I. INTRODUCTION

Different fields of relativistic plasmas are under current development due to the description of various astrophysical scenarios and generation of relativistic plasmas in laboratories at the propagation of high-energy density electromagnetic beams [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14].

The degenerate plasmas can be placed in the middle between the classical and quantum plasmas. It can be considered as the quasi-classical object described by the classic hydrodynamic or kinetics, where the equation of state is chosen as the Fermi pressure or the equilibrium distribution function is chosen as the Fermi step (the zero-temperature limit of the Fermi-Dirac distribution function). However, the Fermi pressure and the Fermi step distribution function are caused by the Pauli blocking, which is the quantum effect. But, other quantum effects like the quantum Bohm potential are secondary in the degenerate plasmas for the majority of physical scenarios.

In this paper we focus on relativistic plasmas. Relativistic regime can be reached within different scenarios. Monoenergetic beams of electrons can propagate through the plasmas, with the velocity of beam close to the speed of light. There is the opposite limit of the macroscopically motionless plasmas heated up to temperatures of order of the rest energy of at least lightest of species of plasmas (here and below we consider temperature in the energy units). This regime is rather more complex in compare with monoenergetic beams since it requires to get relation between the momentum density and the velocity field for the large number of particles moving with different velocities (if we use the Euler equation in the form of the momentum balance equation). The degenerate relativistic plasmas is the low-temperature high-density plasmas, where the Fermi velocity is of order of the speed of light. So, we have distribution of particles over quantum states with different energies and we have same complexity of the model like for the thermally distributed plasmas.

Novel model for the description of the relativistically hot plasmas is developed in Refs. [15], [16], [17], [18], [19], [20]. This model is based on the derivation of hydrodynamic equations directly from their microscopic motion described by the corresponding relativistic form of the second Newton’s law as the equations of motion of each particle. This method can be considered as the reformulation of the classical mechanics in the form of equations of field since no probabilistic method is used during derivation. Originally this method is suggested for the derivation of the kinetic Vlasov equation for the relativistic plasmas [21]. Next, its simplification is presented for the hydrodynamic modeling of the nonrelativistic plasmas [22], [23]. Finally, an original structure of the hydrodynamic model for the relativistic plasmas is derived using this method [15], [16], [17], [18], [19], [20].

The model itself consist of equations for evolution of four functions, two three-scalars and two three-vectors, which can be combined in two four-vectors. These functions are the concentration of particles, the velocity field obtained via the evolution of the current of particles, the average reverse relativistic gamma-factor, and the current of the average reverse relativistic gamma-factor. Basically, the average reverse relativistic gamma-factor is the arithmetic average of \( \sqrt{1 - \frac{v_i(t)^2}{c^2}} \) over all particles, where \( v_i(t) \) is the velocity of \( i \)-th particle as the function of time \( t \) in accordance with the microscopic equation of motion, \( c \) is the speed of light.

A question might appear: Why do we use so strange functions like "the average reverse relativistic gamma-factor” in our model? The answer is following. We do not try to choose functions for the description of plasmas, but we follow the structure of hydrodynamic equations as it appears during derivation. The evolution of the concentration \( n \) leads to the current of particles \( j \).
Therefore, we consider the evolution of the current of particles \( j \), which has no direct relation to the momentum density in contrast with the nonrelativistic regime, where these functions coincide. The current of particles \( j \) allows us to introduce the velocity field \( v = j/n \). Hence, the equation for evolution of the current of particles \( j \) gives the equation for evolution of the velocity field \( v \). This equation contains four novel functions. One is the second rank tensor describing the flux of current of particles. Three other functions appear at the presentation of interaction in the self-consistent (mean-field) approximation. Two of them are mentioned above: the average reverse relativistic gamma-factor, and the current of the average reverse relativistic gamma-factor. Another one is the second rank tensor of flux of the current of the average reverse relativistic gamma-factor. Consequently, we use the average reverse relativistic gamma-factor, and the current of the average reverse relativistic gamma-factor in order to continue the set of hydrodynamic equations and find appropriate regime of truncation. The derivation shows that the evolution of the average reverse relativistic gamma-factor, and the current of the average reverse relativistic gamma-factor mostly leads to reappearance of the concentration and the current of particles. Hence, on this stage the mathematical structure of the model shows some tendency to close itself relatively presented set of functions. Obviously, there is no complete closer of the set of equation with no additional truncation due to the large number of degrees of freedom in the many-particle systems.

Truncation is made via applications of equations of state for the second and higher rank tensors. In this paper we consider the degenerate electron gas. Therefore, we derive corresponding equations of state for the zero temperature regime describe by the Fermi step distribution function.

This paper is organized as follows. In Sec. II the relativistic hydrodynamic model is adopted for the degenerate electron gas. In Sec. III the linear and nonlinear analysis of the small amplitude longitudinal waves is given. The accuracy of the suggested model is also analyzed via comparison of the Langmuir wave spectrum with the result obtain in the kinetic model. The ion-acoustic solitons are considered by the reductive perturbation method. In Sec. IV a brief summary of obtained results is presented.

II. RELATIVISTIC HYDRODYNAMIC MODEL

Quasiclassic analysis of the degenerate electron gas can be based on the classical hydrodynamic model, where the equations of state are obtained within the distribution function in form of the Fermi step (the zero temperature limit of the Fermi-Dirac distribution function).

The nonrelativistic classic hydrodynamics obtained in the selfconsistent field approximation requires some equation of state for the pressure \( P^{ab} \). The relativistic hydrodynamic model based on the momentum balance equation requires two equations of state. One equation of state is for the pressure. The second equation of state is necessary for the momentum density in order to make transition to the velocity field which exists in the continuity and Maxwell equations. The relativistic hydrodynamic model with the average reverse gamma factor evolution considered in this paper includes evolution of the concentration, the velocity field, the average reverse relativistic gamma factor, and the flux of the reverse relativistic gamma factor. Therefore, it requires three equations of state. Two equations of state are for the second rank tensors describing the flux of the current of particles \( p^{ab} \) and the current of flux of the reverse relativistic gamma factor. One equation of state is for the fourth rank tensor \( M^{abcd} \), which is the flux of the current of tensor \( p^{ab} \).

Here we follow Refs. [15], [19], and adopt the model for the relativistic degenerate plasmas. Hence, we have the Fermi velocity \( v_F \) comparable with the speed of light \( c \).

First equation in the presented model is the continuity equation [13]

\[
\partial_t n + \nabla \cdot (nv) = 0.
\]

Next, the velocity field evolution equation is [15], [19]

\[
n\partial_t v^\alpha + n(v \cdot \nabla) u^\alpha + \frac{1}{m} \partial^\beta \bar{p}^\beta = \frac{e}{m} \left( \Gamma - \frac{\bar{t}}{c^2} \right) E^a + \frac{e}{mc^2} \varepsilon^{abc} (\Gamma v_b + t_b) B_c - \frac{e}{mc^2} (\Gamma v^a v^b + v^a t^b + v^b t^a) E_b,
\]

where tensor \( p^{ab} = \bar{p} \delta^{ab} \) is the flux of the thermal velocities, and tensor \( v^{ab} = \bar{t} \delta^{ab} \) is the flux of the average reverse gamma-factor. Parameter \( \bar{p} \) is used here instead of \( p \) presented in earlier papers [15] in order to distinguish it from the momentum. Parameters \( m \) and \( e \) are the mass and charge of particle, \( c \) is the speed of light, \( \delta^{ab} \) is the three-dimensional Kronecker symbol, \( \varepsilon^{abc} \) is the three-dimensional Levi-Civita symbol. In equation (2) and below we assume the summation on the repeating index \( v^b_b E_b = \sum_{b=x,y,z} v^b_b E_b \). Moreover, the metric tensor has diagonal form corresponding to the Minkowski space, it has the following sings \( g^{\alpha\beta} = \{-1, +1, +1, +1\} \). Hence, we can change covariant and contravariant indexes for the three-vector indexes: \( v^b_b = v_{b,s} \). The Latin indexes like \( a, b, c \) etc describe the three-vectors, while the Greek indexes are deposited for the four-vector notations. The Latin indexes can refer to the species \( s = e \) for electrons or \( s = i \) for ions. The Latin indexes can refer to the number of particle \( j \) at the microscopic description. However, the indexes related to coordinates are chosen from the beginning of the alphabet, while other indexes are chosen in accordance with their physical meaning.
The equation of evolution of the averaged reverse relativistic gamma factor includes the action of the electric field
\[
\partial_t \Gamma + \partial_b (\Gamma v^b + t^b) = - \frac{e}{mc^2} n (\mathbf{v} \cdot \mathbf{E}) \left( 1 - \frac{1}{c^2} \left( \mathbf{v}^2 + \frac{5\tilde{p}}{n} \right) \right). \tag{3}
\]
Function $\Gamma$ is also called the hydrodynamic Gamma function \[15\].

The fourth and final equation in this set of hydrodynamic equations is the equation of evolution for the temporal part of current of the reverse relativistic gamma factor (the hydrodynamic Theta function):

\[
(\partial_t + \mathbf{v} \cdot \nabla) t^a + \partial^a \mathbf{v} + (\mathbf{t} \cdot \nabla) v^a + \Gamma(\partial_t + \mathbf{v} \cdot \nabla) v^a
\]

\[
= \frac{e}{m} E^a \left[ 1 - \frac{\mathbf{v}^2}{c^2} - \frac{3\tilde{p}}{nc^2} \right] + \frac{e}{mc} \epsilon^{abc} n v_b B_c \left[ 1 - \frac{\mathbf{v}^2}{c^2} - \frac{5\tilde{p}}{nc^2} \right]
\]

\[- \frac{2e}{mc^2} \left[ E^a \tilde{p} \left( 1 - \frac{\mathbf{v}^2}{c^2} \right) + n v^a v^b E_b \left( 1 - \frac{\mathbf{v}^2}{c^2} - \frac{9\tilde{p}}{3nc^2} \right) - \frac{5M_0}{3c^2} E^a \right]. \tag{4}
\]

All hydrodynamic equations are obtained in the mean-field approximation (the self-consistent field approximation). The fourth rank tensor $M^{abc}d$ entering the equation for evolution of the reverse gamma factor via its partial trace $M^{abcc} = M^{cabc}$. In the isotropic limit, we construct tensor $M^{abc}$ of the Kronecker symbols $M^{abc} = (M_0/3)(\delta^{a}bc + \delta^{ac}b + \delta^{ad}c)$. It gives $M^{xxxx} = M^{yyyy} = M^{zzzz} = M_0$. If we have two pairs of different projections we obtain $M^{xyyz} = M^{xyzy} = M_0/3$. Otherwise the element of tensor $M^{abc}$ is equal to zero. So, for the partial trace $M^{abc}$ we find $M^{abc} = (5M_0/3)\delta^{ab}$. For the explicit definition of tensor $M^{abc}$ see equation (17) of Ref. [15].

The equations of electromagnetic field have the traditional form presented in the three-dimensional notations

\[
\nabla \cdot \mathbf{E} = 0, \tag{5}
\]

\[
\nabla \times \mathbf{E} = - \frac{1}{c} \partial_t \mathbf{B}, \tag{6}
\]

\[
\nabla \cdot \mathbf{E} = 4\pi (en_i - en_e), \tag{7}
\]

and

\[
\nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} + \frac{4\pi q_e}{c} n_v \mathbf{v}, \tag{8}
\]

where the ions exist as the motionless background.

### A. Equations of state

Equations (1)-(4) originally obtained for the relativistically hot plasmas. One of its features is the large temperature, so the effective thermal velocities are close to the speed of light. Let us mention that the zero temperature limit of these equations can also be considered to study the propagation of the monoenergetic beams \[19\]. However, the opposite limit is under consideration in this paper. As it is mentioned above we are going to make a substitution in order to consider the degenerate electron gas of high concentration, so the Fermi velocity $v_{Fe} = p_{Fe}/\sqrt{1 + p_{Fe}^2/m^2c^2}$, where $p_{Fe} = (3\pi^2 n)^{1/3}h$.

In order to give a quasi-classic analysis of the high density relativistic degenerate electron gas (or all species of plasmas) we need to find corresponding equations of state for functions $p^{ab}$, $t^{ab}$, and $M^{abcd}$.

Degenerate electrons are described within the zero-temperature limit of the Fermi-Dirac distribution, which is given by the Fermi step distribution

\[
f_0 = \begin{cases} \frac{2}{(2\pi^2)^{3/2}} & \text{for } p \leq p_{Fe} \\ 0 & \text{for } p > p_{Fe} \end{cases}. \tag{9}
\]

Before we consider novel functions $p^{ab}$, $t^{ab}$, and $M^{abcd}$ we present the equation of state for the pressure as a point of reference. The concentration has well-known form in terms of the distribution function

\[
n = \int f_0 d^3 p. \tag{10}
\]

The pressure (the flux of the momentum density) can be written in the following forms:

\[
P^{ab} = \int p^{a} v^{b} f_0 d^3 p = c \int p v^a v^b f_0 d^3 p/p_0, \tag{11}
\]

where $p_0 = \gamma mc = mc/\sqrt{1 - v^2/c^2}$, $p = mv/\sqrt{1 - v^2/c^2}$, $\gamma = \sqrt{1 - v^2/c^2}$, and the second expression is shown in more symmetric form including the covariant element of volume in the momentum space $d^3p/p_0$. Calculation leads to $P^{ab} = P\delta^{ab}$, with

\[
P = \frac{m^4 c^5}{24\pi^2 h^3} \left[ \xi \sqrt{\xi^2 + 1}(2\xi^2 - 3) + 3\text{arsinh}\xi \right], \tag{12}
\]

where $\xi \equiv p_{Fe}/mc$, $\text{arsinh}\xi = \ln |\xi + \sqrt{\xi^2 + 1}|$, and $\text{sinh}(\text{arsinh}\xi) = \xi$.

At this step we are ready to go further and calculate expressions for the novel functions. The first of the is the flux of the current of particles, which has the following representation in form of the distribution function

\[
p^{ab} = \int v^{a} v^{b} f_0 d^3 p. \tag{13}
\]

Next, we obtain the equation of state $p^{ab} = \tilde{p}\delta^{ab}$ with

\[
\tilde{p} = \frac{m^3 c^5}{3\pi^2 h^3} \left[ \frac{1}{3} \xi^3 - \xi + \text{arctan}\xi \right], \tag{14}
\]
where \( m^3 c^5 / 3 \pi^2 \hbar^3 = n_0 c^2 (m^3 c^3 / p_{F_i}^3) \).

The second of required functions is the flux of the current of the average reverse gamma factor

\[
t^{ab} = \int \left( \frac{v^{a} v^{b}}{\gamma} \right) f_0 d^3 p.
\]  

(15)

Our calculation leads to \( t^{ab} = \tilde{\delta}^{ab} \) with

\[
\tilde{\delta} = \frac{m^3 c^5}{6 \pi^2 \hbar^3} \left[ \xi \sqrt{\xi^2 + 1} + \frac{2 \xi}{\sqrt{\xi^2 + 1}} - 3 \text{Arsh} \xi \right].
\]  

(16)

The fourth rank tensor should be also calculated using the Fermi step

\[
M^{abcd} = \int v^a v^b v^c v^d f_0 d^3 p.
\]  

(17)

It leads to the symmetric expression \( M^{ab} = (M_0 / 3) (\delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}) \) with

\[
M_0 = \frac{m^3 c^5}{30 \pi^2 \hbar^3} \left[ 2 \xi (\xi^2 - 1) - \frac{3 \xi}{\xi^2 + 1} + 15 \text{arctan} \xi \right].
\]  

(18)

We also need to find the equilibrium expression for the average reverse gamma factor \( \Gamma_0 \) for the degenerate electron gas

\[
\Gamma = \int \frac{1}{\gamma} f_0 d^3 p.
\]  

(19)

After calculation for the degenerate electron gas we obtain

\[
\Gamma = \frac{m^3 c^5}{2 \pi^2 \hbar^3} \left[ \xi \sqrt{\xi^2 + 1} - \text{Arsh} \xi \right].
\]  

(20)

In this model the relativistic Gamma function \( \Gamma \) is an independent function. Its evolution is described by equation (3). Equation (20) is used as the equation of state for the equilibrium value of the relativistic Gamma function \( \Gamma \).

III. WAVES IN THE RELATIVISTIC MAGNETIZED PLASMAS

A. Equilibrium state and the linearized hydrodynamic equations

We focus on degenerate electron-ion plasmas, where both components are degenerate. Moreover, the concentration of both components \( n_{e0} = n_{i0} \) is high, up to values giving large Fermi velocities \( v_{Fe} \) and \( v_{Fi} \) getting close to the speed of light \( c \).

We consider small perturbations of the equilibrium state while the equilibrium state is characterized by the constant concentrations of electrons and ions \( n_{e0}, n_{i0} \), constant values of the average reverse relativistic gamma factors \( \Gamma_{e0}, \Gamma_{i0} \), zero values of the equilibrium velocity fields of both species, zero values of the current of average reverse relativistic gamma factors, and zero values of the electric and magnetic fields. So, the hydrodynamic functions are \( n_s = n_{e0} + \delta n_e, v_{xs} = \delta v_{xs}, \quad \Gamma_s = \Gamma_{e0} + \delta \Gamma_e, \quad t_{xs} = \delta t_{xs}, \quad E_s = \delta E_x, \) perturbations of the magnetic field are not considered since we consider the longitudinal waves. It is also assumed that the perturbations have the monochromatic form, for instance \( \delta n_s = N_s e^{-i \omega t + ikz} \), where \( N_s \) is the amplitude.

The linearized continuity equation has well-known form for the macroscopically motionless fluids (no equilibrium velocity field)

\[
\partial_t \delta n_s + n_{e0} \partial_x \delta v_{xs} = 0.
\]  

(21)

The second equation appears from equation (2) in the following form

\[
n_{e0} \partial_t v_{xs} + \frac{\delta \tilde{p}_{e0}}{\delta n_{e0}} \partial_x n_s = \frac{q_s}{m_s} \Gamma_{e0} \delta E_x - \frac{q_s}{m_s c^2} \tilde{\ell}_0 \delta E_x,
\]  

(22)

where parameters \( \delta \tilde{p}_{e0} / \delta n_{e0} \) and \( \tilde{\ell}_0 \) appear from equations of state presented above.

We obtain the linearized equations for \( \delta \Gamma \) and \( \delta t_{xs} \) from equations (3) and (4) in the following form

\[
\partial_t \delta \Gamma_s + \Gamma_{e0} \partial_x \delta v_{xs} + \partial_x \delta t_{xs} = 0,
\]  

(23)

and

\[
\partial_t \delta t_{xs} + \partial_x \delta \tilde{\ell}_s = \frac{\Gamma_{e0} \partial_x \delta \tilde{p}_s + q_s \Gamma_{e0}^3 \delta E_x}{m_s n_{e0}} + \frac{q_s}{m_s} \frac{5 q_s}{3 m_s c^2} \tilde{\ell}_0 \delta E_x + \frac{10 q_s}{3 m_s c} M_0 \delta E_x,
\]  

(24)

where \( M_0^{exc} = (5/3) M_{0s} \).

The linearized Poisson equation has the well-known form

\[
\partial_x \delta E_x = 4 \pi (q_s \delta n_e + q_i \delta n_i).
\]  

(25)

Analysis of the set of linearized equations obtained at zero external fields shows that it is enough to use equations (21), (22), (25) to get a closed set of equations for the longitudinal perturbations.

The linearized hydrodynamic equations (21)-(26) contain \( \Gamma_0 \) and \( \tilde{\ell}_0 \), but function \( \tilde{p} \) enters these equations in two forms. It appears as the equilibrium value of this function on the right-hand side of equation (21). However, the Euler equation (22) contains the perturbation of function \( \tilde{p} \): \( \delta \tilde{p}_s = (\delta p / \delta n) \delta n_s \), so \( u_{ps}^2 \equiv (\delta p_s / \delta n_s) \). Therefore, we present the expression for \( \delta p_s / \delta n_s \) obtained from equation (14)

\[
\delta \tilde{p} / \delta n = \frac{1}{3} \bar{\gamma} \frac{\xi^2}{\xi^2 + 1}.
\]  

(26)

It gives \( v_{Fe}^2 / 3 \) in the nonrelativistic limit. While the ultrarelativistic limit leads to \( c^2 / 3 \). Using the relativistic expression for the Fermi velocity we find that \( \delta \tilde{p} / \delta n = v_{Fe}^2 / 3 \) in the ambient relativistic regimes.
B. Spectrum of the Langmuir waves:
Hydrodynamic description

To give a simple illustration of the relativistic effects existing in the presented model we consider one of fundamental wave effects in plasmas – the Langmuir wave spectrum. The relativistic Langmuir waves are considered in Ref. [12] for the thermally distributed electrons with the relativistic temperatures. Here we consider this problem for the degenerate electrons. Equations (21)-(28) allow to find the following spectrum

$$\omega^2 = \left( \frac{\Gamma_0}{n_0} - \frac{u_i^2}{c^2} \right) \omega_{Le}^2 + u_p^2 k_z^2,$$

(27)

where parameters $\Gamma_0$, $u_i^2$, and $u_p^2$ should be obtained from equations (14)-(20). However, they have different appearance. As it is mentioned above $u_i^2$ appears from the perturbation of pressure (14) $p = p_0 + \delta p$ with $\delta p = u_i^2 \delta n$. But parameter $u_p^2$ appears from the equilibrium value of function $\delta u^2 \equiv E_0/n_0$. Necessary substitution leads to the following result

$$\omega^2 = \frac{\omega_{Le}^2}{\gamma_{Fe}} + \frac{1}{3} \frac{m^2 c^2}{p_{Fe}} k_z^2,$$

(28)

where $\gamma_{Fe} = 1/\sqrt{1 - v_{Fe}^2/c^2} = \sqrt{1 + p_{Fe}^2/m^2 c^2}$ is the standard relativistic gamma factor considered for the Fermi velocity. Expression (28) can be represented via the Fermi velocity as follows $\omega^2 = \frac{\omega_{Le}^2}{\gamma_{Fe}} + \frac{1}{3} \frac{m^2 c^2}{p_{Fe}} k_z^2$.

C. Spectrum of the Langmuir waves in the relativistic kinetics for the degenerate electrons

It would be useful to give an analysis of accuracy of the hydrodynamic model presented within equations (11)-(14). To this end we compare the spectrum of the Langmuir waves obtained above (28) with the result of the relativistic kinetics. Therefore, we present the Vlasov kinetic equation

$$\partial_t f_e + \mathbf{v} \cdot \nabla f_e + q_e \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right) \cdot \frac{\partial f_e}{\partial \mathbf{p}} = 0,$$

(29)

with the corresponding form of the Poisson equation

$$\nabla \cdot \mathbf{E} = 4\pi q_e \int f_e(\mathbf{r},\mathbf{p},t)d\mathbf{p} + 4\pi q_e n_{0r},$$

(30)

for the analysis of the Longitudinal waves.

Equilibrium distribution function is given by equation (9). Here we consider the small perturbations of this distribution $f_e = f_0 + \delta f$ with $\delta f = F e^{-\omega t + i\mathbf{k} \cdot \mathbf{r}}$. Moreover, let us consider the small perturbations for the waves propagating in the plasmas being in the external uniform magnetic field, so we have $\mathbf{E} = 0 + \delta \mathbf{E}$ and $\mathbf{B} = B_0 \mathbf{e}_z + \delta \mathbf{B}$.

Therefore, the Vlasov kinetic equation (29) transforms to the following form for the linear approximation

$$-i(\omega - \mathbf{k} \cdot \mathbf{v}) \delta f - m \frac{v_{\perp}}{p_{\perp}} \Omega_e \frac{\partial \delta f}{\partial \mathbf{p}} + q_e \mathbf{E} \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0,$$

(31)

where we use $\Omega_e = q_e B_0/m_e c$, $[\mathbf{v} \times \delta \mathbf{B}] \cdot (\partial f_0/\partial \mathbf{p}) = 0$, $\mathbf{v} = v_{\perp} \mathbf{e}_\perp + v_z \mathbf{e}_z$, and $v_{\perp}/p_{\perp} = \sqrt{1 - v^2/c^2} = 1/\gamma$.

Equation (31) leads to the following solution

$$\delta f = \int_C d\mathbf{q} \left[ \left( \frac{\omega^2}{\Omega_e} \delta \mathbf{E} \cdot \frac{\partial f_0}{\partial \mathbf{p}} \right) \left( e^{i\mathbf{q} \cdot \mathbf{r}} \right) \cdot \exp \left( \frac{\omega \gamma}{\Omega_e} \left( \int_{\mathbf{q}' \cdot (\omega - \mathbf{k} \cdot \mathbf{v}) \phi'' \right) \right).$$

(32)

Integration leads to

$$\delta f = \frac{q_e \gamma_0}{\Omega_e p} \delta f_0 \exp \left( -i \frac{(\omega - k_z v_z) \phi - k_z v_z \sin \phi}{\Omega_e / \gamma} \right).$$

For the longitudinal waves $\delta \mathbf{E} \parallel \mathbf{k}$ propagating parallel to the external magnetic field $\mathbf{k} \parallel B_0$ we obtain

$$\delta f = \frac{q_e \gamma_0}{\Omega_e \sqrt{1 - v_z^2/c^2}} \left( \frac{1}{p} \frac{\partial f_0}{\partial \mathbf{p}} \right) \delta E_z \cdot \exp \left( -i \frac{(\omega - k_z v_z) \phi}{\Omega_e \sqrt{1 - v_z^2/c^2}} \right).$$

(33)

Final integration leads to the following expression for the perturbation of the distribution function

$$\delta f = q_e \frac{p_z \partial f_0}{\partial \mathbf{p}} \frac{\delta E_z}{p \cdot (\omega - k_z v_z)},$$

(35)

where all relativistic effects are placed in $(\omega - k_z v_z)$ via $v_z = p_z/m \gamma$.

Let us use solution (35) for the calculation of the perturbations of the concentration

$$\delta n = \int d\mathbf{p} \delta f(\mathbf{r},\mathbf{p},t),$$

(36)

and substitute it in the Poisson equation to get the dispersion equation

$$1 + 3 \gamma_{Fe} \frac{\omega_{Le}^2}{p_{Fe} k_z^2/m^2} \left[ 1 - \frac{m \gamma_{Fe} \omega}{p_{Fe} k_z} \ln \left( \frac{m \gamma_{Fe} \omega + p_{Fe} k_z}{m \gamma_{Fe} \omega - p_{Fe} k_z} \right) \right] = 0.$$

(37)

In the limit $\omega \gg k_z v_{Fe} = p_{Fe} k_z/m \gamma_{Fe}$ dispersion equation gives the following spectrum of the relativistic Langmuir waves in the degenerate electron gas

$$\omega^2 = \frac{\omega_{Le}^2}{\gamma_{Fe}} + \frac{3 \gamma_{Fe} k_z^2}{5 m^2 \gamma_{Fe}^2}.$$
It can be also represented in the following form \( \omega^2 = \frac{\omega_{Fe}^2 k_z^2}{\gamma_{Fe}^2 + \frac{3}{5} \gamma_{Fe}^2 k_z^2} \). Comparison with the results of hydrodynamic model presented above shows agreement up to the coefficient in front of \( v_{Fe}^2 \). Which is the well-known difference existing in the nonrelativistic limit as well. There is systematic way of solving this problem via the extension of set of hydrodynamic equations to get complete agreement with the kinetic description (see for instance \[24, 25, 26\]).

Presence of the relativistic gamma factor taken for the Fermi velocity \( \gamma_{Fe} \) in the denominator in the first term of equation (48) shows the decrease of the cut-off frequency of the Langmuir waves. Moreover, presence of the square of relativistic gamma factor in the denominator in the second term of equation (48) shows stronger decrease of the change of frequency at the grough of the wave vector. Which reveals in the decrease of the group velocity as well.

D. Spectrum of the ion-acoustic waves

From equations (21), (25) we find the following dispersion equation for the longitudinal waves in the electron-ion plasmas in order to obtain the spectrum of the low frequency ion-acoustic waves

\[
1 = \left( \frac{\Gamma_0 - \frac{u_{Fe}^2}{c^2}}{n_0} \right) \frac{\omega_{Fe}^2}{\omega^2 - u_{Fe}^2 k_z^2/3} + \left( \frac{\Gamma_0 - \frac{u_{ps}^2}{c^2}}{n_0} \right) \frac{\omega_{ps}^2}{\omega^2 - u_{ps}^2 k_z^2/3}.
\]

As it is demonstrated above we can simplify the coefficients in front of the Langmuir frequencies expressing it via the relativistic gamma factor. Moreover, the characteristic velocities \( u_{Fe} \) and \( u_{ps} \) have simple expressions via the Fermi velocities \( u_{Fe}^2 = v_{Fe}^2/3 \) and \( u_{ps}^2 = v_{ps}^2/3 \). Therefore, we give representation of dispersion equation (40) in simplified form

\[
1 = \frac{\omega_{Fe}^2}{\gamma_{Fe}(\omega^2 - v_{Fe}^2 k_z^2/3)} + \frac{\omega_{ps}^2}{\gamma_{ps}(\omega^2 - v_{ps}^2 k_z^2/3)}.
\]

For the frequencies being in interval \( v_{Fe}^2 k_z^2/3 \gg \omega^2 \gg v_{ps}^2 k_z^2/3 \) equation (40) gives the following spectrum of the ion-acoustic waves

\[
\omega^2 = \frac{\omega_{ps}^2}{\gamma_{Fe}(1 + \frac{\gamma_{Fe}^2 k_z^2/3}{\gamma_{ps}^2 k_z^2/3})},
\]

or, in the long wavelength limit,

\[
\omega^2 = \frac{\omega_{ps}^2}{m_e \gamma_{Fe} v_{Fe}^2 k_z^2/3} = \frac{\omega_{ps}^2}{3m_e v_{ps}^2 k_z^2/3}.
\]

The expression of frequency square in terms of the Fermi velocity contains additional factor equal to ratio of the gamma factors for electrons and ions \( \gamma_{Fe}/\gamma_{Fi} \). However, the momentum has expression via the concentration equals to the nonrelativistic expression \( p_{Fe} = (3\pi^2 n_{0e})^{1/3} \hbar \). Hence, the second expression in equation \[12\] gives more clear physical picture, where the frequency square is proportional to product of reverse gamma factors \( \gamma_{Fe}/\gamma_{Fi} \). The minimal frequency square of the Langmuir wave is decreased by factor \( \gamma_{Fe}^{-1} \) (see equation (38)). While the frequency square of the ion-acoustic waves shows stronger decrease since it contains \( (\gamma_{Fe}/\gamma_{Fi})^{-1} \), where additional factor \( \gamma_{Fi} < 1 \) is included.

E. Small amplitude ion-acoustic soliton

The ion-acoustic solitons are considered in the high-density low-temperature electron-ion plasmas. They are studied in the limit of small amplitude of the soliton. Therefore, we apply the reductive perturbation method (see for instance \[27\]). This method includes the expansion of hydrodynamic functions as the series on the small parameter \( \varepsilon \). In accordance with this method we introduce the following couple of variables with the necessary scaling

\[
\xi = \varepsilon \frac{2}{3}(z - Vt), \quad \tau = \varepsilon \frac{2}{3}t,
\]

where parameter \( \tau \) is proportional to the larger degree of small parameter \( \varepsilon \). The parameter \( \tau \) is called the slower time, while faster dependence on time \( t \) is included in parameter \( \xi \).

we introduce an expansion of the hydrodynamic parameters on a small parameter \( \varepsilon \)

\[
\begin{align*}
n_s &= n_{0s} + \varepsilon n_{1s} + \varepsilon^2 n_{2s}, \\
v_{sz} &= 0 + \varepsilon v_{1sz} + \varepsilon^2 v_{2sz}, \\
\Gamma_s &= \Gamma_{0s} + \varepsilon \Gamma_{1s} + \varepsilon^2 \Gamma_{2s}, \\
t_{sz} &= 0 + \varepsilon t_{1sz} + \varepsilon^2 t_{2sz},
\end{align*}
\]

and

\[
\phi = 0 + \varepsilon \phi_1 + \varepsilon^3 \phi_2,
\]

where function \( \Gamma_{0s} \) is given by equation (20), and \( \phi \) is the potential of the electric field \( E = -\nabla \phi \).

Equations of state (12), (13) for functions \( P, p, t, \) and \( M \) allows to get representations of these functions via the different combinations of \( n_{0s}, n_{1s}, n_{2s}, \) and \( n_{2s} \). We find the expressions presented below:

\[
\tilde{p}_s \approx \tilde{p}_{0s} + \varepsilon u_{ps}^2 n_{1s} + \varepsilon^2 u_{ps}^2 n_{2s} + \varepsilon^2 \frac{v_{ps}^2}{3 \gamma_{Fe}^2 n_{0s}} n_{1s}^2,
\]

where \( u_{ps}^2 = v_{ps}^2/3 \), and

\[
\hat{t}_s \approx \tilde{t}_{0s} + \varepsilon \frac{v_{ps}^2}{3 \gamma_{Fe}^2} n_{1s},
\]
for function $M_{s0}$ we need equilibrium expressions only. 
Presented method of expansion leads to the continuity equation considered in the first (lowest) order of the expansion and the second order of expansion
\[ n_{s0} \partial_\xi v_{s1} = U \partial_\xi n_{s1}, \tag{51} \]
and
\[ \partial_r n_{s1} - U \partial_\xi n_{s2} + \partial_\xi (n_{s0} v_{s2} + n_{s1} v_{s1}) = 0. \tag{52} \]
More accurately speaking we have coefficients $\varepsilon^3/2$ and $\varepsilon^5/2$ for the first and second orders of expansion, correspondingly.
We can integrate equation (51) under boundary condition that the perturbation caused by soliton goes to zero at infinite distance from its center $v_{s1} \to 0$ and $n_{s1} \to 0$ at $\xi \to \pm \infty$. We obtain
\[ n_{s0} v_{s1} = Un_{s1}. \tag{53} \]
Next, we consider the Poisson equation
\[ n_{s1} - n_{s1} = 0 \tag{54} \]
and
\[ -\partial^2_\xi \varphi_1 = 4\pi (q_c n_{s2} + q_i n_{i2}). \tag{55} \]
Necessary relation between concentration, velocity field and electric field is found from the Euler equation which is also considered in the first and second orders of expansion
\[ -Un_{s0} \partial_\xi v_{s1} + u_{sp}^2 \partial_\xi n_{s1} = -\frac{q_s}{m_s} n_{s0} \left( \frac{\Gamma_{s0}}{n_{s0}} - \frac{u_{sp}^2}{c^2} \right) \partial_\xi \varphi_1, \tag{56} \]
and
\[ -Un_{s0} \partial_\xi v_{s2} + n_{s0} \partial_r v_{s1} + u_{sp}^2 \partial_\xi n_{s2} + \frac{u_{sp}^2}{3\gamma_{Fs} n_{s0}} \partial_\xi n_{s1}^2 \]
\[ = -\frac{q_s}{m_s} (\Gamma_{s1} - \frac{u_{st}^2}{c^2} n_{s1}) \partial_\xi \varphi_1. \tag{57} \]
The Euler equation obtained in the first order can be integrated. So, necessary relation between $v_{s1}$, $n_{s1}$, and $\varphi_1$ is found. Relation between $v_{s2}$, $n_{s2}$, $v_{s1}$, and $\varphi_1$ presented by equation (57) includes the first order perturbation of the relativistic hydrodynamic gamma function $\Gamma_{s1}$. Therefore, we consider equation (53) in the lowest order of expansion
\[ -U \partial_\xi \Gamma_{s1} + \Gamma_{s0} \partial_\xi v_{s1} + \partial_\xi t_{s1} = 0. \tag{58} \]
Next, we also need equation for the first order perturbation of the flux of the relativistic hydrodynamic gamma function $t_{s1}$:
\[ U \partial_\xi t_{s1} - u_{sp}^2 \partial_\xi n_{s1} + \frac{\Gamma_{s0}}{n_{s0}} u_{sp}^2 \partial_\xi n_{s1} + \frac{q_s}{m_s} \Gamma_{s0} \frac{u_{st}^2}{n_{s0}} \frac{u_{sp}^2}{c^2} \partial_\xi \varphi_1 \]
\[ = \frac{q_s}{m_s} n_{s0} \left( 1 - \frac{5u_{sp}^2}{c^2} + \frac{10u_{sp}^2}{c^4} \right) \partial_\xi \varphi_1, \tag{59} \]
where we introduce the characteristic velocity for function $M_{s0}$ as follows $u_{sp} = M_{s0}/n_{s0}$. Let us to point out that equations (58) and (59) are used in the second order.
In the first order we find the following expressions for the concentration as the function of the potential of the electric field
\[ n_{s1} = \frac{q_s}{m_s} \left( \frac{\Gamma_{s0}}{n_{s0}} - \frac{u_{st}^2}{c^2} \right) \frac{n_{s0}}{U^2 - u_{sp}^2} \partial_\xi \varphi_1. \tag{60} \]
Let us repeat that $\frac{\Gamma_{s0}}{n_{s0}} - \frac{u_{st}^2}{c^2} > 0$ are positive. Hence equation (60) has solution under condition $u_{sp}^2 < U^2 < u_{sp}^2$. Moreover, it is well-known that stable ion-acoustic waves exist at more strict condition $u_{sp}^2 \ll U^2 \ll u_{sp}^2$. It allows us to get corresponding approximate solution of equation (60)
\[ U^2 = \frac{m_s \gamma_{Fe} u_{sp}^2}{m_i \gamma_{Fi} u_{sp}^2}. \tag{62} \]
The second order leads to the nonlinear equation for the electric potential
\[ \partial_\xi^3 \varphi_1 + \sum_{s=e,i} \frac{U \omega_{Ls}^2}{\gamma_{Fs} (U^2 - u_{sp}^2)^2} \partial_\xi \varphi_1 + \sum_{s=e,i} \frac{q_s}{m_s} \frac{2(U^2 + \frac{u_{sp}^2}{3\gamma_{Fs}}) \omega_{Ls}^2}{\gamma_{Fs} (U^2 - u_{sp}^2)^3} \varphi_1 \partial_\xi \varphi_1 \]
\[ + \sum_{s=e,i} \frac{q_s}{m_s} \frac{2U^2 + \frac{u_{sp}^2}{3\gamma_{Fs}} \omega_{Ls}^2}{\gamma_{Fs} (U^2 - u_{sp}^2)^3} \left[ \frac{\Gamma_{s0}}{n_{s0}} \frac{u_{st}^2}{U^2} \right] + \frac{1}{U} \left( \frac{\Gamma_{s0}}{\gamma_{Fs} U^2} - \frac{u_{st}^2}{U^2} \right) \varphi_1 \partial_\xi \varphi_1 \]
ties of ions can be dropped. Let us consider the second contribution of the relativistic effects in the proper-
ties of the electron Langmuir frequency square in compare-
with $u_{\text{ep}}^2$ in equation (64). In general case, we keep all of them.

We consider the electron contribution in the coefficient of the fourth term in equation (64), where we drop 1 in compare with $u_{\text{ep}}^2/U^2$, we can drop $u_{\text{ep}}^2/c^2$ in compare with $u_{\text{ep}}^2/U^2$. Hence, we can drop the contribution of electrons in the coefficient in the sec-
terms in equation (64). In general case, we keep all of them.

In the opposite limit, if both parts of the coefficient in the third term in equation (64) are comparable we can drop the contribution of ions in coefficient in the second term in equation (64). In general case, we keep all of them.

We consider the electron contribution in the coefficient of the fourth term in equation (64), where we drop 1 in compare with $u_{\text{ep}}^2/U^2$, we can drop $u_{\text{ep}}^2/c^2$ in compare with $u_{\text{ep}}^2/U^2$. Hence, we can drop the contribution of electrons in the coefficient in the sec-
terms in equation (64). In general case, we keep all of them.

In the opposite limit, if both parts of the coefficient in the third term in equation (64) are comparable we can drop the contribution of ions in coefficient in the second term in equation (64). In general case, we keep all of them.

We consider the electron contribution in the coefficient of the fourth term in equation (64), where we drop 1 in compare with $u_{\text{ep}}^2/U^2$, we can drop $u_{\text{ep}}^2/c^2$ in compare with $u_{\text{ep}}^2/U^2$. Hence, we can drop the contribution of electrons in the coefficient in the sec-
terms in equation (64). In general case, we keep all of them.
The relativistic effects contribute in the properties of the ion-acoustic soliton described by equation (65) via the electrons. Main indicator of these effects is the gamma factor on the Fermi velocity $\gamma_{Fe}$. It is located in the denominator. Hence, it reduces the contribution of electrons in compare with the ions.

IV. CONCLUSION

General derivation of the hydrodynamic model for the relativistically hot plasmas has been presented in earlier papers [15], [19]. This hydrodynamic model is based on the dynamics of four material fields: the concentration and the velocity field and the average reverse relativistic $\gamma$ factor and the flux of the reverse relativistic $\gamma$ factor. Here we have presented further generalization of this model for the degenerate species. So, we consider temperatures below the Fermi temperature of the chosen species. Moreover, the concentration of the species is large enough so the Fermi velocity getting close to the speed of light. Necessary equations of state for the flux of the particle current $\tilde{p}$, the flux of the average reverse gamma factor $\Gamma_{0}$, and function $M_{i}$, which is the flux of flux of function $\tilde{p}$, have been obtained in the paper. Moreover, the equilibrium average reverse gamma factor $\Gamma_{0}$ has been calculated for degenerate fluid as well.

Major application of the developed model has been made for the small amplitude ion-acoustic soliton. However, in order to illustrate the role of the relativistic effects on the simple example, we have considered the spectrum of the Langmuir waves. Moreover, Langmuir waves are considered in two ways. First, the suggested model has been applied to find the spectrum of the Langmuir waves. Second, the relativistic Vlasov kinetic equation has been used to consider the same problem in order to estimate the accuracy of the suggested model. The properties of the relativistic ion-acoustic solitons are studied analytically in terms of the suggested model.

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VI. DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study, which is a purely theoretical one.

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