POSITIVE TOEPLITZ OPERATORS ON THE BERGMAN SPACES OF THE SIEGEL UPPER HALF-SPACE

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Abstract. We characterize bounded and compact positive Toeplitz operators defined on the Bergman spaces over the Siegel upper half-space.

1. Introduction

Toeplitz operators on Bergman spaces over the unit disk have been well studied. Especially, positive symbols of bounded and compact Toeplitz operators are completely characterized. See for instance [1] or [16, Chapter 7]. These results were further extended to more general settings in [14] and [12]. However, there are only few works on analogous results over unbounded domains. See [2, 3] for a study of positive Toeplitz operators on harmonic Bergman spaces over the upper half space of $\mathbb{R}^n$.

In this paper we study bounded and compact positive Toeplitz operators on Bergman spaces over the Siegel upper half-space.

Let $\mathbb{C}^n$ be the $n$-dimensional complex Euclidean space. For any two points $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$ we write

$z \cdot \overline{w} := z_1 \overline{w}_1 + \cdots + z_n \overline{w}_n,$

and

$|z| := \sqrt{z \cdot \overline{z}} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$

The unit ball of $\mathbb{C}^n$ is given by

$B = \{ z \in \mathbb{C}^n : |z| < 1 \}.$

The set

$\mathcal{U} = \{ z \in \mathbb{C}^n : \Im z_n > |z'|^2 \}$

is the Siegel upper half-space. Here and the throughout the paper, we use the notation

$z = (z', z_n), \text{ where } z' = (z_1, \cdots, z_{n-1}) \in \mathbb{C}^{n-1} \text{ and } z_n \in \mathbb{C}.$

As usual, for $p > 0$, the space $L^p(\mathcal{U})$ consists of all Lebesgue measurable functions $f$ on $\mathcal{U}$ for which

$\|f\|_p := \left\{ \int_{\mathcal{U}} |f(z)|^p dV(z) \right\}^{1/p}$
is finite, where $V$ denotes the Lebesgue measure on $\mathbb{C}^n$. The Bergman space $A^p(\mathcal{U})$ is the closed subspace of $L^p(\mathcal{U})$ consisting of holomorphic functions on $\mathcal{U}$. Note that when $1 \leq p < \infty$ the space $A^p(\mathcal{U})$ is a Banach space with the norm $\| \cdot \|_p$. In particular, $A^2(\mathcal{U})$ is a Hilbert space endowed with the usual $L^2$ inner product. The orthogonal projection from $L^2(\mathcal{U})$ onto $A^2(\mathcal{U})$ can be expressed as an integral operator:

$$P f(z) = \int_{\mathcal{U}} K(z, w)f(w)dV(w),$$

with the Bergman kernel

$$K(z, w) = \frac{n!}{4\pi^n} \left[ \frac{i}{2} (\overline{w}_n - z_n) - z' \cdot \overline{w}' \right]^{-n-1}.$$

See, for instance, [3, Theorem 5.1]. This formula enables us to extend the domain of the operator $P$, which is usually called a Bergman projection, to $L^p(\mathcal{U})$ for all $1 < p < \infty$. Moreover, $P$ is a bounded projection from $L^p(\mathcal{U})$ onto $A^p(\mathcal{U})$ for $1 < p < \infty$, see [3, Lemma 2.8].

Given $\varphi \in L^\infty(\mathcal{U})$, we define an operator on $A^p(\mathcal{U})$ by

$$T\varphi f := P(\varphi f), \quad f \in A^p(\mathcal{U}).$$

$T\varphi$ is called the Toeplitz operator on $A^p(\mathcal{U})$ with symbol $\varphi$. Toeplitz operators can also be defined for unbounded symbols or even positive Borel measures on $\mathcal{U}$. Let $\mathcal{M}_+$ be the set of all positive Borel measures $\mu$ such that

$$\int_{\mathcal{U}} \frac{d\mu(z)}{|z_n + i|^{\alpha}} < \infty$$

for some $\alpha > 0$. Given $\mu \in \mathcal{M}_+$, the Toeplitz operator $T_\mu$ with symbol $\mu$ is given by

$$T_\mu f(z) = \int_{\mathcal{U}} K(z, w)f(w)d\mu(w)$$

for $f \in H(\mathcal{U})$. In general, $T_\mu$ may not even be defined on all of $A^p(\mathcal{U})$, $1 < p < \infty$, but it is always densely defined by the fact that, for each $\alpha > n + 1/p$, holomorphic functions $f$ on $\mathcal{U}$ such that $f(z) = O(|z_n + i|^{-\alpha})$ form a dense subset of $A^p(\mathcal{U})$ (see Section 4). This is inspired by that of Choe et al. [2], in the setting of the upper half space of $\mathbb{R}^n$.

A positive Borel measure $\mu$ is called a Carleson measure for the Bergman space $A^p(\mathcal{U})$ if there exists a positive constant $C$ such that

$$\int_{\mathcal{U}} |f(z)|^p d\mu(z) \leq C \int_{\mathcal{U}} |f(z)|^p dV(z)$$

for all $f \in A^p(\mathcal{U})$. We shall furthermore say that $\mu$ is a vanishing Carleson measure for $A^p(\mathcal{U})$ if the inclusion map $i_p : A^p(\mathcal{U}) \rightarrow L^p(\mathcal{U}, \mu)$, $f \mapsto f$ is compact, that is,

$$\lim_{j \to \infty} \int_{\mathcal{U}} |f_j(z)|^p d\mu(z) = 0$$

whenever $\{f_j\}$ converges to 0 weakly in $A^p(\mathcal{U})$. 
For a positive Borel measure $\mu$ on $\mathcal{U}$, we formally define a function $\tilde{\mu}$ on $\mathcal{U}$ by

$$\tilde{\mu}(z) := \int_{\mathcal{U}} |k_z(w)|^2 d\mu(w), \quad z \in \mathcal{U},$$

where, for $z \in \mathcal{U}$,

$$k_z(w) := K(z, w) / \sqrt{K(z, z)}, \quad w \in \mathcal{U},$$

and call $\tilde{\mu}$ the Berezin transform of $\mu$. For $z \in \mathcal{U}$ and $r > 0$, we define the averaging function

$$\hat{\mu}_r(z) := \frac{\mu(D(z, r))}{|D(z, r)|},$$

where $D(z, r)$ is the Bergman metric ball at $z$ with radius $r$ (see Section 2.3) and $|D(z, r)| := V(D(z, r))$ denotes the Lebesgue measure of $D(z, r)$.

We can now state our main results.

**Theorem 1.1.** Suppose that $r > 0$, $1 < p < \infty$, $0 < q < \infty$ and that $\mu \in \mathcal{M}_+$. Then the following conditions are equivalent:

(i) $T_{\mu}$ is bounded on $A^p(\mathcal{U})$.

(ii) $\tilde{\mu}$ is a bounded function on $\mathcal{U}$.

(iii) $\hat{\mu}_r$ is a bounded function on $\mathcal{U}$.

(iv) $\mu$ is a Carleson measure for $A^q(\mathcal{U})$.

Let $\rho(z) := \text{Im} z_n - |z'|^2$ and $\partial U := \{z \in \mathbb{C}^n : \rho(z) = 0\}$ denotes the boundary of $\mathcal{U}$. Then $\bar{\mathcal{U}} := \mathcal{U} \cup \partial \mathcal{U} \cup \{\infty\}$ is the one-point compactification of $\mathcal{U}$. Also, let $\partial \bar{\mathcal{U}} := \partial \mathcal{U} \cup \{\infty\}$. Thus, $z \to \partial \mathcal{U}$ means $\rho(z) \to 0$ or $\rho(z) \to \infty$. We denote by $C_0(\mathcal{U})$ the space of complex-value continuous functions $f$ on $\mathcal{U}$ such that $f(z) \to 0$ as $z \to \partial \mathcal{U}$.

**Theorem 1.2.** Suppose that $r > 0$, $1 < p, q < \infty$ and that $\mu \in \mathcal{M}_+$. Then the following conditions are equivalent:

(i) $T_{\mu}$ is compact on $A^p(\mathcal{U})$.

(ii) $\tilde{\mu}$ belongs to $C_0(\mathcal{U})$.

(iii) $\hat{\mu}_r$ belongs to $C_0(\mathcal{U})$.

(iv) $\mu$ is a vanishing Carleson measure for $A^q(\mathcal{U})$.

This paper is organized as follows. Section 2 contains the necessary background material and auxiliary results. In Section 3, we characterize Carleson measures and vanishing Carleson measures for the Bergman spaces over the Siegel upper half-space. In Section 4, we show that the Toeplitz operators are well defined on a family of dense subspaces of the Bergman spaces. The proofs of the theorems 1.1 and 1.2 are carried out in Section 5.

Throughout the paper, the letter $C$ will denote a positive constant that may vary at each occurrence but is independent of the essential variables. The letter $C$ with subscripts usually denotes a specific constant.

2. Preliminaries

2.1. Estimates of the Bergman kernel. For simplicity, we write

$$\rho(z, w) := \frac{i}{2} (\overline{w_n} - z_n) - z' \cdot \overline{w'}.$$
With this notation, the Bergman kernel of $\mathcal{U}$

$$K(z, w) = \frac{n!}{4\pi^n} \frac{1}{\rho(z, w)^{n+1}}, \quad z, w \in \mathcal{U}.$$  

Note also that $\rho(z) = \rho(z, z)$.

**Lemma 2.1.** We have

$$|K(z, w)| \leq \frac{2^{n-1}n!}{\pi^n} \min\{\rho(z), \rho(w)\}^{-n-1}$$

for any $z, w \in \mathcal{U}$.

**Proof.** For each $t > 0$, we define the nonisotropic dilation $\delta_t$ by

$$\delta_t(u) = (tu', t^2u_n), \quad u \in \mathcal{U}.$$  

Also, to each fixed $z \in \mathcal{U}$, we associate the following (holomorphic) affine self-mapping of $\mathcal{U}$:

$$h_z(u) := (u' - z', u_n - \Re z_n - 2iu \cdot z' + i|z'|^2), \quad u \in \mathcal{U}.$$  

All these mappings are holomorphic automorphisms of $\mathcal{U}$. See [11, Chapter XII]. Hence the mappings $\sigma_z := \delta_{\rho(z)^{-1/2}} \circ h_z$ are holomorphic automorphisms of $\mathcal{U}$.

Simple calculations show that $\sigma_z(z) = i := (0', 0)$ and

$$\sigma_z(z) = \left(0', \sigma_z(z)\right) = \sigma_z(z) \rho(z)^{-n-1}.$$  

Note that $|K(i, u)| = \frac{n!}{4\pi^n} \frac{2^{n+1}}{|u_n + i|^{n+1}} \leq \frac{2^{n-1}n!}{\pi^n}$ for all $u \in \mathcal{U}$. Hence (2.1) follows immediately from (2.3).

**Lemma 2.2.** Let $1 < p < \infty$. Then for each $z \in \mathcal{U}$, the Bergman kernel function $K_z := K(\cdot, z)$ is in $A^p(\mathcal{U})$, and

$$\|K_z\|_p = C_{n, p} \rho(z)^{-(n+1)/p'},$$

where $p' = p/(p-1)$ and $C_{n, p}$ is a positive constant depending on $n$ and $p$.

**Proof.** This is just an application of the following formula from [10, Lemma 5]:

$$\int_{\mathcal{U}} \frac{\rho(w)^t}{\rho(z, w)^s} dV(w) = \begin{cases} C_{n, s, t} \rho(z)^{t-s-n-1}, & \text{if } t > -1 \text{ and } s - t > n + 1 \\ +\infty, & \text{otherwise} \end{cases}$$

for all $z \in \mathcal{U}$, where

$$C_{n, s, t} := \frac{4\pi^n \Gamma(1 + t) \Gamma(s - t - n - 1)}{\Gamma^2(s/2)}.$$  

We omit the details.
2.2. Cayley transform and the M"obius transformations. Recall that the Cayley transform $\Phi : B \to U$ is given by

$$(z', z_n) \mapsto \left( \frac{z'}{1 + z_n}, i \frac{1 - z_n}{1 + z_n} \right).$$

It is immediate to calculate that

$$\Phi^{-1} : (z', z_n) \mapsto \left( \frac{2iz'}{i + z_n}, \frac{i - z_n}{i + z_n} \right).$$

We refer to [11, Chapter XII] for the properties of the Cayley transform. For the convenience of later reference, we record the following lemma from [9].

**Lemma 2.3.**

(i) The identity

$$(2.6) \quad \rho(\Phi(\xi), \Phi(\eta)) = \frac{1 - \xi \cdot \eta}{(1 + \xi_n)(1 + \eta_n)}$$

holds for all $\xi, \eta \in B$.

(ii) The real Jacobian of $\Phi$ at $\xi \in B$ is

$$(2.7) \quad (J_R \Phi)(\xi) = \frac{4}{|1 + \xi_n|^{2(n+1)}}.$$

(iii) The identity

$$(2.8) \quad 1 - \Phi^{-1}(z) \cdot \Phi^{-1}(w) = \frac{\rho(z, w)}{\rho(z, i) \rho(i, w)}$$

holds for all $z, w \in U$, where $i = (0', i)$.

(iv) The identity

$$(2.9) \quad |\Phi^{-1}(z)|^2 = 1 - \frac{\rho(z)}{|\rho(z, i)|^2}$$

holds for all $z \in U$.

(v) The real Jacobian of $\Phi^{-1}$ at $z \in U$ is

$$(2.10) \quad (J_R \Phi^{-1})(z) = \frac{1}{4|\rho(z, i)|^{2(n+1)}}.$$

The group of all one-to-one holomorphic mappings of $B$ onto $\mathbb{B}$ (the so-called automorphisms of $B$) will be denoted by $\text{Aut}(B)$. It is generated by the unitary transformations on $\mathbb{C}^n$ along with the M"obius transformations $\varphi_\xi$ given by

$$\varphi_\xi(\eta) := \frac{\xi - P_\xi \eta - (1 - |\xi|^2)Q_\xi \eta}{1 - \eta \cdot \xi},$$

where $\xi \in B$, $P_\xi$ is the orthogonal projection onto the space spanned by $\xi$, and $Q_\xi \eta = \eta - P_\xi \eta$. See [13, Section 1.2].

It is easily shown that the mapping $\varphi_\xi$ satisfies

$$\varphi_\xi(0) = \xi, \quad \varphi_\xi(\xi) = 0, \quad \varphi_\xi(\varphi_\xi(\eta)) = \eta.$$ 

Furthermore, for all $\eta, \omega \in \overline{B}$, we have

$$(2.11) \quad 1 - \varphi_\xi(\eta) \cdot \varphi_\xi(\omega) = \frac{(1 - |\xi|^2)(1 - \eta \cdot \omega)}{(1 - \eta \cdot \xi)(1 - \xi \cdot \omega)}.$$
2.3. Bergman metric balls. Let $\Omega$ be a domain in $\mathbb{C}^n$ and $K_{\Omega}(z, w)$ be the Bergman kernel of $\Omega$. We define

$$g_{i,j}^\Omega(z) := \frac{1}{n+1} \frac{\partial^2 \log K_{\Omega}(z,z)}{\partial z_i \partial \bar{z}_j}, \quad i, j = 1, \ldots, n$$

and call the complex matrix $B_{\Omega}(z) = (g_{i,j}^\Omega(z))_{1 \leq i,j \leq n}$ the Bergman matrix of $\Omega$. For a $C^1$ curve $\gamma : [0, 1] \to \Omega$ we define

$$l_{\Omega}(\gamma) = \int_0^1 \left( B_{\Omega}(\gamma(t)) \gamma'(t) \cdot \overline{\gamma'(t)} \right)^{1/2} dt.$$

For any two points $z$ and $w$ in $\Omega$, let $\beta_{\Omega}(z,w)$ be the infimum of the set consisting of all $l_{\Omega}(\gamma)$, where $\gamma$ is a piecewise smooth curve in $\Omega$ from $z$ to $w$. We will call $\beta_{\Omega}$ the Bergman metric on $\Omega$. For $z \in \Omega$ and $r > 0$ we let $D_{\Omega}(z,r)$ denote the Bergman metric ball at $z$ with radius $r$. Thus

$$D_{\Omega}(z,r) := \{ w \in \Omega : \beta_{\Omega}(z,w) < r \}.$$

If $\Omega_1$, $\Omega_2$ are two domains in $\mathbb{C}^n$ and $h$ is a biholomorphic mapping of $\Omega_1$ onto $\Omega_2$, then

$$\beta_{\Omega_1}(z,w) = \beta_{\Omega_2}(h(z),h(w))$$

for all $z, w \in \Omega_1$. See for instance [7, Proposition 1.4.15]. Hence,

$$\beta_{\mathcal{U}}(z,w) = \beta_{\mathcal{B}}(\Phi^{-1}(z),\Phi^{-1}(w)) = \tanh^{-1} \left( |\varphi_{\Phi^{-1}(z)}(\Phi^{-1}(w))| \right).$$

It follows that

$$D_{\mathcal{U}}(z,r) = \Phi(D_{\mathcal{B}}(\Phi^{-1}(z),r))$$

for every $z \in \mathcal{U}$ and $r > 0$. Also, a computation shows that

$$\beta_{\mathcal{U}}(z,w) = \tanh^{-1} \sqrt{1 - \frac{\rho(z)\rho(w)}{|\rho(z,w)|^2}}.$$

In the sequel, we simply write $\beta(z,w) := \beta_{\mathcal{U}}(z,w)$ and $D(z,r) := D_{\mathcal{U}}(z,r)$ if not cause any confusion.

**Lemma 2.4.** For any $r > 0$, there exists a sequence $\{a_k\}$ in $\mathcal{U}$ such that

(i) $\mathcal{U} = \bigcup_{k=1}^\infty D(a_k,r)$;

(ii) There is a positive integer $N$ such that each point $z$ in $\mathcal{U}$ belongs to at most $N$ of the sets $D(a_k,2r)$.

We are going to call $\{a_k\}$ an $r$-lattice in the Bergman metric.

**Proof.** The proof is similar to that of [15, Theorem 2.23], so is omitted. \hfill $\square$

**Lemma 2.5.** For any $z \in \mathcal{U}$ and $r > 0$ we have

$$|D(z,r)| = \frac{4\pi^n}{n!} \frac{\tanh^{2n} r}{(1 - \tanh^2 r)^{n+1}} \rho(z)^{n+1}.\tag{2.14}$$

**Proof.** We first show that

$$|D(z,r)| = \rho(z)^{n+1} |D(i,r)|\tag{2.15}$$

holds for any $z \in \mathcal{U}$ and $r > 0$. 


Since the metric $\beta$ is invariant under the automorphisms, we have
\[
\beta(w, z) = \beta(\sigma_z(w), \sigma_z(z)) = \beta(z_1(w), 1)
\]
for all $z, w \in \mathcal{U}$, where $\sigma_z$ is as in Subsection 2.1. Hence $D(z, r) = \sigma_z^{-1}(D(i, r))$. It follows that
\[
|D(z, r)| = \int_{\sigma_z^{-1}(D(i, r))} dV(w) = \int_{D(i, r)} |(J_C \sigma_z^{-1})(u)|^2 dV(u).
\]
Combining with (2.2), this gives (2.15).

It remains to show that
\[
|D(z, r)| = \frac{4\pi^n}{n!} \frac{\tanh^{2n} r}{(1 - \tanh^2 r)^{n+1}}.
\]
Note that $D(i, r) = \Phi(B(0, R))$, where $R := \tanh(r)$ and $\Phi$ is the Cayley transform. Thus,
\[
|D(i, r)| = \frac{4\pi^n}{n!} \frac{\tanh^{2n} r}{(1 - \tanh^2 r)^{n+1}}.
\]
where the last equality follows a simple calculation, see for instance [8, p.263, (2.12)]. The proof is complete.

**Corollary 2.6.** For any $z \in \mathcal{U}$ and $r > 0$, the averaging function (defined as in (1.2))
\[
\hat{\mu}_r(z) = \frac{n!}{4\pi^n} \frac{(1 - \tanh^2 r)^{n+1} \mu(D(z, r))}{\rho(z)^{n+1}}.
\]

**Lemma 2.7.** Given $r > 0$, the inequalities
\[
1 - \tanh(r) \leq \frac{|\rho(z, u)|}{|\rho(z, v)|} \leq \frac{1 + \tanh(r)}{1 - \tanh(r)}
\]
hold for all $z, u, v \in \mathcal{U}$ with $\beta(u, v) \leq r$.

**Proof.** Let $\eta = \Phi^{-1}(z)$, $\xi = \Phi^{-1}(u)$ and $\zeta = \Phi^{-1}(v)$. We prove only the second inequality; then the first one follows by symmetry. By (2.12), then we have
\[
\beta_\eta(\xi, \zeta) = \beta(u, v) \leq \tanh(r).
\]
Also,
\[
\frac{\rho(z, u)}{\rho(z, v)} = \frac{(1 - \eta \cdot \xi)(1 + \xi_n)}{(1 - \eta \cdot \zeta)(1 + \zeta_n)}.
\]
Let \( \tilde{\eta} = \varphi_\xi(\eta) \) and \( \tilde{\zeta} = \varphi_\xi(\zeta) \). Then again by (2.12), \( |\tilde{\zeta}| \leq \tanh(r) \in (0, 1) \).

Appealing to (2.11), we have
\[
1 - \eta \cdot \tilde{\zeta} = 1 - \varphi_\xi(\tilde{\eta}) \cdot \varphi_\xi(0) = \frac{1 - |\xi|^2}{1 - \tilde{\eta} \cdot \tilde{\zeta}}
\]
and
\[
1 - \eta \cdot \tilde{\zeta} = 1 - \varphi_\xi(\tilde{\eta}) \cdot \varphi_\xi(\tilde{\zeta}) = \frac{(1 - |\xi|^2)(1 - \tilde{\eta} \cdot \tilde{\zeta})}{(1 - \tilde{\eta} \cdot \tilde{\zeta})(1 - \xi \cdot \zeta)}
\]
Thus we get
\[
\frac{1 - \eta \cdot \tilde{\zeta}}{1 - \eta \cdot \zeta} = \frac{1 - \xi \cdot \zeta}{1 - \tilde{\eta} \cdot \tilde{\zeta}}
\]
Likewise,
\[
\frac{1 + \tilde{\zeta}_n}{1 + \zeta_n} = \frac{1 - \tilde{\omega} \cdot \tilde{\zeta}}{1 - \xi \cdot \zeta},
\]
where \( \tilde{\omega} := \varphi_\xi(-e_n) \). Hence,
\[
\frac{\rho(z, u)}{\rho(z, v)} - 1 = \frac{\tilde{\eta} \cdot \tilde{\zeta} - \tilde{\omega} \cdot \tilde{\zeta}}{(1 - \tilde{\eta} \cdot \tilde{\zeta})}.
\]
Since \( |\tilde{\zeta}| \leq \tanh(r) \), we have
\[
|1 - \tilde{\eta} \cdot \tilde{\zeta}| \geq 1 - \tanh(r)
\]
and
\[
|\tilde{\eta} \cdot \tilde{\zeta} - \tilde{\omega} \cdot \tilde{\zeta}| \leq 2|\tilde{\zeta}| \leq 2 \tanh(r).
\]
It follows that
\[
\left| \frac{\rho(z, u)}{\rho(z, v)} - 1 \right| \leq \frac{2 \tanh(r)}{1 - \tanh(r)},
\]
which implies the asserted inequality.

In what follows we use the notation
\[
Q_j := D(i, j)
\]
for \( j = 1, 2, \ldots \). Note that \( Q_j \subset Q_{j+1} \) for all \( j \in \mathbb{N} \) and \( \mathcal{U} = \bigcup_{j=1}^\infty Q_j \).

**Lemma 2.8.** Given \( \lambda \in \mathbb{R} \) and \( j \in \mathbb{N} \), there is a constant \( C = C(n, \lambda, j) > 0 \) such that
\[
\sup_{w \in Q_j} \frac{\rho(w)^\lambda}{|\rho(z, w)|^{n+1+\lambda}} \leq \frac{C}{|\rho(z, i)|^{n+1+\lambda}}
\]
for all \( z \in \mathcal{U} \).

**Proof.** Let \( w \in Q_j \) be fixed. We first show that
\[
\frac{1 - \tanh^2(j)}{4} \leq \rho(w) \leq \frac{4}{1 - \tanh^2(j)}.
\]
Note from (2.13) that
\[
\rho(w) = |\rho(w, i)|^2 \left(1 - \tanh^2 \beta(w, i)\right).
\]
Since $\beta(w, i) < j$ and $|\rho(w, i)|^2 > 1/4$, the first inequality in (2.18) follows. Again, it follows from (2.13) that

$$\rho(w) = \left( \frac{\rho(w)}{|\rho(w, i)|} \right)^2 \frac{1}{1 - \tanh^2 \beta(w, i)}.$$  

The second inequality in (2.18) is then immediate, in view of (2.1). Now, the assertion of the lemma follows from (2.18) and Lemma 2.7. \qed

2.4. Growth rate for functions in $A^p(U)$.

**Lemma 2.9.** Suppose $r > 0$ and $p > 0$. Then there exists a positive constant $C$ depending on $r$ such that

$$|f(z)|^p \leq \frac{C}{\rho(z)^{n+1}} \int_{D(z, r)} |f(w)|^p dV(w)$$

for all $f \in H(U)$ and all $z \in U$.

**Proof.** Let $f \in H(U)$. Then $f \circ \Phi \in H(\mathbb{B})$. Note that $D(i, r) = \Phi(B(0, R))$ with $R = \tanh(r)$. By the subharmonicity of $|f|^p$, we get

$$|f(\Phi(0))|^p \leq \frac{n!}{\pi^n R^{2n}} \int_{B(0, R)} |f(\Phi(\xi))|^p dV(\xi)$$

$$= \frac{n!}{4 \pi^n R^{2n}} \int_{D(i, r)} |f(w)|^p \frac{1}{\rho(w, 1)^{2(n+1)}} dV(w).$$

Note that $f(\Phi(0)) = f(i)$ and $\inf\{ |\rho(w, i)| : w \in U \} \geq 1/2$. Then we have

$$|f(i)|^p \leq C \int_{D(i, r)} |f(w)|^p dV(w),$$

with $C := 4^n n! / (\pi^n R^{2n})$. Replacing $f$ by $f \circ \sigma_z^{-1}$ in the above inequality, we arrive at

$$|f(z)|^p \leq C \int_{D(i, r)} |f(\sigma_z^{-1}(w))|^p dV(w)$$

$$= \frac{C}{\rho(z)^{n+1}} \int_{D(z, r)} |f(u)|^p dV(u).$$

This completes the proof of the Lemma. \qed

**Corollary 2.10.** Suppose $0 < p < \infty$. Then

$$|f(z)| \leq \left( \frac{4^n n!}{\pi^n} \right)^{1/p} \frac{\|f\|_p}{\rho(z)^{(n+1)/p}}, \quad z \in U$$

for all $f \in A^p(U)$. 

2.5. Weak convergence in \( A^p(\mathcal{U}) \).

**Lemma 2.11.** Assume \( \{f_j\} \) is a sequence in \( A^p(\mathcal{U}) \) with \( 1 < p < \infty \). Then \( f_j \to 0 \) weakly in \( A^p(\mathcal{U}) \) if and only if \( \{f_j\} \) is bounded in \( A^p(\mathcal{U}) \) and converges to 0 uniformly on each compact subset of \( \mathcal{U} \).

**Proof.** The proof of the sufficiency is not hard. The proof of the necessity is also a standard normal family argument. We include a proof for reader’s convenience. Suppose \( \{f_j\} \) converges to 0 weakly in \( A^p(\mathcal{U}) \). Then, by the uniform boundedness principle, \( \{f_j\} \) is bounded in \( A^p(\mathcal{U}) \). Together with Corollary 2.10, this implies that \( \{f_j\} \) is uniformly bounded on each compact subset of \( \mathcal{U} \) and thus is a normal family.

Note that \( f_j \to 0 \) pointwise by the reproducing property of the kernel \( K_z \in A_{p'}(\mathcal{U}) \) (see [5, Theorem 2.1]) with \( p' = p/(p - 1) \). (In fact, by (2.5), for every \( z \in \mathcal{U} \), \( K_z \) belongs to \( A_t(\mathcal{U}) \) for all \( 1 < t < \infty \).) Now, by a standard argument, we see that \( f_j \to 0 \) uniformly on each compact subset of \( \mathcal{U} \). The proof is complete. \( \square \)

**Lemma 2.12.** For \( 1 < p < \infty \), we have \( K_z\|K_z\|_p^{-1} \to 0 \) weakly in \( A^p(\mathcal{U}) \) as \( z \to \partial\mathcal{U} \).

**Proof.** In view of lemma 2.11, it suffices to prove that \( K_z\|K_z\|_p^{-1} \) converges to 0 uniformly on every \( Q_j \).

By Lemmas 2.2 and 2.7 there exists a constant \( C > 0 \) such that

\[
\sup_{w \in Q_j} \frac{|K_z(w)|}{\|K_z\|_p} \leq C \frac{\rho(z)^{(n+1)/p'}}{|\rho(z, i)|^{n+1}}
\]

for all \( z \in \mathcal{U} \). Since \( 2|\rho(z, i)| = |z_n + i| \geq 1 \) for all \( z \in \mathcal{U} \), we have

\[
\sup_{w \in Q_j} \frac{|K_z(w)|}{\|K_z\|_p} \leq C \rho(z)^{(n+1)/p'},
\]

which implies that \( K_z\|K_z\|_p^{-1} \to 0 \) uniformly on \( Q_j \) as \( z \to b\mathcal{U} \). On the other hand, by (2.1) and the fact that \( 2|\rho(z, i)| \geq |z| \) for all \( z \in \mathcal{U} \),

\[
\sup_{w \in Q_j} \frac{|K_z(w)|}{\|K_z\|_p} \leq \frac{C}{|z|^{(n+1)/p'}}
\]

which implies that \( K_z\|K_z\|_p^{-1} \to 0 \) uniformly on \( Q_j \) as \( |z| \to \infty \). The proof of the lemma is complete. \( \square \)

3. Carleson measures

**Theorem 3.1.** Suppose \( 0 < p < \infty \), \( r > 0 \) and \( \mu \) is a positive Borel measure. Then the following conditions are equivalent:

(a) \( \mu \) is a Carleson measure for \( A^p(\mathcal{U}) \).

(b) There exists a constant \( C > 0 \) such that

\[
\int_{\mathcal{U}} \frac{\rho(a)^{n+1}}{|\rho(z, a)|^{2(n+1)}} d\mu(z) \leq C
\]

for all \( a \in \mathcal{U} \).
There exists a constant \( C > 0 \) such that
\[
\mu(D(a, r)) \leq C \rho(a)^{n+1}
\]
for all \( a \in \mathcal{U} \).

There exists a constant \( C > 0 \) such that
\[
\mu(D(a_k, r)) \leq C \rho(a_k)^{n+1}
\]
for all \( k \geq 1 \), where \( \{a_k\} \) is an \( r \)-lattice in the Bergman metric.

**Proof.** It is easy to see that (a) implies (b). In fact, setting
\[
f(z) = \left[ \frac{\rho(a)^{n+1}}{\rho(z, a)^{2(n+1)}} \right]^{1/p}
\]
in (a) immediately yields (b).

If (b) is true, then
\[
\int_{D(a, r)} \frac{\rho(a)^{n+1}}{\rho(z, a)^{2(n+1)}} d\mu(z) \leq C
\]
for all \( a \in \mathcal{U} \). This along with Lemma 2.7 shows that (c) must be true.

That (c) implies (d) is trivial.

It remains to prove that (d) implies (a). So we assume that there exists a constant \( C_1 > 0 \) such that
\[
\mu(D(a_k, r)) \leq C_1 \rho(a_k)^{n+1}
\]
for all \( k \geq 1 \). If \( f \) is holomorphic in \( \mathcal{U} \), then
\[
\int_{\mathcal{U}} |f|^p d\mu \leq \sum_{k=1}^{\infty} \int_{D(a_k, r)} |f(z)|^p d\mu(z)
\]
\[
\leq \sum_{k=1}^{\infty} \mu(D(a_k, r)) \sup \{|f(z)|^p : z \in D(a_k, r)| \}
\]
By Lemmas 2.4 and 2.7, there exists a constant \( C_2 > 0 \) such that
\[
\sup \{|f(z)|^p : z \in D(a_k, r)| \} \leq \frac{C_2}{\rho(a_k)^{n+1}} \int_{D(a_k, 2r)} |f(w)|^p dV(w)
\]
for all \( k \geq 1 \). It follows that
\[
\int_{\mathcal{U}} |f|^p d\mu \leq C_2 \sum_{k=1}^{\infty} \frac{\mu(D(a_k, r))}{\rho(a_k)^{n+1}} \int_{D(a_k, 2r)} |f(w)|^p dV(w)
\]
\[
\leq C_1 C_2 \sum_{k=1}^{\infty} \int_{D(a_k, 2r)} |f(w)|^p dV(w)
\]
\[
\leq C_1 C_2 N \int_{\mathcal{U}} |f(w)|^p dV(w),
\]
where \( N \) is as in Lemma 2.4. This completes the proof of the theorem. \( \square \)

It follows from the above theorem that the property of being a Carleson measure for \( A^p(\mathcal{U}) \) is independent of \( p \), so that if \( \mu \) is a Carleson measure for \( A^p(\mathcal{U}) \) for some \( p \), then \( \mu \) is a Carleson measure for \( A^p(\mathcal{U}) \) for all \( p \).
**Theorem 3.2.** Suppose $1 < p < \infty$, $r > 0$ and $\mu$ is a positive Borel measure. Then the following conditions are equivalent:

(a) $\mu$ is a vanishing Carleson measure for $A^p(\mathcal{U})$.
(b) The measure $\mu$ satisfies

\[
\lim_{a \to \partial \mathcal{U}} \int_{\mathcal{U}} \frac{\rho(a)^{n+1}}{|\rho(z, a)|^{2(n+1)}} d\mu(z) = 0
\]

(c) The measure $\mu$ has the property that

\[
\lim_{a \to \partial \mathcal{U}} \frac{\mu(D(a, r))}{\rho(a)^{n+1}} = 0.
\]

(d) For $\{a_k\}$ an $r$-lattice in the Bergman metric, we have

\[
\lim_{k \to \infty} \frac{\mu(D(a_k, r))}{\rho(a_k)^{n+1}} = 0.
\]

**Proof.** If (a) is true, it means that the inclusion map $i_p$ is compact. By Lemma 2.11 and Lemma 2.12, we see that $k_a$ converges to 0 uniformly on each compact subset of $\mathcal{U}$ as $a \to \partial \mathcal{U}$, and so does $g_a := k_a^2/\rho$. It is obvious that $\|g_a\|_p = 1$. Then by Lemma 2.11 again, $g_a$ converges to 0 weakly in $A^p(\mathcal{U})$ as $a \to \partial \mathcal{U}$. Therefore,

\[
\lim_{a \to \partial \mathcal{U}} \int_{\mathcal{U}} \frac{\rho(a)^{n+1}}{|\rho(z, a)|^{2(n+1)}} d\mu(z) = C \int_{\mathcal{U}} |g_a(z)|^p d\mu(z) \to 0
\]
as $a \to \partial \mathcal{U}$. So (b) follows.

If (b) holds, then

\[
\lim_{a \to \partial \mathcal{U}} \int_{D(a, r)} \frac{\rho(a)^{n+1}}{|\rho(z, a)|^{2(n+1)}} d\mu(z) = 0.
\]

By Lemma 2.11 $|\rho(z, a)|$ and $\rho(z)$ are comparable when $z \in D(a, r)$. So (c) follows.

Note that $a_k \to \partial \mathcal{U}$ as $k \to \infty$ if $\{a_k\}$ is an $r$-lattice in the Bergman metric. That (c) implies (d) is immediate.

It remains to prove that (d) implies (a). Assume that (d) is true and $\{f_j\}$ is a sequence in $A^p(\mathcal{U})$ that converges to 0 weakly. We only need to prove that

\[
(3.1) \quad \lim_{j \to \infty} \int_{\mathcal{U}} |f_j(z)|^p d\mu(z) = 0.
\]

By Lemma 2.11 $\{f_j\}$ has the property that $\sup_j \|f_j\|_p \leq M$ for some positive constant $M$ and converges to 0 uniformly on each compact subset of $\mathcal{U}$. By assumption, given $\varepsilon > 0$ there exists a positive integer $N_0$ such that

\[
\frac{\mu(D(a_k, r))}{\rho(a_k)^{n+1}} < \varepsilon \quad \text{for all } k \geq N_0.
\]
By the last part of the proof of Theorem 3.1, there is a constant \( C > 0 \) such that
\[
\sum_{k=N_0}^{\infty} \int_{D(a_k, r)} |f_j(w)|^p \, d\mu(w) \\
\leq C \sum_{k=N_0}^{\infty} \frac{\mu(D(a_k, r))}{\rho(a_k)^{n+1}} \int_{D(a_k, 2r)} |f_j(w)|^p \, dV(w) \\
\leq \varepsilon C N \int_{U} |f_j(w)|^p \, dV(w) \leq \varepsilon C N M^p
\]
for all \( j \), where \( C, N, \) and \( M \) are all independent of \( \varepsilon \). Since
\[
\lim_{j \to \infty} \sum_{k=1}^{N_0-1} \int_{D(a_k, r)} |f_j(z)|^p \, d\mu(z) = 0
\]
by uniform convergence. Therefore,
\[
\limsup_{j \to \infty} \int_{U} |f_j(z)|^p \, d\mu(z) \\
\leq \limsup_{j \to \infty} \left[ \sum_{k=1}^{N_0-1} \int_{D(a_k, r)} |f_j(z)|^p \, d\mu(z) + \sum_{k=N_0}^{\infty} \int_{D(a_k, r)} |f_j(w)|^p \, dV(w) \right] \\
\leq \varepsilon C N M^p.
\]
Since \( \varepsilon \) is arbitrary, (3.1) follows. The proof of the theorem is complete.

It follows from the above theorem that the property of being a vanishing Carleson measure for \( A^p(U) \) depends neither on \( p \) nor on \( r \).

4. Dense subspaces of \( A^p(U) \)

Given \( \alpha \) real, we denote by \( S_\alpha \) the vector space of functions \( f \) holomorphic in \( U \) satisfying
\[
\sup_{z \in U} |z^n + i^\alpha |f(z)| < \infty.
\]

**Theorem 4.1.** If \( 1 \leq p < \infty \) and \( \alpha > n + 1/p \), then \( S_\alpha \) is a dense subspace of \( A^p(U) \).

**Proof.** It is immediate from (2.5) that \( S_\alpha \) is contained in \( A^p(U) \) whenever \( \alpha > n + 1/p \).

We now prove the density of \( S_\alpha \) in \( A^p(U) \). Let \( f \) be arbitrary in \( A^p(U) \). Put
\( f_j = f \cdot \chi_{Q_j} \) for \( j = 1, 2, \ldots \), where \( Q_j := D(1, j) \) and \( \chi_{Q_j} \) is the characteristic function of \( Q_j \). Clearly, \( \|f_j - f\|_p \to 0 \) as \( j \to \infty \).

Given \( \lambda > -1 \), let \( P_\lambda \) be the integral operator given by
\[
P_\lambda g(z) = c_\lambda \int_{U} \frac{\rho(w)^\lambda}{\rho(z, w)^{n+1+\lambda}} g(w) \, dV(w), \quad z \in U,
\]
where \( c_\lambda = \Gamma(n+1+\lambda)/(4\pi^n \Gamma(1+\lambda)) \). It was shown in [3] Theorem 3.1 that \( P_\lambda \) is a bounded projection from \( L^p(U) \) onto \( A^p(U) \), provided that \( \lambda > 1/p - 1 \).
Take $\lambda = \alpha - n - 1$. By Hölder’s inequality, we obtain

$$|P_{\alpha-n-1}f_j(z)| \leq C_{\alpha-n-1} \|f\|_p |Q_j|^{1/p'} \sup_{w \in Q_j} \frac{\rho(w)^{\alpha-n-1}}{\rho(z,w)^{\alpha}}$$

for all $z \in U$, where $p' = p/(p-1)$ and $|Q_j|$ stands for the Lebesgue measure of $Q_j$. Combining this inequality with Lemma 2.8, we obtain

$$|P_{\alpha-n-1}f_j(z)| \leq C \|f\|_p |\rho(z,i)|^\alpha$$

for all $z \in U$, where $C > 0$ is a constant depending on $n$, $\alpha$, $j$ and $p$. Thus, $P_{\alpha-n-1}f_j \in S_{\alpha}$.

Since $f \in A^p(U)$ and $P_{\alpha-n-1}$ is a bounded projection from $L^p(U)$ onto $A^p(U)$,

$$\|P_{\alpha-n-1}f_j - f\|_p = \|P_{\alpha-n-1}(f_j - f)\|_p \leq \|P_{\alpha-n-1}\| \|f_j - f\|_p \to 0$$

as $j \to \infty$. This implies that $S_{\alpha}$ is dense in $A^p(U)$. □

**Corollary 4.2.** The Toeplitz operator $T_\mu$ with symbol $\mu \in M_+$ is densely defined on $A^p(U)$ for every $1 < p < \infty$.

**Proof.** It suffices to show that

$$\int_U |K(z,w)f(w)|d\mu(w) < \infty$$

holds for every $f \in S_{\alpha}$ with $\alpha > 0$, and for each fixed $z \in U$. Indeed, it follows by (2.1) that there exists a constant $C > 0$ depending on $n$ and $\alpha$ such that

$$\int_U |K(z,w)f(w)|d\mu(w) \leq C \rho(z)^{-n-1} \int_U \frac{d\mu(w)}{|w_n+i|^\alpha} < \infty,$$

as desired. □

## 5. Proofs of the Theorems

Just like the cases of the unit disk or bounded symmetric domains, the key step is the justification of the equality

$$\langle T_\mu f, g \rangle = \int_U f\overline{g}d\mu,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $A^p(U)$ and $A^p(U)$. This would enable us to make a connection between Carleson measures and positive Toeplitz operators.

**Lemma 5.1.** Let $0 < \alpha < n+1$. Then there exists a constant $C > 0$ such that

$$\int_U \frac{dV(u)}{|\rho(z,u)|^{n+1}|\rho(u,i)|^\alpha} \leq C \left( 1 + \log \frac{|\rho(z,i)|^2}{\rho(z)} \right)$$

for any $z \in U$. 

Proof. Given \( z \in \mathcal{U} \), let \( \eta := \Phi^{-1}(z) \). Making the change of variables \( u = \Phi(\xi) \) in the integral and using Lemma 2.3 we obtain

\[
\int_{\mathcal{U}} \frac{dV(u)}{|\rho(z,u)|^{n+1} |\rho(u,\imath)|^\alpha} = 4|1 + \eta_n|^{n+1} \int_{\mathcal{B}} \frac{dV(\xi)}{|1 - \eta \cdot \overline{\xi}|^{n+1} |1 + \xi_n|^{n+1-\alpha}}.
\]

By [13, Theorem 3.1], the last integral is dominated by a constant multiple of

\[
\frac{1}{|1 + \eta_n|^{n+1-\alpha}} \log \left( \frac{e}{|1 - \eta \cdot \varphi(-\epsilon_n)|} \right) = \frac{1}{|1 + \eta_n|^{n+1-\alpha}} \left( 1 + \log \frac{|1 + \eta_n|}{1 - |\eta|^2} \right),
\]

where \( \epsilon_n := (0', 1) \). Thus, there exists a constant \( C > 0 \) such that

\[
\int_{\mathcal{U}} \frac{dV(u)}{|\rho(z,u)|^{n+1} |\rho(u,\imath)|^\alpha} \leq C|1 + \eta_n|^\alpha \left( 1 + \log \frac{1}{1 - |\eta|^2} \right) = \frac{C}{|\rho(z,\imath)|^{\alpha}} \left( 1 + \log \frac{|\rho(z,\imath)|^2}{\rho(z)} \right)
\]

as desired. \( \square \)

**Lemma 5.2.** Suppose that \( 1 < p < \infty \), \( n + 1/p < \alpha < n + 1 \) and that \( \mu \in \mathcal{M}_+ \) is a Carleson measure for \( A^p(\mathcal{U}) \). Then \( T_\mu \) maps \( \mathcal{S}_\alpha \) into \( A^p(\mathcal{U}) \).

Proof. The proof of Corollary 4.2 shows that \( T_\mu \) is well defined on \( \mathcal{S}_\alpha \). Let \( f \in \mathcal{S}_\alpha \). We proceed to show that \( T_\mu f \in A^p(\mathcal{U}) \). Since \( \mu \) is a Carleson measure for \( A^1(\mathcal{U}) \), there exists a constant \( C > 0 \) such that

\[
\int_{\mathcal{U}} |K(z,w)f(w)|d\mu(w) \leq C \int_{\mathcal{U}} |K(z,w)f(w)|dV(w)
\]

\[
\leq C \int_{\mathcal{U}} \frac{dV(u)}{|\rho(z,u)|^{n+1} |\rho(u,\imath)|^\alpha}.
\]

This, together with (5.2), implies

\[
|T_\mu f(z)| \leq \frac{C}{|\rho(z,\imath)|^{\alpha}} \left( 1 + \log \frac{|\rho(z,\imath)|^2}{\rho(z)} \right)
\]

for all \( z \in \mathcal{U} \). Since \( \log x < x^\epsilon \) holds for any \( x > 1 \) and any \( \epsilon > 0 \), we have

\[
\log \frac{|\rho(z,\imath)|^2}{\rho(z)} < \frac{|\rho(z,\imath)|^{2\epsilon}}{\rho(z)^\epsilon}
\]

for all \( z \in \mathcal{U} \). Choose \( \epsilon \) small enough so that \( 0 < \epsilon < \min\{1/p, \alpha - (n + 1)/p\} \). Then we have

\[
\int_{\mathcal{U}} |T_\mu f(z)|^p dV(z) \leq C \left( \int_{\mathcal{U}} \frac{dV(z)}{|\rho(z,\imath)|^{p\alpha}} + \int_{\mathcal{U}} \frac{\rho(z)^{-p\epsilon}}{|\rho(z,\imath)|^{p(\alpha - 2\epsilon)}} dV(z) \right),
\]

which along with (5.1) completes the proof. \( \square \)

**Lemma 5.3.** Suppose that \( 1 < p < \infty \), \( n + 1/p < \alpha < n + 1 \), \( \gamma > n + (p - 1)/p \) and that \( \mu \in \mathcal{M}_+ \) is a Carleson measure for \( A^1(\mathcal{U}) \). Then (5.1) holds for all \( f \in \mathcal{S}_\alpha \) and \( g \in \mathcal{S}_\gamma \).
Proof. Let \( f \in \mathcal{S}_\alpha \) and \( g \in \mathcal{S}_\gamma \). In view of Theorem 4.1 and Lemma 5.2, we see that \( f \in A^p(U) \), \( g \in A^{p'}(U) \) and \( T_\mu f \in A^p(U) \). Then both sides of (5.1) are well defined. By Fubini’s theorem,

\[
\langle T_\mu f, g \rangle = \int_U \left( \int_U K(z, w)f(w)d\mu(w) \right) \overline{g(z)}dV(z) = \int_U f(w) \left( \int_U K(w, z)g(z)dV(z) \right) d\mu(w) = \int_U f(w)\overline{g(w)}d\mu(w),
\]

where the last equality follows from [5, Theorem 2.1]. The interchange of the order of integration is justified as follows. By (5.3) and (5.4), with \( \varepsilon \in (0, 1) \), we obtain

\[
\int_U \left( \int_U |K(z, w)f(w)g(z)| d\mu(w) \right) dV(z) \leq C \int_U \frac{1}{|\rho(z, i)|^{\alpha+\gamma}} \left( 1 + \frac{|\rho(z, i)|^{2\varepsilon}}{\rho(z)^\varepsilon} \right)^{\frac{1}{2(\alpha+\gamma)}} dV(z)
\]

\[
\leq C \left( \int_U \frac{dV(z)}{|\rho(z, i)|^{\alpha+\gamma}} + \int_U \frac{\rho(z)^{-\varepsilon}}{|\rho(z, i)|^{\alpha+\gamma-2\varepsilon}} dV(z) \right),
\]

which is finite, in view of (2.5). The proof of the lemma is complete.

Corollary 5.4. Suppose that \( \mu \in M_+ \) is a Carleson measure for \( A^q(U) \) for some \( q > 0 \). Then \( T_\mu \) is densely defined and extends to a bounded operator on \( A^p(U) \) for any \( p > 1 \). Moreover, (5.1) holds for all \( f \in A^p(U) \) and \( g \in A^{p'}(U) \), where \( p' = \frac{p}{p-1} \).

Now we are in the position to prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. Combining Theorem 3.1 with Corollary 2.6, we see that (ii), (iii) and (iv) are equivalent. Also, that (iv) implies (i) is immediate from Corollary 5.4. So we only need to prove that (i) implies (ii).

Assume that \( T_\mu \) is bounded on \( A^p(U) \). For every \( z \in U \), by Lemma 2.2 \( \langle T_\mu k_z, k_z \rangle \) is well defined. Lemma 2.2 also yields the following identity:

\[
\|K_z\|_p\|K_z\|_{p'} = CK(z, z),
\]

where \( C \) is a positive constant depending on \( p \) and \( n \). Hence

\[
|\langle T_\mu k_z, k_z \rangle| \leq \frac{\|T_\mu K_z\|_p\|K_z\|_{p'}}{K(z, z)} = C \left\| T_\mu \left( \frac{K_z}{\|K_z\|_p} \right) \right\|_p \leq C\|T_\mu\|.
\]

On the other hand, again by [5, Theorem 2.1], we have

\[
\langle T_\mu k_z, k_z \rangle = \frac{\langle T_\mu k_z, K_z \rangle}{\sqrt{K(z, z)}} = \frac{(T_\mu k_z)(z)}{\sqrt{K(z, z)}} = \tilde{\mu}(z).
\]

Hence, \( \tilde{\mu} \) is a bounded function on \( U \).
Proof of Theorem 1.3. Combining Theorem 3.2 with Corollary 2.6 we see that (ii), (iii) and (iv) are equivalent. Therefore, it will suffice to prove the implications (i) \( \Rightarrow \) (ii) and (iv) \( \Rightarrow \) (i).

(i) \( \Rightarrow \) (ii). Assume that \( T_\mu \) is compact on \( A^p(\mathcal{U}) \) for some \( p > 1 \). As is shown in the proof of Theorem 1.1,

\[
|\tilde{\mu}(z)| \leq C \left\| T_\mu \left( \frac{K_z}{\|K_z\|_p} \right) \right\|_p
\]

for all \( z \in \mathcal{U} \). This, together with Lemma 2.12 and the compactness of \( T_\mu \), implies \( \tilde{\mu} \in C_0(\mathcal{U}) \).

(iv) \( \Rightarrow \) (i). Assume that \( \mu \) is a vanishing Carleson measure for \( A^q(\mathcal{U}) \). Then \( \mu \) is also a vanishing Carleson measure for \( A^q(\mathcal{U}) \), where \( p' := p/(p - 1) \), hence

\[
\sup \{ \|g\|_{L^{p'}(\mu)} : \|g\|_{p'} = 1 \}
\]

is finite. Also, by Theorem 1.1 \( T_\mu \) is bounded on \( A^p(\mathcal{U}) \). Therefore, by Corollary 5.4 we have

\[
\|T_\mu f\|_p = \sup \{ \langle T_\mu f, g \rangle : \|g\|_{p'} = 1 \}
\]

\[
= \sup \left\{ \int_{\mathcal{U}} f g d\mu : \|g\|_{p'} = 1 \right\}
\]

\[
\leq \|f\|_{L^p(\mu)} \sup \{ \|g\|_{L^{p'}(\mu)} : \|g\|_{p'} = 1 \}
\]

for any \( f \in A^p(\mathcal{U}) \). If \( f_j \to 0 \) weakly in \( A^p(\mathcal{U}) \), then the compactness of \( i_p \) implies that \( \|f_j\|_{L^p(\mu)} \to 0 \), and hence \( \|T_\mu f_j\|_p \to 0 \). This implies that \( T_\mu \) is compact on \( A^p(\mathcal{U}) \). The proof of the theorem is complete. \( \square \)

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