Field Theory of Anisotropic Quantum Hall Gas: Metrology and a Novel Quantum Hall Regime

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The von Neumann lattice representation is a convenient representation for studying several intriguing physics of quantum Hall systems. In this formalism, electrons are mapped to lattice fermions. A topological invariant expression of the Hall conductance is derived and is used for the proof of the integer quantum Hall effect in the realistic situation. Anisotropic quantum Hall gas is investigated based on the Hartree-Fock approximation in the same formalism. Thermodynamic properties, transport properties, and unusual response under external modulations are found. Implications for the integer quantum Hall effect in the finite systems are also studied and a new quantum Hall regime with non-zero longitudinal resistance is shown to exist.

I. INTRODUCTION

A basic physical law is described by a basic equation which includes usually a fundamental physical constant. The following constants are such parameters and play important roles in modern physical science. These constants make

| Physical Constant | Symbol | Physical law |
|-------------------|--------|-------------|
| electric charge   | $e$    | unit of charge |
| light velocity    | $c$    | special relativity |
| Plank constant    | $h$    | quantum mechanics |

the physics of matter, physics of space and time, and physics of microscopic world different completely from the classical world. It is important to measure them precisely.

First constant, $e$, is the electric charge of the electron, and is the unit of charge. Charge of any object is known to be integer multiple of $e$. Millikan found the fact that the charge is not continuous but is quantized. Atom is composed of a nucleus and electrons. Nucleus has positively quantized charge. So by combining nucleus with suitable number of electrons, the system becomes neutral. The fact that the charge is quantized is important for the neutrality of matter.

Concerning second constant, $c$, the fact that the light velocity is finite was known for a long time. It was discovered that the light velocity is universal in the last century. Velocity of light which is emitted from moving matter is $c$ and the velocity of light that is measured by moving apparatus is also $c$. This constant light velocity plays critical roles for special relativity.

The third constant is Plank constant, $h$, which has the unit of action and plays the important role in microscopic world. When the light of a frequency $\nu$ interacts with microscopic systems, the light has an energy, $h \nu$. The light behaves like a particle in microscopic world. The electron which is a particle in classical mechanics becomes a wave of wave length $\lambda$. The Plank constant determines the wave length, $\lambda$, of the electron of definite momentum, $p$, as $\lambda = h/p$. Particle is compatible with wave in quantum mechanics.

Physics which is connected with all these physical constants, $e$, $c$, and $h$ may reveal fundamental dynamical principle of nature. One area of physics which is connected with all of them is quantum electrodynamics (QED). QED has a long history and has been well established by now. However QED is still an important topic of fundamental physics and may shed a new light for understanding our nature. A unique parameter of QED is the fine structure constant, $\alpha$, defined by

$$\alpha = \frac{e^2}{2\varepsilon_0 c h},$$ (1)

where $\varepsilon_0$ is the dielectric constant of vacuum. Another area of physics which is connected with the fine structure constant is the quantum Hall effect (QHE). The QHE is the phenomenon in semiconductors and is connected with fundamental science.
A. Fine structure constant of Quantum Electrodynamics (QED)

QED is the quantum field theory of electron and photon which satisfies requirement of relativistic invariance, and the foundation of electromagnetic force is given from QED. In QED, electron field describes the electron and satisfies relativistic Dirac equation. Dirac equation is the relativistic extension of Schrödinger equation and has a form of first order differential equation with Hermitian Hamiltonian. Hence probability is preserved as is in ordinary quantum mechanics. Peculiar feature of Dirac equation is that the equation has negative energy solutions in addition to positive energy solutions. Transition of electron in positive energy state to that in negative energy state by emitting a photon should have a finite probability if the corresponding negative energy state is empty. Thus a particle state is unstable. If all the negative energy states are occupied in the vacuum and only one electron is allowed in one state, then the stability is ensured. From this idea, Dirac is lead to introduce Dirac sea. Namely the vacuum is not empty but is full of electrons of negative energy states. All the negative energy states are filled with electrons in the vacuum. Then the transition of electron of positive energy state into the electron of negative energy state and photon does not occur. The stability is ensured. Because vacuum is full of negative energy electrons, a hole of Dirac sea behaves like a particle of a positive charge. This particle is anti-particle of electron and is called positron.

Photon is described by Maxwell equations that is invariant under Lorenz transformation and has no negative energy solution. But each mode of a definite wave vector is equivalent to a harmonic oscillator and in quantum theory each mode has finite energy due to zero point oscillation. Thus the ground state, vacuum, of QED is not actually real empty state but has rich ingredients of negative energy electrons and zero point oscillation of photons. Physical phenomena which are generated from these dynamical freedoms present characteristic features of nature in the microscopic world.

One example of physical quantity which is generated from the vacuum fluctuation is seen in an isolated electron in vacuum. One electron has influences from these rich dynamical effects of vacuum. Mass and electric charge are those physical quantities that are affected from these effects. If we knew both of bare values and the real values of these quantities, it would have been possible to observe the interaction effect. However in the real world the electron always interacts with photon and has an influence from the electrons in negative energy states. Bare values of mass and charge are unobservable. Therefore mass and charge are not appropriate to see the vacuum fluctuation effects. The anomalous magnetic moment of the electron is one of the physical quantities that is due to vacuum fluctuation effect. In fact, the anomalous magnetic moment of the electron gives an important information of the vacuum.

The electron’s anomalous magnetic moment is given by,

\[ \mu = g \frac{e \hbar}{2m_e}, \]

where \( m_e \) is the mass of electron. Here the gyro-magnetic ratio, \( g \), is determined to 2 from Dirac equation, if there is no correction from vacuum fluctuation. The real value of \( g \) is deviated from 2 slightly and the \( g-2 \) is due to vacuum fluctuation. The \( g-2 \) is computed theoretically as a power series of the fine structure constant in QED and is measured experimentally with extraordinary precision.

We compare the theoretical value of \( g-2 \) with the experimental value. From higher order perturbative calculation, Kinoshita has obtained,

\[ g - 2 = \frac{\alpha}{2\pi} - 0.328478965(\frac{\alpha}{\pi})^2 + 1.17562(56)(\frac{\alpha}{\pi})^3 + (-1.472(152)(\frac{\alpha}{\pi})^4 + 4.46 \times 10^{-12}. \]

Here, \( \alpha \) is the fine structure constant and the last term in the right-hand side of the above equation is the estimates from weak interaction. Experiment by Van Dyck et al. uses an electron in Penning trap for measuring electron’s magnetic moment. The value of \( g-2 \) is given as,

\[ g - 2 = 1159652188.4(4.3) \times 10^{-12}. \]

Comparing the experimental value of \( g - 2 \) with the theoretical value, we have the \( \alpha^{-1} \) from \( g - 2 \),

\[ \alpha_{g - 2}^{-1} = 137.03599976(50)(3.7ppb). \]

On the other hand, the fine structure constant is determined directly by quantum Hall effect (QHE),

\[ \alpha_{QHE}^{-1} = 137.0360037(33)(0.024ppm). \]

The value of fine structure constant obtained from the electron’s magnetic moment is slightly different from the value that has been obtained from QHE. Actually QHE gives the most accurate value of direct measurement of \( \alpha \). Now
this difference is too small to conclude something definite, but in future it may become possible. So it is important to clarify if QHE gives the precise value of $\alpha$.

QHE’s are physical phenomena in two dimensional electrons with a strong magnetic field realized in semiconductors and are connected also with the above fundamental constants. Integer quantum Hall effect (IQHE) and fractional quantum Hall effect (FQHE) were found nearly twenty years ago and many other interesting phenomena have been found since then. Study of new phase of QHE discovered recently and its implication to the precise determination of the fine structure constant are presented in the present paper.

B. Quantum Hall effects

In interface of special semiconductors such as GaAs-AlGaAs heterojunctions or Silicon MOSFET, electron’s motion are restricted to a plane. When strong perpendicular magnetic field is applied, electron’s two dimensional motion is frozen due to magnetic field. As was solved by Landau first, electron’s energy becomes discrete and degeneracy of each energy level per area, density $\rho_0$, is proportional to the magnetic field,

$$E_l = \frac{\hbar e B}{m} (l + \frac{1}{2}),$$

$$\rho_0 = \frac{eB}{2\pi\hbar},$$

(7)

where $B$ is the magnetic field and $m$ is the electron’s mass and $l$ is a non-negative integer. In semiconductors there exist disorders which make these electrons been bound. Electrons are localized around disorders and do not carry electric current if their energies are away from one of the above discrete values. Conversely electrons with the energy of the above discrete values are not localized but are extended. Because the electrons in the extended states carry electric current, Hall conductance changes its value with a Fermi energy if the Fermi energy is in the energy region of extended state. If the Fermi energy is in the the localized state region, the Hall conductance stays to constant value. The Hall conductance at plateau takes very special quantized value. This is the IQHE. In high mobility samples, on the other hand, interactions make electron system form new condensed states. From these effects Hall resistance and longitudinal resistance are varied as shown in Fig. 1(a). Many structures are seen. When the longitudinal resistance vanishes the Hall resistance is quantized. Vanishing longitudinal resistance means that the ground state at plateau has energy gap. The ground states of the IQHE and the FQHE have the energy gap and are incompressible. In the former system cyclotron energy gap is the origin of gap and in the latter system interactions among electrons is the origin of energy gap.

![FIG. 1: Experiments of the Hall resistance and longitudinal resistance. (a) In IQHE and FQHE, the Hall resistance is quantized as $(\frac{e^2}{h} n)^{-1}$ or as $(\frac{e^2}{h} \frac{n}{q})^{-1}$ and longitudinal resistance vanishes. Compressible gas phases are seen at $\nu = \frac{1}{2}$, $\nu = \frac{3}{2}$ and at higher Landau levels. (b) Huge anisotropic resistances are seen at around half filled higher Landau levels. The solid line shows the resistance for the current in vertical direction and the dashed line show the resistance for the current in horizontal direction.](image)

At plateau Hall conductance is given as

$$\sigma_{xy} = \frac{e^2}{h} n,$$

(8)
around $\nu = n(\text{integer})$, or as

$$
\sigma_{xy} = \frac{e^2 q}{h p},
$$

around $\nu = \frac{2}{p}$, where $\nu = \rho/\rho_0$ and $p$ and $q$ are coprime integers. Here $\rho$ is the electron’s density. The former is the IQHE and the latter is FQHE. The value is quantized exactly despite the fact that the system has disorders and interactions. There is a special reason why quantization is exact in these systems. The Hall conductance is a topological invariant of the mapping defined by propagator of the electrons in Landau levels. This is a reason why the Hall conductance is quantized in the system of interactions and impurities and is given in the next section.

In addition to these quantum Hall states, compressible quantum Hall states have been seen at around $\nu = 1/2$ and $\nu = 3/2$ and higher Landau levels. The compressible states at $\nu = 1/2$ and $\nu = 3/2$ are isotropic and expressed by composite Fermion theory. A new exotic phenomenon at higher Landau levels, where longitudinal resistance $\rho_{xx}$ does not vanish and the Hall resistance $\rho_{xy}$ is not quantized, was found recently as in Fig. 1(b). Extremely anisotropic resistance, where $\rho_{xx}$ is very different from $\rho_{yy}$, was found. This phenomenon is caused by new many-body state which has no energy gap and behaves like anisotropic gas. Although bare kinetic energy was frozen due to magnetic field, effective kinetic energy is generated by interactions in the Hall gas phase.

We concentrate to study on the anisotropic quantum Hall gas phase and its implications to the metrology of QHE. In the anisotropic quantum Hall gas, one-particle kinetic energy which violates orientational and translational symmetries is generated. To study them we apply a Hartree-Fock mean field theory using a special representation which is symmetric in two directions. This representation, the von Neumann lattice representation, is convenient to study field theory of the Quantum Hall systems. Spectrum of Nambu-Goldstone mode and response of the system under external density modulation are found. Hall gas has several peculiar thermodynamic properties which are connected with spontaneous generation of the kinetic energy. As will be seen later in the present work, these properties guarantee a stability and precision of the IQHE in the realistic systems. It is shown that the precise determination of the fine structure constant is possible from QHE.

This paper is organized in the following manner.

In Section 2, the von Neumann lattice representation of non-commutative coordinates and formulation of field theory of quantum Hall systems, i.e., systems of two dimensional electrons in the magnetic field, based on the von Neumann lattice representation are presented. Hamiltonian and current operators are expressed with the electrons in the von Neumann lattice representation and Ward-Takahashi identity are derived. Using these relations low energy theorem on the Hall conductance is derived. The Hall conductance is written as a special topological invariant of the mapping defined by the propagator of Landau levels and agrees with integer multiple of $\frac{2}{h}$ if certain boundary condition is satisfied. Values of the topological invariant in the system of disorders, interactions, and periodic potentials are found.

In Section 3, a mean field theory of anisotropic Hall gas (stripe) is presented. A one-particle kinetic energy is generated by interactions and a Fermi surface of the Hall gas in the von Neumann lattice is identified. Thermodynamic properties and transport properties are reviewed. Several spatial symmetries such as translations and a rotation are broken in the present mean field. Nambu-Goldstone zero energy excitations connected with these symmetry breaking are implied by Goldstone theorem. The excitations are found to have a peculiar low energy behavior based on the single mode approximation. Easy direction, in which the resistance is extremely smaller than the other direction, becomes orthogonal to the external density modulation of long wave length. This behavior is opposite to that of ordinary charge density wave of zero magnetic field.

In Section 4, implications of unusual thermodynamic properties of the Hall gas are summarized. Due to interactions or disorders, strip of compressible states are formed.

In Section 5, implications of stripe and a strip of the Hall gas to IQHE in realistic systems are presented. It is pointed out that the precision and stability of the IQHE in the realistic systems are derived from unusual properties of the Hall gas. As the external current increases, several transport regimes such as quantum Hall regime (QHR), dissipative QHR, collapse, and breakdown appear.

Summary is given in Section 6.

II. THE VON NEUMANN LATTICE REPRESENTATION AND TOPOLOGICAL EXPRESSION OF THE HALL CONDUCTANCE

Under the perpendicular magnetic field, the electron’s motion in the planar space becomes different from that of the zero magnetic field. The representation which preserves characteristic features of the system such as discrete energy with finite degeneracy that is proportional to the magnetic field, magnetic translational invariance, and non-commutative guiding center coordinates is convenient to develop field theories. The von Neumann lattice
representation is such representation that is symmetric in two directions and is invariant under lattice translations. Universal identities are used to give a proof of the IQHE.

A. The von Neumann lattice representation

One-body Hamiltonian for the two-dimensional electrons in the perpendicular magnetic fields is given as,

\[ H_0 = \frac{(\mathbf{p} + e\mathbf{A})^2}{2m}, \]
\[ \partial_x A_y - \partial_y A_x = B. \]  
(10)

Since the velocity is connected with the momentum,

\[ v_x = \frac{(p + eA)_x}{m}, \]
\[ v_y = \frac{(p + eA)_y}{m}, \]
(11)

components of the velocity satisfy

\[ [v_x, v_y] = -\frac{i\hbar eB}{m^2}. \]
(12)

The above Hamiltonian has a form that is equivalent to the Hamiltonian without magnetic field if it is written by velocity. Solutions of classical equation of motion with integration constants, \((X, Y)\) are,

\[ x = \frac{v_y}{\omega_c} + X, \]
\[ y = -\frac{v_x}{\omega_c} + Y, \]
(13)

where \(\omega_c = eB/m\). This is a circular motion around the center \((X, Y)\) as shown in Fig. 2 and the center coordinates are the constants of motion.

![Fig. 2: A schematic illustration of electron's circular motion around the center coordinates \((X, Y)\). The relative coordinates \((\xi, \eta)\) are \((x - X, y - Y)\).](image)

In quantum mechanics, the center coordinates, \((X, Y)\), become operators but satisfy,

\[ [X, v_x] = [X, v_y] = 0, \]
\[ [Y, v_x] = [Y, v_y] = 0, \]
(14)

hence they commute with the \(H_0\) also. They are the constants of motion as in classical mechanics. Since the commutation relation between \(x\) and \(y\), which vanishes, is given as

\[ [x, y] = \frac{[v_x, v_y]}{\omega_c^2} + [X, Y], \]
(15)
and the velocity operators do not commute each other due to the magnetic field, the center coordinates become also non-commutative. They satisfy

\[ [X, Y] = -\frac{[v_x, v_y]}{\omega_c^2} = \frac{i\hbar}{eB}. \] (16)

Hence it is impossible to diagonalize both center variables same time. They satisfy uncertainty relations. The von Neumann lattice coherent state is an eigestate of the operator, \( X + iY \), with a complex eigenvalue, \( z_{mn} = a(m\omega_x + n\omega_y) \), where \( a = \sqrt{\frac{2\pi \hbar}{eB}} \) and \( m \) and \( n \) are integers. A set of coherent states becomes complete under the condition, \( \text{Im}(\omega_x^* \omega_y) = 1 \). They are symmetric in two-directions and has minimum uncertainty allowed by the commutation relation.

The von Neumann lattice coherent state is constructed by \( A = \sqrt{\frac{x}{a}}(X + iY) \) and its conjugate as,

\[ |\alpha_{m,n}\rangle = \exp \left( i\pi(m + n + mn) + \pi^{1/2}(A^\dagger \frac{z_{mn}}{a} - A \frac{z_{mn}}{a}) \right) |0\rangle, \]

\[ [A, A^\dagger] = 1, \] (17)

and satisfy

\[ \langle \alpha_{m+m',n+n'}| \alpha_{m',n'} \rangle = \exp \left( i\pi(m + n + mn) - \frac{\pi}{2} \left| \frac{z_{mn}}{a} \right|^2 \right). \] (18)

The above matrix elements of two coherent states have translational invariant form. Hence momentum states are defined by Fourier transformation:

\[ |\alpha_p\rangle = \sum_{m,n} e^{ip_x m + ip_y n} |\alpha_{m,n}\rangle, \] (19)

\[ \langle \alpha_p| \alpha_{p'} \rangle = \gamma(p) \sum_N (2\pi)^2 \delta(p - p' - 2\pi N), \] (20)

where two components of \( N, N_x \) and \( N_y \), are integer and the fundamental region of \( p \) is defined as, \( |p_x| \leq \pi, |p_y| \leq \pi \).

The normalization constant is given as,

\[ \gamma(p) = \beta(p)^* \beta(p), \] (21)

\[ \beta(p) = (2\text{Im} \tau)\frac{4}{\pi} e^{\tau p_x^2/2}\vartheta_1 \left( \frac{p_x + \tau p_y}{2\pi} \right), \] (22)

\[ \tau = \frac{\omega_x}{\omega_y}, \] (23)

where \( \vartheta_1(z|\tau) \) is a theta function of the first kind, and \( \beta(p) \) obeys a nontrivial boundary condition

\[ \beta(p + 2\pi N) = e^{i\phi(p,N)} \beta(p), \] (24)

where \( \phi(p,N) = \pi(N_x + N_y) - N_y p_x, \). The norm, \( \gamma(p) \), vanishes once in the inside of fundamental region, at \( p = 0 \).

The relative coordinates, \( (\xi, \eta) \), are the coordinates of the electron measured from the center, \( (X, Y) \), and are proportional to the velocity operators,

\[ \xi = x - X = \frac{v_x}{\omega_c}, \quad \eta = y - Y = -\frac{v_x}{\omega_c}. \] (25)

and commute with the center coordinates, \( (X, Y) \). Eigenstates of \( H_0 \) are those of harmonic oscillator with discrete eigenvalues, \( E_l \) of Eq. (7),

\[ H_0|f_l\rangle = E_l|f_l\rangle, \] (26)

For basis of two dimensional electrons in the magnetic field, we use the direct product between eigenstates of \( H_0 \) of eigenvalue \( E_l \), \( |f_l\rangle \), and the above momentum states,

\[ |l, p\rangle = |f_l\rangle \otimes |\alpha_p\rangle. \] (27)
B. Field theory

Using the von Neumann lattice basis we develop a field theory of the quantum Hall system. Hamiltonian is given as a summation of the free term, $H_0$ and the interaction term, $H_{int}$,

$$ H_{total} = H_0 + H_{int}, \quad (28) $$

$$ H_0 = \int d^2x \psi^\dagger(x)H_0\psi(x), \quad (29) $$

$$ H_{int} = \frac{1}{2}\int\frac{d^2k}{(2\pi)^2}\rho(k)V(k)\rho(-k), \quad (30) $$

where $\rho(k)$ is the density operator defined as,

$$ \rho(k) = \int d^2xe^{-ik\cdot x}\psi^\dagger(x,t)\psi(x,t) \quad (31) $$

and $V(k)$ is the Coulomb potential, $V(k) = q^2\frac{2\pi}{k}$, $q^2 = \frac{\epsilon^2}{4\pi}$, $\epsilon$ is the dielectric constant of matter. The disorder term, $H_{disorder}$, is added later.

Action of the system is given as,

$$ S = \int dt d^2x [\psi^\dagger(x,t)i\hbar\frac{\partial}{\partial t}\psi(x,t)] - H_{total}, \quad (32) $$

and is written as,

$$ S = \int dt \int\frac{d^2p}{(2\pi)^2}[b_l^\dagger(p,t)i\hbar\frac{\partial}{\partial t}b_l(p,t)] - H_{total}, \quad (33) $$

when electron field is expanded with the above basis as,

$$ \psi(x,t) = \int\frac{d^2p}{(2\pi)^2}\sum_{l=0}^{\infty} b_l(p,t)\langle x|l,p \rangle, \quad (34) $$

where BZ in the integration region in momentum stands for the integration in the fundamental region. For simplicity, we omit $t$ in the operators unless it is confusing hereafter. Expansion coefficients, $b_l(p)$, and their conjugates, $b_l^\dagger(p)$, are the annihilation and creation operators of electrons in Landau levels which satisfy equal time anti-commutation relations,

$$ \{b_l(p), b_{l'}(p')\} = \delta_{l,l'}\sum_N (2\pi)^2 \delta(p - p' - 2\pi N)e^{i\phi(p',N)}. \quad (35) $$

 Operators $b_l^\dagger(p)$ and $b_l(p)$ operate on the many-body states and satisfy a torus boundary condition with a phase factor, $\phi(p,N) = \pi(N_x + N_y) - N_y p_x$, in the momentum space.

The free many-body Hamiltonian becomes diagonal and is given as,

$$ H_0 = \sum_{l=0}^{\infty}\int\frac{d^2p}{BZ}(2\pi)^2 E_l b_l^\dagger(p)b_l(p). \quad (36) $$

Due to degeneracy of Landau levels, one-particle energy does not depend on its momentum. The density operator is expressed as,

$$ \rho(k) = \sum_{l,l'=0}^{\infty}\int\frac{d^2p}{BZ}(2\pi)^2 b_l^\dagger(p)b_{l'}(p+a\hat{k})f_{l'\dagger}(k) $$

$$ \times \exp[i\frac{a\hat{k}_x(2p_y + a_ky)]}{4\pi}, \quad (37) $$

$$ j(k) = \sum_{l,l'=0}^{\infty}\int\frac{d^2p}{BZ}(2\pi)^2 b_l^\dagger(p)b_{l'}(p+a\hat{k})\langle f_l|\frac{1}{2}\{v_x e^{-i(k_x\xi + k_y\eta)}\}|f_{l'} \rangle $$

$$ \times \exp[i\frac{a\hat{k}_x(2p_y + a_ky)]}{4\pi}, \quad (38) $$

$$ f_{l'\dagger}(k) = \langle f_l|e^{-i(k_x\xi + k_y\eta)}|f_{l'} \rangle \quad (39) $$

here, \( \hat{k}_i = W_{ij}k_j \) with the matrix defined by

\[
W = \begin{pmatrix} \text{Re}[\omega_x] & \text{Im}[\omega_x] \\ \text{Re}[\omega_y] & \text{Im}[\omega_y] \end{pmatrix}.
\] (40)

For rectangular lattice \( \omega_x = r_x, \omega_y = \frac{i}{r_x} \), where \( r_x \) is an asymmetry parameter. It is convenient to introduce a unitary matrix,

\[
U_{lt'}(p) = \langle f_l | e^{i(p_s \xi + \tilde{p}_t \eta)/a - \frac{t' - t}{2} p_s p_t} | f_{l'} \rangle,
\]

here \( \tilde{p}_t = W_{ij}^{-1}p_j \), and to transform electron operators as,

\[
\tilde{b}_l(p) = \sum_{l'} U_{l'l}(p) b_l(p).
\] (42)

The density operator is written in the diagonal form and the current operator are written also in simple forms with the transformed operators,

\[
\rho(k) = \int_{BZ} \frac{d^2 p}{(2\pi)^2} \sum_l \tilde{b}_l^\dagger(p) \tilde{b}_l(p + a\hat{k}),
\]

\[
j(k) = \int_{BZ} \frac{d^2 p}{(2\pi)^2} \sum_{l,l'} \tilde{b}_l^\dagger(p) \tilde{f}_l \tilde{b}_{l'}^\dagger(p + a\hat{k}),
\]

where \( \hat{v} \) is the velocity operator and its components are proportional to the relative coordinates. Commutation relation of charge density with the electron operators is equivalent to that of local field theory when the transformed operators are used and is given as,

\[
[\rho(k), \tilde{b}_l(p)] = -\tilde{b}_l(p + a\hat{k}),
\] (44)

if the momentum \( k \) and \( p \) are in the fundamental region. From these relations, Ward-Takahashi identity\(^\text{14}\) between the vertex part and the propagator is derived.

Let us define the one-particle irreducible vertex part, \( \tilde{\Gamma}^\mu \) from time ordered product as,

\[
\int dz_0 dx_0 dx_0' e^{i q_0 z_0 - i p x_0 + i p_0' x_0'} \langle T(j^\mu(z_0, q) \tilde{b}_l(x_0, p) \tilde{b}_l^\dagger(x_0', p')) \rangle = (2\pi)^3 \delta(p + Q - p') \tilde{S}_{ll'}(p) \tilde{\Gamma}^\mu_{l'l'}(p, p + Q) \tilde{S}_{ll'}(p + Q),
\] (45)

where \( j_\mu(x) = (\rho(x), j(x)) \) and \( Q^\mu = (q_0, a_q x, a_q y) = t^\mu q^\nu \) is a linear combination of \( q^\mu \) and \( \tilde{S} \) is the full propagator defined by

\[
\int dx dx' e^{ip x - i p' x'} \langle T(\tilde{b}_l(x) \tilde{b}_l^\dagger(x')) \rangle = (2\pi)^3 \delta(p - p') \tilde{S}_{l'l'}(p).
\] (46)

From the current conservation and the commutation relation, the Ward-Takahashi identity\(^\text{14}\) between \( \tilde{S} \) and \( \tilde{\Gamma}^\mu \),

\[
\tilde{\Gamma}^\mu(p, p) = t^\nu_{\mu} \frac{\partial \tilde{S}^{-1}(p)}{\partial p^\nu},
\] (47)

is satisfied. Using the current correlation function,

\[
\pi^\mu\nu(q, q') = \int dx dx' e^{i q x - i q' x'} \langle T(j^\mu(x) j^\nu(x')) \rangle = (2\pi)^3 \delta(q - q') \pi^\mu\nu(q) + \pi^\mu_{(2)}(q, q'),
\] (48)

the Hall conductance is defined by the slope of the momentum conserving part, \( \pi^\mu\nu(q) \), at the origin and is written as

\[
\sigma_{xy} = \frac{e^2}{3!} \epsilon_{\mu\nu\rho} \partial_\rho \pi^\mu\nu(q)|_{q=0},
\] (49)

where \( \epsilon_{\mu\nu\rho} \) is anti-symmetric tensor. As we will see later the momentum non-conserving term, \( \pi^\mu_{(2)}(q, q') \), does not contribute to the conductance.
Using the above relations we are able to write the Hall conductance as a particular form of a topological invariant defined,

$$\sigma_{xy} = \frac{e^2}{h} N_w, \quad (50)$$

$$N_w = \frac{1}{24\pi^2} \int_{BZ \times S^1} d^3p \epsilon_{\mu \nu \rho} \text{tr} \left( \partial_\mu \tilde{S}^{-1}(p) \tilde{S}(p) \partial_\nu \tilde{S}^{-1}(p) \tilde{S}(p) \partial_\rho \tilde{S}^{-1}(p) \tilde{S}(p) \right). \quad (51)$$

In the above equation, the integration region in the spatial component is a torus defined from BZ and the integration region in the energy variable is equivalent to $S^1$ when the propagator has imaginary part that is defined from the Fermi energy. The final formula is valid in almost arbitrary quantum Hall systems of interactions and disorders.

It is rather trivial to derive the final expression Eq. (51) in the free system. In the free system, the slope of the current correlation function at the origin is given by the one loop diagram with three vertices with zero external momenta, as shown in Fig. 3. Using the Ward-Takahashi identity, the vertex parts are replaced with the derivatives of the inverse of the propagator, Eq. (47). So we end up with the final formula, Eq. (51). In general systems that have interactions and disorders, it is not so trivial to derive this formula that further arguments will be given.

![Feynman diagram of three external current lines with zero momentum in the lowest order. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current.](image)

FIG. 3: Feynman diagram of three external current lines with zero momentum in the lowest order. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current.

$N_w$ is a winding number of the mapping from momentum space to the space of matrix defined by the propagator, $\tilde{S}(p)$, and agrees with an integer if the integrand is single-valued and has no singularity in the integration region. This condition is satisfied in the gap region where the ground state is isolated in the energy and excited states have energy gaps and in the localized state region where one-particle states have localized wave functions and discrete energies as well.

![Feynman diagram of three external current lines with zero momentum in the second order correction. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current. The dashed line shows the Coulomb interaction and one of the vertex parts is modified.](image)

FIG. 4: Feynman diagram of three external current lines with zero momentum in the second order correction. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current. The dashed line shows the Coulomb interaction and one of the vertex parts is modified.

C. Interactions

In the systems of interactions, it is highly non-trivial to see the fact that the Hall conductance is written by the topological winding number of the propagator. We give a proof of this fact by treating the interaction term...
perturbatively. The electron propagator is defined by $\mathcal{H}_0$ and corrections are caused by interaction Hamiltonian. To use diagrammatic analysis, we assign a solid line for the electron and a dashed line for a longitudinal photon, which expresses Coulomb interaction.

The current correlation function, $\pi_{\mu\nu}(q,q')$, is expressed with electron propagator and vertex parts defined by the current operators and others. It was shown in the previous subsection that the slope of lowest order amplitude was obtained by the above topological formula, if Ward-Takahashi identity is used. Corrections are calculated by using perturbative expansions with respect to $\mathcal{H}_{int}$. The first order correction to the current correlation function and its slope at the origin of the momentum are given in Feynman diagrams of Fig. 4. Thus the total amplitude is written by using the propagator and the vertex part. By the Ward-Takahashi identity, they are connected each other and the slope of the total amplitude is written by the topological winding number of the full propagator.

FIG. 5: Feynman diagram of three external current lines with zero momentum in the second order correction. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current. The dashed line shows the Coulomb interaction and one of the electron propagators is modified.

Similarly the slope of the current correlation function at the origin in higher orders is classified to diagrams of two different topologies. In first type of diagrams, three vertices are attached to three different electron lines, as is given in Fig. 6. Using the Ward-Takahashi identity, integrand of these amplitudes can be written as the third derivative with respect to all three components of internal electron momentum. These amplitudes are symmetric in the Lorenz indecis $\mu$, $\nu$, and $\rho$ of the currents and do not contribute to the asymmetric part in $\mu$, $\nu$, and $\rho$. Since the Hall conductance is the asymmetric part, the Hall conductance is not affected at all from this kind of amplitude.

FIG. 6: First type of Feynman diagram of three external current lines with zero momentum in higher order corrections. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current. The dashed line shows an electron interaction which is connected with all electron propagators.

In second type of diagrams, three external vertices are attached to one electron line which include all higher order effects, as is given in Fig. 7. So this amplitude is written by the full propagator and full vertices. Since the current conservation and the equal time commutation relation between the charge density and electron operators are satisfied, the Ward-Takahashi identity between the full vertex part and the full propagator is satisfied as well. Hence the slope of the amplitude of the second class is written by the topological invariant Eq. (51) of the full propagator.

It is possible to extend this theorem to general systems which have other dynamical freedoms than the electrons such as phonon and others. Namely the Hall conductance is written by the same topological expression in quite general systems. For later convenience, we study the current correlation function with two different momenta in the currents, as is given in Fig. 8.
FIG. 7: Second type of Feynman diagram of three external current lines with zero momentum in higher order corrections. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current. The dashed lines correspond to interactions. The propagators and the vertices are modified by interactions and satisfy extended Ward-Takahashi identity.

FIG. 8: Feynman diagram of momentum non-conserving current correlation function with small different momenta of the currents. The solid lines correspond to the electron propagator and the wavy lines correspond to the electromagnetic current. The dashed lines correspond to dynamical freedoms which are carrying momentum and the momentum are not conserved.

Let us define such current correlation function, $\pi^{\mu\nu}_{(2)}(q, q')$, which is a function of two momenta $q_\mu$ and $q'_\mu$ and satisfies the identities,

$$q_\mu \pi^{\mu\nu}_{(2)}(q, q') = \pi^{\mu\nu}_{(2)}(q, q') q'_\nu = 0.$$  \hfill (52)

Since the two momenta are independent, it is possible to differentiate these equations with respect to $q^\rho$ or $q'^\rho$. Then we have,

$$\pi^{\rho\nu}_{(2)}(q, q') + q_\mu \partial_\rho \pi^{\mu\nu}_{(2)}(q, q') = 0,$n\pi^{\rho\nu}_{(2)}(q, q') + q'_\mu \partial_\rho \pi^{\mu\nu}_{(2)}(q, q') = 0.$$  \hfill (53)

Thus if the amplitude is smooth around the origin in two momenta and the first derivatives are finite, the amplitude is proportional to two momenta, $q^\mu$, and $q'^\mu$. Thus these amplitudes do not contribute to the linear slope at the origin. If a momentum conserving amplitude is written as a limit of momentum non-conserving amplitude, this amplitude also does not contribute to the linear slope.

In situations where the ultraviolet divergences are involved in perturbative expansions, the renormalized propagator and vertex are to be used. They are defined in such manner that satisfy the Ward-Takahashi identity, and thus the Hall conductance is written by the topological expression of the renormalized propagator. Thus the quantized Hall conductance is not affected by interactions.\hfill (15,16)

D. Random disorders

In systems of disorders, translational invariance is lost and momentum becomes not a good quantum number. It seems that the momentum conserving term does not exist in the system of disorders. However we will see that it is not the case and the momentum conserving term exists. To study disorder effects, we use the diagrammatic analysis. It is
shown that the above topological expression is valid also. For the bulk quantity such as conductance which is defined by a ratio between total voltage and total current, momentum conserving term contributes. Momentum becomes a good quantum number even in the systems of disorders, as far as the bulk physical quantities are concerned.

If the momentum is not conserved, the current correlation function, $\pi^{\mu\nu}(q, q')$, becomes a function of two momenta and satisfies the identities, Eq. (52), and Eq. (53). As was seen in the previous part, if the amplitude is smooth around the origin in two momenta and the first derivatives are finite, the amplitude is proportional to two momenta, $q^{\mu}$ and $q'^{\nu}$. The condition of the smoothness around the origin of the momenta is equivalent to the absence of infrared divergence and is satisfied in the energy gap region and in the localized state region since one-particle states have the energy gap or the one-particle wave functions have finite spatial extensions.

Disorder potentials is expressed in the Hamiltonian:

$$H_{\text{dis}} = \int d^2 k \rho(k) V_{\text{dis}}(k).$$

(54)

Spectrum of $H_0 + H_{\text{dis}}$ becomes different from that of $H_0$ and depends on the potential $V_{\text{dis}}$. For random potentials, energy eigenstates appear in the energy regions between the Landau levels. From numerical study it is known that these states have discrete energy and localized wave functions. So they do not contribute to conductance. Hence when the Fermi energy is located in this region, we treat random potential $H_{\text{dis}}$ perturbatively in the calculation of conductance. For the case of periodic potentials, perturbative treatment is not good and nonperturbative treatments for the calculations of the energy spectrum and conductance are to be made.

Now we give a proof that the momentum non-conserving term does not contribute to the conductance. Systems with the random potentials are studied in the following. Expectation value of the current density is connected with the current correlation function as,

$$j_i(x) = \int d^2 x' \pi_{i0}(x, x') A_0(x').$$

(55)

The total current is obtained by integrating the current density in whole spatial region,

$$I_i = \int d^2 x j_i(x).$$

(56)

Let us substitute the Fourier transformation of the current correlation function and the scalar potential for the coordinate space to the momentum space. Then we have,

$$I_i = \frac{\partial}{\partial q'_p} \pi_{i0}(q, q')|_{q=q'=0} V,$$

(57)

where $\rho$ is different from $i$ and $V$ is the potential difference expressed with $A_0$. Consequently the coefficient $\frac{\partial}{\partial q'_p}$ is proportional to the linear term in the momentum. Because the momentum non-conserving term has higher power than linear, the non-conserving term does not contribute to the conductance.

Now $H_{\text{dis}}$ induces a finite flow of momentum and external momenta carried by the two currents is not conserved generally. This amplitude depends on two independent momenta and behaves as bilinear in the momenta. Hence the momentum non-conserving term does not contribute to the conductance. The momentum conserving terms are generated by cancellation of the momentum flow due to disorders and contribute to the conductance. These momentum conserving terms are similar to those of the system of interactions in which dynamical freedoms carry the momentum and the momenta of the currents disagree. Hence by introducing new interaction terms corresponding to each sets of momentum conserving diagrams, the previous arguments of applying current conservation, commutation relation, and Ward-Takahashi identity are applicable also in the systems of disorders.

### E. Periodic potential

If the potential is periodic in space, one-particle states form band structures. Since magnetic field gives a periodic structure by itself, which can be seen with von Neumann lattice representation clearly, band structure is very sensitive to the ratio of the potential period with the magnetic length. Peculiar structure was identified in one particle spectrum. Hall conductance behaves also in a peculiar way.

Generally, a periodic potential lifts the degeneracy of the Landau level. In what follows, we assume that the potential lattice is formed by two linear-independent basis vectors with integer coefficients. The periodic potential
with this property is written as

$$V(x) = \sum_{N \in \mathbb{Z}} v(x + a_{Nx}w_x^{pot} + a_{Ny}w_y^{pot}),$$

(58)

where $w_x^{pot}, w_y^{pot}$ are the basis vectors of the potential lattice with the components $(\text{Re} \omega_x^{pot}, \text{Im} \omega_x^{pot})$ and $(\text{Re} \omega_y^{pot}, \text{Im} \omega_y^{pot})$. As is well-known, a periodic potential problem in a magnetic field is very sensitive to the flux penetrates a unit cell of the potential lattice. The unit cell means a parallelogram spanned by basis vectors of the potential lattice. If the flux $\Phi$ is a rational multiple of the flux quantum $\Phi_0 = \frac{h}{e}$,

$$\frac{\Phi}{\Phi_0} = \text{Im}[(\omega_y^{pot})^* \omega_x^{pot}]$$

$$= \frac{q}{p},$$

(59)

each Landau band splits into $q$ sub-bands. The fundamental region of the momentum is reduced to one $q$-th. The spectrum in the reduced region of the momentum is $p$-fold degenerate.

When the flux is given by Eq. (59), it is convenient to select the basis vectors of the von Neumann lattice as $\omega_x = \omega_x^{pot}/q, \omega_y = \omega_y^{pot}$. The moduli of the von Neumann lattice becomes $\tau = \tau^{pot}/pq$, where the moduli of the potential lattice is defined by $\tau^{pot} = -\omega_y^{pot}/\omega_x^{pot}$. In this case, the potential energy term in the second quantized form becomes

$$H_{pot} = \frac{1}{q} \sum_{l,l'} \sum_{s=0}^{p-1} \sum_{r=0}^{q-1} \int d^2x v(x) \int_{\text{RBZ}} \frac{d^2p}{(2\pi)^2} b^\dagger_l(p) b_{l'}(p_x - \frac{2\pi r}{q}, p_y)$$

$$\times \langle l, p_x - 2\pi \left(\frac{y}{a} + \frac{s}{p}\right), p_y + 2\pi \frac{a}{q} | x = 0 \rangle$$

$$\times \langle x = 0 | l', p_x - 2\pi \left(\frac{y}{a} + \frac{s}{p} + \frac{r}{q}\right), p_y + 2\pi \frac{a}{q} \rangle, \psi_{l'}(p) \rangle = (E - E_l) \psi_{l'}(p),$$

(61)

The function $\psi_{l'}(p)$ is given by the following form:

$$\psi_{l'}(p) = \psi_1(p_x - 2\pi \frac{r}{q}, p_y).$$

(62)

The equation is solved by diagonalizing a $Lq \times Lq$ matrix, where $L$ is the number of Landau levels. Therefore, each Landau band splits into $q$ sub-bands generally. It is easy to see that the spectrum of the eigenvalue equation is invariant under the translations $p_x \rightarrow p_x + 2\pi n/p$ and $p_x \rightarrow p_x + 2\pi n/q$, where $n, m$ are integers. Since $p, q$ are coprime integers, the above symmetry reads

$$E(p) = E(p_x + 2\pi \frac{n}{pq}, p_y).$$

(63)

Thus, it is proven that the spectrum in RBZ are $p$-fold degenerate.

Next, we consider a system with defects and a periodic potential. To extract properties in such a system, let us suppose a defect in a periodic short-range potential. That is, the potential is given by

$$V(x) = \sum_{N \in \mathbb{Z}} V_0 a^2 \delta(x - a_{Nx}w_x^{pot} - a_{Ny}w_y^{pot}) + ga^2 \delta(x - aM_xw_x^{pot} - aM_yw_y^{pot}),$$

(64)
for integers \( M_x, M_y \). The second term breaks the periodicity of the potential. If we neglect the second term, the eigenvalue equation becomes

\[
\sum_{l' r'} \frac{V_0}{q} (D_l^\dagger(p)D_{l'}(p))_{rr'} \psi_{l' r'}(p) = (E - E_l) \psi_{l r}(p),
\]  

(65)

where \( p \times q \) matrix \( D_l \) is given by

\[
(D_l(p))_{sr} = a \langle x = 0 | l, p_x - 2\pi \left( \frac{s}{p} + \frac{r}{q} \right), p_y \rangle \quad (s = 0, \ldots, p - 1; r = 0, \ldots, q - 1).
\]  

(66)

The spectrum in the LLL consists of flat bands and Hofstadter-type bands. The wave function of the flat band at \( E = E_l \) is given by \( \psi_k = \delta_{kl} Ker(D_l) \).

In Fig. 9 the spectrum for the square lattice potential is shown. We observe that self-similar pattern and large gap above the flat bands exist in this figure. Hofstadter-type bands above flat bands tend to a set of bound states as \( t \) becomes infinity.

Incorporating the defect leads to the following additional term in the L.H.S. of Eq. (65):

\[
g \sum_{l' m'} \int_{\text{RBZ}} \frac{d^2p}{(2\pi)^2} (D_l^\dagger(p))_{rm'} \psi_{l' m'}(q)e^{iqM_x(q_x - p_x) + iS(q_y - p_y)} = 1,
\]  

(67)

where \( \text{Mod}(M_y, p) = s \) and \( (M_y - s)/p = S \). Apparently, flat bands is still flat even if the defect exists. Furthermore, in each sub-band gap of the periodic short-range potential problem a bound state appears rearranging eigenstates of Hofstadter-type bands. The equation of the bound state energies is given by

\[
g \sum_{A} \int_{\text{RBZ}} \frac{d^2p}{(2\pi)^2} \frac{\sum_{l} (D_l(p))_{sr} \psi_{l A}(p)e^{iqM_x(p_x - p_x) + iS(p_y)} \psi_{l A}(q)}{E - E_A(p)} = 1,
\]  

(68)

where \( \psi_{l A}(p) \) and \( E_A(p) \) are the eigenfunction and the eigenvalue of Eq. (65). The R.H.S of the above equation generally becomes infinite when \( E \) approaches to the upper edge of the Landau sub-band. Also, it becomes minus infinite when \( E \) approaches to the lower edge. Thus, there exist a solution in each sub-band gap. The solution corresponds to a bound state trapped at the defect. If many defects exist in the periodic potential, many bound states appear in each the sub-band gap. As the number of defects increases, the sub-band gaps tend to be filled with the bound states and the number of the extended states decreases.
F. Value of topological winding number and integer quantum Hall effect

The winding number of the propagator is a topological invariant and is stable under small changes of the systems. Since the Hall conductance is proportional to the winding number that depends upon the propagator and the position of the Fermi energy, the value is computed from Eq. (50) and Eq. (51) and is given in Fig. 10 as a function of the Fermi energy. The value stays at an integer in finite energy region and has step-like behavior as a whole. This occurs because one-particle energy takes discrete values, $E_i$, and there is no state between these energy values in free theory. Hence the plateau in the $E_F$ does not mean the plateau in the density.

![Graph](image)

FIG. 10: The value of the Hall conductance of the free electron in the magnetic field as a function of the Fermi energy.

The situation becomes different and the plateau with respect to the density appears in the system of disorders because the topological invariant is stable and is unchanged under small changes of the system. In systems with disorders, new one-particle states appear in the energy region between Landau level's energies. They are localized states and do not contribute to the topological invariant. By applying the previous arguments we see that the value agrees exactly with that of free theory if the Fermi energy is in the localized state region. Although these localized states do not contribute to the value of topological invariant, they contribute to the total electron number. Consequently the topological invariant stays at one integer as total electron number is varied. Thus the Hall conductance has plateaus as a function of the electron density in the localized state region as is given in Fig. (11). By treating interactions perturbatively, we see that this is applied also to the system of interactions as far as the ground state is in the same phase.

The propagator is modified completely if the periodic potentials is in the system. The value of the topological invariant becomes also very different from that of the free systems. Our method based on von Neumann lattice representation is useful for its calculations. We refer to our work for more details. It is possible to derive Chern number formula of Thouless et. al. for the periodic system by starting from our formula in periodic systems. Advantage of our formula is that our formula is applicable to the systems with interactions and disorders. Fig. 12 gives the Hall conductance in the periodic system.

In the system of Hall gas where one particle energy is not degenerate but has a dependence upon its momentum, the Hall conductance is unquantized. The value changes monotonically with the electron density as is given in Fig. 13.

III. ANISOTROPIC HALL GAS (STRIPE) STATE

In this section we study an exotic compressible Hall state, anisotropic Hall gas state which is compressible and has huge anisotropic resistances. We apply mean field theory in von Neumann lattice representation which is a convenient representation for studying Hall gas state, since the lattice translational symmetry is preserved in compressible state. Charge density of anisotropic Hall gas, however, will be found to be unidirectional i.e., uniform in one direction and periodic in other direction. So the state is equivalent to unidirectional charge density wave obtained by Koulakov, Fogler, and Shklovskii and Moessner and Chalker at around half-filling of higher Landau levels. The state is called stripe state sometimes. Recently there are investigations on several phases such as unidirectional charge density wave, asymmetric charge density wave, and liquid crystals. Numerical methods on small systems have
been applied also\textsuperscript{28}. We concentrate on the anisotropic quantum Hall gas in this article. We set $\hbar = 1$ in this section for simplicity.

### A. Algebraic property

We study the quantum Hall system with the interaction described by the Hamiltonian,

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}. \quad (69)$$

In this system, the generators of translation in the $y$-direction, translation in the $x$-direction, rotation, and total charge are given as,

$$Q_X = r_s \sum_l \int_{\text{BZ}} \frac{d^2 p}{(2\pi)^2} b_l^\dagger(p) \left( i \frac{\partial}{\partial p_x} - \frac{p_y}{2\pi} \right) b_l(p),$$

$$Q_Y = \frac{1}{r_s} \sum_l \int_{\text{BZ}} \frac{d^2 p}{(2\pi)^2} b_l^\dagger(p) \left( i \frac{\partial}{\partial p_y} \right) b_l(p), \quad (70)$$

$$Q_J = \frac{1}{r_s} \sum_l \int_{\text{BZ}} \frac{d^2 p}{(2\pi)^2} b_l^\dagger(p) \left[ l + \frac{1}{2} + \pi \left( r_s^2 \left( i \frac{\partial}{\partial p_x} - \frac{p_y}{2\pi} \right)^2 + r_s^{-2} \left( i \frac{\partial}{\partial p_y} \right)^2 \right) \right] b_l(p),$$

FIG. 11: The value of Hall conductance of the electrons in the disorder potentials in addition to the magnetic field as a function of the filling factor. Plateau appears in localized states region as a function of electron density.

FIG. 12: The value of Hall conductance of the electrons in the periodic potentials as a function of the Fermi energy for $t = 5/4$. Plateau appears and the value becomes integer multiple of $\frac{e^2}{h}$ in a fractional filling factor.
FIG. 13: The value of Hall conductance of the electrons in the compressible Hall gas region as a function of the filling factor. Plateau disappears and the value changes uniformly with the filling factor.

\[ Q = \sum \int_{BZ} \frac{d^2 p}{(2\pi)^2} b_i^*(p)b_i(p), \]

where \( r_s \) is the asymmetry parameter. From the above expressions it is easy to see that magnetic translation operators are actually the generators of translations in the momentum space, and are called \( K \)-transformation. \( Q_X \) translates the momentum in \( x \)-direction and \( Q_Y \) translates the momentum in the \( y \)-direction. They commute with Hamiltonian,

\[ [Q, \mathcal{H}] = [Q_X, \mathcal{H}] = [Q_Y, \mathcal{H}] = [Q_J, \mathcal{H}] = 0. \] (71)

So they are constants of motion of the present system. They satisfy the algebras,

\[ [Q_X, Q_Y] = \frac{i}{eB} Q, \]
\[ [Q_J, Q_X] = iQ_Y, \]
\[ [Q_J, Q_Y] = -iQ_X, \] (72)

and

\[ [Q_X, Q] = [Q_Y, Q] = [Q_J, Q] = 0. \] (73)

The first commutation relation shows that it is impossible to diagonalize \( Q_X \) and \( Q_Y \) same time. Eigenstate of \( Q_X \) breaks \( Q_Y \)-invariance. These conserved charges are integrals of charge densities, \( j^0(x) \), \( j^i_X(x) \), \( j^i_Y(x) \), \( j^i_J(x) \), of Noether currents which are given as,

\[ j^\mu(x) = \text{Re} (\Psi^\dagger v_\mu \Psi), \]
\[ j^i_X(x) = \text{Re} (\Psi^\dagger v_\mu X \Psi) - \frac{1}{eB} \delta^i_y \mathcal{L}, \]
\[ j^i_Y(x) = \text{Re} (\Psi^\dagger v_\mu Y \Psi) + \frac{1}{eB} \delta^i_x \mathcal{L}, \]
\[ j^i_J(x) = \text{Re} (\Psi^\dagger v_\mu J \Psi) + \epsilon_{\mu i x} x^i \mathcal{L}, \] (74)

where \( v^\mu = (1, v) \), \( J = \frac{eB}{2}(\xi^2 + \eta^2 - x^2 - y^2) \), and \( \mathcal{L} \) is the Lagrangian density. The commutation relations between the above charges and the current densities read

\[ [Q_X, j^\mu(x)] = -\frac{i}{eB} \partial_\mu j^\mu(x), \]
Thus we see that $Q_X$ makes a translation in $-y$-direction of coordinate space and $Q_Y$ makes a translation in $x$-direction of coordinate space. As we saw before they are the generators of translations in the perpendicular directions of momentum space. This mixing of the coordinate space and momentum space is caused by non-commuting coordinates, $X$ and $Y$, and is a feature of the quantum Hall system.

As is clear from the eigenvalue, $E_l$, Landau energy gap, $E_{l+1} - E_l$ is proportional to the magnetic field and becomes large in the strong magnetic field. At low temperature, it is sufficient to study many-body state within one Landau level where Fermi energy is located. The free Hamiltonian depends only on total electron number and is irrelevant large in the strong magnetic field. At low temperature, it is sufficient to study many-body state within one Landau level space and obtain the Hall gas state.

The interaction Hamiltonian within one Landau levels is given in the von Neumann lattice representation as,

\[
\mathcal{H} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \rho(k) V(k) \rho(-k),
\]

where the density operator is defined by the operators of the particular Landau levels in Eq. (37). We study the Hartree Fock approximation in the momentum space. It is instructive, however, to write this Hamiltonian in the coordinate space as,

\[
\mathcal{H} = \frac{1}{2} \sum_{\mathbf{x}_i, \mathbf{x}_i'} b_i(\mathbf{X}_1) b_i(\mathbf{X}_1') V(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) b_i(\mathbf{X}_2) b_i(\mathbf{X}_2'),
\]

where the coordinates are defined in the lattice sites. The $\mathbf{K}$ transformation is expressed as,

\[
\begin{align*}
 b_i(\mathbf{p}) &\rightarrow b_i(\mathbf{p} + \mathbf{K}), \\
 b_i(\mathbf{X}) &\rightarrow e^{i\mathbf{K} \cdot \mathbf{X}} b_i(\mathbf{X}), \\
 \rho(\mathbf{k}) &\rightarrow e^{i(K_y \mathbf{k}_y)} \rho(\mathbf{k}).
\end{align*}
\]

Delicate problems connected with the boundary condition in momentum space are resolved using the gauge freedoms of operators and slight modifications are required in the above transformations. Obviously $\mathcal{H}$ is invariant under the $\mathbf{K}$-transformation. An ordinary free Hamiltonian has a momentum dependent one particle energy and is not invariant under $\mathbf{K}$-transformation. So $\mathbf{K}$ symmetry is a characteristic symmetry of the quantum Hall system. Hall gas state has a momentum dependent one-particle energy and breaks $\mathbf{K}$ symmetry spontaneously. By the Coulomb interaction, the broken state is realized as ground state. Hence Goldstone theorem is applied and gapless Nambu-Goldstone-mode emerges. The properties of Nambu-Goldstone mode and related problems will be studied later.

### B. Mean field solution of anisotropic Hall gas (stripes)

Self-consistent mean field of the form,

\[
U_0(\mathbf{X} - \mathbf{X}') = \langle b_i(\mathbf{X}') b_i(\mathbf{X}) \rangle
\]
is obtained from Hartree-Fock approximation of $\mathcal{H}$, where $U_0(X - X')$ is unknown now and is determined later. We first plug the above expectation value into the Hamiltonian and obtain mean field Hamiltonian which is bi-linear in the field. So it is easy to diagonalize the mean field Hamiltonian and to find the ground state by filling electrons up to certain filling factor. Thus the expectation value of two point function is obtained and is required to agree with the initial expectation value. In this way mean field solution which satisfies self-consistency condition is obtained. From $(X - X')$ dependence of the expectation value, lattice translational invariance is preserved and one-particle energy depends upon the momentum. Momentum dependent kinetic energy did not exist in the free Hamiltonian Eq. (36) but is generated self-consistently in the mean field Hamiltonian. Thus $K$ symmetry is broken spontaneously.

Let us obtain the Hall gas state of filling $\nu$ in the momentum space that breaks $K$ symmetry minimally i.e., $K_x$-invariant solution where one-particle energy depends on $p_y$ but does not depend on $p_x$ explicitly. Expectation value is assumed symmetric in $p_y$ and is given as,

$$\langle b_l^\dagger(p)b_l(p') \rangle = (2\pi)^2 \delta(p - p') \theta(\pi \nu' - |p_y|),$$

where $\nu' = \nu - l$. This state preserves $K_x$-symmetry and breaks $K_y$-symmetry spontaneously. The mean field Hamiltonian is expressed as,

$$\mathcal{H}_m = \int_{BZ} \frac{d^2p}{(2\pi)^2} \epsilon_l(p) b_l^\dagger(p)b_l(p) + E_l(0),$$

where $\epsilon(p)$ is one-particle energy and $E_l(0)$ is zero-point energy. Let us assume that $\mu_0(\nu)$ to be the chemical potential. The self-consistency condition is given by,

$$\epsilon(p) = \int_{BZ} \frac{d^2p'}{(2\pi)^2} v_{HF}(p' - p) \theta(\mu_0(\nu) - \epsilon(p')),$$

$$\nu' = \int_{BZ} \frac{d^2p'}{(2\pi)^2} \theta[\mu_0(\nu) - \epsilon(p')].$$

In the above equations $v_{HF}(p' - p)$ is the Hartree-Fock potential energy that combine the Coulomb potential with mean field as,

$$v_{HF}(p) = \sum_N \{ v_l(2\pi \tilde{N}) e^{i p \times N} - v_l(2\pi \tilde{N} - \tilde{p}) \},$$

where $v_l(k) = e^{-k^2/4\pi} [L_l(k^2/4\pi)^2] V(k)$. By solving self-consistency conditions, $\epsilon(p)$ and $E_l(0)$ are obtained. $p_y$ dependent one-particle energy thus emerges. One-particle energy at half filling is given in Fig. 14. As is seen in Fig. 15, Fermi surface is parallel to $p_x$ axis and $K_x$ symmetry is preserved. The asymmetry parameter $r_s$ is determined by minimizing the total energy.

![FIG. 14: One-particle energy spectrum of anisotropic Hall gas at the filling factor $\nu = l + \frac{1}{2}$. The unit of energy is $\frac{\hbar^2}{m}$](image-url)
Because the Coulomb potential decreases slowly at infinity, slope of one-particle energy at the Fermi energy, Fermi velocity, diverges. If the Coulomb potential is screened and screened Coulomb potential is used, the Fermi velocity becomes finite.

From the total energy we compute physical quantities that show thermodynamic properties of this many-body state. Since the kinetic energy is produced by interaction only, it is expected that the present Hall gas reveals peculiar properties that are different from those of ordinary electron gas. Pressure and compressibility of Hall gas with neutralizing uniform background charge are computed and are given in Fig. 16. As is seen in Fig. 16, pressure becomes negative. Negative pressure means that the gas has a tendency to shrink by itself. Due to the neutralizing background charge, the magnitude of size of shrink is normally very small. But it has important effects for the IQHE in realistic systems. Implications to metrology will be discussed in Section 6.

Compressibility becomes also negative and is given in Fig. 17. In ordinary electron gas, compressibility becomes negative only in low density where interaction effects surpass free kinetic energy. In the Hall gas, ordinary kinetic energy is frozen by the magnetic field so interaction effect is enhanced and this phenomenon occurs in arbitrary density. The asymmetry parameter $r_s$ changes slightly with the filling factor as is shown in Fig. 17.
Asymmetric Fermi surface in the momentum space means that the orientational symmetry is broken. To see a direct signal of orientational symmetry breaking, we study the density in coordinate space. Number density and current density in real space become also asymmetric and are given by,

$$
\langle \rho(x) \rangle = \int \frac{d^2k}{(2\pi)^2} e^{i(k_x x + k_y y)} \int_{BZ} \frac{d^2p}{(2\pi)^2} \langle b_i^\dagger(p) b_i(p + a\hat{k}) \rangle |f_i\rangle |e^{-i(k_x \xi + k_y \eta)}|f_i\rangle \times e^{i\frac{\hat{k}_x}{2} (2p_y + a\hat{k}_y)},
$$

(89)

$$
\langle j(x) \rangle = \int \frac{d^2k}{(2\pi)^2} e^{i(k_x x + k_y y)} \int_{BZ} \frac{d^2p}{(2\pi)^2} \langle b_i^\dagger(p) b_i(p + a\hat{k}) \rangle |f_i\rangle \frac{1}{2} \{v, e^{-i(k_x \xi + k_y \eta)}\} |f_i\rangle \times e^{i\frac{\hat{k}_x}{2} (2p_y + a\hat{k}_y)},
$$

(90)

The phase factor in the above equation is due to commutation relation between the guiding center coordinates, X and Y, and causes particle’s motion toward perpendicular to electric field. In the present equation, it leads the density profile in coordinate space to be uniform in y-direction and is periodic in x-direction with the period $r_s a$, Fig. 18. Thus the density in the real space is uniform in the direction orthogonal to Fermi surface in the momentum space. So the present anisotropic Hall gas is equivalent to unidirectional charge density wave of Koulakov, Fogler, and Shklovskii and of Moessner and Chalker.

Number density and current density of the present mean field are given in Fig. 19. The current density in the real space is perpendicular to the gradient of density. Current flows locally in the direction of the uniform density, but the total current vanishes.

From the shape of Fermi surface, all the one-particle states are filled in the $p_x$ direction. Unfilled one-particle state of the lowest energy in the $p_x$ direction is in the next Landau levels, which has a large energy gap in the system of
FIG. 18: Density profile in the coordinate space of anisotropic Hall gas at $\nu = 2 + 1/2$. The unit of density is $1/a^2$ and the unit of coordinates is $a$.

the strong magnetic field. Hence the state behaves like the integer quantum Hall state against the perturbation in this direction. Resistance in this direction vanishes.

FIG. 19: Current profile in the coordinate space of anisotropic Hall gas at $\nu = 2 + 1/2$. Arrows show current vector and shadows show low density area. The unit of coordinates is $a$. 
D. Symmetry breaking and Nambu-Goldstone zero-mode

Translational and orientational symmetry are broken in the anisotropic Hall gas obtained in the previous section. So Nambu-Goldstone zero mode appears. Center coordinates \((X,Y)\) are non-commuting and magnetic translations are non-commutative. Consequently Goldstone theorem is nontrivial and spectrum of Nambu-Goldstone zero mode would reflect these non-commutative natures. From both of the Fermi surface, Fig. 15, and the charge density, Fig. 18, we see that \(Q_Y\) and \(Q_J\) are broken spontaneously.

Since the conserved charges \(Q_Y\) and \(Q_J\) are broken spontaneously, Nambu-Goldstone mode appears. Proof for the theorem is given in the following using the above commutation relations. Let the broken state be the eigenstate of \(H\), a unitary operator that corresponds to \(Q_X\), the unitary operator that translates the system by \(r_s a\) in \(x\)-direction, and the total charge simultaneously,

\[
\mathcal{H}|n\rangle = E_n|n\rangle \tag{91}
\]

\[
e^{i2\pi Q_X / (r_s L_x)}|n\rangle = e^{i2\pi Q_Y / (r_s L_x)}|n\rangle \tag{92}
\]

\[
e^{i2\pi r_s Q_Y / a}|n\rangle = e^{i2\pi r_s Q_Y^\alpha / a}|n\rangle, \tag{93}
\]

\[
Q|n\rangle = N_e|n\rangle, \tag{94}
\]

where the states including vacuum, \(|0\rangle\), is periodic in \(x\)-direction with the periodicity, \(r_s a\), and \(L_x\) is the length of the system in \(x\)-direction.

By taking expectation value of the algebra, a state of a non-uniform charge density or equivalently a state of partially filled state with a definite Fermi surface leads that the symmetry is broken spontaneously. Let us use the algebra, Eq. (75), in the coordinate space and its expectation value,

\[
\frac{i}{eB} \partial_x j^0(r,t) = (0|Q_Y j^0(r,t)|0) = \int d\mathbf{r}' (0|j^0_Y(r',t'),j^0(r,t)|0) \tag{95}
\]

To rewrite the right-hand side, we use

\[
e^{ieB\Delta y Q_X} j^0_Y(r)e^{-ieB\Delta y Q_X} = j^0_Y[r+(0,\Delta y)] - \Delta y j^0_Y[r+(0,\Delta y)], \tag{96}
\]

and decompose the coordinate \(x\) into \(x = N_x r_s a + \tilde{x}\), where \(N_x\) is an integer and \(0 \leq \tilde{x} \leq r_s a\). Using the commutation relation, the definition of the broken state, and the above coordinate expression, the right-hand side of Eq. (95) is written as,

\[
a^3 \sum_{n \geq 0} \int_{0}^{r_s a} \frac{d\tilde{x}'}{r_s} \left[ \left( (0|j^0_Y[(\tilde{x}',0),0]|n)\langle n|j^0[(\tilde{x},0),0]|0\right) e^{i(t'-t)(E_n - E_0)} - H.C. 
\right.
\]

\[
\times \delta(\Delta y Q_X(0) - Q_X(\tilde{x}))\delta_{\tilde{x}/r_s} Q_Y(0) - Q_Y(\tilde{x}) + \frac{1}{eBt} \left( (0|j^0[(\tilde{x}',0),0]|n)\langle n|j^0[(\tilde{x},0),0]|0\right),
\]

\[
\left. \times e^{i(t'-t)(E_n - E_0)} + H.C. \right) \delta(\Delta y Q_X(0) - Q_X(\tilde{x}))\delta_{\tilde{x}/r_s} (Q_Y(0) - Q_Y(\tilde{x})),
\]

where \(\delta_{\tilde{x}/r_s} (Q_Y)\) stands for the delta function with the period \(\frac{1}{r_s}\). Because the left-hand side does not depend on \(t\), the energy gap, \(\Delta E_{NG}(q) = E_n - E_0\), where \(q\) is the momentum of \(|n\rangle\), vanishes in the small \(q\) limit. Thus the existence of gapless excitation mode is proved.

Properties of Nambu-Goldstone mode is complicated because the center coordinates are non-commuting and magnetic translations are also non-commutative. To calculate its spectrum, we apply single mode approximations which were successfully used to study the properties of liquid-helium and fractional quantum Hall states. Let us map the density operator and Hamiltonian into \(l\)-th Landau level space as,

\[
\rho_s(k) = P_l \rho(k) P_l
\]

\[
= \int_{BZ} \frac{d^2p}{(2\pi)^2} b_l^\dagger(p) b_l(p + a\hat{k}) e^{i\frac{\pi}{2}k_x(2p_y + a k_y)}, \tag{98}
\]

\[
H^{(l)} = \frac{1}{2} \int \frac{d^2k}{(2\pi)^2} \rho_s(k) v_l(k) \rho_s(-k), \tag{99}
\]

where \(v_l(k) = e^{-k^2/4\pi [L_l(k^2/4\pi)^2]2\pi q^2/k}\). This density operator satisfies the commutation relations,

\[
[\rho_s(k), \rho_s(k')] = -2i\sin \left( \frac{k \times k'}{4\pi} \right) \rho_s(k + k'), \tag{100}
\]
\[ [Q_x, \rho_\ast(k)] = \frac{k_y}{2\pi} \rho_\ast(k), \quad (101) \]

\[ [Q_y, \rho_\ast(k)] = -\frac{k_x}{2\pi} \rho_\ast(k). \quad (102) \]

The state defined by
\[ |k\rangle = \rho_\ast(k)|0\rangle \quad (103) \]
has a quantum number of Nambu-Goldstone mode and couples with the corresponding current. So we assume that this state is one particle state of Nambu-Goldstone mode and computes its energy from the expectation value,
\[ \Delta(k) = \frac{\langle k | (H^{(l)} - E_0) | k \rangle}{\langle k | k \rangle}. \quad (104) \]

To compute the numerator we use,
\[ \langle k | (H^{(l)} - E_0) | k \rangle = \langle 0 | \rho_\ast(-k), H^{(l)} | \rho_\ast(k) | 0 \rangle = -\langle 0 | [H^{(l)}, \rho_\ast(k)] \rho_\ast(-k) | 0 \rangle, \quad (105) \]

which are obtained from reflection symmetry,
\[ \langle 0 | \rho_\ast(k) \rho_\ast(-k) | 0 \rangle = \langle 0 | \rho_\ast(-k) \rho_\ast(k) | 0 \rangle, \quad (106) \]
\[ \langle 0 | \rho_\ast(k) H^{(l)} \rho_\ast(-k) | 0 \rangle = \langle 0 | \rho_\ast(-k) H^{(l)} \rho_\ast(k) | 0 \rangle, \quad (107) \]
and \( H^{(l)} | 0 \rangle = E_0 | 0 \rangle. \)

Using commutation relations and the Hamiltonian, we find the spectrum from single mode approximation as,
\[ \Delta(k) = |k_y|\{ Ak_y^2 + Bk_y^4 + O(k_y^6)\} \ln|k_y|, \quad (108) \]

where A and B are constants. The energy spectrum is anisotropic and vanishes with higher powers in the small momentum region. At \( k_x = 0 \) the energy depends on fifth power of \( k_y \) and at a finite \( k_x \) the energy depends linearly on the magnitude of \( k_y \). At \( k_y = 0 \), not only the energy vanishes exactly, but also the state \( |k\rangle \) vanishes. This is trivial from the shape of the Fermi surface, Fig. 15. Spectrum in wide range of \( k \) is given in Fig. 20.

![FIG. 20: Energy spectrum of excited states of anisotropic Hall gas at \( \nu = 2 + \frac{1}{2} \) as a function of momentum \( k_y \). The unit of \( k \) is \( a^{-1} \) and the unit of the energy is \( \frac{a^2}{m} \). The solid line is for \( k_x = 0 \) and the dashed line is for \( k_x = 1 \).](image)

**E. Preferred orientation under external density modulation**

Because orientational symmetry is spontaneously broken in the present mean field, by an infinitesimal perturbative term which breaks orientational symmetry, the system is forced to choose one particular direction. This direction is kept even in the limit of vanishing perturbative term. We study which direction is chosen in the presence of
small external density modulation in this section. Our finding for the energy of the system with external density modulation is counterintuitive. Although it is expected that the energy of stripe becomes minimum when the stripe is aligned parallel to the external density modulation, we found that this depends on the magnitude and the wave length of the density modulation. In the case of weak and long wave length modulation, the energy of stripe becomes minimum when the stripe is perpendicular to the external density modulation.

We add small density modulation term of the coupling strength, \( g \), and wave vector, \( \mathbf{K}_{\text{ext}} \), into the Hamiltonian, \( H_{\text{total}} = H + H_{\text{ext}} \), where \( H_{\text{ext}} \) is given by,

\[
H_{\text{ext}} = g \int d^2r \rho(r) \cos(\mathbf{K}_{\text{ext}} \cdot \mathbf{r}).
\]  

(109)

We study the stripe state from the total Hamiltonian. For small \( g \) case, perturbative expansion with respect to \( g \) may be a good approximation. So first we compute the energy correction of ground state up to the square of \( g \) based on perturbative expansion. This energy becomes a function of the angle between the stripe and external modulation, \( \theta \), and of the magnitude of \( \mathbf{K}_{\text{ext}} \) and is given in Fig. 21.

![Figure 21](image)

FIG. 21: Energy gain of the anisotropic Hall gas as a function of the angle between the external periodic density modulation and intrinsic periodic density. The unit of the energy is \( \frac{e^2}{2\pi} \).

The energy becomes minimum when the angle is \( \frac{\pi}{2} \). Thus the stripe is aligned perpendicular to the external modulation. This result is understandable from the shape of Fermi surface. All the one-particle states are filled in the \( p_x \) direction and unfilled one-particle state of the lowest energy in the \( p_x \) direction is in the next Landau levels, which has a large energy gap in the system of the strong magnetic field. Hence the many-body state is hard and does not get any perturbative energy if the external perturbation is in the \( x \)-direction. This state behaves like the integer quantum Hall state against the perturbation in this direction. Contrary to the \( x \)-direction, there are states around the Fermi surface without energy gap and the many-body state is soft and gets a perturbative energy if the external perturbation is in the \( y \)-direction. Thus the orthogonality between the external density modulation and the intrinsic density profile is caused by the shape of Fermi surface, Fig. 15, and is a characteristic feature of the quantum Hall system.

If \( \mathbf{K}_{\text{ext}} \) is equal to the intrinsic wave vector of the stripe, a HF solution of \( H \) becomes a HF solution of total Hamiltonian. Both directions are the same in this case. So the state realized in the present case is different from the state in the previous case. From comparison of the energies of the both states we obtain phase diagram as a function of several parameters, Fig. 22.

In small \( g \), \textbf{orthogonal phase} in which both direction are orthogonal is realized and in large \( g \), \textbf{parallel phase} in which both directions are parallel is realized. In the long wave length limit, the \textbf{orthogonal phase} is always realized. Systems with in-plane magnetic field in \( x \)-direction is expressed with the vector potential, \( \mathbf{A} + \mathbf{A}_{\text{in}} \), \( \mathbf{A} = \frac{B}{2}(y, -x, 0), \mathbf{A}_{\text{in}} = (0, 0, B_{\text{in}}\frac{\mathbf{K}}{\mathbf{K}}) \). In the limit \( K \to 0 \), in-plane magnetic field becomes uniform, and the orthogonal phase, in which the Fermi surface is parallel to \( p_x \) direction is realized. Thus the effect of the in-plane constant magnetic field is almost the same as the effect of external modulation studied in this section. The orthogonal phase is realized. This is consistent with experiments.
FIG. 22: Phase diagram of parallel phase and orthogonal phase for \( l=2 \) and 3.

F. Electronic transport

**Hall conductance:** We study the electric transport of the system in which the anisotropic Hall gas fills whole spatial regions in this section. Using the energy eigenvalue of Section 3.2, \( \varepsilon_l(p) \), the propagator for \( b_l(p) \) is expressed as,

\[
\tilde{S}_{l_1 l_2}^{(c)}(p) = U_{l_1 l_1'}(p) S_{l_1 l_1'}^{(c)}(p) U_{l_1 l_2}^\dagger(p) \delta_{l_1 l_2} \\
S_{l_1 l_1'}^{(c)}(p) = \frac{\delta_{l_1 l_2}}{p_0 - (E_{l_1} + \varepsilon_l(p))} \tag{110}
\]

We substitute Eq. (110) to Eq. (51), and we have

\[
\sigma_{xy} = \frac{e^2}{h}(l + \nu'), \tag{111}
\]

at \( \nu = l + \nu' \). Thus the Hall conductance is unquantized and is proportional to the filling factor in the Hall gas regime, where the many-body state of this Fermi energy is anisotropic quantum Hall gas.

**Longitudinal resistances:** The \( K_x \)-invariant anisotropic Hall gas states at \( \nu = l + 1/2 \) have the Fermi surface parallel to \( p_x \) axis. In \( p_x \) direction, there is no empty state within the same Landau level. The lowest energy unoccupied state is in the next Landau level and has the cyclotron energy gap. This direction is like the integer quantum Hall state. Thus, longitudinal conductance in this direction vanishes,

\[
\sigma_{xx} = 0. \tag{112}
\]

In the \( p_y \)-direction, there are states in the same Landau levels which have no energy gap. The system behaves like a bunch of parallel one-dimensional systems in coordinate space. Each part is like a narrow one-dimensional strip of high density and of finite current density. If the small voltage contact is attached to one of one-dimensional systems as in Fig. 23, the conductance in this situation is computed from Landauer formula. The current in this region is carried by one-dimensional electrons which have left-moving modes and right-moving modes in narrow spatial region. So the situation is different from the edge states of the bulk quantum Hall systems where only one chiral mode exists in one edge.

In Landauer formula to this one-dimensional system, small difference of the chemical potential, the level density, and the velocity are used. The velocity is given as

\[
v_y = \frac{\partial \varepsilon(p_y)}{\partial p_y}. \tag{113}
\]

and number of states of the extended states in the chemical potential difference \( \Delta \varepsilon \) is,

\[
\Delta n = \frac{1}{2\pi} \frac{\partial p_y}{\partial \varepsilon} \Delta \varepsilon (1 + C), \tag{114}
\]
where $C = (\partial \delta_p / \partial p)_{pF} / L$ is the correction term which comes from the phase shift $\delta_p$ due to impurity scattering, and $L$ is the length of the one-dimensional system. Combining Eqs. (113) and (114), we have total current

$$I_y = ev_y \Delta n$$

$$= \frac{e^2}{2\pi} (1 + C)V_y,$$

where $V_y$ is voltage in $y$-direction and the chemical potential difference $\Delta \epsilon$ is replaced with $eV_y$. The precise value of the correction term $C$ is not known. It is, however, expected that $C \to -1$ when the Landau level index goes to infinity from the scaling theory of Anderson localization in the system of zero magnetic field, and $C$ vanishes in the ideal system without impurities. In ideal systems without impurities, the ideal conductance

$$\sigma_{yy}^0 = \frac{e^2}{h},$$

is obtained. In general cases, $\sigma_{yy}$ depends on the disorders and may be different from $\sigma_{yy}^0$.

Because the resistance tensor is the inverse of the conductance tensor, the resistance in $y$ direction vanishes and the resistance in $x$ direction becomes finite value. They are given as,

$$R_{yy} = 0, \quad R_{xx} = \frac{-\sigma_{yy}}{\sigma_{xx} \sigma_{yx}}.$$  \hspace{1cm} (117)

Thus in the measurement of resistance of the present anisotropic quantum Hall gas, the easy direction is the $y$-direction. The density is uniform and the Fermi surface is orthogonal to this direction. In the system of $y$ dependent density modulation of the long wave length and in the system of the in-plane magnetic field of $x$-direction, the easy direction becomes $y$-direction. This agrees with experiments.

### IV. IMPLICATIONS OF GAS OF NEGATIVE PRESSURE

In the previous section we have found that anisotropic quantum Hall gas has unusual thermodynamic properties such as negative pressure and negative compressibility. Due to these properties, the quantum Hall gas shrinks. Electron density becomes nonuniform and charge neutrality is violated locally. States of high density part becomes strip-like shape in a simple case. Shape may become more complicated in ordinary cases but in fact the shape is irrelevant to physical effects. So we call this non-uniform state strip and study its implications. In systems of disorders non-uniform charge state is also created. Physical effects of this non-uniform state is equivalent to the strip due to interactions. So there are two mechanisms of generating strip. One is by interaction and the other is by disorders. We analyze both mechanisms.

#### A. Shape of the anisotropic quantum Hall gas with a negative pressure

We show first that gas of negative pressure shrinks and is stabilized at a certain smaller size than the original size of the electron system. Compressible many-electron states of negative pressure is deformed due to negative pressure effects. We study uniform Hall gas composed of electrons and of uniform neutralizing background charges $\rho_0$, which
are composed of ions, first. The electrons can move but the ions cannot. So when electrons shrink, charge neutrality is violated partly and the energy of the system is increased by a static Coulomb energy. The Hamiltonian of this system then is given as,

\[ H = \frac{1}{2} \int d^2x d^2y (\rho(x) - \rho_0) v(x - y) (\rho(y) - \rho_0) \]

\[ = \frac{1}{2} \int d^2x d^2y (\langle \rho(x) \rangle - \rho_0 + : \rho(x) : ) v(x - y) (\langle \rho(y) \rangle - \rho_0 + \rho(y) : ). \]  

(118)

The right hand side of the above equation is expanded as,

\[ H = \frac{1}{2} \int d^2x d^2y (\langle \rho(x) \rangle - \rho_0) v(x - y) (\langle \rho(y) \rangle - \rho_0) \]

\[ + \frac{1}{2} \int d^2x d^2y : \rho(x) : v(x - y) : \rho(y) : + \frac{1}{2} \int d^2x d^2y : \rho(x) : v(x - y) : \rho(y) : . \]  

(119)

The first term in the right hand side is the static Coulomb energy and vanish when the charge neutrality is preserved. We study the system in which the charge neutrality is violated in a macroscopic scale. The second and third terms are proportional to the normal ordered products of the density. Hence the expectation values of the second and third term vanish when we study the system within Hartree-Fock approximation. From the assumption that the charge distribution becomes non-uniform in macroscopic size, the HF approximation of the previous section is applied to the present system. Hence we ignore the second and third terms and we apply the result of the last section to the last term in the above Hamiltonian.

We study first the static Coulomb energy in the right hand side. We compute the Coulomb energy per particle of the electron system with the initial value of width in the \( x \)-direction, \( L_w \), and infinite length in the \( y \)-direction, and of the filling factor \( l + \nu' \), when the gas shrinks uniformly by \( \delta_x \) in the width as is in Fig. 24. The static Coulomb energy per particle of this situation is found to have the form:

\[ E_{\text{Coulomb}} = \nu' L_w g \left( \frac{\delta_x}{L_w} \right) \frac{q^2}{a}, \]  

(120)

where \( g(x) \) is a monotonically increasing function which behaves as \( -\frac{x^2}{2} \log x \) in \( x \to 0 \) and diverges at \( x = 1 \) logarithmically. The HF energy is a decreasing function of \( x \) due to negative pressure effect. The total energy is expressed as a function of the \( \delta_x \) in Fig. 25. The total energy becomes minimum at one value of the \( \delta_x \). At this value the Hall gas is stabilized. The pressure vanishes and the compressibility becomes positive. This value depends on the filling factor. \( \delta_x \) nearly vanishes when the filling of the Hall gas is large, and \( \delta_x \) becomes finite if the filling is small. Hence the Hall gas shrinks substantially and is deformed to a smaller size if the filling is slightly away from integer.

![FIG. 24: Electron density of anisotropic Hall gas when the charge neutrality is violated slightly.](image)

FIG. 25: The total energy of anisotropic Hall gas at \( t = 0, 1, 2, \) and 3 in the order from below when the charge neutrality is violated slightly as a function of \( \frac{\delta x}{L_w} \). The width of strip is \( L_w = 20a \) and \( \nu' = 0.5 \) and \( \nu' = 0.03 \). The unit of energy is \( q^2 a_B \).

B. Interplay between localization length and finite size

In real systems of GaAs, there are long range disorder potentials. Localized states due to disorders have discrete energies and finite localization lengths. These states have normalizable wave functions that vanish at infinite spatial region. Hence electrons in these states neither carry the bulk electric current nor contribute to electric conductances in the infinite system. In finite systems, however, localized states could behave like extended states if localization lengths are larger than or equal to the system size. They contribute to the electronic conductances if they have finite wave functions at current contact regions. Thus it is convenient to classify the one-particle energy region to several regime depending upon their localization lengths and the systems’ sizes. There are three systems’ sizes. One is the length of the system, \( L_s \), the second one is the width in the Hall probe region, \( L_{w1} \), and the third one is the width in the potential probe region, \( L_{w2} \). Generally the first one is the largest, the second one is medium, and the third one is smallest.

The localization length, \( \lambda(E) \), depends on the localized state’s energy and is known to behave as

\[
\lambda(E) = \lambda_0 |E - E_l|^s, \quad s \approx 2,
\]

where \( \lambda_0 \) is a constant and \( s \) is a critical exponent. The localization length diverges at the center of the Landau levels and becomes minimum at the middle between two Landau levels as is given in Fig. 25. Let \( E_s, E_{w1}, \) and \( E_{w2} \) be the energy values where the localization lengths agree with the three system’s sizes, \( \lambda(E_s) = L_s, \lambda(E_{w1}) = L_{w1}, \lambda(E_{w2}) = L_{w2} \). Then one particle states in the energy range \( |E - E_l| < E_s \) are extended states, and we call this energy region as the extended state region. One particle states in the energy range, \( E_s < |E - E_l| < E_{w1} \), bridge from one edge to the other edge at the potential probe region and at the Hall probe region, and we call this energy region as the collapse regime. States in the energy range, \( E_{w1} < |E - E_l| < E_{w2} \), bridge from one edge to the other edge at the potential probe region, and we call this energy region the dissipative quantum Hall regime (QHR). Finally states in the energy range, \( E_{w2} < |E - E_l| \), are localized, and we call this energy region as the localized states region or QHR. Reasons why we use particular names for the last two regions will become clear later. The energy regions and typical wave functions in configuration space are written in Fig. 26.
C. Strip formation

Non-uniform electron states where charge density becomes high locally is formed by negative pressure effect. A strip of such non-uniform density region is formed also by localization effects. First one is due to interactions and the second one is due to disorders. One-particle wave functions are extended if its energy is in the center of Landau levels and cover whole spatial region from one end to another end in the system of no interactions. Interaction, however, modifies the many-body wave functions and compressible Hall gas of negative pressure is formed. Due to negative pressure, which is obtained in the previous section, this Hall gas shrinks. So the strip of compressible states of high density is formed. They are composed of one-particle states of the energies near the center of Landau levels.
Second mechanism of forming charge non-uniform states is due to disorders in the finite systems. Wave functions of the energy near mobility edge have large localization length of the order of system sizes. These states which have the localization length in the region, \( L_w < \lambda(E) < L_s \), could bridge one edge to another edge in one direction of the sample, which is shorter in this direction than in the other direction but could not bridge in another direction of a sample. Consequently charge density of the localized states in this energy region becomes non-uniform and bridges only one sides. Their shape may not be simple but they behaves effectively like a strip of compressible gas. Thus the strip is formed in the extended states region near the mobility edge. The electric resistances of the strip states thus formed are studied in the next section.

V. CURRENT ACTIVATION FROM UNDERCURRENT: ON PRECISE DETERMINATION OF THE FINE STRUCTURE CONSTANT

In a normal metal the current flows through the extended one-particle states that are connected with the current contacts, and transport properties are determined by these extended states. The one-particle states that are disconnected from the current contacts should not affect transport properties. However in the quantum Hall system there are extended states below the Fermi energy which carry the Hall current. Due to these states, one particle states which are disconnected from the current contacts are affected and contribute to the transport properties of the whole system in characteristic manners.

We consider the situation where the electrons around the Fermi energy are in a strip which is disconnected with the current contacts and the extended states have the energy below the Fermi energy with a finite energy gap. Since extended states below the Fermi energy is connected with external current, the electrons below the Fermi energy carry the electric current. This current dissipates no energy at low temperature. If current is induced in the states around the Fermi energy, which occurs by electron interaction, then the current carried by electrons dissipates energy, since there are many states around the Fermi energy. A small amount of current which is induced in the compressible region gives a small longitudinal resistance. Thus through higher order correlation effect, bulk Hall current activates the electrons around the Fermi energy which are disconnected from the current contacts. Consequently the current is induced in the strip of compressible states. We study physical effects of this induced current in the next part.

A. Current activation from undercurrent as a new tunneling mechanism

As shown in Sec. IV, strip formed from localized electrons of localization lengths between the width, \( L_w \), and length of the system, \( L_s \), \( (L_w < L_s) \), does not reach at least to one current contact. Hence injected current does not flow through the strip at zero temperature. The inject current flows through the electrons in lower Landau levels. A current in strip in this situation is induced by interactions and gives peculiar transport properties. Implications to the IQHE with finite injected currents at finite temperature will be discussed.

We study in the following the situation where several lower Landau levels are filled completely and one Landau level is partially filled as shown in Fig. 28. The strip is formed in the highest occupied Landau level and bridges one edge to the other edge. Although Fermi energy is in the strip state energy region, the strip is disconnected from the current contacts. Injected electric current flows through the lower Landau levels, and does not flow through the isolated compressible states of higher Landau levels if there is no interaction. In reality the interaction exists and modifies the isolated states. At finite temperature the current is activated into the isolated strip states from lower Landau levels as shown in Fig. 28. We study the effects of Coulomb interaction at finite temperature and find that the strip states have small amount of electric current at low temperature. The magnitude of induced current depends on injected current as well as on temperature.

From Fermi-Dirac statistics, one electron fills one state and states are filled from lower energy levels to higher energy levels. Among electrons in the many-body states, only the electrons near the Fermi surface make transitions with low energy perturbations. The dissipation of the energy occurs only from the states near the Fermi surface and does not occur at the lower energy levels. In the quantum Hall system, the bulk current is carried by the electrons in lower levels and in the levels near Fermi surface. So in the present situation shown in Fig. 28, dissipationless current flows in the lower levels and induced current in the strip in the middle of sample near Fermi surface dissipates energy.

The second order correction to the current correlation functions from the Coulomb interaction is expressed in Feynman diagram of Fig. 29 and gives the induced current to the strip states. One electron line (double line) stands for those of lower Landau levels which are connected with current contacts and are carrying current. Thus these states depend on the Hall electric field\(^\uparrow\). The other line (solid line) stands for those of strip states around Fermi energy which are disconnected from current contacts and are not carrying current in the lowest order. The dashed line stands for the Coulomb interaction. To compute induced electric current, the current operator is inserted in the

\( ^\uparrow \)The Hall electric field refers to the electric field induced by the Hall effect.
FIG. 28: (a) A schematic illustration of a potential probe and a Hall probe. A strip of compressible region bridges one edge to the other edge in the potential probe region of the width $L_{w1}$. S and D stand for the source and drain regions. (b) Electrons in the extended states which are in the lower Landau levels carry the current $J_0$. The electrons in the strip are around the Fermi energy and absorb the current $J_1$ from the lower levels.

FIG. 29: Feynman diagram which gives an induced current. The solid line stands for the electron propagator in the strip state and the double line stands for the electron propagator in the lower Landau levels. The dashed line shows the Coulomb interaction.

strip state line and is shown as wavy line in Fig. 29. We express operators, propagators, vertices, and others in the current basis. The induced current is calculated as

\[
\begin{split}
 j_{\text{ind}}^{\pi} &= \frac{1}{\beta} \sum_{\omega=-\infty}^{\infty} \int \frac{d^2p d^2k}{(2\pi)^4} \text{Tr} \left( V(k) S^{(e)}(p-ak) \right.
 \times \bar{\Gamma}^{\pi}(p-ak,p-ak) S^{(e)}(p-ak) S(p;E_H) \bigg) \\
 &= \frac{1}{\beta} \sum_{\omega=-\infty}^{\infty} \int \frac{d^2p d^2k}{(2\pi)^4} V(k) f_{l,l+1}(k)^2 \\
 &\quad \times \partial_x S_{l,l}(p;E_H) S^{(e)}_{l+1,l+1}(p-ak),
\end{split}
\]

where $S(p;E_H)$ is the propagator of the electrons of the energy determined with the Hall electric field, $E_l + eE_H^{\text{eff}} \frac{a}{2\pi} p_x$, which depends on the Hall electric field $E_H$ and $S^{(e)}(p)$ is the propagator for the stripe states, Eq. (110), which does not depend on $E_H$. Actually for the propagator $S^{(e)}(p)$ in the strip due to disorders, we use the same form as the propagator in the strip due to interactions for simplicity. $\omega_n = (2n+1)\pi/\beta$ is the Matsubara frequencies. The chemical potential $\mu$ is in between $E_l$ and $E_{l+1}$. In the above calculation, we considered only the mixing term between the Landau levels $E_l$ and $E_{l+1}$ which is the dominant term in the induced current. The asymmetry parameter is fixed at $r_s = 1$ for simplicity. Then the induced current reads

\[
\begin{split}
 j_{\text{ind}} &= \int \frac{d^2p d^2k}{(2\pi)^4} V(k) f_{l,l+1}(k)^2 v_x \beta e^{\beta \Delta E_l(p-ak)} \theta(-\Delta E_{l+1}(p), \\
 \Delta E_l(p) &= E_l + eE_H^{\text{eff}} \frac{a}{2\pi} p_x - \mu,
\end{split}
\]
\[ \Delta E_2(p) = E_{l+1} + \varepsilon_{l+1}(p) - \mu, \]
\[ v_x = eE_H^\text{eff} \frac{a}{2\pi}, \]

where \( \Delta E_1(p) \) comes from the effective band width of the electrons in the Hall electric field. \( E_H^\text{eff} \) is an effective Hall electric field and is given as \( E_H^\text{eff} = E_H/\gamma \), which is enhanced by the strong localization effect. In experiments, \( 1/\gamma \) is on the order of 10. \( \Delta E_2(p) \) comes from the band electrons which is computed in the previous section.

The induced current is given by,
\[ j_{\text{ind}} = V_i e^{\beta(E_i - \mu)} \frac{\beta e E_H^\text{eff}}{2}, \]
\[ V_i = \frac{4 \pi \nu' q^3 \pi^{3/2}}{a^2(l + 1)} \int_0^\infty dx x^{1/2} \left\{ L^{(1)}_l(x) \right\}^2 e^{-x}. \]

The value depends on the Hall electric field, the total current, and the temperature. The temperature dependence is of activation type and the gap energy is linear in \( E_H \).

**B. Induced current in a strip of compressible Hall gas**

The temperature dependence of the current is of activation type. At relatively high temperature, the electron temperature is the same as that of the whole system. At low temperature, however, the band electron temperature is different from the system’s temperature. Electron temperature is determined from the energy due to the electric resistance and the heat conduction from the electron system to the lattice system. The temperature determines the electric resistance and conversely the electric resistance determines the temperature. Hence they are determined self-consistently.

The induced current of the strip depends on the total injected current through the Hall electric field. Effective band width Eq. (123) of extended Landau levels in the system of Hall electric field increases with Hall electric field. This agrees with the experiments of the breakdown of the IQHE due to the injected current, where the activation energy depends on the injected current. So we use these previous results in the present work.

**C. Longitudinal resistance**

The total longitudinal resistance is the ratio between the induced electric field in the compressible gas region and the current density and is given as,
\[ R_{xx} = \frac{R_{xx}^{(s)} j_{\text{ind}}}{\sigma_{xy} E_H}, \]
\[ R_{xx}^{(s)} = \left( \frac{e^2}{\hbar} \right)^{-1} \]

where \( R_{xx}^{(s)} \) is the longitudinal resistance of the stripe state. The particular dependence of the longitudinal resistance on the Hall field is obtained for the first time. This result are compared with the experiment of Kawaji et al. in Fig. 30.

In the experiment, the filling factor is 4, magnetic field is 5.7 T, and the temperature is 0.75 K. The electron temperature could be different from the above value due to heating effect and is not known experimentally. So we fit the theoretical curve by changing the temperature, \( \gamma \), and \( \nu' \). As shown in Fig. 30, reasonable agreements are obtained when the temperature is 1.576 K, \( \gamma = 0.065 \), \( \nu' = 0.5 \) and \( \mu = E_i + \hbar \omega_c/2 \). Actually the temperature and the \( \gamma \) are almost the same even though \( \nu' \) is varied substantially. This temperature is that of the electron system at the strip and is different from the experimental value which is at the lattice system. The large deviation at large Hall electric fields is caused by the breakdown of IQHE, which will be discussed later.

**D. Various regimes with current increases**

The transport properties of the finite system depend on the system sizes, the Fermi energy, and the magnitude of the induced current. If the width in the Hall probe region is wider than the width in the potential probe region, longitudinal resistance and Hall resistance behave very differently.
As the current increases, band widths become broad from Eq. 123. Boundary between the extended states and localized states, mobility edges, move and localization length of given energy increase. Hence depending on the magnitude of the current, current carrying states occupy the whole area differently as is shown in Fig. 31. In QHR, current carrying states become small islands and are isolated. At a larger current, the second region, dissipative QHR, where the current carrying states become larger and bridge the sample in the potential probe region appear. At a more larger current, the third region, post collapse regime, where the current carrying states become larger and bridge the sample in the potential probe region and in the Hall probe region appear. If the current becomes even larger, whole area of the sample are covered by the current carrying states and QHE does not occur. QHR is breakdown. The resistances show several different behaviors.

**FIG. 30:** Comparison of the theoretical formula of induced longitudinal resistivity with the experiment of Kawaji et al. The solid line stands for the theoretical curve and the circles correspond the experiments.

**FIG. 31:** A schematic illustration of current carrying states. Shaded regions show the current carrying states and become larger as the current is increased.

**Dissipative QHR**

When the localization length of a state reaches the width of only the potential probe region, this state contributes to longitudinal resistance but not to the Hall resistance. Potential probe region has a temperature dependent electric resistance of activation type as is given in Eq. (123) and Eq. (124). In the Hall probe region, all the particle states...
around the Fermi energy are still localized states which have shorter localization lengths than the width. Hence the 
Hall probe region is considered as the QHR.

The Hall resistance measured in the wider Hall probe region is computed by the topological invariant expression 
Eq. (51) with the momentum in the finite system. The momentum becomes discrete in the finite system. As was 
shown in Ref. 41, this topological invariant is determined by the magnetic field and has no dependence on spatial 
component of the momentum. Hence this topological invariant does not change the value even when the integration 
variables are replaced with the discrete values.

Thus the quantized Hall resistance has no finite size corrections in the localized state region. Hence the Hall resis-
tance measured in the present situation agrees with the exactly quantized value. Thus resistivities at low temperature 
are given in our theory as,

\[
\begin{align*}
\rho_{xx} &= \tilde{\rho} e^{-\beta \Delta E_{\text{gap}}}, \\
\rho_{xy} &= \left(\frac{e^2}{\hbar} N\right)^{-1}, \\
\Delta E_{\text{gap}} &= \mu - E_l - \frac{e E_H a}{2},
\end{align*}
\]

(127)

where \(\tilde{\rho}\) is temperature independent. The finite temperature correction to \(\rho_{xy}\) was estimated before. The correction becomes independent from \(E_H\) and is negligible in the current situation. Thus \(\rho_{xy}\) is quantized even though \(\rho_{xx}\) does not vanish.

Equation (127) show that the Hall resistance is quantized even though the longitudinal resistivity does not vanish. 
This is a new regime of IQHE, which has not been expected from the naive picture of IQHE. Equation (127) shows 
that the new dissipative QHR is realized only in the finite system. If the width in the Hall probe region is the same 
as the width in the potential probe region, the dissipative QHR does not exist.

E. Collapse and breakdown of IQHE

If the current increases further, the extended state regions broaden in energy by the amount that is proportional 
to Hall electric field. Mobility edges move with Hall electric field outward from the center of Landau levels. The 
localization lengths of the localized states become larger. At some current value, they reach the spatial width of the 
Hall probe region. Wave functions in this energy region bridge both of the potential probe region and the Hall probe 
region. The electric current in this region is given by the previous form of the activation type in Eq. (124). If the 
Fermi energy is in this region, not only Hall resistance has a small correction but also longitudinal resistances becomes 
finite at low temperature. They are given in our theory as,

\[
\begin{align*}
\rho'_{xx} &= \tilde{\rho}' e^{-\beta \Delta E_{\text{gap}}}, \\
\Delta \rho_{xy} &= \tilde{\rho}' e^{-\beta \Delta E_{\text{gap}}},
\end{align*}
\]

(128) (129)

FIG. 32: A schematic illustration of the Hall resistance and the longitudinal resistance in the QHR, the dissipative QHR, 
the collapse, and the breakdown region as the inject current increases.
where \( \tilde{\rho} \) and \( \tilde{\rho}' \) are independent from the temperature. This corresponds to the collapse of IQHE.

When the current increases further, the localization lengths of the localized states become even larger and reach the spatial length of the system. Wave functions in this energy region cover whole area of the system and reach the current contacts. If the Fermi energy is in this region, the Hall resistance and the longitudinal resistances have finite corrections and are given by,

\[
\rho_{xx} = \frac{\hbar}{e^2} \delta, \quad (130)
\]

\[
\Delta \rho_{xy} = \frac{\hbar}{e^2} \delta', \quad (131)
\]

where \( \delta \) and \( \delta' \) are finite numbers which do not vanish in \( \beta \to \infty \). This corresponds to the breakdown of the IQHE in the finite system. If the current increases further and exceeds a critical value in which the energy gap \( \Delta E_{\text{gap}} \) vanishes completely, the localization length becomes infinite, the longitudinal resistance becomes finite, and Hall resistance deviates from a quantized value substantially. The IQHE in the infinite system disappears then. The critical value of Hall electric field, \( E_c \), is given from Eq. (123) as,

\[
\frac{\hbar \omega_c}{2} = e E_{c}^{\text{eff}} a. \quad (132)
\]

The critical Hall electric field is proportional to,

\[
E_c = \frac{\hbar \omega_c}{2e} \gamma \frac{a}{a}. \quad (133)
\]

\( E_c \) is proportional to \( B^{3/2} \) which is consistent with the experiment.

F. Precise determination of the fine structure constant from IQHE

The unusual properties found in Section 3 lead the systems to have several interesting transport properties depending upon the magnitude of several parameters such as current, system sizes, and others. One of the critical features is that the quantum Hall gas has a tendency to shrink. Hence the electric property of some spatial region could be confined in that region and does not expand to the whole system. From Section 5.4, even though an energy dissipation occurs in some region of the sample, the other part of the same sample could give the exactly quantized value of the Hall conductance. This happens if the other part of the sample is completely occupied by the localized electrons. The Hall conductance measured in this part of the sample is quantized exactly. This is an important result concerning metrology of the quantum Hall effects and is a key factor for the precise determination of the fine structure constant. In fact there is a small energy dissipation in the real measurement from the current contact area because there is a potential drop in the direction parallel to the current. However it is possible to measure the fine structure constant precisely from the IQHE.

VI. SUMMARY

Fundamental physical constants such as the light velocity, the electron charge, and the Plank constant played important roles in establishing modern physics. They are connected with relativity, electromagnetism, and microscopic world of matters. It is important to measure these constants precisely to understand the law of nature. The fine structure constant, \( \alpha \) is the unique parameter which is a combination of these constants and is the expansion parameter of QED. Computations of higher order corrections and the experimental measurements of QED with great precisions have been achieved. Comparisons of the theory with the experiments could supply a useful test of QED and standard model of high energy physics. Since the most accurate value of \( \alpha \) is obtained from the QHE, it is necessary to know if the precise value of \( \alpha \) is known from the QHE. To find out this problem, we formulate field theory of the QHE in such manner that makes derivation of rigorous field theoretical identities transparent. The von Neumann lattice representation of guiding center variables is suitable representation for developing field theory. Based on this representation, we give the proof of the integer quantization of the Hall conductance in the realistic situation and studied quantum Hall gas. Owing to Ward-Takahashi identity between the propagator and the vertex part, the Hall conductance is written by the special topological invariant of the mapping in three dimensional space defined by the propagator in the Landau levels.
The von Neumann lattice representation is useful also for studying anisotropic quantum Hall gas phase. The ground state, the physical properties, the orientation of the anisotropic quantum Hall gas under the small external perturbation, and the excited states are studied. The translational and rotational symmetries are clarified and the broken phase is studied. Implications of the quantum Hall gas state to the metrology of the QHE are also studied at the end.

Gauge invariance and magnetic field are two important ingredients for the topological invariant expression of Hall conductance of section II. We may ask ourself “Does the same topological invariant appear in physical systems without magnetic field?” This problem has been studied and affirmative answer has been obtained. Namely if time reversal is broken, the same topological invariants appear. One simple system is a free massive Dirac field in planer space and the other is time reversal violating condensed matter system such as unusual superconductors and others. In the first case, the spectrum asymmetry of Dirac operator which causes chiral anomaly and fractional charge in even dimensional space-time is the origin. The mass term breaks time reversal invariance, and U(1) symmetry is preserved. Hence the value of the topological invariant is quantized without any correction. On the other hand, in time reversal violating superconductor, U(1) symmetry is broken spontaneously. So topological invariant is not necessarily quantized and has a correction.

Other phases have been seen recently but are not discussed in the present work. They include a series of FQHS around half filling, pairing state at $\nu = 5/2\mathbb{Z}$, reentrant charge density wave, and many others. The von Neumann lattice representation would be useful for studying these states as well.

Acknowledgements

We thank J. L. Birman, G. H. Chen, K. Cooper, A. T. Dorsey, J. Eisenstein, A. Endo, J. Goryo, Y. Hosotani, Y. Iye, R. Jackiw, S. Kawaji, R. M. Lewis, T. Ochiai, W. Pan, K. Tadaki, L. Tevlin, P. Wiegmann, Y. S. Wu, K. Yang, and J. Zhu for useful discussions. This work was partially supported by the special Grant-in-Aid for Promotion of Education and Science in Hokkaido University provided by the Ministry of Education, Culture, Sports, Science, and Technology, Japan and by the Grant-in-Aid for Scientific Research on Priority area (Dynamics of Superstrings and Field Theories) (Grant No.13135201), Ministry of Education, Culture, Sports, Science, and Technology, Japan, Clark Foundation, and Nukazawa Science Foundation.

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