Critical behavior of classical spin models and local cohomology

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Abstract

Using reflection positivity as the main tool, we establish a connection between the existence of a critical point in classical spin models and the triviality of a certain local cohomology class related to the Noether current of the model in the continuum limit. Furthermore we find a relation between the location of the critical point and the momentum space autocorrelation function of the Noether current.

\footnote{Talk given at the XIIth Max Born Symposium, Wroclaw, dedicated to Jan Lopuszanski on the occasion of his 75th birthday.}
1. Introduction: Lattice and Continuum

It is well known that one possible approach to the construction of a Quantum Field Theory (QFT) goes by way of taking the continuum limit of a system of Classical Statistical Mechanics on a lattice, such as the Ising model, the classical Heisenberg model or more generally a classical spin model. Taking the continuum limit means in this context that one has to drive the system to a critical point, that is a point at which the dynamically produced scale(s) become infinite in terms of the lattice spacing; the continuum limit is then obtained by an infinite rescaling of the lattice model (see below; a rather detailed discussion of how this is done is contained in [1]). A bonus of this construction is that the continuum limit inherits certain properties of the lattice model, such as Reflection Positivity (RP) which leads to positivity of the state space metric and the spectrum condition of the QFT.

More precisely we have to distinguish between two kinds of continuum limits:

- The massive continuum limit: one chooses the dynamically generated length (correlation length) $\xi$ of the system as the standard of length, considers the system at length scales that are fixed multiples of that standard, and sends $\xi \to \infty$ by driving the system to criticality.

- The massless continuum limit: the lattice system is put right on a critical point; one then chooses an arbitrary length scale that becomes infinite in lattice units and rescales the system accordingly.

The first option will produce a (Euclidean) QFT with unit mass, the second one a massless QFT, which according to standard lore will also be conformally invariant. In $2D$ it is believed that this allows to classify the critical behaviors according to the well-studied (rational) Conformal QFTs ([2, 3]).

In this talk we want to discuss this connection, and actually close some gaps. In the course of the argument it turns out that one has to prove the triviality of a certain ‘local cohomology class’ related to the Noether current. This is possible with the use of lattice Ward identities (WI) and RP.

The same ingredients lead at the same time to an interesting and maybe unexpected relation between the location of the critical point of the lattice model and the Noether current 2-point function of the continuum model. This leads to a new criterion that allows to discriminate between the ‘conventional wisdom’ about nonabelian spin models in $2D$, which posits that they do not become critical at any temperature and the scenario long advocated by us [4] that they do have a transition to a massless spin wave phase, just as the plane rotator model.
2. Local Cohomology

It has been noted long ago \[5, 6, 7\] that the imposition of locality (local commutativity, Einstein causality) may make the cohomology of Minkowski space nontrivial.

The problem of local cohomolgy may be stated as follows: assume that an anti-symmetric tensor field $\Phi_{\mu_1,\ldots,\mu_k}(x)$ is given, which satisfies Wightman’s axioms and is closed, i.e. satisfies

$$d\Phi \equiv d(\sum \Phi_{\mu_1,\ldots,\mu_k} dx^{\mu_1}\ldots dx^{\mu_k}) = 0$$

(in the notation of alternating differential forms).

The question is then under which conditions the field $\Phi$ is exact, i.e. there exists a local antisymmetric tensor field $\Psi$ such that $\Phi = d\Psi$.

There are some well-known examples where the answer is ‘no’, even though Minkowski space is topologically trivial:

1. the free Maxwell field $F$ in dimension $D \geq 2$ \[5\];

2. the gradient of the massless free scalar field $\phi$ in $2D$, because the field $\phi$ does not exist as a local (Wightman) field.

There is also a simple $2D$ example on which we hit in our analysis of $2D$ classical spin models: let

$$\Phi = \phi_c dx^1 dx^2$$

(2)

where $\phi_c$ has the Euclidean two-point function

$$\langle \phi_c(0)\phi_c(x) \rangle = \frac{1}{(x^2)^2}. \quad (3)$$

Then $\Phi$ is trivially closed in $2D$, but it is not exact, i.e. there is no local vector field $j_\mu$ such that

$$\phi_c = \epsilon_{\mu\nu} \partial_\mu j_\nu \quad (4)$$

This example can be made more explicit by requiring $\phi_c$ to be a generalized free, i.e. Gaussian field, with its two-point function given by eq.(3). If we solve the differential equations that the two-point function of $j_\mu$ has to fulfill in order to satisfy eq.(4) and impose euclidean covariance, we find that there is no scale invariant solution. The covariant solutions are

$$G_{\mu\nu}(x) = -\delta_{\mu\nu} \frac{\ln x^2 + \lambda}{8x^2} + x_\mu x_\nu \frac{\ln x^2 + 1 + \lambda}{4x^2} \quad (5)$$

This is not the two point function of a local vector field, continued to euclidean times: it violates the so-called reflection positivity \[8\], because the logarithm changes sign. Similarly it also does not obey the positivity required for a random field.
3. What is the massless continuum limit of a critical classical spin model?

There is an old argument \[^{[9]}\] that a classical spin model with a continuous symmetry group \(G\) will have a massless continuum limit that has an enhanced \(G \times G\) symmetry; this is supposed to come about due to the splitting of the model into two independent ‘chiral’ theories. Affleck \[^{[9]}\] gave this argument in the framework of Quantum Field Theory in Minkowski space, but it can be easily rephrased in the euclidean setting. In \[^{[10, 11]}\] we pointed out two possible gaps in those arguments coming from hidden assumptions whose validity has to be checked. But in those papers we also showed that these gaps can be closed, using properties like reflection positivity.

The core of the euclidean version of Affleck’s argument is the following: assume that we have a scale invariant continuum theory with a conserved current \(j_{\mu}(x)\). Euclidean covariance requires that the two-point function \(G_{\mu\nu}\) of \(j_{\mu}\) is of the form

\[
G_{\mu\nu} \equiv \langle j_{\mu}(0) j_{\nu}(x) \rangle = \delta_{\mu\nu} \frac{b}{x^2} + \frac{ax_{\mu}x_{\nu}}{(x^2)^2} \quad (x \neq 0)
\]  

(6)

Imposing current conservation means

\[
\partial_{\mu} G_{\mu\nu} = 0
\]  

(7)

for \(x \neq 0\), which implies

\[
a = -2b
\]  

(8)

\[
G_{\mu\nu}(x) = b \left( \frac{\delta_{\mu\nu}}{x^2} - \frac{2x_{\mu}x_{\nu}}{(x^2)^2} \right)
\]  

(9)

This is, up to the factor \(b\), equal to the two point function of \(\partial_{\mu} \phi\) where \(\phi\) is the massless free scalar field (it is irrelevant here that the massless scalar field does not exist as a Wightman field). If we look at the two-point function of the dual current \(\epsilon_{\mu\nu} j_{\nu}\), it turns out to be

\[
\tilde{G}_{\mu\nu} \equiv \epsilon_{\mu\lambda} \epsilon_{\rho\nu} G_{\lambda\rho} = -G_{\mu\nu}
\]  

(10)

so the dual current two point function satisfies automatically the conservation law. Conservation of the two currents \(j\) and \(\tilde{j}\) is equivalent to conservation of the two chiral currents \(j_{\pm} = j_0 \pm j_1\) in Minkowski space.

By general properties of local quantum field theory (Reeh-Schlieder theorem, see \[^{[12, 13]}\] it follows that the dual current is conserved as a quantum field. So the two conservation laws together imply that

\[
j_{\mu} = \sqrt{b} \partial_{\mu} \phi,
\]  

(11)

where \(\phi\) is the massless scalar free field, and also that

\[
j_{\mu} = \sqrt{b} \epsilon_{\mu\nu} \partial_{\nu} \psi,
\]  

(12)

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where $\psi$ is another ‘copy’ of the massless scalar free field.

As presented, this argument is certainly correct. But it depends on the assumption that the Noether currents exist as Wightman fields, and this assumption is in fact nontrivial and could a priori fail in the critical spin models. A simple example of a Quantum Field Theory with a continuous symmetry in which the Noether current does not exist as a Wightman field is given by the two-component free field in $2D$ in the massless limit. It is simply given by a pair of independent Gaussian fields $\Phi^{(1)}, \Phi^{(2)}$, both with covariance

$$C(x) = \frac{1}{(2\pi)^2} \int d^2p \frac{e^{ipx}}{p^2 + m^2}. \quad (13)$$

where we are interested in the limit $m \to 0$. This system has a global $O(2)$ invariance rotating the two fields into each other. It is well known that the massless limit only makes sense for functions of the gradients of the fields. But the Noether current of the $O(2)$ symmetry is given by

$$j_\mu(x) = \Phi^{(1)}(x) \partial_\mu \Phi^{(2)}(x) - \Phi^{(2)}(x) \partial_\mu \Phi^{(1)}(x), \quad (14)$$

and it cannot be written as a function of the gradients. It is also easy to see directly that its correlation functions do not have a limit as $m \to 0$ (see $[11]$). The Noether current itself makes sense as a quantum field only if it is smeared with test functions $f_\mu$ satisfying

$$\int d^2x f_\mu(x) = 0 \quad (15)$$

On the other hand, it is not hard to see that $\phi_c(x) = \text{curl}(j)$ can be written as a function of the gradients:

$$\phi_c(x) = 2((\partial_2 \Phi^{(1)}(x))(\partial_1 \Phi^{(2)}(x)) - (\partial_1 \Phi^{(1)}(x))(\partial_2 \Phi^{(2)}(x))) \quad (16)$$

and its two-point function is of the form

$$\langle \phi_c(0)\phi_c(x) \rangle \propto \frac{1}{(x^2)^2} \quad (17)$$

In other words, in this model we have found exactly the nontrivial local cohomology class described in the previous section. The problem in the massless continuum limits of classical spin models is then the following: it is conceivable that both $\text{curl } j$ and $\text{div } j$ have bona fide continuum limits, but the current itself does not. In other words, it could happen that there is a nontrivial second ‘local cohomology class’ just as in the example discussed above. But it turns out that reflection positivity can be used to rule out such a possibility, provided we are dealing with a model that becomes critical at a finite value of the inverse temperature $\beta$ (this is, however, a
prerequisite for constructing a massless continuum limit anyway). Our arguments show that both \( \text{curl} \ j \) and \( \text{div} \ j \) have correlations that are pure contact terms in the continuum limit; this means that in Minkowski space both the current and its dual are conserved, thereby justifying Affleck’s claim.

For completeness, let us mention an even more exotic possible way in which the conformal classification of the critical behavior of the classical spin models could fail: one could be imagine that the current itself has correlations that are pure contact terms in the continuum, which would mean that the Noether current simply vanishes as a quantum field. Of course this would also imply vanishing of the corresponding charge, and since the commutator of the charge with the (renormalized) spin field should be a component of the (renormalized) spin field, those fields themselves would have to vanish, leading to a totally trivial theory containing only the vacuum. There is a huge body of numerical results that makes this inconceivable, and we also did some numerical simulations to eliminate this possibility directly in the case of the \( O(2) \) model \cite{10, 11}.

4. The Noether Current: Some Generalities

The \( O(N) \) model is determined by its standard Hamiltonian (action)

\[
H = -\sum_{\langle i j \rangle} s(i) \cdot s(j) \tag{18}
\]

where the sum is over nearest neighbor pairs on a square lattice and the spins \( s(.) \) are unit vectors in \( \mathbb{R}^N \). As usual Gibbs states are defined by using the Boltzmann factor \( \exp(-\beta H) \) together with the standard a priori measure on the spins first in a finite volume, and then taking the thermodynamic limit.

It is rigorously known \cite{14} that for \( N = 2 \) the model has a transition to a massless spin wave phase at a certain \( \beta = \beta_{KT} \approx 1.12 \), the so-called Kosterlitz-Thouless transition \cite{13}. This transition separates a high temperature phase with exponential clustering from a low temperature one with only algebraic decay of correlations. For \( N > 2 \) the standard wisdom is that there is no such transition and the model does not become critical at any finite \( \beta \), but is asymptotically free. For many years, however, we have been criticizing the arguments on which this standard wisdom is based and gave arguments for an alternative scenario according to which ALL the \( O(N) \) models have a transition to a spin wave phase \cite{4}.

Here we do not want to enter into this discussion, but we will produce a criterion that distinguishes between these two scenarios.

But at first let us assume that our model has a finite critical point and study the consequences. We are in particular interested in the correlations of the Noether
currents, given by
\[ j^a_{\mu}(i) = \beta \left( s^a(i) s^b(i + \hat{\mu}) - s^a(i) s^b(i + \hat{\mu}) \right) \] (19)

Typically we will restrict ourselves to the case \( a = 1, b = 2 \) and omit the flavor indices on the current.

On a torus the current can be decomposed into 3 pieces, a longitudinal, a transverse and a constant (harmonic) piece. This decomposition is easiest in momentum space, and effected by the projections
\[ P^T_{\mu\nu} = \left( \delta_{\mu\nu} - \frac{(e^{ip_\mu} - 1)(e^{-ip_\nu} - 1)}{\sum_\alpha (2 - 2 \cos p_\alpha)} \right) (1 - \delta_{p0}), \] (20)
\[ P^L_{\mu\nu} = \frac{(e^{ip_\mu} - 1)(e^{-ip_\nu} - 1)}{\sum_\alpha (2 - 2 \cos p_\alpha)} (1 - \delta_{p0}) \] (21)
and
\[ P^h_{\mu\nu} = \delta_{\mu\nu}\delta_{p0}. \] (22)

with \( p_\mu = 2\pi n_\mu/L, n_\mu = 0, 1, 2, ..., L - 1 \).

In the following we will mostly discuss these correlations in momentum space. In particular we study the tranverse momentum space 2-point function
\[ \hat{F}^T(p, L) \equiv \hat{G}(0, p; L) = \langle |\hat{j}_1(0, p)|^2 \rangle \] (23)
(for \( p \neq 0 \); the hat denotes the Fourier transform)
and the longitudinal two-point function
\[ \hat{F}^L(p, L) \equiv \hat{G}(p, 0; L) = \langle |\hat{j}_1(p, 0)|^2 \rangle \] (24)
(for \( p \neq 0 \)).

Because the current is conserved, its divergence in the Euclidean world should be a pure contact term, and for dimensional reasons the two-point function should be proportional to a \( \delta \) function, i.e.
\[ \hat{F}^L(p, L) = \text{const.} \] (25)

The constant is in fact determined by a Ward identity in terms of \( E = \langle s(0) \cdot s(\hat{\mu}) \rangle \): consider (for a suitable finite volume) the partition function
\[ Z = \int \prod_i ds(i) \prod_{\langle ij \rangle} \exp(\beta s(i) \cdot s(j)) \] (26)
where $ds$ denotes the standard invariant measure on the $(N - 1)$-sphere. Replacing under the integral $s(i)$ by $\exp(\alpha L_{12})$, where $L_{12}$ is an infinitesimal rotation in the 12 plane, does not change the integral. So expanding in powers of $\alpha$, all terms except the one of order $\alpha^0$ vanish identically in $\alpha(i)$. This leads in a well-known fashion to Ward identities expressing the conservation of the current. Looking specifically at the second order term in $\alpha$ and Fourier transforming, we obtain for all $p \neq 0$

$$\langle |j_1(p, 0)|^2 \rangle = \hat{F}_L(p, L) = \frac{2}{N} \beta E$$

(27)

This is confirmed impressively by the Monte Carlo simulations [11].

The thermodynamic limit is obtained by sending $L \to \infty$ for fixed $p = 2\pi n/L$, so that in the limit $p$ becomes a continuous variable ranging over the interval $[-\pi, \pi)$. The $O(N)$ models do not show spontaneous symmetry breaking according to the Mermin-Wagner theorem, and presumably have a unique infinite volume limit at any temperature.

The massive continuum limit is constructed as follows: First one takes the thermodynamic limit of the model in its high temperature phase. There is a dynamically generated length scale $\xi$, the correlation length regulating the exponential decay of the correlations. This is now taken as the standard of length, and the fields are rescaled accordingly. In particular the Noether current is rescaled as follows:

$$j_{\mu}^{\text{ren}}(x) = \xi j_{\mu}(i)$$

(28)

with $x = i/\xi$. After that, the system is driven to the critical point, where $\xi \to \infty$.

The massless continuum limit, on the other hand, is obtained as follows: we take the thermodynamic limit of the model right at its critical point. Since there is no dynamically generated scale, we take an arbitrary sequence $l_n$ going to infinity as our standard of length. The currents are then rescaled as

$$j_{\mu}^{\text{ren}}(x) = l_n j_{\mu}(i)$$

(29)

with $x = i/l_n$ and we take the limit $n \to \infty$.

5. The Noether Current: Bounds and Inequalities

The Gibbs measure formed with the standard action on the periodic lattice has the property of reflection positivity (see for instance [10]). Reflection positivity means that expectation values of the form

$$\langle A\theta(A) \rangle,$$

(30)

are nonnegative, where $A$ is an observable depending on the spins in the ‘upper half’ of the lattice ($\{x|x_1 > 0\}$, and $\theta(A)$ is the complex conjugate of the same function
of the spins at the sites with $x_1$ replaced by $-x_1$. Applied to the current two-point functions this yields:

$$F^L(x_1, L) = \sum_{x_2} \langle j_1(x_1, x_2)j_1(0, 0) \rangle \leq 0$$  \hspace{1cm} (31)

for $x_1 \neq 0$ and

$$F^T(x_1, L) = \sum_{x_2} \langle j_2(x_1, x_2)j_2(0, 0) \rangle \geq 0$$  \hspace{1cm} (32)

for all $x_1$. From these two equations and eq.(27) it follows directly that

$$0 \leq \hat{F}^T(p, L) \leq \hat{F}^L(0, L) = \hat{F}^L(0, L) \leq \hat{F}^L(p, L) = \frac{2}{N} \beta E$$  \hspace{1cm} (33)

These inequalities remain of course true in the thermodynamic limit, but we have to be careful with the order of the limits. If we define $\hat{F}^T(p, \infty)$ and $\hat{F}^L(p, \infty)$ as the Fourier transforms of $\lim_{L \to \infty} F^T(x, L)$ and $\lim_{L \to \infty} F^L(x, L)$, respectively, one conclusion can be drawn immediately:

**Proposition:** $\hat{F}^T(p, \infty)$ and $\hat{F}^L(p, \infty)$ are continuous functions of $p \in [-\pi, \pi)$.

The proof is straightforward, because due to the inequalities (32) (33) and (34) together with the finiteness of $\beta_{\text{crt}}$ the limiting functions $F^L$ and $F^T$ in $x$-space are absolutely summable. But it is not assured that the limits $L \to \infty$ and $p \to 0$ can be interchanged, nor that the thermodynamic limit and Fourier transformation can be interchanged. On the contrary, by the numerics presented in [10, 11], as well as finite size scaling arguments, it is suggested that

$$\lim_{p \to 0} \lim_{L \to \infty} \hat{F}^L(p, L) > \lim_{L \to \infty} \hat{F}^L(0, L)$$  \hspace{1cm} (34)

and therefore also

$$\lim_{p \to 0} \lim_{L \to \infty} \hat{F}^L(p, L) > \lim_{p \to 0} \lim_{L \to \infty} \hat{F}^T(p, L).$$  \hspace{1cm} (35)

The fact that these are strict inequalities plays an important role in the justification of Affleck’s claim, as will be seen below.

To continue, let us describe how the two types of continuum limit are taken in Fourier space, concretely for our functions $\hat{F}^T(p, \infty)$, $\hat{F}^L(p, \infty)$.

The massive continuum limit means considering $\hat{F}^T(p)$ etc. for a sequence of $\xi$ values diverging to $\infty$ as functions of $q \equiv p/m = p\xi$, i.e. taking

$$\lim_{\xi \to \infty} \hat{T}(q) \equiv \hat{F}^T(qm)$$  \hspace{1cm} (36)

In this context it is important to note that the functions $\hat{F}^T(p)$ depend explicitly on $\beta$ which is sent to $\beta_{\text{crt}}$, and through this on the correlation length $\xi$, which is sent to $\infty$.  

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The massless continuum limit on the other hand is obtained by going to the critical point and considering $\hat{F}^T(p)$ etc. as a function of $q \equiv p/l_n$, i.e. taking

$$\lim_{n \to \infty} \hat{T}(q) \equiv \hat{F}^T(q/l_n)$$  \hspace{1cm} (37)

In this case we are always dealing with only one function $\hat{F}^T(p)$, not depending on $n$, because $\beta$ is fixed to its critical value.

6. Consequences

For the massless continuum limit the inequalities (33) lead to an important consequence, which closes the main gap in Affleck's argument by showing the triviality of the second local cohomology class defined by the curl of the noether current:

**Proposition:** In the massless continuum limit both $\hat{F}^L(p, \infty)$ and $\hat{F}^T(p, \infty)$ converge to constants for $p \neq 0$.

**Corollary:** The local cohomology class defined by curl($j$) is trivial.

**Proof:** Let $\hat{F}(p)$ be the Fourier transform of either $\hat{F}^T(p, \infty)$ or $\hat{F}^L(p, \infty)$. We consider $\hat{F}(p)$ as a distribution on $[-\pi, \pi)$. We extend $\hat{F}(p)$ to a periodic distribution on the whole real line. The continuum limit of $F(n)$ (the corresponding function in $x$ space) also has to be considered in the sense of distributions. If we change our standard of length to $l_M = M$, the lattice spacing will be $a = 1/M$, respectively. For an arbitrary test function $f$ (infinitely differentiable and of compact support) on the real axis we then have to consider the limit $M \to \infty$ of

$$(F, f)_M \equiv \sum_n f(n/M)F(n).$$  \hspace{1cm} (38)

We claim that the right hand side of this is equal to

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} dq \hat{F}(q/M)\hat{f}(q).$$  \hspace{1cm} (39)

**Proof:** Insert in eq.(38)

$$F(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \hat{F}(p)e^{ipn}$$  \hspace{1cm} (40)

and

$$f(n/M) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp \hat{f}(p)e^{ipn/M}$$  \hspace{1cm} (41)

and use the identity

$$\sum_n e^{ipm+inb} = 2\pi \sum_r \delta(p+qb+2\pi r)$$  \hspace{1cm} (42)
This produces, after carrying out the trivial integral over \( q \) using the \( \delta \) distribution,

\[
\frac{M}{2\pi} \int_{-\pi}^{\pi} dp \sum_{r=-\infty}^{\infty} \hat{F}(-p) \hat{f}((p+2\pi r)M) = \frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \int_{-M\pi}^{M\pi} dq \hat{F}(-q/M) \hat{f}(q+2\pi Mr) \quad (43)
\]

Finally, using the periodic extension of \( \hat{F}(p) \), this becomes what is claimed in eq.(39).

From eq.(39) one sees that what is relevant for the continuum limit is the small momentum behavior of \( \hat{F}(p) \). In particular, if \( \lim_{p \to 0} \hat{F}(p) \equiv \hat{F}(0) \) exists, we obtain

\[
\lim_{M \to \infty} (F,f)_M = \frac{1}{2\pi} \hat{F}(0) \int dq \hat{f}(q) = \frac{1}{2\pi} f(0) \hat{F}(0) \quad (44)
\]

expressing the fact that in this case the limit of \( F \) is a pure contact term. This finishes the proof of the proposition.

In spite of this result, Affleck’s claim could still fail in a different way if \( \hat{F}^T(p,\infty) \) and \( \hat{F}^L(p,\infty) \) converged to the same constant in the continuum limit. Let us denote the continuum limit of \( \hat{F}^T(p,\infty) \) by \( g \). Then the current-current correlation in this limit is

\[
\langle j_{\mu} j_{\nu} \rangle(p) = \beta E P_{\mu\nu}^L + g P_{\mu\nu}^T = g \delta_{\mu\nu} + (\beta E - g) \frac{P_{\mu} P_{\nu}}{p^2}. \quad (45)
\]

So we see that if \( g = \beta E \), the current-current correlation reduces to a pure contact term and vanishes in Minkowski space. Above we proved only that

\[
g \leq \beta E \quad (46)
\]

But if the the current-current correlation were a pure contact term, it would be unavoidable to conclude that also the spin field becomes ultralocal. This can be seen as follows: if the current is ultralocal in the euclidean world, by the Osterwalder-Schrader reconstruction the current field operator in Minkowski space has to vanish, and so does the charge operator \( Q_{12} = \int dx j_a(x,t) \). But if the charge operator generates a global \( O(N) \) symmetry, it has to have the following commutation relation with the (renormalized, Minkowskian) spin field \( s(x) \):

\[
[Q_{12}, s_a(x)] = 0, \quad a > 2 \quad (47)
\]

\[
[Q_{12}, s_1(x)] = s_2(x) \quad (48)
\]

\[
[Q_{12}, s_2(x)] = -s_1(x) \quad (49)
\]

which would then imply that \( s(x) \) vanishes identically. This argument is not fully rigorous, because it assumes eq.(47) as well as the validity of the Osterwalder-Schrader axioms; both have not been proven in full rigor for the continuum limit of the \( O(N) \) models. Also there is only numerical evidence, but no rigorous proof, that the continuum limit of the spin field is not ultralocal. For these reasons we presented in \[11\]
numerical data which (together with finite size scaling arguments) rule out directly ulralocality of the current.

Let us now turn to the massive continuum limit. For this the inequalities (33) yield the announced bound on the transition temperature in terms of the transverse Noether current in momentum space:

**Proposition:** For the $O(N)$ models the critical inverse temperature satisfies

$$\beta_{\text{crt}} \geq \frac{N}{2} \sup_p \hat{F}^T(p)$$

The quantity $J(p) = \hat{F}^T(p) - \hat{F}^T(0)$ satisfies

$$J(p) \leq \frac{2}{N} \beta_{\text{crt}}$$

**Proof:** Both statements follow directly by taking first the thermodynamic and then the massive continuum limit of eq.(33), using also the trivial fact $E \leq 1$.

This result is the announced criterion distinguishing between $\beta_{\text{crt}} < \infty$ and $\beta_{\text{crt}} = \infty$ by the boundedness or unboundedness of $J(p)$ or $\hat{F}^T(p)$. Of course it is a highly nontrivial matter to verify this criterion. Balog and Niedermaier [17] gave arguments that $J(p)$ is unbounded in their form factor construction of the $O(3)$ model, which seems to suggest $\beta_{\text{crt}} = \infty$. But we found by very precise numerical simulations evidence [18] that the form factor construction disagrees with the (massive) continuum limit of the lattice $O(3)$ model, leaving open the possibility that indeed $\beta_{\text{crt}} < \infty$ as long advocated by us.

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