BOUNDARY FEEDBACK AS A SINGULAR LIMIT OF DAMPED HYPERBOLIC PROBLEMS WITH TERMS CONCENTRATING AT THE BOUNDARY

ÁNGELA JIMÉNEZ-CASAS
Grupo de Dinamica No Lineal
Universidad Pontificia Comillas de Madrid
C/Alberto Aguilera 23, 28015 Madrid, Spain

ANÍBAL RODRÍGUEZ-BERNAL∗
Departamento de Análisis Matemático y Matemática Aplicada
Universidad Complutense de Madrid, 28040, Madrid
and
ICMAT§, Instituto de Ciencias Matemáticas
CSIC-UAM-UC3M-UCM, Spain

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ABSTRACT. In this paper we show how solutions of a wave equation with distributed damping near the boundary converge to solutions of a wave equation with boundary feedback damping. Sufficient conditions are given for the convergence of solutions to occur in the natural energy space.

1. Introduction. In this paper we consider some singular perturbation of a forced wave equation with localized damping. The singular perturbation consists in the damping region to be concentrated in a neighborhood of the boundary. This neighborhood is of width ε and shrinks to the boundary as ε → 0. Also we consider that some part of the forcing is also concentrated in that region. Our goal is then to describe the limit equation and to analyze the convergence of solutions.

To be more precise, we consider the following family of hyperbolic problems for λ > 0 and T > 0 fixed,

\[
\begin{cases}
\varepsilon u_{\varepsilon}^{\prime\prime} + \frac{1}{\varepsilon} \lambda \chi_{\varepsilon'} u_{\varepsilon}^{\prime} - \Delta u_{\varepsilon} + \lambda u_{\varepsilon} = f_{\varepsilon} + \frac{1}{\varepsilon} \lambda \chi_{\varepsilon'} g_{\varepsilon} & \text{in } \Omega \times (0,T) \\
\frac{\partial u_{\varepsilon}^{\prime}}{\partial n} = 0 & \text{on } \Gamma \times (0,T) \\
u_{\varepsilon}(0,x) = u_{\varepsilon}^{0}(x), u_{\varepsilon}^{\prime}(0,x) = v_{\varepsilon}^{0}(x) & \text{in } \Omega
\end{cases}
\]

(1)

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∗ Corresponding author: A. Rodríguez–Bernal.

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where $\Omega$ is an open bounded smooth set in $\mathbb{R}^N$ with a $C^2$ boundary $\Gamma = \partial \Omega$ and $\chi_{\omega_{\varepsilon}}$ is the characteristic function of the set $\omega_{\varepsilon}$ defined as

$$\omega_{\varepsilon} = \{ x - \sigma \vec{n}(x), \ x \in \Gamma, \ \sigma \in [0, \varepsilon) \} \subset \Omega$$

for sufficiently small $\varepsilon$, say $0 \leq \varepsilon \leq \varepsilon_0$, where $\vec{n}(x)$ denotes the outward normal vector at a point $x \in \Gamma$. Hence, the set $\omega_{\varepsilon}$ is a neighborhood of $\Gamma$ in $\Omega$, that collapses to the boundary when the parameter $\varepsilon$ goes to zero. The given functions $f_{\varepsilon}$ and $g_{\varepsilon}$ are defined, respectively, in $\Omega \times (0, T)$ and $\omega_{\varepsilon} \times (0, T)$.

As $\omega_{\varepsilon}$ shrinks to the boundary as $\varepsilon \to 0$, the goal in this work is to show that negative feedback boundary conditions can be obtained as a result of this limiting process. More precisely, the main result in this work is to prove that the family of solutions, $u_{\varepsilon}$, converges in some sense, when the parameter $\varepsilon$ goes to zero, to a limit function $u_0$, which is given by the solution of the following hyperbolic problem with boundary feedback damping term

$$\begin{cases}
    u_{0tt} - \Delta u_0 + \lambda u_0 &= f \quad \text{in } \Omega \times (0, T) \\
    u_{0t} + \frac{\partial u_0}{\partial n} &= g \quad \text{on } \Gamma \times (0, T) \\
    u_{0}(0, x) = u_{00}(x), \ u_{0t}(0, x) = u_{00}(x) \quad \text{in } \Omega
\end{cases}$$

(2)

where $u_{00}, v_0, f$ are obtained as the weak limits of initial data $u_{0\varepsilon}, v_{0\varepsilon}$ and $f_{\varepsilon}$, while $g$ is obtained as the limit of the concentrating terms

$$\frac{1}{\varepsilon} \chi_{\omega_{\varepsilon}} g_{\varepsilon} \to g|_{\Gamma} \quad \text{(3)}$$

in some sense that we make precise below. In particular, we will obtain that the time derivative of the solution concentrates to the time derivative of the restriction to the boundary, as $\varepsilon \to 0$. Notice that all concentrating terms in (1) are transferred, in the limit, to the boundary condition in (2).

Different problems with concentrating terms near the boundary have been considered before. For example linear elliptic problems have been studied in [6]. Nonlinear elliptic problems, some including oscillations in the boundary, have been considered in [7], [5], [2]; see also [19]. Linear parabolic problems can be found in [27] while nonlinear ones where considered in [14], [26], [3]. Other type of problems have also been studied. For example, delay nonlinear parabolic problems can be found in
[4], while parabolic dynamic boundary conditions can be found in [15]. Also, asymptotic behavior of non-autonomous damped wave equations have been studied in [1].

On the other hand, note that problems of the form (1) appear naturally in control theory/stabilization of waves, see [9, 20, 30, 22, 23], while problems of the form (2) appear in the boundary control theory, see [17, 18, 16, 31, 24, 11] and references therein.

Finally, similar eigenvalue problems to the ones associated to (1) appear in homogenization of vibration problems with inclusions near the boundary, see [25, 12, 13, 10] and references therein.

In all these examples a common feature is that concentrating terms near the boundary give rise, in the limit, to a boundary term. The form of the boundary term depends on the problem under consideration. Also, this influences the way the solutions of the approximate problem converge to those of the limit one. Notice that one source of difficulties is that in (3) one term is defined in $\omega_\varepsilon \subset \Omega$ while the other is defined on $\Gamma = \partial \Omega$ so that convergence has to be seen in a dual space of regular test functions.

In the present paper, we build up a suitable functional setting for both problems (1) and (2) in Section 2 and we construct mild and strict solutions for both problems. Mild solutions are constructed in such a way that they satisfy the natural energy estimates that control the norm of the solution in the natural energy space $E = H^1(\Omega) \times L^2(\Omega)$ and shows the energy dissipative effect of the damping term of both (1) and (2), given by the term $\frac{1}{\varepsilon} A_{\omega} u_\varepsilon^\prime$ and $u_0^\prime$ on the boundary respectively, see Theorem 2.1, Proposition 2 and Theorem 2.2. Strict solutions require stronger regularity of the data and have additional regularity. Then in Section 3 we rely on energy estimates and classical compactness arguments, e.g. [21], to obtain mild solutions of (2) as weak limits of the mild solutions of (1), see Theorem 3.1. Then under stronger regularity assumptions on the data we prove in Theorem 3.4 and Proposition 5 that strict solutions of (1) actually converge in the energy space $L^2((0,T), E)$ and also that

$$\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u_\varepsilon^\prime|^2 \rightarrow \int_0^T \int_{\Gamma} |\gamma(u_0^\prime)|^2.$$ 

Some required results on the convergence of concentrating terms as in (3) are collected in Appendix A.

2. On the well–posedness of the approximating and limit problems. In this section we prove the well–posedness results for both the approximating and limit problems (1) and (2), respectively. For this we will make use of minor variations of the results in [28, 29].

Here and below $H^s(\Omega)$ denote, for $s \geq 0$, the standard Sobolev spaces and for $s > 0$ we denote their duals as

$$H^{-s}(\Omega) = (H^s(\Omega))^\prime.$$ 

Also, $H^{-1/2}(\Gamma)$ will denote the dual space of $H^{1/2}(\Gamma)$.

Finally, we will consider below traces on $\Gamma$ of functions defined in $\Omega$. Hence, we will denote by $\gamma(u)$ the trace of a function $u$ and denote by $\gamma$ the trace operator on $H^s(\Omega) \rightarrow H^{s-\frac{1}{2}}(\Gamma)$ for $s > \frac{1}{2}$, and we use the embeddings

$$H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma) \subset H^{-\frac{1}{2}}(\Gamma) \subset H^{-1}(\Omega)$$
in the sense that for every \( f \in H^1(\Omega) \), \( \gamma(f) \in H^{-1/2}(\Gamma) \subset H^{-1}(\Omega) \) is defined as
\[
\langle \gamma(f), \phi \rangle_{-1,1} := \int_{\Gamma} \gamma(f) \gamma(\phi), \quad \phi \in H^1(\Omega).
\]

2.1. Well–posedness of approximating damped hyperbolic problem. Here we consider (1) for \( 0 < \varepsilon \leq \varepsilon_0 \) which we write as
\[
\begin{align*}
\frac{du}{dt} + A_{\varepsilon} u &= h_{\varepsilon} \quad \text{in } \Omega \times (0,T) \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \Gamma \times (0,T)
\end{align*}
\]
with \( h_{\varepsilon}(t, x) = f_{\varepsilon}(t, x) + \frac{1}{\varepsilon} \mathcal{X}_\omega g_{\varepsilon}(t, x) \).

This, in turn, can be written as
\[ U_t + A_{\varepsilon} U = H_{\varepsilon}(t), \]
with \( U = (u, u_t)^T, H_{\varepsilon}(t) = (0, h_{\varepsilon}(t))^T \), and the operator \( U(0) = U_0 = (u_0, v_0)^T \) and
\[ A_{\varepsilon} = \begin{pmatrix} 0 & -I \\ -\Delta + \lambda I & \frac{1}{\varepsilon} \mathcal{X}_\omega \end{pmatrix} \]
acting on \( E = H^1(\Omega) \times L^2(\Omega) \) with domain given by \( D(A_{\varepsilon}) = H^2_N(\Omega) \times H^1(\Omega) \) where
\[ H^2_N(\Omega) = \{ u \in H^2(\Omega), \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma \}. \]

Note that in [29] a very similar problem to (1) was considered but with Dirichlet boundary conditions on \( \Gamma \) instead of Neumann ones as in this paper. Then in a similar fashion as in Theorem 5.1, Theorem 5.2 in [29], we have the following result that states the well–posedness of (5). Notice that no proofs are given, as the change in the boundary conditions does not introduce significant differences in the proofs.

**Theorem 2.1.** i) **(Existence of solutions).** If \( h \in L^1((0, T), L^2(\Omega)) \) and \( u_0 = (u_0, v_0)^T \in E = H^1(\Omega) \times L^2(\Omega) \) then there exists a unique mild solution, \( U(t) = (u, v)^T \) of (5) satisfying \( U(0) = U_0 \), which is given by the variation of constants formula
\[ U(t) = U(t, U_0, h) = S_{\varepsilon}(t) U_0 + \int_0^t S_{\varepsilon}(t-s) H(s) \, ds \quad 0 < t < T \]
where \( S_{\varepsilon}(t) \) is a \( C_0 \) semigroup of contractions generated by the operator \( -A_{\varepsilon} \) in \( E \) and \( H(t) = (0, h(t))^T \). In this case, \( U \in C([0, T], E) \) and \( U(0) = U_0 \) or equivalently
\[ u \in C([0, T], H^1(\Omega)), \quad v \in C([0, T], L^2(\Omega)), \quad u(0) = u_0, \quad v(0) = v_0. \]

Moreover, the mapping \( (U_0, h) \mapsto U \) is Lipschitz between \( E \times L^1((0, T), L^2(\Omega)) \) and \( C([0, T], E) \).

ii) **(Further regularity).** If \( h \in W^{1,1}((0, T), L^2(\Omega)) \) or \( h \in C([0, T], H^1(\Omega)) \), and \( U_0 \in D(A_{\varepsilon}) \). Then the mild solution of (5) given in (6) is a strict solution, that is, \( U \in C([0, T], D(A_{\varepsilon})) \cap C^1([0, T], E) \) and satisfies (5) point-wise. Therefore \( v(t) = u_t(t) \) and \( u_t(t) \) is a solution of (4) such that
\[ u \in C([0, T], H^2_N(\Omega)), \quad u_t \in C([0, T], H^1(\Omega)), \quad u_{tt} \in C([0, T], L^2(\Omega)). \]
Moreover, in the first case for \( h \), \( U_t = (u_t, u_{tt})^\perp \) with \( u_t(0) = v_0 \) and \( u_{tt}(0) = -\frac{1}{\varepsilon} \mathcal{X}_{\omega} v_0 + \Delta u_0 - \lambda u_0 + h(0) \), is a mild solution of (5) in \( E \), with right hand side \( H_t \), that is

\[
U_t(t) = S_\omega(t)U_t(0) + \int_0^t S_\omega(t-s)H_s(s)\, ds \quad 0 < t < T.
\]

Observe that in case ii) of Theorem 2.1, \( u \) satisfies the PDE (4) in \( \Omega \) and the boundary condition in \( \Gamma \) in a point-wise sense.

Also, as in Proposition 5.3 in [29], we get the following characterization of the mild solutions of (5) in part i) of Theorem 2.1.

**Proposition 1. (Characterization of mild solutions).** Assume, as above, \( h \in L^1((0, T), L^2(\Omega)) \) and \( U_0 = (u_0, v_0)^\perp \in E = H^1(\Omega) \times L^2(\Omega) \) and consider \( U = (u, v)^\perp \) be the mild solution of (5) given by (6), with \( H = (0, h(t))^\perp \).

Then, \( U \) is characterized by:

\[
U \in C([0, T], E), \quad v = u_t
\]
as a weak derivative in \( L^2(\Omega) \) (that is, for every \( \varphi \in L^2(\Omega) \), \( \frac{d}{dt} \int_\Omega u \varphi = \int_\Omega v \varphi \) in distribution sense in \( (0, T) \) and for every \( \phi \in H^1(\Omega) \), \( \int_\Omega u_t \phi \) is absolutely continuous with

\[
\frac{d}{dt} \int_\Omega u_t \phi + \frac{1}{\varepsilon} \int_\Omega u_t \phi + \int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u \phi = \int_\Omega h \phi
\]
a.e. \( t \in (0, T) \). In particular, \( v_t = u_{tt} \) as a weak derivative in \( H^{-1}(\Omega) \) and \( u_{tt} \in L^1((0, T), H^{-1}(\Omega)) \), that is

\[
u_{tt} + \frac{1}{\varepsilon} \mathcal{X}_\omega u_t + L(u) = h \quad \text{in } H^{-1}(\Omega) \text{ a.e. } t \in (0, T)
\]
where \( L \) is the isometric isomorphism between \( H^1(\Omega) \) and its dual \( H^{-1}(\Omega) \), given by

\[
\left\langle L(u), \phi \right\rangle_{-1,1} = \int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u \phi
\]
for every \( u, \phi \in H^1(\Omega) \).

Observe that (7) or equivalently, (8) and (9), implies that the mild solution of (5) given by (6) is a weak solution of (4).

We now show that mild solutions in Theorem 2.1 also satisfy the energy equality.

**Proposition 2. (Energy estimate).** Assume, as above, \( h \in L^1((0, T), L^2(\Omega)) \), \( U_0 = (u_0, v_0)^\perp \in E = H^1(\Omega) \times L^2(\Omega) \) and let \( U = (u, v)^\perp \) be the mild solution of (5) given by (6), with \( H = (0, h(t))^\perp \).

Then \( U = (u, u_t)^\perp \) satisfies the energy equality

\[
\frac{2}{\varepsilon} \int_0^\tau \int_{\omega_t} |u_t|^2 + E_0(u(\tau), u_t(\tau)) = E_0(u_0, v_0) + 2 \int_0^\tau \int_\Omega h u_t
\]
for \( 0 < \tau < T \), where \( E_0 \) is the energy functional given by

\[
E_0(u, v) = \int_\Omega u^2 + \frac{1}{2} \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2.
\]

Proof. As usual, we argue by density. First, assume the solution is smooth enough such that \( u \in H^2_X(\Omega) \) and \( u_t \in H^1(\Omega) \). Part ii) in Theorem 2.1 gives sufficient conditions on the data for this assumption to hold true.
Then multiplying in (4) by \( u_t \) in \( L^2(\Omega) \) and integrating by parts we have

\[
\frac{1}{\varepsilon} \int \omega_s |u_t|^2 + \frac{1}{2} \frac{d}{dt} \left( \int \omega_s |u_t|^2 + \int |\nabla u|^2 + \lambda \int \Omega |u|^2 \right) = \int \Omega h u_t \tag{12}
\]

and integrating (12) in \((0, \tau)\), with \( \tau \in [0, T] \) we get (10).

Now for \( h \in L^1((0,T), L^2(\Omega))) \), \( U_0 = (u_0, v_0)^T \in E = H^1(\Omega) \times L^2(\Omega) \) consider a sequence of data such that \( U_0^n \in D(A_c) \subset E \), \( h_n \in C^2_s((0,T), L^2(\Omega)) \), such that \( U_0^n \to U_0 \) in \( E \), \( h_n \to h \) in \( L^1((0,T), L^2(\Omega)) \). From part ii) in Theorem 2.1, \( U^n = (u^n, u^n_t) \in C([0,T], D(A_c)) \cap C^1([0,T], E) \), in particular \( U^n_t = (u^n_t, u^n_{tt}) \in C([0,\tau], E) \) and, as above \( U^n \) satisfies (10) for every \( n \).

By the Lipschitz dependence of mild solutions in part i) of Theorem 2.1 we have

\[ U^n \to U = U(\cdot, U_0, h) \text{ in } C([0,\tau], E) \]

which implies in particular \( \|U^n(t)\|_E \to \|U(t)\|_E \), that is

\[ E_0(u^n, u^n_t(t)) \to E_0(u, u_t(t)) \text{ as } n \to \infty. \]

Also, \( \frac{1}{\varepsilon} \omega_s u^n_t \to \frac{1}{\varepsilon} \omega_s u_t \) in \( C([0,T], L^2(\Omega)) \) as \( n \to \infty \) and

\[
\frac{1}{\varepsilon} \int_0^\tau \int \omega_s |u^n_t|^2 \to \frac{1}{\varepsilon} \int_0^\tau \int \omega_s |u_t|^2.
\]

Finally, since \( u_0^n \to u_t \) in \( C([0,T], L^2(\Omega)) \) and \( h_n \to h \) in \( L^1((0,T), L^2(\Omega)) \) we get

\[
\int_0^\tau \int_\Omega h_n u^n_t \to \int_0^\tau \int_\Omega h u_t,
\]

and passing to the limit in (10) we obtain the energy estimate for the mild solution \( U(\cdot, U_0, h) \).

\[ \Box \]

2.2. Well–posedness of the limit problem. Now we consider the hyperbolic problem (2), that is

\[
\begin{cases}
\begin{align*}
u^n_t - \Delta u^n + \lambda u^n &= f &\text{in } \Omega \times (0,T) \\
u^n_t + \frac{\partial u^n}{\partial n} &= g &\text{on } \Gamma \times (0,T) \\
u^n(0,x) &= u_0(x) &\text{in } \Omega \\
u^n_t(0,x) &= v_0(x) &\text{in } \Omega.
\end{align*}
\end{cases}
\tag{13}
\]

Note that a very similar problem was considered [28] where the boundary \( \Gamma \) was assumed to be split into two regular subsets \( \Gamma = \Gamma_1 \cup \Gamma_0 \). Then Dirichlet boundary conditions were assumed on \( \Gamma_0 \) and dynamic boundary conditions on \( \Gamma_1 \). Therefore we adapt here the results in [28] to our setting.

We will often find below some elements in \( H^{-1}(\Omega) \) for which we will employ the notation

\[ h = f_\Omega + g_\Gamma \]

where \( f \) and \( g \) are functions defined in \( \Omega \) and on \( \Gamma \) respectively. This will denote the functional defined by

\[ \langle h, \phi \rangle_{-1,1} = \int_\Omega f \phi + \int_\Gamma g \phi \]

for all sufficiently smooth function \( \phi \) in \( \Omega \).

Thus, we define the normal derivative operator as follows: for every \( u \in Y_0 := \{ z \in H^1(\Omega), \Delta z \in L^2(\Omega) \} \)
the normal derivative \( \frac{\partial u}{\partial n} \in H^{-\frac{1}{2}}(\Gamma) \) is defined as
\[
\langle \frac{\partial u}{\partial n}, \gamma(v) \rangle_{-\frac{1}{2}, \frac{1}{2}} = \int_{\Omega} \Delta uv + \int_{\Omega} \nabla u \nabla v
\]
for every \( v \in H^1(\Omega) \). This can be recast as
\[
\langle \frac{\partial u}{\partial n}, \gamma(v) \rangle_{-\frac{1}{2}, \frac{1}{2}} = \langle L(u), v \rangle_{-1,1} + \int_{\Omega} (\Delta u - \lambda u) v, \quad u \in Y_0, \ v \in H^1(\Omega) \quad (14)
\]
with \( L \) as in (9).

We consider now \( E = H^1(\Omega) \times L^2(\Omega) \) and \( E' = L^2(\Omega) \times H^{-1}(\Omega) \) its dual space with duality pairing
\[
\langle (u, v), (\phi, \varphi) \rangle_{E \times E'} = \langle \varphi, u \rangle_{-1,1} + \int_\Omega v \phi.
\]
In what follows we will denote by \( U = (u, v) \) a generic element of \( E \), while \( U^* = (u, w) \) will denote a generic element in \( E' \).

Then if \( g = 0 \) problem (13) can be written as
\[
U_t + AU = F(t),
\]
where \( U = (u, u_t)^\dagger, \ F(t) = (0, f(t))^\dagger \) and \( U(0) = U_0 = (u_0, v_0)^\dagger \) with
\[
A = \begin{pmatrix} 0 & -I \\ -\Delta + \lambda I & 0 \end{pmatrix} \quad \text{and} \quad D(A) = \{(u, v), u \in Y_0, \ v \in H^1(\Omega), \ v + \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma\}
\]
such that \( -A \) generates a \( C_0 \) semigroup \( S(t) \) in \( E = H^1(\Omega) \times L^2(\Omega) \).

To handle the case \( g \neq 0 \), following [28], we proceed as follows. By transposition, \( -A^* \) generates in \( E' \) the \( C_0 \) semigroup \( S^*(t) \), i.e. the transposed semigroup of \( S(t) \), and is given by
\[
A^* = \begin{pmatrix} \gamma & -I \\ L & 0 \end{pmatrix}, \quad \text{with} \quad D(A^*) = \{(u, w) \in H^1(\Omega) \times H^{-1}(\Omega), \ \gamma(u) - w \in L^2(\Omega)\},
\]
see Lemma 2.1 in [28].

In this way the solution of the limit problem (13) are given by the following result which relates them with the mild solutions in \( E' \) of the dual equation
\[
U_t^* + A^* U^* = H(t), \quad \text{in } E' = L^2(\Omega) \times H^{-1}(\Omega)
\]
with \( H(t) = (0, h(t))^\dagger \) and \( h(t) := f_\Omega(t) + g_\Gamma(t) \). Observe that a strict solution \( U^* = (u, w)^\dagger \) of this equation satisfies
\[
u_t = w - \gamma(u) \quad \text{in } L^2(\Omega),
\]
\[
w_t + L(u) = f_\Omega(t) + g_\Gamma(t) \quad \text{in } H^{-1}(\Omega)
\]
which can be written as
\[
(u_t + \gamma(u))_t + L(u) = h = f_\Omega + g_\Gamma \quad \text{in } H^{-1}(\Omega),
\]
and is a weak formulation of (13) in the sense that for every \( \phi \in H^1(\Omega) \),
\[
\frac{d}{dt} \left( \int_{\Omega} u_t \phi + \int_\Gamma \gamma(u) \gamma(\phi) \right) + \int_{\Omega} \nabla u \nabla \phi + \lambda \int_{\Omega} u \phi = \int_\Omega f \phi + \int_\Gamma g \phi
\]
a.e. \( t \in (0, T) \).

Indeed, as in Theorem 2.3 in [28] we get the following. Again no proofs are given since the arguments follow line by line those in [28].
Theorem 2.2. Let, \( f \in L^1((0, T), L^2(\Omega)), g \in L^2((0, T), L^2(\Gamma)) \) and \( U_0 = (u_0, v_0) \in E = H^1(\Omega) \times L^2(\Omega) \). Let \( U^*(t) = (u, w)^+ \) be the mild solution of the dual equation (15) in \( E' = L^2(\Omega) \times H^{-1}(\Omega) \)

\[
U^*(t) = S^*(t)U^*_0 + \int_0^t S^*(t-s)H(s)\, ds \quad 0 < t < T
\]  
(16)

where \( U^*_0 = (u_0, w_0)^+, \ w_0 = v_0 + \gamma(u_0) \in H^{-1}(\Omega), \ H(t) = (0, h(t))^+ \) and \( h := f_\Omega + g_\Gamma \in L^1((0, T), H^{-1}(\Omega)) \).

Then \( U^* \in C([0, T], E') \), \( w = u_t + \gamma(u) \), and \( U(t) = (u, u_t)^+ \) satisfies

i) (Regularity). \( U \in C([0, T], E), \ \gamma(u) \in C([0, T], H^1(\Gamma)) \cap H^1((0, T), L^2(\Gamma)) \) and \( u_{tt} \in L^1((0, T), H^{-1}(\Omega)) \).

ii) (Energy estimate). \( U \) satisfies the energy equality

\[
E_0(u(\tau), u_t(\tau)) + 2 \int_0^\tau \int_\Omega \gamma(u)_t^2 = E_0(u_0, v_0) + 2 \int_0^\tau \int_\Gamma g\gamma(u)_t + 2 \int_0^\tau \int_\Omega fu_t
\]  
(17)

for \( 0 < \tau < T \), where \( E_0 \) is the energy functional given by (11), i.e.

\[
E_0(u, v) = \int_\Omega v^2 + \int_\Omega |\nabla u|^2 + \lambda \int_\Omega u^2
\]  
(18)

a.e. \([0, T]\), as an equality in \( H^{-1}(\Omega) \).

Remark 1. We observe that from (16) despite \( H(s) \) is not in \( D(A^*) \) (unless \( g = 0 \)), nor is regular in time we still have \( U^*(s) \) is in \( D(A^*) \) for \( 0 < s < T \). This is due to the particular form of \( H \) and a subtle smoothing effect of the semigroup. Also note that \( u_{tt} \in L^1((0, T), L^2(\Omega)) \) and from the energy estimates (17) we get \( \gamma(u)_t \in L^2((0, T) \times \Gamma) \) and therefore in (18) we have

\[
(u_t + \gamma(u))_t = u_{tt} + \gamma(u)_t
\]

where the derivative is to be understood as weak derivative, i.e. as

\[
\frac{d}{dt} \left< u_t + \gamma(u), \phi \right>_1 = \left< u_{tt}, \phi \right>_1 + \left< \gamma(u)_t, \phi \right>_1 = \left< u_{tt}, \phi \right>_1 + \int_\Gamma \gamma(u)_t \gamma(\phi)
\]

for every \( \phi \in H^1(\Omega) \).

Observe that Theorem 2.2 suggest that when going from \( E \) into \( E' \) we employ the following linear injective (not onto) “change of variables”, see [28] for more details,

\[
E \ni U = (u, v) \mapsto U^* = (u, w) \in E', \quad w = v + \gamma(u) \in H^{-1}(\Omega)
\]

From the above theorem we can make the following definition.

Definition 2.3. Let \( f \in L^1((0, T), L^2(\Omega)), g \in L^2((0, T), L^2(\Gamma)) \) and \( U_0 = (u_0, v_0) \in E = H^1(\Omega) \times L^2(\Omega) \).

A function \( u \in C([0, T], L^2(\Omega)) \cap L^1((0, T), L^2(\Gamma)) \) such that \( u_t \in C([0, T], H^{-1}(\Omega)) \) and \( \gamma(u) \in C([0, T], H^1(\Gamma)) \) is a mild solution of (13) if \( u_t + \gamma(u) \) has a weak derivative in \( H^{-1}(\Omega) \) and satisfies a.e. \( t \in [0, T] \)

\[
(u_t + \gamma(u))_t + L(u) = h = f_\Omega + g_\Gamma \quad \text{in} \ H^{-1}(\Omega)
\]

and \( u(0) = u_0, \ u_t(0) = v_0 \).

Now, we will show that mild solutions as in Definition 2.3 are given by Theorem 2.2.
Proposition 3. For given $f \in L^1((0, T), L^2(\Omega))$, $g \in L^2((0, T), L^2(\Gamma))$ and $U_0 = (u_0, v_0) \in E = H^1(\Omega) \times L^2(\Omega)$, a function $u$ is a mild solution of (13) as Definition 2.3 if and only if $U^*(t) = (u, u_t + \gamma(u))^\perp$ is given by (16) in Theorem 2.2.

In particular, this mild solution is unique and satisfies the energy equality (17).

Proof. First, note that if $U^*(t)$ and $U(t) = (u, u_t)^\perp$ are given as in Theorem 2.2, then $u$ is a mild solution of (13) as Definition 2.3.

Conversely if $u$ is a mild solution as in Definition 2.3, we consider $U^*(t) = (u(t), w(t))$ with $w(t) = u_t(t) + \gamma(u(t))$ and we will prove below that $U^*$ is the mild solution of the dual equation (15) and satisfies (16) with $U_0^* = (u_0, v_0) \in E'$, $u_0 = v_0 + \gamma(u_0) \in H^{-1}(\Omega)$ and $H(t) = (0, h(t))^\perp \in L^1((0, T), E')$ with $h = f_\Omega + g_\Gamma \in L^1((0, T), H^{-1}(\Omega))$. For this we will use the characterization in [8] of the variation of constants formula for mild solutions.

First, $U^*(0) = (u_0, v_0) \in E'$ with $U^* = (u, u_t + \gamma(u)) \in C([0, T], E')$ because $u \in C([0, T], L^2(\Omega))$ and $u_t, \gamma(u) \in C([0, T], H^{-1}(\Omega))$.

Second, for every $(\phi, \varphi) \in D(A) \subset E$, we have that
\[
\left\langle U^*(t), (\phi, \varphi) \right\rangle_{E' \times E} = \int_\Omega w \varphi + \left\langle u_t + \gamma(u), \phi \right\rangle_{-1,1},
\]
is absolutely continuous in $[0, T]$. This follows since $u_t \in C([0, T], H^{-1}(\Omega))$ and using (18) $u_t + \gamma(u), \varphi \in H = L^1((0, T), H^{-1}(\Omega))$.

Finally, for every $\tau \in [0, T]$ and $(\phi, \varphi) \in D(A) \subset E$, we will prove that
\[
\frac{d}{dt} \left\langle U^*(t), (\phi, \varphi) \right\rangle_{E' \times E} + \left\langle U^*(t), A(\phi, \varphi) \right\rangle_{E' \times E} = \left\langle H(\phi, \varphi) \right\rangle_{E' \times E},
\]
a.e. $t \in [0, \tau]$. In fact, $\left\langle H(\phi, \varphi) \right\rangle_{E' \times E} = \left\langle h, \phi \right\rangle_{-1,1}$ and
\[
\frac{d}{dt} \left\langle U^*(t), (\phi, \varphi) \right\rangle_{E' \times E} = \frac{d}{dt} \left( \left\langle u, \varphi \right\rangle + \left\langle u_t + \gamma(u), \phi \right\rangle_{-1,1} \right)
\]
\[
\quad = \left\langle u_t, \phi \right\rangle_{-1,1} + \left\langle (u_t + \gamma(u))_t, \phi \right\rangle_{-1,1},
\]
while since $A(\phi, \varphi) = (-\varphi, -\Delta \phi + \lambda \phi)$,
\[
\left\langle U^*(t), A(\phi, \varphi) \right\rangle_{E' \times E} = \int_\Omega u(-\Delta \phi + \lambda \phi) - \left\langle (u_t + \gamma(u)), \varphi \right\rangle_{-1,1} = \int_\Omega \left( L(\phi), u \right)_{-1,1} - \left( u_t, u \right)_{-1,1},
\]
where we have used (14), $(\phi, \varphi) \in D(A) \subset E$ and $(\gamma(u), \varphi)_{-1,1} = (\gamma(\varphi), u)_{-1,1} = \int_\Gamma u \varphi$.

Next, we note that
\[
\left\langle L(\phi), u \right\rangle_{-1,1} = \int_\Omega \nabla u \nabla \phi + \lambda \int_\Omega u \varphi = \left\langle L(u), \phi \right\rangle_{-1,1},
\]
then (19) is equivalent to
\[
\left\langle (u_t + \gamma(u))_t, \phi \right\rangle_{-1,1} + \left\langle L(u), \phi \right\rangle_{-1,1} = \left\langle h, \phi \right\rangle_{-1,1},
\]
a.e. $t \in [0, \tau]$ which holds true because $u$ is a mild solution. Hence we have (19). Now, the result in [8] gives that $U^*$ above satisfies (16).
Remark 2.

Concerning further regularity, as in Theorem 2.4 in [28] we have the following result that allows to construct strict solutions of (13).

**Proposition 4.** Under the assumptions of Theorem 2.1 above, assume moreover (20) is a weak formulation, in Under the assumptions of Theorem 2.1 above, assume moreover (20) is a weak formulation, in

$$u_0, v_0 \in H^1(\Omega) \times H^1(\Omega), \ f \in W^{1,1}((0, T), L^2(\Omega)) \text{ and } g \in H^1((0, T), L^2(\Gamma))$$

are such that

$$\gamma(v_0) + L(u_0) - g(0) \in L^2(\Omega),$$

that is,

$$\gamma(v_0) + \frac{\partial u_0}{\partial n} = g(0) \text{ in } H^{-1/2}(\Gamma). \quad (20)$$

Let $U = (u, u_t)^\perp$ be constructed as in Theorem 2.2. Then $U = (u, u_t)^\perp$ satisfies $U \in C^1([0, T], E)$ and $\gamma(u_t) \in C([0, T], L^2(\Gamma))$ Moreover, $u_t$ is a mild solution of (13) in $E$, with right hand sides $f_t$ and $g_t$ in $\Omega$ and $\Gamma$, respectively.

In particular,

i) (Further regularity). $U_t = (u_t, u_{tt})^\perp \in C([0, T], E)$, $\gamma(u_t) \in C([0, T], H^{3/2}(\Gamma)) \cap H^1((0, T), L^2(\Gamma))$.

ii) (Energy estimate for $U_t$). $U_t$ satisfies the energy equality

$$E_0(u_0(\tau), u_{tt}(\tau)) + 2 \int_0^\tau \int_\Gamma \gamma(u_t)^2 = E_0(v_0, u_{tt}(0)) + 2 \int_0^\tau \int_\Gamma g_t \gamma(u_t) + 2 \int_0^\tau \int_\Omega f_t u_t$$

for $0 < \tau < T$ where $E_0$ is the energy functional given by (11) and $u_{tt}(0) = -\gamma(v_0) - L(u_0) + f(0) + g(0) \in L^2(\Omega)$.

iii) (The equation for $U_t$). Moreover, $u \in C([0, T], Y_0)$, where $Y_0 = \{z \in H^1(\Omega), \Delta z \in L^2(\Omega)\}$, and a.e. $t \in [0, T]$

$$u_{tt} + \gamma(u_t) + L(u) = h(t) = f_t(t) + g_t(t) \text{ in } H^{-1}(\Omega),$$

$$\frac{\partial u}{\partial n} = g - \gamma(u_t) \in L^2(\Gamma) \subset H^{-1/2}(\Gamma),$$

$$(u_{tt} + \gamma(u_t))_t + L(u_t) = h_t(t) \text{ in } H^{-1}(\Omega).$$

**Remark 2.** i) Note that (20) is a weak formulation, in $H^{-1}(\Omega)$, of the condition

$$\left\{ \begin{array}{ll}
-\Delta u_0 + \lambda u_0 & \in L^2(\Omega) \\
v_0 + \frac{\partial u_0}{\partial n} = g(0) & \text{ on } \Gamma
\end{array} \right.$$ 

ii) Under the hypotheses of Proposition 4 we have $u(t) \in Y_0$ and $u_{tt}(t) \in L^2(\Omega)$ then $u$ satisfies (13) in the sense that

$$\left\{ \begin{array}{ll}
u_{tt} - \Delta u + \lambda u & = f \text{ in } \Omega \times (0, T) \\
\gamma(u_t) + \frac{\partial u}{\partial n} & = g \text{ on } \Gamma \times (0, T).
\end{array} \right.$$ 

3. **Passing to the limit** as $\varepsilon \to 0$. We analyze here the limit of the solutions of the hyperbolic problems (1), with $0 < \varepsilon \leq \varepsilon_0$. 

3.1. Convergence of mild solutions. For this we will assume that the data of the problem satisfy the following uniform bounds in $0 < \varepsilon < \varepsilon_0$:
\[
\|u_0^\varepsilon\|_{H^1(\Omega)} \leq C, \quad \|v_0^\varepsilon\|_{L^2(\Omega)} \leq C
\]  
(21)
\[
\int_0^T \|f_\varepsilon\|_{L^2(\Omega)} \leq C, \quad \int_0^T \|f_\varepsilon\|_{H^{-1}(\Omega)}^p \leq C \text{ for some } 1 < p < 2,
\]  
(22)
and
\[
\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \leq C
\]  
(23)
for some constant $C$ independent of $\varepsilon$.

Then, by compactness and the results in Appendix A, we can assume, by taking subsequences if necessary, the following convergences as $\varepsilon \to 0$:

(i) $u_0^\varepsilon \rightharpoonup u_0^0$ weakly in $H^1(\Omega)$, $v_0^\varepsilon \rightharpoonup v_0^0$ weakly in $L^2(\Omega)$ and strongly in $L^2(\Omega)$ and $H^{-1}(\Omega)$ respectively,

\[
\frac{1}{\varepsilon} \mathcal{K}_\varepsilon u_0^\varepsilon \rightarrow \gamma(u_0^0) \text{ weakly in } H^{-1}(\Omega)
\]  
(24)
and
\[
\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u_0^\varepsilon|^2 \rightarrow \int_{\Gamma} |\gamma(u_0^0)|^2,
\]  
see Lemma A.1 iii).

(ii) $f \in L^1((0, T), L^2(\Omega)) \cap L^p((0, T), H^{-1}(\Omega))$ and $f_\varepsilon \rightharpoonup f$ weakly in $L^1((0, T), L^2(\Omega))$, $f_\varepsilon \to f$ weakly in $L^p((0, T), H^{-1}(\Omega))$.

(iii) $g \in L^2((0, T), L^2(\Gamma))$ and

\[
\frac{1}{\varepsilon} \mathcal{K}_\varepsilon g_\varepsilon \rightarrow g \text{ weakly in } L^2((0, T), H^{-s}(\Omega)) \text{ with } s > \frac{1}{2},
\]  
(26)
see Lemma A.2 ii).

With this assumptions consider the mild solutions, $u^\varepsilon(t)$ and $u^0(t)$ of (1) and (2), respectively, constructed in Sections 2.1 and 2.2. That is, let

\[
U^\varepsilon(t) = (u^\varepsilon(t), v^\varepsilon(t))^\perp = S_\varepsilon(t)U_0^\varepsilon + \int_0^t S_\varepsilon(t-s)H^\varepsilon(s)ds, \quad 0 < t < T,
\]  
(27)
with $U_0^\varepsilon = (u_0^\varepsilon, v_0^\varepsilon)^\perp$, $H^\varepsilon(t) = (0, h^\varepsilon(t))^\perp$, $h_\varepsilon = f_\varepsilon + \frac{1}{2} \mathcal{K}_\varepsilon g_\varepsilon \in L^1((0, T), L^2(\Omega))$, as in Theorem 2.1 and Proposition 1, and $U(t) = (u^0(t), u^0(t))^\perp$ where $U^\varepsilon(t) = (u^0(t), u^0(t))^\perp$ with $w^0 = u_0^0 + \gamma(u^0)$ is given by

\[
U^\varepsilon(t) = S^\varepsilon(t)U_0^\varepsilon + \int_0^t S^\varepsilon(t-s)H(s)ds, \quad 0 < t < T,
\]  
(28)
where $U_0^\varepsilon = (u_0^0, u_0^0)^\perp$, $w_0^0 = u_0^0 + \gamma(u_0^0) \in H^{-1}(\Omega)$, $H(t) = (0, h(t))^\perp$ and $h := f_\Omega + g_\Gamma \in L^1((0, T), H^{-1}(\Omega))$ as in Theorem 2.2.

Then we have the following result concerning convergence of mild solutions of (1) to mild solutions of (2).

**Theorem 3.1.** With the notations above, as $\varepsilon \to 0$, we have

\[
u^\varepsilon \rightharpoonup u^0 \text{ weakly in } L^\infty((0, T), H^1(\Omega)) \text{ and strongly in } C([0, T], L^2(\Omega)),
\]  
(29)
\[
u_\varepsilon^t \rightharpoonup u_\varepsilon^t \text{ weakly in } L^\infty((0, T), L^2(\Omega)) \text{ and strongly in } C([0, T], H^{-1}(\Omega)),
\]  
(30)
\[
u_\varepsilon^t \rightarrow u_\varepsilon^t \text{ weakly in } L^p((0, T), H^{-1}(\Omega)),
\]  
(31)
with $1 < p < 2$ as in (22) and
\[
\frac{1}{\varepsilon} K_{\omega}, u^\varepsilon \to \gamma(u^0) \quad \text{in} \quad H^1((0,T), H^{-1}(\Omega)).
\] (32)

Additionally
\[
\frac{1}{\varepsilon} \int_0^T \int_{\omega_s} |u^\varepsilon|^2 \to \int_0^T \int_{\Gamma} |\gamma(u^0)|^2.
\] (33)

In particular,
\[
\int_0^T E_0(u^0, u^\varepsilon) \leq \liminf_{\varepsilon \to 0} \int_0^T E_0(u^\varepsilon, u^\varepsilon) \quad \text{(34)}
\]
with $E_0$ the energy functional defined by (11).

**Proof.** We proceed in several steps. Below we will use $K$ a generic constant that does not depend on $\varepsilon$.

**Step 1.** We will prove that for any $\tau \in [0, T]$
\[
\frac{1}{\varepsilon} \int_0^\tau \int_{\omega_s} |u^\varepsilon(t)|^2 + \|u^\varepsilon(t)\|_{L^2(\Omega)} + \|u^\varepsilon(t)\|_{H^1(\Omega)}^2 \leq K
\] (35)
with $K > 0$ independent of $\varepsilon$.

For this note from Proposition 2, with $\tau \in [0, T]$, the solutions $(u^\varepsilon(t), u^\varepsilon(t))$ satisfy the energy equality (10), i.e.
\[
\frac{2}{\varepsilon} \int_0^T \int_{\omega_s} |u^\varepsilon|^2 + E_0(u^\varepsilon(\tau), u^\varepsilon(\tau)) = E_0(u^0_0, v^0_0) + 2 \int_0^\tau \int_{\Omega} f\cdot u^\varepsilon(\tau) + 2 \int_0^\tau \int_{\omega_s} g\cdot u^\varepsilon(\tau).
\] (36)

Now we obtain some upper bounds to the terms in the right hand side of (36).

First, using the hypotheses on the initial data (21) we get
\[
\|u^0_0, v^0_0\|_{E}^2 = E_0(u^0_0, v^0_0) \leq K.
\]

Next, applying Young’s inequality, we obtain
\[
\frac{2}{\varepsilon} \int_0^T \int_{\omega_s} g\cdot u^\varepsilon \leq \frac{2}{\varepsilon} \int_0^T \int_{\omega_s} |g\cdot|^2 \leq \frac{1}{\varepsilon} \int_0^T \int_{\omega_s} |u^\varepsilon|^2 + \frac{1}{\varepsilon} \int_0^T \int_{\omega_s} |g|^2.
\]

Thus, taking into account (23) we obtain
\[
\frac{2}{\varepsilon} \int_0^T \int_{\omega_s} g\cdot u^\varepsilon \leq \frac{1}{\varepsilon} \int_0^T \int_{\omega_s} |u^\varepsilon|^2 + K.
\]

Now, for every $\tau \in [0, T]$ we have that, using (22),
\[
2 \int_0^\tau \int_{\Omega} f\cdot u^\varepsilon \leq 2 \sup_{0 \leq t \leq \tau} \|f\|_{L^2(\Omega)} \|u^\varepsilon(t)\|_{L^2(\Omega)} \leq 2 \|f\|_{L^1((0,T), L^2(\Omega))} y(\tau) \leq K y(\tau)
\]
where $y(\tau) = \sup_{0 \leq t \leq \tau} \|u^\varepsilon(t)\|_{L^2(\Omega)}$.

Then, from the energy equality (36) we get
\[
\frac{1}{\varepsilon} \int_0^\tau \int_{\omega_s} |u^\varepsilon|^2 + \|u^\varepsilon(t)\|_{L^2(\Omega)} + \|u^\varepsilon(t)\|_{H^1(\Omega)}^2 \leq K(y(\tau) + 1).
\]

In particular for every $0 \leq \tau \leq T$ we have that
\[
\|u^\varepsilon(t)\|^2_{L^2(\Omega)} \leq K(\sup_{0 \leq s \leq t} \|u^\varepsilon(s)\|_{L^2(\Omega)} + 1) \leq K(y(\tau) + 1)
\]
and then $y^2(\tau) \leq K(y(\tau) + 1)$ from where $y^2(\tau) \leq K$, which gives (35).

**Step 2.** As a consequence we obtain the following uniform estimates

\[ \sup_{0 \leq \tau \leq T} \|u^\varepsilon(t)\|_{H^1(\Omega)} = \|u^\varepsilon\|_{L^\infty((0,T),H^1(\Omega))} \leq K, \]  
\[ \sup_{0 \leq \tau \leq T} \|u_i^\varepsilon(t)\|_{L^2(\Omega)} = \|u_i^\varepsilon\|_{L^\infty((0,T),L^2(\Omega))} \leq K, \]  
\[ \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u_i^\varepsilon|^2 \leq K, \]  
\[ \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u_i^\varepsilon \|_{L^2((0,T),H^{-1}(\Omega))} \leq K, \]  
\[ \|u_i^\varepsilon\|_{L^p((0,T),H^{-1}(\Omega))} \leq K \]  

with $1 < p < 2$ as in (22).

First, note that we get (37), (38) and (39) straight from (35). To prove (40) observe that for every $\phi \in H^1(\Omega)$, from Lemma A.1, there exists $C > 0$ independent of $\varepsilon$, such that we have $\frac{1}{\varepsilon} \int_{\omega_\varepsilon} |\phi|^2 \leq C \|\phi\|^2_{H^1(\Omega)}$. Then,

\[ \left| \left\langle \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u_i^\varepsilon, \phi \right\rangle_{-1,1} \right| = \left| \frac{1}{\varepsilon} \int_{\omega_\varepsilon} u_i^\varepsilon \phi \right| \leq \left( \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u_i^\varepsilon|^2 \right)^{\frac{1}{2}} \left( \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |\phi|^2 \right)^{\frac{1}{2}} \leq C \left( \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u_i^\varepsilon|^2 \right)^{\frac{1}{2}} \|\phi\|_{H^1(\Omega)} \]

and then

\[ \|\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u_i^\varepsilon\|_{H^{-1}(\Omega)} \leq C \left( \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |u_i^\varepsilon|^2 \right)^{\frac{1}{2}} \leq K \]

and then (40) follows.

Finally, we prove (41). In effect, since $u^\varepsilon$ satisfies a.e. $t \in [0,T]$

\[ u_i^\varepsilon = h_\varepsilon - L(u^\varepsilon) - \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u_i^\varepsilon, \quad \text{in } H^{-1}(\Omega) \]

with $h_\varepsilon = f_\varepsilon + \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon$, we get

\[ \|u_i^\varepsilon\|_{H^{-1}(\Omega)} \leq \|f_\varepsilon\|_{H^{-1}(\Omega)} + \|\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon\|_{H^{-1}(\Omega)} + \|L(u^\varepsilon)\|_{H^{-1}(\Omega)} + \|\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u_i^\varepsilon\|_{H^{-1}(\Omega)}, \]

and from (22) and (23), (37) and (40), we get (41).

In the remaining steps, we will pass to the limit as $\varepsilon \to 0$.

**Step 3.** Here we will study the convergence of $u^\varepsilon$ to $u^0$.

i) From (37), (39) and Lemma A.3, there exists a subsequence (that we still denote the same) and a function $u^0 \in L^\infty((0,T),H^1(\Omega))$ with $\gamma(u^0) \in H^1((0,T),L^2(\Gamma))$ such that as $\varepsilon \to 0$

\[ u^\varepsilon \rightharpoonup u^0 \text{ \ w-* in } L^\infty((0,T),H^1(\Omega)), \]

\[ \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} u^\varepsilon \rightharpoonup \gamma(u^0) \text{ \ in } H^1((0,T),H^{-1}(\Omega)), \]

\[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 = \int_0^T \int_{\Gamma} |\gamma(u^0)|^2. \]

Thus, we get the weak* convergence in (29), (32) and (33).

ii) Now we prove

\[ u^\varepsilon \to u^0 \text{ in } C([0,T],L^2(\Omega)) \]
which ends the proof of (29). For this, note that from (38) then \( u^\varepsilon : [0, T] \to L^2(\Omega) \) is equicontinuous. Also, for every \( t \in [0, T] \) fixed, from (37) we have \( u^\varepsilon(t, \cdot) \) is uniformly bounded in \( H^{-1}(\Omega) \subset L^2(\Omega) \) with compact embedding. Hence, using Ascoli-Arzelà’s Theorem we have a new subsequence such that \( u^\varepsilon \to u^0 \) in \( C([0, T], L^2(\Omega)) \). In particular, for \( t = 0 \), \( u^\varepsilon = u^0(0, \cdot) \to u^0(0, \cdot) \) in \( L^2(\Omega) \) and then \( u^0(0) = u_0^0 \).

**Step 4.** In this part, we study the convergence of \( u^\varepsilon \) to \( u_0^0 \) and prove (30).

i) Now, we will prove

\[
\left\{ \begin{array}{l}
  u^\varepsilon(t) \to u_0^0 & \text{w-* in } L^\infty((0, T), L^2(\Omega)) \text{ as } \varepsilon \to 0. \\
end{array} \right.
\]

From the convergence of \( u^\varepsilon, u^0 \) and \( u^\varepsilon \) we get

\[
\int_0^T \int_\Omega v^\varepsilon \phi = - \int_\Omega u_0^0(0) - \int_0^T \int_\Omega v^0 \phi. 
\]

Thus, we have \( v^\varepsilon = u_0^0 \in L^\infty((0, T), L^2(\Omega)) \) and we conclude (43) and (30).

ii) In what follows we will prove

\[
u^\varepsilon \to u_0^0 \quad \text{in } C([0, T], H^{-1}(\Omega)), \quad \text{as } \varepsilon \to 0.
\]

First, from (41), we obtain that for \( t_1 \in [0, T] \)

\[
\|u^\varepsilon(t_1) - u_0^0(t_2)\|_{H^{-1}(\Omega)} \leq \int_{t_1}^{t_2} \|u^\varepsilon\|_{H^{-1}(\Omega)} 
\]

\[
\leq \left( \int_{t_1}^{t_2} \|u^\varepsilon\|_{H^{-1}(\Omega)} \right)^{\frac{1}{p}} |t_2 - t_1|^{\frac{1}{p'}} \leq C |t_2 - t_1|^{\frac{1}{p'}},
\]

and we get \( \{u^\varepsilon\}_\varepsilon \) is equicontinuous with values in \( H^{-1}(\Omega) \).

Next, we also note that for every \( t \in [0, T] \) fixed, we have \( u^\varepsilon(t, \cdot) \) is uniformly bounded in \( L^p(\Omega) \subset H^{-1}(\Omega) \) with compact embedding. Therefore from Ascoli-Arzelà’s Theorem there exists a subsequence which converge to a limit function in \( C([0, T], H^{-1}(\Omega)) \). Finally, we note that this limit function must be \( u_0^0 \) and we conclude.

In particular, for \( t = 0 \), we have \( u^\varepsilon = u_0^0(0, \cdot) \to u_0^0(0, \cdot) \). Thus \( u_0^0(0) = v_0^0 \).

**Step 5.** Now we study the convergence of \( u^\varepsilon \) to \( u_0^0 \) and prove (31).

In fact from (41) we obtain a subsequence such that

\[
\left\{ \begin{array}{l}
  u^\varepsilon(t) \to v^* & \text{weakly in } L^p((0, T), H^{-1}(\Omega)), \\
end{array} \right.
\]

with \( 1 < p < 2 \) as in (25). Analogously to (44) we have \( v^* = u_0^0 \).

**Step 6.** We will prove now \( u^0 \) is a mild solution of (13) as in Definition 2.3.

First, we note from (29), (30), (31) and (32) we have

\[
\left\{ \begin{array}{l}
  u^0 \in L^\infty((0, T), H^1(\Omega)) \cap C([0, T], L^2(\Omega)), \\
  \gamma(u^0) \in H^1([0, T], H^{-1}(\Omega)), \\
  u_0^0 \in L^\infty((0, T), \Omega) \cap C([0, T], H^{-1}(\Omega)),
end{array} \right.
\]

Continued...
If \( \varphi \) is a mild solution of (4) as in Proposition 1, then from (42) if \( \varphi(t, x) \) is a smooth function such that \( \varphi \in L^p((0, T), H^1(\Omega)) \), we have

\[
\int_0^T < u_t^0, \varphi >_{-1, 1} + \int_0^T \int_{\Omega} \frac{1}{\varepsilon} \lambda \omega u_t^0 \varphi + \int_0^T \int_{\Omega} \nabla u^0 \nabla \varphi + \lambda \int_0^T \int_{\Omega} u^0 \varphi = \int_0^T \int_{\Omega} f^0 \varphi + \int_0^T \int_{\Gamma} g^0 \varphi. \tag{45}
\]

Then passing to the limit as \( \varepsilon \to 0 \) in (45), from (31), (32) and (29), together with (25) and (26) respectively, we get

\[
\int_0^T \langle u_{t^0}, \varphi \rangle_{-1, 1} + \int_0^T \int_{\Gamma} \gamma(u^0) \varphi + \int_0^T \int_{\Omega} \nabla u^0 \nabla \varphi + \lambda \int_0^T \int_{\Omega} u^0 \varphi = \int_0^T \int_{\Omega} f^0 \varphi + \int_0^T \int_{\Gamma} g^0 \varphi. \tag{46}
\]

Now, if we consider \( \varphi(t, x) = \xi(t) \phi(x) \in L^p((0, T), H^1(\Omega)) \) with \( \phi \in H^1(\Omega) \) and \( \xi \in L^p(0, T) \) in (46), then we get

\[
\int_0^T \xi(t) \langle u_{t^0}, \phi \rangle_{-1, 1} + \int_0^T \int_{\Gamma} \gamma(u^0) \varphi + \int_0^T \int_{\Omega} \nabla u^0 \nabla \varphi + \lambda \int_0^T \int_{\Omega} u^0 \varphi = \int_0^T \int_{\Omega} f^0 \varphi + \int_0^T \int_{\Gamma} g^0 \varphi.
\]

for every \( \xi(t) \in L^p(0, T) \) and we obtain that

\[
\langle u_{t^0}, \phi \rangle_{-1, 1} + \int_0^T \int_{\Gamma} \gamma(u^0) \varphi + \int_0^T \int_{\Omega} \nabla u^0 \nabla \varphi + \lambda \int_0^T \int_{\Omega} u^0 \varphi = \int_0^T \int_{\Omega} f^0 \varphi + \int_0^T \int_{\Gamma} g^0 \varphi.
\]

a.e. \( t \in [0, T] \) and for every \( \phi \in H^1(\Omega) \).

Thus we get (18) i.e.

\[
(u_t^0 + \gamma(u^0)) t + L(u^0) = h := f_\Omega + g_\Gamma \quad \text{a.e. } t \in [0, T]
\]

and as equality in \( H^{-1}(\Omega) \). Moreover, we obtained above that \( u^0(0) = u_0^0 \) and \( u_0^0(0) = v_0^0 \). Hence \( u^0 \) is a mild solution of (13) as Definition 2.3, as claimed.

**Step 7.** Notice that we actually proved above that any sequence of the family \( u^\varepsilon(t) \) has a subsequence that converges to \( u^0(t) \), the unique solution of (13) as Definition 2.3. Hence all the family converges.

**Step 8.** Observing that

\[
\int_0^T E_0(u^0, u_t^0) = \|(u^0, u_{t^0})\|_{L^2((0, T), E)}^2,
\]

from the weak convergence of \( (u^\varepsilon, u_{t}^\varepsilon) \to (u^0, u_t^0) \) in \( L^2((0, T), E) \) we get (34) and we conclude.
3.2. Convergence of strict solutions. Now we impose stronger assumptions than (21)–(23) on the data and obtain stronger convergence of solutions than in Theorem 3.1. In particular, we obtain convergence of strict solutions.

With the notations in Theorem 3.1, we consider the initial data \( u_0^\varepsilon \in H^2_N(\Omega), v_0^\varepsilon \in H^1(\Omega) \) satisfying the following uniform bounds
\[
\|u_0^\varepsilon\|_{H^1(\Omega)} \leq C, \quad \|v_0^\varepsilon\|_{H^1(\Omega)} \leq C.
\]

We also assume that the nonhomogenous terms satisfy
\[
\|f_\varepsilon\|_{W^{1,1}((0,T),L^2(\Omega))} \leq C \quad \text{and} \quad \|f_\varepsilon\|_{W^{1,p}((0,T),H^{-1}(\Omega))} \leq C,
\]
where \( 1 < p < 2 \) and \( g_\varepsilon \in H^1((0,T),L^2(\omega_\varepsilon)) \) with
\[
\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |g_\varepsilon|^2 \leq C \quad \text{and} \quad \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |(g_\varepsilon)_t|^2 \leq C,
\]
where \( C \) is a positive constant independent of \( \varepsilon \).

We will also assume the compatibility condition of the initial data
\[
\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon - \Delta u_0^\varepsilon - \lambda u_0^\varepsilon + f_\varepsilon(0) + \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon(0) \in L^2(\Omega),
\]
and
\[
\|\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon - \Delta u_0^\varepsilon - \lambda u_0^\varepsilon + f_\varepsilon(0) + \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon(0)\|_{L^2(\Omega)} \leq C. \quad (47)
\]

Then by taking subsequences if necessary, we can assume (24) and moreover
\[
u_0^\varepsilon \to u_0^0, \quad v_0^\varepsilon \to v_0^0 \quad \text{weakly in} \quad H^1(\Omega)
\]
strongly in \( L^2(\Omega) \) and
\[
\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon \to \gamma(u_0^0), \quad \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} v_0^\varepsilon \to \gamma(v_0^0) \quad \text{weakly in} \quad H^{-1}(\Omega),
\]
see Lemma A.1 iii).

On the other hand, on \( f_\varepsilon \) and \( g_\varepsilon \) by taking subsequences if necessary, we can assume (25) and (26) and moreover the convergence given in the following result.

Lemma 3.2. i) \( f \in W^{1,1}((0,T),L^2(\Omega)) \cap W^{1,p}((0,T),H^{-1}(\Omega)) \) and
\[
f_\varepsilon \to f \quad \text{weakly in} \quad W^{1,1}((0,T),L^2(\Omega)) \cap W^{1,p}((0,T),H^{-1}(\Omega))
\]
and strongly in \( C([0,T],H^{-1}(\Omega)) \).

ii) \( g \in H^1((0,T),L^2(\Gamma)) \) and for any \( \frac{1}{2} < s < 1 \)
\[
\frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon \to g \quad \text{weakly in} \quad H^1((0,T),H^{-s}(\Omega))
\]
and strongly in \( C([0,T],H^{-s}(\Omega)) \).

Proof. i) The weak convergence is clear. For the strong convergence in \( C([0,T],H^{-1}(\Omega)) \) just note that the bound on the derivative implies \( \{f_\varepsilon\}_\varepsilon \) is equicontinuous in \( H^{-1}(\Omega) \) and pointwise bounded in \( L^2(\Omega) \). Hence, Ascoli-Arzela’s Theorem gives the result.

ii) The weak convergence follows easily as in Lemma A.2 ii). For the strong convergence in \( C([0,T],H^{-s}(\Omega)) \) for \( \frac{1}{2} < s < 1 \) we use again the Ascoli-Arzela’s Theorem. For this, observe that family \( G_\varepsilon = \frac{1}{\varepsilon} \mathcal{X}_{\omega_\varepsilon} g_\varepsilon : [0,T) \to H^{-s}(\Omega) \) is equicontinuous in \( H^{-s}(\Omega) \) since \( \|G_\varepsilon\|_{L^1((0,T),H^{-s}(\Omega))} \leq C \).

At the same time for any \( \frac{1}{2} < s_0 < s < 1 \), \( \{G_\varepsilon\}_\varepsilon \) is bounded in \( H^1((0,T),H^{-s_0}(\Omega)) \subset C([0,T],H^{-s_0}(\Omega)) \) and \( H^{-s_0}(\Omega) \subset H^{-s}(\Omega) \) with compact embedding. Hence from Ascoli-Arzela’s Theorem we conclude. \( \square \)
The following result shows that the mild solutions $u^\varepsilon(t)$ and $u^0(t)$ of (1) and (2), respectively, constructed in Sections 2.1 and 2.2, see in (27) and (28), are actually strict solutions.

**Lemma 3.3.** With the assumptions above, the function $u^\varepsilon(t)$ in Theorem 3.1 is a strict solution of (1) as in part ii) in Theorem 2.1.

Also, the function $u^0(t)$ in Theorem 3.1 is a strict solution of (2) as in Proposition 4.

**Proof.** First note that since $(u_0^\varepsilon, v_0^\varepsilon) \in D(A_\varepsilon) = H^2_{x, t}(\Omega) \times H^1(\Omega)$ and $h_\varepsilon = f_\varepsilon + \frac{1}{\varepsilon} \mathcal{X}_\omega g_\varepsilon \in W^{1,1}((0, T), L^2(\Omega))$, then $U^\varepsilon$ is as in part ii) of Theorem 2.1.

On the other hand, from (47) we have that $u_0^\varepsilon$ satisfies the elliptic problem

\[
\begin{cases}
-\Delta u_0^\varepsilon + \lambda u_0^\varepsilon = W^\varepsilon - \frac{1}{\varepsilon} \mathcal{X}_\omega v_0^\varepsilon + f_\varepsilon(0) + \frac{1}{\varepsilon} \mathcal{X}_\omega g_\varepsilon(0) & \text{in } \Omega \\
\frac{\partial u_0^\varepsilon}{\partial n} = 0 & \text{on } \Gamma
\end{cases}
\]

where $W^\varepsilon \in L^2(\Omega)$ and is uniformly bounded in $L^2(\Omega)$. From this we can assume $W^\varepsilon \to W_0 \in L^2(\Omega)$ weakly in $L^2(\Omega)$.

Also, from Lemma 3.2 i) we have $f_\varepsilon(0) \to f(0)$ weakly in $L^2(\Omega)$ and strongly in $H^{-1}(\Omega)$. On the other hand, from Lemma 3.2 ii) we get $g_\varepsilon(0) \in L^2(\Gamma)$ and $\frac{1}{\varepsilon} \mathcal{X}_\omega g_\varepsilon(0) \to g(0)$ in $H^{-s}(\Omega)$.

Therefore, using these and $\frac{1}{\varepsilon} \mathcal{X}_\omega v_0^\varepsilon \to \gamma(v_0^0)$ weakly in $H^{-1}(\Omega)$, we can apply the Corollary 3.2 pag 194 from [6] with $r = z = 2$ to obtain that $u_0^\varepsilon \to u_0^0$ in the Bessel space $H^{s,2}(\Omega)$ for any $s < \frac{3}{2}$ and $u_0^0$ is the unique solution of limit elliptic problem

\[
\begin{cases}
-\Delta z + \lambda z = W_0 + f(0) & \text{in } \Omega \\
\frac{\partial z}{\partial n} = g(0) - v_0^0 & \text{on } \Gamma
\end{cases}
\]

Thus $\gamma(v_0^0) + L(u_0^0) - g(0) = W_0 + f(0) \in L^2(\Omega)$, i.e. (20) holds, see Remark 2. Since we also have $f \in W^{1,1}((0, T), L^2(\Omega))$ and $g \in H^1((0, T), L^2(\Gamma))$, we get the result about $u_0^0(t)$.

Hence, we have the following result that improves the convergence in Theorem 3.1.

**Theorem 3.4.** With the notations above, as $\varepsilon \to 0$,

\[
u^\varepsilon \to u^0, \quad w- \text{ in } L^\infty((0, T), H^1(\Omega)),
\]

and strongly in $C([0, T], H^s(\Omega))$, $s < 1$,

\[
u^\varepsilon_t \to u^0_t \quad w- \text{ in } L^\infty((0, T), H^1(\Omega)),
\]

and strongly in $C([0, T], L^2(\Omega))$,

\[
u^\varepsilon_{tt} \to u^0_{tt} \quad w- \text{ in } L^\infty((0, T), L^2(\Omega)), \quad \text{weakly in } W^{1,p}((0, T), H^{-1}(\Omega)),
\]

with $1 < p < 2$ as in (22) and strongly in $C([0, T], H^{-1}(\Omega))$.

Also

\[
\frac{1}{\varepsilon} \mathcal{X}_\omega u^\varepsilon \to \gamma(u^0), \quad \frac{1}{\varepsilon} \mathcal{X}_\omega u^\varepsilon_t \to \gamma(u^0_t) \quad \text{in } H^1((0, T), H^{-1}(\Omega))
\]

\[
\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u^\varepsilon|^2 \to \int_0^T \int_{\omega_0} |\gamma(u^0)|^2, \quad \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u^\varepsilon_t|^2 \to \int_0^T \int_{\Gamma} |\gamma(u^0_t)|^2.
\]

In particular,

\[
\int_0^T E_0(u^0_t, u^0_{tt}) \leq \liminf_{\varepsilon \to 0} \int_0^T E_0(u^\varepsilon_t, u^\varepsilon_{tt})
\]

with $E_0$ the energy functional defined by (11).
Proof. By Lemma 3.3, \( u^\varepsilon_t \) is a mild solution

\[
v^\varepsilon_t + \frac{1}{\varepsilon} \mathcal{A}_\varepsilon v^\varepsilon + L(v^\varepsilon) = (f^\varepsilon)_t + \frac{1}{\varepsilon} \mathcal{A}_\varepsilon (g^\varepsilon)_t \quad \text{in } H^{-1}(\Omega)
\]

\[v^\varepsilon(0) = u^\varepsilon_t(0) = v^\varepsilon_0, \quad v^\varepsilon_t(0) = u^\varepsilon_t(0) = -\frac{1}{\varepsilon} \mathcal{A}_\varepsilon v^\varepsilon_0 + \Delta u^\varepsilon_0 - \lambda u^\varepsilon_0 + f^\varepsilon(0) + \frac{1}{\varepsilon} \mathcal{A}_\varepsilon g^\varepsilon(0).
\]

Also Lemma 3.3 implies \( u_0^\varepsilon \) is a mild solution of

\[
(u_t + \gamma(v)_t) + L(v) = (f_t)_\Omega + (g_t)_\Gamma \quad \text{in } H^{-1}(\Omega)
\]

\[v(0) = u^\varepsilon_t(0) = v^\varepsilon_0, \quad v_t(0) = u^\varepsilon_t(0) = -\gamma(v^\varepsilon_0) - L(u^\varepsilon_0) + f(0) + g(0) \in L^2(\Omega).
\]

Then notice that the assumptions in this section, including (47), allow us to apply Theorem 3.1 to these equations and then we get the result except the strong convergence \( u^\varepsilon \rightarrow u^0 \) in \( C([0,T],H^s(\Omega)), s < 1 \). For this, note that \{\( u^\varepsilon \)_\} is bounded in \( H^s(\Omega) \) for \( s < 1 \) and then \{\( u^\varepsilon \)_\} is equicontinuous in \( H^s(\Omega) \) and is pointwise bounded in \( H^1(\Omega) \subset H^s(\Omega) \) with compact embedding. Therefore from Ascoli-Arzela’s Theorem we get that \( u^\varepsilon \) converges to \( u^0 \) also in \( C([0,T],H^s(\Omega)) \) with \( s < 1 \).

Now we show that strong convergence of the initial data in the energy space, \( E = H^1(\Omega) \times L^2(\Omega) \) implies convergence of the solution \( (u^\varepsilon, u^\varepsilon_t) \rightarrow (u^0, u^0_t) \) in \( L^2((0,T),E) \). From the convergence in Theorem 3.4, it is enough to show the convergence \( u^\varepsilon \rightarrow u^0 \) in \( L^2((0,T),H^1(\Omega)) \).

**Proposition 5.** Under the notation and hypothesis of Theorem 3.4, we also assume the initial data satisfies

\[u_0^\varepsilon \rightarrow u_0 \quad \text{strongly in } H^1(\Omega), \quad v_0^\varepsilon \rightarrow v_0 \quad \text{strongly in } L^2(\Omega).
\]

Then

\[u^\varepsilon \rightarrow u^0 \quad \text{in } L^2((0,T),H^1(\Omega)).
\]

Proof. Notice that from the energy estimate (10) we get for \( 0 \leq \tau \leq T \)

\[
\frac{2}{\varepsilon} \int_0^\tau \int_\omega |u^\varepsilon_t|^2 + E_0(u^\varepsilon(\tau), u^\varepsilon_t(\tau)) = E_0(u_0^\varepsilon, v_0^\varepsilon) + 2 \int_0^\tau \int_\Omega f^\varepsilon u^\varepsilon_t + 2 \int_0^\tau \int_\omega s \int g^\varepsilon u^\varepsilon_t \tag{48}
\]

and we prove below that the right hand side converges, uniformly for \( 0 \leq \tau \leq T \).

**Step 1.** First, clearly from the assumption on the initial data we get, as \( \varepsilon \rightarrow 0 \),

\[E_0(u_0^\varepsilon, v_0^\varepsilon) \rightarrow E_0(u_0, v_0) \cdot \]

**Step 2.** Now since \( u^\varepsilon \rightarrow u^0_t \) in \( C([0,T],L^2(\Omega)) \), \( u^0_t \in L^\infty((0,T),H^1(\Omega)) \), and \( f^\varepsilon \rightarrow f \) in \( C([0,T],H^{-1}(\Omega)) \), \{\( f^\varepsilon \)_\} bounded in \( L^\infty((0,T),L^2(\Omega)) \) we get, as \( \varepsilon \rightarrow 0 \),

\[
\sup_{0 \leq \tau \leq T} \int_0^\tau \int_\Omega f^\varepsilon u^\varepsilon_t - \int_0^\tau \int_\Omega f u^0_t \rightarrow 0.
\]

To see this, note that for \( 0 \leq \tau \leq T \)

\[
\int_\Omega f^\varepsilon u^\varepsilon_t - \int_\Omega f u^0_t \leq \int_\Omega f^\varepsilon (u^\varepsilon_t - u_t^0) + \int_\Omega (f - f^\varepsilon) u_t^0
\]

\[
\leq \|f^\varepsilon\|_{L^\infty((0,T),L^2(\Omega))} \|u^\varepsilon_t - u_t^0\|_{C([0,T],L^\infty(\Omega))}
\]

\[
+ \|f - f^\varepsilon\|_{C([0,T],H^{-1}(\Omega))} \|u^0_t\|_{L^\infty((0,T),H^1(\Omega))} \rightarrow 0.
\]

**Step 3.** Now, we prove that, as \( \varepsilon \rightarrow 0 \),

\[
\sup_{0 \leq \tau \leq T} \frac{1}{\varepsilon} \int_\omega g^\varepsilon u^\varepsilon_t - \int_\Gamma g(\varepsilon u^\varepsilon_t) \rightarrow 0.
\]
For this note first that for $\tau \in [0, T]$,
\[
\sup_{0 \leq \tau \leq T} \left| \int_0^\tau \frac{1}{\varepsilon} \int_{\Omega} g_\varepsilon u_\varepsilon^\tau - \int_0^\tau \int_{\Gamma} g_\varepsilon (u_\varepsilon^\tau) \right| = \sup_{0 \leq \tau \leq T} \left| \int_0^\tau \frac{1}{\varepsilon} x_\varepsilon, g_\varepsilon - g, u_\varepsilon^\tau \right| \leq \frac{1}{\varepsilon} x_\varepsilon, g_\varepsilon - g \|C([0,T],H^{-s}(\Omega))\|u_\varepsilon^\tau\|L^\infty([0,T],H^s(\Omega)) \to 0
\]
since $\frac{1}{\varepsilon} x_\varepsilon, g_\varepsilon \to g$ in $C([0,T],H^{-s}(\Omega))$ and $u_\varepsilon \to u^0$ in $C([0,T],H^s(\Omega))$.

Now, integrating by parts, we get
\[
\int_0^\tau \int_{\Gamma} g_\varepsilon (u_\varepsilon^\tau) = - \int_0^\tau \int_{\Gamma} g_\varepsilon (u_\varepsilon^\tau) + \int_{\Gamma} g(0) \gamma(u_\varepsilon^\tau) (\tau) - \int_{\Gamma} g(0) \gamma(u_\varepsilon^\tau) (0)
\]
which converges uniformly in $\tau \in [0, T]$ to
\[
- \int_0^\tau \int_{\Gamma} g(0) \gamma(u_\varepsilon^\tau) + \int_{\Gamma} g(0) \gamma(u^0(\tau)) - \int_{\Gamma} g(0) \gamma(u^0(0)) = \int_0^\tau \int_{\Gamma} g(0) \gamma(u^0)
\]
since $u_\varepsilon \to u^0$ in $C([0,T],H^s(\Omega))$.

**Step 4.** Therefore, the right hand side of (48) converges uniformly in $\tau \in [0, T]$ to
\[
E_0(u_0^0, v_0^0) + 2 \int_0^\tau \int_{\Omega} f u_0^0 + 2 \int_0^\tau \int_{\Gamma} g_\varepsilon (u_\varepsilon^0)
\]
which, by (17), equals
\[
E_0(u_0^0(\tau), u_0^0(\tau)) + 2 \int_0^\tau \int_{\Gamma} |\gamma(u_\varepsilon^0)|^2, \quad 0 \leq \tau \leq T.
\]

Also, from (11), in the left hand side of (48) we have
\[
E_0(u^\varepsilon(\tau), u_\varepsilon^\tau(\tau)) = \int_{\Omega} |u_\varepsilon^\tau(\tau))|^2 + \int_{\Omega} |\nabla u^\varepsilon(\tau)|^2 + \lambda \int_{\Omega} |u^\varepsilon(\tau)|^2
\]
and the convergence of $u_\varepsilon \to u^0$ implies the uniform convergence for $\tau \in [0, T]$
\[
\int_{\Omega} |u_\varepsilon^\tau(\tau))|^2 + \lambda \int_{\Omega} |u^\varepsilon(\tau)|^2 \to \int_{\Omega} |u_0^0(\tau))|^2 + \lambda \int_{\Omega} |u^0(\tau)|^2.
\]
Hence, passing to the limit in (48) we get that
\[
\lim_{\varepsilon \to 0} \left( \int_0^\tau \frac{2}{\varepsilon} \int_{\omega_{\varepsilon}} |u_\varepsilon^\tau|^2 + \int_{\Omega} |\nabla u^\varepsilon(\tau)|^2 \right) = 2 \int_0^\tau \int_{\Gamma} |\gamma(u_\varepsilon^0)|^2 + \int_{\Omega} |\nabla u^0(\tau)|^2,
\]
uniformly for $0 \leq \tau \leq T$.

**Step 5.** Now, since $u_\varepsilon^\tau \to u_0^0$ weakly in $L^2((0, T), H^1(\Omega))$ and $\frac{1}{\varepsilon} x_\varepsilon, u_\varepsilon^\tau \to \gamma(u_\varepsilon^0)$ in $H^1((0, T), H^{-1}(\Omega))$ then we get, for each $0 \leq \tau \leq T$
\[
\frac{1}{\varepsilon} x_\varepsilon, u_\varepsilon^\tau - 1, 1 = \frac{1}{\varepsilon} \int_0^\tau \int_{\omega_{\varepsilon}} |u_\varepsilon^\tau|^2 \to (\gamma(u_0^0), u_0^0) - 1, 1 = \int_0^\tau \int_{\Gamma} |\gamma(u_0^0)|^2
\]
which implies
\[
\int_{\Omega} |\nabla u^\varepsilon(\tau)|^2 \to \int_{\Omega} |\nabla u^0(\tau)|^2 \quad \text{for each } 0 \leq \tau \leq T.
\]
This, and the boundedness of $u_\varepsilon^\tau$ in $L^\infty((0, T), H^1(\Omega))$ implies, using Lebesgue Theorem that
\[
\int_0^T \int_{\Omega} |\nabla u^\varepsilon(\tau)|^2 \to \int_0^T \int_{\Omega} |\nabla u^0(\tau)|^2.
\]
This and the weak convergence $u^\varepsilon \to u^0$ in $L^2((0, T), H^1(\Omega))$ concludes the result. □
Appendix A. Concentrating integrals. In this section we show several results that describe how different concentrated integrals converge to surface integrals. Hereafter we denote by $C > 0$ any positive constant such that $C$ is independent of $\varepsilon$ and $t$. This constant may change from line to line.

The following lemma is a particular case of a result proved in [6] and basically states that concentrated functions behave as traces.

**Lemma A.1.** i) Assume that $v \in H^s(\Omega)$ with $s > \frac{1}{2}$. Then for sufficiently small $\varepsilon_0$, we have, for some positive constant $C$ independent of $\varepsilon$,

$$\frac{1}{\varepsilon} \int_{\omega_{\varepsilon}} |v|^2 \leq C \|v\|_{H^s(\Omega)}^2$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_{\varepsilon}} |v|^2 = \int_{\Gamma} |v|^2.$$

ii) Consider a family $f_{\varepsilon}$ defined on $\omega_{\varepsilon}$, such that for some $1 \leq r < \infty$ and a positive constant $C$ independent of $\varepsilon$,

$$\frac{1}{\varepsilon} \int_{\omega_{\varepsilon}} |f_{\varepsilon}|^2 \leq C.$$

Then, for every sequence converging to zero (that we still denote $\varepsilon \to 0$) there exists a subsequence (that we still denote the same) and a function $f_0 \in L^2(\Gamma)$ such that, for every $s > \frac{1}{2}$ we have that

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_{\varepsilon}} f_{\varepsilon} \to f_0 \quad \text{in } H^{-s}(\Omega) \text{ as } \varepsilon \to 0$$

where $\mathcal{X}_{\omega_{\varepsilon}}$ is the characteristic function of the set $\omega_{\varepsilon}$. In particular, for any smooth function $\varphi$, defined in $\Omega$, we have

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_{\varepsilon}} f_{\varepsilon} \varphi = \int_{\Gamma} f_0 \varphi.$$

Moreover, if $u_{\varepsilon} \to u^0$ weakly in $H^s(\Omega)$ then

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_{\varepsilon}} f_{\varepsilon} u_{\varepsilon} = \int_{\Gamma} f_0 u^0.$$

In particular, assume $\varphi \in H^1(\Omega)$. Then

$$\frac{1}{\varepsilon} \mathcal{X}_{\omega_{\varepsilon}} \varphi \to \gamma(\varphi) \quad \text{in } H^{-s}(\Omega) \text{ as } \varepsilon \to 0$$

for any $s > \frac{1}{2}$.

iii) Assume now

$$\|u_0\|_{H^s(\Omega)}^2 \leq C.$$

Then, by taking subsequences if necessary, there exists $u_0 \in H^1(\Omega)$ such that, as $\varepsilon \to 0$,

$$u_{\varepsilon}^0 \to u_0 \quad \text{weakly in } H^1(\Omega), \quad \frac{1}{\varepsilon} \mathcal{X}_{\omega_{\varepsilon}} u_{\varepsilon}^0 \to \gamma(u_0) \quad \text{weakly in } H^{-1}(\Omega)$$

and

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\omega_{\varepsilon}} |u_{\varepsilon}^0|^2 = \int_{\Gamma} |u_0|^2.$$

Lemma A.1 was extended in [15] to handle concentrating integrals including a time dependence.
Lemma A.2. i) Consider \( v \in L^2((0,T), H^s(\Omega)) \) with \( s > \frac{1}{2} \). Then, 
\[
\int_0^T \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^2 \, dx \leq C \int_0^T \|v(t,\cdot)|^2_{H^s(\Omega)} \, dt = C \|v\|^2_{L^2((0,T), H^s(\Omega))}
\]
and
\[
\lim_{\varepsilon \to 0} \int_0^T \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |v|^2 = \int_0^T \int_{\Gamma} |v|^2 = \|v\|^2_{L^2((0,T), L^2(\Gamma))}.
\]
ii) Consider a family \( g_\varepsilon \) defined on \((0,T) \times \omega_\varepsilon\), such that for some positive constant \( C \) independent of \( \varepsilon \), 
\[
\int_0^T \frac{1}{\varepsilon} \int_{\omega_\varepsilon} |g_\varepsilon(t,x)|^2 \, dx \, dt \leq C.
\]
Then, for every \( s > 1/2 \) and for every sequence converging to zero (that we still denote \( \varepsilon \to 0 \)) there exists a subsequence (that we still denote the same) and a function \( g \in L^2((0,T), L^2(\Gamma)) \) such that
\[
\frac{1}{\varepsilon} \chi_{\omega_\varepsilon} g_\varepsilon \to g \quad \text{in} \quad L^2((0,T), H^{-s}(\Omega)), \quad \text{weakly as} \quad \varepsilon \to 0,
\]
where \( \chi_{\omega_\varepsilon} \) is the characteristic function of the set \( \omega_\varepsilon \). In particular, for any smooth function \( \varphi \), defined in \([0,T] \times \Omega\), we have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} g_\varepsilon \varphi = \int_0^T \int_{\Gamma} g \varphi.
\]
Also, if \( u^\varepsilon \to u^0 \) strongly in \( L^2((0,T), H^s(\Omega)) \) then
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} g_\varepsilon u^\varepsilon = \int_0^T \int_{\Gamma} gu^0.
\]
We also have the following result from [15].

Lemma A.3. We consider a family of functions \( u^\varepsilon : [0,T] \to H^1(\Omega) \) such that for some positive constant \( C \) independent of \( \varepsilon \) and \( t \), we have
\[
\|u^\varepsilon(t,\cdot)|_{H^1(\Omega)} \leq C, \quad t \in [0,T]
\]
and \( u_\varepsilon^\varepsilon \in L^2((0,T) \times \omega_\varepsilon) \) with
\[
\frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} |u_\varepsilon^\varepsilon|^2 \leq C.
\]
Then, there exists a subsequence (that we still denote the same) and a function \( u^0 \in L^\infty((0,T), H^1(\Omega)) \) with \( \gamma(u^0) \in H^1((0,T), L^2(\Gamma)) \) such that as \( \varepsilon \to 0 \), 
\[
u^\varepsilon \to u^0 \quad \text{weakly as} \quad \varepsilon \to 0,
\]
and
\[
\frac{1}{\varepsilon} \chi_{\omega_\varepsilon} u^\varepsilon \to \gamma(u^0) \quad \text{in} \quad H^1((0,T), H^{-1}(\Omega)).
\]
In particular, for every \( \varphi \in L^2((0,T), H^1(\Omega)) \) we have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} u^\varepsilon \varphi = \int_0^T \int_{\Gamma} u^0 \varphi, \quad \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{\omega_\varepsilon} u_\varepsilon^\varepsilon \varphi = \int_0^T \int_{\Gamma} u_\varepsilon^0 \varphi.
\]
Finally
\[
\frac{1}{\varepsilon} \chi_{\omega_\varepsilon} u^\varepsilon \to \gamma(u^0) \quad \text{in} \quad C([0,T], H^{-1}(\Omega)) \quad \text{as} \quad \varepsilon \to 0
\]
and

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^T \int_{|u|} |u^\varepsilon|^2 = \int_0^T \int_{\Gamma} |u^0|^2.
\]

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E-mail address: ajimenez@comillas.edu
E-mail address: arober@ucm.es