VITERBO’S SPECTRAL BOUND CONJECTURE FOR HOMOGENEOUS SPACES

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Abstract. We prove a conjecture of Viterbo about the spectral distance on the space of compact exact Lagrangian submanifolds of a cotangent bundle $T^*M$ in the case where $M$ is a compact homogeneous space: if such a Lagrangian submanifold is contained in the unit ball bundle of $T^*M$, its spectral distance to the zero section is uniformly bounded. This also holds for some immersed Lagrangian submanifolds if we take into account the length of the maximal Reeb chord.

1. Introduction

In 2007, Viterbo conjectured the following result about the spectral distance on exact Lagrangian submanifolds which he defined in 1982 [17] (this is mentioned in [18]). Every Lagrangian submanifold of $T^*\mathbb{T}^n$, Hamiltonian isotopic to the zero section $0_{\mathbb{T}^n}$, and included into the unit codisc bundle of $T^*\mathbb{T}^n$ is at a distance uniformly bounded from $0_{\mathbb{T}^n}$. This could be understood as a tentative to find non trivial compact sets for the spectral distance or at least bounded sets, which is a completely open question. The conjecture has been since generalized for every cotangent bundle of compact manifold and not restricted to Lagrangian isotopic to the zero section. Note that a priori the spectral norm depends on the coefficient ring. We will restrict ourselves to fields.

The main set of applications are in relation with the existence and properties of quasi-morphisms on the Hamiltonian group of cotangent bundles and with Hamiltonian dynamic. It has been anticipated in symplectic homogenization [18], [10].

Biran and Cornea [2] have proved some bound on this distance but depending on the boundary depth of the Lagrangian with any fiber. Conversely, these authors have shown that this boundary depth is bounded in case the Viterbo conjecture holds.
Shelukhin in 2018 [14] and 2019 [15] solves the conjecture in the following cases, with two different approaches:

1. The case of \( M = \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{S}^n \) for the field \( \mathbb{F}_2 \) which has been obtained by SFT techniques applied to closed symplectic manifold.

2. The case of \( M \) “string point invertible” which depends also of the chosen field and contains compact Lie groups according to previous computations of Menichi. This notion introduced by Shelukhin is related to the possibility to construct the fundamental class of the underlying manifold via cohomological classes of the free loop space.

During the preparation of this paper we learned that Viterbo [21] had another proof of the conjecture for manifolds satisfying a cohomological condition, whose typical example is a homogeneous space.

Dimitroglou Rizell proved in [3] that the conjecture is false in the case of immersed Lagrangian by counterexample in \( T^*\mathbb{S}^1 \).

Here, we prove the conjecture in the case of compact homogeneous spaces. First, in the case of compact Lie groups we give a statement valid for any coefficient field in Theorem 3.1. In the case of a general compact homogeneous space we prove Corollary 3.2 with some restriction on the characteristic of the field and with a weaker bound for the spectral distance. Moreover we extend the initial conjecture in two directions: (1) the proof holds for any finite ranked local system over the Lagrangian; (2) we take into account the counterexample of Dimitroglou Rizell by giving a bound depending on the length of the maximal Reeb chord in the case of an immersed Lagrangian for which there exists a “quantization” (a sheaf whose microsupport is our Lagrangian – such a sheaf exists for a Legendrian deformation of the zero section, see Corollary 3.3). A byproduct of the proof is a bound for the boundary depth of the Lagrangian with the zero section.

**Idea of proof.** Let \( M \) be a compact manifold and \( \Lambda \subset T^*M \) an exact compact Lagrangian submanifold. We use the microlocal theory of sheaves, which deals with conic subsets of the cotangent bundle, and we lift \( \Lambda \) to a conic Lagrangian submanifold \( \Lambda' \subset T^*(M \times \mathbb{R}) \setminus 0_{M \times \mathbb{R}} \). According to [4], there exists a sheaf \( F \) with \( SS(F) = \Lambda' \) which is isomorphic to zero near \( M \times \{ -\infty \} \) and to \( k_{M \times \mathbb{R}} \) near \( M \times \{ +\infty \} \).

Tamarkin translates the filtration of the Floer complex into a sheaf morphism \( \tau_c(F): F \to T_cF \) for \( c \geq 0 \), where \( T_c \) is the translation by \( c \) along \( \mathbb{R} \). He also introduces a twisted \( \mathcal{H}om \)-sheaf, \( \mathcal{H}om^* \), with the property that \( Ra_* \mathcal{H}om^*(F, G) \) encodes the filtration on the Floer cohomology, with \( a \) the projection to \( \mathbb{R} \).

We first use a computation of \( \mathcal{H}om^* \) on sheaves over \( \mathbb{R} \). We deduce that \( F' = \mathcal{H}om^*(Ra_*F, Ra_*F) \otimes k_{|0, +\infty|} \) contains the information on the Viterbo norm, \( \gamma(\Lambda, 0_M) \).
and the boundary depth (see Lemma 2.23). More precisely $\tau_c(F')$ is non zero as long as $c < \gamma(\Lambda, 0_M)$.

We use the fact that $M = G$ is a group by seeing $\mathcal{H}\text{om}^*(Ra_\ast F_1, Ra_\ast F_2)$, for any two sheaves $F_1, F_2$, as the “average” of $Ra_\ast \mathcal{H}\text{om}^*(F_1, \mu_g^1 F_2)$ for $g \in G$. More precisely we define

$$\mathcal{H}\text{om}^{*,G}(F_1, F_2) = R\mathcal{P}_{1*} R\mathcal{H}\text{om}(p_2^{-1} F, \mu^1 F'),$$

where $p_1, p_2, \mu: (G \times \mathbb{R})^2 \to G \times \mathbb{R}$ are the projections and the group operation, and we have $Ra_\ast \mathcal{H}\text{om}^{*,G}(F_1, F_2) \simeq \mathcal{H}\text{om}^*(Ra_\ast F_1, Ra_\ast F_2)$. Now, when $F_1 = F_2 = F'$, we can see that the morphism $\tau_c(F')$ we are interested in is the image by $a_\ast$ of the morphism $\tau_c(F'')$ with $F'' = \mathcal{H}\text{om}^{*,G}(F_1, F_2) \otimes k_{M \times ]0, +\infty[}$. We can see that $F''|_{\{e\} \times \mathbb{R}}$ vanishes outside $[0, l_{\text{max}}]$ where $l_{\text{max}}$ is the length of the maximal Reeb chord of $\Lambda$ (see Lemma 4.4).

These morphisms $\tau_c$ can be interpreted as sections of sheaves and the vanishing of $\tau_c$ can be translated as the vanishing of some sections over subsets of the form $M \times ]-\infty, -c[$. Now the microlocal theory of sheaves gives the following general propagation argument. If $G$ is a sheaf on $M \times \mathbb{R}$ with $G = G / G \times \mathbb{R} = M \times \mathbb{R}$ be the quotient map. For $\Lambda \subset T^* M$ we have a natural inverse image $\Lambda' \subset T^* G$. It is easy to see that the spectral norm of $\Lambda'$ is smaller than the one of $\Lambda$, but they are a priori not equal. However in the other direction we have equality: if $F$ is a sheaf on $G \times \mathbb{R}$ with microsupport $\Lambda'$, then the spectral norms of $F$ and $Rq_* F$ are equal. Now we can see that any sheaf $G$ on $M \times \mathbb{R}$ with microsupport $\Lambda$ is a direct summand of some iterated cone between such direct image sheaves $Rq_* F$ (it is even enough to take $m$ cones, where $m$ is the dimension of $M$). We note that each time we take a cone we need to change some $\tau_{kc}$ into $\tau_{(k+1)c}$ and thus get a weaker bound.

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Notations. We follow mostly the notations of [8].
Let \( M \) be a manifold. Our coefficient ring is a field \( k \). We denote by \( \mathcal{D}(k_M) \) the derived category of sheaves of \( k \)-vector spaces on \( M \). We usually work on a product \( M \times \mathbb{R} \) and denote by \((t, \tau)\) the coordinates on \( T^* \mathbb{R} \). The microsupport of a sheaf \( F \) on \( M \) is written \( SS(F) \); it is a closed conic subset of \( T^*M \). We set \( SS(F) = SS(F) \setminus 0_M \). We let \( D_{\tau \geq 0}(k_{M \times \mathbb{R}}) \) be the full subcategory of \( \mathcal{D}(k_{M \times \mathbb{R}}) \) formed by the \( F \) such that \( SS(F) \subseteq T^*M \times \{ \tau \geq 0 \} \).

We recall the bounds given in Propositions 5.4.4-13-14 of [8]. Let \( F, G \in D(k_M), F' \in D(k_{M'}) \) be given. Let \( T^*M \xrightarrow{f} M \times M', T^*M' \xrightarrow{\tau} T^*M' \) be the natural maps. Assuming respectively (i) \( f \) is proper on \( \text{supp}(F) \), (ii) \( SS(F') \cap f_\pi(f_d^{-1}(0_M)) = \emptyset \), (iii) \( SS(F) \cap SS(G) = \emptyset \) and (iv) \( SS(F) \cap SS(G) = \emptyset \), we have
\[
\begin{align*}
SS(Rf_*F) &\subset f_\pi(f_d^{-1}(SS(F))), \\
SS(f^{-1}F') &\subset f_d(f_\pi^{-1}(SS(F'))), \\
SS(F \otimes G) &\subset SS(F) + SS(G), \\
SS(R\text{Hom}(F,G)) &\subset SS(F)^a + SS(G).
\end{align*}
\]

2. Reminder on Tamarkin’s framework

2.1. On Tamarkin’s morphism. In this section we often use the maps
\[
s, q_1, q_2: M \times \mathbb{R} \to M \times \mathbb{R}, \quad T_c: M \times \mathbb{R} \to M \times \mathbb{R},
\]
where \( s, q_1, q_2 \) send \((x, t, t')\) respectively to \((x, t + t'), (x, t), (x, t')\) and \( T_c(x, t) = (x, t + c) \).

Following [8] we define the functor \( P: D(k_{M \times \mathbb{R}}) \to D(k_{M \times \mathbb{R}}) \) by \( P(F) = R\pi_*(F \boxtimes k_{[0, +\infty[}) \). We remark that \( R\pi_*(F \boxtimes k_{(0)}) \) is naturally isomorphic to \( F \). Hence the morphism \( k_{[0, +\infty[} \to k_{(0)} \) gives a natural morphism of functors \( P \to \text{id} \). By [8] we know that \( F \in D_{\tau \geq 0}(k_{M \times \mathbb{R}}) \) if and only if this morphism \( P(F) \to F \) is an isomorphism.

More generally, we have a natural isomorphism \( R\pi_*(F \boxtimes k_{(c)}) \simeq T_{cs}F \). If \( c \geq 0 \) we also have a morphism \( k_{[0, +\infty[} \to k_{(c)} \) and we deduce a natural morphism of functors \( \tau_c: P \to T_{cs} \).

Definition 2.1. For \( F \in D_{\tau \geq 0}(k_{M \times \mathbb{R}}) \) we thus obtain what we call the Tamarkin’s morphism
\[
\tau_c(F): F \to T_{cs}F, \quad \text{for } c \geq 0.
\]
We say that a \( F \) is torsion if \( \tau_c(F) = 0 \) for some \( c \geq 0 \).

The following lemma follows from results of [6] (it is a quantitative version of Lem. 6.3 with the same proof).

Lemma 2.2. Let \( F, F', G \in D_{\tau \geq 0}(k_{M \times \mathbb{R}}) \). We assume that we have a distinguished triangle \( F \to G \to F' \xrightarrow{+1} \) and that \( \tau_a(F) = 0, \tau_b(F') = 0 \), for some \( a, b \geq 0 \). Then \( \tau_{a+b}(G) = 0 \).
Proof. By the functorial properties of the morphism $\tau_c$ (see [6, §6]) we have the following commutative diagram of distinguished triangles

\[
\begin{array}{ccc}
F & \xrightarrow{T_{b*}F} & G \\
\tau_b(F) & \downarrow & \tau_b(G) \\
T_{b*}F & \xrightarrow{T_{b*}(\tau_a(F))} & T_{b*}G \\
\end{array}
\quad
\begin{array}{ccc}
F' & \xrightarrow{T_{b*}F'} & G' \\
\tau_b(F') & \downarrow & \tau_b(G') \\
T_{b*}F' & \xrightarrow{T_{b*}(\tau_a(F'))} & T_{b*}G' \\
\end{array}
\]

where the vertical morphisms are the morphisms $\tau_c$ of the corresponding sheaves or their images by $T_{b*}$. We also have $\tau_{a+b}(H) = T_{b*}(\tau_a(H)) \circ \tau_b(H)$, for any $H \in D_{\tau \geq 0}(k_M \times \mathbb{R})$.

Since $\tau_b(F') = 0$, we have $\beta \circ \tau_b(G) = 0$ and we can factorize $\tau_b(G) = \alpha \circ u$ for some $u: G \to T_{b*}F$ (this follows from the general fact that $\text{Hom}(G, -)$ turns distinguished triangles into long exact sequences). Hence

\[
\tau_{a+b}(G) = T_{b*}(\tau_a(G)) \circ \tau_b(G) = \alpha' \circ T_{b*}(\tau_a(F)) \circ u
\]

and this vanishes since $\tau_a(F) = 0$. □

We will later assume that $M$ is endowed with a metric and we will give conditions so that the vanishing of $\tau_c(F)|_{x_0 \times \mathbb{R}}$, for some $x_0 \in M$, implies the vanishing of $\tau_{c+d}(F)|_{B \times \mathbb{R}}$, for a ball of radius $d$ around $x_0$. For this it is convenient to consider $\tau_c$ as a section of some sheaf and use microsupport estimates to extend this section. Such a sheaf was defined by Tamarkin as follows. For $F, G \in D(k_M \times \mathbb{R})$ we set

\[\text{(3)} \quad \mathcal{H}om^* (F, G) = Rq_{!*} R\mathcal{H}om(q_2^{-1} F, s^! G).\]

We then have

**Lemma 2.3.** Let $U \subset M$ be an open subset and $F, G \in D(k_M \times \mathbb{R})$. Then $\mathcal{H}om^* (F, G)|_{U \times \mathbb{R}} \simeq \mathcal{H}om^* (F|_{U \times \mathbb{R}}, G|_{U \times \mathbb{R}})$ and, for any $c \in \mathbb{R}$,

\[
R\Gamma_{U \times \{c\}} (U \times \mathbb{R}; \mathcal{H}om^* (F, G)) \simeq R\mathcal{H}om(F|_{U \times \mathbb{R}}, T_{cs}(G|_{U \times \mathbb{R}})).
\]

**Proof.** The fist assertion is obvious. For the second isomorphism, the adjunction between $*$ and $\mathcal{H}om^*$ gives

\[
R\Gamma_{U \times \{c\}} (U \times \mathbb{R}; \mathcal{H}om^* (F, G)) \simeq R\mathcal{H}om(k_{U \times \{c\}}, \mathcal{H}om^* (F, G))
\]

\[
\simeq R\mathcal{H}om(k_{U \times \{c\}} \ast F, G)
\]

\[
\simeq R\mathcal{H}om(T_{-cs} F, G).
\]

□
Lemma 2.4. Let $F \in D_{r \geq 0}(k_{M \times \mathbb{R}})$ such that the map $\text{supp}(F) \to M$ is proper. Then $\text{R}_\Gamma(M \times \mathbb{R}; F) \simeq 0$.

Proof. Let $p: M \times \mathbb{R} \to M$ be the projection. It is enough to see that, for any $x \in M$, the stalk $(Rp_*F)_x \simeq \text{R}_\Gamma(\{x\} \times \mathbb{R}; F|_{\{x\} \times \mathbb{R}})$ vanishes. We choose $a < b$ with $\text{supp}(F|_{\{x\} \times \mathbb{R}}) \subset [a, b]$. We have

$$\text{R}_\Gamma(\mathbb{R}; F|_{\{x\} \times \mathbb{R}}) \cong \text{R}_\Gamma([-\infty, b + 1]; F|_{\{x\} \times \mathbb{R}}) \cong \text{R}_\Gamma([-\infty, a - 1]; F|_{\{x\} \times \mathbb{R}}) \simeq 0,$$

where the second isomorphism follows from the Morse result [8, Cor. 5.4.19] (applied with the function $\phi(t) = t$).

Lemma 2.5. Let $F \in D_{r \geq 0}(k_{M \times \mathbb{R}})$. Then, for any $a < b$, we have

$$\text{R}_\Gamma(M \times [a, b]; F)[-1] \cong \text{R}_\Gamma(M \times \{b\}; F).$$

Proof. Applying $\text{RHom}(-, F)$ to the distinguished triangle $k_{M \times [a, b]} \to k_{M \times [a, b]} \to k_{M \times \{b\}} \to k_{M \times [a, b]}[1]$ shows that the cone of the morphism of the lemma is $\text{R}_\Gamma(M \times \mathbb{R}; F')$, where $F' = \text{RHom}(k_{M \times [a, b]}, F)$. By Lemma 2.4 this cone vanishes.

We will have to consider sheaves $F$ on $\mathbb{R}$ for which it is easier to compute the “costalks” $\text{R}_\Gamma(t)(F)$ than directly the stalks $F_t$ and the next result will be useful.

Lemma 2.6. Let $F \in D_{r \geq 0}(k_\mathbb{R})$. We assume $\text{R}_\Gamma(t)(F) \simeq 0$ for all $t > 0$. Then $F|_{[0, \infty]} \simeq 0$.

Proof. We have $H^iF_t \simeq \lim_{\varepsilon \to 0} H^i([t - \varepsilon, t + \varepsilon]; F)$ (see [8, Rem. 2.6.9]). By Lemma 2.5 we have

$$\text{R}_\Gamma([t - \varepsilon, t + \varepsilon]; F) \simeq \text{R}_\Gamma(t + \varepsilon)(\mathbb{R}; F) \simeq \text{R}_\Gamma(\mathbb{R}; \text{R}_\Gamma(t + \varepsilon, F))$$

and this vanishes for all $t \geq 0$ and $\varepsilon > 0$ by the hypotheses of the lemma. Hence $H^iF_t \simeq 0$ for all $t \geq 0$ and $i \in \mathbb{Z}$ and the result follows.

2.2. Viterbo spectral invariants and boundary depth. In this section we recall the definition of Viterbo spectral invariants from the point of view of sheaf theory. Let $\bar{\Lambda} \subset T^*M$ be a compact exact Lagrangian submanifold and let $\Lambda \subset T^*_{r > 0}(M \times \mathbb{R})$ be an $\mathbb{R}_{r \geq 0}$-conic lift of $\bar{\Lambda}$. We know by [1] or [20] that there exists $F \in D(k_{M \times \mathbb{R}})$ such that $\text{SS}(F) = \Lambda$, $F|_{M \times \{t\}} \simeq 0$. Hence it is natural to consider the following conditions, for a general $F \in D(k_{M \times \mathbb{R}})$:

$$\left\{ \begin{array}{l}
\text{SS}(F) \subset T^*(M \times \mathbb{R}) \setminus \{0\}, \text{ for some } A > 0, \\
F|_{M \times \{t\}} \simeq 0, \text{ for } t < -A,
\end{array} \right.$$

The sheaf $F|_{M \times [A, +\infty]}$ is then locally constant; hence

$$F_+ = F|_{M \times \{t\}}, \text{ for } t > A,$$

is well-defined.
We will in fact consider more general \( \mathbb{R}_{>0} \)-conic Lagrangian submanifold of \( \Lambda \subset T^*_{\tau>0}(M \times \mathbb{R}) \), namely those coming from an immersed compact exact Lagrangian submanifold \( \overline{\Lambda} \) of \( T^*M \). We denote by

\[
D_{\Lambda,+}(k_{M \times \mathbb{R}})
\]

the subcategory of \( D(k_{M \times \mathbb{R}}) \) formed by the \( F \) such that \( \mathcal{S}(F) = \Lambda \) and \( F|_{M \times \{t\}} \simeq 0 \) for \( t \ll 0 \) (in particular \( F \) satisfies (4) because the compactness of \( \overline{\Lambda} \) implies \( \Lambda \subset T^*(M \times [-A, A]) \) for some \( A > 0 \)).

**Remark 2.7.** (i) In the case where \( \overline{\Lambda} \) is embedded we have a uniqueness statement in [4] or [20]. Let \( D_{lc}(k_M) \) be the subcategory of \( D(k_M) \) formed by the \( G \) such that \( H^iG \) is a locally constant sheaf, for any \( i \in \mathbb{Z} \). Then \( D_{\Lambda,+}(k_{M \times \mathbb{R}}) \to D_{lc}(k_M) \), \( F \mapsto F_+ \), is an equivalence of categories. In particular there exists a unique \( F \in D_{\Lambda,+}(k_{M \times \mathbb{R}}) \) such that \( F_+ \simeq k_M \).

(ii) By the results of [5] we have an immediate generalization of this equivalence if we replace \( \Lambda \) by a Legendrian deformation \( \Lambda_1 \) of \( \Lambda \) (here we identify \( \mathbb{R}_{>0} \)-conic Lagrangian submanifolds of \( T^*_{\tau>0}(M \times \mathbb{R}) \) with Legendrian submanifolds of \( J^1(M) \)). Indeed, for such a deformation, [5] gives an equivalence \( D_{\Lambda}(k_{M \times \mathbb{R}}) \simeq D_{\Lambda_1}(k_{M \times \mathbb{R}}) \) and we deduce easily that \( D_{\Lambda_1,+}(k_{M \times \mathbb{R}}) \to D(k_M) \), \( F \mapsto F_+ \), is again an equivalence.

**Lemma 2.8.** For \( F \in D(k_{M \times \mathbb{R}}) \) satisfying (4) we have

\[
R\Gamma(M \times \mathbb{R}; F) \simeq R\Gamma(M \times [t; +\infty]; F) \simeq R\Gamma(M; F_+)
\]

for all \( t \) and

\[
R\Gamma(M \times [-\infty, t]; F) \simeq R\Gamma(M \times \mathbb{R}; F) \quad \text{for } t > A,
\]

\[
R\Gamma(M \times [-\infty, t]; F) \simeq 0 \quad \text{for } t < -A.
\]

**Proof.** The cone of the first morphism is \( R\Gamma(M \times \mathbb{R}; R\Gamma[-\infty, t](F)) \), which vanishes by Lemma 2.4 (the support is proper by the hypothesis (4)). The second isomorphism is clear when \( t > A \) because \( F|_{M \times [A, +\infty]} \) is locally constant. The third isomorphism also reduces to Lemma 2.4 using that \( F|_{M \times [A, +\infty]} \) is locally constant. The last isomorphism is clear by (4).

**Definition 2.9.** For \( F \in D(k_{M \times \mathbb{R}}) \) satisfying (4) and \( \alpha \in H^*(M; F_+) \) we set \( c(F, \alpha) = \sup\{t; \alpha|_{M \times [-\infty, t]} = 0\} \). We set \( c_-(F) = \min\{c(F, \alpha); \alpha \in H^*(M; F_+)\} \) and \( c_+(F) = \max\{c(F, \alpha); \alpha \in H^*(M; F_+)\} \).

**Lemma 2.10.** We assume that \( \Lambda \subset T^*_{\tau>0}(M \times \mathbb{R}) \) is as in Remark 2.7 (i) or (ii), and we let \( F \in D_{\Lambda,+}(k_{M \times \mathbb{R}}) \) be the unique sheaf with \( F_+ \simeq k_M \). Then, for \( \alpha \in H^1(M; k_M) \), we have \( c(F, \alpha) = c(\Lambda, \alpha) \) (the usual Viterbo invariants of \( \Lambda \)).
In particular we have \( c_-(F) = c(\Lambda, 1) \) with \( 1 \in H^0(M; k_M) \simeq k \) and, if \( M \) is oriented or \( k \) is of characteristic 2, we have \( c_+(F) = c(\Lambda, \delta_M) \), where \( \delta_M \in H^n(M; k_M) \) is the fundamental class, \( n = \dim M \).

**Proof.** If we assume that \( \Lambda \) has a generating function quadratic at infinity, say \( f: M \times \mathbb{R}^N \to \mathbb{R} \), we have an easy construction of the sheaf as \( F = Rq_* (k_{\{t \geq f(x,v)\}})[\delta] \), where \( q: M \times \mathbb{R}^{N+1} \to M \times \mathbb{R} \) is the projection and \( i \) the index of the quadratic form which coincides with \( f \) at infinity. To see that this gives the correct sheaf we check using (1) that \( \text{SS}(F) \subset \Lambda \) (the map \( q \) is not proper, but, since \( f \) is a fibration at infinity, we can check directly that (1) still holds here); we also see that \( F_+ \simeq k_M \) and we conclude by the uniqueness property that this \( F \) is the same as in the statement of the lemma. Now \( H^i(M \times ]-\infty, t[; F) \simeq H^i(\{ f < t \}, k) \) and we recover the original definition of \( c(\Lambda, \alpha) \) in [17].

When \( \Lambda \) is not given by a generating function, we can still use the construction of \( F \) in [20] to see that we obtain the definition of \( c(\Lambda, \alpha) \) by Floer homology. \( \square \)

**Remark 2.11.** Let \( f: M \to N \) be a proper map and \( f' = f \times \text{id}_{\mathbb{R}} \). By (1) we see that \( Rf_* F \) satisfies (1) if \( F \) does. We also have \( (Rf_* F)_+ \simeq Rf_* F_+ \) and we can identify \( H^*(N; (Rf_* F)_+) \) with \( H^*(M; F_+) \). Since \( R\Gamma(M \times ]-\infty, t[; F) \simeq R\Gamma(N \times ]-\infty, t[; Rf_* F) \), we have

\[
c(F, \alpha) = c(Rf_* F, \alpha).
\]

In particular, when \( N \) is a point we have \( c(F, \alpha) = c(Ra_* F, \alpha) \), where \( a: M \times \mathbb{R} \to \mathbb{R} \) is the projection.

When \( G = Ra_* F \) is constructible (see Lemma 2.12 for a sufficient condition), we have a description of the numbers \( c(G, \alpha) \) through “barcodes”. We will not need constructible sheaves in general and just recall what it means over \( \mathbb{R} \) (see [9] for more details).

An object \( G \in \mathcal{D}(k_{\mathbb{R}}) \) is constructible if there exists finitely many points \( t_1, \ldots, t_N \in \mathbb{R} \) such that the restriction of \( G \) to each component of \( \mathbb{R} \setminus \{t_1, \ldots, t_N\} \) is constant and moreover each stalk \( G_t \in \mathcal{D}(k) \), \( t \in \mathbb{R} \), is a bounded complex with finite dimensional cohomology. It is not difficult to give an equivalence between the category of constructible sheaves with respect to \( \{t_1, \ldots, t_N\} \) and a category of quiver representations for a quiver of type \( A_{2N+1} \). Since we work with coefficient in a field, we can apply Gabriel theorem and deduce that \( G \) is a sum of constant sheaves on intervals. So we can write \( G \simeq \bigoplus_{I \in \mathcal{I}} k_I^{|I|}[d_I] \), where \( \mathcal{I} \) is a finite family of intervals (with ends belonging to \( \{ -\infty, t_1, \ldots, t_N, +\infty \} \)), \( n_I \in \mathbb{N} \) and \( d_I \in \mathbb{Z} \). If we assume moreover, as will always be the case, that \( \text{SS}(G) \subset \{ \tau \geq 0 \} \), we know that these intervals are of the type \( I = [a, b] \) with \( a \in \mathbb{R} \cup \{ -\infty \} \), \( b \in \mathbb{R} \cup \{ +\infty \} \); in this case we call \( G \) a *barcode*. 


Lemma 2.10 is simple.

Proof. G act Lagrangian submanifold \( \{ d\varphi \in T^*M \mid \varphi \circ \pi \} \) transversely for \( \cap \). For any given compact exact Lagrangian submanifold \( \Lambda \subset T^*M \) meets \( 0_M \), we have \( G_t \cong 0 \) for \( t \ll 0 \). It follows from the above discussion that

\[
G \simeq \bigoplus_{i \in I_1} k_{[c_i, +\infty)}[d_i] \oplus \bigoplus_{j \in I_2} k_{[a_j, b_j]}[d_j]
\]

with \( a_i, b_i, c_i \in \mathbb{R}, n_i \in \mathbb{N} \) and \( d_i \in \mathbb{Z} \). Then \( H^*(M; F_+) \simeq \bigoplus_{i \in I_1} k^{n_i}[d_i] \) and the numbers \( c(\alpha, F) \) coincide with the \( c_i \)'s. In particular

\[
c_-(F) = \min \{ c_i; i \in I_1 \}, \quad c_+(F) = \max \{ c_i; i \in I_1 \}.
\]

Let \( \Lambda \subset T^*_{\tau > 0}(M \times \mathbb{R}) \) be as above a \( \mathbb{R}_{>0} \)-conic closed Lagrangian submanifold. Let \( F \in D(k_{M \times \mathbb{R}}) \) with \( SS(F) = \Lambda \) and let \( (z_0; \eta_0) \in \Lambda \). We choose \( \varphi: M \times \mathbb{R} \to \mathbb{R} \) such that \( \Gamma_{\varphi} = \{ (z; d\varphi(z)) \} \) intersects \( \Lambda \) transversely at \( (z_0; \eta_0) \). Then we know by Prop. 7.5.3 and Prop. 7.5.9 of \[8\] that, if \( \Lambda \) is connected, \( (R\Gamma_{\varphi \circ \pi}(L))_{z_0} \in D(k) \) is independent of the choice of \( z_0 \) and \( \varphi \), up to a shift in degree. With a suitable normalization of the shift (see \[8\]) \( (R\Gamma_{\varphi \circ \pi}(L))_{z_0} \) is called the type of \( F \). We say \( F \) is of finite type if the total cohomology of its type is finite dimensional. In \[8\] \( F \) is called simple if the total cohomology of its type is of dimension 1. The sheaf \( F \) of Lemma 2.10 is simple.

Lemma 2.12. We assume that \( \Lambda \) is a \( \mathbb{R}_{>0} \)-conic lift of an immersed compact exact Lagrangian submanifold \( \overline{\Lambda} \) of \( T^*M \) such that \( \overline{\Lambda} \) meets \( 0_M \) transversely. Let \( F \in D_{\Lambda, <}(k_{M \times \mathbb{R}}) \) such that \( F \) is of finite type in the above sense. Then \( Ra_\varphi F \) is constructible.

Proof. We set \( G = Ra_\varphi F \). Then \( SS(G) \) is contained in the projection to \( T^*\mathbb{R} \) of \( L = \Lambda \cap (0_M \times T^*\mathbb{R}) \). Since \( \overline{\Lambda} \) meets \( 0_M \) transversely, the intersection defining \( L \) is also transverse and \( L \) is a finite family of half-lines; this family is in bijection with \( \overline{\Lambda} \cap 0_M \). Hence \( SS(G) = \bigcup_{k=1}^{N} \{ t_k \} \times \{ \tau > 0 \} \), for some \( N \). Moreover \( G \) is of finite type by \[8\] Cor. 7.5.12 at all points of \( SS(G) \) (the type may vary since \( SS(G) \) is not connected).

In particular \( G \) is constant on the intervals of \( \mathbb{R} \setminus \{ t_1, \ldots, t_N \} \) and it only remains to check that the stalks \( G_t \) are finite dimensional. We have \( G_t \cong 0 \) for \( t \ll 0 \). Assuming the \( t_k \)'s are ordered, for \( t \in [t_{k-1}, t_k] \) and \( s \in [t_k, t_{k+1}] \), we have a natural morphism \( G_t \to G_s \) whose cone is the type of \( G \) at \( (t_k; 1) \). Hence \( G_s \) is finite dimensional if \( G_t \) is. By induction we see that all stalks are finite dimensional.

Remark 2.13. We give a series of remarks to check that we can restrict ourselves to the transverse case when we want to bound the spectral invariants.

(1) For any given compact exact Lagrangian submanifold \( \overline{\Lambda} \) of \( T^*M \), we can find a compactly supported Hamiltonian isotopy \( \phi^t, t \in [0, \varepsilon] \), such that \( \phi^t(\overline{\Lambda}) \) meets \( 0_M \) transversely for \( t \neq 0 \).
(2) We can lift the $\phi'(\Lambda)$’s into a family of conic Lagrangian $\Lambda_t$, $t \in [0, \varepsilon]$. To a given $F \in D_{\Lambda_t}(K_{M \times R})$ we can then associate a family $F_t \in D_{\Lambda_t}(K_{M \times R})$ by $[3]$. These $F_t$ tend to $F$ with respect to a distance induced by the morphisms $\tau_c$ of (2), namely, there exist $a > 0$ and morphisms $T_{-at*}F_t \xrightarrow{\alpha} F \xrightarrow{\alpha} T_{at*}F_t$ such that

$$v_t \circ u_t = \tau_{2at}(T_{-at*}F_t), \quad T_{at*}(u_t) \circ v_t = \tau_{2at}(F).$$

For the existence of $u_t$ and $v_t$ we choose $a$ big enough so that $t \mapsto \tilde{\phi}^{-t} \circ T_t$ and $t \mapsto T_t \circ \tilde{\phi}^t$ are non negative isotopies, where $\phi^t$ is a homogeneous lift of $\phi_t$ and we write abusively $\tilde{\phi}^t = (x, t; \xi, \tau) = (x + c; \xi, \tau)$, and we apply $[5]$ Prop. 4.8.

(3) We have $(F_t)_+ \simeq F_+$ for all $t$. We can deduce from (9) that, for any $\alpha \in H^*(M; F_+) \simeq H^*(M; (F_t)_+)$, the function $t \mapsto c(\alpha, F_t)$ is continuous.

2.3. Propagation and torsion. Now we assume $M$ is endowed with a metric. We denote by $|| \cdot ||_x$ (or $| \cdot |$) the norms induced on the tangent and cotangent spaces $T_x M$, $T^*_x M$. For $\alpha > 0$ we define the conic subset of $T^*(M \times R)$:

$$C_\alpha = \{(x, t; \xi, \tau); \tau \geq \alpha ||\xi||\}.$$

**Lemma 2.14.** Let $\alpha > 0$ and $F, G \in D(k_{M \times R})$ such that $SS(F)$ and $SS(G)$ are contained in $C_\alpha$.

(i) Let $i: N \to M$ be the inclusion of a submanifold and $i_+ = i \times id_R$. Then $SS(i_+^{-1} F) \subset C_\alpha$, where $C_\alpha$ is defined on $N \times R$ using the induced metric. We also have

$$i_+^{-1} \text{Hom}^*(F, G) \simeq \text{Hom}^*(i_+^{-1} F, i_+^{-1} G).$$

(ii) $SS(\text{Hom}^*(F, G)) \subset C_{\alpha/2}$.

**Proof.** The first part of (i) follows from (1) and the second part from $[6]$ Cor. 4.15. The bound (ii) is a particular case of $[6]$ Prop. 4.13. \qed

**Proposition 2.15.** Let $F \in D(k_{M \times R})$ such that $SS(F)$ is contained in $C_\alpha$. Let $x_0 \in M$ and $R > 0$ be given with $R$ less than the injectivity radius of $M$. Let $B_R$ be the open ball centered at $x_0$ of radius $R$. For $a < b' \leq b \leq 0$ we have the natural restriction morphisms

$$r_{x_0, a, b'}: H^0(B_R \times ]a, b'; F) \to H^0(\{x_0\} \times ]a, b'; F),$$

$$r_{a, b'}: H^0(B_R \times ]a, b'; F) \to H^0(B_R \times ]a, b'; F).$$

We assume that $b' < b - \alpha^{-1} R$. Then $\ker(r_{x_0, a, b'}) \subset \ker(r_{a, b'})$.

**Proof.** (i) We take $s \in \ker(r_{x_0, a, b'})$ and prove that $s|_{B_R \times ]a, b'}$ vanishes. By Lemma 2.5 the choice of $a$ is not relevant since, for example, $H^0(B_R \times ]a, b'; F) \cong H^0(B_R \times ]a', b'; F)$ for any $a \leq a' < b'$ (at this point we could even take $a = -\infty$, but it is
better to have a proper support to apply the Morse result below). We choose \( a' \) with \( a < a' < b \). Hence it is enough to see that \( s|_{B_R \times ]a',b[} \) vanishes.

(ii) We define \( \varphi(x) = d_M(x_0, x) \) on \( M \) and we choose \( 0 < \beta < \alpha \) such that \( b' < b - \beta^{-1}R \). For \( \varepsilon > 0 \), we define the open “cone” \( C_\varepsilon = \{ (x,t) \in B_R \times \mathbb{R}; t < b - \varepsilon - \beta^{-1}\varphi(x) \} \) (see Fig. 1). For \( \varepsilon \) small enough we have \( B_R \times ]a',b[ \subset C_\varepsilon \) and it is enough to see that the restriction of \( s \) to \( (B_R \times ]a',0[) \cap C_\varepsilon \) vanishes.

Since \( r_{x_0,]a,b[}(s) = 0 \), there exists an open neighborhood \( U \) of \( \{x_0\} \times ]a,b[ \) in \( B_R \times \mathbb{R} \) such that \( s|_U \) vanishes (indeed \( H^0(\{x_0\} \times ]a,b[; F|_{\{x_0\} \times \mathbb{R}}) \simeq \lim_{U} H^0(U; F) \) where \( U \) runs over such open neighborhoods – see [8, Rem. 2.6.9]). Now we can find \( \delta > 0 \) such that \( U \) contains \( (B_\delta \times ]a',0[) \cap C_\varepsilon \). We visualize the restriction morphisms between the subsets introduced so far in the diagram

\[
\begin{align*}
H^0(B_R \times ]a,0[; F) &\xrightarrow{r_1} H^0(\{x_0\} \times ]a,b[; F|_{\{x_0\} \times \mathbb{R}}) \\
H^0((B_R \times ]a',0[) \cap C_\varepsilon; F) &\xrightarrow{r_2} H^0((B_\delta \times ]a',0[) \cap C_\varepsilon; F) \\
H^0(B_R \times ]a',b'[; F). &
\end{align*}
\]
We want to prove that \( r_{\gamma^1, \gamma^0}(s) = 0 \). It is enough to see that \( r_\gamma(s) = 0 \). Since \( s_{\mid \gamma^1} = 0 \), we have \( r_\gamma \circ r_{\gamma^0}(s) = 0 \) and it is enough to see that \( r_\gamma \) is an isomorphism.

Setting \( V = (M \times \{a', 0\}) \cap C_\varepsilon \) and \( \psi(x, t) = \varphi(x) \) we have

\[
H^0((B_{\gamma^1} \times ]a', 0[) \cap C_\varepsilon; F) \simeq H^0(\psi^{-1}([-\infty, \gamma^0]; R\Gamma_V(F))
\]

for any \( \gamma \). By the microlocal Morse result in [9, Cor. 5.4.19], the restriction from \( \psi^{-1}([-\infty, \gamma]) \) to \( \psi^{-1}([-\infty, \delta]) \) is an isomorphism if

\[
(x, t; d\varphi(x), 0) \notin SS(R\Gamma_V(F)) \quad \text{for all } x \in B_R \setminus B_\delta.
\]

(iii) We check (11) which concludes the proof of the proposition. We estimate \( SS(R\Gamma_V(F)) \) using (1) (we recall that \( R\Gamma_V(F) \simeq R\text{Hom}(k_V, F) \)). When we restrict over \( (B_R \setminus B_\delta) \times \mathbb{R} \), the boundary of \( V \) consists of two smooth disjoint hypersurfaces, \( M \times \{a'\} \) and \( \partial C_\varepsilon \). We only check (11) at a point \( (x, t) \in \partial C_\varepsilon \) (the case \( (x, t) \in V \) is obvious since \( R\Gamma_V(F) \simeq F \) near such a point and the case \( (x, t) \in M \times \{a'\} \) is done in the same steps but easier). Let \( A, B \) be the intersection of \( SS(k_V) \), \( SS(F) \) with the fiber at \( (x, t) \). Then \( A = \mathbb{R}_{\geq 0} \cdot (d\varphi(x), \beta) \) and \( B \subset \{ \tau \geq \alpha ||\xi||_x \} \). Since \( ||d\varphi(x)||_x = 1 \) and \( \beta < \alpha \), we have \( A \cap B = \{0\} \) and, by (1), we obtain the bound \( A^\alpha + B \) for \( SS(R\Gamma_V(F)) \) at \( (x, t) \). Hence, if (11) does not hold, there exist \( \lambda \geq 0 \) and \( (\xi, \tau) \in B \) such that \( \lambda(d\varphi(x), 0) = \lambda(d\varphi(x), \beta) + (\xi, \tau) \), which gives \( \lambda \cdot \beta = \tau \geq \alpha ||\xi||_x = \alpha(1 + \lambda) \). This contradicts \( \beta < \alpha \) and proves (11).

In the following corollary we make use of the remark: for any morphism \( i: N \to M \) (for example, \( i \) is the inclusion of a submanifold or an open subset) and \( i_+ = i \times \text{id}_\mathbb{R} \), the inverse image \( i_+^{-1} \) sends \( D_{\tau \geq 0}(k_{M \times \mathbb{R}}) \) to \( D_{\tau \geq 0}(k_{N \times \mathbb{R}}) \) and we have \( i_+^{-1} \circ T_{c^*} = T_{c^*} \circ i_+^{-1} \) and

\[
\tau_{c}(i_+^{-1}F) = i_+^{-1}((\tau_{c}(F)).
\]

**Corollary 2.16.** Let \( F \in D(k_{M \times \mathbb{R}}) \). We assume that \( SS(F) \) is contained in \( C_\alpha \) and that the morphism \( \tau_{c}(F|_{\{x_0\} \times \mathbb{R}}) \) vanishes for some \( x_0 \in M \) and some \( c \geq 0 \). Let \( R > 0 \) be given, less than the injectivity radius of \( M \). Let \( B \) be the open ball centered at \( x_0 \) of radius \( R \). Then the morphism \( \tau_{c}(F|_{B \times \mathbb{R}}) \) vanishes for \( c' > c + 2\alpha^{-1}R \).

**Proof.** We set \( F_1 = \text{Hom}^*(F, F)|[-1] \). By Lemmas [2.3 and 2.5], the morphism \( \text{id}_F \) corresponds to a section \( s \in H^1_{M \times \{0\}}(M \times \mathbb{R}; F_1) \simeq H^0(M \times \{a, 0\}; F_1) \), for any \( a < 0 \), and we have to prove that the image of \( s \) in \( H^1_{B \times \{-c'\}}(B \times \mathbb{R}; F_1) \simeq H^0(B \times \{a, -c'\}; F_1) \) vanishes (assuming \( a < -c' \)).

Lemma [2.14] gives \( SS(F_1) \subset C_{a/2} \) and \( F_1|_{\{x_0\} \times \mathbb{R}} \simeq \text{Hom}^*(F|_{\{x_0\} \times \mathbb{R}}, F|_{\{x_0\} \times \mathbb{R}})|[-1]. \)

The hypotheses say that the image of \( s \) in \( H^0(\{x_0\} \times \{a, -c\}; F_1|_{\{x_0\} \times \mathbb{R}}) \) is zero. Now the result follows from Proposition [2.15].
2.4. Barcodes computations. We refer to the discussion before (7) for the definition of barcode.

We want to compute $\mathcal{H}om^*(F,G)$ where $F, G$ are sums of constant sheaves on intervals $[a, b]$ with $b$ being possibly infinite. We give the following explicit formulas.

**Lemma 2.17.** Let $a, b, c, d$ be real numbers. Then

$$\mathcal{H}om^*(k_{[a,b]}, k_{[c,d]}) \simeq k_{[c-b, \min(c-a, d-b)]}[1] \oplus k_{[\max(c-a, d-b), d-a]}[1],$$

$$\mathcal{H}om^*(k_{[a,\infty]}, k_{[c,\infty]}) \simeq k_{[-\infty, c-a]}[1],$$

$$\mathcal{H}om^*(k_{[a,b]}, k_{[c,\infty]}) \simeq k_{[-c+b, c-a]}[1],$$

$$\mathcal{H}om^*(k_{[a,\infty]}, k_{[c,d]}) \simeq k_{[c-a, d-a]},$$

where $k_{[\alpha,\beta]} = 0$ if $\alpha \geq \beta$.

**Proof.** We only prove the first two isomorphisms, the last two ones being similar to the second one. Let us denote by $i: \mathbb{R} \to \mathbb{R}$ the involution $x \mapsto -x$ and $q_1, q_2: \mathbb{R}^2 \to \mathbb{R}$ the projections on the first and second factor respectively. We use the formula $\mathcal{H}om^*(F,G) \simeq R_s \mathcal{R}\mathcal{H}om(q_2^{-1}i^{-1}F, q_1^1G)$ see [16] or [6, Lem. 4.10]. We obtain

$$\mathcal{H}om^*(k_{[a,b]}, k_{[c,d]}) = R_s \mathcal{R}\mathcal{H}om(q_2^{-1}i^{-1}k_{[a,b]}, q_1^1k_{[c,d])}$$

$$\simeq R_s \mathcal{R}\mathcal{H}om(k_{\mathbb{R} \times [-b,-a]}, k_{c,d}[\times \mathbb{R}])[1]$$

$$\simeq R_s \mathcal{R}\mathcal{H}om(k_{\mathbb{R} \times [-b,-a]}, \mathcal{R}\mathcal{H}om(k_{[c,d] \times \mathbb{R}}, k_{\mathbb{R}^2}))[1]$$

$$\simeq R_s \mathcal{R}\mathcal{H}om(k_{[-b,-a] \times [c,d]}, k_{\mathbb{R}^2}))[1]$$

$$\simeq R_s(k_{[-b,-a] \times [c,d]}[1].$$

We can easily compute local sections of this direct image: it amounts to computing the cohomology of the constant sheaf on $E_I = ([-b, -a] \times [c, d]) \cap s^{-1}(I)$ where $I$ is an (open) interval of $\mathbb{R}$. (To get the stalk at $t$, we choose $I = \{t\}$ and $E_I$ is an interval which is closed, open or half-closed according to $t$.) This gives the first formula.

We perform a similar computation to prove the second formula:

$$\mathcal{H}om^*(k_{[a,\infty]}, k_{[c,\infty]}) \simeq R_s \mathcal{R}\mathcal{H}om(k_{\mathbb{R} \times [-\infty, -a]} \otimes k_{[c,\infty] \times \mathbb{R}}, k_{\mathbb{R}^2}))[1]$$

$$\simeq R_s \mathcal{R}\mathcal{H}om(k_{[-\infty, -a] \times [c,\infty]}, k_{\mathbb{R}^2}))[1]$$

$$\simeq R_s(k_{[-\infty, -a] \times [c,\infty]}[1]$$

and we conclude by the same argument as for the first isomorphism. \hfill $\square$

We now introduce a convenient bound for the spectral norm $c_+(F) - c_-(F)$ (see Lemma 2.23 below).
Definition 2.18. For $F \in \mathcal{D}_{r \geq 0}(k_{\mathbb{R}})$ we define its boundary depth

\begin{equation}
\beta(F) = \min \{ c; \tau_c(F) = 0 \}
\end{equation}

and, for $F, F' \in \mathcal{D}_{r \geq 0}(k_{\mathbb{R}})$ we define

\begin{align*}
V(F, F') &= \mathcal{H}\text{om}^s(F, F') \otimes k_{[0, \infty]}, \\
v(F, F') &= \beta(V(F, F')).
\end{align*}

For a manifold $M$ and $F, F' \in \mathcal{D}_{r \geq 0}(k_{M \times \mathbb{R}})$ we also set $v(F, F') = v(Ra_* F, Ra_* F')$, where $a: M \times \mathbb{R} \to \mathbb{R}$ is the projection.

Remark 2.19. A sheaf $F \in \mathcal{D}_{r \geq 0}(k_{\mathbb{R}})$ satisfying \((\text{H})\) fits in a distinguished triangle

\begin{equation}
F_0 \to F \to E_{[A, \infty]} \xrightarrow{+1},
\end{equation}

where $\text{supp}(F_0)$ is compact and $E$ is a complex of vector spaces. If $F_1$ and $F_2$ are either with compact support or of the form $E_{[A, \infty]}$, it is easy to check that $V(F_1, F_2)$ is torsion (see Definition \((2.1)\)). Using \((13)\) we deduce that $V(F, F')$ is torsion for $F, F'$ satisfying \((\text{H})\), hence that $v(F, F')$ is finite.

Remark 2.20. If $F \in \mathcal{D}_{r \geq 0}(k_{\mathbb{R}})$ is constructible, we have seen in the discussion before \((7)\) that $F \simeq \bigoplus_{f \in \mathcal{I}} k_f$, where $\mathcal{I}$ is a finite family of intervals $[a, b]$, $a$ and $b$ possibly infinite. The boundary depth of $\mathcal{I}$ is usually defined as the longest finite interval in this decomposition. When the barcode $F$ contains no infinite bar, it is torsion and the boundary depth of $\mathcal{I}$ coincides with $\beta(F)$.

Remark 2.21. (1) It is clear from the definition that, for a morphism $f: M \to N$, we have $v(R(f \times \text{id}_{\mathbb{R}})_*(F), R(f \times \text{id}_{\mathbb{R}})_*(F')) = v(F, F')$.

(2) For $F_1, F_2 \in \mathcal{D}_{r \geq 0}(k_{\mathbb{R}})$ we have $V(F_1 \oplus F_2, F') \simeq V(F_1, F') \oplus V(F_2, F')$. Hence $v(F_1, F') \leq v(F_1 \oplus F_2, F')$. In the same way, $v(F', F_1) \leq v(F', F_1 \oplus F_2)$.

Lemma 2.22. Let $F_1, F_2, F, G \in \mathcal{D}_{r \geq 0}^{+l}(k_{\mathbb{R}})$. We assume that there exists a distinguished triangle $F_1 \to F_2 \to F \xrightarrow{+1}$. Then $v(F, G) \leq v(F_1, G) + v(F_2, G)$ and $v(G, F) \leq v(G, F_1) + v(G, F_2)$.

Proof. We have distinguished triangles $V(G, F_1) \to V(G, F_2) \to V(G, F) \xrightarrow{+1}$ and $V(F, G) \to V(F_2, G) \to V(F_1, G) \xrightarrow{+1}$. Now the result follows from Lemma 2.22.

In order to give a motivation to the definition of $v(F, F')$, it is interesting to compute the special case $v(F, F)$. According to the bound of Lemma 2.17 in the transverse case, $v(F, F)$ is an upper bound of the Viterbo spectral distance and of the boundary depth of the couple $(F, k_{[0, \infty]})$. We check that this holds in the general case. As in \S 2.2, we let $\Lambda \subset T^* M$ be a compact exact Lagrangian submanifold and $\Lambda \subset T^*_{r \geq 0}(M \times \mathbb{R})$ a $\mathbb{R}_{r \geq 0}$-conic lift of $\Lambda$. 


Lemma 2.23. Let $F \in D_{\Lambda,+}(k_{M \times \mathbb{R}})$. We assume that $F$ is of finite type. Then $c_+(F) - c_-(F) \leq v(F,F)$.

Proof. By Remark 2.13 we can assume that $\overline{\Lambda}$ meets 0 transversely. By Lemma 2.12 $G := R\alpha_F$ is constructible, hence $G \simeq \bigoplus_{i \in I_1} k_{c_i,\infty}^i [d_i] \oplus \bigoplus_{i \in I_2} k_{a_j,b_j}^i [d_j]$ as in (7).

As in (8) the family $I_1$ contains $i_{\pm}$ such that $c_{i_{\pm}} = c_\pm(F)$ and $d_{i_{\pm}} = 0$, $d_{i_{\pm}} = -\dim M$. It then follows from Lemma 2.17 that $V(G,G)$ decomposes as a sum of sheaves of the type $k_{[a,b]}[c]$ which contains $k_{[0,c_{i_+} - c_{i_-}]}[-\dim M]$ as one summand. The result follows.

3. Statements

We give bounds for the spectral norm in the case where $M = G$ is a compact Lie group and $\Lambda = G/H$ where $H$ is a Lie subgroup of $G$. We endow $G$ with a bi-invariant metric and put the induced metric on $G/H$. In both cases we denote by $B_1(M)$ the unit ball bundle in $T^*M$. Let $\overline{\Lambda} \subset B_1(M)$ be an immersed compact exact Lagrangian submanifold and let $\Lambda \subset \dot{T}^*(M \times \mathbb{R})$ be a Legendrian lift of $\overline{\Lambda}$, seen as a conic Lagrangian submanifold contained in $\{ \tau > 0 \}$.

We set $n = \dim G$, $m = \dim G/H$ and we let $l$ be the diameter of $G$ and $l_{\text{max}}$ the length of the maximal Reeb chord of $\Lambda$. We recall the notations $D_{\Lambda,+}(k_{G\times\mathbb{R}})$ of (6) and $v(F,F')$ of Definition 2.18 which is a bound for the spectral norm.

Theorem 3.1. For any $F,F' \in D_{\Lambda,+}(k_{G\times\mathbb{R}})$ of finite type we have

$$v(F,F') \leq (n+1)(2l + l_{\text{max}}).$$

Corollary 3.2. If $H$ is not connected, we assume moreover that the characteristic of $k$ does not divide $|\pi_0(H)|$. Then, for any $F,F' \in D_{\Lambda,+}(k_{G\times\mathbb{R}})$ of finite type, we have

$$v(F,F') \leq \frac{1}{4}(m+3)^2(n+1)(2l + l_{\text{max}}).$$

We give the proof of Theorem 3.1 in §4 and deduce Corollary 3.2 in §5.

For a general $\Lambda$ it may happen that there is no sheaf in $D_{\Lambda,+}(k_{G\times\mathbb{R}})$. In good cases we have a canonical $F$ and deduce results on the spectral norm of $\Lambda$.

Corollary 3.3. We assume moreover that $\Lambda$ is a Legendrian deformation in $J^1(M \times \mathbb{R})$ of the Legendrian lift of an embedded compact exact Lagrangian submanifold of $T^*M$. Then

$$c_+(\Lambda) - c_-(\Lambda) \leq \begin{cases} (n+1)(2l + l_{\text{max}}) & \text{if } M = G, \\ \frac{1}{4}(m+3)^2(n+1)(2l + l_{\text{max}}) & \text{if } M = G/H \text{ and char}(k) \nmid |\pi_0(H)|. \end{cases}$$

Proof. By Remark 2.7 there exists a unique $F \in D_{\Lambda,+}(k_{M\times\mathbb{R}})$ such that $F_+ \simeq k_M$. By [4] this sheaf is simple, in particular of finite type. Now the result follows from Lemmas 2.10 and 2.23. □
4. Proof of Theorem 3.1 - compact Lie group case

We assume in this section that $M = G$ is a compact Lie group. We let

$$p_1, p_2, \mu : G \times \mathbb{R} \to G \times \mathbb{R}$$

be the projections and the action defined by $p_1(g, t, g', t') = (g', t')$, $p_2(g, t, g', t') = (g, t')$, $\mu(g, t, g', t') = (g, t + t')$ and $a(g, t) = t$. We will use the following variation of the functor $\text{Hom}^*$ (see (3)) which takes into account the group structure. For $F, F' \in D(k_{G \times \mathbb{R}})$ we define a sheaf on $G \times \mathbb{R}$:

$$\text{Hom}^*_G(F, F') = R\mathcal{H}om(p_1^* F, \mu^! F').$$

This defines a functor $\text{Hom}^*_G$ which comes with a left adjoint $*_G$ defined by $F*G F' = R\mu_!(p_1^{-1} F \otimes p_2^{-1} F')$:

$$\text{Hom}(F, \text{Hom}^*_G(F', F'')) \simeq \text{Hom}(F*G F', F'').$$

We have seen in Section 2.4 how to obtain some information about the spectral invariants from $\text{Hom}^*(R a^* F, R a^* F')$. In Proposition 4.1 we express this latter sheaf as the direct image of $\text{Hom}^*_G(F, F')$ by $a$ and in Lemmas 4.2, 4.4 we check that $R\Gamma_{G \times [0, +\infty]} \text{Hom}^*_G(F, F')$ satisfies the hypotheses of Corollary 2.16, which will be used to bound $v(F, F')$. The fact that $\Lambda$ is the Legendrian lift of an immersed exact Lagrangian is actually only used in Lemma 4.3; the others results follows from $\mathfrak{K} \subset B_1(G)$.

**Proposition 4.1.** We have $R a_* \text{Hom}^*_G(F, F') \simeq \text{Hom}^*(R a_* F, R a_* F')$.

**Proof.** Let $a : G \times \mathbb{R} \to \mathbb{R}$ the canonical projection. We introduce some maps and commutative diagrams in order to create Cartesian squares and apply base changes:

$$
\begin{array}{ccc}
(G \times \mathbb{R}) \times (G \times \mathbb{R}) & \xrightarrow{a \times \text{id}} & \mathbb{R} \times (G \times \mathbb{R}) \\
\downarrow p_1 & & \downarrow \pi_1 \\
G \times \mathbb{R} & \xrightarrow{a} & \mathbb{R} \\
\end{array}
\quad
\begin{array}{ccc}
(G \times \mathbb{R}) \times (G \times \mathbb{R}) & \xrightarrow{a \times \text{id}} & \mathbb{R} \times (G \times \mathbb{R}) \\
\downarrow p_2 & & \downarrow \pi_2 \\
G \times \mathbb{R} & \xrightarrow{\text{id}} & G \times \mathbb{R} \\
\end{array}
\quad
\begin{array}{ccc}
(G \times \mathbb{R}) \times (G \times \mathbb{R}) & \xrightarrow{a \times \text{id}} & \mathbb{R} \times (G \times \mathbb{R}) \\
\downarrow \mu & & \downarrow \pi \\
G \times \mathbb{R} & \xrightarrow{a} & \mathbb{R} \\
\end{array}
$$
with $\overline{\mu}$ the sum over the $\mathbb{R}$ factors. Using these notations we obtain the sequence of isomorphisms

$$\text{Ra}_* \text{RHom}(p_2^{-1} F, \mu^i F')$$

$$\simeq \text{Ra}_* \text{R}(\text{id} \times a)_* \text{R}(a \times \text{id})_* \text{RHom}((a \times \text{id})^{-1} \overline{\mu}_2^{-1} F, \mu^i F')$$

$$\simeq \text{Ra}_* \text{R}(\text{id} \times a)_* \text{RHom}(\overline{\mu}_2^{-1} F, R(a \times \text{id}), \mu^i F')$$

$$\simeq \text{Ra}_* \text{R}(\text{id} \times a)_* \text{RHom}(\overline{\mu}_2^{-1} F, R(a \times \text{id}), \mu^i F')$$

$$\simeq \text{Ra}_* \text{RHom}(\overline{\mu}_2^{-1} F, \mu^i F')$$

We have $\text{SS}(\text{RHom}^* G(F, F')) \subset C_1$.

**Lemma 4.2.** For any $F, F' \in D_{\Lambda,+}(\mathbb{R}G \times \mathbb{R})$ we have $\text{SS}(\text{RHom}^* G(F, F')) \subset C_1$.

**Proof.** This is proved in the same way as (ii) of Lemma [2.14]. By [8, Prop. 5.4.5] we have $\text{SS}(p_2^{-1} F) \subset A$, $\text{SS}(\mu^i F') \subset B$, where

$$A = \{(g, t_1, t_2; g, \tau) \in T^* G; \tau \geq \|\gamma\| \},$$

$$B = \{(g, t_1, t_2; \gamma, \gamma_1, \gamma_2, \tau_\gamma); \tau_\gamma = \gamma_0 = \exists \gamma \in T^* G; \gamma \geq \|\gamma\|, \gamma_0 = t(\mu_2^\gamma, \gamma_2, \tau_\gamma), \gamma_1 = \gamma_2 = t(\mu_2^\gamma, \gamma_2, \tau_\gamma),\}$$

where $\mu_2^\gamma, \mu_2^\gamma : G \to G$ are the right and left actions, $g \mapsto gh$, $g \mapsto hg$. Since $A \cap B$ is contained in the zero section, we deduce $\text{SS}(- \text{RHom}(p_2^{-1} F, \mu^i F')) \subset A^a + B$, using [8, Prop. 5.4.14]. Since the metric is bi-invariant, we have the rough bound $B \subset \{\tau_\gamma \geq \|\gamma\| \}$ and hence $A^a + B \subset \{\tau_\gamma \geq \|\gamma\| \}$. By [8, Prop. 5.4.4] $\text{SS}(\text{RHom}^* G(F, F'))$ is contained in the projection of $A^a + B$ to the first factor $T^* (G \times \mathbb{R})$, which concludes the proof. \hfill $\Box$

**Lemma 4.3.** For any $F, F' \in D_{\Lambda,+}(\mathbb{R}G \times \mathbb{R})$ we have $\text{RHom}(F, T_{-c}\epsilon F') \simeq 0$ for all $c > l_{max}$.

**Proof.** By Lemma [2.3] we have $\text{RHom}(F, T_{-c}\epsilon F') \simeq \text{R}(c) \text{Ra}_* \text{Hom}^* (F, F')$. We also have $\text{Ra}_* \text{Hom}^* (F, F') \simeq \text{Ra}_* \text{RHom}(q_2^{-1} F, s^i G)$, where $b = a \circ q_1 : G \times \mathbb{R}^2 \to \mathbb{R}$, $(g, t_1, t_2) \mapsto t_1$ and $q_1, q_2, s$ are the notations of [3]. We give a bound for the microsupport of $\text{Ra}_* \text{Hom}^* (F, F')$. By (11) $\text{SS}(\text{RHom}(q_2^{-1} F, s^i G)) \subset A^a + B$, where

$$A = \{(g, t_1, t_2; \gamma, 0, \tau); (g, t_2; \gamma, \tau) \in \Lambda \} \cup 0_{G \times \mathbb{R}^2},$$

$$B = \{(g, t_1, t_2; \gamma', \tau'); (g, t_1 + t_2; \gamma', \tau') \in \Lambda \} \cup 0_{G \times \mathbb{R}^2}$$
and we are interested in the \((t_1; \tau_1) \in T^* \mathbb{R}\) such that \((g; t_1; t_2; 0; \tau_1, 0) \in A^a + B\) for some \(g, t_2\). Thus \(0 = -\gamma + \gamma', \tau_1 = \tau', 0 = -\tau + \tau'\) and, assuming \(\tau \neq 0\), we find \((g; t_2; \gamma, \tau) \in \Lambda, (g; t_1 + t_2; \gamma, \tau) \in \Lambda\). This implies that \(\Lambda\) meets \(T_{t_1}(\Lambda)\). Hence \(\mathcal{S}(R_{\Lambda} \mathcal{Hom}^* (F, F'))\) does not meet \(T^*_{c(c)} \mathbb{R}\) when \(c \neq 0\) or \(|c|\) is not a Reeb chord length.

In particular \(R_{\Lambda} \mathcal{Hom}^* (F, F')\) is constant on \([l_{\text{max}}, \infty[\) and it is enough to check that \(R \mathcal{Hom}(F, T_{-c} F') \simeq 0\) for \(c > 0\).

For \(c\) big enough, \(T_{-c} F'\) is locally constant on the support of \(F\) by (4). Hence we can assume \(T_{-c} F' \simeq p^{-1}_G(L')\) for some \(L' \in D(k_G)\), where \(p_G\) is the projection to \(G\). Then \(R \mathcal{Hom}(F, T_{-c} F') \simeq R \mathcal{Hom}(R p_G F, L')[−1]\). Now we check that \(R p_G F\) vanishes. By base change \((R p_G F)_g \simeq R \Gamma_c (\mathbb{R}; F|_{(g) \times \mathbb{R}})\) for any \(g \in G\). Since \(F^g := F|_{(g) \times \mathbb{R}}\) is constant on \([A, \infty[\), \(A > 0\), we have \(R \Gamma_c (\mathbb{R}; F^g_{[A, \infty[}) \simeq 0\). Using the distinguished triangle \(F^g_{[−\infty, A]} \rightarrow F^g \rightarrow F^g_{[A, \infty[} \rightarrow F^g_{[−\infty, A]})\), we obtain \((R p_G F)_g \simeq R \Gamma (\mathbb{R}; F^g_{[−\infty, A]})\), which vanishes by Lemma 2.4.

\[\text{Lemma 4.4.}\] Let \(e\) be the neutral element of \(G\). For any \(F, F' \in D_{A, +}(k_G \times \mathbb{R})\) we have \(\mathcal{Hom}^* (G, F) \times [e] \times [l_{\text{max}}, \infty[ \simeq 0\).

\[\text{Proof.}\] We set for short \(F_1 = \mathcal{Hom}^* (G, F)\). Let \(i_e: \{e\} \rightarrow \mathbb{R}, i_e: \mathbb{R} \simeq \{e\} \times \mathbb{R} \rightarrow G \times \mathbb{R}\) be the inclusions and \(i_{(e,c)} = i_e \circ i_c\). By Lemma 12 \(SS(F_1)\) does not meet \(T_{\{e\} \times \mathbb{R}}(G \times \mathbb{R})\). Hence \(SS(i_{e}^{-1} F_1) \subset \{\tau \geq 0\}\) and \(i_{e}^{-1} F_1 \simeq i_{(e,c)}^{-1} F_1[n]\), where \(n\) is the dimension of \(G\), by [3] Prop. 5.4.13]. By Lemma 2.6 it is thus enough to prove that \(i_{(e,c)}^{-1} F_1[n]\) vanishes for all \(c > l_{\text{max}}\).

By the adjunction (15) we have
\[
i_{(e,c)}^* \mathcal{Hom}^* (G, F) \simeq R \mathcal{Hom}(k_{(e,c)}, \mathcal{Hom}^* (G, F'))
\simeq R \mathcal{Hom}(k_{(e,c)} \ast G F, F')
\simeq R \mathcal{Hom}(T_{−c} F, F')
\simeq R \mathcal{Hom}(F, T_{−c} F').
\]

Now the result follows from Lemma 4.3.

Until the end of the section we set for short
\(F_2 = \mathcal{Hom}^* (G, F') \otimes k_{G \times [0, \infty[}\).

By Lemma 4.4 we thus have \(\tau_c (F_2|_{\{c\} \times \mathbb{R}}) = 0\) for \(c \geq l_{\text{max}}\).

\[\text{Proposition 4.5.}\] (i) For any \(g \in G\) the morphism \(\tau_c (F_2|_{\{g\} \times \mathbb{R}})\) vanishes for \(c > 2d(e, g) + l_{\text{max}}\), where \(d(−, −)\) is the distance on \(G\).

(ii) Let \(l\) be the diameter of \(G\). Let \(\varepsilon > 0\) be less than the injectivity radius of \(G\) and let \(B \subset G\) be a ball of radius \(< \varepsilon\). Then the morphism \(\tau_c (F_2|_{B \times \mathbb{R}})\) vanishes for \(c > 2(l + \varepsilon) + l_{\text{max}}\).
Proof. (i) We choose a sequence of points \( g_0, \ldots, g_n \) on a geodesic in \( G \) such that \( g_0 = e, g_n = g, l_i := d(g_i, g_{i+1}) \) is less than the injectivity radius and \( \sum_{i=0}^{n-1} l_i = d(e, g) \). We choose \( \delta > 0 \) and set \( c_i = l_{\max} + 2 \sum_{j=0}^{i} (l_j + \delta) \). Using Corollary 2.16 we see by induction on \( i \) that \( \tau_c(F_2|_{B(g, l_i + \delta) \times \mathbb{R}}) \) vanishes and hence \( \tau_c(F_2|_{(g, l_i + \delta) \times \mathbb{R}}) \) also vanishes: the initial step is given by Lemma 4.4 and, by Lemma 4.2, the coefficient \( \alpha \) in the corollary is 1. Since \( \delta \) is arbitrary, the result follows.

(ii) Let \( g \) be the center of \( B \). By (i) \( \tau_c(F_2|_{(g) \times \mathbb{R}}) \) vanishes for \( c > 2l + l_{\max} \) and the result follows by Corollary 2.16 again. \( \Box \)

**Corollary 4.6.** The morphism \( \tau_c(F_2) \) vanishes for \( c > (n+1)(2l + l_{\max}) \), where \( l \) is the diameter of \( G \) and \( n \) its dimension.

**Proof.** Let \( \varepsilon > 0 \) be given and \( c = 2(l + \varepsilon) + l_{\max} \). We choose a triangulation of \( G \) such that all simplices are contained in balls of radius less than \( \varepsilon \). We denote by \( \Sigma_k \) the set of simplices of dimension \( k \) and by \( S_k \) the union of the simplices of dimension \( \leq k \). For a subset \( S \) of \( G \) we set for short \( S^+ = S \times \mathbb{R} \).

By Proposition 4.5 for any simplex \( \sigma \), the morphism \( \tau_c(F_2|_{\sigma^+}) \) vanishes. We remark that

\[
\text{Hom}(F_2|_{\sigma^+}, T_{c*}(F_2|_{\sigma^+})) \simeq \text{Hom}((F_2)_{\sigma^+}, T_{c*}((F_2)_{\sigma^+})).
\]

Hence \( \tau_c((F_2)_{\sigma^+}) \) vanishes. Let us prove that \( \tau_{c+1}((F_2)_{S_k^+}) \) vanishes, by induction of \( k \). For \( k = 0 \), this follows from Proposition 4.5. The induction step follows from Lemma 2.2 applied to the excision distinguished triangle \( \bigoplus_{\sigma \in \Sigma_k} (F_2)_{\sigma^+} \rightarrow (F_2)_{S_k^+} \rightarrow (F_2)_{S_{k-1}^+} \). For \( k = n \) we obtain the vanishing of \( \tau_{c(n+1)}c(F_2) \). Since \( \varepsilon \) is as small as required, this gives the result. \( \Box \)

**Proof of Theorem 3.1.** By definition \( v(F, F') = \min\{c; \tau_c(F_3) = 0\} \), where \( F_3 = \mathcal{H}\text{Hom}^*(R_{a*}F, R_{a*}F') \otimes k_{[0, \infty]} \). Using Proposition 4.1 and the projection formula we have \( F_3 \simeq R_{a*}F_2 \). By Corollary 4.6 we deduce \( \tau_c(F_3) = 0 \) when \( c > (n+1)(2l + l_{\max}) \), which proves the theorem. \( \Box \)

5. **Proof of Corollary 3.2 — Homogeneous Spaces**

In this section we assume that \( M = G/H \) where \( G \) and \( H \) are compact Lie groups.

(i) We recall that \( D_{A,+}(k_{M \times \mathbb{R}}) \rightarrow D_{\mathbb{R}}(k_M), F \mapsto F_+ \), is an equivalence (see Remark 2.7). More precisely there exists a unique \( F_0 \in D_{A,+}(k_{M \times \mathbb{R}}) \) such that \( (F_0)_+ \simeq k_M \) and, for any \( F \in D_{A,+}(k_{M \times \mathbb{R}}) \) we have \( F \simeq F_0 \otimes p^{-1}F_+ \), where \( p: M \times \mathbb{R} \rightarrow M \) is the projection.
(ii) Let \( q: G \times \mathbb{R} \to G/H \times \mathbb{R} = M \times \mathbb{R} \) be the quotient map. The projection formula gives \( Rq(q^{-1}F) \simeq Rq(q^{-1}F \otimes k_{G\times \mathbb{R}}) \simeq F \otimes L \), where \( L = Rq!(k_{G\times \mathbb{R}}) \). By Theorem 3.1 and Remark 2.21 we have, for any \( F, F' \in D_{\Lambda,+}(k_{M\times \mathbb{R}}) \),

\[
(16) \quad v(F \otimes L, F' \otimes L) = v(q^{-1}F, q^{-1}F') \leq C, \quad \text{where } C := (n + 1)(2l + l_{\max}).
\]

(iii) We first assume that \( H \) is connected. We set

\[
\mathcal{F}_{L,\Lambda} = \{ F \otimes L; F \in D_{\Lambda,+}(k_{M\times \mathbb{R}}) \}, \quad \mathcal{F}_{L,lc} = \{ G \otimes L; G \in D_{lc}(k_{M\times \mathbb{R}}) \}.
\]

By (i) we have \( \mathcal{F}_{L,\Lambda} = \mathcal{F}_{L,lc} \otimes F_0 \), using the notation of Lemma 6.1. Since \( H \) is connected, we have \( H^0L \simeq k_{M\times \mathbb{R}} \). By Lemma 6.2 there exists \( L' \in \mathcal{F}_{L,lc}^{[(m-1)/2]} \) such that \( k_{M\times \mathbb{R}} \) is a direct summand of \( L' \) (locally constant sheaves on \( M \times \mathbb{R} \) are pull-back of sheaves on \( M \) and we can consider we work on \( M \), which is of dimension \( m \)). By Lemma 6.1 it follows that there exists \( F' \in \mathcal{F}_{L,\Lambda}^{[(m-1)/2]} \) such that \( F_0 \) is a direct summand of \( F' \). For \( F \in D_{\Lambda,+}(k_{M\times \mathbb{R}}) \) we can write \( F \simeq F_0 \otimes G \), for some \( G \in D_{lc}(k_{M\times \mathbb{R}}) \). Then \( F \) is a direct summand of an object of \( (\mathcal{F}_{L,\Lambda} \otimes G)^{[(m-1)/2]} \) by Lemma 6.1 again. We remark that \( \mathcal{F}_{L,\Lambda} \otimes G \subset \mathcal{F}_{L,\Lambda} \).

(iv) By (ii) we have \( v(F, F') \leq C \) for any \( F, F' \in \mathcal{F}_{L,\Lambda} \). Let us prove by induction on \( i \) that, for any \( j \leq i \) and \( F_i \in \mathcal{F}_{L,\Lambda}^{(i)} \), \( F_j \in \mathcal{F}_{L,\Lambda}^{(j)} \), we have \( v(F_i, F_j) \leq C(i + 1)(j + 1) \), \( v(F_j, F_i) \leq C(i + 1)(j + 1) \).

For \( i = 0 \) this is known. Let us assume it holds for \( i \) and let us pick \( F \in \mathcal{F}_{L,\Lambda}^{(i+1)} \) given by a distinguished triangle \( F_i \to F_0 \to F' \xrightarrow{+1} \) with \( F_i \in \mathcal{F}_{L,\Lambda}^{(i)} \), \( F_0 \in \mathcal{F}_{L,\Lambda}^{(0)} \). Now we pick \( F' \in \mathcal{F}_{L,\Lambda}^{(j)} \) with \( j \leq i + 1 \).

If \( j \leq i \) the induction hypothesis and Lemma 2.22 give \( v(F, F') \leq v(F_i, F') + v(F_0, F') \leq C(i + 2)(j + 1) \), as required. Similarly for \( v(F', F) \). These inequalities hold for any \( F \in \mathcal{F}_{L,\Lambda}^{(i+1)} \) and \( F' \in \mathcal{F}_{L,\Lambda}^{(i)} \). Hence, for \( F' \in \mathcal{F}_{L,\Lambda}^{(i+1)} \) we obtain \( v(F_i, F') \leq C(i + 1)(i + 2) \) and \( v(F_0, F') \leq C(i + 2) \). Now we can again apply Lemma 2.22 to obtain \( v(F, F') \) and \( v(F', F) \) in the case \( j = i + 1 \).

By (iii) we deduce \( v(F, F') \leq C([(m - 1)/2] + 1)^2 \leq \frac{1}{4}C(m + 3)^2 \) for any \( F, F' \in D_{\Lambda,+}(k_{M\times \mathbb{R}}) \), which proves the corollary in the case \( H \) connected.

(v) If \( H \) is not connected, we let \( H^0 \) be its neutral component and we set \( M' = G/H^0 \). Then \( r: M' \to M \) is a finite cover with group \( \pi_0(H) \). As in (i) we have \( r_!(r^{-1}F) \simeq F \otimes L_0 \), where \( L_0 = r_!(k_M) \). We remark that \( r_! \simeq r_* \) is exact and \( r^{-1} \simeq r^! \) because \( r \) is a cover map with finite fibers. We deduce the adjunction morphisms \( \alpha: k_M \to L_0 \) and \( \beta: L_0 \to k_M \). The composition \( \beta \circ \alpha \) is the multiplication by \( |\pi_0(H)| \). By the hypothesis on the characteristic we deduce that \( k_M \) is a direct summand of \( L_0 \). Now the result follows from (iv) applied to \( r^{-1}F, \ r^{-1}F' \) and Remark 2.21.
6. Appendix

Let $\mathcal{T}$ be a triangulated category. For a family $\mathcal{F}$ of objects of $\mathcal{T}$ we set

$$\mathcal{F}^{(0)} = \{ F \in \mathcal{T}; F \simeq \bigoplus_{i=1}^{n} F_i[d_i] \text{ for some } F_i \in \mathcal{F}, d_i \in \mathbb{Z}, i = 1, \ldots, n \}$$

and we define $\mathcal{F}^{(k)}$ inductively by

$$\mathcal{F}^{(k+1)} = \{ F \in \mathcal{T}; \text{ there exists a distinguished triangle } F_1 \to F_0 \to F \to +1 \text{ with } F_1 \in \mathcal{F}^{(k)} \text{ and } F_0 \in \mathcal{F}^{(0)} \}.$$

We remark that $(\mathcal{F}^{(k)})^{(0)} = \mathcal{F}^{(k)}$.

**Lemma 6.1.** Let $\mathcal{F}$ be a family of objects of $\mathcal{D}(k_M)$ and let $G \in \mathcal{D}(k_M)$. We set $\mathcal{F} \otimes G = \{ F \otimes G; F \in \mathcal{F} \}$. Then, for any $F' \in \mathcal{F}^{(i)}$, we have $F' \otimes G \in (\mathcal{F} \otimes G)^{(i)}$.

**Proof.** We make an induction on $i$. By definition any $F' \in \mathcal{F}^{(i)}$ fits in a distinguished triangle $F_1 \to F_0 \to F' \to +1$ with $F_1 \in \mathcal{F}^{(i-1)}$, $F_0 \in \mathcal{F}^{(0)}$. Tensoring with $G$ we obtain $F_1 \otimes G \to F_0 \otimes G \to F' \otimes G \to +1$ where $F_1 \otimes G \in (\mathcal{F} \otimes G)^{(i-1)}$ by the induction hypothesis and $F_0 \otimes G \in (\mathcal{F} \otimes G)^{(0)}$. Hence $F' \otimes G \in (\mathcal{F} \otimes G)^{(i)}$. \qed

We let $\mathcal{D}_{lc}(k_M)$ be the subcategory of $\mathcal{D}(k_M)$ formed by the $G$ such that $H^iG$ is a locally constant sheaf, for any $i \in \mathbb{Z}$.

**Lemma 6.2.** Let $n$ be the dimension of $M$. Let $L \in \mathcal{D}_{lc}(k_M)$ be such that $H^iL \simeq 0$ for $i < 0$ and $H^nL \simeq k_M$. We set $\mathcal{F} = \{ F; F \simeq L \otimes G \text{ for some } G \in \mathcal{D}_{lc}(k_M) \}$. Then there exists $F \in \mathcal{F}^{(n-1)/2}$ such that $k_M$ is a direct summand of $F$.

**Proof.** (i) For an integer $i$ we let $\mathcal{D}^{\geq i}(k_M)$ be the subcategory formed by the $G$ such that $H^jG \simeq 0$ for $j < i$. We say that $G \in \mathcal{D}^{\geq 0}(k_M)$ is $l$-lacunary if $H^jG \simeq 0$ for $j = 1, \ldots, l$ (any object of $\mathcal{D}^{\geq 0}(k_M)$ is 0-lacunary). We recall the truncation distinguished triangle, for any $G \in \mathcal{D}(k_M)$ and $i \in \mathbb{Z}$ (see for example [5], (1.7.2))

$$\tau_{\geq i}G \to G \to \tau_{\geq i+1}G \to +1.$$

(ii) We first prove: let $F \in \mathcal{D}^{\geq 0}(k_M)$ be $(n-1)$-lacunary and such that $H^0F \simeq k_M$. Then $k_M$ is a direct summand of $F$. Indeed we have a distinguished triangle $k_M \to F \to F' \to \tau_{\geq 1}F$. Since $F$ is $(n-1)$-lacunary, we have $F' \in \mathcal{D}^{\geq n}(k_M)$. We can see $u$ as a morphism from $F'[-1]$ to $k_M$. Now $k_M$ has a flabby resolution of length $n$ and, since $k$ is a field, flabby sheaves are injective (see [5] Ex. II.9 and II.10). Hence we can compute $u$ by replacing $k_M$ with a complex in degrees $0, \ldots, n$. Since $F'[-1] \in \mathcal{D}^{\geq n+1}(k_M)$ we find $u = 0$. Hence $F \simeq k_M \oplus F'$, as required.
(iii) We prove by induction on $i$ that there exists $L_i \in \mathcal{F}^{(i)}$ such that $L_i \in D^{\geq 0}(k_M)$, $H^0 L_i \simeq k_M$ and $L_i$ is $2i$-lacunary. By (ii) applied to $F = L_{\lceil (n-1)/2 \rceil}$ this proves the lemma.

For $i = 0$ we take $L_0 = L$.

We assume that step $i$ is proved and we pick $L_i$ as above. We have the distinguished triangles

\begin{align}
\tag{17} & k_M \to L \to L' \xrightarrow{+1}, \\
\tag{18} & k_M \to L_i \xrightarrow{\alpha} L'_i \xrightarrow{+1},
\end{align}

where $L' = \tau_{\geq 1} L$ and $L'_i = \tau_{\geq 1} L_i$. Tensoring (17) with $L'_i$ we obtain

\begin{align}
\tag{19} & L'_i \xrightarrow{\beta \alpha} L \otimes L'_i \to L' \otimes L'_i \xrightarrow{+1},
\end{align}

and we define $G$ by the distinguished triangle

\begin{align}
\tag{20} & L_i \xrightarrow{\beta \alpha} L \otimes L'_i \to G \xrightarrow{+1}.
\end{align}

We remark that $L'_i \in D_h(k_M)$, hence $L \otimes L'_i \in \mathcal{F}$ and $G \in \mathcal{F}^{(i+1)}$. The octahedron axiom applied to (18)-(20) gives the distinguished triangle

\begin{align}
\tag{21} & k_M[1] \to G \to L' \otimes L'_i \xrightarrow{+1}.
\end{align}

We have $L' \in D^{\geq 1}(k_M)$ by definition and $L'_i \in D^{\geq 2i+1}(k_M)$ because $L_i$ is $2i$-lacunary. Hence $L' \otimes L'_i \in D^{\geq 2i+2}(k_M)$ and $L' \otimes L'_i[-1] \in D^{\geq 2i+3}(k_M)$. Now it follows from (21) that $L_{i+1} = G[-1]$ satisfies the required properties.

\[\square\]

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