EXTENDING HECKE ENDMORPHISM ALGEBRAS AT ROOTS OF UNITY

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ABSTRACT. Hecke endomorphism algebras are endomorphism algebras over a Hecke algebra associated to a finite Weyl group $W$ of certain $q$-permutation modules, the “tensor spaces.” Such a space may be defined for any $W$ in terms of a direct sum of certain cyclic modules associated to parabolic subgroups. The associated algebras have important applications to the representations of finite groups of Lie type. In [6], it is proved that these algebras can be stratified by means of a filtration defined in terms of the subsets of the Coxeter generators. It was conjectured that by enlarging the “tensor space” the new resulting endomorphism algebra has a finer “standard” stratification in terms of left cells of the Coxeter group, with associated “strata” corresponding to two-sided cells of $W$. Using the work [8] on a rational double affine Hecke algebras (RDAHAs)—also known as rational Cherednik algebras—at a key point, we will prove the conjecture in the characteristic zero case at an $e$th root of unity, $e \neq 2$. We further prove that each of the new Hecke endomorphism algebras constructed in the paper is quasi-hereditary and that its representation category is equivalent, after a base change, to the category $O$ associated to a corresponding RDAHA. We do not treat the $e = 2$ case, but expect the conjecture to be true there also (possibly not giving quasi-hereditary algebras, in general).

1. INTRODUCTION

Let $G = \{G(q)\}$ be a family of finite groups of Lie type having irreducible (finite) Coxeter system $(W, S)$ [3 (68.22)]. Let $B(q)$ be a Borel subgroup of $G(q)$. There are index parameters $c_s \in \mathbb{Z}$, $s \in S$, defined by

$[B(q) : ^sB(q) \cap B(q)] = q^{c_s}, \quad s \in S.$

The generic Hecke algebra $\mathcal{H}$ over the ring $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ of Laurent polynomials associated to $G$ has basis $T_w$, $w \in W$, subject to relations

\begin{equation}
T_s T_w = \begin{cases} 
T_{sw}, & \text{if } sw > w; \\
t^{2c_s} T_{sw} + (t^{2c_s} - 1) T_w, & \text{if } sw < w.
\end{cases}
\end{equation}

We call $\mathcal{H}$ a Hecke algebra of Lie type over $\mathcal{Z}$. It is related to the representation theory of the groups in $G$ as follows: for any prime power $q$, let $\mathcal{H}_q = \mathbb{C} \otimes _{\mathcal{Z}} \mathcal{H}$ be the algebra obtained by the map $\mathcal{Z} \to \mathbb{C}$, $t \mapsto \sqrt{q}$. Then $\mathcal{H}_q \cong \text{End}_{G(q)}(\text{ind}_{B(q)}^G \mathbb{C})$. Thus, the generic Hecke algebra $\mathcal{H}_q$ is the quantumization (in the sense of [4 §0.4]) of an infinite family of important endomorphism algebras.

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Let \( \Omega \) be the set of left Kazhdan-Lusztig cells for the Coxeter system \((W, S)\). Thus, \( \Omega \) has a quasi-poset structure \( \leq_L \). (A quasi-poset is a set with a transitive and reflexive relation \( \equiv \) preorder defined on it.) For each \( \omega \in \Omega \), there is a dual left cell module \( S_\omega \). It is the right \( \mathcal{H} \)-module defined to be the \( \mathcal{Z} \)-linear dual of the left cell module defined by \( \omega \). In addition, for \( \lambda \subseteq S \), the induced modules \( x_\lambda \mathcal{H} \) have an increasing filtration with sections \( S_\omega, \omega \in \Omega \).

Let \( T = \bigoplus_\lambda x_\lambda \mathcal{H} \), and \( A := \text{End}_\mathcal{H}(T) \). For \( \omega \in \Omega, \Delta_\mathcal{H}(\omega) := \text{Hom}_\mathcal{H}(S_\omega, T) \in A\text{-mod.} \)

The following conjecture is worded to be the same as \([6, \text{Conj. 2.5.2}]\), but using \( \mathbb{Z}[t, t^{-1}] \) as \( \mathcal{Z} \), whereas the latter conjecture used \( \mathbb{Z}[t^2, t^{-2}] \). \[\text{Conjecture 1.1.} \] There exists a right \( \mathcal{H} \)-module \( X \) such that the following statements hold:

1. \( X \) has an finite filtration with sections of the form \( S_\omega, \omega \in \Omega \).
2. Let \( T^+ = T \oplus X \), and let \( R \) be any commutative \( \mathcal{Z} \)-algebra. Put \( A_r^+ := \text{End}_\mathcal{H}(T_R^+) \) and, for \( \omega \in \Omega, \Delta_\mathcal{H}^+(\omega)_R := \text{Hom}_\mathcal{H}(S_\omega, T^+_R) \). Then \( \{\Delta_\mathcal{H}(\omega)_R\}_{\omega \in \Omega} \) is a strict stratifying system relative to the quasi-poset \( (\Omega, \leq^\text{op}_L) \) for the category \( A_r^+\text{-mod.} \).

The idea of a strict stratifying system was first defined for a finite dimensional algebra over a field in \([2]\). Such algebras with a strict stratifying system are standardly stratified. The systems in the conjecture are \( \mathcal{Z} \)-integral versions \([6]\). Upon base change to a field they become strict stratifying systems there.

The main result of this paper, given in Theorem 5.6, establishes a "special case" of the original conjecture, in the sense that it would be implied by the latter.

A more detailed description of Theorem 5.6 requires some preliminary notation. Throughout this paper, \( e \) is an integer \( > 2 \). Let \( \Phi_e \) denote the (cyclotomic) minimum polynomial for a primitive \( e \)th root of unity \( \zeta \). Fix a modular system \((K, \mathcal{O}, k)\) by letting

\[
\begin{align*}
\mathcal{O} &:= \mathbb{Q}[t, t^{-1}]_m, \text{ where } m = (\Phi_e(t^2)) = \text{ discrete valuation ring;} \\
K &:= \text{frac. field}(\mathcal{O}) = \mathbb{Q}(t); \\
k &:= \mathcal{O}/m = \mathbb{Q}(\zeta^{2/2}) = \text{ residue field}
\end{align*}
\]

The validity of the original formulation over \( \mathbb{Z}[t^2, t^{-2}] \) would imply the validity over \( \mathbb{Z}[t, t^{-1}] \), by easy base change arguments. Using \( \mathcal{Z} = \mathbb{Z}[t, t^{-1}] \) gives us the convenient property that the fraction field of \( \mathcal{Z} \), namely, \( \mathbb{Q}(t) \), is a splitting field for \( \mathcal{H} \). Also, many results on Hecke algebras quoted from the literature are phrased using \( \mathbb{Z}[t, t^{-1}] \).

Standardly stratified algebras are like quasi-hereditary algebras, except that the irreducible head of an indecomposable standard module \( \Delta \) may appear with multiplicity \( > 1 \) in \( \Delta \); see \([2, \text{Defn. 6.4.1}]\). Empirical evidence (e.g., \([6, 5]\)) suggests these algebras play an important role in non-defining representation theory of finite groups of Lie type, especially for small primes \( p \), where the quasi-hereditary notion can be too strong. As in the quasi-hereditary case, the associated module derived categories of a standardly stratified algebra exhibit a “stratification,” though the number of “strata” may be smaller than in the quasi-hereditary case. In the case of algebras of arising from the conjecture and its modification in the Appendix, the number of strata is the number of two-sided cells.
The base changed algebra $H_Q(t)$ is split semisimple, with irreducible modules corresponding to the irreducible modules of the algebra $QW$. The $Q$-algebra

$$\tilde{H} := H \otimes \mathbb{Z}Q$$

has a presentation by elements $T_w \otimes 1$ (which will still be denoted $T_w$, $w \in W$) completely analogous to (1.0.1). Finally, similar remarks apply to $H_k = \tilde{H}_k$, replacing $t^2$ by $\zeta$. Then Theorem 5.6 establishes that there exists a $\tilde{H}$-module $\tilde{X}$ which is filtered by dual left cell modules $\tilde{S}_w$ such that the analogues of conditions (1) and (2) over $Q$ in Conjecture 1.1 hold. The final stratifying system assertion in the conjecture holds with the pre-order $\leq_{LR}$ on $\Omega$ replaced by an explicit refinement pre-order $\leq_f$.

With more work, it can be shown that the $Q$-algebra $\tilde{A}^+$ it actually quasi-hereditary. This is done in §6; see Theorem 6.4. Then Corollary 6.5 identifies the module category for a base-changed version of this algebra with a RDAHA-category $O$ in [8].

Finally, an Appendix discusses several variations on Conjecture 1.1.

Generally speaking, this paper focuses on the “single parameter” case (i.e., all $c_s = 1$ in (1.0.1)), which covers the Hecke algebras relevant to all untwisted finite Chevalley groups. This avoids a number of complications involving Kazhdan-Lusztig basis elements and Lusztig’s algebra $J$. In this context, the critical Proposition 3.2 depends on results of [8] which, in part, were only determined in the equal parameter case. Nevertheless, much of our discussion applies in the unequal parameter cases. In particular, we mention the elementary, but important, Lemma 4.3 is stated and proved using unequal parameter notation. This encourages the authors to believe the main results are also provable in the unequal parameter case, though this has not yet been carried out.

2. SOME PRELIMINARIES

In this section, we recall some mostly well-known facts and fix notation regarding cell theory. Thus, let $W$ be a finite Weyl group associated to a finite root system $\Phi$ with a fixed set of simple roots $\Pi$. Let $S := \{s_\alpha \mid \alpha \in \Pi\}$. Let $H$ is a Hecke algebra over $Q$ defined by (1.0.1). We assume (unless explicitly noted otherwise) that each $c_s = 1$ for $s \in S$.

Let

$$C'_w = t^{-l(w)} \sum_{y \leq w} P_{y,w}(t^2)T_y.$$ 

Then $\{C'_w\}_{w \in W}$ is a Kazhdan–Lusztig (or canonical) basis for $H$. The element $C'_x$ is denoted $c_x$ in [13], a reference we frequently quote. Let $h_{x,y,z} \in \mathbb{N}[t, t^{-1}]$ denote the structure constants. In other words,

$$C'_x C'_y = \sum_{z \in W} h_{x,y,z} C'_z.$$ 

3 From now on in this paper, we will work with the single parameter case, unless stated explicitly otherwise. Thus, we assume that each $c_s = 1$. The unequal parameter situation will be taken up elsewhere.
Using the pre-orders $\leq_L$ and $\leq_R$ on $W$, the positivity (see [4 §7.8]) of the coefficients of the $h_{x,y,z}$ implies

\begin{equation}
(2.0.3) \quad h_{x,y,z} \neq 0 \implies z \leq_L y, z \leq_R x
\end{equation}

The Lusztig function $a : W \rightarrow \mathbb{N}$ is defined as follows. For $z \in W$, let $a(z)$ be the smallest nonnegative integer such that $t^{a(z)}h_{x,y,z} \in \mathbb{N}[t]$ for all $x, y \in W$. It may equally be defined as the smallest nonnegative integer such that $t^{-a(z)}h_{x,y,z} \in \mathbb{N}[t^{-1}]$, as used in [13] (or see [4 §7.8]). In fact, each $h_{x,y,z}$ is invariant under the automorphism $2^\ast \rightarrow 2^\ast$ sending $t$ to $t^{-1}$. It is not difficult to see that $a(z) = a(z^{-1})$. For $x, y, z \in W$, let $\gamma_{x,y,z}$ be the coefficient of $t^{-a(z)}$ in $h_{x,y,z^{-1}}$. Also, by [13 Conjs. 14.2(P8),15.6],

\begin{equation}
(2.0.4) \quad \gamma_{x,y,z} \neq 0 \implies x \sim_L y^{-1}, y \sim_L z^{-1}, z \sim_L x^{-1}.
\end{equation}

The function $a$ is constant on two-sided cells in $W$, and so can be regarded as a function (with values in $\mathbb{N}$ on (a) the set of two-sided cells; (b) the set of left (or right) cells; and (c) the set $\text{Irr}(QW)$ of irreducible $QW$-modules. In addition, $a$ is related to the generic degrees $d_E, E \in \text{Irr}(QW)$. For $E \in \text{Irr}(QW)$, let $d_E = b t^{a_E} + \cdots + ct^{A_E}$, with $a_E \leq A_E$ and $bc \neq 0$, so that $t^{a_E}$ (resp., $t^{A_E}$) is the lowest (resp., largest) power of $t$ appearing nontrivially in $d_E$. Then $a_E = a(E)$; cf. [13 Prop. 20.6]. Also, as noted in [8 §6], $A_E = N - a(E \otimes \text{det})$, where $N$ is the number of positive roots in $\Phi$. Following [8 §6], we will use the “sorting function” $f : \text{Irr}(QW) \rightarrow \mathbb{N}$ defined by

\begin{equation}
(2.0.5) \quad f(E) = A_E + a_E = a(E) + N - a(E \otimes \text{det}).
\end{equation}

This function $f$ is used in [8] to define a poset structure on the set $\text{Irr}(QW)$ of irreducible $QW$-modules by putting $E <_f E'$ provided $f(E) < f(E')$. We use it here to define a pre-order $\leq_f$ on the set $\Omega$ of left cells: First, observe that the function $f$ above is constant on irreducible modules associated to the same left cell (or even the same two-sided cell) and so may be viewed as a function on $\Omega$. We can now define the (somewhat subtle) pre-order $\leq_f$ on $\Omega$ by setting $\omega \leq_f \omega'$ (for $\omega, \omega' \in \Omega$) if and only if either $f(\omega) < f(\omega')$, or $\omega$ and $\omega'$ lie in the same two-sided cell. Note that the pre-orders $\leq_f$ and $\leq_{op}^{LR}$ on $\Omega$ have the same equivalence classes, which identify with the set of two-sided cells. Also, $E <_{LR} E'$ implies that $E' <_f E$; see [8 Lem. 6.6]. Here $E, E'$ are in $\text{Irr}(QW)$, but could just as well have been taken in $\Omega$. Thus, the pre-order $\leq_f$ is a refinement of the pre-order $\leq_{op}^{LR}$ on $\Omega$, and $\leq_f$ induces on the set of two-sided cells a refinement of the partial order $\leq_{op}^{LR}$. For further discussion, see the Appendix.

3. (DUAL) SPECHT MODULES OF GINZBURG–GUAY–OPDAM–ROUQUIER

The asymptotic form $\mathcal{J}$ of $\mathcal{H}$ is a ring with $\mathbb{Z}$-basis \{\(j_x \mid x \in W\)\} and multiplication

\[ j_x j_y = \sum z \gamma_{x,y,z}^{-1} j_z. \]

This ring was originally introduced in [10], but we prefer to follow [13 18.3], though we do not use exactly the same notation. In the same spirit, we also follow [13 18.9] in defining
a $\mathcal{Z}$-algebra homomorphism
\[ \varpi : \mathcal{H} \to \mathcal{J}_\mathcal{Z} = \mathcal{J} \otimes \mathcal{Z}, \quad C_w' \mapsto \sum_{a(z) = a(d) = a} \sum_{g \in \mathcal{D}} h_{w, d, z} z, \]
where $\mathcal{D}$ is the set of distinguished involutions in $W$. This becomes an isomorphism after base change to $\mathbb{Q}(t)$: $\mathcal{H}_{\mathbb{Q}(t)} \cong \mathcal{J}_{\mathbb{Q}(t)}$. In particular, $\varpi$ induces a monomorphism
\[
\varpi : \mathcal{H}_{\mathbb{Q}[t, t^{-1}]} \rightarrow \mathcal{J}_{\mathbb{Q}[t, t^{-1}]} = \mathcal{J}_\mathbb{Q} \otimes \mathbb{Q}[t, t^{-1}].
\]
Moreover, base change to $\mathbb{Q}[t, t^{-1}]/(t - 1)$ induces an isomorphism $\mathbb{Q}W \cong \mathcal{J}_\mathbb{Q}$ (cf., [11, Prop. 1.7]). This allows us to identify irreducible $\mathbb{Q}W$-modules with irreducible $\mathcal{J}_\mathbb{Q}$-modules.

For the irreducible $\mathcal{J}_\mathbb{Q}$-module identified with $E \in \text{Irr}(\mathbb{Q}W)$, the $\mathcal{H}_{\mathbb{Q}[t, t^{-1}]}$-module
\[ S(E) := \varpi^*(E \otimes \mathbb{Q}[t, t^{-1}]) = \varpi^*(E_{\mathbb{Q}[t, t^{-1}]}) \]
is called here a dual Specht module for $\mathcal{H}_{\mathbb{Q}[t, t^{-1}]}$; cf. [8, Cor. 6.10][4]. For any commutative $\mathbb{Q}[t, t^{-1}]$-algebra $R$, let $\varpi_R$ be the homomorphism obtained from (3.0.6) by base change to $R$:
\[ \varpi_R : \mathcal{H}_{\mathbb{Q}[t, t^{-1}]} \otimes R \rightarrow \mathcal{J}_{\mathbb{Q}[t, t^{-1}]} \otimes R. \]
We also define
\[ S_R(E) := \varpi_R^*(E_{\mathbb{Q}[t, t^{-1}]} \otimes R) \quad \text{and} \quad \tilde{S}(E) = S_{\mathcal{Z}}(E). \]
It is clear from the definition that there are $\mathcal{H}_R$-module isomorphisms $S_R(E) \cong S(E) \otimes R$ for all $E \in \text{Irr}(\mathbb{Q}W)$.

**Remark 3.1.** In the main argument for the proof of the theorem, we work with modules $S_R(E)^* = \text{Hom}_R(S_R(E), R)$, $E \in \text{Irr}(\mathbb{Q}W)$, rather than (the possibly larger) dual left cell modules.

The following proposition is proved using RDAHAs, and it is the only ingredient in the proof of Theorem 5.6 where these algebras are used.

**Proposition 3.2.** Suppose $E, E'$ are irreducible $\mathbb{Q}W$-modules. If $E \not= E'$ and
\[ \text{Hom}_{\mathcal{H}_k}(S_k(E), S_k(E')) \not= 0, \]
then $f(E) < f(E')$. Also, $\text{Hom}_{\mathcal{H}_k}(S_k(E), S_k(E)) \cong k$.

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\[ ^4 \] The map $\varpi$ is the composition $\phi \circ \dagger$, where $\phi$ and $\dagger$ are defined in [13, 18.9] and [13, 3.5], respectively. The signs $\tilde{\alpha}_z$ appearing there are all equal to 1, because of the positivity (see [13, §7.8]) of the structure constants appearing in [13, 14.1]. This $\varpi$ is not the same one as defined in [8, p.647] or [10, §2.4], where the $C$-basis was used. Nevertheless, the arguments of [8, §6] go through, using the $C'$-basis and our $\varpi$ (see Remark 5.2 below), so [8, Thm. 6.8] guarantees the modules $S_C(E)$ defined below using our set-up are the same, at least up to a (two-sided cell preserving) permutation of the isomorphism types labeled by the $E$'s, as the modules $S(E)$ defined in [8, Defn. 6.11] with $R = \mathbb{C}$. The proof of [8, Thm. 6.8] also establishes such an identification of the various modules $S_R(E)$ when $R$ is a completion of $\mathbb{C}[t, t^{-1}]$.

\[ ^5 \] In [8, Defn. 6.1], the module $S(E)$ there is called a standard module. Our choice of terminology is justified by the discussion following the proof of Lemma 5.1 below.
Proof. Without loss, we replace $k$ in the statement of the lemma by $\mathbb{C}$, using the analogous definitions of $S_{C}(E)$. In addition, the statement of the proposition is invariant under any two-sided cell preserving permutation of the labeling of the irreducible modules. After applying such a permutation on the right (say) we may assume, by [8, Thm. 6.8] and taking into account fttn. 4, that

$$KZ(\Delta(E)) \cong S_{C}(E),$$

where

1. $\Delta(E)$ is the standard module for a highest weight category $O$ given in [8], having partial order $\leq_f$ (see [8, Lem. 2.9, 6.2.1]) on its set of isomorphism classes of irreducible modules, which are indexed by isomorphism classes of irreducible $\mathbb{Q}W$-modules. We take $k_{H,1} = 1/e > 0$ in [8] above Thm. 6.8 and in Rem. 3.2 there.

2. The functor $KZ : O \to \tilde{O}$ is naturally isomorphic to the quotient map $M \mapsto \tilde{M}$ in [8, Prop. 5.9, Thm. 5.14], the quotient category there identifying with $\mathcal{H}_{C}$-mod.

Using [8 Prop. 5.9], we have, for any irreducible $\mathbb{Q}W$-modules $E, E'$,

$$\text{Hom}_{C}(\Delta(E), \Delta(E')) \cong \text{Hom}_{\tilde{O}}(\tilde{\Delta}(E), \tilde{\Delta}(E')) \cong \text{Hom}_{\mathcal{H}_{C}}(S_{C}(E), S_{R}(E')).$$

If $E \not\cong E'$, then $\Delta(E) \not\cong \Delta(E')$ and $\text{Hom}_{C}(\Delta(E), \Delta(E')) \neq 0$ implies that $E <_f E'$, i.e., $f(E) < f(E')$.

On the other hand, if $E = E'$, then $\text{Hom}_{C}(\Delta(E), \Delta(E')) \cong \mathbb{C}$. This implies

$$\text{Hom}_{\mathcal{H}_{C}}(S_{C}(E), S_{C}(E')) \cong \mathbb{C}.$$

Returning to the original $k = \mathbb{Q}(\zeta^{\frac{1}{e}})$, we may conclude the same isomorphism holds in the original setting as well. \hfill \Box

Corollary 3.3. Let $E, E'$ be irreducible $\mathbb{Q}W$-modules. Then

$$\text{Ext}^1_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E')) \neq 0 \implies f(E) < f(E').$$

In particular, $\text{Ext}^1_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E)) = 0$.

Proof. In $\mathcal{O}$, write $m = (\pi)$, and consider the short exact sequence

$$0 \to \tilde{S}(E') \xrightarrow{\pi} \tilde{S}(E') \to S_{k}(E) \to 0.$$

By the long exact sequence of $\text{Ext}$, there is an exact sequence

$$0 \to \text{Hom}_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E')) \xrightarrow{\pi} \text{Hom}_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E')) \to \text{Hom}_{\mathcal{H}}(S_{k}(E), S_{k}(E'))$$

$$\to \text{Ext}^1_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E')) \xrightarrow{\pi} \text{Ext}^1_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E'))$$

$$\to \text{Ext}^1_{\mathcal{H}}(S_{k}(E), S_{k}(E')).$$

Because $\mathcal{H}_{\mathbb{Q}(\pi)} = \tilde{\mathcal{H}}_{\mathbb{Q}(\pi)}$ is semisimple,

$$\text{Ext}^1_{\mathcal{H}}(\tilde{S}(E), \tilde{S}(E'))_{\mathbb{Q}(\pi)} \cong \text{Ext}^1_{\mathcal{H}_{\mathbb{Q}(\pi)}}(S(E)_{\mathbb{Q}(\pi)}, S(E')_{\mathbb{Q}(\pi)}) = 0.$$
In other words, if it is nonzero, $\text{Ext}^1_R(\tilde{S}(E), \tilde{S}(E'))$ is a torsion module, so that the map $\text{Ext}^1_R(\tilde{S}(E), \tilde{S}(E')) \twoheadrightarrow \text{Ext}^1_R(\tilde{S}(E), S(E'))$ is not injective. Thus, it suffices to prove that when $f(E) \not\in f(E')$, the map

$$\text{Hom}_R(\tilde{S}(E), \tilde{S}(E')) \to \text{Hom}_R(S_k(E), S_k(E'))$$

is surjective. If $E \not\cong E'$, Proposition 3.2 gives $\text{Hom}_R(S_k(E), S_k(E')) = 0$ implying the surjectivity of (3.0.7) trivially. On the other hand, if $E \cong E'$, the proposition gives $\text{Hom}_R(S_k(E), S_k(E')) \cong k$. This also gives surjectivity of the map in (3.0.7), since it becomes surjective upon restriction to $\mathcal{O} \subseteq \text{Hom}_R(\tilde{S}(E), \tilde{S}(E'))$ (taking $E' = E$). □

4. TWO PRELIMINARY LEMMAS

Let $R$ be a commutative ring and let $\mathcal{C}$ be an abelian $R$-category with enough projective objects or enough injective objects. For $A, B \in \mathcal{C}$ and $n \geq 0$, denote $\text{Ext}^n(A, B)$ simply by $\text{Ext}^n(A, B)$. Let $M, Y \in \mathcal{C}$, and suppose that

$$\text{Ext}^1(M, Y) \cong \bigoplus_{i=1}^m R\epsilon_i,$$

where $R\epsilon_i$ is a cyclic module generated by $\epsilon_i$. Let $\chi := \bigoplus_i \epsilon_i \in \text{Ext}^1(M^\oplus m, Y)$ correspond to the short exact sequence $0 \to Y \to X \to M^\oplus m \to 0$.

**Lemma 4.1.** The map $\text{Ext}^1(M, Y) \to \text{Ext}^1(M, X)$, induced by the inclusion $Y \to X$, is the zero map.

**Proof.** Using the long exact sequence of $\text{Ext}^*$, it suffices to show that the map $\delta$ in the sequence

$$\text{Hom}(M, X) \to \text{Hom}(M, M^\oplus m) \to \text{Ext}^1(M, Y)$$

is surjective—equivalently, that each $\epsilon_i \in \text{Ext}^1(M, Y)$ lies in the image of $\delta$. Let $0 \to Y \to X_i \to M \to 0$ correspond to $\epsilon_i \in \text{Ext}^1(M, Y)$. By construction $\epsilon_i$ is the image of $\chi$ under the natural map

$$j^*_i : \text{Ext}^1(M^\oplus m, Y) \to \text{Ext}^1(M, Y),$$

which is the pull-back of the inclusion $j_i$ of $M$ into the $i$th summand of $M^\oplus m$. So there is a natural commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & Y & \rightarrow & X & \rightarrow & M^\oplus m & \rightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow j_i & . \\
0 & \rightarrow & Y & \rightarrow & X_i & \rightarrow & M & \rightarrow & 0
\end{array}$$
There is a corresponding commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(M, X) & \longrightarrow & \text{Hom}(M, M^\oplus m) \\
\uparrow & & \uparrow \\
\text{Hom}(M, X_i) & \longrightarrow & \text{Hom}(M, M) \\
\end{array} \xrightarrow{\delta_i} \text{Ext}^1(M, Y)
\]

where each row is part of a long exact sequence. Then the commutativity of the right-hand square in the diagram above immediately says that \( \epsilon_i \) lies in the image of \( \delta \).

This lemma together with the additivity of the functor \( \text{Ext}^1 \) gives immediately the following.

**Corollary 4.2.** Maintain the set-up above. If \( \text{Ext}^1(M, M) = 0 \), then \( \text{Ext}^1(M, X) = 0 \).

Next, let \( R \) be a commutative ring which is a \( \mathcal{Z} \)-algebra and write \( q = t^2 \cdot 1 \), the image in \( R \) of \( t^2 \in \mathcal{Z} \). For the rest of this section, we allow general parameters \( c_s, s \in S \).

**Lemma 4.3.** Let \( \mathfrak{M} \subseteq \mathfrak{M} \) be left ideals in \( \mathcal{H}_R \), with each spanned by the Kazhdan–Lusztig basis elements \( C_y' \) that they contain. Let \( s \in S \) be a simple reflection and assume either \( \mathfrak{N} = 0 \) or that \( q^{c_s} + 1 \) is not a zero divisor in \( R \). Suppose \( x \in \mathfrak{M}/\mathfrak{N} \) satisfies

\[
(4.0.8) \quad T_s \cdot x = q^{c_s}x.
\]

Then \( x \) is represented in \( \mathfrak{M} \) by an \( R \)-linear combination of Kazhdan–Lusztig elements \( C_y' \) with \( sy < y \).

**Proof.** Let \([m]\) denote the image in \( \mathfrak{M}/\mathfrak{N} \) of \( m \in \mathfrak{M} \). Note that \( \mathfrak{M}, \mathfrak{N} \) and \( \mathfrak{M}/\mathfrak{N} \) are all \( R \)-free, since the \( C_y' \) which belong to \( \mathfrak{M} \) (resp., \( \mathfrak{N} \)) form a basis for \( \mathfrak{M} \) (resp., \( \mathfrak{N} \)). The \( R \)-module \( \mathfrak{M}/\mathfrak{N} \) has a basis consisting of all \([C_y'] \neq 0 \) with \( C_y' \in \mathfrak{M} \).

Write \( x = \sum_y a_y[C_y'] \) with \( a_y[C_y'] \neq 0 \) and \( C_y' \in \mathfrak{M} \). Observe that, for \( y \in W, s \in S \),

\[
(4.0.9) \quad sy < y \implies T_sC_y' = q^{c_s}C_y'.
\]

Therefore, in the above expression for \( x \), it may also be assumed that \( sy > y \) for each nonzero term \( a_y[C_y'] \). Let \( a_w[C_w'] \neq 0 \) be chosen with \( w \) maximal among these \( y \). In general, for \( sy > y \), we have

\[
T_sC_y' = -C_y' + C_{sy'} + \sum_{z < y, z < s} b_z C_z'
\]

for some \( b_z \in R \). If \( \mathfrak{N} = 0 \), we find that \( C_w' \) appears with nonzero coefficient in \( x \), contradicting the (implicit) assumption \( x \neq 0 \) above. If \( \mathfrak{N} \neq 0 \), equating coefficients of \([C_w'] \) gives by \((4.0.8)\) that \((q^{c_s} + 1)a_w = 0 \), since \( C_w' \) does not appear with any coefficient in the expressions \( T_sC_y' \) with \( y \neq w \) and \( sy > y \). Now the nonzero divisor hypothesis forces \( a_w = 0 \), a contradiction. \( \Box \)
Remark 4.4. As observed in (4.0.9) above, elements \( x \in \mathcal{M}/\mathfrak{N} \) satisfying the conclusion of Lemma 4.3 also satisfy its hypothesis (4.0.8). Next, suppose that \( \lambda \subseteq S \) and \( L \) is any \( \mathcal{H}_R \)-module. By Frobenius reciprocity, the \( R \)-module \( \text{Hom}_{\mathcal{H}_R}(\mathcal{H}_Rx_{\lambda}, L) \) identifies with the \( R \)-submodule \( \mathcal{X} \subseteq L \) consisting of all \( x \in L \) satisfying (4.0.8) for all \( s \in \lambda \). Suppose \( L \) can be realized as \( L = \mathcal{M}/\mathfrak{N} \), with \( \mathcal{M}, \mathfrak{N} \) as in the statement of Lemma 4.3. If \( q^s + 1 \) is invertible in \( R \) for all \( s \in \lambda \), then the lemma implies that \( \mathcal{X} \) has an \( R \)-basis consisting of all nonzero \( [C_y] \) in \( L \) with \( sy < y \) for all \( s \in \lambda \).

Thus, if \( R' \) is an \( R \)-algebra, then the \( R' \)-module \( \text{Hom}_{\mathcal{H}_{R'}}(\mathcal{H}_{R'}x_{\lambda}, L_{R'}) \) has essentially the “same basis.” This will be applied in the corollary immediately below with the role of \( R \) played by \( \tilde{R} \).

Corollary 4.5. Suppose \( \tilde{R} \) is a commutative domain with fraction field \( F \), and assume that \( \tilde{R} \) is also a \( \mathcal{X} \)-algebra. Let \( \lambda \subseteq S \). Assume that

1. \( \mathcal{H}_F \) is semisimple;
2. \( q^s + 1 \) is invertible in \( \tilde{R} \), for each \( s \in \lambda \).

Then, for any dual left cell module \( S \) over \( \tilde{R} \),

\[
\text{Ext}^1_{\mathcal{H}_{\tilde{R}}}(S, x_{\lambda}\mathcal{H}_{\tilde{R}}) = 0.
\]

Proof. Using condition (1) and Lemma [6] (1.2.13), it suffices to prove, for each \( R = \tilde{R}/(d) \) (\( d \in \tilde{R} \)), that the map

\[
\text{Hom}_{\mathcal{H}_{\tilde{R}}}(S, x_{\lambda}\mathcal{H}_{\tilde{R}}) \longrightarrow \text{Hom}_{\mathcal{H}_{R}}(S_R, x_{\lambda}\mathcal{H}_{R})
\]

is surjective. Here \( S_R = S \otimes_{\tilde{R}} R \).

By [6] Lem. 2.1.9], the left \( \mathcal{H}_{\tilde{R}} \)-module \( (x_{\lambda}\mathcal{H}_{\tilde{R}})^* := \text{Hom}_{\tilde{R}}(x_{\lambda}\mathcal{H}_{\tilde{R}}, \tilde{R}) \) is naturally isomorphic to \( \mathcal{H}_{\tilde{R}}x_{\lambda} \). By hypothesis, \( S = L^* \) is the dual of a left cell module \( L \), \( \tilde{R} \)-free by definition. Thus, \( L \cong S^* \); also, \( (\mathcal{H}_{\tilde{R}}x_{\lambda})^* \cong x_{\lambda}\mathcal{H}_{\tilde{R}} \). There are similar isomorphisms for analogous \( R \)-modules (for which we use the same notation \((-)^*\)). The functor \((-)^*\) provides a contravariant equivalence from the category of finitely generated and \( \tilde{R} \)-free left \( \mathcal{H}_{\tilde{R}} \)-modules and the corresponding right \( \mathcal{H}_{\tilde{R}} \)-module category. A similar statement holds with \( \tilde{R} \) replaced by \( R \). Finally, there is a natural isomorphism \((-)^* \otimes_{\tilde{R}} R \cong (- \otimes_{\tilde{R}} R)^* \).

Consequently, it is sufficient to prove that

\[
\text{Hom}_{\mathcal{H}_{\tilde{R}}}(\mathcal{H}_{\tilde{R}}x_{\lambda}, L) \longrightarrow \text{Hom}_{\mathcal{H}_{R}}(\mathcal{H}_{R}x_{\lambda}, L_R)
\]

is surjective. (Here \( \mathcal{L}_R \) denotes the left cell module in \( \mathcal{H}_R \) defined by the same left cell as \( \mathcal{L} \) for \( \mathcal{H} \).) However, viewing \( \mathcal{L} \) and \( \mathcal{L}_R \) as cell modules (over \( \mathcal{H}_{\tilde{R}} \) and \( \mathcal{H}_R \), respectively), Lemma 4.3 and the remark above give the “same basis” (over \( \tilde{R} \) and \( R \), respectively). \( \square \)

5. The Construction of \( \tilde{X}_\omega \) and the Main Theorem

Consider a left cell \( \omega \) and let \( J_\omega = \sum_{y \in \omega} \mathbb{Z}j_y \). Then (2.0.4) implies that \( J_\omega \) is a left \( \mathcal{J} \)-module. We can pull back \( J_\omega \otimes \mathcal{X} \) to an \( \mathcal{H} \)-module using the homomorphism \( \varpi \) of §3.
Lemma 5.1. There is an $\mathcal{H}$-module isomorphism
\[ \sigma : \mathcal{W}^*(\mathcal{J}_\omega \otimes \mathcal{Z}) \rightarrow S(\omega) := \mathcal{H}^{<L\omega}/\mathcal{H}^{<L\omega} \]
induced by the map $\sigma : \mathcal{J}_\mathcal{Z} \rightarrow \mathcal{H}$, $j_y \mapsto C'_y$. In particular, $\tilde{S}(\omega) := S(\omega) \otimes \mathcal{Z}$ is a direct sum of modules $\tilde{S}(E) := S_{\mathcal{Z}}(E)$ for some $E \in \text{Irr}(\mathbb{Q}W)$.

Proof. This is a refinement of [13] 18.10. We first observe that the map $\sigma$ clearly induces a $\mathcal{Z}$-module isomorphism. It remains to check for $y \in \omega$ that
\[ \sigma(\mathcal{W}(C'_x)j_y) \equiv C'_x C'_y \mod \mathcal{H}^{<L\omega}, \quad (x \in W) \]
The proof of [13] 18.10(a)\footnote{The main ingredient is [13] 18.9(b). As previously noted, the signs $\hat{n}_z$ may be set equal to 1.} gives the left-hand equality in the expression
\[ (5.0.10) \quad \sigma(\mathcal{W}(C'_x)j_y) = \sigma(\sum_{a(y) = a(u)} h_{x,y,u}j_u) = \sum_{a(y) = a(u)} h_{x,y,u}C'_u \equiv C'_x C'_y \mod \mathcal{H}^{<L\omega}. \]
The middle equality is just the definition of $\sigma$. Finally, the right-hand congruence follows from the fact that, when $h_{x,y,u}C'_u$ is nonzero mod $\mathcal{H}^{<L\omega}$, $u$ must belong to the same left cell $\omega$ as $y$, and hence have the same $a$-value. 

If $W$ is of type $A$ and $\omega$ is the left cell containing the longest word $w_{0,\lambda}$ for a partition $\lambda$. Then $\mathcal{W}^*(\mathcal{J}_\omega \otimes \mathcal{Z})$ is isomorphic to the left cell module whose dual is the Specht module $S_{\lambda}$. So $\tilde{S}(E)$ above could be called a “dual Specht module,” with $\tilde{S}(E)^*$ a “Specht module.” The module $\tilde{S}_\omega$ defined below are also candidates for the name “Specht module” [6] p.198).

Remark 5.2. A completely analogous result to Lemma 5.1 holds if the Kazhdan-Lusztig $C$-basis (instead of the $C'$-basis here) is used, as in [8]. First, it follows from [9] (3.2)] that the map (which we call $\tau$) $\mathcal{Z} \rightarrow \mathcal{Z}$, sending $t \mapsto -t$, takes the coefficients $h_{x,y,z}$ to analogous coefficients for the $C$-basis. Extend $\tau$ to an automorphism, still denoted $\tau$, of $\mathcal{J}_\mathcal{Z}$, taking $j_x$ to its $C$-analogue; we may put $\tau(j_x) = (-1)^{\ell(x)} j_x$. Thus, any expression $h_{x,y,z}j_x$ is sent to a $C$-basis analogue. In particular, $\mathcal{W}(C'_x)$ is sent to $\mathcal{W}(C_x)$, where the latter $\mathcal{W}$ is taken in the $C$-basis set-up. Now it is clear from (5.0.10) that the analogue of Lemma 5.1 holds in the $C$-basis set-up. Note the resulting left cell modules in $\mathcal{H}$ do not depend on which canonical basis is used. This allows an identification of the module $S(\omega)$ in Lemma 5.1 with its $C$-basis counterpart.

An analogous result holds for two-sided cells, e.g., the $\mathcal{H}$-module $\mathcal{W}^*(\mathbb{Z}[t, t^{-1}] \otimes \mathcal{Z} \mathcal{J}_\mathcal{Z})$ in [8] Cor. 6.4] does not depend on the whether the $C'$-basis is used (as in this paper) or the $C$-basis is used (as in [8]). We do not know, however, if the base-change of $\tau$ to $\mathcal{J}_{\mathbb{Q}(t)}$ preserves the isomorphism types of irreducible $\mathcal{J}_{\mathbb{Q}(t)}$-modules, though their associated two-sided cells are preserved. This leads to the “permutation” language used in ftn. 4. In particular, we do not know if the bijection noted below [8] Defn. 6.1] depends on the choice of $C$ of $C'$-basis set-up, and could result in one choice leading to an identification which is a (two-sided cell preserving) permutation of the other.
For a left cell \( \omega \) in \( W \), let \( S(\omega) \) be the corresponding left cell module and \( S_\omega = S(\omega)^* \) the dual left cell module associated with \( \omega \). By Lemma 5.1, \( S(\omega)_{Q[v,w^{-1}]} \) is a direct sum of \( S(E) \)'s. Since the \( a \)-value of elements of \( \omega \) is constant, it follows that \( f(E) \) is constant over all \( E \) occurring in \( S(\omega)_{Q[v,w^{-1}]} \). Thus, we simply set \( f(\omega) = f(E) \).

**Remark 5.3.** For a two-sided cell \( c \), let \( J_c \) be the \( \mathbb{Z} \)-span of the \( j_x, x \in c \). The order \( \leq_{LR}^{op} \) is compatible with the order \( \leq_f \), since \( E \leq_{LR} E' \) with \( E \in \text{Irr}(J_c, q) \) and \( E' \in \text{Irr}(J_{c'}, q) \) implies \( a(c) > a(c') \) and, by [12, Prop.3.3], \( a_E = a(c) \). Thus, for any left cells \( \omega, \omega' \),

\[
f(\omega) = f(\omega') \iff a(\omega) = a(\omega').
\]

Let

\[
\widetilde{S}_\omega = \widetilde{S}(\omega)^* = (S(\omega) \otimes \mathcal{D})^*
\]

Then we have the following result.

**Corollary 5.4.** For left cells \( \omega, \omega' \), we have

\[
\text{Ext}^1_H(\widetilde{S}_\omega, \widetilde{S}_{\omega'}) \neq 0 \implies f(\omega) > f(\omega').
\]

**Proof.** By the lemma above and Corollary 3.3, \( \text{Ext}^1_H(\widetilde{S}(\omega'), \widetilde{S}(\omega)) \neq 0 \) implies \( f(\omega') > f(\omega) \). The result follows.

**Lemma 5.5.** For any subset \( \lambda \subseteq S \), let \( \omega_\lambda \) be the left cell containing the longest element of the parabolic subgroup \( W_\lambda \). Suppose

\[
0 = F^\lambda_0 \subseteq F^\lambda_1 \subseteq F^\lambda_2 \subseteq \cdots \subseteq F^\lambda_m = x_\lambda \mathcal{H}
\]

is a filtration by dual left cell modules \( S_{\omega_i} \cong F^\lambda_i / F^\lambda_{i-1} \) with \( \omega_1 = \omega_\lambda \). Then \( f(\omega_i) > f(\omega_\lambda) \).

**Proof.** Since \( \omega_1 \leq_L \omega_i \) and \( \omega_1 \not\leq_L \omega_i \) for all \( 2 \leq i \leq m \), by [10, Cor. 1.9(b)], we must have \( a(\omega_1) < a(\omega_i) \) for all \( i > 1 \). Let \( w_0 \) be the longest element of \( W \). For two sided cells \( c, c', c <_L c' \) if and only if \( w_0 c' <_L w_0 c \). It follows that \( E \in \text{Irr}(J_{Q,c}) \) if and only if \( E \otimes \text{sgn} \in \text{Irr}(J_{\text{w},Q}, Q) \). Hence, we have \( f(\omega_i) < f(\omega_i) \) for all \( i > 1 \).

In the following construction, all \( \text{Ext}^1 \)-groups will be over \( \widetilde{\mathcal{H}} \).

We now iteratively construct an \( \widetilde{\mathcal{H}} \)-module \( \widetilde{X}_\omega \), filtered by dual left cell modules, such that \( \widetilde{S}_\omega \subset \widetilde{X}_\omega \) and

\[
\text{Ext}^1(\widetilde{S}_\omega', \widetilde{X}_\omega) = 0 \text{ for all left cells } \omega'.
\]

Let \( \Omega \) be the set of all left cells of \( W \). Let

\[
\Omega_i = \{ \omega \in \Omega \mid f(\omega) = i \}
\]

Fix \( \omega \in \Omega \) with \( f(\omega) = i \). Suppose \( \text{Ext}^1(\widetilde{S}_\tau, \widetilde{S}_\omega) \neq 0 \) for some \( \tau \in \Omega \). Then, by the Corollary 5.4, \( f(\tau) > f(\omega) = i \). Assume \( f(\tau) = j \) is minimal with this property. Since \( \mathcal{D} \) is a DVR and \( \text{Ext}^1(\widetilde{S}_\tau, \widetilde{S}_\omega) \) is finitely generated, it follows that \( \text{Ext}^1(\widetilde{S}_\tau, \widetilde{S}_\omega) \) is a direct sum of \( m_\tau (\geq 0) \) nonzero cyclic \( \mathcal{D} \)-modules. Let \( \tilde{Y}_\tau \) be the extension of \( \tilde{S}_\tau^{\oplus m_\tau} \) by \( \tilde{S}_\omega \). Then by Lemma 4.1 and Corollary 4.2,

\[
\text{Ext}^1(\tilde{S}_\tau, \tilde{Y}_\tau) = 0.
\]
Let
\[ \Omega_{j,\omega} = \{ \nu \in \Omega_j \mid \text{Ext}^1(\widetilde{S}_\nu, \widetilde{S}_\omega) \neq 0 \} \]
If \( \nu \in \Omega_{j,\omega} - \{ \tau \} \), then \( \text{Ext}^1(\widetilde{S}_\nu, \widetilde{S}_\omega) \cong \text{Ext}^1(\widetilde{S}_\nu, \widetilde{Y}_\tau) \) by Proposition 3.2 and Corollary 3.3 together with the long exact sequence for Ext. Thus, if \( \widetilde{Y}_{\tau,\nu} \) denote the extension of \( \widetilde{S}_\nu \) by \( \widetilde{Y}_\tau \), then we have
\[ \text{Ext}^1(\widetilde{S}_\nu, \widetilde{Y}_{\tau,\nu}) = 0 \text{ for } \omega' = \tau, \nu. \]
Note that \( \widetilde{Y}_{\tau,\nu} \) is isomorphic to the extension of \( \widetilde{S}_\nu \oplus \widetilde{S}_\tau \) by \( \widetilde{S}_\omega \) and continue. Eventually, we find that, if \( \tilde{Y}_j \) is the extension of \( \oplus_{\tau \in \Omega_j,\omega} \widetilde{S}_\tau \) by \( \widetilde{S}_\omega \), then
\[ \text{Ext}^1(\tilde{S}_\omega', \tilde{Y}_j) = 0 \text{ for all } \omega' \in \bigcup_{1 \leq j} \Omega_i. \]
Thus, \( \text{Ext}^1(\tilde{S}_\omega', \tilde{Y}_j) \neq 0 \) implies \( f(\omega') > j \).
Continuing the above construction with the role \( \tilde{S}_\omega \) replaced by \( \tilde{Y}_{j_1} \) \((j_1 = j)\), we obtain a module \( \tilde{Y}_{j_1,j_2} \) such that \( j_1 < j_2 \) and
\[ \text{Ext}^1(\tilde{S}_\omega', \tilde{Y}_{j_1,j_2}) = 0 \text{ for all } \omega' \in \bigcup_{1 \leq j_1} \Omega_i. \]
Let \( m \) be the maximal \( f \)-value. This construction will stop after a finite number, say \( r = r(\omega) \), steps, resulting in an \( \mathcal{H} \)-module \( \tilde{X}_\omega := \tilde{Y}_{j_1,j_2,\ldots,j_r} \) such that
\[ f(\omega) < j_1 < j_2 < \cdots < j_r < m, \text{ and } \text{Ext}^1(\tilde{S}_\omega', \tilde{X}_\omega) = 0 \text{ for all } \omega' \in \Omega. \]
Let \( \Omega' \) be the set of all left cells that do not contain the longest element of a parabolic subgroup. Put
\[ \tilde{T} = \bigoplus_{\lambda \subseteq S} x_\lambda \mathcal{H} \quad \text{and} \quad \tilde{X} = \bigoplus_{\omega \in \Omega'} \tilde{X}_\omega. \]

We are now ready to prove the following main result of the paper.

**Theorem 5.6.** Let \( \tilde{T}^+ = \tilde{T} \oplus \tilde{X}, \tilde{A}^+ = \text{End}_\mathcal{H}(\tilde{T}^+) \) and \( \tilde{\Delta}(\omega) = \text{Hom}_\mathcal{H}(\tilde{S}_\omega, \tilde{T}^+) \) for \( \omega \in \Omega \). Then \( \{ \tilde{\Delta}(\omega) \}_{\omega \in \Omega} \) is a strict stratifying system for the category \( \tilde{\mathcal{A}}^+ \)-mod with respect to the quasi-poset \((\Omega, \leq_f)\).

**Proof.** For each left cell \( \omega \), put \( \tilde{T}_\omega = x_\lambda \mathcal{H} \) if \( \omega \) contains the longest element of \( W_\lambda \), where \( \lambda \subseteq S \). If there is no such \( \lambda \) for \( \omega \), put \( \tilde{T}_\omega = \tilde{X}_\omega \). In the first case, \( \tilde{T}_\omega \) has a filtration by dual left cell modules, and \( \tilde{S}_\omega \) appears at the bottom. Moreover, \( f(\omega) < f(\omega') \) for any other filtration section \( \tilde{S}_{\omega'} \), by Lemma 5.5. This same property holds also in the case \( \tilde{T}_\omega = \tilde{X}_\omega \) by construction.

Put \( \tilde{\mathcal{T}} = \bigoplus \tilde{T}_\omega \) and note \( \tilde{T}^+ = \tilde{\mathcal{T}} \). We will now apply [6] Thm. 1.2.10] to \( \tilde{\mathcal{T}} \) and the various \( \tilde{T}_\omega \). Our \( \tilde{\mathcal{T}} \) notation has been chosen to agree with the wording of the cited theorem. We are required the check three conditions (1), (2), (3) and to check the first condition (1) in a strong form to obtain the “strict” property. The discussion above of dual left cell filtrations of the various \( \tilde{T}_\omega \) is precisely what is required for the verification of (1) in its strong form.
Condition (2) translates directly to the requirement
\[ \text{Hom}_{\tilde{H}}(\tilde{S}_\mu, \tilde{T}_\omega) \neq 0 \implies \omega \leq f(\mu) \]
for given \( \mu, \omega \). However, if there is nonzero \( \text{Hom}_{\tilde{H}}(\tilde{S}_\mu, \tilde{T}_\omega) \), there must be a nonzero \( \text{Hom}_{\tilde{H}}(\tilde{S}_\mu, \tilde{S}_{\omega'}) \) for some filtration section \( \tilde{S}_{\omega'} \) of \( \tilde{T}_\omega \). In particular, \( f(\omega') \geq f(\omega) \). Also, \( (\tilde{S}_\mu)_F \) and \( (\tilde{S}_{\omega'})_F \) must have a common irreducible constituent, forcing the two-sided cells containing them to agree. This gives \( f(\mu) = f(\omega') \geq f(\omega) \); so (2) holds.

Finally, three is an \( \text{Ext}^1_{\tilde{H}}(-, \tilde{T}) \) vanishing result implied by the (generally stronger) condition
\[ \text{Ext}^1_{\tilde{H}}(\tilde{S}_\mu, \tilde{T}_\omega) = 0 \text{ for all } \mu, \omega. \]
This follows from the construction below Lemma 5.5 for \( \tilde{T}_\omega = \tilde{X}_\omega \) and by Corollary 4.5 in case \( \tilde{T}_\omega = x_\lambda \tilde{H} \). The conclusion of \([6, \text{Thm. 1.2.10}] \) now immediately gives the theorem we are proving here. \( \square \)

6. IDENTIFICATION OF \( \tilde{A}^+ = \text{End}_{\tilde{H}}(\tilde{T}^+) \)

The construction of the modules \( \tilde{X}_\omega \) in the previous section works just as well using the module \( \tilde{S}_E := \tilde{S}(E)^* \) in the role of dual left cell modules \( \tilde{S}_\omega \). This results in modules \( \tilde{X}_E \). As in the case of \( \tilde{X}_\omega \), we have the following property, with the same proof.

**Proposition 6.1.** \( \text{Ext}^1_{\tilde{H}}(\tilde{S}_{E'}, \tilde{X}_E) = 0 \text{ for all } E, E' \in \text{Irr}(W). \)

However, the modules \( \tilde{X}_E \) have strong indecomposability properties, which the modules \( \tilde{X}_\omega \) do not have, in general.

**Proposition 6.2.** The modules \( \tilde{X}_E \) are indecomposable, as is each \( \tilde{X}_E \otimes k \). The endomorphism algebras of all these modules are “local” (completely primary) with radical quotient \( k \).

**Proof.** This can be argued without using RDAHAs, but it is faster to quote Rouquier’s 1-faithful covering theory, especially \([14, \text{Thm. 5.3}] \), which applies to our \( e \neq 2 \) case, over \( \mathcal{R} \), where
\[ \mathcal{R} := \left( \mathbb{C}[t, t^{-1}]_{(t-\zeta_2^\frac{1}{2})} \right)^\wedge \]
is the completion of the localization \( \mathbb{C}[t, t^{-1}]_{(t-\zeta_2^\frac{1}{2})} \) at the maximal ideal \( (t-\zeta_2^\frac{1}{2}) \). Note that \( \mathcal{R} \) is a \( \mathcal{D} \)-module via the natural ring homomorphism \( \mathcal{D} \to \mathcal{R} \).

We write \( (\tilde{X}_E)_{\mathcal{R}} \) as \( \tilde{X}_{E,\mathcal{R}} \), etc. It is clear that \( \tilde{X}_{E,\mathcal{R}} \) can be constructed from \( \tilde{S}_{E,\mathcal{R}} \) in the same way that \( \tilde{X}_E \) is constructed from \( \tilde{S}_E \). Also, the proof of \([14, \text{Thm. 6.8}] \) shows that the \( \mathcal{R} \)-dual of \( \tilde{S}_{E,\mathcal{R}} \) is the \( Z\mathcal{K} \)-image of a standard module in the \( \mathcal{R} \)-version of \( \mathcal{O} \). (Recall the issues in fttn. 4.)
Consequently, by the 1-faithful property \((\tilde{X}_{E,R})^*)^*\) is the image of a dually constructed module \(P\) under the functor KZ, filtered by standard modules, and with \(\text{Ext}^1_{O}(P, -)\) vanishing on all standard modules. Such a module \(P\) is projective in \(O\), by, say, \([14\text{, Lem. 4.22}].\) (We remark that both \(O\) and KZ would be given a subcript \(\mathcal{R}\) in \([8]\) though not in \([14]\).

If we knew \(P\) were indecomposable, we could say \(\tilde{X}_{E,R}\) is indecomposable. However, this requires proof. Essentially, we want to show \(P\) is the projective cover in \(O\) of the standard module \(\Delta(E) = \Delta_O(E)\). We can, instead, inductively show the truncation \(P_i\) associated to the poset ideal of all \(E' \in \text{Irr}(W)\) with \(f(E') \leq i\), is the projective cover of \(\Delta(E)\) in the associated truncation \(O_i\) of \(O\). This requires \(\Delta(E)\) to be an object of \(O_i\), or equivalently \(f(E) \leq i\).

If \(f(E) = i\), then \(P_i = \Delta(E)\) is trivially the projective cover of \(\Delta(E)\). Inductively, \(P_{i-1}\) is the projective cover of \(\Delta(E)\) in \(O_{i-1}\) for some \(i > f(E)\). Let \(P'\) denote the projective cover of \(\Delta(E)\) in \(O_i\). The truncation \((P')_{i-1}\) to \(O_{i-1}\) of \(P'\) — that is, its largest quotient which is an object of \(O_{i-1}\) — is clearly isomorphic to \(P_{i-1}\). Let \(\varphi : P' \to P_{i-1}\) be a homomorphism extending a given isomorphism \(\psi : (P')_{i-1} \to P_{i-1}\) and let \(\tau : P_i \to P'\) be a homomorphism extending \(\psi^{-1}\). Let \(M, M'\) denote the kernels of the natural surjections \(P_i \to P_{i-1}\) and \(P' \to (P')_{i-1}\). The map \(\tau\varphi : P' \to P'\) is surjective and, consequently, it is an isomorphism. It induces the identity on \((P')_{i-1}\). Therefore, the induced map

\[
\tau|_M\varphi|_{M'} : M' \to M'
\]

is an isomorphism, and \(M = M' \oplus M''\) for some object \(M''\) in \(O\). By construction, \(M\) is a direct sum of objects \(\Delta(E')\), with \(f(E') = i\), each appearing with multiplicity \(m_{E'} = \text{rank}(\text{Ext}^1_{O}(P_i, \Delta(E')))).\) However,

\[
\text{Ext}^1_{O}(P_{i-1}, \Delta(E')) \cong \text{Hom}_O(M', \Delta(E')).
\]

It follows that \(M'' = 0\) and \(P_i \cong P'\) is indecomposable.

In particular, \(P'\) is indecomposable and consequently \(\tilde{X}_{E,R}\) is indecomposable, as noted. In turn, this implies \(\tilde{X}_E\) is indecomposable. The 0-faithfulness (or just the covering property itself) of the cover given by \(O\) and KZ imply

\[
\text{End}_{\mathcal{H}_{\mathcal{R}}}(\tilde{X}_{E,R})^{\text{op}} \cong \text{End}_{\mathcal{H}_{\mathcal{R}}}(\tilde{X}_{E,R}^*)^{\text{op}} \cong \text{End}_O(P).
\]

Thus, the base changed module \(P \otimes \mathbb{C}\) (the tensor is over \(\mathcal{R}\)) has endomorphism ring \(\text{End}_O(P \otimes \mathbb{C}) \cong \text{End}_O(P) \otimes \mathbb{C}\), where \(O_C\) is the \(\mathbb{C}\)-version of \(O\). This is a standard consequence of the projectivity of \(P\). By \([14\text{, Thm. 5.3}].\) the \(\mathbb{C}\) versions of KZ and \(O\) give a cover for \(\mathcal{H}_{\mathcal{R}} \otimes \mathbb{C}\). So \(\text{End}_{\mathcal{H}_{\mathcal{C}}}(\tilde{X}_{E,R} \otimes \mathbb{C})^{\text{op}} \cong \text{End}_{O_C}(P \otimes \mathbb{C})\) is local, with radical quotient \(\mathbb{C}\).

However, we have

\[
(\tilde{X}_E \otimes k) \otimes_k \mathbb{C} \cong \tilde{X}_{E,R} \otimes \mathbb{C}.
\]

\(^{7}\)A similar point should be made regarding the uniqueness claim in \([14\text{, Prop. 4.45}].\) which is false without a minimality assumption on \(Y(M)\) there.
HECKE ENDO MORPHISM ALGEBRAS

In particular, \( \widetilde{X}_E \otimes \mathbb{k} \) is indecomposable since (by endomorphism ring consideration) the \( \mathcal{H} \otimes \mathbb{C} \)-module \( \widetilde{X}_{E, \mathcal{H}} \otimes \mathbb{C} \) is indecomposable. So the endomorphism ring of \( \widetilde{X}_E \otimes \mathbb{k} \) over the finite dimensional algebra \( \mathcal{H} \otimes \mathbb{k} \) is local. The radical quotient is a division algebra \( D \) over \( \mathbb{k} \) with base change under \( - \otimes \mathbb{k} \mathbb{C} \) to a semisimple quotient of \( \text{End}_{\mathcal{H}_C}(\widetilde{X}_{E, \mathcal{H}} \otimes \mathbb{C}) \), which could only be \( \mathbb{C} \) itself. Consequently, \( D = \mathbb{k} \).

Finally, the vanishing \( \text{Ext}^1_{\mathcal{H}}(\widetilde{X}_E, \widetilde{X}_E) = 0 \) implies

\[
\text{End}_{\mathcal{H}}(\widetilde{X}_E) \otimes \mathbb{k} \cong \text{End}_{\mathcal{H}_k}(\widetilde{X}_E \otimes \mathbb{k}).
\]

So the ring \( \text{End}_{\mathcal{H}}(\widetilde{X}_E) \) is local with radical quotient \( \mathbb{k} \). This completes the proof. \( \square \)

We remind the reader that we continue to assume \( e \neq 2 \).

**Lemma 6.3.** Let \( E \in \text{Irr}(W) \). Then \( \widetilde{X}_E \) is a direct summand of \( \widetilde{T}^+ \).

**Proof.** Suppose first \( \tilde{S}(E) \) is a direct summand of a left cell module \( \tilde{S}(\omega) =: \tilde{S}^\omega = (\tilde{S}_\omega)^* \) where \( \omega \) contains the longest element of a parabolic subgroup \( W_\lambda, \lambda \subseteq S \). This implies \( \tilde{S}_\omega \) is the lowest term in the dual left cell module filtration of \( x_\lambda \mathcal{H} \). Consequently, there is an inclusion \( \psi : \tilde{S}_E \to x_\lambda \mathcal{H} \) with cokernel filtered by (sections) \( \tilde{S}_E', E' \in \text{Irr}(W) \).

Thus, \( \psi^{-1} : \tilde{S}(E) \to \tilde{X}_E \) may be extended to a map \( \phi : x_\lambda \mathcal{H} \to \tilde{X}_E \) of \( \mathcal{H} \)-modules. Similarly (using \( e \neq 2 \) and Corollary 4.5), there is a map \( \tau : \tilde{X}_E \to x_\lambda \mathcal{H} \) extending \( \psi \). The composite \( \tau \phi \) restricts to the identity on \( \tilde{S}_E \subseteq \tilde{X}_E \).

On the other hand, restriction from \( \tilde{X}_E \) to \( \tilde{S}_E \) defines a homomorphism

\[
\text{End}_{\mathcal{H}}(\tilde{X}_E) \longrightarrow \text{End}_{\mathcal{H}}(\tilde{S}_E)
\]

since \( (\tilde{S}_E)_K \) is a unique summand of the (completely reducible) \( \mathcal{H} \otimes \mathcal{K} \)-module \( \tilde{X}_E \otimes \mathcal{K} \). (Observe \( \tilde{S}_E = \tilde{X}_E \cap (\tilde{S}_E)_K \), since the \( \mathcal{K} \)-torsion module \( (\tilde{X}_E \cap (\tilde{S}_E)_K) / \tilde{S}_E \) must be zero in the \( \mathcal{K} \)-torsion free module \( \tilde{X}_E / \tilde{S}_E \).) Thus, \( \tau \phi \) is a unit in the local endomorphism ring \( \text{End}_{\mathcal{H}}(\tilde{X}_E) \), so \( \tilde{X}_E \) is a direct summand of \( x_\lambda \mathcal{H} \), and hence of \( \tilde{T} \).

Next consider the case in which \( \tilde{S}_E \) is a summand of a dual left cell module \( \tilde{S}_\omega \) (this always happens for some \( \omega \)), but \( \omega \) does not contain the longest element of any parabolic subgroup. In this case, \( \tilde{X}_\omega \) is one of the summands of \( \tilde{X} \) by construction. The argument above may be repeated with \( \tilde{X}_\omega \), playing the role of \( x_\lambda \mathcal{H} \). In the same way, \( \tilde{X}_E \) is a direct summand of \( \tilde{X}_\omega \), and thus of \( \tilde{X} \).

In both cases, we conclude that \( \tilde{X}_E \) is a direct summand of \( \tilde{T} \oplus \tilde{X} = \tilde{T}^+ \). \( \square \)

**Theorem 6.4.** The \( \mathcal{K} \)-algebra \( \tilde{\mathcal{T}}^+ \) is quasi-hereditary, with standard modules \( \tilde{\Delta}(E) = \text{Hom}_{\mathcal{H}}(\tilde{S}_E, \tilde{T}), E \in \text{Irr}(W) \), and partial order \( <_f \).

**Proof.** We have already seen that this algebra is standardly stratified with strict stratifying system \( \{ \tilde{\Delta}(\omega) \}_{\omega \in \Omega} \). Clearly, \( \tilde{\Delta}(\omega) \) is a direct sum of various \( \tilde{\Delta}(E) \)'s, and every \( \tilde{\Delta}(E) \) arises as such a summand.
Put $\tilde{P}(E) = (X_E) = \text{Hom}_{\tilde{R}}(X_E, \tilde{T}^+), E \in \text{Irr}(W)$. Then $\tilde{P}(E)$ is a direct summand of $\tilde{A}^+ = \text{End}_{\tilde{R}}(\tilde{T}^+)$, viewed as a left module over itself. Thus, $\tilde{P}(E)$ is projective as an $\tilde{A}^+$-module, and $\tilde{P}(E)^\circ := \text{Hom}_{\tilde{A}^+}(\tilde{P}(E), \tilde{T}^+)$ is naturally isomorphic to $X_E$. In particular, the contravariant functor $(-)^\circ$ gives an isomorphism

$$\text{End}_{\tilde{A}^+}(\tilde{P}(E)) \cong (\text{End}_{\tilde{R}}(X_E))^\circ$$

Consequently, $\tilde{P}(E)$ also has a local endomorphism ring with radical quotient $k$, as does $\text{End}_{\tilde{A}^+}(\tilde{P}(E) \otimes_{\partial} k)$. It follows that $\tilde{P}(E)$ is an indecomposable projective $\tilde{A}^+$-module with a semisimple head. (The arguments in this paragraph are largely standard, many taken from [6].)

As previously noted in the proof of Theorem 5.6 we have $\text{Ext}_{\tilde{R}}^1(\bar{S}_E, \tilde{T}^+ = 0$ for all dual left cell module $\bar{S}_E$. Consequently, a similar vanishing holds with $\tilde{S}_E$ replaced by any module $\tilde{S}_E', E' \in \text{Irr}(W)$. It follows that the restriction map

$$\tilde{P}(E) = \text{Hom}_{\tilde{R}}(X_E, \tilde{T}^+) \longrightarrow \text{Hom}_{\tilde{R}}(\tilde{S}_E, \tilde{T}^+) = \tilde{\Delta}(E)$$

is surjective. Hence, $\tilde{\Delta}(E)$ has an irreducible head. Also, repeating the argument for filtered submodules of $X_E$, we find that the kernel of the above map has a filtration with sections $\tilde{\Delta}(E'), E' \in \text{Irr}(W)$ (rather than $X_E$ itself), satisfying $f(E') > f(E)$.

Next, we claim that $\tilde{\Delta}(E)^\circ := \text{Hom}_{\tilde{A}^+}(\tilde{\Delta}(E), \tilde{T}^+)$ is naturally isomorphic to $\tilde{S}_E$. More precisely, we claim that the natural map $\tilde{S}_E \xrightarrow{\text{ev}} (\tilde{S}_E)^\circ$ is an isomorphism. We showed above that the sequence

$$0 \longrightarrow (\tilde{X}_E/\tilde{S}_E)^\circ \longrightarrow (\tilde{X}_E)^\circ \longrightarrow (\tilde{S}_E)^\circ \longrightarrow 0$$

is exact. Applying $(-)^\circ$ once more, we get an injection

$$0 \longrightarrow (\tilde{S}_E)^\circ \longrightarrow (\tilde{X}_E)^\circ$$

with $\tilde{X}_E \xrightarrow{\text{ev}} (\tilde{X}_E)^\circ$ an isomorphism. This gives inclusions

$$\tilde{S}_E \cong \text{ev}(\tilde{S}_E) \subseteq (\tilde{S}_E)^\circ \subseteq (\tilde{X}_E)^\circ \cong \tilde{X}_E.$$ 

If $(-) \otimes_{\partial} K$ is applied, the first inclusion becomes an isomorphism. This gives

$$(\tilde{S}_E)^\circ \subseteq (\tilde{X}_E)^\circ \cap (\tilde{S}_E)_K = \tilde{S}_E$$

identifying $\tilde{X}_E$ with $(\tilde{X}_E)^\circ$ and $\tilde{S}_E$ with its image in $(\tilde{X}_E)^\circ$. Consequently, ev$(\tilde{S}_E) = (\tilde{S}_E)^\circ$, proving the claim.

Finally, we suppose $E \neq E' \in \text{Irr}(W)$ and $\text{Hom}_{\tilde{A}^+}(\tilde{P}(E'), \tilde{\Delta}(E)) \neq 0$. Using the identifications $\tilde{P}(E') = (X_{E'})^\circ$, $\tilde{\Delta}(E) = (\tilde{S}_E)^\circ$, $\tilde{P}(E')^\circ \cong X_{E'}$, and $\tilde{\Delta}(E)^\circ \cong \tilde{S}_E$, we have

$$0 \neq \text{Hom}_{\tilde{A}^+}(\tilde{P}(E'), \tilde{\Delta}(E)) \cong \text{Hom}_{\tilde{R}}(\tilde{S}_E, X_{E'}) \subseteq \text{Hom}_{\tilde{R}_K}(\tilde{S}_E \otimes_{\partial} K, X_{E'} \otimes_{\partial} K).$$
This implies \( f(E') < f(E) \). It follows that now from [1] or [14, Thm. 4.15] that \( \tilde{A}^+ \) is quasihereditary over \( \mathcal{O} \).

\[ \square \]

**Corollary 6.5.** The category of left modules over the base-changed algebra

\[ \tilde{A}^+_\mathcal{R} := \tilde{A}^+ \otimes_\mathfrak{R} \mathcal{R} \]

is equivalent to the \( \mathcal{R} \)-category \( \mathcal{O} \) of modules, as defined in [14] for the RDAHA associated to \( W \) over \( \mathfrak{R} \).

**Proof.** Continuing the proof of the theorem above, the projective indecomposable \( \tilde{A}^+ \)-modules are the various \( \tilde{P}(E) = (\tilde{X}_E)^\circ \). Consequently, \( \tilde{T}^+ = (\tilde{A}^+)^\circ \) is the direct sum of the modules, \( \tilde{X}_E \), each with nonzero multiplicities. The modules \( \tilde{X}_E \) remain indecomposable, as observed in the proof of the indecomposability of the modules \( \tilde{X}_E \) above. By construction, \( \text{Ext}^1_{\tilde{R}}(\tilde{S}_E, \tilde{X}_E) = 0 \) for all \( E, E' \in \text{Irr}(W) \). Thus, there is a similar vanishing for \( \tilde{S}_{E,\mathcal{R}} \) and \( \tilde{X}_{E,\mathcal{R}} \), and—in the reverse order—for their \( \mathcal{R} \)-linear duals. Observe that \((\tilde{S}_{E,\mathcal{R}})^* \cong \tilde{S}(E') \otimes_\mathfrak{R} \mathcal{R} \) is \( \text{KZ}(\Delta(E')) \), taking \( \Delta(E') = \Delta_\mathcal{O}(E') \) to be the standard module for the category \( \mathcal{O} \) over \( \mathfrak{R} \) as discussed in [14] together with KZ for this category.

Put

\[ Y = \bigoplus_E (\tilde{X}_{E,\mathcal{R}})^* \]

and set \( Y(\tilde{S}_{E,\mathcal{R}}) = (\tilde{X}_{E,\mathcal{R}})^* \). This notation imitates that of [14, Prop. 4.45]. The first part of this cited proposition is missing a minimality assumption on the rank of \( Y(M) \), in the terminology there (see fn. 3 there). However, this is satisfied for \( M = (\tilde{S}_{E,\mathcal{R}})^* \) and \( Y(M) = (\tilde{X}_{E,\mathcal{R}})^* \) because \( (\tilde{X}_{E,\mathcal{R}})^* \) is indecomposable.

Several other corrections, in addition to the minimality requirement, should be made to [14, Prop. 4.45]:

- \( A' \) should be redefined as \( \text{End}_B(Y)^\text{op} \);
- \( P' \) should be redefined as \( \text{Hom}_B(Y, B)^\text{op} \).

In addition \( B \) in [14, 4.2.1] should be redefined as \( \text{End}_A(P)^\text{op} \). The "op"s here and above insure action on the left, and consistency with [8, Thm. 5.15, Thm. 5.15]. The definition of \( P' \) is given to be consistent with the basis covering property \( \text{End}_{A'}(P'^{\text{op}}) \cong B \), as in [8, Thm. 5.15]. (We do not need this below.)

With these changes, [14, Thm. 5.3, Prop. 4.45, Cor. 4.46] guarantees that \( A'-\text{mod} \) is equivalent to \( \mathcal{O} \), where \( A' = \text{End}_{\tilde{R}}(Y) \). (All we really need for this are the 0- and 1-faithfulness of the \( \mathcal{O} \) version of the KZ functor.) However, \( \text{End}_{\tilde{R}}(Y) \cong \text{End}_{\tilde{A}}(Y^*)^{\text{op}} \), and \( Y^* \) is the direct sum \( \bigoplus_E \tilde{X}_{E,\mathcal{R}} \). Hence,

\[ Y^{*\circ} \cong \bigoplus_E (\tilde{X}_{E,\mathcal{R}})^\circ \cong \bigoplus_E \tilde{P}(E) \otimes_\mathfrak{R} \mathcal{R} \]

Recall that \((\tilde{X}_{E,\mathcal{R}})^\circ \cong \tilde{X}_{E,\mathcal{R}} \), so that the analogous property holds for \( Y^* \). Thus, \( \text{End}_{\tilde{R}}(Y^*)^{\text{op}} \cong \text{End}_{\tilde{A}}(Y^{*\circ}) \). Since the module \( Y^{*\circ} \) as displayed above is clearly a
projective generator for $\tilde{A}_F^+$, there is a Morita equivalence over $R$ of $\tilde{A}_F^+$ with $A'$. Hence, $\tilde{A}_F^+\text{mod}$ is equivalent to $O$, as $R$-categories. □

7. APPENDIX: A REVISED CONJECTURE

Given the present statement of Conjecture 1.1, the most natural way to formulate Theorem 5.5 would have been to use $\leq_{LR}^{op}$ instead of $\leq_f$. There are several remarks to make:

(1) The formulation with $\leq_{LR}$ implies formally the present formulation, given that $\leq_f$ is a refinement of $\leq_{LR}^{op}$.

(2) However, we do not know if the formulation with $\leq_{LR}^{op}$ is true.

(3) Both formulations formally imply the existence of an underlying standardly stratified algebra with strata indexed by the two-sided cells.

The original form of Conjecture 1.1, namely, [6, Conj. 2.5.2], arose in a context focusing on (3) above. The order $\leq_{LR}^{op}$ was used in the statement simply because it was the most obvious way of getting a pre-order on $\Omega$ whose associated equivalence classes were naturally indexed by the set of two-sided cells. We had no idea there were more sophisticated possibilities, like $\leq_f$, that could achieve the same thing. In view of the experience of this paper, we present the following revision of [6, Conj. 2.5.2], eliminating the dependence on $\leq_{LR}^{op}$.

The notation is the same as in Conjecture 1.1, except that $\mathbb{Z}[t^2, t^{-2}]$ is used for $\mathcal{D}$. In particular, the right $H$-module $\mathcal{X}$ has an finite filtration with sections of the form $S_\omega$, $\omega \in \Omega$ (the set of left cells in $W$). Define $T^+ := T \oplus \mathcal{X}$, and let $A^+ := \text{End}_H(T_R^+)$, etc. be as stated for a commutative ring $R$.

Conjecture 7.1. The module $\mathcal{X}$ can be chosen so that there is a pre-order $\leq$ on $\Omega$ such that:

(a) elements $\omega, \omega' \in \Omega$ are equivalent with respect to $\leq$ if and only if $\omega, \omega'$ are contained in the same two-sided cell;
(b) the indexed set $\{\Delta^+(\omega)\}_{\omega \in \Omega}$ is a strict stratifying system relative to the quasi-poset $(\Omega, \leq)$.

Using $\mathcal{D}' = \mathbb{Z}[t, t^{-1}]$ in Conjecture 7.1 gives a modification of Conjecture [7]. Other modifications are possible. For example, Conjecture [7] with either version of $\mathcal{D}$, remains quite interesting if $R$ above is always required to be a field. Nevertheless, the “integral setting,” allowing $R$ to be a commutative ring is quite useful. It has been somewhat vetted in [6], which checked the original conjecture in all rank 2 cases, even allowing “unequal parameters” and twisted types, including type $^2F_4$. These cases are still too simplistic to distinguish $\leq_f$ from $\leq_{LR}^{op}$ as regards to the assertion (b) above. However, in view of Theorem 6.4, such a distinction might well be made in the future, using examples arising from the categories $O$ for RDAHAs.

Finally, we mention that the original conjecture [6, Conj. 2.5.2] was checked in type $A$ for all ranks using $\leq_{LR}^{op}$; see [7].
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