Rényi entropy for spherical monodromy of free scalar fields in even dimensions

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The effect of a co–dimension–2 spherical monodromy defect on the free–field theory of a conformal scalar is investigated. The conformal anomaly is computed (in two ways) and thence the Rényi and entanglement entropies. The $C_T$ central charge is found to be negative for a range of monodromy flux. The calculations are pursued for any even dimensional sphere.

A possible conflict with existing results is raised.

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1. Introduction.

There has been a certain amount of activity concerning the effect of defects in conformal field theories and a comprehensive analysis of the structure of such effects on the Weyl anomaly has recently appeared, [1]. In the present work, rather than deal with any generalities, I give a specific direct calculation for a very particular defect which might prove useful as a special case.

A number of types of defect are considered in [1] but are not treated *ab initio* in a field–theoretic way. Such analysis, however, is provided in the earlier [2], generally for free fields, and, although some results apply for any manifold dimension, \( d \), most explicit expressions are for \( d \leq 6 \) with various defect co–dimensions, usually 2. One aim here is to take \( d \) as high as practicable. 2

Under consideration is a spherical monodromy defect of co–dimension 2 in any even dimension for free, conformal scalars. The setup is described in [3] for several conformally equivalent manifolds of which I am interested only in the spherical one, \( S^d \), with a monodromy (phase change) around one circle.

Actually, free–field theory on this construction was analysed some time earlier in [4] and even earlier in [5] where the notion of charged Rényi entropy (which is what is being calculated here) was first introduced and analysed 3.

In [6], in response to [3], I computed the difference in the free energies with and without the defect flux by an improved technique for any \( d \), odd or even. Here I restrict to even \( d \), the easier calculation. My evaluations are only concerned with the non–singular modes, [1,2].

In [6], the manifold was the full, round sphere. Now I generalise to the \( q \) orbifolded sphere which I used in [4] (and many times earlier). The manifold is then a lune of angle \( 2\pi/q \), \( q \in \mathbb{Z} \). The complex field, \( \phi \), undergoes a phase change of \( 2\pi i \delta \) on circling this lune once.

I remark that the eigenvalue forms derived in [4] are valid for any \( q \). Relatedly, some expressions have already appeared in [6] obtained using an image sum of full sphere quantities having rational monodromy fluxes, to give the quantities on an \( n \)–covered sphere [7]. Part of the present calculation could be considered just a check that images work (as they must) under the replacement \( q \leftrightarrow 1/n \).

Very little basic theory is given and I confine myself mostly to presenting the

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2 Time constraints of the CAS mean that \( d \) is, sensibly, less than about 20.

3 Only the \( d = 3 \) sphere was treated in any detail, the analysis leading to sums of Hurwitz \( \zeta \)–functions. The hyperbolic formulation was also employed.
results of machine evaluation.

2. Conformal anomaly, Rényi and entanglement entropies

Most formalities have been described in [6,4]. The quantity of interest is the effective action (‘free energy’), or, rather, the (universal) coefficient of its logarithmic divergent part. This is the conformal anomaly equalling the value of the relevant $\zeta$–function at 0. Spectral considerations show that, here, this is given by an average of four generalised Bernoulli polynomials,

$$C(d, \delta, q) = \frac{1}{d!q} \sum_{i=1}^{4} B_{d}^{(d)}(a_i),$$

with arguments

$$a_1 = d/2 + q(1 - \delta), \quad a_2 = a_1 - 1, \quad a_3 = d/2 + q\delta, \quad a_4 = a_3 - 1.$$ 

It is a matter of a few moments to evaluate these polynomials. I find ($\sigma \equiv \delta(1 - \delta)$),

$$\frac{1 + q^2 - 6q^2\sigma}{3q}, \quad -\frac{3 - q^4 + 30q^4\sigma^2}{180q}, \quad \frac{31 + 7q^4 + 2q^6 - 210q^4\sigma^2 - 42q^6\sigma^2 - 84q^6\sigma^3}{15120q},$$

for $d = 2, 4$ and 6.

As mentioned in the previous section, on setting $q = 1/n$, these agree with the values obtained by images in [6]. At one level, this is nothing more than a confirmation of properties of the Bernoulli polynomials.

The effect of the defect is measured by the difference, $\Delta C(d, \delta, q) \equiv C(d, \delta, q) - C(d, 0, q)$, for $d = 4, 6, 8$,

$$\frac{q^3\sigma^2}{6}, -\frac{q^3\sigma^2(5 + q^2 + 2q^2\sigma)}{360}, \quad \frac{q^3\sigma^2(56 + 14q^2 + 2q^4 + 28q^2\sigma + 4q^4\sigma + 3q^4\sigma^2)}{30240},$$

which for the full sphere are $\Delta C(d, \delta, 1)$, and equal,

$$\frac{\sigma^2}{6}, -\frac{\sigma^2(3 + \sigma)}{180}, \frac{\sigma^2(72 + 32\sigma + 3\sigma^2)}{30240}, -\frac{\sigma^2(360 + 180\sigma + 25\sigma^2 + \sigma^3)}{907200}, \quad (1)$$

for $d = 4, 6, 8$ and 10.
Various other quantities can be computed from this conformal anomaly. For example the universal log coefficient of the Rényi entropy, $\mathcal{S}$, and thence that of the entanglement entropy, $\mathcal{E}$. The standard definitions are,

$$
\mathcal{S}(d, \delta, q) \equiv \frac{q \mathcal{C}(d, \delta, q) - \mathcal{C}(d, \delta, 1)}{1 - q},
$$

and, using the replica formulation,

$$
\mathcal{E}(d, \delta) \equiv \mathcal{S}(d, \delta, 1),
$$

which refer to the full sphere. The Rényi index $n \in \mathbb{Z}$ equals $1/q$.

I will not give the Rényi expressions (they are easily constructed) but I will display the entanglement entropies for $d = 4, 6, 8, 10$ and $12$,

$$
\begin{align*}
&\frac{1}{45} - \frac{2\sigma^2}{3}, \quad -\frac{1}{378} + \frac{\sigma^2(13 + 6\sigma)}{180}, \quad \frac{23}{56700} - \frac{\sigma^2(81 + 50\sigma + 6\sigma^2)}{7560}, \\
&-\frac{263}{3742200} + \frac{\sigma^2(1656 + 1152\sigma + 205\sigma^2 + 10\sigma^3)}{907200}, \quad \frac{133787}{10216206000} - \frac{\sigma^2(20160 + 14976\sigma + 3175\sigma^2 + 244\sigma^3 + 6\sigma^4)}{59875200}.
\end{align*}
$$

The difference, $\mathcal{E}(d, \delta) - \mathcal{E}(d, 0)$, is obtained by removing the first (standard) term.

All values, here and later, are for a complex scalar field.

3. Alternative derivation via thermodynamics

The results shown in the previous section could have been derived from the expressions in [6]. The input was just the eigenvalues of the conformal Laplacian. In this section I employ a method that is ultimately based on a Green function without overt reference to the eigenproblem. The method, in the way I do it, involves a conformal transformation to de Sitter space and a thermodynamic looking argument to construct the free energy (both in its thermodynamic sense and, in an older terminology as the effective action) and thence the entropy. It was used in [8] and [4] where it was shown that the coefficient of the logarithmic term that arises

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4 I have taken the flux not to vary in the deformation to a covered (or factored) sphere. This has the consequence of ensuring invariance under $\delta \to 1 - \delta$, i.e. a polynomial in $\sigma$. Furthermore, without loss of generality, $\delta$ is restricted to lie between 0 and 1, other values being obtained by the necessary periodicity, $\delta \equiv 1 + \delta$. 

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when the free–energy density is integrated sufficiently close to the de Sitter horizon is given by,

\[ \mathcal{E}(d, \delta, q) = \frac{(-1)^{d/2}}{2^{d-3} \Gamma(d/2)} \int_{q} q P(d, \delta, q) \frac{dq}{q^2}. \quad (3) \]

\( q \) is interpreted thermodynamically as \( 2\pi/\beta \) in terms of the inverse temperature, \( \beta \). The entanglement entropy is obtained by going on–shell, \( q \to 1 \), so that \( \mathcal{E}(d, \delta) = \mathcal{E}(d, \delta, 1) \). \(^5\) \( \mathcal{E} \) is expected to be the same as \( \mathcal{E} \).

\( P(d, \delta, q) \) is a polynomial occurring in the formula for the averaged energy density on de Sitter space which can be obtained by conformal transformation, via Rindler space, from the vacuum average of \( \hat{T}_0^0 \) on flat space with a conical singularity and an Aharonov–Bohm flux as calculated in [7] using a complex contour method to yield the appropriately periodised (thermalised) Green function with a chemical potential.

For no flux the results were given in [4] and confirmed that \( \mathcal{E} \) equals \( \mathcal{E} \) yielding, for example, the entanglement entropies computed by Casini and Huerta [9]. (See also [10].) The entanglement entropy is minus the conformal anomaly.

In this case the integral can be done explicitly for arbitrary (even) \( d \). In the presence of flux, however, this does not appear to be the case. Nevertheless, everything remains explicit and \( P(d, \delta, q) \) can be written in terms of generalised Bernoulli functions so that the integral, (3), can still be evaluated but only dimension by dimension.

Since part of my motivation is to promote awareness of the utility of the present machinery, I draw attention to the development in [11] where relevant analytical objects have been defined and used. I repeat a few things.

The vacuum average of the energy density (defined as a differential operator acting on a Green function) involves the contour integral, for even dimensions,

\[ W_d(q, \delta) = \frac{i}{2} \oint_C dz \frac{\cos(q(2\delta - 1)z)}{\sin^d z \sin qz}, \quad (4) \]

where \( C \) is a small clockwise loop around \( z = 0 \). A closed form for this can be found in terms of Bernoulli polynomials, as detailed in [7]. Most useful are the Nörlund \( D \)–polynomials,

\[ D^{(n)}_\nu(x | \omega) \equiv 2^n B^{(n)}_\nu(x + (\sum \omega_i)/2 | \omega), \quad (5) \]

where \( \omega \) is an \( n \)–vector of components \( \omega_i \). (Refer to Nörlund, [12,13], for notation). In the case here, \( n = d + 1, \nu = d \) and \( \omega = (q, 1_d) \). \( x \) is referred to as the argument, or variable, and \( q \) is a ‘parameter’. Efficient algorithms for the \( B \)s exist.

\(^5\) The flux, \( \delta \), naturally remains constant offshell, \( q \neq 1 \).
The residue evaluation of (4) produces,

\[ W_{d}(q, \delta) = \frac{(-1)^{d/2} \pi}{d! q} D_{d}^{(d+1)}(q (\delta - 1/2) | q, 1_{d}) . \]

The (UV renormalised) vacuum average of the energy density of a conformally coupled scalar involves two terms and leads to the polynomial,

\[ P(d, \delta, q) = (d - 1)\Gamma(d/2) \frac{q}{\pi} \left( W_{d}(q, \delta) - \frac{d - 2}{d - 1} W_{d-2}(q, \delta) \right) , \]

which is readily evaluated explicitly for any given \( d \). Performing the integral in (3) then confirms, in each case, that the entanglement entropies are equal i.e. that \( \mathfrak{E} \) equals the \( \mathfrak{E} \) computed in section 2, as anticipated.

4. A central charge

The \( C_{T} \) energy–momentum two–point function central charge is given by the second derivative of the conformal anomaly, to a factor,

\[ C_{T}(d, \delta) = \left. \frac{(-1)^{d/2+1}(d+1)!}{(d-1)((d/2-1)!)^2} \frac{\partial^2}{\partial n^2} C(d, \delta, 1/n) \right|_{n=1} , \]

and I find it calculates to,

\[
4 - 24\sigma, \quad \frac{8}{3} - 80\sigma^2, \quad \frac{12}{5} - 21\sigma^2(3 + 2\sigma), \quad \frac{16}{7} - \frac{4}{3}\sigma^2(43 + 38\sigma + 6\sigma^2),
\]

\[
\frac{20}{9} - \frac{110}{864}\sigma^2(428 + 424\sigma + 99\sigma^2 + 6\sigma^3)), \quad \frac{24}{11} - \frac{13}{600}\sigma^2(2436 + 2568\sigma + 713\sigma^2 + 68\sigma^3 + 2\sigma^4),
\]

for \( d = 2, 4, 6, 8, 10 \) and 12. The monodromy contribution, \( C_{T}(d, \delta) - C_{T}(d, 0) \), is obtained by dropping the standard piece, \( 2d/(d-1) \), and is always negative so that \( C_{T} \) vanishes for a particular value, \( \sigma_{0}(d) \), of \( \sigma \) which seems to approach a limit of approximately 0.1856 as the dimension increases, although I could not prove this analytically.

It would appear, therefore, that \( C_{T} \) is negative for a range of flux, determined by \( \sigma_{0}(d) < \sigma \leq 1/4 \). This might prove problematic. I give a few values for \( \sigma_{0}(d) \),

\[
\sigma_{0}(4) = 0.1825741858, \quad \sigma_{0}(6) = 0.1841977956, \quad \sigma_{0}(8) = 0.1847483784, \quad \sigma_{0}(10) = 0.1850213867, \quad \sigma_{0}(20) = 0.1854686558, \quad \sigma_{0}(22) = 0.1855038612.
\]

\(^{6}\) This means Minkowski subtracted.
The corresponding values for the flux $\delta$ (there are two of them, $\delta_0$ and $1 - \delta_0$) are easily found. For positive $C_T$, $\delta$ is restricted by $\{0 \leq \delta < \delta_0\} \cup \{(1 - \delta_0) < \delta \leq 1\}$.

The graph below shows the typical behaviour (for $d = 4$) of $C_T$ as the flux is varied. It should be extended by periodicity.

5. Comments and conclusion

The conformal anomaly, and derivatives thereof, have been evaluated by two different techniques. The agreement is gratifying, but, perhaps, not unexpected.

Reference [1] contains (eqn.(5.30)) a prediction for the spherical co-dimension–2 monodromy’s effect on the entanglement entropy in 6 dimensions. Ignoring the contributions of the slightly singular modes, the value there is,$^7$

$$S_A = -\frac{1}{180}\sigma^2,$$

which differs from the value here in (2). At the moment, I am unable to account precisely for this difference.$^8$

The technical extensions of the calculations in the present paper to the Dirac field and to $p$–forms, as well as the generalisation to higher derivative propagation are reserved for another time.

$^7$ I thank the authors of [1] for providing this expression and for useful comments. Regarding any overall sign differences, note that the quantities in the present work are the coefficients of $\log \epsilon$ whereas those in [1] are of $\log 1/\epsilon$.

$^8$ Preliminary indications suggest that the difference is related to the weight multiplier of the second term in the formula for the entanglement entropy given in [14] and used in [1].
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