On a Local Concept of Wave Velocities

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September 14, 2018

Abstract

The classical far field concept of wave velocities has its merits while exhibiting intrinsic difficulties.

A general local approach for the definition of velocities and especially phase velocities for waves avoiding these difficulties is proposed. It includes the classical definitions as particular cases and can be applied to waves of an arbitrary structure, and to arbitrary propagation media as well. Applications of the formalism are elucidated and some basic properties of the local concept defined here are discussed.

1 Some remarks on wave velocities

Waves are conventionally described via propagating harmonic functions containing a periodic factor of the form $e^{i\varphi}$. The argument of these periodic functions, interpreted as the "phase", is usually identified with $(t - x/c)$ via the free wave equation, where $c$ is the "phase velocity" [1].

This canonical approach originates from classical wave optics and analyzes ordinary light waves propagating in the "far field", where the wave is far away from the source. Such a light wave is a solution of a source free wave equation described by the periodic function mentioned above. The Ansatz earned his own merits in the context of several wave phenomena and in particular classical wave optics [2].

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A second fundamental concept of characterizing wave propagation is that of the "group velocity". This concept is - mathematically speaking - a comprehensive definition for a specified (linear) superposition of solutions of the free wave equation with the same periodicity properties usually expressed by the frequency distribution of the constituent periodic waves. This definition is also an inherent far field concept considering "source-free" waves propagating in a strongly homogeneous and isotropic medium. This medium is characterised only by its "dispersion" (supposed to satisfy additionally the Kramers-Kronig relations), i.e. by a dependence of the complex index of refraction \( n(\omega) \) on the periodicity parameter of the wave - the frequency \( \omega \).

This basis for definitions of "phase" and "group" velocities turns out to be appropriate only for the special class of wave propagation mentioned above. The notions of a signal and its velocity developed in this context did accommodate many experimental data (see e.g.\[6\]). It is however by no means obvious, to what extent these far field concepts can be applied to near field problems. This leads to the question, if this classical approach is flexible enough to cover more general wave phenomena apart from the "special cases" mentioned above. A possible answer to this question is the focus of the present work.

Before proceeding we list some actual problems accompanying the classical definitions and suggesting a resolution by a local definition of a propagation velocity \[12\] as proposed here.

Theories developed for general not necessary periodic electromagnetic pulses \[3, 10\], did draw basically on the adopted "canonical" scenario. The analysis for a wave of an arbitrary form, for instance, has been based in general (by analogy to the classical case) on a representation via periodic functions by means of Fourier analysis. A well defined "phase velocity" is assigned to each Fourier component equipped with its frequency; a corresponding "group velocity" has subsequently been defined for the complete Fourier superposition (i.e. for the "wave packet"). As a consequence, the classical wave theory (in all its facets) is based on apparently "canonical" definitions relying on the phase of periodical functions and of groups of such functions. Such a strategy treats this as the inherent kernel of wave phenomena.

A close inspection, however, reveals mathematical deficiencies and several shortcomings with respect to physics thereby rendering the classical "canonical" approach to some extent non-natural as elucidated now.

First of all, a propagating wave is per definition a local space-time distribution which is a solution of a local differential equation. Hence the definition of velocity as a space-time relation should respect this frame and can be expected to be local as well.

The canonical definitions of a velocity ignore this fact and involve a frequency and a wave vector, which are nonlocal parameters in following the space- and time-periodicity, assumed separately in each case. Moreover, these parameters have nothing to do with the propagation process itself, since they do not enter
in the wave equation. Basically, the wave equation admits a propagation of arbitrary shapes without presumption of periodicity. In other words, the canonical approach bears anyway on a representation of an arbitrary solution by a set of periodic harmonic functions, each of that does not obey, generally speaking, the original wave equation (i.e. is not a solution). An attempt to apply it on waves resulting from non-linear equations such as solitons for instance, demonstrates this problem in a clear way since in this case even a linear superposition of solutions is not anymore a solution!

Second, the subject of transmission and velocity of a signal is based on sufficiently local procedures of measurement [4]. Roughly expressed, the intervals between several space-time points are measured, where the attribute under study is detected. Definitions based on periodicity parameters can possess generically neither time- nor space-locality. Moreover, any Fourier transformation is basically a global object, and all manipulations concerned are in general mathematically exact only with integration over the whole space and whole time, as well as over an infinite frequency band for backward transformations.

The "canonical" definitions of the "phase" and "group" velocity mentioned above [1], are based on the special case of a propagation of periodic harmonic waves in homogeneous media. Thus there is already an essential contradiction between the local character of wave propagation and the inherent globality contained in the definitions of wave velocities.

Any attempt to construct a local measurable object using Fourier sums or integrals contain an essential contradiction as outlined above and provides indeed no real locality. This is the source of several problems arising when replacing originally local features by "microglobal" ones [3 7 10].

The validity of this approach has to be checked from the mathematical point of view in the sense of functional analysis as well as from physical consistence and in fact it turns out to be correct only in special cases as mentioned above.

Even if the approach is supported by mathematical consistency (like the assumption of an infinite frequency band [1]), it does not lend itself easily to a transparent physical interpretation an the calculations are cumbersome.

These drawbacks, are for instance, the origin of all problems involved in the theory of signal transmission in media and an interpretation of the results.

The classical papers since [1] and later improvements thereof [10, 14], still contain an essential mismatch between local and global wave features, based on several misleading definitions (like signal velocity, energy velocity and group velocity [5 17]). This ansatz is bound to lead in applications to controversial results. For example, it seems not more to be surprising, that the subject of "group velocity" failes to describe a propagaton of ultra-short pulses [8].

These remarks shall suffice to motivate some alternative concept of wave propaga-tion and wave velocities for the following reasons:
1. The canonical ”harmonic” approach represents local wave features in terms of generically non-local attributes [12], [13]. Hence:

2. The mathematical equipment is not exact for finite physical values.

3. It is therefore difficult to impossible to apply it to for near-field effects, if the size of space-time regions are of an order of magnitude smaller than the wave periodicity parameter;

4. It is poorly suited for inhomogeneous and anisotropic media, as well as for ultra-short pulses of arbitrary form and for nonlinear waves [7], [15].

5. As a consequence, the applicability of this approach for several fields of modern quantum optics, nanooptics and photonics is barely justified, since these topics deal with the parameter areas outside its range of validity [9].

Moreover there is a clear reason to avoid this ”harmonic” approach from the pure technical point of view: on the one hand it turns out to be an essential restriction of generality and universality of the theory; on the other hand it brings to play (especially the representation via Fourier series and integrals) a number of problematical artifacts such as an apparent violation of causality, infinite frequency bands for signals and so forth. These problems enforce further theoretical constructs like the analysis of dispersion relations [11] to resolve these artificial problems.

A first step towards a more general concept should be based on an independent alternative approach without doubting the canonical ”harmonic” criteria, leading to the same (verified) results in known special cases.

The aim of the present paper is to establish a transparent criterion for evaluation of the propagation velocity for a quite arbitrary signal, that does not need an explicit representation in terms of harmonic or exponential functions at all, and with minimal loss of generality in other respects. First of all, we have to recall, what is being measured in experiments and what is meant when speaking about a ”wave velocity”, thereby providing the spectrum of wave velocities. This discussion is presented in Sec.2. Sec.3 is devoted to a test of the definition using some simple well known examples. Some remarkable properties of the behaviour of the wave velocities defined here under relativistic transformation are outlined in Sec.4. The discussion is concluded in Sec.5 by a practical interpretation and conversion from a local to a global evaluation of signal velocities.

2 N-th order phase velocity (PV)

The following discussion is restricted to a 1+1 dimensional space-time for simplicity. To evaluate the speed of a moving matter point one has to check the
change of the space coordinate $\Delta x$ during the time interval $\Delta t$. How can this ansatz be applied to waves?

An arbitrary wave is a function defined on the 2-dimensional manifold $(x, t)$ and one has no *a priori* defined fixed points to follow up as in the former case. Let us therefore analyze a continuous smooth function $\psi(x, t)$ of arbitrary shape.

In a first step towards a local definition of velocity we consider a vicinity $U(x)$ of a certain point $x$ at the certain time $t$. Further we assume the local information about the function $\psi(x, t)$ to be measurable, i.e. we should be able (at least in principle) to evaluate the values of the function $\psi(x, t)$ itself and all its derivatives in the point $x$ as well, and in the vicinity $U(x)$.

A description of propagation is based on monitoring the fate of a certain attribute (“labelled point”) of this shape, i.e. one finds the same attribute at the next moment $t_1$ in the next point $x_1$.

![Diagram of wave propagation](image)

Figure 1: The propagation of a signal $\psi(x, t)$ without a change of shape with the value $\psi_0$ of the amplitude as a traced attribute.

Let this attribute be labelled by some one-point fixed value $\psi_0$ of the function $\psi(x, t)$. Suppose, one follows up this local ”attribute” and manages a local observation of the condition:

$$\psi(x, t) - \psi_0 = 0,$$

which fixes the space-time points $\{x, t\}$ where this condition holds (Fig.1). Thus we have a function $x(t)$ given in an implicit form (2.1).
The first order implicit derivation provides the velocity of this one-point local attribute:

\[ v_{(0)}(x, t) := \frac{dx}{dt} = -\frac{\partial \psi}{\partial x}, \tag{2.2} \]

called from here on the "zero-order" or "one-point" phase velocity (0-PV). It describes in the simplest case the speed of translation of some arbitrary pulse, that can be treated as the signal propagation velocity, provided that the measured value \( \psi_0 \) of the amplitude is the signal considered.

To proceed with the local description of the shape propagation we now consider as the measurable attribute of \( \psi(x, t) \) at some time \( t_0 \) not the single value \( \psi_0 = \psi(x_0; t_0) \) at the point \( x_0 \), but a set of values \( \psi \) in a local neighborhood (like \( U(x) \)) close to this point. Then another local attribute can be constructed to trace their propagation. If, for instance, one looks at a certain value of the first derivative \( \frac{\partial \psi(x; t_0)}{\partial x} := \chi_0 \) of the shape \( \psi \), (like \( \frac{\partial \psi(x; t_0)}{\partial x} = 0 \), i.e. at a maximum or minimum point), one should consider the propagation of the condition:

\[ \frac{\partial \psi(x, t)}{\partial x} = \chi_0; \tag{2.3} \]

this provides the propagation velocity via

\[ v_{(1)}(x, t) := -\frac{\partial^2 \psi}{\partial x^2}, \tag{2.4} \]

called in view of (2.2) the "first order" or "two-point" phase velocity (1-PV) respectively.

When tracking the propagation of a maximum (minimum), the conditions \( \frac{\partial \psi(x, t)}{\partial x} = 0 \) and \( \frac{\partial^2 \psi}{\partial x^2} < 0 \) have to hold simultaneously.

This Ansatz is easily iterated to phase velocities of order \( N \) interpreted as the propagation velocities of higher order local shape attributes via

\[ v_{(N)}(x, t) := -\frac{\partial^{N+1} \psi}{\partial x^{N+1}}, \tag{2.5} \]

leading to \( N \)-th order ("N+1-point") PV, describing the propagation of higher order local shape attributes.

The phase velocities so defined are obviously local features depending on space-time coordinates. It has to be noted, that any given problem at hand might require a particular choice of a PV allowed for by the definitions given above. The following examples elucidate these requirements and demonstrate the flexibility of this definitions.

Before proceeding to it should be recalled, that the PV-spectrum has been obtained wanting to describe the propagation of an arbitrary pulse in terms of
local attributes in a medium whose properties are dependent on several variables, especially time and space coordinates being the most prominent and natural examples thereof. Any initial shape \( \psi(x_0, t_0) \) thus should be deformed during the propagation (or evolution, as typically encountered for dispersive media). The ordinary phase velocity \( v_0 \) therefore is not a relevant criterion to characterize the shape propagation and one has to choose an appropriate PV \( v(N) \).

For example, let the pulse \( \psi(x, t_0) \) be subject to damping (Fig.2). As a consequence, the zeroth order PV measured in the point \( x_1 \) gives a magnitude much smaller as the same magnitude measured in the point \( x_2 \).

For an amplified signal (Fig.3, like a signal propagating in a laser excited medium), by comparison, the zero order PV from the point \( x_1 \) provides altogether even a backward propagation. In both cases an appropriate approach would be to apply the first order PV \( v(I) \) which describes the propagation of the maximum up from the point \( x_0 \) properly.

![Figure 2: A damping deformation of a signal \( \psi(x, t) \). A relevant propagation attribute is the peak location (maximum).](image)

For a propagation of a kink front that experiences a deformation, one can check the translation of the second derivative of the shape, keeping track of the turning-point of the kink shape (Fig.4). In this case the second order (“three-point”) PV turns out to be the relevant velocity of propagation.

Finally, it should be noted, that the definition of phase velocities of order zero and one, the \( v(0) \) and \( v(I) \) respectively, admits a straightforward generalization to two-, three-, and higher-dimensional propagation, while the phase velocities of
second \( \psi_{(II)} \) and higher orders inherently contain a certain element of ambiguity in definition, since a possibility to choose a second-order traced attribute is not unique \cite{16}.

3 Examples

Let us consider the conventional 1+1-dimensional wave equation

\[
\left[ \frac{1}{a^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right] \psi(x, t) = 0 \tag{3.1}
\]

possessing translational solutions of the form

\[
\psi(x, t) = \psi(t \pm \frac{x}{a}). \tag{3.2}
\]

It is easy to check that in this case the PV's of all orders defined above are identical and read

\[
v_{(N)}(x, t) = a, \quad N = 0, 1, 2, ...
\]

which is nothing else but the classical phase velocity \( \pm a \), thereby satisfying the \textit{a priori} definition of "phase velocity" itself as a medium constant in (3.1).

Let a propagating shape \( \Psi \) now be subject to a temporal damping similar to (Fig.2) with

\[
\Psi(x, t) = \psi(t - \frac{x}{a})e^{-\lambda t} \equiv \psi(\phi)e^{-\lambda t} \tag{3.4}
\]
Figure 4: The propagation of a growing kink ("tsunami model"). The turning point is chosen to trace the propagation

The ordinary 0-PV velocity reads

\[ v_{(0)} = a \left( 1 - \lambda \frac{\psi}{\psi'} \right), \quad (3.5) \]

where the prime denotes the derivative of \( \psi \) with respect to its argument \( \phi \equiv t - x/a \).

This result provides a velocity with bad physical features: the velocity grows for a descending shape, for an ascending shape it decreases, can even be negative, and it diverges exactly for the peak point (under the condition that the (measured) amplitude \( \Psi \) as well as parameter \( a, \lambda \) have positive values).

A relevant physical velocity in this case is for instance the 1-PV

\[ v_{(1)} = a \left( 1 - \lambda \frac{\psi'}{\psi''} \right) \quad (3.6) \]

providing for a peak being traced exactly the canonical phase velocity.

For this shape further PV’s of higher orders are given by (2.5):

\[ v_{(N)} = a \left( 1 - \lambda \frac{\psi^{(n)}}{\psi^{(n+1)}} \right) \equiv a \left( 1 - \lambda (\log' \psi^{(n)})^{-1} \right) \quad (3.7) \]

where \( \psi^{(n)} \) denotes the \( n \)-th derivative of \( \psi \) with respect to its argument as mentioned above. For an amplified signal as in Fig 3 we can e.g. change the sign of \( \lambda \). Especially, for the case of a kink (Fig. 4)

\[ \psi \left( t - \frac{x}{a} \right) \equiv \psi(\phi) = \arctan \phi, \quad (3.8) \]

the spectrum of phase velocities reads by comparison:
\[ v(0) = a \left( 1 + \lambda (1 + \phi^2) \arctan \phi \right), \]
\[ v(I) = a \left( 1 - \frac{\lambda + \phi^2}{2\phi} \right), \]
\[ v(II) = a \left( 1 - \frac{\lambda \phi^3 + \phi}{3\phi^2 - 1} \right). \quad (3.9) \]

The ordinary 0-PV has an oscillating sign at \( \lambda \) and is multiple defined because of the \( \arctan \) periodic function. Therefore it cannot be interpreted as a "well-defined" physical velocity. If the point being traced should be "labelled" by a derivative attribute, it turns out to be a physically inconvenient choice since the shape possesses no real peaks, their propagation could be traced. Moreover the velocity \( v(I) \) diverges at \( \phi = 0 \).

The possible "labelled" attribute is also the turning-point traced by the 2-PV. The velocity \( v(II) \) also possesses two singularities at \( \phi = \pm 1/\sqrt{3} \) which do not coincide with the labelled point \( \phi = 0 \), so it can be successfully followed up at the measurement.

**Historical remark**

Canonically the definition of velocity should have proceeded with an originally artificial extension of a translational solution of (3.1):

\[ \psi(x, t) = \psi(t - \frac{x}{a}) \equiv \psi\left[ \frac{1}{\omega}(\omega t - \frac{\omega}{a} x) \right], \equiv \phi(\omega t - kx), \ k \equiv \frac{\omega}{a} \quad (3.10) \]

and the parameter \( \omega \) is further understood in restricted sense as a "frequency" of a "necessary" periodic harmonic function \( \phi \), usually \( e^{\pm i x} \) as mentioned above in the introduction.

It is not surprising that the 0-order PV provides in this case the value \( \omega/k \), canonically interpreted as a "phase velocity of periodic wave".

In the case of any adopted interrelations between \( k \) and \( \omega \) that are not encountered in the wave equation, in particular any so called "dispersion relation" between frequency and wave-number, the PV \( v(0) \) is in fact a proportionality factor

\[ d\omega = v(0) dk, \quad (3.11) \]

which is identical with the classical ("canonical") definition of a "group velocity" [1].

At this point it should be recalled, that the original idea of the "group velocity" \( U \), as summarized e.g. in [18],

\[ \frac{\partial \lambda}{\partial t} + U \frac{\partial \lambda}{\partial x} = 0 \quad (3.12) \]
was of the similar form as the recent definition (2.2) of 0-PV; the equation (3.12) is however intrinsically controversial since being constructed of local derivatives of the non-local parameter $\lambda$ (wavelength). For a variable wavelength much smaller than a vicinity of the point, this ”group velocity” provides therefore a natural approximation to the zero-order phase velocity $v_{(0)}$. This was for a long time a reason to take it for a granted relevant physical concept.

In the present approach this feature appears as a physical phase velocity following in a straightforward way from the interpretation of phase propagation and does not require an interpretation of $\omega$ and $k$ as a ”frequency” and ”wave number”, as well as a constancy of some ”group” respectively.

Note, that in the case of kink there are no suitable definitions of a ”wave-group” and of a concerned ”group velocity” for this solution since a Fourier decomposition does not work on a non-compact support.

4 Lorentz covariance

The ordinary zero order PV (2.2) possesses a important property, i.e. a local covariance in the sense of special relativity, as shown below. This is not the case for PV’s of higher orders.

Actually the ordinary PV $v_{(0)}$ measured in some stationary system $X$ takes in some other system $X'$, moving with a constant speed $V$, via the Lorentz transformations

\[ x' = \frac{x - Vt}{\sqrt{1 - \frac{V^2}{c^2}}}, \quad t' = \frac{t - \frac{Vx}{c^2}}{\sqrt{1 - \frac{V^2}{c^2}}} \]  

the form

\[ v'_{(0)} = \frac{v_{(0)} + V}{1 + \frac{v_{(0)}V}{c^2}}. \]  

This means that the zero order PV respects the relativistic velocity addition. Especially, a subluminal zero order PV remains also subluminal in any other moving system $X'$.

This result should not be a surprise, since the definition of the $v_{(0)}$ is implied by the condition:

\[ \psi(x, t) = \text{const}, \]  

which remains to be of the same form under arbitrary transormations $x = x(x', t'); t = t(x', t')$, implying

\[ d\psi(x, t) \equiv \frac{\partial \psi}{\partial t} dt + \frac{\partial \psi}{\partial x} dx = 0 \]  

for the first order differential form (or simply first differential), which possesses a form-invariance property under transformations.
Since
\[ v_{(0)}(x, t) = \frac{dx}{dt} \] (4.5)
per definition, the equation (4.4) implies the definition (2.2) of 0-PV, so it should behave under space-time transformations as a usual velocity of a matter point.

The first order PV (2.4), by comparison, evaluated in some system \( X \) transforms in the moving system \( X' \) to:
\[
v'_{(I)} = -\left\{ \left(1 + \frac{V^2}{c^2}\right) \frac{\partial^2}{\partial x \partial t} - V \left( \frac{1}{c^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial t^2} \right) \right\} \psi.
\] (4.6)

If the pulse \( \psi(x, t) \) obeys the free wave equation (3.1), the transformation becomes
\[
v'_{(I)} = -\left(1 + \frac{V^2}{c^2}\right) v_{(I)} + 2V \] (4.7)
i.e. a relation that should be called the ”first order velocity addition”. It differs obviously from the corresponding transformation of \( v_{(0)} \), since the definition of the velocity \( v_{(I)} \) results from the condition
\[ d \left( \frac{\partial \psi(x, t)}{\partial x} \right) = 0 \] (4.8)
whose form is explicitely non-invariant under space-time transformations.

It can be shown that a subluminality of the first order PV is nevertheless still preserved by this transformation as well. Note, that for a signal which does not obey the free equation (3.1), this restriction is in general not guaranteed anymore.

5 A global velocity of signal transmission and ”dynamic separation”

The discussion of local velocities was aimed towards an evaluation of global features of signal propagation, namely the propagation through a finite spatial interval during a finite temporal interval. In other words, a global velocity, practically measured, means roughly the length of the distance \( \Delta x \) traveled by a traced attribute divided by the time interval \( \Delta t \).

The local PV’s analyzed above can be interpreted as first order differential equations of the form
\[ v_{(N)}(x, t) = \frac{dx}{dt}, \] (5.1)
Figure 5: Phase velocities as a family of isoclines \( v(N)(x,t) \) and averaged global velocities between two events (measurements) that can be illustrated graphically as a field of isoclines (Fig.5).

Here the local PV is the tangent function of the tangent vector of the isocline, and the averaged (total) velocity between \((t_0, x_0)\) and \((t_1, x_1)\) is represented by the tangent of the hypotenuse of the triangle \( \{ (t_0, x_0), (t_1, x_1), (t_0, x_1) \} \), see Fig.5.

This procedure is elucidated best by some clear and well known examples: Consider the propagation of a translation mode of the form

\[
\psi(x,t) = \psi(\xi t - k(x)x) \tag{5.2}
\]

describing, for instance, an electromagnetic wave propagation in an inhomogeneous dielectric medium. The \( k(x) \) depends now on the space coordinate, resulting from the optical inhomogeneity:

\[
k(x) = \xi \frac{n(x)}{c}, \tag{5.3}
\]

where \( n(x) \) is the spatially dependent index of refraction. Then the local zero order PV is provided by Eq. (2.2) and results in the first order equation

\[
v(0)(x,t) \equiv \frac{dx}{dt} = \frac{c}{n'x + n}, \quad n' \equiv \frac{\partial n(x)}{\partial x} \tag{5.4}
\]

An explicit integration of the (5.4) leads to

\[
c(t_1 - t_0) = x_1 n(x_1) - x_0 n(x_0) \tag{5.5}
\]
for the interval of two events, where the signal attribute chosen for tracing has been checked at the time $t_0$ in the point $x_0$ and afterwards in $x_1$ at $t_1$. Assumed, the $(t_0 = 0, x_0 = 0)$ is not a critical point (i.e. of the knot type) of the equation (5.1). Then $t_1 = \Delta t$, $x_1 = \Delta x$ and the global transition velocity between two points $x_0, x_1$ is

$$v_{(0)}(\Delta x) = \frac{c}{n(\Delta x)}$$ (5.6)

For the 1-PV velocity $v_{(I)}$ of eq.(2.4) the same procedure leads to

$$\frac{c}{v_{(I)}(x,t)} = \frac{c}{\xi} \log'(\{(xn(x))'\} - (xn(x))').$$ (5.7)

This, equation straightforward to integrate, provides for the averaged velocity of two-point signal attribute along the distance $\Delta x$:

$$\frac{c}{v_{(I)}} = n(\Delta x) - \frac{c}{\xi \Delta x} \log(n'(\Delta x)\Delta x + n(\Delta x))$$ (5.8)

It is worth to notice, that for the first order PV in media with a variable refraction index $n(x)$ an essential dependence on the parameter $\xi$ enters.

The meaning of $\xi$ can be derived from the given form of solution $\psi$. It can be e.g. interpreted as a frequency factor for a periodic mode or a damping factor for evanescent one etc. This parameter simply means an enumeration of solutions of a solution family (space of solutions). The merit gained above is the separation of these solutions on the parameter $\xi$ through the different first (and higher)-order PV through the inhomogeneity of media.

In the case of a periodic wave for instance, it can be interpreted as a ”dispersion” although an explicit frequency-dependence of $n(x)$ was not assumed. Moreover, the parameter $\xi$ has not necessarily to be interpreted as a ”frequency” of some time periodic oscillation, but rather as a time component of the time-spatial wave vector $\{\omega, k\}$ for a translation mode (3.2) in a general form, thus the considered phenomenon has another origin and is much more general as a conventional non-localized frequency dispersion $n(\omega)$

Thus we established for the first (and higher) order PV in optical inhomogeneous media the essential dependence on the t-component $\omega$ of the wave translation vector, especially frequency, even for local non dispersive media, which could be called ”dynamic separation”. The global averaged dynamic separation of eq. (5.8) survives as a corollary of the local separation of (5.7). This phenomena does not occur for the ordinary zero order PV.

### 6 Concluding remarks

The inherent inconsistency of the classical subjects of ”phase velocity” and ”group velocity”, as well as ”signal velocity” based therein has been discussed. The
inapplicability of these concepts for actual studies in photonics, near-field and nano-optics has been shown to result from the essential non-locality of these terms.

An alternative approach for description of a propagation velocity has been proposed. It is strictly local and is based on the natural assumption of an ordinary measurement procedure. It does not need any \textit{a priori} condition of periodicity, frequency, groups and packets or other canonical attributes.

The definition elaborated is applicable in a natural way for arbitrary. In a mathematical sense it describes a propagation of perturbation in any geometrical field. Examples could be: an acoustic wave as a pressure perturbation, a gravitation wave on a fluid surface, a spin wave in a solid state, etc. In particular the formalism is very suited for the description of particle propagation in field theory, where particles are considered as field perturbations.

This approach results in the series of measurable propagation velocities. In zeroth order the propagation velocity coincides with the ordinary phase velocity and appears to be ordinary Lorentz covariant; further application gives rise to generate "Lorentz covariance of higher orders".

For propagation in inhomogeneous media, for instance for light in a medium with a space-dependent index of refraction phase velocities of higher orders exhibit an essential dependence on time component of the wave vector solely as a result of inhomogenity, treated canonically as a "dispersion". This appears to be a more general phenomena, and does not presuppose any periodic frequency and dispersive properties of media. It should be called for this reason "dynamic separation".

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