Very elementary interpretations of the Euler-Mascheroni constant from counting divisors in intervals

David Feldman

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1 Introduction

After $\pi$, and then $e$, or perhaps the golden ratio $\phi$, the Euler-Mascheroni number $\gamma$ stands among the most famous mathematical constants. We aim here for a formulation of $\gamma$ that makes it accessible to the widest possible public.

Now the public knows $\pi$ best by dint of its connection to computing circular perimeters and areas. Though a mathematician would analyze these computations by employing the apparatus of limits à la Cauchy, merely communicating the meaning of $\pi$ should not depend on Cauchy’s sophisticated, abstract, universal, rigorous formulation of limits. Cauchy’s approach synthesized diverse mathematical discourses, but it did not abolish them. The public face of a mathematical constant should preferably not depend on familiarity with Cauchy style limits.

While a mathematical constant will possess a single, definite value, it may admit many interpretations according to the diverse contexts where it arises. For example, formulated properly, we may say that, with probability $6/\pi^2$, two natural numbers chosen at random share no factor greater than 1. That $\pi$ occurs here despite the lack of any apparent connection to circles beautifully exemplifies the sort of excitement associated with pure mathematics!

The public does not know $e$ as well as it knows $\pi$, but $e$ too admits accessible narrative interpretations. If the public knew hyperbolas as well as it knows circles, one could effectively characterize $e$ as that number (greater
than 1) such that the area under \( y = \frac{1}{x} \) over the interval \([1, e]\) equals 1. Closer to practical concerns, one can observe that $1 left in the bank for a year at 100% interest, compounded continuously, grows to $e. A seemingly very different take on \( e \) involves *derangements*. Supposing that \( n \) people participate in a Christmas party grab bag, we can ask for the probability that no one gets their own gift back. All the probabilities with \( n \) even exceed all the probabilities with \( n \) and only the number \( 1/e \) lies in between.

The Euler-Mascheroni constant \( \gamma \) cries out for a canonical narrative interpretation suitable for public consumption. Steven R. Finch’s encyclopedic *Mathematical Constants* lists several candidates, where the most compelling takes the form

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left\{ \frac{n}{k} \right\} = 1 - \gamma
\]

a result of de la Vallée Poussin. As Finch paraphrases de la Vallée Poussin’s result:

\[
\ldots \text{if a large integer } n \text{ is divided by each integer } 1 \leq k \leq n, \text{ then the average fraction by which the quotient } n/k \text{ falls short of the next integer is not } 1/2, \text{ but } \gamma!
\]

As an elementary interpretation of \( \gamma \), de la Vallée Poussin result has two nice features not shared by Finch’s other examples. First, \( \gamma \) occurs more or less directly, rather than embedded in a formula such as \( e^{-\gamma} \). Second, de la Vallée Poussin’s formula for \( \gamma \) refers only to basic arithmetic and in particular avoids mention of natural logarithms.

We offer here a novel elementary interpretation (indeed a vast family of such interpretations) of \( \gamma \) sharing the stated advantages of de la Vallée Poussin’s and the additional advantage, perhaps, that it arises very naturally if one considers a very modest variation on a very familiar mathematical situation.

We mean to address two sorts of readers at once, namely those who have had (or remember) only high school mathematics and would like to learn about the Euler-Mascheroni constant from scratch, and those who know enough calculus to digest the usual definition and wish to understand its equivalence with our reformulation. The former may just skip without loss some remarks obviously directed at the latter, who should exercise patience with details spelled out for the former.
We begin by recalling the usual formula for the Euler-Mascheroni and then offer an alternative formula in the same spirit which nevertheless eliminates the explicit appearance of natural logarithms. Our first attempt at attaching a very simple, compelling narrative interpretation to our formula for $\gamma$ produced only a fallacy, albeit an instructive one. Rather than suppress this initial failure, we start there, so that the reader will appreciate the mildly technical but unavoidable modification required for a valid interpretation.

## 2 The standard definition of the Euler-Mascheroni constant

We begin by explaining in elementary terms the usual definition

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right).$$

We wish to interpret $\gamma$ geometrically. For this purpose it does no harm to make the modification

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n-1} \frac{1}{k} - \ln(n) \right),$$

or (after reindexing)

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) \right).$$

Note that the general term of the original sequence and of the (first formulation of the) new sequence differ by $1/n$, and the difference approaches 0 as $n$ grows, justifying the modification.

The (reindexed) new sequence leads to the area of the shaded region in the following diagram:
Here the curve in the diagram represents the graph of $y = 1/x$. Indeed the first term of the new sequence gives the area of the leftmost black wedge, the second term the area of the two leftmost wedges, and generally, the $n$th term the area of the $n$ leftmost wedges. Explicitly, the sum $\sum_{k=1}^{n} \frac{1}{k}$ gives the area of the $n$ leftmost rectangles and $\ln(n + 1)$ means the area under the curve and within these rectangles. Taking the limit gives the area of all the wedges.

Now imagine all the wedges sliding horizontally to the left until we have the stacked vertically within the square, our original leftmost rectangle:
From this picture we see (or at least glean the tools needed to prove) the finiteness of $\gamma$. As a subset of the unit square it must have an area between 0 and 1, and the picture even makes clear that the area of all the wedges must exceed $.5$. Moreover the area of the first $n$ wedges falls short of $\gamma$ by no more than $1/n$ (since the remaining wedges fit in 1 by $1/n$ rectangle), but also by at least $1/2n$ (since they fill more than half of that rectangle).

Observe that, as with $\pi$, we can interpret $\gamma$ as the area of a region in the plane that can construct explicitly. Of course this region seems highly artificial compared with the unit circle. To a student of integral calculus the region should seem less unnatural. In that context, $\gamma$ bounds the error that occurs when approximating the areas defining natural logarithms of natural numbers by means of upper sums. Of course one can approximate a give area by many different upper sums, but these upper sums often arise in their own right, as harmonic sums $\sum_{k=1}^{n} 1/k$. One often has occasion to turn the story around, and using (sophisticated but easily manageable natural logarithms to approximate (elementary but awkward) harmonic sums. As an a priori estimate of the error involved, $\gamma$ can help us improve such approximations, and in this role it enters many formulas.

3 Getting rid of the logarithms

The following pictures suggest some calculations to approximate $\gamma$ which don’t involve logarithms, and thus lead to a way of framing $\gamma$ for an audience that doesn’t know about logarithms (and doesn’t want to hear about them):
By way of explanation, we would like to estimate the total area of all the wedges without computing exactly the area under the curve. We do this by now also approximating the region under the graph by a union of rectangles, but we let these approximations get more refined as we go.

As far as concerns estimating $\gamma$, we now have two sources of error. First, the $n^{th}$ picture only takes account of the first $n$ wedges. Second, we have unwanted area now below the various wedges.

We have already bounded the magnitude of the first type of error by $1/n$. We can also approximate the second type of error by sliding wedges, this time the new wedges we have created under the graph. In the $n^{th}$ pictures, these will all slide horizontally to fit inside a $1/n$ by 1 rectangle. So $1/n$ also bounds the second type of error. These two types of error, moreover, carry opposite signs, so certainly $1/n$ bounds the total error.

Numerically, the area of the regions in the three diagrams equal, as the

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1 In actually the two types of error tend to cancel. It turns out that $n^2$ times the error approaches $2/3$. Our reformulation converges to $\gamma$ rather must faster than the original definition.
reader may easily check,

\[
1 - \frac{1}{3} - \frac{1}{4},
\]

\[
1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9},
\]

\[
1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} - \frac{1}{13} - \frac{1}{14} - \frac{1}{15} - \frac{1}{16},
\]

or in general

\[
\sum_{k=1}^{n-1} \frac{1}{k} - \sum_{k=n+1}^{n^2} \frac{1}{k}
\]

for the \( n - 1 \)st picture.

In words, to approximate \( \gamma \), for \( q = n^2 \), we sum the reciprocals of numbers less than \( \sqrt{q} \) and subtract off the reciprocals of all numbers greater than \( \sqrt{q} \) up to \( q \). Indeed we need not require making \( q \) a perfect square. We see this easily by comparing the recipe applied to a general \( q \) with the recipe applied to the largest square below it.

So, as a slogan, for large numbers \( q \), \( \gamma \) approximates the sum of the reciprocals of the numbers below the square root of \( q \) minus the sum of the reciprocals of the numbers above the square root of \( q \), up to \( q \).

4 A fallacy, first

Roughly speaking, for random \( q \), the probability that \( d \) divides \( q \) equals \( 1/d \) (since dividing \( q \) by \( d \) can leave \( d \) possible remainders, all equally likely, with 0 just one among them).\(^2\) In probability theory one typically introduces a quantity that equals 1 when an event occurs and 0 when it doesn’t. The expectation of this sort of quantity (intuitively, its value on the average) coincides with its probability. The virtue of working with expectations rather than directly with probabilities lies in the linearity of expectation: the expectation of a sum equals the sum of the expectations.

\(^2\)We must say “roughly speaking” because we cannot make literal sense of “for random \( q \)” since the set of all natural numbers does not carry any uniform probability distribution. We may of course speak of a random \( q \) between 1 and \( B \), but depending upon the \( B \), the probability may not equal exactly \( 1/d \); the larger the \( B \), though, the smaller the error.
So suppose we have a set \( D = \{d_1, \ldots, d_j\} \). Again, roughly speaking, the expected number of elements of \( D \) that divide a random \( q \) should equal

\[
\frac{1}{d_1} + \cdots + \frac{1}{d_j}.
\]

Notice that when \( D \) consists of many consecutive natural numbers, the expected number of elements of \( D \) that divide a random \( q \) has the form of the sort of quantities that come into our approximations for \( \gamma \).

This perhaps suggests asking if \( \gamma \) approximates the expectation of \( Z \), defined as the number of divisors of \( q \) below \( \sqrt{q} \) minus the number of divisors of \( q \) above \( \sqrt{q} \).

\( Z \) does indeed have an expectation, but its expectation turns out equal to 0, not \( \gamma \)!

Indeed, if \( d \) divides \( q \), so does \( q/d \), and if one lies below \( \sqrt{q} \) the other lies above, and vice versa. For example, if \( d < \sqrt{q} \) and also \( q/d < \sqrt{q} \), we have

\[
q = d \cdot (q/d) < (\sqrt{q})^2 = q,
\]

a contradiction, and similarly for \( d, q/d > \sqrt{q} \). Thus every number \( q \) has exactly the same number of divisors below \( \sqrt{q} \) as above.

Of course, the reader already trained to refuse even to hear all but the most rigorous analysis will find no fallacy here. However, in mathematics, our type of heuristic reasoning does often lead, after careful formulation, to true statements, albeit often these statements turn out much harder to prove than the heuristics suggest. So even though we made clear when we left the realm of rigorous reasoning, perhaps it still comes as a surprise that we have failed so badly, that the gaps we left do not admit any repair.

The reader may well wish to think upon the question of what sort of burden a failed heuristic imposes. We have proved that it leads us to a wrong conclusion. Generally speaking we don’t feel we need to explain why erroneous proofs lead to false conclusions! Nevertheless, when an erroneous proof depends on the unproved assumption that certain quantities vary independently when in fact they don’t, we ought enquire into the nature of their interdependence. Alternatively, and we take this approach here, we can see if can rescue the heuristic by some slight change of the situation.

We surely can make perfect sense of “the expected number of elements of \( D = \{d_1, \ldots, d_j\} \) that divide a random \( q \) equals \( \frac{1}{d_1} + \cdots + \frac{1}{d_j} \)” provided that

\[\text{We do not have to end with “up to \( q \)” since no number larger than \( q \) divides \( q \).}\]
we keep $D$ fixed, bound $q$, and accept some small error that tends to vanish as the size of the bound on $q$ grows. But our purported interpretation of $\gamma$ had the “$D$” varying along with $q$.

This suggests a first, but admittedly ugly, fix. First fix $q$. Now given another quantity $Q$, consider, $Z_q$, the number of divisors of $Q$ minus than $\sqrt{q}$ minus the number of divisors of $Q$ between $\sqrt{q}$ and $q$. The expectation of $Z_q$ takes the form of one of our approximates to $\gamma$, but we must let $q$ grow and take a bald limit to get $\gamma$ itself, so not the stuff of a popular interpretation.

5 A surprisingly satisfactory fix

We shall now formulate a family of valid probabilistic interpretations of $\gamma$, all very much in the spirit of the fallacious one, albeit just slightly more complicated.

Theorem 1 Let $F : \mathbb{N} \to \mathbb{R}$ stand for any function which
a) $F$ monotonically weakly increases;
b) $F$ tends to infinity; and
c) such that $q/F(q)$ tends to infinity.

Let $Z_F(q)$ equal the number of divisors of $q$ less than $\sqrt{F(q)}$ minus the number of divisors of $q$ between $\sqrt{F(q)}$ and $F(q)$.

Then, on the average, $Z_F(q)$ equals $\gamma$.

Considering our original goal, a popular interpretation of $\gamma$, we could perhaps just set $F(x) = \sqrt{x}$. We then get $\gamma$ means the average by which the count of divisors of a number that sit below its fourth root exceeds the count of divisors that lie between the fourth root and the square root.

The gist of the previous section consists in telling us that we cannot entirely dispense with the condition that $n/F(n)$ tends to infinity, since the conclusion fails when taking $F(n) = n$.

Proof

In the following diagram,

4Of course by “on the average” we mean taking the limit of averages that arise with $q$ bounded by $B$ as $B$ increases.
circles in row \( r \) (counting up) have area \( 1/r \). We have colored green those circles in column \( q \) having row number less than \( \sqrt{F(n)} \), and those with row number in the half-open interval \([\sqrt{F(n)}, F(n))\) red. (While we have in mind a general \( F \), satisfying the conditions of Theorem 1, the diagram shows the situation specifically for \( F(x) = \sqrt{x} \).)

Consider a particular column. By our previous work\(^5\), the excess of the green area over the red area takes the form of an approximation to \( \gamma \) with the approximations approaching perfection as we move to the right, on account of the assumption that \( F(q) \) grows without bound. So certainly if we consider together all the columns up to column \( B \), the total green area less the total red area divided by \( B \) approaches \( \gamma \) as \( B \) tends toward infinity.

Now compare the following diagram with the previous:

Here all circles now have area 1, but this time we only color circles if the row number divides the column number.

For the second diagram, for a given column, the excess of green area over red area constitutes just the sort of quantity we have claimed averages to \( \gamma \) in the long run.

It suffices to show that if we consider together all the columns up to column \( B \), the total green area less the total red area divided by \( B \) approaches \( \gamma \) as \( B \) tends toward infinity.

While the two diagrams appear quite different column-by-column, a row-

\(^5\) Just for the sake of simplicity now, here we choose to approximate \( \gamma \) by the sum of the reciprocals of the numbers below the square root of \( q \) minus the sum of the reciprocals of the numbers \( \text{equal to or above} \) the square root of \( q \).
by-row comparison works out quite simply, as follows.

Fix a row number, say \( r \), and consider the corresponding \( r \)-rows in the two diagrams, with the aim of estimating the discrepancy between, first, the total red areas they hold, and second, their total green areas.

In the \( r \)-row of the second diagram, consider any colored circle if one occurs. Call it \( C_1 \); \( C_1 \) has area 1. Write \( C_2 \) for the next colored circle to its right (in the infinite version of the second diagram). Next, consider the circle \( c_1 \) in the first diagram corresponding position-wise to \( C_1 \) together with the \( r \) circles in the first diagram in positions corresponding to those circles strictly between \( C_1 \) and \( C_2 \) (all these diagram 1 circles together have total area 1).

The previous paragraph shows that if the total red areas in the \( r \)-rows of the diagrams differ, they differ on account of what happens when, moving left to right say, as we enter and leave the first diagram’s “red island”.

Thus the red area discrepancy in row \( r \) cannot exceed magnitude 1, and likewise for the green area discrepancy.

As for the green area minus the red area in the two \( r \)-rows, the discrepancy between the diagram one difference and the diagram two difference cannot exceed magnitude 2.

For rows with no colored circles in either diagram we obviously have no discrepancy at all, and at most \( F(B) \) rows have colored circles.

We have now bounded the total green area minus red area discrepancy (for all rows) between the two diagrams by \( 2F(B) \). By assumption, \( 2F(B)/B \) approaches 0 as \( B \) grows. Thus, as \( B \) increases, the values for the average green area minus red area per column for two types of diagrams converge.

Since this average approaches \( \gamma \) for diagrams of the first type, it also does for diagrams of the second type, as desired.

### 6 The case of \( F(x) = \alpha x \)

Theorem 1 does not speak to the case of \( F(x) = \alpha x \) for any \( \alpha \in (0, 1) \); such an \( F \) could produce as many as \( \alpha x \) rows that exhibit a discrepancy. Nevertheless we can make the proof technique yield up a complete analysis.

**Theorem 2** Fix \( \alpha \in (0, 1) \). Write \( A \) for the average number of divisors of \( n \) that lie in \( (0, \sqrt{\alpha n}) \) minus the number of that lie in \( (\sqrt{\alpha n}, \alpha n) \). Then

\[
A = \sum_{i=1}^{\left\lfloor \frac{1-\alpha}{\alpha} \right\rfloor} \frac{1}{i} - \ln \left( \frac{1}{\alpha} \right).
\]
Before turning to the proof, we offer a few remarks.

First, except when $1/\alpha$ has integral value, $\lceil \frac{1 - \alpha}{\alpha} \rceil = \lfloor \frac{1}{\alpha} \rfloor$, which looks a bit simpler.

The formula correctly predicts a balance between divisors above and below the square root of $n$, the $\alpha = 1$ case. Moreover, as $\alpha$ approaches 0, the values of the formula converge to $\gamma$, just as one might hope based on Theorem 1.

The discontinuities in the graph below come as no surprise. As $\alpha$ shrinks past $1/k$, we lose, from the second diagram, divisors of $n$ of the form $n/k$ when they occur, which they do for one $n$ out of $k$. For those $n$ large compared to $k$ we will have these divisors colored red (since $n/k$ will exceed $\sqrt{(\alpha n)}$), so we expect the graph to jump up (as we move to the left) by $1/k$.

For all $\alpha < 1$ we have $A > 0$, so we expect, on the average, more divisors in $(0, \sqrt{\alpha n})$ than in $(\sqrt{\alpha n}, \alpha n)$. This leads us to guess that numbers $n$ with more divisors in $(0, \sqrt{\alpha n})$ than in $(\sqrt{\alpha n}, \alpha n)$ should occur with a positive density. But this does not follow immediately. Logically speaking, relatively rare numbers with many more divisors in $(0, \sqrt{\alpha n})$ than in $(\sqrt{\alpha n}, \alpha n)$ might possibly make all the necessary contribution to the average behavior. Nevertheless, such number cannot occur too rarely, since, overall, relatively few numbers $n$ possess even a total number of divisors large compared with $\ln n$.

Because the graph oscillates about the value $\gamma$, for infinitely many special values of $\alpha$ (namely those of the form $e^{\gamma-(1+\ldots+1/k)}$), $A$ takes the value $\gamma$, the right answer for the wrong reason, if you will. Note that this characterizes
\( \gamma \): the only average realized for infinitely many values of \( \alpha \).

One might wonder about the average value of the average if we choose \( \alpha \) from a uniform distribution on \((0,1)\). Curiously, integrating \( A \) as \( \alpha \) varies over \((0,1)\) gives \( \zeta(2) - 1 = \pi^2/6 - 1 = .644934068 \ldots \).

**Proof of Theorem 2** We refer here to the same two sorts of diagrams as the last proof, but now we assume them square, just so that the average per row excess of green area over red area equals the average per column excess.

We wish to compare, asymptotically, the average per row excess of green area over red area in the two types of square diagrams.

Since we have a uniform bound on the excess that occurs in any single row, we can safely ignore the green circles entirely! The green circles occur in only \( \sqrt{\alpha n} \) rows, so the variation in green areas between the two diagrams will tend to vanish when we divide by \( n \) and let \( n \) grow. (Compare with the previous proof, where the condition on \( F \) meant that switching to a row-by-row analysis ultimately allowed us to ignore everything. Even with \( F = \alpha x \), the old reasoning still applies to \( \sqrt{F} \).)

As for the variation in the red area between the two diagrams, we employ a straightforward integral approximation, getting

\[
\int_0^\alpha (\lfloor 1/y \rfloor - \lfloor 1/\alpha \rfloor) - \frac{1 - \frac{y}{\alpha}}{y} dy.
\]

The first term, in parentheses, captures the contribution for the second diagram, and from this we subtract off the contribution from the first diagram. Specifically, we have estimated the average per row excess of the red area in the second diagram over red area in the first.

Please note, for clarity, that since red dots count negatively, and by the remark above concerning the possibility of ignoring the green area, this expression also estimates the amount by which the average per row excess of green area over red area in the first diagram exceeds the average per row excess of green area over red area in the second diagram.
After some routine calculation, the integral in question evaluates to

\[ \gamma \ln \left( \frac{1}{\alpha} \right) - \sum_{i=1}^{\left\lceil \frac{1}{\alpha} \right\rceil} \frac{1}{i}. \]

Since we know that the first diagram has an average per row excess of green area over red area equal to \( \gamma \), while we seek the corresponding information for the second diagram, the result follows when we subtract this quantity from \( \gamma \).

7 Final Remark

Even though we set as our original goal the crafting of novel interpretations for \( \gamma \), a great variety of curious statements arise when we force \( \gamma \) to leave the story. Here we give just one example. By Theorem 1, a number \( n \) tends to have \( \gamma \) more divisors in \((0, n^{1/4})\) than in \((n^{1/4}, n^{1/2})\), and likewise \( \gamma \) more divisors in \((0, n^{1/8})\) than in \((n^{1/8}, n^{1/4})\). Subtract these two differences, we see that:

- on the average, \( n \) has exactly twice as many divisors in \((n^{1/4}, n^{1/2})\) as it does in \((n^{1/8}, n^{1/4})\).

Since \( \gamma \) no longer appears in the statement, one should naturally enquire about the possibility of a \( \gamma \)-free proof.

\[ \text{footnote} \]

In the case of \( \alpha = 1 \), mechanical evaluation of this integral constitutes the essence of a proof of the theorem of de la Vallée Poussin mentioned in the introduction – \( \gamma \) emerges directly from the definition in the form of its usual definition. But from the pairing of divisors of \( n \) above and below \( \sqrt{n} \) we actually know the value of the integral in advance, albeit just in this case. That means we have actually have in hand two independent proofs of de la Vallée Poussin’s theorem.