Linear operators in distribution spaces

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Abstract
In this paper we define the \( S \)-linear operators (Schwartz linear operators) among spaces of tempered distributions. These operators are the analogous of linear continuous operators among separable Hilbert spaces, but in the case of spaces endowed with Schwartz bases having a continuous index set. The Schwartz linear operators enjoy properties very similar to those enjoyed by linear operators in the finite dimensional case. The \( S \)-operators are one possible rigorous mathematical model for the operators and observables used in Quantum Mechanics.

1 Introduction

Let \( X \) and \( Y \) be two vector spaces on the field \( K \) (the real field \( \mathbb{R} \) or the complex one \( \mathbb{C} \)). A function \( f \) from \( X \) into \( Y \) is called linear if, for any two points \( x, y \) of the space \( X \) and for each scalar \( \lambda \in \mathbb{K} \), the equality

\[
  f(\lambda x + y) = \lambda f(x) + f(y),
\]

holds true. Equivalently, a mapping \( f \) from \( X \) into \( Y \) is linear if and only if for every integer \( k \in \mathbb{N} \), for any \( k \)-tuple \( x = (x_i)_{i=1}^k \) of points of the space \( X \) and for any \( k \)-tuple of scalars \( \lambda = (\lambda_i)_{i=1}^k \) in \( \mathbb{K} \), setting

\[
  \sum \lambda x := \sum_{i=1}^k \lambda_i x_i
\]

and \( f(x) := (f(x_i))_{i=1}^k \), we have

\[
  f \left( \sum \lambda x \right) = \sum \lambda f(x),
\]
i.e., the image of the λ-linear combination of a family \( x \) is the λ-linear combination of the image family \( f(x) \) of the family \( x \) under the function \( f \); in indexed notation, we have

\[
f \left( \sum_{i=1}^{k} \lambda_i x_i \right) = \sum_{i=1}^{k} \lambda_i f(x_i).
\]

The aim of this chapter is to extend the last definition to the class of \( S \) families of tempered distributions indexed by the Euclidean space \( \mathbb{R}^k \), using, as coefficient systems, locally summable maps from \( \mathbb{R}^k \) to \( \mathbb{K} \) and, more generally, Schwartz tempered distributions from \( \mathbb{R}^k \) into \( \mathbb{K} \) (which, as we already have seen, are so viewed as “non-pointwise defined” families in the field \( \mathbb{K} \) indexed by the Euclidean space \( \mathbb{R}^k \)). If \( v = (v_i)_{i \in \mathbb{R}^k} \) is an \( S \) family in the distribution space \( S'_n \), i.e. if for every test function \( \phi \in S_n \), the function

\[
v(\phi) : \mathbb{R}^k \to \mathbb{K} : i \mapsto v_i(\phi),
\]

belongs to the test function space \( S_k \), and if \( \lambda \in S'_k \) is a tempered distribution defined on the index set of the family \( v \), we put

\[
\int_{\mathbb{R}^k} \lambda v := \lambda \circ \hat{v} = ^t(\hat{v})(\lambda),
\]

where \( ^t(\hat{v}) \) is the (topological) transpose of the continuous operator

\[
\hat{v} : S_n \to S_k : \phi \mapsto v(\phi).
\]

The idea is very natural:

- an operator \( L : S'_n \to S'_m \) is said to be \( S \)-linear if, for every integer \( k \in \mathbb{N} \), for every distribution coefficient \( \lambda \) in \( S'_k \) and for every family of distributions \( v \) in \( S(\mathbb{R}^k, S'_n) \), the image of the \( S \) family \( v \) is an \( S \) family too and the equality

\[
L \left( \int_{\mathbb{R}^k} \lambda v \right) = \int_{\mathbb{R}^k} \lambda L(v),
\]

holds true.
First of all we have to transform a family of tempered distributions by means of operators defined on spaces of tempered distributions, the definition is pointwise and absolutely straightforward.

**Definition (image of a family of distributions).** Let $W$ be a subset of the distribution space $S'_n$, let $A : W \to S'_m$ be an operator (not necessarily linear) and let $v = (v_p)_{p \in \mathbb{R}^k}$ be a family of tempered distributions belonging to the subset $W$, i.e. a family with trace (trajectory) set $\{v_p\}_{p \in \mathbb{R}^k}$ contained in the subset $W$. The image of the family $v$ by means of the operator $A$ is, by definition, the family $A(v)$ in $S'_m$ defined by

$$A(v) := (A(v_p))_{p \in \mathbb{R}^k},$$

i.e., the family $A(v)$ such that, for all index $p \in \mathbb{R}^k$, we have $A(v)_p = A(v_p)$.

We can read the above definition saying that:

- the image (under an operator) of a family of vectors is the family of the images of vectors.

**Definition (operator of class $S$).** Let $W$ be a subset of the space $S'_n$ and let $L : W \to S'_m$ be an operator (not necessarily linear). The operator $L$ is said to be an $S$-operator or operator of class $S$ if, for each natural $k$ and for each $S$-family $v \in S(\mathbb{R}^k, S'_n)$ with trajectory contained in $W$, the image $L(v)$ of the family $v$ is an $S$-family too (that is, if the image $L(v)$ belongs to the space $S(\mathbb{R}^k, S'_m)$).

We can read the above definition as follows:

- an operator $L$ is of class $S$ if the image by $L$ of any $S$-family is an $S$-family too.
3 \textit{Operators defined on }\mathcal{S}'_n\

The following property proves that the class of linear $\mathcal{S}$ operators defined
on the entire space of tempered distribution contains the class of weakly* continuous linear operators on that space.

**Theorem (the transpose of an operator).** The transpose of a weakly continuous linear operator defined among two spaces of Schwartz test functions is an $\mathcal{S}$ operator. Consequently, every weakly* continuous linear operator defined among two spaces of tempered distributions is an $\mathcal{S}$ operator. Moreover, the operator associated with the image of a family $v$ by the transpose of a weakly continuous operator $A$ is the composition $\tilde{v} \circ A$, that is, we have

$$tA(v)^\wedge = \tilde{v} \circ A.$$

**Proof.** Let $A : \mathcal{S}_n \to \mathcal{S}_m$ be a continuous linear operator with respect to the pair of weak topologies $(\sigma (\mathcal{S}_n), \sigma (\mathcal{S}_m))$. Then, the operator $A$ is (topologically) transposable (i.e., for every tempered distribution $a \in \mathcal{S}'_m$, the functional $a \circ A$ lies in the space $\mathcal{S}'_n$) and its (topological) transpose is (by definition) the operator

$$tA : \mathcal{S}'_m \to \mathcal{S}'_n : a \mapsto a \circ A.$$

Let $v \in \mathcal{S}(\mathbb{R}^k, \mathcal{S}'_n)$ be an $\mathcal{S}$ family of distributions, we have, by definition of image of a family,

$$tA(v)_p = tA(v_p),$$

and hence we deduce

$$tA(v)(\phi)(p) = tA(v)_p(\phi) = tA(v_p)(\phi) = v_p(A(\phi)) = v(A(\phi))(p),$$

so, taking into account that the family $v$ is an $\mathcal{S}$ family, we deduce that the image

$$tA(v)(\phi) = \tilde{v}(A(\phi))$$
belongs to the space $S_k$. Concluding, the image family $^t A(v)$ is a family of class $S$ belonging to the space $S(\mathbb{R}^k, S'_n)$, and thus the transpose operator $^t A$, sending $S$ families into $S$ families, is an $S$ operator. By the way, we proved also that the operator associated with the image family $^t A(v)$ is the composition $\hat{\nu} \circ A$, that is $^t A(v)^\wedge = \hat{\nu} \circ A$. ■

**Application.** Let $L : S'_n \to S'_n$ be a differential operator with constant coefficients and let $v$ be an $S$ family in the space $S'_n$. Then $L(v)$ is an $S$ family, in fact the operator $L$ is the transpose of some differential operator on the space $S_n$. For instance, the Dirac family $(\delta_x)_{x \in \mathbb{R}^n}$ is obviously an $S$ family, and so the family of $i$-th derivatives $(\delta^{(i)}_x)_{x \in \mathbb{R}^n}$ is an $S$ family too, for every multi-index $i$.

## 4 Transposability of linear $S$ operators on $S'_n$

The following property proves that linear $S$ operators defined among distribution spaces are weakly* continuous upon any subspace generated (in the usual algebraic sense) by an $S$ basis.

**Theorem (transposability of $S$ operators).** A linear $S$ operator defined among two spaces of tempered distributions is weakly* continuous on any algebraic linear hull of $S$ basis.

**Proof.** If $L : S'_n \to S'_n$ is a linear $S$ operator, and if $v$ is an $S$ basis in $S'_n$ indexed by the $k$-dimensional real Euclidean space, then the image $L(v)$ of the $S$ basis $v$ is a family of class $S$ in the space $S'_m$ indexed by $\mathbb{R}^k$. So for every test function $h$ in $S_m$, the image of the function $h$ by the family $L(v)$ is a function of class $S$ (belonging to the space $S_k$), namely the function

$$L(v)(h) : \mathbb{R}^k \to \mathbb{K} : q \mapsto L(v_q)(h).$$

We claim that the restriction $M$ of the operator $L$ to the pair $(E, F)$, where $E$ is the linear hull of the family $v$ and $F$ is the linear hull of the family $L(v)$, is weakly topologically transposable with respect to the weak dual pair.
(E, S_n) and (F, S_m). Indeed, we claim that its weak transpose \(^t M\) is the operator

\[
T : S_m \rightarrow S_n : h \mapsto v^-(L(v)(h)),
\]

where \(v^-\) is the inverse of the \(S\) basis \(v\). We have to prove, by the classic definition of weak transpose, that

\[
\langle u, T(h) \rangle_n = \langle L(u), h \rangle_m,
\]

for every tempered distribution \(u\) in the space \(E\) and for every test function \(h\) in the space \(S_m\). But, since \(u\) is a finite linear combination of the family \(v\), we can prove the above duality condition only for the elements of the family \(v\), and we indeed have, for every \(q\) in \(\mathbb{R}^k\),

\[
\langle v_q, T(h) \rangle_n = \langle v_q, v^-(L(v)(h)) \rangle_n = v(v^-(L(v)(h)))(q) = L(v)(h)(q) = L(v_q)(h) = \langle L(v_q), h \rangle_m,
\]

as we desire. Since every weak topologically transposable operator is weakly continuous, we conclude that every linear \(S\) operator is weakly continuous on the linear span of an \(S\) basis. ■

5 \(S\) Linear operators on \(S'_n\)

In this section we shall introduce the main concept of the chapter.

**Definition (**\(S\) linear operators on the entire \(S'_n\))**. Let \(L : S'_n \rightarrow S'_m\) be an \(S\) operator (not necessarily linear). The operator \(L\) is called **\(S\) linear operator** if, for each positive integer \(k\), for each \(S\) family \(v \in S(\mathbb{R}^k, S'_n)\) and for every tempered distribution \(a\) in the space \(S'_k\), the equality

\[
L \left( \int_{\mathbb{R}^k} av \right) = \int_{\mathbb{R}^k} aL(v)
\]
holds true.

Utterly, an \( S \)-linear operator must be linear, as we prove below.

**Property (linearity of the \( S \)-linear operators).** An \( S \)-linear operator is linear.

**Proof.** Indeed, in the conditions of the above definition, for each couple of scalars \( b, c \) and any couple \( u, w \) of tempered distributions in the space \( S' \), if \( \delta \) is the Dirac basis of the space \( S' \), we have

\[
L(au + bw) = L\left( \int_{\mathbb{R}^k} (au + bw)\delta \right) = \\
= \int_{\mathbb{R}^k} (au + bw)L(\delta) = \\
= a\int_{\mathbb{R}^k} uL(\delta) + b\int_{\mathbb{R}^k} wL(\delta) = \\
= aL\left( \int_{\mathbb{R}^k} u\delta \right) + bL\left( \int_{\mathbb{R}^k} w\delta \right) = \\
= aL(u) + bL(w),
\]

as we desired. ■

But we will see more than this preliminary remark about \( S \)-linear operators.

**Remark.** The above definition and property can be immediately be generalized to the case of operators defined on the \( S \)-linear hull \( E \) of an \( S \)-basis.

### 6 Examples of \( S \)-linear operators

In this section we propose two important examples of \( S \)-linear operators. We note that the first is a particular case of the second one, and indeed we shall see that every \( S \)-linear operator defined on the entire \( S' \) is of the type presented in the second example.
6.1 The superposition operator of an $S$ family

Recall that if $v \in s(\mathbb{R}^k, S'_m)$ is any family of tempered distributions in $S'_m$ indexed by an Euclidean space $\mathbb{R}^k$ and if $w \in S(\mathbb{R}^m, S'_n)$ is any $S$ family of tempered distributions in $S'_n$, the family in $S'_n$ indexed by $\mathbb{R}^k$ and defined by

$$\int_{\mathbb{R}^m} vw := \left( \int_{\mathbb{R}^m} v_p w \right)_{p \in \mathbb{R}^k},$$

is called the superposition of the $S$ family $w$ with respect to the family $v$.

We have already proved that, if the family $v$ belongs to the space $S(\mathbb{R}^k, S'_m)$ then the superposition $\int_{\mathbb{R}^m} vw$ belongs to the space $S(\mathbb{R}^k, S'_n)$ and the operator associated with this superposition is the composition of the operators associated with the two families $v$ and $w$, precisely we have

$$\left( \int_{\mathbb{R}^m} vw \right)^\wedge = \hat{v} \circ \hat{w}.$$

In this case, sometimes, it is also convenient to denote the superposition

$$\int_{\mathbb{R}^m} vw$$

by the product notation $v.w$ and to call it also the $S$ product of the family $v$ by the family $w$.

**Proposition.** Let $w \in S(\mathbb{R}^m, S'_n)$ be an $S$ family of distributions and let $L : S'_m \to S'_n$ be the superposition operator of the family $w$, defined by

$$L(a) = \int_{\mathbb{R}^m} aw,$$

for all tempered distribution $a \in S'_m$. Then, the operator $L$ is an $S$ linear operator.

**Proof.** The operator $L$ is an $S$ operator, indeed we know that $L$ is the transpose of the continuous linear operator associated with $v$ and then it is weakly* continuous. But we decide to see this fact directly too. If $v \in$
$S(\mathbb{R}^k, S'_m)$ is an $S$ family then its image by the operator $L$ is $L(v) = v.w$ and the product of two $S$ families is an $S$ family. Let $a \in S'_k$ be a tempered distribution and let $v \in S(\mathbb{R}^k, S'_m)$ be an $S$ family, we have

$$L\left( \int_{\mathbb{R}^k} av \right) = \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^k} av \right) w = \int_{\mathbb{R}^k} a \left( \int_{\mathbb{R}^m} vw \right) = \int_{\mathbb{R}^k} aL(v),$$

applying the already known property of "S linearity" of superpositions. Note in fact that, for each index $p \in \mathbb{R}^k$, we have

$$L(v)_p = L(v_p) = \int_{\mathbb{R}^m} v_p w = \left( \int_{\mathbb{R}^m} vw \right)_p,$$

and the proof is completed. ■

### 6.2 Transpose operators

**Lemma (the image under a transpose operator).** Let $B \in \mathcal{L}(S_n, S_m)$ be a linear continuous operator and let $v \in S(\mathbb{R}^k, S'_m)$ be an $S$ family. Then, the image of the family $v$ by the transpose operator $^tB$ is the product of the family $v$ by the family generated by the operator $B$, in symbol we have

$${^tB}(v) = \int_{\mathbb{R}^k} vB^\vee,$$

so in particular, the transpose operator $^tB$ is an $S$ operator.

**Proof.** For each index $p \in \mathbb{R}^k$, we have

$$\left( \int_{\mathbb{R}^m} vB^\vee \right)_p = \int_{\mathbb{R}^m} v_p B^\vee =$$
\[
= v_p \circ (B^\vee)^\wedge = \\
= v_p \circ B = \\
= tB(v_p) = \\
= tB(v)(p),
\]
and hence
\[
\int_{\mathbb{R}^m} vB^\vee = tB(v),
\]
as we desired. \qed

**Theorem (\(s\)linearity of a transpose operator).** Let \(B \in \mathcal{L}(S_n, S_m)\) be a linear and continuous operator and let \(v \in S(\mathbb{R}^k, S'_m)\) be an \(s\) family. Then, for each tempered coefficient system \(a \in S'_k\), we have
\[
tB \left( \int_{\mathbb{R}^k} av \right) = \int_{\mathbb{R}^k} a^tB(v).
\]

**Proof.** We have
\[
tB \left( \int_{\mathbb{R}^k} av \right) = \left( \int_{\mathbb{R}^k} av \right) \circ B = \\
= (a \circ \tilde{v}) \circ B = \\
= a \circ (\tilde{v} \circ B) = \\
= \int_{\mathbb{R}^k} a(\tilde{v} \circ B)^\vee = \\
= \int_{\mathbb{R}^k} a \left( \int_{\mathbb{R}^m} vB^\vee \right) = \\
= \int_{\mathbb{R}^k} a^tB(v),
\]
as we desired. \qed

**Application (derivatives of a distribution).** As a simple application, we prove the formula
\[
u' = \int_{\mathbb{R}} u\delta',
\]
where \(\delta'\) is the \(s\) family in \(S'_1\) defined by \(\delta' = (\delta'_p)_{p \in \mathbb{R}}\). Recall that the differential operators on the space of tempered distributions are transpose of linear
continuous operators and then they are $S$-linear operators. Let $\delta$ be the Dirac family of the space $S_1'$, then for each tempered distribution $u \in S_1'$, we have

$$u = \int_{\mathbb{R}} u \delta,$$

and consequently

$$u' = \partial \left( \int_{\mathbb{R}} u \delta \right) = \int_{\mathbb{R}} u \partial(\delta) = \int_{\mathbb{R}} u \delta'.$$

More generally, in the space $S_n'$, we have (by the same proof)

$$L(u) = \int_{\mathbb{R}} u L(\delta),$$

for every differential operator $L$, and every tempered distribution $u$.

7 Characterization of $S$-linear operators

Now, we can show the true nature of the $S$-linear operators defined on $S_n'$.

**Theorem (characterization of $S$-linearity).** Let $L : S_n' \rightarrow S_m'$ be an operator. Then, $L$ is $S$-linear if and only if there exists a linear and continuous operator $B \in L(S_m, S_n)$ such that $L = \iota(B)$.

**Proof.** Sufficiency. It follows from the above theorem. Necessity. Let $\delta$ be the Dirac family in the space $S_n'$, we have

$$L(u) = L \left( \int_{\mathbb{R}^n} u \delta \right) = \int_{\mathbb{R}^n} u L(\delta) = \iota \left( L(\delta) \right)(u),$$
so the operator $L$ is the transpose of the operator generated by the Schwartz family $L(\delta)$, that is

$$L = ^t (L(\delta)^\wedge),$$

since the operators generated by Schwartz families are continuous, we conclude the proof. ■

Before to give the last complete characterization of $S$-linear operators, we recall the following classical definition from Linear Functional Analysis.

**Definition (of transposable operator).** A linear operator $L : S'_n \to S'_m$ is said to be transposable with respect to the canonical pairings $(S_n, S'_n)$ and $(S_m, S'_m)$ if and only if there exists a linear continuous operator $B \in \mathcal{L}(S_m, S_n)$ such that $L = ^t (B)$.

Recalling that the operator $L$ is weakly continuous if and only if it is strongly continuous if and only if it is transposable, we derive the following definitive characterization.

**Theorem (characterization of $S$-linearity).** Let $L : S'_n \to S'_m$ be an operator. Then, the following assertions are equivalent

1) the operator $L$ is $S$-linear;
2) there exists an operator $B \in \mathcal{L}(S_m, S_n)$ such that $L = ^t (B)$;
3) the operator $L$ is linear and weakly continuous;
4) the operator $L$ is linear and strongly continuous;
5) the operator $L$ is linear and topologically transposable.
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