Tunable Cooper pair splitting via Andreev bound states

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Two spatially separated quantum dots connected by a common superconductor has been studied extensively for Cooper pair splitting, and is predicted to be the building block of an emulated Kitaev chain. The effective couplings between quantum dots mediated by the middle superconductor is crucial for observing these intriguing physical phenomena, but unfortunately, a method of tuning the effective couplings is still lacking, obstructing further experimental progressions in relevant studies. Here we propose to mediate tunable effective couplings via Andreev bound states in a semiconductor-superconductor nanowire. We show explicitly how the coupling strengths depend on the physical parameters of the Andreev bound states, e.g., chemical potential, induced Zeeman spin splitting and spin-orbit interaction. Furthermore, we propose to extract the effective coupling strengths from resonant current in a three-terminal junction. Our findings will facilitate future experiments on Cooper pair splitting as well as the experimental realization of the Kitaev chain in quantum dots.

Introduction.—Two quantum dots connected by a common superconductor is an ideal platform for exploring fundamental physics in various fields. As a Cooper pair splitter, electrons therein are separated in space while maintaining quantum entanglement [1–6], which can be used to perform the Bell inequality test [7–9] and has potential applications in quantum cryptography [10, 11] and quantum teleportation [12]. Despite many experimental progresses [13–29], the splitting efficiency of the Cooper pair splitters nowadays is still not high enough for the Bell inequality test and other applications. On the other hand, a chain of alternative dots and superconductors can function as an effective Kitaev chain and host Majorana zero modes [30–32]. These exotic quasiparticles are non-Abelian anyons, and might be utilized to implement topological quantum computation [33–40]. However, a decade has elapsed since the original proposal, but experimentally realizing the Kitaev chain—even one with only two dots—remains a major challenge.

All the aforementioned physical phenomena primarily rely on the effective couplings between quantum dots mediated by the middle superconductor. The main origin of the current dilemma in experiments is the missing of control over the effective couplings, especially the ratio between different types of couplings. As a result, in a Cooper pair splitter undesirable physical processes such as elastic co-tunneling are not fully mitigated, making the splitting efficiency below the Bell inequality test threshold, and no physical tuning parameter can function as a knob to drive the Kitaev chain from the topologically trivial to the nontrivial phase. In order for the experiments to proceed, a method of controlling the effective couplings between quantum dots is desperately needed. In most of the existing proposals, effective couplings are mediated by the quasiparticle continuum of the superconductor [1, 3, 5, 30, 41], and thus it is not experimentally feasible to vary the coupling strength, since varying the electronic properties of the quasiparticle continuum is extremely hard. Moreover, the coupling strengths are suppressed exponentially by the dot distance on the scale of the superconducting coherence length. Merely adjusting the tunnel barrier between the superconductor and the quantum dot does not vary the ratio between different types of effective couplings, and neither does it eliminate the exponential suppression.

In this Letter we propose to mediate tunable effective couplings via Andreev bound states in a semiconductor-superconductor nanowire, based on the fact that control over hybrid nanowires has been very accurate with the state-of-the-art experimental techniques, e.g., by tuning a nearby electrostatic gates. We show that the effective couplings are no longer subject to the detrimental suppression, and that they are highly tunable by the physical parameters of the Andreev bound states, e.g., chemical potential, induced Zeeman spin splitting, and spin-orbit interaction. In addition, we propose to experimentally extract the coupling strengths from resonant current in a three-terminal junction.

Model and Hamiltonian.—The Hamiltonian of the dot-

FIG. 1. Left: (a) Schematic of the device. The middle part is a short semiconductor-superconductor hybrid nanowire, hosting Andreev bound states. The nanowire tunnel couples to two separate quantum dots at the opposite ends. Right: (b) Schematic of cross Andreev reflection and (c) elastic co-tunneling between quantum dots. The red (black) horizontal line denotes the Andreev bound state (dot level), and the grey line represents the Fermi energy of the superconductor.
superconductor-dot system, as shown in Fig. 1(a), is
\[
H = H_S + H_D + H_{SD},
\]
\[
H_S \approx E_1 \gamma_1 \gamma_1 + E_2 \gamma_2 \gamma_2,
\]
\[
H_D = \varepsilon d^\dagger d + \varepsilon_s d^\dagger_{\sigma} d_{\sigma},
\]
\[
H_{SD} = -t_{\text{r}} c_{x,\sigma}^\dagger d_{\sigma} - t_{\text{r}} c_{x,\sigma} d_{\sigma} + \text{H.c.}
\]  
(1)

Here \( H_S \) is the Hamiltonian for the hybrid nanowire. In the short-wire limit where the level spacing is larger than the superconducting gap, we consider only two normal states closest to the Fermi energy (which form a degenerate Kramer’s pair in the presence of time-reversal invariance). With an induced s-wave pairing, the pair of normal states are gapped and become two Andreev bound states defined as \( \gamma_i = \sum_{x, \sigma = \uparrow, \downarrow} [u_i(x) c_{x \sigma} + v_i(x) c_{x \sigma}^\dagger] \), where the wavefunctions and excitation energies are obtained by solving the Bogoliubov-de Gennes equation
\[
h_{\text{BDG}}(x) (u_i, v_i)^T = E_i (u_i, v_i)^T
\]  
In order to make our discussions as generic as possible, we do not assume any specific form of \( h_{\text{BDG}}(x) \), and mainly focus on the symmetry properties, e.g., time-reversal and spin-rotation symmetry. \( H_D \) describes the quantum dots located near the Fermi energy, denoted by \( d_{l\eta} \) and \( d_{r\sigma} \), respectively. Here the dot spins are either up or down \( (\eta, \sigma = \uparrow, \downarrow) \), and they have a common polarization axis, which can be realized by applying a global magnetic field. \( H_{SD} \) describes the spin-conserved electron tunneling between the dot levels and the ends of the hybrid nanowire at \( x = x_{l,r} \).

**Effective couplings between dots.**—In the tunneling limit \( t_{l,r} < \Delta \), we can apply a Schrieffer-Wolff transformation to obtain an effective Hamiltonian for the coupled quantum dots. That is, \( H_{\text{eff}} = H_D + H_{\text{interdot}} \), with
\[
H_{\text{interdot}} = -PH_{SD} \left( 1 - \frac{1}{P} \right) \frac{H_S + H_D + H_{SD} + O(t_{l,r}/\Delta^2)}{H_D + H_{SD}} + H_{\text{interdot}} + \text{H.c.}
\]
\[
= -\Gamma_{\text{CAR}} d^\dagger d_{l\eta} d^\dagger_{r\sigma} - \Gamma_{\text{ECT}} d^\dagger_{l\eta} d_{r\sigma} + \text{H.c.}
\]  
(2)

Here \( P \) is the projection operator onto the ground state of the uncoupled dot-superconductor system. \( \Gamma_{\text{CAR}} \) and \( \Gamma_{\text{ECT}} \) are the Andreev bound states-mediated effective couplings between two spin-polarized dot levels, with
\[
\Gamma_{\text{CAR}} \eta\sigma = \frac{t_{l,r}}{\Delta} \sum_{m=1,2} \frac{u_m(x,\eta)v_m(x,\sigma) - u_m(x,\sigma)v_m(x,\eta)}{E_m/\Delta},
\]
\[
\Gamma_{\text{ECT}} \eta\sigma = \frac{t_{l,r}}{\Delta} \sum_{m=1,2} \frac{u_m(x,\eta)v_m(x,\sigma) - u_m(x,\sigma)v_m(x,\eta)}{E_m/\Delta}.
\]  
(3)

Here \( \Gamma_{\text{CAR}} \) is a superconducting-like effective coupling between dots, and physically is induced by a coherent crossed Andreev reflection (CAR) process, where an incoming electron with spin-\( \sigma \) from the right dot is reflected nonlocally into a hole with spin-\( \eta \) in the left dot, as schematically shown in Fig. 1(b). In contrast, \( \Gamma_{\text{ECT}} \) is a normal effective coupling, and is induced by elastic co-tunneling (ECT), where a single electron hops from the right dot to the left via the Andreev bound states, as schematically shown in Fig. 1(c). Equation (3) is the most general expression, and indicates that a finite effective coupling requires the Andreev bound state wavefunctions to extend to both ends of the nanowire. In what follows we will analyze the dependence of the effective coupling strengths on the physical parameters of the Andreev bound states. We define \( P^\eta\sigma = |\Gamma_{\text{CAR}}^\eta\sigma/\Delta(t_{l,r})|/2 \) to characterize the coupling strengths, with \( a = \text{CAR} \) or ECT. As we will see, \( P^\eta\sigma \) is proportional to the experimentally measurable current \( I_{\eta\sigma} \).

**Energy and angle dependence.**—We first consider a time-reversal invariant hybrid nanowire. Physically, this corresponds to a situation where the Zeeman splitting induced from the global magnetic field is negligible compared to the spin-orbit interaction. The excitation energies of the Andreev bound states are degenerate \( E_{1,2} = E_n = \sqrt{\gamma_n^2 + \Delta^2} \) with \( \gamma_n = \gamma_n - \mu \) being the normal-state energy. The Bogoliubov-de Gennes wavefunctions are
\[
u_1(x,\sigma) = u_0 \psi_n(x,\sigma), \quad v_1 = v_0 \psi^\ast_n \]
and
\[
u_2 = -u_0 \psi_n, \quad v_2 = v_0 \psi^\ast_n, \]
where \( \psi_n, \psi^\ast_n \) are the normal-state wavefunctions, and \( v_0^2 = 1 - v_0^2 = 1/2 + \gamma_n/2E_n \) are the coherence factors. From Eq. (3), we then obtain
\[
P_{\text{CAR}}^\eta\sigma = C_0(\gamma_n/\Delta) |\psi_n(x,\eta)\psi^\ast_n(x,\sigma) - \psi_n(x,\sigma)\psi^\ast_n(x,\eta)|^2,
\]
\[
P_{\text{ECT}}^\eta\sigma = \xi_0(\gamma_n/\Delta) |\psi_n(x,\eta)\psi^\ast_n(x,\sigma) + \psi_n(x,\sigma)\psi^\ast_n(x,\eta)|^2.
\]
(4)
where \( C_0(z) = \left( \frac{2\nu_0\nu_0}{E_n/\Delta} \right)^2 = (z^2 + 1)^{-2} \), \( E_0(z) = \left( \frac{u_0^2 - z^2}{E_n/\Delta} \right)^2 = z^2(z^2 + 1)^{-2} \) with \( z = \xi_n/\Delta \). Equation (4) shows that \( P^a \) has a separable dependence on the energy \( \xi_n \) and on the wavefunctions \( \psi_{n,\pi} \) of the bound states. In particular, the energy dependence is universal because it only depends on the coherence factors \( \nu_0, \nu_v \). As shown in Fig. 2(a), \( C_0(z) \) of crossed Andreev reflection has a single peak centered at \( z = 0 \) (\( \xi_n = 0 \)) and decays as \( z^{-2} \) at large \( |z| \), while \( E_0(z) \) of elastic cotunneling has double peaks located at \( z = \pm 1 \), and decays as \( z^{-2} \) at large \( |z| \). Interestingly, \( E_0(z) \) has a dip at \( z = 0 \) due to the destructive interference between two virtual paths with a \( \pi \)-phase shift. The strikingly different profiles of \( C_0(z) \) and \( E_0(z) \) is the first main finding in this work, which indicates that one can vary the ratio between the two types of effective couplings by tuning the chemical potential of the Andreev bound state. The wavefunction part in Eq. (4) gives an overall prefactor to the energy-dependence profiles. Time-reversal invariance, i.e., \( \psi_n(x,\sigma) = T\psi_n(x,\sigma) = -i\sigma_y \psi_n^*(-x,\sigma) \), gives the following symmetry relations between different dot-spin channels

\[
P^a_{\uparrow \uparrow} = P^a_{\downarrow \downarrow}, \quad P^a_{\uparrow \downarrow} = P^a_{\downarrow \uparrow},
\]

for \( a = \text{CAR} \) or \( \text{ECT} \). Thus we can focus on two spin channels \( \uparrow \uparrow \) and \( \uparrow \downarrow \) in the following discussions.

If spin-orbit field is the only or the dominant spinful field in the hybrid nanowire and is constant throughout the wire [42], we can further specify the wavefunctions to be \( \psi_n(x,\sigma) = \phi_n(x)e^{-i\kappa_n x} \times [\cos(\theta/2), \sin(\theta/2)]^T \) [43]. Here \( \phi_n(x) \) is the eigenfunction without spin-orbit interaction, \( \kappa_n = m\alpha_R/\hbar^2 \) is the spin-orbit wave-vector, and \( \theta \) is the angle between the spin-orbit field and the spin axis in dots. Without loss of generality, we fix the dot spin axis along \( z \) and rotate the spin-orbit field in the \( xz \)-plane. Plugging the wavefunctions into Eq. (4), we obtain

\[
P_{\uparrow \uparrow}^{\text{CAR}} = C_0(z) \cdot g(\theta), \quad P_{\uparrow \downarrow}^{\text{CAR}} = C_0(z) \cdot f(\theta),
\]

\[
P_{\uparrow \uparrow}^{\text{ECT}} = E_0(z) \cdot f(\theta), \quad P_{\uparrow \downarrow}^{\text{ECT}} = E_0(z) \cdot g(\theta).
\]

Here \( f(\theta) = p^2 + q^2 \cos^2 \theta, \quad g(\theta) = q^2 \sin^2 \theta \), with \( (p, q) = (\phi_n(x_1)\phi_n(x_2) \cos(k_m L), \sin(k_m L)) \), and \( L = x_2 - x_1 \) is the nanowire length. As shown in Fig. 2(b), \( P^a \) has a sinusoidal dependence on the angle \( \theta \). In particular, \( \text{CAR}-\uparrow \downarrow \) and \( \text{ECT}-\uparrow \uparrow \) are more favorable channels with \( f(\theta) \geq p^2 \). While in the \( \text{CAR}-\uparrow \uparrow \) and \( \text{ECT}-\uparrow \downarrow \) channels, \( P^a \) vanishes at \( \theta = 0 \) or \( \pi \), i.e., equal-spin \( \text{CAR} \) and opposite-spin \( \text{ECT} \) processes are not allowed in the presence of spin conservation. Thereby, in order to have finite effective couplings of both types in a particular dot spin channel, it is crucial to have a spin-orbit field misaligned with the dot spin axis. More surprisingly, although \( P^a_{\eta \sigma} \) depends on both the energy and the wavefunction of the Andreev bound states via \( p^2 \) and \( q^2 \), the ratio between angle-averaged \( P^a \) in favorable and unfavorable channels is universal, i.e.,

\[
\langle P_{\uparrow \uparrow}^{\text{CAR}} \rangle = \langle P_{\uparrow \downarrow}^{\text{CAR}} \rangle = \langle P_{\uparrow \uparrow}^{\text{ECT}} \rangle = \langle P_{\uparrow \downarrow}^{\text{ECT}} \rangle = \frac{\sin^2(k_m L)}{2 - \sin^2(k_m L)},
\]

where \( \langle P_{\eta \sigma}^a \rangle = (2\pi)^{-1} \int_0^{2\pi} d\theta P_{\eta \sigma}^a(\theta) \). The ratio is determined only by the spin procession through the nanowire, thus providing a new way to extract the induced spin-orbit coupling strength in the semiconducting nanowire.

**Effect of Zeeman spin splitting.**—We now consider the angle-averaged \( P^a \) in favorable and unfavorable channels is universal, i.e.,

\[
\langle P_{\uparrow \uparrow}^{\text{CAR}} \rangle = \langle P_{\uparrow \downarrow}^{\text{CAR}} \rangle = \langle P_{\uparrow \uparrow}^{\text{ECT}} \rangle = \langle P_{\uparrow \downarrow}^{\text{ECT}} \rangle = \frac{\sin^2(k_m L)}{2 - \sin^2(k_m L)},
\]

with \( \langle P_{\eta \sigma}^a \rangle = (2\pi)^{-1} \int_0^{2\pi} d\theta P_{\eta \sigma}^a(\theta) \). The ratio is determined only by the spin procession through the nanowire, thus providing a new way to extract the induced spin-orbit coupling strength in the semiconducting nanowire. Therefore, the CAR couplings be-
come

\[
P_{\uparrow\uparrow}^{\text{CAR}}(E_Z) = P_{\downarrow\downarrow}^{\text{CAR}}(E_Z) = C_3(z) \cdot q^2 \sin^2 \theta,
\]
\[
P_{\uparrow\downarrow}^{\text{CAR}}(E_Z) = P_{\downarrow\uparrow}^{\text{CAR}}(E_Z) = C_3(z) \cdot (p^2 + q^2 \cos^2 \theta),
\]

where \( C_3(z) = (z^2 + 1 - \delta^2)^{-2} \) is the modified energy-dependence profile with \( \delta = E_Z \cos \theta / \Delta < 1 \). Figures 3(a) and 3(b) show the energy dependence of \( P_{\uparrow\downarrow}^{\text{CAR}} \) for different values of \( E_Z / \Delta \). Both \( P_{\uparrow\downarrow}^{\text{CAR}} \) and \( P_{\downarrow\uparrow}^{\text{CAR}} \) increase with \( E_Z \), with their profiles remaining symmetric about \( z = 0 \). The interdot ECT couplings are

\[
P_{\uparrow\downarrow}^{\text{ECT}}(E_Z) = P_{\downarrow\uparrow}^{\text{ECT}}(-E_Z) = \frac{(z - \delta \cos \theta)^2 p^2 + (z \cos \theta - \delta)^2 q^2}{(z^2 + 1 - \delta^2)^2},
\]
\[
P_{\uparrow\downarrow}^{\text{ECT}}(E_Z) = P_{\downarrow\uparrow}^{\text{ECT}}(E_Z) = \frac{z^2 q^2 + \delta^2 p^2}{(z^2 + \Delta^2)^2} \cdot \sin^2 \theta.
\]

As shown in Fig. 3(c), the profile of \( P_{\uparrow\downarrow}^{\text{ICT}} \) becomes asymmetric when \( E_Z > 0 \), with one peak being lifted and the other suppressed. According to the symmetry relation in Eq. (9), the profile of \( P_{\uparrow\downarrow}^{\text{ICT}}(E_Z) \) would become asymmetric in the opposite way. On the other hand, \( P_{\uparrow\downarrow}^{\text{ICT}} \) remains symmetric at a finite Zeeman field, with the dip at \( z = 0 \) being elevated to a finite value, as shown in Fig. 3(d).

*Extracting \( \Gamma^a \) experimentally.—*To reach the optimal parameter regime for the desired application, it is necessary to be able to extract the strengths of the effective interdot couplings experimentally. For this purpose, we propose a three-terminal junction, where two quantum dots are now connected with two external normal electrodes, respectively, as shown by the schematics in Fig. 4. The strengths of \( \Gamma^{\text{CAR}} \) and \( \Gamma^{\text{ECT}} \) can be extracted from measuring the resonant current.

Our considerations and calculations follow the same spirit as those in Refs. [1, 2], which focused on the current due to crossed Andreev reflection in a similar setup. Compared to the previous works, the differences made in our calculations include: (1) We now consider Andreev bound states instead of quasiparticle continuum in the superconducting segment. (2) Spin-orbit interaction in the opposite way. On the other hand, the bias voltage window needs to be large enough to include the full width of the broadened dot states, i.e., \( \delta \mu > \Gamma_{DL} \approx \pi \nu (|t_1|^2 + |t_r|^2) \) with \( \nu \) being the number density of states. The dot-lead coupling should be stronger than the superconductor-dot coupling \( \Gamma_{DL} > \Gamma_{SD}, \varepsilon_1, \varepsilon_r \approx \mu_S \).

![Fig. 4](https://example.com/f4.png)

**Fig. 4.** (a) and (b) Schematic for the three-terminal junctions. (c) and (d) Resonant current in the \((\varepsilon_1, \varepsilon_r)\)-plane. The currents have a Breit-Wigner resonance form, with the broadening width being the dot-lead coupling strength \( \Gamma_{DL} \). CAR current assumes the maximum value \( I_{\text{max}}^{\text{CAR}} \) when \( \varepsilon_1 = \pm \varepsilon_r \), respectively. The strengths of the effective couplings can be extracted by \( \Gamma^a = \sqrt{\Gamma_{\text{max}}^{\text{DL}} \hbar / e} \)

generating resonant current is [1, 2]

\[
\Delta, U > \delta \mu > \Gamma_{DL}, k_B T, \Gamma_{DL} > \Gamma_{SD}, \varepsilon_1, \varepsilon_r \approx \mu_S.
\]

Here \( \delta \mu \) is the applied bias voltage, with \( \delta \mu = \mu_S - \mu_{r,S} > 0 \) for generating CAR current, and \( \delta \mu / 2 = \mu_r - \mu_S = \mu_S - \mu > 0 \) for ECT current, as shown schematically in Figs. 4(a) and 4(b). The bias voltage being smaller than the induced gap \( \Delta \) and the dot-charging energy \( U \) helps to suppress the undesired processes such as local Andreev reflection and inelastic cotunneling. On the other hand, the bias voltage window needs to be large enough to include the full width of the broadened dot states, i.e., \( \delta \mu > \Gamma_{DL} = \pi \nu (|t_1|^2 + |t_r|^2) \) with \( \nu \) being the number density of states.

The currents are calculated using the rate equation approach [1, 2, 44]. Calculation details are in the supplemental material [43], and we only show the results here. When \( \mu_S > \mu_{r,S} \), Cooper pairs from the superconducting lead would split into two electrons, which flow to two separate normal leads via dots, respectively, giving the following spin-selective CAR current

\[
I_{\nu a}^{\text{CAR}} = \frac{e}{\hbar} \cdot \frac{\Gamma_{DL}^2}{(\varepsilon_1 + \varepsilon_r)^2 + \Gamma_{DL}^2} \cdot \frac{|I_{\nu a}^{\text{CAR}}|^2}{\Gamma_{DL}^2},
\]

The total Hamiltonian for the three-terminal junction, as shown in Fig. 4, is \( H_{\text{tot}} = H + H_L + H_{DL} \). \( H \) is the dot-superconductor-dot system introduced by Eq. (1). \( H_L = \sum_k (\varepsilon_k - \mu_l) a_{l\downarrow k} a_{l\uparrow k} + (\varepsilon_k - \mu_r) a_{r\downarrow k} a_{r\uparrow k} \) are the normal leads, which are conventional Fermi liquids with electrons filled up to the Fermi energy \( \mu_{l,r} \).

\[
H_{DL} = \sum_k \left( -\varepsilon_k d_{l\downarrow k}^\dagger d_{l\uparrow k} - \varepsilon_r d_{r\downarrow k}^\dagger d_{r\uparrow k} \right) + \text{H.c.}
\]

The relevant parameter regime for...
with $\Gamma_{\text{CAR}}^\varepsilon$ being the effective coupling defined in Eq. (3). As shown in Fig. 4(c), in the $(\varepsilon_l, \varepsilon_r)$-plane this current has a Breit-Wigner resonance form with its broadening width being the dot-lead coupling $\Gamma_{DL}$ and the maximum value reached when $\varepsilon_l = -\varepsilon_r$. On the other hand, in exactly the same setup but with a different bias voltage: $\mu_1 < \mu_2 < \mu_r$, now a single electron flows from one to the other normal lead, giving the spin-selective ECT current

$$I_{\eta\sigma}^{\text{ECT}} = \frac{e}{\hbar} \frac{\Gamma_{DL}^2}{(\varepsilon_l - \varepsilon_r)^2 + \Gamma_{DL}^2} \cdot \frac{|I_{\eta\sigma}^{\text{ECT}}|^2}{\Gamma_{DL}^2}, \quad (12)$$

where $\Gamma_{\eta\sigma}^{\text{ECT}}$ is defined in Eq. (3). The ECT current has the same Breit-Wigner form, but now assumes the maximum value when $\varepsilon_l = \varepsilon_r$, as shown in Fig. 4(d).

Equations (11) and (12) indicate that resonant current is proportional to the square of the corresponding interdot coupling strength. Thus, experimentally one can extract the coupling strengths using the formula $\Gamma = \sqrt{\Gamma_{\text{max}}^r \Gamma_{DL} / e}$, where $\Gamma_{DL}$ is read off from the resonance broadening width in gate voltage times the lever arm, and $\Gamma_{\text{max}}^r$ is the current value along $\varepsilon_l = -\varepsilon_r$ for CAR and $\varepsilon_l = \varepsilon_r$ for ECT.

**Discussions.**—We have given a proposal for mediating tunable effective couplings between quantum dots via Andreev bound states, which provides a previously missing but all-important experimental knob for fine-tuning the physical system into the desirable parameter regime. In particular, the Cooper pair splitting efficiency now can be enhanced by tuning the energy to $z = 0$ in Fig. 2(a), where the crossed Andreev reflection is strengthened and simultaneously the unwanted elastic cotunneling processes are strongly suppressed. The minimal Kitaev chain now become tunable and can host Majorana zero mode at $z \approx \pm 1$ in Fig. 2(a), where crossed Andreev reflection and elastic cotunneling between dots have equal strength. In practice, this tuning protocol can be implemented by controlling the electrostatic gate near the semiconductor-superconductor segment, eliminating the need of non-collinear magnetic fields [31]. This makes our proposal especially appealing, since all the necessary ingredients, i.e., spin-polarized quantum dots [45], gated hybrid nanowire with spin-orbit interaction [46, 47], are within reach of existing materials and technologies. We thus expect that following our proposal, the future Cooper pair splitters will be able to pass the Bell inequality test threshold, becoming ready for fundamental quantum entanglement researches. Also, a tunable Kitaev chain will be experimentally accessible, providing an exciting platform for studying topological superconductivity and non-Abelian statistics.

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[42] For clear illustration, we assume a constant spin-orbit coupling strength $\alpha_R$ throughout the nanowire in the main text. Actually, all the calculations and conclusions carry over to the more general scenarios of a spatially varying $\alpha_R(x)$, with only a minimal substitution of $k_{so}L \rightarrow \int_{x_1}^{x_2} k_{so}(x) \, dx$. See supplemental materials.

[43] The supplemental material includes: 1. Derivation of the Bogliubov-de Gennes Hamiltonian of the hybrid segment with and without time-reversal invariance. 2. Calculation of eigenfunctions in a disordered nanowire with Rashba spin-orbit ineraction, and the ratio of angle averaged effective couplings. 3. Calculation of the resonant currents in a three-terminal junction.

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Supplemental Material for “Tunable Cooper pair splitting via Andreev bound states”

I. HAMILTONIAN OF THE HYBRID NANOWIRE

In the presence of time-reversal invariance and induced superconducting pairing being s-wave, Anderson’s theorem guarantees the following form of BdG Hamiltonian in the terms of Kramers’ pair:

$$H_S = \xi_n (a_n^\dagger a_n + a_{\pi n}^\dagger a_{\pi n}) + \Delta (a_n^\dagger a_{\pi n} + a_{\pi n}^\dagger a_n),$$

where $a_n, a_{\pi}$ denote the time-reversed normal states with wavefunctions $\psi_n$ and $\psi_{\pi} = -i\sigma_y \psi_n^*$. Thus the eigenenergies are

$$E_{1,2} = E_n = \sqrt{\xi_n^2 + \Delta^2},$$

and the BdG eigenfunctions are

$$\gamma_1 = u_0 a_n + v_0 a_{\pi n}^\dagger, \quad \gamma_2 = v_0 a_n^\dagger - u_0 a_{\pi n}.$$

Here $\xi_n = \epsilon_n - \mu$ is the normal-state energy relative to the chemical potential, $u_0^2 = 1 - v_0^2 = 1/2 + \xi_n/E_n$. We further assume that spin-orbit field is the only or the dominant spinful field in the hybrid segment, thus the normal-state wavefunctions can be written as

$$\psi_n(x, \sigma) = \phi_n(x) e^{-ik_{\sigma,\sigma}x} \left(\cos(\theta/2) + \sin(\theta/2)\right),$$

where $\theta$ is the angle between the spin-orbit field and the dot spin axis.

We now introduce a Zeeman field $H_Z = E_Z \int dx \left[c_n^\dagger(x)c_1(x) - c_1^\dagger(x)c_n(x)\right]$ in the hybrid segment, with its direction parallel to the dot spin axis. In the weak-field limit, i.e., $E_Z < E_{so}, \Delta$, the effect can be treated perturbatively by projecting it onto the eigenfunctions in the presence of time-reversal invariance. In particular,.

$$\begin{align*}
(H_Z)_{nn} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_n^*(x, \sigma) (\sigma_{\sigma}^\dagger)_{\sigma,\sigma'} \psi_n(x, \sigma') = E_Z \cos \theta, \\
(H_Z)_{\pi\pi} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_{\pi n}^*(x, \sigma) (\sigma_{\sigma}^\dagger)_{\sigma,\sigma'} \psi_{\pi n}(x, \sigma') = -E_Z \cos \theta, \\
(H_Z)_{\pi n} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_{\pi n}^*(x, \sigma) (\sigma_{\sigma}^\dagger)_{\sigma,\sigma'} \psi_n(x, \sigma') = E_Z \sin \theta \times \int [\phi(x)]^2 e^{-2ik_{\sigma,\sigma}x} dx, \\
(H_Z)_{n\pi} &= E_Z \sum_{\sigma,\sigma'} \int dx \psi_n^*(x, \sigma) (\sigma_{\sigma}^\dagger)_{\sigma,\sigma'} \psi_{\pi n}(x, \sigma') = E_Z \sin \theta \times \int [\phi^*(x)]^2 e^{2ik_{\sigma,\sigma}x} dx. \tag{S-5}
\end{align*}$$
In the strong spin-orbit interaction regime with the spin-orbit length being smaller than the nanowire length, the phase of $e^{-2ik_{so}x}$ oscillate fast, making $\left| \int [\phi(x)]^2 e^{-2ik_{so}x} dx \right| \ll 1$. We can thus neglect $(H_Z)_{n\pi}$ and $(H_Z)_{\pi n}$. Now the Bogoliubov-de Gennes Hamiltonian of the hybrid segment becomes

$$H_S = \begin{pmatrix} a_n^\dagger & a_{\pi} \end{pmatrix} \begin{pmatrix} \xi_n + E_Z \cos \theta & \Delta \\ \Delta & -\xi_n + E_Z \cos \theta \end{pmatrix} \begin{pmatrix} a_n \\ a_{\pi} \end{pmatrix}. \tag{S-6}$$

The eigenenergies are now split as

$$E_{1,2} = \sqrt{\xi_n^2 + \Delta^2 \pm E_Z \cos \theta}, \tag{S-7}$$

and the Bogoliubov-de Gennes eigenfunctions remain the same as the time-reversal invariant case because the effective Zeeman term is an identity in the Nambu space.

II. EIGENFUNCTIONS IN A HYBRID NANOWIRE AND RATIO BETWEEN ANGLE AVERAGED $P^\alpha$

In this section, we derive the eigenfunctions in a one-dimensional hybrid nanowire with Rashba spin-orbit interaction. We assume that both the chemical potential and the strength of the spin-orbit coupling are spatially inhomogeneous, but the direction of the spin-orbit field is constant throughout the nanowire. Furthermore, we show that the the ratio of the angle averaged $P_{\eta\sigma}^\alpha$ in favorable and unfavorable channels has a universal value which depends only on the spin-orbit interaction and nanowire length. The normal Hamiltonian of the hybrid nanowire has the following form

$$H_n = -\frac{\hbar^2}{2m} \partial_x^2 + V(x) - i \frac{1}{2} [\alpha_R(x) \partial_x + \partial_x \alpha_R(x)] \hat{\sigma} = \frac{\hbar^2}{2m} [-i \partial_x + k_{so}(x) \hat{\sigma}] [-i \partial_x + k_{so}(x) \hat{\sigma}] + V(x) - \frac{m \alpha_R^2(x)}{2\hbar^2} \tag{S-8}$$

where $V(x)$ is the spatially varying chemical potential, $k_{so}(x) = \frac{m \alpha_R(x)}{\hbar^2}$ is the local spin-orbit wave-vector, $\hat{\sigma} = \sigma_z \cos \theta + \sigma_x \sin \theta$ with $\theta$ the angle between the spin-orbit field and the spin axis in quantum dots. To find the eigenfunction of the $H_n$, i.e., $H_n \psi_n(x) = E_n \psi_n(x)$, we use the following ansatz

$$\psi_n(x) = \phi_n(x) e^{-i \int_0^x k_{so}(x') dx'} \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{pmatrix}, \tag{S-9}$$

and the Kramers’ partner is $\psi_{\pi}(x) = -i \sigma_y \psi_n^\ast(x)$. Here $\psi_n$ is the eigenfunctions of $H_n$ with eigenenergies $E_n$, if $\phi_n(x)$ is the eigenfunction of the following eigenequation

$$\left[ -\frac{\hbar^2}{2m} \partial_x^2 + V(x) - \frac{m \alpha_R^2(x)}{2\hbar^2} \right] \phi_n(x) = E_n \phi_n(x). \tag{S-10}$$

We thus find the most generic form of the eigenfunction in a one-dimensional nanowire. When the spin-orbit coupling strength is constant throughout the nanowire, i.e., $k_{so}(x) \to k_{so}$, we get the eigenfunction used in the main text. Plugging $\psi_n$ and $\psi_{\pi}$ into Eq. (4) in the main text, we obtain

$$P_{++}^{\text{CAR}}(\theta) = C_0 (\xi_n / \Delta) \phi_n^2(x_1) \phi_{\pi}^2(x_{\pi}) \sin^2 \theta \sin^2(k_{so} L), \tag{S-11}$$

$$P_{+\pi}^{\text{CAR}}(\theta) = C_0 (\xi_n / \Delta) \phi_n^2(x_1) \phi_{\pi}^2(x_{\pi}) \left[ \cos^2(k_{so} L) + \cos^2 \theta \sin^2(k_{so} L) \right]$$

where the averaged spin-orbit wave-vector is defined as

$$\overline{k_{so}} = \frac{1}{L} \int_0^L k_{so}(x') dx'.$$ \tag{S-12}

We now define the angle averaged $P$, i.e.,

$$\langle P_{\eta \sigma}^{\text{CAR}} \rangle = \int_0^{2\pi} P_{\eta \sigma}^{\text{CAR}}(\theta) d\theta. \tag{S-13}$$
Thus the ratio between \( P \)'s in the unfavorable and favorable channels is
\[
  r = \frac{\langle \text{CAR} \rangle_{n+1}}{\langle \text{CAR} \rangle_{n+1}} = \frac{\sin^2(k_{so}L)}{2 - \sin^2(k_{so}L)}.
\] (S-14)

Note that since \( C_0, \phi_n \) in the denominator and the numerator cancel, the ratio does not depend on the energy or the wavefunction details of the bound state, instead it depends only on the ratio between nanowire length and the averaged spin-orbit length. The ratio is no greater than one, i.e., \( r \leq 1 \), and it reaches one (zero) when the nanowire length is half-integer (integer) of the spin-orbit length. We therefore can extract the averaged spin-orbit coupling strength in a hybrid nanowire using the above formula.

### III. RESONANT CURRENTS

In this section, we give details of how we calculate the resonant current in the normal-dot-superconductor-dot-normal junction. The methods we use include the \( T \)-matrix approach and the rate equation, which are standard for resonant current calculations [1, 2] in such mesoscopic systems. In terms of the parameter regime for generating resonant current, we consider
\[
  \Delta, U > \delta \mu > \Gamma_{DL}, k_B T, \quad \Gamma_{DL} > \Gamma_{SD}, \quad \varepsilon_i \approx \varepsilon_r \approx \mu_S,
\] (S-15)

which applies to both crossed Andreev reflection and elastic co-tunneling processes.

#### A. Crossed Andreev reflection

We first consider the CAR current, where two electrons pass from the superconductor via the virtual dot states to two different leads. The whole tunneling process can be decomposed into two main parts. In the first part, a Cooper pair breaks up, where one electron tunnels from superconductor to one dot level, leaving behind a quasiparticle in the second part, the second electron tunnels from superconductor to the other dot level before the first electron escapes from the dot to the electrode, because the relevant time scale is \( h/\Delta < h/\Gamma_{DL} \). Tunneling back to the superconductor is unlikely because \( \Gamma_{SD} < \Gamma_{DL} \). The amplitude for the transition from the initial to the final state is thus
\[
  \langle f|T|\varepsilon_i\rangle \approx \langle f|T(0)|\varepsilon_i\rangle = \langle f|T_2|DD\rangle \langle DD|T_1|i\rangle,
\] (S-16)

where \( T(0) = T(\varepsilon_i = 0) \). The initial state is \( |i\rangle = |0_S\rangle|0_D\rangle|\mu_1\rangle \), where the superconducting is in its ground state with no quasiparticle excitations, both dots levels are vacant and the normal leads are Fermi liquids filled up to its Fermi energy. \( |DD\rangle = |0_S\rangle|1_t, 1_r\rangle|\mu_1\rangle \) is the intermediate state with dot states being occupied by one electron each. \( T_1 \) is the \( T \)-matrix for the tunneling process in the first part is of second order in \( H_{SD} \), with
\[
  T_1 = \frac{1}{i\eta - H_0} H_{SD} \frac{1}{i\eta - H_0} H_{SD}.
\] (S-17)

Thus using the second-order perturbation theory we immediately have
\[
  \langle DD|T_1|i\rangle = \frac{1}{i\eta - (\varepsilon_1 + \varepsilon_2)} \Gamma_{\eta \sigma}^{\text{CAR}},
\] (S-18)

with
\[
  \Gamma_{\eta \sigma}^{\text{CAR}} = \frac{t_1 t_r}{\Delta} \sum_{m=1,2} \frac{u_m(\eta) v_m^*(\sigma) - u_m(\sigma) v_m^*(\eta)}{E_m/\Delta}.
\] (S-19)

Here the spin in the dots are polarized in the \( \sigma_z \) direction. Since \( \varepsilon_i \approx \varepsilon_r \approx \mu_S = 0 \), the energy denominator diverges as \( 1/\eta \), indicating that the tunneling between dots and leads is resonant. Thus for the second part of the tunneling process, we must include the tunnel Hamiltonian to all orders, i.e.,
\[
  T_2 = H_{DL} \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_0} H_{DL} \right)^{2n+1},
\] (S-20)
and thus the transition amplitude for the second part is

$$
\langle f|T_2|DD \rangle = \left\{ \langle pq|H_{DL1}|Dq \rangle \left\langle DD \left| \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_{DL1}} \right)^{2n} \right| DD \right\rangle \langle DD \left| \frac{1}{i\eta - H_{DL2}} \right| DD \right\rangle + \langle pq|H_{DL2}|pD \rangle \left\langle DD \left| \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_{DL2}} \right)^{2n} \right| pD \right\rangle \langle pD \left| \frac{1}{i\eta - H_{DL1}} \right| DD \right\rangle \right\} \times \left\langle DD \left| \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle.
$$

(S-21)

Among all the above terms, the geometrical summation are calculated in a similar manner, and here we only focus on $$\left\langle DD \left| \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle$$, which is

$$
\left\langle DD \left| \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle \approx 1 + \sum_{n=1}^{\infty} \left( \left\langle DD \left| \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle \right)^n
$$

$$
= \frac{1}{1 - \left\langle DD \left| \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle},
$$

(S-22)

with

$$
\left\langle DD \left| \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle = \frac{1}{i\eta - (\varepsilon_1 + \varepsilon_2)} \left( \frac{\Gamma_{DL2}}{2\pi} \int d\xi_k \frac{1}{i\eta - (\varepsilon_1 + \varepsilon_k)} + \frac{\Gamma_{DL1}}{2\pi} \int d\xi_k \frac{1}{i\eta - (\varepsilon_2 + \varepsilon_k)} \right)
$$

$$
= \frac{1}{\varepsilon_1 + \varepsilon_2 - i\eta} (\text{Re}\Sigma + i\Gamma_{DL}),
$$

(S-23)

where $$\text{Re}\Sigma = \ln(E_{c1}/E_{c2})$$ is a logarithmic divergence with the cutoff energy in the conduction band in the normal lead, and $$\Gamma_{DL} = \Gamma_{DL1} + \Gamma_{DLr}$$. Note that here we assume that the dominant process is $$|DD\rangle \rightarrow |pD\rangle \rightarrow |DD\rangle$$ or $$|DD\rangle \rightarrow |Dq\rangle \rightarrow |DD\rangle$$, with $$|pD\rangle$$ denoting that the left electron is in the lead and the right electron in the dot. So the geometrical summation is

$$
\left\langle DD \left| \sum_{n=0}^{\infty} \left( \frac{1}{i\eta - H_{DL}} \right)^{2n} \right| DD \right\rangle = \frac{\varepsilon_1 + \varepsilon_2 - i\eta}{\varepsilon_1 + \varepsilon_2 - i\Gamma_{DL}}.
$$

(S-24)

The key finding here is that the divergent denominator now becomes the numerator and the new denominator has a finite imaginary part which is proportional to the dot-lead coupling. Finally the transition amplitude for the second part is

$$
\langle f|T_2|DD \rangle = \left( \frac{\mathcal{t}'_r}{\varepsilon_1 + \varepsilon_2 - i\Gamma_{DL}} \frac{\mathcal{t}'_q}{\varepsilon_1 + \varepsilon_2 - i\Gamma_{DL}} \right) + \left( \frac{\mathcal{t}'_r}{\varepsilon_2 + \varepsilon_2 - i\Gamma_{DL}} \frac{\mathcal{t}'_q}{\varepsilon_2 + \varepsilon_2 - i\Gamma_{DL}} \right)
$$

$$
= \frac{\mathcal{t}'_r \mathcal{t}'_q}{\varepsilon_1 + \varepsilon_2 - i\Gamma_{DL}} |(\varepsilon_2 + \varepsilon_2 - i\Gamma_{DL})| \varepsilon_1 + \varepsilon_2 - i\Gamma_{DL}.
$$

(S-25)

Plugging it into the formula for CAR current, we have

$$
I_{\text{CAR}} = \frac{e}{h} \mathcal{t}'_r \mathcal{t}'_q \int d\varepsilon_p \int d\varepsilon_q \langle pq|T(0)|q \rangle^2 \delta(\varepsilon_p + \varepsilon_q)
$$

$$
= \frac{e}{h} \Gamma_{DL} \Gamma_{DLr} |I_{\text{CAR}}|^2 \int d\varepsilon_p \left[ \frac{1}{(\varepsilon_1 - \varepsilon_p)^2 + \Gamma_{DL1}^2} |(\varepsilon_2 + \varepsilon_p)^2 + \Gamma_{DLr}^2| \right]
$$

$$
= \frac{e}{h} \frac{\Gamma_{DL}}{(\varepsilon_1 + \varepsilon_2)^2 + \Gamma_{DL}^2} |I_{\text{CAR}}|^2.
$$

(S-26)

So the CAR current has the form of a Breit-Wigner resonance profile, which assumes its maximum value at $$\varepsilon_1 + \varepsilon_2 = 0$$. 
B. Elastic co-tunneling

For the ECT current, a single electron passes from the lead with higher chemical potential via the dot and superconductor states to the other lead with lower chemical potential. Here we make three assumptions in our derivation. First, when calculating the transition rate $W_{f_i}$ between particular initial and final state, we assume that both normal leads are vacant. Second, the Fermi-Dirac distribution will be taken into account only in the final step for $\rho_i$. Third, when the chemical potential in leads are equal, the current flowing in the opposite directions cancel with each other.

Under these assumptions, we first calculate the transition rate, focusing on the scenario of a single electron passing from the right lead to the left lead. The total tunneling process can be separated into three parts.

\[
\langle p| T(\xi_1 = \xi_q)\rangle = \langle p| T_3| 1\rangle \langle 1| T_2| 2\rangle \langle 2| T_1| q\rangle, \quad (S-27)
\]

where $|l\rangle$ or $|r\rangle$ means an electron is in the left or right dot. For the first step, we need to include the resonant tunneling between dot and lead, such that

\[
\langle r| T_1| q\rangle = \left( r \sum_{n=0}^{\infty} \left( \frac{1}{\imath \eta - H_{DLr}} \right)^{2n} \right) \langle r| 1\rangle \langle 1| \left( \frac{1}{\imath \eta - H_{DLr}} \right) q\rangle
\]

\[
= \frac{t'_r}{\xi_r - \xi_q - \imath \Gamma_{DLr}}. \quad (S-28)
\]

And

\[
\langle l| T_2| r\rangle = \left( l \left( \frac{1}{\imath \eta - H_{SD}} \right)^2 \right) \langle l| r\rangle
\]

\[
= \frac{\Gamma_{ECT}^{\eta\sigma}}{\imath \eta - (\xi_l - \xi_q)}, \quad (S-29)
\]

with

\[
\Gamma_{ECT}^{\eta\sigma} = \frac{t_l t_r}{\Delta} \sum_{m=1,2} \frac{u_m(\imath \eta) u_m^\ast(\imath \eta) - v_m(\imath \eta)}{E_m/\Delta} \langle r\rangle, \quad (S-30)
\]

For the third step, we have

\[
\langle p| T_3| 1\rangle = \langle p| H_{DLl}| 1\rangle \left( l \sum_{n=0}^{\infty} \left( \frac{1}{\imath \eta - H_{DLl}} \right)^{2n} \right) \langle 1| r\rangle
\]

\[
= t'_l \frac{\xi_l - \xi_q - \imath \eta}{\xi_l - \xi_q - \imath \Gamma_{DLl}}. \quad (S-31)
\]

Therefore we have the transition amplitude and the transition rate to be

\[
\langle p| T| q\rangle = t'_l t'_r \Gamma_{ECT}^{\eta\sigma},
\]

\[
W_{pq} = 2\pi\langle p| T| q\rangle^2 \delta(\xi_p - \xi_q). \quad (S-32)
\]

The ECT current now is

\[
I = \frac{e}{\hbar} \sum_{f_i} W_{f_i}\rho_i
\]

\[
= \frac{e}{\hbar} \imath \nu_f \int d\xi_p d\nu_r \int_{-\delta\mu/2}^{\delta\mu/2} d\xi_q \langle p| T| q\rangle^2 \delta(\xi_p - \xi_q)
\]

\[
= \frac{e}{\hbar} \int_{-\delta\mu/2}^{\delta\mu/2} d\xi_q \left( \frac{\gamma_{DL1} \gamma_{DL2} |\Gamma_{ECT}^{\eta\sigma}|^2}{\Gamma_{DL}^2} \right)
\]

\[
= \frac{e}{\hbar} \cdot \frac{\Gamma_{DL}}{\langle \xi_l - \xi_r \rangle^2 + \Gamma_{DL}^2} \cdot |\Gamma_{ECT}^{\eta\sigma}|^2. \quad (S-33)
\]
Note that the integral of the outgoing electron energy $\varepsilon_p$ disappear because of the energy conservation. The integral window of the incoming electron energy $\varepsilon_q$ is because when $\delta\mu = 0$ the system is in equilibrium and left-flowing and right-flowing currents cancel and $I = 0$. So the net current is due to the window of the biased voltage. The ECT current is also in the form of the Breit-Wigner resonance, and assumes its maximum value at $\varepsilon_l = \varepsilon_r$. 