A Proof Technique for Skewness of Graphs

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Abstract

The skewness of a graph $G$ is the minimum number of edges in $G$ whose removal results in a planar graph. By appropriately introducing a weight to each edge of a graph, we determine, among other thing, the skewness of the generalized Petersen graph $P(4k, k)$ for odd $k \geq 9$. This provides an answer to the conjecture raised in [3].

Let $G$ be a graph. The skewness of $G$, denoted $\mu(G)$, is defined to be the minimum number of edges in $G$ whose removal results in a planar graph. Skewness of graph was first introduced in the 1970's (see [6] - [8] and [11]). It was further explored in [5], [9] and [10]. More about the skewness of a graph can be found in [2] and [4].

Let $n$ and $k$ be two integers such that $1 \leq k \leq n - 1$. Recall that the generalized Petersen graph $P(n, k)$ is defined to have vertex-set $\{u_0, u_1, \ldots, u_{n-1}\}$ and edge-set $\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \ldots, n-1\}$ with the operations reduced modulo $n$.

Earlier, the authors in [3] showed that $\mu(P(4k, k)) = k + 1$ if $k \geq 4$ is even. In the same paper, they conjectured that $\mu(P(4k, k)) = k + 2$ if $k \geq 5$ is odd. We shall prove that the conjecture is true for $k \geq 9$.

We shall first describe a family of graphs which is very much related to the generalized Petersen graph $P(sk, k)$.

Let $k$ and $s$ be two integers such that $k \geq 1$ and $s \geq 3$. The graph $Q_s(k)$ is defined to have vertex-set $\{0, 1, \ldots, sk - 1, x_0, x_1, \ldots, x_{k-1}\}$ and edge-set $\{i(i+1), jx_j, (j+k)x_j, (j+2k)x_j, \ldots, (j+(s-1)k)x_j : i = 0, 1, \ldots, sk - 1, j = 0, 1, \ldots, k - 1\}$ with the operations reduced modulo $sk$ and those on the subscripts reduced modulo $k$. 
Drawings for the graphs $Q_3(k)$ and $Q_4(k)$ can be found in the papers [1] and [3] respectively. A drawing for the graph $Q_5(8)$ is depicted in Figure 1.

![Figure 1: A drawing of the graph $Q_5(8)$.](image)

Throughout this paper, if $G$ has skewness $r$, then we let $\mathcal{R}(G)$ denote a set of $r$ edges in $G$ whose removal results in a planar graph.

In [1] and [3] it was shown that $\mu(Q_3(k)) = \lceil k/2 \rceil + 1$ and $\mu(Q_4(k)) = k + 1$ respectively. A more general result is now established by employing a proof technique that has not been used before.

Suppose $G$ is a weighted graph and $e$ is an edge of $G$ with weight $w(e)$. If $H$ is any subgraph (proper or improper) of $G$, we let $W(H) = \sum_{e \in E(H)} w(e)$ denote the weight of $H$. In particular, if $G$ is a plane graph and $F$ is an
face of $G$, we let $W(F) = \sum_{e \in E(F)} w(e)$ denote the weight of $F$ in $G$.

**Theorem 1** $\mu(Q_s(k)) = \lceil (s - 2)k/2 \rceil + 1$ if $k \geq 4$.

**Proof:** Let $e$ be an edge of $Q_s(k)$ and let

$$w(e) = \begin{cases} 
2 & \text{if } e = i(i + 1), \\
k - 2 & \text{otherwise}
\end{cases}$$

for $i = 0, 1, \ldots, sk - 1$. Then $W(Q_s(k)) = sk^2$.

Hence, if $J$ is a graph obtained from $Q_s(k)$ by deleting a set of $t$ edges, then $W(J) \leq sk^2 - 2t$ (because $k - 2 \geq 2$).

Let $H$ denote a planar graph obtained from $Q_s(k)$ by deleting a set of $t' = \mu(Q_s(k))$ edges. Then the number of faces in $H$ is $sk - k - t' + 2$. It is easy to see that $W(F) \geq 4k - 4$ for any face $F$ in $H$. Since each edge is contained in 2 faces, we have the following inequality

$$2(sk^2 - 2t') \geq (4k - 4)(sk - k - t' + 2)$$

which gives $t' \geq (s - 2)k/2 + 1$. This proves the lower bound.

To prove the upper bound, we show the existence of a spanning planar subgraph $H_s(k)$ obtained by deleting a set of $\lceil (s - 2)k/2 \rceil + 1$ edges from $Q_s(k)$.

Figures 2, 3 and 4 depicts three drawings of $H_s(k)$ according to the parities on $s$ and $k$. When $k$ is even, the set of edges that have been deleted from $Q_s(k)$ is given by $(2i - 1)(2i), i = k/2, k/2 + 1, \ldots, k/2 + k(s - 2)/2$ (see Figure 2). Note also that if the 9 thick edges in the graph $Q_5(8)$ (as depicted in Figure 1) are deleted, we obtain the graph $H_5(8)$.

When $k$ is odd, the set of edges that have been deleted from $Q_s(k)$ is given by $(sk - 1)0, (2i - 1)(2i), i = (k - 1)/2 + 1, (k - 1)/2 + 2, \ldots, (k - 1)/2 + \lceil k(s - 2)/2 \rceil$ (see Figures 3 and 4).

This completes the proof. \hfill $\square$

**Theorem 2** $\mu(P(4k, k)) = k + 2$ if $k \geq 9$ is odd.

**Proof:** The upper bound $\mu(P(4k, k)) \leq k + 2$ for odd $k \geq 5$ was established in [3]. Hence we just need to show that $\mu(P(4k, k)) \geq k + 2$ for odd $k \geq 9$.

First we shall show that $\mu(P(4k, k)) \geq k + 1$ if $k \geq 9$ is odd. The proof presented here employed a new technique (which involves assigning
appropriate weights to its edges) and hence is different from the one given in [3].

Let \( e \) be an edge of \( P(4k, k) \) and let

\[
w(e) = \begin{cases} 
4 & \text{if } e = u_iu_{i+1}, \\
k - 3 & \text{if } e = u_iv_i, \\
2k - 2 & \text{if } e = v_iv_{i+k} 
\end{cases}
\]

for \( i = 0, 1, \ldots, n - 1 \). Then \( W(P(4k, k)) = 4k(3k - 1) \). Moreover, it is easy to see that, for any cycle \( C \) in \( P(4k, k) \), \( W(C) \geq 8k - 8 \) and that equality holds if and only if \( C \) is any of the following types.

(i) \( u_iu_{i+1}u_{i+2} \ldots u_{i+k-1}u_{i+k}v_{i+k}v_iu_i \),
(ii) \( u_iu_{i+1}v_{i+1}v_{i+k+1}u_{i+k+1}u_{i+k}v_iu_i \),
(iii) \( u_iu_{i+1}v_{i+1}v_{i-k-1}u_{i-k-1}u_{i-k}v_{i-k}v_iu_i \),
(iv) \( v_iv_{i+k}v_{i+2k}v_{i+3k}v_i \).

Let \( H \) denote a planar graph obtained by deleting a set of \( t = \mu(P(4k, k)) \) edges from \( P(4k, k) \). Then \( W(H) \leq 4k(3k - 1) - 4t \) (since \( k \geq 9 \) implies that \( k - 3 > 4 \) and \( 2k - 2 > 4 \)).

From Euler’s formula for plane graph, we see that the number of faces in \( H \) is \( 4k - t + 2 \). Since each edge is contained in only 2 faces, we have the following inequality

\[ 2(4k(3k - 1) - 4t) \geq (4k - t + 2)(8k - 8) \]

which gives \( t \geq k + 1 \).

Now suppose \( \mu(P(4k, k)) = k + 1 \). The fact that equality is tight implies the following.

(a) Only faces of the types (i) to (iv) are found in the planar graph \( H \).
(b) The tight equality \( 4k(3k - 1) - 4t \) implies that \( R(P(4k, k)) \) consists of edges of the form \( u_iu_{i+1} \).
(c) Since \( P(4k, k) \) is a 3-regular graph, removing any two adjacent edges yields a pendant vertex, resulting in \( W(F) > 8k - 8 \). Hence we conclude that \( R(P(4k, k)) \) consists of only independent edges of the form \( u_iu_{i+1} \).

Now for any edge \( e \) of \( P(4k, k) \), let

\[
w'(e) = \begin{cases} 
0 & \text{if } e = u_iu_{i+1}, \\
1 & \text{if } e = u_iv_i, \\
2 & \text{if } e = v_iv_{i+k} 
\end{cases}
\]
for \( i = 0, 1, \ldots, n - 1 \). We note that, if \( F \) is a face of \( H \), then \( W'(F) = \sum_{e \in E(F)} w'(e) \) is equal to 4 if \( F \) is of type (i) and \( W'(F) \) is equal to 8 if \( F \) is of type (ii), (iii) or (iv).

Let \( x \) be the number of faces of type (i) in \( H \) and let \( \mathcal{F}_H \) denote the set of all faces in \( H \). Then \( \sum_{F \in \mathcal{F}_H} W'(F) = 4x + 8(3k + 1 - x) \). Since \( \mathcal{R}(P(4k, k)) \) consists of only independent edges of the form \( u_iu_{i+1} \), we see that \( W'(H) = 12k \). As such, we have

\[
4x + 8(3k + 1 - x) = 2(12k)
\]

which gives \( x = 2 \).

Now let the two faces of type (i) be

\[
F_1 = u_yu_{y+1}u_{y+2} \ldots u_{y+k-1}u_{y+k}u_y \quad \text{and} \quad F_2 = u_zu_{z+1}u_{z+2} \ldots u_{z+k-1}u_{z+k}u_z.
\]

Let \( A = \{ y + 1, y + 2, \ldots, y + k - 1 \} \cap \{ z + 1, z + 2, \ldots, y + k - 1 \} \). We assert that \( A = \emptyset \).

Suppose \( h \in A \). Then the edges \( u_{h-1}u_h, u_hu_{h+1} \) are in \( F_1 \cup F_2 \) implying that \( u_hv_h \in \mathcal{R}(P(4k, k)) \), a contradiction.

Call a vertex \( u_i \) of \( P(4k, k) \) a good vertex if all edges incident to it are not in \( \mathcal{R}(P(4k, k)) \); otherwise it is called a bad vertex. Since \( \mathcal{R}(P(4k, k)) \) consists of independent edges of the form \( u_iu_{i+1} \), the number of bad vertices is \( 2\rho(P(4k, k)) \).

Since \( y + 1, y + 2, \ldots, y + k - 1, z + 1, z + 2, \ldots, y + k - 1 \) are distinct, and since \( F_1 \) and \( F_2 \) share at most one common edge (of the form \( u_nv_1 \)) in the planar graph \( H \), the number of good vertices in \( H \) is at least \( 2k - 2 \) (given by the distinct vertices \( u_{y+1}, u_{y+2}, \ldots, u_{y+k-1}, u_{z+1}, u_{z+2}, \ldots, u_{z+k-1} \)) and at most \( 2k - 1 \) (if \( F_1 \) and \( F_2 \) have an edge in common). Hence \( H \) has at least \( 2k + 1 \) bad vertices which are either consecutive vertices of the form \( u_{n}, u_{n+1}, \ldots, u_{n+2k-1} \) or are separated into 2 sets of consecutive vertices by \( V(F_1) \cup V(F_2) \). In any case, we can find at least \( k + 1 \) consecutive bad vertices. Since we can relabel the vertices if necessary, we may assume without loss of generality that these vertices are \( u_0, u_1, \ldots, u_k \), and that \( \mathcal{R}(P(4k, k)) \) contains \( u_0u_1, u_2u_3, \ldots, u_{k-1}u_k \).

Let \( J \) denote the subgraph obtained from \( P(4k, k) \) by deleting \( u_1, u_2, \ldots, u_{k-1}, v_1, v_2, \ldots, v_{k-1} \) together with the edge \( v_{3k}v_k \). Then \( J \) is a subdivision of \( Q_3(k) \). Note that the vertices of degree-2 in \( J \) are \( u_0, v_0, u_k, v_{k+1}, v_{k+2}, \ldots, v_{2k-1}, v_{4k-1}, v_{4k-2}, \ldots, v_{3k} \). When these degree-2 vertices are suppressed, the resulting graph is isomorphic to \( Q_3(k) \) with \( v_ku_{k+1}u_{k+2} \ldots u_{4k-1}v_k \) playing role of the 3k-cycle in \( Q_3(k) \).
Recall that \( \mu(Q_3(k)) = (k + 3)/2 \). The preceding arguments imply that

\[
\mu(P(4k, k)) \geq |\{u_0u_1, u_2u_3, \ldots, u_{k-1}u_k\}| + \mu(Q_3(k)) = k + 2
\]

which is a contradiction. Hence \( \mu(P(4k, k)) = k + 2 \).

\[\square\]

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Figure 2: A drawing of the graph $H_s(k)$, $k \geq 4$ is even.
Figure 3: A drawing of the graph $H_s(k)$, $k \geq 5$ is odd and $s \geq 4$ is even.
Figure 4: A drawing of the graph $H_s(k)$, $k \geq 5$ is odd and $s \geq 3$ is odd.