ODONI’S CONJECTURE AND SIMULTANEOUS PRIME SPECIALIZATION OVER LARGE FINITE FIELDS

SUSHMA PALIMAR

Abstract. Let $F_q$ be a finite field of odd cardinality $q$. Given a monic, irreducible, separable polynomial $F(t, x) \in F_q[t][x]$ of degree $n \geq 2$ in $x$ and a generic monic polynomial $\Phi(a, t)$ of positive degree $d$ in $t$, with coefficient vector $(a)$ consisting of linearly independent variables over $F_q$, we find that, for $n, d$ of fixed degree and $q \to \infty$, the Galois group of $F(t, \Phi(a, t))$ over $F_q(a)$ is $S_{nd}$. Under this setting, we show that, the Galois group of the $n$-th iterate of $F(t, \Phi(a, t))$ over $F_q(a)$ is $[S_{nd}]^n$, the $n$-fold iterated wreath product of the symmetric group $S_{nd}$. This is the function field analogue of Odoni’s conjecture over $F_q(a)$, which was originally stated over a Hilbertian field $F$ of characteristic 0 and, for polynomials of degree $d \geq 2$. We further see that, $F(t, \Phi(a, t)) \in F_q(a)[t]$ is a stable polynomial. This indicates, function field analogue of the Odoni’s conjecture implies the function field analogue of Schinzel Hypothesis H. Thus, a natural connection between the Odoni’s conjecture in $F_q(a)[t]$ and the number of prime polynomial tuples of the given form in $F_q[t]$ is established via an explicit form of the Chebotarev Density theorem over function fields.

1. Introduction

Birch and Swinnerton Dyer showed that the Galois group of a polynomial $x^n + x + t$, $n > 1$, over the field $\overline{F}_p(t)$, (the algebraic closure of prime field $F_p$ of $p$ elements of rational functions in the variable $t$), is the symmetric group $S_n$ of order $n$, unless, $p \mid 2n(n - 1)$ while answering a question posed by Chowla. In 1966 Sarvadaman Chowla [9] conjectured that, number of polynomials of the form

$$x^n + x + d$$

which are irreducible modulo $p$ is asymptotic to $\frac{2}{n}$, as $p \to \infty$ for fixed $n$. This was proved independently by S.D. Cohen[10] in 1970 and R. Ree[34] in 1971. In fact, they proved for any prime power $q$ that, the number of $d$ such that $x^n + x + d$ is irreducible is indeed $\frac{2}{n} + O(q^{\frac{1}{2}})$ with the implied constant depending only on $n$, while using the function field analogue of Chebotarev density theorem or Weil’s theorem or the Riemann hypothesis for function field over a finite field and the fact that, Galois group of the polynomial $x^n + x + t$ over the function field $\overline{F}_q(t)$ is the symmetric group $S_n$ of order

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We call the extension that Galois group of \( \phi \) group \( S \). Let start with the basic concept and definitions. The following is taken from preimages of \( H \) call 1.1.

function fields. are the roots of \( \phi \) that, Aut(\( T \)) of vertices up to and including \( n \) all \( K \) be identified with \( \text{Gal}(n) \), and number of irreducible tuples of given for m over \( F_p \), as \( q \to \infty \) using an explicit form of the Chebotarev density theorem over function fields.

1.1. Galois groups of iterates. For the convenience of the reader, we start with the basic concept and definitions. The following is taken from Jones, [13]. Let \( K \) be a Hilbertian field and \( \phi(x) \in K[x] \) be a polynomial of degree \( d > 1 \). For \( n \in \mathbb{N} \), let \( \phi^n = \phi \circ \cdots \circ \phi \) denote the \( n \)-th iterate of \( \phi \), that is, the \( n \)-fold composition of \( \phi \) with itself. Let \( t \in K \), the set of preimages of \( t \) is the set of roots of \( \phi^n(x) - t = 0 \).

\[
\phi^{-n}(t) = \{ \alpha \in K | \phi^n(\alpha) = t \}
\]

For all \( n \geq 1 \), let \( K_n/K \) denote the splitting field of \( \phi^n(x) - t \) over \( K \). \( K_n \) is obtained from \( K_{n-1} \) by adjoining the roots of \( \phi(x) - \beta_1 \), where \( \beta_1, \ldots, \beta_{d^{n-1}} \) are the roots of \( \phi^{n-1}(x) \). Denote, \( G_n := \text{Gal}(K_n/K) \), and the Galois group of relative extension \( K_n/K_{n-1} \) be denoted as \( H_n := \text{Gal}(K_n/K_{n-1}) \). Clearly \( H_n \subset G_n \).

Maximality of \( K_n/K \) and \( K_n/K_{n-1} \). The roots of \( n \)th iterate \( \phi^n \) of \( \phi \) can be identified with \( d^n \) vertices at level \( n \) of the \( d \)-ary rooted tree \( T \), in such a way that Galois group of \( \phi^n \) embeds in Aut(\( T_n \)) where \( T_n \) consists of subtree of vertices up to and including \( n \). Aut(\( T_n \)) is the group of automorphism of the \( d \)-ary rooted tree of height \( n \). A standard result in group theory is that, Aut(\( T_n \)) \( \cong [S_d]^n \), the \( n \)-fold iterated wreath product of the symmetric group \( S_d \).

We call the extension \( K_n/K \) maximal if \( \text{Gal}(K_n/K) = \text{Aut}(T_n) \). We call \( H_n \) is maximal if, \( H_n = S_d^{d^{n-1}} \) and \( d^{n-1} \) is the degree of \( \phi^{n-1}(x) \) for all \( n \in \mathbb{N} \). Then it follows by induction and comparison of degrees that \( \text{Gal}(K_n/K) \cong [S_d]^n \) for all \( n \) and, \( [S_d]^n \) is the \( n \)-fold iterated wreath product of the symmetric group \( S_d \). Thus, \( K_n/K \) is maximal.

Known results. The study of Galois groups of the form \( \text{Gal}(K_n/K) \) was initiated by R.W.K. Odoni around 1980. Over a Hilbertian field \( F \) of characteristic zero, Odoni conjectured that, for any positive integer \( n \), there is a polynomial \( f \in F[x] \) of degree \( n \) such that, each iterate \( f^{nk} \) of \( f \) is irreducible and the Galois group of the splitting field of \( f^{nk} \) is isomorphic to the automorphism group of the regular \( n \)-gon branching tree of height \( k \).

Conjecture 1.1. [Conjecture 7.5, Odoni, [29]] For any Hilbertian field \( F \) of characteristic 0 and any \( d > 2 \), there is a monic polynomial \( f(x) \in F[x] \) of degree \( d \) such that,

\[
\text{Gal}(f^{on}(x), F) \cong [S_d]^n \quad (1.2)
\]
Odani (29) further confirmed this result in (1.2) for any generic monic polynomial $\Phi(a, t)$ of degree $d$, defined over a field $F$ of characteristic zero given by,

$$\text{Gal}(\Phi^m(a, t), K) \cong [S_d]^n$$

Later, this result was generalized by Juul, (20) over a field $F$ of characteristic $p \neq 2$ and $d \neq 2$. Looper, (26) established this result over the field $F = \mathbb{Q}$, when the degree $d$ of the polynomial is a prime number. Kadets, (21) showed that Odani’s conjecture holds for any number field. Looper, (26) and Kadets, (21) proved $K_{n+1}/K_n$ is maximal by showing that any element of $K_{n+1}$ is not a square in $K_n$. Benedetto and Juul, established this conjecture for a polynomial $f$ over a number field $K$, when both the degree of $f$ and $[K : \mathbb{Q}]$ are odd.

2. Preliminaries

**Generic monic polynomials.** Denote by

$$\Phi(a, t) = t^d + a_{d-1}t^{d-1} + \ldots + a_0, \quad \text{deg}_t \Phi(a, t) = d \geq 2$$

the generic monic polynomial with algebraically independent coefficients $(a) = (a_0, a_1, \ldots, a_{d-1})$ over $\mathbb{F}_q$. Then we have the following standard result:

$$\text{Gal}(\Phi(a, t), \mathbb{F}_q(a)) = S_d, \text{ the symmetric group } S_d \text{ of order } d.$$

2.1. Composition of polynomials. For the rest of this paper, we consider the following description. Let $\mathbb{F}_q$ be a finite field with odd cardinality $q = p^\nu, \nu > 2$ and $F(t, x) \in \mathbb{F}_q[t][x]$ be a monic, irreducible and separable polynomial in $x$ of $\text{deg}_x F(x, t) = n \geq 1$, which satisfies the appropriate Bunyakovsky condition.

By appropriate we mean, the Bunyakovsky conjecture proposed by B. Conrad, K. Conrad and R. Gross in [17], (also stated in Pollack’s thesis [31], Conjecture 1.3.8, pp, 22.). In the “naive” Bunyakovsky conjecture over function fields (Conjecture 1.3.5, pp. 17 [31]), separability of $F(t, x)$ over $\mathbb{F}_q(t)$ is not considered.

**Conjecture 2.1.** (Bunyakovsky conjecture) over $\mathbb{F}_q[t]$ [B.Conrad, K.Conrad and R.Gross [17].] Suppose $\mathbb{F}_q$ is a finite field of odd characteristic and let $F(t, x) \in \mathbb{F}_q[t][x]$ has $\text{deg}_x F(t, x) > 0$. Then $F(g(t))$ is irreducible for infinitely many $g(t) \in \mathbb{F}_q[t]$ if and only if the following conditions hold:

- $F(t, x)$ is irreducible in $\mathbb{F}_q(t)[x]$.
- no irreducible $\pi(t) \in \mathbb{F}_q[t]$ divides $F(g(t)) \in \mathbb{F}_q[t]$.
- $F(t, x)$ is separable over $\mathbb{F}_q(t)$

2.2. $F(t, \Phi(a, t))$ is irreducible over $\mathbb{F}_q(a)$. We suppose that, $F(t, x)$ in $\mathbb{F}_q[t][x]$ is monic in $x$, irreducible, separable over $\mathbb{F}_q(t)$, which satisfies the Bunyakovsky Conjecture 2.1. Then for infinitely many polynomials $g(t) \in \mathbb{F}_q[t], F(g(t))$ is irreducible in $\mathbb{F}_q[t]$. Thus, for any generic monic polynomial,
\( \Phi(t) \in \mathbb{F}_q[t] \) defined in \((2.1)\), \( F(t, \Phi(a, t)) \) is irreducible over \( \mathbb{F}_q(a) \). Denote, \( F(a, t) := F(t, \Phi(a)) \); we follow the following convention.

1. Denote \( r = \text{deg}_t(F(a, t)) = nd \), \( r \) is fixed and \( q \to \infty \).
2. \( \text{char}\mathbb{F}_q = p > r, r > 2 \) and \( p \mid r(r-1) \).
3. number of linearly independent variables are taken to be at least 2.

Thus, for \( F(a, t) \in \mathbb{F}_q(a)[t] \), defined in (1), (2) and (3), we establish Odoni’s conjecture \(1.1\) over the function field \( \mathbb{F}_q(a) \), where \( (a) = (a_0, \ldots, a_{d-1}) \) are linearly independent variables over \( \mathbb{F}_q \).

By hypothesis, \( a \) is a vector of algebraically independent variables \( (a_0, \ldots, a_{d-1}) \) over \( \mathbb{F}_q \) and \( a \) may be considered as a single variable over the finite field \( \mathbb{F}_q \). Then, \( \mathbb{F}_q(a)/\mathbb{F}_q \) is a global function field with full constant field \( \mathbb{F}_q \). Since \( (a_0, \ldots, a_{d-1}) \) are algebraically independent over \( \mathbb{F}_q, \mathbb{F}_q(a)/\mathbb{F}_q \) is a regular extension. We will discuss these in \(3.3\) Proposition \(3.1\).

**Statement of Results.**

**Theorem 2.1.** Let \( \mathbb{F}(a, t) \) in \( \mathbb{F}_q(a)[t] \) be defined as in \((2.2)\) and, \( \text{deg}_t(F(a, t)) = r \geq 3 \). Then the Galois group of the polynomial \( F(a, t) \) over \( \mathbb{F}_q(a) \) is the symmetric group \( S_r \) on \( r \) letters.

\[
\text{Gal}(F(a, t), \mathbb{F}_q(a)) = S_r. \tag{2.2}
\]

Theorem 2.1 is proved by showing that, the Galois group of \( F(a, t) \) is a primitive subgroup of \( S_r \) containing a transposition. Separability of \( F(a, t) \) over \( \mathbb{F}_q(a) \) is shown in Proposition 2.1. The double transitivity of the Galois group of \( F(a, t) \) over \( \mathbb{F}_q(a) \) is discussed in \(2.4\). Existence of a transposition is proved in Proposition 2.2.

The following result, is the function field analogue of Odoni’s conjecture over \( \mathbb{F}_q(a) \).

**Theorem 2.2.** Let \( \mathbb{F}(a, t) \in \mathbb{F}_q(a)[t] \) be a polynomial in \( t \) with linearly independent coefficients \( a = (a_0, \ldots, a_{d-1}) \) over \( \mathbb{F}_q \) as defined in \((2.2)\) and \( \text{Gal}(F(a, t), \mathbb{F}_q(a)) \cong S_r \). For \( n \in \mathbb{N} \), let \( K_{n+1} \) be the splitting field of \( F^{\circ n}(a, t) \). Then the Galois group of \((n+1)\)-th iterate of \( F(a, t) \) over \( \mathbb{F}_q(a) \) is

\[
\text{Gal}(F^{\circ n+1}(a, t), \mathbb{F}_q(a)) \cong [S_r]^n \tag{2.3}
\]

i.e.,

\[
\text{Gal}(K_{n+1}/\mathbb{F}_q(a)) \cong [S_r]^n \tag{2.4}
\]

In Proposition 4.6, we show that, \( K_{n+1} \) is the compositum of linearly disjoint Galois extensions \( M'_1 \ldots M'_{n+1} \) over \( K_n \), with \( \text{Gal}(M'_i/K_n) \cong S_r \). Maximality of Galois extension \( K_{n+1}/\mathbb{F}_q(a) \) for all \( n \in \mathbb{N} \) is proved in Corollary 4.1. This establishes the function field analogue of Odoni’s conjecture stated in Theorem 2.2. We may note that, the results stated in the above Theorem 2.1 and Theorem 2.2 holds true in an algebraically closed field \( k = \mathbb{F}_q \).

**Some applications.** In \(5\) we discuss the following applications of the result in Theorem 2.2.
**Stable polynomial.** Theorem 2.2 implies \( F(a,t) \) is a stable polynomial over \( \mathbb{F}_q(a) \), this is discussed in §4.5.

**Function field analogue of the Odoni’s Conjecture and the function field analogue of Schinzel Hypothesis.** The maximality of the Galois extension \( K_{n+1}/\mathbb{F}_q(a) \) is equivalent to the function field analogue of Schinzel’s classical Hypothesis H over \( \mathbb{F}_q[t] \). Maximal of \( K_{n+1}/\mathbb{F}_q(a) \) implies the polynomial \( F(a,t) \in \mathbb{F}_q(a)[t] \) is stable. This lets us find the number of specializations, \( a \mapsto A = (A_0, \ldots, A_{d-1}) \in \mathbb{F}_q^d \) (quantitative estimate) for which, “specialized polynomial tuples” \( (F(A,t), \ldots, F^{n+1}(A,t)) \in \mathbb{F}_q[t] \), are simultaneously irreducible over \( \mathbb{F}_q \). We discuss this result in Theorem 3.2.

**Hilbert Irreducibility Theorem and Separably Hilbertian fields.**

**Definition 1** (Fried & Jarden [28], chapter 12). A Hilbertian field is a field \( K \) in which for any irreducible polynomial \( f(t_1, \ldots, t_r, x) \in K[t_1, \ldots, t_r, x] \) by virtue of Hilbert Irreducibility theorem, there exists infinitely many specializations \( \mathcal{H} := \alpha = (a_1, \ldots, a_r) \in K^r \) such that \( f(a_1, \ldots, a_r, x) \) is irreducible in \( K[x] \). \( \mathcal{H} \) is called the basic Hilbert set. Examples of Hilbertian fields include \( \mathbb{Q} \), number fields, and finite extensions of \( k(t) \) for any field \( k \). In general, if \( k \) is a field and \( t \), an indeterminate, then \( k(t) \) is Hilbertian.

**Definition 2.** (Uchida, [40]) Separably Hilbertian fields: Let \( k \) be a Hilbertian field, \( t, X \) be indeterminates over \( k \). Let \( f(t, X) \in k(t, X) \) be a polynomial which is irreducible in \( X \) and separable over \( k(t) \), so the discriminant \( D_{f(t)} \) is not zero. Then, there exists an element \( s \) of \( k \) such that \( f(s, X) \) is irreducible with \( D_{f(s)} \neq 0 \) and \( f(s, X) \) is separably irreducible, i.e., any Hilbertian field is separably Hilbertian. A field \( k \) of non-zero characteristic is Hilbertian if and only if it is separably Hilbertian and non-perfect.

**Proposition 2.1.** With the above suppositions, the irreducible polynomial \( F(a,t) \) is separable over \( \mathbb{F}_q(a) \) so that \( F(a,t)/\mathbb{F}_q(a) \) is a Galois extension.

**Proof.** The proof follows from Definition 2 of separable Hilbertian field. By hypothesis, \( F(t,x) \in \mathbb{F}_q[t][x] \) is monic, irreducible and separable in \( x \). Then by the definition of separably Hilbertian fields, there exists infinitely many \( s \in \mathbb{F}_q \) such that, \( F(t,s) \) is separable over \( \mathbb{F}_q \). This implies, for almost all monic polynomials \( g(t) \in \mathbb{F}_q(t) \), the polynomial \( F(t,g(t)) \) is separable (square-free) and do not have multiple roots in any extension of \( \mathbb{F}_q(a) \). Thus, for any generic monic polynomial \( \Phi(a,t) \) of degree \( d \) in \( t \) with algebraically independent variables \( a_0, \ldots, a_{d-1} \) as coefficients, the discriminant \( D \) of \( F(t, \Phi(a,t)) \) with respect to \( t \) is a non zero function of indeterminates, \( a \). Hence, the polynomial \( F(a,t) \) is separable over \( \mathbb{F}_q(a) \). Thus, \( \text{Gal}(F(a,t), \mathbb{F}_q(a)) \) is a transitive subgroup of \( S_r \), symmetric group on \( r \) letters. \( \square \)
2.3. **Critical points of $F(a, t)$ are distinct.** Separability of $F(a, t)$ gives

$$D = \pm \text{Resultant}(F(a, t), \frac{d}{dt} F(a, t)) \neq 0. \quad (2.5)$$

That is, the system of equations

$$\begin{cases} 
\frac{d}{dt} F(a, t)|_{t = \rho} = 0 \\
\frac{d}{dt} F(a, t)|_{t = \eta} = 0 \\
F(a, t)|_{t = \rho} = F(a, t)|_{t = \eta}
\end{cases} \quad (2.6)$$

do not have a solution for distinct roots $\rho, \eta$ of $F(a, t)$ in the algebraic closure $\Omega$ of $\mathbb{F}_q(a)$. Thus, the critical points of $F(a, t)$ are distinct and, $F(a, t)$ has no multiple (double) roots, in any extension of $\mathbb{F}_q(a)$. That is the polynomial function $\frac{d}{dt} F(a, t)$ has only simple roots. This result is done in detail in [Rudnik [35], Theorem 1.3].

$G = \text{Gal}(F(a, t)/\mathbb{F}_q(a))$ **contains a transposition.** This result is an adaptation of [Uchida, Theorem 1 [39] or Lemma 5 of Smith[37]]. We recall the following standard theorem on local fields.

**Theorem 2.3** (Theorem 11.7, Sutherland, [35]). Let $K$ be a global field with a nontrivial absolute value $|\cdot|$, and let $\hat{K}$ be the completion of $K$ with respect to $|\cdot|$. Every finite separable extension $\hat{L}$ of $\hat{K}$ is the completion of a finite separable extension $L$ of $K$ with respect to an absolute value that restricts to $|\cdot|$. Moreover, one can choose $L$ so that $\hat{L}$ is the compositum of $L$ and $\hat{K}$ and $[\hat{L} : \hat{K}] = [L : K]$.

Every finite extension of local fields $\hat{L}/\hat{K}$ necessarily corresponds to an extension of global fields $L/K$.

**Proposition 2.2.** Let $L$ denote the minimal splitting field of $F(a, t)$ over $\mathbb{F}_q(a)$. The Galois group $G$ of $F(a, t)$ over $\mathbb{F}_q(a)$ contains a transposition.

**Proof.** It is clear from the above discussion that, $F(a, t)$ is irreducible and separable over $\mathbb{F}_q(a)$, so that, $L/\mathbb{F}_q(a)$ is a Galois extension and $F(a, t)$ does not have double roots, hence, every irreducible factor of $F(a, t)$ mod $P$ has ramification index less than 3. Hence, the only primes ramifying in $L$ are the primes dividing $D$. Let $P \neq 0$ be any prime dividing the discriminant $D$ of $F(a, t)$. Clearly, $P$ is a ramified prime in the splitting field $L$ of $F(a, t)$. Thus, factorization of $F(a, t)$ modulo $P$ is,

$$\bar{F} \equiv \bar{f}_1 \cdot \ldots \bar{f}_n \mod P \quad (2.7)$$

Each of $\bar{f}_i, i \geq 2$ is monic and relatively prime to $\bar{f}_1 \mod P$ and $\bar{f}_2, \ldots, \bar{f}_n$ are irreducible. By Hensel’s Lemma, $F(a, t)$ is factored over the completion $\hat{K} = \mathbb{F}_q((a))$ of $\mathbb{F}_q(a)$ wrt non trivial absolute value (prime ideal) $P$ as

$$F = g_1 \cdot \ldots \cdot g_n \quad (2.8)$$

with each $g_i$ having coefficients in $\hat{K}$ and, $g_i \equiv \bar{f}_i \mod P, i \geq 2$ and $g_1 \equiv \bar{f}_1^2 \mod P$. Let $\mathfrak{F}$ be a prime of $L$ lying above $P$ and let $\hat{L}$ be an extension
of $\mathbb{F}_q((a))$. Clearly, $\hat{L}$ is the completion of $L$ wrt $\mathfrak{p}$ containing $\mathbb{F}_q((a))$, and the extension of local fields $\hat{L}/\bar{K}$ is a Galois extension. For $i \geq 2$, the roots of $g_j$ generate unramified extension over $\mathbb{F}_q((a))$ and $g_1$ is irreducible of degree 2 over $\mathbb{F}_q((a))$. Thus, the inertia group of $\hat{L}$ over $\mathbb{F}_q(a)$ is generated by transposition of the roots of $g_1 = 0$ and leaves the roots of $g_i$ fixed for $i \geq 2$. Thus, $G$ contains a transposition. □

Remark 1. This result holds true for any specialization (separable, irreducible) $a_i \to c \in \mathbb{F}_q$. The specialized polynomial $F_c$ is now defined over $\mathbb{F}_q(\tilde{a}) := F_q(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{d-1})$ is irreducible and separable, and the Galois group $\tilde{G}$ of the specialized polynomial $F_c$ contains a transposition. By the fact that, Galois groups can not increase under (separable) specialization of parameters, $\tilde{G} \subset G = \text{Gal}(F(a, t), \mathbb{F}_q(a))$ implies $G$ contains a transposition.

2.4. Galois group $G$ of $F(a, t)$ over $\mathbb{F}_q(a)$ is doubly transitive. A permutation group $G$ acting on a set $\Omega$ is transitive if any $k$-tuple of distinct points can be mapped, by some element of $G$, to any other $k$-tuple of distinct points. Clearly a $k$-transitive group is also, $k-1$ transitive. We know that $F(a, t)$ is irreducible and separable over $\mathbb{F}_q(a)$, and $\deg(F(a, t)) = r > 2$. Separability of $F(a, t)$ implies the roots of $F(a, t)$ are distinct. Let $\alpha_1, \ldots, \alpha_r$ be the $r$ distinct roots of $F(a, t)$. Let $L$ be the splitting field of $F(a, t)$ over $\mathbb{F}_q(a)$, that is $L = \mathbb{F}_q(a)(\alpha_1, \ldots, \alpha_r)$ and $F(a, t)$ splits into $r$ linear factors $(t-\alpha_1)\ldots(t-\alpha_r)$ in $L[t]$. Since the roots are distinct, any two linear factors $(t-\alpha_i)$ and $(t-\alpha_j)$, ($i \neq j$) are pairwise coprime in $\mathbb{F}_q(a)[t]$ and each of $\alpha_i$ is transcendental over $\mathbb{F}_q(a)$. To prove double transitivity of the Galois group of $F(a, t)$ over $\mathbb{F}_q(a)$, we follow the method of “throwing away roots” by Abhyankar in [III §4]. Since $F(a, t)$ is irreducible in $\mathbb{F}_q(a)[t]$, we throw away a root of $F(a, t)$, say $\alpha_1$ and get

$$F_1(a, t) = \frac{F(a, t)}{(t-\alpha_1)} \in \mathbb{F}_q(a)(\alpha_1)[t] \quad (2.9)$$

Now, $F_1(a, t)$ is irreducible in $\mathbb{F}_q(a)(\alpha_1)[t]$. Thus $F(a, t)$ and $F_1(a, t)$ are irreducible in $\mathbb{F}_q(a)[t]$ and $\mathbb{F}_q(a)(\alpha_1)[t]$ respectively. Thus, the Galois group of $F(a, t)$ over $\mathbb{F}_q(a)$ is doubly transitive. Since there is no field between $\mathbb{F}_q(a)[t]$ and $\mathbb{F}_q(a)(\alpha_1)[t]$, the group $\text{Gal}(F(a, t), \mathbb{F}_q(a))$ is primitive §7, [III].

Proposition 2.3. The Galois group $G$ of $F(a, t)$ over $\mathbb{F}_q(a)$ is primitive.

Proof. Since any doubly transitive group action is primitive, $G$ is primitive. □

A result due to Marggrafr (Marggraf or Marggraaf) stated in [W.Burnside, §140, p.197] is that a primitive permutation group $G$, which contains a cycle fixing $k$ points is $k + 1$-fold transitive [§3, G.A. Jones [19]]. This is stated in Wielandt, [III, Theorem 13.8] as below.
Theorem 2.4. Suppose, $G$ is a primitive group of degree $n$, which contains a $d$– cycle, where $1 < d < n$. Then $G$ is $(n - d + 1)$–ply transitive.

Theorem 2.5. [41, Theorem 10.1] Let $k = 1, 2, 3\ldots$ Every $(k+1)$–fold transitive group is $k$–fold primitive. Every $k$–fold primitive group is $k$–fold transitive and every group of which is a subgroup is $k$–fold primitive.

Theorem 2.6. [41, Theorem 13.3] If a primitive permutation group contains a transposition, it is a symmetric group.

Cohen, [12, Corollary 3] proved, over any field $K$ of characteristic $p > 0$, that, Galois group of the trinomials $f(a_0, a_1, x) = x^r + a_0 x^s + a_1$, where $(r, s) = 1$ with two indeterminates $a_0$ and $a_1$, is the symmetric group $S_r$, unless $p$ divides $r(r-1)$ and $s = 1$ or $s = n-1$.

Proof of Theorem 2.1

Proof. We have taken, number of linearly independent coefficients of $F(t, \Phi(t))$ to be at least 2 and $\deg_t(F(a, t)) = r \geq 3$ in $t$. The monic polynomial $F(a, t)$ is monic, irreducible and separable in $t$. Primitivity of $G$, follows from the fact that $G$ is doubly transitive, ($\S$ 2.4, $\deg_t(F(a, t)) = r > 2$). Thus by Theorems 2.4, 2.5 and 2.6

$G = \text{Gal}(F(a, t), \mathbb{F}_q(a)) \cong S_r$

□

The quantitative result for the number of irreducible specializations $a \rightarrow A \in \mathbb{F}_q^d$, for which, the specialized polynomials $F(A, t) \in \mathbb{F}_q[t]$, are irreducible in $\mathbb{F}_q[t]$ is stated here.

Theorem 2.7. Suppose $F(a, t)$ be defined as in Theorem 2.1. Then, the number of $A = \#\{(A_0, \ldots, A_{d-1})\} \in \mathbb{F}_q^d$ for which the specialized polynomial $F(A, t)$ irreducible in $\mathbb{F}_q[t]$ is given by:

$$N_q(F(A, t)) = \frac{q^d}{r} + O_r(q^{d-\frac{1}{2}}), \quad q \rightarrow \infty.$$ (2.10)

where the implied constants depend only on $r$.

Proof. The result on irreducible specialization in equation (2.10) follows from Theorem 1.1 [5]. The estimate in equation (2.10) is attained by Cohen, [Theorem 3, 11] to settle Chowla’s conjecture [9] by using Lang-Weil estimate or function field analogue of Čebotarov density theorem or Weil’s theorem on the Riemann hypothesis for function field over a finite field. This result is also mentioned in §Introduction [14] Cohen.

□

3. Linearly disjoint Galois extensions

Definition 3. Two algebraic sub-extensions $K, L$ of $\Omega/k$ are linearly disjoint if and only if $K \otimes_k L$ is a field.
Definition 4. A field extension $L/k$ is said to be regular if $k$ is algebraically closed in $L$ equivalently, $L \otimes_k \overline{k}$ is an integral domain, $\overline{k}$ is the algebraic closure of $k$ i.e., $L$, $k$ are linearly disjoint over $k$.

Theorem 3.1. [Cohn [15], Theorem 5.5, Theorem 5.5′]

1. Let $K/k$, $L/k$ be two extensions of which one is normal and one is separable. Then $K \otimes_k L$ is a field, if and only if $K \cap L = k$.
2. Further, $K \otimes_k L$ is a field, if and only if $K$, $L$ have no isomorphic subfield properly containing $k$.

3.1. Linearly disjoint Galois extensions under different specializations. As before, let $\text{Gal}(L/k)$. Given $L$ containing $F$ over $k$. Let $c, c'$ be two specializations at $a_{d-1}$ such that, the specialized polynomials $F_c$ and $F_{c'}$ in $F_q(\tilde{a})[t]$ are monic, non-associate, irreducible and separable, over $F_q(\tilde{a})$ and are of degree $r$ in $t$. Denote, the splitting fields of $F_c$ and $F_{c'}$ as $L_c$ and $L_{c'}$ respectively. Clearly $L_c/F_q(\tilde{a})$ and $L_{c'}/F_q(\tilde{a})$ are Galois extensions.

Proposition 3.1. Under these conditions, the following holds:

1. Splitting fields of $F_c$ and $F_{c'}$ namely $L_c/F_q(\tilde{a})$ and $L_{c'}/F_q(\tilde{a})$ are regular over $F_q$.
2. The splitting fields, $L_c/F_q(\tilde{a})$ and $L_{c'}/F_q(\tilde{a})$ are linearly disjoint.

Proof. (1) Given $\tilde{a} = (a_0, \ldots, a_{d-2})$ are algebraically independent variables over $F_q$ implies, $F_q(a_0, \ldots, a_{d-2})/F_q$ is a regular extension. Denote by $\Omega$, the algebraic closure of $F_p(\tilde{a})$ and $\hat{L}$ be any smallest extension of $F_p(\tilde{a})$ inside $\Omega$ containing $L_c$ and $L_{c'}$, then, $\hat{L}$ is finitely generated over $F_p$ and the algebraic closure of $F_p$ in $\hat{L}$ is $F_q$. Thus, $\hat{L}$ being an extension of algebraically closed field $F_q$, is regular over $F_q$. Hence, $\hat{L}$, $\overline{F_q}$ are linearly disjoint over $F_q$. Thus, $L_c$ and $L_{c'}$ being subfields of $\hat{L}$, are regular over $F_q$.

(2) Since $L_c$ and $L_{c'}$ have distinct ramification implies, $L_c \cap L_{c'} = F_q(\tilde{a})$, i.e., $L_c$ and $L_{c'}$ are linearly disjoint over $F_q(\tilde{a})$. □

4. Galois groups of iterates of composition of polynomials

Following the discussion in §2.1 Denote,

$F(a, t) := F(t, \Phi(a, t))$

and

$\deg, F(a, t) = r$

We recall from §2.1 that $F(a, t)$ is a monic, irreducible and separable polynomial of degree $r$ in $t$ with its coefficients as algebraically independent variables $a = (a_0, \ldots, a_{d-1})$, and,

$\text{Gal}(F(a, t), F_q(a)) = S_r$
We name here $F(a, t)$ as $F^{01}(a, t)$. Let us recall, for $n \in \mathbb{N}$, the $n$th iterate of $F(a, t)$ is

$$F^{on}(a, t) := F^{on-1}(F(a, t))$$

By hypothesis, for $n \geq 1$, $F^{on}(a, t) \in \mathbb{F}_q(a)[t]$ is a polynomial in $t$ of degree $r^n$ and $K_n$ is the splitting field of $F^{on}(a, t)$ over $\mathbb{F}_q(a)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{r^n}$ be the distinct roots of $F^{on}(a, t)$, (each $\alpha_i$ is a function of $a_0, \ldots, a_{d-1}$). $K_{n+1}$ is obtained from $K_n$ by adjoining the roots of $F(a, t) - \alpha_i$.

The critical points of $F(a, t)$ over $\mathbb{F}_q(a)$ are distinct (discussed in §2.3). Since $q$ is very large, we can choose a sufficiently large $k \in \mathbb{N}$ such that for $i \neq j \in N$ and $i, j < k$, $F^{oi}(a, t) \neq F^{oj}(a, t)$. Under this assumption, we see in subsequent results that, for any two critical points $\gamma$ and $\delta$ the iterated polynomials $F^{oi}(a, \gamma) \neq F^{oj}(a, \delta)$ (Proposition 4.1 and Proposition 4.2).

### 4.1 Irreducibility and separability of $F^{on}(a, t)$ over $\mathbb{F}_q(a)$.

**Lemma 4.1.** [Capelli’s Lemma] Let $K$ be a Hilbertian field and $f(x)$ and $g(x)$ be in $K[x]$. Let $\beta$ be a root of $f(x)$. Then every root of $g(x) - \beta$ is a root of $f(g(x))$. If $\alpha$ is a root of $f(g(x))$, then $g(\alpha)$ is a root of $f(x)$.

1. Then $f(g(x))$ is irreducible in $K[x]$ if and only if $f(x)$ is irreducible in $K[x]$ and $g(x) - \beta$ is irreducible in $K(\beta)[x]$ for every root $\beta$ of $f(x)$.
2. $f(g(x))$ is separable in $K[x]$ if and only if $f(x)$ is separable in $K[x]$ and $g(x) - \beta$ is separable in $K(\beta)[x]$ for every root $\beta$ of $f(x)$.

**Condition** (2) is proved in Choi, [8]

**Proposition 4.1.** The $n$th iterated polynomial $F^{on}(a, t) \in \mathbb{F}_q(a)[t]$ is irreducible and separable for all $n \in \mathbb{N}$.

**Proof.**

1. The function $F(a, t)$ is irreducible and separable over $\mathbb{F}_q(a)$. By definition,

$$F^{o2}(a, t) := F \circ F(a, t)$$

Let $K_1$ be the splitting field of $F(a, t)$ over $\mathbb{F}_q(a)$ and $\alpha_1, \ldots, \alpha_r$ be the distinct roots of $F(a, t)$, (each of $\alpha_i$ is a function of independent indeterminates $a_0, \ldots, a_{d-1}$). Since $\alpha_i$ is transcendental over $\mathbb{F}_q(a)$, each of $F(a, t) - \alpha_i$ is irreducible over $\mathbb{F}_q(a)(\alpha_i)$ for $i = 1, 2, \ldots, r$. Thus, $r$ factors $F(a, t) - \alpha_i$ are monic, irreducible, separable and relatively prime over $K_1$, none of which are ramified over $K = \mathbb{F}_q(a)$. Thus, $F^{o2}(a, t)$ is irreducible over $\mathbb{F}_q(a)$. Irreducibility of $F^{on}(a, t)$ for higher values on $n \in \mathbb{N}$, follows similarly by induction.

2. By Proposition 2.1 $F(a, t)$ is separable.

$$\frac{d}{dt}(F(a, t) - \alpha_i) \neq 0 \text{ for all roots } \alpha_i \text{ of } F(a, t)$$

implying, $F^{o2}(a, t)$ is separable over $\mathbb{F}_q(a)$. Separability of $F^{on}(a, t)$ over $\mathbb{F}_q(a)$ for $n \geq 3$ follows by induction. Converse follows easily in both the cases.
From now on, we follow the notations from Juul, [20].

**Proposition 4.2.** Let $K_n$ be the splitting field of $F^{\circ n}(a,t)$ over $\mathbb{F}_q(a)$ and, $\alpha_1, \alpha_2, \ldots, \alpha_{r^n}$ be the distinct roots of $F^{\circ n}(a,t)$ and $M_i/\mathbb{F}_q(a)(\alpha_i)$ be the splitting field of $F(a,t) - \alpha_i$, then $\text{Gal}(M_i/\mathbb{F}_q(a)(\alpha_i)) = S_r$.

**Proof.** We note that, each of $r^n$ polynomials $F(a,t) - \alpha_1, \ldots, F(a,t) - \alpha_{r^n}$ are monic, irreducible, separable and relatively prime to each other in $K_n[t]$. Let $M_i$ be the splitting field of $F(a,t) - \alpha_i$ over $\mathbb{F}_q(a)(\alpha_i)$, and

$$\deg_t(F(a,t) - \alpha_i) = r = \deg_q(F(a,t)).$$

We know that,

$$\text{Gal}(F(a,t), \mathbb{F}_q(a)) = S_r$$

Since $\alpha_i$ is transcendental over $\mathbb{F}_q(a)$, it follows that,

$$\text{Gal}(F(a,t) - \alpha_i, \mathbb{F}_q(a)(\alpha_i)) = \text{Gal}(M_i/\mathbb{F}_q(a)(\alpha_i)) \cong \text{Gal}(F(a,t), \mathbb{F}_q(a))$$

Hence,

$$\text{Gal}(M_i/\mathbb{F}_q(a)(\alpha_i)) \cong S_r \text{ for all } i = 1, 2, \ldots, r^n.$$

4.2. $K_n$ is the compositum of linearly disjoint Galois extensions. In the above Proposition 4.2 since, the roots of $F^{\circ n}(a,t)$ are distinct, for $i \neq j$, $M_i$ and $M_j$ have distinct ramification. Hence, $M_i \cap M_j = \mathbb{F}_q(a)$ for $i \neq j$. Thus, the Galois extensions (splitting fields) $M_1, \ldots, M_{r^n}$, are linearly disjoint over $\mathbb{F}_q(a)$.

Let $\beta_1, \ldots, \beta_{r^n-1}$ be the distinct roots of $F^{\circ n-1}(a,t) \in \mathbb{F}_q(a)[t]$. Let $L_i$ be the splitting fields of $F(a,t) - \beta_i$ over $\mathbb{F}_q(a)[\beta_i]$, for each $i = 1, 2, \ldots, r^{n-1}$. Then, $L_1, \ldots, L_{r^n-1}$ are linearly disjoint over $\mathbb{F}_q(a)$ with

$$\text{Gal}(L_i/\mathbb{F}_q(a)[\beta_i]) \cong S_r.$$

Denote

$$L_i' := L_iK_{n-1}. \quad (4.2)$$

It is useful to note that,

$$F^{\circ n}(a,t) = \prod_{i=1}^{r^{n-1}} (F(a,t) - \beta_i), \quad F(a,t) - \beta_i \in \mathbb{F}_q(a)[\beta_i][t], \quad i = 1, \ldots, r^{n-1}$$

and

$$K_n = \prod_{i=1}^{r^{n-1}} L_i' \quad (4.3)$$
The Galois extensions $L'_1, \ldots, L'_{r-1}$ are linearly disjoint over $K_{n-1}$ and $K_n$ is the compositum,

$$K_n = L'_1 \ldots L'_{r-1} \quad \text{(4.4)}$$

Denote,

$$M'_i := M_i K_n$$

$$\widehat{M}_i := K_n \prod_{j \neq i} M_j$$

The Galois extensions $M'_1, \ldots, M'_{r-1}$ are linearly disjoint over $K_n$, that is $M'_i \cap M'_j = K_n, i \neq j$, and $K_{n+1}$ is the compositum of $M'_1 \ldots M'_{r-1}$, i.e.,

$$K_{n+1} = \prod_{i=1}^{r-1} M'_i \quad \text{(4.5)}$$

and

$$F^{\sigma_{n+1}}(a, t) = \prod_{i=1}^{r-1} F(a, t) - \alpha_i, \quad F(a, t) - \alpha_i \in \mathbb{F}_q(a)(\alpha_i)[t]$$

Thus by above discussion, any $K_i$, for $i \geq 1$ is the compositum of linearly disjoint Galois extensions.

4.3. $M'_i$ and $\widehat{M}_i$ are linearly disjoint over $K_n$. Distinct ramification in $M_i$ and $\widehat{M}_i$ implies, $M_i$ and $\widehat{M}_i$ are linearly disjoint over $\mathbb{F}_q(a)(\alpha_i)$, and $M'_i$ and $\widehat{M}_i$ are linearly disjoint over $K_n$.

4.4. **Linearly disjoint Galois extensions.** We consider the field extension in figure 1 and figure 2, and discuss the linear disjointness of splitting fields and the Galois groups of relative extension, (This is a standard result.) found in [Proposition 3.1, VIII §3, Lang, Algebra [24]], [Proposition 2.6 in K. Conrad [16]], [Theorem 6.2.2, Robert B. Ash [3] (last two references are expository article on Galois theory)].

**Figure 1.**

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From Proposition 4.2, \( \text{Gal}(M_i/F_q(a)(\alpha_i)) \cong S_r \) for \( i = 1, 2, \ldots, r^n \). Thus, \( M_i/F_q(a)(\alpha_i) \) is a Galois extension, implies, \( M_iK_n/K_n \) is also a Galois extension [Theorem 6.2.2, Ash, [3]]. For \( n \in \mathbb{N} \), \( K_n/F_q(a)(\alpha_i) \) is a normal extension and \( \text{Gal}(M_iK_n/K_n) \) is a normal subgroup of \( \text{Gal}(M_i/F_q(a)(\alpha_i)) \).

So, this is evident:

\[
\text{Gal}(M_iK_n/K_n) \cong \text{Gal}(M_i/M_i \cap K_n) = \text{Gal}(M_i/F_q(a)(\alpha_i)) = S_r. \quad (4.6)
\]

Hence this forces in Figure 2.,

\[
\text{Gal}(M_iK_n/K_n) \cong \text{Gal}(K_{n+1}/\hat{M}_i) = S_r \quad (4.7)
\]

Since, \( M_i/F_q(a)(\alpha_i) \) is a finite Galois extension, we have, \( \text{Gal}(M_i/M_i \cap \hat{M}_i) \) is a normal subgroup of \( \text{Gal}(M_i/F_q(a)(\alpha_i)) \) which means \( (M_i \cap \hat{M}_i)/F_q(a)(\alpha_i) \) is a Galois extension.

Now from the above diagram in Figure 1, and equation (4.7) we have,

\[
\text{Gal}(K_{n+1}/\hat{M}_i) \cong \text{Gal}(M_i/M_i \cap \hat{M}_i) = S_r \quad \text{for each } i, \quad 1 \leq i \leq r^n \quad (4.8)
\]

**Square-free discriminants.** From equation (4.7), \( \text{Gal}(M_i'/K_n) \cong S_r \). Denote, the unique quadratic sub-extension of \( M_i' \) as \( E_i' \) and,

\[
E_i' = K_n(\sqrt{\text{disc}_t(F(a, t) - \alpha_i)})
\]

and

\[
\delta_t(F(a, t) - \alpha_i) := \sqrt{\text{disc}_t(F(a, t) - \alpha_i)}
\]

Since, each of \( \text{Gal}(F(a, t) - \alpha_i, F_q(a)(\alpha_i)) \cong S_r \), for \( i = 1, 2, \ldots, r^n \) implies, the product \( \text{disc}_t(F(a, t) - \alpha_1) \cdots \text{disc}_t(F(a, t) - \alpha_{r^n}) \) square-free. The following Proposition 4.3 is an application of [Lemma 3.2 and Lemma 3.3, Bary-Soroker [4]]

**Proposition 4.3.** \( \delta_t(F(a, t) - \alpha_1), \ldots, \delta_t(F(a, t) - \alpha_{r^n}) \) are linearly independent in the \( \mathbb{F}_2 \) vector space \( K_n^\times/(K_n^\times)^2 \), if and only if \( E_1', \ldots, E_{r^n}' \) are linearly disjoint over \( K_n \) (i.e., the product \( \prod_{i \neq j} \text{disc}_t(F(a, t) - \alpha_i) \) is non square, hence square independent in \( \mathbb{F}_q(a) \)).
From the above discussion, we have the following.

**Proposition 4.4.**  
(1) \(E'_1, \ldots, E'_{r^n}\) are linearly disjoint over \(K_n\).  
(2) \(\prod_{i \neq j} \operatorname{disc}_i(F(a,t) - \alpha_i)\) are square independent, i.e., \(\delta_i(F(a,t) - \alpha_1), \ldots, \delta_i(F(a,t) - \alpha_{r^n})\) are linearly independent in \(K_n^\times/(K_n^\times)^2\).

**Proof.**  
(1) Since, distinct \(M'_i\) have distinct ramification and \(E'_i\) is the unique quadratic sub-extension of \(M'_i\), \(E'_i \cap E'_j = K_n\), for \(i \neq j\).

(2) The monic, separable polynomials \(F(a,t) - \alpha_1, \ldots, F(a,t) - \alpha_{r^n}\) are irreducible and relatively prime implies, polynomials do not have a common root and it follows, for \(i \neq j\), the product of \(\operatorname{disc}_i(F(a,t) - \alpha_i) \cdot \operatorname{disc}_i(F(a,t) - \alpha_i)\) is square-free. \(\square\)

**Proposition 4.5.** Any prime \(P\) of \(\mathbb{F}_q(a)\) ramifying in \(K_{n+1}\), does not ramify \(K_n\).

**Proof.** The proof is a simple consequence of linearly disjoint splitting fields discussed in §4.4. The \(n\)th iterated polynomial \(F^{\circ n}(a,t)\) is monic, irreducible and separable over \(\mathbb{F}_q(a)\) and \(K_n\) is the splitting field of \(F^{\circ n}(a,t)\) over \(\mathbb{F}_q(a)\). As before, let \(\alpha_1, \ldots, \alpha_{r^n}\) be the distinct roots of \(F^{\circ n}(a,t)\). By (4.3),

\[
F^{\circ n+1}(a,t) = \prod_{i=1}^{r^n} F(a,t) - \alpha_i.
\]

The \(r^n\) factors on the right \(F(a,t) - \alpha_1, \ldots, F(a,t) - \alpha_{r^n}\) are distinct, monic, relatively prime, irreducible and separable over \(K_n\), which does not ramify in \(K_n\). \(M_i\) is the splitting field of \(F(a,t) - \alpha_i, 1 = 1, 2, \ldots, r^n\) and, by Proposition 4.2,

\[\operatorname{Gal}(M_i/\mathbb{F}_q(a)(\alpha_i)) \cong S_r\]  

(4.9)

The \(r^n\) extensions, \(M_1, \ldots, M_{r^n}\) of \(K_n\) are linearly disjoint over \(\mathbb{F}_q(a)\) (4.12). Hence, the “only” primes ramifying in \(M_i, i = 1, 2, \ldots, r^n\) are the prime divisors of \(\operatorname{disc}_i(F(a,t) - \alpha_i)\) and these primes ramify in \(M'_i = M_iK_n\) as well and do not ramify in \(M_j\) and \(M'_j\) for \(i \neq j\). Since,

\[
K_{n+1} = \prod_{i=1}^{r^n} M'_i,
\]

the primes ramifying in \(K_{n+1}\) are the primes dividing

\[
\prod_{i=1}^{r^n} \operatorname{disc}_i(F(a,t) - \alpha_i)
\]

As a fact, we know that \(M'_1, \ldots, M'_{r^n}\) are linearly disjoint over \(K_n\) and \(K_n\) is the compositum of linearly disjoint splitting fields \(L'_1, \ldots, L'_{r^n-1}\) over \(K_{n-1}\). If a prime \(P\), ramifying in \(K_{n+1}\) ramifies in \(K_n\), then it ramifies in one of the \(L_i\) and \(L'_i\), which is a contradiction to \(M'_1, \ldots, M'_{r^n}\) being linearly disjoint over \(K_n\) (4.3) as well as a contradiction to

\[\operatorname{Gal}(M'_i/K_n) \cong \operatorname{Gal}(K_{n+1}/M_i) = S_r\]  

in (4.7).
Thus, any prime $P$ of $\mathbb{F}_q(a)$, which ramifies in $K_{n+1}$, does not ramify in $K_n$. □

$K_{n+1}/K_n$ is maximal.

**Proposition 4.6.** The linear disjointness of $M'_1, \ldots, M'_{r \uparrow}$ implies

$$\text{Gal}(K_{n+1}/K_n) \cong S_{r \uparrow}^{r \uparrow}$$

**Proof.** Proof of this proposition follows from [Lang, Algebra [24], VI, §1.14 1.15]. From the above discussion, $K_{n+1} = M'_1 \cdots M'_{r \uparrow}$ is the compositum of $M'_1, \ldots, M'_{r \uparrow}$ and $M'_1, \ldots, M'_{r \uparrow}$ are linearly disjoint over $K_n$ and $\text{Gal}(M'_i/K_n) \cong S_r$. This is best visualized in the diamond diagram below.

$$
\begin{array}{c}
K_{n+1} \\
M'_1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad M'_{r \uparrow} \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{array}
$$

Then,

$$\text{Gal}(K_{n+1}/K_n) \cong \text{Gal}(M'_1/K_n) \times \cdots \times (M'_{r \uparrow}/K_n)$$

$$\sigma \mapsto (\sigma|_{M'_1}, \ldots, \sigma|_{M'_{r \uparrow}})$$

is an isomorphism

and,

$$\text{Gal}(K_{n+1}/K_n) \cong S_{r \uparrow}^{r \uparrow} \quad (4.10)$$

Thus $\text{Gal}(K_{n+1}/K_n)$ is maximal. Maximality of $K_{n+1}/K_n$ implies, $K_{n+1}$ has degree $(r!)^r \uparrow$ over $K_n$. □

**Corollary 4.1.** The Galois extension $K_{n+1}/\mathbb{F}_q(a)$ is maximal.

**Proof.** We know that, $K_{n+1} = M'_1 \cdots M'_{r \uparrow}$ is the compositum of linearly disjoint Galois extensions $M'_1$ over $K_n$ and $K_{n+1}/K_n$ is maximal. Also, a prime of $\mathbb{F}_q(a)$, which ramifies in $K_{n+1}$, does not ramify in $K_n$. Linear disjointness of $M'_1, \ldots, M'_{r \uparrow}$ over $\mathbb{F}_q(a)$, implies, every quadratic subextension of $M'_1, \ldots, M'_{r \uparrow}$ given by $E'_1, \ldots, E'_r \uparrow$ are linearly disjoint over $\mathbb{F}_q(a)$. Hence, $\delta_t(F(a,t) - \alpha_1), \ldots, \delta_t(F(a,t) - \alpha_{r \uparrow})$ are linearly independent in $\mathbb{F}_q(a)^{\times}/(\mathbb{F}_q(a)^{\times})^2$, that is the product $\prod_{i \neq j} \text{disc}_t(F(a,t) - \alpha_i)$ is non square, hence $\delta_t(F(a,t) - \alpha_1), \ldots, \delta_t(F(a,t) - \alpha_{r \uparrow})$ are square independent in $\mathbb{F}_q(a)$. Also, each of $[K_{i+1} : K_i]$ is maximal for $i = 1, 2, \ldots, n$. Hence, from the expression,

$$[K_{n+1} : \mathbb{F}_q(a)] = [K_{n+1} : K_n][K_n : K_{n-1}] \cdots [K_1 : \mathbb{F}_q(a)]$$

it follows, $\phi : \text{Gal}(K_{n+1}/\mathbb{F}_q(a)) \to \prod_{i=0}^{r \uparrow} S_{r \uparrow}^{r \uparrow}$ is an isomorphism.
that is,

\[ \text{Gal}(\mathbb{K}_{n+1}/\mathbb{F}_q(a)) \cong \left[ S_r \right]^{n+1}. \quad (4.11) \]

Thus, \( \mathbb{K}_{n+1}/\mathbb{F}_q(a) \) is maximal and

\[ |G_{n+1}| = \prod_{i=0}^{n} S_r^i \quad (4.12) \]

Thus, Proposition 4.6 and its Corollary 4.1 together completes Odoni’s conjecture for the polynomial \( F(a, t) \) of degree \( r \geq 3 \) in \( t \) over \( \mathbb{F}_q(a) \) states in Theorem 2.2. We may note that, all these results discussed holds true for \( k = \mathbb{F}_p \).

4.5. **Stable polynomial.** If any iterate \( F^{\circ n+i}(a, t) \) of \( F(a, t) \) is reducible in \( \mathbb{F}_q(a)[t] \), then all successive iterates \( F^{\circ n+i+j}(a, t) \) are also reducible over \( \mathbb{F}_q(a) \). Following the definition of Odoni, we say that, \( F(a, t) \) is stable over \( \mathbb{F}_q(a) \), if \( F^{\circ n}(a, t) \) is irreducible over \( \mathbb{F}_q(a) \) for every \( n \geq 1 \). From Proposition 4.6 and its Corollary 4.1 it is clear that, \( F(a, t) \) is a stable polynomial over \( \mathbb{F}_q(a) \).

5. **Application**

The Schinzel Hypothesis, originally stated over \( \mathbb{Z} \) claims that, for finitely many irreducible polynomials \( f_1(x), \ldots, f_r(x) \) with integer coefficients such that, if no integer \( n > 1 \) divides \( f_1(a), \ldots, f_r(a) \) for all integer \( a \), then there should exist infinitely many \( a \) such that, \( f_1(a), \ldots, f_r(a) \) are simultaneously prime. Bateman-Horn conjectured the quantitative version of the Schinzel hypothesis, specifying the proportion of integers for which the certain collection of irreducible and separable polynomials assume simultaneous prime values generalizing the result of Hardy and Littlewood on prime tuple conjecture. Given the strong analogies between the ring of rational integers and the ring of univariate polynomials over a finite field, it is natural to ask these questions, in the function field setup.

**Function field analogue of Schinzel Hypothesis and the Bateman-Horn conjecture.** The function field analogue of the Schinzel Hypothesis and the Bateman-Horn Conjecture over \( \mathbb{F}_q[t][x] \) is resolved by Pollack, in \[31\], \[32\] and \[33\]. Pollack established the quantitative version of Schinzel hypothesis \( H \) over the function field \( \mathbb{F}_{-q}[t] \), which is the polynomial analogue of Bateman-Horn Conjecture over the large finite field \( \mathbb{F}_q \) [Theorem 2.32], using the methods employed by Cohen to settle Chowla’s conjecture. (see for more details, Theorem 3[31], Cohen[13] §Introduction [13]). This is also resolved over an algebraically closed field \( K \) of characteristic \( p > 2 \) by Bary-Soroker in [1] and [5], to settle the function field analogue of the Hardy-Littlewood prime tuple conjecture. The function field analogue of the Schinzel Hypothesis is stated below:
**Conjecture 5.1.** Analogue of Schinzel Hypothesis in $\mathbb{F}_q[t]$ Let $f_1(t, x), \ldots, f_r(t, x)$ be non constant separable polynomials in $\mathbb{F}_q[t][x]$ which are irreducible in $\mathbb{F}_q(t)[x]$. Suppose that, there is no prime $P \in \mathbb{F}_q[t]$ for which every $g(t) \in \mathbb{F}_q[t]$, satisfies

$$f_1(g(t)), \ldots, f_r(g(t)) \equiv 0 \mod P$$

Then the specializations $f_1(g(t)), \ldots, f_r(g(t))$ are simultaneously irreducible for infinitely many monic polynomials $g(t) \in \mathbb{F}_q[t]$.

For the conditions stated in Conjecture 5.1, the quantitative result is the function field analogue of Bateman-Horn conjecture, proved by Pollack in [33].

**Theorem 5.1 (Pollack, [33]).** Let $n, B$ be positive integers, $p$ a prime such that $p \nmid 2n$, $q$ a power of $p$, $f_1(x), \ldots, f_s(x) \in \mathbb{F}_q[x]$ non associate irreducible polynomials such that, $\sum \deg(f_i) \leq B$. Then the number of degree $n$ monic $g(t) = t^n + \cdots \in \mathbb{F}_q[t]$ for which all $f_i(g(t))$ irreducible in $\mathbb{F}_q[t]$ is

$$\frac{q^n}{n^r} + O_{n, B}(q^n - 1/2)$$

**Function field analogue of Schinzel Hypothesis $H$ and the Odoni’s conjecture.** It follows that,

$$\text{Gal}(F^{n+1}(a, t), \mathbb{F}_q(a)) \cong [S_r]^n$$

by assumption we have,

$$\text{Gal}(F^{n+1}(a, t), \overline{\mathbb{F}_q}(a)) \cong [S_r]^n$$

as well. Thus, $K_{n+1}/\mathbb{F}_q(a)$ is a geometric extension.

From, §4.5 $F(a, t) \in \mathbb{F}_q(a)[t]$ is a stable polynomial. The iterated polynomials

$$F(a, t), \ldots, F^{n+1}(a, t)$$

are irreducible over $\mathbb{F}_q(a)$. Thus, polynomials in (5.1) satisfy the Schinzel Hypothesis over $\mathbb{F}_q(a)$. In (5.1) $i$-th iterated polynomial, $F^{oi}(a, t)$ is of degree $r^i$ in $t$.

By the function field analogue of Schinzel Hypothesis, there exists specializations $a \mapsto A = \{A_0, \ldots, A_{d-1}\} \in \mathbb{F}_q$ for which

$$(F(A, t), \ldots, F^{n+1}(A, t)) \in \mathbb{F}_q[t]$$

are simultaneously irreducible over $\mathbb{F}_q$. Thus, we have the following version of Bateman-Horn conjecture over $\mathbb{F}_q$ for the specialized polynomials in (5.2).

**Theorem 5.2.** Suppose that, $K_{n+1}/\mathbb{F}_q(a)$ is maximal. Then the number of tuples $A = \{A_0, \ldots, A_{d-1}\} \in \mathbb{F}_q^d$ for which the set of iterated polynomials in
Any iterate \( F^\alpha \), specialized at \( A \), are irreducible in \( \mathbb{F}_q[t] \) is
\[
\# \{ A \in \mathbb{F}_q^d | F(A,t), F^{\alpha_2}(A,t), \ldots, F^{\alpha_{n+1}}(A,t) \text{ are irreducible in } \mathbb{F}_q[t] \} = \\
\frac{q^d}{\prod_{i=1}^{n+1} \deg_t F^{\alpha_i}(A,t)} + O_{d,n+1,\deg_t F^{\alpha_i}}(q^{d-1/2})
\]
(5.3)
the implicit constant in the O-notation depends only on \( d, n + 1, \deg_t F^{\alpha_i} \).

Any iterate \( F^{\alpha_{j-1}}(A,t) \) is of degree \( r^{j-1} \) in \( t \) and has the following decomposition in to distinct monic, irreducible and separable polynomials:
\[
F^{\alpha_{j-1}}(A,t) = \prod_{i=1}^{r^j} F(A,t) - \alpha_i, \quad F(A,t) - \alpha_i \in \mathbb{F}_q^{r^j}[A][t].
\]
(5.4)
where, \( \mathbb{F}_q^{r^j} \) is the full constant field of the splitting field \( K_j \) and \( \alpha_1, \ldots, \alpha_{r^j} \) are the roots of \( F^{\alpha_i}(A,t) \in \mathbb{F}_q(A)[t] \), and \( \deg_t(F(A,t) - \alpha_i) = \frac{r^{j+1}}{r_i} = r \).

**Conjugacy classes of symmetric groups.** The conjugacy classes of any \( S_n \) are determined by its cycle type. That is, two permutations \( \sigma \) and \( \rho \) are conjugate, if and only if they have the same cycle type. Let \( f \in \mathbb{F}_q[t] \) be a separable polynomial of degree \( n > 0 \). The analogy between factorization of random elements of \( \mathbb{F}_q[t] \) into primes and the factorizations of random permutations into cycles is that, the degrees of prime factors of \( f \) correspond to cycle type of permutations in \( S_n \). The Frobenius map \( \text{Frob}_q \) given by \((x \mapsto x^q)\), defines a permutation of the roots of \( f \), hence can be identified with an element (permutations) of \( S_n \), which defines a well defined conjugacy class of \( S_n \). \( f \) is irreducible if and only if \( \sigma \) is a full cycle. If \( f_1, \ldots, f_m \) are separable polynomials with \( \deg f_i = N_i \), then, \( \text{Frob}_q \) permutes the roots of each \( f_i \), hence can be identified with \( m \)-tuple of permutations, \((\sigma_1, \ldots, \sigma_m)\) in \( S_{N_1} \times \cdots \times S_{N_m} \), which defines a well defined conjugacy class in \((\sigma_1, \ldots, \sigma_m) \) in \( S_{N_1} \times \cdots \times S_{N_m} \) [Andrade et al. §3, Theorem 3.1 [2]].

5.1. **Bateman-Horn Conjecture for the iterates of \( F(A,t) \).** To prove the function field analogue of Bateman-Horn Conjecture, the strategy is to apply the Chebotarev density theorem to each individual \( F^{\alpha_{i+1}}(A,t) \), \( i = 0, 1, \ldots, n + 1 \), such that all the specialized polynomials in (5.2) are simultaneously irreducible over \( \mathbb{F}_q \).

Let \( C \) be a conjugacy class of \((S_\ell)^n \). For \( A = (A_0, \ldots, A_{d-1}) \in \mathbb{F}_q^d \), we write \( P_A \) for a prime of \( \mathbb{F}_q(A) \), that is of the form, \( P_A = (a_0 - A_0, \ldots, a_{d-1} - A_{d-1}) \). Then the simultaneous irreducibility of \( F(A,t), \ldots, F^{\alpha_{i+1}}(A,t) \) is equivalent to \( C \) coinciding with Frobenius conjugacy class \( \left( \frac{K_{i+1}/P_A}{\mathbb{F}_q(A)} \right) \).

Thus, each of \( F(A,t), \ldots, F^{\alpha_{i+1}}(A,t) \) are simultaneously irreducible in \( \mathbb{F}_q[t] \) if and only if \( P_A \) stays prime in each of \( K_i, i = 1 \ldots, n + 1 \).
5.2. **Explicit Chebotarev density theorem.** We state a particular version of Chebotarev density theorem, which is stated and proved in [28 Proposition 6.4.8]. Similar version of this is considered in [33]. We need the following explicit version of Chebotarev density theorem, which is adapted from [Theorem 3.1 §3, [2]], to hold true for our condition.

**Proposition 5.1.** Let \( C \) be any conjugacy class of \( \text{Gal}(K_{n+1}/\mathbb{F}_q(a)) \), we know that, \( \text{Gal}(K_{n+1}, \mathbb{F}_q(a)) \cong [S_d]^n \) and \( \mathbb{F}_q \) is algebraically closed in \( K_{n+1} \).

Then there exists a constant \( c(r, \deg F^{o_i}, d) \) such that, for every conjugacy class \( C \subseteq [S_d]^n \), we have

\[
\left| \{ A \in \mathbb{F}_q^d : \left( \frac{K_{n+1}/\mathbb{F}_q(a)}{P_A} \right) = C \} \right| - \left| \frac{|C|}{|S_r|^n} \cdot q^d \right| \leq cq^{d-1/2}
\]

For completeness, we need another lemma, we state them here. The following Lemma is a direct adaptation of [Pollack, Lemma 13, [33]].

**Lemma 5.1.** With the suppositions in Proposition 4.6 and Corollary 4.1, the Galois group, \( \text{Gal}(K_{n+1}/\mathbb{F}_q(a)) \) contains a conjugacy class \( C \) of size \( r! + r^2 + \ldots + r^n \)

\[
\prod_{i=1}^{n+1} \deg_t F^{o_i}(a, t)
\]

with the following property. If \( A = (A_0, \ldots, A_{d-1}) \) is any element of \( \mathbb{F}_q^d \), which is not a zero of any of the polynomial:

\[
\text{disc}_t(F(a, t) - \alpha_i), \quad i = 1, 2, \ldots, r^j, j = 1, 2, \ldots, n
\]

Then \( F^{o_j+1}(A, t), j = 1, \ldots, n \) is irreducible over \( \mathbb{F}_q \) if and only if \( C \) coincides with Frobenius conjugacy class \( \left( \frac{K_{n+1}/\mathbb{F}_q(a)}{P_A} \right) \).

Now, we apply Lemma 5.1 and Proposition 5.1, to Theorem 5.2. We note that, by equation (5.5)

\[
\frac{|C|}{|S_r|^n} = \frac{1}{\prod_{i=1}^{n+1} \deg_t F^{o_i}(a, t)}
\]

Thus,

\[
\# \{ A \in \mathbb{F}_q^d : (F^{o_1+1}(A, t), F^{o_2}(A, t), \ldots, F^{o_2}(A, t), F(A, t)) \in \mathbb{F}_q[t] \text{ are irreducible} \}
\]

is

\[
q^d \prod_{i=1}^{n+1} \deg_t F^{o_i}(a, t) + O_{d, r, \deg_t F^{o_i}}(q^{d-1/2}), \quad q \to \infty
\]

This completes Theorem 5.2.

**Concluding remarks.** Chebotarev density theorem, establishes a connection between the function fields and finite fields. In this paper, we have established a connection between the two classical conjectures, namely the Schinzel Hypothesis II and the Odoni’s conjecture over the function field \( \mathbb{F}_q(a) \). This opens a new avenue to find the number of prime tuples of the given form over \( \mathbb{F}_q \), as an application of Chebotarev Density Theorem over function fields.
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Odoni’s conjecture and prime specialization

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Department of Mathematics, Indian Institute of Science, Bangalore, Karnataka, India.

Email address: psushma@iisc.ac.in, sushmapalimar@gmail.com.