A Note on Meromorphic Functions Associated With Bessel Function Defined by Hilbert Space Operator

1 Introduction

Let Σ be denote the class of function $u(z)$ of the form

$$u(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_{\eta}z^\eta, \eta \in \mathbb{N} = \{1, 2, 3, \cdots\}.$$ (1)

Which are analytic in the punctured unit disc $U^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$. Analytically a function $u \in \Sigma$ given by (1) is said to be meromorphically starlike of order $\alpha$ if it satisfies the following:

$$Re \left\{ - \left( \frac{zu'(z)}{u(z)} \right) \right\} > \alpha, \ (z \in \mathbb{U}) \ for some \ \alpha (0 \leq \alpha < 1).$$

We say that $u$ is in the class $\Sigma'(a)$ of such functions. Similarly, a function $u \in \Sigma$ given by (1) is said to be meromorphically convex of order $a$ if it satisfies the following:

$$Re \left\{ - \left( 1 + \frac{zu''(z)}{u'(z)} \right) \right\} > \alpha, \ (z \in \mathbb{U}) \ for some \ \alpha (0 \leq \alpha < 1).$$

Keywords: Meromorphic functions, Bessel Functions, Coefficient estimates, Hadamard product, Hilbert space operators

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Abstract: In this paper, we introduce and study a new subclass of meromorphic functions associated with a certain differential operator on Hilbert space. For this class, we obtain several properties like the coefficient inequality, growth and distortion theorem, radius of close-to-convexity, starlikeness and meromorphically convexity and integral transforms. Further, it is shown that this class is closed under convex linear combinations.

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We say that \( u \) is in the class \( \Sigma_k(\alpha) \) of such functions. For a function \( u \in \Sigma \) is given by (1) is said to be meromorphically close to convex of order \( \beta \) and \( \alpha \) if there exists a function \( v \in \Sigma^*(\alpha) \) such that
\[
\Re \left\{ -\left( \frac{zu'(z)}{v(z)} \right) \right\} > \alpha, \quad (0 \leq \alpha < 1, \ 0 \leq \beta < 1, \ z \in U).
\]

We say that \( u \) is in the class \( K(\beta, \alpha) \). The class \( \Sigma^*(\alpha) \) and various other subclasses of \( \Sigma \) having been studied rather extensively by Clunie [3], Miller [8], Pommerenke [11], Royster [12], Akgul [1, 2], Venkateswarlu [13], Sakar and Guney [9, 10] et al. In recent years, many authors investigated the subclass of meromorphic functions with positive coefficient Juneja and Reddy [7] class \( \Sigma_p \) functions of the form
\[
u(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_\eta z^\eta, \ (a_\eta \geq 0).
\]
(2)

Which are regular and univalent in \( U \), the function in this class are said to be meromorphic functions with positive coefficients.

For functions \( u \in \Sigma_p \) given by (1) and \( v \in \Sigma_p \) given by
\[
v(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} b_\eta z^\eta, \ (b_\eta \geq 0).
\]

We define the Hadamard product (or convolution) of \( u \) and \( v \) by
\[
(u \ast v) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_\eta b_\eta z^\eta.
\]

We recall here the generalized Bessel function of the first kind of order \( \gamma \) (see[5]), denote by
\[
w(z) = \sum_{\eta=0}^{\infty} \frac{(-c)^\eta}{\eta!} \Gamma(\gamma + \eta + \frac{b+1}{2}) \left( \frac{z}{2} \right)^{2\eta+\gamma}, \ (z \in U).
\]

(Where \( \Gamma \) stands for the Gamma Euler function) which is the particular solution of the second-order linear homogeneous differential equation (see, for details, [14])
\[
z^2 w''(z) + bw'w(z) + [cz^2 - \gamma^2 + (1 - b)\gamma]w(z) = 0, \text{ where, } \gamma, \ b \in \mathbb{C}.
\]

We introduce the function \( \psi \) defined, in terms of the generalized Bessel function \( w \) by
\[
\psi(z) = 2^{\gamma} \frac{w(z)}{\Gamma(\gamma + \frac{b+1}{2})} z^{-(1+\frac{b}{2})} w(\sqrt{z}).
\]

By using the well-known Pochhammer symbol \( (x)_\mu \) defined for \( x \in \mathbb{C} \) and in terms of the Euler gamma function by
\[
(x)_\mu = \frac{\Gamma(x + \mu)}{\Gamma(x)} = \begin{cases} 1, & \mu = 0 \\ x(x+1)(x+2) \cdots (x+\eta-1), & \mu = \eta \in \mathbb{N} \end{cases}
\]

We obtain the following series representation for the function \( \psi(z) \)
\[
\psi(z) = \frac{1}{z} + \sum_{\eta=0}^{\infty} \frac{(-c)^{\eta+1}}{4^{\eta+1}(\eta+1)!} \omega^{\eta+1} z^\eta, \text{ where } \omega = \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0 = \{0, -1, -2, \cdots \}.\]
Corresponding to the function $\psi$ defined the Bessel operator $S^c_\lambda$ by the following Hadamard product
\[
S^c_\lambda u(z) = (\psi * u)(z) = \frac{1}{z} + \sum_{\eta=0}^{\infty} \frac{(\frac{-c}{z})^\eta}{(\eta+1)!} a_\eta z^\eta
\]

where $\phi(\eta, \lambda, c) = \frac{(-c)^\eta}{(\eta+1)!}$.

Let $H$ be Hilbert space on the complex field and $L(H)$ denote the algebra of all bounded linear operators on $H$, for a complex valued function $u$ analytic in a domain $E$ of the complex plane containing the spectrum $\sigma(T)$ of the bounded linear operator $T$. Let $u(T)$ denote the operator on $H$ defined by the Riesz-Dunford integral [4]
\[
u(T) = \frac{1}{2\pi i} \int (zI - T)^{-1} u(z) \, dz.
\]

Where $I$ is the identity operator on $H$ and $\mathbb{C}$ is positively oriented simple closed Rectifiable closed contour containing the spectrum $\sigma(T)$ in the interior domain [4]. The operator $u(T)$ can also be defined by the following series
\[
u(T) = \sum_{\eta=0}^{\infty} \frac{\nu^\eta(0)}{\eta!} T^\eta.
\]

This converges in the norm topology. The class of all functions $u \in \Sigma, a_\eta \geq 0$ is defined by $\Sigma_p$. The object of the present paper is to investigate the following subclass $\Sigma_p$ associated with the differential operator $S^c_\lambda u(z)$.

**Definition 1.1.** For $0 \leq \beta < 1$ and $0 \leq \alpha < 1$, a function $u \in \Sigma_p$ given by (1) is in the class $\sigma_p(\alpha, \beta, \lambda, c, T)$ if
\[
\|T(S^c_\lambda u(T)) - ((\beta - 1)S^c_\lambda u(T) + \beta T(S^c_\lambda u(T))')\|
\leq \|T(S^c_\lambda u(T))' + (1 - 2\alpha)((\beta - 1)S^c_\lambda u(T) + \beta T(S^c_\lambda u(T))')\|.
\]

The main object of the paper is to study usual properties of the geometric function theory such as coefficients bounds, growth and distortion properties, a radius of convexity, convex linear combination, convolution properties, integral operators and $\delta$-neighborhoods for the class $\sigma_p(\alpha, \beta, \lambda, c, T)$.

### 2 Coefficient Bounds

We first give a characterization of the class $\sigma_p(\alpha, \beta, \lambda, c, T)$ by finding necessary and sufficient condition for a functions in the class. This characterization implies coefficient estimates.

**Theorem 2.1.** A function $u \in \Sigma_p$ given by (2) is in the class $\sigma_p(\alpha, \beta, \lambda, c, T)$ for all contraction $T$ with $T \neq 0$ if and only if
\[
\sum_{\eta=1}^{\infty} (\eta + \alpha - \alpha \beta(\eta + 1)) \phi(\eta, \lambda, c) a_\eta \leq (1 - \alpha).
\]

The result is sharp for the function
\[
u(z) = \frac{1}{z} + \frac{1 - \alpha}{(\eta + \alpha - \alpha \beta(\eta + 1)) \phi(\eta, \lambda, c)} z^\eta, (\eta \geq 1).
\]
Proof. Suppose that (4) is true for $0 \leq \beta < 1$ and $0 \leq \alpha < 1$. Then

$$
\| T(S^\lambda_1 u(T))' - ((\beta - 1)S^\lambda_1 u(T) + \beta T(S^\lambda_1 u(T))') \|
- \| T(S^\lambda_1 u(T))' + (1 - 2\alpha)((\beta - 1)S^\lambda_1 u(T) + \beta T(S^\lambda_1 u(T))') \|
\leq \left( \sum_{\eta = 1}^{\infty} |\eta + 1|/(1 - \beta)\phi(\eta, \lambda, c)a_\eta T^{\eta + 1} \right) - 2(1 - \alpha)T^{-1} \sum_{\eta = 1}^{\infty} |\eta + 1 - 2\alpha(\beta - 1 + \beta\eta)|\phi(\eta, \lambda, c)a_\eta T^{\eta + 1}
\leq 2(1 - \alpha) - 2(1 - \alpha) = 0, by using (4)
$$

and so $u \in \Sigma_\rho$ is in the class $\sigma_\rho(\alpha, \beta, \lambda, c, T)$. Conversely suppose that $u \in \sigma_\rho(\alpha, \beta, \lambda, c, T)$ satisfies the coefficients inequality (4). Since $u \in \sigma_\rho(\alpha, \beta, \lambda, c, T)$ then

$$
\| T(S^\lambda_1 u(T))' - ((\beta - 1)S^\lambda_1 u(T) + \beta T(S^\lambda_1 u(T))') \| < \| T(S^\lambda_1 u(T))' + (1 - 2\alpha)((\beta - 1)S^\lambda_1 u(T) + \beta T(S^\lambda_1 u(T))') \|
$$

From this inequality, it is obtained that

$$
\left( \sum_{\eta = 1}^{\infty} |\eta + 1|/(1 - \beta)\phi(\eta, \lambda, c)a_\eta T^{\eta + 1} \right) - 2(1 - \alpha) - \sum_{\eta = 1}^{\infty} |\eta + 1 - 2\alpha(\beta - 1 + \beta\eta)|\phi(\eta, \lambda, c)a_\eta T^{\eta + 1} < 0.
$$

By choosing $T = rI(0 < r < 1)$ in above inequality, we get

$$
\sum_{\eta = 1}^{\infty} |\eta + 1|/(1 - \beta)\phi(\eta, \lambda, c)a_\eta r^{\eta + 1} < 2(1 - \alpha) - \sum_{\eta = 1}^{\infty} |\eta + 1 - 2\alpha(\beta - 1 + \beta\eta)|\phi(\eta, \lambda, c)a_\eta r^{\eta + 1}.
$$

Letting $r \to 1$ in the above inequality, we obtain the assertion (4). This completes the proof of the theorem.

From Theorem 4, we have the following result.

**Corollary 2.2.** If a function $u(z) \in \Sigma_\rho$ given by (2) is in the class $\sigma_\rho(\alpha, \beta, \lambda, c, T)$ then

$$
a_\eta \leq \frac{(1 - \alpha)}{\sum_{\eta = 1}^{\infty} |\eta + 1 - a\beta|\phi(\eta, \lambda, c)}, \quad (\eta \geq 1).
$$

The result is sharp for the function $u$ of the form (5).

## 3 Distortion Bounds

In this section we obtain growth and the distortion bounds for the class $\sigma_\rho(\alpha, \beta, \lambda, c, T)$.

**Theorem 3.1.** If $u \in \sigma_\rho(\alpha, \beta, T)$ then $0 < |z| = r < 1$,

$$
\| u(T) \| \geq \frac{1}{\| T \|} - \frac{(1 - \alpha)}{[1 + \alpha - 2a\beta]\phi(\eta, \lambda, c)} \| T \|,
$$

$$
\| u(T) \| \leq \frac{1}{\| T \|} + \frac{(1 - \alpha)}{[1 + \alpha - 2a\beta]\phi(1, \lambda, c)} \| T \|.
$$

(7)
The result is sharp for

\[ u(z) = \frac{1}{z} + \frac{(1 - \alpha)}{\phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta]} z. \]  

(8)

Proof. Suppose \( u(z) \) is in \( \sigma_p(\alpha, \beta, \lambda, c, T) \). By Theorem 4, we have

\[ \phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta] \sum_{\eta=1}^{\infty} a_\eta \leq \sum_{\eta=1}^{\infty} \phi(\eta, \lambda, c)[\eta + \alpha - \alpha\beta(\eta + 1)]a_\eta \leq (1 - \alpha). \]

Therefore

\[ \sum_{\eta=1}^{\infty} a_\eta \leq \frac{1 - \alpha}{\phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta]}. \]

Also, if \( u(T) = T^{-1} + \sum_{\eta=1}^{\infty} a_\eta T^\eta \), then

\[ \frac{1}{\|T\|} - \sum_{\eta=1}^{\infty} a_\eta \|T\|^\eta \leq \|u(T)\| \leq \frac{1}{\|T\|} + \sum_{\eta=1}^{\infty} a_\eta \|T\|^\eta. \]  

(9)

Since \( \|T\| < 1 \), the above inequality becomes

\[ \frac{1}{\|T\|} - \|T\| \sum_{\eta=1}^{\infty} a_\eta \leq \|u(T)\| \leq \frac{1}{\|T\|} + \|T\| \sum_{\eta=1}^{\infty} a_\eta. \]  

(10)

Using (9), we get the result. \( \square \)

Theorem 3.2. If \( u(z) \in \sigma_p(\alpha, \beta, \lambda, c, T) \) then

\[ \|u'(T)\| \geq \frac{1}{\|T\|^2} - \frac{(1 - \alpha)}{\phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta]}, \]

\[ \|u'(T)\| \leq \frac{1}{\|T\|^2} + \frac{(1 - \alpha)}{\phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta]}. \]  

(11)

The result is sharp for

\[ u(z) = \frac{1}{z} + \frac{(1 - \alpha)}{\phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta]} z. \]  

(12)

4 Extreme points

In this section we obtain extreme bounds for the class \( \sigma_p(\alpha, \beta, \lambda, c, T) \).

Theorem 4.1. Let \( u_0(z) = \frac{1}{z} \) and

\[ u_\eta(z) = \frac{1}{z} + \frac{(1 - \alpha)}{[\eta + \alpha - \alpha\beta(\eta + 1)][\eta + 2]\eta} z^\eta, \quad (\eta = 1, 2, \ldots). \]  

(13)

Then \( u(z) \in \sigma_p(\alpha, \beta, \lambda, c, T) \) if and only if can be expressed in the form

\[ u(z) = \sum_{\eta=0}^{\infty} \zeta_\eta u_\eta(z), \quad \text{here } \zeta_\eta \geq 0, \sum_{\eta=0}^{\infty} \zeta_\eta = 1. \]
Proof. Assume that \( u(z) = \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z) \), here \( \zeta_\eta \geq 0, \sum_{\eta=0}^{\infty} \zeta_\eta = 1 \). Then we have

\[
\begin{align*}
  u(z) &= \sum_{\eta=0}^{\infty} \zeta_\eta u_\eta(z) \\
  &= \zeta_0 u_0(z) + \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z) \\
  &= \frac{1}{z} + \sum_{\eta=1}^{\infty} \frac{1 - \alpha}{[\eta + \alpha - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c)} z^\eta.
\end{align*}
\]

Therefore

\[
\sum_{\eta=1}^{\infty} [\eta + \alpha - \alpha (\eta + 1)] \phi(\eta, \lambda, c) \zeta_\eta \frac{1 - \alpha}{[\eta + \alpha - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c)} = (1 - \alpha) \sum_{\eta=1}^{\infty} \zeta_\eta = (1 - \alpha)(1 - \zeta_0) \leq (1 - \alpha).
\]

Hence by Theorem 4, \( u(z) \in \sigma_p(\alpha, \beta, \lambda, c, T) \).

Conversely, suppose that \( u \in \sigma_p(\alpha, \beta, \lambda, c, T) \). Since, by Corollary 4,

\[
\begin{align*}
  a_\eta &\leq \frac{1 - \alpha}{[\eta + \alpha - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c)}, \quad (\eta \geq 1), \\
  \text{setting} \quad \zeta_\eta &= \frac{[\eta + \alpha - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha} a_\eta, \\
  \text{here} \quad \eta \geq 1, \quad \zeta_0 = 1 - \sum_{\eta=1}^{\infty} \zeta_\eta.
\end{align*}
\]

We obtain \( u(z) = \zeta_0 u_0(z) + \sum_{\eta=1}^{\infty} \zeta_\eta u_\eta(z) \).

This completes the proof of the theorem. \( \square \)

**Theorem 4.2.** The class \( u \in \sigma_p(\alpha, \beta, \lambda, c, T) \) is closed under convex combination.

Proof. Let the functions \( u(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} a_\eta z^\eta \) and \( v(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} b_\eta z^\eta \) be in the class \( \sigma_p(\alpha, \beta, \lambda, c, T) \). Then by Theorem 4, we have

\[
\sum_{\eta=1}^{\infty} [\eta + \alpha - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c) a_\eta \leq (1 - \alpha),
\]

\[
\sum_{\eta=1}^{\infty} [\eta + \alpha - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c) b_\eta \leq (1 - \alpha).
\]

To \( 0 \leq \zeta \leq 1 \), define the function \( h(z) \) as \( h(z) = \zeta u(z) + (1 - \zeta)v(z) \).

Then we get \( h(z) = \frac{1}{z} + \sum_{\eta=1}^{\infty} [\zeta a_\eta + (1 - \zeta)b_\eta] z^\eta \), now we obtain
Hence completes the proof.

Proof. Let

\[ \sum_{\eta=1}^{\infty} [\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c) [\zeta a_\eta + (1 - \zeta)b_\eta] \]

\[ = \sum_{\eta=1}^{\infty} [\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c) a_\eta \]

\[ + (1 - \zeta) \sum_{\eta=1}^{\infty} [\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c) b_\eta \]

\[ \leq \zeta(1 - \alpha) + (1 - \zeta)(1 - \alpha) \]

\[ = (1 - \alpha). \]

Therefore \( h \in \sigma_p(\alpha, \beta, \lambda, c, T) \).
Hence completes the proof. \( \square \)

## 5 Radii of close-to-convexity, starlikeness and convexity

**Theorem 5.1.** Let \( u \in \sigma_p(\alpha, \beta, \lambda, c, T) \). Then \( u \) is meromorphically close-to-convex of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in the disc \( |z| < r_1 \), where

\[
r_1 = \inf_{\eta \in \mathbb{N}} \left[ \frac{(1 - \gamma)[\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c)}{\eta(1 - \alpha)} \right]^{\frac{1}{\gamma}}.
\]  

(14)
The result is sharp for the extremal function given by (5).

**Proof.** It sufficient to show that

\[
\|u'(T)^2 + 1\| < (1 - \gamma).
\]  

(15)
By Theorem 4, we have

\[
\sum_{\eta=1}^{\infty} \frac{[\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c) a_\eta}{1 - \alpha} \leq 1.
\]
So the inequality

\[
\left\| u'(T)^2 + 1 \right\| = \left\| \sum_{\eta=1}^{\infty} \eta a_\eta T^{\eta-1} \right\| \leq \sum_{\eta=1}^{\infty} \eta a_\eta \|T\|^{\eta-1} < (1 - \gamma)
\]
holds if

\[
\eta \|T\|^{\eta-1} \leq \frac{(1 - \gamma)[\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha}.
\]
Then (13) holds if

\[
\|T\|^{\eta-1} \leq \frac{(1 - \gamma)[\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c)}{\eta(1 - \alpha)}, \quad (\eta \geq 1).
\]
This yields the close-to-convexity of the function and completes the proof. \( \square \)

**Theorem 5.2.** Let \( u \in \sigma_p(\alpha, \beta, \lambda, c, T) \). Then \( u \) is meromorphically starlike of order \( \gamma \) (\( 0 \leq \gamma < 1 \)) in the disc \( |z| < r_2 \), where

\[
r_2 = \inf_{\eta \in \mathbb{N}} \left[ \frac{(1 - \gamma)[\eta + \alpha - \beta(\eta + 1)] \phi(\eta, \lambda, c)}{(\eta + 2 - \gamma)(1 - \alpha)} \right]^{\frac{1}{\gamma}}.
\]  

(16)
The result is sharp for the extremal function given by (5).

**Proof.** Let \( u(T) = T^{-1} + \sum_{\eta=1}^{\infty} a_\eta z^\eta \). Since \( u \in \sigma_p(\alpha, \beta, \lambda, c, T) \) is meromorphically starlike of order \( \gamma \),

\[
\left\| \frac{u'(T)T}{u(T)} + 1 \right\| \leq (1 - \delta).
\]  

(17)
Substituting for $u$, the above inequality becomes,
\[
\sum_{\eta=1}^{\infty} \left( \frac{\eta + 2 - \gamma}{1 - \gamma} \right) \|T\|^{\eta+1} a_{\eta} \leq 1.
\] (18)

By Theorem 4,
\[
\sum_{\eta=1}^{\infty} \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} a_{\eta} \leq 1.
\] (19)

Then, (18) will be true if
\[
\left( \frac{\eta + 2 - \gamma}{1 - \gamma} \right) \|T\|^{\eta+1} \leq \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} \right]
\]

That is
\[
\|T\| \leq \left[ \frac{(1 - \gamma)[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{(\eta + 2 - \gamma)(1 - \alpha)} \right]^{\frac{1}{\eta+1}}.
\]

Theorem 5.3. Let $u \in \sigma_p(\alpha, \beta, \lambda, c, T)$. Then $u$ is meromorphically convex of order $\gamma$ ($0 \leq \gamma < 1$) in the disc $|z| < r_3$, where
\[
r_3 = \inf_{\eta \in \mathbb{N}} \left[ \frac{(1 - \gamma)[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{\eta(\eta + 2 - \gamma)(1 - \alpha)} \right]^{\frac{1}{\eta+1}}.
\]
The result is sharp for the extremal function given by (5).

Proof. By using the technique employed in the proof of the Theorem 5.1, we can show that
\[
\left\| \frac{Tu''(T)}{u'(T)} + 2 \right\| < 1 - \gamma
\]
for $|z| < r_3$ and prove that the assertion of the theorem is true. \qed

6 Hadamard product

Theorem 6.1. For function $u, v \in \Sigma_p$ defined by (1) and (3) respectively, let $u, v \in \sigma_p(\rho, \beta, \lambda, c, T)$. Then the Hadamard product $u \ast v \in \sigma_p(\rho, \beta, T)$, where
\[
\rho \leq 1 - \frac{(1 - \alpha)^2(\eta + 1)(1 - \beta)}{(1 - \alpha)^2[1 - \beta(\eta + 1)] + [\eta + \alpha - \alpha \beta(\eta + 1)]^2 \phi(\eta, \lambda, c)}.
\]

Proof. We need to find the largest $\rho$ such that
\[
\sum_{\eta=1}^{\infty} \frac{[\eta + \rho - \rho \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \rho} a_{\eta} b_{\eta} \leq 1.
\]

Since $u, v \in \sigma_p(\alpha, \beta, \lambda, c, T)$, by Theorem 4, we have
\[
\sum_{\eta=1}^{\infty} \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} a_{\eta} \leq 1
\] (20)

and
\[
\sum_{\eta=1}^{\infty} \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} b_{\eta} \leq 1.
\] (21)
From (20) and (21), we find, utilizing the Cauchy-Schwartz inequality,

$$\sum_{\eta=1}^{\infty} \frac{[\eta + a - \alpha \beta (\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha} \sqrt{a_\eta b_\eta} \leq 1. \tag{22}$$

We want only to show that

$$\frac{[\eta + \rho - \rho \beta(\eta + 1)] \phi(\eta, \lambda, c)}{1 - \rho} a_\eta b_\eta$$

$$\leq \frac{[\eta + \alpha - \alpha \beta(\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha} \sqrt{a_\eta b_\eta}$$

$$\Rightarrow \sqrt{a_\eta b_\eta} \leq \frac{(1 - \rho)[\eta + \alpha - \alpha \beta(\eta + 1)]}{(1 - \alpha)[\eta + \rho - \rho \beta(\eta + 1)]}. \tag{23}$$

On the other hand, from (22), we have

$$\sqrt{a_\eta b_\eta} \leq \frac{(1 - \alpha)}{[\eta + \alpha - \alpha \beta(\eta + 1)] \phi(\eta, \lambda, c)}. \tag{24}$$

Therefore in view of (23) and (24), it is enough to find the largest \( \rho \) that

$$\frac{(1 - \alpha)}{[\eta + \alpha - \alpha \beta(\eta + 1)] \phi(\eta, \lambda, c)} \leq \frac{(1 - \rho)[\eta + \alpha - \alpha \beta(\eta + 1)]}{(1 - \alpha)[\eta + \rho - \rho \beta(\eta + 1)]}$$

which yields

$$\rho \leq \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]^2 \phi(\eta, \lambda, c) - n(1 - \alpha)^2}{[\eta + \alpha - \alpha \beta(\eta + 1)]^2 \phi(\eta, \lambda, c) + (1 - \alpha)^2[1 - \beta(\eta + 1)]}$$

$$\Rightarrow \rho \leq 1 - \frac{(1 - \alpha)^2(\eta + 1)(1 - \beta)}{(1 - \alpha)^2[1 - \beta(\eta + 1)] + [\eta + \alpha - \alpha \beta(\eta + 1)]^2 \phi(\eta, \lambda, c)}.$$

\( \square \)

**Theorem 6.2.** For function \( u, v \in \Sigma_p \) defined by (1) and (3) respectively, let \( u, v \in \sigma_p(\alpha, \beta, \lambda, c, T) \).

Then the function \( f(z) = \frac{1}{2} + \sum_{\eta=1}^{\infty} (a_\eta^2 + b_\eta^2)z^\eta \) is in the class \( \sigma_p(\alpha, \beta, \lambda, c, T) \), where \( \rho \leq 1 - \frac{2(1 - \alpha)^2[1 - \beta(\eta + 1)] + [\eta + \alpha - \alpha \beta(\eta + 1)]^2 \phi(\eta, \lambda, c)}{[\eta + \alpha - \alpha \beta(\eta + 1)]^2[1 - \beta(\eta + 1)] + 2(1 - \alpha)^2[1 - \beta(\eta + 1)]} \).

**Proof.** Since \( u, v \in \sigma_p(\alpha, \beta, \lambda, c, T) \), we have

$$\sum_{\eta=1}^{\infty} \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha} a_\eta \right]^2 \leq 1 \tag{25}$$

and

$$\sum_{\eta=1}^{\infty} \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha} b_\eta \right]^2 \leq 1. \tag{26}$$

Combining the last two inequalities, we get

$$\sum_{\eta=1}^{\infty} \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)] \phi(\eta, \lambda, c)}{1 - \alpha} \right]^2 (a_\eta^2 + b_\eta^2) \leq 1. \tag{27}$$

But we need to find the largest \( \rho \) such that

$$\sum_{\eta=1}^{\infty} \left[ \frac{[\eta + \rho - \rho \beta(\eta + 1)] \phi(\eta, \lambda, c)(a_\eta^2 + b_\eta^2)}{1 - \alpha} \right] \leq 1. \tag{28}$$
The inequity (28) would hold if
\[
\frac{[\rho + \rho \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \rho} \leq \frac{1}{2} \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} \right]^2.
\]
Then we have
\[
\rho \leq \frac{\frac{1}{2} \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} \right]^2 - 2\eta(1 - \alpha)^2\phi(\eta, \lambda, c)}{\frac{1}{2} \left[ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} \right]^2 + 2(1 - \alpha)^2\phi(\eta, \lambda, c)[1 - \beta(\eta + 1)]}
= 1 - \frac{2(1 - \alpha)^2\phi(\eta, \lambda, c)[1 - \beta(\eta + 1)]}{\left\{ \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} \right\}^2 + 2(1 - \alpha)^2\phi(\eta, \lambda, c)[1 - \beta(\eta + 1)]}
\]
\]

\[\square\]

### 7 Integral operators

In this section, we consider integral transforms of functions in the class \(\sigma_p(\alpha, \beta, \lambda, c, T)\) of the type considered by Goel and Sohi [11].

**Theorem 7.1.** Let the function \(u \in \Sigma_p\) given by (1) be in the class \(\sigma_p(\alpha, \beta, \lambda, c, T)\). Then the integral operator
\[
U(z) = c \int_0^1 q^c u(qz)\,dq, \quad 0 < q \leq 1, \quad 0 < c < \infty
\]
(29)
is in the class \(\sigma_p(\alpha, \beta, \lambda, c, T)\), where
\[
\rho = 1 - \frac{(1 - \alpha)(1 + 2\beta) + c}{(1 + \alpha - 2\alpha\beta)(c + 2) + (1 - \alpha)(1 - 2\beta)c}.
\]
The result is sharp for the function
\[
u(z) = \frac{1}{z} + \frac{(1 - \alpha)}{\phi(1, \lambda, c)[1 + \alpha - 2\alpha\beta]} z.
\]

**Proof.** Let \(u \in \Sigma_p\) given by (1) be in the class \(\sigma_p(\alpha, \beta, \lambda, c, T)\). Then
\[
U(z) = c \int_0^1 q^c u(qz)\,dq = \frac{1}{z} + \frac{c}{c + \eta + 1} a_\eta z^\eta.
\]
(30)
We have to show that
\[
\sum_{\eta=1}^\infty \left[ \frac{c[\eta + \rho - \rho \beta(\eta + 1)]\phi(\eta, \lambda, c)}{(1 - \rho)(c + \eta + 1)} \right] a_\eta \leq 1.
\]
(31)
Since \(u \in \sigma_p(\alpha, \beta, \lambda, c, T)\), we have
\[
\sum_{\eta=1}^\infty \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]\phi(\eta, \lambda, c)}{1 - \alpha} a_\eta \leq 1.
\]
The inequality (31) satisfies if
\[
\frac{c[\eta + \rho - \rho \beta(\eta + 1)]}{(1 - \rho)(c + \eta + 1)} \leq \frac{[\eta + \alpha - \alpha \beta(\eta + 1)]}{1 - \alpha}.
\]
Then we get
\[
\rho \leq \frac{[\eta + \alpha - \alpha \beta(\eta + 1)](\eta + c + 1) - (1 - \alpha) c\eta}{[\eta + \alpha - \alpha \beta(\eta + 1)](\eta + c + 1) + c(1 - \alpha)[1 - \beta(\eta + 1)]}
= 1 - \frac{(1 - \alpha)(1 + \beta(\eta + 1) + c\eta)}{[\eta + \alpha - \alpha \beta(\eta + 1)](\eta + c + 1) + c(1 - \alpha)[1 - \beta(\eta + 1)]}.
\]
since
\[
\phi(\eta) = 1 - \frac{(1 - \alpha)(1 + \beta(\eta + 1) + c \eta)}{[\eta + \alpha - \alpha\beta(\eta + 1)((\eta + c + 1) + c(1 - \alpha)[1 - \beta(\eta + 1)]]}
\]
is an increasing function of $\eta$, $\eta \geq 1$. We obtained the desired result. □

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