I. INTRODUCTION

As radar and communication systems scale up in bandwidth, the cost and power consumption of high-precision (e.g., 10 – 12 b) analog-to-digital converters (ADCs) become the limiting factor for various applications, such as cognitive radio and radar. As a remedy, low resolution quantization has been gradually drawn attention in recent years [1], [2], [3], [4], [5], [6], [7], [8]. In addition, in military radar systems incorporating very large arrays, low resolution quantization is a promising technique as it provides a reasonable estimation accuracy, and it also enables a very high sampling rate for a wideband system. Moreover, it dramatically reduces the memory and transmission requirements for the samples.

For low resolution quantization, the loss of information per sample is large, and new, low-resolution quantization adapted digital signal processing algorithms should be developed [9], [10]. From the linear signal processing point of view, the spectrum of 1-b line spectral signal is analyzed. As shown in [6], [11], the binary data contains plentiful self-generated [11] and cross-generated harmonics [6]. For low signal to noise (SNR) scenario, the strengths of harmonics decay quickly and conventional fast Fourier transform (FFT) performs well. While for high SNR scenario, FFT will overestimate the model order. From the nonlinear signal processing point of view, the direction of arrival (DOA) estimation from 1 b data is treated as a nonlinear parameter estimation problem. In [12], the frequency estimation from 1 b quantized samples has been addressed, and the deterministic signal modeling by treating the signal samples as a deterministic, yet unknown set of parameters that has to be estimated is adopted. By comparing the derived Cramèr-Rao bound (CRB) for 1-b quantization with that of infinite quantization, it is found that 1 b quantization gives a dramatic increase of variance at certain frequencies, and a slightly worse performance for other frequencies. In [13], a single stochastic Gaussian point source model is assumed to study the DOA estimation, which is a spatial analogy to the problem of temporal line spectral estimation (LSE) with multiple measurement vectors (MMVs), and the CRB for a two-sensor array case is derived. It is shown that the estimation error has weak dependency on SNR, and there exist two singular DOA angles 0° and 30° for which higher SNR results in better estimation performance.

In this article, we utilize expectation propagation (EP) [14] to present an approach named multisnapshot VALUE-EP (MVALUE-EP) for LSE with MMVs from coarsely quantized measurements [15], [16]. Compared to [13], we use a deterministic signal model by treating the amplitudes of the signal as unknown deterministic parameters, instead of stochastic Gaussian model. For the stochastic signal model, the spatial signal samples are clearly dependent (as the sensors are correlated), and the covariance matrix is not diagonal. In order to apply the arcsine law coarsely quantized scenarios. Numerical experiments are conducted to demonstrate the effectiveness of MVALUE-EP, including real dataset.
for LSE [17], Gaussian assumption for the signal amplitude must be adopted. This implies that when the signal amplitude distribution violates the Gaussian assumption, the estimation performance of the covariance matrix based approach may degrade significantly. Besides, the number of snapshots is usually needed to be large to obtain accurate reconstruction of the covariance matrix. By using the deterministic signal model, the signal samples can be regarded as independent identically distributed (i.i.d.) variables, and consequently, performing approximate Bayesian inference becomes tractable [14, [18], [19], as we show later. Compared to [19], which proposes the VALSE-EP to tackle the LSE in the single snapshot case, we extend it to address the multisnapshot LSEs. This is meaningful especially in coarsely quantized scenarios as model may become unidentifiable in the single snapshot case in certain scenarios [5], while multisnapshots may make the estimation problem unlikely unidentifiable and benefits the estimation performance [20].

In this article, the LSE estimation with MMVs from coarsely quantized samples is studied. First, to provide a bench mark performance of the LSE, the CRB is derived. It is shown that under 1 b quantization, the CRB is inversely proportional to the number of snapshots and the cubic of the number of samples of a snapshot. For lower SNR scenario, the CRB is inversely proportional to SNR, while the CRB is inversely proportional to the square root of the SNR for high SNR scenario. Second, the MVALID-EP is proposed to estimate the DOAs and the complex weight. Since MVALSE-EP performs Newton step to refine the frequencies, it overcomes the model mismatch [21] issue incurred by on-grid assumptions [22]. The expectation maximization (EM) algorithm is incorporated to estimate the noise variance automatically for bit-depth greater than 1. Finally, numerical and real experiments are conducted to demonstrate the effectiveness of MVALID-EP, including real dataset.

The rest of this article is organized as follows. Section II describes the system model. Section III derives CRB. The MVALID-EP algorithm and the details of the updating expressions are presented in Section IV. Substantial numerical and real experiments are provided in Section V and finally, Section VI concludes this article.

For a complex vector $\mathbf{x} \in \mathbb{C}^M$, let $\Re\{\mathbf{x}\}$ and $\Im\{\mathbf{x}\}$ denote the real and imaginary part of $\mathbf{x}$, respectively, and let $|\mathbf{x}|$ and $\angle \mathbf{x}$ denote the componentwise amplitude and phase of $\mathbf{x}$, respectively. For the square matrix $\mathbf{A}$, let $\text{diag}(\mathbf{A})$ return a vector with elements being the diagonal of $\mathbf{A}$. While for a vector $\mathbf{a}$, let $\text{diag}(\mathbf{a})$ return a diagonal matrix with the diagonal being $\mathbf{a}$, and thus $\text{diag}(\text{diag}(\mathbf{A}))$ returns a diagonal matrix. Let $j$ denote the imaginary unit. Let $\mathcal{S} \subseteq \{1, \ldots, N\}$ be a subset of indices and $|\mathcal{S}|$ denote its cardinality. For the matrix $\mathbf{J} \in \mathbb{C}^{N \times N}$, let $\mathbf{J}_\mathcal{S}$ denote the submatrix by choosing both the rows and columns of $\mathbf{J}$ indexed by $\mathcal{S}$. Similarly, let $\mathbf{h}_\mathcal{S}$ denote the subvector by choosing elements of $\mathbf{h}$ indexed by $\mathcal{S}$. Let $(\cdot)_{\mathcal{S}}^\mathcal{T}$, $(\cdot)_{\mathcal{S}}^\mathcal{H}$ and $(\cdot)_{\mathcal{S}}^\mathcal{H}$ be the conjugate, transpose, and Hermitian transpose operator of $(\cdot)_{\mathcal{S}}$, respectively. For the matrix $\mathbf{A}$, let $|\mathbf{A}|$ denote the elementwise absolute value of $\mathbf{A}$. Let $\mathbf{I}_T$ denote the identity matrix of dimension $L$. Let $CN(x; \mu, \Sigma)$ denote the complex normal (CN) distribution of $\mathbf{x}$ with mean $\mu$ and covariance $\Sigma$. For a frequency $\theta$, let $\mathcal{V}(\mu, \kappa)$ be the von Mises distribution with $\mu$ and $\kappa$ being the mean and concentration parameters. For a random vector $\mathbf{x}$ with probability density function (PDF) $p(\mathbf{x})$, let $\text{Proj}[p(\mathbf{x})]$ denote the projection of $p(\mathbf{x})$ onto Gaussian PDF with diagonal covariance matrix, where the means and variances are matched with that of $p(\mathbf{x})$. Let $\phi(\mathbf{x}) = \exp(-x^2/2)/\sqrt{2\pi}$ and $\phi(\mathbf{x}) = \int_{-\infty}^{\mathbf{x}} \phi(t)dt$ denote the standard normal PDF and cumulative distribution function (CDF), respectively.

II. PROBLEM SETUP

Consider a line spectral estimation problem with MMVs. For the $t$th snapshot, the signal received before quantization can be modeled as

$$\mathbf{r}(t) = \mathbf{A}(\omega)\mathbf{x}(t) + \mathbf{w}(t), \quad t = 1, 2, \ldots, T$$ (1)

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_K(t)]^T$ denotes the complex coefficients of the unknown frequencies $\omega = [\omega_1, \omega_2, \ldots, \omega_K] \in \mathbb{R}^K$, $\omega_i \in (0, 2\pi)$, and $K$ denotes the number of frequencies

$$\mathbf{A} = [\mathbf{a}(\omega_1), \mathbf{a}(\omega_2), \ldots, \mathbf{a}(\omega_k), \ldots, \mathbf{a}(\omega_K)] \in \mathbb{C}^{N \times K}$$ (2)

denotes the nominal array manifold matrix, with the steering vectors $\mathbf{a}(\omega_k) = [1, e^{j\omega_k}, \ldots, e^{j(N-1)\omega_k}]^T$ as its columns, $\mathbf{w}(t) \in \mathbb{C}^N$ denotes the additive internal (e.g., thermal) receiver noise modeled as spatially and temporally i.i.d. zero-mean circular CN with a covariance matrix $\mathbf{R}_w \triangleq \mathbb{E}[\mathbf{w}(t)\mathbf{w}^H(t)] = \sigma^2\mathbf{I}_N$.

The signal $\mathbf{r}(t)$ is then coarsely quantized to obtain

$$\mathbf{y}(t) = \mathcal{Q}\{\Re\{\mathbf{r}(t)\}\} + j\mathcal{Q}\{\Im\{\mathbf{r}(t)\}\}, \quad t = 1, 2, \ldots, T$$ (3)

where $\mathcal{Q}(\cdot)$ is a quantizer which maps the continuous-valued observations into a finite number of bits. Note that for a quantizer with bit-depth $B$, the cardinality of the output of the quantizer is $|\mathcal{D}| = 2^B$. Assume that the quantization intervals for the quantizer $\mathcal{Q}(\cdot)$ are $\{[\tau_i, \tau_{i+1}]\}_{i=0}^{[D]-1}$, where $\tau_0 = -\infty$, $\tau_{[D]} = \infty$, $\bigcup_{i=0}^{[D]-1} [\tau_i, \tau_{i+1}] = \mathbb{R}$. Given a real number $a \in [\tau_i, \tau_{i+1})$ input, the quantizer output is

$$\mathcal{Q}(a) = \tau_i.$$ (4)

For example, one-bit quantization refers to $B = 1$, $D = 2$, $\tau_0 = -\infty$, $\tau_1 = 0$, and $\tau_2 = \infty$, $\mathcal{Q}(\cdot)$ reduces to the signum function, i.e., $\mathcal{Q}(\cdot) = \text{sign}(\cdot)$. Note that the noise variance $\sigma^2$ is assumed to be known for one bit quantization, and for $B \geq 2$, the noise variance is estimated via the EM algorithm.

The SNR of the $k$th target in dB is defined as

$$\text{SNR}_k = 10\log_{10} \frac{\sum_{i=1}^{T} |x_k(t)|^2 / T}{\sigma^2}$$ (5)

which corresponds to the sample SNR. Note that after coherent integration, the sample SNR can be improved up to $10\log_{10} N$ dB.
It is worth noting that model (1) and model (3) has various applications such as range, velocity, and angle estimation in millimeter wave LFMCW radar [23]. Below, we present the details as follows:

1) Range estimation: here \( \omega_k = 2\pi \times \frac{2aT}{c}r_k \) where \( \mu \), \( T_r \), \( c \) and \( r_k \) denote the chirp rate, sampling interval, electromagnetic speed, and radial distance of the \( k \)th target.

2) Velocity estimation: here \( \omega_k = 2\pi \times \frac{2\pi}{T}v_k \) where \( T_r \), \( \lambda \) and \( v_k \) denote the repetition interval, wavelength, and radial velocity of the \( k \)th target.

3) DOA estimation: here \( \omega_k = 2\pi \times \frac{d}{\lambda} \sin \theta_k \) where \( d \), \( \lambda \) and \( \theta_k \) denote the element spacing, wavelength, and DOA of the \( k \)th target.

4) Vital signs detection [26]: The vital sign extraction from multiple independent self-injection locking doppler radars can be formulated as a LSE problem with MMVs, see [26, eq. (5)].

III. CRAMÉR-RAO BOUND

The CRB is a lower bound of unbiased estimators and provides a benchmark against which we can compare the performance of the proposed algorithm [28], [29]. To derive the CRB, \( K \) is assumed to be known, the frequencies \( \omega \in \mathbb{R}^K \) and weights \( X = [x(1), \ldots, x(T)] \in C^K \times T \) are treated as unknown deterministic parameters. As for the quantizer \( Q(\cdot) \), the quantization intervals are \( \{[t_i, t_{i+1})\}_{i=0}^{D-1} \), where \( t_0 = -\infty \), \( t_D = \infty \), \( D^{-1} = \bigcup_{i=0}^{D-1} [t_i, t_{i+1}) = \mathbb{R} \).

The CRB equals to the inverse of the Fisher information matrix (FIM). For calculating the FIM, the following lemma can be utilized.

**Lemma 1** (see [19], [27]) Let \( \kappa \in \mathbb{R}^p \) denote the set of unknown deterministic parameters. Note that in the case of quantized observations \( \mathbf{y} = Q(\mathbf{r}) \in \mathbb{R}^N \), where \( r \sim \mathcal{N}(\mu(\kappa), \sigma^2 I_N/2) \), the FIM is given by

\[
I(\kappa) = \frac{2}{\sigma^2} \left[ \frac{\partial \mu(\kappa)}{\partial \kappa^T} \right]^T \Lambda \left[ \frac{\partial \mu(\kappa)}{\partial \kappa^T} \right] \tag{6}
\]

where

\[
\frac{\partial \mu(\kappa)}{\partial \kappa^T} = \left[ \frac{\partial \mu(\kappa_1)}{\partial \kappa_1}, \ldots, \frac{\partial \mu(\kappa_p)}{\partial \kappa_p} \right] \in \mathbb{R}^{N \times p} \tag{7}
\]

and \( \Lambda \) is a diagonal matrix with the \((i, i)\)th element

\[
\Lambda_{i,i} = h(\mu_i(\kappa), \sigma^2) \tag{8}
\]

and \( h(\cdot, \sigma^2) \) is

\[
h(x, \sigma^2) = \sum_{d=0}^{D-1} \left[ \phi \left( \frac{z_d - i\sigma^2}{\sqrt{\sigma^2}} \right) - \phi \left( \frac{z_d - i\sigma^2}{\sqrt{\sigma^2}} \right) \right]^2 \tag{9}
\]

For one-bit quantization where \( |D| = 2 \), \( t_0 = -\infty \), \( t_1 = 0 \), \( t_2 = \infty \), \( h(x, \sigma^2) \) simplifies to be

\[
h(x, \sigma^2) = \frac{\phi^2 \left( \frac{x}{\sqrt{\sigma^2}} \right) \phi \left( \frac{-x}{\sqrt{\sigma^2}} \right)}{2\pi \phi \left( \frac{x}{\sqrt{\sigma^2}} \right) \phi \left( \frac{-x}{\sqrt{\sigma^2}} \right)} \tag{10}
\]

For unquantized system, the FIM (6) is obtained with \( \Lambda = I_N \).

In our setting, the observations are \( \mathbb{R} \{y(1); \ldots; y(T)\} \). Note that \( \kappa \in \mathbb{R}^{2KT+K} \) and \( \mu(\kappa) \in \mathbb{R}^{2MT} \) are

\[
\mu(\kappa) = \begin{bmatrix} \mathbb{R}\{x(1)\} \\ \mathbb{R}\{x(2)\} \\ \vdots \\ \mathbb{R}\{x(T)\} \\ \mathbb{R}\{y(1)\} \\ \mathbb{R}\{y(2)\} \\ \vdots \\ \mathbb{R}\{y(T)\} \end{bmatrix}, \tag{11}
\]

Define

\[
\bar{A} = \begin{bmatrix} \mathbb{R}\{A\} & -\mathbb{R}\{A\} \\ \mathbb{R}\{A\} & \mathbb{R}\{A\} \end{bmatrix}, \tag{12a}
\]

\[
D_{\omega} = \begin{bmatrix} \frac{\partial \omega(t)}{\partial \omega_1} & \cdots & \frac{d \omega(t)}{d \omega_K} \end{bmatrix}, \tag{12b}
\]

\[
X(t) \triangleq \text{diag}(x(t)), \tag{12c}
\]

\[
\dot{\tilde{D}}(t) = \begin{bmatrix} \mathbb{R}\{\tilde{D}(t)\} \\ \mathbb{R}\{\tilde{D}(t)\} \end{bmatrix}, \tag{12d}
\]

\[
\lambda(t) = \begin{bmatrix} h(\mathbb{R}\{A(x(t), \sigma^2)\}) \\ h(\mathbb{R}\{A(x(t), \sigma^2)\}) \end{bmatrix}, \tag{12e}
\]

\[
\Lambda(t) = \text{diag}(\lambda(t)), \tag{12f}
\]

\[
\mathbf{H}(t) = \bar{A}^T \Lambda(t) \bar{A}, \tag{12g}
\]

\[
\Delta(t) = \bar{A}^T \Lambda(t) \bar{A}(t), \tag{12i}
\]

As shown in Appendix VII-A, according to Lemma 1, the CRB(\( \omega \)) is

\[
\text{CRB}(\omega) = \frac{2}{\sigma^2} \sum_{t=1}^{T} \left( \frac{\dot{\tilde{D}}^T(t) \Lambda(t) \dot{\tilde{D}}(t) - \dot{\Delta}^T(t) \mathbf{H}^{-1}(t) \dot{\Delta}(t)}{\sigma^2} \right)^{-1}. \tag{13}
\]

Note that for the DOA estimation problem, where \( \omega = \frac{2\pi}{\lambda} \sin \theta \) with \( d \) and \( \lambda \) being the element spacing and wavelength, by using vector parameter CRB for transformations [28, eq. (3.30)], the CRB of DOAs is

\[
\text{CRB}(\theta) = \text{diag} \left( \frac{\lambda}{2\pi d \cos(\theta)} \right)^2 \text{CRB}(\omega). \tag{14}
\]

We now hope to provide insight into the relationship between the system parameters and the estimation performance. As a result, we provide an asymptotic analysis of the CRB of a single target under one bit quantization for both low SNR and high SNR scenarios. It is assumed that the amplitude of the frequency \( g = |x(t)| \) is known and is the
same for all the snapshots. Besides, the phase \( \psi(t) = \angle x(t) \) of the frequency at the \( t \)th snapshot is unknown. We assume that \( \psi(t) \) is uniformly drawn from \([0, 2\pi)\). Define \( n = [0, 1, \ldots, N - 1]^{T} \). Similar to the derivation of the CRB, here the unknown deterministic parameters \( \kappa \) and \( \mu(\kappa) \) are

\[
\kappa = \begin{bmatrix}
\psi(1) \\
\psi(2) \\
\vdots \\
\psi(T-1) \\
\psi(T)
\end{bmatrix}, \quad \mu(\kappa) = \begin{bmatrix}
g \cos(n \omega + \psi(1)) \\
g \cos(n \omega + \psi(2)) \\
\vdots \\
g \cos(n \omega + \psi(T))
g \sin(n \omega + \psi(T))
\end{bmatrix}
\]

(15)

Then

\[
\frac{\partial \mu(\kappa)}{\partial \kappa} = \begin{bmatrix}
\ddot{\alpha}(1) \\
\ddot{\alpha}(2) \\
\vdots \\
\ddot{\alpha}(T)
\end{bmatrix}, \quad \ddot{\alpha}(t) = \begin{bmatrix}
-g \sin(n \omega + \psi(t)) \\
g \cos(n \omega + \psi(t))
\end{bmatrix}
\]

(16)

where \( \ddot{\alpha}(t) \) are defined as

\[
\ddot{\alpha}(t) = \begin{bmatrix}
\frac{\partial}{\partial \omega} \dddot{\alpha}(t) \\
\frac{\partial}{\partial t} \dddot{\alpha}(t)
\end{bmatrix} = g \begin{bmatrix}
-n \odot \sin(n \omega + \psi(t)) \\
n \odot \cos(n \omega + \psi(t))
\end{bmatrix}
\]

(17)

with \( \odot \) being defined in (11). It can be shown that the CRB for a single frequency is

\[
\text{CRB}^{-1}(\omega) = \frac{2}{\sigma^2} \sum_{n=0}^{N-1} \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) - \frac{\Delta^2(t)}{H(t)} \right]
\]

(20)

where \( \Delta(t) \) and \( H(t) \) are

\[
\Delta(t) = \ddot{\alpha}(t) \text{diag}(\lambda(t)) \ddot{\alpha}(t) = g^2 \sum_{n=0}^{N-1} \chi \left( n, \omega, \psi, \frac{\sigma}{\sigma} \right) n
\]

(21)

\[
H(t) = \ddot{\alpha}(t) \text{diag}(\lambda(t)) \ddot{\alpha}(t) = g^2 \sum_{n=0}^{N-1} \chi \left( n, \omega, \psi, \frac{\sigma}{\sigma} \right)
\]

(22)

with

\[
\chi \left( n, \omega, \psi, \frac{\sigma}{\sigma} \right) = \sin^2(n \omega + \phi(t)) h(g \cos(n \omega + \psi(t)), \sigma^2)
\]

\[
\cos^2(n \omega + \phi(t)) h(g \sin(n \omega + \psi(t)), \sigma^2)
\]

(23)

Note that

\[
\text{ECRB}^{-1}(\omega) = \frac{2}{\sigma^2} \sum_{n=0}^{N-1} \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) - \frac{\Delta^2(t)}{H(t)} \right]
\]

(24)

It can be seen that given that the number of snapshots is large, \( \text{ECRB}^{-1}(\omega) \) (24) provides a good approximation for \( \text{CRB}^{-1}(\omega) \) (20). Now, suppose that \( \psi(t) \sim U(0, 2\pi) \). We define the expected CRB \( \text{ECRB}^{-1}(\omega) \) as

\[
\text{ECRB}^{-1}(\omega) = \frac{2}{\sigma^2} \text{E}_\psi \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) - \frac{\Delta^2(t)}{H(t)} \right]
\]

(25)

i.e., \( \text{ECRB}^{-1}(\omega) \) is also equal to the CRB \( \text{CRB}^{-1}(\omega) \) for a single frequency. Another property is that the integration

\[
E_{\psi} \left[ \chi \left( n, \omega, \psi, \frac{\sigma}{\sigma} \right) \right] = E_{\psi} \left[ \sin^2(\psi) h(g \cos(\psi), \sigma^2) + \cos^2(\psi) h(g \sin(\psi), \sigma^2) \right]
\]

\[
\triangleq r \left( \frac{\sigma}{\sigma} \right)
\]

(26)

holds, i.e., the integration is independent of \( n \omega \).

It is hard to directly calculate \( \frac{2}{\sigma^2} \text{E}_\psi \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) \right] \). Instead, we use the Jensen inequality to obtain a bound

\[
\frac{2}{\sigma^2} \text{E}_\psi \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) \right] \geq \frac{2}{\sigma^2} \text{E}_\psi \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) \right]
\]

(27)

Here,

\[
E_{\psi}[H] = \sum_{n=0}^{N-1} E \left[ \chi \left( n, \omega, \psi, \frac{\sigma}{\sigma} \right) \right] = N g^2 r \left( \frac{\sigma}{\sigma} \right)
\]

(29)

Furthermore, we have

\[
E_{\psi} \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) \right] = g^4 E_{\psi} \left[ \sum_{n=0}^{N-1} \chi \left( n, \omega, \psi, \frac{\sigma}{\sigma} \right) n \right]
\]

\[
= g^4 \sum_{n=0}^{N-1} n^2 \left( \frac{\sigma}{\sigma} \right)
\]

(30)

Thus,

\[
\frac{2}{\sigma^2} \text{E}_\psi \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) \right] = \frac{2 g^2}{\sigma^2} \sum_{n=0}^{N-1} n^2 \left( \frac{\sigma}{\sigma} \right)
\]

(31)

According to (27), (28), and (31), \( \text{ECRB}^{-1}(\omega) \) (24) can be approximated as

\[
\text{ECRB}^{-1}(\omega) = \frac{2}{\sigma^2} \text{E}_\psi \left[ \ddot{\alpha}(t) \ddot{\alpha}(t) - \frac{\Delta^2(t)}{H(t)} \right]
\]
\[ \leq \left( \approx \right) \frac{2 \left| g \right|^2 T}{\sigma^2} \left( \sum_{n=0}^{N-1} n^2 \right) r \left( \frac{g}{\sigma} \right) - \frac{2g^2T}{\sigma^2} \left( \frac{N^2 - n}{2N} \right) r \left( \frac{g}{\sigma} \right) \]

\[ = \frac{2 \left| g \right|^2}{\sigma^2} r \left( \frac{g}{\sigma} \right) \frac{N(N-1)(N+1)}{12} \]  

(32)

due to \( \sum_{n=0}^{N-1} n^2 = \frac{(N-1)(2N-1)}{6} \) and \( \sum_{n=0}^{N-1} n = \frac{N(N-1)}{2} \).

The Chernoff bound [31] \( \Phi(x) \Phi(-x) \leq \frac{1}{2} e^{-x^2/\pi} \) can be used to approximate \( h(x; \sigma^2) \) as

\[ h(x; \sigma^2) = \frac{1}{2\pi} \phi \left( \frac{x}{\sigma} \sqrt{\pi} \right) \Phi \left( \frac{-x}{\sigma} \sqrt{\pi} \right) \geq \left( \approx \right) \frac{1}{2} e^{-x^2/\pi} \]  

(33)

which is also a very tight approximation. Utilizing the integration [30]

\[ \int_{0}^{2\pi} e^{-x \cos^2 \gamma} \sin^2(\gamma) d\gamma = \int_{0}^{2\pi} e^{-x \sin^2 \gamma} \cos^2(\gamma) d\gamma = \pi e^{-\frac{x^2}{2}} \left( I_0 \left( \frac{x}{2} \right) + I_1 \left( \frac{x}{2} \right) \right) \]  

(34)

where \( I_n(x) \) are the modified Bessel functions of the first kind, \( r \left( \frac{g}{\sigma} \right) \) can be approximated as

\[ r \left( \frac{g}{\sigma} \right) = I_0 \left( \frac{g^2}{2\sigma^2} \right) + I_1 \left( \frac{g^2}{2\sigma^2} \right) \]  

(35)

where \( a \) is due to (33). Substituting (35) in (32), ELCRB\(^{-1}\)(\( \omega \)) can be further approximated as

\[ \text{ELCRB}^{-1}(\omega) \leq \left( \approx \right) \frac{2 \left| g \right|^2 T}{\sigma^2} r \left( \frac{g}{\sigma} \right) \frac{N(N-1)(N+1)}{12} \]

\[ \leq \left( \approx \right) \frac{4 \left| g \right|^2}{\pi \sigma^2} e^{-\frac{\omega^2}{2}} \left( I_0 \left( \frac{g^2}{2\sigma^2} \right) + I_1 \left( \frac{g^2}{2\sigma^2} \right) \right) \]

\[ \times \frac{N(N-1)(N+1)}{12} \triangleq \text{ELCRB}^{-1}(\omega). \]  

(36)

Therefore,

\[ \text{ELCRB}(\omega) \geq \left( \approx \right) \text{ELCRB}(\omega) \]  

(37)
i.e., ELCRB(\( \omega \)) is a lower bound of ECRB(\( \omega \)) and approximates ECRB(\( \omega \)) well.

For low SNR scenario, \( I_0(x) \approx 1 \) and \( I_1(x) \approx 0 \), ELCRB\(^{-1}\)(\( \omega \)) (36) is approximated as

\[ \text{ELCRB}^{-1}(\omega) \approx \frac{4 \left| g \right|^2}{\pi \sigma^2} \frac{N(N-1)(N+1)}{12} \]  

(38)

and

\[ \text{ELCRB}(\omega) = \frac{\pi}{2} \text{CRB}_\infty(\omega) \]  

(39)

where CRB\(_\infty\)(\( \omega \)) is defined in (25). For high SNR scenario, one has

\[ I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}. \]  

(40)

Substituting (40) into (36) yields

\[ \text{ELCRB}^{-1}(\omega) \approx \frac{8gN(N-1)(N+1)}{\pi \sqrt{\pi} \sigma} \approx \frac{2g}{3\pi^2 \sigma} N^3. \]  

(41)

According to (38), (41) and SNR = \( g^2/\sigma^2 \), we have

\[ \text{ELCRB}(\omega) \approx \begin{cases} \frac{\pi^3}{2} N^3 \sqrt{N}, & \text{low SNR} \\ \frac{\pi^3}{2} \frac{1}{\sqrt{\text{SNR}N^2}}, & \text{high SNR}. \end{cases} \]

(42)

Because ELCRB(\( \omega \)) approximates CRB(\( \omega \)) well, it can be concluded that under one-bit quantization, the CRB is inversely proportional to the number of snapshots and the cubic of the number of antennas. For lower SNR scenario, the CRB is inversely proportional to SNR, while the CRB is inversely proportional to the square root of the SNR for high SNR scenario.

IV. MVALSE-EP ALGORITHM

In this section, MVALSE-EP algorithm is developed to estimate the frequencies. The key step in designing the MVALSE-EP is to introduce the suitable hidden variable and expand the factor graph. Here, we define

\[ z(t) = A(\omega) x(t) \]  

(43)

and the EP is adopted to iteratively approximate the nonlinear measurement model as a pseudolinear measurement model. In addition, EM and VALSE are adopted to estimate the nuisance parameters and the frequencies, respectively. In the following text, we first present the modeling setup. Then, we adopt the modularized point of view to design MVALSE-EP algorithm based on existing superresolution VALSE approach.

A. Modeling Setup

Since the number of targets \( K \) is usually unknown, an overcomplete model in which the signal consisting of \( N \) targets is assumed [15]

\[ z(t) = \sum_{k=1}^{N} a(\omega_k) x_k(t) = A(\omega) x(t) \]  

(44)

where \( A(\omega) = [a(\omega_1), \ldots, a(\omega_K)] \) and satisfies \( N > K \). To model the unknown nature of \( K \), the binary hidden variables \( s(t) \) are introduced, where \( s_k(t) = 1 \) means that the \( k \)th target is active in the \( t \)th snapshot, otherwise inactive (\( s_k(t) = 0 \)). In addition, it is required that the number of sources are the same across the snapshots, which requires

\[ \sum_{k=1}^{N} s_k(t) = K. \]  

(45)
The probability mass function of $s_k(t)$ is
\begin{equation}
p(s_k(t)) = \rho^{s_k(t)}(1 - \rho)^{(1-s_k(t))}, \quad s_k(t) \in \{0, 1\}.
\end{equation}
Given that $s_k(t) = 1$, we assume that $x_k(t) \sim \mathcal{CN}(x_k(t); 0, \tau)$. Thus, $(s_z(t), x_z(t))$ follows a Bernoulli-Gaussian distribution, that is
\begin{equation}
p(x_z(t)|s_z(t); \tau) = (1-s_z(t))\delta(x_z(t)) + s_z(t)\mathcal{CN}(x_z(t); 0, \tau).
\end{equation}
From (46) and (47), it can be seen that the parameter $\rho$ denotes the probability of the $k$th component being active and $\tau$ is a variance parameter. The variable $\omega = [\omega_1, \ldots, \omega_N]^T$ has the prior PDF $p(\omega) = \prod_{k=1}^{N} p(\omega_k)$. Without any knowledge of the frequency $\omega$, the uninformative prior distribution $p(\omega_k) = 1/(2\pi)$ is used [15]. For encoding the prior distribution, please refer to [15] and [16] for further details.

Note that the modeling setup of (47) is different from that of [16]. In [16], only binary variables $s_k \in \{0, 1\}$ are introduced, and such modeling makes the algorithm overestimate the model order.

Given $Z$, the PDF $p(Y|Z; \sigma_w^2) = \prod_{t=1}^{T} p(y(t)|z(t); \sigma^2)$ of $Y$ can be easily calculated through (3). Let
\begin{equation}
\Theta = (\omega, \rho, \tau, K)
\end{equation}
be the set of all random variables and the model parameters, respectively. According to the Bayes rule, the joint PDF $p(Y, Z; \Theta; \beta)$ is
\begin{equation}
p(Y, Z; \Theta; \beta) = p(Y|Z)\delta(Z - A(\Theta)X) \prod_{k=1}^{N} p(\omega_k)
\prod_{t=1}^{T} p(x_z(t)|s_z(t))p(s_z(t))\delta\left(\sum_{k=1}^{N} s_z(t) - K\right).
\end{equation}
Given the above joint PDF (50), the type II maximum likelihood (ML) estimation of the model parameters $\hat{\beta}_{ML}$ is
\begin{equation}
\hat{\beta}_{ML} = \arg \max_{\beta} p(Y; \beta) = \arg \max_{\beta} \int p(Y, Z; \Theta; \beta)dzd\Theta.
\end{equation}
Then, the minimum mean squared error (MMSE) estimates of the parameters $(Z, \Theta)$ is
\begin{equation}
\hat{Z}, \hat{\Theta} = \mathbb{E}[(Z, \Theta)|Y; \hat{\beta}_{ML}]
\end{equation}
where the expectation is taken with respect to
\begin{equation}
p(Z, \Theta|Y; \hat{\beta}_{ML}) = \frac{p(Z, \Theta, Y; \beta_{ML})}{p(Y; \beta_{ML})}
\end{equation}
Directly solving the ML estimate of $\beta$ (51) or the MMSE estimate of $(Z, \Theta)$ (52) are both intractable. As a result, an iterative algorithm is designed in the ensuing text.

B. Algorithm Design

Below we describe the details of the algorithm and show that how to utilize the basic VALSE algorithm to perform the LSE estimation. For brevity, the iteration index is omitted.

The factor graph of the joint PDF (50) is shown in Fig. 1, and the algorithm is designed according to Fig. 2.

1) Componentwise MMSE: According to EP, the message $m_{Z\rightarrow\delta}(Z)$ transmitted from the variable node $Z$ to the factor node $\delta(Z - AX)$ can be calculated as [14]
\begin{equation}
m_{Z\rightarrow\delta}(Z) \propto \frac{\text{Proj}[m_{\delta\rightarrow Z}(Z)p(Y|Z)]}{\prod_{t=1}^{T} \text{Proj}[m_{\delta\rightarrow Z}(Z(t))p(y(t)|z(t))]} \propto \prod_{t=1}^{T} \text{Proj}[q_{\beta}(z(t))]
\end{equation}
where $\propto$ denotes identity up to a normalizing constant. First, the MMSE estimate of $z(t)$ can be obtained, i.e.,
\begin{equation}
z_{B}^{\text{post}}(t) = \mathbb{E}[z(t)|q_{B}(z(t))]
\end{equation}
\begin{equation}
v_{B}^{\text{post}}(t) = \mathbb{V}[z(t)|q_{B}(z(t))]
\end{equation}
Substituting (58) in (55), the message $m_{Z\rightarrow\delta}(Z)$ from the variable node $Z(t)$ to the factor node $\delta(Z(t) - A(\omega)x(t))$ is calculated as
\begin{equation}
m_{Z(t)\rightarrow\delta}(Z(t)) \propto \mathcal{CN}(z(t)|z_{B}^{\text{post}}(t), \mathbb{V}_{B}^{\text{post}}(t)), \mathbb{V}_{A}^{\text{ext}}(t), \delta(\mathbb{V}_{A}^{\text{ext}}(t)))
\end{equation}
where $z_{B}^{\text{ext}}(t)$ and $v_{B}^{\text{ext}}(t)$ are [18]
\begin{equation}
v_{B}^{\text{ext}}(t) = \left(\begin{array}{c}
\mathbb{V}_{B}^{\text{post}}(t) - \mathbb{V}_{B}^{\text{ext}}(t) \\
\mathbb{V}_{A}^{\text{ext}}(t)
\end{array}\right)^{-1}
\end{equation}
\begin{equation}
z_{B}^{\text{ext}}(t) = v_{B}^{\text{ext}}(t) \odot \left(\begin{array}{c}
\mathbb{V}_{B}^{\text{post}}(t) - \mathbb{V}_{B}^{\text{ext}}(t) \\
\mathbb{V}_{A}^{\text{ext}}(t)
\end{array}\right)
\end{equation}
where $\odot$ denotes componentwise multiplication. Consequently, we have
\begin{equation}
m_{Z\rightarrow\delta}(Z) \propto \prod_{t=1}^{T} m_{Z(t)\rightarrow\delta}(Z(t))
\end{equation}
\begin{equation}
\propto \prod_{t=1}^{T} \mathcal{CN}(z(t); z_{B}^{\text{ext}}(t), \delta(\mathbb{V}_{A}^{\text{ext}}(t))).
\end{equation}
2) MVALIDE Module: According to (61), the message $m_{Z \rightarrow \delta}(Z)$ transmitted from the variable node $Z$ to the factor node $\delta(Z - AX)$ is Gaussian distributed and is independent of the snapshot $t$. Based on the definition of the factor node $\delta(Z - AX)$, $T$ pseudolinear observation models

$$\tilde{y}(t) = A(\omega)x(t) + \tilde{w}(t), \quad t = 1, \ldots, T$$

are obtained, where $\tilde{w}(t) \sim \mathcal{CN}(0, \text{diag}(\tilde{\sigma}^2(t)))$, $\tilde{y}(t) = z^\text{ext}(t)$ and $\tilde{\sigma}^2(t) = v^\text{ext}(t)$. For the $r$th equation in (62), the variances of the heteroscedastic noise $\tilde{w}(t)$ are different. In addition, All the snapshots share the same frequency $\omega$. Then, we run the MVALIDE under heteroscedastic noise algorithm [32] with known noise variances. Note that we could average the noise variance with respect to snapshots or measurements. Such operation simplifies the computation especially when averaging over snapshots. The only difference is that for this modeling setup, the $k$th target is active only when

$$\Delta_k = \frac{1}{T} \sum_{t=1}^{T} \left( \ln \frac{v_k(t)}{\tau} + \frac{|u_k(t)|^2}{\tau} \right) + \ln \frac{\rho}{1 - \rho} > 0 \quad (63)$$

where $v_k(t)$ and $u_k(t)$ are

$$v_k(t) = \left( \frac{\text{tr}(\Sigma^{-1})(t)}{\tau} + \frac{1}{\tau} J_{S,k}^H(t) \left( J_{H,S,k}^H(t) + \frac{1}{\tau} I_{|S|} \right)^{-1} J_{S,k}^H(t) \right)^{-1}$$

$$u_k(t) = v_k(t) \left( h_k(t) - J_{S,k}^H(t) \left( J_{S,S}^H(t) + \frac{1}{\tau} I_{|S|} \right)^{-1} h_S(t) \right) \quad (64)$$

$\Sigma(t) = \text{diag}(\tilde{\sigma}^2(t))$, $J(t)$ and $h(t)$ are

$$J_{i,j}(t) = \begin{cases} \text{tr}(\Sigma^{-1}(t)), & i = j \\ \tilde{a}_i^H(t) \Sigma^{-1}(t) \tilde{a}_j, & i \neq j \end{cases}, \quad i, j \in \{1, \ldots, N\} \quad (65a)$$

$$h(t) = \tilde{A}^H(t) \Sigma^{-1}(t) \tilde{y}(t) \quad (65b)$$

where $J_{i,j}(t)$ denotes the $(i, j)$th element of $J(t)$ and $\tilde{a}_j$ denotes the posterior mean of $a_j$. The deactive case can be obtained in a similar way. Once the posterior PDF $q(\omega|\tilde{Y})$, $q(X|\bar{Y})$ and the model parameters $\tau$ and $\rho$ are obtained, the posterior means and variances of $z(t) = A(\omega)x(t)$ are also obtained as [19]

$$\tilde{z}_A^{\text{post}}(t) = \tilde{A}_S \tilde{x}_S(t) \quad (66)$$

$$v^\text{post}_A(t) = \text{diag}(\tilde{A}_S \tilde{C}_S(t) \tilde{A}_S^H(t)) + \left( \tilde{x}_S^2(t) \tilde{C}_S(t) \right) I - |\tilde{A}_S|^2 |\tilde{x}_S(t)|^2 \quad (67)$$

where $\tilde{x}_S(t)$ and $\tilde{C}_S(t)$ are the posterior means and covariance matrix of $x(t)$, respectively. $\tilde{A}_S$ is the estimate of $A_S$. The noise variance $\tilde{\sigma}^2$ can be obtained via the EM algorithm as

$$\tilde{\sigma}^2 = \frac{1}{NT} \sum_{t=1}^{T} \left( ||\tilde{y}(t) - \tilde{z}_A^{\text{post}}(t)||^2 + \frac{1}{NT} v^\text{post}_A(t) \right). \quad (68)$$

Then, we calculate the message $m_{z \rightarrow x(t)}(z(t))$ as

$$m_{z \rightarrow x(t)}(z(t)) = \frac{\text{Proj}[q_A(z(t))]}{m_{x(t) \rightarrow \delta}(z(t))} \quad (69)$$

where \text{Proj}[q_A(z(t))] is

$$\text{Proj}[q_A(z(t))] = \mathcal{CN}(z(t)); z_A^{\text{post}}(t), \text{diag}(v^\text{post}_A(t))) \quad (70)$$

According to (69), $m_{z \rightarrow x(t)}(z(t))$ is calculated to be

$$m_{z \rightarrow x(t)}(z(t)) = \mathcal{CN}(z(t); z_A^{\text{post}}(t), \text{diag}(v^\text{post}_A(t))) \quad (71)$$

where the extrinsic mean $z_A^{\text{ext}}(t)$ and variance $v_A^{\text{ext}}(t)$ are given by [18]

$$\frac{1}{v_A^{\text{ext}}(t)} = \frac{1}{v^\text{post}_A(t)} - \frac{1}{\tilde{\sigma}^2(t)} \quad (72)$$

$$z_A^{\text{ext}}(t) = v_A^{\text{ext}}(t) \circ \left( \frac{v^\text{post}_A(t)}{v_A^{\text{ext}}(t)} - \frac{\tilde{y}(t)}{\tilde{\sigma}(t)} \right) \quad (73)$$
 obtain the posterior PDFs of the frequencies. The complexity of the MMSE module per iteration is $O(NT)$. As for the MVALIDE algorithm, its complexity per iteration is dominated by calculating $\Delta_k$ (63), whose complexity is $O(NK^3 T)$ per iteration. Thus, the overall computation complexity of the MVALIDE-EP is $O((N^2 + NK^3 \times \text{Iter}_{\text{max}})T)$.

D. Further Discussion

It is worth noting that the $k$th target is active only when (63) is satisfied. This is different from the MVALIDE algorithm [16] where the $k$th target is active only when $\Delta_k' = 1_k - 10^{-\rho} > 0$.

Because $\ln \frac{\rho}{1-\rho} < 0$ as $\rho < 0.5$ in general, it can be seen that with all the parameters being the same, the $k$th target is more likely to be active for the previous MVALIDE algorithm [16]. We have found that if (74) is adopted as the criterion for activating the frequency, it is more likely to generate false alarms. Thus, the proposed criterion (63) reduces the false alarms.

It is also numerically found that MVALIDE-EP tends to fit some spurious components and overestimates the number of sources for real data. As a consequence, we make the active criterion more harsh when we activate the $k$th

C. Computation Complexity

Let $\text{Iter}_{\text{max}}$ denote the number of iterations. For the proposed MVALIDE-EP algorithm shown in Algorithm 1, it consists of three main steps: The initialization, the componentwise MMSE operation, the MVALIDE algorithm. For initialization, the computation complexity is $O(N^2 T)$ to

and we input them to module B. The algorithm is closed and the algorithm iterates until convergence or the maximum number of iterations is reached. The MVALIDE-EP algorithm is summarized as Algorithm 1. For further details, please refer to [19] and the code https://github.com/RiverZhu/MVALIDE-EP.

![Graph](image1)

**Fig. 3.** CRB of a single source at $30^\circ$ and its asymptotic results.

![Graph](image2)

**Fig. 4.** NMSE($\hat{Z}$) versus the number of iterations, here $\text{SNR}_{\text{max}} = 0$ dB, $\Delta_{\text{dB}} = 5$ dB, $N = 100$, $K = 5$, $T = 5$. 

![Graph](image3)

![Graph](image4)

![Graph](image5)
Algorithm 1: MVALE-EP Algorithm.

1: Initialize $v_{\text{ext}}^A(t)$ and $z_{\text{ext}}^A(t), t = 1, \ldots, T$.
2: Initialize the noise variance $\sigma^2$.
3: Perform the MMSE estimate of $Z$ in module B and calculate the extrinsic message from module B to module A.
4: Initialize $\rho$, $K$ and $\tau$.
5: Initialize $q(\omega|Y)$ and obtain $J(t)$, $h(t)$.
6: Set the number of outer iterations $\text{Iter}_{\max}$;
7: for $\text{Iter} = 1, \ldots, \text{Iter}_{\max}$ do
8: Update the support $s$.
9: Update $\tau$ and $\rho$.
10: Refine the mean and concentration parameters of the frequencies $\omega_i, i \in S$.
11: Calculate the posterior means $z_{\text{post}}^A(t)$ (66) and variances $v_{\text{post}}^A(t)$ (67).
12: Compute the extrinsic mean and variance of $z(t)$ as $z_{\text{ext}}^A(t)$ (73) and $v_{\text{ext}}^A(t)$ (72).
13: Compute the post mean and variance of $z(t)$ as $z_{B}^A(t)$ (56) and $v_{B}^A(t)$ (57), $t = 1, \ldots, T$.
14: Compute the extrinsic mean and variance of $z(t)$ as $z_{\text{post}}^B(t)$ (60b) and $v_{\text{post}}^B(t)$ (60a), and set $\hat{\sigma}^2(t) = v_{\text{post}}^A(t)$ and $\hat{y}(t) = z_{\text{post}}^B(t)$.
15: Estimate the noise variance $\sigma^2$ (68).
16: Update $J(t)$ and $h(t)$.
17: end for
18: Return $\hat{\omega}$, $\hat{K}$, $\hat{Z}$ and $K$.

For multibit quantization, a uniform quantizer is adopted and the quantization interval is restricted to $[-5\sigma_z, 5\sigma_z]$, where $\sigma_z$ is the standard deviation of the signal $y(t|z_k)$ or $3(z_k)$. In our setting, it can be calculated that $\sigma_z^2 = \sum_{k=1}^K g_k^2 + \sigma^2$. For one-bit quantization, zero is chosen as the threshold. For each target, its SNR (dB) is uniformly drawn from $[\text{SNR}_{\min}, \text{SNR}_{\max} + \Delta_{\text{SNR}}]$ (dB). The amplitude of the target is determined from the SNRs and is fixed across the snapshots, the phase is uniformly drawn from $[0, 2\pi)$. The DOAs are generated as $\theta = [-30^\circ, -20^\circ, 20^\circ, 30^\circ, 75^\circ]^T + \delta\theta$ unless stated otherwise, where the elements of $\delta\theta$ are i.i.d. from the uniform distribution $U(-0.25^\circ, 0.25^\circ)$. According to the grid spacing of the Gr-SBL, the RMSE of the Gr-SBL is less than $1/\sqrt{48} \approx 0.144$, as shown later.

The noninformative prior, i.e., $p(\omega) = 1/(2\pi)$ is adopted for the MVALE-EP algorithms. The number of maximum iterations is set as $\text{Iter}_{\max} = 100$. The normalized MSE (NMSE) of signal $\hat{Z}$ (for unquantized and multi-bit quantized system) and the root MSE (RMSE) of the DOAs $\hat{\theta}$ are defined as $\text{NMSE}(\hat{Z}) = \|Z - \hat{Z}\|^2_2/\|Z\|^2_2$ and $\text{RMSE}(\hat{\theta}) = \sqrt{\|\hat{\theta} - \theta\|^2_2/K}$, respectively. Please note that, due to magnitude ambiguity, it is impossible to recover the exact magnitude of $x_k(t)$ from one-bit measurements in the noiseless scenario. Thus, for one-bit quantization, the de-biased NMSE $\text{dNMSE}(\hat{Z}) = \min_{c} \|Z - \text{diag}(c)\hat{Z}\|^2_2/\|Z\|^2_2$ is used. As for the DOA error, we average only the trials in which all those algorithms estimate the correct model order. The empirical probability of correct model order estimation $P(\hat{K} = K)$ is adopted as a performance metric. The RMSE of the DOAs is calculated only when the model order is correctly estimated, i.e., $\hat{K} = K$. For the Gr-SBL and one-bit MUSIC approaches, the RMSEs of the DOAs are calculated by assuming the number of sources is known. All the results are averaged over 500 Monte Carlo (MC) trials unless stated otherwise.

V. NUMERICAL SIMULATION AND REAL EXPERIMENT

In this section, numerical and real experiments are conducted to evaluate the effectiveness of the proposed algorithm, through comparing with the generalized sparse Bayesian learning (Gr-SBL) [24], one-bit MUSIC approach [25] and the CRBs. For the numerical simulation part, we conduct the DOA estimation and the element spacing of the linear array is half wavelength. Thus, the DOA $\theta$ is related to the frequency $\omega$ via $\omega = \pi \sin(\theta)$. For the Gr-SBL algorithm, it is an on-grid based approach. The number of grids is 361 and the grid spacing is 0.5°. We evaluate the signal $Z$ estimation error, the DOA estimation error, the correct model order estimation probability under quantized measurements in numerical simulations. For the real data case, we evaluate the range estimation performance for mmWave FMCW system. The number of grids is 501 and the grid spacing is 0.1 m for Gr-SBL to perform range estimation. The number of maximum iterations of Gr-SBL is set as 250.

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C. Performance Versus SNR\textsubscript{min}

The performance versus SNR\textsubscript{min} is shown in Fig. 5. Fig. 5(a) shows that with bit depth \( B \) and SNR\textsubscript{min} fixed, MVALSE-EP performs significantly better than Gr-SBL in terms of \( \text{NMSE}(\hat{Z}) \). Besides, the \( \text{NMSE}(\hat{Z}) \) performances of both algorithms under 3 b quantization are close to their unquantized settings. The model order estimation probability \( P(\hat{K} = K) \) shown in Fig. 5(b) increases as SNR\textsubscript{min} increases. The required SNR\textsubscript{min} for model order estimation probability close to 1 under 1 b quantization, 3 b quantization are \(-8\) and \(-10\) dB, respectively. For the RMSE(\( \hat{\theta} \)) shown in Fig. 5(c) and (d), the RMSEs(\( \hat{\theta} \)) of all the algorithms decrease as the number of snapshots \( T \) increases. The RMSE(\( \hat{\theta} \)) of MVALSE-EP decreases as the number of snapshots \( T \) increases and MVALSE-EP performs better than Gr-SBL. The successful model order estimation probability increases as the number of snapshots \( T \) increases and approaches 1. The RMSEs RMSE(\( \hat{\theta} \)) of all the algorithms decrease as the number of snapshots \( T \) increases. This demonstrates that increasing the number of snapshots benefits the DOA estimation. Besides, MVALSE-EP performs better than Gr-SBL and MUSIC under 1 b quantization, 3 b quantization, and no quantization, and its performance is close to the CRB computed for the true model order.

D. Performance Versus Number of Snapshots

The performance of MVALSE-EP versus the number of snapshots \( T \) is investigated and the results are plotted in Fig. 6. It can be seen that the NMSE of the signal \( \text{NMSE}(\hat{Z}) \) decreases as the number of snapshots \( T \) increases and MVALSE-EP performs better than Gr-SBL. The successful model order estimation probability increases as the number of snapshots \( T \) increases and approaches 1. The RMSEs RMSE(\( \hat{\theta} \)) of all the algorithms decrease as the number of snapshots \( T \) increases. This demonstrates that increasing the number of snapshots benefits the DOA estimation. Besides, MVALSE-EP performs better than Gr-SBL and MUSIC under 1 b quantization, 3 b quantization, and no quantization, and its performance is close to the CRB computed for the true model order.

E. Real Data

This section uses the LFMCW AWR1642 Single-Chip to perform the range estimation. The chirp rate is set as \( \kappa = 29.982 \times 10^{12} \text{M/Hz}^2 \), the sampling frequency is \( F_s = 10 \text{ MHz} \), and the maximum distance is \( r_{\text{max}} = \frac{c F_s}{2 \kappa} = 50 \text{ m} \) where \( c = 3 \times 10^8 \text{ denotes the speed of the electromagnetic wave} \). \( \zeta (75) \) is set as 6 and 15 for 1 b quantization and 12 b quantization, respectively. Given that MVALSE-EP outputs the frequencies estimates \( \hat{\omega} \), the range estimates are \( \hat{r} = \frac{\hat{\omega}}{2\pi} r_{\text{max}} \). The number of fast time samples is \( N = 128 \) and the number of snapshots is 16. We first put corners as targets and test the algorithms in three experiments, as shown in Fig. 7. For the first experiment in Fig. 7(a), corner 1 is set just in front of the radar with the measured radial distance being about 2.25 m. For experiment 2, corner 1 is put with the measured horizontal and vertical distance being 0.25 and 2.5 m, and the radial
distance can be calculated as $\sqrt{0.25^2 + 2.5^2} \approx 2.51$ m. In addition, a little smaller corner named corner 2 is added in front of the radar with the measured radial distance being about 2.25 m. For experiment 3, the position of corner 1 is kept unchanged, and the measured horizontal and vertical distances of corner 2 are 1 and 3.75 m, corresponding to the radial distance 3.88 m. A much larger corner named corner 3 is added with the measured horizontal and vertical distance being 1 and 5.25 m, and the radial distance can be calculated as $\sqrt{0.25^2 + 5.25^2} \approx 5.34$ m. We compare MVA-LE-EP with MUSIC and Gr-SBL.

For experiment 1, the normalized spectrum and the results are shown in Fig. 8 and Table I, respectively. It can be seen that for both 1 and 12 b quantization, the highest peak of the spectrum occurs at 2.34 m. For 1 b quantization, the second peak occurs at $42.96 \approx 50 - 2.34 \times 3 = 42.98$ m, corresponding to the third harmonic and is 10 dB lower than the highest peak. All the three algorithms estimate the range of corner 1 about 2.3 m, and their reconstructed amplitudes are about $45 \sim 47$ dB. For the running time, MUSIC is the fastest, followed by MVA-LE-EP.

For experiment 2, Fig. 9 shows that FFT approach cannot resolve the two targets. Table II shows that results of MVA-LE-EP and Gr-SBL are similar. For MUSIC under 1 b quantization, it misses corner 2 with distance about 2.25 m in a single snapshot scenario, and its estimation...
error for corner 1 is larger than MVALSE-EP and Gr-SBL in multisnapshot scenario. Also, the amplitude of corner 1 is about 2~4 dB lower than that of corner 2.

For experiment 3, the spectrum in Fig. 10 shows that after 1 b quantization, the peak corresponding to corner 3 disappears. For the reconstruction results shown in Table III, all the algorithms detect corner 1 and 2, and miss corner 3 under 1 b quantization for single snapshot and multiple snapshot scenarios. The reconstructed amplitudes of corner 1 of MVALSE-EP under 1 and 12 b quantization and MU-SIC under 12 b quantization are 5~6 dB lower than that of corner 2. This demonstrates that the accuracy of amplitude estimates of MVALSE-EP is higher than MUSIC and Gr-SBL under 1 b quantization. For high resolution data, MVALSE detects the three corners, and also the leakage component whose amplitude is comparable to that of corner 3. This demonstrates that under 1 b quantization, algorithms cannot detect the weak target in the presence of the strong

**Table I**

**Range Estimation Performance of the Algorithms for Experiment 1**

| Algorithm     | B/T | T=1 (range (m), Amp. (dB)) | Time(s) | T=16 (range (m), Amp. (dB)) | Time(s) |
|---------------|-----|----------------------------|---------|-----------------------------|---------|
| MVALSE-EP     | 1 b | (2.32, 46.9) (2.47, 25.6) | 0.556   | (2.32, 46.6) (2.47, 25.3) | 3.66    |
|               | 12 b| (2.32, 46.9) (2.47, 25.6) | 2.86    | (2.32, 46.6) (2.47, 25.3) | 3.66    |
| MUSIC         | 1 b | (2.32, 46.9) (2.47, 25.6) | 0.006   | (2.32, 46.6) (2.47, 25.3) | 0.009   |
|               | 12 b| (2.32, 46.9) (2.47, 25.6) | 0.005   | (2.32, 46.6) (2.47, 25.3) | 0.008   |
| Gr-SBL        | 1 b | (2.32, 46.9) (2.47, 25.6) | 0.582   | (2.32, 46.9) (2.47, 25.3) | 70.87   |
|               | 12 b| (2.32, 46.9) (2.47, 25.6) | 8.95    | (2.32, 46.9) (2.47, 25.3) | 8.72    |

**Table II**

**Range Estimation Performance of the Algorithms for Experiment 2**

| Algorithm     | B/T | T=1 (range (m), Amp. (dB)) | Time(s) | T=16 (range (m), Amp. (dB)) | Time(s) |
|---------------|-----|----------------------------|---------|-----------------------------|---------|
| MVALSE-EP     | 1 b | (2.32, 46.9) (2.47, 25.6) | 0.77    | (2.32, 46.9) (2.47, 25.3) | 3.56    |
|               | 12 b| (2.32, 46.9) (2.47, 25.6) | 2.74    | (2.32, 46.9) (2.47, 25.3) | 17.5    |
| MUSIC         | 1 b | (2.32, 46.9) (2.47, 25.6) | 0.01    | (2.32, 46.9) (2.47, 25.3) | 0.02    |
|               | 12 b| (2.32, 46.9) (2.47, 25.6) | 0.01    | (2.32, 46.9) (2.47, 25.3) | 0.02    |
| Gr-SBL        | 1 b | (2.32, 46.9) (2.47, 25.6) | 0.582   | (2.32, 46.9) (2.47, 25.3) | 70.87   |
|               | 12 b| (2.32, 46.9) (2.47, 25.6) | 8.95    | (2.32, 46.9) (2.47, 25.3) | 8.72    |

Fig. 8. Normalized spectrum of the 1 and 12 b signal for experiment 1.

Fig. 9. Normalized spectrum of the 1 and 12 b signal for experiment 2.

Fig. 10. The normalized spectrum of the 1 b and 12 b signal for experiment 3.
target, where the weak target (about 23 dB) is 26 dB lower than the strongest target (about 48.6 dB).

For the fourth experiment shown in Fig. 11(a), the ranges of people 1 and people 2 are about 3.42 and 2.27 m, respectively. The reconstruction results are shown in Table IV. It can be seen that even under 1 b quantization, MVALSE-EP detects two people and outputs the range estimates as 3.42 and 2.35 m for single snapshot case, 3.44 and 2.36 m for multisnapshot case. Besides, the reconstructed amplitudes of people 1 and people 2 of MVALSE-EP are about 27 and 21 dB, respectively. According to the results provided by MVALSE-EP and MUSIC under 12 b quantization, the amplitude of people 1 is \( \sim 4 \) dB higher than that of people 2, consistent with the results provided by MVALSE-EP under 1 b quantization. This demonstrates that the amplitude estimation accuracy of MVALSE-EP is higher than that of MUSIC and Gr-SBL under 1 b quantization.

For the last experiment shown in Fig. 11(b), the ranges of the bicycle and people 1 are about 3.41 and 2.77 m, respectively. Results are shown in Table V. The bicycle and people 1 are detected for all the three algorithms under 1 and 12 b quantization. Besides, the reconstructed amplitudes of the bicycle and people 1 of MVALSE-EP are about 31 and 28 dB, respectively. Note that the amplitude of people 1 is consistent with the results in experiment 4. It can also be concluded that the amplitude estimation accuracy of MVALSE-EP is the highest.
VI. CONCLUSION

In this article, the multisnapshot LSE problem from coarsely quantized measurements is studied. The CRB is derived and the effects of the system parameters such as SNR, number of measurements and snapshots on the estimation accuracy are revealed in a single frequency scenario. Then, an MVALUE-EP algorithm which automatically estimates the frequencies, the model order and noise variance is proposed. Substantial numerical experiments including real dataset are conducted to show the effectiveness of the MVALUE-EP.

APPENDIX

A. CRB for the General Case

Note that

\[
\frac{\partial \mu(\kappa)}{\partial \kappa^T} = \begin{bmatrix}
\hat{A} & \hat{D}^{(1)} \\
\hat{A} & \hat{D}^{(2)} \\
\vdots & \\
\hat{A} & \hat{D}(T)
\end{bmatrix}
\tag{76}
\]

where \( \hat{A} \) and \( \hat{D}(t) \) are defined in (12a) and (12d), respectively.

Substituting (76) in (6) yields

\[
I(\kappa) = \frac{2}{\sigma^2} \begin{bmatrix}
\hat{H}^{(1)} & \hat{A}^{(1)} \\
\hat{H}^{(2)} & \hat{A}^{(2)} \\
\vdots & \\
\hat{H}^{(T)} & \hat{A}(T)
\end{bmatrix}
\tag{77}
\]

where \( \hat{H}(t) \) and \( \hat{A}(t) \) are defined in (12h) and (12i), respectively, \( \hat{\Gamma} \) is

\[
\hat{\Gamma} = \sum_{t=1}^{T} \hat{D}^{T}(t) \hat{A}(t) \hat{D}(t).
\tag{78}
\]

Note that for the inverse of a block matrix, one has

\[
\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BS^{-1}B^TA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}B^TA^{-1} & S^{-1} \end{bmatrix}
\tag{79}
\]

where \( S = C - B^TA^{-1}B \). Therefore, CRB(\( \omega \)) is

\[
\text{CRB}(\omega) = \frac{\sigma^2}{2} \left( \hat{\Gamma} - [\hat{A}^{T}(1), \hat{A}^{T}(2), \ldots, \hat{A}^{T}(T)] \right)^{-1} \begin{bmatrix}
\hat{H}^{(1)} \\
\hat{H}^{(2)} \\
\vdots \\
\hat{H}^{(T)}
\end{bmatrix}^{-1}
\]

and performing further simplification yields (13).

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