Real Quartic Surfaces in $\mathbb{RP}_3$ containing 16 Skew Lines

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Abstract

In [BN] the authors construct a special complex of degree 20 over $M$, which for an open three dimensional set parametrizes smooth complex surfaces of degree four invariant which are Heisenberg invariant, and each member of the family contains 32 lines but only 16 skew lines. The coordinates of the lines however need not be real. For a point $l$ in a Zariski open set of $M$ an algorithm is presented which evaluates the real coefficients of $l$ in terms of the K- coordinates of $l$. The author uses a code in Maple which allows him to construct very explicit examples of real smooth Heisenberg invariant Kummer surfaces containing the special configuration of 16 real skew lines. An example is presented at the end of the paper.

MSC (2000): 14Nxx,14Lxx,54Dxx,53Cxx,32Mxx.

1 Introduction

In the variables $x_0, x_1, x_2, x_3, x_4, x_5$ consider the equation :

\begin{align*}
x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 &= 0 \quad (1) \\
1/x_0^2 + 1/x_1^2 + 1/x_2^2 + 1/x_3^2 + 1/x_4^2 + 1/x_5^2 &= 0. \quad (2)
\end{align*}

This is a rational form of the equation which must be written in regular form! Let us consider a solution to these set of equations. For this let $u_i = x_i^2$ and fixing $u_2, u_3, u_4, u_5$ then in [Nie] we have found an equation of the form $u_0^2 + \lambda u_0 + \lambda/\mu = 0$ where $\lambda, \mu$ are constants depending on the above fixed variables. The solutions can however be complex. If we let $\iota = \sqrt{-1}$ and introducing

\begin{align*}
p_{01} &= x_0 - \iota x_1, & p_{02} &= x_2 - \iota x_3, & p_{03} &= x_4 - \iota x_5, \\
p_{23} &= -(x_0 + \iota x_1), & p_{13} &= x_2 + \iota x_3, & p_{12} &= -(x_4 + \iota x_5)
\end{align*}
if all \{p_{i,j}\} are real, the \(p_{i,j}\) coordinates will determine a real line \(l\) in \(\mathbb{P}_3\) contained in a real quartic surface, which is invariant under the well known group \(H^t\) the Heisenberg group of level 2. Such surfaces form a three dimensional parameter space within the thirty four dimensional space of quartic surfaces in \(\mathbb{RP}_3\) and are therefore very special. Here we present two algorithms for writing the equation for the surface and the lines lying on them together with one example illustrating the second algorithm. These examples are surfaces of degree four embedded as Kummer surfaces associated to abelian surfaces of polarization type \((1, 3)\) (c.f. \([BN]\) and \([Nie]\)). We give an elementary solution to the problem stated in the beginning of section 2 which solution is the first algorithm. In section 3 we introduce the second algorithm. In section 4 we briefly recall the basic concepts of Line Geometry and Quartic Surfaces in \(\mathbb{RP}_3\) to solve our problem, use the second algorithm in section 6 to construct one example and briefly sketch how the code was written in Maple. Additionally we produce the equation of the quartic surface and the set of skew lines lying on it.

2 An elementary Problem

We consider the following elementary problem which first solution was communicated by the author by W. Barth in an unpublished manuscript: Find solutions to the set of equations 1 and 2 such that \(x_0, x_2, x_4\) are real and \(x_1, x_3, x_5\) are purely imaginary. For later use we let

\[
x_1 = \iota y_1, x_3 = \iota y_3, x_5 = \iota y_5
\]

where \(\iota = \sqrt{-1}\). By substitution the new form of equation 2 is

\[
x_0^2 + x_2^2 + x_4^2 = y_1^2 + y_3^2 + y_5^2
\]

and similarly for the second equation

\[
1/x_0^2 + 1/x_2^2 + 1/x_4^2 = 1/y_1^2 + 1/y_3^2 + 1/y_5^2.
\]

We will normalize equation 1 as

\[
x_0^2 + x_2^2 + x_4^2 = 1 = y_1^2 + y_3^2 + y_5^2
\]

and set

\[
q = 1/x_0^2 + 1/x_4^2 + 1/x_2^2 = 1/y_1^2 + 1/y_3^2 + 1/y_5^2.
\]

If we introduce the following change of variables:

\[
q_i = \begin{cases} 
  x_i^2 & \text{for } i \text{ even}, \\
  y_i^2 & \text{otherwise}.
\end{cases}
\]

we obtain the following
Proposition 1 Choose \( q_4 \neq q_5 \) real such that
\[
0 \leq q_4, q_5 \leq 1
\]
and
\[
q \geq \max(1/q_4 + 4/(1 - q_4), 1/q_5 + 4/(1 - q_5))
\]
then there exist \( x_0, x_2, x_4, y_1, y_3, y_5 \) real satisfying equation (3).

Proof.- Considering
\[
1/q_0 + 1/q_2 = q - 1/q_4
\]
and setting \( s = 1 - q_4 = q_0 + q_2 \) we obtain the formal equation
\[
p = q_0 q_2 = (q_0 + q_2)/(q - 1/q_4),
\]
\[
= q_4 s/(q q_4 - 1),
\]
\[
= q_4 (1 - q_4)/(q q_4 - 1).
\]
Assuming that \( s, p \) are given then the roots of
\[
X^2 - sX + p = 0
\]
determine \( q_0, q_2 \) whenever \( q \) is given. In fact, \( q_0 = (s + \sqrt{s^2 - 4p})/2, q_2 = (s - \sqrt{s^2 - 4p})/2 \). Similarly if \( s' = 1 - q_5 = q_1 + q_3 \) we obtain the quadratic equation:
\[
X^2 - s'X + p' = 0
\]
and the condition \( q_4 \neq q_5 \) implies that \( s \neq s' \). Equation (3) has the elementary solutions
\[
(s \pm \sqrt{s^2 - 4p})/2.
\]
A necessary condition that equation (3) has different solutions is that
\[
s^2 \geq 4p = 4sq_4/(q q_4 - 1).
\]
By assumption \( s \geq 0 \) thus \( s \geq 4q_4/(q q_4 - 1) \) which gives
\[
q \geq 1/q_4 + 4/s.
\]
It also follows from the last inequality and equation (2) that
\[
p \geq 0, q_0, q_2 \geq 0.
\]
Similarly if \( q \geq 1/q_5 + 4/(1 - q_5) \) and \( 0 \leq q_5 \leq 1 \) then \( y_1, y_3 \) are roots of positive numbers.

\[\diamondsuit\]

Remark 2 For this condition the normalization of equation (3) is not needed. Moreover, the condition chosen for convenience \( q_4 = q_5 \) is indeed necessary since it implies that
\[
x_0^2 + x_1^2 = x_4^2 + x_5^2 = x_2^2 + x_3^2 = 0
\]
and it follows from [Nie1], §5 and §6] that the surface is elliptic ruled or singular along four nodes.
3 The Algorithm

Once again we fix the normalization of equation 4 in the coordinates \( q_i \), which is \( 1 = q_0 + q_2 + q_4 \). We are to find real positive solutions to the non-linear system of polynomial equations given in rational form as:

\[
\begin{align*}
\lambda &= 1/q_2 + 1/q_0 + 1/q_4 \\
1 &= q_0 + q_2 + q_4
\end{align*}
\]

For the following we let

\[
\omega = (\lambda q_0^2 - q_0(\lambda - 3) + 1)/\lambda \quad \text{and} \quad N = (\lambda q_0 - 1)(1 - q_0),
\]

\[
\Delta^2 = N(N - 4q_0).
\]

Note that these are polynomial expressions in \( q_0 \). We fix in the sequel the numerical value for \( q_0 \).

**Proposition 3** The roots of equation 6 for \( q_2 \) are:

\[
q_2 = (N \pm \Delta)/2(\lambda q_0 - 1) \quad \text{with} \quad \Delta^2 = \lambda^2 \omega(1/\lambda - q_0)(1 - q_0).
\]

**Proof.** Claim:

\[
q_2 = (\lambda q_0 - 1)(1 - q_0) \pm \sqrt{(\lambda q_0 - 1)^2(1 - q_0)^2 - 4q_0(1 - q_0)(\lambda q_0 - 1)/2(\lambda q_0 - 1)}
\]

To prove the claim from the above given equation one obtains

\[
q_0q_4 + q_2q_4 + q_0q_2 - \lambda q_0q_2q_4 = 1 - q_0 - q_2 - q_4 = 0
\]

By substituting the value of \( q_4 \) in the last equation one obtains the quadratic equation

\[
q_2^2(\lambda q_0 - 1) - q_2(\lambda q_0 - 1)(1 - q_0) + q_0(1 - q_0) = 0
\]

which solution is as claimed.

To finish the proof we write \( \Delta^2 = NM \) where \( M = (\lambda q_0 - 1)(1 - q_0) - 4q_0 \). The last equality is equal to \(- (\lambda q_0^2 + q_0(\lambda - 3) + 1)\).

**Remark 4** The condition \( 1 = q_0 + q_2 + q_4 \) still has to be fulfilled which does not follow from Proposition 3.

**Remark 5** Let \( q_0 > 1/\lambda \) then

\[
q_0 + q_2 < 1 \iff N - \epsilon \Delta > 0 \quad \text{with} \quad \epsilon \in \{\pm 1\}.
\]

**Proof of the remark** An easy computation shows that

\[
q_0 + q_2 = ((\lambda q_0 - 1)(q_0 + 1) + \epsilon \Delta)/2(\lambda q_0 - 1).
\]

Hence obtain \((\lambda q_0 - 1)(2 - (q_0 + 1)) + \epsilon \Delta > 0\).

We introduce more notation. Let \( \sigma = (\lambda - 3)/2\lambda, \rho = (\lambda - 9)(\lambda - 1)/4\lambda^2 \).
Proposition 6  \( \Delta^2 > 0 \iff \omega > 0, q_0 < 1/\lambda \) or \( \omega < 0, 1/\lambda < q_0 \).

Proof.- This is a consequence of Proposition 3.

Lemma 7  \( \omega > 0 \iff q_0 > \sigma + \sqrt{\rho} \) or \( q_0 < \sigma - \sqrt{\rho} \).

Proof.- By writing:

\[
\omega = q_0^2 - q_0(\lambda - 3)/\lambda + 1/\lambda,
\]

\[
= (q_0^2 - 2(\lambda - 3)/2\lambda q_0 + (\lambda - 3)^2/4\lambda^2) + 1/\lambda - (\lambda - 3)^2/4\lambda^2,
\]

\[
= (q_0 - (\lambda - 3)/2\lambda)^2 - \rho.
\]

The cases claimed in the lemma follow immediately from it.

Lemma 8  \( \omega < 0 \iff q_0 \in (\sigma - \sqrt{\rho}, \sigma + \sqrt{\rho}) \).

Lemma 9  If \( N > 0 \) then \( N - \Delta > 0 \). If additionally \( q_0 \in (\sigma - \sqrt{\rho}, \sigma + \sqrt{\rho}) \) then \( q_0 + q_2 < 1 \).

Proof.- \( \Delta^2 = -N\lambda\omega \). We have to evaluate \( N - \sqrt{-N\lambda\omega} \). But

\[
(N - \sqrt{-N\lambda\omega})(N + \sqrt{-N\lambda\omega}) = N(N + \lambda\omega)
\]

From \( N + \lambda\omega = 4q_0 > 0 \) thus \( N - \Delta > 0 \). Since \( N > 0 \) if and only if \( q_0 > 1/\lambda \) the last remark follows from Remark 3.

We shall need the following easy arithmetic:

Lemma 10  Assume \( \lambda > 9 \) then \( \sigma > \sqrt{\rho} \) and \( 1/\lambda < \sigma - \sqrt{\rho} \).

Proof.- To prove the first inequality:

\[
\lambda > 0 \iff (\lambda - 3)^2 > (\lambda - 9)(\lambda - 1) \iff (\lambda - 3) > \sqrt{(\lambda - 9)(\lambda - 1)}.
\]

To prove the second inequality:

\[
2 < (\lambda - 3) - \sqrt{(\lambda - 9)(\lambda - 1)}
\]

\[
\iff (\lambda - 5) > \sqrt{(\lambda - 9)(\lambda - 1)}
\]

\[
\iff (\lambda - 5)^2 > \lambda^2 - 10\lambda + 9
\]

and the last inequality is always true.

Corollary 11  Let \( q_0 < 1/\lambda \). \( q_0 + q_2 < 1 \iff -N + \Delta > 0 \) or \( -N - \Delta > 0 \).
Proof.- By using the proof of Remark 11, \( q_0 + q_2 < 1 \Longleftrightarrow -1(q_0 - 1)(q_0 + 1) + \epsilon\Delta < 2(1 - q_0) \Longleftrightarrow +\epsilon\Delta < -N \Longleftrightarrow 0 < -N \pm \Delta \) with \( \epsilon \in \{\pm 1\} \). \( \diamond \)

Proposition 12 Assume \( \lambda > 9 \) and \( \lambda q_0 < 1 \) then \( q_0 + q_2 < 1 \).

Proof.- Clearly \( \lambda q_0 < 1 \Longleftrightarrow \lambda q_0 - 1 < 0 \) thus \( N < 0 \). By lemma 11 to prove \( \omega > 0 \) we verify one of the inequalities: \( q_0 < \sigma - \sqrt{\rho} \) which is true by lemma 11 hence \( 1/\lambda < \sigma - \sqrt{\rho} \). By Prop. 11 one of the inequalities is satisfied thus \( \Delta^2 > 0 \) and \( -N + \Delta = -N + \sqrt{-N}\lambda \omega > 0 \) which is one of the inequalities to be satisfied in Corollary 11. \( \diamond \)

Proposition 13 Assume \( \lambda > 9 \) and \( q_0 \in (\sigma - \sqrt{\rho}, \sigma + \sqrt{\rho}) \) then \( q_0 > 1/\lambda, q_0 + q_2 < 1 \).

Proof.- If \( \lambda > 9 \) then \( 1/\lambda < \sigma - \sqrt{\rho} \) by lemma 11 but then \( q_0 > 1/\lambda \). In this case \( N > 0 \) hence by lemma 11 the inequality \( q_0 + q_2 < 1 \) is true. \( \diamond \)

Remark 14 The cases \( \lambda = 1, 9 \) in the above proposition give no criteria to find \( q_0 \) as a solution to equation 3. This can be explained as follows: fix the affine plane \( H = \{g = q_0 + q_2 + q_4 - 1 = 0\} \) in the \( \mathbb{R}^3 \) defined by the coordinates \( q_0, q_2, q_4 \) and define for each \( \lambda \in \mathbb{R}_{>0} \):

\[
\lambda = q_2q_4 + q_0q_4 + q_0q_2 - \lambda q_0q_2q_4
\]

which is a surface in the \( \mathbb{A}_3 \) defined by the coordinates \( q_0, q_2, q_4 \). Let \( C_\lambda = H \cap \{f_\lambda = 0\} \).

Proposition 15 The linear system of curves \( \{C_\lambda = \{f_\lambda = g = 0\}\}_{\lambda \in \mathbb{R}_{>0}} \) is always smooth except for \( \lambda = 1, 9 \) with singularities:

\[
\text{Sing}(C_0) = \{(1/3, 1/3, 1/3)\}, \quad \text{Sing}(C_1) = \{Q = (-1, 1, 1)\}.
\]

Proof.- To simplify the computations let \( \partial_i f = \partial f / \partial q_i \). Recall that \( f_\lambda = q_2q_4 + q_0q_2 + q_0q_4 - \lambda q_0q_2q_4 \) and \(-g = 1 - q_0 - q_2 - q_4\). Note that \( \partial_i f = q_j + q_k - \lambda q_jq_k \) for \( i, j, k \in \{0, 2, 4\} \) with \( i \neq j \neq k \) and \( \partial_i g = -1 \). The jacobian matrix is:

\[
\begin{pmatrix}
\partial_2 f_\lambda & \partial_0 f_\lambda & \partial_4 f_\lambda \\
-1 & -1 & -1
\end{pmatrix},
\]

It is of rank less than one if and only if:

\[
0 = (\partial_i - \partial_j)f_\lambda = 0 \quad i, j \in \{0, 2, 4\}.
\]

It follows that: \((q_i - q_j)(1 - \lambda q_k) = 0\) for some \( i, j, k \in \{0, 2, 4\} \). Due to the symmetries of the indices it is enough to consider the following cases:

I. \( 1 - \lambda q_4 = q_0 - q_4 = q_2 - q_4 = 0 \). \( q_2 = q_4 = q_0 \). \( q_4 = 1/\lambda \). From \( 0 = g = 1 - 3q_0 \). Therefore \( q_0 = 1/3 \). Substituting \( 0 = f_\lambda = 3 - \lambda/3 \). Thus \( \lambda = 9 \).
II. $1 - \lambda q_4 = 1 - \lambda q_2 = q_2 - q_4 = 0$. Therefore $q_2 = q_4 = 1/\lambda$. $g = 1 - 2/\lambda - q_0 = 0$. Substituting $q_0 = 1 - 2/\lambda$ in $f_\lambda = 0$ one obtains $0 = f_\lambda = (1 + \lambda - 2)/\lambda = 0$. Thus $\lambda = 1$.

III. $1 - \lambda q_4 = 1 - \lambda q_2 = 1 - \lambda q_0 = 0$. Therefore $q_0 = q_2 = q_4 = 1/\lambda$. Hence $f_\lambda = 2/\lambda^2 = 0$ which is impossible.

IV. $0 = q_2 - q_0 = q_0 - q_4 = q_2 - q_4$. It follows that $q_0 = q_4 = q_2$. Hence $g = 1 - 3q_0 = 0$. Substituting $q_0 = 1/3$ in $f_\lambda$ one obtains $f_\lambda = 3/9 - \lambda/27 = 0$ therefore $\lambda = 9$. Fix once again the $\mathbb{R}^3$ defined by the coordinates $x, y, z$.

Lemma 16 The equation of $\{C_\lambda = \{0 = f_\lambda(x, y, z) = 1 - (x + y + z)\}\}_{\lambda \in \mathbb{R} > 0}$ can be written for $\lambda = 9$ as:

$$
\begin{align*}
  f_9 &= 9(x^2y + xy^2) + x + y - 10xy - (x^2 + y^2), \\
  f_1 &= x^2y + xy^2 + x + y - 2xy - (x^2 + y^2).
\end{align*}
$$

The proof of the lemma is left as an easy exercise. For the next lemma recall that $P = (1/3, 1/3, 1/3)$ is a singular point for $C_9$ and $Q = \text{Sing}(C_1)$.

Lemma 17 The equation for the tangent cone of $C_9$ (resp. of $C_1$) passing through $P$ (resp. $Q$ of $C_1$) is $x - 1/3)^2 + (x - 1/3)(y - 1/3) + (y - 1/3)^2$ (resp. $4(y - 1)(x + y)$). In particular, $P$ (resp. $Q$) is a non-ordinary double point of $C_9$ at $P$ (resp. $Q$ of $C_1$).

Proof.- The second partial derivatives at $P$ are given as $\partial_x^2 f_9 = -2 + 18y$, $\partial_{xy} f_9 = -10 + 18x + 18y$, $\partial_y^2 f_9 = -2 + 18x$. Summarizing: $\partial_x^2 f_9(P) = 4$, $\partial_y^2 f_9(P) = 4$, $\partial_{xy} f_9(P) = 2$. The equation for the tangent cone at $P$ is then

$$
4(x - 1/3)^2 + 4(x - 1/3)(y - 1/3) + 4(y - 1/3)^2.
$$

The calculation for $C_1$ can be done analogously.

Remark 18 A direct computation shows that our $f_9 = 0$ is irreducible over $\mathbb{R}$. Under the linear change of coordinates $u = x - 1/3, v = y - 1/3$ the equation for $f_9 = 0$ is transformed to $f_9 = 9uv(u + v) + 2uv + 2(u^2 + v^2)$. Under this linear change of coordinates the cubic curve $C_9$ is transformed to a real cubic with isolated singularity at the origin which is to be expected from the classification of irreducible cubic curves over the real number field.

4 The 32 lines on the quartic surface.

We start our discussion with some well known facts on Line Geometry and Group theory. Fix the three dimensional real projective space $\mathbb{RP}^3$ with coordinates $z_0, z_1, z_2, z_3$. Introduce the quartic surface $X_f = \{f(z_0, z_1, z_2, z_3) = 0\}$ given by the homogeneous polynomial $f$ of degree four in the variables $z_0, z_1, z_2, z_3$. A line in $\mathbb{RP}^3$ is generated by a two
plane in \( \mathbb{R}^4 \) represented by a two by four matrix

\[
\begin{pmatrix}
1 & 0 & * & *
\end{pmatrix}
\begin{pmatrix}
0 & 1 & * & *
\end{pmatrix}
\]

(There are in fact 6 ways of choosing the matrix with this property !) In studying problems related to the study of the Geometry of hypersurfaces of the grassmanian of lines in \( \mathbb{P}_3 \), known as Line Geometry, there exist various choices for the coordinates to use. In studying the line complex it was typical in the nineteenth and the early twentieth century to introduce special coordinate systems such as elliptic, pentaspherical coordinates (c.f. [Je, chap. VIII, §130 and chap. XII, §221]. A coordinatefree approach uses only linear algebra to characterize properties of the line complex (c.f. [GH, chap. 6]). Here we choose the first approach due to the symmetries of our problem ( c.f.[BN] for the relation of this approach to polarized abelian surfaces). More precisely let

\[
\Lambda = \begin{pmatrix}
z_0 & z_1 & z_2 & z_3 \\
z_0' & z_1' & z_2' & z_3'
\end{pmatrix}
\]

be a two by four matrix and introduce the following coordinates

\[
p_{i,j} = z_i z_j' - z_j z_i' \quad i,j \in \{0,1,2,3\}
\]

where \( i \neq j \). These are the Plücker coordinates or P-coordinates for short ; In such coordinates the condition that a matrix \( \Lambda \) defines a two-plane is given as:

\[
p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0.
\]

In terms of multilinear algebra this is nothing else than giving a two form \( \omega \) such that \( \omega \wedge \omega = 0 \) (c.f. [GH, chap. 1]). In the coordinates \( \{p_{i,j}\} \) this is a hypersurface of degree two (the Plücker quadric). Let \( H^2 \) be the subgroup of \( \text{SL}(4, \mathbb{R}) \) spanned by the transformations

\[
\sigma_1 = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} \quad \sigma_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

\[
\tau_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \quad \tau_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

which satisfy the relations:

\[
\sigma_i^2 = \tau_i^2 = \text{id}, \quad \sigma_i \tau_i = -\tau_i \sigma_i
\]
for $i = 1, 2$. One obtains a central exact sequence of groups:

$$1 \rightarrow \{\pm 1\} \rightarrow H^t \rightarrow G' \rightarrow 0$$

where $G' \simeq \mathbb{Z}_2^4$. The explicite action of $H^t$ on $f$ for a polynomial as above on the P-coordinates is induced by the usual linear action induced on the polynomials of degree four, in particular

$$\sigma(z_0 z_1 z_2 z_3) = z_0 z_1 z_2 z_3$$

for all $\sigma \in H^t$.

The action of $H^t$, the (unique up to a constant) Schrödinger representation of degree four on $\mathbb{R}^4$ induces a representation on $\wedge^2 \mathbb{R}^4$ as given in [BN, p. 178]. Introduce the following coordinate transformation in $\mathbb{RP}^4$:

$$x_0 = p_{01} - p_{23}, \quad x_2 = p_{02} + p_{13}, \quad x_4 = p_{03} - p_{12};$$

$$x_1 = \sqrt{-1}(p_{01} + p_{23}), \quad x_3 = \sqrt{-1}(p_{02} - p_{13}), \quad x_5 = \sqrt{-1}(p_{03} + p_{12}).$$

Note that

$$x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = -2(p_{01} p_{23} - p_{02} p_{13} + p_{03} p_{12})$$

which is the equation for the Plücker quadric, which parametrizes the set of lines in $\mathbb{RP}^3$. The importance of the coordinates $\{x_i\}$ called the Klein coordinates , hereafter K-coordinates, is that these are eigenfunctions for the action of the Heisenberg group $H^t$ on them. These eigenfunctions are useful for studying the configuration of lines as in [BN, section 4]. Equation 7 parametrises the set of lines in $\mathbb{RP}^3$. By inverting the transformation induced by equation 4 we obtain the coordinates for the P-coordinates in terms of the K-coordinates as presented in the introduction. Using equation 3 the statement of the PROBLEM stated in section 2 is formulated equivalently as:

The line with P-coordinates $\{p_{i,j}\}$ defines a real line if and only if $x_0, x_2, x_4$ are real and $x_1, x_3, x_5$ purely imaginary.

The P-coordinates can be expressed in terms of the $\{q_i\}$-coordinates as:

$$\begin{pmatrix}
  p_{01} \\
  p_{03} \\
  p_{13}
\end{pmatrix} = \begin{pmatrix}
  \sqrt{q_0} + \sqrt{q_1} \\
  \sqrt{q_4} + \sqrt{q_5} \\
  \sqrt{q_2} - \sqrt{q_3}
\end{pmatrix},$$

$$\begin{pmatrix}
  p_{02} \\
  p_{23} \\
  p_{12}
\end{pmatrix} = \begin{pmatrix}
  \sqrt{q_1} - \sqrt{q_0} \\
  \sqrt{q_3} - \sqrt{q_2} \\
  -(\sqrt{q_0} - \sqrt{q_1})
\end{pmatrix}.$$

Let us consider a line $l \subset X$ with coordinates $\{p_{i,j}\}$ such that $p_{01} \neq 0$ (which is equivalent to $\sqrt{q_0} \neq -\sqrt{q_1}$). The two points with coordinates

$$\begin{pmatrix}
  p_a \\
  p_b
\end{pmatrix} = \begin{pmatrix}
  1 & 0 & x & y \\
  0 & 1 & u & v
\end{pmatrix}$$
in $\mathbb{RP}_3$ are determined by the P-coordinates as:

$$
\begin{align*}
  u &= p_{02}, & v &= p_{03}, \\
  y &= -p_{13}, & x &= -p_{12}.
\end{align*}
$$

The line spanned by $p_a, p_b$ will be by definition $l_{a,b}$.

Let us consider quartic surfaces which are invariant under the action of $H^t$. We fix once again the quartic surface $X = \{ f = 0 \}$ then $\sigma f = \lambda_\sigma f$ for all $\sigma \in H^t$ such that $\lambda_\sigma \in \{ \pm 1 \}$. One sees immediately \cite[Prop. 4.1.1 ii)]{Nie} that the set of real quartic forms invariant under $H^t$ is a real vector space of dimension five. By fixing such a form it depends on the real values $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ which are the coefficients of that form. If a form of degree four contains a line it imposes five linear conditions on the coefficients. More precisely, the restriction of $f$ to a line $l_{a,b}$ induces five linear equations in the coordinates of the two points on the line and the coefficients $\lambda_i$ for $i = 0, \ldots, 4$. It follows \cite[§4.3]{BN} that each line $l_{ab}$ with K-coordinates satisfying equation (9) belongs to at least one quartic surface defined by a quartic form invariant under $H^t$. Therefore the five linear equations are linearly dependent and projectively four of these equations determine the set of coefficients $\{ \lambda_i \}$ of the quartic form which depend on the equivalence class of $l_{a,b}$ modulo the action of $H^t$. We will assume in section §6 that $\lambda_4 = 1$. The Maple program given in section §6 allows us to determine two other points of the line given two of the points of the line $l_{ab}$ hence by the previous discussion determine the coefficients $\lambda_i$ for $i = 0, \ldots, 4$ of the quartic surface. If an $H^t$ invariant quartic contains a line $l$ then the orbit of $l$ under $G'$ contains 16 skew lines. In the K-coordinates $\{ x_i \}$ the involution

$$
(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (-1/x_0 : 1/x_1 : 1/x_2 : 1/x_3 : 1/x_4 : 1/x_5)
$$

which is well defined away from the fourfolds $\{ x_i = 0 \}$ applied to $l$ gives a line $l'$. Writing the P-coordinates associated to this line (for this let $q' = -1/\sqrt{q_0} - 1/\sqrt{q_1}$):

$$
\begin{align*}
  \begin{pmatrix}
    p'_{01} \\
    q' \cdot p'_{03} \\
    q' \cdot p'_{13}
  \end{pmatrix} &= \begin{pmatrix} 1 \\
    1/\sqrt{q_4} - 1/\sqrt{q_5} \\
    1/\sqrt{q_2} + 1/\sqrt{q_3}
  \end{pmatrix}, \\
  \begin{pmatrix}
    q' \cdot p'_{02} \\
    q' \cdot p'_{23} \\
    q' \cdot p'_{12}
  \end{pmatrix} &= \begin{pmatrix} 1/\sqrt{q_2} - 1/\sqrt{q_3} \\
    1/\sqrt{q_0} - 1/\sqrt{q_1} \\
    -1/\sqrt{q_4} + 1/\sqrt{q_5}
  \end{pmatrix}.
\end{align*}
$$

The previous P-coordinates allows us in the Maple program given in section §6 to determine very explicitly the parametric equation of the transerval lines once given the line $l_{ab}$. Using the K-coordinates $l'$ cuts exactly 10 lines of the Heisenberg orbit of $l$ \cite[prop. 4.2 b)]{BN}. By reasons of degree, $l' \subset X$. Hence the orbit of $l'$ under $G'$ is contained in $X$. The two orbits of lines that are in $X$ can be grouped as the “even” lines as those having an even number of minus signs in its K-coordinates and the “odd” lines as having
an odd number of minus signs in its K-coordinates (in fact using these K-coordinates we have studied in [BN, prop. 4.2] group-theoretical properties of the configuration of these lines).

Classically, the set of odd and even lines form a configuration of type $(16_{10}, 16_{10})$ on the quartic surface forming a double 16 (as is referred in [BN, loc. cit.]). By [Se] a smooth quartic surface can contain at most 48 lines so if it contains at least 32 lines it must contain exactly 32 lines.

5 A brief description of the program

The program written in Maple IV release 4 defines the following local variables:

\[ R, S, q_1, q_0, M, N, Sq_1, Sq_2, q_2, q_3, q_4, q_5, M_1, N_1, rr, ss, zz, com, ww, m, n, K, quarn \]

and the global variable \( d \). It uses lemma 5 given in section 3. The initial value for the program is \( \lambda \) (in the program this value is \( d \)). It then evaluates \( q_0 \) and \((R, S)\) using prop. 8; this calculation is performed in the procedures Var, Vas. In order to evaluate the variable \( q_2 \) one needs to introduce the variables \( M, N \) in terms of \( d \) and \( q_0 \) and finally evaluate \( Sq_1 = \sqrt{MN} \). The algorithm then evaluates the positive root of \( q_2 \) in terms of \( N, Sq_1 \) and \( dq_0 - 1 \). It follows that \( q_4 = 1 - (q_0 + q_2) \). Analogously, if one gives the value for \( q_1 \) the program evaluates \( q_3, q_5 \) in exactly the same way.

In order to draw the lines one evaluates the parametric equation for the lines. The procedure to evaluate them in the program has been given in section 4. We introduce the variables \( rr, ss \) in terms of \( q_4, q_5, q_2, q_3 \). The last variables are used to give the parametric equation of the line \( l \) as given by equation 9. The orbit of \( l \) under \( H^t \) is evaluated using the procedure graf. The parametric equation of the transversal to \( l \) is evaluated using the variables \( m, n \). For these one uses the variables \( rr, zz, ww \) and uses equation 10. The coefficients for the quartic surface are evaluated by means of a matrix array using the elementary method of the previous section. The equation of the quartic surface in a fixed solid hereafter named the “clipping solid” is evaluated in a procedure and saved as the variable quarn. The procedure graf used to draw the surface (resp. the lines lying on this surface) is based on the command (resp. spacecurve) display3d of the library plots of Maple.

6 An Example

To illustrate the above theory we will introduce a typical example. The value for this example was chosen from a series of 43 test examples for \( d \in [9.3, 18.00] \) on very different intervals, on rectangular grids of \( 30^3 = 27,000 \) points (using the command
grid). The program was run on a Samsung pentium II. The average compilation time per image for these examples averaged to three minutes. We introduce the value for $d$ which in this case is $18.00$ (this is the upper bound!). The necessary condition for $q_0$ is given in lemma [3]. For practical purposes we introduce two small routines written in Maple to evaluate $q_0, q_1$ in terms of $d$. For this example, $q_0 = 0.4168, q_1 = 0.1713$. Using the value for $q_0$ and two other procedures written on the program one obtains the values $M = 2.124999680, N = 3.792199680$. Using $S_{q_1} = 2.83874$ one evaluates $q_2 = 0.5101, q_4 = 0.0731$. The values for $M_1, N_1, S_{q_2}, q_3, q_5, rr, ss, zz, m, ww, n$ are (respectively) $1.041313580, 1.762513580, 1.3408, 0.7364, 0.0923, -0.0334, 1.5724, 0.14393, -0.10242, 0.57418, -0.059153$. Using the elementary theory described in section [4] one obtains a four by four matrix $A$ with entries polynomials of degree four in the variables $x, y, u, v$ and a four by one vector such that the extended matrix $M = (A, b)$ gives the equation $M \cdot \lambda = 0$ meaning that the quartic surface with vector coefficients $\lambda$ contains a line. Written in non-homogeneous form one obtains: $A \cdot \lambda = b$. The procedure mpoly substitutes the variables $x, y, u, v$ for the obtained values $rr, ss, zz, ww$. For this example $\lambda = (-0.366, -1.44, 0.614, 0.526)$. We have chosen the clipping solid as $\{ x+y+u+v = 0 \}$. In this solid the equation for the quartic polynomial is given by:

\[
\begin{align*}
\text{quarn} &= 0.320x^4 + 0.496y^4 + -3.612u^4 \\
&-2.796(x^2y^2 + x^2u^2 + y^2u^2) + uy(-u^27.224 + 0.992y^2) \\
&+xy(0.640x^2 + y^20.992) + xu(x^20.640 - u^27.224) \\
&+xyu(-y5.936 - 14.152u - x6.288). \\
\end{align*}
\]

In the image produced by the procedure graf of the Maple program in all the examples, the line $l$ has been drawn in red. The remaining colors for the disjoint lines are: blue, yellow, sienna, cyan, khaki, pink, turquoise, aquamarine, magenta, plum, violet, braun, green, navy, gold. The ten transversals to $l$ have been drawn in color grey. In the plot3d command of Maple one has to specify a pair of values $(u, v)$ for the orientation i.e. the direction from which the object is to be viewed. For the same group of test examples described in the interval mentioned at the beginning of this paragraph we tested pairs $u \in (0, 45)$ and $v \in (0, 90)$ showing in each image at most 7 transversals, with no further improvement. We illustrate an example at the end of the references for the surface defined by the above given polynomial with a grid of 27,000 points allowing us to see seven of the transversal lines. It is for $d = 18.00$ and at $(u, v) = (20, 36)$.

**References**

[BN] Barth, W. and Nieto, I. Abelian surfaces of type (1,3) and quartic surfaces with 16 skew lines *J. Alg. Geom.*, 3 (1994), pp. 173-222.

[GH] Griffiths, P. and Harris, J. *Principles of Algebraic Geometry*, N.Y. Wiley (1978).
[Je] Jessop, C.M.A. *Treatise on the Line Complex*, New York, Chelsea (1903).

[Nie] Nieto, I. *Invariante Quartiken Unter der Heisenberg Gruppe T*, PhD Thesis, U. of Erlangen, (1989).

[Nie1] Nieto, I. Examples of abelian surfaces with polarization type (1,3) in *Algebraic Geometry and Singularities* (ed. C. Lopez and N. Macarro), Progress in Mathematics, vol. 134, pp. 319-337, Birkhauser, Basel (1996).

[Se] Segre, B. The maximum number of lines lying on a quartic surface *Oxf. Quar. Journ.*, 14 (1943), pp. 87-96.

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Figure 1: The quartic surface for $d = 18.00$ at (20, 36).