Topological mechanisms as classical spinor fields

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A mechanism is a zero-energy motion of a mechanical structure that does not stretch or compress any of its components. Here, we focus on a special class of mechanisms that we dub topological because they are insensitive to smooth changes in material parameters. Topological mechanisms do not arise from local under-coordination, but they can be localized to solitons in the underlying structure. In this letter, we exploit supersymmetry to develop a real-space formalism whereby a topological mechanism can be described as a classical spinor whose real components are the soliton-induced displacement and stress fields. Our analytical approach goes beyond topological band theory by addressing the non-linearity and inhomogeneity of the underlying structure key to the very definition of a mechanism. We apply this general method to an activated mechanism, inspired by the organic molecule polyacetylene, that can propagate down an assembly line without deploying the whole structure.

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In engineering, the term mechanism denotes a free motion of a mechanical structure that does not deform any of its components and hence costs zero elastic energy. As an example, consider the collective motion that arises in networks of bars or plates joined together by pivots or hinges around which two adjacent components can freely rotate. When activated by motors or external fields, such mechanisms become the building blocks of robots and smart metamaterials. Thus, the hard problem of predicting the effect of constraints on an interacting many-body system is as deeply rooted in mechanical design and robotic control theory as it is in modern theoretical physics.

Here, we study a special class of mechanisms that we dub topological because they are insensitive to smooth changes in material parameters that preserve the connectivity of the structure. Unlike trivial dangling bonds, topological mechanisms do not arise from a local deficiency in the number of constraints relative to the degrees of freedom. Nonetheless, they can still be localized to solitons in the underlying mechanical structure. Figure 1 shows a mechanism or zero-energy mode located at a domain wall, which is protected against changes in the length or spacing of the bars comprising the structure.

The first step to study a zero-energy mode in an arbitrary mechanical structure is to write the rigidity matrix that relates site displacements to bond tensions. A zero mode causes no deformation of the elastic bonds even when some of the sites are displaced – hence it is a zero eigenvalue of the rigidity matrix. Conversely a state of self-stress is an assignment of bond tensions that does not result in site displacements. The generalized Maxwell relation stipulates that the number of zero modes, \( n_m \) minus the number of states of self-stress, \( n_{ss} \) is equal to the number of degrees of freedom \( N_{df} \) minus the number of constraints \( N_c \)

\[
\nu \equiv n_m - n_{ss} = N_{df} - N_c
\]

Isostatic structures defined by the condition \( N_{df} = N_c \) lie right at the threshold of a rigidity transition.

Once the connectivity is fixed (which fixes the right-hand side of Eq. (1), the index \( \nu \) can be viewed as a topological charge, invariant under smooth deformations of the local bond length. Just as Gauss’s law yields the net charge enclosed in a region from the flux of the electric polarization through its boundary, the net value of \( \nu \) in an arbitrary portion of an isostatic lattice is given by the flux of a topological polarization through its boundary. The power of this approach lies in its ability to count zero-energy edge modes, which are themselves inhomogeneous excitations, simply from the band theory.
of the undistorted lattice plus topology \[14\].

Despite its elegance, the topological band theory of linear vibrations does not explicitly address two issues. First, a mechanism is inherently a non-linear object — while its presence can be detected from counting zero-energy modes in the linear vibrational spectrum — its properties cannot \[11\]. Second, topological mechanisms are often localized to a soliton or vortex which is by definition a place where the order is disrupted, while preserving the local connectivity. This is not easily captured analytically within topological band theory which assumes a periodic lattice. For example, the dynamics of how a topological zero mode couples to deformations of the underlying structure, that in turn enable the zero mode to propagate, remains hard to tackle.

In this Letter, we exploit a supersymmetric structure present in this problem to develop a real-space formalism whereby a topological mechanism is described as a classical spinor field whose real components are the soliton-induced displacement and stress fields. The mass of this quasiparticle is given by a soliton solution of the non-linear field theory describing the underlying structure. Our analytical approach goes beyond topological band theory by addressing the non-linearities and structural inhomogeneities associated with the existence of a topological mechanism. It also applies to collective buckling modes in metamaterials composed of slender beams for which the frequency (or critical strain) of the instability approaches zero \[20\].

**Mechanical analogue of polyacetylene** To make the exposition more concrete, we present our general calculational framework in the context of a quasi one-dimensional mechanism inspired by the organic molecule polyacetylene \[8, 9\]. A Lego realization shown in Fig. 1 consists of a dimerized arrangement of (yellow) rigid rotors of length \(r\) separated by a distance \(a\) and constrained to rotate about fixed pivot points (white bolts). The local orientation of each rotor with respect to the \(\hat{x}\)-axis is denoted by \(\theta(x)\) and its projection by \(\phi(x) = r \sin \theta(x)\). These rigid pendula are coupled by (gray) bars that can be viewed as Hookean springs with elastic constant \(k\). There is exactly one fewer constraint than degrees of freedom in the chain, thus there is one zero-energy mode \[10\]. The isostatic condition is exactly fulfilled by putting the chain on a ring. Once an edge is created, a zero-energy edge mode and associated mechanism arise.

The zero mode is not merely localized at the edge \[9\]. It can propagate down the chain if the joints at the base are sequentially activated by motors, gravity (if tilted) or simply by hand. As a result, the zero mode will move, without deploying the whole structure, at a speed determined by the external driving. Fig. 1 (a-b) shows that the moving zero mode is coupled to a moving soliton, separating left- and right-leaning rotors, in analogy to polyacetylene where the electron hops by coupling to a moving domain wall in the bond distortion \[22\].

**Soliton-fermion systems** The following 1+1 dimensional Lagrangian, \(\mathcal{L}\), describes coupled soliton-fermion systems like polyacetylene \[25, 26\]

\[
\mathcal{L} = \frac{1}{2} \int dx \left[ (\partial_\mu \phi)^2 - V^2(\phi) + i \bar{\Psi} \gamma^\mu \partial_\mu \Psi - V'(\phi) \bar{\Psi} \Psi \right] \tag{2}
\]

where the subscript \(\mu\) denotes partial derivatives with respect to both space, \(x\), and time, \(t\), and \(\gamma^\mu\) denotes the 2D representation of the gamma matrices. Here \(\phi(x, t)\) is a bosonic field with an onsite potential \(V^2(\phi)\) and \(\Psi(x, t)\) is the fermionic field, coupled to \(\phi\) through \(V'(\phi)\). The form of the Lagrangian, \(\mathcal{L}\), in Eq. (2) remains invariant under a set of supersymmetry transformations between the \(\phi\) and \(\Psi\) fields \[21, 23, 28\].

In polyacetylene, the bosonic field \(\phi\) represents the distortions of the molecular bond length that gives rise to the domain wall (soliton) interpolating between the two minima of \(V^2(\phi)\). An electronic state \(\Psi\) is coupled to the domain wall by the last term in Eq. (2) that gives the fermion an inhomogeneous mass. The first and third terms in Eq. 1 capture the kinetic energies of the moving soliton-electron pair. As the electron hops, it shifts the position of the double bond, hence displacing the domain wall to which it is bound. This coupling is responsible for the electronic conductivity of polyacetylene \[22\].

Similarly, the kink configuration in Fig. 1 is described by a displacement field \(\phi(x, t)\), the projection of the rotor on the \(\hat{x}\)-axis, that minimizes the elastic potential \(V^2(\phi)\) \[9\]. Can one push this mapping further and identify a fermion-like field \(\Psi\) in such a purely mechanical structure? A classical spinor field \(\Psi\) can be constructed using the translational invariance of the soliton solution. Its two entries are the real displacement and stress fields induced by the moving soliton, which are themselves related to each other by supersymmetry transformations, like \(\phi\) and \(\Psi\) in Eq. (2) \[21\].

In this language, the mechanical zero mode is localized around the domain wall because its “mass” \(V'(\phi)\) and the corresponding energy in Eq. (2) are minimized there. However, from the perspective of elasticity, \(\phi\) and \(\Psi\) are not physically distinct bosonic and fermionic fields. They merely provide a redundant (albeit practically convenient) description of a single bosonic distortion field and are related to each other by the translation invariance of the soliton. This constraint can be interpreted as a symmetry between \(\phi\) and \(\Psi\) in Eq. (2). The Lagrangian \(\mathcal{L}\) can be further generalized to describe fermions bound to a two-dimensional soliton, i.e. a vortex of the Abrikosov, Nielsen, Olesen type \[13, 21, 23, 27, 28\]. With this generalization in mind, our approach can describe 2D classical topological mechanisms bound to dislocations recently studied in Ref. \[10\] and more generally purely mechanical structures that mimic the electronic properties of graphene.

**Non-linear mechanics** The defining property of a generic mechanism is that there is a zero-stretching energy deformation of finite amplitude in the system. This property is naturally embedded in the mathematical structure of a non-linear Lagrangian similar to the
where for simplicity, we consider $\phi(x,t)$ to be a scalar field that, for instance, describes the rotors’ projection in the chain shown in Fig. 1 (for more complex mechanisms such as dislocated kagome lattices considered in Ref. [10], the field could be tensorial). In Eq. [3] $x$ denotes a generalized spatial coordinate, $\rho$ is the mass density and $k$ is the elastic constant of the bonds that, when taken to infinity, produces the limit of a linkage. For example, upon introducing a length scale $\lambda = a / \sqrt{2}$ and choosing $V(\phi) = -1/(\hat{\phi}^2 - \phi^2)$ (SUSY potential), the soliton excitations of Eq. [3] describe the zero-energy excitations bound to the domain wall in the mechanical structure of Fig. 1 [9]. In what follows, we use this $\hat{\phi}^4$ potential as a paradigmatic example, even though the supersymmetric formalism presented here is generic.

The static soliton solution $\phi_s(x)$ to Eq. [3] is obtained by setting the second (potential energy) term to zero, i.e.

$$\partial_x \phi_s(x) = V(\phi_s).$$

This condition enforces the absence of bond stretching in the mechanism, and it is a mechanical realization of a field whose solution saturates the BPS bound [21]. The complete time-dependent behaviour is obtained from solutions of the Euler-Lagrange equation of motion,

$$\rho \ddot{\phi}(x,t) - k \partial_x^2 \phi(x,t) = -\frac{k}{2} \frac{\partial V^2}{\partial \phi^2}. \tag{4}$$

To describe the zero mode bound to the soliton, we linearize Eq. [4] around the soliton solution $\phi_s(x,t)$ and write $\phi(x,t) = \phi_s(x,t) + \psi(x,t)$ and look for small distortions of the soliton field in the form $\psi(x,t) = \psi_n^{(1)}(x) \exp(i \omega_n t)$. This yields a Schrödinger-like equation for $\psi(x,t)$

$$\mathcal{H}_1 \psi_n^{(1)} = k \left( -\frac{\partial^2}{\partial x^2} + U_1(x) \right) \psi_n^{(1)} = \rho \omega_n^2 \psi_n^{(1)}, \tag{5}$$

where $U_1(x) = \frac{1}{2} [V^2(\phi)]''(\phi_s(x))$ and we have defined $\mathcal{H}_1$ to be the second-order differential operator in Eq. [5].

**Supersymmetry (SUSY)** The SUSY structure is obtained by factorizing $\mathcal{H}_1$ as a product of two first-order operators, $\mathcal{H}_1 = A \dagger A$, where

$$A = -\frac{d}{dx} + w(x), \quad A \dagger = \frac{d}{dx} + w(x) \tag{6}$$

and the supersymmetry potential reads

$$w(x) = V'(\phi)|_{\phi(x)}. \tag{7}$$

The potential in the Schrödinger Eq. [3] can be written as $U_1(x) = w^2(x) - w'(x)$. The site displacements’ eigenfunctions $\psi_n^{(1)}$ are bound states of $U_1(x)$, plotted in Fig. 2 for the $\hat{\phi}^4$ potential.

Upon applying the lowering operator once, we obtain the bond extensions, $\psi_n^{(2)}(x) = A \psi_n^{(1)}$, or equivalently the tensions $k \psi_n^{(2)}(x)$ that would be measured in an elastic structure. The operator $A$ is thus a continuum version of the discrete rigidity matrix $R_{ij}$, for instance, consider adjacent sites labelled by $\{i,j\}$ in the chain of Fig. 1. The corresponding bond extensions are $\delta_{ij} = \frac{\cos \theta}{\ell}(2r \sin \theta (\delta \theta_i + \delta \theta_j) + a (\delta \theta_i - \delta \theta_j))$, where $r$ is the length of the rotors, $\ell$ is the equilibrium length of the bonds, and $a$ is the space between adjacent (passive) joints and $\delta \theta_i, \delta \theta_j$ are small angular displacements around the homogeneous background $\theta = \overline{\theta}$. A continuum limit of the distortion field $\delta \theta_i \rightarrow \phi(x)$, $\delta \theta_i - \delta \theta_j \rightarrow \partial_x \phi$, reproduces the operator $A$ in Eq. [6] for the special case of a constant potential $w(x) = \frac{a}{2} \sin 2\theta$ (proportional to the gap in the linear spectrum). However, when we expand around the soliton field as in Eq. [5] we obtain bond extensions over an inhomogenous zero-energy state. Thus in general, the operator $A$ is a continuum limit of the real-space rigidity matrix, around a specific solution of the non-linear field theory, that can be explicitly determined using Eqs. [6] and [7].

Note that the bond extensions $\psi_n^{(2)}(x)$ are bound states of the partner potential $U_2(x) = w^2(x) + w'(x)$ obtained from factorizing the Hamiltonian in Eq. 5 as $\mathcal{H}_2 = AA \dagger$. The Hamiltonians $\mathcal{H}_1, \mathcal{H}_2$ are said to be partners because, once the eigenvalues (eigenfunctions) of $\mathcal{H}_1$ are known the corresponding ones for $\mathcal{H}_2$ can be easily obtained, except for the zero-energy eigenvalue that we assume to be part of $\mathcal{H}_1$. Thus, for example if $\mathcal{H}_1 \psi_n^{(1)} = E_n^{(1)} \psi_n^{(1)}$, then $\mathcal{H}_2(A \psi_n^{(1)}) = AA \dagger A \psi_n^{(1)} = E_n^{(1)}(A \psi_n^{(1)})$. Each eigenfunction in $\mathcal{H}_1$ has a partner in the spectrum of $\mathcal{H}_2$ except for the ground state defined via $\mathcal{H}_1 \psi_0 = A \dagger A \psi_0 = 0$. The site-displacement field is obtained from the bond-stretching field by applying the raising operator, i.e.

$$\psi_n^{(1)}(x) = A \dagger \psi_n^{(2)}(x).$$

This construct is illustrated schematically in Fig. 2.

Combining $\mathcal{H}_1, \mathcal{H}_2$, we get the matrix $\mathcal{H}$ that together with the operators $\mathcal{Q}$ and $\mathcal{Q} \dagger$ given by

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_1 & 0 \\ 0 & \mathcal{H}_2 \end{pmatrix}, \quad \mathcal{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad \mathcal{Q} \dagger = \begin{pmatrix} 0 & A \dagger \\ 0 & 0 \end{pmatrix} \tag{8}$$

satisfies the super-algebra, $[\mathcal{H}, \mathcal{Q}] = [\mathcal{H}, \mathcal{Q} \dagger] = 0$, $\{\mathcal{Q}, \mathcal{Q} \dagger\} = \mathcal{H}$ and $\{\mathcal{Q}, \mathcal{Q} \dagger\} = \{\mathcal{Q} \dagger, \mathcal{Q} \dagger\} = 0$ [21, 29]. The two-component field $\Psi_n$, formed by combining $\psi_n^{(1)}$ and $\psi_n^{(2)}$, can itself be viewed as a “fermion” field, as evidenced by the anticommuting algebra of the $\mathcal{Q}$ and $\mathcal{Q} \dagger$ operators.

We now define the index $\nu$ of the operator $A$ as the difference in the dimension of the kernel of $A$ and $A \dagger$. Using the identities $\ker A = \ker A \dagger A$ and $\ker A \dagger = \ker A A \dagger$, we obtain the Witten index [29]

$$\nu = \dim \ker \mathcal{H}_1 - \dim \ker \mathcal{H}_2. \tag{9}$$

The mechanical interpretation of this field-theoretic
inhomogenous mass obtained from the Dirac Lagrangian of a theory in the previous section can be compactly interacting. The bond-extension eigenfunctions, \( \phi \), of the domain wall for the chain in Fig. 1 (described by \( \phi^4 \) theory) or at the core of topological defects in more complex 2D structures.

We now seek solutions of Eq. (11) with the form

\[
\psi(x,t) = \Psi_n(x) \exp(-i\omega_n t)
\]

and obtain

\[
\gamma^0 \omega_n \Psi_n(x) + i\gamma^1 \frac{d\Psi_n}{dx} - w(x) \Psi_n(x) = 0
\]

where the spinor field \( \Psi_n(x) \) and the 2D representation of the gamma matrices \( \{\gamma^0, \gamma^1\} \) read

\[
\Psi_n(x) = \begin{pmatrix} \psi_n^{(1)} \\ \psi_n^{(2)} \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
\]

Thus, the Dirac equation can be decomposed into two coupled linear equations of the form,

\[
A^0 \psi_n^{(1)}(x) = \omega_n \psi_n^{(2)}(x), \\
A^1 \psi_n^{(2)}(x) = \omega_n \psi_n^{(1)}(x)
\]

from which we recover the same eigenvalue problem

\[
A^0 \psi_n^{(1)}(x) = \omega_n^2 \psi_n^{(1)}(x)
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in Eq. (8). For the chain mechanism \( \nu = 1 \) there is only one normalizable zero-energy eigenstate \( \psi_0 \) that we associate with \( H_1 \).

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Mechanical Spinors As hinted by the fermionic character of \( \Psi \), the same results we have derived from the interacting theory in the previous section can be compactly obtained from the Dirac Lagrangian of a free fermion with inhomogenous mass \( V'(\phi) \) given by the last two terms in Eq. (1):

\[
\mathcal{L} = i\bar{\Psi} \gamma^\mu \partial_\mu \Psi - \bar{\Psi} \Psi w(x),
\]

where \( w(x) = V'(\phi)|_{\phi_n(x)} \) is set by the potential \( V^2(\phi) \) of the non-linear field theory prescribing the underlying structure. The Euler-Lagrange equation of motion,

\[
i\gamma^\mu \partial_\mu \Psi - w(x) \Psi = 0,
\]

is a Dirac equation where the constant mass term is replaced by the inhomogenous field \( w(x) \). The classical spinor minimizes its energy by localizing where \( w(x) = V'(\phi)|_{\phi_n(x)} \) is vanishingly small, i.e. in the middle of the domain wall for the chain in Fig. 1 described by \( \phi^4 \) theory) or at the core of topological defects in more complex 2D structures.

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systems.

To sum up, we presented a real-space approach that can describe topological mechanisms and protected modes bound to solitons (vortexes) in two steps. First, a non-linear field theory is written down whose solution is a soliton (vortex) that captures the inhomogeneous underlying structure. Second, the bound topological mode is treated as a classical spinor whose two components are mechanical displacements and stress fields. This quasi-particle satisfies a Dirac equation whose mass depends on the structure to which it is coupled. This classical-to-quantum dictionary may facilitate the design of topologically protected metamaterials and robotic mechanisms.

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