On the derived invariance of cohomology theories for coalgebras

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Abstract

We study the derived invariance of the cohomology theories \( \text{Hoch}^* \), \( H^* \) and \( HC^* \) associated to coalgebras over a field. We prove a theorem characterizing derived equivalences. As particular cases, it describes the two following situations: 1) \( f : C \to D \) a quasi-isomorphism of differential graded coalgebras, 2) the existence of a “cotilting” bicomodule \( cT_D \). In these two cases we construct a derived-Morita equivalence context, and consequently we obtain isomorphisms \( \text{Hoch}^*(C) \cong \text{Hoch}^*(D) \) and \( H^*(C) \cong H^*(D) \). Moreover, when we have a coassociative map inducing an isomorphism \( H^*(C) \cong H^*(D) \) (for example when there is a quasi-isomorphism \( f : C \to D \)), we prove that \( HC^*(C) \cong HC^*(D) \).

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Introduction

This work is devoted to the study of cohomology theories for coalgebras and their relations with the derived categories associated to coalgebras. These cohomology theories are called \( \text{Hoch}^* \), \( H^* \) (defined by Y. Doi in [1]) and \( HC^* \) (defined by A. Solotar and myself in [5]). They are important categorical invariants, for instance \( H^* \) and \( \text{Hoch} \) measure coseparability, and \( H^* \) measures extensions, and hence it is a useful object when studying classification problems. Also in some examples they have a nice geometric interpretation: in [3] we prove that the (topological) coalgebra \( D(X) \) of distributions over a real compact smooth manifold \( X \) has the \( n \)-currents (i.e. the continuous dual of the \( n \)-differential forms) as the \( n \)-th \( \text{Hoch} \) group; we also show in [3], that the cyclic cohomology of \( D(X) \) can be computed in terms of the De Rahm cohomology of the manifold \( X \).

The notion of coalgebra in a monoidal category \( (\mathcal{C}, \otimes) \) coincides with the concept of algebra in the category \( (\mathcal{C}^{op}, \otimes) \), so under this duality, one can translate "algebraic" statements in order to get "coalgebraic" statements and vice versa. But, even if the notion of (abelian and) monoidal category is selfdual, in practice, a category \( \mathcal{C} \) is

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usually far from being equivalent to $C^{op}$. If the category $C$ is fixed, simple dualization of “algebraic” proofs needs not lead to “coalgebraic” proofs, because one wants theorems to hold in $C$ and not in $C^{op}$. The most familiar example of this situation is $k\text{-}Vect$, the category of vector spaces over some fixed ground field $k$, which is not equivalent to its opposite category (it satisfies Grothendieck’s axiom AB5 and it is well known that if an abelian category satisfies AB5 and AB5* then every object is the zero object).

One of the guiding lines of this work is the following informal idea: “the derived category associated to a coalgebra keeps the homological relevant information, so if two coalgebras have equivalent derived categories, they should have isomorphic cohomology”. A precise statement of this idea is Theorem 5.6, together with our two main examples: quasi-isomorphisms (Proposition 4.1) and cotilting bimodules (Theorem 3.2). The language of differential graded objects appears naturally in the context of derived categories, so we treat with differential graded coalgebras, even also when we are looking for results on usual coalgebras.

We remark that, for a fixed field $k$, the category of differential $\mathbb{Z}$-graded $k$-vector spaces is isomorphic (as monoidal category) to a category of comodules over a Hopf algebra (see the first part of section 1), an so it is not equivalent to its opposite category (same reasons that $k\text{-}Vect \not\cong k\text{-}Vect^{op}$).

The organization of this work is the following:

In Section 1 we introduce the notion of a differential graded coalgebra $C$ and the category of differential graded comodules, denoted by $\text{Chain}(C)$. This category has a natural notion of homotopy, so we define the homotopy category $\mathcal{H}(C)$ and its corresponding localization by quasi-isomorphisms $\mathcal{D}(C)$. We define the notion of closed object, which plays the role of injective resolution. The main result of this section is Theorem 1.4, which states that, when $C$ is positively graded, and we restrict to the category $\text{Chain}^+(C)$ of left-bounded differential graded comodules, then the standard resolution $C(M)$ is a closed object, quasi-isomorphic to $M$.

Section 2 deals with derived equivalences. We begin with a general characterization of equivalences given by Proposition 2.8 and we prove Theorem 2.1. It characterizes equivalences induced by derived cotensor products.

If we specialize Theorem 2.1 to the case of usual coalgebras and comodules, the hypothesis lead us to the notion of cotilting bimodule. In Section 3 we define the notion of cotilting bimodule and prove that they induce equivalences of derived categories (Theorem 3.3), and moreover, that they induce derived Morita contexts (Theorem 3.4).

On the other side, concerning non trivial differential graded structures, we prove in Section 4 that if $f : C \to D$ is a quasi-isomorphism of positively graded differential coalgebras, then their derived categories are equivalent, and they fit into a derived Morita context.

Finally Section 5 is devoted to the extension of the definition of $\text{Hoch}^*$, $\text{H}^*$ and $\text{HC}^*$ to the differential graded case extending some properties of the non differential graded case (Proposition 5.2). We prove Theorem 5.6. This result implies the invariance of $\text{Hoch}^*$ and $\text{H}^*$ under quasi-isomorphisms and cotilting equivalences, and Proposition 5.7, which implies the invariance of $\text{HC}^*$ under quasi-isomorphisms.

We mention that the dualization of the notion of tilting modules was introduced in this work with the aim of understanding all possible (or at least most of the) derived equivalences for usual coalgebras. We wanted to answer the following question: given two $k$-coalgebras $C$ and $D$ such that $D^+(C) \cong D^+(D)$ as triangulated categories, is it true that there exists an equivalence (may be another) given by a derived cotensor product? One may view Proposition 2.8 and Theorem 3.2 as partial results in this direction.

On the other hand, the example of quasi-isomorphisms of differential graded coalgebras has a computational motivation: if $C$ is a subcoalgebra of $C'$, one can try to find a differential graded coalgebra of the form $C' \otimes \Lambda$ where $\Lambda_0 = k$ and a quasi-isomorphism $C \to C' \otimes \Lambda$, in other words, find a positively graded coalgebra $\Lambda$ and a differential $d$ on $C' \otimes \Lambda$ such that the cohomology of $(C' \otimes \Lambda, d)$ is zero for positive degrees and the kernel of $d_0$ is $C$. In this situation we can interpret $(C' \otimes \Lambda, d)$ as a “model” for $C$, and Theorem 5.6 and Proposition 5.7 tell us that, in order to compute $\text{H}^*$, $\text{Hoch}^*$ or $\text{HC}^*$, one can replace the coalgebra $C$ by its differential graded model.

Throughout this paper $k$ will denote a field of arbitrary characteristic, all unadorned tensor products will be taken over $k$.

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1 A triangulated category associated to a differential graded coalgebra

1.1 Differential graded coalgebras and differential graded comodules

The category of differential \( \mathbb{Z} \)-graded \( k \)-vector spaces is a monoidal category with \( \otimes_k \) as tensor product, so the notion of coalgebra in this category makes sense. By definition, a **differential graded \( k \)-coalgebra** is a coalgebra in the monoidal category of differential graded \( k \)-vector spaces, namely, it is the data \((C, \Delta)\) where \( C = \bigoplus_{n \in \mathbb{Z}} C_n, d\) is a differential graded vector space, \( d_n : C_n \to C_{n+1} \), and

\[
\Delta : C \to C \otimes C
\]

is a coassociative map (in the category of differential graded \( k \)-vector spaces) admitting a counit, i.e. \( \Delta(C_n) \subset (C \otimes C)_n = \bigoplus_{p+q=n} C_p \otimes C_q \) and \( \Delta d = d \Delta \) where \( d \Delta \) is defined on homogeneous elements by:

\[
d(c_1 \otimes c_2) := d(c_1) \otimes c_2 + (-1)^{|c_1|} c_1 \otimes d(c_2)
\]

It is well-known that, for any group \( G \), the category of \( G \)-graded vector spaces is isomorphic to the category of \( k[G] \)-comodules as monoidal categories (with the diagonal structure in the tensor product), where \( k[G] \) denotes the group-algebra. Also, the category of vector spaces together with an square zero endomorphism is isomorphic to the category of \( k[d]/(d^2) \)-modules. From these two remarks, the following is a natural way of describing the category of differential graded \( k \)-vector spaces as the category of comodules over a Hopf algebra:

Let \( H \) denote the \( k \)-algebra generated by \( X, X^{-1}, d \) with relations

\[
XX^{-1} = X^{-1}X = 1
\]

\[
d^2 = 0
\]

\[
Xd +Xd = 0
\]

Then, \( H \) admits a Hopf algebra structure determined by

\[
\Delta(X) = X \otimes X
\]

\[
\Delta(d) = 1 \otimes d + d \otimes X
\]

The counit is given by evaluation of \( X \) in 1 and \( d \) in 0, the antipode by \( S(X) = X^{-1} \) and \( S(d) = X^{-1}d = -dX^{-1} \).

Now the definition of differential graded coalgebra can be given in terms of \( H \), because a coalgebra in \( H \) is the same as an \( H \)-comodule coalgebra. The natural corepresentations of a differential graded coalgebra \( C \) are the so-called differential graded comodules (i.e. \( C \)-comodules which are differential graded vector spaces and the structure map is a map of differential graded vector spaces). This category can be identified with \( C \otimes \mathbb{C} \mathcal{M} \cong \mathbb{C} \mathcal{M} \), where \( C \# H \) is the smash product of \( C \) with \( H \), and the equivalence \( C \otimes \mathbb{C} \mathcal{M} \cong \mathbb{C} \mathcal{M} \) preserves the underlying vector space and it is the identity on the arrows.

As a corollary of this simple remark, we have that the category of differential graded left comodules over a differential graded coalgebra \( C \), which will be denoted by \( \text{Chain}(C) \), is a comodule-category (over a field). Therefore it is a Grothendieck category, in particular it has arbitrary sums and products, projective limits, filtered inductive limits are exact, and each object is the union of its finite dimensional subobjects.

1.2 The categories \( \mathcal{H}(C) \) and \( \mathcal{D}(C) \)

The category \( \text{Chain}(C) \) has a natural notion of homotopy. We will say that two maps \( f, g : M \to N \) in \( \text{Hom}_{\text{Ch}(C)}(M, N) \) are **homotopic** and we will write \( f \sim g \), in case that there exists a graded \( C \)-colinear map \( h : M \to N[1] \) such that \( f - g = h d_M + d_N h \). As usual, this is an equivalence relation in \( \text{Hom}_{\text{Ch}(C)}(M, N) \), compatible with addition and composition, so we can define the quotient category \( \mathcal{H}(C) \). By definition, the objects of \( \mathcal{H}(C) \) are the same as the objects of \( \text{Chain}(C) \), and the maps are homotopy classes of maps, i.e. \( \text{Hom}_{\mathcal{H}(C)}(M, N) = \text{Hom}_{\text{Ch}(C)}(M, N)/\sim \).
Let’s denote by $C^*$ the graded dual of $C$; $C^*$ is a differential graded algebra. The category $\text{Chain}(C)$ embeds in $\text{Chain}(C^*)$, the category of differential graded (right) $C^*$-modules. The notion of a Cone of a morphism can be defined as the Cone in $\text{Chain}(C^*)$, namely, if $f \in \text{Hom}_{\text{Ch}(C)}(M, N)$, $\text{Co}(f) \in \text{Chain}(C)$ is defined by:

$$\text{Co}(f)_n := M[-1]_n \oplus N_n$$

$$d_{\text{Co}(f)} := d_{M[-1]} + f + d_N$$

The cone of the identity map is not the zero object, but homotopically equivalent to it, so $\text{Chain}(C)$ is never a triangulated category (with cones as distinguished triangles). The additive category $\mathcal{H}(C)$ is triangulated, taking as distinguished triangles the uples isomorphic to

$$(M, N, \text{Co}(f), f : M \rightarrow N, \pi : \text{Co}(f) \rightarrow M[-1], f[-1] : M[-1] \rightarrow N[-1])$$

The embedding $\mathcal{H}(C) \rightarrow \mathcal{H}(C^*)$ is an embedding of triangulated categories.

The ‘derived’ category associated to $C$, denoted by $\mathcal{D}(C)$, is the localization of $\mathcal{H}(C)$ by the class of quasi-isomorphisms (i.e. morphisms in $\mathcal{H}(C)$ inducing an isomorphism in homology). The category $\mathcal{D}(C)$ is also a triangulated category with the triangulated structure induced by the localization functor and the structure of $\mathcal{H}(C)$.

As usual, we will denote $\text{Chain}^{+, -b}(C)$, $\mathcal{H}^{+, -b}(C)$ and $\mathcal{D}^{+, -b}(C)$ the full subcategories of $\text{Chain}(C)$, $\mathcal{H}(C)$ and $\mathcal{D}(C)$ consisting on objects which are isomorphic to left (resp. right, both sided) bounded complexes.

We will use the following Lemma of triangulated categories whose proof is obtained adapting that of [7] (Lemma 13, chap. IV, §1) to the differential graded case. The only thing to check is that every morphism defined in the proof of their Lemma belongs to the category $\text{Chain}(C)$.

**Lemma 1.1** Let $C$ be a differential graded coalgebra and

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

a short exact sequence in $\text{Chain}(C)$ which splits as a sequence of graded $C$-comodules. Then the sequence fits into a triangle $X \rightarrow Y \rightarrow Z \rightarrow X[-1] \rightarrow \ldots$

1.3 **Characterization of $\mathcal{D}^+(C)$**

From now on, $C$ will denote a differential graded $k$-coalgebra.

**Definition 1.2** We will say that an object $X \in \text{Chain}(C)$ is **closed** if, for all $M \in \text{Chain}(C)$, the localization functor gives an isomorphism

$$\text{Hom}_{\mathcal{H}(C)}(M, X) \cong \text{Hom}_{\mathcal{D}(C)}(M, X)$$

As a consequence of the definition, being closed is stable under products, traslation, and (by a five Lemma argument) triangles, namely if two of the three objects in a triangle are closed objects, then so is the third one.

We will see later that there exist enough closed objects, at least when $C$ is positively graded and we restrict ourselves to the subcategory $\text{Chain}^+(C)$. We begin with an adjunction:

**Proposition 1.3** Let $V \in \text{Chain}(k)$, $M \in \text{Chain}(C)$, and consider $C \otimes V \in \text{Chain}(C)$ with the structure of $C$ comodule given by $C$ and the standard graded differential structure of the tensor product, then

$$\text{Hom}_{\mathcal{H}(C)}(M, C \otimes V) \cong \text{Hom}_{\mathcal{H}(k)}(M, V) = \text{Hom}_{\mathcal{D}(k)}(M, V) \cong \text{Hom}_{\mathcal{D}(C)}(M, C \otimes V)$$

**Proof:** The well-known adjunction in the standard comodule context can be carried out in the differential graded case, defining maps

$$\text{Hom}_{\text{Ch}(C)}(M, C \otimes V) \rightarrow \text{Hom}_{\text{Ch}(k)}(M, V)

f \mapsto (\epsilon \otimes \text{id}_V) \circ f$$
and

\[ \text{Hom}_{\mathcal{C}(k)}(M, V) \to \text{Hom}_{\mathcal{C}(k)}(M, C \otimes V) \]
\[ g \mapsto (id \otimes g) \circ \rho_M \]

where \( \rho_M : M \to C \otimes M \) denotes the structure map of \( M \).

If \( f, f' : M \to C \otimes V \) are homotopic maps by means of an homotopy \( h \), then it is easily checked that \( (\epsilon \otimes id)h \) is an homotopy between \( (\epsilon \otimes id)f \) and \( (\epsilon \otimes id)f' \) (one uses that \( \epsilon \) is necessarily a graded morphism and that \( \epsilon \circ d = 0 \)). Conversely, if \( g, g' : M \to V \) are two homotopic maps, let’s call again \( h \) the homotopy between \( g \) and \( g' \), then \( (id_C \otimes h) \circ \rho_M \) gives the desired homotopy between \( (id_C \otimes h) \circ g \) and \( (id_C \otimes h) \circ g' \), and the same formula as above gives a well-defined isomorphism

\[ \text{Hom}_{\mathcal{H}(k)}(M, C \otimes V) \cong \text{Hom}_{\mathcal{H}(k)}(M, V) \]

Now if \( f : M \to M' \) is a quasi-isomorphism of differential graded \( C \)-comodules, then obviously the forgetful functor gives a quasi-isomorphism, and considering the other variable of the \( \text{Hom} \), if \( \phi : V \to V' \) is a quasi-isomorphism of differential graded vector spaces, then by the Künneth formula \( id_C \otimes \phi : C \otimes V \to C \otimes V' \) is also a quasi-isomorphism. As a consequence, the adjointness remains valid after localization giving

\[ \text{Hom}_{\mathcal{D}(k)}(M, C \otimes V) \cong \text{Hom}_{\mathcal{D}(k)}(M, V) \]

Finally, we remark that every quasi-isomorphism in \( \text{Chain}(k) \) is an homotopy equivalence, then \( \mathcal{D}(k) = \mathcal{H}(k) \).

As a consequence of the above proposition, for every differential graded vector space \( V \), the object \( C \otimes V \in \text{Chain}(C) \) is closed.

We will state a theorem which characterizes the category \( \mathcal{D}^+(C) \) for positively graded differential coalgebras:

**Theorem 1.4** Let \( C \) be a positively graded differential coalgebra and let \( M \) be an object in \( \text{Chain}^+(C) \). Then, there is a functorial way of assigning to \( M \) a closed object \( C(M) \) and a quasi-isomorphism \( M \to C(M) \); the object \( C(M) \) also belongs to \( \text{Chain}^+(C) \).

As a consequence of this theorem, the category \( \mathcal{D}^+(C) \) can be described as the full subcategory of \( \mathcal{H}^+(C) \) consisting of closed objects. The proof of the theorem is achieved by exposing an explicit standard resolution, and proving that this resolution is a closed object. In order to do that, we need two Lemmas. We begin by defining the standard resolution:

Let \( M \in \text{Chain}(C) \) where \( C \) is any differential graded coalgebra, we define the object \( C(M) \in \text{Chain}(C) \) by \( C(M) := \oplus_{n \geq 0} C^\otimes n \otimes M \) with the following graduation and differential:

\[ C(M)_p = \bigoplus_{i_1 + \ldots + i_r + j + r - 1 = p} C_{i_1} \otimes \ldots \otimes C_{i_r} \otimes M_j \]

and, for \( (c_{i_1}, \ldots, c_{i_r}, m) \in C_{i_1} \otimes \ldots \otimes C_{i_r} \otimes M \),

\[ \partial(c_{i_1}, \ldots, c_{i_r}, m) = (-1)^r \sum_{k=1}^{r+1} (-1)^{|c_{i_1}| + \ldots + |c_{i_{k-1}}|} (c_{i_1}, \ldots, d(c_{i_k}), \ldots, c_{i_r}, m) + \]
\[ + \sum_{k=1}^{r+1} (-1)^{k+1} (c_{i_1}, \ldots, \Delta(c_{i_k}), \ldots, c_{i_r}, m) \]

where we use the convention \( c_{i_{r+1}} = m \), and \( \Delta(m) = \rho_M(m) \). In an abridged way, we write

\[ \partial(c_{i_1}, \ldots, c_{i_r}, m) = d(c_{i_1}, \ldots, c_{i_r}, m) + b(c_{i_1}, \ldots, c_{i_r}, m) \]
Lemma 1.5 With the above notations, $\partial^2 = 0$, and the structure morphism $\rho_M : M \to C \otimes M \subseteq C(M)$ is a quasi-isomorphism in $\text{Chain}(C)$.

**Proof:** Let us define the extended complex $\hat{C}(M) = (\bigoplus_{n \geq 0} C^\otimes n \otimes M, b', d)$ with differential given by the same formulae as in $C(M)$ and the $k$-linear map $h : C(M) \to C(M)$ by

$$h(c_1, \ldots, c_r, m) = \epsilon(c_1)(c_2, \ldots, c_r, m) \quad \text{for } n \geq 1$$

and $h(m) = 0$ for $m \in M$.

It is well known that $b'^2 = 0, d'^2 = 0$ and that $b'h + hb' = id$, then it is sufficient to see that $b'd + db' = 0 = dh + hd$. Since $\hat{C}(M)$ is the cone of the map $\rho_M : M \to C(M)$, the equality $b'd + db' = 0$ says that $\partial$ is in fact a square zero operator, and the equality $0 = dh + hd$ means that $\partial h + h\partial = id$, which implies that $\hat{C}(M)$ is acyclic, or equivalently that $\rho_M : M \to C(M)$ is a quasi-isomorphism.

We omit the computation of $b'd + db'$ and $dh + hd$, to see that they equal zero is tedious but straightforward.

It is clear that the assignment $M \mapsto C(M)$ is functorial, and by the previous Lemma, $M$ is quasi-isomorphic to $(C(M), \partial)$. We do not know if $C(M)$ is a closed object in general, but we succeeded to prove it (as it was enounced in Theorem [4]) for $C$ positively graded and $M$ bounded below, i.e. for $M$ and object in $\text{Chain}^+(C)$. The proof of this result follows from the following Lemma:

Lemma 1.6 Let $\{M_n\}_{n \in \mathbb{N}}$ be an inverse system in $\text{Chain}(C)$ with morphisms $p_{n+1} : M_{n+1} \to M_n$ which are, as morphisms of graded $C$-comodules, split epimorphisms. Then, the short exact sequence

$$0 \to \lim_{\leftarrow n} M_n \to \prod_{n \in \mathbb{N}} M_n \to \prod_{n \in \mathbb{N}} M_n \to 0$$

fits into a triangle in the category $\mathcal{H}(C)$ where the products (and the inverse limit) are taken in the category $\text{Chain}(C)$, the map between the products is

$$\{m_n\}_{n \in \mathbb{N}} \mapsto \{m_n - p_{n+1}(m_{n+1})\}_{n \in \mathbb{N}}$$

**Proof:** let us call $s_n : M_n \to M_{n+1}$ the splittings of the maps $p_n$, and define $s : \prod_n M_n \to \prod_n M_n$ by

$$s(m_0, m_1, \ldots, m_n, \ldots) = (m_0, 0, -s_1(m_1), -s_2(s_1(m_1)), \ldots, -s_3(s_2(s_1(m_1))), \ldots)$$

This proves that the sequence

$$0 \to \lim_{\leftarrow n} M_n \to \prod_{n \in \mathbb{N}} M_n \to \prod_{n \in \mathbb{N}} M_n \to 0$$

splits as a sequence of graded $C$-comodules. Now use Lemma [3] and the proof is completed.

Now consider, given $M \in \text{Chain}^+(C)$, the system $C_n(M) := (\bigoplus_{i=1}^n C^\otimes i \otimes M, b', d)$ with differential induced by the projection $C(M) \to C_n(M)$. For every $n \in \mathbb{N}$ there is a graded $C$-split short exact sequence

$$0 \to C^\otimes n+1 \otimes M \to C_{n+1}(M) \to C_n(M) \to 0$$

which implies two facts:

- When $n = 1$, $C_1(M) = C \otimes M$ is a closed object; for $n > 1$, $C^\otimes n \otimes M = C \otimes (C^\otimes n-1 \otimes M)$ is also a closed object, and hence, the above exact sequence implies inductively that $C_n(M)$ is closed for all $n \in \mathbb{N}$.

- The maps of the system $C_{n+1}(M) \to C_n(M)$, viewed as morphisms of graded $C$-comodules, are split surjections, and the hypothesis of Lemma [3] are fulfilled.
As a consequence, we can conclude that \( C(M) = \lim_{n} C_{n}(M) \) is a closed object, and the proof of Theorem 1.4 is complete.

We remark that if \( C \) has nonzero components in negative degrees, or if \( M \) is a left unbounded complex (i.e., has infinitely many nonzero components of negative degree), then \( C(M) \neq \lim_{n} C_{n}(M) \). In general \( \lim_{n} C_{n}(M) = \text{Tor} \prod_{n} (C^{\otimes n} \otimes M, b', d) \), which is closed (by the same arguments) but unfortunately is not necessarily quasi-isomorphic to \( M \).

Theorem 1.4 gives also a way of defining derived functors for each functor between the homotopy categories. We define in this way the derived cotensor product, noticing that, given \( D \) and \( C \) are differential graded coalgebras, \( X \sqcup C: \text{Chain}(C) \to \text{Chain}(D) \) preserves homotopies, then it defines a functor \( X \sqcup C: \text{H}(C) \to \text{H}(D) \). In order to have a derived cotensor product between \( D^+(C) \) and \( D^+(D) \) we need \( C \) and \( D \) to be positively graded, and \( X \in \text{Chain}^+(D \otimes C^{\text{op}}) \).

Also, following the proof of Theorem 1.4, we have the following corollary:

**Corollary 1.7** Let \( C \) be a positively graded differential coalgebra, \( X \in \text{Chain}^+(C) \) a closed object and \( V \in \text{Chain}^+(k) \), then \( X \otimes V \) is also a closed object.

**Proof:** we know after Theorem 1.4 that, for any object \( M \in \text{Chain}^+(C) \), \( C(M) \) is closed. Since \( \rho_{M}: M \to C(M) \) is always a quasi-isomorphism, we can conclude that \( M \) is closed if and only if \( \rho_{M}: M \to C(M) \) is an homotopy equivalence. Using this characterization, by functoriality of \( C(-) \), assuming that \( \rho_{X}: X \to C(X) \) is an homotopy equivalence it is clear that \( \rho_{X} \otimes \text{id}_{V} = \rho_{X \otimes V}: X \otimes V \to C(X \otimes V) = C(X) \otimes V \) is also an homotopy equivalence, and consequently \( X \otimes V \) is closed.

**Notation:** We will say that a subcategory \( \mathcal{C} \) of \( D^+(C) \) is closed by extensions of objects of \( \text{Chain}^+(k) \) if, whenever \( X \) belongs to \( \mathcal{C} \) and \( V \in \text{Chain}^+(k) \), then \( X \otimes V \) also belongs to \( \mathcal{C} \).

As a particular case, taking \( V = k^{(I)} \) with \( I \) an arbitrary set, the above corollary implies that if \( X \) is closed, then \( X^{(I)} \) is closed. We can also prove the following Lemma:

**Lemma 1.8** Let \( C \) be a positively graded differential coalgebra and \( X \in \text{Chain}^+(C) \) a direct summand of \( Y \in \text{Chain}^+(C) \) with \( Y \) closed. Then \( X \) is closed.

**Proof:** Let us call \( Z \) a complement of \( X \) in \( Y \), i.e. \( X \oplus Z \cong Y \). Then we have a short exact sequence

\[
0 \to X \to ((X \oplus Z) \oplus (X \oplus Z) \oplus \ldots) \to (Z \oplus (X \oplus Z) \oplus \ldots) \to 0
\]

The middle and right term (after rearranging parenthesis) of this short sequence are isomorphic to \( Y^{(\mathbb{N})} \), then they are closed objects. Also the sequence splits in \( \text{Chain}(C) \) (and then it obviously splits as a sequence of graded \( C \)-comodules), then using Lemma 1.4 the sequence fits into a triangle, which proves that \( X \) is closed.

## 2 Derived equivalences

### 2.1 Equivalences induced by cotensor product and inverse limits

The main result of this section is the following Theorem, that can be considered a partial dualization of a result of B. Keller on differential graded algebras.

**Theorem 2.1** Let \( C \) and \( D \) be two positively graded differential coalgebras, \( X \in \text{Chain}(D \otimes C^{\text{op}}) \) with \( C \) and \( X \) bounded complexes. Call \( F := X \sqcup C: \text{D}^+(C) \to \text{D}^+(D) \) and assume that \( X \) is a closed object in \( \text{Chain}(D) \). Then the following are equivalent:

1. \( F \) is an equivalence.
2. (a) For all $n \in \mathbb{Z}$, $F$ induces an isomorphism $\text{Hom}_{\mathcal{D}(C)}(C, C[n]) \cong \text{Hom}_{\mathcal{D}(D)}(D X, D X[n])$

(b) $F$ commutes with arbitrary products.

(c) The smallest triangulated full subcategory of $\mathcal{D}^+(D)$ which is stable under arbitrary products, extensions by objects of $\text{Chain}^+(k)$ and contains $D X$, contains $D D$.

Proof: $1 \Rightarrow 2$). It is clear that (a) and (b) are necessary conditions. If we consider the smallest triangulated subcategory of $\mathcal{D}^+(C)$ which is stable under arbitrary products, extensions by objects of $\text{Chain}^+(k)$ and contains $C C$, then it contains every standard resolution, so it is equivalent to $\mathcal{D}^+(C)$. We recall that the standard resolution $C(M)$ is an inverse limit in $\text{Chain}(C)$ that can be viewed as a part of a triangle, the other two objects in the triangle are products of $C_n(M)$. Each $C_n(M)$ can be constructed inductively with triangles by “adding” objects like $C \otimes (C^\otimes n-1 \otimes M)$, and all this constructions involve products, triangles, and extensions of $C$ by objects of $\text{Chain}^+(k)$.

Since $F$ commutes with all the operations mentioned above, and $F(C) = D X$, it follows that the smallest triangulated subcategory of $\mathcal{D}^+(D)$ which is stable under arbitrary products, extensions by objects of $\text{Chain}^+(k)$ and contains $D X$ is equivalent to $\mathcal{D}^+(D)$, in particular it contains $D$.

$2 \Rightarrow 1$). By the same arguments of the above paragraph, (b) + (c) implies that $F$ is quasi-surjective, because (c) states that $D D \cong F(Y)$ for some $Y \in \mathcal{D}^+(C)$. Then, given $N \in \mathcal{D}^+(D)$, its standard resolution $D(N) = \text{Tot}(\oplus_{n \geq 1} D(C^\otimes n \otimes N, b', d))$ is isomorphic to some object in the image of $F$, since the image of $F$ is stable under products, triangles, extension by objects of $\text{Chain}^+(k)$ and contains $D$.

The hardest part is to see that $F$ is fully faithful. We will need the notion of a certain kind of inverse limits, and the corresponding behaviour of these limits with respect to the $\text{Hom}$ functor. The proof of $2 \Rightarrow 1$) will finish after Lemma 2.4 and 2.5.

Let $M \in \text{Chain}^+(C)$ and call $M_{\leq n}$ the truncated complex at degree $n$, we have that $M = \lim_{\leftarrow n} M_{\leq n}$ in $\text{Chain}(k)$, and there is, for all $X \in \text{Chain}(k)$, a natural injective morphism

$$\text{Hom}_{\text{Chain}(k)}(M, X) \hookrightarrow \lim_{\leftarrow n} \text{Hom}_{\text{Chain}(k)}(M_{\leq n}, X)$$

In case that $X_p = 0$ for $p \gg 0$, the above morphism is clearly an isomorphism.

The problem is that in general, $M_{\leq n}$ is not an object of $\text{Chain}(C)$ (unless for example when $C$ is a concentrated coalgebra). So we will define a class of inverse limits satisfying a kind of Mittag-Leffler condition, with analogous properties with respect to the $\text{Hom}$ functor, but in the category $\text{Chain}(C)$:

**Definition 2.2** Let $\{U^{n+1} \rightarrow U^n\}_{n \in \mathbb{N}}$ be an inverse system in $\text{Chain}(C)$. We call the system **locally finite** if, given $j \in \mathbb{N}$, there exists $n_0(j)$ such that the map $U^{n+1}_{\leq j} \rightarrow U^n_{\leq j}$ is the identity for all $n \geq n_0(j)$.

**Example:** Let $C$ be a positively graded differential coalgebra and $M \in \text{Chain}^+(C)$, then $\{C_n(M)\}_{n \geq 1} = \{\oplus_{j=1}^n C^\otimes j \otimes M, b', d')\}_{n \geq 1}$ is a locally finite inverse system.

**Lemma 2.3** Let $\{U^n\}_{n \in \mathbb{N}}$ be a locally finite inverse system in $\text{Chain}^+(C)$ (i.e. $U^n \in \text{Chain}^+(C)$ and $U := \lim_{\leftarrow n} U^n \in \text{Chain}^+(C)$) and let $W \in \text{Chain}^+(C)$. Then there is a canonical isomorphism

$$\text{Hom}_{\text{Chain}(C)}(U, W) \cong \lim_{\rightarrow n} \text{Hom}_{\text{Chain}(C)}(U^n, W)$$

**Proof:** Let $P \in \mathbb{N}$ be a number such that $W_p = 0 \forall |p| \geq P$. Since $\{U^n\}_{n \geq 1}$ is a locally finite inverse system, there exists $m_0(P)$ such that the maps $(U^{n+1})_{\leq m} \rightarrow (U^n)_{\leq m}$ are the identity for all $m \geq m_0(P)$. We remark that if $f : U^n \rightarrow W$ is a map in $\text{Chain}(C)$ and $f_p$ denotes the $p$-th component of $f$, then $f_p = 0 \forall |p| > P$. This remark, together with the locally finiteness condition implies that if $\{f_n : U^n \rightarrow W\}$ is a system of maps, compatible with the maps of the system $\{U^n\}$, then the map on the limit is determined by $f^m : U^m \rightarrow W$ for any $m \geq m_0$, then

$$\text{Hom}_{\text{Chain}(C)}(U, W) = \text{Hom}_{\text{Chain}(C)}(U^{m_0}, W) = \lim_{\rightarrow n} \text{Hom}_{\text{Chain}(C)}(U^n, W)$$
Lemma 2.4 Let $X \in \text{Chain}^b(C)$, $F : \text{Chain}^+(C) \to \text{Chain}^+(D)$ a functor commuting with locally finite inverse limits and assume $F(X) \in \text{Chain}^b(D)$. Let $V = \lim_{\leftarrow i} V^i \in \text{Chain}^+(C)$ a limit of a locally inverse system in $\text{Chain}^+(C)$ such that, for every $i \in \mathbb{N}$, $F$ induces an isomorphism

$$F : \text{Hom}_{\text{Chain}(C)}(V^i, X) \to \text{Hom}_{\text{Chain}(C)}(F(V^i), F(X))$$

Then $F$ induces an isomorphism

$$F : \text{Hom}_{\text{Chain}(C)}(V, X) \to \text{Hom}_{\text{Chain}(C)}(F(V), F(X))$$

Proof: Given two complexes $Y, Z \in \text{Chain}(C)$, we denote by $\text{Hom}_C(Y, Z)_*$ the $\text{Hom}$ complex (for details see for example [2]). This complex has in each degree

$$\text{Hom}_C(Y, Z)_n = \text{Hom}_{ZC}(Y, Z[n])$$

i.e. graded $C$-colinear morphism from $Y$ to $Z[n]$. The cohomology of this complex in degree zero are $\mathbb{Z}$-graded $C$-colinear maps which commute with differentials (i.e. morphisms of $\text{Chain}(C)$), modulo the homotopy equivalence relation. We then obtain:

$$\text{Hom}_{\text{Chain}(C)}(V, X) = H^0(\text{Hom}_C(V, X))$$

$$= H^0(\text{Hom}_C(\lim_{\leftarrow i} V^i, X))$$

$$= H^0(\lim_{\leftarrow i} \text{Hom}_C(V^i, X))$$

$$= \lim_{\leftarrow i} H^0(\text{Hom}_C(V^i, X))$$

$$= \lim_{\leftarrow i} \text{Hom}_{\text{Chain}(C)}(V^i, X)$$

$$= \lim_{\leftarrow i} \text{Hom}_{\text{Chain}(D)}(F(V^i), F(X))$$

$$= \lim_{\leftarrow i} H^0(\text{Hom}_{\text{Chain}(D)}(F(V^i), F(X)))$$

$$= H^0(\lim_{\leftarrow i} \text{Hom}_{\text{Chain}(D)}(F(V^i), F(X)))$$

$$= H^0(\text{Hom}_{\text{Chain}(D)}(F(V), F(X)))$$

$$= \text{Hom}_{\text{Chain}(D)}(F(V), F(X))$$

Lemma 2.5 Let $F : \text{Chain}(C) \to \text{Chain}(D)$ be a functor and $\{U^n\}_{n \in \mathbb{N}}$ an inverse system in $\text{Chain}(C)$ such that the sequences

$$0 \to U \to \prod_n U^n \to \prod_n U^n \to 0$$

$$0 \to F(U) \to \prod_n F(U^n) \to \prod_n F(U^n) \to 0$$

are triangles in $\mathcal{H}(C)$ (for example if the maps $U^{n+1} \to U^n$ and $F(U^{n+1}) \to F(U^n)$ are split surjections as morphisms of graded comodules) where $U = \lim_{\leftarrow} U^n$. Assume that for every $n \in \mathbb{N}$, the functor $F$ induces an isomorphism $F : \text{Hom}_{\mathcal{H}(C)}(Z, U^n) \cong \text{Hom}_{\mathcal{H}(D)}(F(Z), F(U^n))$, then

$$F : \text{Hom}_{\mathcal{H}(C)}(Z, U) \cong \text{Hom}_{\mathcal{H}(D)}(F(Z), F(U))$$
**Proof:** we apply the $\Hom_H$ functor to the triangles of the hypothesis and we obtain a morphism of long exact sequences:

\[
\begin{array}{ccccccccc}
\ldots & \rightarrow & \Hom_H(C)(Z,U) & \rightarrow & \Hom_H(C)(Z, \prod_n U^n) & \rightarrow & \Hom_H(C)(Z, \prod_n U^n) & \rightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ldots & \rightarrow & \Hom_H(C)(Z,U) & \rightarrow & \prod_n \Hom_H(C)(Z,U^n) & \rightarrow & \prod_n \Hom_H(C)(Z,U^n) & \rightarrow & \ldots \\
\downarrow & & & & \downarrow & & & & \downarrow \\
\ldots & \rightarrow & \Hom_{H(D)}(F(Z), F(U)) & \rightarrow & \prod_n \Hom_{H(D)}(F(Z), F(U^n)) & \rightarrow & \prod_n \Hom_{H(D)}(F(Z), F(U^n)) & \rightarrow & \ldots
\end{array}
\]

Then by the five Lemma, $F : \Hom_H(C)(Z, U) \cong \Hom_{H(D)}(F(Z), F(U))$.

Now we study the behaviour of the locally finite inverse systems with respect to the tensor and cotensor product:

**Lemma 2.6** Let $X \in \text{Chain}^+(k)$ and $\{U^n\}_{n \geq 1}$ be a locally finite inverse system in $\text{Chain}^+(k)$, then

$$
\lim_{\leftarrow} (X \otimes U^n) = X \otimes U
$$

**Proof:** We first remark that if $X$ and $Y$ are objects of $\text{Chain}^+(k)$, then only a finite number of summands appear in each component of the tensor product $(X \otimes Y)_r = \oplus_{p+q=r} X_p \otimes Y_q$, then $X \otimes Y = \lim_{r \rightarrow} (X \otimes Y_{\leq j})$. Now following the definition of locally finite inverse system we have that

$$
X \otimes U = \lim_{\leftarrow} (X \otimes U_{\leq j}) = \lim_{\leftarrow} (X \otimes \lim_{\leftarrow} U_{\leq j}^n) =
$$

$$
= \lim_{\leftarrow} (X \otimes U_{\leq j}^n) = \lim_{\leftarrow} (X \otimes \lim_{\leftarrow} U_{\leq j}^n) =
$$

$$
= \lim_{\leftarrow} (X \otimes U_{\leq j}^n) = \lim_{\leftarrow} (X \otimes U^n)
$$

**Corollary 2.7** Let $D X_C \in \text{Chain}^+(C \otimes D^{op})$ and $\{U^n\}_{n \geq 1}$ be a locally finite inverse system in $\text{Chain}^+(C)$, where $C$ and $D$ are positively graded differential coalgebras. Then

$$
\lim_{\leftarrow} (X \Box_C U^n) = X \Box_C \lim_{\leftarrow} U^n
$$

**Proof:** once we know that $\Box$ commutes with this particular class of inverse limits, we only notice that $X \Box_C -$ is defined as a kernel of the form

$$
0 \rightarrow X \Box_C U \rightarrow X \otimes U \rightarrow X \otimes C \otimes U
$$

and kernels commutes with arbitrary inverse limits.

Now we come back to the proof of Theorem 2.4, we have to prove that if the functor $D X_C \Box_R -$ verifies (a) (b) and (c) then it is fully faithful.

Let us call $\mathcal{C}_1$ the full subcategory of $\mathcal{D}^+(C)$ whose objects are complexes $U$ such that, for all $n \in \mathbb{Z}$, $F$ induces an isomorphism

$$
F : \Hom_{\mathcal{D}(C)}(U, C[n]) \rightarrow \Hom_{\mathcal{D}(D)}(F(U), D X[n])
$$

It is clear that $\mathcal{C}_1$ is a triangulated subcategory of $\mathcal{D}^+(C)$ and by (a) it contains $C C$. Also by Lemma 2.4 it is stable under locally finite inverse limits, then $\mathcal{C}_1$ coincides with $\mathcal{D}^+(C)$.

Let us now consider $\mathcal{C}_2$ the full subcategory $\mathcal{D}^+(C)$ whose objects are complexes $V$ such that, for all $U \in \mathcal{D}^+(C)$, $F$ induces an isomorphism

$$
F : \Hom_{\mathcal{D}(C)}(U, V) \rightarrow \Hom_{\mathcal{D}(D)}(F(U), F(V))
$$
It is clear that \( \mathcal{C}_2 \) is a triangulated subcategory of \( \mathcal{D}^+(C) \), stable by direct summands, and by (b) \( \mathcal{C}_2 \) is stable under arbitrary products. By the above discussion \( \mathcal{C}_2 \) contains \( C \), in order to see that \( \mathcal{C}_2 \) is equivalent to \( \mathcal{D}^+(C) \) it is enough to see that \( C \otimes V \in \mathcal{C}_2 \) for all \( V \in \text{Chain}^+(k) \).

Since every object \( V \in \text{Chain}^+(k) \) is a locally finite inverse limit of objects of \( \text{Chain}^b(k) \), by Lemma 2.5 one can assume that \( V \) is bounded. Assume that \( V \) is of the form

\[
\cdots \to 0 \to V_n \xrightarrow{d_n} V_{n+1} \xrightarrow{d_{n+1}} \cdots \xrightarrow{d_{m-1}} V_m \xrightarrow{d_m} 0 \to \cdots
\]

Let us call \( V_{\leq m-1} \) the complex which has the same components as \( V \) on degree except in degree \( m \), and the same differentials (except \( d_{m-1} \)), then \( d_{m-1} : V_{\leq m-1} \to V_m \) is a morphism of complexes, and \( V \cong \operatorname{Co}(d_{m-1}) \).

Now the functor \( C \otimes - : \mathcal{D}^+(k) \to \mathcal{D}^+(C) \) commutes with mapping cones, then inductively, since \( \mathcal{C}_2 \) is stable under mapping cones, it is enough to see that \( C(i) \) belongs to \( \mathcal{C}_2 \), for every set \( I \), and this is easy, because \( k(i) \) is (as \( k \)-vector space) a direct summand of \( k^I \), so \( C(i) \) is a direct summand of \( C \otimes k^I = C^I \) (the product in the category \( \text{Chain}^+(C) \)).

2.2 On derived equivalences of concentrated coalgebras

Let us consider the case when \( C \) is a usual coalgebra and we view it as a differential graded one with trivial differential graded structure.

Free comodules are closed objects because they are of the form \( C(i) = C \otimes k(i) \), and direct summands of closed objects are also closed, then injective comodules are closed. Being closed is also stable by triangles, then inductively one can easily prove that a bounded complex with injective components is closed (for example the complex \( \cdots 0 \to X_n \to \cdots \) is the mapping cone of the map \( d : (\cdots 0 \to X_n \to \cdots) \to (\cdots 0 \to X_{n+1} \to \cdots) \)).

Also if \( M \in \text{Chain}^+(C) \) is a complex with injective components, then \( M = \lim_{\to n} M_{\leq n} \) where \( M_{\leq n} \) is the complex truncated at degree \( n \), and the surjections \( M_{\leq n+1} \to M_{\leq n} \) clearly split as morphisms of graded \( C \)-comodules, then by Lemma 1.6 \( M \) is a closed object.

On the other hand, if \( M \) is any object in \( \text{Chain}^+(C) \), the standard resolution \( C(M) \) has, in each degree, free \( C \)-comodules. We characterize then, in this special case, the class of closed objects as those ones homotopy equivalent to (left bounded) complexes with injective components.

We can state the following proposition:

**Proposition 2.8** Let \( C \) and \( D \) be two concentrated coalgebras. Let \( \mathcal{H}_I \) (resp. \( \mathcal{H}_I^+, \mathcal{H}_I^b \)) be the subcategory of \( \mathcal{H} \) (resp. \( \mathcal{H}^+, \mathcal{H}^b \)) consisting of complexes with injective components. Then \( \mathcal{D}^+(C) \cong \mathcal{D}^+(D) \) (as triangulated categories) if and only if \( \mathcal{H}_I^+(C) \cong \mathcal{H}_I^+(D) \), and any triangulated equivalence \( \mathcal{H}_I^+(C) \cong \mathcal{H}_I^+(D) \) restricts to an equivalence \( \mathcal{H}_I^+(C) \cong \mathcal{H}_I^+(D) \).

**Proof:** we know in general that for any positively graded differential coalgebra, the category \( \mathcal{D}^+ \) is equivalent to the subcategory of \( \mathcal{H}^+ \) consisting of closed objects, denoted by \( \mathcal{H}_c^+ \). The discussion above proves that when the coalgebra is concentrated, the inclusion \( \mathcal{H}_c^+ \to \mathcal{H}_c^+ \) is an equivalence, then for \( C \) and \( D \) as in the hypothesis we have that \( \mathcal{D}^+(C) \cong \mathcal{H}_c^+(C) \) and \( \mathcal{D}^+(D) \cong \mathcal{H}_c^+(D) \), and the first assertion is clear.

In order to see that any triangulated equivalence \( \mathcal{H}_I^+(C) \cong \mathcal{H}_I^+(D) \) restricts to an equivalence \( \mathcal{H}_I^+(C) \cong \mathcal{H}_I^+(D) \) we will need the following characterization of bounded complexes:

**Lemma 2.9** Let \( C \) be a concentrated coalgebra and \( X \in \mathcal{H}^+(C) \). Then \( X \in \mathcal{H}^b(C) \) if and only if, for all \( Y \in \mathcal{H}^+(C) \) there exists \( n_0 \in \mathbb{N} \) such that \( \operatorname{Hom}_{\mathcal{H}(C)}(X[n], Y) = 0 \) for all \( n \leq n_0 \).

**Proof:** Let us first see that the condition is necessary.

Assume \( X \in \mathcal{H}^b(C) \), then \( \exists m \in \mathbb{N} \) such that \( X_p = 0 \) \( \forall p \) \( |p| > m \). Now given \( Y \in \text{Chain}^+(C) \) there exists \( m' \in \mathbb{Z} \) such that \( Y_p = 0 \) \( \forall p < m' \). If we take \( n \leq m - m' \) then \( \operatorname{Hom}_{\text{Chain}(C)}(X[n], Y) = 0 \) which obviously implies \( \operatorname{Hom}_{\mathcal{H}(C)}(X[n], Y) = 0 \).

Let us now see that the condition is sufficient:
Let $X \in \mathcal{H}^+(C)$ and consider $Y = C$. By hypothesis there exists $n_0 \in \mathbb{N}$ such that $\text{Hom}_{\mathcal{H}(C)}(X[n], C) = 0$ for all $n \leq n_0$. By Proposition 1.3,

$$\text{Hom}_{\mathcal{H}(C)}(X[n], C) = \text{Hom}_{\mathcal{H}(k)}(X[n], k) = X^*_n$$

then $X_n = 0$ for all $n \geq n_0$, i.e. $X \in \mathcal{H}^-(C)$, then $X \in \mathcal{H}^h(C)$.

Remarks:

1. After this Lemma, the proof of Proposition 2.8 is completed.

2. Proposition 2.8 tells us that, given an equivalence $F: \mathcal{D}^+(C) \rightarrow \mathcal{D}^+(D)$ where $C$ and $D$ are concentrated coalgebras, one can always assume that $F(C)$ is quasi-isomorphic to a bounded complex with $D$-injective components, so the assumption on the $X \in \text{Chain}(D \otimes C^{op})$ (about being bounded and $D$-closed) in Theorem 2.1 is superfluous when $C$ and $D$ are concentrated coalgebras.

3. Let $F: \mathcal{H}^+_I(C) \rightarrow \mathcal{H}^+_I(D)$ a functor commuting with products and extensions by objects of $\text{Chain}^+(k)$, where $C$ and $D$ are concentrated coalgebras. Suppose that $F$ restricts to a functor $\mathcal{H}^+_I(C) \rightarrow \mathcal{H}^+_I(D)$, and assume that this restriction is an equivalence, then re-writing the proof of Theorem 2.1 we have that the original functor $F$ is an equivalence. In that sense, derived equivalences between concentrated coalgebras can be “checked” looking at the category $\mathcal{H}^+_I$.

3 Cotilting theory

If we specialize Theorem 2.1 to the non differential graded case when $C$ and $D$ are usual coalgebras, and $X$ is a $D$-$C$-bicomodule, condition (a) is a vanishing condition of the $\text{Ext}$ groups, and condition (b) is always satisfied when $X$ is quasi-finite (because in that case $X \square_C -$ admits a left adjoint). Condition (c) of 2.1 can be verified if for example $D$ fits into some exact sequence where the other components of the sequence are obtained by some operations on $T$.

Definition 3.1 Let $C$ and $D$ be two (concentrated) coalgebras and $DT_C$ a $D$-$C$-bicomodule. We will call $T$ a cotilting comodule if

1. $DT$ is quasi-finite, $e_D(T) \cong C$ and $e_C(T) \cong D$.
2. $\text{Ext}^n_D(T, T) = 0 \ \forall \ n \geq 1$.
3. There exists an exact sequence of type

$$0 \rightarrow DT_n \rightarrow \ldots \rightarrow DT_0 \rightarrow DT \rightarrow 0$$

with $T_i \in \text{Add}(T)$.
4. There exists an exact sequence of $D$-$C$-bicomodules

$$0 \rightarrow DT_C \rightarrow I_0 \rightarrow \ldots \rightarrow I_r \rightarrow 0$$

with $I_i$ injective and quasifinite as $D$-comodules.

The main result of this section is the following:

Theorem 3.2 Let $C$ and $D$ be two (concentrated) coalgebras admiting a cotilting bicomodule $DT_C$, and call $Q := h_D(T, D)$. Then there are isomorphisms in $\mathcal{D}(D^e)$ and $\mathcal{D}(C^e)$:

$$T \square_D^R Q \cong D$$

$$Q \square_D^R T \cong C$$

In particular, $\mathcal{D}^+(C) \cong \mathcal{D}^+(D)$, as triangulated categories.
Since we will not only prove that $T \boxtimes_C^R$ is an equivalence, but that its quasi-inverse is also a derived cotensor product, the proof of the above Theorem is a little more complicated than just applying Theorem 2.1. Nevertheless, we begin with a Lemma in the direction of condition (c) of 2.1.

**Lemma 3.3** Let $C$ and $D$ be two concentrated coalgebras, $DTC$ a bicomodule which is quasi-finite as $D$-comodule, then for all $X \in \text{Chain}(C)$ and $Y \in \text{Chain}(D)$ we have natural isomorphisms:

$$\text{Hom}_{\text{CH}(C)}(h_D(T, Y), X) \cong \text{Hom}_{\text{CH}(D)}(Y, T \square_C X)$$

$$\text{Hom}_{\mathcal{H}(C)}(h_D(T, Y), X) \cong \text{Hom}_{\mathcal{H}(D)}(Y, T \square_C X)$$

where the differential of $h_D(T, Y)$ and $T \square_C X$ are respectively $h_D(T, d_Y)$ and $id \square_C d_X$.

**Proof:** we will make use of the complex $\mathcal{H}om$. We recall that the component in degree $p$ of $\mathcal{H}om$ is the set of homogeneous $C$-colinear maps of degree $p$, more precisely, we have that

$$\mathcal{H}om_C(h_D(T, Y), X)_p = \prod_{n \in \mathbb{Z}} \mathcal{H}om_C(h_D(T, Y_n), X_n[p]) \cong$$

$$\cong \prod_{n \in \mathbb{Z}} \mathcal{H}om_D(Y_n, T \square_C X_n[p]) = \mathcal{H}om_D(Y, T \square_C X)_p$$

The fact that this isomorphism commutes with the differential in $\mathcal{H}om$ comes from the naturality of the isomorphism applied to $d_X^n : X_n \to X_{n+1}$ and $d_Y^n : Y_n \to Y_{n+1}$.

The complexes $\mathcal{H}om$ being isomorphic, their cocycles in degree zero are isomorphic and they have the same cohomology in degree zero, and the proof of the Lemma is completed.

**Corollary 3.4** Under the same hypothesis of Lemma 3.3, $T \square_C^R : D^+(C) \to D^+(D)$ commutes with products.

**Proof:** the functor $T \square_C : \mathcal{H}(C) \to \mathcal{H}(D)$ commutes with products because it admits a left adjoint. Let $\{M_i\}_{i \in I}$ a family of objects in $\text{Chain}^+(C)$ such that $\prod_{i \in I} M_i \in \text{Chain}^+(C)$, in order to compute $T \square_C^R(\prod_{i \in I} M_i)$ we need to find a closed object in $\text{Chain}^+(C)$ quasi-isomorphic to $\prod_{i \in I} M_i$. One option is to consider $C(\prod_{i \in I} M_i)$, but we consider instead $\prod_{i \in I} C(M_i)$, which is closed because it is a product of closed objects, and it is also quasi-isomorphic to $\prod_{i \in I} M_i$ because each $M_i$ is $k$-homotopically equivalent to $C(M_i)$, then the product of the $M_i$’s is $k$-homotopically equivalent to the product of the $C(M_i)$’s. We obtain then

$$T \square_C^R(\prod_{i \in I} M_i) = T \square_C(\prod_{i \in I} C(M_i)) =$$

$$= \prod_{i \in I} T \square_C C(M_i) = \prod_{i \in I} T \square_C^R M_i$$

We can now prove a different version of theorem 3.2.

**Theorem 3.5** Let $C$ and $D$ be two concentrated coalgebras and $DTC$ a bicomodule satisfying conditions 1. 2. and 3. of the definition of a cotilting bicomodule, then $T \square_C^R : D^+(C) \to D^+(D)$ is an equivalence of triangulated categories.

**Proof:** we use the characterization given in Theorem 2.1.

(a) $\text{Hom}_{D(D)}(C, C[n]) = \text{Ext}^n_C(C, C)$ which is zero unless $n = 0$, in that case $\text{Ext}^n_C(C, C) \cong \text{Com}_C(C, C) = C^*$. On the other hand, $\text{Hom}_{D(D)}(T, T[n]) = \text{Ext}^n_C(T, T)$ which by hypothesis is zero for $n \geq 1$, and $\text{Ext}^n_C(T, T) = \text{Com}_D(T, T) = (e_D(T))^* \cong C^*$.

(b) it is Corollary 3.4.
(c) if a triangulated subcategory of $D^+(D)$ contains $T$ and is stable by extensions by objects of $\text{Chain}^+(k)$, then it contains $T^{(j)}$ and (with same proof of Lemma 1.8) it contains its direct summands, then it contains $\text{Add}(T)$. Now consider the exact sequence

$$0 \to DT_n \to \ldots \to DT_0 \to DT \to 0$$

with $T_i \in \text{Add}(T)$. Inductively, the complex

$$\begin{array}{c}
0 \rightarrow T_n \xrightarrow{d_n} \cdots \xrightarrow{d_i} T_{i-1} \rightarrow 0 \\
\end{array}$$

is the mapping cone of the map $d_i$:

$$\begin{array}{c}
0 \rightarrow T_n \xrightarrow{d_n} \cdots \xrightarrow{d_{i+1}} T_{i+1} \rightarrow T_i \rightarrow 0 \\
\end{array}$$

viewed as a map of complexes. So the complex $0 \rightarrow DT_n \rightarrow \ldots \rightarrow DT_0 \rightarrow 0$ belongs to the triangulated subcategory of $D^+(D)$ stable under extensions by objects of $\text{Chain}^+(k)$, and it is quasi-isomorphic to $D$.

Now coming back to Theorem 3.2, we will prove a Lemma of comodule theory which may be probably considered part of the folklore of the theory; we include the proof for completeness:

**Lemma 3.6** Let $C$ and $D$ be two (concentrated) coalgebras and $DI_C$ a bicomodule such that, viewed as $D$-comodule, is injective and quasi-finite. Then for all $D$-comodule $M$ there is an isomorphism of $C$-comodules

$$h_D(I, M) \cong h_D(I, D)\square_D M$$

**Proof:** The functor $h_D(I, -)$ is left adjoint to the functor $I\square_C -$, then it commutes with direct sums, in particular, if $W$ is any $k$-vector space, we have that $h_D(I, X \otimes W) = h_D(I, X) \otimes W$ for every $D$-comodule $X$.

On the other hand, the functor $h_D(I, -)^* = \text{Com}_D(-, I)$, and since $I$ is assumed to be an injective $D$-comodule, then $h_D(I, -)$ is exact. Now let $M$ be any $D$-comodule and apply the funtor $h_D(I, -)$ to the exact sequence

$$0 \rightarrow M \xrightarrow{\rho M} D \otimes M \xrightarrow{b'} D \otimes D \otimes M$$

By the above discussion we have the identifications

$$\begin{array}{c}
0 \rightarrow h_D(I, M) \rightarrow h_D(I, D \otimes M) \rightarrow h_D(I, D \otimes D \otimes M) \\
\end{array}$$

$$\begin{array}{c}
0 \rightarrow h_D(I, D)\square_D M \rightarrow h_D(I, D) \otimes M \rightarrow h_D(I, D) \otimes D \otimes M \\
\end{array}$$

giving $h_D(I, M) = h_D(I, D)\square_D M$.

**Lemma 3.7** Let $C$ and $D$ be two positively graded differential coalgebras and $T \in \text{Chain}(D \otimes C^\op)$ such that, for every $Y \in \text{Chain}(C)$ there exists an object $h_D(T, Y)$ in $\text{Chain}(D)$ which is functorial in $Y$, and an isomorphism of complexes $\text{Hom}_D(h_D(T, Y), X) \cong \text{Hom}_C(Y, T\square_C X)$ which is natural in $X$. Then $e_D(T) := h_D(T, T)$ is a differential graded coalgebra.

**Proof:** We proceed in the same way as in the non-differential graded case: from the identity map of $e_D(T)$, which is an element of $\text{Hom}_D(e_D(T), e_D(T))$, one gets an element of $\text{Hom}_C(T, T\square_C e_D(T)) \subset \text{Hom}_C(T, T \otimes e_D(T))$, and iterating the procedure one get an homogeneous map from $T$ into $T\square_C e_D(T) \otimes e_D(T)$. Now considering $X = e_D(T) \otimes e_D(T)$ and $Y = T$, we get an element in $\text{Hom}_D(e_D(T), e_D(T) \otimes e_D(T))$. 

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The coassociativity of $e_D(T)$ is proved exactly in the same way as in the non-differential graded case. We will only remark that the coalgebra structure and the differential graded structure fit together because the graded dual object is an associative differential algebra, namely

$$e_D(T)^* = \mathcal{H}om_k(h_D(T,T),k) \cong \mathcal{H}om_C(h_D(T,T),C) \cong \mathcal{H}om_D(T \square_C C) \cong \mathcal{H}om_D(T,T)$$

Remarks: 1. the hypothesis of this Lemma include the case when $T$ is a bicomodule over two concentrated coalgebras $C$ and $D$ and $T$ is quasi-finite viewed as $D$-comodule.

2. If $T$ and $T'$ are complexes satisfying the hypothesis of this Lemma and $f : T \to T'$ is a morphism in $\text{Chain}(C)$, then $\text{Co}(f)$ also satisfies the same hypothesis.

By the second remark, the class of objects satisfying an adjoint-type hypothesis like the above one is closed under finite direct sums and shifting of degree. This implies that if $C$ and $D$ are two concentrated coalgebras, then any bounded complex of $D$-$C$-bicomodules with $D$-quasi-finite components satisfies the adjoint-type hypothesis.

Consider now $C$ and $D$ two concentrated coalgebras admitting a cotilting bicomodule $DTC$. By condition 4. of Definition 3.1, there exists an exact sequence of $D$-$C$-bicomodules

$$0 \to DTC \to I_0 \to \ldots \to I_r \to 0$$

with $I_i$ injective and quasi-finite as $D$-comodules.

Lemma 3.8 Keeping notations, let us call $T_*$ the complex $0 \to I_0 \to \ldots \to I_r \to 0$. Then the differential graded coalgebra $h_D(T_*,T_*)$ is quasi-isomorphic to $C$ as $D$-$C$-bicomodule.

Proof: it is enough to see that the graded dual algebras are quasi-isomorphic:

$$H^n(\mathcal{H}om_D(T_*,T_*)) = H^0(\mathcal{H}om_D(T_*,T_*[n])) = \mathcal{H}om_{\mathcal{H}(D)}(T_*,T_*[n])$$

But $T_*$ is a bounded complex of $D$-injective objects, then it is a closed object and so

$$\mathcal{H}om_{\mathcal{H}(D)}(T_*,T_*[n]) = \mathcal{H}om_D(T_*,T_*[n]) \cong \mathcal{H}om_D(T,T[n]) \cong \mathcal{E}xt^D_T(T,T)$$

and $\mathcal{E}xt^D_T(T,T) = 0$ unless $n = 0$, and in this case $\mathcal{E}xt^D_0(T,T) = \mathcal{C}om_D(T,T) = e_D(T)^* \cong C^*$.

Corollary 3.9 Let $C$ and $D$ be two (concentrated) coalgebras and $DTC$ a cotilting bicomodule. Let us call $T_*$ the complex $0 \to I_0 \to \ldots \to I_r \to 0$ of the definition of cotilting comodule, which is quasi-isomorphic to $DTC$. Then there is an isomorphism in $\mathcal{D}(C^\circ)$:

$$h_D(T_*,D) \square_D T_* \cong C$$

Proof: Since $h_D(T_*,D)_n = h_D(I_{-n},D)$, we have that

$$(h_D(I_*,D) \square_D T_*)_n = \bigoplus_{p+q=n} h_D(I_{-p},D) \square_D I_q = \bigoplus_{p+q=n} h_D(I_{-p},I_q)$$

Then $h_D(T_*,D) \square_D T_* = h_D(T_*,T_*)$ which, by the two Lemmata above, is quasi-isomorphic to $C$.

We now prove our last Lemma needed for the proof ot Theorem 3.2.

Lemma 3.10 Let $C$ and $D$ be two differential positively graded coalgebras and $F : \mathcal{D}^+(C) \to \mathcal{D}^+(D)$ an equivalence with inverse $G : \mathcal{D}^+(D) \to \mathcal{D}^+(C)$. Then, for any positively graded differential coalgebra $E$, one can extend the functors $F$ and $G$ in order to have an equivalence between $\mathcal{D}^+(C \otimes E)$ and $\mathcal{D}^+(D \otimes E)$.
**Proof:** Let $X \in \text{Chain}^+(C \otimes E)$, then using the standard resolution of $X$ with respect to $E$, there is a $C \otimes E$-colinear quasi-isomorphism
\[
C_E X \rightarrow \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes C X, b'_E, d \right)
\]
We define then $\hat{F}(X) := \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes D F(X), b'_E, d \right)$. We remark that we can apply $F$ because the $E$-structure map of the above complex is a $C$-colinear map. In an analogous way we define $\hat{G}$, let us see that they are inverse to each other:
\[
\hat{G}(\hat{F}(X)) = \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes C G(\hat{F}(X)) \right) = \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes C G \left( \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes D F(X) \right) \right) \right)
\]
But $\text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes D F(X), b'_E, d \right)$ is quasi-isomorphic to $F(X)$ as $D$-comodules, then $G \left( \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes D F(X) \right) \right)$ is isomorphic to $G(F(X))$, obtaining
\[
\hat{G}(\hat{F}(X)) \cong \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes C G(F(X)) \right) \cong \text{Tot} \left( \bigoplus_{n \geq 0} E^{\otimes n} \otimes C X \right) \cong C_E X
\]
The other isomorphism is proved identically.

As a Corollary, we can write down a complete proof of Theorem 3.2.

After Corollary 3.9 we know that $h_D(T_\ast, D) \square_D T_\ast \cong C$. We have then isomorphisms in $\mathcal{D}(C^\circ)$:
\[
C \cong h_D(T_\ast, D) \square_D T_\ast \cong h_D(T, D) \square_D T_\ast \cong h_D(T, D) \square_D T
\]
We also know that $T \square_D^{C^\circ} \!$ is an equivalence, let us call $F := T \square_D^{C^\circ} \!$ and consider $\hat{F} : \mathcal{D}^+(C \otimes D^{op}) \rightarrow \mathcal{D}^+(D^\circ)$, the extension given in Lemma 3.10. Since $\hat{F}$ is an equivalence, there must exist an object $S \in \text{Chain}(C \otimes D^{op})$ such that $\hat{F}(S) \cong D_D$. We have then the following isomorphisms in $\mathcal{D}(D^\circ)$:
\[
D_D D \cong \hat{F}(S) = \text{Tot} \left( \bigoplus_{n \geq 0} D^{\otimes_n}_D \otimes D_T \square_C^{D^\circ} S, b'_D, d \right) = \text{Tot} \left( \bigoplus_{n \geq 0} D^{\otimes_n}_D \otimes D_T \square_C^{D^\circ} S, b'_D, d \right) \cong D_T \square_C S_D
\]
Now we recall that $h_D(T, D) \square_D^{C^\circ} \! T \cong C_C$, and this implies that $S$ must be isomorphic (in $\mathcal{D}(D^\circ)$) to $h_D(T, D)$, as it is shown below:
\[
S \cong C \square_C^{D^\circ} S \cong h_D(T, D) \square_D^{C^\circ} \! T \square_D^{C^\circ} \! T \cong h_D(T, D) \square_D^{C^\circ} \! T \square_D^{C^\circ} \! T \cong h_D(T, D)
\]
**Remarks:**
1. If $C$ and $D$ are two (concentrated) coalgebras which are Morita - Takeuchi equivalent (i.e. the category of $C$-comodules is equivalent to the category of $D$-comodules) then their derived categories are equivalent, and we know after 3.8 that there exists an injective cogenerator $pI_C$ such that $I_C h_D(I, D) \cong D$ and $h_D(I, D) \square_D I \cong C$ (isomorphisms of bicomodules). This implies that the notion of cotilting bicomodule generalizes the Morita - Takeuchi equivalence relation, and the generalization is strict, because it is enough to take a cotilting bicomodule which is not injective and so it can never give an equivalence at the level of comodule categories, nevertheless it gives an equivalence at the level of derived categories.
2. If $C$ and $D$ are two concentrated coalgebras and $F : \mathcal{D}^+(C) \rightarrow \mathcal{D}^+(D)$ is an equivalence of triangulated categories, one can ask if there exists an object $D_X C$ in $\text{Chain}^+(D \otimes C^{op})$ such that $X \square_D^{C^\circ} \! : \mathcal{D}^+(C) \rightarrow \mathcal{D}^+(D)$ is
an equivalence. If this were the case, it is clear that \( \rho X \) is the value of \( X \square B \) in \( C \), if we put \( \rho X = F(C) \) then one can suppose that \( X \) is a bounded complex with \( D \)-injective components (Proposition 2.8). But in order to have the \( C \)-structure, one should need to equip \( F(C) \) with a structure of \( \operatorname{C} \operatorname{op} \)-comodule. In general, if some \( X \in \operatorname{Chain}^+(D) \) admits a coendomorphisms coalgebra \( \mathcal{E} := hD(X, X) \), then \( X \) becomes a \( D \otimes \mathcal{E}^{\operatorname{op}} \)-comodule in a canonical way. But the condition of admitting a coendomorphism coalgebra is quite restrictive, for example this is the case if \( X \) is \( D \)-quasi-finite, and we do not know if “quasi-finiteness” is a property preserved by derived equivalences. This discussion is the reason that led us to include condition 4. in the definition of a cotilting comodule \( T \).

4 A quasi-isomorphism, an example of derived equivalence

Let \( C \) and \( D \) be two positively graded differential coalgebras and \( f : C \to D \) be a quasi-isomorphism of differential graded coalgebras. Composing \( f \) with the comultiplication of \( C \)

\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
& \overset{id \otimes f}{\searrow} & C \otimes D
\end{array}
\]

we can consider \( C \) as an object in \( \operatorname{Chain}(C \otimes D^{\operatorname{op}}) \). We will denote \( C_f \) this \( C-D \)-bicomodule, and similarly \( fC \) and \( fC_f \). The main result of this section is that the derived category associated to \( C \) is equivalent to the derived category associated to \( D \), more precisely, we have the following result:

**Proposition 4.1** With the above notations, there are isomorphisms in \( \mathcal{D}(D^c) \) and \( \mathcal{D}(C^c) \):

\[
\begin{align*}
\rho fC_C &= D \\
\rho fC_f &= C
\end{align*}
\]

**Proof:** The easiest part is to show that \( \rho fC_C \cong D \). It is clear that \( C_f \) is a closed object in \( \operatorname{Chain}(C) \), then \( \rho fC_C = \rho fC_Cf = fC_f \). But by hypothesis \( f : C \to D \) is a quasi-isomorphism of coalgebras, then \( f : fC_f \to D \) is a quasi-isomorphism of \( D \)-bicomodules, hence an isomorphism in the derived category.

In order to compute \( \rho f fC_C \), we have to find an object in \( \operatorname{Chain}(C \otimes D^{\operatorname{op}}) \) quasi-isomorphic to \( fC_f \) and closed as right \( D \)-comodule. We consider the standard resolution of \( C_f \) as right \( D \)-comodule, then

\[
\begin{align*}
\rho fC_f &= \operatorname{Tot}(\oplus_{n \geq 1} C \otimes D^{\otimes n}, d, b') \square_{DfC} C \\
&\cong \operatorname{Tot}(\oplus_{n \geq 1} C \otimes D^{\otimes n-1} \otimes C, d, b')
\end{align*}
\]

On the other hand, using the standard resolution of \( C \) as \( C \)-comodule, we have that \( C \) is quasi-isomorphic (as \( C \)-bicomodule) to \( \operatorname{Tot}(\oplus_{n \geq 1} C \otimes C^{\otimes n+1}, d, b') \). A \( C^{\operatorname{op}} \)-colinear morphism of complexes

\[
\operatorname{Tot}(\oplus_{n \geq 1} C \otimes C^{\otimes n+1}, d, b') \to \operatorname{Tot}(\oplus_{n \geq 1} C \otimes D^{\otimes n-1} \otimes C, d, b')
\]

is obtained applying \( f \) in the middle terms. Next we will see that the above map is a quasi-isomorphism.

We know by the Künneth formula that \( \rho d \otimes f^{\otimes r} \otimes \rho id : (C^{\otimes r+2}, d) \to (C \otimes D^{\otimes r} \otimes C, d) \) is a quasi-isomorphism, looking at the double complexes

\[
\begin{array}{ccc}
\vdots & \vdots & \vdots \\
\end{array}
\]

\[
\begin{array}{ccc}
(C \otimes C)_2 & \xrightarrow{b'} (C \otimes C \otimes C)_2 & \xrightarrow{b'} (C \otimes C^{\otimes 2} \otimes C)_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
(C \otimes C)_1 & \xrightarrow{b'} (C \otimes C \otimes C)_1 & \xrightarrow{b'} (C \otimes C^{\otimes 2} \otimes C)_1 \\
\end{array}
\]

\[
\begin{array}{ccc}
(C \otimes C)_0 & \xrightarrow{b'} (C \otimes C \otimes C)_0 & \xrightarrow{b'} (C \otimes C^{\otimes 2} \otimes C)_0 \\
\end{array}
\]
and

\[
\begin{array}{ccc}
(C \otimes C)_2 & \xrightarrow{b'} & (C \otimes D \otimes C)_2 & \xrightarrow{b'} & (C \otimes D \otimes C)_2 \\
\downarrow d & & \downarrow d & & \downarrow d \\
(C \otimes C)_1 & \xrightarrow{b'} & (C \otimes D \otimes C)_1 & \xrightarrow{b'} & (C \otimes D \otimes C)_1 \\
\downarrow d & & \downarrow d & & \downarrow d \\
(C \otimes C)_0 & \xrightarrow{b'} & (C \otimes D \otimes C)_0 & \xrightarrow{b'} & (C \otimes D \otimes C)_0 \\
\end{array}
\]

we see that our map is a quasi-isomorphism on the columns, so filtering by the columns, by a standard spectral sequence argument we have that the total complexes are quasi-isomorphic.

5 \textit{Hoch}, \textit{H} and \textit{HC} for differential graded coalgebras

Given a (concentrated) \(k\)-coalgebra \(C\) and a bicomodule \(M\), Doi defined in [1] two cohomology theories \(Hoch^*(M, C)\) and \(H^*(M, C)\) which play the role of Hochschild homology and cohomology, respectively. They are defined in terms of standard complexes, but also have an interpretation as derived functors, namely

\[
\begin{align*}
Hoch^*(M, C) &= \text{Cotor}^*_C(M, C) = H^*(M \square_C^R, C) \\
H^*(M, C) &= \text{Ext}^*_C(M, C)
\end{align*}
\]

In [3], A. Solotar and myself defined a cyclic cohomology theory for coalgebras (denoted by \(HC^*\)) and proved some of its fundamental properties, including Morita - Takeuchi invariance (see [4] for the invariance of \(Hoch^*\) and \(H^*\) and [5] for the invariance of \(HC^*\) and a more general proof of the invariance of \(Hoch^*\)). More precisely we proved that given two \(k\)-coalgebras \(C\) and \(D\) such that there exists a \(k\)-linear equivalence \(F : C\text{-comod} \rightarrow D\text{-comod},\) then there exists an equivalence \(\hat{F} : C^e\text{-comod} \rightarrow D^e\text{-comod}\) with \(\hat{F}(C) = D\) such that, given a \(C\)-bicomodule \(M\)

\[
\begin{align*}
Hoch^*(M, C) &\cong Hoch^*(\hat{F}(M), D) \\
H^*(M, C) &\cong H^*(\hat{F}(M), D) \\
HC^*(C) &\cong HC^*(D)
\end{align*}
\]

The purpose of this section is to extend the definition of the cohomology theories to the differential graded case, generalizing also the invariance results to derived equivalences.

Let \(C\) be a differential graded coalgebra and consider, for each \(n \in \mathbb{N}_0\), the vector space \(C^\otimes n + 1\). The following natural operators are defined on them:

- The differential \(d : C^\otimes n + 1 \rightarrow C^\otimes n + 1\)

\[
(c_0, \ldots, c_n) \mapsto (-1)^n \sum_{k=0}^{n} (-1)^{|c_0| + \ldots + |c_{k-1}|} (c_0, \ldots, d_C(c_k), \ldots, c_n)
\]

- The cyclic operator \(T\)

\[
(c_0, \ldots, c_n) \mapsto (-1)^n (-1)^{|c_0| + \ldots + |c_{n}|} (c_1, \ldots, c_n, c_0)
\]

where \(|(c_1, \ldots, c_n)|\) is the standard degree on the tensor product: \(|(c_1, \ldots, c_n)| = \sum_{i=1}^{n} |c_i|\).
The differential \( b' := \sum_{i=0}^{n} (-1)^i \Delta_i \) and the differential \( b := \sum_{i=0}^{n+1} (-1)^i \Delta_i \).

- The norm \( N := \sum_{i=0}^{n} T^i : C^{\otimes n+1} \to C^{\otimes n+1} \)

In the same way as in the non-differential graded case, we have the following Lemma and Theorem:

**Lemma 5.1** With the above notations:

\[
T^{n+1}_{C^{\otimes n+1}} = \text{id}_{C^{\otimes n+1}} ; \quad N b' = b N \quad (1 - T) N = 0 = (1 - T) ; \quad (1 - T) b = b'(1 - T)
\]

**Proof:** They are formal consequence of the relations

\[
\begin{align*}
T \Delta_1 &= -\Delta_{i-1} T \quad i = 1, \ldots, n \\
T \Delta_0 &= (-1)^{n+1} \Delta_{n+1}
\end{align*}
\]

We also have that the signs on \( T \) are defined in such a way that \( d \) commutes with \( T \), so \( d \) commutes with \((1 - T)\) and \( N \). Then there is a well-defined double complex, denoted by \( C^{**}(C) \):

\[
\cdots \longrightarrow C_{-4}^{\text{Hoch}}(C) \overset{N}{\longrightarrow} C_{-3}^{\text{Hoch}}(C) \overset{1 - T}{\longrightarrow} C_{-2}^{\text{Hoch}}(C) \overset{N}{\longrightarrow} C_{-1}^{\text{Hoch}}(C) \overset{1 - T}{\longrightarrow} C_{0}^{\text{Hoch}}(C)
\]

where \( C_{\text{Hoch}}(C) = \text{Tot}(\oplus_{n \geq 0} C^{\otimes n+1}, b, d) \) and \( \hat{C}(C) = \text{Tot}(\oplus_{n \geq 0} C^{\otimes n+1}, b', d) \). We define, for a differential graded coalgebra \( C \), \( \text{Hoch}^*(C) := H^*(C_{\text{Hoch}}(C)) \), and \( \text{HC}^*(C) := H^*(\text{Tot}(C^{**}(C))) \).

**Remark:** The difference between \( \hat{C}(C) \) and the standard resolution \( C(C) \) is that \( \hat{C}(C) = \text{Tot}(\oplus_{n \geq 0} C^{\otimes n+1}, b', d) \) while \( C(C) = \text{Tot}(\oplus_{n \geq 1} C^{\otimes n+1}, b', d) \). In fact \( \hat{C}(C) \) can be identified with the mapping cone of \( \rho_C : C \to C(C) \) which is a quasi-isomorphism, then \( \hat{C}(C) \) is an acyclic complex.

**Theorem 5.2** Let \( C \) be a differential graded coalgebra, then there is an SBI-type long exact sequence

\[
\cdots \longrightarrow \text{HC}^n(C) \overset{S}{\longrightarrow} \text{HC}^{n+2}(C) \overset{I}{\longrightarrow} \text{Hoch}^{n+2}(C) \overset{B}{\longrightarrow} \text{HC}^{n+1}(C) \longrightarrow \cdots
\]

**Proof:** we proceed in the same way as in the non differential graded case, noticing the 2-periodicity in the definition of \( C^{**}(C) \) we obtain a short exact sequence of double complexes

\[
0 \to C^{**}(C)[-2] \to C^{**}(C) \to \text{Co}(1 - T)[1] \to 0
\]

and hence a long exact sequence on cohomology.

Since \( \hat{C}(C) \) is acyclic, it follows that \( \text{Co}(1 - T)[1] \) is quasi-isomorphic to \( C_{\text{Hoch}}(C) \).

**Lemma 5.3** Let \( C \) be a positively graded differential coalgebra, then the standard resolution \( C(C) \) is a closed object in \( \text{Chain}(C^\circ) \).

**Proof:** It follows the lines of the proof that \( C(M) \) is a closed object in \( \text{Chain}(C) \) for all \( M \in \text{Chain}^+(C) \). Since \( C \) is positively graded, then \( C(C) = \lim_{p \to -p} C_{\leq p}(C) \) where \( C_{\leq p}(C) = \text{Tot}(\oplus_{j \geq 1} C^{\otimes j+1}, b', d) \) is the quotient of \( C(C) \) by the subcomplex \( \text{Tot}(\oplus_{j \geq 1} C^{\otimes j+1}, b', d) \), and we have short exact sequences

\[
0 \to (C^{\otimes p+2}, d) \to C_{\leq p+1}(C) \to C_{\leq p}(C) \to 0
\]

which clearly split as sequences of graded \( C \)-bicomodules. By the isomorphism \( (C^{\otimes p+2}, d) \cong C^\circ \otimes C^{\otimes p} \), it follows that \( (C^{\otimes p+2}, d) \) is closed in \( \text{Chain}(C^\circ) \) and \( C_{\leq 1}(C) = C \otimes C = C^\circ \) which is also \( C^\circ \)-closed, then inductively \( C_{\leq p}(C) \) is a closed \( C^\circ \)-object for all \( p \geq 1 \). Now using again the above exact sequence together with Lemma 5.6 we can conclude that \( \lim_{p \to -p} C_{p}(C) \) is closed in \( \text{Chain}(C^\circ) \).

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Corollary 5.4 Let $C$ be a positively graded differential coalgebra, then $C_{Hoch}(C) = C \boxtimes_R C$.

Proof: in the same way as in the non-differential graded case, it is easy to see that $C_{Hoch}(C) \cong C \otimes C^*$, the equality $C \boxtimes_R C = C \otimes C^*$ comes from the facts that $C$ is quasi-isomorphic to $C^*$ (as $C$-bicomodule) and that $C$ is a closed object in $\text{Chain}(C^*)$.

Definition 5.5 Let $C$ be a differential graded coalgebra and $M \in \text{Chain}(C^*)$, we define

$$H^n(M, C) := H^*(C(C) \otimes C.M)$$

$$H^n(M, C) := \text{Hom}_{D(C^*)}(M, C[n])$$

Theorem 5.6 Let $C$ and $D$ be two positively graded coalgebras such that there exist $P \in \text{Chain}^+(C \otimes D^op)$ and $Q \in \text{Chain}^+(D \otimes C^op)$ with $P \otimes_R C \cong C$ and $Q \otimes_R C \cong D$ (isomorphisms in $D(C^*)$ and $D(D^*)$ respectively). Then, for all $M \in \text{Chain}^+(C)$

$$H^*(M, C) \cong \text{Hoch}(Q \otimes_R C, M \otimes_R C, P, D)$$

$$H^*(M, C) \cong \text{Hoch}(Q \otimes_R C, M \otimes_R C, P, D)$$

In particular $H^*(C) \cong H^*(D)$ and $H^*(C) \cong H^*(D)$.

Proof: It is clear that $Q \otimes_R C \cong C$ and $Q \otimes_R C \cong D$ is an equivalence of triangulated categories, then

$$H^n(M, C) = \text{Hom}_{D(C^*)}(M, C[n]) \cong \text{Hom}_{D(D^*)}(Q \otimes_R C \otimes_R C \otimes_R C, P, Q \otimes_R C \otimes_R C \otimes_R C) \cong \text{Hom}_{D(D^*)}(Q \otimes_R C \otimes_R C \otimes_R C, D[n]) = H^n(Q \otimes_R C \otimes_R C \otimes_R C, D)$$

For the other cohomology theory,

$$\text{Hoch}^*(Q \otimes_R C \otimes_R C, P, D) = H^*((Q \otimes_R C \otimes_R C) \otimes_R C, D) \cong H^*(M \otimes_R C, P, D) \cong H^*(M \otimes_R C, P, D) = \text{Hoch}^*(M, C)$$

Remark: $H^*(C)$ is a graded algebra, with multiplication given by composition in $D(C^*)$. In the situation of the above theorem, since the isomorphism $H^*(C) \cong H^*(D)$ is given by a functor, it follows that the isomorphism is not only an isomorphism of $k$-vector spaces but also of graded $k$-algebras.

Examples: 1. Let $C$ and $D$ be two usual coalgebras such that they admit a cotilting bicomodule $DT_C$, then $Hoch^*(C) \cong Hoch^*(D)$ and $Hoch^*(C) \cong Hoch^*(D)$.

2. Let $f : C \to D$ be a quasi-isomorphism of positively graded differential coalgebras, then $Hoch^*(M, C) \cong Hoch^*(fM, D)$ and $Hoch^*(M, C) \cong Hoch^*(fM, D)$ (where $fM$ is $M$, but viewed as $D$-bicomodule via $f$).

In the case of the second example, we also have that the quasi-isomorphism $C \otimes_R C \cong D \otimes_R D$ is induced by $f$, we will see next that this implies the invariance of $HC^*$:

Proposition 5.7 Let $C$ and $D$ be two differential graded coalgebras and $f : C \to D$ be a differential graded and coassociative map such that $f_* : C_{Hoch}(C) \to C_{Hoch}(D)$ is a quasi-isomorphism, then $f$ induces an isomorphism $HC^*(C) \cong HC^*(D)$.
Proof: we begin by noticing that the complex $C^{**}(C)$ (also $C(C)$ and $C_{Hoch}(C)$) is defined using the differential graded structure and the comultiplication, so $f$ induces a morphism of complexes $C^{**}(C) \to C^{**}(D)$ which is natural with respect to the SBI-long exact sequence, and we obtain a morphism of long exact sequences:

\[ \cdots \to HC^n(C) \overset{S}{\to} HC^{n+2}(C) \overset{I}{\to} \text{Hoch}^{n+2}(C) \overset{B}{\to} HC^{n+1}(C) \to \cdots \]

\[ \cdots \to HC^n(D) \overset{S}{\to} HC^{n+2}(D) \overset{I}{\to} \text{Hoch}^{n+2}(D) \overset{B}{\to} HC^{n+1}(D) \to \cdots \]

So the proposition is proved by induction, using that $HC^0 = Hoch^0$ and the five Lemma for the inductive step.

As a corollary we have that if $f : C \to D$ is a quasi-isomorphism of positively graded differential coalgebras then $HC^*(C) \cong HC^*(D)$. We remark that if $C$ or $D$ have some nonzero component in negative degrees, we do not know if the standard resolution is a closed object, we cannot interpret $C(C) \Box C$ as $C \Box C$, and so we cannot apply Proposition 4.3. In fact we do not know how to define $- \Box C$ in general when the coalgebra $C$ is not positively graded because we do not know if there exist “enough” closed objects.

On the other hand, it would be interesting to know if $HC^*$ is invariant under cotilting equivalence. This would be implied if one shows that the coassociative maps $e_D(T \oplus D) \to e_D(T) = C$ and $e_D(T \oplus C) \to e_D(D) = D$ (where $DT C$ is a cotilting bicomodule) induce quasi-isomorphisms in $C_{Hoch}$. We know (see [8]) that this is the case when $C$ and $D$ are Morita - Takeuchi equivalent.

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