Frame-like off-shell dualization for mixed-symmetry gauge fields

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Abstract
We construct a purely frame-like parent action that allows us to dualize, at the off-shell level, an arbitrary mixed-symmetry bosonic massless field in Minkowski background of dimensions $d$. Starting from any massless mixed-symmetry gauge field in the standard Skvortsov frame-like formulation and following an off-shell dualization procedure, we obtain dual theories which are on-shell related by $so(d-2)$ Hodge duality. The Hodge dualization can be done on any column of the Young diagram characterizing the generalized spin of the original frame-like field. Dualization with respect to the first column of the Young diagram leads to a standard frame-like action for the dual field. Any other dualization results in an action which cannot be described by the standard frame-like formalism, as the on-shell field is not $so(d-2)$ traceless. Instead, the latter field is given by the product of an irreducible traceless tensor and a certain number of $so(d-2)$-invariant metrics, and the corresponding dual frame-like action is new. Such actions require supplementary fields, which naturally arise along the lines of the approach that we propose.

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1. Introduction

Mixed-symmetric gauge fields have attracted a lot of attention in recent years, partly because the totally symmetric case is fairly well understood by now even at the full nonlinear level [1–3], but also because in dimension higher than 4, mixed-symmetry fields are allowed from the point of view of representation theory of the corresponding spacetime isometry algebra. They also appear in string theory, albeit at the massive level, see e.g. [4] for related discussions.

In the context of string-field theory, mixed-symmetry fields were studied in the eighties and Lagrangians in flat space were explicitly given for some cases in a metric-like fashion, see e.g. [5–7]. An action describing arbitrary mixed-symmetry gauge fields was first proposed by Siegel
and Zwiebach [8]. It was derived via an approach that allows us to systematically covariantize degrees of freedom given in the lightcone gauge. Using string-field-like techniques, Labastida proposed a minimal version of equations of motion [9] (with T Morris) and a Lagrangian [10] describing an arbitrary free, $gl(d)$-irreducible, mixed-symmetry gauge field in a flat background. This formulation uses a single off-shell field of the same Young-symmetry type as that of the tensor that encodes the on-shell degrees of freedom in the lightcone gauge and is relevant for the construction presented in this paper. It was proved much later [11, 12] that the Labastida action indeed propagates the correct degrees of freedom. Still in a flat background and for metric-like fields, more recent works can be found in [13, 14] where, among various results, the equivalent of the Labastida action but for arbitrary tensor–spinor fields was obtained. Let us mention an earlier formulation describing arbitrary tensor–spinor gauge fields [15]. See also the two books [16].

Important progress was made [17, 18] within the frame-like and unfolded approach, when full advantage of the Cartan formulation of gauge theories was taken in order to describe, both on-shell [17] and off-shell [18], arbitrary mixed-symmetry gauge fields freely propagating in flat spacetime. Fermionic field are treated along the same lines in [19]. See [20–29] and references therein for more works on mixed-symmetry gauge fields in flat background.

Mixed-symmetry gauge fields can also appear via dual formulations of totally symmetric fields, see e.g. [11, 20, 21, 30–37]. An off-shell and covariant description of the double-dual graviton, a field first introduced in [33], was obtained in a recent paper [38]. This was done in the metric-like approach, and the purpose of this paper is to give a frame-like treatment of the off-shell dualization procedure, thereby allowing us to treat the cases of arbitrary mixed-symmetry gauge fields via the frame-like formulation [17, 18].

The plan of the paper is as follows. In section 2, we review the off-shell dualization of linearized gravity on the first column of the gauge field in the metric-like formalism. Then, in section 3 we translate the previous analysis to the frame-like approach. We then perform the second nontrivial off-shell dualization of the graviton in a frame-like and first-order fashion. The latter analysis is generalized to the arbitrary mixed-symmetry case in section 4. Finally, we give some conclusions and perspectives in section 5, followed by an appendix summarizing our notation.

The unfolded approach was introduced in [39, 40], and we refer the reader to the works [41–43] for notions relevant to this paper. The unfolded formulation for massless mixed-symmetry fields in flat background was developed in [17, 18]. Generalities of unfolded on-shell dynamics including duality can be found in section 3 of [44].

2. Metric-like dualizations of gravity

Two massless fields in flat spacetime are said to be dual to each other if, on-shell, they describe representations of the Wigner little group\(^1\) \(so(d−2)\) which are related by Hodge dualization, meaning that they are actually equivalent. More precisely, for any massless field propagating in Minkowski spacetime and carrying an irreducible representation of the Wigner little group given by an \(so(d−2)\) Y-shaped tensor, one can trivially generate, on-shell and in the light-cone gauge, other equivalent irreducible representations by Hodge dualizing any (number of) columns with totally antisymmetric Levi–Civita rank \((d−2)\) tensor \(ε_{[d−2]}\). In this paper, we will only dualize fields such that their corresponding \(so(d−2)\) representation on-shell is described by a tensor of the same shape as the tensor used for their \(gl(d)\) covariant representation off-shell, as appearing inside the covariant action.

\(^1\) In this paper, we consider helicity fields for which the action of the translation subalgebra \(t_{d−2}\) of Wigner’s massless little algebra \(so(d−2)\) \(\subset t_{d−2}\) is trivial.
For example [33], consider a massless spin-2 particle, on-shell given by the symmetric traceless tensor $h_{mn}$ of $so(d - 2)$. It can be Hodge-dualized to give a traceless $so(d - 2)$-tensor of shape $Y[d - 3, 1]$

$$T^{[d-3]} = \epsilon^{[d-3]} p h_{np},$$

which obviously gives an equivalent $so(d - 2)$-irrep to the one corresponding to the original $h_{mn}$ field. On the other hand, the representation of $so(d - 2)$ given by $T$ can be uplifted off-shell, in terms of a $gl(d)$ field of the same shape, for which the action is known and can be generated either in the Labastida [10] or in the Skvortsov [18] formulation.

It is of interest to generate the dual action from the action for the original field through the so-called parent action, containing fields associated with both equivalent formulations. One ends up with one or another dual action, depending on the way one eliminates fields through their equations of motion and fixing gauges.

One of the ways to write a parent action for a spin-2 field and its dual is as follows [32, 35]. One starts from the first-order action for linearized gravity, formulated in terms of their equations of motion and fixing gauges. One of the ways to write a parent action for a spin-2 field and its dual is as follows

$$S[\epsilon_{ab}] = 4 \int d^d x \left[ C_{aab} C^{[b]} - \frac{1}{2} C_{abc} C^{[b]} - \frac{1}{4} C_{[d][b]} C^{[d][c]} \right],$$

where $C_{abc} = \delta_{[d]} \epsilon_{b[c]}. The \lambda$-symmetry with $\lambda^{ab} = -\lambda^{ba}$ inherited from (1) can be used to gauge away the antisymmetric part of $\epsilon_{d[b}$, so the action (2) depends only on $h_{ab} = \epsilon_{(ab)}$. The action (2) is just a rewriting of the linearized action of general relativity.

To pass to the parent action, we add one term

$$S[C_{abc}], Y_{abc} = Y_{abc} = 4 \int d^d x \left[ -\frac{1}{2} C_{abc} C^{[b]} + C_{aab} C^{[b]} - \frac{1}{2} C_{abc} C^{[b]} - \frac{1}{4} C_{[d][b]} C^{[d][c]} \right].$$

where $Y_{abc} = Y_{abc}|d$ and $C_{abc}$ is no longer thought as a derivative of $\epsilon$. The field $Y$ can be treated as a Lagrange multiplier for the constraint $\partial_d C_{abc} = 0$, which can be solved as $C_{abc} = \partial_d \epsilon_{b[c]}$, thus recovering (2). On the other hand, by examining the equation of motion for $C$, one sees that $C$ is auxiliary and can be eliminated from the action to give

$$S[Y_{abc}] = 4 \int d^d x \left[ Z_{abc} Z^{abc} - \frac{1}{d-2} Z_{abc} Z^{abc} \right].$$

where $Z^{abc} = \partial_d Y^{abc}$. It is convenient to rewrite it in terms of the Hodge dual field $T^{[d-3]} = \epsilon^{[d-3]} Y_{(3)l}^{[3]},$ which up to an overall factor gives

$$S[T^{[d-3]}] = \int d^d x \left[ X^{[d-2][b]} X^{[d-2][b]} - \frac{(d-2)}{(d-3)} X^{[d-3][b]} X^{[d-3][c]} + \frac{(d-2)}{(d-3)} X^{[d-3][b]} X^{[d-3][c]} \right].$$

where $X^{[d-2][b]} = \partial_{[b} T^{[d-3][b]}.$ This action is the analogue of (2) for dual graviton. Through the parent action, it inherits the $\lambda$-symmetry of (2). Its Hodge dual $(\ast \lambda)^{[d-2]}$ can be used to gauge away the totally antisymmetric part of $T^{[d-3]}$ leading to Labastida’s metric-like formulation for the massless spin-$Y[d - 3, 1]$ field.

On the other hand, (5) can be recast into the form (1), where $T^{[d-3]}$ plays the role of the lowest grade field $\epsilon_{d-3}$. The algebraic symmetry $(\ast \lambda)^{[d-2]}$ signals that the first auxiliary
field is a 1-form valued in $Y[d-2]$-shaped tensors, which is exactly the second-grade field required by the first-order approach [17].

In the light-cone gauge, on-shell, it is possible to dualize the double graviton so as to produce the double dual graviton described on-shell by $Y[d-3, d-3]$-shaped tensor of $so(d-2)$ [33]:

$$Y^{m[d-3], n[d-3]} = \varepsilon^{m[d-3]pr} \varepsilon^{n[d-3]qr} h_{pr}.$$  

The crucial difference with the first dualization is that $Y$ is no longer traceless. Indeed, the product of two antisymmetric tensors can be rewritten in terms of $\delta$-symbols (A.3), so

$$Y^{m[d-3], n[d-3]} = \sigma(d-2) \delta^{m[d-3]}_{[nl[d-3]} h^{p]}.$$  

(6)

The identity (10) can then be formulated as $\text{Tr} \{ \ast_i \ast_j \tilde{K} \} = 0$, $\forall i, j : \ell_j \leq \ell_i$. 

(9)

It can be proved [20] that the algebraic Bianchi identities together with the field equations (8) imply the relations

$$\text{Tr}_{ij} \{ \ast_i (\ast_j K) \} = 0, \quad \forall i, j : \ell_j \leq \ell_i,$$

(10)

where $\ell_i$ is the length (9) of the $i$th column of $\ast_i K$. One defines $\tilde{K}_j$ to be the multiform obtained after reordering the columns of $\ast_j K$, such that the heights of the columns of $\tilde{K}_j$ are non-increasing. The identity (10) can then be formulated as $\text{Tr}_{ij} \{ \ast_i \tilde{K}_j \} = 0$, $\forall i, j : 1 \leq i < j \leq s$. 

More generally, in the case a tensor $T$ is represented by the direct product of $m$ metric tensors and another traceless tensor, we say that $T$ is $m$-fold pure-trace. 

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implying that \( \tilde{K}_I \) is \( gl(d) \)-irreducible. The differential Bianchi identities \( diK = 0 \) together with the field equations (8) imply that
\[
\triangledown_i \ast_i K = 0,
\]
which entails
\[
d_i (\ast_i K) = 0, \quad \forall i \in \{1, \ldots, s\}. \tag{11}
\]
As a result of the generalized Poincaré lemma proved in [20], one has \( \tilde{K}_I = d_1 d_2 \ldots d_s \tilde{\kappa}_I \) for some gauge field \( \tilde{\kappa}_I \). The Hodge operators \( \ast_i \) therefore relate different free field theories of arbitrary tensor gauge fields, extending the electric–magnetic duality property of electrodynamics. In the same way, one obtains the field equations of the dual theory:
\[
\text{Tr}_{ij}^{\mu/} \{ \ast_i K \} = 0, \quad \forall i, j : i < j \tag{11}
\]
where
\[
m_{ij} = \begin{cases} 1 + D - l_i - l_j & \text{if } i \text{ and } j \in I, \\ 1 & \text{if } i \text{ or } j \not\in I. \end{cases} \tag{12}
\]
On-shell formulations involving higher powers of the trace operation generically are not Lagrangian, and the double-dual spin-2 case is the paradigmatic example discussed in [45]. In order to set up the \( gl(d) \)-covariant formalism needed for the description of propagating gauge fields that become pure \( so(d - 2) \)-trace on-shell, some work has to be done since we explained that the Labastida–Skvortsov representation for the covariant field is not suitable in those cases. In a previous work [38], a metric-like action for the double-dual graviton was given, and the purpose of this paper is to give a frame-like action that allows us to treat the arbitrary mixed-symmetry cases as well.

Staying at the on-shell level for the moment, one can use unfolded equations of motion written in [17] (to which we refer for precise definitions),
\[
R^g := dW^g + \sigma_{-}(h)W^{g+1} = 0, \tag{13}
\]
with properly modified trace constraints in order to describe double dual linearized gravity. As we mentioned above, for a propagating spin-Y\([d - 3, d - 3]\) gauge field in flat spacetime of dimension \( d \), the shapes of the Weyl-module tensors given in [17] are related, by the double Hodge dualization with rank-\( d \) antisymmetric tensor in fibre space, to the Weyl-module tensors of gravity. In order for the two Weyl modules to describe the same degrees of freedom, their trace constraints should also be related by the same fibre-space Hodge dualization\(^3\) in a way that generalizes equations (11) and (12). Analogously to the previous discussion, it implies that the Weyl-module tensors for the double dual graviton should be \( (d - 4) \)-fold pure-trace. The question is what gauge potentials should be used to make the theory Lagrangian. One can see that gauge potentials given in [17] are not suitable because, by construction, the lower grade field describes traceless (not \( (d - 4) \)-fold pure-trace as it should be) \( so(d - 2) \) \( Y[d - 3, d - 3] \)-shaped tensor after all the gauge degrees of freedom are factored out.

3. Frame-like dualizations of linearized gravity

In the previous section, we have shown how the metric-like theories can be dualized through the concept of the parent action. It appears that the same can be done in a more economic way purely in terms of frame-like fields and frame-like actions. In this section, we will illustrate the frame-like dualization in all details for the case of the first and the second dualizations of gravity.

\(^3\) Let us note that if we allow \( \sigma_- \) to contain Levi–Civita symbols we can always replace any field of the unfolded system by its dual if this is accompanied by the proper \( \sigma_- \) redefinition. So, one can make Weyl modules of dual theories exactly coinciding.
3.1. The dual gravity

In the particular case of linearized gravity, the spin is $\mathbf{Y}[1,1]$ and the first two equations of (13) acquire the form

$$T_{2}^{a} := \delta e_{1}^{a} + h_{b} \omega^{ab} = 0, \quad (14)$$

$$R_{2}^{(2)} := \delta \omega^{(2)} + h_{b} h_{c} C_{0}^{(2),abc} h^{(2)} = 0. \quad (15)$$

They come from the action (1)

$$S = \int \left( \frac{1}{2} h_{b} \omega^{ab} \right) \omega_{1}^{cd} H_{abcd}. \quad (16)$$

The parent action for the first dualization is

$$S = \int \left[ \left( \frac{1}{2} h_{b} \omega^{ab} + \tau_{2}^{a} \right) \omega_{1}^{cd} H_{abcd} + \tau_{2} \varepsilon \tilde{\varepsilon}_{d-3} \right]. \quad (17)$$

Here, $\tau_{2}^{a}$ is a torsion-like auxiliary field and $\varepsilon \tilde{\varepsilon}_{d-3}$ will be identified with the frame-field for the dual graviton. This action is invariant under the following gauge transformations:

$$\delta e_{1}^{a} = \delta \omega^{[a]} + h_{b} \lambda_{b}^{a} - \psi_{1}^{a}, \quad \delta \omega_{1}^{a[2]} = \psi_{0}^{a}, \quad (18)$$

$$\delta \omega_{1}^{2} = \psi_{0}^{2}, \quad (19)$$

$$\delta \omega_{1}^{3} = \psi_{0}^{3}. \quad (20)$$

$$\delta \varepsilon_{d-3}^{a} = \varepsilon_{d-4}^{a} - h_{\ell[d-3]} (\ast \omega_{1})^{a[\ell[d-3]]}, \quad \delta \varepsilon_{d-4}^{a} = \varepsilon_{d-5}^{a}, \quad \ldots \quad (21)$$

In order to show that the action (17) is equivalent to the original action (16), one should treat $\varepsilon \tilde{\varepsilon}_{d-3}$ as a Lagrange multiplier for the constraint $d \delta_{1}^{a} = 0$, which can be solved as $\tau_{2}^{a} = d \beta_{1}^{a}$. Then, $\tau_{2}^{a}$ can be set to zero by performing a gauge transformation (20) with appropriate parameter $\psi_{1}^{a}$, leading to the linearized gravity action (16).

The field equations derived from the action (17) can be promoted to the following unfolded form:

$$T_{2}^{a} := \delta e_{1}^{a} + h_{b} \omega^{ab} + \tau_{2}^{a} = 0, \quad (22)$$

$$R_{2}^{(2)} := \delta \omega_{1}^{ab} + h_{b} h_{c} C_{0}^{(2),abc} h^{(2)} = 0, \quad (23)$$

$$\tau_{3}^{a} := d \beta_{1}^{a} = 0. \quad (24)$$

$$\tilde{T}_{d-2}^{a} := d \varepsilon_{d-3}^{a} + (-1)^{d-3} h_{\ell[d-3]} (\ast \omega_{1})^{a[\ell[d-3]]} = 0. \quad (25)$$

where

$$(\ast \omega_{1})^{a[\ell[d-3]]} := \varepsilon_{\ell[d-2][\ell[2][2]]}. \quad (26)$$

The corresponding unfolded equations for the zero-forms are identical to those for linearized gravity [46]:

$$d \varepsilon_{0}^{(k, h[2])} + e_{c} \varepsilon_{0}^{e[(k, h[2])]} = 0, \quad k = 2, 3, \ldots \quad (27)$$

where the curly brackets ‘(’) denote projection on the symmetry of the tensor appearing under the differential. Equations (22)–(26) are manifestly gauge invariant with respect to the gauge transformations (18)–(21), where the zero-forms do not transform.

The algebraic $\psi$-symmetry in (18) can be used to gauge away the frame field $e_{1}^{a}$. The gauge transformation for the gauge parameter $\xi_{0}^{a}$, namely $\delta \xi_{0}^{a} = \psi_{0}^{a}$, implies that the $\xi_{0}^{a}$ gauge parameter can be shifted to zero. Equation (22) can be used to express $\tau$ in terms of $\omega_{1}^{a}$:

$$\tau_{2}^{a} = - h \xi_{0}^{ab}. \quad (28)$$

Substituting this back in (17) gives

$$S = \frac{(-1)^{d}}{2!(d-2)!} \int \left( \frac{1}{2} d \varepsilon_{d-3}^{a} + \frac{1}{2} (-1)^{d-3} h_{\ell[d-3]} (\ast \omega_{1})^{a[\ell[d-3]]} \right) (\ast \omega_{1})^{a[\ell[d-3]]} H_{\ell[d-2]}. \quad (29)$$

Identifying $(-1)^{d-3} (\ast \omega_{1})^{a[\ell[d-3]]}$ as a connection $\tilde{\omega}$ of the dual theory, we recover the lower grade fields and the action of the frame-like formulation for $\mathbf{Y}[d - 3, 1]$-spin field.
3.2. The double-dual graviton

The second dualization can be performed analogously. We start from the frame-like action of dual gravity\(^4\)

\[
S = \int \left( d \epsilon^a_{d-3} + \frac{1}{2} h b_{b[d-3]} \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]} \right).
\]

(29)

Following the strategy of the first dualization we should introduce an auxiliary field \(t\) such that its differential symmetry should act on the lowest grade field \(e\) of the original theory in an algebraic way and can be used to gauge it away. Then, we should introduce a new frame field \(\tilde{e}\) such that \(t \tilde{e}\) is a \(d\)-form scalar.

The first option is to take \(t\) to be a \((d-2)\)-form with one fibre index \(t^a_{d-3}\). The associated gauge parameter \(\psi_{[d-3]}^a\) has the same form degree as the form degree of the frame field \(e^a_{d-3}\) and takes its values in the same representation space, so it is appropriate to gauge \(e^a_{d-3}\) away. The parent action is

\[
S = \int \left[ \left( d \epsilon^a_{d-3} + \frac{1}{2} h b_{b[d-3]} \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]} \right) + t^a_{d-2} d \tilde{e}^a_1 \right].
\]

(30)

It is easy to see that it dualizes the dual graviton back to the usual Fierz–Pauli graviton.

To make the second non-trivial dualization, one should choose the auxiliary field \(t\) to be \(2\)-form valued transforming like a \(Y[d-3]\)-type Lorentz tensor. Its differential gauge parameter \(\psi_1^{[d-3]}\) contains enough degrees of freedom to gauge away the frame field completely. The corresponding parent action is now

\[
S[\epsilon^a_{d-3}, \omega^a_1 \omega^{[d-3]} e^{[d-3]} e_{d-3}] = \int \left[ \left( d \epsilon^a_{d-3} + \frac{1}{2} h b_{b[d-3]} \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]} \right)
+ t^a_{2[d-3]} H_{[a[c]} e_{d-2]} \right],
\]

(31)

\[
+ \left( -1 \right)^{d-1} \int 2^a_{d-3} H_{[a[c]} e_{d-2]} \right],
\]

(32)

where

\[
\left( \omega^a_1 \right)^{[d-3]} = \left( \omega^a_{2[d-2]} \right)^{[d-2]}, \quad \left( \omega^a_1 \right)^{[d-3]} = \left( \omega^a_{2[d-3]} \right)^{[d-3]}.
\]

(33)

Note the last term bilinear in the auxiliary field \(t\), which has no analogue in the parent action \((17)\). The coefficient \(\alpha\) is arbitrary although it has two special values which will briefly be discussed later.

The manifest gauge symmetries of the above action read

\[
\delta \epsilon^a_{d-3} = d \tilde{e}^a_{d-4} + \left( -1 \right)^{d-2} h b_{b[d-3]} \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]} + \left( -1 \right)^{d-3} h b_{b[d-4]} \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]}.
\]

(34)

\[
\delta \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]} = d \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]},
\]

(35)

\[
\delta \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]} = d \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2]},
\]

(36)

\[
\delta \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2}} = d \omega^a_1 \omega^{[d-3]} e^{[d-2]} H_{[a[c]} e_{d-2}},
\]

(37)

To show the equivalence with dual gravity, one should treat \(t^a_{d-3}\) as a Lagrange multiplier for the constraint \(d t^a_{d-3} = 0\), which entails \(t^a_{d-3} = 0\). Using the gauge symmetry \((36)\), one can set \(t^a_{2[d-3]}\) to zero and recuperate the frame-like formulation for linearized dual gravity.

\(^4\) Note that we reset the notation, so as to avoid cumbersome ‘double-tildes’ \(\tilde{\tilde{e}}\) notation.

\(^5\) It cannot be constructed there because of the form degree and Lorentz symmetries of \(t^a_{d-3}\).
On the other hand, the field equations for (33) can be promoted to their unfolded form:

\[
T^{a}_{d-2} := d\tilde{e}^{a}_{d-3} + h_b[d-3]\omega^{a}_{1} + h_{b[d-4]}\omega^{a}_{2} = 0, \tag{38}
\]

\[
R^{[d-2]}_2 := d\phi^{[d-2]}_1 + h_a[h_0\phi^{[d-2]}_0,b][2] = 0, \tag{39}
\]

\[
\tau^{[d-3]}_3 := dr^{[d-3]}_2 = 0, \tag{40}
\]

\[
\tilde{R}^{[d-3]}_d := d\phi^{[d-3]}_d + h^a[d-4]h_0(\star^a_1 + \alpha h^{[d-5]}h_0(\star h)^{[2]}_b = 0. \tag{41}
\]

Complemented with the unfolded equations for the Weyl module of massless spin-$Y[d-3, 1]$, the resulting set of equations is Cartan-integrable. Taking into account both fibre and form indices, $e^{a}_{d-3}$ and $\psi^{[d-3]}$ carry the representation of the Lorentz group given by the outer product of $Y[1]$ and $Y[d-3]$, which can be decomposed into irreducible parts$^6$

\[
Y[1] \otimes Y[d-3] = Y[d-2] \oplus Y[d-3, 1] \oplus Y[d-4]. \tag{42}
\]

According to this decomposition, $\psi^{[d-3]}$ can be presented in the form

\[
\psi^{[d-3]}_1 = h_0\phi^{[d-3]}_0 + h_b\phi^{[d-3]}_0,b + h^a\phi^{[d-4]}_d,
\]

with each term being irreducible. It is easy to check that none of these terms are annihilated by the operator in front of $\psi^{[d-3]}_1$ in (34), so $e^{a}_{d-3}$ can be completely gauged away by the $\psi$-symmetry.

An important difference with the first dualization case is that, in spite of the fact that the frame field $e^{a}_{d-3}$ of the original theory can be completely gauged away, some gauge parameters associated with $e^{a}_{d-3}$ survive and play an important role, as we explain now. The gauge transformations for the gauge parameter $\xi^{a}_{d-4}$ are

\[
\delta\xi^{a}_{d-4} = d\tilde{\xi}^{a}_{d-5} + h_{b[d-4]}\tilde{\psi}^{[d-4]}_b,
\]

where $\tilde{\psi}^{[d-3]}_0$ is the second-level gauge parameter associated with $\tilde{r}^{[d-3]}_2$. The gauge parameter $\xi^{a}_{d-4}$ decomposes into the following irreducible representations of the Lorentz group:

\[
Y[1] \otimes Y[d-4] \cong Y[d-3] \oplus Y[d-4, 1] \oplus Y[d-5].
\]

Only the $Y[d-3]$ component can be gauged away by the algebraic second order $\psi$-symmetry (43).

The remaining components of $\xi^{a}_{d-4}$ transform $e^{a}_{d-3}$ as in (34) and should be accompanied by the proper algebraic $\psi$-shifts $\psi(\xi) \sim d\xi$ in order to preserve the gauge $e^{a}_{d-3} = 0$. These compensating $\psi(\xi)$ transformations act on the new frame field $e^{a}_{d-3}$ because of the last term in (37), thus leading to differential gauge transformations containing divergences of the gauge parameter, and not only curls as for usual mixed-symmetry fields traceless on-shell.

Another difference between the second and the first dualizations is that the auxiliary field $r^{[d-3]}_2$ cannot be fully expressed in terms of $\omega^{[d-2]}_1$. Indeed, it is easy to see that some irreducible components of $r^{[d-3]}_2$ are annihilated in (38). So, the action (33) cannot be further simplified to the first-order form where only the frame-like field $e^{a}_{d-3}$ of double-dual gravity and the connection one-form $\omega^{a}_{1}$ would enter. Actually, it is possible to trade $\omega^{a}_{1}$ for the connection $e^{[d-2]}_3$ associated with the frame field $e^{a}_{d-3}$ in the approach of [17], by following the same Hodge dualization steps that we used in order to go from the frame fields $e^{a}_{d-3}$ to $r^{[d-3]}_2$.

The auxiliary field $r^{[d-3]}_2$ plays the role of a supplementary connection for $e^{a}_{d-3}$: Some components of it can be expressed in terms of one derivative of $e^{a}_{d-3}$ via a projection of

$^6$ The multiplicity rules for $GL(d)$ and $SO(m, n)$ representations are given, for example, in [47–49].
which annihilates $\omega^{[d-2]}_1$. Then, plugging the result in (40) gives second-order dynamical equations for $\xi^{[d-3]}_d$ additional to those given by (39).

Although the parent action and unfolding principles guarantee that the action (33) can indeed be used to describe linearized gravity by means of the double dual field $\tilde{e}^{[d-3]}_d$, it is instructive to show it more explicitly by a direct counting of degrees of freedom. First, one should exhibit the dynamical fields and differential gauge parameters of the unfolded system (38)–(41). Then, one can perform a counting of degrees of freedom as follows. (For simplicity we consider here the $d = 5$ case.)

The full set of fields and gauge parameters is

$$
e^a_2 \rightarrow \xi^a_1 \rightarrow \tilde{\xi}^a_0, \quad \omega^{[3]}_1 \rightarrow \lambda^{[3]}_0, \quad t^a_2 \rightarrow \psi^{[2]}_1 \rightarrow \bar{\psi}^{[2]}_0,$$

$$e^a_2 \rightarrow \tilde{\xi}^{[2]}_1 \rightarrow \bar{\xi}^{[2]}_0. \quad (44)$$

The dynamical field is given by $\tilde{e}^{[2]}_2$ modulo pure gauge shifts $\lambda^{[3]}_0$, see (37). As a result (we use Hodge dualizations in order to simplify the Young diagrams),

$$\text{dyn. field: } (Y^{[2]} \otimes Y^{[2]}) \oplus Y^{[3]} \quad (45)$$

By decomposing the zero-torsion-like equations (38) and (41) into irreducible Lorentz components, it is straightforward to see that the fields $t^a_2$ and $\omega^{[3]}_1$ are auxiliary, being fully expressed in terms of the derivative of $\tilde{e}^{[2]}_2$ and $e^a_2$, the latter being pure gauge. The first-order differential gauge parameters are given by $\tilde{\xi}^{[2]}_1$ plus $\xi^a_1$ modulo redundancy $\bar{\psi}^{[2]}_0$. Altogether it yields

$$\text{diff. 1st: } (Y^{[2]} \otimes Y^{[1]}) \oplus [(Y^{[1]} \otimes Y^{[1]}) \oplus Y^{[2]}] \quad (47)$$

$$\cong Y^{[2]} \oplus Y^{[1]} \oplus Y^{[2]} \oplus Y^{[1]} \oplus Y^{[1]} \oplus Y^{[0]}. \quad (48)$$

The second-order gauge parameters are $\xi^a_0$ and $\tilde{\xi}^a_0$ and decompose as follows:

$$\text{diff. 2nd: } Y^{[2]} \oplus Y^{[1]} \quad (49)$$

This reproduces the set of fields and gauge symmetries found in [38].

Let us stress that once the dynamical fields and differential gauge symmetries (45)–(48) are known, by taking advantage of the property of unfolding where all the gauge symmetries are manifest, it is then possible to recover unambiguously the complete set of frame-like fields (44) entering the unfolded formulation. In other words, knowing the fields and full set of gauge parameters entering the metric formulation in [38], one can build the spectrum (44).

This set of fields gives a hint how to construct the frame-like parent action (33) for the second dualization. Although the auxiliary field $r^a_2$ is not required for the unfolding of dual gravity, see [17], the pattern of auxiliary fields for the second dualization of linearized gravity can be directly generalized to the first dualization case, thereby providing the parent action (17).

By looking at the Weyl module entering equations (39), the principle of unfolding guarantees that the propagating degrees of freedom (contained in the zero-form module) precisely correspond to those of linearized gravity via the dual $C^{[0,\lambda]}_\theta^{[2,\lambda]}$ of the primary Weyl tensor $C^{[d-2,\lambda]}_\theta$ of dual gravity. It is nevertheless instructive to reproduce this counting using explicitly the $p$-form modules with $p > 0$ and the experience acquired from the Hamiltonian analysis of constrained systems [50]. The fields, first-order and second-order differential gauge
parameters contribute to the counting of degrees of freedom with the multiplicities 1, 2 and 3 respectively:

\[
\begin{align*}
SO(4, 1) : & \quad 1 \times (Y[2, 2] \oplus Y[2, 1] \oplus Y[1, 1] \oplus Y[1] \oplus Y[0]) \\
& \quad 2 \times (Y[2, 1] \oplus Y[2] \oplus Y[1, 1] \oplus Y[1] \oplus Y[0]) \\
& \quad 3 \times (Y[2] \oplus Y[1]).
\end{align*}
\]

Then, continuing with this heuristic procedure, one performs dimensional reduction of \(SO(4, 1)\) tensors to \(SO(4)\) and makes pairwise cancellations between two adjacent levels, thereby obtaining

\[
\begin{align*}
SO(4) : & \quad Y[2, 1] \oplus Y[1] \\
& \quad Y[2] \oplus Y[1].
\end{align*}
\]

A dimensional reduction to the Wigner little group \(SO(3)\) finally gives \(Y[2, 1]\), which is equivalent to the gravitational spin-\(Y[1, 1]\) field in \(d = 5\).

Let us now find the dynamical equations. As we have already mentioned, equation (41) implies that

\[
d\omega^{ab}_{d-3} + h^b_{(d-4)} h_b (\omega^{0})^{ab} + \alpha h^b_{(d-5)} h_b (\omega^{5})^{a\{b} = 0
\]

is a constraint in the sense that it expresses \(\omega^{d-2}_a\) or \(\tau^{d-3}_2\) in terms of derivatives of \(\dot{e}\). In \(d = 5\), (39) carries the \(Y[2] \oplus Y[3]\) representation of the Lorentz group. Its projection to the representation \(Y[3, 2]\) of the Lorentz group is a constraint because it expresses the primary Weyl tensor \(C\) in terms of the lower grade fields. This equations gives the gluing of the \(p\)-form module to the zero-form, Weyl module. In addition, the curvature \(R\) contributes to the Bianchi identity

\[
d\tau^{2}_{a b} + h^{0}_{(d-4)} h_b (s R)_{2}^{a \{b} \neq 0
\]

carrying the \(Y[4] \otimes Y[2]\) representation of the Lorentz group, which means that all the projections of \(R = 0\) to \(Y[4] \otimes Y[2]\) are consequences of other equations appearing at the lower grade. As a result, the dynamical part of \(R = 0\) is a projection to

\[
(Y[2] \otimes Y[3]) \otimes Y[3, 2] \oplus (Y[4] \otimes Y[2]) = Y[1, 1] \oplus Y[0]
\]

similarly to ordinary gravity.

As we discussed previously, equation (40) also imposes dynamical equations on \(\dot{e}\). It carries the \(Y[3] \otimes Y[2]\) tensor representation of the Lorentz group. The remaining Bianchi identity

\[
dT_a^b - h_{0(2)} R_2^{a\{b} \neq 0
\]

implies that projections of \(\tau = 0\) to \(Y[4] \otimes Y[1]\) are consequences of other equations that have already been taken into account. So, in addition it yields second-order dynamical equations on \(\dot{e}\), which take their values in the following representations of the Lorentz group:

\[
(Y[3] \otimes Y[2]) \oplus (Y[4] \otimes Y[1]) = Y[2, 2] \oplus Y[2, 1] \oplus Y[1].
\]

Equations (52) and (53) transform in the same representation as the field \(\dot{e}\), see equation (45), and give equations whose left-hand side contains the D’Alembertian of \(\dot{e}\) plus other second-order derivative terms that ensure gauge-invariance of the equations. This was to be expected from equations derived from an action.

Although the action (33) cannot be simplified in such a way as to remain frame-like and at the same time with all the fields of the dual-graviton sector eliminated, it is possible to formulate the action in a metric-like way in terms of the dynamical fields valued in the
representations (45). The independent derivation of this metric-like action was done explicitly in [38].

There are two special values for \( \alpha \). When \( \alpha = 0 \), the last term in (33) vanishes. It implies that the last term on the right-hand side of (37) vanishes, which means, in particular, that \( \tilde{e} \) loses its divergence-like \( \xi \)-symmetry that, as we explained, appears from the compensating mechanism between \( \xi_{d-4} \) and \( \psi_{d-3}^{\psi} \) needed for preserving the gauge condition \( e_{d-3}^{\tilde{e}} = 0 \). On the other hand, the last term on the right-hand side of (41) also disappears, which entails that (41) contains not only constraint pieces, but also a first-order, Proca-like, dynamical equation. As for the second special value \( \alpha = 3/2(-1)^d \), the projections of \( e_{d-3}^{\psi} \) that can be expressed in terms of \( \omega_{1}^{[d-2]} \) from (38) cancel the \( \omega \)-term, when plugged into (41). Let us note, however, that both these cases still propagate the same number of degrees of freedom as dual graviton, which is ensured by the construction and by inspection of the zero-form Weyl module that is left unchanged. We do not consider these two special cases in more details here.

4. Dualization of arbitrary massless fields

Let us now discuss the dualization of general massless mixed symmetry field described by the frame-like formulation. As explained in [18] to which we refer for precise explanations, the action for a massless spin-Y[1, 2, 3, ...] field is given by

\[
S = \langle d e + \frac{1}{2} \sigma \cdot \omega \rangle,
\]

where the frame-like field \( e \) is an \( h_1 \)-form, valued in \( Y^1 \) = \( Y[h_2, h_3, ...] \)-shaped traceless tensors, while the spin-connection-like field \( \omega \) is an \( h_2 \)-form, valued in \( Y^2 \) = \( Y[h_1 + 1, h_3, ...] \)-shaped traceless tensors.

As can be noted from the gravity dualization examples, there are two different types of dualizations. The first and second dualizations of gravity are of the first and second types, respectively. The first type of dualizations entails Hodge conjugation with the antisymmetric rank-(d - 2) tensor in the first column of the Young diagram representing the spin of the particle. This operation maps allowed Young diagram to the allowed ones on-shell. Indeed, for the allowed Young diagram \( h_1 + h_2 \leq d - 2 \) and \( h_1 \geq h_2 \): after dualizations of the first column, one obtains a Young diagram with the first column of height \( h'_1 = d - 2 - h_1 \), while the heights of other columns remain unchanged. It is easy to see that

\[
\begin{align*}
\begin{array}{c}
h_1 & \geq h_2 \\
\Leftrightarrow & \\
\Leftrightarrow & \\
& h_1 + h_2 \leq d - 2 \Rightarrow h'_1 \geq h_2.
\end{array}
\end{align*}
\]

So, the first column, after dualizations, remains the highest one, and moreover the dual Young diagram is also allowed.

In order to perform the first column off-shell dualizations, we add a torsion-like field \( t \) being an \( h_1 + 1 \) differential form, valued in \( Y^1 \)-shaped traceless tensors. It is chosen such that its gauge symmetries of different levels can be used to gauge away the original frame-like field \( e \) together with all its gauge symmetries. We also add the dual frame-like field \( \tilde{e} \), valued in the same space \( Y^1 \) as \( t \) but carrying a differential form degree \( d - h_1 - 2 \). The parent action is given by

\[
S = \langle d e + \frac{1}{2} \sigma \cdot \omega + t \rangle + \int t \cdot d\tilde{e},
\]

where ‘\( \cdot \)’ implies that all the fibre indices are contracted and of course, as always in this work, only the wedge product is used for multiplication of differential forms. The field \( \tilde{e} \) can be treated as a Lagrange multiplier for the constraint \( dt = 0 \). It can be solved in the form
In order to construct the dual action, one should gauge away the original frame-like field $e$ using the gauge parameters of $t$. Then, $t$ can be completely expressed in terms of $\omega$. Plugging this back into the action we end up with an action formulated in terms of the dual frame-like field $\tilde{e}$ and the connection-like field $\omega$. To recast this dual action into the usual first-order form (54), one should define the dual connection-like field $\tilde{\omega}$ as the Hodge dual in the first column, taken with respect to the fibre epsilon symbol with $d$ indices, of the original connection $\omega$. So, $\tilde{\omega}$ is an $h_2$-form transforming in the traceless $Y[d - h_1 - 1, h_3, \ldots]$-representation of the Lorentz algebra. The form degrees and fibre space types of $\tilde{e}$ and $\tilde{\omega}$ are exactly those required in order to describe a spin-$Y[d - h_1, h_2, h_3, \ldots]$ particle in the first-order formulation [18]. The parent action (55) then reduces to (1) for a spin-$Y[d - h_1, h_2, h_3, \ldots]$ particle.

The second type of dualization is a dualization which on-shell Hodge dualizes any column—including an empty column, which can formally be added to the right of the Young diagram—of the Young diagram except for the first one. Suppose one is going to Hodge dualize the $i$th column of the traceless tensor of allowed form $Y[h_1, h_2, \ldots, h_i, \ldots]$, so that one has $h_1 + h_i \leq d - 2$. After dualization, the $i$th column gives rise to the column with $h_i' = d - 2 - h_i$ boxes of the dual Young diagram. Then

$$h_1 + h_i \leq d - 2 \iff h_1 \leq h_i'.$$

So the column with height $h_i'$ appears to be the highest one in the dual Young diagram and therefore should be reshuffled to the first place on the left, such that the heights of the dual Young diagram columns do not increase. Moreover

$$h_1 \geq h_i \iff h_1 + h_i' \geq d - 2,$$

which implies that the dual Young diagram is not allowed. It means in turn that the dual tensor of $so(d - 2)$ is not traceless, so to describe the dual theories of this kind one needs an action of a form different from (1).

In order to perform such a dualization at the level of the action, we will generalize in a straightforward way the procedure followed in the previous section for the double-dual spin-2 field. We first introduce an auxiliary field $t$, being an $h_1 + 1$ form valued in traceless $so(d - 1, 1)$ tensors of shape $Y' = Y[h_1, \ldots, h_{i-1}, h_{i+1} \ldots]$, which can be obtained from the Young diagram characterizing the spin of the particle by cutting off the $i$th column. We also introduce a dual frame-like field $\tilde{e}$ such that $t \cdot d\tilde{e}$ is a $d$-form scalar. In other words, $\tilde{e}$ is a $d - h_1 - 2$ form valued in $Y'$. The parent action is

$$S = \langle dt + \frac{1}{2} \sigma_{\omega} + \Sigma_{-t} | \omega \rangle + \int (t \cdot d\tilde{e} + \alpha t^2),$$

where $\Sigma_\omega$ is an operator constructed from the product of $h_1 - h_i$ background vielbeins and mapping $Y'$-shaped traceless tensors to $Y$-shaped traceless tensors in fibre, which defines it uniquely, while $\alpha t^2$ represents all possible contractions, each coming with an arbitrary coefficient, of two $t$ fields by means of $d - 2h_i - 2$ background vielbeins and the $so(d - 1, 1)$ epsilon tensor. Such contractions always exist for $h_1 \geq h_i + 1$.

Performing the same manipulations as before one can show that (56) is equivalent to (54).

To pass to the dual formulation we should gauge away the dynamical field of the original theory (54). In the examples considered in section 3, the gauge parameter $\psi$ associated with the field $t$ was used to completely gauge away the frame-like field $e$ of the original theory. However, it is not possible in the general case. Indeed, taking into account both base and fibre space indices, $e$ and $\psi$ carry the following tensor representations of the Lorentz group:

$$e : \ Y[h_1] \otimes Y[h_2, h_3 \ldots],$$

$$\psi : \ Y[h_i] \otimes Y[h_1, \ldots, h_{i-1}, h_{i+1} \ldots].$$

$$t = \partial \beta, \text{ implying that } t \text{ can be set to zero by gauge fixing, which shows that the action (55) is equivalent to (54).}$$
In general, these representations are different, so $\psi$ cannot be used to gauge away $e$ completely. Indeed, on the one hand we know from the frame-like formulation—from which we borrow all the gauge symmetries for $e$—that the action can be shown to contain only the component of $e$ transforming as a double-traceless $Y$ representation, where the double trace is taken with respect to four indices sitting in the same row. This is $\varphi^Y$, the Labastida metric-like gauge field off-shell. There are enough algebraic gauge symmetries for the frame field $e$ to reach that gauge.

On the other hand, it is clear that the parameter $\psi$ contains an irreducible $gl(d)$ representation with shape $Y$, but in general with more trace constraints than the one characterizing the Labastida field $\varphi^Y$, meaning that $\psi$ possesses fewer components than the Labastida field and hence cannot be used to completely gauge away the $e$ field inside the action. This can be done, however, by resorting to the remaining differential gauge invariance of the Labastida field $\varphi^Y$ and reaching the gauge where $\varphi^Y$ becomes traceless, $\hat{\varphi}^Y$, at the expense of leaving an action invariant under transverse gauge parameters, see [51] for the totally symmetric spin-$s$ cases. Then, at that stage, the gauge parameter $\psi$ can be used in order to completely gauge away the resulting traceless field $\hat{\varphi}^Y$.

The fact that the elimination of the original field from the parent action requires the use of differential gauge symmetries may not seem elegant. To overcome this difficulty, one may introduce a set of auxiliary fields $t$ (instead of a single $t$-field), such that the gauge symmetries associated with them can be used to eliminate the original frame field just by algebraic gauge shifts. We leave this issue for further investigations.

5. Conclusions

In this paper, we performed an off-shell Hodge dualization for massless mixed-symmetry fields in the Minkowski space of arbitrary dimension $d$. The dual fields are related on-shell by $so(d - 2)$ Hodge conjugation on a group of indices associated with one column of the Young diagram describing the generalized spin of the initial field. We built the dual actions by introducing a parent action which, depending on the way one fixes gauges and eliminates fields by equations of motion, reduces to either the initial standard action [18] or to the new, dual theory. The parent action procedure guarantees that both theories propagate the same number of degrees of freedom.

The frame-like approach has the advantage that it allows us to promote the field equations to their unfolded formulation, and the latter formulation requires the introduction of auxiliary fields that are precisely those needed in order to build a frame-like parent action. The parent actions built within the frame-like approach are considerably simpler compared to their metric-like counterpart. The frame-like action also makes the gauge symmetries manifest.

As far as the counting of physical degrees of freedom is concerned, another great advantage of the unfolded formulation is brought by the Weyl module representation which appears in the unfolded equations. This representation contains an infinite set of zero-forms that precisely carry the propagating degrees of freedom and therefore makes their counting straightforward, avoiding all the gauge-fixing difficulties.

We start from the standard first-order frame-like action [18], which on-shell describes irreducible tensors of $so(d - 2)$ characterized by some Young diagram $Y$. Performing the first-column off-shell dualization of such a theory in the way we proposed produces a dual theory which, on-shell, also gives an $so(d - 2)$ traceless tensor characterized by a Young diagram $\tilde{Y}$ related to $Y$ by $so(d - 2)$ Hodge dualization in the first column. The dual action thereby obtained is the standard frame-like action [18] for the dual field.
On the other hand, dualizing the initial action along the lines that we proposed on a column of $Y$ which is not the first one, we obtained a dual theory which describes, on-shell, a dual $gl(d-2)$-irreducible field which turns out to be proportional to the metric tensor of $so(d-2)$. We call such an on-shell field ‘pure-trace’. The corresponding $gl(d)$-covariant field equations can be expressed in terms of higher traces of the generalized curvature tensor, and not via a single trace as is the case [11, 12] for an on-shell gauge field which is not proportional to the $so(d-2)$ metric tensor in the light-cone gauge.

Such ‘higher trace’ theories had been studied on-shell in [45, 33, 20], but so far, no $gl(d)$-invariant off-shell formulations have been found. This work together with [38] fills this gap. Indeed, the action [18] is not suitable for such theories as, for the corresponding field of the dual types we considered, it does not propagate any pure-trace fields on-shell.

The frame-like, dual actions producing a pure-trace field on-shell contain two extra fields $(e, t)$ on top of the fields $(\tilde{e}, \tilde{\omega})$ that one could expect to arise in a first-order approach. We constructed and analysed in detail the frame-like action for the double-dual graviton, as this case already contains all the features of the general, mixed-symmetry case. The double-dual graviton in the $d$-dimensional Minkowski spacetime is given on-shell by a $Y[d-3, d-3]$-shaped tensor being $(d-4)$-fold pure-trace. In particular, it is shown that such theories admit exotic differential symmetries containing divergences of gauge parameters.

In the dualization procedure considered in this paper, we replaced the lowest grade frame-like field $e$ by the dual frame-like field $\tilde{e}$, but the first connection $\omega$ together with all the other higher grade fields of the unfolded approach [17] remain the same. It would be interesting to develop dualization schemes which involve non-trivial dualizations of some higher grade fields.

One can consider the results obtained in this paper as a continuation of the programme consisting in building covariant actions for all the possible irreducible particles propagating freely in flat spacetime, in all the possible dual representations. From this point of view, it would be interesting to generalize the results of the paper to the multiple dualization case. In general, one can study representations given by the arbitrary irreducible trace constraints, not necessarily obtained by the Hodge conjugation of $so(d-2)$-traceless on-shell tensors. One could also make the theory nonlinear, along the lines of [52, 53] in the spin-1 case, and [54] for the dual spin-2 case.

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Appendix

We deal with the $d$-dimensional Minkowski space parametrized by coordinates $x^\mu$. The differential form indices are denoted by Greek letters $\mu, \nu, \ldots$. At each point of the spacetime, the vielbein $h^\mu_a(x)$ defines a local free falling basis with a flat metric $\eta^{ab} = \text{diag}(1, -1, \ldots, -1)$, which is an invariant tensor of $so(d-1, 1)$. The tensor indices in this basis are denoted by Latin letters from the beginning of the alphabet $a, b, \ldots$ and often referred to as fibre indices. Choosing Cartesian coordinates $h^\mu_a = \delta^\mu_a$ one identifies base and fibre indices. We also use Latin letters from the middle of the alphabet $m, n, \ldots$ to denote tensor indices of the Wigner little group.
Differential form degree is often indicated as a lower index written in bold. The square bracket with the indices placed inside implies antisymmetrization of respective indices, while the round bracket denotes symmetrization. Both operations are supplied with overall factors making them projectors, e. g. $A^{[ab]} B^{[cd]} = 1/2 (A^{ab} B^{cd} + A^{cd} B^{ab})$. For a group of $n$ (anti)symmetric indices, we often use notation $(a[n]) a(n)$. We will also use the convention whereby tensors whose Lorentz indices are denoted by the same Latin letter are implicitly (anti)symmetrized on these indices. For example, $A^{aB} = A^{aB}$.

It will be clear from the context whether one is working with the manifestly symmetric or antisymmetric convention. To indicate that a tensor possesses a symmetry of a Young diagram written in a (anti)symmetric basis, we separate groups of (anti)symmetric indices by commas, e.g. $Y[2, 2, 1]$-shaped tensor can be written either as $T^{a[2],b[2],c}$ in antisymmetric basis or as $T^{a(3),b(2)}$ in the symmetric basis. If tensor indices are divided into groups and a tensor does not possess any symmetries with respect to permutations of indices between the groups, then these groups of indices are separated by vertical lines. For example, the differential form index of $V^n$ can be transformed to the fibre one, which yields the second rank tensor $V^{[m]a}$ with indefinite symmetry with respect to permutation of indices.

We will also use notations

$$h^{[k]} = h^a \cdots h^k \text{ and } H_{[k]} = e_{a[k]} h^{a[d-k]} h^{[d-k]},$$

where $e$ is a totally antisymmetric rank $d$ tensor.

We also denote

$$e_{[a[k]n[d-k]} = e_{[a[m] \cdots e_{[a[n] \cdots e_{[a[k]n[d-k]} = \det(\eta),}

then

$$e_{a[k]n[d-k]} e_{a[k]n[d-k]} = \sigma k!(d - k)! e_{[a[k]n[d-k]}$$

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