On Jensen-Type and Hölder-Type Inequality for Interval-Valued Choquet Integrals

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Abstract

Wang (Journal of Applied Mathematics and Computing, vol. 35, no. 1-2, pp. 305-321, 2011) studied Jensen-type and Hölder-type inequality for Choquet integral. In this paper, we consider the interval-valued Choquet integral with respect to a fuzzy measure and investigate Jensen-type and Hölder-type inequality for interval-valued Choquet integrals.

Keywords: Choquet integral, Jensen-type and Holder-type inequality, Interval-valued function

1. Introduction

The Choquet integrals have been studied by many researchers (see [1-6]). Aumann [7], Jang and his colleagues [8-13], and Zhang et al. [14] also have been studying the interval-valued Choquet integrals which are related with some properties and applications of them. Various integral inequalities, such as Jensen’s inequality, Hölder’s inequality, Minkowski’s inequality, and Chebyshev’s inequality for some integrals were developed by the authors in [3, 5, 15, 16].

The main idea of this paper is to prove two-types integral inequalities of the interval-valued Choquet integral which was defined by Jang and his colleagues [8, 11] and Zhang et al. [14].

In Section 2, we list definitions and basic properties of the Choquet integral as follows:

Definition 2.1 ([2, 4, 5]). Let $X$ be a nonempty set, $\Omega$ be a $\sigma$-algebra of subsets of $X$, and $\mu : \Omega \to [0, \infty)$ be a nonnegative real-valued set function. $\mu$ is said to be a fuzzy measure if

1. $\mu(\emptyset) = 0$;
2. for any $A, B \in \Omega$, $A \subseteq B$ implies $\mu(A) \leq \mu(B)$;

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(3) for \( \{A_n\} \subset \Omega \), \( A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \) implies
\[
\lim_{n \to \infty} \mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \quad \text{(continuity from below)};
\]
(4) for \( \{A_n\} \subset \Omega \), \( A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \) and
\[
\mu(A_1) < \infty \implies \lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcap_{n=1}^{\infty} A_n\right) \quad \text{(continuity from above)}.
\]

Note that \((X, \Omega, \mu)\) is called a fuzzy measure space. Let \(F(X)\) be the set of all real-valued nonnegative measurable functions defined on \(X\).

**Definition 2.2.** Let \((X, \Omega, \mu)\) be a fuzzy measure. \(\mu\) is said to be submodular if
\[
\mu(A \cap B) + \mu(A \cup B) \leq \mu(A) + \mu(B)
\]
for any \(A, B \in \Omega\).

**Definition 2.3.** Let \(f, g \in F(X)\). \(f\) and \(g\) are said to be comonotonic if and only if
\[
f(x) < f(x') \implies g(x) \leq g(x')
\]
for any \(x, x' \in X\).

**Definition 2.4.** Let \((X, \Omega, \mu)\) be a fuzzy measure space, \(f \in F(X)\) and \(A \in \Omega\).

1. The Choquet integral of \(f\) with respect to \(\mu\) on \(A\) is defined by
\[
(C) \int_A f \, d\mu = \int_0^\infty \mu(A \cap \{x \mid f(x) \geq \alpha\}) \, d\alpha.
\]
2. If \((C) \int_X f \, d\mu < \infty\), then \(f\) is called \((C)\)-integrable.
3. \(L_1(\mu)\) is the set of all \((C)\)-integrable functions.

**Theorem 2.1 [2,6,8,14,15].** Let \((X, \Omega, \mu)\) be a fuzzy measure space, \(\{f_1, f_2, f\} \subset F(X)\), \(A, B \in \Omega\) and \(c\) be a nonnegative real number. Then, the Choquet integral has the following properties:

1. If \(\mu(A) = 0\), then \((C) \int_A f \, d\mu = 0\);
2. \((C) \int_A c \, d\mu = c \mu(A)\);
3. If \(f_1 \leq f_2\), then \((C) \int_A f_1 \, d\mu \leq (C) \int_A f_2 \, d\mu\);
4. If \(A \subset B\), then \((C) \int_A f \, d\mu \leq (C) \int_B f \, d\mu\);
5. \((C) \int_A (f + c) \, d\mu = (C) \int_A f \, d\mu + c \cdot \mu(A)\);
6. \((C) \int_A c \cdot f \, d\mu = c \cdot (C) \int_A f \, d\mu\).

**Theorem 2.2 [5].** Let \((X, \Omega, \mu)\) be a fuzzy measure space and \(f, g \in F(X)\).

1. If \(f\) and \(g\) are comonotonic, then for \(A \in \Omega\),
\[
(C) \int_A (f + g) \, d\mu = (C) \int_A f \, d\mu + (C) \int_A g \, d\mu.
\]
2. If \(\mu\) is submodular, then for any \(A \in \Omega\),
\[
(C) \int_A (f + g) \, d\mu \leq (C) \int_A f \, d\mu + (C) \int_A g \, d\mu.
\]

**Theorem 2.3** (Markov-type inequality [5]). Let \((X, \Omega, \mu)\) be a fuzzy measure space and \(f \in F(X)\). If \(c\) is a positive real number, then
\[
\mu(A \cap \{x : f(x) \geq c\}) \leq \frac{1}{c} (C) \int_A f \, d\mu
\]
for any \(A \in \Omega\).

**Theorem 2.4** (Jensen-type inequality [5]). Let \((X, \Omega, \mu)\) be a fuzzy measure space and \(f \in L_1(\mu)\). If \(\mu\) is a regular fuzzy measure and \(\phi : [0, \infty) \to [0, \infty)\) is a convex function, then
\[
\phi\left((C) \int_X f \, d\mu\right) \leq (C) \int_X \phi(f) \, d\mu.
\]

**Theorem 2.5** (Hölder-type inequality [5]). Let \((X, \Omega, \mu)\) be a fuzzy measure and \(f, g \in F(X)\). If \(\mu\) is a submodular fuzzy measure and \(1 < p, q < \infty\) with \(\frac{1}{p} + \frac{1}{q} = 1\), then
\[
(C) \int_X f \, d\mu \leq \left((C) \int_X f^p \, d\mu\right)^\frac{1}{p} \left((C) \int_X g^q \, d\mu\right)^\frac{1}{q}.
\]

## 3. Interval-Valued Choquet Integrals

In this section, we introduce the interval-valued Choquet integral with respect to a fuzzy measure on the space \(IF(X)\) of all interval-valued measurable functions as follows [2,8,14].

A set-valued function is a mapping \(G : X \to p(\mathbb{R}^+\cup \{-\infty\})\). Let \(\mathbb{R}^+ = [0, \infty)\) and
\[
I(\mathbb{R}^+) = \{a = [a^-, a^+] \mid a^- \in \mathbb{R}^+, a^+ \in \mathbb{R}^+\}.
\]

\(IF(X)\) be the space of all interval-valued functions from \(X\) to \(I(\mathbb{R}^+)\).

**Definition 3.1.** If \(\bar{a}, \bar{b} \in I(\mathbb{R}^+)\), \(k \in \mathbb{R}^+\), then we denote the following operations:

1. \(\bar{a} + \bar{b} = [a^- + b^-, a^+ + b^+]\),
2. \(k\bar{a} = [ka^-, ka^+]\),
3. \(\bar{a} \wedge \bar{b} = [a^- \wedge b^-, a^+ \wedge b^+]\),
4. \(\bar{a} \vee \bar{b} = [a^- \vee b^-, a^+ \vee b^+]\),
5. \(\bar{a} \leq \bar{b}\) if and only if \(a^- \leq b^-\) and \(a^+ \leq b^+\),
6. \(\bar{a} \leq \bar{b}\) if and only if \(a^- \leq b^-\) and \(a^+ \leq b^+\),
7. \(\bar{a} \leq \bar{b}\) if and only if \(b^- \leq a^-\) and \(a^+ \leq b^+\).
Definition 3.2. Let $G$ be a set-valued function and $A \in \Omega$. Then the Choquet integral of $G$ on $A$ is defined by

$$(C) \int_A Gd\mu = \left\{ (C) \int_A g d\mu \mid g \in S(G) \right\},$$

where $S(G)$ is the family of $\mu$-a.e. measurable selections of $G$.

Recall that instead of $(C) \int_X Gd\mu$, we write $(C) \int_X Gd\mu$ and a set-valued function $G$ is said to be integral existing if $(C) \int_X Gd\mu \neq \emptyset$, and $G$ is said to be Choquet integrable if $(C) \int Fd\mu$ exists and does not include $\infty$.

Definition 3.3 (9). A set-valued function $G$ is said to be Choquet integrable bounded if there is a Choquet integrable function $h$ such that

$$\|G(x)\| \sup_{r \in G(x)} |r| \leq h(x).$$

Theorem 3.1 (9). If a set-valued function $G$ is Choquet integrably bounded, then $G$ is Choquet integrable.

Theorem 3.2 (9). (1) If $G$ is a Choquet integrable set-valued function and $A, B \in \Omega$ with $A \subset B$, then

$$(C) \int_A Gd\mu \leq (C) \int_B Gd\mu.$$

(2) If $G_1$ and $G_2$ are Choquet integrably bounded set-valued functions and $G_1 \leq G_2$, then

$$(C) \int G_1 d\mu \leq (C) \int G_2 d\mu.$$

(3) If $G$ is a Choquet integrable set-valued function, then

$$(C) \int aGd\mu = a(C) \int Gd\mu$$

for all $a \geq 0$.

Theorem 3.3 (9[14]). If $G = [g^-, g^+] \in IF(X)$ is a measurable and Choquet integrably bounded, then

$$(C) \int Gd\mu = \left[ (C) \int g^- d\mu, (C) \int g^+ d\mu \right].$$

4. Three-TYPES Inequalities for the Interval-Valued Choquet Integrals

In this section, we investigate Markov-type inequality, Jensen-type inequality and H"older-type inequality for the interval-valued Choquet integrals. By Definition 3.3, we have that if $G = [g^-, g^+] \in IF(X)$ and $G$ is Choquet integrably bounded, then there exists Choquet integrable function $h$ such that $g^- \leq g^+ \leq h$ and

$$(C) \int Gd\mu = \left[ (C) \int g^- d\mu, (C) \int g^+ d\mu \right] \in I(\mathbb{R}^+).$$

Theorem 4.1 (Markov-type inequality for the interval-valued Choquet integrals). Let $(X, \Omega, \mu)$ be a fuzzy measure space. If $G \in IF(X)$ is integrably bounded and $c$ is a positive real number, then

$$\mu \left( A \cap \{ x \mid G(x) \geq c \} \right) \leq \frac{1}{c} (C) \int_A Gd\mu.$$

Proof. Note that if $G(x) = [g^-(x), g^+(x)] \geq c$, then $g^-(x) \geq c$ and $g^+(x) \geq c$. By Theorem 2.3, we get

$$\mu \left( A \cap \{ x \mid g^-(x) \geq c \} \right) \leq \frac{1}{c} (C) \int_A g^- d\mu, \tag{1}$$

and

$$\mu \left( A \cap \{ x \mid g^+(x) \geq c \} \right) \leq \frac{1}{c} (C) \int_A g^+ d\mu, \tag{2}$$

for any $A \in \Omega$. Since $g^-(x) \leq g^+(x)$ and $\mu$ is monotonic, we get

$$\mu( A \cap \{ x \mid g^-(x) \geq c \} ) \leq \mu( A \cap \{ x \mid g^+(x) \geq c \} ). \tag{3}$$

By Eqs. (1), (2), (3), and Theorem 2.3, we get

$$\mu( A \cap \{ x \mid G(x) = [g^-(x), g^+(x)] \geq c \} ) \leq \mu( A \cap \{ x \mid g^-(x) \geq c \} ) \leq \mu( A \cap \{ x \mid g^-(x) \geq c \}, \mu( A \cap \{ x \mid g^+(x) \geq c \} ) ) \leq \left[ \frac{1}{c} (C) \int_A g^- d\mu, \frac{1}{c} (C) \int_A g^+ d\mu \right]$$

$$= \frac{1}{c} \left[ (C) \int_A g^- d\mu, (C) \int_A g^+ d\mu \right]$$

$$= \frac{1}{c} (C) \int_A Gd\mu. \quad \square$$

We denote that

$$IL_1(\mu) = \left\{ G \in IF(X) \mid G = [g^-, g^+] \text{ and } g^-, g^+ \in L_1(\mu) \right\}.$$
is called a convex function if and only if $\phi^-$ and $\phi^+$ are convex functions.

**Theorem 4.2** (Jensen-type inequality for the interval-valued Choquet integrals). Let $(X, \Omega, \mu)$ be a fuzzy measure space. If $G \in IL_1(\mu)$ is Choquet integrably bounded, $\mu$ is a regular fuzzy measure and

$$\Phi = [\phi^-, \phi^+] : [0, \infty) \times [0, \infty) \to [0, \infty) \times [0, \infty)$$

by $\Phi(u, v) = [\phi^-(u), \phi^+(v)]$ is a convex function, then

$$\Phi \circ \left( (C) \int_X Gd\mu \right) \leq (C) \int_X \Phi \circ Gd\mu,$$

where $\Phi \circ [a, b] = [\phi^-(a), \phi^+(b)]$ and

$$\Phi \circ G = [\phi^-(g^-), \phi^+(g^+)].$$

**Proof.** Since $G \in IL_1(\mu)$, $g^-, g^+ \in L_1(\mu)$ and $\phi^-, \phi^+$ are convex functions. By Theorem 2.4, we get

$$\phi^- \left( (C) \int_X g^-d\mu \right) \leq (C) \int_X \phi^-(g^-)d\mu,$$

and

$$\phi^+ \left( (C) \int_X g^+d\mu \right) \leq (C) \int_X \phi^+(g^+)d\mu.$$

By Eqs. (4), (5), and Theorem 2.4, we get

$$\Phi \circ \left( (C) \int_X Gd\mu \right) = \Phi \circ \left( \left( (C) \int_X g^-d\mu, (C) \int_X g^+d\mu \right) \right) \leq \left[ \phi^- \left( (C) \int_X g^-d\mu \right), \phi^+ \left( (C) \int_X g^+d\mu \right) \right] \leq \left[ (C) \int_X \phi^-(g^-)d\mu, (C) \int_X \phi^+(g^+)d\mu \right] = (C) \int_X [\phi^-(g^-), \phi^+(g^+)]d\mu = (C) \int_X \Phi \circ Gd\mu.$$

**Theorem 4.3** (Hölder-type inequality for the interval-valued Choquet integrals). Let $(X, \Omega, \mu)$ be a fuzzy measure and $G = [g^-, g^+]$, $H = [h^-, h^+] \in IF(X)$ be Choquet integrably bounded. If $\mu$ is a submodular fuzzy measure and $1 < p,

$$\int_X G^p d\mu \leq \left( \int_X (g^-)^p d\mu \right)^{\frac{1}{p}} \left( \int_X (h^-)^q d\mu \right)^{\frac{1}{q}},$$

$$\int_X H^q d\mu \leq \left( \int_X (g^+)^p d\mu \right)^{\frac{1}{p}} \left( \int_X (h^+)^q d\mu \right)^{\frac{1}{q}},$$

where $GH = [g^-h^-, g^+h^+]$, $G^p = [g^-p, g^+p]$, and $H^p = [h^-p, h^+p]$.

**Proof.** Note that if $G$ and $H$ are Choquet integrably bounded then $GH$ is Choquet integrably bounded. Since $GH = [g^-h^-, g^+h^+]$, by Theorem 3.3, we have

$$\int_X GH d\mu = \left[ \int_X g^-h^- d\mu, \int_X g^+h^+ d\mu \right].$$

By Theorem 2.5, we have

$$\int_X g^-h^- d\mu \leq \left( \int_X (g^-)^p d\mu \right)^{\frac{1}{p}} \left( \int_X (h^-)^q d\mu \right)^{\frac{1}{q}},$$

and

$$\int_X g^+h^+ d\mu \leq \left( \int_X (g^+)^p d\mu \right)^{\frac{1}{p}} \left( \int_X (h^+)^q d\mu \right)^{\frac{1}{q}}.$$
\[(C) \int G^p d\mu \right)^\frac{1}{p} \left( (C) \int H^q d\mu \right)^\frac{1}{q}. \]

\[ \square \]

**Conflict of Interest**

No potential conflict of interest relevant to this article was reported.

**References**

[1] L. C. Jang, “A note on Jensen type inequality for Choquet integrals,” *International Journal of Fuzzy Logic and Intelligent Systems*, vol. 9, no. 2, pp. 71-75, 2009. [https://doi.org/10.5391/IJFIS.2009.9.2.071](https://doi.org/10.5391/IJFIS.2009.9.2.071)

[2] R. J. Aumann, “Integrals of set-valued functions,” *Journal of Mathematical Analysis and Applications*, vol. 12, no. 1, pp. 1-12, 1965.

[3] L. C. Jang, “On properties of the Choquet integral of interval-valued functions,” *Journal of Applied Mathematics*, vol. 2011, article ID. 492149, 2011. [https://doi.org/10.1155/2011/492149](https://doi.org/10.1155/2011/492149)

[4] L. C. Jang, “A note on convergence properties of interval valued capacity functionals and Choquet integrals,” *Information Sciences*, vol. 183, no. 1, pp. 151-158, 2012. [https://doi.org/10.1016/j.ins.2011.09.011](https://doi.org/10.1016/j.ins.2011.09.011)

[5] D. Zhang, C. Guo, and D. Liu, “Set-valued Choquet integrals revisited,” *Fuzzy Sets and Systems*, vol. 147, no. 3, pp. 475-485, 2004. [https://doi.org/10.1016/j.fss.2004.04.005](https://doi.org/10.1016/j.fss.2004.04.005)

[6] H. Agahi, Y. Ouyang, R. Mesiar, E. Pap, and M. Stroboja, “Holder and Minkowski type inequalities for pseudo-integral,” *Applied Mathematics and Computation*, vol. 217, no. 21, pp. 8630-8639, 2011. [https://doi.org/10.1016/j.amc.2011.03.100](https://doi.org/10.1016/j.amc.2011.03.100)

[7] G. Choquet, “Theory of capacities,” *Annales de l’institut Fourier*, vol. 5, pp. 131-295, 1954. [https://doi.org/10.5802/aif.53](https://doi.org/10.5802/aif.53)

[8] B. Girotto and S. Holzer, “Chebyshev type inequality for Choquet integral and comonotonicity,” *International Journal of Approximate Reasoning*, vol. 52, no. 8, pp. 1118-1123, 2011. [https://doi.org/10.1016/j.ijar.2011.06.001](https://doi.org/10.1016/j.ijar.2011.06.001)

[9] T. Murofushi and M. Sugeno, “Some quantities represented by Choquet integral,” *Fuzzy Sets and Systems*, vol. 56, no. 2, pp. 229-235, 1993. [https://doi.org/10.1016/0165-0114(93)90148-B](https://doi.org/10.1016/0165-0114(93)90148-B)

[10] J. Wood and L. C. Jang, “A study on the Choquet integral with respect to a capacity and its applications,” *Global Journal of Pure and Applied Mathematics*, vol. 12, no. 2, pp. 1593-1599, 2016.

[11] L. C. Jang, B. M. Kil, Y. K. Kim, and J. S. Kwon, “Some properties of Choquet integrals of set-valued functions,” *Fuzzy Sets and Systems*, vol. 91, no. 1, pp. 95-98, 1997. [https://doi.org/10.1016/S0165-0114(96)00124-8](https://doi.org/10.1016/S0165-0114(96)00124-8)

[12] L. C. Jang and J. S. Kwon, “On the representation of Choquet integrals of set-valued functions and null sets,” *Fuzzy Sets and Systems*, vol. 112, no. 2, pp. 233-239, 2000. [https://doi.org/10.1016/S0165-0114(98)00184-5](https://doi.org/10.1016/S0165-0114(98)00184-5)

[13] L. C. Jang, “The application of interval-valued Choquet integrals in multicriteria decision aid,” *Journal of Applied Mathematics and Computing*, vol. 20, no. 1-2, pp. 549-556, 2006.

[14] H. Roman-Flores, A. Flores-Franulic, and Y. Chalco-Cano, “A Jensen type inequality for fuzzy integral,” *Information Sciences*, vol. 177, no. 15, pp. 3192-3201, 2007. [https://doi.org/10.1016/j.ins.2007.02.006](https://doi.org/10.1016/j.ins.2007.02.006)

[15] R. S. Wang, “Some inequalities and convergence theorems for Choquet integrals,” *Journal of Applied Mathematics and Computing*, vol. 35, no. 1-2, pp. 305-321, 2011. [https://doi.org/10.1007/s12190-009-0358-y](https://doi.org/10.1007/s12190-009-0358-y)

[16] L. C. Jang, “A note on the pseudo Stolarsky type inequality for the g-integral,” *Applied Mathematics and Computation*, vol. 269, pp. 809-815, 2015. [https://doi.org/10.1016/j.amc.2015.07.087](https://doi.org/10.1016/j.amc.2015.07.087)

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