Supplemental Material For “Primal-Dual Q-Learning Framework for LQR Design”

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Abstract

Recently, reinforcement learning (RL) is receiving more and more attentions due to its successful demonstrations outperforming human performance in certain challenging tasks. In our recent paper ‘primal-dual Q-learning framework for LQR design,’ we study a new optimization formulation of the linear quadratic regulator (LQR) problem via the Lagrangian duality theories in order to lay theoretical foundations of potentially effective RL algorithms. The new optimization problem includes the Q-function parameters so that it can be directly used to develop Q-learning algorithms, known to be one of the most popular RL algorithms. In the paper, we prove relations between saddle-points of the Lagrangian function and the optimal solutions of the Bellman equation. As an application, we propose a model-free primal-dual Q-learning algorithm to solve the LQR problem and demonstrate its validity through examples. It is meaningful to consider additional potential applications of the proposed analysis. Various SDP formulations of Problem 5 or Problem 2 of the paper can be derived, and they can be used to develop new analysis and control design approaches. For example, an SDP-based optimal control design with energy and input constraints can be derived. Another direction is algorithms for structured controller designs. These approaches are included in this supplemental material.

I. Notation

The adopted notation is as follows: \( \mathbb{N} \) and \( \mathbb{N}_+ \): sets of nonnegative and positive integers, respectively; \( \mathbb{R} \): set of real numbers; \( \mathbb{R}_+ \): set of nonnegative real numbers; \( \mathbb{R}_{++} \): set of positive real numbers; \( \mathbb{R}^n \): \( n \)-dimensional Euclidean space; \( \mathbb{R}^{n \times m} \): set of all \( n \times m \) real matrices; \( A^T \): transpose of matrix \( A \); \( A \succ 0 \) (\( A \prec 0 \), \( A \succeq 0 \), and \( A \preceq 0 \), respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix.

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II. Projected Gradient Descent Method for Structured Optimal Control Design

In this section, we study a projected gradient descent algorithm to approximately solve the structured control design problems \[1,2\] (e.g., the output feedback, decentralized, and distributed control designs). The main ideas originate from \[2,3\]. For any fixed \(F \in \mathbb{R}^{n \times m}\) and initial state \(z \in \mathbb{R}^n\), consider the discounted cost

\[
J_\alpha(F, z) := \sum_{k=0}^{\infty} \alpha^k \begin{bmatrix} x(k; F, z) \\ Fx(k; F, z) \end{bmatrix}^T \Lambda \begin{bmatrix} x(k; F, z) \\ Fx(k; F, z) \end{bmatrix}.
\] (1)

As in the previous sections, the cost \(J_\alpha(F, z)\) can be expressed as \(J_\alpha(F, z) = \text{Tr}(\Lambda S)\), where \(S\) satisfies

\[
\alpha A_F S A_F^T + \begin{bmatrix} I \\ F \end{bmatrix} z z^T \begin{bmatrix} I \\ F \end{bmatrix}^T = S.
\]

Define the structured identity \[2\]

\[
[I_K]_{ij} = \begin{cases} 
0 & \text{if } [F]_{ij} = 0 \text{ is required} \\
1 & \text{otherwise}
\end{cases}
\]

where \([F]_{ij}\) indicates its element in \(i\)-th row and \(j\)-th column. In addition, define the subspace

\[
\mathcal{K} := \{ F \in \mathbb{R}^{n \times m} : F \circ I_K = F \},
\]

where \(\circ\) denotes the entry-wise multiplication of matrices (Hadamard product). The structured optimal control design problem can be stated as follows.

**Problem 1.** Solve

\[
\inf_{S \in \mathbb{S}^{n+m}, F \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S)
\]

subject to \(F \in \mathcal{K},
\]

\[
\alpha A_F S A_F^T + \begin{bmatrix} I_n \\ F \end{bmatrix} z z^T \begin{bmatrix} I_n \\ F \end{bmatrix}^T = S.
\]
Now, we will compute the gradient of $J_{\alpha}(F, z)$. We follow the main ideas from [2]–[4]. For any matrix $X$, let $dX$ denote an infinitesimal change of the variable $X$. For a matrix valued function $f : \mathbb{R}^{n \times m} \to \mathbb{R}$, we define the differential $df$ as the part of $f(X + dX) - f(X)$ that is linear in $dX$. From the proof of [Lemma 4, main document], we can easily derive the following result.

**Lemma 1.** Let $F \in \mathcal{F}$ be given. If $S \in \mathbb{S}_{++}^{n + m}$ satisfies $S = \alpha A_F S A_F^T + \begin{bmatrix} I \\ F \end{bmatrix} zz^T \begin{bmatrix} I \\ F \end{bmatrix}^T$, then,

$$\alpha \begin{bmatrix} A \\ B \end{bmatrix} S \begin{bmatrix} A \\ B \end{bmatrix}^T + zz^T = S_{11}.$$

Based on the lemma, we can calculate the gradient of the cost in (1).

**Proposition 1.** We have

$$\nabla_F J_{\alpha}(F, z) = 2P_{12}^T S_{11} + 2P_{22} S_{12},$$

where $S$ and $P$ are solutions to

$$S = \alpha A_F S A_F^T + \begin{bmatrix} I \\ F \end{bmatrix} zz^T \begin{bmatrix} I \\ F \end{bmatrix}^T.$$  \hspace{1cm} (2)

for $S$ and $\alpha A_F^T PA_F - P + \Lambda = 0$ for $P$, respectively.

**Proof.** Consider the cost function $J_{\alpha}(F, z) = \text{Tr}(\Lambda S)$, where $S$ solves (2). It is importance to notice that $S$ is a function of $F$. Its differential with respect to $F$ is $dJ_{\alpha}(F, z) = \text{Tr}(\Lambda dS)$, where $dS = \alpha A_F dSA_F^T + N + N^T$ and

$$N := \begin{bmatrix} 0 & 0 \\ dFS_{11} & dFS_{11}^T \\ \end{bmatrix}.$$

Since $dS$ satisfies the Lyapunov equation $dS = \alpha A_F dSA_F^T + N + N^T$, it can be rewritten by

$$dS = \sum_{k=0}^{\infty} \alpha^k (A_F)^k (N + N^T)(A_F^T)^k = 2 \sum_{k=0}^{\infty} \alpha^k (A_F)^k N^T (A_F^T)^k.$$

Plug the above equation into $dJ_{\alpha}(F, z) = \text{Tr}(\Lambda dS)$ to have

$$dJ_{\alpha}(F, z) = \text{Tr}(\Lambda dS) = 2 \text{Tr} \left( \Lambda \sum_{k=0}^{\infty} \alpha^k (A_F)^k N^T (A_F^T)^k \right)$$

$$= 2 \text{Tr} \left( \sum_{k=0}^{\infty} \alpha^k (A_F^T)^k \Lambda (A_F)^k N^T \right) = 2 \text{Tr}(PN^T),$$
Noting that
\[ PN^T = \begin{bmatrix} 0 & P_{11}S_{11}dF_T + P_{12}FS_{11}dF_T \\ 0 & P_{12}^T S_{11}dF_T + P_{22}2FS_{11}dF_T \end{bmatrix}, \]
we have \( dJ_\alpha(F, z) = \text{Tr}(dF^T(2P_{12}^T S_{11} + 2P_{22}2FS_{11})) \). Plugging \( F = S_{12}S_{11}^{-1} \) in [Lemma 4, main document] into the last equation and using \( df(F) = \text{Tr}(dF^TX) \iff \nabla_F f(F) = X \) [4, pp. 840] for any \( X \) leads to the desired result.

To compute the gradient in Proposition 1, we need to solve (2) for \( S \) and \( \alpha \) for \( P \). If the model is known, then both \( S \) and \( P \) can be approximated from simulations. To this end, define the adjoint system
\[ \xi(k+1) = A_F^T \xi(k), \quad \xi(0) = \xi \in \mathbb{R}^{n+m}, \tag{3} \]
We denote \( \xi(k; F, \xi) \) by the solution of (3) starting from \( \xi(0) = \xi \) and let \( \{ \xi_i \}_{i=1}^r \) be a set of initial states such that \( \Lambda = \sum_{i=1}^r (\xi_i \xi_i^T) \), where \( \Lambda := \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} \). Denote by \( \Pi_K \) the projection onto \( K \).

An approximate gradient descent algorithm to find a suboptimal solution to Problem 1 is given in Algorithm 1, where \( (\gamma_t)_{t=0}^\infty \) is a step-size sequence. For any \( F_t \in \mathbb{R}^{n \times m} \), the projection \( \Pi_K \) can be performed easily by \( \Pi_K(F_t) = F_t \circ I_K \). We note that similar and effective gradient descent algorithms have been studied previously [2]–[4]. The proposed Algorithm 1 has a different form, and can be extended to a model-free method presented in the next section. The convergence of Algorithm 1 to a stationary point can be proved based on the standard projected gradient descent algorithm [5, Section 2.3]. The constant, diminishing, or Armijo-Goldstein step size rules can be applied to guarantee the convergence.

**Example 1.** Consider [Example 1, main document] with identical assumptions except for the fact that 1) the system model \((A, B)\) is known, and 2) only the indoor air temperature \( x_1(k) \) (°C) and the reference temperature \( x_4(k) \) (°C) can be measured. Therefore, we want to design an output feedback control policy with the output \( y(k) = Cx(k) \), \( C := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \). Algorithm 1 was applied with \( \alpha = 0.9 \), \( z := \begin{bmatrix} 1 & -1 & 5 & 2 \end{bmatrix}^T \), and \( \gamma_t = 0.001 \). After 10000 iterations, we obtained \( F_t = \begin{bmatrix} -1.8359 & 0 & 0 & 1.8120 \end{bmatrix} \), and the simulation results are depicted in Figure 1.
Algorithm 1 Projected Gradient Descent Algorithm for Structured Optimal Control Design

1: Initialize $F_0$ and set $t = 0$.

2: repeat
3: For a sufficiently large integer $M > 0$, compute
   $$
   \tilde{S}(F_t) := \sum_{k=0}^{M} \alpha^k \begin{pmatrix}
   x(k; F_t, z) \\
   F_t x(k; F_t, z)
   \end{pmatrix}
   \begin{pmatrix}
   x(k; F_t, z) \\
   F_t x(k; F_t, z)
   \end{pmatrix}^T.
   $$
4: Compute
   $$
   d_t := 2 \tilde{S}_{11}(F_t) \tilde{P}_{12}(F_t) + 2 \tilde{S}_{12}(F_t) \tilde{P}_{22}(F_t).
   $$
5: $F_{t+1} = \Pi_K (F_t - \gamma_t d_t)$.
6: $t \leftarrow t + 1$
7: until a certain stopping criterion is satisfied.

![Fig. 1](image1.png)

Fig. 1. Evolution of $J_\alpha(F_t, z)$.

and Figure 2, where Figure 1 includes the evolution of the cost in Algorithm 1 and Figure 2 illustrates the state trajectory with the design control policy for several different initial states, demonstrating the validity of the algorithm.

III. Model-Free Projected Gradient Descent Method

In this section, we propose a model-free version of Algorithm 1. In [Algorithm 2, main document], one can observe that once $\tilde{S}(F_t)$ is obtained, then $\tilde{P}(F_t)$ can be computed by solving the linear matrix equation [(25), main document] if $\tilde{S}(F_t) \succ 0$. Similarly to [Section IV, main
Consider the augmented state vector $v(k) := \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$ and assume that we know the initial state $v(0) = v_0 \in \mathbb{R}^{n+m}$. Denote by $v(k; F, v_0)$ the state trajectory of the augmented system [(3), main document] at time $k$ starting from the initial augmented $\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = v_0$. As in [Section IV, main document], $u(0)$ can be freely chosen, and the control policy $u(k) = Fx(k)$ is valid from $k = 1$. One can choose $v_i \in \mathbb{R}^{n+m}, i \in \{1, 2, \ldots, r\}$, such that $\sum_{i=1}^{r} v_i v_i^T = \Gamma > 0$, where $\Gamma \in \mathbb{S}^{n+m}$. Define

$$J_\alpha(F, \Gamma) := \sum_{i=1}^{r} \sum_{k=0}^{\infty} \alpha^k v(k; F, v_i)^T \Lambda v(k; F, v_i).$$

The cost $J_\alpha(F, \Gamma)$ can be expressed as $J_\alpha(F, \Gamma) = \text{Tr}(\Lambda S)$, where $S$ satisfies

$$\alpha A_F S A_F^T + \Gamma = S.$$

The structured optimal control design problem can be stated as follows.

**Problem 2.** Solve

$$\inf_{S \in \mathbb{S}^{n+m}, F \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S)$$

subject to $F \in \mathcal{K}, \quad \alpha A_F S A_F^T + \Gamma = S.$

We will compute the gradient of $J_\alpha(F, \Gamma)$ following the main ideas of [2]–[4]. The following lemma can be easily proved.

**Lemma 2.** Let $F \in \mathcal{F}$ and $\Gamma > 0$ be given. If $S \in \mathbb{S}^{n+m}_+$ satisfies $S = \alpha A_F S A_F^T + \Gamma$, then $\alpha \begin{bmatrix} A & B \end{bmatrix} \Lambda A \begin{bmatrix} A & B \end{bmatrix}^T + \Gamma_{11} = S_{11}$, where $\Gamma_{11} \in \mathbb{S}^{n \times n}_+ \text{ is the first } n\text{-by-}n \text{ block diagonal of } \Gamma \in \mathbb{S}^{(n+m) \times (n+m)}_+.$
Proposition 2. We have

\[ \nabla_F J(F, \Gamma) = 2P_{12}^T(S_{11} - \Gamma_{11}) + 2P_{22}F(S_{11} - \Gamma_{11}), \]

where \( S \) and \( P \) are solutions to

\[ S = \alpha A_F S A_F^T + \Gamma. \quad (4) \]

for \( S \) and

\[ \alpha A_F^TPA_F - P + \Lambda = 0 \quad (5) \]

for \( P \), respectively.

Proof. Consider the cost function \( J_\alpha(F, \Gamma) = \text{Tr}(\Lambda S) \), where \( S \) solves (4). Its differential with respect to \( F \) is

\[ dJ_\alpha(F, \Gamma) = \text{Tr}(\Lambda dS), \]

where \( dS = \alpha A_F dSA_F^T + N + N^T, \)

\[ N = \begin{bmatrix} 0 & 0 \\ dFH & dFF^T \end{bmatrix}, \]

and \( H := S_{11} - \Gamma_{11} \). Since \( dS \) satisfies the Lyapunov equation \( dS = \alpha A_F dSA_F^T + N + N^T \), it can be rewritten by

\[ dS = \sum_{k=0}^{\infty} \alpha^k(A_F)^k(N + N^T)(A_F^T)^k = 2 \sum_{k=0}^{\infty} \alpha^k(A_F)^k N^T(A_F^T)^k. \]

Plug the above equation into \( dJ_\alpha(F, z) = \text{Tr}(\Lambda dS) \) to have

\[ dJ_\alpha(F, z) = \text{Tr}(\Lambda dS) = 2\text{Tr} \left( \Lambda \sum_{k=0}^{\infty} \alpha^k(A_F)^k N^T(A_F^T)^k \right) = 2\text{Tr} \left( \sum_{k=0}^{\infty} \alpha^k(A_F^T)^k \Lambda(A_F)^k N^T \right) = 2\text{Tr}(PN^T), \]

Noting that

\[ PN^T = \begin{bmatrix} 0 & P_{11}HDdF^T + P_{12}FDdF^T \\ 0 & P_{12}HDdF^T + P_{22}FDdF^T \end{bmatrix}, \]

we have \( dJ_\alpha(F, \Gamma) = \text{Tr}(dF^T(2P_{12}^TH + 2P_{22}2F)). \) Using \( df(F) = \text{Tr}(dF^TX) \Leftrightarrow \nabla_F f(F) = X \) [4, pp. 840] for any \( X \) leads to the desired result.

As in [Section IV, main document], \( S \) solving (4) can be obtained from state trajectories, \( P \) solving (5) can be obtained from the linear matrix equation [(25), main document], and the gradient in Proposition 2 can be calculated. A model-free algorithm based on these procedures is introduced in Algorithm 2.
Algorithm 2 Model-free Projected Gradient Descent Algorithm for Structured Optimal Control Design

1: Initialize $F_0$ and set $t = 0$.
2: repeat
3:   Compute
4:     \[
        \tilde{S}(F_t) := \sum_{i=1}^{M} \sum_{k=0}^{r} \alpha^k v(k; F_t, v_i) v(k; F_t, v_i)^T.
        \]
5:   Compute $P(F_t)$ by solving for $P$
6:     \[
        \alpha W(F_t)^T P W(F_t) + \tilde{S}(F_t)(\Lambda - P) \tilde{S}(F_t) = 0,
        \] where
7:     \[
        W(F_t) = \sum_{i=1}^{r} \sum_{k=0}^{M} \alpha^k v(k + 1; F_t, v_i) v(k; F_t, v_i)^T.
        \]
8:   Compute
9:     \[
        d_t = 2P_{12}(F_t)^T(\tilde{S}_{11}(F_t) - \Gamma_{11}) \\
        + 2P_{22}(F_t)F_t(\tilde{S}_{11}(F_t) - \Gamma_{11}),
        \]
10: $F_{t+1} = \Pi_K(F_t - \gamma_t d_t)$.
11: $t \leftarrow t + 1$
12: until a certain stopping criterion is satisfied.

IV. SEMIDEFINITE PROGRAMMING ALGORITHMS

In this section, we study several semidefinite programming problem (SDP) formulations of the LQR design problem with constraints. We first introduce a modified version of the extended Schur complement lemma in [6, Theorem 1].

Lemma 3. The following conditions are equivalent:

1) There exists a symmetric matrix $P \succ 0$ such that $A^T P A - P' \preceq 0$ hold.

2) There exist a matrix $G$ and a symmetric matrix $P \succ 0$ such that

\[
\begin{bmatrix}
    P' & A^T G^T \\
    G A & G + G^T - P
\end{bmatrix} \succeq 0.
\]
Proof. 1) ⇒ 2): The proof is a modification of the proof of [6, Theorem 1]. If the condition 1) holds, then by the Schur complement [Lemma 3, main document], we have
\[
\begin{bmatrix}
    P' & A^T P
    \\
    PA & P
\end{bmatrix}
\succeq 0 \iff
\begin{bmatrix}
P' & A^T P
\end{bmatrix}
\succeq 0. By setting \( G = G^T = P \), the condition 2) is satisfied.

2) ⇒ 1): Assume that the condition 2) holds. Pre- and post-multiplying it by \( [I - A^T] \) and its transpose yield \( A^T PA - P' \preceq 0 \). Since \( P \) is already positive definite, the desired result follows. \( \square \)

As a first step, we will introduce an SDP relaxation of [Problem 3, main document]. By [Proposition 3, main document], the optimization problem in [Problem 3, main document] is equivalent to the following problem.

**Problem 3.** Solve

\[
J_p := \inf_{S \in \mathbb{S}^{n+m}, F \in \mathbb{R}^{m \times n}} \text{Tr}(AS)
\]
subject to \( S \succeq 0 \),
\[
A_F S A_F^T + \begin{bmatrix} I_n \\ F \end{bmatrix} Z \begin{bmatrix} I_n \\ F \end{bmatrix}^T = S.
\]

By replacing the equality constraint into an inequality constraint, we obtain the following problem.

**Problem 4.** Solve

\[
J_{p,1} := \inf_{S \in \mathbb{S}^{n+m}, F \in \mathbb{R}^{m \times n}} \text{Tr}(AS)
\]
subject to \( S \succeq 0 \),
\[
A_F S A_F^T + \begin{bmatrix} I_n \\ F \end{bmatrix} Z \begin{bmatrix} I_n \\ F \end{bmatrix}^T \preceq S.
\]

We can prove that Problem 4 is equivalent to [Problem 3, main document].

**Proposition 3.** The optimal value of Problem 4 is equivalent to that of [Problem 3, main document], i.e., \( J_p = J_{p,1} \).
Proof. Since Problem 4 has a larger feasible set than Problem 3, we have \( J_{p,1} \leq J_p \). To prove the reversed inequality, let \((S_{p,1}, F_{p,1})\) be an optimal solution to Problem 4 and \((S_p, F_p)\) be an optimal solution to Problem 3. We will prove \( J_p = \text{Tr}(\Lambda S_p) \leq \text{Tr}(\Lambda S_{p,1}) = J_{p,1} \). If we define the operator \( \mathcal{H}(S) := A_{F_{p,1}} S A_{F_{p,1}}^T + \begin{bmatrix} I_n \\ F \\ F \end{bmatrix} \begin{bmatrix} Z \\ I_n \\ F \end{bmatrix}^T \), then we can easily prove that \( \mathcal{H} \) is \( S_+^{n+m} \)-monotone, i.e., \( P \succeq P' \Rightarrow \mathcal{H}(P) \succeq \mathcal{H}(P') \). Repeatedly applying the operator to both sides of \( \mathcal{H}(S_{p,1}) \preceq S_{p,1} \) leads to \( \mathcal{H}(S_{p,1})^k \preceq \cdots \preceq \mathcal{H}(S_{p,1})^2 \preceq \mathcal{H}(S_{p,1}) \preceq S_{p,1} \). Since the sequence \( \{ \mathcal{H}(S_{p,1})^k \}_{k=0}^\infty \) is monotone and bounded, it converges, i.e., \( \lim_{k \to \infty} \mathcal{H}(S_{p,1})^k =: \bar{S} = \mathcal{H}(\bar{S}) \). Then, \( \bar{S} \preceq S_{p,1} \), and hence, \( \text{Tr}(\Lambda \bar{S}) \leq \text{Tr}(\Lambda S_{p,1}) = J_{p,1} \). Now, note that \((\bar{S}, F_{p,1})\) is a feasible solution to Problem 3, we have \( J_p = \text{Tr}(\Lambda S_p) \leq \text{Tr}(\Lambda \bar{S}) \). Therefore, \( J_p = \text{Tr}(\Lambda S_p) \leq \text{Tr}(\Lambda \bar{S}) \leq \text{Tr}(\Lambda S_{p,1}) = J_{p,1} \), and the proof is completed. \( \square \)

Note that even though the optimal values of Problem 3 and Problem 4 are identical, their solutions may be different. Let \((S_p, F_p)\) and \((S_{p,1}, F_{p,1})\) be optimal solutions to [Problem 3, main document] and Problem 4, respectively. By Proposition 3, \( \text{Tr}(\Lambda S_p) = \text{Tr}(\Lambda S_{p,1}) \). By [Assumption 1, main document], \( Q \succeq 0 \) implies that there may exist \( S_{p,1} \succeq 0 \) with \( A_F S_{p,1} A_F^T \begin{bmatrix} I_n \\ F \end{bmatrix} \begin{bmatrix} Z \\ I_n \\ F \end{bmatrix}^T \neq S_{p,1} \).

To proceed, we study a modification of Problem 4, where \( S \succeq 0 \) is replaced with the strict inequality \( S \succ 0 \).

**Problem 5.** Solve

\[
J_{p,2} := \inf_{S \in S_+^{n+m}, F \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S)
\]

subject to \( S \succ 0 \),

\[
A_F S A_F^T + \begin{bmatrix} I_n \\ F \end{bmatrix} \begin{bmatrix} Z \\ I_n \\ F \end{bmatrix}^T \preceq S.
\]

Since the semidefinite cone \( \{ S \in S_+^{n+m} : S \succeq 0 \} \) is the closure of \( \{ S \in S_+^{n+m} : S \succ 0 \} \), the optimal value of Problem 5 and Problem 4 are identical. It is stated in the following proposition.

**Proposition 4.** The optimal value of Problem 4 is equivalent to that of Problem 5, i.e., \( J_p = J_{p,1} = J_{p,2} \).
Proposition 4 is now in the form for which Lemma 3 can be applied. In the following proposition, we propose an SDP-based optimal control synthesis condition.

**Problem 6.** Solve

\[(S^*, G^*, K^*) := \arg \inf_{S \in \mathbb{S}^{n+m}, G \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S) \]

subject to \( S \succ 0, \)

\[
\begin{bmatrix}
S & [G^T] \\
G & K^T
\end{bmatrix} \begin{bmatrix} G + G^T - [A \ B] S [A \ B]^T - Z \end{bmatrix} \succeq 0
\]

**Proposition 5.** If \((S^*, G^*, K^*)\) is an optimal solution to Problem 6, then \(G^*\) is nonsingular, and \(F^* = K^*((G^*)^T)^{-1}\) is the optimal gain in [(4), main document].

**Proof.** We apply Lemma 3 to prove that Problem 5 is equivalent to

\[(S^*, F^*, G^*) := \arg \inf_{S \in \mathbb{S}^{n+m}, G \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S) \]

subject to \( S \succ 0, \)

\[
\begin{bmatrix}
S & [I] \\
G & F^T
\end{bmatrix} \begin{bmatrix} G^T \\
F
\end{bmatrix} \succeq 0.
\]

For any feasible \((S, F)\), because \(Z \succ 0\), we have \([A \ B] S [A \ B]^T + Z \succ 0\), meaning that \(G + G^T \succ 0\) holds. This implies that \(G\) is positive definite and nonsingular. Letting \(G F^T = K\), we can obtain Problem 6. Since \(G^*\) is nonsingular, \(F^*\) can be always recovered from the solution \((S^*, G^*, K^*)\) by \(F^* = K^*((G^*)^T)^{-1}\). From the reasoning, one concludes that \((S^*, F^*)\) is also an optimal solution to Problem 5. Therefore, we have \(J_p = J_{p,1} = J_{p,2} = \text{Tr}(\Lambda S^*)\). Since \((S^*, F^*)\) is feasible for Problem 5, we also have

\[A_{F^*} S^* A_{F^*}^T + \begin{bmatrix} I_n \\ F^* \end{bmatrix} Z \begin{bmatrix} I_n \\ F^* \end{bmatrix}^T \preceq S^*\]

If we define the operator

\[\mathcal{H}(S) := A_{F^*} S^* A_{F^*}^T + \begin{bmatrix} I_n \\ F^* \end{bmatrix} Z \begin{bmatrix} I_n \\ F^* \end{bmatrix}^T,\]
then the last matrix inequality can be written by $\mathcal{H}(S^*) \preceq S^*$. It can be easily proved that $\mathcal{H}$ is $\mathbb{S}^{n+m}_+$-monotone, i.e., $P \succeq P' \Rightarrow \mathcal{H}(P) \succeq \mathcal{H}(P')$. Repeatedly applying the operator to both sides of $\mathcal{H}(S^*) \preceq S^*$ leads to $\mathcal{H}(S^*)^k \preceq \cdots \preceq \mathcal{H}(S^*)^2 \preceq \mathcal{H}(S^*) \preceq S^*$. Since the sequence $\{\mathcal{H}(S^*)^k\}_{k=0}^\infty$ is monotone and bounded, it converges, i.e., $\lim_{k \to \infty} \mathcal{H}(S^*)^k =: \bar{S} = \mathcal{H}(\bar{S})$. Then, $\bar{S} \preceq S^*$, and hence, $J_p = J_{p,1} = J_{p,2} = \text{Tr}(\Lambda S) \succeq \text{Tr}(\Lambda \bar{S})$. However, since $(\bar{S}, F^*)$ is a feasible solution to Problem 3, we obtain $J_p = J_{p,1} = J_{p,2} = \text{Tr}(\Lambda S^*) \succeq \text{Tr}(\Lambda \bar{S}) \succeq J_p$. Therefore, $(\bar{S}, F^*)$ is an optimal solution to Problem 3.

From now on, we focus on optimal LQR control design with energy and input constraints as stated in the following problem.

**Problem 7.** Given constants $\rho > 0, \gamma_i > 0, i \in \{1, 2, \ldots, n+m\}$, solve

$$J_c := \inf_{F \in F} \sum_{i=1}^r J(F, z_i)$$

subject to

1. $\sum_{i=1}^r \sum_{k=0}^\infty x_j(k; F, z_i)^2 \leq \gamma_j, \quad j \in \{1, 2, \ldots, n\}, \quad (6)$
2. $\sum_{i=1}^r \sum_{k=0}^\infty u_{j-n}(k; F, z_i)^2 \leq \gamma_j, \quad j \in \{n+1, n+2, \ldots, n+m\}, \quad (7)$
3. $\|u(k; F, z)\|^2 \leq \rho \|x(k; F, z)\|^2, \quad \forall k \in N, \forall z \in \mathbb{R}^n, \quad (8)$

where for any vector $x$, $x_j$ indicates the $j$-th element of $x$ and $(u(k; F, z_i))_{k=0}^\infty$ denotes the input trajectory with state feedback gain $F$ starting from the initial state $z_i$.

The constraint (6) is related to the energy of the state trajectories, and (7) corresponds to the energy of the input trajectories. The constraint (8) is a state-dependent input constraint. We note that similar energy constraints were considered in [7] as well. However, the SDP condition in [7] cannot address the state-dependent input constraint. Similarly to Problem 4, we first propose an alternative form of Problem 7.

**Problem 8.** Given constants $\rho > 0, \gamma_i > 0, i \in \{1, 2, \ldots, n+m\}$, solve

$$J_{c,1} = \inf_{F \in F} \text{Tr}(\Lambda S)$$

subject to
Proposition 6. The optimal value of Problem 7 is equivalent to that of Problem 8, i.e., \( J_c = J_{c,1} \).

Proof. Following similar lines of the proof of [Proposition 1, main document], Problem 8 can be equivalently expressed as

\[
J_c = \inf_{F \in \mathcal{F}} \text{Tr}(\Lambda S) \tag{9}
\]

subject to

\[
S \succeq 0,
A_F S A_F^T + \begin{bmatrix} I_n \\ F \end{bmatrix} Z \begin{bmatrix} I_n \\ F \end{bmatrix}^T \preceq S
\]

\[
e_i^T S e_i \leq \gamma_i, \quad i \in \{1, 2, \ldots, n + m\},
\]

\[
F^T F \preceq \rho I_n.
\]

where \( S = \sum_{i=1}^r \sum_{k=0}^\infty \begin{bmatrix} x(k; F, z_i) \\ F x(k; F, z_i) \end{bmatrix} \begin{bmatrix} x(k; F, z_i) \\ F x(k; F, z_i) \end{bmatrix}^T \). Note that the constraints \( \|u(k)\|^2 \leq \rho \|x(k)\|^2, k \in \mathbb{N}, \) in Problem 7 can be expressed as \( x(k; F, z)^T (F^T F - \rho I_n) x(k; F, z) \leq 0, \forall k \in \mathbb{N}, \forall z \in \mathbb{R}^n, \) and since \( x(k; F, z) \forall k \in \mathbb{N}, \forall z \in \mathbb{R}^n \) spans \( \mathbb{R}^n \), the last inequality is equivalent to the linear matrix inequality (LMI) \( F^T F \preceq \rho I_n. \) Now, we follows arguments similar to the proof of Proposition 3. The feasible set of Problem 8 is larger than that of (9), we have \( J_{c,1} \leq J_c \). To prove the reversed inequality, one can follow similar arguments of Proposition 3. This completes the proof. \[\]

We propose an SDP-based design algorithm to solve Problem 7.

Problem 9. Given constants \( \rho > 0, \gamma_i > 0, i \in \{1, 2, \ldots, n + m\} \), solve

\[
(\bar{S}, \bar{G}, \bar{K}) := \arg \inf_{S \in \mathbb{S}^{n+m}, G \in \mathbb{R}^{n \times n}, K \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S)
\]
subject to \( S \succ 0, \)

\[
\begin{bmatrix}
S & \begin{bmatrix} G^T \\ K \end{bmatrix} \\
\begin{bmatrix} G & K^T \end{bmatrix} & G + G^T - \begin{bmatrix} A & B \end{bmatrix} S \begin{bmatrix} A & B \end{bmatrix}^T - Z
\end{bmatrix} \succeq 0,
\]

\[
\begin{bmatrix}
\rho I & K \\
K^T & G + G^T - I
\end{bmatrix} \succeq 0,
\]

\( e_i^T S e_i \leq \gamma_i, \quad i \in \{1, 2, \ldots, n + m\}. \)

\textbf{Proposition 7.} If \((\bar{S}, \bar{G}, \bar{K})\) is an optimal solution to Problem 9, then \(\bar{G}\) is nonsingular, and \(F = \bar{K}((\bar{G})^T)^{-1}\) is a suboptimal gain that solves Problem 9. In other words, the state-input trajectories with the feedback gain \(\bar{F}\) satisfy the constraints (6),(8) and \(\text{Tr}(\Lambda \bar{S})\) is an upper bound on \(J_c\).

\textbf{Proof.} We first convert Problem 8 into

\[
J_{c, 2} = \inf_{F \in \mathcal{F}} \text{Tr}(\Lambda S)
\]

subject to

\( S \succ 0, \)

\[
A_F S A_F^T + \begin{bmatrix} I_n \\ F \end{bmatrix} Z \begin{bmatrix} I_n \\ F \end{bmatrix}^T \preceq S,
\]

\( e_i^T S e_i \leq \gamma_i, \quad i \in \{1, 2, \ldots, n + m\}, \)

\( F^T F \preceq \rho I_n. \)

Since the semidefinite cone \(\{ S \in \mathbb{S}^{n+m} : S \succeq 0 \}\) is the closure of \(\{ S \in \mathbb{S}^{n+m} : S \succ 0 \}\), the optimal value of (10) and Problem 8 are identical, i.e., \(J_{c, 2} = J_{c, 1} = J_c\). Applying Lemma 3 yields the equivalent condition

\[
(S, G_1, G_2, F) := \arg \inf_{S \in \mathbb{S}^{n+m}, G_1 \in \mathbb{R}^{n \times n}, G_2 \in \mathbb{R}^{n \times n}, F \in \mathbb{R}^{m \times n}} \text{Tr}(\Lambda S)
\]

subject to \( S \succ 0, \)
Fig. 3. The optimal cost obtained using Problem 9 with $\gamma_1 = \cdots = \gamma_{n+m} = 5$ and different $\rho \in [1.2, 5]$.

$$
\begin{bmatrix}
S & \begin{bmatrix} G_1^T \\ FG_1^T \end{bmatrix} \\
\begin{bmatrix} G_1 & G_1F^T \end{bmatrix} G_1 + G_1^T - \begin{bmatrix} A & B \end{bmatrix} S \begin{bmatrix} A & B \end{bmatrix}^T - Z \\
\begin{bmatrix} \rho I_m & FG_2^T \\
G_2F^T & G_2 + G_2^T - I_n \end{bmatrix}
\end{bmatrix} \succeq 0,
$$

$$
e_i^T S e_i \leq \gamma_i, \quad i \in \{1, 2, \ldots, n+m\}.
$$

Letting $G_1 = G_2$ and introducing the change of variables $FG^T = K$, we obtain the SDP in Problem 9. Finally, we note that since we introduced an additional constraint $G_1 = G_2$, the optimal value of Problem 9 is an upper bound on $J_c$ in Problem 7 and Problem 8. Finally, using similar lines as in the proof of Problem 5, we can easily prove that the state-input trajectories satisfy the constraints (6) and (8). This completes the proof.

**Example 2.** Consider the system $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with $Q = I_2$, $R = 0.1$, $Z = I_2$. Solving Problem 6 results in the gain $F^* = \begin{bmatrix} -0.5792 & -1.5456 \end{bmatrix}$ and the cost $\text{Tr}(\Lambda S^*) = 5.5499$. Both are identical to the results from the standard LQR design approach using ARE. In addition, we solved Problem 9 with $\gamma_1 = \cdots = \gamma_{n+m} = 5$ and different $\rho \in [1.2, 5]$, and the resulting optimal costs $\text{Tr}(\Lambda \bar{S})$ are depicted in Figure 3. At some point around $\rho = 1.2$, Problem 9 becomes infeasible.
APPENDIX A

EXPLICIT DUAL PROBLEM

In this section, we derive an explicit optimization form of the dual problem [(12), main document], which is a convex semidefinite programming problem (SDP).

Proposition 8. Consider the problem

\[
\sup_{P \in S^n_{+}, P_{22} > 0} \text{Tr}(Z(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)) \text{ subject to } \\
\begin{bmatrix}
[A\ B] & P_{11} & [A\ B] \\
P_{12}^T & [A\ B]
\end{bmatrix} \succeq 0.
\]

It is a convex optimization problem and is an explicit form of the dual problem [(12), main document].

Proof. Consider the dual function in [(14), main document], and notice that the dual optimal solution \(P^*\) of [(15), main document] satisfies \(P^*_{22} \succ 0\) and \(-(P^*_{22})^{-1}(P^*_{12})^T \in F\). Therefore, one can restrict the set \(\mathcal{P}\) in [(14), main document] to the subset \(\mathcal{P}' := \{P \in S^n_{+} : A_F^T P A_F - P + \Lambda \succeq 0, \forall F \in F, P_{22} \succ 0, -P_{22}^{-1}P_{12}^T \in F\}\) without changing the optimal dual function value. In this case, the infimum in [(14), main document] is attained at \(F = -P_{22}^{-1}P_{12}^T\). By plugging \(-P_{22}^{-1}P_{12}^T\) into \(F\) in the dual problem [(15), main document], it is equivalently converted to

\[
\sup_{P \in S^n_{+}, P_{22} > 0} \text{Tr}(Z(P_{11} - P_{12}P_{22}^{-1}P_{12}^T)) \text{ subject to } \\
\begin{bmatrix}
[A\ B] & P_{11} & [A\ B] \\
P_{12}^T & [A\ B]
\end{bmatrix} \succeq 0.
\]

The linear matrix inequality (11) can be obtained by using the Schur complement in [Lemma 3, main document]. The problem is a convex optimization problem because the objective function can be replaced by \(t \in \mathbb{R}\) with additional constraints \(\sum_{i=1}^{r} z_i^T (P_{11} - P_{12}P_{22}^{-1}P_{12}^T) z_i \geq t\), and the last inequality can be converted to a linear matrix inequality using the Schur complement \(r\) times. In particular, applying the Schur complement leads to

\[
\begin{bmatrix}
-t + \sum_{i=1}^{r} z_i^T P_{11} z_i + \sum_{i=2}^{r} z_i^T P_{12} P_{22}^{-1} P_{12}^T z_i & z_1^T P_{12} \\
P_{12}^T z_1 & P_{22}
\end{bmatrix} \succeq 0.
\]
The left-hand side can be decomposed into

\[
\begin{bmatrix}
-t + \sum_{i=1}^{r} z_i^T P_{11} z_i + \sum_{i=3}^{r} z_i^T P_{12} P_{22}^{-1} P_{12}^T z_i + z_1^T z_1 P_{11}

P_{12}^T z_1

P_{22}

+ \begin{bmatrix} z_2^T P_{12} \\ 0 \end{bmatrix} P_{22}^{-1} \begin{bmatrix} z_2^T P_{12} \\ 0 \end{bmatrix}^T
\end{bmatrix},
\]

and the Schur complement can be applied again. Repeating this \( r - 1 \) times, one gets a linear matrix inequality constraint. This completes the proof.

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