POISSON STRUCTURES ON TWISTOR SPACES OF HYPERKÄHLER AND HKT MANIFOLDS

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Abstract. We characterize HKT structure in terms of nondegenerate complex Poisson bivector on hypercomplex manifold. We extend the characterization to the twistor space. After considering the flat case in detail, we show that the twistor space of hyperkähler manifold admits a holomorphic Poisson structure.

1. Introduction

HKT structures (an abbreviation from hyperkähler with torsion) were first introduced in String Theory (see [8]) as the structures induced on the target manifolds of (4, 0)-supersymmetric sigma models with Wess-Zumino term. From a mathematical viewpoint, compact HKT manifolds share many properties with the Kähler ones. They have local potential functions [1, 4] and well defined Hodge theory [11], which for spaces with $SL(n,\mathbb{H})$-holonomy, leads to a characterization in dimension eight similar to the topological characterization of Kähler compact complex surfaces [6]. On a hypercomplex manifold $M$, an HKT structure is given by a real positive $(2,0)$-form $\Omega$, which is $\partial$ closed. This also has a description in terms of a $\partial$-closed 2-form on its twistor space [1, 4]. As is well known, when $\Omega$ is closed, the structure is hyperkähler and $\Omega$ is holomorphic symplectic. A characterization of a hyperkähler structure from a twistor space perspective is given in [7] in terms of existence of a twisted holomorphic 2-form that is nondegenerate on each fiber of the twistor projection.

It is known that holomorphic symplectic forms are dual to holomorphic Poisson bivectors of maximal rank. Holomorphic Poisson structures have been studied from different perspectives. Recently, the interest in such structures is growing due to their connection to generalized complex geometry. One purpose of this note is to find characterizations of HKT structures on hypercomplex manifolds and their twistor spaces in terms of Poisson structures.

To this end, we need the notion of complex Poisson structures. Lichnerowicz has introduced complex Poisson structures in [9]. However, the definition in [9] is more restrictive than the one we need. We collect the necessary facts about complex Poisson structures in Section 2. They are straightforward analogs of the properties of the real Poisson structures. We note in particular that a nondegenerate complex Poisson structure is dual to a $\partial$-closed nondegenerate $(2,0)$ form. A direct consequence is the characterization in Section 3 of an HKT structure in terms of a complex Poisson one. Since a characterization in terms of twistor spaces is given in [1, 4], we formulate a similar existence result for complex Poisson structures on the twistor spaces of HKT manifolds. In Section 4, we consider in details the
flat case using local coordinates. In particular we observe that the twistor space has many commuting holomorphic vector fields, so there are also many holomorphic Poisson structures. Moreover, the complex Poisson structure found in Section 3 is also holomorphic. In the last section we prove that this structure is also holomorphic on the twistor space of any (possibly curved) hyperkähler manifold. Intuitively, this follows from the fact that the holomorphic structure on the twistor space, which is determined by the Chern connection, depends on the Levi-Civita connection on the hyperkähler base, but not on its curvature. This is not true for general twistor spaces.

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2. Complex Poisson structures and complex symplectic forms

Although Poisson structures on complex and almost complex manifolds have been introduced earlier in [9, 3], we need slightly different terminology, adapted to our purposes. A complex Poisson structure is a bivector $P$ on a (almost) complex manifold of type $(2, 0)$, such that $[P, P] = 0$. This definition is useful when the manifold is complex and implicit in [10]. In this note, this definition will be applied in the hypercomplex case. Note that it is weaker than the one given by Lichnerowicz [9] since it doesn’t imply that the real and imaginary part of $P$ commute: $[ReP, ImP] \neq 0$.

Let $M$ be a complex manifold and $T^{p,q}, \Lambda^{p,q}$ the spaces of $(p, q)$-vectors and $(p, q)$-forms, respectively. Then $\partial = d + id^c, \overline{\partial} = d - id^c$ are the standard operators defined by $\partial|_{\Lambda^{p,q}} = \pi_{p+1,q} \circ d$ for the projection $\pi_{p+1,q} : \Lambda^{p+q+1} \rightarrow \Lambda^{p+1,q}$. We have the following simple observation, similar to the real case:

Lemma 2.1. If $P$ is a $(2,0)$ bivector on a complex manifold, given by $P = \frac{1}{2} P^{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}$ in local complex chart $(z_1, z_2, \ldots, z_n)$, then for $i < j < k$

$$[P, P](dz^i, dz^j, dz^k) = 2 \left( P^{ih} \partial_h P^{jk} + P^{jh} \partial_h P^{ki} + P^{kh} \partial_h P^{ij} \right)$$

Proof. A direct calculation, similar to the real case gives

$$[P, P] = \frac{1}{2} P^{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} + \frac{1}{2} P^{kl} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l}$$

$$= \frac{1}{4} \left[ P^{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} + P^{kl} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l} - \left[ P^{ij} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} + P^{kl} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l} \right] \wedge \left( P^{kl} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l} \right) \right]$$

$$= \frac{1}{4} \left[ -P^{ij} \frac{\partial P^{kl}}{\partial z^i} \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l} + P^{ij} \frac{\partial P^{kl}}{\partial z^j} \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^l} - P^{kl} \frac{\partial P^{ij}}{\partial z^k} \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^l} \right]$$

$$+ \frac{1}{4} \left( P^{kl} \frac{\partial P^{ij}}{\partial z^l} \frac{\partial}{\partial z^k} \wedge \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} \right)$$

$$= 2 \left( P^{ih} \partial_h P^{jk} + P^{jh} \partial_h P^{ki} + P^{kh} \partial_h P^{ij} \right) \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j} \wedge \frac{\partial}{\partial z^k}$$
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for $i < j < k$. From here the Lemma follows.

As in the real case, $P$ defines a linear map $P : \Lambda^{1,0} \to T^{1,0}$ from the space of $(1,0)$-forms to the space of $(1,0)$ vectors at each point. For each complex valued function $f$, denote by $P(df) = P(\partial f) = X_f$ its Hamiltonian vector field. Then $\{f, g\} := X_f(g) = -X_g(f)$ defines a bracket operation on complex functions, just like in the real case. From Lemma 2.1, it satisfies the Jacobi identity (or equivalently $[X_f, X_g] = X_{\{f,g\}}$) iff $P$ is (complex) Poisson. Note that for $f$ and $g$ holomorphic, $\{f, g\}$ is not necessarily holomorphic, so there is no analog of the notion of "symplectic foliation" in this case. The bracket $\{f, g\}$ is holomorphic only when the functions $P^{ij}$ are holomorphic and $P$ is called holomorphic Poisson in this case. As is well known, a holomorphic Poisson structure defines a holomorphic symplectic foliation.

Suppose now that the complex $(2,0)$ bivector $P$ is of maximal rank at each point. Then $P : \Lambda^{1,0} \to T^{1,0}$ is invertible and $\omega = P^{-1} : T^{1,0} \to \Lambda^{1,0}$ defines a $(2,0)$ form via $\omega(P(\partial f), P(\partial g)) = P(\partial f, \partial g)$. Once more similar to the real case, we have:

**Lemma 2.2.** For nondegenerate complex Poisson structure $P$ and $\omega = P^{-1}$ as above

$$\partial \omega(X_f, X_g, X_h) = 3/2[P, P](df, dg, dh)$$

**Proof.**

\[
d\omega(X_f, X_g, X_h) = X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) - \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g)
\]

\[
= X_f(\{g, h\}) + X_g(\{h, f\}) + X_h(\{f, g\}) + [X_f, X_g](h) + [X_g, X_h](f) + [X_h, X_f](g)
\]

\[
= 3(\{f, g, h\} + \{g, h, f\} + \{h, f, g\})
\]

Since $X_f, X_g, X_h$ are $(1,0)$ vectors, $d\omega(X_f, X_g, X_h) = \partial \omega(X_f, X_g, X_h)$. Now applying Lemma 3.1 we have

\[
[P, P](dz_i, dz_j, dz_k) = 2(\{z_i, \{z_j, z_k\}\} + \{z_j, \{z_k, z_i\}\} + \{z_k, \{z_i, z_j\}\})
\]

So

\[
d\omega(X_{z_i}, X_{z_j}, X_{z_k}) = 3/2[P, P](dz_i, dz_j, dz_k)
\]

and the statement follows.\[\]

From here we obtain:

**Theorem 2.3.** A complex non-degenerate $(2,0)$ bivector is Poisson iff its dual 2-form $\omega = P^{-1}$ is $\partial$-closed, that is, $\partial \omega = 0$.

3. POISSON STRUCTURES ON HKT MANIFOLDS AND THEIR TWISTOR SPACE

We begin by establishing some notations in the presence of a metric. Suppose $g$ is a Riemannian metric and $I$ a complex structure on a manifold $M$ such that $\omega(X, Y) = g(IX, Y)$ is the fundamental form. Let $\sharp : T^* \to T$ be the isomorphism given by $g(\sharp \alpha, Y) = \alpha(Y)$ and denote by $\omega$ and $\omega^{-1}$ the maps $\omega : T \to T^*$ and $\omega^{-1} : T^* \to T$ given as $\omega(\omega^{-1}(\alpha), Y) = \alpha(Y)$ and $\omega(X) = i_X \omega$. Then $\omega(\omega^{-1}) = Id|_{T^*}$, $\omega^{-1}(\omega) = Id|_T$. Thus $\alpha(Y) = \omega(\omega^{-1}(\alpha), Y) = g(I\omega^{-1}(\alpha), Y) = g(\sharp \alpha, Y)$, and $\omega^{-1} = -I \circ \#$. Extending $I$ on $T^*$ in a standard way,
The integrability of imaginary parts of the $\partial$ family

Recall that a hypercomplex structure is a triple $I, J, K$ of complex structures satisfying the quaternionic identities $IJ = -JI = K$. Let $I, J, K, g$ be a hyperhermitian structure and $\omega_I, \omega_J, \omega_K$ be the corresponding 2-forms. From above

$$\# = I\omega_I^{-1} = J\omega_J^{-1} = K\omega_K^{-1}$$

and from here

$$I\omega_J^{-1} = \omega_K^{-1}$$

We want to consider the “complexified” version of the Poisson and symplectic structures on a Hermitian manifold. Let $\Omega = \omega + i\omega_K$ and denote by $\Omega : T^{1,0} \to T^{*1,0}$ the map given by $\Omega(X^{1,0}) = iX\Omega$. Since $\Omega|_{T^{0,1}} = 0$, it can be extended to a map on the whole complexified tangent space. We want to find the real and imaginary parts of $\Omega^{-1}|_{\Lambda^{1,0}}$. First we calculate $\omega_I^{-1}$ via $\omega_I^{-1}(\alpha)(Y) = -JK\alpha(Y) = -I\alpha(Y)$. We see that $\omega_I\omega_K^{-1} = -\omega_K\omega_I^{-1}$. Then we have $(\omega_I + i\omega_K)(\omega_I^{-1} - i\omega_K^{-1})(\alpha) = 2\alpha + 2i\alpha = 2\alpha^{1,0}$. Thus, on $\Lambda^{1,0}$ we have $\Omega^{-1} = \frac{1}{4} (\omega_I^{-1} - i\omega_K^{-1})$ and $\omega_J^{-1} - i\omega_K^{-1}|_{\Lambda^{0,1}} = 0$, so the complex bivector $P$ given by

$$P(\partial f, \partial g) = \Omega(\Omega^{-1}(\partial f), \Omega^{-1}(\partial g))$$

satisfies $P = \frac{1}{4} (\omega_I^{-1} - i\omega_K^{-1})$, where $\partial$ is the operator defined by $I$.

A hyperhermitian metric $g$ is called HKT if $d_I^c\omega_I = d_J^c\omega_J = d_K^c\omega_K$ where $d_I^c, d_J^c, d_K^c$ are the imaginary parts of the $\partial$ operators for $I, J, K$. The definition is equivalent to the existence of a hyperhermitian connection with skew-symmetric torsion, which is the original one proposed by [8]. In [1], it is shown that the HKT condition is equivalent to:

$$\partial(\omega_I + i\omega_K) = \partial \Omega = 0$$

As a direct consequence of Theorem [23] we obtain:

**Theorem 3.1.** Let $(M, I, J, K, g)$ be a hyperhermitian manifold and $P$ as in [11]. Then $g$ is HKT if $P$ is complex Poisson.

Let $Z = M \times S^2$ be endowed with the “tautological” complex structure $\mathcal{I}$ defined as $\mathcal{I}_{(a, b)} = (I_a, I_{S^2})$ where for $a = (a, b, c)$, $I_a = aI + bJ + cK$. In these terms, $I_{S^2}$ is the canonical complex structure on $S^2 \cong \mathbb{CP}^1$. It is well known that the structure $\mathcal{I}$ is integrable and the complex manifold $(Z, \mathcal{I})$ is called the twistor space of $(M, I, J, K)$. The identification of $S^2$ with $\mathbb{CP}^1$ is given by the stereographic projection:

$$St: \lambda \to a = \left( \frac{|\lambda|^2 - 1}{1 + |\lambda|^2}, \frac{i(\overline{\lambda} - \lambda)}{1 + |\lambda|^2}, \frac{-\lambda + \overline{\lambda}}{1 + |\lambda|^2} \right) \in S^2.$$ 

If $\lambda$ corresponds to $a = (a, b, c)$ via the inverse map $\lambda = St^{-1}(a, b, c)$, then $\lambda = 0$ corresponds to $I_{(-1,0,0)} = I$, $\lambda = i$ corresponds to $I_{(0,1,0)} = J$, and $\lambda = -1$ corresponds to $I_{(0,0,-1)} = K$.

**Remark 3.2.** The integrability of $\mathcal{I}$ is equivalent to the fact that all structures in the $S^2$-family $aI + bJ + cK$, for $a = (a, b, c) \in S^2$, are integrable and have the same properties. Therefore, instead of using $I, J, K$ in Theorem [3.1] we may as well use $I_a, I_b, I_c$ for $a, b, c$ an orthonormal triple in $S^2$ with $c = a \times b$. 

\[\omega^{-1} = \sharp \circ I\text{.} \] The map $\omega^{-1}$ defines a bivector $\omega^{-1} \in \Lambda^2 T$ as $\omega^{-1}(\alpha, \beta) = \beta(\omega^{-1}(\alpha))$ and it is the same as the map defined by a Poisson bivector $P$ above.
Suppose now that \( g \) is a hyperhermitian metric on \( M \) and \((Z_1, W_1, Z_2, W_2, \ldots, Z_n, W_n)\) is a unitary basis of the space \( T^1_2(M) \) of \((1, 0)\)-vectors for the structure \( I \) at a point \( x \) such that \( J(Z_i) = \overline{W_i}, J(W_i) = -Z_i. \) We call such basis quaternionic - Hermitian. For the structure \( I_a \) via the stereographic projection, the vectors:

\[
Z_i^\lambda = \frac{1}{\sqrt{1 + |\lambda|^2}}(\lambda Z_i - \overline{W_i}), W_i^\lambda = \frac{1}{\sqrt{1 + |\lambda|^2}}(\lambda W_i + \overline{Z_i})
\]

form a unitary basis for \((g, I_a)\). Let \((\delta_i, \sigma_i), i = 1, \ldots, n\) be the dual quaternionic-Hermitian basis of the cotangent \((1, 0)\)-bundle at \( x \) of \((Z_i^\lambda, W_i^\lambda)\). Then it is given by

\[
\sigma_i^\lambda = \frac{1}{\sqrt{1 + |\lambda|^2}}(\lambda \sigma_i - \overline{\delta_i}), \quad \delta_i^\lambda = \frac{1}{\sqrt{1 + |\lambda|^2}}(\lambda \delta_i + \overline{\sigma_i})
\]

In these terms we have that \( F_a = aF_I + bF_J + cF_K \) is also given by \( F_a = i/2 \sum (\sigma_i^\lambda \wedge \overline{\delta_i^\lambda} + \delta_i^\lambda \wedge \overline{\sigma_i^\lambda}) \) and the dual 2-vector is given by \( P_a = -2i \sum (Z_i^\lambda \wedge \overline{Z_i^\lambda} + W_i^\lambda \wedge \overline{W_i^\lambda}) \). We want to calculate the \((2, 0)\)-part \( P_t|_{I_a}^{(2, 0)} \) of

\[
P_t = -2i \sum (Z_i \wedge \overline{Z_i} + W_i \wedge \overline{W_i})
\]

with respect to \( I_{bfa} \). First, we see that

\[
Z_i = \frac{1}{\sqrt{1 + |\lambda|^2}}(\lambda Z_i^\lambda + \overline{W_i^\lambda}), W_i = \frac{1}{\sqrt{1 + |\lambda|^2}}(\lambda W_i^\lambda - \overline{Z_i^\lambda})
\]

and similar expressions hold for the conjugates \( \overline{Z_i}, \overline{W_i} \).

Now we have

\[
P_t = \frac{-2i}{1 + |\lambda|^2} \sum (\lambda Z_i^\lambda + \overline{W_i^\lambda}) \wedge (\lambda Z_i^\lambda + W_i^\lambda) + (\lambda W_i^\lambda - \overline{Z_i^\lambda}) \wedge (\lambda W_i^\lambda - \overline{Z_i^\lambda})
\]

\[
= \frac{-2i}{1 + |\lambda|^2} \sum (|\lambda|^2 - 1)(Z_i^\lambda \wedge \overline{Z_i^\lambda} + W_i^\lambda \wedge \overline{W_i^\lambda}) + 2\lambda Z_i^\lambda \wedge W_i^\lambda - 2\lambda \overline{Z_i^\lambda} \wedge \overline{W_i^\lambda}
\]

So

\[
P_t|_{I_a}^{(2, 0)} = -\frac{4i\lambda}{1 + |\lambda|^2} Z_i^\lambda \wedge W_i^\lambda
\]

We can also consider \( b = \frac{1}{1 + |\lambda|^2}(i(\overline{\lambda} - \lambda), 1 + \frac{1}{2}(\lambda^2 + \overline{\lambda}^2), -\frac{1}{2}(\lambda^2 - \overline{\lambda}^2)) \) and \( c = \frac{1}{1 + |\lambda|^2}(\overline{\lambda} + \lambda), -\frac{1}{2}(\lambda^2 - \overline{\lambda}^2), 1 - \frac{1}{2}(\lambda^2 + \overline{\lambda}^2)) \) and notice that \( F_b + IF_c = \sum \sigma_i^\lambda \wedge \delta_i^\lambda \) as well as \( c = a \times b \) with \( b \) orthogonal to \( a \). Specifically, the matrix \( A = (a, b, c) \) is a special orthogonal matrix whose inverse is its transpose. Then for the corresponding structures \( I_b \) and \( I_a \) we can find that \( P_b + iP_c = \sum Z_i^\lambda \wedge W_i^\lambda \). Thus we have the following:

\[
(2) \quad P_t|_{I_a}^{(2, 0)} = -\frac{4i\lambda}{1 + |\lambda|^2} (P_b + iP_c)
\]

Denote also by \( \pi : Z \to S^2 \) and \( \pi_1 : Z \to M \) the two projections - \( \pi \) is holomorphic , but \( \pi_1 \) is not. Let also \( s_a : M \to Z \) be the section of \( \pi \) defined as \( s_a(x) = (x, a) \) for every \( x \). Then we can formulate the following:
Theorem 3.3. Let $g$ be a hyperhermitian metric on $M$ and $P_I$ be the bivector dual to the fundamental form for $I$. Let $P_{(x,a)} = (s_a)_*P_I$ be the bivector on $Z$ defined as the $(2,0)$ component of $(s_a)_*(P_I)$ with respect to $I$. Then $g$ is HKT iff $P$ is complex Poisson.

Proof: First we notice that

$$((s_a)_*P_I)^{(2,0)} = ((s_a)_*P_I)^{(2,0)}$$

where $P_I$ is as above the $(2,0)$ part of $P_I$ with respect to $I$. This follows from the definition of $I$ at a point $(x,a)$. From (2) we have $P_I^{(2,0)} = \frac{-4\lambda}{1+|\lambda|^2}(P_b + iP_c)$. Then Theorem 3.1 and Remark 3.2 give that $[P_b + iP_c, P_b + iP_c] = 0$ iff $g$ is HKT. We notice that

$$((s_a)_*P_I)^{(2,0)} = (s_a)_*[P_{(x,a)}^{(2,0)}, P_{(x,a)}^{(2,0)}].$$

Therefore, the theorem follows because $\frac{-4\lambda}{1+|\lambda|^2} \neq 0$ for almost all $a$, and by continuity $[P, P]$ vanishes everywhere.

Q.E.D.

Remark 3.4. The statement is dual to the characterization of the HKT structure by Banos and Swann ([3]) in terms of the twistor space: If $F$ is the fundamental form for the Hermitian structure $(g, I)$ then $g$ is HKT iff $\partial(\pi^*F)^{(2,0)} = 0$ where $\partial$ is the operator with respect to $I$ on $Z$. Banos and Swann used it to prove that every HKT structure has a local HKT-potential. In [3], there was a slightly different twistor characterization of the HKT condition, which is based on Paredes of the hyperkähler structures in [7].

4. Poisson structures on the twistor space of a flat hyperkähler space

In this section we use the set up and notations from [3]. Choose linear coordinates $(z^a_1, z^a_2), a = 1, \ldots, m$, for $\mathbb{C}^m = \mathbb{R}^{4m}$, related to the real coordinates by

$$z^a_1 = x_{2a-1} + ix_{2a}, \quad z^a_2 = y_{2a-1} + iy_{2a}$$

The twistor space $Z = Z(\mathbb{R}^{4m})$ of $\mathbb{R}^{4m}$ is the bundle $\mathbb{C}^m \otimes O(1)$ on $\mathbb{C}P^1$ [2] Example 13.64 and Example 13.66). On $\mathbb{C}P^1$, the homogeneous coordinates are given as $[\lambda_1, \lambda_2]$ and on the open $U_i$ given by $\lambda_i \neq 0$, $i=1,2$, hence we have local coordinates $\lambda = \frac{\lambda_1}{\lambda_2}$ and $\mu = \frac{\lambda_2}{\lambda_1}$ respectively.

On $U_2$, the product coordinates $\{z^a_1, z^a_2, \lambda\}$ are not holomorphic. The holomorphic coordinates are

$$w^a_1 = \lambda z^a_1 - \overline{w}^a_2, \quad w^a_2 = \lambda z^a_2 + \overline{w}^a_1, \quad \zeta = \lambda.$$  

The inverse coordinate change is

$$z^a_1 = \frac{1}{1 + |\zeta|^2} (\overline{\zeta}w^a_1 + \overline{w}^a_2), \quad z^a_2 = \frac{1}{1 + |\zeta|^2} (-\overline{w}^a_1 + \overline{\zeta}w^a_2), \quad \mu = \zeta.$$  

In particular,

$$\frac{\partial}{\partial w^a_1} = \frac{1}{1 + |\lambda|^2} \frac{\partial}{\partial z^a_1} - \frac{1}{1 + |\lambda|^2} \frac{\partial}{\partial z^a_2}, \quad \frac{\partial}{\partial w^a_2} = \frac{1}{1 + |\lambda|^2} \frac{\partial}{\partial z^a_2} + \frac{1}{1 + |\lambda|^2} \frac{\partial}{\partial z^a_1}$$

are local holomorphic vector fields on $Z$ defined whenever $\lambda \neq \infty$. We notice also that

$$\frac{\partial}{\partial \lambda} = \sum_a \left( -\frac{\overline{z}^a_2}{(1 + |\lambda|^2)^2} \frac{\partial}{\partial z^a_1} + \lambda \frac{\partial}{\partial z^a_2} \right) - \frac{\overline{z}^a_1}{(1 + |\lambda|^2)^2} \frac{\partial}{\partial z^a_2} - \lambda \frac{\partial}{\partial z^a_1} \right) + \frac{\partial}{\partial \lambda}$$
Then \( \frac{\partial}{\partial \lambda} = \frac{\partial}{\partial \zeta} + \sum e(\zeta^2 \frac{\partial}{\partial \omega^2} - \zeta \frac{\partial}{\partial \omega^1}) \) and we see that \( \frac{\partial}{\partial \lambda} \) is a smooth \((1,0)\)-vector field but not holomorphic on \( Z \).

When one changes coordinates from \( \lambda_2 \neq 0 \) to \( \lambda_1 \neq 0 \), \( \lambda \tilde{w}^a_j = w^a_j \). Therefore,

\[
V_j^a = \frac{1}{\lambda_2} \frac{\partial}{\partial w^a_j} = \frac{1}{\lambda_1} \frac{\partial}{\partial \tilde{w}^a_j}
\]

are globally defined as vector fields on \( \mathbb{R}^{4m} \times \{ \mathbb{C}^2 - (0,0) \} \). The vector fields on \( Z \)

\[
W_k^a = \frac{1}{2} \left( I_k - \frac{\partial}{\partial x_2a-1} - i \lambda_a I_k \frac{\partial}{\partial x_2a-1} \right).
\]

are well defined at a point \((x,a) \in Z\). These vector fields can also be identified as

\[
W_0^a = \lambda_1 V_1^a + \lambda_2 V_2^a, \quad W_1^a = i(\lambda_1 V_1^a - \lambda_2 V_2^a),
W_2^a = \lambda_1 V_2^a - \lambda_2 V_1^a, \quad W_3^a = i(\lambda_1 V_2^a + \lambda_2 V_1^a).
\]

Clearly, \( W_i^a \) are global holomorphic vector fields on \( Z \) which also commute. In particular if \( W = \text{span}W_k^a \), then any nonzero element in \( \Lambda^2(W) \) is a Poisson structure on \( Z \). In particular one can see that

\[
\lambda_1^2 \sum V_1^a \wedge V_2^a, \lambda_1 \lambda_2 \sum V_1^a \wedge V_2^a, \lambda_2^2 \sum V_1^a \wedge V_2^a
\]

define holomorphic Poisson structures on \( Z \). Moreover we see that the bivector \( \mathcal{P} \) from Theorem 3.3 is given by \( \mathcal{P} = \lambda_1 \lambda_2 \sum V_1^a \wedge V_2^a \). Finally we notice that all vector and bivector fields descend to the quotients of \( \mathbb{R}^{4m} \) by a commutative lattice induced by translations. So they are globally defined also on the torus and its twistor space. As a result of this discussion we obtain:

**Theorem 4.1.** Let \( M = T^{4m} \) be endowed with its flat hyperkähler structure and \( \mathcal{P} = \mathcal{P}_{(x,a)} \) be the bivector defined in (2) on its twistor space \( Z \). Then \( \mathcal{P}, \frac{1}{\lambda} \mathcal{P}, \lambda \mathcal{P} \) are globally defined holomorphic Poisson structures on \( Z \).

In the next section we partially extend this result to the twistor space of arbitrary hyperkähler manifolds.

5. **Holomorphic Poisson structures on twistor spaces of hyperkähler manifolds**

Unlike the complex case, "quaternionic" coordinates like \((w^a_i)\) exist only in the flat case. However, the 2-vector \( \mathcal{P} \) is a global complex Poisson structure and doesn’t depend on existence of such coordinates. To check whether it is holomorphic we can use the Chern connection defined as the only metric connection for which the \((0,1)\) part coincides with the \( \bar{\partial} \) operator on the tangent space. For a Hermitian manifold with metric \( g \) and complex structure \( J \), this connection is determined by

\[
g(\nabla^{Ch}_X Y, Z) = g(\nabla^{LC}_X Y, Z) - \frac{1}{2} dF(JX, Y, Z)
\]

where \( \nabla^{LC} \) is the Levi-Civita connection and \( F(X,Y) = g(JX,Y) \) is the fundamental form.

In general, \( dF \) for the twistor space contains the component of the curvature of the base manifold \( M \). However, in the hyperkähler case it does not, so one expects that the flat
and the general curved case will not be different. We confirm this observation with explicit calculations.

**Theorem 5.1.** If $M$ is hyperkähler, then $\mathcal{P}$ is a holomorphic Poisson structure which vanishes on the two fibers of $\pi : Z \to \mathbb{C}P^1$ corresponding to $I$ and $-I$ or $\lambda = 0, \infty$. The leaves of the symplectic foliation are given by the fibers of $\pi : Z - \{\pi^{-1}(0, \infty)\} \to \mathbb{C}P^1 - \{0, \infty\}$ and the points of $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$. The converse also holds: if $g$ is a hyperhermitian metric on $M$ and $\mathcal{P}$ is holomorphic Poisson on $Z = Z(M)$, then $g$ is hyperkähler.

**Proof:** Endow the twistor space $Z = M \times \mathbb{C}P^1$ with the product metric $g_Z = (g, g_{FS})$ where $g_{FS}$ is the canonical (Fubini-Studii) metric on $\mathbb{C}P^1 = S^2$. Then $(g_Z, I)$ is a Hermitian structure and we have to show that $\nabla^{0,1}_Z \mathcal{P} = 0$ for every $(0, 1)$ vector $X^{0,1}$ on $Z$. Consider as before a local quaternionic-hermitian frame $Z_i, W_i$ on $M$. The local frame

$Z_i^\lambda = \frac{1}{\sqrt{1 + |\lambda|^2}}(XZ_i - \overline{W}_i), W_i^\lambda = \frac{1}{\sqrt{1 + |\lambda|^2}}(\overline{XW}_i + Z_i), \frac{\partial}{\partial \lambda}$

consists of smooth $(1, 0)$ vectors on $Z$ which which is orthogonal, but not orthonormal because $\frac{\partial}{\partial \lambda}$ is not normalized. Then $\mathcal{P} = \sum_i Z_i^\lambda \wedge W_i^\lambda$. We use for $X^{0,1}$ the vectors of the conjugate $(0, 1)$ basis of the basis above. We first check that $\nabla^{Ch}_Z \mathcal{P} = 0$. To this end we use that

$\nabla^{Ch}_{X^{0,1}} Y^{1,0} = [X^{0,1}, Y^{1,0}]^{1,0}$

where superscript $\{1, 0\}$ means the $(1, 0)$-component and same for $\{0, 1\}$. From here we have also that $\nabla^{Ch}_{X^{0,1}} Y^{1,0} \wedge Z^{1,0} = [X^{0,1}, Y^{1,0}]^{1,0} \wedge Z^{1,0} + Y^{1,0} \wedge [X^{0,1}, Z^{1,0}]^{1,0} = [X^{0,1}, Y^{1,0} \wedge Z^{1,0}]^{2,0}$. Since $Z = M \times \mathbb{C}P^1$ as a smooth manifold, $\frac{\partial}{\partial \lambda}, Z_i = \left[ \frac{\partial}{\partial \lambda}, W_j \right] = 0$ and we obtain

\begin{align*}
\left[ \frac{\partial}{\partial \lambda}, Z_i^\lambda \right] &= \frac{1}{(1 + |\lambda|^2)^2} (Z_i + \lambda \overline{W}_i) = \frac{1}{1 + |\lambda|^2} \overline{W}_i^\lambda \\
\text{as well as} \\
\left[ \frac{\partial}{\partial \lambda}, W_i^\lambda \right] &= - \frac{1}{1 + |\lambda|^2} \overline{Z}_i^\lambda \\
\text{and} \\
\left[ \frac{\partial}{\partial \lambda}, \mathcal{P} \right] &= - \frac{1}{1 + |\lambda|^2} \left( Z_i^\lambda \wedge \overline{Z}_i^\lambda + W_i^\lambda \wedge \overline{W}_i^\lambda \right) \\
\text{So} \\
\nabla^{Ch}_Z \mathcal{P} &= 0
\end{align*}

Now for the other vectors we need to use the definition and the fact that $\sum Z_i \wedge W_i, \sum Z_i \wedge \overline{Z_i} + W_i \wedge \overline{W}_i$ are parallel with respect to the Levi-Civita connection on $Z$, since they are parallel on $M$ and the metric on $Z$ is the direct product of the metric on $M$ and the Fubini-Studii metric. In particular

$\nabla^{LC}_{Z_i} \mathcal{P} = \nabla^{LC}_{W_j} \mathcal{P} = 0$

since $Z_i^\lambda$ is a linear combination of $Z_i, W_j, \overline{Z}_i, \overline{W}_j$ and $\mathcal{P}$ is a linear combination of $\sum Z_i \wedge W_i, \sum Z_i \wedge \overline{Z}_i + W_i \wedge \overline{W}_i$ with coefficients depending only on $\lambda$. 


The difference between the Chern and Levi-Civita connection is proportional to the differential of
\[ F_\lambda = \sum \sigma^\lambda_i \wedge \bar{\sigma}^\lambda_i + \delta^\lambda_i \wedge \bar{\delta}^\lambda_i. \]
We use again that
\[
d \sum (\sigma_i \wedge \bar{\sigma}_i + \delta_i \wedge \bar{\delta}_i) = d \sum \sigma_i \wedge \delta_i = d \sum \bar{\sigma}_i \wedge \bar{\delta}_i = 0
\]
to obtain
\[
d F_\lambda = \frac{2d\lambda}{(1 + |\lambda|^2)^2} \wedge \left( \lambda \sum (\sigma_i \wedge \bar{\sigma}_i + \delta_i \wedge \bar{\delta}_i) - \sigma_i \wedge \delta_i - \lambda^2 \sum \bar{\sigma}_i \wedge \bar{\delta}_i \right) + \\
\frac{2d\bar{\lambda}}{(1 + |\lambda|^2)^2} \wedge \left( \bar{\lambda} \sum (\sigma_i \wedge \bar{\sigma}_i + \delta_i \wedge \bar{\delta}_i) + \lambda^2 \sum \sigma_i \wedge \delta_i + \sum \bar{\sigma}_i \wedge \bar{\delta}_i \right)
\]
and after substitution we get
\[
d F_\lambda = \frac{2d\lambda}{1 + |\lambda|^2} \left( \sum \sigma^\lambda_i \wedge \bar{\delta}^\lambda_i \right) - \frac{2d\bar{\lambda}}{1 + |\lambda|^2} \left( \sum \sigma^\lambda_i \wedge \delta^\lambda_i \right)
\]
From here we see that
\[
g(\nabla^\text{Ch}_{Z_i} W^\lambda_j, W^\lambda_j, X) = 0
\]
and similarly for \( \bar{W}^\lambda_j \). Hence
\[
\nabla^\text{Ch}_{Z_i} P = \nabla^\text{Ch}_{W^\lambda_j} P = 0
\]
which proves that \( P \) is holomorphic. It is of maximal rank on all points of \( Z \) except \( \pi^{-1}(0) \) and \( \pi^{-1}(\infty) \), so the Theorem follows.

Q.E.D.

**Remark 5.2.** In the the local basis \( Z_i = X_i - iIX_i, W_i = JX_i - iKX_i \), the local vector fields from (8) given as \( W^i_k = \frac{1}{2}(I_k X_i - iI_a I_k X_i) \) again are well defined for all \( a \in S^2 \) so \( \frac{1}{\lambda} P, \lambda P \) also can be expressed via \( W^i_k \). As a consequence we obtain that they are also globally defined and holomorphic Poisson as in Theorem [11].

In [7] the twistor space \( Z \) of a hyperkähler manifold is characterized in terms of the real structure on \( Z \) and the twisted holomorphic symplectic form on the fibers of \( \pi \). A similar characterization could be found in terms of the holomorphic Poisson structure \( P \) above. We leave the details of this discussion and some applications as future work.

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