We address the problem of how simple a solution can be for a given quantum local consistency instance. More specifically, we investigate how small the rank of the global density operator can be if the local constraints are known to be compatible. We prove that any compatible local density operators can be satisfied by a low rank density operator. Then we study both fermionic and bosonic versions of the $N$-representability problem as applications. After applying the channel-state duality, we prove that any compatible local channels can be obtained through a global quantum channel with small Kraus rank.

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I. INTRODUCTION

Understanding the various correlations and relationships amongst the different parts of a many body quantum system is one of the most difficult challenges in quantum theory. It is well-known that the reduced density operators defined by partial traces characterize subsystems. A system of three parties $A$, $B$ and $C$: If all three two-particle density operators $\rho_{AB}$, $\rho_{AC}$ and $\rho_{BC}$ are consistent with some global density operator $\rho_{ABC}$, they must satisfy $\text{Tr}_A(\rho_{AB}) = \text{Tr}_C(\rho_{BC})$, $\text{Tr}_B(\rho_{AB}) = \text{Tr}_C(\rho_{AC})$ and $\text{Tr}_A(\rho_{AC}) = \text{Tr}_B(\rho_{BC})$. This is a necessary but not sufficient condition for the existence of $\rho_{ABC}$. As the particle number $N$ becomes very large, the correlations between local subsystems become much more complicated. In general, local consistency is the problem of deciding whether a given collection of subsystem descriptions is consistent with some state of the global system, or the problem of finding necessary and sufficient condition for consistency of subsystem descriptions. It is also called the quantum marginal problem in literature[1–3]. The community observed the relation between the spectrum of bipartite quantum states and certain representations of the symmetric groups very recently. The consistency conditions for some special classes of quantum states were then given in [1–4]. For general states with overlapping margins, the situation remains unclear.

If the particles under consideration are fermions instead of qubits, the local consistency problem has been known as the $N$-representability problem, which arose initially in the 1960’s in connection with calculating the ground-state energies of general interacting electrons[5].

It was only recently shown in [6, 7] that both deciding the local consistency and deciding the local consistency for fermions are QMA-complete, meaning both the consistency problem and the $N$-representability problem are computationally at least as hard as any other problem in the complexity class QMA. Here, QMA is the quantum analogue of the complexity class NP. Consequently, it is unlikely to have efficient algorithms for local consistency problems, even on a quantum computer. And very recently, Wei, et al. proved that the bosonic version of the $N$-representability problem is also QMA-complete[8].

Though the local consistency problem is theoretically hard in the worst case, it is still worth exploring the potential solutions. There are various approaches scattered throughout the literature on this subject. Linden, et al. proved that almost every three-qubit pure state can be uniquely determined among all states by their two-party reduced states[9]. A related fermionic version was discussed in [10], whose results indicate that almost any three fermion pure state is uniquely determined among all states by their two-particle reduced states, though it was not stated explicitly. Linden et al.’s result was later generalized to $N$-particle systems[11].

In this paper, we will focus on another direction of the local consistency problem. We are interested in how simple the solution can be, or more specifically, how small the rank of a solution can be. The same question for bipartite quantum system without overlapping margins was discussed in [1], but their approach seems technically difficult to generalize to the case with overlapping margins. In this work, we consider this problem regarding the rank of the solution in a very general setting, for multipartite quantum systems with overlapping margins. We provide a rank reduction based approach. We also show that some useful results from convex analysis can be applied to this problem directly, though it leads to a slightly weaker bound than ours[17]. Then we will apply our results to fermionic and bosonic systems. Finally local consistency problem for quantum channels will be addressed.

We now state our main result. For a given finite-dimensional Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ will denote the space of bounded linear operators acting on $\mathcal{H}$. For the $n$-fold tensor product $\mathcal{H}^{\otimes n}$ and any integer $i \leq n$, $X$, and $Z_i$ are general Pauli $X$-gate and general Pauli $Z$-gate for $i$-th qudit respectively. Formally, the local consistency problem can be stated as follows.
Problem 1. Consider a multipartite quantum system $A_1A_2\cdots A_n$ with Hilbert space $\mathcal{H}_{A_1A_2\cdots A_n} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$. Given a set of reduced density operators $\rho_i (i = 1, 2, \cdots, k)$ where each $\rho_i$ acts on a subsystem $I_i$ of $A_1A_2\cdots A_n$, or $I_i \subseteq \{A_1, A_2, \cdots, A_n\}$. The local consistency problem is to address the existence of a global density operator $\rho \in B(\mathcal{H}_{A_1A_2\cdots A_n})$ satisfying $\text{Tr}_{I_i} \rho = \rho_i$ for any $1 \leq i \leq k$.

An instance of the local consistency problem is a collection of pairs: a reduced density operator $\rho_i$ and corresponding subsystem $I_i$. For any given instance $\{(\rho_i, I_i)\}_{i=1}^k$ of the local consistency problem, if there is a global density operator $\rho \in B(\mathcal{H}_{A_1A_2\cdots A_n})$ satisfying $\text{Tr}_{I_i} \rho = \rho_i$ for any $1 \leq i \leq k$, then we say $\{(\rho_i, I_i)\}_{i=1}^k$ is a compatible instance.

In this paper, we will show that compatible local density operators are reduced states of some simple (low rank) global density operator. More specifically, we have following theorem.

Theorem 1. For any compatible instance of local consistency problem $\{(\rho_i, I_i)\}_{i=1}^k$, there is a solution with rank no more than $\sqrt{\sum_{i=1}^k (\text{rank} \rho_i)^2}$.

We are primarily interested in instances where the whole system is $\mathcal{H}^{\otimes n}$ and only no more than $c$-particle reduced states are known. In this case, the number of reduced states should be no more than $\binom{n}{c}$, the rank is bounded by polynomial in $n$ while the rank of a general density operator should be exponential in $n$.

The paper is structured as follows. After introducing the requisite background material in section I, we give a proof of above theorem in section II. We then apply these ideas to the $N$-representability problem in section III. In section IV we introduce the notion of consistency of quantum channels. Some examples are illustrated in section V.

II. PROOF OF MAIN THEOREM

Proof of Theorem 1. Since the local density operators of this instance are known to be compatible, we can start from some solution $\rho$, which is a density operator acting on $\mathcal{H}_{A_1A_2\cdots A_n}$ and $\text{Tr}_{I_i} \rho = \rho_i$ for any $1 \leq i \leq k$. Using a spectral decomposition of $\rho$ we may write

$$\rho = \sum_{j=1}^r p_j \vert \psi_j \rangle \langle \psi_j \vert$$

where $r = \text{rank}(\rho)$ and $\vert \psi_1 \rangle, \vert \psi_2 \rangle, \cdots, \vert \psi_r \rangle$ are mutually orthogonal unit vectors. We will show that when $r$ is large, then we can find another solution $\rho'$ to the same instance and $\text{rank}(\rho') < \text{rank}(\rho)$.

Consider the spectral decompositions again and write

$$\rho_i = \sum_{j=1}^{r_i} p_j^{(i)} \vert \psi_j^{(i)} \rangle \langle \psi_j^{(i)} \vert \text{ for all } 1 \leq i \leq k$$

where $r_i = \text{rank} \rho_i$ and $\vert \psi_1^{(i)} \rangle, \vert \psi_2^{(i)} \rangle, \cdots, \vert \psi_{r_i}^{(i)} \rangle$ are mutually orthogonal unit vectors for any $i$.

Consider the following set:

$$\mathcal{M} = \{ X \in B(\text{supp} \rho) : \Pi_i (\text{Tr}_{I_i} X) \Pi_i = O \text{ for any } 1 \leq i \leq k \}.$$ 

Here, for each $i$, $\Pi_i$ is a projector on $\text{supp} \rho_i$.

Now, $\mathcal{M}$ is a subspace of dimension at least $r^2 - \sum_{i=1}^k \text{rank}^2 \Pi_i$.

If $r^2 - \sum_{i=1}^k \text{rank}^2 \Pi_i \geq 1$, then $\mathcal{M}$ is not empty. Let’s say there is a non-zero $X \in \mathcal{M}$ which implies $X^\dagger \in \mathcal{M}$ too. Thus both $H_1 = X + X^\dagger$ and $H_2 = i(X - X^\dagger)$ are traceless Hermitian operators in $\mathcal{M}$. Note that $H_1$ and $H_2$ cannot be zero simultaneously when $X$ is not the zero operator. Without loss of generality, let us assume $H = H_1 (or H_2)$ is non-zero.

$H$ is chosen from $B(\text{supp} \rho)$, or equivalently, there is some $\epsilon$ such that $\rho \pm \epsilon H \geq 0$ which follows $\rho_i \pm \epsilon \text{Tr}_{I_i} (H) \geq 0$ for any $i$. Thus Hermitian operator $\text{Tr}_{I_i} (H)$ lies completely in $B(\text{supp} \rho_i)$ that implies $\Pi_i (\text{Tr}_{I_i} H) \Pi_i = \text{Tr}_{I_i} H$ and then $\text{Tr} H = \text{Tr} \text{Tr}_{I_i} H = 0$.

Since the operator $H \neq 0$, $\text{Tr} H = 0$ contains both positive and negative eigenvalues, the same holds for $\rho - \lambda H$ for $\lambda >> 1$. Hence there exists an intermediate value $0 < \lambda < \infty$ for which the operator $\rho - \lambda H$ is nonnegative, but not strictly positive, i.e. $\rho - \lambda H$ is a degenerate density matrix we are looking for.

Now, for any solution $\rho$ to a given instance $\{(\rho_i, I_i)\}_{i=1}^k$, if $r^2 - \sum_{i=1}^k \text{rank}^2 \Pi_i \geq 1$, we can always find a non-zero traceless Hermitian operator $H \in \mathcal{M}$ and then another solution $\rho' = \rho - \lambda H$ to the same instance with rank less than $\text{rank} \rho$. Thus,
by repeating this procedure until above quadratic inequality doesn’t hold anymore, we will finally end with a solution $\sigma$ with rank $\sigma \leq \left\lfloor \sqrt{\sum_{i=1}^{k} (\text{rank } \rho_i)^2} \right\rfloor$.

\[ \sum_{i=1}^{k} (\text{rank } \rho_i)^2 \]

**Corollary 1.** For any instance of the local consistency problem with given local systems $\{ I_i \}_{i=1}^{k}$, if the solution set is nonempty, then there is a solution with rank no more than $\left\lfloor \sqrt{\sum_{i=1}^{k} (\dim I_i)^2} \right\rfloor$.

**Remark 1.** Barvinok proved that if there is a positive semidefinite matrix $X$ satisfying

$$ (Q_i, X) = q_i, \quad \forall 1 \leq i \leq k $$

where $Q_1, \cdots, Q_k$ are symmetric matrices and $q_1, \cdots, q_k$ are complex numbers, then there is a positive semidefinite matrix $X^*$ satisfying the same equation system and additionally $\text{rank } X^* \leq \lfloor \sqrt{\frac{\pi}{2}} \rfloor$ [12]. The main ingredients of his proof are the duality for linear programming in the quadratic form space. After applying Barvinok’s theorem to the local consistency problem, we will have a similar rank reduction which will lead to a solution with rank no more than $\sqrt{2 \sum_{i=1}^{k} \dim^2 I_i}$. Thus this result is weaker than ours.

### III. APPLICATION: $N$-REPRESENTABILITY PROBLEM

In this section, we will study the $N$-representability problem, which is a fermionic analogue of the local consistency problem. The bosonic version of $N$-representability is also addressed later.

We first restate the $N$-representability problem as follows.

**Problem 2.** Given a system of $N$ fermions where each particle has $d$ energy levels, and a $k$-fermion state $\rho$ of size $\binom{d}{k} \times \binom{d}{k}$, determine whether there exists an $N$-fermion state $\sigma$ such that $\text{Tr}_{k+1, \cdots, N}(\sigma) = \rho$.

According to the Pauli exclusion principle, no two particles can occupy the same state, thus we can always assume $d \geq N$.

The space of $N$-fermion pure states is mathematically described as the $N$-th antisymmetric tensor product of $\mathbb{C}_d$ with dimension $\binom{d}{N}$, denoted as $\wedge^N \mathbb{C}_d$. It is the span of all $N$-fold antisymmetric tensor products of vectors $x_1, x_2, \cdots, x_N$ in $\mathbb{C}_d$ which is defined as

$$ x_1 \wedge x_2 \wedge \cdots \wedge x_N = \frac{1}{\sqrt{N!}} \sum_P \varepsilon_P x_{P(1)} \otimes x_{P(2)} \otimes \cdots \otimes x_{P(N)}. $$

Here, $P$ goes through all permutations of $N$ indices and $\varepsilon_P$ is the signature of $P$. So $\varepsilon_P$ is 1, if the number of even-order cycles in $P$’s cycle type is even, and $-1$ otherwise.

Similarly, the space of $N$-boson pure states with $d$ energy levels corresponds to the $N$-th symmetric tensor product of $\mathbb{C}_d$ with dimension $\binom{d}{N}$, denoted as $\vee^N \mathbb{C}_d$.

For more information about $N$-th symmetric/antisymmetric tensor product, please refer to [13].

For the $N$-representability problem, there is a similar rank reduction as follows.

**Theorem 2.** Suppose we are given a system of $N$ fermions where each particle has $d$ energy levels and a $k$-fermion density operator $\rho$ of size $\binom{d}{k} \times \binom{d}{k}$. Assume there exists an $N$-fermion state $\sigma$ such that $\text{Tr}_{k+1, \cdots, N}(\sigma) = \rho$. Then there also exists an $N$-fermion density operator $\sigma'$ with $\text{Tr}_{k+1, \cdots, N}(\sigma') = \rho$ and rank $\sigma' \leq \text{rank } \rho \leq \binom{d}{k}$.

The proof is similar to the proof provided in Section II, with minor modifications. Observe that the whole rank reduction in our approach is processed in $\text{supp}(\rho)$. After introducing additional symmetry to the global system, the rank reduction also works by replacing $\otimes^N \mathbb{C}_d$ with $\wedge^N \mathbb{C}_d$ or $\vee^N \mathbb{C}_d$.

Similarly, we will also get the following theorem for the bosonic version.

**Theorem 3.** Suppose we have a system of $N$ bosons where each particle has $d$ energy levels and a $k$-boson density operator $\rho$ of size $\binom{d+k-1}{k} \times \binom{d+k-1}{k}$. Assume there exists an $N$-boson state $\sigma$ such that $\text{Tr}_{k+1, \cdots, N}(\sigma) = \rho$, then there also exists an $N$-boson density operator $\sigma'$ with $\text{Tr}_{k+1, \cdots, N}(\sigma') = \rho$ and rank $\sigma' \leq \text{rank } \rho \leq \binom{d+k-1}{k}$.

### IV. LOCAL CONSISTENCY PROBLEM FOR QUANTUM CHANNELS

In this section, we will investigate a new type of consistency – the consistency of quantum channels. A quantum channel is a device which transmits classical bits or quantum states. Mathematically, it is a linear map which maps any quantum state
on some Hilbert space $\mathcal{H}_1$ to another state on some Hilbert space $\mathcal{H}_2$. Furthermore, a quantum channel can be described by a completely-positive, trace-preserving map $\Phi$.

Generally, for a quantum channel from system $\mathcal{H}_1$ to $\mathcal{H}_2$, we shall think of $\mathcal{H}_1$ as part of a closed composite system $\mathcal{H}_1 \otimes \mathcal{H}_1'$ and $\mathcal{H}_2$ as part of another closed composite system $\mathcal{H}_2 \otimes \mathcal{H}_2'$ which has same dimension as $\mathcal{H}_1 \otimes \mathcal{H}_1'$. Therefore, the evolution from $\mathcal{H}_1 \otimes \mathcal{H}_1'$ to $\mathcal{H}_2 \otimes \mathcal{H}_2'$ can be described by some unitary operator $U$. The quantum channel $\Phi$ is then described as

$$\Phi(\rho) = \text{Tr}\mathcal{H}_2' (U (\rho \otimes |0\rangle \langle 0|_{\mathcal{H}_1'}) U^\dagger).$$

(6)

By Stinespring’s dilation theorem on completely positive maps, $\Phi$ must take following form

$$\Phi(X) = \sum_{i=1}^{N} K_i A K_i^\dagger$$

(7)

where $K_i$ are some operators, called Kraus operators of $\Phi$. Trace preservation of $\Phi$ is equivalent to the sum $\sum_{i=1}^{N} K_i^\dagger K_i$ equaling the identity operator. The number of Kraus operators $N$ is no more than $\dim \mathcal{H}_1 \dim \mathcal{H}_2$, and the minimum number of $N$ is called the Kraus rank of $\Phi$. In some sense, the smaller the Kraus rank is, the simpler the channel is.

The concept of channel consistency is quite intuitive. Consider the following scenario, in which there is a channel from some large system $\mathcal{H}_1$ to some large system $\mathcal{H}_2$. Here, the mapping is from $B(\mathcal{H}_1)$ to $B(\mathcal{H}_2)$. A local observer Alice can only gather information from part of $\mathcal{H}_1$, say $\mathcal{H}_1^i$ and part of $\mathcal{H}_2$, say $\mathcal{H}_2^i$; therefore, she has information about the partial mapping from $B(\mathcal{H}_1^i)$ to $B(\mathcal{H}_2^i)$. Another observer Bob has his information about the partial mapping from some $B(\mathcal{H}_1^B)$ to some $B(\mathcal{H}_2^B)$. So do any other observers. Several questions naturally arise. How much information about the global quantum channel can be known when Alice, Bob and other observers disclose their local information? If every observer has a description of some channel-state duality, we can define sub-channel of some quantum channel as the following.

Given any channel $\Psi : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_B)$, we can always write a corresponding state of $\Psi$ to be

$$\sigma_\Psi = \frac{1}{\dim \mathcal{H}_1} \sum_{p,q=1}^{\dim \mathcal{H}_1} |p\rangle \langle q| \otimes \Psi(|p\rangle \langle q|)$$

(8)

where $\{|i\rangle\}_{i=1}^{\dim \mathcal{H}_1}$ is an orthonormal basis of $\mathcal{H}_1$. The state $\sigma_\Psi$ is called the Choi-Jamiołkowski state of $\Psi$ and the association above defines an isomorphism between linear maps from $B(\mathcal{H}_A)$ to $B(\mathcal{H}_B)$ and operators in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$, called the Choi-Jamiołkowski isomorphism. Its rank is equal to the Kraus rank of $\sigma_\Psi$.

Therefore, for any quantum channel $\Psi : B(\mathcal{H}_A \otimes \mathcal{H}_B) \rightarrow B(\mathcal{H}_A' \otimes \mathcal{H}_B')$, we can define channel $\Psi^A : B(\mathcal{H}_A) \rightarrow B(\mathcal{H}_A')$ by taking the reduced density operator of the Choi-Jamiołkowski state $\sigma_\Psi$ as its Cho-Jamiołkowski state $\sigma_{\Psi^A}$.

Observe that

$$\Psi^A(\rho) = \text{Tr}_A \left( \left( \frac{1}{\dim \mathcal{H}_{AB}} \sum_{p,q=1}^{\dim \mathcal{H}_{AB}} \text{Tr}_B |p\rangle \langle q| \otimes \Psi(|p\rangle \langle q|) (\rho \otimes I_{A'}) \right) \right)$$

(9)

$$= \frac{1}{\dim \mathcal{H}_{AB}} \sum_{p,q=1}^{\dim \mathcal{H}_{AB}} \text{Tr}_{AB} \left( (|p\rangle \langle q| (\rho \otimes I_B)) \otimes \Psi(|p\rangle \langle q|) \right)$$

(10)

$$= \text{Tr}_B (\rho \otimes I_B) \frac{1}{\dim \mathcal{H}_{B'}},$$

(11)

$\Psi^A$ acts exactly the same as $\Psi$ does between $B(\mathcal{H}_A)$ and $B(\mathcal{H}_A')$. Hence we call $\Psi^A$ sub-channel of $\Psi$ from $B(\mathcal{H}_A)$ to $B(\mathcal{H}_A')$.

By adopting above definition, we will address the following question: how simple the global channel can be, or more specifically, how small its Kraus rank can be if the sub channels are known to be compatible. Mathematically, the local consistency problem for quantum channels can be stated as follows.

**Problem 3.** Consider two multipartite quantum systems $A_1 A_2 \cdots A_n$ and $B_1 B_2 \cdots B_m$ with Hilbert spaces $\mathcal{H}_{A_1 A_2 \cdots A_n} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \cdots \otimes \mathcal{H}_{A_n}$ and $\mathcal{H}_{B_1 B_2 \cdots B_m} = \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes \cdots \otimes \mathcal{H}_{B_m}$ respectively. Assume a set of local quantum channels
{Φ_l : l = 1, 2, · · · , k} is given. Each Φ_l maps states on Hilbert space H_l = ⊗_A_l ∈ I_l H_{A_l} to states on Hilbert space H_{i_l} = ⊗_{B_l ∈ J_l} H_{B_l}, where I_l ⊆ {A_1, A_2, · · · , A_n} and J_l ⊆ {B_1, B_2, · · · , B_m}. The local consistency problem for quantum channels is to address the existence of a quantum channel \( \Phi : B(\mathcal{H}_{A_1 A_2 · · · A_n}) \to B(\mathcal{H}_{B_1 B_2 · · · B_m}) \) satisfying \( \Phi(\rho) = \text{Tr}_{J_l} \Phi(\rho \otimes \frac{I}{\dim H_{i_l}}) \) for any \( \rho \in D(\mathcal{H}_{i_l}) \) for any \( 1 \leq l \leq k \).

By taking the Choi-Jamiolkowski states of each \( \Phi_l \), Problem 3 can be converted to the existence of the global density operator. Then we can apply Theorem 1 to the quantum system \( A_1 A_2 · · · A_n B_1 B_2 · · · B_m \) with Hilbert space \( \mathcal{H}_{A_1 A_2 · · · A_n B_1 B_2 · · · B_m} = \mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes · · · \otimes \mathcal{H}_{A_n} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{B_2} \otimes · · · \otimes \mathcal{H}_{B_m} \). Since any set of reduced density operators \( \sigma_{\Phi_l}(l = 1, 2, · · · , k) \) is given and each \( \sigma_{\Phi_l} \) acts on a subsystem \( I_l J_l \subseteq \{A_1, A_2, · · · , A_n, B_1, B_2, · · · , B_m\} \). According to Theorem 1, we can always find a density operator \( \sigma \) satisfying the local consistencies and rank \( \sigma \leq \sqrt{\sum_l (\dim I_l \dim J_l)^2} \).

**Theorem 4.** Assume there exists a global quantum channel \( \Phi : B(\mathcal{H}_{A_1 A_2 · · · A_n}) \to B(\mathcal{H}_{B_1 B_2 · · · B_m}) \) satisfying \( \Phi(\rho) = \text{Tr}_{J_l} \Phi(\rho \otimes \frac{I}{\dim H_{i_l}}) \) for any \( \rho \in D(\mathcal{H}_{i_l}) \) for any \( 1 \leq l \leq k \). Then there also exists a quantum channel \( \Phi' \) satisfying the same local constraints such that \( \Phi' \) can be expressed with no more than \( \sqrt{\sum_l (\dim I_l \dim J_l)^2} \) Kraus operators.

**V. SOME EXAMPLES**

**Example 1.** Consider an \( n \)-qubit quantum system \( A_1 A_2 · · · A_n \) with Hilbert space \( \mathcal{H}_{A_1 A_2 · · · A_n} = \mathbb{C}^2 \otimes \mathbb{C}^2 \). We are interested in the \( n \)-qubit states \( \rho \) such that any \( k \)-qubit local density operator of \( \rho \) is \( \frac{I_k}{2^k} \).

Obviously, \( \rho = \frac{I_n}{2^n} \) is a trivial candidate with the maximal rank \( 2^n \).

As a corollary, when \( k = 2 \) is fixed, then rank(\( \rho \)) \( \in O(n) \).

Indeed, for \( k = 2 \), there are always pure state (i.e. rank = 1) solutions for any \( n \geq 5 \). One such example could be a graph state \( |\psi_n\rangle \) on a ring, which is a common eigenstate of eigenvalue 1 of the Pauli operators \( g_i = \{Z_{i-1}X_iZ_{i+1}\} \) for \( i = 1, 2, · · · , n \), where \( Z_0 = Z_n, Z_{n+1} = Z_1 \). That is, \( g_i |\psi_n\rangle = |\psi_n\rangle \) for \( i = 1, 2, · · · , n \). Note that

\[
\rho_n = |\psi_n\rangle \langle \psi_n| = \frac{1}{2^n} \prod_{i=1}^{n} (I + g_i).
\]

(12)

It is then straightforward to see that any 2-local density operator of the \( n \)-particle state \( \rho_n \) is \( \frac{I_n}{2^n} \).

In general, for any fixed \( k \), there do exist \( n \)-qubit graph states such that any \( k \)-local density operator of the graph state is \( \frac{I_k}{2^k} \) for large enough \( n \) [16].

**Example 2.** Consider a system of \( N \) bosons where each particle has 2 energy levels. The 2-boson maximally mixed state is defined as

\[
M_B^{(2)} = \frac{1}{3} \sum \langle 00 | 00 \rangle + |11\rangle \langle 11 | + \frac{01 + 10}{\sqrt{2}} \langle 01 + 10 \rangle.
\]

(13)

Obviously there exists a non-degenerate \( N \)-boson maximally mixed operator \( M_B^{(N)} \) such that \( \text{Tr}_{3, · · · , N} (M_B^{(N)}) = M_B^{(2)} \). Then it follows from Theorem 3 that there exists another \( N \)-boson density operator \( \sigma \) with \( \text{Tr}_{3, · · · , N} (\sigma) = M_B^{(2)} \) and rank \( \sigma \leq 3 \).

We can choose \( \frac{N-1}{3} \leq p \leq \frac{2N+1}{3} \) and let \( \sigma_p \) to be

\[
\frac{3p + 1 - N}{6p} |00\rangle \langle 00 | + \frac{2N - 3p + 1}{6(N-p)} |11\rangle \langle 11 | + \frac{\langle \sum_{i_1+\cdots+i_N = p} |i_1i_2\cdots i_N\rangle \langle \sum_{i_1+\cdots+i_N = p} |i_1i_2\cdots i_N| \rangle}{6(N-p)} 
\]

(14)

Notice that

\[
\sum_{i_1+\cdots+i_N = p} |i_1i_2\cdots i_N\rangle = |00\rangle + \sum_{i_3+\cdots+i_N = p} |i_3\cdots i_N\rangle + \langle 01 | + 10 \rangle \sum_{i_3+\cdots+i_N = p-1} |i_3\cdots i_N\rangle + |11\rangle \sum_{i_3+\cdots+i_N = p-2} |i_3\cdots i_N\rangle,
\]

(15)
and then
\[
\text{Tr}_{3,\ldots,N}(\sigma_p) = \begin{cases} 
3p + 1 - N & \text{if } N \equiv 0 \pmod{3} \\
6N - 3p + 1 & \text{if } N \equiv 1 \pmod{3} \\
6N - 3p - 2 & \text{if } N \equiv 2 \pmod{3}
\end{cases}
\]

(17)

\[
= \frac{1}{3}(|00\rangle\langle 00| + |11\rangle\langle 11| + |01 + 10\rangle\langle 01 + 10|)/\sqrt{2}.
\]

(18)

Therefore, \(\{\sigma_p\}_{\frac{N-1}{3} \leq p \leq \frac{2N+1}{3}}\) is a family of \(N\)-boson density operators with rank 3 and every 2-local density operator of any \(\sigma_p\) is the bosonic maximally mixed state \(M^{(2)}_B\). Furthermore, when \(N \equiv 1 \pmod{3}\), we will have \(\text{rank}(\sigma_{N-1}) = 2\).

VI. CONCLUSION AND FUTURE WORKS

In this paper we addressed the problem of how simple a solution can be for any given local consistency instance. More specifically, how small the rank of a global density operator can be if the local constraints are known to be compatible. We provided a reduction based approach to this problem and proved that any compatible local density operators can be satisfied with a global density operator with bounded rank. Then we studied both fermionic and bosonic versions of the \(N\)-representability problem as applications. After applying the channel-state duality, we proved that any compatible local channels can be satisfied with a global quantum channel which can be expressed with a small number of Kraus operators.

This paper represents a preliminary step toward understanding the structure of solutions to the local consistency problem. There are many open questions from this approach deserving further investigation. For example, though the local consistency is known to be QMA-complete in general, efficient algorithms are still possible for some classes of instances. Since we know now the existence of solutions is equivalent to the existence of simple solutions, we can ask if it is possible to find more efficient algorithms for these classes? Further, we could ask if only spectra or other descriptions are known for subsystems, how simple can a solution be?

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[17] See Remark 1 for details.