Are there really ‘black holes’ in the Atlantic Ocean?

Angus Prain and Valerio Faraoni

1 Physics Department and STAR Research Cluster, Bishop’s University, 2600 College St., Sherbrooke, Québec, Canada J1M 1Z7

In this letter we point out some interpretational difficulties associated with concepts from General Relativity in a recent article which appeared in J. Fluid Mech. 731 (2013) R4 where a Lorentzian metric was defined for turbulent fluid flow and interpreted as being analogous to a black hole metric. We show that the similarity with black hole geometry is superficial at best while clarifying the nature of the black hole geometry and the work above with some examples.

I. INTRODUCTION

In a recent article ‘Coherent Lagrangian vortices: the black holes of turbulence’ [1] the authors draw parallels between certain fluid dynamical configurations and certain properties of black hole geometry. The goal the article [1] is to characterize vortex structures which remain coherent over long times. Some such vortices (known as the Agulhas rings) are observed in the South Atlantic and are believed to be relevant for the long range transport of water with relatively high salinity and temperature, and possibly also as moving oases for the food chain ([1] and references therein).

Specifically, the authors of [1] describe how to associate a 1+1 dimensional Lorentzian effective metric tensor to spatial (constant time) snapshots of a fluid flow. One claim made in [1] is that certain closed spatial curves (at fixed time) in the fluid flow are analogous to the ‘photon spheres’ that exist around black holes. A photon sphere necessarily occurs in a black hole space-time and here we draw intuition (for a subsequent two dimensional treatment) from the simplest black hole, which is described (in four dimensions) by the Schwarzschild metric (see below).

While we do not take issue with the interesting fluid-dynamical subject matter of [1], we wish to point out a number of conceptual difficulties associated with the geometric interpretation of the results and hopefully elucidate some of the (perhaps counter-intuitive) features of Lorentzian geometry and in particular of black hole geometries. Specifically, we make the following clarifications:

- The circular photon orbit (associated with the photon sphere) around a Schwarzschild black hole is not a closed null geodesic.
- Closed null curves in General Relativity are extremely pathological and probably forbidden by reasonable physics arguments (for example, globally hyperbolic spacetimes do not admit closed null curves).
- The existence of a photon sphere is not a necessary nor is it a sufficient condition for the existence of a black hole.
- The circular photon orbit is not circular - its 3-dimensional projection onto the fixed time slices of the Schwarzschild geometry in spherical polar coordinates is.
- A singularity in the metric does not imply a singularity of the geometry. A singularity of a metric coefficient in one coordinate system is a weak condition and does not indicate the presence of a real physical singularity.

Below we will expand on these points.

In Sec. II we explore the construction of the ‘fluid-metric’ as defined in [1] and its geometry. In Sec. III we review some relevant aspects of the Schwarzschild geometry, in particular the circular photon orbit and the photon sphere, while in Sec. IV we present some examples of fluid flows which give rise to interesting Lorentzian geometries and which serve to illustrate our observations.

II. DESCRIBING FLUIDS WITH LORENTZIAN GEOMETRY

In this section we briefly review the main theoretical result of Ref. [1] by deriving the 1 + 1-dimensional Lorentzian metric associated with certain fluid configurations.

The primary mathematical result of the article [1] is the identification and characterisation of certain closed curves along which the fluid flow is ‘coherent’: closed curves of material flow which remain closely associated. Such curves are shown in [1] to satisfy a differential equation which is interpreted as an integrability condition of a vector field which is null with respect to an auxiliary metric tensor. That is, one discovers a null vector field and subsequently integrates it to yield the integral curves.

1 The circular photon orbit around a black hole, which defines the photon sphere, is a space-like circular curve along which a photon (or any massless particle) can travel under the influence of gravity alone.
2 The so-called ‘Schwarzschild black hole’ is the unique static spherically symmetric vacuum solution to Einstein’s gravitational field equations and is known to describe the geometry outside of a spherically symmetric gravitating material body, including a non-rotating black hole [2][3].
which are null curves. We shall introduce some concepts from continuum mechanics to this end.

Consider a sufficiently smooth fluid flow velocity \( \mathbf{v}(t, \mathbf{x}) \) (for example, \( \mathbf{v} \) could be a solution of the Navier-Stokes equations, but this is not required). Then a fluid element propagates in time along the flow lines with a world line \( \mathbf{x}(t) \) which satisfies

\[
\dot{\mathbf{x}} = \mathbf{v}(t, \mathbf{x}).
\]

The flow given by \( \Phi \) represents a map \( \phi_t \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) (we restrict attention to a 2-dimensional fluid flow) where the reference configuration given by the initial condition \( \mathbf{x}(0) = \mathbf{x}_0 \) gets mapped to the solution to \( \Phi \) at time \( t \)

\[
\phi_t : \mathbf{x}_0 \mapsto \mathbf{X}(t).
\]

Here we have denoted the coordinates in the plane at time \( t \) with a capital letter \( \mathbf{X} \) for clarity while reserving the lower case variable names for the initial un-deformed coordinates \( \mathbf{x} \). In general, the flow \( \phi_t \) will deform the lines of constant initial coordinate label, say \( x = \text{const.} \) or \( y = \text{const.} \), where \( x = (x, y) \) are Cartesian coordinates, so that lines of constant initial condition \( x_0 \) or \( y_0 \) will, at some fixed time \( t \), be deformed to some curved lines in the plane. We may characterise such a deformation with the so-called ‘deformation gradient’ mixed two-point tensor \( \mathbf{F} \) whose components with respect to an initial and final coordinate basis are given by

\[
F^i_j := \frac{\partial X^i(t)}{\partial x^j_0}.
\]

Note the mixed-coordinate definition (two-point tensor structure) by observing that these components are given with respect to the two coordinate bases, initial and final, in a specific way. We have, in coordinate-independent notation,

\[
\mathbf{F} = F^i_j \left( \frac{\partial}{\partial X^i} \otimes dx^j \right).
\]

For example, and for later reference, in a polar coordinate basis we have

\[
F^i_j = \begin{pmatrix}
\frac{\partial R(t)}{\partial r_0} & \frac{\partial R(t)}{\partial \theta_0} \\
\frac{\partial \theta(t)}{\partial r_0} & \frac{\partial \theta(t)}{\partial \theta_0}
\end{pmatrix},
\]

which is written in the more convenient orthonormal polar basis \( e_r := \partial_r \) and \( e_\theta := \partial_\theta / r \) (\( e_R := \partial_R \) and \( e_\theta := \partial_\theta / R \)) as

\[
F^a_b = \begin{pmatrix}
\frac{\partial R(t)}{\partial r_0} & 1 & \frac{\partial R(t)}{\partial \theta_0} \\
\frac{R(t)}{r_0} & \frac{\partial \theta(t)}{\partial \theta_0} & \frac{R(t)}{r_0}
\end{pmatrix}.
\]

The orthonormal basis \( \mathbf{e} \) is preferable to the polar coordinate basis due to the possibility to raise and lower indices

\[\text{with the identity matrix, which coincides with the matrix of metric coefficients in such a basis.}\]

The ‘Right Cauchy-Green’ deformation tensor \( \mathbf{C} \) carries the same deformation information modulo transformations which merely shift or rotate an initial configuration without genuine deformation and is defined as

\[
C^i_j := F^k_i F^k_j = G_{km} g^{in} \frac{\partial X^m(t)}{\partial x^k_0} \frac{\partial X^n(t)}{\partial x^k_0},
\]

where \( g_{ij} \) and \( G_{ij} \) are the metric coefficients in the two coordinate bases, respectively. In orthonormal bases we have

\[
C_{ab} = \sum_e \frac{\partial X^e(t)}{\partial x^0_0} \frac{\partial X^e(t)}{\partial x^0_0}.
\]

We will be concerned with the eigenvectors \( \xi_i \), and the eigenvalues \( \lambda_i \) of \( \mathbf{C} \). Note that \( \mathbf{C} \) as well as the eigenvectors and eigenvalues depend on the initial point \( \mathbf{x}_0 \) as well as the time \( t \) at which we calculate the strains. Being the ‘square’ of another tensor \( \mathbf{F} \) it is simple to show that the eigenvalues \( \lambda_i(\mathbf{x}_0, t) \) of \( \mathbf{C} \) satisfy

\[
0 \leq \lambda_1(\mathbf{x}_0, t) \leq \lambda_2(\mathbf{x}_0, t) \quad \text{for all } \mathbf{x}_0, t
\]

and that \( \{\xi_i\} \) form an orthonormal basis for \( \mathbb{R}^2 \).

Let \( \lambda > 0 \) and define the so-called ‘generalised Green-Lagrange tensor’ \( \mathbf{E}_\Lambda \) in an orthonormal frame by

\[
E_{ab} := \frac{1}{2} \left[ C_{ab} - \lambda \delta_{ab} \right].
\]

This object can be thought of as acting as a time-dependent bilinear form on the initial conditions space and depends on the constant \( \lambda \). In the ‘moving orthonormal frame’ defined by the eigenvectors of \( \mathbf{C} \) we can write

\[
E_{ab} = \frac{1}{2} \begin{pmatrix}
\lambda_1 - \lambda & 0 & 0 \\
0 & \lambda_2 - \lambda & 0 \\
0 & 0 & \lambda_2 - \lambda
\end{pmatrix}.
\]

The authors of \( \Phi \) define an effective metric tensor induced by the action of \( \mathbf{E} \) in \( (10) \) on the space of initial conditions which, remarkably, can have Lorentzian signature (although the tensor \( \mathbf{E} \) is of the wrong type to be a metric, we gloss over this and talk about \( \mathbf{E} \) as ‘being’ the metric tensor). In principle, this fact allows one to draw parallels between this structure and a black hole metric with an associated photon sphere.

A. The meaning and geometry of the metric \( E_{ab} \)

The first thing to notice about the tensor \( \mathbf{E} \) is that its signature can change from point to point and over time depending on the relative magnitude of the eigenvalues \( \lambda_i(\mathbf{x}_0, t) \) and is only Lorentzian at time \( t \) if \( \lambda_1(\mathbf{x}_0, t) < \lambda < \lambda_2(\mathbf{x}_0, t) \) for all points \( \mathbf{x}_0 \). Assuming this ‘Lorentzian signature'}
condition’ on the eigenvalues at some time \( t \), we can construct the two independent ‘E-null’ vector fields \( \eta \) in the orthonormal eigen-basis of \( \mathbf{C} \) as

\[
\eta_{\pm} := \frac{1}{\sqrt{\lambda_2 - \lambda_1}} \left( \sqrt{\lambda_2 - \lambda} \pm \sqrt{\lambda - \lambda_1} \right) \xi_2 \sigma_1 = \sqrt{\lambda_2 - \lambda_2 - \lambda_1} \xi_2 \pm \sqrt{\lambda - \lambda_1} \xi_1 \eta_1 \,
\]

where we have (arbitrarily but without loss of generality) normalised the vector fields to have unit norm in the background Euclidean metric. The integral curves of these vector fields are null curves with respect to the metric \( \mathbf{E} \). It is shown in \( \mathbf{II} \) that these null curves are curves for which the flow up to time \( t \) uniformly scales the tangent vectors to the curve. That is, given an \( \mathbf{E} \)-null curve \( \gamma(s) \) which gets mapped under the flow \( \phi_t \) to the curve \( \phi_t(\gamma)(s) \), we have

\[
||\phi_t(\gamma)(s)||^2 = \lambda ||\gamma'(s)||^2
\]

uniformly for all \( s \) parametrizing the curve and where the prime denotes taking the tangent vector at the given point on the curve. Such closed curves which possess the uniform stretching property are intuitively ‘invariant curves’ of the flow \( \phi_t \); their fluid dynamical relevance is discussed in \( \mathbf{II} \) where the outermost curve in a family of such curves is said to define the boundary of a ‘coherent material vortexex’ in 2-dimensional flow.

III. PHOTON SPHERES AND BLACK HOLES - WHAT THEY ARE AND WHAT THEY ARE NOT

The authors of \( \mathbf{I} \) refer to a photon sphere in an analogy with vortices through the Lorentzian metric \( \mathbf{E} \). A photon sphere is associated with a black hole spacetime metric. In General Relativity there exists a vast array of black hole and black hole-like solutions with varying degrees of physical sensibility. In the vacuum, spherically symmetric case in 3+1-dimensions, there is a uniqueness theorem (Birkoff’s theorem \( \mathbf{I I I} \)) which points us to the static Schwarzschild black hole metric

\[
ds^2 = - \left( 1 - \frac{2GM}{r} \right) dt^2 + \left( 1 - \frac{2GM}{r} \right)^{-1} dr^2 + r^2 d\Omega^2 (2)
\]

here written in standard spherical polar coordinates, where \( d\Omega^2 (2) = d\theta^2 + \sin^2 \theta d\phi^2 \) is the metric on the unit 2-sphere, \( G \) is Newton’s constant, \( M \) is the black hole mass, and units in which the speed of light is unity are used, as is common in relativity. The metric \( \mathbf{1 5} \) describes the simplest black hole spacetime in 3+1 dimensions. A crucial feature of this metric is the existence of an horizon – a lower-dimensional region on which the metric is degenerate and at which time and space ‘swap their roles’. Without getting into technical details, a precise statement is that the norm of the timelike Killing vector field associated with the time symmetry changes sign, becoming a space-like vector field at a co-dimension 2 hypersurface generated by null geodesics of the metric \( \mathbf{1 5} \) and known as the event horizon. The event horizon is the key structure which, if present in a gravitational configuration, indicates the presence of a black hole. In the above Schwarzschild case, the time-like Killing vector is simply the time direction \( K_t := \partial_t \), with norm

\[
||K_t||^2 = - \left( 1 - \frac{2GM}{r} \right)
\]

which changes sign at \( r = 2GM \), a value of the radial coordinate known as the Schwarzschild radius or black hole radius \( R \). This is the radius of the event horizon. Without an event horizon, a spacetime cannot be sensibly said to contain a black hole\(^4\).

A. The ‘circular’ null geodesic of the Schwarzschild black hole

In a careful relativistic treatment \( \mathbf{2} \), one can show that the effective potential for ‘free’ (geodesic) radial motion in the Schwarzschild geometry is given by the effective 1-dimensional potential

\[
V(r) = - \epsilon \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{r^3} , \quad (17)
\]

where \( \epsilon = +1 \) for massive particles and \( \epsilon = 0 \) for massless particles, and \( L \) is the angular momentum \( \mathbf{I I I} \) of the particle. The first two terms on the right hand side of eq. (17) are identical to the two terms which appear in the effective potential of the Newtonian treatment \( \mathbf{I I I} \), while the third one is a correction due to General Relativity. Circular orbits exist at radii \( r_{\text{orbit}} \) for which \( V'(r_{\text{orbit}}) = 0 \) and we have

\[
r_{\text{orbit}} = \left( L \pm \sqrt{L^2 - 12 \epsilon G^2 M^2} \right) \frac{L}{2\epsilon GM} . \quad (18)
\]

In the massless limit \( \epsilon = 0 \), there exists a single orbit at \( 3GM \), which is unstable, while in the massive case there exist two orbits, one stable and the other (at smaller radius) unstable. In contrast, in Newtonian gravity (in which the term proportional to \( r^{-3} \) is absent in the effective potential \( V(r) \)), there exists a single stable circular orbit for massive particles and no circular orbit for massless particles.

It is important to note that this closed circular orbit is not a closed null geodesic, such curves being highly

\(^4\) A slight caveat to this classification is that it might be possible for black hole horizons to evaporate through a semi-classical process known as Hawking evaporation. In this case the horizon is not a true event horizon since it lives for only a finite amount of time.

\(^5\) For massive particles \( L \) represents the angular momentum per unit mass.
FIG. 1. The projection of the helical null geodesic photon worldline onto the spatial ‘circular photon orbit’.

pathological and most likely unphysical in General Relativity. Indeed, it is not even a geodesic nor is it a null curve. Instead it is the spatial projection of an open and infinitely extended null geodesic curve in space and time as shown in Fig. 1.

The tangent vector to the null geodesic is given by
\[ \frac{dx^\mu(s)}{ds} = \left(3E, 0, 0, \frac{L}{9G^2M^2}\right) \quad (19) \]
in the spherical coordinate basis whose integral curve is the helical null geodesic in spacetime
\[ x^\mu(\lambda) = \left(3E\lambda, 0, 0, \frac{L}{9G^2M^2}\lambda\right) \quad (20) \]

A photon sphere is a sphere of radius equal to that of the closed circular photon orbit and coincides with a 2-sphere of symmetry of the spacetime. The photon sphere (a spacelike hypersurface) of radius \(3GM\) has nothing to do with the black hole horizon (a null hypersurface) of radius \(2GM\), which traps all particles.

**IV. EXAMPLES OF FLUID FLOWS AND THEIR ASSOCIATED LORENTZIAN GEOMETRIES**

In this section we consider various background fluid flows and compute the associated null curves of the auxiliary metric as defined in (10). The purpose of this section is to use examples to highlight the distinction between the closed null curves and any kind of photon sphere. In an attempt to construct simple examples possessing closed curves uniformly stretched by a flow, we consider rotational symmetry. It is clear that circles concentric about the origin are uniformly stretched in circularly symmetric flows and hence constitute examples of the curves sought in Ref. [1] as solutions to the optimization problem and shown to define ‘coherent material vortices’. We consider sequentially more complex flows: rigid body, irrotational, and some rotating and draining vortex flow.

**A. Rigid body rotation**

Consider a rigid body flow profile
\[ \phi_t : (r_0, \theta_0) \rightarrow (r_0, \theta_0 + \omega t) \quad (21) \]
where \(\omega\) is a fixed angular frequency. Then \(C = 1\) coincides with the identity, \(\lambda_1 = \lambda_2 = 1\), and the effective metric \(E_\lambda\) is not of Lorentzian signature for any \(\lambda\). Hence we must move to a more complicated flow in order to explore the construction. In a certain sense, every possible curve is null in this geometry, highlighting the total uniform and coherent (non-deforming) nature of the flow. Such a flow is highly unphysical and does in no way invalidate the results of [1]: it serves only as a first simple example which shows that the metric construction can be non-trivial.

**B. Irrotational vortex**

Consider instead the differential rotating flow
\[ \phi_t : (r_0, \theta_0) \rightarrow \left(r_0, \theta_0 + \frac{t}{r_0}\right) \quad (22) \]
Such a flow is irrotational and is commonly referred to as a ‘vortex line’ flow, in this case of strength \(\Gamma = 2\pi\).

Then the eigenvalues of \(C\) are easily calculable as
\[ \lambda_{\pm}(r_0, t) = \frac{1}{2r_0^2} \left[2r_0^2 + t^2 \pm t\sqrt{4r_0^2 + t^2}\right] \quad (23) \]
and we see that they are independent of the angular variable \(\theta_0\). They satisfy, as functions of \(t\) and \(r_0\), the bounds
\[ 1 < \lambda_+ \rightarrow +\infty \quad \text{as} \quad t \rightarrow \infty, \quad (24) \]
\[ 1 > \lambda_- \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty, \quad (25) \]
as well as the bounds at fixed \(t\)
\[ \lambda_+ \rightarrow 1^+ \quad \text{as} \quad r_0 \rightarrow \infty, \quad (26) \]
\[ \lambda_- \rightarrow 1^- \quad \text{as} \quad r_0 \rightarrow \infty. \quad (27) \]
(see Fig. 2). Hence the only value of \(\lambda\) for which \(E_\lambda\) is of Lorentzian signature in the entire plane is \(\lambda = 1\).

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6 A closed time-like or null geodesic in 4-dimensional spacetime is associated with time travel ([2] [3], see Ref. [5] for a popular exposition).
The eigenvalues $\lambda_\pm$ of the matrix $C$ as a function of $r_0$ at finite time $t$ for the irrotational vortex flow \cite{22}. As time proceeds the red solid curve diverges to $+\infty$ while the blue dashed curve converges to $0$ – neither curve crosses the black dot-dashed line $\lambda = 1$ at any time $t$.

In the orthonormal polar basis for this choice of $\lambda$ we have

$$E_{ab} = \frac{1}{2} \begin{pmatrix} \left(\frac{t}{r}\right)^2 & -\frac{t}{r} \\ -\frac{t}{r} & 0 \end{pmatrix}$$

In this case, one of the two null vector fields of $E$ is time-independent and tangent to circles concentric on the origin (see Fig. 3), in the orthonormal polar basis,

$$\eta_+ = (0, 1)$$

while the other is time-dependent,

$$\eta_- = \left(1, \frac{t}{2r}\right)$$

tending to a purely angular pointing (in the $e_\theta$ direction) vector field as $t \to +\infty$ (see Fig. 4).

It is straightforward to show that the geodesic equation (one might find it easier to work in the non-orthonormal polar coordinate basis at this stage) becomes the pair

$$\frac{d^2r(s)}{ds^2} = 0$$

$$\frac{d^2\theta(s)}{ds^2} + \frac{t}{r^3} \left(\frac{dr(s)}{ds}\right)^2 = 0$$

from which we see that our null curves of constant $r$ are indeed null geodesics. Furthermore one can simply show that also the second set of null curves are geodesics.

Using standard techniques one can show that there are three Killing vector fields, two of which are ‘spacelike’ for all $t$ and $x_0$ and one of which is null for all $t$ and $x_0$. Here ‘spacelike’ is an arbitrary definition which we take to mean positive norm (with respect to the metric $E$).

The null Killing vector field is purely rotational being tangent to circles about the origin. Since there is no 1-dimensional line on which any of the Killing vectors change norm we can conclude that there does not exist a
horizon in this geometry\footnote{Note that in lower dimensions a horizon is still defined by the change in norm of a time-like Killing vector, but is not necessarily given by a co-dimension 2 hypersurface.}

Again, this flow is highly idealised being unbounded at the origin whereas in practice when working with real turbulent flows one would have a region $U_\lambda$ which might even be time dependent.

\section*{C. Irrotational draining vortex}

One might consider also a draining vortex type fluid flow

\begin{equation}
\phi_t : (r_0, \theta_0) \rightarrow \left( \alpha(t) r_0, \theta_0 + \frac{t}{r_0} \right).
\end{equation}

where $\alpha(t)$ is a function which parametrises the radial flow, in the hopes of unearthing some Lorentzian geometry which more closely resembles that of a black hole. Indeed, with the addition of radial flow one might hope to have some kind of “trapped region” inside of which all time-like vectors point towards the origin, reminiscent to have some kind of “trapped region” inside of which all time-like vectors point towards the origin, reminiscent of a similar construction in the singularity theorems of Hawking and Penrose\cite{Hawking:1971uw}. For example one might choose $\alpha(t)$ such that the flow describes a draining ‘bathtub vortex’ which to a first order approximation can be described by $\alpha(t) = \sqrt{1 - \lambda t^2}$ so that $v_r(t) = -A/r(t)$ and the radial velocity is inversely proportional to the radial position as a function of time.

The flow and geometry given here are more complex than in the previous example and, for brevity, we shall not present them in any depth here. It can be shown, however, that at fixed time $t$ the matrix $C$ in this case has two eigenvalues which behave analogously to the irrotational vortex case but separated now by the time-dependent ‘constant’ $\alpha^2(t)$. In this case, and in line with our intuition, circles concentric about the origin are invariant curves and are null geodesics of the metric $E_{\alpha^2(t)}$: a circle of radius $r_0$ gets mapped under the flow to circles of radius $\alpha(t) r_0$ so that the tangent vectors squared are uniformly scaled by $\alpha^2$. This is intuitive since an experimentalist who drops ink droplets in a perfect ring into such a flow will observe a shrinking of the ring (or an expansion, depending on the character of $\alpha$) but it will be \textit{coherent} (not deforming) as time progresses.

\section*{V. CONCLUSION AND OUTLOOK}

While the characterisation of fluid vortices in terms of an auxiliary Lorentzian metric on fixed time slices of a fluid flow should be of practical utility, as pointed out in \cite{Brill:1973}, the interpretation of the metric and its Lorentzian geometry as being ‘close to’ or analogous to that of a black hole is lacking. We have shown that no event horizons exist for simple flows which possess the characteristic features discussed in \cite{Brill:1973} as being ‘photon sphere’ or ‘black-hole-like’. Further, we have clarified the nature of the circular photon orbit in the Schwarzschild geometry (the prototypical black hole geometry containing an horizon surface which traps massive and massless particles) and shown it to be very different from the closed null curves which herald the ‘coherent material vortices’ discussed in \cite{Brill:1973}. Specifically the $\lambda$-lines discussed in \cite{Brill:1973} as being analogous to the photon sphere are closed null curves while the photon sphere contains closed space-like curves.

It is interesting to point out that the use of Lorentzian geometry and auxiliary metrics in fluid dynamics in fact has a healthy and vibrant research community and literature associated with it known as the ‘Analogue Gravity’ program \cite{1} where the effective metric discussed there has a sound physical basis, being the metric which describes the real causal structure of a physical system for various kinds of physical propagating signals such as sound waves or small perturbations. In this manner, the metrics in analogue gravity are expected to be sufficiently ‘physical’, for example they cannot contain closed null or closed time-like curves. The cutting edge of that discipline is the construction and observation of analogue event horizons or analogue photon spheres in analogue systems (see for example \cite{8,10}).

In still other corners of the literature the introduction of an effective metric in a fictitious space is the key feature of the Jacobi form of the Maupertuis variational principle in point particle mechanics \cite{4}. While in principle intriguing, the introduction of such effective metrics in that context (see also \cite{11,12} for effective metrics in different contexts) has, thus far, been of little utility in developing theories and solving practical problems. It is hoped that the characterization of eddies by means of an effective Lorentzian geometry will break this spell.

\section*{ACKNOWLEDGMENTS}

We thank G. Haller and F.J. Beron-Vera for comments given by a co-dimension 2 hypersurface on a previous version of the manuscript. This work is supported by the Natural Sciences and Engineering Research Council of Canada and by Bishop’s University.
