Temporal enhancement of super-horizon curvature perturbations from decays of two curvatons and its cosmological consequences

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Abstract

If more than one curvaton dominate the Universe at different epochs from each other, curvature perturbations can be temporarily enhanced to a value much larger than the observed one $10^{-5}$. The traces of the enhancement may be left as higher order correlation functions, that is, as non-Gaussianity, the stochastic gravitational waves that are sourced by scalar-scalar mode couplings, as well as the primordial black holes that are formed by the gravitational collapse of the enhanced curvature perturbations. We first confirm that such a temporal enhancement indeed occurs by solving the linearized perturbation equations both numerically and analytically. We then derive an analytic expression of the full-order curvature perturbation which does not rely on the frequently used sudden decay approximation and is exact on super-horizon scales. By using this analytic formula, we provide expressions of the non-linearity parameters $f_{\text{NL}}$, $\tau_{\text{NL}}$, and $g_{\text{NL}}$. If both two curvatons contribute to the final curvature perturbations, the strong non-Gaussianity appears in the trispectrum rather than in the bispectrum. We also find a unique consistency relation between $\tau_{\text{NL}}$ and $g_{\text{NL}}$ without $f_{\text{NL}}$. By using the second-order perturbation theory, we numerically show that the spectrum of the induced gravitational waves has a plateau corresponding to duration of the enhancement and such gravitational waves can be probed by ultimate-DECIGO and space-based atomic interferometers. We finally calculate the abundance of the primordial black holes and put a constraint on the amplitude of the enhanced curvature perturbations.
1 Introduction

Cosmic observations \[1\] are now strongly supporting the idea of primordial inflation \[2–4\]. According to the inflationary scenario, the primordial perturbations, which are seeds for the inhomogeneous structure of our Universe, are created from the field fluctuations that are generated quantum mechanically during inflation \[6–9\]. In the simplest inflationary scenario, the fluctuations of the inflaton, where energy drives accelerated expansion, turn into the curvature perturbations when its wavelength becomes larger than the Hubble length during inflation. Once generated, the curvature perturbation remains constant and this is the primordial perturbation that can be compared with CMB observations.

This simple scenario, however, may not be the one that actually happened in our Universe. For example, in the curvaton scenario \[10–12\], a curvaton, a scalar field other than the inflaton, plays a role of creating the primordial perturbation. In this scenario, the curvaton perturbations turn into the curvature perturbation after inflation when the curvaton decays into the radiation. There are many other models in which the primordial perturbations are created after inflation such as inhomogeneous end of inflation \[13–17\], modulated reheating scenario \[18, 19\] etc.. A common feature among all these models is that the curvature perturbation, once generated from zero remains constant outside the horizon.

Although these nonstandard scenarios are already contrived, the real Universe may have evolved even more complicated manner, say, with multiple curvatons dominating at different epochs. Such a scenario has been studied in \[20\] and \[21\], the former focusing on the power spectrum and the latter on the bispectrum. In \[21\], although not emphasized, it was implicitly shown that if the two curvatons dominate the Universe at different epochs from each other, the curvature fluctuations may evolve in a dramatically different way than the standard case. That is, they can grow to an amplitude much larger than the observed value, \(10^{-5}\), when the first curvaton dominates and decays, and then they are moderated to the observed amplitude when the second curvaton dominates and decays. Thus in this scenario the curvature perturbation can be temporarily enhanced.

At first glance, this temporal enhancement seems to have little effect on observables since it must occur, if at all, much before the big-bang nucleosynthesis, not to mention the observation time. But this is not the case. We can provide at least three possible interesting consequences from this effect.

The first possibility is that, as pointed out in \[21\], large possibly detectable non-Gaussian perturbations can be generated. If we denote by \(\zeta_{\text{max}}\) the maximum amplitude of the curvature perturbation when it is enhanced, the so-called \(f_{\text{NL}}\) parameter is given by \(f_{\text{NL}} \simeq \zeta_{\text{max}}/10^{-5} \gg 1\), under the assumption that the curvature perturbation is sourced only by the first decaying curvaton fluctuations. For example, if \(\zeta_{\text{max}} = 10^{-3}\), then we get \(f_{\text{NL}} \simeq 100\).

The second possibility is generation of stochastic gravitational waves (GWs) whose peak frequency can fall into a range of the GW detectors. It is well known that at second order in perturbation the scalar-scalar coupling can source the GWs, most efficiently when the
scalar mode reenters the horizon \[22\ 23\]. Since the amplitude of GWs is proportional to the square of the curvature perturbations, we expect that large amplitude of GWs can be generated at the horizon scales when the curvature perturbation is being enhanced.

The third possibility is a formation of primordial black holes (PBHs) \[25\ 27\]. If \(\zeta_{\text{max}}\) is very large, the perturbation mode which reenters the horizon when the curvature perturbation is being enhanced may undergo a gravitational collapse to form a black hole \#1. Since the abundance of PBHs on various masses are tightly constrained from cosmic observations (see \[28\] for the latest results), we can limit a range of the parameters of the two curvaton model by using such constraints.

The present paper aims to take up this temporal enhancement of the curvature perturbation in two curvaton model, to provide a detailed analysis of the generation and evolution of fluctuations and to discuss three observational implications mentioned above.

2 Basic picture

In this paper, we are interested in a situation where both curvatons dominate the Universe at different epochs. Before going to the detailed analysis of the scenario, let us first explain four non-trivial assumptions needed for the designing scenario to work, along with the basic history of the universe under consideration.

The first assumption we will take in this paper is that, other than the inflaton, there are two light free scalar fields (we call them \(\sigma_1\)-field and \(\sigma_2\)-field, respectively.) having VEVs smaller than the Planck scale in the early universe. By light, we mean that masses of both curvatons are much smaller than the expansion rate of the Universe when it is reheated by the decay of the inflaton, \(m_1, m_2 \ll H_{\text{reh}}\), with \(m_1\) and \(m_2\) being masses of the two curvatons and \(H_{\text{reh}}\) the Hubble parameter at the time of reheating. This condition means that both curvatons are almost massless during inflation. It is well known that such a light scalar field acquires classical fluctuation of order \(H_{\text{inf}}/2\pi\) on super-horizon scales. The requirement of VEVs smaller than the Planck scale is to avoid the second inflation caused by a curvaton. Although this requirement is not essential, we take it for simplicity.

After inflation ends, the inflaton oscillates around the minimum of the potential and finally decays into radiation to complete reheating. Then the radiation dominated universe starts. At this stage, both curvatons are still subdominant. Therefore, those fluctuations have not yet contributed to the curvature perturbation.

The next assumption is that both curvatons start to oscillate during this radiation dominated epoch. If the secondly decaying curvaton starts to oscillate after the first decaying curvaton decays, the temporal enhancement of the curvature perturbation, which is the main focus of this paper, does not occur. Therefore, we do not consider such a case.

\#1Some papers, such as \[29\ 35\], also considered the generation of large amplitude of the curvature perturbations on particular scales and the PBH formation. Contrary to the two curvaton model considered in this paper, the curvature perturbations in those models do not show the temporal enhancement on super-horizon scales.
Since the oscillating free scalar field can be treated as non-relativistic particles, we can assume that the universe consists of the radiation coming from inflaton, two non-relativistic particles that are not interacting. During this epoch, the fraction of energy densities of both curvatons grows in proportion to the scale factor while the ratio between energy densities of the two curvatons stays constant. We call this era epoch A.

Without a loss of generality, we can assume that $\sigma_1$-field decays first and $\sigma_2$-field decays later, $\Gamma_1 \gg \Gamma_2$. Then, the third assumption is that the energy density of $\sigma_1$-field in epoch A is much larger than that of $\sigma_2$-field and there is a period when $\sigma_1$-field dominates the universe. We call such a period epoch B. Since the $\sigma_1$-field is dominating the universe, it is this epoch when the $\sigma_1$-field perturbation is mostly converted to the curvature perturbation. By exactly the same reasoning as the single curvaton case, the curvature perturbation at this epoch is given by $\delta_1 \equiv \delta \rho_1 / \rho_1$ (precise definition of the curvature perturbation and rigorous calculations will be given later.). Since the $\sigma_2$-field is subdominant during this epoch, keeping its energy fraction constant, the $\sigma_2$-field fluctuations have not been converted to the curvature perturbations yet. When the Hubble parameter becomes $\Gamma_1$, the $\sigma_1$-field decays into the radiation and the universe is again dominated by the radiation. We call a period dominated by such radiation epoch C.

Our last assumption is that the $\sigma_2$-field finally dominates the universe before it decays. In other words, $\Gamma_2$ should be small enough to allow the $\sigma_2$-field to dominate the universe. We call a period dominated by the $\sigma_2$-field epoch D. During this epoch, since the universe evolves like a matter dominated universe, the radiation generated from the decay of the $\sigma_1$-field is diluted. Because of this dilution, the curvature perturbation coming from the radiation perturbation is reduced by a factor $\Omega_r \ll 1$, where $\Omega_r$ is a fraction of the radiation energy density to the total one. Meanwhile, the $\sigma_2$-field contributes to the curvature perturbation by $\delta_2$. When the Hubble parameter becomes equal to $\Gamma_2$, the $\sigma_2$-field decays into radiation and the Universe is again dominated by the radiation. Since there are no isocurvature perturbations any more, the curvature perturbation remains constant and this should be regarded as the final perturbation that can be compared with the cosmological observations such as CMB. Thus, the primordial perturbation at the linear order is estimated as

$$\text{final perturbation} \simeq \Omega_r \delta_1 + \delta_2,$$  \hspace{1cm}(1)$$

where $\Omega_r$ must be evaluated at the time when $\sigma_2$-field decays. From observations, we know that this is about $10^{-5}$.

From those arguments, we find that the curvature perturbation evolves from zero to $\delta_1$ at epoch B and then decays to $\Omega_r \delta_1 + \delta_2$. Just for an illustration, let us choose $\delta_1 = 10^{-2}, \Omega_r = 10^{-4}$ and $\delta_2 = 10^{-5}$. In this case, curvature perturbation is temporarily amplified to $10^{-2}$ and then decays to the observed value $10^{-5}$.

It is worthwhile to mention here that the existence of more than a single curvaton is essential to make the curvature perturbations decay in time. One may consider a simpler situation where the inflaton generates large amplitude of the curvature perturbations by the standard mechanism and a curvaton that acquires little quantum fluctuations during
inflation dominates the Universe at late time. In this case, attenuation of the curvature perturbation due to the curvaton dominance does not happen. Instead, the large amplitude of the curvature perturbations of the inflaton origin is taken over by the curvaton because the epoch when the curvaton starts its oscillations is modulated. We need at least two fields other than the inflaton to have a sensible model in which the temporal enhancement happens.

In the following sections, we will give more quantitative discussions of this scenario and some interesting consequences.

3 Decays of two curvatons and generation of perturbations

3.1 Evolution of the background quantities

In this subsection, we see how the background quantities evolve in time. The periods we focus on are from epochs A to D whose definitions are given in sec.2.

The background spacetime is the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe whose metric is given by

\[ ds^2 = a^2(\eta) \left(-d\eta^2 + \delta_{ij}dx^i dx^j\right) = -dt^2 + a^2(t)\delta_{ij}dx^i dx^j. \]

Here \( \eta \) is the conformal time. As we mentioned in the last subsection, we can treat the dynamics of the curvatons as collections of non-relativistic particles. Therefore, the energy-momentum tensor for each curvaton can be written as

\[ T_{\mu\nu}^Z = \rho_Z u_\mu^Z u_\nu^Z, \]

where \( Z \) runs 1 and 2 and \( u_\mu^Z \) is the four-velocity of \( Z \)-field with a normalization condition \( g_{\mu\nu}u_\mu^Z u_\nu^Z = -1 \).

While the total energy-momentum tensor obeys the conservation law, each energy-momentum tensor is no longer conserved because curvaton fields decay into radiation \[36,38;\]

\[ \nabla_\mu T_{(A)\mu\nu}^\mu = Q_{(A)\nu}, \]

where \( A \) now represents radiation, 1 or 2. The r.h.s. represents the transfer of the energy and momentum of the fluid.

At the background order, \( Q_{(a)\nu} \) for each fluid is given by

\[ Q_{(1)0} = a\Gamma_1 \rho_1, \]
\[ Q_{(2)0} = a\Gamma_2 \rho_2, \]
\[ Q_{(r)0} = -a\Gamma_1 \rho_1 - a\Gamma_2 \rho_2. \]

Because of the isotropy of the background spacetime, all the spatial components are zero.
Therefore, the evolution equations for the background quantities are given by

\[
\begin{align*}
\rho'_1 + 3\mathcal{H}\rho_1 &= -a\Gamma_1\rho_1, \\
\rho'_2 + 3\mathcal{H}\rho_2 &= -a\Gamma_2\rho_2, \\
\rho'_r + 4\mathcal{H}\rho_r &= a\Gamma_1\rho_1 + a\Gamma_2\rho_2, \\
\mathcal{H}^2 &= \frac{8\pi G}{3}(\rho_1 + \rho_2 + \rho_r)a^2,
\end{align*}
\]

where a prime and an overdot denote differentiation with respect to \(\eta\) and \(t\), respectively, with \(\mathcal{H} \equiv \frac{a'}{a}\) and \(H \equiv \frac{\dot{a}}{a}\). From the first two equations, we have

\[
\rho_1(t) = \rho_{1,*}\left(\frac{a_*}{a(t)}\right)^3 e^{-\Gamma_1(t-t_*)}, \\
\rho_2(t) = \rho_{2,*}\left(\frac{a_*}{a(t)}\right)^3 e^{-\Gamma_2(t-t_*)},
\]

where \(t_*\) is an arbitrary time in the epoch A. Substituting these solutions into the third equation, we get

\[
\rho_r(t) = \rho_{r,*}\left(\frac{a_*}{a(t)}\right)^4 + \Gamma_1 \int_{t_*}^t \left(\frac{a(t')}{a(t)}\right)^4 \rho_1(t')dt' + \Gamma_2 \int_{t_*}^t \left(\frac{a(t')}{a(t)}\right)^4 \rho_2(t')dt'.
\]

The designing situation in this paper is that both two curvatures dominate the universe at different epochs. We give in Fig.1 a typical evolution of \(\Omega_1\) and \(\Omega_2\). We see that their evolution can be clearly divided into four epochs explained in the last section. During epoch A, both of them grows as \(\propto a\). During epoch B, \(\Omega_1 \simeq 1\) and \(\Omega_2 \ll 1\) stays constant. During epoch C, \(\Omega_2\) starts to grow again like \(\propto a\). During epoch D, \(\Omega_2 \simeq 1\) and this epoch terminates by the decay of the \(\sigma_2\)-field.

### 3.2 Linear order perturbation equations

Before discussing perturbation behaviors and their consequences, we give basic evolution equations for the scalar perturbations.

We write the perturbed metric of the scalar type in the conformal Newtonian gauge:

\[
ds^2 = a^2(\eta)\left\{- (1 + 2\psi)d\eta^2 + (1 - 2\phi)\delta_{ij}dx^i dx^j \right\}.
\]

For the matter perturbations, we introduce density contrast and the velocity perturbation by

\[
\delta_A = \frac{\delta \rho_A}{\rho_A}, \quad u_{Ai} = av^i_A.
\]

Note that \(u_0 = -a(1 + \psi)\) is completely determined by the metric perturbation by using the normalization condition.
Figure 1: The evolution of $\Omega_1$ and $\Omega_2$. We chose $\Omega_{1,ini}/\Omega_{2,ini} = 10$ and $\Gamma_2/\Gamma_1 = 10^{-10}$.

At the linear order in the perturbation, $Q_{(A)\nu}$ for each fluid is given by

\[ \delta Q_{(1)0} = a\Gamma_1 \delta \rho_1 + a\Gamma_1 \rho_1 \psi, \]  
\[ \delta Q_{(2)0} = a\Gamma_1 \delta \rho_2 + a\Gamma_2 \rho_2 \psi, \]  
\[ \delta Q_{(r)0} = -a\Gamma_1 \delta \rho_1 - a\Gamma_1 \rho_1 \psi - a\Gamma_2 \delta \rho_2 - a\Gamma_2 \rho_2 \psi, \]
\[ \delta Q_{(1)i} = -a\Gamma_1 \rho_1 v_1^i, \]  
\[ \delta Q_{(2)i} = -a\Gamma_2 \rho_2 v_2^i, \]  
\[ \delta Q_{(r)i} = a\Gamma_1 \rho_1 v_1^i + a\Gamma_2 \rho_2 v_2^i. \]

The second terms in (17) and (18) represent an effect due to the modulation of time. $Q_{(r)\mu}$ for the radiation is obtained from the conservation law for the total energy momentum. Using these equations, we find that the continuity equation and the Euler equation for each fluid are given by

\[ \delta'_1 - 3\phi' + kv_1 = -a\Gamma_1 \psi, \]  
\[ \delta'_2 - 3\phi' + kv_2 = -a\Gamma_2 \psi, \]  
\[ \delta'_r - 4\phi' + \frac{4}{3}k v_r = a\Gamma_1 \frac{\rho_1}{\rho_r} (\delta_1 - \delta_r + \psi) + a\Gamma_2 \frac{\rho_2}{\rho_r} (\delta_2 - \delta_r + \psi), \]

and

\[ v'_1 + \mathcal{H} v_1 - k \psi = 0, \]  
\[ v'_2 + \mathcal{H} v_2 - k \psi = 0, \]  
\[ v'_r - \frac{k}{4} \delta_r - k \psi = a\Gamma_1 \frac{\rho_1}{\rho_r} \left( \frac{3}{4} v_1 - v_r \right) + a\Gamma_2 \frac{\rho_2}{\rho_r} \left( \frac{3}{4} v_2 - v_r \right). \]
respectively, where \( k \) is the comoving wavenumber. From the perturbed Einstein equations, we can derive the following evolution equations:

\[
k^2 \phi + 3 \mathcal{H} \phi' + 3 \mathcal{H}^2 \psi = -4 \pi G (\rho_1 \delta_1 + \rho_2 \delta_2 + \rho_r \delta_r) a^2, \tag{29}
\]

\[
k (\mathcal{H} \psi + \phi') = 4 \pi G a^2 \left( \rho_1 v_1 + \rho_2 v_2 + \frac{4}{3} \rho_r v_r \right), \tag{30}
\]

\[
\phi - \psi = 0. \tag{31}
\]

### 3.3 Perturbation evolution on super-horizon scales

Since understanding the evolution of the curvature perturbation on super-horizon scales is important for our purposes, let us next consider the perturbation behavior on super-horizon scales, postponing the analysis on sub-horizon scales of perturbations. The evolution equations on super-horizon scales can be obtained by setting \( k = 0 \) in the perturbation equations we gave in the last subsection.

In Fig. 2, we show a typical evolution of \( \phi \) which is obtained by solving numerically the perturbation equations with \( k = 0 \). We started the calculation at a time when the expansion rate \( H \) is much larger than \( \Gamma_1 \), with the initial conditions in the epoch A given by the analytic approximation:

\[
\delta_1(\eta) = \delta_{1,\text{ini}} - \frac{3}{8} (\delta_{1,\text{ini}} \Omega_1(\eta) + \delta_{2,\text{ini}} \Omega_2(\eta)) + \cdots, \tag{32}
\]

\[
\delta_2(\eta) = \delta_{2,\text{ini}} - \frac{3}{8} (\delta_{1,\text{ini}} \Omega_1(\eta) + \delta_{2,\text{ini}} \Omega_2(\eta)) + \cdots, \tag{33}
\]

\[
\delta_r(\eta) = -\frac{1}{2} (\delta_{1,\text{ini}} \Omega_1(\eta) + \delta_{2,\text{ini}} \Omega_2(\eta)) + \cdots, \tag{34}
\]

\[
\phi(\eta) = -\frac{1}{8} (\delta_{1,\text{ini}} \Omega_1(\eta) + \delta_{2,\text{ini}} \Omega_2(\eta)) + \cdots, \tag{35}
\]

where \( \cdots \) represent terms that are suppressed at early time with higher powers of \( \Omega_1 \) and/or \( \Omega_2 \). Since we assume that inflaton fluctuation contributes little to the curvature perturbation, we have imposed the condition that both \( \delta_r \) and \( \phi \) vanish at the outset. The parameters used in Fig. 2 are such that \( \delta_{1,\text{ini}} = 10^{-2} \) and \( \delta_{2,\text{ini}} = 10^{-4} \). The parameters for the background are the same as the ones used in Fig. 1.

From Fig. 2, we can clearly see that \( \phi \) is enhanced during epoch B. The order of \( \phi \) at this time is roughly \( \delta_{1,\text{ini}} \) (we will provide exact analytic expression later.). This enhancement still persists during epoch C until the \( \sigma_2 \)-field dominates the universe. As the \( \sigma_2 \)-field dominates, \( \phi \) decays to \( \Omega_r \delta_{1,\text{ini}} + \delta_{2,\text{ini}} \), where \( \Omega_r \ll 1 \) is \( \Omega_r \) evaluated at the time when the \( \sigma_2 \)-field decays. Therefore, the enhanced value of the curvature perturbation is determined by \( \delta_{1,\text{ini}} \) and the final value is determined by either \( \Omega_r \delta_{1,\text{ini}} \) or \( \delta_{2,\text{ini}} \), whichever is greater.

We can analytically derive time evolution of curvature perturbations in the super-horizon regime solving perturbation equations for gauge-invariant variable, \( \zeta_A \), which is
Figure 2: The evolution of $\phi$. Since $\phi$ is negative, we plot $-\phi$. We chose $\Omega_{1,\text{ini}}/\Omega_{2,\text{ini}} = 10, \delta_{1,\text{ini}} = 10^{-2}, \delta_{2,\text{ini}} = 10^{-4}, \Gamma_2/\Gamma_1 = 10^{-10}$.

defined by $\zeta_A \equiv -\phi - H \frac{\delta \rho}{\rho_A}$ and represents curvature perturbation on the uniform energy density surface of the component $A$. In the presence of interactions their evolution equations in the super-horizon ($k \to 0$) limit read

\begin{align}
\dot{\zeta}_1 &= -\frac{1}{3} \Gamma_1 \zeta_1, \\
\dot{\zeta}_2 &= -\frac{1}{3} \Gamma_2 \zeta_2, \\
\dot{\zeta}_r &= \frac{\Gamma_1 \rho_1}{4 \rho_r} \left[ -(4 + 3 \gamma_1) \zeta_r + (3 + 3 \gamma_1) \zeta_1 \right] + \frac{\Gamma_2 \rho_2}{4 \rho_r} \left[ -(4 + 3 \gamma_2) \zeta_r + (3 + 3 \gamma_2) \zeta_2 \right],
\end{align}

where $\gamma_A \equiv \Gamma_A/(3H)$. Using the background solutions for $\rho_1(t), \rho_2(t),$ and $\rho_r(t)$, namely (12), (13), and (14), these equations are solved as

\begin{align}
\zeta_1(t) &= \zeta_{1*} e^{-\frac{1}{3} \Gamma_1 (t-t*)}, \\
\zeta_2(t) &= \zeta_{2*} e^{-\frac{1}{3} \Gamma_2 (t-t*)}, \\
\zeta_r(t) &= \int_{t_*}^{t} \left[ \frac{3 \Gamma_1 \rho_1(t')} {4 \rho_r(t')} (1 + \gamma_1) \zeta_{1*} e^{-\frac{1}{3} \Gamma_1 (t'-t*')} + \frac{3 \Gamma_2 \rho_2(t')} {4 \rho_r(t')} (1 + \gamma_2) \zeta_{2*} e^{-\frac{1}{3} \Gamma_2 (t'-t*)} \right] \\
&\quad \times \exp \left\{ - \int_{t_*}^{t} \left[ \frac{\Gamma_1 \rho_1(t'')} {4 \rho_r(t'')} (4 + 3 \gamma_1) + \frac{\Gamma_2 \rho_2(t'')} {4 \rho_r(t'')} (4 + 3 \gamma_2) \right] dt'' \right\} dt'.
\end{align}

In terms of these solutions the curvature perturbation in the uniform total density surface, $\zeta$, is expressed as

$$
\zeta \equiv -\phi - H \frac{\delta \rho}{\rho} = \frac{3 \rho_1 \zeta_1 + 3 \rho_2 \zeta_2 + 4 \rho_r \zeta_r + 3 \gamma_1 \rho_1 (\zeta_1 - \zeta_r) + 3 \gamma_2 \rho_2 (\zeta_2 - \zeta_r)} {3 \rho_1 + 3 \rho_2 + 4 \rho_r}.
$$

The key quantity to understand the time evolution of $\zeta$ is $\zeta_r$. In the regimes A and B, only the terms involving $\rho_1$ is important in the both integrands in (41) because by assumption
\( \rho_2 \) is much smaller than \( \rho_1 \) then. Furthermore, in the regime B when \( \rho_1 \gg \rho_r \) the last exponential factor in the right-hand-side of (41) takes an appreciable value of order of unity only for a short time interval

\[
\Delta t'' = t - t' \approx \frac{4\rho_r}{\Gamma_1\rho_1(4 + 3\gamma_1)},
\]

which also limits the range of \( t' \) integral. As a result we find

\[
\zeta_r(t) \approx \frac{3 + 3\gamma_1}{4 + 3\gamma_1} \zeta_1(t).
\]

We can estimate \( \zeta_r \) at the end of the regime B by the contribution at the epoch when the ratio \( \rho_1/\rho_r \) is the largest. This is just before \( \sigma_1 \)'s decay when \( \gamma_1 \) was still negligible with \( \zeta_1(t) \approx \zeta_{1*} \). We therefore find \( \zeta_r \approx 3\zeta_{1*}/4 \) at the end of the regime B.

The behavior of \( \zeta_r \) in the regime D can also be understood similarly replacing the suffix 1 by 2. When \( \rho_2 \) dominates over \( \rho_r \), we find \( \zeta_r(t) \approx 3\zeta_{1*}/4 + 3\zeta_{2*}/4 \), so that the total curvature perturbation reads

\[
\zeta(t) \approx \frac{3\rho_2(t)\zeta_{2*} + 3\rho_r(t)(\zeta_{1*} + \zeta_{2*})}{3\rho_2(t) + 4\rho_r(t)},
\]

which clearly shows that as the fraction \( \rho_2(t)/\rho_r(t) \) increases the large curvature perturbation due to \( \zeta_{1*} \) is regulated to a smaller value to reach the final value

\[
\zeta \approx \left(1 + \frac{\rho_r}{\rho_2}\right) \zeta_{2*} + \frac{\rho_r}{\rho_2} \zeta_{1*} \approx \zeta_2 + \left(\frac{\Gamma_2}{\Gamma_1}\right)^{2/3} \left(\frac{\Omega_{1*}}{\Omega_{2*}}\right)^{4/3} \zeta_{1*}.
\]

Here \( \frac{\rho_r}{\rho_2} \equiv \left(\frac{\Gamma_2}{\Gamma_1}\right)^{2/3} \left(\frac{\Omega_{1*}}{\Omega_{2*}}\right)^{4/3} \) denotes the ratio of radiation energy from \( \rho_1 \) to \( \rho_2 \) upon decay of \( \sigma_2 \). Note that \( \zeta_{1*} = \delta_{1*}/3 \) and \( \zeta_{2*} = \delta_{2*}/3 \) hold because \( \phi \) is negligible and \( \Gamma_1, \Gamma_2 \ll \dot{H} \) at the outset.

### 3.4 Nonlinear super-horizon perturbations

Eq. (46) clarifies how the final curvature perturbation after its temporal enhancement is related to the initial amplitudes of the two curvatons. Although an essential point is completely manifested by Eq. (46), the approximations we have made to derive it neglects a \( \mathcal{O}(1) \) constant factor in front of the second term in Eq. (46).

In this subsection, by using \( \delta N \) formalism [39][43], we derive the exact analytic expressions for the enhanced and the final amplitudes of the curvature perturbation not only to linear order in perturbation but also to any higher order. According to this formalism, the curvature perturbation on the uniform total energy density hypersurface at a point \( \vec{x} \) is given by the perturbation of the e-folding number:

\[
\zeta(\eta, \vec{x}) = N(\sigma_1(\eta, \vec{x}), \sigma_2(\eta, \vec{x})) - \text{spatial average},
\]

\[
= \int_{\eta_*}^\eta d\eta' \mathcal{H}(\eta', \sigma_1(\eta', \vec{x}), \sigma_2(\eta', \vec{x})) - \text{spatial average},
\]

(47)
where \( \eta_* \) is an arbitrary earlier time than \( \eta \) and the hypersurface at \( \eta_* \) should be the flat slicing. The evolutions of \( \sigma_1, \sigma_2 \) and \( \mathcal{H} \) are determined by solving the background equations [5]-[11].

In the appendix, we show that the number of e-folds from \( \eta_* \) at the epoch \( A \) to \( \eta_f \) well after the \( \sigma_2 \)-field decay is given by

\[
N(\sigma_1(\eta_f, \vec{x}), \sigma_2(\eta_f, \vec{x})) = \frac{1}{4} \log \left\{ (1 + \epsilon_r) \left( \frac{\mathcal{H}_*}{a_* \Gamma_1} \right)^{2/3} \Omega_{1,*}^{4/3}(\eta_*, \vec{x}) + \left( \frac{\mathcal{H}_*}{a_* \Gamma_2} \right)^{2/3} \Omega_{2,*}^{4/3}(\eta_*, \vec{x}) \right\} \\
+ \frac{1}{4} \log \left\{ \left( \frac{9}{4} \right)^{1/3} a_* \epsilon_r \right\} + \frac{1}{4} \log \frac{\rho_*}{\rho_f},
\]

(48)

where \( \epsilon_r \approx 1.183 \) is a numerical value and \( \rho_*/\rho_f \) is the total energy density at \( \eta_*/\eta_f \). \( \epsilon_r \) is another numerical value. Eq. (48) is obtained without using the sudden decay approximation which is frequently adopted in the literature. In the limiting case where both curvatons dominate the universe at different epochs separated far enough, (48) is exact.

Therefore, by combining (48) with (47), we can derive the exact expression of \( \zeta \) which is correct to any order in the perturbation. We can relate \( \Omega_{1,*}(\eta_*, \vec{x}) \) and \( \Omega_{2,*}(\eta_*, \vec{x}) \) with the density contrast for each field as

\[
\Omega_{1,*}(\eta_*, \vec{x}) = \Omega_{1,*} [1 + \delta_1(\eta_*, \vec{x})], \quad \Omega_{2,*}(\eta_*, \vec{x}) = \Omega_{2,*} [1 + \delta_2(\eta_*, \vec{x})].
\]

(49)

If we choose \( \eta_* \) to be deep in the epoch \( A \), each density contrast can be approximated with the initial density contrast (see Eqs. (32) and (33)):

\[
\Omega_{1,*}(\eta_*, \vec{x}) = \Omega_{1,*} [1 + \delta_{1,ini}(\vec{x})], \quad \Omega_{2,*}(\eta_*, \vec{x}) = \Omega_{2,*} [1 + \delta_{2,ini}(\vec{x})].
\]

(50)

The initial density contrast is determined by quantum fluctuations generated when the mode crossed the Hubble length during inflation.

Using these equations, we find that the full non-linear expression of \( \zeta \) is given by

\[
\zeta(\eta_f, \vec{x}) = \frac{1}{4} \log \left\{ (1 + \epsilon_r) \Omega_1^{4/3}(\eta_*)[1 + \delta_{1,ini}(\vec{x})]^{4/3} + \left( \frac{\Gamma_1}{\Gamma_2} \right)^{2/3} \Omega_2^{4/3}(\eta_*)[1 + \delta_{2,ini}(\vec{x})]^{4/3} \right\} \\
- \frac{1}{4} \log \left\{ (1 + \epsilon_r) \Omega_1^{4/3}(\eta_*) + \left( \frac{\Gamma_1}{\Gamma_2} \right)^{2/3} \Omega_2^{4/3}(\eta_*) \right\}.
\]

(51)

At linear order in the density contrasts, this equation reduces to

\[
\zeta(\eta_f, \vec{x}) = \frac{1}{3} \left( 1 + \epsilon_r \right) \left( \frac{\Gamma_2}{\Gamma_1} \right)^{2/3} \Omega_2^{4/3}(\vec{x}) + \frac{1}{3} \Omega_1^{4/3}(\vec{x}) \delta_{1,ini}(\vec{x}) + \frac{1}{3} \epsilon_r \right) \left( \frac{\Gamma_2}{\Gamma_1} \right)^{2/3} \Omega_2^{4/3}(\vec{x}) + \frac{1}{3} \Omega_1^{4/3}(\vec{x}) \delta_{1,ini}(\vec{x}) + \frac{1}{3} \epsilon_r \right)
\]

≈ \frac{1}{3} \left( 1 + \epsilon_r \right) \left( \frac{\Gamma_2}{\Gamma_1} \right)^{2/3} \Omega_2^{4/3}(\vec{x}) + \frac{1}{3} \Omega_1^{4/3}(\vec{x}) \delta_{1,ini}(\vec{x}) + \frac{1}{3} \epsilon_r \right).
\]

(52)
To get the second equation, we have used an inequality \((\frac{\Gamma_2}{\Gamma_1})^{2/3} \Omega_{1,*}^{4/3} \ll \Omega_{2,*}^{4/3}\) which is equivalent to the condition that the \(\sigma_2\)-field dominates the Universe eventually. Therefore, it is useful to introduce a parameter \(s\) defined by

\[
s \equiv (1 + \epsilon_{\Gamma}) \left( \frac{\Gamma_2}{\Gamma_1} \right)^{2/3} \left( \frac{\Omega_{1,*}}{\Omega_{2,*}} \right)^{4/3},
\]

which roughly represents the fraction of the radiation from \(\sigma_1\)-field decay at the time of \(\sigma_2\)-field decay. With \(s\), \(\zeta\) becomes

\[
\zeta(\eta_f, \vec{x}) \approx \frac{1}{3} s \delta_{1,\text{ini}}(\vec{x}) + \frac{1}{3} \delta_{2,\text{ini}}(\vec{x}),
\]

which is a more rigorous expression of (54). We see that the transfer coefficient of \(\delta_1\) is proportional to \(s\). Apart from the numerical factors, Eq. (54) is nothing more than the rough estimation (1) and also agrees with the result of [21].

We can also derive the maximum magnitude of the enhanced \(\zeta\) during the epochs B and C. Since the \(\sigma_2\)-field is subdominant during those epochs, \(\zeta_{\text{max}}\) is completely sourced by the \(\sigma_1\)-field perturbation. This means that \(\zeta_{\text{max}}\) is equal to the final curvature perturbation in the single curvaton model in which the curvaton dominates the universe before its decay. Therefore, \(\zeta_{\text{max}}\) can be obtained by the second term in (54) with \(\delta_2\) replaced by \(\delta_1\):

\[
\zeta_{\text{max}}(\vec{x}) = \frac{1}{3} \delta_{1,\text{ini}}(\vec{x}).
\]

We can convert \(\zeta(\eta_f)\) and \(\zeta_{\text{max}}\) into the corresponding \(\phi(\eta_f)\) and \(\phi_{\text{max}}\). In the linear perturbation theory, it is well known that \(\phi\) is related to \(\zeta\) by [5]

\[
\phi = -\frac{2}{3} \zeta, \quad \text{(for the radiation dominated universe),}
\]

\[
\phi = -\frac{3}{5} \zeta, \quad \text{(for the matter dominated universe).}
\]

Therefore, \(\phi(\eta_f)\) becomes

\[
\phi(\eta_f) = -\frac{2}{9} s \delta_{1,\text{ini}} - \frac{2}{9} \delta_{2,\text{ini}}.
\]

On the other hand, \(\phi_{\text{max}}\) at the epoch B/C is given by

\[
\phi_{\text{max}} = -\frac{1}{5} \delta_{1,\text{ini}}, \quad \text{(in the epoch B),}
\]

\[
\phi_{\text{max}} = -\frac{2}{9} \delta_{1,\text{ini}}, \quad \text{(in the epoch C).}
\]
4 Implications

We have shown that the curvature perturbation can be temporarily enhanced in two curvaton models. At first glance, it seems that such an enhancement has nothing to do with observations since what we observe is the curvature perturbation at or after the time of last scattering epoch when it has already settled down to the observed value $10^{-5}$. This naive guess is true for the power spectrum. However, the trace of the enhancement enters the game when we consider the black hole formation, higher order correlation functions of the curvature perturbation (non-Gaussianity), and the gravitational waves generated by scalar-scalar mode couplings.

4.1 Non-Gaussianity

Eq. (51) is the fully non-linear expression of the curvature perturbation. By using this equation, we can calculate correlation functions of any order. In this paper, we calculate the three and four-point functions (bi- and tri-spectra) which are now becoming important observables to extract information of the early universe.

The curvature perturbation given by Eq. (51) is the so-called local type for which the curvature perturbation depends on the source fields at the same point $\zeta(\vec{x}) = N_a \delta \sigma_a(\vec{x}) + \frac{1}{2} N_{ab} \delta \sigma_a(\vec{x}) \delta \sigma_b(\vec{x}) + \frac{1}{6} N_{abc} \delta \sigma_a(\vec{x}) \delta \sigma_b(\vec{x}) \delta \sigma_c(\vec{x}) + \cdots$, (61)

where $\delta \sigma_a(\vec{x})$ is the Gaussian field fluctuation at a point $\vec{x}$ at some initial time and $N_a = \partial N/\partial \sigma_a$.

The power spectrum, $P_\zeta$, bispectrum, $B_\zeta$, and trispectrum, $T_\zeta$, of the curvature perturbation are defined by

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \rangle = (2\pi)^3 P_\zeta(k_1) \delta(\vec{k}_1 + \vec{k}_2),$$

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle = (2\pi)^3 B_\zeta(k_1, k_2, k_3) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3),$$

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \zeta_{\vec{k}_4} \rangle = (2\pi)^3 T_\zeta(k_1, k_2, k_3, k_4) \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4),$$

and respectively. For the case of local type curvature perturbation, $B_\zeta$ and $T_\zeta$ can be written as

$$B_\zeta(k_1, k_2, k_3) = \frac{6}{5} f_{\text{NL}} (P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1)), \quad (65)$$

$$T_\zeta(k_1, k_2, k_3, k_4) = \tau_{\text{NL}} (P_\zeta(k_1) P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + 11 \text{ perms.}) + \frac{54}{25} g_{\text{NL}} (P_\zeta(k_2) P_\zeta(k_3) P_\zeta(k_4) + 3 \text{ perms.}). \quad (66)$$
with \( k_{13} = \left| \vec{k}_1 + \vec{k}_3 \right| \). Here the constant parameters \( f_{NL}, \tau_{NL} \) and \( g_{NL} \) are the so-called non-linearity parameters and given by [43]

\[
\frac{6}{5} f_{NL} = \frac{N_a N_b N^{ab}}{(N_c N_c)^2},
\]

(67)

\[
\tau_{NL} = \frac{N_a N_b N^{ac} N_c}{(N_d N_d)^3},
\]

(68)

and

\[
\frac{54}{25} g_{NL} = \frac{N^{abc} N^a N^b N_c}{(N_d N_d)^3}.
\]

(69)

Let us define a new parameter \( r \) by

\[
r \equiv s \sigma_1 / \sigma_2 \approx \delta_{2,ini} / (s \delta_{1,ini}),
\]

(70)

which represents the contribution of \( \delta_{2,ini} \) to \( \zeta \) compared to that of \( \delta_{1,ini} \). If \( r = 0 \), then \( \zeta \) is solely sourced by the fluctuations in \( \sigma_1 \). If \( r \gg 1 \), then \( \zeta \) is mostly sourced by the fluctuations in \( \sigma_2 \). With this parameter, the non-linearity parameters for two curvaton case are given by

\[
\frac{6}{5} f_{NL} = \frac{N_a N_b N^{ab}}{(N_c N_c)^2} - 25 \frac{15 r^2 + 80 r^2 s - 25}{12 (r^2 + 1)^2 s} + O(1),
\]

(71)

\[
\tau_{NL} = \frac{N_a N_b N^{ac} N_c}{(N_d N_d)^3} - 25 \frac{9 r^4 + 112 r^2 + 64 r^2 s^2 - 80 r^2 s + 25}{4 (r^2 + 1)^3 s^2} + O \left( \frac{1}{s} \right),
\]

(72)

\[
g_{NL} = \frac{N^{abc} N^a N^b N_c}{(N_d N_d)^3} - 25 \frac{225 r^4 + 3300 r^2 s^2 - 1500 r^2 s + 125}{108 (r^2 + 1)^3 s^2} + O \left( \frac{1}{s} \right).
\]

(73)

Since \( s \) enters \( f_{NL} \) in the denominator, \( s \ll 1 \), which is satisfied in the situation we are interested in, is a necessary condition to have large \( f_{NL} \). This condition can be qualitatively understood by expanding (54) to second order in \( \delta \sigma_1 \):

\[
\zeta = \frac{1}{3} s \left( 2 \frac{\delta \sigma_1}{\sigma_1} + \left( \frac{\delta \sigma_1}{\sigma_1} \right)^2 \right) = \zeta_g + \frac{3}{4 s} \zeta_g^2,
\]

(74)

where we have set \( \delta_2 = 0 \) and \( \zeta_g \equiv \frac{2}{3} s \frac{\delta \sigma_1}{\sigma_1} \) is the Gaussian part of \( \zeta \). We see that the second order coefficient which is, apart from the numerical factor, nothing more than \( f_{NL} \) contains an enhancement factor \( 1/s \) [42]. This mechanism to get large \( f_{NL} \) is exactly the same as the single curvaton case in which \( f_{NL} \) is inversely proportional to a fraction of the curvaton energy density at the time when it decays into the radiation [46].

\#2 Strictly speaking, at second order perturbation, \( \delta_2^2 \) terms should also appear in (74). However, such terms only yield \( f_{NL} \) of \( O(1) \) and can be safely neglected when \( s \ll 1 \).
Since $f_{NL}$ is bounded to be $|f_{NL}| \lesssim 100$ from the observations \[1\], $s$ cannot be smaller than $10^{-2}$ if $\delta_2 = 0$. Correspondingly, the observationally allowed maximum curvature perturbation when it is enhanced is at most $10^{-5}/s \simeq 10^{-3}$. However, things change when $\delta_2$ is also allowed to take non-zero amplitude. From (71), we find that if fluctuations in $\sigma_2$ contribute more to the final curvature perturbation than the $\sigma_1$-field fluctuations, i.e. $r \gtrsim 1$, then $f_{NL}$ is suppressed by a factor $r^{-4}$ compared to a case with $r = 0$. Therefore, $s$ smaller than $10^{-2}$ can satisfy the bound $|f_{NL}| \lesssim 100$ if $r$ is suitably chosen. In particular, if $r \gtrsim s^{-1/4} \gg 1$, then $f_{NL}$ becomes as small as $\mathcal{O}(1)$. At this level of $f_{NL}$, non-linear evolutionary effects become important and it will not be easy to extract primordial $f_{NL}$ from observations. Interestingly, in this case, we find from (72) and (73) that $\tau_{NL}$, $g_{NL} \gtrsim \mathcal{O}(s^{-1/2}) \gg 1$. Therefore, strong non-Gaussianity appears in the trispectrum but not in the bispectrum\[3\]. In such a case, the trispectrum would be useful to search for non-Gaussianity.

From (72) and (73), we can also derive a unique relation between $\tau_{NL}$ and $g_{NL}$ as

\[
\frac{\tau_{NL}}{g_{NL}} = \frac{27}{5} + \mathcal{O}\left(\frac{1}{f_{NL}^2}\right),
\]

which will be useful to observationally discriminate the two curvaton model from the other models that generate large non-Gaussianity. Consistency relations between $f_{NL}$ and $g_{NL}$, $f_{NL}$ and $\tau_{NL}$ or among the three parameters have been obtained for various models. As far as we know, (75) is the first example that gives the unique relation between $\tau_{NL}$ and $g_{NL}$ without using $f_{NL}$ (for consistency relations for other models, see \[47\]).

### 4.2 Stochastic gravitational waves

It is well known that second order tensor perturbations are induced by the first order scalar perturbations by the mode-mode couplings. This means, $\Omega_{GW}$, the energy density of GWs per unit logarithmic interval of frequency, is proportional to quartic of the scalar perturbations, motivating us to consider the generation of GWs in the two-curvaton model.

A basic picture of the production of GWs in the two curvaton model is that the scalar perturbations are temporarily enhanced equally on all the superhorizon scales and large magnitude of $\Omega_{GW}$ with a frequency equal to the Hubble parameter is produced at each time while the enhanced modes are re-entering the horizon. Therefore, we expect that the resulting $\Omega_{GW}$ will have a broad peak of an interval of frequencies whose corresponding modes re-enter the horizon during the curvature perturbation is being enhanced. To get this kind of result, we have to solve the equation of motion for the second order

\[\text{\#3}\text{It is generally true that the trispectrum becomes relatively stronger than the bispectrum if more than one field contribute to the curvature perturbations. The local-type single field model yields a relation } \tau_{NL} = \frac{56}{3} f_{NL}^2. \text{ In } \[23\], it was shown that an inequality } \tau_{NL} \geq \frac{56}{3} f_{NL}^2 \text{ holds for any local-type multi-field model. In } \[34\], a scenario that both inflaton and curvaton contribute to the curvature perturbations was considered. It was shown that } \tau_{NL} \text{ is enhanced by a factor } 1 + r^2 \text{ compared to } \frac{56}{3} f_{NL}^2, \text{ where } r \text{ represents the contribution of the inflaton fluctuations to the curvature perturbations.}\]
tensor perturbations as well as the perturbation equations for the scalar modes and the background equations. Although all the numerical calculations to arrive at the final result are straightforward, it takes a long computation time due to multi-integrations. To avoid this, we stick ourselves to cases where $\sigma_1$-field immediately decays soon after it dominates the total energy density. In other words, we assume that the epoch B terminates in a moment and the effects of the enhancement of the curvature perturbation shows up only at the epoch C which is radiation dominated. In such cases, instead of numerically solving the perturbation equations for the scalar modes, we can assume that the scalar modes which are to re-enter the horizon during the epoch C are already enhanced since the epoch A and can use the analytic transfer function for the radiation dominated universe to evolve the scalar modes. By these assumptions, we do not need to numerically integrate the background equations and the linearized equations for the scalar modes, which drastically diminishes the task of the numerical computations. What is then left is to solve the evolution equations for the second order tensor perturbations sourced by the first order scalar perturbations.

In what follows, we first briefly review the general formalism to calculate $\Omega_{GW}$ induced by the scalar-scalar couplings \[22–24\] and then provide our results in two curvaton case. We basically follow the notations of \[50\]. The metric that is relevant to our purpose is given by the metric (15) plus the second order tensor perturbations:

$$ds^2 = a^2(\eta)[ - (1 + 2\psi)d\eta^2 + (1 - 2\phi) \delta_{ij} + h_{ij}]dx^i dx^j, \quad (76)$$

where $h_{ij}$ are the second order tensor perturbations which satisfy the transverse-traceless conditions:

$$h_{ij,j} = \delta^{ij} h_{ij} = 0. \quad (77)$$

In the following, we set $\psi = \phi$. Since the gravitational waves are transverse waves, we can Fourier-decompose $h_{ij}$ as

$$h_{ij}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} \left( e_{ij}(\vec{k}) h_{\vec{k}}(\eta) + \tilde{e}_{ij}(\vec{k}) \tilde{h}_{\vec{k}}(\eta) \right), \quad (78)$$

where $e_{ij}(\vec{k})$ and $\tilde{e}_{ij}(\vec{k})$ are the polarization tensors orthogonal to $\vec{k}$. Introducing the unit vectors $e_i(\vec{k})$ and $\tilde{e}_i(\vec{k})$ orthogonal to $\vec{k}$, they are given by

$$e_{ij}(\vec{k}) = \frac{1}{\sqrt{2}} \left( e_i(\vec{k}) e_j(\vec{k}) - \tilde{e}_i(\vec{k}) \tilde{e}_j(\vec{k}) \right), \quad \tilde{e}_{ij}(\vec{k}) = \frac{1}{\sqrt{2}} \left( e_i(\vec{k}) \tilde{e}_j(\vec{k}) + \tilde{e}_i(\vec{k}) e_j(\vec{k}) \right). \quad (79)$$

Then the evolution equation for $h_{\vec{k}}$ is given by

$$h''_{\vec{k}} + 2\mathcal{H}h'_{\vec{k}} + k^2 h_{\vec{k}} = S_{\vec{k}}, \quad (80)$$

where the source term is given by

$$S_{\vec{k}} = 2e^{ij}(\vec{k}) \int \frac{d^3q}{(2\pi)^3} q_i q_m \left( 3\phi_{\vec{q}} \phi'_{\vec{q} - \vec{k}} + \frac{2}{\mathcal{H}} \phi_{\vec{q}} \phi'_{\vec{q} - \vec{k}} + \frac{1}{\mathcal{H}^2} \phi'_{\vec{q}} \phi'_{\vec{q} - \vec{k}} \right) \quad (81)$$
In the radiation dominated universe, the solution of Eq. (80) is given by

\[
\tilde{u_k}(\eta) = \int_{-\infty}^{\eta} d\eta' \ g_k(\eta, \eta') a(\eta') S_k(\eta'),
\]  

(82)

where \( u_k = ah_k \) and \( g_k(\eta, \eta') \) is the retarded Green’s function:

\[
g_k(\eta, \eta') = \frac{1}{k} \sin(k(\eta - \eta')).
\]  

(83)

Then the two-point function of \( h_k \) becomes

\[
\langle h_{k_1} h_{k_2} \rangle = \frac{1}{a^2(\eta)} \int_{-\infty}^{\eta} d\eta_1 \int_{-\infty}^{\eta} d\eta_2 \ g_{k_1}(\eta, \eta_1) g_{k_2}(\eta, \eta_2) a(\eta_1) a(\eta_2) \langle S_{k_1}(\eta_1) S_{k_2}(\eta_2) \rangle.
\]  

(84)

We need to evaluate \( \langle S_{k_1}(\eta_1) S_{k_2}(\eta_2) \rangle \). To this end, we need time evolution of \( \phi_k \). Its evolution in the radiation dominated universe is given by

\[
\phi_k(\eta) = D_k(\eta) \phi_0(\tilde{k}), \quad D_k(\eta) = \frac{9}{(k\eta)^2} \left( \frac{\sqrt{3}}{k\eta} \sin \left( \frac{k\eta}{\sqrt{3}} \right) - \cos \left( \frac{k\eta}{\sqrt{3}} \right) \right).
\]  

(85)

Using this, the two-point function of the source can be written as

\[
\langle S_{k_1}(\eta_1) S_{k_2}(\eta_2) \rangle = 8 e^{ij}(\tilde{k}_1)e_{mn}(\tilde{k}_2)(2\pi)^3 \delta(\tilde{k}_1 + \tilde{k}_2) \int \frac{d^3q}{(2\pi)^3} P_\phi(|\tilde{k}_1 - \tilde{q}|) P_\phi(q) \\
\times q_i q_j q_m q_n f(|\tilde{k}_1 - \tilde{q}|, q, \eta_1) f(|\tilde{k}_1 - \tilde{q}|, q, \eta_2),
\]  

(86)

where

\[
\langle \phi_0(\tilde{k}_1) \phi_0(\tilde{k}_2) \rangle = (2\pi)^3 P_\phi(k_1) \delta(\tilde{k}_1 + \tilde{k}_2),
\]  

(87)

\[
f(k_1, k_2, \eta) \equiv 3 D_{k_1}(\eta) D_{k_2}(\eta) + \eta \left( 2 D_{k_1}(\eta) + \eta D'_{k_1}(\eta) \right) D'_{k_2}(\eta).
\]  

(88)

We define the power spectrum of the tensor perturbation by

\[
\langle h_{k_1}(\eta) h_{k_2}(\eta) \rangle = (2\pi)^3 P_h(\eta, k) \delta(\tilde{k}_1 + \tilde{k}_2)
\]  

(89)

When the source term can be neglected, the energy density of the tensor perturbation can be written in terms of their power spectrum well inside the horizon,

\[
\Omega_{GW}(\eta, k) = \frac{1}{6} \left( \frac{k}{H} \right)^2 P_h(\eta, k),
\]  

(90)

where \( P_h \equiv k^3 P_h/(2\pi)^2 \) is the dimensionless power spectrum.
Given the general formalism to calculate $\Omega_{GW}$, we are ready to apply it to the two curvaton model. Due to the reasoning we mentioned at the beginning of this subsection, we use top-hat type function with a width $\Delta$ for $P_{\phi}(k)$:

$$P_{\phi}(k) = \begin{cases} A^2/2\Delta & \text{for } |\ln(k/k_\text{p})| < \Delta, \\ 0 & \text{otherwise}, \end{cases}$$ \hspace{1cm} (91)

where $P_{\phi} \equiv k^3P_{\phi}/(2\pi)^2$, $k_\text{p}$ is the peak wavenumber and $A^2$ corresponds to the total power of the spectrum. Then $\Omega_{GW}$ can be written as \[50\]

$$\Omega_{GW}(\eta, k) = \frac{k}{6} \int_{q_{\text{min}}}^{e^\Delta} dq \int_{-\delta_{\text{max}}}^{\delta_{\text{max}}} d\delta \mathcal{F}^2(k, \eta, q, \delta),$$ \hspace{1cm} (92)

where

$$q_{\text{min}} = \max\{k/2, e^{-\Delta}\}, \; \delta_{\text{max}} = \min\left\{1, \frac{2}{k} \sinh \Delta\right\}.$$ \hspace{1cm} (93)

The function $\mathcal{F}(k, \eta, q, \delta)$ is defined by

$$\mathcal{F}(k, \eta, q, \delta) = \frac{k}{2\Delta} \frac{(4q^2 - k^2)(1 - \delta^2)}{4q^2 - k^2\delta^2} I \left( k, q + \frac{k\delta}{2}, q - \frac{k\delta}{2}, \eta \right),$$ \hspace{1cm} (94)

where

$$I(k, k_1, k_2, \eta) = k \int_0^\eta d\eta_1 a(\eta_1)g_k(\eta, \eta_1)f(k_1, k_2, \eta_1).$$ \hspace{1cm} (95)

In Fig. 3 we show numerically calculated $A^{-4}\Omega_{GW}(k, \eta)/(s\Omega_{\text{rad}})$ for various values of $\Delta$ \[\#4\]. A suppression factor $s$ is due to the decay of $\Omega_{GW}$ during the epoch D where the

\[\#4\] The results presented here look somewhat different from the ones given in [50] which also obtained $\Omega_{GW}$ for the top-hat type power spectrum, although the qualitative features are the same. This is due to there being a couple of mistakes in the manipulation in [50].
universe expands like matter dominated regime and the other one \( \Omega_{\text{rad}} \simeq 8 \times 10^{-5} \) comes from the matter dominated universe after the matter-radiation equality. We see that for \( \Delta \lesssim 0.5 \), \( \Omega_{\text{GW}} \) has a strong peak at \( k = k_p \) and decays as \( k^{-3} \) for \( k \ll k_p \). In all the cases, \( \Omega_{\text{GW}} \) has a sharp drop at \( k/k_p \simeq e^\Delta \). This is because the momentum conservation prohibits a generation of the tensor mode whose wavenumber is greater than \( 2k_p e^\Delta \). For \( \Delta \gtrsim 0.5 \), \( \Omega_{\text{GW}} \) has a plateau between a range \( e^{-\Delta} \lesssim k/k_p \lesssim e^\Delta \) and decays as \( k^{-3} \) for \( k \lesssim e^{-\Delta} k_p \). The magnitude of the plateau is well fitted with \( \simeq 2.0 \times \Delta^{-2} \). Therefore, the magnitude of the plateau at present is given by

\[
\Omega_{\text{GW,plateau}} \simeq 2 \frac{A^4}{\Delta^2} s \Omega_{\text{rad}} \simeq 2 \times 10^{-17} \left( \frac{A}{10^{-2}} \right)^4 \left( \frac{s}{10^{-4}} \right) \left( \frac{\Delta}{3} \right)^{-2}.
\]

The current frequency corresponding to the upper limit of the plateau is

\[
f_{\text{max}} \simeq \sqrt{\frac{\Gamma_1 H_{\text{eq}}}{2 \pi z_{\text{eq}}}} s^{1/4} \simeq 10 \text{ Hz} \left( \frac{s}{10^{-4}} \right)^{1/4} \left( \frac{\Gamma_1}{100 \text{ GeV}} \right)^{1/2},
\]

where \( z_{\text{eq}} \simeq 3200 \) and \( H_{\text{eq}} \) are the redshift and the Hubble parameter at the time of the matter-radiation equality respectively. The lowest frequency is

\[
f_{\text{min}} = \left( \frac{\Omega_{2, *}}{\Omega_{1, *}} \right)^2 f_{\text{max}}.
\]

It is interesting to derive an upper bound on \( \Omega_{\text{GW,plateau}} \). From the expression of \( f_{\text{NL}} \) given by Eq. (71) and the WMAP normalization [1],

\[
\mathcal{P}_\phi(\eta_f) = (1 + r^2) s^2 \frac{A^2}{2 \Delta} \simeq 10^{-9},
\]

Figure 4: Left panel shows contours of \( \Omega_{\text{GW,plateau}} \) and the right one shows contours of \( f_{\text{max}} \) and \( f_{\text{min}} \). The other free parameters \( A \) and \( \Gamma_1 \) are fixed to be \( A^2 = 5 \times 10^{-4} \) and \( \Gamma_1 = 100 \text{ GeV} \).
Figure 5: This panel shows contours of $f_{NL}$ and $\tau_{NL}$. The other free parameters $A$ and $\Gamma_1$ are fixed to be $A^2 = 5 \times 10^{-4}$ and $\Gamma_1 = 100$ GeV.

we can write $s$ and $r$ in terms of $f_{NL}$ and $A^2/\Delta$. By using these relations, we find that $\Omega_{GW,\text{plateau}}$ is bounded from above as

$$\Omega_{GW,\text{plateau}} \simeq 5 \times 10^{-10} \left( \frac{A^2}{2\Delta} \right)^{4/3} f_{NL}^{1/3} < 10^{-13},$$

(100)

where in the last inequality, we used $\Delta \gtrsim 0.5$, $f_{NL} \lesssim 50$ and $A \lesssim 0.05$. As we will see in the next subsection, larger value than $A = 0.05$ over-produces black holes, which is excluded by the observations. Therefore, $\Omega_{GW,\text{plateau}} = 10^{-13}$ is the possible maximal amplitude that can be achieved in principle. This amplitude is much smaller than the one given in [50]. This is mainly because $\Omega_{GW,\text{plateau}}$ in the two curvaton model considered here is multiplied by the suppression factor $s$ (see Eq. (96)) due to the existence of matter dominance epoch (epoch D). On the other hand, [50] does not consider such an epoch and hence there is no additional suppression on $\Omega_{GW}$. As can be seen from Fig. 6, $\Omega_{GW,\text{plateau}} = 10^{-13}$ is slightly lower than the sensitivity of LISA [51] and DECIGO/BBO [52,53]. But it is much higher than the sensitivities achieved by ultimate-DECIGO and space-based AGIS [54].

As an example, we show contour plots of $f_{\text{max}}$, $f_{\text{min}}$ and $\Omega_{GW,\text{plateau}}$ in Fig. 4 and $f_{NL}$ and $\tau_{NL}$ in Fig. 5 as functions of $r$ defined by Eq. (70) and $\Omega_{2,*}/\Omega_{1,*}$. The other free parameters $A$ and $\Gamma_1$ are fixed to be $A^2 = 5 \times 10^{-4}$ and $\Gamma_1 = 100$ GeV. $\Delta$ is determined by a relation

$$\Delta = \log \left( \frac{\Omega_{1,*}}{\Omega_{2,*}} \right),$$

(101)

and $s$ is determined by the WMAP normalization (99).

We find that $\Omega_{GW,\text{plateau}}$ is $\mathcal{O}(10^{-14})$ in the frequency band $10^{-4}$ Hz $- 30$ Hz. We also find that in this case, $r \lesssim 1$ is ruled out by the observational bound on $f_{NL}$, $|f_{NL}| < 100$. This is because $s$ is typically $10^{-3}$ in this case and we have $f_{NL} = \mathcal{O}(10^3)$ for $r \lesssim 1$. As we
have seen in the previous subsection, $\tau_{NL}$ becomes very large as $O(10^{3}) - O(10^{4})$. Hence the strong non-Gaussian signal appears in the trispectrum rather than in the bispectrum.

In Fig. 6, we show plots of $\Omega_{GW}$ for three cases: $\Delta = 1.0$, 3.0, 5.0, assuming $\Gamma_1 = 100$ GeV, $A^2 = 5 \times 10^{-4}$ and $r = 3$ with expected sensitivity of the future GW detectors such as LISA, DECIGO/BBO and AGIS. As mentioned before, we see that maximum of $\Omega_{GW}$ is below the LISA and DECIGO/BBO sensitivities. On the other hand, ultimate-DECIGO and space-based AGIS will be able to probe GWs generated in two curvaton models if the model parameters are suitably chosen. If we lower $\Gamma_1$ as small as 0.1 meV, then $\Omega_{GW,\text{plateau}}$ does not change but $f_{\text{max}}$ becomes $O(10^{-6}$ Hz). Hence $\Omega_{GW,\text{plateau}}$ enters the frequency region of the pulsar timing. However, as is clear from Fig. 6, $\Omega_{GW,\text{plateau}}$ is far below the upper limit coming from the pulsar timing observations.

### 4.3 Primordial black holes

PBHs are formed if there exist density fluctuations of the order unity and when such modes re-enter the Hubble radius $[25,27]$. The mass of the PBHs is roughly equal to the horizon mass at the time of formation,

$$GM_{\text{PBH}} \simeq \frac{1}{H}. \quad (102)$$
To simplify the analysis and also to make it consistent with the study of the last subsection, we again assume that the epoch B terminates in a moment. Then, PBHs would be efficiently formed during the epoch C and the mass range of the resultant PBHs is estimated as

$$\frac{1}{\Gamma_1} \lesssim GM_{\text{PBH}} \lesssim \frac{1}{\Gamma_1} \left( \frac{\Omega_{1,*}}{\Omega_{2,*}} \right)^2.$$  \hfill (103)

Let us write the fraction of the energy density of PBHs of mass between \((M, M + dM)\) at the time of formation as

$$\frac{d\beta_{\text{PBH}}}{dM} dM.$$ \hfill (104)

Then the total fraction of the energy density of PBHs is given by the integral of the above quantity,

$$\beta_{\text{PBH}} = \int dM \frac{d\beta_{\text{PBH}}}{dM},$$ \hfill (105)

where the range of integration is given by Eq. (103). The purpose of this subsection is to provide \(\beta_{\text{PBH}}\) for the two curvaton model and to discuss its cosmological implications. Since most of the necessary formulae and their detailed derivations are written in \([50]\), we omit the intermediate calculations and provide only results.

Assuming that the gravitational potential \(\phi\) smoothed over the horizon size is Gaussian, \(d\beta_{\text{PBH}}/dM\) can be written as

$$\frac{d\beta_{\text{PBH}}}{dM} dM = \mathcal{P}_\phi(R_M^{-1}) \frac{\phi_c}{2\sqrt{2\pi} \sigma_{R_M}^2} \exp\left( -\frac{\phi_c^2}{2\sigma_{R_M}^2} \right) \frac{dM}{M},$$ \hfill (106)

where \(R_M\) is the comoving horizon length, \textit{i.e.} \(aR_M = GM\), with \(a\) being the scale factor at the time of horizon crossing, \(\phi_c\) is the threshold value of \(\phi\) for black hole formation and

$$\sigma_{R_M}^2 \equiv \int_0^{R_M^{-1}} \frac{dk}{k} \mathcal{P}_\phi(k),$$ \hfill (107)

is the variance of \(\phi\) smoothed over the horizon size. Although \(\phi_c\) depends on the initial configuration of the perturbations \([55, 56]\), we simply use \(\phi_c = 0.5\) \([57]\).

For the top-hat type of \(\mathcal{P}_\phi(k)\) which we have considered in this paper, \(\sigma_{R_M}\) becomes

$$\sigma_{R_M}^2 = \frac{A^2}{2\Delta} \log \left( \frac{\epsilon \Delta}{k_p R_M} \right).$$ \hfill (108)

Having these, we are ready to do the integration (105). The result is

$$\beta_{\text{PBH}} \approx \sqrt{2} \frac{\Delta \phi_c^2}{\pi} \frac{\sigma_{R_M}^2}{A^2} \exp \left( -\frac{\phi_c^2}{2A^2} \right).$$ \hfill (109)

Fig. 7 shows \(\Omega_{\text{PBH}} h^2\) as a function of \(A\). \(\Delta\) is chosen to be \(\Delta = 1.0\). It is clear that \(\beta_{\text{PBH}}\) is very sensitive to change of \(A\). Since various cosmological observations put different
constraints on the abundance of PBHs for different mass, using $\beta_{\text{PBH}}$ to constrain $A$ is not correct in a precise sense. However, because of the strong sensitivity of $\beta_{\text{PBH}}$ to $A$, even the change of many order of magnitude of $\beta_{\text{PBH}}$ corresponds to the change of a factor of a few in $A$. This fact validates the use of $\beta_{\text{PBH}}$ to constrain $A$ as a first approximation. According to [28], the constraint on $\beta_{\text{PBH}}$ varies from $10^{-30}$ to $10^{-10}$. This can be converted to a constraint on $A$ as $A \lesssim 0.05$.

Since $\sigma_{R,M}$ is a monotonically decreasing function of $M$, the mass of PBHs that dominantly contribute to (109) is the lower limit of Eq. (103), which is estimated as

$$M_{\text{PBH}} \simeq 8 \times 10^{12} \, g \left( \frac{\Gamma_1}{100 \, \text{GeV}} \right)^{-1}. \quad (110)$$

Combining this with Eq. (97) gives a relation between BH mass and the corresponding frequency of GWs,

$$M_{\text{PBH}} \simeq 1.5 \times 10^{15} \, g \left( \frac{f_{\text{max}}}{10 \, \text{Hz}} \right)^{-2} \left( \frac{s}{10^{-4}} \right)^{1/2}. \quad (111)$$

5 Summary

If more than one curvaton dominates the Universe at different epochs, the curvature perturbations can be temporarily enhanced to a value much larger than the observed one $10^{-5}$. In this paper, we studied in detail the evolution of the curvature perturbations in two curvaton models. By solving the linearized perturbation equations both numerically and analytically, we confirmed that the curvature perturbations are indeed enhanced during the period from the time when the first decaying curvaton dominates the Universe until the second decaying curvaton dominates the Universe. The amplitude of the enhanced
curvature perturbation is roughly equal to the density perturbation of the first decaying curvaton. We then provided an analytic expression of the full-order curvature perturbation which does not rely on the sudden decay approximation and is exact on super-horizon scales. At the linear order, we compared with the analytic results with the numerical ones and found they agree very well.

The temporal enhancement of the curvature perturbations leaves its traces as the strong non-Gaussian perturbations, stochastic gravitational waves generated by the scalar-scalar mode couplings and the primordial black holes. By using the analytic formula for the full-order curvature perturbations, we gave the expressions for the so-called non-linearity parameters $f_{\text{NL}}, \tau_{\text{NL}}$ and $g_{\text{NL}}$. If both two curvatons contribute to the final curvature perturbations, then the strongest non-Gaussian signal comes from $\tau_{\text{NL}}$ and $g_{\text{NL}}$ rather than from $f_{\text{NL}}$. We also gave a consistency relation between $\tau_{\text{NL}}$ and $g_{\text{NL}}$ without using $f_{\text{NL}}$. If non-Gaussianity is detected in the future, this relation would be useful to discriminate this model from the others that also generate large non-Gaussian perturbations.

We next studied the generation of GWs sourced by the enhanced curvature perturbations. The spectrum of $\Omega_{\text{GW}}$ has a plateau corresponding to the duration of the enhancement. Because of the existence of the period where the secondly decaying curvaton dominates the Universe and the Universe expands like the matter dominated universe, $\Omega_{\text{GW}}$ today is accompanied by a suppression factor which represents the fraction of the radiation energy density coming from the first decaying curvaton at the time when the secondly decaying curvaton decays. Due to this suppression factor, the possible maximal amplitude of $\Omega_{\text{GW}}$ at the plateau is at most $10^{-13}$, which is below the LISA and DECIGO/BBO sensitivities, but above the ultimate-DECIGO and space-based AGIS sensitivities. Actually, if the decay rate of the first decaying curvaton is around 100 GeV, then the frequency interval of the plateau can be $10^{-3}$ Hz $\sim$ 10 Hz. Such a case can be a target of the ultimate-DECIGO and AGIS.

We finally calculated the abundance of PBHs which are formed by the gravitational collapse of the enhanced curvature perturbations. We then provided an upper bound on the amplitude of the enhanced curvature perturbation by using the observational upper bounds on the abundance of PBHs.

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A Derivation of Eq. (48)

In this appendix, we derive Eq. (48). In the following, we frequently use epoch A, epoch B, epoch C and epoch D which are defined in Sec 2. For convenience, we use the cosmic
time \( t \) as a time variable. The basic equations we use are the background equations \( (8), (9) \) for the curvatons and \( (10) \) for the radiation. In the spirit of the \( \delta N \) formalism, the solutions of these equations, \( (12), (13), \) and \( (14) \) depend on the space coordinate \( \vec{x} \) and evolve independently of other points, although we do not show their dependence explicitly. Also, \( t \) should be interpreted as the local proper time at \( \vec{x} \) hereafter.

If we choose \( t \) such that \( \Gamma_2 t \gg 1 \), i.e., an epoch well after the \( \sigma_2 \)-field decay, then \( \rho_{\text{rad}} \) can be safely replaced by the total energy density \( \rho(t) \). Then from \( (14) \) we find

\[
a^4(t)\rho(t) = \int_0^t dt' \left( \Gamma_1 a(t') a_s^3 \rho_{1,*} e^{-\Gamma_1 t'} + \Gamma_2 a(t') a_s^3 \rho_{2,*} e^{-\Gamma_2 t'} \right). \tag{112}
\]

Also since the integrand is exponentially suppressed for \( t \gg 1/\Gamma_2 \), we can push the upper limit of integration to infinity. Then, the e-folding number from the initial time to the time when the total energy density becomes \( \rho_f \) is given by

\[
N = \frac{1}{4} \log \left( \Gamma_1 \Omega_{1,*} F_1 + \Gamma_2 \Omega_{2,*} F_2 \right) + \frac{1}{4} \log \left( \frac{\rho_s}{\rho_f} \right), \tag{113}
\]

where \( F_1 \) and \( F_2 \) are defined by

\[
F_1 \equiv \int_0^\infty dt \ a(t) e^{-\Gamma_1 t}, \tag{114}
\]

\[
F_2 \equiv \int_0^\infty dt \ a(t) e^{-\Gamma_2 t}. \tag{115}
\]

We see that the e-folding number depends on the field values of \( a \) and \( b \) through \( F_1 \) and \( F_2 \) as well as \( \Omega_{1,*} \) and \( \Omega_{2,*} \). Now the problem is reduced to deriving the analytic expression of \( F_1 \) and \( F_2 \).

Let us first evaluate \( F_1 \). To evaluate it, we need to know the evolution of the scale factor. Since the integrand of \( F_1 \) has an exponential cutoff for \( t \gg 1/\Gamma_1 \), the dominant contribution to the integral comes from a time interval where \( \sigma_1 \)-field dominates the universe (epoch B). During that epoch, we can neglect the \( \sigma_2 \)-field because its energy density is tiny compared to the total one. Then, the background equations in terms of \( \Omega_1 \) and \( H \) are given by

\[
\frac{d\Omega_1}{dt} = \left\{ (1 - \Omega_1) H - \Gamma_1 \right\} \Omega_1, \tag{116}
\]

\[
\frac{dH}{dt} = -\frac{1}{2} (4 - \Omega_1) H^2. \tag{117}
\]

We want to solve these equations from the time \( t = t_b \) when the \( \sigma_1 \)-field is dominating the universe, but still well before the \( \sigma_1 \)-field decay, to the time well after the \( \sigma_1 \)-field decay. To know the magnitude of the scale factor at \( t_b \), we need to connect the scale factor in that epoch with the initial scale factor \( a_* \). We can make the connection by solving the differential equation for \( a(t) \):

\[
H^2 = H_*^2 \left\{ \left( \frac{a}{a_*} \right)^{-4} (1 - \Omega_{1,*}) + \left( \frac{a}{a_*} \right)^{-3} \Omega_{1,*} \right\}. \tag{118}
\]
We can exactly integrate this equation. The result is
\[
\frac{2}{3\Omega_{1,*}^2}\left\{ 2 - 3\Omega_{1,*} + \sqrt{1 + \left(\frac{a}{a_*} - 1\right)\Omega_{1,*}} \left( -2 + \left(\frac{a}{a_*} + 2\right)\Omega_{1,*} \right) \right\} = H_*(t - t_*). \quad (119)
\]
Neglecting \(t_*\), we find that \(a(t)\) at \(t_b\) is given by
\[
a(t_b) = a_*\left(\frac{9}{4}\Omega_{1,*}\right)^{1/3} (H_* t_b)^{2/3} + \frac{a_*}{\Omega_{1,*}} + \mathcal{O}\left(\frac{1}{t_b}\right). \quad (120)
\]
For large \(t_b\), the first term dominates the scale factor. The second term represents the contribution from the radiation being diluted faster than the \(\sigma_1\)-field. As we will see, keeping only the first term gives the desired expression of \(F_1\).

Things become clearer if we introduce the new dimensionless variables by
\[
b(t) \equiv \frac{a(t)}{a_*\left(\frac{9}{4}\Omega_{1,*}\right)^{1/3} (H_* t)^{2/3}}, \quad (121)
\]
\[
E(t) \equiv \frac{3t}{2}H(t). \quad (122)
\]
Defining the dimensionless time \(s\) by \(s \equiv \Gamma_1 t\), the evolution equations for the new variables are given by
\[
\frac{db}{ds} = \frac{2b}{3s}(E - 1), \quad (123)
\]
\[
\frac{dE}{ds} = \frac{E}{s}\left\{ 1 - (4 - \Omega_1)\frac{E}{3} \right\}, \quad (124)
\]
\[
\frac{d\Omega_1}{ds} = \left\{ \frac{2E}{3s}(1 - \Omega_1) - 1 \right\}\Omega_1. \quad (125)
\]
The initial conditions are \(b(0) = E(0) = \Omega_1(0) = 1\). The new set of differential equations plus the initial conditions are free of model parameters, which means that we have extracted the dependence of \(F_1\) on the model parameters. Indeed, in terms of the new variables, \(F_1\) can be written as
\[
F_1 = a_*\left(\frac{9}{4}\Omega_{1,*}H_*^2\right)^{1/3} \frac{c_\Gamma}{\Gamma_1^{5/3}}, \quad (126)
\]
where \(c_\Gamma\) is a purely numerical value defined by
\[
c_\Gamma \equiv \int_0^\infty ds s^{2/3}b(s)e^{-s}. \quad (127)
\]
We find numerically that \(c_\Gamma \approx 0.830\).
Let us next evaluate \( F_2 \). For \( F_2 \), the dominant contribution to the integral comes from a time interval when the \( \sigma_2 \)-field dominates the universe (epoch D). Since \( \sigma_1 \)-field has completely decayed into the radiation by that epoch, we can only consider the radiation and the \( \sigma_2 \)-field in the following analysis. To make an argument similar to the case of \( F_1 \), first we have to connect the scale factor at the epoch C with \( a_* \). The scale factor at the epoch C is given by \( a(t_c) = a_\ast \left( \frac{9}{4} \Omega_{1,\ast} \right)^{1/3} \frac{d_\ast}{d_\ast^{1/6} t_\ast^{1/2}} \) for \( s \gg 1 \). From (123) and (124), we find that the asymptotic form of \( b(s) \) for \( s \gg 1 \) is given by

\[
b(s) = d_\ast s^{-1/6} \left( 1 + \mathcal{O}(s^{-1}) \right), \tag{128}
\]

where \( d_\ast \) is a numerical constant. We find numerically \( d_\ast \approx 1.101 \). Substituting this into (121), the scale factor at a time \( t = t_c \) in the epoch C is given by

\[
a(t_c) = a_\ast \left( \frac{9}{4} \Omega_{1,\ast} \right)^{1/3} \frac{H_*^2}{\Gamma_1^{1/6} t_\ast^{1/2}}. \tag{129}
\]

Using this result, the \( \Omega_2 \) at this time is found to be

\[
\Omega_2(t_c) = \frac{16 \Omega_{2,\ast} \Gamma_1^{1/2}}{9 \Omega_{1,\ast} d_\ast^3 t_\ast^{1/2}}. \tag{130}
\]

Now it may be expected that the things can be exactly mapped to what we did for \( F_1 \) by replacing the epoch A/B by the epoch C/D. To be more precise, simply replacing \( a_\ast, \Omega_{1,\ast}, H_* \) and \( \Gamma_1 \) appearing in (126) by \( a(t_c) \) given by (129), \( \Omega_2 \) given by (130), \( H(t_c) \) and \( \Gamma_2 \) would give the desired expression for \( F_2 \). However, as it will come out, we need to include a term in the scale factor corresponding to the second term of (120) to have the correct answer, which makes the calculations more complicated.

If we denote by \( t_d \) the time when the \( \sigma_2 \)-field is dominating the universe, but still well before the \( \sigma_2 \)-field decay, the scale factor at that time is given by

\[
a(t_d) = a(t_c) \left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} t_d^{2/3} + \frac{a(t_c)}{\Omega_2(t_c)} + \mathcal{O} \left( \frac{1}{t_d} \right), \tag{131}
\]

which is obtained by exactly the same argument as what we used to derive (120). After the example of \( F_1 \), let us introduce the new dimensionless variables by

\[
d(t) \equiv \frac{a(t)}{a(t_c) \left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} t_d^{2/3}}. \tag{132}
\]

To include the second term of (131), let us decompose the scale factor as

\[
a(t) = \bar{a}(t) + \delta a(t). \tag{133}
\]

Here \( \bar{a}(t) (\delta a(t)) \) is a part of the scale factor which reduces to the first(second) term in (131) at \( t_d \). Correspondingly, we can define \( \bar{d}(t), \delta d(t), \bar{E}(t), \delta E(t), \bar{\Omega}_2(t) \) and \( \delta \Omega_2(t) \).
The evolution equations for \( \bar{d}(t) \), \( \bar{E}(t) \) and \( \bar{\Omega}_2(t) \) are given by (123)-(125) with \( b, E, \Omega_1 \) and \( s \) being replaced by \( \bar{d}, \bar{E}, \bar{\Omega}_1 \) and \( u \equiv \Gamma_2 t \). The initial conditions for those variables are \( \bar{d}(0) = \bar{E}(0) = \bar{\Omega}_2(0) = 1 \).

The evolution equations for \( \delta d \), \( \delta E \) and \( \delta \Omega_2 \) at linear order are obtained by perturbing (123)-(125):

\[
\frac{d}{du} \delta d = \frac{2(\bar{E} - 1)}{3u} \delta d + \frac{2 \bar{d}}{3u} \delta E, 
\]

(134)

\[
\frac{d}{du} \delta E = \frac{1}{u} \left\{ 1 - (4 - \bar{\Omega}_2) \frac{\bar{E}}{3} \right\} \delta E + \frac{\bar{E}}{u} \left\{ - \frac{4 - \bar{\Omega}_2}{3} \delta E + \frac{\bar{\Omega}_2}{3} \delta \Omega_2 \right\},
\]

(135)

\[
\frac{d}{du} \delta \Omega_2 = \left\{ (1 - \bar{\Omega}_2) \frac{2 \bar{E}}{3u} - 1 \right\} + \bar{\Omega}_2 \left\{ \frac{2(1 - \bar{\Omega}_2)}{3u} \delta E - \frac{2 \bar{E}}{3u} \delta \Omega_2 \right\}. 
\]

(136)

Behaviors of \( \delta d \), \( \delta E \) and \( \delta \Omega_2 \) for \( u \ll 1 \) are obtained as follows. Combining (131) and (132) yields the time evolution of \( \delta d(s) \) for \( u \ll 1 \) as

\[
\delta d(u) = \frac{\Gamma_2^{2/3}}{\left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} \Omega_2(t_c) u^{2/3}}.
\]

(137)

For \( \delta E \), by using a relation \( \delta E = \frac{3u}{2} \delta H \), we find that

\[
\delta E(u) = -\frac{\Gamma_2^{2/3}}{\left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} \Omega_2(t_c) u^{2/3}}.
\]

(138)

For \( \delta \Omega_2 \), since it is sourced by the residual radiation, we have

\[
\delta \Omega_2(u) = \frac{\rho_e(t)}{\rho_1(t)} = \frac{a(t_c)}{a(t_d) \Omega_2(t_c)} = -\frac{\Gamma_2^{2/3}}{\left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} \Omega_2(t_c) u^{2/3}}.
\]

(139)

To simplify the system further, let us introduce the new variables by

\[
X \equiv \left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} \Omega_2(t_c) u^{2/3} \delta d,
\]

(140)

\[
Y \equiv \left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} \Omega_2(t_c) u^{2/3} \delta E,
\]

(141)

\[
Z \equiv \left( \frac{9}{4} \Omega_2(t_c) H^2(t_c) \right)^{1/3} \Omega_2(t_c) u^{2/3} \delta \Omega_2.
\]

(142)
Then the evolution equations for these variables become

\[
\frac{dX}{du} = \frac{2}{3u} \left( \bar{E}X + \bar{d}Y \right),
\]

(143)

\[
\frac{dY}{du} = \frac{2}{3u} Y + \frac{1}{u} \left\{ 1 - \frac{\bar{E}}{3} (4 - \bar{\Omega}_2) \right\} Y + \frac{\bar{E}}{3u} \left( -(4 - \bar{\Omega}_2)Y + \bar{E}Z \right),
\]

(144)

\[
\frac{dZ}{du} = \frac{2}{3u} Z + \left\{ \frac{2\bar{E}}{3u} (1 - \bar{\Omega}_2) - 1 \right\} Z + \frac{2\bar{\Omega}_2}{3u} \left( (1 - \bar{\Omega}_2)Y - \bar{E}Z \right).
\]

(145)

The corresponding initial conditions are \(X(0) = -Y(0) = -Z(0) = 1\). It is now clear that the new set of differential equations plus the initial conditions are free of model parameters, which means that we have extracted the dependence of \(F_2\) on the model parameters. Indeed, we find that \(F_2\) can be written as

\[
F_2 = a_* \left( \frac{9}{4} \Omega_{2,*} H_*^2 \right) ^{1/3} \frac{c_\Gamma}{\Gamma_2^{5/3}} + a_* \left( \frac{9}{4} \Omega_{1,*} H_*^2 \right) ^{1/3} \frac{9d_{\Gamma,f} f_\Gamma}{16} \frac{\Omega_{1,*}}{\Gamma_1^{2/3} \Gamma_2 \Omega_{2,*}},
\]

(146)

where \(f_\Gamma\) is a constant defined by

\[
f_\Gamma \equiv \int_0^\infty du X(u)e^{-u} \approx 1.18716.
\]

(147)

The second term in (146) comes from \(\delta a\). Substituting (126) and (146) into (113), we get

\[
N = \frac{1}{4} \log \left\{ (1 + \epsilon_\Gamma) \left( \frac{H_*}{\Gamma_1} \right)^{2/3} \Omega_{1,*}^{1/3} + \left( \frac{H_*}{\Gamma_2} \right)^{2/3} \Omega_{2,*}^{1/3} \right\} + \frac{1}{4} \log \left\{ \left( \frac{9}{4} \right)^{1/3} a_* c_\Gamma \right\} + \frac{1}{4} \log \frac{\rho_*}{\rho_f},
\]

(148)

where \(\epsilon_\Gamma\) is a numerical constant defined by

\[
\epsilon_\Gamma \equiv \frac{9d_{\Gamma,f} f_\Gamma}{16c_\Gamma}.
\]

(149)

Numerically, we find that \(\epsilon_\Gamma \approx 1.183\). Since the last two terms are merely constants, only the first term contributes to the curvature perturbation.

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