Commensurate mixtures of ultra-cold atoms in one dimension

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We study binary mixtures of ultra-cold atoms, confined to one dimension in an optical lattice, with commensurate densities. Within a Luttinger liquid description, which treats various mixtures on equal footing, we derive a system of renormalization group equations at second order, from which we determine the rich phase diagrams of these mixtures. These phases include charge/spin density wave order, singlet and triplet pairing, polaron pairing, and a supersolid phase. Various methods to detect our results experimentally are discussed.

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I. INTRODUCTION

Recent advances in controlling ultra cold atoms lead to the realization of truly one dimensional systems, and the study of many-body effects therein. Important benchmarks, such as the Tonks-Girardeau gas and the Mott transition in one dimension, have been achieved by trapping bosonic atoms in tight tubes formed by an optical lattice potential. Novel transport properties of one dimensional lattice bosons have been studied using these techniques. More recently, a strongly interacting one dimensional Fermi gas was realized using similar trapping methods. Interactions between the fermion atoms were controlled by tuning a Feshbach resonance in these experiments. On the theory side, numerous proposals were given for realizing a variety of different phases in ultracold Fermi systems, as well as Bose-Bose mixtures. In , commensurate mixtures in higher dimensions were studied.

In this paper we explore the behavior of ultracold atomic mixtures, confined to one-dimensional (1D) motion in an optical lattice, that exhibit different types of commensurability, by which we mean that the atomic densities and/or the inverse lattice spacing have an integer ratio. Commensurable fillings arise naturally in many ultracold atom systems, because the external trap potential approximately corresponds to a sweep of the chemical potential through the phase diagram, and therefore passes through points of commensurability. At these points the system can develop an energy gap, which fixes the density commensurability over a spatially extended volume. This was demonstrated in the celebrated Mott insulator experiment by Greiner et al., where Mott phases with integer filling occurred in shell-shaped regions in the atom trap. These gapped phases gave rise to the well-known signature in the time-of-flight images, and triggered the endeavor of ‘engineering’ many-body states in optical lattices. Further examples include the recently created density-imbalanced fermion mixtures, in which the development of a balanced, i.e. commensurate, mixture at the center of the trap is observed.

In 1D, this phenomenon is of particular importance, because it is the only effect that can lead to the opening of a gap, for a system with short-range interactions. In contrast to higher dimensional systems, where, for instance, pairing can lead to a state with an energy gap, in 1D only discrete symmetries can be broken, due to the importance of fluctuations. Orders that correspond to a continuous symmetry can, at most, develop quasi long range order (QLRO), which refers to a state in which an order parameter $O(x)$ has a correlation function with algebraic scaling, $\langle O(x)O(0) \rangle \sim |x|^{-(2-\alpha)}$, with a positive scaling exponent $\alpha$.

Due to its importance in solid state physics, the most thoroughly studied commensurate 1D system is the SU(2) symmetric system of spin-1/2 fermions. This system develops a spin gap for attractive interaction and remains gapless for repulsive interaction, as can be seen from a second order RG calculation. However, the assumed symmetry between the two internal spin states, which is natural in solid state systems, does not generically occur in Fermi-Fermi mixtures (FFMs) of ultra-cold atoms, where the ‘spin’ states are in fact different hyperfine states of the atoms. An analysis of the generic system is therefore highly called for. Furthermore, we will extend this analysis to both Bose-Fermi (BFMs) and Bose-Bose mixtures (BBMs), as well as to the dual commensurability, in which the charge field, and not the spin field, exhibits commensurate filling, as will be explained below.

The main results of this paper are the phase diagrams shown in Fig. We find that both attractive and repulsive interactions can open an energy gap. For FFMs the entire phase diagram is gapped, except for the repulsive SU(2) symmetric regime (cp. ), for BFMs or BBMs the bosonic liquid(s) need(s) to be close to the hard core limit, otherwise the system remains gapless. Furthermore, we find a rich structure of quasi-phases, including charge and spin density wave order (CDW, SDW), singlet and triplet pairing (SS, TS), polaron pairing, and a supersolid phase, which is the first example of a supersolid phase in 1D. These results are derived within a Luttinger liquid (LL) description, which treats bosonic and fermionic liquids on equal footing.

This paper is organized as follows: In Section we classify the different types of commensurate mixtures that can occur, and in Section we discuss the effective action of the mixtures with the most relevant com-
mcrsurable term. In Section IV we discuss the set of
renormalization group equations for such systems, and in
Section V, VI and VII we apply these results to Fermi-
Fermi, Bose-Bose, and Bose-Fermi mixtures, respectively.
In Section VIII we discuss the experimental detectibility,
and in Section IX we conclude.

II. CLASSIFICATION OF COMMENSURATE MIXTURES

We will now classify the types of commensurability
that can occur in a system with short-ranged
density-density interaction. We consider Haldane’s represen-
tation of the densities for the two species:

\[ n_{1/2} = [\nu_{1/2} + \Pi_{1/2}] \sum_{\nu} e^{2\nu i\theta_{1/2}} \]  

(1)

\( \nu_1 \) and \( \nu_2 \) are the densities of the two liquids, \( \Pi_{1/2}(x) \) are the low-k parts (i.e. \( k \ll 1/\nu \)) of the density fluct-
uations; the fields \( \Theta_{1/2}(x) \) are given by \( \Theta_{1/2}(x) =
\pi \nu_{1/2} x + \theta_{1/2}(x) \), with \( \theta_{1/2} = \pi \int d\Pi_{1/2}(y) \).
These expressions hold for both bosons and fermions. If we
use this representation in a density-density interaction
term \( U_{12} \int dx \nu_1(x) \nu_2(x) \), we generate to lowest order a
term of the shape \( U_{12} \int dx \Pi_1(x) \Pi_2(x) \), but in addition
an infinite number of nonlinear terms, corresponding to
tall harmonics in the representation. However, only the
terms for which the linear terms (\( 2\pi \nu_{1/2} \theta_{1/2} x \)) cancel,
can drive a phase transition. For a continuous system
this happens for \( m_1 \nu_1 - m_2 \nu_2 = 0 \), whereas for a system
on a lattice we have the condition \( m_1 \nu_1 - m_2 \nu_2 = m_3 \),
where \( m_1, m_2 \) and \( m_3 \) are integer numbers. In general,
higher integer numbers correspond to terms that are less
relevant, because the scaling dimension of the non-linear
term scales quadratically with these integers. We are
therefore lead to consider small integer ratios between
the fillings and/or the lattice if present. In \( V \), we con-
sidered two cases of commensurabilities: a Mott insula-
tor transition coupled to an incommensurate liquid, and a
fermionic liquid at half-filling coupled to an incommensu-
rate bosonic liquid. In both cases the commensurability
occurs between one species and the lattice, but does not
involve the second species. In this paper we consider the
two most relevant, i.e. lowest order, cases which exhibit
a commensurability that involves both species. The first
case is the case of equal filling \( \nu_1 = \nu_2 \), the second is the
case of the total density being unity, i.e. \( \nu_1 + \nu_2 = 1 \),
where the densities \( \nu_1 \) and \( \nu_2 \) themselves are incommen-
surate. The first case can drive the system to a spin-
gapped state, the second to a charged gapped state.
We will determine in which parameter regime these transi-
tions occur, and what type of QRLO the system exhibits
in the vicinity of the transition. These two cases can be
mapped onto each other via a dual mapping, which
enables us to study only one case and then infer the
results for the second by using this mapping. We will write
out our discussion for the case of equal filling and merely
state the corresponding results for complementary filling.

III. EFFECTIVE ACTION

The action of a two-species mixture with equal filling
in bosonized form is given by:

\[ S = S_{0,1} + S_{0,2} + S_{12} + S_{\text{int}}. \]  

(2)

The terms \( S_{0,j} \), with \( j = 1, 2 \), are given by

\[ S_{0,j} = \frac{1}{2\pi K_j} \int d^2r \left( \frac{1}{v_j} (\partial_r \theta_j)^2 + v_j (\partial_\theta \theta_j)^2 \right) \]  

(3)

Each of the two types of atoms, regardless of being
bosonic or fermionic, are characterized by a Luttinger
parameter \( K_{1/2} \) and a velocity \( v_{1/2} \). Here we integrate
over \( r = (v_0, \tau, x) \), where we defined the energy scale
\( v_0 = (v_1 + v_2)/2 \). The term \( S_{12} \) describes the acoustic
coupling between the two species, and is bilinear:

\[ S_{12} = \frac{U_{12}}{\pi^2} \int d^2 r \partial_r \theta_1 \partial_r \theta_2 + \frac{V_{12}}{\pi^2} \int d^2 r \partial_\theta \theta_1 \partial_\theta \theta_2. \]  

(4)

The second term is created during the RG flow; its pref-
actor therefore has the initial value \( V_{12}(0) = 0 \). We define

FIG. 1: Phase diagram of a commensurate FFM or a BBM of
hardcore bosons (with the replacement \( TS_2 \rightarrow SS \)), in terms of
the interaction \( U_{12} \) and the parameter \( z = |v_1 - v_2|/(v_1 + v_2) \).
For both attractive and repulsive interactions a spin gap
opens, except for \( z = 0 \) and positive interaction. In the at-
tractive regime, a FFM or a BBM shows either singlet pairing
or CDW order, or a coexistence of these phases. For repulsive
interaction these mixtures show SDW ordering, with FFM
and BBMs showing subdominant triplet or singlet pairing,
respectively, for a large range of \( z \). In the gapless regime, a
FFM shows degenerate SDW and CDW order, and a BBM
shows SF with subdominant CDW, i.e. supersolid behavior.
For very large positive values of \( U_{12} \) the system undergoes
phase separation (PS); for very large negative values it col-
lapses (CL).
$S_0 = S_{0,1} + S_{0,2} + S_{12},$ which is the diagonalizable part of the action. $S_{int}$ corresponds to the non-linear coupling between the two liquids, which we study within an RG approach:

$$S_{int} = \frac{2\theta_1}{(2\pi \alpha)^2} \int d^2 r \cos(2\theta_1 - 2\theta_2). \quad (5)$$

This bosonized description applies to a BBM, a BFM, and a FFM. Depending on which of these mixtures we want to describe we either construct bosonic or fermionic operators according to Haldane’s construction:

$$f/b = [\nu_0 + \Pi]^{1/2} \sum_{m \text{ odd/even}} e^{m\nu_0} e^{i\Phi}. \quad (6)$$

$\nu_0$ is the zero-mode of the density, $\Phi(x)$ is the phase field, which is the conjugate field of the density fluctuations $\Pi(x)$. The action for a mixture with complementary filling, $\nu_1 + \nu_2 = 1$, is of the form $S_0 + S'_{int}$, where the interaction $S'_{int}$ is given by:

$$S'_{int} = \frac{2\theta_1}{(2\pi \alpha)^2} \int d^2 r \cos(2\theta_1 + 2\theta_2). \quad (7)$$

To map the action in Eq. (2) onto this system we use the mapping: $\theta_2 \rightarrow -\theta_2$, $\phi_2 \rightarrow -\phi_2$, and $g_{12} \rightarrow -g_{12}$, which evidently maps a mixture with complementary filling and attractive (repulsive) interaction and onto a mixture with equal filling with repulsive (attractive) interaction.

**IV. RENORMALIZATION GROUP**

To study the action given in Eq. (2), we perform an RG calculation along the lines of the treatment of the sine-Gordon model in [21]. In our model, a crucial modification arises: the linear combination $\theta_1 - \theta_2$ that appears in the non-linear term, is not proportional to an eigenmode of $S_0$, and therefore the RG flow does not affect only one separate sector of the system, as in an SU(2)-symmetric system. The RG scheme that we use here proceeds as follows: First, we diagonalize $S_0$ through the transformation

$$\theta_1 = B_1 \tilde{\theta}_1 + B_2 \tilde{\theta}_2, \quad (8)$$
$$\theta_2 = D_1 \tilde{\theta}_1 + D_2 \tilde{\theta}_2. \quad (9)$$

The coefficients $B_{1/2}$ and $D_{1/2}$ are given in the Appendix. The fields $\tilde{\theta}_{1/2}$ are the eigenmode fields with velocities $\tilde{v}_{1/2}$ (see Appendix). As the next step, we introduce an energy cut-off $\Lambda$ on the fields $\tilde{\theta}_{1/2}$ according to $\omega^2/\tilde{v}_{1/2}^2 + \tilde{v}_{1/2}^2 k^2 < \Lambda^2$. We shift this cut-off by an amount $d\Lambda$, and correct for this shift up to second order in $g_{12}$. At first order, only $g_{12}$ is affected, its flow equation is given by:

$$\frac{dg_{12}}{dl} = \left(2 - K_1 - K_2 - \frac{2 U_{12} + V_{12} v_1 v_2}{v_1 + v_2}\right)g_{12}. \quad (10)$$

with $dl = d\Lambda/\Lambda$. At second order several terms are created that are quadratic in the original fields $\theta_1$ and $\theta_2$. We undo the diagonalization, Eq. (8) and (9), and absorb these terms into the parameters of the action, which concludes the RG step. By iterating this procedure we obtain these flow equations at second order in $g_{12}$:

$$\frac{dK_{1/2}}{dl} = -\frac{g_{12}^2}{16\pi^2} \left(2 + \left(\frac{v_2}{v_1} + \frac{v_1}{v_2}\right)\right) \quad (11)$$
$$\frac{dv_1}{dl} = v_1 \frac{g_{12}^2}{16\pi^2} \left(\frac{v_2}{v_1} - \frac{v_1}{v_2}\right) \quad (12)$$
$$\frac{dv_2}{dl} = v_2 \frac{g_{12}^2}{16\pi^2} \left(\frac{v_1}{v_2} - \frac{v_2}{v_1}\right) \quad (13)$$
$$\frac{dU_{12}}{dl} = -\frac{g_{12}^2}{8\pi} (v_1 + v_2) \quad (14)$$
$$\frac{dV_{12}}{dl} = -\frac{g_{12}^2}{8\pi} (1/v_1 + 1/v_2) \quad (15)$$

A similar set of equations has been derived in [7] for a FFM in non-bosonized form. The difference between our result and the result in [7] is the renormalization of the velocities, that we find here, which is due to different types of expansions: In [7] only one-loop contributions are taken into account, whereas here we use a cumulant expansion in $g_{12}$, which at second order includes contributions that are two-loop for the renormalization of the velocities. These contributions, which would integrate to
zero for equal velocities, as can be seen from Eqs. [12] and [13] leads to the discrepancy between the expansion in the number of loops and the cumulant expansion, and gives a small quantitative correction of the velocities. As mentioned before, the advantage of the current approach is that the QRLO of the system can be directly determined from the resulting renormalized parameters, and that the same action can be used to study BBMs and BFM.

The system of differential equations, Eqs. [10] to [13], can show two types of qualitative behavior: The coefficient $g_{12}$ of the non-linear term [3] can either flow to zero, i.e. $S_{int}$ is irrelevant, or it diverges, leading to the formation of an energy gap. In the first case, the system flows to a fixed point that is described by a renormalized diagonalizable action of the type $S_0$, from which the quasi-phases can be determined.

When $S_{int}$ is relevant, we introduce the fields $\theta_{p/σ} = \frac{1}{\sqrt{2}}(θ_1 ± θ_2)$, which define the charge and the spin sector of the system. In this regime, these sectors decouple. Each of the two sectors is characterized by a Luttinger parameter and a velocity, $K_{p/σ}$ and $v_{p/σ}$, which are related to the original parameters in $S_0$ in a straightforward way. Using the numerical solution of the flow equations, we find that $K_{σ} → 0$, as can be expected for an ordering of the nature of a spin gap, leaving $K_{p}$ the only parameter characterizing the QRLO in this phase.

In order to determine the QRLO in the system we determine the scaling exponents of various order parameters. For that purpose, we use the bosonization representation of these order parameters, which contain the fields $θ_{1/2}$ and $φ_{1/2}$, and use the diagonalization, Eqs. [15] and [16], for the fields $θ_{1/2}$, as well as the dual transformation for the fields $φ_{1/2}$:

$$φ_1 = C_1 \tilde{φ}_1 + C_2 \tilde{φ}_2, \quad (16)$$

$$φ_2 = E_1 \tilde{φ}_1 + E_2 \tilde{φ}_2. \quad (17)$$

The coefficients $C_{1/2}$ and $D_{1/2}$ are given in the Appendix. Since the order parameters are now written in terms of the eigenfields $\tilde{θ}_{1/2}$ and $\tilde{φ}_{1/2}$, the correlation functions can be evaluated in a straightforward manner. The scaling exponents are given by various quadratic expressions of the parameters in Eqs. [15], [16], [17], and [18]. In [19], we give an extensive list of correlation functions, which can be transferred to the system considered here, with the formal replacement: $β_{1/2} → B_{1/2}$, $γ_{1/2} → C_{1/2}$, $δ_{1/2} → D_{1/2}$, and $ε_{1/2} → E_{1/2}$. The order parameter with the largest positive scaling exponent shows the dominant order, whereas other orders with positive exponents are subdominant.

V. FERMI-FERMI MIXTURES

We will now apply this procedure to the different types of mixtures. For a FFM we find that the system always develops a gap, with the exception of the repulsive SU(2) symmetric regime (cp. [2]). To determine the QRLO we introduce the following operators [19, 20]:

$$O_{SS} = \sum_{σ,σ′} \delta f_{R,σ} δ_{σ,σ′} f_{L,3−σ′}, \quad (18)$$

$$O_{TS}^{a} = \sum_{σ,σ′} \tilde{δ} f_{R,σ} δ_{σ,σ′} f_{L,3−σ′}, \quad (19)$$

$$O_{CDW} = \sum_{σ,σ′} f_{R,σ} δ_{σ,σ′} f_{L,σ′}, \quad (20)$$

$$O_{SDW}^{a} = \sum_{σ,σ′} \tilde{δ} f_{R,σ} δ_{σ,σ′} f_{L,σ′}, \quad (21)$$

with $σ,σ′ = 1,2$, $δ = 3−2σ$, and $a = x,y,z$. In the gapless SU(2) symmetric regime, both CDW and SDW show QRLO, with both scaling exponents of the form $α_{CDW} = 1 − K_{p}^{19}$, which shows that these orders are algebraically degenerate. Within the gapped regime the scaling exponents of these operators are given by $α_{SS,TS} = 2 − K_{p}^{−1}$ and $α_{CDW,SDW} = 2 − K_{p}$. As discussed in [20], the sign of $g_{12}$ determines whether CDW or SDW, and SS or TS, appears. In Fig. [1] we show the phase diagram based on these results. In addition to these phases we indicate the appearance of the Wenzel-Bardeen instability, shown as phase separation for repulsive interaction and collapse for attractive interaction.

We will now use the dual mapping to obtain the phase diagram of a FFM with complementary filling from Fig. [1]. Under this mapping, the attractive and repulsive regimes are exchanged with the following replacements:
CDW→SDW\textsubscript{z}, SDW\textsubscript{z}→CDW, SS, TS\textsubscript{z}→SDW, and SDW→SS. Note that the gapless regime is now on the attractive side, with degenerate CDW and SS pairing.

VI. BOSE-BOSE MIXTURES

For BBMs we proceed in the same way as for FFM. We introduce the following set of order parameters:

\begin{align}
O_{CDW} &= b_1^\dagger b_1 + b_2^\dagger b_2, \\
O_{SS} &= b_1 b_2, \\
O_{SDW_x} &= b_1^\dagger b_1 - b_2^\dagger b_2, \\
O_{SDW_y} &= b_1^\dagger b_2 + b_2^\dagger b_1, \\
O_{SDW_z} &= -i(b_1^\dagger b_2 - b_2^\dagger b_1),
\end{align}

and in addition the superfluid (SF) order parameters \(b_1\) and \(b_2\). In Fig. 1 we show the phase diagram of a mixture of a BBM of hardcore bosons, which is almost identical to the one of a FFM. The phase diagram of the mixture with complementary filling, as obtained from the dual mapping, is also of the same form as its fermionic equivalent, with the exception of the gapless regime, in which BBMs show supersolid behavior (coexistence of SF and CDW order), and with the replacement \(TS\textsubscript{z}→SS\).

In Fig. 2 we show the phase diagram of a mixture of hardcore bosons (species 1) and bosons in the intermediate to hardcore regime (species 2). If species 2 is sufficiently far away from the hardcore limit, the system remains gapless. However, in the vicinity of the transition the scaling exponents of the liquids are affected by the RG flow. As indicated, the effective scaling exponent of the hardcore bosons is renormalized to a value that is smaller than 1, and therefore we find both SF and CDW order, i.e. supersolid behavior. The phase diagram of the dual mixture is of the following form: the attractive and repulsive regime are exchanged, and in the gapped phase we again have the mapping: \(CDW→SDW\textsubscript{z}, SDW\textsubscript{z}→CDW, SS→SDW\textsubscript{x,y},\) and \(SDW\textsubscript{x,y}→SS\). The gapless regime is unaffected.

The paired SF state discussed in \(\text{[13]}\) corresponds to the SS phase discussed here, whereas the dual \(SDW\textsubscript{x,y}\) phase that appears for complementary filling corresponds to the super-counter-fluid phase described therein. Note that here these orders compete with either CDW or \(SDW\textsubscript{z}\) order, and only appear as QLRO, not LRO, as in higher dimensions. Both of these insights can only be gained by the using the LL description and RG that is used in this paper.

VII. BOSE-FERMI MIXTURES

For a BFM we find that the order parameters \(O_{CDW}, O_{SDW}\), the polaron pairing operator

\[
O_{f-PP} = f_R f_L e^{-2i\lambda \Phi_5},
\]

(see \(\text{[9,10]}\)), and \(b\) can develop QLRO in the gapless regime. In the gapped regime, the order parameters

\[
O_{PP} \equiv f_R b f_L b, \quad O_{PP'} \equiv f_R b^\dagger f_L b^\dagger,
\]

in addition to \(O_{CDW}\), show QLRO. \((O_{PP}/PP')\) are special cases of the polaron pairing operator \(\text{(27)}\), extensively discussed in \(\text{[9] and [10]}\). In Fig. 3 we show the phase diagram of a BFM with hardcore bosons, and in Fig. 4 we vary the Luttinger parameter of the bosons. In both the gapless phase and the gapped phase, we find that CDW and \(f-PP\) or PP, respectively, are mutually exclusive and cover the entire phase diagram, cp. \(\text{[9,10]}\). The dual mapping again maps attractive and repulsive regimes onto each other. Within the gapped phase we find the mapping \(CDW→SDW\textsubscript{z}, SDW\textsubscript{z}→CDW,\) and \(PP→PP'\), the gapless regime is unaffected.

VIII. EXPERIMENTAL DETECTION

The phase diagrams that have been derived and shown in Figs. 1-4 are given in terms of the parameters that appear in the effective action. With such a field theoretical approach we can find the correct qualitative long-range behavior, such as the functional form of the correlation functions. However, it is also intrinsic to this approach...
that the effective parameters appearing can only be qualitatively related to the underlying microscopic parameters. Based on a phase diagram such as Fig. 2, for instance, the following features for a mixture of bosonic atoms with a short-range interaction can be expected: If one species is in the hardcore limit, and the other is in between an intermediate interaction regime and the hardcore limit, then for attractive interaction between them, a gapped state can be created, in which there is a competition of SS pairing and CDW order. For repulsive interaction, and the second species being very close to the hardcore regime, one can also expect a gapped phase, in which we find SDW order. For the intermediate regime we expect a supersolid phase.

Before we conclude, we discuss how the predictions presented in this paper could be measured experimentally. Since the appearance of a gapped state has already been demonstrated for the MI-SF transition in 1D [3], and since it constitutes a significant qualitative change in the system, this would be the first feature predicted in this paper to look for. As demonstrated in [23], RF spectroscopy can be used to determine the presence and size of an energy gap. To detect the rich structure of QLRO the following approaches can be taken: CDW order will create additional peaks in TOF images, corresponding to a wavevector $Q = 2k_f$. As demonstrated and pointed out in [22], the noise in TOF images allows to identify the different regimes of both gapped and gapless phases. As discussed in [9,10], a laser stirring experiment could determine the onset of CDW order for fermions, or the supersolid regime for bosons.

IX. CONCLUSION

In conclusion, we have studied mixtures of ultra-cold atoms in 1D with commensurate filling. We used a Luttinger liquid description which enables us to study FFMs, BFMs, and BBMs in a single approach. We find that FFMs are generically gapped for both attractive and repulsive interactions, whereas for BFMs and BBMs the bosons need to be close to the hardcore limit. We find a rich structure of quasi-phases in the vicinity of these transitions, in particular a supersolid phase for BBMs, that occurs close to the hardcore limit. Experimental methods to detect the predictions were also discussed.

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APPENDIX A

Here we give the coefficients that appear in the transformations (8), (9), (10), and (17), that map the original fields on the eigenfields at each point in the RG flow. The coefficients $B_{1/2}$ and $D_{1/2}$ are given by:

$$
B_1 = \beta_1 \zeta_1 + \beta_2 \kappa_1, \quad B_2 = \beta_1 \zeta_2 + \beta_2 \kappa_2, \quad (A1)
$$

$$
D_1 = \delta_1 \zeta_1 + \delta_2 \kappa_1, \quad D_2 = \delta_1 \zeta_2 + \delta_2 \kappa_2. \quad (A2)
$$

The coefficients $\beta_{1/2}$ and $\delta_{1/2}$ are given in [9,10], where the indices 'f' and 'b' need to be replaced by '1' and '2', respectively. The other coefficients are given by:

$$
\zeta_1 = \sqrt{v_1/v_A} \cos \theta, \quad \zeta_2 = \sqrt{v_2/v_A} \sin \theta, \quad (A3)
$$

$$
\kappa_1 = -\sqrt{v_1/v_a} \sin \theta, \quad \kappa_2 = \sqrt{v_2/v_a} \cos \theta, \quad (A4)
$$

where the angle $\theta$ is given by:

$$
tan 2\theta = -\tilde{V}_{12}/\sqrt{v_A v_a (\tilde{v}_A^2 - \tilde{v}_a^2)}, \quad (A5)
$$

where we used:

$$
\tilde{v}_A = v_A/\sqrt{1 + 2V_{12} v_A \beta_1 \delta_1/\pi}, \quad (A6)
$$

$$
\tilde{v}_a = v_a/\sqrt{1 + 2V_{12} v_A \beta_2 \delta_2/\pi}. \quad (A7)
$$

The velocities $\tilde{v}_{1/2}$, corresponding to the eigenmodes $\tilde{\theta}_{1/2}$, are given by

$$
\tilde{v}_{1/2}^{-2} = 1/2(\tilde{v}_A^{-2} + \tilde{v}_a^{-2}) \pm 1/2 \sqrt{(\tilde{v}_A^{-2} - \tilde{v}_a^{-2})^2 + \tilde{V}_{12}^2 v_A^{-1} v_a^{-1}} \quad (A8)
$$

where $\tilde{V}_{12}$ is given by

$$
\tilde{V}_{12} = 2V_{12}(\beta_1 \delta_2 + \beta_2 \delta_1)/\pi. \quad (A9)
$$

$v_{A,a}$ are defined as in [9,10]. The coefficients $C_{1/2}$ and $D_{1/2}$, that appear in the dual transformation, Eqs. (10) and (17), are given by:

$$
C_1 = \gamma_1 \eta_1 + \gamma_2 \lambda_1, \quad C_2 = \gamma_1 \eta_2 + \gamma_2 \lambda_2, \quad (A10)
$$

$$
E_1 = \epsilon_1 \eta_1 + \epsilon_2 \lambda_1, \quad E_2 = \epsilon_1 \eta_2 + \epsilon_2 \lambda_2. \quad (A11)
$$

$\gamma_{1/2}$ and $\epsilon_{1/2}$ are given in [9,10], with 'f' and 'b' replaced by '1' and '2', and $\eta_{1/2}$ and $\lambda_{1/2}$ given by:

$$
\eta_1 = \sqrt{v_A/v_1} \cos \theta, \quad \eta_2 = \sqrt{v_A/v_2} \sin \theta, \quad (A12)
$$

$$
\lambda_1 = -\sqrt{v_a/v_1} \sin \theta, \quad \lambda_2 = \sqrt{v_a/v_2} \cos \theta. \quad (A13)
$$
Due to the confining trap the transition is, while visible, 'blurred' into a gradual cross-over, due to finite size effects, and, more importantly, due to the coexistence of several phases in the trap. This can also be expected for the phase transitions predicted in this paper.