Diagonalizing "compact" operators on Hilbert W*-modules

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Abstract. For W*-algebras $A$ and self-dual Hilbert $A$-modules $M$ we show that every self-adjoint, "compact" module operator on $M$ is diagonalizable. Some specific properties of the eigenvalues and of the eigenvectors are described.

Keywords: diagonalization of "compact" operators, Hilbert W*-modules, W*-algebras, eigenvalues, eigenvectors

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The goal of the present short note is to consider self-adjoint, "compact" module operators on self-dual Hilbert W*-modules (which can be supposed to possess a countably generated W*-predual Hilbert W*-module, in general) with respect to their diagonalizability. Some special properties of their eigenvalues and eigenvectors are described. A partial result in this direction was recently obtained by V. M. Manuilov [10,11] who proved that every such operator on the standard countably generated Hilbert W*-module $l_2(A)$ over finite W*-algebras $A$ can be diagonalized on the respective $A$-dual Hilbert $A$-module $l_2(A)'$. The same was shown to be true for every self-adjoint bounded module operator on finitely generated Hilbert C*-modules over general W*-algebras by R. V. Kadison [5,6,7] and over commutative AW*-algebras by K. Grove and G. K. Pedersen [4] sometimes earlier. M. Frank has made an attempt to find a generalized version of the Weyl-Berg theorem in the $l_2(A)'$ setting for some (abelian) monotone complete C*-algebras which should satisfy an additional condition, as well as a counterexample, cf. [2]. Further results on generalizations of the Weyl-von Neumann-Berg theorem can be found e. g. in papers of G. J. Murphy [12], S. Zhang [15,16] and H. Lin [9].

We go on to investigate situations where non-finite W*-algebras appear as coefficients of the special Hilbert W*-modules under consideration (Proposition 5), and where arbitrary self-dual Hilbert W*-modules are considered (Theorem 9). The applied techniques are rather different from that in [10,11]. By the way, the results of V. M. Manuilov in [10,11] are obtained to be valid for arbitrary self-adjoint, "compact" module operators on the self-dual Hilbert $A$-module $l_2(A)'$ over finite W*-algebras (Proposition 3). This generalizes [10] since in the situation of finite W*-algebras $A$ the set of "compact" operators on $l_2(A)$ may be definitely smaller than that on $l_2(A)'$, and the latter may not contain all bounded module operators on $l_2(A)'$, in general. We characterize the role of self-duality for getting adequate results in the finite W*-case (Proposition 4). The final
result of our investigations is Theorem 9 describing the diagonalizability of "compact" operators on self-dual Hilbert W*-modules in a great generality.

We consider Hilbert W*-modules \( \{\mathcal{M}, \langle ., . \rangle\} \) over general W*-algebras \( A \), i.e. (left) \( A \)-modules \( \mathcal{M} \) together with an \( A \)-valued inner product \( \langle ., . \rangle : \mathcal{M} \times \mathcal{M} \to A \) satisfying the conditions:

(i) \( \langle x, x \rangle \geq 0 \) for every \( x \in \mathcal{M} \).

(ii) \( \langle x, x \rangle = 0 \) if and only if \( x = 0 \).

(iii) \( \langle x, y \rangle = \langle y, x \rangle^* \) for every \( x, y \in \mathcal{M} \).

(iv) \( \langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle \) for every \( a, b \in A, x, y, z \in \mathcal{M} \).

(v) \( \mathcal{M} \) is complete with respect to the norm \( \|x\| = \|\langle x, x \rangle\|_A^{1/2} \).

We always suppose, that the linear structures of the W*-algebra \( A \) and of the (left) \( A \)-module \( \mathcal{M} \) are compatible, i.e. \( \lambda(ax) = (\lambda a)x = a(\lambda x) \) for every \( \lambda \in \mathbb{C}, a \in A, x \in \mathcal{M} \). Let us denote the \( A \)-dual Banach \( A \)-module of a Hilbert \( A \)-module \( \{\mathcal{M}, \langle ., . \rangle\} \) by \( \mathcal{M}' = \{r : \mathcal{M} \to A : r - A - \text{linear and bounded}\} \).

Hilbert W*-modules have some very nice properties in contrast to general Hilbert C*-modules: First of all, the \( A \)-valued inner product can always be lifted to an \( A \)-valued inner product on the \( A \)-dual Hilbert \( A \)-module \( \mathcal{M}' \) via the canonical embedding of \( \mathcal{M} \) into \( \mathcal{M}' \), \( x \to \langle ., x \rangle \), turning \( \mathcal{M}' \) into a (left) self-dual Hilbert \( A \)-module, \( (\mathcal{M}')' \).

Moreover, one has the following criterion on self-duality:

**Proposition 1.** [1, Thm. 3.2] Let \( A \) be a W*-algebra and \( \{\mathcal{M}, \langle ., . \rangle\} \) be a Hilbert \( A \)-module. Then the following conditions are equivalent:

(i) \( \mathcal{M} \) is self-dual.

(ii) The unit ball of \( \mathcal{M} \) is complete with respect to the topology \( \tau_1 \) induced by the semi-norms \( \{f(\langle ., . \rangle)^{1/2}\} \) on \( \mathcal{M} \), where \( f \) runs over the normal states of \( A \).

(iii) The unit ball of \( \mathcal{M} \) is complete with respect to the topology \( \tau_2 \) induced by the linear functionals \( \{f(\langle ., x \rangle)\} \) on \( \mathcal{M} \) where \( f \) runs over the normal states of \( A \) and \( x \) runs over \( \mathcal{M} \).

Furthermore, on self-dual Hilbert W*-modules every bounded module operator has an adjoint, and the Banach algebra of all bounded module operators is actually a W*-algebra. And last but not least, every bounded module operator on a Hilbert W*-module \( \{\mathcal{M}, \langle ., . \rangle\} \) can be continued to a unique bounded module operator on its \( A \)-dual Hilbert W*-module \( \mathcal{M}' \) preserving the operator norm. (Cf. [13].)

We want to consider (self-adjoint,) "compact" module operators on Hilbert W*-modules. By G. G. Kasparov [8] an \( A \)-linear bounded module operator \( K \) on a Hilbert \( A \)-module \( \{\mathcal{M}, \langle ., . \rangle\} \) is "compact" if it belongs to the norm-closed linear hull of the elementary operators

\[ \{\theta_{x,y} : \theta_{x,y}(z) = \langle z, x \rangle y, x, y \in \mathcal{M}\} \]

The set of all "compact" operators on \( \mathcal{M} \) is denoted by \( K_A(\mathcal{M}) \). By [13, Thm. 15.4.2] the C*-algebra \( K_A(\mathcal{M}) \) is a two-sided ideal of the set of all bounded, adjointable module
operators $\text{End}_A^*(\mathcal{M})$ on $\mathcal{M}$, and both these sets coincide if and only if $\mathcal{M}$ is algebraically finitely generated as an $A$-module, (cf. also [3, Appendix]). This will be used below. Since we are going to investigate single "compact" operators we make the useful observation that both the range of a given "compact" operator and the support of it are Hilbert C*-modules generated by countably many elements with respect to the norm topology or at least with respect to the $\tau_1$-topology, (cf. Proposition 1). Hence, without loss of generality we can restrict our attention to countably generated Hilbert W*-modules and their W*-dual Hilbert W*-modules.

We are especially interested in the Hilbert W*-module

$$l_2(A) = \{ \{ a_i : i \in \mathbb{N} \} : a_i \in A, \sum_i a_i a_i^* \text{ converges with respect to } \| \cdot \|_A \}$$

$$\langle \{ a_i \}, \{ b_i \} \rangle = \| \cdot \|_A - \lim_{N \to \infty} \sum_{i=1}^{N} a_i b_i^* ,$$

and in its $A$-dual Hilbert W*-module

$$l_2(A)' = \{ \{ a_i : i \in \mathbb{N} \} : a_i \in A, \sup_{N \in \mathbb{N}} \left\| \sum_{i=1}^{N} a_i a_i^* \right\| < \infty \}$$

$$\langle \{ a_i \}, \{ b_i \} \rangle = w^* - \lim_{N \to \infty} \sum_{i=1}^{N} a_i b_i^* ,$$

because of G. G. Kasparov’s stabilization theorem [8], stating that every countably generated Hilbert C*-module over a unital C*-algebra $A$ is a direct summand of $l_2(A)$.

**Definition 2.** Let $A$ be a W*-algebra and let $\{ \mathcal{M}, \langle \cdot, \cdot \rangle \}$ be a self-dual Hilbert $A$-module possessing a countably generated Hilbert $A$-module as its $A$-predual. A bounded module operator $T$ on $\mathcal{M}$ is diagonalizable if there exists a sequence $\{ x_i : i \in \mathbb{N} \}$ of non-trivial elements of $\mathcal{M}$ such that:

(i) $T(x_i) = \Lambda_i x_i$ for some elements $\Lambda_i \in A, (i \in \mathbb{N})$,

(ii) The Hilbert $A$-submodule generated by the elements $\{ x_i \}$ inside $\mathcal{M}$ has a trivial orthogonal complement.

(iii) The elements $\{ x_i : i \in \mathbb{N} \}$ are pairwise orthogonal, and the values $\{ p_i = \langle x_i, x_i \rangle : i \in \mathbb{N} \}$ are projections in $A$.

(iv) The equality $\Lambda_i p_i = \Lambda_i$ holds for the projection $p_i, (i \in \mathbb{N})$.

Note, that the eigenvalues and the eigenvectors are not uniquely determined for the operator $T$ since $T(x) = \Lambda x$ implies $T(y) = \Lambda' y$ for $\Lambda' = u \Lambda u^*$ and $y = ux$ for all unitaries $u \in A$. Moreover, the eigenvalues of $T$ do not belong to the center of $A$, in general. Consequently, $T(ax) = a(\Lambda x) \neq \Lambda(ax)$, in general. That is, eigenvectors are often not one-to-one related to $T$-invariant $A$-submodules of the Hilbert $A$-module $\mathcal{M}$ under consideration.

Now, we start our investigations decomposing $A$ into components of prescribed type with respect to its direct integral representation. Denote by $p$ that central projection of $A$ dividing $A$ into a finite part $pA$ and into an infinite part $(1 - p)A$. That means, that with respect to the direct integral decomposition of $A$ the fibers are almost everywhere factors of type $I_n, n < \infty$, or $II_1$ inside $pA$ and almost everywhere factors of type $I_\infty$ or $II_\infty$ or III inside $(1 - p)A$. Analogously, the Hilbert $A$-module $l_2(A)$ decomposes into the direct sum of two Hilbert $A$-modules $l_2(A) = l_2(pA) \oplus l_2((1 - p)A)$, and every bounded $A$-linear operator $T$ on $l_2(A)$ splits into the direct sum $T = pT \oplus (1 - p)T$. 
where each part acts only on the respective part of the Hilbert A-module non-trivially and at the same time as an A-linear operator. Consequently, we can proceed considering W*-algebras A of coefficients of prescribed type. Our first goal is to revise the case of finite W*-algebras investigated by V. M. Manuilov. There the set $K_A(l_2(A)'')$ does not coincide with the set $\text{End}_A(l_2(A)'')$, and there are always self-adjoint, bounded module operators $T$ on $l_2(A)''$ which can not be diagonalized. For example, consider a self-adjoint, bounded linear operator $T_o$ on a separable Hilbert space $H$ being non-diagonalizable, (cf. Weyl’s theorem). Using the decomposition $l_2(A) = H \otimes A$ one obtains a self-adjoint, bounded module operator $T$ on $l_2(A)$ by the formula $T(a \otimes h) = a \otimes T_o(h), \ (a \in A, \ h \in H)$. The operator $T$ extends to an operator on $l_2(A)'$, and $T$ can not be diagonalizable by assumption. Surprisingly, V. M. Manuilov proved that every self-adjoint, ”compact” operator on the standard countably generated Hilbert W*-module $l_2(A)$ over finite W*-algebras $A$ can be diagonalized on the respective A-dual Hilbert A-module $l_2(A)'$. A careful study of his detailed proofs at [10], [11] brings to light that for finite W*-algebras with infinite center the continuation of the ”compact” operators to the respective A-dual Hilbert A-module is not only a proof-technical necessity, but it is of principal character. Self-duality has to be supposed to warrant the diagonalizability of all self-adjoint ”compact” module operators on $\mathcal{M} \subseteq l_2(A)'$ in the finite case, and the key steps of the proof can be repeated one-to-one. Consequently, we give the generalized formulation of V. M. Manuilov’s diagonalization theorem for the finite case, and we show additionally that self-duality is an essential property of Hilbert W*-modules for finding a (well-behaved) diagonalization of arbitrary ”compact” module operators on them, in general.

**Proposition 3.** (cf. [10], [11, Thm.4.1]) Let $A$ be a W*-algebra of finite type. Then every self-adjoint, ”compact” module operator $K$ on $l_2(A)'$ is diagonalizable. The sequence of eigenvalues $\Lambda_n : n \in \mathbb{N}$ of $K$ has the property $\lim_{n \to \infty} \|\Lambda_n\| = 0$. The eigenvalues $\Lambda_n : n \in \mathbb{N}$ of $K$ can be chosen in such a way that $\Lambda_2 \leq \Lambda_4 \leq \ldots \leq 0 \leq \ldots \leq \Lambda_3 \leq \Lambda_1$. Moreover, for positive operators $K$ without kernel the eigenvectors $\{x_n : n \in \mathbb{N}\}$ may possess the property $\langle x_n, x_n \rangle = 1_A, \ (n \in \mathbb{N})$, in addition.

For the detailed (but extended) proof of this proposition see [11], (also [10]). The proving technique relies mainly on spectral decomposition theory of operators and on the center-valued trace on the finite W*-algebra $A$.

**Proposition 4.** Let $A$ be a finite W*-algebra with infinite center. Consider a Hilbert A-module $\mathcal{M}$ such that $l_2(A) \subset \mathcal{M} \subseteq l_2(A)'$. Then the following two statements are equivalent:

(i) $\mathcal{M} = l_2(A)'$, i.e., $\mathcal{M}$ is self-dual.

(ii) Every positive ”compact” module operator is diagonalizable inside $\mathcal{M}$ with comparable inside the positive cone of $A$ eigenvalues.

**Proof.** Note, that $l_2(A) \neq l_2(A)'$ by assumption. Denote the standard orthonormal basis of $l_2(A)$ by $\{e_n : n \in \mathbb{N}\}$. If the center of $A$ is supposed to be infinite dimensional then one finds a sequence of pairwise orthogonal non-trivial projections $\{p_n : n \in \mathbb{N}\} \in Z(A)$ summing up to $1_A$ in the sense of $w^*$-convergence. Fix a sequence of positive non-
zero numbers \( \{\alpha_n : n \in \mathbb{N}\} \) monotonically converging to zero. The bounded module operator \( K \) defined by

\[
K(e_1) = (\sum_{n=1}^{\infty} \alpha_n p_n e_n), \quad K(e_j) = \alpha_j p_j e_1 \quad \text{for} \quad j \neq 1
\]

is a ”compact” operator on \( l_2(A) \). It easily continues to a ”compact” operator on \( \mathcal{M} \).

As an exercise one checks that the eigenvalues of \( K \) are \( \{\alpha_1 p_1, \alpha_2 p_2, \cdots, 0, \cdots, -\alpha_2 p_2\} \) (ordering by sign and norm and taking into account (iii) and (iv) of Definition 2), and that the appropriate eigenvectors are

\[
\{p_1 e_1, 1/\sqrt{2}p_2(e_1 + e_2), 1/\sqrt{2}p_3(e_1 + e_3), \cdots, (1_A - p_n)e_n : n \in \mathbb{N}\}, \cdots
\]

\[
\cdots, 1/\sqrt{2}p_3(e_1 - e_3), 1/\sqrt{2}p_2(e_1 - e_2)\}.
\]

The only way of making the eigenvalues comparable inside the positive cone of \( A \) preserving Definition 2, (iii)-(iv) is to sum up the positive and the negative eigenvalues separately. But, then the resulting eigenvector

\[
x = (1_A + (1 + 1/\sqrt{2})(1_A - p_1), 1/\sqrt{2}p_2, 1/\sqrt{2}p_3, \cdots, 1/\sqrt{2}p_n, \cdots),
\]

corresponding to the only positive eigenvalue \( \sum_{n=1}^{\infty} \alpha_n p_n \) of \( K \) does not belong to \( \mathcal{M} \) any longer by assumption. This shows one implication. The converse implication follows from Proposition 6.

The second big step is to investigate the case of infinite \( W^* \)-algebras as coefficients of the Hilbert \( W^* \)-modules under consideration. The result is characteristic for the situation in self-dual Hilbert \( W^* \)-modules over infinite \( W^* \)-algebras, and quite different from that in the finite \( W^* \)-case, and elsemore, from the classical Hilbert space situation.

**Proposition 5.** Let \( A \) be a \( W^* \)-algebra which possesses infinitely many pairwise orthogonal, non-trivial projections \( \{p_i : i \in \mathbb{N}\} \) equivalent to \( 1_A \) and summing up to \( 1_A \) in the sense of \( w^* \)-convergence of the sum \( \sum_i p_i = 1_A \). Then the Hilbert \( A \)-module \( l_2(A)' \) equipped with its standard \( A \)-valued inner product is isomorphic to the Hilbert \( A \)-module \( \{A, (\cdot, \cdot)_A\} \), where \( (a, b)_A = ab^* \).

**Proof.** Suppose, the equivalence of the projections \( \{p_i : i \in \mathbb{N}\} \) with \( 1_A \) is realized by partial isometries \( \{u_i : i \in \mathbb{N}\} \in A, p_i = u_i u_i^*, 1_A = u_i^* u_i \). Then the mapping

\[
S : l_2(A)' \to A, \quad \{a_i\} \to w^* - \text{lim(finite } \sum_i a_i u_i^*)
\]

with the inverse mapping

\[
S^{-1} : A \to l_2(A)' , \quad a \to \{au_i\}
\]

realizes the isomorphism of \( l_2(A)' \) and of \( A \) as Hilbert \( A \)-modules because of Proposition 1.

**Corollary 6.** Let \( A \) be a \( W^* \)-algebra of infinite type. Then every bounded module operator \( T \) on \( l_2(A)' \) is diagonalizable, and the formula

\[
T(\{a_i\}) = \langle\{a_i\}, \{u_i\} \rangle \Lambda_T \{u_i\}
\]

holds for every \( \{a_i\} \in l_2(A)' \), some \( \Lambda_T \in A \) and the partial isometries \( \{u_i\} \in A \) described in the previous proof.

**Proof.** Every \( W^* \)-algebra of type \( I_\infty, II_\infty \) or III possesses a set of partial isometries with properties described at Proposition 3. The same is true for \( W^* \)-algebras consisting only of parts of these types. Now, translate the operator \( T \) on \( l_2(A)' \) to an operator
Let \( A \) and vice versa using Proposition 3, and take into account that every bounded module operator on \( A \) is a multiplication operator with a concrete element (from the right).

**Corollary 7.** Let \( A \) be a \( W^* \)-algebra without any fibers of type \( \text{I}_n \), \( n < \infty \), and \( \Pi_1 \) in its direct integral decomposition. Let \( \mathcal{M} \) be a self-dual Hilbert \( A \)-module possessing a countably generated \( A \)-predual Hilbert \( A \)-module. Then every bounded module operator \( T \) on \( \mathcal{M} \) is diagonalizable, and the formula
\[
T(x) = (x, u)\Lambda_T u
\]
holds for every \( x \in \mathcal{M} \), some \( \Lambda_T \in A \) and a universal for all \( T \) eigenvector \( u \in \mathcal{M} \).

**Proof.** Since \( \mathcal{M} \) has a countably \( A \)-Hilbert module as its \( A \)-predual, \( \mathcal{M} \) is a direct summand of the Hilbert \( A \)-module \( l_2(A)' \) by G. G. Kasparov's stabilization theorem ([8]). Hence, one has to show the assertion for the self-dual Hilbert \( A \)-module \( l_2(A)' \) only. For further use denote the projection from \( l_2(A)' \) onto \( \mathcal{M} \) by \( P \). Consider the direct integral decomposition of \( A \) over its center. Therein every fiber is a \( W^* \)-factor of type \( \text{I}_\infty \), \( \Pi_\infty \) or \( \text{III} \) by assumption. Putting it into the \( l_2(A)' \)-context one obtains that \( A \) is isomorphic to \( l_2(A)' \) either applying Corollary 6 fiberwise or constructing a suitable set of partial isometries \( \{u_i\} \in A \) to make use of Proposition 5. Then in the same way as there the diagonalization result turns out for arbitrary bounded module operators \( T \) on \( l_2(A)' \). To get the formula of Corollary 7 one has only to set \( u = P(\{u_i\}) \).

**Remark.** Let \( A \) be a \( \text{I}_\infty \)-factor, for example. Then there are self-adjoint elements \( \Lambda_T \) in \( A \) which cannot be diagonalized in a stronger sense. More precisely, there is no way of representing any such operator as a sum \( \sum \lambda_i P_i \) with \( \lambda_i \in C = Z(A) \) and \( P_i = P_i^* = P_i^2 \in A \) because of Weyl’s theorem. Therefore, the Corollaries 6 and 7 are the strongest results one could expect.

**Example 8.** Consider the \( C^* \)-algebra \( A \) of all \( 2 \times 2 \)-matrices on the complex numbers. Set \( \mathcal{M} = A^2 \) with the usual \( A \)-valued inner product. Consider the ("compact") bounded module operator \( K = \theta_{x,x} + \theta_{y,y} \) for
\[
x = \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & 3 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \right), \ y = \left( \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 2 & 0 \\ 0 & 2 \end{array} \right) \right) .
\]

Eigenvectors of \( K \) are \( x, y \in A^2 \), for example, and the respective eigenvalues are
\[
\Lambda_x = \left( \begin{array}{cc} 1 & 0 \\ 0 & 9 \end{array} \right), \ \Lambda_y = \left( \begin{array}{cc} 4 & 0 \\ 0 & 4 \end{array} \right) .
\]

Remark, that one can not compare these eigenvalues as elements of the positive cone of \( A \). But, making another choice one arrives at that situation described at Proposition 6:
\[
x_1 = \left( \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right) \right), \ x_2 = \left( \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \right) .
\]
Then the respective eigenvalues are
\[
\Lambda_1 = \left( \begin{array}{cc} 1 & 0 \\ 0 & 4 \end{array} \right), \ \Lambda_2 = \left( \begin{array}{cc} 4 & 0 \\ 0 & 9 \end{array} \right) .
\]
and they can be ordered as well as the eigenvectors \( x_1, x_2 \) are units. Last but not least, dropping out condition (iv) of Definition 2 one can correlate \( K \)-invariant submodules of \( \mathcal{M} \) and eigenvectors of \( K \). Simply, set

\[
x_1 = \left( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right),
\]

\[
x_2 = \left( \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \right).
\]

In this case the corresponding eigenvalues are

\[
\Lambda_1 = \left( \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 0 & 4 \\ 0 & 9 \end{pmatrix} \right).
\]

They can be ordered in the positive cone of \( A \). But, the eigenvectors corresponding to the \( K \)-invariant submodules of \( \mathcal{M} \) can not be selected to be units any longer.

**Theorem 9.** Let \( A \) be a \( W^* \)-algebra and \( \mathcal{M} \) be a self-dual Hilbert \( A \)-module. Then every self-adjoint, ”compact” module operator on \( \mathcal{M} \) is diagonalizable. The sequence of eigenvalues \( \{ \Lambda_n : n \in \mathbb{N} \} \) of \( K \) has the property \( \lim_{n \to \infty} \| \Lambda_n \| = 0 \). The eigenvalues \( \{ \Lambda_n : n \in \mathbb{N} \} \) of \( K \) can be chosen in such a way that \( \Lambda_2 \leq \Lambda_4 \leq \ldots \leq 0 \leq \ldots \leq \Lambda_3 \leq \Lambda_1 \), and that \( \{ \Lambda_n : n \geq 3 \} \) are contained in the finite part of \( A \).

**Proof.** Both the \( \tau_1 \)-closure of the range and of the support of \( K \) are self-dual Hilbert \( C^* \)-modules possessing countably generated \( A \)-predual Hilbert \( A \)-modules because of the ”compact”ness of \( K \). Hence, without loss of generality one can restrict the attention to self-dual Hilbert \( W^* \)-modules with countably generated \( W^* \)-modules formed as the \( \tau_1 \)-completed direct sum of range and support of \( K \). As usual, on the kernel of \( K \) one has the eigenvalue zero and a suitable system of eigenvectors. Now, gluing Corollary 4 and Proposition 6 together the theorem turns out to be true in the special case \( \mathcal{M} = l_2(A)' \), (cf. the remarks in the beginning of the present note). The only loss may be that the eigenvalues are not units, in general. Because of G. G. Kasparov’s stabilization theorem ([8]) \( \mathcal{M} \) possesses an embedding into \( l_2(A)' \) as a direct summand by assumption. Therefore, every self-adjoint, ”compact” module operator \( K \) on \( \mathcal{M} \) can be continued to a unique such operator on \( l_2(A)' \) preserving the norm, simply applying the rule \( K|_{\mathcal{M}^\perp} = 0 \). The eigenvectors of this extension are elements of \( \mathcal{M} \). The Hilbert \( A \)-module \( \mathcal{M}^\perp \) belongs to its kernel. This shows the theorem.

**Remark.** For commutative \( AW^* \)-algebras \( A \) the statement of Theorem 9 is still true by [4]. The general \( AW^* \)-case is open at present because of two crucial unsolved problems in the \( AW^* \)-theory: (i) Are the self-adjoint elements of \( M_n(A) \), \( n \geq 2 \), diagonalizable for arbitrary (monotone complete) \( AW^* \)-algebras \( A \), or not? (ii) Does every finite (monotone complete) \( AW^* \)-algebra possess a center-valued trace, or not?

**Remark.** One can extend the statement of Theorem 9 to the case of normal, ”compact” module operators dropping out only the ordering of the eigenvalues. To see this note that for normal elements \( K \) of the \( C^* \)-algebra \( K_A(\mathcal{M}) \) there always exists a self-adjoint element \( K' \in K_A(\mathcal{M}) \) such that \( K \) is contained in that \( C^* \)-subalgebra of \( End_A(\mathcal{M}) \) generated by \( K' \) and by the identity operator. Applying functional calculus inside the \( W^* \)-algebra \( End_A(\mathcal{M}) \) the result turns out. Beside this, it would be interesting to investigate some more general variants of the Weyl-von Neumann-Berg theorem for appropriate bounded module operators on (self-dual) Hilbert \( W^* \)-submodules over (finite) \( W^* \)-algebras \( A \) as those obtained by H. Lin, G. J. Murphy and S. Zhang.
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