Bayesian inference in high-dimensional models

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Abstract Models with dimension more than the available sample size are now commonly used in various applications. A sensible inference is possible using a lower-dimensional structure. In regression problems with a large number of predictors, the model is often assumed to be sparse, with only a few predictors active. Interdependence between a large number of variables is succinctly described by a graphical model, where variables are represented by nodes on a graph and an edge between two nodes is used to indicate their conditional dependence given other variables. Many procedures for making inferences in the high-dimensional setting, typically using penalty functions to induce sparsity in the solution obtained by minimizing a loss function, were developed. Bayesian methods have been proposed for such problems more recently, where the prior takes care of the sparsity structure. These methods have the natural ability to also automatically quantify the uncertainty of the inference through the posterior distribution. Theoretical studies of Bayesian procedures in high-dimension have been carried out recently. Questions that arise are, whether the posterior distribution contracts near the true value of the parameter at the minimax optimal rate, whether the correct lower-dimensional structure is discovered with high posterior probability, and whether a credible region has adequate frequentist coverage. In this paper, we review these properties of Bayesian and related methods for several high-dimensional models such as many normal means problem, linear regression, generalized linear models, Gaussian and non-Gaussian graphical models. Effective computational approaches are also discussed.
1 Introduction

Advances in technology have resulted in massive datasets collected from all aspects of modern life. Very large datasets appear from internet searches, mobile apps, social networking, cloud-computing, wearable devices, as well as from more traditional sources such as bar-code scanning, satellite imaging, air traffic control, banking, finance, and genomics. Due to the complexity of such datasets, flexible models are needed involving many parameters, routinely exceeding the sample size. In such a situation, a meaningful inference is possible only if there is a hidden lower-dimensional structure involving far fewer parameters. This will happen in a linear or a generalized linear regression model if the vector of regression coefficients mostly consists of zero entries. In this situation, sensible inference will be possible by forcing the estimated coefficient to be sparse through an automatic mechanism determined by the data. Another important problem is studying the interrelationship among a large class of variables. It is often sensible to think that only a few pairs have some intrinsic relations between them, when the effects of other variables are eliminated, that is, most pairs of variables are conditionally independent given other variables. The underlying sparsity is very conveniently described by a graph, where the variables are represented by the nodes of a graph, and an edge connecting a pair is present only if they are conditionally dependent given other variables. Hence the resulting model is called a graphical model. When the variables are jointly Gaussian, the absence of an edge is equivalent to having a zero-entry in the precision (inverse of the covariance) matrix. Thus learning the structural relationship in such a Gaussian graphical model is possible by forcing the estimated precision matrix to have zero-entries in most places. Other problems which effectively use a lower-dimensional structure include matrix completion problems (where many entries of a matrix are missing and it is assumed that the underlying true matrix has a sparse plus a low-rank structure), and stochastic block models (where the extent of interaction between two nodes is determined solely by their memberships in certain hidden blocks of nodes).

Numerous methods of estimating parameters in the high-dimensional setting have been proposed in the literature, most of which use the penalization approach. The idea is to add a suitable penalty term to the loss function to be optimized so that the resulting solution is forced to be sparse. The most familiar method is the least absolute deviation shrinkage and selection operator, abbreviated as LASSO (Tibshirani \[117\]): \( \hat{\beta} = \arg \min \sum_{i=1}^{n} (Y_i - \sum_{j=1}^{p} \beta_j X_{ij})^2 + \lambda \sum_{j=1}^{p} |\beta_j| \) for a linear regression model \( Y_i = \sum_{j=1}^{p} \beta_j X_{ij} + \varepsilon_i, i = 1, \ldots, n \). The sharp corners of the contours of the \( \ell_1 \)-penalty function \( \sum_{j=1}^{p} |\beta_j| \) in this case force sparsity in the resulting estimate, where the extent of the sparsity depends on the tuning parameter \( \lambda \) and the data. While there are many other important penalization procedures for this problem and many other related problems, we shall not explicitly refer to them except to introduce and compare with a Bayesian method, which is the primary object of interest in this review. Excellent sources of information on the frequentist literature for high-dimensional models are Bühlmann and van de Geer [21], Giraud [50], Wainwright
Bayesian inference in high-dimensional models has seen a lot of recent interest. A hidden lower-dimensional structure such as sparsity can be easily incorporated in a prior distribution, for instance, by allowing a point-mass (or some analog of that) at zero. In addition to providing a point estimate, a Bayesian method has the natural ability to assess the uncertainty in structure-learning and provide credible sets with attached probabilities for uncertainty quantification. Good Bayesian procedures should have desirable frequentist properties, like minimax optimal rate of contraction, consistency of variable selection (or more generally, structure learning), and asymptotic frequentist coverage of credible sets. Investigating these properties is a primary research objective, which is particularly important in high-dimensional settings because identifying appropriate priors is more challenging. Continued research has shown that under suitable choices of prior distributions, Bayesian methods can asymptotically perform very well. However, a Bayesian method is typically also very computationally intensive, especially if a traditional Markov chain Monte Carlo (MCMC) procedure is used, since the MCMC iterations will have to explore a lot of possible models. In recent years, effective computational strategies using continuous shrinkage priors, expectation-maximization (EM) algorithm, variational approaches, Hamiltonian Monte Carlo and Laplace approximation, have been proposed. Our review will address some aspects of computation in the supplementary materials part.

We shall cover the following models of interest: normal sequence model (Section 3); high-dimensional linear regression model (Section 4); high-dimensional nonparametric and other regression models (Subsection 4.3); Gaussian graphical model (Subsection 5.2); high-dimensional classification (Subsection 5.3); Ising and other non-Gaussian graphical models (Subsection 5.4); nonparanormal graphical models (Subsection 5.5); signal on a large graph (Section 6); matrix models such as structured sparsity and stochastic block models (Section 7). We shall address the issues of posterior contraction, variable or feature selection, distributional approximation and coverage of credible sets, as well as modifications to the Bayesian approach using fractional posterior and PAC-Bayesian methods.

Notations: A generic statistical model indexed by a parameter $\theta$ is written as $\mathcal{P} = \{P^{(n)}_{\theta}, \theta \in \Theta\}$, with $\Theta$ a subset of a (typically, here, high-dimensional) metric space equipped with a distance $d$. Let $\Pi$ stand for a prior distribution on $\Theta$, and suppose that one observes data $Y^{(n)} \sim P^{(n)}_{\theta_0}$. Let the true value of $\theta$ be denoted by $\theta_0$ so that we study convergence under the distribution $P^{(n)}_{\theta_0}$. We say that $\varepsilon_n$ is a contraction rate (at $\theta_0$, with respect to a metric $d$ on $\Theta$) if $E_{\theta_0}\Pi[d(\theta, \theta_0)] \leq M_n \varepsilon_n |X^{(n)}| \to 1$ as $n \to \infty$ for every sequence $M_n \to \infty$; see Ghosal and van der Vaart [48]. Let $N(0, \Sigma)$ stand for the centered Gaussian distribution with covariance $\Sigma$ and $\text{Lap}(\lambda)$ denote the Laplace distribution of parameter $\lambda > 0$. Let $I_d$ be the identity matrix of dimension $d \geq 1$. $\|\cdot\|_p$, $p \geq 1$, be the $\ell_p$-norm $\|\cdot\|$ stand for the Euclidean i.e., $\|\cdot\|_2$-norm, and $\|\cdot\|_F$ for the Frobenius (Euclidean) norm on matrices.
2 Priors for sparsity

2.1 Spike-and-slab priors

As sparsity plays a very important role in high-dimensional modeling, priors inducing sparsity are of special interest. While an entry $\theta$ may likely be zero, the possibility of a non-zero value, even a large value, cannot be ruled out. This can be thought of as being two regimes superimposed — one corresponding to zero or very small values of $\theta$, and the other to possibly large values, and the prior should give a probabilistic mechanism to choose between the regimes. To address this, a spike-and-slab prior has been considered (Mitchell and Beauchamp [85], Ishwaran and Rao [57]):

$$\pi(\theta) = (1 - w)\phi_0(\theta) + w\phi_1(\theta),$$  \hspace{1cm} (1)

where $\phi_0$ is a density highly concentrated at 0, $\phi_1$ is a density (usually symmetric about 0) allowing intermediate and large values of $\theta$, and $w$ is a small parameter thus inducing sparsity in the mixture. For instance, $\phi_0$ may be the normal density with mean 0 and a small variance $\nu_0$ and $\phi_1$ the normal density with mean zero and a relatively large variance $\nu_1$. The parameters in $\phi_0$ and $\phi_1$ as well as $w$ may be given further priors. Another choice such that both $\phi_0$ and $\phi_1$ are Laplace densities, was proposed and called the spike-and-slab LASSO (Ročková and George [108]). The primary motivation is to identify the posterior mode as a sparse vector as in the LASSO. Generally, the spike part of the prior induces a shrinkage towards zero, which can be limited by using a heavier tailed density $\phi_1$ for the slab such as a $t$-density, or at least as heavy-tailed as the Laplace density. For the spike part, an extreme choice is the distribution degenerate at 0, which corresponds to exact sparsity, and the resulting prior will be referred to as the hard-spike-and-slab prior, while the term soft-spike-and-slab prior will be used if the spike has a density. Johnson and Rossell [62, 63] argued in favor of non-local priors, which make the spike as separated as possible from the slab around zero, by choosing slab distributions that have very little mass close to 0.

2.2 Continuous shrinkage priors

Computation using a spike-and-slab prior involves a latent indicator of the mixture component. Replacing the indicator by a continuous variable leads to the so-called continuous shrinkage priors, typically obtained as scale mixtures of normal. Alternative terms like global-local priors or one-component priors are also used. An early example is the Laplace prior (Park and Casella [97], Hans [56]), which is an exponential scale-mixture of normal, often called the Bayesian LASSO because the corresponding mode is interpreted as the LASSO. However, the obvious drawback of these priors is that there is not sufficient concentration near the value 0 for the entire posterior to concentrate near 0 whenever a coefficient is 0, even though
the posterior mode may be sparse or nearly sparse. The prior concentration near 0 should be high while still maintaining a thick tail, by letting the scale parameter to have a more spiked density at 0. A popular continuous shrinkage prior meeting these requirements is the horseshoe prior (Carvalho et al. [26]), which is a half-Cauchy scale mixture of normal:

$$
\theta | \lambda \sim N(0, \lambda^2), \quad \lambda \sim \text{Cauchy}^+(0, \tau),
$$

where \( \text{Cauchy}^+(0, \tau) \) is the half-Cauchy distribution with scale \( \tau \). The corresponding marginal density of \( \theta \) has a pole at 0 and Cauchy-like tails. A further prior may be put on \( \tau \), such as another half-Cauchy prior leading to the Horseshoe+ prior (Bhadra et al. [17]).

One may consider different priors on the scales \( \lambda \) in (2) with a high concentration near 0. The scale parameter \( \lambda \) is entry-specific and is called the local shrinkage prior. The scale parameter \( \tau \) is common for all entries and is called the global shrinkage parameter. The local shrinkage parameter should have a heavy tail while the global shrinkage parameter should have a high concentration at 0 (Polson and Scott [102]), respectively controlling the tail and the sparsity. Various choices of mixing distributions lead to the introduction of many continuous shrinkage priors such as normal-inverse-Gaussian prior (Caron and Doucet [24]), normal-gamma prior (Griffin and Brown [52]), the generalized double Pareto priors (Armagan et al. [3]). Another possibility is the Dirichlet-Laplace prior (Bhattacharya et al. [19]):

$$
\theta_i | \phi, \tau \sim \text{Lap}(\phi_i \tau), \quad \phi = (\phi_1, \ldots, \phi_p) \sim \text{Dir}(a, \ldots, a),
$$

where choosing \( a \in (0, 1) \) leads to a pole at 0 for the marginal distribution of \( \theta_i \)'s enforcing (near-)sparsity, and \( \tau \) is a parameter which is given a gamma distribution.

### 3 Normal sequence model

The simplest high-dimensional model is given by the normal sequence model:

$$
Y_i = \theta_i + \epsilon_i, \ i = 1, \ldots, n,
$$

where \( \epsilon_i \) are independent and identically distributed (i.i.d.) \( \mathcal{N}(0, 1) \), the parameter set \( \Theta \) for \( \theta = (\theta_1, \ldots, \theta_n) \) is \( \mathbb{R}^n \). The posterior contraction rate will be obtained uniformly for parameters belonging to a nearly-black class \( \ell_0[s] = \{ \theta \in \mathbb{R}^n : \#\{i : \theta_i \neq 0\} \leq s \} \), \( 0 \leq s = s(n) \leq n \), where \( \# \) stands for the cardinality of a finite set. In this model the minimax risk for the squared error loss is equivalent to \( 2s \log(n/s) \), see Donoho et al. [40]. Let \( \theta_0 \) stand for the true value of the vector \( \theta \) and \( s_0 \) the cardinality of the support of \( \theta_0 \), i.e., the set \( \{ i : \theta_{0i} \neq 0 \} \).
3.1 Recovery using hard-spike-and-slab priors

It is clear that if each $\theta_i$ is given Lap($\lambda$)-prior independently, the resulting posterior mode will be the LASSO, which, with the choice $\lambda = \sqrt{\log n}$, converges to the true $\theta$ at the nearly optimal rate $s \log n$, but the whole posterior has a suboptimal contraction property (Castillo et al. [31]). This is because, without having sufficient prior concentration at 0, the posterior cannot contract sufficiently fast near the truth. A remedy is to assign an additional point-mass at zero using a hard-spike-and-slab prior: for $w \in [0, 1]$ and $\Gamma$, a distribution on $\mathbb{R}$, $\Pi_n = \Pi_{n, \Gamma} = \bigotimes_{i=1}^n \left\{ (1-w)\delta_0 + w\Gamma \right\}$. The weight parameter $w$ in the ‘oracle’ situation (i.e. $s$ is known) can be taken to be $s/n$, leading to the optimal rate $s \log(n/s)$. Realistically, the choice $w = c/n$ is possible with a constant $c > 0$, but this leads to a slightly suboptimal rate $s \log n$. In this case, under the prior, the expected number of nonzero coefficients is of the order of a constant. To obtain an improved data fit, options performing better in practice include an empirical Bayes choice $\hat{w}$ of $w$ (George and Foster [44], Johnstone and Silverman [64]). One possibility is to use an ad-hoc estimator of the number of non-zero coordinates of $\theta$, for instance by keeping only coordinates above the expected noise level $\hat{w} = n^{-1}\sum_{i=1}^n 1\{|Y_i| > \sqrt{2\log n}\}$. However, this choice may be too conservative in that signals below the universal $\sqrt{2\log n}$ threshold may not be detected. The marginal maximum likelihood empirical Bayes approach (MMLE) consists of integrating out $\theta$ and maximizing with respect to $w$: $\hat{w} = \arg\max_{w} \prod_{i=1}^n \left\{ (1-w)\phi(Y_i) + wg(Y_i) \right\}$, where $g = \gamma * \phi$. The plug-in posterior $\Pi_\hat{w}(\cdot|Y_1, \ldots, Y_n)$ was advocated and studied in George and Foster [44], Scott and Berger [112], Johnstone and Silverman [64], Castillo and Mismer [27] among others, and is shown to possess the optimal contraction rate for heavy enough tail of the slab. Castillo and van der Vaart [33] considered the hierarchical Bayes using a prior $w \sim \text{Beta}(1, n+1)$. More generally, they also considered a subset-selection prior: if $\pi_n$ a prior on the set $\{0, \ldots, n\}$ and $\mathcal{S}_s$ the collection of all subsets of $\{1, \ldots, n\}$ of size $s$, let $\Pi$ be constructed as

$$s \sim \pi_n, \quad S|s \sim \text{Unif}(\mathcal{S}_s), \quad \theta|S \sim \bigotimes_{i \in S} \Gamma \otimes \bigotimes_{i \not\in S} \delta_0. \quad (4)$$

The hard-spike-and-slab prior is a special case where $\pi_n$ is the binomial distribution Bin$(n, w)$ (see also van Erven and Szabó [127] for further discussion). Through the prior $\pi_n$, it is possible to chose priors on the dimension that ‘penalize’ large dimensions more than the binomial and achieve the optimal rate $s_0 \log(n/s_0)$ uniformly over $\ell_0[\delta_0]$, that is, when the true value of sparsity is $s_0$, for instance, by using the complexity prior $\pi_n(s) \propto \exp[-as \log(bn/s)]$; see Castillo and van der Vaart [33]. Finally, deriving the optimal constant 2 in the minimax sense for posterior contraction is possible for some carefully designed priors (Castillo and Mismer [28]).

It may be mentioned that the convergence speeds of the entire posterior distribution and those of aspects of it such as the posterior mean, median or mode may not match. Like the posterior mode for the Laplace prior, the (empirical or hierarchical Bayes) posterior mean for the hard-spike-and-slab prior may also be lo-
cated ‘relatively far’ from the bulk of the posterior mass in terms of the distance  
\[ d_q(\theta, \theta') = \sum_{i=1}^{n} |\theta_i - \theta'_i|^q, \quad 0 < q < 1 \]  
(Johnstone and Silverman [64], Castillo and van der Vaart [33]). In contrast, the posterior distribution concentrates at the minimax rate for the \( d_q \)-loss for any \( q \in (0, 2) \) (Castillo and van der Vaart [33]). Further, the coordinate-wise median of the MMLE empirical Bayes posterior for the hard-spike-and-slab prior converges at the minimax rate (Johnstone and Silverman [64]) but the plug-in posterior \( \Pi_{\hat{w}}(\cdot|Y_1, \ldots, Y_n) \) converges at a suboptimal rate for certain sparse vectors (Castillo and Mismer [27]) if a Laplace slab is chosen. This is due to an excessively large dispersion-term in the plug-in posterior when \( \hat{w} \) is slightly above \( s_0/p \). The problem is avoided in the empirical Bayes approach if a slab with heavy enough tails is chosen (e.g. Cauchy), and in the hierarchical approach through an additional regularization by a well-chosen Beta prior on \( w \).

Computations using hard-spike-and-slab priors in the sequence model are possible in polynomial time. Implementation of the MMLE empirical Bayes approach amounts to a root-finding algorithm and is done in Johnstone and Silverman [64]; it can handle very large \( n \)’s (e.g. \( n = 10^6 \)). Castillo and van der Vaart [33] proved that aspects of the subset-selection posterior induced by (4), including hierarchical hard-spike-and-slab priors, are computable in polynomial time. Szabó and van Erven [127] provided fast algorithms for \( n \) up to the order of \( 10^5 \).

### 3.2 Uncertainty quantification

If the slab distribution \( \Gamma \) is chosen normal, then the posterior is conjugate, conditional on the selection, and hence allows certain explicit expressions, which are beneficial. However, due to quickly decaying tails of normal distributions, this leads to over-shrinkage of large values leading to suboptimality, or even inconsistency. The problem can be avoided by introducing a mean parameter for the slab distribution of each entry and estimating those by empirical Bayes, which is the same as plugging in the observation itself; see Martin and Walker [83] who used a fractional-posterior, and Belitser and Nurushev [15], who considered the usual posterior distribution. Belitser and Nurushev [15] obtained the optimal contraction rate following an oracle approach to optimality, and also obtained credible regions with frequentist coverage and adaptive optimal size under an “Excessive Bias Restriction” (EBR) condition that controls bias by a multiple of variability at a parameter value. Restricting parameter space in such a manner is essential as honest coverage with adaptive size ball is impossible to obtain by any method, Bayesian or not, in view of impossibility results such as Baraud [10]. Castillo and Szabó [32] considered the MMLE-empirical Bayes approach using heavy-tailed slabs and an estimated value of the hyperparameter \( w \) in the prior for the sparsity level and provided similar coverage results for adaptive credible sets, obtaining results in terms of \( d_q \) distances, \( q \leq 2 \). They also showed that the EBR condition is necessary in a minimax sense.
3.3 Alternative shrinkage priors

While subset selection priors are particularly appealing because of their natural model selection abilities, to address less strict sparsity (such as weak or strong $\ell_p$-classes), one may use a soft-spike-and-slab prior, to get essentially the same posterior contraction rate, provided that the spike distribution is sufficiently concentrated, such as by the spike-and-slab-LASSO with the spike distribution Lap($\lambda_0$) for $\lambda_0 \to \infty$ with $n$ and the slab distribution Lap($\lambda_1$) for constant $\lambda_1$ (Ročková and George [108]). For the horseshoe prior, van der Pas et al. [124, 122] obtained explicit expressions for the posterior mean in terms of degenerate hypergeometric functions, and used that to show posterior contraction in terms of the Euclidean distance at any vector in $\ell_0[s_0]$ at the rate $\sqrt{s_0}\log n$, respectively under known sparsity and unknown sparsity regimes with $0 \leq s_0 \leq n$ possibly depending on $n$; see also Ghosh and Chakraborti [49] for similar results. The optimal posterior contraction rate using the Dirichlet-Laplace prior was obtained by Bhattacharya et al. [19] under a growth condition on the norm $||\theta_0||$. The optimal contraction rates for the spike-and-slab-LASSO in its hierarchical form was obtained in Ročková [106] up to a side-condition on signals, and by Castillo and Mismer [27] for the empirical Bayes form. Van der Pas et al. [125] unified results for scale-mixture of normals with known sparsity parameter, obtaining sufficient and almost necessary conditions for the optimal convergence. Naturally, to detect the sparsity structure using continuous shrinkage priors, a thresholding procedure identifying an entry essentially zero is needed. Credible $\ell_2$-balls for the parameter and marginal credible with asserted frequentist coverages were obtained in van der Pas et al. [123].

3.4 Multiple testing

The problem of identifying a sparse structure can also be thought of as a multiple testing problem, in which case measures like false discovery rate (FDR) (Benjamaini and Hochberg [16]) may be considered instead of the usually adopted family-wise error rate for Bayesian model selection. The FDR control may fail for many priors. In Castillo and Roquain [30], a uniform FDR control over $\ell_0[s_0]$ is derived for procedures based on both $\ell$-values $\Pi(\theta_i = 0|Y_1, \ldots, Y_n)$ and $q$-values $\Pi(\theta_i = 0|Y_i \geq y_i)$, when the prior distribution has suitably heavy slab tails. It is shown that thresholding the $\ell$-values at a given level $t$ leads to the FDR going to 0 uniformly at a logarithmic rate. The procedure of thresholding $q$-values at a level $t$, on the other hand, controls the FDR at level a constant times $t$ uniformly, and exactly $t$ asymptotically when signals are all above the threshold $a\sqrt{2\log(n/s_0)}$, for some $a > 1$. Simultaneous control of FDR and FNR (False Negative Rate, the proportion of false nulls) is considered from a minimax perspective in Castillo and Roquain [30] and Abraham et al. [1], where a method based on averages of ordered $\ell$-values as a way to set the rejection threshold is studied. Early results for continuous shrinkage priors including the horseshoe were obtained in Salomond [111]; see also van der Pas et al. [123] for
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a simulation study. In the Bayesian setting, a very appealing measure is provided by the Bayes risk for testing. With a hard-spike-and-(normal) slab prior with variance \(v_1\) and known \(w\), the oracle Bayes rule for rejecting \(H_0: \theta_i = 0\) can be shown to be a thresholding procedure \(|Y_i|^2 > (1 + v_1^{-1})[\log(1 + v_1) + 2\log((1 - w)/w)];\) see Datta and Ghosh [37] for details.

4 High-dimensional regression

A natural generalization of the normal sequence model is the situation where the mean \(\theta_i\) depends on a (high-dimensional) covariate \(X_i\) associated with the observation \(Y_i, i, \ldots, n\). The most popular statistical model in this setting is given by the normal linear regression model \(Y_i = \beta^TX_i + \epsilon_i, i = 1, \ldots, p\), where \(X_i \in \mathbb{R}^p\) is a deterministic \(p\)-dimensional predictor, \(\beta \in \mathbb{R}^p\) is the linear regression coefficient, and \(\epsilon_i\) are i.i.d. \(N(0, \sigma^2)\) error variables, as before. When \((p^4 \log p)/n \rightarrow 0\), posterior concentration and a Bernstein-von Mises theorem were obtained in Ghosal [46] without assuming any sparsity. We are mainly interested in the high-dimensional setting where \(p\) is very large (possibly even much larger than \(n\)), and \(\beta\) is sparse with only \(s\) co-ordinates of it being non-zero, \(s \ll n\) and \(n \rightarrow \infty\).

4.1 Linear regression with hard-spike-and-Laplace slab

Let \(X_{i,j}\) be the \(j\)th column of the matrix \(X := (X_{ij})\), and consider the norm \(\|X\| = \max \|X_{i,j}\|\). We consider the prior \(\Phi\), with \(g_S\) the product of \(|S|\) Laplace densities \(\beta \rightarrow (\lambda/2)\exp(-\lambda |\beta|)\). We set \(\lambda = \mu \|X\|\), for \(p^{-1} \leq \mu \leq 2\sqrt{\log p}\) and put a prior \(\pi_p(s) \propto e^{-\lambda p^{-as}}\) on the number of non-zero entries of \(\beta\). For \(p > n\), clearly \(\hat{\beta}\) cannot be uniquely recovered from \(X\beta\) (even in the noiseless case). However, if \(\beta\) is assumed to be sparse (with only \(s \ll n\) components non-zero) and the submatrix of \(X\) corresponding to the active predictors is full rank \(s\), it turns out that \(\hat{\beta}\) can be recovered. Define the compatibility number of model \(S \subset \{1, \ldots, p\}\) by 
\[
\Phi(S) := \inf \left\{ \|X\beta\| |S|^{1/2} / \|X\| \|\beta_S\|_1 : \|\beta_S\|_1 \leq \lambda \|X\|, \beta_S \neq 0 \right\},
\]
where \(\beta_S = (\beta_i : i \in S)\), and the \(\epsilon_r\)-compatibility number in vectors of dimension \(s\) by 
\[
\Phi_r(S) := \inf \left\{ \|X\beta\| |S|^{1/2} / \|X\| \|\beta_s\|_r : 0 \neq |S| \leq s \right\}, r = 1, 2,
\]
where \(S = \{ \beta_j \neq 0 \}\), the support of a sparse vector \(\beta\). For \(\beta_0\) the true vector of regression coefficients and \(S_0 = S_{\beta_0}\) and \(s_0 = |S_0|\), we assume that, with \(r = 1\) or \(2\) depending on the context, \(\min (\Phi_r(S_0), \Phi_r(CS_0)) \geq d > 0\), where \(C\) is a suitably large constant depending on \(S_0, a, \mu\). More information about compatibility numbers may be found in van de Geer and Bühlmann [121]. When the entries of \(X\) are sampled randomly, the compatibility numbers are often very well-behaved with high probability. In the leading example where \(X_{ij}\) are independently sampled from \(N(0, 1)\), the
compatibility numbers above of models up to the dimension a multiple of $\sqrt{n/\log p}$ are bounded away from zero (Cai and Jiang [22], van de Geer and Muro [120]).

The following conclusions were derived in Castillo et al. [31]. First, a useful property is that the posterior of $\beta$ sits on models of dimensionality at most a constant multiple of the true dimension $s_0$, thereby returning models at least of the same order of sparsity as the true one. Further, we can recover $\beta_0$ in terms of the $\ell_1$-norm:

$$E_{\beta_0} \Pi(\|\beta - \beta_0\|_1 > M s_0 \sqrt{\log p / \|X\|}, Y_1, \ldots, Y_n) \to 0.$$  

The corresponding rate in terms of the Euclidean distance is $\sqrt{s_0 \log p / \|X\|}$, assuming that the $\ell_2$-compatibility numbers are bounded away from 0. The corresponding rate is $\sqrt{\log p / \|X\|}$ with respect to the maximum-likelihood under a stronger condition, called mutual coherence. The rate for prediction, i.e., bound for $\|X\beta - X\beta_0\|$, is of the order $\sqrt{s_0 \log p}$ under a slightly adapted compatibility condition. The convergence results are uniform over the parameter space under the boundedness conditions on the compatibility numbers, and match with those of celebrated estimators in the frequentist literature.

For sparse regression, variable selection, which is the identification of non-zero coefficients, is extremely important, because that allows simpler interpretation and a better understanding of relations. To address the issue, Castillo et al. [31] developed a technique based on a distributional approximation under relatively low choices of the parameter $\lambda$, known as the small-$\lambda$ regime. This is similar to the normal approximation to the posterior distribution in the Bernstein-von Mises theorem, but the difference is that in this context, the approximating distribution is a mixture of sparse normal distributions over different dimensions. Then the problem of variable selection can be transferred to that for the approximate posterior distribution, and it can be shown that no proper superset of the correct model can be selected with appreciable posterior probability. It is, however, possible to miss signals that are too small in magnitude. If all non-zero signals are assumed to be at least of the order $\sqrt{s_0 \log p / \|X\|}$, then none of these signals can be missed, because missing any of them introduces an error of the magnitude of the contraction rate. Thus the selection consistency follows.

Also, under this situation, distributional approximation reduces to a single normal component with sparsity exactly as in the true coefficient. Ning et al. [92] considered a useful extension of the setting of Castillo et al. [31] by letting each response variable $Y$ to be $d$-dimensional with $d$ also increasing to infinity (at a sub-polynomial growth with respect to $n$), and having completely unknown $d \times d$-covariance matrix $\Sigma$ and the regression has group-sparsity. They used a prior on $\Sigma$ using a Cholesky decomposition and used a hard-spike-and-slab prior with multivariate Laplace slab on the group of regression coefficients selected together. Applying the general theory of posterior contraction (Ghosal et al. [47]) using exponentially consistent tests for separation based on the Rényi divergence, they obtained squared posterior contraction rate

$$\xi^2_n = \max \left\{ \frac{(s_0 \log G)}{n}, \frac{(s_0 p_{\text{max}} \log n)}{n}, \frac{(d^2 \log n)}{n} \right\},$$  

where $G$ stands for the total number of groups of predictors, $s_0$ the number of active groups and $p_{\text{max}}$ the maximum number of predictors in a group, provided that the
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regression coefficients and the covariance matrix are appropriately norm-bounded. Ning et al. [92] further extended the technique of distributional approximation of Castillo et al. [31] to show selection consistency. A general setup of sparse linear regression $Y = X_\beta + \xi_\eta + \epsilon_\eta \sim N_{m_\eta}(0, \Delta_{\eta, \cdot})$ independently with possibly varying dimension $m_\eta$ and covariance matrices $\Delta_{\eta, \cdot}$, allowing a nuisance parameter $\eta$ and an additional term $\xi_{\eta, \cdot}$ to incorporate various departure from the simple linear model $Y = X_\beta + \epsilon_\eta$, was considered in Jeong and Ghosal [60]. They showed optimal recovery for the regression coefficient $\beta$ under standard conditions on compatibility numbers. They established recovery rates from the posterior contraction rate for both the regression coefficient $\beta$ in terms of the Euclidean norm and for the nuisance parameters in terms of the squared distance $\frac{1}{n} \sum_{i=1}^{n} \left( \| \xi_{\eta, i} - \xi_{\eta', i} \|^2 + \| \Delta_{\eta, i} - \Delta_{\eta', i} \|^2 \right)$, which they derived by applying the general theory of posterior contraction in terms of the Rényi divergence as in Ning et al. [92]. Further, extending the distributional approximation technique, Jeong and Ghosal [60] also showed selection consistency. Their setup accommodates a variety of practical extensions of the basic linear model including multidimensional response, partially sparse regression, multiple responses with missing observations, (multivariate) measurement errors, parametric correlation structure, mixed-effect models, graphical structure in precision matrix (see Subsection 5.2), nonparametric heteroscedastic regression and partial linear models.

While the previous framework enables one to obtain asymptotic normality for all coordinates in $S_0$ if these are suitably large, if one is interested in just one coordinate say $\beta_1$, it is possible to build an estimator (or marginal posterior) thereof that is asymptotically normal and efficient, under say only a compatibility–type condition: this has been obtained by Zhang and Zhang [143] for a debiased–LASSO estimator in the regime $s \log p = o(\sqrt{n})$. In Yang [139], the author derives an analogous Bayesian asymptotic normality result for the marginal posterior $\beta_1$ given $Y$. This is achieved by first defining a suitable prior distribution on $\beta_1$ given the remaining coordinates $\beta_{-1}$, and then assigning a parsimonious prior to $\beta_{-1}$ as before.

### 4.2 Linear regression using other priors

Instead of the hard-spike-and-slab prior with a Laplace slab, we may use a normal slab, but use an undetermined mean selected by the empirical Bayes method, as in the sequence model. The advantage is that posterior conjugacy given the selected subset can be retained, allowing explicit expressions. Belitser and Ghosal [13] followed the oracle approach of Belitser [12] to quantify risk locally at every point to define the optimal concentration and established results analogous to Castillo et al. [31]. Moreover, uniformly over the collection of all parameter values satisfying the excessive bias restriction condition, they showed that an appropriately inflated Bayesian credible ball of the optimal size has adequate frequentist coverage simultaneously for any sparsity level. Other references of Bayesian high-dimensional linear regression includes Narisetty and He [91], who considered an alternative to the posterior measure called the skinny Gibbs posterior that avoids an important com-
putational bottleneck, and established its contraction at the true coefficient at the optimal rate.

An alternative to using independent hard-spike-and-Laplace slab is to use an elliptical Laplace prior on the selected coefficients through a set selection prior in the spirit of (4), but adjusted for the normalizing constant appearing in the elliptical Laplace distribution. Gao et al. [43] considered this prior for a structured linear model \( Y = L_X \beta + \varepsilon \), where \( L_X \) is a linear operator depending on \( X \) and \( \varepsilon \) is a vector of errors assumed to have only sub-Gaussian tails, and obtained minimax contraction rates. Apart from linear regression, their setup includes stochastic block models, biclustering, linear regression with group sparsity, multi-task learning, dictionary learning, and others.

In spite of the attractive theoretical properties, posterior computation based on two-component priors is computationally intensive. A faster computation may be possible using continuous shrinkage priors (Bhattacharya et al. [18]). Under certain general conditions on prior concentration near zero, the thickness of the tails and additional conditions on the eigenvalues of the design matrix, Song and Liang [114] derived posterior contraction and variable selection properties. Their results cover a wide variety of continuous shrinkage priors such as the horseshoe, Dirichlet-Laplace, normal-gamma, and t-mixtures.

### 4.3 Regression beyond linear

Regression beyond the linear setting in the high-dimensional context is of substantial interest. The most common extension is a generalized linear model (GLM). A conjugate prior for Bayesian inference in the GLM setting was introduced by Chen and Ibrahim [35]. The prior was further used for variable selection in a high-dimensional GLM by Chen et al. [34]. One of the first papers on convergence properties of posterior distributions in the high-dimensional Bayesian setting is Jiang [61], who derived consistency of the posterior distribution in a GLM with the dimension \( p \) possibly much larger than the sample size \( n \), needing only \( \log p = o(n^{1-\delta}) \) for some \( \delta > 0 \), but only in terms of the Hellinger distance on the underlying densities. Posterior concentration and a Bernstein–von Mises theorem for the parameter were obtained earlier by Ghosal [45] without assuming any sparsity structure but under the restricted growth condition \( (p^4 \log p)/n \rightarrow 0 \). Atchadé [6] considered a pseudo-posterior distribution in a general setting assuming only a certain local expansion of the pseudo-likelihood ratio and derived posterior contraction rates for hard-spike-and-slab priors by constructing certain test functions. Jeong and Ghosal [59] obtained results on recovery rates for the regression coefficients in a sparse generalized linear model for the actual posterior distribution using hard-spike-and-slab priors. For the special case of logistic regression, Wei and Ghosal [136] established posterior contraction using a variety of continuous shrinkage priors using Atchadé’s test construction and prior concentration bounds of Song and Liang [114]. Both posterior contraction rates and selection consistency were established in a more general
multidimensional logistic regression model in Jeong [58] using hard spike-and-slab priors.

Bayesian high-dimensional regression in the nonparametric setting has been addressed for an additive structure with random covariates. Yang and Tokdar [142] considered Gaussian process priors for each selected component function aided by a variable selection prior on the set of active predictors and showed that the minimax rate $\max\{\sqrt{\frac{s_0 \log p}{n}}, \sqrt{\frac{\alpha}{\sqrt{2} \alpha + 1}}\}$ is obtained up to a logarithmic factor, where $s_0$ is the number of active predictors and $\alpha$ is the smoothness level of each component function. See also Suzuki [115] for earlier results in a PAC-Bayesian setting, where ‘PAC’ stands for ‘Probably Approximately Correct’. Using an orthogonal basis expansion technique, Belitser and Ghosal [13] extended their oracle technique from linear to additive nonparametric regression and obtained the minimax rate with hard-spike-and-normal slab prior, with the means of the slabs selected by the empirical Bayes technique. They also obtained the coverage of adaptive size balls of functions under an analog of the excessive bias restriction (EBR) condition for random covariates, called the $\epsilon$-EBR condition. Wei et al. [137], extending the work of Song and Liang [114], obtained the optimal contraction rate and consistency of variable selection for additive nonparametric regression using a B-spline basis expansion prior with a multivariate version of the Dirichlet-Laplace continuous shrinkage prior on the coefficients. The multivariate version is needed to maintain the group selection structure for the coefficient vector corresponding to the same component function.

A high-dimensional single-index nonparametric regression model where the regression function is represented as $a_0(X^T \beta, Z^T \eta) - F(t)$, with $X$ a high-dimensional predictor, $Z$ a low-dimensional predictor and $a_0, a_1, F$ smooth functions, was considered by Roy et al. [110] to model the atrophy of different brain regions over time. For identifiability of the model, the coefficients $\beta$ and $\eta$ need to be unit vectors in respective dimensions, and $\beta$ should be sparse as well in an appropriate sense. To address the problem, Roy et al. [110] introduced a notion of sparsity in the polar co-ordinate system using a soft spike-and-slab prior with a uniform slab and spike distributions with spikes at appropriate multiples of $\pi/2$. They characterized the posterior contraction rate in terms of the average squared distance on the functions using the general theory of posterior contraction essentially as the weakest of the rates $n^{-i/(2i+2)}$, $n^{-t'/(2t'+2)}$ and $\sqrt{\frac{s_0 \log p}{n}}$, up to a logarithmic factor, where $i$ is the smoothness of the functions $(a_0, a_1)$, $t'$ is the smoothness of the function $F$, and $s$ is the sparsity of the true $\beta$.

Shen and Ghosal [113] considered the problem of estimating the conditional density of a response variable $Y \in [0, 1]$ on a predictor $X \in [0, 1]^p$ with $\log p = O(n^\alpha)$ for some $\alpha < 1$, assuming that only $s_0$ predictors are active, where $s_0$ is unknown but does not grow. They put a prior on the conditional density $p(y|x)$ through tensor products of B-splines expansion. First, a subset selection prior with enough control on the effective dimension is used to select the active set, and then priors on the lengths of the spline bases are put. Finally, the coefficients arrays are given independent Dirichlet distributions of the appropriate dimensions. Shen and Ghosal [113] showed the double adaptation property — the posterior contraction rate adapts to
both the underlying dimension of the active predictor set and the smoothness of each function $\alpha$, i.e., $n^{-\alpha/(2\alpha+s_0+1)}$. In fact, the conditional density can have anisotropic smoothness (i.e., different smoothness in different directions), and then the harmonic mean $\alpha^*$ will replace $\alpha$. Norets and Pati [94] obtained analogous results for the conditional density of $Y \in \mathbb{R}$ on a predictor $X \in \mathbb{R}^p$ using Dirichlet process mixtures of normal densities.

5 Learning structural relationship among many variables

In this section, we review methods for finding relations among a large number of variables by studying some features on their joint distributions, and the associated convergence results. In Subsection 5.1, large covariance matrices are considered. In Subsection 5.2, Bayesian estimation of sparse precision matrices, which may have specific structures like banding or can be unstructured, is considered. Classification using a high-dimensional predictor is considered in Subsection 5.3. Relations among non-Gaussian variables are studied in Subsection 5.4, and semiparametric models for finding structural relations are considered in Subsection 5.5.

5.1 Estimating large covariance matrices

Understanding the dependence among a large collection of variables $X = (X_i : i = 1, \ldots, p)$ is an important problem to study. The simplest measure of dependence is given by the pairwise covariances between these variables, which leads to the problem of estimating a large covariance matrix. In the setting of a large collection of variables, it is natural to postulate that most of these variables are pairwise independent, or at least uncorrelated, meaning that most off-diagonal entries are zero, or at least close to zero. This introduces a useful sparsity structure that allows meaningful inference with relatively fewer observations. One particularly prominent structure is (approximate) banding, which means that when the variables are arranged in some natural order, the pairwise covariances are zero (or decay quickly) if the lag between the corresponding two indexes is larger than some value. For example, an exact banding structure arises in a moving average (MA) process, while in an autoregressive (AR) or autoregressive moving average (ARMA) process, the covariances exponentially decay with the lag. Banding or tapering of the sample covariance is often used to estimate such a large covariance matrix. When the sparse structure does not have any special pattern, threshold methods are often used, but positive definiteness may not be preserved. Another important low-dimensional structure is given by a sparse plus low-rank decomposition of a matrix $\Sigma = D + \Lambda \Lambda^T$, where $D$ is a scalar matrix and $\Lambda$ is a “thin matrix”, that is, $\Lambda$ is $p \times r$, where $r \ll p$. Moreover, the columns of $\Lambda$ themselves may be sparse, allowing further dimension reduction. Such a structure arises in structural linear models $X_i = \Lambda \eta_i + \epsilon_i$, where
\( \Lambda \) is a \( p \times r \) sparse factor loading matrix and \( \eta_i \sim N_p(0, I) \) are the latent factors independent of the error sequence \( \varepsilon_i \sim N_p(0, \sigma^2 I) \). In this setting, a Bayesian method was proposed by Pati et al. [99]. They considered independent hard-spike-and-slab prior with a normal slab on entries of \( \Lambda \) and inverse-gamma prior on \( \sigma^2 \), along with a Poisson-tailed prior on the rank \( r \). They showed that if the number of entries in each column of \( \Lambda \) is bounded by \( s_0 \) and the true \( \sigma^2 \) and the number of latent factors are bounded, then the posterior contraction rate \( \sqrt{(s_0 \log n \log p)/n} \) can be obtained.

5.2 Estimating large precision matrices and graphical models

An intrinsic relation among a set of variables is described by conditional dependence of a pair when the effects of the remaining variables are eliminated by conditioning on them. In a large collection of variables, most pairs may be assumed to be conditionally independent given others. It is convenient to describe this structure using a graph, where each variable stands for a node and an edge connects a pair of nodes if and only if the two are conditionally dependent. Therefore such models are popularly called graphical models. An introduction to graphical models is given in the supplementary material part. If \( X_i \) and \( X_j \) are conditionally independent given \( X_{-i, -j} := (X_k : k \neq i, j) \), then it follows that \( \omega_{ij} = 0 \), where \( (\omega_{ij}) = \Sigma^{-1} \) is the precision matrix of \( X \), to be denoted by \( \Omega \). In a Gaussian graphical model (GGM), i.e., when \( X \) is distributed as jointly normal, \( X \sim N_p(0, \Omega^{-1}) \), then the converse also holds, namely \( \omega_{ij} = 0 \) implies that \( X_i \) and \( X_j \) are conditionally independent given \( X_{-i, -j} \). Thus in a Gaussian graphical model, the problem of learning the intrinsic dependence structure reduces to the problem of estimating the precision matrix \( \Omega \) under sparsity (i.e., most off-diagonal entries of \( \Omega \) are 0).

5.2.1 Banding and other special sparsity patterns

A special type of sparsity is given by an approximate banding structure of \( \Omega \). Note that this is different from banding of the covariance matrix \( \Sigma \), as the inverse of a banded matrix is only approximately banded. Among familiar times series, an AR process has a banded precision matrix while an MA process has an approximate banding structure. The graph corresponding to a banded precision matrix has edge set \( \{(i, j) : |i - j| \leq k, i \neq j\} \), \( k \) is the size of the band, which always corresponds to a decomposable graph, allowing the use of structurally rich priors in this context. In particular, a conjugate graphical Wishart (G-Wishart) prior (see the Supplementary Material part for its definition and properties) can be put on \( \Omega \) together with a choice of \( k \), which may even be given a prior. Banerjee and Ghosal [8] showed that with this prior, the posterior contraction rate in terms of the spectral norm at an approximately banded true \( \Omega \) with eigenvalues bounded away from 0 and infinity is given by \( \max\{k^{3/2}/n, k^{3/2} \gamma(k)\} \), where \( \gamma(k) \) is the banding approximation rate given by the total contribution of all elements outside the band. This, in
particular, implies that for a $k_0$-banded true precision matrix with fixed $k_0$, the rate is nearly parametric $\sqrt{\log p}/n$. The rate calculation extends to general decomposable graphs, with $k$ standing for the maximal cardinality of a clique, as shown by Xiang et al. [138], who used distributional results on the Cholesky factor of decomposable G-Wishart matrices. Another proof of the same result using the original technique of Banerjee and Ghosal [8] was given by Banerjee [7].

Lee and Lee [71] considered a class of bandable precision matrices in the high-dimensional setup, using a modified Cholesky decomposition (MCD) approach $\Omega = (I_p - A)D^{-1}(I_p - A)$, where $D$ is a diagonal matrix, and $A$ is lower-triangular with zero diagonal entries. This results in the $k$-banded Cholesky prior with non-zero entries of $A$ getting the improper uniform prior and the diagonal entries a truncated polynomial prior. They considered a decision-theoretic framework and showed convergence in terms of posterior expected loss. Their rate is sharper with $k^{3/2}$ replaced by $k^{3/2}$, but the result is not exactly comparable as their class of true precision matrices is different. The two become comparable only for near-bandable matrices with exponentially decreasing decay functions, in which case the rates are essentially equivalent. Lee and Lin [72] proposed a prior distribution that is tailored to estimate the bandwidth of large bandable precision matrices. They established strong model selection consistency for the bandwidth parameter along with the consistency of Bayes factors.

A natural fully Bayesian approach to graphical structure selection is to put a prior $p(G)$ on the underlying graph $G$ and then a G-Wishart prior $p(\Omega|G)$ on the precision matrix $\Omega$ given $G$. The joint posterior distribution of $(\Omega, G)$ is then given by $p(\Omega, G|X^{(n)}) \propto p(X^{(n)}|\Omega, G)p(\Omega|G)p(G)$, where $X^{(n)}$ stand for $n$ i.i.d. observations on $X$. Taking a discrete uniform distribution over the space $G$ of decomposable graphs for $p$ variables, we get the joint distribution of the data and the graph $G$, after integrating out $\Omega$ as $p(X^{(n)}, G) = ((2\pi)^{np/2}G)^{-1}I_G(\delta + n, D + nS_n)/I_G(\delta, D)$, where $I_G$ is defined in (17). This immediately leads to an expression for the Bayes factor, on the basis of which model selection may be implemented. Computational techniques in graphical models are discussed in the Supplementary Materials part.

5.2.2 Models without specific sparsity patterns

When the graph does not have an orderly pattern, the precision matrix may be estimated by introducing sparsity in any off-diagonal entries. The most commonly known non-Bayesian method is given by the graphical LASSO (Friedman et al. [42]), which is obtained by maximizing the log-likelihood subject to the $\ell_1$-penalty on the off-diagonal elements. A Bayesian analog, called the Bayesian graphical LASSO, was proposed by Wang [131] by imposing independent Laplace priors on the off-diagonal elements and exponential priors on the diagonal elements, subject to a positive definiteness restriction on $\Omega$. As in a Bayesian LASSO, a block Gibbs sampling method based on the representation of Laplace as a scale mixture of normal is used. Clearly, the Bayesian graphical LASSO is motivated by the desire to make the posterior mode the graphical LASSO, but the main drawback is that the
whole posterior is never supported on sparse precision matrices, and the posterior concentration is suboptimal, as in the case of the Bayesian LASSO. The forceful restriction to positive definiteness also leads to an intractable normalizing constant in the prior, although Wang [131, 132] developed a clever computational trick to avoid the computation of the normalizing constant in posterior sampling, known as scaling-it-up. More details of the computational algorithm are described in the Supplementary Materials part. To alleviate the problem of the lack of sparsity in the posterior, Wang [131] used a thresholding approach proposed in Carvalho et al. [26]. Li et al. [78] proposed using the horseshoe prior on the off-diagonal elements instead of Laplace, thus leading to the ‘graphical horseshoe’ procedure. They also developed an analogous block Gibbs sampling scheme using a variable augmentation technique for half-Cauchy priors proposed in Makalic and Schmidt [82]. The resulting procedure seems to have a better posterior concentration in terms of the Kullback-Leibler divergence, and smaller bias for non-zero elements. Other shrinkage priors based on scale mixture of uniform distributions were explored in Wang and Pillai [135].

The convergence properties of the posterior distribution of a sparse precision matrix with an arbitrary graphical structure were studied by Banerjee and Ghosal [9]. They considered a prior similar to the Bayesian graphical LASSO except that a large point-mass at 0 is added to the prior distribution of off-diagonal entries, and the total number of non-zero off-diagonal entries is restricted. They derived posterior contraction rate in terms of the Frobenius norm as $\sqrt{(p + s_0) \log p / n}$, where $s_0$ is the number of true non-zero off-diagonal entries of the precision matrix, which is the same as the convergence rate of the graphical LASSO and is optimal in that class. The proof uses the general theory of posterior contraction (Ghosal et al. [47]) by estimating the prior concentration near the truth and the entropy of a suitable subset of precision matrices, which receives most of the prior mass. The sparsity built in the prior helps control the effective dimension, and thus the prior concentration and entropy. Control over the eigenvalues allows linking the Frobenius norm with the Hellinger distance on the normal densities. Their technique of proof extends to soft-spike-and-slab and continuous shrinkage priors, as long as the prior concentration at zero off-diagonal values is sufficiently high.

Niu et al. [93] addressed the problem of graph selection consistency under model misspecification. In the well-specified case where the true graph is decomposable, strong graph selection consistency holds under certain assumptions on graph size and edges, using a G-Wishart prior. For the misspecified case, where the true graph is non-decomposable, they showed that the posterior distribution of the graph concentrates on the set of minimum triangulations of the true graph.

5.3 High dimensional discriminant analysis

Consider a classification problem based on a high-dimensional predictor $X = (X_1, \ldots, X_p) \sim N_p(\mu, \Omega^{-1})$, where $(\mu, \Omega) = (\mu_1, \Omega_1)$ for an observation from the
first group and $(\mu, \Omega) = (\mu_2, \Omega_2)$ when the observation comes from the second group. Linear and quadratic discriminant analyses are popular model-based classification methods. The oracle Bayes classifier, which uses the true values of $\mu_1, \mu_2$ and $\Omega_1, \Omega_2$, has the lowest possible misclassification error. Using a Bayesian procedure with priors on $\mu_1, \mu_2$ and $\Omega_1, \Omega_2$, the performance can be nearly matched with the oracle if the posterior distributions of $\mu_1, \mu_2$ and $\Omega_1, \Omega_2$ contract near the true values sufficiently fast. In the high-dimensional setting, this is possible only if there is some lower-dimensional structure like sparsity. Du and Ghosal [41] considered a prior on a precision matrix $\Omega$ based on a sparse modified Cholesky decomposition $\Omega = LDL^T$, where $L$ is sparse and lower triangular with diagonal entries 1 and $D$ is a diagonal matrix. The greatest benefit of using a Cholesky decomposition in the sparse setting is that the positive definiteness restriction on $\Omega$ is automatically maintained. However, a drawback is that the sparsity levels in the rows of $\Omega$ are dependent on the ordering of the coordinates, which may be somewhat arbitrary. To alleviate the problem, with a hard-spike-and-slab prior for the entries of $L$, the probability of a non-zero entry should be decreased with the row index $i$ proportional to $i^{-1/2}$. Then it follows that for $i$ and $j$ roughly proportional and both large, the probability of a non-zero at the $i$th and $j$th rows of $\Omega$ are roughly equal. Du and Ghosal [41] used a soft-spike-and-slab or the horseshoe priors, for which also a stationary (approximate) sparsity in the off-diagonal elements of $\Omega$ can be maintained by making the slab probability decay like $i^{-1/2}$ for the former and the local parameter decay at this rate for the horseshoe prior. Du and Ghosal [41] showed that the misclassification rate of the Bayes procedure converges to that of the oracle Bayes classifier for a general class of shrinkage priors when $p^2(\log p)/n \to 0$, provided that the number of off-diagonal entries in the true $\Omega$ is $O(p)$. This is a substantial improvement over the requirement $p^4/n \to 0$ needed to justify this convergence without assuming sparsity of $\Omega$.

5.4 Exponential family with a graphical structure

The representation of various graphical models as exponential families facilitates the inferential problem of graph selection and structure learning. Suppose $X = (X_1, \ldots, X_p)$ is a $p$-dimensional random vector and $G = (V,E)$ is an undirected graph. The exponential family graphical model (with respect to $G$) has the joint density (with respect to a fixed dominating measure $\mu$ like the Lebesgue or the counting measure) of the form

$$p(X; \theta) = \exp\left\{ \sum_{r \in V} \theta_r B(X_r) + \sum_{(r,t) \in E} \theta_{rt} B(X_r)B(X_t) + \sum_{r \in V} C(X_r) - A(\theta) \right\}, \quad (6)$$

for sufficient statistics $B(\cdot)$, base measure $C(\cdot)$ and log-normalization constant $A(\theta)$. Note that for the choice of $B(X) = X/\sigma, C(X) = -X^2/2\sigma^2$, we get the Gaussian graphical model.
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\[ p(X; \theta) \propto \exp\left\{ \sum_{r \in V} \frac{1}{\sigma_r} \theta_r X_r + \sum_{(r,t) \in E} \frac{1}{\sigma_r \sigma_t} \theta_r X_r X_t - \sum_{r \in V} \frac{1}{\sigma_r^2} X_r^2 \right\}; \quad (7) \]

here \( \{\theta_r\} \) are the elements of the corresponding precision matrix.

An alternative formulation using a matrix form, called an exponential trace-class model, was recently proposed by Zhuang et al. [145]. The family of densities for \( X \) is indexed by a \( q \times q \) matrix \( M \), and is given by the expression

\[ f(X|M) = \exp[-\langle M, T(X) \rangle + \xi(X) - \gamma(M)], \]

where \( \langle \cdot, \cdot \rangle \) stands for the trace inner product of matrices, and \( T: \mathbb{R}^p \to \mathbb{R}^{q \times q} \). \( q \) may or may not be the same as \( p \), and \( \gamma(M) \) is the normalizer for the corresponding exponential family given by

\[ \gamma(M) = \log \int \exp[-\langle M, T(X) \rangle + \xi(X)] d\mu(X). \]

The Gaussian graphical model is included as \( q = p, T(X) = XX^T \). In this case, \( M \) agrees with the precision matrix and is positive definite. The most interesting feature in an exponential trace class model is that the conditional independence of \( X_i \) and \( X_j \) given others is exactly characterized by \( M_{ij} = 0 \). In the high-dimensional setting, it is sensible to impose sparsity of the off-diagonal entries. In the Bayesian setting, this can be addressed by spike-and-slab or continuous shrinkage-type priors.

5.4.1 Ising model

For Bernoulli random variables defined over the nodes of the graph \( G \), the exponential family representation of the graphical model with the choice of \( B(X) = X \) and the counting measure as the base measure gives the distribution

\[ p(X; \theta) = \exp\left\{ \sum_{r \in V} \theta_r X_r + \sum_{(r,t) \in E} \theta_r X_r X_t - A(\theta) \right\}, \quad (8) \]

which is popularly known as the Ising model. The conditional distribution of the nodes gives a logistic regression model. The model may also be represented as an exponential trace class model. Graph selection can be achieved via neighborhood-based variable selection method (Meinshausen and Bühlmann [84]) using the \( \ell_1 \)-penalty. In the high-dimensional setting, Barber and Drton [11] used the Bayesian Information Criteria (BIC) in the logistic neighborhood selection approach. Sparsity based conditional MLE approach was proposed by Yang et al. [140]. In the Bayesian setting, continuous shrinkage priors or spike-and-slab priors may be used for inference on the graphical structure. Variational methods for inference on the parameters and evaluation of the log-partition function \( A(\theta) \) were discussed in Wainright and Jordan [130].

5.4.2 Other exponential family graphical models

Count data are often obtained from modern next-generation genomic studies. To study relations in count data, variables can be modeled via a Poisson distribution on the nodes, giving the Poisson graphical model.
\[
p(X; \theta) = \exp\left\{ \sum_{r \in V} (\theta_r X_r - \log(X_r!)) + \sum_{(r,t) \in E} \theta_r X_t - A(\theta) \right\}.
\]

Integrability forces \(\theta_{rt} \leq 0\), so that only negative conditional relationships between variables are captured. An alternative formulation of a Poisson graphical model as an exponential trace class model can accommodate positive interactions between variables in the Poisson graphical model, where marginal distributions are Poisson although the conditional distributions are not (Zhuang et al. [145]). Other useful exponential families include the multinomial Ising model.

### 5.5 Nonparanormal graphical model

A semiparametric extension of GGMs that can model continuous variables located at nodes of a graph is given by the nonparanormal model (Liu et al. [79]), where it is assumed that the vector of variables \(X = (X_1, \ldots, X_p) \in [0, 1]^p\) reduces to a multivariate normal vector through \(p\) monotone increasing transformations: for some monotone functions \(f_1, \ldots, f_p\), \(f(X) := (f_1(X_1), \ldots, f_p(X_p)) \sim N_p(\mu, \Sigma)\) for some \(\mu \in \mathbb{R}^p\) and positive definite matrix \(\Sigma\). The model is not identifiable and needs to fix the location and scale of the functions or the distribution. Liu et al. [79] developed a two-step estimation process in which the functions were estimated first using a truncated empirical through the relations \(f_j(x) = \Phi^{-1}(F_j(x))\), where \(F_j\) stands for the cumulative distribution function of \(X_j\). Two different Bayesian approaches were considered by Mulgrave and Ghosal [90, 89, 88] — based on imposing a prior on the underlying monotone transforms in the first two papers, and based on a rank-likelihood which eliminates the role of the transformations in the third. In the first approach, a finite random series based on a B-spline basis expansion is used to construct a prior on the transformation. The advantage of using a B-spline basis is that in order to maintain monotonicity, the prior on the coefficients only needs to be made increasing. A multivariate normal prior truncated to the cone of ordered values can be conveniently used. However, to ensure identifiability, two constants \(f(0) = 1/2\) and \(f(3/4) - f(1/4) = 1\) are imposed, which translate to linear constraints, and hence the prior remained multivariate normal before imposing the order restriction. Samples from the posterior distribution of the ordered multivariate normal coefficients can be efficiently obtained using the exact Hamiltonian MCMC (Packman and Paninski [96]). In Mulgrave and Ghosal [89], a normal soft-spike-and-slab prior was put on the off-diagonal elements of \(\Omega = \Sigma^{-1}\) and the scaling-it-up technique (Wang [132]) was utilized to avoid the evaluation of the intractable normalizing constant arising from the positive definiteness restriction. They also proved a consistency result for the underlying transformation in terms of a pseudo-metric given by the uniform distance on a compact subinterval of \((0, 1)\). The approach in Mulgrave and Ghosal [89] used a connection with the problem of regression of a component given the following ones, where these partial regression coefficients form the basis of a Cholesky decomposition for the precision matrix. Sparsity in these coef-
ficients was introduced through continuous shrinkage priors by increasing sparsity with the index as in Subsection 5.3. The resulting procedure is considerably faster than the direct approach of Mulgrave and Ghosal [90]. Mulgrave and Ghosal [89] also considered the mean-field variational approach for even faster computation. In Mulgrave and Ghosal [88], the posterior distribution is altered by replacing conditioning on the data by conditioning on the ranks, which are the maximal invariants under all monotone transformations. The benefit is that then the likelihood is free of the transformations, eliminating the need to assign prior distributions on these, and may be considered as fixed, even though those are unknown. The absence of the nonparametric part of the parameter thus makes the procedure a lot more efficient and robust. The posterior can be computed by Gibbs sampling using a simple data-augmentation technique with the transformed variables. Also the arbitrary centering and scaling means that only a scale-invariant of $\Omega$ is identifiable. Mulgrave and Ghosal [88] also derived a consistency result in a fixed dimensional setting using Doob’s posterior consistency theorem (see Ghosal and van der Vaart [48]).

An alternative way of expressing a joint distribution is by using the notion of a copula function $C(x_1, \ldots, x_p)$, which is a joint distribution function with uniform marginals and unspecified one-dimensional distribution functions $F_1, \ldots, F_p$ on $\mathbb{R}$ (Trivedi and Zimmer [118]). It may be noted that the structure of nonparanormality may be equivalently expressed by a Gaussian copula model with $C(x_1, \ldots, x_p) = \Phi_p(\Phi^{-1}(x_1), \ldots, \Phi^{-1}(x_p); R)$, where $\Phi_p(\cdot; R)$ is the joint multivariate normal distribution with mean zero and a given covariance matrix $R$, by taking $C$ the joint distribution function of $(\Phi(f_1(X_1) - \mu_1), \ldots, \Phi(f_p(X_p) - \mu_p))$. Pitt et al. [101] constructed a prior distribution and proposed an efficient posterior sampling technique in a Gaussian copula setting to make inference about conditional dependence. A convenient method of constructing a multidimensional copula is by using a vine copula, which combines pairwise distributions to form a joint distribution, allowing to introduce conditional independence sequentially while constructing a prior; see Gruber and Czado [54] for a method and the corresponding posterior computing technique.

6 Estimating a long vector smoothly varying over a graph

Let us revisit the normal sequence model $X_i = \theta_i + \epsilon_i, i = 1, \ldots, n$, where $\epsilon_i \sim N(0, \sigma^2)$ independently, as in Section 2, but the sparsity assumption on the vector of means $f = (\theta_1, \ldots, \theta_n)$ is not appropriate. Instead, the values are assumed to ‘smoothly vary over locations’ in some appropriate sense. The simplest situation is that these values lie over a linear lattice, but more generally, the positions may stand for the nodes on a graph. Smooth variation of $\theta_i$ with respect to $i$ over a graph can be mathematically quantified through the graph Laplacian $L = D - A$, where $D$ is the diagonal matrix of node degrees, and $A$ stands for the adjacency matrix. More precisely, a vector $f = (\theta_1, \ldots, \theta_n)$ over a graph of size $n$ with graph Laplacian $L$, is said to belong to a Hölder ball of $\beta$-smoothness of radius $Q$ if $\langle f, (I + (n^2/L)^\beta) f \rangle \leq Q^2$, where
where \( r \) stands for the “dimension” of the graph defined through the growth of eigenvalues of the graph Laplacian \( L \): the \( i \)th eigenvalue grows like \( (i/n)^{2/r} \). For lattice graphs, the dimension truly agrees with the physical dimension, but in general, it can be a fractional number. In this setting, Kirichenko and van Zanten [68] showed that the minimax rate of recovery is \( n^{-\beta/(2\beta+r)} \), which depends on the smoothness \( \beta \) as well as the dimension \( r \) of the graph. A Bayesian procedure for this problem was developed by Kirichenko and van Zanten [67] using a multivariate normal prior: 

\[
 f \sim N_n(0, (n/c)^{(2\alpha+r)/r}(L + n^{-2}I)^{(2\alpha+\beta)/r}),
\]

where \( c \) is given the standard exponential prior. Using van der Vaart-van Zanten posterior contraction rate theory for a Gaussian process prior (Ghosal and van der Vaart [48], Chapter 11), they showed that the posterior contracts at the optimal rate \( n^{-\beta/(2\beta+r)} \) up to a logarithmic factor, whenever the true smoothness is \( \beta \leq \alpha + r/2 \). An analogous result also holds for binary regression. Kirichenko and van Zanten [67] also showed that the limited range adaptation can be strengthened to full range adaptation using an exponential of Laplacian covariance kernel.

An extension of the setup of Kirichenko and van Zanten [67] for functional data with the means taking values in some Hilbert space was considered by Roy and Ghosal [109]. They introduced a notion of Sobolev smoothness classes indexed by two smoothness indexes, \( \beta \) for the graphical smoothness and \( \gamma \) for the functional smoothness, and showed that the minimax rate of recovery is \( n^{-\beta \gamma/(2\beta r + \beta + r)} \), where as above, \( r \) stands for the dimension of the graph. They designed Bayes procedures which achieve this rate exactly, and also adaptively within a logarithmic factor. In a non-adaptive setting, they also showed that a slightly inflated Bayesian credible ball of optimal size has adequate frequentist coverage.

### 7 Matrix models

A number of high-dimensional problems involve unknown matrices instead of vectors. In Section 5, we discussed the literature on Bayesian inference for covariance and precision matrices of a high dimensional observation. In this section, we review results about observations that are obtained in the form of a large matrix. Some examples include multiple linear regression with group sparsity, multi-task learning, matrix completion, stochastic block model and biclustering.

#### 7.1 Generic results in structured linear models

Gao et al. [43] derived results on posterior dimensionality and contraction rates for many models simultaneously in a general ‘signal plus noise’ model when the signal is a vector or a matrix having some specific structure. One example is multiple linear regression with group sparsity, where one observes \( Y = XB + W \), with \( X \) being an \( n \times p \) matrix, \( B \) a \( p \times m \) matrix, and the columns of \( B \) are assumed to share
common support of size $s$. They obtained the minimax rate $s(m + \log(ep/s))$ for the prediction problem of estimating $XB$ in Frobenius norm. This model is a special case of so-called multi-task learning problems, where the columns of $B$ share some specific structure. For instance, instead of assuming a joint sparsity pattern among columns, one may assume that columns of $B$ can only be chosen from a given list of $k$ possible columns, with $k$ typically much smaller than $m$. The posterior rate $pk + m\log k$ was derived for prediction in Gao et al. [43] (it is optimal as soon as $k < pm$). Dictionary learning, on the other hand, assumes that the signal matrix of size $n \times d$ is $\theta = QZ$, for $Q$ an $n \times p$ dictionary matrix and $Z$ a discrete matrix of size $p \times d$ with sparse columns. Gao et al. [43] derived the adaptive rate $np + ds\log(ep/s)$ for estimating $\theta$, which is optimal if smaller than $nd$. Kim and Gao [66] proposed an EM-type algorithm to simulate from posterior aspects corresponding to the prior considered in Gao et al. [43], recovering as a special case the EMVS algorithm of Ro˘ckova and George [107] for linear regression. Belitser and Nurushev [14] also followed this general framework, considering a re-centered normal prior extending the approach in Belitser and Nurushev [15] and Belitser and Ghosal [13], and derived both local oracle posterior rates, as well as optimal-size credible balls with guaranteed coverage.

7.2 Stochastic block model and community detection

In a stochastic block model (SBM) with $k$ groups, one observes $Y = \theta + W$ with $\theta_{ij} = Qz(i)z(j) \in [0,1]^{n \times n}$ for some matrix $Q$ of size $k \times k$ of edge probabilities, a labeling map $z \in \{1, \ldots, k\}^n$ and $W$ a centered Bernoulli noise. Gao et al. [43] treated the question of the estimation of $\theta$ in Frobenius norm within their general framework, getting the adaptive minimax rate $k^2 + n\log k$ with unknown $k$. A similar result and coverage of credible sets were derived by Belitser and Nurushev [14]. Pati and Bhattacharya [98] obtained the near-optimal rate $k^2 \log(n/k) + n\log k$ for known $k$ using independent uniform prior on coordinates of $Q$ and Dirichlet prior probabilities for the label proportions. The biclustering model can be viewed as an asymmetric extension of the SBM model, where $\theta$ is a $n \times m$ rectangular matrix and rows and columns of $Q$ have their own labeling, with $k$ and $l$ groups respectively. This model was handled in Gao et al. [43] and Belitser and Nurushev [14] using similar techniques as before, with the optimal adaptive posterior rate $kl + n\log k + m\log l$.

Another popular question is that of the recovery of the vector of labels $z$ (possibly up to a permutation), also known as community detection. This is a more ambitious task compared to the estimation of $\theta$, and it can only be guaranteed either asymptotically (Bickel et al. [20]), or non-asymptotically imposing some separation between classes (Lei and Zhu [73]); see Castillo and Orbanz [29] for a discussion on uniformity issues. Van der Pas and van der Vaart [126] showed that the posterior mode corresponding to a beta prior on edge probabilities and Dirichlet prior probabilities for the label proportions asymptotically recovers the labels when the number of groups $k$ is known, provided that the mean degree of the graph is at least
of order \( \log^2 n \). Their approach relies on studying the Bayesian modularity, that is, the marginal likelihood of the class labels given the data, when the edge probabilities are integrated out. Kleijn and van Waaij [69] and van Waaij and Kleijn [128] derived estimation and uncertainty quantification results for posterior distributions in the special case of the planted multi-section model, where the matrix \( Q \) has only two different values, one for diagonal elements and another for off-diagonal ones.

### 7.3 Matrix completion problem

In the matrix completion model (also known as the “Netflix problem”), one observes \( n \) noisy entries of an unknown \( m \times p \) matrix \( M \) typically assumed to be of low-rank \( r \) (or well-approximated by a low-rank matrix). The entries are sampled at random from a distribution on entries indices. The rate \( (m + p)r \log \max(m, p) \), minimax optimal up to the logarithmic term, was obtained by Mai and Alquier [81] in the PAC-Bayesian setting for a prior sitting close to small-rank matrices and a class of tempered posteriors (see the discussion section below for more on this topic) while Suzuki [116] derived similar results for posterior distributions (under assumptions on the law of the design) and also obtained results in the more general case of tensors.

### 8 Further topics

#### 8.1 Variational Bayes

Variational Bayes (VB) methods have gained much popularity recently as a possible way to compute certain posterior approximations efficiently. The VB method consists of first choosing a class \( \mathcal{Q} \) of computationally tractable distributions. For instance, \( \mathcal{Q} \) can be chosen to be the set of distributions that make the coordinates independent, the so-called mean-field class; another choice of \( \mathcal{Q} \) is the set of parametric distributions of a given dimension. Second, one chooses a loss, very often the Kullback-Leibler divergence, leading to the random distribution

\[
\tilde{\Pi}_X = \arg\min_{Q \in \mathcal{Q}} \text{KL}(Q|| \Pi[\cdot|X]).
\]

In other words, the variational posterior \( \tilde{\Pi}_X \) is the best approximation from the class \( \mathcal{Q} \) of the given posterior \( \Pi[\cdot|X] \). A theory for convergence rates of variational posteriors and \( \alpha \)-posteriors (\( \alpha < 1 \), corresponding to a likelihood in Bayes’ formula raised to the power \( \alpha \)) was obtained in Alquier and Ridgway [2], Zhang and Gao [144] (where a link between VB and a class of empirical Bayes procedures was made) and Yang et al. [141]. The work Chérief-Abdellatif [36] investigated con-
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Convergence of variational posteriors in a number of settings, including mixture models and deep neural nonparametric regression.

In the context of high-dimensional linear regression of Section 4, variational posteriors are very popular and have been considered among others in Carbonetto and Stephens [23] and Ormerod et al. [95], taking a set of spike-and-slab distributions as variational class. Ray and Szabó [104] have analyzed the convergence rate of the corresponding VB–posterior based on a prior with Laplace slabs with parameter $\lambda$ and the mean–field variational class

$$Q = \left\{ \prod_{i=1}^{p} \left[ \gamma_i N(\mu_i, \sigma_i^2) + (1 - \gamma_i) \delta_0 \right], \ (\mu_i, \sigma_i^2, \gamma_i) \in \mathbb{R} \times \mathbb{R}^+ \times [0,1] \right\}.$$ 

One can obtain the same convergence rates and variable selection properties as for spike–and–slab posteriors, under similar conditions for the parameter $\lambda$ and the design matrix $X$. Laplace slabs should be preferred to Gaussian slabs in the prior distribution (but not so in $Q$), for the same reasons as for plain spike–and–slab. The computation of the VB posterior is an optimization problem which can be treated using the coordinate–ascent variational inference (CAVI) algorithm: the parameters $\gamma_i, \mu_i, \sigma_i^2$ are sequentially updated while approximating at each step the posterior by the variational sub–class that keeps all parameters but one fixed. One main advantage of VB is computational speed. The tuning of the procedure needs some care though — the slab parameter $\lambda$ needs to be chosen, and the output of the CAVI algorithm is particularly sensible in the order in which parameters are updated. Ray and Szabó [104] proposed a prioritized update scheme to handle this issue. Similar results for VB–posteriors for the logistic regression were derived by Ray et al. [105].

### 8.2 Link with PAC-Bayesian approaches

A popular approach in machine learning is using a pseudo- (or generalized) Bayes measure by replacing the likelihood by a small power (also called the ‘temperature’) of it in the standard Bayes posterior distribution. This can attenuate the dependence in the data, providing more robustness in misspecified settings. The log-likelihood may also be replaced by an empirical risk, depending on the loss-function of interest. PAC-Bayesian inequalities refer to the theory delivering in-probability bounds for such pseudo-posterior measures. Seminal contributions to PAC-Bayes theory include those by McAllester, Catoni, and T. Zhang. Theory and algorithms for high-dimensional data were considered among others by Audibert, Alquier, Dalalyan and Tsybakov, Martin and Walker and Grünwald and co-authors. We refer to the survey by Guedj [55] for a recent overview of the field. We note that although such methods are sometimes referred to as generalized Bayes, they often assume that the temperature parameter is strictly smaller than 1 (or going to 0 in some cases), which often separates the corresponding results from those on original Bayes posteriors. Obtain-
ing a unified theory for different ranges of temperature parameters including the original Bayes is an interesting direction for future work.

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Appendix

9.1 Undirected Graphs

An undirected graph $G = (V,E)$ consists of a non-empty set of vertices or nodes $V = \{1, \ldots, p\}$ and a set of edges $E \subseteq \{(i,j) \in V \times V : i < j\}$. Nodes which are connected by an edge are termed as adjacent nodes. If all the nodes of a graph are adjacent to each other, we have a complete graph. A subgraph $G' = (V',E')$ of $G$, denoted by $G' \subseteq G$ is such that $V' \subseteq V$ and $E' \subseteq E$. A subset $V' \subseteq V$ induces the subgraph $G_{V'} = (V', (V' \times V') \cap E)$. If a subgraph $G_{V'}$ is not contained in any other complete subgraph of $G$, that is, if $G_{V'}$ is a maximal complete subgraph of $G$, then, $V' \subseteq V$ is called a clique of $G$.

A finite subcollection of adjacent edges $(v_0, \ldots, v_{k-1})$ in $G$ forms a path of length $k$. If $v_0 = v_{k-1}$, that is, if the end-points of the path are identical, then we have a $k$-cycle. A chord of a cycle is a pair of nodes in that cycle which are not consecutive nodes in that path, but are adjacent in $G$. For subgraphs $G_1, G_2$ and $G_3$ of $G$, if every path from a node $v_1 \in G_1$ to a node $v_2 \in G_2$ contains a node in $G_3$, then $G_3$ is said to separate $G_1$ and $G_2$ and is called a separator of $G$.

A major portion of our discussion would focus on decomposable graphs. A graph $G$ is said to be decomposable if every cycle in $G$ having length greater than or equal to four has a chord. Decomposability of a graph can also be characterized by the existence of a perfect ordering of the cliques. An ordering of the cliques $(C_1, \ldots, C_k) \in \mathcal{C}$ and separators $(S_2, \ldots, S_k) \in \mathcal{S}$ is perfect if it satisfies the running intersection property, that is, there exists an $i < j$ such that $S_j = H_{j-1} \cap C_j \subseteq C_i$, where $H_j = \cup_{i=1}^{j} C_j$. For more details on decomposability and perfect ordering of cliques, we refer the readers to Lauritzen [70].

9.2 Graphical models

An undirected graph $G$ equipped with a probability distribution $P$ on the node-set $V$ is termed as an undirected graphical model. We shall particularly focus on Gaussian data, hence leading to the concept of Gaussian graphical models. To define the same, we first discuss the Markov property, or the conditional independence property of a $p$-dimensional random variable. A $p$-dimensional random variable $X = (X_1, \ldots, X_p)$
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is said to be Markov with respect to an undirected graph \( G \) if the components \( X_i \) and \( X_j \) are conditionally independent given the rest of the variables, whenever \( (i, j) \notin E \). Thus, an undirected graph \( G \) with a Gaussian distribution on the components of the above random variable \( X \) corresponding to the nodes \( V = \{1, \ldots, p\} \) is called a Gaussian graphical model (GGM). Without loss of generality, we can assume that the mean of the Gaussian distribution to be zero. For a GGM, there exists a correspondence between the edge set \( E \) and the inverse covariance matrix (or, the precision matrix) \( \Omega = \Sigma^{-1} \) of \( X \) owing to the Markov property. To be precise, whenever \( (i, j) \notin E \), the \((i, j)\)th element of \( \Omega \), given by \( \omega_{ij} \) is exactly zero and vice versa. This leads us to consider the cone of positive definite matrices

\[
\mathcal{P}_G = \{ \Omega \in \mathcal{M}_p^+ : \omega_{ij} = 0, (i, j) \notin E \},
\]

which defines the parameter space of the precision matrix for a Gaussian distribution defined over a graph \( G \). For a decomposable graph \( G \), the parameter space for the covariance matrix \( \Sigma = (\sigma_{ij}) \) is defined by the set \( \mathcal{D}_G \) of partially positive definite matrices, which is a subset of \( I_G \); the set of incomplete matrices with missing entries \( \sigma_{ij} \) whenever \( (i, j) \notin E \). Then,

\[
\mathcal{D}_G = \{ B \in \mathcal{I}_G : B_{Gi} > 0, i = 1, \ldots, k \}.
\]

Gröne et al. [53] showed that there is a bijection between the spaces \( \mathcal{P}_G \) and \( \mathcal{D}_G \). To be precise, for \( \Omega \in \mathcal{P}_G \), we can define \( \mathcal{D}_G \) as the parameter space for the GGM where \( \Sigma = \kappa(\Omega^{-1}) \), and \( \kappa : \mathcal{M}_p^+ \to \mathcal{I}_G \) is the projection of \( \mathcal{M}_p^+ \) into \( I_G \). Thus, a GGM Markov with respect to a graph \( G \) is given by the family of distributions

\[
\mathcal{N}_G = \{ N_\mathcal{P}(0, \Sigma), \Sigma \in \mathcal{D}_G \} = \{ N_\mathcal{P}(0, \Omega^{-1}), \Omega \in \mathcal{P}_G \}.
\]

### 9.3 Hyper Inverse-Wishart and G-Wishart priors

The inverse-Wishart distribution \( \text{IW}_p(\delta, D) \) with degrees of freedom \( \delta \) and a \( p \)-dimensional positive definite fixed scale matrix \( D \) is the conjugate prior for the covariance matrix \( \Sigma \) in case of a complete graph. We denote \( \Sigma \sim \text{IW}_p(\delta, D) \) having density

\[
p(\Sigma \mid \delta, D) = c(\delta, p)\{\det(\Sigma)\}^{-(\delta+p+1)/2} \exp\{-\text{tr}(\Sigma^{-1}D)/2\},
\]

where \( c(\delta, p) = \{\det(D)/2\}^{(\delta+p-1)/2}/\Gamma_p((\delta + p - 1)/2) \) is the normalizing constant and \( \Gamma_p(t) = \pi^{p(p-1)/4}\prod_{j=1}^p \Gamma(t - (j - 1)/2) \) is the \( p \)-variate gamma function.

For decomposable graphs, recall that a perfect set of cliques always exists. In that case, we can write the density as

\[
p(x \mid \Sigma, G) \propto \frac{\prod_{C \in G} p_{C}(x_C \mid \Sigma_C)}{\prod_{S \in \mathcal{F}} p_S(x_S \mid \Sigma_S)},
\]
which is the Markov ratio of respective marginal Gaussian distributions corresponding to the cliques and separators of $G$. Dawid and Lauritzen [38] came up with a generalization of the inverse-Wishart, called the hyper inverse-Wishart, which is conjugate to the Gaussian distribution Markov with respect to the graph $G$. The form of the prior distribution depends on the clique-marginal covariance matrices $\Sigma_C$, such that $\Sigma_C \sim IW_{|C|}(\delta, D_C)$, where $D_C$ is the submatrix of the scale matrix $C$ induced by the clique $C \in \mathcal{C}$. The hyper inverse-Wishart is thus constructed on the parameter space $\mathcal{Q}_G$ with density given by

$$p_G(\Sigma \mid \delta, D) = \prod_{C \in \mathcal{C}} p(\Sigma_C \mid \delta, D_C) \prod_{S \in \mathcal{S}} p(\Sigma_S \mid \delta, D_S),$$

(14)

where $p(\cdot)$ refers to the density of the respective inverse-Wishart distribution.

For a complete graph, the inverse-Wishart prior on the covariance matrix induces a conjugate prior for the precision matrix $\Omega$, namely, the Wishart distribution, with density

$$p(\Omega \mid \delta, D) = c(\delta, p) (\det(\Omega))^{(\delta - 2)/2} \exp\{-\text{tr}(\Omega D)/2\},$$

(15)

with identical normalizing constant as in inverse-Wishart. Similar in lines with this, for a decomposable graphical model, the hyper inverse-Wishart prior induces a prior distribution on the precision matrix $\Omega$ with density

$$p_G(\Omega \mid \delta, D) = I_G(\delta, D)^{-1} (\det(\Omega))^{(\delta - 2)/2} \exp\{-\text{tr}(D\Omega)/2\}$$

(16)

where $I_G(\delta, A)$ is the normalizing constant given by

$$I_G(\delta, D) = \frac{\prod_{S \in \mathcal{S}} \{\det(D_S)\}^{(\delta + |S| - 1)/2} / \Gamma(|S|) ((\delta + |S| - 1)/2)}{\prod_{C \in \mathcal{C}} \{\det(D_C)\}^{(\delta + |C| - 1)/2} / \Gamma(|C|) ((\delta + |C| - 1)/2)}.$$  

(17)

This distribution on $\Omega$ is called the G-Wishart distribution $W_G$. It forms the Diaconis-Ylvisaker conjugate prior for the precision matrix of a Gaussian distribution Markov with respect to the decomposable graph $G$. However, unlike the hyper inverse-Wishart prior, the G-Wishart prior can be generalized to non-decomposable graphical models as well, although there is no closed form analytical expression of the normalizing constant $I_G(\delta, D)$, except for a result by Uhler et al. [119].

Letac and Massam [76] introduced a more general class of conjugate priors, namely the $W_{\mathcal{G}_G}$-Wishart class of priors for the precision matrix $\Omega$ with three sets of parameters — $\alpha = (\alpha_1, \ldots, \alpha_p)$ and $\beta = (\beta_1, \ldots, \beta_p)$ which are suitable functions defined on the cliques and separators of the graph, and a scale matrix $D$. We note that the above class of prior distribution is the Wishart distribution for a fully connected graph, and includes the G-Wishart distribution for decomposable graphs as a special case with suitable choices of $\alpha$ and $\beta$. 

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9.4 Posterior computation in graphical models

9.4.1 Exact expressions under conjugacy

Denoting by $X$ the $n \times p$ data matrix corresponding to a random sample from a Gaussian distribution $N_p(0, \Omega^{-1})$, conditional on a specific graph $G$, the posterior density $p(\Omega \mid X, G)$ of $\Omega$ is given by

$$I_G(\delta + n, D + S_n)^{-1/2} \det(\Omega)^{\delta + n - 2/2} \exp[-\text{tr}((D + S_n)\Omega)/2], \quad (18)$$

where $S_n = XX^T$.

Carvalho et al. [25] proposed a sampler for the hyper inverse-Wishart distribution corresponding to a decomposable graph based on distributions of submatrices of the covariance matrix $\Sigma$. Samples taken from the hyper inverse-Wishart can be inverted to get samples from the corresponding G-Wishart distribution.

Rajaratnam et al. [103] provided a closed form expression of the posterior mean $E(\Omega \mid S_n)$ of $\Omega$ with a $W_{\cal P}$-G-Wishart prior as

$$-2 \left[ \sum_{j=1}^k (\alpha_j - n/2)((D + \kappa(nS_n)^{-1})_{ji}^{-1})^0 - \sum_{j=2}^k (\beta_j - n/2)((D + \kappa(nS_n)^{-1})_{ji}^{-1})^0 \right], \quad (19)$$

where for a $p \times p$ matrix $A = ((a_{ij}))$, $A_T^{-1}$ denotes the inverse $(A_T)^{-1}$ of the submatrix $A_T$ and $(A_T)^0 = ((a_{ij}^0))$ denotes a $p$-dimensional matrix such that $a_{ij}^0 = a_{ij}$ for $(i, j) \in T \times T$, and 0 otherwise. Here $\alpha_j$ and $\beta_j$ are the $j$th component of the parameters $\alpha$ and $\beta$ respectively, $j = 1, \ldots, p$. They also obtained the expression for the Bayes estimator with respect to the Stein’s loss $L(\hat{\Omega}, \Omega) = \text{tr}(\hat{\Omega} - \Omega)^2$ under the $W_{\cal P}$-G-Wishart prior.

9.4.2 MCMC sampling methods for G-Wishart prior

Madigan and York [80] proposed a Metropolis-Hastings algorithm based approach to traverse the space of decomposable graphs $G$, while Giudicci and Green [51] used a reversible jump MCMC sampler also including the precision matrix as one of the state variables.

When the graph is not necessarily decomposable, the G-Wishart prior is still conjugate, but the normalizing constant $I_G(\delta, D)$ does not have a simple expression. Only recently Uhler et al. [119] derived an expression, but it is still extremely difficult to compute for use in inference. Lenkoski and Dobra [75] developed a Laplace approximation $I_G(\delta, D) = \exp(\ell(\hat{\Omega})) \{(2\pi)^{p/2} \det(H(\hat{\Omega}))^{-1/2},$ where $\ell = ((\delta - 2)\log \det(\Omega) - \text{tr}(D\Omega))/2$, the matrix $\hat{\Omega}$ is the mode of the G-Wishart density and $H$ is the Hessian matrix associated with $\ell$, but it lacks accuracy unless the clique sizes are small.
Atay-Kayis and Massam [5] proposed a Monte Carlo based method to sample from the G-Wishart distribution as well as to compute the prior normalizing constant. They considered the Cholesky decomposition $D^{-1} = Q^T Q$ and $\Omega = \Phi^T \Phi$, and defined $\Psi = \Phi Q$. Since for $(i, j) \notin E$, $\omega_{ij} = 0$, and hence the elements $\psi_{ij}$ of $\Psi$ for $(i, j) \notin E$ are functions of $\psi_{ij}$, $(i, j) \in E$ and $\psi_{ii}$, $i = 1, \ldots, p$. Thus the free elements appearing in the Cholesky decomposition of $\Omega$ are given by $\Psi^E = (\psi_{ij}, (i, j) \in E; \psi_{ii}, i = 1, \ldots, p)$. They showed that the free elements have density $p(\Psi^E) \propto f(\Psi^E) h(\Psi^E)$, where $f(\Psi^E) = \exp\left(-\sum_{(i,j) \notin E} \psi_{ij}^2 / 2\right)$ is a function of the non-free elements of $\Psi$, which in turn can be uniquely expressed in terms of $\Psi^E$. Furthermore, $h(\Psi^E)$ is the product of densities of random variables $\psi_{ii} \sim \chi^2_{\delta+1}$, $i = 1, \ldots, p$, and $\psi_{ij} \sim N(0, 1)$, $(i, j) \in E$, where $\nu_i$ stands for the cardinality of $\{ j : j > i, (i, j) \in E \}$. Generating samples for $\Omega$ then uses an acceptance-rejection method based on the above density function of the free elements $\Psi^E$. The normalizing constant $I_{G}(\delta, D)$ can be expressed as the product of a known constant and the expected value of $f(\Psi^E)$. Using samples from the distribution of $\Psi^E$, straightforward Monte Carlo computation gives $I_G(\delta, D)$. However, the Monte Carlo integration method is computationally expensive for non-complete large prime components of the graph, owing to a matrix completion step involving the non-free elements of $\Psi$. 

To alleviate this problem, Wang and Carvalho [133] used a prime-component decomposition so that sampling is individually carried over in lower-dimensional subgraphs of $G$. However, owing to the dependencies of the sampler on $(\delta, D, G)$, the acceptance rate of the MCMC procedure can be very low. Mitsakakis et al. [86] developed an independent Metropolis-Hastings algorithm for sampling from $\mathcal{W}_G(\delta, D)$ based on the density of $\Psi^E$. This method, though improves the acceptance rate over that proposed by Wang and Carvalho [133], suffers from issues of low acceptance rate and slow mixing in large graphs. Similar in lines with Mitsakakis et al. [86], Dobra et al. [39] used a random walk Metropolis-Hastings sampler, where only one entry of $\Psi^E$ is perturbed in a single step compared to changing all the entries of $\Psi^E$ as in the former approaches. Though this method results in increased efficiency of the sampler, it still suffers from the matrix completion problem for non-free elements, thus requiring a time complexity of $O(p^3)$ for each Monte Carlo samples. This results in painfully slow computations for large graphs.

Jones et al. [65] used the method of Atay-Kayis and Massam [5] to compute prior and posterior normalizing constants of the G-Wishart distribution to traverse the space of graphs using a method called the Stochastic Shotgun Search Algorithm using the uniform or Erdős-Renyi type priors on the space of graphs. Though the method performs well in low dimensions, it fails to scale up in high dimensions owing to large search space.

Lenkoski and Dobra [75] and Mitsakakis et al. [86] proposed to use the Bayesian iterative proportional proportional scaling algorithm developed by Piccioni [100] and Asci and Piccioni [4] to sample from the G-Wishart distribution. Their method requires enumeration of maximum cliques of $G$, which is an NP-hard problem, and also requires the inversion of large matrices, which are computationally burdensome.
As discussed before, Giudicci and Green [51] developed a reversible jump MCMC method to learn the graphical structure by sampling over the joint space of \((\Omega, G)\). Dobra et al. [39] also developed a reversible jump MCMC method over \((\Omega, G)\), thus avoiding the issue of searching over a large space of graphs. But their method cannot avoid the problems involving the computation of the prior normalizing constants and matrix completion. The crucial bottleneck in computing the acceptance probabilities of the MCMC based procedures developed for joint exploration of \((\Omega, G)\) is the ratio of prior normalizing constants 

\[
I_{G-e}(\delta, D)/I_G(\delta, D),
\]

where \(G-e\) is the graph obtained from \(G = (V, E)\) by deleting a single edge \(e \in E\). For the choice of \(D = I_p\), Letac et al. [77] showed that the above ratio can be reasonably approximated by constant times the ratio of two gamma functions as

\[
I_{G-e}(\delta, D)/I_G(\delta, D) \approx (2\sqrt{\pi})^{-1}\Gamma\left(\frac{\delta + d}{2}\right)/\Gamma\left(\frac{\delta + d + 1}{2}\right),
\]

(20)

where \(d\) is the number of paths of length two between the two nodes in edge \(e\). They showed that under certain conditions and graph configurations, the approximation performs reasonably good.

To circumvent the problems of computing the normalizing constant \(I_G(\delta, D)\) for arbitrary graphs, especially in the high dimensional scenario, Wang and Li [134] proposed a double reversible jump MCMC algorithm using the partial analytic structure (PAS) of the G-Wishart distribution. Their proposed algorithm involves a Metropolis-Hastings step to move from \(G = (V, E)\) to \(G' = (V, E')\), where \(E'\) differs in only one edge from \(E\), with the acceptance probability for the Metropolis-Hastings step having exact analytical expressions which are easy to compute.

A completely new approach was introduced by Mohammadi and Wit [87], called the Birth-and-Death MCMC method (BDMCMC) for graphical model selection, which explores the space of graphs by addition (a birth) or deletion (a death) of an edge in the graph. They determine a continuous time birth-death Markov process on \(\Omega\), more specifically, independent Poisson process on \(\Omega\) such that under suitable conditions, the process has stationary distribution \(p(\Omega, G | X^{(n)})\). The birth and death rates are given by

\[
\beta_e(\Omega) = \frac{P(G^+e, \Omega \setminus \omega_{jj}, \omega_{jj} | X^{(n)})}{P(G, \Omega \setminus \omega_{jj} | X^{(n)})}, \quad \text{for each } e \in \bar{E},
\]

\[
\delta_e(\Omega) = \frac{P(G-e, \Omega \setminus \omega_{jj}, \omega_{jj} | X^{(n)})}{P(G, \Omega \setminus \omega_{jj} | X^{(n)})}, \quad \text{for each } e \in E; \quad (21)
\]

here \(G^+e, G-e\) are graphs obtained from \(G\) after addition or removal of an edge \(e\) from \(G\). The direct sampler method proposed in Lenkoski [74] is used to sample from the posterior distribution of \(\Omega\). The BDMCMC algorithm is easy to implement and scales well in high-dimensional problems, and also has superior performance in structure learning compared with Bayesian methods discussed earlier.
9.4.3 MCMC sampling methods for the Bayesian graphical LASSO and variants

Wang [131] developed a block Gibbs sampler to simulate from the posterior distribution of the Bayesian graphical LASSO though a reparameterization \( T = (\{\tau_{ij}\}) \) with zeros as diagonal entries and \( \tau \) in upper diagonal entries. Then, the matrices \( \Omega, S \) and \( T \) are partitioned with respect to the last column and row as

\[
\Omega = \begin{pmatrix} \Omega_{11} & \omega_{12} \\ \omega_{12}^T & \omega_{22} \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & s_{12} \\ s_{12} & s_{22} \end{pmatrix}, \quad T = \begin{pmatrix} T_{11} & T_{12} \\ T_{12}^T & 0 \end{pmatrix}.
\] (22)

With the change of variable \((\omega_{12}, \omega_{22}) \rightarrow (\beta = \omega_{12}, \gamma = \omega_{22} - \omega_{12}^T \Omega_{11}^{-1} \omega_{12})\), the conditional posterior distribution of \((\beta, \gamma)\) given the rest is given by

\[
p(\beta, \gamma, \Omega_{11}, T, X^{(s)}, \lambda) \propto \gamma^{p/2} \exp\left( \frac{s_{12} + \lambda}{2} \right) \times \exp\left( -\frac{1}{2} \beta^T \{D_{T}^{-1} + (s_{22} + \lambda) \Omega_{11}^{-1}\} \beta + 2s_{12}^T \beta \right),
\]

where \( D_{T} = \text{diag}(\tau_{12}) \), so that the conditional distributions of \( \gamma \) and \( \beta \) are independent gamma and a normal respectively, whereas the inverse of the latent scale parameters \( \tau_{ij} \) are independent inverse-Gaussian distributions. The resulting block Gibbs sampler iteratively samples one column of \( \Omega \) at a time. It is interesting to note that the positive definiteness constraint is also maintained throughout owing to the fact that \( \gamma \) is always positive definite. The posterior samples hence obtained cannot be directly used for structure learning though, since the Bayesian graphical LASSO prior puts zero mass on the event \( \{\omega_{ij} = 0\} \).

As discussed in the previous sections, computation poses a great deal of challenge in Bayesian graphical models for high dimensional situations. To deal with large dimensions in arbitrary graphs, Wang [132] developed a new technique called stochastic search structure learning for precision matrices as well as covariance matrices in a graphical model set up, based on soft-spike-and-slab priors on the elements of the matrix. This work also focuses on the issue of structure learning of graphs using a fully Bayesian model by specifying priors on binary variables \( Z = (z_{ij})_{i<j} \), which are indicators for the edges of the underlying graph \( G \). The hierarchical prior is then specified as

\[
p(\Omega \mid Z, \theta) = C(Z, \nu_0, \nu_1, \lambda)^{-1} \prod_{i<j} N(\omega_{ij} \mid 0, \nu_{ij}^2) \prod_{i=1}^p \text{Exp}(\omega_i \mid \lambda/2),
\]

\[
p(Z \mid \theta) = C(\theta)^{-1} C(Z, \nu_0, \nu_1, \lambda) \prod_{i<j} \pi^{z_{ij}} (1 - \pi)^{1-z_{ij}},
\] (23)

where \( \theta = (\nu_0, \nu_1, \pi, \lambda) \), \( N(\cdot \mid 0, \nu^2) \) is the density of a normal distribution with mean zero and variance \( \nu^2 \), \( C(\theta) \) and \( C(Z, \nu_0, \nu_1, \lambda) \) are normalizing constants, and \( \nu_{ij} \) is \( \nu_0 \) or \( \nu_1 \) according as \( z_{ij} \) is 0 or 1. The hyperparameter \( \pi \in (0, 1) \) controls the prior distribution of the binary edge indicators in \( Z \). The hierarchical prior specifi-
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cation above leads to the following prior specification on $\Omega$:

$$
p(\Omega) = C(\theta)^{-1} \prod_{i<j}(1-\pi)N(\omega_{ij} \mid 0, \nu_0^2) + \pi N(\omega_{ij} \mid 0, \nu_1^2) \\
\times \prod_{i=1}^p \text{Exp}(\omega_{ii} \mid \lambda/2) 1(\Omega \in \mathcal{M}^+),
$$

(24)

A small $\nu_0 > 0$ and a large value of $\nu_1 > 0$ induces a soft-spike-and-slab prior on the elements of $\Omega$. The above prior having a two-component mixture of normals, facilitates in graph structure learning via the latent binary indicators $Z$. The sampling procedure for generating samples from the posterior distribution $p(\Omega \mid X^{(n)}, Z)$ is identical to the block Gibbs sampler proposed in Wang [131], via introduction of the $p$-dimensional symmetric matrix $V = (V_{ij}^2)$ with zeros in diagonal and $(V_{ij}^2 : i < j)$ as the upper-diagonal entries, where $\nu_{ij}^2 = \nu_{ij}^2$. The conditional posterior distributions of the binary indicator variables $Z$ given the data and $\Omega$ are independent Bernoulli with success probability

$$
p(z_{ij} = 1 \mid \Omega, X^{(n)}) = \frac{\pi N(\omega_{ij} \mid 0, \nu_1^2)}{\pi N(\omega_{ij} \mid 0, \nu_1^2) + (1-\pi)N(\omega_{ij} \mid 0, \nu_0^2)}.
$$

(25)

Though the structure learning accuracy of the above method is comparable to other Bayesian structure learning methods, the block Gibbs sampler based computational approach makes it computationally simple and faster, especially for scaling up in large dimensions.

9.4.4 Laplace approximation to compute posterior model probabilities

For graphical structure learning, in the absence of explicit expressions, one may compute the posterior probabilities of various models by reversible jump Markov chain Monte Carlo, which is computationally expensive. Banerjee and Ghosal [9] proposed to directly compute the marginal posterior probabilities of models using a Laplace approximation method. The main idea is to expand the log-likelihood around the posterior mode, which can be easily identified as the graphical LASSO within that model and computed through efficient algorithms, and integrate the resulting approximate likelihood function. The resulting computation is very fast, but a drawback is that the approach works only for ‘regular models’, where none of the components of the posterior mode is 0, because the log-likelihood function is singular at any point having a coordinate 0. The problem is partly alleviated by the fact that for every ‘non-regular model’, there is a regular submodel with higher posterior probability. Hence, at least for model selection, the approach can be restricted to the latter class.
References

[1] K. Abraham, I. Castillo, and E. Roquin. FDR control of a posterior-based FDR procedure under arbitrary alternatives. 2020+. Manuscript in preparation.
[2] P. Alquier and J. Ridgway. Concentration of tempered posteriors and of their variational approximations. The Annals of Statistics, 48(3):1475–1497, 2020.
[3] A. Armagan, D. B. Dunson, and J. Lee. Generalized double Pareto shrinkage. Statistica Sinica, 23(1):119, 2013.
[4] C. Asci and M. Piccioni. Functionally compatible local characteristics for the local specification of priors in graphical models. Scandinavian Journal of Statistics, 34(4):829–840, 2007.
[5] A. Atay-Kayis and H. Massam. A Monte-Carlo method for computing the marginal likelihood in nondecomposable Gaussian graphical models. Biometrika, 92(2):317–335, 2005.
[6] Y. A. Atchadé. On the contraction properties of some high-dimensional quasi-posterior distributions. The Annals of Statistics, 45(5):2248–2273, 2017.
[7] S. Banerjee. Posterior convergence rates for high-dimensional precision matrix estimation using G-Wishart priors. Stat, 6(1):207–217, 2017.
[8] S. Banerjee and S. Ghosal. Posterior convergence rates for estimating large precision matrices using graphical models. Electronic Journal of Statistics, 8(2):2111–2137, 2014.
[9] S. Banerjee and S. Ghosal. Bayesian structure learning in graphical models. Journal of Multivariate Analysis, 136:147–162, 2015.
[10] Y. Baraud. Non-asymptotic minimax rates of testing in signal detection. Bernoulli, 8(5):577–606, 2002.
[11] R. F. Barber and M. Drton. High-dimensional Ising model selection with Bayesian information criteria. Electronic Journal of Statistics, 9(1):567–607, 2015.
[12] E. Belitser. On coverage and local radial rates of credible sets. The Annals of Statistics, 45(3):1124–1151, 2017.
[13] E. Belitser and S. Ghosal. Empirical Bayes oracle uncertainty quantification for regression, 2020. Advance publication.
[14] E. Belitser and N. Nurushev. General framework for projection structures. 2019. Preprint arXiv:1904.01003.
[15] E. Belitser and N. Nurushev. Needles and straw in a haystack: robust empirical Bayes confidence for possibly sparse sequences. Bernoulli, 26:191–225, 2020.
[16] Y. Benjamini and Y. Hochberg. Controlling the false discovery rate: a practical and powerful approach to multiple testing. Journal of the Royal Statistical Society. Series B., 57(1):289–300, 1995.
[17] A. Bhadra, J. Datta, N. G. Polson, and B. Willard. The horseshoe+ estimator of ultra-sparse signals. Bayesian Analysis, 12(4):1105–1131, 12 2017.
[18] A. Bhattacharya, A. Chakraborty, and B. K. Mallick. Fast sampling with Gaussian scale mixture priors in high-dimensional regression. *Biometrika*, 103:985–991, 2016.

[19] A. Bhattacharya, D. Pati, N. S. Pillai, and D. B. Dunson. Dirichlet-Laplace priors for optimal shrinkage. *Journal of the American Statistical Association*, 110(512):1479–1490, 2015.

[20] P. Bickel, D. Choi, X. Chang, and H. Zhang. Asymptotic normality of maximum likelihood and its variational approximation for stochastic blockmodels. *The Annals of Statistics*, 41(4):1922–1943, 2013.

[21] P. Bühlmann and S. van de Geer. *Statistics for High-dimensional Data*. Springer, Berlin, 2011.

[22] T. T. Cai and T. Jiang. Limiting laws of coherence of random matrices with applications to testing covariance structure and construction of compressed sensing matrices. *The Annals of Statistics*, 39(3):1496–1525, 2011.

[23] P. Carbonetto and M. Stephens. Scalable variational inference for Bayesian variable selection in regression, and its accuracy in genetic association studies. *Bayesian Analysis*, 7(1):73–107, 2012.

[24] F. Caron and A. Doucet. Sparse Bayesian nonparametric regression. In *ICML*, pages 88–95, 2008.

[25] C. Carvalho, H. Massam, and M. West. Simulation of hyper-inverse Wishart distributions in graphical models. *Biometrika*, 94(3):647–659, 2007.

[26] C. M. Carvalho, N. G. Polson, and J. G. Scott. The horseshoe estimator for sparse signals. *Biometrika*, 97(2):465–480, 2010.

[27] I. Castillo and R. Mismer. Empirical Bayes analysis of spike and slab posterior distributions. *Electronic Journal of Statistics*, 12(2):3953–4001, 2018.

[28] I. Castillo and R. Mismer. Sharp asymptotic minimaxity of spike and slab posterior distributions. 2020. Manuscript in preparation.

[29] I. Castillo and P. Orbanz. Uniform estimation of a class of random graph functionals. 2017. Preprint arXiv:1703.03412.

[30] I. Castillo and E. Roquain. On spike and slab empirical Bayes multiple testing. *The Annals of Statistics*, 48(5):2548–2574, 2020.

[31] I. Castillo, J. Schmidt-Hieber, and A. van der Vaart. Bayesian linear regression with sparse priors. *The Annals of Statistics*, 43(5):1986–2018, 2015.

[32] I. Castillo and B. Szabó. Spike and slab empirical Bayes sparse credible sets. *Bernoulli*, 26(1):127–158, 2020.

[33] I. Castillo and A. W. van der Vaart. Needles and straw in a haystack: posterior concentration for possibly sparse sequences. *The Annals of Statistics*, 40(4):2069–2101, 2012.

[34] M.-H. Chen, L. Huang, J. G. Ibrahim, and S. Kim. Bayesian variable selection and computation for generalized linear models with conjugate priors. *Bayesian Analysis*, 3(3):585–614, 2008.

[35] M.-H. Chen and J. G. Ibrahim. Conjugate priors for generalized linear models. *Statistica Sinica*, pages 461–476, 2003.
[36] B.-E. Cherief-Abdellatif. Contributions to the Theoretical Study of Variational Inference and Robustness. Thesis, Institut Polytechnique de Paris, June 2020.

[37] J. Datta and J. K. Ghosh. Asymptotic properties of Bayes risk for the horseshoe prior. *Bayesian Analysis*, 8(1):111–132, 2013.

[38] A. Dawid and S. Lauritzen. Hyper Markov laws in the statistical analysis of decomposable graphical models. *The Annals of Statistics*, 21(3):1272–1317, 1993.

[39] A. Dobra, A. Lenkoski, and A. Rodriguez. Bayesian inference for general Gaussian graphical models with application to multivariate lattice data. *Journal of the American Statistical Association*, 106(496):1418–1433, 2011.

[40] D. L. Donoho, I. M. Johnstone, J. C. Hoch, and A. S. Stern. Maximum entropy and the nearly black object. *Journal of the Royal Statistical Society. Series B.*, 54(1):41–81, 1992. With discussion and a reply by the authors.

[41] X. Du and S. Ghosal. Bayesian discriminant analysis using a high dimensional predictor. *Sankhya, Series A*, 80:1–34, 2018.

[42] J. Friedman, T. Hastie, and R. Tibshirani. Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441, 2008.

[43] C. Gao, A. W. van der Vaart, and H. H. Zhou. A general framework for Bayes structured linear models. *The Annals of Statistics*, 48:2848–2878, 2020.

[44] E. I. George and D. P. Foster. Calibration and empirical Bayes variable selection. *Biometrika*, 87(4):731–747, 2000.

[45] S. Ghosal. Normal approximation to the posterior distribution for generalized linear models with many covariates. *Mathematical Methods of Statistics*, 6(3):332–348, 1997.

[46] S. Ghosal. Asymptotic normality of posterior distributions in high-dimensional linear models. *Bernoulli*, 5(2):315–331, 1999.

[47] S. Ghosal, J. K. Ghosh, and A. W. van der Vaart. Convergence rates of posterior distributions. *The Annals of Statistics*, 28(2):500–531, 2000.

[48] S. Ghosal and A. van der Vaart. Fundamentals of Nonparametric Bayesian Inference. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2017.

[49] P. Ghosh and A. Chakrabarti. Asymptotic optimality of one-group shrinkage priors in sparse high-dimensional problems. *Bayesian Analysis*, 12(4):1133–1161, 2017.

[50] C. Giraud. Introduction to High-dimensional Statistics, volume 139 of *Monographs on Statistics and Applied Probability*. CRC Press, Boca Raton, FL, 2015.

[51] P. Giudici and P. Green. Decomposable graphical Gaussian model determination. *Biometrika*, 86(4):785–801, 1999.

[52] J. E. Griffin and P. J. Brown. Inference with normal-gamma prior distributions in regression problems. *Bayesian Analysis*, 5(1):171–188, 03 2010.

[53] R. Gröne, C. R. Johnson, E. M. Sá, and H. Wolkowicz. Positive definite completions of partial Hermitian matrices. *Linear Algebra and Applications*, 58:109–124, 1984.
[54] L. Gruber and C. Czado. Sequential Bayesian model selection of regular vine copulas. *Bayesian Analysis*, 10(4):937–963, 2015.

[55] B. Guedj. A primer on PAC-Bayesian learning. 2019. Preprint arXiv:1901.05353.

[56] C. Hans. Bayesian lasso regression. *Biometrika*, 96(4):835–845, 2009.

[57] H. Ishwaran and J. S. Rao. Spike and slab variable selection: frequentist and Bayesian strategies. *The Annals of Statistics*, 33(2):730–773, 2005.

[58] S. Jeong. Posterior contraction in group sparse logit models for categorical responses. *arXiv preprint arXiv:2010.03513*, 2020.

[59] S. Jeong and S. Ghosal. Posterior contraction in sparse generalized linear models. *Biometrika*, 2020+.

[60] S. Jeong and S. Ghosal. Unified Bayesian asymptotic theory for sparse linear regression. *arXiv preprint arXiv:2008.10230*, 2020.

[61] W. Jiang. Bayesian variable selection for high dimensional generalized linear models: convergence rates of the fitted densities. *The Annals of Statistics*, 35(4):1487–1511, 2007.

[62] V. E. Johnson and D. Rossell. On the use of non-local prior densities in Bayesian hypothesis tests. *Journal of the Royal Statistical Society: Series B*, 72(2):143–170, 2010.

[63] V. E. Johnson and D. Rossell. Bayesian model selection in high-dimensional settings. *Journal of the American Statistical Association*, 107(498):649–660, 2012.

[64] I. M. Johnstone and B. W. Silverman. Needles and straw in a haystacks: empirical Bayes estimates of possibly sparse sequences. *The Annals of Statistics*, 32(4):1594–1649, 2004.

[65] B. Jones, C. Carvalho, A. Dobra, C. Hans, C. Carter, and M. West. Experiments in stochastic computation for high-dimensional graphical models. *Statistical Science*, 20(4):388–400, 2005.

[66] Y. Kim and C. Gao. Bayesian model selection with graph structured sparsity. *Journal of Machine Learning Research*, 21(109):1–61, 2020.

[67] A. Kirichenko and H. van Zanten. Estimating a smooth function on a large graph by Bayesian Laplacian regularisation. *Electronic Journal of Statistics*, 11(1):891–915, 2017.

[68] A. Kirichenko, H. van Zanten, et al. Minimax lower bounds for function estimation on graphs. *Electronic Journal of Statistics*, 12(1):651–666, 2018.

[69] B. J. K. Kleijn and J. van Waaij. Recovery, detection and confidence sets of communities in a sparse stochastic block model. 2018.

[70] S. Lauritzen. *Graphical Models*. Clarendon Press, Oxford, 1996.

[71] K. Lee and J. Lee. Estimating large precision matrices via modified Cholesky decomposition. *Preprint arXiv:1707.01143*, 2017.

[72] K. Lee and L. Lin. Bayesian bandwidth test and selection for high-dimensional banded precision matrices. *Bayesian Analysis*, 15:737–758, 2020.
[73] J. Lei and L. Zhu. Generic sample splitting for refined community recovery in degree corrected stochastic block models. *Statistica Sinica*, 27(4):1639–1659, 2017.

[74] A. Lenkoski. A direct sampler for G-Wishart variates. *Stat*, 2(1):119–128, 2013.

[75] A. Lenkoski and A. Dobra. Computational aspects related to inference in Gaussian graphical models with the G-Wishart prior. *Journal of Computational and Graphical Statistics*, 20(1):140–157, 2011.

[76] G. Letac and H. Massam. Wishart distributions for decomposable graphs. *The Annals of Statistics*, 35(3):1278–1323, 2007.

[77] G. Letac, H. Massam, and R. Mohammadi. The ratio of normalizing constants for Bayesian graphical Gaussian model selection. Preprint arXiv:1706.04416, 2017.

[78] Y. Li, B. A. Craig, and A. Bhadra. The graphical horseshoe estimator for inverse covariance matrices. *Journal of Computational and Graphical Statistics*, 28(3):1–24, 2019.

[79] H. Liu, J. Lafferty, and L. Wasserman. The nonparanormal: semiparametric estimation of high dimensional undirected graphs. *Journal of Machine Learning Research*, 10:2295–2328, 2009.

[80] D. Madigan, J. York, and D. Allard. Bayesian graphical models for discrete data. *International Statistical Review*, 63:215–232, 1995.

[81] T. T. Mai and P. Alquier. A Bayesian approach for noisy matrix completion: optimal rate under general sampling distribution. *Electronic Journal of Statistics*, 9(1):823–841, 2015.

[82] E. Makalic and D. F. Schmidt. A simple sampler for the horseshoe estimator. *IEEE Signal Processing Letters*, 23(1):179–182, 2015.

[83] R. Martin and S. G. Walker. Asymptotically minimax empirical Bayes estimation of a sparse normal mean vector. *Electronic Journal of Statistics*, 8(2):2188–2206, 2014.

[84] N. Meinshausen and P. Bühlmann. High-dimensional graphs and variable selection with the lasso. *The Annals of Statistics*, 34(3):1436–1462, 2006.

[85] T. J. Mitchell and J. J. Beauchamp. Bayesian variable selection in linear regression. *Journal of the American Statistical Association*, 83(404):1023–1036, 1988. With comments by James Berger and C. L. Mallows and with a reply by the authors.

[86] N. Mitsakakis, H. Massam, and M. D. Escobar. A Metropolis-Hastings based method for sampling from the G-Wishart distribution in Gaussian graphical models. *Electronic Journal of Statistics*, 5:18–30, 2011.

[87] A. Mohammadi and E. C. Wit. Bayesian structure learning in sparse Gaussian graphical models. *Bayesian Analysis*, 10(1):109–138, 2015.

[88] J. J. Mulgrave and S. Ghosal. Rank likelihood for Bayesian nonparanormal graphical models. 2018. Preprint arXiv:1812.02884.

[89] J. J. Mulgrave and S. Ghosal. Regression based Bayesian approach for nonparanormal graphical models. 2018. Preprint arXiv:1812.04442.
[90] J. J. Mulgrave and S. Ghosal. Bayesian inference in nonparanormal graphical models. *Bayesian Analysis*, 15(2):459–475, 2020.

[91] N. N. Narisetty and X. He. Bayesian variable selection with shrinking and diffusing priors. *The Annals of Statistics*, 42(2):789–817, 2014.

[92] B. Ning, S. Jeong, and S. Ghosal. Bayesian linear regression for multivariate responses under group sparsity. *Bernoulli*, 26(3):2353–2382, 2020.

[93] Y. Niu, D. Pati, and B. Mallick. Bayesian graph selection consistency under model misspecification. *Preprint arXiv:1901.04134*, 2019.

[94] A. Norets and D. Pati. Adaptive Bayesian estimation of conditional densities. *Econometric Theory*, 33(4):980–1012, 2017.

[95] J. T. Ormerod, C. You, and S. Müller. A variational Bayes approach to variable selection. *Electronic Journal of Statistics*, 11(2):3549–3594, 2017.

[96] A. Pakman and L. Paninski. Exact Hamiltonian Monte Carlo for truncated multivariate Gaussians. *Journal of Computational and Graphical Statistics*, 23(2):518–542, 2014.

[97] T. Park and G. Casella. The Bayesian lasso. *Journal of the American Statistical Association*, 103(482):681–686, 2008.

[98] D. Pati and A. Bhattacharya. Optimal Bayesian estimation in stochastic block models. 2015. *Preprint arXiv:1505.06794*.

[99] D. Pati, A. Bhattacharya, N. S. Pillai, and D. Dunson. Posterior contraction in sparse Bayesian factor models for massive covariance matrices. *The Annals of Statistics*, 42(3):1102–1130, 2014.

[100] M. Piccioni. Independence structure of natural conjugate densities to exponential families and the Gibbs’ sampler. *Scandinavian Journal of Statistics*, 27(1):111–127, 2000.

[101] M. Pitt, D. Chan, and R. Kohn. Efficient Bayesian inference for Gaussian copula regression models. *Biometrika*, 93(3):537–554, 2006.

[102] N. G. Polson and J. G. Scott. On the half-Cauchy prior for a global scale parameter. *Bayesian Analysis*, 7(4):887–902, 2012.

[103] B. Rajaratnam, H. Massam, and C. Carvalho. Flexible covariance estimation in graphical Gaussian models. *The Annals of Statistics*, 36(6):2818–2849, 2008.

[104] K. Ray and B. Szabó. Variational Bayes for high-dimensional linear regression with sparse priors. *Journal of the American Statistical Association*, 115:1–31, 2020+.

[105] K. Ray, B. Szabó, and G. Clara. Spike and slab variational Bayes for high dimensional logistic regression. *NeurIPS*, 2020.

[106] V. Ročková. Bayesian estimation of sparse signals with a continuous spike-and-slab prior. *The Annals of Statistics*, 46(1):401–437, 2018.

[107] V. Ročková and E. I. George. EMVS: the EM approach to Bayesian variable selection. *Journal of the American Statistical Association*, 109(506):828–846, 2014.

[108] V. Ročková and E. I. George. The spike-and-slab LASSO. *Journal of the American Statistical Association*, 113(521):431–444, 2018.
[109] A. Roy and S. Ghosal. Optimal Bayesian smoothing of functional observations over a large graph. *Preprint*, 2020.

[110] A. Roy, S. Ghosal, and K. Roy Choudhury. High dimensional single-index Bayesian modeling of brain atrophy. *Bayesian Analysis*, 15:1229–1249, 2020.

[111] J.-B. Salomond. Risk quantification for the thresholding rule for multiple testing using g.

[112] J. G. Scott and J. O. Berger. Bayes and empirical-Bayes multiplicity adjustment in the variable-selection problem. *The Annals of Statistics*, 38(5):2587–2619, 2010.

[113] W. Shen and S. Ghosal. Adaptive Bayesian density regression for high-dimensional data. *Bernoulli*, 22(1):396–420, 2016.

[114] Q. Song and F. Liang. Nearly optimal Bayesian shrinkage for high dimensional regression. *Preprint arXiv:1712.08964*, 2017.

[115] T. Suzuki. PAC-Bayesian bound for Gaussian process regression and multiple kernel additive model. volume 23 of *Proceedings of Machine Learning Research*, pages 8.1–8.20, Edinburgh, Scotland, 25–27 Jun 2012. JMLR Workshop and Conference Proceedings.

[116] T. Suzuki. Convergence rate of Bayesian tensor estimator and its minimax optimality. volume 37 of *Proceedings of Machine Learning Research*, pages 1273–1282, Lille, France, 07–09 Jul 2015. PMLR.

[117] R. Tibshirani. Regression shrinkage and selection via the LASSO. *Journal of the Royal Statistical Society, Series B*, 58:267–288, 1996.

[118] P. K. Trivedi and D. M. Zimmer. *Copula Modeling: An Introduction for Practitioners*. Now Publishers Inc, 2007.

[119] C. Uhler, A. Lenkoski, and D. Richards. Exact formulas for the normalizing constants of wishart distributions for graphical models. *The Annals of Statistics*, 46(1):90–118, 2018.

[120] S. van de Geer and A. Muro. On higher order isotropy conditions and lower bounds for sparse quadratic forms. *Electronic Journal of Statistics*, 8(2):3031–3061, 2014.

[121] S. A. van de Geer and P. Bühlmann. On the conditions used to prove oracle results for the lasso. *Electronic Journal of Statistics*, 3:1360–1392, 2009.

[122] S. van der Pas, B. Szabó, and A. van der Vaart. Adaptive posterior contraction rates for the horseshoe. *Electronic Journal of Statistics*, 11(2):3196–3225, 2017.

[123] S. van der Pas, B. Szabó, and A. van der Vaart. Uncertainty quantification for the horseshoe (with discussion). *Bayesian Analysis*, 12(4):1221–1274, 2017. With a rejoinder by the authors.

[124] S. L. van der Pas, B. J. K. Kleijn, and A. W. van der Vaart. The horseshoe estimator: posterior concentration around nearly black vectors. *Electronic Journal of Statistics*, 8(2):2585–2618, 2014.

[125] S. L. van der Pas, J.-B. Salomond, and J. Schmidt-Hieber. Conditions for posterior contraction in the sparse normal means problem. *Electronic Journal of Statistics*, 10(1):976–1000, 2016.
[126] S. L. van der Pas and A. W. van der Vaart. Bayesian community detection. *Bayesian Analysis*, 13(3):767–796, 2018.
[127] T. van Erven and B. Szabo. Fast exact Bayesian inference for sparse signals in the normal sequence model. *Bayesian Analysis*, 2020. To appear.
[128] J. van Waaaij and B. J. K. Kleijn. Uncertainty quantification in the stochastic block model with an unknown number of classes. 2020.
[129] M. J. Wainwright. *High-dimensional Statistics: A Non-asymptotic Viewpoint*, volume 48 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, Cambridge, 2019.
[130] M. J. Wainwright and M. I. Jordan. Graphical models, exponential families, and variational inference. *Foundations and Trends® in Machine Learning*, 1(1–2):1–305, 2008.
[131] H. Wang. Bayesian graphical lasso models and efficient posterior computation. *Bayesian Analysis*, 7(4):867–886, 2012.
[132] H. Wang. Scaling it up: Stochastic search structure learning in graphical models. *Bayesian Analysis*, 10(2):351–377, 2015.
[133] H. Wang and C. M. Carvalho. Simulation of hyper-inverse wishart distributions for non-decomposable graphs. *Electronic Journal of Statistics*, 4:1470–1475, 2010.
[134] H. Wang and S. Z. Li. Efficient Gaussian graphical model determination under G-Wishart prior distributions. *Electronic Journal of Statistics*, 6:168–198, 2012.
[135] H. Wang and N. S. Pillai. On a class of shrinkage priors for covariance matrix estimation. *Journal of Computational and Graphical Statistics*, 22(3):689–707, 2013.
[136] R. Wei and S. Ghosal. Contraction properties of shrinkage priors in logistic regression. *Journal of Statistical Planning and Inference*, 207:215–229, 2020.
[137] R. Wei, B. J. Reich, J. A. Hoppin, and S. Ghosal. Sparse Bayesian additive nonparametric regression with application to health effects of pesticides mixtures. *Statistica Sinica*, 30:55–79, 2020.
[138] R. Xiang, K. Khare, and M. Ghosh. High dimensional posterior convergence rates for decomposable graphical models. *Electronic Journal of Statistics*, 9(2):2828–2854, 2015.
[139] D. Yang. Posterior asymptotic normality for an individual coordinate in high-dimensional linear regression. *Electronic Journal of Statistics*, 13(2):3082–3094, 2019.
[140] E. Yang, P. Ravikumar, G. I. Allen, and Z. Liu. Graphical models via univariate exponential family distributions. *The Journal of Machine Learning Research*, 16(1):3813–3847, 2015.
[141] Y. Yang, D. Pati, and A. Bhattacharya. α-variational inference with statistical guarantees. *The Annals of Statistics*, 48(2):886–905, 2020.
[142] Y. Yang and S. T. Tokdar. Minimax-optimal nonparametric regression in high dimensions. *The Annals of Statistics*, 43(2):652–674, 2015.
[143] C.-H. Zhang and S. S. Zhang. Confidence intervals for low dimensional parameters in high dimensional linear models. *Journal of the Royal Statistical Society, Series B*, 76(1):217–242, 2014.

[144] F. Zhang and C. Gao. Convergence rates of variational posterior distributions. *The Annals of Statistics*, 48(4):2180–2207, 2020.

[145] R. Zhuang, N. Simon, and J. Lederer. Graphical models for discrete and continuous data. *Preprint arXiv:1609.05551*, 2016.