Bayesian Hierarchical Spatial Model for Small Area Estimation with Non-ignorable Nonresponses and Its Application to the NHANES Dental Caries Data

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Abstract

The National Health and Nutrition Examination Survey (NHANES) is a major program of the National Center for Health Statistics, designed to assess the health and nutritional status of adults and children in the United States. The analysis of NHANES dental caries data faces several challenges, including (1) the data were collected using a complex, multistage, stratified, unequal-probability sampling design; (2) the sample size of some primary sampling units (PSU), e.g., counties, is very small; (3) the measures of dental caries have complicated structure and correlation, and (4) there is a substantial percentage of nonresponses, for which the missing data are expected to be not missing at random or non-ignorable. We propose a Bayesian hierarchical spatial model to address these analysis challenges. We develop a two-level Potts model that closely resembles the caries evolution process and captures complicated spatial correlations between teeth and surfaces of the teeth. By adding Bayesian hierarchies to the Potts model, we account for the multistage survey sampling design and also enable information borrowing across PSUs for small area estimation. We incorporate sampling weights by including them as a covariate in the model and adopt flexible B-splines to achieve robust inference. We account for non-ignorable missing outcomes and covariates using the selection model. We use data augmentation coupled with the noisy exchange sampler to obtain the posterior of model parameters that involve doubly-intractable normalizing constants. Our analysis results show strong spatial associations between teeth and tooth surfaces and that dental hygienic factors, fluorosis and sealant reduce the risks of having dental diseases.

Keywords: National Health and Nutrition Examination Survey, Dental Caries, Potts Models, Non-ignorable Nonresponse, Selection Model, Small Area Estimation

1 Introduction

The National Health and Nutrition Examination Survey (NHANES) is a major program of the National Center for Health Statistics (NCHS) and focuses on understanding the health and nutrition of adults and children in the United States. NHANES is composed of two parts. The first survey is administered each year to a random sample of the population with about 5,000 people, and the participants are asked a set of demographic, socioeconomic, dietary, and health-related questions. The second part consists of a physical exam that includes medical, dental, physiological measurements, and laboratory testing.
administered to people located in 15 selected counties across the country. The data collected from this survey is used to determine the nutritional status and the prevalence of major diseases and their risk factors. The findings from NHANES are also used for setting national standards on height, weight, and blood pressure, and for understanding the health status of Americans, developing nutritional guidelines, and forming better health policies (U.S. Department of Health and Human Services, Centers for Disease Control and Prevention, 2005).

In this paper, we focus on the NHANES dental caries data collected from participants aged from 20 to 34 who had information on dental fluorosis and sealants. Dental caries, also known as tooth decay, is one of the most prevalent chronic diseases worldwide (Selwitz et al., 2007). Although preventable, people remain susceptible to the disease throughout their lifetime (Featherstone, 2000; Pitts, 2004); hence it remains a major global oral health burden and is prevalent in the United States. Caries are triggered by acids produced during bacterial fermentation of food debris that accumulates on the tooth surface. This causes localized dissolution of the tooth's hard tissues and leads to the development of cavities or holes in the teeth (Kidd et al., 2003). The four main factors influencing the formation of dental caries are the person's age, the health of the tooth surface, the presence of cariogenic bacteria, and the presence of fermentable carbohydrates (Soames and Southam, 1993). The degree of caries progression varies by individual, depending on the shape of the teeth, oral hygiene habits, and the buffering capacity of saliva. If left untreated, caries can spread to supporting tissues and the jaws, and result in advanced conditions that are often painful (Jamison et al., 2006) and may lead to tooth loss. In this analysis, we focus on two major dental caries outcomes – presence vs. absence of teeth at the tooth level, and healthy vs. non-healthy tooth surfaces at the surface level, and aim to identify the demographic or dental hygienic factors that relate to the dental caries outcomes.

There are several challenges to analyze the NHANES dental caries data: (1) the data were collected using a complex, multistage, stratified, unequal-probability sampling design. It is important to incorporate the sampling design feature, as well as sampling weights, into the model and inference (Breidt and Opsomer, 2000; Zheng and Little, 2003; 2004; 2005; Opsomer and Miller, 2005; Chen et al., 2010; Zhang et al., 2015). (2) Some of the primary sampling units (PSU), e.g., counties, have very small sample sizes, making PSU-level inference highly unreliable or sometimes impossible. This is known as the “small area estimation” problem in the survey sampling literature. (3) The collected data have complicated structure and correlation. The outcomes are consisted of tooth-level measurements and also (tooth) surface-level measurements, where the surface measurements are nested within the tooth-level measurements and they are spatially correlated (Garcia-Zattera et al., 2007). For example, the health status of a surface on a particular tooth might be influenced by the disease status of proximal surfaces or teeth, and the absence of a tooth might relate to the absence/presence of nearby teeth. (4) There is substantial number of nonresponses in both outcomes and covariates (e.g., household income), and the resulting missing data are potentially nonignorable.

To address these challenges, we develop a Bayesian hierarchical spatial model for small area estimation with non-ignorable nonresponse. We account for the multistage sampling scheme using the Bayesian hierarchical model structure, which also enables information borrowing across PSUs to improve the efficiency of small area estimation. We incorporate sampling weights by including them as a covariate in the model and adopt flexible B-splines to achieve robust inference. We capture the feature that the surface measurements are nested within the tooth-level measurements and they are spatially correlated by using a two-level spatial model. At the first level of hierarchy, the trinary probability of a tooth being present, absent due to the dental disease, or absent due to other reasons is modeled via a Potts model. Conditional on the tooth being present, we model the probability of a decayed, filled or healthy surface via a second Potts model. We employ the selection model to account for non-ignorable missing outcome and covariates. Estimation of the proposed Bayesian hierarchical spatial model is challenging because of the presence of doubly-intractable normalizing constant and non-ignorable missing data. We use the
noisy exchange algorithm coupled with data argumentation to make posterior inference.

There is a rich body of literature on modeling caries outcomes. Garcia-Zattera et al. (2007) analyzed the caries experience data with the conditionally specified logistic regression model (Joe and Liu, 1996) and a multivariate probit model (Chib and Greenberg, 1998). Afroughi et al. (2010) modeled experienced caries of deciduous teeth in 3-5 years-old children using the spatial autologistic regression and identified a risk pattern of decayed dents in these children. Bandyopadhyay et al. (2011) developed a multivariate spatial beta-binomial model for the total count of decayed, missing or filled surfaces in a tooth that accommodates both over-dispersion as well as latent spatial associations. Mustvari et al. (2013) used a spatially referenced multilevel autologistic model to investigated (i) if caries experience outcomes recorded at surface level were spatially associated; and (ii) if the dental examiners exhibited some spatial behavior while scoring caries experience at surface level. Jin et al. (2016) developed a Bayesian hierarchical two-level framework that closely resembles the caries evolution process in humans.

For dental data that collected from a survey with small sub-populations, various small area estimation techniques have been employed for parameter estimation. Ghosh and Rao (1994; Rao, 2015). Leroux et al. (1996) is one of the first papers that used small area estimation on dental data. Specifically, the data was collected from an oral health survey in the state of Washington and small area estimation was used for analyzing dental disease and sealant utilization. However, this early study was small in scale and did not include any spatial components. Antunes et al. (2002) used small area estimation and spatial models to describe the epidemiological measurements collected from small sub-population (districts in Sao Paulo, Brazil) and to examine the association between tooth decay and dental treatments in children. However, the spatial analysis was applied to the geographic districts and did not consider the spatial correlations among teeth or surfaces. Gentili et al. (2015) developed a small area estimation spatial model to analyze the access to pediatric primary care. Similarly, the spatial model was with regard to the geographic location of the pediatric primary care, rather than on dental data per samples.

The remainder of the paper is structured as follows. In Section 2, we describe the NHANES study design and caries data. In Section 3, we present the exploratory data analysis results which help us to build models for the NHANES data. In Section 4, we propose a Bayesian hierarchical spatial model for the outcomes that incorporates the sampling design and weights, while accounting for the non-ignorable missing covariates and outcomes. We present the Bayesian computational framework using the noisy exchange algorithm in Section 5. In Section 6, we apply the Bayesian model to the NHANES dataset and summarize analysis results. We provide our conclusions and future developments in Section 7.

2 NHANES Dental Caries Data

The dental caries data analyzed here were collected from 1999 to 2004 as part of the publicly available NHANES survey. We also examined the dental health data from the NHANES survey in other years, but they either did not collect information on some important covariates or focused on a different population rather than young adults, which is our targeted population for this analysis. For example, the 2015-2016 data did not collect fluorsis, an important dental health feature; and the 2011 data focused on children aged from 6 to 14 years, whose teeth haven’t fully developed yet. We thus focus on the 1999-2004 wave of the NHANES data that provides information on PSU, sampling weights, and relevant demographic and dental covariates in subjects aged 20 to 34 years. Specifically, the data came from 87 PSUs via a complex, multistage and stratified sampling design. The sampling procedure consists of four stages:

- Stage 1: PSUs, mostly single counties, are selected with Probability Proportional to its Size (PPS), from strata defined by geography and proportions of minority populations. We denote the probability that PSU $i$ is sampled by $p_i$. 

Stage 2: The PSUs are divided into segments, generally city blocks or their equivalents. The segments are sampled with PPS. We denote the probability of segment \( j \) in PSU \( i \) being sampled by \( p_{ji} \).

Stage 3: A household is randomly sampled from each segment selected in Stage 2. NHANES oversampled certain subgroups of particular public health interest. In the 1999-2004 surveys, African Americans, Mexican Americans, and persons age 60+ are examples of over-sampled subgroups. We denote the probability that household \( k \) in segment \( j|i \) is sampled by \( p_{k|ji} \).

Stage 4: Individuals in a selected household in Stage 3 are randomly chosen from designated age-sex-race/ethnicity screening sub-domains. In other words, all eligible members in a household were listed, and a sub-sample of individuals were chosen based on sex, age, and race or ethnicity. Denote the probability that individual \( l \) in household \( k|ji \) is sampled by \( p_{l|kji} \).

A base sampling weight \( \left( p_{i}p_{ji}p_{k|ji}p_{l|kji} \right)^{-1} \) is first calculated for each individual in the survey incorporating the above 4 sampling stages. The base weight is then adjusted for the nonresponse in the in-home interview and in the physical exam (including the dietary interviews, body measurements, blood work except for young children, dental exam, and other tests), and post-stratified to match the 2000 US Census population total for each sampling sub-domain to obtain the final weight \( w \).

Survey participants either received a oral dental exam or not. Those who received an oral dental exam but did not have any teeth or any molar teeth are excluded from the analysis. The number of respondents who participated in both the oral health survey and the oral exams is 3595; and the number of participants who only answered the oral health survey is 321, leading to a total number of participants is 3916 for our analysis. Figure 1 illustrates the different surfaces of permanent dentition within a human mouth. Following Darby and Walsh (1995), the entire dentition can be divided into four quadrants, two on each jaw bone, the mandible (lower jaw) and maxilla (upper jaw). Each quadrant consists of a cluster of 7 teeth excluding wisdom teeth: the non-anterior teeth (two molars and two premolars) and the anterior teeth (one incisors and two canine). In the study of dental caries, each non-anterior tooth contributes five surfaces (occlusal, mesial, distal, facial, and lingual), while each anterior tooth contributes four of these surfaces without an occlusal surface. Because dental data consist of a two-level hierarchy — a tooth level and a surface level, the primary response variable is recorded differently according to the level of hierarchy. An assessment of the current status for caries progression at the tooth level is a trinary indicator for the presence of a particular tooth, absence of a particular tooth due to dental disease, and absence of a particular tooth due to other reasons. Next, conditional on the tooth being present, an assessment of the current status of caries progression at the surface level of a tooth is a trinary indicator that each surface is either healthy (H), decayed (D), or filled (F). If the whole tooth is missing, then all the surfaces are considered missing. The reason for a missing tooth was determined from the questionnaire administered to the study participants. We acknowledge that this self-reported information may be inaccurate, but it is the best information available.

Several individual-level covariates were also collected, including gender (0 = male, 1 = female), poverty-income ratio (0 = below poverty line, 1 = above poverty line), race (1 = Non-Hispanic White, 2 = Non-Hispanic Black, 3 = Other Races including Mexican American and Hispanics). Across all PSUs, female participated at a slight higher rate (57%) than male and approximately 22% of survey participants are below the poverty line. About 8% of participants declined to answer the questions regarding income/poverty. 43.4% of participants are Non-Hispanic White, 20.0% of participants are Non-Hispanic Black, and the rest is Other Races (including 26.4% Mexican American), which is coded as the reference group in our analysis. There are two tooth-level covariates: sealant for occlusal teeth (1 = if there is a sealant, 1 = if there is no sealant) and fluorosis level for each tooth (0 = normal, 1 = very mild, 2 = mild, 3 = moderate, and 4 = severe). If a participant did not receive an oral dental
In the subsequent analysis, we standardize the covariates using the method proposed by Gelman et al. (2008). Specifically, Binary inputs are shifted to have a mean of 0 and to differ by 1 in their lower and upper conditions (For example, since female respondents in our study is 57% and male is 43%, we would define the centered “gender” variable to take on the values 0.43 and -0.57). Other inputs are shifted to have a mean of 0 and scaled to have a standard deviation of 0.5.

3 Exploratory Data Analysis

We performed exploratory data analysis on the data to guide us through the development of models for the outcome variables (presence vs. absence of teeth and healthy vs. unhealthy tooth surface for each PSU), the incorporation of the sampling weights in the models, and the development of the selection models to model the missingness in the outcome and covariates with missing values.

3.1 Sampling Weights

To incorporate sampling weights in the data analysis, we employ the model-based approach by including the weight as a covariate in the model of the outcome variables (Zheng and Little, 2003, 2004, 2005; Chen et al., 2010). The weight has a wide range (465.59, 69220.78) across the PSUs with magnitude dramatically different from the rest of covariates. Therefore, we standardized weight such that it has a mean of 0 and a standard deviation of 0.5.

The key issue is how to determine the function form between the caries outcomes and sampling weights. We plotted the logit of the proportion of surfaces with cavity in an individual vs. the individual’s weight by PSU and applied the locally weighted scatter-plot smoothing (LOWESS) to smooth out the relationship. The plug-in bandwidth selection method (Sheather and Jones, 1991) was used to select the bandwidth for the LOWESS. Figure 2 shows the relationship from a few PSUs (the rest of the plots are available in Section B of the supplementary materials). This analysis suggests that the relationship differs by PSU and also has various nonlinear trends. We observed similar trends when plotting the logit of the
proportion of absent teeth in an individual vs. weight (see Section B of the supplementary materials). Therefore, we employ the nonparametric regression with B-splines to model the curvature relationship between the caries outcomes and weight. Breidt and Opsomer (2000) previously used nonparametric regression approach (e.g., smoothing spline) to incorporate sampling weights for analyzing survey data.

![Figure 2: LOWESS plots between the logit of the proportion of surfaces with cavity vs. weight](image)

### 3.2 Small Area Estimation Issue

We employed a Bayesian hierarchical model to reflect the multistage sampling design of the NHANES study and overcome the small area estimation issue. Table 1 lists the sample sizes in the PSUs. Most of the PSUs have 30 ~ 69 subjects, 8 PSUs have 20 ~ 29 subjects, and the PSU with the largest sample size has 84 participants. In addition, we observe data sparsity and unbalancedness in the distribution of covariates. Let $x_{ijk}$ be the trinary variable indicating whether the $k$th tooth of $j$th individual at $i$th PSU is absent due to the dental disease ($y_{ijks} = 1$) ($m$), absent not due to the dental disease ($y_{ijks} = 2$) ($\bar{m}$), or present ($y_{ijks} = 3$), and let $y_{ijks}$ be the binary variable of the surface condition on a non-missing tooth ($y_{ijks} = 3$), indicating whether the $s$th surface of $k$th tooth of $j$th individual at the $i$th PSU is either healthy ($y_{ijks} = 1$), decayed ($y_{ijks} = 2$), or filled surfaces ($y_{ijks} = 3$) with $i = 1, \ldots, I$, $j = 1, \ldots, n_i$, $k = 1, \ldots, 28$, and $s = 1, \ldots, 4$ or $s = 1, \ldots, 5$ depending on the tooth type (incisor and canine teeth have four surfaces while molar and pre-molar teeth have five surfaces). Tables 2 and 3 show the summary statistics on the standardized covariates from a few PSUs. The results for all other PSUs are available in Section A of the supplementary materials. Summary statistics at the tooth and surface levels are the weighted sum of standardized covariates where the weights at the tooth level are counts of missing teeth due to disease and due to other reasons and the weights at the surface level are counts of decayed and filled surfaces and are calculated by $\sum_{j,k} I(x_{ijk} = x) z_{ij}$ and $\sum_{j,k,s} I(y_{ijks} = y) z_{ij}$, respectively, where $z_{ij}$ is a standardized covariate for individual $j$ from PSU $i$.

Table 1: PSU Sizes in NHANES Data

| PSU Size | 20 – 29 | 30 – 39 | 40 – 49 | 50 – 59 | 60 – 69 | 70 – 79 | ≥ 80 |
|----------|---------|---------|---------|---------|---------|---------|------|
| # PSU    | 8       | 22      | 33      | 12      | 8       | 2       | 2    |

Tables 2 and 3 suggest two important observations. First, the distribution of covariate by outcome category is highly unbalanced across the PSUs, suggested by the wide spread of the sufficient statistics. Second, there is a data sparsity issue, implied by the 0’s in the tables. For example, the sufficient
statistics for Non-Hispanic White and Non-Hispanic Black at PSU 6 are zero because PSU 6 does not have any individuals from these two race groups who had missing teeth, or decayed and filled surfaces. Similar interpretation can be obtained for other PSUs and other covariates. The full tables of the sufficient statistics from all PSUs are given in Section A of the supplementary material. The data unbalancedness and sparsity in the covariates across the levels of the outcome variables, together with the small sample size in some PSUs, suggest importance of using Bayesian hierarchical model to borrow information across the PSUs (i.e., small areas).

| PSU | Tooth Absence Due to Disease | Decayed Surfaces | Tooth Absence from Other Reasons | Filled Surfaces |
|-----|-------------------------------|------------------|---------------------------------|----------------|
|     | gender | poverty | race | race | sealant | flurosis | gender | poverty | race | race | sealant | flurosis |
| 1   | 22.65  | 7.80    | 2.76 | 0.00 | -2.76  | -12.27  | -0.76  | 3.73    | 1.82 | 0.00 | -1.82  | 1.38    |
| 6   | 3.38   | -0.71   | 0.00 | 0.00 | 0.00   | -1.26   | -0.38  | 0.00    | 0.00 | 0.00 | 0.00   | -0.19   |
| 7   | -12.09 | -19.00  | 21.94 | 0.00 | -2.13  | -10.55  | -1.45  | 0.78    | -0.65| 0.00 | -0.94  | -1.16   |
| 8   | -2.10  | -6.59   | -0.71 | 0.00 | -2.13  | -1.20   | 1.45   | 0.78    | -0.65| 0.00 | -0.94  | -1.16   |
| 12  | -1.00  | 0.59    | -1.50 | 0.00 | 1.50   | -0.00   | -5.33  | -2.94  | 1.00 | 0.00 | -1.33  | -1.32   |
| 26  | 0.11   | -0.65   | 0.00 | -1.44 | 0.00   | -0.52   | -0.89  | -0.48  | 0.00 | 0.56 | 0.00   | -0.13   |
| 27  | -1.54  | -0.67   | 0.00 | 3.15  | 0.00   | -1.38   | -3.08  | 2.67    | 0.00 | -1.69| 0.00   | -1.26   |

Table 2: Sufficient Statistics on the Covariates in the Measurement Model at the Tooth Level for Selected PSUs

| PSU | Decayed Surfaces | Filled Surfaces |
|-----|------------------|----------------|
|     | gender | poverty | race | race | sealant | flurosis | gender | poverty | race | race | sealant | flurosis |
| 1   | 21.24  | 6.43    | 3.82 | 0.00 | 7.18   | -21.61  | -20.53 | 30.43   | 10.65| 0.00 | 1.35   | -3.04   |
| 6   | 13.15  | 4.33    | 0.00 | 0.00 | 0.00   | -3.95   | 2.85   | -9.38   | 0.00 | 0.00 | 0.00   | 3.95    |
| 7   | -2.74  | -12.00  | 3.17 | -0.97| 0.00   | -0.48   | 32.60  | 52.67   | 11.40| 0.40 | 0.00   | 14.53   |
| 8   | 7.23   | -17.59  | -1.68| 0.00 | -5.03  | 1.32    | -11.45 | 36.74   | -4.65| 0.00 | -8.94  | -5.04   |
| 12  | 1.00   | -1.53   | -3.25| 0.00 | 1.25   | 2.23    | -47.67 | 3.94    | -15.50| 0.00 | 1.83   | -4.14   |
| 26  | -9.56  | -14.74  | 0.00 | -4.78| 0.00   | 13.05   | -17.78 | 26.87   | 0.00 | 8.11 | 0.00   | -13.64  |
| 27  | -9.69  | -1.00   | 0.00 | -0.23| 0.00   | -1.07   | -29.15 | -11.33  | 0.00 | 10.62| 0.00   | -12.22  |

Table 3: Sufficient Statistics formed from the Covariates in the Measurement Model at the Surface Level for Selected PSUs

We conducted a preliminary analysis of the two outcome variables based on the model described in Section 4.1 but without considering hierarchical model structure and missing data, in each of the 87 PSUs separately with complete cases only. Due to sparse data, out of 87 PSUs, only 68 PSUs are estimable. Figure 3 shows the density of the estimated regression coefficients obtained from 68 estimable PSUs. The empirical distributions of each estimated coefficients across the PSUs are roughly bell-shaped, providing some empirical evidence for the adoption of a hierarchical model by assuming PSU-level parameters follow a Gaussian distribution.

3.3 Missing values in outcome and covariate

The NHANES data has two major sources of missing data: (1) subjects that elected not to provide family income information in the survey, which is used to calculate “poverty”, referred to as non-responders in poverty (NORP); (2) subjects that did not take the dental exam, referred to non-responders in dental exam (NORD). The histograms on the missing percentages of NORP and NORD are given in Figure 4. The NORP missing percentage ranges from 0% to 31% across the 87 PSUs, while that in NORD ranges from 0% to 24%. Only 18 PSUs out of 87 do not have NORP and 6 PSUs do not have NORD.
It is widely known that income, if collected in a survey, is subject to missingness not at random (MNAR) – compared with respondents who reported income, respondents with missing income information generally appeared younger, less educated, and of lower parity (Kim et al., 2007). The NORD subjects have missing values in surface cavity and absence/presence of teeth and in the covariates sealant and fluorosis in NORD. These missing values are likely to be non-ignorable in the sense that not having a dental exam might correlate with the individual’s oral health status and thus the outcomes of interest (surface with cavity and absence tooth, or not): people who have very good or very bad oral health might not feel as necessary to go to the oral exam. To account for these non-ignorable missing data, it is imperative to model the missing data mechanism (Little and Rubin, 2002). In what follows, we first describe our measurement model, followed by the missing-data model.
4 Methodology

4.1 Measurement Model

In this section, we introduce a Bayesian hierarchical spatial model that accommodates the spatial interactions in dental structures and the sampling weights from the complex design. At the tooth level, we consider only one spatial interaction, i.e., the interaction with neighboring teeth. Denote the corresponding parameter for this interaction as $\psi_t \in [0, \infty)$. At the surface level, we consider three types of spatial interactions: 1) non-occlusal surfaces on the same tooth (type-A interaction), 2) surfaces on adjacent teeth on the same jaw (type-B interaction), and 3) contact surfaces on the opposite jaw (type-C interaction). For the sake of simplicity and ease of interpretation, we eliminated the interactions between non-neighboring surfaces such as the interaction between the facial and lingual surfaces of the same tooth.

We illustrate all the different types of interactions in Figure 5. Type-A interactions consist of two categories characterized by two spatial interaction parameters, $\psi_{p,1}$ and $\psi_{p,2}$. Specifically, $\psi_{p,1}$ denotes associations between the occlusal surface and the other four surfaces on the same tooth, while $\psi_{p,2}$ denotes associations between adjacent non-occlusal surfaces on the same tooth, i.e., between mesial - facial, mesial - lingual, distal - facial, and distal - lingual surfaces. Type-B interactions consist of two categories characterized by two spatial parameters, $\psi_{p,3}$ and $\psi_{p,4}$. While $\psi_{p,3}$ denotes interactions between the contacting mesial and distal surfaces of adjacent teeth on the same jaw, $\psi_{p,4}$ quantifies the interactions between adjacent occlusal surfaces, facial surfaces, and lingual surfaces of teeth on the same jaw. Finally, $\psi_{p,5}$ is the parameter that captures the spatial correlation between the contacting occlusal surfaces on opposite jaws (Type-C interaction). We denote the vector of the spatial association parameters by $\psi_p = \{\psi_{p,1}, \ldots, \psi_{p,5}\}$ where $\psi_{p,1}, \ldots, \psi_{p,5} \in [0, \infty)$. The defined spatial interactions $\psi_t$ and $\psi_p$ are incorporated in the Potts models for tooth and surface outcomes.

4.1.1 Model for the Presence/Absence of Teeth for Each PSU

Recall that $x_{ijk}$ denotes the trinary variable indicating whether the $k$th tooth of $j$th individual at $i$th PSU is absent due to the dental disease ($x_{ijk} = 1$) or absent not due to the dental disease ($x_{ijk} = 2$).
\( (m), \) or present \( (x_{ijk} = 3) \) with \( i = 1, \cdots, I, j = 1, \cdots, n_i, \) and \( k = 1, \cdots, 28. \) We assume that \( x_i = \{x_{ijk}\} \) follows a multinomial distribution via the following Potts model, 

\[
f(x_i | \theta_i) = \frac{1}{\kappa(\theta_i)} \exp \left[ \psi_{ti} \sum_{(j,k) \sim (j,k)'} I(x_{ijk} = x_{ij(k)'} \right] + \sum_{(j,k)} I(x_{ijk} = 2) \left\{ \alpha_{m} + \sum_{r=1}^{6} \beta_{r,m} z_{r,ij} \right\} \\
+ \sum_{(j,k)} I(x_{ijk} = 3) \left\{ \alpha_{m} + \sum_{r=1}^{6} \beta_{r,m} z_{r,ij} \right\} + \sum_{(j,k)} \left\{ I(x_{ijk} = 2) + I(x_{ijk} = 3) \right\} \sum_{q=1}^{k+2} \beta_{q,t_i} B_q(\pi_{ij}) \right]
\]

(1)

where \( \psi_{p_i} \) determines the intensity of interaction between \( x_{ijk} \) and its neighbors, represented by \( (j,k) \sim (j,k)' \), at the \( i \)th PSU; \( z_{r,ij} \) is the \( r \)th individual-level covariate with \( r = 1, \cdots, 6 \) denoting gender, poverty level, race (Non-Hispanic White), race (Non-Hispanic Black), sealant (the binary indicator of having sealants in each individual’s eligible teeth) and fluorosis (the mean of all fluorosis values for all teeth that are present); \( \beta_{r,m} \) and \( \beta_{r,m} \) measure the effect of covariate \( r \) for the missing teeth due to the dental disease and those not due to the dental disease, with \( \alpha_{m} \) and \( \alpha_{m} \) as the intercepts, respectively; \( \pi_{ij} \) is the inclusion probability (i.e., the inverse of sampling weight) for individual \( j \) at the \( i \)th PSU; and, \( B_q(\pi_{ij}) \) is the quadratic B-spline basis function for the inclusion probability with its corresponding parameter \( \beta_{q,t_i} \). We chose to use B-splines to model the effect of sampling weights on \( x_{ij} \) because the basis functions of B-splines are linearly independent \cite{Hastie1992} and thus mitigate the multicollinearity issue. In this Potts model, \( \kappa(\theta_i) \), with \( \theta_i = \{\psi_{ti}, \alpha_{m}, \alpha_{m}, \beta_{m} = \{\beta_{r,m}\}, \beta_{m} = \{\beta_{r,m}\}, \beta_{q,t_i} = \{\beta_{q,t_i}\}\) is a doubly-intractable normalizing constant, which involves the sum over all possible realization of \( x_i \) and the normalizing constant itself is a function of the parameters. Because of this doubly-intractable normalizing constant, the standard Markov Chain Monti Carlo (MCMC) algorithm cannot be applied to fit the model. In Section 5 we employ a new algorithm to handle this issue.

### 4.1.2 Model for the Health/Non-health Surface for Each PSU

As defined previously, \( y_{ijks} \) is the binary variable of the surface condition on a non-missing tooth \( (y_{ijks} = 3) \), indicating whether the \( s \)th surface of \( k \)th tooth of \( j \)th individual at the \( i \)th PSU is either healthy \( (y_{ijks} = 1) \), decayed \( (y_{ijks} = 2) \), or filled surfaces \( (y_{ijks} = 3) \) with \( i = 1, \cdots, I, j = 1, \cdots, n_i, \) \( k = 1, \cdots, 28, \) and \( s = 1, \cdots, 4 \) or \( s = 1, \cdots, 5 \) depending on the tooth type (incisor and canine teeth have four surfaces while molar and pre-molar teeth have five surfaces). We assume that the joint distribution of \( y_i = \{y_{ijks}\} \) follows a Potts model, given by

\[
f(y_i | \theta_{p_i}, x_{ijk} = 3) = \frac{1}{\kappa(\theta_{p_i})} \exp \left[ \sum_{h=1}^{5} \psi_{h,p_i} s_h(y_i) + \sum_{(j,k,s)} I(y_{ijks} = 2) \left\{ \alpha_{d_i} + \sum_{r=1}^{6} \beta_{r,d_i} z_{r,s} \right\} \\
+ \sum_{(j,k,s)} I(y_{ijks} = 3) \left\{ \alpha_{f_i} + \sum_{r=1}^{6} \beta_{r,f_i} z_{r,s} \right\} \right. \\
+ \sum_{(j,k,s)} \left\{ I(y_{ijks} = 2) + I(y_{ijks} = 3) \right\} \sum_{q=1}^{k+2} \beta_{q,p_i} B_q(\pi_{ij}) \right],
\]

(2)

where \( \psi_{h,p_i} \) \( (h = 1, \cdots, 5) \) represent the five spatial interaction parameters; \( \beta_{r,d_i} \) and \( \beta_{r,f_i} \) measures the effects of covariates for the decayed and filled surfaces with \( \alpha_{d_i} \) and \( \alpha_{f_i} \) as the intercept, respectively, and \( \beta_{q,p_i} \) is a regression coefficient of the quadratic B-spline for the inclusion probability. In this Potts model, \( \kappa(\theta_{p_i}) \) is a doubly-intractable normalizing constant, where \( \theta_{p_i} = \{\psi_{p_i}, \{\psi_{h,p_i}\}, \alpha_{d_i}, \alpha_{f_i}, \beta_{d_i} = \{\beta_{r,d_i}\}, \beta_{f_i} = \{\beta_{r,f_i}\}, \beta_{p_i} = \{\beta_{q,p_i}\}\}. \) With a one-to-one correspondence to \( \psi_{h,p_i} \), the five spatial terms \( s_h(y_i) \) are defined as follows:

- \( s_1(y_i) = \sum_{(j,k)} \sum_{s \neq 1} I(y_{ijks} = y_{ij1}) I(x_{ijk} = 3) \), corresponding to \( \psi_{1,p_i} \), which represents the associations between the occlusal surface and the other surfaces on the same tooth;
\[ s_2(y_i) = \sum_{(j,k)} \sum_{s=4,5} \left\{ I(y_{ijks} = y_{ijk2}) + I(y_{ijks} = y_{ijk3}) \right\} I(x_{ijk} = 3), \] corresponding to \( \psi_{2,p_i} \), which represents the association between adjacent non-occlusal surfaces on the same tooth;

\[ s_3(y_i) = \sum_{(j,k)} \sum_{m=k} I(y_{ijkm} = y_{ijm3}) I(x_{ijk} = 3) I(x_{ijm} = 3), \] where \( m \sim k \) represents \( m = k - 1 \) for \( k = 2, \ldots, 7, 16, \ldots, 21 \); and \( m = k + 1 \) with \( k = 8, \ldots, 13, 22, \ldots, 27 \). This corresponds to \( \psi_{3,p_i} \), representing the association between the mesial and distal surfaces of adjacent teeth on the same jaw;

\[ s_4(y_i) = \sum_{(j,k)} \sum_{s=1,4,5} \sum_{m=k} I(y_{ijkm} = y_{ijm3}) I(x_{ijkl} = 3) I(x_{ijm} = 3), \] where \( m \sim k \) represent \( m = k + 1 \) for \( k = 1, 13, 15, \ldots, 27 \) when \( s = 2, 5 \) and \( m = k + 1 \) for \( k = 1, 2, 3, 11, 12, 13, 15, 16, 17, 25, 26, 27 \) when \( s = 1 \). This corresponds to \( \psi_{4,p_i} \), which represents the association between the adjacent occlusal, facial and lingual surfaces of teeth on the same jaw; and

\[ s_5(y_i) = \sum_{(j,k)} \sum_{o=k} I(y_{ijko} = y_{ijol}) I(x_{ijk} = 3) I(x_{ij0} = 3), \] where \( o \leftrightarrow k \) denotes the contacting teeth \( o \) and \( k \) on opposite jaws, corresponding to \( \psi_{5,p_i} \), which represents the association between the occlusal surfaces of these teeth.

### 4.2 Models for Non-ignorable Missingness

As discussed in Section 3.3, there are two types of nonresponses: NORD (i.e., the subject who did not provide poverty information) and NORP (i.e., the subject who missed the oral dental exam), and both are likely to induce non-ignorable missing data (i.e., NMAR). Let \( r_{1,ij} \) and \( r_{2,ij} \) be nonresponse indicators indicating whether individual \( j \) in PSU \( i \) is NORD or NORP, respectively, \( i = 1, \ldots, I \) and \( j = 1, \ldots, n_i \). If a subject is NORP (i.e., \( r_{1,ij} = 0 \)), the subject’s poverty status is missing; and if a subject is NORD (i.e., \( r_{2,ij} = 0 \)), the subject’s oral health data (including tooth and surface outcomes and two covariates sealant and fluorosis) are missing. Let \( r_{kj,ij} \) generically denote the variables that are missing due to \( r_{k,ij} = 0, k = 1, 2, \) and \( u_{ij} \) generically denotes other completely observed covariates. We model the non-ignorable missing data using the selection model (Little and Rubin, 2002; Little, 2008) as follows

\[
 f(r_{k,ij}, v_{k,ij} \mid u_{k,ij}, \gamma_k, \vartheta_k) = f(r_{k,ij} \mid v_{k,ij}, u_{k,ij}, \vartheta_k) f(v_{k,ij} \mid u_{k,ij}, \gamma_k), \quad k = 1, 2, \tag{3}
\]

where \( \vartheta_k \) and \( \gamma_k \) are the model parameters.

#### 4.2.1 Selection Model for NORP

For NORP, we assume that \( f(r_{1,ij} \mid u_{1,ij}, v_{1,ij}, \vartheta_{1,i}) \) in Eqn. 3 for the missingness of poverty follows

\[
 f(r_{1,ij} \mid u_{1,ij}, v_{1,ij}, \vartheta_{1,i}) = \text{Bernoulli} \left\{ \logit \left( \vartheta_{1,i}^T u_{1,ij} + \vartheta_{r,1,i} v_{1,ij} \right) \right\}, \tag{4}
\]

where \( u_{1,ij} = (1, z_{ij}, s(x_{ij}), s_h(y_{ij}))^T \); \( z_{ij} \) is the baseline individual-level covariates other than poverty; \( s(x_{ij}) \) contains the statistics representing the spatial interaction at the tooth level; and \( s_h(y_{ij}) \) contains statistics for the five spatial interactions at the surface level for individual \( j \) in the PSU \( i \); and \( \vartheta_{1,i} \) contains the corresponding regression coefficients. \( f(v_{1,ij} \mid u_{1,ij}, \gamma) \) in Eqn. 3 for modelling the binary poverty is a logistic regression model; that is,

\[
 \logit \left\{ \Pr(v_{1,ij} = 1) \mid u_{1,ij}, \gamma_{1,i} \right\} = \gamma_{1,i}^T u_{1,ij}. \tag{5}
\]
4.2.2 Selection model for NORD

To modeling the non-ignorable missingness of NORD, we assume that \( f(r_{2,ij} \mid u_{2,ij}, r_{1,ij}, \vartheta_{2,i}) \) in Eqn. 3 follows

\[
f(r_{2,ij} \mid u_{2,ij}, r_{1,ij}, \vartheta_{2,i}) \sim \text{Bernoulli}\left\{ \logit\left( \vartheta_{2,i}^T u_{2,ij} + \vartheta_{r_2,i} r_{1,ij} \right) \right\}
\]

(6)

where \( u_{2,ij} = (1, z_{ij}, s(x_{ij}), s_h(y_{ij}))^T \); \( z_{ij} \) includes all the baseline individual-level covariates including sealant and fluorosis; and \( s(x_{ij}) \) and \( s_h(y_{ij}) \) are defined in the same way as for Eqn. 4. Note that we also included \( r_{1,ij} \), the poverty missing indicator, in the regression model as \( r_1 \) and \( r_2 \) might be correlated.

For NORD, both outcome variables (i.e., tooth outcome \( x_{ijks} \) and surface outcome \( y_{ijks} \)) and two covariates (i.e., sealant and fluorosis) are missing. For dental outcomes \( x_{ijks} \) and \( y_{ijks} \), \( f(v_{2,ij} \mid u_{2,ij}, \gamma_{2,i}) \) in Eqn. 3 are provided by the Potts models, i.e., Eqn. 1 and 2. For sealant (a binary indicator variable taking values of yes or no), denoted as \( v_{s,ij} \), we assume \( f(v_{s,ij} \mid u_{s,ij}, \gamma_{s,i}) \) Eqn. 3 follows a logistic regression model,

\[
\logit\left\{ Pr(v_{s,ij} = 1) \mid u_{s,ij}, \gamma_{s,i} \right\} = \gamma_{s,i}^T u_{s,ij}
\]

(7)

where \( u_{s,ij} = (1, z_{ij}, s(x_{ij}), s_h(y_{ij}))^T \) and \( z_{ij} \) is the baseline individual-level covariates excluding sealant. Fluorosis used in our model is defined as the average fluorosis values for all the present teeth. Given that an individual usually have about 28 teeth, it is reasonable to assume that fluorosis, denoted as \( u_{f,ij} \), is approximately normal and follows a linear regression model,

\[
v_{f,ij} \mid u_{f,ij}, \gamma_{f,i} = N(\gamma_{f,i}^T u_{f,ij}, \varphi_i^2)
\]

(8)

where \( u_{f,ij} = (1, z_{ij}, s(x_{ij}), s_h(y_{ij}))^T \), and \( z_{ij} \) is the baseline individual-level covariates excluding fluorosis and sealant, and \( \varphi_i^2 \) is the variance parameter.

4.3 Hierarchical Modelling for Small Area Estimation

As described in Section 3.2, the sample size in some PSUs is small and cannot provide sufficient information to reliably estimate some model parameters in each PSU separately. We employ a hierarchical modelling framework as a small area estimation technique to borrow information across PSUs. In addition, the hierarchical structure also naturally accounts for the multistage sampling scheme (Skinner et al. 1989). The exploratory results in Figure 3 suggest that it is plausible to use normal distributions as the prior for the model parameters in the outcome measurement models. Specifically, for each of the regression coefficient \( k \) from the measurement model in PSU \( i \), we define

\[
\theta_{k,i} \sim N(\delta_{\theta_k}, \sigma_{\theta_k}^2), \text{ where } \delta_{\theta_k} \sim N(\lambda_{\theta_k}, \tau_{\theta_k}^2); \sigma_{\theta_k}^2 \sim \text{Inverse-Gamma}(a_{\theta_k}, b_{\theta_k}).
\]

(9)

We also applied similar priors and hyper-priors to the parameters from the selection models and the imputation models for imputing missing values in the covariates;

\[
\gamma_{k,i} \sim N(\delta_{\gamma_k}, \sigma_{\gamma_k}^2) \text{ where } \delta_{\gamma_k} \sim N(\lambda_{\gamma_k}, \tau_{\gamma_k}^2); \sigma_{\gamma_k}^2 \sim \text{Inverse-Gamma}(a_{\gamma_k}, b_{\gamma_k})
\]

(10)

\[
\vartheta_{k,i} \sim N(\delta_{\vartheta_k}, \sigma_{\vartheta_k}^2) \text{ where } \delta_{\vartheta_k} \sim N(\lambda_{\vartheta_k}, \tau_{\vartheta_k}^2); \sigma_{\vartheta_k}^2 \sim \text{Inverse-Gamma}(a_{\vartheta_k}, b_{\vartheta_k})
\]

(11)

We set \( \lambda_{\theta_k} \in \{ \theta, \delta, \vartheta \} \) at 0.5 for the spatial interaction parameters and at 0 for the regression coefficients associated with other covariates, \( \tau_{\theta_k} = 5 \), and \( a_{\theta_k} = b_{\theta_k} = 0.001 \) for all \( k \).
In principle, we could introduce more model hierarchies to mirror each of the sampling stages (i.e., sampling segments within PSU, sampling households within the segment, and sampling subjects within the household). We did not do it because the published NHANES data do not contain the segment and household identifiers, i.e., there is no information to identify which segment and household a specific subject belongs to. As the sampling weights somewhat already contain the segment and household sampling information (see Section 2), ignoring these sampling procedures might have little impact on the inference.

5 Posterior inference

We make posterior inference for the proposed model using the MCMC method, consisted of three main steps as follows:

1. Use the data augmentation (Tanner and Wong, 1987) technique to impute missing covariates and observations,

2. Use the noisy exchange algorithm (Liang and Jin, 2013; Alquier et al., 2016) to estimate $\theta$ from the measurement model in each PSU, and

3. Update hyperparameters $\delta$ and $\sigma^2$ in Eqns. (9), (10), and (11). $\delta$ contains the population-level covariate effects in the two outcome Potts model and the imputations models for the covariates and is thus of the primary interest and reported in Section 5.

5.1 Step 1: Data Augmentation

To impute the missing covariates and observations, we first assign initial values to all model parameters $\theta^{(0)}$, $\vartheta^{(0)}$, and $\gamma^{(0)}$ and missing values, including the outcome variables for 28 teeth $x_{mis}^{(0)}$ (tooth level) and up to 128 surfaces $y_{mis}^{(0)}$ (surface level), and the covariates for poverty $z_{mis,p}^{(0)}$, sealant $z_{mis,s}^{(0)}$, and fluorosis $z_{mis,f}^{(0)}$ (individual-level). The MCMC steps in each iteration for imputing all the missing values are provided below.

Algorithm 1: Data Augmentation for Imputing Missing Covariates and Observations

1. Draw parameters $\vartheta_1$ and $\gamma_1$ associated with the selection model for poverty (Eqn. 4 and 5) via a conventional MCMC algorithm.

2. Impute missing poverty values $z_{mis,p}^{(0)}$ from Eqn. 5 given the drawn $\gamma_1$ in step 1, imputed sealant $z_{mis,s}^{(0)}$, fluorosis $z_{mis,f}^{(0)}$, tooth outcomes $x_{mis}^{(0)}$, surface outcomes $y_{mis}^{(0)}$, and other covariates.

3. Draw the parameters $\vartheta_2$ associated with the selection model for the outcome variables (Eqn. 6) via a conventional MCMC algorithm.

4. Impute missing tooth outcomes $x_{mis}^{(0)}$ and surface outcomes $y_{mis}^{(0)}$ from conditional unnormalized density of Eqn. 1 and 2 using the Gibbs sampler given parameters in measurement models $\theta$, imputed poverty $z_{mis,p}^{(0)}$, sealant $z_{mis,s}^{(0)}$, fluorosis $z_{mis,f}^{(0)}$, and other imputed tooth and surface outcomes.

5. Draw the parameters $\gamma_s$ associated with the selection model for sealant (Eqn. 7) via a conventional MCMC algorithm.
6. Impute missing sealant values \( z_{mis,s} \) from Eqn. 7, given the drawn \( \gamma_s \) in step 5, imputed poverty \( z_{mis,p} \), fluorosis \( z_{mis,f} \), tooth outcomes \( x_{mis} \), surface outcomes \( y_{mis} \), and other covariates.

7. Draw the parameters \( \gamma_f \) associated with the selection model for fluorosis (Eqn. 8) via a conventional MCMC algorithm.

8. Impute missing fluorosis values \( z_{mis,f} \) from Eqn. 8, given the drawn \( \gamma_f \) in step 7, imputed poverty \( z_{mis,p} \), sealant \( z_{mis,s} \), tooth outcomes \( x_{mis} \), surface outcomes \( y_{mis} \), and other covariates.

The standard MCMC algorithms cannot be applied to simulate posterior samples from the Potts models in Eqn. 1 and 2 because the acceptance probability depends on an unknown intractable normalizing constant ratios. To circumvent this difficulty, we use a noisy exchange algorithm (Liang and Jin, 2013; Alquier et al., 2016) as an approximate version of the exchange algorithm (Murray et al., 2006) by replacing the unknown normalizing constant with a Monte Carlo estimate.

5.2 Step 2: Noisy Exchange Algorithm

Let \( \theta^{(t)} \) denote the current draw of \( \theta \) by the algorithm. Let \((x_1^{(t)}, y_1^{(t)}), \ldots, (x_m^{(t)}, y_m^{(t)})\) denote the auxiliary samples simulated from the distribution \( f(x, y | \theta^{(t)}) \), which can be drawn by any conventional MCMC algorithm. One iteration of the noisy exchange algorithm for generating \( \theta \) is given below.

Algorithm 2: Noisy Exchange Algorithm for Estimating Parameters in Measurement Models

1. Draw \( \theta' \) from a proposal distribution \( Q(\theta^{(t)}, \theta') \).

2. Estimate the normalizing constant ratio \( R(\theta^{(t)}, \theta') = \kappa(\theta^{(t)})/\kappa(\theta') \) by

   \[
   \hat{R}_m\left(\theta^{(t)}, \theta', x^{(t)}, y^{(t)}\right) = \frac{1}{m} \sum_{i=1}^{m} \frac{f(x_i^{(t)}, y_i^{(t)} | \theta')} {f(x_i^{(t)}, y_i^{(t)} | \theta^{(t)})},
   \]
   (12)

   where \( (x_i^{(t)}, y_i^{(t)}) = \{(x_1^{(t)}, y_1^{(t)}), \ldots, (x_m^{(t)}, y_m^{(t)})\} \) denotes the collection of auxiliary samples.

3. Calculate the acceptance ratio

   \[
   \hat{r}_n\left(\theta^{(t)}, \theta', x^{(t)}, y^{(t)}\right) = \frac{1}{\hat{R}_m(\theta^{(t)}, \theta', x^{(t)}, y^{(t)})} \frac{f(x, y | \theta') \pi(\theta') \theta Q(\theta', \theta^{(t)})} {f(x, y | \theta^{(t)}) \pi(\theta^{(t)}) Q(\theta^{(t)}, \theta')}
   \]
   (13)

4. Set \( \theta^{(t+1)} = \theta' \) with probability \( \hat{r}_n(\theta^{(t)}, \theta', x^{(t)}, y^{(t)}) \) and set \( \theta^{(t+1)} = \theta^{(t)} \) with the remained probability.

5.3 Step 3: Posterior sampling of hyperparameters

Suppose \( I \) is the total numbers of PSUs. Then, one iteration of hyperparameter update using normal-inverse-gamma conjugacy is given below.

Algorithm 3: Hyperparameter Update

1. Update \( \delta \) from Normal

   \[
   \left( \frac{1}{\tau^2} + \left( \frac{1}{I} \sum_{i=1}^{I} \theta_i \right) \left( \frac{1}{\sigma^2} \right) \right) \left( \frac{1}{\tau^2} + \left( \frac{1}{I} \sum_{i=1}^{I} \theta_i \right. \left. - \delta \right)^2 \right),
   \]

2. Update \( \sigma^2 \) from Inv-Gamma

   \[
   \left( a + \frac{1}{2}, b + \frac{1}{2} \sum_{i=1}^{I} \left( \theta_i - \delta \right)^2 + \frac{I}{2(I+1)} (\delta - \lambda)^2 \right).
   \]
6 Application: NHANES Dental Survey Dataset

In this section, we apply our method to the NHANES dental survey data described in Section 2. We employed the noisy Monte Carlo sampler to generate posterior samples with random starting values. Our MCMC run consisted of 30,000 iterations, with 20 auxiliary samples for each iteration to evaluate the normalizing constants ratios. We discarded the first 5,000 iterations for the burn-in process, and used a thinning of 5 iterations to collect 5,000 samples from the remaining iterations.

6.1 Covariate effects on dental caries

Table 4 summarizes the posterior means and 95% highest posterior density (HPD) intervals for the parameters from Potts models for the tooth and surface outcomes, quantifying the effects of various covariates on the carious conditions, that is, missing teeth due to the disease, missing teeth due to the other reason, decayed and filled surfaces. For example, the parameter corresponding to Gender represents the difference between Female and Male in the log odds of having a missing tooth due to either dental disease or other reason vs. no missing tooth in the Potts model for the tooth outcome, and in the log

| Covariate | Outcome | Condition | Posterior mean | 95% HPD |
|-----------|---------|-----------|----------------|---------|
| Intercept | Missing (Disease) | -1.5313 | (-2.1193, -0.8921) |
|           | Missing (Other)    | -1.9772 | (-2.7531, -1.1871) |
|           | Decayed            | 1.3158  | (-0.0331, 2.5716)  |
|           | Filled             | 3.6222  | (2.4099, 4.7270)   |
| Gender    | Missing (Disease)  | -0.0622 | (-0.4962, 0.3773)  |
|           | Missing (Other)    | -0.3947 | (-0.9575, 0.1362)  |
|           | Decayed            | 0.3134  | (0.0305, 0.6003)   |
|           | Filled             | -0.1121 | (-0.2043, -0.0153) |
| Poverty   | Missing (Disease)  | -0.1366 | (-0.5911, 0.2938)  |
|           | Missing (Other)    | 0.9306  | (0.2354, 1.5709)   |
|           | Decayed            | -0.2239 | (-0.5713, 0.1870)  |
|           | Filled             | 0.4793  | (0.2291, 0.7615)   |
| Race (White) | Missing (Disease) | -1.2013 | (-2.2308, -0.2440) |
|           | Missing (Other)    | 1.1509  | (0.2075, 2.1915)   |
|           | Decayed            | -0.5098 | (-1.3837, 0.3716)  |
|           | Filled             | 0.1112  | (-0.5248, 0.7493)  |
| Race (Black) | Missing (Disease) | 0.3192  | (-0.7758, 1.3666)  |
|            | Missing (Other)    | -0.7758 | (-1.9332, 0.4796)  |
|            | Decayed            | 0.0158  | (-0.9314, 0.9272)  |
|            | Filled             | 0.0944  | (-0.7201, 0.9193)  |
| Sealant   | Missing (Disease)  | -2.9429 | (-3.8822, -1.9144) |
|           | Missing (Other)    | -1.2722 | (-2.2364, -0.3144) |
|           | Decayed            | -2.9894 | (-3.9718, -2.1118) |
|           | Filled             | -1.1967 | (-1.7942, -0.6432) |
| Fluorosis | Missing (Disease)  | -2.2475 | (-2.7493, -1.7687) |
|           | Missing (Other)    | -1.9646 | (-2.4873, -1.4677) |
|           | Decayed            | -0.7888 | (-1.0625, -0.4959) |
|           | Filled             | -0.1653 | (-0.2457, -0.0923) |

Table 4: Posterior means and 95% HPD intervals of the pooled covariate-effect parameters.
odds of having a decayed or filled surface vs a healthy surface at the same spatial location in the Potts model for the surface outcome, conditional on that the other covariates and spatial referencing for that spatial location remain the same. The effects of other covariates can be interpreted in a similar fashion. If the 95% HPD interval of a parameter do not include 0, we could claim the covariate corresponding to that parameter has substantial effects on the caries outcomes.

The posterior means according to sealant and fluorosis were all negative, suggesting having sealant and fluorosis reduce the risks of having dental caries overall, as expected. The result also shows that females are less likely to have filled surfaces (log odds $= -0.11210$) and more likely to have decayed surfaces (log odds $= 0.3134$). People above the poverty line have increased odds of losing teeth from other reasons (log odds $= 0.9272$) and filling surfaces after decayed (log odds $= 0.4793$). Non-Hispanic White tends to have more missing teeth from other reasons (log odds $= 1.1509$) and has less missing teeth due to disease (log odds $= -1.2013$) compared to the reference race group (most of them are Hispanic) while the differences in caries outcomes between Non-Hispanic Black and Hispanic are insignificant.

The two intercept terms from the two Potts model can be interpreted as the conditional log odd-ratios of having missing teeth due to disease and those due to other reasons with non-missing teeth as the reference, and the conditional log odd-ratios of having decayed or filled surfaces with healthy surfaces as the reference, respectively. The results suggest that having missing teeth due to the dental disease (log odds $= -1.5313$) and due to other reasons (log odds $= -1.9772$) are less likely than preserving teeth among survey participants, and missing teeth from dental disease are more common than due to other reasons; and filled (log odds $= 3.6222$) and decayed (log odds $= 1.3158$) surfaces are more common than healthy surfaces.

### 6.2 Spatial Association Parameters

Table 5 summarizes the posterior means and 95% HPD intervals of the spatial association parameters. Usually, in the Potts model specification, a value of 1.0 for the spatial association parameters $\psi$ amounts to a very high degree of associations (Green and Richardson, 2002). The estimate at the tooth level was 0.6074, implying a moderate-high level of association. At the surface level, the posterior estimates of five spatial association parameters suggest the strongest association was between adjacent non-occlusal surfaces on the same tooth (Type-A$_2$), followed by the association between the mesial and distal (contacting) surfaces of adjacent teeth on the same jaw (Type-B$_1$), and the association between the adjacent occlusal, facial, and lingual (non-contacting) surfaces of teeth on the same jaw (Type-B$_2$), while that of contacting occlusal surfaces on opposite jaws (Type-C) and that between the occlusal surface and the other surfaces on the same tooth (Type-A$_1$) are negligible. To summarize, there exist high associations between non-occlusal surfaces, while those with occlusal surfaces are less likely. In other words, the caries outcomes of the occlusal surfaces are unlikely to influence those of non-occlusal surfaces. This observation is consistent with the results in Jin et al. (2016).

| parameter | post. mean | 95% HPD     | parameter | post. mean | 95% HPD     |
|-----------|------------|-------------|-----------|------------|-------------|
| Tooth     | 0.6074     | (0.5663, 0.6528) | Type-A$_1$ | 0.0964     | (0.0821, 0.1111) |
| Type-A$_2$ | 1.2626     | (1.2262, 1.3007) | Type-B$_1$ | 0.8711     | (0.8164, 0.9284) |
| Type-B$_2$ | 0.6440     | (0.6134, 0.6746) | Type-C     | 0.0003     | (0.0000, 0.0010) |

Table 5: Posterior mean estimates and 95% HPD intervals of the pooled spatial association parameters and pooled B-spline parameters.
6.3 Parameters in Imputation Models for Sealant, Fluorosis and Poverty

Table 6, Table 7, and Table 8 summarize the posterior means and 95% HPD intervals of the parameters from the imputation model for sealant, fluorosis and poverty given other covariates, respectively. These parameters measure the effects of individual-level covariates and spatial association among teeth and surfaces on the tendency of having sealant in the molar teeth, the fluorosis level in teeth, and the likelihood of above poverty line, respectively. The other parameters in the selection models from modelling the nonignorable missingness on these these covariates are summarized in Section C of the supplementary materials.

The results in Table 6 suggest that Non-Hispanic White tends to have preventive sealant treatments for their molar teeth compared to the reference race group (log odds = 1.1951); Female is less likely to have sealants compared to Male (log odds = −0.4317). The type-A spatial associations (spatial associations within a single tooth) shows a positive relationship while the type-C spatial association (spatial association of contacting occlusal surfaces on opposite jaws) has a negative relationship with sealant.

| Covariate   | Estimates | 95% HPD       | Associations | Estimates | 95% HPD       |
|-------------|-----------|---------------|--------------|-----------|---------------|
| Intercept   | -3.2172   | (-3.5479, -2.8764) | Tooth        | 0.1155    | (-0.2376, 0.4417) |
| Gender      | -0.4317   | (-0.7485, -0.1077) | Type-A1      | 0.4494    | (0.1284, 0.7746) |
| Poverty     | 0.4514    | (-0.0531, 1.0134)  | Type-A2      | 0.9076    | (0.3809, 1.4436) |
| Race (White)| 1.1951    | (0.8052, 1.5926)  | Type-B1      | -0.0779   | (-0.6041, 0.4758) |
| Race (Black)| -0.6521   | (-1.3502, -0.1183) | Type-B2      | -0.0620   | (-0.5935, 0.4632) |
| Fluorosis   | -0.1602   | (-0.4615, 0.1243)  | Type-C       | -0.4485   | (-0.7107, -0.1800) |

Table 6: Posterior means and 95% HPD intervals of the pooled parameters in the imputation model for sealant

The results in Table 7 suggest that Non-Hispanic Blacks tend to have a higher level of fluorosis in their teeth compared to the reference race group (log odds = 0.1270). While the existence of Type-A2 and Type-B2 spatial association tends to promote the fluorosis level, while the Type-A1 association tends to decrease the fluorosis-level in teeth.

| Covariate   | Estimates | 95% HPD       | Associations | Estimates | 95% HPD       |
|-------------|-----------|---------------|--------------|-----------|---------------|
| Intercept   | -0.1856   | (-0.2335, -0.1383) | Tooth        | 0.0013    | (-0.0372, 0.0391) |
| Gender      | 0.0081    | (-0.0493, 0.0603)  | Type-A1      | -0.0545   | (-0.0991, -0.0127) |
| Poverty     | 0.0182    | (-0.0500, 0.0893)  | Type-A2      | 0.0985    | (0.0324, 0.1606) |
| Race (White)| -0.0054   | (-0.0667, 0.0578)  | Type-B1      | 0.0302    | (-0.0423, 0.1031) |
| Race (Black)| 0.1270    | (0.0375, 0.2072)   | Type-B2      | 0.2091    | (0.1398, 0.2817) |
| Fluorosis   | 0.0270    | (-0.0567, 0.1168)  | Type-C       | 0.0300    | (-0.0134, 0.0697) |

Table 7: Posterior means and 95% HPD intervals for the parameters in the imputation model for fluorosis

The results in Table 8 suggest that Female as compared to Male (log odds = 0.4688) and Non-Hispanic White as compared to the Reference Race group (log odds = 0.9748) are more likely to be above the poverty line.
Covariate Estimates 95% HPD Associations Estimates 95% HPD
Intercept 0.9320 (0.7996, 1.0640) Tooth -0.0636 (-0.1929, 0.0643)
Gender 0.4688 (0.2770, 0.6245) Type-A1 -0.7260 (-0.8333, -0.6310)
Race (White) 0.9748 (0.7452, 1.2194) Type-A2 0.5259 (0.3299, 0.7704)
Race (Black) 0.0793 (-0.2537, 0.4213) Type-B1 -0.1564 (-0.4330, 0.1571)
Sealant -0.1364 (-0.4604, 0.1247) Type-B2 0.0380 (-0.3203, 0.2967)
Fluorosis -0.0418 (-0.1466, 0.0594) Type-C 0.2801 (0.1372, 0.4384)

Table 8: Posterior means and 95% HPD intervals for the parameters in the imputation model for poverty

6.4 Small Area Estimation Results
Since some of PSU-level parameter estimates are highly unreliable or sometimes impossible due to the small sample sizes, we use a Bayesian hierarchical spatial model for small area estimation. In order to check our PSU-level parameters are well-estimated under our model, we draw boxplots for PSU-level posterior mean estimates of covariates, as shown in Figure 6. Red marks in boxplots represent the overall posterior mean estimates of covariates and all PSU-level posterior mean estimates of covariate-effects parameters at the tooth and surface levels are summarized in Section D of the supplementary material. Figure 6 shows that all PSU-level parameter are reliably estimated under our modeling framework.

7 Discussion
In this paper, we proposed a new model to analyze dental caries outcomes collected in NHANES from participants aged from 20 to 34. To analyze the dental outcomes in each PSU, we refine a Bayesian spatial hierarchical model proposed by Jin et al. (2016), which closely resembles the caries evolution process in human. At the tooth level, a Potts model was used to model the trinary probability of a tooth being present, missing due to dental disease, and missing due to other reasons. At the surface level, conditional on the non-missing tooth, we model the probability of a decayed, filled or healthy surface via a second Potts model. To take into the account the effect of the complex survey design on the dental outcomes, we employed the B-splines on the sampling weights in the outcome models. To handle the sparse information on some model parameters for some PSUs, we used the Bayesian hierarchical framework to borrow information across the PSUs in the survey. We exploited the selection model to model the non-ignorable missingness both in the covariates and in the outcome variables. We combined the data augmentation method and the noisy exchange sampler to estimate parameters from the proposed mode.

The analysis results suggest that there exists strong spatial associations between teeth, between adjacent non-occlusal surfaces on the same tooth, and between the contacting and non-contacting surfaces of adjacent teeth on the same jaw. The dental hygienic factors fluorosis and sealant reduce the risks of having dental diseases. Females are more likely to have decayed surfaces but are less likely have filled surfaces. If respondents are above the poverty line, then they tend to fix their cavities. Non-Hispanic White loses less teeth due to disease compare to other race groups (most of them are Hispanic) whereas there are insignificant differences in caries outcomes between Non-Hispanic Black and other races. The imputation models suggests Non-Hispanic White gets more preventive sealant treatment for their molar teeth, whereas female tends to have less sealant treatment. In addition, Non-Hispanic Black tends to have fluorosis in their teeth; and female and Non-Hispanic Whites are more likely to be above the poverty line.

As an alternative to the proposed measurement model, one may also utilize a multinomial framework
Figure 6: Boxplots for PSU-level posterior mean estimates of covariate-effect parameters at the tooth and surface levels. The red marks in boxplots represent the pooled posterior mean estimates of covariate-effect parameters. (a) Covariates for tooth absence due to disease, (b) Covariates for tooth absence from other reasons, (c) Covariates for decayed surfaces, and (d) Covariates for filled surfaces.
in the spatial generalized linear models (SGLM) that uses a latent Gaussian Markov random field to model spatial dependence. We chose to use the Potts models because spatial dependence can be easily interpreted in the Potts model while choosing the cut-off values for the latent Gaussian Markov random fields is often challenging in the SGLM.

If there are many covariates collected from the participants in a survey, variable selection can be incorporated in our measurement model, which needs to take into account the doubly-intractable normalizing constants (Caimo and N. Friel, 2013; Bouranis et al., 2018).

Though we developed the measurement model in the framework of a dental survey data set, it can be easily extended to other survey data that include general bivariate spatial outcomes with mixed binary and multinomial outcomes in their measurements.

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