Renormalization Group Flow in large $N_c$

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Abstract

We calculate renormalization group flow equations for the linear $\sigma$-model in large $N_c$ approximation. The flow equations decouple and can be solved analytically. The solution is equal to a self consistent solution of the NJL model in the same approximation, which shows that flow equations are a promising method to extend the calculation to higher order in $1/N_c$. Including explicit chiral symmetry breaking, the large $N_c$ approximation describes physics reasonably well. We further compare the analytic solution to the usually used polynomial truncation and find consistency.
1 RG flow equations of the linear $\sigma$-model

The physics of strong interactions in the nonperturbative strong coupling regime cannot be obtained directly from QCD. Especially, it is not yet possible to integrate out the gluonic degrees of freedom in a rigorous way \[1\]. From 1-gluon exchange or instanton models one is lead to a local 4-fermion interaction described by the Nambu–Jona-Lasinio (NJL) model. The NJL model is non-renormalizable and therefore includes an explicit cutoff scale. It is commonly considered in large $N_c$ approximation \[2\]. If fluctuations in the quark condensate are taken into account one needs an additional cutoff \[3\], which reduces the predictive power of the model. Such an additional cutoff is not needed in the renormalization group (RG) framework, when the NJL-model is embedded into the linear $\sigma$-model. The main aim of this work is to clarify the connection between the NJL model and the RG flow equations for the associated linear $\sigma$-model in the large $N_c$ approximation.

RG flow equations describe the average of an effective action and represent the continuum analogue of a block spin transformation. Various kinds of exact flow equations have proved successful in describing a huge set of different systems \[4\]. For an introduction to the RG method we refer to the review \[5\]. The flow equations are ultraviolet (UV) and infrared (IR) finite through the introduction of a scale ($k$) - dependent cutoff function. Degrees of freedom with momenta larger than the coarse–graining scale $k$ are integrated out. The solution of the flow equations includes summations of different diagrams and is similar to the solution of a coupled set of Schwinger Dyson equations. We use the Schwinger proper time regularization method \[6\], which preserves all symmetries of the theory and keeps the physical interpretation of the evolution equations particularly simple for phenomenological applications. This simplicity makes these RG improved 1-loop equations interesting, although they are not exact RG flow equations \[7\].

The lagrangian of the NJL model with an explicit current quark mass $m_c$ reads in Euclidian spacetime

$$\mathcal{L}_{NJL} = \bar{q}(i\partial + m_c) q - G \left( (\bar{q} q)^2 + (\bar{q} i\gamma_5 \pi) q \right).$$  \hspace{1cm} (1)$$

A partial bosonization by the introduction of auxiliary scalar fields $\sigma$ and $\pi$ gives the associated linear $\sigma$-model \[8\]

$$\mathcal{L}_{\sigma \pi} = \bar{q} \phi q + g \bar{q} \left( \sigma + i\gamma_5 \pi \right) q + \frac{m^2}{2} \left( \sigma^2 + \pi^2 \right) - \delta \sigma.$$  \hspace{1cm} (2)$$

The respective couplings are connected by

$$2G = \frac{g^2}{m^2}, \quad m_c = \frac{g \delta}{m^2}.$$  \hspace{1cm} (3)$$

In order to solve the theory it is sufficient to compute the effective action, which is the generating functional of one particle irreducible Feynman diagrams. It
involves in principle arbitrary higher order terms consistent with the symmetry of the initial action. Using RG methods we are going to calculate the effective action in a suitable truncation. Surely we cannot account for all possible terms, so our approximation is to assume that the linear \( \sigma \)-model with quarks is a valid description of nature at a scale of \( 0.2 \text{ GeV} \lesssim k \lesssim 1 \text{ GeV} \). In this region the gluon degrees of freedom are supposed to be already frozen out, but confinement is not yet relevant. It remains to be investigated whether the effects of other \( \bar{q}q \)-resonances are sufficiently taken into account by quarks coupling to \((\sigma, \vec{\pi})\) fields.

The physics of the model is governed by chiral symmetry, which is spontaneously broken in the IR. The Euclidian UV action of the general linear \( \sigma \)-model is given by

\[
S[\Phi, \bar{q}, q] = \int d^4x \left( Z_\Phi \bar{q}q + \frac{1}{2} Z_\Phi (\partial \mu \Phi) (\partial \nu \Phi) + g q M q + U \right). \tag{4}
\]

The quark- and meson fields \( \bar{q}, q, \Phi, M \) are bare fields, where \( \Phi = (\sigma, \vec{\pi}) \) is the \( O(4) \)- and \( M = \sigma + i \vec{\pi} \gamma_5 \) the chiral \( SU(2)_L \otimes SU(2)_R \)-representation of the meson fields. We consider this action to be preserved during the evolution and choose the starting values

\[
Z_\Phi = 0, \tag{5}
\]
\[
Z_q = 1, \tag{6}
\]

and the UV potential

\[
U_{uv} = \frac{m^2}{2} \Phi^2 - \delta \sigma, \tag{7}
\]

in order to obtain Eq. (3) as UV lagrangian. For comparison with the standard NJL approximation we will work in the following in the chiral limit \( m_c = \delta = 0 \), but since the term linear in \( \sigma \) does not get renormalized \( 3 \), the following computation is also valid for explicit symmetry breaking. This case will be discussed in Sec. 3. The effective action \( \Gamma \), which depends only on averaged fields, can be computed by doing a saddle point expansion corresponding to a one loop approximation (cf. 4):

\[
\Gamma[\Phi, \bar{q}, q] = S[\Phi, \bar{q}, q] - \frac{1}{2} \text{Tr} \log \left( \frac{\delta^2 S[\Phi, \bar{q}, q]}{\delta \bar{q}(x) \delta q(y)} \right) \tag{8}
\]

\[
+ \frac{1}{2} \text{Tr} \log \left( \frac{\delta^2 S[\Phi, \bar{q}, q]}{\delta \Phi^i(x) \delta \Phi^j(y)} - 2 \frac{\delta^2 S[\Phi, \bar{q}, q]}{\delta \Phi^i(x) \delta q(y)} \frac{\delta^2 S[\Phi, \bar{q}, q]}{\delta \bar{q}(x) \delta q(y)} \right) \frac{1}{\delta \bar{q}(x) \delta \Phi^j(y)} \right). \]

The first logarithm results from fermion loop fluctuations, whereas the two terms in the second logarithm are the contributions of the bosonic and the mixed loop respectively.
The logarithms are transformed into Schwinger proper time integrals regularized by a heat kernel cutoff $f(k^2 \tau)$ (cf. \cite{6})

$$\log(A) = - \int_{\frac{1}{k^2}}^{\infty} \frac{d\tau}{\tau} e^{-\tau A f(k^2 \tau)} . \tag{9}$$

As cutoff the following set of functions is used

$$f^{(n)}(k^2 \tau) = \sum_{i=0}^{n} \frac{(k^2 \tau)^i}{i!} \exp(-k^2 \tau) . \tag{10}$$

The heat kernel cutoff functions $f^{(n)}$ suppress fluctuations with momenta below the cutoff scale $k$. Going to $k \to 0$ means to include more and more IR modes. Finally all modes are included, because of the limiting behaviour $f^{(n)} \to 1$. Due to the extra $\tau$ terms in front of the exponential the integral diverges in the UV region only through an additional term $\sim \log \Lambda^2$. To keep the following computation simple, we will consider the cutoff $f^{(2)}$ used in \cite{6}. It has been shown for the case of a $O(N)$-model, that the RG equation derived with the cutoff $f^{(2)}$ is equivalent to an exact RG equation \cite{10}.

The detailed computation of the effective action is based on a derivative expansion and will be presented in a separate paper \cite{11}. The purely bosonic part was investigated in \cite{12}.

The derivative of the regularized effective action with respect to the cutoff scale $k$ yields a flow equation for the effective action. Identifying terms in $\frac{d\Gamma}{dk}$ with similar terms in $S[\Phi, \bar{q}, q]$ we obtain evolution equations for the effective potential $U$, coupling constant $g$ and the wave function renormalization factors $Z_q$ and $Z_\Phi$.

To improve the one loop equation we substitute the masses and couplings of the classical action by the running masses and running couplings of the effective action $\Gamma$. This replacement turns the one loop equations into RG improved flow equations, which include higher loop terms successively into the proper time integral. Therefore the flow equations can treat nonperturbative physics in the strong coupling region \cite{5}.
We find the following coupled set of equations for the parameters of the model:

\[
\begin{align*}
\frac{\partial U}{\partial k} &= -\frac{N_f N_c}{8\pi^2 k^2 + M_q^2} \frac{k^6}{k^2 + M_q^2} + \frac{1}{32\pi^2} \left( \frac{k^6}{k^2 + M_q^2} + 3 \frac{k^6}{k^2 + M_\sigma^2} \right), \quad (11) \\
\frac{k \partial g}{\partial k} &= \frac{1}{16\pi^2} \frac{g^3}{Z_q^2 Z_\Phi} \left( \frac{k^6 (2k^2 + M_q^2 + M_\sigma^2)}{(k^2 + M_q^2)^2 (k^2 + M_\sigma^2)^2} \right. \\
&\quad \left. - 3 \frac{k^6 (2k^2 + M_q^2 + M_\sigma^2)}{(k^2 + M_q^2)^2 (k^2 + M_\sigma^2)^2} \right), \quad (12) \\
\frac{k \partial Z_q}{\partial k} &= -\frac{1}{16\pi^2} \frac{g^2}{Z_q Z_\Phi} \left( \frac{k^6 (2k^2 + M_q^2 + M_\sigma^2)}{(k^2 + M_q^2)^2 (k^2 + M_\sigma^2)^2} \right. \\
&\quad \left. + 3 \frac{k^6 (2k^2 + M_q^2 + M_\sigma^2)}{(k^2 + M_q^2)^2 (k^2 + M_\sigma^2)^2} \right), \quad (13) \\
\frac{k \partial Z_\Phi}{\partial k} &= -\frac{N_f N_c}{4\pi^2} \frac{g^2}{Z_q^3 (k^2 + M_q^2)^3} - \frac{1}{8\pi^2} \frac{Z_\Phi}{Z_q^2} \frac{k^6 (M_q^2 - M_\sigma^2)}{(k^2 + M_q^2)^2 (k^2 + M_\sigma^2)^2}, \quad (14)
\end{align*}
\]

where the effective masses are given by

\[
\begin{align*}
M_q^2(\Phi^2) &= \frac{g^2}{Z_q^2} \Phi^2, \\
M_\sigma^2(\Phi^2) &= \frac{2}{Z_\Phi} \left( \frac{\delta U(\Phi^2)}{\delta \Phi^2} + 2 \Phi^2 \frac{\delta^2 U(\Phi^2)}{(\delta \Phi^2)^2} \right), \quad (15) \\
M_\pi^2(\Phi^2) &= \frac{2}{Z_\Phi} \frac{\delta U(\Phi^2)}{\delta \Phi^2},
\end{align*}
\]

and the effective 4-meson coupling reads

\[
\Lambda(\Phi^2) = \frac{2}{Z_\Phi} \frac{\delta^2 U(\Phi^2)}{(\delta \Phi^2)^2}. \quad (16)
\]

In this set of equations all the masses and couplings are functions of the scale \( k \). The heat kernel method gives these equations in a very simple form. Since the IR parameter \( k \) acts like a mass cutoff, the propagator structure of the different loop contributions is still recognizable.

Although the effective masses and the effective 4-meson coupling in Eqs. (11-14) are fully renormalized quantities, the potential \( U \) and the Yukawa coupling

\footnote{Contrary to the equations for \( U(\Phi^2) \) and \( Z_\Phi(\Phi^2) \), which are valid for general \( \Phi^2 \)-dependent functions, the given equations for the Yukawa coupling and the quark wave function renormalization factor are only valid for \( \Phi \)-independent \( g \) and \( Z_q \). The general expressions get correction terms which do not contribute in large \( N_c \).}
are still bare ones. In order to obtain physical couplings one has to express the theory by renormalized fields \( q_R, \bar{q}_R \) and \( \Phi_R \). This is done by rescaling the fields to absorb the wave function renormalization factors in front of the kinetic terms:

\[
\Phi_R = (Z_{\Phi}(\Phi_0^2))^{\frac{1}{2}} \Phi, \quad q_R = Z_{\xi}^{\frac{1}{2}} q.
\]  

(17)

where \( \Phi_0 \) denotes the vacuum expectation value of the field \( \Phi \). The renormalized potential is a function of the renormalized meson field \( U_R(\Phi_R^2) = U((Z_{\Phi}(\Phi_0^2))^{-1} \Phi_R^2) \) and the renormalized Yukawa coupling is related to the bare coupling as \( g_R = Z_{\xi}^{-1} (Z_{\Phi}(\Phi_0^2))^{-\frac{1}{2}} g \). Finally, the physical masses and the physical 4-meson coupling are given at the vacuum expectation value of the meson field

\[
m_i = M_i(\Phi_0^2), \quad i \in \{\sigma, \pi, q\}; \quad \lambda = \Lambda(\Phi_0^2).
\]  

(18)
2 RG in large $N_c$ and NJL gap-equation

In the large $N_c$ limit only the fermion loop is relevant. Since the NJL effective action involves a trace over quark fields it is proportional to $N_c$ and therefore the NJL coupling $G$ is proportional to $1/N_c$. This also has to hold in the bosonized case Eq. (4). Assuming that the meson masses are independent of $N_c$, one obtains from Eq. (3) that $g$ behaves $\propto 1/\sqrt{N_c}$, in the same way as the QCD coupling constant. This implies that the meson fields $\Phi$ scale as $O(\sqrt{N_c})$, which in turn means that the effective 4-meson coupling $\lambda$ should be proportional to $1/N_c$.

Due to the large $N_c$ behaviour, the quark contribution in the flow equation of the potential $U$ Eq. (13) is of order $N_c$, whereas the meson loop term is of order one and can be neglected. Also, for $Z_\Phi$ Eq. (14) the second term can be dropped since it is of order $1/N_c$ compared to the first one which is of order one. The wave function renormalization parameter $Z_q$ Eq. (13) and the bare Yukawa-coupling $g$ Eq. (12) do not evolve, since the right hand sides of the evolution equations are of higher order in $1/N_c$ than the left hand sides, which reflects that the corresponding fluctuations are given by mixed loops.

The large $N_c$ flow equations read ($Z_q = g = 1$)

$$\frac{k}{c} \frac{\partial U}{\partial k} = -\frac{N_f N_c}{8\pi^2} \frac{k^6}{k^2 + M_q^2} , \quad (19)$$

$$\frac{k}{c} \frac{\partial Z_\Phi}{\partial k} = -\frac{N_f N_c}{4\pi^2} \frac{k^6}{(k^2 + M_q^2)^3} . \quad (20)$$

The right hand sides of these equations only depend on the quark mass Eq. (15), but not on the potential $U$ or the wave function renormalization parameter $Z_\Phi$, therefore they completely decouple and can be integrated analytically. In particular, these equations do not contain a RG improvement. Note that in the large $N_c$ limit we have $M_q^2 = \Phi^2$ and the integrals read

$$\int_{k_{uv}}^{k(k)} \frac{U(k)}{dU} = -\frac{N_f N_c}{8\pi^2} \int_{k_{uv}}^{k} \frac{k^5}{\kappa^2 + \Phi^2} d\kappa , \quad (21)$$

$$\int_{Z_\Phi(k)}^{Z_\Phi(k)} \frac{dZ_\Phi}{dZ_\Phi} = -\frac{N_f N_c}{4\pi^2} \int_{k_{uv}}^{k} \frac{\kappa^5}{(\kappa^2 + \Phi^2)^3} d\kappa . \quad (22)$$

Neglecting $\Phi$-independent terms, the integral for the potential $U$ yields

$$U(\Phi, k) = U_{uv}(\Phi) + \frac{N_f N_c}{16\pi^2} \left( \Phi^4 \log \left( \frac{k_{uv}^2 + \Phi^2}{k^2 + \Phi^2} \right) - \Phi^2 (k_{uv}^2 - k^2) \right) , \quad (23)$$

where the UV potential is given by Eq. (7). Using $Z_{uv}^\Phi = 0$ as starting value, the equation for $Z_\Phi$ can be integrated in the same way giving a slightly longer
expression

\[
Z_\Phi(\Phi, k) = \frac{3}{8\pi^2} \left( 2 \log \left( \frac{k_{uv}^2 + \Phi^2}{k^2 + \Phi^2} \right) \right.
\]
\[
- \left( \frac{k_{uv}^2 - k^2}{k^2 + \Phi^2} \right) \Phi^2 \left( 4 k_{uv}^2 k_{uv}^2 + 3 \left( k^2 + k_{uv}^2 \right) \Phi^2 + 2 \Phi^4 \right) \left( k^2 + \Phi^2 \right)^2 \left( k_{uv}^2 + \Phi^2 \right)^2
\].

(24)

Eqs. (23, 24) are the exact analytic solution of the large \( N_c \) flow equations for a given scale \( k \). To obtain the physical vacuum expectation value \( \Phi_0(k) \) of the field, the potential has to be minimized with respect to \( \Phi \):

\[
\frac{\delta U(\Phi, k)}{\delta \Phi^2} \bigg|_{\Phi=\Phi_0} = \frac{m^2}{2} + \frac{N_f N_c}{16\pi^2} \left( 2\Phi_0^2 \log \left( \frac{k_{uv}^2 + \Phi_0^2}{\Phi_0^2} \right) - \frac{k_{uv}^2 (k_{uv}^2 + 2 \Phi_0^2)}{k_{uv}^2 + \Phi_0^2} \right) = 0.
\]

(25)

This equation cannot be solved analytically due to the non algebraic logarithmic term. As we will show, it corresponds to the self consistent NJL gap equation at the IR scale \( k = 0 \), determining the physical quark mass \( m_q = \Phi_0(k = 0) \) as the minimum of the potential Eq. (23). The full potential provides more information than the NJL gap equation, since it gives arbitrary higher order moments of \( \Phi \) besides its vacuum expectation value. This allows to compute for instance the spectrum of the Dirac operator which will be presented in a forthcoming paper [13].

The NJL model in large \( N_c \) approximation is standardly considered in a self consistent approach which leads to the NJL gap equation [8]. This equation determines the constituent quark mass and is usually regularized by a hard cutoff. Furthermore, to make contact with chiral degrees of freedom there is an additional equation that determines the mesonic wave function renormalization factor \( Z_\Phi \), once the constituent quark mass is known [8]. To compare with our flow equations we regularize these equations in a Schwinger proper time scheme using the heat kernel cutoff. Since the heat kernel cutoff \( f^{(n)}(k^2\tau) \) acts like a IR cutoff function, the desired UV regulation is given by the function \( 1 - f^{(n)}(k_{uv}^2\tau) \), which cuts off momenta larger than \( k_{uv} \). The gap equation and the determining equation for \( Z_\Phi \) read

\[
1 = 8G N_c N_f \int_0^\infty d\tau \int \frac{d^4p}{(2\pi)^4} e^{-\tau (p^2 + m_q^2)} \left( 1 - f^{(n)}(k_{uv}^2\tau) \right),
\]

(26)

\[
Z_\Phi = 2N_c N_f \int_0^\infty \tau d\tau \int \frac{d^4p}{(2\pi)^4} e^{-\tau (p^2 + m_q^2)} \left( 1 - f^{(n)}(k_{uv}^2\tau) \right).
\]

(27)

The integrals appearing in these equations can be integrated analytically. With Eq. (3) the NJL coupling \( G \) is connected to the UV meson mass \( m_{uv}^2 \). For the cutoff \( f^{(2)} \) the integrated form of the gap equation Eq. (24) is analytically equal to Eq. (25). Similarly, the determining equation for \( Z_\Phi \) Eq. (27) yields Eq. (24).
evaluated at $\Phi^2 = \Phi_0^2$ after integration. This connection between the equations in the RG scheme and the equations in the self consistent NJL approach holds for all cutoff functions $f^{(n)}$ Eq. (10), provided both schemes are regularized with the same cutoff. This shows that the RG flow result is exactly identical to the standard NJL approach in large $N_c$ approximation.

Although the vacuum has to be computed numerically, there are two quantities, that allow an analytic solution. The first one is the scale $k_{\chi_{SB}}$, where the system evolving from the UV enters the chiral broken regime, which is given as the solution of the equation

$$\frac{\delta U(\Phi,k)}{\delta \Phi^2} \bigg|_{\Phi^2 = 0} = 0 ,$$

and takes the simple form

$$k_{\chi_{SB}} = \sqrt{k_{uv}^2 - \frac{8\pi^2 m_{uv}^2}{N_c N_f}}.$$  

(29)

The other one is the ratio $m_{\sigma}/m_q$, which results from Eqs. (13, 18) using the analytic expressions for $U$ and $Z_\Phi$ at $k = 0$ and $\Phi = \Phi_0$. It is exactly 2 in large $N_c$, as known from the standard NJL approach [8].

For a numerical solution, we adjust the UV parameters to reproduce physical values of $m_q = 320$ MeV and $f_{\pi} = 93$ MeV in the case of an explicit chiral symmetry breaking, discussed in the next chapter. The initial parameters for the RG evolution read

$$k_{uv} = 1037 \text{ MeV} ,$$

$$m_{uv} = 228.0 \text{ MeV} ,$$

$$Z_{\Phi_{uv}} = 0 ,$$

(30)

and $Z_{q} = g = 1 = \text{const}$. This corresponds to a four fermion coupling of $G = 9.62 \text{ GeV}^{-2}$. With these initial values we calculate the effective potential from Eq. (23) and the wave function renormalization from the corresponding equation for $Z_{\Phi}$. The result is the renormalized effective potential $U_R$, which is shown in Fig. 1 as a function of the field $\Phi_R$ and the scale $k$. From this potential we compute the scale dependent pion decay constant $f_{\pi}(k)$ shown as white curve in Fig. 1.

The quark mass and meson masses, which are given from Eq. (13) are plotted in Fig. 2 (solid lines). In the chiral symmetric regime for scales $k > k_{\chi_{SB}} = 626$ MeV (cf. Eq. (29)) both meson masses are identical, while the quark mass is zero. Below the symmetry breaking scale the pions become massless while the $\sigma$-meson mass rises again. Due to the nontrivial vacuum of the effective potential, the quarks acquire a finite mass. In the IR limit $k = 0$ we find the physical values for the the quark mass $m_q = \Phi_0(k = 0)$ and the $\sigma$-meson mass Eqs. (13, 18).
Figure 1: The evolution for the renormalized meson potential in large $N_c$ approximation. In the UV the renormalized potential is infinite for all $\Phi_R$ due to the vanishing wave function renormalization constant. The quadratic potential, which arises when $Z_\Phi$ gets finite, evolves to a potential with broken symmetry in the IR. The white curve shows the evolution of the vacuum expectation value, which corresponds to $f_\pi = 92$ MeV at $k = 0$, denoted by the endpoint.

The Yukawa coupling is given by $g_R = Z_\Phi^{-1/2}(k=0)$ and the pion decay constant is $f_\pi = Z_\Phi^{1/2}(k=0) \Phi_0(k=0)$. The IR values are summarized in Table 1 (second column).

3 Explicit symmetry breaking

Although the physical number of three colors is not really large, we know from standard NJL computations, that the large $N_c$ limit gives a reasonable approximation. We next compare our model with the physical case of an explicit chiral symmetry breaking due to a finite current quark mass $m_c$.

The value of the mean current quark mass $m_c = \frac{m_u + m_d}{2} = 7$ MeV is taken from chiral perturbation theory \[14\], which has been performed at our UV scale of 1
Figure 2: The renormalized masses of the exact solution as functions of the scale \( k \) in the chiral limit (solid) and with an explicit symmetry breaking due to a current quark mass of 7 MeV (dashed). The quark mass in the symmetric regime and the pion mass in the broken regime vanish in the chiral limit, whereas the pion acquires a mass of \( m_\pi = 116 \text{ MeV} \) in the case of a finite current quark mass.

GeV, although there is some uncertainty because of the different schemes. From Eq. (3) the parameter \( \delta \) can be computed resulting in \( \delta = 3.64 \cdot 10^{-4} \text{(GeV)}^3 \). As already discussed, the solution Eq. (23) obtains in the case of explicit symmetry breaking, with a term linear in \( \sigma \) in the UV potential Eq. (7). The vacuum expectation value is now given by minimization with respect to \( \sigma \). The evolution of the renormalized physical masses is shown as dashed curves in Fig. 2, which exhibits a very reasonable splitting of the mass scales. There is a continuous transition in the \( k \) evolution from the current quark to the constituent quark regime, since the system is in a nontrivial vacuum right from the beginning of the evolution. The quark mass assumes its current quark mass in the UV and the pion looses its role as an exact Goldstone boson and acquires a small finite mass in the IR.

The IR values of the physical quantities are shown in the first column of Table 1. The effect of the explicit chiral symmetry breaking is a minor change in the other parameters and a finite pion mass of 116 MeV which is reasonable close to the physical value. The change of the pion decay constant compared to the chiral limit amounts only to 1 MeV and is rather small in large \( N_c \) approximation compared to chiral perturbation theory computations \( \text{[14]} \). However, this
Table 1: The IR values for different approximations are shown. All computations are done with the UV initial conditions $k_{uv} = 1037 \text{ MeV}$, $m_{uv} = 228.0 \text{ MeV}$ and $Z_{uv}^2 \Phi = 0$. The first column shows the analytic RG result with explicit chiral symmetry breaking due to a finite current quark mass of 7 MeV. In the second column the analytic RG result in the chiral limit is given, which is equal to the result of the NJL gap-equation regularized with the proper time regulator. In the last column the results of a simple quartic truncation of the potential are shown.

| $f_\pi$ [MeV] | $m_c = 7 \text{ MeV}$ | $\chi$-limit | $\Phi^4$-truncation |
|---------------|----------------------|--------------|-------------------|
| 93.0          | 92.0                 | 105          |
| 23.6          | 22.6                 | 14.9         |
| 3.44          | 3.36                 | 2.73         |
| 320           | 309                  | 285          |
| 650           | 618                  | 570          |
| 116           | 0                    | 0            |

The evolution equations for the couplings $m^2$ and $\lambda$ can be derived from Eq. (19). We find for large $N_c$ in the symmetric case ($g = Z_q = 1$):

\begin{align*}
    k \frac{\partial m^2}{\partial k} &= + \frac{N_c N_f}{4\pi^2} k^2, \\
    k \frac{\partial \lambda}{\partial k} &= - \frac{N_c N_f}{2\pi^2}, \\
    k \frac{\partial Z_\Phi}{\partial k} &= - \frac{N_c N_f}{4\pi^2},
\end{align*}

4 Truncated potential

The analytic solution of Eqs. (19,20) can test the usual parametrizations of the model. One relies on such parametrizations, if higher corrections are regarded, e.g. mesons are taken into account. A standard approach is to expand the model in powers of $\Phi^2$ up to a given order $p_{\text{max}}$. The potential $U$ evolves under the renormalization flow from a trivial to a nontrivial form. Therefore we consider two parametrizations. In the simplest quartic truncation with $p_{\text{max}} = 2$, they read in the chiral symmetric regime $U(\Phi^2, k) = m_f^2(k)\Phi^2 + \lambda_f(k)(\Phi^2)^2$ and in the broken regime $U(\Phi^2, k) = \lambda_f(k)(\Phi^2 - \Phi_0^2(k))^2$. Since the wave function renormalization factor $Z_\Phi$ arises together with the dynamical term of mass dimension four, it is correspondingly expanded up to order $p_{\text{max}} - 2$ and is therefore $\Phi$-independent in the simplest approximation. The evolution equations for the couplings $m^2$ and $\lambda$ can be derived from Eq. (19). We find for large $N_c$ in the symmetric case ($g = Z_q = 1$):
and for the broken regime:

\[
k \frac{\partial \Phi_0^2}{\partial k} = -\frac{N_c N_f}{4\pi^2} \frac{1}{\lambda} \frac{k^2}{(1 + \Phi_0^2/k^2)^3},
\]

(34)

\[
k \frac{\partial \lambda}{\partial k} = -\frac{N_c N_f}{2\pi^2} \frac{1}{(1 + \Phi_0^2/k^2)^3},
\]

(35)

\[
k \frac{\partial Z_\Phi}{\partial k} = -\frac{N_c N_f}{4\pi^2} \frac{1}{(1 + \Phi_0^2/k^2)^3}.
\]

(36)

To solve these equations numerically, we use again the initial values Eq. (30). In addition we choose \(\lambda_{uv}=0\) to have the same UV potential as before (cf. Eq. (7)). In the symmetric regime the exact and approximate solutions are identical. In the broken regime we switch to the second parametrization and follow the flow given by Eqs. (34-36) to \(k=0\). Now we have a set of coupled equations. This comes from the fact that we are tracking the evolution of the minimum \(\Phi_0\) of the potential \(U\), which has no analytical solution from Eq. (23). In the IR we find a pion decay constant \(f_\pi = 105\) MeV compared to a value of 92 MeV for the exact solution (cf. Fig. 3). The IR masses and couplings from these truncated flow equations are given in the third column of Table 1.

![Graph](image)

**Figure 3:** The renormalized meson mass \(m_R = m_\sigma = m_\pi\) in the symmetric regime and the pion decay constant in the broken regime as functions of \(k\) for the exact solution (solid) and the quartic truncation (dotted). In the symmetric regime the two solutions are equal, but the vacuum results for \(f_\pi\) differ by 14%.
When the truncation is extended to higher orders $p_{\text{max}}$, the IR value of $f_\pi$ converges fast to the exact solution with a pion decay constant of 92 MeV, as indicated by the thick points in Fig. 4. On the other hand, if only the effective potential is expanded, the flow still converges, however to a strongly deviating result of 111 MeV, shown by the thin points in Fig. 4. This is in contrast to a naive dimensional analysis, which would suggest that the higher order momenta of $Z_\Phi$ should be suppressed by their high mass dimension in the vicinity of an IR fixed point. Although this does not seem to be the case in large $N_c$ it may be the case when meson corrections are included, where a partial fixed point behaviour occurs. Nevertheless, in order to obtain a credible solution it is necessary to choose the truncation order in such away, that convergence with respect to all model parameters is achieved. Indeed, at order $p_{\text{max}} = 7$ the results are to the given accuracy exactly the ones of the exact solution in the second column of Table 4.

Figure 4: The dependence of the physical pion decay constant $f_\pi$ and the 4-meson coupling $\lambda$ on the truncation order $p_{\text{max}}$ of a polynomial parametrization. Thin points give the behaviour when only the potential $U$ is expanded and the wave function renormalization factor $Z_\Phi$ is considered $\Phi$-independent. They do not converge to the exact results given by the dashed lines. Thick points show the dependence when $U$ and $Z_\Phi$ are expanded and exhibit a fast convergence to the exact results.
5 Conclusion

We have derived RG flow equations for the linear $\sigma$-model in large $N_c$ approximation as a different approach to solve the NJL model. It was shown that in large $N_c$ the RG approach reproduces the NJL gap equation and therefore in this limit the solutions of the two schemes are equal. In addition one finds from the flow equations the whole effective potential, instead of just its minimum. In the case of explicit symmetry breaking due to a finite current quark mass we find very reasonable meson- and constituent quark masses. A comparison of the full flow equations with a polynomial truncation has shown a deviation of $\approx 15\%$ in lowest order, but a fast convergence to the full result.

Both, the exact and truncated calculation use two parameters $k_{uv}$ and $m_{uv}$ to fix the quark mass $m_q$ and the pion decay constant $f_\pi$. This is the conventional method in NJL theory. An advantage of RG equations is the reduction of the relevant UV parameters, if the theory exhibits a partial IR fixed point behaviour. Such a fixed point behaviour indeed occurs when meson loops are considered.

In addition, meson loop corrections can be included systematically without the problem of introducing additional cutoffs, since the effective action is renormalizable. The analytic large $N_c$ solution presented here serves as a reference to the analysis of the full flow equations with meson fluctuations, which will be published in a forthcoming paper [11].

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