ESTIMATION OF WEIGHTED $L^2$ NORM RELATED TO DEMAILLY’S STRONG OPENNESS CONJECTURE

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Abstract. In the present article, we obtain an estimation of the weighted $L^2$ norm near the singularities of plurisubharmonic weight related to Demailly’s strong openness conjecture, which implies the convergence of the weighted $L^2$ norm.

1. Introduction

Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, $o \in D$ the origin of $\mathbb{C}^n$ and $\varphi \in Psh(D)$ a plurisubharmonic function on $D$. The multiplier ideal sheaf $\mathcal{I}(\varphi)$ consists of germs of holomorphic functions $f$ such that $|f|^2 e^{-\varphi}$ is locally integrable, which is a coherent sheaf of ideals (see [1]).

Demailly’s strong openness conjecture (SOC) [2]: If $(f,o) \in \mathcal{I}(\varphi)_o$, then there exists $\varepsilon > 0$ such that $(f,o) \in \mathcal{I}((1 + \varepsilon)\varphi)_o$.

Note that $\mathcal{I}(\varphi)_o$ is finitely generated by $(f_j)_{j=1,\ldots,k_0}$. Let $\mathcal{I}(\varphi)_o = (f_1,\ldots,f_{k_0})$. The truth of SOC implies that there exists $\varepsilon_j > 0$ such that $(f_j,o) \in \mathcal{I}((1 + \varepsilon_j)\varphi)_o$ for any $1 \leq j \leq k_0$. Then, SOC is equivalent to $\mathcal{I}(\varphi)_o = \mathcal{I}_+(\varphi)_o$, where $\mathcal{I}_+(\varphi)_o = \bigcup_{\varepsilon > 0} \mathcal{I}((1 + \varepsilon)\varphi)_o \subset \mathcal{I}(\varphi)_o$.

In [8], Guan and Zhou proved the above SOC. Moreover, they also established an effectiveness about $\varepsilon$ of the conjecture in [9].

Let $L^2(D)$ be the Hilbert space of homomorphic functions on $D$ with finite $L^2$ norm

$$L^2(D) := \{ f \in \mathcal{O}(D) | ||f||^2_D = \int_D |f|^2 d\lambda_n < \infty \},$$

whose inner product is defined to be $(f,g) = \int_D f \cdot \overline{g} d\lambda_n$.

Let $I \subset \mathcal{O}_o$ be an ideal and $(e_k)_{k \in \mathbb{N}^+}$ an orthonormal basis of $\mathcal{H}_I := \{ f \in L^2(D) | (f,o) \in I \}$, a closed subspace of $L^2(D)$. It is known that there exists a neighborhood $U_0 \subset \subset D$ of $o$, integer $k_0 > 0$ and some constant $C_0 > 1$ such that

$$\sum_{k=1}^{\infty} |e_k|^2 \leq C_0 \cdot \sum_{k=1}^{k_0} |e_k|^2 \quad \text{on} \ U_0.$$
one can see the detail in Lemma 2.1.

Put

\[ C = C_{\varepsilon_0}(\varphi) := \left[ \left( \frac{e^{e_0+1}(t_0+1)}{\varepsilon_0} \right) C_0 \sum_{k=1}^{k_0} \int_D \mathbb{I}_{\{\varphi < -t_0\}} |e_k|^2 d\lambda_n \right]^{-1}, \]

where \( t_0, \varepsilon_0 \) are two positive numbers and \( \varphi \) is negative on \( D \) with \( \varphi(0) = -\infty \).

In the present article, we obtain the following estimation of the weighted \( L^2 \) norm near the singularities of plurisubharmonic weight related to SOC:

**Theorem 1.1.** Assume that \( \mathcal{I}(\varphi) \subset I \subset \mathcal{O}_o \). If \( C > 0 \), then

\[ \int_{U_0 \cap \{\varphi < -(t_0+1)\}} \left( \sum_{k=1}^{k_0} |e_k|^2 \right) e^{-\varphi} d\lambda_n < C^2. \]

**Corollary 1.1.** Let \( e_k \) \((1 \leq k \leq k_0)\) be generators of \( I = \mathcal{I}(\varphi)_o \) with bounded \( \sum_{k=1}^{k_0} |e_k| \) on \( D \), which is in the unit ball \( B(o;1) \) and

\[ \sum_{k=1}^{k_0} \int_D |e_k|^2 e^{-\left((1+\varepsilon_0)\varphi\right)} d\lambda_n < \infty. \]

Then, for any \( M > 0 \), there exists \( t_0 \gg 0 \) such that for any negative plurisubharmonic function \( \psi \) on \( D \) with \( \mathcal{I}(\psi)_o \subset \mathcal{I}(\varphi)_o \) and

\[ \sum_{k=1}^{k_0} \int_D \mathbb{I}_{\{\tilde{\varphi} < -t_0\}} |e_k|^2 d\lambda_n \leq 2 \sum_{k=1}^{k_0} \int_D \mathbb{I}_{\{\tilde{\varphi} < -t_0\}} |e_k|^2 d\lambda_n, \]

we have

\[ \int_{U_0 \cap \{|\tilde{\varphi}| < e^{-\left((1+\varepsilon_0)(1+\varepsilon_0/2)\right)} \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-\tilde{\varphi}} d\lambda_n < M, \]

where

\[ \tilde{\varphi} = \varphi + \frac{\varepsilon_0/2}{(1+\varepsilon_0)(1+\varepsilon_0/2)} \log|z|, \quad \psi = \psi + \frac{\varepsilon_0/2}{(1+\varepsilon_0)(1+\varepsilon_0/2)} \log|z|. \]

By the truth of SOC and the above Corollary, we have the following convergence of the weighted \( L^2 \) norm related to SOC:

**Corollary 1.2.** Let \((\varphi_j)_{j \in \mathbb{N}^+}\) be a sequence of negative plurisubharmonic functions on \( D \), which is convergent to \( \varphi \) in Lebesgue measure, and \( \mathcal{I}(\varphi)_o \subset \mathcal{I}(\varphi)_o \). Let \((F_j)_{j \in \mathbb{N}^+}\) be a sequence of holomorphic functions on \( D \) with \((F_j, o) \in \mathcal{I}(\varphi)_o \), which is compactly convergent to a holomorphic function \( F \). Then, \( |F_j|^2 e^{-\varphi_j} \) converges to \( |F|^2 e^{-\tilde{\varphi}} \) in the \( L^p_{\text{loc}} \) norm near \( o \). In particular, there exists \( \varepsilon_0 > 0 \) such that \( \mathcal{I}(\varphi_j)_o = \mathcal{I}((1+\varepsilon_0)\varphi)_o = \mathcal{I}(\varphi)_o \) for any large enough \( j \).

The last conclusion in the above Corollary can be obtained by Proposition 1.8 in [3] and finite generation of \( \mathcal{I}(\varphi)_o \).

**Remark 1.1.** Let \((\varphi_j)_{j \in \mathbb{N}^+}\) be a sequence of negative plurisubharmonic functions on \( D \). If \( \varphi_j \) is convergent to \( \varphi \) in Lebesgue measure, then \( \varphi_j \) converges to \( \varphi \) in the \( L^p_{\text{loc}} \) \((0 < p < \infty)\) norm.
Proof. It suffices to prove \( p \in \mathbb{N}^+ \). By a small enough multiplication, it is enough to assume the Lelong number \( \nu(c,o) < 1 \). Thus, \( \mathcal{I}(\varphi_j)_o \subset \mathcal{I}(\varphi)_o = \mathcal{O}_o \). Then, the desired result follows from Corollary 1.2 and the inequality
\[
\frac{1}{p!} \int_D |\varphi_j - \varphi|^p d\lambda_n \leq \int_D |e^{-\varphi_j} - e^{-\varphi}| d\lambda_n.
\]
which follows from the inequality \( \frac{1}{p!}(a - b)^p \leq (e^{a-b} - 1)e^b \), for any \( a \geq b \geq 0 \). \( \square \)

2. Lemmas used in the proof of main results

We are now in a position to prove the following Lemma.

**Lemma 2.1.** Let \( I \subset \mathcal{O}_o \) be an ideal and \((e_k)_{k \in \mathbb{N}^+}\) an orthonormal basis of
\[
\mathcal{H}_I := \{f \in L^2_0(D) | (f,o) \in I\},
\]
a closed subspace of \( L^2_0(D) \). Then, there exists a neighborhood \( U_0 \subset D \) of \( o \), integer \( k_0 > 0 \) and some constant \( C_0 > 1 \) such that
\[
\sum_{k=1}^{\infty} |e_k|^2 \leq C_0 \cdot \sum_{k=1}^{k_0} |e_k|^2 \quad \text{on } U_0.
\]

**Proof.** It follows from the strong Noetherian property of coherent analytic sheaves that the sequence of ideal sheaves generated by the holomorphic functions
\[
(e_k(z)e_k(\overline{w}))_{k \leq N}, \quad N = 1, 2, ..., \]
on \( D \times D \) is locally stationary.

Let \( U \subset D \) be a neighborhood of the origin \( o \). Then there exists \( k_0 > 0 \) such that for any \( N \geq k_0 \) we have \((e_k(z)e_k(\overline{w}))_{k \leq N} = (e_k(z)e_k(\overline{w}))_{k \leq k_0} \) on \( U \). Since
\[
|\sum_{k=1}^{\infty} e_k(z)e_k(\overline{w})| \leq \left( \sum_{k=1}^{\infty} |e_k(z)|^2 \right) \sum_{k=1}^{\infty} |e_k(\overline{w})|^2,
\]
then \( \sum_{k=1}^{\infty} e_k(z)e_k(\overline{w}) \) is uniformly convergent on every compact subset of \( D \times D \). By the closedness of coherent ideal sheaves under the topology of compact convergence (see [3]), \( \sum_{k=1}^{\infty} e_k(z)e_k(\overline{w}) \) is a section of the coherent ideal sheaf generated by \((e_k(z)e_k(\overline{w}))_{k \leq k_0} \) over \( U \times U \). Then, there exists a neighborhood \( U_0 \subset U \) of \( o \) and functions \( a_k(z,w) \in \mathcal{O}(U_0 \times U_0) \), \( 1 \leq k \leq k_0 \), such that on \( U_0 \times U_0 \)
\[
\sum_{k=1}^{\infty} e_k(z)e_k(\overline{w}) = \sum_{k=1}^{k_0} a_k(z,w)e_k(z)e_k(\overline{w}).
\]
Finally, by restricting to the conjugate diagonal \( w = \overline{z} \), we get
\[
\sum_{k=1}^{\infty} |e_k|^2 \leq C_0 \cdot \sum_{k=1}^{k_0} |e_k|^2 \quad \text{on } U_0.
\]
\( \square \)

To prove Theorem [4], we also need the following Lemma, whose various forms already appear in [5] [6] [7] [9].
Lemma 2.2. Let $B_0 \in (0,1]$ be arbitrarily given and $t_0$ a positive number. Let $D_v$ be a strongly pseudoconvex domain relatively compact in $\Delta^n$ containing $\alpha$. Let $F$ be a holomorphic function on $\Delta^n$. Let $\varphi, \psi$ be two negative plurisubharmonic functions on $\Delta^n$, such that $\varphi(\alpha) = \psi(\alpha) = -\infty$. Then there exists a holomorphic function $F_{v,t_0}$ on $D_v$, such that,

$$ (F_{v,t_0} - F, \varphi) \in \mathcal{J}(\varphi + \psi)_v $$

and

$$ \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\psi))F|^2 e^{-\varphi} d\lambda_n $$

$$ \leq (1 - e^{-(t_0+B_0)}) \int_{D_v} \frac{1}{B_0} (\mathbb{1}_{(-t_0-B_0<\psi<-t_0)}) |F|^2 e^{-\varphi-\psi} d\lambda_n, $$

(1)

where $b_{t_0}(t) = \int_{-\infty}^{t} \frac{1}{B_0} \mathbb{1}_{(-t_0-B_0<s<-t_0)} ds$.

In particular, given $\varepsilon_0 > 0$ and replacing $B_0, t_0, \psi$ by $\varepsilon_0, \varepsilon_0 t_0, \varepsilon_0 \varphi$ respectively, we have

$$ \int_{D_v} |F_{v,t_0} - (1 - b_{t_0}(\varepsilon_0 \varphi))F|^2 e^{-\varphi} d\lambda_n $$

$$ \leq \frac{1 - e^{-(t_0+B_0)}}{\varepsilon_0} \int_{D_v} \mathbb{1}_{(-t_0+1)<\psi<-t_0} |F|^2 e^{-\varphi-\varepsilon_0 \varphi} d\lambda_n $$

$$ \leq \frac{1}{\varepsilon_0} \int_{D_v} \mathbb{1}_{(-t_0+1)<\psi<-t_0} |F|^2 e^{(\varepsilon_0+1)(t_0+1)} d\lambda_n. $$

(2)

The following Lemma is well known in real analysis.

Lemma 2.3. Let $(f_j)_{j \in \mathbb{N}^+}$ be a sequence of functions in $L^p_{loc}(D)$ $(p > 1)$, which is convergent to $f$ in Lebesgue measure. If there exists some constant $M > 0$ such that

$$ \left( \int_D |f_j|^p d\lambda_n \right)^{\frac{1}{p}} < M, $$

then

$$ \int_D |f_j - f| d\lambda_n \to 0 \quad (j \to \infty). $$

3. THE PROOF OF MAIN RESULTS

Proof of Theorem 1.1. Following from Lemma 2.2 for any $1 \leq k \leq k_0$, there exists a holomorphic function $F_k \in \mathcal{O}(D)$ such that

$$ \int_D |F_k - (1 - b_{t_0}(\varepsilon_0 \varphi))e_k|^2 e^{-\varphi} d\lambda_n $$

$$ \leq \frac{1}{\varepsilon_0} \int_D \mathbb{1}_{(-t_0+1)<\psi<-t_0} |e_k|^2 e^{(\varepsilon_0+1)(t_0+1)} d\lambda_n. $$

(3)

By Minkowski’s inequality, we obtain

$$ \left( \sum_{k=1}^{k_0} \int_D |F_k|^2 d\lambda_n \right)^{\frac{1}{2}} $$

$$ \leq \left( \sum_{k=1}^{k_0} \int_D |F_k - (1 - b_{t_0}(\varepsilon_0 \varphi))e_k|^2 e^{-\varphi} d\lambda_n \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{k_0} \int_D (1 - b_{t_0}(\varepsilon_0 \varphi))e_k|^2 d\lambda_n \right)^{\frac{1}{2}} $$

(4)
It follows from (3) and $0 \leq 1 - b_{t_0}(\varepsilon_0 \varphi) \leq 1_{\{\varphi < -t_0\}}$ that

$$
\left( \sum_{k=1}^{k_0} \int_D |F_k|^2 d\lambda_n \right)^{\frac{1}{2}} 
\leq \left( \frac{1}{\varepsilon_0} \sum_{k=1}^{k_0} \int_D 1_{\{(1 + \varepsilon_0)\varphi < -t_0\}} |e_k|^2 e^{(\varepsilon_0 + 1)(t_0 + 1)} d\lambda_n \right)^{\frac{1}{2}} + \left( \sum_{k=1}^{k_0} \int_D 1_{\{\varphi < -t_0\}} |e_k|^2 d\lambda_n \right)^{\frac{1}{2}} 
\leq \left( \frac{e^{(\varepsilon_0 + 1)(t_0 + 1)}}{\varepsilon_0} \right)^{\frac{1}{2}} + 1 \left( \sum_{k=1}^{k_0} \int_D 1_{\{\varphi < -t_0\}} |e_k|^2 d\lambda_n \right)^{\frac{1}{2}}.
$$

(5)

By Lemma 2.2 we know that $(F_k - e_k, o) \in \mathcal{F}((1 + \varepsilon_0)\varphi) \subset \mathcal{F}(\varphi) \subset I$ and $(F_k, o) \in I$.

Hence, we have

$$
F_k = \sum_{j=1}^{\infty} a_k^j e_j, \quad a_k^j \in \mathbb{C}, \quad 1 \leq k \leq k_0,
$$

and

$$
\int_D |F_k|^2 d\lambda_n = \sum_{j=1}^{\infty} |a_k^j|^2, \quad 1 \leq k \leq k_0.
$$

By Lemma 2.1 the following holds on $U_0$,

$$
\left( \sum_{k=1}^{k_0} |F_k - e_k|^2 \right)^{\frac{1}{2}} \geq \left( \sum_{k=1}^{k_0} |e_k|^2 \right)^{\frac{1}{2}} - \left( \sum_{k=1}^{k_0} |F_k|^2 \right)^{\frac{1}{2}} 
\geq \left( \frac{1}{C_0} \right)^{\frac{1}{2}} \left( \sum_{k=1}^{k_0} |e_k|^2 \right)^{\frac{1}{2}} - \left( \sum_{k=1}^{k_0} \left( \sum_{j=1}^{\infty} |a_k^j|^2 \right) \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |e_k|^2 \right)^{\frac{1}{2}} 
\geq \left( \frac{1}{C_0} \right)^{\frac{1}{2}} - \left( \sum_{k=1}^{k_0} \int_D |F_k|^2 d\lambda_n \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |e_k|^2 \right)^{\frac{1}{2}} 
\geq \left( \frac{1}{C_0} \right)^{\frac{1}{2}} - \left( \frac{e^{(\varepsilon_0 + 1)(t_0 + 1)}}{\varepsilon_0} \right)^{\frac{1}{2}} + 1 \left( \sum_{k=1}^{k_0} \int_D 1_{\{\varphi < -t_0\}} |e_k|^2 d\lambda_n \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} |e_k|^2 \right)^{\frac{1}{2}}.
$$

(6)

Denote by

$$
A := \left( \frac{1}{C_0} \right)^{\frac{1}{2}} - \left( \frac{e^{(\varepsilon_0 + 1)(t_0 + 1)}}{\varepsilon_0} \right)^{\frac{1}{2}} + 1 \left( \sum_{k=1}^{k_0} \int_D 1_{\{\varphi < -t_0\}} |e_k|^2 d\lambda_n \right)^{\frac{1}{2}}.
$$

Since $C_{c_0}(\varphi) > 0$ and

$$
A \cdot C_{c_0}(\varphi) = \left( \frac{e^{(\varepsilon_0 + 1)(t_0 + 1)}}{\varepsilon_0} \right)^{\frac{1}{2}} \sum_{k=1}^{k_0} \int_D 1_{\{\varphi < -t_0\}} |e_k|^2 d\lambda_n \right)^{\frac{1}{2}} > 0,
$$

it follows that $A > 0$. 
Then from (6) we obtain

\[
A^2 \cdot \left( \int_{\{\phi < -(t_0+1)\} \cap U_0} \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-\phi} d\lambda_n \right)
\]

\[
\leq \int_{\{\phi < -(t_0+1)\} \cap U_0} \left( \sum_{k=1}^{k_0} |F_k - e_k|^2 \right) e^{-\phi} d\lambda_n
\]

\[
= \sum_{k=1}^{k_0} \int_{\{\phi < -(t_0+1)\} \cap U_0} |F_k - e_k|^2 e^{-\phi} d\lambda_n.
\]

Note that

\[
\sum_{k=1}^{k_0} |F_k - (1 - b_{t_0}(\varepsilon_0 \phi)) e_k|^2 \mathbb{1}_{\{\phi < -(t_0+1)\} \cap U_0} = \sum_{k=1}^{k_0} |F_k - e_k|^2.
\]

It follows from Lemma 2.2 that

\[
\sum_{k=1}^{k_0} \int_{D} |F_k - (1 - b_{t_0}(\varepsilon_0 \phi)) e_k|^2 e^{-\phi} d\lambda_n
\]

\[
\leq \frac{1}{\varepsilon_0} \sum_{k=1}^{k_0} \int_{D} \mathbb{1}_{\{-(t_0+1) < \phi < -t_0\}} |e_k|^2 e^{(\varepsilon_0+1) (t_0+1)} d\lambda_n
\]

\[
\leq e^{(\varepsilon_0+1) (t_0+1)} \frac{1}{\varepsilon_0} \sum_{k=1}^{k_0} \int_{D} \mathbb{1}_{\{\phi < -t_0\}} |e_k|^2 d\lambda_n.
\]

Combining inequalities (7) and (8), we have

\[
\int_{\{\phi < -(t_0+1)\} \cap U_0} \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-\phi} d\lambda_n
\]

\[
\leq \frac{1}{A^2} \cdot \frac{e^{(\varepsilon_0+1) (t_0+1)}}{\varepsilon_0} \sum_{k=1}^{k_0} \int_{D} \mathbb{1}_{\{\phi < -t_0\}} |e_k|^2 d\lambda_n
\]

\[
= \left[ \frac{e^{(\varepsilon_0+1) (t_0+1)}}{\varepsilon_0} C_0 \sum_{k=1}^{k_0} \int_{D} \mathbb{1}_{\{\phi < -t_0\}} |e_k|^2 d\lambda_n \right] - \frac{1}{2} \left( 1 + \frac{e^{(\varepsilon_0+1) (t_0+1)}}{\varepsilon_0} \right)^{-\frac{1}{2}}
\]

\[
= C_\varepsilon^2_0(\phi).
\]

Remark 3.1. If \( I = \mathcal{I}(\phi) = \mathcal{I}(1+\varepsilon_0 \phi) \) in the above theorem and

\[
\sum_{k=1}^{k_0} \int_{D} |e_k|^2 e^{-(1+\varepsilon_0) \phi} d\lambda_n < \infty,
\]

then for any \( \varepsilon_1, \varepsilon_2 > 0 \), there exists \( t_0 \gg 0 \) such that

\[
\sum_{k=1}^{k_0} \int_{D} \mathbb{1}_{\{-(t_0+1) < \phi < -t_0\}} |e_k|^2 e^{(\varepsilon_0+1) (t_0+1)} d\lambda_n < \varepsilon_1
\]
and
\[ \sum_{k=1}^{k_0} \int_D |(1 - b_{t_0}(\varepsilon_0 \varphi))e_k|^2 d\lambda_n < \varepsilon_2. \]

Furthermore, we can get
\[ \int_{\{\varphi < -(t_0+1)\} \cap t_0} \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-\varphi} d\lambda_n \leq \left( \left( \frac{\varepsilon_1}{\varepsilon_0} \cdot C_0 \right)^{-1/2} - (1 + \left( \frac{\varepsilon_1}{\varepsilon_0} \right)^{-1/2})^{-2} \right)^2. \]

**Proof of Corollary 1.1.** By Hölder inequality, we have
\[
\int_U |F|^2 e^{-\left(1+\varepsilon_0/2\right)\tilde{\varphi}} d\lambda_n \\
\leq \left( \int_U |F|^2 e^{-\left(1+\varepsilon_0\right)\varphi} d\lambda_n \right)^{1+\varepsilon_0/2} \left( \int_U |F|^2 e^{-\log|z|/2} d\lambda_n \right)^{\varepsilon_0/2},
\]
which implies \( \mathcal{J}((1 + \varepsilon_0)\varphi) \subset \mathcal{J}((1 + \varepsilon_0/2)\tilde{\varphi}) \subset \mathcal{J}(\tilde{\varphi}) \subset \mathcal{J}(\varphi) \), i.e.,
\[ \mathcal{J}((1 + \varepsilon_0/2)\tilde{\varphi})_o = \mathcal{J}(\tilde{\varphi})_o = \mathcal{J}(\varphi)_o. \]

As
\[ \sum_{k=1}^{k_0} \int_D |e_k|^2 e^{-\left(1+\varepsilon_0\right)\varphi} d\lambda_n < \infty, \]
there exists \( t_0 \gg 0 \) such that \( 0 < C_{\varepsilon_0/2} / \sqrt{M} < \varepsilon_0/2 \), and \( 0 < C_{\varepsilon_0/2}(\tilde{\varphi}) \leq 2 \cdot C_{\varepsilon_0/2}(\tilde{\varphi}) \) by (†).

Since \( \bar{\psi} \leq \frac{e^{\varepsilon_0/2}}{(1+\varepsilon_0)(1+\varepsilon_0/2)} \log|z| \) on \( D \), we have
\[ \{|z| < e^{-\left[(1+\varepsilon_0)(1+\varepsilon_0/2)(t_0+1)\right]} \} \subset \{|\bar{\psi}| < -(t_0+1)\}. \]

Then, we obtain that
\[ \int_{U_0 \cap \{|z| < e^{-\left[(1+\varepsilon_0)(1+\varepsilon_0/2)(t_0+1)\right]}} \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-\bar{\psi}} d\lambda_n \\
\leq \int_{U_0 \cap \{\bar{\psi} < -(t_0+1)\}} \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-\bar{\psi}} d\lambda_n \leq C^2_{\varepsilon_0/2}(\tilde{\psi}) < M. \]

\[ \square \]

**Proof of Corollary 1.2.** As every sequence which is convergent in Lebesgue measure has a subsequence which is convergent almost everywhere, then it is sufficient to prove the result for the case that \( \varphi_j \) is convergent to \( \varphi \) almost everywhere.

By the truth of SOC, there exists \( \varepsilon_0 > 0 \) such that \( \mathcal{J}(\varphi) = \mathcal{J}((1 + \varepsilon_0)\varphi) \) on a neighborhood \( D \) of \( o \). Without loss of generality, we assume the unit ball \( B(\alpha; 1) \supset D \).

Since \( F_j \) is compactly convergent to a holomorphic function \( F \), by shrinking \( D \), we can assume that \( \int_D |F_j|^2 d\lambda_n \) is uniformly bounded. Let \( e_k, 1 \leq k \leq k_0, \) be as in Corollary 1.1. Then, we infer from \( (F_j, o) \in \mathcal{J}(\varphi)_o \) and Lemma 2.1 that there exist complex numbers \( a_j^k \) such that \( F_j = \sum_{k=1}^{\infty} a_j^k e_k \) and \( \sum_{k=1}^{\infty} |a_j^k|^2 = \int_D |F_j|^2 d\lambda_n \) is uniformly bounded.
Since \( \varphi_j \) is convergent to \( \varphi \) almost everywhere, it follows from the dominated convergence theorem that
\[
\sum_{k=1}^{k_0} \int_D \mathbb{I}_{(\varphi_j < -t_0)} |e_k|^2 d\lambda_n \leq 2 \sum_{k=1}^{k_0} \int_D \mathbb{I}_{(\varphi < -t_0)} |e_k|^2 d\lambda_n,
\]
where \( \varphi_j = \varphi + \frac{\log |z|}{(1+\varepsilon_0)(1+\varepsilon_0/j)^2} \). By Corollary 1.1 there exists a neighborhood \( V_0 \subset \subset D \) of \( o \) and \( M > 0 \) such that
\[
\int_{V_0} \sum_{k=1}^{\infty} |e_k|^2 e^{-\varphi_j} d\lambda_n < M.
\]

Let \( \varepsilon \in (0, \varepsilon_0) \). Replacing \( \varphi \) by \((1+\varepsilon/2)\varphi \) and \( \varphi_j \) by \((1+\varepsilon/2)\varphi_j \), we have
\[
\int_{V_0} \sum_{k=1}^{\infty} |e_k|^2 e^{-(1+\varepsilon/2)\varphi_j} d\lambda_n < \tilde{M},
\]
for some neighborhood \( \tilde{V}_0 \supset o \) and some constant \( \tilde{M} \) which are independent of \( \varphi_j \).

As \( \sum_{k=1}^{\infty} |a_j^k|^2 \) is uniformly bounded, by Schwarz inequality, it follows that
\[
\int_{\tilde{V}_0} |F_j|^2 e^{-(1+\varepsilon/2)\varphi_j} d\lambda_n \leq \int_{\tilde{V}_0} \left( \sum_{k=1}^{\infty} |a_j^k|^2 \right) \cdot \left( \sum_{k=1}^{\infty} |e_k|^2 \right) e^{-(1+\varepsilon/2)\varphi_j} d\lambda_n
\]
is uniformly bounded. Then, by Lemma 2.3 we obtain that \( F_j e^{-\varphi_j} \) converges to \( F e^{-\varphi} \) as \( j \) goes to infinity in the \( L^1_{loc} \) norm on \( \tilde{V}_0 \).

Replacing \( \varphi_j \) by \((1+\varepsilon_0)\varphi_j \), we obtain the second assertion from the first one. \( \square \)

4. Relation to semi-continuity of complex singularity exponents

In [3], Demailly and Kollár proved the following semi-continuity of complex singularity exponents.

**Theorem 4.1.** (Main Theorem 0.2, [3]). Let \( X \) be a complex manifold, \( K \subset X \) a compact subset and \( \varphi \) a plurisubharmonic function on \( X \). If \( c < c_K(\varphi) \) and \( (\varphi_j) \) is a sequence of plurisubharmonic functions on \( X \) which is convergent to \( \varphi \) in \( L^1_{loc} \) norm, then \( e^{-2c\varphi_j} \) converges to \( e^{-2c\varphi} \) in \( L^1 \) norm over some neighborhood \( U \) of \( K \).

Indeed, by subtracting a constant, we can assume \( \varphi \) is negative on \( K \). As
\[
\int_K \varphi_j d\lambda_n \leq \int_K |\varphi - \varphi_j| d\lambda_n + \int_K \varphi d\lambda_n,
\]
we obtain that \( \varphi_j \) is also negative on \( K \). Then, **Theorem 4.1** is a special case of Corollary 1.2 when \( \mathcal{F}(\varphi)_o = O_o \). With additional condition \( \varphi_j \leq \varphi \), **Theorem 4.1** can be referred to [10] for multiplier ideals, which is also a special case of Corollary 1.2.

If \( \varphi = \log |g| \) for \( J \)-vector holomorphic functions \( g(z,c) = (g_1, ..., g_J) \) on a polydisk \( \Delta^n \times \Delta \), then it follows that

**Theorem 4.2.** (Main Theorem, [11]). Assume that \( \int_{\Delta^n} |g(z,0)|^{-\delta} < \infty \). Then there exists a smaller polydisk \( \Delta'^n \times \Delta' \) so that the function \( c \mapsto \int_{\Delta'^n} |g(z,c)|^{-\delta} \) is finite and continuous for \( c \in \Delta' \).
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