Ribbon Biquandles and Virtual Knotted Surfaces

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Abstract

We introduce a type of biquandle called a ribbon biquandle which satisfies the oriented ribbon-pass moves (also called band-pass moves). We use these biquandles to define an invariant of virtual knotted surfaces represented by ch-diagrams.

KEYWORDS: ch-diagrams, knotted surfaces, biquandles, ribbon biquandles, virtual knotted surfaces
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1 Introduction

Biquandles are algebraic structures with axioms motivated by the oriented Reidemeister moves. In particular, the set of labelings of the semiarcs in a oriented knot diagram by elements of a finite biquandle satisfying a certain labeling condition forms a computable invariant known as the biquandle counting invariant. Biquandles were introduced in [2] and have been much studied in recent years; see [1, 8] etc. for more.

Ch-diagrams are a type of planar diagram similar to ordinary knot diagrams which encode a knotted surface in $\mathbb{R}^4$. It has been recently established ([4]) that two ch-diagrams represent ambient isotopic surfaces in $\mathbb{R}^4$ if and only if they are related by a sequence of the Yoshikawa moves introduced in [10]. Ch-diagrams can be used to represent both closed knotted surfaces and cobordisms between knots and have advantages over some other popular methods of representing knotted surfaces in $\mathbb{R}^4$: they are easier to draw than broken surface diagrams and require only a single diagram, unlike movie diagrams which require multiple diagrams. See [3, 9, 10] for more.

Virtual knots, also known as abstract knots, are a combinatorial generalization of knots including virtual crossings representing genus in the ambient space of the knot. Including virtual crossings in ch-diagrams yields virtual knotted surfaces; see [4, 6, 7] for more.

In this paper, we introduce a labeling scheme for naively oriented ch-diagrams by elements of certain biquandles we call ribbon biquandles. We define a multiset-valued invariant of ch-diagrams under Yoshikawa moves, yielding an invariant of knotted surfaces and virtual knotted surfaces.

The paper is organized as follows. In Section 2 we review the basics of biquandles and introduce ribbon biquandles. In Section 3 we review knotted surfaces and ch-diagrams. In Section 4 we introduce the new invariant and compute some examples, and we end in Section 5 with some questions for future research.

2 Biquandles

We begin with a definition.

Definition 1. Let $X$ be a set. A biquandle structure on $X$ is a pair of binary operations $(x, y) \mapsto x^y, x_y$ such that for all $x, y, z \in X$ we have

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(i) \( x^x = x_x \),

(ii) the maps \( \alpha_y, \beta_y : X \to X \) and \( S : X \times X \to X \times X \) defined by \( \alpha_y(x) = x_y, \beta_y(x) = x^y \) and \( S(x,y) = (y_x, x^y) \) are bijective, and

(iii) the exchange laws
\[
\begin{align*}
(x^y)^{(z^y)} &= (x^z)^{(y^z)}, \\
(x^y)_{(z^y)} &= (x_z)^{(y_z)}, \\
(x^y)_{(z^y)} &= (x_z)^{(y_z)}.
\end{align*}
\]

A biquandle such that \( x_y = x \) for all \( x, y \) is a quandle. A biquandle such that \( (x^y)_y = x = (x_y)_y, x^{yy} = x^y \) and \( y_{x^y} = y_x \) for all \( x, y \in X \) is a bikei.

**Remark 1.** We are using the original notation for biquandles from [2]; much recent work uses similar-looking but different notation resulting in less symmetric axioms.

**Example 1.** Let \( n \in \mathbb{Z} \). Any group \( G \) is a biquandle with \( x^y = y^{-n} xy^n \) and \( x^y = x \) as well as with \( x_y = y^{-n} xy^n \) and \( x^y = x \).

**Example 2.** Any abelian group \( A \) with invertible elements \( s, t \) is a biquandle with
\[
x^y = tx + (1 - s^{-1} t)y, \quad x^y = s^{-1} x.
\]

More generally, any module over the two-variable Laurent polynomial ring \( \mathbb{Z}[t^{\pm 1}, s^{\pm 1}] \) is a biquandle under
\[
x^y = tx + (1 - s^{-1} t)y, \quad x^y = s^{-1} x.
\]

Such a biquandle is called an Alexander biquandle.

Biquandle structures on a set \( X = \{ x_1, \ldots, x_n \} \) can be specified using an \( n \times 2n \) matrix encoding the operation tables of the biquandle operations. More precisely, let \( M_{i,j} = k \) where
\[
x_k = \begin{cases} 
(x_i)^{x_j} & 1 \leq j \leq n \\
(x_i)^{x_j} & n + 1 \leq j \leq 2n
\end{cases}
\]

**Example 3.** The Alexander biquandle \( X = \mathbb{Z}_4 = \{ x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 0 \} \) with \( s = 3 \) and \( t = 1 \) has biquandle matrix
\[
\begin{bmatrix}
3 & 1 & 3 & 1 \\
4 & 2 & 4 & 2 \\
1 & 3 & 1 & 3 \\
2 & 4 & 2 & 4
\end{bmatrix}
\]

The biquandle axioms come from the oriented Reidemeister moves. We think of elements of \( X \) as labels for the semiarcs in an oriented knot or link diagram with operations as depicted.

Then the Reidemeister I move requires that for all \( x \in X \), we have \( x^x = x_x \):
The direct Reidemeister II moves require right invertibility, while the reverse II moves require that the map of pairs $(x, y) \mapsto (y_x, x^y)$ is invertible. Together these imply the adjacent pairs rule: any adjacent labels at a crossing determine the other two.

The Reidemeister III move implies the exchange laws:

We will be interested in biquandles which satisfy an additional move, the ribbon pass move or band pass move.

First, we note that in the presence of the oriented Reidemeister moves, we can obtain the other oriented band-pass moves from the one pictured above. If one strand is reversed, we can locally reverse it with a type I move. Then we have:

If both strands are anti-parallel, a similar trick works where we make use of the above single reversed strand.
The other cases are similar.

**Definition 2.** A biquandle \( X \) is ribbon if it satisfies for all \( x, y, z, w \in X \)

- \((x^y)^z = (x_y)_z\) and
- \((x^y)^z(x^v) = (x^{y^z})(w^v)\).

**Proposition 1.** Biquandle labelings are preserved by the ribbon move in the sense that labelings before the move correspond to unique labelings after the move iff \( X \) is ribbon.

**Proof.** We verify with a diagram:

A natural question is to ask when an Alexander biquandle is ribbon. We have

**Proposition 2.** An Alexander biquandle is ribbon only if \( s^{-1} = t \) and \( t^2 = 1 \).

**Proof.** Setting \((x^y)^z = (x_y)_z\), we have

\[
t^2x + (t - s^{-1})y + (1 - s^{-1}t)z = s^{-2}x;
\]
then comparing coefficients, we have \( t^2 = s^{-2} \), \( t - s^{-1} = 0 \) and \( 1 - s^{-1}t = 0 \). The second equation gives us \( t = s^{-1} \), which also satisfies the first; the last then requires that \( t^2 = 1 \).

Despite this, there are many examples of nontrivial ribbon biquandles which can be found by computer search.
Example 4. The biquandle with operation matrix

\[
\begin{pmatrix}
1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 \\
4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 \\
3 & 4 & 2 & 1 & 3 & 4 & 2 & 1 \\
2 & 1 & 3 & 4 & 2 & 1 & 3 & 4
\end{pmatrix}
\]

is ribbon.

3 Knotted Surfaces

A knotted surface is a smoothly embedded compact surface \( \Sigma \subset \mathbb{R}^4 \) with finitely many components. Knotted surfaces can be represented with broken surface diagrams analogous to knot diagrams where a broken sheet indicates the sheet crosses under in the fourth dimension much as a broken strand in a knot diagram indicates the strand crossing under in the third dimension. The self-intersection set in the projection into \( \mathbb{R}^3 \) can include endpoints and triple points as well as closed curves.

![Diagram of a knotted surface]

Every knotted surface can be moved by ambient isotopy into a position such that all of its maxima in the \( x_3 \) (say) direction are in the \( x_3 = 1 \) hyperplane, all of its minima are in the \( x_3 = -1 \) hyperplane and all of its saddle points are in the \( x_3 = 0 \) hyperplane; such a position is known as a hyperbolic splitting. In particular each cross-section of the surface by a hyperplane of the form \( x_3 = \epsilon \) for \( \epsilon \in (-1,0) \cup (0,1) \) is an unlink, with the \((x_1,x_2)\) cross-sections forming Reidemeister move sequences ending with crossingless unlink diagrams near \( \epsilon = \pm 1 \).

We can represent a knotted surface in such a position with a ch-diagram, a diagram consisting of ordinary crossings together with saddle crossings representing saddle points:

![Diagram of a ch-diagram]

To recover the knotted surface diagram from a ch-diagram, we resolve the saddle crossings into saddles with the crossings resolving into crossed sheets; near the \( x_3 = 0 \) hyperplane, we have a cobordism between unlinks. These unlinks then resolve over the intervals \( x_3 \in (-1,0) \cup (0,1) \) into disjoint circles which we cap off with
maxima and minima.

Remark 2. Ch-diagrams which yield nontrivial knots or links after smoothing do not correspond to knotted closed surfaces but to corbordisms between these knots and links.

In [10], Yoshikawa introduced a set of moves now known as Yoshikawa moves and conjectured that two knotted surfaces are ambient isotopic iff their ch-diagrams are related by the Yoshikawa moves. This conjecture has recently been established [5]. There are eight moves, the first three of which are the usual Reidemeister moves:
For labeling the semiarcs of a ch-diagram with biquandle elements, we need an orientation on the diagram.

**Definition 3.** A naïve orientation of a ch-diagram is a choice of orientation for each unicursal component of the diagram.

In particular, a naïvely oriented diagram treats saddle points as a new type of oriented combinatorial crossing; thus, a naïvely oriented ch-diagram has crossing of the four types below:

![Crossing Types](image)

We note that naïve orientations do no correspond to orientations of the knotted surface described by the ch-diagram; instead, they are combinatorial tools which we will use to define invariants of the unoriented knotted surface determined by the diagram. We will use the convention that all strands in portions of naïvely oriented ch-diagrams with boundary points are oriented downward, i.e., from the top boundary point to the bottom boundary point, unless otherwise specified.

In [10], ch-diagrams are required to yield unlink diagrams after smoothing the saddles in order to obtain closed knotted surfaces; as we have observed, relaxing this requirement yields cobordisms between the links obtained after smoothing. In [7], this idea is used to extend the notion of cobordism to the case of virtual knots, knots and links with virtual crossings, which represent genus in the ambient space rather than points where the knot is close to itself within the ambient space. We will not break our semiarcs at virtual crossings, so the bibundle labeling rule is as depicted.

![Virtual Crossings](image)

Virtual crossings interact with classical and saddle crossings via the detour move

![Detour Move](image)

which says that any portion of the knot with only virtual crossings can be replaced with any other strand with the same endpoints and only virtual crossings.
In particular, ch-diagrams with virtual crossings represent cobordisms between the virtual knots obtained by smoothing the saddle crossings.

4 Ribbon Biquandle Invariants

Let \( X \) be a ribbon biquandle and \( L \) a ch-diagram. An \( X \)-labeling of \( L \) is an assignment of elements of \( X \) to the semiarcs of \( L \) (the portions of the diagram between the crossing points) satisfying the rules pictured below:

Checking the Yoshikawa moves, we obtain:

**Theorem 3.** If \( X \) is a ribbon biquandle and \( L \) and \( L' \) are na"ıvely oriented ch-diagrams related by Yoshikawa moves I,II,IV,V,VII and VIII, then for every \( X \)-labeling of \( L \) there is a unique corresponding \( X \)-labeling of \( L' \).

If \( X \) is a finite ribbon biquandle, then for each naïve orientation of \( L \), there are finitely many \( X \)-labelings of \( L \). As mentioned in the previous section, the naïve orientations of \( L \), unfortunately, do not correspond to oriented knotted surfaces in \( \mathbb{R}^4 \); however, the set of all naïve orientations of \( L \) does correspond to the unoriented knotted surface. We note further that there is a way to orient the semiarcs of \( L \) to correspond with oriented knotted surfaces, described in [9].

An additional issue is that unlike the case with biquandle colorings of knotted circles in \( \mathbb{R}^3 \), the number of biquandle colorings of a naïvely oriented ch-diagram is not preserved by all of the Yoshikawa moves. Specifically, a saddle-introducing type VI move multiplies the number of \( X \)-labelings by a factor of \(|X|\), while a saddle-removing VI move divides the number of labelings by \(|X|\). The remaining Yoshikawa moves do not change the number of \( X \)-labelings provided \( X \) is a ribbon biquandle. Then for each naïve orientation \( L \to \) of \( L \), the quantity

\[
\phi(L \to) = \frac{|\mathcal{L}(L \to, X)|}{|X|^s}
\]

where \( s \) is the number of saddle points of \( L \) is unchanged by Yoshikawa moves. Thus, we define

**Definition 4.** Let \( X \) be a ribbon biquandle and \( L \) a ch-diagram. Then then ribbon biquandle invariant of \( L \) is the multiset

\[
\Phi^R_X \mathcal{M}(L) = \left\{ \frac{|\mathcal{L}(L \to, X)|}{|X|^s} : L \to \in NO(L) \right\}
\]

or, in polynomial form,

\[
\Phi^R_X (L) = \sum_{L \to \in NO(L)} u^{\phi(L \to)}
\]

where \( NO(L) \) is the set of naïve orientations of \( L \) and \( L \to \) indicates a choice of naïve orientation for \( L \).

By construction, we have

**Theorem 4.** For any ribbon biquandle \( X \), \( \Phi^R_X \) and \( \Phi^R_X \) are invariants of unoriented knotted surfaces in \( \mathbb{R}^4 \).

**Example 5.** Let \( X \) be the ribbon biquandle with matrix

\[
\begin{bmatrix}
1 & 2 & 4 & 3 & 1 & 2 & 4 & 3 \\
4 & 3 & 1 & 2 & 4 & 3 & 1 & 2 \\
3 & 4 & 2 & 1 & 3 & 4 & 2 & 1 \\
2 & 1 & 3 & 4 & 2 & 1 & 3 & 4
\end{bmatrix}
\]
and consider the ch-diagram below.

There are four possible naïve orientations with indicated labelings.

For each naïve labeling we can choose $x$ arbitrarily and then need $y = x^x = x_x$; hence each naïve orientation contributes $u^\frac{x}{4} = u^1$ to the invariant. Thus we get ribbon biquandle invariant $\Phi^R_X(K) = 4u$.

**Example 6.** Now consider the virtual ch-diagram below.

Using the same ribbon biquandle from example [5] two of the naïve orientations have 16 labelings while two have only two valid $X$-labelings, resulting in a ribbon biquandle invariant value of $\Phi^R_X(K) = 2u + 2u^\frac{1}{2}$. Hence, the invariant distinguishes thus virtual knotted surface from the one in example [5].

**Example 7.** Let $X$ be the ribbon biquandle with operation matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 3 & 3 & 2 & 3 \\
3 & 2 & 2 & 3 & 2
\end{bmatrix}.
$$

Consider the virtual ch-diagrams below.

Then we find that $\Phi^R_X(K_1) = 2u^\frac{1}{2}$ while $\Phi^R_X(K_2) = 2u^\frac{1}{2}$. 
Remark 3. If $X$ is a bikei, then changing the orientation of $K$ does not change the number of $X$-labelings. In this case $\Phi_X^R(K)$ will always have the form $Cu^{\phi(K)}$ where $C$ is the number of components of $K$ and $\phi(K)$ is the number of $X$-labelings of any naive orientation of $K$.

5 Questions

We conclude with some questions for future research.

- What kinds of enhancements are there for the ribbon biquandle invariant?
- What algebraic structures can be defined using the orientation scheme in [9]?
- What happens when we add nontrivial operations at the virtual crossings?
- If a biquandle satisfies the condition $x y = x y$ for all $x, y \in X$, i.e., if the upper and lower operations are the same, then the biquandle is ribbon. The converse is true for constant action biquandles and Alexander biquandles; is it true in general?

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