Raising of a heavy semi-circular vault on a rigid centering

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Abstract. The modelling of a process of a semi-circular vault raising on a rigid cylindrical centering is conducted. Within the bounds of accreted solids mechanics the evolution of the vault stress-strain state under gravity action during the raising and after its completion is investigated. The vault material is described with defining relationships for elastic or aging viscoelastic materials. The formulation of the non-classical quasi-static initial-boundary accretion problem is given. Its analytical solution in series and quadratures is built. An effective method of calculation of residual stresses in the finished vault after supporting centering removing is offered.

1. Introduction
Everybody knows that proper weight of large constructions makes an essential, and more often than not determinative, contribution to their stress-strain state. So when designing such constructions one cannot but take into account gravity action.

It is also known that vast majority of large-size constructions is not installed ready-made on a base prepared but is being installed gradually, element by element. It is easily seen that the weight of every element newly added is sure to cause a supplementary strain of the present construction. That is why the stress-strain state of a construction gradually raised under continuous gravity action must greatly differ from the state of this construction if we would install it on the base and then expose the entire construction to gravity action.

Besides, it is also clear that if the structural material used possesses rheological properties than both any actual and the final state of the raised object should considerably depend on the raising time mode. During a raising process of an object of such kind there are two tendencies that are continuously interacting. The first one is the tendency of permanent loading the construction with weight of additional elements and the second one is the stress rearrangement in elements so far added due to strain procrastination under changing construction geometry. The type and the result of this interaction are determined by various particular factors.

A certain technological process of raising may prove to cause the situation when stresses having initially arisen concentrate in some areas of the construction while progressing and result in inadmissibly high level of technological stresses in some zones of the finished object. In this case it is necessary to take special measures to control the stress-strain state of the being raised object.

From the speculations above it becomes obvious that not only the organization of the control mentioned but even an ordinary strength calculation of gradually raised heavy objects cannot be
implemented in principle by means of a design model dealing only with the final configuration of the object considered. Reasonable results cannot be obtained either if we use classical equations and boundary conditions of mechanics in a time-varying spatial domain. It is caused by the fundamental fact that deformation process of an accreted solid greatly differs from that of a regular solid. Consequently, description of the former process should be performed not by traditional approaches of deformable solids mechanics that do not take into consideration features of accretion processes but in the framework of accreted solids mechanics concepts.

See the information about the current state and some solved problems of accreted solids mechanics, for instance, in [1–9]. And the present paper is intended to one problem of gradual raising of a heavy semi-circular vault that manifests properties of creep and aging. So far this problem has not been solved.

2. Statement of the problem

Let the process of raising of a semi-circular vault supporting on a smooth rigid base start at time \( t = t_1 \). The raising is implemented by means of vault erection due to continuous attachment of uniformly thick elementary layers of additional material to its external surface. During the process the internal surface of the vault is supported by a rigid circular centering which was previously placed on the base. The very first layer of the material is directly attached to the centering and adhere to it. The process lasts up to time \( t = t_2 > t_1 \) and after it is completed the centering is removed. During the process and after its completion the vault is attached to the base by a sliding mount that prevents the feet of the vault to be separated from the base but does not prevent their sliding along it. The material that is originally free of stresses is assumed homogeneous isotropic aging linear and viscoelastic.

Let us state a problem to study the process of creating of the stress-strain state of the vault in question under gravity action during its raising procedure considered above and the evolution of this state from the instant of raising completion to the instant of centering removing. Besides, let us study the residual stresses taking place in the completed vault even if the centering were removed long ago.

The process of erecting vault deformation is assumed to be quasistatic. We perform our consideration in small strains in the case of a plane strain state.

Let \( a \) be the internal vault radius (the external centering radius) and \( b(t) \) its external radius at time \( t \), \( b(t_1) = a \) (Fig. 1). Taking into account the fact that the strain is small suppose the strictly monotone increasing function of time \( b(t) \) is known. As we consider the process of continuous accretion of the vault, the function \( b(t), t \in [t_1, t_2] \) is to be continuous. Moreover, suppose this function is piecewise continuously differentiable. Let \( b(t) \equiv b_{\text{fin}} = b(t_2) \) when \( t > t_2 \).

We associate the vault transverse cross-section plane with the circular cylindrical coordinate system \((\rho, \varphi, z)\) with the right orthonormal local frame \( \{e_\rho, e_\varphi, k\} \) where \( \rho \) is the polar radius counted from the arch central axis, \( \varphi \) is the polar angle counted upwards from the base (Fig. 1) and \( z \) is the longitudinal coordinate. By \( j \) denote the unit normal to the base. Then the cubic density of the gravity forces acting on the vault equals to \( f = -j f \), where \( f = \text{const} \) is the material specific weight.
We describe the material used to erect the vault in the framework of the linear theory of viscoelasticity of homogeneously aging isotropic media [1,10], i.e. we start from the state equation
\[
\mathbf{T}(r, t) = G(t)(\mathcal{I} + \mathcal{N}_s(r))\left[2\mathbf{E}(r, t) + (\kappa - 1) \mathbf{1} \text{ tr } \mathbf{E}(r, t)\right].
\] (1)

Here \(\mathbf{T}(r, t)\) and \(\mathbf{E}(r, t)\) are the stress and small strain tensors, respectively, \(\mathbf{1}\) is the unit tensor of rank 2; \(G(t)\) is the elastic shear modulus and \(\kappa = (1 - 2\nu)^{-1}\), \(\nu = \text{const}\) is Poisson’s ratio. The viscoelasticity operator \(\mathcal{I} + \mathcal{N}_s\) in (1) is determined by the relations
\[
\mathcal{I} + \mathcal{N}_s = (\mathcal{I} - \mathcal{L}_s)^{-1},
\]
\[
\mathcal{L}_s f(t) = \int_{s}^{t} f(\tau) K(t, \tau) d\tau, \quad \mathcal{N}_s f(t) = \int_{s}^{t} f(\tau) R(t, \tau) d\tau,
\]
\[
K(t, \tau) = G(\tau) \frac{\partial \Delta(t, \tau)}{\partial \tau}, \quad \Delta(t, \tau) = G(\tau)^{-1} + \omega(t, \tau),
\]
where \(\mathcal{I}\) is the identity operator, \(\mathcal{L}_s\) and \(\mathcal{N}_s\) are the integral Volterra operators with a parameter \(s\); \(K(t, \tau)\) and \(R(t, \tau)\) are the creep and relaxation kernels, respectively, and \(\Delta(t, \tau)\) and \(\omega(t, \tau)\) are the specific strain function and the creep measure in the case of pure shear. \(\tau_0(r)\) is the time at which stresses appear at the point with the position vector \(r\). Time \(t\) is assumed to be counted from the material fabrication instant.

It should be noted that the equation (1) was suggested by the academician of the Armenian Academy of Sciences N. Kh. Arutyunyan especially for description of creeping processes in concrete and is widely used for this purpose. Meanwhile, it also well comes to agreement with experimental data for creep of some kinds of polymers, rocks, soil, and ice.

Let us introduce the integral linear operator \(\mathcal{H}_s = (\mathcal{I} - \mathcal{L}_s)G(t)^{-1}\) and its reverse operator \(\mathcal{H}^{-1}_s = G(t)(\mathcal{I} + \mathcal{N}_s)\) with the parameter \(s\) and for an arbitrary function \(g(r, t)\) of time \(t\) and point \(r\) of the erected body denote \(g^0(r, t) = \mathcal{H}^{-1}_s g(r, t)\). Then constitutive relation (1) can be written in another form [2]
\[
\mathbf{T}^0(r, t) = 2\mathbf{E}(r, t) + (\kappa - 1) \mathbf{1} \text{ tr } \mathbf{E}(r, t).
\] (2)

Note that the specific strain function for the pure shear \(\Delta(t, \tau)\) describes the increase during time \(t\) of the angle of shear in the sample when we create at time instant \(\tau\) and then hold a constant stress state of pure shear of unit intensity. The summand \(G(\tau)^{-1}\) determines the sample’s instantaneous elastic reaction to the force action indicated and the summand \(\omega(t, \tau)\) is the specific creep strain taking place in the sample at considered time. Thus, by definition, we get the identity \(\omega(\tau, \tau) \equiv 0, \tau \geq 0\). Taking into account this identity the specific strain function can be presented as follows:
\[
\Delta(t, \tau) = \mathcal{H}_s 1.
\] (3)

In the problem under consideration stresses appear in the body points at the time we include these points in the body structure. It means that
\[
\tau_0(r) \equiv \tau_s(\rho),
\] (4)
where \(\tau_s(\rho)\) is the time of attachment of a layer with radius \(\rho\) to the vault. It is clear that the equation \(\tau_s(\rho) = t\) describes the external surface of the continuously erecting vault \(\{\rho = b(t)\}\), i.e. its instantaneous erecting surface. In other words, we get the identity
\[
\tau_s(b(t)) \equiv t.
\] (5)
3. Boundary value problem for the raised vault

During the process of accretion the elements of additional material are added to the growing solid in the course of its strain motion in space. It is clear that the entire body formed in such a way cannot in general have the original unstrained configuration. It is this fact that is decisive in the strain process for any accreted solid and essentially distinguishes the mechanical behavior of such solids from the behavior of solids of constant composition (classical bodies in continuum mechanics) and of solids whose boundary is variable because of the removal of the material. Owing to this characteristic property, it is impossible to define the strain measure of an accreted solid by the method usually adopted in continuum mechanics; therefore, the Cauchy formulas do not hold for the total strain tensor components, and hence the Saint-Venant conditions of their compatibility are not satisfied.

Note, however, that the particles of additional material after its adhesion to the surface of growth continue their motion further as part of the solid. This means that, in the space region occupied by the entire accreted solid at a given time, a sufficiently smooth velocity field of its particles is uniquely determined. Therefore, we can expect that the problem of deforming of such a body can be well defined for the displacement velocities.

Starting from this, we write out the velocity analog of the constitutive relation (2) for velocities of changes of tensor quantities this relation ties together:

\[ S(r, t) = 2D(r, t) + (\kappa - 1) \text{tr} D(r, t). \]  

(6)

Here we introduce the tensor \( S = \partial T^0 / \partial t \) [2] and the strain rate tensor

\[ D = (\nabla v^T + \nabla v)/2, \]

(7)

where \( v(r, t) = e_\rho v_\rho + e_\phi v_\phi + k v_z \) the velocity field of the accreted body particles. In the considered case of plane strain, we have

\[ k \cdot v = 0, \quad \partial v/\partial z = 0. \]

(8)

The equation for the introduced above tensor \( S \) the region occupied at the current time by the accreted body can be obtained by applying the linear operator \( H_{\tau_0}(r) \) to the equilibrium equation \( \nabla \cdot T + f = 0 \) and by differentiating the result with respect to time \( t \). Due to the fact that the lower limit in the integral tensor mentioned depends on a point of the body this integral operator does not commute with the divergence operator. But one can show [2] that in the accretion processes studied in the present paper, for the stress tensor we can write \( (\nabla \cdot T)^0 = \nabla \cdot T^0 \). In this case the following analog of the equilibrium equation is satisfied in the entire accreted solid both during and after the process of its erecting:

\[ \nabla \cdot T^0 + f^0 = 0. \]

(9)

Differentiate it with respect to \( t \): \( \nabla \cdot S + \partial F^0 / \partial t = 0 \). By calculating with (3) the vector-function \( \partial F^0 / \partial t = -j h(\rho, t) \), where \( h(\rho, t) = f \partial \omega(t, \tau_*(\rho))/\partial t \), we obtain the equation

\[ \nabla \cdot S = j h(\rho, t). \]

(10)

Note that according to properties the creep measure should have (see, for example, [1]), \( h(\rho, t) \geq 0 \) when \( t > \tau_*(\rho) \).

Consider now a condition that is to be given on the erection surface of the body. As we suppose the material to be attached is originally free of stresses the complete stress tensor is required to be equal to zero on the accreted body external surface [11]:

\[ T = 0, \quad \rho = b(t), \quad t \in [t_1, t_2]. \]

(11)
A remarkable fact is that (11) can be transformed to the boundary condition for components of the tensor $S$ which has the form analogous to a classical condition for forces on the solid surface [2]. The set of such conditions (11) for all instants of the continuous growth as an initial condition for the stress tensor in points of having been accreted solid can be written as follows:

$$T(r, \tau_s(\rho)) = 0, \quad \rho \in [a, b_{\text{fin}}].$$

(12)

Noticing that from (4) the equality

$$g^\circ(r, \tau_s(\rho)) = g(r, \tau_s(\rho))/G(\tau_s(\rho))$$

(13)

follows for the stress tensor $T$, taking (12) into account, we get

$$T^\circ(r, \tau_s(\rho)) = 0, \quad \rho \in [a, b_{\text{fin}}].$$

(14)

Apply the divergence operator to (14):

$$[\nabla \cdot T^\circ(r, t)\bigg|_{t=\tau_s(\rho)} + \nabla \tau_s(r) \cdot S(r, \tau_s(\rho)) = 0.$$

Then by determining the gradient $\nabla \tau_s(r) = e_\varphi(\varphi) \tau_s^\prime(\rho)$, and using (9), and taking (13) into account we obtain $-f/G(\tau_s(\rho)) + e_\varphi(\varphi) \cdot S(r, \tau_s(\rho)) \tau_s^\prime(\rho) = 0$. The last thing to do is to single out from this conditions family on surfaces $\{ \rho = \text{const} \}$ the one that corresponds to the considered time instant $t \in [t_1, t_2]$, i.e. to suppose $\rho = b(t)$. To do it we use the identity (5) as well as the same identity having been differentiated with respect to $t$, i.e. the identity $\tau_s^\prime(b(t)) \equiv b'(t)^{-1}$. After all we have the following condition for the tensor $S$ on the instantaneous erecting surface at $t \in [t_1, t_2]$:

$$e_\varphi \cdot S = f b'(t)/G(t), \quad \rho = b(t).$$

(15)

After we stop erecting the vault its external surface remains unloaded. Therefore we can use an ordinary condition of stress absence for $t > t_2$. An evident analog of this condition for tensor $S$ is the boundary condition $e_\varphi \cdot S = 0$, $\rho = b_{\text{fin}}$. As we suppose that $b(t) \equiv b_{\text{fin}}$ when $t > t_2$ the form (15) of the condition for the external surface of the vault remains formally valid even after erecting completing and consequently can be used for any $t \geq t_1$. However, it is necessary to understand that nature of this condition during the vault continuous erecting stage and after its completing is different in essence.

Consider now conditions on other parts of boundary surface of erected vault. At its internal surface $\{ \rho = a \}$ we have to require meeting of kinematic condition of tough adhesion of the material and the centering. To set a problem in velocities this condition should be

$$v = 0, \quad \rho = a.$$ 

(16)

For the feet of the vault $\{ \varphi = 0, \pi \}$, as we mentioned when stating the problem, we use a mixed condition for a sliding mount $e_\varphi \cdot T \cdot e_\rho = 0$, $e_\varphi \cdot v = 0$. Its analog for the tensor $S$ is

$$e_\varphi \cdot S \cdot e_\rho = 0, \quad e_\varphi \cdot v = 0, \quad \varphi = 0, \pi.$$ 

(17)

Thus combining the relations (10), (6), (7), (8), (16), (15), (17), we get the following boundary value problem which describes a process of erected vault deforming up to the instant of removing the centering some time later after the erecting process has been completed:

$$\nabla \cdot S = j h(\rho, t), \quad \rho \in (a, b(t)), \quad \varphi \in (0, \pi), \quad t > t_1;$$

$$S = 2D + (\varphi - 1) 1 \times D, \quad D = (\nabla v^2 + \nabla v)/2; \quad k \cdot v = 0, \quad \partial v/\partial z = 0;$$

$$v\big|_{\rho=a} = 0; \quad e_\varphi \cdot S\big|_{\rho=b(t)} = -j q(t); \quad e_\varphi \cdot S\big|_{\varphi=0,\pi} \cdot e_\rho = 0, \quad e_\varphi \cdot v\big|_{\varphi=0,\pi} = 0.$$

(18)

Here we introduce the notation $q(t) = f b'(t)/G(t) \geq 0$. The form of this task is equivalent to a classical boundary value problem of linear elasticity theory where the vector $v$ and the tensors $D$ and $S$ play parts of the displacement vector, the small strain tensor and the stress tensor divided into shear modulus. Time $t$ is this problem parameter.
4. Solution of the problem. Determining of stresses in the raised vault

Let us look for the radial and circular components of the velocity vector field \(v\), which is the solution of the boundary value problem (18), as the cosine and sine Fourier series expansions of angles that are multiples of \(\varphi \in [0, \pi]\) respectively. Because of the structure symmetry we have

\[
v_\rho = v_\rho|_{\pi - \varphi}, \quad v_\varphi = -v_\varphi|_{\pi - \varphi}.
\]

Therefore

\[
v_\rho = c_0(\rho, t) + \sum_{n=1}^{\infty} c_n(\rho, t) \cos 2n\varphi, \quad v_\varphi = \sum_{n=1}^{\infty} d_n(\rho, t) \sin 2n\varphi.
\] (19)

Note that these expansions satisfies to the homogenous mixed boundary conditions on the arch feet \(\{\varphi = 0, \pi\}\) contained in (18).

After we use (19) in (18) that satisfies to the Lame equations we are to solve systems of linear inhomogeneous ordinary differential equations with respect to \(\rho\) of the second order with variable coefficients for unknown functions \(c_0(\rho, t), c_n(\rho, t), d_n(\rho, t)\). On solving them we get

\[
2 c_0/\rho = -\zeta^{-1}B_{40} - B_{10},
\]

\[
2 c_n/\rho = [(n + \zeta^{-1})B_{2n} - B_{3n}]/(2n - 1) + [(n - \zeta^{-1})B_{4n} - B_{1n}]/(2n + 1),
\]

\[
2 d_n/\rho = [(n - 1 - \zeta^{-1})B_{2n} + B_{3n}]/(2n - 1) - [(n + 1 + \zeta^{-1})B_{4n} + B_{1n}]/(2n + 1);
\]

\[
S = e_{\rho}e_{\rho} S_{\rho} + e_{\varphi}e_{\varphi} S_{\varphi} + k k \nu(S_\rho + S_\varphi) + (e_{\rho}e_{\varphi} + e_{\varphi}e_{\rho}) S_{\rho\varphi},
\]

\[
S_{\rho\varphi} = \pm B_{10} - B_{40} + \sum_{n=1}^{\infty} \left[ \pm B_{1n} \mp (n \pm 1)B_{2n} \mp B_{3n} \pm (n \mp 1)B_{4n} \right] \cos 2n\varphi,
\]

\[
S_{\rho\rho} = \sum_{n=1}^{\infty} B_{1n} - nB_{2n} + B_{3n} - nB_{4n} \sin 2n\varphi.
\]

Here we have introduced the functions

\[
B_{1m}(\rho, t) = K^+ \Phi_m+1(\rho, t) + (b(t)/\rho)^{2m+2} \alpha_{1m}(t), \quad B_{2m}(\rho, t) = \zeta \Phi_m(\rho, t) + (b(t)/\rho)^{2m} \alpha_{2m}(t),
\]

\[
B_{3m}(\rho, t) = K^- \Psi_m-1(\rho, t) + (\rho/a)^{2m-2} \alpha_{3m}(t), \quad B_{4m}(\rho, t) = \zeta \Psi_m(\rho, t) + (\rho/a)^{2m} \alpha_{4m}(t);
\]

\[
\Phi_m(\rho, t) = \frac{k}{2m - 1} \int_{\rho}^{\infty} (\xi/\rho)^{2m} h(\xi, t) d\xi, \quad \Psi_m(\rho, t) = \frac{k}{2m + 1} \int_{\rho}^{\infty} (\rho/\xi)^{2m} h(\xi, t) d\xi.
\]

\(K^\pm = [m(2m \pm 3)/(2m \mp 1)] \zeta \pm 1\) and \(k = 2/[\pi(\zeta + 1)]\) are numerical multipliers. Functions \(\alpha_{jm}(t)\) are to be found from the system of linear equations that we obtain on satisfying the boundary conditions for (18) on the internal and external erected vault surfaces. We do not present the expressions for \(\alpha_{jm}\) because they are extremely cumbersome.

After the problem (18) is solved, the tensor \(T^0\) evolution in every point of the completed vault can be restored from the found velocity of changes of this tensor by means of the initial condition (14):

\[
T^0(\mathbf{r}, t) = \int_{\tau_*(\rho)}^{t} \mathbf{S}(\mathbf{r}, \tau) d\tau, \quad t \geq \tau_*(\rho).
\]

Then using the known evolution of the tensor \(T^0\) it is possible according to its definition to determine the evolution of the stress tensor \(T\), by solving the integral Volterra equation of the second kind

\[
g(\mathbf{r}, t) - \int_{\tau_*(\rho)}^{t} g(\mathbf{r}, \tau) K(t, \tau) d\tau = T^0(\mathbf{r}, t)
\]

with the parameter \(\mathbf{r}\) for the unknown tensor-function of time \(g(\mathbf{r}, t) = T(\mathbf{r}, t)/G(t)\). In analytical form will be

\[
T(\mathbf{r}, t) = H^{\mathcal{L}}_{\tau_*(\rho)} T^0(\mathbf{r}, t) = G(t) \left[ T^0(\mathbf{r}, t) + \int_{\tau_*(\rho)}^{t} T^0(\mathbf{r}, \tau) R(t, \tau) d\tau \right], \quad t \geq \tau_*(\rho).
\] (21)

However, if the expression for the resolvent \(R(t, \tau)\) of the kernel \(K(t, \tau)\) is too complicated to calculate using (21), then it seems more desirable to solve (20) numerically, for example, by the quadrature method [12].
5. Residual stresses in the finished vault after centering removing

Above we constructed the solution of the problem on the deformation of a viscoelastic aging vault raised under gravity action on a smooth rigid base on a supporting rigid centering which remains adhered to the internal vault surface. This problem makes sense for any arbitrary large time \( t > t_1 \). Let it be determined by known functions \( \mathbf{v}(r, t), \mathbf{D}(r, t), \mathbf{S}(r, t), \mathbf{T}(r, t) \). Knowing the last one we can determine the centering reaction, that is the evolution of distribution of contact stresses on the internal vault surface \( t_a(\varphi, t) = -e_p \cdot \mathbf{T} \mid_{\varphi=a} \). Then the problem (18) for functions \( \mathbf{v}(r, t), \mathbf{D}(r, t), \mathbf{S}(r, t) \) can be replaced by equivalent boundary value problem

\[
\nabla \cdot \mathbf{S} = \mathbf{j} \rho(\varphi, t), \quad \varphi \in (0, \pi), \quad t > t_1; \\
\mathbf{S} = 2\mathbf{D} + (\sigma - 1) \mathbf{1} \text{ tr } \mathbf{D}, \quad \mathbf{D} = (\nabla \mathbf{v}^T + \nabla \mathbf{v})/2; \quad \mathbf{k} \cdot \mathbf{v} = 0, \quad \partial \mathbf{v}/\partial z = 0; \\
e_p \cdot \mathbf{S} \mid_{\varphi=a} = -p(\varphi, t); \quad e_p \cdot \mathbf{S} \mid_{\varphi=b(t)} = -j q(t); \quad e_\varphi \cdot \mathbf{S} \mid_{\varphi=0, \pi} \cdot e_p = 0, \quad e_\varphi \cdot \mathbf{v} \mid_{\varphi=0, \pi} = 0. 
\]

Here we introduce the notation \( \mathbf{p} = \partial t_a^2 / \partial t \). Adjusted for equality \( \tau_a(a) = t_1 \) we write down

\[
\mathbf{p}(\varphi, t) = \frac{\partial \mathbf{H}_t}{\partial t} = \frac{\partial}{\partial t} \left[ \frac{t_a(\varphi, t)}{G(t)} \right] - \int_{t_1}^{t} \frac{t_a(\varphi, \tau)}{G(\tau)} K(t, \tau) d\tau. 
\]

Now we shall assume that, starting from some time \( t = t_{\text{det}} > t_2 \), when the formation of the vault has been already completed, its internal surface stresses start changing in a way that differs from that prescribed by a presence of a rigid centering in the structure. It changes arbitrary according to a law \( \dot{t}_a(\varphi, t), t > t_{\text{det}} \). Suppose this loading is a continuous time function that has piecewise continuous derivative at any time period and \( \dot{t}_a(\varphi, t_{\text{det}} + 0) = t_a(\varphi, t_{\text{det}}) \). When \( t \leq t_{\text{det}} \) we assume \( \dot{t}_a(\varphi, t) = t_a(\varphi, t) \).

By \( \dot{\mathbf{v}}(r, t), \dot{\mathbf{D}}(r, t), \dot{\mathbf{S}}(r, t), \dot{\mathbf{T}}(r, t) \) denote functions that describe the stress-strain state changing of the vault which internal surface is loaded from the instant of starting of its erecting to the arbitrary large instant with the distributed loading \( \mathbf{t} \). The first three of these functions are evidently solutions of the boundary value problem

\[
\nabla \cdot \dot{\mathbf{S}} = \mathbf{j} \rho(\varphi, t), \quad \varphi \in (a, b(t)), \quad \varphi \in (0, \pi), \quad t > t_1; \\
\dot{\mathbf{S}} = 2\dot{\mathbf{D}} + (\sigma - 1) \mathbf{1} \text{ tr } \dot{\mathbf{D}}, \quad \dot{\mathbf{D}} = (\nabla \dot{\mathbf{v}}^T + \nabla \dot{\mathbf{v}})/2; \quad \dot{\mathbf{k}} \cdot \mathbf{v} = 0, \quad \partial \mathbf{v}/\partial z = 0; \\
e_p \cdot \dot{\mathbf{S}} \mid_{\varphi=a} = -\dot{p}(\varphi, t); \quad e_p \cdot \dot{\mathbf{S}} \mid_{\varphi=b(t)} = -\dot{j} q(t); \quad e_\varphi \cdot \dot{\mathbf{S}} \mid_{\varphi=0, \pi} \cdot e_p = 0, \quad e_\varphi \cdot \dot{\mathbf{v}} \mid_{\varphi=0, \pi} = 0. 
\]

Here \( \dot{\mathbf{p}} = \partial \dot{t}_a^2 / \partial t \). The extensive variant of this equality is analogous to (23).

Introduce the vector-functions \( t_{\alpha} = t_a - t_a, p_{\alpha} = \dot{p} - p, v_{\alpha} = \dot{v} - v, D_{\alpha} = \dot{D} - D, S_{\alpha} = \dot{S} - S, T_{\alpha} = T - T \). As \( t_{\alpha} = 0 \) when \( t \leq t_{\text{det}} \) we get

\[
\begin{align*}
p_{\alpha}(\varphi, t) &= \frac{\partial}{\partial t} \left[ \frac{t_{\alpha}(\varphi, t)}{G(t)} \right] - \int_{t_1}^{t} \frac{t_{\alpha}(\varphi, \tau)}{G(\tau)} K(t, \tau) d\tau = \begin{cases} 0, & t < t_{\text{det}}, \\
\mathbf{H}_{t_{\text{det}}} t_{\alpha}(\varphi, t), & t > t_{\text{det}}; \end{cases} \\
\int_{t_{\text{det}}}^{t} p_{\alpha}(\varphi, \tau) d\tau &= \mathbf{H}_{t_{\text{det}}} t_{\alpha}(\varphi, t) = \mathbf{H}_{t_{\text{det}}} t_{\alpha}(\varphi, t) - \frac{t_{\alpha}(\varphi, t_{\text{det}})}{G(t_{\text{det}})} = \\
&= \mathbf{H}_{t_{\text{det}}} t_{\alpha}(\varphi, t), & t > t_{\text{det}}. \end{align*}
\]

Subtracting relations (22) from respective relations (24) we obtain the problem

\[
\begin{align*}
\nabla \cdot \mathbf{S}_{\alpha} &= 0, \quad \varphi \in (a, b(t)), \quad \varphi \in (0, \pi), \quad t > t_1; \\
\mathbf{S}_{\alpha} &= 2\mathbf{D}_{\alpha} + (\sigma - 1) \mathbf{1} \text{ tr } \mathbf{D}_{\alpha}, \quad \mathbf{D}_{\alpha} = (\nabla v_{\alpha}^T + \nabla v_{\alpha})/2; \quad \mathbf{k} \cdot v_{\alpha} = 0, \quad \partial v_{\alpha}/\partial z = 0; \\
e_p \cdot \mathbf{S}_{\alpha} \mid_{\varphi=a} = -p_{\alpha}(\varphi, t); \quad e_p \cdot \mathbf{S}_{\alpha} \mid_{\varphi=b(t)} = 0; \quad e_\varphi \cdot \mathbf{S}_{\alpha} \mid_{\varphi=0, \pi} \cdot e_p = 0, \quad e_\varphi \cdot v_{\alpha} \mid_{\varphi=0, \pi} = 0. \end{align*}
\]
We obtain the following boundary value problem:

\[ T_{\Delta}^{\varphi}(r,t) = \int_{r_{\varphi}(\rho)}^{t} S_{\Delta}(r,\tau) \, d\tau = \begin{cases} 0, & \tau_{\varphi}(\rho) < t < t_{\text{det}}, \\ \int_{t_{\text{det}}}^{t} S_{\Delta}(r,\tau) \, d\tau, & t \geq t_{\text{det}}; \end{cases} \]

\[ T_{\Delta}(r,t) = \mathcal{H}_{\tau_{\varphi}(\rho)}^{-1} T_{\Delta}^{\varphi}(r,t) + \int_{t_{\text{det}}}^{t} T_{\Delta}(r,\tau) R(t,\tau) \, d\tau = \mathcal{H}_{\tau_{\varphi}(\rho)}^{-1} T_{\Delta}^{\varphi}(r,t), \quad t \geq t_{\text{det}}. \]

In the region occupied by the having been erected vault we determine the functions \( u_{\Delta}(r,t) = \int_{t_{\text{det}}}^{t} v_{\varphi}(r,\tau) \, d\tau \), \( E_{\varphi}(r,t) = \int_{t_{\text{det}}}^{t} D_{\Delta}(r,\tau) \, d\tau \), \( t \geq t_{\text{det}} \). By integrating all relations in problem (26) with respect to the parameter from \( t_{\text{det}} \) to \( t \geq t_{\text{det}} \), and by applying the operator \( \mathcal{H}_{\tau_{\varphi}(\rho)}^{-1} \) to the integrating result those of them that originally contained the tensor \( S_{\Delta} \), and taking into account (27), (25) we obtain the following boundary value problem:

\[ \nabla \cdot T_{\Delta} = 0, \quad \rho \in (a,b_{\text{fin}}), \quad \varphi \in (0,\pi), \quad t \geq t_{\text{det}}; \]

\[ T_{\Delta} \bigg|_{G(t)} = (I + N_{t_{\text{det}}}) \begin{bmatrix} 2E_{\varphi} + (\kappa - 1) \mathbf{1} \mathbf{1}' E_{\varphi} \end{bmatrix}, \quad E_{\varphi} = \frac{\nabla u_{\Delta} + \nabla u_{\Delta}}{2}; \quad k \cdot u_{\Delta} = 0, \quad \partial u_{\Delta} / \partial z = 0; \quad (28) \]

\[ e_{\rho} \cdot T_{\Delta} |_{\rho = a} = -t_{\text{a}}(\varphi,t); \quad e_{\rho} \cdot T_{\Delta} |_{\rho = b_{\text{fin}}} = 0; \quad e_{\varphi} \cdot T_{\Delta} |_{\varphi = 0,\pi} = 0, \quad e_{\varphi} \cdot u_{\Delta} |_{\varphi = 0,\pi} = 0. \]

We see that the problem is a classical problem of plain deforming for a weightless vault that satisfies the constitutive relation (1), attached to a smooth rigid base by a sliding mount and loaded by a time-dependent distributed loading at its internal surface at time \( t = t_{\text{det}} \). In this problem, the displacement vector, the small strain tensor, and the stress tensor are replaced by the variables \( u_{\Delta} \), \( E_{\varphi} \), \( T_{\Delta} \), respectively. According to the known correspondence principle of the linear theory of aging solids viscoelasticity [1] the tensor \( T_{\Delta} \), representing the solution of problem (28), can be found at each time \( t \geq t_{\text{det}} \) from the solution of the corresponding elastic problem which should be got by substituting in (28) the operator \( I + N_{t_{\text{det}}} \) with the identity operator \( I \) and contains \( t \) as a parameter. If \( t \to +\infty \) we get that the tensor \( T_{\Delta}(r,+\infty) \) satisfies to the elastic problem with the loading \( t_{\text{a}}(\varphi,+\infty) \) and the shear modulus \( G(+\infty) \).

If the new law \( t_{\text{a}}(\varphi,t) \) of changing stresses at the internal surface of the originally considered erected vault leads to the exemption of this surface, i.e. to the centering removing, in other words if \( t_{a}(\varphi,+\infty) = 0 \), then \( t_{\text{a}}(\varphi,+\infty) = -t_{\text{a}}(\varphi,+\infty) \). In this case the tensor \( T(r,+\infty) = T(r,+\infty) + T_{\Delta}(r,+\infty) \) is a tensor of residual stresses in the completed vault after centering removing. Thus we have proved

**Proposition.** Let the stress tensors \( T_{\text{res}}(r) \) and \( T_{\infty}(r) \) define the long-term states that take place in a heavy aging viscoelastic vault under consideration after the process of its raising is completed and the centering is removed or, respectively, the vault is exposed on this centering for a long time. In the first case we have \( e_{\rho} \cdot T_{\text{res}} |_{\rho = a} = 0 \), and in the second case \( t_{\text{a}}(\varphi) = -e_{\rho} \cdot T_{\infty} |_{\rho = a} \) is the steady-state centering reaction. Let the stress tensor \( T_{\text{el}}(r) \) define the state of a weightless elastic vault that is built without residual stresses, has the same dimensions as the having been erected viscoelastic vault in question and the same elastic characteristics as this vault has after a long exposure. The elastic vault mentioned is loaded at its internal surface by the distributed loading \( t_{\text{a}}(\varphi) \), with the conditions of feet attaching being the same as for the erected vault. Then \( T_{\text{res}} = T_{\infty} - T_{\text{el}} \).

6. Conclusions

In the present paper a process of gradual raising of a heavy semi-circular vault on a smooth rigid base is considered. The raising is implemented by means of continuous attachment of
additional material layers to the vault external surface while its internal surface is supported by a rigid circular centering. The material used for the constructing possesses properties of creep and aging.

We have formulated a linear quasistatic problem of accreted solids mechanics that describes plane deforming of the vault under gravity action during its raising and after the raising stops but the completed vault is still being supported by the centering. This problem is transformed to a boundary value problem which mathematical form matches with that of the classical problem in elasticity theory. An analytical solution of the latter problem in series is built. By means of this solution the evolution of the stress-strain state in every point of the completed vault is exposed by time integrating procedure and by solving a Volterra integral equation of the second kind.

A proposition about structure of residual stresses in the completed vault after the centering removing is proved. According to the proposition these stresses can be determined as difference between the stresses corresponding to the accretion problem solved in the paper and stresses corresponding to a certain classical problem of elasticity theory with zero mass forces.

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