CRITICAL ONE COMPONENT ANISOTROPIC REGULARITY FOR 3-D NAVIER-STOKES SYSTEM

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Abstract. Let us consider an initial data $v_0$ for the classical 3D Navier-Stokes equation with vorticity belonging to $L^{3/2} \cap L^2$. We prove that if the solution associated with $v_0$ blows up at a finite time $T^*$, then for any $p \in [4, \infty]$, $q_1 \in [1, 2]$, $\mu > 0$, $q_2 \in \|2, (1/p + \mu)^{-1}\|$, $\kappa \in [1, \infty]$, and any unit vector $e$, the $L^p$ estimate in time of $\|v(t)|e\|_{L^{3/2}}$ blows up at $T^*$.

Keywords: Navier-Stokes Equations, Blow-up criteria, Anisotropic Littlewood-Paley Theory

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1. Introduction

In this paper, we investigate necessary conditions for breakdown of regularity of regular solutions to the following 3-D incompressible Navier-Stokes system

\[
\begin{aligned}
\partial_t v + \text{div}(v \otimes v) - \Delta v + \nabla p &= 0, \\
\text{div } v &= 0, \\
v|_{t=0} &= v_0,
\end{aligned}
\]

where $v = (v^1, v^2, v^3)$ stands for the fluid velocity and $p$ for the scalar pressure function, which guarantees the divergence free condition of the velocity field.

Let us sum up the fact about this theory introduced in [8] that will be relevant in our work.

Theorem 1.1. Let $v_0$ be in the homogenous Sobolev space $\dot{H}^{3/2}$. Then there exists a unique maximal solution $v$ in the space $C([0, T^*]; \dot{H}^{3/2}) \cap L^2_{\text{loc}}([0, T^*]; \dot{H}^{3/2})$. Moreover, if $T^*$ is finite, then for any $p$ in $[2, \infty]$, there holds

\[
\int_0^{T^*} \|v(t, \cdot)\|_{\dot{H}^{3/2}}^p dt = \infty.
\]

The endpoint case when $p = \infty$ in the above theorem was proved by Kenig and Koch in [10], that is if the lifespan $T^*$ is finite, then $\limsup_{t \to T^*} \|v(t)\|_{\dot{H}^{3/2}}$ is infinite. This end point case can also be viewed as a consequence of the work [7] of Escauriaza, Seregin and Sverák, where the authors proved that if the lifespan $T^*$ is finite, then $\limsup_{t \to T^*} \|v(t)\|_{L^3}$ is infinite.

Before preceding, we recall the following family of spaces from [6]

Definition 1.1. For $r$ in $[\frac{3}{2}, 2]$, we denote by $\mathcal{V}^r$ the space of divergence free vector fields with the vorticity of which belongs to $L^3 \cap L^r$.

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In [5], Chemin and the second author proved the following component-reduction version of Theorem 1.1:

**Theorem 1.2.** Let \( v \) be the unique maximal solution \( v \) of \((NS)\) associated with \( v_0 \in \mathcal{V}^2 \). If its lifespan \( T^* \) is finite, then for any \( p \in [4, 6] \) and any unit vector \( e \), we have

\[
\int_0^{T^*} \| (v(t)|e)_{\mathbb{R}^2} \|_{H_0^{1/2 + 2/p}}^p \, dt = \infty.
\]

Recently in [6], the blow-up criteria (1.2) was improved to \( p \in [4, \infty] \) provided that the initial data \( v_0 \) belongs to a more regular space \( \mathcal{V}^3 \cap \mathcal{V}^2 \). While the authors in [9] dealt with the remaining case \( p \in [2, 4] \). One may check [5] for more references concerning the regularity criteria of solutions to three-dimensional Navier-Stokes system.

Another improvement of Theorem 1.1 is the well-known Ladyzhenskaya-Prodi-Serrin criteria, more precisely, if the life-span \( T^* \) for smooth enough solutions of \((NS)\) is finite, then

\[
\int_0^{T^*} \| (v(t)|e)_{\mathbb{R}^2} \|_{L^p}^p \, dt = \infty, \quad \text{where} \quad \frac{2}{p} + \frac{3}{q} = 1.
\]

We point out that, in (1.3), there is no requirement on the derivative estimate of the solution \( v \). Whereas the one component version (1.2) requires more than half derivative estimate on \( v \). The purpose of this paper is to reduce the order of derivative estimate in (1.2).

Let us mention that, as in [3], [4], [5] and [12], the definitions of the function spaces we are going to work with require anisotropic dyadic decomposition of the Fourier variables. Let us first recall some basic facts on anisotropic Littlewood-Paley theory from [1]

\[
\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\hat{a}), \quad \Delta^k_j a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{k\ell}|)\hat{a}), \quad \Delta_\ell a = \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_{\ell}|)\hat{a}),
\]

\[
S_j a = \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{a}), \quad S^k_j a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi_k|)\hat{a}), \quad S^k_\ell a = \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_\ell|)\hat{a}) \quad \text{and} \quad \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1} \quad \tilde{\Delta}^k_\ell = \Delta^k_{\ell-1} + \Delta^k_\ell + \Delta^k_{\ell+1},
\]

where \( \xi_{k\ell} = (\xi_1, \xi_2) \), \( \mathcal{F} a \) and \( \hat{a} \) denote the Fourier transform of the distribution \( a \), \( \chi(\tau) \) and \( \varphi(\tau) \) are smooth functions such that

\[
\text{Supp} \ \varphi \subset \left\{ \tau \in \mathbb{R} / \ 3/4 \leq |\tau| \leq 8/3 \right\} \quad \text{and} \quad \forall \tau > 0, \ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1,
\]

\[
\text{Supp} \ \chi \subset \left\{ \tau \in \mathbb{R} / \ |\tau| \leq 4/3 \right\} \quad \text{and} \quad \forall \tau \in \mathbb{R}, \ \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1.
\]

**Definition 1.2.** Let us define the space \( \dot{B}_{p_1, r_1}^{s_1}(\dot{B}_{p_2, r_2}^{s_2})_h \) as the space of distribution in \( \mathcal{S}'_h \) such that

\[
\| u \|_{\dot{B}_{p_1, r_1}^{s_1}(\dot{B}_{p_2, r_2}^{s_2})_h} \overset{\text{def}}{=} \left( \sum_{k \in \mathbb{Z}} 2^{r_1 k s_1} \left( \sum_{l \in \mathbb{Z}} 2^{r_2 l s_2} \| \Delta^k_l u \|_{L^p_h L^{r_2}} \right)^{r_1/r_2} \right)^{1/r_1}
\]

is finite. When \( p_1 = p_2 = p \), \( r_1 = r_2 = r \), we briefly denote \( \dot{B}_{p, r}^{s_1, s_2} \).

The main result of this paper states as follows:

**Theorem 1.3.** Let \( v \) be the unique maximal solution of \((NS)\) associated with initial data \( v_0 \in \mathcal{V}^2 \). If its lifespan \( T^* \) is finite, then for any \( p \in [4, \infty] \), \( q_1 \in [1, 2] \), \( \mu > 0 \), \( q_2 \in [2, (1/p + \mu)^{-1}] \), \( \kappa \in [1, \infty] \), and any unit vector \( e \), there must hold

\[
\int_0^{T^*} \left( \left\| (v(t)|e)_{\mathbb{R}^2} \right\|_{L^p_h L^{q_1}_v} + \left\| (v(t)|e)_{\mathbb{R}^2} \right\|_{(\dot{B}_{p_1, \mu^+ + \mu^{-1}}^{1/2 + 1/\mu})_h (\dot{B}_{p_2, \kappa}^{1/2})_v} \right) \, dt = \infty.
\]
Remark 1.1. We mention that the first term in (1.5) corresponds to the one component version of $L^p_t L^{3+2/3}_x$ Ladyzhenskaya-Prodi-Serrin criteria. While the second term in (1.5) requires $\mu + \frac{2}{p} + \frac{2}{q_1} - 1$ order derivative estimate of $(v|e)$ for horizontal variables, which can be arbitrarily close to zero provided that we choose $q_1$ sufficiently close to $2$, $\mu$ small enough and $p$, $q_2$ large enough. And similar comment for the vertical derivative estimate of $(v|e)$. Yet we can not succeed in the derivative estimate to be zero-th order.

At the end of this section, let us give some notations which will be used throughout this paper. $C$ stands for some real positive constant which may be different in each occurrence. Sometimes we use the notation $a \lesssim b$ for the inequality $a \leq Cb$. For a Banach space $B$, we shall use the shorthand $L^p_T B$ for $\| \cdot \|_{L^p(0, T; dt)}$. We always denote $(c_k, \ell)_{k, \ell \in \mathbb{Z}}$ to be a generic element of $l^2(\mathbb{Z}^2)$ so that $\sum_{k, \ell \in \mathbb{Z}} q_k^2 = 1$, and $(d_k, \ell)_{k, \ell \in \mathbb{Z}}$ to be a generic element of $l^1(\mathbb{Z}^2)$ so that $\sum_{k, \ell \in \mathbb{Z}} c_k, \ell = 1$. And for any index $p \in [1, \infty]$, we shall use $p'$ to denote its conjugate index, i.e. $1/p + 1/p' = 1$.

2. Strategy of the proof of Theorem 1.3

In what follows, we always denote

$$a_p \equiv \frac{1}{p} - \frac{1}{2}, \quad B_{q_1, q_2, r} \equiv \left( B_{q_1, \frac{2}{q_1}} \right)^{q_2, \frac{2}{q_2}}$$

Without loss of generality, we may assume that the unit vector $e$ in Theorem 1.3 is the vertical vector $e_3 \equiv (0, 0, 1)$. Note that for any given $p \in [4, \infty]$, $\mu > 0$, $q_2 \in [2, (\frac{1}{p} + \mu)^{-1}]$, $\kappa \in [1, \infty]$, we can find some $r < 2$ which is arbitrarily close to 2, such that $p \in [4, \frac{2r}{r-2}]$, $\mu > a_p$, $q_2 \in [2, (1/p + 3a_p + \mu)^{-1}]$, and $\kappa < \frac{2}{r-2}$. As a convention in the rest of the paper, we always assume that $r$ satisfies the above assumptions. Then Theorem 1.3 is a direct consequence of the following one:

**Theorem 2.1.** Let $r \in [9/5, 2]$, $p \in [4, 2r/[r-2]$, $q_1 \in [1, 2]$, $\mu \in [\alpha(r), 1/2 - 1/p - 3\alpha(r)]$, $q_2 \in [2, (1/p + 3\alpha(r) + \mu)^{-1}]$, and $v_0$ in $V^r$. If the lifespan $T^*$ of the unique maximal solution $v$ of $(NS)$ is finite, then there holds

$$\int_0^{T^*} \| v \|^{p}_{SC} \, dt = \infty \quad \text{with} \quad \| v \|^{p}_{SC} \equiv \| v \|^{3}_{L^{\frac{3p}{4-r}}} + \| v \|_{B_v, q_1, q_2, r}^{2}.$$  

In the rest of this section, we shall present the strategy of the proof to Theorem 2.1.

Before preceding, we denote $\Omega \equiv \text{curl} v$ to be the vorticity of the velocity field, $\omega = \partial_1 v^2 - \partial_2 v^1$ to be the third component of $\Omega$, and $\nabla \Delta_h = (-\partial_2, \partial_1)$, $\Delta_h = \partial_1^2 + \partial_2^2$. Due to $\text{div} v = 0$, we deduce the following version of Biot-Savart’s law in the horizontal variables

$$\nu^h = v^h + \Delta_h^{-1} \nabla \Delta_h^{-1} \omega, \quad \text{with} \quad v^h = \nabla \Delta_h^{-1} \omega, \quad \Delta_h = \partial_1^2 + \partial_2^2.$$

Then we can reformulate the Navier-Stokes equations $(NS)$ in terms of $\omega$ and $\partial_3 v^3$:

$$\nabla \Delta_h^{-1} \omega = \partial_t \omega + \omega \cdot \nabla \omega - \Delta \omega = \partial_3 v^3 \omega + \partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_2 v^1,$$

$$\partial_t \partial_3 v^3 + v \cdot \nabla \partial_3 v^3 - \Delta \partial_3 v^3 + \partial_3 v \cdot \nabla v^3 = -\partial_3 \Delta^{-1} \left( \sum_{\ell, m=1}^3 \partial_\ell v^m \partial_m v^\ell \right).$$

The first step of the proof of Theorem 2.1 is the following proposition:
Proposition 2.1. Under the assumptions of Theorem 2.1 and for any \( \theta \in [0, \alpha(r)] \), a constant \( C \) exists such that, for any \( t < T^* \), we have

\[
\frac{1}{r} \|\omega_\Sigma(t)\|^2_{L^2} + \frac{r - 1}{r^2} \int_0^t \|\nabla \omega_\Sigma(t')\|^2_{L^2} dt' \\
\leq \left( \frac{1}{r} \|\omega_0\|^2_{L^r} + C \left( \int_0^t \|\partial_3^2 v^3(t')\|_{H^{1,1}, \Theta, r}^2 dt' \right)^{2} \right) \times \exp \left( C \int_0^t \|v^3(t')\|^p_{SC, dt'} \right).
\]

The proof of Proposition 2.1 will be the purpose of Section 3.

We remark that when \( p_1 = p_2 = r_1 = r_2 = 2 \), the anisotropic Besov space \( \dot{B}^{s_1, s_2}_{p_1, p_2} \), given by Definition 1.2 coincides with the classical homogeneous anisotropic Sobolev space \( \dot{H}^{s_1, s_2} \) defined as follows:

**Definition 2.1.** For \((s_1, s_2) \in \mathbb{R}^2\), \( \dot{H}^{s_1, s_2} \) denotes the space of tempered distribution a such that

\[
\|a\|^2_{\dot{H}^{s_1, s_2}} = \int_{\mathbb{R}^3} \left| \xi \right|^{2s_1} \left| \xi_3 \right|^{2s_2} \left| \hat{a}(\xi) \right|^2 d\xi < \infty \quad \text{with} \quad \xi_3 = (\xi_1, \xi_2).
\]

For \( \alpha(r) \) given by (2.1) and \( \theta \in [0, \alpha(r)] \), we denote \( \mathcal{H}^{\theta, r} \) defined by

\[
\mathcal{H}^{\theta, r} = \dot{H}^{-3\alpha(r) + \theta, -\theta}.
\]

Next we are going to deal with the estimate of \( \|\partial_3^2 v^3\|_{L^2_t(\mathcal{H}^{r, r})} \). Due to \( \Omega_0 \) defined by \( \Omega_0 = \Omega_{|t=0} \in L^r \) and \( \text{div} v_0 = 0 \), we have

\[
\|\partial_3 v_0^3\|_{\mathcal{H}^{r, r}}^2 = \int_{|\xi_3| \leq |\xi_h|} |\xi_h|^{-6\alpha(r) + 2\theta} |\xi_3|^{-2\theta} |\mathcal{F}(\partial_3 v_0^3)(\xi)|^2 d\xi \\
\quad + \int_{|\xi_3| \leq |\xi_h|} |\xi_h|^{-6\alpha(r) + 2\theta} |\xi_3|^{-2\theta} |\mathcal{F}(-\text{div} v_0^3)(\xi)|^2 d\xi \\
\leq \|v_0^3\|^2_{\mathcal{H}^{1-3\alpha(r)}} \leq \|\Omega_0\|^2_{L^r}.
\]

By performing \( \mathcal{H}^{\theta, r} \)-norm energy estimate to the \( \partial_3 v^3 \) equation of \((NS)\), we shall prove the following proposition in Section 4:

**Proposition 2.2.** Under the assumption of Proposition 2.1, for any \( t < T^* \), we have

\[
\|\partial_3 v^3(t)\|^2_{\mathcal{H}^{r, r}} + \int_0^t \|\nabla \partial_3 v^3(t')\|_{\mathcal{H}^{r, r}}^2 dt' \leq C \exp \left( C \int_0^t \|v^3(t')\|^p_{SC, dt'} \right)
\]

\[
\times \left( \|\Omega_0\|^2_{L^r} + \int_0^t \left( \|v^3(t')\|^4_{SC, \omega_\Sigma^2(t')} \left\| \nabla \omega_\Sigma(t') \right\|^2_{L^2} \right)\right)
\]

With Propositions 2.1 and 2.2, we can repeat the arguments in Section 6 of [6] to complete the proof of Theorem 2.1. For completeness, we shall sketch the proof below.

**Proposition 2.3.** Under the assumption of Proposition 2.1, for any \( t < T^* \), we have

\[
\|\omega_\Sigma(t)\|^2_{L^2} \leq C \|\Omega_0\|^2_{L^r} \cdot \varepsilon(t),
\]

and

\[
\|\partial_3 v^3(t)\|^2_{\mathcal{H}^{r, r}} \leq C \|\Omega_0\|^2_{L^r} \cdot \varepsilon(t) \quad \text{with} \quad \varepsilon(t) \overset{\text{def}}{=} \exp \left( C \int_0^t \|v^3(t')\|^p_{SC, dt'} \right).
\]
Proof. For $p \in [1, \infty]$ and any $t \in [0, T^*]$, let us denote

$$p' = \frac{p}{p - 1}$$

and $e(t) \overset{\text{def}}{=} C \exp(C \int_0^t \|v^3(t')\|^p_{SC} dt')$,

where the constant $C$ may change from line to line. Then it follows from Proposition 2.2 that

$$\left( \int_0^t \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}} e(t) \leq e(t) \left( \|\Omega_0\|^r_{L^r} + II_1(t) + II_2(t) \right)$$

where

$$II_1(t) = \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2_{L^2} \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}}$$

and

$$II_2(t) = \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2_{H^{\theta, r}} \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}}.$$

Applying Hölder’s and Young’s inequalities yields

$$e(t) II_1(t) \leq e(t) \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}} \left( \int_0^t \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}}$$

and

$$e(t) II_2(t) \leq e(t) \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}} \left( \int_0^t \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}}.$$

It is easy to observe that

$$\left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{1}{1+p(\alpha(r))}} \leq \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{1}{1+p(\alpha(r))}}.$$

Thus we achieve

$$e(t) II_2(t) \leq \frac{r - 1}{3^2} \int_0^t \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' + e(t) \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{1}{1+p(\alpha(r))}}.$$

Inserting the above inequality and (2.11) into (2.10) gives, for any $t$ in $[0, T^*]$

$$\left( \int_0^t \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{r}{2}} e(t) \leq e(t) \|\Omega_0\|^r_{L^r} + \frac{2(r - 1)}{3^2} \int_0^t \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' + e(t) \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{1}{1+p(\alpha(r))}}.$$

Substituting (2.12) into (2.4), we infer that

$$\frac{1}{r} \|\omega^2(t)\|^2_{L^2} + \frac{r - 1}{3^2} \int_0^t \|\nabla \partial_3 v^3(t')\|^2_{L^2} dt' \leq e(t) \|\Omega_0\|^r_{L^r}$$

$$+ e(t) \left( \int_0^t \|v^3(t')\|^p_{SC} \|\omega^2(t')\|^2 \|\nabla \partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 v^3(t')\|^2_{H^{\theta, r}} \|\partial_3 \partial_3 v^3(t')\|^2_{H^{\theta, r}} dt' \right)^{\frac{1}{1+p(\alpha(r))}}.$$
Taking the power $1 + 2\alpha_0(r)$ to the above inequality, and then applying Gronwall’s lemma leads to Inequality (2.7).

On the other hand, it follows from Proposition 2.2 that, for any $t \in [0, T^*[$,

$$
\|\partial_3 v^3(t)\|_{H^{\theta, r}}^2 + \int_0^t \|\nabla \partial_3 v^3(t')\|_{H^{\theta, r}}^2 \, dt' \\
\leq e(t) \left( \|\Omega_0\|_{L^r}^2 + \|v^3\|_{L^p_r(\Omega)}^2 \|\omega_2^r\|_{L^p_r(L^2)}^2 \|\nabla \omega_2^r\|_{L^p_r(L^2)}^2 \\
+ \|v^3\|_{L^p_r(\Omega)}^2 \|\omega_2^r\|_{L^p_r(L^2)}^2 \|\nabla \omega_2^r\|_{L^p_r(L^2)}^2 \right). \tag{2.14}
$$

Inserting the Estimate (2.7) into (2.14) concludes the proof of (2.8) and thus Proposition 2.3. □

Before proceeding, let us recall the following regularity criteria from [5]:

**Theorem 2.2** (Theorem 2.1 of [5]). Let $v$ be a solution of (NS) in the space $C([0, T^*]; \dot{H}^{\frac{3}{2}} \cap L^2_{loc}(0, T^*; \dot{H}^{\frac{3}{2}}))$. If $T^*$ is the maximal time of existence and $T^* < \infty$, then for the norm $B_p \overset{def}{=} B_{\infty, \infty}^p$, and any $(p_{k, \ell})$ in $]1, \infty[^9$, we have

$$
\sum_{1 \leq k, \ell \leq 3} \int_0^{T^*} \|\partial_\ell v^k(t)\|_{B_{p_{k, \ell}}}^p \, dt = \infty.
$$

Now we are in a position to complete the proof of Theorem 2.1.

**Proof of Theorem 2.1.** If we assume that $\int_0^{T^*} ||v^3(t)||_{\Omega}^p \, dt$ is finite, we deduce from Proposition 2.3 that the following quantities

$$
\|\omega\|_{L^\infty([0, T^*]; L^r)}, \int_0^{T^*} \|\nabla \omega_2^r(t)\|_{L^2}^2 \, dt, \text{ and } \int_0^{T^*} \|\nabla \partial_3 v^3(t)\|_{H^{\theta, r}}^2 \, dt
$$

are finite.

It follows from Lemma A.2 that

$$
\max_{1 \leq \ell \leq 3} \|\partial_\ell v^3\|_{B_p} \lesssim \sup_{j \in \mathbb{Z}} 2^j (-1 + \frac{2}{\alpha}) \|\Delta_j v^3\|_{L^\infty} \lesssim \sup_{j \in \mathbb{Z}} \|\Delta_j v^3\|_{L^{\frac{4}{\alpha}}_{\Omega}} \lesssim \|v^3\|_{L^{\frac{4}{\alpha}_{\Omega}}},
$$

which together with (2.2) ensures that

$$
\max_{1 \leq \ell \leq 3} \int_0^{T^*} \|\partial_\ell v^3(t)\|_{B_p}^p \, dt \lesssim \int_0^{T^*} \|v^3(t)\|_{\Omega}^p \, dt < \infty. \tag{2.16}
$$

Let us turn the estimate of the horizontal components of the velocity field. For $v^h_{\text{div}} = -\nabla_h \Delta_h^{-1} \partial_3 v^3$, we get, by using (2.16), that

$$
\int_0^{T^*} \|\nabla_h v^h_{\text{div}}(t)\|_{B_p}^p \, dt \lesssim \int_0^{T^*} \|\partial_3 v^3(t)\|_{B_p}^p \, dt < \infty. \tag{2.17}
$$
While for any distribution $a$, we deduce from Lemma A.2 that
\[
\|\Delta_j a\|_{L^\infty} \lesssim \sum_{k \leq j+1} \sum_{\ell \leq j+1} 2^k 2^j 2^{j+\frac{\ell}{2}} \|\Delta_k^\h \Delta_\ell^\h a\|_{L^2}
\]
(2.18)
\[
\lesssim \|a\|_{\dot{H}^{1-3\alpha(r)+\theta,-\theta}} \sum_{k \leq j+1} \sum_{\ell \leq j+1} 2^{k(3\alpha(r)-\theta)} 2^{j+\frac{\ell}{2}}
\]
\[
\lesssim 2^j \left(2^{-\frac{\alpha}{2}}\right) \|a\|_{\dot{H}^{1-3\alpha(r)+\theta,-\theta}}.
\]

Let $q(r) \overset{\text{def}}{=} \frac{2\alpha}{3}$, (2.18) implies
\[
\|\partial_3 v^h_{\text{div}}\|_{B^{q(r)}_{2,\infty}} \lesssim \|\partial_3 v^h_{\text{div}}\|_{\dot{H}^{1-3\alpha(r)+\theta,-\theta}} = \|\nabla h \Delta_\h^{-1} \partial_3^2 v^3\|_{\dot{H}^{1-3\alpha(r)+\theta,-\theta}} \lesssim \|\partial_3^2 v^3\|_{H^{\theta,r}}.
\]

Due to $r \in ]\frac{3}{2}, 2[$, $q(r) \in ]\frac{4}{3}, 2[$ then we get, by applying (2.19) and Hölder inequality, that
\[
\int_0^{T^*} \|\partial_3 v^h_{\text{div}}(t)\|_{B^{q(r)}_{2,\infty}} dt \lesssim T^* \left(1-\frac{2q(r)}{3}\right) \left(\int_0^{T^*} \|\partial_3^2 v^3(t)\|_{H^{\theta,r}}^2 dt\right)^{\frac{q(r)}{2}} < \infty.
\]

On the other hand, we deduce from Lemma A.2 that
\[
\|\nabla h v^h_{\text{curl}}\|_{B^{q(r)}_{2,\infty}} \lesssim \|\partial_3^2 \Delta_\h^{-1} \nabla^3 \omega\|_{\dot{H}^{1-3\alpha(r)}} \lesssim \|\nabla \omega\|_{\dot{H}^{1-3\alpha(r)}} \lesssim \|\nabla \omega\|_{L^r}.
\]

Applying Lemma A.2 once again and using the fact that $r < 2$, we infer
\[
2^j \left(2^{-\frac{\alpha}{2}}\right) \|\Delta_j \partial_3 v^h_{\text{curl}}\|_{L^\infty} \lesssim 2^j \left(2^{-\frac{\alpha}{2}}\right) \sum_{k \leq j+1} \sum_{\ell \leq j+1} \|\Delta_k^\h \Delta_\ell^\h \partial_3 \nabla_h^3 \Delta_\h^{-1} \omega\|_{L^\infty}
\]
\[
\lesssim \|\partial_3 \omega\|_{L^r} 2^j \left(2^{-\frac{\alpha}{2}}\right) \sum_{k \leq j+1} \sum_{\ell \leq j+1} 2^{k+\frac{\ell}{2}} \lesssim \|\partial_3 \omega\|_{L^r}.
\]

This together with the Estimate (2.21) and Lemma A.7 ensures that
\[
\|\nabla v^h_{\text{curl}}(t)\|_{B^{q(r)}_{2,\infty}} \lesssim \|\nabla \omega(\cdot - t)\|_{L^r} \lesssim \|\omega^\h_{T^*}\|_{L^\infty([0,T^*];L^2)} \|\nabla \omega^\h_{T^*}(t)\|_{L^2}.
\]

Using again the fact that $q(r) \in ]\frac{4}{3}, 2[$, we get, by using the Hölder inequality, that
\[
\int_0^{T^*} \|\nabla v^h_{\text{curl}}(t)\|_{B^{q(r)}_{2,\infty}} dt \lesssim T^* \left(1-\frac{2q(r)}{3}\right) \|\omega^\h_{T^*}\|_{L^\infty([0,T^*];L^2)} \left(\int_0^{T^*} \|\nabla \omega^\h_{T^*}(t)\|_{L^2}^2 dt\right)^{\frac{q(r)}{2}} < \infty.
\]

With the estimates (2.16), (2.17), (2.20) and (2.22), Theorem 2.1 is a direct consequence of Theorem 2.2.

Finally in the Appendix A, we shall collect some basic facts on Littlewood-Paley theory from [1] and some technical lemmas from [5, 6]. While in Appendix B, we present some technical details which will be used in the proof of Proposition 3.1.

3. PROOF OF THE ESTIMATE FOR THE HORIZONTAL VORTICITY

The purpose of this section to present the proof of Proposition 2.1. Let us first recall the $\omega$ equation of (NS) that
\[
\partial_t \omega + v \cdot \nabla \omega - \Delta \omega = \partial_3 v^3 \omega + \partial_2 v^3 \partial_3 v^1 - \partial_1 v^3 \partial_3 v^2.
\]
By applying Lemma 3.1 of [5], we obtain
\[
\frac{1}{r} \left\| \omega_2^r(t) \right\|_{L^2}^2 + \frac{4(r-1)}{r^2} \int_0^t \left\| \nabla \omega_2^r(t') \right\|_{L^2}^2 \, dt' = \frac{1}{r} \left\| \omega_0^r \right\|_{L^2}^2 + \sum_{\ell=1}^3 F_\ell(t) \quad \text{with}
\]
\[
F_1(t) \overset{\text{def}}{=} \int_0^t \int_{\mathbb{R}^3} \partial_3 v^3 |\omega|^r \, dx \, dt',
\]
\[
F_2(t) \overset{\text{def}}{=} \int_0^t \int_{\mathbb{R}^3} \left( \partial_2 v^3 \partial_3 v^1_{\text{curl}} - \partial_1 v^3 \partial_3 v^2_{\text{curl}} \right) \omega_{\tau-1} \, dx \, dt' \quad \text{and}
\]
\[
F_3(t) \overset{\text{def}}{=} \int_0^t \int_{\mathbb{R}^3} \left( \partial_2 v^3 \partial_3 v^1_{\text{div}} - \partial_1 v^3 \partial_3 v^2_{\text{div}} \right) \omega_{\tau-1} \, dx \, dt',
\]
where \( v^h \) (resp. \( v^h_{\text{div}} \)) corresponds to the horizontal divergence free (resp. curl free) part of the horizontal vector \( v^h = (v^1, v^2) \), which is given by (2.3), and where \( \omega_{\tau-1} \overset{\text{def}}{=} |\omega|^{-2} \omega \).

Let us start with the easiest term \( F_1 \). We first get, by using integration by parts, that
\[
|F_1(t)| \leq r \int_0^t \int_{\mathbb{R}^3} |v^3(t', x)| \left| \partial_3 \omega(t', x) \right| |\omega(t', x)|^{r-1} \, dx \, dt'
\]
\[
\leq r \int_0^t \int_{\mathbb{R}^3} |v^3(t', x)| \left| \partial_3 \omega(t', x) \right| \left\| \omega_2^r(t') \right\|_{L^2}^2 \, dt'.
\]
Notice that
\[
\frac{p-2}{3p} + \frac{1}{r} + \frac{2pr-3p+2r}{6p(r-1)} \times \frac{2}{r^2} = 1,
\]
we get, by applying Hölder inequality, that
\[
|F_1(t)| \leq r \int_0^t \left\| v^3(t') \right\|_{L^{\frac{3p}{2}}(x)} \left\| \partial_3 \omega(t') \right\|_{L^r} \left\| \omega_2^r(t') \right\|_{L^2}^2 \, dt'.
\]
As \( p \) is in \( [4, \frac{2r}{2-r}[ \), we observe that \( r' \frac{p-2}{2p} \) belongs to \( ]0, 1[ \). Then Sobolev embedding and interpolation inequality implies that
\[
\left\| \omega_2^r(t') \right\|_{L^{\frac{6p(r-1)}{2p-3p+2r}}} \leq \left\| \omega_2^r(t') \right\|_{H^{r'(\frac{p-2}{2p})}} \leq \left\| \omega_2^r(t') \right\|_{L^2}^{2p(r-1)} \left\| \nabla \omega_2^r(t') \right\|_{L^2}^{1-\frac{2}{p}} \left\| \omega_2^r(t') \right\|_{L^2}^{1-2\left(\frac{1}{r} - \frac{1}{p}\right)} \, dt'.
\]
From which and (A.1), we infer
\[
|F_1(t)| \lesssim \int_0^t \left\| v^3(t') \right\|_{L^{\frac{3p}{2}}(x)} \left\| \partial_3 \omega(t') \right\|_{L^r} \left\| \omega_2^r(t') \right\|_{L^2}^2 \left\| \nabla \omega_2^r(t') \right\|_{L^2}^\frac{2}{1-r} \left\| \omega_2^r(t') \right\|_{L^2}^{1-2\left(\frac{1}{r} - \frac{1}{p}\right)} \, dt'.
\]
Applying Young’s inequality gives rise to
\[
|F_1(t)| \lesssim \int_0^t \left\| v^3(t') \right\|_{L^{\frac{3p}{2}}(x)} \left\| \omega_2^r(t') \right\|_{L^2}^\frac{2}{1-r} \left\| \nabla \omega_2^r(t') \right\|_{L^2}^{\frac{2}{1-r}} \, dt'
\]
\[
\leq \frac{r-1}{r^2} \int_0^t \left\| \nabla \omega_2^r(t') \right\|_{L^2}^2 \, dt' + C \int_0^t \left\| v^3(t') \right\|_{L^{\frac{3p}{2}}(x)}^p \left\| \omega_2^r(t') \right\|_{L^2}^2 \, dt'.
\]

The other two terms in (3.1) require a refined way to describe the regularity of \( \omega_2^r \) and demand a detailed study of the anisotropic operator \( \nabla_h \Delta_h^{-1} \) associated with the Biot-Savart’s law in horizontal variables. We first modify Lemma 4.1 of [6] to the following one.
Proposition 3.1. Under the assumptions of Proposition 2.1 and let $\sigma = r^{\frac{1}{2} - \frac{1}{p}}$, we have

$$\left| \int_{\mathbb{R}^3} \partial_h \Delta_h^{-1} f \cdot \partial_h a \omega_{r-1} \, dx \right| \lesssim \min \left\{ \| f \|_{L^{r'}}, \| f \|_{H^{\theta,r}} \right\} \| a \|_{B_{q_1,q_2,r}^{\mu,p}} \| \omega_2 \|_{H^\sigma}^{\frac{2}{\theta}},$$

where the norm $\| \cdot \|_{B_{q_1,q_2,r}^{\mu,p}}$ is given by (2.1).

Proof. Observe that $\omega_{r-1} = G(\omega_2^2)$ with $G(z) \equiv z|z|^{-2\alpha(r)}$. It follows from Lemma A.5 that

$$\| \omega_{r-1} \|_{B_{r',r}^{2p}} \lesssim \| \omega_2 \|_{H^\sigma}^{\frac{2}{\theta}} \quad \forall \sigma \in ]0,1[.$$

Let us study the product $\partial_h a \omega_{r-1}$. By applying Bony’s decomposition in the horizontal variables, we write

$$\partial_h a \omega_{r-1} = T^h(\partial_h a, \omega_{r-1}) + R^h(\partial_h a, \omega_{r-1}) + T^h(\omega_{r-1}, \partial_h a)$$

$$= \partial_h T^h(\omega_{r-1}, a) + A(a, \omega) \quad \text{with}$$

$$A(a, \omega) \equiv T^h(\partial_h a, \omega_{r-1}) + R^h(\partial_h a, \omega_{r-1}) - T^h(\partial_h \omega_{r-1}, a).$$

In view of Lemma A.2, it is obvious that we only need to prove (3.2) for $q_1 \in ]r,2[ \text{ and } q_2 \in ]r', \frac{r}{2} + 3\alpha(r) + \mu \text{ }^{-1}[$. Then we can estimate the above term by term as follows:

$$\| T^h(\omega_{r-1}, a) \|_{L^{r'}} + \| T^h(\omega_{r-1}, a) \|_{H^{3\alpha(r) - \theta,\theta}} \lesssim \| a \|_{B_{q_1,q_2,r}^{\mu,p}} \| \omega_2 \|_{H^\sigma}^{\frac{2}{\theta}},$$

$$\| (T^h(\partial_h a, \omega_{r-1}), T^h(\partial_h \omega_{r-1}, a)) \|_{(B_{\frac{q_1+3}{q_1+r},2}^{\mu+\frac{2}{q_1+r},2})_h(H^\delta_2 + \alpha(r) - \mu)_v} \lesssim \| a \|_{B_{q_1,q_2,r}^{\mu,p}} \| \omega_2 \|_{H^\sigma}^{\frac{2}{\theta}},$$

$$\| R^h(\partial_h a, \omega_{r-1}) \|_{(B_{\frac{q_1+3}{q_1+r},2}^{\mu+\frac{2}{q_1+r},2})_h(H^\delta_2 + \alpha(r) - \mu)_v} \lesssim \| a \|_{B_{q_1,q_2,r}^{\mu,p}} \| \omega_2 \|_{H^\sigma}^{\frac{2}{\theta}},$$

where $\delta_1 \in ]\mu - \alpha(r), 1 - 2/p[$ and $\delta_2 \in ]0, 0, 1 - 2/q_1 [$. The proofs of (3.4)-(3.6) will be postponed to Appendix B. Let us continue our proof of the proposition.

Note that $q_1 \in ]1,2[$, we have $B_{q_1,q_2,r}^{\mu,p} \rightarrow B_{q_1+3-\alpha(r),r}^{\mu,p}$. Then we deduce from (3.4), that

$$\left| \int_{\mathbb{R}^3} \partial_h \Delta_h^{-1} f \cdot \partial_h T^h(\omega_{r-1}, a) \, dx \right| = \int_{\mathbb{R}^3} \partial_h \Delta_h^{-1} f \cdot T^h(\omega_{r-1}, a) \, dx,$$

$$\lesssim \min \left\{ \| f \|_{L^{r'}}, \| T^h(\omega_{r-1}, a) \|_{L^{r'}}, \| f \|_{H^{\theta,r}}, \| T^h(\omega_{r-1}, a) \|_{H^{3\alpha(r) - \theta,\theta}} \right\}$$

$$\lesssim \min \left\{ \| f \|_{L^{r'}}, \| f \|_{H^{\theta,r}} \right\} \| a \|_{B_{q_1,q_2,r}^{\mu,p}} \| \omega_2 \|_{H^\sigma}^{\frac{2}{\theta}}.$$
And it follows from Lemma A.2 once again that
\[
(\hat{B}^{\alpha}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}})_{v}(\hat{H}^{\alpha}_{v})_{v} \leadsto \hat{B}^{1,0}_{v}, 
(\hat{B}^{\frac{\alpha}{2}+\frac{1}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}})_{h}(\hat{H}^{\alpha}_{v})_{v} \leadsto \hat{B}^{1,0}_{v}. 
\]
Using (3.5), (3.6) with \(\delta_1 = \delta_2 = \mu\), and (3.8), we achieve
\[
|\int_{\mathbb{R}^3} \partial_t \Delta_{h}^{-1} f \cdot \left( T^{h}(\partial_t a, \omega_{r-1}) + T^{h}(\partial_t \omega_{r-1}, a) + R^{h}(\partial_t a, \omega_{r-1}) \right) dx |
\leq \|\partial_t \Delta_{h}^{-1} f\|_{H^{1,0}_{v}} \left\| \left( T^{h}(\partial_t a, \omega_{r-1}), T^{h}(\partial_t \omega_{r-1}, a), R^{h}(\partial_t a, \omega_{r-1}) \right) \right\|_{H^{1,2}_{v}}
\lesssim \|f\|_{L^{r}} \|a\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{H^2_{v}}. 
\]
Combining the above estimate with (3.10), we conclude that
\[
\left| \int_{\mathbb{R}^3} \partial_t \Delta_{h}^{-1} f \cdot \partial_t a \omega_{r-1} dx \right| \lesssim \|f\|_{L^{r}} \|a\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{H^2_{v}}.
\]
On the other hand, it follows from Lemma A.2 once again that
\[
(\hat{B}^\alpha_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}})_{v}(\hat{H}^{\alpha} H^{\alpha})_{v} \leadsto \hat{H}^{1+3\alpha(r)-\theta,\theta}_v, 
(\hat{B}^{\frac{\alpha}{2}+\frac{1}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}})_{h}(\hat{H}^{\alpha} H^{\alpha})_{v} \leadsto \hat{H}^{1+3\alpha(r)-\theta,\theta}_v.
\]
Using (3.5), (3.6) with \(\delta_1 = \delta_2 = \mu + \theta - \alpha(r)\) yields
\[
|\int_{\mathbb{R}^3} \partial_t \Delta_{h}^{-1} f \cdot \left( T^{h}(\partial_t a, \omega_{r-1}) + T^{h}(\partial_t \omega_{r-1}, a) + R^{h}(\partial_t a, \omega_{r-1}) \right) dx |
\leq \|\partial_t \Delta_{h}^{-1} f\|_{H^{1+3\alpha(r)-\theta,\theta}_v} \left\| \left( T^{h}(\partial_t a, \omega_{r-1}), T^{h}(\partial_t \omega_{r-1}, a), R^{h}(\partial_t a, \omega_{r-1}) \right) \right\|_{H^{1+3\alpha(r)-\theta,\theta}_v}
\lesssim \|f\|_{H^{\theta,\theta}_v} \|a\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{H^2_{v}}, 
\]
which together with (3.10) gives rise to
\[
\left| \int_{\mathbb{R}^3} \partial_t \Delta_{h}^{-1} f \partial_t a \omega_{r-1} dx \right| \lesssim \|f\|_{H^{\theta,\theta}_v} \|a\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{H^2_{v}}.
\]
Combining the Estimates (3.9) and (3.10), we complete the proof of this proposition. \(\square\)

The estimate of \(F_2(t)\) uses the Biot-Savart's law in the horizontal variables (namely (2.3)) and Proposition 3.1 with \(f = \partial_3 \omega, a = v^3\) and \(\sigma = \frac{(p-2)r'}{2p}\), which is in \([1, 1]\) provided \(p \in [4, 2r']\}. This gives for any time \(t < T^*\) that
\[
I_{\omega}(t) \overset{\text{def}}{=} \left| \int_{\mathbb{R}^3} (\partial_3 v^3(t, x) \partial_3 v^1(t, x) - \partial_1 v^3(t, x) \partial_3 v^2(t, x)) \omega_{r-1}(t, x) dx \right|
\leq \|v^3(t)\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \|\partial_3 \omega(t)\|_{L^r} \left\| \omega_{v}^{r} \right\|_{H^{\frac{p}{r} - 1} \left( \mathbb{R} \right)}. 
\]
By virtue of (A.1) and of the interpolation inequalities between \(L^2\) and \(\dot{H}^1\), (3.11) implies
\[
I_{\omega}(t) \lesssim \|v^3(t)\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{L^2} \left\| \nabla \omega_{v}^{r} \right\|_{L^2} \left\| \omega_{v}^{r} \right\|_{L^2} \left\| \omega_{v}^{r} \right\|_{L^2} \left\| \nabla \omega_{v}^{r} \right\|_{L^2} \left\| \nabla \omega_{v}^{r} \right\|_{L^2
d\lesssim \|v^3(t)\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{L^2} \left\| \nabla \omega_{v}^{r} \right\|_{L^2} \left\| \nabla \omega_{v}^{r} \right\|_{L^2} \left\| \omega_{v}^{r} \right\|_{L^2}
\]
Then by using Young's inequality and integrating in time, we get
\[
|F_2(t)| \leq \frac{r-1}{r^2} \int_{0}^{t} \left\| \nabla \omega_{v}^{r} \right\|_{L^2}^2 dt' + C \int_{0}^{t} \|v^3(t')\|_{B^{\frac{\alpha}{2}}_{\frac{\alpha}{2}+\frac{r}{\alpha}+\frac{1}{2}}} \left\| \omega_{v}^{r} \right\|_{L^2}^2 dt'.
\]
The estimate of $F_3(t)$ uses (2.3) and Proposition 3.1 with $f = \partial_3^2 v^3$, $a = v^3$:

$$|F_3(t)| = \left| - \int_0^t \int_{\mathbb{R}^3} (\partial_2 v^3(t') \cdot \partial_1 \Delta_h^{-1} \partial_3^2 v^3(t') - \partial_1 v^3(t') \cdot \partial_2 \Delta_h^{-1} \partial_3^2 v^3(t')) \omega_{r-1}(t') dx dt' \right|$$

$$\lesssim \int_0^t \| \partial_3^2 v^3(t') \|_{\mathcal{H}^\theta, r} \| v^3(t') \|_{B_0^1, p_{1/2}, r} \| \omega_2(t') \|_{H_{1/2}^{2r}} dt'$$

$$\lesssim \int_0^t \| \partial_3^2 v^3 \|_{\mathcal{H}^\theta, r} \| v^3 \|_{B_0^1, p_{1/2}, r} (\| v^3 \|^2_{L^2})^{1/2} \| \nabla \omega_2 \|^2_{L^2} dt'$$

Applying Hölder’s inequality and then Young’s inequality leads to

$$|F_3(t)| \leq C \left( \int_0^t \| \partial_3^2 v^3 \|^2_{\mathcal{H}^\theta, r} dt' \right)^{\frac{1}{2}} \left( \int_0^t \| v^3 \|^p_{B_0^1, p_{1/2}, r} dt' \right)^{\frac{1}{2} - \frac{1}{p}} \times \left( \int_0^t \| v^3 \|^p_{B_0^1, p_{1/2}, r} \| \omega_2 \|^2_{L^2} dt' \right)^{\frac{1}{2} - \frac{1}{p}}$$

$$\leq \frac{r - 1}{r^2} \int_0^t \| \nabla \omega_2 \|^2_{L^2} dt' + C \int_0^t \| v^3 \|^p_{B_0^1, p_{1/2}, r} \| \omega_2 \|^2_{L^2} dt'$$

Substituting the estimates (3.2), (3.12) and (3.13) into (3.1), we obtain

$$\frac{1}{r} \| \omega_2(t) \|^2_{L^2} \leq \frac{r - 1}{r^2} \int_0^t \| \nabla \omega_2(t') \|^2_{L^2} dt' \leq \frac{1}{r} \| \omega_0 \|^2_{L^2}$$

Then using Gronwall’s inequality and the elementary inequality that $x^{1-\frac{2}{p}} e^{C_\xi x} \leq e^{C' x}$ for some constant $C' > C$ and any $x \geq 0$ yields (2.4), which is the desired result.

4. PROOF OF THE ESTIMATE FOR $\partial_3^2 v^3$

In this section, we shall present the proof of Proposition 2.2. Recall the $\partial_3 v^3$ equation of $(\widetilde{NS})$ that

$$\partial_t \partial_3 v^3 + v \cdot \nabla \partial_3 v^3 - \Delta \partial_3 v^3 + \partial_3 v \cdot \nabla v^3 = -\partial_3^2 \Delta^{-1} \left( \sum_{\ell, m=1}^3 \partial_\ell v^m \partial_m v^3 \right).$$

Let $\mathcal{H}^\theta, r$ be given by Definition 2.1. Taking $\mathcal{H}^\theta, r$ inner product of the (4.1) with $\partial_3 v^3$, gives

$$\frac{1}{2} \frac{d}{dt} \| \partial_3 v^3(t) \|^2_{\mathcal{H}^\theta, r} + \| \nabla \partial_3 v^3(t) \|^2_{\mathcal{H}^\theta, r} = -3 \sum_{n=1}^3 (Q_n(v, v) | \partial_3 v^3)_{\mathcal{H}^\theta, r}$$

with

$$Q_1(v, v) \overset{\text{def}}{=} \left( \text{Id} + \partial_3^2 \Delta^{-1} \right) (\partial_3 v^3)^2 + \partial_3^2 \Delta^{-1} \left( \sum_{\ell, m=1}^2 \partial_\ell v^m \partial_m v^3 \right),$$

$$Q_2(v, v) \overset{\text{def}}{=} \left( \text{Id} + 2 \partial_3^2 \Delta^{-1} \right) \left( \sum_{\ell=1}^2 \partial_\ell v^3 \partial_\ell v^3 \right)$$

and $Q_3(v, v) \overset{\text{def}}{=} v \cdot \nabla \partial_3 v^3$.

The estimate of $(Q_1(v, v) | \partial_3 v^3)_{\mathcal{H}^\theta, r}$ relies on the following lemma:
Lemma 4.1. Let $L(D)$ be an $L^q$ bounded Fourier multiplier for any $q \in ]1, \infty]$. Let $r \in ]3/2, 2[$, $\theta \in ]0, \alpha(r)[$, $p \in ]4, \infty[$, and $s_1, s_2 \in ]1, \infty]$ satisfy

\begin{equation}
\frac{2}{s_1} + \frac{1}{s_2} = \frac{1}{p'} + 3\alpha(r) \quad \text{and} \quad \frac{1}{s_2} < \frac{1}{p'} - 3\alpha(r) + \theta.
\end{equation}

Then we have

\begin{equation}
\left| (L(D)(fg) | \partial_3 v^3)_{H^\theta,r} \right| \lesssim \|f\|_{H^\theta_{p,s_1}} \|g\|_{H^\theta_{p,s_2}} \cdot \|v^3\|_{(\tilde{B}^{0}_{s_1,\infty})_{v}} \left( \tilde{B}^{\frac{1}{2}+3\alpha(r)}_{s_2,\infty} \right)_{v},
\end{equation}

where we denote $\|f\|_{H^\theta_{p,s_1}} \overset{\text{def}}{=} \|f\|_{H^\theta_{p,s_1}} + \|f\|_{H^\theta_{p', \frac{1}{2} - 3\alpha(r) + \theta}} + \|f\|_{H^\theta_{p', \frac{1}{2} - 3\alpha(r) + \theta}}$.

Proof. Recall that $H^\theta_{p,s_1} = (\tilde{B}^{\frac{1}{2}+3\alpha(r)+\theta}_{2,2})_{v}$, we write

\begin{equation}
(L(D)(fg) | \partial_3 v^3)_{H^\theta,r} = \sum_{k, \ell \in \mathbb{Z}} 2^{2k(-3\alpha(r)+\theta)} 2^{-2\theta} \left( \Delta_{k}^{h} \Delta_{\ell}^{v} (L(D)(fg)) | \Delta_{k}^{h} \Delta_{\ell}^{v} \partial_3 v^3 \right)_{L^2}.
\end{equation}

Applying Lemma A.2 yields

\begin{equation}
\left| (L(D)(fg) | \partial_3 v^3)_{H^\theta,r} \right| \lesssim \sum_{k, \ell \in \mathbb{Z}} 2^{2k(-3\alpha(r)+\theta)} 2^{-2\theta} \| \Delta_{k}^{h} \Delta_{\ell}^{v} (f g) \|_{L^1_{h} L^2_{v}} \| \Delta_{k}^{h} \Delta_{\ell}^{v} v^3 \|_{L^1_{h} L^2_{v}}
\end{equation}

\begin{equation}
\lesssim \| f \|_{(\tilde{B}^{-6\alpha(r)+2\theta}_{1,1})_{h}} \left( \tilde{B}^{\frac{1}{2}+3\alpha(r)-2\theta}_{2,1} \right)_{v} \| \Delta_{k}^{h} \Delta_{\ell}^{v} v^3 \|_{(\tilde{B}^{0}_{1,\infty})_{h}} \left( \tilde{B}^{\frac{1}{2}+3\alpha(r)}_{2,\infty} \right)_{v}.
\end{equation}

So that it remains to verify

\begin{equation}
\| f \|_{(\tilde{B}^{-6\alpha(r)+2\theta}_{1,1})_{h}} \left( \tilde{B}^{\frac{1}{2}+3\alpha(r)-2\theta}_{2,1} \right)_{v} \lesssim \| \Delta_{k}^{h} \Delta_{\ell}^{v} \partial_3 v^3 \|_{H^\theta_{p,s_1}} \| g \|_{H^\theta_{p,s_2}}.
\end{equation}

In order to do so, we get, applying Bony’s decomposition in both horizontal and vertical variables, that

\begin{equation}
fg = \left( T^h + R^h + \bar{T}^h \right) \left( T^v + R^v + \bar{T}^v \right) (f,g).
\end{equation}

We first get, by applying Lemma A.2 and (4.3), that

\begin{equation}
\| \Delta_{k}^{h} \Delta_{\ell}^{v} f \|_{L^2_{h} L^{2+2\theta}_{v}} \lesssim \sum_{\ell' \leq \ell - 2} c_{k, \ell'} 2^{-k} \left( \frac{1}{p'} \frac{1}{s_2} - 3\alpha(r) + \theta \right) 2^{\ell' \theta} \| f \|_{H^{\theta}_{p', \frac{1}{2} - 3\alpha(r) + \theta}} \frac{1}{s_2} - 3\alpha(r) + \theta \frac{1}{2} - \theta,
\end{equation}

and

\begin{equation}
\| S_{k-1}^{h} \Delta_{\ell}^{v} f \|_{L^{2\theta}_{h} L^{2\theta}_{v}} \lesssim \sum_{k' \leq k - 2} c_{k', \ell} 2^{k'} \left( \frac{1}{p'} \theta \right) 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right) \| f \|_{H^{\theta}_{p', \frac{1}{2} - 3\alpha(r) - \theta}} \frac{1}{2} - \theta,
\end{equation}

and

\begin{equation}
\| S_{k-1}^{h} \Delta_{\ell}^{v} f \|_{L^{2\theta}_{h} L^{2\theta}_{v}} \lesssim \sum_{k' \leq k - 2} c_{k', \ell} 2^{k'} \left( \frac{1}{p'} \theta \right) 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right) \| f \|_{H^{\theta}_{p', \frac{1}{2} - 3\alpha(r) - \theta}} \frac{1}{2} - \theta.
\end{equation}
And applying Lemma A.2 and (4.7) gives rise to
\[
\|S_{k-1}^{\nu}S_{\ell-1}^{\nu} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim \sum_{k' \leq k-2} \sum_{\ell' \leq \ell-2} 2^{2k'} \|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim \sum_{k' \leq k-2} 2^{2k'} \|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))}
\]
(4.9)

Considering the support to the Fourier transform of the terms in \(T^h_T^v(f, g)\), we have
\[
\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim \sum_{|k'|-\ell' \leq 4} \left( \|S_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \|\Delta_{k'}^{-1}S_{\ell'}^{-1} g\|_{L^2_h(L_{v_1}^{2\nu}))} \right) \lesssim d_{k', \ell} 2^{2k(3\alpha(r)-\theta)} 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right)
\]

By symmetry, we obtain
\[
\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim d_{k', \ell} 2^{2k(3\alpha(r)-\theta)} 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right) \lesssim \|f\|_{H^{\theta, p'}_{p, \frac{1}{p'} - 3\alpha(r) - \theta}} \|g\|_{H^{\theta, p'}_{p, \frac{1}{p'} - 3\alpha(r) - \theta}}
\]

While we deduce from Lemma A.2 that
\[
\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim \sum_{|k'|-\ell' \leq 4} \left( \|S_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \|\Delta_{k'}^{-1}S_{\ell'}^{-1} g\|_{L^2_h(L_{v_1}^{2\nu}))} \right) \lesssim d_{k', \ell} 2^{2k(3\alpha(r)-\theta)} 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right)
\]

from which, 4.3 and (4.7), we deduce that
\[
\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim \sum_{|k'|-\ell' \leq 4} d_{k', \ell} 2^{2k(3\alpha(r)-\theta)} 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right) \lesssim \|f\|_{H^{\theta, p'}_{p, \frac{1}{p'} - 3\alpha(r) - \theta}} \|g\|_{H^{\theta, p'}_{p, \frac{1}{p'} - 3\alpha(r) - \theta}}
\]

By symmetry, the term \(\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \) shares the above estimate.

Again we deduce from Lemma A.2 that
\[
\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \lesssim \sum_{|k'|-\ell' \leq 4} \left( \|S_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \|\Delta_{k'}^{-1}S_{\ell'}^{-1} g\|_{L^2_h(L_{v_1}^{2\nu}))} \right) \lesssim d_{k', \ell} 2^{2k(3\alpha(r)-\theta)} 2^{-\ell} \left( \frac{1}{p'} - 3\alpha(r) - \theta \right)
\]

By symmetry, the term \(\|\Delta_{k'}^{-1}S_{\ell'}^{-1} f\|_{L^2_h(L_{v_1}^{2\nu}))} \) shares the above estimate.
which together with (4.3) and (4.8) ensures that
\[
\|\Delta_k^h \Delta^\gamma T^h R^v (f, g)\|_{L^q_h (L^s_x)} \lesssim 2^{\frac{\ell}{s_2}} \sum_{k' \geq k-3 \atop \ell' \geq \ell-3} d_{k', \ell} 2^{-k\left(\frac{1}{p'} - \frac{1}{s_2} - 3\alpha + 2\theta\right) + \ell\left(\frac{1}{p'} + \frac{1}{s_2} - 3\alpha - \theta\right)}
\times \frac{\|f\|_{H^{\frac{1}{p} - 3\alpha + \theta}}}{H^{\frac{1}{p} - \frac{1}{s_2} - 3\alpha - \theta}} \frac{\|g\|_{H^{\frac{1}{p'} - \frac{1}{s_2} - 3\alpha - \theta}}}{H^{\frac{1}{p'} - 3\alpha + \theta}}
\lesssim d_{k, \ell} 2^{2k(3\alpha - \theta) + \ell} \|f\|_{H^\theta_h, r} \|g\|_{H^\theta_h, r}.
\]

By symmetry, the same estimate holds for \(\Delta_k^h \Delta^\gamma T^h R^v (f, g)\).

Finally, we get, by applying Lemma A.6 and A.7, that
\[
\|\Delta_k^h \Delta^\gamma R^v (f, g)\|_{L^q_h (L^s_x)} \lesssim 2^{\frac{\ell}{s_2}} \sum_{k' \geq k-3 \atop \ell' \geq \ell-3} d_{k', \ell} 2^{-k\left(\frac{1}{p'} - \frac{1}{s_2} - 3\alpha + 2\theta\right) + \ell\left(\frac{1}{p'} + \frac{1}{s_2} - 3\alpha - \theta\right)}
\times \frac{\|f\|_{H^{\frac{1}{p} - 3\alpha + \theta}}}{H^{\frac{1}{p} - \frac{1}{s_2} - 3\alpha - \theta}} \frac{\|g\|_{H^{\frac{1}{p'} - \frac{1}{s_2} - 3\alpha - \theta}}}{H^{\frac{1}{p'} - 3\alpha + \theta}}
\lesssim d_{k, \ell} 2^{2k(3\alpha - \theta) + \ell} \|f\|_{H^\theta_h, r} \|g\|_{H^\theta_h, r}.
\]

By summing up the above estimates, we obtain (4.5), and thus the lemma. \(\square\)

Applying Lemma 4.1 with \(f\) and \(g\) being of the forms \(\partial_t v_{1h}^\text{curl}, \partial_t v_{1h}^\text{div}\) or \(\partial_t v_3^3\) gives:
\[
\left| (Q_1(u, v)|\partial_t v^3)_{H^\theta_h, r} \right| \lesssim \|v^3\| \left( B_{s_1, \infty}^\theta_h \right) \left( B_{s_2, \infty}^{1 + 3\alpha(h)} \right) \left( \|\omega\|_{H^\theta_h, r}^2 + \|\partial_t v^3\|_{H^\theta_h, r}^2 \right).
\]

Due to \(s_2\) satisfying (4.3), we get, by applying Lemma A.6 and A.7, that
\[
\|\omega\|_{H^\theta_h, r}^2 \lesssim \|\omega\|_{H^{\frac{1}{p} - 3\alpha} \bar{H}^\theta_h} \|\omega\|_{L^2} \|\nabla \omega\|_{L^2}^{1 - \frac{1}{p}}.
\]

While for any function \(a\), it follows from Definition 2.1 that
\[
\|a\|^2_{H^{\frac{1}{p'} - 3\alpha - \theta}} = \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2 \left( |\xi|^{2\left(\frac{1}{q} - 3\alpha\right)} |\xi|^{2\left(\frac{1}{q'} - 3\alpha\right)} \right) \cdot |\xi|^{2\left(-\frac{3\alpha}{q'} + \theta\right)} \hat{a}(\xi) d\xi \leq \int_{\mathbb{R}^3} |\hat{a}(\xi)| \left( |\xi|^2 |\hat{a}(\xi)|^2 \right) \frac{1}{|\xi|^2} \cdot |\xi|^{2\left(-\frac{3\alpha}{q'} + \theta\right)} |\xi|^{-2\theta} d\xi,
\]
and similarly due to \(\frac{2}{s_1} + \frac{1}{s_2} = \frac{1}{p'} + 3\alpha\), we have
\[
\|a\|^2_{H^{\frac{1}{p} + 3\alpha} \bar{H}^\theta_h} = \int_{\mathbb{R}^3} |\hat{a}(\xi)|^2 \left( |\xi|^{2\left(\frac{1}{s_1} - 3\alpha\right)} |\xi|^{s_2} \right) \cdot |\xi|^{2\left(-\frac{3\alpha}{s_2} + \theta\right)} |\xi|^{-2\theta} d\xi \leq \int_{\mathbb{R}^3} |\hat{a}(\xi)| \left( |\xi|^2 |\hat{a}(\xi)|^2 \right) \frac{1}{|\xi|^2} \cdot |\xi|^{2\left(-\frac{3\alpha}{s_2} + \theta\right)} |\xi|^{-2\theta} d\xi.
\]

Applying Hölder’s inequality with measure \(|\xi|^{2\left(-\frac{3\alpha}{q'} + \theta\right)} |\xi|^{-2\theta}\) gives
\[
\|a\|_{H^{\frac{1}{p'} - 3\alpha - \theta}} + \|a\|_{H^{\frac{1}{p} + 3\alpha} \bar{H}^\theta_h} \lesssim \|a\|_{H^{\theta_h, r}} \|\nabla a\|_{H^{\theta_h, r}}^{1 - \frac{1}{p}}.
\]
As a result, it comes out

\[(4.12) \quad \|\partial_3 v^3\|_{\mathcal{H}^{\theta, r}_{v, 2}} \leq \|\partial_3 v^3\|_{\mathcal{H}^{\theta, r}_{v}}^{\frac{1}{p}} \|\nabla \partial_3 v^3\|_{\mathcal{H}^{\theta, r}_{v}}^{\frac{1}{p}}.\]

Substituting (4.11), (4.12) into (4.10), and using Young's inequality, we obtain

\[(4.13) \quad \left|\left(\mathcal{Q}_1(v, v) \partial_3 v^3\right)_{\mathcal{H}^{\theta, r}_{v}}\right| \leq \frac{1}{6} \|\nabla \partial_3 v^3\|_{\mathcal{H}^{\theta, r}_{v}}^{\frac{2}{p}} + C\|v^3\|_{\mathcal{B}_{1, \infty}^{0}} \left(\mathcal{B}_{1, \infty}^{0}\right)_{v}^{\frac{1}{p} + 3 \alpha(r)} \|\partial_3 v^3\|_{\mathcal{H}^{\theta, r}_{v}}^{\frac{2}{p}} \|\omega^2_{T}\|_{L^2}^{2} \left(\frac{2}{p}(2 \alpha(r) + \frac{1}{3}) \|\omega^2_{T}\|_{L^2}^{2} \right),\]

with \(s_1, s_2\) satisfying (4.3).

- The estimate of \(\left|\left(\mathcal{Q}_2(v, v) \partial_3 v^3\right)_{\mathcal{H}^{\theta, r}_{v}}\right|\)

We first get, by applying Bony's decomposition, that

\[
\partial_3 v^h \cdot \nabla_h v^3 = (T^h + R^h + \mathcal{T}^h) (T^v + R^v + \mathcal{T}^v) (\partial_3 v^h, \nabla_h v^3).
\]

Applying Lemma A.2 gives

\[
\|s_{k-1}^{h} \Delta_{v}^y \partial_3 v^h\|_{L^2_{h}(L^2)} \lesssim \sum_{k' \leq k-1} 2^{k' \ell} 2^{\ell} \|\Delta_{v}^{h} \Delta_{v}^y v^h\|_{L^2}
\]

Using this and Lemma A.1, we obtain

\[
\|\Delta_{v}^{h} \Delta_{v}^y R^h \mathcal{T}(\partial_3 v^h, \nabla_h v^3)\|_{L^2} \lesssim \sum_{|k' - k| \leq 4} \|s_{k-1}^{h} \Delta_{v}^y \partial_3 v^h\|_{L^2_{h}(L^2)} 2^{k'} \|\Delta_{k'} \mathcal{S}_{v}^y v^{3}\|_{L^2_{h}(L^2)}
\]

This shows that

\[
\|T^h \mathcal{T}(\partial_3 v^h, \nabla_h v^3)\|_{\mathcal{H}^{1+\theta - \mu - \frac{2}{p}, 1+\mu - 3\alpha(r) - \theta}} + \|R^h \mathcal{T}(\partial_3 v^h, \nabla_h v^3)\|_{\mathcal{H}^{1+\theta - \mu - \frac{2}{p}, 1+\mu - 3\alpha(r) - \theta}}
\]

(4.14)
While note that
\[
\|\Delta_h^k \Delta_h^l T^h T^v (\partial_3 v^h, \nabla_h v^3)\|_{L^2} \lesssim \sum_{|k'-k| \leq 4} \sum_{|\ell'-\ell| \leq 4} \|\Delta_h^{k'} \Delta_h^l v^h\|_{L^2} \|S_{k'-1}^h S_{\ell'-1}^v \nabla_h v^3\|_{L^\infty},
\]
yet it follows from Lemma A.2 that
\[
\|S_{k-1}^h S_{\ell-1}^v \nabla_h v^3\|_{L^\infty} \lesssim \sum_{k' \leq k-2} \sum_{\ell' \leq \ell-2} 2^{k} (\frac{2}{3} - \frac{4}{3p}) 2^{\ell} (\frac{1}{3} - \frac{2}{3p}) \|\Delta_h^{k'} \Delta_h^l v^3\|_{L^{\frac{3p}{5}}}.
\]
As a result, it comes out
\[
\|\Delta_h^k \Delta_h^l T^h T^v (\partial_3 v^h, \nabla_h v^3)\|_{L^2} \lesssim c_k \ell 2^{k} (\frac{2}{3} - \frac{4}{3p}) 2^{\ell} (3\alpha_0 (r) + \frac{1}{3} + \frac{4}{3p}) \|v^h\|_{H^{1-3\alpha_0 (r) - \frac{2}{p}}} \|v^3\|_{L^{\frac{3p}{5}}},
\]
and hence
\[
(4.15) \quad \|T^h T^v (\partial_3 v^h, \nabla_h v^3)\|_{H^{-\frac{2}{3} + \frac{4}{3p}, -\frac{4}{3p} - 3\alpha_0 (r)} \lesssim \|v^h\|_{H^{1-3\alpha_0 (r) - \frac{2}{p}}} \|v^3\|_{L^{\frac{3p}{5}}}.
\]

Observing that \(\partial_3\) is applied on the low-frequency part in \(T^v (\partial_3 v^h, \nabla_h v^3)\), but on the high-frequency part in \(T^h (\partial_3 v^h, \nabla_h v^3)\), hence naturally we believe that the estimates (4.14), (4.15) still hold when \(T^v\) is replaced by \(T^v\). Indeed, exactly along the same line of the proof of these two estimates, we can verify
\[
(4.16) \quad \|T^h T^v (\partial_3 v^h, \nabla_h v^3)\|_{H^{-\frac{2}{3} + \frac{4}{3p}, -\frac{4}{3p} - 3\alpha_0 (r)} \lesssim \|v^h\|_{H^{1-3\alpha_0 (r) - \frac{2}{p}}} \|v^3\|_{L^{\frac{3p}{5}}}.
\]
and
\[
(4.17) \quad \|T^h T^v (\partial_3 v^h, \nabla_h v^3)\|_{H^{-\frac{2}{3} + \frac{4}{3p}, -\frac{4}{3p} - 3\alpha_0 (r)} \lesssim \|v^h\|_{H^{1-3\alpha_0 (r) - \frac{2}{p}}} \|v^3\|_{L^{\frac{3p}{5}}}.
\]

On the other hand, by using Lemma A.1 and interpolation inequality, we have
\[
\|S_{k-1}^h \Delta^l \nabla_h v^3\|_{L^2} \lesssim c_k \ell 2^{k} (\frac{4}{3} + \frac{6\alpha_0 (r) - 2\theta}{3p} + \alpha_0 (r) - 2\theta) \|v^3\|_{H^{1-\frac{2}{3}, \frac{2}{3} + \frac{6\alpha_0 (r) - 2\theta}{3p} + \alpha_0 (r) - 2\theta}} \|\nabla_h \partial_3 v^3\|_{H^0, r} \|\partial_3 v^3\|_{H^\theta, r} \lesssim c_k \ell 2^{k} (\frac{4}{3} + \frac{6\alpha_0 (r) - 2\theta}{3p} + \alpha_0 (r) - 2\theta) \|\nabla_h \partial_3 v^3\|_{H^0, r} \|\partial_3 v^3\|_{H^\theta, r}.
\]
While it is easy to observe from Lemma A.2.2 that
\[
\|\Delta_h^k \Delta_h^l \partial_3 v^3\|_{L^2} \lesssim c_k \ell 2^{-k} \ell (2^{\frac{4\alpha_0 (r)}{3}+\alpha_0 (r)}) \|v^h\|_{H^{1-\frac{2}{3}, \frac{2}{3} + \alpha_0 (r)}}.
\]
As a consequence, we obtain
\[
\|\Delta_h^k \Delta_h^l R^v (\partial_3 v^h, \nabla_h v^3)\|_{L_{\alpha+1}^\frac{3p}{5}} \lesssim 2^{\ell (1-\frac{2}{3})} \sum_{|k'-k| \leq 4} \sum_{|\ell'-\ell| \leq 3} d_{k'} \ell 2^{k} (\frac{6\alpha_0 (r) - 2\theta}{3p} + \alpha_0 (r) - 2\theta) \|\nabla_h \partial_3 v^3\|_{H^\theta, r} \|v^h\|_{H^{1-\frac{2}{3}, \frac{2}{3} + \alpha_0 (r)}} \lesssim d_{k} \ell 2^{k} (\frac{6\alpha_0 (r) - 2\theta}{3p} + \alpha_0 (r) - 2\theta) \|\nabla_h \partial_3 v^3\|_{H^\theta, r} \|v^h\|_{H^{1-\frac{2}{3}, \frac{2}{3} + \alpha_0 (r)}}.
\]
Along the same line, due to $p > 4$, $r > \frac{2}{\theta}$, $\theta > 0$, there holds

$$
(4.18) \quad \frac{2}{3} - \frac{4}{3p} - 6\alpha(r) + 2\theta > 0.
$$

Then we have

$$
\left\| \Delta_k^h \Delta_k^v R^h R^v (\partial_3 v^h, \nabla_h v^3) \right\|_{L^{3p/(p+1)}} \lesssim 2^{k/3} \left( \frac{1}{r} - \frac{2}{3} \right) \sum_{k' \geq k-3} \sum_{k' \geq k-3} \left\| \Delta_k^h \Delta_k^v \nabla_h v^3 \right\|_{L^2} \left\| \Delta_k^h \Delta_k^v \nabla_h v^3 \right\|_{L^2}
$$

$$
\lesssim 2^{k/3} \left( \frac{1}{r} - \frac{2}{3} \right) \sum_{k' \geq k-3} \sum_{k' \geq k-3} d_{k',k} 2^{-k'} \left( \frac{2}{p} + 6\alpha(r) + 2\theta \right) 2^{-k' \left( \frac{2}{p} + 6\alpha(r) + 2\theta \right)} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r)}.
$$

We thus obtain

$$
(4.19) \quad \left\| \left( \mathcal{T}^h + R^h \right) R^v (\partial_3 v^h, \nabla_h v^3) \right\|_{L^{3p/(p+1)}} \lesssim \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r)}.
$$

Similarly, by using interpolation inequality, we have

$$
\left\| \Delta_k^h \Delta_k^v \nabla_h v^3 \right\|_{L^2} \leq c_k \left( \frac{2}{p} + 6\alpha(r) + 2\theta \right) 2^{-\left( \frac{2}{p} + 6\alpha(r) + 2\theta \right)} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}}.
$$

Then we deduce from Lemma A.1 and Lemma A.2 that

$$
\left\| \Delta_k^h \Delta_k^v T^h R^v (\partial_3 v^h, \nabla_h v^3) \right\|_{L^2 \left( L^2 \right)} \lesssim \sum_{|k'-k| \leq 4} \sum_{|k'-k| \leq 4} \left\| S_{k',k} \Delta_k^v \partial_3 v^h \right\|_{L^2 \left( L^2 \right)} \left\| \Delta_k^h \Delta_k^v \partial_3 v^h \right\|_{L^2} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r) + \mu}
$$

$$
\lesssim \sum_{|k'-k| \leq 4} \sum_{|k'-k| \leq 4} \sum_{|k'-k| \leq 4} \sum_{|k'-k| \leq 4} d_{k',k'} \left( \frac{2}{p} + 6\alpha(r) + 2\theta \right) 2^{-\left( \frac{2}{p} + 6\alpha(r) + 2\theta \right)} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r) + \mu},
$$

which implies

$$
(4.20) \quad \left\| T^h R^v (\partial_3 v^h, \nabla_h v^3) \right\|_{\left( B_{2,1}^{3p/(p+2)} \right) \left( B_{1,1}^{1,2} \right)} \lesssim \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r) + \mu}.
$$

Now we are in position to completes the estimate of $\left| \left( Q_2 (v, v) \right) \partial_3 v^3 \right|_{H^{\theta,r}}$. We first get, by using the estimates (4.14)-(4.17), that

$$
(4.21) \quad \left| \left( \text{Id} + 2\partial_3^3 \Delta^{-1} \right) T^h \left( T^v + T^v \right) (\partial_3 v^h, \nabla_h v^3) \partial_3 v^3 \right|_{H^{\theta,r}} \lesssim \left\| T^h \left( T^v + T^v \right) \right\|_{H^{\theta,r}} \left\| \partial_3 v^h \right\|_{H^{\theta,r}} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r)} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}}.
$$

$$
\lesssim \left\| v^3 \right\|_{L^{3p/(p+2)}} \left\| v^h \right\|_{H^{1,1} - \frac{2}{3} \alpha(r)} \left\| \nabla_3 v^3 \right\|_{H^{\theta,r}}.
$$
and
\[
\left| (\text{Id} + 2\partial_3^2 \Delta^{-1}) (T^h + R^h) (T^v + \mathcal{T}^v) \left( \partial_3 v^h, \nabla_h v^3 \right) \right|_{H^{\theta, r}} \\
\lesssim \left\| T^h (T^v + \mathcal{T}^v) \left( \partial_3 v^h, \nabla_h v^3 \right) \right\|_{H^{1-\frac{2}{p} - \frac{2}{q}, \frac{2}{q}, -3\alpha(r) - 2\theta}} \left\| \partial_3 v^3 \right\|_{H^{1-\frac{2}{p} - \frac{2}{q}, \frac{2}{q}, -3\alpha(r) - 2\theta}} \\
+ \left\| R^h (T^v + \mathcal{T}^v) \left( \partial_3 v^h, \nabla_h v^3 \right) \right\|_{H^{1-\theta, -\theta - 3\alpha(r)}} \left\| \partial_3 v^3 \right\|_{H^{1-\theta, -\theta - 3\alpha(r)} + \theta + 3\alpha(r) - \theta}
\] (4.22)
\[
\lesssim \| v^3 \|_{\left( B^{\frac{2}{p} + \mu}_{\infty, \infty} \right)_h (B^{-\mu}_{2, 1})} \| \nabla \partial_3 v^3 \|_{H^{\theta, r}} \\
\times \left( \| v^h \|_{B^{1-\frac{2}{p} - \frac{2}{q}, -3\alpha(r) + \mu}_{2, 1}} + \| v^h \|_{B^{1-\frac{2}{p} + \theta - \mu, -3\alpha(r) - \theta + \mu}_{2, 1}} \right).
\]

Using (4.19) and (4.20), we obtain
\[
\left| (\text{Id} + 2\partial_3^2 \Delta^{-1}) (T^h + R^h) R^v \left( \partial_3 v^h, \nabla_h v^3 \right) \right|_{H^{\theta, r}} \\
\lesssim \left\| (T^h + R^h) R^v \left( \partial_3 v^h, \nabla_h v^3 \right) \right\|_{B^{\frac{2}{p} + \mu}_{2, 1} \rightarrow \frac{\mu}{p} - 2, \frac{1}{2} - 3\alpha(r) \| v^3 \|_{L^{\frac{3}{p}, \frac{d}{d + 2}}}}.
\] (4.23)
\[
\lesssim \| \nabla \partial_3 v^3 \|_{H^{\theta, r}} \| v^3 \|_{B^{1-\frac{2}{p} - \frac{2}{q}, -3\alpha(r) + \mu}_{2, 1}} \| v^3 \|_{\left( B^{\frac{2}{p} + \mu}_{2, 1} \right)_h (B^{-\mu}_{\infty, \infty})}.
\]

and
\[
\left| (\text{Id} + 2\partial_3^2 \Delta^{-1}) T^h R^v \left( \partial_3 v^h, \nabla_h v^3 \right) \right|_{H^{\theta, r}} \\
\lesssim \| T^h R^v \left( \partial_3 v^h, \nabla_h v^3 \right) \|_{\left( B^{\frac{2}{p} + \mu}_{1, 1} \right)_h (B^{-\mu}_{1, 1})} \| \partial_3 v^3 \|_{\left( B^{\frac{2}{p} + \mu}_{2, 1} \right)_h (B^{-\mu}_{\infty, \infty})}.
\] (4.24)
\[
\lesssim \| \nabla \partial_3 v^3 \|_{H^{\theta, r}} \| v^f \|_{B^{1-\frac{2}{p} - \frac{2}{q}, -3\alpha(r) + \mu}_{2, 1}} \| v^3 \|_{\left( B^{\frac{2}{p} + \mu}_{2, 1} \right)_h (B^{-\mu}_{\infty, \infty})}.
\]

In view of Lemma A.3, we have \( (B_{2, 1}^{\frac{2}{p} + \mu}, -\mu) \rightarrow (B_{2, 1}^{-\mu}) \), \( \Theta \rightarrow (B_{2, 1}^{-\mu}) \). Thus we get, by combining (4.21)-(4.24) and using Lemma A.8, that
\[
\left| (Q_2 (v, v) \left| \partial_3 v^3 \right) \right|_{H^{\theta, r}} \\
\leq C \| v^3 \|_{SC} \| \nabla \partial_3 v^3 \|_{H^{\theta, r}} \left( \| \omega \|_{2, 1}^{2(\alpha + 2\theta) + \frac{\mu}{p}} \| \omega \|_{L^2}^{1-\frac{2}{p}} + \| \partial_3 v^3 \|_{H^{\theta, r}} \| \nabla \partial_3 v^3 \|_{H^{\theta, r}} \right)
\leq \frac{1}{6} \| \nabla \partial_3 v^3 \|_{H^{\theta, r}}^2 \| \omega \|_{L^2}^{2(\alpha + 2\theta) + \frac{\mu}{p}} \| \omega \|_{L^2}^{(1-\frac{2}{p})} + C \| v^3 \|_{SC} \| \partial_3 v^3 \|_{H^{\theta, r}}^2.
\] (4.25)

- The estimate of \( \left| (Q_3 (v, v) \left| \partial_3 v^3 \right) \right|_{H^{\theta, r}} \)

Let us first deal with the estimate of \( \left| (v^h \cdot \nabla_h \partial_3 v^3 \mid \partial_3 v^3 \right) \|_{H^{\theta, r}} \). Applying Bony’s decomposition in the vertical variable for \( v^h \cdot \nabla_h \partial_3 v^3 \) yields
\[
v^h \cdot \nabla_h \partial_3 v^3 = (T^v + R^v + \mathcal{T}^v) (v^h \cdot \nabla_h \partial_3 v^3).
\]

We first observe that \( \partial_3 \) is applied on the low-frequency part in \( \mathcal{T}^v (v^h, \nabla_h \partial_3 v^3) \), but on the high-frequency part in \( \mathcal{T}^v (\partial_3 v^h, \nabla_h v^3) \), hence naturally we believe that the estimates for \( \mathcal{T}^v (\partial_3 v^h, \nabla_h v^3) \) still hold for \( \mathcal{T}^v (v^h, \nabla_h \partial_3 v^3) \). For the same reason, we also believe that \( R^v (v^h, \nabla_h \partial_3 v^3) \) shares the same estimates as \( R^v (\partial_3 v^h, \nabla_h v^3) \). Indeed, exactly along the same
line of the estimate for \( |(Q_2(v,v)|\partial_3 v^3\rangle_{\mathcal{H}_h} |, \) we achieve
\[
((T^v + R^v)(v^h, \nabla_h \partial_3 v^3)|\partial_3 v^3\rangle_{\mathcal{H}_h} | \leq \frac{1}{6} \|\nabla_\partial_3 v^3\|^2_{\mathcal{H}_h} + C\|v^3\|^2_{\mathcal{H}_h} \|\partial_3 v^3\|^2_{\mathcal{H}_h} \\
+ C\|v^3\|^2_{\mathcal{H}_h} \|\nabla_\partial_3 v^3\|_{L^2}^2 \cdot \|\nabla_\partial_3 v^3\|_{L^2}^{2(1-\frac{\alpha}{2})}.\]
(4.26)

It remains to deal with the estimate of \( T^v(v^h, \nabla_h \partial_3 v^3) \). By using Bony's decomposition in the horizontal variables for \( T^v(v^h, \nabla_h \partial_3 v^3) \), we write
\[
T^v(v^h, \nabla_h \partial_3 v^3) = (T^h + R^h + \mathbb{T}) \cdot T^v(v^h, \nabla_h \partial_3 v^3).
\]

We first write
\[
(4.27) \quad (T^h T^v(v^h, \nabla_h \partial_3 v^3)|\partial_3 v^3\rangle_{\mathcal{H}_h} = \sum_{k,\ell \in \mathbb{Z}} 2^{2k(-3\alpha(r)+\theta)} 2^{-2\ell(1+\theta)} (I_{k,\ell} + I_{k,\ell}^2 + I_{k,\ell}^3), \quad \text{with}
\]
\[
I_{k,\ell}^1 \quad \text{def} \quad \sum_{\|k'-k\| \leq 4, |\ell'-\ell| \leq 4} \|\Delta_k^h \Delta_{k'} \cdot S_{k'-1}^h \cdot S_{\ell'-1}^v \cdot v^h\| \Delta_k^h \Delta_{k'} \cdot \Delta_{\ell'} \cdot \nabla_h \partial_3 v^3 \| L^2 \|\Delta_k^h \Delta_{k'} \cdot \partial_3 v^3\|^2_{L^2} |\Delta_{\ell'}^h \Delta_{\ell'} \cdot \partial_3 v^3\|_{L^2}^2,
\]
\[
I_{k,\ell}^2 \quad \text{def} \quad \sum_{\|k'-k\| \leq 4, |\ell'-\ell| \leq 4} \|S_{k'-1}^h \cdot S_{\ell'-1}^v \cdot v^h - S_{k'-1}^h \cdot S_{\ell'-1}^v \cdot v^h\| \Delta_k^h \Delta_{k'} \cdot \Delta_{\ell'} \cdot \nabla_h \partial_3 v^3 \| L^2 \|\Delta_k^h \Delta_{k'} \cdot \partial_3 v^3\|^2_{L^2} |\Delta_{\ell'}^h \Delta_{\ell'} \cdot \partial_3 v^3\|_{L^2}^2,
\]
\[
I_{k,\ell}^3 \quad \text{def} \quad \frac{1}{2} (S_{k'-1}^h \cdot S_{\ell'-1}^v \cdot \nabla v^h \cdot \Delta_k^h \Delta_{k'} \cdot \partial_3 v^3 \| |\Delta_{\ell'}^h \Delta_{\ell'} \cdot \partial_3 v^3\|_{L^2}^2.
\]

It follows from a standard commutator's estimate (see for instance [1]) that
\[
|I_{k,\ell}^1| \lesssim \sum_{\|k'-k\| \leq 4, |\ell'-\ell| \leq 4} \left( 2^{-k} \|S_{k'-1}^h \cdot S_{\ell'-1}^v \cdot \nabla v^h\|_{L^6 \ast L^4} \|\Delta_k^h \Delta_{k'} \cdot \Delta_{\ell'} \cdot \nabla_h \partial_3 v^3\|_{L^2} \|\Delta_k^h \Delta_{k'} \cdot \partial_3 v^3\|^2_{L^2} \right) \left( 2^{-\ell} \|S_{k'-1}^h \cdot S_{\ell'-1}^v \cdot \partial_3 v^h\|_{L^6 \ast L^4} \|\Delta_k^h \Delta_{k'} \cdot \Delta_{\ell'} \cdot \nabla_h \partial_3 v^3\|_{L^2} \|\Delta_k^h \Delta_{k'} \cdot \partial_3 v^3\|^2_{L^2} \right)
\]
\[
\quad \quad \quad \text{def} \quad I_{k,\ell}^{1.1} + I_{k,\ell}^{1.2}.
\]

Noting that \(-3\alpha(r) + \theta < 0, -\theta < 0\), we use Lemmas A.1 and A.2 to get
\[
I_{k,\ell}^{1.1} \lesssim c_{\ell,\ell} 2^{k(1-3\alpha(r)+\theta)} 2^{\ell(1-\theta)} \|\nabla v^h\|_H^{\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta} \|\partial_3 v^3\|_{H^{-3\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta}} 2^{-\ell(1+\theta)} \|v^3\|_{L^\infty_{\mathcal{H}_h}}
\]
\[
\quad \quad \quad \quad \quad \quad \quad \text{and}
\]
\[
I_{k,\ell}^{1.2} \lesssim c_{\ell,\ell} 2^{k} \|\partial_3 v^3\|_{H_{\mathcal{H}_h}^{\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta}} \|\partial_3 v^3\|_{H_{\mathcal{H}_h}^{\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta}} \|v^3\|_{L^\infty_{\mathcal{H}_h}}
\]
\[
\quad \quad \quad \quad \quad \quad \quad \text{and}
\]
\[
I_{k,\ell}^{2.1} \quad \text{def} \quad \sum_{k,\ell \in \mathbb{Z}} 2^{2k(3\alpha(r)-\theta)} 2^{2\ell \|v^h\|_{H_{\mathcal{H}_h}^{1-\mu, 1-3\alpha(r)+\theta, 1-\theta}} \|\partial_3 v^3\|_{H_{\mathcal{H}_h}^{\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta}} \|\partial_3 v^3\|_{H_{\mathcal{H}_h}^{\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta}} \|\nabla v^3\|_{H_{\mathcal{H}_h}^{\alpha(r)+\theta, 1-3\alpha(r)+\theta, 1-\theta}}.
\]
Then by summing up in $k, \ell$, using (2.3), Lemmas A.8 and A.7, we obtain
\[
\sum_{k,\ell \in \mathbb{Z}} 2^{2k(-3\alpha+\theta)}2^{-2\theta} |I_{1,k,\ell}^1|
\]
\[
(4.28) \lesssim \|v^3\|_{SC} \|\nabla \partial_3 v^3\|_{H^\theta,r} \cdot \left( \|\omega, \partial_3 v^3\|_{H^{\frac{3}{2} - \frac{1}{3\alpha+\theta}-\frac{1}{2}+\epsilon}} + \|v^h\|_{B^{\frac{1}{2}+3\alpha(r)-\frac{1}{2}+\theta}_{2,1}} \right)
\]
\[
(4.32) \lesssim \|v^3\|_{SC} \|\nabla \partial_3 v^3\|_{H^\theta,r} \cdot \left( \|\omega_2^r\|_{L^2}^{2(\alpha+\theta)+\frac{2}{3}} \|\nabla \omega_2^r\|_{L^2}^{1-\frac{2}{3}} + \|\partial_3 v^3\|_{H^\theta,r}^{\frac{2}{3}} \|\nabla \partial_3 v^3\|_{H^\theta,r}^{1-\frac{2}{3}} \right).
\]

The estimate for $I_{2,k,\ell}^2$ is similar to $I_{1,k,\ell}^2$, whereas the estimate for $I_{2,k,\ell}^1$ is similar to $I_{1,k,\ell}^1$. Then we conclude, by using (4.27) and Young’s inequality, that
\[
|\left( T^h T^v (v^h, \nabla_h \partial_3 v^3) \right) | \lesssim \frac{1}{36} \|\nabla \partial_3 v^3\|_{H^\theta,r}^{2} + C \|v^3\|_{SC} \|\partial_3 v^3\|_{H^\theta,r}^{2}
\]
\[
(4.29) + C \|v^3\|_{SC}^{2} \|\omega_2^r\|_{L^2}^{2(\alpha+\theta)+\frac{2}{3}} \|\nabla \omega_2^r\|_{L^2}^{2(1-\frac{2}{3})}.
\]

Noting that $\nabla_h$ is applied on the high-frequency part in $T^h (v^h, \nabla_h \partial_3 v^3)$, so exactly along the same line of the proof of (4.29), we get
\[
|\left( (\tilde{T}^h + R^h) T^v (v^h, \nabla_h \partial_3 v^3) \right) | \lesssim \frac{1}{36} \|\nabla \partial_3 v^3\|_{H^\theta,r}^{2} + C \|v^3\|_{SC} \|\partial_3 v^3\|_{H^\theta,r}^{2}
\]
\[
(4.30) + C \|v^3\|_{SC}^{2} \|\omega_2^r\|_{L^2}^{2(\alpha+\theta)+\frac{2}{3}} \|\nabla \omega_2^r\|_{L^2}^{2(1-\frac{2}{3})}.
\]

Combining (4.26), (4.29) and (4.30) gives rise to
\[
|v^h \cdot \nabla_h \partial_3 v^3 | \lesssim \frac{2}{9} \|\nabla \partial_3 v^3\|_{H^\theta,r}^{2} + C \|v^3\|_{SC} \|\partial_3 v^3\|_{H^\theta,r}^{2}
\]
\[
(4.31) + C \|v^3\|_{SC}^{2} \|\omega_2^r\|_{L^2}^{2(\alpha+\theta)+\frac{2}{3}} \|\nabla \omega_2^r\|_{L^2}^{2(1-\frac{2}{3})}.
\]

To estimate $|v^3 \cdot \partial_3^2 v^3 | H^\theta,r |$, we first use integration by parts to get
\[
(v^3 \cdot \partial_3^2 v^3 | \partial_3 v^3 | H^\theta,r | = -\frac{1}{2} (\partial_3 v^3 \cdot \partial_3 v^3 | v^3 | H^\theta,r |),
\]
then by applying Lemma 4.1 and interpolation inequality, we obtain
\[
\|v^3 \cdot \partial_3^2 v^3 | \partial_3 v^3 | H^\theta,r | \lesssim \|\partial_3 v^3\|_{H^{\frac{3}{2}+3\alpha(r)-\frac{1}{2}+\epsilon}} \|v^3\|_{B^{\frac{1}{2}+3\alpha(r)-\frac{1}{2}+\theta}_{2,1}}
\]
\[
(4.32) \lesssim \|\partial_3 v^3\|_{H^{\frac{3}{2}+3\alpha(r)-\frac{1}{2}+\epsilon}} \|\nabla \partial_3 v^3\|_{H^{\theta,r}}^{2(1-\frac{1}{2})} \|v^3\|_{B^{\frac{1}{2}+3\alpha(r)-\frac{1}{2}+\theta}_{2,1}}.
\]

Combining the estimates (4.31) and (4.32), and using Young’s inequality, we arrive at
\[
|Q_3 (v, v) | H^\theta,r | \lesssim \frac{1}{3} \|\nabla \partial_3 v^3\|_{H^\theta,r}^{2} + C \|v^3\|_{SC} \|\omega_2^r\|_{L^2}^{2(\alpha+\theta)+\frac{2}{3}} \|\nabla \omega_2^r\|_{L^2}^{2(1-\frac{2}{3})}
\]
\[
(4.33) + C \left( \|v^3\|_{SC}^{2} + \|v^3\|_{B^{\frac{1}{2}+3\alpha(r)-\frac{1}{2}+\theta}_{2,1}} \right) \|\partial_3 v^3\|_{H^\theta,r}^{2}.
\]

Now we are in a position to complete the proof of Proposition 2.2.
Proof of Proposition 2.2. By the assumptions of Proposition 2.2, we have $q_2 < \left(\frac{1}{p} + 3a(r) + \mu\right)^{-1}$, so that we can choose $s_1$, and $s_2$ with

$$\frac{1}{s_2} = \frac{1}{p} + 3a(r) + \mu < \frac{1}{q_2}, \quad \frac{2}{s_1} = 1 - \frac{2}{p} - \mu < \frac{2}{q_1}.$$

Then in view of Lemma A.3, there holds $\left(\tilde{B}_{q_1, \infty}^{\frac{1}{q_1} + \frac{1}{q_2} - 1 + \mu}\right)_V \left(\tilde{B}_{q_2, \infty}^{\frac{1}{q_2}}\right)_V \mapsto \left(\tilde{B}_{s_1, \infty}^0\right)_V \left(\tilde{B}_{s_2, \infty}^0\right)_V$.

Substituting (4.13), (4.25) and (4.33) into (4.2) leads to

$$\frac{d}{dt} \|\partial_3 v^3(t)\|_{H^{s_2, r}}^2 + \|\nabla v^3(t)\|_{H^{s_2, r}}^2 \leq C \|v^3\|_{SC}^p \|\partial_3 v^3(r)\|_{H^{s_2, r}}^2$$

$$+ C \|v^3\|_{SC}^p \|\nabla v^3\|_{L^2} \left(2\frac{(2a(r) + \frac{1}{2})}{\rho} \right) \|\nabla v^3\|_{L^2}^2 \left(1 - \frac{1}{\rho}\right) + C \|v^3\|_{SC}^p \|\nabla v^3\|_{L^2} \left(2\frac{(2a(r) + \frac{1}{2})}{\rho} \right) \|\nabla v^3\|_{L^2}^2 \left(1 - \frac{1}{\rho}\right).$$

Applying Gronwall’s inequality yields (2.6). □

APPENDIX A. TOOL BOX ON FUNCTIONAL SPACES

We first recall the definition of homogeneous Besov space:

**Definition A.1.** Let $(p, q, r)$ be in $[1, \infty]^3$ and $s$ in $\mathbb{R}$. Let us consider $u$ in $\mathcal{S}'_h(\mathbb{R}^d)$, which means that $u$ is in $\mathcal{S}'(\mathbb{R}^d)$ and satisfies $\lim_{j \to -\infty} \|S_j u\|_{L^\infty} = 0$. We set

$$\|u\|_{\dot{B}^s_{p, r}} \overset{def}{=} \left\| \left(2^j \|\Delta_j u\|_{L^p}\right)_j \right\|_{\ell^r(\mathbb{Z})}.$$

- For $s < \frac{d}{p}$ (or $s = \frac{d}{p}$ if $r = 1$), we define $\dot{B}^s_{p, r}(\mathbb{R}^d) \overset{def}{=} \{ u \in \mathcal{S}'_h(\mathbb{R}^d) \mid \|u\|_{\dot{B}^s_{p, r}} < \infty \}$.

- If $k \in \mathbb{N}$ and if $\frac{d}{p} + k \leq s < \frac{d}{p} + k + 1$ (or $s = \frac{d}{p} + k + 1$ if $r = 1$), then we define $\dot{B}^s_{p, r}(\mathbb{R}^d)$ as the subset of $u$ in $\mathcal{S}'_h(\mathbb{R}^d)$ such that $\partial^\beta u$ belongs to $\dot{B}^s_{p, r}(\mathbb{R}^d)$ whenever $|\beta| = k$.

We remark that $\dot{B}^s_{2, 2}$ coincides with the classical homogeneous Sobolev spaces $\dot{H}^s$.

When $s < 0$, we also have the following characterization of Besov spaces $\dot{B}^s_{p, r}$:

**Lemma A.1** (Proposition 2.33 of [1]). Let $s < 0$, $1 \leq p, r \leq \infty$ and $u \in \mathcal{S}'_h(\mathbb{R}^d)$. Then $u$ belongs to $\dot{B}^s_{p, r}(\mathbb{R}^d)$ if and only if

$$\left(2^j \|\dot{S}_j u\|_{L^p}\right)_{j \in \mathbb{Z}} \in \ell^r.$$

Moreover, for some constant $C$ depending only on the dimension $d$, we have

$$C^{-|s|+1} \|u\|_{\dot{B}^s_{p, r}} \leq \left\| \left(2^j \|\dot{S}_j u\|_{L^p}\right)_{j \in \mathbb{Z}} \right\|_{\ell^r} \leq C \left(1 + \frac{1}{|s|}\right) \|u\|_{\dot{B}^s_{p, r}}.$$

For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [4, 12]:

**Lemma A.2.** Let $\mathcal{B}_h$ (resp. $\mathcal{B}_v$) a ball of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$), and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring of $\mathbb{R}^2_h$ (resp. $\mathbb{R}_v$); let $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$. Then there holds:

If the support of $\tilde{a}$ is included in $2^k \mathcal{B}_h$, then

$$\left\| \partial^\alpha a \right\|_{L^ {p_1}_h(L^{q_1}_v)} \lesssim 2^k (|\alpha| + 2(1/p_2 - 1/p_1)) \|a\|_{L^{p_2}_v(L^{q_2}_v)}.$$

If the support of $\tilde{a}$ is included in $2^k \mathcal{B}_v$, then

$$\left\| \partial^\beta a \right\|_{L^ {p_1}_h(L^{q_1}_v)} \lesssim 2^l (|\beta| + (1/q_2 - 1/q_1)) \|a\|_{L^{p_1}_h(L^{q_1}_v)}.$$
If the support of $\hat{a}$ is included in $2^k C_h$, then
\[ \|a\|_{L^p_h(L^q_h)} \lesssim 2^{-kN} \sup_{|a| = N} \|\partial_x a\|_{L^p_h(L^q_h)}. \]

If the support of $\hat{a}$ is included in $2^l C_v$, then
\[ \|a\|_{L^p_h(L^q_h)} \lesssim 2^{-lN} \|\partial_x a\|_{L^p_h(L^q_h)}. \]

Lemma A.3 (Proposition 2.20 of [1]). Let $1 \leq p_1 \leq p_2 \leq \infty$, and $1 \leq r_1 \leq r_2 \leq \infty$. Then for any $1 \leq s \leq \infty$, the space $B^s_{p_1,r_1}$ is continuously embedded in $B^s_{p_2,r_2}$.

Lemma A.4 (Theorem 2.40 and 2.41 of [1]). For any $p \in [2, \infty]$, we have
\[ \hat{B}_{p,2}^0 \hookrightarrow L^p \hookrightarrow \hat{B}_{p,p}^0 \quad \text{and} \quad \hat{B}_{p',2'}^0 \hookrightarrow L^{p'} \hookrightarrow \hat{B}_{p',p'}^0. \]

Lemma A.5 (Lemma 5.1 of [5]). Let $(s, \alpha)$ be in $[0,1]^2$ and $(p,q)$ in $[1,\infty]^2$. For any function $G$ from $\mathbb{R}$ to $\mathbb{R}$ which is Hölderian of exponent $\alpha$, and any $a \in \dot{B}_{s,p,q}^0$, one has
\[ \|G(a)\|_{\dot{B}_{s,p,q}^0} \lesssim \|G\|_{C^\alpha} \|\|a\|_{\dot{B}_{s,p,q}^0}\|^{\alpha} \quad \text{with } \|G\|_{C^\alpha} \overset{\text{def}}{=} \sup_{r \neq r'} \frac{|G(r) - G(r')|}{|r - r'|^{\alpha}}. \]

Lemma A.6 (Lemma 4.3 of [5]). For any $s$ positive and any $\theta$ in $[0,s]$, we have
\[ \|f\|_{(\dot{B}_{s,p,q}^0)_{\theta}(\dot{B}_{s,p}^0)} \lesssim \|f\|_{\dot{B}_{s,p,q}^0}. \]

We shall frequently use the following non-linear interpolation inequalities.

Lemma A.7 (Lemma 3.1 of [6]). For $r$ in $[3/2,2]$, we have
\begin{equation}
\|\nabla a\|_{L^r} \lesssim \|\nabla a\|_{L^2} \|a\|_{L^2}^{\frac{2}{r} - 1}.
\end{equation}

Moreover, for $s$ in $[-3\alpha(r),1-\alpha(r)]$, we have
\begin{equation}
\|a\|_{H^s} \leq C\|a\|_{H^s}^{1-\alpha(r) - s} \|\nabla a\|_{L^2}^{3\alpha(r) + s}.
\end{equation}

The following lemma is one of the main motivations of using anisotropic Besov space. It can be viewed as a generalization of Proposition 2.1 in [11] and Proposition 3.1 in [6], and its proof follows immediately by combining the proofs of these two propositions together.

Lemma A.8. Let $\theta \in ]0,\alpha(r)]$ and $s < \frac{2}{3} \alpha(r)$, $\beta < \min\{1 - \frac{2}{3} \alpha(r), 1 - \alpha(r) + \theta\}$ satisfy $s + \beta > 0$. Let $v$ be a divergence free vector field and $\omega = \partial_1 v^2 - \partial_2 v^1$. Then one has
\[ \|v^h\|_{\left(\dot{B}_{2,1}^{1-3\alpha(r) - \beta}\right)_{\theta}} \lesssim \omega_{\theta}^{\frac{2\alpha(r)+\beta}{\theta}} \|\nabla \omega\|_{L^2}^{1-s-\beta} \|\nabla v^3\|_{H^{s+\beta}} \|\nabla v^3\|_{H^{1-s-\beta}}. \]

At the end of this section, let us recall the para-differential decomposition (Bony’s decomposition) from [2]: let $a$ and $b$ be in $S'(\mathbb{R}^3)$, then we have the following decomposition
\begin{equation}
ab = T(a,b) + \bar{T}(a,b) + R(a,b) \quad \text{with}
\end{equation}
\[ \begin{align*}
T(a,b) &= \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b, \\
\bar{T}(a,b) &= T(b,a), \\
R(a,b) &= \sum_{j \in \mathbb{Z}} \Delta_j a \bar{\Delta}_j b.
\end{align*} \]

We shall also use Bony’s decomposition in horizontal variables or vertical variable, in order to study product laws between distributions in anisotropic Besov spaces.
Lemma B.1. Let \( q_3 > r' \), \( \alpha(r) < \min\{ \frac{1}{p}, \frac{1}{q_1} \} \) and \( \delta \in ]0, \frac{1}{q_3} - \alpha(r)[ \). Then we have
\[
\left\| S_k^h \Delta_k^\nu \omega_{r-1} \right\|_{L_h^\infty(L_{q_1}'')} \lesssim c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_2\|_{H^{2\nu}'} \,
\]
\[
\left\| S_k^h S_k^\nu \omega_{r-1} \right\|_{L_h^\infty(L_{q_3}'')} \lesssim c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_2\|_{H^{2\nu}'} \,
\]
\[
\left\| S_k^h S_k^\nu \omega_{r-1} \right\|_{L_h^\infty(L_{q_3}'')} \lesssim c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_2\|_{H^{2\nu}'} \,
\]
Here and in all that follows, we always denote \( (c(k, \ell, r))_{k, \ell \in \mathbb{Z}^2} \) to be a generic element in \( \ell^r(\mathbb{Z}^2) \) so that \( \sum_{(k, \ell) \in \mathbb{Z}^2} c(k, \ell, r) = 1 \). In particular, when \( r = 2 \), \( (c(k, \ell, 2))_{k, \ell \in \mathbb{Z}^2} \) is the same to the \( (c(k, \ell))_{k, \ell \in \mathbb{Z}^2} \) defined before.

Proof. Note that \( \alpha(r) < \frac{1}{p} \) and \( \delta > 0 \), we get, by applying Lemma A.2 and Lemma A.6, that
\[
\left\| S_k^h \Delta^\nu_k^l \omega_{r-1} \right\|_{L_h^\infty(L_{q_1}'')} \lesssim \sum_{k' \leq k-1} 2^{k'} \left( \frac{4}{q_3} - \frac{1}{r} \right) \left\| \Delta^\nu_k^l \Delta_k^\nu \omega_{r-1} \right\|_{L_{r'}^\infty(L_{q_1}'')}
\]
\[
\lesssim \sum_{k' \leq k-1} c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_{r-1}\|_{B_{r', \epsilon}^{1-\frac{4}{q_3} - \alpha(r)}} \lesssim c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_2\|_{H^{2\nu}'}.
\]
This proves the first inequality of the lemma. Applying Lemma A.2 once again yields
\[
\left\| S_k^h S_k^\nu \omega_{r-1} \right\|_{L_h^\infty(L_{q_3}'')} \lesssim \sum_{k' \leq k-1} 2^{k'} \left( \frac{4}{q_3} - \frac{1}{r} \right) \left\| S_k^h \Delta^\nu_k^l \omega_{r-1} \right\|_{L_h^\infty(L_{q_3}'')}
\]
\[
\lesssim \sum_{k' \leq k-1} c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_{r-1}\|_{B_{r', \epsilon}^{1-\frac{4}{q_3} - \alpha(r)}} \lesssim c(k, \ell, r') 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell \delta} \|\omega_2\|_{H^{2\nu}'}.
\]
This proves the second inequality of the lemma. The remaining one can be proved along the same line.

Proof of (3.4). We first get, by applying Bony’s decomposition in the vertical variable, that
\[
T_h^\nu(\omega_{r-1}, a) = T_h^\nu(T^\nu + \bar{T}^\nu + R^\nu)(\omega_{r-1}, a).
\]
By using Lemma B.1 and Lemma A.2, we obtain
\[
\|\Delta^\nu_k^l \Delta_k^\nu T^\nu(\omega_{r-1}, a)\|_{L_{r'}^\infty} \lesssim \sum_{k' \leq k-1} \sum_{\ell' \leq \ell-1} \left\| S_k^h S_{k'}^\nu \omega_{r-1} \right\|_{L_h^\infty(L_{q_3}'')} \left\| \Delta^\nu_k^l \Delta_k^\nu a \right\|_{L_h^\infty(L_{q_3}'')}
\]
\[
\lesssim \sum_{k' \leq k-1} \sum_{\ell' \leq \ell-1} 2^{k' \alpha(r)} 2^{-k' \left( \frac{2}{p} + \mu \right)} 2^{-\ell' \left( \frac{4}{q_3} - \frac{\mu}{2} \right)} \|a\|_{B_{2^{4(p+\mu)}} \left( \frac{4}{q_3} - \frac{\mu}{2} \right)} \times 2^{k'} \left( \delta \left( \frac{4}{q_3} - \alpha(r) \right) \right) 2^{-\ell' \delta} \|\omega_2\|_{H^{2\nu}'} \cdot c(k, \ell').
\]

where $m_1$ satisfies $\frac{1}{m_1} = \frac{1}{q_2} - \frac{1}{q}$. Due to $q_2 \in [2, (\mu + 3\alpha(r) + 1/p)^{-1}]$, one has $\mu < \frac{1}{q_2} - \alpha(r)$. So that taking $\delta = \mu$ in the above inequality leads to

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{r'}} \lesssim c_{k, \ell} \|a\|_{B^p_{2q_2, r}} \|\omega^r_{q_2}\|_{H^s}^\frac{2}{p}.$$  

Similarly, we have

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{r'}} \lesssim \sum_{|k' - k| \leq 4} \sum_{|\ell' - \ell| \leq 4} \|S^h_{k' - 1} \Delta^y_{k'} \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \|\Delta^h_k \Delta^y_k \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})},$$

from which, Lemma B.1, we deduce that $T^h T^v (\omega_r - 1, a)$ shares the same estimate as $T^h T^v (\omega_r - 1, a)$.

Whereas applying Lemma A.2 and then Lemma B.1 yields

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{r'}} \lesssim 2\frac{\ell}{q_2} \sum_{|k' - k| \leq 4} \sum_{|\ell' - \ell| \leq 4} \|S^h_{k' - 1} \Delta^y_{k'} \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \|\Delta^h_k \Delta^y_k \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \lesssim 2\frac{\ell}{q_2} \sum_{|k' - k| \leq 4} \sum_{|\ell' - \ell| \leq 4} \|a\|_{B^p_{2q_2, r}} \|\omega^r_{q_2}\|_{H^s}^\frac{2}{p}.$$  

As a result, it comes out

$$(B.1) \quad \|T^h (\omega_r - 1, a)\|_{L^{r'}} \lesssim \|T^h (\omega_r - 1, a)\|_{B^0_{r', 2}} \lesssim \|a\|_{B^p_{2q_2, r}} \|\omega^r_{q_2}\|_{H^s}^\frac{2}{p}.$$  

Along the same line to proof of (B.1), we deduce from Lemmas A.2 and B.1 that

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{q_2}} \lesssim \sum_{|k' - k| \leq 4} \sum_{|\ell' - \ell| \leq 4} \|S^h_{k' - 1} S^y_{\ell' - 1} \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \|\Delta^h_k \Delta^y_k \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \lesssim \sum_{|k' - k| \leq 4} \sum_{|\ell' - \ell| \leq 4} c_{k', \ell'} 2^{-k(2\alpha(r) + \mu + \delta) - \ell'\theta} \|a\|_{B^p_{2q_2, r}} \|\omega^r_{q_2}\|_{H^s}^\frac{2}{p},$$

where $m_2$ satisfies $\frac{1}{m_2} = \frac{1}{2} - \frac{1}{q_2}$. Due to $\mu \in ] \alpha(r), \frac{1}{q_2} - 3\alpha(r) - \frac{1}{p} [, \ and \ \theta \in ] 0, \alpha(r) [, \ we \ have \ \mu - \alpha(r) + \theta \in ] 0, \frac{1}{q_2} - \alpha(r) [, \ T \ hus \ we \ can \ take \ \delta = \mu - \alpha(r) + \theta \ in \ the \ above \ inequality \ to \ get

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{q_2}} \lesssim c_{k, \ell'} 2^{-k(3\alpha(r) - \theta) - \ell'\theta} \|a\|_{B^p_{2q_2, r}} \|\omega^r_{q_2}\|_{H^s}^\frac{2}{p}.$$  

Similarly, for $\rho_1 = 1/\alpha(r)$, we write

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{q_1}} \lesssim \sum_{|k' - k| \leq 4} \sum_{|\ell' - \ell| \leq 4} \|S^h_{k' - 1} \Delta^y_k \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \|\Delta^h_k \Delta^y_k \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})},$$

and it is easy to observe from Lemma A.1, Lemma A.2 and $\mu > \alpha(r)$ that

$$\|\Delta^h_k \Delta^y_k \omega_{r - 1}\|_{L^\infty_h(L^q_{\ell'}^{q_2})} \lesssim c_{k', \ell'} 2^{-k(\mu + \frac{3}{p}) - \ell'\theta} \|a\|_{B^p_{2q_2, r}},$$

from which and Lemma B.1 with $\delta = \mu - \alpha(r) + \theta$, we infer

$$\|\Delta^h_k \Delta^y_k T^h T^v (\omega_r - 1, a)\|_{L^{q_1}} \lesssim c_{k, \ell'} 2^{-k(3\alpha(r) - \theta) - \ell'\theta} \|a\|_{B^p_{2q_2, r}} \|\omega^r_{q_2}\|_{H^s}^\frac{2}{p}.\]
Finally applying Lemma A.2 and then Lemma B.1 with \( \delta = \mu - \alpha(r) + \theta \) yields

\[
\| \Delta_k^h \Delta_k^v R_h^v (\omega_{r-1}, a) \|_{L^2} \lesssim 2^\ell \left( \frac{1}{2^\ell} - \alpha(r) \right) \sum_{|k'-k| \leq 4} \sum_{|\ell'| \geq \ell - 3} \| S^h_{k'-1} \Delta^v_{\ell'} \omega_{r-1} \|_{L^2_h(L^2_{\ell'})} \| \Delta_k^h \Delta_k^v \omega \|_{L^2_h(L^2_{\ell'})} \\
\lesssim 2^\ell \left( \frac{1}{2^\ell} - \alpha(r) \right) \sum_{|k'-k| \leq 4} \sum_{|\ell'| \geq \ell - 3} c_{k', \ell'} 2^{-k'(3\alpha(r) - \theta)} 2^{-\ell' \left( \theta - \alpha(r) + \frac{1}{2^\ell} \right)} \| a \|_{B^{\mu, p}_{2, q_2, r}} \| \omega \|_2^{\frac{2}{2} \frac{2}{2} \frac{2}{2}}. \\
\lesssim c_{k, \ell} 2^{k(3\alpha(r) - \theta)} 2^{-\ell \theta} \| a \|_{B^{\mu, p}_{2, q_2, r}} \| \omega \|_2^{\frac{2}{2} \frac{2}{2} \frac{2}{2}}.
\]

Hence, we achieve

\[
\| T_h^v (\omega_{r-1}, a) \|_{H^{3\alpha(r) - \theta, \theta}} \lesssim \| a \|_{B^{\mu, p}_{2, q_2, r}} \| \omega \|_2^{\frac{2}{2} \frac{2}{2} \frac{2}{2}}.
\]

Together with (B.1), we obtain (3.4).

\[\square\]

**Proof of (3.5).** We first get, by applying Bony’s decomposition in the vertical variable, that

\[
T_h^v (\partial_h a, \omega_{r-1}) = T_h^v (T^v + \tilde{T}^v + R^v) (\partial_h a, \omega_{r-1}) \quad \text{and} \\
T_h^v (\partial_h \omega_{r-1}, a) = T_h^v (T^v + \tilde{T}^v + R^v) (\partial_h \omega_{r-1}, a).
\]

Since the estimates of the above terms are similar, we only present the estimates to the typical terms above. Noting \( \delta_1 \in [\mu - \alpha(r), 1 - 2/p] \), we can use Lemma B.1 to obtain

\[
\| \Delta_k^h \Delta_k^v T_h^v (\partial_h a, \omega_{r-1}) \|_{L^{2\ell'}_{2, q_2, r}(L^2)} \lesssim \sum_{|k'-k| \leq 4} \sum_{|\ell'| \geq \ell - 3} \| S^h_{k'-1} \Delta^v_{\ell'} \partial_h a \|_{L^2_h(L^2_{\ell'})} \| \Delta_k^h \Delta_k^v \omega_{r-1} \|_{L^2_h(L^2_{\ell'})} \\
\lesssim \sum_{|k'-k| \leq 4} \sum_{|\ell'| \geq \ell - 3} c_{k', \ell'} 2^{-k' \left( \mu - \delta_1 \right)} 2^{-\ell' \left( \delta_1 + \alpha(r) - \mu \right)} \| \partial_h a \|_{L^{\frac{1}{2} + \frac{2}{2} + \mu}_{2, q_2, r}} \| \omega_{r-1} \|_{L^{\frac{1}{2} + \frac{2}{2} - \delta_1 \delta_1}_{2, q_2, r}} \\
\lesssim c_{k, \ell} 2^{-k \left( \mu - \delta_1 \right)} 2^{-\ell' \left( \delta_1 + \alpha(r) - \mu \right)} \| a \|_{B^{\mu, p}_{2, q_2, r}} \| \omega \|_2^{\frac{2}{2} \frac{2}{2} \frac{2}{2}}.
\]

Along the same line, for \( \rho_1 = \frac{1}{\alpha(r)} \), we infer

\[
\| \Delta_k^h \Delta_k^v T_h^v (\omega_{r-1}, \partial_h a) \|_{L^{2\ell'}_{2, q_2, r}(L^2)} \lesssim \sum_{|k'-k| \leq 4} \sum_{|\ell'| \geq \ell - 3} \| S^h_{k'-1} \Delta^v_{\ell'} \partial_h \omega_{r-1} \|_{L^2_h(L^2_{\ell'})} \| \Delta_k^h \Delta_k^v a \|_{L^2_h(L^2_{\ell'})} \\
\lesssim \sum_{|k'-k| \leq 4} \sum_{|\ell'| \geq \ell - 3} c_{k', \ell'} 2^{-k' \left( \mu - \delta_1 \right)} 2^{-\ell' \left( \delta_1 + \alpha(r) - \mu \right)} \| a \|_{B^{\mu, p}_{2, q_2, r}} \| \omega_{r-1} \|_{L^{\frac{1}{2} + \frac{2}{2} - \delta_1 \delta_1}_{2, q_2, r}} \\
\lesssim c_{k, \ell} 2^{-k \left( \mu - \delta_1 \right)} 2^{-\ell' \left( \delta_1 + \alpha(r) - \mu \right)} \| a \|_{B^{\mu, p}_{2, q_2, r}} \| \omega \|_2^{\frac{2}{2} \frac{2}{2} \frac{2}{2}}.
\]

The remaining terms can be handled along the same line. \[\square\]

**Proof of (3.6).** Applying Bony’s decomposition in the vertical variable gives

\[
R_h^v (\partial_h a, \omega_{r-1}) = R_h^v (T^v + \tilde{T}^v + R^v) (\partial_h a, \omega_{r-1}).
\]
We just present the estimate to the typical last term. Indeed, we have
\[
\| \Delta_k^h \Delta^y R^k R^{y^*} (\partial_h a, \omega_y - 1) \|_{L^q_{h+1} (L^2 \omega_y)} \lesssim \sum_{k' \geq k-3} \sum_{\ell' \geq \ell-3} \| \Delta_k^h \Delta^y \partial_h a \|_{L^q_{h+1} (L^2 \omega_y)} \| \tilde{\Delta}_k^h \Delta^y \omega_y - 1 \|_{L^q_{h+1} (L^2 \omega_y)}
\]
\[
\lesssim \sum_{k' \geq k-3} \sum_{\ell' \geq \ell-3} c_{k', \ell'} 2^{-k' (\mu + \frac{2}{q_1} - \delta_2 - 1)} 2^{-\ell' (\alpha(r) + \delta_2 - \mu)} \| a \|_{B^{0, p}_{q_2, r}} \| \omega_y - 1 \|_{B^{1, \frac{2}{q_2}, r}} \| \tilde{\Delta}_k^h \Delta^y \omega_y - 1 \|_{L^q_{h+1} (L^2 \omega_y)}
\]
\[
\lesssim c_{k, \ell} 2^{-k (\mu + \frac{2}{q_1} - \delta_2 - 1)} 2^{-\ell (\alpha(r) + \delta_2 - \mu)} \| a \|_{B^{0, p}_{q_2, r}} \| \omega_y - 1 \|_{H^{r, \frac{2}{q_2}}}.
\]

The remaining terms can be handled along the same line. \qed

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**References**

[1] H. Bahouri, J. Y. Chemin and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften 343, Springer-Verlag Berlin Heidelberg, 2011.

[2] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, *Annales de l’École Normale Supérieure*, 14, 1981, pages 209-246.

[3] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, *Modélisation Mathématique et Analyse Numérique*, 34, 2000, pages 315-335.

[4] J.-Y. Chemin and P. Zhang, On the critical one component regularity for 3-D Navier-Stokes equations, *Communications in Mathematical Physics*, 272, 2007, pages 529-566.

[5] J.-Y. Chemin and P. Zhang, On the critical one component regularity for 3-D Navier-Stokes system, *Annales de l’École Normale Supérieure* (4), 49, 2016, pages 131-167.

[6] J.-Y. Chemin, P. Zhang and Z. Zhang, On the critical one component regularity for 3-D Navier-Stokes system: general case, *Archive for Rational Mechanic Analysis*, 224, 2017, pages 871-905.

[7] L. Escauriaza, G. Seregin and V. Sverák, $L^3, \infty$-solutions of Navier-Stokes equations and backward uniqueness, (Russian) *Uspekhi Mat. Nauk*, 58, 2003, no. 2(350), pages 3-44; translation in *Russian Math. Surveys*, 58 , 2003, pages 211-250.

[8] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, *Archive for Rational Mechanic Analysis*, 16, 1964, pages 269-315.

[9] B. Han, Z. Lei, D. Li and N. Zhao, Sharp one component regularity for Navier-Stokes, arXiv:1708.04119v1 [math.AP].

[10] C. E. Kenig and G. S. Koch, An alternative approach to regularity for the Navier-Stokes equations in critical spaces, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 28 (2011), 159-187.

[11] Y. Liu and P. Zhang, Global well-posedness of 3-D anisotropic Navier-Stokes system with large vertical viscous coefficient, arXiv:1708.04731v2[math.AP].

[12] M. Paicu, Équation anisotrope de Navier-Stokes dans des espaces critiques, *Revista Matemática Iberoamericana*, 21, 2005, pages 179-235.

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