ON ADMISSIBLE STRATEGIES IN ROBUST UTILITY MAXIMIZATION

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The existence of optimal strategy in robust utility maximization is addressed when the utility function is finite on the entire real line. A delicate problem in this case is to find a “good definition” of admissible strategies to admit an optimizer. Under certain assumptions, especially a kind of time-consistency property of the set \( \mathcal{P} \) of probabilities which describes the model uncertainty, we show that an optimal strategy is obtained in the class of those whose wealths are supermartingales under all local martingale measures having a finite generalized entropy with one of \( P \in \mathcal{P} \).

1. INTRODUCTION

This paper analyzes a qualitative aspect of the problem of robust utility maximization. Given a utility function \( U \) and a set \( \mathcal{P} \) of probabilities which describes the model uncertainty, the basic problem of this paper is to maximize the robust utility functional

\[
X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]
\]

over all terminal wealths \( x + \theta \cdot S_T = x + \int_0^T \theta dS \) of admissible strategies \( \theta \), where \( S \) is an underlying semimartingale. When \( U \) is finite only on the positive half-line, the duality theory for this problem in the spirit of [15, 16] has been studied in both quantitative and qualitative aspects (e.g. [23], [22], [8]). In the case of utility taking finite values for all \( x \in \mathbb{R} \), [18] shows the key duality, while [8] and [17] give a partial result on the existence of optimal strategy which we shall complete in this paper. See also [9] for more comprehensive reference and the background of the robust utility maximization problem.

A key subtlety intrinsic to the case of utility on \( \mathbb{R} \) is the “good definition” of admissible strategies \( \theta \), which will constitute the central theme of this paper. In this case, a universal and conceptually natural definition of admissibility is that \( \theta \cdot S \) is uniformly bounded from below by some constant, which completely determines the quantitative nature of the problem. This class, however, typically fails to admit an optimizer. On the other hand, if \( U \) is \(-\infty \) on \( \mathbb{R}_- \), the only natural (non-redundant) definition of admissibility is that the stochastic integral \( \theta \cdot S \) is bounded from below by \(-x \), and an optimal strategy is indeed obtained in this class under certain mild assumptions (see [23, 22]).

In the classical case (i.e., \( \mathcal{P} = \{P\} \), say), the question of the good definition of admissibility is closely analyzed by [21] following the observation by [7] and [14] in the case of

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exponential utility. [21] shows that a “good definition” which yields us an optimal strategy is that \( \theta \cdot S \) is a supermartingale under all local martingale measures \( Q \) which has a “finite entropy” with the physical probability \( P \). We denote the class of such \( \theta \) by \( \Theta_V(P) \) (see Section 2 for precise definitions including the meaning of “finite entropy”). Note that this class contains the usual admissible class, and the supermartingale property is consistent to the “No-Arbitrage philosophy”. Thus \( \Theta_V(P) \) is acceptably natural choice when a single physical probability is specified.

In the general robust case with \( \mathcal{P} \) containing (infinitely) many elements, [8] (see also [17] for a slight generalization) provides a partial analogue of the above result which states that, under certain stronger assumptions, an optimal strategy is obtained in the class of \( \theta \) with \( \theta \cdot S \) being a supermartingale under all local martingale measures \( Q \) having a finite entropy w.r.t. a certain element \( \hat{P} \in \mathcal{P} \) called a least favorable measure, i.e., in the class \( \Theta_V(\hat{P}) \). Here a dissatisfaction comes of course from the dependence of admissibility on \( \hat{P} \). In philosophy, \( \mathcal{P} \) is the set of candidates of real world models, and we do not know which one is true. Thus an “admissible strategy” should be universally admissible for all candidates \( P \in \mathcal{P} \). Also, the least favorable probability \( \hat{P} \) is a part of solution to the dual problem of robust utility maximization, hence the class \( \Theta_V(\hat{P}) \) is not a priori available.

In this view, a seemingly natural admissible class is \( \bigcap_{\theta \in \mathcal{P}} \Theta_V(P) \) which is universal and contains all \( \theta \) whose stochastic integrals are bounded below. Thus our central question in this paper is:

**Question 1.** Does the class \( \bigcap_{\theta \in \mathcal{P}} \Theta_V(P) \) admit an optimal strategy?

The main result (Theorem 3.2) states that this is indeed the case if (in addition to standard assumptions) the set \( \mathcal{P} \) of candidate models has a time-consistency property. We proceed as follows. The first step is to construct a so-called “optimal claim” for the abstract version of robust utility maximization, from which a candidate of optimal strategy \( \hat{\theta} \) is derived through a predictable representation argument. This part is mostly standard excepting some technicality, but we give a slightly better description of optimal claim. Note that the additional time-consistency assumption is not required at this stage. The crucial step is to verify the supermartingale property of \( \hat{\theta} \cdot S \) under all local martingale measures \( Q \) which has a finite entropy with some \( P \in \mathcal{P} \) but its entropy with \( \hat{P} \) is infinite. We shall do this by a (slight surprisingly) simple trick.

### 2. Formulation

We fix a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) as well as a filtration \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]} \) satisfying the usual conditions, where \( T \in (0, \infty) \) is a fixed time horizon. Though many probabilities on \( (\Omega, \mathcal{F}) \) will appear in the sequel, the probability \( \mathbb{P} \) plays the role of reference probability, i.e., every probabilistic notion is defined under \( \mathbb{P} \) unless other probability is explicitly specified as \( E_\mathbb{P}[] \), \( L^1(P) \) etc. In particular, the underlying asset prices \( S \) is a \( d \)-dimensional \( \mathbb{P} \)-càdlàg semimartingale, and we assume:

- (A1) \( S \) is \( \mathbb{P} \)-locally bounded.

Let \( \mathcal{P} \) be a set of probabilities \( P \ll \mathbb{P} \), which we can (and do) embed into \( L^1 \) via the mapping \( P \mapsto dP/d\mathbb{P} \). In this sense, we assume:

- (A2) \( \mathcal{P} \) is convex and \( \sigma(L^1, L^\infty) \)-compact.

We work with a utility function \( U : \mathbb{R} \rightarrow \mathbb{R} \) which we assume

- (A3) \( U \) is differentiable, strictly concave on \( \mathbb{R} \), and \( U'(\infty) = \infty, U'(-\infty) = 0 \),
and satisfies the condition of reasonable asymptotic elasticity:

\[(A4)\] \[\liminf_{x \to -\infty} \frac{x U'(x)}{U(x)} > 1 \text{ and } \limsup_{x \to \infty} \frac{x U'(x)}{U(x)} < 1.\]

The conjugate of utility function \(U\) is denoted by \(V\), i.e.,

\[(2.1)\] \[V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy), \quad y \in \mathbb{R}.\]

The assumptions (A3) and (A4) guarantee that \(V\) is a “nice” convex function (see [10], [19] for details). Using this function, we introduce a generalized entropy:

\[(2.3)\] \[V(Q|P) := \inf_{P \in \mathcal{P}} V(Q|P) < \infty.\]

Let \(\mathcal{M}_{loc}\) be the set of all local martingale measures for \(S\), i.e., probabilities \(Q \ll P\) such that \(S\) is a local martingale. We then set

\[(2.4)\] \[\mathcal{M}_V := \{Q \in \mathcal{M}_{loc} : V(Q|P) < \infty\}.\]

Generically, for any set \(Q\) of probabilities \(Q \ll P\), we denote by \(Q'\) the set of \(Q \in Q\) with \(Q \sim P\). We assume the existence of equivalent local martingale measure with finite entropy in the following sense:

\[(A5)\] \[\mathcal{M}_V := \{Q \in \mathcal{M}_V : Q \sim P\} \neq \emptyset.\]

In particular, this implies the existence of \((Q, P) \in \mathcal{M}_V \times \mathcal{P}\) such that \(Q \sim P \sim P\) and \(V(Q|P) < \infty\). See [17] for detail and other consequences of these assumptions.

Let \(L(S)\) be the totality of all \((S, P)\)-integrable \(d\)-dimensional predictable processes, \(L_0(S) := \{\theta \in L(S) : \theta_0 = 0\}\), and we denote by \(\theta \cdot S\) the stochastic integral of \(\theta \in L(S)\) w.r.t. \(S\). See e.g., [12] or [13] for more information. When the utility function is finite on the entire real line, a conceptually natural choice of \(\Theta\) is

\[(2.5)\] \[\Theta_{\Theta} := \{\theta \in L_0(S) : \theta \cdot S_0 = 0, \theta \cdot S \text{ is bounded from below}\}.\]

Then the value function of the robust utility maximization problem is given by

\[(2.6)\] \[u(x) := \sup_{\theta \in \Theta_{\Theta}} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S)], \quad x \in \mathbb{R}.\]

When we seek an optimal strategy, however, the class \(\Theta_{\Theta}\) is typically too small to admit an optimal strategy. We thus have to enlarge the admissible class. Our choice is the following.

\[(2.7)\] \[\Theta_V := \{\theta \in L_0(S) : \theta \cdot S \text{ is a } Q\text{-supermartingale}, \forall Q \in \mathcal{M}_V\}.\]

**Remark 2.1** (Another equivalent formulation). We have defined the classes \(\mathcal{M}_V\) and \(\Theta_V\) through the robust generalized entropy \(Q \mapsto V(Q|P)\). But the following equivalent formulation is sometimes useful for comparison. For each \(P \in \mathcal{P}\), we set

\[(2.8)\] \[\mathcal{M}_V(P) := \{Q \in \mathcal{M}_{loc} : V(Q|P) < \infty\},\]

\[(2.9)\] \[\Theta_V(P) := \{\theta \in L_0(S) : \theta \cdot S \text{ is a } Q\text{-supermartingale} \forall Q \in \mathcal{M}_V(P)\}.\]
When a single $P \in \mathcal{P}$ is fixed as the physical probability, the class $\Theta_V(P)$ is shown to be an appropriate domain of utility maximization in [21]. Recalling (2.3), our choices $\mathcal{M}_V$ and $\Theta_V$ are rewritten respectively as

$$\mathcal{M}_V = \bigcup_{P \in \mathcal{P}} \mathcal{M}_V(P), \quad \Theta_V = \bigcap_{P \in \mathcal{P}} \Theta_V(P).$$

Thus our definition (2.7) is consistent to what we wrote in introduction.

Under the assumptions (A1) – (A5), a duality result (Theorem 2.3 of [18]) is applicable, which states in our case that for any $\Theta$ with $\Theta_{bb} \subset \Theta \subset \Theta_V$, we have

$$u(x) = \sup_{\theta \in \Theta} \inf_{P \in \mathcal{P}} E_P[U(x + \theta \cdot S_T)] = \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} (V(\lambda Q|P) + \lambda x).$$

In particular, the value function is unchanged if we replace $\Theta_{bb}$ by the larger class $\Theta_V$. Under the same assumptions, the right hand side, the dual problem of (2.6), admits a solution $(\hat{\lambda}, \hat{Q}) \in (0, \infty) \times \mathcal{M}_V$, and the infimum $V(\hat{\lambda} \hat{Q}|P) = \inf_{P \in \mathcal{P}} V(\hat{\lambda} \hat{Q}|P)$ is attained by a $\hat{P} \in \mathcal{P}$ since $\mathcal{P}$ is weakly compact, and $V(\cdot |\cdot)$ is lower semicontinuous. Thus the right hand side of (2.10) is also written as $V(\hat{\lambda} \hat{Q}|\hat{P})$, and we call the triplet $(\hat{\lambda}, \hat{Q}, \hat{P})$ a dual optimizer.

A way of proving (2.10) and the existence of a solution $(\hat{\lambda}, \hat{Q})$ is to closely analyze the robust utility functional $X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)]$ on $L^\infty$ characterizing $V(\cdot |\cdot)$ as its conjugate. Then the duality and the existence of $(\hat{\lambda}, \hat{Q})$ follow simultaneously from Fenchel’s duality theorem. See [18] for detail. Alternatively, one can separate the dual problem into the minimization of $\hat{\lambda} \mapsto \inf_{Q \in \mathcal{M}_V} V(\lambda Q|P) + \lambda x$ and of $Q \mapsto V(\lambda Q|P)$ for each $\lambda$. For the latter problem, called the robust $f$-projection, [8] proves the existence by establishing a uniform integrability criterion in terms of $V(\cdot |\cdot)$ in the spirit of the de la Vallée-Poussin theorem.

In contrast to the standard utility maximization, neither the uniqueness of $(\hat{\lambda}, \hat{Q})$ (hence of the triplet $(\hat{\lambda}, \hat{Q}, \hat{P})$) nor the equivalence $\hat{Q} \sim \mathbb{P}$ hold in the robust case, as the following trivial example illustrates:

**Example 2.2.** Suppose $\mathcal{M}_V^{loc} \neq \emptyset$, and that $\mathcal{M}_V^{loc}$ contains an element $Q_0$ which is not equivalent to $\mathbb{P}$. Then we take $\mathcal{P}$ so that $Q_0 \in \mathcal{P} \subset \mathcal{M}_V^{loc}$. In this case, $\hat{\lambda}$ is uniquely determined as the minimizer of $\hat{\lambda} \mapsto V(\hat{\lambda})$. Then a triplet $(\hat{\lambda}, \hat{Q}, \hat{P})$ is a dual optimizer if (and only if) $\hat{P} = Q \in \mathcal{P} \subset \mathcal{M}_V^{loc}$. Indeed, by Jensen’s inequality and the strict convexity of $V$, $V(\lambda Q|P) = E_P[V(\lambda dQ/dP)] \geq V(\lambda)$ whenever $Q \ll \mathbb{P}$, and the “equality” holds if and only if $Q = P$. Hence $(\hat{\lambda}, \hat{Q}, \hat{P})$ is not unique, and $(\hat{\lambda}, Q_0, Q_0)$ is a solution with $Q_0 \not\sim \mathbb{P}$.

As for the equivalence, we still have $\hat{Q} \sim \hat{P}$ whenever $(\hat{\lambda}, \hat{Q}, \hat{P})$ is a dual optimizer (see [17], Theorem 2.7). Also, by an exhaustion argument, there exists a maximal solution $(\hat{\lambda}, \hat{Q}, \hat{P})$ in the sense that if $(\hat{\lambda}, \hat{Q}, \hat{P})$ is another dual optimizer, then $\hat{P} \ll \hat{P}$ (hence $Q \ll \hat{Q}$) and $\hat{\lambda} dQ/dP = \hat{\lambda} dQ/d\hat{P}$, $P$-a.s., where the density $d\hat{Q}/d\hat{P}$ is defined $P$-a.s. in the sense of Lebesgue decomposition. In particular, if $(\hat{\lambda}, \hat{Q}, \hat{P})$ and $(\hat{\lambda}, \hat{Q}, \hat{P})$ are two maximal solution, then

$$\hat{\lambda} d\hat{Q}/d\hat{P} = \hat{\lambda} d\hat{Q}/d\hat{P}, \quad P\text{-a.s.}. \quad (2.11)$$

See [17, Theorem 2.5 and Proposition 4.7]. This uniqueness is still useful in our purpose. Note finally that even such a maximal $\hat{Q}$ may fail to be equivalent to the reference probability $\mathbb{P}$. See [23, Example 2.5] for a counter example. In the sequel, we fix such a maximal dual optimizer, and call $\hat{P}$ a least favorable measure.

The duality (2.10) completely characterizes the quantitative nature of the problem (2.6). But our aim in this paper is to discuss the qualitative nature, especially the existence of
optimal strategy in $\Theta_V$. To do this, assumptions (A1) – (A5) are not enough, and we assume additionally

$$(A6) \quad \sup_{\theta \in \Theta_A} E_p[U(\theta \cdot S_T)] < \infty, \quad \forall P \in \mathcal{P}^e.$$ 

**Remark 2.3.** Several remarks on assumption (A6) are in order.

1. This assumption is automatically satisfied if $U(\infty) := \sup_x U(x) < \infty$ as exponential utility, and in this case, $U(X) \in \bigcap_{P \in \mathcal{P}} L^1(P)$ for any random variable $X$. Therefore, the robust utility functional $X \mapsto \inf_{P \in \mathcal{P}} E_p[U(X)]$ is well-defined on $L^0$ as $[\infty, \infty]$-valued concave functional.

2. If $U(\infty) = \infty$, [2, Th. 1.1 and Remark 1.2] show under (A4) that (A6) is equivalent to:

$$\forall P \in \mathcal{P}^e, \exists Q \in \mathcal{M}_V \text{ such that } V(Q|P) < \infty.$$ 

This is further equivalent to saying that $v_P(y) < \infty$ for all $y > 0$ and $P \in \mathcal{P}^e$, where $v_P$ is the dual value function

$$v_P(y) := \inf_{Q \in \mathcal{M}_V} V(yQ|P), \quad y > 0.$$ 

3. We could state (A6) with the whole $\mathcal{P}$ rather than $\mathcal{P}^e$. But for our purpose, (A6) is enough. Recall that (A5) implies in particular $\mathcal{P}^e \neq \emptyset$. If $\hat{P} \in \mathcal{P}^e$, we have, for instance, \(\bigcap_{P \in \mathcal{P}} L^1(P) = \bigcap_{P \in \mathcal{P}} L^1(\hat{P})\), \(a\) if \((X^n)\) is bounded in \(L^1(P)\) for all \(P \in \mathcal{P}^e\), the same is true for all \(P \in \mathcal{P}\).

In particular, (A6) (hence (2.12)) guarantees even in the case $U(\infty) = \infty$ that

$$X \in \bigcap_{Q \in \mathcal{M}_V} L^1(Q) \Rightarrow U(X) \in \bigcap_{P \in \mathcal{P}} L^1(P).$$

In fact, if $V(Q|P) < \infty$ and $X \in L^1(Q)$, Young’s inequality implies $U(X) \leq V(dQ/dP) + (dQ/dP)X \in L^1(P)$, and we can take such a $Q \in \mathcal{M}_V$ by (2.12) for all $P \in \mathcal{P}^e$.

**Remark 2.4.** (Continuation of Remark 2.1). We give a brief comparison of admissible classes considered in literature. In [17], the class $\Theta_V(\hat{P})$ is used to discuss the existence of optimal strategy, while [8] considered (implicitly) a slightly smaller class:

$$\mathcal{M}_V^0(\hat{P}, \hat{P}) := \{Q \in \mathcal{M}_V : V(\alpha Q + (1 - \alpha)\hat{P}) < \infty, \forall \alpha \in (0, 1)\},$$

$$\Theta_V^0(\hat{P}, \hat{P}) := \left\{ \theta \in L_0(S) : \theta \cdot S \text{ is a } Q\text{-supermartingale, } \forall Q \in \mathcal{M}_V^0(\hat{P}, \hat{P}) \right\},$$

Note that $\Theta_V^0(\hat{P}, \hat{P}) \subset \Theta_V(\hat{P})$ since $\mathcal{M}_V(\hat{P}) \subset \mathcal{M}_V^0(\hat{P}, \hat{P})$, while if we set $\Theta_m(\hat{Q}) := \{\theta \in L_0(S) : \theta \cdot S \text{ is a } \hat{Q}\text{-martingale}\}$,

$$\Theta_V \cap \Theta_m(\hat{Q}) \subset \Theta_V(\hat{P}) \cap \Theta_m(\hat{Q}) = \Theta_V^0(\hat{Q}, \hat{P}) \cap \Theta_m(\hat{Q}).$$

Thus $\Theta_V(\hat{P})$ and $\Theta_V^0(\hat{Q}, \hat{P})$ are essentially equivalent for the existence of optimal strategy (see Theorem 3.2). We just emphasize here that our class $\Theta_V$ depends neither on particular $P \in \mathcal{P}$ nor $Q \in \mathcal{M}_V$, while $\Theta_V(\hat{P})$ and $\Theta_V^0(\hat{Q}, \hat{P})$ do.

We conclude this section by recalling a stability property of a set of probability measures, called $m$-stability, which will be used in Theorem 3.2 below.
Definition 2.5 ([5], Definition 1). A set $Q$ of probability measures is said to be $m$-stable (multiplicatively stable) if for any $Q \in Q$, $Q' \in Q'$ with the density processes $Z_t = (dQ'/dP)|_{F_t}$ and $Z_t' = (dQ'/dP)|_{F_t}$, as well as any stopping time $\tau \leq T$, a new probability $Q$ defined by $dQ/dP := Z_t(Z_t'/Z_t')$ is an element of $Q$.

This property is equivalent to the time-consistency of the corresponding dynamic coherent monetary utility function $\theta_\tau(X) := \text{essinf}_{Q \in Q} E_Q[X|F_\tau]$ for any $X,Y \in L^\infty$ and stopping times $\sigma \leq \tau$, $\phi_\tau(X) \leq \phi_\tau(Y)$ implies $\phi_\sigma(X) \leq \phi_\sigma(Y)$. This is further equivalent (under (A3)) to the time-consistency of the dynamic robust utility functional $U_\tau(X) := \text{essinf}_{Q \in Q} E_Q[U(X)|F_\tau]$. See [5, Theorem 12] for details and precise formulation. Note that the set $M_{\text{loc}}$ of all local martingale measures is m-stable.

3. MAIN RESULTS

We first state a result on a “weak solution” to the problem (2.6), which yields a candidate $M$ that the set $\text{essinf}_{Q \in Q} E_Q[X|F_\tau]$ to the time-consistency of the dynamic robust utility functional $X$ by (A6) and Remark 2.3. Thus the robust utility functional $X \mapsto \text{inf}_{P \in P} E_P[U(x + X)]$ is well-defined on $X$.

Theorem 3.1. Suppose (A1) – (A6), and let $x \in \mathbb{R}$ and $(\hat{\lambda}, \hat{Q}, \hat{P})$ be a maximal dual optimizer. Then there exists an $\hat{X} \in X$ such that $U(x + \hat{X}) \in \bigcap_{P \in P} L^1(P)$ and

$$u(x) = \sup_{X \in X} \inf_{P \in P} E_P[U(x + X)] = \inf_{P \in P} E_P[U(x + \hat{X})],$$

where the infimum is attained by $\hat{P}$. Moreover, there exists an $(S, \hat{Q})$-integrable predictable process $\hat{\theta}$ with $\hat{\theta}_0 = 0$ such that $\hat{\theta} \cdot S$ is a $\hat{Q}$-martingale (not only local) and

$$x + \hat{X} = -V'(\hat{\lambda} d\hat{Q}/d\hat{P}) = x + \hat{\theta} \cdot S_T, \hat{Q}-\text{a.s.}$$

In particular, $\hat{X}$ is $\hat{Q}$-a.s. unique, and $\hat{\theta}$ is unique in the sense that $\hat{\theta} \cdot S$ is unique up to $\hat{Q}$-indistinguishability.

The proof is given in Section 4. The first equality in (3.2) states that the robust utility maximization over terminal wealths $x + \theta \cdot S_T$ is (quantitatively) equivalent to the indirect utility maximization:

$$u_{M_\mathfrak{V}}(x) = \sup_{X \in X} \inf_{P \in P} E_P[U(x + X)],$$

while the random variable $\hat{X}$ is the so-called optimal contingent claim. Such arguments are quite standard in (non-robust) utility maximization, and also in the robust case, [8, Theorem 3.11] shows a similar result: under (A1) – (A5), the assertions of Theorem 3.1 hold true except that the sets $M_\mathfrak{V}$ (in the definition (3.1)) and $P$ are replaced by $M_\mathfrak{V}^0(\hat{Q}, \hat{P})$ and $P^\alpha(\hat{Q}, \hat{P})$ defined respectively by Remark 2.4 and

$$P^\alpha(\hat{Q}, \hat{P}) := \{P \in P : V(\hat{Q}|\alpha P + (1 - \alpha)\hat{P}) < \infty, \exists \alpha \in (0, 1)\}.$$ 

Note that our finite utility assumption (A6) is automatic if $P$ is replaced by $P^\alpha(\hat{Q}, \hat{P})$. Also, when $U(\infty) < \infty$, the set $P^\alpha(\hat{Q}, \hat{P})$ actually coincides with the whole set $P$ ([8], Remark 3.10). However, $M_\mathfrak{V}^0(\hat{Q}, \hat{P})$ still depends on $(\hat{Q}, \hat{P})$ which is the solution to the dual problem, hence not a priori available. On the other hand, our formulation is universal, which is a slight, but qualitatively crucial contribution.
Theorem 3.1 suggests that the “strategy” \( \hat{\theta} \) is a candidate of optimal strategy. However, we still have to prove that this strategy is indeed \textit{admissible}.

\textbf{Theorem 3.2.} In addition to (A1) – (A6), we assume that \( \hat{Q} \sim P \) and \( P \) is m-stable. Then \( \hat{\theta} \) is \((S,\hat{P})\)-integrable (hence \( (S,P)\)-integrable for all \( P \in \mathcal{P} \)), and \( \hat{\theta} \cdot S \) is a supermartingale under all \( Q \in \mathcal{M}_V \). In particular, \( \hat{\theta} \) belongs to \( \Theta_V \) and is an optimal strategy.

The proof is given in Section 5. When \( \mathcal{P} = \{P\} \), the question of \textit{uniform supermartingale property} of this type goes back to the “six-author paper” [7] which shows that the optimal wealth in \textit{exponential utility} maximization is a martingale under all local martingale measures having a finite relative entropy with \( \mathbb{P} \), under an additional assumption on reverse Hölder inequality which is later removed by [14]. Although this uniform martingale property is no longer true for other utility functions, [21] shows that the optimal wealth is a supermartingale under all \( Q \in \mathcal{M}_V(\mathbb{P}) \), for any utility functions on \( \mathbb{R} \) with reasonable asymptotic elasticity. There are also some extensions to the case where the semimartingale \( S \) is not locally bounded. See e.g. [3] and [4].

In the robust case, the \( Q \)-supermartingale property for all \( Q \in \mathcal{M}_V(\hat{P}) \) (hence all \( Q \in \mathcal{M}_{\mathcal{Q}}(\hat{Q},\hat{P}) \) since \( \hat{\theta} \cdot S \) is a \( \hat{Q} \)-martingale) is shown by [8] (see also [17] for a slight extension). We emphasize that the difference between \( \mathcal{M}_V(\hat{P}) \) and \( \mathcal{M}_V \) is essential here. Note that \( \hat{X} \) is also optimal for the utility maximization problem under the fixed measure \( \hat{P} \), and the same is true for \( (\hat{\lambda},\hat{Q}) \) in the dual side. Thus the result of [21] cited in the previous paragraph is still applicable (under the assumption \( \hat{Q} \sim P \)) for \( Q \) with \( V(Q|\hat{P}) < \infty \), while we have to consider the case where \( V(Q|P) < \infty \) for some \( P \in \mathcal{P} \) but possibly \( V(Q|\hat{P}) = \infty \).

To grasp the situation, we try to describe the heuristics behind the argument in [21] (from our point of view), and our idea of extending it. In what follows in this section, we suppose all the assumptions of Theorem 3.2, especially \( \hat{Q} \sim P \).

For a moment, we \textit{suppose} that \( \hat{\theta} \cdot S \) is a \( Q \)-supermartingale for some \( Q \in \mathcal{M}_V \). Then the \( \hat{Q} \)-martingale property and the representation (3.3) imply: for any stopping time \( \tau \leq T \),

\begin{equation}
E[QV(\hat{\lambda}d\hat{Q}/d\hat{P})|\mathcal{F}_\tau] \leq E[QV(\hat{\lambda}d\hat{Q}/d\hat{P})|\mathcal{F}_\tau], \ Q\text{-a.s.}
\end{equation}

On the other hand, Ansel-Stricker’s lemma [1] shows that \( \hat{\theta} \cdot S \) is a \( Q \)-supermartingale if and only if there exists a \textit{Q-martingale lower bound}, i.e., a \( Q \)-martingale \( M^\mathcal{Q} \) such that \( \hat{\theta} \cdot S \geq M^\mathcal{Q}, \ Q\text{-a.s.} \). In particular, if (3.4) holds true for any stopping time \( \tau \leq T \), the process defined by \( M^\mathcal{Q} = -E[QV(\hat{\lambda}d\hat{Q}/d\hat{P})|\mathcal{F}_\tau] \) provides a desired lower bound, hence (3.4) is a necessary and \textit{sufficient} condition for \( \hat{\theta} \cdot S \) to be a \( Q \)-supermartingale.

When \( V(Q|\hat{P}) < \infty \), the inequality (3.4) is obtained as the \textit{variational inequality} which characterizes \( \hat{Q} \) as a minimizer of the functional \( Q \mapsto V(\hat{\lambda}Q|\hat{P}) \) when \( \tau = 0 \), and a “Bellman-type” principle using the m-stability of the set of local martingale measures shows the case of general \( \tau \leq T \).

If \( \inf_{P \in \mathcal{P}} V(Q|P) < \infty \) but \( V(Q|\hat{P}) = \infty \), this argument is no longer applicable at least directly. Mathematically speaking, we loose some important estimates to guarantee the necessary convergences, or more intuitively, any element \( Q \) with \( V(Q|\hat{P}) = \infty \) is in no way optimal at very early stage, and we can not draw further information from the optimality of \( \hat{Q} \) in the minimization of \( Q \mapsto V(\hat{\lambda}Q|\hat{P}) \). However, we have used only a part of information of \( \hat{Q} \) so far, and it is natural to expect that a better information may improve the result. More specifically,
Step 1 the optimality of \((\hat{Q}, \hat{P})\) in the minimization of \((Q, P) \mapsto V(\hat{\lambda}Q|P)\) should yield a variational inequality similar to (3.4) but with an additional term involving \(P\):

\[
E_\hat{Q}[V'(\hat{\lambda}d\hat{Q}/d\hat{P})|F_T] + F_T(\hat{P}) \leq E_\hat{Q}[V'(\hat{\lambda}d\hat{Q}/d\hat{P})|F_T] + F_T(P).
\]

Step 2 Though we may not take \(P = \hat{P}\) in general, it seems natural to expect that we may take \(\hat{P}\) “arbitrarily close to \(\hat{P}\)” keeping \(V(Q|P) < \infty\) with fixed \(Q\).

Step 3 If this is the case, we may expect (3.4) by an approximation argument:

\[
F_T(P) \rightarrow F_T(\hat{P})
\]

The formal inequality in Step 1 will be realized as Proposition 5.4 below, where the m-stability of \(P\) will play an important role. On the other hand, Steps 2 and 3 will be justified in a certain sense by a simple trick which is a consequence of reasonable asymptotic elasticity (Lemma 5.5).

Remark 3.3 (What happens when \(\hat{Q} \neq P\)). The equivalence \(\hat{Q} \sim P\) is automatic if all elements of \(P\) are equivalent to \(P\). When the filtration \(F\) is continuous (i.e., every \((F, P)\)-martingale is continuous, especially if it is generated by a Brownian motion), the latter condition is already implied by the m-stability of \(P\) and (A2) (see [5, Theorem 8]), thus it is not a further restriction in that case.

In general, however, the equivalence \(\hat{Q} \sim P\) may fail (see [23, Example 2.5] for a counter example), thus it is worth asking what happens in that case. When \(U\) is finite only on the positive half-line, the optimal claim \(\hat{X}\) (which does not require the assumption \(\hat{Q} \sim P\)) is super-hedged by some \((S, P)\)-integrable process \(\hat{\theta}\) with \(\hat{\theta} \cdot S = \hat{\theta} \cdot \hat{S}\) \(\hat{Q}\)-a.s. By the monotonicity of robust utility functional, we see that \(\hat{\theta}\) is an optimal strategy without the additional assumption \(\hat{Q} \sim P\) (see [23] and [22]). However, this argument essentially relies on the fact that \(\hat{X}\) is bounded below by \(-x\) (since \(U(x) = -\infty\) for \(x < 0\)), and no longer works when the utility function is finite on the entire real line. Thus we can not drop the assumption \(\hat{Q} \sim P\) (at now).

Remark 3.4 (Random Endowment). The results of this paper may also be stated with a random endowment \(B\) as long as it is an \(F_T\)-measurable random variable satisfying

\[
\forall P \in P, \exists \varepsilon_P > 0 \text{ such that } U(-\varepsilon_PB^+) \in L^1(P),
\]

\[
\exists \varepsilon > 0 \text{ such that } \{U(-(1+\varepsilon)B^-)dP/dP\}_{P \in P} \text{ is uniformly integrable.}
\]

Then the robust utility maximization problem (2.6) reads as

\[
\inf_{\theta \in \Theta_0} \inf_{P \in P} E_P[U(x + \theta \cdot S_T + B)],
\]

Assumption (3.5) implies that \(B \in \int_{Q \in \mathcal{M}_V} L^1(Q)\), and guarantees under (A1) – (A5) that a duality corresponding to (2.10) holds true [18, Theorem 2.3]:

\[
\sup_{\theta \in \Theta_0} \inf_{P \in P} E_P[U(x + \theta \cdot S_T + B)] = \inf_{\lambda > 0} \sup_{Q \in \mathcal{M}_V} \{V(\lambda Q|P) + \lambda x + \lambda E_Q[B]\},
\]

With the same assumptions, the dual problem admits a maximal solution with the unique density in the sense of (2.11). Then Theorems 3.1 and 3.2 remain true with similar proofs, and with obvious modifications, e.g., (3.3) is replaced by \(x + \hat{X} + B = -V'(\hat{\lambda}d\hat{Q}/d\hat{P}) = x + \hat{\theta} \cdot \hat{S}_T + B\) \(\hat{Q}\)-a.s. We omit the details. See [18] for the treatment of random endowment and other implications of (3.5).
We first note that we have only to consider the case \( x = 0 \). Indeed, assumptions (A3) and (A4) on the utility function are invariant under the translation of utility function from \( U \) to \( U(x + \xi) \), and all the results for \( x \neq 0 \) follow from those for \( x = 0 \) applied to the new utility function \( U_x \). Thus we assume \( x = 0 \) in what follows.

The next technical lemma is a collection of several arguments in [4].

**Lemma 4.1 ([4]).** Let \((Q, P)\) be a pair of probabilities with \( V(Q|P) < \infty \), and \((k^n)_n\) a sequence of random variables such that \( E_P[U(k^n)] \) is bounded from below and \( E_Q[k^n] \leq 0 \) for all \( n \). Then

(a) \((k^n)_n\) is bounded in \( L^1(Q)\);
(b) \((U(k^n))_n\) is bounded in \( L^1(P)\);
(c) If in addition \( k^n \) converges a.s. to some \( k \in L^0 \), we have \( k \in L^1(Q) \), \( U(k) \in L^1(P) \) and that

\[
E_Q[k] \leq 0 \quad \text{and} \quad \limsup_n E_P[U(k^n)] \leq E_P[U(k)].
\]

**Proof.** We just fill the gap from [4]. As we are assuming the reasonable asymptotic elasticity (A4), assertions (a) and (b) are contained in Proposition 6.3 of [4]. The assertion (c) also appears (implicitly) in the proof of their Theorem 4.10, which we briefly recall here.

Assume \( k^n \rightarrow k \), \( P\)-a.s. Since \((k^n)\) (resp. \((U(k^n))\)) is bounded in \( L^1(Q)\) (resp. \( L^1(P)\)), Fatou’s lemma applied to the sequence \((k^n)_n\) (resp. \((U(k^n))_n\)) shows that \( k \in L^1(Q) \) (resp. \( U(k) \in L^1(P) \)). By Young’s inequality, we have \( U(k^n) - \lambda(dQ/dP)k^n \leq V(\lambda dQ/dP) \in L^1(P) \) for all \( n \in \mathbb{N} \) and \( \lambda > 0 \), where the \( P\)-integrability of the right hand side for all \( \lambda \) follows from the reasonable asymptotic elasticity. By this integrable upper bound as well as the assumption \( E_Q[k^n] \leq 0 \), (reverse) Fatou’s lemma shows that

\[
\limsup_n E_P[U(k^n)] \leq \limsup_n E_P[U(k^n)] - \lambda(dQ/dP)k^n
\]

\[
\leq E_P[U(k)] - \lambda(dQ/dP)k^n = E_P[U(k)] - \lambda E_Q[k], \forall \lambda > 0.
\]

Letting \( \lambda \downarrow 0 \), we have (4.1), while \( E_Q[k] \leq 0 \) follows by letting \( \lambda \uparrow \infty \). \( \square \)

**Proof of Theorem 3.1.** We choose a maximizing sequence \((\theta^n)_n \subset \Theta_{ab}\), that is

\[
\inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)] \nearrow u(0).
\]

This sequence does not have to converge, thus we appeal to a Komlós type argument. Let \((\hat{Q}, \hat{P}) \in \mathcal{M}_V \times \mathcal{P}\) be such that \( \hat{Q} \sim \hat{P} \sim \mathbb{P} \) and \( V(\hat{Q}|\hat{P}) < \infty \) which exists by (A5). Since \( E_P[U(\theta^n \cdot S_T)] \geq \inf_{P \in \mathcal{P}} E_P[U(\theta^1 \cdot S_T)] \) and \( E_Q[\theta^n \cdot S_T] \leq 0 \) by construction, Lemma 4.1 (a) shows that \((\theta^n \cdot S_T)_n\) is bounded in \( L^1(\hat{Q}) \). Hence Komlós’ theorem (see e.g. [6, Theorem 15.1.3]) yields another sequence \((\bar{k}^n)_n\) such that

\[
\begin{cases}
\bar{k}^n \in \text{conv}(\theta^n \cdot S_T, \theta^{n+1} \cdot S_T, \ldots) \\
\bar{k}^n \text{ converges } \hat{Q}\text{-a.s. (hence } \mathbb{P}\text{-a.s.) to some } \bar{X} \in L^1(\hat{Q}).
\end{cases}
\]

By construction, each \( \bar{k}^n \) is again the terminal value of a stochastic integral \( \bar{\theta}^n \cdot S_T \) where \( \bar{\theta}^n \) is the convex combination of \((\theta^n, \theta^{n+1}, \ldots)\) with the same convex weights as \( \bar{k}^n \), hence \( \bar{\theta}^n \in \Theta_{ab} \) and \( E_Q[\bar{k}^n] \leq 0 \) for each \( n \) and \( Q \), in particular.

Since the robust utility functional \( X \mapsto \inf_{P \in \mathcal{P}} E_P[U(X)] \) is concave as a point-wise infimum of concave functionals, we have \( \inf_{P \in \mathcal{P}} E_P[U(\bar{k}^n)] \geq \inf_{P \in \mathcal{P}} E_P[U(\theta^n \cdot S_T)] \) for each \( n \). Hence we still have \( \lim_n \inf_{P \in \mathcal{P}} E_P[U(\bar{k}^n)] = u(0) \), and the sequence \((E_P[U(\bar{k}^n)]_n)\) is bounded from below for all \( P \in \mathcal{P} \).
If $Q \in \mathcal{M}_V$, there is a $P \in \mathcal{P}$ with $V(Q|P) < \infty$ by the definition of $\mathcal{M}_V$, hence another application of Lemma 4.1 to the sequence $(\tilde{k}_n)$ with the pair $(Q, P)$ shows that $\tilde{X} \in L^1(Q)$ and $E_Q[\tilde{X}] \leq 0$. Hence $\tilde{X} \in \mathcal{X}$.

We next show that $U(\tilde{X}) \in \cap_{P \in \mathcal{P}} L^1(P)$ and

\[(4.3) \quad \limsup_n E_P[U(\tilde{k}_n)] \leq E_P[U(\tilde{X})], \quad \forall P \in \mathcal{P}.
\]

This is immediate from Fatou’s lemma if $U$ is bounded from above. When $U(\infty) = \infty$ and $P \in \mathcal{P}^\ast$, we can take a $Q \in \mathcal{M}_V$ with $V(Q|P) < \infty$ by (2.12), hence Lemma 4.1 shows (4.3) and that $(U(\tilde{k}_n))_n$ is bounded in $L^1(P)$. Then Remark 2.3 shows that $U(\tilde{X}) \in L^1(P)$ and $(U(\tilde{k}_n))_n$ is still bounded in $L^1(P)$ for arbitrary $P \in \mathcal{P}$ which need not be equivalent to $\mathbb{P}$. To prove (4.3) in the case $P \not\sim \mathbb{P}$, we take $(\hat{Q}, \hat{P})$ as above, and set $P_\alpha := \alpha P + (1 - \alpha)\hat{P}$ for $\alpha \in (0, 1)$. Since $P_\alpha \sim \mathbb{P}$, the claim is true for $P_\alpha$ for all $\alpha \in (0, 1)$, while we see that $\sup_n [E_{P_\alpha}[U(\tilde{k}_n)] - E_P[U(\tilde{k}_n)]] \leq 2(1 - \alpha)\sup_n ([U(\tilde{k}_n)]_{L^1(P)} \vee [U(\tilde{k}_n)]_{L^1(\hat{P})}) \to 0$, as $\alpha \uparrow 1$. Thus we deduce

\[
\limsup_n E_P[U(\tilde{k}_n)] = \limsup_{\alpha \uparrow 1} \inf_n E_{P_\alpha[P+(1-\alpha)\hat{P}]}[U(\tilde{k}_n)] \\
\leq \lim_{\alpha \uparrow 1} E_{P_\alpha[P+(1-\alpha)\hat{P}]}[U(\tilde{X})] = E_P[U(\tilde{X})].
\]

Hence (4.3) holds for all $P \in \mathcal{P}$.

We now prove (3.2). Note first that for all $\lambda > 0, X \in \mathcal{X}, Q \in \mathcal{M}_V$ and $P \in \mathcal{P}$,

\[E_P[U(X)] \leq V(\lambda Q|P) + \lambda E_Q[X] \leq V(\lambda Q|P).
\]

In particular,

\[
\inf_{P \in \mathcal{P}} E_P[U(X)] \leq \inf_{\lambda > 0} \inf_{Q \in \mathcal{M}_V} V(\lambda Q|P) \overset{(2.10)}{=} u(0), \quad \forall X \in \mathcal{X},
\]

On the other hand, (4.3) shows

\[
u(0) = \liminf_{P \in \mathcal{P}} E_P[U(\tilde{k}_n)] \leq \limsup_{P \in \mathcal{P}} E_P[U(\tilde{k}_n)] \leq \inf_{P \in \mathcal{P}} E_P[U(\tilde{X})].
\]

This concludes the proof of (3.2).

We proceed to (3.3). Notice that

\[(4.4) \quad U(\hat{X}) = V(\hat{\lambda} d\hat{Q}/d\hat{P}) + \hat{\lambda}(d\hat{Q}/d\hat{P})\hat{X}, \quad \hat{P}\text{-a.s.}
\]

Indeed, “≤” is just a Young’s inequality, while “≥” follows from

\[
u(0) = \inf_{P \in \mathcal{P}} E_P[U(\tilde{X})] \leq E_{\hat{P}}[U(\tilde{X})] \leq E_{\hat{P}} \left[ V\left(\hat{\lambda} \frac{d\hat{Q}}{d\hat{P}}\right) + \hat{\lambda} \frac{d\hat{Q}}{d\hat{P}}\hat{X} \right] \overset{(2.10)}{=} \inf_{P \in \mathcal{P}} E_P[U(\tilde{X})].
\]

Here (i) follows from the “≤” part, and (ii) from $\tilde{X} \in \mathcal{X}$. In particular, $\hat{P}$ attains the infimum in (3.2) and we obtain (4.4). But an elementary knowledge from convex analysis shows that this is possible only if

\[\tilde{X} = -V'(\hat{\lambda} d\hat{Q}/d\hat{P}), \quad \hat{P}\text{-a.s.}
\]

This is the first equality in (3.3), and the $\hat{Q}$-a.s. uniqueness of $\tilde{X}$ follows from that of $\hat{\lambda} d\hat{Q}/d\hat{P}$ (see (2.11)). On the other hand, the existence of $\hat{\theta} \in L(S, \hat{Q})$ with $\theta_0 = 0$ and $\hat{\theta} \cdot S$ being a $\hat{Q}$-martingale, which represents $-V'(\hat{\lambda} d\hat{Q}/d\hat{P})$ as (3.3), follows from Theorem 3.2 of [11] (see also [20, Theorem 2.2 (iv)]). Finally, $\hat{Q}$-a.s. uniqueness of the process $\hat{\theta} \cdot S$ follows from the $\hat{Q}$-a.s. uniqueness of the terminal value $\hat{\theta} \cdot S_T$ and the fact that $\hat{\theta} \cdot S$ is a $\hat{Q}$-martingale. \qed
5. Uniform Supermartingale Property of Optimal Wealth

We now proceed to the uniform supermartingale property of the optimal wealth, that is, we shall show that \( \hat{\theta} \cdot S \) is a supermartingale under all local martingale measures \( Q \) with finite entropy w.r.t. some \( P \in \mathcal{P} \). As outlined in Section 3, this will follow if we can prove the dynamic variational inequality (3.4) for every \( Q \in \mathcal{M}_V \). Therefore, the key of this section is the next proposition which should be compared with [8, Lemma 3.12]. Recall that we have only to consider the case \( x = 0 \). In what follows, all the assumptions of Theorem 3.2 are in force, and we do not cite them in each statement.

Proposition 5.1. We have

1. for all \( Q \in \mathcal{M}_V \), and for all stopping time \( \tau \leq T \),
   \[ E_Q \left[ V' \left( \hat{\lambda} d\hat{Q}/d\hat{P} \right) \right] \leq E_Q \left[ V' \left( \hat{\lambda} d\hat{Q}/d\hat{P} \right) \right], \quad Q \text{-a.s.} \tag{5.1} \]

2. for all \( P \in \mathcal{P} \), and for all stopping time \( \tau \leq T \),
   \[ E_P[U(\hat{\theta} \cdot S_T)|\mathcal{F}_\tau] \leq E_P[U(\hat{\theta} \cdot S_T)|\mathcal{F}_\tau], \quad P \text{-a.s.} \tag{5.2} \]

We introduce some notations. If \( L \) is a strictly positive martingale, we denote \( L_{\tau,T} := L_T/L_\tau \), for any stopping time \( \tau \leq T \). Recall that any probability \( Q \ll P \) is identified with a (uniformly integrable) martingale, namely the density process \( Z^Q = (dQ/dP)|_T \). In what follows, denote by \( \hat{Z} \) (resp. \( \hat{D} \)) the density process of \( \hat{Q} \) (resp. \( \hat{P} \)). Also, when a pair \( (Q,P) \in \mathcal{M}_{loc} \times \mathcal{P} \) is fixed, the density process of \( Q \) (resp. \( P \)) is denoted by \( Z \) (resp. \( D \)), and set:

\[ Z_{\alpha,T} := \alpha Z_{\tau,T} + (1-\alpha)\hat{Z}_{\tau,T}, \quad D_{\alpha,T} := \alpha D_{\tau,T} + (1-\alpha)\hat{D}_{\tau,T}, \quad \alpha \in [0,1]. \tag{5.3} \]

We make a couple of simple reductions. The first one is just a notational reduction. In our purpose, we can assume without loss of generality that \( \hat{\lambda} = 1 \) since we already know \( \hat{\lambda} \). Indeed, \( (\hat{\lambda}, \hat{Q}, \hat{D}) \) minimizes \( \langle v, P \rangle \mapsto V(v|P) \) if and only if \( (\hat{\lambda}, \hat{Q}, \hat{D}) \) minimizes \( \langle v, P \rangle \mapsto V(\hat{\lambda} v|P) \). Next, we have only to prove (5.1) and (5.2) for all \( Q \in \mathcal{M}'_V \) and \( P \in \mathcal{P}' \), respectively. Indeed, if we could show (5.1) for all \( Q \in \mathcal{M}'_V \) for instance, we have \( \hat{Q} := (Q + \hat{Q})/2 \in \mathcal{M}'_V \) for any \( Q \in \mathcal{M}_V \) on the one hand, and on the other hand, Bayes’ formula implies

\[ E_{\hat{Q}}[\Phi|\mathcal{F}_\tau] \leq E_Q[\Phi|\mathcal{F}_\tau] = \frac{Z_\tau}{Z_\tau + \hat{Z}_\tau} E_Q[\Phi|\mathcal{F}_\tau] + \frac{\hat{Z}_\tau}{Z_\tau + \hat{Z}_\tau} E_{\hat{Q}}[\Phi|\mathcal{F}_\tau] \text{ a.s. on } \{Z_\tau > 0\} \]

where \( \Phi = V'(d\hat{Q}/d\hat{P}) \), hence (5.1). A similar argument applies also to (5.2).

The first step is to show a “Bellman-type” principle for a time-consistent optimization. Note that the set \( \mathcal{M}_{loc} \) of all local martingale measures is m-stable, while \( \mathcal{M}_V \) is not. The next simple lemma allows us to avoid this difficulty.

Lemma 5.2. Let \( (Q,P) \in \mathcal{M}'_V \times \mathcal{P}' \) with \( V(Q|P) < \infty \), and \( (Z,D) \) the corresponding density processes as well as \( \alpha \in [0,1] \). Then for any stopping time \( \tau \leq T \), the random variable \( \hat{D}_T D_{\tau,T}^{\alpha} V \left( \frac{Z_{\alpha,T}}{\hat{D}_\tau D_{\tau,T}^{\alpha}} \right) \) is \( \mathcal{F}_\tau \)-locally integrable i.e., there exists an increasing sequence \( A_n \in \mathcal{F}_\tau \) such that

\[ P(A_n) \not\rightarrow 1 \quad \text{and} \quad 1_{A_n} \hat{D}_T D_{\tau,T}^{\alpha} V \left( \frac{Z_{\alpha,T}}{\hat{D}_\tau D_{\tau,T}^{\alpha}} \right) \in L^1, \forall n. \tag{5.4} \]
Lemma 5.3. For any \((Q,P) \simeq (Z,D) \in \mathcal{M}_\alpha \times \mathcal{P}_\nu\) with \(V(Q|P) < \infty\), \(Q \in [0,1]\),

\[
E \left[ \frac{\hat{Z}_t}{D_t} V \left( \frac{\hat{Z}_T}{D_T} \right) \right] \leq E \left[ \tilde{D}_t D_{t,T}^a V \left( \frac{\hat{Z}_T}{D_T} \right) \right] \quad \text{a.s.}
\]

Proof. Recall from [10] that the condition \((A4)\) of reasonable asymptotic elasticity is equivalent to: for any \(a \geq 1\), there exists \(C_a, C'_a > 0\) such that

\[
V(\lambda y) \leq C_a V(y) + C'_a (y + 1), \quad \forall \lambda \in [a^{-1}, a], \forall y > 0.
\]

Since \(V\) is bounded from below by \(U(0)\), we can choose the constant \(C_a\) so that the right hand side is always positive. For the sequence \(A_n\), we take

\[
A_n := \{\hat{Z}_t, Z_t, \tilde{D}_t, D_t \in (n^{-1}, n)\} \in \mathcal{F}_t, \quad \forall n.
\]

Now the formal inequality in Step 1 at the end of Section 3 is realized as follows.

Proposition 5.4. For any \((Q,P) \simeq (Z,D) \in \mathcal{M}_\alpha \times \mathcal{P}_\nu\) with \(V(Q|P) < \infty\),

\[
\hat{Z}_t \left\{ E_Q \left[ V'(d\hat{Q}/d\hat{P}) \right] \mathbb{1}_{\mathcal{F}_t} \right\} - E_{\hat{Q}} \left[ V'(d\hat{Q}/d\hat{P}) \right] \mathbb{1}_{\mathcal{F}_t} + \hat{D}_t \left\{ E_P[U(\hat{X})|\mathcal{F}_t] - E_{\hat{P}}[U(\hat{X})|\mathcal{F}_t] \right\} \geq 0, \quad \text{a.s.}
\]
Proof. Let \((Z, D), \tau, \alpha\) be as above, and set

\[ G_\tau(\alpha) := \hat{D}_\tau D_\tau^\alpha V(\hat{Z}_\tau Z_\tau^\alpha / D_\tau D_\tau^\alpha). \]

Then \(\alpha \mapsto G_\tau(\alpha)\) is convex (a.s.) by (the proof of) Lemma 5.5 below, hence \((G_\tau(\alpha) - G(0))/\alpha\) decreases a.s. to the limit \(\Xi_\tau(Q, P)\) as \(\alpha \searrow 0\). Here \(\Xi_\tau(Q, P)\) is explicitly computed as:

\[ \Xi_\tau(Q, P) = \hat{Z}_\tau V\left( \frac{d\hat{Q}}{dP} \right) (\hat{Z}_\tau - \hat{Z}_\tau) + \hat{D}_\tau U(\hat{X})(\hat{D}_\tau - \hat{D}_\tau), \]

using \(\hat{Z}_\tau\) and \(U(\hat{X}) = V(d\hat{Q}/d\hat{P}) - (d\hat{Q}/d\hat{P}) V'(d\hat{Q}/d\hat{P})\). Since \(G_\tau(1)\) is \(\mathcal{F}_\tau\)-locally integrable and \(E[(G_\tau(\alpha) - G_\tau(0))/\alpha | F_\tau] \geq 0\) a.s. by Lemma 5.3, the (generalized) conditional monotone convergence theorem shows that \(E[\Xi(Q, P) | F_\tau] \geq 0\). Noting that \(V'(d\hat{Q}/d\hat{P}) = -\hat{X} \in L^1(Q)\) and \(U(\hat{X}) \in L^1(P)\) by Theorem 3.1, we deduce (5.7) from Bayes' formula.

We proceed to Step 2. Fixing \(Q \in \mathcal{M}_V\), we want to take \(P\) “arbitrarily close” to \(\hat{P}\). The next simple lemma gives a precise form of this argument.

Lemma 5.5. Let \((Q, P)\) and \((Q', P')\) be any two pairs of probability measures absolutely continuous w.r.t. \(P\). Then for any \(\alpha, \gamma \in (0, 1)\), we have

\[
V(\alpha Q + (1 - \alpha) Q' | \gamma P + (1 - \gamma) P') \\
\leq \gamma V\left( \frac{\alpha Q}{\gamma} \big| P \right) + (1 - \gamma)V\left( \frac{1 - \alpha Q'}{1 - \gamma} \right) .
\]

In particular, \(V(Q/P) < \infty\) and \(V(Q'/P') < \infty\) imply \(V(\alpha Q + (1 - \alpha) Q' | \gamma P + (1 - \gamma) Q') < \infty\) for any \(\alpha, \gamma \in (0, 1)\).

Proof. Note that for any positive numbers \(x, x', y, y'\),

\[
\frac{\alpha x + (1 - \alpha) x'}{\gamma y + (1 - \gamma) y'} = \gamma y \frac{\alpha x}{\gamma y} \frac{1 - \alpha x'}{1 - \gamma y'} + \frac{1 - \gamma y'}{\gamma y} \frac{1 - \gamma y'}{1 - \gamma y'} .
\]

Thus the convexity of \(V\) shows that

\[
(\gamma y + (1 - \gamma) y')V\left( \frac{\alpha x + (1 - \alpha) x'}{\gamma y + (1 - \gamma) y'} \right) \\
\leq \gamma V\left( \frac{\alpha x}{\gamma y} \right) + (1 - \gamma) y' V\left( \frac{1 - \alpha x'}{1 - \gamma y'} \right) .
\]

Putting \(d\hat{Q}/d\hat{P}\) (resp. \(dQ'/dP', dP'/d\hat{P}\), \(dP'/d\hat{P}\)) into \(x\) (resp. \(x', y, y'\), and taking the \(\hat{P}\)-expectation, this implies (5.8). The second claim follows from the fact that \(V(Q/P) < \infty\) \(\Rightarrow V(\lambda Q/P) < \infty\) for any \(\lambda > 0\), as a consequence of reasonable asymptotic elasticity. \(\square\)

Proof of Proposition 5.1. As noted after the statement of Proposition 5.1, we have only to consider the case \((Q, P) \in \mathcal{M}_\tau \times \mathcal{P}_e\) with \(V(Q/P) < \infty\). Fixing such a pair \((Q, P),\) we put \(Q_\alpha := \alpha Q + (1 - \alpha) Q\) and \(P_\gamma := \gamma P + (1 - \gamma) \hat{P}\) for any \(\alpha, \gamma \in (0, 1)\). By Lemma 5.5, the auxiliary variational inequality (5.7) is valid for any \((Q_\alpha, P_\gamma)\) with arbitrary \(\alpha, \gamma \in (0, 1)\).

Noting that \(E_{Q_\alpha}[\Phi | F_\tau] - E_Q[\Phi | F_\tau] = \frac{\alpha Z_\tau}{\alpha Z_\tau + (1 - \alpha) Z_\tau} \{E_{Q_\alpha}[\Phi | F_\tau] - E_Q[\Phi | F_\tau]\} \) etc., we have

\[
\frac{\alpha Z_\tau}{\alpha Z_\tau + (1 - \alpha) Z_\tau} \{E_Q[V'(d\hat{Q}/d\hat{P}) | F_\tau] - E_Q[V'(d\hat{Q}/d\hat{P}) | F_\tau]\} \\
+ \frac{\gamma Z_\tau}{\gamma Z_\tau + (1 - \gamma) Z_\tau} \{E_Q[U(\hat{X}) | F_\tau] - E_Q[U(\hat{X}) | F_\tau]\} \\
\geq 0, \text{ a.s. } \forall \alpha, \gamma \in (0, 1).
\]
Since $\gamma D_\tau / (\gamma D_\tau + (1 - \gamma) \hat{D}_\tau) \overset{\gamma \downarrow 0}{\to} 0$ and $\alpha Z_\tau / (\alpha Z_\tau + (1 - \alpha) \hat{Z}_\tau) \overset{\alpha \downarrow 0}{\to} 0$, we deduce (5.1) and (5.2) by letting $\gamma \downarrow 0$ (resp. $\alpha \downarrow 0$) with $\alpha$ (resp. $\gamma$) being fixed, whenever $V(Q|P) < \infty$.

Finally, any $Q \in \mathcal{M}_V$ (resp. $P \in \mathcal{P}_e$) admits a $P \in \mathcal{P}$ (resp. $Q \in \mathcal{M}_V$) with $V(Q|P) < \infty$ by definition (resp. by Remark 2.3).

□

Proof of Theorem 3.2. Under the assumption $\hat{Q} \sim \hat{P}$, the $(S, \mathbb{P})$-integrability of $\hat{\theta}$ is clear. We verify that $\hat{\theta} \cdot S$ is a supermartingale under each $Q \in \mathcal{M}_V$. Since $V'(d\hat{Q}/d\hat{P}) \in L^1(Q)$, the process defined by $M^Q_T = -E_Q[V'(d\hat{Q}/d\hat{P})|\mathcal{F}_\tau]$ is a $Q$-martingale. Then (3.3), (5.1) as well as the fact that $\hat{\theta} \cdot S$ is a $\hat{Q}$-martingale show that

$$\hat{\theta} \cdot S_\tau = -E_{\hat{Q}}[V'(d\hat{Q}/d\hat{P})|\mathcal{F}_\tau] \geq M^Q_T, \ Q\text{-a.s.}$$

for any stopping time $\tau \leq T$. A stochastic integral w.r.t. a $Q$-local martingale dominated below by a $Q$-(uniformly integrable) martingale is a $Q$-supermartingale by [24, Theorem 1], which is a variant of Ansel-Stricker’s lemma [1, Proposition 3.3].

□

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