Generalized Coherent States for q-oscillator connected with discrete q-Hermite Polynomials

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We are continuing here the study of generalized coherent states of Barut-Girardello type for the oscillator-like systems connected with the given set of orthogonal polynomials. In this work we construct the family of coherent states associated with discrete \(q\)-Hermite polynomials of the II-type and prove the over-completeness of this family of states by constructing the measure for unity decomposition for this family of coherent states.

1 Introduction

It is difficult to overestimate the role played by coherent states in the quantum theory. This explains a great many of works in which discuss the properties of coherent states and their different generalizations as well as their numerous applications in fields of modern physics, first of all in quantum optics. Basic facts relating to the theory of coherent states can be found in works \cite{[1]}—\cite{[9]}. An extensive bibliography (up to early 21st century) was given in the review \cite{[10]}. Only in the last few months there appeared a number of interesting works devoted to generalization of vector coherent states for the case of matrix index (\cite{[11]}—\cite{[14]}), hypergeometric \cite{[15]} and combinatorial \cite{[16]} coherent states, as well as to further development of mathematical \cite{[17], [18]} and physical \cite{[19], [20]} applications of coherent states, including applications to the description of models with exactly solvable potentials \cite{[21]}, super-symmetric conformal field theory \cite{[22]} and to path integral \cite{[23]} in holomorphic representation.

Coherent states were first introduced for the case of the boson-oscillator (connected with Heisenberg group) \cite{[24]} and then they were rediscovered in \cite{[25]}—\cite{[28]} since quantum optics inception. Now coherent states are determined for a wide class of quantum systems (including quantum field ones), and also for systems connected with other groups (including super-groups). Coherent states can be determined also for quantum groups as well as using of various exponential generalizations (deformations).

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Nowadays some variant definition of coherent states are known:

1) As eigenfunctions of the annihilation operator: \( a|z\rangle = z|z\rangle, \) \( z \in \mathbb{C} \) (in this case coherent states are usually called \( \text{Barut - Girardello coherent states} \), so in the work [29] this definition was extended to the case of non-compact groups);

2) As a result of action of the unitary shift operator \( D(z) = e^{za^\dagger - z^*a} \) upon the fixed vector of the state space (usually the Fock vacuum state \( |0\rangle \)): \( |z\rangle = D(z)|0\rangle \) (\textit{coherent states of Perelomov type}, who actively studied these states in the research summarized in [1]);

3) As states minimizing the Heisenberg (or Heisenberg - Robertson) uncertainty relations;

4) As states satisfying several natural conditions - norm-ability, continuity for an index, (over)completeness and the existence of the decomposition of unity connected with it, and sometimes evolutionary stability (the \textit{coherent states of Klauder - Gazeau type}[30, 31]).

In the case of boson- oscillator all these definitions generate the same family of coherent states; however, it is not in a general case.

It is known that standard coherent states of one boson-oscillator are defined by the relation

\[
|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}} |k\rangle, \quad z \in \mathbb{C}.
\]

The generalizations of coherent states connected with this definition (they are known as nonlinear, generalized or deformed coherent states) have the form

\[
|z\rangle = (\mathcal{N}(|z|^2))^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{\rho_k!}} e_k, \quad z \in \mathcal{D} \subseteq \mathbb{C}.
\]

The \( \mathcal{N} \) factor is a normalizing coefficient, \( \{e_k\}_{k=0}^{\infty} \) is orthonormal basis in Hilbert space \( \mathfrak{F}_F \) (considered also as the Fock basis and the Fock space, respectively), \( \rho_k! \) is a generalized factorial on an index \( \rho_k! = \rho_1 \cdot \rho_2 \cdots \cdot \rho_k \), with the \( \rho_0! = 1 \) constraint. The possible choice of the \( \{\rho_k\}_{k=0}^{\infty} \) sequence of positive numbers is limited by the (over)completeness requirement for the family of coherent states

\[
\int_{\mathcal{D}} d\mu(z, \bar{z}) \mathcal{N}(|z|^2) |z\rangle\langle z| = I,
\]

where \( I \) is the identity operator in \( \mathfrak{F}_F \), and the \( d\mu \) measure gives a solution of the moment problem [32] connected with the \( \{\rho_k\}_{k=0}^{\infty} \) sequence (see also [34, 33]). Distinct families of generalized coherent states differ by a distinct choice of these sequences. It is known that if for the chosen sequence (or equivalently for the chosen family of the generalized coherent states) the series

\[
\sum_{k=0}^{\infty} \frac{1}{\sqrt{\rho_k}}.
\]

is divergent, then one can relate with this sequence the appropriate family \( \{p_k\}_{k=0}^{\infty} \) of polynomials orthonormal with respect to the uniquely determined measure \( d\nu \) on the real line, which make up Fock basis \( \{e_k\}_{k=0}^{\infty} \) in Fock space \( \mathfrak{F}_F = L^2(\mathbb{R}; d\nu) \). These polynomials generate an oscillator-like system, for which they play the same role as Hermite polynomials in the case of standard boson-oscillator. The spectrum of the hamiltonian for this system is determined by the \( \{\rho_k\}_{k=0}^{\infty} \) coefficients of the recurrent relations for this family of orthogonal polynomials.
Note that a specific choice of the family of the polynomials orthogonal on the real axis connected with Jacobi matrix (i.e. polynomials defined by three-term recurrent relations) as the Fock basis, extends significantly the list of applied problems of quantum optics, where coherent states are successfully used. Application of generating functions for known polynomials allows to express coherent states through standard special functions explicitly and to prove the completeness of related system of coherent states by solving the appropriate classical moment problem. It is possible also, that interpretation of some of differential and difference the equations arising in this approach as Schrödinger equations for a system of ”generalized oscillators”, will indicate the way to the solution of certain asymptotic and spectral problems, which are well studied for the usual Schrödinger equation.

Such motivation underlies the new approach to construction of generalized coherent states, developed in the works of the authors (see also work, in which the connection of boson-systems with the families of orthogonal polynomials is also noted). The new approach to construction of coherent states rests on the construction of generalized oscillator algebras related with an arbitrary systems of orthogonal polynomials suggested in the work. Namely, given a system of orthogonal polynomials one can define in the canonical way the oscillator-like system; that is the coordinate and momentum operators, the ladder operators of creation and annihilation satisfying the commutation relations of the deformed boson-oscillator algebras and quadratic hamiltonian are defined. Note that the spectrum of the latter is defined by the coefficients of recurrent relations for these orthogonal polynomials. This allows to construct the family of coherent states not only for the case of classical orthogonal polynomials, but for their various q-analogues as well.

In particular situations the basic difficulties are connected with the solution of the proper classical moment problem and with expression of coherent states in terms of the standard special functions. The construction of coherent states families described above is realized in the works of the authors for the coherent states of the Barut - Girardello and Klauder - Gazeau types, connected with the classical Hermite, Laguerre, Legendre and Chebyshev polynomials. The main point in this research, as well as in the given work, is the solution of the proper classical moment problem (see also), necessary for construction of the measure involved in the decomposition of unity relation - one of the most important properties of coherent states. The approach developed for the case of classical polynomials was extended to the case of continuous q-Hermite polynomials. In it was shown also, that for discrete q-Hermite polynomials of the I-type coherent states do not exist. In the given work we shall continue the investigation of the case of deformed polynomials and construct a family of coherent states connected with the last remaining case of discrete q-Hermite polynomials, namely with the q-Hermite polynomials of II-type.
2 q-oscillator, connected with discrete $q$-Hermite polynomials of II-type

2.1 Discrete $q$-Hermite polynomials of II-type $\tilde{h}_n(x; q)$

Discrete $q$-polynomials Hermite of II-type $\tilde{h}_n(x; q)$ ($|q| < 1$) has the form \cite{45, 46}

$$\tilde{h}_n(x; q) = i^{-n}V_n^{-1}(ix; q) = i^{-n}q^{-\frac{n}{2}}2\phi_0\left(\begin{array}{c} q^{-n}, ix \\ 0 \end{array} | q; -q^n \right) =$$

$$= x^n2\phi_1\left(\begin{array}{c} q^{-n}, q^{-n+1} \\ 0 \end{array} | q^2; -\frac{1}{x^2} \right), \tag{4}$$

where the basic hypergeometric series $\phi_s$ by definition is equal to

$$\phi_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} | q; z \right) = \sum_{k=0}^\infty (-1)^k(1+s-r)q^{(1+s-r)(\frac{k}{2})} (a_1; q)_k (b_1; q)_k \cdots (a_r; q)_k (b_s; q)_k z^k, \tag{5}$$

so that

$$2\phi_0\left(\begin{array}{c} a, b \\ - \end{array} | q; z \right) = \sum_{k=0}^\infty (-1)^k q^{-\frac{k}{2}} (a; q)_k (b; q)_k z^k, \tag{6}$$

$$2\phi_1\left(\begin{array}{c} a, b \\ c \end{array} | q; z \right) = \sum_{k=0}^\infty (a; q)_k (b; q)_k (c; q)_k (q; q)_k z^k. \tag{7}$$

Here the shifted $q$-factorials (q-Pochhammer symbols), defined by

$$(a; q)_0 = 1, \quad (a; q)_k = \prod_{s=0}^{k-1} (1 - aq^s), \quad (a; q)_\infty = \prod_{s=0}^\infty (1 - aq^s) \tag{8}$$

are used.

Recurrent relations for polynomials $\tilde{h}_n(x; q)$ have the form

$$x\tilde{h}_n(x; q) = \tilde{h}_{n+1}(x; q) + q^{-2n+1}(1 - q^n)\tilde{h}_{n-1}(x; q) \tag{9}$$

$$\tilde{h}_0(x; q) = 1, \quad \tilde{h}_{-1}(x; q) \equiv 0.$$

The polynomial system $\{\tilde{h}_n(x; q)\}_{n=0}^\infty$ on an interval $(-\infty, +\infty)$ $(-\infty, +\infty)$ satisfies the orthogonality relations ($m \neq n, \ c > 0$)

$$c(1 - q) \sum_{k=-\infty}^\infty \left[ \tilde{h}_m(cq^k; q)\tilde{h}_n(cq^k; q) + \tilde{h}_m(-cq^k; q)\tilde{h}_n(-cq^k; q) \right] W(cq^k) q^k = 0, \tag{10}$$

where

$$W(x) = \frac{1}{(ix; q)_\infty (-ix; q)_\infty}. \tag{11}$$
One can write these orthogonality relations in the term of the Jackson $q$-integral (see, for example, [16])

\[
\int_{-\infty}^{+\infty} \tilde{h}_m(t; q) \tilde{h}_n(t; q) W(t) d_q t = 0. \tag{12}
\]

Let us remind, that the concept of the Jackson $q$-integral is connected naturally with definition of a $q$-derivative

\[
f(x) := (\frac{d}{dx})_q F(x) = \frac{F(x) - F(qx)}{x(1 - q)}, \quad (0 < q < 1). \tag{13}
\]

From this relation it follows that

\[
F(x) - F(qx) = x(1 - q)f(x). \tag{14}
\]

According to this relation, continuous in a point $x = 0$ function $F(x)$ can be reconstructed from $f(x)$ by the formula

\[
F(x) - F(0) = (F(x) - F(qx)) + (F(qx) - F(q^2x)) + \cdots + (F(q^nx) - F(q^{n+1}x)) + \cdots \\
= x(1 - q) \sum_{n=0}^{\infty} q^n f(q^nx). \tag{15}
\]

From (15), using an analogue of the Newton - Leibnitz formula, we obtain

\[
\int_0^x f(t) d_q t := F(x) - F(0) = x(1 - q) \sum_{n=0}^{\infty} q^n f(q^nx). \tag{16}
\]

Similarly, we receive

\[
\int_x^\infty f(t) d_q t := x(1 - q) \sum_{n=-\infty}^{-1} q^n f(q^n x). \tag{17}
\]

Adding (16) and (17), we receive for all $x > 0$

\[
\int_0^\infty f(t) d_q t := x(1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^nx). \tag{18}
\]

In the same way one can to define

\[
\int_{-\infty}^0 f(t) d_q t := x(1 - q) \sum_{n=-\infty}^{\infty} q^n f(-q^nx). \tag{19}
\]

Then for all $x > 0$ we have

\[
\int_{-\infty}^{\infty} f(t) d_q t := x(1 - q) \sum_{n=-\infty}^{\infty} q^n [f(q^nx) + f(-q^nx)]. \tag{20}
\]

Let us note that for simplification of notation, in these formulas one choose often $x = 1$. 

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For the case of the "symmetric" \( q \)-derivative \( (\frac{d}{dx})_q F(x) := \frac{F(qx)-F(q^{-1}x)}{x(q-q^{-1})} \) the definition of the Jackson \( q \)-integral was given in [44]). Below we construct an other variant of the Jackson \( q \)-integral, connected with the deformed derivative \( \left[ \frac{d}{dx} \right]_q \).

Following the construction given in [42], we introduce normalized family of polynomials \( \{\Psi_n(x; q)\}_{n=0}^\infty \), by the relation

\[
\Psi_n(x; q) = \frac{q^{\frac{1}{2}n^2}}{\sqrt{(q; q)_n}} \tilde{h}_n(x; q), \tag{21}
\]

From (9) we obtain a recurrent relation

\[
\Psi_n(x; q) = b_n \Psi_{n+1}(x; q) + b_{n-1} \Psi_{n-1}(x; q), \tag{22}
\]

where

\[
b_n = q^{-\frac{1}{2}(2n+1)} \sqrt{1 - q^{n+1}}, \quad b_{-1} \equiv 0. \tag{23}
\]

The constructed family of polynomials forms orthonormal basis in the space \( H_\mu = L^2(\mathbb{R}; d\mu) \) with respect to some probability measure \( d\mu(x) \). Because of

\[
\sum_{n=0}^\infty b_n^{-1} < \infty, \tag{24}
\]

the moment problem for the Jacobi matrix, defined by the recurrent relations (22), is undetermined (see [32]) and the measure \( d\mu(x) \) is determined not uniquely.

In the following we shall consider the Hilbert space \( H_\mu \) as the Fock space for oscillator - like system (q-oscillator) which will be constructed in the following subsection.

### 2.2 q-oscillator, connected with discrete \( q \)-Hermite polynomials of \( \Pi \)-type

Let us define the position operator \( X_\mu \) in the Hilbert space \( H_\mu \) by its action

\[
X_\mu |n\rangle = X_\mu \Psi_n(x; q) = b_n |n+1\rangle + b_{n-1} |n-1\rangle, \tag{25}
\]

on the basis elements \( |n\rangle := \Psi_n(x; q), \ n = 0, 1, 2, \ldots, \) in full accordance with the recurrent relations (22). It is known (see [32]), that such defined operator \( X_\mu \) is symmetric and has indices of defect equal to \((1, 1)\), so it has the family of the selfadjoint extensions. Following the method described in [34], fixing some selfadjoint extension \( X_\mu^\infty \) of the operator \( X_\mu \), it is possible to construct the completely defined extremal measure \( d\mu_\infty \). Let us recall some related results from [34] for a case when parameter \( \infty \) numbering extensions has the zero value \( \infty = 0 \). In this case the carrier \( \Pi_0 \equiv \{x_k\}_{k=-\infty}^\infty \) of the measure \( d\mu_0 \) consists from the points being the roots of transcendental equation

\[
\Psi_0(x; q) + x \sum_{k=0}^\infty (-1)^k \sqrt{\frac{[2k-2]!!}{[2k-1]!!}} \Psi_{2k-1}(x; q) = 0, \tag{26}
\]
where

\[ [s] = \frac{b_{s-1}}{b_0}, \quad s \geq 1, \quad [0] = 0. \quad (27) \]

Loadings \( \sigma_0(x_k) \) in a points of the carrier \( \Pi_0 \equiv \{ x_k \}_{k=-\infty}^{\infty} \) of the measure \( d\mu_0 \) are equal

\[ \sigma_0(x_k) = \frac{x_k \sum_{j=1}^{\infty} \frac{1}{(2j-1)!!} \sum_{m=0}^{j-1} (-1)^{j+m+1} \beta_{2m, 2j-2x} 2^{j-m-1}}{\left( \frac{d}{dx} \left[ -1 + \sum_{j=1}^{\infty} \frac{1}{(2j-1)!!} \sum_{m=0}^{j-1} (-1)^{j+m-1} \alpha_{2m-1, 2j-2x} 2^{j-m} \right] \right) (x_k)}. \quad (28) \]

Coefficients \( \alpha_{ij} \) and \( \beta_{ij} \) are coefficients of polynomials \( P_n(x; q) = \Psi_n(x; q) \) and \( Q_n(x; q) \) of the 1-st and 2-nd type for the Jacobi matrix connected with the operator \( X_\mu \) \[32\]. We recall their explicit expressions from the work \[34\]

\[ P_n(x; q) = \sum_{m=0}^{\text{Ent}(\frac{1}{2}n)} \frac{(-1)^m}{\sqrt{[n]!!}} \alpha_{2m-1, n-1} x^{n-2m}, \quad n \geq 0, \quad (29) \]

\[ \alpha_{-1, 2n-1} = 1, \quad \alpha_{2m-1, n-1} = \sum_{k_1=2m-1}^{n-1} [k_1] \sum_{k_2=2m-3}^{k_1-2} [k_2] \cdots \sum_{k_m=1}^{k_{m-1}-2} [k_m], \quad m \geq 1. \quad (30) \]

\[ Q_0(x; q) = 0, \quad Q_{n+1}(x; q) = \sum_{m=0}^{\text{Ent}(\frac{1}{2}n)} \frac{(-1)^m}{\sqrt{[n+1]!!}} \beta_{2m, n} x^{n-2m}, \quad n \geq 0, \quad (31) \]

\[ \beta_{0, n} = 1, \quad \beta_{2m, n} = \sum_{k_1=2m}^{n} [k_1] \sum_{k_2=2m-2}^{k_1-2} [k_2] \cdots \sum_{k_m=2}^{k_{m-1}-2} [k_m], \quad m \geq 1. \quad (32) \]

Let us note, that for some problems there is no necessity to consider an extremal measure. In that case it is possible to take the most "convenient" carrier of the measure, for example, a geometrical progression \( \{ cq^k \}_{k=-\infty}^{\infty} \) and pick up loadings so that to receive the solution of the same moment problem (see detailed consideration in \[17\]).

We define the momentum operator \( P_\mu \) by its action on the basic elements of the space \( \mathcal{H}_\mu \)

\[ P_\mu \Psi_n(x; q) = i (b_n \Psi_{n+1}(x; q) - b_{n-1} \Psi_{n-1}(x; q)), \quad (33) \]

or in other notation

\[ P_\mu |n\rangle = i (b_n |n+1\rangle - b_{n-1} |n-1\rangle). \quad (34) \]

This operator as well as \( X_\mu \) is symmetric and also has defect indexes \( (1, 1) \), so it too has a family of self-adjoint extensions. We shall consider that self-adjoint extension of the operator \( P_\mu \) which is correspond to the self-adjoint extension of the operator \( X_\mu \) chosen above so, that the related creation and annihilation operators are adjoined. Let us keep for these extensions the same notation \( X_\mu \) and \( P_\mu \). Following the methods described in \[13\], one can realize the operator \( P_\mu \) in space \( \mathcal{H}_\mu \) as differential - difference operator.
Let us define the ladder operators of creation and annihilation in the space $\mathcal{H}_\mu$ by the relations

$$a_\mu^+ = \frac{1}{2} \sqrt{\frac{q}{1-q}} \left( X_\mu - P_\mu \right), \quad a_\mu^- = \frac{1}{2} \sqrt{\frac{q}{1-q}} \left( X_\mu + i P_\mu \right).$$  \hfill (35)

These operators act on the basic elements in the space $\mathcal{H}_\mu$, according to relations

$$a_\mu^+ \Psi_n(x; q) = \sqrt{\frac{q}{1-q}} b_n \Psi_{n+1}(x; q), \quad n \geq 0; \quad \text{and} \quad a_\mu^- \Psi_{n+1}(x; q) = \sqrt{\frac{q}{1-q}} b_n \Psi_n(x; q), \quad n \geq 1; \quad a_\mu^- \Psi_0(x; q) = 0.$$ \hfill (36)\hfill (37)

Let us define further the operator $N$, numbering the basic states,

$$N \Psi_n(x; q) = n \Psi_n(x; q); \quad (N|n\rangle = n|n\rangle) \quad n \geq 0.$$ \hfill (38)

It is easy to check the validity of the following relations

$$a_\mu^- a_\mu^+ = q^{-2N} [N + I]_q, \quad a_\mu^+ a_\mu^- = q^{-2(N-1)} [N]_q,$$ \hfill (39)

where a symbol $[n]_q := \frac{1}{1-q} \frac{1-q^n}{1-q}$ denotes the standard "mathematical" $q$-number. From formulas (39) one obtains the following commutation relations for the creation and annihilation operators:

$$a_\mu^- a_\mu^+ - q^{-1} a_\mu^+ a_\mu^- = q^{-2N}, \quad a_\mu^+ a_\mu^- - q^{-2} a_\mu^- a_\mu^+ = q^{-N}.$$ \hfill (40)

The polynomials $|n\rangle := \Psi_n(x; q)$, are eigenfunctions

$$H_\mu |n\rangle = \lambda_n |n\rangle,$$ \hfill (41)

corresponding to eigenvalues

$$\lambda_n = \frac{q}{1-q} \left( b_{n-1}^2 + b_n^2 \right) = q^{-2n} [n+1]_q + q^{-2(n-1)} [n]_q, \quad n \geq 0.$$ \hfill (42)

In the same way as in [42] where the case of the classical polynomials is considered, it is possible to prove, that the equation (41) is equivalent to the $q$-difference equation for discrete $q$-Hermite polynomials of the II-nd type:

$$-(1-q^n)x^2 \tilde{h}_n(x; q) = q \tilde{h}_n(x - i; q) - (1 + q + x^2) \tilde{h}_n(x; q) + (1 + x^2) \tilde{h}_n(x + i; q),$$ \hfill (44)

which is analogue of the Schrödinger equation for the $q$-oscillator described above.
3 coherent states for q-oscillator, associated with discrete q-Hermite polynomials of the II-nd type $\tilde{h}_n(x; q)$

Let us define the Barut - Girardello coherent states by the standard way [48]

$$a_{\mu}^\dagger |z\rangle = z |z\rangle; \quad |z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(q_1 - q)b_n^{-1}}} |n\rangle. \quad (45)$$

Taking into account, that

$$\left(\frac{q_1 - q}{1 - q}\right)! = \left(\frac{q}{1 - q}\right)^n q^{-n^2}(q; q)_n \quad (46)$$

one can calculate a normalizing factor

$$\mathcal{N}^2(|z|^2) = \sum_{n=0}^{\infty} \left(\frac{1 - q}{q}\right)^n q^{n^2}(q; q)_n = 0\phi_1 \left(0 \left| q; (1 - q)|z|^2\right\rangle. \quad (47)$$

and overlapping of two coherent states

$$\langle z_1 | z_2 \rangle = \sum_{n=0}^{\infty} \left(\frac{1 - q}{q}\right)^n q^{n^2}(q; q)_n = 0\phi_1 \left(0 \left| q; (1 - q)z_1 z_2\right\rangle. \quad (48)$$

Further, substituting (47) in (15) and using reproducing function for polynomials $\tilde{h}_n(x; q)$ (see [45] (3.296)):

$$\sum_{n=0}^{\infty} \tilde{h}_n(x; q) \tau^n = (i\tau; q)_\infty \phi_1 \left(i\tau \left| q; -i\tau\right\rangle. \quad (49)$$

we receive, taking into account (22), the following explicit expression for the coherent state $|z\rangle$

$$|z\rangle = \mathcal{N}^{-1}(|z|^2) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{(q_1 - q)^n q^{n^2}(q; q)_n}} \frac{\tilde{h}_n(x; q)}{q^{-n^2}\sqrt{(q; q)_n}} =$$

$$= \left(0\phi_1 \left(0 \left| -iq; (1 - q)|z|^2\right\rangle\right)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \left(\sqrt{\frac{1 - q}{q}} zq\right)^n q^{n(n-1)} \tilde{h}_n(x; q) =$$

$$\left(i\sqrt{q(1 - q)} z; q\right)_\infty \phi_1 \left(i\sqrt{q(1 - q)} z \left| q; -i\sqrt{q(1 - q)} z\right\rangle. \quad (50)$$

It is necessary to prove the decomposition of unity formula (a (over)completeness relation)

$$\int_{\mathbb{C}} \hat{W}(|z|^2)|z\rangle\langle z|d^2z = 1 \quad (51)$$
i.e. to construct a measure
\[ \mathrm{d}\mu(|z|^2) = \tilde{W}(|z|^2)\mathrm{d}^2z. \] (52)

It is known \[48, 49\] that for this purpose it is necessary to solve a classical Stieltjes moment problem
\[ \pi \int_0^{\infty} x^n W(x) \mathrm{d}x = \left( \frac{q}{1-q} \right)^n q^{-n^2}(q; q)_n, \quad n \geq 0, \] (53)
\[ W(x) = \frac{\tilde{W}(x)}{N^2(x)}, \quad (x = |z|^2). \] (54)

Under the replacement of variables
\[ x = \frac{q}{1-q} y, \quad \pi q \int_0^{\infty} x^n W(x) \mathrm{d}x = \tilde{W}(y), \] (55)
the moment problem \[53\] takes the form
\[ \int_0^{\infty} y^n \tilde{W}(y) \mathrm{d}y = q^{-n^2}(q; q)_n. \] (56)

4 The solution of the moment problem \[56\]

Let us define generalized exponential
\[ \hat{\bar{e}}_q(x) := \sum_{n=0}^{\infty} \frac{x^n}{b_{n-1}!} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} x^n = \phi_1 \left( \frac{|z|}{q, qx} \right) \] (57)
and the corresponding deformed derivative
\[ \left[ \frac{d}{dx} \right]_q f(x) = \frac{f(q^{-2}x) - f(q^{-1}x)}{q^{-1}x}, \] (58)
so that
\[ \left[ \frac{d}{dx} \right]_q x^n = \begin{cases} b_{n-1}^2 x^{n-1} & n \geq 1; \\ 0 & n = 0 \end{cases}, \quad \left[ \frac{d}{dx} \right]_q \hat{\bar{e}}_q(x) = \hat{\bar{e}}_q(x); \] (59)
and following generalization of Leibnitz rule
\[ \left[ \frac{d}{dx} \right]_q (u(x)v(x)) = \begin{cases} v(q^{-2}x) \left[ \frac{d}{dx} \right]_q u(x) + u(q^{-1}x) \left[ \frac{d}{dx} \right]_q v(x) , \\ v(q^{-1}x) \left[ \frac{d}{dx} \right]_q u(x) + u(q^{-2}x) \left[ \frac{d}{dx} \right]_q v(x) . \end{cases} \] (60)
is hold.

The related q-integral is defined by equality
\[ \int_0^{\infty} f(t)\tilde{\alpha}_q t = \frac{1-q}{q^2} \sum_{k=0}^{\infty} (q^{-k-1} f(q^{-k-1}) + q^{k+2} f(q^{k+2})), \quad |q| < 1. \] (61)
It is not difficult to prove that this integral exists for continuous functions $f(x) = \hat{\frac{d}{dx}}_q F(x)$, such that $F(x) \to 0$ as $x \to \infty$ and that

$$\int_0^a \hat{\frac{d}{dx}}_q f(x) \hat{d}_q x = f(a) - f(0).$$

(62)

After integrating the Leibnitz rule (60) in view of the relation (62), we receive the formula of integration by parts ($0 < a \leq \infty$)

$$\int_0^a u(q^{-1}x) \hat{\frac{d}{dx}}_q v(x) \hat{d}_q x = u(x)v(x) \bigg|_0^a - \int_0^a v(q^{-2}x) \hat{\frac{d}{dx}}_q u(x) \hat{d}_q x;$$

(63a)

$$\int_0^a u(q^{-2}x) \hat{\frac{d}{dx}}_q v(x) \hat{d}_q x = u(x)v(x) \bigg|_0^a - \int_0^a v(q^{-1}x) \hat{\frac{d}{dx}}_q u(x) \hat{d}_q x.$$

(63b)

For a case $a = \infty$ we shall rewrite the relation (63a) in the following form. At first, having done the replacement $q^{-1}x = y$, in the integral in the right hand part of equality (63a) we shall receive

$$\int_0^\infty u(q^{-1}y) \hat{\frac{d}{dx}}_q v(x) \hat{d}_q x = \int_0^\infty u(y) \hat{\frac{d}{dx}}_q v(qy) \hat{d}_q y.$$

(64)

Taking into account independence of integral (61) from a designation of the variable of integration, we shall receive finally

$$\int_0^\infty u(x) \hat{\frac{d}{dx}}_q v(qx) \hat{d}_q x = u(x)v(x) \bigg|_0^\infty - \int_0^\infty v(q^{-2}x) \hat{\frac{d}{dx}}_q u(x) \hat{d}_q x.$$

(65)

Let us consider the function $f(y)$, satisfying a condition

$$\hat{\frac{d}{dy}}_q f(q^2y) = -q^2 f(y).$$

(66)

Using the definition of the deformed derivative (58), we receive from (66) the difference equation

$$\frac{f(qy) - f(q^2y)}{q^{-1}y} = -q^2 f(y)$$

for the function $f(y)$. Rewriting this equation in a form

$$f(qy) - f(q^2y) = -qy f(y),$$

(67)

we find his solution in the form of power series

$$f(y) = \sum_{n=0}^\infty \frac{1}{(q; q)_n} q^{-\binom{n}{2}} (-y)^n = 2\phi_0 \left( \begin{array}{c} 0, 0 \\ q; y \end{array} \right).$$

(68)

Let us consider an integral

$$I_n(q) := \int_0^\infty x^n 2\phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2}x \end{array} \right) \hat{d}_q x, \quad n \geq 0.$$

(69)
From (66) it follows, that
\[ - \frac{d}{dx} q \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^2 x \end{array} \right) = 2 \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2} x \end{array} \right). \] (70)

Applying to an integral \( I_n(q) \) the formula of integration by parts as \( u(x) = x^n \) and \( v(x) = - \left( 2 \phi_0 \left( \begin{array}{c} 0, 0 \\ q; x \end{array} \right) \right) \), we obtain for \( n \geq 1 \)
\[ I_n(q) = \int_0^\infty 2 \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2} x \end{array} \right) b_{n-1}^2 x^n - 1 \tilde{d}_q x. \] (71)

Because the out of integral term vanishes, we obtain finally
\[ I_n(q) = b_{n-1}^2 I_{n-1}(q), \quad n \geq 1. \] (72)

On the other hand, for \( n = 0 \) we have
\[ I_0(q) = - \int_0^\infty \frac{d}{dx} q \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2} x \end{array} \right) \tilde{d}_q x = -2 \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q x \end{array} \right)|^\infty_0 = 1. \] (73)

From the relations (72) and (73) it follows that
\[ I_n(q) = b_{n-1}^2! = q^{-n^2}(q; q)_n. \] (74)

Thus, the classical moment problem (56) has the following solution
\[ \tilde{W}(y) = \frac{1 - q}{q^2} \sum_{k=0}^\infty y \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2} y \end{array} \right) \left( \delta(y - q^{-(k-1)}) + \delta(y - q^{(k+2)}) \right) \] (75)
so that
\[ \tilde{W}(x) = N^2(x) \frac{1 - q}{q^2} \tilde{W}(\frac{1 - q}{q} x) = \frac{1 - q}{q} \phi_0 \left( \begin{array}{c} - \\ 0 \\ q; (1-q)x \end{array} \right) \times \sum_{k=0}^\infty \frac{(1-q)^2}{q^{2k}} x \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2} x \end{array} \right) \left( \delta(\frac{1-q}{q} x - q^{-(k-1)}) + \delta(\frac{1-q}{q} x - q^{(k+2)}) \right). \] (76)

Thus for a measure \( d\mu(|z|^2) \) from the decomposition of unity (51) - (52) we have
\[ d\mu(|z|^2) = \frac{(1-q)^3}{\pi q^4} \phi_0 \left( \begin{array}{c} - \\ 0 \\ q; (1-q)|z|^2 \end{array} \right) \times \sum_{k=0}^\infty |z|^2 \phi_0 \left( \begin{array}{c} 0, 0 \\ q; q^{-2}|z|^2 \end{array} \right) \left( \delta(\frac{1-q}{q} |z|^2 - q^{-(k-1)}) + \delta(\frac{1-q}{q} |z|^2 - q^{(k+2)}) \right). \] (77)

Thus completeness of the constructed system of coherent states is proved.

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