SMOOTH RATIONAL CURVES ON SINGULAR RATIONAL SURFACES

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Abstract. We classify all complex surfaces with quotient singularities that do not contain any smooth rational curves, under the assumption that the canonical divisor of the surface is not pseudo-effective. As a corollary we show that if $X$ is a log del Pezzo surface such that for every closed point $p \in X$, there is a smooth curve (locally analytically) passing through $p$, then $X$ contains at least one smooth rational curve.

1. Introduction

Let $X$ be a projective rationally connected variety defined over $\mathbb{C}$. When $X$ is smooth, it is well known that there are many smooth rational curves on $X$: if $\dim X = 2$ then $X$ is isomorphic to a blowup of either $\mathbb{P}^2$ or a ruled surface $F_r$; if $\dim X \geq 3$, any two points on $X$ can be connected by a very free rational curve, i.e. image of $f : \mathbb{P}^1 \to X$ such that $f^*T_X$ is ample, and a general deformation of $f$ is a smooth rational curve on $X$ (for the definition of rationally connected variety and the above mentioned properties, see [Kol96]). It is then natural to ask about the existence of smooth rational curves on $X$ when $X$ is singular. In this paper, we study this problem on rational surfaces.

There are some possible obstructions to the existence of smooth rational curves. It could happen that there is no smooth curve germ passing through the singular points of $X$ (e.g. when $X$ has $E_8$ singularity) while the smooth locus of $X$ contains no rational curves at all (this could be the case when the smooth locus is of log Calabi-Yau or log general type), and then we won’t be able to find any smooth rational curves on $X$. Hence to produce smooth rational curves on $X$, we will need some control on the singularities of $X$ and the “negativity” of its smooth locus. We will show that these restrictions are also sufficient, in particular, we will prove the following theorem, which is one of the main results of this paper:

**Theorem.** Let $X$ be a surface with only quotient singularities. Assume that

1. $K_X$ is not pseudo-effective;
2. For every closed point $p \in X$, there is a smooth curve (locally analytically) passing through $p$.

Then $X$ contains at least one smooth rational curve.

In fact, we will prove something stronger. By studying various adjoint linear systems on rational surfaces, we show that condition (1) above combined with nonexistence of smooth rational curves has strong implication on the divisor class group of $X$ (Proposition 2.5), which allows us to classify all surfaces with quotient singularities that satisfy condition (1) above but do not contain smooth rational curves (Theorem 2.15). It turns out that all
such surfaces have an $E_8$ singularity, which is the only surface quotient singularity that does not admit a smooth curve germ.

This paper is organized as follows. In section 2 we study the existence of smooth rational curves on rational surfaces with quotient singularities whose anticanonical divisor is pseudo-effective but not numerically trivial and give the proof of the main result. In section 3 we study some examples and propose a few questions. In particular we construct some rational surfaces with quotient singularity and numerically trivial canonical divisor that contain no smooth rational curves.

**Conventions.** We work over the field $\mathbb{C}$ of complex numbers. Unless mentioned otherwise, all varieties in this paper are assumed to be proper and all surfaces normal. A surface $X$ is called log del Pezzo if there is a $\mathbb{Q}$-divisor $D$ on $X$ such that $(X, D)$ is klt and $-(K_X + D)$ is ample.

**Acknowledgement.** The author would like to thank his advisor János Kollár for constant support and lots of inspiring conversations. He also wishes to thank Qile Chen, Ilya Karzhemanov, Brian Lehmann, Yuchen Liu, Chenyang Xu and Yi Zhu for many helpful discussions. Finally he is grateful to the anonymous referee for careful reading of his manuscript and for the numerous constructive comments.

2. **Proof of main theorem**

In this section, we will classify all surfaces with quotient singularities containing no smooth rational curves, under the assumption that the anticanonical divisor is pseudo-effective but not numerically trivial. As a corollary, we will see that if $X$ is a log del Pezzo surface that has no $E_8$ singularity (as $E_8$ is the only surface quotient singularity whose fundamental cycle contains no reduced component, by [GSLJ94] this is equivalent to saying that for every point $p \in X$, there is a smooth curve germ passing through $p$), then $X$ contains at least one smooth rational curve.

We start by introducing a few results on adjoint linear systems that we frequently use to identify smooth rational curves on a surface.

**Lemma 2.1.** Let $X$ be a smooth rational surface and $D$ a reduced divisor on $X$, then $|K_X + D| = \emptyset$ if and only if every connected component of $D$ is a rational tree (i.e. every irreducible component of $D$ is a smooth rational curve and the dual graph of $D$ is a disjoint union of trees).

**Proof.** We have an exact sequence $0 \to \omega_X \to \omega_X(D) \to \omega_D \to 0$ which induces a long exact sequence

$$
\cdots \to H^0(X, \omega_X) \to H^0(X, \omega_X(D)) \to H^0(D, \omega_D) \to H^1(X, \omega_X) \to \cdots
$$

Since $X$ is a smooth rational surface, $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$, hence $H^0(X, \omega_X(D)) = 0$ if and only if $H^0(D, \omega_D) = 0$. We now show that the latter condition holds if and only if every connected component of $D$ is a rational tree. By doing this, we may assume $D$ is connected. Since $D$ is reduced, $H^0(D, \omega_D) = 0$ is equivalent to $p_a(D) = 0$. Let $D_i(i = 1, \cdots, k)$ be the irreducible components of $D$, we have $0 = p_a(D) = \sum_{i=1}^k p_a(D_i) + e - v + 1$ where $e$, $v$ are the number of edges and vertices in the dual graph of $D$. Since each $p_a(D_i) \geq 0$ and $e - v + 1 \geq 0$, we have equality everywhere, hence the lemma follows. □

We also need an analogous result when $X$ is not smooth.
Lemma 2.2. Let $X$ be a projective normal Cohen-Macaulay variety of dimension at least 2 and $D$ a Weil divisor on $X$, then we have an exact sequence
\begin{equation}
0 \to \omega_X \to \mathcal{O}_X(K_X + D) \to \omega_D \to 0
\end{equation}
where $\omega_X$, $\omega_D$ are the dualizing sheaf of $X$ and $D$, and $K_X$ is the canonical divisor of $X$.

Proof. See [Koll13, 4.1] □

Corollary 2.3. Let $X$ be a rational surface with only rational singularities, $D$ an integral curve on $X$, then $D$ is a smooth rational curve if and only if $|K_X + D| = \emptyset$.

Proof. Since $X$ is a normal surface, it is CM, so we can apply the previous lemma to get the exact sequence $(2.1)$, which induces the long exact sequence
\[ H^0(X, \omega_X) \to H^0(X, \mathcal{O}_X(K_X + D)) \to H^0(D, \omega_D) \to H^1(X, \omega_X) \to \cdots \]
As $X$ is a rational surface with only rational singularities, we have $H^0(X, \omega_X) = H^1(X, \omega_X) = 0$, hence $D$ is a smooth rational curve iff $H^0(D, \omega_D) = 0$ if $|K_X + D| = \emptyset$. □

One may notice that the above lemmas only apply to rational surfaces while our main theorem is stated for arbitrary surfaces. This is only a minor issue, as illustrated by the following lemma.

Lemma 2.4. Let $X$ be a surface. Assume that $X$ does not contain any smooth rational curves. Then either $K_X$ is nef or $-K_X$ is numerically ample and $\rho(X) = 1$.

Here since $-K_X$ is in general not $\mathbb{Q}$-Cartier, its nefness or numerical ampleness is understood in the sense of [Sak87]. In particular, if we further assume $X$ has rational singularities (which implies $X$ is $\mathbb{Q}$-factorial) and $K_X$ is not pseudo-effective (as we do in our main theorem), then $-K_X$ is ample and $X$ is a rational surface of Picard number one by [KT09, Lemma 3.1].

Proof. First suppose $X$ is not relatively minimal. By [Sak87, Theorem 1.4], we may run the $K_X$-MMP on $X$. Let $f : X \to Y$ be the first step in the MMP. Since $-K_X$ is $f$-ample, by [Sak85, Theorem 6.3] we have $R^1f_!\mathcal{O}_X = 0$. Let $C \subseteq X$ be an irreducible curve contracted by $f$ and $\mathcal{I}_C$ its ideal sheaf. Since the fibers of $f$ has dimension $\leq 1$ we have $R^2f_!\mathcal{I}_C = 0$ by the theorem of formal functions. It then follows from the long exact sequence associated to $0 \to \mathcal{I}_C \to \mathcal{O}_X \to \mathcal{O}_C \to 0$ that $H^1(C, \mathcal{O}_C) = R^1f_*\mathcal{O}_C = 0$, hence $C$ is a smooth rational curve on $X$, contrary to our assumption.

We may therefore assume that $X$ is relatively minimal. If $K_X$ is not nef then by [Sak87, Theorem 3.2], either $-K_X$ is numerically ample and $\rho(X) = 1$ or $X$ admits a fibration $g : X \to B$ whose general fiber is $\mathbb{P}^1$. However, the latter case cannot occur since $X$ does not contain smooth rational curves. This proves the lemma. □

Now we come to a useful criterion for whether a surface contains at least one smooth rational curve.

Proposition 2.5. Let $X$ be a surface with only rational singularities. Assume $K_X$ is not pseudo-effective, then the following are equivalent:

1. $X$ does not contain any smooth rational curves;
2. The class group $\text{Cl}(X)$ is infinite cyclic and is generated by some effective divisor $D$ linearly equivalent to $-K_X$.
Proof. First assume (2) holds. By [KT09] Lemma 3.1, $X$ is necessarily a rational surface. If $X$ contains a smooth rational curve $C$, then by Corollary $2.3$, $K_X + C = 0$, but by (2), we may write $C \sim kD$ for some integer $k \geq 1$, and $K_X + C \sim (k - 1)D$ is effective, a contradiction, so (1) follows.

Now assume (1) holds. By Lemma $2.3$ and its subsequent remark, $X$ is a rational surface with ample anti-canonical divisor. Let $H$ be an ample divisor on $X$ and assume there exists some effective divisor $C$ on $X$ that is not an integral multiple of $-K_X$ in $\text{Cl}(X)$. Among such divisors we may choose $C$ so that $(H.C)$ is minimal. Clearly $C$ is integral, and by (1) it is not a smooth rational curve, hence by Corollary $2.3$, $K_X + C$ is effective. Since $-K_X$ is ample, we have $(K_X + C.H) < (C.H)$, so by our choice of $C$, $K_X + C$ is an integral multiple of $K_X$, hence so is $C$, a contradiction. It follows that every effective divisor on $X$ is linearly equivalent to a multiple of $-K_X$. Since $\text{Cl}(X)$ is generated by the class of effective divisors, we see that it is infinite cyclic and generated by $-K_X$. Now let $m$ be the smallest positive integer such that $-mK_X$ is effective. Write $-mK_X = \sum a_i D_i$ where $a_i > 0$ and $D_i$ is integral. As $m$ is minimal and each $D_i$ is also a multiple of $-K_X$, we have indeed $-mK_X \sim D$ an integral curve. $D$ is not smooth rational by (1), hence again by Corollary $2.3$, $K_X + D$ is effective, but $K_X + D \sim -(m - 1)K_X$, so by the minimality of $m$ we have $m = 1$, and thus all the assertions in (2) are proved. □

From now on, $X$ will always be a normal surface that satisfies the assumptions and the equivalent conditions (1)(2) of Proposition $2.5$. In particular, $X$ is rational, $\mathbb{Q}$-factorial and has Picard number one, $-K_X$ is ample and $\text{Pic}(X) \cong \mathbb{Z}$ is generated by $-rK_X$ where $r$ is smallest positive integer such that $rK_X$ is Cartier (i.e. the index of $X$). We further assume that $X$ has at worst quotient singularities (or equivalently, klt singularities, as we are in the surface case). Let $X^0$ be the smooth locus of $X$, $\pi : Y \to X$ the minimal resolution and $E \subset Y$ the reduced exceptional locus.

Lemma 2.6. Notation as above. Then we have an exact sequence

$$0 \to \text{Cl}(X)/\text{Pic}(X) \to H^2(E, \mathbb{Z})/H_2(E, \mathbb{Z}) \to H_1(X^0, \mathbb{Z}) \to 0$$

and an isomorphism $H^2(E, \mathbb{Z})/H_2(E, \mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z}$.

Here we identify $H_2(E, \mathbb{Z})$ as a subgroup of $H^2(E, \mathbb{Z})$ by the composition $H_2(E, \mathbb{Z}) \to H_2(Y, \mathbb{Z}) \to H^2(Y, \mathbb{Z}) \to H^2(E, \mathbb{Z})$ where the first and the last map are induced by the inclusion $E \subset Y$ and the second by Poincaré duality. In other words, the intersection pairing on $Y$ induces a nondegenerate pairing $H_2(E, \mathbb{Z}) \times H_2(E, \mathbb{Z}) \to \mathbb{Z}$, hence we may view $H_2(E, \mathbb{Z})$ as a subgroup of $H^2(E, \mathbb{Z})$. Note that the intersection numbers between irreducible components of $E$ only depend on the singularities of $X$, so the quotient $H^2(E, \mathbb{Z})/H_2(E, \mathbb{Z})$ should be considered as a local invariant of the singularities of $X$.

Proof. The existence of the exact sequence follows from [MZSS, Lemma 2]. If $\text{Cl}(X) \cong \mathbb{Z}[-K_X]$, then from what we just said $\text{Pic}(X) \cong \mathbb{Z}[-rK_X]$ hence $\text{Cl}(X)/\text{Pic}(X) \cong \mathbb{Z}/r\mathbb{Z}$. It remains to prove $H_1(X^0, \mathbb{Z}) = 0$. Since the intersection matrix of $E$ is nondegenerate, $H_1(X^0, \mathbb{Z})$ is finite. If it is not zero, $X^0$ will admit a nontrivial étale cyclic covering of degree $d > 1$, hence $\text{Pic}(X^0) \cong \text{Cl}(X)$ would contain $d$-torsion, a contradiction. □

If $p \in X$ is a singular point, we let $r_p$ be the local index of $p$, i.e., the smallest positive integer $m$ such that $mK_X$ is Cartier at $p$, and define $\text{Cl}_p = H^2(E_p, \mathbb{Z})/H_2(E_p, \mathbb{Z})$ in the
same way as in the above lemma with $E_p = \pi^{-1}(p)_{\text{red}}$. As explained in the next lemma, it can be viewed as the “local class group” of $X$ at $p$. Since $(X, p)$ has quotient singularities, locally (in the analytic topology) it is isomorphic to a neighbourhood in $\mathbb{C}^2/G$ of the image of the origin where $G$ is a finite subgroup of $GL(2, \mathbb{C})$, then $\tau_p = |H|$ where $H$ is the image of $G$ under the determinant map det : $G \subset GL(2, \mathbb{C}) \to \mathbb{C}^*$, and $\text{Cl}_p$ is isomorphic to the abelianization of $G$:

**Lemma 2.7.** In the above notations, $\text{Cl}_p \cong G/G'$.

**Proof.** By definition, $\text{Cl}_p$ only depends on the intersection matrix of $E_p$, hence we may replace $X$ by an étale neighbourhood of $p$, in particular we may assume $(X, p) \cong (\mathbb{C}^2/G, 0)$. As before $\pi : Y \to X$ is the minimal resolution, then $E_p$ is a deformation retract of $Y$. As $X$ is affine and has rational singularities, $H^i(Y, \mathcal{O}_Y) = H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$, so by the long exact sequence associated to the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0$ we have $\text{Pic}(Y) \cong H^2(Y, \mathbb{Z}) \cong H^2(E_p, \mathbb{Z})$ and hence the following commutative diagram (where $U = X \setminus p = Y \setminus E_p$ and $E_{p,i}$ are the irreducible components of $E_p$):

\[
\begin{array}{ccc}
\oplus \mathbb{Z}[E_{p,i}] & \xrightarrow{\cong} & \text{Pic}(Y) \\
\downarrow \cong & & \downarrow \cong \\
H_2(E_p, \mathbb{Z}) & \xrightarrow{\cong} & H^2(E_p, \mathbb{Z}) \\
\end{array}
\]

It follows that $\text{Cl}_p \cong \text{Pic}(U)$. Let $V = \mathbb{C}^3 \setminus 0$, then $\text{Pic}(V) = 0$ and giving a line bundle on $U$ is equivalent to giving a $G$-action on the trivial line bundle on $V$ that is compatible with the $G$-action on $V$. Such objects are classified by $H^1(G, \mathcal{O}_V) = H^1(G, \mathbb{C}^*) \cong G/G'$, so the lemma follows. \qed

In particular, $r_p \leq |\text{Cl}_p|$. Since $r$ is the lowest common multiple of all $r_p$ and $H^2(E, \mathbb{Z})/H_2(E, \mathbb{Z}) \cong \mathbb{Z}/r\mathbb{Z}$ is the direct sum of all $\text{Cl}_p$, we obtain

**Corollary 2.8.** $\text{Cl}_p \cong \mathbb{Z}/r_p\mathbb{Z}$ for all $p \in \text{Sing}(X)$.

Quotient surface singularities are classified in [Bri68, Satz 2.11], using the table there together with the well known classification of Du Val singularities (see for example [Dur79]) we see that each singularity of $X$ has to be one of the following: the cyclic singularity $\frac{1}{n}(1, q)$ where $(q, n) = (q + 1, n) = 1$, type $\langle b; 2, 1; 3, 1; 3, 2 \rangle$ (recall from [Bri68, Satz 2.11] that a type $\langle b; n_1, q_1; n_2, q_2; n_3, q_3 \rangle$ singularity is the one whose dual graph is a fork such that the central vertex represents a curve with self intersection number $-b$ and the three branches are dual graph of the cyclic singularity $\frac{1}{n_i}(1, q_i)$ ($i = 1, 2, 3$), or $\langle b; 2; 3; 5 \rangle$ (meaning it is of type $\langle b; 2, r; 3, s; 5, t \rangle$ for some $r, s, t$). In particular, $E_8$ is the only Du Val singularity that appears in the list.

We now turn to the classification of surfaces without smooth rational curves.

**Lemma 2.9.** $X$ has at most one non Du Val singular point.

**Proof.** Since $X$ satisfies (2) of Proposition 2.5 there is an effective divisor $D \in |-K_X|$ (which is necessarily an integral curve). Let $\bar{D}$ be its strict transform on $Y$, we may write

\[K_Y + \bar{D} + \sum a_iE_i = \pi^*(K_X + D) \sim 0\]
Lemma 2.12. \( \tilde{Y} \)

Lemma 2.12.

\( \tilde{Y} \), \( p \) has a cyclic singularity at \( \tilde{p} \) and no other singular points of \( \tilde{X} \). By the classification of Gorenstein log del Pezzo surfaces, \( \tilde{X} \) is a rational tree, in particular \( \tilde{X} \) is Gorenstein, then by the previous discussion it has only \( E_{\tilde{p}} \)-singularities, hence by the classification of Gorenstein log del Pezzo surfaces, \( X \) is one of the two types of \( S(E_8) \) as discussed in [KM98, Lemma 3.6] and it is straightforward to verify that neither of them contain smooth rational curves (e.g. using Proposition 2.4). So from now on we assume \( X \) is not Gorenstein, and by the above lemma, we may denote by \( p \) the unique non Du Val singular point of \( X \) and let \( \Delta = \pi^{-1}(p)_{\text{red}} \). We also get the following immediate corollary from the proof of Lemma 2.9.

Corollary 2.10. Notation as above, then every effective divisor \( D \sim -K_X \) passes through \( p \) and no other singular points of \( X \).

In some cases, the curve \( D \) constructed in the previous proof turns out to be already a smooth rational curve on \( X \). To be precise:

Proposition 2.11. Let \( D \in |-K_X| \) and \( \tilde{D} \) its strict transform on \( Y \). Then either \( X \) has a cyclic singularity at \( p \) and \( K_Y + \tilde{D} + \Delta \sim 0 \), or \( (X, p) \) is a singular point of type \( (b; 2, 1; 3, 1; 3, 2) \) or \( (b; 2, 3; 5) \) with \( b = 2 \).

Proof. We have \( K_Y + \tilde{D} + \sum a_i E_i \sim 0 \) as in (2.2), where \( a_i \in \mathbb{Z}_{\geq 0} \) and \( E_i \subset \text{Supp} \Delta \) by Corollary 2.1. If some \( a_i \geq 2 \), then \( |K_Y + \tilde{D} + \Delta| = \emptyset \), hence by Lemma 2.1 \( \tilde{D} + \Delta \) is a rational tree, in particular (\( \tilde{D}, \Delta \)) = 1, and if in addition \( \Delta \) is the fundamental cycle of \( (X, p) \) (i.e. \( -\Delta \) is \( \pi \)-nef; this is the case if \( (X, p) \) has cyclic singularity or if the central curve of \( \Delta \) has self-intersection at most \(-3\)), then by [KM99, Lemma 4.12], \( D \) is a smooth rational curve on \( X \), but by Corollary 2.3 this contradicts our assumption as \( |K_X + D| \neq \emptyset \). We already know that the singularity of \( X \) at \( p \) is cyclic, \( (b; 2, 1; 3, 1; 3, 2) \) or \( (b; 2, 3; 5) \), hence in the first case all \( a_i = 1 \), and we claim that in the latter two cases at least one \( a_i \geq 2 \), it would then follow that \( b = 2 \). Suppose all \( a_i = 1 \), then \( K_Y + \tilde{D} + \Delta \sim 0 \), but the LHS has positive intersection with the central curve of \( \Delta \), a contradiction.

We need a more careful analysis in the cyclic case, so assume for the moment that \( X \) has cyclic singularity at \( p \). As above, \( D \) is an effective divisor in \( |-K_X| \) and \( \tilde{D} \) its birational transform on \( Y \), while \( \Delta = \pi^{-1}(p)_{\text{red}} \).

Lemma 2.12. \( \tilde{D} \) is a \((-1)\)-curve on \( Y \).
Lemma 2.13. There exists a birational morphism $f : Y \rightarrow \tilde{Y}$ such that

1. $\tilde{Y}$ is an $S(E_8)$;
2. $\text{Ex}(f)$ consists of all but one component of $\tilde{D} + E$;
3. $f(\tilde{D})$ is a smooth point on $\tilde{Y}$.

Proof. Let $Y \rightarrow Y_0$ be the contraction of all curves in $E \setminus \Delta$, then every closed point of $Y_0$ is either smooth or an $E_8$-singularity (every Du Val singularity of $X$ is an $E_8$-singularity). We run the $K$-negative MMP starting with $Y_0$:

$$Y_0 \xrightarrow{\phi_1} Y_1 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_m} Y_m \xrightarrow{g} Z$$

where each step is the contraction of an extremal ray, $\phi_i$’s are birational, and $\dim Z < 2$ ($Y_0$ is a Gorenstein rational surface, so the MMP stops at a Mori fiber space). By [KM99, Lemma 3.3], each $\phi_i$ is the contraction of a $(−1)$-curve contained in the smooth locus of $Y_{i-1}$. If this $(−1)$-curve is not a component of the image of $\tilde{D} + E$, let $C$ be its strict transform in $Y$, then $C$ is a smooth rational curve with negative self-intersection, hence by [Zha88, Lemma 1.3] it is a $(−1)$-curve. Now the same argument as in Lemma 2.12 shows that $(C, \Delta) \leq 1$ and $\pi(C)$ is a smooth rational curve in $X$, a contradiction. So the exceptional locus of $Y_0 \rightarrow Y_m$ is contained in $\tilde{D} + E$. In particular, since $\tilde{D}$ is the only component of $\tilde{D} + E$ that is a $(−1)$-curve, $\phi_1$ is the contraction of $\tilde{D}$.

We claim that $Z$ is a point. Suppose it is not, then $g$ is a $\mathbb{P}^1$-fibration. By [KM99, Lemma 3.4], as $Y_m$ has only singularities of $E_8$ type, it is actually smooth and isomorphic to $\mathbb{P}^e$ for some $e \geq 0$. If $e = 1$, one can choose to contract the $(−1)$-curve from $Y_m$ and then $Y_{m+1} = \mathbb{P}^2$ while $Z$ is a point. So we may assume $e = 0$ or $e \geq 2$. Since $\text{Cl}(X)$ is generated by $−K_X$, we see that $\text{Cl}(Y)$ is freely generated by $−K_Y$ and the components of $E$, or equivalently, by the components of $\tilde{D} + E$. Let $\Gamma$ be the image of $\tilde{D} + E$ on $Y_m$, we have $K_{Y_m} + \Gamma \sim 0$ and the irreducible components of $\Gamma$ freely generate $\text{Cl}(Y_m)$. As $\rho(Y_m) = 2$ in this case, $\Gamma$ has exactly two irreducible components. However, this contradicts the next lemma.

Hence $Z$ is a point and $Y_m$ is a Gorenstein rank one del Pezzo. By construction $\text{Cl}(Y_m)$ is generated by the effective divisor $\Gamma \sim −K_{Y_m}$, in other words, $Y_m$ does not contain any smooth rational curves, hence by the discussion on Du Val case, $Y_m$ is an $S(E_8)$, and the lemma follows by taking $\tilde{Y} = Y_m$. □

The following lemma is used in the above proof.
Lemma 2.14. Let $S = F_{e}$ where $e = 0$ or $e \geq 2$, then $-K_S$ cannot be written as the sum of two irreducible effective divisors that generate $Pic(S)$.

Proof. It is quite easy to see that when $e = 0$ such a decomposition of $-K_S$ is not possible, so we assume $e \geq 2$. Let $C_0$ be the unique section of negative self-intersection and $F$ be a fiber, then $Pic(S)$ is freely generated by $C_0$ and $F$. If $M = aC_0 + bF$ represents an irreducible curve then $M = aC_0$, or $b \geq ae \geq 0$. Suppose $-K_S \sim 2C_0 + (e+2)F \sim M_1 + M_2$ where $M_1$ and $M_2$ are irreducible and generate $Pic(S)$. Then we must have $M_i = C_0 + m_iF$ with $m_i \geq e$ and $m_1 + m_2 = e + 2$, this is only possible when $m_1 = m_2 = e = 2$, but then $M_1 = M_2$ can not generate $Pic(S)$.

Back to the general case. To finish the classification, let us now construct some surfaces that satisfy the conditions in Proposition 2.5. Let $\bar{Y}$ be an $S(E_8)$ with $\Gamma \in |-K_Y|$ a rational curve. We have $\Gamma \subset Y^0$ and $(\Gamma^2) = 1$. Let $q$ be the unique double point of $\Gamma$.

Let $Y \rightarrow \bar{Y}$ be the blowup at $q_1 = q$, $q_2$, $\cdots$, $q_m$ where each $q_i$ is infinitely near $q_{i-1}$ ($i > 1$). If $q$ is a node of $\Gamma$, we also require that $q_i$ always lies on the strict transform of either $\Gamma$ or exceptional curves of previous blowup (there are 2 different choices of $q_i$ for each $i > 1$). If $q$ is a cusp then we require that $m = 1$ or $2$ and that $q_i$ lies on the strict transform of $\Gamma$ for $i = 2, 3$ while $q_1$ is away from $\Gamma$ and previous exceptional curves. Let $E_i$ be the strict transform of the exceptional curve coming from the blowup of $q_i$. We define $X(\bar{Y}, \Gamma; q_1, \cdots, q_m)$ to be the contraction from $\bar{Y}$ of $\Gamma$ and $E_i$ ($i = 1, \cdots, m - 1$). It has two singular points, one of which is an $E_8$ singularity and the other is a cyclic singularity except when $\Gamma$ has a cusp at $q$ and $m = 4$, in which case the second singularity has type $(2; 2; 1; 3; 1; 5, 1)$. Argue inductively, we get $-K_Y \sim \Gamma + \sum_{i=1}^m E_i$ unless $\Gamma$ has a cusp at $q$ and $m = 4$, in which case we have $-K_Y \sim \Gamma + E_1 + E_2 + 2E_3 + E_4$ instead. As $Cl(Y)$ is generated by $\Gamma$ and all the $E_i$, it is not hard to verify that $X(\bar{Y}, \Gamma; q_1, \cdots, q_m)$ satisfies condition (2) in Proposition 2.5.

Theorem 2.15. If $X$ is a surface with only quotient singularities that satisfies the conditions in Proposition 2.5, then it is either an $S(E_8)$ or one of the $X(\bar{Y}, \Gamma; q_1, \cdots, q_m)$ constructed above.

Proof. If $X$ is Gorenstein then it is an $S(E_8)$, so we may assume that $X$ is not Gorenstein. Let $p$ be its unique non Du Val singular point, by Proposition 2.11 there are 3 possibilities for the singularity of $(X, p)$, and we analyse them one by one:

1. $(X, p)$ is a cyclic singularity. Let $Y_0$ be as in Lemma 2.13. By Lemma 2.13 there exists a birational morphism $f : Y_0 \rightarrow \bar{Y}$ where $\bar{Y}$ is an $S(E_8)$ such that $f$ contracts all but one component of $\bar{D} + \Delta$ to a smooth point (we use the same letters for strict transforms of $\bar{D}$ and $\Delta$ on $Y_0$). By Proposition 2.11, $K_{Y_0} + \bar{D} + \Delta \sim 0$, thus the dual graph of $\bar{D} + \Delta$ is a loop. It follows that $\Gamma = f(\bar{D} + \Delta) \sim -K_{\bar{Y}}$ is a rational curve with a double point $q$. In addition, $q$ is a cusp if and only if $\bar{D} + \Delta$ consists of two rational curves that are tangent to each other or three rational curves that intersect at the same point. In particular, $\bar{D} + \Delta$ has at most three components when $q$ is a cusp. Since $f$ is a composition of blowing down of $(-1)$-curves, we recover $Y_0$ as a successive blowup from $\bar{Y}$ of nodes on the images of $\bar{D} + \Delta$. Let $q_1, \cdots, q_m$ be the centers of these blowups. Clearly $q_1 = q$ and if $q$ is a cusp then $m \leq 2$. As $\bar{D}$ is the only $(-1)$-curve among the components of $\bar{D} + \Delta$,
each $q_i$ is infinitely near $q_{i-1}$. It is now easy to see that $X$ is a $X(\bar{Y}, \Gamma; q_1, \cdots, q_m)$ with $\Gamma$ nodal or $\Gamma$ cuspidal and $m \leq 2$.

(2) $(X, p)$ has type $\langle 2; 2, 1; 3, 1; 3, 2 \rangle$. By assumption $H^2(Y, Z) = \text{Pic}(Y)$ is freely generated by $K_Y$ and the components in $E$. Since the intersection paring on $H^2(Y, Z)$ is unimodular, the intersection matrix of $K_Y$ and $E$ has determinant $\pm 1$. Write $K_Y + G = \pi^*K_X$ where $G$ is supported on $E$ (and can be easily computed from the given singularity type). As $\pi^*K_X$ is the orthogonal projection of $K_Y$ to the span of the components of $E$, we must then have $(K_X^2) = ((K_Y + G)^2) = (K_Y^2) + (K_Y \cdot G) = 10 - r + (K_Y \cdot G) = \frac{1}{8}$ where $r = |\det((E_i, E_j))|$ and $r$ is the Picard number of $Y$. It is straightforward to compute that $G = \frac{5}{8}E_1 + \cdots$ where $E_1$ is only component of $E$ with self-intersection $(-3)$ (and this is the only component whose coefficient is relevant to us), $(K_Y \cdot G) = \frac{5}{8}$ and $r = 9$. But $r$ is an integer, so this case cannot occur.

(3) $(X, p)$ has type $\langle 2; 2, 3; 5 \rangle$. A similar computation as in case (2) shows that in order to have $10 - r + (K_Y \cdot G) = \frac{1}{8}$, $(X, p)$ must have type $\langle 2; 2, 1; 3, 1; 5, 1 \rangle$ and $r = 13$. Since the other singularities of $X$ are of $E_8$-type, $X$ has exactly one $E_8$-singularity. By the same proof as the proof of Lemma 2.12 $\bar{D}$ is a $(-1)$-curve. Let $E_1$ be the central curve of $\Delta$ and $E_2, E_3, E_5$ the other three components of $\Delta$ with self-intersections $-2, -3$ and $-5$ respectively. Write $K_Y + \bar{D} + \sum a_iE_i = \pi^*(K_X + D) \sim 0$ as before. We have $a_1 \geq 2$ since otherwise the LHS has positive intersection with $E_1$. By Lemma 2.1 $\bar{D} + \Delta$ is a rational tree, thus $\bar{D}$ intersects transversally with exactly one component of $\Delta$. It is straightforward to find the discrepancies $a_i$ once we know which component $\bar{D}$ intersects. But as $a_i$'s are integers, we find that $\bar{D}$ intersects $E_1$ by enumerating all the possibilities and that $a_2 = a_3 = a_5 = 1$. Now as in Lemma 2.13 we may contract $\bar{D}, E_1, E_2, E_3$ and all components of $E \setminus \Delta$ from $Y$ to obtain $\bar{Y}$, which is an $S(E_8)$, such that the image of $E_5$ is a cuspidal rational curve $\Gamma \sim -K_Y$. Reversing this blowing down procedure we see that $X$ is isomorphic to some $X(\bar{Y}, \Gamma; q_1, \cdots, q_4)$ where $\bar{Y}$ is an $S(E_8)$ and $\Gamma$ is cuspidal.

It is well known that $E_8$ is the only surface quotient singularity that does not admit a smooth curve germ [GSL94]. Hence the following corollary follows immediately from the above theorem.

**Corollary 2.16.** Let $X$ be a surface with only quotient singularities. Assume that

1. $K_X$ is not pseudo-effective;
2. For every closed point $p \in X$, there is a smooth curve (locally analytically) passing through $p$.

Then $X$ contains at least one smooth rational curve.

3. **Examples and Questions**

If $X$ is a log del Pezzo surface, a curve of minimal degree on $X$ seems to be a natural candidate for the smooth rational curve (such a curve is used extensively in the study of log del Pezzo surfaces). However, the following example shows that this is not always the case, even if $X$ is known to contain some smooth rational curve.
Example 3.1. Let $Y \neq S(E_8)$ be a Gorenstein log del Pezzo surface of degree 1 such that the linear system $|−K_Y|$ contains a nodal curve $D$. Let $\tilde{Y} \to Y$ be the blow up of the node of $D$. Let $E$ be the exceptional curve and $\tilde{D}$ the strict transform of $D$. Contract the $(-3)$-curve $\tilde{D}$ to get our surface $X$. It is straightforward to verify that the image of $E$ under the contraction is the only curve of minimal degree on $X$. But since $E$ intersects $\tilde{D}$ at two points, its image on $X$ is not smooth. In fact the smooth rational curves on $X$ are usually given by the strict transform of $(-1)$-curves on the minimal resolution of $Y$. Observe that as $K_X + E \sim 0$, we have that $K_X + C$ is ample for any smooth rational curve $C$ on $X$.

We are also interested in whether the smooth rational curve $C$ we find supports a tiger of the log del Pezzo surface $X$ (i.e. there exists $D \sim _{–} -K_X$ such that Supp$(D) = C$ and $(X,D)$ is not klt. See [KM99, Definition 1.13]). At least when $C$ passes through at most one singular point we have a positive answer:

Lemma 3.2. Let $C$ be a smooth rational curve on a rank 1 log del Pezzo surface $X$. Assume that $C$ passes through at most one singular point of $X$. If $\alpha \in \mathbb{Q}$ is chosen such that $K_X + \alpha C \equiv 0$, then the pair $(X, \alpha C)$ is not klt.

Proof. If $C$ lies in the smooth locus of $X$ then by adjunction $(K_X + C.C) = −2 < 0$, hence $\alpha > 1$ and the result is clear. Otherwise we may assume $C \cap \text{Sing}(X) = \{p\}$. Let $\beta$ be the log canonical threshold of the pair $(X,C)$ and $\pi : \tilde{X} \to X$ the minimal resolution. It suffices to show that $(K_X + \beta C.C) \leq 0$. As $C$ is a smooth rational curve, $\pi$ is also a log resolution of $(X,C)$. Write $\pi^*(K_X + C.C) = K_{\tilde{X}} + \beta \tilde{C} + \sum a_i E_i$ where the $E_i$’s are the exceptional curves of $\pi$. We have $a_i \leq 1$ by the choice of $\beta$ and $\tilde{C}$ only intersects one $E_i$. Now since $X$ is of rank 1 we have $(\tilde{C}^2) \geq −1$ by [Zha88, Lemma 1.3] and $(K_{\tilde{X}} + \tilde{C}.\tilde{C}) = −2$ by adjunction, thus

$$(K_X + \beta C.C) = (K_{\tilde{X}} + \beta \tilde{C} + \sum a_i E_i.C) \leq −1 − \beta + \sum (E_i.\tilde{C}) \leq −\beta < 0$$

On the other hand, once $C$ passes through more singular points of $X$, the situation becomes more complicated. The following example suggests that even if $X$ has a tiger, there is in general no guarantee that the tiger can be supported on $C$.

Example 3.3. Similar to the previous example, let $Y$ be a Gorenstein log del Pezzo surface of degree 1 with an $A_k$-singularity and $D \in |−K_Y|$ a nodal curve. Blow up the node and one of its infinitely near points to get a new surface $\tilde{Y}$ and let $X$ be the contraction of the strict transform of $D$ and the first exceptional curve. The second singularity of $X$ has dual graph of type $(4,2)$. Every smooth rational curve on $X$ is a $(-1)$-curve on the minimal resolution and intersects both singular points of $X$. By direct computation we have $\beta = \frac{1}{2}$ (where $\beta = \text{lct}(X,C)$ as in the proof of the above lemma) and $(K_X + \beta C.C) > 0$, hence by the same reasoning for the above lemma we know that $C$ does not support a tiger. However, $−K_X$ is effective so $X$ does have a tiger.

In view of Proposition 2.5, we may ask for a similar classification of surfaces with rational singularities that do not contain smooth rational curves. The next example shows that we do get additional cases.
Example 3.4. The construction is similar to that of \( X(\bar{Y}, \Gamma; q_1, \cdots, q_m) \). Let \( \bar{Y} \) be an \( S(E_8) \) with \( \Gamma \in | - K_{\bar{Y}} | \) a cuspidal rational curve and let \( q \) be the cusp of \( \Gamma \). Let \( Y \to \bar{Y} \) be the blowup at \( q_1 = q, q_2, \cdots, q_m \) \((m \geq 5)\) where each \( q_i \) is infinitely near \( q_{i-1} \) \((i > 1)\) such that \( q_i \) lies on the strict transform of \( \Gamma \) for \( i < m \) while \( q_m \) is away from \( \Gamma \) and the previous exceptional curves. Let \( E_i \) be the strict transform of the exceptional curve coming from the blowup of \( q_i \). The dual graph of \( \Gamma \) and \( E_i \) \((i = 1, \cdots, m - 1)\) is given as follows:

\[
(-2) \quad (\cdots) \quad (-2) - (-2) - \cdots - (-2) - (-m - 1)
\]

We define \( X \) to be the contraction from \( Y \) of these curves. It has two singular points, one of which is an \( E_8 \) singularity and the other is not a quotient singularity since we assume \( m \geq 5 \). Nevertheless, it is a rational singularity (one way to see this is to attach \( m \) auxiliary \((-1)\)-curves to \( \Gamma \) and notice that the corresponding configuration of curves contracts to a smooth point, hence any subset of these curves also contracts to a rational singularity by \cite[Proposition 1]{Art66}). We also have \(-K_Y \sim \Gamma + E_1 + E_2 + 2 \sum_{i=3}^{m-1} E_i + E_m\) by induction on \( m \) and it follows as before that \( \text{Cl}(X) \) is generated by the image of \( E_m \) which is linearly equivalent to \(-K_X\). By Proposition 2.16, \( X \) is a surface with rational singularities that doesn’t contain any smooth rational curves.

We observe that the surfaces in the above example still contain \( E_8 \) singularities and thus violate the second assumption of Corollary 2.16. In addition the above construction does not seem to have many variants. It is therefore natural to ask the following question:

Question 3.5. Let \( X \) be a surface with rational singularities. Assume that \( K_X \) is not pseudo-effective and every closed point of \( X \) admits a smooth curve germ. Is it true that \( X \) contains a smooth rational curve? More aggressively, classify all surfaces with rational singularities that do not contain smooth rational curves.

Finally we investigate what happens if we remove the assumption on \( K_X \) in our main theorem. Clearly there are many smooth surfaces (e.g. abelian surfaces, ball quotients, etc.) with nef canonical divisors that do not even contain rational curves. Since we are mostly interested in the existence of smooth rational curves, we restrict ourselves to rational surfaces. We will construct some examples of rational surfaces with cyclic quotient singularities that do not contain smooth rational curves. These rational surfaces \( X \) will be the quotient of certain singular K3 surfaces and satisfy \( K_X \sim \mathbb{Q} 0 \), hence the assumption (1) in our main theorem is necessary.

Example 3.6. Let \( T \) be a smooth del Pezzo surface of degree 1. For general choice of \( T \), the linear system \( |-K_T| \) contains at least two nodal rational curves \( C_i \) \((i = 1, 2)\). Let \( Q_i \) be the node of \( C_i \) and \( P = C_1 \cap C_2 \). Let \( \pi : Y \to T \) be the blowup of both \( Q_i \) with exceptional divisors \( E_i \) and let \( \tilde{C}_i \) be the strict transform of \( C_i \) on \( Y \). Then \( K_Y = \pi^* K_T + E_1 + E_2 \) and \( \tilde{C}_i = \pi^* C_i - 2E_i = \pi^*(-K_T) - 2E_i \), thus \( -2K_Y \sim \tilde{C}_1 + \tilde{C}_2 \). We also have \( (\tilde{C}_1^2) = -3 \) and \( (\tilde{C}_1 \cdot \tilde{C}_2) = 1 \), hence we can contract both \( \tilde{C}_i \) simultaneously to get a rational surface \( X \) with a cyclic singularity \( p \) of type \( \frac{1}{8}(1, 3) \). The next three
lemmas tell us that for very general choice of $T$ and $C_1$, such $X$ does not contain any smooth rational curves.

**Lemma 3.7.** Every smooth curve on $X$ is away from $p$.

*Proof.* Let $p \in C$ be a smooth curve on $X$ and $\tilde{C}$ its strict transform on $Y$, then $(\tilde{C}, \tilde{C}_1 + \tilde{C}_2) = 1$. But we have $\tilde{C}_1 + \tilde{C}_2 = -2KY$, so the intersection must be even, a contradiction.

Let $Y' \rightarrow Y$ be the blowup of $P$ and $C'_i$ the strict transform of $\tilde{C}_i$, then $C'_1 + C'_2 = -2KY'$, hence we can take the double cover $f : S \rightarrow Y'$ ramified along $C'_1 + C'_2$. $S$ is smooth as $C'_1$ and $C'_2$ are smooth and disjoint. $S$ is indeed a K3 surface as $K_S = f^*KY'(\frac{1}{2}(C'_1 + C'_2)) \sim 0$ and $H^1(S, O_S) = H^1(Y', O_{Y'}) \oplus H^1(Y', O_{Y'}(KY')) = 0$.

**Lemma 3.8.** For very general choice of $T$ and $C_1$, the above K3 surface $S$ has Picard number 12.

*Proof.* $\rho(S) \geq 12$ as it’s a double cover of $Y'$ and $\rho(Y') = 12$. Since the moduli space of K3 surfaces is 20-dimensional, the locus of those with Picard number at least 13 is a countable union of subvarieties of dimension at most 7. On the other hand, the above construction gives us an 8-dimensional family of K3 surfaces: we have an 8-dimensional family of del Pezzo surfaces of degree 1. Hence for very general choice of $T$, we get a K3 surface $S$ with $\rho(S) = 12$.

It now follows that

**Lemma 3.9.** For very general choice of $C_1$, the rational surface $X$ constructed above does not contain any smooth rational curve.

*Proof.* Suppose $C \subseteq X$ is a smooth rational curve. By Lemma 3.7, $p \not\in C$, hence its strict transform $C'$ in $Y'$ is disjoint from $C'_1 + C'_2$. As $f : S \rightarrow Y'$ is étale outside $C'_1 + C'_2$, $f^{-1}(C')$ splits into a disjoint union of two smooth rational curves $D_1$, $D_2$. This implies $\rho(S) \geq 13$ ($D_1$, $D_2$ and the pullback of the orthogonal complement of $C'$ in Pic($Y'$) generate a sublattice of rank 13), which can not happen for very general choice of $T$ and $C_1$ by Lemma 3.8.

By allowing more singular points, we can give a similar construction with a simpler proof of non-existence of smooth rational curves.

**Example 3.10.** Instead of taking a smooth del Pezzo surface of degree 1, let $T$ be a Gorenstein rank one log del Pezzo surface of degree 1. Assume either $T$ has a unique singular point or it has exactly two $A_n$-type singular points, then a similar argument as the proof of [KM99, Lemma 3.6] implies that for general choice of $T$, $|−K_T|$ contains two nodal rational curves $C_i (i = 1, 2)$ lying inside the smooth locus of $T$. Let $X$ be the surface obtained by the same construction in Example 3.6 (i.e. blow up the nodes $Q_i$ of $C_i$ and contract both $\tilde{C}_i$), then it has the same singularities as $T$ as well as a cyclic singularity $p$ of type $\frac{1}{3}(1, 3)$. Suppose $X$ contains a smooth rational curve $C$. As before we know that $C \subseteq U = X\setminus p$, and as $2K_X \sim 0$, we have a double cover $g : Y \rightarrow X$ that is unramified over $U$ (since $K_X$ is Cartier over $U$). Since $C \cong \mathbb{P}^1$ is simply connected, we see that $g^{-1}(C)$ consists of two disjoint copies of $\mathbb{P}^1$. By construction $X$ has Picard number one, hence $C$ is ample, thus $g^*C$ is also ample on $Y$, but this contradicts [Har77].
III.7.9]. In some cases one can also derive a contradiction without using the double cover. For example suppose $T$ has a unique $A_8$-type singularity $q$, then modulo torsion $\text{Cl}(X)$ is generated by $E$, the strict transform of the exceptional curve over either one of the $Q_i$’s, and $(E^2) = \frac{1}{2}$. It follows that

\[(3.1) \quad \deg(K_C + \text{Diff}_C(0)) = (K_X + C.C) = (C^2) \geq \frac{1}{2}\]

but $\deg K_C = -2$ and as $C$ is smooth at $q$, the dual graph of $(X, C)$ at $q$ is a fork with $C$ being one of the branches. It is then straightforward to compute that $\deg \text{Diff}_C(0) = \left(\frac{1}{m} + \frac{1}{n}\right)^{-1} \leq \frac{20}{9}$ where $m, n$ are the index of the other two branches of the dual graph (i.e. one larger than the number of vertices in the branch), which contradicts (3.1).

Inspired by these examples, we may expect to take certain quotients of Calabi-Yau varieties and construct higher-dimension rationally connected varieties with klt singularities that do not contain smooth rational curves. Unfortunately we are unable to identify such an example, and therefore leave it as a question:

**Question 3.11.** Let $X$ be a rationally connected variety of dimension $\geq 3$ with klt singularities. Does $X$ always contain a smooth rational curve?

We remark that if $X$ is indeed log Fano, then a folklore conjecture predicts that the smooth locus of $X$ is rationally connected and thus contains a smooth rational curve since $\dim X \geq 3$.

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