Nondegenerate Super-Anti-de Sitter Algebra
and a Superstring Action

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Abstract

We construct an Anti-de Sitter(AdS) algebra in a nondegenerate superspace. Based on this algebra we construct a covariant kappa-symmetric superstring action, and we examine its dynamics: Although this action reduces to the usual Green-Schwarz superstring action in flat limit, the auxiliary fermionic coordinates of the nondegenerate superspace becomes dynamical in the AdS background.

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1 Introduction

After the conjecture of AdS/CFT correspondence [1], supersymmetric graded algebras based on [2] have been reexamined [3] and superstring actions in AdS spaces have been studied intensively [4, 5, 6]. In these references the superstring actions are constructed as $\sigma$-models on coset superspaces of suitable graded algebras. These superalgebras are degenerate in a sense that nondegenerate metric for fermionic sector can not be defined using only one kind of supergenerator. Green showed that a nondegenerate superspace can be defined by introducing a fermionic central charge [7] in a flat background. It is an interesting issue to examine how the fermionic “central extension” is incorporated with the super-AdS algebra, since the “fermionic center” can not stay as a center anymore in the AdS space. In this paper we discuss on the issue of “nondegenerate super-AdS algebra”.

In the usual Green-Schwarz (GS) superstring [8] the Wess-Zumino action is pseudo (quasi) invariant under supersymmetry transformations. The Noether charges of the supersymmetry acquire additional contributions from the surface term. As a result the SUSY algebra contains anomaly or topological terms [9] which play important roles in discussions of the BPS properties [10]. On the other hand Siegel has shown that the Wess-Zumino action of the superstring can be obtained by a simple bilinear combination of supercovariant Maurer-Cartan 1-forms for the nondegenerate supertranslation algebra. This method overcomes the difficulty of the Wess-Zumino action on the random lattice, then it gives a second quantized particle superfield theory [11]. The Wess-Zumino action obtained in this way is an element of a trivial class of Chevalley-Eilenberg cohomology on the nondegenerate superspace [9] which is manifestly SUSY invariant. In the AdS space the bilinear Wess-Zumino (WZ) action for the usual GS superstring could be written formally, but the bilinear WZ action contains the AdS radius parameter then the WZ term vanishes in the flat limit. In order to get the bilinear WZ action, the nondegenerate superalgebra which includes the new spinor charge must be constructed.

In this nondegenerate approach no topological term appears in the SUSY algebra. Information of topological charge is contained in the anti-commutator of fermionic constraints [12, 13] both for the usual GS superstring and for the “nondegenerate” superstring. In the anti-commutator of the fermionic constraints the topological term makes a half of fermionic constraints to be first class which generate kappa-symmetry. In the nondegenerate superstring action there appear additional fermionic variables associated with new fermionic charges. In flat space the Lagrangian of the nondegenerate superstring coincides with the Lagrangian of the usual GS superstring up to surface term [11]. Since the new variables appear in the surface term they lead to additional constraints which are almost trivially solved and the constraint set is reduced to the usual one of the GS superstring [14]. There are, however, subtle differences in their SUSY algebra and constraint algebra caused from different canonical variables. Arbitrary p-brane can be also described in the nondegenerate superalgebra approach in the flat background [15, 16, 17, 18].

The equivalence between the usual GS superstring and “the nondegenerate superstring” is no longer hold in general background such as the AdS space, because the
fermionic charge is no more a center. Criteria which were used for constructing the GS superstring actions in AdS spaces \cite{4, 5} are followings \cite{4}:

* the standard $\sigma$-model bosonic part with AdS structure as a target space
* the global AdS supersymmetry
* the local $\kappa$-symmetry
* reducing to the standard Green-Schwarz superstring action in the flat limit

In this paper we will construct a superstring action satisfying these criteria based on a nondegenerate super-AdS algebra \footnote{A superstring action in N=2 AdS$_2 \times S^2$ was suggested as the nondegenerate superalgebra form in the reference \cite{6}. The existence of corresponding nondegenerate superalgebra itself is a nontrivial problem. We thank Nathan Berkovits for useful discussions about these issues.}. In order to make concrete calculation simpler we focus on the AdS space in this paper rather than realistic AdS$\times S$ target spaces. We will examine the difference from the usual GS superstring, such as relation of two supercharges, anti-commutators of the fermionic constraints, and dynamical modes.

This paper is organized as follows. In section 2, we introduce the nondegenerate super-AdS algebra and we give scale dimensions to generators in such a way that this algebra reduces into the nondegenerate supertranslation algebra in the flat limit. In section 3, we calculate Maurer-Cartan 1-forms of a coset $G/H$ where $G =$ (the nondegenerate super-AdS group) and $H =$ (Lorentz group). In section 4, by using these expressions we construct a superstring action in the nondegenerate AdS space, and we examine differences between this nondegenerate superstring approach and the usual GS superstring approach both for the flat limit case and for the AdS case.

## 2 Nondegenerate Super-Anti-de Sitter Algebra

We show that the following algebra is given as the nondegenerate super-anti de Sitter algebra in $d$ dimensions:

\[
[J_{mn}, J_{lk}] = \eta_{[k[l} J_{n]\,]l]} \\
[J_{mn}, J_{lk}] = \eta_{[k[l} J_{n]\,]l]} \\
[P_m, J_{lk}] = \eta_{m[l} P_{k]} \\
[P_m, J_{lk}] = \eta_{m[l} P_{k]} \\
[P_m, P_l] = J_{ml} \\
[P_m, P_l] = J_{ml} \\
[Z, J_{mn}] = -\frac{1}{2} Z \Gamma_{mn} \\
[Z, J_{mn}] = -\frac{1}{2} Z \Gamma_{mn} \\
{\{Q, Q\}} = -2iCP \\
{\{Q, Q\}} = -2iCP \\
{\{Q, Z\}} = C \Gamma^{mn} J_{mn} \\
{\{Q, Z\}} = C \Gamma^{mn} J_{mn} \\
{\{Z, Z\}} = 2iCP \\
{\{Z, Z\}} = 2iCP \\
{\{Z, P_m\}} = \frac{i}{2} Q \Gamma_m \\
{\{Z, P_m\}} = \frac{i}{2} Q \Gamma_m \\
\]

where $J_{mn}$, $P_m$ are AdS$_d$ generators, and $Q_\alpha$ and $Z_\alpha$ are $d$-dimensional Majorana spinor generators. This algebra exists for $d = 3$ and its bosonic symmetry group is $SO(2, 2)$ where the Majorana representation exists with the anti-symmetric charge conjugation matrix, $C^{\sigma}_{\sigma'} = -C_{\sigma'}^\sigma$. 

\footnote{\eta_{[k[l} J_{n]\,]l]} = \eta_{km} J_{nt} - \eta_{kn} J_{mt} - \eta_{lm} J_{nk} + \eta_{ln} J_{mk}$
A flat limit is realized by giving following scale dimensions to the generators as
\[ P_m \to R P_m, \quad Q_\alpha \to R^{1/2} Q_\alpha, \quad Z_\alpha \to R^{3/2} Z_\alpha \] and \[ J_{mn} \to J_{mn} \] (2.2)
and taking \( R \to \infty \). In this limit the algebra (2.1) becomes
\[ \{ Q, Q \} = -2iC P, \quad [ Q, P_m ] = -\frac{i}{2} Z \Gamma_m \] (2.3)
with keeping correct Lorentz spins for the generators
\[ [ J_{mn}, J_{lk} ] = \eta_{[k||m} J_{n||l]} , \quad [ P_m, J_{lk} ] = \eta_{m||l} P_k \] (2.4)
\[ [ Q, J_{mn} ] = -\frac{1}{2} Q \Gamma_{mn} , \quad [ Z, J_{mn} ] = -\frac{1}{2} Z \Gamma_{mn} . \]

\( Z_\alpha \) coincides with fermionic central charges in the supertranslation algebra (2.3) introduced by Green [7].

The algebra (2.1) is also written in a \( SO(d-1,2) \) covariant notation as
\[ [ J_{MN}, J_{LK} ] = \eta_{[K||M} J_{N||L]} , \quad [ Q, J_{MN} ] = -\frac{1}{2} Q \gamma_{MN} , \quad \{ Q, Q \} = C \gamma_{MN} J_{MN} , \] (2.5)
where indices \( M \) run \( M = \{ m, d \} = \{ 0, 1, \ldots, d-1, d \} \) and
\[ J_{MN} = \{ J_{mn}, J_{md} = P_m \} , \quad Q = \{ Q P_+ = Q, \quad Q P_- = Z \} \]
\[ P_\pm = \frac{1}{2}(1 \pm i \gamma_d) . \] (2.6)

3 Maurer-Cartan 1-forms

In this section we construct left invariant one forms of the coset group \( G/H \) with \( G=( \text{the nondegenerate super AdS group (2.1)} ) \) and \( H=( \text{Lorentz group} ) \). An element of the coset is parameterized as
\[ g = g_Z g_P g_Q = e^{\xi_\alpha Z_\alpha} e^{X^m P_m} e^{\theta^\alpha Q_\alpha} . \] (3.1)
Maurer-Cartan 1-forms are defined as
\[ \Omega = g^{-1} dg = L^A(X,\theta,\xi) T_A = L^m P_m + \frac{1}{2} L^m_{J, J mn} + L^\alpha Q_\alpha + L^\alpha_Z Z_\alpha \] (3.2)
and they satisfy the Maurer-Cartan equation \( d \Omega = -\Omega^2 \) whose components are
\[ dL^m = L^l_{J, J l} L^m + i \bar{\Gamma}^m_\xi L - \frac{i}{R^2} \bar{L} Z \Gamma^m L_Z \]
\[ dL^m_{J, J mn} = -\frac{1}{R^2} L^n L^m - L^n_{J, J n} L^m_\xi L - \frac{2}{R^2} \bar{\Gamma}^m L Z \] (3.3)
\[ dL^\alpha = -\frac{1}{4} L^m_{J, J mn} \Gamma mn L + \frac{i}{2R^2} L^m \Gamma_m L_Z \]
\[ dL^\alpha_Z = -\frac{1}{4} L^m_{J, J mn} \Gamma mn L Z - \frac{i}{2} L^m \Gamma_m L \]
with AdS radius or equivalently scaling parameter $R$.

The 2-form potential and the 3-form field strength for a string are given by

$$B = 2iL\mathcal{L}_Z, \quad dB = \mathcal{I}\mathcal{L} + \frac{1}{R^2} \mathcal{L}_Z$$

using with (3.3).

Next we will obtain expression of MC 1-forms of (2.1) where the scale parameter $R$ is not included. Although the parametrization (3.1) leads following three kinds of 1-forms $L^A, \mathcal{L}^A, \ell^A$

$$g_Z^{-1} dg = l^A(\xi)T_A \quad g_P^{-1} g_Q^{-1} dg_P = \mathcal{L} = \mathcal{L}^A(X, \xi)T_A \quad g_Q^{-1} dg_Q = L = L^A(X, \theta, \xi)T_A \ ,$$

the main structure of the total MC 1-forms is governed by $l^A$;

$$g_Q^{-1} dg_Q = l^A(\theta)T_A = d\theta^\alpha (l^A)_\alpha T_A \ .$$

The MC 1-forms are obtained as

$$(L)^m = (L_P)^m + (l_P)^m D\theta^\alpha + \frac{1}{8} (\partial \Gamma^m \phi^{-4} \Gamma_m \theta) (l_J)^m (\Gamma_2 D')^\alpha + 2i (\partial \Gamma^m l_Q)_\alpha (L_Q)^\alpha + i (\partial \Gamma^m \gamma_2 \phi^{-4} l_Z)_\alpha (\Gamma_2 \mathcal{L}_Z)^\alpha$$

$$(L_J)^m = (L_J)^m + (l_J)^m D\theta^\alpha - \frac{1}{8} (\partial \Gamma^m \phi^{-4} \Gamma_2 \gamma_1 l_P) (D')^\alpha - 2 (\partial \Gamma^m l_2)_\alpha (L_Q)^\alpha - (\partial \Gamma^m \gamma_1 l_Q)_\alpha (\phi^{-4} \Gamma_2 \mathcal{L}_Z)^\alpha$$

$$(L)^a = L_Q^a + (l_Q)^a D\theta^\beta + \frac{1}{8} (\gamma_2 \phi^{-4} l_Z)^a (\Gamma_2 D')^\beta - \frac{1}{4} (\Gamma_m \theta)^a (l_P)^m (\Gamma_2 \mathcal{L}_Z)^\beta + \frac{1}{4} (\gamma_1 \Gamma_m \theta)^a (l_J)^m (\phi^{-4} \Gamma_2 \mathcal{L}_Z)^\beta$$

where the covariant derivatives on $\theta$ are

$$D\theta = d\theta + \frac{1}{4} (L_J)^m \Gamma_m \theta, \quad D'\theta = (L_P)^m \Gamma_m \theta \ .$$

In the above expressions $l^A, \ell^A, \mathcal{L}^A$ are defined respectively as

$$\left\{\begin{array}{l}
(l_P)^m = i(\partial \Gamma^m (\cosh \phi - \cos \phi) \phi^{-2} d\theta \\
(l_J)^m = -\partial \Gamma^m \gamma_1 (\cosh \phi + \cos \phi - 2) \phi^{-4} d\theta \\
(l_Q)^a = \frac{1}{2} ((\sinh \phi + \sin \phi) \phi^{-1})^a d\theta^\beta \\
(l_Z)^a = \frac{1}{2} (\gamma_1 (\sinh \phi - \sin \phi) \phi^{-3})^a d\theta^\beta \\
(\phi^4)^\beta = \frac{1}{2} (\gamma_2)_{\gamma}^{a} (\gamma_1)^{\gamma}_{\beta} , \quad (\gamma_1)^{a}_{\gamma} = (\Gamma^m \theta)^a (\partial \Gamma_m)_{\gamma} , \quad (\gamma_2)^{a}_{\beta} = (\Gamma^m \theta)^a (\partial \Gamma_m)_{\beta} \ .
\end{array}\right.$$
A superstring action can be given in the following form using \( MC \) 1-forms \( \phi, \phi' \) and \( L \) functions and

\[
\phi^4 \gamma = \frac{1}{2} (\gamma_2^\alpha (\gamma_1^\alpha \gamma_2^\beta = (\Gamma^m \xi)^\alpha (\xi \Gamma_m)^\beta \), (\gamma_2^2)^\alpha = (\Gamma^{mn} \xi)^\alpha (\xi \Gamma_{mn})^\beta
\]

\[
\{ \begin{array}{l}
(\mathcal{L}_P)^m = (\sinh \sqrt{T}/\sqrt{T})_n (dX^n - X_i (l'_i)^n) + (\cosh \sqrt{T})_n (l'_p)^m \\
(\mathcal{L}_J)^{mn} = 2X^m ((1 - \cosh \sqrt{T}/\sqrt{T})(dX - X l'_j))^n + (l'_j)^{mn} \\
(\mathcal{L}_Q)^\alpha = \cosh(X/2)_\beta (l'_Q)^\beta + i \sinh(X/2)_\beta (l'_Z)^\beta \\
(\mathcal{L}_Z)^\alpha = \cosh(X/2)_\beta (l'_Z)^\beta - i \sinh(X/2)_\beta (l'_Q)^\beta
\end{array}
\]

\( \gamma^{mn} = \eta^{mn} X^2 - X^m X^n \)

where \( l^A \) and \( l'^A \) are written as four-module functions \( \bar{A} \) rather than usual two-module functions and \( \mathcal{L}_P \) and \( \mathcal{L}_J \) are usual vielbein and connection respectively.

4 Superstring in the nondegenerate super-AdS space

The nondegenerate algebra \( T_A = (Q, P, Z) \) given by \( (2.1) \) enables to introduce nondegenerate group metric \( \eta^{mn} \)

\[
tr(T_A T_B) = \begin{pmatrix}
0 & 0 & C_{\alpha \beta} \\
0 & \frac{1}{2} \gamma_{mn} & 0 \\
-C_{\alpha \beta} & 0 & 0
\end{pmatrix}.
\]

A superstring action can be given in the following form using with \( MC \) 1-forms \( g^{-1}dg = L = d\sigma^\mu L_\mu = d\sigma^\mu L_\mu^A T_A \) obtained as \( \{3.7\} \)

\[
S = S_0 + S_{WZ} \\
S_0 = -T \text{ tr } \int d^2 \sigma [h^{\mu \nu} L_\mu L_\nu] = -\frac{T}{2} \int d^2 \sigma h^{\mu \nu} L_\mu^m L_{\nu m} \]

\[
S_{WZ} = T \text{ tr } \int d^2 \sigma [\epsilon^{\mu \nu} L_\mu L_\nu] = 2T \int d^2 \sigma e^{\mu \nu} \tilde{L}_\mu L_{\nu} \]

\[
\sum_{n=0}^{\phi^{4n}} \frac{\phi^{4n}}{(4n)!} = \frac{1}{2} (cosh \phi + cos \phi), \quad \sum_{n=0}^{\phi^{4n+1}} \frac{\phi^{4n+1}}{(4n + 1)!} = \frac{1}{2} (sinh \phi + sin \phi)
\]

\[
\sum_{n=0}^{\phi^{4n+2}} \frac{\phi^{4n+2}}{(4n + 2)!} = \frac{1}{2} (cosh \phi - cos \phi), \quad \sum_{n=0}^{\phi^{4n+3}} \frac{\phi^{4n+3}}{(4n + 3)!} = \frac{1}{2} (sinh \phi - sin \phi)
\]
with \( h^{\mu \nu} = \sqrt{-g} g^{\mu \nu} \) and \( \epsilon^{01} = 1 \). Following to (2.2) these variables are scaled as \( X \rightarrow (1/R)X, \theta \rightarrow (1/\sqrt R)\theta \) and \( \xi \rightarrow (1/\sqrt R^3)\xi \) and also \( L \rightarrow (1/R)L \), \( L \rightarrow (1/\sqrt R)L \) and \( L_Z \rightarrow (1/\sqrt R^3)L_Z \). In the limit \( R \rightarrow \infty \) they reduce to the MC 1-forms in the flat space:

\[
\begin{align*}
(L)_m^\mu &= \partial_\mu X - i \bar{\theta} \Gamma \partial_\mu \theta + \frac{1}{R^2} \left( \frac{L}{\sqrt{m}} \partial_\mu X_n + i \bar{\xi} \Gamma^m \partial_\mu \xi + \bar{\theta} \Gamma^m \bar{X} \partial_\mu \xi + \frac{i}{4} \bar{\theta} \Gamma^m l \partial_\mu X_i \right) + o(\frac{1}{R^3}) \\
(L)_n^\mu &= \partial_\mu \theta + \frac{1}{R^2} \left( \frac{L}{\sqrt{m}} \partial_\mu \xi - \frac{1}{4} X^m \partial_\mu X^n \Gamma_{mn} \theta + \frac{1}{4} \bar{\theta} \Gamma_2 \partial_\mu \xi \right) + o(\frac{1}{R^3}) \\
(L_Z)_p^\mu &= \partial_\mu \xi + \frac{i}{2} (\Gamma_m \theta)^\alpha (\partial_\mu X^m - \frac{i}{2} \bar{\theta} \Gamma^m \partial_\mu \theta) + \frac{1}{R^2} \left( -\frac{1}{2} (\xi \Gamma \partial_\mu \xi) \cdot \Gamma \theta + \frac{1}{2} (\bar{X}^2) \partial_\mu \xi \Gamma + \frac{i}{4} \bar{\theta} \Gamma \partial_\mu X \cdot \Gamma \theta \right) + o(\frac{1}{R^3})
\end{align*}
\]

(4.4)

where \( \theta^3 \) and \( \xi^3 \) vanish in 3-dimensional AdS space.

The supertransformation rules which leave (4.4) invariant up to Lorentz rotation are determined independently of the action. They are calculated by performing the infinitesimal supertransformation on the coset element

\[
g \rightarrow e^{eQ} g = g'h \quad h \in H
\]

\[
\begin{align*}
g &= e^{\xi Z} e^{X.P} e^{\theta Q} \\
g' &= e^{(\xi+\delta \xi) Z} e^{(X+\delta X).P} e^{(\theta+\delta \theta) Q}
\end{align*}
\]

(4.5)

The obtained N=1 AdS_3 supertransformation rules for large \( R \) are

\[
\begin{align*}
\delta \xi^m &= -i \bar{\theta} \Gamma^m \epsilon + \frac{1}{R^2} \left( -\frac{X^2}{8} i \bar{\theta} \Gamma^m \epsilon + \frac{X^2}{6} \partial_\mu \epsilon + \frac{X^2}{2} \bar{\theta} \partial_\mu \epsilon \right) + o(\frac{1}{R^3}) \\
\delta \theta &= \epsilon + \frac{1}{R^2} \left( -\frac{X^2}{8} \epsilon + \frac{X}{2} (\Gamma^m \epsilon \Gamma_m \theta) \right) + o(\frac{1}{R^3}) \\
\delta \xi &= -\frac{X^2}{8} \epsilon \theta - \frac{X^2}{6} \bar{\theta} \Gamma \theta + \frac{1}{R^2} \left( -\frac{X^2}{2} \bar{\xi} \Gamma \xi \right) + o(\frac{1}{R^3})
\end{align*}
\]

(4.6)

The supercharges are obtained independently from the form of the Wess-Zumino action in this approach:

\[
Q \epsilon = \int (\bar{\epsilon} \delta \theta + p \delta \xi + \pi \xi \delta \xi)
\]

(4.7)

where \((X, \theta, \xi)\) are canonical variables and \((p, \bar{\zeta}, \pi \xi)\) are their conjugates. By construction they satisfy the following superalgebra (2.1)

\[
\{Q, Q\} = -2iC \mathcal{P}
\]

(4.8)

In the nondegenerate approach, the Wess-Zumino action does not affect the supercharges and the superalgebra, but does affect the fermionic constraints and their anti-commutator. The canonical conjugates scale are defined as

\[
\begin{align*}
\zeta_\alpha &= \frac{\delta^\epsilon S}{\delta \theta^\alpha} = \frac{\delta S_0}{\delta L_0} \cdot \partial \bar{\theta}^\alpha + \frac{\delta S_{WZ}}{\delta \bar{\theta}^\alpha} \\
p_m &= \frac{\delta^\epsilon S}{\delta X^m} = \frac{\delta S_0}{\delta L_0} \cdot \partial X^m + \frac{\delta S_{WZ}}{\delta X^m}
\end{align*}
\]

(4.9)

(4.10)
where $\delta^r$ denotes the right derivative. From the definition (4.9) fermionic constraints are written as

$$ F = \sum_N \frac{1}{R^{2N}} F^{(N)} = 0 \quad (4.11) $$

which satisfy the anti-commutator of the fermionic constraints

$$ \{F_\alpha(\sigma), F_\beta(\sigma')\} = 2i(C \gamma)_{\alpha\beta} + \cdots \quad (4.12) $$

and the terms “$\cdots$” will be calculated in the following sections. The fermionic local constraints are SUSY invariant.

### 4.1 Flat case ($1/R \to 0$)

In the flat case the Wess-Zumino Lagrangian for nondegenerate superspace approach is given by

$$ \mathcal{L}_{WZ} = \mathcal{L}_{WZ,GS} + \Delta \mathcal{L} \quad (4.13) $$

$$ \mathcal{L}_{WZ,GS} = T \epsilon^{\mu\nu} (i \partial_\mu \bar{\theta} \Gamma \theta \cdot \partial_\nu X - \frac{1}{3} \partial_\mu \bar{\theta} \Gamma \theta \cdot \partial_\nu \bar{\theta} \Gamma \theta) $$

$$ \Delta \mathcal{L}_{WZ} = 2T \epsilon^{\mu\nu} \partial_\mu \bar{\theta} \partial_\nu \xi , $$

where $\mathcal{L}_{WZ,GS}$ is the Wess-Zumino Lagrangian for the Green-Schwarz superstring and $\Delta \mathcal{L}_{WZ}$ is rewritten in total derivative form. Since $\xi$-dependence is only in the surface term, obviously $\xi$ is not dynamical. However $\xi$ is transformed in such a way that the Wess-Zumino action is invariant under the SUSY transformations. The supercharges and the superalgebra are given as

$$ Q\epsilon = \int \mathrm{d}\sigma \left( \zeta - i \bar{\theta} \gamma^\mu \partial_\mu X - \frac{1}{2} \bar{\theta} \Gamma \theta \cdot \bar{\theta} \Gamma X \right) \epsilon \quad (4.14) $$

$$ P_m = \int p_m \ , \ Z_\alpha = \int \pi_{\xi,\alpha} $$

$$ \{Q, Q\} = -2iC \bar{\gamma} \ , \ [Q, P_m] = -\frac{i}{2} Z \Gamma_m . \quad (4.15) $$

In flat case $Z$ is a center, $\{Z, Z\} = \{Z, Q\} = \{Z, P\} = 0$.

On the other hand, the usual GS supercharges and their SUSY algebra are given by

$$ Q_{GS}\epsilon = \int \left( p \delta_\epsilon X + \zeta \delta_\epsilon \theta - U^0_{WZ} \right) \ , \ \delta_\epsilon \mathcal{L}_{WZ,GS} = \partial_\mu U^\mu_{WZ} \quad (4.16) $$

$$ = \int \left( \zeta - i \bar{\theta} \gamma^\mu - T( i \bar{\theta} \gamma^\mu \cdot \bar{\theta} \Gamma ) \right) \epsilon $$

$$ \Sigma_m = T \int X'_m $$

$$ \{Q_{GS}, Q_{GS}\} = -2iC(\bar{\gamma} + \Sigma) \ , \ [Q_{GS}, P_m] = 0 . \quad (4.17) $$
The presence of “ξ” leads to the difference of superalgebras (1.15) and (1.17). Especially the topological term appears in the usual GS superalgebra but not in the nondegenerate superalgebra.

In the nondegenerate approach, the existence of the Wess-Zumino action does not affect the supercharges and the superalgebra, but does affect the fermionic constraints and their anti-commutator. The fermionic constraint set and its algebra are

\[ F = F^{(0)} = (\z + i\bar{\theta}p) + T(\bar{\theta}T' \cdot \bar{\Gamma} - 2\xi' + i\bar{\theta}X') = 0 \] (4.18)
\[ F_Z = \pi_\xi + 2T\bar{\theta}' = 0 \] (4.19)
\[ \{F_\alpha(\sigma), F_\beta(\sigma')\} = 2iC \Gamma \cdot (\bar{p} + TL^{(0)}_1) \delta(\sigma - \sigma') \] (4.20)
\[ \{F_{Z,\alpha}(\sigma), F_{Z,\beta}(\sigma')\} = 0 = \{F_\alpha(\sigma), F_{Z,\beta}(\sigma')\} \] (4.21)

where \( \bar{p} \) is SUSY invariant combination given by

\[ \bar{p} = p + iT\bar{\theta}' \Gamma \] (4.22)

and \( L^{(0)}_1 \) is given by (4.4). The anti-commutator of fermionic constraints (4.20) is the same as one of the Green-Schwarz.

The relation of two supercharges becomes clear by imposing constraints in (4.18) \( F_Z = 0 \rightarrow \pi_\xi = -2T\bar{\theta}' \)

\[ Q\epsilon|_{F_Z=0} = Q_{GS}\epsilon - 2T \int \partial(\bar{\theta}\delta_\epsilon \xi) . \] (4.23)

For a case of an open string the surface term does not vanish, \( \bar{\theta}X\epsilon|_{\sigma=0} \neq 0 \), and it causes breaking of translational invariance giving the \( Z \) charge. For cases such as a closed string and a string with the periodic boundary condition the surface term and the fermionic charge \( Z \) vanish.

Topological charge is an important issue. Although the SUSY algebra does not contain a topological term, the action and the local fermionic constraints contain the topological term information. If we impose \( F = 0 \) of (4.18) in supercharges

\[ Q\epsilon|_{F=0} = Q\epsilon - \int F\epsilon = \int (2p\delta_\epsilon X + \pi_\xi \delta_\epsilon \xi - F_{WZ}\epsilon) \] (4.24)

where \( F_{WZ} \) is \( T \)-dependent part (i.e. the Wess-Zumino action dependent part) of \( F \), they produce the topological term in their bracket

\[ \{Q|_{F=0}, Q|_{F=0}\} = 2iC\z . \] (4.25)

This is a result of the fact that \( \{Q, F\} = 0 \) and the anti-commutator of the fermionic constraints carry the topological information, \( \{\int F, \int F\} = 2iC(\bar{P} + \z) \).

It is stressed that \( Q|_{F_Z=0} \) and \( Q|_{F=0} \) are not conserved Noether charges of the system. Second class parts of \( F_Z = 0 \) and \( F = 0 \) play a role of leading to relations (1.23) and (4.23).
4.2 AdS case (finite $R$)

In this section we calculate the SUSY algebra and the anti-commutator of the fermionic constraints in next to leading order of $1/R$. Then we generalize this result for finite $R$ expression, comparing with the usual GS superstring case. It is straightforward to confirm the SUSY algebra (4.15) in the next to leading order by using (4.6). Next we calculate the anti-commutator of the fermionic constraints. The fermionic constraints are obtained as

\[ F(0) + \frac{1}{R^2} F(1) = 0 \]  
\[ F(1) = -i \frac{p_m}{3!} \gamma^{mn} \bar{\theta} \Gamma_n + T \left\{ i \bar{\theta} \Gamma_m \left( \frac{i}{3} \gamma^{mn} \bar{\theta} \Gamma_n \theta' + \frac{1}{6} \gamma^{mn} X'_n \right) \right. 
+ \left. \frac{1}{2} X^m \xi^i \theta + i \bar{\xi} \gamma^{mn} \theta' + \frac{2}{3} \bar{\theta} \Gamma^m X \xi' - \frac{1}{4} \bar{\xi} X^2 \right\} \]

where $F(0)$ is given in (4.18). The anti-commutator of the fermionic constraints in next to leading order is

\[ \{ F_\alpha(\sigma), F_\beta(\sigma') \} = 2i (C \Gamma)_{\alpha \beta} \cdot (\tilde{p} + TL_1) \delta (\sigma - \sigma') \]  

with $\tilde{p}$ given by

\[ \tilde{p}_m = p_m + Ti \bar{\theta} \Gamma_m \theta 
+ \frac{1}{R^2} \left[ - \frac{1}{3!} \gamma^{mn} p^n - \frac{i}{8} \bar{\theta} \Gamma_{mn} \theta X^np \right] 
+ T \left( - \frac{i}{8} \bar{\theta} \Gamma_{mn} \theta X^n X^m + \frac{i}{3} \gamma_{mn} \bar{\theta} \Gamma^n \theta' + \frac{1}{4} \bar{\theta} \Gamma_m \xi' \right) \]

(4.28)

and $L_1 = L_1^{(0)} + \frac{1}{R^2} L_1^{(1)}$ given in (4.4). $\tilde{p}$ and $L_1$ are SUSY invariant combinations. From the SUSY invariance the form of the anti-commutator of the fermionic constraints, (4.27) should be hold in all order of the $1/R$ expansion. The usual GS superstring case the same form of the anti-commutator of the fermionic constraints (4.27) is expected, although the concrete expression of the fermionic constraints $F = 0$ is completely different. This anti-commutator of the fermionic constraints (4.27) guarantees the $\kappa$-symmetry of the system for both the GS superstring and the nondegenerate superstring: The rank of the matrix in the right hand side of (4.27) is half of its dimension. Since $(\tilde{p} + TL_1)$ is light-like vector, $(\tilde{p} + TL_1)^2 \approx 0$ by using with the diffeomorphism constraints, $(\tilde{p} + TL_1)_m \Gamma^m$ is a projection operator. Projected constraints $F \Gamma \cdot (\tilde{p} + TL_1) = 0$ are first class constraints generating $\kappa$-symmetry.

Analogous to the flat case (4.25), the global SUSY algebra will produce a brane charge if the fermionic constraints $F = 0$ of (4.20) is used in $Q$,

\[ \{ (Q|_{F=0})_\alpha, (Q|_{F=0})_\beta \} = -2i T(C \Gamma)_{\alpha \beta} \cdot \int d\sigma L_1 \]
\[ = -2i T(C \Gamma)_{\alpha \beta} \cdot \int d\sigma \sinh \sqrt{\gamma} \cdot \gamma^{-1/2} \cdot X' \]  

(4.29)
The BPS states are eigenstates of this brane charge.

For finite $R$, $Z$ is not center any more. $Q$ and $Z$ satisfy almost same algebra except their opposite sign of the right hand side in (2.1). The opposite sign of $\{Q, Q\}$ and $\{Z, Z\}$ comes from the dimensional reduction from the larger algebra (2.3) and (2.6). Since the square of the supercharges leads to positivity of the energy, the new supercharge $Z$ looks the wrong sign for unitarity. In order to make the algebras in (2.1) to be consistent, $Z$ must be anti-hermite: The algebras are represented on physical states as $[Q, Q]_+ = 2E = -[Z, Z]_+$. If $Q$ is hermite then the algebra of two $Q$’s is consistent with positivity of the energy $\langle \psi|QQ|\psi \rangle = \sum_n |\langle \psi|Q|n \rangle|^2 = E_\psi \langle \psi|\psi \rangle \geq 0$. On the other hand, positivity requires $0 \leq E_\psi \langle \psi|\psi \rangle = \sum_n |\langle \psi|Z|n \rangle|^2$, and the algebra of $Z$ leads to $-\langle \psi|ZZ|\psi \rangle = -\sum_n \langle \psi|Z|n \rangle \langle n|Z|\psi \rangle = -\sum_n \langle \psi|Z|n \rangle \langle \psi|Z^\dagger|n \rangle^*$, therefore $Z^\dagger = -Z$. The anti-hermiticity of $Z$ is consistent with (2.3) because there can be pure real Majorana representation in 3-dimension. It requires replacing replacing $\xi \to i\xi$ and $\bar{\xi} = \xi^T C \to i\bar{\xi}$.

Now for a case with $R = 1$, $\theta$ and $\xi$ are treated as same and generate $N=2$ supersymmetry. As expected the $\kappa$ symmetry for $\xi$ is guaranteed by the fermionic constraints for $\xi, F_Z = 0$:

$$\{F_{Z,\alpha}(\sigma), F_{Z,\beta}(\sigma')\} = -2i(CT)_{\alpha\beta} \cdot (\bar{\rho} - \mathbf{T}L_1) \delta(\sigma - \sigma') \quad .$$

Therefore dynamical modes of the nondegenerate AdS superstring model is twice of the usual Green-Schwarz’s one. For general finite $R$ case is obtained analogously by rescaling.

It is curious that dynamics of $\theta$ and $\xi$ look the same in a case for AdS with $R = 1$, but they are not the same in the flat limit. In order to see the difference of their dynamics, let us compare the equation of motion in the next to leading order:

$$\frac{\delta S}{\delta \theta} = -2 \left( \left( 1 + \frac{1}{R^2} Y \right) \partial_\mu X \right)^+ (g^{\mu\nu} - \epsilon^{\mu\nu}) i \partial_\nu \bar{\theta} \Gamma^- \right.$$

$$+ \partial_\nu \left\{ \left( 1 + \frac{1}{R^2} Y \right) \partial_\mu X \right)^+ (g^{\mu\nu} - \epsilon^{\mu\nu}) \} i \partial_\nu \Gamma^- \right.$$}

$$+ \frac{1}{R^2} \left[ 2(g^{\mu\nu} - \epsilon^{\mu\nu})(\xi \Gamma^+ \partial_\nu \xi) \partial_\mu \theta \Gamma^- + \partial_\mu \left\{ (g^{\mu\nu} - \epsilon^{\mu\nu}) \bar{\xi} \Gamma^+ \partial_\nu \bar{\xi} \right\} \partial_\nu \Gamma^- \right.$$}

$$- \frac{1}{2} (g^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\nu X^2 \partial_\nu \bar{\xi} \right\} = 0 \quad .$$

$$\frac{\delta S}{\delta \xi} = -\frac{1}{R^2} \left[ 2(g^{\mu\nu} + \epsilon^{\mu\nu})(\partial_\mu X^- - i \partial_\nu \partial_\mu \theta) \partial_\nu \bar{\xi} \Gamma^+ \right.$$}

$$+ \partial_\nu \left\{ (g^{\mu\nu} + \epsilon^{\mu\nu})(\partial_\mu X^- - i \partial_\nu \partial_\mu \theta) \right\} i \xi \Gamma^+ \right.$$}

$$+ \frac{1}{2} \partial_\nu \left\{ (g^{\mu\nu} + \epsilon^{\mu\nu}) \partial_\mu X^2 \partial_\nu \bar{\theta} \right\} = 0 \quad ,$$

where these fermionic variables are gauge fixed as $\Gamma^+ \theta = \Gamma^- \xi = 0$ for simple commutation. In the flat limit $\theta$ satisfies $(\partial_\theta - \partial_\bar{\theta}) \theta = 0$ and the equation for $\xi$ does not exist as expected. In the next to leading order, the AdS effect appears in the equation for $\theta$, and the equation of motion for $\xi$ appears then $\xi$ becomes dynamical.
5 Summary and discussions

We have extended the nondegenerate supertranslation algebra to the one in AdS space. In flat space the fermionic central extension is the way to make a superspace to be non-degenerate, but it is not so in AdS space. Introducing partner supercharges $Z$ is essential to make the superspace to be nondegenerate irrespective of its central property as we have shown in this paper. The algebra (2.1) is almost unique which contain minimal set of the super AdS generators plus additional supercharges and reproduces the fermionic central extended form in flat limit. Further extension, such as extended SUSY and other target spaces like AdS×S, of the nondegenerate super-AdS algebra are non-trivial problems. The difficulty is that more fermionic charges require more bosonic charges which can not be identified with spacetime symmetry generators nor R-symmetry generators. In other words nondegenerate SUSY partners may be just elements in the large multiplet accompanied with brane charges [15, 16, 14, 17, 18]. Interpretation of $Z$ as the fermionic brane charge recently examined [19] may be essential.

We examined the new superstring action based on this nondegenerate super-AdS algebra: The difference of the nondegenerate approach from the usual GS superstring is clarified. The supercharge of the nondegenerate approach is related to the ones of GS superstring as (4.23), and this surface term leads to difference in the global charge algebras (4.15) and (4.17). While the fermionic charge $Z$ is center in the flat space, $Z$ generates another SUSY in the AdS space as discussed in section 4.2. The equation (4.29) gives concrete expression of the brane charge in AdS space which should appear in the super-algebra of the usual GS superstring in AdS. The origin of the nontrivial relation between our action and the Green-Schwarz action is the ambiguity of the new fermion’s scaling weight in the AdS space where the scale parameter exists. For example;

(i) Conventional choice

| Charges in AdS space | scaling weight | Charges in flat space |
|----------------------|----------------|-----------------------|
| $P_m$                | 1              | $P_m$ ...survived     |
| $Q_\alpha$           | 1/2            | $Q_\alpha$ ...survived|
| $Z_\alpha$           | 1/2            | $Z_\alpha$ ...survived|

Conventional choice of the scaling for spinors gives the N=2 GS variables, but the bilinear form Wess-Zumino action vanishes in the flat limit.

(ii) Our choice (2.2)

| Charges in AdS space | scaling weight | Charges in flat space |
|----------------------|----------------|-----------------------|
| $P_m$                | 1              | $P_m$ ...survived     |
| $Q_\alpha$           | 1/2            | $Q_\alpha$ ...survived|
| $Z_\alpha$           | 3/2            | $Z_\alpha$ ...center(auxiliary) |

This choice gives the N=1 GS variables plus an auxiliary spinor in the flat limit, and a simple bilinear Wess-Zimino action can be constructed even in the flat limit.
The nondegenerate superstring should be useful for quantum theory, since fermionic states have their nondegenerate norm keeping canonical pairs $\theta$ and $\xi$. In the AdS space the new fermion $\xi$ is dynamical as same as $\theta$ and both modes contribute to the Hamiltonian. In the flat limit, although $\xi$ dependence is just in a surface term in the classical action, there exist the quantum states of $\xi$ making nondegenerate metric for original fermion $\theta$-only Hamiltonian hides $\xi$ dependence-. The “nondegenerate” approach could give the formulation of the random superstring \[1\] which is worth to translate into the continuum quantization theory. The “nondegenerate” approach manifests symmetrical structure of the system in the continuum action too, so it will be also useful for the continuum quantization. Further studies are required to clarify these issues.

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