A class of index coding problems with rate $\frac{1}{3}$

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Outline

Groupcast Index Coding from the Interference Alignment perspective

Known results for groupcast

Our result for rate 1/3 index codes
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Groupcast Index Coding from the Interference Alignment perspective

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Our result for rate 1/3 index codes
Groupcast index coding

- A broadcast channel between the source and $T$ receivers
- Messages $\mathcal{W} = \{ W_i \in \mathbb{F}, i \in [1 : n] \}$.
- Demand set at receiver $j$: $D(j) \subseteq \mathcal{W}$
- Side information at receiver $j$: $S(j) \subseteq \mathcal{W}\backslash D(j)$

Linear Index Code and its Rate

- Index code: Map from \{ Messages \} $\rightarrow$ \{ L-length codewords \}
- Rate $R = \frac{1}{L}$.
- Our results can be generalised to vector-linear index codes.
Interference alignment framework for index codes

- Index Coding map be $B_{L \times n}$; Transmitted vector is $B\mathbf{W}$.
- Consider a sink $j$ which demands message $W_k$.
- Sink $j$ can cancel the contributions from $S(j)$, obtaining

$$\sum_{i : i \notin S(j)} W_i b_i = W_k b_k + \sum_{i : i \notin S(j) \cup \{k\}} W_i b_i$$

- Let $I(j, k) = \{W_i : i \notin S(j) \cup \{W_k\}\}$.
- Decoding is possible if $b_k$ is independent of space spanned by vectors assigned to $I(j, k)$ (interference constraints).
- Choose matrix $B$ such that this is satisfied with least possible $L$ (alignment opportunities).
Outline

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Known results for groupcast

Our result for rate 1/3 index codes
Tehrani et al, ISIT 2012, “Bipartite Index Coding”.

Maleki et al, “Index Coding - An Interference Alignment Perspective”, ISIT 2012, TIT Sep. 2014.

Jafar, “Topological Interference Management Through Index Coding”, IEEE TIT, Jan 2014 [Jaf].
Known results for groupcast

- **Rate 1**: Each receiver demands exactly one message, and has all others as side-information.
- **Rate $\frac{1}{2}$**: [Jaf] via *alignment* and *conflict* graphs.
Alignment graph and conflict hypergraphs
Alignment graph and conflict hypergraphs
Alignment graph and conflict hypergraphs
Alignment graph and conflict hypergraphs
Alignment sets

Connected component of alignment graph = Alignment set
Rate $\frac{1}{2}$ result from [Jaf]

**Theorem (Rate $\frac{1}{2}$)**

An index coding problem is rate $\frac{1}{2}$ feasible if and only if there are no internal conflicts (conflicts within alignment sets).
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Our result for rate 1/3 index codes
Our contribution

Main result (A rate $\frac{1}{3}$ feasible class of index coding problems)

A rate $\frac{1}{2}$ infeasible IC problem is rate $\frac{1}{3}$ feasible if every alignment set satisfies one of the following properties

- It doesn’t have both forks and cycles (follows from [Jaf]).
- It is a type-2 alignment set with no restricted internal conflicts.

Theorem (Known from [Jaf])

A rate $\frac{1}{2}$ infeasible IC problem is rate $\frac{1}{3}$ feasible if no alignment set has both forks and cycles (i.e., $|I(j, k)| \leq 3, \forall j, k$).
Type-2 alignment set

- **Triangular interfering set:** A set of three messages interfering at some receiver, with at least two of them in conflict.
- Any two triangular interferers are ‘adjacent’ if they are meeting at conflict edges.
- **Type-2 alignment set:** A ‘connected component’ of such triangular interfering sets.
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The condition on the type-2 alignment set

Type-2 alignment set with no restricted internal conflicts.

\[ \Leftrightarrow \]

IC problem restricted to Type-2 alignment set is rate \( \frac{1}{2} \) feasible.

\[ \Leftrightarrow \]

Type-2 alignment set can be assigned vectors from a two dimensional space with all its internal conflicts resolved.
The condition on the type-2 alignment set

Type-2 alignment set with no *restricted internal conflicts*.

\[\uparrow\]

IC problem restricted to Type-2 alignment set is rate \( \frac{1}{2} \) feasible.

\[\downarrow\]

Type-2 alignment set can be assigned vectors from a two dimensional space with all its internal conflicts resolved.

\[\rightarrow\] A *necessary* condition for rate \( \frac{1}{3} \) feasibility
A necessary condition for rate $\frac{1}{3}$

**Theorem A**

An IC problem is rate $\frac{1}{3}$ feasible only if any type-2 alignment can be allocated vectors from a two dimensional vector space with all its internal conflicts resolved.
Proof of the Necessary Condition for rate $\frac{1}{3}$

Has to be 2D!
Proof of the Necessary Condition for rate $\frac{1}{3}$

Intersection has to be 2D
Proof of the Necessary Condition for rate $\frac{1}{3}$

Intersection has to be 2D
$\Rightarrow$ Union is 2D as well
Proof of the Necessary Condition for rate $\frac{1}{3}$
Proof of the Necessary Condition for rate \( \frac{1}{3} \)
Proof of the Necessary Condition for rate $\frac{1}{3}$

Entire Type-2 set has to be 2D!
The condition on the type-2 alignment set

Type-2 alignment set with no *restricted internal conflicts*.

\[ \iff \]

IC problem restricted to Type-2 alignment set is rate $\frac{1}{2}$ feasible

\[ \iff \]

Type-2 alignment set can be assigned vectors from a two dimensional space with all its internal conflicts resolved.

\[ \rightarrow \] A necessary condition for rate $\frac{1}{3}$ feasibility: \checkmark
Restricted IC problem and restricted conflicts

- For \( \mathcal{W}' \subset \mathcal{W} \), the **IC problem restricted to** \( \mathcal{W}' \) considers all demands and side-information only within \( \mathcal{W}' \) at receivers.
- Restricted alignment graphs, Restricted conflict graphs.
- **Restricted internal conflicts**: Conflicts within restricted alignment sets.

![Restricted IC problem and restricted conflicts diagram](image-url)
The condition on the type-2 alignment set

Type-2 alignment set can be assigned vectors from a two dimensional space with all its internal conflicts resolved.

\( \uparrow \uparrow \) (Projection to \( \mathbb{F}^2 \))
IC problem restricted to type-2 alignment set is rate \( \frac{1}{2} \) feasible

\( \downarrow \downarrow \) (Rephrasing)
Type-2 alignment set with no restricted internal conflicts.
How to assign vectors if the conditions in Main Result are met?

- All alignment sets are one of three types. Assign vectors differently in each case.
  - Alignment set which has no three messages interfering at any receiver: Assign a random $3 \times 1$ vector to each message.
  - Alignment set which consists only of three messages interfering at any receiver without any conflicts in-between: Assign the same random vector to all messages.
  - Alignment set which is a type-2 alignment set without restricted internal conflicts: For each restricted alignment set, assign one randomly generated vector from a 2D space.
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Open problems

- Solving the rate $\frac{1}{3}$ problem.
- Generalize to other rates.

*Thank you!*