EXTENDING THE TRACE OF A PIVOTAL MONOIDAL FUNCTOR

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Abstract. We consider a pivotal monoidal functor whose domain is a modular tensor category (MTC). We show that the trace of such a functor naturally extends to a representation of the corresponding tube category. As irreducible representations of the tube category are indexed by pairs of simple objects in the underlying MTC, the simple multiplicities of this representation form a candidate modular invariant matrix. In general, this matrix will not be modular invariant, however it will always commute with the T-matrix. Furthermore, under certain additional conditions on the original functor, it is shown that the corresponding representation of the tube category is a haploid, symmetric, commutative Frobenius algebra. Such algebras are known to be connected to modular invariants, in particular a result of Kong and Runkel implies that the matrix of simple multiplicities commutes with the S-matrix if and only if the dimension of the algebra is equal to the dimension of the underlying MTC. Finally, this procedure is applied to certain pivotal monoidal functors arising from module categories over the Temperley-Lieb category and the associated MTC.

1. Introduction

An important property of a conformal field theory (CFT) is that it has two chiral halves: a holomorphic (or “left-moving”) half and an anti-holomorphic (or “right-moving”) half. In other words, the state space $H$ of the theory decomposes into the direct sum

$$H = \bigoplus_{I,J} Z_{IJ} H_I \otimes H_J$$

(1)

where the $Z_{IJ}$ are multiplicity spaces and the $H_I$ range over the irreducible modules of a vertex operator algebra (VOA) $V$ (we assume that our CFT is non-heterotic, i.e. that $H_I$ and $H_J$ are modules over the same VOA). The physical “uniqueness of the vacuum” assumption imposes that $Z_{1,1} = \mathbb{C}$ where 1 is such that $H_1 = V$. The CFT is called rational if $V$ admits only finitely many irreducible modules; we assume that this is the case from now on. The decomposition of $H$ given by (1) implies that

$$Z(\tau) = \sum_{I,J} \dim Z_{IJ} \chi_I(\tau)\chi_J(\tau)^*$$

(2)

where $Z$ is the partition function of the theory, i.e. $Z(\tau)$ is the value of the theory on the torus corresponding to $\tau \in \mathbb{H}$ and $\chi_I$ is the character of the irreducible VOA module $H_I$. As conformal structures on the torus are parametrized by $\mathbb{H}/\text{PSL}_2(\mathbb{Z})$, we require that $Z$ be invariant under the action of $\text{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$.

The category of modules over a VOA has an extremely rich structure: it forms a modular tensor category (MTC) [Hua05]. MTCs possess many nice properties (they are semisimple, rigid, braided...) and in particular they come equipped with a representation of $\text{PSL}_2(\mathbb{Z})$ given by their modular data. Let $\mathcal{I}$ be (an indexing set for) a complete set of irreducible objects in an MTC. The modular data is composed of two $\mathcal{I} \times \mathcal{I}$-matrices known as the $S$-matrix and the $T$-matrix; they are denoted by $S$ and $T$ respectively.
Using the graphical calculus of MTCs, the entries of these matrices are given as follows,

\[ T_{IJ} := \delta_{I,J}, \quad S_{IJ} := I \otimes J, \]

The condition that the partition function of a CFT is invariant under the action of \( \text{PSL}_2(\mathbb{Z}) \) on \( \mathbb{H} \) may be rephrased as requiring that the \( I \times I \)-matrix with entries \( \dim Z_{IJ} \) commutes with the modular data of the category of modules over the relevant VOA. This motivates the following definition.

**Definition 1.1.** For a modular tensor category with tensor identity \( 1 \) and complete set of simples \( I \), a **modular invariant** is a non-negative integer \( I \times I \)-matrix that commutes with the modular data and whose \((1,1)\)-entry is 1.

A popular strategy when attempting to classify CFTs is to fix a VOA \( \mathcal{V} \) and search for all compatible partition functions. From the above discussion we see that this is related to finding the modular invariants associated to the MTC of modules over \( \mathcal{V} \). An example in which this has been successfully carried out is provided by the VOA constructed from the affine Lie algebra \( A_k^{(1)} \) together with a positive integer \( k \), via the Sugawara construction \[\text{Sug68}\]. The category of modules in this case is the category of integrable highest weight modules of \( A_k^{(1)} \) at level \( k \), denoted \( \text{Rep}_k A_k^{(1)} \). In 1986 Cappelli, Itzykson and Zuber classified all possible modular invariants in this context and, to their surprise, the classification followed an A-D-E pattern \[\text{CIZ87}\]. The appearance of this pattern intrigued many researchers in the field and was the subject of much speculation \[\text{Gan00, Zub02, KO02}\]. The first explanation of the pattern was provided by an operator algebra technique known as \( \alpha \)-induction, due to Böckenhauer and Evans \[\text{BE98}\]. This technique relates the A-D-E classification of Goodman-de la Harpe-Jones subfactors to Cappelli, Itzykson and Zuber’s classification \[\text{BE01, Ocn99}\].

When translating from the operator algebra language to the purely categorical one an inclusion of subfactors corresponds to a module category. Finite modules categories also have a physical interpretation. In 1989 Cardy showed that the algebraic data of an annular partition function in a boundary CFT (as opposed to the toroidal partition function \( Z \)) is given by a finite module category over the corresponding MTC \[\text{Car89}\]. From a physical point of view the correspondence between module categories and modular invariants should therefore be thought of as a “closing up” just as an annulus closes up into a torus. Mathematically we would expect this “closing up” to correspond to taking the *trace*, in some suitable sense, of the module category. A notion of trace does exist for module categories (and more generally for monoidal functors), however it simply produces a representation of the MTC. A priori it is not at all clear how to associate a non-negative integer \( I \times I \)-matrix to this representation. This article presents a solution to this problem by extending the representation to take values on the *tube category* of the underlying MTC.

For a spherical fusion category \( \mathcal{C} \), the tube category, denoted \( T\mathcal{C} \), shares the same objects as \( \mathcal{C} \) but has more morphisms i.e. \( \text{Hom}_\mathcal{C}(X,Y) \leq \text{Hom}_{T\mathcal{C}}(X,Y) \). The intuition is that whereas morphisms in \( \mathcal{C} \) may be represented graphically as diagrams drawn on a bounded region of the plane, morphisms in \( T\mathcal{C} \) are given by diagrams drawn on a cylinder. Section 3 describes how, for \( \mathcal{M} : \mathcal{C} \to \mathcal{D} \) a pivotal monoidal functor, the trace of \( \mathcal{M} \) naturally extends to a representation of \( T\mathcal{C} \), which we denote \( T\mathcal{M} \). As irreducible representations of the tube category are indexed by pairs of elements in \( I \), decomposing...
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$\mathcal{T}\mathcal{M}$ into irreducibles gives a non-negative integer $I \times I$-matrix, $Z(\mathcal{T}\mathcal{M})$. For $F$ a representation of $\mathcal{T}\mathcal{C}$, $F$ is called T-invariant (respectively S-invariant) if $Z(F)$ commutes with $T$ (respectively $S$).

Section 4 gives a graphical characterisation of T-invariance when $\mathcal{C}$ is an MTC. In particular, Theorem 4.4 proves that $F$ is T-invariant if and only if $F(t_X) = \text{id}_{F(X)}$ where $t_X \in \text{End}_{\mathcal{T}\mathcal{C}}(X)$ is the twist morphism on $X$, see (9). An immediate corollary of this is that $\mathcal{T}\mathcal{M}$ is T-invariant; this corollary will later be strengthened to Theorem 7.6 which only assumes that $\mathcal{C}$ is pre-modular. Section 5 starts by showing that, in general, $Z(\mathcal{T}\mathcal{M})$ fails to be S-invariant. Indeed, when $M = \text{id}_C$, $Z(\mathcal{T}\mathcal{M})$ is given by

$$Z(\mathcal{T}\mathcal{M})_{IJ} = \begin{cases} 1 & \text{if } I = J = 1 \\ 0 & \text{else} \end{cases}$$

which doesn’t commute with the S-matrix in general (this will be explained in greater detail in Example 5.2). However, under the assumption that $\mathcal{M}$ is indecomposable and takes value in a category whose idempotent completion is multifusion, Theorem 6.7 proves that $\mathcal{T}\mathcal{M}$ is a haploid, symmetric, commutative, Frobenius algebra. By a result of Kong and Runkel [KR09, Theorem 3.4] this implies that $Z(\mathcal{T}\mathcal{M})$ commutes with the S-matrix if and only if the dimension of $\mathcal{T}\mathcal{M}$ is equal to the dimension of $\mathcal{C}$. This condition on the dimension of $\mathcal{T}\mathcal{M}$ is equivalent to requiring that

$$(S Z(\mathcal{T}\mathcal{M}) S^{-1})_{1,1} = Z(\mathcal{T}\mathcal{M})_{1,1}$$

and is therefore always a necessary condition for S-invariance.

Section 7 describes a categorical formulation of $\alpha$-induction given by Ostrik [Ost03, Section 5]. Let $\mathcal{C}$ be a pre-modular tensor category. A module category may be thought of as a (not necessarily pivotal) monoidal functor $\mathcal{M} : \mathcal{C} \to A, A$-Bimod, where $A$ is a semisimple algebra. Following [Ost03], we define a subspace $\text{Hom}^{\sigma}_{A,A}$-Bimod$(I^\vee, J) \leq \text{Hom}^{\sigma}_{A,A}$-Bimod$(I^\vee, J)$. Ostrik’s categorical formulation of $\alpha$-induction states that, when the dimension of all the objects in $\mathcal{C}$ are positive, the $I \times I$-matrix with entries given by the dimension of $\text{Hom}^{\sigma}_{A,A}$-Bimod$(I^\vee, J)$ is a modular invariant. Theorem 7.5 proves that, when $\mathcal{M}$ induces a pivotal structure on its image, the $\mathcal{T}\mathcal{M}$ construction may be applied and $Z(\mathcal{T}\mathcal{M})$ will produce the same matrix as $\alpha$-induction. Furthermore, this application of the $\mathcal{T}\mathcal{M}$ construction to module categories leads us to Corollary 7.7 which states that, when $\mathcal{M}$ is an indecomposable module category that induces a pivotal structure on its full image, $\mathcal{T}\mathcal{M}$ is a haploid, symmetric, commutative, Frobenius algebra.

Finally, Section 8 applies the $\mathcal{T}\mathcal{M}$ construction to a class of examples arising from module categories over the Temperley-Lieb category. The Temperley-Lieb category may be thought of as a diagrammatic presentation of the previously discussed category $\text{Rep}_k A_1^{(1)}$. It is shown that all module categories over $\text{Rep}_k A_1^{(1)}$ induce a pivotal structure on their full image and so the $\mathcal{T}\mathcal{M}$ construction may be applied. This leads to a new explanation of the A-D-E pattern that appears in the Cappelli-Itzykson-Zuber classification of $A_1^{(1)}$ modular invariants.

There are pre-existing methods for relating module categories to modular invariant $^2$ Frobenius algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. Any module category over $\mathcal{C}$ may be realised (non-uniquely) as the category of modules of an algebra in $\mathcal{C}$ [Ost03]. The full centre construction [FFRS08, Definition 4.9] then associates a modular invariant, commutative, symmetric Frobenius algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$ to a special (as defined in, for example, [KR09]), symmetric Frobenius

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$^2$As defined in [Kon08, Section 6], cf. Remark 5.7.
algebra in \(C\) [KR09, Theorem 3.18]. Furthermore every modular invariant, commutative, symmetric Frobenius algebra in \(\mathcal{C} \boxtimes \mathcal{C}\) may be realised in this way [KR09, Theorem 3.22].

The full centre construction may also be described in terms of the module category directly [DKR15, Section 3.1]. Schaumann has worked on characterising the condition that the module category be equivalent to the category of modules of a special symmetric Frobenius algebra purely in terms of the module category itself. In particular he has shown that it is equivalent to requiring that the module category admits a module trace [Sch13]. It is possible that this could be related to the condition identified in this article: that the module category induce a pivotal structure on its full image.

**Conventions.** For \(V\) and \(W\) vector spaces, we write “\(V = W\)” to indicate that \(V\) and \(W\) are isomorphic under an isomorphism that should be clear from the context. Unless otherwise specified, a sum over a variable object ranges over a complete set of simple objects. Similarly, unless otherwise specified, a sum over a variable morphism ranges over a basis of the appropriate Hom-space. All categories are assumed to be enriched over the category of finite dimensional vector spaces. For a category \(\mathcal{C}\) we use \(\mathcal{RC}\) to denote the category of contravariant functors from \(\mathcal{C}\) to the category of finite dimensional vector spaces. An object in \(\mathcal{RC}\) is called a representation of \(\mathcal{C}\). Much of the work carried out in this article will be done relative to a fixed spherical fusion category \(\mathcal{C}\), when taking tensor products in this category we will omit the “\(\otimes\)” symbol and write \(XY\) for \(X \otimes Y\). However, we will write the “\(\otimes\)” symbol when taking a tensor product in any other category. Many of the arguments in this article exploit the graphical calculus of spherical fusion categories. For an exposition of these techniques see, for example, [Har19, Section 1-6]. All diagrams are read top to bottom.

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2. **Preliminaries on the Tube Category**

We start by recording some results on the tube category, which will be used throughout; for more details on these results see [HK19] and [Har19]. Let \(\mathbb{K}\) be a field and let \(\mathcal{C}\) be a spherical fusion category over \(\mathbb{K}\) with complete set of simples \(I\). The tube category of \(\mathcal{C}\), denoted \(\mathcal{T}\mathcal{C}\), is a category whose objects coincide with those of \(\mathcal{C}\) and whose Hom-spaces are given by

\[
\mathrm{Hom}_{\mathcal{T}\mathcal{C}}(X, Y) := \bigoplus_S \mathrm{Hom}_{\mathcal{C}}(SX, YS)
\]

where, as per our conventions, the direct sum ranges over \(I\) and the monoidal product symbol is suppressed. To depict a morphism in \(\mathcal{T}\mathcal{C}\) using the graphical calculus of spherical fusion categories we take \(\alpha \in \mathrm{Hom}_{\mathcal{C}}(GX, YG)\) and write

\[
\alpha_G = \begin{tikzpicture}[baseline=-.5ex]
\node (X) at (0,1) {$X$};
\node (Y) at (0,0) {$Y$};
\node (G) at (1,0) {$G$};
\draw[->] (X) to node [above] {$\alpha$} (G);
\draw[->] (Y) to (G);
\end{tikzpicture}
\] (4)

as shorthand for \(\bigoplus_S \sum_b (\mathrm{id}_Y \otimes b^*) \circ \alpha \circ (b \otimes \mathrm{id}_X) \in \mathrm{Hom}_{\mathcal{T}\mathcal{C}}(X, Y)\), where \(\{b\}\) is a basis of \(\mathrm{Hom}_{\mathcal{C}}(S, G)\) and \(\{b^*\}\) is the corresponding dual basis of \(\mathrm{Hom}_{\mathcal{C}}(G, S)\) with respect to the perfect pairing given by composition into \(\mathrm{End}_{\mathcal{C}}(S) = \mathbb{K}\), see [HK19, Proposition 3.1].
The intuition is that whereas morphisms in $\mathcal{C}$ may be represented graphically as diagrams drawn on a bounded region of the plane, morphisms in $\mathcal{TC}$ are given by diagrams drawn on a cylinder. In particular, the red lines in (4) should be thought of as being glued; this is compatible with our notation as one may indeed show that

$$
\alpha_{X,Y}^{G_1 G_2 G_2 G_1} = \alpha_{X,Y}^{G_2 G_1 G_1 G_2}
$$

for any $\alpha \in \text{Hom}_\mathcal{C}(G_2 X, Y G_1)$ and $g \in \text{Hom}_\mathcal{C}(G_1, G_2)$. Composition in $\mathcal{TC}$ is then defined following the intuition of vertically stacking the cylinders:

$$
\beta_H \circ \alpha_G := \bigoplus_T \sum_b \beta_T^X \alpha_Y^G G
$$

This intuition, together with the associativity of the tensor product, makes it clear that composition in $\mathcal{TC}$ is associative.

**Remark 2.1.** If we consider the algebra $\text{End}_{\mathcal{TC}}(\bigoplus_S S)$ we recover Ocneanu’s tube algebra [Ocn94]. As $\bigoplus_S S$ is a projective generator in $\mathcal{TC}$, the tube algebra is Morita equivalent to $\mathcal{TC}$.

**Remark 2.2.** Let $\mathcal{K}(\mathcal{C})$ denote the Grothendieck ring of $\mathcal{C}$ and let $\mathcal{K}_\mathbb{K}(\mathcal{C})$ denote $\mathcal{K}(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{K}$. Then $\text{End}_{\mathcal{TC}}(1)$ and $\mathcal{K}_\mathbb{K}(\mathcal{C})$ are canonically isomorphic algebras. Indeed, $\text{End}_{\mathcal{TC}}(1) = \bigoplus_S \text{End}(S) = \bigoplus_S \mathbb{K}$ is precisely the underlying vector space of $\mathcal{K}_\mathbb{K}(\mathcal{C})$. Furthermore, composition in $\text{End}_{\mathcal{TC}}(1)$ corresponds to the tensor product in $\mathcal{K}_\mathbb{K}(\mathcal{C})$.

**Remark 2.3.** The canonical inclusion $\text{Hom}_\mathcal{C}(X, Y) \hookrightarrow \text{Hom}_{\mathcal{TC}}(X, Y)$ realises $\mathcal{C}$ as a wide subcategory of $\mathcal{TC}$.

Remark 2.3 suggests the following natural question: for a given representation of $\mathcal{C}$ (i.e. an object in $\mathcal{RC}$) what additional data could be provided to specify a unique extension to an object $F$ in $\mathcal{RTC}$? This question is answered in [Har19] by considering the value of the extended functor on morphisms in $\mathcal{TC}$ of the form

$$
\varepsilon_{G,X} := \begin{array}{ccc}
G & X \\
G & X \\
\end{array}
$$

**Proposition 2.4.** Let $F$ be in $\mathcal{RC}$ and let $\kappa_{G,X} : F(GX) \to F(XG)$ be a collection of isomorphisms which are natural in $G$ and $X$ and satisfy $\kappa_{H,XG} \circ \kappa_{G,HX} = \kappa_{G,H,X}$. Then there is a unique object $(F, \kappa)$ in $\mathcal{RTC}$ which satisfies $(F, \kappa)(X) = F(X)$ for all $X$ in $\mathcal{C}$, $(F, \kappa)(\alpha) = F(\alpha)$ for all $\alpha \in \text{Hom}_\mathcal{C}(X, Y)$ and $(F, \kappa)(\varepsilon_{G,X}) = \kappa_{G,X}$ for all $G, X$ in $\mathcal{C}$.
Proof. See Proposition 8.1 in [Har19]. □

**Remark 2.5.** As $C$ is a fusion category the Yoneda embedding gives an equivalence between $C$ and $RC$. As described in Section 9 of [Har19], the data required to extend the image of $X$ under the Yoneda embedding to $\mathcal{T}C$ (as given by Proposition 2.4) corresponds to a half braiding on $X$. Combining these facts yields an equivalence between $Z(C)$ and $\mathcal{R}TC$, where $Z(C)$ is the Drinfeld centre of $C$.

We now equip $C$ with a (balanced) braiding, in other words, $C$ is a pre-modular tensor category. Our main tool for studying $\mathcal{T}C$ in this case will be the following endomorphisms:

$$e^X_Y = \frac{1}{d(C)} \bigoplus_S d(S) S \in \text{End}_{\mathcal{T}C}(XY).$$

In particular, Proposition 9.4 in [Har19] proves that the canonical braided functor

$$\Phi: C \boxtimes \overline{C} \to Z(C) \cong \mathcal{R}TC$$

satisfies

$$\Phi(X \boxtimes Y) = (XY, e^X_Y)^\sharp := \text{Hom}_{\mathcal{T}C}(\_,-, e^X_Y).$$

Combining this with the fact that $\Phi$ is an equivalence when $C$ is modular (see [EGNO15, Proposition 8.20.12]) and we obtain the result that, in the modular case, the set $\{e^I_J\}_{I,J \in I}$ forms a complete set of orthogonal primitive idempotents in $\mathcal{T}C$.

**Remark 2.6.** The notation $e^X_Y$ is chosen (as opposed to $e_{XY}$) as $(XY, e^X_Y)^\sharp$ is isomorphic (as an object in $\mathcal{R}TC$) to $(YX, \tilde{e}^X_Y)^\sharp$, where

$$\tilde{e}^Y_X = \frac{1}{d(C)} \bigoplus_S d(S) S \in \text{End}_{\mathcal{T}C}(YX).$$

The isomorphism is in fact given by the embedding of the braiding on $C$ into $\mathcal{T}C$. Therefore the isomorphism class of $e^X_Y$ is really determined by the fact that the $X$ strand is under-braided and the $Y$ strand is over-braided. This motivates the notation.

**Remark 2.7.** We recall from Remark 2.2 that $\mathcal{K}(\mathcal{C}) = \text{End}_{\mathcal{T}C}(\mathbf{1})$. By Proposition 9.4 in [Har19] we have

$$\text{Hom}_{\mathcal{T}C}(\mathbf{1}, e^I_J) = \text{Hom}_C(\mathbf{1}, IJ) = \delta_{IJ} \times \mathbb{K} \quad \forall I, J \in I.$$  

As $\{e^I_J\}_{I,J \in I}$ forms a complete set of primitive idempotents in $\mathcal{T}C$ we may conclude that $\mathcal{K}(\mathcal{C})$ is a commutative semisimple algebra generated by a set of primitive orthogonal idempotents indexed by $I$.

### 3. The $\mathcal{T}M$ Construction

Let $\mathcal{C}$ be spherical fusion category, let $\mathcal{D}$ be a pivotal monoidal category and let $\mathcal{M}: \mathcal{C} \to \mathcal{D}$ be a pivotal monoidal functor. When doing graphical calculus in $\mathcal{D}$ we use blue to
depict the image of objects and morphisms in \( C \) under \( \mathcal{M} \). For example a morphism \( \alpha \in \text{Hom}_D(A, B) \) is depicted in the normal way,

\[
\begin{array}{c}
A \\
\alpha \\
B
\end{array}
\]

whereas, for \( \beta \in \text{Hom}_C(X, Y) \), we depict \( \mathcal{M}(f) \in \text{Hom}_D(\mathcal{M}(Y), \mathcal{M}(X)) \) as

\[
\begin{array}{c}
Y \\
\beta \\
X
\end{array}
\].

Composing \( \mathcal{M} \) with the trace functor gives the following object in \( \mathcal{RC} \)

\[
\text{Tr} \, \mathcal{M} : C \to \text{Vect} \\
X \mapsto \text{Hom}_D(1, \mathcal{M}(X)).
\]

For \( X \) and \( G \) in \( C \) we consider the isomorphism

\[
\kappa_{G,X} : \text{Tr} \, \mathcal{M}(XG) \to \text{Tr} \, \mathcal{M}(GX)
\]

\[
\begin{array}{c}
\alpha \\
X \\
G
\end{array} \mapsto \begin{array}{c}
\alpha \\
G \\
X
\end{array}.
\]

As, for \( f \) and \( g \) morphisms in \( C \),

\[
\begin{array}{c}
\alpha \\
Y \to \to \\
\beta
\end{array} = \begin{array}{c}
\alpha \\
Y \to \to \\
\beta
\end{array}
\]

we have that \( \kappa_{G,X} \) is natural in both \( G \) and \( X \). Furthermore, we have

\[
\kappa_{G,HX} \circ \kappa_{H,XG} = \kappa_{G,HX} \left( \begin{array}{c}
\alpha \\
G \\
X \\
H
\end{array} \right) = \begin{array}{c}
\alpha \\
H \\
G \\
X
\end{array} = \kappa_{GH,X}
\]

and \( \kappa_{1,X} = \text{id}_{\mathcal{F}(X)} \). We can therefore apply Proposition 2.4 to extend \( \text{Tr} \, \mathcal{M} \) to a functor on \( \mathcal{T} \mathcal{C} \). We denote this extension \( \mathcal{T} \, \mathcal{M} \). For a more concrete description of \( \mathcal{T} \, \mathcal{M} \) we consider \( \alpha_G \in \text{Hom}_{\mathcal{T} \mathcal{C}}(X, Y) \). Then we have

\[
\mathcal{T} \, \mathcal{M}(\alpha_G) : \text{Hom}_D(1, \mathcal{M}(Y)) \to \text{Hom}_D(1, \mathcal{M}(X))
\]

\[
\beta \mapsto \begin{array}{c}
\beta \\
Y \\
\alpha \\
G
\end{array} \mapsto \begin{array}{c}
\beta \\
Y \\
\alpha \\
G \\
X
\end{array}.
\]

**Remark 3.1.** It is possible to define the \( \mathcal{T} \, \mathcal{M} \) construction for functors \( \mathcal{M} : C \to D \) that are not pivotal. However, doing so would not be compatible with our graphical conventions: the pivotal structure in \( C \) (which is suppressed from the graphical calculus)
would be mapped to a morphism in $\mathcal{D}$ which could fail to be the corresponding pivotal structure (and thus not be suppressed from the graphical calculus).

**Remark 3.2.** It straightforward to check that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ implies $\mathcal{T}\mathcal{M} = \mathcal{T}\mathcal{M}_1 \oplus \mathcal{T}\mathcal{M}_2$.

We now once again suppose that $\mathcal{C}$ is equipped with a (balanced) braiding and is therefore a pre-modular tensor category. For an object $F$ in $\mathcal{RTC}$ we consider the Hom-space

$$F_X^Y := \text{Hom}_{\mathcal{RTC}}(\langle XY, \epsilon_X^Y \rangle^2, F) = \{ \alpha \in F(XY) \mid F(\epsilon_X^Y)(\alpha) = \alpha \}.$$ 

**Proposition 3.3.** $\mathcal{T}\mathcal{M}_X^Y$ is given by the subspace of $\mathcal{T}\mathcal{M}(XY) = \text{Hom}_D(1, \mathcal{M}(XY))$ defined by the condition that $\alpha \in \text{Hom}_D(1, \mathcal{M}(XY))$ satisfy

$$\begin{align*}
Z \xrightarrow{\alpha} Z & = \begin{array}{c}
\alpha
\end{array}.
\end{align*}
$$

for all $Z$ in $\mathcal{C}$.

**Proof.** Evaluating $\mathcal{T}\mathcal{M}$ on $\epsilon_X^Y$ gives the map

$$\begin{align*}
\mathcal{T}\mathcal{M}(\epsilon_X^Y) : \text{Hom}_D(S, \mathcal{M}(XY)) & \rightarrow \text{Hom}_D(S, \mathcal{M}(XY)) \\
\alpha & \mapsto \frac{1}{d(\mathcal{C})} \sum_S d(S) \begin{array}{c}
\alpha
\end{array}.
\end{align*}$$

Therefore, if $\alpha$ satisfies (8), we have

$$\begin{align*}
\mathcal{T}\mathcal{M}(\epsilon_X^Y)(\alpha) & = \frac{1}{d(\mathcal{C})} \sum_S d(S) \begin{array}{c}
\alpha
\end{array} = \alpha.
\end{align*}$$

Furthermore, for $\alpha \in \mathcal{T}\mathcal{M}_X^Y$, we have

$$\begin{align*}
\begin{array}{c}
\alpha
\end{array} & = \frac{1}{d(\mathcal{C})} \sum_S d(S) \\
\begin{array}{c}
\alpha
\end{array} & = \frac{1}{d(\mathcal{C}) \sum_{S,T,b} d(S) \sum_{S,t,b} d(S)} \\
\begin{array}{c}
\alpha
\end{array} & = \frac{1}{d(\mathcal{C}) \sum_{S,T,b} d(S)}.
\end{align*}$$
where, to make certain string manipulations clearer, we have chosen to write $\hat{b}$ and $\hat{b}^*$ upside-down instead of writing $b^\vee$ and $(b^*)^\vee$ and the penultimate equality uses [HK19, Lemma 3.11]. □

Remark 3.4. We recall from Remark 2.6 that $(X Y, \varepsilon_Y X)^\# = (Y X, \tilde{\varepsilon}_Y X)^\#$ where $\tilde{\varepsilon}_Y X$ is given by (7). Therefore $\mathcal{T} \mathcal{M}_X$ may also be identified with the subspace of $\text{Hom}_\mathcal{D}(1, \mathcal{M}(Y X))$ defined by the condition that $\alpha \in \text{Hom}_\mathcal{D}(1, \mathcal{M}(Y X))$ satisfy

\[
\begin{array}{c}
Z \quad \alpha \\
Z X Y \\
\end{array}
\begin{array}{c}
Z \\
Z X Y \\
\end{array}
\begin{array}{c}
\text{for all } Z \text{ in } \mathcal{C}. \\
\end{array}

\]

Definition 3.5. For any $F$ in $\mathcal{RTC}$ one may consider the $I \times I$ integer matrix.

\[
Z(F) := (\text{dim } F^I_J)_{I,J \in I}.
\]

Remark 3.6. We recall that, if $\mathcal{C}$ is modular, the set $\{(IJ, \epsilon_J^I)^\#\}_{I,J \in I}$ forms a complete set of simples in $\mathcal{RTC}$. Therefore an entry of $Z(F)$ simply gives the multiplicity of the corresponding simple object in $F$.

4. T-Invariance

Definition 4.1. Let $\mathcal{C}$ be a pre-modular tensor category and let $\mathcal{T}$ be the T-matrix of $\mathcal{C}$ as defined by (3). We call an object $F$ in $\mathcal{RTC}$ T-invariant if $Z(F)$ commutes with $\mathcal{T}$.

The principal goal of this section is to give a graphical characterisation of T-invariance when $\mathcal{C}$ is modular. We consider the following automorphism of $X$ in $\mathcal{TC}$,

\[
t_X := \begin{array}{c}
X^\vee \\
X \\
X^{\vee} \\
\end{array}.
\]

Lemma 4.2. For all $\alpha \in \text{Hom}_{\mathcal{TC}}(X, Y)$ we have,

\[
\alpha \circ t_X = t_Y \circ \alpha.
\]
Proof. Let $\alpha_G$ be in $\text{Hom}_{\mathcal{T}C}(X,Y)$. We have

$$\alpha_G \circ t_X = G \alpha \circ X \circ Y = t_X \circ \alpha_G.$$ as desired. \qed

As described in Section 2, if $\mathcal{C}$ is modular then $\epsilon_I^J$ is a primitive idempotent. In particular we have $\text{End}_{\mathcal{T}C}(\epsilon_I^J) = \mathbb{K}$. However, by Lemma 4.2 we have

$$\epsilon_I^J \circ t_{IJ} \circ \epsilon_I^J = \epsilon_I^J \circ \epsilon_I^J \circ t_{IJ} = \epsilon_I^J \circ t_{IJ}$$

so $\epsilon_I^J \circ t_{IJ} \in \text{End}_{\mathcal{T}C}(\epsilon_I^J)$. Therefore $\epsilon_I^J \circ t_{IJ} = \lambda \epsilon_I^J$ for some $\lambda \in \mathbb{K}$. This turns out to also be true in the case when $\mathcal{C}$ is only assumed to be a pre-modular tensor category.

**Proposition 4.3.** Let $\mathcal{C}$ be a pre-modular tensor category and let $I, J$ be in $\mathcal{I}$. Then

$$\epsilon_I^J \circ t_{IJ} = \frac{T_{II}}{T_{JJ}} \epsilon_I^J.$$ 

Proof. We have

$$\epsilon_I^J \circ t_{IJ} = \bigoplus_S \sum_{T,b} d(T).$$

Therefore the $S$-summand of $\epsilon_I^J \circ t_{IJ}$ is given by

$$\sum_{T,b} d(T) = \sum_{T,b} d(T) = \frac{T_{II}}{T_{JJ}} \sum_{T,b} d(T).$$
where the final equality is due to Lemma 3.11 in [HK19]. As this is exactly the $S$-summand of $\frac{T_{IJ}}{T_{JJ}}e^I_J$ we are done. □

We may now prove the main result of this section.

**Theorem 4.4.** Let $C$ be a modular tensor category and let $F$ be an object in $\mathcal{R}TC$. $F$ is $T$-invariant if and only if $F(t_X) = \text{id}_F(X)$ for all $X$ in $C$.

**Proof.** As $C$ is modular the $(IJ, \epsilon^I_J)$ form a complete set of simples. We can therefore decompose $F$ as

$$F = \bigoplus_{IJ} F^I_J \cdot (IJ, \epsilon^I_J)^t.$$

Evaluating this on $t_X$ gives

$$F(t_X) = \bigoplus_{IJ} \text{id}_{F^I_J} \otimes (IJ, \epsilon^I_J)^t(t_X).$$

By Lemma 4.2 and Proposition 4.3 we have, for $\alpha \in \text{Hom}_{\mathcal{T}C}(X, \epsilon^I_J)$,

$$(IJ, \epsilon^I_J)^t(t_X)(\alpha) = \epsilon^I_J \circ \alpha \circ t_X = \epsilon^I_J \circ t_{IJ} \circ \alpha = \frac{T_{IJ}}{T_{JJ}} \epsilon^I_J \circ \alpha = \frac{T_{IJ}}{T_{JJ}} \alpha.$$

Therefore

$$\bigoplus_{IJ} \text{id}_{F^I_J} \otimes (IJ, \epsilon^I_J)^t(t_X) = \bigoplus_{IJ} \frac{T_{IJ}}{T_{JJ}} \text{id}_{F^I_J} \otimes (IJ, \epsilon^I_J)^t(t_X).$$

This is equal to $\text{id}_{F(X)}$ if and only if $F^I_J \neq 0$ implies $\frac{T_{IJ}}{T_{JJ}} = 1$. As $\mathcal{T}$ is diagonal that is precisely the condition that $Z(F)$ commutes with $\mathcal{T}$. □

**Corollary 4.5.** If $C$ is modular then $\mathcal{T} \mathcal{M}$ is $T$-invariant.

**Proof.** For $\alpha \in \mathcal{T} \mathcal{M}(X) = \text{Hom}_{\mathcal{D}}(1, \mathcal{M}(X))$ we have

$$\mathcal{T} \mathcal{M}(t_X): \begin{array}{c} \alpha \\ X \end{array} \mapsto \begin{array}{c} \alpha \\ X \end{array} = \begin{array}{c} \alpha \\ X \end{array}$$

as $\mathcal{M}$ is pivotal. Therefore Theorem 4.4 applies, and $\mathcal{T} \mathcal{M}$ is $T$-invariant. □

### 5. S-Invariance and Frobenius Algebras

**Definition 5.1.** Let $C$ be a pre-modular tensor category and let $S$ be the S-matrix of $C$ as defined by [13]. We call an object $F$ in $\mathcal{R}TC$ $S$-invariant if $Z(F)$ commutes with $S$.

We start with an example which illustrates that, even when $C$ is modular, $\mathcal{T} \mathcal{M}$ is not necessarily $S$-invariant.
Example 5.2. Let $\mathcal{C}$ be a modular tensor category and let $\mathcal{M} = \text{id}_\mathcal{C}$. Then

$$Z(\mathcal{T}\mathcal{M})_{IJ} \leq \text{Hom}_\mathcal{C}(1, IJ) = \delta_{I,J'} \left\langle \begin{array}{c}
Z \\
J^\vee _J
\end{array} \right\rangle_K.$$  

We now suppose $\delta_{I,J'} = 1$. By Proposition 3.3 $Z(\mathcal{T}\mathcal{M})_{J',J}$ is non-trivial if and only if

$$\begin{array}{c}
Z \\
J^\vee _J
\end{array} \left\langle \begin{array}{c}
Z \\
J^\vee _J
\end{array} \right\rangle = \delta_{I,J} \vee _J J.$$  

for all $Z$ in $\mathcal{C}$. Post-composing this equality with $\text{id}_Z \otimes \text{an}_J$ and taking the trace implies $S_{Z,J} = d(Z)d(J)$ for all $Z$ in $\mathcal{C}$.

Therefore the $J$-th column in $S$ is proportional to the 1-th column. As $\mathcal{C}$ is modular this implies $J = 1$. In summary, we have

$$\mathcal{T}\mathcal{M}_{IJ} = \begin{cases} 
1 & \text{if } I = J = 1 \\
0 & \text{else.}
\end{cases}$$

Conjugating this matrix with $S$ and using the fact that $S_{I,J} = d(\mathcal{C})S^{-1}_{I,J'}$ gives us

$$(S \mathcal{T}\mathcal{M} S^{-1})_{11} = \frac{1}{d(\mathcal{C})}$$

implying that $S$-invariance will fail whenever $d(\mathcal{C}) \neq 1$.

However, when $\mathcal{C}$ is a modular tensor category, we have a helpful theorem from [KR09]. Before stating this theorem we recall the definition of a Frobenius algebra.

Definition 5.3. A Frobenius algebra $A$ is an algebra and a coalgebra such that

$$(\text{id}_A \otimes \nabla) \circ (\Delta \otimes \text{id}_A) = \Delta \circ \nabla = (\nabla \otimes \text{id}_A) \circ (\text{id}_A \otimes \Delta)$$  

where $\nabla$ is the product and $\Delta$ is the coproduct.

Remark 5.4. Using the graphical notation

$$\nabla = \begin{array}{c}
A \\
A
\end{array} \text{ and } \Delta = \begin{array}{c}
A \\
A
\end{array}$$

we can rewrite Condition (10) as

$$(A A A) = (A A A) = (A A A).$$  

Theorem 5.5 (Theorem 3.4, [KR09]). Let $A$ be a haploid, symmetric, commutative Frobenius algebra in $\mathcal{C} \boxtimes \overline{\mathcal{C}}$. Then the $I \times I$-matrix with entries $\text{hom}(I \boxtimes J, A)$ commutes with the $S$-matrix of $\mathcal{C}$ (where, as before, $\text{hom}$ denotes the dimension of the relevant Hom-space) if and only if

$$d(A) = d(\mathcal{C}).$$  

(12)
Remark 5.6. We note that
\[ d(A) = \sum_{I,J} \text{hom}_{\mathcal{C} \boxtimes \mathcal{C}}(I \boxtimes J, A)d(I \boxtimes J) \]
\[ = \sum_{I,J} \text{hom}_{\mathcal{C} \boxtimes \mathcal{C}}(I \boxtimes J, A)d(I)d(J) \]
\[ = \sum_{I,J} S_{I,J} \text{hom}_{\mathcal{C} \boxtimes \mathcal{C}}(I \boxtimes J, A)S_{I,J}. \]

As \( S_{I,J} = d(C)S_{I,J}^{-1} \), Condition (12) is precisely the condition that the matrix with entries \( \text{hom}(I \boxtimes J, A) \) commutes with the \( S \)-matrix evaluated at \((1,1)\) for an arbitrary object \( A \) in \( \mathcal{C} \boxtimes \mathcal{C} \). Condition (12) is therefore certainly necessary; the content of the theorem is that, when \( A \) is a haploid, symmetric, commutative Frobenius algebra, it is also sufficient.

Remark 5.7. [KR09, Theorem 3.4] actually proves that when \( A \) is a haploid, symmetric, commutative Frobenius algebra (12) implies an equality which is strictly stronger than the result stated here. In particular, [KR09, Theorem 3.4] proves that \( A \) will be a modular invariant algebra. This notion is defined and motivated in [Kon08, Section 6].

Remark 5.8. As explained in the proof of [KR09, Theorem 3.4], there exists an MTC (the category of of local \( A \)-modules) whose dimension is given by \( d(C)^2d(A)^2 \). Combining this with the fact that any MTC over the complex numbers has dimension at least 1 [ENO05, Theorem 2.3.] tells us that, in the case when \( K = \mathbb{C} \), the dimension of \( A \) cannot exceed \( d(C) \).

We recall that, when \( \mathcal{C} \) is modular, \( \Phi: \mathcal{C} \boxtimes \overline{\mathcal{C}} \to \mathcal{RT}\mathcal{C} \) is an equivalence and \( I \boxtimes J \mapsto (IJ, e_I^J)^\sharp \). Therefore, for \( F \) in \( \mathcal{RT}\mathcal{C} \), the \( I \times I \) matrix with entries \( \text{hom}_{\mathcal{RT}\mathcal{C}}(\Phi(I \boxtimes J), F) \) is precisely \( Z(F) \). The goal of the following section is to prove that \( \mathcal{T}\mathcal{M} \) is a commutative algebra in \( \mathcal{RT}\mathcal{C} \), and then, under a further condition on \( \mathcal{M} \), to show that it is also a haploid, symmetric, commutative Frobenius algebra.

The above stated goal assumes that \( \mathcal{RT}\mathcal{C} \) is a braided pivotal monoidal category; this is indeed the case as \( \mathcal{RT}\mathcal{C} = Z(\mathcal{C}) \) (see Remark 2.5) and \( Z(\mathcal{C}) \) admits a canonical MTC structure [EGNO15, Corollary 8.20.13]. To achieve this goal we are therefore going to have to work with the monoidal product, braiding and pivotal structure that \( \mathcal{RT}\mathcal{C} \) inherits from \( Z(\mathcal{C}) \). In general this is not easy; for instance it is hard to express the tensor product of two generic objects in \( \mathcal{RT}\mathcal{C} \). However, if we restrict our attention to functors coming from idempotents of the form (5) these structures may be described graphically.

**Definition 5.9.** Let \( \otimes_{\mathcal{T}\mathcal{C}}: \mathcal{T}\mathcal{C} \times \mathcal{T}\mathcal{C} \to \mathcal{T}\mathcal{C} \) be the bifunctor given by
\[ X \otimes_{\mathcal{T}\mathcal{C}} Y = XY \]
for \( X,Y \) in \( \mathcal{T}\mathcal{C} \) and

\[ f \otimes_{\mathcal{T}\mathcal{C}} g = d(C) \sum_{S} \frac{1}{d(S)} f_{S}g_{S} \in \text{Hom}_{\mathcal{T}\mathcal{C}}(WY,XZ) \]  

(13)

for \( f \in \text{Hom}_{\mathcal{T}\mathcal{C}}(W,X) \) and \( g \in \text{Hom}_{\mathcal{T}\mathcal{C}}(Y,Z) \).
We note that this product does not give a monoidal product as there is no unit. The tensor identity $1$ in $C$ fails to give a unit as the functor
$$- \otimes_{TC} 1: TC \to TC$$
maps $\alpha \in \text{Hom}_{TC}(X, Y)$ to $d(C)\alpha_1 \in \text{Hom}_C(X, Y)$ and so the unit isomorphisms fail to be natural.

**Remark 5.10.** The scalars appearing in (13) are chosen to guarantee that $\otimes_{TC}$ is well-behaved with respect to idempotents of the form $\epsilon_X^Y$. Indeed, we have
$$(XYAB, \epsilon_X^Y \otimes_{TC} \epsilon_A^B) \sharp = (XAYB, \epsilon_X^Y \otimes_{TC} \epsilon_A^B) \sharp$$
where the isomorphism is once again given by the braiding.

**Proposition 5.11.** Let $C$ be a modular tensor category. For $X, Y$ objects in $C$, we have
$$(XY, \epsilon_X^Y) \sharp \otimes (AB, \epsilon_A^B) \sharp = (XAYB, \epsilon_X^Y \otimes_{TC} \epsilon_A^B) \sharp.$$ 
Furthermore, for $\alpha \in \text{Hom}_{TC}(\epsilon_X^Y, \epsilon_A^B)$ and $\beta \in \text{Hom}_{TC}(\epsilon_X^Y, \epsilon_A^B)$, we have
$$\alpha \otimes \beta = \alpha \otimes_{TC} \beta$$
where $\otimes_{TC}$ is the associative product given by Definition 5.9.

**Proof.** By (6), we have
$$(XY, \epsilon_X^Y) \sharp \otimes (AB, \epsilon_A^B) \sharp = \Phi(X \boxtimes Y) \otimes \Phi(A \boxtimes B)$$
$$= \Phi(XA \boxtimes YB)$$
$$= (XAYB, \epsilon_X^Y \otimes_{TC} \epsilon_A^B) \sharp$$
where $\Phi$ is as defined in Section 2. As described in Remark 5.10, we then have
$$(XAYB, \epsilon_X^Y \otimes_{TC} \epsilon_A^B) \sharp = (XYAB, \epsilon_X^Y \otimes_{TC} \epsilon_A^B) \sharp.$$
where the natural isomorphism is given by the braiding. This proves the first half of the proposition.

Let $f, f', g, g'$ be in $\text{Hom}_C(X, A)$, $\text{Hom}_C(X', A')$, $\text{Hom}_C(Y, B)$ and $\text{Hom}_C(Y', B')$ respectively and let $\alpha$ and $\beta$ be given by
$$\alpha = \Phi(f \boxtimes g) \quad \text{and} \quad \beta = \Phi(f' \boxtimes g').$$
(14)

Then
$$\alpha \otimes \beta = \Phi((f \boxtimes g) \otimes (f \boxtimes g'))$$
$$= \Phi((f \otimes f') \boxtimes (g \otimes g'))$$
$$= \alpha \otimes_{TC} \beta.$$ as desired.

As $\Phi$ is fully faithful any morphism in $TC$ may be written as a sum of morphisms of the form (14). This implies $\alpha \otimes \beta = \alpha \otimes_{TC} \beta$ for arbitrary $\alpha$ and $\beta$. \qed
The braiding between \((XY, \epsilon_X^Y)^\sharp\) and \((AB, \epsilon_A^B)^\sharp\) is then given by the following morphism,

\[
\sigma_{X,A}^{Y,B} = \frac{1}{d(C)} \bigoplus d(T) \in \text{Hom}_{\mathcal{RT}C}(\epsilon_X^Y \otimes_{\mathcal{TC}} \epsilon_A^B, \epsilon_A^B \otimes_{\mathcal{TC}} \epsilon_X^Y)
\]

and the creation and annihilation morphisms for \((XY, \epsilon_X^Y)^\sharp\) and \((XY, \epsilon_Y^X)^\sharp\) are given by

\[
\begin{align*}
\frac{1}{d(C)} \bigoplus d(T) \quad \text{and} \quad \frac{1}{d(C)} \bigoplus d(T)
\end{align*}
\]

respectively. Note the tensor identity in \(\mathcal{RT}C\) is \((1, \epsilon_1)^\sharp\) and not \(\text{Hom}_{\mathcal{TC}}(-, 1)\).

As \(\mathcal{T}\mathcal{M}\) is not of the form \((XY, \epsilon_X^Y)^\sharp\) equipping it with the structure of a Frobenius algebra directly is difficult. However, as we are assuming that \(\mathcal{C}\) is modular then we can decompose \(\mathcal{T}\mathcal{M}\) as follows:

\[
\mathcal{T}\mathcal{M} = \bigoplus_{I,J} \mathcal{T}\mathcal{M}_{I}^{J} \cdot \epsilon_{I}^{J}.
\]

We may then define the Frobenius structure in terms of this decomposition. To illustrate this approach let \(\mathcal{B}\) be a fusion category with complete set of simples \(\text{Irr}(\mathcal{B})\) and let \(A\) be an object in \(\mathcal{B}\). Any morphism \(\nabla\) from \(A \otimes A\) to \(A\) gives rise to the following morphisms,

\[
\nabla_{X}^{X,Z} : \text{Hom}_{\mathcal{B}}(X, YZ) \to \text{Hom}(A_Y \otimes A_Z, A_X)
\]

\[
\alpha \mapsto \left( g \otimes h \mapsto \begin{array}{c}
\alpha \\
\downarrow \\
a \\
\end{array} \right)
\]

where \(X, Y, Z\) are in \(\mathcal{B}\) and \(A_X := \text{Hom}_{\mathcal{B}}(X, A)\).

**Remark 5.12.** The full map \(\nabla\) is determined by \(\nabla_{R}^{S,T}\) for \(R, S, T \in \text{Irr}(\mathcal{B})\). Indeed we can recover it via

\[
\bigoplus \sum_{RST, g,h,\alpha} \nabla_{R}^{S,T}(\alpha)(g \otimes h) \circ \alpha^* \circ (g^* \otimes h^*) = \nabla
\]

where \(g\) ranges over a basis of \(A_S\), \(h\) ranges over a basis of \(A_T\) and \(\alpha\) ranges over a basis of \(\text{Hom}_{\mathcal{B}}(R, ST)\). Similarly any morphism from \(A\) to \(A \otimes A\) can also be decomposed in
the following way

\[ \Delta^R_{ST} : \operatorname{Hom}_B(ST, R) \to \operatorname{Hom}(A_R, A_S \otimes A_T) \]

\[ \beta \mapsto \left( f \mapsto \sum_{g, h} f \otimes g \otimes h \right) \]

and then recovered via

\[ \sum_{\beta, f} \Delta^R_{ST}(\beta)(f) \circ \beta^* \circ f^* = \Delta. \]

The condition that \( \nabla \) gives an algebra product may be rephrased in terms of \( \nabla^R_{ST} \) as described by the following lemma.

**Lemma 5.13.** Let \( A \) be an object in \( \mathcal{B} \) and let \( \nabla \) be in \( \operatorname{Hom}_B(A \otimes A, A) \). Then \( \nabla \) is associative if

\[ \nabla^R_{ST}(id)(\nabla^S_{RS}(id)(f \otimes g) \otimes h) = \nabla^R_{ST}(id)(f \otimes \nabla^S_{ST}(id)(g \otimes h)) \quad (15) \]

for all \( R, S, T \in \operatorname{Irr}(\mathcal{B}) \), \( \alpha \in \operatorname{Hom}_B(R, ST) \), \( f \in A_R \), \( g \in A_S \) and \( h \in A_T \). An element \( u \in A_1 \) is a unit for \( \nabla \) if

\[ \nabla^1_S(id)(u \otimes g) = g \quad \text{and} \quad \nabla^S_1(id)(g \otimes u) = g \quad (16) \]

Furthermore, if \( \mathcal{B} \) is braided then \( \nabla \) is commutative if

\[ \nabla^T_{ST}(\bigotimes)(h \otimes g) = \nabla^S_{ST}(id)(g \otimes h). \quad (17) \]

**Proof.** The first claim follows from the fact that, by decomposing the top of each strand (as in, for example, [HK19, Lemma 3.3]), we have

\[ \sum_{R,S,T \atop f,g,h} \nabla^R_{ST}(id)(\nabla^S_{RS}(id)(f \otimes g) \otimes h) \circ (f^* \otimes g^* \otimes h^*) \]

and

\[ \sum_{R,S,T \atop f,g,h} \nabla^R_{ST}(id)(f \otimes \nabla^S_{ST}(id)(g \otimes h)) \circ (f^* \otimes g^* \otimes h^*). \]

Similarly the second claim follows from

\[ \sum_{S,g} \nabla^S_1(id)(u \otimes g) \circ g^* \quad \text{and} \quad \sum_{S,g} \nabla^S_1(id)(g \otimes u) \circ g^*. \]
and the third claim from

\[
\sum_{S,T \in \mathcal{Irr}(B)} g \triangleright h = \sum_{S,T \in \mathcal{Irr}(B)} h \triangleright g = \sum_{S,T \in \mathcal{Irr}(B)} \nabla_{ST}^{T,S}(h \otimes g) \circ (g^* \otimes h^*).
\]

\[\square\]

An important property of Frobenius algebras is that they naturally carry a self-dual structure. Indeed, it is simple to check that the maps

\[
\begin{align*}
\begin{array}{c}
\mathcal{A} \\
\end{array} & \quad \begin{array}{c}
\mathcal{A} \\
\end{array} \quad \begin{array}{c}
\mathcal{A} \\
\end{array} \\
\end{align*}
\end{align*}
\]

where \(\mathcal{A}\) denotes the unit and \(\mathcal{D}\) denotes the counit, are self-dualizing maps on \(\mathcal{A}\).

Our strategy for identifying Frobenius algebras will be as follows: we start by identifying an algebra \(\mathcal{A}\) together with self-dualizing structure maps on \(\mathcal{A}\). We then ask “What additional condition should be satisfied for this to imply that \(\mathcal{A}\) is a Frobenius algebra?” Well, if \(\mathcal{A}\) were a Frobenius algebra, combining (10) and (18) tells us that the coproduct could be written as both sides of the following condition.

\[
\begin{align*}
\begin{array}{c}
\mathcal{A} \\
\end{array} & \begin{array}{c}
\mathcal{A} \\
\end{array} \begin{array}{c}
\mathcal{A} \\
\end{array} \\
\end{align*}
\end{align*}
\]

(19)

So both of these morphisms being equal is certainly a necessary condition. In fact, it is also sufficient (see, for example, Proposition 2.1 in [Yam04]). The following lemma rewrites this condition in terms of \(\nabla_{ST}^{S,T}\).

**Lemma 5.14.** Let \(\mathcal{B}\) be a spherical fusion category and let \(\mathcal{A}\) be an algebra object in \(\mathcal{B}\) (with product \(\triangleright\)) together with structure maps that make \(\mathcal{A}\) self-dual. Then \(\mathcal{A}\) satisfies (19) if and only if

\[
d(S) \left( g^* \circ \nabla_{S,T}^{R,T} \left( \begin{array}{c}
\beta \\
\end{array} \right) \right) \left( f \otimes (h^*)^\vee \right) = d(T) \left( h^* \circ \nabla_{T,R}^{S,T} \left( \begin{array}{c}
\beta \\
\end{array} \right) \right) \left( (g^*)^\vee \otimes f \right)
\]

(20)

for all \(R, S, T \in \text{Irr}(\mathcal{B})\), \(\beta \in \text{Hom}_{\mathcal{B}}(ST, R)\), \(h, f \in \mathcal{A}_R\), \(g, h \in \mathcal{A}_T\) and \(f, g \in \mathcal{A}_S\).
Proof. Decomposing the coproduct given by the left-hand side of (19) gives
\[
\left( \bigotimes_{s,t} \right)^R_{s,t} (\beta)(f) = \sum_{g,h} R_{\beta}^{g} \otimes h = \frac{1}{d(R)} \sum_{g,h} g \otimes h
\]
\[
= \frac{1}{d(R)} \sum_{g,h} g \otimes h = \frac{1}{d(R)} \sum_{g,h} g \otimes h
\]
\[
= \frac{d(S)}{d(R)} \sum_{g,h} g \otimes h
\]
\[
= \frac{d(S)}{d(R)} \sum_{g,h} \left( g^* \circ \nabla_{s,t}^{S} \left( s_{\beta}^{T} \right) \right) (f \otimes (h^*)^\vee) g \otimes h
\]
In an analogous way, we also have
\[
\left( \bigcap_{s,t} \right)^R_{s,t} (\beta)(f) = \frac{d(T)}{d(R)} \sum_{g,h} h^* \circ \nabla_{s,t}^{S} \left( s_{\beta}^{T} \right) ((g^*)^\vee \otimes f) g \otimes h
\]
which proves the proposition. \qed

Similarly, a self-duality may also be described in terms of simple multiplicity spaces. In particular, we start with the following perfect pairings
\[
\langle -,- \rangle_S: A_S \otimes A_{S^\vee} \rightarrow \mathbb{K}
\]
for all $S \in \text{Irr}(\mathcal{B})$. However, in constructing a self-duality one degree of freedom remains. This is described by the following lemma.

**Lemma 5.15.** Let $c$ be a map from $\text{Irr}(\mathcal{B})$ to $\mathbb{K}\setminus\{0\}$. We consider the morphisms
\[
A \bigcap_A = \sum_{S,b} c(S) \left[ \begin{array}{c} S \vspace{1em} \beta \\ A \end{array} \right]
\]
and
\[
A \bigcup_A = \sum_{S,b} \frac{1}{c(S)} \left[ \begin{array}{c} A \vspace{1em} b^* \\ s \end{array} \right]
\]
where $\{b\}$ is a basis of $A_S$ and $\{b'\}$ is the corresponding dual basis of $A_{S^\vee}$ with respect to $\langle -,- \rangle_S$. Then $\left( A, A \bigcap_A, A \bigcup_A \right)$ is a dual object to $A$. Furthermore, with respect to this duality, we have
\[
\langle f, (g^*)^\vee \rangle = c(S)g^*(f)
\]
for all \( f, g \in A_S \).

**Proof.** We have

\[
\bigcup_{A}^{A} = \sum_{S,b} c(S) \begin{array}{c}
\begin{array}{c}
S^\vee
\end{array}
\begin{array}{c}
S
\end{array}
\end{array} = \sum_{S,b} c(S) \begin{array}{c}
\begin{array}{c}
S^\vee
\end{array}
\begin{array}{c}
S
\end{array}
\end{array}
\]

and, in the same way, we also have

\[
\bigcup_{A}^{A} = \sum_{S,b} c(S) \begin{array}{c}
\begin{array}{c}
S^\vee
\end{array}
\begin{array}{c}
S
\end{array}
\end{array} = \sum_{S,b} c(S) \begin{array}{c}
\begin{array}{c}
S^\vee
\end{array}
\begin{array}{c}
S
\end{array}
\end{array}
\]

To prove the second claim we simply compute

\[
\langle f, (g^*)^\vee \rangle = \langle f, \begin{array}{c}
\begin{array}{c}
S^\vee
\end{array}
\begin{array}{c}
S
\end{array}
\end{array} \rangle = c(S) \sum_{b} \langle f, \begin{array}{c}
\begin{array}{c}
S^\vee
\end{array}
\begin{array}{c}
S
\end{array}
\end{array} \rangle = c(S) \sum_{b} g^*(b) f(b) = c(S) g^*(f).
\]

\[\square\]

6. \( \mathcal{T} \mathcal{M} \) as a Frobenius Algebra

As before let \( \mathcal{C} \) be an MTC and let \( \mathcal{M} \) be a pivotal monoidal functor from \( \mathcal{C} \) to \( \mathcal{D} \) where \( \mathcal{D} \) is a pivotal monoidal category. Our first step is to equip \( \mathcal{T} \mathcal{M} \) with the structure of an algebra. We do this by specifying a map

\[
\nabla_{Y,Z}^{X}: \text{Hom}_{\mathcal{RTC}}(X,YZ) \to \text{Hom}(\mathcal{T} \mathcal{M}_Y \otimes \mathcal{T} \mathcal{M}_Z, \mathcal{T} \mathcal{M}_X)
\]

for all \( X, Y, Z \) in \( \mathcal{RTC} \) of the form \((AB, \epsilon^B_A)^i\). As \( \mathcal{C} \) is modular \{\((IJ, \epsilon^I_J)^i\)\}_{I,J} forms a complete set of simples and this determines a map \( \nabla: \mathcal{T} \mathcal{M} \otimes \mathcal{T} \mathcal{M} \to \mathcal{T} \mathcal{M} \) as described in Remark 5.12. We recall from Proposition 3.3 that \( \mathcal{T} \mathcal{M}^B_A \) is identified with the subspace of \( \text{Hom}_{\mathcal{D}}(1, \mathcal{M}(AB)) \) characterised by \([S] \).

**Definition 6.1.** Let \( X, Y \) and \( Z \) be given by \((AB, \epsilon^B_A)^i, (CD, \epsilon^D_C)^i \) and \((EF, \epsilon^E_F)^i\) respectively. Let \( \alpha \) be in \( \text{Hom}_{\mathcal{RTC}}(X,YZ) = \text{Hom}_{\mathcal{TC}}(\epsilon^B_A, \epsilon^D_C \otimes \mathcal{TC} \epsilon^E_F) \). We consider the map

\[
\text{Hom}_{\mathcal{D}}(1, \mathcal{M}(CD)) \otimes \text{Hom}_{\mathcal{D}}(1, \mathcal{M}(EF)) \to \text{Hom}_{\mathcal{D}}(1, \mathcal{M}(AB))
\]

\[
f \otimes g \mapsto \mathcal{T} \mathcal{M}(\alpha)(f \otimes_D g).
\]

We note that the image of this map is in \( \mathcal{T} \mathcal{M}_X = \mathcal{T} \mathcal{M}^B_A \) as

\[
\mathcal{T} \mathcal{M}(\epsilon^B_A) \circ \mathcal{T} \mathcal{M}(\alpha)(f \otimes_D g) = \mathcal{T} \mathcal{M}(\alpha \circ \epsilon^B_A)(f \otimes_D g) = \mathcal{T} \mathcal{M}(\alpha)(f \otimes_D g).
\]

Therefore restricting this map to the subspace \( \mathcal{T} \mathcal{M}_Y \otimes \mathcal{T} \mathcal{M}_Z \) gives a map

\[
\nabla_{Y,Z}^{X}(\alpha): \mathcal{T} \mathcal{M}_Y \otimes \mathcal{T} \mathcal{M}_Z \to \mathcal{T} \mathcal{M}_X.
\]

Let \( \nabla: \mathcal{T} \mathcal{M} \otimes \mathcal{T} \mathcal{M} \to \mathcal{T} \mathcal{M} \) be the map construed from \( \nabla_{Y,Z}^{X}(\alpha) \) as described in Remark 5.12.
Proposition 6.2. The morphisms $\nabla$ and $u := \text{id}_{1_T} \in \mathcal{M}^1_T = \text{Hom}_{\mathcal{T}C}(1_{\mathcal{T}C}, \mathcal{T}M)$ form a product/unit pair that make $\mathcal{T}M$ a commutative algebra.

Proof. Let $X, Y$ and $Z$ be given by $(AB, \epsilon^B_A), (CD, \epsilon^D_C)$ and $(EF, \epsilon^E_F)$ respectively. To prove the desired result we have to show that (15), (16) and (17) are satisfied. We first note that (16) reduces to a triviality in this case.

To verify (15) we let $f, g$ and $h$ be in $\mathcal{T}M_{A}, \mathcal{T}M_{D}$ and $\mathcal{T}M_{F}$ respectively and compute,

$$\nabla_{X,Y,Z} (\epsilon^B_A \otimes_{\mathcal{T}C} \epsilon^D_C \otimes_{\mathcal{T}C} \epsilon^F_E) (f \otimes g) \otimes h$$

where we are simply using multiple instances of Proposition 3.3. Finally, once again by Proposition 3.3 we have

$$\nabla_{Y,Z} (\otimes) (h \otimes g) = \frac{1}{d(C)^2} \sum_{S,T} d(S) d(T) \nabla_{Y,Z} (\epsilon^D_C \otimes_{\mathcal{T}C} \epsilon^F_E) (g \otimes h)$$

which proves (17). □

The next step is to equip $\mathcal{T}M$ with self-dualizing structure maps. For this to work we need to make some additional assumptions on $\mathcal{M} : \mathcal{C} \to \mathcal{D}$. Firstly we assume that the idempotent completion of $\mathcal{D}$, denoted $\mathcal{D}$, is a multifusion category. Secondly we assume that $\mathcal{M} : \mathcal{C} \to \mathcal{D}$, obtained by composing $\mathcal{M}$ with this embedding, is indecomposable.
other words, that there do not exist functors $\mathcal{M}_1: \mathcal{C} \to \mathcal{D}_1$ and $\mathcal{M}_2: \mathcal{C} \to \mathcal{D}_2$ such that $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, where $\mathcal{D}_1 \leq \mathcal{D}_2$.

As described in [EGNO15, Section 4.3], $\mathcal{D}$ decomposes into $\bigoplus_{i,j \in I} i \mathcal{D}_j$ where $I$ is an indexing set for the primitive idempotents in $\text{End}_{\mathcal{D}}(1)$. Therefore the condition that $\mathcal{M}$ is indecomposable is equivalent to requiring that there exists no subset $K \subset I$ such that $\mathcal{M}(X)_j = j \mathcal{M}(X)_i = 0$ for all $X \in \mathcal{C}$, $i \in K$ and $j \in I \setminus K$.

**Proposition 6.3.** $\mathcal{M}$ is indecomposable if and only if $\mathcal{M}_1 = \mathbb{K}$. Furthermore, in this case, any non-zero $\alpha \in \mathcal{M}_X \leq \text{Hom}_D(1, \mathcal{M}(XY))$ has a left-inverse in $\text{Hom}_D(\mathcal{M}(XY), 1)$ for all $X, Y \in \mathcal{C}$.

**Proof.** By Proposition 3.3, $\mathcal{M}_1$ is given by the subspace of $\text{End}_D(1)$ such that $\alpha \otimes \text{id}_{\mathcal{M}(Z)} = \text{id}_{\mathcal{M}(Z)} \otimes \alpha$ $\forall Z \in \mathcal{C}$.

Embedding this equality into $D$ and decomposing gives $\alpha_i \text{id}_{\mathcal{M}(Z)_j} = \alpha_j \text{id}_{\mathcal{M}(Z)_j}$ $\forall Z \in \mathcal{C}$.

This implies $\alpha_i = \alpha_j$ for all $i, j \in I$ if and only if $\mathcal{M}$ is indecomposable. This proves the first claim.

To prove the second claim we recall the characterisation of $\mathcal{M}_X$ provided by Proposition 3.3 i.e. the subspace of $\text{Hom}_D(1, \mathcal{M}(XY))$ such that $\phi \circ (\alpha \otimes \text{id}_{\mathcal{M}(Z)}) = \text{id}_{\mathcal{M}(Z)} \otimes \alpha$ $\forall Z \in \mathcal{C}$.

where $\phi$ is a certain isomorphism. Embedding this equality into $D$ and decomposing gives $\phi_j \circ (\alpha_i \otimes \text{id}_{\mathcal{M}(Z)_j}) = \text{id}_{\mathcal{M}(Z)_j} \otimes \alpha_j$ $\forall Z \in \mathcal{C}$.

Therefore, if $\mathcal{M}$ is indecomposable, $\alpha_i = 0$ for any $i \in I$ implies $\alpha = 0$. This proves the second claim. □

We are now ready to equip $\mathcal{M}$ with some self-dualizing structure maps. To accomplish this we shall use Lemma 5.15. We therefore first establish the following perfect pairing.

**Lemma 6.4.** Let $X$ and $Y$ be in $\mathcal{C}$. As usual $\mathcal{M}_X$ is identified with a subspace of $\text{Hom}_D(1, \mathcal{M}(XY))$, however, as described in Remark 3.4 we identify $\mathcal{M}_X^\vee$ with a subspace of $\text{Hom}_D(1, \mathcal{M}(Y^\vee X^\vee))$. The map

$$\langle -,- \rangle : \mathcal{M}_X \otimes \mathcal{M}_X^\vee \to \mathcal{M}_1 = \mathbb{K}$$

$$f \otimes g \mapsto \f \g$$

is a perfect pairing.

**Proof.** Given a non-zero $f \in \mathcal{M}_X$, by Proposition 5.3 there exists $g \in \text{Hom}_D(1, \mathcal{M}(Y^\vee X^\vee))$ such that

$$\f \g = \text{id}_1.$$ 

We therefore have

$$\text{id}_1 = \frac{1}{d(C)} \sum_S d(S) \f \g S = \frac{1}{d(C)} \sum_S d(S) \f \g S \quad (22)$$
by Proposition 3.3. We now consider \( \tilde{\eta} = T\mathcal{M}(\tilde{\epsilon}_X^\vee)(g) \in T\mathcal{M}_X^Y \) (where \( \tilde{\epsilon}_X^\vee \) is given by (7)). Then the right-hand side of (22) is \( \langle f, \tilde{\eta} \rangle \) and so we are done.  

**Remark 6.5.** We note that this perfect pairing is symmetric with respect to the pivotal structure, i.e. \( \langle f, g \rangle = \langle g, f \rangle \) where \( f \in T\mathcal{M}_X = T\mathcal{M}_X^{Y\vee} \).

**Proposition 6.6.** We consider \( T\mathcal{M} \) equipped with the algebra structure from Proposition 6.2. We also equip \( T\mathcal{M} \) with the self-dualizing maps given by Lemma 6.4 and Lemma 5.1 with \( c = d \) (the dimension map for \( \mathcal{RTC} \)). Then \( T\mathcal{M} \) satisfies (10), i.e. is a Frobenius algebra.

**Proof.** Let \( R, S \) and \( T \) be given by \((IJ, \epsilon_I^J),(KL, \epsilon_K^L)\) and \((MN, \epsilon_M^N)\) respectively where \( I, J, K, L, M, N \in \mathcal{I} \). Let \( f, g \) and \( h \) be in \( T\mathcal{M}_I, T\mathcal{M}_K \) and \( T\mathcal{M}_M \) respectively and let \( \beta \) be in \( \text{Hom}_{\mathcal{RTC}}(ST, R) = \text{Hom}_{\mathcal{TC}}(\epsilon_K^L \otimes_{\mathcal{TC}} \epsilon_M^N, \epsilon_I^J) \). We have

\[
\begin{align*}
&d(S) \left( g^* \circ \nabla_{S}^{R,T^\vee} \left( \begin{array}{c}
\beta \\
R_T \end{array} \right) \right) (f \otimes (h^*)^\vee) \\
&= d(S) \left( \begin{array}{c}
f \\
(h^*)^\vee \\
\beta \\
K \end{array} \right) \\
&= d(T) \left( \begin{array}{c}
g^* \circ \nabla_{T}^{S^\vee,R} \left( \begin{array}{c}
\beta \\
M \end{array} \right) \\
(f^*)^\vee \\
\beta \\
N \end{array} \right) \end{align*}
\]

where we have used Proposition 3.3 and Lemma 5.15 multiple times.  

**Theorem 6.7.** Let \( \mathcal{C} \) be an MTC and let \( \mathcal{M} \) be a pivotal tensor functor from \( \mathcal{C} \) to \( \mathcal{D} \) such that \( \overline{\mathcal{M}} \) is indecomposable. Then \( T\mathcal{M} \) is a haploid, symmetric, commutative Frobenius algebra.

**Proof.** This result follows from Proposition 6.3, Proposition 6.2, Proposition 6.6 and Remark 6.5.

7. Module Categories and \( \alpha \)-induction

**Definition 7.1.** Let \( \mathcal{C} \) be a monoidal category. A module category over \( \mathcal{C} \) is a monoidal category \( \mathcal{B} \) together with a monoidal (contravariant) functor \( \mathcal{M}: \mathcal{C} \rightarrow \text{End}(\mathcal{B}) \), where \( \text{End}(\mathcal{B}) \) is the category of endofunctors on \( \mathcal{B} \). If \( \mathcal{B} \) is semisimple with finitely many simple objects we call \( \mathcal{M} \) a finite module category over \( \mathcal{C} \).
Let $\mathcal{M}: \mathcal{C} \to \text{End}(\mathcal{B})$ be a finite module category and let $\text{Irr}(\mathcal{B})$ be a complete set of simples in $\mathcal{B}$. We then consider

$$T := \bigoplus_{v \in \mathcal{I}} v \in \text{Obj}(\mathcal{B})$$

and the semisimple algebra $A = \text{End}_\mathcal{B}(T)$.

**Remark 7.2.** As every object in $\mathcal{I}$ is simple and distinct, Schur’s Lemma implies that $A$ is a direct sum of division algebras over $\mathbb{K}$.

As $T$ is a projective generator in $\mathcal{B}$ the (covariant) functor

$$\text{Hom}_\mathcal{B}(T, -): \mathcal{B} \to \text{Mod-A}$$

is an equivalence of categories. A finite module category over $\mathcal{C}$ is therefore equivalent to a monoidal functor

$$\mathcal{M}: \mathcal{C} \to \text{End}(\text{Mod-A}) = A, A\text{-Bimod}.$$

From a physical point of view Cardy [Car89] showed that the algebraic data of an *annular partition function* in a boundary (rational) conformal field theory is given by a finite module category over the corresponding MTC. The process known as $\alpha$-induction is an operator algebra technique developed by Böckenhauer and Evans [BE98] that produces a *toroidal partition function* (as described in the introduction) from an annular partition function. Ostrik [Ost03, Section 5] rephrased $\alpha$-induction using categorical language in the following way.

Let $\mathcal{M}: \mathcal{C} \to \mathcal{D}$ be a finite module category over a PTC $\mathcal{C}$, where $\mathcal{D}$ denotes $A, A\text{-Bimod}$. For $A, B$ in $\mathcal{C}$ we consider the subspace

$$\text{Hom}_\mathcal{C}^\sigma(M, A, B) \leq \text{Hom}_\mathcal{D}(\mathcal{M}(A), \mathcal{M}(B))$$

defined by the condition that $\beta \in \text{Hom}_\mathcal{D}(\mathcal{M}(A), \mathcal{M}(B))$ satisfies, for all $X$ in $\mathcal{C}$,

$$\begin{array}{c}
\mathcal{M}(A) \otimes \mathcal{M}(X) & \xrightarrow{\mathcal{M}(\sigma_{AY})} & \mathcal{M}(Y) \otimes \mathcal{M}(A) \\
\beta \otimes \text{id} & \circ & \text{id} \otimes \beta \\
\mathcal{M}(B) \otimes \mathcal{M}(X) & \xrightarrow{\mathcal{M}(\overline{\sigma}_{BY})} & \mathcal{M}(X) \otimes \mathcal{M}(B)
\end{array}$$

(23)

where $\sigma$ and $\overline{\sigma}$ are the braiding on $\mathcal{C}$ and its opposite respectively. The principal claim of $\alpha$-induction is then as follows. Under the assumption that the dimensions of all the objects in $\mathcal{C}$ are positive, the $I \times I$-matrix whose entries are given by the dimension of $\text{Hom}_\mathcal{C}^\sigma(M, I', J)$ commutes with the modular data of $\mathcal{C}$. Furthermore if $\mathcal{M}$ is irreducible then this matrix is a *modular invariant* (see Definition 1.1).

**Remark 7.3.** The claim found in [Ost03] is actually that the $I \times I$-matrix whose entries are given by the dimension of $\text{Hom}_\mathcal{C}^\sigma(M, I, J)$ commutes with the modular data of $\mathcal{C}$. However as the modular data always commutes with the charge conjugation matrix these statements are equivalent.

In [Ost03] Ostrik also provides the following example to prove the necessity of the condition that the objects in $\mathcal{C}$ have positive dimension.
Example 7.4. Let $\mathcal{C}$ be the fusion category with complete set of simples $\{0, 1\}$, where $0$ is the tensor unit and $1 \otimes 1 = 0$. As this category is rigid we may equip it with the pivotal structure $\delta_1 = -\text{id}_1$ (so that $d(1) = -1$). One may also check that setting $\sigma_{11} = \text{id}_2$ defines a (degenerate) braiding on the category and we obtain a PTC. We then consider the module category

$$\mathcal{M}: \mathcal{C} \rightarrow \mathcal{Vect}$$

$$0 \mapsto \mathbb{K}$$

$$1 \mapsto \mathbb{K}.$$ 

As the braiding is given by the identity, we have $\sigma = \sigma$ and Equation (23) reduces to a tautology. Therefore $\text{Hom}_{\mathcal{M}}^\sigma(0, 1) = \mathbb{K}$ and the resulting dimension matrix fails to commute with the T-matrix

$$\mathcal{T} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

We start by remarking that Condition (23) makes sense even when $\mathcal{D}$ is an arbitrary tensor category. Therefore to connect Ostrik’s formulation of $\alpha$-induction to $\mathcal{T}\mathcal{M}$ we have the following.

**Theorem 7.5.** Let $\mathcal{C}$ be a PTC and let $\mathcal{M}: \mathcal{C} \rightarrow \mathcal{D}$ be a pivotal monoidal functor. Then $\text{Hom}_{\mathcal{M}}^\sigma(I^\vee, J) \cong \mathcal{T}\mathcal{M}_J^I$.

**Proof.** Graphically Condition (23) is given by

$$\beta_{I^\vee X J} = \beta_{I^\vee X J}$$

for all $X$ in $\mathcal{C}$. As $\mathcal{T}\mathcal{M}_J^I$ is a subspace of $\text{Hom}_{\mathcal{D}}(1, \mathcal{M}(IJ)) = \text{Hom}_{\mathcal{D}}(\mathcal{M}(I^\vee), \mathcal{M}(J))$ we only have to check that Condition (24) is equivalent to Condition (8).

Suppose $\beta \in \text{Hom}_{\mathcal{D}}(\mathcal{M}(I), \mathcal{M}(J))$ satisfies Condition (23). Then we have

$$\beta_{XI J} = \beta_{XI J} = \beta_{XI J}.$$ 

Furthermore, for $\alpha \in \mathcal{T}\mathcal{M}_J^I$, we have

$$\alpha_{I^\vee X J} = \alpha_{I^\vee X J} = \alpha_{I^\vee X J}$$

where the final equality uses Proposition 3.3. This is equivalent to Condition (23) as desired. \qed

The alternative characterisation of $\mathcal{T}\mathcal{M}_J^I$ given by Theorem 7.5 allows for the following generalization of Corollary 4.5 to the pre-modular case.
**Theorem 7.6.** Let $\mathcal{C}$ be a pre-modular tensor category, let $\mathcal{D}$ be a pivotal monoidal category and let $\mathcal{M}: \mathcal{C} \to \mathcal{D}$ be a pivotal monoidal functor. Then $T\mathcal{M}$ is $T$-invariant.

**Proof.** Let $I, J$ be such that $T\mathcal{M}^2_I \neq 0$. Then, by Theorem 7.5, there exists a non-zero map $\beta \in \text{Hom}_\mathcal{D}(\mathcal{M}(I^\vee), \mathcal{M}(J))$ that satisfies (24). We have

$$T_{II} \beta = \begin{array}{c} I^\vee \beta \\downarrow \Downarrow \\uparrow \\downarrow \\beta \Downarrow \Downarrow \end{array} = \begin{array}{c} I \vee \beta \\Downarrow \Downarrow \uparrow \\downarrow \\beta \Downarrow \Downarrow \end{array} = T_{JJ} \beta$$

where $T$ denotes the T-matrix and to make certain string manipulations clearer, we have chosen to write $\beta$ upside-down instead of writing $\beta^\vee$. Therefore $Z(T\mathcal{M})_{IJ} \neq 0$ implies $T_{II} = T_{JJ}$. As $T$ is diagonal that is precisely the condition that $Z(F)$ commutes with $T$. □

Our goal is therefore to reinterpret a module category $\mathcal{M}: \mathcal{C} \to A, A\text{-Bimod}$ as a pivotal monoidal functor. Initially, this may seem impossible as $A, A\text{-Bimod}$ admits a canonical pivotal structure and, excluding pathological examples, modules categories fail to be pivotal. However, we can study many interesting examples if we only require that $\mathcal{M}$ induce a pivotal structure on its full image. Let $\mathcal{D}$ be the full image of $\mathcal{M}$ in $A, A\text{-Bimod}$. Clearly $\mathcal{D}$ is a rigid monoidal category. Furthermore, it comes with a natural candidate pivotal structure: $\mathcal{M}(\delta_X)$, where $\delta: \vee \to \vee$ gives the pivotal structure on $\mathcal{C}$. As $\mathcal{M}$ is a functor, $\mathcal{M}(\delta_X)$ is natural with respect to morphisms in $\mathcal{C}$; however, to give a pivotal structure on $\mathcal{D}$ it must be natural with respect all morphisms in $\mathcal{D}$. In other words the diagram

$$\begin{array}{c} \mathcal{M}(Y^\vee) \xrightarrow{\alpha^\vee} \mathcal{M}(X^\vee) \\ \mathcal{M}(\delta_Y) \downarrow \bigcirc \downarrow \mathcal{M}(\delta_X) \\ \mathcal{M}(Y^\vee) \xrightarrow{\vee \alpha} \mathcal{M}(Y X) \end{array}$$

must commute for all $\alpha \in \text{Hom}_\mathcal{D}(\mathcal{M}(X), \mathcal{M}(Y))$. When this is satisfied and $\mathcal{D}$ is equipped with the resulting pivotal structure, the functor $\mathcal{M}: \mathcal{C} \to \mathcal{D}$ is automatically pivotal. We may therefore construct $T\mathcal{M}$ and Theorem 7.5 guarantees that $Z(T\mathcal{M})$ will give the same matrix as $\alpha$-induction. Furthermore, the inclusion $\mathcal{D} \hookrightarrow A, A\text{-Bimod}$ fully embeds $\mathcal{D}$ into a multifusion category. We therefore obtain the following corollary of Theorem 6.7.

**Corollary 7.7.** Let $\mathcal{C}$ be an MTC and let $\mathcal{M}: \mathcal{C} \to S, S\text{-Bimod}$ be an indecomposable module category over $\mathcal{C}$ that induces a pivotal structure on its full image. Then $T\mathcal{M}$ is a haploid, symmetric, commutative Frobenius algebra.

**Remark 7.8.** Ostrik’s Example 7.4 also shows that the condition that $\mathcal{M}$ be pivotal is necessary for the results of Section 4 and Section 6. Indeed, one may check that his example fails to induce a pivotal structure on its full image.

For the remainder of this article we describe a class of interesting examples of module categories that induce a pivotal structure on their full images.

\[\text{EXTENDING THE TRACE OF A PIVOTAL MONOIDAL FUNCTOR} \]
8. A Case Study: The Temperley-Lieb Category

Let $\beta$ be in $\mathbb{C}^*$. The Temperley-Lieb category $\text{TL}(\beta)$ is a $\mathbb{C}$-linear category whose set of objects is given by $\{n\}_{n\in\mathbb{N}}$ where $n$ may be thought of as a collection of $n$ dots along an interval. The space $\text{Hom}(m, n)$ is the span of planar $(m, n)$-tangles modulo the relation $t \sqcup u - \beta t$ where $t$ is a planar $(m, n)$-tangle and $u$ is the unknot. Composition is then given by tangle composition. For a more detailed description, see, for example, [GW02]. $\text{TL}(\beta)$ is a monoidal category whose tensor product satisfies $n \otimes m = n + m$. Furthermore, $\text{TL}(\beta)$ is rigid and every object admits a canonical choice of self dualizing maps denoted $\text{cr}_n : \emptyset \to n \otimes n$ and $\text{an}_n : n \otimes n \to \emptyset$ respectively. In the case when $\beta = -[2]_q$ for a primitive even root of unity $q$, $\text{TL}(\beta)$ admits a unique tensor ideal $\mathcal{N}$ [GW02]. Quotienting $\text{TL}(\beta)$ by $\mathcal{N}$ and idempotent completing the result yields a spherical fusion category we denote $\mathcal{C}$. Using a skein relation we may define a non-degenerate braiding on $\mathcal{C}$ giving us a modular tensor category [Tur16, Theorem 7.5.3]. Let $h$ be the the smallest positive integer such that $q^h = 1$ or equivalently $[h]_q = 0$; $h$ is called the Coxeter number of $\mathcal{C}$. $\mathcal{C}$ turns out to be equivalent to the category of integrable highest weight modules of $A_1^{(1)}$ at level $k = h - 2$, denoted $\text{Rep}_k A_1^{(1)}$ (for this equivalence to be pivotal one must equip $\text{TL}(\beta)$ with a ‘twisted’ pivotal structure, or alternatively, consider the so-called “disoriented” diagrammatic category presented in [CMW09, p. 5]; for further details on this issue see [ST09]). In particular, a complete set of simples in $\mathcal{C}$ has size $h - 1$.

Let $Q$ be a symmetric quiver with non-degenerate eigenvalue $\beta$ (here non-degenerate signifies there exists an eigenvector $(x_i) = x$ with non-zero entries), let $A$ be the basic algebra spanned by vertices in $Q$ and let $B$ be the $A$-bimodule spanned by arrows in $Q$. We can construct a module category over $\text{TL}(\beta)$ as follows

$\mathcal{M} : \text{TL}(\beta) \to A, A\text{-Bimod}$

\[
\mathcal{M}(n) = B^\otimes n
\]

\[
\mathcal{M}(\text{cr}_1)_{ij} = (\phi_{ij} : v \otimes w \mapsto x_j \langle v, w \rangle)
\]

\[
\mathcal{M}(\text{an}_1)_{ij} = (\varphi_{ij} : 1 \mapsto x_i^{-1} \sum b \otimes b^*)
\]

As $\text{cr}_1$ and $\text{an}_1$ tensor generate $\text{TL}(\beta)$ this fully determines $\mathcal{M}$. One advantage of considering module categories of this form is that they induce a pivotal structure on the full image (cf. Section 7). To prove this we first consider the following lemma.

**Lemma 8.1.** Let $\phi^n$ and $\varphi^n$ denote the image of $\mathcal{M}(\text{cr}_n)$ and $\mathcal{M}(\text{an}_n)$ respectively. For $i, j \in Q_0$ and $n \in \mathbb{N}^+$ we have the following

\[
\phi^n_{ji}(w \otimes v) = \frac{x_i}{x_j} \phi^n_{ij}(v \otimes w)
\]

and

\[
\varphi^n_{ji} = \frac{x_i}{x_j} T^n_{ij} \varphi^n_{ij}
\]
Therefore (27) is proved. To prove (28) we proceed more directly.

**Proof.** We proceed by induction on $n$. The base case $n = 1$ is clear. Assuming the hypothesis for all integers up to $n - 1$, we take $b \in s_i B_k$, $v \in s_k B_n^{n-1}$, $w \in s_j B_n^{n-1}$ and compute,

\[
\phi^n_{ij}(b \otimes v \otimes w \otimes b^*) = \phi^n_{kj}(v \otimes w)\phi_{ik}(b \otimes b^*) = x_k \phi^n_{kj}(v \otimes w).
\]

We then also have

\[
\phi^n_{ji}(w \otimes b^* \otimes b \otimes v) = \phi^n_{jk}(w \otimes v)\phi_{ki}(b^* \otimes b) = x_i \phi^n_{jk}(w \otimes v).
\]

\[
x_j \phi^n_{ij}(b \otimes v \otimes w \otimes b^*) = x_i \phi^n_{ij}(b \otimes v \otimes w).
\]

Therefore (27) is proved. To prove (28) we proceed more directly,

\[
\frac{x_i}{x_j} T^n_{ij} \circ \varphi^n_{ij} = \frac{x_i}{x_j} T^n_{ij} \circ \left( \sum_k (\text{id}_{n-1} \otimes \varphi_{kj} \otimes \text{id}_{n-1}) \circ \varphi^n_{ik} \right)
\]

\[
= \frac{x_i}{x_j} \sum_k (\text{id}_{1} \otimes (T^n_{ik} \circ \varphi^n_{jk}) \otimes \text{id}_{1}) \circ (T^n_{kj} \circ \varphi_{kj})
\]

\[
= \frac{x_i}{x_j} \sum_k (\text{id}_{1} \otimes \varphi^n_{kj} \otimes \text{id}_{1}) \circ \varphi_{jk}
\]

\[
= \varphi^n_{ji}
\]

as desired. \qed

**Proposition 8.2.** Let $\mathcal{M}$ be a module category over $\text{TL}(\beta)$ given by (26). Then (25) commutes. In other words, $\mathcal{M}$ induces a pivotal structure on its full image.

**Proof.** As the pivotal structure on $\text{TL}(\beta)$ is given by the identity (25) reduces to $\varphi^\alpha = \alpha^\varphi$ for all $\alpha \in \text{Hom}_D(M(m), M(n))$. For $a \in s_i B_n^j$, by (27), we have

\[
\alpha^\varphi(a) = (\text{id} \otimes \phi^n_{ji}) \circ (\text{id} \otimes \alpha_{ji} \otimes \text{id}) \circ (\varphi^n_{ij} \otimes \text{id})(a)
\]

\[
= \sum_{IJ} \lambda_{I,J}^n \phi^n_{ji}(a_{ji}(b_{ij}) \otimes a) b_I
\]

\[
= \frac{x_i}{x_j} \sum_{IJ} \lambda_{I,J}^n \phi^n_{ij}(a \otimes \alpha_{ji}(b_{ij})) b_I
\]

where the $\lambda_{J,I}^n$, the $b_I$ and the $b_J$ are such that

\[
\varphi^n_{ij}(1) = \sum_{IJ} \lambda_{I,J}^n b_I \otimes b_J \in s_i B_n^j \otimes s_j B_n^i.
\]

However, by (28), we also have

\[
\alpha^\varphi(a) = (\phi^n_{ij} \otimes \text{id}) \circ (\text{id} \otimes \alpha_{ji} \otimes \text{id}) \circ (\text{id} \otimes \varphi^n_{ji})(a)
\]

\[
= \frac{x_i}{x_j} \sum_{IJ} \lambda_{I,J}^n \phi^n_{ij}(a \otimes \alpha_{ji}(b_{ij})) b_I
\]

\[
= \alpha^\varphi(a)
\]
and so we are done. \hfill \Box

Once again, let $Q$ be a symmetric quiver with non-degenerate eigenvalue $\beta$ and let $M$ be given by \([20]\). Under certain additional conditions on $Q$, $M$ will vanish on $N$ and give a module category over $C$. Such modules categories turn out to classify all modules categories over $C$, as described by the following theorem.

**Theorem 8.3** \cite[(EO04), Theorem 3.12]{EO04}. Indecomposable module categories over $C$ are classified by the double Dynkin quivers of type $A, D, E$.

**Corollary 8.4.** Every module category over $C$ induces a pivotal structure on its full image.

**Proof.** This follows immediately from Theorem 8.3 and Proposition 8.2. \hfill \Box

A module category $\mathcal{M}: C \to A$, $A$-Bimod over an arbitrary monoidal category $C$ comes equipped with a natural action of $K_C(C)$ on $K_C(\text{Mod-A})$ given by \([X] \cdot [V] = [\mathcal{M}(X) \otimes_A V]\). However, when $C$ is a spherical fusion category and $M$ induces a pivotal structure on its full image, we may consider the $T\mathcal{M}$ construction. As $\text{End}_{T\mathcal{M}}(1) = K_C(C)$ this defines another action of $K_C(C)$ on $T\mathcal{M}(1) = \text{End}_D(A)$ (where $D$ is the full image of $\mathcal{M}$). Exploiting graphical calculus in $D$, this action is given by

\[
[X] \cdot \alpha = \boxed{\alpha} X .
\]

In the case when $C$ is the semisimple quotient category constructed from $\text{TL}(\beta)$, these two actions coincide.

**Proposition 8.5.** Let $\mathcal{M}$ be a module category over $\text{TL}(\beta)$. W.l.o.g. we suppose that $\mathcal{M}$ is given by \([20]\). For $j \in Q_0$, let $1_j$ be the corresponding idempotent on $A$ and let $V_i$ be the corresponding simple $A$-module. Then the map

\[
\Phi: \text{End}_D(A) \to K_C(\text{Mod-A})
\]

\[
1_j \mapsto x_j [V_j]
\]

is an isomorphism of $K_C(C)$-modules.

**Proof.** As $\{[V_j]\}$ and $\{1_j\}$ form a basis of $K_C(\text{Mod-A})$ and $\text{End}_D(A)$ respectively, $\Phi$ is an isomorphism of vector spaces. As, for $X$ in $C$,

\[
[X] \cdot 1_j = \sum_i \phi_{ij}^X \circ \varphi_{ij}^X 1_i = \sum_i \frac{x_i}{x_j} \dim \mathcal{M}(X) \cdot 1_i
\]

where $\phi_{ij}^X$ and $\varphi_{ij}^X$ is $\mathcal{M}(\text{cr}_X)$ and $\mathcal{M}(\text{an}_X)$ respectively, we have

\[
\Phi
\]

\[
1_j \xrightarrow{[X]} \sum_i \frac{x_i}{x_j} \dim \mathcal{M}(X) \cdot 1_i \xrightarrow{\Phi^{-1}} x_j \sum_i \dim \mathcal{M}(X)_j [V_i]
\]

as desired. \hfill \Box
We are now ready to exploit the $\mathcal{T} \mathcal{M}$ construction to explain a well known pattern in the classification of modular invariants over $\mathcal{C}$.

**Theorem 8.6** (C.I.Z. classification [CIZ87]). The complete list of modular invariants over $\mathcal{C}$ is as follows. To aid legibility, we present these modular invariants as partition functions, cf. [2].

$$A_{h-1} = \sum_{a=1}^{h-1} |\chi_a|^2, \quad \forall h \geq 3$$

$$D_{\frac{h+1}{2}} = \sum_{a=1}^{h-1} \chi_a \chi_{\beta a - 1}, \quad \text{whenever } \frac{h}{2} \text{ is even}$$

$$D_{\frac{h+1}{2}} = |\chi_1 + \chi_{2j_1}|^2 + |\chi_3 + \chi_{2j_3}|^2 + \cdots + 2|\chi_{\frac{h}{2}}|^2, \quad \text{whenever } \frac{h}{2} \text{ is odd}$$

$$\mathcal{E}_6 = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2, \quad \text{for } h = 12$$

$$\mathcal{E}_7 = |\chi_1 + \chi_{17}|^2 + |\chi_5 + \chi_{13}|^2 + |\chi_7 + \chi_{11}|^2$$

$$+ |\chi_9 (\chi_3 + \chi_{15})^* + (\chi_3 + \chi_{15})^* \chi_9^* + |\chi_9|^2, \quad \text{for } h = 18$$

$$\mathcal{E}_8 = |\chi_1 + \chi_{11} + \chi_{19} + \chi_{29}|^2 + |\chi_7 + \chi_{13} + \chi_{17} + \chi_{23}|^2, \quad \text{for } h = 30,$$

where $h$ is the Coxeter number of $\mathcal{C}$ and $J : \{1, 2, ..., h-1\} \rightarrow \{1, 2, ..., h-1\}$ maps a to $h - a$.

As alluded to in the introduction, the classification of modular invariants over $\mathcal{C}$ admits the following A-D-E pattern. Let $\mathcal{X}$ be a double Dynkin quiver of type A, D or E. The eigenvalues of $\mathcal{X}$ form a subset of $\{-2q | q = e^{2\pi i l}/h, 1 \leq l \leq h - 1\}$ for some $h \in \mathbb{N}$. Then, for $l \in \{1, 2, ..., h-1\}$, the $l$th diagonal entry in the modular invariant associated to $\mathcal{X}$ gives the dimension of the corresponding eigenspace of $\mathcal{X}$.

The $\mathcal{T} \mathcal{M}$ construction explains the A-D-E pattern appearing in the classification of $A_1$ modular invariants in the following way. Let $\mathcal{X}$ be an A-D-E double Dynkin quiver and let $\mathcal{M} : \mathcal{C} \rightarrow \mathcal{A}, \mathcal{A}$-Bimod be the corresponding module category over $\mathcal{C}$. It is known that applying $\alpha$-induction, as described in Section 7, to $\mathcal{M}$ yields the modular invariant associated to $\mathcal{X}$ by the list appearing in Theorem 8.6 [BE01 Section 5]. We denote this modular invariant $Z$. By Theorem 7.5 the entries of $Z$ may be thought of as the dimensions of the simple multiplicity spaces in $\mathcal{T} \mathcal{M}$, in other words

$$Z = Z(\mathcal{T} \mathcal{M})$$

where $Z(\mathcal{T} \mathcal{M})$ is given by Definition 8.5. We recall that $\text{End}_{\mathcal{C}}(1)$ is a semisimple commutative algebra generated by the orthogonal primitive idempotents $\{1_I\}_{I \in \mathcal{I}}$ where $(1, 1_I)^2 = (1_I^{\perp}, e_I^{\perp})^2$, see Remark 2.7. The diagonal terms in $Z$ therefore correspond to the dimensions of the weight spaces of the action of $\text{End}_{\mathcal{C}}(1) = \mathcal{K}_\mathcal{C}(\mathcal{C})$ on $\mathcal{T} \mathcal{M}(1)$. However by Proposition 8.5 this action coincides with the natural action of $\mathcal{K}_\mathcal{C}(\mathcal{C})$ on $\mathcal{K}_\mathcal{C}(\text{Mod-A})$. As the weight spaces of this action are given by the eigenspaces of $\mathcal{X}$ this explains the pattern.

**References**

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