AN EXTENSION OF MERCER’S THEOREM TO UNBOUNDED OPERATORS

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Abstract. We give some extensions of Mercer’s theorem to continuous Carleman kernels inducing unbounded integral operators.

INTRODUCTION

The source of the following theorem is [12].

Mercer’s theorem. Let $T$ be a positive, integral operator on $L^2[a, b]$ with continuous kernel $K(s, t) = K(t, s)$ on $[a, b]^2$ $(|a|, |b| < \infty)$. Then the kernel $K(s, t)$ can be represented by the bilinear series

$$
K(s, t) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(s) \overline{\varphi_n(t)}
$$

absolutely and uniformly convergent on $[a, b]^2$, where $\lambda_n \geq 0$ $(n = 1, 2, 3, \ldots)$ are the eigenvalues of operator $T$ and $\varphi_n$ $(n = 1, 2, 3, \ldots)$ are the corresponding orthonormal eigenfunctions.

Within modern theory of integral operators, the theorem just formulated is related to the problem of representing the kernel via spectrum and eigenfunctions (see, for example, [9]). In the works [2], [4], [5], [11], [13]-[15], Mercer’s theorem has been generalized to wider classes of kernels inducing bounded integral operators. In the present paper, we study the problem of spectral representing for a continuous kernel in that case when the kernel induces an unbounded normal operator whose spectrum is in a sector of angle less than $\pi$ with vertex at 0.

The following definitions and notions are needed in what follows. Let $\mathbb{R}_+ = [0, \infty)$, and let $L_2$ be the Hilbert space of complex-valued measurable functions on $\mathbb{R}_+$ that are square integrable in the Lebesgue sense, equipped with the inner product

$$
\langle f, g \rangle = \int_{\mathbb{R}_+} f(s) \overline{g(s)} \, ds
$$
and the norm \( \|f\| = \langle f, f \rangle^{\frac{1}{2}} \).

A linear operator \( T : D_T \to L_2 \), where \( D_T \) is a dense linear manifold in \( L_2 \), is said to be \textit{integral} if there exists a measurable almost-everywhere finite-valued function \( K(s, t) \) on \( \mathbb{R}_+^2 \), a \textit{kernel}, such that, for every \( f \in D_T \),

\[
(Tf)(s) = \int_{\mathbb{R}_+} K(s, t)f(t) \, dt
\]

for almost every \( s \in \mathbb{R}_+ \).

A kernel \( K(s, t) \) is said to be \textit{Carleman} if \( K(s, \cdot) \in L_2 \) for almost every fixed \( s \in \mathbb{R}_+ \).

An integral operator induced by a Carleman kernel is called \textit{Carleman operator}, and it is called \textit{bi-Carleman} if \( K(\cdot, t) \in L_2 \) for almost every \( t \in \mathbb{R}_+ \). In the last case the kernel induces two \textit{Carleman functions} from \( \mathbb{R}_+ \) to \( L_2 \) by \( k(s) = K(s, \cdot) \), \( k^*(t) = K(\cdot, t) \) for all those \( s, t \in \mathbb{R}_+ \) for which \( K(s, \cdot) \in L_2 \) and \( K(\cdot, t) \in L_2 \). We need the following important result of Carleman operator theory

\textbf{The right-multiplication lemma.} (3). Let \( T : D_T \to L_2 \) be a Carleman operator, and let \( R : L_2 \to L_2 \) be a bounded operator. Then the product \( TR \) is a Carleman operator with Carleman function \( R^*(k(s)) \), where \( k(s) \) is Carleman function of \( T \).

If \( X \) is a locally compact space, and if \( B \) is a Banach space with norm \( \|\cdot\|_B \), then let \( C(X, B) \) denote the Banach space (with the norm \( \|f\|_{C(X, B)} = \sup_X \|f(x)\|_B \)) of all continuous functions from \( X \) to \( B \) vanishing at infinity \( \infty \) (the latter means that for each fixed \( f \in C(X, B) \), given an arbitrary \( \varepsilon > 0 \), there exists a compact set \( X(\varepsilon, f) \subset X \) such that \( \|f(x)\|_B < \varepsilon \) if \( x \notin X(\varepsilon, f) \)).

\textbf{Definition.} (15, 16). A kernel \( K \) of a bi-Carleman operator \( T \) is called a \textit{\( K^0 \)-kernel}, if \( K \in C(\mathbb{R}_+^2, \mathbb{C}) \) and its Carleman functions \( k, k^* \in C(\mathbb{R}_+, L_2) \).

In Section 1, we give an “integral” analogue of Mercer’s theorem (Theorem 1) that represents “functions of a \( K^0 \)-kernel” by uniformly convergent principal value Lebesgue-Stieltjes integrals with respect to the spectral function of this kernel. In Section 2, \( K^0 \)-kernels of unbounded diagonal operators are considered. There we prove Theorem 2 on representations of \( K^0 \)-kernels by absolutely and uniformly convergent bilinear series. In the case of bounded integral operators Theorems 1 and 2 were proved in [14, 15].

1. \textbf{Integral representations of \( K^0 \)-kernels}

Let \( N : D_N \to L_2 \) be a normal operator, that is, a closed linear operator with a dense in \( L_2 \) domain \( D_N = D_N^* \) and such that \( NN^* = N^*N \). Let \( E(\cdot) \) be the resolution of identity induced by \( N \) (see [18]). Assume that \( N \) is an integral operator having a \( K^0 \)-kernel \( N(s, t) \) and Carleman functions \( \nu(s) = N(s, \cdot), \nu^*(t) = N(\cdot, t) \).
Let $\Omega$ be the $\sigma$-algebra of Borel sets in $\mathbb{C}$, and let $\Omega_0$ be the set of all $\omega \in \Omega$ whose closures in $\mathbb{C}$ do not contain $0$. If $\chi_\omega$ is the characteristic function of the set $\omega \in \Omega_0$, then $\chi_\omega(z) = zv_\omega(z)$, where $v_\omega(z) = \chi_\omega(z)/z$.

From the multiplicative property of the spectral measure $E(\cdot)$ it follows that $E(\omega) = \chi_\omega(N) = Nv_\omega(N)$ (here and throughout, $f(N)$, given an $\Omega$-measurable function $f$, denote a normal operator defined by

$$\langle f(N)h, g \rangle = \int_C f(z)\langle E(dz)h, g \rangle$$

for all $h, g \in D_f(N)$ with domain

$$D_f(N) = \{x \in L_2 : \int_C |f(z)|^2\langle E(dz)x, x \rangle < \infty\}.$$ 

Since $B(\omega) = v_\omega(N)$ is a bounded operator for each $\omega \in \Omega_0$, it follows by the right-multiplication lemma that the orthogonal projection $E(\omega)$ is a bounded bi-Carleman operator, with the Carleman functions

$$e(s; \omega) = (B(\omega))^*(\nu(s)), \quad e^*(t; \omega) = e(t; \omega)$$

belonging to $C(\mathbb{R}_+, L_2)$. Since $E^2(\omega) = E(\omega)$, the kernel of $E(\omega)$ can be computed as follows

$$E(s, t; \omega) = \langle e(t; \omega), e(s; \omega) \rangle$$

for all $(s, t) \in \mathbb{R}_+^2$, and hence belongs to $C(\mathbb{R}_+^2, \mathbb{C})$.

**Definition.** A set function $E(s, t; \cdot) : \Omega_0 \rightarrow C(\mathbb{R}_+^2, \mathbb{C})$ whose value on every $\omega \in \Omega_0$ is the inducing $K^0$-kernel $E(s, t; \omega)$ of $E(\omega)$ is called a spectral function for the kernel $N(s, t)$.

**Remark 1.** For other definitions of spectral functions of Carleman kernels we refer to the works [6, 3, 10, 19, 11, 14].

Let $\omega_\varepsilon = \mathbb{C} \setminus \{z \in \mathbb{C} : |z| \leq \varepsilon, \varepsilon > 0\}$, and let $\Phi$ denote the family of all functions $\phi$ of the form $\phi(z) = zv(z)$ where $v(z)$ is a bounded $\Omega$-measurable function on $\mathbb{C}$. Let $\Omega_\varepsilon$ be the algebra of all Borel subsets of $\omega_\varepsilon$. Since $E(\omega_\varepsilon)E(\sigma) = E(\sigma)$ for each $\sigma \in \Omega_\varepsilon$, the right-multiplication lemma yields $e(s; \sigma) = E(\sigma)(e(s; \omega_\varepsilon))$ and hence

$$E(s, t; \sigma) = \langle e(t; \sigma), e(s; \sigma) \rangle = \langle E(\sigma)(e(t; \omega_\varepsilon)), e(s; \omega_\varepsilon) \rangle$$

for all $s, t \in \mathbb{R}_+$. Whence it follows that

$$\|E(\cdot; \sigma)\|_{C(\mathbb{R}_+^2, \mathbb{C})} \leq \|e(\cdot; \omega_\varepsilon)\|^2_{C(\mathbb{R}_+, L_2)}$$

for each $\sigma \in \Omega_\varepsilon$. The last inequality implies that, for each $\varepsilon > 0$, the spectral function $E(s, t; \cdot)$ is bounded on $\Omega_\varepsilon$. Furthermore, it is additive; this property follows from that of $E(\cdot)$. Therefore, for each $\phi \in \Phi$, the Lebesgue-Stieltjes integrals

$$\Phi_\varepsilon(s, t) = \int_{\omega_\varepsilon} \phi(z)E(s, t; dz),$$

can be formed for every $\varepsilon > 0$ and for all $s, t \in \mathbb{R}_+$. 
Theorem 1. Let \( N : D_N \to L_2 \) be a normal, integral operator induced by a
\( K^0 \)-kernel \( N(s,t) \), the spectrum of which is in a sector \( \mathbb{S} \) of angle less than
\( \pi \) with vertex at 0. Then for every \( \phi \in \Phi \) operator \( \phi(N) \) is also integral
operator with \( K^0 \)-kernel. Moreover, the \( K^0 \)-kernel \( \Phi(s,t) \) inducing \( \phi(N) \)
can be represented by the principal value integral with singularity at \( z = 0 \):
\[
\Phi(s,t) = \int_{\mathbb{C}} \phi(z) E(s,t; dz) = \lim_{\varepsilon \to 0} \int_{\omega_{\varepsilon}} \phi(z) E(s,t; dz)
\]
for all \( s, t \in \mathbb{R}_+ \), where the integral converges to \( \Phi(s,t) \) in \( C(\mathbb{R}_+^2, \mathbb{C}) \) as
\( \varepsilon \to 0 \) along arbitrary decreasing sequence of positive numbers.

Proof. From multiplicative property it follows that
\[
\langle \phi(N)f, h \rangle = \int_{\mathbb{C}} z \nu(z) \langle E(dz)f, h \rangle = \langle N\nu(N)f, h \rangle \quad \text{for all } f, h \in D\phi(N),
\]
that is, \( \phi(N) = N\nu(N) \) is a Carleman operator with a kernel \( \Phi(s,t) \) and a
Carleman function
\[
\varphi(s) = \bar{\Phi(s,\cdot)} = (\nu(N))^*(\nu(s)), \quad (1.3)
\]
which is in \( C(\mathbb{R}_+, L_2) \). Since \( \phi(N) \) is a normal operator, it is bi-Carleman
(see Corollary 2.19 from [8, p. 131]). In addition, the Carleman function of
the adjoint
\[
\langle (\phi(N))^*f, h \rangle = \int_{\mathbb{C}} \overline{\nu(z)} \langle E(dz)f, h \rangle = \langle N^*(\nu(N))^*f, h \rangle, \quad f, h \in D\phi(N)
\]
has, by the right-multiplication lemma, the form
\[
\varphi^*(t) = \Phi(\cdot, t) = \nu(N)(\nu^*(t)) \quad (t \in \mathbb{R}_+)
\]
and hence belongs to \( C(\mathbb{R}_+, L_2) \) too. Using (1.0) and (1.1), compute the integral (1.2) as follows
\[
\Phi_{\varepsilon}(s,t) = \int_{\omega_{\varepsilon}} \phi(z) \langle E(dz)(e(t;\omega_{\varepsilon})), e(s;\omega_{\varepsilon}) \rangle = \int_{\omega_{\varepsilon}} \phi(z) \langle E(dz)(B(\omega_{\varepsilon}))^*\nu(t), (B(\omega_{\varepsilon}))^*(\nu(s)) \rangle = \langle \phi(N)B(\omega_{\varepsilon})(B(\omega_{\varepsilon}))^*(\nu(t)), \nu(s) \rangle
\]
for all \( s, t \in \mathbb{R}_+ \). It is clear that the function \( \Phi_{\varepsilon}(s,t) \) belongs to \( C(\mathbb{R}_+^2, \mathbb{C}) \)
and that this is a kernel of the integral operator \( \Phi(N)E(\omega_{\varepsilon}) = NE(\omega_{\varepsilon})\nu(N) \)
with Carleman function
\[
\varphi_{\varepsilon}(s) = \bar{\Phi_{\varepsilon}(s,\cdot)} = (\nu(N))^*E(\omega_{\varepsilon})(\nu(s)) \quad (s \in \mathbb{R}_+)
\]
belonging to \( C(\mathbb{R}_+, L_2) \). The function \( E(\{0\})(\nu(s)) \) from \( C(\mathbb{R}_+, L_2) \) is equal
identically to zero on \( \mathbb{R}_+ \), because it is a Carleman function of the integral
operator \( NE(\{0\}) \), which, by the multiplicative property, is the null operator:

\[
\langle NE(\{0\})f, h \rangle = \int_{\mathbb{C}} z\chi_{\{0\}}(z)\langle E(dz)f, h \rangle = 0, \quad f, h \in L_2.
\]

Let \( \{\varepsilon_n\} \subset \mathbb{R}_+ \) be an arbitrary sequence decreasing to 0, and let \( \omega_0 = \mathbb{C}\setminus\{0\} \).

By virtue of (1.3) and (1.5),

\[
\sum_{s, t} \varepsilon_s \cdot \varepsilon_t = \varepsilon \quad (\varepsilon > 0) \quad (\text{see [17, p. 98]}),
\]

whence

\[
\varepsilon \in (\mathbb{R}_+^\ast, \mathbb{R}) \quad \text{is positive if} \quad 0 < \varepsilon < \infty \quad (\text{see [17, p. 98]}).\]

\[
\lim_{n \to \infty} \frac{\|\varphi - \varphi_{\varepsilon_n}\|_{C(\mathbb{R}_+^\ast, L^2)}}{\|\varphi - \varphi_{\varepsilon_n}\|_{C(\mathbb{R}_+^\ast, L^2)}} = 0, \quad \text{(1.6)}
\]

because \( E(\omega_{\varepsilon_n}) \to E(\omega_0) \) whenever \( n \to \infty \) in strong operator topology and the set \( \{\varphi(s) : s \in \mathbb{R}_+\} \) is precompact in \( L_2 \) (see [17, p. 193]). The following property

\[
\lim_{n \to \infty} \frac{\|\varphi - \varphi_{\varepsilon_n}\|_{C(\mathbb{R}_+^\ast, L^2)}}{\|\varphi - \varphi_{\varepsilon_n}\|_{C(\mathbb{R}_+^\ast, L^2)}} = 0, \quad \text{(1.6')}
\]

where \( \varphi_{\varepsilon_n}(t) = \Phi_{\varepsilon_n}(\cdot, t), \quad n = 1, 2, 3, \ldots \) can be proved analogously.

Without loss of generality, assume that the sector \( \mathcal{S} \) is bounded by the rays \( \text{Im} z = \pm l \text{Re} z, \text{Re} z \geq 0, l > 0 \). It is obvious that

\[
\sup_{z \in \mathcal{S}} |v_1(z)| \leq l, \quad \text{where} \quad v_1(z) = \frac{\text{Im} z}{\text{Re} z}. \quad \text{(1.7)}
\]

Consider a self-adjoint integral operator \( X = (N + N^*)/2 \) induced by the \( K^0 \)-kernel \( X(s, t) = \overline{X(t, s)} = (N(s, t) + N(t, s))/2 \) having the Carleman function \( x(s) = \overline{X(s, \cdot)} \). Fix \( \varepsilon > 0 \) and consider the bounded integral operator \( X(I - E(\omega_{\varepsilon})) \) with a \( K^0 \)-kernel \( X(s, t) - X(\varepsilon(s, t), \varepsilon(t, s)) \), where

\[
X(\varepsilon(s, t)) = \int_{\mathcal{S} \cap \omega_{\varepsilon}} \text{Re} z E(s, t; dz) = \int_{\mathcal{S} \cap \omega_{\varepsilon}} \text{Re} z E(s, t; dz) \quad \text{(1.8)}
\]

is the \( K^0 \)-kernel of integral operator \( XE(\omega_{\varepsilon}) \) inducing the Carleman function \( x(\varepsilon(s)) = \overline{X(\varepsilon(s, \cdot))} \). This operator is positive, since

\[
\langle X(I - E(\omega_{\varepsilon}))f, f \rangle = \int_{\mathcal{S} \cap \omega_{\varepsilon}} \text{Re} z X(\varepsilon(s, t)) \langle E(dz)f, f \rangle \geq 0, \quad f \in L_2.
\]

Hence, its \( K^0 \)-kernel satisfies the inequality \( X(s, s) - X(\varepsilon(s, s)) \geq 0 \) for all \( s \in \mathbb{R}_+ \) (see [17, p. 98]), whence

\[
X(\varepsilon(s, s)) \leq X(s, s) \quad \text{for all} \quad s \in \mathbb{R}_+. \quad \text{(1.9)}
\]

The bounded self-adjoint operator

\[
x(\varepsilon_m, \varepsilon_n) = X \left( B(\omega_{\varepsilon_m})(B(\omega_{\varepsilon_n}))^* - B(\omega_{\varepsilon_n})(B(\omega_{\varepsilon_n}))^* \right)
\]

is positive if \( 0 < \varepsilon_m \leq \varepsilon_n \):

\[
\langle x(\varepsilon_m, \varepsilon_n)f, f \rangle = \int_{\mathcal{S} \cap \omega_{\varepsilon_n} \setminus \omega_{\varepsilon_m}} \frac{\text{Re} z}{|z|^2} \langle E(dz)f, f \rangle \geq 0, \quad f \in L_2.
\]
Substituting \( f = \nu(s) \) to the last inequality and taking into account (1.8) we obtain \( X_{\varepsilon_m}(s, s) \geq X_{\varepsilon_n}(s, s) \) for all \( s \in \mathbb{R}_+ \). Conclude via (1.9) that the sequence \( X_{\varepsilon_n}(s, s) \) \( (n = 1, 2, 3, \ldots) \) of functions from \( C(\mathbb{R}_+, \mathbb{C}) \) converges in \( \mathbb{R}_+ \) if \( \varepsilon_n \searrow 0 \) as \( n \to \infty \). Apply the generalized Schwarz inequality \([\Pi \text{ p. 78}]\) to the positive operator \( x(\varepsilon_m, \varepsilon_n) \) to write
\[
|X_{\varepsilon_m}(s, t) - X_{\varepsilon_n}(s, t)|^2 = |\langle x(\varepsilon_m, \varepsilon_n)(\nu(t)), \nu(s) \rangle|^2 \leq \langle x(\varepsilon_m, \varepsilon_n)(\nu(s)), \nu(s) \rangle \langle x(\varepsilon_m, \varepsilon_n)(\nu(t)), \nu(t) \rangle = (X_{\varepsilon_m}(s, s) - X_{\varepsilon_n}(s, s))(X_{\varepsilon_m}(t, t) - X_{\varepsilon_n}(t, t)).
\]
Whence, by (1.9), one can infer that
\[
\lim_{n \to \infty} \sup_{t \in \mathbb{R}_+} |X_{\varepsilon_n}(s, t) - X'(s, t)| = 0 \quad \text{for all } s \in \mathbb{R}_+,
\]
where \( X'(s, \cdot) \in C(\mathbb{R}_+, \mathbb{C}) \) for each \( s \in \mathbb{R}_+ \). Now apply the preceding arguments to the functions \( x, x_{\varepsilon_n}, \frac{\text{Re} z}{z} \) instead of \( \varphi, \varphi_{\varepsilon_n}, v(z) \) respectively to conclude that
\[
\lim_{n \to \infty} \|x_{\varepsilon_n} - x\|_{C(\mathbb{R}_+, L_2)} = 0.
\]
The functions \( X(s, t) \) and \( X'(s, t) \) are continuous with respect to \( t \) and coincide for each fixed \( s \in \mathbb{R}_+ \) and all \( t \in \mathbb{R}_+ \), because from two last limit relations it follows that, for every \( s \in \mathbb{R}_+ \),
\[
X(s, t) = X'(s, t)
\]
for almost every \( t \in \mathbb{R}_+ \) (see \([\Pi \text{ p. 42}]\)). Assume that \( X(\infty, \infty) = 0 \) and that \( X_{\varepsilon_n}(\infty, \infty) = 0 \) \( (n = 1, 2, 3, \ldots) \), and conclude, by the Dini theorem, that the monotone increasing sequence of \( \{X_{\varepsilon_n}(s, s)\} \) uniformly converges on the compactum \([0, \infty]\) to the function \( X(s, s) \). Consequently, one can write
\[
\lim_{n \to \infty} \sup_{s \in \mathbb{R}_+} |X_{\varepsilon_n}(s, s) - X(s, s)| = 0. \tag{1.10}
\]
Consider the bounded operator
\[
\phi(\varepsilon_m, \varepsilon_n) = \phi(N)(B(\omega_{\varepsilon_m})(B(\omega_{\varepsilon_n}))^* - B(\omega_{\varepsilon_n})(B(\omega_{\varepsilon_n}))^*), \quad 0 < \varepsilon_m \leq \varepsilon_n.
\]
By (1.7), we have
\[
\langle \phi(\varepsilon_m, \varepsilon_n) f, h \rangle = \int_{\mathbb{C} \cap (\omega_{\varepsilon_m} \setminus \omega_{\varepsilon_n})} v(z)(1 + i \cdot v_1(z)) \frac{\text{Re} z}{|z|^2} \langle E(dz) f, h \rangle = \langle Px(\varepsilon_m, \varepsilon_n) f, h \rangle, \quad f, h \in L_2,
\]
where \( P = v(N)(I + iv_1(N)) \) is a bounded normal operator that commutes with \( x(\varepsilon_m, \varepsilon_n) \). Put in (1.11) \( f = \nu(t) \), \( h = \nu(s) \), use the generalized Schwarz
inequality as above, and obtain, by (1.4), that
\[
|\Phi_{\varepsilon_n}(s, t) - \Phi_{\varepsilon_n}(s, t)| = |(\phi(\varepsilon_m, \varepsilon_n)(\nu(t)), \nu(s))| = \\
= |\langle x(\varepsilon_m, \varepsilon_n)(\nu(t)), P^*\nu(s) \rangle| \leq \\
\leq \|PP^*\| |\langle x(\varepsilon_m, \varepsilon_n)(\nu(t)), \nu(t)\rangle| |\langle x(\varepsilon_m, \varepsilon_n)(\nu(s)), \nu(s) \rangle| = \\
\leq \|PP^*\| |\langle x(\varepsilon_m, \varepsilon_n)(\nu(t)), \nu(t)\rangle| |\langle x(\varepsilon_m, \varepsilon_n)(\nu(s)), \nu(s) \rangle| = \\
\leq \|PP^*\| (X_{\varepsilon_n}(s, s) - X_{\varepsilon_n}(s, t)) (X_{\varepsilon_n}(t, t) - X_{\varepsilon_n}(t, t)) 
\]
for all \((s, t) \in \mathbb{R}^2_+\) (in the next to the last inequality of (1.12), the Reid inequality \cite{20} p. 59 was used). From (1.12), by (1.10), it follows that there exists \(\Phi' \in C(\mathbb{R}^2_+, \mathbb{C})\) such that
\[
\lim_{n \to \infty} \|\Phi_{\varepsilon_n} - \Phi'\|_{C(\mathbb{R}^2_+, \mathbb{C})} = 0.
\]
Now, exactly as for \(X(s, t)\) and \(X'(s, t)\), we can conclude in view of (1.6) and (1.6)' that, for each \(s \in \mathbb{R}_+\),
\[
\Phi(s, t) = \Phi'(s, t)
\]
for almost every \(t \in \mathbb{R}_+\), and conversely. It follows that the function \(\Phi'(s, t)\) is a \(K^0\)-kernel of the integral operator \(\phi(N)\). \(\square\)

2. Representation of \(K^0\)-kernels by bilinear series

**Theorem 2.** Let \(N : D_N \to L_2\) be a normal integral operator induced by a \(K^0\)-kernel \(N(s, t)\) and having diagonal form
\[
Nf = \sum_{n=1}^{\infty} \alpha_n \langle f, \varphi_n \rangle \varphi_n, \quad f \in D_N,
\]
where \(\{\varphi_n\}_{n=1}^{\infty} \subset L_2\) is an orthonormal set, the numbers \(\alpha_n\) \((n = 1, 2, 3, \ldots)\) are in a sector of angle less than \(\pi\) with vertex at 0, and the series converges in \(L_2\)-norm to \(Nf\). Then the bilinear series
\[
\sum_{n=1}^{\infty} \alpha_n \varphi_n(s) \overline{\varphi_n(t)} \tag{2.0}
\]
converges both absolutely and uniformly in \(\mathbb{R}^2_+\) to the kernel \(N(s, t)\).

**Proof.** Take an \(\alpha \in [0, 2\pi]\) so that the eigenvalues \(\lambda_n = e^{i\alpha} \alpha_n = x_n + iy_n\) of the operator \(P = e^{i\alpha} N\) are in the sector bounded by the rays \(y = \pm lx\), where \(x \geq 0, l > 0\). The diagonal operator
\[
Tf = \frac{P + P^*}{2} f = \sum_{n=1}^{\infty} x_n \langle f, \varphi_n \rangle \varphi_n, \quad f \in D_T = D_N, \tag{2.1}
\]
is an integral one having the \(K^0\)-kernel
\[
K(s, t) = K(s, t) = \frac{e^{i\alpha} N(s, t) + e^{-i\alpha} N(t, s)}{2}
\]
and the Carleman function $k(s) = \overline{K(s, \cdot)}$. Assume, with no loss of generality, that $x_n > 0$, $n = 1, 2, 3, \ldots$. Since $k \in C(R_+, L_2)$, the equivalence class $T \{f \in D_T\}$ contains a unique continuous function; this function has the form $\langle f, k(s) \rangle$ ($s \in R_+$) and belongs to $C(R_+, \mathbb{C})$. Therefore we can assume in what follows that

$$\varphi_n \in C(R_+, \mathbb{C}), \quad n = 1, 2, 3, \ldots,$$

because $\frac{1}{x_n} T \varphi_n = \varphi_n$, $n = 1, 2, 3, \ldots$.

Consider the functions

$$K_m(s, t) = K(s, t) - \sum_{n=1}^{m} x_n \varphi_n(s) \overline{\varphi_n(t)}, \quad m = 1, 2, 3, \ldots,$$

By virtue of (2.2), these functions belong to $C(R_+^2, \mathbb{C})$. Moreover, for every $f \in D_T$,

$$\int_{R_+} \int_{R_+} K_m(s, t) f(t) \overline{f(s)} \, dt \, ds = \sum_{n=1}^{\infty} x_n |\langle f, \varphi_n \rangle|^2 - \sum_{n=1}^{m} x_n |\langle f, \varphi_n \rangle|^2 =

= \sum_{n=m+1}^{\infty} x_n |\langle f, \varphi_n \rangle|^2 \geq 0.$$

Therefore, $K_m(s, s) \geq 0$ for all $m$ and for all $s \in R_+$ (see, for example, [17, p. 263]), and hence

$$K(s, s) \geq \sum_{n=1}^{\infty} x_n |\varphi_n(s)|^2 \quad \text{for all} \quad s \in R_+.$$  

(2.3)

By virtue of the Cauchy inequality,

$$\left( \sum_{n=p}^{q} x_n |\varphi_n(s)| |\varphi_n(t)| \right)^2 \leq M \sum_{n=p}^{q} x_n |\varphi_n(s)|^2 \quad \text{for all} \quad p, q \in \mathbb{N},$$

(2.4)

where $M = \max_{s \in R_+} K(s, s)$. Consequently, the series

$$\sum_{n=1}^{\infty} x_n \varphi_n(s) \overline{\varphi_n(t)}$$

(2.5)

absolutely converges on $R_+^2$ and uniformly converges with respect to $t$. Its sum-function $B(s, t)$ is continuous with respect to $t$ for each fixed $s$, and vice versa. Apply the Bessel inequality to the function $k(s) \in L_2$ and obtain, for each fixed $s \in R_+$,

$$\sum_{n=1}^{\infty} x_n^2 |\varphi_n(s)|^2 = \sum_{n=1}^{\infty} |\langle \varphi_n, k(s) \rangle|^2 \leq \|k\|_{C(R_+, L_2)}^2.$$

(2.6)
It follows from the Riesz-Fisher theorem that \( B(s, \cdot) \in L_2 \) and that
\[
\int_{\mathbb{R}_+} B(s, t) f(t) \, dt = \sum_{n=1}^{\infty} x_n \langle f, \varphi_n \rangle \varphi_n(s) \quad (f \in L_2)
\]
for all \( s \in \mathbb{R}_+ \). The series on right-hand side of the last equation uniformly converges, since, by the Cauchy inequality and by (2.6),
\[
\left| \sum_{n=p}^{q} x_n \langle f, \varphi_n \rangle \varphi_n(s) \right|^2 \leq \sum_{n=p}^{q} x_n^2 |\varphi_n(s)|^2 \cdot \sum_{n=p}^{q} |\langle f, \varphi_n \rangle|^2 \leq \|k\|_{C([\mathbb{R}_+, L_2])}^2 \sum_{n=p}^{q} |\langle f, \varphi_n \rangle|^2.
\]

Note also that if \( f \in D_T \), then by (2.1) the pointwise sum of this series is the exactly continuous function \( (Tf)(s) \). Therefore, one can write
\[
\int_{\mathbb{R}_+} (K(s, t) - B(s, t)) f(t) \, dt \equiv 0 \quad \text{on } \mathbb{R}_+
\]
for each \( f \) from the dense set \( D_T \) in \( L_2 \). Consequently, for every fixed \( s \in \mathbb{R}_+ \),
\[
K(s, t) = B(s, t) \quad (2.7)
\]
for all \( t \in \mathbb{R}_+ \), since both functions are continuous with respect to \( t \). In particular, for each \( s \in \mathbb{R}_+ \),
\[
K(s, s) = B(s, s) = \sum_{n=1}^{\infty} x_n |\varphi_n(s)|^2. \quad (2.8)
\]

The series (2.8) can be considered on the compactum \([0, \infty]\), keeping in mind that \( K(\infty, \infty) = \varphi_n(\infty) = 0, \ n = 1, 2, 3, \ldots \). Hence, by the Dini theorem the series (2.8) converges uniformly on \( \mathbb{R}_+ \). From (2.4) it follows that the series (2.5) uniformly converges on \( \mathbb{R}_+^2 \) to the kernel \( K(s, t) \). Using the estimate \( |\alpha_n| = |\lambda_n| \leq x_n \sqrt{1 + t^2} \ (n = 1, 2, 3, \ldots) \) and the Cauchy inequality, we have
\[
\sum_{n=p}^{q} |\lambda_n| |\varphi_n(s)||\varphi_n(t)| \leq \sqrt{1 + t^2} \left( \sum_{n=p}^{q} x_n |\varphi_n(s)|^2 \right)^{1/2} \left( \sum_{n=p}^{q} x_n |\varphi_n(t)|^2 \right)^{1/2}.
\]

Therefore, the series (2.0) uniformly converges on \( \mathbb{R}_+^2 \). Its sum \( \Omega(s, t) \) is a function belonging to \((\mathbb{R}_+^2, \mathbb{C})\). Now apply preceding reasoning to the functions \( \Omega(s, t) \) and \( N(s, t) \) in place of \( B(s, t) \) and \( K(s, t) \) in (2.7) to infer that \( \Omega(s, t) \equiv N(s, t) \) on \( \mathbb{R}_+^2 \). \( \square \)
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