Recovering the mass profile and orbit anisotropy of mock dwarf galaxies with Schwarzschild modelling

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ABSTRACT
We present a new study concerning the application of the Schwarzschild orbit superposition method to model spherical galaxies. The method aims to recover the mass and the orbit anisotropy parameter profiles of the objects using measurements of positions and line-of-sight velocities usually available for resolved stellar populations of dwarf galaxies in the Local Group. To test the reliability of the method, we used different sets of mock data extracted from four numerical realizations of dark matter haloes. The models shared the same density profile but differed in anisotropy profiles, covering a wide range of possibilities, from constant to increasing and decreasing with radius. The tests were done in two steps, first assuming that the mass profile of the dwarf is known and employing the method to retrieve the anisotropy only, and then varying also the mass distribution. We used two kinds of data samples: unrealistically large ones based on over 270,000 particles from the numerical realizations and small ones matching the amount of data available for the Fornax dwarf. For the large data samples we recover both the mass and the anisotropy profiles with very high accuracy. For the realistically small ones we also find a reasonably good agreement between the fitted and the input anisotropies, however the total density profiles can be significantly biased as a result of their oversensitivity to the available data. Our results therefore provide convincing evidence in favour of the applicability of the Schwarzschild method to break the mass-anisotropy degeneracy in dwarf galaxies.

Key words: galaxies: dwarf – galaxies: fundamental parameters – galaxies: kinematics and dynamics – Local Group – dark matter

1 INTRODUCTION

Dwarf galaxies are believed to be the most dark matter dominated objects in the Universe with dark to baryonic mass ratios even of the order of hundreds (Mateo 1998, Gilmore et al. 2007) so they seem to be the best laboratory for studying this unexplored component of the Universe. For decades astronomers have been running simulations of the behaviour of dark matter (Navarro et al. 1995, Springel 2005, Diemand et al. 2008) in order to describe the structure formation, profiles of dark matter distribution, sizes and shapes of dark haloes and compare the resulting observables with the astronomical data to find the best model describing dark matter. However, we do not have at our disposal reliable tools even to precisely measure the most basic parameter of a dwarf galaxy, its mass. The most commonly used Jeans modelling (Binney & Tremaine 2008) based on fitting of the velocity dispersion is subject to the mass-anisotropy degeneracy (Binney & Mamon 1982), the degeneracy between the underlying mass profile and the anisotropy of orbits of the tracer particles, since the anisotropy profile is generally unknown. The degeneracy can be partially lifted by including kurtosis into the fit (Lokas 2002, Lokas et al. 2005, Richardson & Fairbairn 2013). The method however relies on the predefined form of the anisotropy profile. The standard assumption is then for the anisotropy to be constant with radius. This is much more restrictive than the range of possibilities that are found in simulations (Campbell et al. 2017, El-Badry et al. 2017) e.g. profiles that are monotonically growing or decreasing with radius, and interestingly, also profiles with a pronounced local maximum. A powerful tool to break the mass-anisotropy degeneracy is the application of stellar proper motions. Wilkinson et al. (2002) and Strigari et al. (2007) showed that 100-200 measurements are sufficient in order to lift the degeneracy in Jeans modelling method. Unfortunately, we still do not have such results at our disposal as the Space

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Interferometry Mission (SIM), on which these authors based their studies, has been cancelled. Proper motions of stars in the nearby dwarf galaxies could be also obtained with MICADO instrument (Tripple et al. 2010). However, in this case we may expect first measurements by 2030. Since, as pointed out by Majewski (2008), Gaia mission does not have capabilities necessary to derive proper motions of stars in even the nearest dwarf galaxies with high enough precision, the prospect of breaking the mass-anisotropy degeneracy in this way will not be realized in the near future.

In contrast, the growing availability of sensitive multiobject spectrographs on 6-8m class telescopes is making it possible to greatly increase the number of radial velocity measurements in nearby dwarf spheroidal galaxies. It is therefore worthwhile investigating how well both anisotropy and mass profiles can be recovered with much larger radial velocity samples than currently available.

One method which does not require the prior knowledge of the orbit anisotropy is the orbit superposition modelling, first introduced by Schwarzschild (1979) for constructing distribution functions for triaxial galaxies. It was first applied to modeling kinematical data in spherical galaxies by Richstone & Tremaine (1984) and has been developed since then in the studies of massive early type galaxies and bulges of spiral galaxies, in order to derive their mass distribution and mass-to-light ratios and/or to infer the existence of the black holes in the centres of ellipticals and measure their masses (van der Marel et al. 1998, Cretton et al. 1999, Gebhardt et al. 2003, Valluri et al. 2004, Thomas et al. 2004, Cretton & Emsellem 2004, Cappellari et al. 2006, van den Bosch & de Zeeuw 2010). The first proposed models described simplest, spherical objects (Richstone & Tremaine 1984, Rix et al. 1997) but complexity of studies increased with time (as better data and higher computational power were emerging) going through 3-integral axisymmetric models (van der Marel et al. 1998) up to triaxial (van den Bosch & de Zeeuw 2010).

The application of the method to dwarf galaxies has been attempted only recently. First of all, the methodology needs to be adapted because, in contrast with luminous ellipticals, dwarfs seem to be dominated by dark matter at all scales, its spatial distribution not necessarily following the distribution of the visible tracer (stars). The subject of the form of the dark matter density profile has been a matter of extensive study over the last few decades. It is still under debate whether these profiles should be modelled as cuspy Navarro-Frenk-White (NFW, Navarro et al. 1997) or Einasto (Ludlow et al. 2013) profiles emerging from cosmological, dark matter only simulations or rather by a variety of cored profiles resulting from simulations including baryonic physics (Governato et al. 2010). Moreover, Jeans modelling of dwarf spheroidal (dSph) galaxies tends to suggest that they do not share a universal dark matter profile (Walker et al. 2009). Therefore the determination of not only the total mass but also its distribution in dwarf galaxies is currently one of the hottest topics in galactic dynamics.

The Schwarzschild modelling method has been applied to dSphs of the Local Group (LG) in order to obtain density profiles of their dark matter haloes in the case of Fornax and Draco by Jardel & Gebhardt (2012) and Jardel et al. (2013) and independently by Breddels & Helmi (2013) for Fornax, Sculptor, Carina and Sextans. Unfortunately, the results are not conclusive. Whereas Breddels & Helmi find cuspy profiles to be favoured over cores for all galaxies in their sample (except for Sextans but in this case the fit was done for only two data bins), for Fornax Jardel & Gebhardt reject the NFW profile at a high confidence level. On the other hand, according to Jardel et al. Draco is embedded in an NFW-like halo.

The differences between modelling ellipticals/bulges and dwarfs of the LG also concern the types of data available. In the case of ellipticals/bulges we deal with the integrated light distribution and line-of-sight velocity profiles which need to be extracted from the integrated stellar spectra. In dwarfs we are able to resolve individual stars and measure their positions and line-of-sight velocities, which then require different treatment. Chanamé et al. (2008) developed a maximum-likelihood based version of the Schwarzschild method that allows the orbit libraries to fit these individual positions and velocities. Unfortunately, to date this method has not been applied extensively to real data (however see Breddels 2013). The two main approaches that have been applied to dwarf galaxies resort to binning the data in radius; one relies on using velocity moments (Breddels et al. 2013, Breddels & Helmi 2013) and the other on fitting the full line-of-sight velocity distribution (Jardel & Gebhardt 2012, Jardel et al. 2013). Both have some disadvantages: the former necessarily leads to the loss of some information while the latter has to struggle with large errors.

With so many conflicting results and discrepancies, the reliability of the Schwarzschild orbit superposition method needs to be tested on mock data. Such an experiment has been performed by Breddels et al. (2013) but only for one mock numerical realization of a Sculptor-like galaxy with an adopted anisotropy. In this work we therefore intend to investigate the reliability of the method trying to recover the mass profile of mock dark matter haloes (as a first approximation of a dwarf galaxy) for a larger variety of anisotropy profiles. Moreover, we examine the ability of our orbit superposition code to adequately recover the anisotropy profile as a result rather than an assumption of the modelling method. This will enable us to verify if the Schwarzschild modelling is truly independent of the intrinsic anisotropy and also to determine whether it is capable of recovering the velocity anisotropy profile, thereby breaking the mass-anisotropy degeneracy.

The paper is organized as follows. In section 2 we describe the numerical models used, in section 3 we introduce the Schwarzschild modelling scheme, in section 4 we carry out the recovery of the anisotropy and mass profiles for large data sets and in section 5 we apply the method to a data sample typical for a dwarf galaxy. We summarize our results in section 6 and discuss them in the light of the available literature and current state of knowledge in section 7.

2 MOCK DATA

2.1 Numerical realizations

In our study we use numerical realizations of dark matter haloes containing 10^6 particles and generated using the distribution function of Wojtak et al. (2008). The spherically
symmetric density profile of the haloes is given by the formula:

\[
\rho(r) = \begin{cases} 
\frac{\rho_0}{(r/r_s)(1+r/r_s)} & r < r_v \\
\frac{\rho_0}{(r/r_s)(1+r/r_s)^3} & r > r_v 
\end{cases}
\]

(1)

corresponding to the cuspy, cosmologically motivated NFW profile (with \(\rho(r) \propto r^{-1}\) at the centre and \(\propto r^{-3}\) at infinity) within the virial radius \(r_v\) and the steeper cut-off \(\rho(r) \propto r^{-6}\) beyond. The cut-off is necessary to ensure the finite mass of the halo.

We use models with the virial mass \(M_v = 10^8 M_\odot\) and concentration \(c = 20\), which translate to the following values used in eq. (1): \(\rho_0 = 1.77 \times 10^7 M_\odot\) kpc\(^{-3}\), \(r_v = 25.80\) kpc, \(r_s = 1.29\) kpc, \(N = 1.36 \times 10^4 M_\odot\) kpc\(^{-3}\) and \(r_e = 41.92\) kpc (see Lokas & Mamon 2001 for the discussion on the dependence of the NFW profile on the cosmological model in use). Consequently each dark matter particle has a mass of 1533.83 \(M_\odot\).

The models differ in the underlying orbit anisotropy profile, defined as:

\[
\beta(r) = 1 - \frac{\sigma_\phi^2(r) + \sigma_\theta^2(r)}{2\sigma_r^2(r)}
\]

(2)

where \(\sigma_r, \sigma_\theta, \sigma_\phi\) are the components of the velocity dispersion in the spherical coordinate system with the origin at the centre of the halo.

We use four models in total, two with anisotropy constant with radius, \(\beta = 0\) and \(\beta = 0.5\), and two with varying anisotropy: growing and decreasing from 0 (0.5) at the centre of the halo to 0.5 (0) at infinity, reaching an intermediate value \(\beta = 0.25\) at \(r_s\).

### 2.2 Tracer particles

In the observed dwarf galaxies stars are most probably distributed differently than dark matter. Dark matter haloes are believed to be more extended and in our models we describe them by the NFW profile, whereas the observed (projected) stellar profiles are best fitted with more concentrated profiles like Plummer (Plummer 1911), Sérsic (Sérsic 1968) or King (King 1962).

In order to test our method on more realistic data, we decided to select subsamples of particles following the Sérsic profile:

\[
I(R) = I_0 \exp[-(R/R_s)^{1/m}],
\]

(3)

where \(I_0\) is the normalization, \(R_s\) is the characteristic radius and \(m\) is the Sérsic index.

Introducing the stellar component into a dark matter-only models by taking subset of dark matter particles follows the practice used by other studies. Bullock & Johnston (2005) and Peñarrubia, Navarro & McGaugh (2008) suggested a method of selecting such subsamples in dynamical equilibrium. However, the method is relatively simple only for an isotropic halo. Therefore we applied another approach instead. In each halo we selected 273,078 particles reaching as far as 34 kpc given with the deprojected Sérsic profile (Lima, Gerbal & Marquez 1999):

\[
\nu(r) = \nu_0 \left(\frac{r}{R_s}\right)^{-p} \exp\left[-\left(\frac{r}{R_s}\right)^{1/m}\right]
\]

where:

\[
p = 1 - 0.6097/m + 0.05463/m^2
\]

(5)

with \(R_s = 0.4\) kpc and \(m = 1.6\). The normalization \(\nu_0\) was chosen so that the maximum number of particles could be used. Then, we evolved the haloes in isolation using the N-body code GADGET-2 (Springel 2005) following the selected particles, until equilibrium was reached.

In the following we will assume that the total mass of the stars is at first approximation negligible in comparison with the mass of the dark matter halo. The selected subsamples of particles remain dark matter particles and no particles (of any type) have been added to the systems. Therefore, our simulations do not contain stellar particles and denoting the subsamples as stars is just a convention, since their spacial distribution and kinematics can be identified as the distribution and kinematics of the massless tracer. Nevertheless, the marked particles’ physical mass still contributes to the total mass of the dark matter halo.

In Fig. 1 we compare the number density profiles of the selected tracer particles in red and all particles of the halo (green line). The profiles overlap at around 1 kpc as a result of the choice of normalization for the tracer profile so that the maximum number of particles could be used.

![Figure 1. The number density profiles of the tracer particles (red line) and all particles of the halo (green line). The profiles overlap at around 1 kpc as a result of the choice of normalization for the tracer profile so that the maximum number of particles could be used.](image)

As shown in Gajda et al. (2015) (where the models we use are labelled C1, C3, I2 and D) who evolved the haloes in isolation for 10 Gyr, the models with \(\beta = 0\) and growing anisotropy are stable and remain spherical till the end of the simulations. When considering our subsamples only, the particles achieved equilibrium, i.e. the moment when the profiles of density, radial velocity dispersion and anisotropy stopped changing, after approximately 6 Gyr but we let them evolve for another 2 Gyr and took for further analysis the outputs at 8 Gyr. However, the models with \(\beta = 0.5\) and decreasing anisotropy are subject to radial orbit instability and become significantly non-spherical in less than 1 Gyr. In those cases we took the outputs at 0.6 Gyr, keeping in mind that they are not in exact equilibrium. In all chosen outputs the total density profiles remained unchanged.

After the evolution in isolation the properties of the selected particles differ slightly from the initial values, how-
ever the general behaviour remains the same in each case. We compare the number density profiles in Fig. 2 and the anisotropy profiles calculated with eq. (2) in Fig. 3, where the values from the initial conditions are presented in red and final values in blue for each model separately. The thin dashed vertical lines indicate the adopted upper radial limit for the mock data and the outer radius of the orbit library, respectively.

Figure 2. The number density profiles of the tracer particles in the initial conditions (red lines) and final outputs used for further analysis (blue lines) as a function of the radius from the centre. The thin dashed and dashed-dotted lines indicate the adopted upper radial limit for the mock data and the outer radius of the orbit library, respectively.

Figure 3. The profiles of the anisotropy of the tracer particles in the initial conditions (red lines) and final outputs used for further analysis (blue lines) as a function of the radius from the centre. The thin dashed vertical lines indicate the adopted upper radial limit for the mock data. The asymptotes of the initial varying $\beta$ profiles are marked in green.

3 THE METHOD

In this section we present the steps of the Schwarzschild method and the parameters used for the modelling which need to be adjusted to the data, mainly their size constraining a number of spatial bins which can be used and the spatial distribution constraining the energy range of the orbit library.

Our approach is based on the original Fortran code by Valluri et al. (2004) developed as a complete tool to model elliptical, axisymmetric galaxies. The code generates initial conditions for the orbit library by deriving the multipole expansion of the potential from a given mass profile, integrates the orbits and stores their observables, reads the observational data and fits the constraints. However, the methodology for dwarfs differs so much that we were unable to apply the code in a straightforward way. Therefore, we have used only the first part of the original code which generates the initial conditions and we modified (simplified) it for use in the spherical case. The software necessary for the later stages of the modelling has been written especially for the purpose of this work in C++.

3.1 Orbit library

For the purpose of the application of the method we generated a library of 1 200 orbits sampling the energy and angular momentum spaces. We used 100 values of energy in units of the radius of the circular orbit sampled logarithmically and 12 values of the relative angular momentum $l = L/L_{\text{max}}$, where $L_{\text{max}}$ is the angular momentum of the circular orbit, sampled linearly within the open interval $l \in (0, 1)$ to avoid numerical errors. The initial conditions for the orbits were calculated under the assumption that each particle was placed at the apocentre of its orbit. The apocentres in the library fall between $r_{\text{in}} = 0.081 \text{kpc}$ which is smaller than the upper limit of the innermost bin of the constraints (see the next section) and $r_{\text{out}} = 24.656 \text{kpc}$, which is ~4.1 times larger than the outer boundary of the mock observations and contains over 99.9% of the tracer particles $\left(\frac{N(r_{\text{out}})}{N(r_{\text{inner}})} \geq 0.999\right)$.

The orbits have been integrated simultaneously in two groups using the N-body code GADGET-2, modified to accommodate a constant potential (by adding for each particle in each timestep accelerations calculated from the Gauss theorem for a given mass profile), for $t = 10 \text{ Gyr}$ (the inner 600 orbits) or $t = 100 \text{ Gyr}$ (the remaining ones) in order to cover at least a few full orbital periods even for the most extended orbits. In each case we saved 2001 outputs in equal timesteps. By definition, the library contains a set of orbits of test particles, i.e. the massless tracers of the underlying potential. We applied the N-body code instead of a standard numerical integration scheme because of its numerical convenience and speed. However, as the code requires positive masses of particles, we guaranteed that their gravitational interaction did not affect the resulting orbits by assigning a very small mass to the particles. We confirmed that it
was sufficient and the approach did not need to be changed. Storing the actual orbits and not only observables demands more disk space but allowed us to reuse the orbits and re-calculate the library with different spatial binning, saving computational time.

### 3.2 Extracting observables

In order to generate mock data sets, we observed each galaxy, i.e. the particles marked as stars, along an arbitrarily chosen axis, saving projected positions of the particles in terms of their distances from the centre and the line-of-sight velocities, setting the outer maximum projected radius of the data to \( R = 6 \) kpc to imitate the distribution of stars in a real dwarf galaxy. We binned the data in 30 radial bins spaced linearly. In each radial bin we derived the proper moments of velocity: the second \((m_2)\), third \((m_3)\) and fourth \((m_4)\), calculated with estimators based on the sample of \( N \) line-of-sight velocity measurements \( v_i \) (Lokas & Mamon 2003):

\[
m_{n,l} = \frac{1}{N} \sum_{i=0}^{N} (v_i^n - \bar{v}_l)^n
\]

where

\[
\bar{v}_l = \frac{1}{N} \sum_{i=0}^{N} v_i^l
\]

and \( l \) labels the radial bins. We present the resulting moments in the three panels of Fig. 4 with colours denoting different anisotropy models: red for the isotropic case \( \beta = 0 \), green for \( \beta = 0.5 \), blue for the growing anisotropy and magenta for the decreasing one. The noise, clearly visible for \( m_3 \), is the consequence of the dense spatial binning.

The kinematics of a galaxy can be also expressed in the terms of the Gauss-Hermite moments \( h_i \) (van der Marel & Franx 1993, Gerhard 1993). We have tested their application on the original data i.e. un-evolved dark matter haloes, and presented the results in Kowalczyk et al. (2016). They proved to be useful in recovering the anisotropy profiles but at the same time demanded large amounts of data to derive the moments correctly. As the studies of dwarf galaxies struggle with rather limited data samples, we have abandoned this approach.

In the next step we need to obtain observables for the orbits from the library. As the orbits in the spherical potential are coplanar we randomly rotated each orbit 100 000 times around two axes of the simulation box and combined them to mimic the symmetry. We have noticed that the value of the resulting \( \chi^2 \) (see next section) depends on this random choice, i.e. on a particular set of rotation angles, therefore revealing a numerical defect of the method, as a finite set of rotation angles is not sufficient to achieve the needed level of symmetry. In a set of tests we have established that the optimal number of rotations is 100 000 for which the \( \chi^2 \) varies by less than 0.1 between the selections and the computational time necessary for rotations is reasonably short.

The orbits have been observed along an arbitrarily chosen axis and their observables have been stored on the same grid as the mock data. The velocity moments have been calculated following eq. (6) and (7).

For the purpose of recovering the anisotropy we have also stored the three components of the velocity dispersion in spherical coordinates as a function of the deprojected radius in 30 linearly spaced bins in the range \( r \in [0, 6] \) kpc for the mock data and for the orbits.

### 3.3 Fitting of constraints

The Schwarzschild method is based on the assumption that the observed kinematics of a galaxy (or in our case of marked particles of a dark matter halo) can be reproduced as a function of the deprojected radius in 30 linearly spaced bins in the range \( r \in [0, 6] \) kpc for the mock data and for the orbits.

![Figure 4. The values of the 2nd, 3rd, and 4th velocity moment (top to bottom panels, respectively) for the four models: \( \beta = 0 \) (red), \( \beta = 0.5 \) (green), increasing \( \beta \) (blue) and decreasing \( \beta \) (magenta) measured using all tracer particles.](image-url)
orbital weights. The general function to minimize is:

\[
\chi^2 = \sum_t \sum_{n} \left( \frac{M_{\text{obs}}^{k,l} - \sum_k \gamma_k M_{l}^{k,b} M_{n}^{k,l}}{\Delta(M_{l}^{k,b} M_{n}^{k,l})} \right)^2 \tag{8}
\]

under the constraints that for each \(k\) and each \(l\):

\[
\begin{cases}
|M_{\text{obs}}^{k,l} - \sum_k \gamma_k M_{l}^{k,b}| \leq \Delta M_{l}^{k,b} \\
\gamma_k \geq 0
\end{cases}
\]

where \(M_{\text{obs}}^{k,l}\) and \(M_{l}^{k,b}\) are the fractions of the projected mass of the tracer contained within \(l^{\text{th}}\) bin for \(k^{\text{th}}\) orbit or from the observations and \(m_{l}^{k,b}\) are \(n^{\text{th}}\) proper moments. \(\Delta\) denotes the measurement uncertainty associated with a given parameter. The velocity moments are weighted with the projected masses and to derive the errors we treat both quantities as independent.

We assume that tracer particles are massless and orbit in the potential of their dark matter halo. However, if the mass-to-light ratio of stars is constant with radius, then \(M_{\text{obs}}^{k,l} \equiv N_{l}^{k,\text{obs}}\), where \(N_{l}^{k,\text{obs}}\) is a fraction of tracer particles contained within \(l^{\text{th}}\) bin. Therefore in this study by ‘projected mass’ we mean \(N_{l}^{k,\text{obs}}\).

As it has been already pointed out by Breddels et al. (2013), the 4th velocity moment is not independent of the 2nd moment. Therefore, the kurtosis \((\kappa = m_4/m_2^2)\), which is not correlated with the 2nd moment, is preferred as a kinematic parameter in dynamical studies. However, the kurtosis cannot be used as a constraint for the Schwarzschild modelling, as it would not be linear in the orbital weights. Consequently, we proceed neglecting the possible correlations and using the proper 4th moment \(m_4\).

We used an additional constraint on the sum of the weights. As the weights have the physical meaning of the amount of mass assigned to the corresponding orbits, they should sum up to unity as long as an orbit library covers the whole (or typically in numerical studies \(\geq 99.9\%\), which holds for our library) deprojected mass of the tracer:

\[
\sum_k \gamma_k = 1. \tag{10}
\]

The minimization of the objective function under the equality and inequality constraints has been executed using quadratic programming as implemented in the CGAL\(^1\) library (The Computational Geometry Algorithms Library, The CGAL Project 2015).

We calculate the resulting anisotropy \(\beta\) in the \(l^{\text{th}}\) bin by assuming that:

\[
\beta_l = 1 - \frac{\sum_k \gamma_k M_{lD,l}^k (\sigma_{r,i}^k)^2 + \sum_k \gamma_k M_{lD,l}^k (\sigma_{v,i}^k)^2}{2 \sum_k \gamma_k M_{lD,l}^k (\sigma_{r,i}^k)^2} \tag{11}
\]

where \(\sigma_{r,i}^k, \sigma_{v,i}^k\) are the components of the velocity dispersion in the spherical coordinate system for the \(k^{\text{th}}\) orbit calculated in the \(l^{\text{th}}\) spatial bin.

4 LARGE DATA SAMPLES

In this section we present the application of our method to the data derived from all available stellar particles (see section 2.2) contained within a cylinder of radius \(R = 6\) kpc, corresponding to \((2.11-2.24) \times 10^5\) particles, depending on the model.

4.1 Known mass profile

First, we check how reliably we can recover the anisotropy profile alone. For this purpose we assumed that the correct mass profile (see section 2) was known and performed the fitting of the mock data to the orbit library calculated in the potential generated by this distribution. In each of the 30 radial bins we applied 6 constraints following eq. (9) and (8): the rigid lower and upper boundaries for the mass projected along the line of sight and the \(\chi^2\) fit for the velocity moments 1-4 given with eq. (6) and (7). We assumed Poissonian errors for the projected masses and the theoretical sampling errors (Kendall & Stuart 1977, Harding et al. 2014) of standard deviation:

\[
\Delta \sigma = \frac{\sigma}{\sqrt{2(n-1)}} \tag{12}
\]

where \(n\) is the size of a sample, skewness:

\[
\Delta \gamma = \sqrt{\frac{6n(n-1)}{(n-2)(n+1)(n+3)}} \tag{13}
\]

where \(\gamma = m_3/m_2^{1/2}\) and kurtosis:

\[
\Delta \kappa = 2\Delta \gamma \sqrt{\frac{n^2-1}{(n-3)(n+5)}} \tag{14}
\]

propagated back to the velocity moments \(m_2, m_3\) and \(m_4\). The formulae are derived under the assumption that the parent distribution is normal. A priori it may not be true for our models. However, as the data samples we use are large, the crude estimates of sampling errors are sufficient in order to test the method. We present a precise study of sampling errors and their application in section 5.

By definition, \(m_2(R) \equiv 0\) up to numerical precision (typically \(10^{-14} - 10^{-17}\)) so we applied a fixed value of \(\Delta m_2 = 0.001\) everywhere.

The results are presented with magenta curves in Fig. 5 together with the true values (in red) calculated from the full 6D information about the particles. The four panels refer to the models with different anisotropy. The accuracy of the recovered profiles is remarkable, with only some noise.

We also need to comment on the missing results for the innermost bins in Fig. 5 and the following figures presenting recovered anisotropies. In our opinion the innermost bin is underconstrained, having only one neighbour and many orbits contained entirely within one or two bins (which is not the case for the outermost bin) so one cannot rely on the results in this bin. This effect manifests itself notably for the small data samples leading to enormous (when compared to other bins) and highly non-Gaussian errors. A problem of the innermost bin has been already reported by Breddels et al. (2013), however with a different justification.

4.2 Unknown mass profile

We also examined the possibility of recovering the underlying mass profile for the four models. We constructed a grid of profiles given with eq. (1) with the values of the
The strength of the Schwarzschild modelling with green dots.

Figure 5. The anisotropy parameter profiles for all particles from the simulations for the four models. In red we present the values of direct measurements from the full data and in magenta and blue the fits obtained with the assumption of the correct mass profile and for the best-fitting mass profile, respectively. The shaded regions correspond to the extreme values for the mass profiles recovered within the 1σ confidence level.

Figure 6. The maps of the χ^2 values relative to the minima of fitted surfaces for four anisotropy models on the grids of different mass profiles. The global minima are marked with yellow dots and in each case lie within the 1σ regions. In order to avoid calculating libraries for the minima of the fitted surfaces which do not correspond to any models on the adopted mass grid, we identified the models on the grid closest to the global minima along the contours of equal Δχ^2 plotted with white curves.

virial mass and concentration spaced linearly in the ranges: \( M_\nu \in [0.2, 3] \times 10^8 M_\odot \) and \( \epsilon \in [8, 27] \). The parameters of cut-off in the mass profile, i.e. \( N \) and \( r_c \), were in each case adopted so that the profile and its first derivative at the distance of the virial radius were continuous. For each profile we integrated the library of orbits as explained in section 3.1.

Similarly to the procedure described in the previous section, for each model we used the projected mass and the velocity moments 1-4 in 30 spatial bins as constraints to fit the orbit library. The absolute values of the χ^2 function, eq. (8), have been saved and we compare them in Fig. 6, where each panel corresponds to a different anisotropy model. The logarithmic scale colour represents the differences of the values of χ^2 relative to the minimum (\( \chi^2_{\text{min}} \)) of the fitted two-dimensional surfaces of the 4th order (\( \propto M_\nu^2 \epsilon^2 \)). The minima are marked with yellow dots. The white lines indicate the contours of equal \( \Delta \chi^2 = 2.3, 6.17 \), and 11.8 corresponding to 1, 2, and 3σ confidence levels for two degrees of freedom (Press et al. 1992) also based on the fitted surfaces. The concentration is constrained much more poorly than the virial mass as it is a very sensitive parameter.

We have found the fitting of a surface necessary in order to derive a global minimum and contours of equal \( \Delta \chi^2 \) as the Schwarzschild method is severely influenced by numerical effects and therefore one should consider trends rather than particular values. The discretization appears on many levels and cannot be avoided: a smooth distribution function of a galaxy is represented by a finite set of deltas (orbits), continuous orbits are represented by a finite set of timesteps and the spherical symmetry is represented by a finite set of rotations of the orbits.

The true values of the mass profiles are marked with red dots and in each case lie within the 1σ regions. In order to avoid calculating libraries for the minima of the fitted surfaces which do not correspond to any models on the adopted mass grid, we identified the models on the grid closest to the global minima along the contours of equal \( \Delta \chi^2 \) as the best-fitting models and marked them in Fig. 6 with green dots. For the galaxy with the decreasing anisotropy profile the best-fitting models are plotted in Fig. 5 in blue.

Using the 1σ regions of the recovered mass profiles we estimated errors on the values of the recovered anisotropy by taking in each bin the extreme values among the mass profiles within \( \Delta \chi^2 = 2.3 \). The results are shown in Fig. 5 as the shaded regions, following well the values and shapes of the true anisotropies.

Despite the mass-anisotropy degeneracy, with our method we have recovered the proper values of both the anisotropy and the mass profile: \( M_\nu = 10^8 M_\odot \) and \( \epsilon = 20 \) for each model with high accuracy, independently of the underlying anisotropy.

The anisotropy can be equally well recovered using only the 2nd and 4th velocity moments. However, the addition of the 1st and 3rd moments to the fit, which is not typical, influenced the recovery of the mass profile, providing an estimate with a higher confidence. As it will prove to be of great importance in the next section, we decided to use all the moments.

5 SMALL DATA SAMPLES

Section 4 shows the strength of the Schwarzschild modelling in breaking the mass-anisotropy degeneracy. The weak point
of this argument is the amount of data we used, which is impossible to achieve in the case of dwarf galaxies of the LG. In order to test our method on realistic samples, for each of the galaxy models we randomly chose 100 000 stellar particles with positions (in observations those are stars with only photometry measured) and 2 500 particles with positions and line-of-sight velocity (corresponding to spectroscopic data) contained within the projected radius of 6 kpc and binned them in 10 radial bins spaced linearly.

It is typical for the treatment of small data samples to adjust binning to the data by fixing the number of stars in the bin. However, binning is then based on a particular data sample, not the parent distribution. Therefore, when fixing a number of particles we impose sampling errors on borders of the adopted binning, as different random samples would result in different spatial partitions. Since this effect is impossible to correct for in our method, we decided to keep the predefined binning fixed in radius.

5.1 Sampling errors

For observational data the sampling errors dominate over the measurement errors of line-of-sight velocities for single stars and are therefore the main source of uncertainties imposed on the velocity moments. In order to estimate them properly we ran Monte Carlo (MC) simulations deriving the sampling errors for various parameters (β, m2, m3, m4) for the adopted spatial binning and for each halo model as a function of a number of particles in each bin in the range of sample sizes from 40 to 550 particles with the step of 30. Between the nodes of such a dense grid the errors can be interpolated linearly. This allowed us to apply our method regardless of the size of the current data sample.

We took advantage of it when running the next set of tests. For each halo we randomly selected our data samples, assigned the errors and fitted the orbit library, repeating the procedure 10 000 times. The resulting distributions of the anisotropy profiles were fitted with Gaussians in order to derive the mean values and the 1σ deviations, in each spatial bin separately. The results for all models are shown in Fig. 7 in cyan, dark blue and magenta, depending on the number of random small samples and three-dimensional velocity vectors, i.e. variations in the anisotropy calculated from full deprojected positions and line-of-sight velocity (corresponding to spectroscopic data) contained within the projected radius of 6 kpc and binned them in 10 radial bins spaced linearly.

5.2 Examples of data modelling

In this section we present the results of modelling four sets of mock data for 100 000 (positions) and 2 500 (velocities) randomly selected particles, one for each model. Fig. 8 shows the kinematics of the samples as points with 1σ deviations for comparison. The profiles of the anisotropy are shown in Fig. 9 where the red points correspond to true values from the data sample and blue ones to the recovered anisotropy, both with the errors calculated in section 5.1. The smooth cyan curves present the values for all stellar particles from the simulations for comparison.

5.3 Recovering the mass profile

As for the large data samples, we also studied the reliability of recovering the mass profile for our small data samples, fitting libraries of 1200 orbits integrated in the potentials generated by the mass profiles described in section 4.2. We present the resulting colour maps of $\Delta \chi^2 = \chi^2 - \chi^2_{\text{min}}$ as a function of virial mass and concentration in Fig. 10. As it has been done in section 4.2, we derived the minima and 1, 2, and 3σ confidence levels by fitting two-dimensional surfaces to the $\chi^2$ maps. For the stable models ($\beta = 0$ and $\beta = f_1(r)$, see section 2.2) the virial masses are overestimated whereas the concentrations are underestimated, covering the true profiles within 2σ. This suggests that the particles ‘feel’ a slightly different potential.
For $\beta = 0$ we have confirmed that this is not an outlier, i.e. an unfortunate random sample, in a test in which we studied 10000 different random samples (as for the sampling errors of recovered anisotropy, section 5.1), fitting all the orbit libraries and calculating the mean $\chi^2$ values for each library. This behaviour might be a consequence of our choice of the outer radius of the data sets as the anisotropy grows rapidly outside it (see Fig. 3). The particles which are in large physical distance from the centre still enter our calculations since their projected distances are smaller so that the small samples are contaminated by the particles on radial orbits. Higher values of line-of-sight velocities cause the line-of-sight velocity dispersion to grow, which (under the assumption of the dynamical equilibrium) leads to overestimation of the total mass. Unfortunately, those particles also affect the values of sampling errors, enlarging them. It was not the case for the large samples as ‘contaminants’ were outnumbered by ‘well-behaved’ particles and the errors were calculated analytically. For the unstable models the situations is even worse. Larger sampling errors result in poorly constrained density profile, spanning large area of our grid.

As the $1\sigma$ regions are large and may not be very accurate, we decided not to identify the minima as the best-fitting models. Instead, in Fig. 11 we present only the ranges (shaded regions) of the values of anisotropy spanned by the results for the density profiles within $1\sigma$. The true values of anisotropy are presented with cyan and red lines for all particles and small samples, respectively. The blue lines correspond to the results for the true mass profiles for comparison.

Despite the wide ranges of the similarly plausible density profiles, the derived anisotropy intervals are not much larger than the sampling errors for the known mass distribution. On average the deviations are larger by 35% for the stable models (with $\beta = 0$ and with the growing profile), 70% for $\beta = 0.5$ and 132% for the model with decreasing anisotropy. Nevertheless the intervals include the correct values and follow the general behaviour of the anisotropy profiles. Our approach is simplistic and does not provide a full picture as the recovered anisotropy for each mass profile is additionally subject to the sampling errors as presented in section 5.1.

6 SUMMARY

We have presented a study aimed at determining the efficiency of recovering the anisotropy and density profiles by the application of the Schwarzschild modelling method to a set of four dwarf galaxies obtained from the numerical realizations of NFW dark matter haloes by marking particles described with a Sérsic profile and following their evolution in isolated haloes in order to achieve equilibrium. The models shared the same spherically symmetric density profile but differed in the orbit anisotropy, covering a wide class of possible profiles and therefore allowing for a thorough
test of the scheme. We have tested in total four models of anisotropy, two constant with radius $\beta = 0$ (isotropic model) and $\beta = 0.5$, and two with anisotropy varying with radius, one growing and one decreasing.

We performed our tests applying two different approaches, and in addition using two types of samples, which we called large and small. The large sample contained over 211 000 particles within the projected radius of 6 kpc used as the outer boundary of the mock data. As the small samples we used subsamples of the large ones by randomly choosing 100 000 particles with positions and 2 500 particles with positions and line-of-sight velocities imitating the best data samples currently available for dwarf galaxies of the LG.

First, we assumed we knew the density profile exactly and performed the fitting of the observables in order to retrieve only the anisotropy profile. Our results for the large samples show that the anisotropy can be recovered with very high accuracy independently of its profile. We have demonstrated that also for the small samples our method provides interesting results. We carried out a set of MC simulations in order to determine the sampling errors imposed on the recovered values of the anisotropy, deriving the mean value over the profiles and the radial bins of $\sigma_\beta = 0.2$, only $\sim 2\times$ larger than the mean sampling error for the anisotropy. Such small errors enable us to clearly distinguish between the different models of anisotropy we used, proving the strength of the Schwarzschild method in this respect.

In the second approach we tested how precisely we can recover both the mass and anisotropy profiles. We assumed that the profile was given by the NFW formula with a cut-off at the distance of the virial radius and we constructed a grid of orbit libraries by varying the virial mass and the concentration. For the large data samples we have recovered the true mass profile for each halo model within the confidence level of 1$\sigma$ whereas for the small samples the parameters of the density profiles were strongly degenerated, resulting in extended regions of possible values. However, the correct values were included within at least 2$\sigma$ confidence regions.

Finally we have calculated the uncertainties associated with the anisotropy and coming from the uncertainty of the recovered mass profile. They are not much larger (by only 35% for the two reliable models) than the sampling errors derived from our MC simulations demonstrating that the unknown mass distribution affects anisotropy similarly to the limited amount of data.

7 DISCUSSION

The attempts to recover the anisotropy for dwarf galaxies with Schwarzschild modelling were already made for Fornax dSph (Jardel & Gebhardt 2012) and Draco dSph (Jardel et al. 2013) but without a clear demonstration that the undertaken procedures actually work. Here, we have filled this gap by showing that the anisotropy, regardless of its profile, can indeed be recovered by this method.

For the purpose of simplicity we have tested the method on numerical realizations of dark matter haloes only, therefore neglecting the stellar component. However, as dwarf galaxies are believed to be highly dark matter dominated, at first approximation we may assume that the influence of stars on the dynamics of the system is in fact negligible and
stars move in the potential generated by the distribution of dark matter. However, we need to bear in mind that as a result we obtain the total mass profile in which the orbit library has been integrated and the anisotropy profile of the tracer. We plan to implement the stellar mass to the fit in future extensions of our models by quantifying total mass in terms of the mass-to-light ratio varying with radius $\Upsilon(r)$.

Yet another complication may arise from the stellar mass-to-light ratio $\Upsilon$, varying with radius. However, as there is no strong evidence for its gradients in dwarfs (even for the ones with multiple stellar populations) it is typical to assume that the parameter is constant. It has been derived for many galaxies of the LG (Mateo 1998).

Breddels et al. (2013) did similar work to the one presented here, testing the Schwarzschild modelling on the mock Sculptor dSph with anisotropy assumed to be constant with radius, $\beta = -1$. These authors obtained good estimates of the mass profile, which we were not able to reach for our haloes, so we conclude that the parametrization of the mass profile may play a role in recovering precise values as the stars, concentrated at the centre of the dark matter halo, do not feel the mass distribution at the virial radius. After reparametrization of the density profiles, we found that the mass contained within 6 kpc ($M_{6\text{kpc}}$) was constrained much better (overestimated by no more than 50%) but the characteristic radius of the NFW profile ($r_s$) could be overestimated as much as 4 times for the unstable models and small data samples. Also the underlying anisotropy profile (tangential vs. our radial) may affect the quality of the mass profile recovery.

As the final remark we would like to comment on two tools often applied to the orbit superposition method in order to enforce smoother, more physical distribution function. The first tool is the regularization (van der Marel et al. 1998, Valluri et al. 2004), which imposes a penalty term restraining the values of the weights of the consecutive orbits (in the energy, angular momentum or both) so that they do not differ too much. The expression ‘too much’ is not precise and the strength of the regularization is a moot point. The regularization worsens the quality of the fit and for example Rix et al. (1997) imposed a constraint on the resulting $\chi^2$ value such that it was not to be changed by the regularization by more than $\Delta \chi^2 < 1$ when compared with the non-regularized case. Breddels et al. (2013) used an arbitrary value of the regularization strength which according to them worked well. In our opinion such approaches do not affect the distribution function of the system sufficiently to classify it as smooth while much higher values of regularization strength result in diminishing the role of the observational constraints and cause the regularization to dominate. Therefore, following the example of Yildirim et al. (2015), we have decided not to apply the regularization at all.

The other tool is the dithering (Rix et al. 1997, Breddels et al. 2013) in which one orbit in a library is constructed as a compound of a few (typically 5-8) orbits with close values of energy and angular momentum. It might be treated as a kind of strong regularization on the subgrid level, since ‘suborbits’ are assigned equal weights, but it is much more quantitative and we do not discard the possibility of including it in our modelling in the future.

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