HIGHER ORDER REDUCTION THEOREMS
FOR CLASSICAL CONNECTIONS AND NATURAL
(0,2)-TENSOR FIELDS ON THE COTANGENT BUNDLE

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Abstract. We generalize reduction theorems for classical connections to operators with
values in k-th order natural bundles. Using the first reduction theorem in order two we
classify all (0,2)-tensor fields on the cotangent bundle of a manifold with a linear (non-
symmetric) connection.

1. Introduction

It is well known that natural operators of linear symmetric connections on manifolds and
of tensor fields which have values in bundles of geometrical objects of order one can be
factorized through the curvature tensors, the tensor fields and their covariant differentials.
These results are known as the first (the operators of connections only) and the second
reduction theorems. The history of the first reduction theorem comes back to the paper by
Christoffel, [1], and the history of the second reduction theorem comes back to the paper
by Ricci and Levi Civita, [11]. For further references see [9, 12, 16]. In [12] the proof
for algebraic operators (concomitants) is given. In [5] the first and the second reduction
theorems are proved for all natural differential operators by using the modern approach of
natural bundles and natural differential operators, [8, 10, 15].

In this paper we generalize the reduction theorems for natural operators which have values
in higher order natural bundles.

As an example we discuss natural (0,2)-tensor fields on the cotangent bundle of a manifold.

In this paper we use the terms ”natural bundle” and “natural operator” in the sense of [5, 8, 10, 15]. Namely, a natural operator is defined to be a system of local operators
\( A_M : C^\infty(FM) \to C^\infty(GM) \), such that
\[ A_N(f^*_Fs) = f^*_G A_M(s) \]
for any section \( s : M \to FM \) \( \in C^\infty(FM) \) and any (local) diffeomorphism \( f : M \to N \), where \( F, G \) are two
natural bundles and \( f^*_Fs = Ff \circ s \circ f^{-1} \). A natural operator is said to be of order \( r \) if,
for all sections \( s, q \in C^\infty(FM) \) and every point \( x \in M \), the condition
\( j^r_x s = j^r_x q \) implies
\( A_M s(x) = A_M q(x) \). Then we have the induced natural transformation \( A_M : J^r FM \to GM \)
such that \( A_M(s) = A_M(j^r s) \), for all \( s \in C^\infty(FM) \). The correspondence between natural
operators of order \( r \) and the induced natural transformations is bijective. In this paper we
shall identify natural operators with the corresponding natural transformations.

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Any natural bundle $F$ of order $r$ is given by its standard fibre $S_F$ which is a left $G^r_m$-manifold, where $G^r_m = \text{inv} J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ is the $r$-th order differential group. A classification of natural operators between natural bundles is equivalent to the classification of equivariant maps between standard fibers. Very important tool in classifications of equivariant maps is the orbit reduction theorem, [3, 7, 8]. Let $p : G \to H$ be a Lie group epimorphism with the kernel $K$, $M$ be a left $G$-space, $N, Q$ be left $H$-spaces and $\pi : M \to Q$ be a $p$-equivariant surjective submersion, i.e. $\pi(gx) = p(g)\pi(x)$ for all $x \in M$, $g \in G$. Having $p$, we can consider every left $H$-space $N$ as a left $G$-space by $gy = p(g)y$, $g \in G$, $y \in N$.

**Theorem 1.1.** If each $\pi^{-1}(q)$, $q \in Q$ is a $K$-orbit in $M$, then there is a bijection between the $G$-maps $f : M \to N$ and the $H$-maps $\varphi : Q \to N$ given by $f = \varphi \circ \pi$.

2.** Preliminaries**

Let $M$ be an $m$-dimensional manifold. If $(x^\lambda)$, $\lambda = 1, \ldots, m$, is a local coordinate chart, then the induced coordinate charts on $TM$ and $T^*M$ will be denoted by $(x^\lambda, \dot{x}^\lambda)$ and $(x^\lambda, \dot{x}^\lambda)$ and the induced local bases of sections of $TM$ and $T^*M$ will be denoted by $(\partial_\lambda)$ and $(d\lambda)$, respectively.

**Definition 2.1.** We define a classical connection to be a connection

$$\Lambda : TM \to T^*M \otimes TTM$$

of the vector bundle $p_M : TM \to M$, which is linear and torsion free.

The coordinate expression of a classical connection $\Lambda$ is of the type

$$\Lambda = d\lambda \otimes (\partial_\lambda + \Lambda^\mu_{\nu\rho} \dot{x}^\nu \partial_\mu)$$

with $\Lambda^\mu_{\nu\rho} = \Lambda^\nu_{\rho\mu} \in C^\infty(M, \mathbb{R})$.

Classical connections can be regarded as sections of a 2nd order natural bundle $\text{Cl}aM \to M$, [5]. The standard fibre of the functor $\text{Cl}aM$ will be denoted by $Q = \mathbb{R}^m \otimes \mathbb{R}^m \otimes \mathbb{R}^m$, elements of $Q$ will be said to be formal classical connections, the induced coordinates on $Q$ will be said to be formal Christoffel symbols and will be denoted by $(\Lambda^\mu_{\nu\rho})$.

The action $\alpha : G^2_m \times Q \to Q$ of the group $G^2_m$ on $Q$ is given in coordinates by

$$(\Lambda^\mu_{\nu\rho}) \circ \alpha = (a^\lambda_\rho (\Lambda^\rho_{\sigma\tau} \sigma^\sigma_\mu \tilde{\sigma}^\lambda_{\mu\nu} - \tilde{\sigma}^\rho_{\mu\nu}))$$

where $(a^\lambda_\mu, a^\lambda_{\mu\nu})$ are the coordinates on $G^2_m$ and $\tilde{\sigma}$ denotes the inverse element.

**Note 2.2.** Let us note that the action $\alpha$ gives in a natural way the action

$$\alpha^r : G^{r+2}_m \times T^r_m Q \to T^r_m Q$$

given by the jet prolongation of the action $\alpha$.

**Remark 2.3.** Let us consider the group epimorphism $\pi^{r+2} : G^{r+2}_m \to G^{r+1}_m$ and its kernel $B^{r+1}_m = \text{Ker} \pi^{r+2}$. We have the induced coordinates $(a^\lambda_{\mu_1,\ldots,\mu_{r+2}})$ on $B^{r+2}_m$. Then the restriction $\tilde{\alpha}^r$ of the action $\alpha^r$ to $B^{r+2}_m$ has the following coordinate expression

$$(\Lambda^\mu_{\nu_1,\ldots,\nu_{r+1}}) \circ \tilde{\alpha}^r = (\Lambda^\mu_{\lambda_2,\ldots,\lambda_{r+2}}) (\Lambda^\lambda_{\mu_1}, \ldots, \Lambda^\lambda_{\mu_2,\mu_3,\ldots,\mu_{r+2}}) - \tilde{\sigma}^\lambda_{\mu_1,\ldots,\mu_{r+2}},$$

where $(\Lambda^\mu_{\lambda_2,\ldots,\lambda_{r+2}}, \Lambda^\lambda_{\mu_1,\mu_2,\mu_3,\ldots,\mu_{r+2}})$ are the induced jet coordinates on $T^r_m Q$. 

The curvature tensor of a classical connection is a section \( R[\Lambda] : M \to W M := T^* M \otimes T M \otimes \Lambda^2 T^* M \) with coordinate expression
\[
R[\Lambda] = R^\nu_{\rho \lambda \mu} \ d^\nu \otimes \partial^\rho \otimes d^\lambda \wedge d^\mu ,
\]
where the coefficients are
\[
R^\nu_{\rho \lambda \mu} = \partial^\lambda \Lambda^\rho_{\mu \nu} - \partial^\lambda \Lambda^\nu_{\rho \lambda \mu} + \Lambda^\sigma_{\nu \lambda \rho} - \Lambda^\sigma_{\rho \nu \lambda} .
\]
Let us denote by the same symbol its corresponding \( G^3 \)-equivariant mapping, called the \textit{formal curvature map} of classical connections,
\[
\mathcal{R} : T^1_m Q \to W := S_{T^* \otimes T \otimes \Lambda^2 T^*} M
\]
with coordinate expression
\[
(2.1) \quad (w^\nu_{\rho \lambda \mu}) \circ \mathcal{R} = ((\Lambda^\rho_{\nu \mu} - \Lambda^\rho_{\nu \lambda} + \Lambda^\sigma_{\nu \lambda \rho} - \Lambda^\sigma_{\rho \nu \lambda})),
\]
where \((w^\nu_{\rho \lambda \mu})\) are the induced coordinates on the standard fibre \( W = \mathbb{R}^{m \times} \otimes \mathbb{R}^m \otimes \Lambda^2 \mathbb{R}^{m \times} \).

Let \( V M \) be a first order natural vector bundle over (i.e., \( V M \) is some tensor bundle over \( M \)). Let us put \( V_r M = V M \otimes \otimes^r T^* M \). Let us denote by \( V = \mathbb{R}^m \) or \( V_r \) or \( V^{(k, r)} \) the standard fibres of \( V M \) or \( V_r M \) or \( V^{(k, r)} M \), respectively.

The \( r \)-th order covariant differential of sections of \( V M \) with respect to classical connections is a natural operator
\[
\nabla^r : J^{r-1} \text{ Cla } M \times J^r V M \to V_r M .
\]
We shall denote by the same symbol its corresponding \( G^r_{m+1} \)-equivariant mapping
\[
\nabla^r : T^{r-1}_m Q \times T^r_m V \to V_r .
\]
We shall put
\[
\nabla^{(k, r)} := (\nabla^k, \ldots, \nabla^r) : J^{r-1} \text{ Cla } M \times J^r V M \to V^{(k, r)} M
\]
and the same for the corresponding \( G^{r+1}_{m-1} \)-equivariant mapping. Especially \( \nabla^{(r)} := \nabla^{(0, r)} \).

**Remark 2.4.** For any section \( \sigma : M \to V M \) we have
\[
(2.2) \quad \text{Alt} (\nabla^2 \sigma) = \text{pol} (R[\Lambda], \sigma) ,
\]
where \text{Alt} is the antisymmetrization and \text{pol} \((R[\Lambda], \sigma)\) is a bilinear polynomial. Namely, \text{Alt} \((\nabla^2 R[\Lambda])\) is a quadratic polynomial of \( R[\Lambda] \).

If \((v^A)\) are coordinates on \( V \), then \((v^A, v^A_{\lambda_1}, \ldots, v^A_{\lambda_1...\lambda_r})\) are the induced jet coordinates on \( T^r_m V \) (symmetric in all subscripts) and \((V^A_{\lambda_1...\lambda_r})\) are the canonical coordinates on \( V_r \), then \( \nabla^r \) is of the form
\[
(2.3) \quad (V^A_{\lambda_1...\lambda_r}) \circ \nabla^r = v^A_{\lambda_1...\lambda_r} + \text{pol}(T^r_{m-1} Q \times T^r_{m-1} V) ,
\]
where \text{pol} is a quadratic homogeneous polynomial on \( T^r_{m-1} Q \times T^r_{m-1} V \).
Remark 2.5. Let us recall the 1st and the 2nd Bianchi identities of classical connections given in coordinates by

\[ R_{(\nu^{\rho} \lambda \mu)} = 0, \quad R_{\nu^{\rho} (\lambda \mu ; \sigma)} = 0, \]

respectively, where \( \nu \) denotes the covariant differential with respect to \( \Lambda \) and \( (\ldots) \) denotes the cyclic permutation.

3. The first \( k \)-th order reduction theorem

Let us introduce the following notations.

Let \( W_0 M := W M, W_i M = W M \otimes \otimes^i T^* M, i \geq 0 \). Let us put \( W^{(k,r)} M = W_k M \times_M \ldots \times_M W_r M, k \leq r \). We put \( W^{(r)} M = W^{(0,r)} M \). Then \( W_i M \) and \( W^{(k,r)} M \) are natural bundles of order one and the corresponding standard fibers will be denoted by \( W_i \) and \( W^{(k,r)} \), respectively, where \( W_0 := W, W_i = W \otimes \otimes^i R^{\nu \sigma}, i \geq 0 \), and \( W^{(k,r)} = W_k \times \ldots \times W_r \).

We denote by

\[ \mathcal{R}_i : T^{i+1}_m Q \to W_i \]

the \( C^{i+3}_m \)-equivariant map associated with the \( i \)-th covariant differential of curvature tensors of classical connections

\[ \nabla^i R[\Lambda] : C^\infty(\text{Cla} M) \to C^\infty(W_i M). \]

The map \( \mathcal{R}_i \) is said to be the formal curvature map of order \( i \) of classical connections.

Let \( C_i \subset W_i \) be a subset given by identities of the \( i \)-th covariant differentials of the curvature tensors of classical connections, i.e., by covariant differentials of the Bianchi identities and the antisymmetrization of second order covariant differentials, see Remark 2.4 and Remark 2.5. So \( C_i \) is given by the following system of equations

\begin{align*}
(3.1) & \quad w_{(\nu^{\rho} \lambda \mu ; \sigma_1 \ldots \sigma_i)} = 0, \\
(3.2) & \quad w_{\nu^{\rho} (\lambda \mu ; \sigma_1 \ldots \sigma_i)} = 0, \\
(3.3) & \quad w_{\nu^{\rho} \lambda \mu \sigma_1 \ldots [\sigma_{j-1} \sigma_j] \ldots \sigma_i} + \text{pol}(W^{(i-2)}) = 0,
\end{align*}

where \( j = 2, \ldots, i \) and \( [\ldots] \) denotes the antisymmetrization.

Let us put \( C^{(r)} = C_0 \times \ldots \times C_r \) and denote by \( C_{r(k-1)}^{(k,r)}, k \leq r \), the fiber in \( r^{(k-1)} \in C^{(k-1)} \) of the canonical projection \( \text{pr}^{r-1}_k : C^{(r)} \to C^{(k-1)} \). For \( r < k \) we put \( C_{r(k-1)}^{(k,r)} = \emptyset \). Let us note that there is an affine structure on the projection \( \text{pr}^{r-1}_r : C^{(r)} \to C^{(r-1)} \), \( \mathfrak{I} \).

Then we put

\[ (3.4) \quad \mathcal{R}^{(k,r)} := (\mathcal{R}_k, \ldots, \mathcal{R}_r) : T^{r+1}_m Q \to W^{(k,r)}, \quad \mathcal{R}^{(r)} := \mathcal{R}^{(0,r)}, \]

which has values, for any \( \gamma \in T^{r+1}_m Q \), in \( C_{r(k-1)}^{(k,r)} \). In \( \mathfrak{I} \) it was proved that \( C^{(r)} \) is a submanifold in \( W^{(r)} \) and the restriction of \( \mathcal{R}^{(r)} \) to \( C^{(r)} \) is a surjective submersion. Then we can consider the fiber product \( T_m^k Q \times_{C^{(k-1)}} C^{(r)} \) and denote it by \( T_m^k Q \times C^{(k,r)} \).

First we shall prove the technical

Lemma 3.1. If \( r + 1 \geq k \geq 0 \), then the restricted map

\[ (\pi^{r+1}_k, \mathcal{R}^{(k,r)}) : T^{r+1}_m Q \to T_m^k Q \times C^{(k,r)} \]

is a surjective submersion.
PROOF. To prove surjectivity of \((\pi^{r+1}_k, \mathcal{R}^{(k,r)})\) it is sufficient to consider the commutative diagram

\[
\begin{array}{ccc}
T^{r+1}_m Q & \xrightarrow{\mathcal{R}^{(r)}} & C^{(r)} \\
\pi^{r+1}_k & \downarrow & \downarrow \text{pr}_{r-1} \\
T^{k}_m Q & \xrightarrow{\mathcal{R}^{(k-1)}} & C^{(k-1)}
\end{array}
\]

All morphisms in the above diagram are surjective submersions which implies that for any element \(j^r_0 \gamma \in T^k_m Q\) the restriction of \(\mathcal{R}^{(r)}\) to the fibre \((\pi^{r+1}_k)^{-1}(j^r_0 \gamma)\) is a surjective submersion of the fibre \((\pi^{r+1}_k)^{-1}(j^r_0 \gamma)\) on the fibre \((\text{pr}_{r-1})^{-1}(\mathcal{R}^{(k-1)}(j^r_0 \gamma))\) \(\equiv C^{(k,r)}_{\mathcal{R}^{(k-1)}(j^r_0 \gamma)}\), which proves that the mapping \((\pi^{r+1}_k, \mathcal{R}^{(k,r)})\) is surjective. To prove that \((\pi^{r+1}_k, \mathcal{R}^{(k,r)})\) is a submersion we shall consider the above diagram for \(k = r\). From the formal covariant differentials of \([2.1]\) it follows, that \(\mathcal{R}^{(r,r)} \equiv \mathcal{R}_r\) is an affine morphism over \(\mathcal{R}^{(r-1)}\) (with respect to the affine structures on \(\pi^{r+1}_r : T^{r+1}_m Q \to T^r_m Q\) and \(\text{pr}_{r-1} : C^{(r)} \to C^{(r-1)}\) which has a constant rank. So the surjective morphism \((\pi^{r+1}_r, \mathcal{R}_r) : T^{r+1}_m Q \to T^r_m Q \times C^{(r,r)}\) has a constant rank and hence is a submersion. \((\pi^{r+1}_r, \mathcal{R}^{(k,r)})\) is then a composition of surjective submersions. \(\square\)

Let \(F\) be a natural bundle of order \(k \geq 1\), i.e., \(S_F\) is a left \(G^k_m\)-manifold.

**Theorem 3.2.** Let \(r + 2 \geq k\). For every \(G^{r+2}_m\)-equivariant map

\[ f : T^r_m Q \to S_F \]

there exists a unique \(G^k_m\)-equivariant map \(g : T^{k-2}_m Q \times C^{(k-2,r-1)} \to S_F\) satisfying

\[ f = g \circ (\pi^{r}_{k-2}, \mathcal{R}^{(k-2,r-1)}) . \]

**PROOF.** Let us consider the space \(S_r := \mathcal{R}^m \otimes \mathcal{R}^{mr}^r\) with coordinates \((s^\lambda_{\mu_1 \mu_2 \ldots \mu_r})\). Let us consider the action of \(G^r_m\) on \(S_r\) given by

\[ s^\lambda_{\mu_1 \mu_2 \ldots \mu_r} = s^\lambda_{\mu_1 \mu_2 \ldots \mu_r} - \tilde{a}^\lambda_{\mu_1 \ldots \mu_r}. \]

From Remark 2.3 and (3.5) it is easy to see that the symmetrization map \(\sigma_s : T^r_m Q \to S_{r+2}\) given by

\[ (s^\lambda_{\mu_1 \mu_2 \ldots \mu_{r+2}}) \circ \sigma_s = \Lambda_{\mu_1}^{\lambda} \mu_2 \ldots \mu_{r+2}, \]

is equivariant.

We have the \(G^{r+2}_m\)-equivariant map

\[ \varphi_r := (\sigma_r, \pi^{r}_{r-1}, \mathcal{R}_{r-1}) : T^{r}_m Q \to S_{r+2} \times T^{r-1}_m Q \times W_{r-1}. \]

On the other hand we define a \(G^{r+2}_m\)-equivariant map

\[ \psi_r : S_{r+2} \times T^{r-1}_m Q \times W_{r-1} \to T^r_m Q \]

over the identity of \(T^{r-1}_m Q\) by the following coordinate expression

\[ \Lambda_{\mu}^{\lambda}_{\nu, \rho_1 \ldots \rho_r} = s^\lambda_{\mu \nu \rho_1 \ldots \rho_r} + \text{lin}(w_{\mu}^{\lambda}_{\nu \rho_1 \ldots \rho_r} - \text{pol}(T^{r-1}_m Q)), \]

\[ (s^\lambda_{\mu_1 \mu_2 \ldots \mu_{r+2}}) \circ \sigma_s = \Lambda_{\mu_1}^{\lambda} \mu_2 \ldots \mu_{r+2}, \]

is equivariant.

We have the \(G^{r+2}_m\)-equivariant map

\[ \varphi_r := (\sigma_r, \pi^{r}_{r-1}, \mathcal{R}_{r-1}) : T^{r}_m Q \to S_{r+2} \times T^{r-1}_m Q \times W_{r-1}. \]

On the other hand we define a \(G^{r+2}_m\)-equivariant map

\[ \psi_r : S_{r+2} \times T^{r-1}_m Q \times W_{r-1} \to T^r_m Q \]

over the identity of \(T^{r-1}_m Q\) by the following coordinate expression

\[ \Lambda_{\mu}^{\lambda}_{\nu, \rho_1 \ldots \rho_r} = s^\lambda_{\mu \nu \rho_1 \ldots \rho_r} + \text{lin}(w_{\mu}^{\lambda}_{\nu \rho_1 \ldots \rho_r} - \text{pol}(T^{r-1}_m Q)), \]
where \( \text{lin} \) denotes a linear combination with real coefficients which arises in the following way. We recall that \( R_{r-1} \) gives the coordinate expression, given by formal covariant differentials of \((3.4)\),

\[
(3.7) \quad \Lambda_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_r} - \Lambda_\mu^{\lambda; \rho_1, \nu, \rho_2, \ldots, \rho_r} = w_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_r} - \text{pol}(T_m^{r-1}Q).
\]

We can write

\[
\Lambda_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_r} = s_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_r} + (\Lambda_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_r} - \Lambda_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_r}).
\]

Then the term in brackets can be written as a linear combination of terms of the type

\[
\Lambda_\mu^{\lambda; \nu, \rho_1, \ldots, \rho_i, \rho_{i+1}, \ldots, \rho_r} - \Lambda_\mu^{\lambda; \rho_1, \nu, \rho_2, \ldots, \rho_i, \rho_{i+1}, \ldots, \rho_r},
\]

\(i = 1, \ldots, r\), and from \((3.7)\) we get \((3.6)\).

Moreover,

\[
\psi_r \circ \varphi_r = \text{id}_{T_m^{r}Q}.
\]

Then the map \( f \circ \psi_r : S_{r+2} \times T_m^{r-1}Q \times W_{r-1} \to S_F \) satisfies the conditions of the orbit reduction Theorem \((1.1)\) for the group epimorphism \( \pi_{r+2}^{r+2} : G_{r+2} \to G_{r+1} \) and the surjective submersion \( \text{pr}_{2,3} : S_{r+1} \times T_m^{r-1}Q \times W_{r-1} \to T_m^{r-1}Q \times W_{r-1} \). Indeed, the space \( S_{r+2} \) is a \( B_{r+1}^{r+2} \)-orbit. Moreover, \((3.5)\) implies that the action of \( B_{r+1}^{r+2} \) on \( S_{r+2} \) is simply transitive. Hence there exists a unique \( G_{r+1}^{r+1} \)-equivariant map \( g_{r-1} : T_m^{r-1}Q \times W_{r-1} \to S_F \) such that the following diagram

\[
\begin{array}{ccc}
S_{r+2} \times T_m^{r-1}Q \times W_{r-1} & \xrightarrow{\psi_r} & T_m^{r}Q \\
\text{pr}_{2,3} \downarrow & & \downarrow \text{id}_{T_m^{r}Q} \\
T_m^{r-1}Q \times W_{r-1} & \xrightarrow{\text{id}_{T_m^{r-1}Q \times W_{r-1}}} & T_m^{r-1}Q \times W_{r-1} \xrightarrow{g_{r-1}} S_F
\end{array}
\]

commutes. So \( f \circ \psi_r = g_{r-1} \circ \text{pr}_{2,3} \) and if we compose both sides with \( \varphi_r \), by considering \( \text{pr}_{2,3} \circ \varphi_r = (\pi_{r-1}^{r}, R_{r-1}) \), we obtain \( f = g_{r-1} \circ (\pi_{r-1}^{r}, R_{r-1}) \).

In the second step we consider the same construction for the map \( g_{r-1} \) and obtain the commutative diagram

\[
\begin{array}{ccc}
(S_{r+1} \times T_m^{r-2}Q \times W_{r-2}) \times W_{r-1} & \xrightarrow{\psi_{r-1} \times \text{id}_{W_{r-1}}} & T_m^{r-1}Q \times W_{r-1} \xrightarrow{g_{r-1}} S_F \\
\text{pr}_{2,3} \times \text{id}_{W_{r-1}} \downarrow & & \downarrow \text{id}_{T_m^{r-1}Q \times W_{r-1}} \\
T_m^{r-2}Q \times W^{(r-2,r-1)} & \xrightarrow{\text{id}_{T_m^{r-2}Q \times W^{(r-2,r-1)}}} & T_m^{r-2}Q \times W^{(r-2,r-1)} \xrightarrow{g_{r-2}} S_F
\end{array}
\]

So that there exists a unique \( G_m^{r} \)-equivariant map \( g_{r-2} : T_m^{r-2}Q \times W^{(r-2,r-1)} \to S_F \) such that \( g_{r-1} = g_{r-2} \circ ((\pi_{r-2}^{r-1}, R_{r-2}) \times \text{id}_{W_{r-1}}) \), i.e., \( f = g_{r-2} \circ (\pi_{r-2}^{r-1}, R^{(r-2,r-1)}) \).

Proceeding in this way we get in the last step a unique \( G_m^{k} \)-equivariant map \( g_{k-2} : T_m^{k-2}Q \times C^{(k-2,r-1)} \to S_F \) such that

\[
f = g_{k-2} \circ (\pi_{k-2}^{r}, R^{(k-2,r-1)}).
\]

Putting \( g \) the restriction of \( g_{k-2} \) to \( T_m^{k-2}Q \times C^{(k-2,r-1)} \) we prove Theorem \((3.2)\). \(\square\)
In the above Theorem 3.2 we have find a map \( g \) which factorizes \( f \), but we did not prove, that \( (\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)}) : T_m^r Q \to T_m^{k-2} Q \times C^{(k-2,r-1)} \) satisfy the orbit conditions, namely we did not prove that \( (\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)})^{-1}(j_0^{k-2} \lambda, r^{(k-2,r-1)}) \) is a \( B_{k+2}^r \)-orbit for any \( (j_0^{k-2} \lambda, r^{(k-2,r-1)}) \in T_m^{k-2} Q \times C^{(k-2,r-1)} \). Now we shall prove it.

**Lemma 3.3.** If \( (j_0^r \gamma), (j_0^r \dot{\gamma}) \in T_m^r Q \) satisfy
\[
(\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)})(j_0^r \gamma) = (\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)})(j_0^r \dot{\gamma}),
\]
then there is an element \( h \in B_{k+2}^r \) such that \( (j_0^r h \gamma) = (j_0^r \dot{\gamma}) \).

**Proof.** Consider the orbit set \( T_m^r Q/B_{k+2}^r \). This is a \( G_m^r \)-set. Clearly the factor projection
\[
p : T_m^r Q \to T_m^r Q/B_{k+2}^r
\]
is a \( G_m^r \)-map. By Theorem 3.2 there is a \( G_m^k \)-equivariant map
\[
g : T_m^{k-2} Q \times C^{(k-2,r-1)} \to T_m^r Q/B_{k+2}^r
\]
satisfying \( p = g \circ (\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)}) \). If \( (\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)})(j_0^r \gamma) = (\pi^r_{k-2}, \mathcal{R}^{(k-2,r-1)})(j_0^r \dot{\gamma}) = (j_0^{k-2} \lambda, r^{(k-2,r-1)}) \), then \( p(j_0^r \gamma) = g(j_0^{k-2} \lambda, r^{(k-2,r-1)}) = p(j_0^r \dot{\gamma}) \), so \( j_0^r \gamma \) and \( j_0^r \dot{\gamma} \) are in the same \( B_{k+2}^r \)-orbit.

It is easy to see that \( T_m^{k-2} Q \times C^{(k-2,r-2)} \) is closed with respect to the action of the group \( G_m^k \). The corresponding natural bundle of order \( k \) is \( J^{k-2} \text{ Cla } M \times C^{(k,r)} M \). Then, as a direct consequence of Theorem 3.2 we obtain the first \( k \)-order reduction theorem for classical connections.

**Theorem 3.4.** Let \( F \) be a natural bundle of order \( k \geq 1 \) and let \( r + 2 \geq k \). All natural differential operators \( f : C^\infty(\text{ Cla } M) \to C^\infty(\mathcal{F}) \) which are of order \( r \) are of the form
\[
f(j^r \Lambda) = g(j^{k-2} \Lambda, \nabla^{(k-2,r-1)} R[\Lambda])
\]
where \( g \) is a unique natural operator
\[
g : J^{k-2} \text{ Cla } M \times C^{(k-2,r-1)} M \to F M.
\]

**Remark 3.5.** From the proof of Theorem 3.2 it follows that the operator \( g \) is the restriction of a natural operator defined on the natural bundle \( J^{k-2} \text{ Cla } M \times W^{(k-2,r-1)} M \).

4. **The second \( k \)-th order reduction theorem**

[22] defines for \( r = 2 \) the equation
\[
(E_2) \quad \quad \quad V^A_{\{\mu\}} - \text{pol}(C_0, V) = 0
\]
on \( C_0 \times V_2 \) and for \( r > 2 \) the system of equations
\[
(E_r) \quad \quad \quad V^A_{\mu_1...[\mu_{r-1} \mu_1]...\mu_r} - \text{pol}(C^{(r-2)} V^{(r-2)}) = 0
\]
on \( C^{(r-2)} \times V^{(r)} \).

The \( r \)-th *Ricci subspace* \( Z^{(r)} \subset C^{(r-2)} \times V^{(r)} \) is defined by solutions of \( (E_2), \ldots, (E_r) \), \( r \geq 2 \). For \( r = 0 \) we put \( Z^{(0)} = V \) and for \( r = 1 \) we put \( Z^{(1)} = V^{(1)} \). In [3] it was proved that \( Z^{(r)} \) is a submanifold in \( C^{(r-2)} \times V^{(r)} \) and \( (\mathcal{R}^{(r-2)}, \nabla^{(r)}) : T_m^{r-1} Q \times T_m^r V \to Z^{(r)} \) is a surjective
submersion. For $r > k - 1$ we can consider the projection $\text{pr}_{k-1}^r : Z^{(r)} \to Z^{(k-1)}$ and denote by $Z_{Z^{(k-1)}}^{(k-1)}$ its fiber in $z^{k-1} \in Z^{(k-1)}$. Then we shall denote by $T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(r,k)}$ the fiber product $(T_m^{k-2}Q \times T_m^{k-1}V) \times Z^{(k-1)} Z^{(r)}$.

**Lemma 4.1.** If $r + 1 \geq k \geq 1$, then the restricted map

$$(\pi_{k-2}^{r-1} \times \pi_{k-1}^r) \times (\mathcal{R}^{(k-2,r-2)}, \nabla^{(k,r)}) : T_m^{r-1}Q \times T_m^rV \to T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(k,r)}$$

is a surjective submersion.

**Proof.** The proof of Lemma 4.1 follows from the commutative diagram

$$
\begin{array}{ccc}
T_m^{r-1}Q \times T_m^rV & \xrightarrow{(\mathcal{R}^{(r-2)}, \nabla^{(r)})} & Z^{(r)} \\
\downarrow \pi_{k-2}^{r-1} \times \pi_{k-1}^r & & \downarrow \text{pr}_{k-1}^r \\
T_m^{k-2}Q \times T_m^{k-1}V & \xrightarrow{(\mathcal{R}^{(k-3)}, \nabla^{(k-1)})} & Z^{(k-1)}
\end{array}
$$

where all morphisms are surjective submersions. Hence $(\pi_{k-2}^{r-1} \times \pi_{k-1}^r) \times (\mathcal{R}^{(k-2,r-2)}, \nabla^{(k,r)})$ is surjective. For $k = r$ the map $(\mathcal{R}^{(r-2,r-2)} = \mathcal{R}_{r-2}, \nabla^{(r,r)} = \nabla^{(r)})$ is affine morphism over $(\mathcal{R}^{(r-3), \nabla^{(r-1)}})$ with a constant rank, i.e., $(\pi_{r-2}^{r-1} \times \pi_{r-1}^r) \times (\mathcal{R}_{r-2}, \nabla^{(r)})$ is a submersion. $(\pi_{k-2}^{r-1} \times \pi_{k-1}^r) \times (\mathcal{R}^{(k-2,r-2)}, \nabla^{(k,r)})$ is then a composition of surjective submersions. □

**Theorem 4.2.** Let $S_F$ be a left $G_m$ manifold. If $r + 1 \geq k \geq 1$, then for every $G_m^{r+1}$-equivariant map $f : T_m^{r-1}Q \times T_m^rV \to S_F$ there exists a unique $G_m^r$-equivariant map $g : T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(r,k)} \to S_F$ such that

$$f = g \circ ((\pi_{k-2}^{r-1} \times \pi_{k-1}^r) \times (\mathcal{R}^{(k-2,r-2)}, \nabla^{(k,r)}))$$

**Proof.** Consider the map

$$(\text{id}_{T_m^{r-1}Q} \times \pi_{k-1}^r) \times (\mathcal{R}^{(k,r)}) : T_m^{r-1}Q \times T_m^rV \to T_m^{r-1}Q \times T_m^{k-1}V \times V^{(r,k)}$$

and denote by $\check{V}^{(r,k)} \subset T_m^{r-1}Q \times T_m^{k-1}V \times V^{(r,k)}$ its image. By (2.3), the restricted morphism

$$\check{V}^{(k,r)} : T_m^{r-1}Q \times T_m^rV \to \check{V}^{(k,r)}$$

is bijective for every $j_0^{r-1} \gamma \in T_m^{r-1}Q$, so that $\check{V}^{(k,r)}$ is an equivariant diffeomorphism. Define

$$\mathcal{R}^{(k-2,r-2)} : \check{V}^{(k,r)} \to T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(k,r)}$$

by

$$(\mathcal{R}^{(k-2,r-2)}(j_0^{r-1} \gamma, j_0^{k-1} \mu, v) = (j_0^{k-2} \gamma, j_0^{k-1} \mu, \mathcal{R}^{(k-2,r-2)}(j_0^{r-1} \gamma), v))$$

for

$$(j_0^{r-1} \gamma, j_0^{k-1} \mu, v) \in \check{V}^{(k,r)}.$$ By Lemma 3.1, $\mathcal{R}^{(k-2,r-2)}$ is a surjective submersion.

Thus, Lemma 3.1 and Lemma 3.3 imply that $\mathcal{R}^{(k-2,r-2)}$ satisfies the orbit conditions for the group epimorphism $\pi_{k+1}^{r+1} : G_m^{r+1} \to G_m^k$ and there exists a unique $G_m^k$-equivariant map $g : T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(k,r)} \to S_F$ such that the diagram

$$\begin{array}{ccc}
\check{V}^{(k,r)} & \xrightarrow{(\mathcal{R}^{(k,r)})^{-1}} & T_m^{r-1}Q \times T_m^rV & \xrightarrow{f} & S_F \\
\downarrow \mathcal{R}^{(k-2,r-2)} & & \downarrow (\pi_{k-2}^{r-1} \times \pi_{k-1}^r, \nabla^{(k,r)}) & & \downarrow \text{id}_{S_F} \\
T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(k,r)} & \xrightarrow{\text{id}} & T_m^{k-2}Q \times T_m^{k-1}V \times Z^{(k,r)} & \xrightarrow{g} & S_F
\end{array}$$
commutes. Hence \( f \circ (\nabla^{(k,r)})^{-1} = g \circ \tilde{R}^{(k-2,r-2)} \). Composing both sides with \( \tilde{\nabla}^{(k,r)} \), by considering \( \tilde{R}^{(k-2,r-2)} \) \( \circ \tilde{\nabla}^{(k,r)} = (\pi_{k-2}^{-1} \times \pi_{r-1}^{-1}, \nabla^{(k,r)}) \), we get
\[
f = g \circ (\pi_{k-2}^{-1} \times \pi_{r-1}^{-1}, \nabla^{(k,r)}) .
\]

Then the second \( k \)-order reduction theorem can be formulated as follows.

**Theorem 4.3.** Let \( F \) be a natural bundle of order \( k \geq 1 \) and let \( r + 1 \geq k \). All natural differential operators \( f : C^\infty(C^\infty M \times V M) \to C^\infty(F M) \) of order \( r \) with respect sections of \( V M \) are of the form
\[
f(j^r\Lambda, j^r\Phi) = g(j^{k-2}\Lambda, j^{k-1}\Phi, \nabla^{(k-2,r-2)} R[\Lambda], \nabla^{(k,r)}\Phi)
\]
where \( g \) is a unique natural operator
\[
g : J^{k-2} C^\infty M \times J^{k-1} V M \times Z^{(k,r)} M \to F M .
\]

**Remark 4.4.** The order \((r - 1)\) of the above operators with respect to classical connections is the minimal order we have to use. The second reduction theorem can be easily generalized for any operators of order \( s \geq r - 1 \) with respect to connections. Then
\[
f(j^s\Lambda, j^r\Phi) = g(j^{k-2}\Lambda, j^{k-1}\Phi, \nabla^{(k-2,s-1)} R[\Lambda], \nabla^{(k,r)}\Phi) .
\]

**Remark 4.5.** If \( \Lambda \) is a linear non-symmetric connection on \( M \), then there exists its splitting \( \Lambda = \tilde{\Lambda} + T \), where \( \tilde{\Lambda} \) is the classical connection obtained by the symmetrization of \( \Lambda \) and \( T \) is the torsion tensor of \( \Lambda \). Then all natural operators of order \( r \) defined on \( \Lambda \) are of the form
\[
f(j^r\Lambda) = f(j^2\Lambda) = g(j^{k-2}\tilde{\Lambda}, j^{k-1}T, \tilde{\nabla}^{(k-2,r-1)} R[\tilde{\Lambda}], \tilde{\nabla}^{(k,r)}T) .
\]

**Remark 4.6.** If \( g \) is a metric field on \( M \), then there exists the unique classical Levi Civita connection \( \Lambda \) given by the metric field \( g \). Then, applying the second reduction theorem, we get that all natural operators of order \( r \geq 1 \) defined on \( g \) are of the form
\[
f(j^r g) = f(j^{r-1}\tilde{\Lambda}, j^r g) = h(j^{k-2}\tilde{\Lambda}, j^{k-1}g, \tilde{\nabla}^{(k-2,r-2)} R[\tilde{\Lambda}], \tilde{\nabla}^{(k,r)}T) .
\]

5. **Natural \((0,2)\)-tensor fields on the cotangent bundle**

Typical applications of of higher order reduction theorems are classifications of natural tensor fields on the tangent (or cotangent) bundle of a manifold endowed with a classical connection or lifts of tensor fields to the tangent (or cotangent) bundle by means of a classical connection, see [2] [3] [6] [13] [14].

As a direct consequence of Theorem 3.2, Theorem 4.3 and Remark 4.5 we get

**Corollary 5.1.** Let \((M, \Lambda)\) be a manifold endowed with a linear (non-symmetric) connection \( \Lambda \). Then any natural tensor field \( \Phi \) on \( TM \) or \( T^* M \) of order \( r \) is of the type
\[
\Phi(u, j^r \Lambda) = \Phi(u, \tilde{\Lambda}, j^r T, \tilde{\nabla}^{(r-1)} R[\tilde{\Lambda}], \tilde{\nabla}^{(2,r)} T) ,
\]
where \( u \in TM \) or \( u \in T^* M \), respectively, \( \tilde{\Lambda} \) is the classical connection given by the symmetrization of \( \Lambda \) and \( T \) is the torsion tensor of \( \Lambda \).
Corollary 5.2. Let \((M, \Lambda, \Psi)\) be a manifold endowed with a linear (non-symmetric) connection \(\Lambda\) and a tensor field \(\Psi\). Then any natural tensor field \(\Phi\) on \(TM\) or \(T^*M\) of order \(s\) with respect to \(\Lambda\) and of order \(r\), \(s \geq r - 1\), with respect to \(\Psi\) is of the type
\[
\Phi(u, j^s \Lambda, j^r \Psi) = \Phi(u, \tilde{\Lambda}, j^1 T, j^1 \Psi, \tilde{\nabla}^{(s-1)} R[\tilde{\Lambda}], \tilde{\nabla}^{(2,s)} T, \tilde{\nabla}^{(2,r)} \Psi),
\]
where \(u \in TM\) or \(u \in T^*M\), respectively.

As a concrete example let us classify all \((0,2)\)-tensor fields on \(T^*M\) given by a linear (non-symmetric) connection \(\Lambda\).

Theorem 5.3. Let \((M, \Lambda)\) be a manifold endowed with a linear (non-symmetric) connection \(\Lambda\). Then all finite order natural \((0,2)\)-tensor fields on \(T^*M\) are of order one and they form a 14-parameter family of operators with coordinate expression
\[
(5.1) \quad \Phi = (A \dot{x}_\lambda \dot{x}_\mu + C_1 \dot{x}_\lambda T_{\rho \mu} + C_2 \dot{x}_\mu T_{\rho \lambda} + C_3 \dot{x}_\rho T_{\lambda \mu} + F_1 T_{\rho \lambda} T_{\sigma \mu} + F_2 T_{\sigma \lambda} T_{\rho \mu} + F_3 T_{\rho \sigma} T_{\lambda \mu} + G_1 T_{\rho \lambda} \mu + G_2 T_{\rho \mu} \lambda + G_3 T_{\lambda \rho} \mu + H_1 R_{\rho \lambda} \mu + H_2 R_{\lambda \rho} \mu) d^\lambda \otimes d^\mu + B d^\lambda \otimes (\dot{d}_\lambda + \Lambda_{\rho \mu} \dot{x}_\rho d^\mu) + C (\dot{d}_\lambda + \Lambda_{\rho \mu} \dot{x}_\rho d^\mu) \otimes d^\lambda,
\]
where \(A, B, C, C_1, F_1, G_1, H_1, i = 1, 2, 3, j = 1, 2\), are real constants.

Proof. Let us denote by \(S = \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*} \otimes \mathbb{R}^{m*}\) the standard fibre of \(\otimes^2 T^*(T^*M)\). The coordinates on \(S\) will be denoted by \((\dot{x}_\lambda, \phi_{\lambda \mu}, \phi_{\lambda \tilde{\mu}}, \phi_{\lambda \mu}, \phi_{\lambda \tilde{\mu}})\). Then we have the following action of the group \(G_2^m\) on \(S\)
\[
\begin{align*}
\tilde{\dot{x}}_\lambda &= \tilde{A}_\lambda^\mu \dot{x}_\mu, \\
\tilde{\phi}_{\lambda \mu} &= \tilde{A}_\lambda^\rho \phi_{\rho \sigma} + \alpha_\lambda^\beta a_{\sigma \rho} \tilde{A}_\mu^\alpha \tilde{a}_\alpha \tilde{a}_\kappa \phi_{\rho \sigma} + \alpha_\mu^\alpha a_{\rho \beta} \tilde{A}_\lambda^\kappa \tilde{a}_\kappa \phi_{\beta \sigma}, \\
\tilde{\phi}_{\lambda \tilde{\mu}} &= \tilde{A}_\lambda^\rho \phi_{\rho \phi} + \alpha_\mu^\alpha a_{\rho \beta} \tilde{A}_\lambda^\kappa \tilde{a}_\kappa \phi_{\beta \phi}, \\
\tilde{\phi}_{\lambda \mu} &= \tilde{A}_\lambda^\rho \phi_{\rho \phi} + \alpha_\mu^\alpha a_{\rho \beta} \tilde{A}_\lambda^\kappa \tilde{a}_\kappa \phi_{\beta \phi}, \\
\tilde{\phi}_{\lambda \tilde{\mu}} &= \tilde{A}_\lambda^\rho \phi_{\rho \phi} + \alpha_\mu^\alpha a_{\rho \beta} \tilde{A}_\lambda^\kappa \tilde{a}_\kappa \phi_{\beta \phi}.
\end{align*}
\]

First let us discuss \(\tilde{\phi}_{\lambda \tilde{\mu}}\). We have, by Corollary 5.1
\[
\phi_{\lambda \tilde{\mu}} = \phi_{\lambda \tilde{\mu}}(\dot{x}_\lambda, \Lambda_{\mu \nu}, T_{\mu \nu}, \Lambda_{\nu \sigma}, R_{\nu \rho \lambda \mu \nu \sigma} \cdots, w_{\mu \nu \cdots}),
\]
i = 0, \ldots, r - 1, j = 2, \ldots, r. The equivariance with respect to homotheties \((c \delta_{\mu}^\lambda)\) implies
\[
c^2 \phi_{\lambda \tilde{\mu}} = \phi_{\lambda \tilde{\mu}}(c^{-1} \dot{x}_\lambda, c^{-1} \Lambda_{\mu \nu}, c^{-1} T_{\mu \nu}, c^{-2} T_{\nu \sigma}, c^{-i+2} \tilde{R}_{\nu \rho \lambda \mu \nu \sigma} \cdots, c^{-(j+1)} T_{\nu \rho \lambda \mu \nu \sigma})
\]
which implies, by the homogeneous function theorem, [3], that \(\phi_{\lambda \tilde{\mu}}\) is a polynomial of orders \(a\) in \(\dot{x}_\lambda, b\) in \(\Lambda_{\mu \nu}, c_0\) in \(T_{\mu \nu}, c_1\) in \(T_{\nu \sigma}, d_i\) in \(\tilde{R}_{\nu \rho \lambda \mu \nu \sigma} \cdots, e_j\) in \(T_{\nu \rho \lambda \mu \nu \sigma}\), such that
\[
(5.2) \quad 2 = -a - b - c_0 - 2 c_1 - \sum_{i=0}^{r-1} (i+2) d_i - \sum_{j=2}^{r} (j+1) e_j.
\]
The equation (5.2) has no solution in natural numbers, so we get by the homogeneous function theorem that $\phi^{\lambda\bar{\mu}}$ is independent of all variables and so it have to be absolute invariant, hence

$$\phi^{\lambda\bar{\mu}} = 0.$$ 

For $\phi^{\lambda\bar{\mu}}$ and $\phi^{\bar{\lambda}\mu}$ we get from the equivariancy with respect to the homotheties ($c\delta^\lambda_\mu$) that they are polynomials of orders satisfying

$$0 = -a - b - c_0 - 2c_1 - \sum_{i=0}^{r-1} (i + 2) d_i - \sum_{j=2}^{r} (j + 1) e_j.$$ 

(5.3)

So also $\phi^{\lambda\bar{\mu}}$ and $\phi^{\bar{\lambda}\mu}$ are independent of all variables and they have to be absolute invariant, hence

$$\phi^{\lambda\bar{\mu}} = B\delta^{\mu}_\lambda, \quad \phi^{\bar{\lambda}\mu} = C\delta^\lambda_\mu.$$ 

(5.4)

Finally $\phi_{\lambda\mu}$ has to be a polynomial of orders satisfying

$$-2 = -a - b - c_0 - 2c_1 - \sum_{i=0}^{r-1} (i + 2) d_i - \sum_{j=2}^{r} (j + 1) e_j.$$ 

(5.5)

There are 8 possible solutions of (5.5):

- $a = 2$ and the others exponents vanish;
- $a = 1$, $b = 1$ and the others exponents vanish;
- $a = 1$, $c_0 = 1$ and the others exponents vanish;
- $b = 2$ and the others exponents vanish;
- $b = 1$, $c_0 = 1$ and the others exponents vanish;
- $c_0 = 2$ and the others exponents vanish;
- $c_1 = 1$ and the others exponents vanish;
- $d_0 = 1$ and the others exponents vanish.

It implies that the maximal order of the operator is one and $\phi_{\lambda\mu}$ is of the form

$$\phi_{\lambda\mu} = A^{\rho\sigma}_{\lambda\mu} \dot{x}_\rho \dot{x}_\sigma + B^{\rho\omega\tau}_{\lambda\mu k} \dot{x}_\rho \tilde{\Lambda}_\omega^\kappa \tau + C^{\rho\omega\tau}_{\lambda\mu k} \dot{x}_\rho T_\omega^\kappa \tau$$

$$+ D^{\omega\tau_1\omega_2\tau_2}_{\lambda\mu k_{1} k_{2}} \tilde{\Lambda}_{\omega_1}^{\kappa_{1}} \tau_1 \tilde{\Lambda}_{\omega_2}^{\kappa_{2}} \tau_2 + E^{\omega\tau_1\omega_2\tau_2}_{\lambda\mu k_{1} k_{2}} \tilde{\Lambda}_{\omega_1}^{\kappa_{1}} \tau_1 T_{\omega_2}^\kappa \tau_2$$

$$+ F^{\omega\tau_1\omega_2\tau_2}_{\lambda\mu k_{1} k_{2}} T_{\omega_1}^{\kappa_{1}} \tau_1 T_{\omega_2}^{\kappa_{2}} \tau_2 + G^{\omega\tau\epsilon}_{\lambda\mu k} T_\omega^\kappa \tau_{\epsilon} + H^{\omega\tau\epsilon}_{\lambda\mu k} \tilde{R}_\omega^\kappa \tau_{\epsilon},$$

where $\tilde{\Lambda}_\omega^\kappa \tau$ and $\tilde{R}_\omega^\kappa \tau_{\epsilon}$ are certain tensors.
where \( A^{\rho\sigma}_{\lambda\mu}, \ldots, H^\omega_{\lambda\mu\nu} \) are absolute invariant tensors, i.e.,

\[
\phi_{\lambda\mu} = A \dot{x}_\lambda \dot{x}_\mu + B_1 \dot{x}_\lambda \tilde{\Lambda}_\rho^\rho\mu + B_2 \dot{x}_\mu \tilde{\Lambda}_\rho^\rho\lambda + B_3 \dot{x}_\rho \tilde{\Lambda}_\lambda^\rho_{\lambda\mu} + C_1 \dot{x}_\lambda T^\rho_{\rho\mu} + C_2 \dot{x}_\mu T^\rho_{\rho\lambda} + C_3 \dot{x}_\rho T^\rho_{\lambda\mu} \\
+ D_1 \tilde{\Lambda}_\rho^\rho\lambda \Lambda^\sigma_{\sigma\mu} + D_2 \tilde{\Lambda}_\rho^\rho\lambda \tilde{\Lambda}^\sigma_{\sigma\mu} + D_3 \Lambda^\rho_{\rho\mu} \Lambda^\lambda_{\lambda\mu} \\
+ E_1 \tilde{\Lambda}_\rho^\rho\lambda T^\sigma_{\sigma\mu} + E_2 \tilde{\Lambda}_\rho^\rho\lambda T^\sigma_{\mu\sigma} + E_3 \tilde{\Lambda}^\rho_{\rho\mu} T^\lambda_{\lambda\sigma} \\
+ E_4 \tilde{\Lambda}^\rho_{\rho\mu} T^\sigma_{\sigma\lambda} + E_5 \tilde{\Lambda}^\rho_{\rho\mu} T^\sigma_{\mu\lambda} + E_6 \tilde{\Lambda}^\rho_{\rho\mu} T^\sigma_{\sigma\lambda} \\
+ F_1 T^\rho_{\rho\lambda} T^\sigma_{\sigma\mu} + F_2 T^\rho_{\rho\lambda} T^\sigma_{\mu\sigma} + F_3 T^\rho_{\rho\mu} T^\sigma_{\lambda\sigma} \\
+ G_1 T^\rho_{\rho\lambda} \lambda_{\mu} + G_2 T^\rho_{\rho\lambda} \lambda_{\mu} + G_3 T^\rho_{\rho\lambda} \lambda_{\mu}, \\
+ H_1 \tilde{R}^\rho_{\rho\lambda} \mu + H_2 \tilde{R}^\rho_{\rho\mu} \lambda \mu, \\
\phi^\lambda_{\mu} = B_{\delta^\mu_\lambda}, \quad \phi^\lambda_{\mu} = C_{\delta^\mu_\lambda}, \quad \phi^\lambda_{\mu} = 0,
\]

which is the equivariant mapping corresponding to (5.1). \( \square \)

**Remark 5.4.** Let us note that the canonical symplectic form \( \omega \) of \( T^*M \) is a special case of (5.1). Namely, for \( C = -B \neq 0 \) and the other coefficients vanish we get just the scalar multiple of \( \omega = d^\lambda \otimes d_\lambda - \tilde{d}_\lambda \otimes d^\lambda \).

The invariant description of the tensor fields (5.1) is the following. We have the canonical Liouville 1-form on \( T^*M \) given in coordinates by

\[
\theta = \dot{x}_\lambda d^\lambda.
\]

The operator standing by \( A \) is then \( \theta \otimes \theta \).

\( A \) gives 3-parameter family of (1,2) tensor fields on \( M \), (5.1), given by

\[
(5.6) \quad S(A) = C_1 I_{TM} \otimes \hat{T} + C_2 \hat{T} \otimes I_{TM} + C_3 T,
\]

where \( \hat{T} \) is the contraction of the torsion tensor and \( I_{TM} : M \to TM \otimes T^*M \) is the identity tensor. Then the the evaluation \( \langle S(A), u \rangle \) gives three operators standing by \( C_1, C_2, C_3 \).
The connection $\Lambda$ defines naturally the following 8 parameter family of $(0,2)$-tensor fields on $M$, given by

$$G(\Lambda) = F_1 C_{13}^{12} (T \otimes T) + F_2 C_{31}^{12} (T \otimes T) + F_3 C_{12}^{12} (T \otimes T)$$

$$+ G_1 C_1^1 \tilde{\nabla} T + G_2 C_1^2 \tilde{\nabla} T + G_3 C_1^3 \tilde{\nabla} T$$

$$+ H_1 C_1^1 R[\tilde{\Lambda}] + H_2 C_2^1 R[\tilde{\Lambda}],$$

where $C_{ij}^{kl}$ is the contraction with respect to indicated indices and $C_1^i \tilde{\nabla} T$ denotes the conjugated tensor obtained by the exchange of subindices. The second 8-parameter subfamily of operators from (5.1) is then given by the pullback of $G(\Lambda)$ to $T^* M$.

The last two operators are given by the vertical projection

$$\nu[\Lambda^*] : T^* M \to T^* T^* M \otimes VT^* M, \quad \nu[\Lambda^*] = (\tilde{d}_\lambda + A_\lambda^\rho \tilde{x}_\rho d^{\mu}) \otimes \tilde{\partial}^\lambda$$

associated with the connection $\Lambda^*$ dual to $\Lambda$. Two possible contractions of $\nu[\Lambda^*] \otimes \omega$ give the last two operators.

**Remark 5.5.** For a symmetric connection $\Lambda$ the family (5.1) reduces to 5-parameter family.

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