TD IMPLIES CC_\mathbb{R}

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Abstract. Assuming ZF, we prove that Turing determinacy (TD) implies the countable choice axiom for sets of reals (CC_\mathbb{R}).

1. Introduction

Turing reduction \leq_T is a partial order over reals. It naturally induces an equivalence relation \equiv_T. Given a real x, its corresponding Turing degree x is a set of reals defined by \{y \mid y \equiv_T x\}. We say x \leq y if x \leq_T y. We use D to denote the set of Turing degrees. An upper cone u_x of Turing degrees is the set \{y \mid y \geq x\}.

Definition 1.1. Turing determinacy, or TD, says that for any set A of Turing degrees, either A or D \setminus A contains an upper cone of Turing degrees.

Martin proves the following famous theorem.

Theorem 1.2 (Martin \[5\]). Over ZF, the Axiom of determinacy, or AD, implies TD.

Definition 1.3. \begin{itemize} 
    \item Countable choice axiom for sets of reals, or CC_\mathbb{R}, says that for any countable sequence \{A_n\}_{n \in \omega} of nonempty sets of reals, there is a function f : \omega \to \mathbb{R} so that for every n, f(n) \in A_n.
    \item Dependent choice axiom for sets of reals, or DC_\mathbb{R}, says that for any binary relation R over reals so that \forall x \exists y R(x, y), there is a function f : \omega \to \mathbb{R} so that for every n, R(f(n), f(n+1)).
\end{itemize}

Though AD contradict the Axiom of choice, or AC, Mycielski proves the following theorem.

Theorem 1.4 (Mycielski \[6\]). Over ZF, AD implies CC_\mathbb{R}.

It is a long standing question whether AD implies DC_\mathbb{R}.

The following question has been also circulated among set theorists (for example, see \[2\]).

Question 1.5. Over ZF, does TD imply CC_\mathbb{R}?

In this paper, we answer the question.

We assume that readers have some knowledge of descriptive set theory and recursion theory. The major references are \[1\] and \[4\].

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Throughout the section, we assume ZF + TD.

We identify $\mathbb{R}$ as $2^\omega$ or $\mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$. So if $x \in \mathbb{R}$, then $x(n)$, the $n$-th bit of $x$, belongs to $\{0, 1\}$. The structure of the Turing degrees is an upper semi-lattice. I.e. for any reals $x$ and $y$, $x \oplus y = \{2n \mid x(n) = 1 \land n \in \mathbb{N}\} \uplus \{2n+1 \mid y(n) = 1 \land n \in \mathbb{N}\}$ has the least Turing degree above both $x$ and $y$.

We prove the following theorem.

**Theorem 2.1.** Over ZF, TD implies CC$_\mathbb{R}$

The following reformulation of CC$_\mathbb{R}$ is helpful to understand the idea behind the subsequent proofs.

**Proposition 2.2.** CC$_\mathbb{R}$ is equivalent to that for any function $f : D \to \mathbb{R}$, there is an upper cone on which $f$ is constant.

**Proof.** Assume CC$_\mathbb{R}$. For any $i$, there is a $j_i \in \mathbb{N}$ so that $A_i = \{x \mid f(x)(i) = j_i\}$ contains an upper cone. By CC$_\mathbb{R}$, we can choose, for each $i \in \mathbb{N}$, $z_i$ such that $u_{z_i} \subset A_i$ and some $z$ above all $z_i$. Then $f$ is constant on $u_z$.

Now suppose that $\{A_n\}_{n \in \omega}$ is a sequence of nonempty sets of reals witnessing the failure of CC$_\mathbb{R}$. For any degree $x$, let $i_x$ be the least $i$ so that there is no real in $A_i$ Turing below $x$. By the assumption, $i_x$ exists for any degree $x$. Define a function $f : D \to \mathbb{R}$ as follows:

$$f(x)(n) = \begin{cases} 0, & n < i_x; \\ 1, & \text{otherwise.} \end{cases}$$

By the assumption, $f$ is well defined for every degree. But clearly $f$ cannot be constant on any upper cone, a contradiction. \hfill $\square$

The following technical lemma, which can be proved with ZF, is folklore in recursion theory.

**Lemma 2.3.** For any degree $x$, there is a family Turing degrees $\{y_r \mid r \in \mathbb{R}\}$ satisfying the following property:

1. For any $r \in \mathbb{R}$, $x < y_r$;
2. For any $r_0 \neq r_1 \in \mathbb{R}$ and $z < y_{r_0}, y_{r_1}$, we have that $z \leq x$;
3. For any $z \geq x''$, the Turing double jump of $x$, there is an infinite set $C_z \subset \mathbb{R}$ so that $y''_r = z$ for any $r \in C_z$.

**Proof.** Fix a real $x$. It is routine (see [4]) to prove, by a recursive Sacks forcing relative to $x$, that there is a perfect tree $S \subset 2^{<\omega}$ with $S \leq_T x''$ satisfying the following property:

1. For any $y \in [S]$, the collection of infinite paths through $S$, we have that $x <_T y$ and there is no real $z$ so that $x <_T z <_T y$;
2. For any $y_0 \neq y_1 \in [S]$, $y_0$ is Turing incomparable with $y_1$;
3. For any $y \in [S]$, $y'' \equiv_T y \oplus x''$. 

Then for any reals $y_0 \neq y_1 \in [S]$, $z <_T y_0$ and $z <_T y_1$, we have that $x \leq_T z \oplus x \leq_T y_0$ and $x \leq_T z \oplus x \leq_T y_1$. Thus, by (a) and (b), $x \equiv_T z \oplus x$ and so $z \leq_T x$.

Also for any $z \geq_T x''$, it is clear that there is an infinite set $C_z \subset [S]$ so that for any $y \in C_z$, we have that $y \oplus S \equiv_T z$. Then for any $y \in C_z$, $y'' \equiv_T y \oplus x'' \geq_T y \oplus S \equiv_T z$. Also $y'' \equiv_T y \oplus x'' \leq_T z$. So $y'' \equiv_T z$.

Now let $f$ be a bijection between $\mathbb{R}$ and $[S]$ and set $y_r$ to be the Turing degree of $f(r)$ for any $r \in \mathbb{R}$. Then the collection $\{y_r\}_{r \in \mathbb{R}}$ is exactly what we want. \hfill \qed

Firstly, we prove the following weak form of CC$_\mathbb{R}$.

**Proposition 2.4.** Suppose that $\{A_n\}_{n \in \omega}$ is a countable family of countable sets of reals, then $A = \bigcup_n A_n$ is countable.

**Proof.** Suppose not. Let $\{A_n\}_{n \in \omega}$ is a countable family of countable sets of reals so that $A = \bigcup_n A_n$ is not countable. Given a real $x$, let $n_x$ be the least number $n$ so that there is a real $z \in A_n$ that is not Turing below $x$. Then $n_x$ is defined for every $x$. Moreover, $n_x \leq n_y$ for any $x \leq y$.

Now fix a degree $x$. Let $\{y_r \mid r \in \mathbb{R}\}$ be as in Lemma 2.3. Since $A_{n_x}$ is countable, by (3) of the lemma, there is an uncountable set $Z \subseteq \mathbb{R}$ so that for any $r \in Z$, $n_{y''_r} > n_x$.

We claim that there must be some $r \in Z$ so that $n_{y_r} = n_x$ and hence $n_{y''_r} < n_{y''_r}$. Otherwise, for any $r \in Z$, there is some $s \in A_{n_x}$ so that $s \leq y_r$ but $s \not\leq x$. Then by (1) and (2) of the Lemma, $A_{n_x}$ must be uncountable, which is a contradiction to the assumption.

Then by TD, there is some $x_0$ so that for any $y \in u_{x_0}$, $n_y < n_{y''_r}$. Then $n_{(x_0)(\omega)}$ is undefined where $(x_0)(\omega)$ is the $\omega$-th Turing jump of $x_0$. A contradiction. \hfill \qed

By Proposition 2.4, we have the following conclusion.

**Corollary 2.5.** Every countable set of Turing degrees has an upper bound.

**Definition 2.6.**

- A function $\Phi : D \rightarrow D$ is called almost increasing, if there is an upper cone $u_x$ so that for any $y \in u_x$, $y < \Phi(y)$.

- Fix a function $\Phi : D \rightarrow D$. We say that a function $f : D \rightarrow \mathbb{R}$ is almost injective corresponding to $\Phi$ if there is an upper cone $u_x$ so that for any $y \in u_x$, $f(y) \neq f(\Phi(y))$.

For example, the Turing jump function $J : x \mapsto x'$ is almost increasing.

**Lemma 2.7.** Every function $F : D \rightarrow \text{Ord}$ is non-decreasing over an upper cone.

**Proof.** It suffices to prove that

\[ L = \{x \mid \forall y \geq x(F(x) \leq F(y)) \} \]

is cofinal and hence contains an upper cone. To see this, just note that for any $x$, there is a $y \geq x$ such that

\[ F(y) = \min\{F(z) : z \geq x\}. \]

It is clear that $y \in L$. \hfill \qed
Lemma 2.8. For any almost increasing function \( \Phi \), there is no almost injective function corresponding to it.

Proof. Suppose not. Fix an almost increasing function \( \Phi : D \to D \) witnessed by an upper cone \( u_x \) and an almost injective function \( f \) corresponding to it. For \( y, z \in u_x \), define

\[
l(y, z) = \min \{ l \mid f(y)(l) \neq f(z)(l) \}.
\]

Note that, by the assumption, \( l(y, \Phi(y)) \) is defined for every \( y \in u_x \).

By Lemma 2.7

\[
L_1 = \{ y \mid \forall z \geq y(l(y, \Phi(y)) \leq l(z, \Phi(z))) \}
\]

contains an upper cone \( u_{x_1} \) for some \( x_1 \geq x \).

Also for \( i \in \{0, 1\} \), let

\[
L_i^2 = \{ y \mid f(y)(l(y, \Phi(y)) = i) \}.
\]

Then for some \( i \), \( L_i^2 \) contains an upper cone \( x_2 \geq x_1 \). Then for any \( y \geq x_2 \),

\[
f(y)(l(y, \Phi(y))) = i = f(\Phi(y))(l(\Phi(y), \Phi(\Phi(y)))).
\]

So by the definition of \( l \),

\[
l(y, \Phi(y)) < l(\Phi(y), \Phi(\Phi(y))).
\]

By Corollary 2.5, there is a degree \( z \) which is above \( \Phi^{(n)}(y) \) for every \( n \). Then \( l(z, \Phi(z)) \) is undefined since \( z > x_1 \).

Lemma 2.9. Suppose that \( f : D \to \mathbb{R} \) is a function so that \( f(y) \leq y^1 \) over an upper cone, then the range of \( f \) is at most countable over an upper cone.

Proof. Suppose not.

By Lemma 2.8 and applying \( x \mapsto x' \) to \( \Phi \), we get that \( f(y) = f(y') \) over an upper cone. In particular, \( f(y) \notin y \).

So we can find an upper cone \( u_x \) so that \( f(y') = f(y) < y \) for any \( y \in u_x \). We fix a set \( \{y_r \mid r \in \mathbb{R}\} \) as in Lemma 2.3. By applying the assumption to the upper cone \( u_{x''} \), we may pick a real \( r_0 \) with \( y'' \geq x'' \) so that \( f(y''_r) \notin x \). By (3) of Lemma 2.3, there is another real \( r_1 \neq r_0 \) so that \( y''_r = y''_{r_1} \) and so \( f(y''_{r_0}) = f(y''_{r_1}) \).

However, \( f(y''_{r_1}) = f(y''_{r_1}) = f(y_{r_i}) \leq y_{r_i} \) for both \( i \leq 1 \). By (2) of Lemma 2.3, \( f(y''_{r_0}) \leq x \) for both \( i \leq 1 \), contradicting the selection of \( y_{r_0} \).

Lemma 2.10. Suppose that \( f : D \to \mathbb{R} \) is a function. The range of \( f \) is at most countable over an upper cone.

Proof. Suppose not. Then by Lemma 2.9, we may assume that over an upper cone \( u_x \),

\[
f(y) \notin y. \quad (\star)_y
\]

Let

\[
\Phi(y) = \{ z \mid \exists y_0 \in y(z \equiv_T y_0 \oplus f(y)) \}.
\]

Then \( \Phi \) is a well defined almost increasing function over the upper cone \( u_x \). Then by Lemma 2.8, \( f(y) = f(\Phi(y)) \) over an upper cone \( u_{x_1} \subseteq u_x \). Then over the upper cone,

\[
f(\Phi(y)) = f(y) \leq \Phi(y).
\]

\(^1\)Here we identify a real \( y \) with its degree \( y \).
This contradicts $(\ast)_{\Phi(y)}$. □

Now we are ready to prove the main theorem 2.1.

**Proof.** (of Theorem 2.1.) Suppose that $\{A_n\}_{n \in \mathbb{N}}$ is a countable family of nonempty sets of reals. Without loss of generality, we may assume that $A_n$ is Turing upward closed. Furthermore, we also may assume that $A_{n+1} \subset A_n$ for every $n$ (reset $A_n$ to be $\bigcap_{k \leq n} A_k$ if necessary).

For a contradiction, we assume that for any real $y$, there is some $n$ (and so there are infinitely many $n$’s) so that there is no real in $A_n$ Turing below $y$. Define $B_n = A_n \setminus A_{n+1}$ for every $n$. Note that $B_n$ is nonempty and disjoint from an upper cone for every $n$.

Define a function $f : D \to \mathbb{R}^2$ so that

$$f(y) = \{n \mid \exists z \geq y(z \in B_n)\}.$$  

By the property of $B_n$’s, for every $y$, $f(y)(n) = 1$ for infinitely many $n$’s. By Lemma 2.10, $f$ is countable over an upper cone of Turing degrees. Let $\{a_i\}_{i \in \omega}$ be an enumeration of the range of $f$ over the upper cone. Note that for every $i$, $a_i(n) = 1$ for infinitely many $n$’s. Then there is some $a \subseteq \omega$ so that $a \cap a_i \neq \emptyset$ and $a_i \setminus a \neq \emptyset$ for every $i$. Let $C_0 = \bigcup_{n \in a} B_n$ and $C_1 = \bigcup_{n \notin a} B_n$. Then $C_0 \cap C_1 = \emptyset$ and $C_0 \cup C_1 = \bigcup_{n \in \mathbb{N}} B_n$.

So either $C_0$ or $C_1$ contains an upper cone of Turing degrees.³

If $C_0$ contains an upper cone $u_x$ of Turing degrees, then let $y \in u_x$. Then for any $y_0 \geq y$, $y_0 \notin C_1$ and so $f(y) \subseteq a$. But $a_i \subseteq a$ for every $i$. Thus $f(y) \neq a_i$ for every $i$.

If $C_1$ contains an upper cone $u_x$ of Turing degrees, then let $y \in u_x$. Then for any $y_0 \geq y$, $y_0 \notin C_0$ and so $f(y) \cap a = \emptyset$. But $a_i \cap a \neq \emptyset$ for every $i$. Thus $f(y) \neq a_i$ for every $i$.

So in either case, there is some $y$ so that $f(y)$ is not in the range of $f$, which is absurd. □

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²Here we identify a real as a subset of natural numbers.

³Actually ranges of the $f$ over upper cones generate an ultrafilter as observed by Larson [3].
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