Supporting Information

Network ‘small-world-ness’: a quantitative method for determining canonical network equivalences

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1 Real-world systems — details

The correlations of $S^\triangle$ and $S^{ws}$ with $n$ (see main text) are re-plotted in Figure ST1a and Figure ST1b, respectively, using the indices of the networks from Table 1. The three real-world systems with $S^{ws} < 1$ were omitted from that correlation as these were not small-world networks. Some systems were borderline small-world networks, defined here as $1 \leq S^\triangle, S^{ws} \leq 3$ — 4 systems had $S^{ws}$ in this range, 6 systems had $S^\triangle$ in this range. For these we tested the significance of their small-world-ness scores as detailed in the main text. All had $S$ values greater than the upper 99% confidence limit. We conclude that all other real-world systems were small-world networks.

Figure ST1: Correlation of real-world system properties. Both $S^\triangle$ (a) and $S^{ws}$ (b) scale linearly with network size $n$ across real networks from all domains, and irrespective of other properties. Numbers correspond to entries in Table 1.
The few systems that were not small-world networks were defined as such according to their
$S^{ws}$ values, and not their $S^\Delta$ values. This illustrates the comment made in the main text that
$C^{ws}$ and $C^\Delta$ often considerably differ, and actually describe two different graph properties: one
interpretation is that $C^{ws}$ measures average local edge density and that $C^\Delta$ measures the proportion
of closed loops in the network. We can see the difference clearly in Figure ST2, which shows the
correlation of $C^{ws}$ and $C^\Delta$ for the real-world systems in Table 1 (main text).

Figure ST2: Correlation of clustering coefficients for real-world systems. Linear regression shows
some correlation ($r^2 = 0.65, n = 27$), but $C^{ws}$ and $C^\Delta$ for some systems differ by an order of
magnitude.

2 Finding maximum $S^\Delta$

We wanted to find out how close the particular linear model $S^\Delta \simeq 0.023n$ was to the theoretical
maximum possible value for $S^\Delta$ from the Watts-Strogatz (WS) model, given the corresponding
mean degree $\langle k \rangle \simeq 5$ for that data-set. To do this, we differentiated $S^\Delta_{ws}$ with respect to $p$; as $S^\Delta_{ws}$
has a unique maximum value, finding the value of $p$ for which $dS^\Delta_{ws}/dp = 0$ would thus give us
the theoretical maximum $S^\Delta$ value.

The ratios $\lambda_{ws}$ and $\gamma^\Delta_{ws}$ can be expressed as (see main text):

$$\lambda_{ws} = \frac{n \ln(2K)f(nKp)}{K \ln(n)},$$

$$\gamma^\Delta_{ws} = \frac{3K - 3}{8K^2 - 4K}n(1 - p)^3.$$  

If we assume that the product $nKp \gg 1$, and thus substitute the asymptotic limit $f(x) = \ln(2x)/4x$
into (1), we get the full expression

\[ S^\triangle_{ws} = \frac{\gamma_{ws} \lambda_{ws}}{(8K^2 - 4K) \ln(2K) \ln(2nKp)}. \]  

(3)

We want to differentiate this with respect to \( p \), so gather all constant terms in (3)

\[ S^\triangle_{ws} = \beta (1 - p)^3 \ln(2nKp), \]  

(4)

where

\[ \beta = \frac{4K^2(3K - 3)n \ln(n)}{(8K^2 - 4K) \ln(2K)}, \]  

(5)

and differentiate (4) to obtain

\[ \frac{dS^\triangle_{ws}}{dp} = \beta \left\{ \frac{\ln(2nKp) [(1 - p)^3 - 3p(1 - p)^2] - (1 - p)^3}{\ln(2nKp)^2} \right\}, \]  

(6)

We set \( dS^\triangle_{ws}/dp = 0 \) and re-arrange to find \( p \). No closed form solution exists, but after some algebra we find

\[ 0 = (1 - p) \left( 1 - \frac{1}{\ln(2nKp)} \right) - 3p. \]  

(7)

We use a standard minimisation routine — fzero from MATLAB (The MathWorks, Natick, MA), with an initial value of \( p = 0.5 \) — to find values for \( p \) which satisfy this equality, given \( n \in [10^3, 10^4, \ldots, 10^{20}] \) and \( K = \langle k \rangle / 2 = 2.5 \). These are shown in Figure ST3a. We see that the value for \( p \) that maximises \( S^\triangle_{ws} \) is surprisingly restricted across the whole range of \( n \), converging on an asymptotic value of \( p = 0.246 \) as \( n \to \infty \). If we substitute the values for \( n \) and the resulting \( p \) values into (3) we find the linear relationship \( S^\triangle_{ws} = 0.181n \), shown in Figure ST3b. Thus, as shown in the main text (Figure 2a), the linear rate of real-world scaling does not reach the theoretical maximum.
Figure ST3: Determining maximum possible $S^\Delta$.  

**a** The value for $p$ that maximises $S^\Delta_{ws}$ in the WS model falls in a narrow range over many orders of magnitude of $n$, converging on an asymptotic value of $p = 0.246$ as $n \to \infty$.  

**b** The result is that maximum $S^\Delta_{ws}$ grows linearly with $n$. 

![Graph a](image1.png)

![Graph b](image2.png)