Representations for the restricted Lie color algebras

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1 Introduction

The main goal of the present paper is to develop the nonrestricted representation theory for the restricted Lie color algebras. The restricted Lie color algebras was defined in [1, 2, 3, 6]. Progresses have been made concerning both the structure theory and the representation theory of the restricted Lie color algebras in the literature (see [1, 2, 3, 18, 6]). But the study on the representation theory so far has been solely about the restricted case.

Assume $F$ is an algebraically closed field with char.$F = p > 3$. Let $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a restricted Lie color algebra over $F$, where $\Gamma$ is an abelian group with a bicharacter $(, )$. Unless indicated, we allow $\Gamma$ to be infinite throughout the paper, so we consider both the case $\mathfrak{g}$ is infinite dimensional and the case $\mathfrak{g}$ is finite dimensional. The aim of this paper is primarily about the nonrestricted representations of $\mathfrak{g}$.

By introducing the $p$-characters for locally finite simple $\mathfrak{g}$-modules, we work on both restricted and nonrestricted $\mathfrak{g}$-modules. From Sec. 3 on, we are particularly interested in the modules for a class of restricted Lie color algebras named algebraic Lie color algebras.

The paper is arranged as follows: In Sec. 2 we give the definition of the restricted Lie color algebras, and define the $p$-character $\chi$ for the simple locally finite $\mathfrak{g}$-modules. Then we study the $\chi$-reduced enveloping algebras $u_\chi(\mathfrak{g})$ for $\mathfrak{g}$, and determine the PBW-type of basis of $u_\chi(\mathfrak{g})$. Unlike the Lie algebra case, a $p$-character $\chi$ is not a linear function unless $\chi(\mathfrak{g}_\alpha) = 0$ for all $\alpha \in \Gamma$ with $p\alpha \neq 0$. 
In Sec. 3, we define the algebraic Lie color algebras. We introduce the FP triple, and determine the simplicity for the induced module associated with a FP triple.

In Sec. 4, we are concerned with the applications of the main theorems to the algebraic Lie color algebra $\mathfrak{g} = \text{cgl}(V)$. We then obtain an analogue of the Kac-Weisfeiler theorem. The second application of the main theorem is determining the simplicity of the baby Verma module $Z^\chi(\lambda)$ for $\mathfrak{g} = \text{cgl}(V)$. When $\mathfrak{g}$ is infinite dimensional, we employ the $\mathfrak{u}(\mathfrak{g}) - T$ method given in [8, 13] to work on the modules in the Category $\mathcal{O}$.

In the appendix, we define the infinite dimensional algebraic group $\text{GL}(\{m_i\}, \mathbb{F})$ and its Lie algebra. Since there are no literature accessible to the author on this subject, we give a brief introduction for the reader’s convenience. The conclusion in this section is needed in the definition of algebraic Lie color algebras, and independent of the other results in the paper.

2 Preliminaries

2.1 notions and definitions

Let $\mathbb{F}^\times$ denote the set $\mathbb{F} \setminus 0$, and let $\mathbb{N}$ denote the set of all nonnegative integers.

**Definition 2.1.** [1, 18, 6] Let $\Gamma$ be an abelian group. A bicharacter on $\Gamma$ is a mapping $(,): \Gamma \times \Gamma \to \mathbb{F}^\times$ such that

$$(\alpha, \beta)(\beta, \alpha) = 1, (\alpha, \beta + \gamma) = (\alpha, \beta)(\alpha, \gamma),$$

$$(\alpha + \beta, \gamma) = (\alpha, \gamma)(\beta, \gamma),$$

for all $\alpha, \beta, \gamma \in \Gamma$.

To avoid any confusion, we denote the zero element in $\Gamma$ by $0$.

A Lie color algebra $\mathfrak{g}$ is a $\Gamma$-graded $\mathbb{F}$-vector space $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha + \beta}$ for every $\alpha, \beta \in \Gamma$ and

$(1) \quad [y, x] = - (\beta, \alpha)[x, y],$

$(2) \quad (\gamma, \alpha)[x, [y, z]] + (\alpha, \beta)[y, [z, x]] + (\beta, \gamma)[z, [x, y]] = 0$

for every $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $z \in \mathfrak{g}_\gamma$ and $\alpha, \beta, \gamma \in \Gamma$. (1) is called color skew-symmetry and (2) is called the Jacobi color identity [18, 6].

Let $\Gamma$ be an abelian group with a bicharacter $(,)$. Let $A = \bigoplus_{\alpha \in \Gamma} A_\alpha$ be an associate $\Gamma$-graded $\mathbb{F}$-algebra. We define $[x, y] = xy - (\alpha, \beta)yx$, for any $x \in A_\alpha$, $y \in A_\beta$, then $A$ becomes a Lie color algebra. We denote this Lie color algebra usually by $A^-$. 

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Let $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a Lie color algebra. A color subalgebra of $\mathfrak{g}$ is a $\Gamma$-graded subspace $\mathfrak{g}_\alpha$ which is closed under the Lie bracket operation $[,]$. i.e., $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$. A color ideal of $\mathfrak{g}$ is a $\Gamma$-graded subspace $I$ of $\mathfrak{g}$ such that $[\mathfrak{g}, I] \subseteq I$. Note that the color symmetry implies that $[I, \mathfrak{g}] \subseteq I$. We can also define the the Lie color algebra $\mathfrak{g}/I$ by letting $[x + I, y + I] =: [x, y] + I$ for $x, y \in \mathfrak{g}$ [6, Sec. 3]. Similarly one defines the solvable and nilpotent Lie color algebras [1, 6]. The maximal nilpotent ideal of a Lie color algebra $\mathfrak{g}$ is called the nilradical of $\mathfrak{g}$. If it is zero, then we say that $\mathfrak{g}$ is reductive.

A morphism of Lie color algebras $f : \mathfrak{g} \to \mathfrak{h}$ is a $\mathbf{F}$-linear mapping satisfying:

1. $f(\mathfrak{g}_\alpha) \subseteq \mathfrak{h}_\alpha$ for any $\alpha \in \Gamma$.
2. $f([x, y]) = [f(x), f(y)]$ for any $x, y \in \mathfrak{g}$.

**Lemma 2.2.** [1, Lemma 1.8] Let $\mathfrak{g}$ be a Lie color algebra. We take $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and $z \in \mathfrak{g}_\gamma$ with $\alpha \in \Gamma^+$ and $\beta \in \Gamma^-$. If $2, 3 \in \mathbf{F}^\times$. Then

$$[x, x] = 0, [[y, y], y] = 0, [[y, y], z] = 2[y, [y, z]].$$

We assume that char.$\mathbf{F} = p > 3$ throughout the paper.

Let $\Gamma$ be an abelian group with bicharacter $(,)$. Let $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ be a $\Gamma$-graded vector space. We then define the general linear Lie color algebra $\text{gl}(V) = \bigoplus_{\alpha \in \Gamma} \text{gl}(V)_\alpha$, where

$$\text{gl}(V)_\alpha = \{f \in \text{gl}(V) | f(V_\gamma) \subseteq V_{\gamma + \alpha}, \text{for all } \gamma \in \Gamma\}.$$ 

Then the associative algebra $\text{gl}(V)$ becomes a Lie color algebra with Lie multiplication defined by $[f, g] = fg - (\alpha, \beta)gf$, for all $f \in \text{gl}(V)_\alpha$, $g \in \text{gl}(V)_\beta$.

**Definition 2.3.** Let $A = \bigoplus_{\alpha \in \Gamma} A_\alpha$ be a (not necessarily associative) $\Gamma$-graded algebra and let $d \in \text{gl}(A)_\delta$ satisfy $d(xy) = d(x)y + (\delta, \alpha)xd(y)$ for any $\alpha \in \Gamma$, $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}$, then $d$ is called a derivation of $A$ of degree $\delta$.

Let $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a Lie color algebra. By the Jocobi color identity one sees that $adx$ is a derivation for any homogeneous $x \in \mathfrak{g}$.

Assume $\Gamma$ is a countable set, and let $\dim V_\alpha = m_\alpha < \infty$, for each $\alpha \in \Gamma$. We define a subalgebra $\text{cgl}(V)$ of $\text{gl}(V)$ as

$$\text{cgl}(V) = \{f \in \text{gl}(V) | f(V_\alpha) = 0 \text{ for all but finitely many } \alpha\}.$$ 

Let $\Gamma = \{\alpha_i\}_{i \geq 1}$ be given the order that $\alpha_i < \alpha_j$ if $i < j$. Taking an ordered homogeneous basis $\{v_i\}_{i \geq 1}$ of $V$ such that $\alpha_i \leq \alpha_j$, whenever $v_i \in V_{\alpha_i}$, $v_j \in V_{\alpha_j}$. We define $e_{ij} \in \text{cgl}(V)$ by $e_{ij}(v_k) = \delta_{jk}v_i$. Then $\{e_{ij}| i, j \in \mathbf{Z}^+\}$ is a basis of $\text{cgl}(V)$, and we may identify $\text{cgl}(V)$ with the matrix algebra

$$\{(a_{ij})_{i,j \geq 1} | a_{ij} = 0 \text{ for all but finitely many } i, j\}.$$
If $\Gamma$ is finite, then $\text{cgl}(V) = \text{gl}(V)$. In this case we denote the finite dimensional general linear Lie color algebra $\text{gl}(V)$ also by $\text{gl}(m, \Gamma)$, where $m = \dim V = \sum_{\alpha \in \Gamma} \dim V_\alpha$.

Let $g = \oplus_{\alpha \in \Gamma} g_\alpha$ be a Lie color algebra. A $g$-module $M$ is a $\Gamma$-graded vector space $M = \oplus_{\gamma \in \Gamma} M_\gamma$ such that $g_\alpha \cdot M_\beta \subseteq M_{\alpha+\beta}$ for every $\alpha, \beta \in \Gamma$, and

$$[x, y] \cdot m = x \cdot (y \cdot m) - (\alpha, \beta) y \cdot (x \cdot m)$$

for every $m \in M$, $x \in g_\alpha$, $y \in g_\beta$. We say that the $g$-module $M$ is locally finite if there is a finite dimensional $\Gamma$-graded $x$-invariant subspace $0 \neq M_x \subseteq M$, for any homogeneous $x \in g$. We see from Lemma 2.2 that, $g$ itself is a locally finite $g$-module under the adjoint action and the assumption $p > 3$. A $g$-module $M$ is called simple if it has no nontrivial $\Gamma$-graded submodules.

Let $g$ be a Lie color algebra. A representation of $g$ is a Lie color algebra homomorphism $\rho : g \rightarrow \text{gl}(V)$ for some $\Gamma$-graded $F$-space $V = \oplus_{\alpha \in \Gamma} V_\alpha$. Each representation $\rho$ of $g$ defines a $g$-module structure on $V$ and vice versa. We say that the representation $\rho$ is afforded by the $g$-module $V$.

The bicharacter $(,) \colon \Gamma \times \Gamma \rightarrow \Gamma$ induces a partition of $\Gamma$

$$\Gamma = \Gamma_+ \cup \Gamma_-,$$

where $\Gamma_\pm = \{ \gamma \in \Gamma | (\gamma, \gamma) = \pm 1 \}$. We denote $g^+ = \sum_{\gamma \in \Gamma_+} g_\gamma$ and $g^- = \sum_{\gamma \in \Gamma_-} g_\gamma$. $g^+$ is referred as the even subalgebra of $g$.

A Lie color algebra is called restricted if for every $\gamma \in \Gamma^+$ there exists a map $(,)^{[\gamma]} : g^+_\gamma \rightarrow g^-_{\gamma}$, such that [6]

1. $(x + y)^{[\gamma]} = x^{[\gamma]} + y^{[\gamma]} + \sum_{i=1}^{p-1} s_i(x, y)$, where $s_i(x, y)$ is the coefficient of $t^{i-1}$ in the polynomial $(ad_{\gamma}(x + y))^{p-1}(x) \in g[t]$;
2. $(rx)^{[\gamma]} = r^{p}x^{[\gamma]}$;
3. $ad_{\gamma}x^{[\gamma]} = (ad_{\gamma}x)^{p}$ for every $x, y \in g_\alpha$ and every $r \in F$.

Let $A = \oplus_{\alpha \in \Gamma} A_\alpha$ be a $\Gamma$-graded associative algebra. Then $A^-$ becomes a restricted Lie color algebra if we define $x^{[\gamma]} = x^p$, the $p$-th power, for any $x \in g_\alpha$, $\alpha \in \Gamma^+$. So that $\text{gl}(V)$ is restricted and $\text{cgl}(V)$ is its restricted subalgebra.

The Lie color algebra $\text{cgl}(V)$ has a maximal torus $H = \{ e_i | i \geq 1 \} \subseteq g_0$. Let $\dim V_\alpha = m_\alpha$. Then $g_0$ is a restricted Lie algebra $\oplus_{\alpha \in \Gamma} \text{gl}(m_\alpha)$.

Let $g$ be a Lie color algebra. Consider the two-sided ideal $I(g)$ of the tensor algebra $T(g)$ of $g$ over $F$ generated by $x \otimes y - (\alpha, \beta)y \otimes x - [x, y]$ for $x \in g_\alpha$, $y \in g_\beta$, $\alpha, \beta \in \Gamma$. Then $U(g) = T(g)/I(g)$ is the universal enveloping algebra of $g[6]$.

Similar to the Lie algebra case, there is an equivalence between the category of $g$-modules and the category of unitary $\Gamma$-graded $U(g)$-modules[6, p.114].

Let $A = \oplus_{\alpha \in \Gamma} A_\alpha$ be an associative $\Gamma$-graded algebra, we define the center of $A$ by $Z(A) = \oplus_{\alpha \in \Gamma} Z(A)_\alpha$, where

$$Z(A)_\alpha = \{ a \in A_\alpha | ab = (\alpha|\beta)ba, \text{ for all } b \in A_\beta, \beta \in \Gamma \}.$$
Let $q \in \mathbb{F}^\times$ and $n \in \mathbb{N}$. We denote

$$[n]_q = \frac{1 - q^n}{1 - q} = \begin{cases} 1 + q + \cdots + q^{n-1}, & \text{if } q \neq 1 \\ n, & \text{if } q = 1. \end{cases}$$

Recall the quantum binomial coefficient $Q_n^i =: \frac{[n]_q!}{[i]_q! [n-i]_q!}$. Let $a \in Z(A)_\alpha$ and $b \in A_\beta$. Then by induction, one gets the binomial formula

$$(a + b)^n = \sum_{i=0}^{n} Q_n^i a^i b^{n-i},$$

where $Q_n^i$ is defined with $q = (\beta, \alpha)$.

Let $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a Lie color algebra. By induction, one can show that, in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$,

$$(adx)^k y = \sum_{i=0}^{k} (-1)^i (\alpha|\beta)^i C_k^i x^{k-i} y x^i,$$

for all $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $\alpha, \beta \in \Gamma$. If $\mathfrak{g}$ is restricted, then we have

$$(x^p - x^{[p]}) y = (p\alpha|\beta) y (x^p - x^{[p]})$$

and hence $x^p - x^{[p]} \in Z(U(\mathfrak{g}))_{p\alpha}$, for all $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $\alpha \in \Gamma^+$, $\beta \in \Gamma$.

Let $\mathfrak{g}$ be a Lie color algebra and let $U(\mathfrak{g})$ be its universal enveloping algebra. For each $k \geq 0$, we define a subspace $U_{(k)}$ of $U(\mathfrak{g})$ as follows: If $k = 0$, we let $U_{(0)} = \mathbb{F}$. If $k \geq 1$, we let

$$U_{(k)} =: \langle \Pi_{j=1}^{l} x_j | x_j \in \mathfrak{g}_\alpha, \alpha \in \Gamma; l \geq 0, l \leq k \rangle.$$

Then $\{U_{(k)}\}_{k \in \mathbb{N}}$ is a filtration of $U(\mathfrak{g})$.

Let $I$ be a totally ordered set. We denote

$$N(I) = \{ f : I \to \mathbb{N} | f(i) = 0 \text{ for all but finitely many } i \in I \}.$$
where $z_i \in Z(U(\mathfrak{g})), v_l \in U(k(i) - 1)$ and $t_i \in F$. Then

$$B := \{ z^e s | r, s \in N(I), s(i) < k(i), \text{ for all } i \in I \}$$

is a basis of $U(\mathfrak{g})$.

**Proof.** We prove by induction that

$$B_n = \{ z^e s | \sum_{i \in I} (r(i)k(i) + s(i)) \leq n, s(i) < k(i) \}$$

is a basis of $U_{(n)}$ for any $n \in \mathbb{N}$. The case $n = 0$ is trivial. Assume the claim is true for the case $n - 1$ with $n \geq 1$. By the PBW theorem, $U_{(n)}$ is spanned by the set

$$P_n := \{ e_i^{s_1} e_i^{s_2} \cdots e_i^{s_k} | i_1 < \cdots < i_k, s_1 + \cdots + s_k \leq n \}.$$ 

For any $\Pi_i e_i^{l_i} \in P_n$ with $\sum_{i} l_i = n$, assume $l_i = q_i k(i) + r(i), 0 \leq r(i) < k(i)$, we get

$$\Pi_i e_i^{l_i} = \Pi_{j \in J, i \notin J} e_j^{l_j} \cdots e_i^{q_i k(i) + r(i)} \cdots \equiv \Pi_{j \in J, i \notin J} e_j^{l_j} \cdots (z_i + t_i e_i^{k(i)}) e_i^{r(i)} \cdots \mod U_{(n-1)}$$

(using the binomial formula given earlier)

$$\equiv \sum_{i \notin J, n_i \leq q_i} c(\Pi_{i \notin J} z_i^{n_i})(\cdots e_j^{l_j} + \sum_{j = \theta(i)} (q_i - \eta_k) k(i)) \cdots e_i^{r(i)} \cdots \mod U_{(n-1)}),$$

where for each $i \notin J$ and $j \in J$,

$$l_j + \sum_{j = \theta(i)} (q_i - \eta_k) k(i) = q_{ij} k(j) + r(j),$$

$0 \leq r(j) < k(j)$, and each coefficient $c$ is resulted from the applications of the binomial formula. It is easy to see that each term in the summation lies in $B_n$, so that $U_{(n)}$ is spanned by $B_n$.

To verify the linear independency of $B_n$, we note that

$$z_i^{r(i)} = \begin{cases} e_i^{k(i) r(i)}, & \text{if } i \in J \\ (e_i^{k(i)} - t_i e_i^{\theta(i)}) e_i^{r(i)} & \text{if } i \notin J \ (\mod U(k(i) r(i) - 1)). \end{cases}$$

For each $z^e s \in B_n$, we have

$$z^e s \equiv \Pi_{i \notin J, j \in J} e_j^{k(j) r(j) + s(j)} \cdots (e_i^{k(i)} - t_i e_i^{\theta(i)}) e_i^{s(i)} \cdots \mod U_{(n-1)}$$

$$\equiv \sum_{i \notin J, n_i \leq r(i)} c(\Pi_{j \in J} e_j^{k(j) r(j) + s(j)} + \sum_{j = \theta(i)} (q_{ij} r(i)) \cdots e_i^{k(i) r(i) - \eta(i) + s(i)} \cdots \mod U_{(n-1)}),$$

where each term in the summation is a basis element in $P_n$ multiplied by some $c \in F^x$. Note that the term with the minimal power for all $j \in J$ is $\Pi_{i \in I} e_i^{k(i) r(i) + s(i)}$ so by basic linear algebra we get the linear independency of $B_n$. 

\[\square\]
Definition 2.5. Let \( A = \oplus_{\alpha \in \Gamma} A_{\alpha} \) be an associative \( \Gamma \)-graded \( F \)-algebra. If \( ab = (\alpha, \beta)ba \) for any \( a \in g_{\alpha}, b \in g_{\beta} \) or, equivalently, \( A^{-} \) is abelian, then we say that \( A \) is a color commutative algebra.

Let \( g \) be a restricted Lie color algebra. Suppose \( g = g^{+} \) and \( (e_{i})_{i \in I} \) is a totally ordered homogeneous basis of \( g \). Let \( z_{i} = e_{i}^{p} - e_{i}^{p}[p] \in Z(U(g)) \). Then \( R = F[z_{i}, i \in I] \) is a color commutative polynomial ring with indeterminate’s \( z_{i} \).

2.2 \( p \)-characters for the simple modules

Let \( g = \oplus_{\alpha \in \Gamma} g_{\alpha} \) be a restricted Lie color algebra. We say that \( g \) is \( p \)-finite, if the abelian subalgebra \( \langle x \rangle_{p} =: \langle x^{[p]}, x^{[p]^2}, \cdots \rangle \) is finite dimensional for any homogenous \( x \in g \). Then each finite dimensional restricted Lie color algebra is \( p \)-finite. It is also easy to see that \( g = cg(V) \) is \( p \)-finite.

Let \( g \) be a \( p \)-finite Lie color algebra. In this section we introduce for each locally finite simple \( g \)-module a \( p \)-character \( \chi \).

Proposition 2.6. Let \( g = \oplus_{\alpha \in \Gamma} g_{\alpha} \) be a \( p \)-finite Lie color algebra, and \( V = \oplus_{\alpha \in \Gamma} V_{\alpha} \) be a locally finite simple \( g \)-module. Assume \( \alpha \in \Gamma^{+} \) such that \( p\alpha \) has finite order \( s \). Then there exists a function \( \kappa: g_{\alpha}^{+} \to F \), for each \( \alpha \in \Gamma^{+} \) such that

\[
((x^{p} - x^{[p]} \lambda^{s}) \cdot \kappa(x)) \cdot m = 0,
\]

for any \( x \in g_{\alpha}, m \in V \).

Proof. Let \( x \in g_{\alpha} \). Then we have \( x^{p} - x^{[p]} \in Z(U(g))_{p\alpha} \), and hence \( (x^{p} - x^{[p]} \lambda^{s}) \in Z(U(g))_{0} \). Since \( V \) is locally finite, there is a finite dimensional \( \Gamma \)-graded \( x \)-invariant subspace \( V_{x} \subseteq V \). Since \( g \) is \( p \)-finite, the \( \Gamma \)-graded subspace \( V' = V_{x} + \sum_{i \geq 1} x^{[p]^{i}}V_{x} \) is finite dimensional and invariant under the action of both \( x \) and \( x^{[p]} \). Acting on \( V' \), the element \( (x^{p} - x^{[p]} \lambda^{s}) \) has an eigenvalue \( \lambda \), since \( F \) is algebraically closed.

Let \( V_{\lambda} = \{ v \in V | (x^{p} - x^{[p]} \lambda^{s})v = \lambda v \} \). It is easy to see that \( V_{\lambda} \) is a nonzero \( \Gamma \)-graded submodule of \( V \). The simplicity of \( V \) implies that \( V_{\lambda} = V \). Then the mapping \( x \mapsto \lambda \) defines a function \( \kappa(x) \).

Note that if \( g \) is a finite dimensional restricted Lie color algebra, then [1, Lemma 2.5] says that each simple \( g \)-module is finite dimensional, so that the assumption of the proposition is satisfied.

Definition 2.7. [1, 1.11] Let \( g = \oplus_{\alpha \in \Gamma} g_{\alpha} \) be a Lie color algebra and let \( V = \oplus_{\alpha \in \Gamma} V_{\alpha} \) be a \( g \)-module. Suppose \( \phi \in gl(V)_{\alpha} \). If \( \phi(y) = (\beta, \alpha)\phi(\gamma \cdot m) \) for any \( y \in g_{\beta}, \beta \in \Gamma, m \in V, \) then we say that \( \phi \) is a centralizer of the \( g \)-module \( V \).

Let \( g = \oplus_{\alpha \in \Gamma} g_{\alpha} \) be a \( p \)-finite Lie color algebra, and let \( V = \oplus_{\alpha \in \Gamma} V_{\alpha} \) be a simple \( g \)-module as that in the proposition above. If \( \alpha \in \Gamma^{+} \) and \( p\alpha = 0 \), we let \( \chi(x) = \kappa(x)^{\frac{1}{p}} \). Then we have

\[
(x^{p} - x^{[p]} - \chi(x)^{p}) \cdot m = 0,
\]
for any \( x \in \mathfrak{g}_\alpha \) and \( m \in V \). By [19, Prop 2.1, p.70], the mapping \( x \rightarrow x^p - x^{[p]} \) from \( \mathfrak{g}_\alpha \) to \( U(\mathfrak{g}) \) is \( p \)-semilinear. It follows that \( \chi|_{\mathfrak{g}_\alpha} \in \mathfrak{g}_\alpha^* \). If \( p\alpha \neq 0 \) has finite order \( s \). Let \( x \in \mathfrak{g}_\alpha \) with \( \kappa(x) = 0 \). Then Proposition 2.5 says that \( x^p - x^{[p]} \) is a nilpotent centralizer of \( \mathfrak{g} \)-module \( V \). But \( (x^p - x^{[p]} \cdot V is also a graded submodule of \( V \), so we have \( (x^p - x^{[p]} \cdot V = 0 \).

**Lemma 2.8.** If \( p\alpha \neq 0 \) and \( \kappa|_{\mathfrak{g}_\alpha} \neq 0 \), then there exists an invertible centralizer \( \phi \in \mathfrak{gl}(V)_{p\alpha} \), \( c(x) \in \mathfrak{g}_\alpha^* \) such that

\[
(x^p - x^{[p]} - c(x)^p \phi) \cdot m = 0
\]

for any \( x \in \mathfrak{g}_\alpha \) and \( m \in V \).

**Proof.** Let \( \rho \) be the representation afforded by the \( \mathfrak{g} \)-module \( V \). By assumption, we get \( \kappa(y) \neq 0 \) for some \( y \in \mathfrak{g}_\alpha \). Then the centralizer \( \phi =: \rho(y^p - y^{[p]}) \in \mathfrak{gl}(V)_{p\alpha} \) is invertible. For each \( x \in \mathfrak{g}_\alpha \), we have

\[
\rho(x^p - x^{[p]}) \phi^{-1} \in \rho(Z(U(\mathfrak{g})))_{\emptyset}.
\]

Since the mapping \( x \rightarrow \rho(x^p - x^{[p]}) \) is \( p \)-semilinear, we get, in the light of the proof of Prop. 2.6, a linear function \( c(x) \in \mathfrak{g}_\alpha^* \) such that

\[
[(x^p - x^{[p]}) \phi^{-1} - c(x)^p] \cdot m = 0
\]

for any \( x \in \mathfrak{g}_\alpha \), \( m \in V \). Thus we have \( [(x^p - x^{[p]} - c(x)^p \phi] \cdot m = 0 \).

\( \square \)

Let \( \mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha \) be a restricted Lie color algebra, and \( V = \oplus_{\alpha \in \Gamma} V_\alpha \) be a simple \( \mathfrak{g} \)-module as above. Assume \( \rho \) is the irreducible representation afforded by the \( \mathfrak{g} \)-module \( V \). Suppose \( \alpha \in \Gamma^+ \) has infinite order. Let \( x \in \mathfrak{g}_\alpha \). Then since \( \rho(x^p - x^{[p]}) \) is a centralizer of \( \mathfrak{g} \)-module \( V \), we get either \((x^p - x^{[p]}) \cdot V = 0 \) or \((x^p - x^{[p]}) \cdot V = V \). In the latter case, \( \phi = \rho(x^p - x^{[p]}) \) is invertible. In the light of the proof of Prop. 2.6, we get a linear function \( c(y) \in \mathfrak{g}_\alpha^* \) such that

\[
(y^p - y^{[p]} - c(y)^p \phi) \cdot m = 0
\]

for any \( y \in \mathfrak{g}_\alpha \) and \( m \in V \). In the case \( V \) is a locally finite \( \mathfrak{g} \)-module, the freeness of the subgroup \( (a) \subseteq \Gamma \) implies that \( x^p - x^{[p]} \) acts nilpotently on \( V \), hence annihilates \( V \).

**Proposition 2.9.** Let \( p\alpha \neq 0 \) and \( \kappa|_{\mathfrak{g}_\alpha} \neq 0 \). Assume \( p\alpha \in \Gamma^+ \) has finite order \( s \).

There exist \( 0 \neq \xi \in \mathfrak{g}_\alpha \) and \( c(x) \in \mathfrak{g}_\alpha^* \) such that for any \( x \in \mathfrak{g}_\alpha \) and \( m \in V \), we have:

1. \( [(\xi^p - \xi^{[p]})^s - 1] \cdot m = 0 \).
2. \( \rho(x^p - x^{[p]}) = c(x)^p \rho(\xi^p - \xi^{[p]}) \).
3. \( c(x)^{ps} = \kappa(x) \).
Proof. By assumption $\kappa(x_0) \neq 0$ for some $x_0 \in g_\alpha$. Since $F$ is algebraically closed, we get (1) from the semilinearity of the mapping $x \mapsto x^p - x^{[p]}$ from $g$ into $U(g)$ and Prop. 2.6.

(2) We fix $x \in g_\alpha$ as in (1). Let $\phi = \rho(x^p - x^{[p]})$. Then Lemma 2.8 gives $c(x) \in g^*_\alpha$ as required.

(3) By taking the $s$-th power on both sides of (2) and using Prop. 2.6, one gets $c(x)^{p^s} = \kappa(x)$ for any $x \in g_\alpha$.

Let $\alpha \in \Gamma^+$ such that $p\alpha \neq 0$ has finite order $s$. We define an equivalence relation on the set

\[ \{(\xi, c(x)) | c(x) \in g^*_\alpha, \xi \in g_\alpha, c(\xi) = 1\} \]

as follows:

\[ (\xi, c(x)) \sim (\eta, b(x)) \quad \text{if} \quad c(x) = c(\eta)b(x), b(x) = b(\xi)c(x) \]

and \( c(\eta)^s = b(\xi)^s = 1 \).

Each equivalent class is denoted by $[\xi, c(x)]$. We denote by $\mathcal{F}_\alpha$ the set all such equivalency classes. We shall now define the generalized $p$-character $\chi$.

**Definition 2.10.** Let $g = \oplus_{\alpha \in \Gamma} g_\alpha$ be a restricted Lie color algebra. A $p$-character of $g$ is a set $\{\chi_\alpha\}_{\alpha \in \Gamma^+}$ satisfying the following properties:

1. If $p\alpha = 0$, then $\chi_\alpha \in g^*_\alpha$.
2. If $\alpha$ has infinite order, then $\chi_\alpha = 0$.
3. If $p\alpha \neq 0$ has finite order $s$, then $\chi_\alpha \in \mathcal{F}_\alpha$.

We denote the set $\{\chi_\alpha | \alpha \in \Gamma^+\}$ usually by $\chi$.

**Definition 2.11.** Let $g = \oplus_{\alpha \in \Gamma} g_\alpha$ be a restricted Lie color algebra. A $g$-module $V = \oplus_{\alpha \in \Gamma} V_\alpha$ is said to have a $p$-character $\chi$ if the following statements hold.

1. If $\chi_\alpha \in g^*_\alpha$, then $(x^p - x^{[p]} - \chi_\alpha(x)^p) \cdot m = 0$ for any $x \in g_\alpha$, $m \in V$.
2. If $\chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha$, then Prop. 2.9(1),(2) are satisfied.

It is easy to check that (2) is independent of the representatives for $\chi_\alpha$.

Let $g$ be a $p$-finite Lie color algebra. Then each locally finite simple $g$-module has a $p$-character.

Let $g$ be a restricted Lie color algebra, and let $\chi$ be a $p$-character for $g$. We shall now define the $\chi$-reduced universal enveloping algebra for $g$. Let $U(g)$ be the universal enveloping algebra of $g$. We define a two-sided ideal $I_\chi$ of $U(g)$ generated by the homogeneous elements as follows:

\[ x^p - x^{[p]} - \chi(x)^p \cdot 1, \quad x \in g_\alpha, \quad \text{if} \quad \chi_\alpha \in g^*_\alpha \]
\((\xi^p - \xi^{[p]})^s - 1, (x^p - x^{[p]}) - c(x)^p(\xi^p - \xi^{[p]}), \quad x \in \mathfrak{g}_\alpha, \quad \text{if} \quad \chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha.\)

It is easy to check that the definition of the quotient \(U(\mathfrak{g})/I_\chi\) is independent of the representatives of each \(\chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha.\)

Let \(A = \bigoplus_{\alpha \in \Gamma} A_\alpha\) and \(B = \bigoplus_{\alpha \in \Gamma} B_\alpha\) be two \(\Gamma\)-graded associative algebras. By a homomorphism \(f\) from \(A\) to \(B\) we usually mean that \(f\) is a homomorphism of algebras satisfying \(f(A_\alpha) \subseteq B_\alpha.\)

**Definition 2.12.** Let \(\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha\) be a restricted Lie color algebra, and let \(\chi\) be a \(p\)-character of \(\mathfrak{g}.\) A pair \((u_\chi(\mathfrak{g}), \sigma)\) consisting of an associate \(\Gamma\)-graded \(F\)-algebra with unity and a homomorphism \(\sigma : \mathfrak{g} \longrightarrow u_\chi(\mathfrak{g})^-\) is called a \(\chi\)-reduced universal enveloping algebra if:

1. \(\sigma(x)^p - \sigma(x^{[p]}) = \chi(x)^p, \quad \text{for any} \quad x \in \mathfrak{g}_\alpha, \quad \text{if} \quad \chi_\alpha = \mathfrak{g}_\alpha^*;\)
2. \((\sigma(\xi)^p - \sigma(\xi^{[p]}))^s = 1, \sigma(x^p) - \sigma(x^{[p]}) = c(x)^p(\sigma(\xi)^p - \sigma(\xi^{[p]})), \quad \text{for any} \quad x \in \mathfrak{g}_\alpha, \quad \text{if} \quad \chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha.\)

(2) Given any \(\Gamma\)-graded associate \(F\)-algebra \(A = \bigoplus_{\alpha \in \Gamma} A_\alpha\) with unity and any homomorphism \(g : \mathfrak{g} \longrightarrow A^-\) such that condition (1) is satisfied, there is a unique homomorphism \(\tilde{g} : u_\chi(\mathfrak{g}) \longrightarrow A\) of associate \(\Gamma\)-graded algebras such that \(\tilde{g} \cdot \sigma = g.\)

Let \(\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha\) be a restricted Lie color algebra. Assume \(\{e_i\}_{i \in I}\) (resp. \(\{f_i\}_{i \in M}\)) is a totally ordered homogeneous basis of \(\mathfrak{g}^+\) (resp. \(\mathfrak{g}^\).) Let \(\chi\) be a \(p\)-character of \(\mathfrak{g}\). With a minor adjustment, we may choose a basis \(\{e_i\}_{i \in I}\) of \(\mathfrak{g}^+\) containing each \(\xi\), if \(\chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha\) for some \(\alpha \in \Gamma^+.\) Let \(J\) denote the subset of \(I\)

\[\{i \in I| e_i = \xi, \quad \text{if} \quad \chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha, \xi \in \mathfrak{g}_\alpha\}.\]

For \(\alpha \in \Gamma^+,\) assume \(p\alpha \neq 0\) has finite order \(s.\) \(s\) varies with different \(\alpha.\) We denote all these different orders simply by \(s.\)

Recall the notation \(N(I)\) for each set of indices \(I.\) We have

**Theorem 2.13.** Let \(\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha\) be a restricted Lie color algebra. Assume \(\{e_i\}_{i \in I}\) (resp. \(\{f_i\}_{i \in M}\)) be a totally ordered homogeneous basis of \(\mathfrak{g}^+\) (resp. \(\mathfrak{g}^\).) Let \(\chi\) be a \(p\)-character of \(\mathfrak{g}.\) Then the quotient \(U(\mathfrak{g})/I_\chi\) is the \(\chi\)-reduced universal enveloping algebra having the basis

\[\{\sigma(f)^s\sigma(e)^n| \delta \in N(M), n \in N(I), \delta_j = 0, 1, j \in M; 0 \leq n(i) < \begin{cases} p, & \text{if} \quad i \in I \setminus J \\ ps, & \text{if} \quad i \in J \end{cases}\}.\]

**Proof.** The universal property is obvious. We now apply Lemma 2.4 to get a basis of \(U(\mathfrak{g})/I_\chi\). For each \(i \in J,\) we choose \(k(i) = ps,\) then we have \(e_i^{ps} = v_i + z_i,\) where

\[v_i = e_i^{ps} - (e_i^p - e_i^{[p]})^s + 1 \in U_{(ps-1)}\]
and
\[ z_i = (e_i^p - e_i^{[p]})^* - 1 \in Z(U(g)). \]
Both \( v_i \) and \( z_i \) are homogeneous with the \( \Gamma \)-grading \( 0 \). If \( i \notin J \) and \( e_i \in g_a \) with \( \chi_a = [\xi, c(x)] \in F_a \), we choose \( k_i = p \), then we have \( e_i^p = z_i + v_i + t_i \xi^p \), where
\[ z_i = e_i^p - e_i^{[p]} - c(e_i)^p(\xi^p - \xi^{[p]}) \in Z(U(g)), \]
\[ v_i = e_i^{[p]} - c(e_i)^p \xi^{[p]} \in U_{(p-1)} \quad \text{and} \quad t_i = -c(e_i)^p. \]
If \( i \notin J \) and \( e_i \in g_a \) with \( \chi_a \in g^*_a \), we choose \( k(i) = p \), then we get \( e_i^p = v_i + z_i \), where \( v_i = e_i^{[p]} \in U_{(p-1)} \) and \( z_i = e_i^p - e_i^{[p]} \in Z(U(g)) \).

For each \( i \in M \), let \( f_i \in g_a \), \( \alpha \in \Gamma^- \). Then since \( (\alpha, \alpha) = -1 \) and \( p > 3 \), we have in \( U(g) \) that \( f_i^2 = \frac{1}{2}[f_i, f_i] \in g \), so that we may choose \( k(i) = 2 \),
\[ v_i = \frac{1}{2}[f_i, f_i] \in g = U(1), \quad z_i = 0. \]
Then we have \( f_i^{k(i)} = v_i + z_i \).

Applying Lemma 2.4, we get
\[ I_\chi = \langle z^r f^\delta e^n | r, n \in N(I), |r| \geq 1, n(i) < k(i) \rangle, \]
so that \( U(g)/I_\chi \) has the basis claimed, where \( \sigma \) is the canonical epimorphism: \( U(g) \rightarrow U(g)/I_\chi \).

We will denote \( U(g)/I_\chi \) by \( u_\chi(g) \) in the following. In particular, if \( \chi = 0 \), we simply denote \( u_0(g) \) by \( u(g) \), which is also referred to as the restricted universal enveloping algebra\([6, 5.7]\). Then each locally finite simple \( g \)-module is a simple \( u_\chi(g) \)-module for some \( p \)-character \( \chi \). A \( u(g) \)-module is also referred to as a restricted \( g \)-module.

### 2.3 The Frobenious algebra \( u_\chi(g) \)

**Definition 2.14.** (1) Let \( V = \bigoplus_{\alpha \in \Gamma} V_\alpha \) be a \( \Gamma \)-graded space, and let \( f \) be a bilinear form on \( V \). Then \( f \) is called symmetric if \( f(x, y) = (\alpha, \beta)f(y, x) \) for any \( x \in V_\alpha \), \( y \in V_\beta \).

(2) Let \( A = \bigoplus_{\alpha \in \Gamma} A_\alpha \) be a \( \Gamma \)-graded algebra. A bilinear form \( f \) on \( A \) is called invariant if \( f(xy, z) = f(x, yz) \) for any \( x \in A_\alpha \), \( y \in A_\beta \), \( z \in A_\gamma \), \( \alpha, \beta, \gamma \in \Gamma \).

**Definition 2.15.** An associate \( \Gamma \)-graded algebra is called Frobenious if it has a nondegenerate invariant bilinear form.

Let \( g \) be a restricted finite dimensional Lie color algebra. Let \( \chi = \{ \chi_a \}_{a \in \Gamma^+} \) be a \( p \)-character of \( g \). We now show that the \( \chi \)-reduced universal enveloping algebra \( u_\chi(g) \) is a Frobenious algebra.
Taking for \( g^+ \) (resp. \( g^- \)) a homogeneous basis \( \{e_i | i \leq k\} \) (resp. \( \{e_i | k + 1 \leq i \leq n\} \)) containing each \( \xi \) if \( \chi_\alpha = [\xi, c(x)] \in \mathcal{F}_\alpha \), we denote

\[
J =: \{1 \leq i \leq k | e_i = \xi \quad \text{for some} \quad \chi_\alpha = [\xi, c(x)]\}.
\]

Note that \( J = \phi \) simply means that each \( \chi_\alpha \) is a linear function on \( g_\alpha \), such that \( \chi_\alpha = 0 \) for all \( \alpha \in \Gamma \) with \( p\alpha \neq 0 \).

By the PBW theorem, \( U(g) \) is a free left \( U(g^+) \)-module with a basis

\[
\{\Pi^n_{i=k+1} e_i^{a_i} | 0 \leq a_i \leq 1\}.
\]

Let \( x \in g_\alpha \), \( \alpha \in \Gamma \). For the simplicity, we also use \( \bar{x} \) to denote \( \alpha \) in the following.

Let \( R \) be the \( \Gamma \)-graded subalgebra of \( U(g) \) generated by

\[
z_i = \begin{cases} 
  e_i^p - e_i^{[p]} - \chi(e_i)^p \cdot 1, & \text{if } i \leq k, \text{ and } \chi_{e_i} \text{ is linear} \\
  e_i^p - e_i^{[p]} - c(e_i)^p(\xi^p - \xi^{[p]}) & \chi_{e_i} = [\xi, c(x)], i \notin J \\
  (e_i^p - e_i^{[p]})^s - 1, & \text{if } i \in J.
\end{cases}
\]

By Lemma 2.4, \( R \) is a color commutative polynomial algebra with indeterminate \( z_i \)'s. Then \( U(g^+) \) is free over \( R \) with a basis

\[
\{\Pi^k_{i=1} e_i^{a_i} | 0 \leq a_i \leq p - 1, \text{if } i \notin J, 0 \leq a_i \leq ps - 1 \text{ if } i \in J\}.
\]

Recall that we denote the set of all the mappings from \( \{1, \ldots, n\} \) to \( \mathbb{N} \) by \( N(\{1, \ldots, n\}) \).

We now define \( \tau \in N(\{1, \ldots, n\}) \) by

\[
\tau(i) = \begin{cases} 
  p - 1, & \text{if } i \leq k \text{ and } i \notin J \\
  ps - 1, & \text{if } i \in J \\
  1, & \text{if } i \geq k + 1.
\end{cases}
\]

For two mappings \( a, b \in N(\{1, \ldots, n\}) \), we define \( a \leq b \) if \( a(i) \leq b(i) \) for each \( 1 \leq i \leq n \), and \( a < b \) if \( a \leq b \) and at least there is \( 1 \leq i \leq n \) such that \( a(i) < b(i) \).

Then by Lemma 2.4 \( U(g) \) is free over \( R \) with a basis \( \{e^a | 0 \leq a \leq \tau\} \). As a \( R \)-module,

\[
U(g) = \sum_{a<\tau} Re^a \oplus Re^\tau.
\]

We denote the \( R \)-submodule \( \sum_{a<\tau} Re^a \) by \( V \). We define the \( R \)-linear map \( p_\tau : U(g) \rightarrow R \) by \( p_\tau(v + re^x) = r, v \in V, r \in R \). Then we get a \( R \)-bilinear form \( \mu: U(g) \times U(g) \rightarrow R, \mu(x,y) = p_\tau(xy) \), for any homogeneous \( x,y \in g \). Obviously we have \( \mu(x,y,z) = \mu(x, yz) \).

Let \( g \) be a Lie color algebra. For each homogeneous \( x \in g \), the derivation \( adx \) is naturally extended to a derivation on \( U(g) \) which is also denoted by \( adx \). A straightforward computation shows that

\[
adx(\Pi^n_{i=1} y_i) = \sum_{i=1}^n (\bar{x}, \sum_{l=1}^{i-1} [y_l] y_1 \cdots y_{i-1} [x, y_i] y_{i+1} \cdots y_n),
\]

for any homogeneous \( y_1, \ldots, y_n \in g \).
Lemma 2.16. Let $\mu$ be the bilinear form defined above, and let $I$ be a $\Gamma$-graded ideal of $R$. Then

$$IU(\mathfrak{g}) = \{ u \in U(\mathfrak{g}) | \mu(u, U(\mathfrak{g})) \subseteq I \}.$$ 

Proof. (see [19, Th.4.2(1)] for the Lie algebra case) If $J = \phi$, then the proof for [19, Th. 4.2] also works here. Now we assume $J \neq \phi$. Using the identities given in the proof of Th. 2.13, one can show that if $a(i) > \tau(i)$ for some $i \in J$, then $e^a \in RU_{\{a\}}^{-1}$; if $a(j) \leq \tau(j)$ for all $j \in J$, and $a(i) > \tau(i)$ for some $i \notin J$, then $e^a \equiv c_b e^b (\mod U_{\{a\}}^{-1})$, where $b \leq \tau$, $|b| = |a|$, $c_b \in F^\times$. In the light of the proof of [19, Th.4.2(1)], one gets $V = RU_{\{\tau\}}^{-1}$.

Taking $a, b \leq \tau$ with $|a + b| \leq \tau$ and using the discussions above, we get $e^a e^b \equiv \delta_{a,\tau-b} r(a, b)e^\tau (\mod V)$, where

$$0 \neq r(a, b) = \prod_{i=1}^{n-1} (\sum_{j=i+1}^n a_j \bar{e}_j | b_i \bar{e}_i).$$

Then the arguments leading to [19, Th 4.2 (1)] can be repeated to give the desired result. 

Theorem 2.17. Let $\mathfrak{g}$ be a restricted Lie color algebra, and let $\chi$ be a $p$-character of $\mathfrak{g}$. Then $u_\chi(\mathfrak{g})$ is Frobenius.

Proof. Recall the ideal $I_\chi$ of $U(\mathfrak{g})$ defined earlier. With a minor adjustment of the proof for [19, Coro 4.3, p. 218], we get the induced bilinear form $\bar{\mu}$:

$$\left\{ \begin{array}{l} u_\chi(\mathfrak{g}) \times u_\chi(\mathfrak{g}) \longrightarrow R/I_\chi \cong F \\
(\bar{x}, \bar{y}) \longrightarrow \mu(xy) + I_\chi, \end{array} \right.$$ 

which is well defined and nondegenerate. 

Definition 2.18. Let $\mathfrak{g}$ be a restricted Lie color algebra. For each $x \in \mathfrak{g}_\alpha \subseteq \mathfrak{g}^+$, if there is $n \in \mathbb{Z}^+$ such that $x^{[p]^n} = 0$, then we say that $\mathfrak{g}$ is unipotent.

Lemma 2.19. Let $\mathfrak{g}$ be a finite dimensional unipotent Lie color algebra. Then $u(\mathfrak{g})$ is a symmetric algebra.

Proof. Let $e_1, \ldots, e_n$ be a basis of $\mathfrak{g}$ consisting of homogeneous elements. For a homogeneous $x \in \mathfrak{g}$, suppose that $adx(e_j) = \sum_{i=1}^n k_{ij} e_i$, $1 \leq j \leq n$. A straightforward calculation shows that

$$adx(e^a) = \sum_{i=1}^n (\prod_{j=1}^{i-1} (\bar{x}, \bar{e}_j)) \frac{1 - (\bar{x}, \bar{e}_i)^a}{1 - (\bar{x}, \bar{e}_i)^1} k_{ii} e^a + \sum_{|b|=|a|, b \neq a} c_b e^b (\mod U_{\{a\}}^{-1}).$$

Since $\mathfrak{g}$ is unipotent, $\mathfrak{g}$ is a restricted $\mathfrak{g}$-module under the adjoint action. By [6, Th. 3.2], $\mathfrak{g}$ acts strictly triangulable on $\mathfrak{g}$. Then we can find a homogeneous basis.
of $\mathfrak{g}$, under which the matrix of $adx$ is strictly upper triangular for all homogeneous $x \in \mathfrak{g}$. Using the formula above and the identity $V = RU_{(|r|-1)}$, one can easily get $[x, e^r] \in V$ for any homogeneous $x$. Then using the fact $[x, e^a] \in U_{(|a|)}$ we obtain $xu - (x|u)xu \in V$ for all homogeneous $u \in U(\mathfrak{g})$, and hence $vu - (v|u)uv \in V$ for all homogeneous $v, u \in U(\mathfrak{g})$. Therefore, $\mu$ is symmetric, so is the induced bilinear form $\overline{\mu}$. 

\[\]  

3 The main theorems

Let $\mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a restricted Lie color algebra. Let $\text{Aut}(\mathfrak{g})$ be the group of all the automorphisms for $\mathfrak{g}$. We denote

$$\text{Aut}^{res}(\mathfrak{g}) = \{ f \in \text{Aut}(\mathfrak{g}) | f(x^{[p]}) = f(x)^{[p]} \text{ for any } x \in \mathfrak{g}_\alpha, \alpha \in \Gamma^+ \}. $$

Recall the Lie color algebra $\mathfrak{g} = cgl(V)$ in 2.1. Then $\mathfrak{g}_0 = \oplus_{\alpha_i \in \Gamma} gl(m_{\alpha_i}, \mathbb{F})$. If we denote the linear algebraic group $\Pi_{\alpha_i \in \Gamma} GL(m_{\alpha_i}, \mathbb{F})$ by $GL\{(m_{\alpha_i}, \mathbb{F})\}$, then we get

$$\text{Lie}(GL\{(m_{\alpha_i}, \mathbb{F})\}) = \mathfrak{g}_0$$

(see the Appendix for the infinite dimensional case). Acting on $\mathfrak{g}$ by conjugation, $GL\{(m_{\alpha_i}, \mathbb{F})\}$ becomes a subgroup of $\text{Aut}^{res}(\mathfrak{g})$.

**Definition 3.1.** Let $\mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a restricted Lie color subalgebra of $cgl(V)$. If there is a linear algebraic group $G \subseteq \text{Aut}^{res}(\mathfrak{g}) \cap GL\{(m_{\alpha_i}, \mathbb{F})\}$ such that $\text{Lie}(G) = \mathfrak{g}_0$. Then $\mathfrak{g}$ is referred to as an algebraic Lie color algebra.

Let $\mathfrak{g}$ be an algebraic Lie color algebra. We fix a maximal torus $T$ of $G$. Then $\mathfrak{g}$ has the root space decomposition relative to $T$: $\mathfrak{g} = H \oplus \sum_{\delta \in \Phi} \mathfrak{g}_\delta$. Then we get a triangular decomposition of $\mathfrak{g}$: $\mathfrak{g} = N^- \oplus H \oplus N^+$, where

$$N^+ = \oplus_{\delta \in \Phi^+} \mathfrak{g}_\delta \quad N^- = \oplus_{\delta \in \Phi^-} \mathfrak{g}_\delta \quad H = \text{Lie}(T).$$

For each $\delta \in \Phi^+$, we use $e_\delta$ (resp. $f_\delta$, $H_\delta$) to denote the positive root vector (resp. negative root vector, $[e_\delta, f_\delta]$).

In this section we study the simple modules for the algebraic Lie color algebras. We keep the convention that $\chi \in \mathfrak{g}_0^*$ implies that $\chi \in \mathfrak{g}^*$ and $\chi(\mathfrak{g}_\alpha) = 0$ for all $\bar{0} \neq \alpha \in \Gamma$.

Let $\Gamma$ be an abelian group, and let $V = \oplus_{\alpha \in \Gamma} V_\alpha$ be a $\Gamma$-graded $\mathbb{F}$-vector space. We define the $\mathbb{F}$-linear vector space

$$V_f^* = \{ f \in \text{Hom}_\mathbb{F}(V, \mathbb{F}) | f(V_\alpha) = 0 \text{ for all but finitely many } \alpha \text{'s} \}. $$

Then $V_f^* = V^*$ in case $V$ is finite dimensional.

Let $\mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a Lie color algebra. For each $\alpha \in \Gamma$, we let

$$\mathfrak{g}_{f,\alpha}^* = \{ \psi \in \mathfrak{g}_{f}^* | \psi(\mathfrak{g}_\beta) = 0 \text{ for all } \beta \neq \alpha \}. $$
Then $\mathfrak{g}_f^* = \oplus_{\alpha \in T} \mathfrak{g}_{f,\alpha}^*$ is also a $\Gamma$-graded $F$-vector space.

A linear function $\chi \in \mathfrak{g}_{f,0}^*$ is called in the standard form if $\chi = \chi_s + \chi_n$, if $\chi_s(N^\pm) = 0$, $\chi_n(H + N^+) = 0$, and $\chi(H) \neq 0$ implies that $\chi(\mathfrak{g}_\delta) = \chi(\mathfrak{g}_{-\delta}) = 0$. We say that $\chi$ is standard semisimple(resp. nilpotent), if $\chi = \chi_s$(resp. $\chi = \chi_n$).

### 3.1 The algebraic Lie color algebra $\mathfrak{g} = \text{cgl}(V)$

Let $\mathfrak{g} = \text{cgl}(V)$ and let $G = \text{GL}(\{m_\alpha\}, F)$. If $\Gamma$ is infinite(resp. finite), then the linear algebraic group $G$ has the maximal torus:

$$T = \{ (a_{ii}) | a_{ii} \in F^\times, i \in Z^+ \} \text{(resp.} T = \{ \text{diag}(t_1, \ldots, t_n) | t_i \in F^\times \}, n = \text{dim}V).$$

We have the root space decomposition $\mathfrak{g} = \oplus_{\delta \in \Gamma} \mathfrak{g}_\delta$ relative to $T$, where

$$\Phi = \{ \epsilon_i - \epsilon_j | i \neq j \}, \quad \mathfrak{g}_{\epsilon_i - \epsilon_j} = Fe_{ij}.$$ 

We take the set of positive(resp. negative) roots $\Phi^+ = \{ \epsilon_i - \epsilon_j | i < j \}$ (resp. $\Phi^- = \{ \epsilon_i - \epsilon_j | i > j \}$). Then

$$\Delta = \{ \epsilon_i - \epsilon_{i+1} | i = 1, 2, \ldots \}$$

is a set of simple roots.

For each $\delta = \epsilon_i - \epsilon_j \in \Phi^+$, we denote

$$e_\delta = e_{ij}, \quad f_\delta = e_{ji}, \quad H_\delta = \begin{cases} e_{ii} - e_{jj}, & \text{if } \bar{e}_\delta \in \Gamma^+ \\ e_{ii} + e_{jj}, & \text{if } \bar{e}_\delta \in \Gamma^- \end{cases}.$$ 

Suppose $e_{ij} \in \mathfrak{g}_\alpha$ and $\alpha \in \Gamma^+$. Then the subalgebra $< e_{ij}, e_{ji}, e_{ii} - e_{jj} >$ is isomorphic to the Lie algebra $sl_2$. If $\alpha \in \Gamma^-$, then the subalgebra $< e_{ij}, e_{ji}, e_{ii} + e_{jj} >$ is isomorphic to the Hensenberg algebra. $\mathfrak{g}$ has the triangular decomposition: $\mathfrak{g} = N^+ + H + N^-$, where

$$N^+ = \sum_{i < j} F e_{ij}, \quad N^- = \sum_{i > j} F e_{ij}, \quad H = \sum_{i \geq 1} F e_{ii}.$$ 

Note that if $\bar{e}_{ij} = \alpha \in \Gamma$, then $\bar{e}_{ji} = -\alpha$.

We now define a bilinear form on $\mathfrak{g}$ by setting

$$b(x, y) = (\alpha, \beta)tr(xy), \quad x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta.$$ 

It is easy to see that $b(x, y) = 0$ if $\bar{x} + \bar{y} \neq \bar{0} \in \Gamma$.

**Lemma 3.2.** (1) $b(\cdot)$ is nondegenerate, that is, for $x \in \mathfrak{g}_\alpha$, $\alpha \in \Gamma$, $x = 0$ if and only if $b(x, y) = 0$ for any homogeneous $y \in \mathfrak{g}$.

(2) $b(\cdot)$ is invariant, that is, $b([x, y], z) = b(x, [y, z])$, for any $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$, $z \in \mathfrak{g}_\gamma$. 

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Proof. (1) is obvious.

(2) By definition, we have

\[ b([x, y], z) = (\alpha + \beta, \gamma) tr([x, y] z) = (\alpha + \beta, \gamma) tr(x y z) - (\alpha + \beta, \gamma)(\alpha, \beta) tr(y x z) \]

\[ = \delta_{\alpha + \beta, -\gamma}(\alpha + \beta, \gamma) tr(x y z) - (\alpha + \beta, \gamma)(\alpha, \beta) tr(y x z) \]

\[ = \delta_{-\alpha, \beta + \gamma}(\alpha, \beta + \gamma) tr(x y z) - (\alpha, \beta + \gamma)(\beta, \gamma) tr(x y z) \]

\[ = b(x, [y, z]), \]

where the second last equality is given by the identity

\[ (\alpha + \beta, \gamma)(\alpha, \beta) = (\alpha, \beta + \gamma)(\beta, \gamma). \]

\[ \square \]

Note that the restriction of \( b(\cdot, \cdot) \) to \( \mathfrak{g}_0 \) is the usual trace form, which is again nondegenerate and invariant.

**Lemma 3.3.** Let \( x \in \mathfrak{g}_\alpha, \ y \in \mathfrak{g}_\beta \) and let \( g \in G =: GL(\{m_\alpha\}, \mathbb{F}) \). Then

\[ b(g \cdot x, y) = b(x, g \cdot y). \]

We say that \( b(\cdot, \cdot) \) is \( G \)-invariant.

The proof is a straightforward computation. We leave it to the interested reader.

For each homogeneous \( x \in \mathfrak{g} \), \( b(x, -) \) defines a linear function on \( \mathfrak{g} \). Moreover, \( b(x, -) \in \mathfrak{g}^*_x \). \( \mathfrak{g}^*_x \) is a \( G \)-module defined by the coadjoint action:

\[ g \cdot \chi(x) =: \chi^g(x) = \chi(g^{-1} x g), \]

for any \( g \in G, \chi \in \mathfrak{g}^*_x, x \in \mathfrak{g} \). We define a linear mapping \( \theta: \mathfrak{g} \to \mathfrak{g}^*_x \) such that \( \theta(x) = b(x, -) \). It is easy to see that \( \theta \) is surjective. Then the \( G \)-invariancy of \( b(\cdot, \cdot) \) ensures that \( \theta \) is an isomorphism of \( G \)-modules. Since the restriction of \( b(\cdot, \cdot) \) to \( \mathfrak{g}_0 \) is also \( G \)-invariant, we get \( \theta(\mathfrak{g}_0) = \mathfrak{g}^*_f \).

For each \( x \in \mathfrak{g}_0 \), we say that \( x \) is semisimple (resp. nilpotent), if \( gxg^{-1} \in H \) (resp. \( gxg^{-1} \in N \)) for some \( g \in G \). We claim that each \( x \in \mathfrak{g}_0 \) has a unique Jordan decomposition: \( x = x_s + x_n \), where \( x_s \) (resp. \( x_n \)) is semisimple (resp. nilpotent) and \( [x_s, x_n] = 0 \). This is well known in the case \( \mathfrak{g} \) is finite dimensional. Suppose \( \mathfrak{g} \) is infinite dimensional. Then there is \( n \in \mathbb{Z}^+ \) such that \( x \in \oplus_{i \leq n} gl(m_\alpha) \). Therefore \( x \) has a Jordan decomposition in \( \oplus_{i \leq n} gl(m_\alpha) \). Since the decomposition is unique in any finite dimensional Lie subalgebra of \( \mathfrak{g}_0 \), all these decompositions agree. So we can define it to be the Jordan decomposition for \( x \in \mathfrak{g}_0 \).

Let \( \chi \in \mathfrak{g}^*_f \). By our convention this means \( \chi(\mathfrak{g}_\alpha) = 0 \) for all \( 0 \neq \alpha \in \Gamma \). Then there is \( N \in \mathbb{Z}^+ \) such that \( \chi(gl(m_\alpha)) = 0 \) for all \( i > N \). Applying a similar argument as that for finite dimensional algebraic Lie algebras[16], and using the
Jordan decomposition in \( g \), we get \( \chi = \chi_s + \chi_n \), and there is \( g \in \bigoplus_{i=1}^{N} \text{GL}(m_{\alpha_i}) \subseteq G \), such that \( \chi_{\alpha_i}^2(N^+) = 0 \) and \( \chi_{\alpha_i}^2(H + N^+) = 0 \). Besides, \( \chi_{\alpha_i}^2(H_{\alpha_i}) \neq 0 \) implies \( \chi_{\alpha_i}^2(e_{\alpha_i}) = \chi_{\alpha_i}^2(f_{\alpha_i}) = 0 \), so that \( \chi^g = \chi_s^g + \chi_n^g \) is in the standard form.

Let \( \chi \in \mathfrak{g}^*_f, \bar{\chi} \). For each \( g \in G \), obviously the \( \Gamma \)-graded algebras \( u_\chi(g) \) and \( u_{\chi^2}(g) \) are Morita equivalent, so we can assume that \( \chi \) is in the standard form in the rest of the paper.

Let \( \chi = \chi_s + \chi_n \in \mathfrak{g}_0^* \) be in the standard form. For each \( \alpha \in \Gamma \), we define \( Z_\alpha =: c_{\bar{\chi}_\alpha}(\chi_s) = \{ x \in \mathfrak{g}_{\alpha} | \chi_s([x, y]) = 0 \} \) for any homogeneous \( y \in \mathfrak{g} \).

Then \( Z = \bigoplus_{\alpha \in \Gamma} Z_\alpha \) is a Lie color subalgebra of \( \mathfrak{g} \). In particular, we have

\[
Z = \bigoplus_{\delta \in \Phi, \chi(H_{\delta}) = 0} \mathfrak{g}_{\delta} \oplus H.
\]

Since \( \chi \in \mathfrak{g}_0^* \), the codimension of \( Z \) in \( g \) is finite. Let \( P = Z + N^+ \). Then \( P \) is a parabolic subalgebra of \( \mathfrak{g} \). Let \( \mathcal{N}^+ \) denote \( P^\perp \) with respect to the invariant form \( b(\cdot, \cdot) \). It is easy to check that

\[
\mathcal{N}^+ = \bigoplus_{\delta \in \Phi^+, \chi(H_{\delta}) \neq 0} \mathfrak{g}_{\delta},
\]

and \( \mathcal{N}^+ \) is the finite dimensional nilradical of \( P \).

### 3.2 Category \( \mathcal{O} \)

Let \( \mathfrak{g} \) be an algebraic Lie color algebra, and let \( \chi = \chi_s + \chi_n \in \mathfrak{g}_0^*, \bar{\chi} \) be in the standard form. Recall \( G \subseteq \text{Aut}^{res}(\mathfrak{g}) \) with a maximal torus \( T \) such that \( \text{Lie}(G) = \mathfrak{g}_0 \). We choose \( T_0 \) to be a subgroup of \( T \) such that \( \chi_s(\text{Lie}(T_0)) = 0 \), and \( \chi(\text{Ad}(t)x) = \chi(x) \) for all \( x \in \mathfrak{g}, t \in T_0 \).

Let \( \mathcal{S} \) denote a restricted Lie color subalgebra of \( \mathfrak{g} \) containing \( \text{Lie}(T_0) \). We identify \( u_{\chi}(\mathcal{S}) \) with its canonical image in \( u_{\chi}(\mathfrak{g}) \). Following [8, 14], we define the \( u_{\chi}(\mathcal{S}) - T_0 \) modules. A \( \Gamma \)-graded \( \mathbf{F} \)-vector space \( V = \bigoplus_{\alpha \in \Gamma} V_{\alpha} \) is called a \( u_{\chi}(\mathcal{S}) - T_0 \) module if it is both a \( u_{\chi}(\mathcal{S}) \)-module and a \( T_0 \)-module such that the following conditions hold:

1. We have \( t(xv) = \text{Ad}(t)(x)(tv) \) for all \( t \in T_0, x \in \mathcal{S}_{\alpha}, v \in V_{\beta}, \alpha, \beta \in \Gamma \).
2. The restriction of the \( \mathcal{S} \)-action on \( V \) to \( \text{Lie}(T_0) \) is equal to the derivative of the \( T_0 \)-action on \( V \).

Let \( u_{\chi}(\mathfrak{g}) \) be the \( \chi \)-reduced universal algebra of \( \mathfrak{g} \), and let \( u_{\chi}(\mathfrak{g}) - \text{Mod}_{T_0} \) be the category of \( u_{\chi}(\mathfrak{g}) - T_0 \)-modules. We define the category \( \mathcal{O} \) to be the full subcategory of \( u_{\chi}(\mathfrak{g}) - \text{Mod}_{T_0} \) whose objects satisfying the following conditions([10]):

1. \( M = \bigoplus_{\alpha \in \Gamma} M_{\alpha} \) is a finitely generated \( u_{\chi}(\mathfrak{g}) - T_0 \)-module.
2. \( M = \bigoplus_{\alpha \in \Gamma} M_{\alpha} \) is \( H - T_0 \) -semisimple, that is, \( M \) is a weight module: \( M_{\alpha} = \bigoplus_{\lambda \in \chi(T_0)} M_{\lambda} = \bigoplus_{\lambda \in \chi(T_0)} M_{\lambda} \) for each \( \alpha \in \Gamma \).
Lemma 3.4. for $(5, 7.4)$, we have $u$ see that Recall $N$ for $\chi_{(\operatorname{gl}(m_{\alpha_i})) = 0}$ for all $i \geq N$. Let $T_{\alpha_i}$ denote the maximal torus of $\operatorname{GL}(m_{\alpha_i}, \mathbb{F})$ consisting of diagonal matrices. We choose

$$T_0 = \Pi_{i \geq N} T_{\alpha_i} = \{(A_i)|A_i = E_{\alpha_i} \text{ for all } i \leq N, A_i \in T_{\alpha_i} \text{ for all } i \geq N\}.$$  

Then we see that $\chi(Ad(t)x) = \chi(x)$ for all $t \in T_0$ and $x \in g_0$.

By assumption $\chi(N^+) = 0$, $u(N^+) \subseteq u_{\chi}(g)$ is the reduced enveloping algebra for $N^+$. Then it is easy to see that $g = \operatorname{cgl}(V)$, as a $g$-module under the adjoint action, is in Category $\mathcal{O}$ for $\chi = 0$ and $T_0 = T$.

3.3 The Harish-Chandra homomorphism

Let $g = \sum_{\delta \in \Phi} g_\delta$ be an algebraic Lie color algebra, where $\Phi = \Phi^+ \cup \Phi^-$ is its root system. We define a mapping $\bar{p} : \cup g_\alpha - \{0\} \rightarrow \mathbb{N}$ by

$$\bar{p}(x) = \begin{cases} p, & \text{if } x \in g_\alpha, \alpha \in \Gamma^+ \\ 2, & \text{if } x \in g_\alpha, \alpha \in \Gamma^- \end{cases}.$$  

For the simplicity, we write $\bar{p}$ instead of $\bar{p}(x)$.

Let $\chi \in g^*_r \Phi \bar{0}$ be standard semisimple. For each $h \in H$, the derivation $adh$ of $g$ can be extended naturally to the $\chi$-reduced universal enveloping algebra $u_{\chi}(g)$. Then $u_{\chi}(g)$ has a weight space decomposition: $u_{\chi}(g) = \bigoplus_{\lambda \in H^*} u_{\chi}(g)_\lambda$. It is easy to see that $u_{\chi}(g)_0$ is spanned by

$$e_i^{k_1} \cdots e_i^{k_n} u_{\chi}(H)^{k_1-1} \cdots f_{\delta_n}^{k_n} \delta_i \in \Phi^+, 0 \leq k_1, \ldots, k_n \leq \bar{p} - 1.$$  

Recall $N^+ = \sum_{\delta \in \Phi^+} g_\delta$ and $N^- = \sum_{\delta \in \Phi^-} g_{-\delta}$. Applying a similar argument as that for [5, 7.4], we have

Lemma 3.4. Let $L = u_{\chi}(g)N_+ \cap u_{\chi}(g)_0$. Then

(1) $L = N^- u_{\chi}(g) \cap u_{\chi}(g)_0$, and $L$ is a two-sided ideal of $u_{\chi}(g)_0$.

(2) $u_{\chi}(g)_0 = u_{\chi}(H) \oplus L$.

The projection of $u_{\chi}(g)_0$ onto $u_{\chi}(H)$ with kernel $L$ is called the Harish-Chandra homomorphism, denoted by $\gamma$.  

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3.4 FP triples

Definition 3.5. Let $\Phi$ be the root system of an algebraic Lie color algebra $\mathfrak{g}$. We call a subset $\Phi_1 \subseteq \Phi$ a subsystem (resp. an additive subset) of $\Phi$ if: for any $\delta_1, \delta_2 \in \Phi_1$, $l_1 \delta_1 + \lambda_2 \delta_2 \in \Phi_1$ (resp. $\delta_1 + \delta_2 \in \Phi_1$), whenever $l_1 \delta_1 + l_2 \delta_2 \in \Phi$, $l_1, l_2 \in \mathbb{Z}$ (resp. $\delta_1 + \delta_2 \in \Phi$).

Let $\mathfrak{g}$ be the Lie algebra of a semisimple, simply connected algebraic group $G$. Let $\chi \in \mathfrak{g}^*$ be in the standard form $\chi = \chi_s + \chi_n$. By [7], $\chi$ induces a parabolic subalgebra $P = \mathcal{Z} \oplus N$ of $\mathfrak{g}$, where $\mathcal{Z}$ is a Levi factor of $P$ and $N$ is the nilradical. Let $T$ be the maximal torus of of $G$, and let $\Phi^+, \Phi^1$ and $\Phi_u$ denote the positive roots of $T$ in $\mathfrak{g}$ of $\mathcal{Z}$ and $N$. Then Friedlander and Parshall proved that [7, Lemma 8.4], there is an ordering of $\Phi_u$ as $\delta_1, \ldots, \delta_i$ such that for each $1 \leq i \leq m + 1$,

$$\Phi_i^+ = \Phi_1^+ \cup \{-\delta_1, \ldots, -\delta_{i-1}, \delta_i, \ldots, \delta_1\}$$

is a system of positive roots for $\Phi$ in which $\delta_i$ is a simple root. For each $i$, $\{-\delta_1, \ldots, -\delta_i\}$ is an additive subset of $\Phi$ normalized by $\Phi_1^+$.

Definition 3.6. Let $\mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be an algebraic Lie color algebra with root system $\Phi$, and let $\Phi_1$ be a subsystem of $\Phi$ such that the set $\Phi \setminus \Phi_1$ is finite. Put $\mathcal{Z} = \oplus_{\delta \in \Phi_1} \mathfrak{g}_\delta \oplus \mathfrak{h}$. Let $\Phi^+ - \Phi_1^+ = \{\delta_1, \ldots, \delta_m\}$. For each $i$, suppose

$$\Phi_i^+ =: \Phi_1^+ \cup \{-\delta_1, \ldots, -\delta_{i-1}, \delta_i, \ldots, \delta_m\}$$

is a positive root system of $\mathfrak{g}$, in which $\delta_i$ is a simple root, and $\{-\delta_1, \ldots, -\delta_i\}$ is an additive subset of $\Phi$ normalized by $\Phi_1^+$. Then we call $(\mathfrak{g}, \mathcal{Z}, (\delta_i)_{i \geq 1}^m)$ a Friedlander-Parshall triple, or simply a FP triple.

Examples: (1) Let $\mathfrak{g} = \mathfrak{gl}(m, \Gamma)$ and let $\Phi = \Phi^+ \cup \Phi^-$ be its root system. We denote the root system of $\mathfrak{g}_0$ by $\Phi_0$. Let $N_0^+ = \sum_{\delta \in \Phi_0^+} \mathfrak{g}_\delta$. Assume $\chi \in \mathfrak{g}_0^*$ is in the standard form: $\chi = \chi_s + \chi_n$, where $\chi_s$ (resp. $\chi_n$) is standard semisimple (resp. nilpotent). Let $\mathcal{Z} = c_0(\chi_s)$. Then

$$\mathcal{Z} = \sum_{\chi_s(H_0) = 0} \mathfrak{g}_\alpha + \mathfrak{h}, \quad \mathcal{N}^+ = \sum_{\alpha \in \Phi^+, \chi_s(H_0) \neq 0} \mathfrak{g}_\alpha.$$ 

By [7, 8.4] and Lemma 4.1, there is an order of the roots of $\mathcal{N}^+$: $\delta_1, \ldots, \delta_m$, such that $(\mathfrak{g}, \mathcal{Z}, (\delta_i)_{i = 1}^m)$ is a FP triple.

(2) Let $\mathfrak{g} = \mathfrak{gl}(m, \Gamma)$ and let $\mathfrak{h}$ be its maximal torus as above. By the argument in the proof of [7, 8.4] and Lemma 4.1, we can put $\Phi^+$ in such an order that $(\mathfrak{g}, \mathfrak{h}, \Phi^+)$ is a FP triple.

(3) Let $\mathfrak{g} = \mathfrak{cgl}(V)$ be infinite dimensional, and let $\chi \in \mathfrak{g}_0^f$ be in the standard form. Recall the Lie color subalgebra $\mathcal{Z}$ and $\mathcal{N}^+$. We claim that an order can be given to the roots of $\mathcal{N}^+$: $\delta_1, \ldots, \delta_m$, such that $(\mathfrak{g}, \mathcal{Z}, (\delta_i)_{i = 1}^m)$ is a FP triple. For each
\[ n \in \mathbb{Z}^+, \text{ let us denote } g_n := \text{gl}((\sum_{i \leq n} m_{\alpha_i}, F) \subseteq g. \] Then \( g_n \) is a Lie color subalgebra of \( g \) with the 0 component \( \oplus_{i=1}^{\infty} \text{gl}(m_{\alpha_i}, F) \). Moreover, \((g_n)_{n=1}^{\infty}\) is a filtration for \( g \):

\[ g_1 \subseteq g_2 \subseteq \cdots \subseteq g = \bigcup_{n=1}^{\infty} g_n. \]

Let \( n \) be large enough that the root system of \( g_n \) contains \( \delta_1, \ldots, \delta_m \). Then the claim follows from [7, Lemma 8.4] and Lemma 4.1.

Let \((g, Z, \{\delta_1, \ldots, \delta_m\})\) be a FP triple. For each \( 0 \leq k \leq m \), we denote the subspace of \( g \)

\[ N_0^+ =: \sum_{i=1}^{k} g_{-\delta_i} + \sum_{i=k+1}^{m} g_{\delta_i}. \]

Let \( P_0 \) be the subspace of \( g \) defined by

\[ P_0 := Z + N_0^+ = Z + \sum_{i=1}^{m} g_{\delta_i}. \]

Then we have

**Lemma 3.7.** \( P_0 \) is a parabolic Lie color subalgebra of \( g \) with \( N_0^+ \) as its nilpotent radical.

**Proof.** Since \( \{\delta_1, \delta_2, \ldots, \delta_m\} \) is an additive subset of \( \Phi^+ \), \( N_0^+ \) is a nilpotent subalgebra of \( g \). Let \( \delta \in \Phi_1^+ \). By definition, we have

\[ \{\delta - \delta_1, \delta - \delta_2, \ldots, \delta - \delta_m\} = \{-\delta_1, -\delta_2, \ldots, -\delta_m\}, \]

so that \([f_\delta, N^+] \subseteq N_0^+\). Let \( \delta \in \Phi_1^+ \). Suppose \( \delta + \delta_i \notin \{\delta_1, \delta_2, \ldots, \delta_m\} \) for some \( 1 \leq i \leq m \). Then we get \( \delta + \delta_i \in \Phi_1^+ \), and hence \( \delta_i = (\delta + \delta_i) - \delta \in \Phi_1 \), contrary to the assumption that \( \Phi_1 \) is a subsystem of \( \Phi \). Then we conclude that \([e_\delta, N_0^+] \subseteq N_0^+\).

It follows that \([Z, N_0^+] \subseteq N_0^+\). Thus, \( P_0 \) is a parabolic subalgebra of \( g \) and \( N_0^+ \) is an ideal of \( P_0 \).

Let us denote \( P_0^- := Z + N_m^+ = Z + \sum_{i=1}^{m} g_{-\delta_i} \). Similarly one can prove that \( P_0^- \) is a parabolic Lie color subalgebra of \( g \) with \( N_m^+ \) as its nilpotent radical.

### 3.5 Induced modules associated with the FP triples

Let \( g \) be an algebraic Lie color algebra, and let \((g, Z, (\delta_i)_{i=1}^{m})\) be a FP triple. Assume \( \chi = \chi_s + \chi_n \in g_f \) is in the standard form, and assume \( \chi(g_{±\delta_i}) = 0 \) for all \( 1 \leq i \leq m \). Let \( M \) be a simple \( u_\chi(P_0) \) (resp. \( u_\chi(P_0^-) \)) module. Then by [6, Coro. 3.2], \( M \) is annihilated by \( N_0^+ \) (resp. \( N_m^+ \)).

**Definition 3.8.** Let \( g = N^- + H + N^+ \) be the triangular decomposition of \( g \), and let \( M \) be a \( g \)-module. Assume \( 0 \neq v \in M_\lambda \). If there exists \( \lambda \in H^* \) such that \( N^+ v = 0 \) and \( hv = \lambda(h)v \) for all \( h \in H \), then \( v \) is called a maximal vector with \( H \)-weight \( \lambda \).
Let $B = H + N^+$. Then each maximal vector $v$ defines a 1-dimensional $u_\chi(B)$-module with $\lambda$ satisfying $\lambda(h)^p - \lambda(h) = \chi(h)^p$, $h \in H$, and $v$ also defines an induced $u_\chi(P_0)$-module $u_\chi(P_0) \otimes_{u_\chi(B)} Fv$. We denote each simple quotient of $u_\chi(P_0) \otimes_{u_\chi(B)} Fv$ by $\mathcal{M}(\lambda)$.

Remark: $\mathcal{M}(\lambda)$ is a simple $u_\chi(P_0)$-module generated by a maximal vector $v$ of weight $\lambda$. $v$ may not be unique, so that $\lambda$ is not necessarily unique. But this does no harm to our discussions in the following. If $\chi$ is standard semisimple, one can easily show that $u_\chi(P_0) \otimes_{u_\chi(B)} Fv$ has a unique simple quotient $\mathcal{M}(\lambda)$, and $\mathcal{M}(\lambda)$ contains a unique maximal vector $v$(with $H$-weight $\lambda$).

Suppose $\mathcal{M}(\lambda)$ is a simple $u_\chi(P_0)$-module as that given above. We now define the induced module

$$Z^\chi(\mathcal{M}(\lambda)) = u_\chi(g) \otimes_{u_\chi(P_0)} \mathcal{M}(\lambda).$$

First, let $g$ be finite dimensional. Then each simple $u_\chi(g)$-module $M$ is finite dimensional. We have by using [6, Th.3.2] that each simple $u_\chi(P_0)$-submodule of $M$ contains a maximal vector, and hence in the form $\mathcal{M}(\lambda)$, so that $M$ is a homomorphic image of $Z^\chi(\mathcal{M}(\lambda))$.

Secondly, let $g = cgl(V)$ be infinite dimensional, and let $(g, Z, (\delta_i)_{i=1}^m)$ be the FP triple given in Example (3) of Sec 3.4, with $T_0$ given in Example 3.2. We now have

**Lemma 3.9.** Each simple $u_\chi(P_0)$-module $\mathcal{M}$ in Category $O$ contains a maximal vector.

**Proof.** Recall the definition of $T_0$. Let us denote $\underline{m} =: \sum_{i < N} m_{\alpha_i}$. From 3.2, we have

$$T_0 = \{(t_{ii}) | t_{ii} \in F^\times, t_{ii} = 1 \quad \text{for all} \quad i \leq \underline{m}\}.$$ 

Let us denote the weight lattice of $T_0$ by $\Lambda = \bigoplus_{i=m+1}^\infty \mathbb{Z} \epsilon_i$. For $\lambda = \sum n_i \epsilon_i \in \Lambda$, we define the $T_0$-height of $\lambda$ to be $|\lambda| = \sum n_i$. It should be noted that the weight of the positive root vector $e_{ij}$ relative to $T_0$ is

$$\text{wt}(e_{ij}) = \begin{cases} \epsilon_i - \epsilon_j, & \text{if} \quad \underline{m} < i < j \\ -\epsilon_j, & \text{if} \quad i \leq \underline{m} < j \\ 0, & \text{if} \quad i < j \leq \underline{m}. \end{cases}$$

Let $\mathcal{M}$ be a simple $u_\chi(P_0)$-module in the category $O$ of $u_\chi(P_0) - T_0$-modules. Assume $0 \neq u \in \mathcal{M}$ is a homogeneous weight vector. Since $\mathcal{M}$ is locally $N^+$-finite, $\mathcal{N} =: u(N^+)u \subseteq \mathcal{M}$ is a finite dimensional $u_\chi(B_0) - T_0$-submodule. Recall the $T_0$-action on $\mathcal{N}$ is given by

$$t(xu) = Adt(x)tu.$$ 

Then using the PBW theorem, with a given order for the positive root vector $e_{ij}'s$, $\mathcal{N}$ is spanned by a finite set of $H - T_0$-weight vectors:

$$\mathcal{S} =: \{\Pi_{i<j} e_{ij}^{k_{ij}} u | 0 \leq k_{ij} \leq \bar{p}\}. $$
Assume the $T_0$-weight of $u$ is $\lambda \in X(T_0)$. Then each vector $\Pi_{i<j} e_{ij}^k u$ has the $T_0$-weight
\[ \lambda + \sum_{m<i<j} k_{ij}(\epsilon_i - \epsilon_j) - \sum_{i \leq m<j} k_{ij} \epsilon_j \in \Lambda, \]
and the $T_0$-height $|\lambda| - \sum_{i \leq m<j} k_{ij}$.

Let $S_l$ be the subset of $S$ consisting of elements with the minimal $T_0$-height $l \in \mathbb{Z}$, and let $N_l$ be the subspace of $N$ spanned by $S_l$. Then it is easy to see that $e_{ij} S_l = 0$ for any $i \leq m < j$, and $e_{ij} N_l \subseteq N_l$ for any $m < i < j$ or $i < j \leq m$.

Each element in $S_l$ has the weight
\[ \lambda' + \sum_{m<i<j} k_{ij}(\epsilon_i - \epsilon_j), \quad |\lambda'| = l. \]

Let $S_l^h$ be the subset of $S_l$ consisting of elements with the largest number $\sum_{m<i<j} k_{ij}$, and let $N_l^h$ be the subspace spanned by $S_l^h$. Then we have $e_{ij} S_l^h = 0$ for all $m < i < j$, and $e_{ij} N_l^h \subseteq N_l^h$ for all $i < j \leq m$. This implies that $N_l^h$ is a $u(N^+_n)$-submodule, where $N^+_n$ is the sum of all positive root spaces of $\mathfrak{g}_n$. By [6, Th.3.2], there is a homogeneous weight vector $0 \neq v \in N_l^h$, such that $N^+_n \cdot v = 0$, and hence $N^+ \cdot v = 0$. Thus, $v \in \mathcal{M}$ is a maximal vector.

By the lemma, each simple $u_{\chi}(P_0)$-module $\mathcal{M}$ in Category $\mathcal{O}$ is a homomorphic image of the induced $u_{\chi}(P_0)$-module defined above, and hence each simple $u_{\chi}(\mathfrak{g})$-module in Category $\mathcal{O}$ is a homomorphic image of the induced module $Z^\chi(\mathcal{M}(\lambda))$.

### 3.6 The simplicity of $Z^\chi(\mathcal{M}(\lambda))$

With the same assumption as above. Recall $\mathcal{N}_m^+ = \oplus_{i=1}^m \mathfrak{g}_{-\delta_i}$. Suppose $\delta_i, \ldots, \delta_{im}$ is the roots of $\mathcal{N}_m^+$ in the order of increasing heights. By Th. 2.13, $u(\mathcal{N}_m^+)$ has a basis
\[ \{ f_{\delta_1}^{a_1} \cdots f_{\delta_m}^{a_m} | 0 \leq a_i \leq \hat{p} - 1 \}. \]
For the convenience, we denote $f_{\delta_1}^{a_1} \cdots f_{\delta_m}^{a_m} \in u(\mathcal{N}_m^+)$ by $[a_1, a_2, \ldots, a_m]$, and
\[ f_{\delta_1}^{a_1} \cdots f_{\delta_k}^{a_k} f_{\delta_j} f_{\delta_{k+1}}^{a_{k+1}} \cdots f_{\delta_m}^{a_m} \in u(\mathcal{N}_m^+) \]
by $[a_1, \ldots, a_{k-1}, f_{\delta_j}, \ldots]$. 

**Lemma 3.10.** Let $[a_1, \ldots, a_m] \in u(\mathcal{N}_m^+)$ and let $1 \leq s \leq m$. If $a_{ij} = \hat{p} - 1$ for all $j \geq s$, then $[a_1, \ldots, a_{k-1}, f_{\delta_i}, \ldots] = 0$ for any $1 \leq k \leq m + 1$.

**Proof.** Let us note that if $\mathfrak{g}_\delta \subseteq \mathfrak{g}_\alpha$ and if $\alpha \in \Gamma^-$, then it is possible that $2\delta$ is also a root of $\mathfrak{g}$ (see [16, p.51]). In this case $2\delta$ is a root of $\mathfrak{g}^+$ with the height greater than that of $\delta$. Let $f_{\delta} \in \mathfrak{g}_{-\delta}$. Then we have $f_{\delta}^2 = \frac{1}{2} [f_{\delta}, f_{\delta}] \in \mathfrak{g}_{-2\delta}$. 

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In \( u(\mathcal{M}_m^+) \), we have

\[ (*) \quad [f_{\delta_1}, f_{\delta_2}] = f_{\delta_1} f_{\delta_2} - (f_{\delta_1}, f_{\delta_2}) f_{\delta_1} f_{\delta_2} = \begin{cases} cf_{\delta_i + \delta_j}, & \text{if } \delta_i + \delta_j \text{ is a root} \\ 0, & \text{otherwise}, \end{cases} \]

where \( c \in F^\times \). We shall proceed with induction on \( s \). Suppose \( s = m \). Then since \( \delta_m \) has the greatest height, \([f_{\delta_m}, f_{\delta}] = 0\) for all \( 1 \leq j \leq m \). The fact that \( f_{\delta_m}^2 = 0 \) gives

\[ [a_1, \ldots, a_{k-1}, f_{\delta_m}, \ldots] = 0, \]

for any \( 1 \leq k \leq m + 1 \) whenever \( a_{im} = \bar{p} - 1 \).

Suppose \([a_1, \ldots, a_m] \in u(\mathcal{M}_m^+) \) with \( a_j = \bar{p} - 1 \) for all \( j \geq s \) and \( s < m \). For any \( 1 \leq k \leq m + 1 \), if \( \delta_s = \delta_k \), then

\[ [a_1, \ldots, a_{k-1}, f_{\delta_i}, \ldots] = [a_1, \ldots, a_{k-1}, \bar{p}, \ldots] \]

\[ = \begin{cases} c[a_1, \ldots, a_{k-1}, f_{2\delta_k}, \bar{p} - 2, \ldots], & \text{if } 2\delta_k \text{ is a root} \\ 0, & \text{otherwise}. \end{cases} \]

Since the height of \( 2\delta_k \) is greater than that of \( \delta_k \), the induction hypothesis yields

\[ [a_1, \ldots, a_{k-1}, f_{\delta_i}, \ldots] = 0. \]

The case \( \delta_s = \delta_{k-1} \) is similar.

Now suppose \( \delta_s \notin \{\delta_{k-1}, \delta_k\} \). By repeated applications of the formula \((*)\) and the induction assumption, we have

\[ [a_1, \ldots, a_{k-1}, f_{\delta_j}, \ldots] = c[a_1, \ldots, a_{s-1}, f_{\delta_j}, \ldots] \]

for some \( c \in F \). Then the discussion above applied, we get \([a_1, \ldots, a_{k-1}, f_{\delta_j}, \ldots] = 0. \]

\[ \square \]

It follows from the lemma that

\[ f_{\delta_i} [\bar{p} - 1, \ldots, \bar{p} - 1] = 0 \quad [\bar{p} - 1, \ldots, \bar{p} - 1] f_{\delta_i} = 0, \]

for all \( 1 \leq i \leq m \).

Recall the induced module \( Z^\chi(\mathcal{M}(\lambda)) \). Let \( v \in \mathcal{M}(\lambda) \) be a maximal vector of \( H \)-weight \( \lambda \in H^* \). By applying the Harish-Chandra homomorphism, we get

\[ e^{\bar{p} - 1}_{\delta_1} \ldots e^{\bar{p} - 1}_{\delta_m} f^{\bar{p} - 1}_{\delta_1} \ldots f^{\bar{p} - 1}_{\delta_m} \otimes v = f(h) \otimes v = 1 \otimes f(\lambda(h)) v, \]

where

\[ f(h) = \gamma(e^{\bar{p} - 1}_{\delta_1} \ldots e^{\bar{p} - 1}_{\delta_m} f^{\bar{p} - 1}_{\delta_1} \ldots f^{\bar{p} - 1}_{\delta_m}) \in u(\lambda(H)). \]

**Theorem 3.11.** \( Z^\chi(\mathcal{M}(\lambda)) \) is simple if and only if \( f(\lambda(h)) \neq 0 \).
Proof. Let $\mathcal{N}$ be a simple submodule of $Z^\chi(\mathcal{M}(\lambda))$. Let us take

$$0 \neq v = \sum f_{\delta_1}^{k_1} \cdots f_{\delta_m}^{k_m} \otimes m_{k_1, \ldots, k_m} \in \mathcal{N},$$

where each $m_{k_1, \ldots, k_m} \in \mathcal{M}(\lambda)$ is homogeneous. Let $\delta_1, \ldots, \delta_m$ be in the order of increasing heights. Using Lemma 3.10 and applying $f_{\delta_{im}}, \ldots, f_{\delta_{1m}}$ repeatedly, we get

$$0 \neq f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1} \otimes m \in \mathcal{N}, \text{ with } m \text{ homogeneous.}$$

Then it follows that

$$e_{\delta_1}^{p-1} \cdots e_{\delta_m}^{p-1} f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1} \otimes m \in \mathcal{N}.$$  

By the assumption $\chi(\mathfrak{g}_{\pm\delta_i}) = 0, 1 \leq i \leq m$, we can regard $e_{\delta_1}^{p-1} \cdots e_{\delta_m}^{p-1} f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1} \in u_\chi(\mathfrak{g})$ as an element in some $u_{\chi'}(\mathfrak{g})$ with $\chi'$ being standard semisimple, so that we can apply the Harish-Chandra homomorphism $\gamma$. Using the fact that $e_\delta \cdot \mathcal{M}(\lambda) = 0$, $i = 1, \ldots, m$ and applying $\gamma$, we get $1 \otimes f(\lambda(h))m \in \mathcal{N}$. Then the assumption $f(\lambda(h)) \neq 0$ says that $1 \otimes m \in \mathcal{N}$, hence $\mathcal{M}(\lambda) \subseteq \mathcal{N}$. Thus, we get $\mathcal{N} = Z^\chi(\mathcal{M}(\lambda))$, so that $Z^\chi(\mathcal{M}(\lambda))$ is simple.

Let $(\delta_i)_{i \in I}$ be the positive root system of $\mathcal{Z}$. Then $(\delta_i)_{i \in I} \cup \{\delta_1, \ldots, \delta_m\}$ is a positive root system of $\mathfrak{g}$. For any $j \in I$, if $[f_{\delta_j}, f_{\delta_i}] \neq 0$, then it must be a nonzero multiple of $f_{\delta_i + \delta_j}$. The height of $\delta_i + \delta_j \in \{\delta_1, \ldots, \delta_m\}$ is greater than that of $\delta_i$. Using the formula for $adx|_{U(\mathfrak{g})}$ in 2.3 and Lemma 3.10, we get $[f_{\delta_j}, f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1}] = 0$. If $[e_{\delta_j}, f_{\delta_i}] \neq 0$, then it must be a multiple of $f_{\delta_i - \delta_j}$ with $\delta_i - \delta_j \in \{\delta_1, \ldots, \delta_m\}$, since $\mathcal{N}_m^+$ is an ideal of $P_0^+ = \mathcal{Z} \oplus \mathcal{N}_m^+$. Applying Lemma 3.10, we get

$$[e_{\delta_j}, f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1}] = 0.$$  

This implies that $\mathcal{M}_0 = f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1} \otimes \mathcal{M}(\lambda)$ is a $\mathcal{Z}$-submodule of $Z^\chi(\mathcal{M}(\lambda))$. By Lemma 3.10, $\mathcal{M}_0$ is annihilated by $\mathcal{N}_m^+$, so it is a $u_\chi(P_0^-)$-submodule. Recall $\mathcal{N}_0^+ = \oplus_{i=1}^m \mathfrak{g}_{\delta_i}$. The simplicity of $Z^\chi(\mathcal{M}(\lambda))$ says that

$$u(\mathcal{N}_0^+).\mathcal{M}_0 = u(\mathcal{N}_0^+).u_\chi(P_0^-)\mathcal{M}_0 = u_\chi(\mathfrak{g})\mathcal{M}_0 = Z^\chi(\mathcal{M}(\lambda)).$$

Let $(v_i)_{i \in I}$ be a basis of $\mathcal{M}_0$. Then

$$\{e_{\delta_i}^{k_1} \cdots e_{\delta_m}^{k_m} v_i | 0 \leq k_i \leq p-1, i \in I\}$$

is a basis of $Z^\chi(\mathcal{M}(\lambda))$. Taking $f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1} \otimes v \in \mathcal{M}_0$, where $0 \neq v \in \mathcal{M}(\lambda)$ is a maximal vector of $H$-weight $\lambda$, we get

$$0 \neq e_{\delta_1}^{p-1} \cdots e_{\delta_m}^{p-1} f_{\delta_1}^{p-1} \cdots f_{\delta_m}^{p-1} \otimes v = 1 \otimes f(\lambda(h))v,$$

so that $f(\lambda(h)) \neq 0$. \hfill $\square$

### 3.7 The determination of $f(\lambda(h))$

Recall $\mathcal{N}_k^+ = \sum_{i=1}^k \mathfrak{g}_{-\delta_i} + \sum_{i=k+1}^m \mathfrak{g}_{\delta_i}, 0 \leq k \leq m$. 

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Lemma 3.12. With the same assumption as in 3.6. For each \(0 \leq k \leq m\), we define a subspace of \(Z^\chi(M(\lambda))\) by

\[
M_k =: f_{\delta_k}^{p^k} \cdots f_{\delta_1}^{p^1} \otimes M(\lambda).
\]

If \(2\delta \not\in \{\delta_1, \ldots, \delta_m\}\) for any \(\delta \in \{\delta_1, \ldots, \delta_m\}\), then \(M_k\) is annihilated by \(M_k^+\).

Proof. We proceed with induction on \(k\). The case \(k = 0\) is given by the definition. Assume the claim is true for \(M_{k-1}\) with \(k \geq 1\). By definition, \(M_k = f_{\delta_k}^{p^k-1}M_{k-1}\). Using Lemma 3.10 and the assumption that \(\{-\delta_1, \ldots, -\delta_k\}\) is a closed system, we get \(f_{\delta_k}M_k = 0\) for all \(i = 1, \ldots, k\).

Suppose \(\delta = \delta_i\) and \(i \geq k+1\). Then \(\delta\) is also a positive root of \(M_k^+\) and \(\delta \neq \delta_k\), so it suffices to assume \([e_{\delta}, f_{\delta_k}] \neq 0\). Since \(\delta_k\) is simple in the positive root system \(\Phi_{k-1}^+, \delta - \delta_k\) a positive root for \(\Phi_{k-1}^+\). Then we must have \([e_{\delta}, f_{\delta_k}] = ce_{\delta-\delta_k}\) for some \(c \in \mathbb{F}^\times\).

If \(\delta_k\) is a root for \(g^+\), by assumption we get \(\delta - \delta_k \in \Phi_{k-1}^\pm \setminus \{\delta_k\}\). Then since \(\delta_k\) is simple, we have for any \(i \in \mathbb{N}\), \(\delta - i\delta_k \in \Phi_{k+1}^\pm \setminus \{\delta_k\}\) if \(\delta - i\delta_k \in \Phi\), and so the induction assumption yields

\[
eq [e_{\delta}, f_{\delta_k}]f_{\delta_k}^{p-2} + \cdots + f_{\delta_k}^{p-2}e_{\delta-\delta_k})M_{k-1} = 0.
\]

If \(\delta_k\) is a root for \(g^-\), then we have \(p = 1\), so that

\[
eq e_{\delta} \cdot M_k = e_{\delta} f_{\delta_k} M_{k-1}
\]

where the last equality is given by induction hypothesis. \(\square\)

Let \(A = \bigoplus_{\alpha \in \Gamma} A_{\alpha}\) be a finite dimensional Frobenious algebra. We use the notation \(A_L(\text{resp. } A_R)\) to denote the left(\text{resp. right}) \(\Gamma\)-graded regular \(A\)-module \(A\). For each \(\alpha \in \Gamma\), let

\[
A^*_\alpha = \{ f \in A^*| f(A_\beta) = 0, \text{for all } \beta \neq \alpha\}.
\]

Then we have \(A^* = \bigoplus_{\alpha \in \Gamma} A^*_\alpha\). It is easy to check that \(A^*_L(\text{resp. } A^*_R)\) is a right(\text{resp. left}) \(\Gamma\)-graded \(A\)-module.

Lemma 3.13. [4, Th. 61.3] For a finite dimensional \(\Gamma\)-graded algebra \(A\), the following are equivalent.

1. \(A\) is Frobenious.
2. \(A_L \cong A^*_R, A_R \cong A^*_L\).

Definition 3.14. [6] Let \(g\) be a restricted Lie color algebra. We call \(g\) unipotent if for every \(x \in g_\alpha\) with \(\alpha \in \Gamma^+\), there exists \(r > 0\) such that \(x^{[p]^r} = 0\).
Proposition 3.15. Let $\mathfrak{g} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a finite dimensional unipotent Lie color algebra. Then the left (resp. right) regular $u(\mathfrak{g})$-module $u(\mathfrak{g})$ has a unique 1-dimensional trivial submodule.

Proof. Let $x \in \mathfrak{g}_\alpha$ with $\alpha \in \Gamma^+$. Then we have $(x^n - x^{[\beta]})u(\mathfrak{g}) = 0$. Since $\mathfrak{g}$ is unipotent, there is $n \in \mathbb{Z}^+$ such that $x^{[\beta]^n} = 0$. Hence we get $x^n u(\mathfrak{g}) = 0$. If $x \in \mathfrak{g}_\alpha$, $\alpha \in \Gamma^-$, then we get $x^2 = \frac{1}{2} [x, x] \in \mathfrak{g}_{2\alpha}$, so $x$ is also nilpotent in $u(\mathfrak{g})$, since $2\alpha \in \Gamma^+$. By [6, Th. 3.2], $u(\mathfrak{g})$ has a 1-dimensional trivial submodule $Fv$.

To show the uniqueness, we let $f$ be the nondegenerate invariant bilinear form on $u(\mathfrak{g})$. Then we get $f(x, v) = f(1, xv) = 0$ for all homogeneous $x \in \mathfrak{g}$, so that $v$ is in the homogeneous right orthogonal complement of $u(\mathfrak{g})g$. Since $u(\mathfrak{g})\mathfrak{g}$ has codimension 1 in $u(\mathfrak{g})\mathfrak{g}$, its homogeneous orthogonal complement is 1-dimensional. This implies that $u(\mathfrak{g})$ has a unique trivial submodule $Fv$. \qed

Proposition 3.16. [13] Let $\mathfrak{g}$ be a finite dimensional unipotent Lie color algebra, and let $\chi = \{ \chi_\alpha \mid \alpha \in \Gamma^+ \}$ be a $p$-character of $\mathfrak{g}$. Then $u_\chi(\mathfrak{g})$ has only one (up to isomorphism) simple module.

Proof. Let $M = \bigoplus_{\alpha \in \Gamma} M_\alpha$ and $M' = \bigoplus_{\alpha \in \Gamma} M'_\alpha$ be two simple $u_\chi(\mathfrak{g})$-modules. By [1, Lemma 2.5], we have $\dim M, \dim M' < \infty$. For any $\alpha \in \Gamma$, let

$$\text{Hom}_F(M, M')_\alpha = \{ f \in \text{Hom}_F(M, M') \mid f(M_\beta) \subseteq M'_{\beta + \alpha} \}.$$

Then $\text{Hom}_F(M, M') = \bigoplus_{\alpha \in \Gamma} \text{Hom}_F(M, M')_\alpha$. $\text{Hom}_F(M, M')$ is a $\mathfrak{g}$-module with the $\mathfrak{g}$-action defined by ([6])

$$(x \cdot f)(m) = x \cdot f(m) - (\bar{x} | f) f(x \cdot m)$$

for all homogeneous $x$ and $f$. Then one can easily check that $\text{Hom}_F(M, M')$ is a restricted $\mathfrak{g}$-module.

For each $x \in \mathfrak{g}_\alpha$, by the proof of Proposition 3.15, we see that $x$ acts nilpotently on $\text{Hom}_F(M, M')$. A straightforward computation shows that $\text{Hom}_F(M, M')_0$ is a $u(\mathfrak{g})$-submodule of $\text{Hom}_F(M, M')$. By [6, Th.3.2], $\text{Hom}_F(M, M')_0$ contains an 1-dimensional trivial submodule $Ff$. By definition, $f$ is a homomorphism of $\mathfrak{g}$-modules. Then the simplicity of both $M$ and $M'$ implies that $f$ is an isomorphism. Thus, $M \cong M'$. \qed

Lemma 3.17. Let $Fv_L$ (resp. $Fv_R$) be the unique trivial $u(\mathfrak{g})$-submodule of $u(\mathfrak{g})_L$ (resp. $u(\mathfrak{g})_R$). Then $v_L = cv_R$, for some $c \in F^\times$.

Proof. By Lemma 2.19, $u(\mathfrak{g})$ is a symmetric algebra. Let $f$ be a nondegenerate invariant bilinear form on $u(\mathfrak{g})$. The symmetry of $f$ implies that both $v_L$ and $v_R$ are in the right homogeneous orthogonal complement of $u(\mathfrak{g})\mathfrak{g}$, which is 1-dimensional. Then we get $v_L = cv_R$ for some $c \in F^\times$. \qed
Proof. Since $u$ submodule of the $\Gamma$-graded regular $c$ for some following Lemma 3.10, $F \{ f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} \}$ is a 1-dimensional trivial submodule of the $\Gamma$-graded regular $u(\mathcal{N}_m^+) \text{-module} u(\mathcal{N}_m^+)_{L}$ (resp. $u(\mathcal{N}_m^+)_{R}$), so we get

$$f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} = c f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1}$$

for some $c \in F^\times$.

**Lemma 3.18.** Let $\mathfrak{g} = \oplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha$ be a Lie color algebra, and let $M = \oplus_{\alpha \in \Gamma} M_\alpha$ be a $\mathfrak{g}$-module. We take $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ and assume $[e, f] = h$. Let $v \in M$ such that $e \cdot v = 0$.

1. If $(\alpha, \alpha) = -1$, then $efv = (fe + h)v$.

2. If $(\alpha \mid \alpha) = 1$, then $e^l f^l v = l! \Pi_{i=0}^{l-1} (h - i)v$

for every $l > 1$.

3. If $h \cdot v = \lambda(h)v$, for some $\lambda(h) \in F$, then $e^{\rho-1} f^{\rho-1} v = (\rho - 1)! \Pi_{i=0}^{\rho-2} (\lambda(h) - i)v$.

**Proof.** (1) is obvious. (2) is given by induction on $l$.

(3) From (1) and (2), we get

$$e^{\rho-1} f^{\rho-1} v = (\rho - 1)! \Pi_{i=0}^{\rho-2} (\lambda(h) - i)v.$$ 

Then the claim follows from the identity

$$x^{\rho-1} - 1 = \Pi_{i=1}^{\rho-1} (x - i).$$

**Theorem 3.19.** Let $f(\lambda(h))$ be the polynomial given in Sec. 3.6. Assume $2\delta \notin \{\delta_1, \ldots, \delta_m\}$, for any $\delta \in \{\delta_1, \ldots, \delta_m\}$. Then we have

$$f(\lambda(h)) = c \Pi_{i=1}^{m}[\lambda_i(H_{\delta_i}) + 1]\rho-1 - 1$$

for some $c \in F^\times$, where $\lambda_i = \lambda - [(\rho - 1)\delta_1 + \cdots + (\rho - 1)\delta_{i-1}], 1 \leq i \leq m$.

**Proof.** Since $f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} = k f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} \in u(\mathcal{N}_m^+)$, $k \in F^\times$, we get

$$e_{\delta_1}^{\rho-1} \cdots e_{\delta_m}^{\rho-1} f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} \otimes v$$

$$= ke_{\delta_1}^{\rho-1} \cdots e_{\delta_m}^{\rho-1} f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} \otimes v$$

$$= k e_{\delta_1}^{\rho-1} \cdots e_{\delta_m}^{\rho-1} (e_{\delta_m}^{\rho-1} f_{\delta_m}^{\rho-1}) f_{\delta_m}^{\rho-1} \cdots f_{\delta_1}^{\rho-1} \otimes v$$

(Using Lemma 3.12 and Lemma 3.18)

$$= k(\rho - 1)! [(\lambda_m(H_{\delta_m}) + 1)\rho-1 - 1] e_{\delta_1}^{\rho-1} \cdots e_{\delta_m}^{\rho-1} f_{\delta_1}^{\rho-1} \cdots f_{\delta_m}^{\rho-1} \otimes v$$

$$= \cdots = c \Pi_{i=1}^{m}[\lambda_i(H_{\delta_i}) + 1]^{\rho-1} - 1] v,$$

c \in F^\times. Thus, the claim holds. 

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4 Applications

In this section, we consider the applications of Th. 3.11 and Th. 3.19 to the Lie color algebra \( g = \text{cgl}(V) \). We prove an analogue of Kac-Weisfeiler theorem. Then we determine the condition for the baby Verma module to be simple.

4.1 The Kac-Weisfeiler theorem

Let \( g = \text{cgl}(V) \). The \( \mathbb{F} \)-vector space \( \text{cgl}(V) \) also has a natural \( \Gamma \)-graded restricted Lie algebra structure with the Lie product defined by \([x, y] = xy - yx, x, y \in \bigcup_{\alpha \in \Gamma} \text{cgl}(V)_\alpha\). We denote this Lie algebra by \( \text{cgl}(V)^- \). Recall that \( g \) has a filtration 
\[ g_1 \subseteq g_2 \subseteq \cdots \subseteq \bigcup_{i=1}^{\infty} g_i = g \]
in case \( g \) is infinite dimensional. Then the underlining \( \mathbb{F} \)-vector subspace of each \( g_n \) becomes a Lie subalgebra of \( \text{cgl}(V)^- \), denoted \( g_n^- \).

Recall the linear algebraic group \( G \) with \( \text{Lie}(G) = g \bar{0} \). Then \( G \) is also a group of automorphisms for \( \text{cgl}(V)^- \) which keeps both the \( \Gamma \)-grading and the \( p \)-mapping. Moreover, \( \text{cgl}(V)^- \) has the same root space decomposition relative to \( T \) with that of \( \text{cgl}(V) \). i.e., \( \Phi^+ \) (resp. \( \Delta \)) in 3.1 is the positive (resp. simple) root system for both \( \text{cgl}(V)^- \) and \( \text{cgl}(V) \). Recall the notion \( e_\delta, f_\delta \) for \( \delta \in \Phi^+ \).

We use \([,]_\Gamma\) (resp. \([,]\)) momentarily to denote the Lie color product (resp. Lie product) in \( \text{cgl}(V) \). Then it is easy to see that
\[ [e_\delta, e_\delta]_\Gamma = c_{ij}[e_\delta, e_\delta], [f_\delta, f_\delta]_\Gamma = c_{ij}[f_\delta, f_\delta] \quad \text{for any} \quad \delta_i, \delta_j \in \Phi^+ \]
and
\[ [e_\delta, f_\delta]_\Gamma = k_{ij}[e_\delta, f_\delta] \quad \text{whenever} \quad \delta_i \neq \delta_j, \]
where \( c_{ij}, k_{ij} \in \mathbb{F}^\times \).

4.1.1 The reflections

Recall the simple roots \( \Delta \) for \( \Phi^+ \):
\[ \Delta = \{ \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{n-1} - \epsilon_n, \ldots \} \]
We define a real vector space \( V \) having a basis \( \Delta \). Then \( V \) is infinite dimensional if \( \Gamma \) is infinite. For any \( \delta_i, \delta_j \in \Delta \), we let
\[ \langle \delta_i, \delta_j \rangle = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } i \in \{ j - 1, j + 1 \} \\ 0, & \text{otherwise.} \end{cases} \]
For each $\delta \in \Delta$, we define a linear function $\langle -, \delta \rangle \in V^*$ by
\[
\langle \beta, \delta \rangle = \sum_i c_i \langle \delta_i, \delta \rangle \quad \text{if} \quad \beta = \sum_i c_i \delta_i \in V.
\]

For each $\delta \in \Delta$, we define a reflection on $V$ by
\[
\tau_\delta(\beta) = \beta - \langle \beta, \delta \rangle \delta.
\]
It is easy to see that $\tau_\delta(\delta) = -\delta$, and $\tau_\delta(\beta) = 0$ for any $\beta \in V$ with $\langle \beta, \delta \rangle = 0$. The group $W$ generated by all $\tau_\delta \delta \in \Delta^+$ is called the Weyl group of the Lie algebra $\operatorname{cgl}(V)^-$. Then $W$ is an infinite group in case $\Gamma$ is infinite.

Let $g = \operatorname{cgl}(V)$ be infinite dimensional. Then the Lie algebra $\operatorname{cgl}(V)^-$ has a filtration
\[
g_1^- \subseteq g_2^- \subseteq \cdots \subseteq \bigcup_{i=1}^{\infty} g_i^- = \operatorname{cgl}(V)^-.
\]

Let us denote the root system (resp. positive root system, simple root system, Weyl group) of $g_n$ by $\Phi_n$ (resp. $\Phi_n^+$, $\Delta_n$, $W_n$). Let $V_n$ (resp. $V$) denote the real vector space spanned by $\Phi_n^+$ (resp. $\Phi$). Then we get
\[
\Phi = \bigcup_{n=1}^{\infty} \Phi_n, \quad \Phi^+ = \bigcup_{n=1}^{\infty} \Phi_n^+ \quad \Delta = \bigcup_{n=1}^{\infty} \Delta_n.
\]

By extending the action of each $\sigma \in W_n$ on $V_n$ to that of $V$, we can identify $W_n$ as a subgroup of $W$. Then we get $W = \bigcup_{n=1}^{\infty} W_n$.

Let $\sigma \in W$ and assume $\sigma = \tau_{\delta_1} \cdots \tau_{\delta_i}\delta_1,\ldots,\delta_i \in \Delta$. Let $N$ be large enough that $\Delta_N$ contains each $\delta_i$. Then for each $n \geq N$, we have $\sigma_{\mid V_n} \in W_n$, and hence $\sigma(\Phi_n) \subseteq \Phi_n$. This gives $\sigma(\Phi) = \Phi$, and hence $\sigma(\Phi^+)$ is a system of positive roots of $\operatorname{cgl}(V)^-$ with the set of simple roots $\sigma(\Delta)$.

**Lemma 4.1.** For each $\sigma \in W$, $\sigma(\Phi^+)$ is a system of positive roots of the Lie color algebra $g = \operatorname{cgl}(V)$ with the set of simple roots $\sigma(\Delta)$.

**Proof.** Since $\sigma(\Phi^+)$ is a positive root system of the Lie algebra $\operatorname{cgl}(V)^-$ with simple roots $\sigma(\Delta)$, $\sigma(\Delta)$ is a minimal subset $\sigma(\Phi^+)$ satisfying:

1. $\{e_{\gamma} \mid \gamma \in \sigma(\Delta)\}$ (resp. $\{f_{\gamma} \mid \gamma \in s_{\alpha}(\Delta)\}$) generates $\{e_{\gamma} \mid \gamma \in \sigma(\Phi^+)\}$ (resp. $\{f_{\gamma} \mid \gamma \in \sigma(\Phi^+)\}$).

2. $\{e_{\gamma}, f_{\gamma} \mid \gamma \in \sigma(\Delta)\}$ generates $\operatorname{cgl}(V)^-$.

3. $[e_{\alpha}, f_{\beta}] = \delta_{\alpha,\beta} h \in H$, for any $\alpha, \beta \in \sigma(\Delta)$.

Since each $e_{\gamma}$ above is also a root vector for the Lie color algebra $g = \operatorname{cgl}(V)$, the formulas preceding 4.1.1 implies that $\sigma(\Phi^+)$ is a positive root system of $\operatorname{cgl}(V)^-$ with the set of simple roots $\sigma(\Delta)$.

Let $g = \operatorname{cgl}(V)$ and let $\chi \in g_0^*$ be in the standard form. We consider both the case $g$ is finite dimensional and the case $g$ is infinite dimensional. Let
\[
Z = c_\phi(\chi_s) = \bigoplus_{\alpha \in \Gamma} Z_\alpha,
\]
where $Z_\alpha = \{ x \in g_\alpha | \chi_\alpha([x, -]) = 0 \}$. Then by Example (1), (3) of 3.4, we get a resulted FP triple $(g, Z_\alpha, (\delta_i)_{i=1}^m)$. Recall the parabolic color subalgebra $P_0$ having

$$N_0^+ = \sum_{\delta \in \Phi^+ : \chi(H_\delta) \neq 0} g_\delta = \bigoplus_{i=1}^m g_{\delta_i},$$

as its nilradical.

Recall the definition of $Z^\chi(M(\lambda))$ and $f(\lambda(h))$. Since $2\delta \notin \Phi^+$ for every $\delta \in \Phi^+$, we get by Th. 3.19 that

$$f(\lambda(h)) = c\Pi_{i=1}^m [(\lambda_i(H_{\delta_i}) + 1)^{p-1} - 1], \quad c \in F^\times.$$

For each $1 \leq i \leq m$, we have $H_{\delta_i}^{[p]} = H_{\delta_i}$. This implies that

$$\lambda(H_{\delta_i})^p - \lambda(H_{\delta_i}) = \chi(H_{\delta_i})^p \neq 0,$$

and hence $\lambda(H_{\delta_i}) \notin F_p$. Thus $f(\lambda(h)) \neq 0$. Then we get from Th. 3.11 that

Corollary 4.2. (Kac-Weisfeiler Theorem) Let $g = cgl(V)$. With the assumption as above, then $Z^\chi(M(\lambda))$ is a simple $u\chi(g)$-module.

4.2 The simplicity of the baby Verma module

In this subsection, we consider the simplicity of the baby Verma module for the algebraic Lie color algebra $g = cgl(V)$.

4.2.1 $g$ is finite dimensional

Let $\Gamma$ be finite. Then $cgl(V)$ is the general linear Lie color algebra $g = gl(m, \Gamma)$. Recall the triangular decomposition of $g$: $g = N^+ \oplus H \oplus N^-$. We assume $\chi \in g_0^*$ is standard semisimple. Then by Example (2) of 3.4, there is a resulted FP triple: $(g, H, \Phi^+)$. In the present situation, the parabolic subalgebra $P_0$ is just the Borel subalgebra $B = H \oplus N^+$, and $N_0^+ = N^+$ is its nilradical. By [6, Th. 3.2], each simple $u\chi(B)$-module $M(\lambda)$ is 1-dimensional and defined as follows:

$$M(\lambda) = Fv, N^+v = 0, h \cdot v = \lambda(h)v \text{ for every } h \in H,$$

where $\lambda(h) \in H^*$ satisfies $\lambda(H_{\delta_i})^p - \lambda(H_{\delta_i}) = \chi^p(H_{\delta_i})$ for any $\delta_i \in \Phi^+$. Recall the induced module $Z^\chi(M(\lambda)) =: u\chi(g) \otimes_{u\chi(B^+)} Fv$. We call it the baby Verma module with character $\chi$ and denote it by $Z^\chi(\lambda)$. Then by Th. 3.11, we get

Corollary 4.3. $Z^\chi(\lambda)$ is simple if and only if $f(\lambda) \neq 0$. 

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Note: In the special case that $g$ is a finite dimensional semisimple Lie algebra, a conclusion analogous to Corollary 4.3 was proved by Rudakov in a different approach [17].

By Th. 3.19, we get $f(\lambda) = c \Pi_{\delta \in \Phi^+} [(\lambda_i(H_{\delta_i}) + 1)^{p_i} - 1]$, $c \in F^*$. This formula enables us to conclude with

**Corollary 4.4.** Let $g = gl(m, \Gamma)$ and assume that $\chi \in g^+_0$ is regular semisimple, that is, $\chi(H_{\delta}) \neq 0$ for every $\delta \in \Phi^+$. Then the baby Verma module $Z^\chi(\lambda)$ is simple.

### 4.2.2 $g$ is infinite dimensional

We now let $g = cgl(V)$ be infinite dimensional. Recall the triangular decomposition of $g$ in 3.1: $g = N^+ \oplus H \oplus N^-$. Let $B$ denote the Borel subalgebra $H \oplus N^+$. Assume $\chi \in g^+_0$ is standard semisimple. We take $T_0$ as that in the Example 3.2. By Lemma 3.9, any simple $u_\chi(B) - T_0$ module in Category $\mathcal{O}$ is 1-dimensional. Let us denote it by $F\nu$ and assume $\lambda \in H^*$ is the $H$-weight of $\nu$. Then the $(\Gamma$-graded$)$baby Verma module is defined as

$$Z^\chi(\lambda) = u_\chi(g) \otimes_{u_\chi(B)} F\nu.$$ 

It is easy to see that $Z^\chi(\lambda)$, as a $u_\chi(g) - T_0$ module, is also an object in Category $\mathcal{O}$.

Recall the finite dimensional Lie color subalgebra $g_n$ for each $n \in Z^+$. Let $H_n$ be its maximal torus consisting of diagonal matrices, and let $\delta_1, \ldots, \delta_{k_n}$ be the set of all its positive roots. By the arguments in [7, Lemma 8.4] and Lemma 4.1, the set of roots can be ordered such that $(g_n, H_n, \delta_1, \ldots, \delta_{k_n})$ is a FP triple. Then we obtain a polynomial:

$$f_n(\lambda(h)) = \Pi_{i=1}^{k_n} [\lambda_i(H_{\delta_i}) + 1]^{p_i} - 1].$$

**Theorem 4.5.** $Z^\chi(\lambda)$ is simple if and only if $f_n(\lambda(h)) \neq 0$ for all $n \geq 1$.

**Proof.** Assume $f_n(\lambda(h)) \neq 0$ for all $n \geq 1$. Let $\mathcal{N} \subseteq Z^\chi(\lambda)$ be a simple $u_\chi(g)$-submodule. Let us take an element

$$0 \neq \sum f_{l_1}^{l_1} \cdots f_{l_k}^{l_k} \otimes v \in \mathcal{N}, \quad 0 \leq l_i \leq \bar{p}.$$ 

Let $n$ be large enough that the root system of $g_n$ contains all $\delta_i$'s appeared in the summation. Then by applying a similar argument as that used in the proof of Th. 3.11, we get $1 \otimes f_n(\lambda(h))v \in \mathcal{N}$. This gives $1 \otimes v \in \mathcal{N}$, and hence $\mathcal{N} = Z^\chi(\lambda)$. Thus, $Z^\chi(\lambda)$ is simple.

On the other hand, suppose $f_n(\lambda(h)) = 0$ for some $n$. Then by Coro. 4.3, the baby Verma module $Z_n^\chi(\lambda)$ for the Lie color subalgebra $g_n$ fails to be simple. Here $\chi$ (resp. $\lambda$) is the restriction of that for $g$ (resp. $H$) to $g_n$ (resp. $H_n$). Let us take the parabolic subalgebra $P_n = g_n + B$. We can write $P_n$ in the form

$$P_n = g_n \oplus H_n^c \oplus N(n),$$
where $N^{(n)}$ is the nilradical of $\mathcal{P}_n$ and

$$H^c_n = \sum_{l > \sum_{i \leq n} m_{\alpha_i}} F e_l.$$  

Since $[H^c_n, g_n] = 0$ and $N^{(n)}$ annihilates the maximal vector $v$ of $Z^\chi(\lambda)$, $N^{(n)}$ annihilates the induced $u_\chi(\mathcal{P}_n)$-submodule

$$u_\chi(\mathcal{P}_n) \otimes_{u_\chi(\mathcal{B})} F v \subseteq Z^\chi(\lambda).$$

Also $H^c_n$ acts as scalar multiplications on $u_\chi(\mathcal{P}_n) \otimes_{u_\chi(\mathcal{B})} F v$, so that $u_\chi(\mathcal{P}_n) \otimes_{u_\chi(\mathcal{B})} F v$ is isomorphic to $Z^\chi_n(\lambda)$ as a $u_\chi(g_n)$-module. Conversely, we can regard each $u_\chi(g_n)$-module $Z^\lambda_n(\lambda)$ as the induced $u_\chi(\mathcal{P}_n)$-module above by letting $N^{(n)}$ annihilate $Z^\lambda_n(\lambda)$, and letting $H^c_n$ act on $Z^\lambda_n(\lambda)$ as multiplications by $\lambda(h), h \in H^c_n$. Therefore, there is an isomorphism of $u_\chi(g)$-modules:

$$\Psi : u_\chi(g) \otimes_{u_\chi(\mathcal{P}_n)} Z^\chi_n(\lambda) \rightarrow Z^\chi(\lambda).$$

Since $u_\chi(g) \otimes_{u_\chi(\mathcal{P}_n)}$ is exact, $Z^\chi(\lambda)$ is not simple. This completes the proof.  

5 Appendix: The linear algebraic group $GL(\{m_i\}, F)$ and its Lie algebra

In this section, we define an infinite dimensional algebraic group $GL(\{m_i\}, F)$ and its Lie algebra. We draw most of the notation and standard procedure from [9].

Let $K$ be an algebraically closed filed. The set $K \times \cdots \times K \cdots$ is called an infinite affine space and denoted $A^\omega$. Then each element $a \in A^\omega$ is an infinite sequence $a = (a_i)_{i=1}^{\infty}$, denoted simply by $(a_i)$ in the following.

Let $K[T] = K[T_{i,j}]_{i,j=1}^{\infty}$. For each ideal $I$ in $K[T]$, let $V(I)$ denote the common zeros of all $f \in I$. Then the collection of all $V(I)$'s defines the Zariski topology on $A^\omega$. We call each open or closed set $X$ with coordinate ring $K[X]$ an affine variety. Similarly one can define the morphisms of affine varieties.

Let $(a, b)$ denote the countable set $(a_i, b_j)_{i,j=1}^{\infty}$ for each $a = (a_i), b = (b_j) \in A^\omega$. We define

$$A^\omega \times A^\omega = \{(a, b)|a, b \in A^\omega\}.$$  

The set $A^\omega \times A^\omega$ has the Zariski topology defined with the coordinate ring

$$K[T_{i,j}]_{i,j=1}^{\infty} \cong K[T] \otimes K[U].$$

Then $A^\omega \times A^\omega$ becomes the product of $A^\omega$ with $A^\omega$.

Let $\{m_i|i \in \mathbb{Z}^+\}$ be a sequence of positive integers, and let $GL(m_i)$ be the general linear group consisting of invertible $m_i$ by $m_i$ matrices. We denote the direct product

$$G =: GL(\{m_i\}, F) = \{f : \mathbb{Z}^+ \rightarrow \cup_{i=1}^{\infty} GL(m_i)|f(i) \in GL(m_i)\}.$$
We regard each element $g \in G$ as an infinite diagonal block matrix, with the $i$th block in $GL(m_i)$. Let $I_i$ be the set of all pairs of integers $(i, j)$ such that $\sum_{k=1}^{i-1} m_k + 1 \leq i, j \leq \sum_{k=1}^{i} m_k$. Then

$$GL(m_i) = \{(a_{st})_{m_i \times m_i} | (s, t) \in I_i, \det(a_{st}) \neq 0\}.$$ 

Since the set $\bigcup_{i=1}^{\infty} I_i$ is countable, we can identify $G$ with an open subset of $A^\omega$:

$$G = A^\omega - V(\{\det(T_{ij})_{(i,j)\in I_i} | i \in \mathbb{Z}^+\]).$$

Let $\{A_i | i \in \mathbb{Z}^+\}$ be a sequence of finitely-generated commutative $K$-algebras. We define the infinite tensor product by

$$\otimes_i A_i =: \{\otimes_i a_i | a_i = 1 \text{ for all but finitely many } i's\}.$$ 

Then we can write each element $\otimes_i a_i$ by a finite product, say, $a_{i_1} \otimes \cdots \otimes a_{i_n}$ if all $a_i = 1$ for $i \notin \{i_1, \ldots, i_n\}$. It follows that $\otimes_i A_i$ is a commutative algebra with a countable set of generators consisting of finite products. Then the coordinate ring of the group $G$ is

$$K[G] = \otimes_i K[GL(m_i)] = \otimes_i (K[T_{ij}]_{(i,j)\in I_i})_{\det(T_{ij})}.$$ 

Let us denote the coordinate ring of $K[GL(m_i)]$ by $K[t]_i =: K[t_{sr} | (s,r) \in I_i]$. We identify it with its canonical image in $K[G]$. By identifying $G \times G$ with an open subset of $A^\omega \times A^\omega$ with induced topology, we have that the group multiplication $\phi : G \times G \longrightarrow G$ defined by

$$\phi((A_i), (B_i)) = (A_iB_i)$$

is a morphism of affine varieties, since $\phi = (\phi_{ij})_{(i,j)\in \bigcup_i I_i}$, where for each $(i, j) \in I_i$, we have

$$\phi_{ij} = \sum_{k=1}^{m_i} t_{ik} u_{kj} \in K[t]_i \otimes K[u]_i.$$ 

Similarly, one can show that the map $i : G \longrightarrow G$ defined by $i(x) = x^{-1}$ is also a morphism of affine varieties. Thus, $G$ is an algebraic group in the sense of [9]. One can also define as in [9] the $G$-action on $K[G]$ via left(resp. right) translation $\lambda_x$(resp. $\rho_x$).

We are now ready to define the Lie algebra of $G$. Let $A = K[G]$. Then $\text{Der}A$ is a Lie algebra. We define the Lie subalgebra

$$\text{Lie}(G) = \{\delta \in \text{Der}A | \delta \lambda_x = \lambda_x \delta, \text{ for all } x \in G, \delta(K[GL(m_i)]) = 0, \text{ for all } i \text{ greater than some } N \in \mathbb{Z}^+\},$$

and call it the Lie algebra of $G$. There are also equivalent definitions of the Lie algebra Lie$(G)$ as in [9]. For each $x \in G$, we can write $x$ as a sequence $(x_i)$ with
each $x_i \in GL(m_i)$. Let $\mathcal{O}_{x_i}$ denote the local ring of the finite dimensional group $GL(m_i)$ at $x_i$ with the unique maximal $m_{x_i}$. Let us denote the ideal of $\otimes_i \mathcal{O}_{x_i}$

$$m_{x_i} = \sum_{i=1}^{\infty} \mathcal{O}_{x_1} \otimes \cdots \otimes \mathcal{O}_{x_{i-1}} \otimes m_{x_i} \otimes (\otimes_{j=i+1}^{\infty} \mathcal{O}_{x_j}).$$

Then the local ring $\mathcal{O}_x$ is $\otimes_i \mathcal{O}_{x_i}$ localized by $m_x = \sum_{i=1}^{\infty} m_{x_i}^{(i)}$.

Identifying each $\mathcal{O}_{x_i}$ with its canonical image in $\mathcal{O}_x$, we can define Lie$(G)$ to be the set of all point derivations from $\mathcal{O}_x$ to $\mathbf{F}$ such that $\delta(\mathcal{O}_{x_i}) = 0$ for all $i$ greater than some $N \in \mathbf{Z}^+$. While in terms of tangent spaces, Lie$(G)$ can be defined as

$$(m_x/m_x^2)^*_f = \{ \phi \in m_x^* | \phi(m_x^{(i)}) = 0, \phi(m_x^{(i)}) = 0, \text{for all } i \text{ greater than some } N \in \mathbf{Z}^+ \}.$$ 

By similar arguments as those used in [9], one can show that all these definitions agree.

Then we can define the differentiation for each morphism of algebraic groups. It is easy to see that Lie$(G) = \oplus_{i=1}^{\infty} \mathfrak{gl}(m_i, \mathbf{F})$. Each element $x \in$ Lie$(G)$ is a diagonal block matrix. Taking the differentiation of the automorphism Int$x$ of $G$, $x \in G$, one gets the adjoint action of $G$ on Lie$(G)$:

$$\text{Ad}_g(x) = gxg^{-1}, \quad x \in \text{Lie}(G), g \in G,$$

where the right side of the equality is the product of infinite block matrices.

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