A time-periodic competition model with nonlocal dispersal and bistable nonlinearity: propagation dynamics and stability

Manjun Ma, Wentao Meng and Chunhua Ou

Abstract. This paper is concerned with traveling waves to a time-periodic bistable Lotka–Volterra competition system with nonlocal dispersal. We first establish the existence, uniqueness and stability of traveling wave solutions for this system. By utilizing comparison principle and the stability property, the relationship among the bistable wave speed, the asymptotic propagation speeds of the associated monotone subsystems and the speed of upper/lower solutions is obtained. Explicit sufficient conditions for positive and negative bistable wave speeds are derived. Our explicit results are derived by constructing particular and novel upper/lower solutions with specific asymptotical behaviors, which can be seen as case studies applicable to further investigations and improvements. Finally, the theoretical results are corroborated under weak conditions by direct simulations of the underlying time-periodic system with nonlocal dispersal. The combined impact of competition, dispersal and seasonality on the invasion direction has shed new light on the modelings and analysis of population competition and species invasion in heterogeneous media.

Mathematics Subject Classification. Primary 35K57, 35C07, 37C65, 92D25.

Keywords. Bistable traveling wave, Existence, Stability, Lotka–Volterra competition model, Nonlocal dispersal, Invasion direction.

1. Introduction

In this paper we are concerned with traveling wave solutions to the following nonlocal dispersal system

\[
\begin{align*}
\frac{du}{dt} &= d_1(t) [J_1 * u - u] + u(r_1(t) - a_1(t)u - b_1(t)v), \\
\frac{dv}{dt} &= d_2(t) [J_2 * v - v] + v(r_2(t) - a_2(t)u - b_2(t)v),
\end{align*}
\]

where \( u = u(x, t) \) and \( v = v(x, t) \) stand for the densities of two competitive species at position \( x \) at time \( t \); the functions \( d_i(t) \) are the dispersal coefficients, \( r_i(t) \) are the net birth rates or resource strength, \( r_1(t)/a_1(t) \) and \( r_2(t)/b_2(t) \) are called the carrying capacities, \( b_1(t)/r_1(t) \) and \( a_2(t)/r_2(t) \) are the competition coefficients, \( J_i \) represent the kernel functions, \( i = 1, 2 \). The convolution \( J_i * \omega(x, t) \) means

\[
J_i * \omega(x, t) = \int_{\mathbb{R}} J_i(x - y) \omega(y, t) \, dy
\]

for any continuous function \( \omega(x, t) \). Moreover, we assume that all the coefficients are continuous positive T-periodic functions and satisfy the bistable nonlinearity

\[
\begin{align*}
\int_0^T a_1(t)p(t) - b_1(t)q(t) \, dt < 0, & \quad \int_0^T b_2(t)q(t) - a_2(t)p(t) \, dt < 0,
\end{align*}
\]

Published online: 24 October 2023
where

\[
\begin{align*}
p(t) &= \frac{\int_0^t r_1(s)ds}{p_0 e^{\int_0^t r_1(s)ds}} + 1, \\
q(t) &= \frac{\int_0^t r_2(s)ds}{q_0 e^{\int_0^t r_2(s)ds}} + 1,
\end{align*}
\]

\[0 < \frac{T}{T} \int_0^T r_1(s)ds - 1 > 0, \quad \frac{T}{T} \int_0^T r_2(s)ds - 1 > 0.
\] (1.3)

Competition among species is an eternal topic in nature. In recent years, Lotka–Volterra type systems with nonlocal dispersal have been frequently applied to describe dynamic interactions between two competing species, see e.g., [5,9,14,17,27,32–34]. Among them, Yu and Yuan in [32] established the existence of traveling wave solutions to a nonlocal dispersal competitive–cooperative system by using the Schauder’s fixed-point theorem and a cross-iteration technique. Li and Lin [17] proved the existence of traveling wave solutions to a nonlocal dispersal competitive–cooperative system by using the monotone semiflow theory. When all the coefficients are constant, the system in (1.1) becomes

\[
\begin{align*}
\frac{u_t}{v_t} = d_1 [J_1 * u - u] + u(r_1 - a_1 u - b_1 v), \\
\frac{v_t}{v_t} = d_2 [J_2 * v - v] + v(r_2 - a_2 u - b_2 v),
\end{align*}
\]

\[x \in \mathbb{R}, \ t > 0,
\] (1.4)

which has been studied in [9,33]. Zhang, Ma and Li in [33] showed that the bistable traveling waves with nonzero speed are strictly monotone.

Fang and Zhao [9] showed that the minimal wave speed must be the spreading speed, when monostable-nonlinearity is assumed. In [23], Ma, Yue and Ou studied the speed sign of traveling waves in bistable nonlinearity. Dynamics for related diffusive Lotka-Volterra competitive models have been extensively studied in [1–3,7,10–13,15,16,18,21,22,24,25,29–31,35] in various media.

In this paper, we further study traveling wave solutions of Lotka–Volterra competition model (1.1) when periodicity is coupled with nonlocal dispersal. Throughout this paper, we will use the notation

\[\bar{f} = \frac{1}{T} \int_0^T f(t) dt\]

to denote the average value of a function on the interval \([0,T]\) and always assume the following:

\(A1\) \(J_i\) is nonnegative and Lebesgue measurable for each \(i\);

\(A2\) For any \(\lambda \in \mathbb{R}\), \(\int_{\mathbb{R}} J_i(x)e^{-\lambda x} dx < \infty\);

\(A3\) \(\int_{\mathbb{R}} J_i(x) dx = 1\).

Under the condition (1.2), the corresponding kinetic system of (1.1)

\[
\begin{align*}
\frac{u_t}{v_t} &= u(r_1(t) - a_1(t)u - b_1(t)v), \\
\frac{v_t}{v_t} &= v(r_2(t) - a_2(t)u - b_2(t)v)
\end{align*}
\]

(1.5)

has three nonnegative \(T\)-period solutions \((0,0), (p(t),0), (0,q(t))\) and at least one coexistence solution \((u^*(t),v^*(t))\), where \(p(t)\) and \(q(t)\) are explicitly given by (1.3) and satisfy \(0 < u^*(t) < p(t), 0 < v^*(t) < q(t)\) for all \(t \in \mathbb{R}^+\); it further follows that the two semitrivial periodic solutions \((p(t),0)\) and \((0,q(t))\)
are stable, and \((0,0)\) is unstable. The uniqueness and the linear unstability of the coexistence solution \((u^*, v^*)\) are assured by a stronger condition than \((1.2)\)

\[
\mathcal{F}_1 < \min_{0 \leq t \leq T} \left( \frac{b_1(t)}{b_2(t)} \right), \quad \mathcal{F}_2 < \min_{0 \leq t \leq T} \left( \frac{a_2(t)}{a_1(t)} \right) \mathcal{F}_1.
\]

A detailed argument of the above results can be found in [4].

To study the time-periodic traveling wave of \((1.1)\) connecting \((0, q(t))\) to \((p(t), 0)\), we set

\[
\phi(x, t) = \frac{u(x, t)}{p(t)} \quad \text{and} \quad \psi(x, t) = \frac{q(t) - v(x, t)}{q(t)},
\]

which leads to a cooperative system of the form

\[
\begin{align*}
\phi_t &= a_1(t) \left( \int_{\mathbb{R}} J_1(y) \phi(x-y, t) dy - \phi \right) + \phi \left[ a_1(t) p(t) (1 - \phi) - b_1(t) q(t) (1 - \psi) \right], \\
\psi_t &= a_2(t) \left( \int_{\mathbb{R}} J_2(y) \psi(x-y, t) dy - \psi \right) + (1 - \psi) \left[ a_2(t) p(t) \phi - b_2(t) q(t) \psi \right],
\end{align*}
\]

\[(1.7)\]

where

\[
\phi_0 = \frac{u(x, 0)}{p(0)} \quad \text{and} \quad \psi_0 = \frac{q(0) - v(x, 0)}{q(0)}
\]

are nonnegative real functions.

Under this setting, the trivial solution \((0, 0)\) of the system \((1.1)\) becomes \(\alpha_1 = (0, 1)\), while the other three solutions \((p(t), 0)\), \((0, q(t))\) and \((u^*, v^*)\) becomes \(\beta = (1, 1)\), \(\mathbf{0} = (0, 0)\) and a positive solution \((\phi(t), \psi(t))\), respectively. Therefore, studying the traveling wave connecting \((0, q(t))\) to \((p(t), 0)\) is equivalent to the study of the traveling wave of \((1.7)\) connecting \(\mathbf{0} = (0, 0)\) to \(\beta = (1, 1)\). Here a traveling wave solution of \((1.7)\) is a translation invariant solution of the form

\[
\phi(x, t) = \Phi(z, t), \quad \psi(x, t) = \Psi(z, t), \quad z = x + ct,
\]

\[(1.8)\]

where \(c\) is the bistable wave speed. Thus, \((\Phi(z, t), \Psi(z, t))\) must satisfy the following wave profile system

\[
\begin{align*}
\Phi_t &= d_1(t) \left( \int_{\mathbb{R}} J_1(y) \Phi(z-y, t) dy - \Phi \right) - c\Phi_z + \Phi \left[ a_1(t) p(t) (1 - \Phi) - b_1(t) q(t) (1 - \Psi) \right], \\
\Psi_t &= d_2(t) \left( \int_{\mathbb{R}} J_2(y) \Psi(z-y, t) dy - \Psi \right) - c\Psi_z + (1 - \Psi) \left[ a_2(t) p(t) \Phi - b_2(t) q(t) \Psi \right],
\end{align*}
\]

\[(1.9)\]

with the asymptotic conditions

\[
(\Phi, \Psi)(-\infty, t) = (0, 0), \quad (\Phi, \Psi)(\infty, t) = (1, 1).
\]

\[(1.10)\]

For convenience, we set \(\omega(x, t; \omega_0) = (\phi(x, t; \omega_0), \psi(x, t; \omega_0))\) with \(\omega_0 = (\phi_0, \psi_0)\) and \(\Gamma(x+ct, t) = (\Phi(x+ct, t), \Psi(x+ct, t))\). Moreover, let

\[
f_1(\phi, \psi, t) = \phi \left[ a_1(t) p(t) (1 - \phi) - b_1(t) q(t) (1 - \psi) \right]
\]

\[(1.11)\]

and

\[
f_2(\phi, \psi, t) = (1 - \psi) \left[ a_2(t) p(t) \phi - b_2(t) q(t) \psi \right].
\]

\[(1.12)\]
Thus we can rewrite (1.7) into
\[
\begin{cases}
\phi_t = d_1(t) \left( \int \frac{J_1(y)}{\chi} (\phi(x-y,t)) dy - \phi \right) + f_1(\phi, \psi, t), \\
\psi_t = d_2(t) \left( \int \frac{J_2(y)}{\chi} (\psi(x-y,t)) dy - \psi \right) + f_2(\phi, \psi, t), \\
(\phi(x,0), \psi(x,0)) = (\phi_0, \psi_0)(x).
\end{cases}
\]

Nonlocal dispersal is different from the classical local diffusion, for example, the solution map cannot smooth the initial data, and it also yields challenges in the compactness of solutions as well as in the study of eigenvalue problems. When time-periodicity is incorporated, the eigenvalue problem in a periodic functional space for the solution semiflow needs to be particularly considered. Therefore, compared with our recent work in [23], many new difficulties have arisen. In this paper, we first establish the existence, uniqueness, monotonicity and stability of the traveling waves. Since the sign of wave speed determines which species will win the competition (or which species will die out), more interestingly and importantly, we will study how to obtain criteria to determine the speed sign of the wave. Our results provide possible deep understandings on the combined impact of competition, dispersal and seasonality on the invasion in heterogeneous media. Compared to [23], novel results on the criterion of the wave speed sign are obtained by developing the skill of construction of upper/lower solutions. This new technique can establish the speed sign of the system when the kernel functions are either symmetrical or asymmetrical.

**Remark 1.1.** It is easy to check that condition (1.2) is weaker than (1.6). In what follows, we will prove the existence of a bistable traveling wave solution under (1.6). However, condition (1.2) is enough for us to prove the uniqueness of the bistable traveling wave solution. We conjecture that condition (1.2) can guarantee the existence of a bistable traveling wave solution.

**Remark 1.2.** Here the traveling wave $\phi(x, t) = \Phi(z, t), \psi(x, t) = \Psi(z, t), z = x + ct$ is a special solution of the PDE system (1.7). As a solution, it means that $\frac{\partial \phi}{\partial t} = \frac{\partial \Phi(z+ct, t)}{\partial t}, \frac{\partial \psi}{\partial t} = \frac{\partial \Psi(z+ct, t)}{\partial t}$ must exist. In (1.9), we should understand them as $\Phi_t + c\Phi_z = \frac{\partial \Phi(z+ct, t)}{\partial t}$ and $\Psi_t + c\Psi_z = \frac{\partial \Psi(z+ct, t)}{\partial t}$ so that the wave profile system is well-defined. For this wave, we can further show that $\Phi$ and $\Psi$ are monotone in $z$ (see Theorem 2.1) so that their left and right partial derivatives in $z$ always exist, although the solution map lacks regularity. Therefore, the left and right limits $\Phi(t, z_-), \Phi(t, z_+), \Psi(t, z_-), \Psi(t, z_+)$ must exist. If the solution is not in $C^1$, we understand (1.9) in left or right limit of $z$.

The paper is organized as follows. In Sect. 2, we prove the existence, monotonicity, uniqueness, and stability of the time-periodic traveling wave. In Sect. 3, we establish the value range of the bistable wave speed, which indicates the relationship among the bistable wave speed, the spreading speeds of monostable subsystems, and the speed of upper/lower solutions. In Sect. 4, by constructing upper/lower solutions, we derive explicit conditions to get the positive and negative wave speeds. Examples and numerical simulations are presented in Sect. 5. Section 6 is conclusion and discussion.

2. The bistable traveling wave

2.1. Preliminaries

**Notation.** We suppose that $\chi$ is an ordered Banach space with a norm $\|\cdot\|_\chi$ and its positive cone $\chi^+$ is well defined. Assume that $\text{Int}(\chi^+)$ is not empty. For any $\xi, \varsigma \in \chi$, we say $\xi \geq \varsigma$ if $\xi - \varsigma \in \chi^+$, $\xi > \varsigma$ if
ξ ≥ ς but ξ ≠ ς, and ξ ≈ ς if ξ − ς ∈ \text{Int}(χ^+) . A subset of χ is called totally unordered provided that no two elements are ordered. Let
\[
\mathcal{C} = \{ u \in C(\mathbb{R}, \chi) | u \text{ is a nondecreasing function} \}
\]
and equip \mathcal{C} with a compact open topology. For any \( \rho, \varphi \in \mathcal{C} \), we define
\[
\rho \succ \varphi \quad \text{if} \quad \rho(x) \geq \varphi(x) \quad \text{for all} \quad x \in \mathbb{R}.
\]
Similarly, we define \( \rho \prec \varphi \) if \( \rho \geq \varphi \) but \( \rho \neq \varphi \), and \( \rho \preceq \varphi \) if \( \rho(x) \geq \varphi(x) \) for all \( x \in \mathbb{R} \). Let \( \chi_\sigma = \{ \gamma \in \chi : o \leq \gamma \leq \sigma \} \) and \( \chi_\sigma = \{ \varphi \in \mathcal{C} : o \leq \varphi \leq \sigma \} \) for any \( \sigma, \chi \) with \( \sigma > o \), where \( o \) is the zero element in \( \chi \) or \( \mathcal{C} \).

Assume that \( \beta \in \text{Int}(\chi^+) \) and \( Q \) maps \( \mathcal{C}_\beta \) to \( \mathcal{C}_\beta \). Let \( E \) be the set of all fixed points of \( Q \) restricted to \( \chi_\beta \). Suppose that \( o \) and \( \beta \) are in \( E \). Define a translation operator \( T_y \) on \( \mathcal{C} \) for any \( y \in \mathbb{R} \) by \( T_y[\phi](x) = \phi(x - y), \forall x \in \mathbb{R}, \phi \in \mathcal{C} \). Based on the idea in [4,8,9], we give the assumptions on the map \( Q \).

(H1) (Translation invariance) \( T_y \circ Q[\phi] = Q \circ T_y[\phi], \forall \phi \in \mathcal{C}_\beta, y \in \mathbb{R} \).
(H2) (Continuity) \( Q : \mathcal{C}_\beta \rightarrow \mathcal{C}_\beta \) is continuous in the sense that if \( \phi_n \rightarrow \phi \) in \( \mathcal{C}_\beta \), then \( Q[\phi_n](x) \rightarrow Q[\phi](x) \) in \( \chi_\beta \) for almost all \( x \in \mathbb{R} \).
(H3) (Monotonicity) \( Q \) is order-preserving in the sense that \( Q[\phi] \geq Q[\psi] \) whenever \( \phi \geq \psi \) in \( \mathcal{C}_\beta \).
(H4) (Weak-compactness) For any fixed \( x \in \mathbb{R} \), the set \( Q[\mathcal{C}_\beta] \) is precompact in \( \chi_\beta \).
(H5) (Bistability) Two fixed points \( o \) and \( \beta \) are strongly stable from above and below, respectively. For the map \( Q : \chi_\beta \rightarrow \chi_\beta \), the equilibria set \( E \{ o, \beta \} \) is totally unordered. The definition of strong stability is seen in [4,8].
(H6) (Counter-propagation) For \( \alpha_i \in E \{ o, \beta \}, c^*(\alpha_i, \beta) + c^*_+(0, \alpha_i) > 0, i = 1, 2 \), where \( c^*(\alpha_i, \beta) \) and \( c^*_+(0, \alpha_i) \) are recalled in [4,8,19,20].

Definition 2.1. A family of mappings \( \{ Q_t \}_{t \in \mathbb{R}^+} \) is called a \( T \)-periodic semiflow on space \( \mathcal{C} \) provided that it has the following properties:
(i) \( Q_0[\phi] = \phi, \forall \phi \in \mathcal{C} \).
(ii) \( Q_{t+T}[\phi] = Q_t \circ Q_T[\phi] \) for all \( t \geq 0, \phi \in \mathcal{C} \).
(iii) \( Q_{t_n}[\phi_n](x) \rightarrow Q_t[\phi](x) \) in \( \chi_\beta \) for almost all \( x \in \mathbb{R} \) whenever \( t_n \rightarrow t \) and \( \phi_n \rightarrow \phi \) in \( \mathcal{C}_\beta \).

Moreover, the mapping \( Q_T \) is called the Poincaré map associated with this periodic semiflow.

Definition 2.2. (see [8] Definition 3.3) \( \Gamma(x + ct, t) \) is said to be a time-periodic traveling wave of the semiflow \( \{ Q_t \}_{t \in \mathbb{R}^+} \) with speed \( c \), if \( Q_t[\Gamma(x, 0)](x) = \Gamma(x + ct, t), \Gamma(x, t + \omega) = \Gamma(x, t) \) for all \( x \in \mathbb{R} \) and \( t \in \mathbb{R}^+ \).

Lemma 2.1. (see [8] Theorem 5.4) Let \( \beta(t) \) be a strongly positive periodic fixed point of \( \{ Q_t \}_{t \geq 0} \) restricted to \( \chi_\beta \) with \( Q_t[\beta(0)] = \beta(t) \) and assume that \( \{ Q_t \}_{t \geq 0} \) is a \( T \)-periodic semiflow on \( \mathcal{C}_{\beta(0)} \). Further, assume that the Poincaré map \( Q_T \) satisfies (H1)-(H6) with \( \beta = \beta(0) \). Then there exist \( c \in \mathbb{R} \) and \( \phi(x, t) \) with \( \phi(-\infty, t) = 0 \) and \( \phi(+\infty, t) = \beta(t) \) such that \( Q_t[\phi](x) = \phi(x + ct, t) \) for all \( (x, t) \in \mathbb{R} \times \mathbb{R}^+ \). Furthermore, \( \phi(x, t) \in \mathcal{C}_{\beta(t)} \) is nondecreasing in \( x \) and is \( T \)-periodic in \( t \).

To discuss the dynamical behaviors of the semiflow generated by system (1.7), we first introduce the definition of upper and lower solutions to the wave profile system in (1.9). An upper solution/lower solution of (1.7) can be defined similarly.

Definition 2.3. A pair of bounded function \( (\Phi(z, t), \Psi(z, t)) \) on \( \mathbb{R} \times [0, T] \) is called an upper solution (a lower solution) of (1.9) if \( (\Phi(z, t), \Psi(z, t)) \) is continuous in \( (z, t) \in \mathbb{R} \times [0, T] \), and satisfy
\[
\left\{ \begin{array}{l}
\Phi_t \geq (\leq) \ d_1(t) \ [J_1 \ast \Phi(z, t) - \Phi(z, t)] - c\Phi_z + \Phi[a_1(t) p(t)(1 - \Phi) - b_1(t) q(t)(1 - \Psi)], \\
\Psi_t \geq (\leq) \ d_2(t) \ [J_1 \ast \Psi(z, t) - \Psi(z, t)] - c\Psi_z + (1 - \Psi)(a_2(t) p(t) \Phi - b_2(t) q(t) \Psi)
\end{array} \right.
\]
for all \( (z, t) \in \mathbb{R} \times (0, T) \).
2.2. Existence and monotonicity

Let $\chi = \mathbb{R}^2$. Let $P_t(t)$ be the solution semigroup of the linear nonlocal dispersal equation $u_t = d_i(t)(J \ast u - u)$ as below:

$$P_i(t)[\phi](x) = e^{-\int_0^t d_i(\tau) d\tau} \left[ a_0 + a_1 \int_0^t d_i(\tau) d\tau + a_2 \int_0^t \int_0^\tau d_i(\tau) d_1(s) ds d\tau + \cdots \right](x), \quad i = 1, 2,$$

where $a_0(\phi) = \phi$ and $a_m(\phi) = J_i \ast a_{m-1}(\phi), \forall m \geq 1$. At this point, define

$$P(t) = \begin{pmatrix} P_1(t) & 0 \\ 0 & P_2(t) \end{pmatrix}, \quad f(\omega, t) = \begin{pmatrix} f_1(\phi, \psi, t) \\ f_2(\phi, \psi, t) \end{pmatrix}.$$

Then the solution of system (1.7) (or (1.13)) can be represented in an integral form

$$\omega(x, t; \omega_0) = P(t)[\omega_0](x) + \int_0^t P(t - s)[f(\omega(\cdot, s), s)](x) ds, \quad x \in \mathbb{R}, \ t \geq 0. \quad (2.1)$$

By this, we define a family of operators $Q_t$ associated with system (1.7) by

$$Q_t(\omega_0) = \omega(x, t; \omega_0), \quad \forall x \in \mathbb{R}, \ t \geq 0. \quad (2.2)$$

It is easy to show that $Q_t(\omega_0)$ is a $T$-periodic semiflow. However, the existence of a bistable traveling wave is usually difficult to prove. Here we use the theory of monotone dynamical systems developed in [8] to deal with it. Hence a further condition on the symmetry of the kernel functions is required so that the counter-propagation (H6) is satisfied for the Poincaré map $Q_T$ associated with (2.2), i.e.,

$$Q_T(\omega_0) = \omega(x, T; \omega_0) = P(T)[\omega_0](x) + \int_0^T P(T - s)[f(\omega(\cdot, s), s)](x) ds, \ x \in \mathbb{R}, \ \omega_0 \in C_\beta.$$

**Theorem 2.1.** Assume that $J_i(x) = J_i(-x), i = 1, 2$ and (1.6) holds. Then there exist a constant $c \in \mathbb{R}$ and a $T$-periodic nondecreasing (in $z$) traveling wave profile $\Gamma(z, t) = (\Phi(z, t), \Psi(z, t))$ to (1.9)–(1.10), where $z = x + ct$, $\Gamma(z, t + T) = \Gamma(z, t)$. Moreover, $\frac{\partial}{\partial z} \Phi_\pm(z, t) > 0$ and $\frac{\partial}{\partial z} \Psi_\pm(z, t) > 0$ for $z \in \mathbb{R}$ and $t \in \mathbb{N}_+$. Here, for the symbol $\pm$, we mean the left and right derivatives at $z$.

**Proof.** We can easily verify that $Q_T$ satisfies assumptions (H1)–(H5). If (H6) is true for $Q_T$, then Lemma 2.1 guarantees the first statement in the theorem. In the following, we prove that $Q_T$ satisfies (H6), that is,

$$c^+_{i}(\alpha_i, \beta) + c^+_{i}(0, \alpha_i) > 0, i = 1, 2, \quad (2.3)$$

where $c^+_{i}(\alpha_i, \beta)$ is called the leftward spreading speed of $Q_T$ in the phase space $C_{[\alpha_i, \beta]}$, and $c^+_{i}(0, \alpha_i)$ is called the rightward spreading speed of $Q_T$ in the phase space $C_{[0, \alpha_i]}$ (see (2.3) in [4] or (2.8) in [8]).

Here we only prove inequality (2.3) for the case of $i = 1$, i.e., $\alpha_1 = (0, 1)$, since the other case ($i = 2$) can be similarly handled as in [4] with the assumption of (1.6). Suppose that $(\Phi(x + ct, t), \Psi(x + ct, t))$ is a traveling wave solution of (1.7) connecting $\alpha_1$ to $\beta$. Then $(\phi(x, t), \psi(x, t))$ solves

$$\begin{cases}
\phi_t = d_1(t) \left( \int_{\mathbb{R}} J_1(y) \phi(x - y, t) dy - \phi(x, t) \right) + a_1(t)p(t)\phi(1 - \phi), \\
\psi_t = 0
\end{cases} \quad (2.4)$$
with the initial data \((\phi, \psi)(x, 0) = (\Phi, \Psi)(x, 0)\). Linearizing (2.4) at \(\alpha_1 = (0, 1)\), we have
\[
\begin{align*}
\phi_t &= d_1(t) \left( \int J_1(y) \phi(x - y, t) dy - \phi(x, t) \right) + a_1(t) p(t) \phi, \\
\psi_t &= 0.
\end{align*}
\] (2.5)

Let the solution of (2.5) be of the form \((\eta_1(t), \eta_2(t))e^{\mu x}\). Then \((\eta_1(t), \eta_2(t))\) satisfies the \(\mu\)-parameterized linear system
\[
\begin{align*}
\eta_1'(t) &= \left( d_1(t) \int J_1(y) e^{-\mu y} dy - d_1(t) + a_1(t)p(t) \right) \eta_1(t), \\
\eta_2'(t) &= 0, \\
\eta_i(0) &= \eta_i(T), i = 1, 2.
\end{align*}
\] (2.6)

Then solving (2.6) for \(\eta_1(t)\) and \(\eta_2(t)\), we have
\[
\eta_1(t) = \eta_1(0)e^{\int_0^t d_1(s) \left( \int J_1(y) e^{-\mu y} dy - 1 \right) + a_1(s)p(s) ds}, \quad \eta_2(t) = \eta_2(0).
\]

Hence the \(T\)-periodic semiflow associated with system (2.6) is
\[
\begin{pmatrix}
\int_0^T \int J_1(y) e^{-\mu y} dy - 1 + a_1(t)p(t) dt & 0 \\
0 & 1
\end{pmatrix}.
\]

By this, the principal eigenvalue of (2.4) is
\[
\gamma_1(\mu) = \int_0^T d_1(t) \left( \int J_1(y) e^{-\mu y} dy - 1 \right) + a_1(t)p(t) dt.
\]

Furthermore, by the reference [20], we have
\[
c^*_+(\alpha_1, \beta) = \frac{1}{T} \inf_{0 < \mu < \infty} \frac{\gamma_1(\mu)}{\mu}. \tag{2.7}
\]

The condition \(J_1(x) = J_1(-x)\) implies that it is positive. On the other hand, assume that \((\Phi(x + ct, t), \Psi(x + ct, t))\) is a traveling wave solution of (1.7) connecting 0 to \(\alpha_1\). Then, by (1.7), it is obvious that \((\phi(x, t), \psi(x, t))\) satisfies
\[
\begin{align*}
\phi_t &= 0, \\
\psi_t &= d_2(t) \left( \int J_2(y) \psi(x - y, t) dy - \psi(x, t) \right) - b_2(t)q(t) \psi(1 - \psi)
\end{align*}
\] (2.8)

with the initial data \((\phi, \psi)(x, 0) = (\Phi, \Psi)(x, 0)\).

Repeating the above process, we obtain
\[
c^*_+(0, \alpha_1) = \frac{1}{T} \inf_{0 < \mu < \infty} \frac{\gamma_2(\mu)}{\mu} > 0, \tag{2.9}
\]

where
\[
\gamma_2(\mu) = \int_0^T d_2(t) \left( \int J_2(y) e^{\mu y} dy - 1 \right) + b_2(t)q(t) dt.
\]

By (2.9) and (2.9), we get that (H6) is true.

Next, we verify the second statement in the theorem. By Lemma 2.1, it follows that
\[
\frac{\partial}{\partial z} \Phi(z, t) \geq 0 \quad \text{and} \quad \frac{\partial}{\partial z} \Psi(z, t) \geq 0
\]
for $z \in \mathbb{R}$ and $t \in \mathbb{R}^+$ if the derivatives exist. If they do not exist, here we mean the left and right derivatives. We need only to prove that the equal sign does not appear. Suppose that there exist $z_0, t_0 \in \mathbb{R}$ such that $\frac{\partial}{\partial z} \Phi(z_0, t_0) = 0$. By the periodic property of the wave functions, it would give $\frac{\partial}{\partial z} \Phi(z_0, t_0 + nT) = 0$ for any positive integer $n$. We can assume that $t_0 = 0$ and $z_0 = x_0 + ct_0 = x_0$ for some $x_0$. Therefore, we can re-write (1.13), with initial wavefront profile, as

$$
\begin{align*}
\phi_t &= d_1(t) \left( \int J_1(y) \phi(x - y, t) dy - \phi \right) - \beta_1 \phi + \tilde{f}_1(\phi, \psi, t), \\
\psi_t &= d_2(t) \left( \int J_2(y) \psi(x - y, t) dy - \psi \right) - \beta_2 \psi + \tilde{f}_2(\phi, \psi, t), \\
(\phi(x, 0), \psi(x, 0)) &= (\Phi(x, 0), \Psi(x, 0))
\end{align*}
$$

(2.10)

Here $\tilde{f}_1 = f_1 + \beta_1 \phi, \tilde{f}_2 = f_2 + \beta_2 \psi$ with a proper choice of $\beta_1$ and $\beta_2$ so that both $\tilde{f}_1$ and $\tilde{f}_2$ are monotone in $\phi$ and $\psi$. Taking derivative (left or right derivative if its derivative doesn’t exist) with respect to $x$ at both sides of each equation in (2.10) gives

$$
\begin{align*}
(\phi_x)_t &= d_1(t) \left( \int J_1(y) \phi_x(x - y, t) dy - \phi_x \right) - \beta_1 \phi_x + \tilde{f}_1 \phi_x + \tilde{f}_1 \psi_x, \\
(\psi_x)_t &= d_2(t) \left( \int J_2(y) \psi_x(x - y, t) dy - \psi_x \right) - \beta_2 \psi_x + \tilde{f}_2 \phi_x + \tilde{f}_2 \psi_x, \\
(\phi(x, 0), \psi(x, 0)) &= (\Phi_x(x, 0), \Psi_x(x, 0)),
\end{align*}
$$

(2.11)

where $f_{iy}$ represents the partial derivative of $g_i$ with respective to $y$. Let $\tilde{P}_1(t)$ be the solution semigroup of the linear nonlocal dispersal equation $u_t = d_i(t)[(J * u) - u] - \beta_i u$. As in (2.1), we get from (2.11)

$$
\phi_x(x, t) = \tilde{P}_1(t)[\Phi_x(x, 0)](x) + \int_0^t \tilde{P}_1(t - s)[\tilde{f}_1 \phi_x + \tilde{f}_1 \psi_x] ds \geq \tilde{P}_1(t)[\Phi_x(x, 0)](x).
$$

(2.12)

Recall that the support of $J_1(x)$ contains at least an interval with the length larger than zero, and then this makes $\tilde{P}_1(t)[\Phi_x(x, 0)](x)$ positive for $x = x_0$ when time $t$ is large, say $nT$ for large $n$. This is a contradiction. Thus the supposition is false, and the proof is complete.

\[ \Box \]

**Remark 2.1.** When the wave speed $c$ is not zero, it can be proved that $\frac{\partial}{\partial z} \Phi(z, t)$ and $\frac{\partial}{\partial z} \Psi(z, t)$ are continuous functions in $(z, t)$. However, the smooth property is not clear to us when $c = 0$.

### 2.3. Uniqueness

The following comparison principle can be proved by properly modifying the argument of Lemma 3.2 in [28]. Hence the proof is omitted here.

**Lemma 2.2.** Suppose that $(\phi^-, \psi^-)(x, t)$ and $(\phi^+, \psi^+)(x, t)$ in $C_{\beta}$ are a bounded lower solution and a bounded upper solution of (1.7) on $\mathbb{R} \times [0, T)$. Then we have

(i) if $\phi^-(x, 0) \leq \phi^+(x, 0)$ and $\psi^-(x, 0) \leq \psi^+(x, 0)$ for $x \in \mathbb{R}$, then

$$
\phi^-(x, t) \leq \phi^+(x, t), \quad \psi^-(x, t) \leq \psi^+(x, t), \quad (x, t) \in \mathbb{R} \times [0, \infty).
$$

(2.13)
(ii) if $\phi^-(x,0) \leq \phi_0 \leq \phi^+(x,0)$ and $\psi^-(x,0) \leq \psi_0 \leq \psi^+(x,0)$ for $x \in \mathbb{R}$, where $(\phi_0, \psi_0) = w_0$ is the initial data of (1.7) then
\[
\begin{align*}
\phi^-(x,t) &\leq \phi(x,t;w_0) \leq \phi^+(x,t), \\
\psi^-(x,t) &\leq \psi(x,t;w_0) \leq \psi^+(x,t), \\
(x,t) &\in \mathbb{R} \times [0,\infty).
\end{align*}
\] (2.14)

To proceed, we need the following lemma.

**Lemma 2.3.** Assume that (1.2) holds. There exist two positive pairs $(\lambda_0, (p_1^-(t), p_2^-(t)))$ and $(\lambda_1, (p_1^+(t), p_2^+(t)))$ solving the following problems of eigenvalue inequalities
\[
\begin{align*}
\frac{dp_1^-(t)}{dt} &\geq [a_1(t)p(t) - b_1(t)q(t) + \lambda_0]p_1^-(t), \\
\frac{dp_2^-(t)}{dt} &\geq a_2(t)p(t)p_1^-(t) + [\lambda_0 - b_2(t)q(t)]p_2^-(t), \\
p_1^-(t + T) &= p_1^-(t), \quad p_2^-(t + T) = p_2^-(t)
\end{align*}
\] (2.15)

and
\[
\begin{align*}
\frac{dp_1^+(t)}{dt} &\geq [\lambda_1 - a_1(t)p(t)]p_1^+(t) + b_1(t)q(t)p_2^+(t), \\
\frac{dp_2^+(t)}{dt} &\geq [\lambda_1 + b_2(t)q(t) - a_2(t)p(t)]p_2^+(t), \\
p_1^+(t + T) &= p_1^+(t), \quad p_2^+(t + T) = p_2^+(t),
\end{align*}
\] (2.16)

respectively.

**Proof.** Take two real numbers $\lambda_0$ and $\lambda_1$ satisfying
\[0 < \lambda_0 < \min\{\frac{b_1q - a_1p}{b_2q}, \quad 0 < \lambda_1 < \min\{\frac{a_2p - b_2q}{a_1p}\}.\]

Then define four functions $p_i^-, p_i^+, i = 1, 2$ by
\[
\begin{align*}
p_i^-(t) &= \exp \left( \int_0^t a_1(\tau)p(\tau) - b_1(\tau)q(\tau) \, d\tau + \frac{b_1q - a_1p}{b_2q} \right), \\
p_i^+(t) &= (c_0(t) + p_i^-(0)) \exp \left( - \int_0^t b_2(\tau)q(\tau) \, d\tau + \lambda_0 t \right),
\end{align*}
\]

where
\[
\begin{align*}
p_1^-(0) &= 1, \\
p_2^-(0) &= \int_0^T \frac{a_2(t)p(t)p_1^-(t)\exp \left( \int_0^t b_2(\tau)q(\tau) \, d\tau - \lambda_0 t \right)}{\exp \left( \int_0^t b_2(t)q(t) \, dt - \lambda_0 T \right) - 1} \, dt, \\
c_0(t) &= \int_0^t a_2(s)p(s)p_1^-(s)\exp \left( \int_0^s b_2(\tau)q(\tau) \, d\tau - \lambda_0 s \right) \, ds.
\end{align*}
\]

and
\[
\begin{align*}
p_i^+(t) &= (c_1(t) + p_i^+(0)) \exp \left( - \int_0^t a_1(\tau)p(\tau) \, d\tau + \lambda_1 t \right), \\
p_2^+(t) &= \exp \left( \int_0^t (b_2(\tau)q(\tau) - a_2(t)p(\tau)) \, d\tau + \frac{a_2p - b_2q}{a_1p} t \right).
\end{align*}
\]
Then it is easy to check that (2.15) and (2.16) are true.

We next apply the eigenvalues \( \lambda_0, \lambda_1 \) and eigenfunctions \((p_1^- (t), p_2^- (t))\) and \((p_1^+ (t), p_2^+ (t))\) to construct upper and lower solutions of the system (1.7).

**Lemma 2.4.** Assume that (1.2) holds and there exists \((c, \Phi(z, t), \Psi(z, t))\) as a traveling wave solution of (1.7). Then there exist positive constants \( \delta_0, \sigma_1, \rho, \) real numbers \( \kappa^\pm \in \mathbb{R} \) and positive and bounded functions \( p_1(x, t), p_2(x, t) \) such that, for any \( \delta \in (0, \delta_0) \), the functions \((\phi^+, \psi^+) (x, t)\) and \((\phi^-, \psi^-) (x, t)\) defined by

\[
\begin{align*}
\phi^\pm(x, t) &= \Phi\left(x + ct + \kappa^\pm \pm \sigma_1 \delta (1 - e^{-\rho t}), t \right) \pm \delta p_1\left(x + ct + \kappa^\pm \pm \sigma_1 \delta (1 - e^{-\rho t}), t \right) e^{-\rho t}, \\
\psi^\pm(x, t) &= \Psi\left(x + ct + \kappa^\pm \pm \sigma_1 \delta (1 - e^{-\rho t}), t \right) \pm \delta p_2\left(x + ct + \kappa^\pm \pm \sigma_1 \delta (1 - e^{-\rho t}), t \right) e^{-\rho t},
\end{align*}
\]

are upper-lower solutions of (1.7) for \((x, t) \in \mathbb{R} \times (0, \infty)\).

**Proof.** We first give several notations by

\[
\begin{align*}
\lambda &= \min \{ \lambda_0, \lambda_1 \}, \\
&= \max \left\{ \max_{t \in [0, T]} d_1(t), \max_{t \in [0, T]} d_2(t) \right\}, \\
C_0 &= \max \left\{ \max_{t \in [0, T]} \left| \frac{d}{dt} p_1^+(t) \right|, \max_{t \in [0, T]} \left| \frac{d}{dt} p_1^-(t) \right|, \max_{t \in [0, T]} \left| \frac{d}{dt} p_2^+(t) \right|, \max_{t \in [0, T]} \left| \frac{d}{dt} p_2^-(t) \right| \right\}, \\
C_1 &= \max \left\{ \max_{t \in [0, T]} (a_1(t)p(t) + 2b_1(t)q(t)), \max_{t \in [0, T]} (a_2(t)p(t) + 2b_2(t)q(t)) \right\}, \\
C_2 &= \max \left\{ \max_{t \in [0, T]} p_1^+(t), \max_{t \in [0, T]} p_1^-(t), \max_{t \in [0, T]} p_2^+(t), \max_{t \in [0, T]} p_2^-(t) \right\}, \\
C_3 &= \min \left\{ \min_{t \in [0, T]} p_1^+(t), \min_{t \in [0, T]} p_1^-(t), \min_{t \in [0, T]} p_2^+(t), \min_{t \in [0, T]} p_2^-(t) \right\}.
\end{align*}
\]

For any given positive constant \( \rho \), it follows from (A3) that there exists a large positive constant \( M_0 \) such that

\[
\int_{|y| > M_0} J_i(y) dy < \frac{C_3}{dC_2} \rho, \quad i = 1, 2.
\]

Define a continuous function \( \zeta(x) \) by

\[
\zeta(x) = \begin{cases} 
0, & x < -M_1, \\
1, & x > M_1,
\end{cases}
\]

where \( M_1 \) is a large positive constant and \( 0 \leq \zeta'(x) \leq 1 \) for \( x \in \mathbb{R} \).
Let
\[ \varepsilon_0 := \min \left\{ \frac{1}{2}, \frac{C_3 \lambda}{3C_1(C_2 + C_3)} \right\}, \tag{2.20} \]
where \( \lambda \) is defined in (2.18). It then follows from (1.10) that there exists a large positive constant \( M := M(\varepsilon_0) \geq M_0 + M_1 \) such that
\[ 0 < \Phi(z, t) \leq \varepsilon_0, \quad 0 < \Psi(z, t) \leq \varepsilon_0, \quad \text{if } z < -M; \]
\[ 1 - \varepsilon_0 \leq \Phi(z, t) \leq 1, \quad 1 - \varepsilon_0 \leq \Psi(z, t) \leq 1, \quad \text{if } z > M. \tag{2.21} \]

Then, furthermore, we define
\[ C_4 = \min \left\{ \sup_{x \in [-M, M], t \in [0, T]} \frac{\partial}{\partial x} \Phi(\xi^+, t), \sup_{x \in [-M, M], t \in [0, T]} \frac{\partial}{\partial \xi^+} \Psi(\xi^+, t) \right\}, \tag{2.22} \]
\[ \delta_0 = \min \left\{ \frac{\lambda}{3C_1C_2}, \frac{1}{C_2}, \frac{C_4}{2C_2} \right\}, \tag{2.23} \]
and two functions \( p_1(x, t) \) and \( p_2(x, t) \) by
\[ p_1(x, t) = \zeta(x) p_1^+(t) + (1 - \zeta(x)) p_1^-(t), \quad p_2(x, t) = \zeta(x) p_2^+(t) + (1 - \zeta(x)) p_2^-(t). \]

We now let
\[ \xi^\pm(x, t) = x + ct + \kappa^\pm \pm \sigma_1 \delta(1 - e^{-\rho t}), \]
then (2.17) can be rewritten as
\[ \phi^\pm(x, t) = \Phi(\xi^\pm, t) \pm \delta p_1(\xi^\pm, t)e^{-\rho t} \quad \text{and} \quad \psi^\pm(x, t) = \Psi(\xi^\pm, t) \pm \delta p_2(\xi^\pm, t)e^{-\rho t}. \tag{2.24} \]

We only show that \((\phi^+(x, t), \psi^+(x, t))\) in (2.24) is an upper solution of system (1.7). The proof of the lower solution is similar and is omitted here.

Substituting \( \phi^+ \) into the first equation of system (1.7) gives
\[ d_1(t) [d_1 * \phi^+(x, t) - \phi^+(x, t)] - \phi^+_t + \phi^+[a_1(t)p(t)(1 - \phi^+) - b_1(t)q(t)(1 - \psi^+)] \]
\[ = d_1(t) \int J_1(y) \left[ \Phi(\xi^+ - y, t) - \Phi(\xi^+, t) \right] dy \]
\[ + \delta e^{-\rho t} d_1(t) \int J_1(y) \left[ p_1(\xi^+ - y, t) - p_1(\xi^+, t) \right] dy \]
\[ - \left[ \sigma_1 \rho \delta e^{-\rho t} \Phi_{\xi^+} + c \Phi_{\xi^+} + \Phi_1 + \frac{\partial p_1(\xi^+, t)}{\partial \xi^+} (c + \sigma_1 \rho \delta e^{-\rho t}) \delta e^{-\rho t} + \frac{\partial p_1(\xi^+, t)}{\partial t} \delta e^{-\rho t} - \rho p_1(\xi^+, t) \delta e^{-\rho t} \right] \]
\[ + \Phi \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \psi) \right] + \delta p_1(\xi^+, t) e^{-\rho t} \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \psi) \right] \]
\[ - \delta e^{-\rho t} \left( \Phi + \delta e^{-\rho t} p_1(\xi^+, t) \right) \left[ a_1(t)p(t)p_1(\xi^+, t) - b_1(t)q(t)p_2(\xi^+, t) \right] \]
\[ = \delta e^{-\rho t} d_1(t) \int J_1(y) \left[ p_1(\xi^+ - y, t) - p_1(\xi^+, t) \right] dy \]
\[ + \delta e^{-\rho t} \left[ -\sigma_1 \rho \Phi_{\xi^+} - \frac{\partial p_1(\xi^+, t)}{\partial \xi^+} (c + \sigma_1 \rho \delta e^{-\rho t}) \frac{\partial p_1(\xi^+, t)}{\partial t} + \rho p_1(\xi^+, t) \right] \]
\[ + \delta p_1(\xi^+, t) e^{-\rho t} \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \psi) \right] - \delta e^{-\rho t} \Phi \left[ a_1(t)p(t)p_1(\xi^+, t) - b_1(t)q(t)p_2(\xi^+, t) \right] \]
\[ - (\delta e^{-\rho t})^2 p_1(\xi^+, t) \left[ a_1(t)p(t)p_1(\xi^+, t) - b_1(t)q(t)p_2(\xi^+, t) \right] \]
\[ \overset{\text{def}}{=} I_1(\xi^+, t) + I_2(\xi^+, t) + I_3(\xi^+, t), \]
where the last equality holds by using
\[
\frac{d}{dt} \left( \int_{\mathbb{R}} J_1(y) \Phi(\xi^+ - y, t) dy - \Phi \right) - c\Phi_{\xi^+} - \Phi_1 + \Phi [a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \Psi)] = 0;
\] (2.25)

and
\[
I_1(\xi^+, t) = \delta e^{-\rho t} d_1(t) \int_{\mathbb{R}} J_1(y) \left[ p_1(\xi^+ - y, t) - p_1(\xi^+, t) \right] dy
\]
\[
= \delta e^{-\rho t} d_1(t) \left( p_1^+(t) - p_1^-(t) \right) \int_{\mathbb{R}} J_1(y) \left[ \zeta(\xi^+ - y) - \zeta(\xi^+) \right] dy,
\]
\[
I_2(\xi^+, t) = \delta e^{-\rho t} \left[ -\sigma_1 \rho \Phi_{\xi^+} - \frac{\partial p_1(\xi^+, t)}{\partial \xi^+} (c + \sigma_1 \delta \rho e^{-\rho t}) - \frac{\partial p_1(\xi^+, t)}{\partial t} + \rho p_1(\xi^+, t) \right],
\]
\[
I_3(\xi^+, t) = \delta e^{-\rho t} p_1(\xi^+, t) \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \Psi) \right]
\]
\[
- \delta e^{-\rho t} \Phi \left[ a_1(t)p(t)p_1(\xi^+, t) - b_1(t)q(t)p_2(\xi^+, t) \right]
\]
\[
- (\delta e^{-\rho t})^2 p_1(\xi^+, t) \left[ a_1(t)p(t)p_1(\xi^+, t) - b_1(t)q(t)p_2(\xi^+, t) \right].
\] (2.26)

We next consider three separate cases:

(i) \( \xi^+(x, t) < -M \);

(ii) \( |\xi^+(x, t)| \leq M \);

(iii) \( \xi^+(x, t) > M \).

**Case (i)** According to the definition of the function \( \zeta(x) \), we have
\[
\zeta(\xi^+) = 0, \quad p_1(\xi^+, t) = p_1^-(t), \quad \text{and} \quad p_2(\xi^+, t) = p_2^-(t).
\]

In addition, if \( |y| \leq M_0 \), then
\[
\zeta(\xi^+ - y) = 0, \quad p_1(\xi^+ - y, t) = p_1^-(t), \quad \text{and} \quad p_2(\xi^+ - y, t) = p_2^-(t).
\]

It follows from (2.19) and the definition of \( d, C_2 \) and \( C_3 \) that
\[
I_1(\xi^+, t) = \delta e^{-\rho t} d_1(t) \left( p_1^+(t) - p_1^-(t) \right) \int_{|y| > M_0} J_1(y) \left[ \zeta(\xi^+ - y) - \zeta(\xi^+) \right] dy
\]
\[
\leq \delta e^{-\rho t} d_1(t) \left| p_1^+(t) - p_1^-(t) \right| \int_{|y| > M_0} J_1(y) dy
\]
\[
< \delta e^{-\rho t} d_1(t) \left| p_1^+(t) - p_1^-(t) \right| \frac{C_3}{dC_2} \rho
\]
\[
\leq \delta e^{-\rho t} C_3 \rho.
\] (2.27)
Using Theorem 2.1 and Lemma 2.3 yields

\[
I_2(\xi^+, t) = -\delta e^{-\rho t} \left[ \sigma_1 \rho \Phi_{\xi^+} + \frac{dp^-_1(t)}{dt} - \rho p^-_1(t) \right]
\]

\[
\leq -\delta e^{-\rho t} \left[ \sigma_1 \rho \Phi_{\xi^+} + (a_1(t)p(t) - b_1(t)q(t) + \lambda_0)p^-_1(t) - \rho p^-_1(t) \right]
\]

\[
\leq -\delta e^{-\rho t} p^-_1(t) \left[ a_1(t)p(t) - b_1(t)q(t) + \lambda_0 - \rho \right].
\]

By (2.20) and the definition of \( M \), we have

\[
I_3(\xi^+, t) = \delta e^{-\rho t} p^-_1(t) \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \Psi) \right]
\]

\[
- \delta e^{-\rho t} \Phi \left[ a_1(t)p(t)p^-_1(t) - b_1(t)q(t)p^-_2(t) \right]
\]

\[
- (\delta e^{-\rho t})^2 p^-_1(t) \left[ a_1(t)p(t)p^-_1(t) - b_1(t)q(t)p^-_2(t) \right]
\]

\[
\leq \delta e^{-\rho t} p^-_1(t) \left[ a_1(t)p(t) - b_1(t)q(t)(1 - \varepsilon_0) \right] + \delta e^{-\rho t} b_1(t)q(t)p^-_2(t) \left[ \varepsilon_0 + \delta p^-_1(t) \right]
\]

\[
= \delta e^{-\rho t} p^-_1(t) \left\{ a_1(t)p(t) - b_1(t)q(t) + \varepsilon_0 b_1(t)q(t) \left[ 1 + \frac{p^-_2(t)}{p^-_1(t)} \right] + \delta b_1(t)q(t)p^-_2(t) \right\}.
\]

Note that \( \delta \in (0, \delta_0] \), by the definition of \( \varepsilon_0 \) and \( \delta_0 \), let \( \rho \leq \frac{\lambda}{6} \), and then it follows that

\[
I_1(\xi^+, t) + I_2(\xi^+, t) + I_3(\xi^+, t)
\]

\[
< \delta e^{-\rho t} \left[ C_3 \rho - (a_1(t)p(t) - b_1(t)q(t) + \lambda_0 - \rho)p^-_1(t) \right]
\]

\[
+ \delta e^{-\rho t} p^-_1(t) \left\{ a_1(t)p(t) - b_1(t)q(t) + \varepsilon_0 b_1(t)q(t) \left[ 1 + \frac{p^-_2(t)}{p^-_1(t)} \right] + \delta b_1(t)q(t)p^-_2(t) \right\}
\]

\[
\leq \delta e^{-\rho t} p^-_1(t) \left( \frac{C_3}{p^-_1(t)} + 1 \right) \rho - \lambda_0 + \varepsilon_0 C_1 \left( 1 + \frac{C_2}{C_3} \right) + \delta C_1 C_2
\]

\[
\leq \delta e^{-\rho t} p^-_1(t) \left( 2\rho - \lambda_0 + \frac{2\lambda}{3} \right) \leq 0.
\]
Similarly, for the second equation in system (1.7), we have

\[
\begin{align*}
&d_2(t)[J_2 \ast \psi^+(x, t) - \psi^+(x, t)] = \psi^+_1 + (1 - \psi^+) \left[ a_2(t)p(t) \phi^+ - b_2(t)q(t) \psi^+ \right] \\
= &\delta e^{-\rho t}d_2(t)(p_2^+(t) - p_2^-(t)) \int_{|y| > M_0} J_2(y)[\zeta(\xi^+ - y) - \zeta(\xi^+)]dy \\
- &\delta e^{-\rho t} \left[ \sigma_1 \rho \Psi_{\xi^+} + \frac{dp_2^-(t)}{dt} - \rho p_2^-(t) \right] + \delta e^{-\rho t} \left[ a_2(t)p(t)p_1^-(t) - b_2(t)q(t)p_2^-(t) \right](1 - \Psi) \\
- &\delta e^{-\rho t}p_2^-(t) \left[ a_2(t)p(t)\Phi - b_2(t)q(t)\Psi \right] - (\delta e^{-\rho t})^2p_2^-(t) \left[ a_2(t)p(t)p_1^-(t) - b_2(t)q(t)p_2^-(t) \right] \\
< &\delta e^{-\rho t}d_2(t)(p_2^+(t) - p_2^-(t)) \int_{|y| > M_0} J_2(y)[\zeta(\xi^+ - y) - \zeta(\xi^+)]dy \\
- &\delta e^{-\rho t} \left[ a_2(t)p(t)p_1^+(t) - (\lambda_0 - b_2(t)q(t))p_2^-(t) - \rho p_2^-(t) \right] \\
+ &\delta e^{-\rho t} \left[ a_2(t)p(t)p_1^-(t) - b_2(t)q(t)p_2^-(t)(1 - \varepsilon_0) \right] + \delta e^{-\rho t}p_2^-(t)b_2(t)q(t) \left[ \varepsilon_0 + \delta e^{-\rho t}p_2^-(t) \right] \\
\leq &\delta e^{-\rho t}p_2^-(t) \left[ \left( \frac{C_3}{p_2(t)} + 1 \right) \rho - \lambda_0 + \varepsilon_0 C_1 \left( 1 + \frac{C_2}{C_3} \right) + \delta C_1 C_2 \right] \\
\leq &\delta e^{-\rho t}p_2^-(t) \left( 2\rho - \lambda_0 + \frac{2\lambda}{3} \right) \leq 0.
\end{align*}
\]

**Case(ii)** By the definition of $C_0$ and $C_2$, we have

\[
\left| \frac{\partial p_1(\xi^+, t)}{\partial t} \right| = \left( \zeta(\xi^+ - y) \frac{d}{dt}p_1^+(t) + (1 - \zeta(\xi^+)) \frac{d}{dt}p_1^-(t) \right) \leq C_0,
\]

\[
\left| \frac{\partial p_2(\xi^+, t)}{\partial t} \right| = \left( \zeta(\xi^+ - y) \frac{d}{dt}p_2^+(t) + (1 - \zeta(\xi^+)) \frac{d}{dt}p_2^-(t) \right) \leq C_0,
\]

\[
\left| \frac{\partial p_1(\xi^+, t)}{\partial \xi^+} \right| = \zeta'(\xi^+) |p_1^+(t) - p_1^-(t)| \leq |p_1^+(t) - p_1^-(t)| \leq C_2,
\]

\[
\left| \frac{\partial p_2(\xi^+, t)}{\partial \xi^+} \right| = \zeta'(\xi^+) |p_2^+(t) - p_2^-(t)| \leq |p_2^+(t) - p_2^-(t)| \leq C_2,
\]

\[
p_1(\xi^+, t) = \zeta(\xi^+)p_1^+(t) + (1 - \zeta(\xi^+))p_1^-(t) \leq C_2,
\]

\[
p_2(\xi^+, t) = \zeta(\xi^+)p_2^+(t) + (1 - \zeta(\xi^+))p_2^-(t) \leq C_2.
\]
We now give the estimates of $I_1, I_2$ and $I_3$ in terms of $\delta, C_0, C_1, C_2$ and $C_4$.

\[
I_1(\xi^+, t) = \delta e^{-\rho \int_0^y} \left[ J_1(y) \left[ \zeta(\xi^+ - y) - \zeta(\xi^+) \right] \right] dy
\]

\[
\leq \delta e^{-\rho \int_0^y} \left[ \left| p_1^+ (t) - p_1^- (t) \right| \right] \leq \delta e^{-\rho \int_0^y} d C_2,
\]

\[
I_2(\xi^+, t) = \delta e^{-\rho \int_0^y} \left[ -\sigma_1 \rho \Phi_\xi - \frac{\partial p_1(\xi^+, t)}{\partial \xi} (c + \sigma_1 \rho e^{-\rho t}) - \frac{\partial p_1(\xi^+, t)}{\partial t} + \rho p_1(\xi^+, t) \right]
\]

\[
\leq \delta e^{-\rho \int_0^y} \left[ -\sigma_1 \rho C_4 + C_2 (|c| + \sigma_1 \rho) + C_0 + \rho C_2 \right]
\]

\[
\leq \delta e^{-\rho \int_0^y} \left[ -\frac{1}{2} \sigma_1 \rho C_4 + C_0 + (|c| + \rho) C_2 \right],
\]

\[
(2.31)
\]

Thus, if $\sigma_1$ is chosen as

\[
\sigma_1 \geq \frac{2C_0 + 2C_2 (d + |c| + \rho + C_1)}{C_4 \rho},
\]

then we have

\[
I_1(\xi^+, t) + I_2(\xi^+, t) + I_3(\xi^+, t) < \delta e^{-\rho \int_0^y} \left[ d C_2 - \frac{1}{2} \sigma_1 \rho C_4 + C_0 + (|c| + \rho) C_2 + C_1 C_2 \right] \leq 0.
\]

Likewise, it can be verified that

\[
d_2(t) \left[ J_2 \ast \psi^+(x, t) - \psi^+(x, t) \right] - \psi^+_t + (1 - \psi^+) \left[ a_2(t) p(t) \phi^+ - b_2(t) q(t) \psi^+ \right]
\]

\[
= \delta e^{-\rho \int_0^y} \left[ p_2^+ (t) - p_2^- (t) \right] \int_0^y J_2(y) \left[ \zeta(\xi^+ - y) - \zeta(\xi^+) \right] dy
\]

\[
+ \delta e^{-\rho \int_0^y} \left[ -\sigma_1 \rho \Psi_\xi - \frac{\partial p_2(\xi^+, t)}{\partial \xi} (c + \sigma_1 \rho e^{-\rho t}) - \frac{\partial p_2(\xi^+, t)}{\partial t} + \rho p_2(\xi^+, t) \right]
\]

\[
+ \delta e^{-\rho \int_0^y} \left[ b_2(t) q(t) p_1(\xi^+, t) - b_2(t) q(t) p_2(\xi^+, t) \right] \left( (1 - \psi) - p_2(\xi^+, t) \left( a_2(t) p(t) \Phi - b_2(t) q(t) \Psi \right) \right]
\]

\[
- \delta e^{-\rho \int_0^y} \left[ d C_2 - \frac{1}{2} \sigma_1 \rho C_4 + C_0 + (|c| + \rho) C_2 + C_1 C_2 \right] \leq 0.
\]

Case(iii) can be dealt with as Case(i). Hence $(\phi^+, \psi^+)$ is an upper solution of system (1.7). Then the proof is complete. \hfill \Box

**Remark 2.2.** System (1.7) is monotone only in the phase space

\[
\mathbb{W} = \{ (\phi, \psi) | \phi, \psi \in \mathcal{C}, \phi \geq 0 \text{ and } \psi \leq 1 \}.
\]

When $\phi^- \notin \mathbb{W}$ or $\psi^+ \notin \mathbb{W}$, we can use their truncations $\hat{\phi}^- = \max\{0, \phi^-\}$ and $\hat{\psi}^+ = \min\{1, \psi^+\}$ to replace $\phi^-$ and $\psi^+$, respectively, so that the comparison principle in Lemma 2.2 still works for the upper and lower solutions in Lemma 2.4.
We are in a position now to state and prove the uniqueness of the bistable time-periodic traveling wave if it exists.

**Theorem 2.2.** Assume that (1.2) holds. Then there exists at most one (up to translation) bistable time-periodic traveling wave solution to (1.9)–(1.10).

**Proof.** We first prove the uniqueness of the bistable wave speed by contradiction and assume that (1.9)–(1.10) has two solutions \((\Phi, \Psi)(x + ct, t)\) and \((\Phi_1, \Psi_1)(x + c_1 t, t)\) with speeds \(c\) and \(c_1\). By Lemma 2.4 and the comparison principle, we have

\[
\Phi(x + ct + \kappa^- - \sigma_1 \delta(1 - e^{-\rho t}), t) - Ce^{-\rho t}/2 = \phi^-(x, t) \leq \Phi_1(x + c_1 t, t)
\]

for some \(\kappa^- \in \mathbb{R}\) and \(C > 0\) since (2.32) is true at \(t = 0\). It follows from the above formulas that \(c \leq c_1\). Otherwise, assume that \(\Phi_1(\eta, t) < 1\) for some fixed value \(\eta\) and \(t \in (0, \infty)\). If \(c > c_1\), then on the line \(x + c_1 t = \eta\) we have

\[
\Phi_1(\eta, t) = \Phi_1(x + c_1 t, t) \geq \phi^-(x, t) = \Phi(\eta + (c - c_1) t + \kappa^- - \sigma_1 \delta(1 - e^{-\rho t}), t) - Ce^{-\rho t}/2.
\]

It turns out that \(\Phi_1(\eta, t) \geq 1\) as \(t\) is sufficiently large. This is a contradiction. By using the same idea we can also prove \(c \geq c_1\). Therefore, we have \(c = c_1\).

Now from (2.32) by letting \(t \to \infty\), we get

\[
\Phi(\eta + \kappa^- - \sigma_1 \delta, t) \leq \Phi_1(\eta, t).
\]

Similarly we can get

\[
\Phi_1(\eta, t) \leq \Phi(\eta + \kappa^+ + \sigma_1 \delta, t).
\]

As such, we can easily follow the idea in [6] (see Step 2 on page 133) to prove that the wave profile is unique up to translation. \(\square\)

### 2.4. Stability

This subsection is devoted to discussing the Liapunov stability of the bistable \(T\)-periodic traveling wave solution of system (1.7).

**Theorem 2.3.** Assume that (1.2) holds and there exists \(\Gamma(z, t) = (\Phi(z, t), \Psi(z, t)), z = x + ct\) as the bistable \(T\)-periodic traveling wave profile of system (1.7) connecting \(\alpha\) and \(\beta\). Suppose that \(\omega(x, t) = (\phi(x, t), \psi(x, t))\) is the solution of system (1.7) with the initial data \(\omega_0 = (\phi_0, \psi_0)\) satisfying \((0, 0) \leq \omega_0 \leq (1, 1)\). Then the traveling wave \(\Gamma(z, t)\) is stable in the sense that for arbitrary small \(\epsilon > 0\) there is a constant \(\delta^*\), such that

\[
\|\omega(x, t) - \Gamma(z, t)\| < \epsilon, \ (x, t) \in \mathbb{R} \times \mathbb{R}^+
\]

as long as \(\omega_0\) satisfies

\[
\|\omega_0(x) - \Gamma(x, 0)\| < \delta^*, \ x \in \mathbb{R}.
\]

**Proof.** Let \(\delta\) be defined in Lemma 2.4 with \(\kappa^\pm = 0\), \(\delta_m = \min \{\inf_{x \in \mathbb{R}} \{p_1(x, 0)\}, \inf_{x \in \mathbb{R}} \{p_2(x, 0)\}\}\) and \(\delta^* = \delta m\). Then condition (2.34) means that

\[
\Phi(x, 0) - \delta p_1(x, 0) \leq \phi_0 \leq \delta p_1(x, 0) + \Phi(x, 0), \ x \in \mathbb{R}
\]

and

\[
\Psi(x, 0) - \delta p_2(x, 0) \leq \psi_0 \leq \delta p_2(x, 0) + \Psi(x, 0), \ x \in \mathbb{R}.
\]
By this and (2.24), we have
\[ \phi^-(x,0) \leq \phi_0(x) \leq \phi^+(x,0), \quad \psi^-(x,0) \leq \psi_0(x) \leq \psi^+(x,0), \quad x \in \mathbb{R}. \]

Then lemma 2.2 implies that
\[ \phi^-(x,t) \leq \phi(x,t) \leq \phi^+(x,t), \quad \psi^-(x,t) \leq \psi(x,t) \leq \psi^+(x,t), \quad x \in \mathbb{R}, \quad t \in [0,1) \]
where \((\phi^\pm(x,t),\psi^\pm(x,t))\) are defined in lemma 2.4. Thus we have
\[ |\phi^\pm(z,t) - \Phi(z,t)| \]
\[ \leq |p_1(z \pm \sigma_1 \delta(1 - e^{-\rho t}), t) e^{-\rho t}| + |\Phi(z \pm \sigma_1 \delta(1 - e^{-\rho t}), t) - \Phi(z,t)| \]
\[ \leq \delta |p_1(z \pm \sigma_1 \delta(1 - e^{-\rho t}), t) |e^{-\rho t} + \sigma_1 \delta| \frac{\partial}{\partial z} \Phi(z \pm \theta \sigma_1 \delta(1 - e^{-\rho t}), t) |(1 - e^{-\rho t}) \]
\[ \leq \chi \delta, \]
where \(\theta \in (0,1)\) and \(\chi > 0\) does not depend on \(\delta\). Likewise, it is easy to check that
\[ |\psi^\pm(x,t) - \Psi(z,t)| \leq \chi \delta. \]
By (2.35) and further taking \(\delta < \frac{\epsilon}{\sqrt{2} \chi}\), i.e., \(\delta^* < \delta_m \frac{\epsilon}{\sqrt{2} \chi}\), after a simple computation, we get
\[ \|\omega(x,t) - \Gamma(z,t)\| \leq \sqrt{2} \chi \delta < \epsilon. \]
The proof is complete. \(\Box\)

3. The value interval of the bistable wave speed

In this section, we focus on the estimation of value interval and the determinacy of sign of the bistable wave speed. We will assume that only (1.2) (instead of (1.6)) holds in the following sections.

**Theorem 3.1.** Assume that there exists a bistable \(T\)-periodic traveling wave profile of system (1.7) connecting \(\mathbf{o}\) and \(\beta\). Let \(c\) be the bistable wave speed. Then we have
\[ -c^+_i(0,\alpha_i) \leq c \leq c^+_i(\alpha_i, \beta), \quad i = 1, 2. \]

Particularly, for \(i = 1\) we have
\[ \frac{1}{T} \inf_{0 < \mu < \infty} \frac{\gamma_1(\mu)}{\mu} \leq c \leq \frac{1}{T} \inf_{0 < \mu < \infty} \frac{\gamma_2(\mu)}{\mu}, \]
(3.2)
where
\[ \gamma_1(\mu) = \int_0^T d_1(t) \left( \int_{\mathbb{R}} J_1(y,t) e^{-\mu y} dy - 1 \right) + a_1(t)p(t) dt \]
and
\[ \gamma_2(\mu) = \int_0^T d_2(t) \left( \int_{\mathbb{R}} J_2(y,t) e^{\mu y} dy - 1 \right) + b_2(t)q(t) dt. \]

**Proof.** We shall prove \(c \leq c^+_i(\alpha_i, \beta)\). The remainder of (3.1) can be proved by means of a similar method.

Let \((\Phi_1, \Psi_1)(z,t)\) be the \(T\)-periodic monostable traveling wave profile of (1.9) satisfying
\[ (\Phi_1, \Psi_1)(-\infty, t) = \alpha_1, \quad (\Phi_1, \Psi_1)(\infty, t) = \beta \]
with the wave speed $c^+_{\alpha_1, \beta}$. Then, $(\Phi_1, \Psi_1)(x + c^+_{\alpha_1, \beta}t, t)$ is an exact solution of (1.7) with the initial data as $(\Phi_1, \Psi_1)(x, 0)$. To proceed, we give another initial functions $(\phi, \psi)(x, 0)$ of (1.7), which is continuous, nondecreasing and satisfies

$$\phi(x, 0) = \psi(x, 0) = \begin{cases} 0, & x < -L, \\ 1 - \tau, & x > L, \end{cases} \quad (3.3)$$

for some $L > 0$ and $\tau \in (0, 1)$ such that (2.34) in Theorem 2.3 holds. It is possible (by shift if necessary) to assume that

$$\Phi_1(x, 0) \geq \phi(x, 0) \text{ and } \Psi_1(x, 0) \geq \psi(x, 0), \quad x \in \mathbb{R}.$$ 

By applying the comparison principle, we then obtain

$$(\Phi_1, \Psi_1)(x + c^+_{\alpha_1, \beta}t, t) \geq (\phi, \psi)(x, t), \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+.$$ \quad (3.4)

By Theorem 2.3, we know that $(\phi, \psi)(x, t)$ is sufficiently close to the $T$-periodic bistable traveling wave profile $\Gamma(z, t) = (\Phi, \Psi)(z, t)$ with $z = x + ct$. Then it follows that

$$\Phi_1(x + c^+_{\alpha_1, \beta}t, t) \geq \phi(x, t) \geq \Phi(x + ct, t) - \epsilon,$$ \quad (3.5)

for any $\epsilon > 0$. Let $\Phi_1(\xi, t) < 1$ for the fixed points $(\xi, t), t > 0$. By (3.5), on the line $\xi = x + c^+_{\alpha_1, \beta}t$, if $c > c^+_{\alpha_1, \beta}$, then we have

$$\Phi_1(\xi, t) \geq \Phi(\xi + (c - c^+_{\alpha_1, \beta})t, t) - \epsilon \to 1 - \epsilon \text{ as } t \to \infty.$$ 

This is a contradiction since $\epsilon$ is arbitrary small. Thus $c \leq c^+_{\alpha_1, \beta}$.

The inequality in (3.2) is a straightforward consequence of (3.1) together with the proof process of Theorem 2.1. Then the proof is complete. \qed

We now establish the relationship between the bistable wave speed and the wave speeds of upper/lower solutions of (1.9). The proofs of the following two theorems can be proceeded in the same way as in Theorem 3.1 and are omitted here.

**Theorem 3.2.** Assume that (1.9) has a nonnegative upper solution $(\Phi(z, t), \Psi(z, t))$ with speed $\bar{c}$, nondecreasing in $z$, $T$-periodic in $t$, and satisfying

$$(\Phi, \Psi)(-\infty, t) < (1, 1), \quad (\Phi, \Psi)(\infty, t) \geq (1, 1).$$

Then the speed $c$ of the bistable $T$-periodic traveling wave of (1.9) satisfies

$$c \leq \bar{c}. \quad (3.6)$$

In particular, if $\bar{c} < 0$, then the bistable wave speed $c$ is negative.

**Theorem 3.3.** Suppose that (1.9) has a nonnegative lower solution $(\Phi(z, t), \Psi(z, t))$ with speed $\underline{c}$, nondecreasing in $z$, $T$-periodic in $t$ and satisfying

$$(\Phi, \Psi)(-\infty, t) = (0, 0) < (\Phi, \Psi)(\infty, t) \leq (1, 1).$$

Then the speed $c$ of the bistable $T$-periodic traveling wave of (1.9) satisfies

$$c \geq \underline{c}. \quad (3.7)$$

In particular, if $\underline{c} > 0$, then the bistable wave speed $c$ is positive.

Therefore, based on these two theorems, we could find explicit conditions for determining the sign of the bistable wave speed by seeking the formulas of upper/lower solutions of (1.9).
4. Result on the propagation direction

In this section, we shall give some explicit criteria to determine the sign of the bistable wave speed, which implies the propagation direction of the bistable traveling wave. To this end, we first discuss the characteristic equation of the bistable traveling wave of system (1.9)–(1.10) near the equilibrium points $o$ and $\beta$, which will be used to precisely construct upper solutions and lower solutions of (1.9).

4.1. Eigenvalue problem near $o$ and $\beta$

Linearizing system (1.9) at $o$ yields

\[
\begin{aligned}
&d_1(t) \left( \int_{\mathbb{R}} J_1(y)\Phi(z-y)dy - \Phi \right) - c\Phi_t - \Phi_t + \Phi[a_1(t)p(t) - b_1(t)q(t)] = 0, \\
&d_2(t) \left( \int_{\mathbb{R}} J_2(y)\Psi(z-y)dy - \Psi \right) - c\Psi_t - \Psi_t + a_2(t)p(t)\Phi - b_2(t)q(t)\Psi = 0.
\end{aligned}
\] (4.1)

Let $(\Phi(z,t), \Psi(z,t)) = (\varphi_1(t)e^{\mu z}, \nu_1(t)e^{\mu z})$ solve (4.1). Then this leads to an eigenvalue problem

\[
\begin{aligned}
&\varphi'_1(t) = \left[ d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{-\mu y}dy - 1 \right) - c\mu + a_1(t)p(t) - b_1(t)q(t) \right] \varphi_1(t), \\
&\nu'_1(t) = \left[ d_2(t) \left( \int_{\mathbb{R}} J_2(y)e^{-\mu y}dy - 1 \right) - c\mu - b_2(t)q(t) \right] \nu_1(t) + a_2(t)p(t)\varphi_1(t), \\
&\varphi_1(t) = \varphi_1(t+T), \quad \nu_1(t) = \nu_1(t+T).
\end{aligned}
\] (4.2)

By the first equation in (4.2), we have

\[
I_1(\mu, c) := \int_0^T \left\{ d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{-\mu y}dy - 1 \right) - c\mu + a_1(t)p(t) - b_1(t)q(t) \right\} dt = 0. \tag{4.3}
\]

It is easy to check that $\frac{\partial^2 I_1}{\partial \mu^2}(\mu, c) \geq 0$ and $I_1(0, c) < 0$, where (1.2) is used. Then $I_1(\mu, c) = 0$ has only one positive root denoted by $\mu_1$ or $\mu_1(c)$. If we let $\nu_1(t) = \rho_1(t)\varphi_1(t)$, then the second equation in (4.2) is changed into

\[
\rho'_1(t) - \left( g_1(\mu, t) - \frac{\varphi'_1(t)}{\varphi_1(t)} \right) \rho_1(t) = a_2(t)p(t), \tag{4.4}
\]

where

\[
g_1(\mu, t) = d_2(t) \left( \int_{\mathbb{R}} J_2(y)e^{-\mu y}dy - 1 \right) - c\mu - b_2(t)q(t). \tag{4.5}
\]

Hence the linearized system has a solution such that

\[
(\Phi, \Psi)(z, t) \sim (\varphi_1(t), \rho_1(t)\varphi_1(t))e^{\mu_1 z} \text{ as } z \to -\infty. \tag{4.6}
\]

However, let $(\Phi(z, t), \Psi(z, t)) = (\varphi_1(t)e^{\mu_1 z}, \nu_1(t)e^{\mu_2 z} + \rho_1(t)\varphi_1(t)e^{\mu_1 z})$ solve (4.1), where $\mu_2$ or $\mu_2(c)$ is another positive eigenvalue of (4.2) with $\mu_1 \neq \mu_2$. Then we have that $\rho_1(t)$ still satisfies (4.4) and $\mu_2$ solves the following equation

\[
h_1(\mu, c) := \int_0^T \left\{ d_2(t) \left( \int_{\mathbb{R}} J_2(y)e^{-\mu y}dy - 1 \right) - c\mu - b_2(t)q(t) \right\} dt = 0. \tag{4.7}
\]
Thus we have a solution with
\[(\Phi, \Psi)(z, t) \sim (\varphi_1(t)e^{\mu_1z}, \nu_1(t)e^{\mu_2z} + \rho_1(t)\varphi_1(t)e^{\mu_1z}) \text{ as } z \to -\infty.\] (4.8)

Next linearizing system (1.9) around the equilibrium \(\beta\), we have
\[
\begin{cases}
d_1(t) \left( \int_{\mathbb{R}} J_1(y)\Phi(z - y, t)\mathrm{d}y - \Phi \right) - c\Phi_z - \Phi_t - a_1(t)p(t)\Phi + b_1(t)q(t)\Psi = 0, \\
d_2(t) \left( \int_{\mathbb{R}} J_2(y)\Psi(z - y, t)\mathrm{d}y - \Psi \right) - c\Psi_z - \Psi_t + [b_2(t)q(t) - a_2(t)p(t)]\Psi = 0.
\end{cases}
\] (4.9)

Let (4.9) have solutions in the form of \((\Phi(z, t), \Psi(z, t)) = (\varphi_2(t)e^{-\mu z}, \nu_2(t)e^{-\mu z})\). Then the eigenvalue problem associated with it is
\[
\begin{cases}
\nu_2'(t) = \left[ d_2(t) \left( \int_{\mathbb{R}} J_2(y)e^{\mu y}\mathrm{d}y - 1 \right) + c\mu + b_2(t)q(t) - a_2(t)p(t) \right] \nu_2(t), \\
\varphi_2'(t) = \left[ d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{\mu y}\mathrm{d}y - 1 \right) + c\mu - a_1(t)p(t) \right] \varphi_2(t) + b_1(t)q(t)\nu_2(t), \\
\varphi_2(t) = \varphi_2(t + T), \quad \nu_2(t) = \nu_2(t + T).
\end{cases}
\] (4.10)

Integrating the first equation over the interval \([0, T]\) produces
\[
I_2(\mu, c) := \int_0^T \left\{ d_2(t) \left( \int_{\mathbb{R}} J_2(y)e^{\mu y}\mathrm{d}y - 1 \right) + c\mu + b_2(t)q(t) - a_2(t)p(t) \right\} \mathrm{d}t = 0.\] (4.11)

By a simple computation, we have \(\frac{\partial I_2}{\partial \mu^2}(\mu, c) \geq 0\). Furthermore, (1.2) implies \(I_2(0, c) < 0\). Hence (4.11) has only one positive root denoted by \(\mu_4\) or \(\mu_4(c)\). If assume \(\varphi_2(t) = \rho_2(t)\nu_2(t)\), then the second equation in (4.10) yields
\[
\rho_2'(t) - \left( g_2(\mu, t) - \frac{\nu_2'(t)}{\nu_2(t)} \right) \rho_2(t) = b_1(t)q(t),\] (4.12)

where
\[
g_2(\mu, t) = d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{\mu y}\mathrm{d}y - 1 \right) + c\mu - a_1(t)p(t).\] (4.13)

This means that the solution of system (1.9) may have a behavior
\[(\Phi, \Psi)(z, t) \sim (1, 1) - (\rho_2(t)\nu_2(t), \nu_2(t))e^{-\mu_4z} \text{ as } z \to \infty.\] (4.14)

However, we suppose that \((\Phi(z, t), \Psi(z, t)) = (\varphi_2(t)e^{-\mu_3z} + \rho_2(t)\nu_2(t)e^{-\mu_4z}, \nu_2(t)e^{-\mu_4z})\) is a solution of (4.10), where \(\mu_3\) or \(\mu_3(c)\) is another positive eigenvalue of (4.10) and \(\mu_3 \neq \mu_4\). Then it is easy to verify that \(\rho_2\) still solves (4.12) and \(\mu_3\) satisfies
\[
h_2(\mu, c) := \int_0^T \left\{ d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{\mu y}\mathrm{d}y - 1 \right) + c\mu - a_1(t)p(t) \right\} \mathrm{d}t = 0.\] (4.15)

Then the asymptotical behavior of the bistable traveling wave to system (1.9) near \(\beta\) may become
\[(\Phi, \Psi)(z, t) \sim (1 - \varphi_2(t)e^{-\mu_3z} - \rho_2(t)\nu_2(t)e^{-\mu_4z}, 1 - \nu_2(t)e^{-\mu_4z}) \text{ as } z \to \infty.\] (4.16)

The positivity of \(\varphi_1(t)\) and \(\nu_2(t)\) is easy to verify, which shall be used to construct upper/lower solutions of (1.9).
4.2. Explicit condition for propagation direction

According to the results in the last subsection and Theorems 3.2 and 3.3, under condition (1.2), we shall derive explicit conditions for determining the sign of the bistable wave speed by constructing upper/lower solutions. By (1.8), it is well-known that the bistable traveling wave with positive (negative) speed propagates to the left (right). We begin with introducing two functions

\[ Y_1(\mu(c), t) = d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{\mu(c)y}dy - 1 \right) - d_2(t) \left( \int_{\mathbb{R}} J_2(y)e^{\mu(c)y}dy - 1 \right) \]  

(4.17)

and

\[ Y_2(\mu(c), t) = -d_1(t) \int_{0}^{\infty} J_1(y)(1 + e^{\mu(c)y})(e^{\frac{1}{2}\mu(c)y} - e^{-\frac{1}{2}\mu(c)y})^2dy. \]  

(4.18)

**Theorem 4.1.** Assume that the T-periodic coefficients \( d_i(t), a_i(t), b_i(t) (i = 1, 2) \) satisfy

\[ 0 < \frac{Y_1(\mu_1(0), t) + a_1(t)p(t) - b_1(t)q(t) + b_2(t)q(t)}{a_2(t)p(t)} < \frac{a_1(t)p(t) + Y_2(\mu_1(0), t)}{a_1(t)p(t)}, \quad t \in [0, T], \]  

(4.19)

where \( p(t), q(t) \) are defined in (1.3) and \( \mu_1(0) \) solves

\[ I_1(\mu_1(0), 0) := \int_{0}^{T} \left\{ d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{-\mu_1(0)y}dy - 1 \right) + a_1(t)p(t) - b_1(t)q(t) \right\} dt = 0. \]  

(4.20)

Then the bistable wave speed of system (1.9) is positive.

**Proof.** By (4.19) and \( Y_2(\mu_1(0), t) < 0 \), there exists a real number \( k_1 \) satisfying

\[ \frac{Y_1(\mu_1(0), t) + a_1(t)p(t) - b_1(t)q(t) + b_2(t)q(t)}{a_2(t)p(t)} < k_1 < \frac{a_1(t)p(t) + Y_2(\mu_1(0), t)}{a_1(t)p(t)} < 1. \]  

(4.21)

Define a pair of functions \((\Phi, \Psi)\) by

\[ \Phi(z, t) = \frac{k_1 \varphi_1(t)}{\varphi_1(t) + e^{-\mu_1(z)}}; \quad \Psi(z, t) = \frac{1}{k_1} \Phi \]  

(4.22)

with speed \( c > 0 \). If \((\Phi, \Psi)\) can be proved to be a lower solution of (1.9), then Theorem 3.3 implies the desired result.

Indeed, substituting \((\Phi, \Psi)\) into (1.9), from the first equation it follows that

\[ d_1(t) \left( \int_{\mathbb{R}} J_1(y)\Phi(z - y, t)dy - \Phi \right) - \epsilon \Phi_z - \Phi_t + \Phi [a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \Psi)] \]

\[ = \Phi(1 - \Phi) \left\{ \frac{d_1(t) \left( \int_{\mathbb{R}} J_1(y)\Phi(z - y, t)dy - \Phi \right)}{\Phi(1 - \Phi)} - \epsilon \mu_1(z) - \frac{\varphi_1(t)}{\varphi_1(t)} + H_1(z, t) \right\} \]

\[ \overset{\text{def}}{=} \Lambda_1, \]
where

\[ H_1(z,t) = \frac{a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \frac{\Phi}{k_1})}{1 - \frac{\Phi}{k_1}}. \]

Now applying the first equation in (4.2), we have

\[ \Lambda_1 = \frac{\Phi^2}{k_1} (1 - \frac{\Phi}{k_1}) \left\{ \frac{d_1(t) \int_{\mathbb{R}} J_1(y) \left( \Phi(z - y,t) - \Phi - (e^{-\mu_1(z)y} - 1)\Phi(1 - \frac{\Phi}{k_1}) \right) dy}{\Phi^2(1 - \frac{\Phi}{k_1})} \right\} \]

\[ + \frac{a_1(t)p(t)(1 - k_1)}{1 - \frac{\Phi}{k_1}} \right\} \]

\[ = \frac{\Phi^2}{k_1} (1 - \frac{\Phi}{k_1}) \left\{ d_1(t) \int_{\mathbb{R}} J_1(y) S(\mu_1(z),z,y,t)(2 - e^{\mu_1(z)y} - e^{-\mu_1(z)y})dy + \frac{a_1(t)p(t)(1 - k_1)}{1 - \frac{\Phi}{k_1}} \right\}, \]

where \( S(\mu_1(z),z,y,t) = \frac{\varphi_1(t)e^{\mu_1(z)y} + 1}{\varphi_1(t)e^{\mu_1(z)y} + e^{\mu_1(z)y}} \). It is easy to verify

\[ \begin{cases} 
  e^{-\mu_1(z)y} < S(\mu_1(z),z,y,t) < 1 & \text{for } y \geq 0, \\
  1 < S(\mu_1(z),z,y,t) < e^{-\mu_1(z)y} & \text{for } y \leq 0, 
\end{cases} \quad (z,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{4.23} \]

By this, the first equation in (4.2) and (4.21), we have

\[ \Lambda_1 \geq \frac{\Phi^2}{k_1}(1 - \frac{\Phi}{k_1}) [Y_2(\mu_1(z),t) + a_1(t)p(t) - a_1(t)p(t)k_1] \]

\[ \rightarrow \frac{\Phi^2}{k_1}(1 - \frac{\Phi}{k_1}) [Y_2(\mu_1(0),t) + a_1(t)p(t) - a_1(t)p(t)k_1] > 0 \text{ as } z \rightarrow 0^+. \tag{4.24} \]

For the second equation in (1.9), by (4.23) and (4.21), we have

\[ d_2(t) \left( \int_{\mathbb{R}} J_2(y) \Psi(z - y,t) dy - \Psi \right) - c\Psi - \Psi + (1 - \Psi) [a_2(t)p(t)\Phi - b_2(t)q(t)\Psi] \]

\[ = \frac{\Phi}{k_1}(1 - \frac{\Phi}{k_1}) \left\{ d_2(t) \int_{\mathbb{R}} J_2(y) S(\mu_1(z),z,y,t)(1 - e^{\mu_1(z)y})dy - c\mu_1(z) - \frac{\varphi_1'(t)}{\varphi_1(t)} \right\} \]

\[ + a_2(t)p(t)k_1 - b_2(t)q(t) \]

\[ > \frac{\Phi}{k_1}(1 - \frac{\Phi}{k_1}) \left\{ \int_{\mathbb{R}} J_2(y)(1 - e^{\mu_1(z)y})dy - c\mu_1(z) - \frac{\varphi_1'(t)}{\varphi_1(t)} + a_2(t)p(t)k_1 - b_2(t)q(t) \right\} \]

\[ = \frac{\Phi}{k_1}(1 - \frac{\Phi}{k_1}) \left\{ \int_{\mathbb{R}} J_2(y)(1 - e^{\mu_1(z)y})dy - d_1(t) \left( \int_{\mathbb{R}} J_1(y)e^{-\mu_1(z)y}dy - 1 \right) \right\} \]

\[ - a_1(t)p(t) + b_1(t)q(t) - b_2(t)q(t) + a_2(t)p(t)k_1 \]
Thus, by (4.24) and (4.25), as \( \varepsilon \) is sufficiently close to 0, \((\Phi, \Psi)\) is a lower solution of (1.9), and the proof is complete. \( \square \)

In order to obtain conditions for the negative wave speed, we next construct an explicit upper solution which possesses two piecewise continuous components.

**Theorem 4.2.** Let

\[
F_i (\mu_4(c), s_0, t) = d_i(t) \int_{\mathbb{R}} J_i(y) \left( 2 + \frac{(1 - s_0)(1 - e^{\mu_4(c)y})}{s_0 + (1 - s_0)e^{\mu_4(c)y}} \right) (1 - e^{\mu_4(c)y})dy, i = 1, 2,
\]

where \( s_0 \in (0, 1) \) is a constant. Suppose that there exists \( s_0 \) such that the T-periodic coefficients \( d_i(t), a_i(t), b_i(t) \) \((i = 1, 2)\) satisfy

\[
\max \left\{ \frac{a_2(t)p(t)}{b_2(t)q(t)} + \frac{d_2(t)}{2s_0b_2(t)q(t)}, \Theta_1(s_0, t), \Theta_2(s_0, t) \right\} < \frac{1}{s_0} - \frac{d_1(t)(1 - s_0)}{2s_0^2b_1(t)q(t)} - \frac{a_1(t)p(t)(1 - s_0)}{s_0b_1(t)q(t)}, \quad t \in [0, T],
\]

where

\[
\Theta_1(s_0, t) = \frac{F_1(\mu_4(0), s_0, t) + 2a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)}}{F_1(\mu_4(0), s_0, t) + 2a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} + \frac{d_1(t)}{2}} > 1,
\]

\[
\Theta_2(s_0, t) = \frac{F_2(\mu_4(0), s_0, t) - Y_1(\mu_4(0), t) + a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)}}{F_2(\mu_4(0), s_0, t) - Y_1(\mu_4(0), t) + a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} + \frac{d_2(t)}{2}} > 1,
\]

and \( p(t), q(t), Y_1(\mu_4(0)) \) and \( \rho_2(t) \) are defined in (1.3), (4.17) and (4.12), respectively, and \( \mu_4(0) \) solves

\[
I_2(\mu_4(0), 0) := \int_0^T \left\{ d_2(t) \left( \int_{\mathbb{R}} J_2(y) e^{\mu_4(0)y}dy - 1 \right) + b_2(t)q(t) - a_2(t)p(t) \right\} dt = 0.
\]

Then the bistable wave speed of system (1.9) is negative.

**Proof.** By (4.27), we can take a constant \( k_2 \) such that

\[
\max \left\{ \frac{a_2(t)p(t)}{b_2(t)q(t)} + \frac{d_2(t)}{2s_0b_2(t)q(t)}, \Theta_1(s_0, t), \Theta_2(s_0, t) \right\} < k_2 < \frac{1}{s_0} - \frac{d_1(t)(1 - s_0)}{2s_0^2b_1(t)q(t)} - \frac{a_1(t)p(t)(1 - s_0)}{s_0b_1(t)q(t)}, \quad t \in [0, T].
\]

Now define a pair of continuous and nondecreasing functions by

\[
\Phi(z, t) = \left\{ \begin{array}{ll}
    s_0, & z < z_1(t), \\
    \varphi_2(t), & \varphi_2(t) + e^{-\mu_4(c)\varepsilon}, z \geq \varphi_2(t), \end{array} \right.
\]

\[
\Psi(z, t) = \left\{ \begin{array}{ll}
    k_2\Phi, & z \leq z_2(t), \\
    1, & z > z_2(t), \end{array} \right.
\]

with \( \varepsilon < 0, \Phi(z_1(t), t) = s_0 \) and \( k_2\Phi(z_2(t), t) = 1 \). Obviously, \( z_1(t) < z_2(t) \).
For \( z \geq z_1(t) \), it is easy to check that
\[
\Phi_t = \frac{\varphi_2(t)}{\varphi_2(t)} \Phi(1 - \Phi), \quad \Psi_z = \mu_4(r) \Phi(1 - \Phi).
\] (4.32)

If we prove that \((\Phi, \Psi)\) is an upper solution of system (1.9), then the desired result follows. To this end, we substitute \((\Phi, \Psi)\) into (1.9). When \( z < z_1(t) \), by using the first equation of (4.2) and (4.31), from the first equation in (1.9) we have
\[
d_1(t) \left( \int_{\mathbb{R}} J_1(y) \Phi(z - y, t) \, dy - \Phi(z, t) \right) - c \Phi_z - \Phi_t + \Phi \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \Psi) \right]
\]
and from the second equation in (1.9) it follows that
\[
d_2(t) \left( \int_{\mathbb{R}} J_2(y) \Psi(z - y, t) \, dy - \Psi(z, t) \right) - c \Psi_z - \Psi_t + (1 - \Psi) \left[ a_2(t)p(t)\Phi - b_2(t)q(t)\Psi \right]
\]
\[
= b_1(t)q(t)s_0 \left[ k_2s_0 - \left( 1 - \frac{d_1(t)(1 - s_0)}{2s_0b_1(t)q(t)} - \frac{a_1(t)p(t)(1 - s_0)}{b_1(t)q(t)} \right) \right] \leq 0,
\]
and from the second equation in (1.9) it follows that
\[
d_2(t) \left( \int_{\mathbb{R}} J_2(y) \Psi(z - y, t) \, dy - \Psi(z, t) \right) - c \Psi_z - \Psi_t + (1 - \Psi) \left[ a_2(t)p(t)\Phi - b_2(t)q(t)\Psi \right]
\]
\[
= \left( 1 - k_2s_0 \right)s_0 \int_{\mathbb{R}} J_2(y) q(t) \left[ \frac{a_2(t)p(t)}{b_2(t)q(t)} + \frac{d_2(t)}{2s_0b_2(t)q(t)} - k_2 \right] \leq 0.
\]

To proceed, we first estimate the nonlocal terms in system (1.9). When \( z \geq z_1(t) \), by direct computation, we have
\[
\int_{\mathbb{R}} J_1(y) \Phi(z - y, t) \, dy - \Phi(z, t)
\]
\[
= \int_{-\infty}^{-z_1(t)} J_1(y) \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(r)(z - y)}} \, dy + \int_{z_1(t)}^{+\infty} J_1(y) s_0 \, dy - \Phi(z, t)
\]
\[
= \int_{\mathbb{R}} J_1(y) \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(r)(z - y)}} \, dy + \int_{z_1(t)}^{+\infty} J_1(y) \left( s_0 - \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(r)(z - y)}} \right) \, dy - \Phi(z, t)
\] (4.33)
\[
\leq \int_{\mathbb{R}} J_1(y) \left( \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(r)(z - y)}} - \Phi(z, t) \right) \, dy + \int_{0}^{+\infty} J_1(y) s_0 \, dy
\]
\[
= \int_{\mathbb{R}} J_1(y) \left( \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(r)(z - y)}} - \Phi(z, t) \right) \, dy + \frac{1}{2} s_0.
\]

Similarly, if \( z_1(t) \leq z \leq z_2(t) \), then
\[
\int_{\mathbb{R}} J_2(y) \Psi(z - y, t) \, dy - \Psi(z, t)
\]
\[ \begin{align*}
&= \int_{-\infty}^{z-z_2(t)} J_2(y) dy + \int_{z-z_1(t)}^{z-z_1(t)} J_2(y) \frac{k_2 \varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(\tau)(z-y)}} dy + \int_{z-z_1(t)}^{+\infty} J_2(y) k_2 s_0 dy - \overline{\Psi}(z, t) \\
&\leq k_2 \int_{\mathbb{R}} J_2(y) \left( \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(\tau)(z-y)}} \right) dy + k_2 \int_{z-z_1(t)}^{+\infty} J_2(y) \left( s_0 - \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(\tau)(z-y)}} \right) dy - \overline{\Psi}(z, t) \\
&= k_2 \int_{\mathbb{R}} J_2(y) \left( \frac{\varphi_2(t)}{\varphi_2(t) + e^{-\mu_4(\tau)(z-y)}} - \overline{\Phi}(z, t) \right) dy + k_2 \int_{0}^{+\infty} J_2(y) s_0 dy.
\end{align*} \]

We further give the estimates of \( S(\mu_4(\tau), z, y, t) \) for \( z \in [z_1(t), z_2(t)] \) and \( t > 0 \) by

\[ \left\{ \begin{array}{ll}
1 + \frac{1 - e^{\mu_4(\tau)y}}{\varphi_2(t) e^{\mu_4(\tau)z_1(t)} + e^{\mu_4(\tau)y}} \leq S(\mu_4(\tau), z, y, t) \leq 1 + \frac{1 - e^{\mu_4(\tau)y}}{\varphi_2(t) e^{\mu_4(\tau)z_2(t)} + e^{\mu_4(\tau)y}} & \text{for } y > 0, \\
1 + \frac{1 - e^{\mu_4(\tau)y}}{\varphi_2(t) e^{\mu_4(\tau)z_1(t)} + e^{\mu_4(\tau)y}} \leq S(\mu_4(\tau), z, y, t) \leq 1 + \frac{1 - e^{\mu_4(\tau)y}}{\varphi_2(t) e^{\mu_4(\tau)z_2(t)} + e^{\mu_4(\tau)y}} & \text{for } y < 0.
\end{array} \] (4.35)

Thus, for \( z \in [z_1(t), z_2(t)] \), by (4.33) and (4.35), from the first equation in (1.9) we have

\[ d_1(t) \left( \int_{\mathbb{R}} J_1(y) \Phi(z-y, t) dy - \overline{\Phi} \right) - \overline{\varphi} \Phi_z - \Phi_t + \Phi \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t)(1 - \Psi) \right] \]

\[ = \Phi(1 - \Phi) \left\{ \frac{d_1(t)}{\Phi(1 - \Phi)} \left( \int_{\mathbb{R}} J_1(y) \overline{\Phi}(z-y, t) dy - \overline{\Phi} \right) - \Phi \left[ a_1(t)p(t)(1 - \Phi) - b_1(t)q(t) \frac{1 - \Psi}{1 - \Phi} \right] \right\} \]

\[ \leq \Phi(1 - \Phi) \left\{ d_1(t) \int_{\mathbb{R}} J_1(y) S(\mu_4(\tau), z, y, t)(1 - e^{\mu_4(\tau)y}) dy + \frac{d_1(t) s_0}{2 \Phi(z(t), t)(1 - \Phi(z(t), t))} \right\} \]

\[ - \Phi \left[ a_1(t)p(t) - b_1(t)q(t) \frac{1 - \Psi}{1 - \Phi} \right] \]

\[ \leq \Phi(1 - \Phi) \left\{ d_1(t) \int_{\mathbb{R}} J_1(y) \left( S(\mu_4(\tau), z, y, t) + 1 \right)(1 - e^{\mu_4(\tau)y}) dy + \frac{d_1(t) s_0}{2 \Phi(z_1(t), t)(1 - \Phi(z_2(t), t))} \right\} \]

\[ - 2 \Phi \left[ a_1(t)p(t) - b_1(t)q(t) \frac{1 - \Psi}{\rho_2(t)} \right] \]}

\[ \rightarrow \Phi(1 - \Phi) \left\{ - \left( \frac{d_1(t)}{2} + F_1(\mu_4(0), s_0, t) + 2 a_1(t)p(t) - b_1(t)q(t) \frac{1}{\rho_2(t)} \right) \left( \Theta_1(s_0, t) - k_2 \right) \right\} < 0 \]

and for the second equation in (1.9), by (4.34) and (4.35), we obtain

\[ d_2(t) \left( \int_{\mathbb{R}} J_2(y) \overline{\Psi}(z-y, t) dy - \overline{\Psi} \right) - \overline{\varphi} \overline{\Psi}_z - \left( 1 - \overline{\Psi} \right) \left[ a_2(t)p(t)\overline{\Psi} - b_2(t)q(t)\overline{\Psi} \right] \]
\[
\begin{align*}
= d_2(t) & \left( \int J_2(y)k_2\overline{\Phi}(z - y,t)dy - k_2\overline{\Phi} \right) - \varepsilon k_2\overline{\Phi}_z - k_2\overline{\Phi}_t + \overline{\Phi}(1 - k_2\overline{\Phi}) [a_2(t)p(t) - b_2(t)q(t)k_2] \\
\leq k_2\overline{\Phi}(1 - \overline{\Phi}) & \left\{ \frac{d_2(t) \left( \int J_2(y)\overline{\Phi}(z - y,t)dy - \overline{\Phi} \right)}{k_2\overline{\Phi}(1 - \overline{\Phi})} - \varepsilon\mu_4(\overline{\tau}) - \frac{\varphi'_2(t)}{\varphi_2(t)} \right\} \\
\leq k_2\overline{\Phi}(1 - \overline{\Phi}) & \left\{ d_2(t) \int J_2(y)S(\mu_4(\overline{\tau}), z, y, t)(1 - \epsilon \mu_4(\overline{\tau})y)dy + \frac{d_2(t)s_0}{2\Phi(z(t), t)(1 - \Phi(z(t), t))} \\
& + d_1(t) \int J_1(y)(1 - \epsilon \mu_4(\overline{\tau})y)dy - 2\epsilon\mu_4(\overline{\tau}) + a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right\} \\
\leq k_2\overline{\Phi}(1 - \overline{\Phi}) & \left\{ d_2(t) \int J_2(y)S(\mu_4(\overline{\tau}), z, y, t)(1 - \epsilon \mu_4(\overline{\tau})y)dy + \frac{d_2(t)s_0}{2\Phi(z_1(t), t)(1 - \Phi(z_2(t), t))} \\
& + d_1(t) \int J_1(y)(1 - \epsilon \mu_4(\overline{\tau})y)dy - 2\epsilon\mu_4(\overline{\tau}) + a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right\} \\
\rightarrow k_2\overline{\Phi}(1 - \overline{\Phi}) & \left\{ F_2(\mu_4(0), s_0, t) - Y_1(\mu_4(0), t) + \frac{d_2(t)}{2(1 - \frac{1}{k_2})} + a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right\} \quad \text{as } \overline{\tau} \rightarrow 0^- \\
= \frac{k_2\overline{\Phi}(1 - \overline{\Phi})}{k_2 - 1} & \left\{ \left( \frac{d_2(t)}{2} + F_2(\mu_4(0), s_0, t) - Y_1(\mu_4(0), t) + a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right)(k_2 - \Theta_2(s_0, t)) \right\} < 0,
\end{align*}
\]
where \( F_i(\mu_4(\overline{\tau}), s_0, t), i = 1, 2, \) is defined in (4.26).

For \( z > z_2(t), \) we have \( \overline{\Psi} = 1. \) Then from the first equation in (1.9) it follows that

\[
\begin{align*}
& d_1(t) \left( \int J_1(y)\overline{\Phi}(z - y,t)dy - \overline{\Phi} \right) - \varepsilon\overline{\Phi}_z - \overline{\Phi}_t + \overline{\Phi} [a_1(t)p(t)(1 - \overline{\Phi}) - b_1(t)q(t)(1 - \overline{\Phi})] \\
= \overline{\Phi}(1 - \overline{\Phi}) & \left\{ \frac{d_1(t) \left( \int J_1(y)\overline{\Phi}(z - y,t)dy - \overline{\Phi} \right)}{\overline{\Phi}(1 - \overline{\Phi})} - \varepsilon\mu_4(\overline{\tau}) - \frac{\varphi'_2(t)}{\varphi_2(t)} + a_1(t)p(t) \right\} \\
\leq \overline{\Phi}(1 - \overline{\Phi}) & \left\{ d_1(t) \int J_1(y)S(\mu_4(\overline{\tau}), z, y, t) + 1)(1 - \epsilon \mu_4(\overline{\tau})y)dy + \frac{d_1(t)}{2(1 - \frac{1}{k_2})} \\
& - 2\epsilon\mu_4 + 2a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right\} \\
\leq \overline{\Phi}(1 - \overline{\Phi}) & \left\{ d_1(t) \int J_1(y)(S(\mu_4(\overline{\tau}), z_2(t), y, t) + 1)(1 - \epsilon \mu_4(\overline{\tau})y)dy + \frac{d_1(t)}{2(1 - \frac{1}{k_2})} \\
& - 2\epsilon\mu_4 + 2a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right\} \\
\rightarrow \overline{\Phi}(1 - \overline{\Phi}) & \left\{ F_1(\mu_4(0), s_0, t) + \frac{d_1(t)}{2(1 - \frac{1}{k_2})} + 2a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right\} \quad \text{as } \overline{\tau} \rightarrow 0^- \\
= \frac{\overline{\Phi}(1 - \overline{\Phi})}{k_2 - 1} & \left\{ - \left( \frac{d_1(t)}{2} + F_1(\mu_4(0), s_0, t) + 2a_1(t)p(t) - \frac{b_1(t)q(t)}{\rho_2(t)} \right)(\Theta_1(s_0, t) - k_2) \right\} < 0
\end{align*}
\]
and the second equation in (1.9) becomes

\[
d_2(t) \left( \int J_2(y) \bar{\Psi}(z-y,t)dy \right) - c \bar{\Psi}_z - \bar{\Psi}_t + (1 - \bar{\Psi}) \left[ a_2(t)p(t)\bar{\Psi} - b_2(t)q(t)\bar{\Psi} \right] = 0.
\]

Hence \((\bar{\Psi}, \bar{\Psi})\) is an upper solution of (1.9) as \(c\) is sufficiently close to 0. The proof is complete. \(\square\)

The two theorems in this section imply that the more competitive species will win in the competition. To be specific, when the competition coefficient \(a_2(t)\) of species \(u\) becomes larger, the condition (4.19) in the Theorem 4.1 is more easily satisfied, so that the bistable wave speed is positive, which means that species \(u\) will win the competition. Similarly, the larger the competition coefficient \(b_1(t)\) of the species \(v\) is, the easier the condition (4.27) in Theorem 4.2 is to meet, such that the bistable wave speed is negative, i.e., the species \(v\) will win.

### 5. Examples and simulations

In this section we present two examples to demonstrate the results of Theorems 4.1 and 4.2 when the condition (1.2) is satisfied, but the condition (1.6) is not.

Theorem 4.1 indicates that the condition (4.19) can guarantee that the bistable wave speed is positive, that is, the bistable traveling wave connecting \((0, q(t))\) to \((p(t), 0)\) propagates to the left, which means the stable state \((p(t), 0)\) wins the competition, and thus the species \(u\) will tend to the periodic state \(p(t)\) and the species \(v\) will tend to be extinct as time increases.

Theorem 4.2 shows that if there exists a constant \(s_0\) such that (4.27) are satisfied, the bistable wave speed is negative. Therefore, with the increase of the time, the species \(u\) will become extinct and the species \(v\) will approach the periodic state \(q(t)\).

In the two examples, the kernel functions \(J_i, i = 1, 2\) are taken as

\[
J_1(y) = J_2(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}, \quad -\infty < y < +\infty.
\]

It is easy to verify that \(J_i, i = 1, 2\) satisfy (A1)–(A3). The simulation is to directly integrate the full system (1.1) with the initial data

\[
u(x, 0) = \frac{p_0}{1 + e^{-x}}, \quad v(x, 0) = \frac{q_0}{1 + e^x},
\]

where \(p_0\) and \(q_0\) are defined in (1.3).

In Example 1, the coefficient functions are taken as

\[
d_1(t) = 10, \quad r_1(t) = 3.5, \quad a_1(t) = 3 \sin(2t) + 5, \quad b_1(t) = 3 \sin(2t) + 10;
\]

\[
d_2(t) = 15, \quad r_2(t) = 3, \quad a_2(t) = 3 \cos(2t) + 15, \quad b_2(t) = 3 \cos(2t) + 8.
\]

Then it is easy to check that (1.2)(not (1.6)) and the condition (4.19) in Theorem 4.1 are satisfied. The propagation behavior of the \(\pi\)-periodic bistable traveling wave is displayed in Fig. 1.

In Example 2, the coefficient functions are chosen as

\[
d_1(t) = 0.8, \quad r_1(t) = 1.5, \quad a_1(t) = 0.1 \sin(2t) + 0.40, \quad b_1(t) = 0.3 \sin(2t) + 0.8;
\]

\[
d_2(t) = 0.2, \quad r_2(t) = 1.8, \quad a_2(t) = 0.3 \cos(2t) + 0.78, \quad b_2(t) = 0.1 \cos(2t) + 0.2.
\]

If we take \(s_0 = 0.2\), then we can verify that (1.2) (not (1.6)), and the condition (4.27) in Theorem 4.2 holds. The dynamical behavior of the \(\pi\)-periodic bistable traveling wave is displayed in Fig. 2.
6. Conclusion and discussion

In this work, we have studied the Lotka–Volterra type of competition model with nonlocal dispersal and time periodicity. By applying the theory of monotone dynamical systems, we prove the existence and monotonicity of the bistable $T$-periodic traveling wave solution. The uniqueness, Lyapunov stability, the value range of the wave speed and the general conditions for sign determinacy have been established mainly by means of the comparison principle (the upper and lower solution method). Based on these generic results and the characteristics of the bistable waves, we derive explicit conditions for the speed sign, i.e., Theorems 4.1 and 4.2 guarantee the positive and negative wave speeds, respectively. Moreover, numerical simulations demonstrate our theoretical results even only under the bistable condition (1.2) weaker than (1.6), which reveal the effects of dispersal rate, competition strength, growth rate, seasonality and carrying capacity on the propagation direction of the bistable traveling wave. It should be pointed out that the monotonicity of the wave speed in terms of the function $b_1(t)$ and $a_2(t)$ can be easily shown by way of comparison principle. However, it is challenging to give a complete classification of the speed sign in terms of all parameters. As such, we are particularly interested in obtaining analytic and easy-to-apply formulas for determining the speed sign. Our explicit results are derived by
constructing upper/lower solutions with the asymptotical behavior (4.6) which can be seen as case studies, shedding light on further studies and improvement. We expect that different explicit conditions could be obtained by finding different formulas of upper/lower solutions with the asymptotical behaviors similar to (4.8), (4.14) and (4.16), respectively. The exponential stability of the bistable traveling wave of the system (1.1) has been presented in [26]. In addition, the condition (1.6) is required only for the existence of traveling waves, while the weaker condition (1.2) (i.e., the bistable condition) is sufficient for other results. Hence we presume that the existence result developed in [8] (i.e., Lemma 2.1) could be improved.

Acknowledgements

The authors would like to thank the anonymous referees’ valuable comments which greatly improve the exposition of the paper. We would also thank Ms. Jiajun Yue for the initial discussion on this problem. The work of Manjun Ma and Wentao Meng was supported by the National Natural Science Foundation of China (No. 12071434). The work of Chunhua Ou was supported by the NSERC discovery grants(RGPIN-2016-04709 and RGPIN-2022-03842).

Author contributions CO came up with the proposed problem. All three wrote the main draft and all three reviewed the manuscript and revised it again and again.

Funding The work of Manjun Ma and Wentao Meng was supported by the National Natural Science Foundation of China (No. 12071434). The work of Chunhua Ou was supported by the NSERC discovery grants(RGPIN-2016-04709 and RGPIN-2022-03842).

Data availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Manjun Ma and Wentao Meng
Department of Mathematics, School of Science
Zhejiang Sci-Tech University
Hangzhou 310018 Zhejiang
China
e-mail: mjunm9@zstu.edu.cn

Wentao Meng
e-mail: wentaom@mun.ca

Chunhua Ou
Department of Mathematics and Statistics
Memorial University of Newfoundland
St. John’s NL A1C 5S7
Canada
e-mail: ou@mun.ca

(Received: April 13, 2023; revised: August 7, 2023; accepted: September 25, 2023)