$N = 2$ super $W$ algebra
in half-twisted Landau-Ginzburg model

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**Abstract**

We investigate $N = 2$ extended superconformal symmetry, using the half-twisted Landau-Ginzburg models. The first example is the $D_{2n+2}$-type minimal model. It has been conjectured that this model has a spin $n$ super $W$ current. We checked this by the direct computations of the BRS cohomology class up to $n = 4$. We observe for $n \leq 3$ the super $W$ currents generate the ring isomorphic to the chiral ring of the model with respect to the classical product. We thus conjecture that this isomorphism holds for any $n$. The next example is $CP_n$ coset model. In this case we find a sort of Miura transformation which gives the simple formula for the super $W$ currents of spin $\{1,2,\ldots,n\}$ in terms of the chiral superfields. Explicit form of the super $W$ currents and their Poisson brackets are obtained for $CP_2, CP_3$ case. We also conjecture that as long as the classical product is concerned, these super $W$ currents generate the ring isomorphic to the chiral ring of the model and this is checked for $CP_2$ model.
1 Introduction

\( N = 2 \) superconformal field theories have been intensively studied in recent years \cite{1,22,16,14,14,4,9,11,21,5,8,7,19,6,15,12}. The large class of \( N = 2 \) superconformal theories can be realized as the infra-red fixed point of Landau-Ginzburg models \cite{22,18}. Thus it is important to investigate the structure of \( N = 2 \) superconformal algebras in the framework of the Landau-Ginzburg description. To do so the apparent difficulty is that the action of the Landau-Ginzburg model does not have conformal invariance. However Witten has shown quite recently that under the operation of half-twist \cite{24} the Landau-Ginzburg model turns out to be left-moving conformally invariant \cite{26}. This enable us to extract the essential information about the \( N = 2 \) superconformal algebra realized at the infra-red fixed point of the Landau-Ginzburg model.

In this paper our purpose is to extend the analysis of Witten to the multi-variable case and to analyse the classical aspect of \( N = 2 \) super \( W \) symmetry. In section 2 we clarify the twist operation in the Landau-Ginzburg model. In particular it is argued that the Landau-Ginzburg model with the quasi-homogeneous superpotential \( W \) admits the half-twist operation, so that it gets the conformal symmetry. In section 3 we treat the \( D_{2n+2} \)-type minimal model which consists of two Landau-Ginzburg fields for lower \( n \)s. It is shown that the \( N = 2 \) algebra of this model is extended by an operator of spin \( n \). Moreover we find that super \( W \) currents generate a ring isomorphic to the chiral ring of the model under the classical product. In section 3 the \( CP_n \) coset model is analysed. We point out that there exists a simple transformation law which yields the super \( W \) currents directly from the chiral superfields. The explicit form of the super \( W \) currents and their Poisson brackets are obtained for \( CP_2 \) and \( CP_3 \) cases. It is conjectured that in these examples also the super \( W \) currents generate a ring isomorphic to the chiral ring under the classical product, and this is proved for \( CP_2 \) model. In appendix we collect some useful classical and quantum formulas for the Landau-Ginzburg model.
2 Twists of Landau-Ginzburg models

2.1 Preliminaries

The Landau-Ginzburg model is an $N = 2$ super field theory described by $n$ chiral superfields. The Lagrange density is

$$L = 4\Phi^i \bar{\Phi}^\dagger_i |_{\theta^i} - W(\Phi^i) |_{\theta^i} - \bar{W}(\bar{\Phi}^i) |_{\bar{\theta}^i}$$

$$= \int d\theta^i d\bar{\theta}^i d\theta^+ d\bar{\theta}^+ \Phi^i \bar{\Phi}^\dagger_i - \frac{1}{2} \int d\theta^i d\theta^+ W(\Phi^i) - \frac{1}{2} \int d\bar{\theta}^i d\bar{\theta}^+ \bar{W}(\bar{\Phi}^i)$$

$$= 4(2\sqrt{-1} \psi^i_+ \partial_+ \psi^i_+ + 2\sqrt{-1} \bar{\psi}^i_+ \partial_- \bar{\psi}^i_+ - 4\psi^i_+ \partial_- \psi^i_+ + F^i \bar{F}^i)$$

$$- \frac{\partial W}{\partial \phi^i} F^i - \frac{\partial \bar{W}}{\partial \bar{\phi}^i} \bar{F}^i + \frac{\partial^2 W}{\partial \phi^i \partial \phi^j} \psi^i_+ \psi^j_+ + \frac{\partial^2 \bar{W}}{\partial \bar{\phi}^i \partial \bar{\phi}^j} \bar{\psi}^i_+ \bar{\psi}^j_+.$$ (1)

See Appendix for the convention of the $N = 2$ super formalism. The equations of motion for the Landau-Ginzburg model can be written by superfields as

$$2 \bar{D}_+ D_- \bar{\Phi}^i = \frac{\partial W}{\partial \Phi^i}.$$ (2)

In terms of the component fields we have

$$4F^i - \frac{\partial \bar{W}}{\partial \phi^i} = 0, \quad 4\bar{F}^i - \frac{\partial W}{\partial \bar{\phi}^i} = 0$$ (3)

$$8\sqrt{-1}\partial_+ \psi^i_+ + \partial_i \partial_j \bar{W} \bar{\psi}^j_+ = 0, \quad -8\sqrt{-1}\partial_- \bar{\psi}^i_+ + \partial_i \partial_j W \psi^j_+ = 0$$ (4)

$$8\sqrt{-1}\partial_- \psi^i_+ - \partial_i \partial_j \bar{W} \bar{\psi}^j_+ = 0, \quad -8\sqrt{-1}\partial_+ \bar{\psi}^i_+ - \partial_i \partial_j W \psi^j_+ = 0$$ (5)

$$-16\partial_- \partial_+ \phi^i - \frac{1}{4} \partial_j \partial_k W \partial_i \partial_j \bar{W} + \partial_i \partial_j \partial_k \bar{W} \bar{\psi}^j_+ \bar{\psi}^k_+ = 0$$ (6)

$$-16\partial_- \partial_+ \bar{\phi}^i - \frac{1}{4} \partial_j \partial_k W \partial_i \partial_j W + \partial_i \partial_j \partial_k W \psi^j_+ \psi^k_+ = 0$$ (7)

The canonical energy-momentum tensor of the Landau-Ginzburg model reads

$$T_{++} = \partial_+ \phi^i \partial_+ \bar{\phi}^\dagger_i - \frac{\sqrt{-1}}{4} (\partial_+ \bar{\psi}^i_+ \psi^i_+ - \bar{\psi}^i_+ \partial_+ \psi^i_+)$$

$$T_{+-} = \frac{1}{32} (\partial_+ \partial_j \bar{W} \bar{\psi}^j_+ \bar{\psi}^k_+ + \partial_+ \partial_j W \psi^j_+ \psi^k_+) + \frac{1}{64} \partial_j W \partial_k \bar{W}$$

$$T_{-+} = \partial_- \phi^i \partial_- \bar{\phi}^\dagger_i - \frac{\sqrt{-1}}{4} (\partial_- \bar{\psi}^i_+ \psi^i_+ - \bar{\psi}^i_+ \partial_- \psi^i_+).$$ (8)
2.2 Topological twist of Landau-Ginzburg models

Let us consider a field theory which has an energy-momentum tensor \((T_-, T_+, T_+)\) and a conserved current \((J_-, J_+)\). Then one can define a new energy-momentum tensor as

\[
\begin{align*}
\tilde{T}_- &= T_+ - \frac{1}{2} \partial_- J_-
\tilde{T}_+ &= T_- - \frac{1}{2} \partial_+ J_+ = T_+ + \frac{1}{2} \partial_+ J_-
\tilde{T}_+ &= T_+ + \frac{1}{2} \partial_+ J_+ (9)
\end{align*}
\]

It is clear that the possibility of twist procedure depends upon if the current is conserved. We thus first look for conserved currents in the Landau-Ginzburg model to consider its twists. Two \(U(1)\) transformations, which are essentially the rotations of the super coordinates \((\theta^\pm, \bar{\theta}^\pm)\), can be defined; the axial \(U(1)\) transformation,

\[
(\theta^-, \bar{\theta}^+) \rightarrow e^{\sqrt{-1} \beta}(\theta^-, \bar{\theta}^+), \quad (\bar{\theta}^-, \bar{\theta}^+) \rightarrow e^{-\sqrt{-1} \beta}(\bar{\theta}^-, \bar{\theta}^+),
\]

and the vector \(U(1)\) transformation,

\[
(\theta^-, \bar{\theta}^+) \rightarrow (e^{-\sqrt{-1} \beta} \theta^-, e^{\sqrt{-1} \beta} \bar{\theta}^+), \quad (\bar{\theta}^-, \bar{\theta}^+) \rightarrow (e^{\sqrt{-1} \beta} \bar{\theta}^-, e^{-\sqrt{-1} \beta} \bar{\theta}^+).\]

It is seen that the vector \(U(1)\) symmetry is conserved but the axial \(U(1)\) symmetry is violated by the superpotential term,

\[
\frac{1}{2} d\theta^- d\theta^+ \rightarrow e^{-2\sqrt{-1} \beta} \frac{1}{2} d\theta^- d\theta^+. (12)
\]

The conserved current of the vector \(U(1)\) symmetry is given by

\[
(J^\text{vect}_-, J^\text{vect}_+) = (-\frac{\sqrt{-1}}{2} \psi_+ \bar{\psi}_-^\dagger, -\frac{\sqrt{-1}}{2} \bar{\psi}_+^\dagger \psi_-),
\]

Using the vector \(U(1)\) current \((13)\) we twist the energy-momentum tensor \((8)\) according to \((9)\). Then two of four supercharges \(\bar{Q}_\pm\) turn into Lorentz scalars with respect to the twisted energy-momentum operator \((8)\) and we can define the BRS operator \(Q_{\text{BRS}}\) as

\[
Q_{\text{BRS}} = \bar{Q}_+ + \bar{Q}_- .
\]

(14)
Now the twisted system has the following topological formula,

\[
\tilde{T}^{++} = \{Q_{BRS}, \frac{\sqrt{-1}}{2\sqrt{2}} \partial_+ \bar{\phi} \bar{\psi} \}
\]

\[
\tilde{T}^{+-} = \{Q_{BRS}, -\frac{1}{16\sqrt{2}} \partial_+ \bar{W} \bar{\phi} \}
\]

\[
\tilde{T}^{--} = \{Q_{BRS}, \frac{\sqrt{-1}}{2\sqrt{2}} \partial_- \bar{\phi} \bar{\psi} \}.
\]

This is the Vafa’s topological Landau-Ginzburg model [20]. BRS local observables consist of the lowest component of chiral super fields which generate the chiral ring \( \mathcal{R} \) [10]

\[
\mathcal{R} \cong C[\phi_1, \phi_2, ..., \phi_n]/(\frac{\partial W}{\partial \phi_1}, ..., \frac{\partial W}{\partial \phi_n})
\]

(16)

2.3 Half-twisted Landau-Ginzburg models

We now show that if one twists the energy-momentum tensor (8) by the right-moving chiral \( U(1) \) current then the remaining model has left-moving conformal symmetry. The right chiral \( U(1) \) rotation is defined by

\[
(\theta^-, \bar{\theta}^-) \rightarrow (\theta^-, \bar{\theta}^-), \quad (\theta^+, \bar{\theta}^+) \rightarrow (e^{\sqrt{-1}\beta \theta^+}, e^{-\sqrt{-1}\beta \bar{\theta}^+}).
\]

(17)

Similarly to the axial \( U(1) \) rotation, the superpotential term violates this symmetry. However it is recovered if one can simultaneously transform the chiral superfields so that

\[
W(\Phi^i) \rightarrow W(e^{\sqrt{-1}\alpha_i \beta \Phi^i}) = e^{\sqrt{-1}\beta} W(\Phi^i),
\]

(18)

namely if \( W \) is quasi-homogeneous. Here the weight \( \alpha_i \) is called the quasi-homogeneous degree of \( \Phi^i \) [22]. When the superpotential is quasi-homogeneous, therefore, we have the conserved chiral \( U(1) \) current

\[
J^R_+ = \frac{\alpha_i}{2} (-2 \bar{\phi}^i \partial_+ \phi^i + \sqrt{-1} \bar{\psi}^i_+ \psi^i_+) - \frac{\sqrt{-1}}{2} \bar{\psi}^i_+ \bar{\psi}^i_+ - \frac{\sqrt{-1}}{2} \partial_+ \bar{\phi} \bar{\psi}.
\]

\[
J^R_- = \frac{\alpha_i}{2} (2 \bar{\phi}^i \partial_- \bar{\phi}^i + \sqrt{-1} \bar{\psi}^i_- \psi^i_-).
\]

(19)

Now let us twist the energy-momentum tensor (8) with the right moving chiral \( U(1) \) current [19] according to the formula (9). Define

\[
Q_{BRS} = \bar{Q}_+
\]

(20)
as a BRS operator, then we have the following energy-momentum operator

\[ T_{++} = \frac{\sqrt{-1}}{2\sqrt{2}} \{ Q_{BRS}, \partial_+ \bar{\phi}^i \psi_i^+ - \frac{1}{2} \alpha_i \partial_+ (\bar{\phi}^i \psi_i^+) \} \]

\[ T_{+-} = \{ Q_{BRS}, -\frac{\sqrt{-1}}{16\sqrt{2}} \partial_i W \bar{\psi}_i^- + \frac{\sqrt{-1}}{4\sqrt{2}} \alpha_i \partial_- (\bar{\phi}^i \psi_i^+) \} \]

\[ T_{-+} = \partial_- \bar{\phi}^i \partial_- \bar{\psi}_i^- - \frac{\sqrt{-1}}{4} (\partial_- \bar{\psi}_i^i \psi_i^- - \bar{\psi}_i^i \partial_- \psi_i^-) \]

\[ -\frac{\alpha_i}{4} \partial_- (2\phi^i \partial_\bar{\phi}^i + \sqrt{-1} \bar{\psi}_i^i \psi_i^-) \]

\[ 0 = \{ Q_{BRS}, T_{+-} \}, \partial_+ T_{+-} = \{ Q_{BRS}, * \} \quad (21) \]

Only a (- -) component of \( T^{ht} \) is an observable in the sense of BRS cohomology. Thus in the half-twisted model conformal invariance is realized in the left moving sector at the level of the \( Q_{BRS} \) cohomology \[25\] \[26\]. It can be checked that \( T^{ht} \) generates the Virasoro algebra with \( \hat{c} = \sum_i (1 - 2\alpha_i) \). There are infinite number of the observables of the half-twisted Landau-Ginzburg model and their spectra are represented by the elliptic genus. (See \[26\] \[3\] \[13\] for the computations of the elliptic genera of the various Landau-Ginzburg models.) We conclude that making use of the half-twist operation one can study the \( N = 2 \) superconformal algebra realized at the infra-red fixed point of the Landau-Ginzburg model with a quasi-homogeneous superpotential \( W \). In the following we search for \( N = 2 \) super \( W \) currents in multi-component Landau-Ginzburg models by constructing local observables as commutants of the BRS operator \( Q_{BRS} \) of half-twisted model.

3 The \( D_{2n+2} \)-type minimal model

3.1 General strategy

Our approach of constructing the \( N = 2 \) super \( W \) algebra is as simple as that for the super Toda model \[13\]. What we should do is to find operators expressed as differential polynomials of chiral superfields which are in the \( Q_{BRS} \) cohomology class. According to the equivalence relation \[26\], \( \bar{Q}_+ = e^{-4\sqrt{-1}\theta^+ \bar{\theta}^+ \partial_+} \bar{D}_+ e^{4\sqrt{-1}\theta^+ \bar{\theta}^+ \partial_+} \) we shall consider cohomology classes of \( \bar{D}_+ \) for convenience. To find a BRS cohomology of spin \( m \) and charge 0
we take as the candidates all the monomials
\[ \prod (\Phi^i)^{a(i)}(D_\Phi^i)^{b(i)}(\bar{D}_\Phi^i)^{c(i)}(D_\bar{D}_\Phi^i)^{d(i)} \]
\[ \text{(22)} \]
such that \( \sum \alpha_i(a(i) + b(i) - c(i) - d(i)) = 0 \), \( \sum (c(i) + d(i)) = m \) \( \sum (b(i) + d(i)) = m \). Similarly BRS exact operators are made by operating \( \bar{D}_+ \) to monomials such that \( \sum \alpha_i(a(i) + b(i) - c(i) - d(i)) = -1 \), \( \sum (c(i) + d(i)) = m + 1 \), \( \sum (d(i) + b(i)) = m \). In this paper we only deal with the classical aspect of the algebra and possible quantum corrections to observables are neglected. Before analyzing the multi-variable case we briefly review the 1-variable case. The model is the A-type minimal model with the superpotential
\[ W(X) = \frac{1}{k+2}X^{k+2}. \]
\[ \text{(23)} \]
\[ \text{The equation of motion is} \]
\[ 2\bar{D}_+ D_- X = X^{k+1}. \]
\[ \text{(24)} \]
\[ \text{The energy-momentum superfield obtained by Witten is} \]
\[ J = \frac{k+1}{2(k+2)}D_-X\bar{D}_-\bar{X} - \frac{1}{2(k+2)}XD_-\bar{D}_-\bar{X}, \quad \bar{D}_+ J = 0. \]
\[ \text{(25)} \]
We easily see that \( J^{k+1} \) is a BRS exact operator since
\[ (-2(k+1)J)^{k+1} = \bar{D}_+(2\bar{D}_-\bar{X}(D_-\bar{D}_-\bar{X})^{k+1}). \]
\[ \text{(26)} \]
It is interesting to notice that in (24) the superfield \( X \) raised to the power \( k + 1 \) is BRS exact while in (26) the energy-momentum superfield \( J \) raised to the same power \( k + 1 \) is also BRS exact. In this sense we observe simple correspondence between the chiral superfield and the \( N = 2 \) \( W \) current. This kind of correspondence is not peculiar to the A-type model, but can be extended to other multi-variable cases as we will see below.

3.2 The \( D_{2n+2} \) model

The superpotential and the energy-momentum tensor for the \( D_{2n+2} \)-type minimal model is given by
\[ W(X, Y) = \frac{1}{2n+1}X^{2n+1} + XY^2 \]
\[ 2(2n+1)J = 2nD_-X\bar{D}_-\bar{X} - XD_-\bar{D}_-\bar{X} + (n+1)D_-Y\bar{D}_-\bar{Y} - nYD_-\bar{D}_-\bar{Y} \]
\[ \text{(27)} \]
This model at the infra-red fixed point has central charge \( \hat{c} = \frac{2n}{2n+1} \) [22] and it has a realization as the level 1 supercoset \( SO(2n+2)/SO(2n) \times SO(2) \) model [4] [4]. From the form of the modular invariant partition function one expects that in this model \( N = 2 \) algebra is extended by the operator of spin \( n \) [4]. We construct directly this operator for \( n = 1, 2, 3, 4 \) below. These are all parity odd under the symmetry \( Y \rightarrow -Y \).

**n = 1 case**

One has the following BRS cohomology class of spin 1 with odd \( \mathbb{Z}_2 \) parity,

\[
J_A = \frac{1}{6}(2DX\bar{D}Y + 2DY\bar{D}X - XD\bar{D}Y - YD\bar{D}X)
\]

(28)

It is readily seen that \( J \) and \( J_A \) generate a ring with respect to the classical product

\[
(3J)(3J_A) = \bar{D}_+ \left[ \frac{1}{4}(D\bar{D}X)^2\bar{D}Y + \frac{1}{2}D\bar{D}XDD\bar{D}XDD\bar{D}Y + \frac{1}{4}\bar{D}Y(D\bar{D}Y)^2 \right]
\]

\[
(3J)^2 + (3J_A)^2 = \bar{D}_+ \left[ \frac{1}{2}\bar{D}X(D\bar{D}X)^2 + D\bar{D}XDD\bar{D}Y + \frac{1}{2}\bar{D}X(D\bar{D}Y)^2 \right].
\]

(29)

Notice that this is isomorphic to the chiral ring

\[
2\bar{D}_+\bar{D}_-X = \frac{\partial W}{\partial X} = X^2 + Y^2,
\]

\[
2\bar{D}_+\bar{D}_-Y = \frac{\partial W}{\partial Y} = 2XY.
\]

(30)

Before going to the \( n = 2 \) case, we describe the quantum version of the extended algebra.

Let us renormalize the \( W \) supercurrents as

\[
T = -\frac{4\pi}{8} J, \quad T_A = -\frac{4\pi}{8} J_A
\]

(31)

(see Appendix), then we evaluate the operator product expansions

\[
T(Z_1)T_A(Z_2) = \left[ \frac{\theta_{12}}{Z_{12}^2} + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\bar{\theta}_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12}}{Z_{12}} \partial \right] T_A(Z_2)
\]

\[
T(Z_1)T(Z_2) = -\frac{1}{163} \left[ \frac{\theta_{12}}{Z_{12}^2} + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\bar{\theta}_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12}}{Z_{12}} \partial \right] T(Z_2)
\]

\[
T_A(Z_1)T_A(Z_2) = -\frac{1}{163} \left[ \frac{\theta_{12}}{Z_{12}^2} + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\bar{\theta}_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12}}{Z_{12}} \partial \right] T(Z_2).
\]

This algebra coincides with that of the \( N = 2 \) super Toda field theory associated with the Lie superalgebra \( sl(2 \mid 1) \) [13].
\( n = 2 \) case

This model has \( \hat{c} = \frac{4}{5} \) and can be identified with the level 2 \( CP_2 \) model. In this case we search for a spin 2 BRS cohomology element in the parity odd sector. There exist two BRS closed operators, one of which is BRS exact and the other is not. The BRS exact operator is

\[- 2X^2DY\bar{D}YD\bar{D}Y + 2XY\bar{D}YDXD\bar{D}Y + X^2(D\bar{D}Y)^2 = \bar{D}_+(X\bar{D}Y(D\bar{D}Y)^2), \tag{32}\]

and the BRS observable \( W^{(2)} \) is

\[W^{(2)} = 2DY\bar{D}XDD\bar{X} - \frac{1}{2}Y(DD\bar{X})^2 - X^2Y(D\bar{D}Y)^2 + 3X^2DY\bar{D}YD\bar{D}Y \\
- X^3DD\bar{X}DD\bar{Y} + 4X^2DXDXD\bar{D}Y + 3X^2DXDD\bar{X}D\bar{Y}. \tag{33}\]

We find that \( 10J \) and \( 2W^{(2)} \) form a ring under classical product

\[(10J)(2W^{(2)}) = \bar{D}_+[\cdots] \tag{34}\]

\[(10J)^4 + (2W^{(2)})^2 = \bar{D}_+[\cdots]. \tag{34}\]

Notice again that this is isomorphic to the chiral ring,

\[2\bar{D}_+\bar{D}_-\bar{X} = \frac{\partial W}{\partial \bar{X}} = X^4 + Y^2 \]
\[2\bar{D}_+\bar{D}_-\bar{Y} = \frac{\partial W}{\partial \bar{Y}} = 2XY. \tag{35}\]

The detailed description of the right hand side of (34) is given in Appendix.

\( n = 3 \) case

In this model we obtain the supercurrent of spin 3,

\[W^{(3)} = \frac{1}{3}Y(D\bar{D}X)^3 - 2DY\bar{D}X(D\bar{D}X)^2 \\
+ X^5D\bar{D}Y(D\bar{D}X)^2 - 4X^4DXD\bar{Y}(D\bar{D}X)^2 - 12X^4DX\bar{D}XD\bar{D}XDD\bar{Y} \\
- 6X^4DY\bar{D}X(D\bar{D}Y)^2 - 8X^4DYD\bar{D}Y\bar{D}Y\bar{D}YD\bar{D}X + 3X^4YD\bar{D}X(D\bar{D}Y)^2 \\
- 12X^8DXD\bar{Y}(D\bar{D}Y)^2 + X^9(D\bar{D}Y)^3. \tag{36}\]

Moreover there exist four BRS exact currents of spin 3 in the parity odd sector,

\[\bar{D}_+(X^2Y\bar{D}Y(D\bar{D}Y)^3), \quad \bar{D}_+(X^3\bar{D}X(D\bar{D}Y)^3)\]
\[ \bar{D}_+(X^2 DX \bar{D}X \bar{D}Y (D \bar{D}Y)^2), \quad \bar{D}_+(X^3 \bar{D}Y \bar{D} \bar{X} (D \bar{D}Y)^2). \] (37)

These super \( W \) currents generate the ring under the classical product,

\[
(14J)(3W^{(3)}) = D_+ [\cdots ], \\
(14J)^6 + (3W^{(3)})^2 = \bar{D}_+ [\cdots ],
\]

which is isomorphic to the chiral ring

\[
2D_+ \bar{D}_- \bar{X} = \frac{\partial W}{\partial X} = X^6 + Y^2, \\
2\bar{D}_+ \bar{D}_- \bar{Y} = \frac{\partial W}{\partial Y} = 2XY. 
\] (39)

We represent the right hand side of (38) explicitly in Appendix.

\( n = 4 \) case

We find here that there exists a unique spin 4 BRS cohomology element with the odd parity,

\[
W^{(4)} = \frac{1}{4} Y (D \bar{D}X)^4 - 2DY \bar{D}X (D \bar{D}X)^3 \\
-5X^6 DX \bar{D}Y (D \bar{D}X)^3 - 24X^6 DXD \bar{D}Y \bar{D}X (D \bar{D}X)^2 \\
+X^7 D \bar{D}Y (D \bar{D}X)^3 - 48X^5 DXD \bar{D}Y \bar{D}X \bar{D}Y D \bar{D}X D \bar{D}Y \\
+12X^5 Y DX \bar{D}Y D \bar{D}Y (D \bar{D}X)^2 - 3X^6DY \bar{D}Y D \bar{D}Y (D \bar{D}X)^2 
\] (40)

In addition to this, other 9 BRS exact terms of spin 4 with odd parity exist though we don’t describe here. Now we would like to conjecture that the \( D_{2n+2} \)-type minimal model has a spin \( n \) parity odd current of the form,

\[
W^{(n)} = \frac{1}{n} Y (D \bar{D}X)^n - 2DY \bar{D}X (D \bar{D}X)^{n-1} + X^{2n-1} (D \bar{D}X)^{n-1} D \bar{D}Y \\
-(n + 1)X^{2n-2} DX (D \bar{D}X)^{n-1} \bar{D}Y - 2n(n - 1) X^{2n-2} DX \bar{D}X (D \bar{D}X)^{n-2} D \bar{D}Y + \cdots 
\] (41)

where the remainig terms in the ellipses can be systematically evaluated for each \( n \). We also conjecture that under the classical product the supercurrents \( 2(2n + 1)J, nW^{(n)} \)
generate a ring graded by $L_0$ which is isomorphic to the chiral ring $\mathcal{R}$ of the model generated by $(X,Y)$, i.e.,
\[
(2(2n+1)J)(nW^{(n)}) = \bar{D}_+[\cdots]
\]
\[
(2(2n+1)J)^2 + (nW^{(n)})^2 = \bar{D}_+[\cdots].
\]

It would be interesting if one can find the Miura transformation $(X,Y) \rightarrow (J,W^{(n)})$. For the case of $CP_n$ coset model we have such a transformation as will be explained in the next section.

4 The $CP_n$ coset model

4.1 General aspect

For the $CP_n$ coset model, Ito [12] identified the $N = 2$ super $W$ symmetry of the model with that of super Toda field theory [19] [15] [6]. This super $W$ algebra has the generators of spin $\{1, 2, 3, \ldots, n\}$. In the super Toda approach, these generators are given by the Miura transformation [3], which originates from the fact that the super Toda field equation can be written in the form of super flat curvature equation. In the following we will give simple formulas for the BRS invariant supercurrents of spin $\{1, 2, 3, \ldots, n\}$ in terms of the chiral superfields. (The relation between these two approaches is yet to be clarified.)

The chiral ring of $CP_n^{(k)}$ model is isomorphic to the cohomology ring of the coset space $SU(n+k+1)/SU(n+1) \times SU(k) \times U(1)$ [14]. An important property is the level-rank duality
\[
CP_n^{(k)} \cong CP_k^{(n)}, \quad \hat{c} = \frac{kn}{n+k+1}.
\]

The chiral superfields and the superpotential of $CP_n^{(k)}$ model are given as follows [10],
\[
1 + t\Phi^1 + t^2\Phi^2 + \cdots + t^n\Phi^n = (1 + ta(1))(1 + ta(2))\cdots(1 + ta(n))
\]
\[
\Phi^1 = a(1) + a(2) + \cdots + a(n)
\]
\[
\Phi^2 = a(1)a(2) + a(1)a(3) + \cdots + a(n-1)a(n)
\]
\[
\vdots
\]
\[
\Phi^n = a(1)a(2)\cdots a(n)
\]
\[ W_{(k+n+1)} = \frac{1}{(n + k + 1)} \sum_{l=1}^{n} a(l)^{n+k+1}. \]  

The point is that the right hand side of (46) is a completely symmetric polynomial of \( a(1), a(2), \ldots a(n) \) so that it can be expanded as a polynomial of \( \Phi^1, \Phi^2, \ldots \Phi^n \). Our idea to construct the BRS closed supercurrents of spin 1, 2, \ldots n is very simple. First one writes the equation of motion in terms of chiral superfields \( \{a(1), a(2), \ldots a(n)\} \) as

\[ 2 \bar{D}_{+} \bar{D}_{-} \bar{b}(\bar{i}) = a(i)^{k+n}, \quad 1 \leq i \leq n, \quad \bar{b}(\bar{i}) = \frac{\partial K}{\partial a(\bar{i})}, \quad K = \sum \Phi^i \bar{\Phi}^{\bar{i}} \]  

(47)

It is easily seen that (47) is equivalent to the original equations of motion of \( \Phi^i \)'s. Next we observe that there are \( n \) BRS invariant spin 1 currents

\[ J(i) = (k + n) D a(i) \bar{D} \bar{b}(\bar{i}) - a(i) D \bar{D} \bar{b}(\bar{i}), \quad 1 \leq i \leq n \]  

(48)

\[ D_{+} J(i) = 0, \quad 1 \leq i \leq n \]  

(49)

If one combine these \( n \) currents in a completely symmetric manner in \( i \),

\[ 1 + t W^{(1)} + t^2 W^{(2)} + t^3 W^{(3)} + \cdots + t^n W^{(n)} = (1 + tJ(1))(1 + tJ(2)) \cdots (1 + tJ(n)) \]  

(50)

the resulting spin \( i \) current \( W^{(i)} \) can be rewritten by the original variables \( \Phi \). Then \( W^{(i)} \)'s are BRS invariant by construction. In particular \( W^{(1)} \) coincides with the energy-momentum operator

\[ W^{(1)} = 2(n + k + 1) J = \sum_{i=1}^{n} [(n + k + 1) D \Phi^i \bar{D} \bar{\Phi}^{\bar{i}} - iD(\Phi^i \bar{D} \bar{\Phi}^{\bar{i}})], \]  

(51)

The remaining higher spin currents \( W^{(i)} \) should give an extention of the \( N = 2 \) superconformal algebra under the Poisson bracket. We shall check this for the \( CP_2 \) and \( CP_3 \) models later. Finally we remark that the transformation (50) is the analogue of the Miura transformation.

### 4.2 The \( CP_2 \) model

The superpotential and chiral superfields are given by

\[ W_{k+3} = \frac{1}{k+3} (a(1)^{k+3} + a(2)^{k+3}), \quad X = a(1) + a(2), \quad Y = a(1)a(2). \]  

(52)
The kinetic term (Kähler potential) is written as

\[ K = X \bar{X} + Y \bar{Y} = (a(1) + a(2))(\bar{a}(1) + \bar{a}(2)) + a(1)a(2)\bar{a}(1)\bar{a}(2), \] (53)

so that

\[ \bar{b}(1) = \frac{\partial K}{\partial a(1)} = \bar{X} + a(2)\bar{Y}, \quad \bar{b}(2) = \frac{\partial K}{\partial a(2)} = \bar{X} + a(1)\bar{Y}. \] (54)

Two spin 1 currents are

\[ J(1) = (k + 2)Da(1)\bar{D}\bar{b}(1) - a(1)\bar{D}\bar{D}\bar{b}(1) \]
\[ J(2) = (k + 2)Da(2)\bar{D}\bar{b}(2) - a(2)\bar{D}\bar{D}\bar{b}(2), \] (55)

from which one can construct BRS invariant supercurrent of spin 1 and 2,

\[ W^{(1)} = 2(k + 3)J = (k + 2)DX\bar{D}\bar{X} + (k + 1)DY\bar{D}\bar{Y} - XD\bar{D}\bar{X} - 2YD\bar{D}\bar{Y} \] (56)
\[ W^{(2)} = -(k + 2)^2DXDY\bar{D}\bar{X}\bar{D}\bar{Y} - (k + 2)YDX\bar{D}\bar{X}\bar{D}\bar{Y} - (k + 2)XDY\bar{D}\bar{X} \]
\[ + (k + 3)YDX\bar{D}\bar{X}\bar{D}\bar{Y} + XYD\bar{D}\bar{X}D\bar{D}\bar{Y} - (k + 2)DY\bar{D}\bar{X}D\bar{D}\bar{Y} \]
\[ + Y(D\bar{D}\bar{X})^2 - (k + 1)DYD\bar{D}\bar{Y}D\bar{D}\bar{Y} + Y^2(D\bar{D}\bar{Y})^2 \] (57)

As in the $D_{2n+2}$-type minimal model treated in the previous section, one may suspect that $W^{(1)}$ and $W^{(2)}$ generate a ring isomorphic to the chiral ring with respect to the classical product. Indeed this is true as we show. To this purpose we describe the ideal of the chiral ring [10]. The generators of the ideal of the chiral ring of the level $k$ model can easily described by the variables $(a(1), a(2))$ as

\[ \frac{\partial W_{k+3}}{\partial X} = a(1)^{k+2} + a(1)^{k+1}a(2) + a(1)^ka(2)^2 + \cdots + a(2)^{k+2} \]
\[ - \frac{\partial W_{k+3}}{\partial Y} = a(1)^{k+1} + a(1)^ka(2) + a(1)^{k-1}a(2)^2 + \cdots + a(2)^{k+1}. \] (58)

It is easily found that

\[ (-2(k + 3)J(1))^{k+2} = 2\bar{D}\bar{b}(1)(D\bar{D}\bar{b}(1))^{k+2} \]
\[ (-2(k + 3)J(2))^{k+2} = 2\bar{D}\bar{b}(2)(D\bar{D}\bar{b}(2))^{k+2}. \] (59)
However the (1, 2)—permutation invariant subideal of the ideal \((J(1)^{k+2}, J(2)^{k+2})\) is smaller than the ideal \([58]\) with \(a(1), a(2)\) replaced by \((J(1), J(2))\), so that one need a refinement of \([59]\). The solution is given by the following equation

\[
\bar{D}b(1)(D\bar{D}b(1))^{k+2} - \bar{D}b(2)(D\bar{D}b(2))^{k+2} = +2(k + 3)(J(1) - J(2))M^{(k)} - (a(1) - a(2))\bar{D} [DXDY(D\bar{D}Y)^{k+2}] \tag{60}
\]

where \(M^{(k)}\) is some superdifferential polynomial, i.e., an element of \(C[X, Y, D\bar{X}, D\bar{Y}; D]\). From this we have the needed equation,

\[
2\bar{D}M^{(k)} = \frac{(-2(k + 3)J(1))^{k+2} - (-2(k + 3)J(2))^{k+2}}{-2(k + 3)J(1) + 2(k + 3)J(2)} = (-2(k + 3))^{k+1}[\bar{J}(1)^{k+1} + J(1)J(2) + \cdots + J(2)^{k+1}] \tag{61}
\]

Now it is easy to see that \([61]\) combined with \([59]\) constitute the ideal isomorphic to that of the chiral ring. Thus we can say that \((W^{(1)}, W^{(2)})\) generate a ring isomorphic to the chiral ring under the classical product. For the computation of \(M^{(k)}\) see Appendix.

For example the generators of the ideal of the chiral ring for \(k = 1\) is given by

\[
\frac{\partial W_4}{\partial X} = X^3 - 2XY, \quad \frac{\partial W_4}{\partial Y} = X^2 - Y. \tag{62}
\]

Correspondingly, the spin 2 current \(W^{(2)}\) is BRS equivalent to \((W^{(1)})^2\),

\[
(W^{(2)}) - (W^{(1)})^2 = 2\bar{D} [(D\bar{D}X)^2D\bar{Y} + 3D\bar{X}D\bar{D}XDD\bar{Y} + 3DXD\bar{X}DD\bar{Y} + 3DXD\bar{X}D\bar{D}Y + 3\bar{D}X(D\bar{D}Y)^2 + 3X\bar{D}Y(D\bar{D}Y)^2 + 3\bar{D}Y(D\bar{D}Y)^2 + 3X^2\bar{D}Y(D\bar{D}Y)^2] = 2\bar{D}M^{(1)} \tag{63}
\]

and \((W^{(1)})^3\) is cohomologous to 0,

\[
(W^{(1)})^3 = \bar{D}_+[72X^2DXD\bar{X}D\bar{Y}(D\bar{D}Y)^2 + 24X^3D\bar{D}XDD\bar{Y}(D\bar{D}Y)^2 - 16X^2YD\bar{D}Y(D\bar{D}Y)^3 - 24XDXD\bar{X}DD\bar{Y}D\bar{D}Y + 72YDXD\bar{X}D\bar{Y}(D\bar{D}Y)^2 - 48XDXD\bar{X}D\bar{Y}(D\bar{D}Y)^2 + 24XYD\bar{D}XD\bar{Y}(D\bar{D}Y)^2 - 16XYD\bar{X}(D\bar{D}Y)^3
\]

\[ +16Y^2 D\bar{Y}(D\bar{D}\bar{Y})^3 - 2\bar{D}\bar{X}(D\bar{D}\bar{X})^3 \]
\[ +12DX\bar{D}\bar{X}(D\bar{D}\bar{X})^2\bar{D}\bar{Y} - 4X(D\bar{D}\bar{X})^2\bar{D}\bar{Y} \]
\[ -12X\bar{D}\bar{X}(D\bar{D}\bar{X})^2D\bar{D}\bar{Y}, \]  
(64)

this is consistent with the fact that the level 1 model is isomorphic to the minimal model of level 2 (43). Also in the general \( CP_n^{(k)} \) model, it seems that the chiral ring conjecture about \( \{W^{(1)}, W^{(2)}, \ldots, W^{(n)}\} \) does not conflict with the level-rank duality (43).

### 4.3 The \( CP_3 \) model

The superpotential and the chiral superfields are given by

\[
W_{k+4} = \frac{1}{(k+4)}(a(1)^{k+4} + a(2)^{k+4} + a(3)^{k+4})
\]
\[
X = a(1) + a(2) + a(3)
\]
\[
Y = a(1)a(2) + a(1)a(3) + a(2)a(3)
\]
\[
Z = a(1)a(2)a(3). \]  
(65)

The spin 1 currents are

\[
J(1) = (k + 3)Da(1)\bar{D}\bar{b}(1) - a(1)D\bar{D}\bar{b}(1)
\]
\[
J(2) = (k + 3)Da(2)\bar{D}\bar{b}(2) - a(2)D\bar{D}\bar{b}(2)
\]
\[
J(3) = (k + 3)Da(3)\bar{D}\bar{b}(3) - a(3)D\bar{D}\bar{b}(3)
\]
\[
\bar{b}(1) = \frac{\partial K}{\partial a(1)} = \bar{X} + (a(2) + a(3))\bar{Y} + a(2)a(3)\bar{Z}
\]
\[
\bar{b}(2) = \frac{\partial K}{\partial a(2)} = \bar{X} + (a(1) + a(3))\bar{Y} + a(1)a(3)\bar{Z}
\]
\[
\bar{b}(3) = \frac{\partial K}{\partial a(3)} = \bar{X} + (a(2) + a(1))\bar{Y} + a(2)a(1)\bar{Z} \]  
(66)

We recover the energy-momentum operator as

\[
W^{(1)} = 2(k + 4)J = J(1) + J(2) + J(3)
\]
\[
= (k + 3)DX\bar{D}\bar{X} + (k + 2)DY\bar{D}\bar{Y} + (k + 1)DZ\bar{D}\bar{Z} - XD\bar{D}\bar{X} - 2YD\bar{D}\bar{Y} - 3ZD\bar{D}\bar{Z}. \]
Computations of super $W$ currents $W^{(2)} = J(1)J(2) + J(1)J(3) + J(2)J(3)$, $W^{(3)} = J(1)J(2)J(3)$ are a little complicated. The explicit results are

$$W^{(2)} = -(k + 3)^2 DXDY DXDY - (k + 2)(k + 3) DXDZDXDZ$$
$$- (k + 2)^2 DYDZDYZD$$
$$- (k + 3)DY DXDXDY - (k + 3) YDXDXDY - (k + 3) DZDXDZY$$
$$- 2(k + 3) ZDX DXDXDY - (k + 3) XDY DXDXDY + (k + 4) YDXDXDXDY$$
$$- (k + 2) DZDXDXDY - (k + 2) YDY DXDYDDY - (k + 3) XDXDZYDDY$$
$$+ (k + 5) ZDXDZYDDY - 2(k + 2) ZDY DXDYDDY - (k + 2) XDXDZXDXDZ$$
$$+ (k + 4) ZDXDZXDXDZ - 2(k + 2) YDZDXDYDZ + (k + 5) ZDY DDXDZY$$
$$- 2(k + 1) ZDXDZXDXDZ + Y(DDXX)^2 + XYDXXDXY$$
$$+ 3ZDXDZXDXDZ + 2XZDXXDXDZ + Y^2(DDXY)^2 + XZ(DDXY)^2$$
$$+ 2YZDXXDXDZ + 3Z^2(DDDX)^2$$

$$W^{(3)} = (k + 3)^3 DXDY DZDXDZYD$$
$$+ (k + 3)^2 DXDZDXDXDYD + (k + 3)^2 DYDZXDXDYD$$
$$+ (k + 3)^2(XDZDY + XDYDZ - YDXDZ) DDXDZYD$$
$$+ (k + 3)^2(XDXDZ - DYDZ) DXDXYDDY$$
$$+ (k + 3)^2(YDXDZ - ZDXDY) DXDXDYZDYD + 2(k + 3)^2 DXDYDXDYD$$
$$+ (k + 3)^2 ZDXDZXDXDZYD + (k + 3)^2 ZDY DZDXDYD$$
$$+ (k + 3)^2(YDYDZ - ZDXYZ) DYYDYD$$
$$+ (k + 3) DZXDXDXDYD + (k + 3) DXDZDXDXD$$
$$+ (k + 3)(XDZ + ZDX) DXXDXDYY + (k + 3) ZDY DXDXYD$$
$$+ (k + 3) DYDZXDXDYD$$
$$- (k + 3) ZDXDYDXDYD + (k + 3) (YZDX + ZDZ) DXDYYD$$
$$- (k + 3) ZDXDZXDXDZDYD + (k + 3) Z DXXDXDY$$
$$+ (k + 3) (XDZ - ZDX) (DDDX)^2D$$

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\[\begin{align*}
+ (k + 3) ZDXDYD\bar{D}\bar{X}\bar{D}\bar{Y}D\bar{Z} &+ (k + 3)(XZDY - YZDX + ZDZ)D\bar{D}\bar{X}D\bar{Y}D\bar{D}\bar{Z} \\
+ (k + 3)(XY - Z)DZ\bar{D}\bar{Y}(D\bar{D}\bar{Y})^2 &+ (k + 3)(YDYDZ + 2XZDXDY - 2XYDYZD)D\bar{Y}D\bar{D}\bar{Y}D\bar{D} \\
+ (k + 3)(XZDZ + YZDY - Z^2DX)\bar{D}\bar{Y}D\bar{D}YD\bar{D}\bar{Z} &- 2(k + 3)ZDYDZD\bar{Y}D\bar{Z}D\bar{D} \\
+ (k + 3)Z^2DYD\bar{Y}(D\bar{D}\bar{Y})^2 + (k + 3)(ZDXDY - YDXDZ)D\bar{D}XDYD\bar{Z} &+ (k + 3)(YDY - ZDY)(D\bar{D}\bar{X})^2D\bar{Z} \\
+ (k + 3)(XZDZ - Z^2DX)D\bar{D}\bar{X}D\bar{D}\bar{Z}D\bar{D} &+ (k + 3)(Z^2DX + Y^2DZ - YZDY)(D\bar{D}\bar{Y})^2D\bar{Z} \\
+ (k + 3)(2YDZ - Z^2DY)D\bar{D}YD\bar{Y}D\bar{D}D\bar{Z} &+ (k + 3)Z^2DZD\bar{Z}(D\bar{D}\bar{Z})^2 \\
-Z(D\bar{D}\bar{X})^3 &- 2ZDX(D\bar{D}\bar{X})^2D\bar{Y} - 2XZ(D\bar{D}\bar{X})^2D\bar{D}\bar{Y} \\
-ZDY(D\bar{D}X)^2D\bar{Z} &- ZD(X^2 + Y)D\bar{D}XDYD\bar{D}Y \\
+ZDXDYD\bar{D}\bar{X}D\bar{Y}D\bar{Z} &- Z(YDX + DZ)D\bar{D}\bar{X}D\bar{Y}D\bar{D} \\
-Z(X^2 + Y)D\bar{D}\bar{X}(D\bar{D}\bar{Y})^2 &- Z(2DZ + XYD)D\bar{D}\bar{X}D\bar{D}\bar{Y}D\bar{D} \\
-Z(XY + 3Z)D\bar{D}\bar{X}D\bar{D}\bar{Y}D\bar{D}\bar{Z} &- Z(ZDX + XDX)D\bar{D}\bar{X}D\bar{D}\bar{Y}D\bar{D}\bar{Z} \\
-XZ^2D\bar{D}\bar{X}(D\bar{D}\bar{Y})^2 &- Z(XDY + YDX - DZ)D\bar{Y}(D\bar{D}\bar{Y})^2 \\
+2ZDXDYZD\bar{D}\bar{Y}D\bar{D}\bar{Y}D\bar{D} &- Z(2ZDX + YDY)D\bar{Y}D\bar{D}\bar{Y}D\bar{D} \\
+ZDYDZD\bar{Y}D\bar{D}D\bar{Z} &- Z^2DYD\bar{Y}(D\bar{D}\bar{Z})^2 - Z(XY - Z)(D\bar{D}\bar{Y})^3 \\
-Z(YDY - XDY - ZDX)(D\bar{D}\bar{Y})^2D\bar{Z} &- Z(Y^2 + XZ)(D\bar{D}\bar{Y})^2D\bar{D} \\
-Z(2YDZ + ZDY)D\bar{D}\bar{Y}D\bar{D}D\bar{Z} &- 2YZ^2D\bar{D}\bar{Y}(D\bar{D}\bar{Z})^2 \\
-2Z^2DZD\bar{Z}(D\bar{D}\bar{Z})^2 &- Z^3(D\bar{D}\bar{Z})^3. 
\end{align*}\]

4.4 Poisson brackets between W currents

In this subsection we show that the W supercurrents constructed above indeed close among themselves under the Poisson bracket defined by

\[
\{ \Phi_i(Z_1), \bar{D}\Phi_j(Z_2) \} = \delta^{ij}\frac{4\sqrt{-1}}{4\pi}\frac{\theta_{12}}{Z_{12}}.
\]

We checked that the energy-momentum tensor generates a center-less \( N = 2 \) Virasoro algebra under the Poisson bracket. Thus the structure of the super W algebra is much
simpler than that of the super Toda theory. To compute the Poisson brackets for $W$ currents one only needs the bracket for spin 1 currents $J(i)$ by virtue of the decomposition (50). It can be shown that (59) is equivalent to

$$\{a(i)(Z_1), \bar{D}b(j)(Z_2)\} = \delta^{ij} \frac{4\sqrt{-1}}{4\pi} \frac{\theta_{12}}{Z_{12}}.$$  \hspace{1cm} (70)

Then this induces the following commutation relation

$$-\frac{4\pi}{8} \frac{1}{2(k+n+1)} \{J(i)(Z_1), J(j)(Z_2)\} = \delta^{ij} \left[ \theta_{12} \theta_{12} \frac{\theta_{12}}{(Z_{12})^2} + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial \right] J(i)(Z_2).  \hspace{1cm} (71)$$

Notice that the level $k$ of the model appears only as a normalization constant of the Poisson bracket.

**$C_P^2$ model**

In this case we must show that $(W^{(1)}, W^{(2)})$ form a closed algebra under the Poisson bracket. The results are as follows.

$$-\frac{4\pi}{8} \frac{1}{2(k+3)} \{W^{(1)}(Z_1), W^{(2)}(Z_2)\} = \frac{\theta_{12} \bar{\theta}_{12}}{(Z_{12})^2} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial$$

$$\hspace{3cm} W^{(2)}(Z_2)$$  \hspace{1cm} (72)

$$-\frac{4\pi}{8} \frac{1}{2(k+3)} \{W^{(2)}(Z_1), W^{(2)}(Z_2)\} = \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \frac{\theta_{12}}{Z_{12}} + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial$$

$$\hspace{3cm} \left[ W^{(1)} D W^{(2)} - W^{(2)} D W^{(1)} \right](Z_2) + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} \left[ W^{(1)} \bar{D} W^{(2)} - W^{(2)} \bar{D} W^{(1)} \right](Z_2)$$

$$+ \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} \left[ \bar{D} W^{(2)} D W^{(1)} - \bar{D} W^{(1)} D W^{(2)} - (D \bar{D} + \bar{D} D) W^{(2)} \cdot W^{(1)} \right](Z_2)$$  \hspace{1cm} (73)

**$C_P^3$ model**

In this case we see that $(W^{(1)}, W^{(2)}, W^{(3)})$ close among themselves under the Poisson bracket,

$$-\frac{4\pi}{8} \frac{1}{2(k+4)} \{W^{(1)}(Z_1), W^{(2)}(Z_2)\} = \frac{\theta_{12} \bar{\theta}_{12}}{(Z_{12})^2} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial$$

$$\hspace{3cm} W^{(2)}(Z_2)$$  \hspace{1cm} (74)
\[ -\frac{4\pi}{8} \frac{1}{2(k+4)} \{W^{(1)}(Z_1), W^{(3)}(Z_2)\} = \left[ 3 \frac{\theta_{12} \bar{\theta}_{12}}{(Z_{12})^2} + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} D + \frac{\sqrt{-1}}{4} \frac{\bar{\theta}_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \right] W^{(3)}(Z_2) \] (75)

\[ -\frac{4\pi}{8} \frac{1}{2(k+4)} \{W^{(2)}(Z_1), W^{(2)}(Z_2)\} = \frac{\theta_{12} \bar{\theta}_{12}}{(Z_{12})^2} [W^{(1)}W^{(2)} + 3W^{(3)}](Z_2) \]

\[ + \frac{\sqrt{-1}}{4} \frac{\theta_{12}}{Z_{12}} [W^{(1)}DW^{(2)} - W^{(2)}DW^{(1)} + DW^{(3)}](Z_2) \]

\[ + \frac{\sqrt{-1}}{4} \frac{\bar{\theta}_{12}}{Z_{12}} [W^{(1)}\bar{D}W^{(2)} - W^{(2)}\bar{D}W^{(1)} + 3\bar{D}W^{(3)}](Z_2) \]

\[ + \frac{\sqrt{-1}}{4} \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} [DW^{(2)}DW^{(1)} - \bar{D}W^{(1)}DW^{(2)} - (D\bar{D} + \bar{D}D)W^{(2)}W^{(1)} - 2(D\bar{D} + \bar{D}D)W^{(3)}](Z_2) \] (76)

We can easily evaluate the Poisson brackets for the other \( CP_n \) models by the same method.

To conclude this section we make two remarks. Firstly these algebras depend on the level \( k \) only through the normalization of the brackets. This is in marked contrast with the super Toda case where the central extension term appears even at the classical level. Hence it is much more important to know the quantum regularization of these \( W \) currents in the Landau-Ginzburg model and compute the operator product expansions. Secondly it should be noted that while we used the superfields \( (a(1), a(2), \ldots, a(n)) \) to express the equation of motion and Poisson bracket, we cannot use them in quantum theory because of the nontrivial Jacobian comes out.

\[ \det \frac{\partial \Phi^i}{\partial a(j)} = \prod_{l_1 < l_2} (a(l_1) - a(l_2)). \] (79)
5 Discussion

In this article we investigated the classical aspect of $N = 2$ super $W$ symmetry in Landau-Ginzburg models after clarifying the meaning of the half-twist operation. In the half-twisted Landau-Ginzburg model, $N = 2$ super $W$ currents are realized as BRS cohomology classes in the field space. Remarkable property which we have observed is that these $W$ currents generate a ring isomorphic to the chiral ring so long as the classical product is concerned. In general normal ordered product in CFT is known to be neither commutative nor associative. Thus it is interesting to know the quantum regularizations to $N = 2$ super $W$ currents discussed here and look into their normal ordered product structure. For example in the case of $N = 2$ minimal model of level $k$, $J^{k+1}$ is BRS exact by the classical equation of motion (26). When quantum corrections are made this may represent the existence of the singular vector in the vacuum Verma module. Our method would be used for analyze the $W$ symmetry of other level 1 coset (for example $SO(2n)/U(n)$) models. Finally the $CP_n$ coset models have quantum level description in terms of the Landau-Ginzburg models [14] [10] and the classical level description by super Toda theories [14] [15] [8]. In these models it is important to elucidate the Landau-Ginzburg / Toda correspondence [17].

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A Appendix

A.1 Notation of $N = 2$ superformalism

First we collect some formulas of $N = 2$ superformalism used in this paper. Our convention is that of Wess-Bagger [23] reduced to 2 dimensions $(x^0, x^1, x^2, x^3) \rightarrow (x^0, x^3)$ [23]. Supercovariant differentials are given by

\[
D_\pm = \frac{\partial}{\partial \theta^\pm} - 2\sqrt{-1}\partial^\pm \partial_\pm, \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + 2\sqrt{-1}\bar{\theta}^\pm \bar{\partial}_\pm
\] (80)

\[\footnote{We thank Y. Yamada for pointing out this. Also see [8].}\]
and supercharges are

\[ Q_{\pm} = \frac{\partial}{\partial \theta^\pm} + 2 \sqrt{-1} \partial^\pm \partial_\pm, \quad Q_{\bar{\pm}} = - \frac{\partial}{\partial \theta^{\bar{\pm}}} - 2 \sqrt{-1} \theta^{\bar{\pm}} \partial_\pm \]  \hspace{1cm} (81)

. Right-moving (+) and left-moving (−) coordinates are defined as

\[ \bar{\theta}^\pm = (\theta^\pm)^*, \quad x^\pm = x^0 \pm x^3 \]  \hspace{1cm} (82)

Supertransformations for general superfield \( A \) are defined by

\[ \delta_\xi A = (\xi^+ Q_+ + \xi^- Q_- - \bar{\xi}^+ \bar{Q}_+ - \bar{\xi}^- Q_-) A. \]  \hspace{1cm} (83)

Chiral superfield \( \Phi \) is defined as

\[ \bar{D}_\pm \Phi = 0 \]  \hspace{1cm} (84)

and is expanded as follows,

\[ \Phi = \phi(y^-, y^+) + \sqrt{2} \theta^- \psi_-(y^-, y^+) + \sqrt{2} \theta^+ \psi_+(y^-, y^+) + 2 \theta^+ \theta^- F(y^-, y^+) \]  \hspace{1cm} (85)

\[ y^\pm = x^\pm + 2 \sqrt{-1} \bar{\theta}^\pm \theta^\pm \]  \hspace{1cm} (86)

We also define anti chiral superfield \( \bar{\Phi} \)

\[ D_+ \bar{\Phi} = 0, D_- \bar{\Phi} = 0 \]  \hspace{1cm} (87)

\[ \bar{\Phi} = \bar{\phi}(\bar{y}^-, \bar{y}^+) - \sqrt{2} \bar{\theta}^- \bar{\psi}_-(\bar{y}^-, \bar{y}^+) - \sqrt{2} \bar{\theta}^+ \bar{\psi}_+(\bar{y}^-, \bar{y}^+) + 2 \bar{\theta}^+ \bar{\theta}^- \bar{F}(\bar{y}^-, \bar{y}^+). \]  \hspace{1cm} (88)

\[ \bar{y}^\pm = x^\pm + 2 \sqrt{-1} \theta^\pm \bar{\theta}^\pm \]  \hspace{1cm} (89)

We finally give the super transformations of component fields using the equation of motion of \( F \) field.

\[ \delta \phi^i = \sqrt{2}(\xi^- \psi^i_+ + \xi^+ \psi^i_-), \quad \delta \bar{\phi}^\bar{i} = - \sqrt{2}(\bar{\xi}^- \bar{\psi}^\bar{i}_+ + \bar{\xi}^+ \bar{\psi}^\bar{i}_-) \]  \hspace{1cm} (90)

\[ \delta \psi^i_- = \sqrt{2}(\frac{1}{4} \xi^- \partial_t W - 2 \sqrt{-1} \xi^- \partial_\phi^i), \quad \delta \bar{\psi}^\bar{i}_+ = \sqrt{2}(\frac{1}{4} \bar{\xi}^+ \partial_t W + 2 \sqrt{-1} \bar{\xi}^+ \bar{\psi}^\bar{i}_-) \]  \hspace{1cm} (91)

\[ \delta \psi^i_+ = \sqrt{2}(\frac{1}{4} \xi^- \partial_t W - 2 \sqrt{-1} \xi^- \partial_\phi^i), \quad \delta \bar{\psi}^\bar{i}_- = \sqrt{2}(\frac{1}{4} \bar{\xi}^+ \partial_t W + 2 \sqrt{-1} \bar{\xi}^- \bar{\psi}^\bar{i}_-) \]  \hspace{1cm} (92)
A.2 Operator product expansion formulas

We give operator product expansion formulas in Landau-Ginzburg model which may be
useful in discussing the free field realization of $N = 2$ algebra in terms of the Landau-
Ginzburg superfields. The left moving super coordinates are

\[
\theta = \theta^-, \quad \bar{\theta} = \bar{\theta}^-, \quad z_{12} = x_1^- - x_2^- \\
Z_{12} = z_{12} + 2\sqrt{-1}(\theta_1\bar{\theta}_2 - \theta_2\bar{\theta}_1), \quad \theta_{12} = \theta_1 - \theta_2, \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2
\]

\[
\Phi(Z_1)\bar{\Phi}(Z_2) = \frac{1}{4\pi} \log(Z_{12} - 2\sqrt{-1}\theta_{12}\bar{\theta}_{12}) \\
\Phi(Z_1)\Phi(Z_2) = \frac{1}{4\pi} \log(Z_{12} + 2\sqrt{-1}\theta_{12}\bar{\theta}_{12}) \\
\Phi(Z_1)D\bar{\Phi}(Z_2) = D\bar{\Phi}(Z_1)\Phi(Z_2) = \frac{4\sqrt{-1}}{4\pi} \frac{\theta_{12}}{Z_{12}} \\
D\Phi(Z_1)\bar{\Phi}(Z_2) = \bar{\Phi}(Z_1)D\Phi(Z_2) = -\frac{4\sqrt{-1}}{4\pi} \frac{\bar{\theta}_{12}}{Z_{12}} \\
D\Phi(Z_1)D\bar{\Phi}(Z_2) = -\frac{4\sqrt{-1}}{4\pi} \frac{1}{Z_{12} + 2\sqrt{-1}\theta_{12}\bar{\theta}_{12}} \\
\bar{D}\Phi(Z_1)D\bar{\Phi}(Z_2) = \frac{4\sqrt{-1}}{4\pi} \frac{1}{Z_{12} - 2\sqrt{-1}\theta_{12}\bar{\theta}_{12}}
\]

(93)

The Taylor expansion for $N = 2$ superfield is as follows,

\[
A(z_1, \theta_1, \bar{\theta}_1) = \sum \frac{1}{k!}(Z_{12})^k \left[ 1 + \theta_{12}D_2 - \bar{\theta}_{12}\bar{D}_2 + \frac{1}{2}\theta_{12}\bar{\theta}_{12}(D_2\bar{D}_2 - \bar{D}_2D_2) \right] \partial^k A(z_2, \theta_2, \bar{\theta}_2).
\]

(94)

The energy-momentum tensor is given as

\[
J = \frac{1}{2}(1 - \alpha_i)D\Phi^i\bar{D}\bar{\Phi}^i - \frac{\alpha_i}{2}\Phi^i \bar{D}\Phi^i = \frac{1}{2}D\Phi^i\bar{D}\bar{\Phi}^i - \frac{\alpha_i}{2}D(\Phi^i\bar{D}\bar{\Phi}^i)
\]

(95)

The normalized energy-momentum tensor is $T = \frac{4\pi}{8}J$, and if $A$ is a primary field with
$(L_0, J_0) = (h, q)$, then

\[
T(Z_1)A(Z_2) = \left[ \sqrt{-1} \frac{1}{4} \frac{1}{Z_{12}^2} + h\frac{\theta_{12}\bar{\theta}_{12}}{(Z_{12})^2} + \sqrt{-1} \frac{\theta_{12}}{Z_{12}} D + \sqrt{-1} \frac{\bar{\theta}_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}} \partial \right] A(Z_2)
\]

\[
T(Z_1)T(Z_2) = -\frac{1}{16} \hat{c} \frac{1}{(Z_{12})^2} + \left[ \frac{\theta_{12}\bar{\theta}_{12}}{(Z_{12})^2} + \sqrt{-1} \frac{\theta_{12}}{Z_{12}} D + \sqrt{-1} \frac{\bar{\theta}_{12}}{Z_{12}} \bar{D} + \frac{\theta_{12}\bar{\theta}_{12}}{Z_{12}} \partial \right] T(Z_2)
\]
In particular for the fundamental chiral fields,

\[
T(Z_1)\Phi^i(Z_2) = \left[ \sqrt{-\frac{1}{4}} \frac{1}{Z_{12}} + \frac{\alpha_i}{2} \frac{\theta_{12} \bar{\theta}_{12}}{(Z_{12})^2} + \sqrt{-\frac{1}{4}} \frac{\theta_{12}}{Z_{12}} D + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial \right] \Phi^i(Z_2)
\]

\[
T(Z_1)\bar{D}\bar{\Phi}^i(Z_2) = \left[ \sqrt{-\frac{1}{4}} \frac{1}{Z_{12}} + \frac{1}{2} (1 - \alpha_i) \frac{\theta_{12} \bar{\theta}_{12}}{(Z_{12})^2} + \sqrt{-\frac{1}{4}} \frac{\theta_{12}}{Z_{12}} D + \frac{\theta_{12} \bar{\theta}_{12}}{Z_{12}} \partial \right] \bar{D}\bar{\Phi}^i(Z_2)
\]

**B Ring structure of W currents**

Here we give the complete expression of \([B8]\).

\[
(10J)(2W^{(2)}) = \bar{D}_+[-24XDX\bar{D}\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^2 + 6X^2D\bar{D}\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^2
\]

\[
4XYD\bar{Y} (D\bar{D}\bar{Y})^3 + (D\bar{D}\bar{X})^3 \bar{D}\bar{Y} + \bar{D}\bar{X}(D\bar{D}\bar{X})^2 D\bar{Y}]
\]

\[
(10J)^4 + (2W^{(2)})^2 = \bar{D}_+[96X^3DX\bar{D}\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^3 - 24X^4D\bar{D}\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^3
\]

\[-12X^3YD\bar{Y} (D\bar{D}\bar{Y})^4 - 96XDX\bar{D}\bar{X} (D\bar{D}\bar{X})^2 D\bar{Y} D\bar{D}\bar{Y}
\]

\[+12X^2(D\bar{D}\bar{X})^3 D\bar{Y} D\bar{D}\bar{Y} + 8X^2(D\bar{D}\bar{X})^2 (D\bar{D}\bar{Y})^2
\]

\[+192XD\bar{Y}D\bar{X}D\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^2 + 24XY(D\bar{D}\bar{X})^2 D\bar{Y} (D\bar{D}\bar{Y})^2
\]

\[+64XYD\bar{X}D\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^3 + 192YD\bar{X}D\bar{X}D\bar{Y} (D\bar{D}\bar{Y})^3
\]

\[+32Y^2D\bar{X} (D\bar{D}\bar{Y})^4 + 2Y^2D\bar{X} (D\bar{D}\bar{Y})^4]
\]

The right hand side of \([B8]\) is given by

\[
(14J)(3W^{(3)}) = \bar{D}_+[-9X^8D\bar{Y} (D\bar{D}\bar{Y})^4 + 144X^3DX\bar{D}XDD\bar{X}DY (D\bar{D}\bar{Y})^2
\]

\[ -6X^4(D\bar{D}\bar{X})^2 D\bar{Y} (D\bar{D}\bar{Y})^2 - 18X^4D\bar{X}DD\bar{X} (D\bar{D}\bar{Y})^3
\]

\[ +48X^3DYD\bar{X}DD\bar{Y} (D\bar{D}\bar{Y})^3 - 24X^3YD\bar{X}DD\bar{Y} (D\bar{D}\bar{Y})^3
\]

\[ -(D\bar{D}\bar{X})^4 D\bar{Y} - 6D\bar{X}(D\bar{D}\bar{X})^3 D\bar{D}\bar{Y}]
\]

\[
(14J)^6 + (3W^{(3)})^2 = \bar{D}_+[-8640X^8 DX\bar{D}XDD\bar{Y} (D\bar{D}\bar{Y})^5 + 1512X^{12} D\bar{X} (D\bar{D}\bar{Y})^6
\]

\[ +X^{12}D\bar{X} (D\bar{D}\bar{Y})^6 - 648X^7 DX\bar{D}X (D\bar{D}\bar{X})^2 D\bar{Y} (D\bar{D}\bar{Y})^3
\]

\[ +68X^8(D\bar{D}\bar{X})^3 D\bar{Y} (D\bar{D}\bar{Y})^3 + 36X^8D\bar{X}(D\bar{D}\bar{X})^2 (D\bar{D}\bar{Y})^4
\]

\[ -14832X^7DYD\bar{X}DD\bar{X}DY (D\bar{D}\bar{Y})^4 - 1152X^7Y(D\bar{D}\bar{X})^2 D\bar{Y} (D\bar{D}\bar{Y})^4
\]
\[-2916X^7Y\bar{D}\bar{X}D\bar{D}\bar{X}(DD\bar{Y})^5 - 11664X^6YDY\bar{D}\bar{X}D\bar{Y}(DD\bar{Y})^5\]
\[-1458X^6Y^2\bar{D}\bar{X}(DD\bar{Y})^6 - 576X^3DX\bar{D}\bar{X}(DD\bar{X})^4\bar{D}\bar{Y}D\bar{D}\bar{Y}\]
\[+24X^4(D\bar{D}\bar{X})^5\bar{D}\bar{Y}D\bar{D}\bar{Y} + 18X^4D\bar{X}(D\bar{D}\bar{X})^4(DD\bar{Y})^2\]
\[-3288X^2DYD\bar{X}(D\bar{D}\bar{X})^3\bar{D}\bar{Y}(DD\bar{Y})^2 - 120X^3DY\bar{D}\bar{X}(D\bar{D}\bar{X})^3\bar{D}\bar{Y}(DD\bar{Y})^2\]
\[+144X^3Y(D\bar{D}\bar{X})^4\bar{D}\bar{Y}(DD\bar{Y})^2 - 16X^3DYD\bar{X}(D\bar{D}\bar{X})^3(DD\bar{Y})^3\]
\[+9672XY^2DX\bar{D}\bar{X}(D\bar{D}\bar{X})^2\bar{D}\bar{Y}(DD\bar{Y})^3 + 548X^2Y^2(D\bar{D}\bar{X})^3\bar{D}\bar{Y}(DD\bar{Y})^3\]
\[+2430X^2Y^2\bar{D}\bar{X}(D\bar{D}\bar{X})^2(DD\bar{Y})^4 + 19440XY^2DY\bar{D}\bar{X}D\bar{D}\bar{X}\bar{D}\bar{Y}(DD\bar{Y})^4\]
\[+2916XY^3DYD\bar{X}D\bar{D}\bar{X}(DD\bar{Y})^5 + 11664Y^3DYD\bar{X}D\bar{Y}(DD\bar{Y})^5\]
\[+1458Y^4\bar{D}\bar{X}(DD\bar{Y})^6 + 2\bar{D}\bar{X}(D\bar{D}\bar{X})^6].\quad (99)\]

C  The computation of \(M^{(k)}\)

We first note that the left hand side of (99) can be expanded as

\[
\bar{D}\bar{b}(1)(D\bar{D}\bar{b}(1))^{k+2} - \bar{D}\bar{b}(2)(D\bar{D}\bar{b}(2))^{k+2}
= (a(1) - a(2))N^{(k)} + D(a(1) - a(2))S^{(k)}, \quad (100)
\]

where \(N^{(k)}, S^{(k)}\) are elements of \(C[X, Y, \bar{D}\bar{X}, \bar{D}\bar{Y}; D]\). In the same way we have

\[-2(k + 3)(J(1) - J(2)) = (a(1) - a(2))\left[D\bar{D}\bar{X} + \frac{1}{2}(k + 3)DX\bar{D}\bar{Y}\right]
+ D(a(1) - a(2))\left[-\frac{1}{2}(k + 3)XD\bar{Y} - (k + 2)\bar{D}\bar{X}\right].\quad (101)\]

\(S^{(k)}\) can be directly computed as

\[
S^{(k)} = (k + 2)\bar{D}\bar{X}\bar{D}\bar{Y}
\times \sum_{l=0}^{k+2} \frac{1}{l + 1}k^{k+2}C_l(a(1)^l + a(1)^{l-1}a(2) + \cdots + a(2)^l)(D\bar{D}\bar{X})^{k+2-l}(DD\bar{Y})^l. \quad (102)
\]

Finally we can solve \(M^{(k)}\) by

\[
S^{(k)} = \left[\frac{1}{2}(k + 3)XD\bar{Y} + (k + 2)\bar{D}\bar{X}\right]M^{(k)}. \quad (103)
\]
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