Abstract. In this paper we explore a special class of metric spaces called smocked metric spaces and study their tangent cones at infinity. We prove that under the right hypotheses, the rescaled limits of balls converge in both the Gromov-Hausdorff and Intrinsic Flat sense to normed spaces. This paper will be applied in upcoming work by Kazaras and Sormani concerning Gromov’s conjectures on the properties of GH and SWIF limits of Riemannian manifolds with positive scalar curvature.

1. Introduction

In 1983 Gromov introduced the notion Gromov-Hausdorff (GH) convergence of Riemannian manifolds and metric spaces [6]. Gromov proved all GH limits are geodesic metric spaces but they may not have the same dimension as the sequence. In [10] Sormani and Wenger introduced the notion of intrinsic flat (SWIF) convergence of Riemannian manifolds. SWIF limit spaces are called integral current spaces: they are countably \( H^m \) rectifiable metric spaces with normed tangent spaces almost everywhere of the same dimension as the original sequence. When the SWIF and GH limits agree, then GH limits have far more structure than initially proven by Gromov.

Here we prove that balls in smocked metric spaces are integral current spaces [Theorem 3.12], and we explore the SWIF limits of these balls under rescaling. We prove that under certain hypotheses the rescaled balls converge in both the GH and the SWIF sense to the same limit space and that this space is a normed metric space [Theorem 1.1]. In future work of Kazaras and Sormani [7], the theorems here will be applied to address questions regarding the SWIF and GH limits of Riemannian manifolds with scalar curvature bounds (cf. [5] [8]).

Recall that a smocked metric space is a metric space created by selecting a collection (called a smocking pattern) of disjoint compact subsets (called stitches) in Euclidean space and identifying all the points in each stitch to a single point (cf. Definition 2.1 within). See Figure 1 for a variety of smocking patterns of stitches on a Euclidean plane that have been analyzed by the authors and their collaborators in [9]. The patterns of the stitches need not be periodic but we do require that stitches be separated from one another. The stitches do not have to be one dimensional but we do say the space is “nice” if the volumes of the tubular neighborhoods of the patterns behave well (cf. Definition 2.13).

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Note that any smocked metric space, \((X, d)\), can be identified with a pseudometric on Euclidean space \((\mathbb{E}^N, \bar{d})\) where \(\bar{d}(p, q) = 0\) whenever \(p, q\) lie in the same stitch. The distances between points \(p, q\) that do not lie on the same stitch are found by taking the minimum length of straight line segments running between them jumping across the stitches. We review the definition of these spaces and properties of these spaces in Section \(2\). It is worth noting that these spaces are easy for even undergraduates to understand and there are a variety of undergraduate and masters level research projects suggested at the end of [9].

The notion of an integral current space is far more difficult to understand as it relies on the theory of Ambrosio-Kirchheim developed in [1]. An integral current space is an integer rectifiable metric space (which is a space covered by a countable collection of biLipschitz charts with integer weights) that has a boundary which is also integer rectifiable [10]. In Section \(3\) we review this theory while proving that balls within smocked metric spaces are integral current spaces. See Theorem 3.12.

The SWIF distance between a pair of integral current spaces, \(M_1\) and \(M_2\), is estimated by finding a higher dimensional integral current space, \(Z\), and distance preserving maps \(\varphi_i : M_i \rightarrow Z\), and then estimating the volumes between their images. In Section \(4\) we review this theory while estimating the SWIF distance between balls in smocked metric spaces balls in normed spaces. We prove that under certain hypotheses on the pseudometrics, the rescaled limits of the balls in a smocked metric spaces converge in both the GH and SWIF sense to a unique tangent cone at infinity that is a normed space:

**Theorem 1.1.** Suppose we have a nice smocked metric space, \((X, d)\), such that
\[
|\bar{d}(x, x') - [F(x) - F(x')]| \leq K \quad \forall x, x' \in \mathbb{E}^N
\]
where \(F : \mathbb{E}^N \rightarrow [0, \infty)\) is a norm.

Then \((X, d)\) has a unique GH=SWIF tangent cone at infinity, \((\mathbb{R}^N, d_F)\), where
\[
d_F(x, x') = \|x - x'\|_F = F(x - x').
\]

That is, for any basepoint \(x_0 \in X\), \(r, R > 0\) the balls \(\bar{B}(x_0, Rr)\) viewed as integral current spaces as in rescaled by \(R\) converge in both the GH and SWIF sense to a ball of radius \(r\) in the normed space \((\mathbb{R}^N, d_F)\).
We hope that this paper will be understandable even to the student and that it can be used to provide an introduction to both Gromov-Hausdorff and Intrinsic Flat Convergence of metric spaces. We close the paper by applying the main theorem to the smocked metric spaces that were analyzed by the authors, Kazaras, Afrifa, Antonetti, George, Hepburn, Huynh, Minichiello, and Rendla in [9], [2], and [3].

2. Background on Smocked Spaces

2.1. Review of Smocked Spaces. The notion of a smocked space was recently introduced by Sormani, Kazaras, and their team of students in [9]. The notion is built upon the classical handcraft of smocked fabrics. One has a pattern of intervals on a cloth and each interval is sewn with a thread and pulled to a point. A smocked space is similarly a plane of some dimension with a collection of intervals each of which is identified to a point. See Figure 1 for examples of patterns studied in [9] and here.

Definition 2.1. Given a Euclidean space, $\mathbb{E}^N$, and a finite or countable collection of disjoint connected compact sets called smocking intervals,

$$I = \{I_j : j \in J\},$$

with a positive smocking separation factor,

$$\delta = \min\{|z - z'| : z \in I_j, z' \in I_{j'}, j \neq j' \in J\} > 0,$$

one can define the smocked metric space, $(X, d)$, in which each interval is viewed as a single point.

$$X = \left\{x : x \in \mathbb{E}^N \setminus S\right\} \cup I$$

where $S$ is the smocking set or smocking pattern:

$$S = \bigcup_{j \in J} I_j.$$

We have a smocking map $\pi : \mathbb{E}^N \to X$ defined by

$$\pi(x) = \begin{cases} x & \text{for } x \in \mathbb{E}^N \setminus S \\ I_j & \text{for } x \in I_j \text{ and } j \in J \end{cases}$$

The smocked distance function, $d : X \times X \to [0, \infty)$, is defined for $y, x \notin \pi(S)$, and intervals $I_m$ and $I_k$ as follows:

$$d(x, y) = \min\{d_0(x, y), d_1(x, y), d_2(x, y), d_3(x, y), \ldots\}$$

$$d(x, I_k) = \min\{d_0(x, z), d_1(x, z), d_2(x, z), d_3(x, z), \ldots : z \in I_k\}$$

$$d(I_m, I_k) = \min\{d_0(z', z), d_1(z', z), d_2(z', z), d_3(z', z), \ldots : z' \in I_m, z \in I_k\}$$
where \( d_k \) are the sums of lengths of segments that jump to and between \( k \) intervals:

\[
\begin{align*}
  d_0(v, w) &= |v - w| \\
  d_1(v, w) &= \min(|v - z_1| + |z'_1 - w| : z_1, z'_1 \in I_{j_1}, j_1 \in J) \\
  d_2(v, w) &= \min(|v - z_1| + |z'_1 - z_2| + |z'_2 - w| : z_1, z'_1, z_2, z'_2 \in I_{j_1}, j_1 \neq j_2 \in J) \\
  d_k(v, w) &= \min(|v - z_1| + \sum_{i=1}^{k} |z'_i - z_i| + |z'_k - w| : z_i, z'_i \in I_{j_i}, j_1 \neq \cdots \neq j_k \in J).
\end{align*}
\]

We define the smocking pseudometric \( \bar{d} : \mathbb{B}^N \times \mathbb{B}^N \to [0, \infty) \) to be

\[
\bar{d}(v, w) = d(\pi(v), \pi(w)) = \min(d_k(v', w') : \pi(v) = \pi(v'), \pi(w) = \pi(w'), k \in \mathbb{N}).
\]

We will say the smocked metric space is parametrized by points in the intervals, if

\[
J \subset \mathbb{B}^N \text{ and } \forall j \in J \ j \in I_j.
\]

In \([9]\) it is proven that the minima are achieved:

**Theorem 2.2.** The smocked metric space is a well defined metric space and in fact for any \( v, w \in \mathbb{B}^N \)

\[
\exists N(v, w) \leq |\lfloor v - w \rfloor|\ s.t.\ d_{N(v, w)}(v, w) \leq d_k(v, w) \quad \forall k \geq \mathbb{N}
\]

so the minimum in the definition of the smocking distance and pseudometric is achieved

\[
\bar{d}(v, w) = d_N(v, w) \text{ and } d(\pi(v), \pi(w)) = d_N(v, w).
\]

In that paper the following constants are defined as well:

**Definition 2.3.** The smocking depth, \( h \), is defined to be

\[
h = \inf\{r : \mathbb{B}^N \subset T_r(S)\} \in [0, \infty).
\]

which by definition of tubular neighborhood, is

\[
h = \inf\{r : \forall x \in X \exists j \in J \exists z \in I_j \ s.t.\ |x - z| < r\}.
\]

**Lemma 2.4.** The smocking depth satisfies:

\[
h = \sup\{D(x) : x \in \mathbb{B}^N\}
\]

where \( D : \mathbb{B}^N \to [0, \infty) \) is the distance to the interval set:

\[
D(x) = \min(|x - z| : z \in I_j, j \in J).
\]

**Lemma 2.5.** The infimum in Definition 2.3 is achieved as is the supremum in Lemma 2.4.

**Definition 2.6.** The smocking lengths are defined

\[
L_{\min} = \inf\{L(I_j) : j \in J\} \in [0, \infty)
\]

\[
L_{\max} = \sup\{L(I_j) : j \in J\} \in (0, \infty]
\]

and if \( L_{\min} = L_{\max} \) we call the the smocking length. if the \( I_j \) are not intervals we can replace length with diameter.
Definition 2.7. The smocking separation factor, $\delta = \delta_X$, is defined to be

$$\delta_X = \delta \leq \min \left\{ |z - w| : z \in I_j, w \in I_k, j \neq k \in J \right\}. \quad (17)$$

Lemma 2.8. If a smocked metric space is parametrized by points in intervals as in \[9\], then

$$\mathbb{E}^N \subset T_{h+L}(S) \quad (18)$$

where $S = \bigcup_{j \in J} I_j$ is the smocking set and $h$ is the smocking depth and $L = L_{\text{max}}$ is the maximum smocking length.

2.2. Review of Balls in Smocked Metric Spaces. In \[9\], the students explored the shapes of balls in a variety of smocked metric spaces. To best describe these balls, one looks at their pre-images in Euclidean space:

$$U_r(x) = \pi^{-1}(\bar{B}_r(x)) \subset \mathbb{E}^N. \quad (19)$$

See Figure 2.2 for the balls found in that paper. We will study the balls of additional smocked spaces here using the following propositions and lemmas proven in that paper.

![Figure 2.2](image)

Figure 2. The balls in $X_+, X_T$, and $X_{\Box}$ were studied in \[9\].

In \[9\] the following two propositions were proven:

**Proposition 2.9.** Suppose that $p \in \mathbb{E}^N \setminus S$ and $r < D(x)$ is defined as in Definition 2.7 then

$$\pi^{-1}(B_r(\pi(p))) = B_r(p) = \{x : |x - p| < r\}. \quad (20)$$

**Proposition 2.10.** Suppose that $I_j$ is a smocking interval and $r < \delta_X$ is defined as in Definition 2.7 then

$$U_r(\pi(I_j)) = \pi^{-1}(B_r(\pi(I_j))) = T_r(I_j) \quad (21)$$

2.3. Volumes and Smocked Metric Spaces. In this paper we are finding the SWIF limit and SWIF limits are defined using volumes and a more general notion called mass which can be bounded using weighted volumes. In preparation for this, we will say something here about the volumes of the pre-images of balls which lie in Euclidean space:

**Lemma 2.11.** The volume,

$$\text{Vol}(U_r(x)) = \text{Vol}(\pi^{-1}(\bar{B}_r(x))) \quad (22)$$
is finite and well defined because
\[ \rho_x : \mathbb{B} \rightarrow [0, \infty) \text{ such that } \rho_x(v) = d_X(x, \pi(v)) \]
is a proper Lipschitz function with respect to the Euclidean norm.

**Proof.** We first prove \( \rho_x \) is Lipschitz:
\[
\frac{\rho(v) - \rho(w)}{|v - w|} = \frac{|d_X(x, \pi(v)) - d_X(x, \pi(w))|}{|v - w|} \leq \frac{|d_X(\pi(v), \pi(w))|}{|v - w|} \leq 1.
\]
It is proper because its level sets are compact. Since
\[ U_R(x) = \rho_x^{-1}([0, R]) \text{ where } \rho_x(v) = d_X(x, \pi(v)) \]
the volume may be computed exactly as in vector calculus by breaking up the region into subregions with boundaries and integrating the function 1. \( \square \)

**Lemma 2.12.** If \( \varepsilon \) is less than the separation factor then
\[
\text{Vol}(U_{\varepsilon}(\pi(I))) = \text{Vol}(T_{\varepsilon}(I)) = \omega_N \varepsilon^N + \omega_{N-1} \varepsilon^{N-1} L.
\]
where \( \omega_N = \text{Vol}(B_1(0)) \text{ and } \omega_0 = 2. \)

**Proof.** This follows from the fact that the tubular neighborhood is the union of a cylindrical region around interval and two hemispheres at the tips. \( \square \)

Since we would like the volume of \( U_R(x) \) to relate well with the volume of \( B_R(x) \) we now define a nice smocking set:

**Definition 2.13.** We say that a smocking set \( S \) is nice if for any fixed compact set, \( K \), we have
\[
\lim_{\varepsilon \to 0} \text{Vol}(T_{\varepsilon}(S) \cap K) = 0.
\]
A smocked metric space is nice if it has a nice smocking set.

Combining this definition with the two previous propositions we immediately have the following new lemma:

**Lemma 2.14.** In a smocked space, \( X \), with a nice smocking set,
\[
\lim_{\varepsilon \to 0} \text{Vol}(U_{\varepsilon}(x)) = 0 \quad \forall x \in X.
\]

**Remark 2.15.** Note that not all smocked metric spaces have nice smocking sets. If for example a smocking interval
\[
I = [0, 1] \times [0, 1] \subset \mathbb{B}^2
\]
then
\[
\lim_{\varepsilon \to 0} \text{Vol}(U_{\varepsilon}(\pi(I))) = \lim_{\varepsilon \to 0} T_{\varepsilon}(I) = \text{Vol}(I) = 1 = 0.
\]

**Lemma 2.16.** A smocked metric space with \( L_{\max} < \infty \) has a nice smocking set iff for every interval in the smocking set
\[
\lim_{\varepsilon \to 0} \text{Vol}(T_{\varepsilon}(I_j)) = 0 \quad \forall j \in J.
\]
Proof. If one interval has nonzero limit, just set $K$ to be that interval to see that the smocking set is not nice. Conversely, assume all the intervals have 0 limits and take any $K$. Since the intervals in a smocking set $\{I_j : j \in J\}$ have a separation factor, and $K$ is compact, there are only finitely many intervals in the set
\begin{equation}
\{I_j : j \in J_K\} = \{I_j : I_j \cap K \neq \emptyset j \in J\},
\end{equation}
So we have a finite sum
\begin{equation}
\text{Vol}(T_e(S) \cap K) \leq \sum_{j \in J_K} \text{Vol}(T_e(I_j))
\end{equation}
Taking the limit we have our proof of the converse. □

Remark 2.17. It is easy to see that all four of our smocked spaces, $X_\diamond$, $X_+$, $X_\Box$, and $X_T$ are nice because the smocking intervals are line segments in $X_\diamond$ and $X_T$, they are unions of two line segments in $X_+$, and they are unions of four line segments in $X_\Box$, and the tubular neighborhood of a line segment, $I$, in $\mathbb{R}^N$ of length $L$ has volume
\begin{equation}
\text{Vol}(T_e(I)) = \omega_N e^N + \omega_{N-1}e^{N-1}L.
\end{equation}

2.4. Review of GH Convergence and Tangent Cones at Infinity. Gromov-Hausdorff convergence was first defined by Edwards in [4] and rediscovered by Gromov in [6].

Definition 2.18. We say a sequence of compact metric spaces
\begin{equation}
(X_j, d_j) \xrightarrow{\text{GH}} (X_\infty, d_\infty)
\end{equation}
iff
\begin{equation}
d_{\text{GH}}((X_j, d_j), (X_\infty, d_\infty)) \to 0.
\end{equation}
Where the Gromov-Hausdorff distance is defined
\begin{equation}
d_{\text{GH}}(X_j, X_\infty) = \inf\{d_H^2(\varphi_j(X_j), \varphi_\infty(X_\infty)) : Z, \varphi_j : X_j \to Z\}
\end{equation}
where the infimum is over all compact metric spaces, $Z$, and over all distance preserving maps $\varphi_j : X_j \to Z$:
\begin{equation}
d_Z(\varphi_j(a), \varphi_j(b)) = d_j(a, b) \ \forall a, b \in X_j.
\end{equation}
The Hausdorff distance is defined
\begin{equation}
d_H(A_1, A_2) = \inf\{r : A_1 \subset T_r(A_2) \text{ and } A_2 \subset T_r(A_1)\}.
\end{equation}
Gromov also defined pointed Gromov-Hausdorff convergence:

Definition 2.19. If one has a sequence of complete noncompact metric spaces, $(X_j, d_j)$, and points $x_j \in X_j$, one can define pointed GH convergence:
\begin{equation}
(X_j, d_j, x_j) \xrightarrow{\text{ptGH}} (X_\infty, d_\infty, x_\infty)
\end{equation}
iff for every radius $R > 0$ the closed balls of radius $R$ in $X_j$ converge in the GH sense as metric spaces with the restricted distance to closed balls in $X_\infty$:
\begin{equation}
d_{\text{GH}}((B_R(x_j) \subset X_j, d_j), (B_R(x_\infty) \subset X_\infty, d_\infty)) \to 0.
\end{equation}
One may consider a single unbounded metric space, and take a sequence of rescalings of that metric space. A Gromov-Hausdorff limit of a sequence of rescalings, if it exists, is called a GH tangent cone at infinity:

**Definition 2.20.** A complete noncompact metric space with infinite diameter, \((X, d_X)\), has a GH tangent cone at infinity, \((Y, d_Y)\), if there is a sequence of rescalings, \(R_j \to \infty\), and points, \(x_0 \in X\) and \(y_0 \in Y\), such that

\[
(X, d/R_j, x_0) \xrightarrow{\text{ptGH}} (Y, d_Y, y_0)
\]

There are a variety of theorems in the literature concerning the existence and uniqueness of such tangent spaces at infinity.

2.5. **Review of Tangent Cones at Infinity for Smocked Spaces.** In [9] the GH tangent cones at infinity were found for four smocked spaces. See Figure [2.5]. It was shown that these four spaces had unique tangent cones at infinity that were normed spaces. The proofs of convergence were based upon the lemmas and theorems stated in this subsection (which can be applied more generally to a large class of smocked spaces to prove there exist unique tangent cones at infinity which are normed spaces). Note that these results were proven using only the definitions by Gromov reviewed above.

![Figure 3](image)

**Figure 3.** In [9] the GH tangent cones at infinity were found for \(X_+, X_\square,\) and \(X_T\) and proven to be normed spaces.

The first step towards finding a tangent cone at infinity or any GH limit of a smocked space is to apply the following lemma proven in [9] to obtain an estimate on the smocking pseudometric using only estimates on the distances between intervals.

**Lemma 2.21.** Given an \(N\) dimensional smocked metric space parametrized by points in intervals as in [8], with smocking depth, \(h \in (0, \infty)\), and smocking length \(L = L_{\max} \in (0, \infty)\), if one can find a Lipschitz function \(F : \mathbb{R}^N \to [0, \infty)\) such that

\[
|d(I_j, I_{j'}) - [F(j) - F(j')]| \leq C,
\]
then
\[ |\bar{d}(x, x') - [F(x) - F(x')]| \leq 2h + C + 2dil(F)(h + L) \]

where \(dil(F)\) is the dilation factor or Lipschitz constant of \(F\):
\[ dil(F) = \sup \left\{ \frac{|F(a) - F(b)|}{|a - b|} : a \neq b \in \mathbb{R}^N \right\}. \]

This next theorem, also proven in [9], is crucial for finding the GH limits of rescalings:

**Theorem 2.22.** Suppose we have an \(N\) dimensional smocked metric space, \((X, d)\), as in Definition 2.1 such that
\[ |\bar{d}(x, x') - [F(x) - F(x')]| \leq K \quad \forall x, x' \in \mathbb{R}^N \]
where \(F : \mathbb{R}^N \to [0, \infty)\) is a norm. Then \((X, d)\) has a unique GH tangent cone at infinity, \((\mathbb{R}^N, d_F)\), where
\[ d_F(x, x') = \|x - x'\|_F = F(x - x'). \]

### 3. Balls in Smocked Metric Spaces are Integral Current Spaces

Recall that any compact metric space, \((X, d)\), endowed with a countable collection of biLipschitz charts such that the total volume of the images of all the charts is finite is an integer rectifiable current space [10]. If the boundary of this space is also integer rectifiable, then it is an integral current space. Before we define this in more detail, we will look at bi-Lipschitz maps onto closed balls in our smocked space. We will then find the right collection of biLipschitz charts for a closed ball in a smocked space, and finally prove they are integral current spaces. Before each theorem we will provide the precise definition of what we are proving.

Throughout this section, as above,
\[ U_R(p) = \pi^{-1}(\bar{B}_R(p)) \subset \mathbb{R}^N, \]
is the pre-image of a closed ball under the smocking map.

#### 3.1. Finding bi-Lipschitz maps.

**Definition 3.1.** A map \(f : Y \to Z\) is bi-Lipschitz if it is a bijection with uniform upper bounds on \(dil(f)\) and \(dil(f^{-1})\). That is, there exists \(\lambda > 0\) such that
\[ \frac{1}{\lambda} \leq \frac{d_Z(f(p), f(q))}{d_Y(p, q)} \leq \lambda \quad \forall p, q \in Y. \]

**Remark 3.2.** Observe immediately that even though the smocking map \(\pi : \mathbb{R}^N \to X\) is Lipschitz, it is not bi-Lipschitz onto its image. It is not even bijective. Even if we remove the smocking set and study the restriction
\[ \pi : \mathbb{R}^N \setminus S \to X \setminus \pi(S), \]
we only obtain a bijective map but it is still not bi-Lipschitz map. One can see this by taking \( p_i, q_i \) arbitrarily close to the same interval in the smocking set, but keeping \( \pi^{-1}(p_i) \) and \( \pi^{-1}(q_i) \) a definite distance \( L/2 \) apart:

\[
(52) \quad \frac{|\pi^{-1}(p_i) - \pi^{-1}(q_i)|}{d_X(p_i, q_i)} \to \infty.
\]

**Lemma 3.3.** If we avoid a tubular neighborhood of the smocking set,

\[
(53) \quad \pi : \mathbb{B}^N \setminus T_r(S) \to X \setminus \pi(T_r(S)),
\]

then we do have a bi-Lipschitz map. This map is still bi-Lipschitz when further restricted to a compact ball in \( X \):

\[
(54) \quad \pi : U_R(p) \setminus T_r(S) \to \tilde{B}_R(p) \setminus \pi(T_r(S)).
\]

In fact on \( U_R(p) \) we have

\[
(55) \quad \min \left\{ \frac{r}{\text{Diam}(U_R(p))}, 1 \right\} \leq \frac{d(\pi(p), \pi(q))}{|p - q|} \leq 1 \quad \forall p, q \in U_R(p) \setminus T_r(S)
\]

**Proof:** We already know that \( \text{dil}(\pi) = 1 \). So we have a bi-Lipschitz map, unless there are

\[
(56) \quad p_i, q_i \in X \setminus \pi(T_r(S))
\]

such that

\[
(57) \quad \lim_{i \to \infty} \frac{|\pi^{-1}(p_i) - \pi^{-1}(q_i)|}{d_X(p_i, q_i)} = \infty.
\]

This can only happen if

\[
(58) \quad \lim_{i \to \infty} d_X(p_i, q_i) = 0.
\]

Thus there is an \( N \) sufficiently large that

\[
(59) \quad d_X(p_i, q_i) < r \quad \forall i \geq N.
\]

Since

\[
(60) \quad |p_i - z| > r \text{ and } |q_i - z'| > r \quad \forall z, z' \in S,
\]

we see the minimum in the Definition 2.1 is achieved by a direct segment from \( \pi^{-1}(p_i) \) to \( \pi^{-1}(q_i) \):

\[
(61) \quad d_X(\pi^{-1}(p_i), \pi^{-1}(q_i)) = |(\pi^{-1}(p_i) - \pi^{-1}(q_i))| \quad \forall i \geq N.
\]

Thus

\[
(62) \quad \lim_{i \to \infty} \frac{|\pi^{-1}(p_i) - \pi^{-1}(q_i)|}{d_X(p_i, q_i)} = 1
\]

which is a contradiction.

In fact we know for any \( p, q \in U_R(p) \setminus T_r(S) \) with \( d(p, q) < r \) we have:

\[
(63) \quad \frac{|\pi^{-1}(p) - \pi^{-1}(q)|}{d_X(p, q)} = 1.
\]
If \( p, q \in U_R(p) \setminus T_r(S) \) has \( d(p, q) \geq r \) then we have
\[
\frac{|\pi^{-1}(p) - \pi^{-1}(q)|}{d_X(p, q)} \leq \frac{\text{Diam}(U_R(p))}{r}.
\]
\[\square\]

### 3.2. Review of Integer Rectifiable Currents

The following definitions are in Ambrosio-Kirchheim [1].

**Definition 3.4.** We say \((f_0, f_1, \ldots, f_N)\) is an \(N\)-tuple on a complete metric space \(X\) if \(f_0\) is bounded and if all \(f_i : X \to \mathbb{R}\) are Lipschitz. Note that if
\[
\varphi : A \subset \mathbb{R}^N \to X
\]
is Lipschitz, then
\[
f_j \circ \varphi : A \subset \mathbb{R}^N \to \mathbb{R}
\]
is also Lipschitz and is thus differentiable almost everywhere.

**Definition 3.5.** Given a precompact Borel set \(A \subset \mathbb{R}^N\) and a Lipschitz map
\[
\varphi : A \subset \mathbb{R}^N \to \varphi(A) \subset X
\]
we can define \(\varphi_*[A]\), denoted \(\varphi_*[A]\), which acts on an \(N\)-tuple:
\[
\varphi_*[A](f_0, f_1, \ldots, f_N) = \int_A f_0 \circ \varphi \ d(f_1 \circ \varphi) \wedge \cdots \wedge d(f_N \circ \varphi)
\]
This integration is well defined because Lipschitz functions on Euclidean space are differentiable almost everywhere.

**Definition 3.6.** Given a countable collection of weights \(a_i \in \mathbb{Z}\) and bi-Lipschitz charts from Borel sets \(A_i\) in Euclidean space of dimension \(N\),
\[
\varphi_i : A_i \subset \mathbb{R}^N \to \varphi_i(A_i) \subset X
\]
that are pairwise disjoint
\[
\varphi_i(A_i) \cap \varphi_j(A_j) = \emptyset
\]
whose total weighted volume is finite
\[
\sum_{i=1}^{\infty} |a_i| \text{Lip}^N(\varphi_i) \text{Vol}(A_i) < \infty,
\]
we may define an \(N\) dimensional **integer rectifiable current**, \(T\), acting on \(N\)-tuples
\[
T(f_0, f_1, \ldots, f_N) = \sum_{i=1}^{\infty} \varphi_i_*[A_i](f_0, f_1, \ldots, f_N).
\]
Two collections of weighted charts define the same current if they have the same values when acting on \(N\) tuples.
3.3. Review of the Mass of Currents. Ambrosio-Kirchheim define a mass for an integer rectifiable current which they then prove to be bounded above by a constant multiple of the weighted volume in Section 9 of [1]:

\[ M(T) \leq C_N \sum_{i=1}^{\infty} |a_i| H^N(\phi_i(A_i)) \]

where

\[ C_N = 2^N / \omega_N. \]

The mass is finite of the currents we’ve defined here because

\[ H^N(\phi_i(A_i)) \leq \text{Lip}(\phi_i)^N \text{Vol}(A_i). \]

We do not need the precise definition of mass in this paper.

Remark 3.7. Ambrosio-Kirchheim defined the push forward of a current by a Lipschitz map \( \psi : X \to Y \) to be

\[ \psi_\#(T)(f_0, f_1, \ldots, f_N) = T(f_0 \circ \psi, f_1 \circ \psi, \ldots, f_N \circ \psi) \]

and prove that

\[ M(\psi_\# T) \leq (dil \psi)^N M(T). \]

3.4. Review of Adding and Subtracting Currents. Ambrosio-Kirchheim defined addition of currents,

\[ (T + S)(f_0, f_1, \ldots, f_N) = T(f_0, f_1, \ldots, f_N) + S(f_0, f_1, \ldots, f_N). \]

One can prove that when \( T, S \) are integer rectifiable then so is their sum (by carefully cancelling parts of the various charts in the union of all the charts as necessary to ensure they are pairwise disjoint). They prove that

\[ M(T + S) \leq M(T) + M(S) \]

Note this is not an equality as some charts in the parametrization of \( T \) may cancel with some charts in the parametrization of \( S \). It is easy to verify that

\[ \partial(T + S) = \partial T + \partial S. \]

Thus the sums of integral currents is an integral current. The 0 current has

\[ 0(f_0, \ldots, f_N) = 0 \quad \forall \text{ tuples } (f_0, \ldots, f_N). \]

Note that the negative of a current can be parametrized by the same collection of charts as the original current but with the opposite orientation on all of them:

\[ -T(f_0, f_1, \ldots, f_N) = \sum_{i=1}^{\infty} -\phi_i[A_i](f_0, f_1, \ldots, f_N) \]

\[ = \sum_{i=1}^{\infty} -\int_{A_i} (f_0 \circ \phi_i) d(f_1 \circ \phi_i) \wedge \cdots \wedge (f_N \circ \phi_i) \]
3.5. **Review of Integer Rectifiable Current Spaces.** The following definition first appeared in [10] by Sormani-Wenger:

**Definition 3.8.** Suppose a complete metric space, \((X, d)\), is endowed with a countable collection weights \(a_i \in \mathbb{Z}\) (here \(a_i = 1\)) and bi-Lipschitz charts from Borel sets, \(A_i\), in Euclidean space of dimension \(N\), as in (69) that are pairwise disjoint as in (70) such that total weighted volume is finite as in (71) and such that the complement of the sets has zero measure

\[
\mathcal{H}^N \left( X \setminus \bigcup_{i=1}^{\infty} \varphi_i(A_i) \right) = 0.
\]

Then we say the space has an **integer rectifiable current structure**, \(T = \sum_{k=0}^{\infty} a_i \pi_* [A_k]\), with mass as in (73). We say the space has weight 1 if all the \(a_i = 1\). We say that \((X', d, T)\) is an **integer rectifiable current space** where \(X' \subset \bar{X}\) is defined as the set of positive density of \(T\) (here we need only know \(X'\) is defined).

3.6. **Balls in Normed Spaces are Integer Rectifiable Current Spaces.** Consider a metric space which is a finite dimensional normed vector space, \((E^N, \|\cdot\|_F)\) and compare it to Euclidean space.

**Remark 3.9.** We now use the fact that all norms on a finite dimensional vector space are equivalent. In other words for every norm \(F\) on \(E^n\) there exists \(C\) so that

\[
\frac{1}{C} \leq \frac{|v|}{\|v\|_F} \leq C
\]

for all \(v\) in \(E^n\)

**Lemma 3.10.** There exists

\[
\lambda = \max_{v \neq 0} \frac{|v|}{\|v\|_F} < \infty.
\]

Thus

\[
\bar{B}_R^F(0) = \{ v : \|v\|_F \leq R \} \subset B_{\lambda R}(0)
\]

**Proof.** For the first part, note that

\[
\lambda = \max_{v \neq 0} \frac{|v|}{\|v\|_F} \leq C < \infty
\]

This gives us that \(\frac{|v|}{\|v\|_F} \leq \lambda\), so \(|v| \leq \lambda \|v\|_F\). Supposing \(v\) has \(\|v\|_F \leq R\), we get

\[
|v| \leq \lambda \|v\|_F
\]

\(\square\).
Theorem 3.11. A closed ball $\bar{B}_R^F(0)$ in an oriented finite dimensional normed vector space with norm $\|x\|_F$ is an integral current space with weight one where the integral current structure, $T$, is defined by a single chart that is the identity map and the ball itself as the domain of the chart. Furthermore

\begin{equation}
M(T_R) \leq (\text{dil}(id))^N \omega_N R^N \leq C^N \omega_N R^N
\end{equation}

and

\begin{equation}
M(\partial T_R) \leq (\text{dil}(id))^N \omega_N N R^{N-1} \leq C^N \omega_N N R^{N-1}
\end{equation}

Proof. The identity map

\begin{equation}
\text{id} : (\bar{B}_R^F(0), d_E) \to (\bar{B}_R^F(0), d_F)
\end{equation}

where $d_E(v, w) = |v - w|$ and $d_F(v, w) = \|v - w\|_F$ is biLipschitz with

\begin{equation}
\text{dil}(id) \leq C < \infty
\end{equation}

and

\begin{equation}
\text{dil}(id)^{-1} \leq C \leq \infty
\end{equation}

(Note that $\text{dil}(id)^{-1} = \max_{v \neq 0} \frac{|v|}{\|v\|_F} = \lambda$ and $\text{dil}(id) = \max_{v \neq 0} \frac{\|v\|_F}{|v|}$. By remark 3.9 we know there exists some $C$ which bounds both $\max_{v \neq 0} \frac{|v|}{\|v\|_F}$ and $\max_{v \neq 0} \frac{\|v\|_F}{|v|}$).

So we define

\begin{equation}
T_R = \text{id}_{\#}[\bar{B}_R^F(0)]
\end{equation}

and this has

\begin{equation}
M(T_R) \leq (\text{dil}(id))^N \text{Vol}_E(\bar{B}_R^F(0)) < (\text{dil}(id))^N \omega_N R^N
\end{equation}

by Lemma 3.10.

\[ \square \]

3.7. Balls in Smocked Spaces are Integer Rectifiable Current Spaces. We now prove that a closed ball in a smocked metric space is an integer rectifiable space providing a precise set of charts to define a canonical integer rectifiable current on the space (up to sign/orientation):

Theorem 3.12. A closed ball, $\bar{B}_R(p)$, in a smocked metric space, $(X, d_X)$, with smocking map $\pi : \mathbb{B}^N \to X$, has a pair of natural integer rectifiable structures of weight 1 defined by pushing forward the two oriented local integral current structures, $\pm [U_R(p)]$, on $\mathbb{B}^N$:

\begin{equation}
T = \pi_{\#}[U_R(p)] = \sum_{i=0}^{\infty} \pi_{\#}[A_i]
\end{equation}

where

\begin{equation}
A_k = U_R(p) \cup T_{1/k}(S) \setminus T_{1/(k+1)}(S) \quad \text{and} \quad A_0 = U_R(p) \setminus \left(S \cup \bigcup_{k=1}^{\infty} A_k\right)
\end{equation}
so that for any collection of functions we have

\[ T(f, h_1, ..., h_N) = \sum_{k=0}^{\infty} \int_{A_k} (f \circ \pi) d(h_1 \circ \pi) \wedge \cdots \wedge d(h_N \circ \pi). \]

Furthermore the mass satisfies \(\mathbf{M}(T) = \text{Vol}(U_R(p))\).

**Proof.** By Lemma 3.3 these charts are bi-Lipschitz. They are pairwise disjoint since \(\pi\) is a bijection away from the smocking set and the \(A_k\) are pairwise disjoint.

Observe that (96) holds because

\[ H_N \left( U_R(p) \setminus \bigcup_{k=0}^{\infty} A_k \right) = H_N (S \cap U_R(P)) = 0 \]

because the smocking set \(S\) has zero measure. Note that this also implies that we have (82).

We claim that

\[ H^m(\pi(A_k)) = H^m(A_k). \]

Recall that the Hausdorff measure, \(H^m(A_k)\) is defined using small sets, \(Z_w\) about \(w \in A_k\). Once \(\text{Diam}(Z_w) < 1/(4k)\), we have a isometries \(\pi : Z_w \to \pi(Z_w)\). The Hausdorff measure, \(H^m(\pi(A_k))\) is also defined using small sets, \(Z_p\) about \(p \in \pi(A_k)\). Once \(\text{Diam}(Z_p) < 1/(4k)\), we have a isometries \(\pi : Z_p \to \pi(Z_p)\). Thus we are estimating the Hausdorff measures of these sets with isometric collections of small sets.

As a consequence, taking our weights, \(a_k = 1\), we have weighted volume equal to mass:

\[ \mathbf{M}(T) = \sum_{k=1}^{\infty} H^N(\pi(A_k)) = \sum_{k=1}^{\infty} H^N(A_k) = \text{Vol}(U_R(p)) < \infty. \]

\[ \square \]

### 3.8. Review of Boundaries of Currents.

Ambrosio-Kirchheim defined the boundary of a current as follows in [1]:

**Definition 3.13.** The boundary of an \(N\) dimensional current to be the following \((N - 1)\) dimensional current:

\[ \partial T(f_0, f_1, ..., f_{N-1}) = T(1, f_0, ..., f_{N-1}). \]

**Lemma 3.14.** For nice enough sets, \(A \subset \mathbb{R}^N\), (eg. with piecewise smooth boundary)

\[ \partial \pi[#A] = [\pi(\partial A)] \]
Proof. By the definitions and Stoke’s Theorem we have:

\[ \partial \pi[A](f_0, f_1, ... , f_{N-1}) = \pi[A](1, f_0, f_1, ... , f_{N-1}) \]
\[ = \int_A (f_0 \circ \pi) \wedge (f_1 \circ \pi) \wedge \cdots \wedge (f_{N-1} \circ \pi) \]
\[ = \int_{\partial A} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ = \pi[\partial A](f_0, f_1, ... , f_{N-1}). \]

\[ \square \]

3.9. Boundaries of Balls in Smocked Spaces. In general the boundary of a current can be difficult to compute since there is no reason for the Borel sets to have nice boundaries. However in the boundaries of the charts we found for our smocking sets are very nice and cancel perfectly when the smocking set is nice as in Definition 2.13.

Proposition 3.15. Suppose the smocking set, \( S \), is nice as in Definition 2.13. Then boundary of the current defined in Theorem 3.12 is

\[ (104) \quad \partial T = \pi[\partial U_R(p)] \]

so that for any collection of functions we have

\[ (105) \quad \partial T(f_0, f_1, ... , f_N) = \int_{\partial U_R(p)} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi). \]

Since \( \partial U_R(p) \) is covered by a finite collection of Lipschitz maps \( \varphi_i : H_i \subset \mathbb{B}^{N-1} \rightarrow \partial U_R(p) \) which are used to define what we mean by the integral in (105):

\[ (106) \quad \partial T(f_0, f_1, ... , f_N) = \sum_i \int_{H_i} (f_0 \circ \pi \circ \varphi_i) d(f_1 \circ \pi \circ \varphi_i) \wedge \cdots \wedge d(f_{N-1} \circ \pi \circ \varphi_i). \]

Thus \( \partial T \) is an integer rectifiable current with charts \( \pi \circ \varphi_i : H_i \subset \mathbb{B}^{N-1} \rightarrow \partial B_R(p) \). Furthermore the mass satisfies

\[ (107) \quad M(\partial T) = \text{Vol}(\partial U_R(p)). \]
Proof:
\[ \partial T(f_0, f_1, \ldots, f_{N-1}) = T(1, f_0, f_1, \ldots, f_{N-1}) \]
\[ = \sum_{k=0}^{\infty} \pi_k[A_k](1, f_0, f_1, \ldots, f_{N-1}) \]
\[ = \sum_{k=0}^{\infty} \int_{A_k} 1 \, d(f_0 \circ \pi) \wedge d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ = \sum_{k=0}^{\infty} \int_{A_k} d((f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ = \sum_{k=0}^{\infty} \int_{\partial A_k} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ + \sum_{k=1}^{\infty} \int_{\partial A_k \cap U_R(p)} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ + \int_{\partial T_{1/2}(S) \cap U_R(p)} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ - \int_{\partial T_{1/(k+1)}(S) \cap U_R(p)} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]

Telescoping the second with the third parts of the sum and totaling, we have
\[ \partial T(f_0, f_1, \ldots, f_{N-1}) = \int_{\partial U_R(p)} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ - \lim_{k \to \infty} \int_{\partial T_{1/(k+1)}(S) \cap U_R(p)} (f_0 \circ \pi) d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi). \]

The limit is zero because the integral in the limit is
\[ \leq \int_{T_{k+1}(S) \cap U_R(p)} d(f_0 \circ \pi) \wedge d(f_1 \circ \pi) \wedge \cdots \wedge d(f_{N-1} \circ \pi) \]
\[ \leq \prod_{i=0}^{N-1} \text{Lip}(f_i \circ \pi) \text{Vol}(T_{1/(k+1)}(S) \cap U_R(p)) \to 0 \]

because \( S \) has dimension \( N - 1 \) and \( S \cap U_R(p) \) has finitely many components of finite \( N - 1 \) volume. \( \square \)

3.10. Review of Integral Currents and Spaces. Ambrosio-Kirchheim defined an integral current as follows in [11]:

**Definition 3.16.** An \( N \) dimensional integral current is an \( N \) dimensional integer rectifiable current whose boundary is integer rectifiable. The 0 current

\[ 0(f_0, f_1, \ldots, f_N) = 0 \]
is also included among the integral currents.

This allowed Sormani-Wenger to naturally define an integral current space in [10].

**Definition 3.17. int-cur-space** An N dimensional integral current space is an integer rectifiable metric space whose integral current structure is an N dimensional integral current. We also include $0 = (0, 0, 0)$ as an integral current space of dimension $N$.

3.11. **Balls in Normed Spaces are Integral Current Spaces.** Consider a metric space which is a finite dimensional normed vector space, $(\mathbb{E}^N, \| \cdot \|_F)$ and compare it to Euclidean space.

**Theorem 3.18.** A closed ball $\bar{B}^F_R(0)$ in an oriented finite dimensional normed vector space with norm $\| x \|_F$ is an integral current space with weight one where the integral current structure $T^F_R$ is defined by a single chart that is the identity map and the ball itself as the domain of the chart. Furthermore

$$M(T^F_R) \leq \kappa_F^{2N} \omega_N R^N$$

and

$$M(\partial T^F_R) \leq \kappa_F^{N-1} \alpha R^{N-1}$$

where $\kappa_F > 0$ is defined so that

$$\frac{1}{\kappa_F} \leq \frac{\| v \|_F}{|v|} \leq \kappa_F \quad \forall v \neq 0.$$

**Proof.** Recall that in Theorem 3.11 we defined its integral current structure

$$T = id[\bar{B}^F_R(0)].$$

Here $id$ is the biLipschitz identity map

$$id : (\bar{B}^F_R(0), d_E) \rightarrow (\bar{B}^F_R(0), d_F)$$

where $d_E(v, w) = |v - w|$ and $d_F(v, w) = \| v - w \|_F$ is biLipschitz with

$$\text{dil}(id) = \sup_{v \neq w} \frac{\| v - w \|_F}{|v - w|} \leq \kappa_F.$$

and

$$\text{dil}(id)^{-1} = \sup_{v \neq w} \frac{|v - w|}{\| v - w \|_F} \leq \kappa_F.$$

Thus

$$M(T) \leq (\text{dil}(id))^N \Vol_E(F^{-1}[0, R]) \leq \kappa_F^N \omega_N (\kappa_F R)^N$$

because

$$F^{-1}[0, R] = \{ v \| \| v \|_F \leq R \} \subset \{ v \| v \| \leq \kappa_F R \}.$$
Finally
\[(118) \quad M(\partial T) \leq (\text{dil}(id))^{N-1} \text{Vol}_E(F^{-1}(\{R\})) \leq \kappa_F^{N-1} a R^{N-1}\]
because the volumes of level sets of a norm scale as follows:
\[(119) \quad \text{Vol}_E(F^{-1}(\{R\})) = \text{Vol}_E(F^{-1}(\{1\})) R^{N-1}.\]

\[\square\]

3.12. Balls in Smocked Metric Spaces are Integral Current Spaces.

**Theorem 3.19.** Suppose the smocking set, \( S \), is nice as in 2.13. Then the integer rectifiable current space, \((\bar{B}_R(p), d_X, \pi_* (U_R(p)))\), defined in Theorem 3.12 is an integral current space. Furthermore the rescaled ball \((\bar{B}_R(p), d_X/t, \pi_* (U_R(p)))\) is also an integral current space.

**Proof.** The first part follows from the definition and Proposition 3.15. The second part follows from the observation that any collection of charts which is bi-Lipschitz with respect to \( d_X \) is also bi-Lipschitz with respect to \( d_X/t \) and that the weighted volume will still be bounded. \( \square \)

**Remark 3.20.** It is an open question as to whether this is true or false for arbitrary smocked metric spaces. See the end of the proof of Proposition 3.15 to see where the hypothesis on the smocking set was applied.

4. SWIF Convergence of Smocked Metric Spaces

In this section we prove Theorem 1.1. Before beginning we quickly review the definition of SWIF convergence in one subsection.

4.1. Review of SWIF Convergence. The Sormani-Wenger Intrinsic Flat (SWIF) distance was defined in [10] imitating the Gromov-Hausdorff (GH) distance replacing the Hausdorff distance in Gromov’s infimum with the Flat distance of Federer-Flemming. Since the Federer-Flemming flat distance was defined only for integral currents in Euclidean space, we used the notion of an integral current defined as in Ambrosio-Kirchheim [1] that we have just reviewed above.

**Definition 4.1.** We say a sequence of compact integral current spaces
\[(120) \quad (X_j, d_j, T_j) \xrightarrow{f} (X_\infty, d_\infty, T_\infty)\]
iff
\[(121) \quad d_{\text{SWIF}}((X_j, d_j, T_j), (X_\infty, d_\infty, T_\infty)) \to 0.\]

Where the Sormani-Wenger intrinsic flat distance is defined
\[(122) \quad d_{\text{SWIF}}(X_j, X_\infty) = \inf \{d_F^Z(\varphi, \#(T_j), \varphi_\#(T_\infty)) : Z, \varphi_j : X_j \to Z\}\]
where the infimum is over all complete metric spaces, \( Z \), and over all distance preserving maps \( \varphi_j : X_j \to Z \):
\[(123) \quad d_Z(\varphi_j(a), \varphi_j(b)) = d_j(a, b) \forall a, b \in X_j.\]
The flat distance between two integral currents in $Z$ is defined

$$d^F_Z(S_1, S_2) = \inf \{ M(A) + M(B) : S_1 - S_2 = A + \partial B \}$$

where the infimum is over all integral currents $A, B$ in $Z$ such that

$$S_1(f_0, f_1, ..., f_N) - S_2(f_0, f_1, ..., f_N) = A(f_0, f_1, ..., f_N) + B(1, f_1, f_2, ..., f_N)$$

for all tuples $(f_0, f_1, ..., f_N)$ on $Z$.

**Remark 4.2.** Examples in [10] demonstrate that GH and SWIF limits need not agree and that SWIF limits may exist when there is no GH limit.

In [10], Sormani and Wenger proved that:

**Theorem 4.3.** If $(X_j, d_j, T_j)$ converge to $(X_\infty, d_\infty, T_\infty)$ in the Lipschitz sense:

$$\exists \text{ bi-Lip } F_j : X_j \to X_\infty \text{ with } \frac{1}{\lambda_j} < \text{dil}(F_j) < \lambda_j \text{ where } \lambda_j \to 1$$

and if

$$F_j \# T_j = T_\infty$$

(which holds if they are all weight 1 and oriented in the same way), then

$$\lim_{R \to \infty} (X_j, d_j, T_j) \overset{F_j}{\to} (X_\infty, d_\infty, T_\infty).$$

**Remark 4.4.** Our rescaled smocked metric spaces do not converge in the Lipschitz sense to their tangent cones at infinity. In fact, there does not even exist a bi-Lipschitz map between a smocked metric space with a nonempty smocking set and a Euclidean space endowed with a definite norm. We won’t prove this claim in general but will observe that the estimate obtained in (2.21):

$$|\bar{d}(x, x') - [F(x) - F(x')]| \leq 2h + C + 2\text{dil}(F)(h + L)$$

are not able to control ratios of distances between pairs of points $x, x'$ which lie on a common smocking interval.

### 4.2. The set up

Take any $x_0 \in X$. By shifting the smocking set, $S$, we may assume that $\pi(0) = x_0$ where $\pi : \mathbb{E}^N \to X$ is the pulled thread map. Let

$$\lambda = \max_{v \neq 0} |v| / |F(v)| < \infty,$$

be as in Lemma 3.10.

We need to show that for all $r > 0$

$$\bar{B}_{Rr}(x_0, d_X/R) \to (\bar{B}_r(0), d_F) \text{ as } R \to \infty$$

where

$$\bar{B}_r(0) = F^{-1}([0, r]) = \{ v : F(v) \leq r \} \subset \{ v : |v| \leq \lambda r \} \subset B_{\lambda r}(0) \subset \mathbb{E}^N.$$

and

$$\bar{B}_r^X(x_0) = \{ x \in X : d_X(x, x_0) \leq r \} \subset X$$

Since we are rescaling this ball in $X$, using the metric $d_X/R$, there is an isometry

$$F_R : (\bar{B}_{Rr}(x_0), d_X/R) \to (\bar{B}_r^X(x_0), d_R)$$
whose image is a ball is the rescaled smocked metric space:

\[(135) \quad \bar{B}_R^r(x_0) = \{ x \in X_R : d_R(x, x_0) \leq r \} \subset X_R \]

where \((X_R, d_R)\) is the rescaled smocked metric space defined with rescaled smocking intervals:

\[(136) \quad I_R = \{ I_j/R : j \in J \} \text{ where } I_j/R = \{ z/R : z \in I_j \} \]

and a rescaled smocking map,

\[(137) \quad \pi_R : \mathbb{B}^N \to X_R \]

Observe that

\[(138) \quad \pi_R(v) = F_R(\pi(Rv)) \]

See Figure 2.5 to see how the smocking intervals rescale in a variety of smocked metric spaces as \(R \to \infty\).

We set

\[(139) \quad U_R^r(x_0) = \pi_R^{-1}(\bar{B}_R^r(x_0)) \subset \mathbb{B}^N. \]

Observe that for \(K/R < r\) we have

\[(140) \quad U_R^r(x_0) = \{ v : d_R(\pi_R(v), x_0) \leq r \} \]
\[(141) \quad = \{ v : d_X(\pi(Rv), x_0) \leq rR \} \]
\[(142) \quad = \{ v : \bar{d}_X(Rv, 0) \leq rR \} \subset \{ v : \|Rv\|_F < rR + K \} \]
\[(143) \quad = \{ v : \|v\|_F < r + (K/R) \} \subset \{ v : |v| < \lambda(r + (K/R)) \} \subset B_{2,\lambda\rho}(0) \subset \mathbb{B}^N. \]

Note that the current structure, \(T_R\), is

\[(146) \quad T_R = \pi_B[U_R^r(x_0)] \]

as in Theorem 3.12 and

\[(147) \quad T_F = id_B[F^{-1}([0, r])] \]

where \(id\) is the identity map as in Theorem 3.18.

To prove the theorem, we will show that as \(R \to \infty\)

\[(148) \quad d_{GH}\left( (\bar{B}_R^r(x_0), d_R), (F^{-1}([0, r]), d_F) \right) \to 0 \]

and

\[(149) \quad d_{SWIF}\left( (\bar{B}_R^r(x_0), d_R, T_R), (F^{-1}([0, r]), d_F, T_F) \right) \to 0. \]
4.3. **Constructing a metric space, Z.** As in Figure 4.3, we take Z to be a smocked space defined in one dimension higher, $\mathbb{E}^{N+1}$, with smocking intervals

$$I_Z = \{i_j/R \times \{0\} : j \in J \} \subset \mathbb{E}^N \times \{0\}$$

and a smocking map $\pi_Z : \mathbb{E}^{N+1} \to Z$, and smocking metric

$$d^S_Z : Z \times Z \to [0, \infty).$$

Thus we have a distance preserving map

$$\varphi^S : (X_R, d_R) \to (Z, d^S_Z)$$

defined by

$$\varphi^S(x) = \pi_Z(\pi_R^{-1}(x) \times \{0\}).$$

The smocking preimage of

$$\varphi^S(\bar{B}_R(x_0)) = \pi_Z(U_R(0) \times \{0\})$$

is depicted in Figure 4.3 as a purple set in the lower plane with the intervals.

![Figure 4](image.png)

**Figure 4.** The metric space Z is a smocked space defined using a collection of smocking intervals in the plane $\mathbb{E}^N \times \{0\}$. Also depicted here are the images of our distance preserving maps.

4.4. **Choosing a height $H$:** We would like to show the map:

$$\varphi^H : (\mathbb{R}^N, d_F) \to \pi_Z(\mathbb{E}^N \times \{H\}) \subset Z$$

defined by

$$\varphi^H(w) = \pi_Z(w, H)$$

to be distance preserving. This will not work with $d_Z$, so we do not claim this is distance preserving. For intuition, you will see the image of $F^{-1}(\{0, r\})$ under this map as a green diamond in the upper plane in Figure 4.3.

We choose $H = H_r > 0$ such that

$$H = \sqrt{8\lambda r (K/R) + (K/R)^2}$$
so that
\[
H \geq \sqrt{2a(K/R) + (K/R)^2} \quad \forall a \in [0, 4l_r],
\]
which implies
\[
\sqrt{H^2 + a^2} \geq a + K/R \quad \forall a \in [0, 4l_r].
\]
Thus
\[
|(v, H) - (z, 0)| \geq |v - z| + K/R \quad \forall v, z \in B_{2l_r}(0) \subset \mathbb{E}^N
\]
Since the smocking distance between points are sums over segments between intervals and all the intervals lie in \(\mathbb{E}^N \times [0, 1]\), for any \(x, y \in B_{2l_r}(0)\) such that \(|x - y| > \delta\)
where \(\delta\) is the smocking separation factor, there is \(z \in B_{2l_r}0\) such that \((z, 0) \in S_Z\) so that
\[
\bar{d}_Z^S((x, H), (y, 0)) = |(x, H) - (z, 0)| + \bar{d}_Z^S((z, 0), (y, 0))
\]
\[
\geq |x - z| + K/R + \bar{d}_Z^S((z, 0), (y, 0))
\]
\[
\geq \bar{d}_Z^S((x, 0), (y, 0)) + K/R
\]
\[
= \bar{d}_X(x, y) + K/R
\]
\[
\geq \|x - y\|_F
\]
by the hypothesis rescaled.
Thus for all \(x_i, y_i \in B_{2l_r}(0) \subset \mathbb{E}^N\)
\[
\bar{d}_Z^S((y_1, H), (x_1, 0)) + \bar{d}_Z^S((x_1, 0), (x_2, 0)) + \bar{d}_Z^S((x_2, 0), (y_2, H)) \geq
\]
\[
\|y_1 - x_1\|_F + |x_1 - x_2| + \|x_2 - y_2\|_F \geq
\]
\[
\|y_1 - x_1\|_F + \|x_1 - x_2\|_F + \|x_2 - y_2\|_F \geq
\]
\[
\|y_1 - y_2\|_F
\]
and since all the smocking intervals lie in \(\mathbb{E}^N \times [0, 1]\) and \(\|y_1 - y_2\|_F \leq \|y_1 - y_2\|\), we have
\[
\|y_1 - y_2\| \leq \bar{d}_Z^S((y_1, H), (y_2, H)) \quad \forall y_1, y_2 \in B_{2l_r}(0) \subset \mathbb{E}^N.
\]
Thus
\[
\varphi^H : (\mathbb{R}^N, d_F) \rightarrow (Z, d_Z).
\]
is distance nonincreasing. Sadly it is not a distance preserving map.

4.5. **Constructing a better metric space**, \(Z_R\). We define a new metric space:
\[
Z_R = \pi_Z(B_{2l_r}(0) \times [0, H]) \subset Z
\]
with a new metric
\[
d_Z^R(p_1, p_2) = \min\{d_Z^S(p_1, p_2), d_Z^F(p_1, p_2)\}
\]
where
\[
d_Z^F(p_1, p_2) = \min\{d_Z^S(p_1, \pi_Z(y_1, H)) + \|y_1 - y_2\|_F + d_Z^S(\pi_Z(y_2, H), p_2) : y_i \in \mathbb{E}^N\}.
\]
Intuitively we are just shrinking the distances between points exactly enough to make \(\varphi^H\) a distance preserving function.
4.6. **Proof that \( d_Z^R \) is a metric on \( Z_R \).** It is easy to see that \( d_Z^R \) is symmetric and definite because both \( d_Z^F \) and \( d_Z^S \) are symmetric and

\[
\text{(175)} \quad d_Z^F(x, y) = 0 \iff x = y \quad \text{and} \quad d_Z^S(x, y) = 0 \iff x = w = w' = y.
\]

We claim the triangle inequality,

\[
\text{(176)} \quad d_Z^R(p_1, p_3) \leq d_Z^R(p_1, p_2) + d_Z^R(p_3, p_2) \quad \forall p_1, p_2, p_3 \in Z_R.
\]

**Case I:** Suppose both \( d_Z^R(p_i, p_2) = d_Z^S(p_i, p_2) \) for \( i = 1, 3 \). Then the triangle inequality holds because \( d_Z^S \) is a metric and

\[
\text{(177)} \quad d_Z^R(p_1, p_2) \leq d_Z^S(p_1, p_2).
\]

**Case II:** Suppose only one is the smocking length:

\[
\text{(178)} \quad d_Z^R(p_1, p_2) = d_Z^S(p_1, p_2) \quad \text{and} \quad d_Z^R(p_3, p_2) = d_Z^S(p_3, p_2)
\]

Then there exists \( y_1 \in \mathbb{B}^N \) such that

\[
\text{(179)} \quad d_Z^R(p_1, p_2) = d_Z^S(p_1, \pi_Z(y_1, H)) + ||y_1 - y_2||_F + d_Z^S(\pi_Z(y_2, H), p_2).
\]

Then we can apply these \( y_1 \) to estimate:

\[
\text{(180)} \quad d_Z^R(p_1, p_3) \leq d_Z^S(p_1, p_3) \leq d_Z^S(p_1, \pi_Z(y_1, H)) + ||y_1 - y_2||_F + d_Z^S(\pi_Z(y_2, H), p_3).
\]

**Case III:** Suppose neither is the smocking length

\[
\text{(181)} \quad d_Z^R(p_1, p_2) = d_Z^S(p_1, \pi_Z(y_{i,j}, H)) + ||y_{i,j} - y_{i,2}||_F + d_Z^S(\pi_Z(y_{i,2}, H), p_2).
\]

Then we can apply these \( y_{1,1}, y_{3,3} \) to estimate:

\[
\text{(182)} \quad d_Z^R(p_1, p_3) \leq d_Z^S(p_1, p_3) \leq d_Z^S(p_1, \pi_Z(y_{1,1}, H)) + ||y_{1,1} - y_{3,3}||_F + d_Z^S(\pi_Z(y_{3,3}, H), p_3).
\]

Since \( \cdot ||F \) is a norm, we have

\[
\text{(183)} \quad ||y_{1,1} - y_{3,3}||_F \leq ||y_{1,1} - y_{1,2}||_F + ||y_{1,2} - y_{3,2}||_F + ||y_{3,2} - y_{3,3}||_F.
\]

Combining this with the previous two equations we have

\[
\text{(184)} \quad d_Z^R(p_1, p_3) \leq d_Z^S(p_1, p_2) - d_Z^S(\pi_Z(y_{1,2}, H), p_2) + ||y_{1,2} - y_{3,2}||_F + d_Z^R(p_3, p_2) - d_Z^S(\pi_Z(y_{3,2}, H), p_2).
\]

So we need only show

\[
\text{(185)} \quad ||y_{1,2} - y_{3,2}||_F \leq d_Z^S(\pi_Z(y_{1,2}, H), p_2) + d_Z^S(\pi_Z(y_{3,2}, H), p_2).
\]

This holds by the norm being less than the smocking length in \[170\] and the triangle inequality for \( d_Z^S \). Thus \((Z_R, d_Z^R)\) is a metric space.
4.7. Proof that $\varphi_H$ of (156) is now distance preserving. Let $\varphi_H$ be the map

$$\varphi_H : (B_{2,1b}(0), d_F) \to (Z, d_Z^R)$$

defined by

$$\varphi_H(w) = \pi_Z(w, H)$$

Note that the image of $\varphi_H$ is depicted as a green diamond shaped region in the upper plane in Figure 4.3.

We want to show for any $x_1, x_2 \in B_{2,1b}(0) \subset \mathbb{E}^N$

$$d_F(x_1, x_2) = d_Z^R(\varphi_H(x_1), \varphi_H(x_2)).$$

"≤" Equation 170 gives

$$d_F(x_1, x_2) \leq d_Z^S(\varphi_H(x_1), \varphi_H(x_2)).$$

All that remains to show is that

$$d_F(x_1, x_2) \leq d_Z^F(\varphi_H(x_1), \varphi_H(x_2)).$$

By triangle inequality and equation 187 we have that for any $y_1, y_2 \in B_{2,1b}(0),$

$$d_F(x_1, x_2) \leq d_F(x_1, y_1) + d_F(y_1, y_2) + d_F(y_2, x_2)$$

$$\leq d_Z^S(\pi_Z(x_1, H), \pi_Z(y_1, H)) + ||y_1 - y_2||_F$$

$$\leq d_Z^F(\pi_Z(y_2, H), \pi_Z(x_2, H)).$$

By passing to minimum of the right hand side of inequality 189 over all $y_i \in \mathbb{E}^N$, we get 188 as required.

"≥" By the definition of $d_Z^R$ we have

$$d_Z^R(\varphi_H(x_1), \varphi_H(x_2)) = d_Z^R(\pi_Z(x_1, H), \pi_Z(x_2, H))$$

$$= \min\{d_Z^F(\pi_Z(x_1, H), \pi_Z(x_2, H)), d_Z^F(\pi_Z(x_1, H), \pi_Z(x_2, H))\}$$

$$\leq d_Z^F(\pi_Z(x_1, H), \pi_Z(x_2, H))$$

$$\leq ||x_1 - x_2||_F = d_F(x_1, x_2)$$

where the last step follows from taking $y_1 = x_1$ and $y_2 = x_2$ in definition 173.

4.8. Proof that $\varphi_S$ of (153) is distance preserving. Let $\varphi_S$ be the map

$$\varphi_S : (B_{R}^R(x_0), d_R) \to (Z, d_Z^R)$$

defined by

$$\varphi_S(w) = \pi_Z(\pi_R^{-1}(w) \times \{0\})$$

Note that the image of $\varphi_S$ is depicted as a purple region in the lower plane in Figure 4.3. Recall that $\varphi_S$ was distance preserving with respect to $d_Z^S$.

Therefore, by definition 173 it suffices to show that

$$d_Z^S(\varphi_S(w_1), \varphi_S(w_2)) \leq d_Z^F(\varphi_S(w_1), \varphi_S(w_2)).$$
Let \( w_i = \pi_R(x_i) \in \mathcal{B}_R(x_0) \) for some \( x_i \in \mathbb{E}^N \). Let \( p_i = \pi_Z(x_i, 0) \). We wish to show

(195) \[ d_Z^S(p_1, p_2) \leq d_Z^F(p_1, p_2). \]

Intuitively, it means that to go from \( p_1 \) to \( p_2 \), it is better to stay on the 0-plane rather than jumping to the \( H \)-plane and coming back to the 0-plane (Fig. 5).

\[
\text{By definition of } d_Z^F \text{ in equation } 174, \text{ there exists } y_1, y_2 \in \mathbb{E}^N \text{ such that}
\]

(196) \[ d_Z^F(p_1, p_2) = d_Z^S(p_1, q_1) + ||y_1 - y_2||_F + d_Z^S(q_2, p_2) \]

where \( q_i = \pi_Z(y_i, H) \). Write \( q_i = \pi_Z(y_i, 0) \).

Let \( \tilde{p}_i = \pi_Z(\tilde{x}_i, 0) \) be points such that

(197) \[ d_Z^S(p_i, q_i) = d_Z^S(p_i, \tilde{p}_i) + ||(\tilde{x}_i, 0) - (y_i, H)||. \]

By triangle inequality,

(198) \[ d_Z^S(p_1, p_2) \leq d_Z^S(p_1, \tilde{p}_1) + d_Z^S(\tilde{p}_1, \tilde{p}_2) + d_Z^S(\tilde{p}_2, p_2). \]

By equation [196] and [197]

(199) \[ d_Z^F(p_1, p_2) = d_Z^S(p_1, q_1) + ||y_1 - y_2||_F + d_Z^S(q_2, p_2) \]

(200) \[ = d_Z^S(p_1, \tilde{p}_1) + ||(\tilde{x}_1, 0) - (y_1, H)|| + ||y_1 - y_2||_F + ||(y_2, H) - (\tilde{x}_2, 0)|| + d_Z^S(\tilde{p}_2, p_2) \]

Therefore, it is enough to show that

(201) \[ d_Z^S(\tilde{p}_1, \tilde{p}_2) \leq ||(\tilde{x}_1, 0) - (y_1, H)|| + ||y_1 - y_2||_F + ||(y_2, H) - (\tilde{x}_2, 0)||. \]
Notice
\[
d_Z^S(\hat{p}_1, \hat{p}_2) \leq d_Z^S(\bar{p}_1, q_1) + d_Z^S(q_1, q_2) + d_Z^S(q_2, \hat{p}_2) \quad \text{(Triangle Inequality)}
\]
\[
\leq |(\bar{x}_1, 0) - (y_1, 0)| + d_Z^S(q_1, q_2) + |y_2, 0) - (\bar{x}_2, 0)|
\leq |(\bar{x}_1, 0) - (y_1, 0)| + \|y_1 - y_2\|_F + \frac{K}{R} + |(y_2, 0) - (\bar{x}_2, 0)|
\leq |(\bar{x}_1, 0) - (y_1, H)| + \|y_1 - y_2\|_F + \frac{K}{R} + |(y_2, H) - (\bar{x}_2, 0)|
\leq |(\bar{x}_1, 0) - (y_1, H)| + \|y_1 - y_2\|_F + |(y_2, H) - (\bar{x}_2, 0)|
\]
where we use the inequality in theorem \([1.1]\) and the last inequality is an application of \([159]\).

4.9. **Proof of GH convergence as** \( R \to \infty \).

(202) \[ d_{GH}\left((\bar{B}^R_H(x_0), d_R), (F^{-1}([0, r]), d_F)\right) \to 0 \]

We need only show the Hausdorff distances:

(203) \[ d_H(\varphi_S(\bar{B}^R_H(x_0)), \varphi_H(F^{-1}([0, r]))) \leq \delta_R \]

where \( \delta_R \to 0 \).

First we show

(204) \[ \varphi_S(\bar{B}^R_H(x_0)) \subset T_\delta(\varphi_H(F^{-1}([0, r]))) \]

where

(205) \[ \delta = H + K/R \]

Taking any \( x \in \bar{B}^R_H(x_0) \) we know from the set up that there exists

(206) \[ v \in U^R_H(x_0) \subset \{ v : \|v\|_F < r + (K/R) \} \subset B_{2, H}(0) \subset \mathbb{E}^N. \]

such that \( \pi_R(v) = x \). Let

(207) \[ \bar{w} = rv/(r + K/R) \in F^{-1}[0, r] \]

so we have

(208) \[ \|w\|_F < (r/(r + K/R))(r + (K/R)) = r. \]

Thus

(209) \[ d_Z^S(\varphi_S(x), \varphi_F(w)) = d_Z^S(\pi_Z(v, 0), \pi_Z(w, H)) \]

(210) \[ \leq d_Z^S(\pi_Z(v, 0), \pi_Z(v, H)) + d_Z^S(\pi_Z(v, H), \pi_Z(w, H)) \]

(211) \[ \leq H + \|v - w\|_F \]

(212) \[ \leq H + \left(1 - \frac{r}{r + K/R}\right)\|v\|_F \]

(213) \[ \leq H + \left(1 - \frac{r}{r + K/R}\right)(r + K/R) \]

(214) \[ = H + K/R = \delta \]

Next we show
\[(215) \quad \varphi_H(F^{-1}([0, r]) \subset T_\delta(\varphi_S(\bar{B}_R(x_0)))) :\]
Consider any \(v \in F^{-1}([0, r]).\) Let
\[(216) \quad w = \frac{(r - K/R)v}{r} \in F^{-1}[0, r]\]
so we have
\[(217) \quad ||w||_F < \frac{(r - K/R)}{r}r = r - K/R.\]

By the definition of \(K,\) we have
\[(218) \quad |\bar{d}_R(w, 0) - ||w||_F| \leq K/R\]
which implies that
\[(219) \quad \bar{d}_R(w, 0) \leq r.\]
Therefore, \(x = \pi_R(w) \in \bar{B}_R(x_0).\) Thus
\[(220) \quad d^R_Z(\varphi_S(x), \varphi_F(v)) = d^R_Z(\pi_Z(w, 0), \pi_Z(v, H))\]
\[(221) \quad \leq d^R_Z(\pi_Z(w, 0), \pi_Z(w, H)) + d^R_Z(\pi_Z(w, H), \pi_Z(v, H))\]
\[(222) \quad \leq H + ||v - w||_F\]
\[(223) \quad = H + \left(1 - \frac{(r - K/R)}{r}\right)||v||_F\]
\[(224) \quad \leq H + \left(1 - \frac{(r - K/R)}{r}\right)r\]
\[(225) \quad = H + K/R = \delta.\]

Taking \(\delta_R = \delta\) we have
\[(226) \quad d_H(\varphi_S(\bar{B}_R(x_0)), \varphi_H(F^{-1}([0, r]))) \leq \delta_R\]
and
\[(227) \quad \lim_{R \to \infty} \delta_R = \lim_{R \to \infty} (H + K/R)\]
which converges to 0 as \(R \to \infty.\) So
\[(228) \quad d_{GH}(\bar{B}_R(x_0), d_R, (F^{-1}([0, r]), d_F)) \to 0\]

4.10. **Proof of SWIF Convergence as \(R \to \infty.** To prove
\[(229) \quad d_{SWIF}(\bar{B}_R(x_0), d_R, T_R, (F^{-1}([0, r]), d_F, T_F)) \leq M_R \to 0\]
we use the same distance preserving maps, \(\varphi_H\) and \(\varphi_S\) into the same \((Z_R, d^R_Z).\)

So we need only prove there exists integral currents \(A\) and \(B\) such that
\[(230) \quad \partial B + A = \varphi_S#T_X - \varphi_H#T_F.\]
with
\[(231) \quad M(B) + M(A) \leq M_R\]
so that
\[(232) \quad d^{Z}_F(\varphi_{S\#T_X}, \varphi_{H\#T_F}) \leq M(B) + M(A) \leq M_R\]
and we will have our claim.

We begin by defining the current \(B\) which is depicted as the lightly shaded diamond prism in Figure 4.3.

Recall that by the definition of \(Z\) our smocking map
\[(233) \quad \pi_{Z} : B_{2R}(0) \times [0, H] \subset \mathbb{B}^{N+1} \to Z_{R}\]
is surjective and by the definition of \(d^{Z}_{R}\) and \(d^{Z}_{S}\) is distance non-increasing
\[(234) \quad d^{Z}_{R}(\pi_{Z}(v_1, v_2)) \leq d^{Z}_{S}(\pi_{Z}(v_1, v_2)) \leq |v_1 - v_2|.
Thus it is Lipschitz with \(\text{dil}(\pi_Z) \leq 1\). We apply it to define \(B\) using a single chart
\[(235) \quad B = \pi_{Z\#}[F^{-1}([0, r]) \times [0, H]].\]
Then
\[(236) \quad M(B) \leq \text{dil}(\pi_{Z})^N \text{Vol}(F^{-1}([0, r]) \times [0, H]) \leq H \text{Vol}(F^{-1}([0, r])) \leq H(\text{dil}(F^{-1}))^N \omega_{N} r^N.\]
The boundary of \(B\) is
\[(238) \quad \partial B = S_{\text{top}} + S_{\text{bottom}} + S_{\text{around}}\]
where
\[(239) \quad S_{\text{top}} = \pi_{Z\#}[F^{-1}([0, r]) \times \{H\}]\]
\[(240) \quad S_{\text{bottom}} = -\pi_{Z\#}[F^{-1}([0, r]) \times \{0\}]\]
\[(241) \quad S_{\text{around}} = \pi_{Z\#}[F^{-1}(\{r\}) \times [0, H]].\]
Note that
\[(242) \quad S_{\text{bottom}} = \varphi_{H\#T_F} \text{ where } T_F = [F^{-1}([0, r])]\]
is the integral current structure on \(F'([0, r])\) with weight 1.

If \(T_X\) is the standard integral current structure on the rescaled closed ball \(\bar{B}_r(x_0)\) is our smocked metric space, then
\[(243) \quad \varphi_{S\#T_X} = \varphi_{S\#\pi_{X\#}[\pi_{X}^{-1}(\bar{B}_r(x_0))]} = \pi_{Z\#}[\pi_{X}^{-1}(\bar{B}_r(x_0) \times \{0\}).\]
Since this is not \(S_{\text{top}}\) we set
\[(245) \quad A_{\text{top}} = \varphi_{S\#T_X} - S_{\text{top}} = \pi_{Z\#}[U_1 \times \{0\}] - \pi_{Z\#}[U_2 \times \{0\}].\]
where
\[(246) \quad U_1 = \{v \in \mathbb{B}^N : F(v) > r \text{ and } d^{R}_{X}(v, 0)/R < r\}\]
and
\[(247) \quad U_2 = \{v \in \mathbb{B}^N : F(v) < r \text{ and } d^{R}_{X}(v, 0)/R > r\}.
Since \(\text{dil}(\pi_{Z}) \leq 1\) we can estimate the mass
\[(248) \quad M(A_{\text{top}}) \leq \text{Vol}(U_1) + \text{Vol}(U_2).\]
Setting $A = A_{top} - S_{around}$ (which is a disjoint difference) we have
\begin{equation}
M(A) \leq \text{Vol}(U_1) + \text{Vol}(U_2) + H \text{Vol}(F^{-1}(r))
\end{equation}
which gives us
\begin{equation}
\partial B + A = \varphi_S \# T_X - \varphi_H \# T_F.
\end{equation}
Now take
\begin{equation}
M_R = M(A) + M(B)
\end{equation}
\begin{equation}
\leq \text{Vol}(U_1) + \text{Vol}(U_2) + H \text{Vol}(F^{-1}(r)) + H(\text{dil}(F^{-1}))^{N_0} N^r r^N
\end{equation}
We need only show this converges to 0 for fixed $r$ and $F$ as $R \to \infty$.
Recall that by (157)
\begin{equation}
\lim_{R \to \infty} H = \lim_{R \to \infty} \sqrt{4rK/R + (K/R)^2} = 0
\end{equation}
so the first and last terms which are constants multiplied by $H$ converge to 0.
Finally we estimate $\text{Vol}(U_i)$ using the hypotheses to show that
\begin{equation}
U_1 \subset F^{-1}(r, r + (K/R)) \text{ and } U_2 \subset F^{-1}(r - (K/R), r).
\end{equation}
So that
\begin{equation}
\text{Vol}(U_1) + \text{Vol}(U_2) \leq \text{Vol}\left(F^{-1}(r - (K/R), r + (K/R))\right)
\end{equation}
\begin{equation}
\leq 2 (K/R) \text{dil}(F^{-1}) \text{Vol}(F^{-1}(r)) \to 0.
\end{equation}
Thus we have proved intrinsic flat convergence.
This completes the proof of Theorem 1.1
\hfill $\Box$

5. The SWIF Tangent Cones of $X_+, X_0$ and $X_T$

We can now apply Theorem 1.1 to find the SWIF tangent cones of $X_+, X_0$, and $X_T$. Recall that in Remark 2.17 we explained that these are all nice smocked spaces. While it is not easy to find the norm, $\| \cdot \|_F$ for a given smocked metric space, and it requires a length double inductive proof to estimate $\bar{d}_+(x, x')$, this has already been done for these three spaces.

Example 5.1. The SWIF and GH tangent cone at $\infty$ of $X_+$ is a taxicab plane $(\mathbb{R}^2, \| \cdot \|_{F_+})$ where
\begin{equation}
F_+(x) = (|x_1| + |x_2|)/3
\end{equation}
because it was proven by Shanell George, Vishnu Rendla, and Hindy Drillick in [9] that
\begin{equation}
|\bar{d}_+(x, x') - [F_+(x) - F_+(x')]| \leq K \quad \forall x, x' \in \mathbb{B}^N.
\end{equation}

Example 5.2. The SWIF and GH tangent cone at $\infty$ of $X_T$ is also isometric to a taxicab plane, $(\mathbb{R}^2, \| \cdot \|_{F_T})$, where
\begin{equation}
F_T(x) = (|x_1| + |x_2|)/2
\end{equation}
because it was proven by Kazaras, Dinowitz, and Afrifa in [9] that
\begin{equation}
|\bar{d}_T(x, x') - [F_T(x) - F_T(x')]| \leq K \quad \forall x, x' \in \mathbb{B}^N.
\end{equation}
Example 5.3. The SWIF and GH tangent cone at $\infty$ of $X_\omega$ is a normed space whose unit ball is an octagon, $(\mathbb{R}^2, \|\cdot\|_\omega)$, where

$$F_\omega(x) = 2\sqrt{2} \min\{|x_1/3|, |x_2/3|\} + 2\|x_1/3 - |x_2/3|$$

because it was proven by Huynh, Minichiello, and Hepburn in [9] that

$$|\tilde{d}_+(x, x') - [F_\omega(x) - F_\omega(x')]| \leq K \quad \forall x, x' \in \mathbb{E}^N.$$

Example 5.4. Drillick and Mujo discovered a smocked space whose tangent cone at infinity is a normed space whose unit ball is the convex hull of a countable collection of points located where lines of integer slope cross the ellipse \((x/2)^2 + y^2 = 1\). Their smocked space, $X_\omega$, is defined by a periodic lattice of unit length horizontal line segments with left endpoints located at \((2i, j) : i, j \in \mathbb{Z}\) as in [9]. We believe this tangent cone is also a SWIF limit but do not have access to their estimates to prove this. We cite their presentation [3].

Example 5.5. Other sequences of smocked metric spaces are studied in work of Antonetti, Farahzad, and Yamin [2]. We expect their limits should also be SWIF limits but leave the verification to them.

A list of open problems on the various limits of smocked metric spaces has been included at the end of [9] some of which are at the level of undergraduates while others are more advanced and might be tackled by doctoral students. Any work on the construction of three dimensional smocked metric spaces would be of significant interest as such spaces could be applied to better understand the SWIF limits of sequences of manifolds with almost nonnegative scalar curvature [8].

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