The Ordered Matrix Dirichlet for Modeling Ordinal Dynamics

Niklas Stoehr
ETH Zurich

Benjamin J. Radford
UNC Charlotte

Ryan Cotterell
ETH Zurich

Aaron Schein
University of Chicago

Abstract

Many dynamical systems exhibit latent states with intrinsic orderings such as “ally”, “neutral” and “enemy” relationships in international relations. Such latent states are evidenced through entities’ cooperative versus conflictual interactions which are similarly ordered. Models of such systems often involve state-to-action emission and state-to-state transition matrices. It is common practice to assume that the rows of these stochastic matrices are independently sampled from a Dirichlet distribution. However, this assumption discards ordinal information and treats states and actions falsely as order-invariant categoricals, which hinders interpretation and evaluation. To address this problem, we propose the Ordered Matrix Dirichlet (OMD): rows are sampled conditionally dependent such that probability mass is shifted to the right of the matrix as we move down rows. This results in a well-ordered mapping between latent states and observed action types. We evaluate the OMD in two settings: a Hidden Markov Model and a novel Bayesian Dynamic Poisson Tucker Model tailored to political event data. Models built on the OMD recover interpretable latent states and show superior forecasting performance in few-shot settings. We detail the wide applicability of the OMD to other domains where models with Dirichlet-sampled matrices are popular (e.g. topic modeling) and publish user-friendly code.

1 Introduction

In many modeling settings and application domains, some aspect of the observation space has an intrinsic ordering. For example, observed political interactions between countries can be ordered by conflict-cooperation intensity, ranging from “provide aid” to “fight” (Goldstein, 1992; Schrodt, 2008). This ordering should ideally be reflected in the latent states driving the observed actions. For example, conflictual actions like “fight” should more likely be observed if countries are in a latent “enemy” state (Schrodt, 2006). In this setting, latent states represent relationship statuses that are ordered due to observed actions, ranging from “ally” to “enemy” states. An additional structural assumption may be that these latent states transition step-by-step through adjacent states. Enemies rarely become allies from one moment to the other, but transition smoothly. These assumptions are a subject of study in international relations (Davis and Stan, 1984; Kalyvas, 2006) and exemplified in Fig. 1.

![Emission Matrix PMF](image1)

Figure 1: We are dealing with observed states that are naturally ordered such as action types $a$. Emission matrix: We expect cooperative actions (e.g. "provide aid") to be associated with more cooperative latent states $k$ (e.g. "ally"). Transition matrix: we expect latent states to transition to adjacent states (e.g. $k = 0$ to $k = 1$). To model these assumptions, the Ordered Matrix Dirichlet (OMD) ensures that each discrete probability distribution per state $k$ has first-order stochastic dominance (FSD) over the next distribution $k + 1$. This constraint is exemplified by non-overlapping cumulative distribution functions for each latent $k$. 
Many models of dynamical systems rely on the notion of a state-to-action emission matrix and a state-to-state transition matrix such as the prominent Hidden Markov Model (HMM) (Baum and Petrie, 1966) (§2). The matrices’ rows describe discrete distributions that are often sampled independently from a Dirichlet distribution. This however results in the states and actions to be categorical without particular ordering. We refer to the distribution over this kind of stochastic matrices as Standard Matrix Dirichlet (SMD) (§3.1). It not only discards ordinal information, but also leads to a problem called label switching (Stephens, 2000): the order-invariance of states complicates interpretability and evaluation with conventional classification metrics.

So how can we reflect the assumption of action and state ordering in our model? One existing idea is to simply constrain the non-zero entries of a matrix to be “banded” along its diagonal (§3.2). However, this Banded Matrix Dirichlet (BMD) is inflexible, ruling out a large number of structures such as upper diagonal matrices. Instead, we propose the Ordered Matrix Dirichlet (OMD), a distribution over a subset of stochastic matrices with uniquely ordered latent states (§3.3). In particular, the OMD ensures that row $k$ has first-order stochastic dominance over row $k + 1$. This means that probability mass is shifted to the right of the matrix as we move down rows as outlined in Fig. 1.

The OMD as a modeling motif has wide applicability to a broad family of dynamical systems with Dirichlet-sampled datasets of human-annotated, ordered interactions as well as theoretically generated data with known ordinal dynamics (§5). Many models of dynamical systems rely on the notion of a state-to-action emission matrix representing a discrete distribution over a set of action types. For this kind of data, we describe a canonical family of dynamical system comprising emission and transition matrices. In these models, each observation $y_{at}$ is modeled as $E[y_{at}] = \mu_{at}$ and decomposes according to

$$\mu_{at} = \sum_{k=1}^{K} \lambda_{k}^{(t)} \phi_{ka}$$

The parameter $\phi_{ka}$ represents the relevance of action $a$ in latent state $k$ and forms a state-to-action emission matrix $\Phi \in [0, 1]^{K \times A}$. $\lambda_{k}^{(t)}$ describes a latent parameter over $K$ latent states and transitions over time based on

$$\lambda_{k}^{(t)} = \sum_{k_{1}=1}^{K} \lambda_{k_{1}}^{(t-1)} \pi_{k_{1}k}$$

where $\pi_{k'k}$ is an element of a state-to-state transition matrix $\Pi \in [0, 1]^{K \times K}$ that evolves latent states from time step $t - 1$ to $t$. This formulation can be related to a Hidden Markov Model (HMM) or more complex dynamical systems as presented in §6.2.

**Ordered State-to-Action Mapping.** Note that each row $\phi_{a}$ of the emission matrix represents a discrete distribution over $A$ ordered action types on the $(A - 1)$-simplex — i.e., $\sum_{a=1}^{A} \phi_{ka} = 1$ and $\phi_{ka} > 0$. Unlike O’Connor et al. (2013); Schein et al. (2015, 2016b), who treat action types and latent states as categorical, we explicitly model ordering: on average, latent state $k$ should be associated with higher ranked actions than state $k - 1$ (Stoehr et al., 2022).

**Ordered State-to-State Transitions.** Each row $\pi_{k}$ in the transition matrix defines a $(K - 1)$-probability simplex of transitioning to all $K$ latent states from state $k$. One may wish to constrain latent states to transition smoothly to neighboring states and to itself instead of immediately jumping from low-rank to high-rank states within a single time step. This gradual (de-)escalation is typically achieved by constraining the transition matrix $\Pi$ (Schrodt, 2006; Netzer et al., 2008; Anders, 2020; Randahl and Vegelius, 2022).

## 3 Stochastic Matrix Construction

### 3.1 Standard Matrix Dirichlet (SMD)

The conventional way of constructing a stochastic matrix is to sample each row $\phi_{a}$ independently from a Dirichlet distribution. We term the distribution over this standard kind of stochastic matrices the Standard Matrix Dirichlet (SMD). As outlined in Fig. 2, SMD does not necessarily adhere to any desired ordering constraints, previously outlined in §2.

### 3.2 Banded Matrix Dirichlet (BMD)

A simple trick to obtain some row ordering is to limit the non-zero elements to be *banded* along the matrix’ diagonal...
Figure 2: Stochastic matrices sampled from the Standard Matrix Dirichlet (SMD), Banded Matrix Dirichlet (BMD) and Ordered Matrix Dirichlet (OMD) distribution. Neither the SMD nor the OMD are necessarily adhering to the First-order Stochastic Dominance (FSD) constraint. This can be seen from overplotting cumulative distribution functions (CDF) for each row / latent state. In both, the SMD and BMD, rows can be switched.

(see the middle plot of Fig. 2). This is referred to as Banded Matrix Dirichlet (BMD) or left-to-right matrix (Schrodt, 2006; Subakan et al., 2015). For instance, in the transition matrix proposed by Schrodt (2006); Randahl and Vegelius (2022), the $k$th state can only be excited by its adjacent states, $(k-1)^{th}$ and $(k+1)^{th}$, as well as by itself. We can consider a wider bandwidth $b \geq 1$ so that components $k' \in \{k-b, \ldots, k+b\}$ all excite $k$. However, the BMD is still very restrictive and a unique, monotonic ordering of rows is not ensured as can be seen in Fig. 2.

### 3.3 Ordered Matrix Dirichlet (OMD)

What is a more flexible construction realizing the structural constraints described in §2? We propose a constraint based on First-order Stochastic Dominance (FSD). Then, we bake this constraint into a stick-breaking construction for

$$
\sum_{a=1}^{\ell} \phi_{ka} \geq \sum_{a=1}^{\ell} \phi_{(k+1)a} \text{ for all } \ell \in \{1, \ldots, A\} \quad (3)
$$

In particular, $\phi_k$ dominates $\phi_{k+1}$ if its cumulative distribution function $CDF_k(a)$ at each point $a$ is higher than or equal to $CDF_{k+1}(a)$ (Massey, 1987) following

$$
CDF_k(a) = \sum_{a'=1}^{a} \phi_{ka'} \quad (4)
$$

$$
CDF_k(a) \geq CDF_{k+1}(a) \text{ for all } a \quad (5)
$$

**Stick-Breaking Construction.** We can generate a Dirichlet random variate by relying on a stick-breaking algorithm (Gelman et al., 2013, p. 585) (Algorithm 1). Stick-breaking (Sethuraman, 1994) iteratively partitions (breaks) a line (stick) into parts. The size (length) of the broken part is determined by sampling from a Beta distribution.

Using this construction, we can independently sample $K$ Dirichlet distributions. This is exemplified by the for-loop of length $K$ in lines 2, 7 and 15 of Algorithm 1. In fact, if we omitted lines 5 and 10 in Algorithm 1, the algorithm

```
Algorithm 1 Ordered Matrix Dirichlet
1: Input: alpha concentration prior $\alpha \in \mathbb{R}_+^A$
2: for $k = 1, \ldots, K$ do
3: $\phi_{k1} \sim Beta(\alpha_1, \sum_{a=2}^{A} \alpha_a)$
4: end for
5: $(\phi_{11}, \ldots, \phi_{K1}) \leftarrow \text{SORT}((\phi_{11}, \ldots, \phi_{K1}))$
6: for $a = 2, \ldots, A$ do
7: for $k = 1, \ldots, K$ do
8: $\beta_{ka} \sim Beta(\alpha_a, \sum_{a'=a+1}^{A} \alpha_{a'})$
9: end for
10: $(\beta_{1a}, \ldots, \beta_{Ka}) \leftarrow \text{SORT}((\beta_{1a}, \ldots, \beta_{Ka}))$
11: for $k = 1, \ldots, K$ do
12: $\phi_{ka} \leftarrow (1 - \sum_{a'=a+1}^{A} \phi_{ka'}) \beta_{ka}$
13: end for
14: end for
15: for $k = 1, \ldots, K$ do
16: $\phi_{kA} \leftarrow 1 - \sum_{a'=1}^{A-1} \phi_{ka'}$
17: end for
18: Output: OMD variate $\Phi \in \mathbb{R}^{K \times A}$
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by Algorithm 1 would be all ones.
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would return a sample from the Standard Matrix Dirichlet (SMD). Yet, the sorting causes the \( K \) discrete distributions to be dependent. Note that sorting only marginally increases time complexity from \( O(A \times K) \) to \( O(A \times K \log K) \). Importantly, the sorting ensures that the \( (k + 1) \)th distribution is bounded by the \( k \)th. Due to this constraint, distributions \( \phi_k \) are not sampled from a Dirichlet anymore, but are constrained simplex vectors. The algorithm returns a variance from a distribution over FSD-constrained matrices.

**Alpha Concentration Prior.** The OMD is parametrized by an alpha concentration vector \( \alpha \in \mathbb{R}_+^A \). Fig. 3 visualizes the effect that different configurations of \( \alpha \) have on \( \Phi \). For symmetric \( \alpha \), one typically expects the probability mass to be symmetric as in the Dirichlet distribution. In the case of the OMD, we observe that probability mass is slightly asymmetric to the right. This is due to the sorting operation that returns the \( K \)th order statistics of a Beta-distributed variable as elaborated in App. A. However, this does not mean that the OMD is inherently asymmetric. There exist infinitely many settings of \( \alpha \) values that induce symmetry. These values can be learned during posterior inference. Thus, the asymmetry has no negative consequences on the flexibility of the OMD.

**Label switching proposition.** The First-order Stochastic Dominance constraint mitigates the label switching problem (Stephens, 2000) of order-invariant latent states (see appendix Fig. 11). Previous work has shown that the parameters of Hidden Markov Models (and admixture models (Hillar and Sommer, 2011)) are identifiable up to permutations of the latent states (Allman et al., 2009, p. 3123). By definition, there exists a single ordering of random variables under the FSD constraint (Hadar and Russell, 1969). In turn, the FSD-bounded rows of the OMD have a unique ordering. Importantly, we consider imposing structure in both the emission and the transition matrix. However, to ensure this unique ordering of latent states, it suffices that the rows of the emission matrix \( \Phi \) are ordered. The ordering perpetuates and affects the ordering of rows and columns in the transition matrix \( I \) alike. Constraining \( I \) serves the purpose of imposing (de-)escalatory transition dynamics.

4 Technical Details

4.1 Implementation and Inference

The Ordered Matrix Dirichlet integrates nicely with the available tools in probabilistic programming. We implement the OMD in a few lines using the probabilistic programming API Pyro (Bingham et al., 2018; Phan et al., 2019). Our construction algorithm only involves the sorting of samples from a Beta distribution. This does not obstruct the computation of gradients (Blondel et al., 2020) and enables gradient-based optimization methods, such as Hamiltonian Markov Chain Monte Carlo (HMC) and Stochastic Variational Inference (SVI) for posterior inference. To infer posterior parameters, we run the HMC-variant No-U-Turn Sampler (NUTS) (Duane et al., 1987; Homan and Gelman, 2014) to take \( S = 1000 \) posterior after 200 warm-up samples. This takes about 30 minutes on a single NVIDIA TITAN RTX GPU core.

4.2 Predictive Evaluation Methodology

**Prediction Tasks.** We evaluate the inferred posterior parameters in two predictive tasks on a held-out testing set:

1. Imputation: we randomly select, remove and mask 30% of the entries in the full count tensor. For evaluation, we consider a testing set which consists of the full, unmasked tensor. However, we evaluate model predictions only on the previously masked event entries.
2. Forecasting: we split the time series by time: the first 70% serve as training and the latter 30% as testing data.

**Evaluation Metrics.** We consider two evaluation metrics:

1. As a density-based metric, we compute the posterior predictive density (PPD) (Gelman et al., 1996) following \( \exp \left( \frac{1}{N} \sum_{n=0}^{N} \log \left( \sum_{s} p(y_{n} | \theta^{(s)}) \right) \right) \). PPD ranges between 0 and \( \infty \) and measures the averaged log-likelihood of the held-out data given the inferred posterior parameters \( \theta^{(s)} \).
2. As a point estimate-based metric, we consider the mean absolute error (MAE). To predict the posterior mean \( \tilde{y}_n \), we compute \( E[y_{n}^{(s)} | \theta^{(s)}] \approx \frac{1}{S} \sum_{s=1}^{S} y_{n}^{(s)} \).

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3. We open-source our code jointly with tutorials and examples on how to use the OMD within popular models.

4. \( y_{n} \) or in the case of the 4-mode count tensor introduced in §6.1, \( n = (i, j, a, t) \) is a multi-index such that \( y_{n} = y_{i \to j}^{(a, t)} \).
Figure 4: Forecasting comparison between the SMD and the OMD as transition and emission matrix in an Hidden Markov Model (HMM). We synthetically generate data using different ground truth transition matrices (left column). We use a noisy diagonal matrix as ground truth emission matrix. The OMD does a better job at recovering ground truth transitions as the SMD (center (left)). The OMD performs en par with SMD at predicting observed states and even wins in few-shot settings ($N = 100$) (center right). OMD consistently outperforms SMD in predicting latent states due to label switching (right).

5 Synthetic Data Experiments

We generate synthetic data using a Hidden Markov Model with known parameters. For the emission matrix, we consider a noisy diagonal matrix. For the transition matrix, we experiment with different transition patterns: “banded”, “bonbon” and “triangle”, all displayed in the left column of Fig. 4. The “bonbon” pattern, for instance, represents a realistic scenario for political event data, where “neutral” states fluctuate but “ally” and “enemy” states are absorbing. Using these matrices as ground truth parameters, we generate $N = 10000$ sequences of length $T = 10$.

For training and testing, we consider two settings: SMD + SMD and OMD + OMD. For instance, SMD + SMD refers to the parametrization of the HMM’s emission and transition matrix using the SMD. Next, we fit the model and inspect whether learned transition patterns recover the ground truth, displayed in columns 2 and 3 of Fig. 4. Considering the ground truth of the “triangle transition” in the bottom row for instance: SMD confuses the correct upper triangle structure with a lower triangle. This exemplifies the label switching problem. In contrast, all ground truth transition patterns are clearly captured by the OMD.

Generating synthetic data has the advantage that we know the ground truth latent states at each time step, allowing for a quantitative evaluation of inferred latent states. We consider one setting with all $N = 10000$ sequences and a few-shot setting with $N = 100$ sequences. Inspecting columns 4 and 5 of Fig. 6, two results are standing out: Firstly, when forecasting observed states, the OMD + OMD consistently outperforms the more flexible SMD + SMD in the few-shot setting of $N = 100$. The inductive bias of the OMD is picking up the ground truth structures more quickly.

Secondly, when predicting latent states, OMD + OMD clearly wins in all settings. The reason for this lies in the label switching of latent states. This is also reflected in the high standard deviation of the MAE over the 10 runs with different random seeds. For SMD + SMD, the labels of the latent states are switched for every run, resulting in volatile results. We repeat these synthetic data experiments for the imputation setting (see Fig. 12 in the appendix).
6 Case Study: International Relations

6.1 Dyadic Political Events Dataset

Ordered Action Types. We consider the political event dataset Integrated Crisis Early Warning System (ICEWS) (Boschee et al., 2015), in particular the ICEWS Coded Event Data. ICEWS is one of the largest collections of country-level “who-did-what-to-whom-at-what-time” quadruplets. All events are annotated with a timestamp, source and target country and an action type. Action types and actors are machine-coded by Raytheon BBN’s ACCENT software into the Conflict and Mediation Event Observations (CAMEO) ontology (McClelland, 1984; Schrod et al., 2008; Schrod, 2012). CAMEO specifies 20 high-level action types, listed on the left in Fig. 5. The Goldstein Scale (Goldstein, 1992) provides an expert-based conflict-cooperation intensity ranking of these action types. For instance, “use unconventional mass violence” and “fight” are ranked most conflictual (both −10.0) and “provide aid” (+7.0) most cooperative.

4-mode Count Tensor. We omit all self-targeted events where the source and target country are the same and select the time period from 2015 to 2020 which leaves us with 1,833,552 events in total. The data can be organized into a 4-mode count tensor $Y \in \mathbb{Z}_{0}^{I \times J \times A \times T}$ where an element $y_{i \rightarrow j}^{(t)}$ describes the number of times country $i$ took action $a$ to country $j$ during time step $t$. We consider $I = 249$ source and $J = 249$ target countries, $A = 20$ action types ordered by the Goldstein scale and $T = 72$ time steps, aggregating onto a temporal granularity of months. A slice of this count tensor showing interactions between $i =$ Armenia and $j =$ Azerbaijan is presented in Fig. 5.

6.2 Dynamic Poisson Tucker Model

We present the OMD as the empowering motif within a highly expressive model tailored to count-based political event data. To this end, we extend the dynamical system presented in §2 by augmenting the likelihood of Eq. (1) and the transition dynamics of Eq. (2). We model the counts $y_{i \rightarrow j}^{(t)}$ of the 4-mode tensor $Y$ to be Poisson-distributed:

$$y_{i \rightarrow j}^{(t)} \sim \text{Pois} \left( \delta_{a} \delta(t) \sum_{k=1}^{K} \lambda_{i \rightarrow j}^{(t)} \phi_{ka} \right)$$

(6)

$\delta_{a} \in \mathbb{R}_{+}^{A} \sim \text{Gamma}(\alpha_{a}, \beta_{a})$ and $\delta(t) \in \mathbb{R}_{+}^{T} \sim \text{Gamma}(\alpha_{t} \delta^{-1}, \alpha_{t} \beta_{t})$ are (optional) action and time scaling coefficients. $\delta(t)$ accounts for changes in the number of reported events over time. The model is said to be stationary if $\delta(t) = \delta$ for $\forall t \in \{1, ..., T\}$. Instead of modeling latent states $k$ between countries $i$ and $j$, we can model the rate at which countries in community $c_1$ take actions in state $k$ towards countries in community $c_2$ during time step $t$. To this end, we apply Tucker Decomposition (Tucker, 1964):

$$\sum_{k=1}^{K} \lambda_{i \rightarrow j}^{(t)} \phi_{ka} = \sum_{c_{1}=1}^{C} \sum_{c_{2}=1}^{C} \sum_{k=1}^{K} \psi_{c_{1}} \psi_{c_{2}} \phi_{ka} \lambda_{c_{1} \rightarrow c_{2}}^{(t)}$$
where $\lambda_{c_1 \rightarrow c_2}^{(t)}$ forms the core tensor $\Lambda^{(t)} \in \mathbb{R}_{+}^{C \times C \times K}$. The parameter $\psi_{c_1, i}$ represents the rate at which country $i$ acts as a source in community $c_1$, analogously $\psi_{c_2, j}$ represents country $j$ as a target in community $c_2$. Both parameters are gamma-distributed, e.g. $\psi_{c_1, i} \sim \text{Gamma}(\alpha_i, \beta_i)$, and describe a community-country matrix $\Psi \in \mathbb{R}_{+}^{C \times V}$.

Finally, we adjust the transition dynamics expressed in Eq. (2) to be Gamma-distributed. In doing so, we follow the Poisson–Gamma Dynamical System (Schein et al., 2016b, 2019) and set the hyperparameter $\tau_0$ to 1 as shown below.

$$\lambda_{c_1 \rightarrow c_2}^{(t)} \sim \text{Gam} \left( \tau_0 \sum_{k_1=1}^{K} \lambda_{c_1 \rightarrow c_2}^{(t-1) k_1 \rightarrow k_2} \pi_{kk} \right)$$

### 6.3 Experiments and Results

**Predictive Evaluation.** We quantitatively evaluate the performance of the SMD, BMD and OMD within the Dynamic Poisson Tucker Model with $C = 5$ and $K = 3$ to the ICEWS event dataset and inspect the posterior parameters of the latent variables. Importantly, we parameterize the emission and transition matrix using the Ordered Matrix Dirichlet. Fig. 7 shows the time period of 2020. Fig. 7A visualizes country-community activity $\Psi$. We observe that Armenia and Azerbaijan are predominantly involved in community C3. This community is mostly interacting with itself as can be seen in Fig. 7B. Moreover, the state of interaction is conflictual as shown in Fig. 7C. We know that latent state $k = 2$ corresponds to a conflictual relationship thanks to the well-ordered emission matrix $\Phi$ of state-to-action probabilities (see Fig. 7D). Without this ordering constraint, the conflictual latent state could be label switched, e.g. $k = 0$ or $k = 1$. Moreover, the OMD ensures that latent states (de-)escalate step-wise through adjacent states in the transition matrix (see Fig. 7E).

**Descriptive Evaluation.** We fit the DPT model with $C = 5$ and $K = 3$ to the ICEWS event dataset and inspect the posterior parameters of the latent variables. Importantly, we parameterize the emission and transition matrix using the Ordered Matrix Dirichlet. Fig. 7 shows the time period of 2020. Fig. 7A visualizes country-community activity $\Psi$. We observe that Armenia and Azerbaijan are predominantly involved in community C3. This community is mostly interacting with itself as can be seen in Fig. 7B. Moreover, the state of interaction is conflictual as shown in Fig. 7C. We know that latent state $k = 2$ corresponds to a conflictual relationship thanks to the well-ordered emission matrix $\Phi$ of state-to-action probabilities (see Fig. 7D). Without this ordering constraint, the conflictual latent state could be label switched, e.g. $k = 0$ or $k = 1$. Moreover, the OMD ensures that latent states (de-)escalate step-wise through adjacent states in the transition matrix (see Fig. 7E).

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*Fig. 9 in the appendix presents the posterior parameters of a model fitted to the full time range from 2000 to 2020.*
7 Applicability to other Models / Domains

The Ordered Matrix Dirichlet as a modeling motif is widely applicable to models with Dirichlet-sampled stochastic matrices. We review other models and discuss application domains as summarized in Fig. 8 in the appendix.

Poisson-Gamma Dynamical System (PGDS). The Poisson-Gamma Dynamical System (Schein et al., 2016a, 2019) has been applied to ICEWS and GDELT (Leetaru and Schrödt, 2013) political events modeling latent states between countries instead of latent communities. It could similarly model ordinal product life cycles (Arvidsson, 2019): the ordinal latent describes the level of product popularity driving purchase counts in different stores per month.

Hidden Markov Model (HMM). Non-ordered, categorical HMMs are widely applied: customer engagement could be modeled as an ordinal latent while purchase counts are naturally ordered and observed (Netzer et al., 2008). The observed change of temperature may be modeled using an ordinal latent indicative of “warming” and “cooling” periods (Perry and Hsu, 2000).

Markov Chain (MC). In Markov Chains, we have a single state transition matrix that could model sleep cycles which transition step-by-step from wake to sleep stages (Pan et al., 2012). Also, rankings in sports or macroeconomics typically involve transitioning to adjacent ranks.

Latent Dirichlet Allocation (LDA). Despite not being a time series model, many topic models feature a Dirichlet-sampled topic-word matrix. The Latent Dirichlet Allocation (LDA) (Blei et al., 2003) is a prominent example. We could semantically order the vocabulary of words, for instance from negative to positive, or boring to funny words. Then, we may want the inferred latent topics to mirror this observed ordering. We can also imagine ordinal topic models beyond textual data such as ordinal topics over ranked athletes from different sports teams.

8 Related Work

Stochastic Matrix Variants To the best of our knowledge, there exists no work constructing row-ordered stochastic matrices. The work by Griffin and Steel (2006) is related to the extent that they modify the weights defining a stick-breaking Dirichlet process to be dependent on co-variates. Relatedly, Massey (1987) studies FSD-constrained Markov processes. The term “ordered Dirichlet” first appears in Yamada and Matsunawa (2000); Huillet (2005) who investigate order statistics of the Dirichlet distribution.

Ordinal Latent Variables A large branch of ordinal methods such as ordered logistic regression (McCullagh, 1980) is based on the concept of latent cut-off points (Chu and Ghahramani, 2005; Virtanen and Girolami, 2015; Terechshenko, 2020; Gouvert et al., 2020). Stoehr et al. (2022) present an ordinal latent variable model that relies on a smooth bijective ordering transform of priors. Their model of conflict intensity incorporates the Goldstein scale, however not a temporal dimension. Netzer et al. (2008) build a Hidden Markov Model with a transition matrix over latent, ordered states representing customer relationship dynamics. They constrain the matrix to adjacent state moves and learn transition probabilities with an ordered logit model. The ordinal characteristics behind the OMD are conceptually different as we tweak a stick-breaking process by repeatedly sorting Beta-sampled vectors.

Latent Conflict Intensity We impose an intensity ordering on event types which sets us apart from O’Connor et al. (2013); Schein et al. (2015, 2016b); Minhas et al. (2016) who assume unordered categorical event types, but still evaluate their latent states against the ordered Goldstein scale in the case of O’Connor et al. (2013). Terechshenko (2020) measure interstate hostility at the country dyad-quarter level. They use an item response theory (IRT) model based on an ordered logistic regression. Treating international relations as latent, unordered states in an Hidden Markov Model is an established modeling idea (Schrödt, 2006; Anders, 2020; Randahl and Vegelius, 2022). Randahl and Vegelius (2022) pre-specify four latent states: peace, conflict, escalation, and de-escalation, and strictly band transitions. This is reminiscent of Kahn’s Escalation Ladder (Kahn, 1962; Davis and Stan, 1984) that defines stages and thresholds of conflict. Neural network-based approaches focus on modeling friend-enemy relationships (Han et al., 2019; Stoehr et al., 2021). The tracing of conflict escalation has also been studied in word embeddings (Kutuzov et al., 2017) and language models (Lefebvre* and Stoehr*, 2022; Hu et al., 2022).

9 Conclusion

This work is fundamentally concerned with imposing a structural constraint onto a (latent) matrix random variable. In doing so, we are facing a trade-off between interpretability and performance which is related, but conceptually different to the bias-variance trade-off in Bayesian model selection (Stoica and Selen, 2004). We are not forfeiting modeling flexibility to reduce model parameters and mitigate overfitting. Instead, the constraint serves the purpose of increasing interpretability of latent states by enforcing a single unique ordering thereof. We test the OMD within a new expressive Dynamic Poisson Tucker Model (DPT) and capture interpretable transition and emission patterns. The OMD may benefit or worsen model fit depending on the true underlying structure of the data. In our experiments, we observe that the OMD serves as a rightful inductive bias making training more robust in few-shot and forecasting settings.
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Figure 8: Different models with Dirichlet-sampled latent matrices fitted on data exhibiting ordinal dynamics. The Latent Dirichlet Allocation (LDA) is not a dynamical system, but similarly comprises a stochastic matrix describing word distributions per latent topic. If we order the observed vocabulary of words by the words’ sentiment score, the Ordered Matrix Dirichlet (OMD) can recover topics representative of sentiment levels. In all settings, we find that the OMD yields more easily interpretable stochastic matrices than the Standard Matrix Dirichlet (SMD).

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Impact Statement

We emphasize that our models are intended for research purposes and empirical insights. They should not be blindly deployed for automated decision-making processes. The used ICEWS data may contain biases that are potentially reinforced by our modeling assumptions. The experiments with real-world event data in §6.3 were conducted on an NVIDIA TITAN RTX GPU. The experiments with synthetically generated data in §5 can be run on a local M1 CPU with 64 GB of RAM in less than 10 minutes. Limiting factors are the selected hyperparameter sizes for the latent states $K$ and communities $C$, as well as the number of time series and their length. We discuss further model limitations in §9 and §3.3.
A  Expected Value of the Ordered Matrix Dirichlet

The following behavior has been observed:

$$
E[\phi_{11} | \alpha_1 = \cdots = \alpha_A] \neq E[\phi_{K1} | \alpha_1 = \cdots = \alpha_A]
$$

(7)

This is unappealing. Let’s get an analytic form for these expectations to understand why and to understand how to set \(\alpha_1 \ldots \alpha_A\) to get this form of symmetry.

What is the expectation \(E[\phi_{11} | \alpha_1 = \cdots = \alpha_A]\)? This will involve looking at the order statistics of the Beta distribution. From Algorithm 1, we know that \(\phi_{11}\) is drawn

$$
P(\phi_{11} | \alpha_1, \ldots, \alpha_A) = \text{Beta}^{(1)} \left( \alpha_1, \sum_{a=2}^A \alpha_a \right)
$$

(8)

where Beta\(^{(k)}\)(\(\alpha_1, \alpha_2\)) denotes the \(k\)th order statistic of a Beta distribution with parameters \(\alpha_1\) and \(\alpha_2\). The order statistic of the Beta is known to be:

\[
\text{Beta}^{(k)}(\phi; \alpha_1, \alpha_2) = k(1 - \phi)^{a_2-1} \phi^{a_1-1} I_\phi(\alpha_1, \alpha_2) k^{-1} \left(1 - \frac{I_\phi(\alpha_1, \alpha_2) \Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)}\right)^{n-k} \frac{\Gamma(n+1)}{\Gamma(k+1) \Gamma(n-k+1)} \beta(\alpha_1, \alpha_2)
\]

(9)

\[
= \frac{\phi^{a_1-1}(1 - \phi)^{a_2-1}}{\beta(\alpha_1, \alpha_2)} \left(1 - \frac{I_\phi(\alpha_1, \alpha_2)}{\beta(\alpha_1, \alpha_2)}\right)^{n-k} \left(I_\phi(\alpha_1, \alpha_2)\right)^{k-1} \frac{1}{k}
\]

(10)

B  Supplementary Technical Details

B.1 Details of the Banded Matrix Dirichlet (BMD).

In this section, we elaborate on the Banded Matrix Dirichlet (BMD) introduced in §3.2. For simplicity, we consider a square matrix \(\Pi \in [0,1]^{K \times K}\), but the BMD can be non-square as well. We assume that the \(k\)th state can only be excited by its directly adjacent states, \((k-1)\)th and \((k+1)\)th, as well as by itself (Schrodt, 2006; Randahl and Vegelius, 2022). This results in a matrix whose non-zero elements are banded along the diagonal following:

\[
\pi_{kk} = \begin{cases} 
\pi_k^{(\uparrow)} & \text{if } k = k + 1 \text{ (escalating)} \\
\pi_k^{(\downarrow)} & \text{if } k = k - 1 \text{ (descending)} \\
\pi_k^{(\text{st})} & \text{if } k = k \text{ (steady)} \\
0 & \text{otherwise}
\end{cases}
\]

(11)

Finally, we place a Dirichlet prior over the three non-zero elements in each \(k\)th row

\[
(\pi_k^{(\uparrow)}, \pi_k^{(\downarrow)}, \pi_k^{(\text{st})}) \sim \text{Dir}(\alpha_0^{(\uparrow)}, \alpha_0^{(\downarrow)}, \alpha_0^{(\text{st})})
\]

(12)

Moreover, we can consider a wider bandwidth \(b \geq 1\) so that components \(k' \in \{k - b, \ldots, k + b\}\) all excite \(k\). An example of the full vector might then look like

\[
\pi_k = (0, \ldots, 0, \pi_k^{(\downarrow)}, \pi_k^{(\text{st})}, \pi_k^{(\uparrow)}, 0, \ldots, 0)
\]

(13)

Relevant Links.
Integrated Crisis Early Warning System (ICEWS)
https://dataverse.harvard.edu/dataverse/icews
Goldstein Scale
https://parusanalytics.com/eventdata/cameo.dir/CAMEO.SCALE.txt
Conflict and Mediation Event Observations (CAMEO)
https://parusanalytics.com/eventdata/cameo.dir/CAMEO.09b6.pdf
Figure 9: Posterior parameters of Dynamic Poisson Tucker Model, with $K = 6$ latent states and $C = 12$ latent communities, fitted to full temporal range (2000-2020) of ICEWS data. We find that the probability mass of the transition matrix is centered along the diagonal revealing step-wise (de-)escalatory dynamics. Particularly state 5 has high probability representing an “absorbing state” of conflict that is hard to escape. The country-community affiliation matrix $\Psi$ provides no information on whether communities represent allies or enemies per se. To obtain this information, we interact the country-community matrix with the core tensor $\psi_{c_1c_2}^{t\rightarrow t'} \sum_{c_2=1}^{C} \sum_{j=1}^{J} \psi_{c_2j}^{t\rightarrow t'} \lambda_{c_1k}^{t}$ for specific choice of $k$ and $t$.

Figure 10: Descriptive statistics showing total number of interactions between countries in ICEWS data from 2015 to 2020. The rows and columns are sorted by the total number of actions a country is involved in. Note that we omit self-targeted actions.

Figure 11: Recovering ground truth structures in transition and emission matrices of a dynamical system. Conventionally, rows are samples independently from a (standard) Dirichlet distribution. This can result in label switching making the latent states (topics) difficult to interpret. This is particularly problematic if states are ordinal, e.g. representing “ally”, “neutral” and “enemy” relations.
Figure 12: Imputation results of synthetic data experiments. As discussed in §5, we generate time series with different ground truth transition structures: “banded”, “bonbon”, “triangle”. We fit a Hidden Markov Model (HMM) to a train set of these data and evaluate imputation performance on a test set. In contrast to the forecasting experiments (Fig. 4), SMD + SMD outperforms OMD + OMD in two out of three cases on observed states. In contrast to forecasting, imputation does not necessarily require a model with temporal dynamics and the ordered transition matrix does not help. As expected, OMD + OMD performs better at imputing latent states because it circumvents label switching.

Figure 13: Ordered CAMEO action types with assigned Goldstein values. We order action types by Goldstein value first and, in case of a tie, by CAMEO ID second.

Figure 14: Illustration of the Dynamic Poisson Tucker Model (DPT) that combines the Poisson Tucker Decomposition and the Poisson-Gamma Dynamical System. The 4-mode count tensor of dyadic country interactions $Y$ is factorized into a country-community $\Psi$ and community-interaction core tensor $\Lambda$. 