GROWTH OF SOBOLEV NORMS FOR 2d NLS WITH HARMONIC POTENTIAL

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Dedicated to Professor Vladimir Georgiev for his 65’s birthday

Abstract. We prove polynomial upper bounds on the growth of solutions to 2d cubic NLS where the Laplacian is confined by the harmonic potential. Due to better bilinear effects our bounds improve on those available for the 2d cubic NLS in the periodic setting: our growth rate for a Sobolev norm of order \( s = 2k \), \( k \in \mathbb{N} \), is \( \varepsilon^{2(s-1)/3+s} \). In the appendix we provide an direct proof, based on integration by parts, of bilinear estimates associated with the harmonic oscillator.

1. Introduction

In recent years, growth of Sobolev norms for solutions to nonlinear dispersive equations generated a huge interest, in relation with weak turbulence phenomena. Concerning upper bounds, we quote the pioneering work of Bourgain [3] and its extension in a series of subsequent papers ([7], [8], [9], [14], [18], [19], [21]) to quote only a few of them). On the other end, growth of Sobolev norm cannot occur in settings where the dispersive effect is too strong. For instance consider the translation invariant cubic defocusing NLS on \( \mathbb{R}^2 \). Then [11] proved the long standing conjecture that nonlinear solutions scatter to free waves when time goes to infinity and hence no growth phenomena is possible in such setting.

We are interested in the growth of solutions to the following nonlinear Schrödinger equation:

\[
\begin{align*}
    i \partial_t u + Au &\pm u|u|^2 = 0, \\
    u(0, x) &= \varphi(x) \in \mathcal{H}^s
\end{align*}
\]

where \( x = (x_1, x_2) \), the operator \( A \) is the usual Laplacian with an harmonic potential,

\[ A = -\Delta + |x|^2, \quad \text{where} \quad \Delta = \partial_{x_1}^2 + \partial_{x_2}^2, \quad |x|^2 = x_1^2 + x_2^2 \]

and \( \|\varphi\|_{\mathcal{H}^s} = \|A^{s/2}\varphi\|_{L^2} \), where in general we use the notation \( L^p = L^p(\mathbb{R}^2) \). We shall also denote \( L^p_t, x = L^p(\mathbb{R} \times \mathbb{R}^2) \) to emphasize the Lebesgue space of space-time dependent functions.

Let us first comment briefly about the local Cauchy theory associated with (1.1). By combining preservation of regularity for the linear flow, \( \|e^{itA}\varphi\|_{\mathcal{H}^s} = \|\varphi\|_{\mathcal{H}^s} \) and that \( \mathcal{H}^s \) is an algebra for \( s > 1 \), one proves existence of a local solution to (1.1) by fixed point; its local time of existence depends on the \( \mathcal{H}^s \) norm of the initial datum. Moreover the solution map is Lipschitz continuous. In order to globalize our solution one can rely on the Brezis-Gallouët inequality (see [4]) provided that

\[
\sup_{t \in (-T_{\min}(\varphi), T_{\max}(\varphi))} \|u(t, x)\|_{\mathcal{H}^1} < \infty
\]

where \((-T_{\min}(\varphi), T_{\max}(\varphi)), \) with \( T_{\min}(\varphi), T_{\max}(\varphi) > 0 \), is the maximal time interval of existence of the solution associated with (1.1). In particular, assuming (1.2), \( T_{\max}(\varphi) = T_{\min}(\varphi) = \infty \) and a double exponential bound holds:

\[
\|u(t, x)\|_{\mathcal{H}^s} \leq C \exp(C \exp(C|t|)).
\]

Solutions to (1.1) satisfy the conservation of the Hamiltonian

\[
\frac{1}{2} \|u(t, x)\|_{\mathcal{H}^1}^2 \pm \frac{1}{4} \|u(t, x)\|_{L^4}^4 = \text{const},
\]

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therefore, in the defocusing case, (1.2) is automatically satisfied, while in the focusing case it is not granted for free. Of course, by using more sophisticated tools, e.g. Bourgain’s spaces $X^{s,b}$ associated with $i\partial_t + A$, one can deal with initial data at lower regularity than $H^{1+\varepsilon}$. These $X^{s,b}$ spaces will play a key role in our analysis as they allow us to exploit a bilinear effect associated with the propagator $e^{itA}$. They will be defined in Section 3 where we also provide more useful facts about Cauchy theory.

Our main goal is to improve (1.3) and prove polynomial upper bounds for the quantity $\|u(t,x)\|_{H^s}$ when $t \to \pm \infty$ with $s > 1$. Along the rest of the paper the following equivalence of norms will be useful: for every $s \geq 0$ there exist $C > 0$ such that

$$\frac{1}{C}(\|D^s u\|_{L^2}^2 + \|\langle x \rangle^s u\|_{L^2}^2) \leq \|\varphi\|_{H^s}^2 \leq C(\|D^s u\|_{L^2}^2 + \|\langle x \rangle^s u\|_{L^2}^2)$$

where $D^s$ is the operator associated with the Fourier multiplier $|\xi|^s$ and $\langle x \rangle = \sqrt{1 + x_1^2 + x_2^2}$. The proof of the equivalence (1.4) is a special case of a more general result proved in [11]. In particular establishing growth upper bounds on $H^s$ norm of the solution is equivalent to establish polynomial bounds on the classical Sobolev norms $H^s$ and the corresponding moment of order $s$. We now state our main result.

**Theorem 1.1.** Let $\varepsilon > 0$ and $k \in \mathbb{N}$. For every global solution $u$ to (1.1) such that $u(t,x) \in C(\mathbb{R}, H^{2k})$ and

$$\sup_{t \in \mathbb{R}} \|u(t,x)\|_{H^1} < \infty$$

there exists a constant $C$ such that

$$\|D^{2k} u(t,x)\|_{L^2} + \|\langle x \rangle^{2k} u(t,x)\|_{L^2} \leq C(t)^{\frac{2(2k-1)}{2} + \varepsilon}.$$

Our bound may be compared to the corresponding bound for solutions to NLS on a generic compact $2 - d$ manifold $M^2$ and more specifically on the torus $\mathbb{T}^2$. In fact at the best of our knowledge the best known upper bound available on the growth of the classical Sobolev norm $H^{2k}(\mathbb{T}^2)$ for solutions to cubic NLS on $\mathbb{T}^2$ is $(1 + t)^{2k-1 + \varepsilon}$, as proved in [21], [14]. Notice also that in our case we control the growth of the moments as well (see also [20] for a different perspective on the moments).

Theorem 1.1 may also be compared with [7] Theorem 2, where the same bound on the growth of Sobolev norm was achieved for the translation invariant cubic NLS posed on $\mathbb{R}^2$, at a time where Dodson’s definitive result was not available. As already mentioned, unlike the situation considered in Theorem 1.1 where in general scattering theory is not available, in the euclidean setting one can deduce uniform boundedness of high order Sobolev norms, at least in the defocusing situation. Nevertheless bounds provided in Theorem 1.1 are still meaningful and non trivial in the flat case either, if one considers solutions to the focusing NLS such that the $H^1$ norm is uniformly bounded. In fact under this assumption it is not true in general that the solutions scatter to a free wave and hence the uniform boundedness of Sobolev norms is not granted.

It would be very interesting to construct solutions to the defocusing (1.1) such that the $H^k$ norms do not remain bounded in time for some $k > 1$. Unfortunately such results are rare in the context of canonical dispersive models (with the notable exception of [12]).

## 2. $X^{s,b}$ Framework and Linear Estimates

We first define $X^{s,b}$ spaces associated with the harmonic oscillator in dimension two: the spectrum of the harmonic oscillator is given by the following set of integers $\{2n + 2, n \in \mathbb{N}\}$. For every $n \in \mathbb{N}$ we shall denote by $\Pi_n$ the orthogonal projector on the eigenspace associated with the eigenvalue $2n + 2$. Then the $X^{s,b}$ norm is given by the expression

$$\|u\|_{X^{s,b}}^2 = \sum_{n \in \mathbb{N}} (2n + 2)^s \|\tau + 2n + 2)^b \mathcal{F}_{t \to \tau}(\Pi_n u(t,x))\|_{L^2_{\tau,x}}^2$$

where $u(t,x)$ is a function globally defined on space-time and $\mathcal{F}_{t \to \tau}$ denotes the Fourier transform with respect to the time variable. Along with the $X^{s,b}$ spaces, which are defined for global space-time functions, we also introduce its localized version for every $T > 0$. More precisely for functions $v(t,x)$ on the strip $(-T,T) \times \mathbb{R}^2$ we define:

$$\|v\|_{X^{s,b}_T} = \inf_{\tilde{v} \in X^{s,b}} \|\tilde{v}\|_{X^{s,b}}$$

The main result of this section is the continuity of suitable linear operators in the Bourgain’s spaces $X^{s,b}_T$. 

Proposition 2.1. For every $\delta \in (0, \frac{1}{2})$, $b \in (0, 1)$ there exists $C > 0$ such that we have the following estimate for every $T > 0$:

\begin{equation}
\| Lu \|_{X_T^\infty, 2^1/2 + 2b} \leq C \| u \|_{X_T^\infty, 2^1/2 + 2b}
\end{equation}

(2.1)

\begin{equation}
\| Lu \|_{X_T^{\infty, (1-\delta)b}} \leq C \| u \|_{X_T^{\infty, (1-\delta)b}}
\end{equation}

(2.2)

where $L$ can be either $\partial_{x_i}$, $i = 1, 2$ or multiplication by $\langle x \rangle$.

Proof. We prove Proposition 2.1 without the time localization. The corresponding version in localized Bourgain’ spaces is straightforward. We will prove the following bounds:

\begin{equation}
\| Lu \|_{X_0^{0,b}} \leq C \| u \|_{X_1^{1,0}}, \quad b \in [0, 1]
\end{equation}

(2.3)

\begin{equation}
\| Lu \|_{X_1^{1,0}} \leq C \| u \|_{X_0^{0,0}}
\end{equation}

(2.4)

Notice that (2.2) follows by interpolation between (2.3) and (2.4). Moreover we get

\begin{equation}
\| Lu \|_{X_T^{\infty, 2, 1/2}} \leq C \| u \|_{X_T^{1,2, 1/2}}
\end{equation}

(2.5)

by duality from (2.3) for $b = 0$, and we also get

\begin{equation}
\| Lu \|_{X_T^{\infty, -1, 1/2}} \leq C \| u \|_{X_T^{1,2, 1/2}}
\end{equation}

(2.6)

by interpolation between (2.5) and (2.3) for $b = 1$. Then (2.1) follows, interpolating (2.3) and (2.6). Hence we focus on (2.3) and (2.4). Since the proof is slightly different depending from the operator $L$ that we consider, we consider two cases.

First case: proof of (2.3) and (2.4) for $Lu = \partial_{x_i} u$

First we prove that, for space-time dependent functions $u(t, x)$ we have

\begin{equation}
\| \partial_{x_i} u \|_{X_0^{0,0}} \leq C \| u \|_{X_1^{1,0}}.
\end{equation}

(2.7)

This estimate is a consequence of the following one for time independent functions $v(x)$:

\begin{equation}
\| \partial_{x_i} v \|_{L^2} \leq C \| \sqrt{Av} \|_{L^2}
\end{equation}

(2.8)

that in turn follows by $\| \sqrt{Av} \|_{L^2} = \| v \|_{H^1}$ and by recalling (1.4) for $s = 1$. Next we prove

\begin{equation}
\| \partial_{x_i} u \|_{X_0^{0,1}} \leq C \| u \|_{X_1^{1,1}},
\end{equation}

(2.9)

and by interpolation with (2.7), (2.3) will follow for $L = \partial_{x_i}$. As $\| w(t, x) \|_{X_0^{0,1}}$ is equivalent to $\| (i\partial_t + A)u \|_{L^2_{t,x}} + \| u \|_{L^2_{t,x}}$, in order to get (2.8) we estimate

\begin{equation}
\| (i\partial_t + A)\partial_{x_i} u \|_{L^2_{t,x}} + \| \partial_{x_i} u \|_{L^2_{t,x}} = \| \partial_{x_i} (i\partial_t + A)u + [\| x \|^2, \partial_{x_i} u]_{L^2_{t,x}} + \| \partial_{x_i} u \|_{L^2_{t,x}}
\end{equation}

\begin{equation}
\leq \| \partial_{x_i} (i\partial_t + A)u \|_{L^2_{t,x}} + 2\| x \|_{L^2_{t,x}} + \| \partial_{x_i} u \|_{L^2_{t,x}}.
\end{equation}

(2.10)

By combining (2.7) with the following identity

\begin{equation}
\| \sqrt{Av} \|_{L^2}^2 = (Av, v) = \| \nabla v \|_{L^2}^2 + \| x \| \| v \|_{L^2}^2
\end{equation}

(2.11)

we can continue (2.9) as follows:

\begin{equation}
(\ldots) \leq \| (i\partial_t + A)u \|_{X_1^{1,0}} + 3\| \sqrt{Av} \|_{L^2_{t,x}} \leq \| u \|_{X_1^{1,1}} + 3\| u \|_{X_1^{1,0}} \leq 4\| u \|_{X_1^{1,1}}.
\end{equation}

(2.12)

and (2.8) for $L = \partial_{x_i}$ follows. Next we prove (2.4) (where $L = \partial_{x_i}$), namely

\begin{equation}
\| \partial_{x_i} u \|_{X_1^{1,0}} \leq C \| u \|_{X_2^{0,0}}.
\end{equation}

(2.13)

This estimate is a consequence of the following one for time independent functions $v(x)$:

\begin{equation}
\| \sqrt{A} \partial_{x_i} v \|_{L^2} \leq C \| Av \|_{L^2},
\end{equation}

(2.14)

that in turn is equivalent to

\begin{equation}
(A \partial_{x_i} v, \partial_{x_i} v) \leq C(Av, Av).
\end{equation}

(2.15)
As on the r.h.s. we get \( \|v\|^2_{H^2} \), by (1.4) and elementary considerations it is sufficient to prove
\[
(2.10) \quad \int |x|^2 |\partial_x v|^2 \leq C \left( \|D^2 v\|^2_{L^2} + \|\langle x \rangle^2 v\|^2_{L^2} \right).
\]
In turn this last inequality follows by combining integration by parts and the Cauchy-Schwarz inequality:
\[
\int |x|^2 |\partial_x v|^2 = - \int |x|^2 \partial_x^2 v \bar{v} - 2 \int x_i \partial_x v \bar{v} \leq \|\partial_x^2 v\|_{L^2} \|\langle x \rangle^2 v\|_{L^2} + 2 \|x|\partial_x v\|_{L^2} \|v\|_{L^2}
\]
\[
\leq \frac{1}{2} \|D^2 v\|^2_{L^2} + \frac{1}{2} \|\langle x \rangle^2 v\|^2_{L^2} + \frac{1}{2} \|\langle x \rangle^2 v\|^2_{L^2} + 2 \|\langle x \rangle^2 v\|^2_{L^2},
\]
from which we easily conclude moving \( \frac{1}{2} \|\langle x \rangle^2 v\|^2_{L^2} \) to the left-hand side.

Second case: proof of (2.3) and (2.4) for \( Lu = \langle x \rangle u \)

The proof follows the same steps as in the case \( L = \partial_x \), with minor modifications. First notice that we have for space-time dependent functions \( u(t,x) \) the following estimate:
\[
(2.11) \quad \|\langle x \rangle u\|_{X^{0,0}} \leq C \|u\|_{X^{1,0}}.
\]
This is a consequence of the following estimate for time independent functions \( v(x) \):
\[
\|\langle x \rangle v\|_{L^2} \leq C \|\sqrt{A} v\|_{L^2}
\]
that in turn follows by noticing that \( \|\sqrt{A} v\|_{L^2} = \|v\|_{H^1} \) and recalling (1.4) for \( s = 1 \). Moreover we have
\[
\|\langle x \rangle u\|_{X^{0,1}} \leq C \|u\|_{X^{1,1}}.
\]
that by interpolation with (2.11) implies (2.3) for \( L = \langle x \rangle \). In order to prove this estimate recall again that \( \|u(t,x)\|_{X^{0,1}} \) is equivalent to \( \|i\partial_t u + A u\|_{L^2_{t,x}} + \|u\|_{L^2_{t,x}} \) and hence we compute
\[
\|\langle x \rangle (i\partial_t + A)u\|_{L^2_{t,x}} + \|\langle x \rangle u\|_{L^2_{t,x}} = \|\langle x \rangle (i\partial_t + A)u + [\Delta, \langle x \rangle] u\|_{L^2_{t,x}} + \|\langle x \rangle u\|_{L^2_{t,x}}
\]
\[
\leq \|\langle x \rangle (i\partial_t + A)u\|_{L^2_{t,x}} + \|2\nabla (\langle x \rangle) \cdot \nabla u + \Delta (\langle x \rangle) u\|_{L^2_{t,x}} + \|\langle x \rangle u\|_{L^2_{t,x}}
\]
\[
\leq C \|\langle x \rangle (i\partial_t + A)u\|_{L^2_{t,x}} + \|\nabla u\|_{L^2_{t,x}} + \|\langle x \rangle u\|_{L^2_{t,x}}.
\]
By combining (2.3) with the identity \( \|\sqrt{A} u\|_{L^2_{t,x}} = \|u\|_{H^1} \) and by recalling (1.4) for \( s = 1 \), we can proceed with our estimate above,
\[
\|\langle x \rangle u\|_{X^{0,0}} \leq C \|u\|_{X^{1,1}}.
\]
Next we prove (2.4) (where \( L = \langle x \rangle \)), namely
\[
\|\langle x \rangle u\|_{X^{1,0}} \leq C \|u\|_{X^{2,0}}.
\]
This estimate is a consequence of the following one for time independent functions \( v(x) \):
\[
\|\sqrt{A} (\langle x \rangle v)\|_{L^2} \leq C \|Av\|_{L^2}
\]
that in turn is equivalent to
\[
(\langle x \rangle v, \langle x \rangle v) \leq C \|v\|^2_{H^2}.
\]
By (1.4) it is equivalent to
\[
\|\nabla (\langle x \rangle v)\|^2_{L^2} + \|\langle x \rangle v\|^2_{H^2} \leq C (\|D^2 v\|^2_{L^2} + \|\langle x \rangle^2 v\|^2_{L^2}).
\]
In turn, developing the gradient on the l.h.s. the estimate above follows from
\[
\int (\langle x \rangle^2 |\nabla v|^2 \leq C (\|D^2 v\|^2_{L^2} + \|\langle x \rangle^2 v\|^2_{L^2})
\]
whose proof proceeds by integration by parts and Cauchy-Schwarz inequality as we did for (2.10). \( \square \)
3. The Cauchy theory in $X^{s,b}$ and consequences

We first obtain a trilinear estimate, whose proof heavily relies on the analysis of [10] (also available as [17]); for the sake of completeness, we provide a relatively elementary proof of the crucial bilinear estimate from [16] in the appendix, using the bilinear virial techniques from [15]. The only novelty in our trilinear estimate is that we prove a tame estimate, while such an estimate was not needed for the low regularity analysis of [16]. We first recall the following key bilinear estimate [10, Theorem 2.3.13]. There exists $\delta_0 \in (0, \frac{1}{2})$ such that for every $\delta \in (0, \delta_0)$ there exists $b' < \frac{b}{2}$ and $C > 0$ such that:

\[
(3.1) \quad \|\Delta_N(u)\Delta_M(v)\|_{L^2((0,T);L^2)} \leq C(\min(M,N))^\delta \left( \frac{\min(M,N)}{\max(M,N)} \right)^{\frac{1}{2}-\delta} \|\Delta_N(u)\|_{X_T^{0,b'}} \|\Delta_M(v)\|_{X_T^{0,b'}}
\]

where $\Delta_N, \Delta_M$ are the Littlewood-Paley localization associated with $A$ and $N$, $M$ are dyadic integers.

**Proposition 3.1.** Let $0 < T < 1$ and $\epsilon > 0$ be fixed. Then there exist $C > 0$, $b > 1/2$ and $\gamma > 0$ such that for $s \geq \epsilon$:

\[
(3.2) \quad \left\| \int_0^1 e^{i(t-\tau)A} (u_1(\tau)u_2(\tau)\bar{u}_3(\tau)) d\tau \right\|_{X_T^{s,b}} \leq CT^{\gamma} \sum_{\sigma \in S_3} \|u_{\sigma(1)}\|_{X_T^{s,b}} \|u_{\sigma(2)}\|_{X_T^{s,b}} \|u_{\sigma(3)}\|_{X_T^{s,b}}.
\]

**Proof.** Using standard arguments (see for instance [5, Proposition 3.3]), it suffices to prove that

\[
\|u_1u_2\bar{u}_3\|_{X_T^{s,0}} \leq C \sum_{\sigma \in S_3} \|u_{\sigma(1)}\|_{X_T^{s,b}} \|u_{\sigma(2)}\|_{X_T^{s,b}} \|u_{\sigma(3)}\|_{X_T^{s,b}}
\]

for some $b > 1/2$, $b' < 1/2$ such that $b + b' < 1$. Using duality, the last estimate is equivalent to:

\[
\left| \int \int \int u_1u_2\bar{u}_3 \right| \leq C \|u_0\|_{X_T^{s,0}} \sum_{\sigma \in S_3} \|u_{\sigma(1)}\|_{X_T^{s,b}} \|u_{\sigma(2)}\|_{X_T^{s,b}} \|u_{\sigma(3)}\|_{X_T^{s,b}}.
\]

where $\int \int$ denotes a space-time integral on $\mathbb{R}^2 \times \mathbb{R}$ with respect to the Lebesgue measure $dxdt$. We now perform a Littlewood-Paley decomposition in the left-hand side of the last inequality and using a symmetry argument, we are reduced to obtaining a bound on

\[
\left| \sum_{N_0 \geq N_2 \geq N_3} \sum_{N_0} \int \Delta_{N_0}(\bar{u}_0)\Delta_{N_1}(u_1)\Delta_{N_2}(u_2)\Delta_{N_3}(\bar{u}_3) \right|
\]

where the summation is meant over dyadic values of $N_1$, $N_2$, $N_3$ and $N_0$. The other possible orders of magnitudes of $N_1$, $N_2$ and $N_3$ provide all permutations involved in the sum of the right hand-side of (3.2).

**First case: $N_0 \geq N_1^{1+\delta}$ for some $\delta > 0$**

In this case, we can apply the 2d version of [10, Lemme 2.1.23] to obtain that for every $K$ there is $C_K$ such that

\[
\left| \int \int \Delta_{N_0}(\bar{u}_0)\Delta_{N_1}(u_1)\Delta_{N_2}(u_2)\Delta_{N_3}(\bar{u}_3) \right| \leq C_K N_0^{-K} \|\Delta_{N_0}u_0\|_{X_T^{0,b'}} \|\Delta_{N_1}u_1\|_{X_T^{0,b'}} \|\Delta_{N_2}u_2\|_{X_T^{0,b'}} \|\Delta_{N_3}u_3\|_{X_T^{0,b'}}.
\]

where $b' < \frac{1}{2}$. Now we can readily perform the $N_0$, $N_1$, $N_2$, $N_3$ summations thanks to the large negative power of $N_0$.

**Second case: $N_0 \leq N_1^{1+\delta}$, with $\delta > 0$ to be chosen later depending on $\epsilon$**

Combining Cauchy-Schwarz and (3.1), we write

\[
\left| \int \int \Delta_{N_0}(\bar{u}_0)\Delta_{N_1}(u_1)\Delta_{N_2}(u_2)\Delta_{N_3}(\bar{u}_3) \right| \leq \|\Delta_{N_1}(u_1)\Delta_{N_2}(u_2)\|_{L^2((0,T);L^2)} \|\Delta_{N_0}(u_0)\Delta_{N_3}(u_3)\|_{L^2((0,T);L^2)}
\]

\[
\leq C(N_2N_3)^{\delta} \left( \frac{N_2N_3}{N_0N_1} \right)^{\frac{1}{2}-\delta} \prod_{j=1}^{4} \|\Delta_{N_j}(u_j)\|_{X_T^{0,b'}}.
\]
A normalization yields that it suffices to prove the following inequality:

\[
\sum_{N_1 \geq N_2 \geq N_3, N_0 \leq N_1^{1+\delta}} (N_2 N_3)^\delta \left( \frac{N_2 N_3}{N_0 N_1} \right)^{\frac{4}{N_0 N_1}} N_0^{-1-s} (N_2 N_3)^{-\varepsilon} \times \| \Delta u_0 \|_{X_T^+, \nu} \| \Delta u_1 \|_{X_T^+, \nu} \prod_{j=3}^4 \| \Delta u_j \|_{X_T^+, \nu} \leq C \left( \sum_{N} \| \Delta u_0 \|_{X_T^+, \nu}^2 \right)^{1/2} \left( \sum_{N} \| \Delta u_1 \|_{X_T^+, \nu}^2 \right)^{1/2} \prod_{j=3}^4 \left( \sum_{N} \| \Delta u_j \|_{X_T^+, \nu}^2 \right)^{1/2}.
\]

In the range of summation,

\[
(N_2 N_3)^\delta \left( \frac{N_2 N_3}{N_0 N_1} \right)^{\frac{4}{N_0 N_1}} N_0^{-1-s} (N_2 N_3)^{-\varepsilon} = N_0^{-s-\frac{4}{N_0 N_1}+\delta} (N_2 N_3)^{\frac{4}{N_0 N_1}+\delta} N_1^{-1-s-\frac{4}{N_0 N_1}+\delta} \leq N_1^{-\kappa}
\]

where at the last step we have chosen \( \delta > 0 \) small enough enough in such a way that \( \kappa > 0 \), allowing us to sum over \( N_0, N_1, N_2, N_3 \). This completes the proof of Proposition 3.1. \( \square \)

As a standard consequence of Proposition 3.1 (see e.g. [5, Proposition 3.3]), we can obtain the following well-posedness result.

**Proposition 3.2.** Let \( R > 0 \) and \( s_0 \geq 1 \) be given. Then there exists \( T > 0 \) and \( b > \frac{1}{2} \) such that (1.1) has a unique local solution in \( X_T^{s_0, b} \) for every \( \varphi \in H^{s_0} \) with \( \| \varphi \|_{H^1} < R \). Moreover,

\[
\| u(t, x) \|_{X_T^{s_0, b}} \leq 2 \| \varphi \|_{H^{s_0}}.
\]

**Remark 3.1.** While the Proposition is stated above regularity \( H^1 \), it can be extended to lower regularity \( H^\varepsilon \) with \( \varepsilon > 0 \), however we do not need such a low regularity later on.

Our next proposition reduces studying the growth of the \( H^{2k} \) norm of the solution \( u(t, x) \) to the analysis of the growth of \( \| \partial_t^{h} u(t, x) \|_{L^2} \). In fact this last quantity is easier to handle, as \( \partial_t \) has better commutation properties with the nonlinear Schrödinger flow than the operator \( A \).

**Proposition 3.3.** Let \( k, s \in \mathbb{N} \) and \( R, \delta > 0 \) be given. Let \( T > 0 \) be associated with \( R \) and \( s_0 = 2k + s \) as in Proposition 3.2 and let \( u(t, x) \in X_T^{2k+s, b} \) be the unique local solution to (1.1) with initial condition \( \varphi \in H^{2k+s} \). Assume moreover that \( \sup_{t \in (-T, T)} \| u(t, x) \|_{H^1} < R \). Then there exists \( C > 0 \) such that

\[
\forall t \in (-T, T), \quad \| \partial_t^h u(t) - i^k A^h u(t) \|_{H^\varepsilon} \leq C \| u(t) \|_{H^{2k+s+1}}^{1+\delta}.
\]

**Proof.** We temporarily drop dependence on \( t \) since the estimates we prove are pointwise in time. In the sequel we shall also use without further comment that \( \sup_{t \in (-T, T)} \| u(t) \|_{H^1} < R \). We shall denote by \( \delta > 0 \) an arbitrary small number that can change from line to line. We start from the identity

\[
\partial_t^h u = i^h A^h u + \sum_{j=0}^{h-1} c_j \partial_t^j A^{h-j-1} (u | u|^2),
\]

available for every integer \( h \geq 1 \) and for suitable coefficients \( c_j \in \mathbb{C} \). Its elementary proof follows by induction on \( h \), using the equation solved by \( u(t, x) \).

Next we argue by induction on \( k \) in order to establish (3.4). More precisely by assuming (3.4) we shall prove that the same estimate is true if we replace \( k \) by \( k + 1 \). Indeed by (3.5), where we choose \( h = k + 1 \), the estimate (3.3) for \( k + 1 \) reduces to

\[
\| \partial_t^j (u | u|^2) \|_{H^{2k-2j+1}} \leq C \| u \|_{H^s}^{1+\delta} H^{s+2k+1}, \quad j = 0, \ldots, k.
\]

Hence we prove (3.0), assuming (3.4). Recalling (1.3), we have to prove

\[
\| D^{2k-2j+1} \partial_t^j (u | u|^2) \|_{L^2} \leq C \| u \|_{H^{s+2k+1}}^{1+\delta}, \quad j = 0, \ldots, k,
\]

(3.8) \( \| (\tilde{u}^s)^{2k-2j+1} \partial_t^j (u | u|^2) \|_{L^2} \leq C \| u \|_{H^{s+2k+1}}^{1+\delta}, \quad j = 0, \ldots, k \).

To prove (3.7) we expand time and space derivatives on the left-hand side. Since \( s \) is an integer and we never work with \( L^1 \) and \( L^\infty \) norms, we may replace the operator \( D \) by the usual gradient operator \( \nabla \), and
in particular, use the Leibniz rule. Hence by expanding space-time derivatives and by using Hölder, we can estimate as follows the l.h.s. in (3.7):

\[
\begin{aligned}
&\sum_{j_1+j_2=j} \prod_{l=1,2,3} \|\partial^j_t u\|_{W^{s+l}} \leq C \sum_{j_1+j_2=j} \prod_{l=1,2,3} \|\partial^j_t u\|_{H^{s+l+1}} \\
&\quad \leq C \sum_{j_1+j_2=j} \prod_{l=1,2,3} \|u\|_{H^{s+l\theta+1}}
\end{aligned}
\]

where we used the Sobolev embedding and the induction hypothesis at the last step. We proceed with a trivial interpolation argument,

\[
(\ldots) \leq C \|u\|_{H^{s+2k+1}} \left( \prod_{l=1,2,3} \|u\|_{H^{s+2k+l+1}} \right),
\]

where \(\theta_l = \frac{s + 2k + 1}{l} + (1 - \theta_l) = 2j_l + s_l + 1\). We conclude to (3.7) since by direct computation we have \(\sum_{l=1}^{3} \theta_l = 1\) for every \(j = 0, \ldots, k\).

We now turn to (3.8): by Leibniz rule and Hölder, we estimate the l.h.s.

\[
\sum_{j_1+j_2+j_3=j \atop j_2,j_3<j} \|\langle x\rangle^{2k-2j+s} \partial^j_t u\|_{L^1} \|\partial^j u\|_{L^\infty} \|\partial^j \delta u\|_{L^\infty} \leq C \sum_{j_1+j_2+j_3=j \atop j_2,j_3<j} \|\partial^j_t u\|_{H^{2k-2j+s}} \|\partial^j u\|_{H^{1+s}} \|\partial^j \delta u\|_{H^{1+s}}
\]

where we used Sobolev embedding. By interpolation we proceed with

\[
(\ldots) \leq C \sum_{j_1+j_2+j_3=j \atop j_2,j_3<j} \|\partial^j_t u\|_{H^{2k-2j+s}} \|\partial^j u\|_{H^{1+s}} \|\partial^j \delta u\|_{H^{1+s}} \|\partial^j \delta u\|_{H^{1+s}}
\]

where we used the inductive assumption (3.4) to estimate \(\|\partial^j u\|_{H^s}\) for \(l = 2, 3\) and \(s = 1, 2\). By a further interpolation step and using again (3.4) we get

\[
(\ldots) \leq C \|u\|_{H^{s+2k+2}} \left( \prod_{l=1,2,3} \|u\|_{H^{s+2k+l+1}} \right),
\]

where we have chosen

\[
\theta_l(s + 2k + 1) + (1 - \theta_l) = 2j_l + 2k - 2j + s, \quad \theta_l(s + 2k + 1) + (1 - \theta_l) = 2j_l + 1, \quad l = 2, 3.
\]

We conclude to (3.8) since one can check \(\sum_{l=1}^{3} \theta_l < 1\) for \(j = 0, \ldots, k\).

The next proposition will be crucial in the sequel. It allows to estimate the norm of time derivatives of the solution in the localized \(X^{s,b}_T\) spaces, by using suitable Sobolev norms of the initial datum.

**Proposition 3.4.** Let \(l \in \mathbb{N}, R, \delta > 0\) and \(s \in (0,2]\) be given. Let \(T > 0\) be associated with \(R\) and \(s_0 = 2l + 2\) as in Proposition (3.2) and \(u(t,x) \in X^{2l+2,b}_T\) be the unique local solution to (1.1) with initial condition \(\varphi \in H^{2l+2}\) and \(\|\varphi\|_{H^1} < R\). Assume moreover that \(\text{sup}_{t \in (-T,T]} \|u(t,x)\|_{H^1} < R\), then there exists \(C > 0\) such that:

\[
\begin{aligned}
&\|\partial^j_t u\|_{X^{s,b}_T} \leq C \|\varphi\|_{H^{2l+1}} \|\varphi\|_{H^{2l+1}} \|\varphi\|_{H^{2l+1}}, \quad \text{if } s \in (0,1]
\end{aligned}
\]

and

\[
\begin{aligned}
&\|\partial^j_t u\|_{X^{s,b}_T} \leq C \|\varphi\|_{H^{2l+1}} \|\varphi\|_{H^{2l+1}} \|\varphi\|_{H^{2l+1}}, \quad \text{if } s \in (1,2].
\end{aligned}
\]

**Proof.** We shall prove separately (3.9) and (3.10) by induction on \(l\).

**Proof of (3.9)**
We consider the integral formulation of the equation solved by $\partial_t^{l} u$,

$$\partial_t^{l} u(t) = e^{t A} \partial_t^{l} u(0) + \int_0^t e^{(t-\tau) A} \partial_t^{l} (u(\tau)|u(\tau)|^2) d\tau$$

and then by standard properties of the $X^{s,b}$ spaces,

$$\|\partial_t^{l} u\|_{X_t^{s,b}} \leq C \left( \|\partial_t^{l} u(0)\|_{H^s} + \left\| \int_0^t e^{(t-\tau) A} \partial_t^{l} (u(\tau)|u(\tau)|^2) d\tau \right\|_{X_t^{s,b}} \right).$$

Expanding the time derivative and using Proposition 3.1 we get

$$\|\partial_t^{l} u(t)\|_{X_t^{s,b}} \leq C \left( \|\partial_t^{l} u(0)\|_{H^s} + T^n \sum_{l_1+l_2+l_3=l} \|\partial_t^{l_1} u\|_{X_t^{s,b}} \|\partial_t^{l_2} u\|_{X_t^{s,b}} \|\partial_t^{l_3} u\|_{X_t^{s,b}} \right).$$

By interpolation and Proposition 3.3 we also have

$$\|\partial_t^{l} u(0)\|_{H^s} \leq \|\partial_t^{l} u(0)\|_{H_t^s} \leq C \|u(0)\|_{H_t^{s+b}} \leq C \|\varphi\|_{H_t^{s+b}} \leq C |\varphi|_{H_t^{s+b}}.$$ 

Therefore, estimating the second term on the r.h.s. in (3.11) is sufficient. We split the proof into two cases.

**First case:** $0 < \min\{l_1, l_2, l_3\} \leq \max\{l_1, l_2, l_3\} < l$

We use induction on $l$ and estimate

$$\sum_{\max\{l_1, l_2, l_3\} < l} \|\partial_t^{l_1} u\|_{X_t^{s,b}} \|\partial_t^{l_2} u\|_{X_t^{s,b}} \|\partial_t^{l_3} u\|_{X_t^{s,b}} \leq C \sum_{\max\{l_1, l_2, l_3\} < l} \|\varphi\|_{H_t^{s+b}} \leq C |\varphi|_{H_t^{s+b}}.$$ 

**Second case:** $0 = \min\{l_1, l_2, l_3\} \leq \max\{l_1, l_2, l_3\} < l$

We can assume $l_1 = 0$. Then we argue exactly as above except that, since $\|\varphi\|_{H_t^{s+b}} = \|\varphi\|_{L^2}$ is bounded since we assume a control on the $H^1$ norm of the initial datum, it is not necessary to introduce the parameter $\eta_1$. Hence we need only $\eta_2, \eta_3, \theta_1, \theta_2, \theta_3$. The conclusion is the same as above.

**Third case:** $\max\{l_1, l_2, l_3\} = l$

We estimate the terms in the sum at the r.h.s. of (3.11) as follows

$$\|u\|_{X_t^{s,b}} \|\partial_t^{l_1} u\|_{X_t^{s,b}} \leq C \|\varphi\|_{H_t^{s+b}} \|\partial_t^{l_1} u\|_{X_t^{s,b}} \leq C |\varphi|_{H_t^{s+b}} \|\partial_t^{l_1} u\|_{X_t^{s,b}}$$

where we have used (3.3) for $s_0 = 1$. 

We first notice that by interpolation and Proposition 3.3 (see the proof of (3.12)),
\[ \eta \]
By direct computation we have to consider three cases.

Next we estimate the sum on the r.h.s. of (3.14) by considering two cases.

First case: \( 0 < \min\{l_1, l_2, l_3\} \leq \max\{l_1, l_2, l_3\} < l \)

By combining the inductive assumption on \( l \) and (3.13) we get:

\[
\sum_{\max\{l_1, l_2, l_3\} < l} \|\partial_t^l u(t)\|_{X_T^{s,b}} \leq C \sum_{\max\{l_1, l_2, l_3\} < l} \left\{ \begin{array}{ll}
\|\partial_t^l u(t)\|_{X_T^{s,b}} & \leq C \sum_{\max\{l_1, l_2, l_3\} < l} \left( \|\partial_t^l u(0)\|_{X_T^{s,b}} + T^\gamma \sum_{l, l_1, l_2, l_3} \|\partial_t^l u\|_{X_T^{s,b}} \right) \right. \\
\end{array} \right.
\]

Second case: \( 0 = \min\{l_1, l_2, l_3\} \leq \max\{l_1, l_2, l_3\} < l \)

If \( l_1 = 0 \), then our previous proof is valid since we have to deal with the norm \( \|\varphi\|_{H^{l+1}} \) and hence we have regularity \( H^{l} \) and the interpolation argument above can be applied. However, in the cases \( l_2 = 0 \) or \( l_3 = 0 \) the proof needs to be slightly modified. We can assume \( l_2 = 0 \) then in this case \( \|\varphi\|_{H^{l+1}} = \|\varphi\|_{L^2} \) is bounded since we assume a control on the \( H^{l} \) norm of the initial datum, hence it is not necessary to introduce the parameter \( \eta_2 \) in the interpolation step. The conclusion is the same as above.

Third case: \( \max\{l_1, l_2, l_3\} = l \)

We have to consider three cases \( (l_1, l_2, l_3) = (l, 0, 0) \), \( (l_1, l_2, l_3) = (0, l, 0) \) and \( (l_1, l_2, l_3) = (0, 0, l) \) (the last two cases are similar). In the first case we have

\[
\|\partial_t^l u\|_{X_T^{s,b}} \leq C \|\partial_t^l u\|_{X_T^{s,b}}
\]
where we used the estimate

\begin{equation}
\|u\|_{X^{2-s,b}_T} \leq C\|\varphi\|_{H^1},
\end{equation}

which is a consequence of (3.3) where we choose \( s_0 = 1 \). We conclude this case by choosing \( T \) small enough, exactly as we did along the proof of (3.3). Finally, for \((l_1, l_2, l_3) = (0, l, 0)\) by Proposition 3.7, (3.10) and (3.9),

\begin{align*}
\|u\|_{X^{2-s,b}_T} \|\partial_t u\|_{X^{2-s,b}_T} \|u\|_{X^{2-s,b}_T} & \leq C\|\varphi\|_{H^s} \|\partial_t u\|_{X^{2-s,b}_T} \leq C\|\varphi\|_{H^s} \|\varphi\|_{H^{s-1}}^{\frac{1}{2}} \|\varphi\|_{H^{1}}^{\frac{1}{2}} \|\varphi\|_{H^{2s+2}}^{\frac{1}{2}} \\
& \leq C\|\varphi\|_{H^{2s+2}} \|\varphi\|_{H^{s-1}}^{\frac{1}{2}} \|\varphi\|_{H^{1}}^{\frac{1}{2}} \|\varphi\|_{H^{2s+2}}^{\frac{1}{2}}
\end{align*}

where we used interpolation and the a priori bound on the \( H^1 \) norm of the initial datum. This concludes the proof of (3.10).

\section{4. Modified energies and proof of Theorem 1.1}

The aim of this section is to introduce suitable energies and to measure how far they are from being exact conservation laws. Those energies are the key tool in order to achieve the growth estimate provided in Theorem 1.1. Along this section we denote by \( \int \) the integral on \( \mathbb{R}^2 \) with respect to the Lebesgue measure \( dx \), and \( \int \int \) the integral on \( \mathbb{R}^2 \times \mathbb{R} \) with respect to the Lebesgue measure \( dx dt \).

\begin{proposition}
Let \( u(t,x) \in C((-T, T); H^{2k+2}) \) be a local solution to (1.1) with initial datum \( \varphi \in H^{2k+2} \). Then we have:

\begin{equation}
\frac{d}{dt} \left( \int \frac{1}{2} \|\partial_t^k Au(t,x)\|_{L^2}^2 + S_{2k+2}(u(t,x)) \right) = R_{2k+2}(u(t,x))
\end{equation}

where \( S_{2k+2}(u(t,x)) \) is a linear combination of terms of the following type:

\begin{equation}
\int \partial_t^k L_0 \partial_t^{m_1} L_1 \partial_t^{m_2} u_2 \partial_t^{m_3} u_3, \quad m_1 + m_2 + m_3 = k, \quad m_1 < k.
\end{equation}

and \( R_{2k+2}(u(t,x)) \) is a linear combination of terms of the following type:

\begin{equation}
\int \partial_t^k L_0 \partial_t^{l_1} L_1 \partial_t^{l_2} u_2 \partial_t^{l_3} u_3, \quad l_1 + l_2 + l_3 = k + 1, \quad l_1 \leq k.
\end{equation}

where in (1.1) and (1.2) we have \( v_0, u_1, u_2, u_3 \in \{u, \bar{u}\} \) and \( L \) can be any of the following operators:

\begin{align*}
Lu &= \partial_x u & \text{for } i = 1, 2, & \quad Lu = (x) u, & \quad Lu = u.
\end{align*}

\begin{proof}
We have

\begin{equation}
\partial_t \left( \partial_t^k \sqrt{A}u \right) + A(\partial_t^k \sqrt{A}u) \pm \partial_t^k \sqrt{A}(|u|^2) = 0.
\end{equation}

Next we multiply the equation above by \( \partial_t^{k+1} \sqrt{A}u \) and we take the real part,

\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( \|\partial_t^k Au\|_{L^2}^2 \right) = \Re \int \partial_t^k \sqrt{A}(|u|^2) \partial_t^{k+1} \sqrt{A}u.
\end{align*}

By symmetry of the operator \( \sqrt{A} \) we have

\begin{align*}
\Re \int \partial_t^k \sqrt{A}(|u|^2) \partial_t^{k+1} \sqrt{A}u &= \Re \int \partial_t^k (|u|^2) \partial_t^{k+1} A\bar{u} \\
&= -\Re \int \partial_t^k (|u|^2) \partial_t^{k+1} \Delta \bar{u} + \Re \int \partial_t^k (|u|^2) \partial_t^{k+1} (|x|^2 \bar{u})
\end{align*}

and we proceed by integration by parts

\begin{equation}
(\ldots) = \sum_{i=1}^2 \Re \int \partial_t^k \partial_x (|u|^2) \partial_t^{k+1} \partial_x \bar{u} + \Re \int |x|^2 \partial_t^k (|u|^2) \partial_t^{k+1} \bar{u}.
\end{equation}

\end{proof}
Next notice that the first term on the r.h.s. in (4.3) can be written as follows

\[
\sum_{i=1}^{2} \text{Re} \int \partial_t^k \partial_x(u|u|^2) \partial_t^{k+1} \partial_x \bar{u} \\
= \sum_{i=1}^{2} \left(2 \text{Re} \int |u|^2 \partial_t^k \partial_x u \partial_t^{k+1} \partial_x \bar{u} + \text{Re} \int u^2 \partial_t^k \partial_x \bar{u} \partial_t^{k+1} \partial_x \bar{u} \right) \\
+ \sum_{l_1+l_2+l_3=k \atop \max \{l_1,l_2,l_3\} < k} \text{Re} \left( a_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} + b_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} \right),
\]

where \(a_{l_1,l_2,l_3},b_{l_1,l_2,l_3}\) are suitable real numbers. Rewriting

\[
(\ldots) = \frac{d}{dt} \sum_{i=1}^{2} \left( \int |\partial_t^k \partial_x u|^2 |u|^2 + \frac{1}{2} \text{Re} \int (\partial_t^k \partial_x u)^2 u^2 \right) \\
- \sum_{i=1}^{2} \left( \int |\partial_t^k \partial_x u|^2 \partial_t (|u|^2) + \frac{1}{2} \text{Re} \int (\partial_t^k \partial_x u)^2 \partial_t (u^2) \right) \\
+ \sum_{l_1+l_2+l_3=k \atop l_1 < k} \text{Re} \left( a_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} + b_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} \right),
\]

by elementary manipulations on the last two lines we get

\[
(\ldots) = \frac{d}{dt} \sum_{i=1}^{2} \left( \int |\partial_t^k \partial_x u|^2 |u|^2 + \frac{1}{2} \text{Re} \int (\partial_t^k \partial_x u)^2 u^2 \right) \\
- \sum_{i=1}^{2} \left( \int |\partial_t^k \partial_x u|^2 \partial_t (|u|^2) + \frac{1}{2} \text{Re} \int (\partial_t^k \partial_x u)^2 \partial_t (u^2) \right) \\
+ \frac{d}{dt} \sum_{l_1+l_2+l_3=k \atop l_1 < k} \text{Re} \left( a_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} + b_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} \right) \\
+ \sum_{l_1+l_2+l_3=k+1 \atop l_1 \leq k} \text{Re} \left( a_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} + b_{l_1,l_2,l_3} \partial_t^l \partial_x u \partial_t^{l_1} u \partial_t^{l_2} u \partial_t^{l_3} \bar{u} \partial_t^{k+1} \partial_x \bar{u} \right).
\]

This last expression is a sum of a linear combination of terms with structure (4.2) with \(Lu = \partial_x u\) plus a time derivative of a linear combination of terms of type (4.2) with \(Lu = \partial_x u\).

Notice that the second term on the r.h.s. in (4.3) rewrites

\[
\text{Re} \int |x|^2 \partial_t^k (\langle x \rangle u|u|^2) \partial_t^{k+1} \bar{u} = \text{Re} \int \partial_t^k (\langle x \rangle u|u|^2) \partial_t^{k+1} (\langle x \rangle \bar{u}) - \text{Re} \int \partial_t^k (u|u|^2) \partial_t^{k+1} \bar{u}
\]

and arguing as above one checks, by developing first the derivative of order \(k\) with respect to time, that this expression is a time derivative of terms of type (4.1), where \(Lu = \langle x \rangle u\) or \(Lu = u\), plus a linear combination of terms with structure (4.2) where \(Lu = \langle x \rangle u\) or \(Lu = u\).

Next we estimate the energy \(R_{2k+2}\) introduced in Proposition 4.1.

**Proposition 4.2.** Let \(k \in \mathbb{N}, R > 0\) be given and \(u(t,x) \in X_T^{2k+2,\varphi}\) be the unique local solution to (1.1) with initial condition \(\varphi \in H^{2k+2}\) and \(\|\varphi\|_{H^k} < R\), where \(T > 0\) is associated with \(R\) and \(s_0 = 2k + 2\) as in Proposition 4.1. Assume moreover that \(\sup_{t \in (-T,T)} \|u(t,x)\|_{H^k} < R\). Then for every \(\delta > 0\) there exists \(C > 0\) such that:

\[
\left| \int_0^T R_{2k+2}(u(\tau,x))d\tau \right| \leq C\|\varphi\|_{H^{2k+2}}^{\delta + \frac{k+1}{2k+2}}.
\]
Proof. We have to estimate integrals like (4.2). After a Littlewood-Paley decomposition we are reduced to estimating

$$\sum_{N_0,N_1,N_2,N_3} \int \int \Delta_{N_0}(\partial^k_t L_{u_0})\Delta_{N_1}(\partial^{l_1}_t L_{u_1})\Delta_{N_2}(\partial^{l_2}_t u_2)\Delta_{N_3}(\partial^{l_3}_t u_3)$$

where

$$l_1 + l_2 + l_3 = k + 1, \quad l_3 \leq k.$$  

Here we have used the compact notation \( \int \int \) to denote the space-time integral on the strip \((-T, T) \times \mathbb{R}^2\) and \(\Delta_N\) denotes the Littlewood-Paley localization associated with the operator \(A\) at dyadic frequency \(N\). We split the sum in several pieces depending on the frequencies \(N_0, N_1, N_2, N_3\) and we shall make extensively use of the following bilinear estimate (see [10] Proposition 2.3.15]). For every \(\delta \in (0, \frac{1}{2})\), \(b > \frac{1}{2}\) there exists \(C > 0\) such that:

\[
\|\Delta_N u(\Delta_M v)\|_{L^2((0,T);L^2)} \leq C \left( \min \left\{ N, M \right\} \right)^{\frac{1}{2}-\frac{\delta}{2}} \|\Delta_N u\|_{L^2_T X^{0,\delta}} \|\Delta_M u\|_{L^2_T X^{0,b}}.
\]

Using the equation solved by \(u\) and noticing that, with the imposed conditions on \(l_1, l_2, l_3\) we may assume \(l_2 \geq 1\) (otherwise \(l_3 \geq 1\) and it is symmetric) we get

\[
\left| \int \int (\partial^k_t L_{u_0})(\partial^{l_1}_t L_{u_1})\partial^{l_2}_t u_2\partial^{l_3}_t u_3 \right| \leq C \left\{ \left| \int \int (\partial^k_t L_{u_0})(\partial^{l_1}_t L_{u_1})(\partial^{l_2}_t u_2)\partial^{l_3}_t u_3 \right| + \int \int (\partial^k_t L_{u_0})(\partial^{l_1}_t L_{u_1})(\partial^{l_2}_t-1(u|u|^2))\partial^{l_3}_t u_3 \right\}.
\]

The second term on the right hand side is estimated by Cauchy-Schwarz with

\[
\left( \int_0^T \|\partial^k_t L_{u_0}(\tau)\|_{L^2} \|\partial^{l_1}_t L_{u_1}\|_{L^2} \|\partial^{l_2}_t u_2\|_{L^2} \|\partial^{l_3}_t u_3\|_{L^2} d\tau \right) \||\partial^{l_2}_t-1(u|u|^2)\|_{L^\infty((0,T);L^2)} \|\partial^{l_3}_t u_3\|_{L^\infty((0,T);L^2)}
\]

where we used Sobolev embedding. Using (8.9) we proceed with

\[
(\ldots) \leq C \|\varphi\|_{H^{2k+1}} \|\varphi\|_{H^{2l_1+1}} \|\varphi\|_{H^{2l_2+1}} \|\varphi\|_{H^{2k+2}} \|\partial^{l_2}_t-1(u|u|^2)\|_{L^\infty((0,T);H^{1+\delta})}
\]

where

\[
\begin{align*}
\eta_2(2k + 2) + (1 - \eta) &= 2k + 1 \\
\eta_2(2k + 2) + (1 - \eta_2) &= 2l_1 + 1, \quad i = 1, 3
\end{align*}
\]

and hence, by using the bound assumed on \(\|\varphi\|_{H^1}\) we can continue the estimate above as follows

\[
\cdots \leq C \|\varphi\|_{H^{2k+2}} \|\partial^{l_2}_t-1(u|u|^2)\|_{L^\infty((0,T);H^{1+\delta})}.
\]

Expanding \(\partial^{l_2}_t-1(u|u|^2)\) and using that \(H^{1+\delta}\) is an algebra we get

\[
\|\partial^{l_2}_t-1(u|u|^2)\|_{L^\infty((0,T);H^{1+\delta})} \leq C \sum_{j_1+j_2+j_3 = 2l_2-1} \|\partial^{l_1}_t u\|_{L^\infty((0,T);H^{1+\delta})} \|\partial^{l_2}_t u\|_{L^\infty((0,T);H^{1+\delta})} \|\partial^{l_3}_t u\|_{L^\infty((0,T);H^{1+\delta})}
\]

\[
\leq C \sum_{j_1+j_2+j_3 = 2l_1-1} \|\varphi\|_{H^{2l_1+1}} \|\varphi\|_{H^{2l_2+1}} \|\varphi\|_{H^{2l_3+1}} \|\varphi\|_{H^{2k+2}}
\]

\[
\leq C \sum_{j_1+j_2+j_3 = 2l_1-1} \|\varphi\|_{H^{2k+2}} \|\varphi\|_{H^{2k+2}} \|\varphi\|_{H^{2k+2}}
\]

where

\[
\theta_i(2k + 2) + (1 - \theta_i) = 2j_i + 1, \quad i = 1, 2, 3
\]

and recalling the a priori bound assumed on \(\|\varphi\|_{H^1}\) we conclude that

\[
\|\partial^{l_2}_t-1(u|u|^2)\|_{L^\infty((0,T);H^{1+\delta})} \leq C \|\varphi\|_{H^{2k+2}}^{2k+2}.
\]
By combining the estimates above we get that the second term on the right-hand side in (4.3) can be estimated up to a constant by
\[ \| \varphi \|_{H^{2k+2}}^{2+2|b|+2|a|+2} = \| \varphi \|_{H^{2k+2}}^{2+\delta}. \]
We now focus on the first term on the right-hand side in (4.5) and by Littlewood-Paley decomposition we are reduced to estimating
\[ \sum_{N_0,N_1,N_2,N_3} \int \int \Delta N_0 (\partial_t^k L u_0) \Delta N_1 (\partial_t^l L u_1) \Delta N_2 (\partial_t^{l-1} A u_2) \Delta N_3 (\partial_t^m u_3). \]
Here we used again \( \int \int \) to denote the space-time integral on the strip \((-T,T) \times \mathbb{R}^2\) and \( \Delta N \) still denotes the Littlewood-Paley localization associated with the operator \( A \) at dyadic frequency \( N \). We split the sum in several pieces depending on the frequencies \( N_0, N_1, N_2, N_3 \) and we shall make again extensive use of the bilinear estimate (4.4). Next we consider several subcases.

**First subcase:** \( \min \{N_0, N_2\} \geq \max \{N_1, N_3\} \)

By Cauchy-Schwarz,
\[ \left| \int \int \Delta N_0 (\partial_t^k L u_0) \Delta N_1 (\partial_t^l L u_1) \Delta N_2 (\partial_t^{l-1} A u_2) \Delta N_3 (\partial_t^m u_3) \right| \leq \| \Delta N_0 (\partial_t^k L u_0) \|_{L^2((0,T);L^2)} \| \Delta N_1 (\partial_t^l L u_1) \|_{L^2((0,T);L^2)} \| \Delta N_2 (\partial_t^{l-1} A u_2) \|_{L^2((0,T);L^2)} \| \Delta N_3 (\partial_t^m u_3) \|_{L^2((0,T);L^2)} \]
and by (4.4) we can continue as follows
\[ \ldots \leq C \frac{(N_1 N_3)^{\frac{1}{2}-\delta}}{(N_0 N_2)^{\frac{1}{2}}-\delta} \| \Delta N_0 (\partial_t^k L u) \|_{X^{0,s}_T} \| \Delta N_1 (\partial_t^l L u) \|_{X^{0,s}_T} \| \Delta N_2 (\partial_t^{l-1} A u_2) \|_{X^{0,b}_T} \| \Delta N_3 (\partial_t^m u_3) \|_{X^{0,b}_T}. \]

Summarizing,
\[ \sum_{N_0,N_1,N_2,N_3, \begin{align*} \min(N_0,N_2) &\geq \max(N_1,N_3) \end{align*}} \left| \int \int \Delta N_0 (L u_0) \Delta N_1 (L u_1) \Delta N_2 (A u_2) \Delta N_3 (u_3) \right| \leq C \| L \partial_t^k u \|_{X^{0,s}_T} \| L \partial_t^l u \|_{X^{0,s}_T} \| A \partial_t^{l-1} u \|_{X^{0,b}_T} \| \partial_t^m u \|_{X^{0,b}_T} \]
\[ \leq C \| \partial_t^k u \|_{X^{0,s}_T} \| \partial_t^l u \|_{X^{0,s}_T} \| \partial_t^{l-1} u \|_{X^{0,b}_T} \| \partial_t^m u \|_{X^{0,b}_T}, \]
where we used Lemma 2.1 at the last step, assuming we chose \( b > \frac{1}{2} \) in such a way the estimate at the last line fits with Lemma 2.1.

**Second subcase:** \( \min \{N_1, N_2\} \geq \max \{N_0, N_3\} \)

We can argue as above and we are reduced to the previous case by noticing that we have the inequality \( \frac{N_0 N_3}{N_1 N_2} \leq \frac{N_1}{N_2} \) since in this subcase \( N_0 \leq N_1 \).

**Third subcase:** \( \min \{N_3, N_2\} \geq \max \{N_0, N_1\} \)

We can argue as above and we are reduced to the first subcase by noticing that we have the inequality \( N_0 N_1 \leq N_3^2 \) and hence \( \frac{N_0 N_3}{N_2 N_3} \leq \frac{N_1}{N_2} \).

**Fourth subcase:** \( \min \{N_1, N_3\} \geq \max \{N_0, N_2\} \)

We can argue as above and we are reduced to the first subcase by noticing that we have the inequality
We conclude by computing $\theta, \theta_\gamma$ with initial condition $\min_{14} F_{ABRICE PLANCHON, NIKOLAY TZVETKOV, AND NICOLA VISCIGLIA}$. We are therefore left with proving $(\ldots)$ this concludes the proof.

We prove the desired estimate for every expression with type (4.1). Indeed by Hölder we have for $N \leq N_1 + 2 \gamma_1 + (1 - \gamma_2) = 2 \eta_2 + 1, 2 \eta_3 + 1$.

We conclude by computing $\theta, \theta_1, \theta_3, \gamma_2, \eta_2, \eta_3$ and noticing that

\[
\theta + \theta_1 + \frac{1}{2} \gamma_2 + \frac{1}{2} \eta_2 + \frac{1}{2} \theta_3 + \frac{1}{2} \eta_3 = \frac{8k + 1}{4k + 2},
\]

and this concludes the proof.

Next we estimate the terms involved in the expression of $S_{2k+2}$ introduced in Proposition 4.1.

**Proposition 4.3.** Let $k \in \mathbb{N}, R > 0$ be given and $u(t,x) \in X^{2k+2,b}_T$ be the unique local solution to (1.1) with initial condition $\varphi \in \mathcal{H}^{2k+2}$ and $\|\varphi\|_{\mathcal{H}^{1}} < R$, where $T > 0$ is associated with $R$ and $s_0 = 2k + 2$ as in Proposition 3.2. Assume moreover that $\sup_{t \in (-T, T)} \|u(t,x)\|_{\mathcal{H}^{1}} < R$. Then for every $\delta > 0$ there exists $C > 0$ such that

\[
\sup_{t \in (-T, T)} \|S_{2k+2}(u(t,x))\| \leq C \|\varphi\|_{\mathcal{H}^{2k+2}}^{\frac{8k + 1}{4k + 2} + \delta}.
\]

**Proof.** We prove the desired estimate for every expression with type (1.1). Indeed by Hölder we have for every fixed $t \in (-T, T)$

\[
\int \partial_t^k L u_0 \partial_{t}^{m_1} L u_1 \partial_{t}^{m_2} u_2 \partial_{t}^{m_3} u_3 \leq \|\partial_t^k L u_0\|_{L^2} \|\partial_{t}^{m_1} L u_1\|_{L^2} \|\partial_{t}^{m_2} u_2\|_{L^\infty} \|\partial_{t}^{m_3} u_3\|_{L^\infty},
\]

and by Sobolev embedding and (1.4) we proceed with

\[
(\ldots) \leq C \|\partial_t^k L u\|_{L^2} \|\partial_{t}^{m_1} L u\|_{L^2} \|\partial_{t}^{m_2} u\|_{L^2} |1 - \delta| \|\partial_{t}^{m_2} u\|_{H^2} \|\partial_{t}^{m_3} u\|_{H^1} \|\partial_{t}^{m_3} u\|_{H^2} \|\partial_{t}^{m_3} u\|_{H^2} \|\partial_{t}^{m_3} u\|_{H^2} \|\partial_{t}^{m_3} u\|_{H^2}.
\]
Then we can use (3.4) to get
\[
(\ldots) \leq C \|u\|_{H^{2k+1}} \|u\|_{H^{2m_1+1}} \|u\|_{H^{2m_2+1}} \|u\|_{H^{2m_3+1}} \|u\|_{H^{2k+2}}^\delta \\
\leq C \|\varphi\|_{H^{2k+1}} \|\varphi\|_{H^{2m_1+1}} \|\varphi\|_{H^{2m_2+1}} \|\varphi\|_{H^{2m_3+1}} \|\varphi\|_{H^{2k+2}}^\delta
\]
where \( \delta > 0 \) is a small constant that can change from line to line and the last estimate follows from the embedding \( X_T^b \subset C((-T,T);H^b) \) for \( b > \frac{1}{2} \) and (3.3). Next we choose \( \theta, \theta_1, \theta_2, \theta_3 \in [0,1] \) such that
\[
\begin{aligned}
\theta(2k + 2) + (1 - \theta) = 2k + 1 \\
\theta_i(2k + 2) + (1 - \theta_i) = 2m_i + 1, \quad i = 1, 2, 3
\end{aligned}
\]
and by interpolation, we proceed with
\[
\int \partial_t^k L u_0 \partial_{t_1}^{m_1} L u_1 \partial_{t_2}^{m_2} u_2 \partial_{t_3}^{m_3} u_3 \leq C \|\varphi\|_{H^{2k+2}}^\delta
\]
and we conclude by computing explicitly \( \theta + \theta_1 + \theta_2 + \theta_3 \).

**Proof of Theorem 1.1** It will follow as a consequence of Propositions 4.1, 4.2, 4.3. Let
\[
\sup_{t \in (-\infty, \infty)} \|u(t,x)\|_{H^1} = R
\]
then \( R < \infty \) by (4.6). By integration on the strip \((0,T)\) of the identity (4.1) we get
\[
\frac{1}{2} \|\partial_t^k Au(T, x)\|_{L^2}^2 + S_{2k+2}(u(T, x)) \leq \frac{1}{2} \|\partial_t^k Au(0, x)\|_{L^2}^2 + S_{2k+2}(u(0, x)) + \int_0^T R_{2k+2}(u(\tau, x))d\tau
\]
where \( T \) is the one defined in Propositions 4.2 and 4.3 with \( R \) is defined as above. Then we get
\[
\frac{1}{2} \|\partial_t^k Au(T, x)\|_{L^2}^2 - \frac{1}{2} \|\partial_t^k Au(0, x)\|_{L^2}^2 \leq C \left( \|u(0, x)\|_{H^{2k+2}}^\delta + \|u(0, x)\|_{H^{2k+2}}^{\delta + 1}\right)
\]
Next notice that if we assume that \( u(t, x) \) is the nontrivial solution (otherwise the conclusion is trivial) then \( \|u(t, x)\|_{H^{2k+2}} \geq \|u(t, x)\|_{L^2} = \text{const} > 0 \) and the second term on the r.h.s. in (4.7) may be absorbed by the first one provided that we modify the multiplicative constant; then
\[
\frac{1}{2} \|\partial_t^k Au(T, x)\|_{L^2}^2 - \frac{1}{2} \|\partial_t^k Au(0, x)\|_{L^2}^2 \leq C \|u(0, x)\|_{H^{2k+2}}^{\delta + 1}\delta
\]
One easily checks that by (4.3) the bound above can be iterated with the same constants, namely
\[
\frac{1}{2} \|\partial_t^k Au((n + 1)T, x)\|_{L^2}^2 - \frac{1}{2} \|\partial_t^k Au(nT, x)\|_{L^2}^2 \leq C \|u(nT, x)\|_{H^{2k+2}}^{\delta + 1} + \delta
\]
for every \( n \in \mathbb{N} \). By summing up for \( n \in [0, N - 1] \) we obtain
\[
\|\partial_t^k Au(NT, x)\|_{L^2}^2 \leq C \sum_{n \in \{0, \ldots, N - 1\}} \|u(nT, x)\|_{H^{2k+2}}^{\delta + 1} + \delta
\]
and it implies
\[
\sup_{n \in [0, N]} \|\partial_t^k Au(nT, x)\|_{L^2}^2 \leq CN \left( \sup_{n \in [0, N]} \|u(nT, x)\|_{H^{2k+2}} \right)^{\delta + 1} + \delta
\]
which implies by (3.4) the estimate
\[
\sup_{n \in [0, N]} \|u(nT, x)\|_{H^{2k+2}} \leq CN^{2(2k+1)}
\]
and \( \forall N \in \mathbb{N} \),
\[
\|u(NT, x)\|_{H^{2k+2}} \leq CN^{2(2k+1) - \frac{1}{2}}.
\]
By using (3.3) it is easy to deduce that the estimate above implies
\[
\sup_{t \in [NT, (N+1)T]} \|u(t, x)\|_{H^{2k+2}} \leq CN^{2(2k+1) - \frac{1}{2}}
\]
provided that we suitably modify the multiplicative constant \( C \). Summarizing we get that, for all \( t > 0 \),
\[
\|u(t, x)\|_{H^{2k+2}} \leq C t^{\frac{2(2k+1)}{1}}
\]
The same argument works for \( t < 0 \).
We intend to provide a direct proof, based on integration by parts, of the crucial bilinear estimate from [16], for solutions to
\begin{equation}
\tag{4.8}
i\partial_t u - \Delta u + |x|^2 u = 0 .
\end{equation}

**Theorem 4.1.** Let \( 1 \leq M \leq N \) be dyadic numbers; for \( T \in (0, \infty) \) there exists \( C_T \) such that
\begin{equation}
\tag{4.9}
\| u_N v_M \|_{L^2((0,T);L^2)}^2 \leq C_T MN^{-1} \| u_N(0) \|_{L^2}^2 \| v_M(0) \|_{L^2}^2
\end{equation}
where \( u_N \) and \( v_N \) are spectrally localized solutions to \( (4.8) \) (namely \( \Delta_N u_N = u_N , \Delta_M v_M = v_M \)) respectively with initial datum \( u_N(0), v_M(0) \).

Such bilinear estimates were first obtained for solutions to the classical linear Schrödinger equation in [2], using direct computations in Fourier variables. In [6], so-called interaction Morawetz interaction estimates were introduced for the 3D nonlinear Schrödinger equation, relying on a bilinear version of the classical Morawetz estimate. Here, we rely on the bilinear computation from [15] that non only extended such bilinear virial estimates to low dimensions but also allowed to recover Bourgain’s estimates from [2]. We will follow the strategy from [13] where bilinear estimates on bounded domains were obtained, bypassing the need for Fourier localization. We split the proof in several steps. First we prove that, for a given \( T \in (0, \infty) \):
\begin{equation}
\tag{4.10}
\int_0^T \left( \int \int_{|x-y|<\frac{1}{4T}} M |u_N(x) \nabla_y \bar{v}_M(y) + \bar{v}_M(y) \nabla_x u_N(x)|^2 \, dx \, dy \right) dt \leq C_T N \| u_N(0) \|_{L^2}^2 \| v_M(0) \|_{L^2}^2 .
\end{equation}

Next we deduce from (4.10) that
\begin{equation}
\tag{4.11}
\int_0^T \left( \int |\nabla_x (v_M u_N)|^2 \, dx \right) dt \leq C_T MN \| u_N(0) \|_{L^2}^2 \| v_M(0) \|_{L^2}^2 .
\end{equation}

Estimate (4.11), along with a companion easier estimate for \( \int_0^T \left( \int |x|^2 |v_M u_N|^2 \, dx \right) dt \), implies
\begin{equation}
\tag{4.12}
\int_0^T \| v_M \bar{u}_N \|_{H^1}^2 \, dt \leq C_T MN \| u_N(0) \|_{L^2}^2 \| v_M(0) \|_{L^2}^2 .
\end{equation}

Finally, by a spectral localization argument, we prove that (4.12) implies (4.9).

**Proof of (4.10).** We first remark for later use that once (4.10) will be established, then we are allowed to replace \( v_N \) by \( A u_N \) (which is still a localized solution to (4.8)) and we get:
\begin{equation}
\tag{4.13}
\int_0^T \left( \int \int_{|x-y|<\frac{1}{4T}} M |u_N(x) \nabla_y (A \bar{v}_M)(y) + (A \bar{v}_M)(y) \nabla_x u_N(x)|^2 \, dx \, dy \right) dt \\
\leq C_T NM^2 \| u_N(0) \|_{L^2}^2 \| v_M(0) \|_{L^2}^2 .
\end{equation}

Next we focus on the proof of (4.10) and from now on, \( T \) is fixed in \( (0, +\infty) \). Let \( \rho : \mathbb{R}^2 \to \mathbb{R} \) be a \( C^1 \) function whose derivative is piecewise differentiable, with \( H_\rho \) denoting the bilinear form associated to its Hessian (as a distribution), \( H_\rho(a, b) = \sum_{i,j} \partial_{ij} \rho(a_k b_j) \); all \( \partial_{ij} \rho \) are actually piecewise continuous functions and under such assumptions all subsequent integrations by parts are fully justified in the classical sense. We claim that, for any couple of solutions \( u, v \) of (1.8),
\begin{equation}
\tag{4.14}
\int_0^T \left( \int \int H_\rho(x-y)(\bar{v}(y) \nabla_x u(x) + u(x) \nabla_y \bar{v}(y), v(y) \nabla_x \bar{u}(x) + \bar{u}(x) \nabla_y v(y)) \, dx \, dy \right) dt \\
\leq C_T \| \nabla \rho \|_{L^\infty} (\| v(0) \|_{L^2}^2 \| u(0) \|_{H^1}^2 + \| u(0) \|_{L^2}^2 \| v(0) \|_{L^2}^2 + \| v(0) \|_{H^1}^2 )
\end{equation}
where we dropped time dependence for notational simplicity. Following [13] we define a convex function \( \rho_M : \mathbb{R} \to \mathbb{R} \),
\[ \rho_M(z) = \begin{cases} 
\frac{M}{2} z^2 + \frac{1}{2M}, & z \leq \frac{1}{M} \\
z, & z > \frac{1}{M}
\end{cases} \]
and we use (4.14) with \( \rho(x - y) = \rho_M(x_1 - y_1) \): we get, by direct computation of the Hessian \( H_{\rho_p} \),
\[
\int_0^T \left( \int \int_{|x - y| < \frac{1}{T}} M_1(\tilde{v}(y) \partial_x u(x) + u(x) \partial_y \tilde{v}(y))^2 \, dx \, dy \right) \, dt \leq C_T \| \rho_0 \|_{L^2_x}^2 \| u_0 \|_{L^2}^2 \| u_0 \|_{H^1_t}^2 + \| u(0) \|_{L^2}^2 \| v(0) \|_{L^2}^2 \| v(0) \|_{H^1_t}^1
\]
where there is no contribution in the region \( |x - y| > \frac{1}{T} \) as \( H_{\rho_p} = 0 \) there, and we used that \( \| \rho_M(z) \|_{L^\infty} \leq 1 \). Of course by choosing \( \rho(x - y) = \rho(x_2 - y_2) \) we get a similar estimate where \( x_1, y_1 \) are replaced by \( x_2, y_2 \) and by combining the two estimates we get (4.11) where we noticed that \( |x - y| < \frac{1}{T} \subset \max \{|x_1 - y_1|, |x_2 - y_2|\} < \frac{1}{T} \).
Replacing \( u \) and \( v \) by \( u_N \) and \( v_M \) and using spectral localization we get (4.10).

Next we focus on the proof of (4.14). We compute the second derivative w.r.t. time of the functional
\[
I_\rho(t) = \int \int |u(x)|^2 \rho(x - y) |v(y)|^2 \, dx \, dy
\]
where for for simplicity we have dropped the time dependence of \( u, v \). In order to do so, recall that by the classical virial computation we get for a solution \( w(t, x) \) to (4.8) (we drop again time-dependence of \( w \) and set \( \rho_g(x) = \rho(x - y) \) to emphasize that \( y \) is a fixed base point here):
\[
\frac{d}{dt} \int \rho_g(x) |w(x)|^2 \, dx = 2 \int \nabla \rho_g(x) \cdot \text{Im} (\nabla \bar{v}(x) w(x)) \, dx
\]
(4.15) \[
\frac{d^2}{dt^2} \int \rho_g(x) |w|^2 \, dx = 4 \int H_{\rho_g}(\nabla w(x), \nabla \bar{v}(x)) - \int \Delta \rho_g(x) \Delta (|w|^2) \, dx - 4 \int x \cdot \nabla \rho_g(x) |w(x)|^2 \, dx,
\]
where we emphasize that we will not be using more than two derivatives on \( \rho_g \). Next, using (4.15) we get
\[
\frac{d}{dt} I_{\rho}(t) = 2 \int \nabla \rho(x - y) \cdot \text{Im} \left( \nabla_x \bar{u}(x) u(x) \right) |v(y)|^2 \, dx \, dy - 2 \int \nabla \rho(x - y) \cdot \text{Im} \left( \nabla_y \bar{v}(y) v(y) \right) |u(x)|^2 \, dx \, dy.
\]
Using that \( \| \nabla w \|_{L^2_x} \leq C \| w \|_{H^1_t} \) at fixed time followed by conservation of mass and energy for (4.8), we get
\[
\frac{d}{dt} I_{\rho}(t) \leq 2 \| \nabla \rho \|_\infty (\| v \|_{L^2_x}^2 \| u \|_{L^2} \| \nabla u \|_{L^2} + \| u \|_{L^2}^2 \| \nabla \|_{L^2} \| \nabla \|_{L^2}) \leq C \| \nabla \rho \|_{L^\infty} (\| v \|_{L^2}^2 \| u \|_{L^2}^2 \| u \|_{H^1} + \| u \|_{L^2}^2 \| v \|_{L^2} \| v \|_{H^1}^1).
\]
For later use notice also that using (4.15) on both mass densities,
\[
\int (x - y) \frac{d}{dt} |u(x)|^2 \, dx \frac{d}{dt} |v(y)|^2 \, dy = 2 \int (x - y) \cdot \text{Im} (\nabla_x \bar{u}(x) u(x)) \, dx \frac{d}{dt} |v(y)|^2 \, dy = -4 \int H_{\rho} (\text{Im} (\nabla_x \bar{u}(x) u(x)), \text{Im} (\nabla_y \bar{v}(y) v(y))) \, dx \, dy.
\]
On the other hand by combining (4.16) and (4.18),
\[
\frac{d^2}{dt^2} I_{\rho}(t) = 4 \int H_{\rho}(x - y) (\nabla u(x), \nabla \bar{u}(x)) |v(y)|^2 \, dx \, dy + 4 \int H_{\rho}(x - y) (\nabla \bar{v}(y), \nabla \bar{v}(y)) |u(x)|^2 \, dx \, dy - 4 \int \Delta \rho(x - y) \Delta (|u(x)|^2) |v(y)|^2 \, dx \, dy - 4 \int \Delta \rho(x - y) |u(x)|^2 \Delta (|u(x)|^2) \, dx \, dy
\]
\[- 8 \int H_{\rho}(x - y) (\text{Im} (\nabla \bar{u}(x) u(x)), \text{Im} (\nabla \bar{v}(y) v(y))) \, dy \, dx
\]
\[- 4 \text{Re} \int (x - y) \cdot \nabla \rho(x - y) |u(x)|^2 |v(y)|^2 \, dx \, dy = I + II + III + IV + V + VI.
\]
Following (4.9), we rewrite \( \Delta \rho(x - y) = -\nabla_x \cdot \nabla_y \rho(x - y) \) and integrate by parts w.r.t. to \( x \) and \( y \):
\[
III + IV = 8 \int \int H_{\rho}(x - y) (\text{Re} (\bar{u}(x) \nabla u(x)), \text{Re} (\bar{v}(y) \nabla v(x))) \, dx \, dy.
\]
Now, thinking about just one direction of derivation, we have the following identity
\[
4|v|^2(y) |\partial u|^2(x) + 4|\partial x|^2(x) |\partial u|^2(y) + 8 \frac{(v \bar{v} + \bar{v} \bar{v})(y)}{2} (u \partial u + u \partial u)(x) - 8 \frac{(v \bar{v} - \bar{v} \bar{v})(y)}{2} (\bar{u} \partial u - u \partial \bar{u})(x)
\]
\[
= 4|v|^2(y) |\partial u|^2(x) + 4|\partial x|^2(x) |\partial u|^2(y) + 4v \bar{v}(y) u \partial u(x) + 4\bar{v} \bar{v}(y) u \partial \bar{u}(x) = 4\bar{v}(y) \partial u(x) + u(x) \partial \bar{v}(y))^2,
\]
which allows to recombine $I + II + III + IV + V$, to get
\[
\frac{d^2}{dt^2}I_\rho(t) = 4 \int \int H_\rho(x - y)(\bar{v}(y)\nabla u(x) + u(x)\nabla \bar{v}(y), v(y)\nabla \bar{u}(x) + \bar{u}(x)\nabla v(y)) \, dxdy
\]
\[-4 \text{Re} \int \int \nabla \rho(x - y) \cdot (x - y)|v(y)|^2|u(x)|^2 \, dxdy.
\]

After integration in time of the identity above and by recalling (4.17), we get
\[
4 \int_0^T \left( \int \int \nabla \rho(x - y)(\bar{v}(y)\nabla u(x) + u(x)\nabla \bar{v}(y), v(y)\nabla \bar{u}(x) + \bar{u}(x)\nabla v(y)) \, dxdy \right) \, dt
\]
\[\leq C\parallel \nabla \rho \parallel_{L^\infty} (\parallel v(0) \parallel_{L^2}^2 \parallel u(0) \parallel_{H^1} + \parallel u(0) \parallel_{L^2}^2 \parallel v(0) \parallel_{H^1})
\]
\[+ 4 \int_0^T \left( \int \int |\nabla \rho(x - y)||y - x||v(y)||^2|u(x)|^2 \, dxdy \right) \, dt
\]
\[\leq C\parallel \nabla \rho \parallel_{L^\infty} (\parallel v(0) \parallel_{L^2}^2 \parallel u(0) \parallel_{H^1} + \parallel u(0) \parallel_{L^2}^2 \parallel v(0) \parallel_{H^1})
\]
\[+ C_T \parallel \nabla \rho \parallel_{L^\infty} \sup_{t \in (0, T)} (\parallel u(t) \parallel_{L^2}^2 \parallel v(t) \parallel_{L^2} \parallel v(t) \parallel_{L^2} + \parallel v(t) \parallel_{L^2}^2 \parallel u(t) \parallel_{L^2} \parallel u(t) \parallel_{L^2}).
\]

Using $\parallel y \parallel_{L^2}^2 \leq \parallel w \parallel_{H^1}$ and, again, conservation of mass and energy for (1.8), this estimate implies (4.14).

**Proof of (4.10) ⇒ (4.11).** We need a suitable local elliptic estimate for our operator $A = -\Delta + |x|^2$ to reproduce the computation from [13]. The next lemma is a modification of Lemma 4.2 in [13].

**Lemma 4.1.** There exists $C > 0$ and $\lambda_0 \geq 1$ such that, for any smooth function $\phi$ in $\mathbb{R}^2$ and $\lambda \geq \lambda_0$, the following pointwise estimate holds:
\[
|\phi(x)|^2 \leq C \lambda^{-2} \int_{|x - y| < \lambda^{-1}} |A\phi|^2 \, dy + C \lambda^2 \int_{|x - y| < \lambda^{-1}} |\phi|^2 \, dy, \quad \forall x \in \mathbb{R}^2.
\]

**Proof.** Without loss of generality we may restrict to real-valued $\phi$. The lemma is proved in [13] if we replace in the r.h.s. the operator $A$ by $-\Delta$ and the domain of integration by the smaller domain $|x - y| < (4\lambda)^{-1}$ (this fact follows from classical elliptic theory and Sobolev embedding for $\lambda = 1$ and then any $\lambda > 0$ by rescaling). Thus we conclude that provided we prove
\[
\lambda^{-2} \int_{|x - y| < (4\lambda)^{-1}} |\Delta \phi|^2 \, dy \leq C \lambda^{-2} \int_{|x - y| < \lambda^{-1}} |A\phi|^2 \, dy + C \lambda^2 \int_{|x - y| < \lambda^{-1}} |\phi|^2 \, dy.
\]

In order to prove this estimate we expand the square $\int |\Delta f|^2 = \int |Af - |y||^2f^2$ and after integrations by parts we get
\[
\int (|\Delta f|^2 + |y|^4f^2 + 2|y|^2|\nabla f|^2) \, dy = \int (|Af|^2 + 4|f|^2) \, dy
\]
for any real-valued function $f \in C_0^\infty(\mathbb{R}^2)$. Next we pick $f(y) = \chi_\lambda(y)\phi(y)$, where $\chi_\lambda(y) = \chi(\lambda(y - x))$, with $\chi(|z|) = 1$ on $|z| < \frac{\lambda}{4}$ and $\chi(|z|) = 0$ on $|z| > \frac{\lambda}{2}$. All subsequent cutoffs w.r.t. $y$ will be centered at $x$. Expanding $\int |\Delta(\chi_\lambda\phi)|^2$ and $\int |A(\chi_\lambda\phi)|^2$ and replacing in the previous identity we get:
\[
\int (|\chi_\lambda|^2|\Delta \phi|^2 + |y|^4|\chi_\lambda|^2|\phi|^2 + 2|y|^2|\nabla (\chi_\lambda \phi)|^2) \, dy
\]
\[= \int (|\chi_\lambda|^2|Av|^2 + 4|\chi_\lambda|^2|\phi|^2) \, dy - 2 \int \chi_\lambda|y|^2\phi(2\nabla \chi_\lambda \cdot \nabla \phi + \Delta \chi_\lambda \phi) \, dy.
\]

By Cauchy-Schwarz and elementary manipulations we estimate the last term on the r.h.s. as follows, for every $\mu > 0$ and with a universal constant $C > 0$:
\[
\left| \int \chi_\lambda|y|^2\phi(2\nabla \chi_\lambda \cdot \nabla \phi + \phi \Delta \chi_\lambda) \, dy \right|^2 \leq C \mu \int |y|^4|\chi_\lambda|^2|\phi|^2 \, dy + \frac{C}{\mu} \int (|\nabla \chi_\lambda \cdot \nabla \phi|^2 + |\Delta \chi_\lambda|^2|\phi|^2) \, dy.
\]

If we choose the constant $\mu$ small enough then we can absorb $\int |y|^4|\chi_\lambda v|^2 \, dx$ on the l.h.s. in (4.21) and by neglecting some positive terms we get, abusing notation for the constant $C$,
\[
\int |\chi_\lambda|^2|\Delta \phi|^2 \, dy \leq C \int (|\chi_\lambda|^2|A\phi|^2 + 4|\chi_\lambda|^2|\phi|^2 + |\nabla \chi_\lambda \cdot \nabla \phi|^2 + |\Delta \chi_\lambda|^2|\phi|^2) \, dy
\]
and by elementary considerations
\[
\int_{|z| < (2\lambda)^{-1}} |\Delta \phi|^2 dy \leq C \int_{|z| < (2\lambda)^{-1}} |A\phi|^2 dy + C\lambda^2 \int |\tilde{\chi}_\lambda \nabla \phi|^2 dy + C(1 + \lambda^2) \int |\phi|^2 dy
\]
where \(\tilde{\chi}_\lambda\) is a suitable enlargement of \(\chi_\lambda\), namely \(\tilde{\chi}_\lambda(y) = \bar{\chi}(\frac{y - z}{\lambda})\), with \(\bar{\chi}(|z|) = 1\) on \(|z| < \frac{1}{2}\) and \(\bar{\chi}(|z|) = 0\) on \(|z| > 1\). Then (4.19) follows provided that
\[
\int \tilde{\chi}_\lambda |\nabla \phi|^2 dy \leq C\lambda^{-2} \int |\phi|^2 dy + C\lambda^2 \int |\phi|^2 dy.
\]
In order to do that we write (either integrating by parts or replacing \(-\Delta\) by \(A - |x|^2\))
\[
-2 \int \tilde{\chi}_\lambda \phi \Delta \phi dy = 2 \int \tilde{\chi}_\lambda |\nabla \phi|^2 dy - \int \Delta \tilde{\chi}_\lambda |\phi|^2 dy = 2 \int \tilde{\chi}_\lambda \phi A\phi dy - 2 \int |\phi|^2 \tilde{\chi}_\lambda |\phi|^2 dy
\]
and hence
\[
2 \int \tilde{\chi}_\lambda |\nabla \phi|^2 dy + 2 \int |\phi|^2 \tilde{\chi}_\lambda |\phi|^2 dy = -2 \int \tilde{\chi}_\lambda \phi A\phi dy + \int \Delta \tilde{\chi}_\lambda |\phi|^2 dy
\]
\[
\leq C\lambda^{-2} \int |\phi|^2 dy + C(1 + \lambda^2) \int |\phi|^2 dy
\]
where we used Cauchy-Schwarz at the last step. □

We now proceed to prove that (4.10) \(\Rightarrow\) (4.11). The first term in the square at the l.h.s. of (4.10) turns out to be lower order: we compute by change of variable, Cauchy-Schwarz inequality and Strichartz estimate,
\[
(4.22) \quad \int_0^T \left( \int \int_{|x| < \frac{1}{2\lambda}} |u_N(x) \nabla_y \bar{v}_M(y)|^2 dx dy \right) dt = \int_0^T \int \int_{|z| < \frac{1}{2\lambda}} |u_N(x) \nabla_x \tilde{v}_M(x - z)|^2 dx dz dt
\]
\[
\leq \int_{|z| < \frac{1}{2\lambda}} \|u_N\|_{L^2((0,T);L^4)} \|\nabla v_M\|_{L^2((0,T);L^4)} dz \leq C_T \|u_N(0)\|_{L^2} \|v_M(0)\|_{L^2}
\]
where at the last step we used that \(\|\nabla v_M\|_{L^1((0,T);L^4)} \leq C_T \|v_M(0)\|_{L^2}\). In turn this bound follows by noticing that \(\nabla v_M\) is solution to the inhomogeneous equation associated with (4.8) with forcing term \(2xv_M\). Hence by the inhomogeneous Strichartz estimate, placing the forcing term in \(L^1((0,T);L^2)\),
\[
(4.23) \quad \|\nabla v_M\|_{L^1((0,T);L^4)} \leq C \|\nabla v_M(0)\|_{L^2} + C \|xv_M\|_{L^1((0,T);L^2)} \leq C_T \|v_M(0)\|_{H^1} \leq C_T M \|v_M(0)\|_{L^2},
\]
where we used conservation of energy for (4.8) and the bound \(\|xw\|_{L^2} \leq C \|w\|_{H^1}\) for every time independent function.

Recall that (4.22) holds with \(v_M\) replaced by \(Av_M\) (it is still a solution to (4.8)), hence we get:
\[
(4.24) \quad \int_0^T \left( \int \int_{|x| < \frac{1}{2\lambda}} |u_N(x) \nabla_y (A\bar{v}_M)(y)|^2 dx dy \right) dt \leq C_T M^4 \|u_N(0)\|_{L^2}^2 \|v_M(0)\|_{L^2}^2.
\]
We now proceed using the Lemma 4.1 and we get
\[
\int_0^T \left( \int M^2 \bar{v}_M(y) \nabla_x u_N(x)^2 \right) dt \leq C \int_0^T \left( \int \int_{|x| < \frac{1}{2\lambda}} M^2 |\bar{v}_M(y)\nabla_x u_N(x)|^2 \right. \right.
\]
\[
+ \left. \frac{1}{M^2} |A\bar{v}_M(y)\nabla_x u_N(x)|^2 dx dy \right) dt
\]
that by (4.10) and (4.13) implies
\[
(\ldots) \leq C \int_0^T \left( \int \int_{|x| < \frac{1}{2\lambda}} M^2 |u_N(x)\nabla_y \bar{v}_M(y)|^2 + \frac{1}{M^2} |u_N(x)\nabla_y (A\bar{v}_M)(y)|^2 dx dy \right) dt
\]
\[
+ C_T M \|u_N(0)\|_{L^2}^2 \|v_M(0)\|_{L^2}^2.
\]
Combining the above estimate with (4.22) and (4.24) we obtain
\[
\int_0^T \left( \int |\tilde{v}_M(x)\nabla_x u_N(x)|^2 \, dx \right) dt \leq C_T(M^2 + NM)\|u_N(0)\|_{L^2}^2 \|v_M(0)\|_{L^2}^2.
\]
On the other hand by Cauchy-Schwarz, Strichartz estimate and (4.24) we have
\[
\int_0^T \left( \int |u_N(x)\nabla_x \tilde{v}_M(x)|^2 \, dx \right) dt \leq \|u_N\|_{L^4((0,T);L^4)}^2 \|\nabla v_M\|_{L^4((0,T);L^4)}^2 \leq C_T M^2 \|v_M(0)\|_{L^2}^2 \|u_N(0)\|_{L^2}^2.
\]
Therefore combining this last estimate with (4.25) we get (4.11).

**Proof of (4.12).** Due to (4.11), it suffices to prove
\[
\int_0^T \left( \int |x|^2 |v_M u_N|^2 \, dx \right) dt \leq C_T M N \|u_N(0)\|_{L^2}^2 \|v_M(0)\|_{L^2}^2.
\]
By Hölder inequality we have
\[
\int_0^T \left( \int |x|^2 |v_M \tilde{u}_N|^2 \, dx \right) dt \leq \|x|v_M\|_{L^4((0,T);L^4)}^2 \|u_N\|_{L^4((0,T);L^4)}^2.
\]
Next notice that $|x|^2 v_M$ is solution to the inhomogeneous equation associated with (4.8), with source term $-4v_M - 2x \cdot \nabla v_M$. Again, using Strichartz and placing the source term in $L^3((0,T);L^2)$,
\[
|||x|^2 v_M||_{L^6((0,T);L^4)} \leq C(||x|^2 v_M(0)||_{L^2} + C||v_M||_{L^6((0,T);L^2)} + C|x| \cdot \nabla v_M||_{L^6((0,T);L^2)}
\]
where we used the time independent estimate $\|x \cdot \nabla u\|_{L^2} \leq C\|u\|_{H^2}$ (see (4.20)) and conservation of the $H^2$ norm for (4.8). Interpolation between (4.25) and the Strichartz estimate $\|v_M\|_{L^6((0,T);L^2)} \leq C\|v_M(0)\|_{L^2}$ implies $\|x|v_M||_{L^4((0,T);L^4)} \leq CM\|v_M(0)\|_{L^2}$. Combining this estimate, Strichartz for $u_N$ and (4.24) we obtain (4.26) (actually, a stronger version of (4.26) as on the r.h.s. we get $M^2$).

**Proof of the implication (4.12) \implies (4.9).** We can write
\[
\|v_M u_N\|_{L^2((0,T);L^2)} = \sum_{K \in 2^{\mathbb{N}}} \|\Delta_K(v_M u_N)\|_{L^2((0,T);L^2)}
\]
If $K > N$, we may forget about $\Delta_K$ and use (4.12) in order to get
\[
\sum_{K \geq N} ||\Delta_K(v_M u_N)||_{L^2((0,T);L^2)} \leq C \sum_{K \geq N} (1 + K)^{-2} \|v_M u_N\|_{L^2((0,T);H^1)}^2 \leq C_T M N^{-1} \|u_N(0)\|_{L^2}^2 \|v_M(0)\|_{L^2}^2.
\]
For $K \leq N$, denote $S_N = \sum_{K \leq N} \Delta_K$ and write directly
\[
\|S_N(v_M u_N)\|_{L^2}^2 \leq \|S_N v_M\|_{L^2}^2 \leq CN^{-2} \|S_N \sqrt{\Delta} S_N(v_M \tilde{u}_N)\|_{L^2}^2 + CN^{-4} \|\tilde{u}_N \Delta v_M\|_{L^2}^2 + \|\nabla v_M \cdot \nabla \tilde{u}_N\|_{L^2}^2.
\]
After integration in time, using Strichartz estimates to control $L^4$ norms (use (4.23) to control $\|\nabla v_M\|_{L^4}$ and a similar argument to control $\|\nabla \tilde{u}_N\|_{L^4}$) and (4.12), we get
\[
\int_0^T \|S_N(v_M u_N)\|_{L^2}^2 \, dt \leq C_N^{-2} \int_0^T \|v_M \tilde{u}_N\|_{H^1}^2 \, dt + C_T M^2 N^{-4}(M^2 + N^2) \|v_M(0)\|_{L^2}^2 \|\tilde{u}_N(0)\|_{L^2}^2 \leq C_T M N^{-1} \|v_M(0)\|_{L^2}^2 \|\tilde{u}_N(0)\|_{L^2}^2 + C_T M^2 N^{-2} \|v_M(0)\|_{L^2}^2 \|\tilde{u}_N(0)\|_{L^2}^2,
\]
and we complete the proof with $\|\tilde{u}_N(0)\|_{L^2} \leq C\|u_N(0)\|_{L^2}$. \qed
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