A Gaussian model for survival data subject to dependent censoring and confounding

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Abstract

This paper considers the problem of inferring the causal effect of a variable $Z$ on a survival time $T$. The error term of the model for $T$ is correlated with $Z$, which leads to a confounding issue. Additionally, $T$ is subject to dependent censoring, that is, $T$ is right censored by a censoring time $C$ which is dependent on $T$. In order to tackle the confounding issue, we leverage a control function approach relying on an instrumental variable $\tilde{W}$. Further, it is assumed that $T$ and $C$ follow a joint regression model with bivariate Gaussian error terms and an unspecified covariance matrix, allowing us to handle dependent censoring in a flexible manner. We derive conditions under which the model is identifiable, a two-step estimation procedure is proposed and we show that the resulting estimator is consistent and asymptotically normal. Simulations are used to confirm the validity and finite-sample performance of the estimation procedure. Finally, the proposed method is used to estimate the effectiveness of the Job Training Partnership Act (JTPA) programs on unemployment durations.

Keywords: dependent censoring, causal inference, instrumental variable, control function, survival analysis.

1 Introduction

The purpose of this article is to infer the causal effect of a variable $Z$ on a survival time $T$, subject to both dependent censoring and confounding. Precisely, let $T$ depend on a vector of covariates $X$, a confounded variable $Z$ and a confounding variable $u_T$. There is a confounding issue due to $Z$ and $u_T$ being correlated. A common example of this is when the data is not experimental. This problem implies that the causal effect of $Z$ on $T$ cannot be identified from the conditional distribution of $T$ on $Z$. A naive estimator would assume that the confounded variable $Z$ and the error term $u_T$ are uncorrelated, resulting in biased estimates. We also introduce a right censoring mechanism through the censoring time $C$, such that only the minimum of $T$ and $C$ is observed through the follow-up time $Y = \min\{T, C\}$ and the censoring indicator $\Delta = 1(T \leq C)$. We do not assume that $T$ and $C$ are independent. This possible dependence creates an additional statistical issue since the distribution of $T$ cannot be recovered from that of $(Y, \Delta)$ without further assumptions.

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To tackle the confounding issue, a control function approach is used. This method uses an instrumental variable \( \tilde{W} \) and the vector of covariates \( X \) to split \( u_T \) into two parts, one which is correlated with \( Z \) and one which is not. It is assumed that \( \tilde{W} \) is a valid instrumental variable for \( Z \), which means that \( \tilde{W} \) is independent of \( u_T \), sufficiently correlated with \( Z \) and only affects \( T \) through \( Z \). The part of \( u_T \) that is correlated with \( Z \) is the control function \( V \). This control function is assumed to be a function of \( Z, X \) and \( \tilde{W} \) that captures all confounding. This implies that conditioning on \( V \) allows us to estimate the causal effect of \( Z \) on \( T \). To model the dependent censoring, we assume that \( T \) and \( C \) follow a joint Gaussian regression model with an unspecified covariance matrix. It is shown that the model is identifiable, which means that we can identify not only the causal effect of \( Z \) on \( T \), but also the correlation parameter between \( T \) and \( C \). This can be seen as surprising, since we only observe the minimum of \( T \) and \( C \) through the follow-up time \( Y \). To estimate the model parameters, a two-step estimation method is proposed. The first step estimates a parameter \( \gamma \) required to construct \( V \). The second step uses maximum likelihood and includes the estimated \( V \) from the first step. A correction for the randomness coming from the first step needs to be applied to get asymptotically valid standard errors. To implement this correction, we treat the two steps as a joint generalized method of moments estimator with their moment conditions stacked in one vector. This allows us to prove consistency and asymptotic normality of the parameter estimates. The estimator demonstrates excellent finite sample performances in simulations. We illustrate the procedure to evaluate the effect of Job Training Partnership Act (JTPA) funded programs on unemployment duration.

**Related literature.** This paper is first related to the literature on dependent censoring. In survival analysis, it is usually assumed that the survival time \( T \) is independent of the right censoring time \( C \), which is called independent censoring. However, it is easy to think of situations where this assumption is not a reasonable one to make. A common example of the independent censoring assumption being doubtful can be found in transplant studies. The survival time (time to death) is likely dependent on the censoring time (time to transplant), since selection for transplant is based on the patient’s medical condition. In this case we would expect a positive dependence between \( T \) and \( C \), as usually the most ill patients are selected for transplant (Staplin et al., 2015). In the literature, many methods have been proposed to handle dependent censoring. An important result comes from Tsiatis (1975), who proved that it is impossible to identify the joint distribution of two failure times by their minimum in a fully nonparametric way. Because of this, more information about the dependence and/or marginal distributions of \( T \) and \( C \) is needed to identify their joint distribution. The most popular approaches are based on copulas, and Zheng and Klein (1995) were the first to apply this idea. Under the assumption of a fully known copula for the joint distribution of \( T \) and \( C \), a nonparametric estimator of the marginals was proposed. This estimator is called the copula-graphic estimator, which extends the Kaplan and Meier (1958) estimator to the dependent censoring case. Rivest and Wells (2001) further investigated the copula-graphic estimator for Archimedean copulas. Note that all of these methods rely on a completely known copula, which means in particular that the association parameter specifying the dependence between \( T \) and \( C \) is assumed to be known. However, this is often not the case in practice. The copula methods were extended to include covariates by Braekers and Veraverbeke (2005), Huang and Zhang (2008) and Sujica and Van Keilegom (2018) among others. Nevertheless, these methods still rely on a fully known copula. More recently a new method was proposed by Czado and Van Keilegom (2021), which does not require the association parameter to be known. As a trade-off, this requires the marginals to be fully parametric for the association parameter to be identifiable. Deresa and Van Keilegom (2020c) and Deresa and Van Keilegom (2020a) propose a semiparametric and parametric transformed joint regression model respectively, where the transformed variables \( T \) and
follow a bivariate normal distribution after adjusting for covariates. Deresa and Van Keilegom (2020b) extends the parametric transformed joint regression model to allow for different types of censoring. The present paper relies on a similar Gaussian model as Deresa and Van Keilegom (2020a) but nevertheless differs from it because we allow for confounding.

Next, the present work falls within the instrumental variable and control function literature. A confounding issue could occur due to a multitude of reasons such as noncompliance (Angrist et al., 1996), sample selection (Heckman, 1979), measurement error or omitting relevant variables. The control function approach used in this paper has been discussed extensively in the literature on confounding and endogeneity by Lee (2007), Navarro (2010) and Wooldridge (2015) among others. The idea is that adding an appropriate control function to the regression, which is estimated in the first stage using a valid instrument, solves the confounding issue. The advantages of this approach are that it is computationally simple and it can handle complicated models that are nonlinear in the confounded variable in a parsimonious manner.

Finally, the last string of research linked to this paper is that of instrumental variable methods for right censored data. We first discuss methods assuming that the censoring mechanism is independent. Some papers follow a nonparametric approach assuming that both \( Z \) and \( \tilde{W} \) are categorical: Frandsen (2015), Sant’Anna (2016) and Beyhum et al. (2021b). Other approaches are semiparametric as Bijwaard and Ridder (2005), Li et al. (2015), Tchetgen Tchetgen et al. (2015) and Chernozhukov et al. (2015). Note that Tchetgen Tchetgen et al. (2015) also proposes a control function approach. Centorrino and Florens (2021) study nonparametric estimation with continuous regressors. Confounding has also been discussed in a competing risks framework by Richardson et al. (2017), Zheng et al. (2017), Martinussen and Vansteelandt (2020) and Beyhum et al. (2021a). Research on confounding within a dependent censoring framework is sparser. Firstly, Robins and Finkelstein (2000) looked at a correction for noncompliance and dependent censoring. However, they make the strong assumption that conditional on the treatment arm and the recorded history of six time-dependent covariates, \( C \) does not further depend on \( T \). It is clear that this assumption is violated if there is a variable affecting both \( T \) and \( C \) that is not observed. Secondly, Khan and Tamer (2009) discusses an endogenously censored regression model, but they make a strong assumption (IV2) regarding the relationship between the instruments and the covariates. The assumption is not met when the support of \( C \), conditional on \( \tilde{W} \), is not finite. An example of this assumption being violated is when the distribution of \( C \) given \( Z \) and \( X \) has infinite support, which is allowed in our model. Finally, Blanco et al. (2020) looks at treatment effects on duration outcomes under censoring, selection, and noncompliance. However, they derive bounds on the causal effect of \( Z \) on \( T \) instead of point estimates.

Outline. This paper is structured as follows: Section 2.1 specifies the model to be studied and, in Section 2.2, some distributions are derived such that the expected log-likelihood can be defined. Section 3.1 derives the identification results and Section 3.2 outlines the estimation procedure. Section 3.3 shows consistency and asymptotic normality for the estimator described in Section 3.2. Section 3.4 describes how the asymptotic variance can be estimated. The technical details for the three theorems outlined in Section 3 can be found in Appendix C. Simulation results and an empirical application regarding the impact of JTPA programs on time until employment are described in Sections 4 and 5 respectively. The code used for both of these sections can be found on https://github.com/GillesCrommen.
2 The model

2.1 Model specification

Let \( T \) and \( C \) be the logarithm of the survival time and the censoring time respectively. The variables \( T \) and \( C \) are dependent on each other, even when conditioning on the covariates. Because \( T \) and \( C \) censor each other, only one of them is observed through the follow-up time \( Y = \min\{T, C\} \) and the censoring indicator \( \Delta = 1 (T \leq C) \). The covariates that have an influence on both \( T \) and \( C \) are given by \( X = (1, X^\top)^\top \) and \( Z \), where \( X \) and \( Z \) are of dimension \( m \) and \( 1 \) respectively. We are interested in estimating the causal effect of \( Z \) on \( T \). A joint regression model can be specified as follows:

\[
\begin{align*}
T &= X^\top \beta_T + Z\alpha_T + u_T \\
C &= X^\top \beta_C + Z\alpha_C + u_C
\end{align*}
\]  

(1)

We do not assume that \( Z \perp \perp (u_T, u_C) \), where \( \perp \perp \) denotes statistical independence, hence there may be a confounding issue. A common example of this is when \( Z \) is non-randomized, which implies that the causal effect of \( Z \) on \( T \) cannot be identified from the conditional distribution of \( T \) on \( Z \). Therefore, using model (1) to estimate the causal effect of \( Z \) on \( T \) would lead to biased estimates. To tackle this problem, we leverage a control function approach. Let \( \tilde{W} \) be a valid scalar instrument for \( Z \), meaning that \( \tilde{W} \) is independent of \( (u_T, u_C) \), sufficiently correlated with \( Z \) and only affects \( (T, C) \) through \( Z \). Since it is assumed that only \( Z \) is possibly confounded, we define \( W = (X^\top, \tilde{W})^\top \) such that \( W \perp \perp (u_T, u_C) \). Further, we assume that there exists a vector \( (\lambda_T, \lambda_C) \in \mathbb{R}^2 \), an unobserved control function \( \nu \) and error terms \( (\epsilon_T, \epsilon_C) \) such that

\[
\begin{align*}
\epsilon_T &= \lambda_T \nu_T + \epsilon_T \quad \text{and} \quad \epsilon_C = \lambda_C \nu + \epsilon_C.
\end{align*}
\]

The control function depends on \( (Z, W) \) and allows to capture all confounding (see Lee (2007), Navarro (2010) and Wooldridge (2015)). There exists a mapping \( g \), depending on a parameter \( \gamma \), such that \( V = g_\gamma(Z, W) \). The function \( g \) is known (specified by the analyst), but the parameter \( \gamma \) is unknown. Later in the current section, we discuss specific choices of \( g \). It is also assumed that:

(A1) \( (\epsilon_T, \epsilon_C) \sim N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma_\epsilon = \begin{pmatrix} \sigma_T^2 & \rho \sigma_T \sigma_C \\ \rho \sigma_T \sigma_C & \sigma_C^2 \end{pmatrix}\right) \),

with \( \Sigma_\epsilon \) positive definite \((\sigma_T, \sigma_C > 0 \text{ and } |\rho| < 1)\).

(A2) \( (\epsilon_T, \epsilon_C) \perp \perp (W, Z, V) \).

(A3) The covariance matrix of \( (X^\top, Z, V) \) is full rank.

(A4) The probabilities \( P(Y = T \mid W, Z) \) and \( P(Y = C \mid W, Z) \) are both strictly positive almost surely.

This results in the following proposed joint regression model:

\[
\begin{align*}
T &= X^\top \beta_T + Z\alpha_T + V\lambda_T + \epsilon_T \\
C &= X^\top \beta_C + Z\alpha_C + V\lambda_C + \epsilon_C
\end{align*}
\]  

(2)

But how do we specify the control function? In the literature, different specifications of the control function are proposed which depend on a model for the relationship between \( Z \) and \( W \). Following Wooldridge (2010) and Navarro (2010), we give two examples of possible control functions. Consider first the case where \( Z \) is a continuous random variable and the relation between \( Z \) and \( W \) follows a linear model, that is

\[
Z = W^\top \gamma + \nu \quad \text{with} \quad \mathbb{E}[\nu W] = 0,
\]

(3)
where \( \nu \) is an unobserved error term and \( \gamma \in \mathbb{R}^{m+2} \). In this setting, it is natural to set \( V = g_\gamma(Z,W) = Z - W^\top \gamma \), such that \( V \) is the confounded part of \( Z \), that is the one that does not depend on \( W \). Another, more involved, example is when \( Z \) is a binary random variable and the relation between \( Z, W \) and \( \nu \) is specified as

\[
Z = 1(W^\top \gamma - \nu > 0) \quad \text{with} \quad \nu \perp \perp W. 
\]

Since we cannot directly separate \( \nu \) from \( Z \) and \( W \), we follow the literature and let

\[
V = g_\gamma(Z,W) = Z \mathbb{E}[\nu \mid W^\top \gamma > \nu] + (1 - Z) \mathbb{E}[\nu \mid W^\top \gamma < \nu].
\]

Then, the function \( g \) is known when the distribution of \( \nu \) is known. This specification of the control function is discussed and justified in Section 19.6.1 and Section 21.4.2 of Wooldridge (2010). The value of the function \( g \) in (5) depends on the distribution of \( \nu \). If \( \nu \sim N(0,1) \), then we have a probit model and if \( \nu \) follows a standard logistic distribution, we have a logit model for \( Z \). Specific expressions of \( g \) for the probit and logit model can be found in Appendix A. Throughout the paper, we will use \( V \) and \( g_\gamma(Z,W) \) interchangeably.

### 2.2 Useful distributions and definitions

Using the assumptions that have been made so far, some conditional distributions and densities are derived. They are useful in proving the identification results and to define the estimator. The expected log-likelihood function is also defined. For a given \( \theta = (\beta_T, \alpha_T, \lambda_T, \beta_C, \alpha_C, \lambda_C, \sigma_T, \sigma_C, \rho)^\top \) and \( \gamma \), we define \( F_{T|W,Z}(\cdot \mid w, z, \gamma; \theta) \) and \( F_{C|W,Z}(\cdot \mid w, z, \gamma; \theta) \) as the conditional distribution function of \( T \) and \( C \) given \( W = w \) and \( Z = z \), respectively. Thanks to Assumptions (A1) and (A2), we have that:

\[
F_{T|W,Z}(t \mid w, z, \gamma; \theta) = \Phi\left( \frac{t - x^\top \beta_T - z\alpha_T - g_\gamma(z,w)\lambda_T}{\sigma_T} \right),
\]

\[
F_{C|W,Z}(c \mid w, z, \gamma; \theta) = \Phi\left( \frac{c - x^\top \beta_C - z\alpha_C - g_\gamma(z,w)\lambda_C}{\sigma_C} \right),
\]

with \( \Phi \) the cumulative distribution function of a standard normal variable. It follows that for a given \( \gamma \) and \( \theta \), the conditional density functions of \( T \) and \( C \) given \( W = w \) and \( Z = z \) are, respectively,

\[
f_{T|W,Z}(t \mid w, z, \gamma; \theta) = \sigma_T^{-1} \phi\left( \frac{t - x^\top \beta_T - z\alpha_T - g_\gamma(z,w)\lambda_T}{\sigma_T} \right),
\]

\[
f_{C|W,Z}(c \mid w, z, \gamma; \theta) = \sigma_C^{-1} \phi\left( \frac{c - x^\top \beta_C - z\alpha_C - g_\gamma(z,w)\lambda_C}{\sigma_C} \right),
\]

where \( \phi \) is the density function of a standard normal variable. For ease of notation, define \( b_C = y - x^\top \beta_C - z\alpha_C - g_\gamma(z,w)\lambda_C \) and \( bt = y - x^\top \beta_T - z\alpha_T - g_\gamma(z,w)\lambda_T \). The sub-distribution function \( F_{Y,\Delta|W,Z}(\cdot, 1 \mid w, z, \gamma; \theta) \) of \((Y, \Delta)\) given \((W, Z)\) and \((\gamma, \theta)\) can be derived as follows:

\[
F_{Y,\Delta|W,Z}(y, 1 \mid w, z, \gamma; \theta) \\
= P(Y \leq y, \Delta = 1 \mid W = w, Z = z) \\
= P(Y \leq y, T \leq C \mid W = w, Z = z) \\
= P(\epsilon_T \leq bt, x^\top (\beta_T - \beta_C) + z(\alpha_T - \alpha_C) + g_\gamma(z,w)(\lambda_T - \lambda_C) + \epsilon_T \leq \epsilon_C).
\]
This expression is equivalent to
\[
\int_{-\infty}^{b_T} P(\epsilon_C \geq x^T(\beta_T - \beta_C) + z(\alpha_T - \alpha_C) + g_\gamma(z, w)(\lambda_T - \lambda_C) + e \mid \epsilon_T = e) f_{\epsilon_T}(e) \, de.
\]
Since \((\epsilon_C \mid \epsilon_T = e) \sim N\left(\rho_{\epsilon_T} \sigma_C \epsilon, \sigma_C^2(1 - \rho^2)\right)\) and \(\epsilon_T \sim N(0, \sigma_T^2)\), it follows that
\[
f_{Y,\Delta|W,Z}(y, 1 \mid w, z, \gamma; \theta) = \frac{1}{\sigma_T} \left[1 - \Phi\left(\frac{b_T - \rho_{\epsilon_T} \sigma_C b_T}{\sigma_C(1 - \rho^2)^{1/2}}\right)\right] \phi\left(\frac{b_T}{\sigma_T}\right).
\]
Using the same arguments, it can be shown that
\[
f_{Y,\Delta|W,Z}(y, 0 \mid w, z, \gamma; \theta) = \frac{1}{\sigma_C} \left[1 - \Phi\left(\frac{b_T - \rho_{\epsilon_T} \sigma_C b_T}{\sigma_T(1 - \rho^2)^{1/2}}\right)\right] \phi\left(\frac{b_C}{\sigma_C}\right).
\]
Note that since
\[
P(Y \leq y) = P(T \leq y) + P(C \leq y) - P(T \leq y, C \leq y),
\]
we have that
\[
F_{Y|W,Z}(y \mid w, z, \gamma; \theta) = \Phi\left(\frac{b_T}{\sigma_T}\right) + \Phi\left(\frac{b_C}{\sigma_C}\right) - \Phi\left(\frac{b_T}{\sigma_T}, \frac{b_C}{\sigma_C}; \rho\right),
\]
with \(\Phi(\cdot, \cdot, \rho)\) the distribution function of a bivariate normal distribution with covariance matrix \(\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\). Further, let
\[
S = (Y, \Delta, \bar{X}, \bar{W}, Z) \text{ with distribution function } G \text{ on } \mathcal{G} = \mathbb{R} \times \{0, 1\} \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R},
\]
and
\[
\ell : \mathcal{G} \times \Gamma \times \Theta \to \mathbb{R} : (s, \gamma, \theta) \mapsto \ell(s, \gamma, \theta) = \log f_{Y,\Delta|W,Z}(y, \delta \mid w, z, \gamma; \theta),
\]
where \(\Theta \subset \{\theta : (\beta_T, \alpha_T, \lambda_T, \beta_C, \alpha_C, \lambda_C) \in \mathbb{R}^{2m+6}, (\sigma_T, \sigma_C) \in \mathbb{R}_+^2, \rho \in (-1, 1)\}\) is the parameter space of \(\theta\) and \(\Gamma\) the parameter space of \(\gamma\) (usually \(\Gamma \subseteq \mathbb{R}^{m+2}\)). The expected log-likelihood can be defined as follows:
\[
L(\gamma, \theta) = \mathbb{E} \left[\ell(S, \gamma, \theta)\right] = \int_{\mathcal{G}} \ell(s, \gamma, \theta) \, dG(s).
\]

3 Model identification and estimation

3.1 Identification of the model

We will start by showing that model (2) is identifiable in the sense that two different values of the parameter vector \((\gamma, \theta)\) result in two different distributions of \(S\). Let \((\gamma^*, \theta^*)\) denote the true parameter vector. In order to prove the identifiability of the model, it will be assumed that:

(A5) \(\gamma^*\) is identified.

Considering again the examples from Section 2.1, when \(Z\) is a continuous random variable for which (3) holds, it is well known that the assumption that the covariance matrix of \((\bar{X}, \bar{W})\) is full rank implies Assumption (A5). When \(Z\) is a binary random variable for which (4) holds, the assumption that the covariance matrix of \((\bar{X}, \bar{W})\) is full rank together with a known distributional assumption on \(\nu\) (e.g. \(\nu \sim N(0, 1)\) or \(\nu \sim \text{Logistic}(0, 1)\)) implies Assumption (A5) as shown by Manski (1988).
Theorem 1. Under Assumptions (A1)-(A5), suppose that \((T_1, C_1)\) and \((T_2, C_2)\) satisfy model (2) with \((\gamma, \theta)\) and \((\gamma^*, \theta^*)\) as parameter vectors respectively. If \(f_{Y_1, \Delta_1|W, Z}(\cdot, k \mid w, z; \gamma, \theta) \equiv f_{Y_2, \Delta_2|W, Z}(\cdot, k \mid w, z; \gamma^*, \theta^*)\) for almost every \((w, z)\), then

\[
\gamma = \gamma^* \quad \text{and} \quad \theta = \theta^*.
\]

The proof of the theorem can be found in Appendix C. It is based on the proof of Theorem 1 by Deresa and Van Keilegom (2020a). The fact that the proposed joint regression model is identifiable can be seen as surprising, since this means that we can identify the relationship between \(T\) and \(C\) while only observing their minimum through the follow-up time \(Y\).

3.2 Estimation of the model parameters

We consider estimation when the data consist of an i.i.d. sample \(\{Y_i, \Delta_i, W_i, Z_i\}_{i=1,...,n}\). Further, it is assumed that:

(A6) There exists a known function \(m : (w, z, \gamma) \in \mathbb{R}^{m+2} \times \mathbb{R} \times \Gamma \mapsto m(w, z, \gamma)\) twice differentiable with respect to \(\gamma\) such that the estimator

\[
\hat{\gamma} \in \arg \max_{\gamma \in \Gamma} n^{-1} \sum_{i=1}^{n} m(W_i, Z_i, \gamma)
\]

is consistent for the true parameter \(\gamma^*\).

Using the first-order conditions of program (6), we obtain that \(n^{-1} \sum_{i=1}^{n} \nabla_\gamma m(W_i, Z_i, \hat{\gamma}) = 0\). Hence, Assumption (A6) implies that we possess a consistent \(Z\)-estimator of \(\gamma\). The theory on \(M\)-estimators (Newey and McFadden, 1994) allows us to find sufficient conditions for the assumption that \(\hat{\gamma}\) is consistent. Assumption (A6) will hold when (i) the true parameter \(\gamma^*\) belongs to the interior of \(\Gamma\), which is compact, (ii) \(\mathcal{L}(\gamma) = \mathbb{E}[m(W, Z, \gamma)]\) is continuous and uniquely maximized at \(\gamma^*\) and (iii) \(\hat{\mathcal{L}}(\gamma) = n^{-1} \sum_{i=1}^{n} m(W_i, Z_i, \gamma)\) converges uniformly (in \(\gamma \in \Gamma\)) in probability to \(\mathcal{L}(\gamma)\). In the case where \(\hat{\mathcal{L}}(\cdot)\) is concave, (i) can be weakened to \(\gamma^*\) being an element of the interior of a convex set \(\Gamma\), while (iii) is only required to hold pointwise rather than uniformly. Returning again to the examples given in Section 2.1, when \(Z\) is a continuous random variable for which (3) holds, it is well known that ordinary least squares is an extremum estimation method that consistently estimates \(\gamma\) under the assumption that the covariance matrix of \((\tilde{X}, \tilde{W})\) is full rank. In this case, we can define \(m(W, Z, \gamma) = -(Z - W^\top \gamma)^2\). When \(Z\) is a binary random variable for which (4) holds and the distribution of \(\nu\) is known, maximum likelihood estimation can be used to consistently estimate \(\gamma\) under weak regularity conditions that can be found in Aldrich and Nelson (1991). In this case, we can define \(m(W, Z, \gamma) = Z \log P(W^\top \gamma > \nu) + (1 - Z) \log P(W^\top \gamma < \nu)\). After obtaining \(\hat{\gamma}\) from (6), the parameters from model (2) can be estimated using maximum likelihood with the estimates given by the second-step estimator:

\[
\hat{\theta} = (\hat{\beta}_T, \hat{\alpha}_T, \hat{\lambda}_T, \hat{\beta}_C, \hat{\alpha}_C, \hat{\lambda}_C, \hat{\sigma}_T, \hat{\sigma}_C, \hat{\rho}) = \arg \max_{\theta} \hat{\mathcal{L}}(\hat{\gamma}, \theta),
\]
with $\Theta$ the parameter space as defined before and

$$
\tilde{L}(\gamma, \theta) = \frac{1}{n} \sum_{i=1}^{n} \log f_{Y, \Delta|W,Z}(Y_i, \Delta_i | W_i, Z_i, \gamma; \theta)
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left\{ \Delta_i \log f_{Y, \Delta|W,Z}(Y_i, 1 | W_i, Z_i, \gamma; \theta) + (1 - \Delta_i) \log f_{Y, \Delta|W,Z}(Y_i, 0 | W_i, Z_i, \gamma; \theta) \right\}
$$

$$
= \frac{1}{n} \sum_{i=1}^{n} \left\{ \Delta_i \left( - \log(\sigma_T) + \log \left[ 1 - \Phi \left( \frac{b_{C_i} - \rho \frac{\sigma_C}{\sigma_T} b_{T_i}}{\sigma_C(1 - \rho^2)^{1/2}} \right) \right] + \log \left[ \phi \left( \frac{b_{T_i}}{\sigma_T} \right) \right] \right) 
+ (1 - \Delta_i) \left( - \log(\sigma_C) + \log \left[ 1 - \Phi \left( \frac{b_{T_i} - \rho \frac{\sigma_T}{\sigma_C} b_{C_i}}{\sigma_T(1 - \rho^2)^{1/2}} \right) \right] + \log \left[ \phi \left( \frac{b_{C_i}}{\sigma_C} \right) \right] \right) \right\},
$$

with

$$
b_{C_i} = Y_i - X_i^\top \beta_C - Z_i \alpha_C - g_\gamma(W_i, Z_i) \lambda_C,
$$

$$
b_{T_i} = Y_i - X_i^\top \beta_T - Z_i \alpha_T - g_\gamma(W_i, Z_i) \lambda_T.
$$

### 3.3 Consistency and asymptotic normality

In this section, it will be shown that the parameter estimates $\hat{\gamma}$, as defined in (7), are consistent and asymptotically normal. Theorems 2 and 3 show consistency and asymptotic normality respectively. The proofs can be found in Appendix C. We start by providing some definitions and assumptions that will be useful in stating these theorems. Let

$$
h_\ell(S, \gamma^*, \theta^*) = \nabla_\theta \ell(S, \gamma^*, \theta^*),
$$

$$
h_m(W, Z, \gamma^*) = \nabla_\gamma m(W, Z, \gamma^*),
$$

$$
M = \mathbb{E} \left[ \nabla_\gamma h_m(W, Z, \gamma^*) \right],
$$

$$
\tilde{h}(S, \gamma^*, \theta^*) = \left( h_m(W, Z, \gamma^*)^\top, h_\ell(S, \gamma^*, \theta^*)^\top \right)^\top,
$$

$$
H = \mathbb{E} \left[ \nabla_\gamma, \theta \tilde{h}(S, \gamma^*, \theta^*) \right].
$$

The following assumptions will be used in the proofs of Theorems 2 and 3:

(A7) The parameter space $\Theta$ is compact and $\theta^*$ belongs to the interior of $\Theta$.

(A8) There exists a function $D(s)$ integrable with respect to $G$ and a neighborhood $\mathcal{N}_\gamma \subseteq \Gamma$ of $\gamma^*$ such that $|\ell(s, \gamma, \theta)| \leq D(s)$ for all $\gamma \in \mathcal{N}_\gamma$ and $\theta \in \Theta$.

(A9) $\mathbb{E} \left[ \left\| \tilde{h}(S, \gamma^*, \theta^*) \right\|^2 \right] < \infty$ and $\mathbb{E} \left[ \sup_{(\gamma, \theta) \in \mathcal{N}_\gamma, \theta} \left\| \nabla_\gamma, \theta \tilde{h}(S, \gamma, \theta) \right\| \right] < \infty$, with $\mathcal{N}_\gamma, \theta$ a neighborhood of $(\gamma^*, \theta^*)$ in $\Gamma \times \Theta$.

(A10) $H^\top H$ is nonsingular.

Note that $\|\cdot\|$ represents the Euclidean norm. Assumption (A8) is necessary to show the consistency and asymptotic normality of the parameter estimates. A sufficient condition for this assumption is that the support of $S$ is bounded. Assumptions (A7), (A9) and (A10) are regularity conditions that are commonly made in a maximum likelihood context. We have the following consistency theorem.

**Theorem 2.** Under Assumptions (A1)-(A8), suppose that $\hat{\gamma}$ and $\hat{\theta}$ are parameter estimates as described in (6) and (7) respectively, then

$$
\hat{\gamma} \xrightarrow{p} \gamma^* \text{ and } \hat{\theta} \xrightarrow{p} \theta^*.
$$
The challenge in proving this theorem comes from the fact that we are using a two-step estimation method, meaning that the results from the first step are used in the second step. To ensure consistency of \( \hat{\theta} \), in the proofs, we show uniform convergence (in \( \theta \in \Theta \)) in probability of the empirical likelihood function \( \hat{L}(\hat{\gamma}, \theta) \) in (7) to the true likelihood of the model at \( \gamma^* \).

We also have the following asymptotic normality result:

**Theorem 3.** Under Assumptions (A1)-(A10), suppose that \( \hat{\theta} \) is a parameter estimate as described in (7), then

\[
\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{d} N(0, \Sigma_\theta)
\]

with

\[
\Sigma_\theta = H_\theta^{-1} \mathbb{E} \left[ \{h_\ell(S, \gamma^*, \theta^*) + H_\gamma \Psi \} \{h_\ell(S, \gamma^*, \theta^*) + H_\gamma \Psi \}^\top \right] (H_\theta^{-1})^\top.
\]

The difficulty in proving this theorem is related to the fact that the randomness coming from the first step inflates the asymptotic variance of \( \hat{\theta} \). Hence, ignoring the first step would lead to inconsistent standard errors and confidence intervals that are not asymptotically valid. To obtain correct standard errors, we treat the two steps as a joint generalized method of moments (GMM) estimator with their moment conditions stacked in one vector (Newey and McFadden, 1994). Indeed, given that \( \hat{\gamma} \) and \( \hat{\theta} \) are consistent by Theorem 2, they are the unique solutions to the first order conditions of their respective objective functions in a neighborhood of \( \gamma^* \) and \( \theta^* \) (with probability going to 1). Therefore, the two-step estimator is asymptotically equivalent to the GMM estimator corresponding to the following moments:

\[
\mathbb{E}[h_m(W, Z, \gamma)] = 0 \quad \text{and} \quad \mathbb{E}[h_\ell(S, \gamma, \theta)] = 0,
\]

for the first and second step respectively. As a last remark, if we were to remove the correction \( H_\gamma \Psi \) for the first step, the covariance matrix simplifies to the inverse of Fisher’s information matrix (assuming the model is correctly specified).

### 3.4 Estimation of the asymptotic variance

Using the result from Theorem 3, we can construct a consistent estimator \( \hat{\Sigma}_\theta \) for the covariance matrix of the parameters in \( \theta \) in the following way:

\[
\hat{\Sigma}_\theta = \hat{H}_\theta^{-1} \left[ n^{-1} \sum_{i=1}^{n} \{h_\ell(S_i, \hat{\gamma}, \hat{\theta}) + \hat{H}_\gamma \hat{\Psi}_i\} \{h_\ell(S_i, \hat{\gamma}, \hat{\theta}) + \hat{H}_\gamma \hat{\Psi}_i\}^\top \right] (\hat{H}_\theta^{-1})^\top,
\]

where \( S_i = (Y_i, \Delta_i, \tilde{X}_i, \tilde{W}_i, Z_i) \) and

\[
h_\ell(S_i, \hat{\gamma}, \hat{\theta}) = \nabla_\theta \ell(S_i, \hat{\gamma}, \hat{\theta}), \quad h_m(W_i, Z_i, \hat{\gamma}) = \nabla_\gamma m(W_i, Z_i, \hat{\gamma}), \quad \hat{H}_\theta = n^{-1} \sum_{i=1}^{n} \nabla_\theta h_\ell(S_i, \hat{\gamma}, \hat{\theta}),
\]

\[
\hat{H}_\gamma = n^{-1} \sum_{i=1}^{n} \nabla_\gamma h_\ell(S_i, \hat{\gamma}, \hat{\theta}), \quad \hat{M} = n^{-1} \sum_{i=1}^{n} \nabla_\gamma h_m(W_i, Z_i, \hat{\gamma}), \quad \hat{\Psi}_i = -\hat{M}^{-1} h_m(W_i, Z_i, \hat{\gamma}).
\]

Thanks to the asymptotic normality and the consistent estimator for the variance of the estimators, confidence intervals can easily be constructed. Note that since \( \sigma_T, \sigma_C > 0 \) and \( \rho \in (-1, 1) \), their confidence intervals will be constructed using a logarithm and a Fisher’s z-transformation respectively. These transformations project the estimates on the real line, after which the delta method can be used to obtain their standard errors. The confidence intervals can then be constructed and
transformed back to the original scale. This procedure makes sure that our confidence intervals are reasonable (e.g. no negative values for the confidence limits of the standard deviation estimates). Also note that instead of calculating \( h_\ell(S_i, \hat{\gamma}, \hat{\theta}) \), \( \hat{H}_\theta \) and \( \hat{H}_\gamma \) using their analytical expressions, they are approximated. This is due to the complexity of these expressions and the amount of them that would have to be derived. For example, \( \hat{H}_\theta \) is already a \((2m + 9) \times (2m + 9)\) matrix of derivatives where \( m \) is the dimension of \( \tilde{X} \). The calculation of these approximations is done by making use of Richardson’s extrapolation (Richardson, 1911), resulting in more accurate estimates. A general description of the method to approximate the Jacobian matrix can be given as repeated calculations of the central difference approximation of the first derivative with respect to each component of \( \theta \), using a successively smaller step size. Richardson’s extrapolation uses this information to estimate what happens when the step size goes to zero. A similar description can be given for the approximation of the Hessian matrices. Note that these calculations can be quite time consuming depending on the required level of accuracy.

4 Simulation study

In this section, a simulation study is performed to investigate the finite sample performance of the proposed two-step estimator. We consider the four combinations of the cases where \( Z \) and \( \tilde{W} \) are continuous or binary random variables. It is assumed that when \( Z \) is binary, it follows a logit model. The proposed estimator is compared to three other estimators: one which does not account for the confounding issue, one which assumes \( T \) and \( C \) are independent and one which uses the proposed method but treats \( V \) as observed. The parameters are estimated for samples of 250, 500 and 1000 observations. The first step of the data generating process is as follows:

\[
\begin{pmatrix}
\epsilon_T \\
\epsilon_C
\end{pmatrix} \sim N_2 \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 1.1^2 & 0.75 \cdot 1.1 \cdot 1.4 \\ 0.75 \cdot 1.1 \cdot 1.4 & 1.4^2 \end{pmatrix} \right), \quad \tilde{X} \sim N(0,1).
\]

We have 4 different designs depending on whether \( Z \) and \( \tilde{W} \) are assumed to be a continuous or binary random variable:

| Design 1 | \( \tilde{W} \sim U[0, 2], \quad \nu \sim N(0, 2) \) | \( Z = W^\top \gamma + \nu \) |
| Design 2 | \( \tilde{W} \sim \text{Bernoulli}(0.5), \quad \nu \sim N(0, 2) \) | \( Z = 1(W^\top \gamma - \nu > 0) \) |
| Design 3 | \( \tilde{W} \sim U[0, 2], \quad \nu \sim \text{Logistic}(0, 1) \) | \( Z = (1 + \exp\{W^\top \gamma\}) \log \{1 + \exp\{W^\top \gamma\}\} - W^\top \gamma \exp\{W^\top \gamma\} \) |
| Design 4 | \( \tilde{W} \sim \text{Bernoulli}(0.5), \quad \nu \sim \text{Logistic}(0, 1) \) | \( Z = (1 - Z) \left[ (1 + \exp\{-W^\top \gamma\}) \log \{1 + \exp\{-W^\top \gamma\}\} - W^\top \gamma \exp\{-W^\top \gamma\} \right] \) |

Where \( W = (1, \tilde{X}, \tilde{W})^\top \) and \( \gamma = (-1, 0.6, 2.3)^\top \). It is also assumed that

\[
(\epsilon_T, \epsilon_C) \perp (\tilde{X}, \tilde{W}, \nu), \quad \tilde{W} \perp (\tilde{X}, \nu) \quad \text{and} \quad \tilde{X} \perp \nu.
\]

From this, we can construct:

| Design 1 | \( V = Z - W^\top \gamma \) |
| Design 2 | \( V = (1 - Z) \left[ (1 + \exp\{W^\top \gamma\}) \log \{1 + \exp\{W^\top \gamma\}\} - W^\top \gamma \exp\{W^\top \gamma\} \right] \) |
| Design 3 | \( V = (1 - Z) \left[ (1 + \exp\{-W^\top \gamma\}) \log \{1 + \exp\{-W^\top \gamma\}\} - W^\top \gamma \exp\{-W^\top \gamma\} \right] \) |
| Design 4 | \( V = -Z \left[ (1 + \exp\{-W^\top \gamma\}) \log \{1 + \exp\{-W^\top \gamma\}\} + W^\top \gamma \exp\{-W^\top \gamma\} \right] \) |
Finally, $T$ and $C$ can be constructed for each design in the following way:

\[
\begin{align*}
T &= \beta_{T,0} + \tilde{X}\beta_{T,1} + Z\alpha_T + V\lambda_T + \epsilon_T \\
C &= \beta_{C,0} + \tilde{X}\beta_{C,1} + Z\alpha_C + V\lambda_C + \epsilon_C
\end{align*}
\]

where

\[
(\beta_{T,0}, \beta_{T,1}, \alpha_T, \lambda_T) = (2.5, 2.6, 1.8, 2) \quad \text{and} \quad (\beta_{C,0}, \beta_{C,1}, \alpha_C, \lambda_C) = (2.8, 1.9, 1.5, 1.2).
\]

It follows that $Y = \min\{T, C\}$ and $\Delta = 1(T \leq C)$.

This data generating process was repeated 2500 times for the four possible designs. The parameter values were chosen such that there is between 45% and 50% censoring for each design. For each sample size, there are four different estimators. The first, which we call the naive estimator, ignores the confounding issue and therefore does not include $V$ in the model (no estimates for $\lambda_T$ and $\lambda_C$). The second, which we call the independent estimator, assumes that $T$ and $C$ are independent from each other (no estimates for $\rho$ as it is assumed to be zero). The third, which we call the oracle estimator, uses the control function approach to handle the confounding issue but treats $V$ as if it were observed. The fourth and last, which we call the two-step estimator, uses the two-step estimation method proposed in this article. This means that $V$ is constructed using $\hat{\gamma}$ from the first step. The estimation is performed in R and uses the package \textit{nloptr} to maximize certain functions and the package \textit{numDeriv} for computing the necessary Hessian and Jacobian matrices. The package \textit{MASS} is used to generate the bivariate normal variables.

For each estimator, the bias of each parameter estimate is given together with the empirical standard deviation (ESD) and the root mean squared error (RMSE). Note that, as the bias decreases, these last 2 statistics should converge to the same value. To better explain how these statistics are calculated, we give the formulas for $\alpha_T$ as an example. Let $N$ represent the total amount of simulations with $j = 1, ..., N$ and $(\hat{\alpha}_T)_j$ the estimate of $\alpha_T$ for the $j$'th simulation. The ESD and RMSE for $\alpha_T$ are given as follows:

\[
\begin{align*}
\text{ESD} &= \sqrt{(N - 1)^{-1} \sum_{j=1}^{N} [(\hat{\alpha}_T)_j - \bar{\alpha}_T]^2}, \quad \text{with} \ \bar{\alpha}_T = N^{-1} \sum_{j=1}^{N} (\hat{\alpha}_T)_j. \\
\text{RMSE} &= \sqrt{N^{-1} \sum_{j=1}^{N} [(\hat{\alpha}_T)_j - \alpha_T]^2}, \quad \text{with} \ \alpha_T \ \text{the real parameter value}.
\end{align*}
\]

Lastly, the coverage rate (CR) shows in which percentage of the simulations the real parameter value is included in the estimated 95% confidence interval.

Table 1 shows the results for design 4, meaning that both $Z$ and $W$ are binary. The results show a very noticeable bias for the naive estimator for almost each parameter estimate. Note that this bias remains the same as the sample size increases. The table also shows that the estimated standard errors are not asymptotically valid as the CR is inconsistent and does not converge to the expected 95%. We find the same results when looking at the independent estimator but to a lesser extent. The bias is lower compared to the naive estimator but is still noticeable. A difference with the naive estimator is that there is a lot less bias for $\alpha_T$ and $\alpha_C$. This is to be expected as the independent estimator does take the confounding issue into account. However, the estimated standard errors for $\alpha_T$ and $\alpha_C$ are not asymptotically valid as the CR is inconsistent and does not seem to converge to 95%. As with the naive estimator, the bias does not decrease when the sample size increases. The bias for the proposed two-step estimation method is very close to 0.
and is clearly an improvement over the naive and independent estimator. It is also very close to that of the oracle estimator, which treats $V$ as observed. The bias decreases when the sample size increases and the ESD and RMSE converge to the same value, which also decreases as the sample size increases. The CR is around 95%, meaning that we have asymptotically valid standard errors and confidence intervals. From this, it is clear that the two-step estimator performs well, even for small sample sizes. The results for the other designs are similar and can be found in Appendix D.
Table 1: Estimation results for design 4 with 46% censoring and 2500 simulations. Given are the bias, the empirical standard deviation (ESD), the root mean squared error (RMSE) and the confidence rate (CR).

|                | Bias  | ESD   | RMSE  | CR   | Bias  | ESD   | RMSE  | CR   | Bias  | ESD   | RMSE  | CR   |
|----------------|-------|-------|-------|------|-------|-------|-------|------|-------|-------|-------|------|
|                |       |       |       |      |       |       |       |      |       |       |       |      |
| $n = 250$      | $\beta_{T,0}$ | 2.482 | 0.524 | 2.537 | 0.004 | 2.479 | 0.355 | 2.504 | 0.000 | 2.487 | 0.261 | 2.501 | 0.000 |
|                | $\beta_{T,1}$ | 0.531 | 0.250 | 0.587 | 0.344 | 0.534 | 0.175 | 0.562 | 0.096 | 0.538 | 0.127 | 0.553 | 0.004 |
|                | $\alpha_T$    | -4.566 | 0.488 | 4.592 | 0.000 | -4.572 | 0.312 | 4.583 | 0.000 | -4.577 | 0.229 | 4.583 | 0.000 |
|                | $\beta_{C,0}$ | 1.467 | 0.237 | 1.486 | 0.006 | 1.472 | 0.151 | 1.480 | 0.000 | 1.475 | 0.111 | 1.479 | 0.000 |
|                | $\beta_{C,1}$ | 0.230 | 0.254 | 0.342 | 0.817 | 0.230 | 0.184 | 0.295 | 0.728 | 0.225 | 0.134 | 0.262 | 0.559 |
|                | $\alpha_C$    | -2.608 | 0.470 | 2.650 | 0.004 | -2.625 | 0.323 | 2.645 | 0.000 | -2.614 | 0.231 | 2.624 | 0.000 |
|                | $\sigma_T$    | 0.584 | 0.114 | 0.595 | 0.001 | 0.581 | 0.078 | 0.586 | 0.000 | 0.582 | 0.058 | 0.584 | 0.000 |
|                | $\sigma_C$    | 0.222 | 0.101 | 0.244 | 0.278 | 0.222 | 0.066 | 0.232 | 0.028 | 0.223 | 0.047 | 0.228 | 0.000 |
|                | $\rho$        | -0.067 | 0.263 | 0.272 | 0.921 | -0.040 | 0.182 | 0.186 | 0.929 | -0.035 | 0.134 | 0.139 | 0.916 |
|                | $n = 500$     |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\beta_{T,0}$ | 0.439 | 0.504 | 0.668 | 0.829 | 0.454 | 0.342 | 0.569 | 0.698 | 0.467 | 0.238 | 0.524 | 0.446 |
|                | $\beta_{T,1}$ | 0.214 | 0.226 | 0.312 | 0.982 | 0.223 | 0.156 | 0.273 | 0.957 | 0.225 | 0.109 | 0.250 | 0.876 |
|                | $\alpha_T$    | 0.054 | 0.869 | 0.871 | 0.774 | 0.027 | 0.584 | 0.584 | 0.764 | 0.004 | 0.405 | 0.405 | 0.746 |
|                | $\lambda_T$   | 0.232 | 0.337 | 0.409 | 0.881 | 0.220 | 0.226 | 0.315 | 0.844 | 0.211 | 0.160 | 0.265 | 0.751 |
|                | $\beta_{C,0}$ | 0.567 | 0.364 | 0.673 | 0.844 | 0.566 | 0.247 | 0.618 | 0.659 | 0.579 | 0.179 | 0.606 | 0.280 |
|                | $\beta_{C,1}$ | -0.293 | 0.176 | 0.342 | 0.978 | -0.289 | 0.119 | 0.312 | 0.955 | -0.285 | 0.084 | 0.298 | 0.850 |
|                | $\alpha_C$    | 0.039 | 0.622 | 0.623 | 0.945 | 0.034 | 0.424 | 0.425 | 0.949 | 0.011 | 0.304 | 0.304 | 0.924 |
|                | $\lambda_C$   | -0.244 | 0.241 | 0.343 | 0.952 | -0.241 | 0.167 | 0.293 | 0.942 | -0.250 | 0.119 | 0.277 | 0.872 |
|                | $\sigma_T$    | 0.049 | 0.069 | 0.084 | 0.954 | 0.053 | 0.049 | 0.072 | 0.908 | 0.056 | 0.036 | 0.067 | 0.779 |
|                | $\sigma_C$    | 0.192 | 0.106 | 0.219 | 0.929 | 0.201 | 0.074 | 0.214 | 0.782 | 0.207 | 0.053 | 0.214 | 0.383 |
|                | $n = 1000$    |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\beta_{T,0}$ |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\beta_{T,1}$ |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\alpha_T$    |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\lambda_T$   |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\beta_{C,0}$ |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\beta_{C,1}$ |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\alpha_C$    |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\lambda_C$   |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\sigma_T$    |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\sigma_C$    |       |       |       |      |       |       |       |      |       |       |       |      |
|                | $\rho$        |       |       |       |      |       |       |       |      |       |       |       |      |
5 Data application

In this section, we apply the outlined methodology to estimate the effect of Job Training Partnership Act (JTPA) services on time until employment. The data come from a large-scale randomized experiment known as The National JTPA Study and have been analyzed by Bloom et al. (1997), Abadie et al. (2002) and Frandsen (2015) among others. This study was performed to evaluate the effectiveness of more than 600 federally funded services, established by the Job Training Partnership Act of 1982, that were intended to increase the employability of eligible adults and out-of-school youths. These services included classroom training, on-the-job training and job search assistance. The JTPA started to fund these programs in October of 1983 and continued funding up until the late 1990’s. Between 1987 and 1989, a little over 20,000 adults and out-of-school youths who applied for JTPA were randomly assigned to be in either a treatment or a control group. Treatment group members were eligible to receive JTPA services, while control group members were not eligible for 18 months. However, due to local program staff not always following the randomization rules closely, about 3% of the control group members were able to participate in JTPA services. It is important to note that we are not comparing JTPA services to no services but rather JTPA services versus no and other services, since control group members were still eligible for non-JTPA training. Between 12 and 36 months after randomization, with an average of 21 months, the participants were surveyed by data collection officers. Next, a subset of 5,468 subjects participated in a second follow-up survey, which focused on the period between the two surveys. The second survey took place between 23 and 48 months after randomization.

In this application, we will focus our attention on the effect of JTPA programs on the sample of 1,298 fathers who reported having no job at the time of randomization, for which participation data is available. The outcome of interest is the time between randomization and employment. For the individuals that were only invited to the first interview, the outcome is measured completely if an individual is employed by the time of the survey and censored at the time of the interview otherwise. For the fathers that were invited to the second follow-up interview and participated, the outcome is measured completely if an individual is employed by the time of the second follow-up interview, but is otherwise censored at the second interview date. If the individual does not participate in the second survey after being invited, they will be censored at the time of the first interview. Therefore, there could be some dependence between $T$ and $C$ when this decision to go to the second follow-up interview is influenced by them having found a job between the two interview dates. This possible dependence combined with the fact that the data suffer from two-sided noncompliance makes it an appropriate application of the proposed methodology.

The instrument $\hat{W}$ will be a binary variable indicating whether an individual is in the control or treatment group (0 and 1 respectively). The confounded variable $Z$ indicates whether they actually participated in a JTPA program (0 for no participation and 1 otherwise). This participation variable is confounded due to individuals moving themselves between the treatment and control group in a non-random way. The covariates include the participant’s age, race (white or non-white), marital status and whether they have a high school diploma or GED. We expect $\hat{W}$ to be a valid instrument because it is randomly assigned, correlated with JTPA participation and should have no impact on time until employment other than through participation in a JTPA funded program. Rows 2 through 5 from Table 2 show that the individual characteristics are balanced across the control and treatment group. This indicates satisfactory random assignment. The first row shows that about 31% of the total sample was assigned to the control group.

The last 3 rows of table 2 show summary statistics for variables observed after randomization. It is interesting to note that 13% of the fathers in the control group were nevertheless able to participate in JTPA services compared to 3% for the entire control group. The mean time to
employment also seems to be about 30 days shorter for the individuals assigned to the treatment group compared to the control group. The censoring rate is similar for both groups. Figure 1 plots a histogram of the observed follow-up time $Y$, where darker shading indicates a higher censoring rate. A lot of the censored observations are around the 600 days mark, at which time most of the first follow-up interviews took place. Since everyone in the sample participated in the first follow-up interview, an observation before the date of the first follow-up survey cannot be censored.

|                  | total sample | control group | treatment group |
|------------------|--------------|---------------|-----------------|
| sample size      | 1298         | 402           | 896             |
| age              | 32.7 years   | 33.3 years    | 32.5 years      |
| white            | 0.66         | 0.64          | 0.67            |
| married          | 0.77         | 0.76          | 0.77            |
| GED              | 0.47         | 0.48          | 0.47            |
| $Z$              | 0.52         | 0.13          | 0.70            |
| $(Y, \Delta = 1)$ | 160.9 days  | 181.6 days    | 151.9 days      |
| $\Delta$        | 0.87         | 0.84          | 0.88            |

Table 2: Summary statistics for the sample of fathers unemployed at the time of random assignment. The means are shown for the total sample, the control group and the treatment group.

Figure 1: Histogram of $Y$, in days, starting from random assignment. The sample includes fathers unemployed at the time of random assignment. The darker the shade of the bin, the higher the censoring rate.

The results of applying the two-step estimator (using a logit model for $Z$), compared to other estimators, can be found in Table 3. The naive estimator, which does not treat $Z$ as a confounded
variable, seems to underestimate the effect of JTPA services on time until employment compared to the proposed two-step estimator. At a 5% significance level, both of these estimators find a significant effect of JTPA training reducing time until employment. However, the two-step estimate is almost twice the naive estimate which indicates that the individuals participating in the treatment are those with a lower ability to find employment. The independent estimator, which assumes independent censoring, seems to slightly overestimate the size of the effect compared to the proposed two-step estimator, but is not significant at a 5% significance level. Age seems to be (borderline) significant across the estimators as does marriage status. Being older seems to increase time until employment, while being married reduces it. Both the naive and the two-step estimator seem to agree that there is quite a strong negative correlation of about -0.43 between $T$ and $C$.

| $T$          | naive estimator | independent estimator | two-step estimator |
|--------------|-----------------|-----------------------|--------------------|
| Intercept    | 4.753 0.223 0.000 | 4.949 0.360 0.000 | 4.866 0.370 0.000 |
| Age          | 0.015 0.006 0.007 | 0.013 0.006 0.026 | 0.015 0.008 0.056 |
| White        | -0.197 0.108 0.068 | -0.197 0.122 0.107 | -0.187 0.622 0.764 |
| Married      | -0.331 0.123 0.007 | -0.330 0.140 0.018 | -0.331 0.132 0.012 |
| GED          | -0.166 0.102 0.104 | -0.179 0.119 0.133 | -0.172 0.217 0.430 |
| $\alpha_T$  | -0.218 0.102 0.033 | -0.483 0.298 0.105 | -0.428 0.210 0.041 |
| $\lambda_T$ | -0.113 0.136 0.403 | -0.097 0.087 0.268 |

| $C$          | naive estimator | independent estimator | two-step estimator |
|--------------|-----------------|-----------------------|--------------------|
| Intercept    | 6.866 0.134 0.000 | 6.672 0.209 0.000 | 6.848 0.403 0.000 |
| Age          | -0.001 0.002 0.436 | -0.001 0.002 0.607 | -0.001 0.003 0.605 |
| White        | -0.063 0.042 0.137 | -0.069 0.046 0.134 | -0.062 0.177 0.728 |
| Married      | 0.012 0.033 0.713 | -0.005 0.064 0.932 | 0.010 1.206 0.993 |
| GED          | 0.010 0.032 0.746 | -0.006 0.101 0.950 | 0.013 0.661 0.984 |
| $\alpha_C$  | -0.062 0.042 0.138 | -0.052 0.156 0.737 | -0.028 1.335 0.983 |
| $\lambda_C$ | 0.006 0.056 0.908 | 0.015 0.511 0.976 |

Table 3: Estimation results for the naive, independent and two-step estimator. Given are the parameter estimate, standard error (SE) and the $p$-value.

**Conflict of interest**

The authors have declared no conflict of interest.

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Appendix

A Control function examples

In this first section of the Appendix, we expand on the examples of \(g_\gamma(Z,W)\) given in Section 2.1. More specifically, given the distribution of \(\nu\), we give explicit expressions for

\[ g_\gamma(Z,W) = Z \mathbb{E}[\nu \mid \nu > W^\top \gamma] + (1 - Z) \mathbb{E}[\nu \mid \nu < W^\top \gamma]. \]

If we assume that \(\nu\) follows a standard normal distribution, it can be derived that

\[ g_\gamma(Z,W) = (1 - Z) \frac{\phi(W^\top \gamma)}{\Phi(-W^\top \gamma)} - Z \frac{\phi(W^\top \gamma)}{\Phi(W^\top \gamma)}. \]

Similarly, if we assume that \(\nu\) follows a standard logistic distribution, we find:

\[ g_\gamma(Z,W) = (1 - Z) \left(1 + \exp\left\{W^\top \gamma\right\}\right) \log\left(1 + \exp\left\{-W^\top \gamma\right\}\right) - W^\top \gamma \exp\left\{-W^\top \gamma\right\}. \]

Both of these expressions follow from the following conditional expectations. Let \(\nu \sim N(0,1)\), it follows that:

\[ \mathbb{E}[\nu \mid \nu < a] = \frac{1}{\Phi(a)} \int_{-\infty}^{a} \nu \phi(\nu) \, d\nu = \frac{-1}{\Phi(a)} \int_{-\infty}^{a} \phi'(\nu) \, d\nu = \frac{-\phi(a)}{\Phi(a)}, \]

\[ \mathbb{E}[\nu \mid \nu > a] = \frac{1}{1 - \Phi(a)} \int_{a}^{+\infty} \nu \phi(\nu) \, d\nu = \frac{-1}{\Phi(-a)} \int_{a}^{+\infty} \phi'(\nu) \, d\nu = \frac{\phi(a)}{\Phi(-a)}. \]

If \(\nu \sim \text{Logistic}(0,1)\), it follows that:

\[ \mathbb{E}[\nu \mid \nu < a] = (1 + e^{-a}) \int_{-\infty}^{a} \frac{e^{-\nu}}{(1 + e^{-\nu})^2} \, d\nu = -(1 + e^{-a}) \log(1 + e^{-a}) - ae^{-a}, \]

\[ \mathbb{E}[\nu \mid \nu > a] = (1 + e^{a}) \int_{a}^{+\infty} \frac{e^{-\nu}}{(1 + e^{-\nu})^2} \, d\nu = (1 + e^{a}) \log(1 + e^{a}) - ae^{a}. \]

B Technical lemmas

In this section, we prove three lemmas of which Lemma 3 is needed to prove Theorem 2 in Appendix C. Lemma 1 is used in the proof of Lemma 2 and Lemma 2 is used in the proof of Lemma 3. Note that the proofs of the first two lemmas are inspired by the proof of Lemma 1 in Tauchen (1985). For all of these proofs, it is useful to define the open cube \(I(\gamma, \theta, d)\) as

\[ I(\gamma, \theta, d) = \left\{ (\tilde{\gamma}, \tilde{\theta}) \in \mathbb{N}_\gamma \times \Theta : ||(\tilde{\gamma}, \tilde{\theta}) - (\gamma, \theta)||_\infty < d \right\}, \]

where ||·||_\infty is the sup-norm and \(d \in \mathbb{R}_{>0}\).
Lemma 1. If Assumptions (A7)-(A8) hold, then for all \( \varepsilon > 0 \) and \((\gamma, \theta)\) in \( \mathcal{N}_\gamma \times \Theta \), there exists \( d > 0 \) such that

\[
E \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I(\gamma, \theta, d)} |\ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma, \theta)| \right] < \frac{\varepsilon}{4}.
\]

Proof. Define

\[
\kappa(s, \gamma, \theta, d) = \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I(\gamma, \theta, d)} |\ell(s, \tilde{\gamma}, \tilde{\theta}) - \ell(s, \gamma, \theta)|,
\]

with \((s, \gamma, \theta, d) \in \mathcal{G} \times \mathcal{N}_\gamma \times \Theta \times \mathbb{R}_{>0}\). By Assumption (A8), for all \((s, \gamma, \theta, d) \in \mathcal{G} \times \mathcal{N}_\gamma \times \Theta \times \mathbb{R}_{>0}\), we have

\[
0 \leq \kappa(s, \gamma, \theta, d) \leq \sup_{\tilde{\gamma} \in \mathcal{N}_\gamma, \tilde{\theta} \in \Theta} |\ell(s, \tilde{\gamma}, \tilde{\theta}) - \ell(s, \gamma, \theta)| \leq 2 \cdot D(s).
\]

Let \( d_n \) be a sequence that converges to 0 when \( n \to \infty \) and define \( \kappa_n(s, \gamma, \theta) = \kappa(s, \gamma, \theta, d_n) \). The continuity of \( \ell \), implies that (use Heine-Cantor theorem) for all \((s, \gamma, \theta) \in \mathcal{G} \times \mathcal{N}_\gamma \times \Theta \) it holds that

\[
\lim_{n \to \infty} \kappa_n(s, \gamma, \theta) = 0.
\]

Since \( E[D(s)] < \infty \), the dominated convergence theorem states that

\[
\lim_{n \to \infty} E \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I(\gamma, \theta, d_n)} |\ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma, \theta)| \right] = 0.
\]

From this we can infer that for all \( \varepsilon > 0 \) and \((\gamma, \theta) \in \mathcal{N}_\gamma \times \Theta \), there exists \( d > 0 \) such that

\[
E \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I(\gamma, \theta, d)} |\ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma, \theta)| \right] < \frac{\varepsilon}{4}.
\]

Lemma 2. Under Assumptions (A7)-(A8), there exists a neighborhood \( B \) of \( \gamma^* \) such that for all \( \varepsilon > 0 \) and \( \gamma \in B \), we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{\gamma \in \mathcal{B}, \theta \in \Theta} |\hat{L}(\gamma, \theta) - L(\gamma^*, \theta)| < \varepsilon \right) = 1.
\]

Proof. Take \( \gamma \in \mathcal{N}_\gamma \) (with \( \mathcal{N}_\gamma \) defined in Assumption (A8)). By Lemma 1, for all \( \theta \in \Theta \), there exists \( d_\theta > 0 \) such that

\[
E \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I(\gamma^*, \theta, d_\theta)} |\ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma^*, \theta)| \right] < \frac{\varepsilon}{4}.
\]

Given these open cubes, we can define

\[
I = \bigcup_{\theta \in \Theta} I(\gamma^*, \theta, d_\theta),
\]
which is clearly an open cover of \( \{\gamma^*\} \times \Theta \). Because \( \Theta \) is assumed to be compact and \( \gamma^* \) fixed, the open cover definition of a compact space tells us that since \( I \) is a collection of open subsets that cover \( \{\gamma^*\} \times \Theta \), there must exist a finite subcollection \( \{I_1, I_2, ..., I_K\} \) such that

\[
\{\gamma^*\} \times \Theta \subseteq \bigcup_{k=1}^{K} I_k,
\]

with

\[
I_k = I(\gamma^*, \theta_k, d_k), \quad k = 1, \ldots, K,
\]

for some \( d_k > 0 \). Thanks to the subcollection being finite, we can define \( \bar{d} = \min_{k=1}^{K} d_k > 0 \). Let \( B_{\gamma} \) be the open ball in \( \Gamma \) centered around \( \gamma^* \) with a radius of \( \bar{d} \). For all \( k = 1, \ldots, K \) and \( \gamma \in B_{\gamma} \), it holds that \( (\gamma, \theta_k) \in I_k \), which means that \( F \) is also a finite open cover of \( B_{\gamma} \times \Theta \). Let \( B = B_{\gamma} \cap \mathcal{N}_{\gamma} \), \( \gamma \in B \) and \( (\gamma, \theta) \in I_k \). It follows from (9) that

\[
\frac{\varepsilon}{4} \geq \mathbb{E} \left[ |\ell(S, \gamma, \theta) - \ell(S, \gamma^*, \theta_k)| \right] \geq |L(\gamma, \theta) - L(\gamma^*, \theta_k)|.
\]

This means that

\[
|L(\gamma, \theta) - L(\gamma^*, \theta_k)| < \frac{\varepsilon}{4} \quad \text{when} \quad (\gamma, \theta) \in I_k.
\]

We have

\[
\begin{align*}
|\hat{L}(\gamma, \theta) - L(\gamma^*, \theta)| & = \frac{1}{n} \sum_{i=1}^{n} \ell(S_i, \gamma, \theta) - \ell(S_i, \gamma^*, \theta_k) + \ell(S_i, \gamma^*, \theta_k) - L(\gamma^*, \theta_k) + L(\gamma^*, \theta_k) - L(\gamma^*, \theta) \\
 & \leq \frac{1}{n} \sum_{i=1}^{n} \left| \ell(S_i, \gamma, \theta) - \ell(S_i, \gamma^*, \theta_k) \right| - \mathbb{E} \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I_k} \left| \ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma^*, \theta_k) \right| \right] \\
 & \quad + \mathbb{E} \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I_k} \left| \ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma^*, \theta_k) \right| \right] + |\hat{L}(\gamma^*, \theta_k) - L(\gamma^*, \theta_k)| + L(\gamma^*, \theta_k) - L(\gamma^*, \theta) \\
 & \leq \frac{1}{n} \sum_{i=1}^{n} \left| \ell(S_i, \tilde{\gamma}, \tilde{\theta}) - \ell(S_i, \gamma^*, \theta_k) \right| - \mathbb{E} \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I_k} \left| \ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma^*, \theta_k) \right| \right] \\
 & \quad + \mathbb{E} \left[ \sup_{(\tilde{\gamma}, \tilde{\theta}) \in I_k} \left| \ell(S, \tilde{\gamma}, \tilde{\theta}) - \ell(S, \gamma^*, \theta_k) \right| \right] + |\hat{L}(\gamma^*, \theta_k) - L(\gamma^*, \theta_k)|. 
\end{align*}
\]

The terms in (12) are bounded by \( \frac{\varepsilon}{2} \) due to (9) and (10). For (11) and (13), the strong law of large numbers can be applied such that they go to 0 in probability.

Because of this, we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{(\gamma, \theta) \in I_k} |\hat{L}(\gamma, \theta) - L(\gamma^*, \theta)| < \varepsilon \right) = 1,
\]

for all \( k = 1, \ldots, K \). Since \( F \) is a finite open covering of \( B \times \Theta \), this leads to (8).
Lemma 3. If Assumptions (A6)-(A8) hold, then
\[
\sup_{\theta \in \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| \overset{p}{\to} 0.
\]

Proof. Note that for all \(\varepsilon > 0\):
\[
P\left( \sup_{\theta \in \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| > \varepsilon \right) = P\left( \sup_{\theta \in \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| > \varepsilon \cap \{\hat{\gamma} \in B\} \right) + P\left( \sup_{\theta \in \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| > \varepsilon \cap \{\hat{\gamma} /\in B\} \right).
\]

(14)

Assumption (A6) implies that \(\hat{\gamma} \overset{p}{\to} \gamma^*\). Because of this we know that
\[
\lim_{n \to \infty} P\left( \sup_{\theta \in \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| > \varepsilon \cap \{\hat{\gamma} /\in B\} \right) \leq \lim_{n \to \infty} P(\hat{\gamma} /\in B) = 0.
\]

Moreover, we have
\[
P\left( \sup_{(\gamma, \theta) \in B \times \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| > \varepsilon \cap \{\hat{\gamma} \in B\} \right) \leq P\left( \sup_{(\gamma, \theta) \in B \times \Theta} |\hat{L}(\hat{\gamma}, \theta) - L(\gamma^*, \theta)| > \varepsilon \right) \to 0,
\]
by Lemma 2. We obtain the result using (14).

C Proofs of theorems

In this section of the Appendix, the three main theorems regarding identifiability, consistency and asymptotic normality of the estimators are proven.

Proof of Theorem 1

Because of Assumption (A6) we know that \(\gamma\) is identified, meaning that \(\gamma = \gamma^*\). For ease of notation, let \(\theta_1 = \theta\) and \(\theta_2 = \theta^*\). From the model specification, we know that:
\[
f_{Y_j,\Delta_j|W,Z}(y, 1 \mid w, z, \gamma; \theta_j) = \left[ 1 - \Phi\left( \frac{y - x^\top \beta_{C_j} - z\alpha_{C_j} - v\lambda_{C_j} + \rho_j \sigma_{T_j} \sigma_C (x^\top \beta_{T_j} + z\alpha_{T_j} + v\lambda_{T_j})}{\sigma_{C_j}(1 - \rho_j^2)^{1/2}} \right) \right] \times \frac{1}{\sigma_{T_j}} \phi\left( \frac{y - x^\top \beta_{T_j} - z\alpha_{T_j} - v\lambda_{T_j}}{\sigma_{T_j}} \right), \quad j = 1, 2.
\]
We will consider a number of cases that are dependent on the values of the following \( \pi \)'s:

\[
\pi_{11} = 1 - \rho_1 \frac{\sigma_{C_1}}{\sigma_{T_1}}, \quad \pi_{12} = 1 - \rho_1 \frac{\sigma_{T_1}}{\sigma_{C_1}}, \quad \pi_{21} = 1 - \rho_2 \frac{\sigma_{C_2}}{\sigma_{T_2}}, \quad \pi_{22} = 1 - \rho_2 \frac{\sigma_{T_2}}{\sigma_{C_2}}.
\]

**Case 1:** All \( \pi_{jk} \) with \( j, k = 1, 2 \) are strictly positive. Note that the positivity of \( \pi_{j1} \) (\( j = 1, 2 \)) allows us to rewrite the argument of \( \Phi(\cdot) \) as:

\[
y - \frac{x^T \beta_{C_j} + z\alpha_{C_j} + v\lambda_{C_j} - \rho_j \frac{\sigma_{C_j}}{\sigma_{T_j}} (x^T \beta_{T_j} + z\alpha_{T_j} + v\lambda_{T_j})}{\sqrt{\frac{\sigma_{C_j}^2 (1 - \rho_j^2)}{\pi_{j1}^2}}}.
\]

Therefore, we define \( \xi_{j1} \) (\( j = 1, 2 \)), whose distribution for a given \((W, Z)\) is specified as:

\[
(\xi_{j1} \mid W, Z) \sim N\left( \frac{x^T \beta_{C_j} + z\alpha_{C_j} + v\lambda_{C_j} - \rho_j \frac{\sigma_{C_j}}{\sigma_{T_j}} (x^T \beta_{T_j} + z\alpha_{T_j} + v\lambda_{T_j})}{\pi_{j1}}, \frac{\sigma_{C_j}^2 (1 - \rho_j^2)}{\pi_{j1}^2} \right).
\]

This allows us to rewrite the sub-density as follows:

\[
f_{Y_j,\Delta_j \mid W, Z}(y, 1 \mid w, z, \gamma; \theta_j) = P(\xi_{j1} > y \mid W = w, Z = z) f_{T_j \mid W, Z}(t \mid w, z, \gamma; \theta_j).
\]

(15)

Since \( f_{Y_1,\Delta_1 \mid X, Z, V}(y, 1 \mid w, z, \gamma; \theta_1) = f_{Y_2,\Delta_2 \mid W, Z}(y, 1 \mid w, z, \gamma; \theta_2) \) for almost every \((w, z)\), it follows from (15) that

\[
\lim_{y \to -\infty} f_{T_1 \mid W, Z}(t \mid w, z, \gamma; \theta_1) = \lim_{y \to -\infty} f_{T_2 \mid W, Z}(t \mid w, z, \gamma; \theta_2)
\]

for almost every \((w, z)\). Because of Proposition A.1. in Deresa and Van Keilegom (2020a), it is implied that:

\[
\beta_{T_1} = \beta_{T_2}, \quad \alpha_{T_1} = \alpha_{T_2}, \quad \lambda_{T_1} = \lambda_{T_2}, \quad \sigma_{T_1} = \sigma_{T_2}.
\]

Putting \( \Delta_j = 0 \) and repeating the same arguments (with \( \pi_{j2} > 0, \ j = 1, 2 \)), we get that:

\[
\beta_{C_1} = \beta_{C_2}, \quad \alpha_{C_1} = \alpha_{C_2}, \quad \lambda_{C_1} = \lambda_{C_2}, \quad \sigma_{C_1} = \sigma_{C_2}.
\]

Using the expression for \( F_{Y \mid W, Z}(y \mid w, z, \gamma; \theta) \) from Section 2.2, we know that:

\[
\Phi \left( \frac{y - x^T \beta_{T_1} - z\alpha_{T_1} - v\lambda_{T_1}}{\sigma_{T_1}}, \frac{y - x^T \beta_{C_1} - z\alpha_{C_1} - v\lambda_{C_1}}{\sigma_{C_1}}; \rho_1 \right) = \Phi \left( \frac{y - x^T \beta_{T_2} - z\alpha_{T_2} - v\lambda_{T_2}}{\sigma_{T_2}}, \frac{y - x^T \beta_{C_2} - z\alpha_{C_2} - v\lambda_{C_2}}{\sigma_{C_2}}; \rho_2 \right),
\]

for almost every \((w, z)\). From this it is clear that \( \rho_1 = \rho_2 \) and thus \( \theta_1 = \theta_2 \).

**Case 2:** One of \( \pi_{jk} \) with \( j, k = 1, 2 \) is strictly negative and the others are strictly positive. Firstly, we assume that \( \pi_{11} < 0 \), and therefore it must be the case that \( \pi_{21} > 0 \). From Case 1 we know that the positivity of \( \pi_{21} \) implies that

\[
f_{Y_2,\Delta_2 \mid W, Z}(y, 1 \mid w, z, \gamma; \theta_2) = P(\xi_{21} > y \mid W = w, Z = z) f_{T_2 \mid W, Z}(t \mid w, z, \gamma; \theta_2).
\]
Note that the negativity of $\pi_{11}$ allows us to rewrite the argument of $\Phi(\cdot)$ as

$$-y - \frac{-\frac{x^T \beta_{C_1} + z \alpha_{C_1} + v \lambda_{C_1} - \rho_j \sigma_{C_j} (x^T \beta_{T_j} + z \alpha_{T_j} + v \lambda_{T_j})}{\pi_{j1}}}{\sqrt{\frac{\sigma_{C_j}^2 (1 - \rho_j^2)}{\pi_{j1}}}}.$$ 

As before, we define a variable $\zeta_{j1}$ ($j = 1, 2$) whose distribution for given $(W, Z)$ is specified as:

$$(\zeta_{j1} \mid W, Z) \sim N\left(-\frac{-\frac{x^T \beta_{C_j} + z \alpha_{C_j} + v \lambda_{C_j} - \rho_j \sigma_{C_j} (x^T \beta_{T_j} + z \alpha_{T_j} + v \lambda_{T_j})}{\pi_{j1}}}{\sigma_{C_j}^2 (1 - \rho_j^2)}\right).$$

This can be used to find that:

$$f_{Y_1, \Delta_1 \mid W, Z}(y, 1 \mid w, z, \gamma; \theta_1) = P(\xi_{11} < y \mid W = w, Z = z) f_{T_1 \mid W, Z}(t \mid w, z, \gamma; \theta_1),$$

$$= P(\xi_{11} < y \mid W = w, Z = z) f_{T_1 \mid W, Z}(t \mid w, z, \gamma; \theta_1),$$

$$= P(\xi_{11} < y \mid W = w, Z = z) f_{T_1 \mid W, Z}(t \mid w, z, \gamma; \theta_1).$$

Because $f_{Y_1, \Delta_1 \mid W, Z}(y, 1 \mid w, z, \gamma; \theta_1) = f_{Y_2, \Delta_2 \mid W, Z}(y, 1 \mid w, z, \gamma; \theta_2)$, we have that:

$$P(\xi_{21} > y \mid W = w, Z = z) = P(\xi_{11} < y \mid W = w, Z = z) \times \frac{f_{T_1 \mid W, Z}(t \mid w, z, \gamma; \theta_1)}{f_{T_2 \mid W, Z}(t \mid w, z, \gamma; \theta_2)}.$$ 

Taking the limit on both sides when $y$ approaches $-\infty$, it follows that the left-hand side goes to 1. However, Lemma 2.4 in Basu and Ghosh (1978) shows that the right hand side does not go to 1, leading to a contradiction. Using the same arguments, it can be shown that one strictly negative $\pi$ and three strictly positive $\pi$'s always leads to a contradiction.

**Case 3:** Two of $\pi_{jk}$ with $j, k = 1, 2$ are strictly negative and the other two are strictly positive. When $\pi_{11}$ and $\pi_{12}$ are both either strictly positive or strictly negative, **Case 2** shows that this leads to a contradiction. Therefore, one of $\pi_{11}, \pi_{12}$ and one of $\pi_{21}, \pi_{22}$ is strictly positive. Assuming $\pi_{11} > 0$ and $\pi_{12} < 0$, we know from **Case 2** that if $\pi_{21} < 0$ we get a contradiction. Hence, we further assume that $\pi_{21} > 0$ and $\pi_{22} < 0$. We now define

$$(\zeta_{j2} \mid W, Z) \sim N\left(-\frac{-\frac{x^T \beta_{T_j} + z \alpha_{T_j} + v \lambda_{T_j} - \rho_j \sigma_{C_j} (x^T \beta_{C_j} + z \alpha_{C_j} + v \lambda_{C_j})}{\pi_{j2}}}{\sigma_{C_j}^2 (1 - \rho_j^2)}\right).$$

$$(\xi_{j2} \mid W, Z) \sim N\left(-\frac{-\frac{x^T \beta_{T_j} + z \alpha_{T_j} + v \lambda_{T_j} - \rho_j \sigma_{C_j} (x^T \beta_{C_j} + z \alpha_{C_j} + v \lambda_{C_j})}{\pi_{j2}}}{\sigma_{C_j}^2 (1 - \rho_j^2)}\right).$$

Since $f_{Y_1, \Delta_1 \mid W, Z}(y, 0 \mid w, z, \gamma; \theta_1) = f_{Y_2, \Delta_2 \mid W, Z}(y, 0 \mid w, z, \gamma; \theta_2)$, the same arguments as before can be used to show that:

$$P(\xi_{12} < y \mid W = w, Z = z) f_{C_1 \mid W, Z}(c \mid w, z, \gamma; \theta_1)$$

$$= P(\xi_{22} < y \mid W = w, Z = z) f_{C_2 \mid W, Z}(c \mid w, z, \gamma; \theta_2).$$

It is clear that $\lim_{y \to -\infty} f_{C_1 \mid W, Z}(y \mid w, z, \gamma; \theta_1) = \lim_{y \to -\infty} f_{C_2 \mid W, Z}(y \mid w, z, \gamma; \theta_2)$ for almost every $(w, z)$. Application of Proposition A.1. in Deresa and Van Keilegom (2020a) implies that

$$\beta_{C_1} = \beta_{C_2}, \ \alpha_{C_1} = \alpha_{C_2}, \ \lambda_{C_1} = \lambda_{C_2}, \ \sigma_{C_1} = \sigma_{C_2}.$$
The result for $\Delta_j = 1$ and $\pi_{11}, \pi_{21} > 0$ was already discussed in Case 1. Combining these, it is clear that $\theta_1 = \theta_2$. It follows that this argument can be replicated for $\pi_{11}, \pi_{21} < 0$ and $\pi_{12}, \pi_{22} > 0$.

**Case 4:** Three or four of $\pi_{jk}$ with $j,k = 1,2$ are strictly negative and one or none of the $\pi$’s are strictly positive, respectively. This immediately leads to a contradiction since

$$\rho_1 \frac{\sigma_{C_1}}{\sigma_{T_1}} \rho_1 \frac{\sigma_{T_1}}{\sigma_{C_1}} = \rho_1^2 < 1,$$

which implies that

$$\rho_1 \frac{\sigma_{C_1}}{\sigma_{T_1}} < 1 \text{ or } \rho_1 \frac{\sigma_{T_1}}{\sigma_{C_1}} < 1,$$

meaning that at least one of $\pi_{11}$ and $\pi_{12}$ is strictly positive. This also is the case for $\pi_{21}$ and $\pi_{22}$. Hence, there are always at least two strictly positive $\pi$’s and we have a contradiction.

**Case 5:** One of $\pi_{jk}$ with $j,k = 1,2$ is equal to zero. We assume $\pi_{11} = 0$, which implies that $\pi_{21} > 0$. Note that $\pi_{11} = 0$ implies that: Because $y$ is no longer included in the argument of $\Phi(\cdot)$, we can rewrite this as

$$f_{Y_1, \Delta_1 | W,Z}(y, 1 | w, z, \gamma; \theta_1) = p \times \frac{1}{\sigma_{T_1}} \phi \left( \frac{y - x^T \beta T_1 - z \alpha T_1 - v \lambda T_1}{\sigma_{T_1}} \right), \text{ with } 0 < p < 1.$$

Using what we have learned from the previous cases, we get

$$p \times f_{T_1 | W,Z}(t | w, z, \gamma; \theta_1) = P(\xi > y | W = w, Z = z) f_{T_2 | W,Z}(t | w, z, \gamma; \theta_2).$$

Taking the limit where $y \to -\infty$ we get that

$$p = \lim_{y \to -\infty} \frac{f_{T_2 | W,Z}(y | w, z, \gamma; \theta_2)}{f_{T_1 | W,Z}(y | w, z, \gamma; \theta_1)}.$$

Note that this is a contradiction since, according to Lemma 2.3 in Basu and Ghosh (1978), the right hand side of the equation can only be equal to $0, 1$ or $\infty$. It follows that this argument can be replicated for the other $\pi$’s.

**Case 6:** Two of $\pi_{jk}$, with $j,k = 1,2$ are equal to zero. Note that it cannot be the case that $\pi_{11}$ and $\pi_{12}$ are both zero or that $\pi_{21}$ and $\pi_{22}$ are both zero since $|\rho_j| < 1$ with $j = 1,2$. Therefore, one of $\pi_{11}, \pi_{12}$ and one of $\pi_{21}, \pi_{22}$ needs to be zero. To avoid contradictions, as seen in Case 5, the only possibilities are $\pi_{11} = \pi_{21} = 0$ or $\pi_{12} = \pi_{22} = 0$. Without loss of generality, we will assume that $\pi_{11} = \pi_{21} = 0$. From Case 5, we also know that $f_{Y_1, \Delta_1 | W,Z}(y, 1 | w, z, \gamma; \theta_1) = f_{Y_2, \Delta_2 | W,Z}(y, 1 | w, z, \gamma; \theta_2)$ for almost every $(w, z)$ implies that

$$\frac{p_1}{p_2} = \lim_{y \to -\infty} \frac{f_{T_2 | W,Z}(y | w, z, \gamma; \theta_2)}{f_{T_1 | W,Z}(y | w, z, \gamma; \theta_1)},$$

with $0 < p_1, p_2 < 1$. As discussed before in Case 5, the right hand side can only be equal to $0, 1$ or $\infty$. Therefore, the only way this does not lead to a contradiction is if the right hand side is equal to 1 and $p_1 = p_2$. According to Proposition A.1. in Deresa and Van Keilegom (2020a), this means that

$$\beta T_1 = \beta T_2, \quad \alpha T_1 = \alpha T_2, \quad \lambda T_1 = \lambda T_2, \quad \sigma_{T_1} = \sigma_{T_2}.$$ 

Combining this with $p_1 = p_2$, it can easily be shown that $\theta_1 = \theta_2$.

**Case 7:** Three or four of $\pi_{jk}$, with $j,k = 1,2$ are equal to zero. Note that if $\pi_{11} = 0$, it must be the case that $\pi_{12} > 0$. The same holds for $\pi_{21}$ and $\pi_{12}$, leading to a contradiction. □
Proof of Theorem 2

Note that (i) $\ell$ is a continuous function, since it consists of well known other continuous functions. Because of the continuity of $\ell$ and Assumptions (A7)-(A8), it is clear that $L(\gamma, \theta)$ is continuous. Furthermore, Lemma 3 shows that (ii) $\hat{L}(\gamma, \theta)$ converges uniformly in probability to $L(\gamma^*, \theta)$ under Assumptions (A6)-(A8). Moreover, we also have (iii) $L(\gamma, \theta)$ is uniquely maximized at $L(\gamma, \theta^*)$. Indeed, let $\theta \in \Theta$ such that $\theta \neq \theta^*$ and define

$$R = \frac{f_{Y,\Delta|X,Z,V}(Y, \Delta | X, Z, W, \gamma; \theta)}{f_{Y,\Delta|X,Z,V}(Y, \Delta | X, Z, W, \gamma; \theta^*)}.$$ 

Assumptions (A1)-(A5) guarantee that $\theta^*$ is identifiable according to Theorem 1. Hence, $R$ is not 1 almost everywhere. As a result, by the strict version of Jensen’s inequality, we have

$$-\log(\mathbb{E}[R]) < \mathbb{E}[-\log(R)].$$

This means that

$$L(\gamma, \theta^*) - L(\gamma, \theta) = \mathbb{E}[-\log(R)] > -\log(\mathbb{E}[R]) = -\log(1) = 0,$$

which shows (iii). Given (i)-(iii) and Assumption (A7), Theorem 2.1 from Newey and McFadden (1994) tells us that

$$\hat{\theta} \xrightarrow{p} \theta^*.$$

Combining this with Assumption (A6), this leads to the desired result. \hfill $\square$

Proof of Theorem 3

Since Assumptions (A1)-(A8) guarantee that Theorem 2 holds, we know that (i) $\hat{\theta} \xrightarrow{p} \theta^*$ and $\hat{\gamma} \xrightarrow{p} \gamma^*$. By dominated convergence arguments, it can be shown that (ii) $\hat{h}(S, \gamma, \theta) = (h_m(W, Z, \gamma)^\top, h_\ell(S, \gamma, \theta)^\top)^\top$ is continuously differentiable in a neighborhood $N_{\gamma, \theta}$ of $(\gamma^*, \theta^*)$ with probability approaching one. Theorem 2 shows that $L(\gamma, \theta)$ is uniquely maximized at $\theta^*$ for a given $\gamma$, which combined with Assumptions (A6) and (A7) implies that (iii) $\mathbb{E}[\hat{h}(S, \gamma^*, \theta^*)] = 0$. Given (i)-(iii), combined with Assumptions (A7), (A9) and (A10), Theorem 6.1 from Newey and McFadden (1994) tells us that

$$\sqrt{n}(\hat{\theta} - \theta^*) \xrightarrow{d} N(0, \Sigma_\theta)$$

with

$$\Sigma_\theta = H_\theta^{-1} \mathbb{E}\left[\{h_\ell(S, \gamma^*, \theta^*) + H_{\gamma}\Psi\} \{h_\ell(S, \gamma^*, \theta^*) + H_{\gamma}\Psi\}^\top\right] (H_\theta^{-1})^\top.$$

\hfill $\square$

D Tables
Table 4: Estimation results for design 1 with 51% censoring and 2500 simulations. Given are the bias, the empirical standard deviation (ESD), the root mean squared error (RMSE) and the confidence rate (CR).

|          | Bias  | ESD   | RMSE | CR  | Bias  | ESD   | RMSE | CR  | Bias  | ESD   | RMSE | CR  |
|----------|-------|-------|------|-----|-------|-------|------|-----|-------|-------|------|-----|
| $n = 250$ |       |       |      |     | $n = 500$ |       |      |     | $n = 1000$ |       |      |     |
| naive estimator |       |       |      |     | independent estimator |       |      |     | oracle estimator |       |      |     | two-step estimator |       |      |     |
| $\beta_{T,0}$ | -1.798 | 0.221 | 1.812 | 0.000 | $\beta_{T,0}$ | 0.478 | 0.416 | 0.634 | 0.904 | $\beta_{T,0}$ | 0.000 | 0.154 | 0.154 | 0.952 | $\beta_{T,0}$ | 0.017 | 0.385 | 0.385 | 0.949 |
| $\beta_{T,1}$ | -0.830 | 0.179 | 0.850 | 0.009 | $\beta_{T,1}$ | 0.137 | 0.318 | 0.346 | 1.000 | $\beta_{T,1}$ | 0.001 | 0.100 | 0.100 | 0.949 | $\beta_{T,1}$ | 0.010 | 0.299 | 0.299 | 0.949 |
| $\alpha_T$ | 1.385 | 0.114 | 1.390 | 0.000 | $\alpha_T$ | 0.038 | 0.230 | 0.233 | 0.536 | $\alpha_T$ | -0.001 | 0.066 | 0.066 | 0.941 | $\alpha_T$ | -0.001 | 0.066 | 0.066 | 0.941 |
| $\beta_{C,0}$ | -1.093 | 0.353 | 1.148 | 0.132 | $\beta_{C,0}$ | 0.160 | 0.238 | 0.287 | 0.701 | $\beta_{C,0}$ | 0.000 | 0.088 | 0.088 | 0.946 | $\beta_{C,0}$ | -0.003 | 0.210 | 0.210 | 0.941 |
| $\beta_{C,1}$ | -0.499 | 0.144 | 0.519 | 0.062 | $\beta_{C,1}$ | 0.353 | 0.276 | 0.602 | 0.722 | $\beta_{C,1}$ | -0.001 | 0.122 | 0.122 | 0.946 | $\beta_{C,1}$ | -0.001 | 0.122 | 0.122 | 0.946 |
| $\alpha_C$ | 0.836 | 0.099 | 0.842 | 0.000 | $\alpha_C$ | 0.160 | 0.238 | 0.287 | 0.701 | $\alpha_C$ | 0.002 | 0.078 | 0.078 | 0.946 | $\alpha_C$ | 0.000 | 0.104 | 0.104 | 0.949 |
| $\sigma_T$ | 1.353 | 0.132 | 1.359 | 0.000 | $\sigma_T$ | 0.160 | 0.238 | 0.287 | 0.701 | $\sigma_T$ | -0.001 | 0.122 | 0.122 | 0.946 | $\sigma_T$ | -0.019 | 0.079 | 0.081 | 0.939 |
| $\sigma_C$ | 0.511 | 0.101 | 0.521 | 0.000 | $\sigma_C$ | 0.160 | 0.238 | 0.287 | 0.701 | $\sigma_C$ | -0.010 | 0.063 | 0.064 | 0.942 | $\sigma_C$ | -0.019 | 0.079 | 0.081 | 0.938 |
| $\rho$ | 0.103 | 0.074 | 0.127 | 0.818 | $\rho$ | 0.103 | 0.074 | 0.127 | 0.818 | $\rho$ | 0.103 | 0.074 | 0.127 | 0.818 | $\rho$ | 0.103 | 0.074 | 0.127 | 0.818 |
Table 5: Estimation results for design 2 with 46% censoring and 2500 simulations. Given are the bias, the empirical standard deviation (ESD), the root mean squared error (RMSE) and the confidence rate (CR).

| n   | naive estimator | independent estimator | oracle estimator | two-step estimator |
|-----|-----------------|-----------------------|------------------|--------------------|
|     | Bias ESD RMSE CR | Bias ESD RMSE CR      | Bias ESD RMSE CR | Bias ESD RMSE CR   |
| n = 250 |                     |                       |                   |                    |
| β_{T,0} | 3.050 0.659 3.120 0.011 | 3.077 0.442 3.109 0.001 | 3.085 0.308 3.100 0.000 | -0.064 0.619 0.622 0.957 |
| β_{T,1} | 0.360 0.247 0.436 0.629 | 0.362 0.177 0.403 0.389 | 0.363 0.126 0.384 0.121 | -0.009 0.205 0.205 0.944 |
| α_T   | -4.379 0.567 4.415 0.002 | -4.416 0.353 4.430 0.001 | -4.424 0.240 4.431 0.000 | 0.071 0.804 0.807 0.914 |
| β_{C,0} | 1.965 0.322 1.991 0.006 | 1.979 0.172 1.987 0.000 | 1.984 0.113 1.987 0.000 | 0.001 0.181 0.181 0.947 |
| β_{C,1} | 0.303 0.255 0.396 0.732 | 0.304 0.187 0.357 0.572 | 0.301 0.135 0.330 0.337 | 0.001 0.252 0.252 0.945 |
| α_C   | -2.853 0.446 2.887 0.003 | -2.877 0.303 2.893 0.001 | -2.874 0.215 2.882 0.000 | 0.005 0.071 0.071 0.946 |
| σ_T   | 0.482 0.110 0.494 0.002 | 0.474 0.069 0.479 0.000 | 0.473 0.047 0.476 0.000 | -0.011 0.146 0.146 0.972 |
| σ_C   | 0.182 0.109 0.212 0.480 | 0.178 0.066 0.190 0.144 | 0.180 0.047 0.186 0.006 | 0.004 0.071 0.071 0.972 |
| ρ     | 0.031 0.229 0.231 0.964 | 0.056 0.160 0.169 0.976 | 0.067 0.109 0.128 0.985 |                    |

| n = 500 |                     |                       |                   |                    |
| β_{T,0} |                       |                       |                   |                    |
| β_{T,1} |                       |                       |                   |                    |
| α_T   |                       |                       |                   |                    |
| β_{C,0} |                       |                       |                   |                    |
| β_{C,1} |                       |                       |                   |                    |
| α_C   |                       |                       |                   |                    |
| σ_T   |                       |                       |                   |                    |
| σ_C   |                       |                       |                   |                    |
| ρ     |                       |                       |                   |                    |

| n = 1000 |                     |                       |                   |                    |
| β_{T,0} |                       |                       |                   |                    |
| β_{T,1} |                       |                       |                   |                    |
| α_T   |                       |                       |                   |                    |
| β_{C,0} |                       |                       |                   |                    |
| β_{C,1} |                       |                       |                   |                    |
| α_C   |                       |                       |                   |                    |
| σ_T   |                       |                       |                   |                    |
| σ_C   |                       |                       |                   |                    |
| ρ     |                       |                       |                   |                    |
Table 6: Estimation results for design 3 with 46% censoring and 2500 simulations. Given are the bias, the empirical standard deviation (ESD), the root mean squared error (RMSE) and the confidence rate (CR).

|        | Bias    | ESD    | RMSE   | CR       | Bias    | ESD    | RMSE   | CR       | Bias    | ESD    | RMSE   | CR       |
|--------|---------|--------|--------|----------|---------|--------|--------|----------|---------|--------|--------|----------|
|        | n = 250 |        |        |          | n = 500 |        |        |          | n = 1000|        |        |          |
|        | naive estimator |        |        |          | oracle estimator |        |        |          | two-step estimator |        |        |          |
| **β_{T,0}** | -0.230  | 0.239  | 0.332  | 0.817    | -0.232  | 0.169  | 0.287  | 0.700    | -0.232  | 0.121  | 0.261  | 0.506    |
| **β_{T,1}** | -0.898  | 0.167  | 0.913  | 0.002    | -0.904  | 0.115  | 0.912  | 0.000    | -0.901  | 0.081  | 0.905  | 0.000    |
| **α_T**    | 1.497   | 0.102  | 1.500  | 0.000    | 1.498   | 0.072  | 1.499  | 0.000    | 1.499   | 0.050  | 1.500  | 0.000    |
| **β_{C,0}**| -0.149  | 0.276  | 0.314  | 0.910    | -0.150  | 0.194  | 0.245  | 0.878    | -0.148  | 0.140  | 0.204  | 0.809    |
| **β_{C,1}**| -0.541  | 0.142  | 0.559  | 0.026    | -0.540  | 0.098  | 0.549  | 0.000    | -0.540  | 0.070  | 0.545  | 0.000    |
| **α_C**    | 0.907   | 0.106  | 0.913  | 0.000    | 0.907   | 0.072  | 0.910  | 0.000    | 0.906   | 0.052  | 0.908  | 0.000    |
| **σ_T**    | 1.167   | 0.107  | 1.172  | 0.000    | 1.172   | 0.076  | 1.174  | 0.000    | 1.177   | 0.053  | 1.178  | 0.000    |
| **σ_C**    | 0.424   | 0.097  | 0.435  | 0.000    | 0.432   | 0.068  | 0.437  | 0.000    | 0.436   | 0.049  | 0.438  | 0.000    |
| **ρ**      | 0.091   | 0.078  | 0.120  | 0.858    | 0.095   | 0.053  | 0.109  | 0.701    | 0.095   | 0.038  | 0.102  | 0.478    |
|        | independent estimator |        |        |          |        |        |        |          | oracle estimator |        |        |          |
| **β_{T,0}** | 0.436   | 0.307  | 0.533  | 0.904    | 0.431   | 0.216  | 0.482  | 0.752    | 0.431   | 0.149  | 0.456  | 0.420    |
| **β_{T,1}** | 0.132   | 0.328  | 0.353  | 1.000    | 0.112   | 0.224  | 0.250  | 1.000    | 0.114   | 0.159  | 0.195  | 1.000    |
| **α_T**    | 0.018   | 0.260  | 0.260  | 0.492    | 0.037   | 0.178  | 0.182  | 0.467    | 0.043   | 0.123  | 0.130  | 0.446    |
| **λ_T**    | 0.159   | 0.264  | 0.308  | 0.604    | 0.139   | 0.183  | 0.230  | 0.518    | 0.133   | 0.126  | 0.183  | 0.419    |
| **β_{C,0}**| 0.568   | 0.207  | 0.605  | 0.596    | 0.563   | 0.147  | 0.582  | 0.240    | 0.560   | 0.104  | 0.569  | 0.014    |
| **β_{C,1}**| -0.127  | 0.217  | 0.251  | 0.853    | -0.132  | 0.149  | 0.199  | 0.778    | -0.133  | 0.106  | 0.170  | 0.627    |
| **α_C**    | -0.072  | 0.167  | 0.182  | 0.849    | -0.065  | 0.113  | 0.130  | 0.812    | -0.060  | 0.079  | 0.099  | 0.772    |
| **λ_C**    | -0.143  | 0.183  | 0.233  | 0.833    | -0.147  | 0.121  | 0.191  | 0.735    | -0.151  | 0.087  | 0.174  | 0.533    |
| **σ_T**    | 0.051   | 0.068  | 0.071  | 0.970    | 0.024   | 0.048  | 0.054  | 0.953    | 0.029   | 0.035  | 0.045  | 0.910    |
| **σ_C**    | 0.081   | 0.097  | 0.127  | 0.934    | 0.092   | 0.069  | 0.115  | 0.801    | 0.098   | 0.049  | 0.110  | 0.540    |
| **ρ**      | 0.002   | 0.098  | 0.098  | 0.937    | -0.001  | 0.067  | 0.067  | 0.942    | -0.001  | 0.046  | 0.046  | 0.950    |

Bias, ESD, RMSE and CR refer to the bias, empirical standard deviation, root mean squared error and confidence rate, respectively.