Stability of Heisenberg Isoperimetric Profiles

Francescopaolo Montefalcone

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Abstract

In the context of sub-Riemannian Heisenberg groups $H^n$, $n \geq 1$, we shall study Isoperimetric Profiles, which are closed compact hypersurfaces having constant horizontal mean curvature, very similar to ellipsoids. Our main goal is to study the stability of Isoperimetric Profiles.

Key words and phrases: Carnot groups; Sub-Riemannian geometry; hypersurfaces.

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1 Introduction

In the last few years sub-Riemannian Carnot groups have become a large research field in Analysis and Geometric Measure Theory; see, for instance, [2], [5], [8], [10], [11], [15], [16], [23], [31], [24], [27], [38], but the list is far from being exhaustive. For a general overview of sub-Riemannian (or Carnot-Charactéodory) geometries, we refer the reader to Gromov, [18], Pansu, [37], and Montgomery, [30].

In this paper, our ambient space is the Heisenberg group $H^n$, $n \geq 1$, which can be regarded as $C^n \times \mathbb{R}$ endowed with a polynomial group law $*: \mathbb{H}^n \times \mathbb{H}^n \rightarrow \mathbb{H}^n$. Its Lie algebra $\mathfrak{h}_n$ identifies with the tangent space $T_0 \mathbb{H}^n$ at the identity $0 \in \mathbb{H}^n$. Later on, $(z,t) \in \mathbb{R}^{2n+1}$ will denote exponential coordinates of a generic point $p \in \mathbb{H}^n$. Now, take a left-invariant frame $\mathcal{F} = \{X_1, Y_1, \ldots, X_n, Y_n, T\}$ for the tangent bundle $T \mathbb{H}^n$, where $X_i = \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}$, $Y_i = \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}$, and $T = \frac{\partial}{\partial t}$. Denoting by $[,]$ the usual Lie bracket of vector fields, one has $[X_i, Y_i] = T$ for every $i = 1, \ldots, n$ and all other commutators vanish. Hence, $T$ is the center of $\mathfrak{h}_n$ and $\mathfrak{h}_n$ turns out to be nilpotent and stratified of step 2, i.e. $\mathfrak{h}_n = H \oplus H_2$ where $H := \text{span}_\mathbb{R}\{X_1, Y_1, \ldots, X_n, Y_n\}$ is the horizontal bundle and $H_2 = \text{span}_\mathbb{R}\{T\}$ is the 1-dimensional vertical bundle associated with the center of $\mathfrak{h}_n$. From now on, $\mathbb{H}^n$ will be endowed with the (left-invariant) Riemannian metric $h := \langle \cdot, \cdot \rangle$ which makes $\mathcal{F}$ an orthonormal frame. In particular, this metric induces a corresponding metric $h_H$ on $H$ which is used in order to measure the length of horizontal curves. Note that the natural distance in sub-Riemannian geometry is the Carnot-Carathéodory distance $d_C$, defined by minimizing the (Riemannian) length of all piecewise smooth horizontal curves joining two different points. This definition makes sense because, in view of Chow’s Theorem, different points can be joined by (infinitely many) horizontal curves.

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The stratification of $\mathfrak{h}_n$ is related with the existence of a 1-parameter group of automorphisms, called Heisenberg dilations, defined by $\delta_{\rho}(z,t) := (sz, s^2t)$, for every $p \equiv (z,t) \in \mathbb{R}^{2n+1}$. The intrinsic dilations play an important role in this geometry. In this regard, we stress that the integer $Q = 2n + 2$, that is the “homogeneous dimension” of $\mathbb{H}^n$ with respect to these anisotropic dilations, turns out to be the dimension of $\mathbb{H}^n$ as a metric space with respect to the CC-distance $d_{\mathcal{C}}$.

Let us define a key notion: that of $H$-perimeter\(^1\). So let $S \subset \mathbb{H}^n$ be a smooth hypersurface and let $\nu$ the (Riemannian) unit normal along $S$. The $H$-perimeter measure $\sigma_{\nu}^2$ is the $(Q - 1)$-homogeneous measure, with respect to the intrinsic dilations, given by $\sigma_{\nu}^2 := |\mathcal{P}_\nu| \sigma_{\nu}^2$, where $\mathcal{P}_\nu : TG \rightarrow H$ is the orthogonal projection operator onto $H$ and $\sigma_{\nu}^2$ denotes the Riemannian measure on $S$. The $H$-perimeter is in fact the natural measure on hypersurfaces and it turns out to be equivalent, up to a density function called metric factor (see, for instance, [24]), to the spherical $(Q - 1)$-dimensional Hausdorff measure associated with $d_{\mathcal{C}}$ (or to any other homogeneous distance on $\mathbb{H}^n$).

Our main interest concerns “Isoperimetric Profiles”, that are compact closed hypersurfaces which can be described in terms of CC-geodesics (even if they are not CC-balls). In Heisenberg groups, they play an equivalent role of spheres in Euclidean spaces and for this reason it seems interesting to study some basic geometric features of these sets from an intrinsic point of view; see also [28].

Isoperimetric Profiles, henceforth denoted by the symbol $\mathcal{S}_{\mathbb{H}^n}$, turn out to be constant horizontal mean curvature hypersurfaces (i.e. $\mathcal{H}_H = - \text{div}_H \nu_\gamma$ is constant; in particular, this implies that they are critical points of the $H$-perimeter functional) whose importance comes from a long-standing conjecture, usually attributed to Pansu, claiming that they minimize the $H$-perimeter in the class of finite $H$-perimeter sets (in the variational sense) having fixed volume, or in other words, they solve the sub-Riemannian isoperimetric problem in $\mathbb{H}^n$. There is a wide literature on this subject; see, for instance, [5], [10], [11], [12], [23], [31], [32], [33], [34], [35], [36], [37] and references therein.

The plan of the paper is the following. In Section 1.1 we review the sub-Riemannian geometry of Heisenberg groups $\mathbb{H}^n$. We then discuss some basics about smooth hypersurfaces endowed with the $H$-perimeter measure $\sigma_{\nu}^2$ and we prove some important geometric facts; see Section 1.2. Section 1.3 provides some horizontal integration by parts formulas. In Section 2 we study Isoperimetric Profiles and compute some of their geometric invariants appearing in the 2nd variation formula of the $H$-perimeter. Section 3 gives a self-contained account of variational formulas for the $H$-perimeter measure $\sigma_{\nu}^2$ along the lines of [27], but in addition we consider the case of non-empty characteristic sets. These formulas are then used as a tool to study the (homogeneous) sub-Riemannian Isoperimetric Functional

$$J(D) := \frac{\sigma_{\nu}^2(\partial D)}{(\sigma_{\nu}^{2n+1}(D))^{\frac{1}{1 - n}}} ,$$

where $D \subset \mathbb{H}^n$ varies among $C^2$-smooth compact domains. In Section 3.1 we calculate 1st and 2nd variation of the top-dimensional volume form $\sigma_{\nu}^{2n+1}$. This allow us to state the notion of stability for smooth domains bounded by constant horizontal mean curvature hypersurfaces; see Definition 3.14 and Definition 3.17. If $D$ has radial symmetry with respect to a barycentric vertical axis, we also consider a restricted family of (normal) radial variations. In this case, the functional $J(D)$ becomes 1-dimensional and stability becomes radial stability; see Remark 3.16. We also introduce a localized notion of stability. Roughly speaking, being locally stable means that for each point $p \in \partial D$ there exists a neighborhood of $p$ which is stable in the previous sense; see Definition 3.13. For completeness, in the Appendix A we shall discuss the simpler (but less general) case of $T$-graphs. Moreover, in the Appendix B, we shall discuss some further properties which are related to stability.

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1. Since we deal with smooth boundaries, we do not define the $H$-perimeter from a variational point of view.
2. This point can be interpreted as the “cut point” of $S$ along $\gamma$. In fact this is the end-point of all CC-geodesics starting from $S$ with same slope. Note however that, strictly speaking the cut locus of any point in $\mathbb{H}^n (n \geq 1)$ coincides with the vertical $T$-line over the point.
3. Let $\gamma_0 \subset \mathbb{H}^2$ denote the circle given by orthogonal projection of $\gamma$ onto $\mathbb{R}^2$ and let $r$ be its radius. Then, it turns out that $d_{\mathcal{C}}(\mathcal{S}, N) = \pi r^2$. 

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Then, in Section [1] we begin the study of the stability of Isoperimetric Profiles, or the positivity of the 2nd variation of the isoperimetric functional $J(\cdot)$. Our approach was somehow motivated by the Riemannian case described here below; see, for instance, [1].

We recall that the 2nd variation (under normal variations) of a compact closed bounding hypersurface $S$ embedded in Euclidean space $\mathbb{R}^n$ is provided by the formula

$$II(\varphi, S) = \int_S (-\varphi \Delta_{S^\text{n}} \varphi - \varphi^2 \|B\|_{G^r}^2) \sigma_{R}^{n-1}$$

for every (piecewise) smooth $\varphi : S \rightarrow \mathbb{R}$, where $\Delta_{S^\text{n}}$ denotes the Laplace-Beltrami operator and $\|B\|_{G^r}$ is the Gram norm of the 2nd fundamental form $B$ of $S$. So let $S^{n-1} \subset \mathbb{R}^n$ be the unit sphere and let us apply the Rayleigh principle; see [6, 7]. We have

$$\lambda_1 := \lambda_1(S^{n-1}) \leq \frac{\int_{S^{n-1}} |\text{grad}_{S^\text{n}} \varphi|^2 \sigma_{S^\text{n}}^{n-1}}{\int_{S^{n-1}} \varphi^2 \sigma_{R}^{n-1}}$$

for every smooth function $\varphi : S^{n-1} \rightarrow \mathbb{R}$ such that $\int_{S^{n-1}} \varphi \sigma_{R}^{n-1} = 0$, where $\lambda_1$ denotes the first non-trivial eigenvalue of (the closed eigenvalue problem on) $S^{n-1}$. It is well-known that $\lambda_1 = n - 1$ and that $\|B\|_{G^r}^2 = n - 1$. Therefore,

$$II(\varphi, S^{n-1}) = \int_{S^{n-1}} (-\varphi \Delta_{S^\text{n}} \varphi - \varphi^2 \|B\|_{G^r}^2) \sigma_{R}^{n-1} = \int_{S^{n-1}} (|\text{grad}_{S^\text{n}} \varphi|^2 - (n-1)\varphi^2) \sigma_{R}^{n-1} \geq 0,$$

where we have used the Divergence Theorem. This proves the stability of $S^{n-1}$, i.e. $II(\varphi, S^{n-1}) \geq 0$ for every differentiable function $\varphi : S^{n-1} \rightarrow \mathbb{R}$ such that $\int_{S^{n-1}} \varphi \sigma_{R}^{2n-1} = 0$.

However, a such strategy does not work verbatim in the framework of Heisenberg groups and our methods, although similar in spirit, are very different. Actually, the main analogy here is that the positivity of the 2nd variation formula can be studied in terms of an eigenvalue equation associated with the (2nd variation) functional

$$\mathfrak{F}(\varphi) := \int_{S^{2n}} \left(|\text{grad}_{S^2} \varphi|^2 - \frac{Q-4}{\rho^2} \varphi^2\right) \sigma_{S^2}^{2n}$$

subject to the condition $\int_{S^{2n}} \varphi \sigma_{S^2}^{2n} = 0$. In this formula, $\text{grad}_{S^2}$ denotes the horizontal tangent gradient operator and $\rho$ stands for the (Euclidean) distance from the vertical $T$-line passing through the barycenter of $S^{2n}$; for a detailed discussion, see Section 3.

Our main results concerning stability of Isoperimetric Profiles can be summarized as follows:

- Let $n = 1$. The Isoperimetric profile $\mathbb{S}^{2n+1}$ is a stable bounding hypersurface in the sense of both Definition 3.14 and Definition 3.17.

- Let $n > 1$. The Isoperimetric profile $\mathbb{S}^{2n}$ is a radially stable bounding hypersurface in the sense of Remark 3.10. Furthermore, the Isoperimetric Profile $\mathbb{S}^{2n}$ turns out to be a locally stable bounding hypersurface in the sense of Definition 3.15.

This paper is part of a project aiming to study constant and minimal horizontal mean curvature hypersurfaces, in the setting of Heisenberg groups; see also [28]. I would like to express my gratitude to Prof. N. Garofalo and to Prof. A. Parmeggiani for many interesting conversations about these topics over the past few years.

1.1 Heisenberg group $\mathbb{H}^n$

The $n$-th Heisenberg group ($\mathbb{H}^n, \star$), $n \geq 1$, is a connected, simply connected, nilpotent and stratified Lie group of step 2 on $\mathbb{R}^{2n+1}$, with respect to a polynomial group law $\star$. The Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}^n$ is a $(2n+1)$-dimensional real vector space henceforth identified with the tangent space $T_0\mathbb{H}^n$ at the identity $0 \in \mathbb{H}^n$. We adopt exponential coordinates of the 1st kind in such a way that every point $p \in \mathbb{H}^n$ can be written out as $p = \exp(x_1, y_1, \ldots, x_i, y_i, \ldots, x_n, y_n)$. The Lie algebra $\mathfrak{h}_n$ can be described by means of a frame $\mathcal{F} := \{X_1, Y_1, \ldots, X_i, Y_i, \ldots, X_n, Y_n, T\}$ of left-invariant vector fields for $T\mathbb{H}^n$, where $X_i(p) := \frac{\partial}{\partial x_i} - \frac{y_i}{2} \frac{\partial}{\partial t}$, $Y_i(p) := \frac{\partial}{\partial y_i} + \frac{x_i}{2} \frac{\partial}{\partial t}$, $i = 1, \ldots, n$, $T(p) := \frac{\partial}{\partial t}$, for every $p \in \mathbb{H}^n$. More precisely,
denoting by \([\cdot, \cdot]\) the Lie bracket of vector fields, we get that \([X_i, Y_i] = T\) for every \(i = 1, \ldots, n\), and all other commutators vanish. In other words, \(T\) is the center of \(h_n\) and \(h_n\) turns out to be a nilpotent and stratified Lie algebra of step 2, i.e. \(h_2 = H \oplus H_2\). The first layer \(H\) is called horizontal whereas the complementary layer \(H_2 = \text{span}_2\{T\}\) is called vertical. A horizontal left-invariant frame for \(H\) is given by \(\mathcal{F}_H = \{X_1, Y_1, \ldots, X_n, Y_n\}\). The group law \(*\) on \(h_n\), i.e. \(\exp X \ast \exp Y = \exp(X \circ Y)\) for every \(X, Y \in h_n\), where \(\circ : h_n \times h_n \to h_n\) is defined by \(X \circ Y = X + Y + \frac{1}{2}([X, Y])\). Thus, for every \(p = \exp(x_1, y_1, \ldots, x_n, y_n, t)\), \(p' = \exp(x'_1, y'_1, \ldots, x'_n, y'_n, t')\in \mathbb{H}^n\) we have

\[
p \ast p' := \exp \left( x_1 + x'_1, y_1 + y'_1, \ldots, x_n + x'_n, y_n + y'_n, t + t' + \frac{1}{2} \sum_{i=1}^{n} (x_i y'_i - x'_i y_i) \right).
\]

The inverse of \(p \in \mathbb{H}^n\) is given by \(p^{-1} := \exp(-x_1, -y_1, \ldots, -x_n, -y_n, -t)\) and \(0 = \exp(0_{2n+1})\). Later on, we shall set \(z := (x_1, y_1, \ldots, x_n, y_n) \in \mathbb{R}^{2n}\) and identify each point \(p \in \mathbb{H}^n\) with its exponential coordinates \((z, t) \in \mathbb{R}^{2n+1}\).

**Definition 1.1.** We call sub-Riemannian metric \(h_H\) any symmetric positive bilinear form on \(H\). The CC-distance \(d_{CC}(p, p')\) between \(p, p' \in \mathbb{H}^n\) is defined by

\[
d_{CC}(p, p') := \inf \int \sqrt{h_H(\dot{\gamma}, \dot{\gamma})} dt,
\]

where the inf is taken over all piecewise-smooth horizontal curves \(\gamma\) joining \(p\) to \(p'\). We shall equip \(T\mathbb{H}^n\) with the left-invariant Riemannian metric \(h := \langle \cdot, \cdot \rangle\) making \(\mathcal{F}\) an orthonormal -abbreviated o.n.- frame and assume \(h_H := h|_H\).

By Chow’s Theorem it turns out that every couple of points can be connected by a horizontal curve, not necessarily unique, and for this reason \(d_{CC}\) turns out to be a metric on \(\mathbb{H}^n\). Moreover, the \(d_{CC}\)-topology is equivalent to the Euclidean topology on \(\mathbb{R}^{2n+1}\); see [18, 30]. The so-called structural constants (see [19, 25] or [26, 27]) of \(h_n\) are described by the skew-symmetric \((2n \times 2n)\)-matrix

\[
C_{2n+1}^{2n+1} := \begin{pmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 0
\end{pmatrix},
\]

which is the matrix associated with the skew-symmetric bilinear map \(\Gamma_H : H \times H \to \mathbb{R}\) given by \(\Gamma_H(X, Y) = \langle [X, Y], T\rangle\).

**Notation 1.2.** We set \(z := -C_{2n+1}^{2n+1}z = (-y_1, x_1, \ldots, -y_n, x_n) \in \mathbb{R}^{2n}\) and \(X := -C_{2n+1}^{2n+1}X\) for every \(X \in H\).

Given \(p \in \mathbb{H}^n\), we shall denote by \(L_p : \mathbb{H}^n \to \mathbb{H}^n\) the left translation by \(p\), i.e. \(L_p p' = p \ast p'\), for every \(p' \in \mathbb{H}^n\). \(L_p\) is a group homomorphism and its differential \(L_{p*} : T_p \mathbb{H}^n \to T_p \mathbb{H}^n\) is given by the matrix

\[
L_{p*} = \left. \frac{\partial (p \ast p')}{\partial p'} \right|_{p=0} = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\
-\frac{B}{2} & -\frac{B}{2} & \cdots & -\frac{B}{2} & -\frac{B}{2} & \cdots & -\frac{B}{2} & -\frac{B}{2} & +\frac{B}{2} & +\frac{B}{2}
\end{pmatrix}.
\]

Equivalently, one has \(L_{pa} = \text{col}[X_1(p), Y_1(p), \ldots, X_n(p), Y_n(p), T(p)]\).

There exists a 1-parameter group of automorphisms \(\delta_s : \mathbb{H}^n \to \mathbb{H}^n\) \((s \geq 0)\), called Heisenberg dilations, defined by \(\delta_s p := \exp(sz, s^2t)\) for every \(s \geq 0\), where \(p = \exp(z, t) \in \mathbb{H}^n\). We recall that the homogeneous dimension of \(\mathbb{H}^n\) is the integer \(Q := 2n + 2\). By a well-known result of Mitchell (see, for instance, [30]), this number coincides with the Hausdorff dimension of \(\mathbb{H}^n\) as metric space with respect to the CC-distance \(d_{CC}\); see, for instance, [18, 30].
We shall denote by $\nabla$ the unique left-invariant Levi-Civita connection on $T\mathbb{H}^n$ associated with the metric $h = \langle \cdot, \cdot \rangle$. We observe that, for every $X, Y, Z \in \mathfrak{x} := C^\infty(\mathbb{H}^n, T\mathbb{H}^n)$ one has

$$\langle \nabla_X Y, Z \rangle = \frac{1}{2}(\langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle).$$

For every $X, Y \in \mathfrak{x}_h := C^\infty(\mathbb{H}^n, H)$, we shall set $\nabla^h X := \mathcal{P}_h(\nabla_X Y)$, where $\mathcal{P}_h$ denotes the orthogonal projection operator onto $H$. The operation $\nabla^h$ is a vector-bundle connection later called $H$-connection; see [27] and references therein. It is not difficult to see that $\nabla^h$ is flat, compatible with the sub-Riemannian metric $h$ and torsion-free. These properties follow from the very definition of $\nabla^h$ and from the corresponding properties of the Levi-Civita connection $\nabla$.

**Definition 1.3.** For any $\psi \in C^\infty(\mathbb{H}^n)$, the $H$-gradient of $\psi$ is the horizontal vector field $\text{grad}_h \psi \in \mathfrak{x}_h$ such that $\langle \text{grad}_h \psi, X \rangle = d\psi(X) = X \psi$ for every $X \in H$. The $H$-divergence $\text{div}_h X$ of $X \in \mathfrak{x}_h$ is defined, at each point $p \in \mathbb{H}^n$, by

$$\text{div}_h X(p) := \text{Trace}(Y \rightarrow \nabla^h \psi(X)(p)) \quad (Y \in H_p).$$

The $H$-Laplacian $\Delta_h$ is the 2nd order differential operator given by

$$\Delta_h \psi := \text{div}_h (\text{grad}_h \psi) \quad \text{for every } \psi \in C^\infty(\mathbb{H}^n).$$

Having fixed a left-invariant Riemannian metric $h$ on $T\mathbb{H}^n$, one defines by duality a global coframe $\mathcal{F} := \{X_1^*, Y_1^*, ..., X_n^*, Y_n^*, T^\star \}$ of left-invariant 1-forms for the cotangent bundle $T^*\mathbb{H}^n$, where $X_i^* = dx_i, Y_i^* = dy_i \ (i = 1, ..., n)$ and

$$\theta := T^\star = dt + \frac{1}{2} \sum_{i=1}^n (y_i dx_i - x_i dy_i).$$

The differential 1-form $\theta$ is called contact form of $\mathbb{H}^n$. The Riemannian left-invariant volume form $\sigma_{2n+1}^h \in \Lambda_2^{2n+1}(T^*\mathbb{H}^n)$ is given by $\sigma_{2n+1}^h := (\bigwedge_{i=1}^n dx_i \wedge dy_i) \wedge \theta$ and the measure obtained by integrating $\sigma_{2n+1}^h$ is the Haar measure of $\mathbb{H}^n$.

### 1.2 Hypersurfaces and some geometric calculations

Let $S \subset \mathbb{H}^n$ be a $C^1$-smooth hypersurface and let $\nu$ be the (Riemannian) unit normal along $S$. Remind that the Riemannian measure $\sigma_{2n}^h \in \Lambda_2^n(T^*S)$ on hypersurfaces can be defined by contraction of the top-dimensional volume form $\sigma_{2n+1}^h$ with the unit normal $\nu$ along $S$, i.e. $\sigma_{2n}^h \bigwedge S := (\nu \bigwedge \sigma_{2n+1}^h)|_S$.

We say that $p \in S$ is a characteristic point if $\dim H_p = \dim(H_p \cap T_p S)$. The characteristic set of $S$ is the set of all characteristic points, i.e. $C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_x S)\}$. It is worth noticing that $p \in C_S$ if, and only if, $|\mathcal{P}_h \nu(p)| = 0$. Since $|\mathcal{P}_h \nu(p)|$ is continuous along $S$, it follows that $C_S$ is a closed subset of $S$, in the relative topology. We stress that characteristic points are few. More precisely, under our current assumptions the $(Q-1)$-dimensional Hausdorff measure of $C_S$ vanishes, i.e. $H^{Q-1}_S(C_S) = 0$; see [2], [24].

**Remark 1.4.** Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth hypersurface. By using Frobenius’ Theorem about integrable distributions, it can be shown that the topological dimension of $C_S$ is strictly less than $(n+1)$; see also [13]. For deeper results about the size of $C_S$ in $\mathbb{H}^n$, see [3], [5].

Throughout this paper we make use of a homogeneous measure on hypersurfaces, called $H$-perimeter measure; see also [15], [17], [10], [11], [24], [25], [27], [35], [38].

**Definition 1.5 ($\sigma_{2n}^h$-measure).** Let $S \subset \mathbb{H}^n$ be a $C^1$-smooth non-characteristic hypersurface and let $\nu$ be the unit normal vector along $S$. The unit $H$-normal along $S$ is defined by $\nu_h := \mathcal{P}_h \nu$. Then, the $H$-perimeter form $\sigma_{2n}^h \in \Lambda_2^n(T^*S)$ is the contraction of the volume form $\sigma_{2n+1}^h$ of $\mathbb{H}^n$ by the horizontal unit normal $\nu_h$, i.e.

$$\sigma_{2n}^h \bigwedge S := (\nu_h \bigwedge \sigma_{2n+1}^h)|_S.$$
If $C_S \neq \emptyset$ we extend $\sigma^2_n|_S$ up to $C_S$ by setting $\sigma^2_n|_{C_S} = 0$. It turns out that $\sigma^2_n|_S S = |H| \nu |H|_2 \sigma^2_n|_S$. At each non-characteristic point $p \in S \setminus C_S$ one has $H_p = \text{span}_R \{ \nu_i(p) \} \oplus H_p S$, where $H_p S := H_p \cap T_p S$. This allows us to define, in the obvious way, the associated subbundles $H S \subset TS$ and $\nu_i S$ called horizontal tangent bundle and horizontal normal bundle along $S \setminus C_S$, respectively. On the other hand, at each characteristic point $p \in C_S$, only the subbundle $H S$ turns out to be defined, and in this case $H_p S = H_p$. Another important geometric object is given by $\varpi := \frac{\nu \nu}{|H|_2^2}$; see [26, 27, 11]. Although the function $\varpi$ is not defined at $C_S$, we have $\varpi \in L^1_{loc}(S, \sigma^2_n)$. 

**Notation 1.6.** Let $S \subset \mathbb{H}^n$ be a $C^k$-smooth hypersurface. We shall denote by $C^k_{\text{loc}}(S)$, $i = 1, 2, ..., k$, the space of functions whose $i$-th $H$-derivatives are continuous. An analogous notation will be used for open subsets of $S$.

The following definitions can also be found in [27], for general Carnot groups. Below, unless otherwise specified, we shall assume that $S \subset \mathbb{H}^n$ is a $C^2$-smooth non-characteristic hypersurface. Let $\nabla^H_S$ be the connection on $S$ induced from the Levi-Civita connection $\nabla$ on $\mathbb{H}^n$. As for the horizontal connection $\nabla^H$, we define a “partial connection”$\nabla^H_S$ associated with the subbundle $H S \subset TS$ by setting

$$\nabla^H_X Y := P_{H N} (\nabla^H_X Y)$$

for every $X, Y \in \mathfrak{X}^1_{\text{HS}} := C^1(S, H S)$, where $P_{H N} : TS \to H S$ denotes the orthogonal projection operator of $TS$ onto $H S$. Starting from the orthogonal splitting $H = \nu H S \oplus H S$, it can be shown that

$$\nabla^H_X Y = \nabla^H_X Y - (\nabla^H_X \nu, \nu) \nu$$

for every $X, Y \in \mathfrak{X}^1_{\text{HS}}$.

**Definition 1.7.** Given $\psi \in C^2_{\text{loc}}(S)$, we define the $H$-Laplacian $\Delta^H : C^2_{\text{loc}}(S) \to C(S)$ is the 2nd order differential operator given by

$$\Delta^H \psi := \text{div} \text{grad}^H \psi$$

for every $\psi \in C^2_{\text{loc}}(S)$. The horizontal mean curvature is the trace of $B^H$, i.e. $H^H := \text{Tr} B^H$. We shall set

$$W^H X := \nabla^H_X \nu, \quad \text{for every } X \in \mathfrak{X}^1_{\text{HS}}.$$

The torsion $T^H$ of $\nabla^H$ is given by $T^H(X, Y) := \nabla^H_X Y - \nabla^H_Y X - P_{H N} [X, Y]$ for every $X, Y \in \mathfrak{X}^1_{\text{HS}}$.

If $n = 1$, the horizontal tangent space $H S$ is 1-dimensional and the torsion vanishes, but if $n > 1$ this is no longer true in general, because $B^H$ is not symmetric; see [27]. Therefore, it is convenient to represent $B^H$ as a sum of two operators, one symmetric and the other skew-symmetric, i.e. $B^H = S^H + A^H$. It turns out that $A^H = \frac{x}{2} C^H_{2n+1}|_{HS}$; see [27]. The linear operator $C^H_{2n+1}$ only acts on horizontal tangent vectors and hence we shall set $C^H_{2n+1} := C^H_{2n+1}|_{HS}$.

**Definition 1.8.** In analogy with the Riemannian case, the eigenvalues $\kappa_i, i \in I_{\text{HS}}$, of the symmetric linear map $S^H$ are called principal horizontal curvatures.

**Definition 1.9.** Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth non-characteristic hypersurface. We call adapted frame along $S$ any o.n. frame $F := \{ \tau_1, ..., \tau_{2n+1} \}$ for $T \mathbb{H}^n$ such that:

$$\tau_1|_S = \nu, \quad H_p S = \text{span}_R \{ \tau_2(p), ..., \tau_{2n}(p) \} \quad \text{for every } p \in S, \quad \tau_{2n+1} := T.$$

Furthermore, we shall set $I_0 := \{ 1, 2, 3, ..., 2n \}$ and $I_{\text{HS}} := \{ 2, 3, ..., 2n \}$.

---

*We are requiring that all $i$-th $H$-derivatives be continuous at each characteristic point $p \in C_S$.}
Lemma 1.10. Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth non-characteristic hypersurface and fix \( p \in S \). We can always choose an adapted o.n. frame \( F = \{ \tau_1, \ldots, \tau_{2n+1} \} \) along \( S \) such that \( \langle \nabla_X \tau_i, \tau_j \rangle = 0 \) at \( p \) for every \( i, j \in I_{HS} \) and every \( X \in H_p S \).

For a proof, see Lemma 3.8. in [27]. We end this section by stating some useful technical lemmata. In the next proofs we shall make use of an adapted o.n. frame \( F \) along \( S \).

Lemma 1.11. Under the previous assumptions, let us further suppose that \( S \) has constant horizontal mean curvature \( H \). Then

\[
\| B_H \|^2_{|G_r|} = - \sum_{i \in I_{HS}} \langle \nabla_{\tau_i}^H \nabla_{\tau_i}^H \nu_H, \nu_H \rangle,
\]

where \( \| \cdot \|_{G_r} \) is the “Gram norm of a linear operator”; see [6].

Proof. Since \( \langle \nu_H, \nu_H \rangle = 1 \) we get that \( \langle \nabla_{\tau_i}^H \nu_H, \nu_H \rangle = 0 \) for every \( i \in I_{HS} \). Therefore

\[
\sum_{i \in I_{HS}} \langle \nabla_{\tau_i}^H \nabla_{\tau_i}^H \nu_H, \nu_H \rangle = - \sum_{i \in I_{HS}} \langle \nabla_{\tau_i}^H \nu_H, \nabla_{\tau_i}^H \nu_H \rangle = - \sum_{i,j,k \in I_{HS}} \langle \nabla_{\tau_i}^H \nu_H, \tau_j \rangle \langle \nabla_{\tau_i}^H \nu_H, \tau_k \rangle \langle \tau_j, \tau_k \rangle = - \sum_{i,j \in I_{HS}} \langle \nabla_{\tau_i}^H \nu_H, \tau_j \rangle^2 = - \| B_H \|^2_{|G_r|}.
\]

Lemma 1.12. Under our previous assumptions, we have:

(i) \( \text{Tr}(B_H (\cdot, A_H \cdot)) = \| A_H \|^2_{|G_r|} = \frac{n-1}{2} \omega^2 \);

(ii) \( \text{Tr}(B_H (\cdot, C_{HS}^{2n+1} \cdot)) = (n-1) \omega \).

Proof. We claim that \( \text{Tr}(S_H (\cdot, A_H \cdot)) = 0 \). In order to prove this identity we compute

\[
\langle S_H \tau_i, A_H \tau_i \rangle = \frac{1}{4} \langle (B_H + B_H^*) \tau_i, (B_H - B_H^*) \tau_i \rangle = \frac{1}{4} \left\{ \langle B_H \tau_i, B_H \tau_i \rangle - \langle B_H^* \tau_i, B_H^* \tau_i \rangle + \langle B_H \tau_i, B_H^* \tau_i \rangle - \langle B_H^* \tau_i, B_H \tau_i \rangle \right\} = 0
\]

for any \( i \in I_{HS} \). Summing up over \( i \in I_{HS} \) yields \( \text{Tr}(S_H (\cdot, A_H \cdot)) = \| B_H \|^2_{|G_r|} - \| B_H^* \|^2_{|G_r|} = 0 \) and the claim follows. Now let us compute

\[
\text{Tr}(B_H (\cdot, A_H \cdot)) = \sum_{i \in I_{HS}} \langle (S_H + A_H) \tau_i, A_H \tau_i \rangle = \sum_{i \in I_{HS}} \langle A_H \tau_i, A_H \tau_i \rangle = \| A_H \|^2_{|G_r|} = \frac{n-1}{2} \omega^2.
\]

This proves (i). Finally, (ii) follows from (i) by using the identity \( A_H = \frac{1}{2} \omega C_{HS}^{2n+1} \).

In the preceding proof we have used the identity \( \| A_H \|^2_{|G_r|} = \frac{n-1}{2} \omega^2 \); see [27], Example 4.11, p. 470. This identity can easily be proved by making use of an adapted o.n. frame \( F \) along \( S \). Furthermore, we observe that \( \nu_H^+ \in \text{Ker} A_H \), where \( \nu_H^+ = - C_{HS}^{2n+1} \nu_H \).
Remark 1.13. The following holds

\[ \langle W_h X, Y \rangle - \langle W_h Y, X \rangle = -\varpi \langle C^{2n+1}_{	ext{hs}} X, Y \rangle \quad \text{for every } X, Y \in \mathfrak{X}^1_{\text{hs}}. \]

The proof of this identity uses the fact that the bracket of tangent vector fields is tangent to \( S \).

Lemma 1.14. For every \( X, Y \in \mathfrak{X}^1_{\text{hs}} \) one has \( S_h (X, Y) = -\langle W_h X, Y \rangle - \frac{\varpi}{2} \langle C^{2n+1}_{\text{hs}} X, Y \rangle \).

Proof. Let \( X, Y \in \mathfrak{X}^1_{\text{hs}} \) and compute

\[
S_h (X, Y) = \frac{1}{2} (\langle \nabla^h X, Y \rangle + \langle \nabla^h Y, X \rangle) \quad \text{(by definition of } S_h) \\
= -\frac{1}{2} (\langle \nabla^h Y, X \rangle + \langle \nabla^h X, Y \rangle) \quad \text{(compatibility of } \nabla^h \text{ with the metric } \langle \cdot, \cdot \rangle) \\
= -\frac{1}{2} \big( 2 \langle W_h X, Y \rangle + \varpi \langle C^{2n+1}_{\text{hs}} X, Y \rangle \big) \quad \text{(by Remark 1.13)} \\
= -\langle W_h X, Y \rangle - \frac{\varpi}{2} \langle C^{2n+1}_{\text{hs}} X, Y \rangle.
\]

Remark 1.15 (Characteristic direction and CC-geodesics). Let \( S \) be a smooth hypersurface and assume that there exists a CC-geodesic \( \gamma : ]-\epsilon, \epsilon[ \to \mathbb{H}^n \) such that \( \gamma \subset S \) and \( \frac{\, d\gamma}{d s} = \nu^\perp_h (\gamma) \), \( s \in ]-\epsilon, \epsilon[ \); see also Remark 2.1. As a consequence of Remark 1.15, we have \( \frac{\, d\nu^\perp_h}{d s} = -\lambda C^{2n+1}_{\text{hs}} \nu^\perp_h = -\lambda \nu^\perp_h \) for every \( s \in ]-\epsilon, \epsilon[ \).

for some constant \( \lambda \). It follows that at each point of \( \gamma \cap S \) one must have \( \nabla^h_{\nu^\perp_h} \nu^\perp_h = -\lambda \nu^\perp_h \) or, in other words, this shows that \( S_h (\nu^\perp_h, \nu^\perp_h) = -\lambda \).

1.3 Homogeneous measure on \( \partial S \) and horizontal integration by parts

Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth compact hypersurface with boundary. Let \( \partial S \) be a \((2n-1)\)-dimensional (piecewise) \( C^1 \)-smooth manifold, oriented by its unit normal vector \( \eta \in TS \), and denote by \( \sigma^{2n-1}_\partial \) the Riemannian measure on \( \partial S \) defined by setting \( \sigma^{2n-1}_\partial \) the Riemannian manifold \( (\eta \cup \sigma^{2n} \partial S) = \langle \eta \cup \sigma^{2n} \partial S \rangle |_{\partial S} \). Note that

\[
\langle X \cup \sigma^{2n} \partial S \rangle |_{\partial S} = \langle X, \eta \rangle |_{|\mathcal{P}_h \nu| \sigma^{2n} \partial S} \]

for every \( X \in C(S, TS) \). The characteristic set \( C_{\partial S} \) is defined as \( C_{\partial S} := \{ p \in \partial S : |\mathcal{P}_h \nu| \sigma^{2n} \partial S = 0 \} \). The unit HS-normal along \( \partial S \) is given by \( \eta_{\text{hs}} := \frac{\mathcal{P}_h \eta}{|\mathcal{P}_h \eta|} \). As for the \( H \)-perimeter measure, we define a homogeneous measure \( \sigma^{2n-1}_\text{hs} \) along \( \partial S \) by setting

\[
\sigma^{2n-1}_\text{hs} \partial S := \langle \eta_{\text{hs}} \cup \sigma^{2n} \partial S \rangle |_{\partial S}.
\]

We have \( \sigma^{2n-1}_\text{hs} \partial S = |\mathcal{P}_h \nu| \sigma^{2n} \partial S \sigma^{2n-1}_\text{hs} \partial S \); furthermore \( \langle X \cup \sigma^{2n}_\text{hs} \partial S \rangle |_{\partial S} = \langle X, \eta_{\text{hs}} \rangle \sigma^{2n-1}_\text{hs} \partial S \) for every \( X \in \mathfrak{X}^0_{\text{hs}} \).

Definition 1.16 (Horizontal tangential operators). For simplicity, let us assume that \( S \subset \mathbb{H}^n \) is non-characteristic. Later on, we shall denote by \( \mathcal{D}_{\text{hs}} : \mathfrak{X}^1_{\text{hs}} \to C(S) \) be the 1st order differential operator given by

\[
\mathcal{D}_{\text{hs}} (X) := \text{div}_{\text{hs}} X + \varpi \langle C^{2n+1}_{\text{hs}} \nu^\perp_h, X \rangle = \text{div}_{\text{hs}} X - \varpi \langle \nu^\perp_h, X \rangle \quad \text{for every } X \in \mathfrak{X}^1_{\text{hs}}.
\]

Moreover, we shall denote by \( \mathcal{L}_{\text{hs}} : \mathfrak{C}^2_{\text{hs}} (S) \to C(S) \) the 2nd order differential operator given by

\[
\mathcal{L}_{\text{hs}} \varphi := \mathcal{D}_{\text{hs}} (\text{grad}_{\text{hs}} \varphi) = \Delta_{\text{hs}} \varphi - \varpi \frac{\partial \varphi}{\partial \nu^\perp_h} \quad \text{for every } \varphi \in \mathfrak{C}^2_{\text{hs}} (S).
\]
Remark 1.19. A simple way to state Stokes formula is the following:

\[ \int_S D_{\text{hs}}(X) \sigma_{ij}^{2n} = - \int_S \mathcal{H}_n \langle X, \nu \rangle \sigma_{ij}^{2n} + \int_{\partial S} \langle X, \eta_{\text{hs}} \rangle \sigma_{ij}^{2n-1} \quad \text{for every } X \in \mathfrak{X}_n = C^1(\mathbb{H}^n, H). \]

Note that, if \( X \in \mathfrak{X}_n \), the first integral on the right hand side vanishes and, in this case, the formula is referred to as “horizontal divergence formula”. We collect below some useful Green’s formulas for the operator \( \mathcal{L}_{\text{hs}} \).

Corollary 1.18. Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth compact hypersurface with piecewise \( C^1 \)-smooth boundary \( \partial S \). If \( n = 1 \) assume further that \( C_S \) is contained in a finite union of \( C^1 \)-smooth horizontal curves. Under the previous notation, the following hold:

(i) \( \int_S L_{\text{hs}} \varphi \sigma_{ij}^{2n} = 0 \) for every compactly supported \( \varphi \in C^2_{\text{hs}}(S) \);
(ii) \( \int_S L_{\text{hs}} \varphi \sigma_{ij}^{2n} = \int_{\partial S} \partial \varphi / \partial \eta_{\text{hs}} \sigma_{ij}^{2n-1} \) for every \( \varphi \in C^2_{\text{hs}}(S) \);
(iii) \( \int_S \psi L_{\text{hs}} \varphi \sigma_{ij}^{2n} = \int_S \varphi L_{\text{hs}} \psi \sigma_{ij}^{2n} \) for every compactly supported \( \varphi, \psi \in C^2_{\text{hs}}(S) \);
(iv) \( \int_S (\psi L_{\text{hs}} \varphi - \varphi L_{\text{hs}} \psi) \sigma_{ij}^{2n} = \int_{\partial S} (\psi \partial \varphi / \partial \eta_{\text{hs}} - \varphi \partial \psi / \partial \eta_{\text{hs}}) \sigma_{ij}^{2n-1} \) for every \( \varphi, \psi \in C^2_{\text{hs}}(S) \);
(v) \( \int_S \psi L_{\text{hs}} \varphi \sigma_{ij}^{2n} = - \int_S \langle \text{grad}_{\text{hs}} \varphi, \text{grad}_{\text{hs}} \psi \rangle \sigma_{ij}^{2n} + \int_{\partial S} \psi \partial \varphi / \partial \eta_{\text{hs}} \sigma_{ij}^{2n-1} \) for every \( \varphi, \psi \in C^2_{\text{hs}}(S) \);
(vi) \( \int_S L_{\text{hs}} (\varphi^2 \sigma_{ij}^{2n}) = 2 \int_S \varphi L_{\text{hs}} \varphi \sigma_{ij}^{2n} + 2 \int_S \text{grad}_{\text{hs}} \varphi^2 \sigma_{ij}^{2n} = \int_{\partial S} \partial \varphi / \partial \eta_{\text{hs}} \sigma_{ij}^{2n-1} \) for every \( \varphi \in C^2_{\text{hs}}(S) \).

The proof of the characteristic case follows from the non-characteristic one by dominated convergence by a family of subsets \( \{ U_\epsilon \}_{\epsilon > 0} \) such that: (i) \( \text{Cap}_{\epsilon} \subset U_\epsilon \) for every \( \epsilon > 0 \); (ii) \( \sigma_{ij}^{2n}(U_\epsilon) \rightarrow 0 \) for \( \epsilon \rightarrow 0^+ \);
(iii) \( \int_{\partial S} |\nu| \psi \sigma_{ij}^{2n-1} \rightarrow 0 \) for \( \epsilon \rightarrow 0^+ \); see also [28]. It is not difficult to see that, under the previous assumptions, such a family does exist. Later on this idea will be used in order to extend the variational formulas for the \( H \)-perimeter measure proved in [26, 27], to characteristic hypersurfaces.

Remark 1.19. A simple way to state Stokes formula is the following:

Let \( M \) be an \( n \)-dimensional manifold of class \( C^2 \) with boundary \( \partial M \). Then

\[ \int_M d\alpha = \int_{\partial M} \alpha \]

for every compactly supported \((k-1)\)-form \( \alpha \) of class \( C^1 \).

Without much effort, it is possible to extend this formula to the case where:

(\( \ast \)) \( M \) is of class \( C^1 \) and \( \alpha \) is a \((k-1)\)-form such that \( \alpha \) and \( d\alpha \) are continuous.

For a more detailed discussion see [17].

The previous condition (\( \ast \)) can be used to extend the previous formulas to vector fields (and functions) possibly singular at the characteristic set \( C_S \). So let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth hypersurface with (piecewise) \( C^1 \)-smooth boundary \( \partial S \) and let \( X \in C^1(S \setminus C_S, HS) \). Set

\[ \alpha_X := (X \downarrow \sigma_{ij}^{2n})|_S. \]

Then, condition (\( \ast \)) requires that \( \alpha_X \) and \( d\alpha_X \) be continuous on \( S \). Note that \( X \) is of class \( C^1 \) out of \( C_S \) but may be singular at \( C_S \). For later purposes, we also define the space of “admissible” functions for the horizontal Green’s formulas (iii)-(vi) of Corollary 1.18.

Definition 1.20. Let \( X \in C^1(S \setminus C_S, HS) \) and set \( \alpha_X := (X \downarrow \sigma_{ij}^{2n})|_S. \) We say that \( X \) is admissible (for the horizontal divergence formula) if, and only if, the differential forms \( \alpha_X \) and \( d\alpha_X \) are continuous on all of \( S \). We say that \( \varphi \in C^2_{\text{hs}}(S \setminus C_S) \) is admissible (for the horizontal Green’s formulas (iii)-(vi)) stated in Corollary 1.18 if, and only if, \( \psi \text{grad}_{\text{hs}} \varphi \) is admissible (for the horizontal divergence formula) for every \( \psi \in C^2_{\text{hs}}(S \setminus C_S) \) such that \( \psi \text{grad}_{\text{hs}} \psi \) is admissible (for the horizontal divergence formula). We shall denote by \( \Phi(S) \) the space of all admissible functions.
2 Isoperimetric Profiles

Remark 2.1 (CC-geodesics and Isoperimetric Profiles). By definition, CC-geodesics are horizontal curves which minimize the CC-distance. In Heisenberg groups, they are obtained by solving the following system of O.D.E.s:

\[
\begin{align*}
\dot{\gamma} &= P_u \\
\dot{P}_u &= -P_{2n+1}C_u^{2n+1}P_u \\
P_{2n+1} &= 0.
\end{align*}
\]

Equivalently, the 2nd equation to solve is given by \(\dot{P}_u = P_{2n+1}P_u^T\), where \(P_u = (P_1, ..., P_{2n})^T\), \(|P_u| = 1\). The quantity \(P_{2n+1}\) turns out to be a constant parameter along \(\gamma\). The vector \(P = (P_u, P_{2n+1})\) in \(\mathbb{R}^{2n+1}\) can be regarded as a vector of “Lagrangian multipliers”. Solutions of (1) are called normal CC-geodesics. We stress that (1) can be deduced by minimizing the constrained Lagrangian \(L(t, \gamma, \dot{\gamma}) = |\dot{\gamma}| + P_{2n+1}^T\dot{\gamma};\) see [28] and references therein, or [30]. Unlike the Riemannian case, CC-geodesics in \(\mathbb{H}^n\) depend not only on the initial point \(\gamma(0)\) and on the initial direction \(P_u(0)\), but also on the parameter \(P_{2n+1}\). Now if \(P_{2n+1} = 0\), CC-geodesics are Euclidean horizontal lines. Furthermore, if \(P_{2n+1} \neq 0\), any CC-geodesic turns out to be a “helix”. To be more precise, the horizontal projection of any CC-geodesic \(\gamma\) onto \(H_0 = \mathbb{R}^n\) belongs to a sphere whose radius only depends on \(P_{2n+1}\). Now take a point \(S \in \gamma\) and consider the positively oriented, vertical \(T\)-line over this point. On this line, there exists a first consecutive point \(N\) to \(S\) belonging to \(\gamma\). It can be proved that these two points, henceforth called South and North poles, determine a minimizing connected subset of \(\gamma\). By rotating this curve around the \(T\)-axis passing from \(S\) we obtain a closed convex surface, which is the so-called Isoperimetric Profile.

Later on we shall study some features of a model Isoperimetric Profile, having barycenter at 0 \(\in \mathbb{H}^n\). It goes without saying that any other Isoperimetric Profile can be obtained from this one, by left-translations and intrinsic dilations.

Let \(\rho := |z| = \sqrt{\sum_{i=1}^{2n}(x_i^2 + y_i^2)}\) be the norm of \(z = (x_1, y_1, ..., x_i, y_i, ..., x_n, y_n) \in \mathbb{R}^{2n}\) and let \(u_0 : B_1(0) := \{z \in \mathbb{R}^{2n} : 0 \leq \rho \leq 1\} \to \mathbb{R}\) be the radial function given by

\[
u_0(z) = \frac{\pi}{8} + \frac{\rho}{4} \sqrt{1 - \rho^2} - \frac{\rho}{4} \arcsin \rho =: u_0(\rho).
\]

Setting

\[
S_{\mathbb{H}^n}^\pm := \left\{ p = \exp (z, t) \in \mathbb{H}^n : t = \pm u_0(z), \forall z \in B_1(0) \right\},
\]

we call Heisenberg unit Isoperimetric Profile \(S_{\mathbb{H}^n}^\pm\) the compact hypersurface built by gluing together \(S_{\mathbb{H}^n}^+\) and \(S_{\mathbb{H}^n}^-\), i.e. \(S_{\mathbb{H}^n} = S_{\mathbb{H}^n}^+ \cup S_{\mathbb{H}^n}^-\). Since \(\nabla_{\mathbb{R}^n} u_0 = u_0'(\rho) \frac{\rho}{2}\), it follows that the Euclidean unit normal along \(S_{\mathbb{H}^n}^\pm\) is given by \(n_{\mathbb{E}^n} = \frac{(-\nabla_{\mathbb{R}^n} u_0, \pm 1)}{\sqrt{1 + \|
abla_{\mathbb{R}^n} u_0\|^2 + \frac{\rho^2}{4}}}\). This implies that

\[
\nu^\pm = \frac{\left(-\nabla_{\mathbb{R}^n} u_0 \pm \frac{z}{2}\right)}{\sqrt{1 + \|
abla_{\mathbb{R}^n} u_0\|^2 + \frac{\rho^2}{4}}}, \quad |\mathcal{P}_u(\nu^\pm)| = \frac{\sqrt{\|
abla_{\mathbb{R}^n} u_0\|^2 + \frac{\rho^2}{4}}}{\sqrt{1 + \|
abla_{\mathbb{R}^n} u_0\|^2 + \frac{\rho^2}{4}}}.
\]

Using \(u_0'(\rho) = \frac{-\rho^2}{2\sqrt{1 - \rho^2}}\), we get that

\[
\nu^\pm = \frac{\left(-\nabla_{\mathbb{R}^n} u_0 \pm \frac{z}{2}\right)}{\sqrt{1 + \|
abla_{\mathbb{R}^n} u_0\|^2 + \frac{\rho^2}{4}}} = z \pm \frac{\sqrt{1 - \rho^2}}{\rho} z^\perp.
\]

Hence, by definition, it follows that

\[
\sigma_{\mathbb{H}^n}^2 \cdot S_{\mathbb{H}^n}^\pm = |\mathcal{P}_u(\nu^\pm)| \sigma_{\mathbb{E}^n}^2 \cdot S_{\mathbb{H}^n}^\pm = \sqrt{\|
abla_{\mathbb{R}^n} u_0\|^2 + \frac{\rho^2}{4}} \, ds \cdot B_1(0) = \frac{\rho}{2\sqrt{1 - \rho^2}} \, dz \cdot B_1(0).
\]

7If \(n = 1\), \(\gamma(t)\) is a circular helix with axis parallel to the vertical direction \(T\) and whose slope depends on \(P_3\). We stress that the projection of \(\gamma(t)\) onto \(\mathbb{R}^2\) turns out to be a circle whose radius explicitly depends on \(P_3\).

8This point is a sort of “cut point” of \(S\) along \(\gamma\). Actually, this is the end-point of all CC-geodesics starting from \(S\) with same slope. However the properly said cut locus of any point in \(\mathbb{H}^n\) \((n \geq 1)\) is the vertical \(T\)-line over that point.
It is not difficult to compute the horizontal mean curvature of $S_{H^n}$. In fact $H_u = -\text{div}_H \nu_u = -2n$. The so-called “characteristic direction” along $S$ is the horizontal tangent vector field given by

$$(\nu^+_H)^\perp \left( -z \pm \sqrt{1 - \rho^2} \frac{z}{\rho} \right) = \frac{z \mp \sqrt{1 - \rho^2}}{\rho}.$$ \par

**Remark 2.2.** In the 1st Heisenberg group $\mathbb{H}^1$, the geometric quantity $(\nabla^n_{\nu^+_H}, \nu^+_H, \nu_H)$, which coincides with the horizontal mean curvature $H_u$ of $S$, turns out to be the geodesic curvature (see [20], p. 203) of any smooth horizontal path $\gamma : [0, \epsilon] \subseteq \mathbb{R} \rightarrow S$ such that $\dot{\gamma} = \nu^+_H(\gamma)$; see also Remark [1.1].

The (weighted) vertical component of the Riemannian normal is given by

$$\mp = \frac{\nu_1^H}{|\nu_1^H|} = \frac{1}{\|\nabla R^2 u_0 + \frac{z^\perp}{\rho}\|} = \frac{1}{\sqrt{\|\nabla R^2 u_0\|^2 + \frac{z^2}{4}}} = \frac{1}{\sqrt{\frac{\rho^4}{4(1 - \rho^2)^2} + \frac{z^2}{4}}} = \frac{\sqrt{1 - \rho^2}}{\rho}. \tag{3}$$

Note that $\text{grad}_u \phi = \nabla_{R^2} \phi$ for every function $\phi : \mathbb{H}^n \rightarrow \mathbb{R}$ independent of $t$. Therefore

$$\text{grad}_u \mp = \nabla_{R^2} \mp = \pm \frac{\partial}{\partial \rho} \left( 2 \frac{\sqrt{1 - \rho^2}}{\rho} \right) \frac{z}{\rho} = \mp \left( \frac{2}{\rho^2 \sqrt{1 - \rho^2}} \right) \frac{z}{\rho}$$

and

$$\frac{\partial \mp}{\partial \rho} = \mp \frac{2}{\rho^2 \sqrt{1 - \rho^2}} \nabla \left( \frac{z}{\rho}, (\nu^+_H)^\perp \right) = \frac{1}{\rho \sqrt{1 - \rho^2}} \frac{2 \sqrt{1 - \rho^2}}{\rho^2} \frac{z}{\rho} = \frac{2}{\rho^3}. \tag{4}$$

**Notation 2.3.** Let $\kappa : S_{H^n} \rightarrow \mathbb{R}$, where $\kappa^\pm := \kappa|_{S_{H^n}} = \pm \frac{1}{\rho \sqrt{1 - \rho^2}}$. Moreover, we set $g_u := (z, \nu_u)$ and $g_u^+ := (z, \nu^+_H)$. The function $g_u$ is called horizontal support function associated with $S_{H^n}$.

Throughout the next proofs, we shall choose an adapted o.n. frame centered at a point $p \in S_{H^n}$ as in Lemma [1.10]. For the sake of simplicity, we only consider the case of the north hemisphere $S^+_H$. In this case, one has $\nu_u = \nu^+_H = z + \kappa z^\perp$ and $\kappa' = \frac{\partial z^\perp}{\partial \rho} = -\frac{1}{\rho \sqrt{1 - \rho^2}}$. Let us state a key-identity of this paper.

**Lemma 2.4.** We have $\Delta_{H^2} \kappa = -\frac{2n}{\rho^2} \kappa$.

**Proof.** Setting $z_i := (z, \tau_i)$, we have $\tau_i(\kappa) = \kappa' \frac{z_i}{\rho^3 \sqrt{1 - \rho^2}} = -\frac{z_i}{\rho^3 \sqrt{1 - \rho^2}}$. So we compute

$$\Delta_{H^2} \kappa = \sum_{i \in I_{H^2}} \tau_i(\tau_i(\kappa)) = -\sum_{i \in I_{H^2}} \left( \text{grad}_u \left( \frac{z_i}{\rho^3 \sqrt{1 - \rho^2}} \right), \tau_i \right)$$

$$= -\sum_{i \in I_{H^2}} \left( \frac{\tau_i(z_i)}{\rho^3 \sqrt{1 - \rho^2}} + z_i \tau_i \left( \frac{1}{\rho^3 \sqrt{1 - \rho^2}} \right) \right)$$

$$= -\sum_{i \in I_{H^2}} \left( \frac{\delta_{ii} + \nabla^2 \tau_i, z_i}{\rho^3 \sqrt{1 - \rho^2}} + \frac{z_i^2}{\rho^3 \sqrt{1 - \rho^2}} - \frac{3(1 - \rho^2)}{\rho^5 \sqrt{1 - \rho^2}} \right) \quad (\delta_{ij} \text{ is the Kronecker delta})$$

$$= -\left( \frac{2n - 1 + g_u H_u}{\rho^3 \sqrt{1 - \rho^2}} - |z_{H^2}|^2 \frac{3(1 - \rho^2)}{\rho^3 \sqrt{1 - \rho^2}} \right)$$

$$= -\left( \frac{2n - 1 - 2n \rho^2}{\rho^3 \sqrt{1 - \rho^2}} - \frac{3 - 4 \rho^2}{\rho^3 \sqrt{1 - \rho^2}} \right) \kappa = \frac{2n - 4}{\rho^2} \kappa,$$

where we have used the identity $|z_{H^2}|^2 = \rho^2(1 - \rho^2)$. This achieves the proof. \(\square\)

**Lemma 2.5.** Let $B_u$ be the horizontal 2nd fundamental form of $S_{H^n}$. Then $\|B_u\|_{G, r}^2 = 4 + \frac{2n - 2}{\rho^2}$ and $\|S_u\|_{G, r}^2 = Q = 2n + 2$. 
Proof. By applying (i) of Lemma 2.1 we have \( \|B_u\|_{G_r}^2 = -\sum_{i \in I_{HS}} \langle \nabla^u_{\tau_i} \nabla^u_{\nu_i} \nu_i, \nu_i \rangle \) and since

\[
\nabla^u_{\tau_i} \nu_i = \nabla^u_{\tau_i} \left( z + \kappa z^\perp \right) = \tau_i + \tau_i(\kappa) z^\perp - \kappa C^2_{u} \tau_i,
\]

we get that

\[
\|B_u\|_{G_r}^2 = - \sum_{i \in I_{HS}} \langle \nabla^u_{\tau_i} \nabla^u_{\nu_i} \nu_i, \nu_i \rangle
\]

\[
= - \sum_{i \in I_{HS}} \langle \nabla^u_{\tau_i} \left( \tau_i + \tau_i(\kappa) z^\perp - \kappa C^2_{u} \tau_i \right), \nu_i \rangle
\]

\[
= \sum_{i \in I_{HS}} \left( -\langle \nabla^u_{\tau_i} \tau_i, \nu_i \rangle - \Delta_{HS} \kappa (z^\perp, \nu_i) + 2\tau_i(\kappa) \langle C^2_{u} \tau_i, \nu_i \rangle + \kappa \langle C^2_{u} \nabla^u_{\nu_i} \tau_i, \nu_i \rangle \right)
\]

\[
= \left( -\mathcal{H}_u + g^\perp_u \Delta_{HS} \kappa + 2\langle C^2_{u} \nabla^u_{\nu_i} \kappa, \nu_i \rangle \right),
\]

(4)

where we have used the identity \( \langle C^2_{u} \nabla^u_{\tau_i}, \nu_i \rangle = 0 \), which holds for every \( i \in I_{HS} \). Since \( S_{u}^2 + 1 + 1 \in HS \), the last identity can be proved by using an adapted horizontal frame, as in Lemma 2.10. Moreover

\[
\langle C^2_{u} \nabla^u_{\nu_i} \kappa, \nu_i \rangle = \frac{\kappa'}{\rho} \langle C^2_{u} \nabla^u_{\nu_i} \kappa, \nu_i \rangle = \frac{\kappa'}{\rho} \langle C^2_{u} \nabla^u_{\nu_i} \kappa, \nu_i \rangle = -\rho \kappa' \rho.
\]

(5)

At this point, using Lemma 2.4 together with (4), (5) and the identity \( g^\perp_u = -\rho \sqrt{1 - \rho^2} \), yields

\[
\|B_u\|_{G_r}^2 = 2n + \rho \sqrt{1 - \rho^2}, \quad \|A_u\|_{G_r}^2 + \|A_u\|_{G_r}^2 = 2n^2 - 2z \rho^2, \quad \text{using}\quad \rho^2 = 4 \frac{1 - z^2}{\rho^2} \text{yields } \|S_u\|_{G_r}^2 = Q = 2n + 2.
\]

\[
Q \quad \square
\]

Note that we have found the Gram-norm of \( S_u \) in an indirect way. However, we can be more precise. To this aim, we first compute

\[
\mathcal{J}_u \nu^\perp_u = \mathcal{J}_u (z + \kappa z^\perp) = \text{Id} + \frac{\kappa'}{\rho} \otimes z + \kappa C^2_{u} z^\perp.
\]

By Lemma 2.11, we have \( S_u (X, Y) = -\langle W_u X, Y \rangle - \frac{\omega}{2} \langle C^2_{u} X, Y \rangle \) for every \( X, Y \in \mathfrak{h}^1_{HS} \). Furthermore, let \( \mathcal{F}_{HS} = \{ \tau_i : i \in I_{HS} \} \) be any horizontal tangent o.n. frame. Then

\[
-S_u (\tau_i, \tau_j) = W_u (\tau_i, \tau_j) + \frac{\omega}{2} \langle C^2_{u} \tau_i, \tau_j \rangle
\]

\[
= \langle \mathcal{J}_u \nu^\perp_u \tau_i, \tau_j \rangle + \langle \mathcal{J}_u \nu^\perp_u \tau_i, \tau_j \rangle
\]

\[
= \langle \mathcal{J}_u \nu^\perp_u \tau_i, \tau_j \rangle \pm \kappa (C^2_{u} \tau_i, \tau_j)
\]

\[
= \delta_{ij} \pm \frac{\kappa'}{\rho} \langle (z^\perp \otimes z) \tau_i, \tau_j \rangle
\]

\[
= \delta_{ij} \pm \frac{1}{\rho^2} \langle z^\perp \otimes z \rangle \langle z^\perp \otimes z \rangle.
\]

With no loss of generality, take \( \tau_2 := \nu^\perp_u \). By a simple computation, we get \( -S_u (\tau_2, \tau_2) = 2 \). Furthermore, note that for any \( X \in HS \) such that \( \langle X, \nu^\perp_u \rangle = 0 \), one has must have \( \langle X, z \rangle = \langle X, z^\perp \rangle = 0 \). Hence, for any o.n. frame \( \mathcal{F}_{HS} = \{ \tau_2, ..., \tau_{2n} \} \) for \( HS \) such that \( \tau_2 = \nu^\perp_u, S_u \) turns out to be the diagonal matrix of order \( (2n-1) \) given by \( S_u := \text{Diag}(-2, -1, -1, ..., -1) \). Thus we have the following:

**Proposition 2.6.** The principal horizontal curvatures of the Heisenberg unit Isoperimetric profile \( \mathbb{S}_{u}^n \) are the numbers \( \kappa_2 = -2, \kappa_3 = ..., \kappa_{2n} = -1 \). Furthermore, we have that any horizontal tangent o.n. frame \( \mathcal{F}_{HS} = \{ \tau_i : i \in I_{HS} \} \) such that \( \tau_2 = \nu^\perp_u \) turns out to be a system of eigenvectors of \( S_u \).
The principal horizontal curvatures reflect the geometric construction of the unit Isoperimetric profile $S^n_{2n}$. Indeed, any Isoperimetric profile is generated by rotating a CC-geodesic $\gamma$ joining two consecutive points belonging to a vertical $T$-line; these points are the South and North poles. Note that the number $\kappa_2$ just express a “curvature parameter” which uniquely determines all CC-geodesics joining the South pole to the North pole; see also Remark 1.5. As already said, this parameter is a special feature of CC-geodesics. Note also that the other principal horizontal curvatures express the rotational symmetry is a (2
umber \)

Remark 2.7. Let $\varphi : S^2 \to \mathbb{R}$ be a smooth function and consider its restrictions to the hemispheres $S^2_+ = \{ p \equiv (z,t) \in \mathbb{R}^n : t = \pm u_0(z), z \in B_1(0) \}$. Since $\varphi(x) = \varphi(\exp(z, \pm u_0(z)))$, we may thinking of $\varphi^\pm := \varphi|_{S^2_\pm}$ as functions of the variable $z \in B_1(0)$. Now let $\varphi : \overline{B_1(0)} \setminus \{0\} \to \mathbb{R}$ be a smooth function and fix spherical coordinates on $\overline{B_1(0)} \setminus \{0\}$, i.e. $(\rho, \xi) \in [0,1] \times S^{2n-1}$. With a slight abuse of notation, every function $\varphi : S^2_+ \setminus \{N,S\} \to \mathbb{R}$ will be regarded as a function of the variables $(\rho, \xi) \in [0,1] \times S^{2n-1}$.

Setting $\zeta := \frac{\xi}{\rho} \in S^{2n-1}$, the operator $L_{\text{HS}}$ on $S^2_\pm$ takes the following form:

$$L_{\text{HS}} \varphi = \begin{cases} (1 - \rho^2) \varphi''_\rho + \frac{2n - (2n + 1) \rho^2}{\rho} \varphi'_\rho - 2\rho \sqrt{1 - \rho^2} \varphi''_\rho + \frac{1}{\rho} \Delta_{S^{2n-1}} \varphi - (1 - \rho^2) \varphi''_\zeta - (Q - 1) \sqrt{1 - \rho^2} \varphi'_\zeta, & \text{Angular Operator} \\
\text{Mixed Derivatives} & \text{Radial Operator}
\end{cases}$$

where $\Delta_{S^{2n-1}}$ denotes the Laplace operator on the Sphere $S^{2n-1}$; see [28]. Note also that

$$\text{div}_{T^{S^{2n-1}}} \zeta = 0;$$

see Lemma 2.15 in [28].

3 1st and 2nd variation of $\sigma^{2n}_H$ along compact hypersurfaces

Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth compact closed hypersurface oriented by its unit normal vector $\nu$ and let $U \subset S \setminus C_S$ be a non-characteristic open set in the relative topology. We assume that the boundary $\partial U$ is a $(2n - 1)$-dimensional (piecewise) $C^1$-smooth submanifold oriented by its outward unit normal vector $\eta$. We say that a smooth map $\vartheta : [-\epsilon, \epsilon] \times U \to \mathbb{H}^n$ is a variation of $U$ if the following hold: (i) every $\vartheta_t := \vartheta(t, \cdot) : U \to \mathbb{H}^n$ is an immersion; (ii) $\vartheta_0 = 1D_U$. By definition, the variation vector of $\vartheta$ is given by $W := \frac{\partial W}{\partial t}|_{t=0} = \dot{\vartheta}_t|_{t=0}$, where $\dot{\vartheta}_t = \vartheta_t \frac{\partial}{\partial t}$. Let $(\sigma^{2n}_H)$ denote the $H$-perimeter measure along $\vartheta_t(U)$. Let $\Gamma(t) := \vartheta_t^\ast (\sigma^{2n}_H) \in \Lambda^{2n} (\tau_t^u)$, for every $t \in [-\epsilon, \epsilon]$. Note that $\Gamma(t)$ is a 1-parameter family of $2n$-forms along $U$. The 1st and 2nd variation formulas of the $H$-perimeter $\sigma^{2n}_H$ under the variation $\vartheta$ are given by $I_{\vartheta}(W, \sigma^{2n}_H) = \int_{\vartheta(0)} \Gamma(0)$ and $II_{\vartheta}(W, \sigma^{2n}_H) = \int_{\vartheta(0)} \Gamma(0)$.

In [27] we proved in a more general context, the following:

Theorem 3.1 (see [27]). Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth hypersurface oriented by its unit normal vector $\nu$ and let $U \subset S \setminus C_S$ be a non-characteristic relatively compact open set having piecewise $C^1$-smooth boundary $\partial U$ oriented by its outward unit normal vector $\eta$. Let $\vartheta_t$ be a variation of $U$ with variation vector $W$ and set $w := \frac{\overline{W}_{\nu}}{|\overline{W}_{\nu}|}$. Then

$$I_{\vartheta}(W, \sigma^{2n}_H) = - \int_U H_w w \sigma^{2n}_H + \int_{\partial U} \langle W, \eta \rangle |\overline{P}_{\nu} \nu| \sigma^{2n-1}_H$$

Moreover, if $U$ has constant horizontal mean curvature $H_w$, then

$$II_{\vartheta}(W, \sigma^{2n}_H) = \int_U \left( -H_w \overline{W}_w \big|_{t=0} + \|\text{grad}_{H_w} w\|^2 + w^2 \left( (H_w)^2 - \|S_H\|_C^2 + 2 \frac{\partial S_H}{\partial \nu} \frac{n + 1}{2} \right) \right) \sigma^{2n}_H + \int_{\partial U} \left( \left( -w \text{grad}_{H_w} w + \overline{\nabla}_{\nu'} \overline{W}_{\nu'} \big|_{t=0} \right) \right) \langle \overline{P}_{\nu} \nu | \overline{(W_{\nu'})} - H_w \langle W, \eta \rangle \langle W^{\nu'}, \eta \rangle \rangle \sigma^{2n-1}_H,$$

where $W_{\nu'}$ denotes the tangential component of $W$ along $U$ and $\overline{W}_{\nu'}$ denote, respectively, tangential and normal components of $\overline{W}$ along $U$. Moreover, we have set $w_t := \frac{\overline{W}_{\nu'}}{|\overline{W}_{\nu'}|}$, where $\nu'$ is the Riemannian unit normal vector along $\vartheta_t(U)$ and $\overline{P}_{\nu}$ is the orthogonal projection onto $H$ at $\vartheta_t(p)$, for every $p \in S$. 

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In [27] we stated all results for $C^\infty$-smooth hypersurfaces. If instead we assume that $S$ is only in $C^2$, some of the computations in the proof of the previous theorem should be understood in the sense of the distribution theory. More precisely, the proof uses explicitly the so-called curvature 2-forms associated with an adapted o.n. frame along $S$; see [3], [39], or [20, 27]. However, one can also assume that $S$ is in $C^3$ and then use an approximation argument in order to extend the final formula, where no third derivatives occur, to the $C^2$ case.

**Remark 3.2.** Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth compact closed hypersurface. In this case it turns out that $\dim_{\text{Haus}}(\text{CarrS}) \leq n$; see [3]. Just for the case $n=1$, we shall further suppose that $C_S$ is contained in a finite union of $C^1$-smooth horizontal curves. As already said, under these assumptions one can show that there exists a family $\{U_\epsilon\}_{\epsilon > 0}$ of open subsets of $S$, with piecewise $C^1$-smooth boundaries, such that:

1. $C_S \subset U_\epsilon$ for every $\epsilon > 0$;
2. $\sigma^{2n}(U_\epsilon) \rightarrow 0$ for $\epsilon \rightarrow 0^+$;
3. $\int_{\partial U_\epsilon} \lfloor P_n \nu \rfloor \sigma^{2n-1}_\mathcal{R} \rightarrow 0$ for $\epsilon \rightarrow 0^+$.

Set $S_\epsilon := S \setminus U_\epsilon$. Later on we shall discuss the validity of Theorem 3.1 for $C^2$-smooth closed compact hypersurfaces $S$ having non-empty characteristic set. To this end, let us consider the following limits (if they exist):

$$I_S(W, \sigma^{2n}_\mathcal{R}) := \lim_{\epsilon \rightarrow 0^+} I_{S_\epsilon}(W, \sigma^{2n}_\mathcal{R}), \quad II_S(W, \sigma^{2n}_\mathcal{R}) := \lim_{\epsilon \rightarrow 0^+} II_{S_\epsilon}(W, \sigma^{2n}_\mathcal{R}).$$

Note that $I_{S_\epsilon}(W, \sigma^{2n}_\mathcal{R})$ and $II_{S_\epsilon}(W, \sigma^{2n}_\mathcal{R})$ represent the 1st and 2nd variation of $\sigma^{2n}_\mathcal{R}$ along the 1-parameter family $\{S_\epsilon\}_{\epsilon > 0}$ of non-characteristic hypersurfaces (with boundary). Since $C_S$ is a null set with respect to the $\sigma^{2n}_\mathcal{R}$-measure (see Remark 3.2), it is clear that, if they exist, the limits $I_S(W, \sigma^{2n}_\mathcal{R})$ and $II_S(W, \sigma^{2n}_\mathcal{R})$ express the 1st and 2nd variation of $\sigma^{2n}_\mathcal{R}$ along $S$. For every $\epsilon > 0$ one has

$$I_S(W, \sigma^{2n}_\mathcal{R}) = I_{S_\epsilon}(W, \sigma^{2n}_\mathcal{R}) + I_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R}), \quad II_S(W, \sigma^{2n}_\mathcal{R}) = II_{S_\epsilon}(W, \sigma^{2n}_\mathcal{R}) + II_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R}).$$

**Remark 3.3.** Although the quantities appearing in these formulas are not well defined at $C_S$, we could alternatively compute $I_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R})$ and $II_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R})$, by using the representation formula $\sigma^{2n}_\mathcal{R} = |P_n \nu| \sigma^{2n}_\mathcal{R}$, and then show that:

$$\lim_{\epsilon \rightarrow 0^+} I_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R}) = 0, \quad \lim_{\epsilon \rightarrow 0^+} II_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R}) = 0. \quad (8)$$

Since $I_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R}) := \int_{U_\epsilon} \hat{\Gamma}(0), \quad II_{U_\epsilon}(W, \sigma^{2n}_\mathcal{R}) := \int_{U_\epsilon} \hat{\Gamma}(0)$, where $\Gamma(t) := \partial^*_t (\sigma^{2n}_\mathcal{R})_t$, we need to compute

$$\hat{\Gamma}(0) = \left( \frac{d}{dt} |P_n \nu| \right) \bigg|_{t=0} \sigma^{2n}_\mathcal{R} + |P_n \nu| \left( \frac{d}{dt} (\sigma^{2n}_\mathcal{R})_t \right) \bigg|_{t=0},$$

$$\hat{\Gamma}(0) = \left( \frac{d^2}{dt^2} |P_n \nu| \right) \bigg|_{t=0} \sigma^{2n}_\mathcal{R} + 2 \left( \frac{d}{dt} |P_n \nu| \right) \bigg|_{t=0} \left( \frac{d}{dt} (\sigma^{2n}_\mathcal{R})_t \right) \bigg|_{t=0} + |P_n \nu| \left( \frac{d^2}{dt^2} (\sigma^{2n}_\mathcal{R})_t \right) \bigg|_{t=0}.$$

Note that $\frac{d}{dt} (\sigma^{2n}_\mathcal{R})_t$ and $\frac{d^2}{dt^2} (\sigma^{2n}_\mathcal{R})_t$, evaluated at $t = 0$, express the “infinitesimal” 1st and 2nd variation of the Riemannian Area $\sigma^{2n}_\mathcal{R}$; see [37] or [6]. Nevertheless, this analysis goes beyond the scopes of this paper and below we shall discuss a different approach and other results valid in Heisenberg groups.

By applying the 1st variation of $\sigma^{2n}_\mathcal{R}$ to $U = S_\epsilon$ ($\epsilon > 0$) and (iii) of Remark 3.2 we see that the boundary integral tends to 0 as long as $\epsilon \rightarrow 0^+$, i.e.

$$\lim_{\epsilon \rightarrow 0^+} \int_{\partial S_\epsilon} \langle W, \eta \rangle |P_n \nu| \sigma^{2n-1}_\mathcal{R} = 0.$$

Therefore, it remains to study the convergence of the integral along the interior of $S_\epsilon$ as long as $\epsilon \rightarrow 0^+$. By definition $\mathcal{H}_\nu = - \text{div}_{\text{res}} \nu$ and it turns out that $\text{div}_{\text{res}} \nu = \text{div}_H \nu$. So we have

$$- \mathcal{H}_\nu = \text{div}_H \nu = \text{div}_H (|P_n \nu|) = \frac{\text{div}_H (P_n \nu) - \langle \text{grad}_H |P_n \nu|, \nu \rangle}{|P_n \nu|}. \quad (9)$$

By noting that $|P_n \nu|$ is Lipschitz continuous at $C_S$, it follows that $\mathcal{H}_\nu \in L^1(S, \sigma^{2n}_\mathcal{R})$. More precisely, for every $\epsilon > 0$ one has

$$\int_{U_\epsilon} |\mathcal{H}_\nu| \sigma^{2n}_\mathcal{R} = \int_{U_\epsilon} \left| \text{div}_H (P_n \nu) - \langle \text{grad}_H |P_n \nu|, \nu \rangle \right| \sigma^{2n}_\mathcal{R} \leq C \sigma^{2n}_\mathcal{R} (U_\epsilon).$$

\[\text{Since } S \text{ is } C^2\text{-smooth, } \nu \text{ is of class } C^1 \text{ everywhere on } S.\]
where $C$ is a constant only dependent on (the Lipschitz constant of) $\mathcal{P}_\nu$. So $I_{U_\epsilon}(W, \sigma_{n_\epsilon}^2) \to 0$ as long as $\epsilon \to 0^+$ and we finally get that

$$I_S(W, \sigma_{n_\epsilon}^2) = \lim_{\epsilon \to 0^+} I_{S_\epsilon}(W, \sigma_{n_\epsilon}^2) = - \int_S \mathcal{H}_\nu \, \omega_{\sigma_{n_\epsilon}^2}. \quad (10)$$

We also need the following fact:

**Lemma 3.4.** Let $\varphi : S \to \mathbb{R}$ be a piecewise smooth function such that $\int_S \varphi \, \omega_{\sigma_{n_\epsilon}^2} = 0$. Then there exists a volume-preserving normal variation whose variation vector is $W = \varphi \nu$. If $\varphi = 0$ along $\partial S$ we can always assume that the variation fixes the boundary.

**Proof.** This fact is well-known in the Euclidean setting and its proof applies as well to our case; see Lemma 2.4.

**Remark 3.5.** As in Riemannian Geometry, the $1$st variation of $\sigma_{n_\epsilon}^2$ along a compact closed hypersurface $S$ will depends on the normal component of the variation vector. For this reason, in the sequel only normal variations of $S$ will be considered. Without loss of generality, we shall also assume that $W := \varphi \mathcal{P}_\nu \nu$ for some smooth function $\varphi : S \to \mathbb{R}$. Since $w = \frac{W}{\mathcal{P}_\nu}$, we have $w = \varphi$.

Let $S$ be of constant horizontal mean curvature $\mathcal{H}_\nu$ (at each point of $S \setminus C_S$). Under the previous assumptions, the $2$nd variation of $\sigma_{n_\epsilon}^2$ along $S_\epsilon$ is given by

$$I_{S_\epsilon}(W, \sigma_{n_\epsilon}^2) = \int_{S_\epsilon} \left( -\mathcal{H}_\nu \tilde{W}(\varphi_t) \big|_{t=0} + (\text{grad}_{\mathcal{N}} \varphi)^2 + \varphi^2 \left( \mathcal{H}_{\sigma_\epsilon}^2 - ||S_{\sigma_\epsilon}||_{C_{\sigma_\epsilon}}^2 + 2 \frac{\partial \varphi}{\partial \nu_{\sigma_\epsilon}} - \frac{n+1}{2} \omega^2 \right) \right) \sigma_{n_\epsilon}^2 - \int_{\partial S_\epsilon} \varphi (\text{grad}_{\mathcal{N}} \varphi, \eta) \mathcal{P}_\nu ||\sigma_{n_\epsilon}^2 - 1||.$$

**Lemma 3.6.** Let $S \subset \mathbb{R}^n$ be a $C^2$-smooth compact closed hypersurface with constant horizontal mean curvature $\mathcal{H}_\nu$. If $\frac{1}{|\mathcal{P}_\nu|} \in L^1(S, \sigma_{n_\epsilon}^2)$, then $I_{S_\epsilon}(W, \sigma_{n_\epsilon}^2) = \lim_{\epsilon \to 0^+} I_{S_\epsilon}(W, \sigma_{n_\epsilon}^2)$, where the variation vector $W$ is chosen as in Remark 3.5.

Notice that $\frac{1}{|\mathcal{P}_\nu|} \in L^1(S, \sigma_{n_\epsilon}^2) \iff \frac{1}{|\mathcal{P}_\nu|} \in L^2(S, \sigma_{n_\epsilon}^2)$.

**Proof.** We claim that the boundary integral tends to $0$ as long as $\epsilon \to 0^+$, i.e.

$$\lim_{\epsilon \to 0^+} \int_{\partial S_\epsilon} \varphi (\text{grad}_{\mathcal{N}} \varphi, \eta) \mathcal{P}_\nu ||\sigma_{n_\epsilon}^2 - 1|| = 0.$$

Since $\partial S_\epsilon = \partial U_\epsilon$, using (iii) of Remark 3.2 we get $\int_{\partial S_\epsilon} \mathcal{P}_\nu ||\sigma_{n_\epsilon}^2 - 1|| \to 0$ for $\epsilon \to 0^+$ and the claim follows since $|\varphi (\text{grad}_{\mathcal{N}} \varphi, \eta)| \leq \frac{1}{2} ||\text{grad}_{\mathcal{N}} \varphi^2||_{L^\infty(S)}$. Let us study the integral along the interior of $S_\epsilon$.

- Since $-\mathcal{H}_\nu \tilde{W}(\varphi_t) \big|_{t=0} = -\mathcal{H}_\nu \varphi \mathcal{P}_\nu \mathcal{P}_\nu (\varphi_t \big|_{t=0})$, and since $\mathcal{H}_\nu$ is constant along $S \setminus C_S$, it follows that the $1$st addend can be integrated over all of $S$; since $|\text{grad}_{\mathcal{N}} \varphi| \leq ||\text{grad}_{\mathcal{N}} \varphi||_{L^\infty(S)}$, the same holds true for the $2$nd addend.

- One has $||S_{\sigma_\epsilon}||_{C_{\sigma_\epsilon}} = \sum_{i,j \in I_{HS}} \left( \frac{\langle \nabla_{\tau_i}^2 \nu_{\sigma_\epsilon} \nu_{\sigma_\epsilon}, \tau_i \rangle + \langle \nabla_{\tau_j}^2 \nu_{\sigma_\epsilon} \nu_{\sigma_\epsilon}, \tau_j \rangle}{2} \right)^2$, where $\{\tau_i : i \in I_{HS}\}$ is an o.n. basis of $HS$; see Definition 1.9. Note also that $\langle \nabla_{\tau_i}^2 \nu_{\sigma_\epsilon} \nu_{\sigma_\epsilon}, \tau_j \rangle = \frac{\langle \nabla_{\tau_i}^2 \mathcal{P}_\nu \nu, \tau_j \rangle^2}{|\mathcal{P}_\nu|}$ for every $i, j \in I_{HS}$. Since $\frac{1}{|\mathcal{P}_\nu|} \in L^1(S, \sigma_{n_\epsilon}^2)$, it follows that $||S_{\sigma_\epsilon}||_{C_{\sigma_\epsilon}} \in L^1(S, \sigma_{n_\epsilon}^2)$.

- The $5$th addend can be integrated over all of $S$, since the following estimate

$$\left| \frac{\partial \varphi}{\partial \nu_{\mathcal{N}}} \right| \leq \frac{|\nu_{\mathcal{N}}^\perp (\mathcal{P}_\nu \nu) - \nu_{\mathcal{N}}^\perp (|\mathcal{P}_\nu \nu| \nu)|}{|\mathcal{P}_\nu \nu|^2} \lesssim \frac{1}{|\mathcal{P}_\nu \nu|^2}$$

holds true near $C_S$. Finally, the $6$th term satisfies $\omega^2 \lesssim \frac{1}{|\mathcal{P}_\nu \nu|^2}$.
Using the previous remarks and the smoothness of \( \varphi \) (over all of \( S \)), the thesis easily follows.

**Corollary 3.7.** Let \( S \subset \mathbb{H}^n \) be a \( C^2 \)-smooth compact closed hypersurface. Let \( \partial_t \) be a normal variation, having variation vector \( W = \varphi \left| \mathcal{P}_n \nu \right| \nu \), for some smooth function \( \varphi : S \rightarrow \mathbb{R} \). Then

\[
I_S(W, \sigma_{H}^{2n}) = - \int_S \mathcal{H}_W \varphi \sigma_{H}^{2n}.
\]

If \( \frac{1}{|\mathcal{P}_n \nu|} \in L^2(S, \sigma_{H}^{2n}) \) and \( S \) has constant horizontal mean curvature, then

\[
II_S(W, \sigma_{H}^{2n}) = \int_S \left( -\mathcal{H}_W \widetilde{W}(\varphi_t) \right)_{t=0} + |\text{grad}_{\nu} \varphi|^2 + \varphi^2 \left( \mathcal{H}_W^2 - \| S_H \|_{\mathcal{G}^\nu}^2 + 2 \frac{\partial \varphi}{\partial \nu} - \frac{n+1}{2} \omega^2 \right) \sigma_{H}^{2n}.
\]

Finally, we discuss another point of view, which can be used in order to extend Theorem 3.1 even in more general situations. In particular, we would like to use (normal) variations which can be (possibly) singular at \( C_S \). For the sake of simplicity, we only consider the case \( n > 1 \). In view of Remark 3.8 this implies that \( \dim C_S < 2n - 2 \). As already said, the validity of the 1st and 2nd variation formulas for \( \sigma_{H}^{2n} \) up to the characteristic set \( C_S \), can be formulated in terms of a limit procedure. More precisely, let \( S \) be a compact hypersurface of class \( C^2 \) without boundary and let us set \( S_\epsilon = S \setminus \{ U \}_\epsilon \), where \( \{ U \}_\epsilon \subset \mathbb{R} \) is a family of open subsets of \( S \), with (piecewise) \( C^1 \)-smooth boundaries, such that: (i) \( C_S \subset \cup U \) for every \( \epsilon > 0 \); (ii) \( \sigma_{H}^{2n}(\cup U) \rightarrow 0 \) for \( \epsilon \rightarrow 0^+ \); (iii) \( \sigma_{H}^{2n-1}(\partial \cup U) \rightarrow 0 \) for \( \epsilon \rightarrow 0^+ \). Note that these sets shrink around \( C_S \), as long as \( \epsilon \rightarrow 0^+ \). Furthermore, let \( \partial_t \) be a normal variation of \( S \) with variation vector \( W = \varphi |\mathcal{P}_n \nu| \nu \), for some function \( \varphi : S \rightarrow \mathbb{R} \). Nevertheless, we do not assume that \( \varphi \) is smooth over all \( S \), but only on \( S \setminus C_S \). Under these assumptions, for every \( \epsilon > 0 \) the 1st and 2nd variation formulas on the non-characteristic hypersurfaces \( S_\epsilon \) are given by

\[
I_{S_\epsilon}(W, \sigma_{H}^{2n}) = - \int_{S_\epsilon} \mathcal{H}_W \varphi \sigma_{H}^{2n},
\]

\[
II_{S_\epsilon}(W, \sigma_{H}^{2n}) = \int_{S_\epsilon} \left( -\mathcal{H}_W \widetilde{W}(\varphi_t) \right)_{t=0} + |\text{grad}_{\nu} \varphi|^2 + \varphi^2 \left( \mathcal{H}_W^2 - \| S_H \|_{\mathcal{G}^\nu}^2 + 2 \frac{\partial \varphi}{\partial \nu} - \frac{n+1}{2} \omega^2 \right) \sigma_{H}^{2n} - \int_{\partial S_\epsilon} \langle \varphi \text{grad}_{\nu} \varphi, \eta \rangle |\mathcal{P}_n \nu| \sigma_{H}^{2n-1}.
\]

This follows from Theorem 3.1 since \( \partial_t \) is a normal variation. Now consider the limits (if they exist):

\[
I_S(W, \sigma_{H}^{2n}) := \lim_{\epsilon \rightarrow 0^+} I_{S_\epsilon}(W, \sigma_{H}^{2n}), \quad II_S(W, \sigma_{H}^{2n}) := \lim_{\epsilon \rightarrow 0^+} II_{S_\epsilon}(W, \sigma_{H}^{2n}).
\]

What is a sufficient condition for the existence of these limits? The previous analysis showed that if \( \varphi \) is smooth on all of \( S \), then the limits exist (For what concerns the 2nd variation, we also have to assume \( \frac{1}{|\mathcal{P}_n \nu|} \in L^2(S, \sigma_{H}^{2n}) \)). However, it is enough to require that all integrands are continuous over all of \( S \).

**Notation 3.8.** Let us set

\[
\chi_1 := -\mathcal{H}_W \varphi \sigma_{H}^{2n},
\]

\[
\chi_2 := \left( -\mathcal{H}_W \widetilde{W}(\varphi_t) \right)_{t=0} + |\text{grad}_{\nu} \varphi|^2 + \varphi^2 \left( \mathcal{H}_W^2 - \| S_H \|_{\mathcal{G}^\nu}^2 + 2 \frac{\partial \varphi}{\partial \nu} - \frac{n+1}{2} \omega^2 \right) \sigma_{H}^{2n},
\]

\[
\chi_3 := -\langle \varphi \text{grad}_{\nu} \varphi, \eta \rangle |\mathcal{P}_n \nu| \sigma_{H}^{2n-1}.
\]

**Proposition 3.9.** Let \( n > 1 \). Let \( S \subset \mathbb{H}^n \) be a compact hypersurface of class \( C^2 \) without boundary and let \( \partial_t \) be a normal variation of \( S \). Let \( W \) be the vector variation of \( \partial_t \) and assume that \( W = \varphi |\mathcal{P}_n \nu| \nu \), for some function \( \varphi : S \rightarrow \mathbb{R} \) which is \( C^2 \)-smooth on \( S \setminus C_S \). Furthermore, assume that the differential forms \( \chi_1, \chi_2, \chi_3 \) are continuous on all of \( S \). Then

\[
I_S(W, \sigma_{H}^{2n}) = - \int_S \mathcal{H}_W \varphi \sigma_{H}^{2n},
\]

\[
II_S(W, \sigma_{H}^{2n}) = \int_S \left( -\mathcal{H}_W \widetilde{W}(\varphi_t) \right)_{t=0} + |\text{grad}_{\nu} \varphi|^2 + \varphi^2 \left( \mathcal{H}_W^2 - \| S_H \|_{\mathcal{G}^\nu}^2 + 2 \frac{\partial \varphi}{\partial \nu} - \frac{n+1}{2} \omega^2 \right) \sigma_{H}^{2n}.
\]

**Proof.** It is enough to note that, if the differential forms \( \chi_1, \chi_2, \chi_3 \) are continuous on all of \( S \), then the integrals \( I_S(W, \sigma_{H}^{2n}), II_S(W, \sigma_{H}^{2n}) \), turn out to be well-defined (and finite) for every \( \epsilon \geq 0 \). The thesis follows since the boundaries \( \partial S_\epsilon \) converge to the lower dimensional set \( C_S \), as long as \( \epsilon \rightarrow 0 \).
3.1 1st and 2nd variation of volume, isoperimetric functional and the notion of stability

Let $D \subset \mathbb{H}^n$ be a relatively compact domain with $C^2$-smooth boundary $S := \partial D$. Let $\iota_D : D \rightarrow \mathbb{H}^n$ be the inclusion of $D$ in $\mathbb{H}^n$ and let $\vartheta : (-\varepsilon, \varepsilon) \times D \rightarrow \mathbb{H}^n$ be a smooth map. We say that $\vartheta$ is a variation of $\iota_D$ if the following hold: (i) every $\vartheta_t := \vartheta(t, \cdot) : D \rightarrow \mathbb{H}^n$ is an immersion; (ii) $\vartheta_0 = \iota_D$.

Let $\vartheta_t$ be a variation of $D$ with variation vector $W = \vartheta_t \frac{\partial}{\partial t} |_{t=0}$ and set $\tilde{W} := \vartheta_t \frac{\partial}{\partial t}$. The 1st and 2nd variation formulas of $\sigma^{2n+1}_R \mathbb{L} D$, denoted as $I_D(W, \sigma^{2n+1}_R)$ and $II_D(W, \sigma^{2n+1}_R)$, are given by

$$I_D(W, \sigma^{2n+1}_R) := \frac{d}{dt} \left( \int_D \vartheta_t^* \left( \sigma^{2n+1}_R \right) \right) \Big|_{t=0}, \quad II_D(W, \sigma^{2n+1}_R) := \frac{d^2}{dt^2} \left( \int_D \vartheta_t^* \left( \sigma^{2n+1}_R \right) \right) \Big|_{t=0}.$$  

Setting $(\sigma^{2n+1}_R)_t := \vartheta_t^* \left( \sigma^{2n+1}_R \right)$, we see that

$$\frac{d}{dt} \left( \sigma^{2n+1}_R \right) = \mathcal{L}_{\tilde{W}} \left( \sigma^{2n+1}_R \right)_t, \quad \frac{d^2}{dt^2} \left( \sigma^{2n+1}_R \right) = \mathcal{L}_{\tilde{W}} \left( \mathcal{L}_{\tilde{W}} \left( \sigma^{2n+1}_R \right)_t \right).$$

By Cartan’s formula for the Lie derivative we compute

$$\mathcal{L}_{\tilde{W}} \left( \sigma^{2n+1}_R \right)_t = \tilde{W} \mathcal{L}_{\vartheta_t} \left( \sigma^{2n+1}_R \right)_t + \mathcal{L}_{\tilde{W}} \tilde{W} \mathcal{L}_{\vartheta_t} \left( \sigma^{2n+1}_R \right)_t = \text{div} \tilde{W} \left( \sigma^{2n+1}_R \right)_t,$$

and by applying Stokes’ Theorem we get that

$$I_D(W, \sigma^{2n+1}_R) = \int_D \text{div} W \sigma^{2n+1}_R = \int_S w \sigma^{2n}_R, \quad (13)$$

where $w = \frac{(W, \nu)}{|\nu|}$. For what concerns the 2nd variation of $\sigma^{2n+1}_R \mathbb{L} D$, let us compute

$$\mathcal{L}_{\tilde{W}} \left( \mathcal{L}_{\tilde{W}} \left( \sigma^{2n+1}_R \right)_t \right) = \tilde{W} \mathcal{L}_{\vartheta_t} \left( \mathcal{L}_{\tilde{W}} \left( \sigma^{2n+1}_R \right)_t \right) = \mathcal{L}_{\tilde{W}} \left( \text{div} \tilde{W} \left( \sigma^{2n+1}_R \right)_t \right),$$

where $w_t := \frac{(\tilde{W}, \nu)}{|\nu|}$, $\nu^t$ is the Riemannian unit normal along $S_t := \vartheta_t(S)$ and $(\mathcal{H}_t)_t$ is the horizontal mean curvature of $S_t$. Note that the last identity follows from the infinitesimal 1st variation of $\sigma^{2n}_R$.

Using again Stokes’ Theorem, it turns out that the 2nd variation of $\sigma^{2n+1}_R \mathbb{L} D$ can be written out as a boundary integral along $S$, i.e.

$$II_D(W, \sigma^{2n+1}_R) = \int_S \left( \tilde{W} \mathcal{L}_{\vartheta_t} \left( \sigma^{2n+1}_R \right)_t \right) dS_t \sigma^{2n}_R. \quad (14)$$

**Corollary 3.10.** Let $D \subset \mathbb{H}^n$ be a $C^2$-smooth compact domain. Suppose $S = \partial D$ has constant horizontal mean curvature and let $\frac{1}{|\nu|} \in L^2(S, \sigma^{2n}_R)$. Let $\vartheta_t$ be a volume preserving normal variation of $S$ having variation vector $W = \varphi |\nu \nu| \nu$ for some smooth function $\varphi : S \rightarrow \mathbb{R}$. Then

$$II_S(W, \sigma^{2n}_R) = \int_S \left( |\text{grad}_{\nu \nu} \varphi|^2 + \varphi^2 \left( -\|S_t\|_{\sigma^{2n}_R}^2 + 2 \frac{\partial \varphi}{\partial S_t} - \frac{n+1}{2} \varphi^2 \right) \right) \sigma^{2n}_R. \quad (15)$$

**Proof.** For volume preserving variations, using the 2nd variation formula of volume yields

$$\int_S \left( \mathcal{H}_t \tilde{W} (\varphi \nu) \right) |_{t=0} - \mathcal{H}_t^2 \varphi^2 \sigma^{2n}_R = 0$$

and the thesis follows by substituting the last identity into (12). \qed

We also have the following “alternative” version.
Corollary 3.11. Let $n > 1$. Let $D \subset \mathbb{H}^n$ be a $C^2$-smooth compact domain with boundary $S = \partial D$ of constant horizontal mean curvature. Let $\vartheta_t$ be a volume preserving normal variation of $S$, having variation vector $W = \varphi |\mathcal{P}_n \nu| \nu$, for some smooth function $\varphi : S \rightarrow \mathbb{R}$ on $S \setminus C_S$. Furthermore, assume that the differential forms $\chi_1, \chi_2, \chi_3$ are continuous on all of $S$; see Notation 3.8. Then

$$II_S(W, \sigma_{n}^{2n}) = \int_S \left( |\text{grad}_{\nu} \varphi|^2 + \varphi^2 \left(-\|S_{\nu}\|_{\nu}^2 + \frac{\partial \sigma_{\nu}^n \varphi}{\partial \nu_{\nu}^n} - \frac{n+1}{2} \varphi^2 \right) \right) \sigma_{\nu}^{2n}.$$

The Isoperimetric Functional in our context can naturally be defined by

$$J(D) := \frac{\sigma_{n}^{2n}(\partial D)}{(\sigma_{n}^{2n+1}(D))^{1-\frac{n}{2}}},$$

where $D$ varies over bounded domains in $\mathbb{H}^n$ having $C^2$-smooth boundaries. So let $\vartheta_t$ be a variation of $D$ with variation vector $W$. Differentiating (16) along the flow $\vartheta_t$, using (13) and (11), yields

$$\frac{d}{dt} J(\vartheta_t(D)) \bigg|_{t=0} = -\frac{1}{(\sigma_{n}^{2n+1}(D))^{1-\frac{n}{2}}} \int_{\partial D} \mathcal{H}_u w \sigma_{n}^{2n} - \frac{Q-1}{Q} \frac{\sigma_{n}^{2n}(D)}{(\sigma_{n}^{2n+1}(D))^{2-\frac{n}{2}}} \int_{\partial D} w \sigma_{n}^{2n}. \quad (17)$$

By choosing a volume-preserving variation, this means that the flow $\vartheta_t$ associated with $W$ does not change the volume, i.e. $\sigma_{n}^{2n+1}(\vartheta_t(D)) = \sigma_{n}^{2n+1}(D)$ for every $t \in [-\epsilon, \epsilon]$. It follows that the last integral vanishes and, by means of the Fundamental Lemma of Calculus of Variations, we obtain the following:

Corollary 3.12. Let $D \subset \mathbb{H}^n$ be a relatively compact domain with $C^2$-smooth boundary and assume that $D$ is a critical point of the functional $J(D)$ under volume-preserving variations. Then $\mathcal{H}_u$ must be constant on $\partial D \setminus C_{\partial D}$.

Remark 3.13. Let $D$ be a critical point of $J(\cdot)$ under volume-preserving variations. Using (17) yields

$$\sigma_{n}^{2n+1}(D) \mathcal{H}_u = -\frac{Q-1}{Q} \sigma_{n}^{2n}(S),$$

which implies that there are no closed compact $H$-minimal hypersurfaces in $\mathbb{H}^n$; see [21], [38].

Using volume-preserving normal variations, we also get that

$$\frac{d^2}{dt^2} J(\vartheta_t(D)) \bigg|_{t=0} = \frac{II_{\partial D}(W, \sigma_{n}^{2n})}{(\sigma_{n}^{2n+1}(D))^{1-\frac{n}{2}}}.$$ 

The last computation motivates the following:

Definition 3.14 (Stability I). Let $D \subset \mathbb{H}^n$ be a compact domain with $C^2$-smooth boundary $S = \partial D$ such that $\frac{1}{\mathcal{P}_n \nu} \mathcal{P}_n \nu \in L^2(S, \sigma_{n}^{2n})$. We assume that $D$ is a critical point of $J(D)$ under volume-preserving variations. We say that $S$ is a stable bounding hypersurface if $II_O(W, \sigma_{n}^{2n}) > 0$ for every non-zero volume-preserving normal variation $\vartheta_t$ of $S$ having variation vector $W = \varphi |\mathcal{P}_n \nu| \nu$, where $\varphi : S \rightarrow \mathbb{R}$ is any smooth function on $S$.

Moreover, we propose a weakening of the notion of stability.

Definition 3.15 (Local Stability). Let $D \subset \mathbb{H}^n, n > 1$, be a compact domain with $C^2$-smooth boundary $S = \partial D$ and assume that $D$ is a critical point of $J(D)$ under volume-preserving variations. We say that $S$ is a locally stable bounding hypersurface if for each $p \in S$ there exists a neighborhood $\Omega \subseteq S$ of $p$ such that $II_{\Omega}(W, \sigma_{n}^{2n}) > 0$ for every non-zero volume-preserving normal variation $\vartheta_t$ of $\Omega$ having variation vector $W = \varphi |\mathcal{P}_n \nu| \nu$ such that $\varphi : \Omega \rightarrow \mathbb{R}$ is any smooth compactly supported function on $\Omega$.

Remark 3.16 (Radial variations). Let $D \subset \mathbb{H}^n$ be a compact domain with radial symmetry with respect to the vertical direction $T$. In this case, a “natural” class of normal variations can be defined by using radially symmetric functions on $S = \partial D$. More precisely, a useful “stability test” for the domain $D$ is that of being stable in the sense of Definition 3.14 for all smooth radial function $\varphi : S \rightarrow \mathbb{R}$. In this case, we shall say that $D$ is radially stable. Clearly, radial stability is just a necessary condition for stability.

By applying Corollary 3.11, the notion of stability can be further generalized.

Definition 3.17 (Stability II). Let $D \subset \mathbb{H}^n, n > 1$, be a compact domain with $C^2$-smooth boundary $S = \partial D$ and assume that $D$ is a critical point of $J(D)$ under volume-preserving variations. We say that $S$ is a stable bounding hypersurface if $II_{\Omega}(W, \sigma_{n}^{2n}) > 0$ for every non-zero volume-preserving normal variation $\vartheta_t$ of $S$ having variation vector $W = \varphi |\mathcal{P}_n \nu| \nu$, where $\varphi : S \rightarrow \mathbb{R}$ is any smooth function on $S \setminus C_S$ such that the differential forms $\chi_1, \chi_2, \chi_3$ turn out to be continuous on all of $S$; see Notation 3.8.
4 Isoperimetric Profiles and Stability

We already know that \( S_{\mathbb{H}^n} \) is (the boundary of) a critical point under volume preserving variations of the isoperimetric functional \( J(\cdot) \). Furthermore, it is not difficult to see that \( \frac{1}{|\nabla_H \varphi|} \in L^2(S_{\mathbb{H}^n}, \sigma_n^{2n}) \); see also Remark 4.17. We start with the following:

**Theorem 4.1.** Let \( n = 1 \). The Heisenberg Isoperimetric profile \( S_{\mathbb{H}^1} \) is a stable bounding hypersurface in the sense of both Definition 3.14 and Definition 3.17.

**Proof.** Let \( \partial_t \) be any non-zero volume-preserving normal variation having variation vector \( W \). Thus, using Lemma 2.5 together with (3) and the fact that \( (\varphi^\pm)^2 = 4\frac{1-n^2}{r^2} \), yields

\[-\|S_n\|^2_\n + 2 \frac{\partial \varphi}{\partial \nu^{\perp}_n} - \varphi^2 = 0.\]

By applying formula (15) we obtain

\[ H_{S_{\mathbb{H}^1}}(W, \sigma_n^{2n}) = \int_{S_{\mathbb{H}^1}} |\text{grad}_H \varphi|^2 \sigma_n^{2n} \geq 0. \]

If \( \int_{S_{\mathbb{H}^1}} |\text{grad}_H \varphi|^2 \sigma_n^{2n} = 0 \), then \( |\text{grad}_H \varphi| = |\nu_H^\perp \varphi| = 0 \). In particular, it follows that \( \varphi \) is constant along any leaf of the so-called characteristic foliation of \( S_{\mathbb{H}^1} \). Note also that each leaf joins together North and South poles of \( S_{\mathbb{H}^1} \). Hence \( \varphi \) is constant along \( S_{\mathbb{H}^1} \) and since \( \int_{S_{\mathbb{H}^1}} \varphi \sigma_n^{2n} = 0 \), we finally get that \( \varphi = 0 \). Therefore

\[ H_{S_{\mathbb{H}^1}}(W, \sigma_n^{2n}) > 0 \]

for every non-zero normal variation, as wished; see also Proposition 1.22 in [28].

*From now on, we shall study the case \( n > 1 \).* In the general case, we are not able to give a complete proof of the statement valid for \( n = 1 \). Nevertheless, below we shall obtain some partial stability results.

We choose a (non-zero) volume-preserving normal variation \( \partial_t \) with variation vector \( W \). This means that \( \tilde{W} \) is parallel to \( \nu^t \) for every \( t \in [-\epsilon, +\epsilon[ \). As in Remark 3.5, we already know that \( \tilde{W} = \varphi \) on \( \mathcal{P}_H \nu \nu \) for some (at least piecewise) smooth function \( \varphi : S \rightarrow \mathbb{R} \). As already said, this choice implies that \( w = \frac{\partial \varphi}{\partial \nu^{\perp}_n} = \varphi \). From (3), Lemma 2.5 and the identity \( (\varphi^\pm)^2 = 4\frac{1-n^2}{r^2} \), it follows that

\[-\|S_n\|^2_\n + 2 \frac{\partial \varphi}{\partial \nu^{\perp}_n} - \frac{n+1}{2} \varphi^2 = -\frac{Q-4}{\rho^2}.\]

Hence, using formula (15) yields

\[ H_{S_{\mathbb{H}^n}}(W, \sigma_n^{2n}) = \int_{S_{\mathbb{H}^n}} \left( |\text{grad}_H \varphi|^2 - \frac{Q-4}{\rho^2} \varphi^2 \right) \sigma_n^{2n} \]

for every smooth function \( \varphi : S_{\mathbb{H}^n} \rightarrow \mathbb{R} \) such that \( \int_{S_{\mathbb{H}^n}} \varphi \sigma_n^{2n} = 0 \). Note that \( Q - 4 = 2n - 2 \).

**Remark 4.2.** In order to study the positivity of the 2nd variation of \( S_{\mathbb{H}^n} \), we shall set

\[ \mathcal{F}(\varphi) := \int_{S_{\mathbb{H}^n}} \left( |\text{grad}_H \varphi|^2 - \frac{Q-4}{\rho^2} \varphi^2 \right) \sigma_n^{2n} \]

and study the sign of the functional \( \mathcal{F}(\cdot) \) for functions belonging to the class \( \Phi(S_{\mathbb{H}^n}) \) of admissible functions; see Definition 1.20.

Roughly speaking, we are considering functions \( \varphi \in C_0^\infty(S_{\mathbb{H}^n} \setminus \{N, S\}) \) which can be “integrated by parts” on \( S_{\mathbb{H}^n} \). In using this class, we are including possibly singular solutions at the poles \( N, S \) of \( S_{\mathbb{H}^n} \), which are the only characteristic points of \( S_{\mathbb{H}^n} \). Nevertheless, we may apply the horizontal Green’s formulas (iii)-(vi) stated in Corollary 1.18. Moreover, it is not difficult to realize that the “right functional class” where studying this problem is given by

\[ \Phi_0(S_{\mathbb{H}^n}) := \left\{ \varphi \in L^2(S_{\mathbb{H}^n}, \sigma_n^{2n}) : \varphi \neq 0, \ |\text{grad}_H \varphi| \in L^2(S_{\mathbb{H}^n}, \sigma_n^{2n}), \ \int_{S_{\mathbb{H}^n}} \varphi \sigma_n^{2n} = 0 \right\} \]
For simplicity, in the sequel we shall restrict our study to functions belonging to the class
\[ \Phi_1(S_{\mathbb{H}^n}) := \Phi(S_{\mathbb{H}^n}) \cap \Phi_0(S_{\mathbb{H}^n}). \]

**Remark 4.3.** Integration by parts in (19) yields
\[ S(\varphi) = - \int_{S_{\mathbb{H}^n}} \varphi \left( \mathcal{L}_{HS} \varphi + \frac{2n - 2}{\rho^2} \varphi \right) \sigma_0^{2n} \]
for every \( \varphi \in \Phi_1(S_{\mathbb{H}^n}). \) Hence, it becomes natural to study the associated equation
\[ \mathcal{L}_{HS} \varphi + \frac{2n - 2}{\rho^2} \varphi = C \quad (C \in \mathbb{R}). \]
Note that we can also consider a non-zero constant \( C \in \mathbb{R}, \) because \( \int_{S_{\mathbb{H}^n}} \varphi \sigma_0^{2n} = 0. \) We already know a solution to this equation when \( C = 0. \) Indeed, Lemma 2.4 says that \( \Delta_{HS} \kappa = -\frac{2n - 4}{\rho^2} \kappa, \) where we recall that \( \kappa = \kappa_\pm = \pm \sqrt{\frac{1 - \rho^2}{\rho}} \) along \( S_{\mathbb{H}^n}^\pm. \) Since \( w = 2\kappa, \) using (3) yields
\[ \mathcal{L}_{HS} \kappa + \frac{2n - 2}{\rho^2} \kappa = 0 \]
which shows that \( \kappa \) is an eigenfunction, with eigenvalue \( \mu = 2n - 2, \) of the closed “singular” eigenvalue problem:
\[ \mathcal{L}_{HS} \varphi + \frac{\mu}{\rho^2} \varphi = 0 \quad \text{for} \ \varphi \in \Phi_1(S_{\mathbb{H}^n}), \ (\mu \in \mathbb{R}_+). \]
For the sake of completeness, let us first compute 1st and 2nd variation of \( S(\cdot). \) So let \( t \in ]-\epsilon, \epsilon[, \) let \( \varphi_1, \varphi_2 \in \Phi_1(S_{\mathbb{H}^n}) \) and consider the “perturbed functional” \( \tilde{S}(\varphi + t \varphi_1 + \frac{t^2}{2} \varphi_2). \) Then, the 1st and 2nd variation of \( S(\cdot) \) can easily be obtained by computing the following derivatives:
\[ \tilde{S}'(\varphi) := \frac{d}{dt} \tilde{S}(\varphi + t \varphi_1 + \frac{t^2}{2} \varphi_2) \bigg|_{t=0}, \quad \tilde{S}''(\varphi) := \frac{d^2}{dt^2} \tilde{S}(\varphi + t \varphi_1 + \frac{t^2}{2} \varphi_2) \bigg|_{t=0}. \]
We thus have
\[ \tilde{S}'(\varphi) = 2 \int_{S_{\mathbb{H}^n}} \left( \langle \text{grad}_{HS} \varphi, \text{grad}_{HS} \varphi_1 \rangle - \frac{2n - 2}{\rho^2} \varphi \varphi_1 \right) \sigma_0^{2n}, \]
\[ \tilde{S}''(\varphi) = 2 \int_{S_{\mathbb{H}^n}} \left( \langle \text{grad}_{HS} \varphi, \text{grad}_{HS} \varphi_2 \rangle + \langle \text{grad}_{HS} \varphi_1, \text{grad}_{HS} \varphi_1 \rangle - \frac{2n - 2}{\rho^2} (\varphi_1^2 + \varphi_2^2) \right) \sigma_0^{2n}, \]
and, by integrating by parts, we obtain
\[ \tilde{S}'(\varphi) = -2 \int_{S_{\mathbb{H}^n}} \varphi_1 \left( \mathcal{L}_{HS} \varphi + \frac{2n - 2}{\rho^2} \varphi \right) \sigma_0^{2n}, \]
\[ \tilde{S}''(\varphi) = 2 \int_{S_{\mathbb{H}^n}} \left( -\varphi_1 \left( \mathcal{L}_{HS} \varphi + \frac{2n - 2}{\rho^2} \varphi \right) + \left| \text{grad}_{HS} \varphi_1 \right|^2 - \frac{2n - 2}{\rho^2} \varphi_1^2 \right) \sigma_0^{2n}. \]
It follows that any critical point of \( S \) (i.e. any solution \( \varphi \in \Phi_1(S_{\mathbb{H}^n}) \) to \( S'(\varphi) = 0) \) solves the equation:
\[ \mathcal{L}_{HS} \varphi + \frac{2n - 2}{\rho^2} \varphi = C \]
for some constant \( C \in \mathbb{R}. \) Hence, a critical point of \( S \) is a stable minimum if, and only if, one has
\[ \tilde{S}''(\varphi) = 2 \int_{S_{\mathbb{H}^n}} \left( \left| \text{grad}_{HS} \varphi_1 \right|^2 - \frac{2n - 2}{\rho^2} \varphi_1^2 \right) \sigma_0^{2n} \geq 0 \]
for all \( \varphi_1 \in \Phi_1(S_{\mathbb{H}^n}). \) As a straightforward consequence, positivity of \( S \) is equivalent to positivity of \( S''. \)
Remark 4.4. Unlike the Riemannian case, for which we refer the reader to [1], the knowledge of the minimum eigenvalue of $L_{\mathbb{R}^n}$ on $\mathbb{S}_{\mathbb{R}^n}$ is not sufficient to solve this problem. More precisely, Rayleigh’s Inequality says that

$$
\lambda_1 \leq \frac{\int_{\mathbb{S}_{\mathbb{R}^n}} |\text{grad}_{\mathbb{R}^n} \varphi|^2 \sigma_\mu^{2n}}{\int_{\mathbb{S}_{\mathbb{R}^n}} \varphi^2 \sigma_\mu^{2n}} \quad \forall \, \varphi \in \Phi(\mathbb{S}_{\mathbb{R}^n}), \, \int_{\mathbb{S}_{\mathbb{R}^n}} \varphi \sigma_\mu^{2n} = 0,
$$

where $\lambda_1$ denotes the first non-trivial eigenvalue of $L_{\mathbb{R}^n}$ on $\mathbb{S}_{\mathbb{R}^n}$. This implies that

$$
II_{\mathbb{S}_{\mathbb{R}^n}}(W, \sigma_\mu^{2n}) \geq \int_{\mathbb{S}_{\mathbb{R}^n}} \varphi^2 \left( \lambda_1 - \frac{Q - 4}{\rho^2} \right) \sigma_\mu^{2n},
$$

with strict inequality unless $\varphi$ is an eigenfunction associated to $\lambda_1$. But the last integral it is not necessarily greater than zero.

From now on, we will study the closed eigenvalue problem (singular at $\mathcal{N}, \mathcal{S}$):

$$
L_{\mathbb{R}^n} \varphi + \frac{\mu}{\rho^2} \varphi = 0 \quad \forall \, \varphi \in \Phi_1(\mathbb{S}_{\mathbb{R}^n}).
$$

(20)

Here we have to remark that all solutions to this equation must satisfy the following further compatibility condition:

$$
\int_{\mathbb{S}_{\mathbb{R}^n}} \varphi \rho^2 \sigma_\mu^{2n} = 0.
$$

To see this, it is sufficient to integrate (20) over $\mathbb{S}_{\mathbb{R}^n}$ and use $\int_{\mathbb{S}_{\mathbb{R}^n}} L_{\mathbb{R}^n} \varphi \sigma_\mu^{2n} = 0$. Furthermore, it is a simple consequence of the horizontal Green formulas discussed in Section 1.3, that the following hold:

- all eigenvalues are positive real numbers;
- all eigenfunctions can be chosen to be real-valued;
- eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the “weighted” inner product $(\phi_1, \phi_2)_0 := \int_{\mathbb{S}_{\mathbb{R}^n}} \frac{\phi_1 \phi_2}{\rho^2} \sigma_\mu^{2n}$;
- all eigenfunctions can be chosen to be orthogonal (note that eigenvalues with multiplicity will have several eigenfunctions) with respect to $\langle \cdot, \cdot \rangle_0$.

Set now

$$
\mathcal{G}(\varphi) := \frac{\int_{\mathbb{S}_{\mathbb{R}^n}} |\text{grad}_{\mathbb{R}^n} \varphi|^2 \sigma_\mu^{2n}}{\int_{\mathbb{S}_{\mathbb{R}^n}} \frac{\varphi^2}{\rho^2} \sigma_\mu^{2n}}, \quad \forall \, \varphi \in \Phi_1(\mathbb{S}_{\mathbb{R}^n}).
$$

Lemma 4.5. Let $\mu_1$ be the first eigenvalue of (20) and consider the minimization problem:

$$
m := \min_{\varphi \in \Phi_1(\mathbb{S}_{\mathbb{R}^n})} \mathcal{G}(\varphi).
$$

Assume that the minimum is achieved, but probably not unique. Then $m$ is the first eigenvalue of (20) and any minimizer $f$ of $\mathcal{G}(\cdot)$ is a corresponding eigenfunction.

Proof. If $f : \mathbb{S}_{\mathbb{R}^n} \to \mathbb{R}$ is a minimizer in $\Phi_1(\mathbb{S}_{\mathbb{R}^n})$, then $\mathcal{G}(f) \leq \mathcal{G}(\varphi)$ for all $\varphi \in \Phi_1(\mathbb{S}_{\mathbb{R}^n})$. In this case, the real-valued function $g(\epsilon) = \mathcal{G}(f + \epsilon \varphi)$ has a minimum at $\epsilon = 0$ and hence $g'(0) = 0$. We have

$$
g'(0) = 2 \left( \int_{\mathbb{S}_{\mathbb{R}^n}} \frac{f^2}{\rho^2} \sigma_\mu^{2n} \right) \left( \int_{\mathbb{S}_{\mathbb{R}^n}} \langle \text{grad}_{\mathbb{R}^n} f, \text{grad}_{\mathbb{R}^n} \varphi \rangle \sigma_\mu^{2n} \right) - \left( \int_{\mathbb{S}_{\mathbb{R}^n}} \frac{f^2}{\rho^2} \sigma_\mu^{2n} \right) \left( \int_{\mathbb{S}_{\mathbb{R}^n}} |\text{grad}_{\mathbb{R}^n} f|^2 \sigma_\mu^{2n} \right) = 0.
$$

(21)

Therefore

$$
\int_{\mathbb{S}_{\mathbb{R}^n}} \langle \text{grad}_{\mathbb{R}^n} f, \text{grad}_{\mathbb{R}^n} \varphi \rangle \sigma_\mu^{2n} = m \int_{\mathbb{S}_{\mathbb{R}^n}} \frac{f \varphi}{\rho^2} \sigma_\mu^{2n}.
$$
and since \(-\int_{S^{2n}} \varphi \mathcal{L}_{HS} f \sigma_{H}^{2n} = \int_{S^{2n}} (\text{grad}_{HS} f, \text{grad}_{HS} \varphi) \sigma_{H}^{2n}\), it follows that
\[
\int_{S^{2n}} \varphi \left( \mathcal{L}_{HS} f + \frac{m f}{\rho^2} \right) \sigma_{H}^{2n} = 0
\]
for all \(\varphi \in \Phi_1(S_{H}^{n})\). Hence
\[
\mathcal{L}_{HS} f + \frac{m f}{\rho^2} = 0,
\]
i.e. \(f\) is eigenfunction of \((20)\) with eigenvalue \(m\). In order to prove the last claim, let \(\mu_i\) be another eigenvalue of \((20)\) with corresponding eigenfunction \(f_i\). Then
\[
m \leq \Theta(f_i) = -\frac{\int_{S^{2n}} f_i \mathcal{L}_{HS} f_i \sigma_{H}^{2n}}{\int_{S^{2n}} \frac{\mu_i f_i}{\rho^2} \sigma_{H}^{2n}} = \mu_i
\]
and hence \(m = \mu_1\). 

We already know that the function \(\kappa = \kappa = \pm \sqrt{\frac{1-\rho^2}{\rho^2}}\) is an eigenfunction of \((20)\) with corresponding eigenfunction \(\mu = 2n - 2\); see Remark 4.3. Below we shall show that \(\mu = \mu_{1}^{rad}\), where \(\mu_{1}^{rad}\) denotes the 1st (radial) eigenvalue of \((20)\) in the class of all radial \(\varphi \in \Phi_1(S_{H}^{n})\).

Along the lines of [28], where a similar method is adopted to study the “closed eigenvalue problem on \(S_{H}^{n}\)” for the equation
\[
\mathcal{L}_{HS} \varphi + \lambda \varphi = 0,
\]
as a first step, we shall study the equation \((20)\) for radial functions on \(S_{H}^{n}\). Hence, we have to solve the following O.D.E:
\[
\varphi''(1 - \rho^2) + \frac{\varphi'}{\rho} (2n - (2n + 1) \rho^2) + \frac{\mu}{\rho^2} \varphi = 0,
\]
where \(\varphi\) is now a radial function belonging to \(\Phi_1(S_{H}^{n})\); see Remark 4.6. This can be done, exactly as in [28], by studying the restrictions \(\varphi^{\pm}\) of \(\varphi\) to the hemispheres \(S_{H}^{\pm}\), together with some suitable boundary conditions. For the sake of simplicity, in doing this, we shall assume \(\varphi \in C^2([0, 1])\).

Remark 4.6. Remind that
\[
\sigma_{H}^{2n} \left| S_{H}^{\pm}\right| = \frac{\rho}{2\sqrt{1 - \rho^2}} d\varepsilon_{B_1(0)},
\]
and using spherical coordinates \((\rho, \xi) \in [0, 1] \times S^{2n-1}\) on \(B_1(0) \subseteq \mathbb{R}^{2n}\) yields
\[
d\varepsilon = \rho^{2n-1} dp \wedge d\sigma_{S^{2n-1}}(\xi).
\]
The integral conditions required for belonging to \(\Phi_1(S_{H}^{n})\) can then be rephrased in terms of one-dimensional integrals over the interval \([0, 1]\), endowed with a “weight-function” induced by \(\sigma_{H}^{2n}\).

Lemma 4.7. Let \(\psi\) be an eigenfunction of \((20)\) with corresponding eigenvalue \(\mu\) and denote by \(\psi_0\) its spherical mean, i.e.
\[
\psi_0 := \int_{S^{2n-1}} \psi \, d\sigma_{S^{2n-1}}.
\]
If \(\psi_0 \neq 0\), then \(\psi_0\) is an eigenfunction of \((22)\) with corresponding eigenfunction \(\mu\).

Proof. Analogous to the proof of Claim 2.23 in [28]. First, note that
\[
\int_{S^{2n}} \psi_0 \sigma_{H}^{2n} = \int_{S^{2n}} \int_{S^{2n-1}} \psi \left( d\sigma_{S^{2n-1}} \wedge \sigma_{H}^{2n} \right) = \int_{S^{2n-1}} \int_{S^{2n}} \psi \sigma_{H}^{2n} \, d\sigma_{S^{2n-1}} = 0.
\]
By making use of \((21)\) (see Remark 4.6) we have
\[
\mathcal{L}_{HS} \psi = (1 - \rho^2)\psi'' + \frac{2n - (2n + 1) \rho^2}{\rho} \psi' - 2\rho \sqrt{1 - \rho^2} \psi'' - \frac{1}{\rho^2} \Delta_{S^{2n-1}} \psi - (1 - \rho^2) \psi'' - (Q - 1) \sqrt{1 - \rho^2} \psi'.
\]
and its associated eigenfunction is given, up to constants, by

\[
\Phi \quad \text{such that} \quad \int_{\mathbb{S}^{2n-1}} \mathcal{L}_{HS} \psi \, d\sigma_{\mathbb{S}^{2n-1}} = -\mu \int_{\mathbb{S}^{2n-1}} \frac{\psi}{\rho^2} \, d\sigma_{\mathbb{S}^{2n-1}} = -\mu \psi_0 \quad \text{for all} \quad \mu \neq 0.
\]

Furthermore, one has

\[
\mathcal{L}_{HS} \psi_0 = \frac{(1 - \rho^2) \partial^2 \psi_0}{\partial \rho^2} + \frac{2n - (2n + 1) \rho^2}{\rho} \partial_\rho \psi_0
\]

\[
= \int_{\mathbb{S}^{2n-1}} \left( (1 - \rho^2) \psi''_0 + \frac{2n - (2n + 1) \rho^2}{\rho} \psi'_0 \right) \, d\sigma_{\mathbb{S}^{2n-1}}.
\]

We therefore get that

\[
\mathcal{L}_{HS} \psi_0 = \int_{\mathbb{S}^{2n-1}} \mathcal{L}_{HS} \psi \, d\sigma_{\mathbb{S}^{2n-1}} = -\mu \int_{\mathbb{S}^{2n-1}} \frac{\psi}{\rho^2} \, d\sigma_{\mathbb{S}^{2n-1}} = -\mu \psi_0
\]

which proves the thesis. \(\square\)

Unfortunately, the spherical mean \(\psi_0\) of any eigenfunction \(\psi\) can be equal to 0 and so, in general, we cannot conclude that there are no other eigenvalues, apart from the radial eigenvalues. We need something different.

In the next Lemma 4.8 we will use Frobenius’ Method; see [42]. Note that if we attempt to find the solution of equation (22) in the form of a power series, we have to employ the Laurent expansion around \(t = 0\). So let \(m \in \mathbb{Z}\) and assume that

\[
\varphi(\rho) = \rho^{-m} \sum_{l=0}^{+\infty} a_l \rho^l = \sum_{l=0}^{+\infty} a_l \rho^{l-m} = 0.
\]

**Lemma 4.8.** There exist solutions to (22) of the form (24) if, and only if, one has

\[
\mu_m = m(2n - (m + 1)) \quad \text{for every} \quad m = 1, \ldots, 2n - 1.
\]

These numbers can be eigenvalues of (22), associated with radial eigenfunctions belonging to the class \(\Phi_1(\mathbb{S}^{2n})\), only if \(m = 1, \ldots, n - 1\). In particular, the first eigenvalue of (22) turns out to be \(\mu_1 = Q - 4\) and its associated eigenfunction is given, up to constants, by \(\varphi_1 = \kappa\); see Remark 3.3.

**Proof.** Since \(\varphi'(\rho) = \sum_{l=0}^{+\infty} (l - m) a_l \rho^{l-m-1}\) and \(\varphi''(\rho) = \sum_{l=0}^{+\infty} (l - m)(l - m - 1) a_l \rho^{l-m-2}\), substituting these expressions into (22) yields

\[
\sum_{l=0}^{+\infty} (l - m)(l - m - 1) a_l (1 - \rho^2) + (l - m) a_l ((2n - (2n + 1) \rho^2) + \mu a_l) \rho^{l-m-2} = 0.
\]

So we get that

\[
\sum_{l=0}^{+\infty} a_l ((l - m)(l - m - 1 + 2n) + \mu) \rho^{l-m-2} = \sum_{l=0}^{+\infty} a_l ((l - m)(l - m + 2n) \rho^{l-m}.
\]

From this identity we infer a system of necessary conditions on the coefficients of the Laurent expansion of \(\varphi\). More precisely, we must have

\[
a_0 = a_1 = 0 \quad \text{and} \quad \alpha_{l+2} = \beta_l \quad \text{for all} \quad l \in \mathbb{N}.
\]

(25)

Since \(a_0 = a_0 (m(m + 1 - 2n) + \mu)\) and \(a_1 = a_1 (-m - 1)(2n - m) + \mu\) we obtain either

\[
a_0 = 0 \quad \text{or} \quad \mu = m(2n - 1 - m),
\]

(26)

or

\[
a_1 = 0 \quad \text{or} \quad \mu = (m - 1)(2n - m); \quad (27)
\]
Furthermore,
\[ a_{l+2} = a_l \frac{(l-m)(l+2n-m)}{(l+2-m)(l+2n+1-m)+\mu} \quad \text{for all } l \in \mathbb{N}. \]

Note that this procedure makes possible to write down the solutions in terms of recurrence relations. From (25) we get that, if \( a_0 = a_1 = 0 \), then all coefficients must be zero. Since \( \mu > 0 \), assuming \( a_0 \neq 0 \) (and therefore, \( a_1 = 0 \)), yields \( m \in [2, 2n-1] \), while assuming \( a_1 \neq 0 \) (and therefore, \( a_0 = 0 \)) yields \( m \in [1, 2n-2] \). This proves the first claim.

For what concerns the second claim, note that any (radial) solution \( \varphi \) belongs to \( \Phi_1(S_{\mathbb{H}^n}) \) only if
\[ \varphi \in L^2 \left( [0,1], \frac{\rho^{2n-2}}{2\sqrt{1-\rho^2}} d\rho \right), \quad \varphi' \in L^2 \left( [0,1], \frac{\rho^{2n}}{2\sqrt{1-\rho^2}} d\rho \right). \]

Therefore, by an elementary computation, we get that it must be
\[ 2n - 2(m + 1) > -1 \iff m \leq n - 1. \]

Finally, the last claim was already known; see Remark 4.3. \( \square \)

Let us state a first consequence. By making use of Lemma 4.5 and Lemma 4.8 we get that
\[ \int_{S_{\mathbb{H}^n}} |\text{grad}_{\mathbb{H}^n} \varphi|^2 \sigma_{n}^{2n} \geq (2n-2) \int_{S_{\mathbb{H}^n}} \frac{\varphi^2}{\rho^2} \sigma_{n}^{2n} \quad \text{for all radial } \varphi \in \Phi_1(S_{\mathbb{H}^n}). \]

Therefore
\[ I_{S_{\mathbb{H}^n}}(W, \sigma_{n}^{2n}) = \int_{S_{\mathbb{H}^n}} \left( |\text{grad}_{\mathbb{H}^n} \varphi|^2 - \frac{Q-4}{\rho^2} \varphi^2 \right) \sigma_{n}^{2n} \geq 0 \]
for all radial function \( \varphi \in \Phi_1(S_{\mathbb{H}^n}) \) and, more precisely, \( I_{S_{\mathbb{H}^n}}(W, \sigma_{n}^{2n}) > 0 \), unless \( \varphi \) is an eigenfunction of \( \mu_1 = Q - 4 \). The radial eigenfunction of \( \mu_1 \), up to constants, is the function \( \kappa \pm = \pm \sqrt{1 - \rho^2} / \rho \) which does not satisfy Definition 3.14 because it is singular at the poles. This proves the following:

**Proposition 4.9** (Radial stability). The Isoperimetric Profile \( S_{\mathbb{H}^n} \) turns out to be radially stable in the sense of Remark 3.10. More precisely, let \( \delta \nu \) be any normal variation of \( S_{\mathbb{H}^n} \), having variation vector \( W = \varphi |P_{\mathbb{H}} \nu| \nu \), where \( \varphi \in \Phi_1(S_{\mathbb{H}^n}) \) is radial. Then, we have \( I_{S_{\mathbb{H}^n}}(W, \sigma_{n}^{2n}) > 0 \) and \( I_{S_{\mathbb{H}^n}}(W, \sigma_{n}^{2n}) = 0 \) if, and only if, \( \varphi \) is an eigenfunction for the first eigenvalue \( \mu_1^{\text{rad}} := \mu_1 = Q - 4 \) of equation (22).

We now discuss some other features of the general case. We start from a lemma which is well-known in the classical setting; see [14].

**Lemma 4.10.** Let \( S \subset \mathbb{H}^n \) be a hypersurface of class \( C^2 \) and let \( \Omega \subset S \) be any bounded domain. If there exists a function \( \psi > 0 \) on \( \Omega \) satisfying the equation \( L_{\mathbb{H}^n} \psi = q \psi \), then
\[ \int_{\Omega} (|\text{grad}_{\mathbb{H}^n} \varphi|^2 + q \varphi^2) \sigma_{n}^{2n} \geq 0 \]
for all smooth function \( \varphi \) compactly supported on \( \Omega \).

**Proof.** If \( \psi > 0 \) satisfies \( L_{\mathbb{H}^n} \psi = q \psi \) on \( \Omega \), let us define a new function \( \phi := \log \psi \). By an elementary calculation we see that \( L_{\mathbb{H}^n} \phi = q - |\text{grad}_{\mathbb{H}^n} \phi|^2 \). Indeed, one has
\[
L_{\mathbb{H}^n} \phi = \text{div}_{\mathbb{H}^n} \left( \text{grad}_{\mathbb{H}^n} \phi \right) - \nabla \left( \nabla_{\mathbb{H}^n} \phi, \text{grad}_{\mathbb{H}^n} \phi \right) \\
= \text{div}_{\mathbb{H}^n} \left( \text{grad}_{\mathbb{H}^n} \psi \right) - \nabla \left( \nabla_{\mathbb{H}^n} \psi, \text{grad}_{\mathbb{H}^n} \psi \right) \\
= \left( \Delta_{\mathbb{H}^n} \psi \right) - \nabla \left( \nabla_{\mathbb{H}^n} \psi, \text{grad}_{\mathbb{H}^n} \psi \right) - \frac{|\text{grad}_{\mathbb{H}^n} \psi|^2}{\psi^2} \\
= \frac{L_{\mathbb{H}^n} \psi}{\psi} - |\text{grad}_{\mathbb{H}^n} \phi|^2 \\
= q - |\text{grad}_{\mathbb{H}^n} \phi|^2.
\]
Now let \( \varphi \) be any smooth function with compact support on \( \Omega \). Multiplying by \( -\varphi^2 \) both sides of this equation and integrating by parts, yields

\[
- \int_\Omega \varphi^2 \left( q - |\text{grad}_{\text{HS}} \varphi|^2 \right) \sigma_n^2 = \int_\Omega \varphi^2 L_{\text{HS}} \varphi \sigma_n^2 = \int_\Omega 2\varphi \langle \text{grad}_{\text{HS}} \varphi, \text{grad}_{\text{HS}} \varphi \rangle \sigma_n^2. \tag{28}
\]

Now since

\[
2|\varphi \langle \text{grad}_{\text{HS}} \varphi, \text{grad}_{\text{HS}} \varphi \rangle| \leq 2|\varphi||\text{grad}_{\text{HS}} \varphi||\text{grad}_{\text{HS}} \varphi| \leq |\varphi|^2|\text{grad}_{\text{HS}} \varphi|^2 + |\text{grad}_{\text{HS}} \varphi|^2,
\]

the thesis follows by inserting this inequality into (28) and then by cancelling the terms \( \int_\Omega \varphi^2 |\text{grad}_{\text{HS}} \varphi|^2 \sigma_n^2 \).

\[\square\]

**Corollary 4.11.** Let \( \Omega \subseteq \mathbb{S}^*_{3n} \) or \( \Omega \subseteq \mathbb{S}^*_{2n} \). Then, the following inequality holds

\[
\int_\Omega \left( |\text{grad}_{\text{HS}} \varphi|^2 - \frac{2n-2}{\rho^2} \varphi^2 \right) \sigma_n^2 \geq 0
\]

for all smooth function \( \varphi \) compactly supported on \( \Omega \).

**Proof.** Setting \( q = -\frac{2n-2}{\rho^2} \), the thesis follows by applying Lemma 4.10 with \( \psi := \sqrt{\frac{|\rho^2}{\rho}} \).

Another easy consequence of Lemma 4.10 is contained in the next:

**Corollary 4.12.** Set \( S^* := \left\{ p = \exp(z, t) \in \mathbb{S}^*_{3n} : \rho = |z| \geq \sqrt{\frac{2n-3}{2n-2}} \right\} \). Then, for every \( \Omega \subseteq S^* \), the following inequality holds

\[
\int_\Omega \left( |\text{grad}_{\text{HS}} \varphi|^2 - \frac{2n-3}{\rho^2} \varphi^2 \right) \sigma_n^2 \geq 0
\]

for all smooth function \( \varphi \) compactly supported on \( \Omega \).

**Proof.** Choose \( q = -\frac{2(2n-3)}{\rho^2} \). Note that the function \( \psi := \frac{2n-3}{2n-2} - \frac{2n-3}{\rho^2} \) is strictly positive on every \( \Omega \subseteq S^* \). Furthermore, it turns out that \( L_{\text{HS}} \psi = 0 \). Then, the thesis follows by applying Lemma 4.10.

**Remark 4.13.** In the inequalities of both Corollary 4.11 and Corollary 4.12, the function \( \varphi \) is not necessarily zero-mean. In other words, we do not require the validity of the condition \( \int_\Omega \varphi \sigma_n^2 = 0 \). In a sense, these inequalities are stronger than what one might expect.

Putting together Corollary 4.11 and Corollary 4.12, we immediately get the following:

**Theorem 4.14 (Local Stability).** The Isoperimetric profile \( S^*_{2n} \) is a locally stable bounding hypersurface in the sense of Definition 3.17.

We end this section with the following:

**Remark 4.15 (Question).** If \( \mu \) denotes the 1st eigenvalue of (20), then is it true that \( \mu = 2n - 2 \)? Roughly speaking, is the 1st eigenvalue of (20) equal to the first eigenvalue of the radial case?

Note that a negative answer to this question would automatically imply that Isoperimetric Profiles are unstable.

### 4.1 Appendix A: the case of T-graphs

Below we overview the variational formulas for the \( H \)-perimeter \( \sigma_n^2 \), in the case of smooth \( T \)-graphs.

Let \( \Omega \subseteq \mathbb{R}^{2n} \) be an open set, let \( u \in C^2(\Omega) \) and set \( S := \{ \exp(z, t) : t = u(z) \quad \forall \ z \in \Omega \} \), i.e. \( S \) is the \( T \)-graph associated with \( u \). Then, \( \nu = \left( -\frac{\nabla_{\mathbb{R}^{2n}} u + z}{\|\nabla_{\mathbb{R}^{2n}} u + z\|} \right) \) is the unit normal along \( S \) and we have \( \nu^\perp = \frac{-\nabla_{\mathbb{R}^{2n}} u + z}{\|\nabla_{\mathbb{R}^{2n}} u + z\|} \) and \( \varpi = \frac{1}{\|\nabla_{\mathbb{R}^{2n}} u - z\|} \). The \( H \)-perimeter measure \( \sigma_n^2 \) on \( S \) turns out to be given by

\[
\sigma_n^2 \mathcal{L} S = |P_h \nu| \sigma_n^2 \mathcal{L} S = \frac{1}{\varpi} \int_\Omega \left\| \nabla_{\mathbb{R}^{2n}} u - \frac{z}{2} \right\| \, dz \mathcal{L} \Omega.
\]
The computation of the 1st and 2nd variation of $\sigma^2_n \mathbf{L} = \int_{\Omega} \left\| \nabla_{\mathbb{R}^n} u - \frac{u^t}{2} \right\| \, dz$ can be done by using variations $\varphi_t$, which only act along the $T$-direction. So in particular we are here assuming that the variation vector is given by $W = \varphi T$ for some $\varphi \in C^4(S)$. Since $p = \exp(z, u(z)) \in S$ for every $z \in \Omega$, with a slight abuse of notation, we shall also assume that $\varphi : \Omega \to \mathbb{R}$. Hence, using $T$-variations one has

$$I_S(\varphi T, \sigma^2_n) = \frac{d}{ds} \bigg|_{s=0} \int_{\Omega} \left\| \nabla_{\mathbb{R}^n} (u + s \varphi) - \frac{z^t}{2} \right\| \, dz,$$

$$II_S(\varphi T, \sigma^2_n) = \frac{d^2}{ds^2} \bigg|_{s=0} \int_{\Omega} \left\| \nabla_{\mathbb{R}^n} (u + s \varphi) - \frac{z^t}{2} \right\| \, dz,$$

for every $\varphi \in C^1(\Omega)$. An elementary calculation shows that:

$$I_S(\varphi T, \sigma^2_n) = - \int_{\Omega} \left( \nabla_{\mathbb{R}^n} \varphi \right) \left( \frac{-\nabla_{\mathbb{R}^n} u + \frac{z^t}{2}}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|} \right) \, dz,$$

$$II_S(\varphi T, \sigma^2_n) = - \int_{\Omega} \frac{\left\| \nabla_{\mathbb{R}^n} \varphi \right\|^2 \left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|^2}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|^3} \, dz.$$

**Remark 4.16.** Let $\Omega \subset \mathbb{R}^n$ be a relatively compact open set having piecewise $C^4$-smooth boundary and let $\varphi \in C^4(\Omega)$. Since the integrand in (29) is everywhere bounded by $\left\| \nabla_{\mathbb{R}^n} \varphi \right\|$, the 1st variation formula (29) makes sense for every $T$-graph $u : \Omega \to \mathbb{R}$ of class $C^2$. Furthermore, by using the standard Divergence Theorem, we get

$$I_S(\varphi T, \sigma^2_n) = \int_{\Omega} \varphi \, \text{div}_{\mathbb{R}^n} \left( \frac{-\nabla_{\mathbb{R}^n} u + \frac{z^t}{2}}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|} \right) \, dz,$$

for every $\varphi \in C^4_0(\Omega)$, where $\mathcal{H}_u = -\text{div}_{\mathbb{R}^n} \left( \frac{-\nabla_{\mathbb{R}^n} u + \frac{z^t}{2}}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|} \right)$. The integrand in (30) can be estimated by

$$2 \frac{\left\| \nabla_{\mathbb{R}^n} \varphi \right\|^2}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|}.$$ 

Thus, by assuming

$$\frac{1}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|} \in L^1(\Omega, dz),$$

the 2nd variation formula (30) makes sense for every $T$-graph $u : \Omega \to \mathbb{R}$ of class $C^2$. It is not difficult to show that (31) is equivalent to $\frac{1}{\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\|} \in L^1(S, \sigma^2_n)$; compare Lemma 3.6. Finally, under the same assumptions, one has $II_S(\varphi T, \sigma^2_n) \geq 0$ for every $\varphi \in C^4(\Omega)$.

**Remark 4.17.** For radial $T$-graphs of class at least $C^4$, condition (31) is satisfied. Indeed, one has

$$\left\| \nabla_{\mathbb{R}^n} u - \frac{z^t}{2} \right\| = \sqrt{(u^t)'^2 + \frac{z^t}{4}},$$

and the claim follows since

$$\frac{1}{\sqrt{(u^t)'^2 + \frac{z^t}{4}}} \leq \frac{2}{\rho} \in L^1(\Omega, dz).$$

In the case of Isoperimetric Profiles, we use (29) and (30) to give a heuristic proof of their stability.

**Remark 4.18.** Let $S = S^+_{\mathbb{R}^n}$ and $u = \pm u_\rho$; see Section 2. In this case $\Omega = B_1(0) \subset \mathbb{R}^2$ and we have

$$I_{S^+_{\mathbb{R}^n}}(\varphi T, \sigma^2_n) = - \int_{B_1(0)} \left( \nabla_{\mathbb{R}^n} \varphi, z \right) \, dz = \int_{B_1(0)} \varphi \left( \text{div}_{\mathbb{R}^n} z - \text{div}_{\mathbb{R}^n}(\varphi z) \right) \, dz,$$

where the second equality follows by applying the usual Divergence Theorem. Therefore

$$I_{S^+_{\mathbb{R}^n}}(\varphi T, \sigma^2_n) = 2n \int_{B_1(0)} \varphi \, dz,$$

for every $\varphi \in C^4_0(B_1(0))$. This shows that each hemisphere $S^+_{\mathbb{R}^n}$ is a critical point of $\sigma^2_n$ belonging to the class $\Psi = \{ \varphi \in C^4_0(B_1(0)) : \int_{B_1(0)} \varphi \, dz = 0 \}$. Note that the last integral condition gives a volume constraint on the functional $\sigma^2_n \mathbf{L} \subset C^4_{B_1}$. In other words, we are using “volume preserving variations”; see Section 3.7. By Cauchy-Schwarz we obtain

$$II_{S^+_{\mathbb{R}^n}}(\varphi T, \sigma^2_n) \geq 2 \int_{B_1(0)} \left\| \nabla_{\mathbb{R}^n} \varphi \right\|^2 \left( 1 - \rho^2 \right) \frac{1}{\rho} \, dz \geq 0,$$

or, in other words, the stability of each hemisphere in the class $\Psi$.  

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4.2 Appendix B: further remarks about stability

For future purposes, we discuss two conditions concerning stability: a sufficient condition and a necessary condition. Let $D \subset \mathbb{H}^n$ be a compact domain with boundary $S = \partial D$ satisfying either the hypotheses of Corollary 3.11 or those of Corollary 3.11. Integration by parts yields

$$II_S(W, \sigma^2_{n}) = \int_S \left( |\text{grad}_H \varphi|^2 + \varphi^2 \left( -\| S_H \|_{G_r}^2 + 2 \frac{\partial \varphi}{\partial n_H} - \frac{n+1}{2} \varphi^2 \right) \right) \sigma^2_{n}$$

$$= - \int_S \varphi \left( L_{HS} \varphi - \varphi \left( -\| S_H \|_{G_r}^2 + 2 \frac{\partial \varphi}{\partial n_H} - \frac{n+1}{2} \varphi^2 \right) \right) \sigma^2_{n}.$$ 

Therefore, we can turn our attention to a suitable eigenvalue problem for the operator $L_{HS}$. More precisely, let us consider the following:

$$(P) \begin{cases} L_{HS} \varphi = \lambda \varphi \left( -\| S_H \|_{G_r}^2 + 2 \frac{\partial \varphi}{\partial n_H} - \frac{n+1}{2} \varphi^2 \right) & \text{on } S \\ \int_S \varphi \sigma^2_{n} = 0 \end{cases}$$ 

whenever $\varphi \in \Phi(S)$, $\varphi \neq 0$; see Definition 1.20. Thus we see that:

- A sufficient condition for the stability of $D$ is that the first eigenvalue of this problem is greater than, or equal to, one.

For what concerns the necessary condition, we first state an integral identity.

**Lemma 4.19.** Let $S \subset \mathbb{H}^n$ be a $C^2$-smooth compact hypersurface without boundary. Then

$$\int_S \left( 2 \omega^2 \frac{\partial \varphi}{\partial n} - \frac{2n}{3} \omega^4 \right) \sigma^2_{n} = 0,$$

where $\omega \nu^i_H$ is admissible (for the horizontal divergence formula); see Definition 1.20.

**Proof.** We have

$$\int_S D_{HS} (\omega \nu^i_H) \sigma^2_{n} = \int_S (\text{div}_H (\omega \nu^i_H) - \omega (\nu^i_H, \omega \nu^i_H)) \sigma^2_{n} = \int_S \omega \sigma^2_{n} = 0.$$ 

Since

$$\text{div}_H (\omega \nu^i_H) = 3 \omega^2 \frac{\partial \varphi}{\partial n_H} \omega^3 \text{div}_H (\nu^i_H) + \omega^3 \left( \sum_{i \in I_{HS}} \langle \nabla_{H \tau i} \nu^i_H, \tau_i \rangle \right)$$

$$= 3 \omega^2 \frac{\partial \varphi}{\partial n_H} - \omega^4 \text{Tr} (B_H (\cdot, C^0_{HS} \cdot)),$$

where $C^0_{HS} = C^0_{HS} |_{HS}$, we get that

$$\int_S \left( 2 \omega^2 \frac{\partial \varphi}{\partial n_H} - \omega^4 \right) \sigma^2_{n} = \int_S \left( \text{Tr} (B_H (\cdot, C^0_{HS} \cdot)) \right) \omega \sigma^2_{n}.$$ 

By Lemma 1.12 we have $\text{Tr} (B_H (\cdot, C^0_{HS} \cdot)) = (n-1) \omega$ and (32) easily follows from (33). \qed

Now we apply Lemma 4.19 together with a special choice of the variation vector $W$. Here we have to assume the validity of Corollary 3.11 for a variation $\vartheta$, having variation vector $W = \omega [\vartheta, \nu].$ Note that $\omega$ is a 0-mean function on $S$ with respect to the measure $\sigma^2_{n}$ and that $\omega$ is smooth out of $C_S$. Moreover, let us suppose that the vector field $\omega \nu^i_H$ is admissible (for the horizontal divergence formula); see Definition 1.20. It follows that

$$II_S(W, \sigma^2_{n}) = \int_S \left( |\text{grad}_H \omega|^2 + \omega^2 \left( -\| S_H \|_{G_r}^2 + 2 \frac{\partial \omega}{\partial n_H} - \frac{n+1}{2} \omega^2 \right) \right) \sigma^2_{n}$$

$$= \int_S \left( |\text{grad}_H \omega|^2 - \omega^2 \left( \| S_H \|_{G_r}^2 + \frac{3-n}{6} \omega^2 \right) \right) \sigma^2_{n}.$$
• Under our current assumptions, a necessary condition for the stability of $D$ is given by the following geometric inequality:

$$
\int_S |\text{grad}_H \omega|^2 \sigma_H^{2n} \geq \int_S \omega^2 \left( \|S_H\|^2_{G^r} + \left( \frac{3-n}{6} \right) \omega^2 \right) \sigma_H^{2n}.
$$

We stress that, in the case of the Isoperimetric Profile $S_{\mathbb{H}^n}$, this inequality is in fact an identity.

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Francescopaolo Montefalcone:
Dipartimento di Matematica Pura e Applicata
Università degli Studi di Padova,
Via Trieste, 63, 35121 Padova (Italy)
E-mail address: montefal@math.unipd.it