} \left( e^{\frac{4\sqrt{1-\gamma^2}}{\gamma}} - 1 \right).$$  (45)

For $\beta < 0$, the solution is wormhole connecting the AdS spacetime at $\rho \to -\infty$ to the flat spacetime at $\rho \to +\infty$. The function $h$ in terms of the Ellis wormhole coordinate $\rho$ is depicted in figure 6.

When $\beta > 0$, the effective cosmological constant $\Lambda_-$ is positive and hence there is an cosmic horizon in the $\rho < 0$ region, as can be seen in the left graph of figure 7. In the right graph, we also changed the $\alpha$ parameter so that the wormhole connects the dS spacetime ($\Lambda_- < 0$) to the AdS spacetime ($\Lambda_+ > 0$).
Figure 7. The left one is $h$ of the wormhole connecting the dS and flat spacetimes, with parameters $\beta = 0.1, \alpha = -11.1318, \gamma = 0.8, q = 1$ then $\Lambda_+ = 140.927$. The cosmic horizon is located at $\rho_H = -0.291$. The right one of the wormhole connecting dS to AdS spacetimes, with $\beta = 0.1, \alpha = -10, \gamma = 0.8, q = 1$ and the cosmic horizon is located at $\rho_H = -0.322$. The effective cosmological constants of the two asymptotic regions are $\Lambda_- = 126.597$ and $\Lambda_+ = -0.129$ respectively.

Figure 8. For $\alpha = -1113.18, \beta = 10, q = 0.5, \gamma = 0.8$, then $\Lambda_+ = 14092.7$ we have an asymptotically flat black hole with horizon located at $\rho = 0.525$; for $\alpha = -1112.878, \beta = 10, q = 0.5, \gamma = 0.8$, we have an asymptotically AdS black holes with horizon $\rho_H = 0.496$. The effective cosmological constants of the two asymptotic regions are $\Lambda_- = 14029.4$ and $\Lambda_+ = -0.569$ respectively. In both cases, inside the horizon is a cosmological bouncing dS universes, with no curvature singularity.

4.5. A bouncing Universe inside a black hole

In the previous subsection, for positive $\Lambda_-$, the root $\rho_H < 0$ of the vanishing $h(\rho)$ is a cosmic horizon in the $\rho < 0$ world of the $\rho = 0$ wormhole throat. If we can adjust the parameters such that the root becomes positive, then it becomes a black hole horizon $\rho_H$ in the $\rho > 0$ world. We find that such parameters do exist and the result is plotted in figure 8.

It is now appropriate to address the comments under (33). In all our solutions, we have $a = R_{\text{min}} > 0$; therefore, our spacetimes have no curvature singularity that is typically associated with $R(\rho) = 0$. When the function $h$ has no zeros, we have the usual wormhole with no horizon. Our parameter space allows the possibility that $h$ can have a root at some $\rho_0$, which describes a null Killing horizon. We assume that ‘our’ world is at $\rho > 0$ that is asymptotic to Minkowski, AdS, or even the dS spacetime whose the cosmic horizon is much greater than the wormhole throat. When $\rho_0$ is negative, which we discussed in the previous subsection, the horizon is behind the wormhole, and the world behind the wormhole with $\rho_0 < \rho < 0$ is surrounded by
this null Killing horizon, which is therefore called the cosmic horizon. In this subsection, we have obtained solutions with $\rho_0 > 0$, and therefore the Killing horizon is in our side, it is thus an event horizon, giving rise to a black hole. Inside the black hole $\rho < \rho_0$, $\rho$ becomes timelike and the solution is cosmological, just like the inside of the Schwarzschild black hole. However, there is one big difference since now there is no singularity at $\rho = 0$. Instead, $\rho = 0$ now describes a cosmological bounce. Specifically, as the time coordinate $\rho$ runs from $\rho_0 > 0$ to $-\infty$, the scaling factor $R$ of the two-sphere first shrinks to its minimum $a$ at $\rho = 0$, and then bounces back to infinity, which is a cosmological dS spacetime. During this evolution, the $t$ coordinate is spacelike, and its scaling factor $\sqrt{-h}$ increases monotonously, as illustrated in figure 8.

5. Conclusions

In this paper, we first studied the global structure of the Bronnikov wormhole that is the spherically-symmetric and static solution of Einstein gravity coupled to a free massless phantom scalar. We demonstrated that the wormhole connected two Minkowski worlds with different speeds $c_{\pm}$ of light. Although in each world, one can scale the time coordinate such that the speed of light is unit; however, when they are connected by the wormhole, there is no globally defined time such that the speeds of light can be set to unit simultaneously. The two worlds are thus not connected by the Poincare transformations, but only through the wormhole. This should be contrasted with the Ellis wormhole, which can be a shortcut for travelling between two locations in the same Minkowski spacetime.

The Bronnikov wormhole involves two parameters ($M, q$), parameterizing the radius of the wormhole throat and the mass. The wormhole is asymmetric and the mass $M_{\pm}$ in the two sides of the wormhole are different, with one positive and one negative. In other words, the wormhole appears attractive in one world, but repulsive in the other. By contrast, the Ellis wormhole is symmetric and massless. An intriguing feature of the Bronnikov wormhole is that it can be used as a time machine since the time flow of the two asymptotic worlds are different, with the time flowing faster in the world where the wormhole is attractive.

In the Bronnikov wormhole Universe, for the two fixed asymptotic Minkowski spacetimes, the wormhole involves actually only one integration constant since $c_+/c_-$ must be fixed. In other words, the dimensionless parameter $\gamma$ (15) must be fixed. This led to the second part of the this paper, where we constructed a general class of the scalar potential involving the parameter $\gamma$. This allowed us to construct a more general class of the Bronnikov type of exact wormhole solutions that connect not only the flat spacetimes, but also the (A)dS spacetimes. We analysed the global structures of these wormhole solutions, and classified the parameters for different types of asymptotic structures. We also obtained examples of regular black holes with a bouncing dS Universe inside its horizon.

The general wormholes involve four parameters, but the effective cosmological constants $\Lambda_{\pm}$ of both worlds and the parameter that determines ratio of the speeds of light are all fixed by the Einstein-scalar theory itself. Thus our exact solutions involve only one integration constant, with no independent scalar hair parameter. It is of interest to study how the new scalar hair would affect the wormhole properties. Furthermore, it is interesting to examine how our discussion can be generalized to more general Einstein–Maxwell-scalar theories.
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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

H Li https://orcid.org/0000-0001-7100-2466
Jinbo Yang https://orcid.org/0000-0001-8316-2926

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