A rational parametrization of Bezier like curves

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Résumé

Une classe de paramétrisation rationnelle a été développée puis utilisée pour générer des fonctions de Bernstein. La nouvelle famille de bases obtenue dépend d’un indice \( \alpha \in (-\infty, 0) \cup (1, +\infty) \), et pour un degré \( k \in \mathbb{N}^* \), est formée de fonctions rationnelles dont le numérateur et le dénominateur sont de degré \( k \). Ces bases de Bernstein rationnelles vérifient toutes les propriétés classiques notamment la positivité, la partition de l’unité et constituent de véritables bases d’approximation de fonctions continues.

Les courbes de Bézier associées vérifient les propriétés classiques et on a obtenu les algorithmes d’évaluation comme celui de deCasteljau et de subdivision avec une complexité équivalente. Pour le même degré \( k \) et le même polygone de contrôle ces algorithmes convergent vers la même courbe de Bézier que le cas standard.

Les bases de Bernstein polynomiales classiques se revèlent comme un cas asymptotique de la nouvelle classe.

Mots clés : Fonctions de Bernstein rationnelles, Approximation de fonctions, Courbe de Bézier, Algorithme de deCasteljau.

Abstract

In this paper, we construct a family of Bernstein functions using a class of rational parametrization. The new family of rational Bernstein basis on an index \( \alpha \in (-\infty, 0) \cup (1, +\infty) \), and for a given degree \( k \in \mathbb{N}^* \), these basis functions are rational with a numerator and a denominator are polynomials of degree \( k \). All of the classical properties as positivity, partition of unity are hold for these rational Bernstein basis and they constitute approximation basis functions for continuous functions spaces.

The Bézier curves obtained verify the classical properties and we have the classical computational algorithms like the deCasteljau Algorithm and the algorithm of subdivision with the similar accuracy.

Given a degree \( k \) and a control polygon points all of these algorithms converge to the same Bézier curve as the classical case. That means the Bézier curve is independent of the index \( \alpha \).

The classical polynomial Bernstein basis seems a asymptotic case of our new class of rational Bernstein basis.

Keywords : Rational Bernstein functions, Functions approximation, Bézier curves, deCasteljau Algorithm.

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1 Introduction

We know that, for a Bézier curve \( B \) of degree \( n \in \mathbb{N}^* \) in \( \mathbb{R}^d \) with \( d \in \mathbb{N}^* \) and \( 1 \leq d \leq 3 \), the polynomial Bernstein basis \( (B_n^i)_{i=0}^n \) can be defined for \( a < b \in \mathbb{R} \) and a parametrization \( x \in [a, b] \), by:

\[
B_n^i(x) = \begin{cases} 
\left( \frac{b-x}{b-a} \right)^n & \text{if } i = 0 \\
\binom{n}{i} \left( \frac{x-a}{b-a} \right)^i \left( \frac{b-x}{b-a} \right)^{n-i} & \text{if } 1 \leq i \leq n-1 \\
\left( \frac{x-a}{b-a} \right)^n & \text{if } i = n \\
0 & \text{otherwise}
\end{cases}
\]  

(1)

where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) and \( 0! = 1 \).

Let \( (d_i)_{i=0}^n \) be the control points of \( B \), \( d_i \in \mathbb{R}^d \) for all \( i \) then we have

\[ B(x) = \sum_{i=0}^n d_i B_n^i(x), \forall x \in [a, b] \]

In the same way we define the rational Bernstein basis \( (R_n^i)_{i=0}^n \) of degree \( n \in \mathbb{N}^* \) on \( [a, b] \) by

\[ R_i(x) = \frac{\omega_i B_n^i(x)}{\sum_{j=0}^n \omega_j B_j^i(x)} \]

where \( \omega_i > 0, \forall i = 0, \ldots, n \).

We can then define the rational Bézier curves by substituting the polynomial basis by the rational one.
We observe that, by putting \( w(x) = \frac{x-a}{b-a} \) we have \( 1-w(x) = \frac{b-x}{b-a} \). Therefore,

\[
B_i^n(x) = \begin{cases} 
C_n^i (w(x))^i (1-w(x))^{n-i} & \text{if } 0 \leq i \leq n \text{ and } a \leq x \leq b \\
0 & \text{otherwise}
\end{cases}
\]

This function \( w \) we defined is increasing on \([a, b]\) with \( w(a) = 0 \) and \( w(b) = 1 \).

The goal here is to retain these properties while requiring that \( w \) is homographic, to get a basis Bernstein naturally formed, of rational functions of degree \((k, k)\), i.e., a numerator of degree \(k\) and a denominator of degree \(k\).

## 2 New class of rational Bézier like basis

### 2.1 A class of rational parametrizations

We start with a lemma who fixed the foundation of our new class rational Bezier curves and can be expressed as follows:

**Lemma 2.1** Let \( a, b \in \mathbb{R} \) such that \( a < b \). There exists \( H([a, b]) \) a family of the homographic functions strictly increasing \( f \) on \([a, b]\) such that \( f(a) = 0 \) and \( f(b) = 1 \).

More precisely, for all \( f \in H([a, b]) \) there exists a unique \( \alpha \in ]-\infty, 0[\cup [1, +\infty[ \) such that

\[
f(x) = \frac{\alpha(x-a)}{x + (\alpha-1)b-\alpha a}, \quad \forall x \in [a, b]
\]

**Proof**

(Establishment)

Since \( f \) is homographic such that \( f(a) = 0 \), then there exist \( \alpha \neq 0 \) and \( c \in \mathbb{R}\setminus\{-a, -b\} \) such that for all \( x \in [a, b] \) we have : \( f(x) = \frac{\alpha(x-a)}{x + c} \). We have \( f(b) = 1 \) then \( 1 = \frac{\alpha(b-a)}{b + c} \). We get \( c = (\alpha-1)b-\alpha a \). Since \( c \notin \{-a, -b\}\), then we have \( \alpha \notin \{0, 1\} \). By strictly increasing of \( f \) we have \( \alpha(\alpha-1) > 0 \), and \( \alpha \in ]-\infty, 0[\cup [1, +\infty[ \).

Therefore, we have

\[
H([a, b]) = \left\{ f_{\alpha} | f_{\alpha}(x) = \frac{\alpha(x-a)}{x + (\alpha-1)b-\alpha a}, \alpha \in ]-\infty, 0[\cup [1, +\infty[, x \in [a, b] \right\}
\]

(Uniqueness)
Let \( \alpha, \beta \in ]-\infty, 0[ \cup ]1, +\infty[ \) and \( f_\alpha, f_\beta \in \mathcal{H}([a, b]) \) the associate homogeneous functions

\[
f_\alpha = f_\beta \iff f_\alpha(x) = f_\beta(x) \quad \forall x \in [a, b]
\]

\[
= \frac{\alpha(x-a)}{x + (\alpha - 1)b - \alpha a} \quad \forall x \in [a, b]
\]

\[
\Rightarrow \frac{\alpha(x-a)}{x + (\alpha - 1)b - \alpha a} = \frac{\beta(x-a)}{x + (\beta - 1)b - \beta a} \quad \forall x \in [a, b]
\]

\[
\Rightarrow (\alpha - \beta)(x-b) = 0 \quad \forall x \in [a, b]
\]

\[
\Rightarrow (\alpha - \beta)(x-b) = 0 \quad \forall x \in [a, b]
\]

\[
\Rightarrow \alpha = \beta
\]

**Remark 2.1** Let \( x \in [a, b] \) and \( \alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \).

We have \( D = x + (\alpha - 1)b - \alpha a \neq 0 \).

Indeed???, we observe that \( D = x - b + \alpha(b-a) = x - a + (\alpha - 1)(b-a) \),

then we have \( (\alpha - 1)(b-a) \leq D \leq \alpha(b-a) \). We can now write

\[
\left\{ \begin{array}{l}
0 < \alpha(\alpha - 1)(b-a) \leq \alpha D \leq \alpha^2(b-a) & \text{if } \alpha > 1 \\
0 < \alpha^2(b-a) \leq \alpha D \leq \alpha(\alpha - 1)(b-a) & \text{if } \alpha < 0
\end{array} \right.
\]

Then we conclude that \( D \neq 0 \forall x \in [a, b] \).

**Remark 2.2** Let \( \alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \) and \( a < b \).

Let \( f_\alpha \in \mathcal{H}([a, b]) \) continuous and increasing strictly on \( [a, b] \) with \( f_\alpha([a, b]) = [0, 1] \).

Moreover, for \( \lambda \in ]0, 1[ \) and \( x = a + \lambda(b-a) \in [a, b] \) we have

\[
f_\alpha(x) = \frac{\lambda \alpha}{\lambda + \alpha - 1} \in [0, 1]
\]

Thus obtaining the standard case as a asymptotic one :

\[
\lim_{|\alpha| \to \infty} f_\alpha(x) = \lambda = \frac{x-a}{b-a}
\]

### 2.2 A new class of rational Bernstein functions

**Definition 2.1** Let \( a, b \in \mathbb{R} \) such that \( a < b, \alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \), \( f_\alpha \in \mathcal{H}([a, b]) \) and \( n \in \mathbb{N}^* \).

A rational Bernstein basis of \( \alpha \) index and degree \( n \) on \( [a, b] \), is a family \( (\alpha B_n^\alpha)_i=0 \) of real functions defined for all \( x \in [a, b] \):

\[
\alpha B_n^\alpha(x) = \begin{cases} (1 - w(x))^n & \text{if } i = 0 \\
C_n^i (w(x))^i (1-w(x))^{n-i} & \text{if } 1 \leq i \leq n-1 \\
(w(x))^n & \text{if } i = n \\
0 & \text{otherwise}
\end{cases}
\]

where \( w(x) = f_\alpha(x) \).

It was agreed that \( \alpha B_0^\alpha(x) = 1, \forall x \in [a, b] \) and \( [a, b] \) the parametrization space ( or parameter space).
Definition 2.2 Let \( n, d \in \mathbb{N}^* \) and \( a, b \in \mathbb{R} \) such that \( a < b \).
Let \( \alpha \in [\infty, 0] \cup [1, +\infty] \) and \( (d_i)_{i=0}^n \subset \mathbb{R}^d \).

A rational Bézier curve of \( \alpha \) index, degree \( n \) on \([a, b]\) and corresponding control polygon points \((d_i)_{i=0}^n\), the function \( B_\alpha \) defined on \([a, b] \) valued in \( \mathbb{R}^d \) by

\[
B_\alpha(x) = \sum_{i=0}^n d_i \alpha^n B_i^n(x), \forall x \in [a, b]
\]

where \((\alpha^n B_i^n)_{i=0}^n\) is the rational Bernstein basis of \( \alpha \) index and degree \( n \) as defined by the equation 2.

Remark 2.3 Observe that \((\alpha^n B_i^n)_{i=0}^n\) is the classical polynomial Bernstein basis defined in 1.

Illustration 2.1 Before any deep analysis, we show throughout the figures 1, 2, 3, 4 and 5 the qualitative effect of the index \( \alpha \) on the behavior of the new Bernstein basis functions.

Figure 1 – Basis function \( \alpha^n B_i^n \) for \( \alpha \in \{-1, 2, 5, \infty\} \)

A flash exploration of these figures shows that the parameter \( \alpha \) has a notable effect on the functions of the Bernstein basis for any degree. We now refine this first reading by the check of the classical properties of Bernstein functions.

2.3 Properties of the new Bernstein basis

Proposition 2.1 Let \( n \in \mathbb{N}^* \) and \( a, b \in \mathbb{R} \) such that \( a < b \).
Let \( \alpha \in [\infty, 0] \cup [1, +\infty] \), \( f_\alpha \in \mathcal{H}([a, b]) \) and \((\alpha^n B_i^n)_{i=0}^n\) the rational Bernstein basis of \( \alpha \) index and degree \( n \) defined by the equation 3.

The following properties hold:

1. The positivity: For all \( x \in [a, b] \), \( \alpha^n B_i^n(x) \geq 0 \)

2. The partition of unity: For all \( x \in [a, b] \), \( \sum_{i=0}^n \alpha^n B_i^n(x) = 1 \)
Figure 2 – Basis function $\alpha B_i^2$ for $\alpha \in \{-1, 2, 5, \infty\}$

Figure 3 – Basis function $\alpha B_i^3$ for $\alpha \in \{-1, 2, 5, \infty\}$

3. The symmetry : For all $x \in [a, b]$, for all $i = 0, \ldots, n$,
\[ \alpha B_i^n(a + b - x) = 1 - \alpha B_{n-i}^n(x) \]

4. The recursion : For all $x \in [a, b]$, for all $i = 0, \ldots, n + 1$,
\[ \alpha B_i^{n+1}(x) = w(x) \alpha B_i^n(x) + (1 - w(x)) \alpha B_{i-1}^n(x) \]
where $w(x) = f_\alpha(x)$.

**Proof**
Soit $\alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \ et \ f_\alpha \in \mathcal{H}([a, b])$.

1. *(The positivity)*

By remark 2.2, for all $x \in [a, b]$ $w(x) = f_\alpha(x) \in [0, 1]$. Then $1 - w(x) \in [0, 1]$ so that
\[ \alpha B_i^n(x) = C_i^n w^i(x)(1 - w(x))^{n-i} \geq 0 \ \forall i = 0, \ldots, n \]
2. (The partition of unity)
Using the binomial formula of Newton’s, we have
\[
\sum_{i=0}^{n} {^\alpha B_i}^n(x) = \sum_{i=0}^{n} C_n^i w^i(x)(1-w(x))^{n-i} = (w(x) + (1-w(x)))^n = 1
\]

3. (The symmetry)
Let \( i = 0, \ldots, n \) and \( x \in [a, b] \). We have \( y = a + b - x \in [a, b] \) and
\[
\begin{align*}
  w(y) &= f_\alpha(a + b - x) = \frac{\alpha(b-x)}{-x + \alpha b + (1-\alpha)a} \\
  1 - w(y) &= f_{1-\alpha}(x) = \frac{(1-\alpha)(x-a)}{x - \alpha b - (1-\alpha)a}
\end{align*}
\]
Then
\[ \alpha \mathcal{B}^n_{n-i}(a + b - x) = \alpha \mathcal{B}^n_{n-i}(y) = C^{n-i}_n w^{n-i}(y)(1 - w(y))^i \]
\[ = C^n_{n-i}(1 - f_{1-a}(x))^{n-i}(f_{1-a}(x))^i \]
\[ = C^n_{n}(f_{1-a}(x))^i(1 - f_{1-a}(x))^{n-i} \]
\[ = 1 - \alpha \mathcal{B}^n_{i}(x) \]

4. (The recursion)
\[ \alpha \mathcal{B}^{n+1}_{i+1}(x) = C^{i+1}_n w^{i+1}(x)(1 - w(x))^{n-i} \]
\[ = (C^n_{i} + C^{i+1}_n) w^{i+1}(x)(1 - w(x))^{n-i} \]
\[ = C^n_{i} w^{i+1}(x)(1 - w(x))^{n-i} + C^{i+1}_n w^{i+1}(x)(1 - w(x))^{n-i} \]
\[ = w(x)C^n_{i} w^{i}(x)(1 - w(x))^{n-i} + (1 - w(x))C^{i+1}_n w^{i+1}(x)(1 - w(x))^{n-i-1} \]
\[ = w(x)\alpha \mathcal{B}^n_{i}(x) + (1 - w(x))\alpha \mathcal{B}^{n+1}_{i+1}(x) \]

**Remark 2.4** For all \( n \in \mathbb{N}^* \), \( a, b \in \mathbb{R} \) such that \( a < b \) and \( \alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \), \( \alpha \mathcal{B}^n_{i} \) the rational Bernstein basis of \( \alpha \) index and degree \( n \) on the parameter space \([a, b]\), satisfies:

\[
\begin{aligned}
\alpha \mathcal{B}^n_{0}(a) &= \alpha \mathcal{B}^n_{0}(b) = 1 \\
\alpha \mathcal{B}^n_{i}(a) &= 0 \quad \forall i = 1, \ldots, n \\
\alpha \mathcal{B}^n_{i}(b) &= 0 \quad \forall i = 0, \ldots, n-1 \\
\alpha \mathcal{B}^n_{i}(x) &> 0 \quad \forall i = 0, \ldots, n, \forall x \in [a, b] 
\end{aligned}
\] (3)

**Proposition 2.2** Let \( n \in \mathbb{N}^* \) and \( a, b \in \mathbb{R} \) such that \( a < b \).
Let \( \alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \) and \( f_a \in H([a, b]) \).

Consider the rational Bernstein basis \( \alpha \mathcal{B}^n_{i} \) and \( \alpha \mathcal{B}^{n+1}_{i} \) of \( \alpha \) index, of degrees \( n \) and \( n + 1 \) with parameters in \([a, b]\).

For all \( x \in [a, b] \) and \( w(x) = f_a(x) \), we have:

1. \((1 - w(x))\alpha \mathcal{B}^n_{i}(x) = \frac{n + 1 - i}{n + 1} \alpha \mathcal{B}^{n+1}_{i}, \forall i = 0, \ldots, n \)
2. \( w(x)\alpha \mathcal{B}^n_{i}(x) = \frac{i + 1}{n + 1} \alpha \mathcal{B}^{n+1}_{i+1}, \forall i = 0, \ldots, n \)

**Proof**
By a direct computation.

**Proposition 2.3** Let \( n \in \mathbb{N}^* \) and \( a, b \in \mathbb{R} \) such that \( a < b \).
Let \( \alpha \in ]-\infty, 0[ \cup ]1, +\infty[ \) and \( f_a \in H([a, b]) \).

The rational Bernstein basis \( \alpha \mathcal{B}^n_{i} \) of \( \alpha \) index and degree \( n \) on the parameter space \([a, b]\), is formed of rational functions of degree \( (n, n) \) and infinitely differentiable on \([a, b]\). Moreover, we have:
1. First order derivative: For all $x \in [a, b]$ and $i = 0, \ldots, n$ we have

$$
\frac{d}{dx} \alpha B^n_i(x) = \begin{cases} 
-n \frac{dw}{dx} (1 - w(x))^{n-1} & i = 0 \\
(i - nw(x)) \frac{dw}{dx} \times C_n w^{i-1}(x)(1 - w(x))^{n-i-1} & 1 \leq i \leq n-1 \\
\frac{dw}{dx} w^{n-1}(x) & i = n 
\end{cases}
$$

(4)

with $w(x) = f_\alpha(x) = \frac{\alpha(x-a)}{x + (\alpha-1)b - \alpha a}$ and $\frac{dw}{dx}(x) = \frac{\alpha(\alpha-1)(b-a)}{(x + (\alpha-1)b - \alpha a)^2}$

2. Second order Derivative: For all $x \in [a, b]$ and $i = 0, \ldots, n$

$$
\frac{d^2}{dx^2} \alpha B^n_i(x) = \begin{cases} 
n(1 - w(x))^{n-2} \left[ \frac{d^2 w}{dx^2} (1 - w(x)) + (n-1) \left( \frac{dw}{dx} \right)^2 \right] & i = 0 \\
n(1 - w(x))^{n-3} \left[ \frac{d^2 w}{dx^2} (1 - w(x))(1 - nw(x)) \right. \\
- (n-1) \left( \frac{dw}{dx} \right)^2 (2 - nw(x)) \left. \right] & i = 1 \\
C_n w^{i-2}(x)(1 - w(x))^{n-i-2} \times \\
\left[ (i - nw(x)) w(x)(1 - w(x)) \right] \frac{d^2 w}{dx^2} \left. \right] + \{(n-1)(nw(x) - 2i)w(x) + i(i - 1)\} \left( \frac{dw}{dx} \right)^2 & 2 \leq i \leq n-2 
\end{cases}
$$

and

$$
\frac{d^2}{dx^2} \alpha B^n_i(x) = \begin{cases} 
n w^{n-3}(x) \left[ \frac{d^2 w}{dx^2} w(x)(n-1 - nw(x)) \right. \\
+ (n-1) \left( \frac{dw}{dx} \right)^2 (n - 2 - nw(x)) \left. \right] & i = n-1 \\
n w^{n-2}(x) \left[ \frac{d^2 w}{dx^2} w(x) + (n-1) \left( \frac{dw}{dx} \right)^2 \right] & i = n 
\end{cases}
$$

where $\frac{d^2 w}{dx^2} = \frac{-2\alpha(\alpha-1)(b-a)}{(x + (\alpha-1)b - \alpha a)^3}$

**Proof**

$\alpha B^n_i$ is a polynomial of $f_\alpha$. Since $f_\alpha \in C^\infty([a, b])$, it is the same for $\alpha B^n_i$.

We complete the proof by simple computation.

**Remark 2.5** For all $n \in \mathbb{N}^*$ and $a, b \in \mathbb{R}$ such that $a < b$ and $\alpha \in ]-\infty, 0] \cup ]1, +\infty[$, using the relation (4) we observe that the derivatives of
elements of the rational Bernstein basis of \( \alpha \) index and degree \( n \) on the parameter space \([a, b]\), \((\alpha \mathcal{B}_n^\alpha)_{n=0}^\infty\) satisfy the following relations:

\[
\begin{aligned}
\frac{d}{dx} \alpha \mathcal{B}_0^\alpha(a) &= -n \frac{\alpha}{(\alpha - 1)(b - a)} \\
\frac{d}{dx} \alpha \mathcal{B}_1^\alpha(a) &= n \frac{\alpha}{(\alpha - 1)(b - a)} \\
\frac{d}{dx} \alpha \mathcal{B}_{n-1}^\alpha(b) &= n \frac{\alpha - 1}{\alpha(b - a)} \\
\frac{d}{dx} \alpha \mathcal{B}_n^\alpha(b) &= -n \frac{\alpha - 1}{\alpha(b - a)}
\end{aligned}
\]  

(5)

Proposition 2.4 Let \( n \in \mathbb{N} \) and \( a, b \in \mathbb{R} \) such that \( a < b \) and \( \alpha \in ]-\infty, 0[ \cup [1, +\infty[ \). Let \((\alpha \mathcal{B}_n^\alpha)_{n=0}^\infty\) be the rational Bernstein basis of \( \alpha \) index and degree \( n \) on the parameter space \([a, b]\),

1. Uniqueness of the maximum: For \( i = 0, \ldots, n \), \( \alpha \mathcal{B}_i^\alpha \) has a unique maximun in \( x_i \in [a, b] \), such that \( w(x_i) = \frac{i}{n} \). The value of this maximun is independant of \( \alpha \). 
   More precisely, we have
   \[
x_i = a + \frac{i(\alpha - 1)}{n\alpha - i} (b - a) \quad \text{and} \quad \alpha \mathcal{B}_i^\alpha(x_i) = C_n^i \frac{i(i(n - i))^{n-i}}{n^n}
\]

2. Maximum symmetrical property: For \( i = 0, \ldots, n \), \( \alpha \mathcal{B}_i^\alpha \), \( \alpha \mathcal{B}_{n-i}^\alpha \) and \( 1^{\alpha} \mathcal{B}_i^\alpha \) have the same maximun.

Proof

1. Existence and uniqueness of the maximum
   — \( \alpha \mathcal{B}_0^\alpha \) is decreasing strictly on \([a, b]\), then it has a maximun in \( a \).
   — \( \alpha \mathcal{B}_n^\alpha \) is increasing strictly on \([a, b]\), then it has a maximun in \( b \).
   — For \( i = 1, \ldots, n - 1 \) the sign of \( \frac{d}{dx} \alpha \mathcal{B}_i^\alpha \) and \( i - nw(x) \) are the same on \([a, b]\). This is the sign same of \( i - nf(x) \).
   
   Since \( f(x) \) is increasing strictly on \([a, b]\) then there exist a unique \( x_i \in [a, b] \) such that \( f(x_i) = \frac{i}{n} \in [0, 1] \). We have
   \[
   \begin{aligned}
   i - nw(x) > 0 & \quad \forall x < x_i \\
   i - nw(x) < 0 & \quad \forall x > x_i
   \end{aligned}
   \]

   Thus \( \alpha \mathcal{B}_i^\alpha \) has a maximun in \( x_i \) satisfying \( w(x_i) = \frac{i}{n} \) and \( \alpha \mathcal{B}_i^\alpha(x_i) \) is independant of \( \alpha \). and it can be obtained by simple computation.

2. Maximum symmetrical property:
   By simple computation, we have
   \[
   \max_{x \in [a, b]} \alpha \mathcal{B}_i^\alpha(x) = \max_{x \in [a, b]} \alpha \mathcal{B}_{n-i}^\alpha(x) = \max_{x \in [a, b]} 1^{\alpha} \mathcal{B}_i^\alpha(x) \quad \forall i = 0, \ldots, n
   \]
Proposition 2.5 Let $n \in \mathbb{N}^*$ and $a, b \in \mathbb{R}$ such that $a < b$.
Let $\alpha \in ]-\infty, 0[\cup]1, +\infty[,$ and $f_\alpha \in \mathcal{H}([a, b])$.

The rational Bernstein basis of $\alpha$ index and degree $n$ on $[a, b]$, $(\alpha B^n_i)_{i=0}^n$ is a linear independent family.

Proof
We proceed by induction on $n$.
Let $x \in [a, b]$ and $w(x) = f_\alpha(x)$

For $n = 1$, if $\sum_{i=0}^{n} \lambda_i \alpha B^n_i = 0$ then for all $x \in [a, b]$ we have $\lambda_0(1 - w(x)) + \lambda_1 w(x) = 0$.

In particular :
\[
\begin{aligned}
\lambda_0 &= \lambda_0(1 - w(a)) + \lambda_1 w(a) = 0 \\
\lambda_1 &= \lambda_0(1 - w(b)) + \lambda_1 w(b) = 0
\end{aligned}
\]

Therefore $(\alpha B^1_i)_{i=0}^1$ is a linear independent family

Assume that the property holds up to an order $n \in \mathbb{N}^*$, and show that $(\alpha B^{n+1}_i)_{i=0}^{n+1}$ is a linear independent family.

If $\sum_{i=0}^{n+1} \lambda_i \alpha B^{n+1}_i = 0$ then for all $x \in [a, b]$ we have

\[
0 = \sum_{i=0}^{n+1} \lambda_i \alpha B^{n+1}_i (x)
= \sum_{i=0}^{n+1} \lambda_i \left( w(x) \alpha B^n_{i-1}(x) + (1 - w(x)) \alpha B^n_i(x) \right)
= \sum_{i=0}^{n} \left( w(x) \lambda_{i+1} + (1 - w(x)) \lambda_i \right) \alpha B^n_i(x)
\]

Using the assumption : $(\alpha B^n_i)_{i=0}^n$ is a linear independent system we have

\[
w(x) \lambda_{i+1} + (1 - w(x)) \lambda_i = 0, \quad \forall i = 0, \ldots, n, \forall x \in [a, b]
\]
(6)

Otherwise

\[
\lambda_{n+1} = \sum_{i=0}^{n+1} \lambda_i \alpha B^{n+1}_i (b) = 0
\]
(7)

The linear system from the equation [6], completed by equation [7] is inversible triangular upper since the terms of the diagonal are nonzero. We deduce that $\lambda_i = 0, \quad \forall i = 0, \ldots, n + 1$. Then we can conclude that $(\alpha B^{n+1}_i)_{i=0}^{n+1}$ is a linear independent system.

Remark 2.6 Since $(\alpha B^n_i)_{i=0}^n$ is a linear independent system then there is a basis of an approximation space of continuous real functions.
3 Properties of the new class of Bézier curves

Let $n \in \mathbb{N}^*$ and $a, b \in \mathbb{R}$ such that $a < b$.

Let $\alpha \in ]-\infty , 0[ \cup ]1, +\infty [$, and $f_\alpha \in \mathcal{H}([a, b])$.

Let $(^{\alpha}\mathbf{B}_i^n)_{i=0}^n$ be the rational Bernstein basis of $\alpha$ index and degree $n$ on $[a, b]$.

Consider the rational Bézier curve $B_\alpha$ of $\alpha$ index and degree $n$ with control polygon points $(d_i)_{i=0}^n \subset \mathbb{R}^d$ defined for all $x \in [a, b]$ by $B_\alpha(x) = \sum_{i=0}^n d_i^{\alpha}\mathbf{B}_i^n(x)$

3.1 Geometric properties

The curves of this new class check the properties of the classical Bézier curves. The proposition below lists the most important of these properties.

Proposition 3.1 Let $B_\alpha$ the rational Bézier curve of $\alpha$ index and degree $n$ with control polygon points $(d_i)_{i=0}^n$ on a parameter space $[a, b]$.

The following properties hold :

1. Extremities interpolation properties : The Bézier curve $B_\alpha$ interpolates the extremities points of its control polygon; it means that $B_\alpha(a) = d_0$ and $B_\alpha(b) = d_n$

2. Extremities tangents properties : The Bézier curve $B_\alpha$ is tangent to its control polygon to the extremities. More precisely, we have

$$\left\{ \begin{array}{l} \frac{dB_\alpha}{dx}(a) = \frac{\alpha}{(\alpha-1)(b-a)}(d_1 - d_0) \\ \frac{dB_\alpha}{dx}(b) = \frac{(\alpha-1)}{\alpha(b-a)}(d_n - d_{n-1}) \end{array} \right.$$  

3. Convex hull property : $B_\alpha$ is in the convex hull of its control points $(d_i)_{i=0}^n$. In the other way, for all $x \in [a, b]$, there exists $(\lambda_i)_{i=0}^n \subset \mathbb{R}^+$ such that $B_\alpha(x) = \sum_{i=0}^n \lambda_i d_i$ with $\sum_{i=0}^n \lambda_i = 1$

4. Invariance property of affine transformation : For all affine transformation $T$ in $\mathbb{R}^d$, we have

$$T(B_\alpha(x)) = \sum_{i=0}^n T(d_i)^{\alpha}\mathbf{B}_i^n(x)$$

Proof

Consider the rational Bézier curve $B_\alpha$ of $\alpha$ index and degree $n$ with control polygon points $(d_i)_{i=0}^n$ on the parameter space $[a, b]$. 
1. Using the remark [3], we have

\[
B_\alpha(a) = \sum_{i=0}^{n} d_i^\alpha B_i^a(a)
\]
\[
= d_0^\alpha B_0^n(a) = d_0
\]

and

\[
B_\alpha(b) = \sum_{i=0}^{n} d_i^\alpha B_i^b(b)
\]
\[
= d_n^\alpha B_n^n(b) = d_n
\]

2. Using the remark [5], we have

\[
\frac{d}{dx}B_\alpha(a) = \sum_{i=0}^{n} d_i \frac{d}{dx}^\alpha B_i^a(a)
\]
\[
= d_0 \frac{d}{dx}^\alpha B_0^n(a) + d_1 \frac{d}{dx}^\alpha B_1^n(a)
\]
\[
= (d_1 - d_0) \frac{d}{dx}^\alpha B_1^n(a)
\]

and

\[
\frac{d}{dx}B_\alpha(b) = \sum_{i=0}^{n} d_i \frac{d}{dx}^\alpha B_i^b(b)
\]
\[
= d_{n-1} \frac{d}{dx}^\alpha B_{n-1}^b(b) + d_n \frac{d}{dx}^\alpha B_n^n(b)
\]
\[
= (d_n - d_{n-1}) \frac{d}{dx}^\alpha B_n^n(b)
\]

3. For all \(x \in [a, b]\), we have

\[
\begin{cases}
B_\alpha(x) = \sum_{i=0}^{n} d_i^\alpha B_i^n(x) \\
= \sum_{i=0}^{n} \lambda_i d_i \\
\text{where } \lambda_i = ^\alpha B_i^n(x) \in \mathbb{R}_+ \forall i
\end{cases}
\]

Using a unity partition property we obtain \(\sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} ^\alpha B_i^n(x) = 1\). Then \(B_\alpha(x)\) is in the convex hull of its control polygon points. \((d_i)_{i=0}^{n}\)

4. Let \(T\) be an affine transformation in \(\mathbb{R}^d\). There is a square matrix \(M\) of order \(d\) and a point \(C \in \mathbb{R}^d\) such that for all \(X \in \mathbb{R}^d\), we have
\( T(X) = MX + C \). Let \( x \in [a, b] \). Since \( B_\alpha(x) \in \mathbb{R}^d \) we have

\[
\begin{align*}
T(B_\alpha(x)) &= T\left( \sum_{i=0}^{n} d_i \alpha B_i^n(x) \right) \\
&= M\left( \sum_{i=0}^{n} d_i \alpha B_i^n(x) \right) + C \\
&= \sum_{i=0}^{n} M d_i \alpha B_i^n(x) + \left( \sum_{i=0}^{n} \alpha B_i^n(x) \right) C \\
&= \sum_{i=0}^{n} (Md_i \alpha B_i^n(x)) + \sum_{i=0}^{n} (C \alpha B_i^n(x)) \\
&= \sum_{i=0}^{n} (Md_i + C) \alpha B_i^n(x) = \sum_{i=0}^{n} T(d_i) \alpha B_i^n(x)
\end{align*}
\]

This proves the property.

**Proposition 3.2 (Degree elevation algorithm)** Given the rational Bézier curve \( B_\alpha \) of \( \alpha \) index and degree \( n \) with control polygon points \( (d_i)_{i=0}^{n} \subset \mathbb{R}^d \) on the parameter space \([a, b]\), then there exists a rational Bézier curve \( C_\alpha \) of \( \alpha \) index and degree \( n + 1 \) with control polygon points \( (\hat{d}_i)_{i=0}^{n+1} \subset \mathbb{R}^d \) on the parameter space \([a, b]\) identical to \( B_\alpha \).

More precisely we have : \( B_\alpha(x) = \sum_{i=0}^{n+1} \hat{d}_i \alpha B_i^{n+1}(x) \) with

\[
\begin{align*}
\hat{d}_0 &= d_0 \\
\hat{d}_i &= \frac{i}{n+1} d_{i-1} + \left( 1 - \frac{i}{n+1} \right) d_i \quad 1 \leq i \leq n \\
\hat{d}_{n+1} &= d_n
\end{align*}
\]

**Proof**

For all \( \alpha \in ]-\infty, 0[\cup[1, +\infty[ \) and \( f_\alpha \in \mathcal{H}([a, b]) \), putting \( w(x) = f_\alpha(x) \) for all \( x \in [a, b] \). Using the proposition 2.2 we have :

\[
\begin{align*}
\alpha B_i^n(x) &= (1 - w(x))\alpha B_i^n(x) + w(x)\alpha B_i^n(x) \quad \forall i = 0, \ldots, n \\
&= \frac{n + 1 - i}{n + 1} \alpha B_i^{n+1}(x) + \frac{i + 1}{n + 1} \alpha B_{i+1}^{n+1}(x) \quad \forall i = 0, \ldots, n
\end{align*}
\]

14
Then

\[ B_\alpha(x) = \sum_{i=0}^{n} d_i \alpha B^i_n(x) \]

\[ = \sum_{i=0}^{n} d_i \left[ \frac{n+1-i}{n+1} \alpha B^{i+1}_n(x) + \frac{i+1}{n+1} \alpha B^{i+1}_{n+1}(x) \right] \]

\[ = \sum_{i=0}^{n} d_i \frac{n+1-i}{n+1} \alpha B^{i+1}_n(x) + \sum_{i=0}^{n} d_i \frac{i+1}{n+1} \alpha B^{i+1}_{n+1}(x) \]

\[ = \sum_{i=0}^{n} d_i \frac{n+1-i}{n+1} \alpha B^{i+1}_n(x) + \sum_{i=1}^{n+1} \frac{i}{n+1} \alpha B^{i+1}_n(x) \]

\[ = d_0 \alpha B^{n+1}_0(x) + \sum_{i=1}^{n} \left[ \frac{n+1-i}{n+1} d_i + \frac{i}{n+1} d_{i-1} \right] \alpha B^{n+1}_i(x) + d_n \alpha B^{n+1}_{n+1}(x) \]

This proves the result.

**Illustration 3.1** For illustration we show here the Bézier curves \( B_\alpha \) of degree 3 for \( \alpha \in \{-1, 2, 5, \infty\} \). The control polygons of the respective curves are as follow:

- \( \Pi_a = \{(0, 2), (3.5, 0), (3.5, 4), (0, 0)\} \)
- \( \Pi_b = \{(0, 1), (3.5, 0), (3.5, 4), (0, 1)\} \)
- \( \Pi_c = \{(0, 1), (4.5, 4), (5.5, 0), (3.5, 1)\} \)
- \( \Pi_d = \{(0, 1), (4, 0.5), (2.5, 3), (6, 3)\} \)
- \( \Pi_e = \{(0, 1.5), (4, 0.5), (5, 4), (3, 2)\} \)
- \( \Pi_f = \{(0, 3.5), (4, 0.5), (5, 4), (0, 0)\} \)
- \( \Pi_g = \{(0, 3.5), (4, 0.5), (4.5, 2.5), (0, 0)\} \)
- \( \Pi_h = \{(0, 0), (2, 2.5), (4.5, 3), (6.5, 1.5)\} \)
- \( \Pi_i = \{(0, 3.5), (5, 1), (5, 1), (0, 0)\} \)

![Figure 6 – The curves \( B_{-1} \)](image-url)
A comparative analysis of figures 6, 7, 8, and 9 suggest that the Bézier curves are independent in index $\alpha$, but each curve is function of its control polygon.

### 3.2 Algorithms for computing the Bézier curve

These algorithms show that it is possible to calculate a point of the Bézier curve or all without the need to build the basis of Bernstein. The most fundamental is deCasteljau algorithm which can be formulate by the following proposition:

**Proposition 3.3 (deCasteljau algorithm)** *For all* $r = 0, \ldots, n$ *and* $x \in \mathbb{R}$ *the deCasteljau algorithm*
Figure 9 – The curves $B_\infty$

\[ [a, b] \text{ we have : } \]
\[ B_\alpha(x) = \sum_{i=0}^{n-1} d_i^\alpha(x)^n B_{i-r}^n(x) \]

with
\[
\begin{align*}
  d_0^\alpha(x) &= d_i \quad \forall i = 0, \ldots, n \\
  d_i^{r+1}(x) &= w(x) d_{i+1}^r(x) + (1 - w(x)) d_i^r(x) \quad \forall r = 0, \ldots, n-1 \\
  d_i^r(x) &= w(x) d_i^{r-1}(x) + (1 - w(x)) d_i^{r-1}(x) \quad \forall i = 0, \ldots, n-r-1
\end{align*}
\]

where $w(x) = f_\alpha(x)$

Moreover we have $B_{10}(x) = d_0^n(x)$

**Proof**

As in [1], we proceed by induction to prove this algorithm.

Let $x \in [a, b]$ and $w(x) = f_\alpha(x)$.

Since $\alpha^n B_i^n(x) = w(x) \alpha^n B_{i-1}^{n-1}(x) + (1 - w(x)) \alpha^n B_{i-1}^{n-1}(x)$ and $\alpha^n B_1^{n-1}(x) = \alpha^n B_{n+1}^{n}(x) \equiv 0$

then

\[
B_\alpha(x) = \sum_{i=0}^{n} d_i^\alpha B_{11}^n(x) = \sum_{i=0}^{n} d_i [w(x) \alpha B_{i-1}^{n-1}(x) + (1 - w(x)) \alpha B_{i}^{n-1}(x)]
\]

\[
= \sum_{i=0}^{n} d_i w(x) \alpha B_{i-1}^{n-1}(x) + \sum_{i=0}^{n} d_i (1 - w(x)) \alpha B_{i}^{n-1}(x)
\]

\[
= \sum_{i=0}^{n-1} [w(x) d_{i+1} + (1 - w(x)) d_i] \alpha B_{i}^{n-1}(x) = \sum_{i=0}^{n-1} d_i^1(x) \alpha B_{i}^{n-1}(x)
\]

with $d_1^1(x) = (1 - w(x)) d_1^0(x) + w(x) d_1^{i+1}(x)$ where $d_i^1(x) = d_i \quad \forall i$
Now we will show that for all \( 1 \leq r \leq n \) we have \( B_\alpha(x) = \sum_{i=0}^{n-r} d_i^\alpha B_i^{n-r}(x) \) with \( d_i^r(x) = (1 - w(x))d_i^{r-1}(x) + w(x)d_{i+1}^{r-1}(x) \).

Assume that for \( 1 \leq r < n \), we have \( B_\alpha(x) = \sum_{i=0}^{n-r} d_i^\alpha B_i^{n-r}(x) \) with \( d_i^r(x) = (1 - w(x))d_i^{r-1}(x) + w(x)d_{i+1}^{r-1}(x) \). Then

\[
B_\alpha(x) = \sum_{i=0}^{n-r} d_i^\alpha B_i^{n-r}(x)
\]

\[
= \sum_{i=0}^{n-r} d_i^r [w(x)^\alpha B_{i-1}^{n-r-1}(x) + (1 - w(x))^\alpha B_i^{n-r-1}(x)]
\]

\[
= \sum_{i=0}^{n-r-1} d_i^r w(x)^\alpha B_{i-1}^{n-r-1}(x) + \sum_{i=0}^{n-r} d_i^r (1 - w(x))^\alpha B_i^{n-r-1}(x)
\]

\[
= \sum_{i=0}^{n-r-1} d_{i+1}^r w(x)^\alpha B_i^{n-r-1}(x) + \sum_{i=0}^{n-r-1} d_i^r (1 - w(x))^\alpha B_i^{n-r-1}(x)
\]

\[
= \sum_{i=0}^{n-r-1} [w(x)d_{i+1}^r + (1 - w(x))d_i^r] B_i^{n-r-1}(x) = \sum_{i=0}^{n-r-1} d_i^{r+1} B_i^{n-r-1}(x)
\]

with \( d_i^{r+1} = (1 - w(x))d_i^r(x) + w(x)d_{i+1}^r(x) \) and we deduce the result.

**Proposition 3.4 (subdivision algorithm)** Let \( c \in ]a, b[ \). The point \( B_\alpha(c) = d_0^\alpha(c) \) computed by the relation

\[
d_0^i(c) = d_i, \quad \forall i = 0, \ldots, n
\]

\[
d_r^{i+1}(c) = w(c)d_{i+1}^r(c) + (1 - w(c))d_i^r(c) \quad \forall r = 0, \ldots, n - 1, \quad \forall i = 0, \ldots, n - r - 1
\]

where \( w(x) = f_\alpha(x) \) divides the curve \( B_\alpha \) in two rational Bézier curves of \( \alpha \) index and degree \( n \) on the parameter space \( ]a, b[ \) whose the control polygon points are respectively \( (d_0^i(c))_{i=0}^n \) and \( (d_{r+i}^n(c))_{i=0}^n \).

The proof of this proposition uses the following three lemmas:

**Lemma 3.1** The points \( d_k^j(c) \) defined by the algorithm of the proposition 3.4 satisfy for all \( 0 \leq k \leq j \leq n \), the relation

\[
d_k^j(c) = \sum_{i=0}^{j} d_{i+k}^\alpha B_i^j(c)
\]

**Proof**

Let \( c \in ]a, b[ \) and \( w = f_\alpha(c) \). From the proposition 3.4 for all \( 0 \leq k \leq j \leq n \),
we have \( d_k^r(c) = (1 - w)d_{k+1}^{r-1}(c) + wd_{k+1}^{r-1}(c) \). we proceed by induction to prove that for all \( 1 \leq r \leq j \), \( d_k^r(c) = \sum_{i=0}^{r} d_{k+i}^{r-i}(c)\alpha B_i^r(c) \) and we deduce the result for \( r = j \).

For all \( j \geq 1 \) we have
\[
d_k^j(c) = (1 - w)d_k^{j-1}(c) + wd_k^{j-1}(c) = \alpha B_0^r(c)d_k^{j-1}(c) + \alpha B_1^r(c)d_k^{j-1}(c) = \sum_{i=0}^{1} \alpha B_i^{r-1}(c)d_{k+i}^{j-1}(c)
\]

Assume that for all \( 1 \leq r < j \), we have \( d_k^r(c) = \sum_{i=0}^{r} d_{k+i}^{r-i}(c)\alpha B_i^r(c) \) then using the proposition 2.2, we have
\[
d_k^j(c) = \sum_{i=0}^{r} d_{k+i}^{j-i}(c)\alpha B_i^r(c)
\]
\[
= \sum_{i=0}^{r} \left[ (1 - w)d_{k+i}^{j-i-1}(c) + wd_{k+i}^{j-i-1}(c) \right] \alpha B_i^r(c)
\]
\[
= \sum_{i=0}^{r} d_{k+i}^{j-i-1}(c) \left[ (1 - w)\alpha B_i^r(c) \right] + \sum_{i=0}^{r} d_{k+i}^{j-i-1}(c) \left[ w\alpha B_i^r(c) \right]
\]
\[
= \sum_{i=0}^{r} d_{k+i}^{j-i-1}(c) \left[ \frac{r+1-i}{r+1} \alpha B_i^{r+1}(c) \right] + \sum_{i=0}^{r} d_{k+i}^{j-i-1}(c) \left[ \frac{r+1+i}{r+1} \alpha B_i^{r+1}(c) \right]
\]
\[
= \sum_{i=0}^{r} d_{k+i}^{j-i-1}(c) \left[ \frac{r+1-i}{r+1} \alpha B_i^{r+1}(c) \right] + \sum_{i=1}^{r+1} d_{k+i}^{j-i-1}(c) \left[ \frac{i}{r+1} \alpha B_i^{r+1}(c) \right]
\]
\[
= \sum_{i=0}^{r} d_{k+i}^{j-i-1}(c)\alpha B_i^{r+1}(c)
\]

This completes the proof.

Lemma 3.2 Let \( \alpha \in ]-\infty, 0[ \cup ]1, \infty[ \), and \( a, b \in \mathbb{R} \) with \( a < b \). Let \( c \in ]a, b[ \) and \( f_{\alpha} \in \mathcal{H}(]a, b[) \). There exists a bijective mapping \( u \) from \( ]a, b[ \) to \( ]a, c[ \) such that \( f_{\alpha} \circ u = f_{\alpha}(c)u_{\alpha} \)

Moreover, for all \( n \in \mathbb{N}^* \) and \( i \in \mathbb{N} \) with \( i \leq n \) we have
\[
\alpha B_i^a(u(t)) = \sum_{j=0}^{n} \alpha B_j^a(c)\alpha B_i^a(t), \forall t \in ]a, b[.
\]

Proof

Step 1 First of all let us prove the existence of \( u \)

For \( c \in ]a, b[ \), we have \( f_{\alpha}(c) \in ]0, 1[ \). For all \( t \in ]a, b[ \), we have \( f_{\alpha}(t) \in ]0, 1[ \).

Since \( f_{\alpha} \) is increasing strictly then
\[
w = f_{\alpha}(c)f_{\alpha}(t) \in ]0, f_{\alpha}(c)[ = f_{\alpha}(]a, c[) \subset ]0, 1[.
\]
Using the fact that $f_\alpha$ is a bijection we have an unique $u(t) \in [a, c]$ such that $f_\alpha(u(t)) = w = f_\alpha(c)f_\alpha(t)$.

This implies the mapping $t \in [a, b] \mapsto u(t) = f_\alpha^{-1}(f_\alpha(c)f_\alpha(t)) \in [a, c]$.

Let $x, y \in [a, c]$ such that $x < y$, we have $f_\alpha(x) < f_\alpha(y)$. Then $f_\alpha(c)f_\alpha(x) < f_\alpha(c)f_\alpha(y)$ and

$$u(x) = f_\alpha^{-1}(f_\alpha(c)f_\alpha(x)) < f_\alpha^{-1}(f_\alpha(c)f_\alpha(y)) = u(y)$$

$u$ is increasing strictly. Then $u$ is an increasing injection.

$$u(a) = f_\alpha^{-1}(f_\alpha(c)f_\alpha(a)) = f_\alpha^{-1}(0) = a$$

$$u(b) = f_\alpha^{-1}(f_\alpha(c)f_\alpha(b)) = f_\alpha^{-1}(f_\alpha(c)) = c$$

Since $f_\alpha$ and $f_\alpha^{-1}$ are continuous then $u$ is continuous and we have

$$u([a, b]) = [u(a), u(b)] = [a, c]$$

Therefore $u$ is a bijection from $[a, b]$ to $[a, c]$ and satisfies $f_\alpha \circ u = f_\alpha(c)f_\alpha$.

**Step 2**

Let $t \in [a, b]$, $w(t) = f_\alpha(t)$ and $w(c) = f_\alpha(c)$ we have

$$\tilde{w} = f_\alpha \circ u(t) = w(c)w(t)$$

$$1 - \tilde{w} = 1 - w(c)w(t) = (1 - w(t)) + w(t)(1 - w(c))$$

We deduce

$$\left(1 - \tilde{w}\right)^{n-i} = \left[\left(1 - w(t)\right) + w(t)(1 - w(c))\right]^{n-i}$$

$$= \sum_{j=0}^{n-i} C_j^n\left(1 - w(t)\right)^j[w(t)(1 - w(c))]^{n-i-j}$$

$$\tilde{w}^i(1 - \tilde{w})^{n-i} = \sum_{j=0}^{n-i} C_j^n[w(t)w(c)]^i\left(1 - w(t)\right)^j[w(t)(1 - w(c))]^{n-i-j}$$

$$= \sum_{j=0}^{n-i} C_j^n \left[w^{n-j}(t)(1 - w(t))^j\right][w^i(c)(1 - w(c))^{n-i-j}]$$

$$\alpha B_i^n(u(t)) = C_i^n \tilde{w}^i(1 - \tilde{w})^{n-i}$$

$$= \sum_{j=0}^{n-i} C_i^n C_j^n \left[w^{n-j}(t)(1 - w(t))^j\right][w^i(c)(1 - w(c))^{n-i-j}]$$

$$= \sum_{j=0}^{n-i} C_j^n \frac{\alpha B_n^{n-j}(t)}{C_n^{n-j}} \frac{\alpha B_i^n(c)}{C_i^n} (1 - w(c))^{-j}$$

$$= \sum_{j=0}^{n-i} \frac{C_j^n}{C_{n-j}^i} \alpha B_n^{n-j}(t) \alpha B_i^n(c) (1 - w(c))^{-j}$$

By an iterative use of the proposition 2.2 we have

$$\alpha B_i^n(c) (1 - w(c))^{-j} = \prod_{l=0}^{j-1} \left(\frac{n-l}{n-i-l}\right) \alpha B_i^n(c) = \frac{C_j^n}{C_{n-i}^j} \alpha B_i^n(c)$$

20
We deduce that

\[ \alpha B_i^n(u(t)) = \sum_{j=0}^{n-i} \alpha B_{n-j}^n(t) \alpha B_{i-j}^n(c) = \sum_{j=0}^{n-i} \alpha B_{j}^n(t) \alpha B_{i}^n(c) = \sum_{j=0}^{n-i} \alpha B_{i}^n(c) \]

Since for all \( i > j \) we have \( \alpha B_i^n(c) = 0 \), the expected result follows.

**Lemma 3.3** Let \( \alpha \in [-\infty, 0] \cup [1, \infty) \), and \( a, b \in \mathbb{R} \) and \( a < b \). Let \( c \in [a, b] \) and \( f_\alpha \in \mathcal{H}([a, b]) \). There exists a bijective mapping \( v \) from \([a, b]\) to \([c, b]\) such that for all \( t \in [a, b] \)

\[ f_\alpha \circ v(t) = 1 - (1 - f_\alpha(c))(1 - f_\alpha(t)) \]

Moreover, for all \( n \in \mathbb{N}^* \) and \( i \in \mathbb{N} \) with \( i \leq n \)

\[ \alpha B_i^n(v(t)) = \sum_{j=0}^{n} \alpha B_{i-j}^n(c) \alpha B_{j}^n(t), \forall t \in [a, b] \]

**Proof**

**Step 1** First of all we prove the existence of \( v \)

Let \( c \in [a, b] \). We have \( f_\alpha(c) \in [0, 1] \) since \( 1 - f_\alpha(c) \in [0, 1] \). By same way, for all \( t \in [a, b] \), we have \( f_\alpha(t) \in [0, 1] \) and \( 1 - f_\alpha(t) \in [0, 1] \).

Since \( f_\alpha \) is increasing strictly we deduce that, for all \( t \in [a, b] \), we have an unique

\[ w(t) = 1 - (1 - f_\alpha(c))(1 - f_\alpha(t)) \in [f_\alpha(c), 1] = f_\alpha([c, b]) \subset [0, 1] \]

Using the fact that \( f_\alpha \) is a bijection we have an unique \( v(t) \in [a, c] \) such that \( f_\alpha(v(t)) = w(t) \).

This implies a mapping

\[ t \in [a, b] \mapsto v(t) = f_\alpha^{-1} [1 - (1 - f_\alpha(c))(1 - f_\alpha(t))] \in [c, b] \]

Let \( x, y \in [c, b] \) such that \( x < y \), we have \( (1 - f_\alpha(c))(1 - f_\alpha(x)) > (1 - f_\alpha(c))(1 - f_\alpha(y)) \).

Then

\[ v(x) = f_\alpha^{-1} [1 - (1 - f_\alpha(c))(1 - f_\alpha(x))] < f_\alpha^{-1} [1 - (1 - f_\alpha(c))(1 - f_\alpha(y))] = v(y) \]

\( v \) is increasing strictly. \( v \) is an increasing injection from \([a, b]\) to \([v(a), v(b)] = [c, b] \) since \( v \) is continuous.

Then \( v \) is a bijection from \([a, b]\) to \([c, b]\) and satisfies, for all \( t \in [a, b] \)

\[ f_\alpha \circ v(t) = 1 - (1 - f_\alpha(c))(1 - f_\alpha(t)) \]
Step 2
Let \( t \in [a, b] \), \( w(t) = f_\alpha(t) \) and \( w(c) = f_\alpha(c) \) we have
\[
\bar{w} = f_\alpha \circ v(t) = 1 - (1 - w(c)) (1 - w(t)) = w(t) + w(c) (1 - w(t))
\]
Therefore we have
\[
1 - \bar{w} = (1 - w(t)) (1 - w(c))
\]
Then we have
\[
(\bar{w})^i = [w(t) + w(c) (1 - w(t))]^i
= \sum_{j=0}^{i} C_i^j (w(t))^j [w(c) (1 - w(t))]^{i-j}
\]
\[
\bar{w}^i (1 - \bar{w})^{n-i} = \sum_{j=0}^{i} C_i^j \left[ (w(t))^j (1 - w(t))^{n-j} \right] \left[ (w(c))^{i-j} (1 - w(c))^{n-i} \right]
\]
\[
\alpha B^n_i (u(t)) = C_i^n \bar{w}^i (1 - \bar{w})^{n-i}
\]
\[
= \sum_{j=0}^{i} C_i^n C_i^j \left[ (w(t))^j (1 - w(t))^{n-j} \right] \left[ (w(c))^{i-j} (1 - w(c))^{n-i} \right]
\]
\[
= \sum_{j=0}^{i} C_i^j \left[ (w(t))^j (1 - w(t))^{n-j} \right] C_i^j (w(c))^{i-j} (1 - w(c))^{n-i}
\]
\[
= \sum_{j=0}^{i} C_i^j \frac{\alpha B^n_i (t)}{C_i^n} \alpha B^n_i (c) (w(c))^{-j}
\]
\[
= \sum_{j=0}^{i} C_i^j \frac{\alpha B^n_i (t)}{C_i^n} \alpha B^n_i (c) (w(c))^{-j}
\]
An iterative use of the proposition 22 implies
\[
\alpha B^n_i (c) (w(c))^{-j} = \prod_{l=0}^{j-1} \left( \frac{n-l}{j-l} \right) \alpha B^n_{i-l} (c) = \frac{C_i^n \alpha B^n_{i-j} (c)}{C_i^n}
\]
We deduce that
\[
\alpha B^n_i (v(t)) = \sum_{j=0}^{i} \alpha B^n_j (t) \alpha B^n_{i-j} (c)
\]
\[
= \sum_{j=i}^{n} \alpha B^n_{i-j} (c) \alpha B^n_j (t)
\]
because that, for all \( i < j \) we have \( \alpha B^n_{i-j} (c) = 0 \)

Proof
Proof of the proposition 3,4

22
Let $\bar{B}_\alpha$ and $\hat{B}_\alpha$ be the restrictions of $B_\alpha$ on the respective intervals $[a, c] \subset [a, b]$ and $[c, b] \subset [a, b]$ mapping the parameter space $[a, b]$.

For all $t \in [a, b]$, using the lemma 3.2 we have $u(t) \in [a, c]$ and

$$
\bar{B}_\alpha(t) = B_\alpha(u(t)) = \sum_{i=0}^{n} d_i \alpha B_i^\alpha(u(t))
= \sum_{i=0}^{n} d_i \left[ \sum_{j=0}^{n} \alpha B_i^\alpha(c) B_j^\alpha(t) \right]
= \sum_{i=0}^{n} \sum_{j=0}^{n} \left( d_i \alpha B_i^\alpha(c) \right) \alpha B_j^\alpha(t)
= \sum_{j=0}^{n} \sum_{i=0}^{n} d_i \alpha B_i^\alpha(c) \alpha B_j^\alpha(t)
= \sum_{j=0}^{n} d_j^\alpha(c) \alpha B_j^\alpha(t)
$$

That is the expected result by using the lemma 3.1.

With the lemma 3.3 for all $t \in [a, b]$ we have

$$
\hat{B}_\alpha(t) = B_\alpha(v(t)) = \sum_{i=0}^{n} d_i \alpha B_i^\alpha(v(t))
= \sum_{i=0}^{n} d_i \left[ \sum_{j=0}^{n} \alpha B_i^\alpha(c) B_j^\alpha(t) \right]
= \sum_{i=0}^{n} \sum_{j=0}^{n} \left( d_i \alpha B_i^\alpha(c) \right) \alpha B_j^\alpha(t)
= \sum_{j=0}^{n} \sum_{i=0}^{n} d_i \alpha B_i^\alpha(c) \alpha B_j^\alpha(t)
= \sum_{j=0}^{n} d_j^\alpha(c) \alpha B_j^\alpha(t)
$$

using the lemma 3.4.
Corollary 3.1 (Algorithm of Bézier curve construction) Let $c = \frac{a + b}{2}$.

The point $B_\alpha(c) = d_0^0$ computed by the relation

$$
\begin{align*}
&d_0^0 = d_i \\
&d_i^{r+1} = \frac{\alpha}{2\alpha - 1} d_i^{r+1} + \frac{\alpha - 1}{2\alpha - 1} d_i^r \\
&\forall r = 0, \ldots, n-1, \forall i = 0, \ldots, n
\end{align*}
$$

subdivides the rational Bézier curve $B_\alpha$ in two rational Bézier curves of $\alpha$ index and degree $n$, on the parameter space $[a, b]$ with the respective control polygon points: $(d_0^i)_{i=0}^n$ and $(d_i^{r-1})_{i=0}^n$

Proof

A direct application of the proposition 3.4

Illustration 3.2 The figures 10, 11, 12 and 13 point out some alternatives of the subdivision algorithm iterate on 4 levels in the construction of the Bézier curve $B_\alpha$ with control polygon $\Pi = \{(0, 3.5), (4, 0.5), (4.5, 2.5), (0, 0)\}$ with $\alpha \in \{-1, 2, 5, \infty\}$

![Figure 10](image)

Figure 10 – Bézier curve $B_{-1}$ and control sub-polygons on levels up to 4

From the analysis of the figures 10 and 11 we can observe that the chronological repartition of the control sub-polygons points of the Bézier curves $B_{-1}$ et $B_2$, points out the symmetrical effect of the indexes $\alpha$ and $1 - \alpha$. In the Bézier curve $B_{-1}$ the points concentration converge to the endpoint of parameter 0, in the Bézier curve $B_2$ the points concentration converge to the endpoint of parameter 1.

The points equirepartition in the figure 13 confirms the self symmetry shown in the standard case which corresponds to $\alpha = \infty$.

Remark 3.1 In conducting evidence geometric properties Bézier curves discussed above, the nature of the functions homographic $f_\alpha \in \mathcal{H}([a, b])$ was not
Figure 11 – Bézier curve $B_2$ and control sub-polygons on levels up to 4

Figure 12 – Bézier curve $B_5$ and control sub-polygons on levels up to 4

exploited. We only used the fact that are an increasing bijection of $[a, b]$ into $[0, 1]$.

The following proposition shows that the curves $B_\alpha$ are completely defined by their points of control. Which reinforces this observation. In this proof we only used the fact that $f_\alpha$ is a diffeomorphism of the class $C^2$.

**Proposition 3.5** Let $n, d \in \mathbb{N}$ and we consider $n+1$ points $(d_i)_{i=0}^n \in \mathbb{R}^d$. Let $\alpha, \beta \in ]-\infty, 0[ \cup [1, \infty[$ and $a, b, a_1, b_1 \in \mathbb{R}$ such that $a < b$ and $a_1 < b_1$. The rational Bézier curve $B_\alpha$ of $\alpha$ index and degree $n$ with control polygon points $(d_i)_{i=0}^n$ on the parameter space $[a, b]$ is same to the rational Bézier curve $B_\beta$ of $\beta$ index and degree $n$ with control polygon points $(d_i)_{i=0}^n$, the parameter space $[a_1, b_1]$

**Proof**
Let $M \in B_\alpha$ and $f_\alpha \in \mathcal{H}([a, b])$, then there exists $x \in [a, b]$ such that

$w = f_\alpha(x) \in [0, 1]$ and $M = B_\alpha(x) = \sum_{i=0}^{n} d_i^{\alpha} B_i^n(x)$ with $\alpha B_i^n(x) = C^n_i w^i (1 - w)^{n-i}$.

Since $f_\beta \in \mathcal{H}([a_1, b_1])$ is a bijection from $[a_1, b_1]$ into $[0, 1]$ then

$y = f_\beta^{-1}(w) \in [a_1, b_1]$ and $\beta B_i^n(y) = C^n_i w^i (1 - w)^{n-i} = \alpha B_i^n(x)$. This implies

$M = B_\alpha(x) = \sum_{i=0}^{n} d_i^{\alpha} B_i^n(x) = \sum_{i=0}^{n} d_i^{\beta} B_i^n(y) = B_\beta(y) \in B_\beta$

We conclude that $M \in B_\alpha \Rightarrow M \in B_\beta$

In the same way we show that $M \in B_\beta \Rightarrow M \in B_\alpha$

More precisely we have $\alpha B_i^n = \beta B_i^n \circ f_\beta^{-1} \circ f_\alpha$ and $\forall x \in [a, b] B_\alpha(x) = B_\beta(y)$ with $y = f_\beta^{-1} \circ f_\alpha(x) \in [a_1, b_1]$.

Otherwise one can observe that $\forall f \in \mathcal{H}([a, b])$ we have $f, f^{-1} \in \mathcal{C}^\infty([a, b])$.

Then we have $f_\beta^{-1} \circ f_\alpha \in \mathcal{C}^\infty([a, b])$.

For all $x \in [a, b]$ we have

\[
\frac{dB_\alpha}{dx} = \frac{dB_\beta}{dy}(y) \frac{dy}{dx}
\]

\[
\frac{d^2B_\alpha}{dx^2} = \frac{d^2B_\beta}{dy^2}(y) \left( \frac{dy}{dx} \right)^2 + \frac{dB_\beta}{dy}(y) \frac{d^2y}{dx^2}
\]

with $y = f_\beta^{-1} \circ f_\alpha(x)$

The curves $B_\alpha$ and $B_\beta$ have the same tangente at the common point $B_\alpha(x)$ for all $x \in [a, b]$.

Therefore

\[
\frac{dB_\alpha}{dx} \times \frac{d^2B_\alpha}{dx^2}(x) = \frac{dB_\beta}{dy}(y) \times \frac{d^2B_\beta}{dy^2}(y) \left( \frac{dy}{dx} \right)^3
\]
and
\[ \left\| \frac{dB_\alpha(x)}{dx} \times \frac{d^2 B_\alpha(x)}{dx^2} \right\| = \left\| \frac{dB_\beta(y)}{dy} \times \frac{d^2 B_\beta(y)}{dy^2} \right\| \left\| \frac{dy}{dx} \right\|^3 \]

We deduce that \( \kappa_\alpha(x) \) and \( \kappa_\beta(y) \) the respective curvatures of \( B_\alpha \) and \( B_\beta \) at the common point \( B_\alpha(x) = B_\beta(y) \) satisfy

\[ \kappa_\alpha(x) = \left\| \frac{dB_\alpha(x)}{dx} \times \frac{d^2 B_\alpha(x)}{dx^2} \right\| \left\| \frac{2B_\alpha(x)}{dx^2} \right\|^3 = \left\| \frac{dB_\beta(y)}{dy} \times \frac{d^2 B_\beta(y)}{dy^2} \right\| \left\| \frac{dy}{dx} \right\|^3 = \kappa_\beta(y) \]

This completes the proof.

4 Conclusion

We have an approximation basis of rational functions which unfortunately cannot create the functions \( t \mapsto t + t^2 \) and \( t \mapsto 1 - t^2 \). Thus, this new family of Bézier curves do not resolve the primary motivation of rational Bézier curves, which consists of generating planar conics exactly. This new class, however, gives another alternative construction of Bézier curves using algorithms as soon as accurate than the standard one with a large conservation of usual properties of Bernstein functions and Bézier curves.

The analysis of the approximation properties of this new class of Bernstein functions can be of some interest.

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Appendix

Rational Bernstein basis functions of degree 2 and derivatives:
\[ \alpha \mathcal{B}_i^2(t) = \begin{cases} 
\frac{(\alpha-1)^2 (b-t)^2}{(a-b-\alpha a+t)^2} & \text{if } i = 0 \\
\frac{-2(\alpha-1) \alpha (a-t) (b-t)}{(a-b-\alpha a+t)^2} & \text{if } i = 1 \\
\frac{\alpha^2 (a-t)^2}{(a-b-\alpha a+t)^2} & \text{if } i = 2
\end{cases} \]

\[ \frac{d}{dt} \alpha \mathcal{B}_i^2(t) = \begin{cases} 
\frac{-2(\alpha-1)^2 \alpha (b-t) (b-a)}{(a-b-\alpha a+t)^3} & \text{if } i = 0 \\
\frac{2(\alpha-1) \alpha (b-a) (a-b-\alpha a-2 \alpha t+t)}{(a-b-\alpha a+t)^4} & \text{if } i = 1 \\
\frac{-2(\alpha-1) \alpha^2 (a-t) (b-a)}{(a-b-\alpha a+t)^3} & \text{if } i = 2
\end{cases} \]

Rational Bernstein basis functions of degree 3 and derivatives:

\[ \alpha \mathcal{B}_i^3(t) = \begin{cases} 
\frac{(\alpha-1)^3 (b-t)^3}{(a-b-\alpha a+t)^3} & \text{if } i = 0 \\
\frac{-3(\alpha-1)^2 \alpha (a-t) (b-t)^2}{(a-b-\alpha a+t)^3} & \text{if } i = 1 \\
\frac{3(\alpha-1) \alpha^2 (a-t)^2 (b-t)}{(a-b-\alpha a+t)^4} & \text{if } i = 2 \\
\frac{-\alpha^3 (a-t)^3}{(a-b-\alpha a+t)^4} & \text{if } i = 3
\end{cases} \]

\[ \frac{d}{dt} \alpha \mathcal{B}_i^3(t) = \begin{cases} 
\frac{-3(\alpha-1)^3 \alpha (b-t)^2 (b-a)}{(a-b-\alpha a+t)^4} & \text{if } i = 0 \\
\frac{3(\alpha-1)^2 \alpha (b-t) (b-a) (a-b+2 \alpha a-3 \alpha t+t)}{(a-b-\alpha a+t)^4} & \text{if } i = 1 \\
\frac{-3(\alpha-1) \alpha^2 (a-t) (b-a) (2 a b-2 b+\alpha a-3 \alpha t+2 t)}{(a-b-\alpha a+t)^4} & \text{if } i = 2 \\
\frac{3(\alpha-1) \alpha^3 (a-t)^2 (b-a)}{(a-b-\alpha a+t)^4} & \text{if } i = 3
\end{cases} \]