Asymptotic multiplicity scaling: a renormalization group perspective

S. Hegyi

KFKI Research Institute for Particle and Nuclear Physics of the Hungarian Academy of Sciences,

H-1525 Budapest 114, P.O. Box 49. Hungary

(September 12, 2018)

Abstract

A generalization of the Polyakov-Koba-Nielsen-Olesen scaling law of the multiplicity distributions $P(n, s)$ is developed. It states that a suitable change in the normalization point of $P(n, s)$ compensated by a rescaling can restore data collapsing onto a universal curve if the original scaling rule is violated. We show that the iteratively executed transformation of $P(n, s)$ can be viewed as varying the collision energy. The $e^+e^-$ and $p\bar{p}$ multiplicity data at top energies are found to exhibit a fixed point property of the iteration.

PACS numbers: 13.85.Hd, 05.40.+j, 11.10.Hi
This year is the 25th anniversary year of the influential work of Koba, Nielsen and Olesen concerning the asymptotic scaling behavior of multiplicity distributions [1]. They put forward the hypothesis that at “sufficiently high” energies $s$ the probability distributions $P(n, s)$ of detecting $n$ final-state hadrons in a certain collision process should exhibit the scaling relation

$$
P(n, s) = \frac{1}{\langle n(s) \rangle} \psi \left( \frac{n}{\langle n(s) \rangle} \right)$$

where $\langle n(s) \rangle$ is the average multiplicity at collision energy $s$. According to Eq. (1), $P(n, s)$ is a homogeneous function of degree $-1$ of $n$ and $\langle n(s) \rangle$. The homogeneity rule states that the multiplicity distributions are simple rescaled copies of the universal function $\psi$, i.e., the change of collision energy $s$ amounts only to a change of scale in the shape of $P(n, s)$.

The above prediction caused immediately a great deal of activity in the experimental and theoretical analysis of multiplicity distributions which remained immense during the past 25 years. It should be mentioned here that Polyakov arrived at the scaling law Eq. (1) already in 1970 postulating a similarity hypothesis for strong interactions in $e^+e^- \rightarrow$ hadrons annihilation [2]. Despite of this fact, the phenomenon became known as KNO scaling in the high-energy physics community.

The main goal of the present Letter is to extend the domain of data collapsing behavior of $P(n, s)$ beyond the scaling relation Eq. (1). In particular, we propose an iterative procedure capable of finding homogeneity rules for $P(n, s)$ if the scaling hypothesis Eq. (1) is violated. We shall consider in detail the fixed points of the iteration, their domains of attraction and the constraints they put on the asymptotic scaling form of $P(n, s)$. Our approach is inspired by the renormalization group methods in the theory of critical phenomena. In the beginning of the paper we will review some basic results of Ref. [3] which should be considered as the first part of the present work; it is cited as Paper 1 in the followings.

We start with some elementary properties of the scaling function $\psi(z)$, $z$ denoting the scaled multiplicity $n/\langle n(s) \rangle$. Finite energy discreteness effects at moderate $\langle n(s) \rangle$ are usually taken into account via Poisson transform which is discussed in Paper 1. The continuous
probability density $\psi(z)$ fulfills the normalization conditions $\int_0^\infty \psi(z)dz = \int_0^\infty z\psi(z)dz = 1$. The latter one, yielding the constraint $\langle z \rangle = 1$, defines a second properly normalized scaling function: $\psi'(z) = z\psi(z) = nP(z)$. Obviously, the moments of $\psi(z)$ coincide with the normalized moments of $P(n,s)$,

$$
C_q = \int_0^\infty z^q\psi(z)dz = \frac{\langle n^q(s) \rangle}{\langle n(s) \rangle^q}.
$$

They are independent of collision energy $s$ if the scaling hypothesis Eq. (1) holds valid. The moments of $\psi'(z)$ are given by $\mathcal{C}'_q = \int_0^\infty z^q\psi'(z)dz = \mathcal{C}_{q+1}$ i.e. the difference between the two scaling functions amounts to a shift in the ranks of their moments.

Let us now recapitulate a scaling argument first presented in Paper 1. Assume that violation of the original scaling hypothesis Eq. (1) is observed and we measure energy dependent “scaling” functions $\psi(z,s)$ and $\psi'(z,s)$. A key point to the later developments is the observation that $\psi'(z,s)$ enables one to perform a rescaling of type Eq. (1) yet again to arrive at data collapsing behavior. The modified scaling hypothesis reads

$$
\psi'(z,s) = \frac{1}{\mathcal{C}'_1(s)}\psi_1\left(\frac{z}{\mathcal{C}_1'(s)}\right)
$$

where the subscript of the new scaling function refers to the first step of an iterative sequence of transformations that can be performed on $P(n,s)$. For quantities corresponding to the 0th step the subscript will be omitted. Our new scaling variable is $z_1 = z/\mathcal{C}_1'(s) = z/\mathcal{C}_2(s)$ and the first-iterate moments $\mathcal{C}_{q,1} = \int_0^\infty z_1^q\psi_1(z_1)dz_1$ are related to the original ones through

$$
\mathcal{C}_{q,1} = \frac{\mathcal{C}'_q(s)}{[\mathcal{C}'_1(s)]^q} = \frac{\mathcal{C}_{q+1}(s)}{[\mathcal{C}_2(s)]^q}.
$$

Expressing $\psi_1(z_1)$ in a similar manner we get $\psi_1(z_1) = \mathcal{C}_2(s)n\mathcal{P}_1[z/\mathcal{C}_2(s)]$ which can be rewritten according to

$$
\psi_1(z_1) = \frac{\langle n^2(s) \rangle}{\langle n(s) \rangle} \frac{n}{\langle n(s) \rangle} P\left(n \left/ \frac{\langle n^2(s) \rangle}{\langle n(s) \rangle} \right. \right).
$$

It is seen that $\psi_1(z_1)$ obeys the same structure as the original scaling function $\psi(z)$, namely, $\psi_1(z_1) = \langle n(s) \rangle_1 \cdot P_1[n/\langle n(s) \rangle_1]$. The first-iterate multiplicity distribution $P_1(n,s)$ and its
moments $\langle n^q(s) \rangle_1$ are obtained by changing the normalization point of $P(n, s)$ from the 0th moment to $\langle n(s) \rangle$ and rescaling by $\langle n(s) \rangle$ to maintain the overall normalization. The distribution $P_1(n, s)$ is known in the mathematical literature as the first order moment distribution of $P(n, s)$.

If $\psi_1(z_1)$ is not independent of collision energy $s$, the transformation rule described above can be repeated iteratively until the appearance of data collapsing onto a universal scaling curve. The details are presented in Paper 1, here we recall only the final result. In the $i$th step of the iteration the connection between $P_i(n, s)$ and the original multiplicity distribution $P(n, s)$ is provided by

$$P_i(n, s) = \frac{n^i}{\langle n^i(s) \rangle} P(n, s)$$

i.e. the normalization point of $P(n, s)$ is changed to $\langle n^i(s) \rangle$ and a rescaling by $\langle n^i(s) \rangle$ is made to preserve proper normalization. Thus the necessary condition of performing the $i$th iteration step is the existence of the moments of $P(n, s)$ up to $i$th order. $P_i(n, s)$ is the moment distribution of order $i$ of $P(n, s)$. Data collapsing of $P_i(n, s)$ onto the scaling function $\psi_i(z_i) = \langle n(s) \rangle_i \cdot P_i[n/\langle n(s) \rangle_i]$ occurs when the normalized moments at the previous iteration step exhibit the “monofractal” type behavior $C_{q,i-1}(s) \propto [C_{2,i-1}(s)]^{q-1}$ with constants of proportionality independent of collision energy $s$, see Eq. (4). It is worth mentioning here that for discrete probability laws such as $P(n, s)$ the factorial moment distributions, involving factorial powers of $n$, arise more naturally as discussed in Paper 1.

It is also argued in Paper 1 that the iterative procedure acting on $P(n, s)$ bears some similarity with the renormalization group (RG) transformations. They correspond in many important applications to a change in the norm of the parameters characterizing a physical system [4]. On the one hand, Eq. (6) can be viewed as a Gell-Mann - Low type relationship in which a multiplicative transformation (here multiplication by $n^i$) is compensated by a rescaling and a suitable change in the renormalized parameters (here in the moments). Since the operation $P_i(n, s) \rightarrow P_{i-1}(n, s)$ does not exist the iteration steps constitute a semigroup. In statistical physics the Kadanoff - Wilson type RG transformations perform a systematic
reduction in the number of degrees of freedom via e.g. spin decimation. This makes possible to eliminate the small-scale fluctuations from the problem which are irrelevant to critical point behavior such as the homogeneity of thermodynamic functions [4]. In the iterative procedure described above the elimination of small-scale fluctuations corresponds to the elimination of the low-order moments of $P(n, s)$ via moment-shifting and rescaling until one observes a homogeneity rule of type Eq. (1) for $P_t(n, s)$.

The suggested analogy between the transformation Eq. (6) of multiplicity distributions and the RG methods in field-theory and statistical physics may seem too remote at first glance. But there is a very close relationship between RG ideas and certain concepts of probability theory [5,6] which naturally fits into our approach to asymptotic multiplicity scaling. Let us demonstrate it by considering some properties of Eq. (6) in more detail.

**Property #1: Form-invariance under size-biasing**

If we view the Gell-Mann - Low or Kadanoff - Wilson version of the RG as a transformation acting on a probability distribution, it is a transformation that does not change the form of the distribution [3]. Weighting a probability law according to Eq. (6) is known in statistics as size-biasing of order $i$. Members of the log-exponential family of distributions, such as the beta, gamma, Pearson type V, Pareto and log-normal to mention but a few, are form-invariant under size-biasing, i.e. they retain functional form and only their parameters are affected [7]. Therefore if $P(n, s)$ belongs to the log-exponential family, our iterative procedure is in accordance with this particular aspect of the Gell-Mann - Low and Kadanoff - Wilson type RG transformations.

**Property #2: Fixed points and automodel distributions**

A probability distribution is called a scaling- or automodel distribution if it is invariant under the action of the RG, or, in other words, if it is a fixed point of the RG transformation [3]. In our case automodel distributions are those members of the log-exponential family which are affected by size-biasing through a scale-change. A theorem of Ref. [8] states that size-biasing amounts to a change of scale in the original distribution if its normalized moments have the form $C_q = C_2^{q(q-1)/2}$. This is a well-known property of the log-normal
law whose parameters are transformed by Eq. (6) according to $\nu \rightarrow \nu + i\sigma^2$ with $\nu$ and $\sigma^2$ being the mean and variance of the unbiased distribution. A log-normal $P(n, s)$ displays fixed point behavior in the following manner: although the transformation Eq. (6) changes its first moment, this change is scaled out by constructing $\psi_i(z_i)$ and one arrives at an unaffected scaling function at each iteration step. Therefore a log-normally shaped $\psi(z)$ is a fixed point of the iterative procedure. Let us emphasize that the log-normal law is not a unique fixed point of the iteration because the distribution is not uniquely determined by its moments.

**Property #3: Domains of attraction of the fixed points**

Besides finding the fixed points of the RG transformation, it is important to specify their domains of attraction. A domain of attraction is the set of initial probability laws which converge to a given automodel distribution under the action of the RG transformation [6]. The domains of attraction are analogous to the universality classes of critical phenomena. Here we are interested in those probability laws which converge to the log-normal fixed point under the action of Eq. (6). Let us choose $\psi(z)$ to be the generalized gamma density

$$
\psi(z) = \frac{\mu}{\Gamma(k)} \lambda^{\mu k} z^{\mu k - 1} \exp \left(-\mu z\right) \quad (7)
$$

with shape parameter $k > 0$, scaling exponent $\mu > 0$ and scale parameter $\lambda$ restricted to $\lambda = \Gamma(k + 1/\mu)/\Gamma(k)$ by the normalization condition $\langle z \rangle = 1$. In the limit $\mu \rightarrow 0$ and $k \rightarrow \infty$ the scaling function given by Eq. (7) converges to a log-normal $\psi(z)$ of variance $\sigma^2$ in such a way that $k\mu^2 \rightarrow 1/\sigma^2$ [6]. The generalized gamma distribution is a member of the log-exponential family and thus it is form-invariant under size-biasing. The shape parameter $k$ is changed by Eq. (6) according to $k \rightarrow k + i/\mu$ whereas the scaling exponent $\mu$ remains unchanged [10]. The $\mu \rightarrow 0$, $k \rightarrow \infty$ log-normal limit can be achieved by requiring $C_{2,i} = C_2$ in the course of the iteration; the increase of $k$ is compensated by a decrease of $\mu$ so that the second moment of $\psi(z)$ is not affected by the variation of the parameters. Thus suitably standardized generalized gamma distributions constitute a domain of attraction of the log-normal fixed point of the iteration.
Property #4: An inequality for the first moments

The transformation rule Eq. (6) is a special case of weighting a probability distribution $f(x)$ with the non-negative weight-function $w(x)$ according to $w(x)f(x)/\langle w(x) \rangle$. The first moment of the weighted distribution is greater or smaller than the first moment of $f(x)$ depending on whether $w(x)$ is a monotonously increasing or decreasing function of $x$. In the case $w(x) = x$, i.e. for size-biasing of order one, we have the inequality $\langle x \rangle_1 > \langle x \rangle$ and in the general case $\langle x \rangle_{i+1} > \langle x \rangle_i$.

Consequences

In many applications of renormalization group methods the transformations establish correspondences between physically different states of the same system. For example, in the Ising model the iterative repetition of the RG transformation can be viewed as varying the temperature. Similar correspondence can be established for asymptotic multiplicity scaling. According to the above inequality of the first moments, the iterative sequence of normalization point changing transformations acting on $P(n, s)$ can be viewed as increasing the collision energy $s$. The asymptotic scaling relation Eq. (1), which states that the increase of $s$ amounts to a change of scale in the shape of $P(n, s)$, is therefore a fixed point property. An important consequence of our reasoning is the fact that the asymptotic scaling function $\psi(z)$ can not be arbitrarily shaped: the above fixed point behavior is satisfied only by those probability laws which are equivalent in their moments to the log-normal distribution.

In the light of the previous findings it is of interest to estimate the degree of deviation between the log-normal distribution and the shape of $\psi(z)$ at asymptotic energies. Making use of the generalized gamma density Eq. (7) the parameter pair $(k, \mu)$ is well suited to measure the departure from fixed point behavior corresponding to the $\mu \to 0, k \to \infty$ limit. The shape of the asymptotic $\psi(z)$ can be reconstructed by fitting the Poisson transform of the generalized gamma distribution, let us call it HNBD for short, to the experimental data for $P(n, s)$ available at the highest collision energies $s$. The HNBD was invented and developed in Refs. [11,12] by the present author. The analytic form of $P(n, s)$ can be expressed in terms of generalized special functions:
\[ P(n, s) = \mathcal{N} H_{1,1}^{1,1} \left[ \frac{1}{\theta} \begin{bmatrix} (1, 1) \\ (K, 1/\mu) \end{bmatrix} \right] \text{ for } 0 < \mu < 1 \] (8)

\[ P(n, s) = \mathcal{N} \Gamma(K) (1 + \theta)^{-K} \text{ for } \mu = 1 \text{ (NBD)} \] (9)

\[ P(n, s) = \mathcal{N} \Psi_0 \left[ \begin{bmatrix} (K, 1/\mu) \\ -\theta \end{bmatrix} \right] \text{ for } \mu > 1 \] (10)

where \( K = k + n/\mu, \mathcal{N}^{-1} = n! \Gamma(k) \theta^{-n} \) and \( \theta = \langle n(s) \rangle \Gamma(k)/\Gamma(k + 1/\mu) \); of course the shape parameter \( k \) and scaling exponent \( \mu \) may also depend on \( s \). The functions \( H_{1,1}^{1,1}(\cdot) \) and \( \Psi_0(\cdot) \) are particular cases of the Fox- and Wright hypergeometric functions, respectively [13]. The negative binomial distribution given by Eq. (8) is the \( \mu = 1 \) marginal case of Eq. (8) for \( \langle n(s) \rangle > k \) and of Eq. (10) for \( \langle n(s) \rangle < k \). The Poisson transformed log-normal limit (\( \mu = 0 \)) lacks a representation in terms of known functions.

We have investigated two full phase-space data sets for \( P(n, s) \): the Delphi data at \( \sqrt{s} = 91 \text{ GeV} \) in \( e^+e^- \) annihilations [14] and the UA5 data at \( \sqrt{s} = 900 \text{ GeV} \) in \( p\bar{p} \) collisions [15]. For each data set we have performed several HNBD fits with different scaling exponents \( \mu \). The value of \( \mu \) was varied in the interval \( 0.1 \leq \mu \leq 2 \) by stepsize \( \Delta \mu = 0.1 \). It is worth considering how the \( \chi^2 \) and the best-fit shape parameter \( k \) depend on \( \mu \). The trends are displayed in Fig. 1. The top right inset shows the variation of fit quality. As is seen the \( \chi^2 \) decreases monotonously towards \( \mu = 0 \), approximately as an exponential, for both data sets. The shape parameter \( k \) increases according to a power-law as \( \mu \to 0 \) with slopes being the same for the two reactions (the estimated errors of \( k \) are too small to be seen). Let us recall that the convergence to a log-normal law of variance \( \sigma^2 \) is such that \( \mu \to 0 \) and \( k \to \infty \) with \( k\mu^2 \to 1/\sigma^2 \). Thus the \( \mu \)-dependence of \( k \) allows us to estimate the value of \( \sigma \) by fitting \( k = (\sigma\mu)^{-2} \) to the data points. The fits are represented by the straight lines in Fig. 1. The quality of fits and the best-fit value of \( \sigma \) are shown in the first row of Table I for each data set. The second row quotes the same numbers obtained in [12] by fitting the Poisson transform of the log-normal distribution to \( P(n, s) \). All these results
suggest the same conclusion: the asymptotic scaling function $\psi(z)$, as can be guessed from pre-asymptotic multiplicity data, is log-normally shaped both in $e^+e^-$ annihilations and in $p\bar{p}$ collisions. Interestingly, this is just the fixed point behavior we would expect on the basis of the RG approach to asymptotic multiplicity scaling.

Summarizing our results, we have developed an iterative procedure well suited to find homogeneity rules of type Eq. (1) for the multiplicity distributions $P(n, s)$. The quoted scaling law states that the energy dependence of $P(n, s)$ is due entirely to the energy dependence of its first moment $\langle n(s) \rangle$. Thus rescaling $P(n, s)$ by the average multiplicity according to Eq. (1) a universal scaling function emerges whose shape is independent of $s$. This scaling argument can be extended to the more general case when the $s$-dependence of $P(n, s)$ can not be attributed to $\langle n(s) \rangle$ exclusively. To arrive at data collapsing onto a universal scaling curve, a modified scaling transformation is needed capable of eliminating the $s$-dependence of higher-order moments. An obvious solution is to “shift out” the moments of the multiplicity distributions up to the required order. This can be achieved by changing the normalization point of $P(n, s)$ from the 0th moment to $\langle n(s) \rangle$, rescaling by $\langle n(s) \rangle$ to maintain the overall normalization and repeating the two-step transformation until the appearance of the scaling law Eq. (1) for the iterated distribution. In other words, we have the possibility that the increase of collision energy $s$ amounts to a change of scale not in $P(n, s)$, rather, in a moment distribution $P_i(n, s)$ with $i > 0$. Accordingly, the phenomenon of data collapsing onto a universal scaling curve should be checked for $i > 0$ as well in Eq. (6). During the past 25 years this was traditionally done only for $i = 0$.

Besides a whole family of new scaling relations for the multiplicity distributions, Eq. (6) provides a close analogy with certain RG ideas. In this respect there is a distinguished role of those probability laws which are form-invariant under the action of Eq. (6) and the moment distributions $P_i(n, s)$ are simple rescaled copies of $P(n, s)$. These distributions are the only ones which may exhibit the asymptotic scaling behavior Eq. (1). It is known that all probability laws having the above property are equivalent in their moments to the log-normal law. Remarkably, the multiplicity data in $e^+e^-$ annihilations and in $p\bar{p}$ collisions
at the highest available energies indicate log-normality of the asymptotic scaling function \( \psi(z) \) — just as one would expect on the basis of RG arguments. To decide whether this is purely accidental (as already happened with results obtained by related arguments [16]) or is an intrinsic feature of multiparticle production, measurements at even higher energies are needed. The forthcoming \( p\bar{p} \) data at \( \sqrt{s} = 1800 \) GeV at Tevatron are awaited with keen interest.

This research was supported by the Hungarian Science Foundation under Grant No. OTKA-T024094/1997.
REFERENCES

[1] Z. Koba, H.B. Nielsen and P. Olesen, Nucl. Phys. B40, 314 (1972).

[2] A.M. Polyakov, Zh. Eksp. Teor. Fiz. 59, 542 (1970).

[3] S. Hegyi, hep-ph/9612309, to appear in Phys. Lett. B.

[4] S.K. Ma, Modern Theory of Critical Phenomena (Benjamin, Reading, MA, 1976).

[5] G. Jona-Lasinio, Nuovo Cim. B26, 99 (1975).

[6] Ya.G. Sinai, Theor. Probab. Appl. 21, 63 (1976).

[7] G.P. Patil and J.K. Ord, Sankhyä B38, 48 (1976).

[8] Y. Vardi, L.A. Shepp and B.F. Logan, Z. Warsch. verw. Gebiete 56, 415 (1981).

[9] T.S. Ferguson, Ann. Math. Statist. 33, 986 (1962).

[10] J. Dias de Deus, C. Pajares and C.A. Salgado, US-FT-6-97 and hep-ph/9702398.

[11] S. Hegyi, Phys. Lett. B387, 642 (1996), ibid. B388, 837 (1996).

[12] S. Hegyi, hep-ph/9707322 and 9708241, to appear in Phys. Lett. B.

[13] A.M. Mathai and R.K. Saxena, The H-Function with Applications in Statistics and Other Disciplines (Wiley Eastern, 1978).

[14] P. Abreau et al., Z. Phys. C50, 185 (1991).

[15] R.E. Ansorge et al., Z. Phys. C43, 357 (1989).

[16] W. Ernst and I. Schmitt, Nuovo Cim. A31, 109 (1976).
| Data set         | $\sigma$      | $\chi^2$/d.o.f. |
|------------------|---------------|-----------------|
| $e^+e^-, \sqrt{s} = 91$ GeV | $0.199 \pm 0.001$ | 12.0/18         |
|                  | $0.201 \pm 0.004$ | 32.4/24         |
| $p\bar{p}, \sqrt{s} = 900$ GeV | $0.527 \pm 0.003$ | 17.3/18         |
|                  | $0.538 \pm 0.014$ | 32.7/52         |

**TABLE I.** For each data set the first row displays the result of fitting the power-law $k = (\sigma \mu)^{-2}$ to the $\mu$-dependence of the HNBD shape parameter $k$. The fits are represented by the straight lines in Fig. 1. The second row quotes the outcome of Poisson transformed log-normal fits to $P(n, s)$, see Ref. [12] for details.
FIG. 1. Variation of the best-fit shape parameter $k$ of the HNBD as a function of scaling exponent $\mu$ for $e^+e^-$ annihilations at $\sqrt{s} = 91$ GeV (open circles) and for $p\bar{p}$ collisions at $\sqrt{s} = 900$ GeV (solid squares). The straight lines are power-law fits, see the text and Table I for details. The inset in the top right displays the $\mu$-dependence of $\chi^2$ corresponding to the HNBD fits.