FLAT INFORMATION GEOMETRIES
IN BLACK HOLE THERMODYNAMICS

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Abstract

The Hessian of either the entropy or the energy function can be regarded as a metric on a Gibbs surface. For two parameter families of asymptotically flat black holes in arbitrary dimension one or the other of these metrics are flat, and the state space is a flat wedge. The mathematical reason for this is traced back to the scale invariance of the Einstein-Maxwell equations. The picture of state space that we obtain makes some properties such as the occurrence of divergent specific heats transparent.

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1. Introduction

It has been known for a long time that black holes can be described as thermodynamic systems. The connection to ordinary thermodynamics was clinched by Hawking’s calculation, showing that the surface gravity of the event horizon equals $2\pi$ times the ordinary temperature of the radiation emitted by the black hole to infinity [1]. The original derivation of the laws of black hole thermodynamics has nothing in common with statistical mechanics, but there is a general belief that a connection nevertheless exists at the quantum gravity level. Still there are questions that can be pursued without any quantum theory at hand. Thus we can observe that there are many thermodynamical systems, but—due to uniqueness theorems—only a few kinds of stationary black holes. Hence the Gibbs surfaces that arise in black hole thermodynamics are of necessity very special. We can ask: in what way?

In this paper we will employ a device that was introduced in thermodynamics by Weinhold [2] and Ruppeiner [3]. Their observation is that the Hessian matrix of the second derivatives of the energy, or alternatively the entropy, can be regarded as a Riemannian metric on the space of thermodynamical states. When energy is used as a potential this metric is called the Weinhold metric, when the entropy is used it is called the Ruppeiner metric. It is essential that the energy and the entropy are regarded as functions of the extensive variables, such as volume and particle number. Ruppeiner’s proposal is related to the use of the canonical ensemble, and his metric is closely connected to thermodynamic fluctuation theory. For self–gravitating systems it is natural to work with the microcanonical ensemble, and extensivity does not hold. But we can still demand that energy and entropy are functions of mechanically conserved control parameters, such as angular momentum and electric charge, and then proceed as before. The interesting thing is that very special metrics ensue—the Weinhold metric turns out to be flat for the Kerr black hole in arbitrary spacetime dimension (provided that only one spin parameter is used), while the Ruppeiner metric is flat for the Reissner–Nordström black hole in arbitrary spacetime dimension [4] [5]. This is so provided that the cosmological constant is kept to zero. When a (negative) cosmological constant is turned on the thermodynamical metrics develop curvature [4]. (The 2+1 dimensional BTZ black hole is an exception; it has a flat Ruppeiner metric.)

There is more to this story. There are numerous studies of soluble models
in statistical mechanics which suggest that the detailed behaviour of the curvature of the Ruppeiner metric carries information about phase transitions, and indeed about the underlying statistical mechanical model (see Ruppeiner [3], and a more recent review [6]). Something similar may be true in black hole thermodynamics; in particular Arcioni and Lozano-Tellechea argue that it is relevant for fluctuations around near extremal black holes [7]. Let us also take note of the suggestion that the thermodynamical metrics may provide us with aspects of black hole physics that are safe against quantum corrections [8]. In a different direction we observe that the use of a Hessian matrix as a Riemannian metric arises in other contexts. The obvious example is mathematical statistics, where such metrics are known as information metrics. What thermodynamics and mathematical statistics have in common—the connections at the level of statistical mechanics apart—is a preferred affine structure with respect to which the second derivatives are defined.

The present paper has two purposes. First, to investigate the mathematical requirements for having a flat Ruppeiner metric. The conclusion is that the black hole examples have flat thermodynamic geometries, with wedge shaped state spaces, because of the special quasi–homogeneity properties of their fundamental relations. A second purpose is to explore the picture of state space offered by the Ruppeiner theory, and incidentally to comment on some criticism directed against our earlier work [9].

2. Flat information metrics

We begin with some generalities. We study metrics that are defined, in some preferred affine coordinate system, by

\[ g_{ij} = \partial_i \partial_j \psi . \] (1)

The potential \( \psi \) can be any reasonable function. Examples occur in mathematical statistics, where a favoured choice of potential is

\[ \psi = \sum_{i=1}^{N} x^i \ln x^i , \quad x^i > 0 . \] (2)

As it stands this is a flat metric on the positive cone, but if the condition that the positive numbers \( x^i \) sum to unity is imposed it becomes round—and the potential becomes equal to the Shannon entropy with sign reversed, while
the metric itself is known as the Fisher information matrix [10]. This serves to explain why such metrics are called information metrics.

In thermodynamics the potential is either the entropy with sign reversed, or the energy function. If

$$\psi = -S(M, Q)$$

the corresponding metric is known as the Ruppeiner metric, if

$$\psi = M(S, Q)$$

it is the Weinhold metric. These two metrics are related by a conformal factor equal to the temperature,

$$ds^2_W = Tds^2_R, \quad T \equiv \left( \frac{\partial M}{\partial S} \right)_Q.$$  \hspace{1cm} (5)

The preferred affine coordinate systems are provided by the mechanically conserved control parameters, including energy or entropy. Our choice of the letter $M$ for the energy is of course dictated by our interest in black hole thermodynamics. For the Reissner–Nordström black hole $Q$ denotes the electric charge, while for Kerr it stands in for angular momentum. In this paper we will consider two dimensional state spaces only.

Our question is: when is an information metric flat? One possibility is that

$$\psi = \sum_{i=1}^{N} f_i(x^i).$$

The Fisher metric on the positive cone comes from such a potential. Moreover the Ruppeiner metric of the ideal gas at fixed particle number is of this type. However, the black hole information metrics are not.

Next consider potentials that have the quasi–homogeneity property

$$\lambda^{a_3} \psi(x, y) = \psi(\lambda^{a_1} x, \lambda^{a_2} y).$$

We can afford to assume that $x > 0$ and $\psi > 0$. The property is equivalent to

$$\psi(x, y) = x^a f(x^b y),$$

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where $f$ is some function and $a$, $b$ are some exponents. (To show this [11], choose $\lambda^{a_1}x = 1$. One finds $a = a_3/a_1$, $b = -a_2/a_1$.) Note that

$$S(M, Q) = M^a f(M^b Q) \iff M = S^{1/a} h(S^{b/a} Q). \quad (9)$$

We can now state a small theorem, namely: If $\psi(x, y) = x^a f(y/x)$ then the information metric is flat. The converse does not hold.

The most straightforward proof of this theorem is to change to the new coordinates

$$\psi = x^a f(x^b y) \quad \text{and} \quad \sigma = x^b y. \quad (10)$$

(To avoid misunderstanding: the metric is always defined using differentiation with respect to the preferred coordinates—but once the metric is given we can use any coordinates we please.) An explicit calculation shows that

$$ds^2 = \left(\frac{a - 1}{a} - \frac{b(b + 1)\sigma f'}{a^2 f}\right) \frac{d\psi^2}{\psi} + 2(b + 1) \left(\frac{1}{a f} + \frac{b}{a^2 f^2}\right) d\psi d\sigma +$$

$$\left(\frac{f''}{f} - \frac{2b + a + 1}{a} f^2 - \frac{b(b + 1)\sigma f'^2}{a^2 f^3}\right) d\sigma^2. \quad (11)$$

This is diagonal if $b = -1$. If we introduce the new coordinate $r = \sqrt{\psi}$ it is also manifestly a flat metric, and it covers a wedge shaped region. Given the function $f$ we can reparametrize $\sigma$ so that we end up with polar coordinates, or Rindler coordinates if the metric is Lorentzian, and read off the opening angle of the wedge. Anyway the small theorem is proved. There is an exception if $b = -1$ and $a = 1$, since then the metric is degenerate. This is so for homogeneous potentials in any dimension.

An intermediate step in the calculation is of interest as well. Using $x$ and $\sigma$ as coordinates we get

$$ds^2 = x^{a-2} \left( (a-1)f - b(b+1)\sigma f' \right) dx^2 + 2(a+b)x f' dx d\sigma + x^2 f'' d\sigma^2. \quad (12)$$

If $a + b = 0$ this is diagonal, and can be written as

$$ds^2 = \psi_x \left( (a-1) \frac{dx^2}{x} + \frac{x f''}{a f - \sigma f'} d\sigma^2 \right), \quad (13)$$
where $\psi_x$ is the derivative with respect to $x$ of $\psi(x,y)$. This is the metric on a flat wedge multiplied with the conformal factor $\psi_x$, and should be compared to eq. (5).

We can restate the small theorem in thermodynamical language: Let

$$ S = M^a f(M^b Q) . $$

(14)

If $b = -1$ the Ruppeiner metric is flat. If $a + b = 0$ the Weinhold metric is flat.

It is instructive to look at the Riemann curvature tensor as well. In the preferred coordinate system the Christoffel symbols (with one index lowered using the metric) are given by

$$ \Gamma_{ijk} = \frac{1}{2} \partial_i \partial_j \partial_k \psi . $$

(15)

In mathematical statistics this is also known as the skewness tensor [10]. The expression for the Riemann curvature tensor then simplifies to

$$ R_{ijkl} = \Gamma_{ikm} g^{mn} \Gamma_{njl} - \Gamma_{ilm} g^{mn} \Gamma_{njk} . $$

(16)

For our special choice of potential, eq. (8), setting the Riemann tensor to zero results in a non–linear third order ODE of somewhat frightening aspect. To be precise about it, the curvature scalar is

$$ R = \frac{(b + 1)x^{3a + 4b - 4}}{2g^2} [a(a - 1)(a + b)f'f''m - 2a(a - 1)(a + 2b)f f''m^2 - $$

$$ - ab(a - 1)\sigma f f''m + (a + b)^2(a + b - 1)f'^2f'' + $$

$$ + b(a + b)(2a + b - 1)\sigma f'^2f'' + $$

$$ + b(2b - a^2 - 3ab)\sigma f'^2f''^2 + b^2(b + 1)\sigma^2(f' f''m - f''^3)] $$

(17)

where $g$, the determinant of the metric, is

$$ g = x^{2(a+b-1)}[a(a - 1)f'f'' - (a + b)^2 f'^2 - b(b + 1)\sigma f'f''] . $$

(18)
For our present purposes the main feature of $R$ is that it has $(b + 1)$ as a prefactor, and therefore the metric is flat for $b = -1$ whatever the form of the function $f$.

Regardless of why it happens, flatness has an interesting mathematical consequence. Suppose that the Riemann tensor vanishes. A somewhat similar structure arises in the theory of Frobenius manifolds [12], where it is observed that the resulting equation can be used to define an algebra through

$$
\partial_i \circ \partial_j = \Gamma_{ijm} g^{mk} \partial_k .
$$

This algebra is commutative by construction, and associative because of eq. (16). Such algebras are used to describe the moduli space of topological conformal field theories; this sounds as if it might, through some back door, have some connection to black hole thermodynamics, but in fact the two settings are very different. In the theory of Frobenius manifolds the metric tensor used in eq. (16) is a fixed quadratic form, and the skewness tensor (15) is not a Christoffel symbol of any relevant metric. Thus, reluctantly, we conclude that the theory of Frobenius manifolds is irrelevant to us.

### 3. Black hole examples

For black holes the fundamental relation relates the area of the event horizon to the ADM charges of the black hole. More precisely we set $S = kA/4$, where $A$ is the area of the event horizon and $k$ is Boltzmann’s constant. We adjust the numerical value of the latter to simplify the resulting expression.

When the cosmological constant vanishes, the Einstein-Maxwell equations are scale invariant. This has consequences for the solutions, which can be deduced by dimensional analysis. Using length as the only basic unit, the black hole control parameters have dimensions

$$
[S] = L^{d-2}, \quad [M] = L^{d-3}, \quad [Q] = L^{d-3}, \quad [J] = L^{d-2},
$$

(20)

where $d$ is the dimension of spacetime. It follows that the fundamental relation will have quasi-homogeneity properties, with definite exponents. Indeed

$$
L^{d-2} S(M, Q, J) = S(L^{d-3} M, L^{d-3} Q, L^{d-2} J).
$$

(21)

Hence, by the result quoted in the previous section, in the two parameter cases the fundamental relations must be
\[ S = M^{\frac{4-d}{d-3}} f \left( \frac{Q}{M} \right) \quad \text{and} \quad S = M^{\frac{4-d}{d-3}} f \left( \frac{J}{M^{\frac{4-d}{d-3}}} \right). \]  

Finally the theorem proved in section 2 implies that the Ruppeiner geometry of the Reissner–Nordström black holes will be flat in any dimension, and similarly the Weinhold geometry of the Kerr black holes is flat in any dimension. This will be true also for “exotic” Kerr black holes such as the “black ring” in five dimensions [13].

Some explicit examples are as follows. The Reissner–Nordström black hole in arbitrary spacetime dimension \( d \) has the fundamental relation

\[ S = M^c \left( 1 + \sqrt{1 - \frac{c Q^2}{2 M^2}} \right)^c, \quad c \equiv \frac{d-2}{d-3}. \]  

The Ruppeiner geometry is a timelike wedge in a flat Minkowski space, with an opening angle that grows with \( d \). It is a black hole if the integer \( d \geq 4 \). The Kerr black hole in spacetime dimension \( d \) has the fundamental relation

\[ M = \frac{d-2}{4} S^{d-4} \left( 1 + \frac{4 J^2}{S^2} \right)^{1/(d-2)}. \]  

The Weinhold geometry is a timelike wedge in Minkowski space for \( d = 4 \) and \( d = 5 \), while it fills the entire forwards light cone when \( d \geq 6 \) (due to the absence of extremal Kerr–Myers–Perry black holes in these dimensions [14]). An explicit form of the fundamental relation for the black ring can be found in the literature [7]. For the three dimensional Kerr–Newman family the Ruppeiner and Weinhold metrics are both curved [4].

Our dimensional argument fails in the presence of a cosmological constant. In spite of this the 2+1 dimensional BTZ black hole [15] has a fundamental relation of the form \( S = M^a f(J/M) \), and hence its Ruppeiner state space is a flat wedge (in an Euclidean space) [4]. In higher dimensions anti-de Sitter black holes have curved thermodynamic geometries.

It is instructive to compare the black hole examples to the ideal gas, which has the fundamental relation

\[ S = N \ln \left[ \frac{V}{N} \left( \frac{U}{N} \right)^c \right] + k_1 N \quad \Leftrightarrow \quad U = k_2 \frac{N(c+1)/c}{V^{1/c}} e^{S/(cN)}, \]  

\[ (25) \]
where \(c\) is the ratio of specific heats, \(k_1\) and \(k_2\) are constants, and we use \(U\) for energy. Since \(S = S(U,V,N)\) is a homogeneous function the three dimensional Ruppeiner metric is actually degenerate, but if we consider the ideal gas at fixed volume \(V\) it belongs to the class \((8)\). For the Ruppeiner case we have \(a + b = 0\), while for the Weinhold case \(b = -1\). Hence the Weinhold geometry is flat; the opening angle of its wedge turns out to go between zero and infinity, so it is actually an infinite covering of the punctured plane. But more is true: because of the quite special function involved the Ruppeiner metric also is flat. Similarly both the Weinhold and Ruppeiner geometries of the ideal gas at fixed particle number \(N\) are flat; the latter describes a flat plane \([16]\). This illustrates that our small theorem gives a sufficient but not necessary condition for flatness, and it shows that the ideal gas is even more special than our black hole examples. Another case where both the Weinhold and the Ruppeiner metrics are flat is given by the Kerr formula \((24)\), for the unphysical value \(d = 3\).

4. The Reissner–Nordström black hole

In this section we focus on the Reissner–Nordström family of black holes in four spacetime dimensions. The Gibbs surface is defined by the fundamental relation

\[
S = M^2 \left(1 + \sqrt{1 - \frac{Q^2}{M^2}}\right)^2. \tag{26}
\]

The Ruppeiner metric can be obtained from eq. \((12)\). Actually it is convenient to trade the coordinate \(\sigma = Q/M\) for

\[
u \equiv \frac{\sigma}{1 + \sqrt{1 - \sigma^2}} = \frac{Q}{\sqrt{S}} = \left(\frac{\partial M}{\partial Q}\right)_S, \quad -1 \leq u \leq 1. \tag{27}
\]

This coordinate is conjugate to the charge, and equals the electric potential at the event horizon. We now get the metric in the form

\[
ds^2 = -\frac{dS^2}{2S} + \frac{4Sdu^2}{1 - u^2} = -d\tau^2 + \tau^2 d\chi^2. \tag{28}
\]

In the last step we traded our coordinates for
Figure 1: The state space of the Reissner-Nordström black hole is a wedge inside the forwards light cone of a 1+1 dimensional Minkowski space. We show curves of constant entropy (spacelike hyperbolas), constant mass (also spacelike), constant temperature, and constant charge. The latter two become null at the “Davies point”, which is given by a dashed line of constant electric potential.

\[ \tau = \sqrt{2S} \quad \chi = \sqrt{2 \arcsin u} . \]  

(29)

The coordinates \( \tau \) and \( \chi \) are the usual Rindler coordinates on the forward lightcone in Minkowski space. If we like we can introduce the inertial coordinates

\[ t = \tau \cosh \chi \quad x = \tau \sinh \chi . \]  

(30)

A picture of the state space as a flat wedge is given in Fig. 1, with some details added. Our picture for the Kerr case (based on the Weinhold metric) is qualitatively similar, although the wedge is thinner and curves of constant \( Q \) are replaced by curves of constant \( J \).
In a recent paper it was argued that the entropy ought to be expressed as a function of the enthalphy and the electric potential, before the derivatives are taken [9]. We do not wish to appear dogmatic on this, or any other point. But we are concerned with the consequences of the definitions that we have stated. There was also a more specific criticism. It was argued (long ago) by Davies [17] that charged black holes suffer a second order phase transition at \( Q/M = \sqrt{3}/2 \). The argument was based on the observation that the specific heat

\[
c_Q \equiv T \frac{\partial S}{\partial T} \bigg|_Q = \frac{2S(S - Q^2)}{3Q^2 - S} \tag{31}
\]

diverges there, and then changes sign. Although this argument was quickly challenged [18], it resurfaces now and then. Thus Penrose [19] has used it to suggest that it should be qualitatively easier to understand the entropy of an extremal black hole in state counting terms, because unlike its Schwarzschild counterpart such a black hole has positive specific heat. It has also been used to suggest that the Ruppeiner metric as defined by us must be irrelevant since “a statistical model without any interaction cannot reproduce thermodynamic properties of the RN black hole” [9]. Now we never claimed that the flat Ruppeiner metric proves that the “statistical model” is non-interacting. Nor are we worried by any phase transition at \( Q/M = \sqrt{3}/2 \). As explained by Sorkin [20] and others [21], a microcanonical instability would occur only if the specific heat changed sign through zero.

From our present point of view let us observe that the specific heat is not only a property of the Gibbs surface, it is also a property of the special curve along which we evaluate the specific heat. As one can see in the picture, what happens at \( Q/M = \sqrt{3}/2 \) is that the particular curve defined by constant \( Q \) changes from timelike (negative \( c_Q \)) to spacelike (positive \( c_Q \)). At \( Q/M = \sqrt{3}/2 \) the curve is null, which means that the heat capacity related to this curve diverges [2]. Similarly, curves of constant \( J \) become null at the “Davies’ point” of the Kerr black hole. Indeed this kind of behaviour will occur, for some curves, in every point on a Gibbs surface where the entropy function is non-concave—as it typically will be for self-gravitating systems [22]. For the question of stability in the microcanonical ensemble this is quite irrelevant.
5. Conclusions

We have investigated the fact that some black hole families have either flat Ruppeiner metrics or flat Weinhold metrics. We also used this fact to draw a simple picture of the state space of Reissner–Nordström black holes.

Our main result is that eqs. (1) and (8), with \( b = -1 \), always give a flat information metric defined on a wedge in a flat space. All our black hole examples belong to this class; in the asymptotically flat case this is a consequence of the scale invariance of the Einstein equations. It is perhaps a little disappointing that we do not obtain any restriction on the free function contained in the potential.

Although we have understood why certain thermodynamical metrics of some black holes are flat, there are many questions that remain to be investigated. There may be more to the fact that Reissner-Nordström black holes have a flat Ruppeiner geometry, while that of the Kerr black holes is curved. After all for the former there are quantum gravity based calculations of the entropy, but not so far for the latter. The broader question about the nature of the Gibbs surfaces that appear in black hole thermodynamics has many aspects; in this paper we have investigated only one of them.

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