CYCLIC HOMOGENEOUS RIEMANNIAN MANIFOLDS

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ABSTRACT. Cyclic homogeneous Riemannian manifolds are a kind of homogeneous Riemannian manifolds admitting invariant metrics which are in some way far from being naturally reductive metrics. In this paper, we give some characterizations and properties of these manifolds. We extend to the general case, Kowalski and Tricerri’s classification of simply-connected traceless cyclic homogeneous Riemannian manifolds for dimensions less than or equal to four.

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1. INTRODUCTION

Naturally reductive homogeneous Riemannian manifolds are known to be (see [1], [10], [11], [13], [17] and [22], among many others) the simplest manifolds after Riemannian symmetric spaces, concerning a number of properties.

There is another kind of homogeneous Riemannian manifolds, which we call cyclic homogeneous Riemannian manifolds, admitting invariant metrics which are in some way far from being naturally reductive metrics. They can be characterized (under suitable topological conditions) as those manifolds admitting some nontrivial homogeneous Riemannian structure of type $T_1 \oplus T_2$, in Tricerri and Vanhecke’s [22] classification of geometric types. In turn, naturally reductive homogeneous Riemannian manifolds are known to be related to the last basic Tricerri-Vanhecke’s geometric type, $T_3$. We should underline that the property of a homogeneous Riemannian manifold $(M,g)$ being either naturally reductive or cyclic does not depend only on the metric $g$, but also on a quotient expression $G/K$ and a given reductive decomposition $g = \mathfrak{k} \oplus \mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of $G$.

The study of cyclic homogeneous Riemannian manifolds was started by Tricerri and Vanhecke [22] and continued by Kowalski and Tricerri [15], Pastore and Verroca [19], Bieszk [4], Falcitelli and Pastore [6], and the present authors in [8], where we named them “cotorsionless manifolds.”

Naturally reductive homogeneous Riemannian manifolds arise as the homogeneous version of Lie groups equipped with bi-invariant metrics, so we have similarly studied properties of Lie groups endowed with cyclic left-invariant metrics in [7], where we call them cyclic metric Lie groups.

In the present paper we give some properties and characterizations of cyclic homogeneous Riemannian manifolds, and classify the simply-connected ones for dimensions less or equal than four.
The paper is organized as follows. In Section 2, some preliminaries on homogeneous Riemannian structures are given.

In Section 3 we show that in the study of cyclic homogeneous Riemannian manifolds, the fundamental one-form of a homogeneous Riemannian manifold \( G/K \) plays a central role. This is the \( G \)-invariant one-form determined by the covector \( \eta \) on \( \mathfrak{m} \) given by \( \eta(X) = \text{tr} \, \text{Ad}_X \), for all \( X \in \mathfrak{m} \). Then \( \eta \) is identically zero if and only if \( G \) is unimodular.

Moreover, we prove that \( \eta \) is closed. This answers a question discussed by Pastore and Verroca in [19] (see Remark 3.2 below).

The existence of a nonvanishing fundamental one-form implies (Theorem 3.1) that the homogeneous Riemannian manifold admits a codimension one and Riemannian homogeneous fibration

\[
\pi : L/K \to M = G/K \to \mathbb{R} = G/L,
\]

where \( L \) is the identity component of \( \text{Ker} \det \text{Ad} \) and (Corollary 3.5) the manifold \( G/K \) is not compact.

When \( M \) is moreover simply-connected, we prove in Theorem 3.10 that \( (M, g) \) is a semidirect Riemannian product (see Definition 3.8)

\[
\mathbb{R} \rtimes \pi (L/K) := \mathbb{R} \rtimes \pi L/K,
\]

where \( L \) is a simply-connected unimodular Lie group, \( K \) is connected and \( \pi \) is a one-parameter subgroup of \( \text{Aut}(L) \) such that \( \pi_*(d/dt) \neq 0 \).

Cyclic homogeneous Riemannian manifolds equipped with a homogeneous structure of class \( T_2 \), also called traceless cyclic homogeneous Riemannian manifolds, are those admitting a vanishing fundamental one-form and those with a nontrivial homogeneous structure of class \( T_1 \), the vectorial homogeneous Riemannian manifolds, are spaces of negative constant curvature. The classification of simply-connected traceless cyclic homogeneous Riemannian manifolds and dimension \( \leq 4 \) was given by Kowalski and Tricerri in [15].

We are especially interested in the study of cyclic homogeneous Riemannian manifolds with nonvanishing fundamental one-form. They must be locally isometric to semidirect Riemannian products of the form (1.1).

In Section 4, among other results, we obtain a formula for the curvature of any cyclic homogeneous Riemannian manifold with a nonvanishing fundamental one-form \( \eta \), from which it follows (Proposition 4.7) that then strictly negative sectional curvatures exist.

In Section 5, we extend the above-mentioned Kowalski and Tricerri classification, adding the manifolds corresponding to a nonunimodular Lie group \( G \) (Theorems 5.1 and 5.3, respectively).

In dimension three, any simply-connected cyclic homogeneous Riemannian manifold with nonvanishing fundamental one-form is a cyclic metric Lie group. The first examples which are not cyclic metric Lie groups appear in dimension four. They are a particular case of a more general family of examples described in Example 4.6.

2. Preliminaries

A homogeneous structure on a Riemannian manifold \( (M, g) \) is a tensor field \( S \) of type \( (1, 2) \) satisfying \( \nabla g = \nabla R = \nabla S = 0 \), where \( \nabla \) is (see [22]) the connection \( \nabla = \nabla - S \), \( \nabla \) being the Levi-Civita connection of \( g \). The condition \( \nabla g = 0 \)
is equivalent to $S_{XYZ} = -S_{XZY}$, where $S_{XYZ} = g(S_X Y, Z)$. Then $S$ can be expressed in terms of the torsion $\tilde{T}$ of the connection $\nabla$ as follows ([14, p. 83]):

$2S_{XYZ} = \tilde{T}_{YZX} + \tilde{T}_{XYZ} + \tilde{T}_{XYZ},$

where $\tilde{T}_{XYZ} = g(\tilde{T}_X Y, Z)$. When it be necessary to refer to the metric, we shall say that $\tilde{T}$, as a tensor field of type $(0, 3)$, is the $g$-torsion of $\nabla$. Conversely, we have

$\tilde{T}_{XYZ} = S_{YXZ} - S_{XYZ}.$

Hence, one gets

$2 \mathcal{S}_{XYZ} S_{XYZ} = - \mathcal{S}_{XYZ} \tilde{T}_{XYZ}$ and $c_{12}(S)(X) = \text{tr} \tilde{T}_X,$

where $(c_{12})_x(S)(X) = \sum_i S_{e_i e_i X}$, for an arbitrary local orthonormal basis $\{e_i\}$ of $T_x M$, $x \in M$.

Any homogeneous Riemannian manifold $(M, g)$ admits a homogeneous structure. More precisely, $S = \nabla - \tilde{\nabla}$ is a homogeneous structure on $(M, g)$ (see [22, Theorem 1.12] for more details), where $\tilde{\nabla}$ is the canonical connection with respect to a reductive decomposition. Ambrose and Singer [2] gave the following characterization for homogeneous Riemannian manifolds: A connected, simply-connected and complete Riemannian manifold $(M, g)$ is homogeneous if and only if it admits a homogeneous structure $S$.

In [22], Tricerri and Vanhecke studied the decomposition of the space of all (algebraic) tensors $S$ satisfying the same symmetries as a homogeneous structure into irreducible components under the action of the orthogonal group. In this way, they found three irreducibles classes $\mathcal{T}_i$, $i = 1, 2, 3$, of homogeneous structures. A homogeneous structure $S$ is then of type $\mathcal{T}_i$ if there exists a vector field $\xi$ on $M$ such that $S_X Y = g(X, Y)\xi - g(\xi, Y)X$. $S$ is of type $\mathcal{T}_1 \oplus \mathcal{T}_2$ if the cyclic sum $\mathcal{S}_{XYZ} S_{XYZ}$ vanishes. If moreover $c_{12}(S)(X) = 0$, for all vector fields $X$, it is of type $\mathcal{T}_2$. $S$ is of type $\mathcal{T}_3$ if $S_{XYZ} = -S_{YXZ}$.

From (2.2), (2.3) and (2.4) and according with the terminology in [21, p. 222], the torsion $\tilde{T}$ is said to be

- **vectorial** if there exists an one-form $\varphi$ on $M$ such that
  
  $\tilde{T}_X Y = \varphi(Y)X - \varphi(X)Y$

  or equivalently, taking $\varphi(X) = g(\xi, X)$, if $S \in \mathcal{T}_1$;

- **cyclic** if $\mathcal{S}_{XYZ} \tilde{T}_{XYZ} = 0$, or equivalently if $S \in \mathcal{T}_1 \oplus \mathcal{T}_2$;

- **traceless** if $\text{tr} \tilde{T}_X = 0$, or equivalently if $S \in \mathcal{T}_2 \oplus \mathcal{T}_3$;

- **traceless cyclic** if $\tilde{T}$ is traceless and cyclic, or equivalently if $S \in \mathcal{T}_2$;

- **totally skew-symmetric** if $\tilde{T}_{XYZ} = -\tilde{T}_{XZY}$, or equivalently if $S \in \mathcal{T}_3$.

Note that the properties of being vectorial or traceless do not depend on the metric $g$.

3. The fundamental one-form on a homogeneous Riemannian manifold

A homogeneous Riemannian manifold $(M, g)$ can be described as a quotient manifold $G/K$, where $G$ is a Lie group, which is supposed to be connected, acting transitively and effectively on $M$, $K$ is the isotropy subgroup of $G$ at some point $o \in M$, the origin of $G/K$, and $g$ is a $G$-invariant Riemannian metric. Moreover,
$G$ can be considered as a closed subgroup of the full isometry group $I(M, g)$ (see [3, Chapter 7] for more details). This implies that $K$ is a compact subgroup of $G$ and that $M$ is compact if and only if $G$ is compact. We can also assume that $G/K$ is a reductive homogeneous Riemannian manifold, i.e., there is an $\text{Ad}(K)$-invariant subspace $\mathfrak{m}$ of the Lie algebra $\mathfrak{g}$ of $G$ such that one has the vector space direct sum $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{k}$ being the Lie algebra of $K$.

Let $\nabla$ be the canonical connection with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. For each $X \in \mathfrak{m}$, let $X^*$ be the fundamental vector field on $M = G/K$ associated to $X$. This is a complete Killing vector field satisfying $X^*_* = X$. Under the identification $\mathfrak{m} \cong T_oM$, the $G$-invariant connection $\tilde{\nabla}$ is uniquely determined ([22, p. 20]) by

$$\tilde{\nabla}_X Y^* = [X^*, Y^*]_o = -[X, Y]^* = -[X, Y]_m$$

and the torsion $\tilde{T}$ and the curvature $\tilde{R}$ of $\tilde{\nabla}$ by

$$\tilde{T}(X, Y) = -[X, Y]_m, \quad \tilde{R}(X, Y) = \text{ad}_{[X,Y]}.$$

The Levi-Civita connection $\nabla$ of $(M, g)$ is also $G$-invariant and it is given (cf. [3, 7.27, 7.21]) by

$$2(\nabla_X Y^*, Z) = -\langle [X, Y]_m, Z \rangle - \langle Y, [Z, X]_m, X \rangle + \langle [Z, X]_m, Y \rangle,$$

for all $X, Y, Z \in \mathfrak{m}$, where $\langle \cdot, \cdot \rangle$ denotes the $\text{Ad}(K)$-invariant inner product on $\mathfrak{m}$ induced by $g$. Let $\mathcal{U}: \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ be the symmetric bilinear mapping defined by

$$2(\mathcal{U}(X, Y), Z) = \langle [Z, X]_m, Y \rangle + \langle [Z, Y]_m, X \rangle.
$$

Since any $G$-invariant tensor field on $M$ is parallel with respect to $\tilde{\nabla}$ (see [14, Proposition I.11]), it follows that $S = \nabla - \tilde{\nabla}$ determines a homogeneous structure on $M$. We say that $S$ is the homogeneous structure associated to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. At the origin, it is given by

$$S_X Y = \frac{1}{2} [X, Y]_m + \mathcal{U}(X, Y), \quad X, Y \in \mathfrak{m}.$$

Note that $S = 0$ if and only if $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$. Then $(G/K, g)$ is locally symmetric and, in general, it is locally symmetric for any $G$-invariant metric (see [12, Chapter IV, Proposition 3.6]). Using [12, Chapter IV], if $G$ is simply-connected and $K$ is connected then $M = G/K$ is a simply-connected symmetric space and $(G, K)$ is a symmetric pair.

The homogeneous structure $S$ is of type $T_3$ if and only if $\mathcal{U} = 0$, or equivalently, $(M = G/K, g)$ is naturally reductive with respect to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$.

Denote by $\xi$ the vector in $\mathfrak{m}$ given by $\xi = \sum_{i=1}^n \mathcal{U}(e_i, e_i)$, where $\{e_i\}$ is an orthonormal basis of $(\mathfrak{m}, \langle \cdot, \cdot \rangle)$. Since $\mathfrak{m}$ is Ad($K$)-invariant, $\xi$ is also Ad($K$)-invariant and so, it determines a $G$-invariant vector field on $M$ which we denote by the same letter $\xi$. The one-form $\eta$ dual to $\xi$ with respect to the metric $g$ is called the fundamental one-form of $(M = G/K, g)$ (associated to the reductive decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$). Because $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, it follows from (3.2) and (3.3) that $\eta$ satisfies

$$\eta(X) = c_{12}(S)(X) = \text{tr} \text{ad}_X, \quad X \in \mathfrak{m}.$$

Hence, $S$ is of type $T_2 \oplus T_3$ if and only if $\eta = 0$ and, using [22, Theorem 3.2], $S$ is of type $T_2$ if and only if

$$S_X Y = \frac{1}{n-1}(\langle X, Y \rangle \xi - \langle \xi, Y \rangle X).$$
If $\xi \neq 0$, the subspace $\mathcal{D} = \{X \in \mathfrak{m} : \eta(X) = 0\}$ of $\mathfrak{m}$ is also $\text{Ad}(K)$-invariant and then $\mathcal{D}$ determines a $(n - 1)$-dimensional $G$-invariant distribution on $M$, which is also denoted by $\mathcal{D}$, defined by $\eta = 0$.

We have the next theorem.

**Theorem 3.1.** The fundamental one-form $\eta$ on $M = G/K$ is closed. Moreover, we have:

(i) $G$ is unimodular if and only if $\eta = 0$.

(ii) If $G$ is not unimodular, then

1. The $(n - 1)$-dimensional distribution $\mathcal{D}$ is integrable and the corresponding $G$-invariant foliation $\mathcal{F}_\mathcal{D}$ is Riemannian.

2. $\xi$ is a $G$-invariant geodesic but not a Killing vector field on $(M, g)$.

3. The embedded submanifold $N = L/K$ of $M$, where $L$ is the connected component of the identity of $\text{Ker}(\text{det Ad})$, is the leaf through the origin of $\mathcal{F}_\mathcal{D}$.

4. If moreover $G$ is simply-connected and $K$ is connected, $M$ can be expressed as the product manifold $\mathbb{R} \times N$.

**Remark 3.2.** Before proving Theorem 3.1, note that for nonunimodular $G$, it follows from (3) that the leaf $L_p$ of the $G$-invariant foliation $\mathcal{F}_\mathcal{D}$ through a point $p = g \cdot o \in M$, is given by $L_p = g \cdot (L/K)$. Because $L$ is closed, one can consider the natural projection map $G/K \to G/L, gK \mapsto gL$. Then the leaves of $\mathcal{F}_\mathcal{D}$ are the fibres of the homogeneous fibration

$$N = L/K \to M = G/K \to G/L.$$ 

Moreover, from (1) and (2), the integral curves of $\xi$ are geodesics which meet orthogonally each leaf. Nevertheless, these leaves cannot be totally geodesic as Riemannian submanifolds of $(M, g)$. In fact, $\mathcal{F}_\mathcal{D}$ is totally geodesic if and only if $\mathcal{L}(D, \mathcal{D}) \subset \mathcal{D}$ (see [9, Lemma 4.1]), or equivalently, $\xi$ is a Killing vector field, which contradicts (2). In particular, the simply-connected Riemannian manifold $(M, g)$ in (4) is diffeomorphic to $\mathbb{R} \times N$, but not isometric to it as a Riemannian product.

On the other hand, we recall that Pastore and Verroca [19] studied homogeneous structures in $T_1 \oplus T_2$ whose fundamental one-form is closed and said that no examples of homogeneous structures of type $T_1 \oplus T_2$ whose fundamental one-form is not closed are known. Since $\eta$ is always closed, it follows that their results actually hold true in the general case.

For the proof of Theorem 3.1 we need the following result.

**Lemma 3.3.** Let $M_1 = G_1/K_1$ and $M_2 = G_2/K_2$ be two homogeneous Riemannian manifolds and consider a homomorphism $\pi : G_1 \to \text{Aut}(G_2)$, satisfying $\pi(g_1)(K_2) \subset K_2$, for all $g_1 \in G_1$. Then the mapping

$$\phi : (G_1 \ltimes G_2)/(K_1 \ltimes K_2) \to M_1 \times M_2,$$

given by $\phi((g_1, g_2)(K_1 \ltimes K_2)) = (g_1K_1, g_2K_2)$, is a diffeomorphism, $\pi_{K_1}$ being the homomorphism $\pi_{K_1} : K_1 \to \text{Aut}(K_2)$, restriction of $\pi$ to $K_1$.

**Proof.** Consider the mapping $\theta : (G_1 \ltimes G_2) \times (M_1 \times M_2) \to M_1 \times M_2$ given by

$$\theta((g_1, g_2), (g_1'K_1, g_2'K_2)) = (g_1'K_1, g_2\pi(g_1)(g_2'K_2)).$$

Then:
The mapping $\theta$ is well-defined: If $h_1' = g_1'k_1$ and $h_2' = g_2'k_2$, for $k_1 \in K_1$ and $k_2 \in K_2$, then $g_1h_1'K_1 = g_1g_2'K_1$ and, using that $\pi(g_1)(K_2) \subset K_2$, we obtain

$$g_2\pi(g_1)(h_2')K_2 = g_2\pi(g_1)(g_2')(g_1)(k_2)K_2 = g_2\pi(g_1)(g_2')K_2.$$ 

The mapping $\theta$ is a $G_1 \ltimes_\pi G_2$-transitive action on $M_1 \times M_2$. By a direct checking one can see that $\theta$ defines a smooth action and, since given $(g_1'K_1, g_2'K_2)$ and $(h_1'K_1, h_2'K_2) \in M_1 \times M_2$, one gets

$$\theta((h_1'g_1'^{-1}, h_2'\pi(h_1'g_1'^{-1})(g_2'^{-1})), (g_1'K_1, g_2'K_2)) = (h_1'K_1, h_2'K_2),$$

so $\theta$ is transitive. Finally, the isotropy subgroup of $G_1 \ltimes_\pi G_2$ at $(e_1K_1, e_2K_2)$ with respect to $\theta$ is clearly the semidirect product $K_1 \ltimes_{\pi_{K_1}} K_2$. Hence, taking into account that $\phi((g_1, g_2)(K_1 \ltimes_{\pi_{K_1}} K_2)) = \theta(g_1, g_2)(e_1K_1, e_2K_2)$, we conclude that $\phi$ is a diffeomorphism.

We identify as usual $M_1 \times M_2$, via $\phi$, with the quotient manifold $(G_1 \ltimes_\pi G_2)/(K_1 \ltimes_{\pi_{K_1}} K_2)$.

Proof of Theorem 3.1. First, we prove (i). From (3.4), $\eta = 0$ if $G$ is unimodular. For the converse, we only need to show that $\text{tr} \, \text{ad} \, \eta = 0$, for all $W \in \mathfrak{k}$. But this equality holds since $\langle \cdot, \cdot \rangle$ is $\text{Ad}(K)$-invariant on $\mathfrak{m}$ and $K$ is unimodular, because if it is compact.

If $G$ is not unimodular then $\xi$ is a nonvanishing vector in $\mathfrak{m}$ and the unimodular kernel $\mathfrak{l}$ of the Lie algebra $\mathfrak{g}$ of $G$, that is,

$$\mathfrak{l} = \{X \in \mathfrak{g} : \text{tr} \, \text{ad} \, X = 0\}$$

is an ideal of codimension one and $\mathfrak{t} \subset \mathfrak{l}$. From (3.4), one gets

$$\mathfrak{g} = \mathbb{R}\xi \oplus \mathfrak{l} = \mathbb{R}\xi \oplus (\mathfrak{t} \oplus \mathfrak{d}).$$

Then $[X, Y]_\mathfrak{m} \in \mathfrak{d}$ for all $X, Y \in \mathfrak{d}$, and so $\eta([X, Y]_\mathfrak{m}) = 0$, or equivalently, $d\eta(\mathfrak{d}, \mathfrak{d}) = 0$. Hence, the distribution $\mathfrak{d}$ is integrable. Using again that $\mathfrak{l}$ is an ideal, one gets moreover that $\eta([\mathfrak{m}, \mathfrak{m}]_\mathfrak{m}) = 0$. Then, $\eta$ is closed and, since $\mathfrak{u}(\xi, \xi) = 0$, $\mathfrak{d}$ determines a Riemannian foliation (see [9, Lemma 4.1]). Hence $\xi$ must be geodesic.

For each $X \in \mathfrak{m}$, let $X^+$ be the $G$-invariant vector field defined in a neighborhood of $o$ in $M = G/K$ such that $X^+_o = X$. Then we have [17, (7.3)]

$$[X^+, Y^+]_o = [X, Y]_\mathfrak{m}, \quad X, Y \in \mathfrak{m}.$$ 

This implies that $(\mathcal{L}_\xi g)(X^+, Y^+)_o = -((\langle \xi, X \rangle_\mathfrak{m}, Y) + \langle \xi, Y \rangle_\mathfrak{m}, X))$. If $\xi$ were a Killing vector field, from the homogeneity of $M$ we would equivalently have

$$\langle \xi, X \rangle_\mathfrak{m}, Y) + \langle \xi, Y \rangle_\mathfrak{m}, X) = 0, \quad X, Y \in \mathfrak{m},$$

and, from (3.4), we would have $\eta(\xi) = 0$, which is a contradiction. So, using (1), one has that $G$ must be unimodular.

Next, we prove (3). $L$ is a closed subgroup of $G$ with Lie algebra the unimodular kernel $\mathfrak{l}$ of $\mathfrak{g}$. Because the Lie algebra $\mathfrak{t}$ of $K$ is in $\mathfrak{l}$, it follows that the quotient $L/K$ is a homogeneous manifold and, moreover, $\mathfrak{l} = \mathfrak{t} \oplus \mathfrak{d}$ is a reductive decomposition. This implies that $N = L/K$ is an integral submanifold of $\mathfrak{d}$.

Finally, we show (4). Since $G$ is simply-connected, it is isomorphic to the semidirect product $\mathbb{R} \ltimes_\pi L$, where the subgroup $L$ of $G$ is also simply-connected (cf. [16, p. 519]) and, for each $t \in \mathbb{R}$, $\pi(t)$ denotes the (unique) automorphism of $L$ such that $(\pi(t))_* = \text{Ad} \, \exp(t \xi) : \mathfrak{l} \to \mathfrak{l}$. 

On the other hand, using that $\xi$ is $\text{Ad}(K)$-invariant, one gets that $(\pi(t))_\ast = \exp\text{ad}_\xi$ acts as the identity on $\mathfrak{k}$, that is, 

$$k \exp t \xi = (\exp t \xi) k, \quad k \in K,$$

and hence $\pi(t)$ must be the identity map on $K$. From Lemma 3.3, the quotient manifold $G/K = (\mathbb{R} \times L)/K$ is naturally diffeomorphic to the product manifold $\mathbb{R} \times (L/K)$, so concluding.

From Theorem 3.1 (i), the following result is immediate.

**Corollary 3.4.** On homogeneous manifolds $M = G/K$ such that $G$ is unimodular, the torsion of the canonical connection with respect to any adapted reductive decomposition is traceless.

**Corollary 3.5.** Every homogeneous Riemannian manifold with nonvanishing fundamental one-form is not compact and the integral curves of $\xi$ are one-to-one geodesics.

**Proof.** Suppose that $(M = G/K, g)$ is a compact homogeneous Riemannian manifold. Then $G$ must be compact. This implies the existence of an $\text{Ad}(G)$-invariant inner product on its Lie algebra. Hence, $G$ is unimodular and, using Theorem 3.1 (i), the fundamental one-form $\eta$ of $(M = G/K, g)$, associated to any reductive decomposition, vanishes. This proves the first part of the corollary.

Because $G$ is not compact, the base space $G/L$ of the homogeneous fibration determined by $F_D$ is then diffeomorphic to $\mathbb{R}$. Since every maximal geodesic on $G/K$ is either one-to-one or periodic (cf. [5, Lemma 1]), it follows from Theorem 2.1 (2) that the integral curves of $\xi$ must be one-to-one geodesics. □

Hence, we have the next result.

**Corollary 3.6.** The homogeneous structure associated to any reductive decomposition of a compact homogeneous Riemannian manifold is of type $T_2 \oplus T_3$.

**Corollary 3.7.** If the fundamental one-form $\eta$ does not vanish, we have:

(i) The second fundamental form of the $(n - 1)$-dimensional distribution $\mathcal{D}$ is determined by the (symmetric) bilinear mapping $h: \mathcal{D} \times \mathcal{D} \to \mathbb{R}$, given by

$$h(X, Y) = \frac{1}{\langle \xi, \xi \rangle} \eta(U(X, Y)) \xi, \quad \text{for all} \quad X, Y \in \mathcal{D};$$

(ii) $\xi$ is the mean curvature vector field of $\mathcal{D}$.

**Proof.** Because $\xi$ is $G$-invariant, it must be $\nabla$-parallel. Hence, one gets $\nabla \xi = S \xi$. Then, using that $\eta$ is closed and (3.3), we have

$$h(X, Y) = \frac{1}{\langle \xi, \xi \rangle} \langle S_X Y, \xi \rangle \xi = \frac{1}{\langle \xi, \xi \rangle} \langle U(X, Y), \xi \rangle \xi,$$

for all $X, Y \in \mathcal{D}$. This proves (3.6). Moreover, from (3.2),

$$\text{tr} h = \frac{1}{\langle \xi, \xi \rangle} \sum_{i=1}^{n-1} \langle [\xi, e_i]_m, e_i \rangle \xi = \xi,$$

where $\{e_1, \ldots, e_{n-1}\}$ is an orthonormal basis of $\mathcal{D}$. Then (ii) is proved. □
The notion of orthogonal semidirect product of Lie groups equipped with left-invariant metrics can be extended to homogeneous Riemannian manifolds as follows. Let \((M_1 = G_1/K_1, g_1)\) and \((M_2 = G_2/K_2, g_2)\) be two homogeneous Riemannian manifolds with reductive decompositions \(g_i = \mathfrak{t}_i \oplus \mathfrak{m}_i\), and corresponding Ad\((K_i)\)-invariant inner products \(\langle \cdot, \cdot \rangle_i\) on \(\mathfrak{m}_i\), for \(i = 1, 2\). Consider a homomorphism \(\pi: G_1 \to \text{Aut}(G_2)\).

**Definition 3.8.** The semidirect Riemannian product \(M = M_1 \ltimes G_2\) is the product manifold \(M_1 \times M_2\) equipped with the metric tensor \(g_\pi\) such that \(g_\pi(g_1, g_2) = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2\), under the identification \(\mathfrak{m}_1 \oplus \mathfrak{m}_2 \cong T_{(e_1,e_2)}M\), and the semidirect product \(G_1 \ltimes G_2\) is a transitive subgroup of the group of all isometries \(I(M, g_\pi)\) of \((M, g_\pi)\).

This means that \(M = M_1 \ltimes G_2\) is a \(G_1 \ltimes G_2\)-homogeneous Riemannian manifold.

In the sequel we shall suppose that \(\pi\) satisfies \(\pi(g_1)(K_2) \subset K_2\), for all \(g_1 \in G_1\). Then, from Lemma 3.3, the Lie group \(G_1 \ltimes G_2\) acts transitively on \(M_1 \times M_2\). Moreover, if the decomposition

\[
\mathfrak{g}_1 \oplus_{\pi_\ast} \mathfrak{g}_2 = (\mathfrak{t}_1 \oplus_{\pi_\ast} \mathfrak{t}_2) \oplus (\mathfrak{m}_1 \oplus \mathfrak{m}_2)
\]

is reductive, the existence of the metric \(g_\pi\) depends on whether the inner product \(\langle \cdot, \cdot \rangle_\pi\) on \(\mathfrak{m}_1 \oplus \mathfrak{m}_2\), given by

\[
\langle \cdot, \cdot \rangle_\pi = \langle \cdot, \cdot \rangle_1 + \langle \cdot, \cdot \rangle_2,
\]

is Ad\((K_1 \ltimes_{\pi_\ast} K_2)\)-invariant. Here, \(\pi_\ast\) is the differential of the homomorphism, which is also denoted by \(\pi, \pi: G_1 \to \text{Aut}(\mathfrak{g}_2)\), given by \(\pi(g_1) = (\pi(g_1))_\ast\), for all \(g_1 \in G_1\). Then \(\pi_\ast\) is a Lie algebra homomorphism \(\mathfrak{g}_1 \to \text{Der}(\mathfrak{g}_2)\). Note that if \(G\) is simply-connected, the Lie groups \(\text{Aut}(G)\) and \(\text{Aut}(\mathfrak{g})\) are naturally isomorphic (cf. [20, p. 234]).

Now, we have the following lemma.

**Lemma 3.9.** If \(K_1\) and \(K_2\) are connected and the homomorphism \(\pi: G_1 \to \text{Aut}(G_2)\) satisfies the following conditions:

(i) \(\mathfrak{t}_2 \subset \text{Ker} \pi_\ast(X_1)\).

(ii) \(\pi_\ast(U_1)\mathfrak{m}_2 \subset \mathfrak{m}_2\).

(iii) The linear transformation \(\pi_\ast(U_1): \mathfrak{m}_2 \to \mathfrak{m}_2\) is skew-symmetric with respect to \(\langle \cdot, \cdot \rangle_2\),

for all \(X_1 \in \mathfrak{m}_1\) and \(U_1 \in \mathfrak{t}_1\), then the inner product \(\langle \cdot, \cdot \rangle_\pi\) defines a \((G_1 \ltimes G_2)\)-invariant metric \(g_\pi\) on \(M = M_1 \times M_2\), and \((M, g_\pi)\) is the semidirect Riemannian product \(M_1 \ltimes G_2\).

**Proof.** From the connectedness of \(K_1\) and \(K_2\), the conditions (i) and (ii) are equivalent to the decomposition (3.7) being reductive. Moreover, because one gets

\[
\langle [U_1, U_2], (X_1, X_2), (Y_1, Y_2) \rangle = \langle [U_1, X_1], Y_1 \rangle_1 + \langle [U_2, X_2], Y_2 \rangle_2 + \langle \pi_\ast(U_1)(X_2), Y_2 \rangle_2,
\]

(iii) is equivalent to \(\langle \cdot, \cdot \rangle_\pi\) being Ad\((K_1 \ltimes_{\pi_\ast} K_2)\)-invariant.

\(\square\)

**Theorem 3.10.** A simply-connected homogeneous Riemannian manifold admits a nonvanishing fundamental one-form if and only if it is isometric to a semidirect Riemannian product \(\mathbb{R} \ltimes (L/K)\), where \(L\) is a simply-connected unimodular Lie group, \(K\) is connected and \(\text{tr} \pi_\ast(d/dt) \neq 0\).
Proof. Let $M = G/K$ be a simply-connected homogeneous Riemannian manifold. Then $K$ must be connected and we can assume that $G$ is simply-connected. In fact, $M$ can be expressed as the coset $\tilde{G}/\tilde{K}$ where $(\tilde{G}, \Phi)$ is the universal covering of $G$ and $\tilde{K}$ is the identity component of $\Phi^{-1}(K)$.

From Theorem 3.1, if $(M, g)$ admits a homogeneous structure which is not of type $T_2 \oplus T_3$, then $G$ is not unimodular, $M$ is diffeomorphic to $\mathbb{R} \times (L/K)$ and $G$ is the semidirect product $G = \mathbb{R} \ltimes \pi L$, where $L$ is the identity component of $\ker(\det \text{Ad})$. Moreover, using (3.4), one gets $\xi = (\text{tr} \pi \cdot (d/dt))d/dt$. Hence, it follows that $\mathfrak{t} \subset \ker\pi \cdot (d/dt)$, and so, using Lemma 3.9, $(M, g)$ is isometric to $\mathbb{R} \ltimes \pi (L/K)$.

The converse in immediate, taking into account that the semidirect product $G = \mathbb{R} \ltimes \pi L$ is not unimodular.

4. CYCLIC HOMOGENEOUS RIEMANNIAN MANIFOLDS

Definition 4.1. A homogeneous Riemannian manifold $(M, g)$ is said to be vectorial, cyclic or traceless cyclic if there exists a homogeneous description $M = G/K$ such that the $g$-torsion of the canonical connection with respect to a reductive decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ is so. In this case, we also say that $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ is a cyclic reductive decomposition and that $g$ is a cyclic homogeneous metric.

Then, from (3.1) and (3.5), an $n$-dimensional homogeneous Riemannian manifold $(M = G/K, g)$, with respect to $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$, is vectorial if, for all $X, Y \in \mathfrak{m}$, $[X, Y]_\mathfrak{m} = \frac{1}{n-1}(\eta(X)Y - \eta(Y)X)$.

Using (3.4) and Theorem 3.1 (i), $(M = G/K, g)$ is cyclic if the inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{m}$ induced by $g$ satisfies

$$\mathcal{S}_{XYZ} \langle [X, Y]_\mathfrak{m}, Z \rangle = 0, \quad X, Y, Z \in \mathfrak{m},$$

and it is traceless cyclic if moreover $G$ is unimodular. Any cyclic left-invariant metric [7] on a Lie group is cyclic. Actually, cyclic metric Lie groups are precisely the cyclic homogeneous Riemannian manifolds whose isotropy subgroup for the associated quotient expression is trivial.

Proposition 4.2. Let $(M, g)$ be a connected, simply-connected and complete Riemannian manifold equipped with a homogeneous structure $S$ of type $T_1 \oplus T_2$. Then $(M, g)$ is a cyclic homogeneous Riemannian manifold.

Proof. Let $\mathfrak{M} = (\mathfrak{m} = T_o M, \bar{T}, \bar{R}, \langle \cdot, \cdot \rangle = g_o)$ be the infinitesimal model associated to the homogeneous structure $S$ at a point $o \in M$ (see [23]), where $\bar{T}$ and $\bar{R}$ are the torsion and the curvature tensors of the connection $\bar{\nabla} = \nabla - S$ at $o$, $\nabla$ being the Levi Civita connection of $(M, g)$.

Following [22, Chapter 1], $M$ can be expressed as a quotient manifold $G/K$, where $G$ is a Lie group of isometries of $(M, g)$ acting effectively with Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$, $\mathfrak{t}$ being the Lie subalgebra of $\mathfrak{so}(n)$ generated by the skew-symmetric endomorphisms $\bar{R}_{XY}$, for all $X, Y \in \mathfrak{m}$, and equipped with the following brackets:

$$[A, B] = AB - BA, \quad [A, X] = AX, \quad [X, Y] = S_X Y - S_Y X + \bar{R}_{XY}$$

for all $A, B \in \mathfrak{t}$ and $X, Y \in \mathfrak{m}$. Since $S \in T_1 \oplus T_2$, $(M = G/K, g)$ is cyclic and $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{m}$ is a cyclic reductive decomposition.
Next, we consider the case of cyclic homogeneous Riemannian manifolds of a nonunimodular Lie group $G$.

**Proposition 4.3.** Let $(M = G/K, g)$ be a homogeneous Riemannian manifold whose fundamental one-form associated to a reductive decomposition $g = \mathfrak{k} \oplus \mathfrak{m}$ does not vanish. Then, it is cyclic with respect to $g = \mathfrak{k} \oplus \mathfrak{m}$ if and only if the Riemannian submanifold $N = L/K$ of $(M, g)$ is cyclic with respect to the decomposition $l = \mathfrak{k} \oplus \mathfrak{d}$ and

\[
[\langle [\xi, X]\mathfrak{m}, Y \rangle = \langle [\xi, Y]\mathfrak{m}, X \rangle, \quad X, Y \in \mathfrak{m}.
\]

**Proof.** Because from Theorem 3.1 the fundamental one-form is closed, one gets

\[
\langle [\xi, X]\mathfrak{m}, Y \rangle + \langle [Y, \xi]\mathfrak{m}, X \rangle + \langle [X, Y]\mathfrak{m}, \xi \rangle = \langle [\xi, X]\mathfrak{m}, Y \rangle + \langle [Y, \xi]\mathfrak{m}, X \rangle,
\]

for all $X, Y \in \mathfrak{m}$, and moreover

\[
\mathcal{S}_{XYZ} \langle [X, Y]\mathfrak{m}, Z \rangle = \mathcal{S}_{XYZ} \langle [X, Y]\mathfrak{d}, Z \rangle, \quad X, Y, Z \in \mathfrak{d},
\]

where $\langle \cdot, \cdot \rangle_\mathfrak{d}$ is the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{d}$. Then the result follows taking into account that $\langle \cdot, \cdot \rangle_\mathfrak{d}$ determines the metric induced on $N$ by $g$. \hfill $\square$

**Remark 4.4.** Since $L$ is unimodular, the associated homogeneous structure on $L/K$ is of type $T_2$. Moreover, taking into account that $l$ is an ideal of $g$ and $\mathfrak{m} = \mathbb{R} \xi \oplus \mathfrak{d}$, the condition (4.2) is equivalent to the following one:

\[
\langle [\xi, X]\mathfrak{d}, Y \rangle = \langle [\xi, Y]\mathfrak{d}, X \rangle, \quad X, Y \in \mathfrak{d}.
\]

Using Proposition 4.3 and the same arguments that in the proof of Theorem 3.10, we have the next result.

**Theorem 4.5.** A simply-connected homogeneous Riemannian manifold is cyclic with nonvanishing associated fundamental one-form if and only if it is isometric to a semidirect product $\mathbb{R} \ltimes \pi(L/K)$, where:

(i) $L$ is a simply-connected unimodular Lie group and $K$ is connected.

(ii) $L/K$ is traceless cyclic with respect to a reductive decomposition $l = \mathfrak{k} \oplus \mathfrak{d}$ and an $\text{Ad}(K)$-invariant inner product $\langle \cdot, \cdot \rangle_\mathfrak{d}$ on $\mathfrak{d}$.

(iii) The one-parameter subgroup $\pi: \mathbb{R} \to \text{Aut}(L)$ of $\text{Aut}(L)$ satisfies:

1. $\text{tr} \pi_*(d/dt) \neq 0$.
2. $\mathfrak{t} \subset \ker \pi_*(d/dt)$.
3. $\langle \pi_*(d/dt)X|\mathfrak{d}, Y\rangle_\mathfrak{d} = \langle \pi_*(d/dt)Y|\mathfrak{d}, X\rangle_\mathfrak{d}, \quad X, Y \in \mathfrak{d}$.

**Example 4.6.** Let $U/K$ be a traceless cyclic homogeneous Riemannian manifold and let $G_0$ be an $r$-dimensional connected metric abelian Lie group. Consider the semidirect Riemannian product

\[
(M, g) = \mathbb{R} \ltimes \pi \left(G_0 \times (U/K)\right) \cong \mathbb{R} \ltimes \pi \left(G_0 \times U/\right) K,
\]

where the homomorphism $\pi: \mathbb{R} \to \text{Aut}(G_0 \times U)$ is determined by the corresponding homomorphism into $\text{Aut}(g_0 \oplus U)$ given by

\[
\pi(t)(x, u) = \left( \sum_{i=1}^{r} x^i e^{\alpha_i t} e_i, u \right),
\]
for all \( x = \sum_{i=1}^{r} x^i e_i \in \mathfrak{g}_0 \) and \( u \in \mathfrak{u} \), where \( \{e_1, \ldots, e_r\} \) is an orthonormal basis of the Lie algebra \( \mathfrak{g}_0 \) of \( G_0 \) and \( (\alpha_1, \ldots, \alpha_r) \in \mathbb{R}^r \). Then,

\[
[e_0, e_i] = \alpha_i e_i, \quad i = 1, \ldots, r,
\]

\( e_0 \) being the unit generator \( d/dt \) of \( \mathbb{R} \). Here, the vector field \( \xi \) takes the form \( \xi = (\sum_{i=1}^{r} \alpha_i) e_0 \). If \( \sum_{i=1}^{r} \alpha_i \neq 0 \), it follows from Proposition 4.3, that the Riemannian manifold \((M, g)\) is cyclic with nonvanishing fundamental one-form.

Under the identification \( m \cong T_oM \), the curvature \( R \) at the origin \( o \) of an arbitrary homogeneous Riemannian manifold \((M = G/K, g)\) satisfies (see [3])

\[
(R_{XY} X, Y) = -\frac{1}{2} \|[X, Y]_m\|^2 - \frac{1}{2} \langle [X, [X, Y]]_m, Y \rangle - \frac{1}{2} \langle [Y, [Y, X]]_m, X \rangle
+ \|[\mathfrak{u}(X, Y)]\|^2 - \langle \mathfrak{u}(X, X), \mathfrak{u}(Y, Y) \rangle,
\]

for all \( X, Y \in \mathfrak{m} \). Then, using (4.1) in the second and third summands of the above equality, we have the following formula for the curvature of cyclic homogeneous Riemannian manifolds:

\[
(R_{XY} X, Y) = -\frac{1}{2} \|[X, Y]_m\|^2 + \langle [X, Y]_m, [X, Y]_m \rangle
+ \|[\mathfrak{u}(X, Y)]\|^2 - \langle \mathfrak{u}(X, X), \mathfrak{u}(Y, Y) \rangle.
\]

**Proposition 4.7.** On any cyclic homogeneous Riemannian manifold with nonvanishing associated fundamental one-form, we have

\[
\langle R_{X\xi} X, \xi \rangle = -\|[X, \xi]_m\|^2, \quad X \in \mathfrak{m}.
\]

Consequently, there exist strictly negative sectional curvatures.

**Proof.** From (3.2) and (4.1), using that \( \eta \) is closed, it follows that

\[
\|[\mathfrak{u}(X, \xi)]_m\|^2 = \frac{1}{2} \langle [\mathfrak{u}(X, \xi), X]_m, \xi \rangle + \langle [\mathfrak{u}(X, \xi), \xi]_m, X \rangle = \frac{1}{2} \langle [\mathfrak{u}(X, \xi), \xi]_m, X \rangle
- \frac{1}{2} \langle [\xi, X]_m, [\mathfrak{u}(X, \xi)]_m, \xi \rangle = \frac{1}{2} \|[X, \xi]_m\|^2
\]

and, since both the inner product \( \langle \cdot, \cdot \rangle \) and \( \xi \) are \( \text{Ad}(K) \)-invariant, one gets \( \langle [X, \xi]_m, [X, \xi]_m \rangle = 0 \). Then, taking into account that \( \mathfrak{u}(\xi, \xi) = 0 \), the result is obtained by substituting in (4.3). For the last part of the statement, we use that there exist \( X \in \mathfrak{m} \) such that \( [X, \xi]_m \neq 0 \). In fact, if \( [X, \xi]_m \) were null for all \( X \in \mathfrak{m} \), then \( \mathfrak{u} \) would be identically zero, and so \( \eta \) would be zero. \( \square \)

From Corollary 3.7, the distribution \( \mathcal{D} \) is umbilical if and only if the second fundamental form \( h \) can be expressed as \( h = (\langle \cdot, \cdot \rangle_{\mathcal{D}}/(n-1))\xi \).

**Proposition 4.8.** The leaves of the \( (n-1) \)-dimensional distribution \( \mathcal{D} \) on a cyclic homogeneous Riemannian manifolds with associated nonvanishing fundamental one-form are totally umbilical hypersurfaces if and only if the sectional curvature \( K(X, \xi) \) is constant, for all \( X \in \mathcal{D} \). Then,

\[
K(X, \xi) = -\frac{\langle \xi, \xi \rangle}{(n-1)^2}.
\]

**Proof.** From (3.6) and (4.2), it follows that \( h(X, Y) = (1/\langle \xi, \xi \rangle)\langle [X, \xi]_m, Y \rangle \xi \), for all \( X, Y \in \mathcal{D} \). Then \( \mathcal{D} \) is umbilical if and only if \( [X, \xi]_m = (1/(n-1))\langle \xi, \xi \rangle X \), for all \( X \in \mathcal{D} \). From (4.4), the sectional curvature \( K(X, \xi) \) satisfies (4.5).

Conversely, according with Proposition 4.3, the mapping \( X \in \mathcal{D} \mapsto [\xi, X]_m \) is a selfadjoint operator, so there exists an orthonormal basis \( \{e_1, \ldots, e_{n-1}\} \) of \( \mathcal{D} \) such
that \([\xi, e_i]_m = \lambda_i e_i\), for some \((\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n-1} \setminus \{0\}\). Then, using (4.4), we have \(\lambda_1 = \cdots = \lambda_n = (1/(n-1)) \langle \xi, \xi \rangle\). Hence, \(\mathcal{D}\) must be umbilical. \(\square\)

**Remark 4.9.** Consider the \(n\)-dimensional Poincaré half-space \((H^n(c), g)\), where \(H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}\) and the metric \(g\), with constant curvature \(-c^2\), is given by \(g = (cx_n)^{-2} \sum_{i=1}^n dx_i^2\). From Proposition 4.8, one gets the well-known result that the hyperplanes \(x_n = \lambda\), for \(\lambda > 0\), are totally umbilical (actually, totally geodesic, cf. [18, pp. 35–37] and [13, vol. II, p. 271]).

Because \(\text{ad}_X(t) \subset \mathfrak{m}\) for all \(X \in \mathfrak{m}\), one gets

\[
\text{tr} (W \in \mathfrak{t} \mapsto (\text{ad}_X \circ \text{ad}_X(W))|_t) = \sum_{i=1}^n \langle [X, [X, e_i]]_m, e_i \rangle,
\]

where \([e_i]\) is an orthonormal basis of \((\mathfrak{m}, \langle \cdot, \cdot \rangle)\). Hence, it follows that the Killing form \(B\) of \(g\) satisfies

\[
B(X, X) = \sum_{i=1}^n \langle [X, [X, e_i]]_m, e_i \rangle + \langle [X, [X, e_i]]_m, e_i \rangle,
\]

**Proposition 4.10.** The Ricci curvature of a cyclic homogeneous Riemannian manifold is given by

\[
\text{Ric}(X, X) = -B(X, X) + \sum_{i=1}^n \langle [X, [X, e_i]]_m, e_i \rangle - \eta(U(X, X)).
\]

**Proof.** The Ricci curvature of a homogeneous Riemannian manifold can be expressed at the origin in terms of \(B\) as (see [3, Chapter 7])

\[
\text{Ric}(X, X) = -\frac{1}{2} \sum_i \|X, e_i\|_m^2 - \frac{1}{2} B(X, X)
\]

\[
+ \frac{1}{2} \sum_{i,j} \langle [e_i, e_j]_m, X \rangle^2 - \langle [\xi, X]_m, X \rangle.
\]

Using that the metric is cyclic, we have

\[
\sum_{i,j} \langle [e_i, e_j]_m, X \rangle^2 = \sum_{i,j} \left( \langle [X, e_i]_m, e_j \rangle + \langle [X, e_j]_m, e_i \rangle \right)^2
\]

\[
= 2 \sum_i \left( \|X, e_i\|_m^2 + \langle [X, e_i]_m, X \rangle_\mathfrak{m}, e_i \rangle \right).
\]

Substituting then this equality in (4.8), we have

\[
\text{Ric}(X, X) = -\frac{1}{2} B(X, X) + \frac{1}{2} \sum_{i=1}^n \langle [X, e_i]_m, X \rangle_\mathfrak{m}, e_i \rangle - \langle [\xi, X]_m, X \rangle.
\]

Applying now (3.2) and (4.6), the result follows. \(\square\)

Because \(K\) is trivial for cyclic metric Lie groups and \(\eta \circ U\) is a symmetric bilinear form, we have the following corollary.

**Corollary 4.11.** The Ricci curvature of a cyclic left-invariant metric on a Lie group is given by

\[
\text{Ric} = -(B + \eta \circ U).
\]

**Corollary 4.12.** There is no nonabelian unimodular Lie group equipped with an Einstein cyclic left-invariant metric.
Proof. From Corollary 4.11, the Ricci curvature of a cyclic left-invariant metric on a unimodular Lie group \( G \) satisfies \( \text{Ric} = -B \). So, if the metric is Einstein, the corresponding inner product \( \langle \cdot, \cdot \rangle \) on the Lie algebra \( g \) of \( G \) must be proportional to the Killing form and, in particular, \( G \) is semisimple. Hence the decomposition of the Lie algebra \( g \) of \( G \) into its simple ideals \( g = g_1 \oplus \cdots \oplus g_r \) is an orthogonal direct sum with respect to \( \langle \cdot, \cdot \rangle \) and its restriction to each \( g_i \) defines again cyclic left-invariant metrics. From [7, Theorem 4.4], each \( g_i, i = 1, \ldots, r \), is isomorphic to \( \text{sl}(2, \mathbb{R}) \). But this contradicts our assumption that the metric is Einstein, because the set of cyclic left-invariant metrics on \( \widetilde{\text{SL}}(2, \mathbb{R}) \) form a two-parameter family with signature of the Ricci form \((-,-,+)\). So there is no Einstein cyclic left-invariant metric on this unimodular Lie group. \( \square \)

5. Classifications of cyclic homogeneous Riemannian manifolds of dimension \( n \leq 4 \)

Any two-dimensional homogeneous Riemannian manifold \( (M, g) \) obviously has constant curvature and its homogeneous structures are of type \( T_1 \) (see, for example [22, Theorem 3.1]). For the simply-connected case, it admits a nonvanishing homogeneous structure if and only if it is isometric to the hyperbolic plane, considered as a semidirect Lie group [22, Theorem 4.3].

In [15], Kowalski and Tricerri gave the classification of simply-connected cyclic homogeneous Riemannian manifolds of dimension three and four with fundamental one-form zero. In the three-dimensional general case we have the following theorem.

Theorem 5.1. A three-dimensional simply-connected cyclic homogeneous Riemannian manifold with nontrivial associated homogeneous structure is isometric to one of the following homogeneous Riemannian manifolds:

1. The universal covering group \( \widetilde{\text{SL}}(2, \mathbb{R}) \) of \( \text{SL}(2, \mathbb{R}) \), equipped with a cyclic left-invariant metric. Such metrics form a two-parameter family with signature of the Ricci form \((-,-,+)\).

2. The orthogonal semidirect product \( \mathbb{R} \ltimes \mathbb{R}^2 \), both factors with the additive group structure and where the action \( \pi \) of \( \mathbb{R} \) on \( \mathbb{R}^2 \) is \( \pi(t) = \text{diag}(e^{\alpha t}, e^{\beta t}) \), \((\alpha, \beta) \in \mathbb{R}^2 \setminus \{(0,0)\}\). This Lie group can be described as the matrix group \( G^3(\alpha, \beta) \) of matrices of the form

\[
\begin{pmatrix}
e^{\alpha z} & 0 & x \\
0 & e^{\beta z} & y \\
0 & 0 & 1
\end{pmatrix}
\]

with the left-invariant metric \( g = dz^2 + e^{-2\alpha z}dx^2 + e^{-2\beta z}dy^2 \).

3. The unimodular Lie groups \( G : \widetilde{\text{SL}}(2, \mathbb{R}), \text{SU}(2) \) and the Heisenberg group \( H_3 \), equipped with a suitable left-invariant metric. They admit an adapted quotient expression of the type \( (\text{SO}(2) \ltimes \mathbb{R})/\text{SO}(2) \), where the homomorphism \( \pi : \text{SO}(2) \to \text{Aut}(G) \) is given by

\[
\pi(e^{i\theta}) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

with respect to the coordinate system on \( G \) induced by the exponential map. The admissible metrics on \( G \) depend on two parameters.
Proof. According with [7, Theorem 6.1], the Riemannian manifolds in (1) and (2) are all the three-dimensional simply-connected nonabelian cyclic metric Lie groups. Moreover, from [15, Theorem 2.1], the metric Lie groups $SL(2, \mathbb{R})$ in (1), $E(1, 1) \cong G^4(\alpha, -\alpha)$ in (2) and the metric unimodular Lie groups $(G, g)$ in (3), are all the three-dimensional cyclic homogeneous Riemannian manifolds admitting a nonvanishing homogeneous structure of type $T_2$. Each Lie algebra $\mathfrak{g}$ of $G$ for this last case admits an orthonormal basis $\{e_1, e_2, e_3\}$ such that

\begin{equation}
[e_1, e_2] = ae_3, \quad [e_2, e_3] = be_1, \quad [e_3, e_1] = be_2,
\end{equation}

for some $a, b \in \mathbb{R}$, $a \neq 0$. If sign($b$) = $-\text{sign}(a)$, $G$ is $SL(2, \mathbb{R})$; if sign($b$) = sign($a$), $G$ is $SU(2)$; and $G$ is $H_3$ if $b = 0$. For the particular case $a = b$, $G$ is the ordinary sphere $S^3(c) \cong SU(2)$, with sectional curvature $c = a^2/4$.

From Lemma 3.3, $G$ can be identified with the homogeneous manifold $(SO(2) \ltimes \pi G)/SO(2)$, where $\pi$ is given in (5.1). Then,

$$
\pi_*(e_0)e_1 = e_2, \quad \pi_*(e_0)e_2 = -e_1, \quad \pi_*(e_0)e_3 = 0,
$$

where $e_0 = d/d\theta \in \mathfrak{so}(2)$. Putting

\begin{equation}
f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_3 - \frac{a + 2b}{2} e_0,
\end{equation}

and denoting by $\mathfrak{m}_{a,b}$ the subspace generated by $f_1$, $f_2$ and $f_3$, one gets that the decomposition $\mathfrak{so}(2) \oplus_{\pi_0} \mathfrak{g} = \mathbb{R}e_0 \oplus \mathfrak{m}_{a,b}$ is reductive and the inner product on $\mathfrak{m}_{a,b}$ making $\{f_1, f_2, f_3\}$ an orthonormal basis determines a $(SO(2) \ltimes \pi G)$-invariant cyclic metric on $(SO(2) \ltimes \pi G)/SO(2)$, which is then isometric to $(G, g)$.

From Theorem 4.5 (see also [12, Chapter IV, Proposition 3.6]), any three-dimensional cyclic homogeneous Riemannian manifold of a nonunimodular Lie group have to be a semidirect Riemannian product $\mathbb{R} \ltimes \pi (L/K)$, where $L/K$ is a two-dimensional simply-connected Riemannian symmetric space and $(L, K)$ is a symmetric pair, satisfying the conditions (iii) of Theorem 4.5 for $\pi$. So, they may be written as one of the following semidirect Riemannian products:

(i) $\mathbb{R} \ltimes \pi S^2(c) \cong (\mathbb{R} \ltimes \pi SO(3))/SO(2),$

(ii) $\mathbb{R} \ltimes \pi \mathbb{R}^2,$

(iii) $\mathbb{R} \ltimes \pi H^2(c) \cong (\mathbb{R} \ltimes \pi SO(1, 2))/SO(2),$

where the homomorphism $\pi$ satisfies the conditions in (iii) of Theorem 4.5. The case (ii) corresponds exactly with the case (2).

For (i) and (iii), the Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ of $G_+ = \mathbb{R} \ltimes \pi SO(3) \supset g_+ = \mathbb{R} \ltimes \pi SO(1, 2)$ admit a basis $\{e_0 = d/dt, u, e_1, e_2\}$, adapted to the reductive decomposition $\mathfrak{g}_\pm = \mathfrak{so}(2) \oplus \mathfrak{m} = \mathbb{R}u \oplus \mathbb{R}\{e_0 = d/dt, e_1, e_2\}$, respectively, such that $\{e_1, e_2\}$ is an orthonormal basis of $\mathcal{D} \cong T_0 S^2(c)$ (resp. $\mathcal{D} \cong T_0 H^2(c)$). Then

$$
[e_1, e_2] = \pm cu, \quad [e_2, u] = cc_1, \quad [u, e_1] = cc_2,
$$

and $[e_0, u] = 0$. Since $\text{ad}_{e_0}$ is selfadjoint on $\mathcal{D}$, it can be expressed as

$$
\text{ad}_{e_0} = \begin{pmatrix}
0 & \mu_1 & \mu_2 \\
0 & \alpha_1 & \lambda \\
0 & \lambda & \alpha_2
\end{pmatrix}
$$
with respect to the basis \( \{ u, v \} \) of \( \mathfrak{so}(2) \). Because \( \text{ad}_v \) acts as a derivation, one gets that the Lie algebras \( g_{\pm} \) are direct products and the corresponding homogeneous structures vanish. \( \square \)

**Remark 5.2.** In general, if \( H \ltimes G \) is a semidirect product of connected Lie groups \( H \) and \( G \), and \( H \) is closed, the map \( \theta \) given by

\[
\theta: (H \ltimes G) \times G \rightarrow G, \quad ((h, g_1), g) \mapsto g\pi(h)g_1,
\]

is a transitive action of \( H \ltimes G \) on \( G \). Moreover, \( H \), considered as a subgroup of \( H \ltimes G \), is the isotropy subgroup at the identity element of \( G \). Hence, \( G \) can be identified with the homogeneous manifold \( (H \ltimes G)/H \).

**Theorem 5.3.** A four-dimensional simply-connected cyclic homogeneous Riemannian manifold with nontrivial associated homogeneous structure is isometric to one of the following homogeneous Riemannian manifolds:

1. \( \widetilde{SL(2, \mathbb{R})} \times \mathbb{R} \), where the universal covering group \( \widetilde{SL(2, \mathbb{R})} \) of \( SL(2, \mathbb{R}) \) is equipped with a cyclic left-invariant metric.
2. The Lie group \( G^4(\alpha, \beta, \gamma) \) of matrices of the form

\[
\begin{pmatrix}
  e^{\alpha u} & 0 & 0 & x \\
  0 & e^{\beta u} & 0 & y \\
  0 & 0 & e^{\gamma u} & z \\
  0 & 0 & 0 & 1
\end{pmatrix}, \quad (\alpha, \beta, \gamma) \in \mathbb{R}^3 \setminus \{(0, 0, 0)\},
\]

with the left-invariant metric \( g = du^2 + e^{-2\alpha u}dx^2 + e^{-2\beta u}dy^2 + e^{-2\gamma u}dz^2 \).
3. The orthogonal semidirect product \( \mathbb{R}^2 \ltimes \mathbb{R}^2 \) under \( \pi: \mathbb{R}^2 \rightarrow \text{Aut}(\mathbb{R}^2) \) given by \( \pi(s, t) = \text{diag}(e^{2\rho s}, e^{2\rho t}) \), \( \rho + \sigma \neq 0, \lambda > 0 \). This Lie group can be described as the group \( H^4(\rho, \sigma; \lambda) \) of matrices of the form

\[
\begin{pmatrix}
  e^{\rho u + \lambda v} & 0 & 0 & x \\
  0 & e^{\sigma u - \lambda v} & 0 & y \\
  0 & 0 & 0 & 1
\end{pmatrix}, \quad \rho + \sigma \neq 0, \quad \lambda > 0,
\]

with the left-invariant metric \( g = du^2 + dv^2 + e^{-2(\rho u + \lambda v)}dx^2 + e^{-2(\sigma u - \lambda v)}dy^2 \).
4. The Riemannian product \( (G, g) \times \mathbb{R} \), where \( (G, g) \) is one of the metric unimodular Lie groups given in (3) of Theorem 5.1.
5. The Cartesian space \( \mathbb{R}^4(x, y, u, v) \) equipped with a Riemannian metric of the form

\[
g = (-x + \sqrt{x^2 + y^2 + 1})du^2 + (x + \sqrt{x^2 + y^2 + 1})dv^2 - 2y(dudv)
\]

\[
+ \lambda^2(1 + x^2 + y^2)^{-1}((1 + y^2)dx^2 + (1 + x^2)dy^2) - 2xydxdy,
\]

where \( \lambda > 0 \) is a real parameter.
6. The semidirect Riemannian products \( \mathbb{R} \ltimes (\mathbb{R} \times S^2(c)) \) and \( \mathbb{R} \ltimes (\mathbb{R} \times H^2(c)) \), with corresponding adapted quotient expressions

\[
\mathbb{R} \ltimes (\mathbb{R} \times SO(3))/SO(2), \quad \mathbb{R} \ltimes (\mathbb{R} \times SO(1, 2))/SO(2),
\]

where \( \pi: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \times SO(3)) \) and \( \pi: \mathbb{R} \rightarrow \text{Aut}(\mathbb{R} \times SO(1, 2)) \) are given by \( \pi(t)(\lambda, x) = (\lambda e^{\alpha t}, x) \), for \( \alpha \in \mathbb{R} \setminus \{0\} \). The admissible metrics depend on the parameter \( \alpha \).
Proof: The Riemannian manifolds in (1), (2) and (3) are the four-dimensional simply-connected nonabelian cyclic metric Lie groups \([7, \text{Theorem 6.2}]\). Moreover, \(SL(2, \mathbb{R}) \times \mathbb{R}\) in (1); \(G^4(\alpha, \beta, \gamma)\), with \(\alpha + \beta + \gamma = 0\), in (2); and the Riemannian manifolds in (4) and (5), are all the four-dimensional cyclic homogeneous Riemannian manifolds with a nonvanishing homogeneous structure of type \(T_2\) (see \([15, \text{Theorem 3.1}]\)).

Next, let \((G, g)\) be one of the three-dimensional unimodular Lie groups equipped with a left-invariant metric \(g\) in (3) of Theorem 5.1. Then \((SO(2) \ltimes \pi G)/SO(2)\) is a cyclic quotient expression for \(G\) and \(\mathbb{R}e_0 \oplus \mathbb{R}\{f_1, f_2, f_3\}\) is an adapted reductive decomposition, where \(f_1, f_2, f_3\) are given in (5.3). From (5.2) and (5.3), the corresponding multiplication scheme is given by

\[
[f_1, f_2] = a f_3 + \frac{a(a + 2b)}{2} e_0, \quad [e_0, f_1] = f_2,
\]

(5.4)

\[
[f_2, f_3] = -\frac{a}{2} f_1, \quad [e_0, f_2] = -f_1,
\]

\[
[f_3, f_1] = -\frac{a}{2} f_2, \quad [e_0, f_3] = 0.
\]

Consider the semidirect Riemannian products \(\mathbb{R} \ltimes \pi G\) such that the derivation \(\pi_*(\alpha/\alpha t)\) of the Lie algebra \(g\) of \(G\) satisfies (iii) in Theorem 4.5. Then \(\text{ad}_{\alpha/\alpha t}\) can be expressed as

\[
\text{ad}_{\alpha/\alpha t} = \begin{pmatrix}
0 & \mu_1 & \mu_2 & \mu_3 \\
0 & \alpha_1 & \beta & \gamma \\
0 & \beta & \alpha_2 & \delta \\
0 & \gamma & \delta & \alpha_3
\end{pmatrix}
\]

(5.5)

in terms of the basis \(\{e_0 = \alpha/\alpha t, e_1, e_2, e_3\}\). Since \(\text{ad}_{\alpha/\alpha t}\) acts as a derivation on \(g\), one gets, using (5.4) and by a straightforward calculation, that \(\text{ad}_{\alpha/\alpha t} = 0\). Hence, \(\mathbb{R} \ltimes \pi G\) must be a direct Riemannian product and then one of the Riemannian manifolds considered in (4).

As for cyclic homogeneous manifolds \(G/K\) of a nonunimodular Lie group \(G\), arguing as in the three-dimensional case (see the proof of Theorem 5.1), we have the following possible Riemannian manifolds:

(i) \(\mathbb{R} \ltimes \pi \mathbb{R}^3\),

(ii) \(\mathbb{R} \ltimes \pi S^3(c) \cong (\mathbb{R} \ltimes \pi SO(4))/SO(3)\),

(iii) \(\mathbb{R} \ltimes \pi H^3(c) \cong (\mathbb{R} \ltimes \pi SO(1,3))/SO(3)\),

(iv) \(\mathbb{R} \ltimes \pi S^3(c) \cong (\mathbb{R} \ltimes \pi (SU(2) \times SU(2)))/\Delta(SU(2))\),

(v) \(\mathbb{R} \ltimes \pi H^3(c) \cong (\mathbb{R} \ltimes \pi SL(2,\mathbb{C}))/SU(2)\),

(vi) \(\mathbb{R} \ltimes \pi (\mathbb{R} \times S^2(c)) \cong (\mathbb{R} \ltimes \pi (\mathbb{R} \times SO(3)))/SO(2)\),

(vii) \(\mathbb{R} \ltimes \pi (\mathbb{R} \times H^2(c)) \cong (\mathbb{R} \ltimes \pi (\mathbb{R} \times SO(1,2)))/SO(2)\).

(For (iv) and (v), cf. \([3, 7.103, 7.105]\).) The case (i) corresponds with the case (2), cf. \([7, \text{Example 5.6}]\). For (ii) and (iii), the Lie algebras \(g_+\) and \(g_-\) of \(G_+ = \mathbb{R} \ltimes \pi SO(4)\) and \(G_- = \mathbb{R} \ltimes \pi SO(1,3)\), respectively, admit bases \(\{e_0 = \alpha/\alpha t, u_i, e_i\}\), \(i = 1, 2, 3\), adapted to the reductive decompositions \(g_\pm = \mathfrak{so}(3) \oplus \mathfrak{m} = \mathbb{R}\{u_1, u_2, u_3\} \oplus \mathbb{R}\{e_0, e_1, e_2, e_3\}\), such that \(\{e_1, e_2, e_3\}\) is an orthonormal basis of \(D \cong T_o S^3(c)\), (resp. \(D \cong T_o H^3(c)\)). Then

\[
[u_2, u_3] = cu_1, \quad [u_3, u_1] = cu_2, \quad [u_1, u_2] = cu_3,
\]
Similarly to cases (ii) and (iii), one gets a basis of $\mathfrak{g}$ and $\text{ad}/dt|_{\mathfrak{m}}$ is given by a matrix like (5.5), hence null, we have for $[e_0, u_i] \in \mathfrak{m}, i = 1, 2, 3$, that $\langle [e_0, u_i], e_j \rangle = -\langle u_i, [e_0, e_j] \rangle = 0$, for $i, j = 1, 2, 3$. That is, $[e_0, u_i] = 0, i = 1, 2, 3$.

In the cases (iv) and (v), we also denote by $\mathfrak{g}_+$ and $\mathfrak{g}_-$ the Lie algebras of $G_+ = \mathbb{R} \ltimes \pi(SU(2) \times SU(2))$ and $G_- = \mathbb{R} \ltimes \pi(S \times SL(2, \mathbb{C}))$, respectively, where $SL(2, \mathbb{C})$ is considered as a real Lie group. They admit bases $\{e_0 = d/dt, u_i, e_i\}, i = 1, 2, 3$, adapted to the corresponding reductive decompositions, such that $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{D}$ and

$$ [u_2, u_3] = \pm [e_2, e_3] = cu_1, \quad [u_3, u_1] = \pm [e_3, e_1] = cu_2, \quad [u_1, u_2] = \pm [e_1, e_2] = cu_3, \quad [u_i, e_i] = 0, \quad [u_2, e_3] = [e_2, u_3] = ce_1, \quad [u_3, e_1] = [e_3, u_1] = ce_2, \quad [u_1, e_2] = [e_1, u_2] = ce_3. $$

We have, as in cases (ii) and (iii), that $[e_0, u_i] = 0, i = 1, 2, 3$. Using again that $\text{ad}_{e_0}$ is selfadjoint on $\mathbb{D}$ and it acts as a derivation, one can prove by a direct computation that $\text{ad}_{e_0} = 0$, so, in particular, $\text{tr} \ ad_{e_0} = 0$ in these cases. Hence the Riemannian manifolds in (ii)-(v) do not admit any cyclic structure with nonvanishing fundamental one-form.

Finally, we consider the cases (vi) and (vii). Here, the Lie algebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$ of $G_+ = \mathbb{R} \ltimes \pi(SU(2) \times SO(3))$ and $G_- = \mathbb{R} \ltimes \pi(S \times SO(1, 2))$, respectively, admit a basis $\{e_0 = d/dt, u_i, e_i\}, i = 1, 2, 3$, adapted to the reductive decomposition $\mathfrak{g}_\pm = \mathfrak{so}(2) \oplus \mathfrak{m} = \mathbb{R}u \oplus \mathbb{R}\{e_0, e_1, e_2, e_3\}$, such that $\{e_1, e_2, e_3\}$ is an orthonormal basis of $\mathbb{D}$ and we have the following brackets:

$$ [e_2, e_3] = \pm cu_1, \quad [e_3, u] = ce_2, \quad [u, e_2] = ce_3. $$

Similarly to cases (ii) and (iii), one gets $[e_0, u] = 0$. Furthermore, as $[u, e_1] \in \mathfrak{m}$ and $\langle [u, e_1] , e_0 \rangle = -\langle e_1, [u, e_0] \rangle = -\langle e_1, [u, e_1] \rangle = -\langle e_1, [u, e_2] \rangle = 0$, $\langle [u, e_1], e_3 \rangle = -\langle e_1, [u, e_3] \rangle = 0$, we have $[e_1, u] = 0$. Applying now that $[e_1, e_2] = [e_1, e_3] \in \mathfrak{t}$ and $[e_1, u] = 0$, we have $[e_1, e_2] = [e_1, e_3] = 0$. On the other hand, we have $(\pi(t))_*|_{\mathfrak{m}} = \text{ad}_{e_0}|_{\mathfrak{m}} \equiv \text{diag}(0, \alpha, 0, 0)$, that is, $\langle [e_0, e_1] = \alpha e_1$ and $[e_0, e_2] = [e_0, e_3] = 0$, for some $\alpha \in \mathbb{R}\setminus\{0\}$. This gives the case (6) of the theorem.}

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