TRANSVERSE IN Variant MEASURES EXTEND TO THE AMBIENT SPACE

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Abstract. It is shown that any transverse invariant measure of a foliated space can be considered as a measure on the ambient space.

1. Introduction

Transverse invariant measures of foliated spaces play an important role in the study of their transverse dynamics. They are measures on transversals invariant by holonomy transformations. There are many interpretations of transverse invariant measures; in particular, they can be extended to generalized transversals, which are defined as Borel sets that meet each leaf in a countable set [4]. Here, we show that indeed invariant measures can be extended to the σ-algebra of all Borel sets becoming an “ambient” measure (a measure on the ambient space). Precisely, the following result is proved.

Theorem 1.1. Let X be a foliated space with a transverse invariant measure Λ. There exists a Borel measure ˜Λ on X such that ˜Λ(T) = Λ(T) for all generalized transversal T.

This ˜Λ is constructed as a “pairing” of the transverse invariant measure Λ with the counting measure on the leaves. Connes has proved that this pairing is coherent for generalized transversals but it can not be directly extended to any Borel set since the local projection of a Borel set is not necessarily a Borel set [6, 7]. The solution of this problem is the main difficulty of the proof. In fact, this result is proved in the more general setting of foliated measurable spaces [1, 2]. The uniqueness of this extension is also discussed.

The extension ˜Λ will be used in study of the concept of Λ-LS category, which will appear elsewhere.

2. Foliated measurable spaces

A Polish space is a completely metrizable and separable topological space. A standard Borel space is a measurable space isomorphic to a Borel subset of a Polish space. A measurable topological space or MT-space X is a set equipped with a σ-algebra and a topology. Usually, measure theoretic concepts will refer to the σ-algebra of X, and topological concepts will refer to its topology. Notice that the σ-algebra does not necessarily agree with the Borel σ-algebra associated with the topology. An MT-map between MT-spaces is a measurable continuous map. An MT-isomorphism is a map which is a measurable isomorphism and a homeomorphism, simultaneously.

Let T be a standard Borel space. On T × R^n, we consider the σ-algebra generated by products of Borel subsets of T and R^n, and the product of the discrete topology
on $T$ and the Euclidean topology on $\mathbb{R}^n$. $T \times \mathbb{R}^n$ will be endowed with the structure of MT-space defined by this $\sigma$-algebra and this topology.

A **foliated measurable chart** on $X$ is an MT-isomorphism $\varphi : U \to T \times \mathbb{R}^n$, where $U$ is open and measurable in $X$. A **foliated measurable atlas** on $X$ is a countable family of foliated measurable charts whose domains cover $X$. The sets $\varphi^{-1}(\{\ast\} \times \mathbb{R}^n)$ are the **plaques** of the foliated chart $\varphi$ and the sets $\varphi^{-1}(T \times \{\ast\})$ are called **transversals** induced by $\varphi$. A **foliated measurable space** is an MT-space that admits a foliated measurable atlas. Observe that we always consider countable atlases. The connected components of $X$ are called its **leaves**. An example of foliated measurable space is a foliated space with its Borel $\sigma$-algebra and the leaf topology. According to this definition, the leaves are second countable connected manifolds but they may not be Hausdorff.

A measurable subset $T \subset X$ is called a **generalized transversal** if its intersection with each leaf is countable; these are slightly more general than the transversals of $\mathbb{R}$, which are required to have discrete and closed intersection with each leaf. Let $T(X)$ be the set of generalized transversals of $X$. This set is closed under countable unions and intersections, but it is not a $\sigma$-algebra.

A **measurable holonomy transformation** is a measurable isomorphism $\gamma : T \to T'$, for $T, T' \in T(X)$, which maps each point to a point in the same leaf. A **transverse invariant measure** on $X$ is a $\sigma$-additive map $\Lambda : T(X) \to [0, \infty]$ which is invariant by measurable holonomy transformations. The classical definition of transverse invariant measure in the context of foliated spaces is a measure on usual transversals invariant by usual holonomy transformations [3]. Both definitions are equivalent in the case of foliated measurable spaces induced by foliated spaces [4].

### 3. Case of a Product Foliated Measurable Space

In this section, we take foliated measurable spaces of the form $T \times P$, where $T$ is a standard measurable space and $P$ a connected, separable and Hausdorff manifold. Indeed, the results of this section hold when $P$ is any Polish space. We assume that a new topology is given in this space as follows. All standard Borel spaces are Borel isomorphic to a finite set, $\mathbb{Z}$ or the interval $[0, 1]$ (see [7, 8]). Identify $T \times P$ with $\mathbb{Z} \times P$, $[0, 1] \times P$ or $\mathbb{A} \times P$ ($A$ finite), via a Borel isomorphism. We work with two topologies in $T \times P$. On the one hand, the topology of the MT-structure is the product of discrete topology on $T$ and the topology of $P$. On the other hand, the topology is the product of the topology of $[0, 1]$, $\mathbb{N}$ or $P$ with the topology of $P$; the term “open set” is used with this topology. The $\sigma$-algebra of the MT-structure on $T \times P$ is generated by these “open sets”. Let $\pi : T \times P \to T$ be the first factor projection.

**Proposition 3.1** (R.Kallman [5]). If $B \subset T \times P$ is a Borel set such that $B \cap (\{t\} \times P)$ is $\sigma$-compact for all $t \in T$, then $\pi(B)$ is a Borel set. Moreover there exists a Borel subset $B' \subset B$ such that $\#(B' \cap (\{t\} \times P)) = 1$ if $B \cap (\{t\} \times P) \neq \emptyset$, and $\#(B' \cap (\{t\} \times P)) = 0$ otherwise.

For any measurable space $(X, \mathcal{M}, \Lambda)$, the **completion** of $\mathcal{M}$ with respect to $\Lambda$ is the $\sigma$-algebra

$$\mathcal{M}_\Lambda = \{ Z \subset X \mid \exists A, B \in \mathcal{M}, \ A \subset Z \subset B, \ \Lambda(B \setminus A) = 0 \}.$$  

The measure $\Lambda$ extends in a natural way to $\mathcal{M}_\Lambda$ by defining $\Lambda(Z) = \Lambda(A) = \Lambda(B)$ for $Z, A, B$ as above.
Now, let $\Lambda$ be a Borel measure on $T$. Define

$$\pi(B^*, \mathcal{B}_\Lambda) = \{ B \subset B^* \mid \pi(B \cap U) \in \mathcal{B}_\Lambda \forall U \subset T \times P \} ,$$

where $\mathcal{B}$ and $B^*$ are the Borel $\sigma$-algebras of $T$ and $T \times P$, respectively.

**Proposition 3.2.** $\pi(B^*, \mathcal{B}_\Lambda)$ is closed under countable unions.

**Proof.** This is obvious since, for any countable family $\{B_n\} \subset B^*$, we obtain $\bigcup_n B_n \in B^*$ and

$$\pi\left(\left(\bigcup_n B_n\right) \cap U\right) = \bigcup_n \pi(B_n \cap U) \in \mathcal{B}_\Lambda$$

for any open subset $U \subset T \times P$. \quarem

**Remark 3.3.** If $\Lambda$ is $\sigma$-finite (i.e., $T$ is a countable union of Borel sets with finite $\Lambda$-measure), then $\pi(B^*, \mathcal{B}_\Lambda) = \mathcal{B}^*$: by Exercise 14.6 in [6], any set in $\mathcal{B}^*$ projects onto an analytic set, which is $\Lambda$-measurable since $\Lambda$ is $\sigma$-finite [7, Theorem 4.3.1].

**Remark 3.4.** If $B \in \pi(B^*, \mathcal{B}_\Lambda)$ and $U$ is an open set, then $B \cap U \in \pi(B^*, \mathcal{B}_\Lambda)$. By Proposition 3.1, $\pi(B^*, \mathcal{B}_\Lambda)$ contains the Borel sets with $\sigma$-compact intersection with the plaques $\{t\} \times P$.

Now, we want to extend $\Lambda$ to all Borel sets satisfying the conditions of a measure. Let $\mathcal{B}^{**}$ denote the Borel $\sigma$-algebra of $T \times P \times T \times P$, $\bar{\pi}$ the natural projection $T \times P \times T \times P \to T \times T$, and $\langle \mathcal{B}_\Lambda \times \mathcal{B}_\Lambda \rangle$ the $\sigma$-algebra generated by sets of the form $A \times B$ for $A, B \in \mathcal{B}_\Lambda$.

**Lemma 3.5.** If $B, B' \in \pi(B^*, \mathcal{B}_\Lambda)$, then $B \times B' \in \bar{\pi}(\mathcal{B}^{**}, \langle \mathcal{B}_\Lambda \times \mathcal{B}_\Lambda \rangle)$.

**Proof.** Since $B, B' \in B^*$, we have $B \times B' \in \mathcal{B}^{**}$. Observe that every open set $U \subset T \times P$ is a countable union of products of open sets. Write $U = \bigcup_{n=1}^{\infty} (U_n \times V_n)$ with $U_n$ and $V_n$ open subsets of $T$ and $P$, respectively. Then

$$\bar{\pi}\left((B \times B') \cap U\right) = \bar{\pi}\left((B \times B') \cap \bigcup_{n=1}^{\infty} (U_n \times V_n)\right)$$

$$= \bar{\pi}\left(\bigcup_{n=1}^{\infty} ((B \cap U_n) \times (B' \cap V_n))\right) = \bigcup_{n=1}^{\infty} \bar{\pi}((B \cap U_n) \times (B' \cap V_n))$$

$$= \bigcup_{n=1}^{\infty} (\pi(B \cap U_n) \times \pi(B' \cap V_n)) ,$$

which is in $\langle \mathcal{B}_\Lambda \times \mathcal{B}_\Lambda \rangle$. \quarem

**Definition 3.6.** For $B \in \pi(B^*, \mathcal{B}_\Lambda)$, let

$$\bar{\Lambda}(B) = \int_T \#(B \cap (\{t\} \times P)) \, d\Lambda(t) = \int_T \left(\int_{\{t\} \times P} \chi_{B \cap (\{t\} \times P)}(x) \, d\nu\right) \, d\Lambda(t) ,$$

where $\nu$ denotes the counting measure and $\chi_X$ the characteristic function of a subset $X \subset \{t\} \times P$.

**Remark 3.7.** A measure on $T$ induces a transverse invariant measure on $T \times P$. When $B$ is a generalized transversal, $\bar{\Lambda}(B)$ is the value of this transverse invariant measure on $B$. Therefore Definition 3.6 defines an extension of this transverse invariant measure to a map $\bar{\Lambda} : \pi(B^*, \mathcal{B}_\Lambda) \to [0, \infty]$. 
Proposition 3.8. On $B \in \pi(B^*, B_\Lambda)$, $\tilde{\Lambda}$ is well defined and satisfies the following properties:

(a) $\tilde{\Lambda}(\emptyset) = 0$.

(b) $\tilde{\Lambda}(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n=1}^{\infty} \tilde{\Lambda}(B_n)$ for every countable family of disjoint sets $B_n \in \pi(B^*, B_\Lambda), n \in \mathbb{N}$.

Proof. $\tilde{\Lambda}$ is well defined if and only if the function $h : T \rightarrow \mathbb{R} \cup \{\infty\}$, $h(t) = \#(B \cap (\{t\} \times P))$, is measurable with respect to the $\sigma$-algebra $B_\Lambda$ in $T$; i.e., if $h^{-1}(\{n\}) \in B_\Lambda$ for all $n \in \mathbb{N}$. To prove this property, we proceed by induction on $n$. It is clear that $h^{-1}(\{0\}) = T \setminus \pi(B)$ belongs to $B_\Lambda$ since $B \in \pi(B^*, B_\Lambda)$. Now, suppose $h^{-1}(\{i\}) \in B_\Lambda$ for $i \in \{0, ..., n-1\}$ and let us check that $h^{-1}(\{n\}) \in B_\Lambda$. Let

$$C_n = \{((t, p_1), (t, p_2), ..., (t, p_{n+1})) | t \in T, p_1, ..., p_{n+1} \in P\},$$

which is a closed in $(T \times P)^{n+1}$. Observe that $C_n$ is the set of $(n+1)$-uples in $T \times P$ that lie in the same plaque. We remark that $C_n$ is homeomorphic to $\Delta_T \times P^{n+1}$, where $\Delta_T$ is the diagonal of the product $T^{n+1}$, and the projection $\pi_T : \Delta_T \rightarrow T$, $(t, ..., t) \mapsto t$ is a homeomorphism. The measure $\Lambda$ becomes a measure on $\Delta_T$ via $\pi_T$. The intersection $B^{n+1} \cap C_n$, denoted by $D_n$, is the set of $(n+1)$-uples in $B$ that lie in the same plaque. Let

$$\Delta_n = \{((t, p_1), (t, p_2), ..., (t, p_{n+1})) \in C_n | \exists i, j \text{ with } i \neq j \text{ and } p_i = p_j\},$$

which is closed in $C_n$. This set consists of the $(n+1)$-uples in each plaque such that two components are equal. The set $D_n \setminus \Delta_n$ consists of the $(n+1)$-uples of different elements in $B$ that lie in the same plaque. Therefore, $\pi_T \circ \pi_{\Delta}(D_n \setminus \Delta_n)$ consists of the points $t \in T$ such that the corresponding plaque $\{t\} \times P$ contains more than $n$ points of $B$, where $\pi_{\Delta} : C_n \rightarrow \Delta_T$ is the natural projection.

Now, let us prove that $\pi_{\Delta}(D_n \setminus \Delta_n) \in \pi^{-1}_T(B_\Lambda) = \pi^{-1}_T(B)$. By Lemma 3.5, $\tilde{\pi}(B^{n+1} \setminus \Delta_n) \in \langle B^{n+1}_\Lambda \rangle$, where $\tilde{\pi} : (T \times P)^{n+1} \rightarrow T^{n+1}$ is the natural projection. Therefore

$$\pi_{\Delta}(D_n \setminus \Delta_n) = \Delta_T \cap \tilde{\pi}(B^{n+1} \setminus \Delta_n) \in \langle B^{n+1}_\Lambda \rangle, \quad \text{where } \langle B^{n+1}_\Lambda \rangle|_{\Delta_T} \text{ denotes the restriction of the } \sigma\text{-algebra } \langle B^{n+1}_\Lambda \rangle \text{ to } \Delta_T.$$
Moreover

\[
\Lambda \left( \left( \prod_{k=1}^{n+1} B_k \right) \cap \Delta_T \right) \setminus \left( \left( \prod_{k=1}^{n+1} A_k \right) \cap \Delta_T \right) \\
= \Lambda \left( \bigcap_{k=1}^{n+1} (\pi_T^{-1}(B_k) \setminus \pi_T^{-1}(A_k)) \right) = \Lambda \left( \pi_T^{-1} \left( \bigcap_{k=1}^{n+1} (B_k \setminus A_k) \right) \right) \\
= \Lambda \left( \bigcap_{k=1}^{n+1} (B_k \setminus A_k) \right) = 0 .
\]

This shows that \( \pi_\Delta(D_n \setminus \Delta_n) \in \pi_T^{-1}(B) \). By induction, we have

\[
h^{-1}(n) = T \setminus \left( (\pi_T \circ \pi_\Delta(D_n \setminus \Delta_n) \cup h^{-1}(\{0, \ldots, n-1\}) \right) \in B_\Lambda .
\]

Property (a) is obvious. To show property (b), observe that \( \chi_{\bigcup B_n} = \sum \chi_{B_n} \), and then use the monotone convergence theorem.

**Definition 3.9.** If \( B \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, B_\Lambda) \), then define \( \tilde{\Lambda}(B) = \infty \).

**Proposition 3.10.** \((T \times P, \mathcal{B}^*, \tilde{\Lambda})\) is a measure space and \( \tilde{\Lambda} \) extends \( \Lambda \).

**Proof.** We only have to prove that \( \tilde{\Lambda}(\bigcup_n B_n) = \sum_n \Lambda(B_n) \) for every countable family of disjoint sets \( B_n, n \in \mathbb{N} \), in \( \mathcal{B}^* \). By Proposition 3.8, this holds if \( B_n \in \pi(\mathcal{B}^*, B_\Lambda) \) for all \( n \in \mathbb{N} \). If \( \bigcup_n B_n \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, B_\Lambda) \), then the above equality is obvious. So we only have to consider the case where some \( B_j \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, B_\Lambda) \) and, however, \( \bigcup_n B_n \in \pi(\mathcal{B}^*, B_\Lambda) \). We can suppose \( B_j = B_1 \), and let \( B = \bigcup_n B_n \). Let

\[
B^\infty = \{ t \in T \ | \ \#(B \cap \{t\} \times P) = \infty \} ,
\]

which belongs to \( B_\Lambda \) by Proposition 3.8. The proof will be finished by checking that \( \Lambda(B^\infty) > 0 \). We have \( B_1^\infty \subset B^\infty \), where

\[
B_1^\infty = \{ t \in T \ | \ \#(B_1 \cap \{t\} \times P) = \infty \} .
\]

Suppose \( \Lambda(B^\infty) = 0 \). Since \( B^\infty \in B_\Lambda \), there is some \( A \in \mathcal{B} \) such that \( B^\infty \subset A \) and \( \Lambda(A) = 0 \). The Borel set \( \pi^{-1}(A) \) satisfies \( B_1 \cap \pi^{-1}(A) \in \pi(\mathcal{B}^*, B_\Lambda) \) since \( \emptyset \subset \pi(B_1 \cap \pi^{-1}(A) \cup A) \subset A \) and \( \Lambda(A) = 0 \) for each open set \( U \subset T \times P \). On the other hand, \( B_1 \setminus \pi^{-1}(A) \) is a Borel set meeting every plaque in a finite set, which is \( \sigma \)-compact, and therefore projects to a Borel set by Proposition 3.1. Hence \( B_1 \in \pi(\mathcal{B}^*, B_\Lambda) \) by Proposition 3.2 which is a contradiction. \( \square \)

We have constructed an extension of each transverse invariant measure in a product foliated measurable space, but its uniqueness was not proved. This uniqueness is false in general. For instance, take the foliated product \( \mathbb{R} \times \{*\} \) and let \( \Lambda \) be the null measure on the singleton \( \{*\} \); our extension \( \tilde{\Lambda} \) is the zero measure in the total space. Now, let \( \mu \) be the measure defined by

(i) \( \mu(B) = 0 \) for all countable set \( B \); and
(ii) \( \mu(B) = \infty \) for all uncountable Borel set \( B \).

This measure \( \mu \) extends \( \Lambda \) too and is quite different from \( \tilde{\Lambda} \). In order to solve this problem, we require some conditions to the extension. These conditions have the spirit of coherency with the concept of transverse invariant measures. We will prove that our extension is the unique coherent extension.
Definition 3.11. Let $\mu$ be an extension of a transverse invariant measure $\Lambda$ on $T \times P$. The measure $\mu$ is called a coherent extension of $\Lambda$ if satisfies the following conditions:

(a) If $B \in \mathcal{B}^*$, $B \notin \pi^{-1}(S)$ for any $S \in \mathcal{B}$ with $\Lambda(S) = 0$, and $\#B \cap \{t\} \times P = \infty$ for each plaque $\{t\} \times P$ which meets $B$, then $\mu(B) = \infty$.

(b) If $\Lambda(S) = 0$ for some $S \in \mathcal{B}$, then $\mu(\pi^{-1}(S)) = 0$.

(c) If $B \in \mathcal{B}^*$ and $\Lambda(S) = \infty$ for all $S \in \mathcal{B}$ with $B \subset \pi^{-1}(S)$, then $\mu(B) = \infty$.

Remark 3.12. Condition (a) determines $\mu$ on Borel sets with infinite points in plaques which are not contained in the saturation of a $\Lambda$-null set. Condition (b) means certain coherency between the support of $\Lambda$ and the support of the extension $\mu$. Condition (c) determines $\mu$ on any Borel set so that any Borel set containing its projection has infinity $\Lambda$-measure.

Proposition 3.13. $\tilde{\Lambda}$ is the unique coherent extension.

Proof. We prove that every coherent extension has the same values as $\tilde{\Lambda}$ on $\mathcal{B}^*$. First, we consider the case $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$. Let

$$B^\infty = \{ t \in T \mid \#(B \cap \{t\} \times P) = \infty \}.$$ 

This set belongs to $\mathcal{B}_\Lambda$ by Proposition 3.8. Therefore there exist Borel sets $A, C$ such that $A \subset B^\infty \subset C$ and $\Lambda(C \setminus A) = 0$. Let $B^\infty = B \cap \pi^{-1}(C)$. The Borel set $B \setminus B^\infty$ is a generalized transversal and hence $\mu(B \setminus B^\infty) = \Lambda(B \setminus B^\infty)$. On the other hand, if $\Lambda(\pi(B^\infty)) = 0$, then $\Lambda(C) = 0$ and $\mu(B^\infty) = \leq \mu(\pi^{-1}(C)) = 0$ by (b). If $\Lambda(B^\infty)) > 0$, let $B^\infty = B \cap \pi^{-1}(A)$ and $B^\infty = B \cap \pi^{-1}(C \setminus A)$. Then $\mu(B^\infty) = \infty$ by (a), and $\mu(B^\infty) = 0$ by (b). Therefore $\mu$ equals $\tilde{\Lambda}$ on $\pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$.

The case $B \in \mathcal{B}^* \setminus \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ is similar. The set $B^\infty$ is not a Borel set in this case, but observe that $(B \cap \pi^{-1}(B^\infty)) \notin \pi^{-1}(S)$ with $\Lambda(S) < \infty$ or we obtain $\pi(B \cap \pi^{-1}(S) \cap U) \in \mathcal{B}_\Lambda$ for all open set $U \subset T \times P$ by Remark 3.8, since $B \cap \pi^{-1}(S) \cap U$ is a Borel set in $S \times P$ and $\Lambda$ is finite in $S$. Hence $B \in \pi(\mathcal{B}^*, \mathcal{B}_\Lambda)$ by Propositions 3.2 and 3.1. Therefore $\mu(B) = \infty$ by (c). This proves that $\mu$ and $\tilde{\Lambda}$ agree on $\mathcal{B}^*$, as desired. 

4. The general case

In this section, we prove the following theorem.

Theorem 4.1. Let $X$ be a foliated measurable space with a transverse invariant measure $\Lambda$. There exists a measure $\tilde{\Lambda}$ on $X$ that extends $\Lambda$.

Let $\{U_i, \varphi_i\}_{i \in \mathbb{N}}$ be a foliated measurable atlas with $\varphi_i(U_i) = T_i \times \mathbb{R}^n$, where $T_i$ is a standard Borel space. It is clear that $\varphi_i^{-1}(T_i \times \{\ast\})$ is a generalized transversal and, via $\varphi_i$, we obtain a Borel measure $\Lambda_i$ on $T_i$. Proposition 3.10 provides a measure $\tilde{\Lambda}_i$ on $U_i \approx T_i \times \mathbb{R}^n$ that extends $\Lambda_i$. Moreover Proposition 3.1 gives $\tilde{\Lambda}_i(T) = \Lambda(T)$ for all generalized transversal $T \subset U_i$. Let $\pi_i : U_i \rightarrow \varphi_i^{-1}(T_i \times \{\ast\})$ denote the natural projections.

We begin with a description of the change of foliated measurable charts.

Theorem 4.2 (Kumugui, Novikov [7]). Let $\{V_n\}_{n \in \mathbb{N}}$ be a countable base of open sets for a Polish space $P$. Let $B \subset T \times P$ be a Borel set such that $B \cap (\{t\} \times P)$ is
Lemma 4.3. For \( j \in \mathbb{N} \), there exists a sequence of Borel subsets of \( T \) such that

\[
B = \bigcup_n (B_n \times V_n) .
\]

We take a countable base \( \{V_m\}_{m \in \mathbb{N}} \) of \( \mathbb{R}^n \) by connected open sets.

Lemma 4.4. Let \( B \) be a Borel subset of \( U_i \cap U_j \), \( i, j \in \mathbb{N} \). Then

\[
B \in \pi_i(B^*, B\Lambda) \iff B \in \pi_j(B^*, B\Lambda) .
\]

Proof. By Lemma 4.3 there exists a countable family of measurable holonomy transformations from \( \varphi_i^{-1}(T_i \times \{\ast\}) \) to \( \varphi_j^{-1}(T_j \times \{\ast\}) \) whose domains and ranges cover \( \pi_i(U_i \cap U_j) \) and \( \pi_j(U_i \cap U_j) \), respectively. Therefore, if \( A \) is a Borel set contained in \( U_i \cap U_j \) and \( \pi_i(A) \) is a Borel set, then \( \pi_j(A) \) is a Borel set and

\[
\text{\Lambda} (\pi_i(A)) = 0 \iff \text{\Lambda} (\pi_j(A)) = 0 .
\]

Lemma 4.5. \( \tilde{\Lambda}_i(B) = \tilde{\Lambda}_j(B) \) for all Borel set \( B \subset U_i \cap U_j \), \( i, j \in \mathbb{N} \).

Proof. We remark that \( \tilde{\Lambda}_i \) and \( \tilde{\Lambda}_j \) have the same values in generalized transversals of \( U_i \cap U_j \). By Lemma 4.4 we only consider Borel sets in \( \pi_i(B^*, B\Lambda) \). Suppose that \( \pi_i(B) \) is a Borel set; otherwise, \( \pi_i(B) \) is \( \Lambda \)-measurable and we can choose a Borel set \( A \subset \pi_i(B) \) with \( \text{\Lambda} (\pi_i(B) \setminus A) = 0 \). We take the Borel set \( B = B \cap \pi_i^{-1}(A) \).

This Borel set projects onto the Borel set \( A \) and \( \tilde{\Lambda}_i(B \setminus \tilde{B}) = \tilde{\Lambda}_j(B \setminus \tilde{B}) = 0 \) by Definition 3.6 and Lemma 4.4 hence \( \tilde{\Lambda}_i(B) = \tilde{\Lambda}_j(B) \) and \( \tilde{\Lambda}_j(B) = \tilde{\Lambda}_j(B) \). Let

\[
B^k = \{ t \in T_i \mid \#(\varphi_i(B) \cap (\{t\} \times \mathbb{R}^n)) = k \}, \quad k \in \mathbb{N} \cup \{\infty\} .
\]
These are \( \Lambda \)-measurable sets by Proposition 3.8 and we assume that they are Borel sets by the same reason as above. Let \( \tilde{B}_k \) denote \( B \cap \pi_i^{-1}(\varphi_i^{-1}(B^k \times \{\ast\})) \), which is a Borel set. It is obvious that \( \bigcup_{i=1}^{\infty} \tilde{B}_k \) is a generalized transversal, hence \( \Lambda_i(\bigcup_{k=1}^{\infty} \tilde{B}_k) = \Lambda_j(\bigcup_{k=1}^{\infty} \tilde{B}_k) \). Now consider \( \tilde{B}_\infty^l = \{ x \in \tilde{B}_\infty \mid \#(B \cap P_j^l) = l \} \), \( l \in \mathbb{N} \cup \{\infty\} \), where \( P_j^l \) denotes the plaque of \( U_j \) that contains \( x \). The proof is finished in the case \( \Lambda(\pi_i(\tilde{B}_\infty^\infty)) = 0 \) (we can restrict to the case of a generalized transversal). If \( \Lambda(\pi_i(\tilde{B}_\infty^\infty)) > 0 \), then \( \Lambda(\pi_j(\tilde{B}_\infty^\infty)) > 0 \). Therefore we obviously obtain \( \widetilde{\Lambda}_i(B) = \infty = \widetilde{\Lambda}_j(B) \). \( \square \)

**Definition 4.6.** Let \( B \) be a measurable set in \( X \), and \( B_1 = B \cap U_1 \), \( B_k = (B \cap U_k) \setminus (B_1 \cup \ldots \cup B_{k-1}) \), for \( k \geq 2 \). Define

\[
\tilde{\Lambda}(B) = \sum_{i=1}^{\infty} \tilde{\Lambda}_i(B_i).
\]

By Lemma 4.5 it is easy to prove that Definition 4.6 does not depend neither on the ordering of the charts nor on the choice of the countable foliated measurable atlas. It is also easy to prove that \( \tilde{\Lambda} \) extends \( \Lambda \) since both of them have the same values on generalized transversals contained in each chart and, hence, in every generalized transversal. Theorem 4.1 is now established.

**Definition 4.7.** Let \( \mu \) be an extension of a transverse invariant measure \( \Lambda \) on a foliated measurable space \( X \). The measure \( \mu \) is called a **coherent extension** of \( \Lambda \) if it is a coherent extension on each foliated measurable chart with the induced transverse invariant measure.

**Corollary 4.8.** The extension \( \tilde{\Lambda} \) is the unique coherent extension of \( \Lambda \).

Theorem 1.1 gives a new interpretation of transverse invariant measures. It can be also used to introduce the following version of the concept of transversal for foliated measurable spaces with transverse invariant measures.

**Definition 4.9.** Let \( X \) be a foliated measurable space with a transverse invariant measure \( \Lambda \). A Borel subset of \( X \) with finite \( \tilde{\Lambda} \)-measure is called a \( \Lambda \)-generalized transversal.

**Remark 4.10.** In Section 3, we have only used that the plaques are Polish spaces. We can weaken the conditions of foliated measurable spaces taking charts with the form \( T \times P \), where \( P \) is any connected and locally connected Polish space. In this way our result can be extended to other interesting cases like measurable graphs [1].

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