Quantitative uniqueness for fractional heat type operators

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Abstract
In this paper we obtain quantitative bounds on the maximal order of vanishing for solutions to $(\partial_t - \Delta)^s u = Vu$ for $s \in [1/2, 1)$ via new Carleman estimates. Our main results Theorems 1.1 and 1.3 can be thought of as a parabolic generalization of the corresponding quantitative uniqueness result in the time independent case due to Rüland and it can also be regarded as a nonlocal generalization of a similar result due to Zhu for solutions to local parabolic equations.

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1 Introduction and the statement of the main result

We say that the vanishing order of a function $u$ is $\ell$ at $x_0$, if $\ell$ is the largest integer such that $D^\alpha u = 0$ for all $|\alpha| \leq \ell$, where $\alpha$ is a multi-index. In the papers [19, 20], Donnelly and...
Fefferman showed that if \( u \) is an eigenfunction with eigenvalue \( \lambda \) on a smooth, compact and connected \( n \)-dimensional Riemannian manifold \( M \), then the maximal vanishing order of \( u \) is less than \( C \sqrt{\lambda} \), where \( C \) only depends on the manifold \( M \). Using this estimate, they showed that if the Riemannian metric is real analytic, then \( H^{n-1}(\{ x : u_{\lambda}(x) = 0 \}) \leq C \sqrt{\lambda} \), where \( u_{\lambda} \) is the eigenfunction corresponding to \( \lambda \) and therefore gave a complete answer to a famous conjecture of Yau [52] in the analytic setting. It is to be mentioned that in recent times, there has been some very interesting developments in the smooth setting as well, thanks to some breakthrough works of Logunov and Malinnikova in [38–40]. We notice that the zero set of \( u_{\lambda} \) is referred to as the nodal set. This order of vanishing is sharp. If, in fact, we consider \( M = S^n \subset \mathbb{R}^{n+1} \), and we take the spherical harmonic \( Y_{\kappa} \) given by the restriction to \( S^n \) of the function \( f(x_1, \ldots, x_n, x_{n+1}) = \Re(x_1 + i x_2)^\kappa \), then one has \( \Delta_{S^n} Y_{\kappa} = -\lambda_{\kappa} Y_{\kappa} \), with \( \lambda_{\kappa} = \kappa(\kappa + n - 2) \), and the order of vanishing of \( Y_{\kappa} \) at the North pole (0, \ldots, 0, 1) is precisely \( \kappa = C \sqrt{\lambda_{\kappa}} \). We refer to [28] for related results on Hausdorff measure estimates of nodal sets of solutions to parabolic equations. In his work [33], Kukavica considered the more general problem

\[
\Delta u = V(x)u,
\]

where \( V \in W^{1,\infty} \), and showed that the maximal vanishing order of \( u \) is bounded above by \( C(1 + ||V||_{W^{1,\infty}}) \). He also conjectured that the rate of vanishing order of \( u \) is less than or equal to \( C(1 + ||V||_{L^{1/\infty}}^{1/2}) \), which agrees with the Donnelly-Fefferman result when \( V = -\lambda \). Employing Carleman estimates, Kenig in [32] showed that the rate of vanishing order of \( u \) is less than \( C(1 + ||V||_{L^{2/\infty}}^{2/3}) \), and furthermore the exponent \( \frac{2}{3} \) is sharp for complex potentials \( V \) based on a counterexample of Meskho (see [42]).

Not so long ago, the rate of vanishing order of \( u \) has been shown to be less than \( C(1 + ||V||_{W^{1,\infty}}^{1/2}) \) independently by Bakri in [6] and Zhu in [53]. Bakri’s approach is based on an extension of the Carleman method in [19]. On the other hand, Zhu’s approach is based on a variant of the frequency function approach employed by Garofalo and Lin in [26, 27]), in the context of strong unique continuation problems. The approach of Zhu has been subsequently extended in [12] to variable coefficient principal part with Lipschitz coefficients where a similar quantitative uniqueness result at the boundary of \( C^{1, \text{Dini}} \) domains has been obtained. We would also like to mention that in [54], an analogous quantitative uniqueness result has been established for solutions to parabolic equations of the type

\[
\text{div}(A(x, t)\nabla u) - u_t = Vu,
\]

where \( V \in C^1 \) and \( A(x, t) \in C^2 \) by an adaption of an approach due to Vessella in [51] (see also [23]). Now for nonlocal equations of the type

\[
(-\Delta)^s u = Vu,
\]

Rüland in [45] showed that the vanishing order is proportional to \( C_1 ||V||_{C^{1}}^{1/2s} + C_2 \) which in the limit as \( s \to 1 \), exactly reproduces the result of Donnelly and Fefferman. See also [55] for vanishing order estimates for Steklov eigenfunctions which via the extension approach of Caffarelli and Silvestre in [17], is relevant to the case \( s = 1/2 \). See also [15] for earlier results on quantitative uniqueness for Steklov eigenvalue problems. We also refer to [24, 46] for other qualitative strong unique continuation results in the nonlocal elliptic setting.

In this work, we are interested in quantitative uniqueness for the following nonlocal equation

\[
(\partial_t - \Delta)^s u = V(x, t)u \text{ in } B_4 \times (-16, 16), \quad s \in [1/2, 1), \quad (1.2)
\]
where
\[
\begin{align*}
V & \in C^{1}_{x,t}(B_{4} \times (-16, 16)) \quad \text{for } s > 1/2 \\
\text{and } V & \in C^{1,\alpha}_{x,t}(B_{4} \times (-16, 16)) \cap C^{1}_{x,t}(B_{4} \times (-16, 16)) \quad \text{for some } \alpha > 0 \text{ when } s = 1/2.
\end{align*}
\]
(1.3)

We notice that the qualitative $C^{1,\alpha}$ assumption on $V$ that we enforce is only to ensure that the solution to the extended problem (2.10) is $C^{2}$ in the tangential directions. More precisely, we study functions $u \in \text{Dom}(H^{s}) = \{u \in L^{2}(\mathbb{R}^{n} \times \mathbb{R}) \mid H^{s} u \in L^{2}(\mathbb{R}^{n} \times \mathbb{R})\}$ that satisfy the nonlocal Eq. (1.2) above locally in $B_{4} \times (-16, 16)$. Our main result is the following.

**Theorem 1.1** Let $u$ be a non-trivial solution of (1.2) with $||u||_{L^{2}(\mathbb{R}^{n} \times \mathbb{R})} \leq 1$ and where $V$ satisfies the assumptions in (1.3). Then there exist constants $C_{1} = C_{1}(n, s, ||V||_{C^{1}})$, $C_{2} = C_{2}(n, s)$, $C_{3} = C_{3}(n, s, u)$ and $\tilde{R} = \tilde{R}(n, s)$ such that for all $\rho \leq \tilde{R}$ we have
\[
||u||_{C^{2}(B_{\rho} \times (-2, 2))} \geq C_{1} \rho^{A_{0}},
\]
(1.4)
where $A_{0} = C_{2}||V||_{C^{1}_{x,t}(B_{4} \times (-16, 16))}^{1/2} + C_{3}$. Over here, the $C^{2}$ norm of $u$ is the parabolic $C^{2}$ norm defined as
\[
||u||_{C^{2}(B_{\rho} \times (-1, 0))} \overset{\text{def}}{=} ||u||_{L^{\infty}(B_{\rho} \times (-1, 0))} + ||\nabla x u||_{L^{\infty}(B_{\rho} \times (-1, 0))} + ||\nabla^{2} u||_{L^{\infty}(B_{\rho} \times (-1, 0))} + ||u_{t}||_{L^{\infty}(B_{\rho} \times (-1, 0))}.
\]

**Remark 1.2** We notice that the qualitative $C^{1,\alpha}$ assumption on $V$ that we impose when $s = 1/2$ is required in order to ensure that the solution to (1.2) is $C^{2}$ for the estimate in Theorem 1.1 to make sense. In this context, we refer to Lemma 2.2 below.

In the case when $V \in C^{2}_{x,t}(B_{4} \times (-16, 16))$, we have the following improvement of Theorem 1.1 where the $C^{2}$ norm of $u$ in the estimate (1.4) can be replaced by the $L^{2}$ norm.

**Theorem 1.3** Let $u$ be a non-trivial solution of (1.2) with $V \in C^{2}_{x,t}(B_{4} \times (-16, 16))$, such that $||u||_{L^{2}(\mathbb{R}^{n} \times \mathbb{R})} \leq 1$. Then there exist constants $C_{1} = C_{1}(n, s, ||V||_{C^{1}})$, $C_{2} = C_{2}(n, s)$, $C_{3} = C_{3}(n, s, u)$ and $\tilde{R}$ as in Theorem 1.1 above such that for all $\rho \leq \tilde{R}$ we have
\[
||u||_{L^{2}(B_{\rho} \times (-2, 2))} \geq C_{1} \rho^{M},
\]
(1.5)
where $M = C_{2}||V||_{C^{2}_{x,t}(B_{4} \times (-16, 16))}^{1/2} + C_{3}$.

**Remark 1.4** It is to be mentioned that the constants $C_{3}$ and $C_{3}$ that appear in Theorems 1.1 and 1.3 respectively, depend in a rather explicit way on values of the function $U$ that solves the extension problem (2.9) corresponding to $u$ at larger scales. More precisely, the constants depend on
\[
\frac{1 + \int_{B_{2} \times (-4, 4)} U^{2} y^{a} \, dX \, dt}{\int_{B_{R_{0}/8} \times (-1, 1)} U^{2} y^{a} \, dX \, dt},
\]
(1.6)
where $a = 1 - 2s$ and $R_{0} < 1$ is some universal constant that depends only on $n$ and $s$. This is optimal as examples in the plane such as
\[
u(z) = \Re(x_{1} + i x_{2})^{k},
\]
(1.7)
show that even in the local case (when $V \equiv 0$), the vanishing order can be arbitrarily large. However, taking into account such an example, it turns out that if one factors out a similar
quantity as in (1.6) above which depends on $\int_{B_1} u^2$ and $\int_{B_{1/2}} u^2$, then one can deduce a vanishing order estimate of the Donnelly-Fefferman type (say for $r < 1/2$) which only depends on $||V||^{1/2}_{C^1}$. See for instance [12, 53].

We also mention that from our proofs of Theorems 1.1 and 1.3, it can be seen that it is possible to derive estimates in (1.4) and (1.5) where the constants $C_3$ and $C_3$ would only depend on a doubling ratio of the type $\int_{B_2} \times (-4,4) U^2 y^a dX dt$ instead of the expression in (1.6) above. However, in such a case, the corresponding constants $C_1$ and $C_1$ in Theorems 1.1 and 1.3 has to additionally depend on $u$. This later fact can be seen by multiplying a harmonic function as in (1.7) above by an arbitrary small constant.

Remark 1.5 It is to be noted that in general, solutions to parabolic equations can vanish to infinite order at a point in the local case $s = 1$. We notice that an example of Frank Jones in [31] shows that in fact, there exist non-trivial solutions to the heat equation in $\mathbb{R}^n \times \mathbb{R}$ which are supported in a strip of the type $\mathbb{R}^n \times (0, 1)$ and vanishes to infinite order at every point in the strip. However, (1.4) and (1.5) corresponds to vanishing order estimates in space which are averaged over time and moreover it is implicit in both the estimates that the solution $u$ is such that $\int_{B_{R/8} \times (-1,1)} U^2 y^a dX dt \neq 0$, where $U$ is the extended function corresponding to $u$. Therefore, the solutions to (1.2) that we consider are normalized in a way that such an example as in [31] gets ruled out.

Our results in Theorems 1.1 and 1.3 can thus be viewed as a generalization of the results in [45] and [54]. To the best of our knowledge, this is the first quantitative uniqueness result for fractional heat type operators. To provide some further perspective, we mention that for global solutions of the nonlocal Eq. (1.2) a backward space-time strong unique continuation theorem was previously established by one of us with Garofalo in [11]. Such result represented the nonlocal counterpart of the one first obtained by Poon in [44] for the local case $s = 1$. Very recently in [1], a space like unique continuation property for local solutions to equations of the type (1.2) has been established by both of us in a joint work with Danielli and Garofalo which constitutes the nonlocal counterparts of the space like strong unique continuation results in [21, 22] for the local case $s = 1$.

1.1 Key ideas in the proofs of the main results

The following are the key steps in the proof of our main result.

Step 1: We first establish a quantitative Carleman estimate for solutions to the extension problem (3.1) in Lemma 3.1. The key new feature of such an estimate is the precise quantitative dependence of the weight parameter $\alpha$ on the $C^1$ norm of $V$ as in (3.2). Using such an estimate, we derive a quantitative vanishing order estimate at the bulk in Lemma 3.2 below.

Step 2: We then generalize to the parabolic setting, a Carleman estimate due to Rüland and Salo in [47]. See Lemma 4.1 below. It turns out that such an estimate only holds when $s \geq 1/2$. This is precisely where we require the restriction that $s \geq 1/2$. Using such an estimate, we obtain a propagation of smallness estimate from the boundary to the bulk in Lemma 4.3.

Step 3: Then by combining the bulk quantitative estimate for the extension problem in Step 1 with the propagation of smallness estimate in Step 2, we establish a quantitative vanishing order estimate at the boundary for the extension problem (2.10) which implies Theorem 1.1.
Step 4: Theorem 1.3 follows from Theorem 1.1 by means of new interpolation type inequalities that we prove in Lemma 2.4 combined with the regularity estimates in Lemma 2.2. We notice that similar interpolation type inequalities have previously appeared in [15, 45, 47]. We believe that the inequalities in Lemma 2.4 that we derive are of independent interest and can have applications in other scenarios.

The reader will find that although the proof of Theorem 1.1 is inspired by ideas in [6, 13, 45, 51, 54], it has nevertheless required some delicate adaptations in our situation because of new novel challenges in the nonlocal parabolic case.

The paper is organized as follows. In Sect. 2, we introduce some basic notations and gather some preliminary results that are relevant to our work. In Sect. 3, we derive the quantitative vanishing order estimate in Step 1 using the Carleman estimate that we derive in Lemma 3.1. In Sect. 4, we establish the propagation of smallness estimate for the extension problem in Lemma 4.3. In Sect. 5, we finally prove our main results Theorems 1.1 and 1.3.

In closing, we would like to mention that the study of the fractional heat type operators as well as the related extension problem has received a lot of attention in recent times, see for instance [3–5, 8–10, 14, 16, 18, 25, 29, 35, 41].

2 Notations and preliminaries

In this section we introduce the relevant notation and gather some auxiliary results that will be useful in the rest of the paper. Generic points in \( \mathbb{R}^n \times \mathbb{R} \) will be denoted by \((x_0, t_0), (x, t)\), etc. For an open set \( \Omega \subset \mathbb{R}^n_+ \times \mathbb{R} \) we indicate with \( C^\infty_0(\Omega) \) the set of compactly supported smooth functions in \( \Omega \). The symbol \( \mathcal{S}(\mathbb{R}^{n+1}_+) \) will denote the Schwartz space of rapidly decreasing functions in \( \mathbb{R}^{n+1}_+ \). For \( f \in \mathcal{S}(\mathbb{R}^{n+1}_+) \) we denote its Fourier transform by

\[
\hat{f}(\xi, \sigma) = \int_{\mathbb{R}^n \times \mathbb{R}} e^{-2\pi i (\langle \xi, x \rangle + \sigma t)} f(x, t) dx dt = \mathcal{F}_{x \to \xi} (\mathcal{F}_{t \to \sigma} f).
\]

The heat operator in \( \mathbb{R}^{n+1}_+ = \mathbb{R}^n_+ \times \mathbb{R}_+ \) will be denoted by \( H = \partial_t - \Delta_x \). Given a number \( s \in (0, 1) \) the notation \( H^s \) will indicate the fractional power of \( H \) that in [48, formula (2.1)] was defined on a function \( f \in \mathcal{S}(\mathbb{R}^{n+1}_+) \) by the formula

\[
H^s f(x, t) = -s \frac{1}{\Gamma(1-s)} \int_0^\infty \tau^{1+s} \left( P_{\tau}^H f(x, t) - f(x, t) \right) d\tau,
\]

with the understanding that we have chosen the principal branch of the complex function \( z \to z^s \). We then introduce the natural domain for the operator \( H^s \).

\[
\mathcal{H}^{2s} = \text{Dom}(H^s) = \{ f \in \mathcal{S}'(\mathbb{R}^{n+1}_+) \mid f, H^s f \in L^2(\mathbb{R}^{n+1}_+) \} = \left\{ f \in L^2(\mathbb{R}^{n+1}_+) \mid (\xi, \sigma) \rightarrow (4\pi^2|\xi|^2 + 2\pi i \sigma)^s \hat{f}(\xi, \sigma) \in L^2(\mathbb{R}^{n+1}) \right\},
\]

where the second equality is justified by (2.1) and Plancherel theorem. It is important to keep in mind that definition (2.1) is equivalent to the one based on Balakrishnan formula (see [49, (9.63) on p. 285])

\[
H^s f(x, t) = -s \frac{1}{\Gamma(1-s)} \int_0^\infty \frac{1}{\tau^{1+s}} \left( P_{\tau}^H f(x, t) - f(x, t) \right) d\tau,
\]
where we have denoted by
\[ P^H_t f(x, t) = \int_{\mathbb{R}^n} G(x - y, t) f(y, t) dy = G(\cdot, t) * f(\cdot, t - \tau)(x) \] (2.4)
the *evolutive semigroup*, see [49, (9.58) on p. 284]. We refer to Section 3 in [11] for relevant
details.

Henceforth, given a point \((x, t) \in \mathbb{R}^{n+1}\) we will consider the thick half-space \(\mathbb{R}^{n+1} \times \mathbb{R}^+_y\).
At times it will be convenient to combine the additional variable \(y > 0\) with \(x \in \mathbb{R}^n\) and
denote the generic point in the thick space \(\mathbb{R}^+_x \times \mathbb{R}^+_y\) with the letter \(X = (x, y)\). For \(x_0 \in \mathbb{R}^n\)
and \(r > 0\) we let \(B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}, \mathbb{B}_r(x_0, 0) = \{X = (x, y) \in \mathbb{R}^n \times \mathbb{R}^+ \mid
|x - x_0|^2 + y^2 < r^2\}\) (notice that this is the upper half-ball). When the center \(x_0\) of \(B_r(x_0)\) is not
explicitly indicated, then we are taking \(x_0 = 0\). Similar agreement for the thick half-balls \(\mathbb{B}_r(x_0, 0)\). We will also use the \(Q_r\) for the set \(\mathbb{B}_r \times (t_0 - r^2, t_0 + r^2)\) and \(Q_{r, t}\) for the set \(B_r \times (t_0 - r^2, t_0 + r^2)\). For notational ease \(\partial U\) and \(\text{div } U\) will respectively refer to the
quantities \(\partial X U\) and \(\text{div } X U\). The partial derivative in \(t\) will be denoted by \(\partial_t U\) and also at
times by \(U_t\). The partial derivative \(\partial_y U\) will be denoted by \(U_t\). At times, the partial derivative \(\partial_y U\) will be denoted by \(U_{n+1}\).

We next introduce the extension problem associated with \(H^s\). Given a number \(a \in (-1, 1)\) and a \(u : \mathbb{R}^n_x \times \mathbb{R}^t \to \mathbb{R}\) we seek a function \(U : \mathbb{R}^n_x \times \mathbb{R}^t \times \mathbb{R}^+_y \to \mathbb{R}\) that satisfies the boundary-value problem
\[
\begin{align*}
L_a U &\overset{\text{def}}{=} \partial_t (y^a U) - \text{div}(y^a \nabla U) = 0, \\
U((x, t), 0) &= u(x, t), \quad (x, t) \in \mathbb{R}^{n+1}. \tag{2.5}
\end{align*}
\]

The most basic property of the Dirichlet problem (2.5) is that if \(s = \frac{1 - a}{2} \in (0, 1)\) and
\(u \in \text{Dom}(H^s)\), then we have the following convergence in \(L^2(\mathbb{R}^{n+1})\)
\[
2^{-a} \frac{\Gamma \left( \frac{1-a}{2} \right)}{\Gamma \left( \frac{1+a}{2} \right)} \partial_y^a U((x, t), 0) = -H^s u(x, t), \tag{2.6}
\]
where \(\partial_y^a\) denotes the weighted normal derivative
\[
\partial_y^a U((x, t), 0) \overset{\text{def}}{=} \lim_{y \to 0^+} y^a \partial_y U((x, t), y). \tag{2.7}
\]

Throughout our discussion, we let
\[
a = 1 - 2s. \tag{2.8}
\]

When \(a = 0\) (\(s = 1/2\)) the problem (2.5) was first introduced in [30] by Frank Jones, who
in such case also constructed the relevant Poisson kernel and proved (2.6). More recently
Nyström and Sande in [43] and Stinga and Torrea in [50] have independently extended the
results in [30] to all \(a \in (-1, 1)\).

Therefore, if \(u\) is a solution to (1.2), then it follows that the extended \(U\) is a weak solution to
\[
\begin{align*}
L_a U &= 0 \quad \text{in } \mathbb{R}^{n+1} \times \mathbb{R}^+_y, \\
U((x, t), 0) &= u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^{n+1}, \\
\partial U((x, t), 0) &= V(x, t) u(x, t) \quad \text{for } (x, t) \in \mathbb{R}^{n+1}. \tag{2.9}
\end{align*}
\]
We notice that the \(V\) in (2.9) differs from the \(V\) in (1.2) by a multiplicative constant. For the
precise notion of weak solutions to (2.9) above, we refer to [11, Definition 4.3]. For notational
purposes it will be convenient to work with the following backward version of problem (2.9)

\[
\begin{cases}
y^a \partial_t U + y^a \text{div}(y^a \nabla U) = 0 \text{ in } \{y > 0\}, \\
U((x, t), 0) = u(x, t) \\
\partial_y^a U((x, t), 0) = V(x, t)u(x, t).
\end{cases}
\]

(2.10)

We notice that the former can be transformed into the latter by changing \( t \rightarrow -t \).

We now record a regularity result which is crucial to our analysis. Prior to that, we would like to mention that for relevant notions of parabolic \( C^k \) and \( C^{k, \alpha} \) spaces, we refer the reader to chapter 4 in [36]. Before proceeding further, we make the following discursive remark.

**Remark 2.1** From now on, by a universal constant \( C \), we refer to a constant that depends only on \( n \) and \( s \).

**Lemma 2.2** Let \( U \) be a weak solution of (2.10) where \( V \) satisfies (1.3). Then there exists \( \alpha' > 0 \) such that one has up to the thin set \( \{y = 0\} \)

\[
U_i, \ U_t, \ y^a U_y \in C^{\alpha'}_{loc}.
\]

Moreover, the relevant Hölder norms over a compact set \( K \) are bounded by \( \int U^2 y^a dX dt \) over a larger set \( K' \) which contains \( K \). We also have that \( \nabla^2 U \in C^{\alpha'}_{loc} \) up to the thin set \( \{y = 0\} \).

Furthermore, when \( V \in C^2 \), we have that the following estimate holds for \( i, j = 1, \ldots, n \)

\[
\int_{B_1 \times (-1, 0]} (U_i^2 + U_t^2)y^a + \int_{B_1 \times (-1, 0]} |\nabla U_i|^2 y^a + \int_{B_1 \times (-1, 0]} |\nabla U_{ij}|^2 y^a \\
\leq C(1 + ||V||_{C^2}) \int_{B_2 \times (-4, 0]} U^2 y^a,
\]

(2.11)

where \( C \) is some universal constant.

**Proof** Notice that in the case when \( s = 1/2 \) where the extension operator is the heat operator, it follows from the classical theory as in Chapter 6 in [36]. Therefore we only focus on the case \( s > 1/2 \). In such a case, by arguing as in the proof of Lemma 5.5 in [11] using repeated incremental quotients, we deduce that for each \( i = 1, \ldots, n, U_i = w \) solves

\[
\begin{cases}
\text{div}(y^a \nabla w) + y^a w_t = 0, \\
\partial_y^a w = V w + V_i U = f \in L^\infty_{loc}.
\end{cases}
\]

(2.12)

We can then apply the arguments in [2] (see also [16]) based on compactness methods to assert that \( \nabla w \in H^\alpha \) for some \( \alpha' > 0 \) which implies the desired conclusion.

Now when \( V \) is additionally \( C^2 \), then we can take further incremental quotients and finally assert that \( w = U_{ij} \) solves

\[
\begin{cases}
\text{div}(y^a \nabla w) + y^a w_t = 0, \\
\partial_y^a w = V w + V_i U_j + V_j U_i + V_{ij} U.
\end{cases}
\]

(2.13)

Similarly \( w = U_t \) solves

\[
\begin{cases}
\text{div}(y^a \nabla w) + y^a w_t = 0, \\
\partial_y^a w = V w + V_t U.
\end{cases}
\]

(2.14)
Also since $V$ is $C^2$ in time, we can likewise assert that $w = U_{tt}$ solves
\begin{equation}
\begin{cases}
\nabla(y^a \nabla w) + y^a w_t = 0, \\
\partial_y^a w = V w + V_t U_t + V_{tt} U,
\end{cases}
\end{equation}
and moreover
\begin{equation}
\|U_{tt}\|_{L^\infty(\mathbb{B}_1 \times (-1, 0))} \leq C (1 + \|V\|_{C^2}) \|y^{a/2} U\|_{L^2(\mathbb{B}_2 \times (-4, 0))}
\end{equation}
Thus from the energy estimate as in the proof of Theorem 5.1 in [11], the estimates for $\nabla U, \nabla^2 U, U_t$ in terms of $\|y^{a/2} U\|_{L^2(\mathbb{B}_2 \times (-4, 0))}$ and also by using (2.16), we find that (2.11) follows.

We now state and prove an elementary Rellich type identity that is required in our analysis. This can be regarded as a slight variant of the one in [17].

**Lemma 2.3** (Rellich type identity) Let $F$ be a smooth function with supp($F$) $\subset (\mathbb{B}_R \setminus \{0\}) \times (0, 1)$. Then for any $k$ we have
\begin{equation}
\int |X|^k \nabla(y^a \nabla F) (\nabla F, X) = -k \int |X|^{k-2} (\nabla F, X)^2 y^a + \frac{(n + a - 1 + k)}{2} \int |X|^k |\nabla F|^2 y^a - \int_{\{y = 0\}} \int |x|^k \partial_x^a F (\nabla F, x)
\end{equation}
and
\begin{equation}
\int |X|^k \nabla(y^a \nabla F) F_t = -k \int |X|^{k-2} F_t (\nabla F, X) y^a - \int_{\{y = 0\}} \int |x|^k \partial_x^a F F_t
\end{equation}

**Proof** For notational convenience, we let $r = |X|$ throughout the proof. Since supp($F$) $\subset \mathbb{B}_R \setminus \{0\} \times (0, 1)$, we will only have the boundary term at $\{y = 0\}$. Now by integrating by parts, we have
\begin{align*}
\int \nabla(y^a \nabla F) r^k (\nabla F, X) &= - \int \nabla F, \nabla (r^k \nabla F, X)) y^a - \int r^k \partial_x^a F (\nabla F, x) \\
&= - \int \nabla F, \nabla (r^k \nabla F, X) y^a - \int r^k \nabla F, \nabla (\nabla F, X) y^a - \int r^k \partial_x^a F (\nabla F, x) \\
&= -k \int \langle \nabla F, X \rangle^2 r^{k-2} y^a - \int r^k |\nabla F|^2 y^a - \int r^k \langle \nabla F, \nabla^2 F X \rangle y^a - \int r^k \partial_x^a F (\nabla F, x) \\
&= -k \int \langle \nabla F, X \rangle^2 r^{k-2} y^a - \int r^k |\nabla F|^2 y^a - \frac{1}{2} \int r^k \langle \nabla F, \nabla (|\nabla F|^2) \rangle y^a - \int r^k \partial_x^a F (\nabla F, x).
\end{align*}
By applying integration by parts to the term
\begin{equation}
\frac{1}{2} \int r^k \langle X, \nabla (|\nabla F|^2) \rangle y^a
\end{equation}
and by using $< X, \nabla r^k > = kr^k$, lim$_{\gamma \to 0}$ $y^{1+a} r^k |\nabla F|^2 = 0$, we get
\begin{equation}
\int \nabla(y^a \nabla F) r^k (\nabla F, X)
\end{equation}
We now apply Young's inequality

\[ -k \int \langle \nabla F, X \rangle^2 r^{k-2} y + \int r^k |\nabla F|^2 y + \frac{k}{2} \int r^k (|\nabla F|^2) y \]

\[ + \frac{n+a+1}{2} \int r^k |\nabla F|^2 y - \int r^k \partial_y^a F \langle \nabla_x F, x \rangle \]

This completes the proof of (2.17).

Now for the proof of (2.18), we again apply integration by parts to get

\[ \int r^k \text{div}(y^a \nabla F) F_t = - \int \langle \nabla (r^k F_t), \nabla F \rangle y^a - \int_{\{y=0\}} r^k \partial_y^a F F_t \]

\[ = -k \int r^{k-2} F_t \langle \nabla F, X \rangle y^a - \int r^k \langle \nabla F_t, \nabla F \rangle y^a - \int_{\{y=0\}} r^k \partial_y^a F F_t \]

\[ = -k \int r^{k-2} F_t \langle \nabla F, X \rangle y^a - \frac{1}{2} \int \partial_t (r^k |\nabla F|^2 y) - \int_{\{y=0\}} r^k \partial_y^a F F_t. \]

Now using the fundamental theorem of calculus in the \( t \)-variable, we deduce that the second integral on the right hand side in the expression above is zero which consequently finishes the proof of (2.18).

\[ \square \]

In order to derive Theorem 1.3 from Theorem 1.1, we will need the following interpolation inequality. The proof of such an inequality is inspired by some ideas developed in [47].

**Lemma 2.4** Let \( s \in (0, 1) \) and \( f \in C_0^1(\mathbb{R}^n \times \mathbb{R}_+) \). Then there exists a universal constant \( C \) such that for any \( 0 < \eta < 1 \) the following holds

\[ ||\nabla_x f||_{L^2(\mathbb{R}^n)} \leq C \eta^s \left( ||y^{a/2} \nabla_x f||_{L^2(\mathbb{R}^n \times \mathbb{R}_+)} + ||y^{a/2} \nabla_x f||_{L^2(\mathbb{R}^n \times \mathbb{R}_+)} \right) + C \eta^{-1} ||f||_{L^2(\mathbb{R}^n)}. \]

(2.19)

In particular when \( n = 1 \), we get

\[ ||f_t||_{L^2(\mathbb{R})} \leq C \eta^s \left( ||y^{a/2} \partial_y f_t||_{L^2(\mathbb{R} \times \mathbb{R}_+)} + ||y^{a/2} f_t||_{L^2(\mathbb{R} \times \mathbb{R}_+)} + ||y^{a/2} f_t||_{L^2(\mathbb{R} \times \mathbb{R}_+)} \right) + C \eta^{-1} ||f||_{L^2(\mathbb{R})}. \]

(2.20)

**Proof** In the proof, we use the notation \( \langle \xi \rangle := \sqrt{1 + |\xi|^2} \). From Plancherel theorem, we have

\[ \int_{\mathbb{R}^n} |\nabla_x f|^2 = \int_{\mathbb{R}^n} |\nabla_x f|^2. \]

(2.21)

Now we write \( |\nabla_x f|^2 = \left( \langle \xi \rangle^{2s} |\nabla_x f|^2 \right)^{1-s} \left( \langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2 \right)^s \), then (2.21) becomes

\[ \int_{\mathbb{R}^n} |\nabla_x f|^2 = \int_{\mathbb{R}^n} \left( \langle \xi \rangle^{2s} |\nabla_x f|^2 \right)^{1-s} \left( \langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2 \right)^s. \]

(2.22)

We now apply Young’s inequality

\[ AB \leq \frac{\mu^p A^p}{p} + \frac{B^q}{p^q q}, \]

(2.23)
with $A = \left(\langle \xi \rangle^{2s} |\nabla_x f|^2\right)^{1-s}$, $B = \left(\langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2\right)^s$, $p = 1/(1 - s)$ and $q = 1/s$ in the right hand side of (2.22) to get

$$
\int_{\mathbb{R}^n} |\nabla_x f|^2 \leq (1-s)\mu^{1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\nabla_x f|^2 + s\mu^{-1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2.
$$

(2.24)

Now we estimate the second term in the right hand side of (2.24). We first use $|\nabla_x f|^2 = |\xi|^2 |\hat{f}|^2 \leq \langle \xi \rangle^2 |\hat{f}|^2$ to get

$$
s\mu^{-1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2 = s\mu^{-1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\hat{f}|^2 \left(|\hat{f}|^{2-2s}\right).
$$

(2.25)

Then we again apply the Young’s inequality (2.23) with $\mu = \epsilon$, $A = (\langle \xi \rangle |\hat{f}|)^{2s}$, $B = |\hat{f}|^{2-2s}$, $p = 1/s$, and $q = 1/(1 - s)$ to obtain

$$
s\mu^{-1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2 \leq s^2 \mu^{-1/s} \epsilon^{1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^2 |\hat{f}|^2 + s(1-s)\mu^{-1/s} \epsilon^{-1/s} \int_{\mathbb{R}^n} |\hat{f}|^2.
$$

(2.26)

Since $|\nabla_x f| = |\xi||\hat{f}|$, therefore $\langle \xi \rangle^2 |\hat{f}|^2 = |\hat{f}|^2 + |\nabla_x f|^2$. We now choose $\epsilon$ such that

$$
s^2 \mu^{-1/s} \epsilon^{1/s} = \frac{1}{2} \implies \epsilon = \frac{\mu}{(2s^2)^s}.
$$

(2.27)

By substituting this value of $\epsilon$ in (2.26), we get

$$
s\mu^{-1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{-2(1-s)} |\nabla_x f|^2 \leq \frac{1}{2} \int_{\mathbb{R}^n} |\nabla_x f|^2 + \frac{1}{2} \int_{\mathbb{R}^n} |\hat{f}|^2 + s(1-s)\mu^{-1/s} \mu^{-1/s} \int_{\mathbb{R}^n} |\hat{f}|^2.
$$

(2.28)

Then by using (2.28) in (2.24) we find

$$
\frac{1}{2} \int_{\mathbb{R}^n} |\nabla_x f|^2 \leq (1-s)\mu^{1/s} \int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\nabla_x f|^2 + \frac{1}{2} \int_{\mathbb{R}^n} |\hat{f}|^2 + (1-s)^2 \mu^{1/s} s^{1/s} \mu^{-1/s} \int_{\mathbb{R}^n} |\hat{f}|^2.
$$

(2.29)

Now using the trace inequality as in Lemma 4.4 in [47], we can estimate $\int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\nabla_x f|^2$ as follows

$$
\int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |\nabla_x f|^2 \leq C \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla \nabla_x f|^2 y^a + C \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla_x f|^2 y^a.
$$

(2.30)

The conclusion follows by employing the estimate (2.30) in (2.29) and subsequently by letting $\mu^{1/(1-s)} = \eta^s$. □

In our analysis, we will require the following “surface” trace inequality which can be found in [45, Lemma 3.1].
Lemma 2.5 (Surface trace inequality) Let $g : S^n_+ := \{X = (x, y) : |x| = 1, y > 0\} \to \mathbb{R}$ be smooth. Then there exists $C = C(n, s) > 0$ such that for all $\tau > 1$, one has
\[
\|g\|_{L^2(S^{n-1})} \leq C \left( \tau^{1-s} \left\| \omega_n^{\alpha/2} g \right\|_{L^2(S^n_+)} + \tau^{-s} \left\| \omega_n^{\alpha/2} \nabla_{S^n_+} g \right\|_{L^2(S^n_+)} \right).
\] (2.31)
Here $S^{n-1} = \{(x, 0) : |x| = 1\}$ and $\nabla_{S^n_+} g$ is the Riemannian gradient of $g$.

We will also require the following Hardy inequality which can be found in [47, Lemma 4.6].

Lemma 2.6 If $\alpha \neq 1/2$ and if $u(x, y)$ vanishes for $y$ large, then
\[
\|y^{-\alpha} u\|_{L^2(\mathbb{R}^{n+1}_+)}^2 \leq \frac{4}{(2\alpha - 1)^2} \|y^{1-\alpha} u\|_{L^2(\mathbb{R}^{n+1}_+)}^2 + \frac{2}{2\alpha - 1} \lim_{\epsilon \to 0} \|y^{1/2 - \alpha} u\|_{L^2(\mathbb{R}^n \times \{\epsilon\})}^2.
\] (2.32)

3 Quantitative estimate in the bulk

3.1 Carleman estimate I

We first state and prove our main Carleman estimate. We use such an estimate to prove an upper bound on the vanishing order in the bulk. This is a generalization of the estimate in [23, Theorem 2]. However, the Carleman weight that we use is similar to the one used in [6, Theorem 2.1]. The proof of this estimate in (3.1) below is also partly inspired by ideas developed in [13].

Lemma 3.1 Given $R < 4$, let $\tilde{W}$ be a solution to
\[
\begin{cases}
\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t = g & \text{in } \mathbb{R}^{n+1} \times \mathbb{R}_y^+
\end{cases}
\] (3.1)
such that $\tilde{W}$ is compactly supported in $(\mathbb{R}^n_\gamma \\setminus \{0\}) \times (0, 1)$. Assume that $V \in C^1$. Then there exist universal constants $C = C(n, s)$ and $R_0$ such that for all $R < R_0$ and any
\[
\alpha > C \left( 1 + \|V\|_{C^1}^{1/2s} \right),
\] (3.2)
the following estimate holds
\[
\alpha^3 \int |X|^{-2\alpha - 4 + \epsilon} e^{2\alpha |X|^{\epsilon}} \tilde{W}^2 y^a \leq C \int |X|^{-2\alpha} e^{2\alpha |X|^{\epsilon}} g^2 y^{-\alpha},
\] (3.3)
where $\epsilon = \frac{1 - \alpha}{4}$.

Proof Again for notational convenience we let $r = |X|$ and also denote
\[
\|V\|_{C^1_{\epsilon, t}} := \|V\|_1.
\] (3.4)
We then set $W = r^{-\beta} e^{\alpha r^s} \tilde{W}$ where $\epsilon = (1 - \alpha)/4$ and $\beta$ will be chosen depending on $\alpha$ later. Since $\tilde{W} = r^\beta e^{-\alpha r^s} W$, from direct calculations we have
\[
g = \text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t
\]
\[
= (\beta(n + a - 1)r^{\beta - 2} - \alpha (2\beta + \epsilon + n + a - 1)r^{\beta + \epsilon - 2} + \alpha^2 e^{2\beta + 2\epsilon - 2}) y^a e^{-\alpha r^s} W.
\]
Now using the inequality \((A + B)^2 \geq A^2 + 2AB\) with
\[
A = 2(\beta r^{\beta-2} - \alpha e r^{\beta+\epsilon-2})e^{-\alpha r^\epsilon} \langle \nabla W, X \rangle y^a + r^\beta e^{-\alpha r^\epsilon} (\text{div}(y^a \nabla W) + y^a W_t),
\]
and with \(B\) being the rest of the terms in the expression for \(g\) above, we get
\[
\int r^{-2a} e^{2\alpha r^\epsilon} g^2 y^{-a} \geq \int r^{-2a} (2(\beta r^{\beta-2} - \alpha e r^{\beta+\epsilon-2})\langle \nabla W, X \rangle + r^\beta W_t)^2 y^a + 4\beta(\beta + n + a - 1) \int r^{2\beta-2} (\beta r^{\beta-2} - \alpha e r^{\epsilon-2}) \langle \nabla W, X \rangle y^a + 4\alpha(2\beta + \epsilon + n + a - 1) \int r^{2\beta-2\alpha+2\epsilon-2}(\beta r^{\beta-2} - \alpha e r^{\epsilon-2}) W\langle \nabla W, X \rangle y^a + 4 \int r^{2\beta-2\alpha}(\beta r^{\beta-2} - \alpha e r^{\epsilon-2}) \langle \nabla W, X \rangle \text{div}(y^a \nabla W)

- 2\alpha(2\beta + n + \epsilon + a - 1) \int r^{2\beta-2\alpha+\epsilon-2} W W_t y^a + 2 \alpha \langle \nabla W, X \rangle y^a + 2 \sqrt{2} \int r^{2\beta-2\alpha+2\epsilon-2} W W_t y^a + 2 \int r^{2\beta-2\alpha} \text{div}(y^a \nabla W) W_t.
\]

We now simplify each of the integrals separately. The ensuing computations are rather involved and therefore we divide the rest of the proof into the following steps. In what follows, the constant \(C = C(n, s)\) will be an all purpose universal constant that changes from line to line.

\textit{Step 1:} We first show that there exists a universal constant \(C = C(n, s)\) such that for all \(\alpha > C\) and \(R_0\) with
\[
16 R_0^\epsilon < \frac{1}{2},
\]
we have
\[
I_2 + I_3 + I_4 \geq 5\alpha^2 \epsilon^2 \int r^{-n-a-1+\epsilon} W^2 y^a.
\]

This is seen as follows. We first observe by an application of the divergence theorem that the following holds
\[
I_2 = 2\beta(\beta + n + a - 1) \int r^{2\beta-2} (\beta r^{\beta-2} - \alpha e r^{\epsilon-2}) \langle \nabla W^2, X \rangle y^a - 2\beta(\beta + n + a - 1) \int \text{div}(y^a r^{2\beta-2\alpha-2}(\beta r^{\beta-2} - \alpha e r^{\epsilon-2})) W^2

= -2\beta^2(\beta + n + a - 1)(2\beta - 2\alpha - 4 + n + a + 1) \int r^{2\beta-2\alpha-4} W^2 y^a

+ 2\alpha \beta \epsilon(\beta + n + a - 1)(2\beta - 2\alpha + \epsilon - 4 + n + a + 1) \int r^{2\beta-2\alpha+\epsilon-4} W^2 y^a.
\]
We now choose $\beta$ in terms of $\alpha$ in such a way so that the first term on the right hand side of (3.8) equals 0. This in turn forces $\beta$ to be related $\alpha$ in the following way

$$2\beta - 2\alpha - 4 + n + a + 1 = 0. \quad (3.9)$$

Therefore, after substituting the value of $\beta$ from (3.9) in (3.8), we conclude that there exists $C = C(n, s)$ such that

$$I_2 \geq (2\alpha^3 - C\alpha^2)\epsilon^2 \int r^{-n-a-1+\epsilon} W^2 y^a. \quad (3.10)$$

To estimate the $I_3$, we use (3.9) and $2W(\nabla W, X) = \langle \nabla W^2, X \rangle$ to get

$$I_3 = -2\alpha \epsilon (2\alpha + 2 + \epsilon) \int (\beta - \alpha \epsilon r^\epsilon) r^{-n-a-1+\epsilon} \langle \nabla W^2, X \rangle y^a \quad (3.11)$$

$$= 2\alpha \epsilon (2\alpha + 2 + \epsilon) \int \text{div}((\beta - \alpha \epsilon r^\epsilon) r^{-n-a-1+\epsilon} y^a X) W^2$$

$$= 2\alpha \epsilon (2\alpha + 2 + \epsilon) \left[ \beta \epsilon \int r^{-n-a-1+\epsilon} W^2 y^a - 2\alpha \epsilon^2 \int r^{-n-a-1+2\epsilon} W^2 y^a \right],$$

where in the second equality we have used divergence theorem. Now use (3.9) and $R < R_0$ to conclude that for $\alpha > 3$ the following holds

$$I_3 \geq (4\alpha^3 - C\alpha^2 - 12\alpha^3 \epsilon R_0^\epsilon)\epsilon^2 \int r^{-n-a-1+\epsilon} W^2 y^a. \quad (3.12)$$

We estimate $I_4$ in the similar fashion. We use (3.9) and $2W(\nabla W, X) = \langle \nabla W^2, X \rangle$ to write $I_4$ in the following manner

$$I_4 = 2\alpha^2 \epsilon^2 \int (\beta - \alpha \epsilon r^\epsilon) r^{-n-a-1+2\epsilon} \langle \nabla W^2, X \rangle y^a \quad (3.13)$$

$$= -2\alpha^2 \epsilon^2 \int \text{div}((\beta - \alpha \epsilon r^\epsilon) r^{-n-a-1+2\epsilon} y^a X) W^2$$

$$= -4\alpha^2 \beta \epsilon^3 \int r^{-n-a-1+2\epsilon} W^2 y^a + 6\alpha^3 \epsilon^4 \int r^{-n-a-1+3\epsilon} W^2 y^a,$$

where in the second equality we have used divergence theorem. Also, the second term in the right hand side of (3.13) is positive. Therefore using $\beta < \alpha$ (from (3.9)) and $r = |x| \leq R < R_0$, we obtain

$$I_4 \geq -4\alpha^3 \epsilon^3 R_0^\epsilon \int r^{-n-a-1+\epsilon} W^2 y^a. \quad (3.14)$$

On adding (3.10), (3.12) and (3.14), we find

$$I_2 + I_3 + I_4 \geq (6\alpha^3 - 2C\alpha^2 - 16\alpha^3 \epsilon R_0^\epsilon)\epsilon^2 \int r^{-n-a-1+\epsilon} W^2 y^a. \quad (3.15)$$

We now choose $\alpha > 3$ such that

$$6\alpha - 2C > \frac{11\alpha}{2}$$

and $R_0$ small enough such that

$$16R_0^\epsilon \epsilon < 1/2.$$
Consequently, (3.15) becomes

\[ I_2 + I_3 + I_4 \geq 5\alpha^2 \epsilon^2 \int r^{-n-\alpha+1+\epsilon} W^2 y^a. \]

This completes the proof of Step 1.

Step 2: We show that there exists \( C = C(n, s) \) such that for \( \alpha > C \) and \( R_0 \) sufficiently small such that

\[ CR_0^\epsilon \leq \frac{1}{4}, \quad (3.16) \]

we have

\[ I_5 \geq 4\beta(n + a - 1) \int r^{-n-a-1}(\nabla W, X)^2 y^a - 4\alpha\epsilon(n + a - 1 + \epsilon) \int r^{-n-a+1+\epsilon}(\nabla W, X)^2 y^a \]

\[ - 2\epsilon^2 \int e^{2\alpha r^\epsilon} r^{-2\alpha} g^{-1/a} - \frac{22}{5} \alpha^3 \epsilon^2 \int r^{-n-a-1+\epsilon} W^2 y^a - 2\alpha^2 \int r^{-n-a+1+\epsilon} V W^2 \]

\[ - 4\beta \int_{\{y = 0\}} r^{-n-a+1} W \langle \nabla x W, x \rangle + 4\alpha\epsilon \int_{\{y = 0\}} r^{-n-a+1+\epsilon} V W \langle \nabla x W, x \rangle. \]

This is seen as follows. We first notice that from (3.1) it follows that \( \partial_y^\alpha W = V W \). Using this along with (2.17), we obtain

\[ I_5 = 4 \int r^{2\beta - 2\alpha}(\beta r^{-2} - \alpha\epsilon r^{-\epsilon}) (\nabla W, X) \text{ div}(y^a \nabla W) \quad (3.17) \]

\[ = 4 \int r^{-n-a+3}(\beta r^{-2} - \alpha\epsilon r^{-\epsilon}) (\nabla W, X) \text{ div}(y^a \nabla W) \]

\[ = 4\beta(n + a - 1) \int r^{-n-a-1}(\nabla W, X)^2 y^a - 4\alpha\epsilon(n + a - 1 + \epsilon) \int r^{-n-a+1+\epsilon}(\nabla W, X)^2 y^a \]

\[ - 2\alpha^2 \int r^{-n-a+1+\epsilon} V W^2 y^a - 4\beta \int_{\{y = 0\}} r^{-n-a+1} W \langle \nabla x W, x \rangle \]

\[ + 4\alpha\epsilon \int_{\{y = 0\}} r^{-n-a+1+\epsilon} V W \langle \nabla x W, x \rangle. \]

We now estimate the integral

\[ \int r^{-n-a+1+\epsilon} V W^2 y^a. \]

Recall \( W = r^{-\beta} e^{\alpha r^\epsilon} \tilde{W} \), therefore we have \( \nabla W = r^{-\beta} e^{\alpha r^\epsilon} \nabla \tilde{W} - \beta r^{-\beta-2} e^{\alpha r^\epsilon} \tilde{W} X + \alpha \epsilon r^{-\beta+\epsilon-2} e^{\alpha r^\epsilon} X \tilde{W} \). Thus

\[ |\nabla W|^2 = \langle \nabla W, \nabla W \rangle \]

\[ = \langle r^{-\beta} e^{\alpha r^\epsilon} \nabla \tilde{W} - \beta r^{-\beta-2} e^{\alpha r^\epsilon} \tilde{W} X + \alpha \epsilon r^{-\beta+\epsilon-2} e^{\alpha r^\epsilon} X \tilde{W}, r^{-\beta} e^{\alpha r^\epsilon} \nabla \tilde{W} \]

\[ - \beta r^{-\beta-2} e^{\alpha r^\epsilon} \tilde{W} X + \alpha \epsilon r^{-\beta+\epsilon-2} e^{\alpha r^\epsilon} X \tilde{W} \rangle \]

\[ = r^{-2\beta} e^{2\alpha r^\epsilon} \langle |\nabla \tilde{W}|^2 - 2\beta r^{-2}(\nabla \tilde{W}, X) \tilde{W} + 2\alpha \epsilon r^\epsilon (\nabla \tilde{W}, X) \tilde{W} + \beta^2 r^{-2} \tilde{W}^2 \]

\[ - 2\alpha \beta \epsilon r^{-\epsilon-2} \tilde{W}^2 + \alpha^2 \epsilon^2 r^\epsilon \tilde{W}^2 \rangle. \]
Now we estimate this term by term. Notice that from (3.9),\(-2B - n - a + 1 + \epsilon = -2\alpha - 2 + \epsilon\) and using divergence theorem, we get

\[
\int r^{-2\alpha - 2 + \epsilon} e^{2ar^2} |\nabla \tilde{W}|^2 y^a \quad (3.20)
\]

\[
= -\int \text{div}(r^{-2\alpha - 2 + \epsilon} e^{2ar^2} y^a \nabla \tilde{W}) \tilde{W} - \int e^{2ar^2} r^{-2\alpha - 2 + \epsilon} V \tilde{W}^2 \quad (y=0)
\]

\[
= -\int r^{-2\alpha - 2 + \epsilon} e^{2ar^2} \text{div}(y^a \nabla \tilde{W}) \tilde{W} - \int (\nabla \tilde{W}, \nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2})) W_y^a
\]

\[
- \int e^{2ar^2} r^{-2\alpha - 2 + \epsilon} V \tilde{W}^2. \quad (y=0)
\]

Also from the fundamental theorem of calculus in the \(t\)-variable we have

\[
\int r^{-2\alpha - 2 + \epsilon} e^{2ar^2} \tilde{W} \tilde{W}_t y^a = \frac{1}{2} \int r^{-2\alpha - 2 + \epsilon} e^{2ar^2} (\tilde{W}^2)_t y^a = 0. \quad (3.21)
\]

From (3.20) and (3.21) it thus follows

\[
\int r^{-2\alpha - 2 + \epsilon} e^{2ar^2} |\nabla \tilde{W}|^2 y^a = -\int r^{-2\alpha - 2 + \epsilon} e^{2ar^2} (\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t) \tilde{W}
\]

\[
- \int (\nabla \tilde{W}, \nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2})) W_y^a - \int e^{2ar^2} r^{-2\alpha - 2 + \epsilon} V \tilde{W}^2. \quad (y=0)
\]

Now by applying Cauchy-Schwarz inequality, writing \(\tilde{W}\) in terms of \(W\) and also by using (3.1) and (3.9) we get

\[
\int (\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t) \tilde{W} r^{-2\alpha - 2 + \epsilon} e^{2ar^2} \leq \frac{1}{\alpha} \int e^{2ar^2} r^{-2\alpha} (\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t)^2 y^{-a}
\]

\[
+ \alpha \int r^{-2\alpha - 4 + 2\epsilon} e^{2ar^2} \tilde{W}^2 y^a
\]

\[
= \frac{1}{\alpha} \int e^{2ar^2} r^{-2\alpha} g^2 y^{-a} + \alpha \int r^{-\alpha - a - 1 + 2\epsilon} W^2 y^a.
\]

Now we estimate the second term in (3.20). We have

\[
\int (\nabla \tilde{W}, \nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2})) W_y^a = \frac{1}{2} \int (\nabla \tilde{W}^2, \nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2})) y^a
\]

\[
= -\frac{1}{2} \int \tilde{W}^2 \text{div}(\nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2})) y^a - \frac{1}{2} \int \tilde{W}^2 \lim_{y \to 0^+} ((r^{-2\alpha - 2 + \epsilon} e^{2ar^2}) y^a)
\]

\[
= -\frac{1}{2} \int \tilde{W}^2 \text{div}(\nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2}) y^a).
\]

Over here, we have used that

\[
\lim_{y \to 0^+} ((r^{-2\alpha - 2 + \epsilon} e^{2ar^2}) y^a) = 0.
\]

The later can be seen from the fact that the function \(r^{-2\alpha - 2 + \epsilon} e^{2ar^2}\) is radial and hence symmetric in the \(y\) variable across \(y = 0\). Now from direct calculations we have

\[
\text{div}(\nabla (r^{-2\alpha - 2 + \epsilon} e^{2ar^2}) y^a)
\]
Hence by substituting $\tilde{W} = r^\beta e^{-\alpha r^\epsilon} W$ and also using (3.9), we get for some $C = C(n, s)$ that the following holds

$$\int \langle \nabla \tilde{W}, \nabla (r^{-2\alpha-2+\epsilon} e^{2\alpha r^\epsilon}) \rangle \tilde{W} y^a \quad (3.22)$$

$$= -\frac{1}{2}(2\alpha + 2 - \epsilon)(2\alpha - \epsilon - n - a + 3) \int r^{-n-a-1+\epsilon} W^2 y^a$$

$$- \alpha \epsilon (-4\alpha - 5 + 3\epsilon + n + a) \int r^{-n-a-1+2\epsilon} W^2 y^a - 2\alpha^2 \epsilon^2 \int r^{-n-a-1+3\epsilon} W^2 y^a$$

$$\geq (-2\alpha^2 + C\alpha + CR_0^2 \alpha^2) \int r^{-n-a-1+\epsilon} W^2 y^a$$

$$\geq -\frac{5}{2} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a,$$

provided $\alpha$ is sufficiently large and $R_0$ is small enough such that

$$-2\alpha^2 + C\alpha + CR_0^2 \alpha^2 \geq -\frac{5}{2} \alpha^2.$$

This in turn is guaranteed by ensuring that $R_0$ satisfies (3.16). Hence from (3.20) to (3.22), it follows that for all $\alpha > C$ where $C$ is sufficiently large, we have

$$\int r^{-2\alpha-2+\epsilon} e^{2\alpha r^\epsilon} |\nabla \tilde{W}|^2 y^a \leq \frac{1}{\alpha} \int e^{2\alpha r^\epsilon} r^{-2\alpha} g^2 y^a$$

$$+ \frac{13}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a - \int_{\{y=0\}} r^{-n-a+1+\epsilon} \nabla W^2. \quad (3.23)$$

Now we consider the integrals corresponding to the second and the third term in (3.19). By integrating by parts, we obtain

$$-2\beta \int r^{-2\alpha-4+\epsilon} e^{2\alpha r^\epsilon} \langle \nabla \tilde{W}, X \rangle \tilde{W} y^a + 2\alpha \epsilon \int r^{-2\alpha-4+2\epsilon} e^{2\alpha r^\epsilon} \langle \nabla \tilde{W}, X \rangle \tilde{W} y^a$$

$$= \beta (-2\alpha - 3 + \epsilon + n + a) \int r^{-2\alpha-4+\epsilon} e^{2\alpha r^\epsilon} \tilde{W}^2 y^a - \alpha \epsilon (-2\alpha - 3 + 2\epsilon + n + a)$$

$$\int r^{-2\alpha-4+2\epsilon} e^{2\alpha r^\epsilon} \tilde{W}^2 y^a$$

$$\leq -\frac{3}{2} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a \quad \text{(using} R_0 < 1). \quad (3.24)$$

Now concerning the remaining terms in (3.19), the corresponding integrals can be estimated as follows

$$\int (\beta^2 r^{-2} \tilde{W}^2 - 2\alpha \beta \epsilon r^{-2} \tilde{W}^2 + \alpha^2 \epsilon^2 r^{-2} \tilde{W}^2) r^{-2\alpha-4+\epsilon} r^{-n-a+1+\epsilon} y^a$$

$$= \beta^2 \int r^{-n-a-1+\epsilon} W^2 y^a - 2\alpha \beta \epsilon \int r^{-n-a-1+2\epsilon} W^2 y^a + \alpha^2 \epsilon^2 \int r^{-n-a-1+3\epsilon} W^2 y^a$$

$$\leq \frac{11}{10} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a. \quad (3.25)$$
Hence from (3.19), (3.23), (3.24) and (3.25), we get
\[
\int r^{-n-a+1+\epsilon} |\nabla W|^2 y^a \leq \frac{1}{\alpha} \int e^{2ar^\epsilon} r^{-2a} g^2 y^{-a} + \frac{11}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a - \int r^{-n-a+1+\epsilon} VW^2. \tag{3.26}
\]

Using (3.26) in (3.18), we find that Step 2 follows.

**Step 3**: We now show that with the restrictions on \(\alpha\) and \(R_0\) as in Step 1 and Step 2, one has
\[
I_2 + I_3 + I_4 + I_5 \geq 4\beta(n + a - 1) \int r^{-n-a-1}(\nabla W, X)^2 y^a
- 4\alpha \epsilon(n + a - 1 + \epsilon) \int r^{-n-a-1+\epsilon} (\nabla W, X)^2 y^a + 2\epsilon^2 \int e^{2ar^\epsilon} r^{-2a} g^2 y^{-a}
+ \frac{3}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a - 5\alpha||V||_1 \int r^{-n-a+1} W^2. \tag{3.27}
\]

From (3.7) and (3.17) we obtain
\[
I_2 + I_3 + I_4 + I_5 \geq 4\beta(n + a - 1) \int r^{-n-a-1}(\nabla W, X)^2 y^a
- 4\alpha \epsilon(n + a - 1 + \epsilon) \int r^{-n-a-1+\epsilon} (\nabla W, X)^2 y^a + 2\epsilon^2 \int e^{2ar^\epsilon} r^{-2a} g^2 y^{-a}
+ \frac{3}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a - 2\alpha \epsilon^2 \int r^{-n-a+1+\epsilon} VW^2
- 4\beta \int r^{-n-a+1} VW(\nabla_x W, x) + 4\alpha \epsilon \int r^{-n-a+1+\epsilon} VW(\nabla_x W, x). \tag{3.28}
\]

Now the boundary integral which appears in (3.28) can be rewritten in the following way
\[
\int_{\{y=0\}} r^{-n-a+1} VW(\nabla_x W, x) = \frac{1}{2} \int_{\{y=0\}} r^{-n-a+1} V(\nabla_x W^2, x)
= -\frac{1}{2} \int_{\{y=0\}} \text{div}_x(Vx r^{-n-a+1}) W^2
= -\frac{1}{2} \int_{\{y=0\}} r^{-n-a+1} W^2(\nabla_x V, x) - \frac{1}{2} \int_{\{y=0\}} r^{-n-a+1} VW^2. \tag{3.29}
\]

Hence using (3.29) in (3.28), Step 3 follows.

**Step 4**: We claim that
\[
I_6 = I_7 = I_8 = 0. \tag{3.30}
\]

We only consider \(I_6\) since the arguments for \(I_7\) and \(I_8\) are the same. By integrating by parts in the \(t\)-variable, we notice that
\[
I_6 = -2\alpha \epsilon(2\beta + n + \epsilon + a - 1) \int r^{2\beta-2a+\epsilon-2} W y^a dt dX \tag{3.31}
\]
Similarly we have

\[ I_7 = I_8 = 0. \]

This finishes the proof of Step 4.

**Step 5:** We now show that

\[ I_1 + I_9 \geq 4(\beta^2 - \alpha^2) \int r^{-n-a-1} (\nabla W, X)^2 - ||V||_1 \int_{\{y=0\}} r^{-n-a+3} W^2. \tag{3.32} \]

To see this, using (2.18), we observe that

\[
I_9 = 2 \int r^{2\beta - 2\alpha} \text{div}(y^a \nabla W) W_t \\
= (4\alpha - 4\beta) \int r^{2\beta - 2\alpha - 2} W_t \langle \nabla W, X \rangle - 2 \int_{\{y=0\}} r^{2\beta - 2\alpha} V W W_t \\
= (4\alpha - 4\beta) \int r^{2\beta - 2\alpha - 2} W_t \langle \nabla W, X \rangle + \int_{\{y=0\}} r^{2\beta - 2\alpha} V W W_t \\
\geq (4\alpha - 4\beta) \int r^{2\beta - 2\alpha - 2} W_t \langle \nabla W, X \rangle - ||V||_1 \int_{\{y=0\}} r^{2\beta - 2\alpha} W^2, \tag{3.33}
\]

where in the second line, we integrated by parts in the \( t \)-variable.

The first integral on RHS \((4\alpha - 4\beta) \int r^{2\beta - 2\alpha - 2} W_t \langle \nabla W, X \rangle \) will be absorb in \( I_1 \). For that, we use the following algebraic identity

\[
(A + B + C)^2 + 2yAC = ((1 + y)A + B + C)^2 - y^2 A^2 - 2y A^2 - 2y AB,
\]

with \( A = 2\beta r^{\beta - 2} \langle \nabla W, X \rangle, B = -2\alpha \epsilon r^{\beta + \epsilon - 2} \langle \nabla W, X \rangle, C = r^\beta W_t \) and \( y = (\frac{\alpha}{\beta} - 1) \).

Thus we have

\[
I_1 + (4\alpha - 4\beta) \int r^{-2\alpha+2\beta-2} W_t \langle \nabla W, X \rangle \\
= \int r^{-2\alpha}(2\beta r^{\beta - 2} \langle \nabla W, X \rangle - 2\alpha \epsilon r^{\beta + \epsilon - 2} \langle \nabla W, X \rangle + r^\beta W_t)^2 + (4\alpha - 4\beta) \\
\int r^{-2\alpha+2\beta-2} W_t \langle \nabla W, X \rangle \\
= \int r^{-2\alpha}(2\alpha r^{\beta - 2} \langle \nabla W, X \rangle - 2\alpha \epsilon r^{\beta + \epsilon - 2} \langle \nabla W, X \rangle + r^\beta W_t)^2 + 4(\beta^2 - \alpha^2) \\
\int r^{-n-a-1} \langle \nabla W, X \rangle^2 \\
+ (8\alpha^2 \epsilon - 8\alpha \beta \epsilon) \int r^{-n-a-1+\epsilon} \langle \nabla W, X \rangle^2 \\
\geq 4(\beta^2 - \alpha^2) \int r^{-n-a-1} \langle \nabla W, X \rangle^2, \tag{3.34}
\]

where in the last inequality, we have used \(8\alpha^2 \epsilon - 8\alpha \beta \epsilon \geq 0\) which follows from (3.9). Using this in (3.33), we find that the estimate claimed in (3.32) follows.
Notice that from (3.9), $4(\beta^2 - \alpha^2) < 0$, therefore the term $4(\beta^2 - \alpha^2) \int r^{-n-a-1} \langle \nabla W, X \rangle^2$ in (3.32) above is unfavourable. However at this point we make the crucial and subtle observation that such a term can be absorbed in the term $4\beta(n + a - 1) \int r^{-n-a-1} \langle \nabla W, X \rangle^2 y^a$ that appears in (3.27) because

$$4(\beta^2 - \alpha^2) + 4\beta(n + a - 1) = 8\beta - (n + a - 3)^2 > 6\beta,$$

for all $\beta$. Now by taking $R_0$ small enough, we can ensure that the following term in (3.27), i.e. $4\alpha\epsilon(n + a - 1 + \epsilon) \int r^{-n-a-1-\epsilon} \langle \nabla W, X \rangle^2 y^a$ can be estimated in the following way

$$4\alpha\epsilon(n + a - 1 + \epsilon) \int r^{-n-a-1+\epsilon} \langle \nabla W, X \rangle^2 y^a < 3\beta \int r^{-n-a-1} \langle \nabla W, X \rangle^2 y^a. \tag{3.35}$$

More precisely, in order to ensure that (3.35) holds, we take $R_0$ such that

$$4\alpha\epsilon(n + a - 1 + \epsilon) R_0^\epsilon < 3\beta. \tag{3.36}$$

Hence from (3.27), (3.30), (3.32) and (3.35) we have for all $\alpha$ large

$$\int e^{2ar^s} r^{-2a} g_2 y^{-a} \geq -2\epsilon^2 \int e^{2ar^s} r^{-2a} g_2 y^{-a} + \frac{3\alpha^3\epsilon^2}{4} \int r^{-n-a-1+\epsilon} W^2 y^a \tag{3.37}$$

$$- 6\alpha \|V\|_1 \int_{\{y=0\}} r^{-n-a+1} W^2.$$

**Step 6 (Estimating the boundary contribution in (3.37)):**

Now we estimate the boundary integral $\int_{\{y=0\}} r^{-n-a+1} W^2$ as follows:

$$\int_{\{y=0\}} r^{-n-a+1} W^2 \leq C_T \tau^{2-2s} R_0^{1-a-\epsilon} \int r^{-n-a-1+\epsilon} W^2 y^a$$

$$+ C_T \tau^{-2s} R_0^{1-a-\epsilon} \left( \frac{1}{\alpha} \int e^{2ar^s} r^{-2a} g_2 y^{-a} + \frac{11}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a \right). \tag{3.38}$$

Using polar coordinates, we have

$$\int_{\{y=0\}} r^{-n-a+1} W^2 = \int_{\{y=0\}} \int_{S^{n-1}} r^{-a} W^2 (r\omega', 0) d\omega dr dt.$$

Now by applying the surface trace inequality in Lemma 2.5 and using $|\nabla W|^2 = W^2 + \frac{1}{r^2} |\nabla S^n W|^2$ which implies $|\nabla S^n W|^2 \leq r^2 |\nabla W|^2$, we get that there exists $C_T = C_T(n, a)$ such that for all $\tau > 1$

$$\int r^{-n-a+1} W^2 \leq C_T \tau^{2-2s} \int r^{-a} \int_{S^+} o_{n+1}^a W^2 (r\omega) + C_T \tau^{-2s} \int r^{-a} \int_{S^+} o_{n+1}^a |\nabla S^n W|^2$$

$$\leq C_T \tau^{2-2s} \int r^{-a} \int_{S^+} o_{n+1}^a W^2 (r\omega) + C_T \tau^{-2s} \int r^{-a} \int_{S^+} o_{n+1}^a |\nabla W (r\omega)|^2.$$

Now by transforming it back to Euclidean coordinates, we get

$$\int_{\{y=0\}} r^{-n-a+1} W^2 \leq C_T \tau^{2-2s} \int r^{-2a-n} W^2 y^a + C_T \tau^{-2s} \int r^{-2a-n+2} |\nabla W|^2 y^a. \tag{3.39}$$
\[ \leq C_T \tau^{2-2s} R_0^{1-a-\epsilon} \int r^{-n-a-1+\epsilon} W^2 y^a + C_T \tau^{-2s} R_0^{1-a-\epsilon} \int r^{-n-a+1+\epsilon} |\nabla W|^2 y^a. \]

In (3.39), we have used that since \( \epsilon = (1-a)/4 \) therefore \( 1-a-\epsilon > 0 \) which in particular implies that \( r^{1-a-\epsilon} \leq R_0^{1-a-\epsilon} \). Now using the estimate (3.26) in (3.39) we obtain

\[ \int_{\{y=0\}} r^{-n-a+1} W^2 \leq C_T \tau^{2-2s} R_0^{1-a-\epsilon} \int_{\{y=0\}} r^{-n-a-1+\epsilon} W^2 y^a + C_T \tau^{-2s} R_0^{1-a-\epsilon} \left( \frac{1}{\alpha} \int e^{2ar^+} r^{-2a} y^{-a} + \frac{11}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a \right) + C_T \tau^{-2s} R_0^{1-a-\epsilon} ||V||_1 \int_{\{y=0\}} r^{-n-a+1+\epsilon} W^2. \]

We now take \( \tau \) such that \( \tau^{-2s} ||V||_1 < 1 \). More precisely, we let \( \tau = ||V||_1^{1/2s} + 1 \). Then by taking \( R_0 \) small enough such that

\[ C_T \tau^{-2s} R_0^{1-a-\epsilon} \leq \frac{1}{2}, \tag{3.40} \]

we can ensure that the term \( C_T \tau^{-2s} R_0^{1-a-\epsilon} ||V||_1 \int_{\{y=0\}} r^{-n-a+1+\epsilon} W^2 \) can be absorbed in the left hand side of the above inequality and thus the estimate (3.38) claimed in Step 6 thus follows.

**Step 7 (Conclusion):**

Using (3.38) in (3.37) and also that \( \tau = ||V||_1^{1/2s} + 1 \), we get for \( R_0 \) small enough satisfying (3.6), (3.16), (3.36) and (3.40) that the following holds

\[
2 \int e^{2ar^+} r^{-2a} g^2 y^{-a} \\
\geq \frac{3}{5} \alpha^2 \int r^{-n-a-1+\epsilon} W^2 y^a - 6\alpha C_T \tau^2 R_0^{1-a-\epsilon} \int r^{-n-a-1+\epsilon} W^2 y^a \\
- 6\alpha C_T R_0^{1-a-\epsilon} \left( \frac{1}{\alpha} \int e^{2ar^+} r^{-2a} g^2 y^{-a} + 2.2 \alpha^2 \int W^2 r^{-n-a-1+\epsilon} \right). \tag{3.41}
\]

Now by letting \( \alpha \geq \tau + C \) for a sufficiently large \( C \) and finally by rewriting \( W \) in terms of \( \tilde{W} \) we deduce from (3.41) that for sufficiently small \( R_0 \) the following inequality holds

\[
\alpha^3 \int r^{-2a-4+\epsilon} \tilde{W}^2 y^a < C \int r^{-2a} e^{2ar^+} g^2 y^{-a},
\]

which completes the proof. □

### 3.2 Vanishing order estimate in the bulk

Now given that the Carleman estimate (3.3) in Lemma 3.1 holds for all \( \alpha > C(n, s) (1 + ||V||_1^{1/2s}) \), we can now argue as in the proof of Theorem 1.2 in [13] or as in the proof of Theorem 15 in [51], to assert that the following quantitative vanishing order estimate in the bulk holds. We provide the details for the sake of completeness.
Lemma 3.2 Let $U$ be a weak solution of (2.10) in $\mathbb{B}_4 \times (-16, 16)$. Then there exists $C$ universal such that for all $\rho < R_0/8$ where $R_0$ is as in Lemma 3.1, we have

$$\int_{\mathbb{B}_\rho \times (0, 1)} U^2 y^a \geq C \rho^A,$$

where $A = C ||V||_{C^{1/2}_C} + C \left( 1 + \int_{\mathbb{B}_R \times (0, 1)} y^a U^2 \right) / \int_{\mathbb{B}_{R_0/4} \times (1/4, 3/4)} y^a U^2 + C$.

Proof In the proof, we will denote an all purpose constant by the letter $C$, which might vary from line to line, and will depend only on $n, s, R_0$ and $T$. For simplicity of the computations and notational convenience we work with the symmetric time-interval $(-T, T)$, instead of $(0, T)$, in our case $T = 1$. For notational convenience, we denote $|X|$ by $r$ and we define the following sets:

$$\hat{A}(r_1, r_2) := \{(x, y) \in \mathbb{R}^n \times \{y > 0\} : r_1 < |X| < r_2\}$$

$$A(r_1, r_2) := \{(x, 0) : r_1 < |x| < r_2\}.$$

Let $R_0$ be as in the Lemma 3.1. Let $0 < r_1 < r_2/2 < 2r_2 = R_0/2$ be fixed, and let $\phi(X)$ be a smooth radial function such that $\phi(X) \equiv 0$ in $\mathbb{R}_{r_1} \cup \mathbb{B}_{2r_2}$ and $\phi(X) \equiv 1$ in $\hat{A}(r_1, r_2)$.

We now let $T_1 = 3T/4$ and $T_2 = T/2$, so that $0 < T_2 < T_1 < T$. As in [51], we let $\eta(t)$ be a smooth even function such that $\eta(t) \equiv 1$ when $|t| < T_2$, $\eta(t) \equiv 0$, when $|t| > T_1$. Furthermore, it will be important in the sequel (see (3.42) below) that $\eta$ decay exponentially near $t = \pm T_1$. As in (118) of [51] we take

$$\eta(t) = \begin{cases} 
0 & -T_1 \leq t \leq T_1 \\
\exp \left( -\frac{r^3(T_1+t)^2}{(T_1-t)^3(T_1-t)^3} \right) & -T_1 \leq t \leq T_2, \\
1, & T_2 \leq t \leq 0.
\end{cases}$$

(3.42)

Without loss of generality, we will assume that

$$\int_{\mathbb{B}_{r_2} \times (-T_2, T_2)} U^2 y^a \neq 0.$$  

(3.43)

Otherwise, $U \equiv 0$ in $\mathbb{B}_{r_2} \times (-T_2, T_2)$, which by Theorem 15 (b) in [51] implies $U \equiv 0$ in $\mathbb{B}_{R_0} \times (-T, T)$ and by the assertions that follow we could conclude $U \equiv 0$ in $\mathbb{B}_{R_0} \times (-T, T)$.

We then let $\tilde{W} = \phi \eta U$. Since $U$ is a solution of (2.10) and $\phi$ is a radial function, we have $\partial_y \phi = 0$ on the thin set $\{y = 0\}$ and therefore we find that $\tilde{W}$ solves

$$\begin{cases} 
\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t = 2\eta y^a (\nabla U, \nabla \phi) + \text{div}(y^a \nabla \phi) \eta U + y^a \phi U \eta_t & \text{in } \{y > 0\}, \\
\partial_y \tilde{W} = V \tilde{W} & \text{on } \{y = 0\}.
\end{cases}$$

Since $\eta \phi$ is compactly supported in $(\mathbb{B}_{R_0} \setminus \{0\}) \times (-T, T)$, $\tilde{W}$ is compactly supported in $(\mathbb{B}_{R_0} \setminus \{0\}) \times (-T, T)$. Therefore we apply the Carleman estimate (3.3) to $\tilde{W}$, obtaining

$$\alpha^3 \int r^{-2a-4+\epsilon} e^{2ar^\epsilon} \tilde{W}^2 y^a \leq C \int r^{-2a} e^{2ar^\epsilon} (2\eta) y^a (\nabla U, \nabla \phi)$$

$$+ \text{div}(y^a \nabla \phi) \eta U + y^a \phi U \eta_t)^2 y^{-a}.$$
As a consequence of the algebraic inequality \((A + B + C)^2 \leq 3(A^2 + B^2 + C^2)\), we get
\[
\alpha^3 \int r^{-2\alpha-4+\epsilon} e^{2ar^\epsilon} \bar{W}^2 y^a \leq C \int r^{-2\alpha} e^{2ar^\epsilon} (\eta^2 |\nabla U|^2 |\nabla \phi|^2 y^a + (\text{div}(y^a \nabla \phi))^2 \eta^2 U^2 y^{-a} + (\phi U \eta)^2 y^a).
\] (3.44)

We now recall that the way \(\phi\) and \(\eta\) have been chosen, \(\nabla \phi\) is supported in \(\mathcal{A}(r_1, r_2)\) and in this set \(|\nabla \phi(X)| = |\phi'(x)|/|X| = O(1/r)\) and \(|\nabla^2 \phi(X)| = O(1/r^2)\), which gives
\[
|\text{div}(y^a \nabla \phi)| = |\Delta \phi y^a + ay^{-1} \phi| = |\Delta \phi y^a + ay^{-1}(y\phi')/r| = y^a O(1/r^2),
\]
hence (3.44) becomes
\[
\alpha^3 \int r^{-2\alpha-4+\epsilon} e^{2ar^\epsilon} \bar{W}^2 y^a \leq C \int_{\mathcal{A}(\frac{r_1}{2}, r_1) \times (-T_1, T_1)} e^{2ar^\epsilon} (r^{-2\alpha-2} |\nabla U|^2 y^a + r^{-2\alpha-4} U^2 y^a)
+ C \int_{\mathcal{A}(r_2, 2r_2) \times (-T_1, T_1)} e^{2ar^\epsilon} (r^{-2\alpha-2} |\nabla U|^2 y^a + r^{-2\alpha-4} U^2 y^a)
+ C \int_{\mathcal{A}(\frac{r_1}{2}, 2r_2) \times (-T_1, T_1)} r^{-2\alpha} e^{2ar^\epsilon} \phi^2 \eta^2 U^2 y^a = I + II + III.
\] (3.45)

**Step 1 (Estimating \(I\) and \(II\))**: There exists universal \(\bar{C}\) such that the following estimates hold:
\[
I = C \int_{\mathcal{A}(\frac{r_1}{2}, r_1) \times (-T_1, T_1)} e^{2ar^\epsilon} (r^{-2\alpha-2} |\nabla U|^2 y^a + r^{-2\alpha-4} U^2 y^a)
\leq \bar{C} \alpha^2 \left(\frac{r_1}{2}\right)^{-2\alpha-4} e^{2ar^\epsilon} \int_{\mathcal{A}(\frac{r_1}{2}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a,
\] (3.46)

and
\[
II = C \int_{\mathcal{A}(r_2, 2r_2) \times (-T_1, T_1)} e^{2ar^\epsilon} (r^{-2\alpha-2} |\nabla U|^2 y^a + r^{-2\alpha-4} U^2 y^a)
\leq \bar{C} \alpha^2 r_2^{-2\alpha-4} e^{2ar^\epsilon} \int_{\mathcal{A}(\frac{r_2}{2}, 4r_2) \times (-T, T)} U^2 y^a.
\] (3.47)

In order to prove (3.46) and (3.47), we first notice that since the functions
\[
r \rightarrow r^{-2\alpha-4} e^{2ar^\epsilon}, \quad r \rightarrow r^{-2\alpha-2} e^{2ar^\epsilon} \quad \text{and} \quad r \rightarrow r^{-2\alpha} e^{2ar^\epsilon}
\] (3.48)
are decreasing in \((0, 1)\), we can estimate the first and second integral in the right hand side of (3.45) in the following way
\[
\int_{\mathcal{A}(\frac{r_1}{2}, r_1) \times (-T_1, T_1)} e^{2ar^\epsilon} (r^{-2\alpha-2} |\nabla U|^2 y^a + r^{-2\alpha-4} U^2 y^a)
\]
We start by splitting the last term in the right-hand side of (3.45) as follows:

\[
III = C \int_{\mathcal{A}(\frac{1}{2}, r_1) \times (-T, T)} \frac{r_1}{2} - 2a - 2 e^{2ar_1^2} (3.45) \\
\leq C \left( \frac{r_1}{2} \right)^{-2a} - 2 \frac{2ar_1^2}{e^{2ar_1^2}} \int_{\mathcal{A}(\frac{1}{2}, r_1) \times (-T, T)} |\nabla U|^2 y^a \\
+ C \left( \frac{r_1}{2} \right)^{-2a} - 4 e^{2ar_1^2} \int_{\mathcal{A}(\frac{1}{2}, r_1) \times (-T, T)} U^2 y^a. (3.49)
\]

and

\[
\int_{\mathcal{A}(r_2, 2r_2) \times (-T, T)} e^{2ar_2} (r^{-2a} - 2 |\nabla U|^2 y^a + r^{-2a} - 4 U^2 y_a) (3.50)
\]

\[
\leq C r_2^{-2a} - 2 e^{2ar_2^2} \int_{\mathcal{A}(r_2, 2r_2) \times (-T, T)} |\nabla U|^2 y^a + C r_2^{-2a} - 4 e^{2ar_2^2} \int_{\mathcal{A}(r_2, 2r_2) \times (-T, T)} U^2 y^a.
\]

We then notice that the following energy estimate holds:

\[
\int_{\mathcal{A}(\frac{1}{2}, r_1) \times (-T, T)} |\nabla U|^2 y^a \leq \frac{C \alpha^2}{\frac{r_1}{2}} \int_{\mathcal{A}(\frac{1}{2}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a. (3.51)
\]

(3.51) follows from the energy estimate in the proof of Theorem 5.1 in [11] and also by using that \( \alpha = C \left( ||V||_{1/2s} + 1 \right) \).

Similarly we find

\[
\int_{\mathcal{A}(r_2, 2r_2) \times (-T, T)} |\nabla U|^2 y^a \leq \frac{C \alpha^2}{r_2^2} \int_{\mathcal{A}(\frac{1}{2}, \frac{3r_2}{2}) \times (-T, T)} U^2 y^a. (3.52)
\]

Using (3.51) in (3.49) we deduce that (3.46) follows. Similarly, using (3.52) in (3.50) we obtain (3.47). Thus we have estimated the first and second term in the right-hand side of (3.45) which finishes Step 1.

**Step 2 (Estimating III):** We now estimate the last term in the right-hand side of (3.45) in the following way.

\[
III = C \int_{\mathcal{A}(\frac{1}{2}, 2r_2) \times (-T, T)} r^{-2a} e^{2ar_2} \phi \eta^2 U^2 y^a \\
\leq C \left( \frac{r_1}{2C_1} \right)^{-2a} \int_{\mathcal{A}(\frac{1}{2}, r_1) \times (-T, T)} U^2 y^a + C r_2^{-2a} e^{2ar_2^2} \int_{\mathcal{A}(r_2, 2r_2) \times (-T, T)} U^2 y^a \\
+ C \frac{\alpha^2}{2} \int_{\mathcal{A}(r_1, 2r_2) \times (-T, T)} r^{-2a} - 4 e^{2ar_2^2} \tilde{W}^2 y^a + C \int_{\mathbb{B}_{90} \times (-T, T)} U^2 y^a. (3.53)
\]

We start by splitting the last term in the right-hand side of (3.45) as follows

\[
C \int_{\mathcal{A}(\frac{1}{2}, 2r_2) \times (-T, T)} r^{-2a} e^{2ar_2} \phi \eta^2 U^2 y^a = C \int_{\mathcal{A}(\frac{1}{2}, r_1) \times (-T, T)} r^{-2a} e^{2ar_2} \phi \eta^2 U^2 y^a (3.54)
\]
Using \( |\eta| \leq C/T, \phi \leq 1 \) and (3.48), we observe that the first term of the right hand side of (3.54) can be estimated as

\[
C \int_{\mathbb{R}(r_2, 2r_2) \times (-T_1, T_1)} r^{-2a} e^{2ar} \phi^2 \eta_i^2 U^2 y^a + C \int_{\mathbb{R}(r_1, 2r_1) \times (-T_1, T_1)} r^{-2a} e^{2ar} \phi^2 \eta_i^2 U^2 y^a.
\]

where \( C_1 = e^{R_0} \), which is a consequence of the facts that the exponential function is an increasing function and \( r_1/2 < R_0 \). Since \( R_0 \) is a universal constant, \( C_1 \) is a universal constant. Similarly, the last term in the right hand side of (3.54) can be upper bounded in the following way

\[
C \int_{\mathbb{R}(r_2, 2r_2) \times (-T_1, T_1)} r^{-2a} e^{2ar} \phi^2 \eta_i^2 U^2 y^a \leq C_2 r^{-2a} e^{2ar} \int_{\mathbb{R}(r_2, 2r_2) \times (-T_1, T_1)} U^2 y^a.
\]

Now by arguing as in the proof of [13, (3.12)], we can assert that the following estimate holds for the second term in the right-hand side of (3.54)

\[
C \int_{\mathbb{R}(r_1, 2r_2) \times (-T_1, T_1)} r^{-2a} e^{2ar} \phi^2 \eta_i^2 U^2 y^a \leq \frac{\alpha^3}{2} \int_{\mathbb{R}(r_1, 2r_1) \times (-T_1, T_1)} r^{-2a-4+\epsilon} e^{2ar} \tilde{W}^2 y^a + C \int_{\mathbb{B}_{R_0} \times (-T, T)} U^2 y^a.
\]

It is to be noticed that the proof of the estimate (3.12) in [13] is inspired by ideas in [51]. We nevertheless provide the details for the sake of completeness. Since \( \phi \equiv 1 \) in the region \( \{r_1 < |x| < r_2\} \), and the function \( \eta_i \) is supported in the set \((-T_1, -T_2) \cup (T_2, T_1)\), if we indicate \( Z = \mathbb{R}(r_1, r_2) \times \{(-T_2, -T_1) \cup (T_1, T_2)\} \), we can bound

\[
C \int_{\mathbb{R}(r_1, 2r_2) \times (-T_1, T_1)} r^{-2a} e^{2ar} U^2 \phi^2 \eta_i^2 y^a \leq \frac{\alpha^3}{2} \int_{\mathbb{R}(r_1, 2r_1) \times (-T_1, T_1)} r^{-2a-4+\epsilon} e^{2ar} \tilde{W}^2 y^a
\]

\[
\leq \int_Z r^{-2a-4+\epsilon} e^{2ar} U^2 \left( C r^{4-\epsilon} \frac{\eta_i^2}{\eta^2} - \frac{\alpha^3}{2} \right) y^a
\]

\[
\leq \int_Z r^{-2a-4+\epsilon} e^{2ar} U^2 \eta_i^2 \left( C r^{3} \frac{\eta_i^2}{\eta^2} - \frac{\alpha^3}{2} \right) y^a
\]

if \( \epsilon < < 1/4 \). (3.57) is then equivalent to proving the following estimate,

\[
\int_Z r^{-2a-4+\epsilon} e^{2ar} U^2 \eta_i^2 \left( C r^{3} \frac{\eta_i^2}{\eta^2} - \frac{\alpha^3}{2} \right) y^a \leq C \int_{\mathbb{B}_{R_0} \times (-T, T)} U^2 y^a.
\]

\( \mathbb{B}_{R_0} \times (-T, T) \)
The proof of (3.58) will be accomplished in several steps. First, we notice that it suffices to concern ourselves with the portion of the integral in the left-hand side of (3.58) over the region $Z^- = A(r_1, r_2) \times (-T_1, -T_2)$, since the estimate on $Z^+ = A(r_1, r_2) \times (T_2, T_1)$ is similar. Now, if $-T_1 \leq t \leq -T_2$, keeping in mind that $T_1 - T_2 = \frac{T}{4}, |T_2 + t| \leq T_1 - T_2 = \frac{T}{4}$, and that $\frac{3}{4} T \leq 4T_1 - 3T_2 + t \leq T$, from (3.42) a standard calculation shows

$$\left| \frac{\eta_t}{\eta} \right| = \left| \frac{T^3 (T_2 + t)^3 (4T_1 - 3T_2 + t)}{(T_1 - T_2)^4 (T_1 + t)^4} \right| \leq \frac{4T^3}{|T_1 + t|^4}.$$  

Using this estimate in the above inequality, we obtain

$$\int_{Z^-} e^{2ar^\varepsilon} r^{-2\alpha - 4 + \varepsilon} U^2 \eta^2 \left( Cr^3 \frac{\eta^2}{\eta^2} - \frac{\alpha^3}{2} \right) y^a \leq \int_{Z^-} e^{2ar^\varepsilon} r^{-2\alpha - 4 + \varepsilon} U^2 \eta^2 \left( Cr^3 \frac{T^6}{(T_1 + t)^8} - \frac{\alpha^3}{2} \right) y^a.$$  

Next, we write $Z^- = D \cup (Z^- \setminus D)$, where $D$ is the region in $Z^-$ where the inequality

$$\frac{\alpha^3}{2} \leq Cr^3 \frac{T^6}{(T_1 + t)^8}$$  

holds. Since we clearly have $\int_{Z^- \setminus D} e^{2ar^\varepsilon} r^{-2\alpha - 3} U^2 \eta^2 \left( Cr^3 \frac{T^6}{(T_1 + t)^8} - \frac{\alpha^3}{2} \right) y^a \leq 0$, we obtain

$$\int_{Z^-} e^{2ar^\varepsilon} r^{-2\alpha - 4 + \varepsilon} U^2 \eta^2 \left( Cr^3 \frac{T^6}{(T_1 + t)^8} - \frac{\alpha^3}{2} \right) y^a \leq C \int_{D} e^{2ar^\varepsilon} r^{-2\alpha - 4 + \varepsilon} \eta U^2 y^a \frac{\eta T^6}{(T_1 + t)^8}.$$  

Comparing the right-hand side of (3.60) with that of (3.58), it should be clear to the reader that, in order to establish (3.58), it suffices at this point to be able to bound from above in $D$ the quantity $r^{-2\alpha - 4 + \varepsilon} e^{2ar^\varepsilon} \eta \frac{\eta T^6}{(T_1 + t)^8}$. We accomplish this by first observing that, thanks to the exponential vanishing of $\eta$ at $t = -T_1$, see (3.42), we obtain for $t \in (-T_1, -T_2)$,

$$\frac{\eta T^6}{(T_1 + t)^8} \leq C,$$  

for some universal $C > 0$ (depending on $T$). Secondly, we show that, thanks to the inequality (3.59), the following holds in the region $D$ provided that we choose the parameter $\alpha$ large enough

$$r^{-2\alpha - 4 + \varepsilon} e^{2ar^\varepsilon} \eta \leq 1.$$  

Using the expression (3.42) for $\eta(t)$, we see that (3.62) does hold in $D$ if and only if for $\alpha$ sufficiently large we have in such set

$$(2\alpha + 4 - \varepsilon) \log r + \frac{T^3 (T_2 + t)^4}{(T_1 + t)^8 (T_1 - T_2)^4} - 2ar^\varepsilon \geq 0.$$  

\[\vspace{10pt}\]

\[\vspace{10pt}\]
To prove (3.63) observe that (3.59) can be equivalently written in $D$ as

$$
\frac{T_1 + t}{T} \leq \left( \frac{C}{2T^2} \right)^{1/8} \left( \frac{r}{\alpha} \right)^{3/8} = C \left( \frac{r}{\alpha} \right)^{3/8},
$$

for some universal $C > 0$. Since for $\alpha$ sufficiently large we trivially have

$$
C \left( \frac{r}{\alpha} \right)^{3/8} \leq C \left( \frac{R_0}{\alpha} \right)^{3/8} \leq \frac{1}{12},
$$

we conclude that in $D$ we must have

$$
\frac{T_1 + t}{T} \leq \frac{1}{12}, \tag{3.64}
$$

if $\alpha > 1$ has been chosen large enough. Since $\frac{T}{4} = T_1 - T_2 = T_1 + t + |T_2 + t|$, from (3.64) we conclude that we must have in $D$

$$
|T_2 + t| \geq \frac{T}{6}.
$$

If we now use this bound from below along with (3.59), we find in $D$

$$
(2\alpha + 4 - \varepsilon) \log r + \frac{T^3 (T_2 + t)^4}{(T_1 + t)^3 (T_1 - T_2)^4} - 2\alpha r^\varepsilon
\geq \left( \frac{4}{6} \right)^{3/4} \left( \frac{2}{C} \right)^{3/8} T^{3/4} \left( \frac{\alpha}{r} \right)^{9/8} - (2\alpha + 4 - \varepsilon) \log \frac{1}{r} - 2\alpha r^\varepsilon
= C \left( \frac{\alpha}{r} \right)^{9/8} - (2\alpha + 4 - \varepsilon) \log \frac{1}{r} - 2\alpha r^\varepsilon \geq 0,
$$

provided that $r \leq 1$, and that $\alpha$ is sufficiently large. We stress here the critical role of the power $\alpha^{9/8}$, versus the linear term $2\alpha + 4 - \varepsilon$, in reaching the above conclusion. We have thus proved (3.63), and consequently (3.62). Combining (3.60), (3.61) and (3.62), we conclude that (3.58) holds from which (3.57) follows.

Now using the estimates (3.55), (3.56) and (3.57) in (3.54), we deduce that (3.53) holds. Thus we have estimated all the three terms in the right-hand side of (3.45).

Step 3 (Conclusion): Using (3.46), (3.47) and (3.53) in (3.45), we find

$$
\alpha^3 \int_{\mathcal{A}(r_1, r_2) \times (-T, T)} r^{-2\alpha - 4 + \varepsilon} e^{2\alpha r^\varepsilon} \tilde{W}^2 y^a
\leq C \alpha^2 \left( \frac{r_1}{2} \right)^{-2\alpha - 4} e^{2\alpha r^\varepsilon} \int_{\mathcal{A}(\frac{r_1}{2}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a + C \alpha^2 r_2^{-2\alpha - 4} e^{2\alpha r_2^\varepsilon} \int_{\mathcal{A}(\frac{r_2}{2}, r_2) \times (-T, T)} U^2 y^a
+ C \left( \frac{r_1}{2C_1} \right)^{-2\alpha} \int_{\mathcal{A}(\frac{r_1}{2}, r_1) \times (-T, T)} U^2 y^a + C \int_{\mathcal{B}(r_0) \times (-T, T)} U^2 y^a.
\tag{3.66}
$$

We first observe that the term $\frac{\alpha^3}{T} \int_{\mathcal{A}(r_1, r_2) \times (-T, T)} r^{-2\alpha - 4 + \varepsilon} e^{2\alpha r^\varepsilon} \tilde{W}^2 y^a$ can be absorbed in the left hand side of (3.66). Over here we also remark that since $\tilde{C}$ in (3.46), (3.47) are
universal, therefore it has been replaced by $C$ in (3.66) above. Furthermore, since $r_1/2 < R_0$, and $C_1 = e^{R_0^2}$ we have

$$
\left( \frac{r_1}{2} \right)^{-2\alpha-4} e^{2\alpha r_1^2} \leq \left( \frac{r_1}{2C_1} \right)^{-2\alpha-4}.
$$

(3.67)

Also we notice

$$
A\left( \frac{r_1}{2}, r_1 \right) \subset A\left( \frac{r_1}{4}, \frac{3r_1}{2} \right)
$$

and

$$
A(r_2, 2r_2) \subset A(r_2/2, 4r_2) \subset B_{R_0}.
$$

(3.68)

Therefore using (3.67) and (3.68) in (3.66), we obtain

$$
\frac{\alpha^3}{2} \int_{A(r_1, r_2) \times (-T_1, T_1)} r^{-2\alpha-4+\epsilon} e^{2\alpha r^2} \bar{W}^2 y^a \leq C(\alpha^2 + 1) \left( \frac{r_1}{2C_1} \right)^{-2\alpha-4+\epsilon} \int_{A(\frac{r_1}{4}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a
$$

$$
+ C(\alpha^2 + 2)r_2^{-2\alpha-4} e^{2\alpha r_2^2} \int_{B_{R_0} \times (-T, T)} U^2 y^a.
$$

(3.69)

Using (3.48), the fact that $\eta \equiv 1$ and $\phi \equiv 1$ in $A(r_1, r_2) \times (-T_2, T_2)$, we find that the integral in the left-hand side of (3.69) can be bounded from below in the following way

$$
\frac{\alpha^3}{2} \int_{A(r_1, r_2) \times (-T_2, T_2)} r^{-2\alpha-4+\epsilon} e^{2\alpha r^2} \bar{W}^2 y^a \geq \frac{\alpha^3}{2} \int_{A(\frac{r_1}{4}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a.
$$

(3.70)

Subsequently using the bound (3.70) in (3.69), and then by dividing both sides of the resulting inequality in (3.69) by $a^2 r_2^{-2\alpha-4+\epsilon} e^{2\alpha r_2^2}$, we obtain

$$
\frac{\alpha}{2} \int_{B_{R_0} \times (-T, T)} U^2 y^a \leq C \left( \frac{r_1}{2C_1 r_2} \right)^{-2\alpha-4} \int_{A(\frac{r_1}{4}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a
$$

$$
+ C r_2^{-\epsilon} \int_{B_{R_0} \times (-T, T)} U^2 y^a.
$$

(3.71)

Since $r_2 = R_0/4$ and $\epsilon$ are universal constants, we can replace $C r_2^{-\epsilon}$ by a new universal $C$. We now add $\frac{\alpha}{2} \int_{B_{R_1} \times (-T_2, T_2)} U^2 y^a$ to both sides of the inequality (3.71) to deduce the following

$$
\frac{\alpha}{2} \int_{B_{R_2} \times (-T_2, T_2)} U^2 y^a \leq C \left( \frac{r_1}{2C_1 r_2} \right)^{-2\alpha-4} \int_{A(\frac{r_1}{4}, \frac{3r_1}{2}) \times (-T, T)} U^2 y^a + \frac{\alpha}{2} \int_{B_{R_1} \times (-T_2, T_2)} U^2 y^a
$$

$$
+ C \int_{B_{R_0} \times (-T, T)} U^2 y^a.
$$
\[ \leq 2C \left( \frac{r_1}{2C_1 r_2} \right)^{-2\alpha-4} \int_{B_{3r_1/2} \times (-T,T)} U^2 y^a + C \int_{B_{R_0} \times (-T,T)} U^2 y^a, \]  

(3.72)

where in the second inequality in (3.72), we have used,

\[ \frac{\alpha}{2} \leq \left( \frac{r_1}{2C_1 r_2} \right)^{-2\alpha-4}. \]  

(3.73)

(3.73) can be seen as follows. Since \( 2r_1 < r_2 \) and \( C_1 > 1 \), therefore

\[ \left( \frac{2C_1 r_2}{r_1} \right)^{2\alpha+4} \geq (4C_1)^{2\alpha+4} \geq \frac{\alpha}{2}. \]

Using (3.43), we now choose \( \alpha \) (depends also on \( U \)) such that

\[ \frac{\alpha}{2} \int_{B_{r_2} \times (-T_2,T_2)} U^2 y^a \geq 1 + C \int_{B_{R_0} \times (-T,T)} U^2 y^a. \]  

(3.74)

Using (3.74) in (3.72), we get

\[ 1 + C \int_{B_{R_0} \times (-T,T)} U^2 y^a \leq 2C \left( \frac{r_1}{2C_1 r_2} \right)^{-2\alpha-4} \int_{B_{3r_1/2} \times (-T,T)} U^2 y^a + C \int_{B_{R_0} \times (-T,T)} U^2 y^a. \]  

(3.75)

Now (3.75) can be equivalently written as

\[ \left( \frac{r_1}{C_1 r_2} \right)^{2\alpha+4} \leq 2C \int_{B_{3r_1/2} \times (-T,T)} U^2 y^a. \]  

(3.76)

Let \( \rho = 3r_1/2 \), then (3.76) becomes

\[ \rho^{2\alpha+4} \leq 2C \left( \frac{3C_1 r_2}{2} \right)^{2\alpha+4} \int_{B_{\rho} \times (-T,T)} U^2 y^a. \]  

(3.77)

We now multiply both sides of the inequality (3.77) by \( \rho^{2\alpha+4} \) and noting that since \( \rho < R_0/8 \) and \( C_1 = e^{R_0^2} < e \), it follows that

\[ \frac{3C_1 r_2 \rho}{2} \leq \frac{3e R_0^2}{64} \leq 1. \]  

(3.78)

Using (3.78) in (3.77) we obtain

\[ \rho^{4\alpha+8} \leq 2C \left( \frac{3C_1 r_2 \rho}{2} \right)^{2\alpha+4} \int_{B_{\rho} \times (-T,T)} U^2 y^a \leq 2C \int_{B_{\rho} \times (-T,T)} U^2 y^a \leq 2C \int_{B_{R_0} \times (-T,T)} U^2 y^a. \]  

(3.79)

Now by letting

\[ \alpha = C(n, s) (\| V \|_C^{1/2} + 1) + 2C \frac{\int_{B_{R_0} \times (-T,T)} U^2 y^a}{\int_{B_{r_2} \times (-T_2,T_2)} U^2 y^a} + C, \]  

(3.80)
which guarantees that (3.74) holds, we can now assert that there exists a universal constant $C$ such that the following holds for all $0 < \rho \leq R_0/8$,\[
\int_{B_\rho \times (-T, T)} U^2 y^a \geq C \rho^A.
\]
where $A = C ||V||_{C^1}^{1/2} + C \int_{B_{R_0/4} \times (-T_2, T_2)} U^2 y^a + C$. This completes the proof of Lemma 3.2. $\square$

4 Propagation of smallness estimate

4.1 Carleman estimate II

Our next Carleman estimate is a parabolic generalization of the estimate in [47, Proposition 5.7]. This is precisely where we require that $s \in [1/2, 1)$ similar to that in [47]. Using such an estimate, we derive the propagation of smallness estimate from the boundary to the bulk in Lemma 4.3. As the reader will find, such a propagation of smallness estimate as in Lemma 4.3 below allows passage of quantitative uniqueness from the bulk to the boundary using which we derive our main result Theorem 1.4.

Lemma 4.1 Let $\tilde{W}$ be a solution to

$$
\begin{align*}
\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t &= f \quad \text{in } \{y > 0\} \\
\tilde{W} &= 0 \quad \text{on } \{y = 0\},
\end{align*}
$$

with $\text{supp}(\tilde{W}) \subset [0,1/2] \times (0,1)$. Assume $||y^{-a/2}f||_{L^2} < \infty$ and also assume that $\nabla_x \tilde{W}, \tilde{W}_t$ are continuous up to the thin set $\{y = 0\}$. Then there exist a universal $C = C(n, s)$ such that for any $\alpha > 2$, we have

$$\alpha^3 \int e^{2\alpha \phi} \tilde{W}^2 y^a \leq C \int e^{2\alpha \phi} f^2 y^{-a} + C\alpha \int_{\{y=0\}} (\delta_y^a \tilde{W})^2, \quad (4.1)$$

where

$$
\phi(x, y) = -\frac{|x|^2}{4} + \gamma \left[ -\frac{y^{a+1}}{a+1} + \frac{y^2}{2} \right]. \quad (4.2)
$$

Here $\gamma$ is a fixed large parameter (eventually we will choose $\gamma = 2$ as in [47]).

Proof Similar to that in [47], with $\phi$ as in (4.2) above, we set $W = e^{\alpha \phi} y^{a/2} \tilde{W}$. Thus we have $\tilde{W} = e^{-\alpha \phi} y^{-a/2} W$. Now by direct calculation, we find

$$
\begin{align*}
e^{\alpha \phi} y^{-a/2} \text{div}(y^a \nabla \tilde{W}) + e^{\alpha \phi} y^{a/2} \tilde{W}_t &= \Delta W + \alpha^2 (|x|^2/4 + \gamma^2 (y^{2a} + y^2 - 2 y^{a+1})) W \\
- \alpha (-n/2 - \gamma a y^{a-1} + \gamma) W + \alpha (\nabla_x W, x) - 2 \alpha \gamma (y - y^a) W_y + c_s W y^{-2} + W_t,
\end{align*}
$$

where

$$c_s := \frac{a^2}{2} + \frac{a}{2} + \frac{a^2}{4} = \frac{a^2}{4} + \frac{a}{2} = 1 - 4s^2.$$
Before proceeding further, we mention that in the ensuing computations below, all the integrals will be on the set \( \{ y > \epsilon \} \) but the domain of the integration will not be specified for the simplicity of exposition. In the end, we will take the limit \( \epsilon \to 0 \).

We begin by applying the algebraic inequality \((A + B)^2 \geq A^2 + 2AB\) to
\[
A = \Delta W + \alpha^2 (|x|^2/4 + \gamma^2 (y^a - y)^2) W + c_s W y^{-2},
\]
\[
B = -\alpha (-n/2 - \gamma ay^{a-1} + \gamma) W + \alpha \langle \nabla x W, x \rangle - 2\alpha \gamma (y - y^a) W_y + W_t,
\]
and consequently obtain
\[
\int e^{2\alpha \phi} f^2 y^{-a} \geq \int (\Delta W + \alpha^2 (|x|^2/4 + \gamma^2 (y^a - y)^2) W + c_s W y^{-2})^2
\]
\[
- 2\alpha \int (-n/2 - \gamma ay^{a-1} + \gamma) W \Delta W + 2\alpha \int \langle \nabla x W, x \rangle \Delta W
\]
\[
- 4\alpha \gamma \int (y - y^a) W \Delta W + 2 \int (\alpha^3 |x|^2/4 + \alpha^3 \gamma^2 (y^a - y)^2 + \alpha c_s y^{-2})
\]
\[
\langle \nabla x W, x \rangle W
\]
\[
- 4\alpha \gamma \int (\alpha^2 |x|^2/4 + \alpha^2 \gamma^2 (y^a - y)^2 + c_s y^{-2}) (y - y^a) W_y W_t
\]
\[
+ 2 \int W_t \Delta W + 2 \int (\alpha^2 |x|^2/4 + \alpha^2 \gamma^2 (y^a - y)^2 + c_s y^{-2}) W_t
\]
\[
- 2 \int (\alpha^3 |x|^2/4 + \alpha^3 \gamma^2 (y^a - y)^2 + \alpha c_s y^{-2}) (-n/2 - \gamma ay^{a-1} + \gamma) W^2
\]
\[
= I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9. \tag{4.3}
\]

We estimate each of the integrals above separately. We will use the following inequality which can be found in the proof of Proposition 5.7 in [47].
\[
\lim_{\epsilon \to 0} \int_{\{ y = \epsilon \}} e^{2\alpha \phi} y^{2(a-1)} \tilde{W}^2 \leq C \int_{\{ y = 0 \}} e^{2\alpha \phi} (\partial^a_y \tilde{W})^2. \tag{4.4}
\]
The inequality (4.4) will be crucially used in the estimate for \( I_j \)'s below.

\textit{Step 1 (Estimating \( I_2 \))}: We claim that the following inequality holds:
\[
I_2 = -2\alpha \int (-n/2 - \gamma ay^{a-1} + \gamma) W \Delta W \geq 2\alpha \int (\gamma - \gamma ay^{a-1} - n/2) |\nabla W|^2 \tag{4.5}
\]
\[
+ \alpha \int \gamma a(a-1)(a-2) y^{a-3} W^2 - 6Ca^2 \gamma \int_{\{ y = 0 \}} e^{2\alpha \phi} (\partial^a_y \tilde{W})^2.
\]

To see this, we argue as follows. Using integration by parts and also using the fact that \( \text{supp}(W) \subset B_{1/2} \times (0, 1) \), we get
\[
I_2 = -2\alpha \int (-n/2 - \gamma ay^{a-1} + \gamma) W \Delta W = 2\alpha \int (\gamma - \gamma ay^{a-1} - n/2) |\nabla W|^2
\]
\[
- 2\alpha \int \gamma a(a-1) y^{a-2} W W_y + 2\alpha \int_{\{ y = \epsilon \}} (-n/2 - \gamma ay^{a-1} + \gamma) W W_y
\]

Another application of integration by parts gives
\[
- 2\alpha \int \gamma a(a-1) y^{a-2} W W_y = -\alpha \int \gamma a(a-1) y^{a-2} \partial_y W^2
\]
\[
= \alpha \int \gamma a(a-1)(a-2)\gamma y^{a-3}W^2 + \alpha \int \gamma a(a-1)\gamma y^{a-2}W^2
\]
\[
\geq \alpha \int \gamma a(a-1)(a-2)\gamma y^{a-3}W^2. \tag{4.6}
\]

In the last inequality in (4.6) above, we have used that since we are in the \( s > \frac{1}{2} \), therefore we have that \( a(a-1) \geq 0 \) which in particular implies that
\[
\alpha \int \gamma a(a-1)\gamma y^{a-2}W^2 \geq 0.
\]

Now on substituting \( W = e^{a\phi} y^{a/2} \tilde{W} \), we find that \( W_y = e^{a\phi} (y^{a/2} \tilde{W}_y + a y^{a-2-1} \tilde{W} + \alpha(-\gamma y^a + \gamma y) y^{a/2} \tilde{W}) \) and thus we get for small enough \( \epsilon > 0 \)
\[
2\alpha \int \left( -\frac{n}{2} - \gamma ay^{a-1} + \gamma \right) W W_y
\]
\[
= 2\alpha \int \left( -\frac{n}{2} - \gamma ay^{a-1} + \gamma \right) e^{2a\phi} (y^{a} \tilde{W}_y + \frac{a}{2} y^{a-1} \tilde{W} + \alpha(-\gamma y^a + \gamma y) y^{a} \tilde{W}) \tilde{W}
\]
\[
\geq -4\alpha \gamma \int e^{2a\phi} |y^{a-1} \tilde{W}_y| y^a \tilde{W}_y - 2\alpha \gamma \int e^{2a\phi} y^{2(a-1)} \tilde{W}^2 - 4\alpha^2 \gamma^2
\]
\[
\int e^{2a\phi} y^{2(a-1)} y^{1+a} \tilde{W}^2
\]
\[
\geq -6\alpha \gamma C \int e^{2a\phi} (\partial_y^a \tilde{W})^2, \tag{4.7}
\]
as \( \epsilon \to 0 \). In the first inequality in (4.7) above, we have used that \( a - 1 < 0 \) and thus for small enough \( y \), we have \( y^{a-1} >> 1 \). In the second inequality, we have used (4.4) to estimate the second and the third term on the right hand side of the first inequality in (4.7) in the following way
\[
\begin{align*}
2 \lim_{\epsilon \to 0} \alpha \gamma \int e^{2a\phi} y^{2(a-1)} \tilde{W}^2 &\leq 2\alpha \gamma C \int e^{2a\phi} (\partial_y^a \tilde{W})^2, \\
4 \lim_{\epsilon \to 0} \alpha^2 \gamma^2 \int e^{2a\phi} y^{2(a-1)} y^{1+a} \tilde{W}^2 &\to 0. \tag{4.8}
\end{align*}
\]
Thus letting \( \epsilon \to 0 \), we find that (4.5) follows and this finishes the proof of Step 1.

Step 2 (Estimating \( I_3 \)): We claim the following estimate for \( I_3 \).
\[
I_3 \geq 2\alpha \int \Delta W \langle \nabla_x W, x \rangle = (n-2)\alpha \int |\nabla_x W|^2 + n\alpha \int W_y^2. \tag{4.9}
\]
We first apply Rellich identity (Lemma 2.3) for \( a = 0 \) and \( k = 0 \) to get
\[
2\alpha \int \Delta_x W \langle \nabla_x W, x \rangle = (n-2)\alpha \int |\nabla_x W|^2. \tag{4.10}
\]
Furthermore, by integrating by parts we observe
\[
2\alpha \int W_{yy} \langle \nabla_x W, x \rangle = -2\alpha \int W_y \langle \nabla_x W_y, x \rangle - 2\alpha \int W_y \langle \nabla_x W, x \rangle
\]
In order to estimate the boundary integral in (4.12) above, we substitute $\epsilon \to -\epsilon$.

Hence from (4.10) and (4.11) we have

$$I_3 = 2\alpha \int \Delta W \langle \nabla_x W, x \rangle = (n - 2)\alpha \int |\nabla_x W|^2 + n\alpha \int W_y^2 - 2\alpha \int W_y \langle \nabla_x W, x \rangle.$$  \hspace{1cm} (4.12)

In order to estimate the boundary integral in (4.12) above, we substitute $W$ in terms of $\tilde{W}$ and then by computations as in (4.7) above, we get

$$\begin{align*}
&\int \nabla_y \langle x, \nabla_x W \rangle \\
&= -2\alpha \int \nabla_y \langle \nabla_x W, x \rangle \\
&= -2\alpha \int \nabla_y (\langle \nabla_x W, x \rangle - \frac{\alpha}{2}|x|^2\tilde{W}) + \alpha \int e^{2\alpha \phi} \gamma \tilde{W} (x, \nabla_x \tilde{W}) + 2\alpha^2 \gamma \\
&+ \alpha^2 \int e^{2\alpha \phi} y^2 (y^{a-1} \tilde{W}) \langle \nabla_x \tilde{W}, x \rangle + \alpha \int e^{2\alpha \phi} y^{a-1} \tilde{W} (y^{a-1} \tilde{W})^2 - \alpha^3 \gamma \\
&\int \nabla_y (\tilde{W})^2 + \alpha^2 \int e^{2\alpha \phi} y^2 \gamma \tilde{W} (x, \nabla_x \tilde{W}) > + \alpha^3 \gamma \int y^{1+a} |x|^2\tilde{W}^2. \hspace{1cm} (4.13)
\end{align*}$$

Now by an application of Cauchy-Schwartz, (4.4) and the fact that $\tilde{W}, < \nabla_x \tilde{W}, x \to 0$ as $\epsilon \to 0$ (recall $\tilde{W} \equiv 0$ at $|y| = 0$), we observe that all the integrals in (4.13) go to 0 as $\epsilon \to 0$. Again letting $\epsilon \to 0$, we conclude that (4.9) holds.

Step 3 (Estimating $I_4$): We claim that $I_4$ can be lower bounded in the following way.

$$I_4 \geq -2\alpha \gamma \int \langle 1 - ay^{a-1} \rangle |\nabla_x W|^2 + 2\alpha \gamma \int (1 - ay^{a-1}) W_y^2 - 8C\alpha \gamma \int e^{2\alpha \phi} (\partial^a_y \tilde{W})^2.$$  \hspace{1cm} (4.14)

We have

$$I_4 = -4\alpha \gamma \int \Delta W (y - y^a) W_y = 4\alpha \gamma \int \langle \nabla_x W, (y - y^a) \nabla_x W \rangle - 4\alpha \gamma \int W_y W_y (y - y^a)$$

$$= 2\alpha \gamma \int (y - y^a) \partial_y |\nabla_x W|^2 - 2\alpha \gamma \int (W_y^2)_y (y - y^a).$$
We now estimate the boundary integral in the above expression. We find

\[
\left| 2\alpha\gamma \lim_{\epsilon \to 0} \int_{\{y = \epsilon\}} (y - y^a)W_y^2 \right| \leq 2\alpha\gamma \lim_{\epsilon \to 0} \int_{\{y = \epsilon\}} (y^a)e^{2\alpha\phi}(y^{a/2}\tilde{W}_y
\]

\[
+ \frac{a}{2} y^{a/2 - 1} \tilde{W} + \alpha(-\gamma y^a + \gamma y)y^{a/2}\tilde{W}^2
\]

\[
\leq 4\alpha\gamma \int_{\{y = \epsilon\}} e^{2\alpha\phi}(y^a\tilde{W}_y)^2 + (y^{a-1}\tilde{W})^2 + \alpha y^{2a+2}(y^{a-1}\tilde{W})^2
\]

\[
\leq 8C\alpha\gamma \int_{\{y = 0\}} e^{2\alpha\phi}(\partial_y^a\tilde{W})^2.
\]

Notice that in the last inequality, we again used (4.4). Hence we get as \( \epsilon \to 0 \), that (4.14) holds.

**Step 4 (Estimating I₅):** To estimate I₅, we first write \( W_{x} \ W = \frac{1}{2} \nabla_{x} W^2 \) and then integrate by parts with respect to \( x \)-variable to get

\[
I_5 = 2 \int (\alpha^3|x|^2/4 + \alpha^3\gamma^2(y^a - y)^2 + \alpha c_s y^{-2})W(\nabla_{x} W, x)
\]

\[
= -\frac{\alpha^3}{4} \int (n + 2)|x|^2W - n\alpha^3\gamma^2 \int (y^a - y)^2W^2 - nac_s \int W^2y^{-2}. \tag{4.15}
\]

**Step 5 (Estimating I₆):** We prove the following estimate for I₆.

\[
I_6 \geq \frac{\alpha^3}{2}\gamma \int (1 - ay^{a-1})|x|^2W^2 + 6\alpha^3\gamma^3 \int (y - y^a)^2(1 - ay^{a-1})W^2
\]

\[
- 2\alpha\gamma c_s \int (y^{-2} + (a - 2)y^{a-3})W^2 - C\alpha\gamma c_s \int e^{2\alpha\phi}(\partial_y^a\tilde{W})^2. \tag{4.16}
\]

By writing \( 2W_{y} = \partial_y W^2 \) and then by integrating by parts in the \( y \)-variable, we have

\[
I_6 = -4\alpha\gamma \int (\alpha^2|x|^2/4 + \alpha^2\gamma^2(y^a - y)^2 + c_s y^{-2})(y - y^a)W \ws
\]

\[
= \frac{\alpha^3}{2}\gamma \int (1 - ay^{a-1})|x|^2W^2 + 6\alpha^3\gamma^3 \int (y - y^a)^2(1 - ay^{a-1})W^2
\]

\[
+ 2\alpha^3\gamma \int (|x|^2/4 + \gamma^2(y - y^a)^2)(y - y^a)W^2
\]

\[
- 2\alpha\gamma c_s \int (y^{-2} + (a - 2)y^{a-3})W^2 + 2\alpha\gamma c_s \int (y - y^a)y^{-2}W^2.
\]
Again the first boundary integral in the expression of $I_6$ above can be estimated by writing $W$ in terms of $\tilde{W}$ and by using (4.4) in the following way

$$2 \lim_{\epsilon \to 0} \alpha^3 \gamma \int_{\{y = \epsilon\}} (|x|/4 + \gamma^2 (y - y^a)^2)(y - y^a) W^2 \leq C \lim_{\epsilon \to 0} \alpha^3 \gamma^3 \int_{\{y = \epsilon\}} e^{2\alpha \phi} y^a \tilde{W}_t$$

$$= C \lim_{\epsilon \to 0} \alpha^3 \gamma^3 \int_{\{y = \epsilon\}} e^{2\alpha \phi} y^2 (y^a - 1) \tilde{W}_t^2 \leq 0.$$

Likewise using (4.4), the second boundary integral can be estimated as

$$\left| 2 \lim_{\epsilon \to 0} \alpha \gamma c_s \int_{\{y = \epsilon\}} (y - y^a) y^{-2} W^2 \right| \leq C \alpha \gamma c_s \int_{\{y = 0\}} e^{2\alpha \phi} (\partial_{y^a} \tilde{W})^2. \quad (4.17)$$

Thus (4.16) follows.

**Step 6 (Estimating $I_7$):** We claim that

$$I_7 = 0. \quad (4.18)$$

In order to establish this, using (2.18) in Lemma 2.3 with $k = a = 0$, we find

$$I_7 = 2 \int W_t \Delta W = -2 \int_{\{y = \epsilon\}} W_t W_y.$$

As before, the boundary integral is handled in the following way by writing $W$ in terms of $\tilde{W}$ and by using Cauchy-Schwartz inequality

$$-2 \int_{\{y = \epsilon\}} W_t W_y = -2 \int_{\{y = \epsilon\}} e^{2\alpha \phi} y^a \tilde{W}_t \tilde{W}_y - 2 \alpha \int_{\{y = \epsilon\}} e^{2\alpha \phi} (-\gamma y^a + y y^a) y^a \tilde{W}_t \tilde{W}_y - a \int_{\{y = \epsilon\}} e^{2\alpha \phi} y^a \tilde{W}_t \tilde{W}_t$$

$$\leq C \left( \int_{\{y = \epsilon\}} e^{2\alpha \phi} (\partial_{y^a} \tilde{W})^2 \int_{\{y = \epsilon\}} e^{2\alpha \phi} \tilde{W}_t^2 \right)^{1/2}$$

$$+ C \alpha \left( \int_{\{y = \epsilon\}} e^{2\alpha \phi} (y^a - 1) \tilde{W}_t^2 \int_{\{y = \epsilon\}} e^{2\alpha \phi} (y^1 + a) \tilde{W}_t^2 \right)^{1/2}$$

$$+ C \left( \int_{\{y = \epsilon\}} e^{2\alpha \phi} (y^a - 1) \tilde{W}_t^2 \int_{\{y = \epsilon\}} e^{2\alpha \phi} \tilde{W}_t^2 \right)^{1/2}.$$
All the boundary integrals on the right hand side of the above expression are seen to go to zero as \( \epsilon \to 0 \) using the fact that \( W_t \to 0 \) as \( y \to 0 \) and also by using (4.4). Thus (4.18) follows.

*Step 7 (Estimating \( I_8 \)): By integrating by parts in the \( t \) variable, we notice
\[
I_8 = 2 \int (\alpha^2 |x|^2/4 + \alpha^2 \gamma^2 y^a - y^2 + c_y y^{a-2}) W_t W = c_s \int \partial_t (y^{-2} W^2) = 0. \tag{4.19}
\]

*Step 8 (Conclusion):* Therefore from (4.3), Step 1-Step 8 and by rearranging the terms we have
\[
\int e^{2\alpha \phi} y^{-a} f^2 \geq I_1 - 2\alpha \int |\nabla_x W_t|^2 + 4\alpha \gamma \int (1 - ay^{a-1}) W_t^2 \\
+ (\alpha \gamma a(a - 1)(a - 2) - 2\alpha \gamma c_y(a - 2) + 2\alpha \gamma a c_s) \int y^{a-3} W^2 \\
+ \left( -\left( n + 2 \right) \frac{\alpha^3}{4} + n \frac{\alpha^3}{4} \right) \int |x|^2 W^2 + \left( -\frac{n\alpha^3}{r^2} + \frac{n\alpha^3}{r^2} \right) \int (y - y^a)^2 W^2 \\
+ (\alpha^3 \gamma a - \alpha^3 \gamma a) \int (1 - ay^{a-1}) |x|^2 W^2 + \left( 6\alpha^3 \gamma^3 - 2\alpha^3 \gamma^3 \right) \int (y - y^a)^2 (1 - ay^{a-1}) W^2 \\
+ (-nac_s - 2\alpha \gamma c_y + nac_s - 2\alpha \gamma c_s) \int y^{-2} W^2 - 16C\alpha \gamma \int_{\{y=0\}} e^{2\alpha \phi} (\gamma^{a} \tilde{W}_t)^2. \tag{4.20}
\]

Now from the Hardy’s inequality as in Lemma 2.6, we have
\[
\int W^2 y^{a-3} \leq \frac{4}{(3 - a - 1)^3} \int y^{a-1} W_t^2 + \frac{2}{3 - a - 1} \int y^{a-2} W^2. \tag{4.21}
\]

Notice that \( \alpha \gamma a(a - 1)(a - 2) - 2\alpha \gamma c_y(a - 2) + 2\alpha \gamma a c_s = \alpha \gamma (1 + 2 s)^2 a \leq 0 \), hence
\[
(\alpha \gamma a(a - 1)(a - 2) - 2\alpha \gamma c_y(a - 2) + 2\alpha \gamma a c_s) \int y^{a-3} W^2 \\
\geq 4\alpha \gamma a \int_{\{y=0\}} y^{a-1} W_t^2 + 2\alpha \gamma a (2 - a) \int_{\{y=0\}} y^{a-2} W^2. \tag{4.22}
\]

Now writing \( W \) in terms of \( \tilde{W} \) and then again by using (4.4) we get
\[
\left| 2\alpha \gamma a (2 - a) \right| \int_{\{y=0\}} y^{a-2} W^2 \leq 8\alpha \gamma \int_{\{y=0\}} e^{2\alpha \phi} y^{2a-2} W^2 \leq 8C\alpha \gamma \int_{\{y=0\}} e^{2\alpha \phi} (\partial_y^a \tilde{W})^2. \tag{4.23}
\]

Also since \( a \leq 0 \), we have
\[
(y - y^a)^2 (1 - ay^{a-1}) \geq (y - y^a)^2. \tag{4.24}
\]

Hence using (4.21)-(4.24) in (4.20) we obtain the following estimate
\[
\int e^{2\alpha \phi} y^{-a} f^2 \geq I_1 - 2\alpha \int |\nabla_x W_t|^2 + 4\alpha \gamma \int W_t^2 - \frac{\alpha^3}{2} \int |x|^2 W^2 + 4\alpha \gamma W^2 \int (y - y^a)^2 W^2. \tag{4.25}
\]
Also, by substituting $W$ holds

$$- 4\alpha y c_s \int y^{-2} W^2 - 24 C \alpha y \int_{\{y=0\}} e^{2\alpha \phi} (\beta^\alpha_y \tilde{W})^2.$$  

Since supp$(W(\cdot, t)) \subset \overline{B_{1/2}}$ and $c_s \leq 0$, we get from the above inequality that the following holds

$$\int e^{2\alpha \phi} y^{-a} f^2 \geq I_1 - 2\alpha \int |\nabla_x W|^2 - \frac{\alpha^3}{8} \int W^2 + 4\alpha^3 \gamma^3 \int (y - y^a)^2 W^2$$

$$- 24 C \alpha y \int_{\{y=0\}} e^{2\alpha \phi} (\beta^\alpha_y \tilde{W})^2.$$  

(4.26)

Now we absorb the term $-2\alpha \int |\nabla_x W|^2$ using $I_1$. Notice that from an integration by parts argument

$$- \alpha \int (\Delta W + \alpha^2 (|x|^2/4 + y^2 (y^a - y^2)) W + c_s W y^{-2}) W$$

$$= \alpha \int |\nabla W|^2 + \alpha \int W W_y - \alpha^3 \int (|x|^2/4 + y^2 (y^a - y^2)) W^2 - \alpha c_s \int W^2 y^{-2}.$$  

(4.27)

Now since supp$(W(\cdot, t)) \subset \overline{B_{1/2}}$, we get the following lower bound on the last two terms in (4.27) above

$$- \alpha^3 \int (|x|^2/4 + y^2 (y^a - y^2)) W^2 \geq - \frac{\alpha^3}{16} \int W^2 - \alpha^3 \gamma^2 \int (y^a - y^2) W^2.$$  

Also, by substituting $W$ in terms of $\tilde{W}$ and by using (4.4) it is seen that $\int_{\{y=0\}} W W_y = 0$.

Now using Cauchy-Schwarz inequality we get

$$- 2\alpha y \int (\Delta W + \alpha^2 (|x|^2/4 + y^2 (y^a - y^2)) W + c_s W y^{-2}) W$$

$$\leq \int (\Delta W + \alpha^2 (|x|^2/4 + y^2 (y^a - y^2)) W + c_s W y^{-2})^2 + \alpha^2 \gamma^2 \int W^2$$

$$= I_1 + \alpha^2 \gamma^2 \int W^2.$$  

(4.28)

Hence from (4.27)-(4.28) we get the lower bound for $I_1$

$$I_1 = \int (\Delta W + \alpha^2 (|x|^2/4 + y^2 (y^a - y^2)) W + c_s W y^{-2})^2$$

$$\geq -2\alpha y \int (\Delta W + \alpha^2 (|x|^2/4 + y^2 (y^a - y^2)) W + c_s W y^{-2}) W - \alpha^2 \gamma^2 \int W^2$$

$$\geq 2\alpha y \int |\nabla W|^2 - \frac{2\gamma \alpha^3}{16} \int W^2 - 2\alpha^3 \gamma^3 \int (y^a - y^2) W^2 - \alpha^2 \gamma^2 \int W^2.$$  

(4.29)

Using (4.29) in (4.26) we thus obtain

$$\int e^{2\alpha \phi} y^{-a} f^2 \geq -2\alpha \int |\nabla_x W|^2 - \frac{\alpha^3}{8} \int W^2 + 4\alpha^3 \gamma^3 \int (y - y^a)^2 W^2$$

$$+ 2\alpha y \int |\nabla W|^2 - \frac{2\gamma \alpha^3}{16} \int W^2 - 2\alpha^3 \gamma^3 \int (y^a - y^2) W^2 - \alpha^2 \gamma^2 \int W^2$$  

(4.30)
Therefore from the definition of \( \supp(W(t, \cdot)) \subset B_{1/2} \), we have \( (y^a - y)^2 \geq 1/4 \). At this point, by letting \( \gamma = 2 \), from (4.30) we can now assert that there exist \( C = C(n, \delta) \) such that for all \( \alpha \geq 2 \) we have

\[
\alpha^3 \int e^{2 \alpha \phi} y^a \tilde{W}^2 \leq C \int e^{2 \alpha \phi} y^{-a} \tilde{f}^2 + C \alpha \int_{\{y = 0\}} e^{2 \alpha \phi} (\partial^a_y \tilde{W})^2.
\]

This completes the proof. \( \square \)

### 4.2 Propagation of smallness estimate

Using the Carleman estimate in Lemma 4.1, we now establish a quantitative propagation of smallness from the boundary to the bulk as in Lemma 4.2 below. This is a parabolic generalization of Proposition 5.10 in [47].

Similar to that in [47], we first define the following sets for \( s \in [1/2, 1) \) which are tailored to the geometry of Carleman weights employed in Lemma 4.1. For a given \( r > 0 \), we let

\[
\begin{align*}
P_r^*(x_0) &:= \left\{ x \in \mathbb{R}^n \times \{ y \geq 0 \} : y \leq ((1-s) (r-|x-x_0|^2/4))^{1/2} \right\}, \\
P_r(x_0) &:= P_r^*(x_0) \cap \{ y = 0 \}.
\end{align*}
\]

**Lemma 4.2** Let \( s \in [1/2, 1) \) and \( \tilde{W} \) be a solution to

\[
\text{div}(y^a \nabla \tilde{W}) + y^a \tilde{W}_t = 0 \quad \text{in} \quad B_1 + \times (0, 1),
\]

such that \( \tilde{W} = 0 \) in \( B_{1/4} \times (1/8, 7/8) \cap \{ y = 0 \} \). Also assume that \( \nabla_x \tilde{W}, \tilde{W}_t \) are continuous up to the thin set \( \{ y = 0 \} \). Then there exists \( \theta \in (0, 1) \) and \( R_1 < 1/2 \) such that

\[
||y^{a/2} \tilde{W}||_{L^2(B_{1} \times (1/4,3/4))} \leq C ||y^{a/2} \tilde{W}||_{L^2(B_1 \times (0,1))}^{\theta} ||\partial^a_y \tilde{W}||_{L^2(B_1 \times (0,1))}^{1-\theta} + 2 ||\partial^a_y \tilde{W}||_{L^2(B_1 \times (0,1))}.
\]

**Proof** We work with \((-T, T)\) instead of \((0, T)\). Also we will write \( \phi_+(r) = \inf_{X \in \partial P_r^*} \phi(X) \) and \( \phi_-(r) = \sup_{X \in \partial P_r^*} \phi(X) \). We notice that on \( \partial P_r^* \), we have that

\[
\frac{2y_{a+1}}{a+1} + \frac{|x|^2}{4} = r.
\]

Therefore from the definition of \( \phi \) in Lemma 4.1, it follows that

\[
\inf_{\partial P_r^*} \phi = \inf_{P_r^*} \phi = -r.
\]

Moreover, it is also seen that \( \phi_+(r) \) is a decreasing function of \( r \) for all \( r \leq C(a) \).

Consider \( W = \eta(t) \psi(X) \tilde{W} \) where as before, \( \eta \) is a defined as

\[
\eta(t) = \begin{cases} 
0 & \text{if } -T \leq t \leq T_1 \\
\exp\left(-\frac{T_3(T_2+t)^4}{(T_1+t)^4(T_1-T_2)^4}\right) & \text{if } -T_1 \leq t \leq T_1 \\
1 & \text{if } -T_2 \leq t \leq 0
\end{cases}
\]
\( \eta(t) = \eta(-t) \) for \( t > 0 \), where \( T_1 = 3T/4 \) and \( T_2 = T/2 \), and \( \psi \) is a smooth cut-off function symmetric in \( y \) such that \( \psi = 1 \) in \( P^*_a \), \( \psi = 0 \) in \( \mathbb{R}^{n+1}_+ \setminus P^*_a \). Notice that one can ensure that \( |\partial_y \psi(x,y)| \leq C(r_0)y \). We also notice that \( W \) solves
\[
\begin{align*}
\text{div} (y^a \nabla W) + y^a W_t &= \eta \tilde{W} \text{div} (y^a \nabla \psi) + 2 \eta (\nabla \tilde{W}, \nabla \psi) y^a + \eta \psi \tilde{W} y^a \quad \text{in} \quad \mathbb{R}^{n+1}_+ \times \mathbb{R} \\
W &= 0 \quad \text{on} \quad \{ y = 0 \}.
\end{align*}
\]
(4.33)

Also we observe that \( \text{supp}(W) \subset P^*_a \times (-T, T) \), where \( r_0 \) is chosen such that \( P^*_a \subset B_{1/2} \). Now by applying the Carleman estimate (4.1) in Lemma 4.1 to \( W \), we obtain
\[
\alpha^3 \int_{P^*_a \times (-T, T)} e^{2\alpha \phi} W^2 y^a \leq C \int_{P^*_a \times (-T, T)} e^{2\alpha \phi} g^2 y^a + C \int_{P^*_a \times (-T, T)} e^{2\alpha \phi} \eta^2 \tilde{W}^2 y^a
\]
\[+ C\alpha \int_{P^*_a \times (-T, T)} e^{2\alpha \phi} (\partial_y \tilde{W})^2 \eta^2 \tilde{W}^2 y^a = I_1 + I_2 + I_3. \quad (4.34)\]

where \( g = \eta \tilde{W} \text{div} (y^a \nabla \psi) + 2 \eta (\nabla \tilde{W}, \nabla \psi) y^a. \quad (4.35) \)

**Step 1 (Estimating \( I_1 \))**: We first observe that \( g \) is supported on \( P^*_a \setminus P^*_a \). Also since \( \psi \) is smooth and symmetric in \( y \) across \( \{ y = 0 \} \), we have \( |\psi_y| \leq C(r_0)y \). Moreover \( \nabla^2 \psi \) is bounded. Furthermore, since \( \tilde{W} = 0 \) on \( \{ y = 0 \} \), given the expression of \( g \) as in (4.35), we get from a Caccioppoli type energy estimate that the following inequality holds
\[
I_1 = \int_{(P^*_a \setminus P^*_a) \times (-T, T)} e^{2\alpha \phi} g^2 y^a \leq C e^{2\alpha \phi + (r_0)} \int_{B_1 \times (-T, T)} \tilde{W}^2 y^a. \quad (4.36)\]

Here we have used that \( \phi_+ \) is a decreasing function of \( r \).

**Step 2 (Estimating \( I_2 \))**: In order to estimate \( I_2 \), we argue as in the proof of the estimate (3.57) above. As before, we have that \( \eta_t \) is supported in \( (-T_1, -T_2) \cup (T_2, T_1) \). We will only estimate \( I_2 \) in the region \( P^*_a \times (-T_1, -T_2) \) since the other estimate is similar. We write \( P^*_a \times (-T_1, -T_2) = D \cup (P^*_a \times (-T_1, -T_2) \setminus D) \), with \( D \) being the set where
\[
\frac{C \eta^2_t}{\eta^2} > \frac{\alpha^3}{2}.
\]

Now by a direct computation, it is easily seen that the above inequality is equivalent to insisting
\[
2C \frac{16T^6}{(T_1 + t)^8} > \alpha^3. \quad (4.37)
\]

Hence for large \( \alpha \) (depends also on \( T \), \( t \in (-T_1, -T_2) \)), we get that (4.37) implies
\[
\frac{T_1 + t}{T} < \frac{1}{12}.
\]

Now since \( T_1 - T_2 = T/4 \), we consequently obtain from the above inequality
\[
|T_2 + t| > \frac{T}{6}.
\]
Moreover, as $\eta$ decays as exponential near $T_1$ we get that there exists a universal constant $C_1$ such that
\[
\frac{\eta^2}{\eta^2} \eta \leq C_1.
\] (4.38)

Also, notice that in $D$, we have for large $\alpha$
\[
\ln CC_1 - 2\alpha \phi_+ (r_0) + \ln \eta
\]
\[
= \ln CC_1 - 2\alpha \phi_+(r_0) - \frac{T^3 (T_2 + t)^4}{(T_1 + t)^3 (T_1 - T_2)^4}
\]
\[
\leq \ln CC_1 - 2\alpha \phi_+(r_0) - C(T) \alpha^{9/8} \leq 0.
\] (4.39)

We would like to emphasise the last inequality is possible because the exponent of $\alpha$ in $C(T) \alpha^{9/8}$ is strictly greater than 1 and this is precisely where the presence of the power $\alpha^3$ in front of the integral $\int e^{2\alpha \phi} W^2 y^a$ in (4.1) plays a crucial role. Thus for all large $\alpha$’s, we have
\[
CC_1 \eta \leq e^{2\alpha \phi_+(r_0)}.
\] (4.40)

Using this, we estimate $I_2$ as follows.
\[
I_2 = C \int_{P_{2r_0}^* \times (-T_1, -T_2)} e^{2\alpha \phi} \eta^2 \tilde{W}^2 \psi^2 y^a = C \int_D e^{2\alpha \phi} \eta^2 \tilde{W}^2 \psi^2 y^a
\]
\[
+ C \int_{P_{2r_0}^* \times (-T_1, -T_2) \setminus D} e^{2\alpha \phi} \eta^2 \tilde{W}^2 \psi^2 y^a
\]
\[
\leq C \int_D \left( \frac{\eta^2}{\eta^2} \eta \right) \eta \tilde{W}^2 y^a + C \int_{P_{2r_0}^* \times (-T_1, -T_2) \setminus D} e^{2\alpha \phi} \eta^2 \tilde{W}^2 \psi^2 y^a
\]
\[
\leq e^{2\alpha \phi_+(2r_0)} \int_{P_{2r_0}^* \times (-T, T)} \tilde{W}^2 y^a + \frac{\alpha^3}{2} \int_{P_{2r_0}^* \times (-T, T)} e^{2\alpha \phi} \tilde{W}^2 y^a
\]
(4.41)

Using (2.11) and (4.39), we thus obtain
\[
\frac{\alpha^3}{2} \int_{P_{2r_0}^* \times (-T, T)} e^{2\alpha \phi} \tilde{W}^2 y^a \leq C e^{2\alpha \phi_+(r_0)} \int_{B_1 \times (-T, T)} \tilde{W}^2 y^a + C \alpha \int_{P_{2r_0}^* \times (-T, T)} (\tilde{\phi}^a \tilde{W})^2.
\] (4.42)

We now minorize the integral on the left hand side in (4.42) over the set $P_{r_0/2}^* \times (-T_2, T_2)$. Furthermore, we notice that from (4.32) it follows that on the set $P_{r_0/2}^*$, we have
\[
e^{2\alpha \phi} \geq e^{2\alpha \phi_-(r_0/2)}.
\] (4.43)

Thus using (4.43) in (4.42) we deduce that the following holds for all $\alpha$ large enough
\[
\int_{P_{r_0/2}^* \times (-T_2, T_2)} \tilde{W}^2 y^a \leq e^{2\alpha \phi_+(r_0) - \phi_-(r_0/2)} \int_{B_1^* \times (-T, T)} \tilde{W}^2 y^a + e^{-2\alpha \phi_-(r_0/2)} \int_{B_1 \times (-T, T)} (\tilde{\phi}^a \tilde{W})^2.
\] (4.44)
We now observe that \( \phi_+(r_0) - \phi_-(r_0/2) < 0 \). This is seen as follows.

\[
\phi_+(r_0) \leq -r_0 + y^2 \leq -r_0 + r_0^2/4 \leq -3r_0/4 \leq -r_0/2 = \phi_-(r_0/2) \leq 0. \tag{4.45}
\]

In (4.45) we have used that since \( s \geq 1/2 \), therefore we have \( y^2 \leq ((1-s)(r_0 - |x|^2/4))^{1/1-s} \leq r_0^2/4 \). We now split the rest of the argument into two cases.

**Case 1:** When \( \int_{B_1 \times (-T, T)} \tilde{W}^2 y^a \leq 2 \int_{B_1 \times (-T, T)} (\partial_y^a \tilde{W})^2 \). In this case, the desired estimate (4.31) is seen to hold.

**Case 2:** When \( \int_{B_1 \times (-T, T)} \tilde{W}^2 y^a > 2 \int_{B_1 \times (-T, T)} (\partial_y^a \tilde{W})^2 \). In this case, we let

\[
\alpha = \frac{1}{-2\phi_+(r_0)} \ln \left( \frac{\int_{B_1 \times (-T, T)} \tilde{W}^2 y^a}{\int_{B_1 \times (-T, T)} (\partial_y^a \tilde{W})^2} \right) + C_0
\]

in (4.44) above, where \( C_0 \) is a universal constant, we again conclude that (4.31) holds in view of the fact that \( P_{r_0/2}^* \) contains \( B_{R_1} \) for some \( R_1 \) depending only \( r_0, n, a \). This completes the proof. \( \square \)

From Lemma 4.2, the following propagation of smallness estimate follows which relates smallness in the bulk to the \( W^{2,2} \) norm in the boundary. The proof of such a lemma is inspired by ideas in [37].

**Lemma 4.3** Let \( U \) solve (2.10). Then there exist universal \( C = C(n, s) \) such that

\[
||y^{a/2} U||_{L^2(B_{r_0} \times (1/4, 3/4))} \leq C ||u||_{L^2(B_1 \times (0, 1))}^{1-\theta} \left( ||Vu||_{L^2(B_1 \times (0, 1))}^{\theta} + ||u||_{W^{2,2}(B_2 \times (0, 1))}^{\theta} \right) + C \left( ||Vu||_{L^2(B_1 \times (0, 1))} + ||u||_{W^{2,2}(B_2 \times (0, 1))} \right),
\]

where \( R_1 \) and \( \theta \) are as in Lemma 4.2 and \( W^{2,2} \) norm over a domain \( \Omega \) in space-time is defined as

\[
||u||_{W^{2,2}(\Omega)} := ||u||_{L^2(\Omega)} + ||\nabla_x u||_{L^2(\Omega)} + ||\nabla_x^2 u||_{L^2(\Omega)} + ||u_t||_{L^2(\Omega)}.
\]

**Proof** Let \( \eta \in C^\infty_0(B_2 \times (0, 1)) \) be a cutoff function such that \( \eta = 1 \) in \( B_1 \times (1/8, 7/8) \).

First notice that if \( u \in W^{2,2} \) then \( \eta u \in \text{Dom}(H^s) \) as

\[
\int (1 + |4\pi^2 |\xi|^2 + 2\pi i \sigma|)^{2s}|\hat{\eta u}(\xi, \sigma)|^2 \leq C \int (1 + |\xi|^4 + \sigma^2)|\hat{\eta u}(\xi, \sigma)|^2 \leq C (||u||_{W^{2,2}(B_2 \times (0, 1))})^2.
\]

Now let \( W \) be a solution of

\[
\text{div}(y^a \nabla W) + y^a W_t = 0 \quad \text{in} \quad \mathbb{R}_+^{n+1} \times \mathbb{R}
\]

\[
W = \eta u \quad \text{on} \quad \{y = 0\}.
\]

Since \( \eta u \in \text{Dom}(H^s) \), we have

\[
\left( \int_{\{y=0\}} (\partial_y^a W)^2 \right)^{1/2} = ||H^s(\eta u)||_{L^2} \leq ||u||_{W^{2,2}}. \tag{4.46}
\]
Now consider $\tilde{W} = U - W$ then we have $\tilde{W} = 0$ on $B_1$. We notice that by an odd reflection of $\tilde{W}$ across $\{y = 0\}$ and by arguments as in Section 5 in [11], one can show that $\tilde{W}$ satisfies regularity assumptions in Lemma 4.2. Therefore by applying the estimate (4.31) in Lemma 4.2 to $\tilde{W}$ we get
\[
||y^{a/2}\tilde{W}||_{L^2(\mathbb{B}_R)} \leq C||y^{a/2}\tilde{W}||_{L^2(\mathbb{B}_R)}^{1-\theta}||\partial_y^a\tilde{W}||_{L^2(B_1)}^{\theta} + 2||\partial_y^a\tilde{W}||_{L^2(B_1)}
\]

We now deduce from the above inequality
\[
||y^{a/2}U||_{L^2(\mathbb{B}_R)} \leq ||y^{a/2}\tilde{W}||_{L^2(\mathbb{B}_R)} + ||y^{a/2}W||_{L^2(\mathbb{B}_R)}
\]

(4.47)

Also from [11, Lemma 4.5] we have
\[
||y^{a/2}W||_{L^2(\mathbb{B}_R)} \leq C||\eta u||_{L^2}\leq C||u||_{L^2}\n\]

(4.48)

and
\[
||y^{a/2}U||_{L^2(\mathbb{B}_R)} \leq C||u||_{L^2}\n\]

(4.49)

Thus from (4.46) to (4.49) we have
\[
||y^{a/2}U||_{L^2(\mathbb{B}_R)} \leq C||u||_{L^2}^{1-\theta}||\partial_y^aU||_{L^2(B_1)}^{\theta} + ||u||_{W^{2,2}(B_1)}^{\theta} + C(||\partial_y^aU||_{L^2(B_1)} + ||u||_{W^{2,2}(B_1)})
\]

from which the desired estimate in the Lemma follows using $\partial_y^aU = Vu$. □

By a translation in time, the following corollary follows from Lemma 4.3 which will be more convenient to use in the proof of Theorem 1.1.

Corollary 4.4 Let $U$ be as in Lemma 4.3. Then the following inequality holds
\[
||y^{a/2}U||_{L^2(\mathbb{B}_R)} \leq C||u||_{L^2}^{1-\theta}||Vu||_{L^2(B_1)}^{\theta} + ||u||_{W^{2,2}(B_1)}^{\theta}
\]

(4.50)

where $R_1$ and $\theta$ are as in Lemma 4.2.

5 Proof of main results

With Lemma 3.2 and Corollary 4.4 in hand, we now proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1 Let $U$ be the solution to the corresponding extension problem (2.10). We choose $\rho$ such that $\rho < \min\{R_0/8, R_1^2/8\}$ where $R_0$ and $R_1$ are as in Lemmas 3.1 and 4.3 respectively. Since $\rho < R_0/8$, from Lemma 3.2, we have
\[
\int_{B_\rho(0,1)} U^{2,\alpha} > C\rho^A
\]

(5.1)
where $A$ is as in Lemma 3.2. The estimate (5.1) implies that there exist $t_0 \in (\rho^2, 1 - \rho^2)$ such that

$$\int_{\mathbb{B}_\rho \times (t_0 - \rho^2, t_0 + \rho^2)} U^2 y^a \geq C \rho^{A + 2}. \quad (5.2)$$

We now rescale $U$ to $\tilde{U}$ as follows

$$\tilde{U}(x, y, t) = U \left( \frac{\rho}{R_1} x, \frac{\rho}{R_1} y, \left( \frac{\rho}{R_1} \right)^2 t + t_0 \right),$$

notice that $\tilde{U}$ solves

$$\text{div}(y^a \nabla \tilde{U}) + y^a \tilde{U}_t = 0 \quad \text{in} \quad \mathbb{R}^{n+1} \times \mathbb{R},$$

$$\tilde{U} = \tilde{u}(x, t) \quad \text{on} \quad \{ y = 0 \},$$

where $\tilde{u}(x, y, t) = u \left( \frac{\rho}{R_1} x, (\frac{\rho}{R_1})^2 t + t_0 \right)$. Since $\partial_y^a U = Vu$, we have

$$\partial_y^a \tilde{U} = \left( \frac{\rho}{R_1} \right)^{1-a} (Vu) \left( \frac{\rho}{R_1} x, \left( \frac{\rho}{R_1} \right)^2 t + t_0 \right). \quad (5.3)$$

Then from Corollary 4.4, we have

$$||y^a/2 \tilde{U}||_{L^2(\mathbb{B}_{R_1} \times (-R_1^2, R_1^2))} \leq C ||\tilde{u}||_{L^2(B_1 \times (-1, 1))} \left( ||\partial_y^a \tilde{U}||_{L^2(B_1 \times (-1, 1))} + ||\tilde{u}||_{W^{2, 2}(B_2 \times (-1, 1))} \right) + C \left( ||\partial_y^a \tilde{U}||_{L^2(B_1 \times (-1, 1))} + ||\tilde{u}||_{W^{2, 2}(B_2 \times (-1, 1))} \right). \quad (5.4)$$

By change of variables and the assumption $||u||_{L^2(\mathbb{R}^{n+1})} \leq 1$, we get

$$||\tilde{u}||_{L^2(\mathbb{R}^{n+1})} = (\rho/R_1)^{-1-\theta} ||u||_{L^2(\mathbb{R}^{n+1})} \leq (\rho/R_1)^{-1-\theta} \left( \frac{\rho}{R_1} \right)^{1-\theta} \frac{n+2}{2}. \quad (5.5)$$

Now using (5.3) and change of variables we find

$$||\partial_y^a \tilde{U}||_{L^2(B_1 \times (-1, 1))} \leq ||Vu||_{L^\infty} \left( \frac{\rho}{R_1} \right)^{(1-a)} \left( \frac{\rho}{R_1} \right)^{-\frac{n+2}{2}} \leq \left( \frac{\rho}{R_1} \right)^{(1-a)} \frac{n+2}{2}.$$

Also, using change of variable we have

$$||\tilde{u}||_{W^{2, 2}(B_2 \times (-1, 1))} \leq (\rho/R_1)^{-(n+2)/2} ||u||_{W^{2, 2}(B_{2R_1} \times (-\rho/R_1)^2 + 0, (\rho/R_1)^2 + 0)}.$$

Now by using (5.5), (5.6), (5.7) in (5.4) and also that $\left( \frac{\rho}{R_1} \right)^{(1-a)} \leq 1$, we obtain

$$\text{div}(y^a/2 U) \leq 2C \left( C_\theta a + ||V||_{L^\infty} \right) \left( \frac{\rho}{R_1} \right)^{-(n+3+a)/2} ||u||_{L^2(\mathbb{B}_\rho \times (-\rho^2, \rho^2 + t_0))} \leq 2C \left( C_\theta + ||V||_{L^\infty} \right) \left( \frac{\rho}{R_1} \right)^{-(n+3+a)/2} ||u||_{L^2(\mathbb{B}_{2R_1} \times (-\rho/R_1)^2 + 0, (\rho/R_1)^2 + 0)}$$

$$\leq 2C \left( C_\theta + ||V||_{L^\infty} \right) \left( \frac{\rho}{R_1} \right)^{-(n+3+a)/2} \left( \frac{\rho}{R_1} \right)^{-(n+2)/2} \leq 2C \left( C_\theta + ||V||_{L^\infty} \right) ||u||_{W^{2, 2}(B_{2R_1} \times (-\rho/R_1)^2 + 0, (\rho/R_1)^2 + 0)}.$$

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Then by multiplying the above inequality on both sides with \((\rho/R_1)^{(\alpha+2)/2}\) and using \(\rho/R_1 \lesssim 1\), \((\rho/R_1)^2 + t_0 < 1 + t_0 < 2\), \(- (\rho/R_1)^2 + t_0 > -1 + t_0 > -1\), we get

\[
\|y^{a/2}U\|_{L^2(B_2 \times (-\rho^2 + t_0, \rho^2 + t_0))}^\theta \\
\leq 2C(C_n^\theta + \|V\|_{L^\infty}^\theta)\|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))}^\theta \\
+ 2C(C_n + \|V\|_{L^\infty})\|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))}^\theta.
\]

(5.8)

Now there exists universal constant \(C = C(\theta, s)\) such that we have

\[
(C_n^\theta + \|V\|_{L^\infty}^\theta) \leq C2\|V\|_{L^\infty}^{1/2s}
\]

and also

\[
(C_n + \|V\|_{L^\infty}) \leq C2\|V\|_{L^\infty}^{1/2s}.
\]

Using this in (5.8) we get

\[
\|y^{a/2}U\|_{L^2(B_2 \times (-\rho^2 + t_0, \rho^2 + t_0))}^\theta \\
\leq C(\theta, s)2\|V\|_{C^1}^{1/2s} \left(\|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))}^\theta \right.
\]

\[
+ \|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))}^\theta\right).
\]

(5.9)

Using the bulk quantitative estimate (5.2) in the above inequality we obtain

\[
C(\theta, s)\rho^{A/2+1} \\
\leq 2\|V\|_{C^1}^{1/2s} \left(\|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))}^\theta \right.
\]

\[
+ \|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))}^\theta\right).
\]

(5.10)

Now since we are interested in a vanishing order estimate from below for small \(\rho\), without loss of generality, we may assume that

\[
\|u\|_{W^{2,2}(B_{2\rho/R_1} \times (-\rho/R_1)^2 + t_0, (\rho/R_1)^2 + t_0))} < 1
\]

(5.11)

since otherwise the desired estimate in Theorem 1.1 holds trivially. Therefore by using (5.10) in (5.9) and by letting \(2\rho/R_1\) as our new \(\rho\), we find that the following estimate holds for all \(\rho \leq \bar{R}\), where \(\bar{R}\) is some universal constant depending on \(R_0\) and \(R_1\)

\[
\|u\|_{W^{2,2}(B_p \times (-\rho^2, t_0 + \rho^2))} \geq C_1 \rho^{A_0},
\]

(5.12)

where \(C_1 = C_1(n, s, \|V\|_{C^1})\) and \(A_0\) is as in Theorem 1.1. The conclusion of the theorem now follows by noting that

\[
\|u\|_{C^2(B_p \times (-2, 2))} \geq \|u\|_{W^{2,2}(B_p \times (-\rho^2, t_0 + \rho^2))}.
\]

**Proof of Theorem 1.3** Before we proceed with the proof, we would like to alert the reader that throughout the proof, we will use the letter \(C\) to denote all purpose universal constant which might vary from line to line.

We first notice while proving Theorem 1.1, we showed that there exists \(t_0\) such that

\[
\|u\|_{W^{2,2}(B_p \times (-\rho^2, t_0 + \rho^2))} \geq C_1 \rho^{A_0},
\]

(5.13)
where $C_1$ and $A_0$ are as in the Theorem 1.1. See (5.11) above. Over here, $W^{2.2}$ norm of $u$ refers to
\[
||u||_{W^{2.2}} \overset{def}{=} ||u||_{L^2} + ||\nabla_x u||_{L^2} + ||\nabla^2_x u||_{L^2} + ||u_t||_{L^2}.
\]
In the rest of the proof, for notational convenience we will denote $B_\rho \times (t_0 - \rho^2, t_0 + \rho^2)$ by $Q_\rho$ and $\mathbb{B}_\rho \times (t_0 - \rho^2, t_0 + \rho^2)$ by $\mathbb{Q}_\rho$. Also $||V||_{C^2_{\rho}}$ will be denoted by $||V||_{C^2}$. Then from the rescaled version of the estimate (2.11) in Lemma 2.2 we have
\[
y^{a/2}_\rho \nabla U_{ij} ||L^2(Q_\rho) \leq r^{-3} C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} \tag{5.13}
\]
and also
\[
r^2 ||y^{a/2}_\rho U_t||_{L^2(Q_\rho)} + r^3 ||y^{a/2}_\rho \nabla U_t||_{L^2(Q_\rho)} + r^4 ||y^{a/2}_\rho U_t||_{L^2(Q_\rho)} \leq C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})}. \tag{5.14}
\]
Let $\phi$ be a smooth function supported in $\mathbb{B}_\rho \times (t_0 - (2\rho)^2, t_0 + (2\rho)^2)$ such that $\phi \equiv 1$ in $\mathbb{B}_{\rho} \times (t_0 - \rho^2, t_0 + \rho^2)$. We now apply interpolation inequality (2.19) to $f = \phi U$ and obtain for any $0 < \eta_1 < 1$ that the following estimate holds
\[
||\nabla f||_{L^2(\mathbb{R}^{n+1})} \leq C \left( \eta_1^{\frac{1}{2}} ||y^{a/2}_\rho \nabla f||_{L^2(\mathbb{R}^{n+1} \times \mathbb{R}^+)} + ||y^{a/2}_\rho \nabla f||_{L^2(\mathbb{R}^{n+1} \times \mathbb{R}^+)} + \eta^{-1}_1 ||f||_{L^2(\mathbb{R}^{n+1})} \right)
\]
\[
\leq C \left( \eta_1^{\frac{1}{2}} (||y^{a/2}_\rho \nabla U||_{L^2(Q_{2\rho})} + (||\nabla \phi|| + ||\phi||) ||y^{a/2}_\rho \nabla U||_{L^2(Q_{2\rho})}) + (||\nabla \phi|| + ||\phi||) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + \eta^{-1}_1 ||u||_{L^2(Q_{2\rho})} \right)
\]
\[
\leq \eta_1^{\frac{1}{2}} \rho^{-2} C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + C \eta^{-1}_1 ||u||_{L^2(Q_{2\rho})}. \tag{5.15}
\]
where in the last inequality, we have used the rescaled versions of the regularity estimate in Lemma 2.2. From (5.15) it follows
\[
||\nabla f||_{L^2(Q_\rho)} \leq C \eta_1^{\frac{1}{2}} \rho^{-2} (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + C \eta^{-1}_1 ||u||_{L^2(Q_{2\rho})}. \tag{5.16}
\]
Similarly by applying (2.19) to $\nabla f$ and also by using (5.13) we get for any $0 < \eta < 1$
\[
||\nabla u||_{L^2(Q_\rho)} \leq C \eta^{\frac{1}{2}} \rho^{-3} (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + C \eta^{-1} ||\nabla f||_{L^2(\mathbb{R}^{n+1})}. \tag{5.17}
\]
Now using (5.15) in (5.17) we obtain
\[
||\nabla u||_{L^2(Q_\rho)} \leq C \eta^{\frac{1}{2}} \rho^{-3} C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + \eta^{-1} \eta_1^{\frac{1}{2}} \rho^{-2} C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + C (\eta \eta_1)^{-1} ||u||_{L^2(Q_{2\rho})}. \tag{5.18}
\]
We now take $\eta_1 = \eta^3$. Then we find
\[
\eta^{-1}_1 \eta_1^{\frac{1}{2}} = \eta^{3s-1}. \tag{5.19}
\]
Also observe that $3s - 1 \geq s$ as $s \geq 1/2$. Eventually we take $\eta < < 1$. In view of this, we thus obtain from (5.18)
\[
||\nabla^2 u||_{L^2(Q_\rho)} \leq \eta^{\frac{1}{2}} \rho^{-3} C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + C \eta^{-4} ||u||_{L^2(Q_{2\rho})}. \tag{5.20}
\]
Similarly by applying (2.20) to $f = U \phi$ and also by using the estimate (5.14) we get
\[
||u_t||_{L^2(Q_\rho)} \leq \eta^{\frac{1}{2}} \rho^{-4} C (1 + ||V||_{C^2}) ||y^{a/2}_\rho U||_{L^2(Q_{2\rho})} + C \eta^{-1} ||u||_{L^2(Q_{2\rho})}. \tag{5.21}
\]
Therefore from (5.16), (5.20) and (5.21) we have
\[ ||u||_{W^{2,2}(Q_\rho)} \leq C \eta^{-4} ||u||_{L^2(Q_{4\rho})} + \left( \frac{C}{\rho^4} \right) \eta^s \frac{||V||}{c^2}^{1/2s}. \tag{5.22} \]

Notice that in (5.22) we have used that \[ ||y^{\alpha/2}U||_{L^2(Q_{4\rho})} \leq C. \] If we now take \( \eta^s = (C_1/2C) \rho^{A_0 + 4 + ||V||_c^{1/2s}} \) and use (5.12), we get
\[ C_1 \rho^{A_0} \leq C \eta^{-4} ||u||_{L^2(Q_{4\rho})} + \frac{C_1}{2} \rho^{A_0} (2\rho) ||V||_c^{1/2s}. \tag{5.23} \]

In (5.23) above, we can now absorb the term \( \frac{C_1}{2} \rho^{A_0} (2\rho) ||V||_c^{1/2s} \) in the left hand side and then the desired estimate (1.5) is seen to follow.

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**Data availability**  
Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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