SPECTRAL ANALYSIS AND STABILIZATION OF THE DISSIPATIVE SCHRÖDINGER OPERATOR ON THE TADPOLE GRAPH

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Abstract. We consider the damped Schrödinger semigroup $e^{-i \frac{t}{\hbar} \Delta}$ on the tadpole graph $\mathcal{R}$. We first give a careful spectral analysis and an appropriate decomposition of the kernel of the resolvent. As a consequence and by showing that the generalized eigenfunctions form a Riesz basis of some subspace of $L^2(\mathcal{R})$, we prove that the corresponding energy decay exponentially.

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1. Introduction

In the last few years various physical models of multi-link flexible structures consisting of finitely or infinitely many interconnected flexible elements such as strings, beams, plates, shells have been of great interest. See the references [2] as well as [7, 4, 6] and the references therein. The spectral analysis of such structures has some applications to control or stabilization problems (cf. [3]).

In this paper we give a careful spectral analysis of the dissipative Schrödinger operator on the tadpole graph (sometimes also called lasso graph) and as application we study the corresponding stabilization problem:

![Tadpole graph](image)

**Figure 1. Tadpole graph**

Before a precise statement of our main result, let us introduce some notations which will be used throughout the rest of the paper.

Let $R_i, i = 1, 2$, be two disjoint sets identified with a closed path of measure equal to $L > 0$ for $R_2$ and to $(0, +\infty)$, for $R_1$, see Figure 1. We set $\mathcal{R} := \bigcup_{k=1}^{2} R_k$. We denote by $f = (f_k)_{k=1,2} = (f_1, f_2)$ the functions on $\mathcal{R}$ taking their values in $\mathbb{C}$ and let $f_k$ be the restriction of $f$ to $R_k$.

Define the Hilbert space

$$
\mathcal{H} := \bigoplus_{k=1}^{2} L^2(R_k) = L^2(\mathcal{R})
$$

with inner product

$$( (u_k), (v_k) )_{\mathcal{H}} = \sum_{k=1}^{2} (u_k, v_k)_{L^2(R_k)}$$

and introduce the following transmission conditions (see [6, 1]):

1.1) $$(u_k)_{k=1,2} \in \bigoplus_{k=1}^{2} C(R_k)$$ satisfies $u_1(0) = u_2(0) = u_2(L),$

1.2) $$(u_k)_{k=1,2} \in \bigoplus_{k=1}^{2} C^1(R_k)$$ satisfies

$$\frac{2}{\alpha} \sum_{k=1}^{2} \frac{d u_k}{dx}(0^+) - \frac{d u_2}{dx}(L^-) = i \alpha u_1(0),$$

where $\alpha$ is a positive constant.

Let $H : \mathcal{D}(H) \subset \mathcal{H} \rightarrow \mathcal{H}$ be the linear operator on $\mathcal{H}$ defined by:

$$\mathcal{D}(H) = \{(u_k)_{k=1,2} \in \bigoplus_{k=1}^{2} H^2(R_k); (u_k)_{k=1,2} \text{ satisfies (1.1), (1.2)} \},$$

$$H(u_k)_{k=1,2} = (H_k u_k)_{k=1,2} = \left( -\frac{d^2 u_k}{dx^2} \right)_{k=1,2} = -\Delta_{\mathcal{R}}(u_k)_{k=1,2}.$$

The operator $H$ generates a $C_0$ semigroup of contractions on $\mathcal{H}$, denoted by $(e^{tH})_{t \geq 0}$. We have in particular, for all $u_0 \in \mathcal{H}$, the following Schrödinger system

$$
\begin{align*}
\begin{cases}
u' = iH u, & \mathcal{R} \times (0, +\infty), \\
u(0) = u_0, & \mathcal{R},
\end{cases}
\end{align*}
$$

admits a unique mild solution $u \in C([0, +\infty); \mathcal{H})$ and satisfies the energy identity:
Let us use the notation \( \omega \) for
\[
\omega(2.6)
\]
The resolvent.

2.1. Theorem. For all \( t \geq 0 \),
\[
\|e^{itH}\|_{L(H^{-\infty})} \leq C e^{-\omega t},
\]
where \( C \) is a positive constant independent of \( t \), \( \omega = \frac{8\alpha}{3L} \) and \( H^{-\infty} \) is a subspace of \( H \) which will be specified later (Theorem 4.1).

The proof of this result is based on an appropriate decomposition of the kernel of the resolvent in meromorphic terms which proves the absence of a singular continuous spectrum. Moreover, a careful spectral analysis (Theorem 2.2 and Proposition 2.4) of the damped Schrödinger operator will be specified later (Theorem 4.1).

The paper is organized as follows. The kernel of the resolvent is given in Section 2 (Theorem 2.1). Further all eigenfunctions of the dissipative Hamiltonian on the tadpole are constructed. They correspond to the confined modes on the head of the tadpole, which do not interact with the queue. The interaction is described by the absolutely continuous spectrum. Technically the main point is a decomposition of the kernel of the resolvent in meromorphic terms (Theorem 2.5). The poles are shown to be the eigenvalues of the operator, the continuous term creates the absolutely continuous spectrum. The absence of further terms proves the absence of a singular continuous spectrum. Moreover, a careful spectral analysis (Theorem 2.2 and Proposition 2.4) of the damped Schrödinger operator are given in Section 2. In Section 3 we prove that the generalized eigenfunctions form a Riesz basis (in Theorem 3.1) of some subspace of \( H \). The precise formulation and a proof of Theorem 1.1 (Theorem 4.1) is given in Section 4.

2. The Resolvent and the spectrum

2.1. The resolvent. Given \( z \in \{ z \in \mathbb{C} : \Re z > 0, \Re z \leq 0 \} \) and \( g \in L^2(\mathcal{R}) \), we are looking for \( u \in D(H) \) solution of
\[
-\Delta u - z^2 u = g \text{ in } \mathcal{R}.
\]
Let us use the notation \( \omega = -iz \).

Hence we look for \( u \) in the form
\[
\begin{align*}
\omega(2.6) & \quad u_1(x) = \int_0^{+\infty} \frac{g_1(y)}{2\omega} \left( e^{-\omega|x-y|} - F_1(\omega)e^{-\omega(y+x)} \right) dy \\
& \quad - \int_0^L \frac{g_2(y)}{2\omega} \left( F_2(\omega)e^{-\omega(y+x)} + F_3(\omega)e^{\omega(y-x)} \right) dy,
\end{align*}
\]
\[
\begin{align*}
\omega(2.7) & \quad u_2(x) = \int_0^{+\infty} \frac{g_1(y)}{2\omega} \left( e^{-\omega|x+y|} + H_1(\omega)e^{-\omega(y-x)} \right) dy \\
& \quad + \int_0^L \frac{g_2(y)}{2\omega} \left( e^{-\omega|x+y|} + G_2(\omega)e^{-\omega(y+x)} + G_3(\omega)e^{\omega(y-x)} \\
& \quad \quad \quad + H_2(\omega)e^{\omega(x+y)} + H_3(\omega)e^{\omega(x+y)} \right) dy,
\end{align*}
\]
where \( F_i(\omega), G_i(\omega) \) and \( H_i(\omega) \), \( i = 1, 2, 3 \) are constants calculated below in order to satisfy \( 1.1 \), \( 1.2 \). Indeed from these expansion, we clearly see that for \( k = 1 \) or 2:
\[
u_k'' + \omega^2 u_k = g_k \text{ in } R_k.
\]
Now we see that the continuity condition (1.1) is satisfied if and only if
\[ G_1 + H_1 = G_1 e^{-\omega L} + H_1 e^{\omega L} = 1 - F_1, \]
\[ 1 + G_2 + H_2 = G_2 e^{-\omega L} + H_2 e^{\omega L} = -F_2, \]
\[ G_3 + H_3 = (1 + G_3) e^{-\omega L} + H_3 e^{\omega L} = -F_3, \]
while Kirchhoff condition (1.2) holds if and only if
\[ (1 + \frac{\alpha}{\omega}) F_1 + G_1 (e^{-\omega L} - 1) + H_1 (1 - e^{\omega L}) = -1 + \frac{\alpha}{\omega}, \]
\[ (1 + \frac{\alpha}{\omega}) F_2 + G_2 (e^{-\omega L} - 1) + H_2 (1 - e^{\omega L}) = -1, \]
\[ (1 + \frac{\alpha}{\omega}) F_3 + G_3 (e^{-\omega L} - 1) + H_3 (1 - e^{\omega L}) = e^{-\omega L}. \]

These equations correspond to three linear systems in \( F_i, G_i, H_i, \ i = 1, 2, 3, \) whose associated matrix has a determinant \( D(\omega) \) given by
\[ D(\omega) = (1 - \frac{\alpha}{\omega}) e^{\omega L} (1 - e^{-\omega L}) \left( e^{-\omega L} - \frac{3\omega + i\alpha}{\omega - i\alpha} \right). \]
Since this determinant is different from zero (as \( \Im z > 0, \Re z \leq 0 \)), we deduce the following expressions:

\[ F_1(\omega) = 1 + \frac{2\omega}{i\alpha - \omega} \left( e^{-\omega L} + 1 - \frac{3\omega + i\alpha}{\omega - i\alpha} \right), \]
\[ G_1(\omega) = \frac{2\omega}{(i\alpha - \omega) \left( e^{-\omega L} - \frac{3\omega + i\alpha}{\omega - i\alpha} \right)}, \]
\[ H_1(\omega) = \frac{2\omega e^{-\omega L}}{(i\alpha - \omega) \left( e^{-\omega L} - \frac{3\omega + i\alpha}{\omega - i\alpha} \right)}, \]
\[ F_2(\omega) = \frac{2\omega}{(i\alpha - \omega) \left( e^{-\omega L} - \frac{3\omega + i\alpha}{\omega - i\alpha} \right)}, \]
\[ G_2(\omega) = \frac{w + i\alpha}{\omega - i\alpha} \frac{1}{(e^{-\omega L} - 1) \left( e^{-\omega L} - \frac{3\omega + i\alpha}{\omega - i\alpha} \right)}, \]
\[ H_2(\omega) = \frac{2\omega + (i\alpha - \omega) e^{-\omega L}}{\omega D_\alpha(\omega)}, \]
\[ F_3(\omega) = -\frac{2e^{-\omega L} (e^{-\omega L} - 1)}{D_\alpha(\omega)}, \]
\[ G_3(\omega) = \frac{(\omega + i\alpha) e^{-\omega L}}{\omega D_\alpha(\omega)}, \]
\[ H_3(\omega) = \frac{e^{-\omega L}}{D_\alpha(\omega)} \left( 2e^{-\omega L} - \frac{3\omega + i\alpha}{\omega} \right). \]

Inserting these expressions in (2.6)-(2.7), we have obtained the next result.

2.1. Theorem. Let \( f \in H. \) Then, for \( x \in \mathcal{R} \) and \( z \in \mathbb{C} \) such that \( \Im z > 0, \Re z \leq 0, \) we have
\[ [R(z^2, H)f](x) = \int_{\mathcal{R}} K(x, x', z^2) f(x') \, dx', \]
where the kernel $K$ is defined as follows:

\begin{align*}
(2.18) K(x, y, z^2) &= \frac{1}{2iz} \left( e^{iz|x-y|} - F_1(-iz)e^{iz(x+y)} \right), \forall x, y \in R_1, \\
(2.19) K(x, y, z^2) &= -\frac{1}{2iz} \left( F_2(-iz)e^{iz(y+x)} + F_3(-iz)e^{-iz(y-x)} \right), \forall x \in R_1, y \in R_2, \\
(2.20) K(x, y, z^2) &= \frac{1}{2iz} \left( e^{iz|x-y|} + G_2(-iz)e^{iz(y+x)} + G_3(-iz)e^{-iz(y-x)} \right) \\
&\quad + H_2(-iz)e^{-iz(x-y)} + H_3(-iz)e^{-iz(x+y)}, \forall x, y \in R_2, \\
(2.21) K(x, y, z^2) &= \frac{1}{2iz} \left( G_1(-iz)e^{iz(y+x)} + H_1(-iz)e^{iz(y-x)} \right), \forall x \in R_2, y \in R_1.
\end{align*}

2.2. The spectrum. As usual, to obtain the resolution of the identity of $H$, we want to use the limiting absorption principle that consists to pass to the limit in $K(x, y, z^2)$ as $\Re z$ goes to zero. But in view of the presence of the factor $e^{izL} - 1$ in the denominator of $G_2, G_3, H_2, H_3$, this limit is a priori not allowed. This factor comes from the circle $R_2$ and suggests that the real point spectrum is distributed in the whole continuous spectrum. This is indeed the case has the next results will show.

2.2. Theorem. The spectrum of $H$ is given by the disjoint union of $[0, +\infty]$ and $\Sigma_p^-(H)$ where

\begin{equation}
\Sigma_p^-(H) = \left\{ \lambda^2; \lambda \in \mathbb{C}_+^* \text{ and } e^{i\lambda L} = 3 - \frac{4\alpha}{\lambda + \alpha} \right\},
\end{equation}

where $\mathbb{C}_+^* := \{ z \in \mathbb{C}; \Re z > 0, \Re z > 0 \}$.

Indeed, for all $k \in \mathbb{N}^*$, the number $\lambda_{2k}^2 = \frac{4k^2\pi^2}{L^2}$ is an eigenvalue of $H$ and an associated eigenvector $\varphi^{(2k)} \in \mathcal{D}(H)$ given by

\begin{align}
\varphi_1^{(2k)} &= 0 \text{ in } R_1, \\
\varphi_2^{(2k)}(x) &= \frac{\sqrt{3}}{L} \sin(\lambda_{2k}x), \forall x \in R_2.
\end{align}

Furthermore $\Sigma_p^-(H)$ is a discrete set of eigenvalues of $H$.

Proof. We start by proving that the semi-axis $[0, +\infty]$ is a part of the spectrum of $H$ and that a number $\lambda^2 \in [0, +\infty]$ is an eigenvalue if and only if there exists $k \in \mathbb{N}^*$ such that $\lambda^2 = \frac{4k^2\pi^2}{L^2}$.

Let $\lambda^2 \geq 0$ and $\chi : \mathbb{R} \to [0, 1]$ a smooth function such that $\chi(x) = 1$ for $|x| \leq 1/2$ and $\chi(x) = 0$ for $|x| \geq 1$. For $n \in \mathbb{N}^*$ we define the ansatz $\theta_{n,\lambda} \psi_\lambda(x) = \frac{e^{i|x|}x}{\sqrt{n}} \chi \left( \frac{x}{n} - 1 \right)$, for $x \geq 0$. We indeed define the sequence $\varphi^{(n)} \in \mathcal{D}(H), \forall n \in \mathbb{N}^*, \|\varphi^{(n)}\|_H \neq 0$ and $\|(H + \lambda^2)\varphi^{(n)}\|_H \to 0$ when $n \to +\infty$. Then the Weyl criterion proves that $\lambda^2$ is in the spectrum of $H$. It is not hard to check that $\varphi^{(n)} \in \mathcal{D}(H)$ for all $n \geq 1$ and that

\begin{equation}
\|\varphi^{(n)}\|_H = \|\theta_n\|^2_{L^2} \geq \frac{1}{n} \oint_{n/2} \frac{dx}{x^2} = 1.
\end{equation}

Moreover, we remark that $\theta_{n,\lambda}(x) = e^{i|x|}x \psi_\lambda(x)$ for $n \geq 1$ and $x \geq 0$ and

\begin{equation}
\|\theta_n''\|^2 \leq \frac{2}{n^2} \|\lambda''\|^2, \quad \|\theta_n''\|^2 \leq \frac{2}{n^2} \|\lambda''\|^2.
\end{equation}

Hence we obtain

\begin{align}
(2.25) \quad \|H + \lambda^2\varphi^{(n)}\|_H &= \|\theta_n''\|^2 + \lambda^2 \theta_n'' \|L^2 \leq 2\lambda \|\theta_n''\|^2 + \|\theta_n''\|^2 \|L^2 \\
(2.26) &= 2\lambda \left( \sqrt{2} \lambda^2 \theta_n'' \|\chi''\| + \|\lambda''\| \right).
\end{align}
To prove that the numbers $\lambda^2_{2k}, k \in \mathbb{N}^*$ are eigenvalues is direct since we readily check that $\varphi^{(2k)}$ defined by (2.23) and (2.24) is indeed in $\mathcal{D}(H)$ and satisfies $H \varphi^{(2k)} = \lambda^2_{2k} \varphi^{(2k)}$.

For the second assertion, we simply remark that if $\varphi$ is an eigenvector of $H$ of eigenvalue $\lambda^2$, then for $\lambda > 0$, we have

$$\varphi_1(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x), \forall x \in R_1,$$

with $c_i \in \mathbb{C}$. But the requirement that $\varphi_1$ belongs to $L^2(R_1)$ directly implies that $c_1 = c_2 = 0$. Hence $\varphi$ has to be in the form of the first assertion. In the case $\lambda = 0$, $\varphi_1$ has to be zero and therefore

$$\varphi_2(x) = c_1 + c_2 x, \forall x \in R_2,$$

with $c_i \in \mathbb{C}$. By the continuity property at 0, we get $c_1 = c_1 + c_2 L = 0$, hence $c_1 = c_2 = 0$. Now we look for other eigenvalues. Let $\lambda \in \mathbb{C}^+$ and let us find a solution of $Hu = \lambda^2 u$ in the domain $\mathcal{D}(H)$. The parameter $\lambda$ can be considered as a complex branch of the square root of $\lambda^2$. Thus we will often choose the branch $\lambda$ such that $\Re(i\lambda^2) < 0$. This is because of the dissipativity of $iH$.

A fundamental system of solution of the differential equation $-\Delta f = \lambda^2 f$ is $\{e^{i\lambda x}, e^{-i\lambda x}\}$. Hence if we let $u_k(x) = A_k e^{i\lambda x} + B_k e^{-i\lambda x}, A_k, B_k \in \mathbb{C}$ for $k = 1, 2$ then $u = (u_1, u_2) \in \mathcal{D}(H)$ yields

- the continuity condition
  $$A_1 + B_1 = A_2 + B_2 = A_2 e^{i\lambda L} + B_2 e^{-i\lambda L}$$
- and the Kirchhoff transmission condition
  $$(1 - \frac{\alpha}{\lambda}) A_1 - (1 + \frac{\alpha}{\lambda}) B_1 + (1 - e^{i\lambda L}) A_2 - (1 - e^{-i\lambda L}) B_2 = 0.$$

Since the eigenvector $(u_1, u_2)$ has to be in the domain of $H$ and $\Re \lambda \geq 0$, we can see that $B_1 = 0$. Therefore $A_1 = A_2 + B_2$ and $A_2, B_2$ are the solutions of the linear system

$$\begin{cases}
(1 - e^{i\lambda L}) A_2 + (1 - e^{-i\lambda L}) B_2 = 0 \\
(2 - \frac{\alpha}{\lambda} - e^{i\lambda L}) A_2 + (e^{-i\lambda L} - \frac{\alpha}{\lambda}) B_2 = 0.
\end{cases}$$

The determinant of the system is

$$d(\lambda) := e^{-i\lambda L} \left( \left( \frac{\alpha}{\lambda} + 1 \right) e^{2i\lambda L} - 4 e^{i\lambda L} + 3 - \frac{\alpha}{\lambda} \right).$$

The reals $\lambda_{2k}, k \in \mathbb{N}^*$ are roots of $d(\lambda) = 0$ and they correspond to the eigenvalues given in the first assertion.

Now if we set $T = e^{i\lambda L}$ then $d(\lambda) = 0$ if and only if $(\frac{\alpha}{\lambda} + 1) T^2 - 4 T + 3 - \frac{\alpha}{\lambda} = 0$. The two roots of the previous polynomial are $T_1 = 1$ and $T_2 = 3 - \frac{4\alpha}{\lambda + \alpha}$. Thus $\lambda$ is a zero of the function $h_\alpha(\lambda) = e^{i\lambda L} + \frac{4\alpha}{\lambda + \alpha} - 3$ in the domain $\mathbb{C}^+$. Since this is an open subset of $\mathbb{C}$ and $h_\alpha$ is analytic on it, then $h_\alpha$ has a discrete set of zeros with no accumulation point. \(\square\)

2.3. Remark. (1) We shall see below that the eigenvalues $(\lambda^2_{2k})_{k \in \mathbb{N}^*}$ are embedded in the continuous spectrum with corresponding eigenfunctions $\varphi^{(2k)}$ which are confined in the circle.

(2) For the non real eigenvalues $\lambda^2_{\alpha,k}$ we can see that there exists $c > 0$ such that $\Re \lambda^2_{\alpha,k} \geq c, \forall k \in \mathbb{N}$. For such eigenvalues, we already know that $\lambda^2_{\alpha,k}$ satisfies the relation

$$e^{i\lambda_{\alpha,k} L} = 3 - \frac{4\alpha}{\lambda_{\alpha,k} + \alpha}.$$

Setting $\lambda_{\alpha,k} = \sigma_k + i\mu_k$ then $\Re \lambda^2_{\alpha,k} = 2\sigma_k \mu_k$. If we assume that there exists a subsequence of $\lambda_k$ such that $\Re \lambda^2_{\alpha,k} \rightarrow 0$ when $k \rightarrow \infty$ then by taking the modulus in the relation (2.29) and passing to the limit when $k \rightarrow \infty$ we get $1 = 3$. This is a contradiction.
(3) Using the two remarks above, the spectrum of \( H \) is the union of two separated sets \( \sigma_1 \) and \( \sigma_2 \) where \( \Sigma^-_p = \{ \lambda_{a,k}^2, k \in \mathbb{N} \} \) and \( \sigma_2 = [0, +\infty) = \sigma_c(H) \cup \{ \lambda_{2k}^2; k \in \mathbb{N}^* \} \) and \( \sigma_c \) is the continuous spectrum of \( H \) and contains the numbers \( \lambda_{2k}^2, k \in \mathbb{N}^* \) as a sequence of embedded eigenvalues.

2.3. Asymptotic of the point spectrum. Let \( \alpha > 0 \) and recall that we have the following disjoint decomposition of the point spectrum

\[
\Sigma^-_p(H) = \Sigma^+_p(H) \cup \Sigma^-_p(H)
\]

with \( \Sigma^-_p(H) \) is given by (2.22) and

\[
\Sigma^+_p(H) = \left\{ \frac{4\pi^2 k^2}{L^2}; k \in \mathbb{N}^* \right\}.
\]

We want to find a precise localization of \( \Sigma^-_p(H) \). We consider the meromorphic function \( h_\alpha \) defined by

\[
h_\alpha(\lambda) = e^{i\lambda L} + \frac{4\alpha}{\lambda + \alpha} - 3.
\]

We already know that \( \lambda^2 \) is an eigenvalue of \( H \) if \( h_\alpha(\lambda) = 0 \) and \( \lambda \in \mathbb{C}_+ \). The function \( h_\alpha \) is analytic on \( \mathbb{C}_+ \) so it has a discrete set of roots. Let’s denote by \( (\lambda_n(\alpha))_n \) the sequence of these roots. When \( \alpha = 0 \), it is a straightforward calculus to find that the numbers \( \lambda_n(0) = \frac{2n\pi}{L} + \frac{j\ln(3)}{L}, n \in \mathbb{N} \) are roots of \( h_0 \). Hence we have

\[
\lambda_n^2(0) = \frac{4n^2\pi^2 - \ln^2(3)}{L^2} + \frac{4in\pi \ln(3)}{L^2}.
\]

Now, we consider the operator family \( \{ H; \alpha \in \mathbb{R} \} := (H_\alpha)_{\alpha \in \mathbb{R}} \). This is an analytic family of type \( B \) in the sense of Kato \([5]\), chap. VII.4. By perturbation arguments, there exists, for small \( \alpha \), an analytic function \( \alpha \rightarrow \lambda_n^2(\alpha) \) such that \( \lambda_n^2(\alpha) \) is in the spectrum of \( H_\alpha \). Let us write

\[
\lambda_n(\alpha) = \lambda_n(0) + a\alpha + b\alpha^2 + O(\alpha^3)
\]

where \( a \) and \( b \) are complex numbers we are going to compute.

On one hand, we have, for small \( \alpha \),

\[
e^{iL\lambda_n(\alpha)} = e^{i\lambda_n(0)L} e^{iaL\alpha + ib\alpha^2 + O(\alpha^3)}
= 3 \left(1 + i\alpha L + \frac{a^2 L^2 \alpha^2}{2} + O(\alpha^3)\right)
\]

(2.30)

\[
= 3 + 3iaL\alpha + 3 \left(\frac{a^2 L^2}{2}\right) \alpha^2 + O(\alpha^3).
\]

On the other hand, we can develop the following

\[
3 - \frac{4\alpha}{\lambda_n(\alpha) + \alpha} = 3 - \frac{4\alpha}{\lambda_n(0) + (a + 1)\alpha + b\alpha^2 + O(\alpha^3)}
\]

(2.31)

\[
= 3 - \frac{4\alpha}{\lambda_n(0)} \frac{4(a + 1)}{\lambda_n^2(0)} \alpha^2 + O(\alpha^3).
\]

Using (2.29) and identifying (2.30) and (2.31), yield

\[
a = \frac{4i}{3L\lambda_n(0)}
\]

and

\[
b = \frac{16}{9L^2 \lambda_n^2(0)} - \frac{4i}{9L\lambda_n^2(0)}.
\]

Now, we want to prove that the asymptotic expansion valid for small \( \alpha \) remains true for large \( n \) and any fixed \( \alpha \). The \( \lambda_n(\alpha) \) satisfies the relation \( (\alpha + \lambda_n)e^{i\lambda_n L} = 3\lambda_n - \alpha \). Deriving this equation with respect to \( \alpha \) yields

\[
\lambda_n'(\alpha) = \frac{4i\lambda_n(\alpha)}{3L\lambda_n^2(\alpha) + 2\alpha L\lambda_n(\alpha) - \alpha^2 L + 4i\alpha}.
\]
2.5. Theorem.\[\textit{Theorem.}\] The spectrum and
in Theorem 2.1.

The function while
The function while $z$

\[\{\text{set}\cup\text{namely for any}\quad f\]

Thus
\[
\lambda_n(\alpha) = \frac{2n\pi}{L} + i \frac{\ln(3)}{L} + O\left(\frac{1}{n}\right), \quad n \to +\infty.
\]

Then we obtain the asymptotic expansion of $\lambda_n(\alpha)$ for large $n$ in the same way we did for small $\alpha$. In conclusion, we have the following result.

2.4. Proposition. Let $\alpha > 0$ fixed. Then when $n$ is large enough, the eigenvalue $\lambda_n^2(\alpha)$ has the following expansion:

\[\lambda_n^2(\alpha) = \frac{4\pi^2}{L^2} n^2 - \frac{\ln^2(3)}{L^2} + i \left(\frac{8\alpha}{3L} + \frac{4\pi \ln(3)}{L^2} n\right) + O\left(\frac{1}{n}\right).
\]

At this stage we define the projection $P_{pp}$ on the closed subspace spanned by the $\varphi^{(2k)}$’s, namely for any $f \in \mathcal{H}$, we set

\[P_{pp}^+ f = \sum_{k=0}^{+\infty} (f, \varphi^{(2k)})_\mathcal{H} \varphi^{(2k)}.
\]

Note that $P_{pp}^+ f$ is different from $f$ on $R_2$ because $L^2(R_2)$ is spanned by the set of eigenvectors of the Laplace operator with Dirichlet boundary conditions at 0 and $L$, that are the set $\{\varphi^{(t)}\}_{t \in \mathbb{N}^*}$, where

\[\varphi^{(t)}(x) = \frac{\sqrt{2}}{\sqrt{L}} \sin(\lambda t x), \forall x \in R_2,
\]

and $\lambda_t = \frac{\pi t}{L}$. Hence

\[f - P_{pp}^+ f = \sum_{k=0}^{+\infty} \left( \int_0^L f(x) \varphi^{(2k+1)}(x) dx \right) \varphi^{(2k+1)}.
\]

We show, according to [1, Corollary 2.6], that our operator has no singular continuous spectrum and $P_{pp} f = f - (P_{pp}^+ + P_{pp}^-) f, \forall f \in \mathcal{H}$.

2.5. Theorem. For all $z \in \{z \in \mathbb{C} : \text{Re} z > 0\}$, and all $x, y, z \in \mathcal{R}$, the kernel $K(x, y, z^2)$ defined in Theorem 2.1 admits the decomposition

\[K(x, y, z^2) = K_c(x, y, z^2) + K_{pp}^+(x, y, z^2) + K_{pp}^-(x, y, z^2),
\]

where for $x, y \in R_2$ and $X = e^{i\pi L}$ we have

\[K_c(x, y, z^2) = -\frac{\sin z y \sin z x}{2iz} + \frac{1}{2iz} \left( \frac{i \alpha + \omega}{i \alpha - \omega} e^{-iz y e^{iz x}} + \frac{2\omega}{i \alpha - \omega} \left( X - 2\left(\frac{i \alpha + \omega}{i \alpha - \omega}\right) e^{-iz y e^{iz x}} \right) \right),
\]

\[K_{pp}^-(x, y, z^2) = -\frac{i X \sin(z y) e^{-iz x} - \sin(z y) \sin z x - 2(\frac{i \alpha + \omega}{i \alpha - \omega}) e^{-iz y e^{iz x}} - \frac{e^{-iz y e^{iz x}}}{i \alpha - \omega} e^{-iz y e^{iz x}}}{iz (X - \omega_e)}
\]

and

\[K_{pp}^+(x, y, z^2) = -\frac{\cos \left(\frac{\pi x}{2}\right) \sin(z y) \sin z x}{2z \sin \left(\frac{\pi x}{2}\right)}.
\]

The function $z \mapsto K_c(x, y, z^2)$ is continuous on $\mathbb{R} z \geq 0, \mathbb{R} z \leq 0$ except at $z = 0$.

The function while $z \mapsto K_{pp}^+(x, y, z^2)$ is meromorphic in $\mathbb{C}$ with poles at the points $\lambda_{2k}$, $k \in \mathbb{N}^*$ and at $z = 0$.

The function while $z \mapsto K_{pp}^-(x, y, z^2)$ is meromorphic in $\mathbb{C}$ with poles at the points $\lambda_{\alpha, k}$, $k \in \mathbb{N}$.

For $x \notin R_2$ or $y \notin R_2$ we have $K_p(x, y, z^2) = 0$ and $K_c(x, y, z^2) = K(x, y, z^2)$ as defined in Theorem 2.1.
Proof. Let us recall that

\[ D_\alpha(w) = \left( \frac{\omega - i\alpha}{\omega} \right) e^{\omega L} (1 - e^{-\omega L})(e^{-\omega L} - w_c) \]

with \( \omega = -iz \) and \( \omega_c = \frac{3\omega + 4i\alpha}{\omega + 4i\alpha} \). The problem in the decomposition of resolvent kernel \( K(x, y, z^2) \) only appears for \( x \) and \( y \) in \( R_2 \), since in the other cases, \( K \) has no poles and therefore in that cases we simply take \( K^+_{pp} = K^-_{pp} = 0 \). Hence we need to perform this splitting for \( x, y \in R_2 \) using the formula (2.20). On one hand we obtain

\[ G_2(\omega)e^{izy} + G_3(\omega)e^{-izy} = \left( \frac{i\alpha + \omega}{i\alpha - \omega} \right) \left( \frac{X + 1}{X - \omega_c} e^{izy} - \frac{2i}{(X - 1)(X - \omega_c)} \sin zy \right). \]

On the other hand we get

\[ H_2(\omega)e^{izy} + H_3(\omega)e^{-izy} = \frac{1}{X - \omega_c} \left( Xe^{izy} + \frac{i\alpha + \omega}{i\alpha - \omega} e^{izy} \right) + \frac{2\omega}{i\alpha - \omega} \left( \frac{X^2 e^{-izy} - \omega + i\alpha}{X - 1} e^{-izy} \right) \]

\[ = \left( \frac{i\alpha + \omega}{i\alpha - \omega} \right) \frac{2i\sin zy}{(X - 1)(X - \omega_c)} + \frac{2iX \sin zy}{X - \omega_c} + \frac{2\omega}{i\alpha - \omega} \left( \frac{X^2 - X - \frac{i\alpha + \omega}{2\omega}}{X - \omega_c} \right) e^{-izy}. \]

Now an elementary calculus gives

\[ \frac{X^2 - X - \frac{i\alpha + \omega}{2\omega}}{X - \omega_c} = \frac{X}{X - \omega_c} + \frac{2(\omega + i\alpha)}{\omega - i\alpha} - A(\alpha) \frac{1}{X - \omega_c}, \]

\[ \left( \frac{i\alpha + \omega}{i\alpha - \omega} \right) \frac{2i}{(X - 1)(X - \omega_c)} = i \left( \frac{1}{X - 1} - \frac{1}{X - \omega_c} \right), \]

\[ \left( \frac{i\alpha + \omega}{i\alpha - \omega} \right) \frac{X + 1}{X - \omega_c} = \frac{i\alpha + \omega}{i\alpha - \omega} \frac{4\omega(i\alpha + \omega)}{(\omega - i\alpha)^2} \frac{1}{X - \omega_c}. \]

and

\[ \frac{2iX}{X - \omega_c} = 2i + \frac{2i\omega_c}{X - \omega_c}, \]

where we have set

\[ A(\alpha) = \frac{(\omega + i\alpha)(11\omega^2 + \alpha^2 + 6i\alpha)}{2\omega(\omega - i\alpha)^2}. \]

Thus,

\[ Q := (G_2(\omega)e^{izy} + G_3(\omega)e^{-izy})e^{izx} + (H_2(\omega)e^{izy} + H_3(\omega)e^{-izy})e^{-izx} = \]

\[ \frac{i\alpha + \omega}{i\alpha - \omega} 4 \sin zy \sin zx \left( \frac{X}{X - 1}(X - \omega_c) + \frac{2i\omega_c}{X - \omega_c} \right) e^{-izx} + \]

\[ \frac{2i}{i\alpha - \omega} \sin zy e^{-izx} + \frac{2\omega}{i\alpha - \omega} \left( \frac{X - \frac{A(\alpha)}{X - \omega_c} - \frac{2(i\alpha + \omega)}{i\alpha - \omega}}{X - \omega_c} \right) e^{-izx}. \]

We decompose the last expression into two terms

\[ K_0(x, y, z) := \frac{i\alpha + \omega}{i\alpha - \omega} \frac{4 \sin zy \sin zx}{(X - 1)(X - \omega_c)} \]

and

\[ K_1(x, y, z) := \left( \frac{i\alpha + \omega}{i\alpha - \omega} \right) \left( 1 - \frac{4\omega}{X - \omega_c} X - \frac{A(\alpha)}{X - \omega_c} - \frac{2(i\alpha + \omega)}{i\alpha - \omega} \right) e^{-izx} \]

\[ + \frac{2iX}{X - \omega_c} \sin zy e^{-izx} + \frac{2\omega}{i\alpha - \omega} \left( X - \frac{A(\alpha)}{X - \omega_c} - \frac{2(i\alpha + \omega)}{i\alpha - \omega} \right) e^{-izx}. \]

We go further in the calculus and we find the following splitting of \( Q \)

\[ K_0(x, y, z) = -i \cos \left( \frac{zL}{2} \right) \sin zy \sin zx - \left( 1 + \frac{2}{X - \omega_c} \right) \sin zy \sin zx. \]
Here we used the fact that \( X = e^{izL} \) and thus
\[
\frac{1}{X - 1} = -\frac{1}{2} + \frac{\cos \left( \frac{zL}{2} \right)}{2i \sin \left( \frac{zL}{2} \right)}.
\]

Going back to the formula for \( Q \) and assembling together the terms with \( \frac{1}{X - \omega_c} \), then \( Q \) is the sum of three terms
\[
Q = Q_c + Q_{pp}^+ + Q_{pp}^-
\]
with
\[
Q_c = -\sin z y \sin zx + \frac{i\alpha + \omega}{i\alpha - \omega} e^{-izy} e^{izx} + \frac{2\omega}{i\alpha - \omega} \left( X - \frac{2(i\alpha + \omega)}{i\alpha - \omega} \right) e^{-izy} e^{-izx},
\]
\[
Q_{pp}^- = \frac{2iX \sin(zy)e^{-izx} - 2\sin(zy) \sin zx - \frac{4(i\alpha + \omega)}{i\alpha - \omega} e^{-izy} e^{izx} - \frac{2\omega A(\alpha)}{i\alpha - \omega} e^{-izy} e^{izx}}{X - \omega_c},
\]
and
\[
Q_{pp}^+ = -i \frac{\cos \left( \frac{zL}{2} \right)}{\sin \left( \frac{zL}{2} \right)} \sin(zy) \sin zx.
\]

\[\square\]

3. Riesz basis

In this section, it is proved that the generalized eigenfunctions of the dissipative operator \( iH \) associated to the eigenvalues in \( i\Sigma_p^-(\lambda) \) of the subspace of \( H^- \) span (denoted by \( \mathcal{H}^+ \)). To this end, we recall that a sequence \( (\Lambda_n)_{n \in \mathbb{N}} \) is a Riesz basis in a Hilbert space \( V \) if there exist a Hilbert space \( \tilde{V} \), and an orthonormal basis \( (\psi_n)_{n \in \mathbb{N}} \) of \( \tilde{V} \) and an isomorphism \( \Theta : \tilde{V} \to V \) such that \( \Theta \psi_n = \psi_n, \forall n \in \mathbb{N} \).

We start by computing the eigenfunctions of \( iH \) corresponding to the eigenvalues \( \lambda_n^2(\alpha) \). So let \( n \) and \( \alpha \) be fixed. The eigenfunction associated to \( \lambda_n^2(\alpha) \) is of the form \( \psi_n = (\psi_{1,n}^-, \psi_{2,n}^-) \) with
\[
\psi_{1,n}^-(x) = A_{1,n} e^{i\lambda_n(\alpha)x} , \quad \text{for } x \in [0, +\infty[
\]
and
\[
\psi_{2,n}^- = A_{2,n} e^{i\lambda_n(\alpha)x} + B_{2,n} e^{-i\lambda_n(\alpha)x} , \quad \text{for } x \in [0, L]
\]
where \( A_{1,n}, A_{2,n}, B_{2,n} \) are complex constants. Since \( \psi_n^- \) has to be in the domain of \( H \), we obtain the following
\[
(3.35) \quad \psi_{1,n}^-(x) = C_n \frac{4\lambda_n(\alpha)}{\lambda_n(\alpha) + \alpha} e^{i\lambda_n(\alpha)x}
\]
and
\[
(3.36) \quad \psi_{2,n}^-(x) = C_n \left( e^{i\lambda_n(\alpha)x} + \frac{3\lambda_n(\alpha)}{\lambda_n(\alpha) + \alpha} e^{-i\lambda_n(\alpha)x} \right)
\]
where \( C_n \) is a constant such that \( \| \psi_n^- \|_{L^2} = 1 \).

3.1. Theorem. \([Riesz basis for the operator \( iH \)]\) The generalized eigenfunctions of \( iH \) form a Riesz basis of \( \mathcal{H}^- \).

We recall that \( \lambda_n(\alpha) \) and \( \lambda_n^2(\alpha) \) have the asymptotic expansions given in (2.32) and (2.33). Thus one can prove the following behavior of the family \( (\psi_{2,n}^-)_{n \in \mathbb{N}} \).

3.2. Lemma. There exists a constant \( C > 0 \) such that for all \( n, m \in \mathbb{N} \) with \( n < m \) we have
\[
(3.37) \quad < \psi_{2,n}^-, \psi_{2,m}^- >_{L^2(0,L)} \leq \frac{C}{< n > (m - n)} , \quad \text{where } < n > = (1 + n^2)^{1/2}.
\]
We conclude using the asymptotic \( \| \phi_n \|_{L^2(0,L)}^2 = \| \beta_n \|_{L^2(0,L)}^2 = \frac{4L}{\ln(3)} + O\left(\frac{1}{n}\right) \), and
\[
< \phi_n, \beta_n >_{L^2(0,L)} = O\left(\frac{1}{n}\right).
\]
Indeed, for \( n \neq m \), we obtain
\[
< \phi_n, \phi_m >_{L^2(0,L)} = \frac{e^{i(\lambda_n - \lambda_m)L} - 1}{\lambda_n - \lambda_m}
\]
and
\[
< \beta_n, \beta_m >_{L^2(0,L)} = \frac{e^{-i(\lambda_n - \lambda_m)L} - 1}{\lambda_m - \lambda_m}.
\]
We know that \( \lambda_n - \lambda_m = (n-m)\nu + O\left(\frac{1}{n}\right) \) for \( n \) large, with \( \nu = \frac{2\pi}{L} \). Hence
\[
|< \phi_n, \phi_m >| + |< \beta_n, \beta_m >| \leq \frac{1}{m-n},
\]
and
\[
|< \phi_n, \phi_m >| + |< \beta_n, \beta_m >| \leq \frac{1}{n > (m-n)}.
\]
By the same arguments we get
\[
|< \phi_n, \beta_m >| + |< \beta_n, \phi_m >| \leq \frac{1}{m+n},
\]
and
\[
|< \phi_n, \beta_m >| + |< \phi_n, \beta_m >| \leq \frac{1}{n > (m+n)}.
\]
Going back to \( \psi_n^- \), we write \( \psi_n^- = C_n \left( \phi_n + \frac{3\lambda_n(\alpha)}{\lambda_n(\alpha) + \alpha} \beta_n \right) \) and we obtain
\[
|< \psi_{2,n}^-, \psi_{2,m}^- >| = \left| C_n \bar{C}_n \left( < \phi_n, \phi_m > + < \phi_n, \frac{3\lambda_n(\alpha)}{\lambda_n(\alpha) + \alpha} \beta_m > + \frac{3\lambda_n(\alpha)}{\lambda_n(\alpha) + \alpha} < \phi_n, \beta_m > + \frac{3\lambda_m(\alpha)}{\lambda_m(\alpha) + \alpha} \frac{3\lambda_m(\alpha)}{\lambda_m(\alpha) + \alpha} < \beta_n, \beta_m > \right) \right|.
\]
We conclude using the asymptotic \( |\lambda_n(\alpha)| = 2\pi + O\left(\frac{1}{n}\right) \) for \( n \) large enough. \( \square \)

**Proof of Theorem 3.1.** Consider now the map \( \Theta : \ell^2(\mathbb{C}) \rightarrow L^2(0, +\infty) \times L^2(0, L) \) given by
\[
\Theta((a_n)_{n \in \mathbb{N}}) = \sum_{n=0}^{+\infty} a_n \psi_n^-.
\]
We claim that \( \Theta \) is an isomorphism from \( \ell^2(\mathbb{C}) \) to \( V_p := sp(\Psi_n^-) \) where \( \Psi_n^- = (\psi_{1,n}^-, \psi_{2,n}^-) \).
Using the relations (3.35) and (3.36), it is enough to consider the functions \( \phi_n \) and \( \beta_n \). Let \( a = (a_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{C}) \). For \( N, p \) fixed in \( \mathbb{N} \) we have
\[
\left(3.38\right) \quad \left| \sum_{n=N}^{N+p} a_n \phi_n \right|^2 - \sum_{n=N}^{N+p} |a_n|^2 = 2 \sum_{j=N+1}^{N+p} \sum_{k=j+1}^{N+p} 2 \Re(a_j \bar{a}_k < \phi_j, \phi_k >) \leq 2 \tilde{C} \sum_{j=0}^{+\infty} |a_j |^2 \sum_{k=1}^{+\infty} \left| \frac{a_{k+j}}{k} \right| \leq \tilde{C} \|a\|_{\ell^2(\mathbb{C})}^2.
\]
\[
\left(3.39\right)
\]
where \( C \) is the constant given in (3.37) and \( \tilde{C} > 0 \). This proves that the series \( \sum_{n=0}^{+\infty} a_n \phi_n \) converges in \( L^2(0, L) \) as \( a = (a_n)_n \in \ell^2(\mathbb{C}) \). Moreover, letting \( p \to +\infty \) and taking \( N = 0 \) give

\[
\left\| \sum_{n=0}^{+\infty} a_n \phi_n \right\|^2 \leq \tilde{C}||a||^2_{\ell^2(\mathbb{C})}.
\]

Thus \( \Theta \) is continuous. It remains to prove that \( \Theta \) is injective. For this, let \( a \) give \( \lim_{n \to +\infty} a_n \phi_n \) in \( \ell^2(\mathbb{C}) \). Let \( a_n \phi_n \) be the normalized (in \( \ell^2(\mathbb{C}) \)) \( \phi_n \). Moreover, assume that \( \sum_{n=0}^{+\infty} a_n \phi_n = 0 \) in \( L^2(0, L) \). Let \( n \neq m \in \mathbb{N} \). First, we have

\[
\lambda^2_n(\alpha) \phi_n, \phi_m > = \phi_n, H^* \phi_m > = \phi_n, \lambda^2_m(\alpha) \phi_m > = \lambda^2_m(\alpha) \phi_n, \phi_m >.
\]

This proves that \( < \phi_n, \phi_m > = 0 \) for \( n \neq m \). Next, we can see from (3.35, 3.36) that \( \lambda\phi_n, \phi_n > \neq 0 \) for \( n > N \) with \( N \) large enough. Hence, taking the inner product of \( \sum_{n=0}^{+\infty} a_n \phi_n = 0 \) by \( \tilde{\phi}_p \) with \( p \) fixed yields \( 0 = \sum_{n=0}^{N} a_n \phi_n = 0 \) in \( L^2(0, L) \). Applying the operator \( H \) iteratively \( N \) times gives the following sequence of relations:

\[
0 = \sum_{n=0}^{N} \lambda^2_n(\alpha) a_n \phi_n = 0, \quad \text{in } L^2(0, L), \text{ for } k = 0, \ldots, N.
\]

This is a Vandermonde problem. Since the \( \lambda^2_n(\alpha) \) are pairwise different, we deduce that \( a_n \phi_n = 0 \) and hence \( a_n = 0 \) for all \( n \).

\[\square\]

4. Energy decreasing

Using the Riesz basis constructed in the latest section, the energy is proved to decrease exponentially to a non-vanishing value depending on the initial datum. The decay rate is explicitly given at the end of Theorem 4.1 below since the \( \lambda \)'s satisfying \( i\lambda^2 \in i \Sigma^-_p(H) \). The decay rate is computable numerically.

4.1. Energy decreasing using the Riesz basis.

4.1. Theorem. [Energy decreasing of the solution] Let \( E(t) := \frac{1}{2} \|u(t)\|^2_H \) be the energy, \( H^-_p \) (respectively \( H^+_p \)) be the subspace of \( H \) spanned by the \( \psi^-(\lambda, \cdot) \)'s (resp. \( \psi^+(\omega, \cdot) \)'s), which are the normalized (in \( H \)) eigenfunctions of \( iH \) associated to the eigenvalues \( i\lambda^2 \) in \( i\Sigma^-_p(H) \) (resp. \( i\Sigma^+_p(H) \)).

\( \lambda \)

1. \( H^-_p \) is orthogonal to \( H^+_p \).
2. Let \( u_0 \) in \( H^-_p \oplus H^+_p \) be the initial condition of the boundary value problem given in the introduction and \( u_0^+ \) its orthogonal projection onto \( H^+_p \).
Then \( E(t) \) decreases exponentially to \( E(0) := \frac{1}{2} \|u_0^+\|^2_H \) when \( t \) tends to \( +\infty \). More precisely

\[
E(t) = E^+(t) + E^-(t) = E^+(0) + E^-(t) \leq E^+(0) + e^{-2\omega t}E^-(0)
\]

where

\[
\omega := \sup_{\{i\lambda^2 \in i\Sigma^-_p(H)\}} \Re(i\lambda^2) < 0.
\]
Proof. First part: Above all, it is easy to see that the operator \((iH)^*\) is obtained by changing \(i\) by \(-i\) in \(iH\). Thus, if \(iH\psi^+(\lambda,.) = i\lambda^2\psi^+(\lambda,.)\), then \((iH)^*\psi^+(\lambda,.) = -i\lambda^2\psi^+(\lambda,.)\).

Now, to prove that \(\mathcal{H}_p^+\) is orthogonal to \(\mathcal{H}_p^-\), it suffices to check that any generalized eigenfunction \(\psi^-(\lambda',.)\) of \(\mathcal{H}_p^-\) is orthogonal to any eigenfunction \(\psi^+(\lambda,.)\) of \(\mathcal{H}_p^+\).

First we assume that \(\psi_p^-(\lambda',.)\) is an eigenfunction, i.e

\[
iH\psi^-(\lambda',.) = i\lambda^2\psi^-(\lambda',.).
\]

Therefore, since \(i\lambda^2\) is purely imaginary,

\[
i\lambda^2 < \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = < iH\psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = < \psi^-(\lambda',.),(iH)^*\psi^+(\lambda,.) >_\mathcal{H} = -< \psi^- (\lambda',.),iH\psi^+(\lambda,.) >_\mathcal{H} = -i\lambda^2 < \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = i\lambda^2 < \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H}.
\]

Consequently \(< \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = 0\).

Secondly, we assume that \(\lambda'\) is not simple. Let \(\psi^2(\omega',.)\) be an associated generalized eigenfunction of order \(p \geq 2\), in the sense that

\[
(iH - i\lambda^2)p\psi^-(\lambda',.) = 0,\quad (iH - i\lambda^2)p^{-1}\psi^-(\lambda',.) \neq 0.
\]

Setting \(\psi = (iH - i\lambda^2)p\psi^-(\lambda',.)\), then \(\psi\) is a generalized eigenfunction associated to \(i\lambda^2\) of order \(p - 1\), so arguing by iteration with respect to the order \(p\) we can assume that \(< \psi,\psi^+(\lambda,.) >_\mathcal{H} = 0\).

Therefore

\[
i\lambda^2 < \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = < iH\psi^-(\lambda',.) + \psi,\psi^+(\lambda,.) >_\mathcal{H} = < iH\psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = i\lambda^2 < \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H},
\]

as previously. Consequently \(< \psi^-(\lambda',.),\psi^+(\lambda,.) >_\mathcal{H} = 0\).

Second part: the purely point spectrum of the dissipative operator \(iH\) is the union of \(i\Sigma_+^+(H)\) set of the purely imaginary eigenvalues and \(i\Sigma_-^-(H)\) set of the other eigenvalues.

The initial condition \(u_0\) is written as a sum of two terms:

\[
u_0 := \sum_{i\lambda^2 \in \Sigma_+^+(H)} u_0^+(\lambda,.)\psi^+(\lambda,.) + \sum_{i\lambda^2 \in \Sigma_-^-(H)} u_0^-\psi^-(\lambda,.)
\]

where \(\psi^+(\lambda,.)\) (respectively \(\psi^-(\lambda,.)\)) is a normalized (in \(\mathcal{H}\)) eigenfunction of \(iH\) associated to the eigenvalue \(i\lambda^2\) in \(i\Sigma_+^+(H)\) (resp. \(i\Sigma_-^-(H)\)). Note that the sum takes into account the multiplicities of the eigenvalues here.

Thus the solution of the boundary value problem given in the introduction is:

\[
u(t) := \sum_{i\lambda^2 \in \Sigma_+^+(H)} u_0^+(\lambda,.)e^{i\lambda^2 t}\psi^+(\lambda,.) + \sum_{i\lambda^2 \in \Sigma_-^-(H)} u_0^-\psi^-(\lambda,.)e^{i\lambda^2 t}.
\]

The energy \(E(t) = E^+(t) + E^-(t)\) with

\[
E^+(t) := 1/2 \sum_{i\lambda^2 \in \Sigma_+^+(H)} \|u_0^+(\lambda,.)\|^2 e^{i\lambda^2 t} + \|u_0^-(\lambda,.)\|^2 e^{i\lambda^2 t}.
\]
\[
E^-(t) := \frac{1}{2} \sum_{i\lambda^2 \in \Sigma^+_{p}(H)} \|u_0^- (\lambda, \cdot)\|^2_H |e^{i\lambda^2 t}|^2.
\]

Now, since \(i\Sigma^+_{p}(H)\) contains only purely imaginary eigenvalues, \(|e^{2i\lambda^2 t}| = 1\), for any \(\lambda\) such that \(i\lambda^2 \in i\Sigma^+_{p}(H)\) and any \(t > 0\). Thus \(E^+(t) = E^+(0)\) for any \(t > 0\).

The real part of \(i\lambda^2\) is a non-positive real number if \(\lambda\) is such that \(i\lambda^2 \in i\Sigma^-_{p}(H)\). This real part is proved to be equal to \(-\frac{8\alpha}{3L}\) if \(i\lambda^2 \in i\Sigma^-_{p}(H)\).

It holds \(E^-(t) \leq e^{-2\omega t} E^-(0)\). Thus \(E^-(t)\) decreases exponentially to 0 when \(t\) tends to \(+\infty\) and the total energy \(E(t)\) decreases exponentially to \(E^+(0)\) when \(t\) tends to \(+\infty\). □

4.2. Energy decreasing: a numerical example. The purely point spectrum of the conservative operator \(iH\), with \(\alpha = 0\), is given by \(i\Sigma^+_{p}(H)\).

Then the set \(i\Sigma^-_{p}(H)\) which is a part of the spectrum of the dissipative operator \(iH\) has a vertical asymptote:

\[
\Re(i\lambda^2) = -\frac{8\alpha}{3L},
\]

which is consistent with the numerical computation of the spectrum, see Figure 2.

![Figure 2](image)

**Figure 2.** \(\lambda_n(\alpha)\), for \(n = 1, \ldots, 30\) and \(L = 2\pi\)

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