THE GEOMETRIC STRUCTURE OF SYMPLECTIC CONTRACTION

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ABSTRACT. We show that the symplectic contraction map of Hilgert-Manon-Martens [1] – a symplectic version of Popov’s horospherical contraction – is simply the quotient of a Hamiltonian manifold \( M \) by a “stratified null foliation” that is determined by the group action and moment map. We also show that the quotient differential structure on the symplectic contraction of \( M \) supports a Poisson bracket. We end by proving a very general description of the topology of fibers of Gelfand-Zeitlin systems on multiplicity free Hamiltonian \( U(n) \) and \( SO(n) \) manifolds.

1. INTRODUCTION

Degenerations and their gradient-Hamiltonian flows are a major theme in recent studies of interactions between algebraic geometry, representation theory, and symplectic geometry. Although it can be difficult to precisely describe the gradient-Hamiltonian flow of a given degeneration – for the simple reason that the defining differential equation can be quite complicated – an improved understanding of this flow is desirable, since it often leads to interesting new results lying at the interface between algebraic and symplectic geometry (cf. [2, 3, 4, 5, 6]).

Recent work by Hilgert-Manon-Martens (HMM) provides an algebraic formula for the time-1 flow of Popov’s degeneration of a semi-projective variety equipped with an action by a connected complex reductive group \( G \) [7], to its horospherical contraction [1]. To this end, HMM define more generally, the symplectic contraction, \( M^{\text{sc}} \), of any Hamiltonian \( K \)-manifold \( (M, \omega, \mu) \) (\( K \) compact, connected) and the symplectic contraction map, \( \Phi_M : M \to M^{\text{sc}} \). HMM prove that if \( M \) is semi-projective and \( G = K^\mathbb{C} \), then \( \Phi_M \) coincides with the time-1 flow of the gradient-Hamiltonian flow of Popov’s degeneration [1 Corollary 5.12].

Although a formula for \( \Phi_M \) – and thus, when everything is sufficiently algebraic, a formula for the time-1 flow of horospherical degeneration –
presents significant progress, both the definition of $M^{sc}$ as a diagonal reduction of a product of symplectic imploded spaces, and the formula for $\Phi_M$, are somewhat opaque from the perspective of symplectic geometry.

In this note we study the geometry of symplectic contraction in more detail. We observe that:

i) there is a naturally defined decomposition of any Hamiltonian $K$ manifold $M$ into coisotropic submanifolds (whose definition only depends on the action of $K$ and the moment map),

ii) the quotient of $M$ by the null foliation of these coisotropic submanifolds is isomorphic to $M^{sc}$ (i.e. there is a stratification preserving $K \times T$ equivariant homeomorphism of the two spaces whose restriction to the symplectic strata is a symplectomorphism), and

iii) with this identification, $\Phi_M : M \to M^{sc}$ is simply the quotient map for the stratified null foliation of $M$.

In addition to demonstrating symplectic contraction as a natural geometric quotient of a Hamiltonian manifold, in many ways analogous to Marsden and Weinstein’s symplectic reduction and Guillemin-Jeffreys-Sjamaar’s symplectic implosion, this perspective has several immediate consequences. First, it is obvious that the restriction of the map $\Phi_M$ to the open dense piece of $M$ is a symplectomorphism onto its image: the null foliation of a symplectic manifold is trivial! Second, from this perspective one observes that $M^{sc}$ has a naturally defined Poisson algebra of smooth functions, which endows it with the structure of a symplectic stratified space in the sense of [8]. Finally, from this perspective we see that the symplectic contraction map (and in the algebraic case, the time-1 gradient-Hamiltonian flow of horospherical degeneration) is not just a continuous map that extends a smooth map on an open dense set: it is smooth in a stratified sense and Poisson in a differential sense.

In Section 2 and 3 we describe the results outlined above. In Section 4 we discuss this geometric perspective in relation to branching contraction (iterated symplectic contraction) and Gelfand-Zeitlin systems. In particular, we prove that the symplectic pieces of the branching contraction corresponding to a Gelfand-Zeitlin system (on an arbitrary multiplicity free $U(n)$ or $SO(n)$ manifold) are all toric manifolds (Theorem [25]). It follows from this fact that the fibers of Gelfand-Zeitlin systems can be described geometrically as iterated bundles of coisotropic homogeneous spaces over isotropic tori. This provides a more general version of results obtained for Gelfand-Zeitlin systems on $U(n)$ coadjoint orbits by [9], which only came to the attention of the author of this paper after completing this paper.
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2. The “stratified null foliation” of a Hamiltonian $K$-manifold

Let $K$ be a compact, connected Lie group with Lie algebra $\mathfrak{k}$. In this section we show that any Hamiltonian $K$-manifold $(M, \omega, \mu)$ has a “stratified null foliation” determined by the action of $K$ and the moment map $\mu$ and describe the quotient of $M$ by this foliation. The stratified null foliation of $M$ is closely related to its symplectic implosion and, as a result, this section is inspired by many ideas from [10].

If one fixes a maximal torus $T$ and a positive Weyl chamber $\Delta \subseteq \mathfrak{t}^*$, there is a maximal slice for the coadjoint action of $K$ at each stratum $\sigma \subseteq \Delta$ given by $S_\sigma := K_\sigma \cdot \text{star}(\sigma) \subseteq \mathfrak{k}^*$, where $K_\sigma$ is the stabilizer subgroup of points in $\sigma$ for the coadjoint action of $K$ (cf. [10]). For a Hamiltonian $K$-manifold $(M, \omega, \mu)$, let $M_\sigma := \mu^{-1}(S_\sigma)$ denote the symplectic cross-section of $M$ at $\sigma$. We recall the symplectic cross-section theorem, as stated in [10, Theorem 2.5], which we will use below.

**Theorem 1** (The symplectic cross-section theorem). Let $(M, \omega, \mu)$ be a Hamiltonian $K$-manifold. Then,

1. $M_\sigma$ is a $K_\sigma$-invariant symplectic submanifold of $M$ and the restriction of $\mu$ to $M_\sigma$ is a moment map for the $K_\sigma$ action.
2. The map $K \times M_\sigma \to M$ given by $(k, m) \mapsto k \cdot m$ induces a symplectomorphism $K \times K_\sigma \cdot M_\sigma \cong KM_\sigma$ (with respect to the symplectic structure on $K \times K_\sigma \cdot M_\sigma$ that will be described below) onto its image, which is open and dense in $M$.
3. If $\sigma_{\text{prin}}$ is the principal stratum of $\Delta$ corresponding to $(M, \omega, \mu)$, then $[K_{\sigma_{\text{prin}}}, K_{\sigma_{\text{prin}}}]$ acts trivially on $M_{\sigma_{\text{prin}}}$.

By part (1) of the symplectic cross-section theorem, the action of $K' = [K_\sigma, K_\sigma]$ on $M_\sigma$ is Hamiltonian, generated by $p \circ \mu$ where $p : \mathfrak{t}_{\sigma}^* \to (\mathfrak{t}_{\sigma}^*)^*$ is the dual projection map. Following [10], observe that the zero level set of this moment map

$$(p \circ \mu)^{-1}(0) = \mu^{-1}(\sigma).$$

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1. As a polyhedral cone, the positive Weyl chamber $\mathfrak{t}_{\sigma}^* = \Delta$ has a natural stratification by relative interiors of faces.

2. Recall that the principal stratum of $\Delta$ corresponding to a Hamiltonian $K$-manifold $M$ is the unique stratum such that $\mu(M) \cap \sigma$ is dense in $\mu(M) \cap \Delta$.
It follows by [8, Theorem 2.1], that for every closed subgroup $H \leq K'$ the intersection of $(p \circ \mu)^{-1}(0)$ with the orbit-type stratum of $H$ for the Hamiltonian action of $K'$ on $M$ is a coisotropic submanifold,

$$(p \circ \mu|_{M})(0) \cap M_{\sigma}(H) = \mu^{-1}(\sigma) \cap M_{\sigma}(H) \subseteq M$$

and, moreover, the leaves of the null foliation of this coisotropic submanifold equal the orbits of the action of $K'$. We denote this coisotropic submanifold $Q_{\sigma}(H)$ (note that $Q_{\sigma}(H)$ may have multiple connected components of varying dimension).

Define

$$(2) 
W_{\sigma}(H) := K \cdot Q_{\sigma}(H).$$

Note that

$$W_{\sigma}(H) = \left\{ m \in \mu^{-1}(\Sigma_{\sigma}): \text{Stab}_{K'}_{\mu(m)}(m) \in (H) \right\}$$

where $(H)$ denotes the conjugacy class of $H$ in $K$. By part (2) of the symplectic cross-section theorem (cf. [11, Theorem 41.1])

$$W_{\sigma}(H) \cong K \times K_{\sigma} Q_{\sigma}(H) \subseteq K \times K_{\sigma} M_{\sigma} \cong (K \times \mathfrak{g}_{\sigma} \times M_{\sigma}) \sslash 0 K_{\sigma}$$

where the space on the right is equipped with the symplectic form $\Omega|_{K \times \mathfrak{g}_{\sigma} + \omega|_{M_{\sigma}}}$ and the reduction by $K_{\sigma}$ is diagonal, generated by the moment map $\mu_{\mathcal{R}} + \mu|_{M_{\sigma}}$ (here $\Omega$ is the canonical symplectic form on $T^* K \cong K \times \mathfrak{k}^*$ and $\mu_{\mathcal{R}}(k, \lambda) = \lambda$ is the moment map for the cotangent lift of the right action of $K$ on itself).

**Proposition 3.** Each $W_{\sigma}(H)$ is a coisotropic submanifold of $M$. For every $m \in W_{\sigma}(H)$, the leaf of the null foliation of $W_{\sigma}(H)$ through $m$ equals the orbit $K'_{\mu(m)} \cdot m$.

**Proof.** We first show that for $q \in Q_{\sigma}(H)$, the leaf of the null foliation of $W_{\sigma}(H)$ through $q$ equals the orbit $K'_{\mu(q)} \cdot q$.

Every element of $TqW_{\sigma}(H)$ can be represented by $Y + X$ for $Y \in \mathfrak{k}$ and $X \in TQ_{\sigma}(H)$. Every element of $TqM_{\sigma}$ can be represented by $Y' + Z$ for $Y' \in \mathfrak{k}$ and $Z \in TM_{\sigma}$.

If for all $Y' \in \mathfrak{k}$,

$$0 = \omega_q(Y, Y' + Z) = \langle d\mu_q(Y' + Z), Y \rangle$$

then it must be true that $Y' + Z \in \ker(d\mu_q) \subseteq TqM_{\sigma}$, in which case $Y' + Z$ can be represented by $Z \in TqM_{\sigma}$.

If for all $X \in TqQ_{\sigma}(H)$,

$$0 = \omega_q(X, Z),$$

then it must be the case that $Z \in (TqQ_{\sigma}(H))^{\omega}$, which by [8, Theorem 2.1] equals $Tq(K'_{\sigma} \cdot q)$. 

For $m = k \cdot q$, the result follows since $K$ acts by symplectomorphisms:

the leaf of the null foliation of $W_{\sigma,(H)}$ through $k \cdot q$ is the set

$$k \cdot (K_{\sigma}' \cdot q) = (kK_{\sigma}'k^{-1}) \cdot (k \cdot q) = K_{\mu(m)}' \cdot m.$$ 

Since $W_{\sigma,(H)}$ is coisotropic, there is a quotient map

$$\pi: W_{\sigma,(H)} \to W_{\sigma,(H)}/\sim$$

defined by the null foliation of $W_{\sigma,(H)}$. However, unlike the situation of symplectic reduction, the leaves of this null foliation do not equal the orbits of a compact group action. Instead, observe the following obvious identifications,

$$(K \times_{K_{\sigma}} Q_{\sigma,(H)}/\sim) \cong \text{Homeo} K \times_{K_{\sigma}} (Q_{\sigma,(H)}/K_{\sigma}') \cong \text{Diff} K/K_{\sigma}' \times Z_{\sigma} (Q_{\sigma,(H)}/K_{\sigma}') .$$

Under the homeomorphism on the left, the quotient map $W_{\sigma,(H)} \to W_{\sigma,(H)}/\sim$ is identified with the smooth submersion

$$K \times_{K_{\sigma}} Q_{\sigma,(H)} \to K \times_{K_{\sigma}} (Q_{\sigma,(H)}/K_{\sigma}'), (k,q) \mapsto (k,[q]).$$

Thus we conclude the following (cf. [11, Theorem 25.2]).

**Proposition 5.** The quotient of each $W_{\sigma,(H)}$ by its null foliation is a smooth manifold equipped with a symplectic form, which we denote $\tilde{\omega}$, defined by the property that

$$\pi^* \tilde{\omega} = \omega|_{W_{\sigma,(H)}} .$$

Combining the results above, we have decomposed $M$ as a disjoint union

$$M = \bigcup_{\sigma \subseteq \Delta} \bigcup_{H \leq K_{\sigma}'} W_{\sigma,(H)}$$

such that each $W_{\sigma,(H)}$ is a smooth, coisotropic submanifold of $M$, invariant under the action of $K$. We define an equivalence relation on $M$ by $m \sim m'$ if $m,m'$ are contained in the same leaf of the null foliation of one of the pieces $W_{\sigma,(H)}$. We call this the stratified null foliation of $(M,\omega,\mu)$.

The quotient of $M$ by the stratified null foliation is a topological space with a decomposition into pieces that are smooth symplectic manifolds:

$$M/\sim = \bigcup_{\sigma \subseteq \Delta} \bigcup_{H \leq K_{\sigma}'} W_{\sigma,(H)}/\sim .$$

Although it is not a manifold, $M/\sim$ has a naturally defined algebra of functions (the quotient differential structure),

$$C^\infty(M/\sim) := \{ f \in C(M/\sim) : \pi^* f \in C^\infty(M) \}$$

where $\pi: M \to M/\sim$ is the quotient map and $C^\infty(M)^\sim \subseteq C^\infty(M)$ is the subalgebra of smooth functions on $M$ that are locally constant on leaves of
the stratified null foliation. The inclusion maps \( \iota: W_{\sigma(H)/\sim} \to M/\sim \) are smooth in the differential sense: for all \( f \in C^\infty(M/\sim) \),
\[
\pi^*(\iota^* f) = (\pi^* f)|_{W_{\sigma(H)}}
\]
so \( \iota^* f \in C^\infty(W_{\sigma(H)/\sim}) \).

**Proposition 6.** The bracket on \( C^\infty(M/\sim) \) defined by the equation
\[
\pi^* \{ f, g \}_{M/\sim} = \{ \pi^* f, \pi^* g \}_M
\]
is a Poisson bracket. Moreover, the inclusion maps \( \iota: W_{\sigma(H)/\sim} \to M/\sim \) are Poisson with respect to \( \{ \cdot, \cdot \}_{M/\sim} \) and the natural symplectic structure on each symplectic piece, \( W_{\sigma(H)/\sim} \).

**Remark 7.** This proposition shows that one may alternately view \( \{ \cdot, \cdot \}_{M/\sim} \) as defined point-wise on \( M/\sim \) by the symplectic structure on each symplectic piece.

**Proof.** To show that \( \{ \cdot, \cdot \}_{M/\sim} \) is a Poisson bracket on \( C^\infty(M/\sim) \), we simply must show that \( C^\infty(M) \) is a Poisson subalgebra of \( C^\infty(M) \) (and therefore \( \{ \cdot, \cdot \}_{M/\sim} \) is well-defined).

Let \( f, g \in C^\infty(M) \) and let \( W \) be one of the coisotropic submanifolds of \( M \) as defined in (2). Since \( f \) and \( g \) are constant on the leaves of the null foliation of \( W \), we have that for all \( \mathbf{X}, \mathbf{Y} \in T_wW \),
\[
\omega(\mathbf{X}_f, \mathbf{Y}) = df(\mathbf{Y}) = L_{\mathbf{Y}} f = 0
\]
and similarly for \( g \). Thus \( \mathbf{X}_f, \mathbf{X}_g \in (TW)^\omega = TW \).

Thus, for \( \mathbf{Y} \in T_wW \), we have that
\[
L_{\mathbf{Y}} \{ f, g \}_M = L_{\mathbf{Y}} \omega(\mathbf{X}_f, \mathbf{X}_g)
= L_{\mathbf{Y}} d(\omega(\mathbf{X}_f, \mathbf{X}_g))
= L_{\mathbf{Y}} \iota^* \mathbf{X}_f \iota^* \mathbf{X}_g \omega
= \omega([\mathbf{X}_f, \mathbf{X}_g], \mathbf{Y})
\]
which equals 0 since \( [\mathbf{X}_f, \mathbf{X}_g] \in TW \). Thus \( \{ f, g \}_M \in C^\infty(M) \). Additionally, one sees that \( fg \in C^\infty(M) \), so \( C^\infty(M) \) is a Poisson subalgebra.

To see that the inclusions of the symplectic pieces are Poisson, it is sufficient to observe that for all \( f, g \in C^\infty(M) \),
\[
\pi^* \iota^* \{ f, g \}_{M/\sim} = \{ \pi^* f, \pi^* g \}_M|_{W_{\sigma(H)}} = \pi^* \{ \iota^* f, \iota^* g \}_{W_{\sigma(H)/\sim}}.
\]
See [13] Theorem 25.3].

**Remark 8.** If \( \sigma_{\text{prin}} \) is the principal stratum of \( \Delta \) corresponding to \( (M, \omega, \mu) \). By part (3) of the symplectic cross-section theorem, the action of \( K'_{\sigma_{\text{prin}}} \) on \( \mu^{-1}(\sigma_{\text{prin}}) \) is trivial, so \( W_{\sigma_{\text{prin}}(K'_{\sigma_{\text{prin}}})} = K \cdot \mu^{-1}(\sigma_{\text{prin}}) \). Since this is an
open subset of $M$, its null foliation is trivial, so the restriction of $\pi$ is a symplectomorphism onto its image.

2.1. The $K \times T$ action on $M/\sim$. Since the leaves of the equivalence relation $\sim$ are invariant under the action of $K$ and $\mu$ is constant on these leaves, both the action of $K$ and the map $\mu$ descend to $M/\sim$ to define a continuous function $\tilde{\mu}$ and a continuous action of $K$. In terms of the diffeomorphism $W_{\omega,(H)/\sim} \cong K \times K_\sigma (Q_{\sigma,(H)}/K'_\sigma)$, we have

$$k \cdot (k', [q]) = (kk', [q]) \quad \text{and} \quad \tilde{\mu}(k, [q]) = k\mu(q).$$

As observed in [10], the action of $T$ on $M$ leaves $\mu^{-1}(\Delta)$ invariant and (since $T$ normalizes each $K'_\sigma$) descends to a continuous action of $T$ on $\mu^{-1}(\Delta)/\sim$. This extends by $K$-equivariance to a continuous action of $T$ on $M/\sim$ defined by the formula

$$t * (k, [q]) = (k, t \cdot [q]) = (kttk^{-1}k, [q]) = (kttk^{-1} \cdot (k, [q])$$

which commutes with the action of $K$. Note that the action of $T$ on $\mu^{-1}(\Delta)$ does not, in general, extend to $M$ by $K$-equivariance.

It follows from the identifications established in the next section that the restriction of this $K \times T$ action to each symplectic piece of $M/\sim$ is Hamiltonian, generated by the map $(\tilde{\mu}, s \circ \tilde{\mu})$ where $s : \mathfrak{t}^* \to \Delta$ is the sweeping map of Thimm’s trick (see [1] for details).

3. Symplectic contraction

In [1], HMM define the symplectic contraction of a connected Hamiltonian $K$-manifold $(M, \omega, \mu)$ as the topological space

$$M^{sc} := (EM \times E_L T^* K) \sslash_0 T$$

where $T$ is a choice of maximal torus of $K$, $EM$ is the symplectic implosion of $(M, \omega, \mu)$ with respect to a choice of positive Weyl chamber $\Delta \subseteq \mathfrak{t}^*$, and $E_L T^* K$ is the symplectic implosion of $T^* K$ with respect to the cotangent lift of the left action of $K$ on itself and the opposite Weyl chamber $-\Delta$ (see [1] [10] for definitions). This choice of $E_L T^* K$ is isomorphic to $E_R T^* K$ taken with respect to $-\Delta$ via the symplectic involution of $T^* K$, $(k, \lambda) \mapsto (k^{-1}, -k\lambda)$, and thus we can rewrite the definition above as

$$M^{sc} := (EM \times E_R T^* K) \sslash_0 T.$$

The definition with $E_R T^* K$ is preferable since $EM \times E_R T^* K$ decomposes nicely into smooth pieces diffeomorphic to

$$(Q_{\sigma,(H)}/K'_\sigma) \times K / K'_\sigma \times (-\sigma),$$

where $Q_{\sigma,(H)}/K'_\sigma$ is the same as in the proof of Proposition 3 equipped with symplectic form $\tilde{\omega} + \tilde{\Omega}$ (cf. [10]).
The “symplectic reduction” by $T$ is taken with respect to the diagonal action of $T$ on $EM \times E_R T^* K$ generated on symplectic strata by the moment map $\mu + \mu_{R,T}$ (or, equivalently, the diagonal action of $T$ on $EM \times E_L T^* K$ generated by $\mu + \mu_{L,T}$). With respect to the description of the symplectic pieces given above, this moment map is explicitly given by the formula

$$(\mu + \mu_{R,T})([q], kK'_\sigma, \lambda) = \mu(q) + \lambda.$$  

The diagonal action of $T$ is given explicitly by the formula

$$t \cdot ([q], kK'_\sigma, \lambda) = ([t \cdot q], kt^{-1}K'_\sigma, t\lambda).$$

Combining the facts above, one sees that the symplectic pieces of $M^{sc}$ are diffeomorphic to

$$(Q_{\sigma,(H)}/K'_\sigma) \times_T K/K'_\sigma.$$  

We record some topological facts about symplectic contraction.

**Proposition 9.** $M^{sc}$ is Hausdorff, second countable, locally compact, and connected.

**Proof.** Since $\mu + \mu_{R,T}$ is continuous, the level set $(\mu + \mu_{R,T})^{-1}(0)$ is closed in $EM \times E_R T^* K$. By [10, Theorem 2.3] this implies that the level set is Hausdorff, locally compact, and second countable.

The action of the compact group $T$ on the level set $(\mu + \mu_{R,T})^{-1}(0)$ is continuous, so the quotient map is open. Thus $M^{sc}$ is locally compact and second countable. Furthermore, any quotient of any locally compact Hausdorff space by a proper group action is Hausdorff, so $M^{sc}$ is Hausdorff.

Finally, HMM prove that the symplectic contraction map $\Phi_M : M \to M^{sc}$ is continuous and surjective (see Proposition 11 below), thus $M^{sc}$ is connected.

HMM define the symplectic contraction map $\Phi_M : M \to M^{sc}$ by the formula

$$m \mapsto [hm, (h, \mu(m))] \in (EM \times E_L T^* K) /_0 T$$

where $h \in K$ such that $h\mu(m) \in \Delta$. HMM note that this map is well defined and $K$-equivariant:

$$\Phi_M(k\cdot m) = [(hk^{-1})km, (hk^{-1}, \mu(km))] = [(hm, (hk^{-1}, k\mu(m))] = R_k \Phi_M(m)$$

where the $K$ action on $M^{sc}$ descends from the right $K$ action on $E_L T^* K$. Using the symplectic involution above, and writing $m = k \cdot q$ for $q \in \mu^{-1}(\Delta)$, the map $\Phi_M$ can be written equivalently as

$$k \cdot q \mapsto [q, (k, -\mu(q))] \in (EM \times E_R T^* K) /_0 T$$

in which case $K$-equivariance is with respect the descended left $K$ action on $E_R T^* K$. HMM prove two main facts:
Proposition 11. \[\Phi_M\] is continuous, proper, and surjective.

Proposition 12. \[\Phi_M\] The restriction of \(\Phi_M\) to the open dense set \(\mu^{-1}(\Sigma_{\sigma_{\text{prin}}})\) is a symplectomorphism onto its image.

We observe that, combined with Proposition 9, Proposition 11 immediately implies the following.

Corollary 13. \(\Phi_M\) is a quotient map.\(^3\)

Proof. Since \(\Phi_M\) is proper and \(M^{sc}\) is locally compact, \(\Phi_M\) is closed. It follows since \(\Phi_M\) is surjective that it is a quotient map. \(\square\)

We can use the definition of \(\Phi_M\) to describe its fibres. For \(k \in K'_{\mu(m)} \) and \(h \in K\) such that \(hk \cdot m \in \Delta\),
\[
\Phi_M(k \cdot m) = [hk \cdot m, (h, \mu(k \cdot m))]
\sim [(hk^{-1}h^{-1})hk \cdot m, (h, \mu(m))]
= [h \cdot m, (h, \mu(m))]
= \Phi_M(m)
\]
so \(\Phi_M\) is constant on the leaves of the stratified null foliation of \(M\). Conversely, if \(\Phi_M(m) = \Phi_M(m')\) then \(\exists g_1, g_2 \in K'_{\sigma}, t \in T\) such that
\[
(tg_1hm, (tg_2h, \mu(m))) = (h'm', (h', \mu(m'))).
\]
This implies that
\[
\mu(m) = \mu(m'), tg_2h = h', \text{ and } h^{-1}g_2^{-1}g_1hm = m'
\]
which implies that \(m\) and \(m'\) lie in the same leaf of the stratified null foliation of \(M\). Thus we conclude by Proposition 3 that,

Proposition 14. The fibres of \(\Phi_M\) coincide with the leaves of the stratified null foliation of \(M\).

Thus, by Corollary 13, there exists a homeomorphism \(\psi\) such that the diagram
\[
\begin{array}{ccc}
M/\sim & \xrightarrow{\pi} & M \\
\Phi_M & \downarrow{\psi} & \leftarrow M^{sc}
\end{array}
\]

\(^3\)Recall, a continuous map \(f : X \to Y\) is a quotient map if it is surjective and a subset \(U \subseteq Y\) is open iff \(f^{-1}(U)\) is open.
commutes. Since $\pi$ and $\Phi_M$ are both $K$-equivariant, $\psi$ is also equivariant (in fact it is $K \times T$ equivariant). This homeomorphism preserves the decompositions of the spaces $M^{sc}$ and $M/\sim$ into pieces indexed by $\sigma \subseteq \Delta$ and $H \leq K_\sigma$. Moreover, the restriction of $\psi$ to each piece respects its smooth and symplectic structures:

**Proposition 16.** The restriction of $\psi$ to each symplectic piece $W_{\sigma,(H)}/\sim$ is a symplectomorphism onto its image (the corresponding symplectic piece in $M^{sc}$).

**Proof.** We have already seen that, considering $W_{\sigma,(H)}/\sim$ with its quotient smooth structure, we have diffeomorphisms

$$W_{\sigma,(H)}/\sim \cong K \times K_\sigma (Q_{\sigma,(H)}/K_\sigma) \cong (Q_{\sigma,(H)}/K_\sigma) \times K/K_\sigma \times (-\sigma) \parallel_0 T$$

given by the maps

$$\pi(k \cdot q) \mapsto [k, q] \mapsto q, (k, -\mu(q))$$

Since $\Phi_M(k \cdot q) = [q, (k, -\mu(q))]$ (cf. equation (10)), we see that this composition of maps equals the restriction of $\psi$ to $W_{\sigma,(H)}/\sim$.

To see that the restriction of $\psi$ to $W_{\sigma,(H)}/\sim$ is a symplectomorphism, it is sufficient to show that

$$\pi^* \psi^* (\tilde{\omega} \vert_{Q_{\sigma,(H)}} + \tilde{\Omega} \vert_{K \times \mathfrak{e}_\sigma}) = \Phi_M^* (\tilde{\omega} \vert_{Q_{\sigma,(H)}} + \tilde{\Omega} \vert_{K \times \mathfrak{e}_\sigma}) = \omega \vert_{W_{\sigma,(H)}}$$

at a point $q \in Q_{\sigma,(H)}$. Here $\tilde{\Omega} \vert_{K \times \mathfrak{e}_\sigma}$ is the symplectic form on the symplectic quotient $K/K_\sigma \times (-\sigma) \cong (K \times \mathfrak{g}_{-\sigma}) \parallel_0 K_\sigma$ (cf. [10, p. 162]). The result then follows by $\tilde{\omega}$-equivariance of $\psi$.

An arbitrary element of $T_qW_{\sigma,(H)}$ can be written as $X + Y$ where $X \in T_qQ_{\sigma,(H)}$ and $Y$ is the image of $Y \in \mathfrak{t}$ at $q$ under the Lie algebra action. One computes that

$$(d\Phi_M)_q(X + Y) = (\pi_*(X), Y, -d\mu_q(X))$$

where we write $Y$ to mean $Y + \mathfrak{t}'_q \in \mathfrak{t}/\mathfrak{t}' = T_e(K/K'_\sigma)$. For $X, X' \in T_qQ_{\sigma,(H)}$ and $Y, Y' \in \mathfrak{t}$, we compute

$$\Phi_M^* (\tilde{\omega} \vert_{Q_{\sigma,(H)}} + \tilde{\Omega} \vert_{K \times \mathfrak{e}_\sigma}) \bigg|_{(X + Y, X', Y')} = \omega_q(X + Y, X', Y')$$

where

$$\omega_q(X + Y, X', Y') = \omega_q(X, X') + \omega_q(Y, Y') + \omega_q(X, Y') + \omega_q(Y, X')$$

and

$$\omega_q(X, X') = \omega_q(X, X') + \omega_q(Y, Y') + \omega_q(Y, X') + \omega_q(Y, Y')$$

for $X, X' \in T_qQ_{\sigma,(H)}$ and $Y, Y' \in \mathfrak{t}$. The result then follows by $\tilde{\omega}$-equivariance of $\psi$. Thus we have shown that $\psi$ is a symplectomorphism.
where in the penultimate equality we have used Hamilton’s equation and the fact that \( \mu \) is Poisson. Thus the restriction of \( \psi \) to \( W_{\sigma, (H)}/\sim \) is a symplectomorphism.

The homeomorphism \( \psi \), along with the algebra \( C^\infty(M/\sim) \) defined in Section 2, shows that \( M^{sc} \) is endowed with a naturally defined algebra of smooth functions equipped with a Poisson bracket. This was not evident from the algebraic definition of HMM via symplectic implosion. Indeed, the symplectic implosion of a Hamiltonian \( K \)-manifold \((M, \omega, \mu)\) is not naturally a symplectic stratified space in the sense of [8] (cf. the comment in [10] on page 167). We end this section with the following observation.

**Theorem 17.** The symplectic contraction of a Hamiltonian \( K \)-manifold \((M, \omega, \mu)\) is a stratified space in the sense of [8]; the decomposition of \( M^{sc} \) into symplectic pieces satisfies the following conditions.

1. (locally finite) Every point in \( M^{sc} \) has a neighbourhood which intersects finitely many of the symplectic strata.
2. (frontier condition) If for two symplectic strata \( X \) and \( Y \), \( X \cap \overline{Y} \neq \emptyset \), then \( X \subseteq \overline{Y} \).
3. (local normal triviality) Every point in \( M^{sc} \) has a neighbourhood homeomorphic to a cone over a lower dimensional stratified space.

Moreover, \( M^{sc} \) is a symplectic stratified space in the sense of [8]; the smooth structure \( C^\infty(M^{sc}) \) defined above satisfies the following conditions.

a) The strata are symplectic manifolds.
b) \( C^\infty(M^{sc}) \) is a Poisson algebra.
c) The inclusions of the strata are smooth Poisson maps.

**Proof.** Conditions i)-iii) follow from the corresponding facts for symplectic imploided spaces [10] by HMM’s definition of \( M^{sc} \) as a reduction of imploided spaces. Conditions a)-c) follow by Propositions [6] and [10].

4. Fibers of Gelfand-Zeitlin systems

In this section, we apply our geometric perspective to describe the fibers of Gelfand-Zeitlin systems, which – as was observed in [1] – can be constructed via contraction.

Given a Hamiltonian \( K_n \)-manifold \((M, \omega, \mu)\), and a chain of group homomorphisms,

\[
K_1 \xrightarrow{\phi_2} K_2 \xrightarrow{\phi_3} \cdots \xrightarrow{\phi_n} K_n,
\]

where the \( K_i \) are connected, compact Lie groups with maximal tori \( T_i \), we obtain a chain of contraction maps

\[
M \xrightarrow{\Phi_n} M_n \xrightarrow{\Phi_{n-1}} \cdots \xrightarrow{\Phi_1} M_1
\]
in the following way. First, by performing symplectic contraction with respect to the $K_n$ action on $M$, we get a symplectic contraction map $\Phi_n$ from $M$ to the Hamiltonian $K_n \times T_n$-space $M_n$. $M_n$ stratifies into symplectic manifolds equipped with a Hamiltonian $K_{n-1} \times T_n$ action coming from the homomorphism $\phi_n: K_{n-1} \to K_n$. Second, we take the quotient of $M_n$ by simultaneously performing symplectic contraction of all the symplectic strata of $M_n$ with respect to the $K_{n-1}$ action (note that since $T_n$ is abelian, this is the same as the symplectic contraction with respect to the $K_{n-1} \times T_n$ action). This results in a continuous map $\Phi_{n-1}: M_n \to M_{n-1}$, generated by a moment map $\tilde{\mu}$ such that the following diagram

\begin{equation}
\begin{array}{c}
M \\
\downarrow \mu \\
\Phi \\
\downarrow \tilde{\mu} \\
\mathfrak{k}^* \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\end{array} \begin{array}{c}
M_1 \\
\downarrow \mu \\
\Phi \\
\downarrow \tilde{\mu} \\
\mathfrak{k}_1^* \times \cdots \times \mathfrak{k}_n^* \\
\end{array}
\end{equation}

commutes, where $F$ is the Gelfand-Zeitlin system on $\mathfrak{k}^*$ constructed from the chain of groups \([1]\) as in \([11]\) and $\Phi = \Phi_1 \circ \cdots \circ \Phi_n$. The space $M_1$ is the branching contraction space considered by HMM in \([1]\).

Following work by \([12]\) and others on collective integrable systems, Guillemin and Sternberg observed in \([13, 14]\) that given a multiplicity free Hamiltonian $K$ manifold for $K = U(n)$ or $SO(n)$, the Gelfand-Zeitlin system constructed from a chain of subgroups \([20]\)

$$
U(1) \leq \cdots \leq U(n) \text{ or } SO(2) \leq \cdots \leq SO(n)
$$

defines a completely integrable torus action on the open dense subset of $M$ where the Gelfand-Zeitlin functions are smooth.

We now show that, in general, if this construction yields a completely integrable system on an open dense subset of $M$, then the action of $T_1 \times \cdots \times T_n$ on each symplectic piece of $M_1$, is completely integrable.

**Lemma 21.** If $(M, \omega, \mu)$ is a multiplicity free Hamiltonian $K$ manifold with connected fibers, then the action of the maximal torus $T$ on each of the symplectic pieces $Q_{\sigma(H)}/K'_\sigma \subseteq EM$ is completely integrable.

**Proof.** The action of $K$ on $M$ is multiplicity free iff the reduced spaces $M/\lambda K$ are points for all $\lambda$ (cf. \([15]\) Proposition A.1). By \([10]\) Theorem
3.4], for $\lambda \in \sigma \subseteq \Delta$,

$$M \sslash \lambda K \cong EM \sslash \lambda T = \left( \bigcup_{H \leq K'_\sigma} Q_{\sigma,(H)}/K'_\sigma \right) \sslash \lambda T$$

is therefore a point. It follows that the action of $T$ on each symplectic piece $Q_{\sigma,(H)}/K'_\sigma$ is multiplicity free, or, in other words, completely integrable. \[ \square \]

**Proposition 22.** Let $(M,\omega,\mu)$ be a multiplicity free Hamiltonian $K$ manifold with connected fibers, and let $M^{sc}$ be its symplectic contraction. Suppose that $S \leq K$ is a connected Lie subgroup such that the action of $S$ on every $K$ coadjoint orbit is multiplicity free. Then, every symplectic stratum of $M^{sc}$ is a multiplicity free Hamiltonian $S \times T$ manifold.

**Proof.** Every symplectic piece of $M^{sc}$ is of the form $K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right)$. We want to show that the symplectic reduced spaces

$$\left( K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right) \right) \sslash (\xi,\lambda) S \times T$$

are all points. By reduction in stages [8], this space is isomorphic to

$$(K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right) \sslash \lambda T) \sslash \xi S$$

By Lemma 21, the symplectic reduction $\left( Q_{\sigma,(H)}/K'_\sigma \right) \sslash \lambda T$ is a point. It follows that

$$K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right) \sslash \lambda T = K \times K_\sigma \{\ast\} \cong K/K_\sigma.$$ 

The Hamiltonian action of $K$ on $K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right)$ commutes with the action of $T$, so it descends to the symplectic quotient. By $K$-equivariance, and the line above, the moment map for the action of $K$ on $K/K_\sigma$ is a symplectomorphism onto the $K$ coadjoint orbit through $\lambda$, so

$$K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right) \sslash \lambda T \cong K \cdot \lambda.$$ 

By our assumption that all $K$ coadjoint orbits are multiplicity free Hamiltonian $S$ manifolds, we conclude that the space

$$\left( K \times K_\sigma \left( Q_{\sigma,(H)}/K'_\sigma \right) \sslash \lambda T \right) \sslash \xi S = \{\ast\},$$

thus $K \times K_\sigma \left( Q_{\sigma,(H)}/\sim \right)$ is a multiplicity free $S \times T$ manifold. \[ \square \]

We require the following fact (cf. [14]).

**Lemma 24.** Every $U(n)$ coadjoint orbit is a multiplicity free $U(n-1)$ manifold for any embedding of $U(n-1)$ as a subgroup of $U(n)$. Respectively, every $SO(n)$ coadjoint orbit is a multiplicity free $SO(n-1)$ manifold for any embedding of $SO(n-1)$ as a subgroup.
With these results in hand, we can conclude the following: every symplectic piece of the iterated symplectic contraction $M_1$ corresponding to a Gelfand-Zeitlin system is a toric manifold.

**Theorem 25.** Suppose $(M, \omega, \mu)$ is a multiplicity free Hamiltonian $U(n)$ or $SO(n)$ manifold. Let $M_1$ be the space constructed from $M$ by iterated symplectic contraction as in (18), using one of the chains (20). Then the action of $T = T_1 \times \cdots \times T_n$ on every symplectic piece of $M_1$ is completely integrable.

**Proof.** We apply Proposition 22 at each stage of the iterated symplectic contraction for the case of $K = U(n)$ (the proof for $K = SO(n)$ is identical).

First, by Proposition 22 and Lemma 24 we have that the symplectic pieces of $M_n$ are multiplicity free Hamiltonian $U(n-1) \times T_n$ manifolds.

If we apply Proposition 22 and Lemma 24 again, to the symplectic pieces of $M_n$, it follows that the symplectic pieces of $M_{n-1}$ are multiplicity free $U(n-2) \times T_{n-1} \times T_n$ manifolds (note that we perform symplectic contraction with respect to the action of $U(n-1) \times T_n$, the maximal torus of which is $T_{n-1} \times T_n$, the result is identical to performing symplectic contraction with respect to the $U(n-1)$ action, except that this way the extra $T_n$ action descends as part of the construction).

Repeating this process, we finally have that the symplectic pieces of $M_1$ are multiplicity free $T_1 \times T_2 \times \cdots \times T_n$ manifolds (note that since $U(1) = T_1$, the last symplectic contraction map $\Phi_1$ is trivial, so $M_2 = M_1$). In other words, the torus action on each symplectic piece is completely integrable. □

This result allows us to give a very general description of the fibers of Gelfand-Zeitlin systems, similar to that of [9].

**Theorem 26.** Suppose $(M, \omega, \mu)$ is a connected Hamiltonian $U(n)$ or $SO(n)$ manifold with $\mu$ proper, equipped with a completely integrable Gelfand-Zeitlin system constructed as above. Then the fibers of the Gelfand-Zeitlin system are the total spaces of sequences of fiber bundles

\[(27) \quad E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_2 \rightarrow E_1 = L \]

where $L$ is an isotropic torus contained in symplectic piece of $M_1$ and the each

$$K_{k-1}/H_{k-1} \rightarrow E_k \rightarrow E_{k-1}$$

is a fiber bundle of homogeneous spaces.

**Proof.** If the Gelfand-Zeitlin construction yields an integrable system, then $(M, \omega, \mu)$ is a multiplicity free Hamiltonian manifold [14], so we are in the setting of Theorem 25. If $\mu$ is proper, then $F \circ \mu$ is proper, so by [16, Theorem 1] the fibers of $F \circ \mu$ are all connected. Since the maps $\Phi_k$ are
all surjective (they are quotient maps), and the diagram (19) commutes, it follows that the fibers of \( \tilde{\mu} \) are connected.

- Since (19) commutes, the fibers of the Gelfand-Zeitlin system \( F \circ \mu \) equal the fibers of the composition \( \tilde{\mu} \circ \Phi \).
- Since the torus actions generated by \( \tilde{\mu} \) on the symplectic pieces of \( M_1 \) are completely integrable, the fibers of the restriction of \( \tilde{\mu} \) to any symplectic piece of \( M_1 \) are isotropic tori.
- Since the fibers of \( \tilde{\mu} \) are connected, the intersection of any fiber of \( \tilde{\mu} \) with a symplectic piece of \( M_1 \) is closed in the symplectic piece, and the symplectic pieces of \( M_1 \) are locally closed in \( M_1 \), each fiber of \( \tilde{\mu} \) is contained in a single symplectic piece of \( M_1 \).
- At each stage of the iterated symplectic contraction, the pre-image under \( \Phi_k \) of a submanifold \( N \) of a symplectic piece of \( M_k \) indexed by \( \sigma \subseteq \Delta_k \) and \( (H) \) is a coisotropic fiber bundle over \( N \) whose fibers are the homogeneous spaces \( K_k/H_k \).

\[ \square \]

Remark 28. Let \( M = \mathcal{O}_\lambda \) be a \( U(n) \) coadjoint orbit and consider the Hamiltonian action of \( U(n) \) generated by the inclusion \( \iota: \mathcal{O}_\lambda \to u(n)^* \). The fibers of Gelfand-Zeitlin systems on \( M \) were studied extensively by Cho-Kim-Oh [9] who prove a more detailed result analogous to Theorem 26. Cho-Kim-Oh show that – in this specific case, \( M = \mathcal{O}_\lambda \) – the only fibers occurring in the bundles \( E_k \to E_{k-1} \) of (27) are points or odd-dimensional spheres. Moreover – in this specific case – they show that the fibers of the Gelfand-Zeitlin system are all isotropic.

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