QED theory of the nuclear recoil effect on the atomic \( g \) factor

V. M. Shabaev\(^{1,2} \)

\(^{1}\)Department of Physics, St.Petersburg State University, Oliianovskaya 1, Petrodvorets, St.Petersburg 198504, Russia
\(^{2}\)Gesellschaft für Schwerionenforschung, 64291 Darmstadt, Germany

The quantum electrodynamic theory of the nuclear recoil effect on the atomic \( g \) factor to all orders in \( \alpha Z \) and to first order in \( m/M \) is formulated. The complete \( \alpha Z \)-dependence formula for the recoil correction to the bound-electron \( g \) factor in a hydrogenlike atom is derived. This formula is used to calculate the recoil correction to the bound-electron \( g \) factor in the order \( (\alpha Z)^2 m/M \) for an arbitrary state of a hydrogenlike atom.

I. INTRODUCTION

High precision direct measurements of the bound-electron \( g \) factor in low-\( Z \) hydrogenlike ions, which are performed by a GSI - Universitäit Mainz collaboration \(^{[12]}\), have triggered a great interest to the calculations of QED and nuclear corrections to this effect \(^{[3–12]}\). It is expected that in nearest future the collaboration will extend these measurements to high-\( Z \) systems. The accurate measurements of the \( g \) factor in heavy ions would provide a good possibility for a test of the magnetic sector of QED in a strong Coulomb field.

In particular, in the case of hydrogenlike bismuth, it was found that including the QED correction to the \( g \) factor is necessary to obtain agreement between theory and experiment (see \(^{[13–14]}\) for details). However, the indirect method can not provide so high accuracy as the direct method if the latter is extended to high-\( Z \) systems.

It is known (see, e.g., \(^{[0–17]}\) that calculations of heavy ions have to be performed without any expansion in the papameter \( \alpha Z \). In particular, it means that, to zeroth-order approximation, the electron in a hydrogenlike atom must be described by the Dirac equation with the Coulomb potential induced by the nucleus. For the point-charge nucleus, the bound-electron \( g \) factor in a hydrogenlike atom is easily calculated analytically (see, e.g., \(^{[18]}\)). The one-loop QED corrections to the \( 1s \) \( g \) factor were calculated to all orders in \( \alpha Z \) in \(^{[19–22]}\) (see also \(^{[23]}\)). The finite nuclear size correction can be found by solving the Dirac equation for an extended nucleus \(^{[13–23]}\). For low- and middle-\( Z \) atoms, this correction can also be found by non-relativistic and approximate relativistic formulas \(^{[19–23]}\). The one-loop QED corrections to the 1s \( g \) factor was evaluated only to the lowest order in \( \alpha Z \) \(^{[20–23]}\) (see also \(^{[24–12]}\)). The methods developed in \(^{[24–23]}\) are suitable for calculations to a few lowest orders in the parameter \( \alpha Z \) in \( \alpha Z \) and are not intended for calculations of the recoil effect to all orders in \( \alpha Z \). Therefore, first of all we need to develop a new method which would provide a systematic treatment of the recoil effect on the \( g \) factor to all orders in \( \alpha Z \) and to first order in \( m/M \) (\( m \) is the electron mass and \( M \) is the nucleus mass).

Such a method is proposed in the present paper. This method is valid for the general case of a many-electron atom. Applying the method to a hydrogenlike atom, the complete \( \alpha Z \)-dependence formula for the recoil correction to the \( g \) factor is derived. To the lowest relativistic order \( \sim (\alpha Z)^2 m/M \), this formula is used to calculate analytically the recoil correction to the \( g \) factor in the case of an arbitrary state of a hydrogenlike atom. For the 1s state, the result obtained in this paper coincides with the related result obtained previously in \(^{[21–23]}\).

The relativistic units \( (\hbar = c = 1) \) and the Heaviside charge unit \( (\alpha = e^2/(4\pi), \ e < 0) \) are used in the paper.

II. QUANTUM ELECTRODYNAMICS TO FIRST ORDER IN \( m/M \) BEYOND THE FURRY PICTURE

To derive the Hamiltonian of the atom to first order in \( m/M \) and to lowest order in \( \alpha \) (but to all orders in \( \alpha Z \)), we start with the relativistic quantum mechanical treatment of the electrons and the non-relativistic treatment of the nucleus. For simplicity, we consider the nucleus as a spinless particle. The electrons and the nucleus interact with each other, with the quantized electromagnetic field, and with the classical homogeneous magnetic field, \( A_{cl}(r) = [H \times r]/2 \).

In the Coulomb gauge and the Schrödinger representation, the Hamiltonian of the system to first order in the interaction with \( A_{cl}(r) \) is

\[
H = \sum_{i=1}^{N'} [\alpha_i \cdot p_i^{(e)} + \beta_i m + V(r_i^{(e)} - r^{(n)}) - e\alpha_i \cdot A(r_i^{(e)})]
\]
\[ + \frac{1}{2} \sum_{i \neq k} | \alpha_i |^2 | \mathbf{r}_i - \mathbf{r}_k | + \frac{1}{2M} \left[ \mathbf{p}_i^{(n)} - |e| Z \mathbf{A}(\mathbf{r}^{(n)}) \right]^2 \]
\[ + \frac{1}{2} \int d\mathbf{x} \left[ \mathcal{E}_i^2(\mathbf{x}) + \mathcal{H}^2(\mathbf{x}) \right] \]
\[ - e \sum_{i=1}^{N'} \alpha_i \cdot \mathbf{A}_{\text{cl}}(\mathbf{r}_i^{(e)}) - |e| Z \frac{\mathbf{p}_i^{(n)}}{M} \cdot \mathbf{A}_{\text{cl}}(\mathbf{r}^{(n)}) \], \tag{1} \]

where \( N' \) is the total number of the positive and negative energy state electrons, \( \mathbf{p}_i^{(e)} \) and \( \mathbf{p}_i^{(n)} \) are the electron and nucleus momentum operators, respectively, \( V(\mathbf{r}_i^{(e)} - \mathbf{r}^{(n)}) \) is the operator of the electron-nucleus Coulomb interaction, \( \mathbf{A}(\mathbf{r}) \) is the vector potential of the quantized electromagnetic field, \( \mathcal{H} = \nabla \times \mathbf{A} \), and \( \mathcal{E}_i = -\partial \mathbf{A}/\partial t \). The last two terms in equation (1) describe the interaction of the electrons and the nucleus with the homogeneous magnetic field.

We introduce the center-of-mass variables for the electron-nucleus subsystem
\[ \mathbf{R} = \frac{1}{M + N'm} \left( M \mathbf{r}^{(n)} + m \sum_{k=1}^{N'} \mathbf{r}_k^{(e)} \right), \tag{2} \]
\[ \mathbf{r}_i = \mathbf{r}_i^{(e)} - \mathbf{r}^{(n)}. \tag{3} \]

From these equations we derive
\[ \mathbf{r}_i^{(e)} = \mathbf{r}_i + \mathbf{R} - \frac{m}{M + N'm} \sum_{k=1}^{N'} \mathbf{r}_k, \tag{4} \]
\[ \mathbf{r}^{(n)} = \mathbf{R} - \frac{m}{M + N'm} \sum_{k=1}^{N'} \mathbf{r}_k. \tag{5} \]

For the corresponding momentum operators \( (\mathbf{P} = -i\nabla \mathbf{R}, \mathbf{p}_i = -i\nabla \mathbf{r}_i) \), we obtain
\[ \mathbf{p}_i^{(e)} = \mathbf{p}_i + \frac{m}{M + N'm} \mathbf{P}, \tag{6} \]
\[ \mathbf{p}_i^{(n)} = \frac{M}{M + N'm} \mathbf{P} - \sum_{k=1}^{N'} \mathbf{p}_k. \tag{7} \]

To keep in the Hamiltonian (1) the terms of zeroth and first orders in \( m/M \), we can replace equations (4)-(7) by the following ones
\[ \mathbf{r}_i^{(e)} = \mathbf{r}_i + \mathbf{R} - \frac{m}{M} \sum_{k=1}^{N'} \mathbf{r}_k, \tag{8} \]
\[ \mathbf{r}^{(n)} = \mathbf{R} - \frac{m}{M} \sum_{k=1}^{N'} \mathbf{r}_k, \tag{9} \]
\[ \mathbf{p}_i^{(e)} = \mathbf{p}_i + \frac{m}{M} \mathbf{P}, \tag{10} \]
\[ \mathbf{p}_i^{(n)} = \mathbf{P} - \sum_{k=1}^{N'} \mathbf{p}_k. \tag{11} \]

With these substitutions, the individual terms in equation (1) are transformed as
\[ \sum_{i=1}^{N'} \alpha_i \cdot \mathbf{p}_i^{(e)} = \sum_{i=1}^{N'} \alpha_i \cdot \mathbf{p}_i + \frac{m}{M} \sum_{i=1}^{N'} \alpha_i \cdot \mathbf{P}, \tag{12} \]
\[ \sum_{i=1}^{N'} V(\mathbf{r}_i^{(e)} - \mathbf{r}^{(n)}) = \sum_{i=1}^{N'} V(\mathbf{r}_i), \tag{13} \]
\[ -e \sum_{i=1}^{N'} \alpha_i \cdot A(r_i^{(e)}) = -e \sum_{i=1}^{N'} \alpha_i \cdot A(r_i + R - \frac{m}{M} \sum_{k=1}^{N'} r_k) \]
\[ = -e \sum_{i=1}^{N'} \alpha_i \cdot A(r_i + R) \]
\[ + e \frac{m}{M} \left[ \left( \sum_{k=1}^{N'} r_k \cdot \frac{\partial}{\partial \xi} \right) \left( \sum_{i=1}^{N'} \alpha_i \cdot A(\xi) \right) \right]_{\xi = r_i + R}, \quad (14) \]
\[ \frac{1}{2} \sum_{i \neq k} \frac{\alpha}{|r_i^{(e)} - r_k^{(e)}|} = \frac{1}{2} \sum_{i \neq k} \frac{\alpha}{|r_i - r_k|}, \quad (15) \]
\[ \frac{1}{2M} [p^{(n)} - |eZ A^{(n)}|^2] = \frac{p^2}{2M} - e \frac{m}{M} \sum_{k=1}^{N'} p_k + \frac{1}{2M} \left( \sum_{k=1}^{N'} p_k \right)^2 \]
\[ - |e|Z \frac{M}{M} P \cdot A(R) + |e|Z \frac{M}{M} \left( \sum_{k=1}^{N'} p_k \right) \cdot A(R) + \frac{e^2Z^2}{2M} A^2(R), \quad (16) \]
\[ -e \sum_{i=1}^{N'} \alpha_i \cdot A_{cl}(r_i^{(e)}) = -e \sum_{i=1}^{N'} \alpha_i \cdot A_{cl}(r_i + R - \frac{m}{M} \sum_{k=1}^{N'} r_k) \]
\[ = -e \sum_{i=1}^{N'} \alpha_i \cdot A_{cl}(r_i) - e \sum_{i=1}^{N'} \alpha_i \cdot A_{cl}(R) \]
\[ + e \frac{m}{M} \sum_{i=1}^{N'} \alpha_i \cdot \sum_{k=1}^{N'} A_{cl}(r_k), \quad (17) \]
\[ - \frac{|e|Z}{M} p^{(n)} \cdot A_{cl}(r^{(n)}) = - \frac{|e|Z}{M} P \cdot A_{cl}(R) + \frac{|e|Z}{M} \sum_{k=1}^{N'} p_k \cdot A_{cl}(R). \quad (18) \]

Here we have disregarded the terms of second and higher orders in \(m/M\). Because the second term in the right hand side of equation (14) contains the factor \(e \frac{m}{M}\), to first order in \(m/M\), it can contribute only to first and higher orders in \(\alpha\). Therefore, to the order under consideration (to zeroth order in \(\alpha\), to first order in \(m/M\), and to all orders in \(aZ\)) this term can be omitted. We obtain

\[ H = \sum_{i=1}^{N'} [\alpha_i \cdot p_i + \beta m + V(r_i)] + \frac{1}{2} \sum_{i \neq k} \frac{\alpha}{|r_i - r_k|} \]
\[ + \frac{1}{2} \int dx \left[ \mathcal{L}^2(x) + \mathcal{H}^2(x) \right] - e \sum_{i=1}^{N'} \alpha_i \cdot A(r_i + R) \]
\[ + \frac{p^2}{2M} + \frac{1}{2M} \left( \sum_{k=1}^{N'} p_k \right)^2 + \frac{m}{M} \sum_{i=1}^{N'} \alpha_i \cdot P \]
\[ - \frac{1}{M} P \cdot \sum_{k=1}^{N'} p_k - \frac{|e|Z}{M} P \cdot A(R) \]
\[ + \frac{|e|Z}{M} \left( \sum_{k=1}^{N'} p_k \right) \cdot A(R) + \frac{e^2Z^2}{2M} A^2(R) \]
\[ - e \sum_{i=1}^{N'} \alpha_i \cdot A_{cl}(r_i) - e \sum_{i=1}^{N'} \alpha_i \cdot A_{cl}(R) \]
\[ + e \frac{m}{M} \sum_{i=1}^{N'} \alpha_i \cdot \sum_{k=1}^{N'} A_{cl}(r_k) - \frac{|e|Z}{M} P \cdot A_{cl}(R) + \frac{|e|Z}{M} \sum_{k=1}^{N'} p_k \cdot A_{cl}(R). \quad (19) \]
In the second quantized form, this Hamiltonian can be written as

\[
H = H_0 + \sum_{s=1}^{13} H_{\text{int}}^{(s)},
\]

\[
H_0 = \int dx \, \psi^\dagger(x)[-i \nabla \cdot \alpha + \beta m + V(x)]\psi(x) - \frac{1}{2M} \int dx \, \phi^\dagger \Delta_x \phi(x)
\]

\[
+ \frac{1}{2} \int dx \, [E_x^2(x) + \mathcal{H}_x^2(x)],
\]

\[
H_{\text{int}}^{(1)} = \frac{\alpha}{2} \int dx \, dy \, \frac{\psi^\dagger(x)\psi(x)\psi^\dagger(y)\psi(y)}{|x - y|},
\]

\[
H_{\text{int}}^{(2)} = -e \int dx \, dy \, \psi^\dagger(x)\alpha \psi(x)A(x + y)\phi^\dagger(y)\phi(y),
\]

\[
H_{\text{int}}^{(3)} = \frac{\alpha}{2M} \int dx \, \psi^\dagger(x)(-i \nabla_x)\psi(x) \int dy \, \psi^\dagger(y)(-i \nabla_y)\psi(y),
\]

\[
H_{\text{int}}^{(4)} = \frac{|e|Z}{M} \int dx \, \psi^\dagger(x)(-i \nabla_x)\psi(x) \int dy \, A(y)\phi^\dagger(y)\phi(y),
\]

\[
H_{\text{int}}^{(5)} = \frac{e^2Z^2}{2M} \int dy \, A^2(y)\phi^\dagger(y)\phi(y),
\]

\[
H_{\text{int}}^{(6)} = \frac{m}{M} \int dx \, \psi^\dagger(x)\alpha \psi(x) \int dy \, \phi^\dagger(y)(-i \nabla_y)\phi(y),
\]

\[
H_{\text{int}}^{(7)} = -\frac{1}{M} \int dx \, \psi^\dagger(x)(-i \nabla_x)\psi(x) \int dy \, \phi^\dagger(y)(-i \nabla_y)\phi(y),
\]

\[
H_{\text{int}}^{(8)} = -\frac{|e|Z}{M} \int dy \, A(y)\phi^\dagger(y)(-i \nabla_y)\phi(y),
\]

\[
H_{\text{int}}^{(9)} = -e \int dx \, \psi^\dagger(x)\alpha \cdot A_{\text{cl}}(x)\psi(x),
\]

\[
H_{\text{int}}^{(10)} = \frac{e}{2M} \int dx \, \psi^\dagger(x)\alpha \psi(x) \int dy \, \psi^\dagger(y)A_{\text{cl}}(y)\psi(y)
\]

\[
+ \frac{e}{2M} \int dy \, \psi^\dagger(y)A_{\text{cl}}(y)\psi(y) \int dx \, \psi^\dagger(x)\alpha \psi(x),
\]

\[
H_{\text{int}}^{(11)} = -e \int dx \, \psi^\dagger(x)\alpha \psi(x) \int dy \, \phi^\dagger(y)A_{\text{cl}}(y)\phi(y),
\]

\[
H_{\text{int}}^{(12)} = -\frac{|e|Z}{M} \int dy \, A_{\text{cl}}(y)\phi^\dagger(y)(-i \nabla_y)\phi(y),
\]

\[
H_{\text{int}}^{(13)} = \frac{|e|Z}{M} \int dx \, \psi^\dagger(x)(-i \nabla_x)\psi(x) \int dy \, \phi^\dagger(y)A_{\text{cl}}(y)\phi(y).
\]

Here \(\psi(x)\) and \(\phi(x)\) are the electron and nucleus field operators in the Schrödinger representation and \(\Delta_x\) is the Laplace operator. The scalar product is implicit in equations (23)-(34). It should be stressed that, despite the nuclear field operator \(\phi(x)\) is introduced, the Hamiltonian (20) has physical sense only in the one nuclear particle sector. The introduction of \(\phi(x)\) will simplify the use of the standard QED methods for calculations by perturbation theory.

The total momentum operator of the system is given by

\[
P = \int dx \, \phi^\dagger(-i \nabla_x)\phi(x) + \int dx \, [E_x(x) \times \mathcal{H}_x(x)].
\]

This operator commutes with the Hamiltonian of the system if the external magnetic field is switched off, \(A_{\text{cl}}(x) = 0\). It also commutes with the operator

\[
H' = H_0 + \sum_{s=1}^{10} H_{\text{int}}^{(s)}.
\]

In the theory without the terms \(H_{\text{int}}^{(11)}, H_{\text{int}}^{(12)},\) and \(H_{\text{int}}^{(13)}\) as well as in the theory without the external magnetic field at all, operators \(\phi(x), A(x),\) and \(\psi(x)\) obey the following space-shift transformation rules
\[ \phi(x + a) = \exp(-i a \cdot \mathbf{P}) \phi(x) \exp(i a \cdot \mathbf{P}), \]
\[ \mathbf{A}(x + a) = \exp(-i a \cdot \mathbf{P}) \mathbf{A}(x) \exp(i a \cdot \mathbf{P}), \]
\[ \psi(x) = \exp(-i a \cdot \mathbf{P}) \psi(x) \exp(i a \cdot \mathbf{P}). \]

For any operator \( F(x) \), the transition to the Heisenberg representation is performed by the substitution
\[ F_h(x) = \exp(-i H t) F_i(t, x) \exp(i H t). \]

The calculations of the atomic energy levels, based on the Hamiltonian (20), can be performed by perturbation theory. It is convenient to consider that in zeroth approximation the Hamiltonian of the system is given by the term \( H_0 \). The terms \( H_{\text{int}}^{(1)} - H_{\text{int}}^{(13)} \) are accounted for by perturbation theory. As usual (see, e.g., [27,28]), the vacuum state is defined as a state in which the negative-energy continuum is occupied by electrons and the electron current operator \( \psi(x) \gamma_{\mu} \psi(x) \) must be replaced by \( \langle e/2 | \psi(x) \gamma_{\mu} \psi(x) | \rangle \) in all the equations given above.

### III. GENERAL FORMULAS FOR THE ENERGY SHIFTS

First, we formulate a procedure for deriving the energy shift due to the terms \( H_{\text{int}}^{(1)} - H_{\text{int}}^{(10)} \). The operator \( H' \) defined by equation (30) commutes with the total momentum of the system (35) and, therefore, the atom can be characterized by a fixed momentum \( \mathbf{K} \). We are interested in the energy shift, \( \Delta E_{(a, \mathbf{K})} = E_{(a, \mathbf{K})} - E_{(a, \mathbf{K})}^{(0)} \) of a single isolated level \( |a, \mathbf{K}\rangle \) of an \( N \)-electron atom. Here \( N \) denotes the number of the atomic (positive-energy-state) electrons, \( \mathbf{K} \) is the total momentum of the system described by the Hamiltonian \( H' \), and \( a \) denotes the set of the other quantum numbers. In zeroth approximation, the energy \( E_{(a, \mathbf{K})}^{(0)} \) is
\[ E_{(a, \mathbf{K})}^{(0)} = \varepsilon_{a_1} + \cdots + \varepsilon_{a_N} + \frac{\mathbf{K}^2}{2M}. \]

The one-electron energies \( \varepsilon_n \) are defined by the Dirac equation
\[ (-i \alpha \cdot \nabla + \beta m + V(x)) \psi_n(x) = \varepsilon_n \psi_n(x), \]
where \( V(x) \) is the Coulomb potential of the nucleus. We note that the formalism considered here allows us to account partially for the finite nuclear size effect by employing in (42) the potential of an extended nucleus. We could start also with the Dirac equation with an effective potential \( V_{\text{eff}}(x) \) (e.g., a local version of the Hartree-Fock potential) which approximately describes the electron-electron interaction. In this case the interaction with the potential \( \Delta V(x) = V(x) - V_{\text{eff}}(x) \) must be accounted for perturbatively to eliminate double counting the interelectronic interaction corrections. The electronic part of the unperturbed wave function is a linear combination of one-determinant functions
\[ u_a(x_1, \ldots, x_N) = \sum_b \frac{C_{ab}}{\sqrt{N!}} \sum_P (-1)^P \psi_{P_{b_1}}(x_1) \cdots \psi_{P_{b_N}}(x_N). \]

We introduce the Green function \( g_a(E, \mathbf{K}) \) by
\[ g_a(E, \mathbf{K}) \delta(E' - E) \delta(\mathbf{K}' - \mathbf{K}) = \frac{1}{2\pi i} \frac{1}{N! (2\pi)^3} \int dx_1 \cdots dx_N dx'_1 \cdots dx'_N dX dX' \]
\[ \times \int_{-\infty}^{\infty} dt dt' \exp(i E' t' - i E t) \exp(-i \mathbf{K}' \cdot \mathbf{X}' + i \mathbf{K} \cdot \mathbf{X}) \]
\[ \times \psi(t, x) \phi(t, x) \psi(t', x') \phi(t', x') \]
\[ \times u_a(x_1', \ldots, x_N') \langle 0| T \psi(t', x'_1) \cdots \psi(t', x'_N) \phi(t', x') \cdots \psi(t, x_1) \langle 0| u_a(x_1, \ldots, x_N), \]
where \( \psi(t, x) \) and \( \phi(t, x) \) are the electron and nucleus field operators in the Heisenberg representation and \( T \) is the time-ordering operator. As one can see from equation (44), \( g_a(E, \mathbf{K}) \) is the Fourier transform of the two-time Green function. The \( g_a(E, \mathbf{K}) \) function contains the complete information about the energy levels of the system described by \( H' \). Defined by equation (44) for real \( E \), it can be continued analytically to the complex \( E \) plane with some cuts (see, e.g., [27,28]). From the spectral representation of \( g_a(E, \mathbf{K}) \), we obtain
\[ g_a(E, \mathbf{K}) = \frac{\varphi_{(a, \mathbf{K})} \varphi_{(a, \mathbf{K})}^*}{E - E_{(a, \mathbf{K})}}, \]
where
where
\[ \varphi(a, \mathbf{K}) = \frac{(2\pi)^{3/2}}{\sqrt{N!}} \int dx_1 \cdots dx_N \ u_a^\dagger(x_1, \ldots, x_N) \langle 0 | \psi(0, x_1) \cdots \psi(0, x_N) \phi(0, 0) | a, \mathbf{K} \rangle. \] (46)

As in [28], using equation (15), we can derive the following formula for the energy shift
\[ \Delta E_{(a, \mathbf{K})} = \frac{1}{2\pi i} \oint_\Gamma dE \left( E - E_{(a, \mathbf{K})}^{(0)} \right) \Delta g_a(E, \mathbf{K}) \]
\[ \frac{1}{1 + \frac{1}{2\pi i} \oint_\Gamma dE \Delta g_a(E, \mathbf{K})}, \] (47)

where \( \Delta g_a(E, \mathbf{K}) = g_a(E, \mathbf{K}) - g_a^{(0)}(E, \mathbf{K}) \) and \( g_a^{(0)}(E, \mathbf{K}) = (E - E_{(a, \mathbf{K})}^{(0)})^{-1} \). The contour \( \Gamma \) surrounds the pole corresponding to the level \( a \) and keeps outside all other singularities of \( g_a(E, \mathbf{K}) \). It is assumed that the contour \( \Gamma \) is oriented anticlockwise. The Green function \( g_a(E, \mathbf{K}) \) is constructed by perturbation theory after the transition in equation (14) to the interaction representation. Since we are interested in the energy shifts in the atom rest system, we must put \( \mathbf{K} = 0 \) in formula (17).

Formula (17) can be used to calculate the corrections due to the terms \( H_{\text{int}}^{(1)} - H_{\text{int}}^{(10)} \). The term \( \Delta H' \), defined as
\[ \Delta H' = H_{\text{int}}^{(11)} + H_{\text{int}}^{(12)} + H_{\text{int}}^{(13)}, \] (48)
does not commute with \( \mathcal{P} \) and, therefore, requires a special treatment. To first order in the interaction with the magnetic field, the energy shift due to this term can be written as (cf., [28])
\[ \Delta E_{(a, \mathbf{K})} \delta(K' - \mathbf{K}) \simeq \langle a, \mathbf{K}' | \Delta H' | a, \mathbf{K} \rangle, \] (49)
where \( |a, \mathbf{K}\rangle \) and \( |a, \mathbf{K}'\rangle \) are the eigenvectors of the operator \( H' \). Since we are interested in the corrections of first order in the magnetic field, in equation (49) we can consider the eigenvectors of the operator \( H'' = H' - H_{\text{int}}^{(9)} - H_{\text{int}}^{(10)} \). We assume that they are normalized by
\[ \langle a, \mathbf{K}' | a, \mathbf{K} \rangle = \delta(K' - K). \] (50)

We introduce the Green function \( g_a(E', E, \mathbf{K}', \mathbf{K}) \) by
\[ g_a(E', E, \mathbf{K}', \mathbf{K}) = \frac{1}{N! (2\pi)^d} \int dx_1 \cdots dx_N dx'_1 \cdots dx'_N dX dX' \]
\[ \times \int_{-\infty}^\infty dt dt' \exp(iE't' - iEt) \exp(-ik' \cdot X' + ik \cdot X) \]
\[ \times u_a^\dagger(x'_1, \ldots, x'_N)(0|T\psi(t', x'_1) \cdots \psi(t', x'_N)\phi(t', X')\Delta H'(0) \]
\[ \times \phi^\dagger(t, X)\psi^\dagger(t, x_1) \cdots \psi^\dagger(t, x_N)|0\rangle u_a(x_1, \ldots, x_N), \] (51)

where, as in equation (14), the Heisenberg representation is used and \( \Delta H'(0) = \Delta H'(t = 0) \). The analytical properties of this type Green function as a function of two complex variables \( E \) and \( E' \) in the region \( E \sim E_{(a, \mathbf{K})}, E' \sim E_{(a, \mathbf{K}')}, \) can be investigated by deriving the double spectral representation (see, e.g., [28]). From this representation, we obtain
\[ g_a(E', E, \mathbf{K}', \mathbf{K}) = \frac{\varphi(a, \mathbf{K}')}{E' - E_{(a, \mathbf{K}')}} \langle a, \mathbf{K}' | \Delta H'(0) | a, \mathbf{K} \rangle \frac{\varphi^\dagger(a, \mathbf{K})}{E - E_{(a, \mathbf{K})}} \]
\[ + \text{terms that are regular functions of } E' \text{ or } E \text{ when } \]
\[ E' \sim E_{(a, \mathbf{K}')} \text{ and } E \sim E_{(a, \mathbf{K})}. \] (52)

As in the case of the photon emission by an atom considered in detail in [28], from equations (13), (52) we derive
\[ \Delta E_{(a, \mathbf{K})} \delta(K' - K) \simeq \frac{1}{2\pi i} \oint_{\Gamma'} dE' \frac{1}{2\pi i} \oint_\Gamma dE \ g_a(E', E, \mathbf{K}', \mathbf{K}) \]
\[ \left[ \frac{1}{2\pi i} \oint_{\Gamma'} dE' \ g_a(E', \mathbf{K}') \right]^{1/2} \left[ \frac{1}{2\pi i} \oint_\Gamma dE \ g_a(E, \mathbf{K}) \right]^{1/2}, \] (53)

where the contours \( \Gamma \) and \( \Gamma' \) surround the poles corresponding to the levels \( E_{(a, \mathbf{K})} \) and \( E_{(a, \mathbf{K}')} \), respectively, and keep outside all other singularities. The Green function \( g_a(E', E, \mathbf{K}', \mathbf{K}) \) is constructed by perturbation theory after the transition in (51) to the interaction representation. Since we are interested in the energy shifts in the atom rest system, at the end of the calculation by formula (53), we must put \( \mathbf{K} = 0 \).
IV. RELATIVISTIC FORMULA FOR THE RECOIL CORRECTION TO THE BOUND-ELECTRON $g$ FACTOR IN A HYDROGENLIKE ATOM

The method formulated in the previous sections can be used to derive the complete $\alpha Z$-dependence expressions for the recoil corrections to the energy levels ($A_{ci}(x) = 0$) and to the atomic $g$ factor to first order in $m/M$. The relevant expression for the recoil correction to the energy levels in a hydrogenlike atom was first derived by a quasipotential method in [30] and was subsequently rederived by other methods in [31–33] (a part of the expression was previously found in [34]). In this section, we derive the corresponding formula for the recoil correction to the bound-electron $g$ factor in a hydrogenlike atom.

In what follows, we will consider that $V(x)$ in the Dirac equation (12) is the potential of an extended-charge nucleus. This will allow us to avoid a special treatment of the degenerate states of different parity (such as 2s and 2p1/2) in the derivation of the contributions from the $\Delta H'$ term. However, as it will be seen from the final formulas obtained below, they have neither singularities nor ambiguities when $V(x) \to V_C(x) = -\alpha Z/|x|$ and, therefore, the pure Coulomb potential can be used in these formulas as well.

We are interested in the energy shifts of first order in the magnetic field and of first order in $m/M$. The contributions of interest arise in a few lowest orders of the perturbation theory. They can be conventionally represented by the following combinations:

$$
\Delta E^{(1)} \sim H_{\text{int}}^{(9)} \times H_{\text{int}}^{(3)} + H_{\text{int}}^{(9)} \times H_{\text{int}}^{(4)} \times H_{\text{int}}^{(2)}
$$

$$
\Delta E^{(2a)} \sim H_{\text{int}}^{(10)}
$$

$$
\Delta E^{(2b)} \sim H_{\text{int}}^{(11)} \times H_{\text{int}}^{(6)} + H_{\text{int}}^{(11)} \times H_{\text{int}}^{(7)}
$$

$$
+ H_{\text{int}}^{(11)} \times H_{\text{int}}^{(8)} \times H_{\text{int}}^{(2)} + \text{permutations},
$$

As to the terms $H_{\text{int}}^{(12)}$ and $H_{\text{int}}^{(13)}$, in the order under consideration, the first of these terms gives a contribution which is independent of the internal atomic quantum numbers while the second one gives zero when it is averaged with the electron wave function of a definite parity. Therefore, the both terms can be omitted.

To calculate $\Delta E^{(1)}$, it is convenient to consider that $H_{\text{int}}^{(9)}$ is included in the unperturbed Hamiltonian. This means that the interaction with the magnetic field, $\delta V(x) = \alpha \mathbf{a} \cdot \mathbf{A}_{ci}(x)$, must be included in the Dirac equation (12). Then the derivation of the formula for this correction reduces to the related derivation of the recoil correction to the energy levels for $A_{ci}(x) = 0$ [30,33]. It follows that to obtain $\Delta E^{(1)}$, we may simply replace $V(x)$ by $V(x) + \delta V(x)$ everywhere in the formula for the recoil correction to the energy level and decompose it to first order in $\delta V(x)$. However, before to do that, let us demonstrate how the recoil correction to the energy level can be derived within the formalism considered here.

Let us consider, for example, the contribution $\sim H_{\text{int}}^{(4)} \times H_{\text{int}}^{(2)} + H_{\text{int}}^{(2)} \times H_{\text{int}}^{(4)}$. In the interaction representation, the related contribution to the two-time Green function is

$$
\Delta G(t', x', X'; t, x, X) = \frac{(-i)^2}{2} \int_{-\infty}^{\infty} dt_1 dt_2 \langle 0| T \psi(t', x') \phi(t', X') \psi^\dagger(t, x) \phi^\dagger(t, X) 
$$

$$
\times \left\{ \frac{(-\epsilon)}{2} \int dx_1 dx_1 \ [\psi^\dagger(t_1, x_1) \alpha^i, \psi(t_1, x_1)] A^i(t_1, x_1 + X_1) 
$$

$$
\times \phi^\dagger(t_1, X_1) \phi(t_1, X_1) \frac{|e| Z}{M} \int dy_1 dy_1 \psi^\dagger(t_2, y_1) 
$$

$$
\times (-i \nabla_{y_1}^k) \psi(t_2, y_1) A_k(t_2, Y_1) \phi^\dagger(t_2, Y_1) \phi(t_2, Y_1) + \frac{|e| Z}{M} \int dy_1 dy_1 \psi^\dagger(t_2, y_1) (-i \nabla_{y_1}^k) \psi(t_2, y_1) A_k(t_2, Y_1) 
$$

$$
\times \phi^\dagger(t_2, Y_1) \phi(t_2, Y_1) \frac{(-\epsilon)}{2} \int dx_1 dx_1 \ [\psi^\dagger(t_1, x_1) \alpha^i, \psi(t_1, x_1)] 
$$

$$
\times A^i(t_1, x_1 + X_1) \phi^\dagger(t_1, X_1) \phi(t_1, X_1) \} \langle 0|_{\text{conf}},
$$

where all the operators are considered in the interaction representation and the summation over the repeated indices $(i, k = 1, 2, 3)$, which enumerate components of the three-dimensional vectors, is implicit. The label "conf" means that contributions containing disconnected vacuum-vacuum terms must be omitted. According to the permutation rules...
for the $T$ product of the boson and fermion operators, second term in the right hand side of equation (57) is equal to first one. Assuming that the time-ordering operator is defined for equal-time fermion operators by 17, we have

$$T[A(t)B(t)] = \frac{1}{2} A(t)B(t) - \frac{1}{2} B(t)A(t),$$

we have

$$\Delta G(t', x', X'; t, x, X) = \frac{e^2 |Z|}{M} \int_{-\infty}^{\infty} dt_1 dt_2 \int dx_1 dX_1 \int dy_1 dY_1 \left[ S(t' - t_1, x', x_1) \right.$$  

$$\times \alpha^i S(t_1 - t_2, x_1, y_1) (\nabla_{x_1}^k) S(t_2 - t, y_1, x)$$  

$$+ S(t' - t_2, x', y_1) (\nabla_{y_1}^k) S(t_2 - t_1, y_1, x_1)$$  

$$\times \alpha^i S(t_1 - t, x_1, x) D^{ik}(t_1 - t_2, x_1 + X_1 - Y_1)$$  

$$\times \delta(t - t_2, Y_1 - X_1) \int dx_1 dX_1 dY_1$$

$$\left. \times \phi^i(t_1, X_1) \phi(t_1, X_1) \phi(t_2, Y_1) \right] \right.$$  

Using the Wick theorem and keeping only the terms which will contribute to the state-dependent energy shift, we obtain

$$\Delta G(t', x', X'; t, x, X) = \frac{e^2 |Z|}{M} \int_{-\infty}^{\infty} dt_1 dt_2 \int dx_1 dX_1 dY_1 [S(t' - t_1, x', x_1)$$  

$$\times \alpha^i S(t_1 - t_2, x_1, y_1) (\nabla_{x_1}^k) S(t_2 - t, y_1, x)$$  

$$+ S(t' - t_2, x', y_1) (\nabla_{y_1}^k) S(t_2 - t_1, y_1, x_1)$$  

$$\times \alpha^i S(t_1 - t, x_1, x) D^{ik}(t_1 - t_2, x_1 + X_1 - Y_1)$$  

$$\times \delta(t - t_2, Y_1 - X_1) \int dx_1 dX_1 dY_1$$

$$\left. \times \phi^i(t_1, X_1) \phi(t_1, X_1) \phi(t_2, Y_1) \right] \right.$$  

Here

$$S(t' - t, x, y) = \frac{i}{2\pi} \int_{-\infty}^\infty d\omega \exp[-i\omega(t' - t)] \sum_n \frac{\psi_n(x)\psi_n^\dagger(y)}{\omega - \epsilon_n(1 - i0)}$$

is the electron propagator in the nuclear potential $V(x)$,

$$D^{ik}(t' - t, x - y) = \frac{i}{2\pi} \int_{-\infty}^\infty d\omega \exp[-i\omega(t' - t)] \int \frac{dk}{(2\pi)^3} \frac{\exp[ik \cdot (x - y)]}{\omega^2 - k^2 + i0} \left( \delta_{il} - \frac{k^i k^l}{k^2} \right)$$

is the transverse part of the photon propagator in the Coulomb gauge, and

$$S_n(t' - t, X - Y) = \frac{i}{2\pi} \int_{-\infty}^\infty d\omega \exp[-i\omega(t' - t)] \int \frac{dK}{(2\pi)^3} \frac{\exp[iK \cdot (X - Y)]}{\omega - K^2 + i0}$$

is the nucleus propagator. In equation (62), the index $n$ runs over all bound and continuum states. Since we are interested in the corrections of first order in $m/M$ and we have already the factor $1/M$ in front of expression (61), we can consider the limit $M \to 0$ in the expression for the nucleus propagator. In this limit, we have

$$S_n(t' - t, X - Y) = \theta(t' - t) \delta(X - Y).$$

Using this expression for $S_n$, we obtain

$$\Delta G(t', x', X'; t, x, X) = \frac{e^2 |Z|}{M} \int_{-\infty}^{\infty} dt_1 dt_2 \int dx_1 dX_1 \left[ S(t' - t_1, x', x_1)$$  

$$\times \alpha^i S(t_1 - t_2, x_1, y_1) (\nabla_{x_1}^k) S(t_2 - t, y_1, x)$$  

$$+ S(t' - t_2, x', y_1) (\nabla_{y_1}^k) S(t_2 - t_1, y_1, x_1)$$  

$$\times \alpha^i S(t_1 - t, x_1, x) D^{ik}(t_1 - t_2, x_1 + X_1 - Y_1)$$  

$$\times \delta((X - X) \theta(t' - t_1) \theta(t_1 - t_2) \theta(t_2 - t)$$  

$$+ \theta(t' - t_2) \theta(t_2 - t_1) \theta(t_1 - t)).$$
The related contribution to the Green function $g_a(E, K)$ defined by equation (44) is

$$\Delta g_a(E, K) = \frac{1}{2\pi i} \frac{e|e|Z}{M} \int_{-\infty}^{\infty} dt \, dt_1 \, dt_2 \int dx \, dx' \, dX \, dX' \times dy_1 \exp{(iE't' - iEt)} \times \exp{(-iK' \cdot X' + iK \cdot X)} \psi_a(x') \langle S(t', t_1, y_1) | S(t_2 - t, t_1, x) \rangle \times \alpha \langle S(t_1 - t_2, x_1, y_1)(-i
abla_{y_1}) S(t_2 - t, y_1, x) + S(t' - t_2, x', y_1)(-i
abla_{y_1}) S(t_2 - t_1, y_1, x) \rangle \times \delta(x' - x) \langle \theta(t' - t_1) \theta(t_2 - t) + \theta(t' - t_2) \theta(t_1 - t) \rangle . \tag{66}$$

Integrating over $X$, $X'$, and over the time variables and setting $K = 0$, we obtain

$$\Delta g_a(E, 0) = -\frac{e|e|Z}{M} \int_{-\infty}^{\infty} d\omega \left\{ \sum_n \frac{\langle a | \alpha^i D^{ik}(\omega) | n \rangle \langle n | (-i
abla^k)a \rangle }{E - \varepsilon_n (1 - i0)} + \sum_n \frac{\langle a | (-i
abla^k)a | n \rangle \langle n | \alpha^i D^{ik}(\omega) a \rangle }{E - \varepsilon_n (1 - i0)} \right\} , \tag{67}$$

where

$$D^{il}(\omega, r) = \int \frac{dk}{(2\pi)^3} \frac{\exp{i(k \cdot r)}}{\omega^2 - k^2 + i0} \left( \delta_{il} + \frac{k^l k^i}{k^2} \right) = -\frac{1}{4\pi} \left\{ \frac{\exp{i|\omega|r}}{r} \delta_{il} + \nabla^i \nabla^l \left( \frac{\exp{i|\omega|r} - 1}{\omega^2 r} \right) \right\} . \tag{68}$$

The corresponding contribution to the energy shift according to equation (47) for $K = 0$ is

$$\Delta E = \frac{4\pi aZ}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_n \left\{ \frac{\langle a | \alpha D^{ik}(\omega) | n \rangle \langle n | (-i
abla^k)a \rangle }{E - \varepsilon_n (1 - i0)} + \frac{\langle a | (-i
abla^k)a | n \rangle \langle n | \alpha D^{ik}(\omega) a \rangle }{E - \varepsilon_n (1 - i0)} \right\} . \tag{69}$$

The energy shifts due to the terms $H^{(3)}_{\text{int}}$ and $H^{(5)}_{\text{int}} \sim H^{(5)}_{\text{int}} \times H^{(2)}_{\text{int}} \times H^{(2)}_{\text{int}}$ (and permutations) can be derived in the same way. The total contribution of the terms $H^{(3)}_{\text{int}}$, $H^{(4)}_{\text{int}} \sim H^{(4)}_{\text{int}} \times H^{(2)}_{\text{int}} \times H^{(2)}_{\text{int}}$ (and permutations) is

$$\Delta E = \frac{2\pi i}{2M} \int_{-\infty}^{\infty} d\omega \langle a | [p^k - D^{k}(\omega)]G(\omega + \varepsilon_a) [p^k - D^{k}(\omega)] | a \rangle , \tag{70}$$

where $p^k = -i
abla^k$,

$$D^{k}(\omega) = -4\pi \alpha Z \alpha^i D^{ik}(\omega) , \tag{71}$$

and

$$G(\omega) = \sum_n \frac{\langle n | n \rangle }{\omega - \varepsilon_n (1 - i0)} . \tag{72}$$

Deriving equation (70) we considered that in the zeroth approximation the electron obeys the Dirac equation with the nuclear potential $V(x)$. As was indicated above, to obtain the desired formula for $\Delta E^{(1)}$, defined by equation (53), we should replace $V(x)$ by $V(x) + \delta V(x)$, where $\delta V(x) = -e\alpha \cdot A_{\alpha}(x)$, in equation (70) and expand it to first order in $\delta V(x)$. As a result of this expansion, we obtain
\[
\Delta E^{(1)} = \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \left\{ \langle \delta a | p^k - D^k(\omega) \rangle G(\omega + \varepsilon_a) [p^k - D^k(\omega)] |a\rangle + \langle a | p^k - D^k(\omega) \rangle [\delta a | p^k - D^k(\omega)] \right\} ,
\]

where

\[
\delta \varepsilon_a = \langle a | \delta V | a \rangle , \quad (74)
\]

\[
|\delta a\rangle = \sum_{\alpha} \langle n | n | \delta V | a \rangle . \quad (75)
\]

For practical calculations, it is convenient to represent expression (73) by the sum of the lower-order term and the higher-order term,

\[
\Delta E^{(1)} = \Delta E_L^{(1)} + \Delta E_H^{(1)} , \quad (76)
\]

\[
\Delta E_L^{(1)} = \frac{1}{2M} \int_{-\infty}^{\infty} d\omega \left\{ \langle \delta a | p^k - 2D^k(0) | p^k | a \rangle + \langle a | p^k - 2D^k(0) | p^k | \delta a \rangle \right\} , \quad (77)
\]

\[
\Delta E_H^{(1)} = \frac{i}{2\pi M} \int_{-\infty}^{\infty} d\omega \left\{ \langle \delta a | \left( D^k(\omega) - \frac{[p^k, V]}{\omega + i0} \right) G(\omega + \varepsilon_a) \left( D^k(\omega) + \frac{[p^k, V]}{\omega + i0} \right) |a\rangle + \langle a | \left( D^k(\omega) - \frac{[p^k, V]}{\omega + i0} \right) G(\omega + \varepsilon_a) (\delta V - \delta \varepsilon_a) G(\omega + \varepsilon_a) \left( D^k(\omega) + \frac{[p^k, V]}{\omega + i0} \right) |a\rangle \right\} , \quad (78)
\]

where \([A, B] = AB - BA\). The term \(\Delta E_L^{(1)}\) contains the non-relativistic and lowest-order relativistic contributions and a part of the higher-order relativistic contributions. The term \(\Delta E_H^{(1)}\) contains all the higher-order relativistic contributions which are not included into \(\Delta E_L^{(1)}\).

The direct evaluation of the term \(H_{\text{int}}^{(10)}\) yields

\[
\Delta E^{(2a)} = e \frac{m}{M} \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \left[ \langle a | \alpha^b G(\omega) A_{12}^b | a \rangle + \langle a | A_{12}^b G(\omega) \alpha^b | a \rangle \right] . \quad (79)
\]

The calculation of the correction \(\Delta E^{(2b)}\) is a more difficult task. Here we derive in detail the contribution which appears as a combination \(H_{\text{int}}^{(11)} \times H_{\text{int}}^{(1)} \times H_{\text{int}}^{(2)} + \text{permutations}\). To find this correction we should use formula (53). The related contribution to the Green function \(g_a(E', E, K', K)\) defined by equation (63) is

\[
\Delta g_a(E', E, K', K) = \frac{1}{(2\pi)^3} (-i)^2 \frac{\varepsilon^2 |Z|}{M} \int_{-\infty}^{\infty} dt \, dt' \, dt_1 \, dt_2 \int d\mathbf{x} \, d\mathbf{x}' \, d\mathbf{X} \, d\mathbf{X}'
\]

\[
\times \exp \left( iE't' - iEt \right) \exp \left( -iK' \cdot \mathbf{X}' + iK \cdot \mathbf{X} \right) \psi_0(t, t') \int d\mathbf{z}_1 \psi_0^\dagger(0, \mathbf{z}_1) \alpha_i^\dagger(0, \mathbf{z}_1) A_{12}^i(\mathbf{z}_1)
\]

\[
\times \phi^\dagger(0, \mathbf{Z}_1) \phi(0, \mathbf{Z}_1) \phi^\dagger(t, \mathbf{X}) \phi^\dagger(t, \mathbf{X}) \int d\mathbf{x}_1 \psi_1^\dagger(t_1, \mathbf{x}_1)
\]

\[
\times \alpha_i^\dagger(t_1, \mathbf{x}_1) A_i^\dagger(t_1, \mathbf{x}_1 + \mathbf{X}_1) \phi^\dagger(t_1, \mathbf{X}_1) \phi(t_1, \mathbf{X}_1)
\]

\[
\times \int d\mathbf{Y}_1 \, A^\dagger(t_2, \mathbf{Y}_1) \phi^\dagger(t_2, \mathbf{Y}_1) (-i \nabla_{\mathbf{Y}_1}^k \phi(t_2, \mathbf{Y}_1)|0\rangle_{\text{con}} \psi_0(\mathbf{x}) , \quad (80)
\]
where we have used equation [38]. To evaluate this expression we employ the Wick theorem and equation [14] for the nucleus propagator. Keeping only the terms which contribute to the state-dependent energy shift, we obtain

$$\Delta g_a(E', E, K', K) = -\frac{\varepsilon^2 |e| Z}{M} \int_{-\infty}^{\infty} dt dt' dt_1 dt_2 \int d\mathbf{x} d\mathbf{x}' d\mathbf{x}_1 d\mathbf{x}_2 \exp{(iE't' - iEt)}$$

$$\times [\psi_i^\dagger(\mathbf{x}') S(t' - t_1, \mathbf{x}', \mathbf{x}_1) \alpha^i S(t_1, \mathbf{x}_1, \mathbf{z}_1) \alpha^I S(-t, \mathbf{z}_1, \mathbf{x}) \psi_a(\mathbf{x})]$$

$$+ [\psi_i^\dagger(\mathbf{x}') S(t', \mathbf{x}, \mathbf{z}_1) \alpha^I S(-t_1 - t, \mathbf{x}_1, \mathbf{x}) \psi_a(\mathbf{x})]$$

$$\times D^{ik}(t_1 - t_2, \mathbf{x}_1) \theta(t' - t_1) \theta(t_1 - t_2) \theta(t_2) \theta(-t) B_1^{kl}(K', K)$$

$$+ \theta(t' - t_1) \theta(t_1) \theta(-t_2) \theta(t_2 - t) B_2^{kl}(K', K)$$

$$+ \theta(t' - t_1) \theta(t_1) \theta(-t_1 - t_2) \theta(t_2) B_1^{kl}(K', K)$$

$$+ \theta(t') \theta(-t_1 - t_2) \theta(t_1 - t_2) B_2^{kl}(K', K)$$

$$+ \theta(t') \theta(-t_2) \theta(t_2 - t_1) \theta(t_1 - t) B_2^{kl}(K', K)$$

$$= \frac{1}{(2\pi)^3} \int d\mathbf{X} \exp{(-iK' \cdot \mathbf{X})} (-i\nabla^k_{\mathbf{X}}) A_{c_1}(\mathbf{X}) \exp{(iK \cdot \mathbf{X})},$$

$$B_1^{kl}(K', K) = \frac{1}{(2\pi)^3} \int d\mathbf{X} \exp{(-iK' \cdot \mathbf{X})} A_{c_1}(\mathbf{X}) (-i\nabla^k_{\mathbf{X}}) \exp{(iK \cdot \mathbf{X})}.$$
\( \times B_1^{kl}(K', K) + \theta(\varepsilon_n) \frac{i}{E' - \varepsilon_a} \frac{i}{E' + \omega - \varepsilon_a + i0} \times \frac{i}{E + \omega - \varepsilon_n + i0} \frac{i}{E - \varepsilon_a} B_1^{kl}(K', K) \)

\[ + \theta(\varepsilon_n) \frac{i}{E' - \varepsilon_a} \frac{i}{E - \omega - \varepsilon_n + i0} \frac{i}{E - \varepsilon_a} \]

\( \times B_2^{kl}(K', K) + \theta(\varepsilon_n) \frac{i}{E' - \varepsilon_a} \frac{i}{E + \omega - \varepsilon_n + i0} \times \frac{i}{E - \varepsilon_n} \frac{i}{E - \varepsilon_a} B_2^{kl}(K', K) \). \quad (84)

Taking into account that \( \langle n|\alpha'|a \rangle = 0 \) if \( \varepsilon_n = \varepsilon_a \), we have

\[
\frac{1}{2\pi i} \oint dE \frac{1}{2\pi i} \oint dE' \Delta g_n(E', E, K', K) = -\frac{e^2|Z|}{2M} \sum_{n} \frac{\langle a|\alpha^i D^{ik}(0)\rangle \langle n|\alpha'|a \rangle}{\varepsilon_a - \varepsilon_n} \left[ B_1^{kl}(K', K) + B_2^{kl}(K', K) \right] \]

\[ - \frac{e^2|Z|}{2\pi} \frac{i}{M} \int_{-\infty}^{\infty} d\omega \sum_{n} \left( \frac{\langle a|\alpha^i D^{ik}(\omega)\rangle \langle n|\alpha'|a \rangle}{(\varepsilon_a - \varepsilon_n)(\varepsilon_a - \omega - \varepsilon_n(1 - i0))} \right) C^{kl}(K', K), \quad (85)\]

where

\[ C^{kl}(K', K) = B_1^{kl}(K', K) - B_2^{kl}(K', K) = -\frac{i}{2} \epsilon_{kl} S H^a \delta(K' - K) \] \quad (86)

and \( \epsilon_{kl} \) is the Levi-Civita symbol (\( \epsilon_{123} = \epsilon_{312} = \epsilon_{231} = 1, \epsilon_{321} = \epsilon_{213} = -1 \), and \( \epsilon_{kl} = 0 \) if at least two of the indices are equal each other). It can easily be shown that the term containing \( [B_1^{kl} + B_2^{kl}] \) in equation (85) is equal to zero. Indeed, using the identity

\[ \alpha^i = i[H_D, x^i], \quad (87) \]

where \( H_D = -i\alpha \cdot \nabla + \beta m + V \), we obtain

\[
\sum_{n} \left( \frac{\langle a|\alpha^i D^{ik}(0)\rangle \langle n|\alpha'|a \rangle}{\varepsilon_a - \varepsilon_n} + \frac{\langle a|\alpha'^i|n\rangle \langle n|\alpha^i D^{ik}(0)\rangle}{\varepsilon_a - \varepsilon_n} \right) \]

\[ = i \sum_{n} \left( \langle a|\alpha^i D^{ik}(0)\rangle \langle n|\alpha'|a \rangle + \langle a|\alpha'^i|n\rangle \langle n|\alpha^i D^{ik}(0)\rangle \right) \]

\[ = i \left( \langle a|\alpha^i D^{ik}(0)\rangle x^i|a \rangle + \langle a|x^i\alpha^i D^{ik}(0)\rangle \right) = 0. \quad (88) \]

Here we have taken into account that for the case of an extended-charge nucleus considered here there are no degenerate states of different parity. Taking into account that to zeroth order the denominator in equation (82) is equal to 1, we have

\[
\Delta E = \frac{e^2|Z|}{M} \frac{i}{2\pi} \epsilon_{kl} S H^a \frac{i}{2\pi} \int_{-\infty}^{\infty} d\omega \sum_{n} \left( \frac{\langle a|\alpha^i D^{ik}(\omega)\rangle \langle n|\alpha'|a \rangle}{(\varepsilon_a - \varepsilon_n)(\varepsilon_a - \omega - \varepsilon_n(1 - i0))} \right) \]

\[ - \frac{\langle a|\alpha^i|n\rangle \langle n|\alpha^i D^{ik}(\omega)\rangle}{(\varepsilon_a - \varepsilon_n)(\varepsilon_a + \omega - \varepsilon_n(1 - i0))} \] \quad (89)

Using identity (87), we obtain
\[
\Delta E = \frac{e}{2M} \varepsilon_{kl} \mathcal{H}^k \frac{i}{2\pi} \int_{-\infty}^{\infty} dw \left\{ \langle a|x^l G(\omega) D^k (\varepsilon_a - \omega)|a \rangle + \langle a|D^k (\varepsilon_a - \omega) G(\omega) x^l |a \rangle \right\},
\]
where \( D^k (\omega) \) is defined by equation (71).

The other contributions to \( \Delta E^{(2b)} \) defined by equation (58) are derived in the same way. As a result of this derivation, keeping only the state-dependent contributions, we have

\[
\Delta E = \frac{e}{2M} \varepsilon_{kl} \mathcal{H}^k \frac{i}{2\pi} \int_{-\infty}^{\infty} dw \left\{ \langle a|x^l G(\omega) (m\alpha^k - p^k)|a \rangle + \langle a|(m\alpha^k - p^k) G(\omega) x^l |a \rangle \right\}. \tag{91}
\]

The sum of expressions (90) and (91) gives the correction \( \Delta E \). For the correction \( \Delta E^{(2)} \equiv \Delta E^{(2a)} + \Delta E^{(2b)} \), where \( \Delta E^{(2a)} \) is defined by equation (79), we obtain

\[
\Delta E^{(2)} = \frac{e}{2M} \varepsilon_{kl} \mathcal{H}^k \frac{i}{2\pi} \int_{-\infty}^{\infty} dw \left\{ \langle a|x^l G(\omega + \varepsilon_a) |p^k - D^k (\omega)|a \rangle \right\} + \langle a||p^k - D^k (\omega)|G(\omega + \varepsilon_a) x^l |a \rangle \right\}. \tag{92}
\]

For practical calculations, it is convenient to represent this correction by the sum of the lower-order and higher-order terms,

\[
\Delta E^{(2)} = \Delta E^{(2)} + \Delta E^{(2)}_H, \tag{93}
\]

\[
\Delta E^{(2)}_L = -\frac{e}{2M} \varepsilon_{kl} \mathcal{H}^k \{ \langle a|x^l |p^k - D^k (0)|a \rangle \}, \tag{94}
\]

\[
\Delta E^{(2)}_H = -\frac{e}{2M} \varepsilon_{kl} \mathcal{H}^k \frac{i}{2\pi} \int_{-\infty}^{\infty} dw \left\{ \frac{1}{\omega + i0} \{ \langle a|x^l G(\omega + \varepsilon_a) |p^k - D^k (\omega)|a \rangle \right\} - \langle a||p^k - D^k (\omega)|G(\omega + \varepsilon_a) x^l |a \rangle \right\}. \tag{95}
\]

Here, as in equations (76)-(78), the term \( \Delta E^{(2)}_L \) contains the non-relativistic and lowest-order relativistic contributions and a part of the higher-order relativistic contributions while the term \( \Delta E^{(2)}_H \) contains all the higher-order relativistic contributions which are not included into \( \Delta E^{(2)} \).

The total energy shift of first order in \( m/M \) and of first order in the interaction with the classical magnetic field is \( \Delta E^{(tot)} = \Delta E^{(1)} + \Delta E^{(2)} \). In the compact form, \( \Delta E^{(1)} \) and \( \Delta E^{(2)} \) are defined by equations (74) and (78), respectively. For practical calculations, it is more convenient to use the representation given by equations (76)-(78) and (93)-(95). To derive these equations, we have assumed that \( V(\mathbf{x}) \) deviates from the pure Coulomb potential. However, since the final formulas for the energy shift exhibit neither singularities nor ambiguities when \( V(\mathbf{x}) \to V_C = -\alpha Z/|\mathbf{x}| \), the pure Coulomb potential can be used in these formulas as well.

The corresponding correction to the bound-electron \( g \) factor is defined as

\[
\Delta g = \frac{\Delta E^{(tot)}}{\mu_0 H m_j}, \tag{96}
\]
where \( \mu_0 = |e|/(2m) \) is the Bohr magneton and \( m_j \) is the angular momentum projection of the state under consideration. Here and below we assume that \( H \) is directed along the \( z \) axis.

**V. RECOIL CORRECTION TO THE BOUND-ELECTRON \( g \) FACTOR TO LOWEST ORDERS IN \( \alpha Z \)**

To the lowest-order relativistic approximation, the recoil correction to the \( g \) factor is given by

\[
\Delta E^{(tot)} = \Delta E^{(1)}_L + \Delta E^{(2)}_L, \tag{97}
\]
where \( \Delta E^{(1)}_L \) and \( \Delta E^{(2)}_L \) are defined by equations (77) and (44), respectively. Let us calculate this correction for an arbitrary state of a hydrogenlike atom. For the case of the point-charge nucleus that we will consider, this calculation can be performed analytically.

Consider first the calculation of \( \Delta E^{(1)}_L \). According to equation (77), it is

\[
\Delta E^{(1)}_L = \frac{1}{M} \langle \delta a | \left[ p^2 - \frac{\alpha Z}{r} (\alpha \cdot p + (\alpha \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{p}) \right] |a \rangle, \tag{98}
\]
where \( \mathbf{n} = r/r \). Taking into account that \( \mathbf{p}^2 = (\mathbf{\alpha} \cdot \mathbf{p})^2 \) and \( (\mathbf{\alpha} \cdot \mathbf{p}) = H_\Omega - \beta m - V_C \), one easily obtains

\[
\langle \delta \mathbf{a} | \mathbf{p}^2 | \mathbf{a} \rangle = \langle \delta \mathbf{a} | (\varepsilon_a + \beta m - V_C) (\varepsilon_a - \beta m - V_C) | \mathbf{a} \rangle + i \langle \delta \mathbf{a} | (\mathbf{\alpha} \cdot \nabla V_C) | \mathbf{a} \rangle.
\]

(99)

The second term in equation (98) can be transformed as (see, e.g., [35])

\[
- \langle \delta \mathbf{a} | \frac{\alpha Z}{r} (\mathbf{\alpha} \cdot \mathbf{n}) (\mathbf{n} \cdot \mathbf{p}) | \mathbf{a} \rangle = - \langle \delta \mathbf{a} | \frac{\alpha Z}{r} [2\varepsilon_a - 2\beta m - 2V_C + i \frac{\alpha Z}{r} (\beta \kappa + 1) | \mathbf{a} \rangle,
\]

(100)

where \( \kappa = (-1)^{j+l+1/2} (j + 1/2) \) is the relativistic angular quantum number of the state \( a \), \( j \) is the total angular momentum, and \( l = j \pm 1/2 \) defines the parity of the state. The wave function correction \( |\delta \mathbf{a} \rangle \) defined by equation (73) can easily be found analytically using the method of the generalized virial relations for the Dirac equation developed in [36]. Since the operator sandwiched between \(| \mathbf{a} \rangle \) and \(| \delta \mathbf{a} \rangle \) in equation for \( \Delta E^{(1)}_L \) conserves the angular quantum numbers, we need only that component of \(| \delta \mathbf{a} \rangle \) which has the same angular quantum numbers as the unperturbed state \(| \mathbf{a} \rangle \). This component is

\[
|\delta \mathbf{a} \rangle_{kmj} = \left( \frac{X(r) \Omega_{kmj}(\mathbf{n})}{iY(r)\Omega_{kmj}(\mathbf{n})} \right),
\]

(101)

where

\[
X(r) = b_0 \left[ \frac{2m\kappa - m + 2\kappa \varepsilon_a}{2m^2} + \frac{\alpha Z}{m^2} \kappa \right] f(r) + \frac{\kappa - 2\kappa^2}{2m^2} g(r),
\]

(102)

\[
Y(r) = b_0 \left[ \frac{2m\kappa + m - 2\kappa \varepsilon_a}{2m^2} - \frac{\alpha Z}{m^2} \kappa \right] g(r) + \frac{\kappa + 2\kappa^2}{2m^2} f(r),
\]

(103)

\[
b_0 = -\frac{\varepsilon}{2} \frac{\mathbf{M} \cdot \kappa}{j(j+1)m_j},
\]

(104)

g(r) and \( f(r) \) are the radial parts of the unperturbed wave function defined as

\[
| \mathbf{a} \rangle = \left( \frac{g(r) \Omega_{kmj}(\mathbf{n})}{i f(r) \Omega_{kmj}(\mathbf{n})} \right).
\]

(105)

Integrating over the angular variables in equations (99) and (100), we find

\[
\Delta E^{(1)}_L = \frac{b_0}{M} \int_0^\infty dr \ r^2 \left\{ X(r) g(r) \left[ -2V_C m - V_C^2 + \varepsilon_a^2 - m^2 \right]
+ Y(r) f(r) \left[ 2V_C m - V_C^2 + \varepsilon_a^2 - m^2 \right]
+ [X(r)f(r) + Y(r)g(r)] \frac{\alpha Z}{r^2} \right\}.
\]

(106)

Substituting expressions (102) and (103) into equation (104), we obtain

\[
\Delta E^{(1)}_L = \frac{b_0}{M} \left\{ \frac{\alpha Z}{m} \left[ \frac{2\kappa \varepsilon_a - m}{m} C^0 + \frac{\alpha Z}{m} \frac{\kappa}{m} C^{-1} + \left( \varepsilon_a^2 - m^2 \right) \frac{\kappa}{m} C^1 \right]
+ \frac{\alpha Z}{2m^2} C^{-2} + \left( \varepsilon_a^2 - m^2 \right) \frac{\kappa}{2m^2} A^0 - \alpha Z \frac{\kappa^3}{m} A^{-1}
- \left( \alpha Z \right)^2 \frac{\kappa}{2m^2} A^{-2} - \left( \varepsilon_a^2 - m^2 \right) \frac{\kappa^2}{m^2} B^0 + \alpha Z \frac{3m\kappa - 2\kappa \varepsilon_a}{2m^2} B^{-1} \right\},
\]

(107)

where we have used the notations [35]

\[
A^s = \int_0^\infty dr \ r^{2+s} (g^2 + f^2), \quad B^s = \int_0^\infty dr \ r^{2+s} (g^2 - f^2), \quad C^s = 2 \int_0^\infty dr \ r^{2+s} g f.
\]

(108)

The integrals \( A^s, B^s, \) and \( C^s \) are easily calculated by the recurrent equations given in [36] (the relevant equations were first derived in [37]). We have
\[
\begin{align*}
C^0 &= \frac{\kappa m^2 - \varepsilon_a^2}{\alpha Z}, \\
C^{-1} &= \frac{(\alpha Z)^2 \kappa m}{N^3 \gamma}, \\
C^1 &= \frac{2\kappa \varepsilon_a - m}{2m^2}, \\
C^{-2} &= \frac{2(\alpha Z)^3 [2\kappa (\gamma + n_r) - N]}{N^4 (4\gamma^2 - 1) \gamma}, \\
A^0 &= 1, \\
A^{-1} &= \frac{\alpha Z (\kappa^2 + n_r \gamma)}{N^3 \gamma}, \\
A^{-2} &= \frac{2(\alpha Z)^2 \kappa [2\kappa (\gamma + n_r) - N] m^2}{N^4 (4\gamma^2 - 1) \gamma}, \\
B^0 &= \frac{\varepsilon_a}{m}, \\
B^{-1} &= \frac{m^2 - \varepsilon_a^2}{\alpha Z m},
\end{align*}
\]

(109)

where \(n_r\) is the radial quantum number, \(N = \sqrt{n^2 - 2n_r(|\kappa| - \gamma)}\), and \(n = n_r + |\kappa|\) is the principal quantum number.

Substituting formulas (109) into equation (107), we obtain

\[
\Delta E^{(1)}_L = -\frac{e}{2M} \frac{\kappa^2}{2j(j+1)} \frac{m^2 - \varepsilon_a^2}{m^2} m_j.
\]

(110)

Consider now the term \(\Delta E^{(2)}_L\). According to equation (94), we have

\[
\Delta E^{(2)}_L = \frac{e}{2M} \frac{\alpha Z}{2r} \frac{(r \times p)_z - (r \times D(0))_z}{|a|}.
\]

(111)

Integrating over the angular and radial variables, we obtain

\[
\Delta E^{(2)}_L = \frac{e}{2M} \frac{\alpha Z}{2r} \frac{(l^2 - \alpha Z (r \times \alpha)_z)}{|a|}.
\]

(112)

For the sum \(\Delta E_L = \Delta E^{(1)}_L + \Delta E^{(2)}_L\), we find

\[
\Delta E_L = \frac{e}{2M} \frac{\alpha Z^2 \varepsilon_a^2 + \kappa m \varepsilon_a - m^2}{2m^2 j (j+1)} m_j.
\]

(113)

The related contribution to the \(g\) factor is

\[
\Delta g_L = \frac{m}{M} \frac{2\kappa^2 \varepsilon_a^2 + \kappa m \varepsilon_a - m^2}{2m^2 j (j+1)}.
\]

(114)

To the two lowest orders in \(\alpha Z\), we have

\[
\Delta g_L = \frac{m}{M} \frac{1}{j (j+1)} \left[ \kappa^2 + \frac{\kappa}{2} - \frac{1}{4} \left( \kappa^2 + \frac{\kappa}{4} \right) \right] \frac{(\alpha Z^2)^2}{n^2}.
\]

(115)

For an ns state, formula (114) yields

\[
\Delta g_L = \frac{m}{M} \frac{2 \varepsilon_a^2 - m \varepsilon_a - m^2}{3 m^2}.
\]

(116)

while formula (114) gives

\[
\Delta g_L = \frac{m}{M} \frac{(\alpha Z)^2}{n^2}.
\]

(117)

Formula (117) agrees with the related result obtained for the 1s state in [20–23].
VI. CONCLUSION

In this paper we formulated the systematic QED method for calculations of the nuclear recoil corrections to the energy levels and to the electronic $g$ factor in atoms to first order in $m/M$ and to all orders in $\alpha Z$. Employing this method, we derived the complete $\alpha Z$-dependence expression for the recoil correction to the bound-electron $g$ factor in a hydrogenlike atom (equations (73), (72)). This expression was also represented as the sum of the lower-order and higher-order terms (equations (74)-(78), (113)-(115)). The lower-order term contains all the non-relativistic and lowest-order relativistic contributions and a part of the higher-order corrections. For an arbitrary state of a hydrogenlike atom, a simple analytical formula was derived for this term (equation (113)). As to the higher-order term, we expect that its numerical evaluation can be performed in the same way as the evaluation of the related contribution to the energy levels [38–40]. This calculation, which is equally important for low- and high-$Z$ systems, is under way and will be published elsewhere.

ACKNOWLEDGEMENTS

Valuable conversations with D. Arbatsky, T. Beier, S. Karshenboim, J. Kluge, W. Quint, and V. Yerokhin are gratefully acknowledged. I am grateful to the Atomic group of GSI and personally to T. Beier and J. Kluge for hospitality during the visit in winter 2001. This work was supported in part by RFBR (Grant No. 01-02-17248) and by the program ”Russian Universities - Basic Research” (project No. 3930).
[1] N. Hermanspahn, H. Häffner, H.-J. Kluge, W. Quint, S. Stahl, J. Verdu, and G. Werth, Phys. Rev. Lett. 84, 427 (2000).
[2] H. Häffner, T. Beier, N. Hermanspahn, H.-J. Kluge, W. Quint, S. Stahl, J. Verdu, and G. Werth, Phys. Rev. Lett. 85, 5308 (2000).
[3] S. A. Blundell, K. T. Cheng, and J. Sapirstein, Phys. Rev. A 55, 1857 (1997).
[4] H. Persson, S. Salomonson, P. Sunnergren, and I. Lindgren, Phys. Rev. A 56, R2499 (1997).
[5] T. Beier, I. Lindgren, H. Persson, S. Salomonson, and P. Sunnergren, Hyperfine Interactions 127, 339 (2000).
[6] T. Beier, I. Lindgren, H. Persson, S. Salomonson, P. Sunnergren, H. Häffner, and N. Hermanspahn, Phys. Rev. A 62, 032510 (2000).
[7] T. Beier, Phys. Rep. 339, 79 (2000).
[8] S. G. Karshenboim, Phys. Lett. A 266, 380 (2000).
[9] S. G. Karshenboim, hep-ph/0008227 (2000) (http://xxx.lanl.gov).
[10] S. G. Karshenboim, V. G. Ivanov, and V. M. Shabaev, Can. J. Phys., in press.
[11] A. Czarnecki, K. Melnikov, and A. Yelkhovsky, Phys. Rev. A 63, 012509 (2001).
[12] R. N. Faustov and A. P. Martynenko, hep-ph/0011344 (2000) (http://xxx.lanl.gov).
[13] V. M. Shabaev, Can. J. Phys. 76, 907 (1998).
[14] P. Seelig, S. Borneis, A. Dax, T. Engel, S. Faber, M. Gerlach, C. Holbrow, G. Huber, T. Kühl, D. Marx, K. Meier, P. Merz, W. Quint, F. Schmitt, M. Tomaselli, L. Völker, M. Würtz, K. Beckert, B. Franzke, F. Nolden, H. Reich, M. Steck, and T. Winkler, Phys. Rev. Lett. 81, 4824 (1998).
[15] H. Winter, S. Borneis, A. Dax, T. Kühl, D. Marx, F. Schmitt, P. Seelig, W. Seelig, V. M. Shabaev, M. Tomaselli, and M. Würtz, in GSI scientific report 1998, edited by U. Grundinger (GSI, Darmstadt, Germany, 1999), p. 87.
[16] V. M. Shabaev, A. N. Artemyev, and V. A. Yerokhin, Phys. Scr. T 86, 7 (2000).
[17] P. J. Mohr, G. Plunien, and G. Soff, Phys. Rep. 293, 227 (1998).
[18] A. S. Zapryagaev, Opt. Spectr. 47, 18 (1979).
[19] D. A. Glazov and V. M. Shabaev, to be published.
[20] R. N. Faustov, Phys. Lett. B 33, 422 (1970).
[21] R. N. Faustov, Nuovo Cimento A 69, 37 (1970).
[22] H. Grotch, Phys. Rev. A 2, 1605 (1970).
[23] H. Grotch and R. A. Hegstrom, Phys. Rev. A 4, 59 (1971).
[24] F. E. Close and H. Osborn, Phys. Lett. B 34, 400 (1971).
[25] A. S. Davydov, Quantum Mechanics (Fiz.-Mat. Lit., Moscow, 1963).
[26] J. D. Bjorken and D. Drell, Relativistic Quantum Fields (McGraw-Hill, New York, 1965).
[27] A. A. Logunov and A. N. Tavkhelidze, Nuovo Cim. 29, 380 (1963).
[28] V. M. Shabaev, Phys. Rep., in press, physics/0009018 (2000) (http://xxx.lanl.gov).
[29] R. N. Faustov, Teor. Mat. Fiz. 63, 394 (1985) (Theor. Math. Phys. 63, 588 (1985)); In: Papers at First Soviet-British Symposium on Spectroscopy of Multicharged Ions (Academy of Sciences, Troitsk, 1986), pp. 238-240.
[30] A. S. Yelkhovsky, Preprint BINP 94-27 (Budker Inst. of Nuclear Physics, Novosibirsk, 1994); hep-th/9403095 (1994).
[31] K. Pachucki and H. Grotch, Phys. Rev. A 51, 1854 (1995).
[32] V. M. Shabaev, Phys. Rev. A 57, 59 (1998).
[33] A. S. Davydov, Zh. Eksp. Teor. Fiz. 64, 413 (1973) (Sov. Phys. JETP 37, 211 (1973)).
[34] V. M. Shabaev and A. N. Artemyev, J. Phys. B 27, 1307 (1994).
[35] V. M. Shabaev, J. Phys. B 24, 4479 (1991).
[36] J. Epstein and S. Epstein, Am. J. Phys. 30, 266 (1962).
[37] A. N. Artemyev, V. M. Shabaev, and V. A. Yerokhin, Phys. Rev. A 52, 1884 (1995); J. Phys. B 28, 5201 (1995).
[38] V. M. Shabaev, A. N. Artemyev, T. Beier, G. Plunien, V. A. Yerokhin, and G. Soff, Phys. Rev. A 57, 4235 (1998); Phys. Scr. T 80, 493 (1999).
[39] V. M. Shabaev, A. N. Artemyev, T. Beier, and G. Soff, J. Phys. B 31, L337 (1998).