**Abstract.**  MHD Turbulence is a critical component of the current paradigms of star formation, particle transport, magnetic reconnection and evolution of the ISM. Progress on this difficult subject is made via numerical simulations and observational studies. However, due to limitations of resolution, scale discrepancies, and complexity of the observations, the best approach for connecting numerics to observations is not always obvious. Here we advocate for an approach that invokes statistical techniques to understand the underlying physics of turbulent astrophysical systems. The wealth of numerical and observational data calls for new statistical tools to be developed in order to study turbulence in the interstellar medium. We briefly review some of the recently developed statistics that focus on characterizing gas compressibility and magnetization and their uses to interstellar studies.

1. Introduction

The paradigm of the interstellar medium has undergone major shifts in the past two decades thanks to the combined efforts of high resolution surveys and the exponential increase in computation power allowing for more realistic numerical simulations. The ISM is now known to be highly turbulent and magnetized, which affects ISM structure, formation, and evolution.

To this date, turbulence has always been understood in a statistical manner - showing the ‘order from chaos.’ In the ISM in particular, statistical studies have proven to be important in characterizing the properties of the magnetized turbulent ISM (Lazarian 2009). Indeed, a statistical description is necessary in any situation where the subject of turbulence arises, as it allows one to study the underlying regularities of the fluid motion (Frisch 1995).

The most common “go-to” statistical tool for both observers and theorist alike is the spatial power spectrum. In fact, most of the attempts to relate observations to models has been by obtaining the spectral index (i.e. the log-log slope of the power spectrum) of column density and velocity. Although the power spectrum is useful for obtaining information about energy transfer over scales, it does not provide a full picture of turbulence. This is partially because it only contains information on Fourier amplitudes and neglects information in the phases. In light of this, many other techniques have been developed to study and parametrize observational magnetic turbulence. For astrophysical settings in particular, these statistical studies have found their uses in the comparison of observations with models of turbulence. These statistics include probability density functions (PDFs), wavelets, the principal component analysis, higher order moments,
Tsallis statistics, spectrum and bispectrum (Brunt & Heyer 2002; Kowal, Lazarian & Beresnyak 2007, Burkhart et al. 2009, Esquivel & Lazarian 2010, Toffelmire, Burkhart, & Lazarian 2011). Wavelets methods, such as the $\Delta$-variance method, have also been shown to be very useful in characterizing structure in data (see Ossenkopf et al. 2008a).

In general, the best strategy for studying a difficult subject like interstellar turbulence is to use a synergetic approach, combining theoretical knowledge, numerical simulations, and observational data via statistical studies. In this way one can obtain the most complete and reliable picture of the physics of turbulence. Here we seek to extend the statistical comparison between numerical and observational turbulence by reviewing statistical tools that can greatly complement the information provided by the power spectrum. In particular we focus the review on tools that can provide information on sonic and Alfvénic Mach numbers. We understand the sonic Mach number to be defined as $M_s \equiv v/C_s$, and the Alfvénic Mach number $M_A \equiv v/v_A$, where $v_A = |B|/\sqrt{\rho}$ is the Alfvénic velocity, $B$ is magnetic field and $\rho$ is density. These parameters are not always easy to characterize observationally, with the Alfvénic Mach number being particularly difficult due to cumbersome observational measurements of vector magnetic field.

This review will highlight several different tools (see Figure 1) studied in the works of Kowal et al. 2007, Burkhart et al. 2009, 2010, 2011, Esquivel et al. 2010, and Toffelmire et al. 2011 which represent a mixture of numerical and observational studies. We focus on statistics that have application for observable data, in this case, particularly PPV or column density.

2. The Sonic Mach Number

The sonic Mach number describes the ratio of the flow velocity to the sound speed, and thus is a measure of the compressibility of the medium. Turbulence that is supersonic displays very different characteristics from subsonic turbulence in terms of the
Figure 2. Uses of PDFs for characterizing turbulence. Top row: Skewness vs. sonic Mach number for six different simulations of MHD turbulence. Similar trends are seen in kurtosis. Dashed line is a linear best fit. From Burkhart et al. 2010

Bottom row: Tsallis of simulations with $M_s = 10, 7, 4, 0.7$ (top left to bottom right respectively). In this case the field $G(x)$ as shown in equation 2 is 3D density. We plot the Tsallis fit parameter $w$ vs. lag. Sub-Alfvénic simulations are denoted with red diamonds while super-Alfvénic simulations are denoted with blue triangles. From Toffelmire et al. 2011.

spectral slope and density/velocity fluctuations. Because the physical environment of compressible turbulence is very different from incompressible, this parameter is extremely important for many different fields of astrophysics including, but not limited to, star formation and cosmic ray acceleration.

### 2.1. Higher Order Moments of Column Density

Moments of the density distribution can be used to roughly determine the gas compressibility through shock density enhancements. As the ISM media transitions from subsonic to supersonic and becomes increasingly supersonic, shocks create enhanced intensity, which is reflected in the mean value and variance of the density and column density PDFs. In addition, as the shocks become stronger, the PDF is skewed and becomes more kurtotic than Gaussian. Thus, one can apply very simple statistical descriptors such as skewness (see Figure 2, top) and kurtosis to astrophysical maps and gain insight into the compressibility of the region. However, Kowal et al. 2007 showed that this method is not very effective for subsonic cases, as these distributions are roughly Gaussian. For areas where Mach numbers approach and exceed unity, higher order moments of column density PDFs can be used as a measure of compressibility.
Figure 3. The amplitude of the bispectrum for the scaled simulated column density. The left plot shows a subsonic model, while the right plot is for a supersonic model. Both models have $M_A=0.7$. These figures show the degree of correlation between wavenumbers $k_1$ and $k_2$. The supersonic model has higher bispectral amplitudes, and more circular isocontours, therefore a stronger correlation between wave modes. From Burkhart et al. 2010

2.2. Bispectrum

While the power spectrum has been used extensively in ISM studies, higher order spectrum have been more rare. The bispectrum, or Fourier transform of the 3rd order autocorrelation function, has been applied to isothermal ISM turbulence simulations and the SMC only recently (Burkhart et al. 2009, 2010) although it is extensively used in other fields including cosmology and biology.

The bispectrum preserves both the amplitude and phase and provides information on the interaction of wave modes. Completely randomized modes will show a bispectrum of zero, while mode coupling will show non-zero bispectrum. Shocks and high magnetic field have been shown to increase mode coupling in the bispectrum (Burkhart et al. 2009, 2010). The bispectrum shows a particular sensitivity to picking out shocks in the medium (see Figure 3). Due to their ability to shallow out the density energy spectrum, shocks greatly enhance the small scale wave-wave coupling.

3. Magnetization of Turbulence

The Alfvén number is the dimensionless ratio of the flow velocity to the Alfvén speed. As the Alfvén speed depends on the magnetic field, this ratio can provide information on the strength of the magnetic field relative to the velocity and density. The Alfvénic number is critical in several fields including interplanetary studies and star formation. The solar wind is known to be a super-Alfvénic flow while the Alfvénic number in star forming regions is still hotly debated.

3.1. Tsallis PDFs of PPV and Column Density

PDFs of increments (of density, magnetic field, velocity etc.) are a classic way to study turbulence since the phenomena is scale dependent. The Tsallis function was formulated in Tsallis 1988 as a means to extend traditional Boltzmann-Gibbs mechanics to fractal and multifractal systems.
Figure 4. Moments of the dendrogram tree (leaves + branches) vs. $M_s$ for eight different simulations spanning a range of sonic numbers from 0.5 to 10. Here we have chosen $\delta=4$. Each sonic number is divided into sub-Alfvénic or super-Alfvénic. Panels show mean, variance, skewness and kurtosis of the distribution. Colors indicate different Alfvén number. From Burkhart et al. 2011.

The Tsallis distribution (Equation 1) can be fit to PDFs of increments, that is,

$$f(x, r) = G(x + r) - G(x)$$  \hspace{1cm} (2)$$

where $G(x)$ is a particular field (for example, turbulent density, velocity or magnetic field) and $r$ is the lag. The Tsallis fit parameters ($q$, $a$, and $w$ in Equation 1) describe the width, amplitude, and tails of the PDF. These parameters have been shown to have dependencies on both sonic and Alfvénic Mach number, however we include Tsallis in this section because it is highly sensitive to the Alfvénic Mach number of turbulence. Tsallis parameters are able to distinguish between subsonic, transsonic and supersonic turbulence as well as gauge whether the turbulence is sub-Alfvénic or super-Alfvénic. We provide an example of the fit parameter $w$ (see Equation 1) in Figure 2 bottom row, taken from Tofflemire, Burkhart, & Lazarian 2011. This parameter describes the width of the PDF and is particularly sensitive to both the Alfvénic and sonic Mach numbers. The amplitude parameter (not shown) $a$ is sensitive to the sonic Mach number, while the kurtotic $q$ parameter (not shown) is not sensitive to either of the Mach numbers.

3.2. Dendrograms of Position-Position-Velocity (PPV) data

The ISM is fractal in nature (Stutzki 1998), and as such, the observed gas structures are often hierarchical. This is especially true in the case of molecular gas or where the sonic number is high and dense filaments develop. A Dendrogram is a hierarchical tree diagram that has been used extensively in other fields, particular galaxy evolution and biology. It is a graphical representation of a branching diagram, and for our particular purposes with PPV data, quantifies how and where local maxima of emission merge.
with each other. The dendrogram was first used on ISM data in Rosolowsky et al. 2008 and Goodman et al. 2009 in order to characterize self-gravitating structures in star forming molecular clouds. The dendrogram picks out local maxima in the data based on a user set threshold value \( \delta \), then contours the data and travels down the contours till it finds a level they merge at.

Burkhart et al. 2011 used the dendrogram on synthetic PPV cubes (all normalized to unity, which makes the threshold parameter \( \delta \) easier to interpret) and found it to be rather sensitive to magnetic density/velocity enhancements. In particular, they studied how the moments of the tree diagram distribution vary with the sonic and Alfvénic Mach number and level of self-gravity. These moments showed clear signs of being dependent on Mach numbers (see Figure 4 for an example with \( \delta = 4 \)). In addition to being sensitive to the Mach numbers of turbulence the dendrogram is also able to distinguish between simulations that show varying degrees of gravitational strength.

4. Conclusions

The last decade has seen major increases in the knowledge of the ISM and of its turbulent nature thanks to high resolution observations and advanced numerical simulations. This calls for new advances in statistical tools in order to best utilize the wealth of observational data in light of numerical and theoretical predictions. Recently several authors have explored new tools for studying turbulence beyond the power spectrum. While these proceedings do not cover nearly all the useful tools in the literature, we attempt to provide some review on tools that describe the gas compressibility and the Alfvénic Mach number by utilizing density fluctuations created by shocks and magnetic density enhancements.

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