Bose-Einstein condensation of magnons

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Abstract

We use the Renormalization Group method to study the Bose-Einstein condensation of the interacting dilute magnons which appears in three dimensional spin systems in magnetic field. The obtained temperature dependence of the critical field $H_c(T) - H_c(0) \sim T^2$ is different from the recent self-consistent Hartree-Fock-Popov calculations [cond-mat/0405422] in which a $T^{3/2}$ dependence was reported. The origin of this difference is discussed in the framework of quantum critical phenomena.
I. INTRODUCTION

Recently, the quantum spin system in magnetic field became of great interest because the experimental data on the ladder systems showed the possibility of a quantum phase transition (QPT) driven by the magnetic field [1]. The QPT in the spin systems have been treated using the quantum rotor model [2]. However, the transition driven by a magnetic field is difficult to be described in such a formalism.

The occurrence of the magnetic order in the spin-gap magnetic compounds has been interpreted as a Bose-Einstein condensation (BEC) of magnons. At the present time it is known that the BEC has been discovered in ultra-cooled dilute atomic gases, but these experiments present various limitations. The analogy between a quantum spin system which presents long-range order and an interacting Bose gas which presents BEC is well known for a long time [3]. In the case of quantum spin system is possible to tune the density of magnons by magnetic field [11, 12] or pressure [20] to observe BEC in these systems.

In this paper we formulate the theory of magnon condensation using the Renormalization Group method (RNG) applied for bosonic systems [4, 5, 6, 7, 8, 9] and recently reconsidered for the spin systems [10].

The outline of this paper is as follows. In Sect. II we present the basic experimental evidence obtained on the material class $X CuCl_3$ ($X = Tl, K, NH_4$) and the main results obtained with the mean field theory. The model based on these data will be presented in Sect. III. The Sect. IV is devoted to the RNG formulation using results from Ref. [10] in order to calculate the low temperature quantum critical properties of the system. Sec. V will present the main results of the paper, the relevant thermodynamic quantities next to the quantum critical point. Finally, in the last section we will discuss the results and compare with other theoretical approaches and experimental evidence.

II. EXPERIMENTAL EVIDENCES

In this section we present the basic experimental data obtained on the $X CuCl_3$ compounds which gave us the possibility to elaborate a model for the critical behavior of this QPT as a BEC driven by the magnetic field.

The compound which offered the possibility of performing many measurements is
$TlCuCl_3$ which is composed of a chemical double chain $Cu_2Cl_6$. The magnetic susceptibility, measured as function of temperature $T$ for three different directions of the magnetic field, exhibit a broad maxima at $T = 38K$, decreasing to zero with the temperature decreasing $[11]$. This result indicates that the compound has an excitation gap $\Delta/k_B = 7.5K$ above the singlet ground state. This gap may by attributed to the antiferromagnetic dimer coupling in the double chain $[12]$. The magnon dispersion has been investigated theoretically $[13]$ and experimentally by inelastic neutron scattering $[14]$. The experiments showed that the magnon modes are split into three by magnetic field with the splittings proportional to the field, and the lower modes becomes soft at the critical field $H_c$. In a magnetic field $H$ the chemical potential is $\Delta - g\mu_B H$ (where $g = 2$) and $H_c = \Delta/\mu_B$.

An important result has been obtained by Sherman et al. $[15]$ by sound attenuation in $TlCuCl_3$ in magnetic field at low temperatures. The occurrence of the sharp peak in the sound attenuation near the BEC critical temperature and the Drude form of the sound dumping suggested a constant weak magnon-magnon coupling.

The sound attenuation suggested a magnons dispersion like $\omega^2 = \Delta^2 + J^2k^2$ (here $J$ is the exchange interaction) for small wave vector $"k"$, approximation valid if $T_c$ and $\Delta$ are of the order of $k_B T_c = 7K$ and $k_B \Delta = 7.5$ as for the case of $TlCuCl_3$.

The existence of a induced magnetic field-spin transition has been proposed by Giamarchi and Tsvelik $[16]$ considering as a possible mechanism the BEC of the soft mode. The Hartree-Fock approximation using the Hamiltonian from the theory of BEC in dilute bosonic gases with effective chemical potential $\mu = \mu_B g(H - H_c)$ has been applied $[17]$ to calculate the temperature dependence of magnetization. Near the critical temperature $T_c$ this approximation breaks down and the magnetization is expected to behave like $M \sim (T - T_c)^\beta$ where $\beta = \frac{3}{2}$. The temperature dependence of the critical field has the form $[H_c(T) - H_c(0)] \sim T^\Phi$ where $\Phi = \frac{3}{2}$ but the best fit is obtained for $\Phi = 2.2$.

The system has some important characteristics. First, we mentions the small three-dimensional (3D) character of the system which leads to a small gap. Another important feature is the decreasing the number of thermal excitations with the increasing of the magnetic field, and as a consequence the driving the condensate to $T = 0$ behavior.

The form of the magnon dispersion (called "relativistic") changes the dynamics of the system and we expect $\Phi = 2$. This conjecture is on of the most important result of Ref. $[15]$ and we will use it for the model of the magnon condensation.
III. MICROSCOPIC MODEL

We introduce the description of the magnon condensation in magnetic field using the action:

\[ S_{\text{eff}} = S_{\text{eff}}^{(2)} + S_{\text{eff}}^{(4)} \] 

where:

\[ S_{\text{eff}}^{(2)} = \frac{1}{2} \sum_{k} \chi^{-1}(k)|\phi(k)|^2 \]  

\[ S_{\text{eff}}^{(4)} = \frac{u_0}{16} \sum_{k_1} \ldots \sum_{k_4} \phi(k_1) \ldots \phi(k_4) \delta(k_1 + \ldots + k_4) \] 

Here we introduced the notations \( k = (k, \omega_n) \), \( \omega_n \) being the bosonic Matsubara frequency and

\[ \sum_{k} = T \sum_{\omega_n} \int \frac{d^d k}{(2\pi)^d} \]

In Eq.\( (2) \) \( \chi(k) \) is the magnon propagator and \( u_0 \) the bare coupling constant. Using \[15\] we write the magnon propagator as

\[ \chi(k, \omega_n) = \frac{1}{\omega_n^2 + k^2 + r_0} \]  

where \( r_0 = \Delta - \mu_B g H \) and \( H \) is the external magnetic field.

IV. RENORMALIZATION GROUP EQUATIONS

The model has the dynamical critical exponent \( z = 1 \) and for this model \( d = 3 \). The Renormalization group equations in one-loop approximation at finite temperature \( T \) for \( d = 3 \) have the form \[10\]:

\[ \frac{dT(l)}{dl} = T(l) \]  

\[ \frac{du(l)}{dl} = -\frac{(n + 8)K_3}{8} u^2(l) F_1[r(l), T(l)] \]  

\[ \frac{dr(l)}{dl} = 2r(l) + \frac{(n + 2)K_3}{8} u(l) F_2[r(l), T(l)] \] 

where \( n \) is the number of components of the fluctuation field \( \Phi(k) \), \( K_3 = \frac{1}{2\pi^2} \). We denote by \( u(l = 0) = u_0, r(l = 0) = r_0 \) and the functions \( F_{1,2} \) are characteristic functions for the model \[10\]. We approximate \( F_2[r(l), T(l)] \simeq \frac{1}{4} \) in the low temperatures limit and the solution of Eq. \[5\] has the form:

\[ u(l) = \frac{1}{C_0(l + l_0)} \]
where $C_0 = \frac{(n+8)K_3}{16}$ and $K_3 = \frac{1}{2\pi^2}$. The physics near the QCP is described by the scaling field $t_r(l)$ defined as:

$$t_r(l) = r(l) + \frac{n+2}{16}K_d u(l)$$ (9)

The expression of $t_r(l)$ has been calculated in Ref. [10] for $d=3$ as:

$$t_r(l) = e^{\Lambda_r(l)} \left\{ t_r(0) + \frac{n+2}{4}K_d \int_0^l dx \frac{e^{-2xu(x)}}{e^{1/T(x)} - 1} \right\}$$ (10)

where $\Lambda_r(l)$ has the expression:

$$\Lambda_r(l) = 2l - \frac{n+2}{n+8} \ln \left( \frac{l}{l_0} + 1 \right)$$ (11)

Using these results we will calculate the thermodynamic quantities in near the critical point. The basic idea of the method is to stop the renormalization procedure close enough to this point that the system can see the influence of the quantum effects. This matching condition is in fact equivalent to the stopping of the renormalization at the scale $l = l^*$, where $l^* \gg 1$. This value is obtained from the condition [10]:

$$t_r(l^*) = 1$$ (12)

In the approximation $l^* > 1$ and $T(l^*) > 1$ we obtain from Eq.(11) the relation:

$$\exp(l^*) = [t_{r_0}(T)]^{\lambda_r}$$ (13)

where $\lambda_r$ is the eigenvalue of the relevant parameter $r$ and is given by the expression:

$$\lambda_r = 2, \epsilon = 0 \ (d = 3)$$ (14)

and $t_{r_0}(T)$ is given by the relation [10]:

$$t_{r_0}(T) = r_0 - r_{0c} + \frac{n+2}{128}u_0 T^2$$ (15)

Following the method from [7, 8] we calculate $l^*$ as:

$$l^* = \frac{1}{2} \ln \left( \frac{1}{T} \right)$$ (16)
V. THERMODYNAMIC QUANTITIES

Using these results we will calculate the relevant thermodynamic quantities near the quantum critical point, but in the disordered state. First, we define the critical line by \( t(T) = 0 \) and we get:

\[
H_c(T) - H_c(0) = C_0 T^2
\]

where \( C_0 \propto u_0 \) and \( H_c = \frac{\Delta}{\mu_B} \). The temperature dependence of the number of magnons is defined as:

\[
n(T) = \exp(-3l^*) \int \frac{d^3k}{(2\pi)^3} f_B[T(l^*)]
\]

This equation gives for \( n(T) \) a temperature dependence of the form:

\[
n(T) \propto T^{3/2}
\]

This result is in agreement with the behavior of magnetization at very low temperatures.

From Eq. (15) we calculate the critical line in the \((r_0, T)\) plane using the condition \( t_{r_0}(T) = 0 \). This gives:

\[
r_{0c}(T) = r_{0c} - \frac{(n + 2)}{128} u_0 T^2
\]

and for \( r_0 \leq r_{0c} \) we get the general equation

\[
T_c(r_0) = \left[ \frac{128}{(n + 2) u_0} \right]^{1/2} (r_{0c} - r_0)^{1/2}
\]

Using now the definition (see Ref. \[10\]) of \( r_{0c} \)

\[
r_{0c} = -\frac{(n + 2)}{32\pi^2} u_0
\]

we obtain

\[
T_c(H) \sim |H - \tilde{H}_c|^{1/\alpha}
\]

with \( \alpha = 2 \) and

\[
\tilde{H}_c = H_c + \frac{(n + 2)}{32\pi^2} u_0
\]

The specific heat can be also calculated for the important case \( r_0 \neq r_{0c} \) but \( T \to T_c^+(r_0) \) using the singular part of the free energy reported in \[10\] as:

\[
F_s(T) \sim [T - T_c(H)]^2 \left| \ln(T - T_c(H)) \right|^{[2(n+2)]}/[n+3]
\]
and it gives us a logarithmic behavior:

\[
\frac{C_v(T)}{T} \sim |\ln(T - T_c(H))|^{[-2\frac{n+2}{n+8}]}
\] (25)

We will discuss these results in connection with the existent theoretical approaches and the experimental data for \textit{TlCuCl}_3.

VI. DISCUSSIONS

The occurrence of BEC induced by the magnetic field in a spin-gap system has been predicted by Giamarki and Tsvelik \cite{16} in order to explain the three-dimensional ordering in coupled ladders. Nikumi \textit{et al.} \cite{17} applied Popov theory (see for example \cite{18}) for the BEC obtaining a temperature dependence for magnetization in agreement with the experimental data, of the form \(T^{3/2}\).

However, as it was mentioned in Ref. \cite{17} the critical exponent \(\Phi\) of the critical field \([H_c(T) - H_c(0)] \sim T^\Phi\) was obtained as \(\Phi = 3/2\), but the experimental data shows \(\Phi = 2.2\).

Recently Misguich and Oshikawa \cite{19} reconsidered the self-consistent Hartree-Fock-Popov (HPF) method using a realistic dispersion for magnons \cite{13} and they reobtained \(\Phi = 3/2\). They also calculated the specific heat and obtained a \(\lambda\)-shape.

We would like to mention several basic points related to this problem:

- in the HPF the effect of fluctuations have been neglected and it appears normal that thermodynamic quantities on the critical line have wrong critical exponents.
- the changing of the dispersion law for the magnon can improve the HPF results, but does not affect the basic approach.
- both papers \cite{17} and \cite{18} considered the QPT as driven by the temperature and in fact it is induced by the magnetic field.

These observations justify the application of the RNG method in the version proposed by Caramico \textit{et al.} \cite{10} to study the magnon condensation. Our main result can be summarized as:

- we obtained the critical exponent \(\Phi = 2\), which is in a better agreement with experimental data (\(\Phi = 2.2\)).
• the magnetization calculated in our approach is also $T^{3/2}$ dependent.

• the critical temperature $T_c(H)$ has a dependence given by Eq. [23] with $\alpha = 2$ in agreement with [12]. The numerical results have been analyzed in [20].

• the specific heat calculated by RNG method has a logarithmic correction as was expected. The $T$ dependence of $C_v$ is not relevant if we take the approach of a magnetic field driven QPT, but we have it to the dependence of $H(T)$. Such an experiment has been done [12] and we can notice a small anomaly in this dependence. But a final decision about the $H(T)$ dependence of our Eq.(25) cannot be taken.

The agreement between the experimental data and the theory presented in [19] appears in our opinion due to the fact that the authors considered, on a very narrow interval a linear dependence between the critical field $H_c(T)$ and the critical density $n_c$. This approximation can be valid for $n_c < 0.002$ which describes a part of the real temperature dependence of $H_c(T)$.

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