CONTACT STRUCTURES ON 5–MANIFOLDS

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ABSTRACT. Using recent work on high dimensional Lutz twists and families of Weinstein structures we show that any almost contact structure on a 5–manifold is homotopic to a contact structure.

1. INTRODUCTION

Though contact structures have been studied for quite some time we still do not know which odd dimensional manifolds support such structures. Recall that (oriented) contact structures only exist on odd dimensional manifolds $M^{2n+1}$ and are described as $\xi = \ker \alpha$ where $\alpha$ is a 1–form for which $\alpha \wedge (d\alpha)^n$ is a volume form on $M$. (We will always take $M$ oriented and assume this volume form gives the preferred orientation.) Noting that $d\alpha$ defines a symplectic form on $\xi$ we see that a contact structure gives a reduction of the structure group of $TM$ to $U(n) \times 1$ (more specifically choosing a complex structure $J$ on $\xi$ that is compatible with $d\alpha$ gives the reduction and one may easily check this reduction is independent of the choice of $J$ or $\alpha$). Any reduction of the structure group of $TM$ to $U(n) \times 1$, or equivalently a choice of hyperplane field $\eta$ in $TM$ with a complex structure $J$ on $\eta$, is called an almost contact structure and the fundamental existence question in contact geometry concerns whether or not almost contact structures always come from contact structures.

For open manifolds Gromov [15] showed that all almost contact structures are homotopic to contact structures, but on closed manifolds much less is known. Previously we only had a complete answer in dimensions 1 and 3. The 1 dimensional result is trivial. In dimension 3 almost contact structures are equivalent to plane fields and so all oriented 3–manifolds $M$ admit almost contact structures. Martinet [20] showed that all closed oriented 3–manifolds admit a contact structure and Lutz [19] showed that every plane field is homotopic to a contact structure. The main result of this paper is to extend this existence result to 5–manifolds.

Theorem 1.1. On any closed oriented 5–manifold any almost contact structure $(\eta, J)$ is homotopic, through almost contact structures, to a contact structure $\xi$.

There have been many partial results towards this theorem. The first breakthrough was due to Geiges [9] who showed that on simply connected 5–manifolds any almost contact structure was homotopic to a contact structure. This result was reproven using open book decompositions by van Koert in [23]. In the papers [12] and [13] Geiges and Thomas extended the existence results to some 5–manifolds with finite fundamental group. In [11] Geiges and Stipsicz were able to prove existence on some other 5–manifolds. The most recent breakthrough is due to Casals, Pancholi, and Presas [1] who proved that all possible first Chern classes of almost contact structures can be realized by contact structures on 5–manifolds. In particular, they established the above theorem for manifolds without 2 torsion and gave a completely general existence result (that is any almost contact 5–manifold admits some contact structure). (Upon receiving a draft of this paper Casals, Pancholi, and Presas have informed the author that their arguments in [1] can be extended to obtain the main theorem above.)

The proof of Theorem 1.1 involves the use of an open book decompositions of a manifold $M$. In Section 3 we show that every almost contact structure on a closed oriented 5–manifold $M$ can be “supported” by an open book decomposition. The open book decomposition allows us to write $M$ as a neighborhood $N = Y \times D^2$ of an embedded 3–manifold and its complement $C$ that can be expressed as a mapping torus of a 4–manifold $X$. We further break $C$ into two pieces $C'$ and $C''$ where $C' = D^4 \times S^1$ and $C''$ is the mapping torus of $X - D^2$. We then use a result of the author and Pancholi [7] to put a contact structure on $C'$ that induces an overtwisted contact structure on
the boundary of $D^4 \times \{\theta\}$ for each $\theta \in S^1$. We then use work of Cieliebak and Eliashberg [4] on Weinstein manifolds to extend this contact structure over $C''$. The final extension over $N$ relies on Eliashberg’s understanding of overtwisted contact structures on 3–manifolds in [5].

We note that this proof is relatively elementary in that the two most sophisticated tools that it uses are the stable $s$-cobordism theorem (when quoting Quinn’s result below on the existence of open book decompositions [21]) and a symplectic analog of the $h$-cobordism theorem due to Cieliebak and Eliashberg [4].

We also note that most of the steps in the proof have obvious analogs in higher dimensions except the last step concerning the extension of the contact structure over the neighborhood $N$ of the binding $Y$, which relies heavily on Eliashberg’s understanding of overtwisted contact structures in dimension 3. So one might hope that a similar approach would yield a general existence result in higher dimensions. This is essentially a program laid out by Giroux independently from the work here, though the author does thank Giroux for comments that simplified the arguments in Section 3.

**Remark 1.2.** In the proof of Theorem 1.1 we make the somewhat surprising observation that any 5–manifold $M$ has a fixed open book decomposition that supports (in a weak sense, see Section 3.3) any almost contact structure on $M$ and more generally any hyperplane field on $M$ can, in some sense, be thought of as a perturbation of the pages of the open book. For a more precise formulation of this fact see Theorem 3.1. This observation is also true in higher dimensions but clearly far from true for open book decompositions of 3–manifolds. The intuitive explanation for this counter-intuitive fact is that the pages of open books in higher dimensional manifolds have more interesting topology than those of 3–dimensional open books.

**Acknowledgments:** The author thanks Kai Cieliebak and Yasha Eliashberg for providing an advanced copy of [4] and especially thanks Cieliebak for several very helpful discussions concerning Weinstein cobordisms. He additionally thanks David Gay, Emmanuel Giroux and Rob Kirby for useful conversations and correspondence. The author is particularly grateful to Dishant Pancholi who has made many helpful comments to improve this paper and helped to identify a potential gap in the original version of this paper. This work was partially supported by NSF grant DMS-0804820.

### 2. Preliminary results and recollections

In this section we first recall the notion of an open book decomposition for a 5–manifold and discuss a special open book decomposition on $S^5$. We then discuss a generalization of Lutz twists on 5–manifolds and finish by recalling Eliashberg’s result on the space of overtwisted contact structures.

#### 2.1. Open book decompositions of 5–manifolds.

An open book decomposition of a manifold $M$ is a pair $(Y, \pi)$ where $Y$ is a co-dimension 2 submanifold of $M$ that has a product neighborhood $N = Y \times D^2$ in $M$ and $\pi: (M - Y) \to S^1$ is a fibration such that $\pi|_{N - Y}$ is the projection onto the $\theta$–coordinate of $D^2$ where we give $D^2$ polar coordinates $(r, \theta)$. We call $Y$ the binding of the open book decomposition and $\pi^{-1}(\theta)$ a page of the open book.

There is another useful view of open books decompositions. If we are given a pair $(X, \phi)$ where $X$ is a compact manifold with boundary and $\phi: X \to X$ is a diffeomorphism that agrees with the identity map near $\partial X$, then we can construct a manifold as follows. Let $T_{\phi}$ be the mapping torus of $\phi$, that is $T_{\phi} = X \times [0, 1]/\sim$ where $(x, 1) \sim (\phi(x), 0)$ for all $x \in X$. Notice that $\partial T_{\phi} = Y \times S^1$ where $Y = \partial X$. We now glue $Y \times D^2$ to $T_{\phi}$ so that the product structures on the boundary agree. This gives a manifold $M_{(X, \phi)}$. We call $(X, \phi)$ an open book decomposition for a manifold $M$ if $M$ is diffeomorphic to $M_{(X, \phi)}$. Notice that the image of $Y \times \{(0, 0)\}$ in $M$ is the binding of an open book decomposition as above and any open book decomposition as above can be constructed from a pair $(X, \phi)$. We will use these notions interchangeably but note that when using the second definition of an open book we must always keep in mind a fixed diffeomorphism between $M$ and $M_{(X, \phi)}$.

We have the following simple observation that we will use later.
Lemma 2.1. Given open book decompositions \((X_i, \phi_i)\) of \(M_i\), \(i = 1, 2\), define \(X\) to be the boundary sum \(X_1 \# X_2\) and \(\phi\) to be the diffeomorphism of \(X\) that restricts to \(\phi_i\) on \(X_i\). Then \(M(X, \phi) \cong M(X_1, \phi_1) \# M(X_2, \phi_2)\).

2.2. An open book decomposition for \(S^5\). In this section we discuss a special open book decomposition of \(S^5\) that we will need to “stabilize” open books of other 5–manifolds to have nice properties that will be needed later.

Theorem 2.2. There is an open book \((Y_{\text{stab}}, \pi_{\text{stab}})\) for \(S^5\) with page \(X_{\text{stab}}\) the manifold \((S^2 \times S^2) \# (S^2 \times S^2)\) with an open disk removed. Let \(\phi_{\text{stab}}\) be the monodromy of this open book. There is a Morse function \(f_{\text{stab}}: X_{\text{stab}} \to \mathbb{R}\) for the page such that \(f_{\text{stab}}\) and \(f_{\text{stab}} \circ \phi_{\text{stab}}\) can be connected by a family of Morse functions having only 0 and 2–handles.

Proof. In [22], Saeki showed there was an open book for \(S^5\) with the given page. We sketch the argument here for the convenience of the reader. Given diffeomorphisms \(\phi\) of \(X_{\text{stab}}\) that is the identity on the boundary we see that the identity map \(I\) on the homology chains of \(X_{\text{stab}}\) and \(\phi\) induce the same map on the homology chains in \(\partial X_{\text{stab}}\). Thus \(I - \phi\) induce a map from \(H_2(X_{\text{stab}}, \partial X_{\text{stab}}) \to H_2(X_{\text{stab}})\). Since \(X_{\text{stab}}\) can be built with just 0 and 2–handles one may easily check that the open book \((X, \phi)\) gives a homotopy sphere if \(I - \phi\) is an isomorphism. Using the obvious product basis for \(H_2(X_{\text{stab}})\) the matrix

\[
\Phi = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}
\]

defines an automorphism of \(H_2(X_{\text{stab}})\) that preserves the intersection pairing. As \(H_1(X_{\text{stab}}) = 0\) this determines a map \(I - \Phi: H_2(X_{\text{stab}}, \partial X_{\text{stab}}) \to H_2(X_{\text{stab}})\) that is easily seen to be an isomorphism.

A result of Wall [24] implies that there is a diffeomorphism \(\phi_{\text{stab}}\) of \(X_{\text{stab}}\) that induces the \(\Phi\). Thus \((X_{\text{stab}}, \phi_{\text{stab}})\) give an open book decomposition for \(S^5\) which we denote by \((Y_{\text{stab}}, \pi_{\text{stab}})\).

One may easily show, cf. [17], that the diffeomorphism given in Wall’s theorem can be induced by a sequence of 2–handle slides. More specifically, there is a family of handle decompositions \(H_t\), \(t \in [0, 1]\), of manifold \(X_t = X_{\text{stab}}\) that starts and ends with the same handle decomposition of \(X_{\text{stab}} = X_0 = X_1\). In addition each \(H_t\) only consists of 0 and 2–handles and the family describes isotopies of the attaching regions and handles slides. We can use the handle decomposition of \(X_t\) to describe a diffeomorphism \(\phi_t: X_0 \to X_1\) and \(\phi_1: X_{\text{stab}} \to X_{\text{stab}}\) is a diffeomorphism that induces the map \(\Phi\) on \(H_2(X_{\text{stab}})\), that is \(\phi_t\) can be taken to be \(\phi_{\text{stab}}\). Now if \(f_{\text{stab}}\) is a Morse function on \(X_0 = X_{\text{stab}}\) that corresponds to \(H_0\) then set \(f_t = f_{\text{stab}} \circ \phi_t\). Thus we have a sequence of Morse functions \(f_t: X_{\text{stab}} \to \mathbb{R}\) that correspond to 2–handle slides and \(f_1 = f_{\text{stab}} \circ \phi_1\). \(\square\)

2.3. A notion of high dimensional Lutz twists. Recall the standard contact structure \(\xi_{\text{std}}\) on \(S^3\) is obtained as the set of complex tangencies to the unit \(S^3\) in \(\mathbb{C}^2\). There is an overtwisted contact structure \(\xi_{\text{ot}}\) on \(\mathbb{C}^2\) in the same homotopy class of plane field as \(\xi_{\text{std}}\). This follows from Eliashberg’s classification of overtwisted contact structures [5], or can easily be constructed by performing a full Lutz twist, see Section 2.3 below, on a transverse unknot in \(\xi_{\text{std}}\).

Proposition 2.3. There is a contact structure \(\xi\) on \(D^4 \times S^1\) such that \(\xi\) induces the overtwisted contact structure \(\xi_{\text{ot}}\) on \(\partial D^4 \times \{\theta\}\) for each \(\theta \in S^1\). More precisely, if \(\alpha_{\text{ot}}\) is a 1–form for which \(\xi_{\text{ot}} = \ker \alpha_{\text{ot}}\) then in a neighborhood \(S^3 \times (1/2, 1] \times S^1\) of the boundary of \(D^4 \times S^1\) the contact structure is given by

\[
\xi = \ker(K d\theta + t \alpha_{\text{ot}}),
\]

where \(t\) is the coordinate on \((1/2, 1]\), \(\theta\) is the coordinate on \(S^1\), and \(K\) is any positive constant. Moreover, \(\xi\) is homotopic through almost contact structures to the almost contact structure coming from the tangents to \(D^4 \times \{\theta\}\), for each \(\theta \in S^1\), where we think of \(D^4\) as the unit disk in \(\mathbb{C}^2\).

This theorem follows easily from a results of Pancholi and the author [7]. To state the result we first establish some notion. Consider \(T^2 \times \mathbb{R}\) with coordinates \((\theta, \varphi, r)\) and the contact structure

\[
\xi_{T^2 \times \mathbb{R}} = \ker(\cos r d\theta + \sin r d\varphi).
\]
We will think of \( \xi_{[a,b]} \) as the contact structure obtained from \( \xi_{T^2 \times \mathbb{R}} \) by restricting it to \( T^2 \times [a,b] \). We also denote \( T^2 \times \{ r \} \) by \( T_r \).

Notice that \( T^2 \times [0, \pi/2] \) has boundary \( T_0 \cup T_{\pi/2} \) and the characteristic foliation induced by \( \xi_{[0,\pi/2]} \) is given by a linear horizontal foliation on \( T_0 \) and linear vertical foliation on \( T_{\pi/2} \). So if we collapse the leaf of the foliation on the boundary of \( T^2 \times [0, \pi/2] \), or more precisely perform a contact cut [18], the we obtain \( S^3 \) and \( \xi_{[0,\pi/2]} \) induces the standard contact structure on \( S^3 \). Similarly performing a contact cut on the boundary of \( T^2 \times [0, 5\pi/2] \) will also produce \( S^3 \) but this time with the contact structure \( \xi_{ot} \). (In particular it is easy to see that this contact structure is obtained from the one above by a full Lutz twist.)

We now define a symplectic structure on a neighborhood of the boundary of \( Y \times [0,1] \), where \( Y = T^2 \times [0,1] \). Using an identification of \([0,1]\) with \([0,\pi/2]\) we construct a diffeomorphism \( \phi_0 : Y \times \{ 0 \} \to T^2 \times [0, \pi/2] \). Then on \((Y \times [0,1/4]) \cup (T^2 \times ([0,1/4] \cup [3/4,1]) \times [0,1])\) we let \( \lambda \) be \( t \phi_0^* \alpha \), where \( t \) is the coordinate on \([0,1]\). Similarly an identification of \([0,1]\) with \([0,5\pi/2]\) can be used to build a diffeomorphism \( \phi_1 : Y \times \{ 1 \} \to T^2 \times [0, 5\pi/2] \). Then on \( Y \times [3/4,1] \) we can define \( \lambda \) to be \( t \phi_1^* \alpha \). If \( \phi_1 \) is chosen correctly, \( \lambda \) will be a well defined 1–form on a closed neighborhood of the boundary of \( Y \times [0,1] \) such that \( d \lambda \) is a symplectic form with \( Y \times \{ 0 \} \) concave, \( Y \times \{ 1 \} \) convex and \( (\partial Y) \times [0,1] \) flat. It is well known that \( \lambda \) cannot be extended over all of \( Y \times [0,1] \) but if there was an extension we could construct a contact from on \( Y \times [0,1] \times S^1 \). Pancholi and the author [7] proved that this contact structure does exist.

**Lemma 2.4** (Etnyre and Pancholi, 2011 [7]). With \( Y \) and \( \lambda \) as defined above, there is a contact structure \( \xi \) on \( Y \times [0,1] \times S^1 \) so that near the boundary \( \xi = \ker(K d \theta + \lambda) \), where \( \theta \) is the coordinate on \( S^1 \) and \( K \) is any positive constant. Moreover, the contact structure \( \xi \) is homotopic through almost contact structures to \( \ker(K d \theta + t \phi_0^* \alpha) \).

With this lemma in hand we return to the proof of our main theorem of this section.

**Proof of Proposition 2.3** Simultaneously performing a contact cut on \( \partial Y \times \{ t \} \) for each \( t \) gives a contact structure on \( S^3 \times [0,1] \times S^1 \). We can glue \( D^4 \times S^1 \) with the contact structure \( K d \theta + \lambda \), where \( \lambda \) is the standard Liouville form on \( D^4 \), to the lower boundary of \( S^3 \times [0,1] \times S^1 \). This results in a contact structure on \( D^4 \times S^1 \) with the desired properties. \( \square \)

### 2.4. Plane fields and contact structures on 3–manifolds

Let \( Y \) be a closed oriented 3–manifolds. A contact structure \( \xi \) on \( Y \) is called overtwisted if there is a disk \( D \) embedded in \( Y \) that is tangent to \( \xi \) along the boundary of \( D \) and at one point on the interior of \( D \) and is transverse to \( \xi \) elsewhere. The disk \( D \) is called an overtwisted disk.

Recall that given any contact structure \( \xi \) on \( Y \) we can always alter it to be overtwisted by a Lutz twist. To define a Lutz twist let \( K \) be a knot in \( Y \) that is transverse to \( \xi \). It is easy to show, for example see [10], that \( K \) has a standard neighborhood \( S \) and the boundary of \( S \) has a neighborhood contactomorphic to \((T^2 \times [a,b], \xi_{[a,b]})\) for some \( a < b \), where we are using the notation from the previous section. Notice that one can replace the contact structure on this neighborhood with \((T^2 \times [a,b], \xi_{[a,b+2\pi]})\) to obtain a new contact structure on \( Y \). We say that this contact structure is obtained from \( \xi \) by a full Lutz twist along \( K \). It is well known, see for example [10] for a nice exposition of this, that the contact structure obtained from \( \xi \) by a full Lutz twist is homotopic as a plane field to \( \xi \); moreover, this homotopy can be done fixing the plane fields along \( K \) (which is called the core of the Lutz tube). It is also simple to see that if full Lutz twists are performed parametrically on a family of contact structure then the resulting family is homotopic to the original family (also fixing the planes along the cores of the Lutz tubes).

We fix a base point \( p \) in \( Y \) and a plane \( P \subset TY \). Then denote by \( P_p(Y) \) the space of plane fields on \( Y \) that agree with \( P \) at \( p \). Let \( D \) be a disk in \( Y \) that is tangent to \( P \) at \( p \) (and \( p \) is on the interior of \( D \)). We let \( C_{ot}(Y) \) be the space of contact structures on \( Y \) for which \( D \) and overtwisted disk. An amazing insight of Eliashberg relates these spaces.

**Theorem 2.5** (Eliashberg, 1989 [5]). The natural inclusion map

\[ i : C_{ot}(Y) \to P_p(Y). \]
is a homotopy equivalence.

3. Almost contact structures

In this section we study hyperplane fields and almost contact structures on 5–manifolds.

3.1. Hyperplane fields on 5–manifolds. Let $M$ be an oriented 5–manifold. To study the homotopy classes of hyperplane fields on $M$ we consider the bundle associated to the tangent bundle $TM$ with fiber the Grassmann manifold of oriented 4–planes in $\mathbb{R}^5$:

\[
\begin{array}{ccc}
G^+(4,5) & \longrightarrow & E \\
\downarrow & & \downarrow \\
M & & 
\end{array}
\]

Hyperplane fields on $M$ correspond to sections of this bundle. So we are interested in studying the homotopy classes of sections of this bundle. Throughout this section we will make no distinction between a hyperplane field and a section of this bundle. Note that $G^+(4,5)$ is diffeomorphic to $S^4$. (One may easily see this by noting that oriented hyperplanes are in one-to-one correspondence with unit vectors in $\mathbb{R}^5$, where the correspondence comes from an inner product on $\mathbb{R}^5$.)

Studying the obstruction theory of this bundle we first observe that $\pi_n(G^+(4,5))$ agrees with $[S^n, G^+(4,5)]$, the set of homotopy classes of maps of $S^n$ to $G^+(4,5)$ (that is $G^+(4,5)$ is $n$–simple). In addition because $TM$ is oriented it is well known that the action of $\pi_2(M)$ on $\pi_n(G^+(4,5))$ coming from the bundle $E$ is trivial. Thus the obstruction to homotoping two sections of $E$ over the $n$–skeleton of $M$ lies in $H^n(M; \pi_n(G^+(4,5)))$. Knowing that

\[
\pi_n(G^+(4,5)) = \begin{cases} 0 & n < 4 \\ \mathbb{Z} & n = 4 \\ \mathbb{Z}/2\mathbb{Z} & n = 5 \end{cases}
\]

it is clear that any section $s_1$ can be homotoped to $s_2$ on the 3–skeleton of $M$ and the obstruction to homotoping them together on the 4–skeleton is an element

\[d_4(s_1, s_2) \in H^4(M; \mathbb{Z})\]

and is a primary obstruction. Moreover if $s_1$ and $s_2$ agree on the 4–skeleton the, non–primary, obstruction to homotoping $s_1$ to $s_2$ over the 5–skeleton is an element

\[d_5(s_1, s_2) \in H^5(M; \mathbb{Z}/2\mathbb{Z}).\]

Given a plane field $\xi$ on a 3–manifold $Y$ and an open book decomposition $(Y, \pi)$ of a 5–manifold $M$ we can construct a hyperplane field as follows. Let $N = Y \times D^2$ be a neighborhood of $Y$ in $M$, so that $\pi|_{(N-Y)}$ is just the projection to the $\theta$–coordinate of $D^2$. On $N$ we take the hyperplane field

\[\ker \{ f(r)d\theta + g(r)\alpha \}\]

where $\alpha$ is a 1–form on $Y$ that defines $\xi$, $(r, \theta)$ are polar coordinates on the unit disk $D^2$, $f$ is a non–decreasing function equal to $r^2$ near 0 and constantly 1 near 1, and $g$ is a non–increasing function equal to 1 near 0 and 0 near 1 for which $f$ and $g$ are never simultaneously zero. This hyperplane field can be extended across $(M-N)$ by $\ker dr$. Denote this hyperplane field $H(\xi)$. It is clear that $H$ gives a well defined map from the set of plane fields on $Y$ to the set of hyperplane fields on $M$. Our main observation is that under certain hypothesis this map is onto.

Theorem 3.1. Let $M$ be any closed oriented 5–manifold and $(Y, \pi)$ any open book decomposition of $M$. Then we have the following:

1. The map $H$ is well defined up to homotopy. That is if $\xi$ and $\xi'$ are to homotopic plane fields on $Y$ then $H(\xi)$ is homotopic to $H(\xi')$.

2. If the pages of $(Y, \pi)$ are handlebodies with handles of index 0,1 and 2, then any homotopy class of hyperplane field is in the image of the map $H$ defined above. More specifically, if $\eta$ is a hyperplane field on $M$ then there is some plane field $\xi$ on $Y$ such that $H(\xi)$ is homotopic to $\eta$. 
Remark 3.2. The obvious generalization of this theorem to higher dimensional manifolds is also true and might be of use in trying to construct contact structures homotopic to almost contact structures on higher dimensional manifolds.

Proof. The first statement is clear as we can apply the same construction used in defining $H$ to the homotopy between $\xi$ and $\xi'$.

The second statement requires a bit more work. We begin by reinterpreting the obstruction cohomology class $d_4$. Let $\eta$ and $\eta'$ be two hyperplane fields on $M$. Thinking of $\eta$ and $\eta'$ as sections of the bundle $G^+(4, 5) \to E \to M$ we homotope them so that they are transverse. Let

$$\gamma(\eta, \eta') = \{ x \in M | \eta(x) = \eta'(x) \}$$

be the locus where $\eta$ and $\eta'$ agree.

Lemma 3.3. With the notation above, the Poincaré dual of $d_4(\eta, \eta')$ is $[\gamma(\eta, -\eta')] \in H_1(M; \mathbb{Z})$.

Proof. We begin by recalling how to compute $d_4(\eta, \eta')$. We can assume that $\eta$ and $\eta'$ agree on the 3–skeleton of $M$. Now given a 4–cell $D$ of $M$ the sections $\eta$ and $\eta'$ each give a map from $D \to G^+(4, 5) \cong S^4$ that agree along the boundary. We can thus use $\eta$ and $\eta'$ to define a map from $S^4 \to S^4$ by using $\eta$ on the upper hemisphere and $\eta'$ on the lower hemisphere. The degree of this map is the value of $d_4(\eta, \eta')$ on the 4–cell $D$.

We may isotope $\eta$ and $\eta'$ so that the 1–manifold $\gamma(\eta, -\eta')$ is disjoint from the 3–skeleton of $M$ and intersect the 4–skeleton transversely. It is clear that we may compute the degree of the above mentioned map on $D$ as the intersection number of $D$ and $\gamma(\eta, -\eta')$. (You can pick a trivialization of $TM$ over $D$ so that $\eta$, say, is constant, then $\gamma(\eta, -\eta') \cap D$ is just the preimage of some point on $S^4$.)

Lemma 3.4. Let $M$ be a closed oriented 5–manifold with open book decomposition $(Y, \pi)$ whose page is a handlebody with handle of index 0,1 and 2. Fix a plane field $\xi$ on a $Y$. As $\xi'$ ranges over all homotopy classes of plane field on $Y$ the class

$$d_4(H(\xi), H(\xi')) \in H^4(M, \mathbb{Z})$$

ranges over all of $H^4(M; \mathbb{Z})$.

In particular, every hyperplane field $\eta$ on $M$ is homotopic, over the 4–skeleton of $M$, to $H(\xi')$ for some plane field $\xi'$ on $Y$.

Proof. Using Lemma 3.3 we prove the first statement by showing that $[\gamma(H(\xi), -H(\xi'))]$ ranges over all of $H_1(M; \mathbb{Z})$ as $\xi'$ ranges over all homotopy classes of plane field on $Y$. To this end we first observe that the inclusion map

$$i : Y \to M$$

induces a surjection

$$i_* : H_1(Y; \mathbb{Z}) \to H_1(M; \mathbb{Z}).$$

To see this let $X$ be a page of the open book $(Y, \pi)$. A neighborhood $V_0$ of $X$ can be identified with $X \times [0, 1]$ and $V_1 = M \setminus V_0$ is also homeomorphic to $X \times [0, 1]$. Notice that $V_0$ and $V_1$ are handlebodies with handles of index less than 3. So $M$ is built from $V_0$ by adding handles of index greater than 2 (that is the handles in $V_1$ turned upside down). Thus the inclusion of $V_0$ into $M$ induces an isomorphism on the first homology $H_1(V_0; \mathbb{Z}) \to H_1(M; \mathbb{Z})$. Of course the inclusion of $X$ into $V_0$ is a homotopy equivalence so $H_1(X; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$. Finally notice that $X$ can be build from $Y = \partial X$ by attaching 4–dimensional handles of index 2, 3 and 4 (the handles in $X$ turned upside down). Thus the inclusion map gives a surjection of $H_1(Y, \mathbb{Z})$ onto $H_1(X; \mathbb{Z})$. The claim follows.

We now define a map $\tilde{H}$ that is a slight variant of $H$. Recall in the definition of $H$ we used the neighborhood $N = Y \times D^2$ of $Y$ in $M$, where $D^2$ was the unit disk. We also had functions $f(r)$ and $g(r)$ where $f(r) = r^2$ near 0 and is constantly 1 near 1 and non-decreasing, and $g(r) = 1$ near 0 and is 0 near 1 and non-increasing. We now specify that $g$ should be zero on only the interval $[1/2, 1]$. Now choose $\tilde{g}(r)$ to be a non-increasing function that is 1 near 0 and 0 near 1, but is 0 only on $[3/4, 1]$. For the moment let $\tilde{f} = f$, but we will perturb it later. Define $\tilde{H}$ in the same manner as
\[ H \text{ except use } \tilde{f} \text{ and } \tilde{g} \text{ in place of } f \text{ and } g. \] Notice that \( H(\xi) \) and \( \tilde{H}(\xi) \) are homotopic as hyperplane fields. Moreover, on \( (Y \times \{(r, \theta) | r \in [1/2, 3/4]\}) \subset M, H(\xi) \) and \( \tilde{H}(\xi') \) give disjoint section. Thus \( \tilde{H}(\xi) \) can be perturbed, relative to the region \( (Y \times \{(r, \theta) | r \in [1/2, 3/4]\}) \subset M, so that it is transverse to \( H(\xi) \) over the complement of \( Y \times \{(r, \theta) | r < 1/2\} \). From now on we will denote by \( \tilde{H}(\xi) \) this perturbed hyperplane field. Notice that \( H(\xi') \) for any other plane field \( \xi' \) agrees with \( H(\xi) \) on the complement of \( Y \times \{(r, \theta) | r < 1/2\} \). Thus \( H(\xi') \) is transverse to \( \tilde{H}(\xi) \) on the complement of \( Y \times \{(r, \theta) | r < 1/2\} \).

We claim with the appropriate choice of \( \tilde{f} \) and \( \tilde{g}, H(\xi') \) will be transverse to \( \tilde{H}(\xi) \) on all of \( M \) if \( \xi \) and \( \xi' \) are transverse on \( Y \). To achieve this we choose \( \tilde{f} \) to be zero only on \([0, 1/2]\) (it is still non-decreasing and 1 near 1). It is now a simple exercise to see that \( \tilde{H}(\xi) \) and \( H(\xi') \) are transverse on \( Y \times \{(r, \theta) | r < 1/2\} \). (Though the exercise is somewhat easier if one uses the dual picture to think of the hyperplane fields as unit vector fields.)

Obstruction theory as discussed above shows \( \xi \) and \( \xi' \) have a difference class \( d_2(\xi, \xi') \) that obstructs homotoping them on the 2–skeleton of \( Y \) and it is Poincaré dual to the 1–dimensional homology class given by the locus where they agree, which we denote by \( \gamma' \). From construction it is clear that \( H(\xi) \) and \( H(\xi') \) on \( Y \times \{(r, \theta) | r < 1/2\} \) only agree along \( \gamma' \). So the Poincaré dual of \( d_4(\tilde{H}(\xi), H(\xi')) \) is given by the homology class of \( \gamma' \cup \gamma'' \) where \( \gamma'' \) is a 1–manifold in the complement of \( Y \times \{(r, \theta) | r < 1/2\} \). Notice that \( \gamma'' \) is independent of \( \xi \). Moreover we know that by the appropriate choice of \( \xi' \) we can realize any homology class in \( H_1(Y, \mathbb{Z}) \) as the Poincaré dual of the difference class between \( \xi' \) and \( \xi \). Thus we see that any element of \( H_1(M; \mathbb{Z}) \) can be realized as the Poincaré dual of the difference class \( d_4(\tilde{H}(\xi), H(\xi')) = d_4(H(\xi), H(\xi')) \).

For the last statement recall that the difference class \( d_4 \) satisfies
\[ d_4(\eta_1, \eta_2) = d_4(\eta_1, \eta_2) + d_4(\eta_2, \eta_3). \]

Given a hyperplane field \( \eta \) the first part of the proof guarantees a plane field \( \xi' \) such that
\[ d_4(H(\xi), H(\xi')) = -d_4(\eta, H(\xi)). \]

So we see that \( d_4(\eta, H(\xi')) = d_4(\eta, H(\xi)) + d_4(H(\xi), H(\xi')) = 0 \). Thus \( H(\xi') \) is homotopic to \( \eta \) over the 4–skeleton of \( M \).

Returning to the proof of Theorem 3.1 we note that given any hyperplane field \( \eta \) on \( M \) we can find some plane field \( \xi \) on \( Y \) such that \( H(\xi) \) is homotopic to \( \eta \) on the 4–skeleton of \( M \). We are left to see that \( \xi \) can be chosen so that the homotopy can be extended over the 5–skeleton. Recall from above that if two hyperplane fields agree on the 4–skeleton then the obstruction to homotoping them on the 5–skeleton is a class in \( H^2(M; \mathbb{Z}/2\mathbb{Z}) \). Thus we see that any element of \( H_1(M; \mathbb{Z}) \) can be realized as the Poincaré dual of the difference class \( d_4(\tilde{H}(\xi), H(\xi')) \).
$H(\xi')$ is the homotopy class of the map $H: S^5 \to S^4$ defined by $H(\xi)$ on the upper hemisphere and by $H(\xi)$ on the lower hemisphere.

We claim the map $H: S^6 \to S^4$ factors through a map homotopic to the map obtained form 
$h: S^3 \to S^2$ by a “double suspension”. Recall that the suspension of a space $X$ is the quotient
of $X \times [0, 1]$ by $X \times \{0\}$ and $X \times \{1\}$. Given a function $f: X \to X'$ there is clearly an induced
suspension map $S(f): S(X) \to S(X')$. It is well known, and easy to see, that $S^n \cong S^{(n-1)}$. So
we are claiming that our map $H$ factors through a map $k: S^5 \to S^4$ that is homotopic to $S^5(h)$, in
other words there is a map $k': S^5 \to S^6$ such that $H = k \circ k'$. Moreover $k': \pi_5(S^5) \to \pi_5(S^6)$ is
an isomorphism. The Freudenthal suspension theorem implies that the double suspension map from 
$\pi_5(S^2) \to \pi_5(S^4)$ is surjective (here we have ignored base points, but due to the simple connectivity
of the spaces their inclusion would not alter the statement). Thus, as $\pi_5(S^2) \cong \mathbb{Z}$ and $\pi_5(S^4) \cong \mathbb{Z}_2$,
we see that $S^5(h)$ is not null homotopic and represents the generator of $\pi_5(S^4)$. Thus $k$, which is
homotopic to $S^5(h)$, induces a surjective map $k_\ast: \pi_5(S^6) \to \pi_5(S^4)$. As $k' \ast$ gives an isomorphism
on $\pi_5(S^2)$ we see $H$ is surjective and hence $H: S^5 \to S^4$ represents the generator of $\pi_5(S^4)$. Thus
d$_3(H(\xi), H(\xi'))$ is the mod 2 reduction of 3–dimensional obstruction $d_3(\xi, \xi')$. This finishes the proof.

So we are left to justify the claim about $H$ and $S^2(h)$. We first note that the suspension can be
described in terms of joins. Specifically, given a space $X$ its suspension is $S(X) \cong X * S^0$. (Recall
the join $X * Y$ of $X$ and $Y$ is obtained from $X \times Y \times [0, 1]$ where each of the sets $X \times \{y\} \times \{0\}$
and $\{x\} \times Y \times \{1\}$ are identified to a point.) As we know that $S^0 \ast S^0 = S^1$ it is clear that the
double suspension can be written $S^3(X) = X \ast S^1$. Now given a map $f: X \to Y$ the double
suspension map $S^2(f): X \ast S^1 \to Y \ast S^1$ is constructed as follows. We begin by defining the map
$X \times S^1 \times [0, 1] \to Y \times S^1 \times [0, 1]$ by $(x, \theta, t) \to (f(x), \theta, t)$. Composition with the quotient map
$Y \times S^1 \times [0, 1] \to Y \ast S^1$. This map clearly descends to a map $X \ast S^1 \to Y \ast S^1$ which we define to be $S^2(f)$.

In our case we start with the map $h: S^3 \to S^2$ and the map $H: S^5 \to S^4$. These maps are
defined with respect to some trivialization of the tangent space $TY$ over the 3–handle $D^3$ and of
the tangent space $TM$ over $D^5$, respectively. Recall $D^3 = D^2 \times D^2$ and over $D^5$ we have $TM|_{D^5} =
TY_{D^3} \oplus TD^3_{D^2}$. Notice that the unit sphere $S^4$ in $\mathbb{R}^5$ is naturally represented as the join of $S^2$ and $S^1$
where $S^2$ is the unit sphere in the first 3 coordinates of $\mathbb{R}^5$ and the $S^1$ is the unit sphere in the last
2 coordinates. Thus we make think of the $S^3$–bundle in $TM|_{D^5}$ as a fiber wise join of the unit sphere
bundle in $TY|_{D^3}$ and the unit sphere bundle in $TD^2$.

Recall that the $S^5$ in the domain of $H$ is obtained by gluing two copies of the 5–handle $D^5$
together. Moreover, $D^5 = D^3 \times D^2$ where $D^3$ is the 3–handle in $Y$ and $D^2$ is the fiber in the
neighborhood $N$ of $Y$. We glue the two copies of $D^3 \times D^2$ together in two stages. We first glue
along $(\partial D^3) \times D^2$. This yields $S^3 \times D^2$. Writing $D_1$ and $D_2$ for the two copies of $D^3$ glued to form
$D^3$ we see that $S^5$ is obtained from $S^3 \times D^2$ by gluing $D_1 \times \{\theta\}$ to $D_2 \times \{\theta\}$ for $\theta \in \partial D^2$. In addition
we see that the join $S^3 \ast S^3$ is obtained from $S^3 \times D^2$ (this is $S^3 \times S^1 \times [0, 1]$) by $\{p\} \times S^1 \times \{t\}$
is collapsed to a point) by collapsing each $S^3 \times \{\theta\}$ to a point. Thus we clearly see that there is a
map $k': S^5 \to S^5$ obtained by thinking of the domain as two copies of $D^3 \times D^2$ glued together
and the image space as $S^3 \ast S^3$. (Specifically the first $S^5$ is obtained from $S^3 \times D^2$ by partially
collapsing each $S^3 \times \{\theta\}$ while the second is obtained by completely collapsing the spheres.) This
is clearly a degree 1 map and hence induces an isomorphism on the homotopy groups of $S^5$. Below
we will denote $S^5$ created via the first method by $S^5_2$ and via the second method as $S^5_2$. Clearly
$S^5_2(h): S^5_2 \to S^4$ and $H: S^5 \to S^4$. We are now left to see that $H = k \circ k'$ where $k$ is homotopic to
$S^2(h)$.

Representing $S^4$ as $S^2 \ast S^1$ from the splitting of $TM|_{D^5}$ above we can consider “coordinates” on
$S^2 \ast S^1$ to be $(p, \theta, t) \in S^2 \times S^1 \times [0, 1]$ as discussed above. Using coordinates $(p, r, \theta)$ on $S^3 \times D^2$,
where $(r, \theta)$ are polar coordinates on $D^2$, we see their map $H$ is given by

$$(p, r, \theta) \mapsto \left( h(p), \theta, \frac{2}{\pi} \tan^{-1} \frac{f(r)}{g(r)} \right).$$
(To see this notice that we can map $S^2 \times S^1 \times [0, 1]$ to the unit sphere in $\mathbb{R}^5$ by the map $(p, \theta, t) \mapsto (p \cos \frac{\pi}{2} t, \theta \sin \frac{\pi}{2} t)$, where we think of $S^2$ is the unit sphere in the first 3 coordinates of $\mathbb{R}^5$ and $S^1$ as the unit sphere in the last 2 coordinates. Now the map $H$ from $S^3 \times D^2$ to $S^4 \subset \mathbb{R}^5$ is given by $(p, r, \theta) \mapsto (g(r)h(p), f(r)(\cos \theta, \sin \theta))$. So if $g$ and $f$ are chosen so that $g^2 + f^2 = 1$ we get the above representation of $H$ when using the “join coordinates” on $S^4$.)

The map $H: S^3_1 \to S^4$ descends to a map $k: S^3_2 \to S^4$ so that $H = k \circ k'$. Now choosing a homotopy from $r \mapsto \frac{2}{\pi} \tan^{-1} \frac{f(r)}{g(r)}$ to the identity map $r \mapsto r$ we have an induced homotopy $k: S^3_2 \to S^4$ to $S^2(h): S^3_2 \to S^4$. Thus completing the proof. $\square$

3.2. Almost contact structures. Recall an almost contact structure on an oriented 5-manifold $M$ is a reduction of the structure group of $TM$ from $SO(5)$ to $U(2) \times 1$. This is equivalent to a choice of homotopy class of hyperplane field $\eta$ on $M$ together with a homotopy class of complex structure $J$ on $\eta$. We will always think of an almost contact structure as a pair $(\eta, J)$, up to homotopy.

To study the homotopy classes of almost contact structures on $M$ we consider the bundle associated to the tangent bundle $TM$ with fiber $SO(5)/U(2)$.

$$SO(5)/U(2) \longrightarrow E \longrightarrow M.$$ 

Homotopy classes of sections of this bundle correspond to homotopy classes of almost contact structures. Using obstruction theory to understand the homotopy classes of sections of this bundle we need to know the homotopy type of the fiber. To this end we recall that $SO(5)/U(2)$ is diffeomorphic to $\mathbb{C}P^3$, see for example [10]. Thus all the relevant homotopy groups are zero except $\pi_2(SO(5)/U(2)) \cong \mathbb{Z}$. From this we see that the first and only obstruction to homotoping one almost contact structure to another is in $H^2(M; \mathbb{Z})$. In particular, we have the following useful observation.

**Lemma 3.5.** If two almost contact structures on the same 5-manifold $M$ are homotopic over the 2-skeleton of $M$ then they are homotopic.

3.3. Almost contact structures and open book decompositions. Recall that in Section 3.1 a map $H_{(Y, \pi)}: \{\text{hyperplanes on } Y\} \to \{\text{hyperplanes on } M\}$ was assigned to an open book decomposition $(Y, \pi)$ of a 5-manifold $M$. We say that an open book $(Y, \pi)$ supports an almost contact structure $(\eta, J)$ if there is some hyperplane $\xi$ on the binding $Y$ such that $\eta$ is (homotopic to) $H_{(Y, \pi)}(\xi)$ and the plane field $\xi$ is a complex sub-bundle of $H_{(Y, \pi)}(\xi)$ on $Y$ (that is $\xi$ on $Y$ is $J$ invariant). Notice that this implies that there is a homotopy of $(\eta, J)$ so that $\eta$ is transverse to $Y$ and outside of a small neighborhood of $Y$ is tangent to the pages of $(Y, \pi)$ and thus defines an almost complex structure on the pages.

**Theorem 3.6.** Given an almost contact structure $(\eta, J)$ on a closed oriented 5-manifold, there is an open book decomposition $(Y, \pi)$ with pages having a handle decomposition with handles of index less than or equal to two that supports the given almost contact structure.

Moreover there is an overtwisted contact structure $\xi$ on $Y$ such that $H(\xi) = \eta$ and one can assume that there is a neighborhood $N = Y \times D^2$ of the binding $Y$ such that $\eta$ is tangent to the pages of the open book outside of $N$ and at each point of $N$ the plane $\xi$ is a $J$-complex sub-bundle of $H(\xi)$.

To prove this proposition we will need a preliminary lemma.

**Lemma 3.7.** If $J$ is a complex structure on the hyperplane field $\eta$ then there is a $C^0$-neighborhood of $\eta$ in the space of hyperplane fields such that $J$ induces a complex structure for all hyperplane fields in this neighborhood. Moreover, the complex structures are well-defined up to homotopy.

**Proof.** Given $\eta$ choose any line field $L$ on $M$ that is transverse to $\eta$. Let $O$ be a $C^0$-neighborhood of $\eta$ in the space of hyperplane fields that consists of hyperplanes transverse to $L$. Note that $L$ and $\eta$ can be used to define a projection $p: TM \to \eta$. It is clear that at any point $x \in M$ the projection $p$ maps $\eta_x$ isomorphically onto $\eta_x$ for any $\eta' \in O$. Thus we can use $p$ to pull $J$ back form $\eta$ to $\eta'$.
The only choice made in this construction was the choice of $L$, but clearly as we vary $L$ this only changes the complex structures induced on $\eta'$ by a homotopy.

\textbf{Proof of Theorem 3.6} Quinn \cite{quinn1996open} proved that there is an open book decomposition $(Y, \pi)$ of $M$ whose pages have a handle decomposition with handles only of index less than or equal to two. Let $(\eta, J)$ be an almost contact structure on $M$. By Proposition 3.1 there is a hyperplane field $\xi$ on $Y$ such that the induced hyperplane field $H(\xi)$ is homotopic to $\eta$. Using Lemma 3.2 we can pull $J$ back along this homotopy so that $H(\xi)$ also has a complex structure $J'$. Thus $(H(\xi), J')$ is an almost contact structure that is homotopic to $(\eta, J)$. To simplify notation we now assume our given almost contact structure is $(H(\xi), J)$.

We are left to see that we can arrange that $\xi$ on $Y$ is $J$–invariant. We will do this by altering $H(\xi)$ on the neighborhood $N$ of the binding $Y$ to construct a new almost contact structure and then show that this is homotopic to $(H(\xi), J)$ through almost contact structures.

Recall $N = Y \times D^2$ where $D^2$ is the unit disk in $\mathbb{R}^2$. We write $D^2$ as the union $D_{1/4} \cup A_{1/4,1/2} \cup A_{1/2,3/4} \cup A_{3/4,1}$, where $D_{1/4}$ is the disk of radius $1/4$ and $A_{a,b}$ denotes the annulus $\{ (r, \theta) : a \leq r \leq b \}$. We can homotope $(H(\xi), J)$ so that it is independent of the coordinates on $D^2$ near $Y \times \{(0,0)\}$. Now one can use a map from $A_{3/4,1}$ to $D$ that is the identity near the outer boundary of the annulus, a diffeomorphism away from the inner boundary and collapses the inner boundary to the origin, to pull $(H(\xi), J)$ back to an almost contact structure on $Y \times A_{3/4,1}$, call the resulting almost contact structure $(\eta', J')$. Notice that $\eta'$ on $Y \times \{ p \}$ for any point on the inner boundary of $A_{3/4,1}$ is homotopic to $TY \oplus \mathbb{R}$ (where we can take $\mathbb{R}$ to point to the radial direction if we like). We can use this homotopy to extend $\eta'$ to a hyperplane field on $A_{1/2,3/4}$ so that on the inner boundary of this annulus the hyperplane field is $TY \oplus \mathbb{R}$. We can also use the homotopy to extend $J'$ over this annulus so that it is independent of the angular coordinate on the annulus.

To continue to define our almost contact structure we need to discuss complex structures on $TY \oplus \mathbb{R}$. Recall that complex structures on a 4–dimensional bundle correspond to reductions of the structure group from $SO(4)$ to $U(2)$ and thus correspond to sections of a principal $SO(4)/U(2)$ bundle. As $SO(4)/U(2)$ is homotopy equivalent to $S^2$ and the bundle $TY \oplus \mathbb{R}$ is trivial, we see that complex structures on this bundle correspond to maps $Y \to S^2$. We can concretely see this by noting that a unit vector in the $\mathbb{R}$ direction of $TY \oplus \mathbb{R}$ will be mapped by the complex structure to a unit vector in $TY$. The span of these two vectors is a complex line in $TY \oplus \mathbb{R}$ and there is a unique, up to homotopy, complex structure on the orthogonal complement of this complex line. Thus a unit vector field on $Y$, or dually an oriented plane field, determines the complex structure on $TY \oplus \mathbb{R}$. Clearly there is some plane field, say $\xi'$, on $Y$ that corresponds to $J'$ on $TY \oplus \mathbb{R}$ (thought of as a hyperplane field along the inner boundary of $Y \times A_{1/2,3/4}$). We can assume that $\xi'$ is an overtwisted contact structure.

We can now define $\eta'$ on $D_{1/4}$ using the form $f(r) \, d\theta + g(r) \, \alpha'$, where $\alpha'$ is a contact form for $\xi'$ and $f$ and $g$ are functions analogous to the ones used in the definition of $H(\xi)$ in Section 3.1. With the appropriate choice of $f$ and $g$ this is a contact form on the interior of $D_{1/4}$ and the complex structure discussed in the previous paragraph thought of as defined on $\eta'$ restricted to $Y \times \partial D_{1/4}$ extends to a complex structure compatible the contact structure $\eta'$ on the interior of $D_{1/4}$. In particular at any point in $Y \times D_{1/4}$ the plane field $\xi \subset TM$ can be assumed to be a $J'$–complex line in $\eta'$. Finally by our choice of $J'$ and $\xi'$ above we can extend $\eta'$ over $Y \times A_{1/4,1/2}$ to be $TY \oplus \mathbb{R}$ and extend $J'$ by the homotopy from the paragraph above.

We finally note that $\eta'$ can be homotoped on $Y \times (A_{1/2,3/4} \cup A_{3/4,1})$ so that it is $TY \times \mathbb{R}$ at every point. Thus $\eta'$ is homotopic to $H(\xi')$ and the complex structure $J'$ has the properties stated in the theorem. We are left to see that $(H(\xi'), J')$ is homotopic to $(H(\xi), J)$. To this end notice that if $U$ is a neighborhood of a page of the open book $(Y, \pi)$ then $M$ has a handle decomposition with $U$ being the union of 0, 1, and 2–handles and the other handles being index 3 and above. Since $(H(\xi'), J')$ and $(H(\xi), J)$ agree along the cores of the 0, 1 and 2–handles we see from Lemma 3.3 that they are homotopic on all of $M$. \qed
4. Cobordisms

This section consists of two subsections. In the first we recall the notions of Weinstein cobordisms and Cieliebak and Eliashberg’s “Weinstein flexibility results”. The following subsection examines Morse functions on the pages of open book decompositions of 5–manifolds.

4.1. Weinstein cobordisms and flexibility. A cobordism $W$ is an compact $n$-manifold with boundary $\partial W = -\partial_- W \cup \partial_+ W$. We say that $W$ is a cobordism from $\partial_- W$ to $\partial_+ W$. A Morse cobordism is a pair $(W, f)$ where $W$ is a cobordism and $f : W \to \mathbb{R}$ is a Morse function having $\partial_\pm W$ as regular level sets.

Recall that a 1–form $\lambda$ on a manifold $W$ is called a Liouville form if $d\lambda$ is a symplectic form. Given such a 1–form there is a unique vector field $v$ such that $\iota_v d\lambda = \lambda$. This vector field is called the Liouville vector field associated to $\lambda$. Notice that if $\omega = d\lambda$ is a symplectic structure on $W$ then $\lambda$ can be recovered from $v$ and $\omega$ by $\lambda = \iota_v \omega$. We call $(\omega, v)$ a Liouville structure on $W$ if $d\iota_v \omega = \omega$.

A Weinstein cobordism is a tuple $(W, \omega, v, f)$ where $W$ is a compact cobordism, $(\omega, v)$ is a Liouville structure on $W$ so that $v$ points out of $W$ along $\partial_+ W$ and into $W$ along $\partial_- W$, and $f$ is a Morse function on $W$ that is constant on each boundary component and for which $v$ is a gradient-like vector field. Recall $v$ is gradient-like for $f$ if there is some constant $\delta > 0$ and some metric so that $df(v) \geq \delta(|v|^2 + |df|^2)$.

It is well known, see for example [2], that the Weinstein structure gives $W$ a handle decomposition with handles of index less than or equal to half the dimension of $W$. We also note that $\iota_v \omega|_{\partial_\pm W}$ is a contact form on $\partial_\pm W$ and thus Weinstein cobordisms are cobordisms between contact manifolds.

Cieliebak and Eliashberg define a flexible Weinstein cobordism on a 4–manifold $W$ to be one with only 0 and 1–handles or one whose lower boundary is an overtwisted contact 3–manifold and all the 2–handles are attached along Legendrian knots that have overtwisted disks in their complement.

**Theorem 4.1** (Cieliebak and Eliashberg 2012, [4]). Let $(W, f)$ be a 4–dimensional Morse cobordism such that $f$ has no critical points of index larger than 2. Let $\eta$ be a non-degenerate 2–form on $W$ and $w$ a vector field near $\partial_- W$ such that $(\eta, w, f)$ is a Weinstein cobordism structure on some neighborhood $N$ of $\partial_- W$. Suppose that the contact structure induced by $\iota_w \eta$ on $\partial_+ W$ is overtwisted. Then there is a flexible Weinstein cobordism structure $(\omega, v, f)$ on $W$ such that

1. $(\omega, v) = (\eta, w)$ on some neighborhood $N'$ of $\partial_- W$, and
2. the non-degenerate 2–forms $\omega$ and $\eta$ are homotopic on $W$ relative to $N'$.

We now discuss a slight refinement of the above theorem that follows directly from the proof of that theorem, [3]. We first establish some notation. A relative cobordism $W$ is a compact $n$–manifold with boundary $\partial W = \partial_- W \cup \partial_+ W \cup \partial_0 W$ where $\partial_0 W \cong [0, 1] \times \partial (\partial_- W) \cong [0, 1] \times \partial (\partial_+ W)$. We say $W$ is a relative cobordism from the manifold with boundary $\partial_- W$ to the manifold with boundary $\partial_+ W$. We say the vertical boundary of $W$ is $\partial_0 W$. (There is of course an analogous notion of relative Morse cobordism.)

A relative Weinstein cobordism structure on a relative cobordism $W$ is a triple $(\omega, v, f)$ as in the ordinary definition of Weinstein cobordism and in addition, $f$ must have no critical points near $\partial_0 W$ and $v$ must be tangent to $\partial_0 W$. As in the case of Weinstein cobordisms a relative Weinstein cobordism is flexible if it only has 0 and 1–handles or if on its lower boundary the induced contact structure is overtwisted and all the 2–handles are attached along Legendrian knots that have overtwisted disks in their complement.

**Theorem 4.2** (Cieliebak and Eliashberg 2012, [3, 4]). Let $(W, f)$ be a 4–dimensional relative Morse cobordism such that $f$ has no critical points of index larger than 2. Let $\eta$ be a non-degenerate 2–form on $W$ and $w$ a vector field defined near $\partial_- W \cup \partial_0 W$ such that $(\eta, w, f)$ is a Weinstein cobordism structure on some neighborhood $N$ of $\partial_- W$ and on some neighborhood $N_0$ of $\partial_0 W$. Suppose that the contact structure induced by $\iota_w \eta$ on $\partial_- W$ is overtwisted. Then there is a flexible Weinstein cobordism structure $(\omega, v, f)$ on $W$ such that
Theorem 4.4. \( (\omega, v) = (\eta, w) \) on some neighborhood \( N' \) of \( \partial W \) and \( N'_v \) of \( \partial_v W \) (\( N' \) can be any subset of \( N \) on which \( (\eta, w) \) gives a Weinstein cobordism structure and similarly for \( N'_v \) and \( N_v \)), and

(2) the non-degenerate 2–forms \( \omega \) and \( \eta \) are homotopic on \( W \) relative to \( N' \cup N'_v \).

A Weinstein homotopy on a cobordism \( W \) is a smooth family of Weinstein structures \( (\omega_t, v_t, f_t), t \in [0, 1] \), were we allow the functions \( f_t \) to have birth-death type degenerations (that is, they form a generic family of functions), and \( (\omega_t, v_t) \) is a smooth family of Liouville structures on \( W \). We now state a slight strengthening of Cieliebak and Eliashberg’s Weinstein flexibility theorem from [4]. Just as for Theorem 4.2 above, the proof of this is the same as for the non-relative version in [4].

Theorem 4.3 (Cieliebak and Eliashberg 2012, [3, 4]). Let \( (\omega_0, v_0, f_0) \) and \( (\omega_1, v_1, f_1) \) be two flexible Weinstein structures on a 4–dimensional relative cobordism \( W \) from an overtwisted contact structure on \( \partial W \) to some contact structure on \( \partial_v W \). Let

(1) \( f_t, t \in [0, 1] \), be a Morse homotopy without critical points of index greater than two, and

(2) \( \beta_t, t \in [0, 1] \), be a homotopy of non-degenerate 2–forms connecting \( \omega_0 \) and \( \omega_1 \) such that there are neighborhoods \( N \) of \( \partial W \) and \( N_v \) of \( \partial_v W \) and vector fields \( w_t \) defined on \( N \cup N_v \) connecting \( \omega_0 |_{\partial N} \) to \( \omega_1 |_{\partial N} \), such that \( (\beta_t, w_t, f_t) \) has the structure of a relative Weinstein cobordism when restricted to \( N \) and when restricted to \( N_v \).

Then there are sub–neighborhoods \( N' \) of \( \partial W \) in \( N \) and \( N'_v \) of \( \partial_v W \) in \( N_v \) and there is a homotopy \( (\omega_t, v_t, f_t), t \in [0, 1] \), of Weinstein structures agreeing with \( (\beta_t, w_t, f_t) \) on \( \partial N' \cup \partial N'_v \) such that the paths of 2–forms \( \beta_t \) and \( \omega_t \), for \( t \in [0, 1] \), are homotopic relative to \( N' \cup N'_v \) and the endpoints.

4.2. Open books and stabilization. Let \((Y, \pi)\) be an open book decomposition of a closed oriented 5–manifold \( M \). Let \( N = Y \times D^2 \) be a closed neighborhood of \( Y \) in \( M \). We can think of \( \pi \) as a fibration of \( M \setminus N \) (and \( \pi \) extends over \( N = Y \) by projection to the \( \theta \)-coordinate of \( D^2 \)).

We denote the pages (that is fibers of \( \pi \)): \( (M \setminus N) \rightarrow S^1 \) of the open book by \( X_\theta = \pi^{-1}(\theta) \). Fixing some \( \theta_0 \in S^1 \), notice that

\[
(M \setminus N) \setminus X_{\theta_0} \cong X \times [0, 1]
\]

where \( X = X_{\theta_0} \). Moreover there is some diffeomorphism \( \phi: X \rightarrow X \) that is the identity near \( \partial X \) so that the mapping torus

\[
T_\phi = X \times [0, 1]/(x, 1) \sim (\phi(x), 0)
\]

is diffeomorphic to \( M \setminus N \).

Let \( f \) be a Morse function for \( X_{\theta_0} \). We denote \( f \) on \( X \times \{0\} \) by \( f_0 \) and think of \( \phi^* f \) is a Morse function on \( X \times \{1\} \), which we denote by \( f_1 \).

Theorem 4.4. Given an open book \((Y, \pi)\) for a closed oriented 5–manifold \( M \) and a Morse function \( f: X \rightarrow \mathbb{R} \) on a page of the open book with critical points of index less than or equal to 2, there is another open book \((Y', \pi')\) with monodromy \( \phi' \) and Morse function \( f': X' \rightarrow \mathbb{R} \) on its pages so that, in the notation above, when \( M \setminus N' \) is cut open along the page \( X' \) to obtain \( X' \times [0, 1] \) the Morse function \( f_0 = f' \) on \( X' \times \{0\} \) and \( f_1 = \phi'^* f' \) on \( X' \times \{1\} \) can be extended to a family of functions \( f_t: (X' \times \{t\}) \rightarrow \mathbb{R}, t \in [0, 1] \), satisfying the following:

1. \( f_t \) is either a Morse function or has a birth/death degenerate critical point,
2. no \( f_t \) has critical points if index larger than 2,
3. each \( f_t \) has a unique index 0 critical point,
4. the index 0 critical is constant in \( t \) and it can be connected to the boundary by a gradient like flow line that is also constant in \( t \) and \( \phi' \) fixes a neighborhood of the index 0 critical point and flow line.

Proof. Let \( f_0 = f \) and \( f_1 = f \circ \phi \). Let \( f_t, t \in [0, 1] \), be a generic family of functions connecting \( f_0 \) to \( f_1 \). It is well known, see [2], that such a family has the following structure. For all but finitely many \( \{t_1, \ldots, t_k\} \subset (0, 1) \) the functions \( f \) are Morse functions whose critical points all have distinct values. At each \( t_i \) there are either precisely two critical points with the same value or all critical points have distinct values, but one of the critical points is a birth or death point.

The Cerf graphic for the family \( \{f_t\} \) consists of the points \((t, u) \in [0, 1] \times \mathbb{R} \) where \( u \) is a critical value of \( f_t \). It is now a standard argument using manipulations of the Cerf graphic, see [2] [10], that we can alter \( f_t \) relative to \( f_0 \) and \( f_1 \) so that there are no births or deaths of index 0 or 4 critical.
points. Thus $f_t$ has a unique index 0 critical point for all $t$. One can isotope $\phi$ and then the functions \{f_t\}, relative to $f_0$ and $f_1$, so that the critical point and a gradient-like flow line is fixed for all $t$; moreover, one can further isotope $\phi$ and then the functions relative to $f_0$ and $f_1$ so that they all agree on a neighborhood of the index 0 critical point and flow line.

We are left to show how to remove the index 3 critical points. We will give a modification of Fenn and Rouke’s extension [8] of Kirby’s arguments in [16] to the non-simply connected setting. We will be considering the family $X_t$ of cobordisms upside down. That is we use the functions \{-f_t\} so in order to eliminate the index 3 critical points of the \{f_t\} we will eliminate the index 1 critical points of the \{-f_t\}.

We begin by identifying a product neighborhood $N = (\partial X) \times \{a, b\}$ of $\partial X$ so that $\partial X = (\partial X) \times \{a\}$ and isoting $\phi$ so that it is the identity map on $N$. We can also assume that $f|N: (\partial X) \times \{a, b\} \to [a, b]$ is just the projection map (and thus $-f_0$ and $-f_1$ will also be the projection map when restricted to $N$). We will denote the copy of $N$ in $X_i$ by $N_i$, $i = 0, 1$.

Following Kirby we can modify \{-f_t\} so that all the index 1 critical points occur as shown on the left of Figure 1. Looking at the bottom most line of index 1 critical points we can isotope \{-f_t\} so that the birth and death points are pushed past the edge of the Cerf graphic as shown on the right of Figure 1. We can assume that the canceling 1/2-handle pair in $X_0$ and $X_1$ is contained a small 4-ball $B_0 \subset N_0$ and $B_1 \subset N_1$, respectively, where $B_0 = B_1$.

More precisely, we can assume that an index 1-handle, $h^1_0$ and canceling index 2-handle, $h^2_0$ are attached to a 3-ball $\widetilde{B}_i$ in $(\partial X) \times \{c\} \subset N_i$, for some $c \in (a, b)$ and $i = 0, 1$; and moreover $\widetilde{B}_0 = \widetilde{B}_1$. The attaching sphere for $h^2_0$ is a circle $A_0$ that is the union of two arcs $a^0_1$ and $a^0_2$ with $a^0_1 \subset \widetilde{B}_0$ and $a^0_2$ contained in $h^0_1$. Let $\gamma_0$ be an arc in $\widetilde{B}_0$ that connects the two components of the attaching region of $h^1_0$ and is parallel to $a_1$. There is an arc $\gamma'_0$ in $h^1_0$ that is parallel to the core of $h^1_0$ and completes $\gamma_0$ into a circle $C_0$. Notice that the 2-handle $h^2_0$ shows that the circle $C_0$ is null-homotopic and in particular bounds an embedded disk $D_0$.

Moving through the cobordisms $(X_i, -f_t)$ the attaching region of the 1-handle changes by an isotope in $(\partial X) \times \{c\}$ (since it is the bottom most 1-handle it does not slide over the other 1-handles). This can be extended to an ambient isotope of $(\partial X) \times \{c\}$ and thus we may push $\gamma_0$ to a curve $\gamma_1$ in $(\partial X) \times \{c\} \subset N_1$. We can further extend this ambient isotope to $X$ so that $D_0$ can be pushed to the disk $D_1$ in $X_1$. In particular we get simple closed curves $C_i$ in $X_i$. Identifying $X$ with $X_0$ we let $X'$ be the result of surgering $C_0$ in $X$. This will replace the 1-handle (that is index 1 critical point) with a 2-handle (that is index 2 critical point). We can use the circles $C_i$ to surger all the $X_t$ simultaneously to get $X'$ that fit together to form $X' \times [0, 1]$. Moreover the functions $-f'_t$ induced from the $-f_t$ and this surgery have one less critical point of index 1. We claim that

1. $X' = X \# (S^2 \times S^2)$
2. we may extend $\phi$ to a diffeomorphism $\phi'$ of $X'$ and
3. we may still assume that $f'_t = f_0' \circ \phi'$.
Once we establish these three claims we will see how to alter the pair \((X', \phi')\) to an open book decomposition \((X'', \phi'')\) of \(M\) with the desired properties.

We first observe that since \(h_1^0, h_0^0\) and \(\gamma_0\) are attached in \(B_0\) and so the surgery on \(C_0\) with the appropriate framing in \(X_0\) clearly results in \(X_0 \# (S^2 \times S^2)\). Since we can surger all the \(C_i\) in \(X_i\) (using the same framing) at the same time and preserve the product structure (we could interpret this as surgering the annulus that is formed as the union of the \(C_i\) in \(X \times [0, 1]\)) we see that the surgered family of cobordisms \(X \times [0, 1]\) becomes \(X' \times [0, 1]\) where \(X' = X \# (S^2 \times S^2)\) and thus establish Claim (1). We note that we can extend the functions \(f_t\) restricted to the complement of the surgered regions to the surgery regions to obtain new functions \(f_t\).

We now establish Claims (2) and (3) by analyzing the curve \(C_1\). Notice that \(C_1\) runs over the 1–handle \(h_1^1\) and no other 1–handles (again since we are considering the bottom most 1–handle it does not slide over the other 1–handles). In particular \(C_1 = \gamma_1 \cup \gamma_1'\), where \(\gamma_1'\) is the analog of \(\gamma_0\) that runs over the 1–handle \(h_1^1\) in \(X_1\).

The curve \(C_1\) bounds the disk \(D_1\) in \(N_1 = (\partial X_1) \times [a, b]\) and so is null-homotopic. The 2–handle \(h_2^0\) is attached along a curve \(a_1^1 \cup a_2^1\) where \(a_1^1\) is in \(B_1\) and \(a_2^1\) is in the handle \(h_1^1\). The arcs \(a_1^1\) and \(\gamma_1\) can be connected in \((\partial X_1) \times \{c\}\) to from a simple closed curve \(c_1\). The 2–handle \(h_2^0\) can be used to show that \(C_1\) and \(c_1\) are isotopic and thus \(c_1\) is null-homotopic in \((\partial X_1) \times \{c\}\) (since \((\partial X_1) \times \{c\}\) is homotopy equivalent to \(N_1\)). We can use this null-homotopy to isotope \(\gamma_1\) in \((\partial X_1) \times \{c\}\) so that it is parallel to \(a_1^1\) (of course this isotopy may cross \(a_1^1\)). The family of functions \(-f_t\) for \(t \in [0, 1]\) can be extended to a family for \(t \in [0, 2]\) that is constant for \(t \in [1, 2]\). The isotopy above allows us to extend \(C_1, t \in [0, 1]\) to a family of curves \(C_1\) in \(X_t\) for \(t \in [0, 2]\) so that \(-f_2 = -f_0 \circ \phi\) and \(C_0 = \phi(C_2)\). Thus the \(C_t\) form a torus in the mapping torus of \(\phi\) and we can perform surgery on \(C_t\) in \(X_t\), simultaneously for all \(t\). Reparameterizing the interval \([0, 2]\) establishes Claims (2) and (3).

Given \(X_t' = -f_t\) and \(\phi'\) as above we construct \(X''\) by connect summing \(X'\) with \(S^2 \times S^2\). More specifically we add two 2–handles to \(X'\) to obtain \(X'' = X' \# S^2 \times S^2\). Since these 2–handles are attached to \(\partial X'\) and \(\phi'\) is the identity map here we can clearly extend \(\phi'\) over \(X''\). Moreover we can extend the \(-f_t'\) over \(X''\) to obtain Morse functions on \(X'' = X''\) that satisfy \(f'' = f''_0 \circ \phi'\).

Recall that \(X'' = X \# Z\) where \(Z = S^2 \times S^2 \# S^2 \times S^2\). Using Theorem 2.2 we can perform handle slides on the \(Z\) part of \(X''\), encoded by Morse functions \(f''_t, t \in [0, 1]\), to obtain a diffeomorphism \(\psi: X'' \rightarrow X''\) that is the identity away from \(Z\) and gives the monodromy for \(S^5\) described in the theorem. Thus setting \(\phi'' = \phi' \circ \psi\) we have a diffeomorphism of \(X''\) and a family of Morse functions \(f''_t, t \in [0, 2]\), such that \(f''_2 = f''_0 \circ \phi''\). Moreover it is clear that \((X'', \phi'')\) describes the open book decomposition obtained from \((X, \phi)\) by boundary summing with the open book from Theorem 2.2. That is \((X'', \phi'')\) is an open book for \(M \# S^5\) according to Lemma 2.1. Moreover the Morse functions \(f''_t\) have one less 3–handle than \(f_t\). Thus continuing the above construction for each 3–handle in \(f_t\) we eventually arrive at an open book and Morse functions as described in the theorem.

5. Almost Contact Structures are Homotopic to Contact Structures

We are now ready to assemble the pieces discussed above to prove that an almost contact structure on a 5–manifold can be homotoped to a contact structure.

**Proof of Theorem 3.1** Let \((\eta, J)\) be any almost contact structure on the closed oriented 5–manifold \(M\). Let \((Y, \pi)\) be an open book decomposition for \(M\) with pages having no handles of index larger than two. Theorem 3.2 allows us to assume that there is a Morse function \(f\) on a page of \((Y, \pi)\) so that \(f\) and \(f \circ \phi\), where \(\phi\) is the monodromy of \((Y, \pi)\), can be connected by a family of functions as described in the theorem. By Theorem 3.6 there is an overtwisted contact structure \(\xi\) on \(Y\) such that the hyperplane field \(H(\xi)\) is homotopic to \(\eta\) and using the homotopy we can think of \(\eta\) as \(H(\xi)\) and \(J\) as a complex structure on \(H(\xi)\). Thus \(J\) induces an almost complex structures \(J_0\) on the pages \(X_\theta = \pi^{-1}(\theta), \theta \in S^1\) of \((Y, \pi)\).

We break \(M\) into several parts. Let \(N = Y \times D^2\) be a neighborhood of the binding \(Y\) in \(M\), as in Theorem 3.6. Let \(C = M \setminus N\) be the complement of the interior of \(N\). We can think of \(\pi\) as a fibration \(C \rightarrow S^1\) (that extends over \(N - Y\) as projection onto the \(\theta\)-coordinate of \(D^2\)). By a slight abuse of notation we will refer to the fibers of \(\pi: C \rightarrow S^1\) as \(X_\theta\) and \(J_0\) will be the almost complex
structures on $X_{\theta}$. Letting $X = X_{\theta_0}$ for some fixed $\theta_0 \in S^1$ we can write

$$C = X \times [0,1]/(x,1) \sim (\phi(x),0)$$

for some diffeomorphism $\phi: X \to X$ that is the identity near the boundary of $X$. According to Theorem 4.4 we can write $X$ as a union of $X'$ and $X''$ where $X'$ is a neighborhood of the fixed index 0 critical point of $f: X \to \mathbb{R}$ and $X'' = X \setminus X'$ is the complement of the interior of $X'$; moreover, $\phi$ is the identity on $X'$. Thus we may decompose $C$ as

$$C = C' \cup C''$$

where $C' = D^4 \times S^1$ is the mapping torus of $\phi|_{X'}$ and $C''$ is the mapping torus of $\phi|_{X''}$.

Glancing at Steps 1 and 2 below will convince the reader that is relatively easy to create a contact structure on $C$ (using Proposition 2.3 for $C'$ and Theorem 4.3 for $C''$). However, it is not easy to extend this over $N$. To achieve this extension we need to be more careful in Steps 1 and 2. For this reason we need to further decompose $X''$. More specifically, we will show below that there is a cylinder $W' = D^3 \times [0,1] \times S^1$ in $X''$ that breaks $X''$ into two relative cobordisms $X'' = W' \cup W$ where $W$ is a relative cobordism from $D^3$ to $Y \setminus B$ (where $B$ is a ball in $Y$). Moreover $\phi$ is the identity on $X'$ and $W'$, the complex structure $J$ restricted to $X'$ is standard and restricted to $W'$ is compatible with (part of) the symplectization of the standard contact structure on the 3–ball $D^3$. Moreover we have a Morse function $f: X'' \to \mathbb{R}$ that satisfies the properties in Theorem 4.4 and has a gradient like vector field that is non-zero on $W'$ and tangent to $\partial W' \cap \partial W$.

We begin by constructing a model situation. Let $B$ be the ball of radius 1 in $\mathbb{C}^2$ with the standard complex structure. The complex tangencies to $S^3 = \partial B$ give the standard contact structure $\xi_{std}$ on $S^3$. Let $D^3$ be a small ball in $S^3 = \partial B$ on which the contact structure is standard (i.e. a Darboux ball). Gluing $D \times [1,2]$ to $B$ where $D \times \{1\}$ is identified with $D \subset \partial B$ gives another ball $R$ and using a portion of the symplectization of $(\xi_{std})_B$ we can extend the standard symplectic structure and complex structure on $B$ to all of $R$. Denote this complex structure $J$.

We now carefully construct the decomposition $C' \cup C''$ on $C$. There is some neighborhood $U$ of $\partial X$ such that $\phi$ is the identity map on $U$ and hence each page $X_{\theta}$ has a corresponding neighborhood $U_{\theta}$ that can be identified with $U$. In particular, $U' = \cup_{\theta} U_{\theta}$ is a neighborhood of $\partial C$ in $C$ and is diffeomorphic to $U \times S^1$. We can assume that $U \cong Y \times [0,2]$ and the construction in Theorem 3.6 allows us to assume that under the identifications of $U$ with $U_{\theta}$ the complex structure $J_{\theta}$ is independent of $\theta$ on $U$. Again referring to the proof of Theorem 3.5 $Y \times \{t\}$ in each $U_{\theta}$ is $J_{\theta}$ convex and the $J_{\theta}$ complex tangencies to $Y \times \{t\}$ are $\xi$ (recall that $\xi$ is the overtwisted contact structure on $Y$ such that $\eta = H(\xi)$).

Looking at the proof of Theorem 4.4 we can assume the ball $X'$ is a small neighborhood of a point in $U'$. Moreover we can take an arc $\gamma$ from $\partial X'$ to $\partial X$ that is transverse to all the $Y \times \{t\}$ in $U$. Clearly there is a diffeomorphism from $R$, constructed above, and $X'$ union a neighborhood of $\gamma$ so that $B$ maps to $X'$ and $D \times [1,2]$ maps to a neighborhood of $\gamma$. We can also arrange that $D \times \{t\}$ maps to a ball in $Y \times \{t\}$ and by (a parametric version of) Darboux’s theorem the $J'$–complex tangencies to $D \times \{t\}$ in $R$ map to the $J$–complex tangencies to the ball in $Y \times \{t\}$. We can now homotope, if necessary, the complex structure $J$ so that it agrees with $J'$ on the image of $R$ and the complex tangencies to $Y \times \{t\}$ are not changed for $t \in (1,2]$ (but we might change the complex tangencies on $[0,1]$). Since $J$ is $\theta$–invariant in $U'$ we can clearly arrange this on each page $X_{\theta}$. We now set $W = X' \setminus R$ and $W' = D \times [1,2]$ to get the desired decomposition of $X''$ and notice that Theorem 4.4 still gives the desired Morse function.

**Step 1:** The contact structure on $C'$. Proposition 2.3 gives a contact structure $\zeta$ on $C' = D^4 \times S^1$ such that $\zeta$ induces the overtwisted contact structure $\xi_{ot}$ on $\partial D^4 \times \{\theta\}$ for each $\theta \in S^1$. The proposition also gives the following specific form for the contact structure near $\partial C'$: if $\alpha_{ot}$ is a 1–form for which $\xi_{ot} = \text{ker} \alpha_{ot}$ then in a neighborhood $S^3 \times (1/2,1] \times S^1$ of the boundary of $D^4 \times S^1$ the contact structure is given by

$$\zeta = \text{ker}(K d\theta + t \alpha_{ot})$$
where \( t \) is the coordinate on \((1/2, 1)\), \( \theta \) is the coordinate on \(S^1\), and \( K \) is any positive constant. Proposition 2.3 allows us to arrange that \( \ker(\alpha_{ot}) \) is standard on the disk \( D^3 \) where \( W' \) intersects \( \partial X' \).

We note that \( \zeta_{ot} \) is homotopic to \( \zeta_{std} \) on \( S^3 \) relative to the disk \( D^3 \) and thus there is a homotopy of the almost complex structure on \((D^4 = X') \cup X''\), relative to \( W' \), so that \( \zeta \cap (\partial D^4 \times \{\theta\}) \) is the set of complex tangencies to \( \partial D^4 = \partial_\ast X'' \). By Proposition 2.3, \( \zeta \) is homotopic, through almost contact structures, to \( \ker(Kd\theta + \lambda) \), where \( \lambda \) is a primitive for the standard symplectic form on \( D^4 \). And this contact structure is homotopic to the almost contact structure \( \eta' \) coming from the tangencies to \( B^4 \) (with the almost complex structure above).

**Step 2: Extending the contact structure to \( C'' \).** Recall that
\[
C'' = X'' \times [0, 1]/(x, 1) \sim (\phi(x), 0).
\]
In addition \( X'' = W' \cup W \) and we can think of \( W' \) as part of the symplectization of the standard contact structure on a 3–ball \( D^3 \), that is \( W' = D^3 \times [1, 2] \) and if \( \alpha \) is a contact form for the standard contact structure on \( D^3 \) then \( d(\alpha) \) is a symplectic form on \( W' \) and it is compatible with the almost complex structure \( J \). Thus we see that \( W' \times S^1 \subset C'' \) already has a contact structure given by \( \ker(Kd\theta + \alpha) \). Note also that \( (d(t\alpha), v = \partial_t, f|_{W'}) \) gives \( W' \) the structure of a relative Weinstein cobordism.

We are left to consider
\[
W \times [0, 1]/(x, 1) \sim (\phi(x), 0).
\]
We will denote \( W \times \{t\} \) by \( W_t \) and the almost complex structure on \( W_t \) by \( J_t \).

Recall that \( f \) and \( f \circ \phi \) are connected by a family of functions as in Theorem 4.4. Thinking of \( f \) as a Morse function on \( W_0 \) and \( f \circ \phi \) as a Morse function on \( W_1 \) the family of functions from Theorem 4.4 can be thought of as giving functions \( f_t : W_t \to \mathbb{R} \). (We are of course considering the restriction of \( f \) to \( W \), but leave this out of the notation for convenience.)

Recall that \( J_t \) are all the same near the lower boundary \( \partial_\ast W_t \) so the complex tangencies on \( S^3 = \partial_\ast X_t \) give the standard overtwisted contact structure \( \zeta_{ot} \). In addition all the \( J_t \) are the same near the vertical boundary \( \partial_t W \). By extending \( W \) into \( W' \) and \( X' \) slightly we can assume that there is a neighborhood of \( \partial_\ast W \) that has the structure of a Weinstein cobordism and there is a neighborhood of \( \partial_t W \) that has the structure of a Weinstein cobordism.

Recall that for any almost complex structure the space of compatible non-degenerate 2–forms is contractible. Thus we can use the almost complex structures \( J_t \) to find a path \( \beta_t \) of non-degenerate 2–forms on \( W_t \) such that \( \beta_0 = \phi^* \beta_0 \). Moreover we can assume that \( \beta_t \) agrees with the 2–forms defining the Weinstein cobordism structures on the neighborhoods of \( \partial_\ast W_t \) and \( \partial_t W_t \).

We now claim that there are vector fields \( v_i \) on \( W_t , i = 0, 1 \) such that \( (\phi^{-1})_* v_1 = v_0 \) and we can assume that \( (\beta_0, v_0, f_0) \) and \( (\beta_1, v_1, f_1) \) are flexible Weinstein structures on \( W_0 \) and \( W_1 \), respectively.

To see this we first note that via homotopy we can assume that \( \beta_t \) and \( f_t \) are fixed near \( t = 0 \) and \( t = 1 \). Now recall that Theorem 1.2 says that there is a homotopy of \( \beta_0 \) to a symplectic form \( \omega \) and there exists a vector field \( v \) such that \( (\omega, v, f_0) \) is a flexible relative Weinstein cobordism structure on \( W_0 \). We can now extend \( W \times [0, 1] \) to, say, \( W \times [-\epsilon, 1] \), for some \( \epsilon > 0 \), and use the homotopy of 2–forms between \( \beta_0 \) and \( \omega \) to define non-degenerate 2–forms on \( W_t \) for \( t \in [-\epsilon, 0] \). We can also extend \( f_t \) so that it is constant on \( [-\epsilon, 0] \). Moreover pulling back the homotopy of 2–forms via \( \phi \) we can also extend our \( \beta_t \) and \( f_t \) over \( [1, 1 + \epsilon] \). We now have \( (\beta_t, f_t) \) defined for all \( t \in [-\epsilon, 1 + \epsilon] \) so that the structures at the endpoints, together with \( \phi \) and \( \phi_* \phi \) define relative Weinstein cobordism structures on \( W \). Moreover all the properties from Theorem 4.4 for the functions \( f_t \) are unchanged and \( W \times [-\epsilon, 1 + \epsilon] \) can be used to recover \( C'' \). Thus re-parameterizing \( [-\epsilon, 1 + \epsilon] \) to \([0, 1]\) establishes the claim. (Notice that the homotopy of the 2–forms can be taken to be fixed in a neighborhood of \( \partial_\ast W \cup \partial_t W \).)

Noting that \( f_t, t \in [0, 1] \), is a Morse homotopy without critical points of index greater than 2, we can now apply Gielisbeck and Eliashberg’s theorem, Theorem 1.3 above, to get a path \( (\omega_t, v_t, f_t) \), \( t \in [0, 1] \), of Weinstein structures such that \( \omega_t \) is homotopic to \( \beta_t \) relative to the end points and a neighborhood of \( \partial_\ast W \) and \( \partial_t W \). Now set \( \lambda_t = v_t \omega_t \) and consider the 1–form
\[
\alpha = Kdt + \lambda_t
\]
on $W$, where $K$ is some constant. Note
\[ \alpha \wedge d\alpha \wedge d\alpha = K\, dt \wedge d\lambda_t \wedge d\lambda_t + 2\, dt \wedge \frac{d\lambda_t}{dt} \wedge \lambda_t \wedge d\lambda_t. \]

So for any $K$ sufficiently large, $\alpha$ defines a contact from on $W \times [0,1]$ that extends the one on $W' \times [0,1]$ and clearly descends to a contact from on $C''$. Since we have a precise formula for the contact form near $\partial - X \times S^1 \subset C''$ that agrees with the one near $\partial C' \subset C'$ we can clearly extend the contact form from Step 1 over $C''$ using $\alpha$. We denote the associated contact structure $\zeta$. Moreover the contact structure $\zeta$ is clearly homotopic through almost contact structures to the tangents to the pages of the open book on $C''$ (just let the constant $K$ go to infinity).

We wish to use Eliashberg’s computation of the homotopy type of overtwisted contact structures, Theorem 2.5 To this end we study the loop of contact structures $\xi_\theta = \ker(\lambda_\theta)$ on $Y$. First notice that the homotopy of 1–forms from $\beta_t$ compatible with the $J_t$, to the symplectic forms $\omega_t$, can be covered by a homotopy of almost complex structures $(J_t)_s, s \in [0,1]$. We can arrange that the complex structures $(J_t)_s$ that are compatible with $\omega_t$ also have $\xi_\theta$ as the set of complex tangencies to $Y = \partial_s X''$.

The original almost complex structures on $X_\theta$ induced the constant loop of plane fields $\xi$ on $Y$ as the complex tangencies to $\partial_s X_\theta'' = Y$. The homotopy of almost complex structures on $(J_t)_s$ induces a homotopy of this loop to $\xi_\theta = \ker(\lambda_\theta)$. Moreover this loop is constant on the ball $\partial_s W' \subset Y$. Thus $\xi_\theta$ is homotopic to a constant loop of plane fields in the space $\mathcal{P}_p(Y)$ (for any $p \in \partial_s W'$). (We are using the notation from Section 2.4.) To apply Theorem 2.5 we need to see that we can assume that $\xi_\theta$ have a fixed overtwisted disk through $p$.

**Step 3:** Create a fixed overtwisted disk in each fiber of $\partial C$. We may identify a neighborhood $N''$ of $\partial C$ in $C$ with $(1/2, 1] \times Y \times S^1$. Under this identification the contact form $\alpha$ can be written $K\, dt + s\alpha_\theta$, where $s$ is the coordinate on $(1/2, 1], \theta$ is the coordinate on $S^1$ and $\alpha_\theta$ is a contact form on $Y$ for each $\theta$. Moreover in $D^3 = \partial_s W' \subset Y$ the contact forms are independent of $\theta$ and induce the standard contact structure on the ball. Let $S$ be a solid torus in $D^3$ that contains the point $p$ discussed in the previous paragraph and let $T^2 \times [0,1]$ be a neighborhood of $\partial S$ that does not contain $p$. We use the contact structure constructed in Lemma 2.4 to remove $(1/2, 1] \times T^2 \times [0,1] \times S^1$ from $C$ and replace it with the contact structure on $(1/2, 1] \times T^2 \times [0,1] \times S^1$ described in the lemma. In particular we can assume that $S$ is a “Lutz tube” and, more to the point, there is a disk $D$ in $D^3$ that is overtwisted in the contact structures $\xi'_\theta$ induced on $Y$ thought of as the boundary of $X_\theta$ with this new contact structure on $C$. Each $\xi'_\theta$ differs from $\xi_\theta$ by a Lutz twist. And by the discussion in Section 2.4 we see that the loop $\xi'_\theta$ is homotopic to $\xi_\theta$ in $\mathcal{P}_p(Y)$. This completes Step 3. For convenience we rename $\xi'_\theta$ to $\xi_\theta$.

**Step 4:** Construct a contact structure on $N$. From the previous step we know that $\xi_\theta = \ker \alpha_\theta$ is a loop of contact structures on $Y$ with a fixed overtwisted disk that is contractible in $\mathcal{P}_p(Y)$. Thus according to Eliashberg’s result, Theorem 2.5 the loop is contractible through contact structures. That is for each $r \in [0,1]$ there is a loop $\alpha_\theta^r$ of contact forms on $Y$ such that $\alpha_\theta^1 = \alpha_\theta$ and $\alpha_\theta^0$ is independent of $\theta$. Moreover, we can assume that $\alpha_\theta^r$ is independent of $r$ near 0 and near 1. Consider the 1–form
\[ \beta = f(r)\, d\theta + g(r)\, \alpha_\theta^r \]
on $Y \times D^2$. One can compute that
\[ \beta \wedge d\beta \wedge d\beta = g(f'g - g'f)\, dr \wedge d\theta \wedge \alpha_\theta^r \wedge d\alpha_\theta^r - g^2\, dr \wedge d\theta \wedge \frac{\partial \alpha_\theta^r}{\partial r} \wedge \left(f\, d\alpha - g\, \alpha \wedge \frac{\partial \alpha_\theta^r}{\partial \theta}\right). \]

Where $\alpha_\theta^r$ is independent of $r$ the last term vanishes. Thus as long as $g$ is positive and $(f'g - g'f) > 0$, $\beta$ will be a contact from. Where $\alpha_\theta^r$ does depend on $r$ we will take $g$ to be constant and $f(r) = e^{cr}$. If $g$ is any fixed constant and $c$ is chosen sufficiently large, $\beta$ will be a contact form in this region too.

Specifically, let $0 < r_1 < r_2 < 1$ be constants such that $\alpha_\theta^r$ is independent of $r$ outside $[r_1, r_2]$. Let $r_3 < r_4$ be any numbers in $(r_2, 1)$. We can choose $f(r)$ to be $r^2$ near 0, $e^{cr}$ on $[r_1, r_2]$, strictly
increasing on $[0, r_4]$ and some large constant $K$ on $[r_4, 1]$. We also choose $g(r)$ to be 2 on $[0, r_3]$, strictly decreasing on $[r_3, 1]$ and equal to $2 - r$ near 1.

If $D_a$ is the disk of radius $a$ then we clearly see that $\beta$ defines a contact structure on $Y \times D_a$ for any $a > 1$.

Using the coordinates $(1/2, 1] \times Y \times S^1$ for a neighborhood of $\partial C$ in $C$ from above, we can glue $Y \times D_a$ to $C$ by a diffeomorphism

$$
\Psi : \{(p, r, \theta) : 1 < r < a\} \to (1/2, 1] \times Y \times S^1 : (p, r, \theta) \mapsto (2 - r, p, \theta).
$$

It is clear that $\Psi$ pulls the contact from from Step 2 back to the one constructed in above. Thus $\Psi$ can be used to glue the contact structures on $N$ and $C$ together, thus extending $\zeta$ on $C$ over $N$.

**Step 5:** See that $\zeta$ is homotopic to $(\eta, J)$. Notice in the construction above $\zeta$ is clearly homotopic to $(\eta, J)$ along a page of the open book $(Y, \pi)$. Thus, just as in the proof of Theorem 3.6 we see that $(\eta, J)$ and $\zeta$ are homotopic as almost contact structures. (Recall this is simply because almost contact structures on a 5–manifolds are determined up to homotopy by their restriction to the 2–skeleton.)

**Remark 5.1.** We note that in [14] Giroux defines the notion of a contact structure being supported by an open book. While we use open books to construct our contact structures, none of the contact structures coming from Theorem 5.6 are supported by the open book used in the proof of the theorem. In particular, it is possible that the given open book does not support any almost contact structures.

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