A Novel Attack Strategy on Entanglement Swapping QKD Protocols

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Li et al. presented a protocol [Int. Journal of Quantum Information, Vol. 4, No. 6 (2006) 899-906] for quantum key distribution based on entanglement swapping. In this protocol they use random and certain bits to construct a classical key and they claim that this key is secure. In our article we show that the protocol by Li et al. is insecure presenting a new type of attack strategy which gives an adversary full information about the key without being detected. This strategy is based on entanglement swapping, too, and manages to preserve the correlation between the measurement results of the legitimate parties. Further we present a modified version of the protocol and show that it is secure against this new attack strategy.

Keywords: Quantum Key Distribution, Quantum Cryptography, Entanglement, Entanglement Swapping, Attack Strategy

1. Introduction

Entanglement swapping is a phenomenon where 2 or more qubits which didn’t interact in the past are brought into an entangled state. For example, two parties Alice and Bob share the Bell states $|\Phi^+\rangle_{12}$ and $|\Phi^+\rangle_{34}$ such that Alice is in possession of qubits 1 and 3 and Bob of qubits 2 and 4. If Alice performs a measurement in the Bell basis on qubits 1 and 3 she will end up with one of the four Bell states. Simultaneously Bob’s qubits 2 and 4 are also brought into a Bell state. This measurement can be described by the equation

$$|\Phi^+\rangle_{12} \otimes |\Phi^+\rangle_{34} = \frac{1}{2} \left( |\Phi^+\rangle_{13} |\Phi^+\rangle_{24} + |\Phi^-\rangle_{13} |\Phi^-\rangle_{24} + |\Psi^+\rangle_{13} |\Psi^+\rangle_{24} + |\Psi^-\rangle_{13} |\Psi^-\rangle_{24} \right)$$

This equation shows that Alice’s result is completely random, i.e. she will obtain any of the four Bell states with equal probability. But the state of the qubits 2 and 4 is completely determined by the result of Alice’s measurement. This correlation is used in a number of key distribution protocols to share a secret key.
between Alice and Bob. Since the practical realization of entanglement swapping is rather complex to achieve with today’s technology none of these QKD protocols has been implemented yet.

Also Li et al. presented a QKD protocol based on entanglement swapping which not only produces random key bits but also certain key bits. This is achieved by introducing a Pauli operation in eq. (1) as it will be described in detail in section 2. As it is shown in [9] this protocol is secure against an intercept/resend attack as well as a collective attack. But we will present an attack strategy in section 3 which is based on a 6-qubit state. This state is more complex than the system of the sender and the receiver, but it provides an adversary with full information about the key shared between the two parties.

The main idea of this attack is that the adversary Eve entangles herself with Alice and Bob using a state which preserves the correlations of Alice’s Bell state measurement. Further this state contains additional information which allows Eve to eavesdrop Bob’s secret result. An attack where Eve is entangled with one party was presented by Zhang, Li and Guo on the protocol of Cabello but it wouldn’t work for the protocol in [9] since Eve wouldn’t stay undetected.

We will also make a proposal how to secure the protocol against this new attack strategy. This can be achieved using a Hadamard operation and we will describe the new version in detail in section 4. In section 5 we will discuss the security of our new version.

2. The Original Protocol

Alice creates 2n EPR pairs, each in a Bell state, e.g. $|\Phi^+\rangle$. In the following we will focus on the simple case $n = 1$, where Alice and Bob share the states $|\Phi^+\rangle_{12}$ and $|\Phi^+\rangle_{34}$, which are publicly known. In [9] Li et al. start the protocol with preshared entangled states which is a very strong assumption because the qubits have to be shared somehow between the two parties. But in the security analysis in [9] an adversary Eve is granted the possibility to interact with the Bell states. Thus we can conclude that there have to be some qubits in transit between Alice and Bob. There are two possible ways to share the Bell states between the legitimate parties: either one party, e.g. Alice, prepares both Bell states and sends two qubits to the other party or both parties prepare one Bell state and they exchange two qubits.

The first scenario is insecure against a simple intercept-resend attack: If Alice prepares the Bell states $|\Phi^+\rangle_{12}$ and $|\Phi^+\rangle_{34}$ and sends qubits 2 and 4 to Bob Eve can intercept these qubits and perform a Bell state measurement on them. This will bring qubits 1 and 3 also into a Bell state which is known to Eve. Then she forwards qubits 2 and 4 to Bob. Thus Eve has full information about Alice’s and Bob’s secret measurement result and is able to eavesdrop the key perfectly. Therefore we will assume that Alice prepares $|\Phi^+\rangle_{12}$ and Bob prepares $|\Phi^+\rangle_{34}$ and that they exchange the qubits 2 and 3 (cf. picture (2) in figure 1).

Alice randomly chooses one of the four Pauli operations, $I$, $\sigma_x$, $\sigma_y$ or $\sigma_z$ and
applies it to qubit 1 (cf. picture (3) in figure 1). Alice’s operation will generally be described as \( \sigma_A^{(i)} \), where the superscript \((i)\) denotes that it is applied on qubit \(i\).

As a next step Alice performs a Bell state measurement on qubits 1 and 3 in her possession (cf. picture (4) in figure 1).

With this measurement the effect of entanglement swapping arises and entangles Bob’s qubits 2 and 4. Following the equation

\[
\sigma_A^{(1)} |\Phi^+\rangle_{12} \otimes |\Phi^+\rangle_{34} = \frac{1}{2} \left( |\Phi^+\rangle_{13}\sigma_A^{(2)}|\Phi^+\rangle_{24} + |\Phi^+\rangle_{13}\sigma_A^{(2)}|\Phi^-\rangle_{24} + |\Psi^+\rangle_{13}\sigma_A^{(2)}|\Psi^+\rangle_{24} + |\Psi^-\rangle_{13}\sigma_A^{(2)}|\Psi^-\rangle_{24} \right) \tag{2}
\]

Alice can determine in which state Bob’s qubits are after her Bell state measurement if she applied the operation \(\sigma_A\) on qubit 1. Additionally, she can compute also what the state of qubits 2 and 4 would be, if she didn’t apply any operation on qubit 1 using eq. (1) from above. Now Alice announces to Bob that she made a Bell state measurement but keeps her result secret.

To infer Alice’s result Bob performs a Bell state measurement on qubit 2 and 4 in his possession. Since he does not know yet which operation Alice applied and in which state qubits 1 and 2 have been before the measurement he does not know the exact state of qubits 1 and 3. With the help of eq. (1) he can at least deduce in which state qubits 1 and 3 should be if Alice didn’t apply anything to qubit 1.

Fig. 1. An illustration of the original protocol by Li et al. 9. Here \(\sigma_x\) is chosen for Alice’s secret operation and \(|\Psi^-\rangle_{13}\) as Alice’s result of the Bell state measurement.
This is called Alice’s imaginary result.

To get the correct state of qubits 1 and 3 Bob asks Alice about her result but keeps his own result secret. Using the information about Alice’s and his own result Bob can deduce which Pauli operation Alice applied. Alice and Bob now share information about Alice’s randomly chosen operation and the result Bob obtained, which is also random. They use these two pieces of information to extract a classical raw key. Therefore they agree beforehand on a mapping of Bell states onto classical 2-bit strings, e.g.

$$|\Phi^+\rangle \rightarrow 00 \quad |\Phi^-\rangle \rightarrow 01 \quad |\Psi^+\rangle \rightarrow 10 \quad |\Psi^-\rangle \rightarrow 11$$ (3)

as well as of Pauli operations onto classical 2-bit strings, i.e.

$$I \rightarrow 00 \quad \sigma_x \rightarrow 01 \quad \sigma_y \rightarrow 10 \quad \sigma_z \rightarrow 11$$ (4)

If we take the general case with $2^n$ EPR pairs Alice and Bob repeat these steps for all the pairs. In the end they publicly compare a certain number of bits of the generated raw key to detect an eavesdropper.

### 3. A Perfect Attack Strategy

In this section we discuss an attack strategy where Eve is able to perfectly eavesdrop the secret key of Alice and Bob without being detected. Therefore she uses the six qubit state (cf. picture (1) in figure 2)

$$|\delta\rangle_{PQRSTU} = \frac{1}{2\sqrt{2}} \left( |000000\rangle_{PQRSTU} + |001101\rangle_{PQRSTU} + |010111\rangle_{PQRSTU} + |011010\rangle_{PQRSTU} + |100110\rangle_{PQRSTU} + |101011\rangle_{PQRSTU} + |110001\rangle_{PQRSTU} + |111100\rangle_{PQRSTU} \right)$$ (5)

This state is rather complex and not easy to generate (cf. appendix) but it has the special property that it can be written as

$$|\delta\rangle_{PQRSTU} = \frac{1}{2} \left( |\Phi^+\rangle_{PR} \otimes |\Phi^+\rangle_{QS} \otimes |\Phi^+\rangle_{TU} + |\Phi^-\rangle_{PR} \otimes |\Phi^-\rangle_{QS} \otimes |\Phi^-\rangle_{TU} + |\Psi^+\rangle_{PR} \otimes |\Psi^+\rangle_{QS} \otimes |\Psi^+\rangle_{TU} + |\Psi^-\rangle_{PR} \otimes |\Psi^-\rangle_{QS} \otimes |\Psi^-\rangle_{TU} \right)$$ (6)

That means, a measurement on qubits $P$ and $R$ as well as $Q$ and $S$ yields in the same result as it would be expected from entanglement swapping in eq. (1). Further the remaining two qubits $T$ and $U$ are in the same state as the qubits $Q$ and $S$. Eve can use this fact to obtain enough information to deduce the secret measurement results of Alice and Bob if she somehow manages that the two parties share the state $|\delta\rangle$ instead of two Bell states. That can be achieved easily using entanglement swapping on qubits 2 and 3 in transit together with her six qubits.
When Alice sends out qubit 2 and Bob qubit 3 Eve preforms Bell state measurements on both of them with respective qubits from $|\delta\rangle_{PQRSTU}$ (cf. picture (2) in figure 2). A problem is that Eve, at first, doesn’t know which initial state Alice prepared. But, as we will show in the next paragraphs, she can overcome this problem fairly easy. Eve’s first Bell state measurement on qubits 2 and $P$ is described in eq. (7). (As already pointed out above, we will use $\sigma_A$ to describe Alice’s secret Pauli operation.)

$$\sigma_A^{(1)} |\Phi^+\rangle_{12} \otimes |\delta\rangle_{PQRSTU} = \sigma_A^{(1)} \frac{1}{\sqrt{2}} \left( |\Phi^+\rangle_{12} + |\Phi^-angle_{12} \right) \otimes \frac{1}{2\sqrt{2}} \left( |00\rangle_{PQRSTU} + |01\rangle_{PQRSTU} + |10\rangle_{PQRSTU} + |11\rangle_{PQRSTU} \right)$$

$$= \frac{1}{2} \left( |\Phi^+\rangle_{2P} \otimes \sigma_A^{(1)} \frac{1}{2\sqrt{2}} \left( |000000\rangle_{1QRSTU} + |001101\rangle_{1QRSTU} + |010111\rangle_{1QRSTU} + |011010\rangle_{1QRSTU} + |100110\rangle_{1QRSTU} + |101011\rangle_{1QRSTU} \right) \right)$$

At this point Eve can not clearly say which state the qubits 1, $Q$, $R$, $S$, $T$, and $U$ are in. All four results of her Bell state measurement are equally likely and she has absolutely no information about the operation $\sigma_A$. Nevertheless, Eve is able to transform the 6-qubit state in a way that it is always in the state $\sigma_A^{(1)} |\delta\rangle_{1QRSTU}$. Therefore she performs a $\sigma_x$ operation on qubits $S$ and $T$ if she gets $|\Psi^+\rangle_{2P}$ or does nothing if her result is $|\Phi^+\rangle_{2P}$. To correct the negative signs in case Eve obtains a $|\Phi^+\rangle_{2P}$ she performs a $\sigma_x$ on qubits $S$ and $U$. In case of $|\Psi^+\rangle_{2P}$ she first has to apply the $\sigma_x$ operator on qubits $S$ and
U.

The second Bell state measurement is performed on qubits 3 and S. Again, the resulting 6-qubit state is not entirely known but it can be one out of four possible states. We won’t describe this Bell state measurements explicitly but we want to stress that Eve is able to bring the 6 qubits in the state $\sigma^{(1)}_{A}|\delta\rangle_{1QR4TU}$. Therefore she uses a similar method as described above applying the Pauli operations $\sigma_x$ and $\sigma_z$ on qubits $R$, $T$ and $U$.

Now Eve sends qubit $R$ to Alice and qubit $Q$ to Bob and keeps qubits $T$ and $U$ by herself (cf. picture (4) in figure 2). Alice and Bob, who are not aware of Eve’s intervention follow the protocol and perform Bell state measurements on their respective particles (cf. pictures (5) and (6) in figure 2). If we look at the 6-qubit state $\sigma^{(1)}_{A}|\delta\rangle_{1QR4TU}$, Alice’s Bell state measurement on qubits 1 and $R$ changes the state in a way that it leaves the qubit pairs $Q$, 4 and $T$, $U$ in an entangled state. This can be seen from the alternative description of the state $\sigma^{(1)}_{A}|\delta\rangle_{1QR4TU}$ in eq. (8) and picture (7) in figure 2).

$$\sigma^{(1)}_{A}|\delta\rangle_{1QR4TU} =$$

$$= \frac{1}{2} \left( \sigma^{(1)}_{A}|\Phi^+\rangle_{1R} \otimes \frac{1}{2} \left( |0000\rangle_{Q4TU} + |0011\rangle_{Q4TU} + |1100\rangle_{Q4TU} + |1111\rangle_{Q4TU} \right) + \sigma^{(1)}_{A}|\Phi^-\rangle_{1R} \otimes \frac{1}{2} \left( |0000\rangle_{Q4TU} - |0011\rangle_{Q4TU} - |1100\rangle_{Q4TU} + |1111\rangle_{Q4TU} \right) + \sigma^{(1)}_{A}|\Psi^+\rangle_{1R} \otimes \frac{1}{2} \left( |0101\rangle_{Q4TU} + |0110\rangle_{Q4TU} + |1001\rangle_{Q4TU} + |1010\rangle_{Q4TU} \right) + \sigma^{(1)}_{A}|\Psi^-\rangle_{1R} \otimes \frac{1}{2} \left( |0101\rangle_{Q4TU} + |0110\rangle_{Q4TU} - |1001\rangle_{Q4TU} - |1010\rangle_{Q4TU} \right) \right)$$

(8)

Alice gets each possible result with equal probability of $\frac{1}{4}$ as she would expect it. Further, Bob’s Bell state measurement will end up with certainty in a specific Bell state, which is fully dependent on Alice’s result. Due to the special choice of the state $|\delta\rangle$ the correlation between his and Alice’s result is preserved as they would expect it if no eavesdropper was present. Thus the presence of Eve won’t be
detected when Alice and Bob compare some bits of the raw key. Additionally, it leaves the qubits $T$ and $U$, which are in Eve’s possession, in the very same state Bob obtains as result (cf. also picture (7) in figure 2). This is easy to see if we look at the alternative description in the Bell basis of the remaining four qubits $Q$, $4$, $T$ and $U$ after Alice’s measurement

$$\frac{1}{2} \left( \sigma_A^{(1)} |\Phi^+\rangle_{1R} \otimes |\Phi^+\rangle_{Q4} \otimes |\Phi^+\rangle_{TU} + \sigma_A^{(1)} |\Phi^-\rangle_{1R} \otimes |\Phi^-\rangle_{Q4} \otimes |\Phi^-\rangle_{TU} \\ + \sigma_A^{(1)} |\Psi^+\rangle_{1R} \otimes |\Psi^+\rangle_{Q4} \otimes |\Psi^+\rangle_{TU} + \sigma_A^{(1)} |\Psi^-\rangle_{1R} \otimes |\Psi^-\rangle_{Q4} \otimes |\Psi^-\rangle_{TU} \right)$$

(9)

At this time, Eve has full information about Bob’s secret measurement. That means, Eve has as much information as Bob and thus can obtain the classical secret raw key in the same way Bob does: From her result of the Bell state measurement Eve can compute Alice’s imaginary result. Moreover, when Alice publicly announces her result, Eve is able to infer which Pauli operation Alice has chosen. Thus Eve knows both parts of the shared secret information. Therefore she easily can infer the classical bit string which is used as a raw key by Alice and Bob and so she is able to obtain the secret key, too.

Instead of performing the Bell state measurements on the qubit pairs 2, $P$ and 3, $S$ as soon as she receives them, Eve has the opportunity to delay her measurement and immediately send qubit $R$ to Alice (cf. picture (2) in figure 5). With this strategy Eve is able to overcome operations Alice applies on qubit 1 but she introduces a much higher error rate as we will describe in section 5. Another, but rather strong, assumption is that Eve is in control of Alice’s and Bob’s EPR source. That means she is able to distribute the state $|\delta\rangle_{PQRSTU}$ between her and the two parties without further Bell state measurements. In this case Alice’s measurement on qubits $P$ and $R$ leaves Bob’s qubits in a correlated state and Eve with the full information about the results, as described above. Since Alice and Bob can prepare their EPR source themselves this assumption is not very practical.

4. The Modified Protocol

The protocol can be improved by performing a Hadamard operation before Alice applies her secret Pauli operation on qubit 1 (cf. picture (3) in figure 4). The Hadamard operation alters the initial state in a way that Eve can’t eavesdrop the secret key without introducing a certain error rate. Thus Alice and Bob can detect her easily as it will be shown in section 5.

Due to the use of the additional Hadamard operation it is not important any more whether Alice prepares both Bell states or each party prepares a Bell state by its own. To stay consistent with the above descriptions of the protocol and the attack we will discuss the scenario where Alice prepares the state $|\Phi^+\rangle_{12}$ and Bob prepares $|\Phi^+\rangle_{34}$ (for the simple case $n = 1$). Alice sends out qubit 2 to Bob and he sends qubit 3 to Alice. When she receives Bob’s qubit she randomly applies either
the identity operator $I$ or the Hadamard operator
\[ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]  

on qubit 1. This alters the initial state to
\[ H^{(1)}|\Phi^+\rangle_{12} \otimes |\Phi^+\rangle_{34} = |\omega^+\rangle_{12} \otimes |\Phi^+\rangle_{34} \]  

where $|\omega^+\rangle_{12}$ is one of the four possible superpositions of Bell states, i.e.
\[ H^{(1)}|\Phi^\pm\rangle_{12} = |\omega^\pm\rangle_{12} = \frac{1}{\sqrt{2}}(|\Phi^\mp\rangle_{12} \pm |\Psi^\pm\rangle_{12}) \]  
\[ H^{(1)}|\Psi^\pm\rangle_{12} = |\chi^\pm\rangle_{12} = \frac{1}{\sqrt{2}}(|\Psi^\mp\rangle_{12} \pm |\Phi^\pm\rangle_{12}) \]  

Then Alice applies her secret Pauli operation $\sigma_A$ on qubit 1 and performs a Bell state measurement on qubits 1 and 3 (cf. picture (5) in figure 3). This Bell state measurement can be described as
\[ |\omega^+\rangle_{12} \otimes |\Phi^+\rangle_{34} = \frac{1}{2} \left( |\Phi^+\rangle_{13} \otimes |\omega^+\rangle_{24} + |\Phi^+\rangle_{13} \otimes |\omega^-\rangle_{24} + |\Psi^+\rangle_{13} \otimes |\chi^+\rangle_{24} + |\Psi^-\rangle_{13} \otimes |\chi^-\rangle_{24} \right) \]  

Further she publicly tells Bob whether she has applied the $H$ operator or not. If she did so, Bob applies the $H$ operator on qubit 2 and otherwise he does nothing (cf. picture (6) in figure 3). As we can see from eq. (12) a repeated application of the Hadamard operator eliminates the superposition and the correlation from the original Bell state measurement is preserved. In the end Bob performs a Bell measurement on qubits 2 and 4 and computes Alice’s actual and imaginary result as it is described in the original protocol.

5. Security Analysis

In the following we will provide a security analysis of the modified protocol. Since the application of the Hadamard operation is just a minor change of the original protocol, the modified version is also secure against an intercept/resend attack as well as a collective attack, as described in 9. Therefore we will first inspect Eve’s probability to stay undetected if she follows the same attack strategy just described in section 3. If Alice does not apply the Hadamard operator Eve’s attack will be successful as we already pointed out above. Thus we will discuss the case where Alice performs the Hadamard operation on qubit 1 (cf. figure 4). Due to the fact that the Hadamard operation alters the initial state as we have seen in eq. (11) Eve’s first Bell state measurement on qubits 2 and $P$ will also change as described
in eq. (14),
\[
\sigma_A^{(1)} H^{(1)} |\Phi^+\rangle_{12} \otimes |\delta\rangle_{PQRSTU} = \\
\frac{1}{\sqrt{2}} \left( |\Phi^+\rangle_{2P} \otimes |\Phi^+\rangle_{A} \right) H^{(1)} \frac{1}{2\sqrt{2}} \left( |000000\rangle_{1QRSTU} + |001101\rangle_{1QRSTU} + |010111\rangle_{1QRSTU} + |110001\rangle_{1QRSTU} + |111100\rangle_{1QRSTU} \right) \\
+ |\Phi^+\rangle_{2P} \otimes |\Phi^+\rangle_{A} \right) H^{(1)} \frac{1}{2\sqrt{2}} \left( |001101\rangle_{1QRSTU} + |010111\rangle_{1QRSTU} + |011010\rangle_{1QRSTU} - |100110\rangle_{1QRSTU} - |101101\rangle_{1QRSTU} - |110001\rangle_{1QRSTU} - |111100\rangle_{1QRSTU} \right) \\
+ |\Phi^-\rangle_{2P} \otimes |\Phi^+\rangle_{A} \right) H^{(1)} \frac{1}{2\sqrt{2}} \left( |000110\rangle_{1QRSTU} + |001011\rangle_{1QRSTU} + |010001\rangle_{1QRSTU} + |011100\rangle_{1QRSTU} + |100000\rangle_{1QRSTU} + |101101\rangle_{1QRSTU} + |110111\rangle_{1QRSTU} + |111010\rangle_{1QRSTU} \right) \\
+ |\Phi^-\rangle_{2P} \otimes |\Phi^+\rangle_{A} \right) H^{(1)} \frac{1}{2\sqrt{2}} \left( |000110\rangle_{1QRSTU} + |001011\rangle_{1QRSTU} + |010001\rangle_{1QRSTU} + |011100\rangle_{1QRSTU} + |100000\rangle_{1QRSTU} + |101101\rangle_{1QRSTU} - |110001\rangle_{1QRSTU} - |111100\rangle_{1QRSTU} \right) \right)
\] (14)

Fig. 2. An illustration of the new attack strategy described in section 3. Again \(\sigma_x\) is chosen as Alice's secret operation and \(\Psi^+\rangle_{1R}\) as Alice's result of the Bell state measurement.
As it has been shown in section 3, Eve is able to change the resulting states of her measurements such that she will always end up with a slight variation of her initial state. A similar argument holds for Eve’s second measurement on qubits 3 and S. Thus the state after these two measurements is (cf. picture (6) in figure 4):

\[ |\delta'\rangle_{1QR4TU} = \sigma_A^{(1)} H^{(1)} |\delta\rangle_{1QR4TU} \]  

Then Eve sends out qubit R to Alice and qubit Q to Bob. When Alice performs her Bell state measurement the state \( |\delta'\rangle_{1QR4TU} \) can be described as in eq. (16) (cf. also picture (7) in figure 4):

\[
\begin{align*}
\sigma_A^{(1)} H^{(1)} |\delta\rangle_{1QR4TU} &= \sigma_A^{(1)} H^{(1)} \frac{1}{2\sqrt{2}} \left( |000000\rangle_{QR4TU} + |011011\rangle_{QR4TU} \\
&\quad + |010110\rangle_{QR4TU} + |011011\rangle_{QR4TU} + |100111\rangle_{QR4TU} \\
&\quad + |101011\rangle_{QR4TU} + |110001\rangle_{QR4TU} + |111100\rangle_{QR4TU} \right) \\
&= \frac{1}{2} \left( \sigma_A^{(1)} |\Phi^+\rangle_{1R} \otimes \frac{1}{2\sqrt{2}} \left( |0000\rangle_{Q4TU} - |0011\rangle_{Q4TU} \\
&\quad - |1100\rangle_{Q4TU} + |1111\rangle_{Q4TU} + |0101\rangle_{Q4TU} \\
&\quad + |0110\rangle_{Q4TU} + |1001\rangle_{Q4TU} + |1010\rangle_{Q4TU} \right) \\
&\quad + \sigma_A^{(1)} |\Phi^-\rangle_{1R} \otimes \frac{1}{2\sqrt{2}} \left( |0000\rangle_{Q4TU} + |0011\rangle_{Q4TU} \\
&\quad + |1100\rangle_{Q4TU} + |1111\rangle_{Q4TU} - |0101\rangle_{Q4TU} \\
&\quad + |0110\rangle_{Q4TU} + |1001\rangle_{Q4TU} - |1010\rangle_{Q4TU} \right) \\
&\quad + \sigma_A^{(1)} |\Psi^+\rangle_{1R} \otimes \frac{1}{2\sqrt{2}} \left( |0101\rangle_{Q4TU} - |0110\rangle_{Q4TU} \\
&\quad - |1001\rangle_{Q4TU} + |1010\rangle_{Q4TU} + |0000\rangle_{Q4TU} \\
&\quad + |0011\rangle_{Q4TU} + |1100\rangle_{Q4TU} + |1111\rangle_{Q4TU} \right) \\
&\quad + \sigma_A^{(1)} |\Psi^-\rangle_{1R} \otimes \frac{1}{2\sqrt{2}} \left( |0101\rangle_{Q4TU} + |0110\rangle_{Q4TU} \\
&\quad + |1001\rangle_{Q4TU} + |1010\rangle_{Q4TU} - |0000\rangle_{Q4TU} \\
&\quad + |0011\rangle_{Q4TU} + |1100\rangle_{Q4TU} - |1111\rangle_{Q4TU} \right) \right) 
\end{align*}
\]  

For reasons of simplicity we will assume that Alice’s result is \( \sigma_A^{(1)} |\Phi^-\rangle_{1R} \) as it is illustrated in picture (8) in figure 4 (the computations for any other result are analog). Bob also applies the Hadamard operator on qubit Q in his possession and
thus alters the state of qubits $Q, 4, T$ and $U$ to

$$\frac{1}{4} \left( |0000\rangle_{QTU} + |1000\rangle_{QTU} + |0011\rangle_{QTU} + |1011\rangle_{QTU} + |0100\rangle_{QTU} - |1100\rangle_{QTU} + |0111\rangle_{QTU} - |1111\rangle_{QTU} - |0101\rangle_{QTU} - |1101\rangle_{QTU} + |0110\rangle_{QTU} + |1110\rangle_{QTU} + |0001\rangle_{QTU} - |1001\rangle_{QTU} - |0010\rangle_{QTU} + |1010\rangle_{QTU} \right)$$

(17)

which can be alternatively written as

$$\frac{1}{\sqrt{2}} \left( |\Phi^-\rangle_{Q4} \otimes |\omega^-\rangle_{TU} + |\Psi^+\rangle_{Q4} \otimes |\chi^+\rangle_{TU} \right)$$

(18)

From this it is easy to see that Bob will get the expected result, $|\Phi^-\rangle_{Q4}$ only with 50% probability. In the other half of the cases he will obtain $|\Psi^+\rangle_{Q4}$. Alice and Bob will detect this error when they check the correlation between their results.

![Diagram](image-url)

Fig. 3. An illustration of the modified protocol presented in section 4. Alice performs the Hadamard operation before she applies her secret operation $\sigma_x$. Again, $|\Psi^-\rangle_{13}$ is assumed to be Alice’s result of the Bell state measurement.
That means in \(\frac{1}{2}\) of the cases, when Alice doesn’t apply the Hadamard operation, Eve will stay undetected with certainty. In the other half of the cases when Alice performs the Hadamard transformation on qubit 1 Eve will be detected with probability \(\frac{1}{2}\). These probabilities are valid for the simple case \((n = 1)\) where Alice and Bob share only 2 Bell states. In general, Alice and Bob share \(2^n\) entangled qubit pairs and then they will detect Eve with the probability

\[
p = 1 - \left( \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} \right)^n = 1 - \left( \frac{3}{4} \right)^n
\]

which can be brought close to 1 for large \(n\).

A possibility for Eve to overcome the effect of the Hadamard operation is to apply a \(H\) operation herself on the qubits \(P\) and \(Q\) before the protocol starts. This changes the initial state \(|\delta\rangle_{PQRSTU}\) to \(H^{(P)}H^{(Q)}|\delta\rangle_{PQRSTU}\) and in the course of the protocol Alice and Bob’s application of the Hadamard operator will invert Eve’s operations. In detail, eq. (15) changes to

\[
|\delta^\prime\rangle_{1QRATU} = \sigma_A^{(1)}H^{(1)}H^{(1)}H^{(Q)}|\delta\rangle_{1QRATU} = \sigma_A^{(1)}H^{(Q)}|\delta\rangle_{1QRATU}
\]

and further, after Bob’s application of the Hadamard operation

\[
H^{(Q)}|\delta^\prime\rangle_{1QRATU} = \sigma_A^{(1)}H^{(Q)}H^{(Q)}|\delta\rangle_{1QRATU} = \sigma_A^{(1)}|\delta\rangle_{1QRATU}
\]

which is equal to the state from eq. (8). Thus, the attack reduces to the version described in section 3 and Eve will obtain full information about the key whenever Alice decides to perform the \(H\) operation. In all the other cases Eve will introduce an error equal to eq. (19) due to her Hadamard operations, which is easy to see following the arguments in the paragraphs above. At this point we want to stress that Alice and Bob will become suspicious if they find errors only in those cases when Alice does (or does not, respectively) perform the Hadamard operation. Thus, a better strategy for Eve is to randomly apply the two Hadamar d operations on qubits \(P\) and \(Q\). She will introduce the same error rate but the errors will be equally distributed.

Another option for Eve to invert Alice’s Hadamard operation is to delay her measurement on the qubit pairs 2, \(P\) and 3, \(S\) until Alice publicly announces whether she applied the Hadamard operation or not. That means, Eve intercepts qubits 2 and 3 in transit between Alice and Bob and immediately forwards qubit \(R\) to Alice (cf. picture (2) in figure 5). Alice, convinced to have received Bob’s qubit, applies the Hadamard operation and performs the Bell state measurement on qubits 1 and \(R\) (cf. pictures (3)-(5) in figure 5). Due to entanglement swapping qubit 2 is now
entangled with qubits $P$, $Q$, $S$, $T$ and $U$, as described in eq. (22)

$$\sigma_A^{(1)}H^{(1)}|\Phi^+\rangle_{12} \otimes |\delta\rangle_{PQRSTU} = \frac{1}{2} \left( |\Phi^+\rangle_{1R} \otimes \sigma_A^{(P)}H^{(2)} \frac{1}{2\sqrt{2}} \left( |000000\rangle_{PQRSTU} + |001101\rangle_{PQRSTU} + |010111\rangle_{PQRSTU} + |011011\rangle_{PQRSTU} + |100101\rangle_{PQRSTU} + |101011\rangle_{PQRSTU} + |110001\rangle_{PQRSTU} + |111100\rangle_{PQRSTU} \right) \right) \right)

$$

Here, Eve does not know in which state her qubits $2$, $P$, $Q$, $S$, $T$ and $U$ are because she does not know Alice’s result. But for our further considerations we assume that Alice’s result is $|\Psi^-\rangle_{1R}$, i.e. Eve is in possession of some state $\sigma_A^{(P)}|\delta'\rangle_{PQRSTU}$ similar to her initial state $|\delta\rangle_{PQRSTU}$. Next, Alice publicly announces that she applied the Hadamard operation. Having that knowledge Eve applies a $H$ operation on qubits $2$ and $Q$ in her possession inverting the effect of Alice $H$ operation and preparing for Bob’s $H$ operation (cf. picture (7) in figure 5). Then she sends qubit $Q$ to Bob and performs a Bell state measurement on qubits $3$ and $S$ which changes the state to $\sigma_A^{(P)}|\delta'\rangle_{PQRSTU}$. Finally, Bob inverts the Hadamard operation due to Alice’s public announcement (cf. picture (8) in figure 5). Similar to eq. (8) Bob’s Bell state measurement on qubits $Q$ and $4$ can be described as

$$\sigma_A^{(P)}|\delta'\rangle_{PQRSTU} = \frac{1}{2} \left( -|\Phi^+\rangle_{Q4} \otimes |\Phi^-\rangle_{P2} \otimes |\Phi^+\rangle_{TU} + |\Phi^-\rangle_{Q4} \otimes |\Phi^+\rangle_{P2} \otimes |\Phi^-\rangle_{TU} + |\Phi^+\rangle_{Q4} \otimes |\Psi^-\rangle_{P2} \otimes |\Psi^+\rangle_{TU} + |\Phi^-\rangle_{Q4} \otimes |\Psi^+\rangle_{P2} \otimes |\Psi^-\rangle_{TU} \right)$$

It is easy to see that Bob’s obtains every Bell state with equal probability of 25%, i.e. his and Alice’s results are completely uncorrelated. In case Alice does not apply
the Hadamard operation Eve will introduce the same error rate. Therefore, the probability Eve will be detected is
\[
p = 1 - \left( \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \right)^n = 1 - \left( \frac{1}{4} \right)^n
\]
which converges much faster to 1 compared to the first strategy described above (cf. eq. (19)).

6. Conclusion
We showed that the protocol presented by Li et al. is open to an attack strategy where Eve entangles herself with both legitimate communication parties. Using a

![Diagram](image)

Fig. 4. An illustration of the attack on the modified protocol as described in section 5. Eve follows the strategy presented in section 3 and Alice has chosen to apply the Hadamard operation. Again, \(\sigma_x\) is Alice’s secret operation and \(|\Psi^-\rangle_{1R}\) is Alice’s result of the Bell state measurement.
special 6-qubit state which preserves the correlated results of entanglement swapping an adversary is able to obtain full information about the key Alice and Bob share in the end. We also discussed an improvement where Alice uses a Hadamard transformation on one qubit to secure the protocol. We showed that due to the change of the initial state caused by the Hadamard operation an adversary will be detected with a high probability.

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Fig. 5. An illustration of the attack strategy with delayed measurement on the modified protocol as described in section 5. Eve waits with her measurement on qubits 2 and P until Alice announces whether she applied the Hadamard operation or not.
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**Appendix. Generation of Eve’s Initial State**

Eve’s initial state $|\delta\rangle_{PQRSTU}$ from eq. (5) is rather complex and it might not be producible due to today’s physical limitations. Nevertheless, we can show that it is, in principle, possible for Eve to generate this state using a sequence of Hadamard operations and controlled Pauli operations.

First we assume that Eve is in possession of 3 sources emitting Bell states. The state of these 6 qubits is

$$|\Phi^\pm\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU}$$  \hspace{1cm} (25)$$

Eve performs a Hadamard operation $H$ on qubits $P$ and $Q$ (indicated by the superscripts $(P)$ and $(Q)$), which alters the initial state first to

$$H^{(P)}|\Phi^\mp\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU}$$

and further to

$$H^{(P)}H^{(Q)}|\Phi^\mp\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} =$$

$$\frac{1}{2} \left( |\Phi^\mp\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} + |\Phi^\pm\rangle_{PR} \otimes |\Psi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU}$$

$$- |\Psi^\pm\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} - |\Psi^\mp\rangle_{PR} \otimes |\Psi^\pm\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} \right)$$ \hspace{1cm} (27)$$

Then Eve applies a $\sigma_z$ operation on qubit $Q$ if qubits $P$ and $R$ are in the Bell state $|\Phi^\mp\rangle_{PR}$. Otherwise the identity operator is applied. This action can be described as

$$|\Phi^\pm\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} + |\Phi^\mp\rangle_{PR} \otimes |\Psi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU}$$

$$+ |\Psi^\pm\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} - |\Psi^\mp\rangle_{PR} \otimes |\Psi^\pm\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} \right)$$ \hspace{1cm} (28)$$

and it changes the state to

$$\frac{1}{2} \left( |\Phi^\pm\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} + |\Phi^\mp\rangle_{PR} \otimes |\Psi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU}$$

$$+ |\Psi^\MP\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} + |\Psi^\MP\rangle_{PR} \otimes |\Psi^\pm\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} \right)$$ \hspace{1cm} (29)$$

Next Eve performs a $\sigma_z$ operation on qubit $P$ and a $\sigma_x$ on qubit $T$ if qubits $Q$ and $S$ are in the state $|\Psi^\pm\rangle_{QS}$. Alternatively, she performs a $\sigma_z$ operation on qubit $P$ and a $\sigma_x$ operation on qubit $T$ if qubits $Q$ and $S$ are in the state $|\Psi^\MP\rangle_{QS}$. If qubits $Q$ and $S$ are in the state $|\Psi^\MP\rangle_{QS}$ Eve applies a $i\sigma_y$ operation on both qubits $P$ and $T$. These three controlled operation can be stated as

$$I \otimes I \otimes |\Phi^\mp\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS} \otimes |\Phi^\mp\rangle_{TU} + \sigma_x \otimes I \otimes |\Phi^\mp\rangle_{PR} \otimes |\Phi^\mp\rangle_{QS}$$

$$+ \sigma_x \otimes I \otimes |\Psi^\MP\rangle_{PR} \otimes |\Phi^\MP\rangle_{QS} + i \sigma_y \otimes I \otimes |\Psi^\MP\rangle_{PR} \otimes |\Psi^\MP\rangle_{QS} \otimes I$$ \hspace{1cm} (30)$$

Using this operator Eve is able to bring the 6 qubits into the desired state from eq. (5).
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