Abstract

We define the compatibility JSJ tree of a group $G$ over a class of subgroups. It exists whenever $G$ is finitely presented and leads to a canonical tree (not just a deformation space) which is invariant under automorphisms. Under acylindricity hypotheses, we prove that the (usual) JSJ deformation space and the compatibility JSJ tree both exist when $G$ is finitely generated, and we describe their flexible subgroups. We apply these results to CSA groups, $\Gamma$-limit groups (allowing torsion), and relatively hyperbolic groups.

1 Introduction

Though self-contained, this paper is a sequel to [GL09]. In that first paper we gave a general definition of the JSJ deformation space $D_{JSJ}$ of a finitely generated group $G$ over a class of subgroups $\mathcal{A}$, by means of a universal maximality property; this definition agrees with the constructions given by Rips-Sela, Dunwoody-Sageev, Fujiwara-Papasoglu [RS97, DS99, FP06] in various contexts. We showed that the JSJ deformation space always exists when $G$ is finitely presented (without any assumption on $\mathcal{A}$).

We also explained that in general the JSJ decomposition is not a single tree (or graph of groups) but a deformation space, i.e. a collection of trees all having the same elliptic subgroups. A trivial example: if $G$ is free, its JSJ deformation space over any $\mathcal{A}$ is Culler-Vogtmann’s (unprojectivized) outer space.

The first main goal of this paper is to introduce another type of JSJ decomposition, which also exists whenever $G$ is finitely presented, and does lead to a well-defined tree $T_{co}$: being canonical, this tree is in particular invariant under any automorphism of $G$ (provided that $\mathcal{A}$ is). This construction is based on compatibility, and we call $T_{co}$ the compatibility JSJ tree. It is similar to the canonical tree constructed by Scott and Swarup [SS03], it dominates it and it may be non-trivial when Scott and Swarup’s decomposition is trivial.

The second main goal is to use acylindricity to construct and describe the JSJ deformation space and the JSJ compatibility tree in various situations, for instance when $G$ is a CSA group, or a $\Gamma$-limit group with $\Gamma$ a hyperbolic group (possibly with torsion), or a relatively hyperbolic group. Being based on acylindrical accessibility, these constructions only require that $G$ be finitely generated.

Compatibility

Given two trees $T_1, T_2$ (always with an action of $G$ and edge stabilizers in a given class $\mathcal{A}$), there does not always exist a map $f : T_1 \rightarrow T_2$ (maps are always assumed to be $G$-equivariant). A necessary and sufficient condition is that every subgroup of $G$ which is elliptic in $T_1$ (i.e. fixes a point in $T_1$) also fixes a point in $T_2$. We then say that $T_1$ dominates $T_2$. They belong to the same deformation space if $T_1$ dominates $T_2$ and $T_2$ dominates $T_1$ (i.e. they have the same elliptic subgroups). Domination is a partial order on the set of deformation spaces.
A tree $T_1$ is elliptic with respect to $T_2$ if all edge stabilizers of $T_1$ are elliptic in $T_2$. As in [GL09], we say that $T_1$ is universally elliptic if it is elliptic with respect to every tree. The JSJ deformation space $D_{JSJ}$ is defined as the maximal deformation space (for domination) containing a universally elliptic tree. Its elements are JSJ trees, i.e. universally elliptic trees which dominate all universally elliptic trees.

To define compatibility, we impose restrictions on maps between trees. A collapse map $f : T_1 \to T_2$ is a map consisting in collapsing edges in certain orbits to points (in terms of graphs of groups, collapsing certain edges of the graph). We then say that $T_2$ is a collapse of $T_1$, and $T_1$ is a refinement of $T_2$.

Two trees $T_1, T_2$ are compatible if they have a common refinement: a tree $\hat{T}$ such that there exist collapse maps $\hat{T} \to T_1$. The standard example is the following: splittings of a hyperbolic surface group associated to two simple closed geodesics $C_1, C_2$ are compatible if and only if $C_1$ and $C_2$ are disjoint or equal.

Compatibility implies ellipticity, as $T_1$ is elliptic with respect to $T_2$ if and only if there exists $\hat{T}$ with maps $f_i : \hat{T} \to T_i$, where $f_1$ is a collapse map but $f_2$ is arbitrary.

We now mimic the definition of the JSJ deformation space given in [GL09], using compatibility instead of ellipticity.

**Definition 1.1.** A tree is universally compatible if it is compatible with every tree. The compatibility JSJ deformation space $D_{co}$ is the maximal deformation space (for domination) containing a universally compatible tree, if it exists.

**Theorem 1.2.** Let $G$ be finitely presented, and let $A$ be any conjugacy-invariant class of subgroups of $G$, stable under taking subgroups. Then the compatibility JSJ deformation space $D_{co}$ of $G$ over $A$ exists.

As mentioned above, the deformation space $D_{co}$ contains a preferred element $T_{co}$ (except in degenerate cases).

**Theorem 1.3.** Assume that $D_{co}$ exists and is irreducible. If $A$ is invariant under automorphisms of $G$, then $D_{co}$ contains a tree $T_{co}$ which is invariant under automorphisms.

This is because a deformation space $D$ containing an irreducible universally compatible tree has a preferred element. We develop an analogy with arithmetic, viewing a refinement of $T$ as a multiple of $T$. We define the least common multiple (lcm) of a family of pairwise compatible trees, and $T_{co}$ is the lcm of the reduced universally compatible trees contained in $D_{co}$.

Being invariant under automorphisms sometimes forces $T_{co}$ to be trivial (a point). This happens for instance when $G$ is free. On the other hand, we give simple examples of virtually free groups, generalized Baumslag-Solitar groups, Poincaré duality groups, with $T_{co}$ non-trivial. For splittings over virtually cyclic groups (or more generally $VPC_n$ groups), $T_{co}$ dominates (sometimes strictly) the tree constructed by Scott and Swarup as a regular neighbourhood. When $G$ is a one-ended hyperbolic group, $T_{co}$ is very close to the tree constructed by Bowditch [Bow98] from the topology of $\partial G$.

More involved examples are given at the end of the paper, using acylindricity. Before moving on to this topic, let us say a few words about the proof of Theorem 1.2. Existence of the usual JSJ deformation space is a fairly direct consequence of accessibility [GL09], but proving existence of the compatibility JSJ deformation space is more delicate. Among other things, we use a limiting argument, and we need to know that a limit of universally compatible trees is universally compatible.

At this point we move into the world of $\mathbb{R}$-trees (simplicial or not), where collapse maps have a natural generalisation as maps preserving alignment: the image of any arc is an arc (possibly a point). Compatibility of $\mathbb{R}$-trees thus makes sense. Recall that the length function of an $\mathbb{R}$-tree $T$ with an isometric action of $G$ is the map $l : G \to \mathbb{R}$ defined by $l(g) = \min_{x \in T} d(x, gx)$.
Theorem 1.4. Two irreducible $\mathbb{R}$-trees $T_1, T_2$ with length functions $l_1, l_2$ are compatible if and only if $l_1 + l_2$ is the length function of an $\mathbb{R}$-tree.

As a warm-up, we give a proof of the following classical facts: a minimal irreducible $\mathbb{R}$-tree is determined by its length function; the equivariant Gromov topology and the axes topology (determined by length functions) agree on the space of irreducible $\mathbb{R}$-trees (following a suggestion by M. Feighn, we extend this to the space of semi-simple trees). Our proof does not use based length functions and extends to a proof of Theorem 1.4.

A useful corollary of Theorem 1.4 is that compatibility is a closed equivalence relation on the space of $\mathbb{R}$-trees. In particular, the space of $\mathbb{R}$-trees compatible with a given tree is closed.

Acylindricity

In the second part of this paper, we prove the existence of the usual JSJ deformation space $D_{JSJ}$, and the JSJ compatibility tree $T_{co}$, under some acylindricity conditions. Since these conditions are rather technical, we first describe some of the applications given in the third part.

Recall that a group $G$ is CSA if centralizers of non-trivial elements are abelian and malnormal. In order to deal with groups having torsion, we introduce $K$-CSA groups (for $K$ a fixed integer), and we show that, for every hyperbolic group $\Gamma$ (possibly with torsion), there exists $K$ such that $\Gamma$-limit groups are $K$-CSA. Also recall that a subgroup of a relatively hyperbolic group is elementary if it is parabolic or virtually cyclic. A group is small if it has no non-abelian free subgroup (see Subsection 2.3 for a better definition).

Theorem 1.5. Let $G$ be a finitely generated group, and $\mathcal{H}$ an arbitrary family of subgroups, with $G$ one-ended relative to $\mathcal{H}$. In the following situations, the JSJ deformation space $D_{JSJ}$ of $G$ over $A$ relative to $\mathcal{H}$, and the compatibility JSJ tree $T_{co}$, exist. Moreover, non-small flexible vertex stabilizers are the same for $T_{co}$ as for trees in $D_{JSJ}$; they are QH with finite fiber.

- $G$ is a torsion-free CSA group, $A$ is the class of abelian (resp. cyclic) subgroups.
- $G$ is a $\Gamma$-limit group (with $\Gamma$ a hyperbolic group), $A$ is the class of virtually abelian (resp. virtually cyclic) subgroups.
- $G$ is a relatively hyperbolic group with small parabolic groups, $A$ is the class of elementary (resp. virtually cyclic) subgroups.

Some explanations are in order. We work in a relative setting. This is important for applications (see e.g. [Pan04, Per09]), and also needed in our proofs. We therefore fix a (possibly empty) family $\mathcal{H}$ of subgroups of $G$, and we only consider trees relative to $\mathcal{H}$, i.e. trees in which every $H \in \mathcal{H}$ fixes a point. The family $\mathcal{H}$ is completely arbitrary (in [GL09] we needed $\mathcal{H}$ to consist of finitely many finitely generated subgroups).

We say that $G$ is one-ended relative to $\mathcal{H}$ if it does not split over a finite group relative to $\mathcal{H}$.

A vertex stabilizer of a JSJ decomposition is flexible if it is not universally elliptic: there is a tree in which it fixes no point. A key fact of JSJ theory is that flexible vertex stabilizers often are quadratically hanging (QH): assuming $G$ to be torsion-free for simplicity, this means in the context of Theorem 1.5 that flexible vertex stabilizers $G_v$ which contain a free group $F_2$ are fundamental groups of compact surfaces, with incident edge groups contained in boundary subgroups; moreover, the intersection of $G_v$ with a group conjugate to a group of $\mathcal{H}$ must also be contained in a boundary subgroup.
Let us now explain how acylindricity is used to prove Theorem 1.5. Recall [Sel97] that a tree is \( k \)-acylindrical if segments of length \( \geq k + 1 \) have trivial stabilizers (when \( G \) has torsion, it is convenient to weaken this definition by allowing long segments to have finite stabilizers; we neglect this issue in this introduction). Acylindrical accessibility [Sel97] gives a bound on the number of orbits of edges and vertices of \( k \)-acylindrical trees. However, one cannot use such a bound directly to prove the existence of JSJ decompositions.

It may happen that all trees under consideration are acylindrical (for instance, cyclic splittings of hyperbolic groups have this property). In this case we prove that JSJ decompositions exist by analyzing a limiting \( \mathbb{R} \)-tree using Sela’s structure theorem [Sel97].

But in general we have to deal with non-acylindrical trees. To prove Theorem 1.5, we have to be able to associate an acylindrical tree \( T^* \) to a given tree \( T \); in applications \( T^* \) is the (collapsed) tree of cylinders of \( T \) introduced in [GL08] (see Subsection 9.1). The tree \( T^* \) must be dominated by \( T \), and not too different from \( T \): groups which are elliptic in \( T^* \) but not in \( T \) should be small (they do not contain \( F_2 \)). We say that \( T^* \) is \textit{smallly dominated} by \( T \).

Figure 1: a JSJ splitting of a toral relatively hyperbolic group and its tree of cylinders

Here is an example (which already appears in [GL08]). Let \( T \) be the Bass-Serre tree of the graph of groups \( \Gamma \) pictured on the left of Figure 1 (all punctured tori have the same boundary subgroup, equal to the edge groups of \( \Gamma \)). It is not acylindrical, so we consider its tree of cylinders \( T^* \) (whose quotient graph of groups \( \Gamma^* \) is pictured on the right of Figure 1). The tree \( T^* \) is acylindrical, but \( \Gamma^* \) has a new (small) vertex stabilizer isomorphic to \( \mathbb{Z}^2 \). The tree \( T \) is a cyclic JSJ tree. The tree \( T^* \) is the cyclic compatibility JSJ tree of \( G \), it is also a cyclic JSJ tree relative to non-cyclic abelian subgroups.

The proof of Theorem 1.5 first consists in constructing a JSJ tree \( T_r \) relative to \( \mathcal{H} \) and to small subgroups which are not virtually cyclic. We then construct a JSJ tree \( T_a \) (only relative to \( \mathcal{H} \)) by refining \( T_r \) at vertices with small stabilizer. The tree \( T_r \) is the collapsed tree of cylinders of \( T_a \). We show that it is (very closely related to) the compatibility JSJ tree.

The paper is organized as follows. In Section 3 we prove Theorem 1.4 and we define lcm’s of compatible simplicial trees. Theorems 1.2 and 1.3 are proved in Section 4, and simple examples of compatibility JSJ trees are given in Section 5. In Sections 7 and 8 we construct JSJ trees and we describe their flexible subgroups, first under the assumption that trees are acylindrical, and then under the assumption that every tree smally dominates an acylindrical tree. We then recall the definition of the tree of cylinders and explain how this leads to small domination (Section 9). The compatibility properties of the tree of cylinders are used in Subsection 9.3 to construct the JSJ compatibility tree. In Sections 11 through 13 we apply the results of Part II to CSA (and \( K \)-CSA) groups and to relatively hyperbolic groups. In Section 14 we consider (virtually cyclic) splittings. In this case one may have to slightly refine \( T_r \) in order to get the compatibility JSJ tree. The last section explains how one can use JSJ trees to describe small actions on \( \mathbb{R} \)-trees. This gives another, more general, approach to the main result of [Gui00b].
2 Preliminaries

2.1 Simplicial trees

Let $G$ be a finitely generated group. We consider actions of $G$ on simplicial trees $T$. We usually assume that $G$ acts without inversion, and $T$ has no redundant vertex: if a vertex has valence 2, it is the unique fixed point of some element of $G$. We denote by $V(T)$ the set of vertices of $T$.

By Bass-Serre theory, the action of $G$ on $T$ can be viewed as a splitting of $G$ as a marked graph of groups $\Gamma$, i.e. an isomorphism between $G$ and the fundamental group of a graph of groups. A one-edge splitting (when $\Gamma$ has one edge) is an amalgam or an HNN-extension.

In order to do JSJ theory, we will fix a family $A$ of subgroups of $G$, which is stable under conjugation and under taking subgroups, and we only consider trees with edge stabilizers in $A$ (splittings over groups in $A$). We say that such a tree is an $A$-tree, or a tree over $A$.

We will also work in a relative situation. We consider an arbitrary family $\mathcal{H}$ of subgroups of $G$, and we restrict to $A$-trees in which every subgroup occurring in $\mathcal{H}$ fixes a point. We say that these trees are relative to $\mathcal{H}$, and we call them $(A, \mathcal{H})$-trees. If $\mathcal{H}$ is empty, one recovers the non-relative situation.

We say that $G$ splits relative to $\mathcal{H}$ if it acts non-trivially on a tree relative to $\mathcal{H}$. It is freely indecomposable (resp. one-ended) relative to $\mathcal{H}$ if it does not split over the trivial group (resp. over a finite group) relative to $\mathcal{H}$.

2.2 Metric trees

When endowed with a path metric making each edge isometric to a closed interval, a simplicial tree becomes an $\mathbb{R}$-tree (we usually declare each edge to have length 1). An $\mathbb{R}$-tree is a geodesic metric space $T$ in which any two distinct points are connected by a unique topological arc. We denote by $d$, or $d_T$, the distance in a tree $T$. All $\mathbb{R}$-trees are equipped with an isometric action of $G$, and considered equivalent if they are equivariantly isometric. The considerations below apply to actions on trees, simplicial or not. Non-simplicial trees will be needed in Subsection 7.1.

A branch point is a point $x \in T$ such that $T \setminus \{x\}$ has at least three components. A non-empty subtree is degenerate if it is a single point, non-degenerate otherwise. If $A, B$ are disjoint closed subtrees, the bridge between them is the unique arc $I = [a, b]$ such that $A \cap I = \{a\}$ and $B \cap I = \{b\}$.

We denote by $G_v, G_e$ the stabilizer of a point $v$ or an arc $e$. If $T$ is simplicial and $v$ is a vertex, the incident edge groups of $G_v$ are the subgroups $G_e \subset G_v$, where $e$ is an edge incident on $v$.

A subgroup $H < G$, or an element $g \in G$, is called elliptic if it fixes a point in $T$. We then denote by $\text{Fix } H$ or $\text{Fix } g$ its fixed point set. An element $g$ which is not elliptic is hyperbolic. It has an axis, a line on which it acts as a translation.

If $g \in G$, we denote by $\ell(g)$ its translation length $\ell(g) = \min_{x \in T} d(x, gx)$. The subset of $T$ where this minimum is achieved is the characteristic set $A(g)$: the fixed point set if $g$ is elliptic, the axis if $g$ is hyperbolic. The map $\ell : G \to \mathbb{R}$ is the length function of $T$; we denote it by $\ell_T$ if there is a risk of confusion. We say that a map $\ell : G \to \mathbb{R}$ is a length function if there is a tree $T$ such that $\ell = \ell_T$. 

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If $\ell$ takes values in $\mathbb{Z}$, the $\mathbb{R}$-tree $T$ is a simplicial tree.

The action of $G$ on $T$ (or $T$ itself) is *trivial* if there is a global fixed point (i.e. $G$ itself is elliptic). If the action is non-trivial, there is a hyperbolic element (so $\ell \neq 0$). This is a consequence of the following basic fact (recall that $G$ is assumed to be finitely generated).

**Lemma 2.1** (Serre’s Lemma [Ser77]). *If two elliptic elements $s_1, s_2$ satisfy $\text{Fix } s_1 \cap \text{Fix } s_2 = \emptyset$, then $s_1 s_2$ is hyperbolic.*

Let $s_1, \ldots, s_n$ be elliptic elements (or subgroups) such that $\text{Fix } s_i \cap \text{Fix } s_j \neq \emptyset$ for all $i, j$. Then $\langle s_1, \ldots, s_n \rangle$ fixes a point.

If the action is non-trivial, there is a smallest invariant subtree $T_{\text{min}}$, the union of all axes of hyperbolic elements. We always assume that the action is *minimal*, i.e. $T_{\text{min}} = T$.

The action of $G$ on $T$ (or $T$ itself) is *irreducible* if there exist hyperbolic elements $g, h$ such that $A(g) \cap A(h)$ is compact. This is equivalent to the existence of hyperbolic elements $g, h$ with $[g, h]$ hyperbolic. It implies that $G$ contains a non-abelian free group.

If a minimal action is neither trivial nor irreducible, there is a fixed end or an invariant line. More precisely, there are three possibilities:

- $T$ is a line, and $G$ acts by translations.
- $T$ is a line, and some $g \in G$ reverses orientation. The action is *dihedral*. In the simplicial case, the action factors through an action of the infinite dihedral group $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$.
- There is a unique invariant end (this happens in particular if $T$ is the Bass-Serre tree of a strictly ascending HNN extension). In this case the length function $\ell$ is *abelian*; it is the absolute value of a non-trivial homomorphism $\phi : G \to \mathbb{R}$ (whose image is infinite cyclic if $T$ is simplicial): in other words, $\ell$ is the length function of a non-trivial action on a line by translations.

These considerations apply to the action of a subgroup $H \subset G$. If $H$ is infinitely generated, its action may be non-trivial although every $h \in H$ is elliptic. In this case $T_{\text{min}}$ is not defined. There is a unique invariant end as in the third assertion of the trichotomy stated above, but $\phi = 0$.

### 2.3 Smallness

We will consider various classes of “small” subgroups of $G$. Each class contains the previous one.

- A group $H$ is VPC$_n$ (resp. VPC$_{\leq n}$) if it is virtually polycyclic of Hirsch length $n$ (resp. at most $n$). VPC$_{\leq 1}$ is the same as virtually cyclic.
- A group $H$ is *slender* if $H$ and all its subgroups are finitely generated. If a slender group acts non-trivially on a tree, there is an invariant line.
- Given a tree $T$, we say (following [BF91] and [GL07]) that a subgroup $H < G$ is *small in $T$* if its action on $T$ is not irreducible. As mentioned above, $H$ then fixes a point, or an end, or leaves a line invariant.

We say that $H$ is *small (in $G$)* if it is small in every simplicial tree on which $G$ acts. Every group not containing $F_2$ (the free group of rank 2) is small.

- If we fix $A$, and possibly $\mathcal{H}$, as above, we say that $H$ is *small in $A$-trees* (resp. *small in $(A, \mathcal{H})$-trees*) if it is small in every $A$-tree (resp. every $(A, \mathcal{H})$-tree) on which $G$ acts. These smallness properties are invariant under commensurability and under taking subgroups. Also note that a group contained in a group of $\mathcal{H}$ is small in $(A, \mathcal{H})$-trees.
2.4 Maps between trees, compatibility, deformation spaces

All maps between trees will be assumed to be $G$-equivariant. A map $f : T \to T'$ preserves alignment, or is a collapse map, if the image of any arc $[x, y]$ is a point or the arc $[f(x), f(y)]$. Equivalently, the preimage of every subtree is a subtree.

When $T$ and $T'$ are simplicial, we only consider maps obtained by collapsing edges to points. By equivariance, the set of collapsed edges is a union of $G$-orbits. In terms of graphs of groups, one obtains $T'/G$ by collapsing edges of $T/G$.

If there is a collapse map $f : T \to T'$, we say that $T'$ is a collapse of $T$, and $T$ is a refinement of $T'$. In the simplicial context, we say that $T$ is obtained by refining $T'$ at those vertices $v$ for which $f^{-1}(v)$ is not a single point.

Two trees $T_1, T_2$ are compatible if they have a common refinement: there exists a tree $\hat{T}$ with collapse maps $g_i : \hat{T} \to T_i$.

In the remainder of these preliminaries, we only consider simplicial trees (with an action of $G$) and we fix $A$ as above.

A tree $T_1$ dominates $T_2$ if there is an equivariant map from $T_1$ to $T_2$. Equivalently, $T_1$ dominates $T_2$ if every vertex stabilizer of $T_1$ fixes a point in $T_2$. Two $A$-trees belong to the same deformation space $D$ over $A$ if they have the same elliptic subgroups (i.e. each one dominates the other). If a tree in $D$ is irreducible, so are all others, and we say that $D$ is irreducible. We say that $D$ dominates $D'$ if trees in $D$ dominate those in $D'$. This induces a partial order on the set of deformation spaces of $G$ over $A$.

A tree $T$ is reduced [For02] if no proper collapse of $T$ lies in the same deformation space as $T$. Equivalently, $T$ is reduced if, for any edge $uv$ such that $\langle G_u, G_v \rangle$ is elliptic, there exists a hyperbolic element $g \in G$ sending $u$ to $v$ (in particular the edge maps to a loop in $T/G$). This may also be read at the level of the graph of groups: $T$ is not reduced if and only if $\Gamma = T/G$ contains an edge $e$ with distinct endpoints $\bar{u}, \bar{v}$ such that $i_e(G_v) = G_u$.

If $T$ is not reduced, one obtains a reduced tree $T'$ in the same deformation space by collapsing certain orbits of edges (but $T'$ is not uniquely defined in general).

2.5 Universal ellipticity and JSJ decompositions [GL09]

We fix $A$ and $H$ as above (with $H$ empty in the non-relative case). For instance, $A$ may be the family of subgroups which are finite, (virtually) cyclic, abelian, VPC$_{\infty}$ for some fixed $n$, slender. We refer to (virtually) cyclic, abelian, slender... splittings (or trees, or JSJ decompositions) when $A$ is the corresponding family of subgroups (we consider the trivial group as cyclic, so that $A$ is subgroup closed).

A tree $T$ is elliptic with respect to a tree $T'$ if edge stabilizers of $T$ are elliptic in $T'$. In this case there is a tree $\hat{T}$ (an $(A, H)$-tree if $T$ and $T'$ are $(A, H)$-trees) which refines $T$ and dominates $T'$ [GL09, Lemma 3.2].

A subgroup $H \subset G$ is universally elliptic (or $(A, H)$-universally elliptic if there is a risk of confusion) if $H$ is elliptic in every $(A, H)$-tree. A tree is universally elliptic if its edge stabilizers are universally elliptic.

A JSJ tree of $G$ over $A$ relative to $H$ is a universally elliptic $(A, H)$-tree $T_J$ which is maximal for domination: every universally elliptic $(A, H)$-tree is dominated by $T_J$. The set of all JSJ trees is a deformation space called the JSJ deformation space. JSJ trees always exist when $G$ is finitely presented and $H$ consists of finitely many finitely generated subgroups [GL09].

If $T_J$ is a JSJ tree, a vertex stabilizer $G_v$ of $T_J$ is rigid if it is universally elliptic. Otherwise, $G_v$ (or $v$) is flexible.

In many situations, flexible vertex stabilizers are quadratically hanging (QH) subgroups: there is a normal subgroup $F \triangleleft G_v$ (called the fiber of $G_v$) such that $G_v/F$ is isomorphic to the fundamental group $\pi_1(\Sigma)$ of a hyperbolic 2-orbifold $\Sigma$ (usually with boundary), and images of incident edge groups in $\pi_1(\Sigma)$ are either finite or contained.

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in a boundary subgroup (a subgroup conjugate to the fundamental group of a boundary component). When \( F \) is trivial and \( \Sigma \) is a surface, we say that \( G_v \) is a QH surface group.

An extended boundary subgroup of \( G_v \) is a group whose image in \( \pi_1(\Sigma) \) is finite or contained in a boundary subgroup.

In a relative situation, with \( T \) relative to \( H \), we define \( G_v \) to be a relative QH-subgroup if, additionally, every conjugate of a group in \( H \) intersects \( G_v \) in an extended boundary subgroup. Since incident edge groups are extended boundary subgroups, it is enough to check this for conjugates of \( H \) that are contained in \( G_v \).

If \( G_v \) is a relative QH-subgroup, we say that a boundary component \( C \) of \( \Sigma \) is used if there exists an incident edge group, or a subgroup of \( G_v \) conjugate to a group of \( H \), whose image in \( \pi_1(\Sigma) \) is contained with finite index in \( \pi_1(C) \).

Part I
Compatibility

3 Length functions and compatibility

The main result of this section is Theorem 3.7, saying that two \( \mathbb{R} \)-trees are compatible if and only if the sum of their length functions is again a length function. This has a nice consequence: the set of \( \mathbb{R} \)-trees compatible with a given tree is closed.

As a warm-up we give a proof of the following known facts: the length function of an irreducible \( \mathbb{R} \)-tree determines the tree up to equivariant isometry, and the Gromov topology agrees with the axes topology (determined by translation lengths).

After proving Theorem 3.7, we show that pairwise compatibility for a finite set of \( \mathbb{R} \)-trees implies the existence of a common refinement. We conclude by defining least common multiple (lcm’s) and prime factors for irreducible simplicial trees.

In Subsections 3.1 to 3.3, the countable group \( G \) does not have to be finitely generated (hypotheses such as irreducibility ensure that \( G \) contains enough hyperbolic elements). We leave details to the reader.

We will use the following facts:

**Lemma 3.1** ([Pau89]). Let \( T \) be an \( \mathbb{R} \)-tree with a minimal action of \( G \). Let \( g, h \) be hyperbolic elements.

1. If their axes \( A(g), A(h) \) are disjoint, then
   \[
   \ell(gh) = \ell(g^{-1}h) = \ell(g) + \ell(h) + 2d(A(g), A(h)) > \ell(g) + \ell(h).
   \]
   The intersection between \( A(gh) \) and \( A(hg) \) is the bridge between \( A(g) \) and \( A(h) \).

2. If their axes meet, then
   \[
   \min(\ell(gh), \ell(g^{-1}h)) \leq \max(\ell(gh), \ell(g^{-1}h)) = \ell(g) + \ell(h).
   \]
   The inequality is an equality if and only if the axes meet in a single point.

**Lemma 3.2** ([Pau89, Lemma 4.3]). If \( T \) is irreducible, any arc \([a, b]\) is contained in the axis of some \( g \in G \).

3.1 From length functions to trees

Let \( T \) be the set of minimal isometric actions of \( G \) on \( \mathbb{R} \)-trees modulo equivariant isometry. Let \( T_{\text{irr}} \subset T \) be the set of irreducible \( \mathbb{R} \)-trees. The following are classical results:
**Theorem 3.3** ([AB87, CM87]). Two minimal irreducible $\mathbb{R}$-trees $T, T'$ with the same length function are equivariantly isometric.

**Theorem 3.4** ([Pau89]). The equivariant Gromov topology and the axes topology agree on $\mathcal{T}_{irr}$.

By Theorem 3.3, the assignment $T \mapsto \ell_T$ defines an embedding $\mathcal{T}_{irr} \rightarrow \mathbb{R}^G$. The axes topology is the topology induced by this embedding. The equivariant Gromov topology on $T$ is defined by the following neighbourhood basis. Given $T \in \mathcal{T}$, a number $\varepsilon > 0$, a finite subset $A \subset G$, and $x_1, \ldots, x_n \in T$, define $N_{\varepsilon, A, (x_1, \ldots, x_n)}(T)$ as the set of trees $T' \in \mathcal{T}$ such that there exist $x'_1, \ldots, x'_n \in T'$ with $|d_{T'}(x'_i, ax'_j) - d_T(x_i, ax_j)| \leq \varepsilon$ for all $a \in A$ and $i, j \in \{1, \ldots, n\}$.

Theorem 3.4 should be viewed as a version with parameters of Theorem 3.3: the length function determines the tree, in a continuous way. As a preparation for the next subsection, we now give quick proofs of these theorems. Unlike previous proofs, ours does not use based length functions.

**Proof of Theorem 3.3.** Let $T, T'$ be minimal irreducible $\mathbb{R}$-trees with the same length function $\ell$. We denote by $A(g)$ the axis of a hyperbolic element $g$ in $T$, by $A'(g)$ its axis in $T'$. By Lemma 3.1, $A(g) \cap A(h)$ is empty if and only if $A'(g) \cap A'(h)$ is empty.

We define an isometric equivariant map $f$ from the set of branch points of $T$ to $T'$, as follows. Let $x$ be a branch point of $T$, and $y \neq x$ an auxiliary branch point. By Lemmas 3.1 and 3.2, there exist hyperbolic elements $g, h$ whose axes in $T$ do not intersect, such that $[x, y]$ is the bridge between $A(g)$ and $A(h)$, with $x \in A(g)$ and $y \in A(h)$. Then $\{x\} = A(g) \cap A(gh) \cap A(hg)$. The axes of $g$ and $h$ in $T'$ do not intersect, so $A'(g) \cap A'(gh) \cap A'(hg)$ is a single point which we call $f(x)$.

Note that $f(x) = \cap_k A'(k)$, the intersection being over all hyperbolic elements $k$ whose axis in $T$ contains $x$: if $k$ is such an element, its axis in $T'$ meets all three sets $A'(g)$, $A'(gh)$, $A'(hg)$, so contains $f(x)$. This gives an intrinsic definition of $f(x)$, independent of the choice of $y$, $g$, and $h$. In particular, $f$ is $G$-equivariant. It is isometric because $d_T(f(x), f(y))$ and $d_{T'}(x, y)$ are both equal to $1/2(\ell(gh) - \ell(g) - \ell(h))$.

We then extend $f$ equivariantly and isometrically first to the closure of the set of branch points of $T$, and then to each complementary interval. The resulting map from $T$ to $T'$ is onto because $T'$ is minimal.

**Proof of Theorem 3.4.** Given $g \in G$, the map $T \mapsto \ell_T(g)$, from $\mathcal{T}_{irr}$ to $\mathbb{R}$, is continuous in the Gromov topology: this follows from the formula $\ell(g) = \max(d(x, g^2x) - d(x, gx), 0)$. This shows that the Gromov topology is finer than the axes topology.

For the converse, we fix $\varepsilon > 0$, a finite set of points $x_i \in T$, and a finite set of elements $a_k \in G$. We have to show that, if the length function $\ell'$ of $T'$ is close enough to $\ell$ on a suitable finite subset of $G$, there exist points $x'_i \in T'$ such that $|d_{T'}(x'_i, a_k x'_j) - d_T(x_i, a_k x_j)| < \varepsilon$ for all $i, j, k$.

First assume that each $x_i$ is a branch point. For each $i$, choose elements $g_i, h_i$ as in the previous proof, with $x_i$ an endpoint of the bridge between $A(g_i)$ and $A(h_i)$. If $\ell'$ is close to $\ell$, the axes of $g_i$ and $h_i$ in $T'$ are disjoint and we can define $x'_i$ as $A'(g_i) \cap A'(g_i h_i) \cap A'(h_i g_i)$. A different choice $\tilde{g}_i, \tilde{h}_i$ may lead to a different point $\tilde{x}'_i$. But the distance between $x'_i$ and $\tilde{x}'_i$ goes to 0 as $\ell'$ tends to $\ell$ because all pairwise distances between $A'(g_i), A'(g_i h_i), A'(h_i g_i), A'(\tilde{g}_i), A'(\tilde{g}_i h_i), A'(\tilde{h}_i g_i)$ go to 0. It is then easy to complete the proof.

If some of the $x_i$'s are not branch points, one can add new points so that each such $x_i$ is contained in a segment bounded by branch points $x_{b_i}, x_{c_i}$. One then defines $x'_i$ as the point dividing $[x'_{b_i}, x'_{c_i}]$ in the same way as $x_i$ divides $[x_{b_i}, x_{c_i}]$. 

\[\square\]
As suggested by M. Feighn, one may extend the previous results to reducible trees. Let $T_{ss}$ consist of all minimal trees which are either irreducible or isometric to $\mathbb{R}$ (we only rule out trivial trees and trees with exactly one fixed end, see Subsection 2.2); these trees are called semi-simple in [CM87]. Every non-zero length function is the length function of a tree in $T_{ss}$, and the set of length functions is projectively compact in $\mathbb{R}^G$ [CM87, Theorem 4.5].

**Theorem 3.5.** Two minimal trees $T, T' \in T_{ss}$ with the same length function are equivariantly isometric. The equivariant Gromov topology and the axes topology agree on $T_{ss}$.

In other words, the assignment $T \mapsto \ell_T$ induces a homeomorphism between $T_{ss}$, equipped with the equivariant Gromov topology, and the space of non-zero length functions.

We note that the results of [CM87] are stated for all trees in $T_{ss}$, those of [Pau89] for trees which are irreducible or dihedral.

**Proof.** We refer to [CM87, page 586] for a proof of the first assertion in the reducible case. Since the set of irreducible length functions is open, it suffices to show the following fact:

**Claim 3.6.** If $T_n$ is a sequence of trees in $T_{ss}$ whose length functions $\ell_n$ converge to the length function $\ell$ of an action of $G$ on $T = \mathbb{R}$, then $T_n$ converges to $T$ in the Gromov topology.

To prove the claim, we denote by $A_n(g)$ the characteristic set of $g \in G$ in $T_n$, and we fix $h \in G$ hyperbolic in $T$ (hence in $T_n$ for $n$ large). We denote by $I_n(g)$ the (possibly empty or degenerate) segment $A_n(g) \cap A_n(h)$.

The first case is when $G$ acts on $T$ by translations. To show that $T_n$ converges to $T$, it suffices to show that, given elements $g_1, \ldots, g_k$ in $G$, the length of $\bigcap_n I_n(g_i)$ goes to infinity with $n$. By a standard argument using Helly’s theorem, we may assume $k = 2$.

We first show that, for any $g$, the length $|I_n(g)|$ goes to infinity. Let $N \in \mathbb{N}$ be arbitrary. Since $g^{-1}h^Ng^{-N}$ is elliptic in $T$, the distance between $I_n(h^Ng^{-N})$ and $I_n(g)$ goes to 0 as $n \to \infty$. But $I_n(h^Ng^{-N})$ is the image of $I_n(g)$ by $h^N$, so $\liminf_{n \to \infty} |I_n(g)| \geq N \ell(h)$.

To show that the overlap between $I_n(g_1)$ and $I_n(g_2)$ goes to infinity, we can assume that the relative position of $I_n(g_1)$ and $I_n(g_2)$ is the same for all $n$. If they are disjoint, $I_n(g_1g_2^{-1})$ or $I_n(g_2g_1^{-1})$ is empty, a contradiction. Since every $I_n(g)$ goes to infinity, the result is clear if $I_n(g_1)$ and $I_n(g_2)$ are nested. In the remaining case, up to changing $g_i$ to its inverse, we can assume that $g_1, g_2$ translate in the same direction along $A_n(h)$ if they are both hyperbolic. Then $I_n(g_1) \cap I_n(g_2)$ equals $I_n(g_1g_2^{-1})$ or $I_n(g_2g_1^{-1})$, so its length goes to infinity.

Now suppose that the action of $G$ on $T$ is dihedral. Suppose that $g \in G$ reverses orientation on $T$. For $n$ large, the axes of $h, g^{-1}hg, ghg^{-1}$ in $T_n$ have a long overlap by the previous argument. On this overlap $h$ translates in one direction, $g^{-1}hg$ and $ghg^{-1}$ in the other (because $\ell_n(hg^{-1}hg)$ is close to 0 and $\ell_n(h^{-1}g^{-1}hg)$ is not). It follows that $g$ acts as a central symmetry on a long subsegment of $A_n(h)$. Moreover, if $g, g'$ both reverse orientation, the distance between their fixed points on $A_n(h)$ is close to $2\ell(gg')$.

The convergence of $T_n$ to $T$ easily follows from these observations. This proves the claim, hence the theorem.

### 3.2 Compatibility and length functions

Recall that two $\mathbb{R}$-trees $T_1, T_2$ are compatible if they have a common refinement: there exists an $\mathbb{R}$-tree $\tilde{T}$ with (equivariant) collapse maps $g_i : \tilde{T} \to T_i$ (see Subsection 2.4).

If $T_1$ and $T_2$ are compatible, they have a standard common refinement $T_s$ constructed as follows.
We denote by $d_i$ the distance in $T_i$, and by $\ell_i$ the length function. Let $\hat{T}$ be any common refinement. Given $x, y \in T$, define

$$\delta(x, y) = d_1(g_1(x), g_1(y)) + d_2(g_2(x), g_2(y)).$$

This is a pseudo-distance satisfying $\delta(x, y) = \delta(x, z) + \delta(z, y)$ if $z \in [x, y]$ (this is also a length measure, as defined in [Gui00a]). The associated metric space $(T_s, d)$ is an $\mathbb{R}$-tree which refines $T_1$ and $T_2$, with maps $f_i : T_s \to T_i$ satisfying $d(x, y) = d_1(f_1(x), f_1(y)) + d_2(f_2(x), f_2(y))$.

The length function of $T_s$ is $\ell = \ell_1 + \ell_2$ (this follows from the formula $\ell(g) = \lim_{n \to \infty} \frac{1}{n} d(x, g^nx)$). In particular, $\ell_1 + \ell_2$ is a length function. We now prove the converse.

**Theorem 3.7.** Two minimal irreducible $\mathbb{R}$-trees $T_1, T_2$ with an action of $G$ are compatible if and only if the sum $\ell = \ell_1 + \ell_2$ of their length functions is a length function.

**Remark 3.8.** If $T_1$ and $T_2$ are compatible, then $\lambda_1 l_1 + \lambda_2 l_2$ is a length function for all $\lambda_1, \lambda_2 \geq 0$.

**Corollary 3.9.** Compatibility is a closed relation on $T_{irr} \times T_{irr}$. In particular, the set of irreducible $\mathbb{R}$-trees compatible with a given $T_0$ is closed in $T_{irr}$.

**Proof.** This follows from the fact that the set of length functions is a closed subset of $\mathbb{R}^G$ [CM87].

**Proof of Theorem 3.7.** We have to prove the “if” direction. Let $T_1, T_2$ be irreducible minimal $\mathbb{R}$-trees with length functions $\ell_1, \ell_2$, such that $\ell = \ell_1 + \ell_2$ is the length function of a minimal $\mathbb{R}$-tree $T$. We denote by $A(g), A_1(g), A_2(g)$ axes in $T$, $T_1, T_2$ respectively.

First note that $T$ is irreducible: hyperbolic elements $g, h$ with $[g, h]$ hyperbolic exist in $T$ since they exist in $T_1$. We want to prove that $T$ is a common refinement of $T_1$ and $T_2$. In fact, we show that $T$ is the standard refinement $T_s$ mentioned earlier (which is unique by Theorem 3.3). The proof is similar to that of Theorem 3.3, but we first need a few lemmas.

**Lemma 3.10.** [[Gui05, lemme 1.3]] Let $S \subset G$ be a finitely generated semigroup such that no point or line in $T$ is invariant under the subgroup $\langle S \rangle$ generated by $S$. Let $I$ be an arc contained in the axis of a hyperbolic element $h \in S$.

Then there exists a finitely generated semigroup $S' \subset S$ with $\langle S' \rangle = \langle S \rangle$ such that every element $g \in S' \setminus \{1\}$ is hyperbolic in $T$, its axis contains $I$, and $g$ translates in the same direction as $h$ on $I$.

**Lemma 3.11.** Let $T_1, T_2, T$ be arbitrary irreducible minimal trees. Given an arc $I \subset T$, there exists $g \in G$ which is hyperbolic in $T_1, T_2$ and $T$, and whose axis in $T$ contains $I$.

**Proof.** Apply Lemma 3.10 with $S = G$ and any $h$ whose axis in $T$ contains $I$. Since $S'$ generates $G$, it must contain an element $h'$ which is hyperbolic in $T_1$: otherwise $G$ would have a global fixed point in $T_1$ by Serre’s lemma (Lemma 2.1). Applying Lemma 3.10 to the action of $S'$ on $T_1$, we get a semigroup $S'' \subset S'$ whose non-trivial elements are hyperbolic in $T_1$. Similarly, $S''$ contains an element $g$ which is hyperbolic in $T_2$. This element $g$ satisfies the conclusions of the lemma.

**Remark 3.12.** More generally, one may require that $g$ be hyperbolic in finitely many trees $T_1, \ldots, T_n$.

**Lemma 3.13.** Let $g, h$ be hyperbolic in $T_1$ and $T_2$ (and therefore in $T$).

- If their axes in $T$ meet, do so their axes in $T_i$. 

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• If their axes in $T$ do not meet, their axes in $T_i$ meet in at most one point. In particular, the elements $gh$ and $hg$ are hyperbolic in $T_i$.

Proof. Assume that $A(g)$ and $A(h)$ meet, but $A_1(g)$ and $A_1(h)$ do not. Then $\ell_1(gh) > \ell_1(g) + \ell_1(h)$. Since $\ell(gh) \leq \ell(g) + \ell(h)$, we get $\ell_2(gh) < \ell_2(g) + \ell_2(h)$. Similarly, $\ell_2(g^{-1}h) < \ell_2(g) + \ell_2(h)$. But these inequalities are incompatible by Lemma 3.1.

Now assume that $A(g)$ and $A(h)$ do not meet, and $A_1(g), A_1(h)$ meet in a non-degenerate segment. We may assume $\ell_1(gh) < \ell_1(g^{-1}h) = \ell_1(g) + \ell_1(h)$. Since $\ell(gh) = \ell(g^{-1}h) > \ell(g) + \ell(h)$, we have $\ell_2(gh) > \ell_2(g^{-1}h) > \ell_2(g) + \ell_2(h)$, contradicting Lemma 3.1.

We can now complete the proof of Theorem 3.7. It suffices to define maps $f_i : T \to T_i$ such that $d(x, y) = d_1(f_1(x), f_1(y)) + d_2(f_2(x), f_2(y))$. Such maps are collapse maps (if three points satisfy a triangular equality in $T$, then their images under $f_i$ cannot satisfy a strict triangular inequality), so $T$ is the standard common refinement $T_s$.

The construction of $f_i$ is the same as that of $f$ in the proof of Theorem 3.3. Given branch points $x$ and $y$, we use Lemma 3.11 to get elements $g$ and $h$ hyperbolic in all three trees, and such that the bridge between $A(g)$ and $A(h)$ is $[x, y]$. Then Lemma 3.13 guarantees that $A_i(g) \cap A_i(hg) \cap A_i(h) = \{x, y\}$ is a single point of $T_i$, which we define as $f_i(x)$; the only new phenomenon is that $A_i(g)$ and $A_i(h)$ may now intersect in a single point.

The relation between $d, d_1, d_2$ comes from the equality $\ell = \ell_1 + \ell_2$, using the formula $d_2(f_i(x), f_i(y)) = 1/2(\ell_i(gh) - \ell_i(g) - \ell_i(h))$. Having defined $f_i$ on branch points, we extend it by continuity to the closure of the set of branch points of $T$ (it is 1-Lipschitz) and then linearly to each complementary interval. The relation between $d, d_1, d_2$ still holds. \qed

3.3 Common refinements

The following result is proved for almost-invariant sets in [SS03, Theorem 5.16].

Proposition 3.14. Let $G$ be a finitely generated group, and let $T_1, \ldots, T_n$ be irreducible minimal $\mathbb{R}$-trees such that $T_i$ is compatible with $T_j$ for $i \neq j$. Then there exists a common refinement $T$ of all $T_i$’s.

Remark 3.15. This statement may be interpreted as the fact that the set of projectivized trees satisfies the flag condition for a simplicial complex: whenever one sees the 1-skeleton of an $n$-simplex, there is indeed an $n$-simplex. Two compatible trees $T_i, T_j$ define a 1-simplex $\ell_i + (1 - \ell)\ell_j$ of length functions. If there are segments joining any pair of length functions $\ell_i, \ell_j$, the proposition says that there is an $(n - 1)$-simplex $\sum t_i\ell_i$ of length functions.

To prove Proposition 3.14, we need some terminology from [Gui05]. A direction in an $\mathbb{R}$-tree $T$ is a connected component $\delta$ of $T \setminus \{x\}$ for some $x \in T$. A quadrant in $T_1 \times T_2$ is a product $Q = \delta_1 \times \delta_2$ of a direction of $T_1$ by a direction of $T_2$. A quadrant $Q = \delta_1 \times \delta_2$ is heavy if there exists $h \in G$ hyperbolic in $T_1$ and $T_2$ such that $\delta_i$ contains a positive semi-axis of $h$ (equivalently, for all $x \in T_i$ one has $h^n(x) \in \delta_i$ for $n$ large). We say that $h$ makes $Q$ heavy. The core $C(T_1 \times T_2) \subset T_1 \times T_2$ is the complement of the union of quadrants which are not heavy.

By [Gui05, Théorème 6.1], $T_1$ and $T_2$ are compatible if and only if $C(T_1 \times T_2)$ contains no non-degenerate rectangle (a product $I_1 \times I_2$ where each $I_i$ is a segment not reduced to a point).

We first prove a technical lemma.

Lemma 3.16. Let $T_1, T_2$ be irreducible and minimal. Let $f : T_1 \to T'_1$ be a collapse map, with $T'_1$ irreducible. Let $\delta'_1 \times \delta_2$ be a quadrant in $T'_1 \times T_2$, and $\delta_1 = f^{-1}(\delta'_1)$. If the quadrant $\delta_1 \times \delta_2 \subset T_1 \times T_2$ is heavy, then so is $\delta'_1 \times \delta_2$. 

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Note that $\delta_1$ is a direction because $f$ preserves alignment.

Proof. Consider an element $h$ making $\delta_1 \times \delta_2$ heavy. If $h$ is hyperbolic in $T'_i$, then $h$ makes $\delta'_1 \times \delta'_2$ heavy and we are done. If not, assume that we can find some $g \in G$, hyperbolic in $T'_i$ and $T_2$ (hence in $T_1$), such that for $i = 1, 2$ the axis $A_i(g)$ of $g$ in $T_i$ intersects $A_i(h)$ in a compact set. Then for $n > 0$ large enough the element $h^ngh^{-n}$ makes $\delta_1 \times \delta_2$ heavy. Since this element is hyperbolic in $T'_1$ and $T'_2$, it makes $\delta'_1 \times \delta'_2$ heavy.

We now prove the existence of $g$. Consider a line $l$ in $T_2$, disjoint from $A_2(h)$, and the bridge $[x, y]$ between $l$ and $A_2(h)$. Let $I \subset l$ be a segment containing $x$ in its interior. By Lemma 3.11, there exists $g$ hyperbolic in $T'_1$ and $T_2$ whose axis in $T_2$ contains $I$, hence is disjoint from $A_2(h)$. Being hyperbolic in $T'_1$, the element $g$ is hyperbolic in $T_1$. Its axis intersects $A_1(h)$ in a compact set because $A_1(h)$ is mapped to a single point in $T'_1$ (otherwise, $h$ would be hyperbolic in $T'_1$). \hfill \Box

Proof of Proposition 3.14. First assume $n = 3$. Let $T_{12}$ be the standard common refinement of $T_1, T_2$ (see Subsection 3.2). Let $C$ be the core of $T_{12} \times T_3$. By [Gui05, Théorème 6.1], it is enough to prove that $C$ does not contain a product of non-degenerate segments $[a_{12}, b_{12}] \times [a_3, b_3]$. Assume otherwise. Denote by $a_1, b_1, a_2, b_2$ the images of $a_{12}, b_{12}$ in $T_1, T_2$. Since $a_{12} \neq b_{12}$, at least one inequality $a_1 \neq b_1$ or $a_2 \neq b_2$ holds. Assume for instance $a_1 \neq b_1$.

We claim that $[a_1, b_1] \times [a_3, b_3]$ is contained in the core of $T_1 \times T_3$, giving a contradiction. We have to show that any quadrant $Q = \delta_1 \times \delta_3$ of $T_1 \times T_3$ intersecting $[a_1, b_1] \times [a_3, b_3]$ is heavy. Denote by $f : T_{12} \to T_1$ the collapse map. The preimage $\tilde{Q} = f^{-1}(\delta_1) \times \delta_3$ of $Q$ in $T_{12} \times T_3$ is a quadrant intersecting $[a_{12}, b_{12}] \times [a_3, b_3]$. Since this rectangle in contained in $C(T_{12} \times T_3)$, the quadrant $\tilde{Q}$ is heavy, and so is $Q$ by Lemma 3.16. This concludes the case $n = 3$. The general case follows by a straightforward induction. \hfill \Box

3.4 Arithmetic of trees

In this subsection, we work with simplicial trees. We let $S_{\text{irr}}$ be the set of simplicial trees $T$ which are minimal, irreducible, with no redundant vertices and no inversion. We also view such a $T$ as a metric tree, by declaring each edge to be of length 1. This makes $S_{\text{irr}}$ a subset of $T_{\text{irr}}$. By Theorem 3.3, a tree $T \in S_{\text{irr}}$ is determined by its length function $\ell$.

Definition 3.17. The prime factors of $T$ are the one-edge splittings $T_i$ obtained from $T$ by collapsing edges in all orbits but one. Clearly $\ell = \sum_i \ell_i$, where $\ell_i$ is the length function of $T_i$.

We may view a prime factor of $T$ as an orbit of edges of $T$, or as an edge of the quotient graph of groups $\Gamma = T/G$. Since $G$ is assumed to be finitely generated, there are finitely many prime factors (this remains true if $G$ is only finitely generated relative to a finite collection of elliptic subgroups).

Lemma 3.18. Let $T \in S_{\text{irr}}$.

1. Any non-trivial tree $T'$ obtained from $T$ by collapses (in particular, its prime factors) belongs to $S_{\text{irr}}$.

2. The prime factors of $T$ are distinct ($T$ is “squarefree”).

Proof. Let $e$ be any edge of $T$ which is not collapsed in $T'$. Since $T$ has no redundant vertex and is not a line, either the endpoints of $e$ are branch points $u, v$, or there are branch points $u, v$ such that $[u, v] = e \cup e'$ with $e'$ in the same orbit as $e$. Using Lemma 3.2, we can find elements $g, h$ hyperbolic in $T$, whose axes are not collapsed to points in $T'$, and such that $[u, v]$ is the bridge between their axes. Since $g, h$ are hyperbolic with
disjoint axes in $T'$, the tree $T'$ is irreducible. It is easy to check that collapsing cannot create redundant vertices, so $T' \in S_{irr}$.

Now suppose that $e$, hence $[u, v]$, gets collapsed in some prime factor $T''$. Then $\ell'(gh) > \ell'(g) + \ell'(h)$ holds in $T'$ but not in $T''$, so $T' \neq T''$. □

Remark 3.19. This lemma shows that collapsing an irreducible simplicial tree yields an irreducible tree (or a point). Collapsing a non-irreducible tree clearly yields a minimal non-irreducible tree belonging to the same deformation space (or a point).

Because of this lemma, a tree of $S_{irr}$ is determined by its prime factors. In particular, $T$ refines $T'$ if and only if every prime factor of $T'$ is also a prime factor of $T$.

If $T_1$ and $T_2$ are compatible, the standard refinement $T_s$ constructed in Subsection 3.2 is a metric tree which should be viewed as the “product” of $T_1$ and $T_2$. We shall now define the lcm $T_1 \lor T_2$ of simplicial trees $T_1$ and $T_2$. To understand the difference between the two, suppose $T_1 = T_2$. Then $T_s$ is obtained from $T_1$ by subdividing each edge, its length function is $2\ell_1$. On the other hand, $T_1 \lor T_1 = T_1$.

Definition 3.20. Consider two trees $T_1, T_2 \in S_{irr}$, with length functions $\ell_1, \ell_2$. We define $\ell_1 \land \ell_2$ as the sum of all length functions which appear as prime factors in both $T_1$ and $T_2$. It is the length function of a tree $T_1 \land T_2$ (possibly a point) which is a collapse of both $T_1$ and $T_2$. We call $T_1 \land T_2$ the gcd of $T_1$ and $T_2$.

We define $\ell_1 \lor \ell_2 = \ell_1 + \ell_2 - \ell_1 \land \ell_2$ as the sum of all length functions which appear as prime factors in $T_1$ or $T_2$ (or both).

Lemma 3.21. Let $T_1$ and $T_2$ be compatible trees in $S_{irr}$. There is a tree $T_1 \lor T_2 \in S_{irr}$ whose length function is $\ell_1 \lor \ell_2$. It is a common refinement of $T_1$ and $T_2$, and no edge of $T_1 \lor T_2$ is collapsed in both $T_1$ and $T_2$.

We call $T_1 \lor T_2$ the lcm of $T_1$ and $T_2$.

Proof. Let $T$ be any common refinement. We modify it as follows. We collapse any edge which is collapsed in both $T_1$ and $T_2$. We then remove redundant vertices and restrict to the minimal subtree. The resulting tree $T_1 \lor T_2$ belongs to $S_{irr}$ (it is irreducible because it refines $T_1$). It is a common refinement of $T_1$ and $T_2$, and no edge is collapsed in both $T_1$ and $T_2$.

We check that $T_1 \lor T_2$ has the correct length function by finding its prime factors. Since no edge is collapsed in both $T_1$ and $T_2$, a prime factor of $T_1 \lor T_2$ is a prime factor of $T_1$ or $T_2$. Conversely, a prime factor of $T_i$ is associated to an orbit of edges of $T_i$, and this orbit lifts to $T_1 \lor T_2$. □

Proposition 3.22. Let $T_1, \ldots, T_n$ be pairwise compatible trees of $S_{irr}$. There exists a tree $T_1 \lor \cdots \lor T_n$ in $S_{irr}$ whose length function is the sum of all length functions which appear as a prime factor in some $T_i$. Moreover:

1. A tree $T \in S_{irr}$ refines $T_1 \lor \cdots \lor T_n$ if and only if it refines each $T_i$.

2. A tree $T \in S_{irr}$ is compatible with $T_1 \lor \cdots \lor T_n$ if and only if it is compatible with each $T_i$.

3. A subgroup $H$ is elliptic in $T_1 \lor \cdots \lor T_n$ if and only if it is elliptic in each $T_i$. If $T_1$ dominates each $T_i$, then $T_1 \lor \cdots \lor T_n$ belongs to the deformation space of $T_1$.

Definition 3.23. $T_1 \lor \cdots \lor T_n$ is the lcm of the compatible trees $T_i$. 

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Proof. First suppose \( n = 2 \). We show that \( T_1 \lor T_2 \) satisfies the additional conditions.

If \( T \) refines \( T_1 \) and \( T_2 \), it refines \( T_1 \lor T_2 \) because every prime factor of \( T_1 \lor T_2 \) is a prime factor of \( T \). This proves Assertion 1.

If \( T_1, T_2, T \) are pairwise compatible, they have a common refinement \( \hat{T} \) by Proposition 3.14 or [SS03, Theorem 5.16] (where one should exclude ascending HNN extensions). This \( \hat{T} \) refines \( T_1 \lor T_2 \) by Assertion 1, so \( T \) and \( T_1 \lor T_2 \) are compatible.

Assertion 3 follows from the fact that no edge of \( T_1 \lor T_2 \) is collapsed in both \( T_1 \) and \( T_2 \), as in the proof of [GL09, Lemma 3.2]: if \( H \) fixes a point \( v_1 \in T_1 \) and is elliptic in \( T_2 \), it fixes a point in the preimage of \( v_1 \) in \( T_1 \lor T_2 \).

The case \( n > 2 \) now follows easily by induction. By Assertion 2, the tree \( T_1 \lor \cdots \lor T_{n-1} \) is compatible with \( T_n \), so we can define \( T_1 \lor \cdots \lor T_n = (T_1 \lor \cdots \lor T_{n-1}) \lor T_n \). \( \square \)

4 The compatibility JSJ tree

We fix a family \( \mathcal{A} \) of subgroups which is stable under conjugation and under taking subgroups. We work with simplicial trees, which we often view as metric trees in order to apply the results from the previous section. We only consider \( \mathcal{A} \)-trees (trees with edge stabilizers in \( \mathcal{A} \)). Note that the lcm of a finite family of pairwise compatible \( \mathcal{A} \)-trees is an \( \mathcal{A} \)-tree.

In [GL09] we have defined a JSJ tree (of \( G \) over \( \mathcal{A} \)) as an \( \mathcal{A} \)-tree which is universally elliptic (over \( \mathcal{A} \)) and dominates every universally elliptic \( \mathcal{A} \)-tree. Its deformation space is the JSJ deformation space \( \mathcal{D}_{JSJ} \). In this section we define the compatibility JSJ deformation space and the compatibility JSJ tree. For simplicity we work in the non-relative case (see Subsection 4.3 for the relative case).

Definition 4.1. An \( \mathcal{A} \)-tree \( T \) is universally compatible (over \( \mathcal{A} \)) if it is compatible with every \( \mathcal{A} \)-tree.

If, among deformation spaces containing a universally compatible tree, there is one which is maximal for domination, it is unique. It is denoted by \( \mathcal{D}_{co} \) and it is called the compatibility JSJ deformation space of \( G \) over \( \mathcal{A} \).

Clearly, a universally compatible tree is universally elliptic. This implies that \( \mathcal{D}_{co} \) is dominated by \( \mathcal{D}_{JSJ} \).

In Subsection 4.2 we shall deduce from [GL07] that \( \mathcal{D}_{co} \), if irreducible, contains a preferred tree \( T_{co} \), which we call the compatibility JSJ tree. It is fixed under any automorphism of \( G \) that leaves \( \mathcal{D}_{co} \) invariant. In particular, if \( \mathcal{A} \) is Aut(\( G \))-invariant, then \( T_{co} \) is Out(\( G \))-invariant. Also note that \( T_{co} \) may be refined to a JSJ tree (Lemma 4.8 of [GL09]).

4.1 Existence of the compatibility JSJ space

Theorem 4.2. If \( G \) is finitely presented, the compatibility JSJ space \( \mathcal{D}_{co} \) of \( G \) (over \( \mathcal{A} \)) exists.

The heart of the proof of Theorem 4.2 is the following proposition.

Proposition 4.3. Let \( G \) be finitely presented. Let \( T_0 \leftarrow T_1 \cdots \leftarrow T_k \leftarrow \cdots \) be a sequence of refinements of irreducible universally compatible \( \mathcal{A} \)-trees. There exist collapses \( \overline{T}_k \) of \( T_k \), in the same deformation space as \( T_k \), such that the sequence \( \overline{T}_k \) converges to a universally compatible simplicial \( \mathcal{A} \)-tree \( T \) which dominates every \( T_k \).

Proof of Theorem 4.2 from the proposition. We may assume that there is a non-trivial universally compatible \( \mathcal{A} \)-tree. We may also assume that all such trees are irreducible: otherwise it follows from Remark 3.19 that there is only one deformation space of \( \mathcal{A} \)-trees, and the theorem is trivially true.
Let \((S_\alpha)_{\alpha \in A}\) be the set of universally compatible \(A\)-trees, up to equivariant isomorphism. We have to find a universally compatible \(A\)-tree \(T\) which dominates every \(S_\alpha\). By Lemma 3.9 of [GL09], we only need \(T\) to dominate all trees in a countable set \(S_k, k \in \mathbb{N}\). We obtain such a \(T\) by applying Proposition 4.3 with \(T_k = S_0 \vee \cdots \vee S_k\); the trees \(T_k\) are universally compatible by Assertion 2 of Proposition 3.22.

**Proof of Proposition 4.3.** By accessibility (see Proposition 4.4 of [GL09]), there exists an \(A\)-tree \(S\) which dominates every \(T_k\) (this is where we use finite presentability of \(G\)). But of course we cannot claim that it is universally compatible.

We may assume that \(T_k\) and \(S\) are minimal, that \(T_{k+1}\) is different from \(T_k\), and that \(S \cap T_k\) is independent of \(k\). We define \(S_k = S \cup T_k\). We denote by \(\Delta_k, \Gamma, \Gamma_k\) the quotient graphs of groups of \(T_k, S, S_k\), and we let \(\pi_k : \Gamma_k \to \Gamma\) be the collapse map (see Figure 2). We denote by \(\rho_k\) the collapse map \(\Gamma_{k+1} \to \Gamma_k\).

![Figure 2: The trees \(T_k, S, S_k, \overline{T}_k, S'_k\), with \(C_1 \supset C_2 \supset \cdots \supset C_k \supset C\), and \(B_k = B \ast_C C_k\).](image)

The trees \(S_k\) all belong to the deformation space of \(S\), and \(S_{k+1}\) strictly refines \(S_k\). In particular, the number of edges of \(\Gamma_k\) grows. Since accessibility holds within a given deformation space (see [GL07] page 147), this growth occurs through the creation of a bounded number of long segments whose interior vertices have valence 2, with one of the incident edge groups equal to the vertex group. We now make this precise.

Fix \(k\). For each vertex \(v \in \Gamma\), define \(Y_v = \pi^{-1}_k(\{v\}) \subset \Gamma_k\). The \(Y_v\)'s are disjoint, and edges of \(\Gamma_k\) not contained in \(\cup_v Y_v\) correspond to edges of \(\Gamma\).

Since \(S_k\) and \(S\) are in the same deformation space, \(Y_v\) is a tree of groups, and it contains a vertex \(c_v\) whose vertex group equals the fundamental group of \(Y_v\) (which is the vertex group of \(v\) in \(\Gamma\)). This \(c_v\) may fail to be unique, but we can choose one for every \(k\) in a way which is compatible with the maps \(\rho_k\). We orient edges of \(Y_v\) towards \(c_v\). The group carried by such an edge is then equal to the group carried by its initial vertex.

Say that a vertex \(u \in Y_v\) is **peripheral** if \(u = c_v\) or \(u\) is adjacent to an edge of \(\Gamma_k\) which is not in \(Y_v\) (i.e. is mapped onto an edge of \(\Gamma\) by \(\pi_k\)). By minimality of \(S_k\), each terminal vertex \(u_0\) of \(Y_v\) is peripheral (because it carries the same group as the initial edge of the segment \(u_0c_v\)).

In each \(\Gamma_k\), the total number of peripheral vertices is bounded by \(2|E(\Gamma)| + |V(\Gamma)|\). It follows that the number of points of valence \(\neq 2\) in \(\cup_v Y_v\) is bounded. Cutting each \(Y_v\) at its peripheral vertices and its points of valence \(\geq 3\) produces the segments of \(\Gamma_k\) mentioned earlier. On the example of Figure 2, there is one segment in \(\Gamma_k\), corresponding to the edges labelled \(C_1, \ldots, C_k\). The point \(c_v \in \Gamma_k\) is the vertex labelled by \(A\). The vertex \(v\) of \(\Gamma\) to which the segment corresponds is the vertex of \(\Gamma\) labelled by \(A\).

Having defined segments for each \(k\), we now let \(k\) vary. The preimage of a segment of \(\Gamma_k\) under the map \(\rho_k\) is a union of segments of \(\Gamma_{k+1}\). Since the number of segments is bounded independently of \(k\), we may assume that \(\rho_k\) maps every segment of \(\Gamma_{k+1}\) onto a segment of \(\Gamma_k\). In particular, the number of segments is independent of \(k\).
Recall that we have oriented the edges of \( Y_v \) towards \( e_v \). Each edge contained in \( \cup_v Y_v \) carries the same group as its initial vertex, and edges in a given segment are coherently oriented. Segments are therefore oriented.

There are various ways of performing collapses on \( \Gamma_k \). Collapsing all edges contained in segments yields \( \Gamma \) (this does not change the deformation space). On the other hand, one obtains \( \Delta_k = T_k / G \) from \( \Gamma_k \) by collapsing some of the edges which are not contained in any segment (all of them if \( S \cap T_k \) is trivial).

The segments of \( \Gamma_k \) may be viewed as segments in \( \Delta_k \), but collapsing the initial edge of a segment of \( \Delta_k \) may now change the deformation space (if the group carried by the initial point of the segment has increased when \( \Gamma_k \) is collapsed to \( \Delta_k \)).

We define a graph of groups \( \overline{\Delta}_k \) by collapsing, in each segment of \( \Delta_k \), all edges but the initial one. The corresponding tree \( \overline{T}_k \) is a collapse of \( T_k \) which belongs to the same deformation space as \( T_k \). Moreover, the number of edges of \( \overline{\Delta}_k \) (prime factors of \( \overline{T}_k \)) is constant: there is one per segment, and one for each common prime factor of \( T_k \) and \( S \).

Let \( \ell_k : G \to \mathbb{Z} \) be the length function of \( \overline{T}_k \).

**Lemma 4.4.** The sequence \( \ell_k \) is non-decreasing (i.e. every sequence \( \ell_k(g) \) is non-decreasing) and converges.

**Proof.** The difference between \( \ell_k \) and \( \ell_{k-1} \) comes from the fact that initial edges of segments of \( \Delta_k \) may be collapsed in \( \Delta_{k-1} \). Fix a segment \( L \) of \( \Delta_k \). Let \( e_k \) be its initial edge.

We assume that \( e_k \) is distinct from the edge \( f_k \) mapping onto the initial edge of the image of \( L \) in \( \Delta_{k-1} \).

Assume for simplicity that \( e_k \) and \( f_k \) are adjacent (the general case is similar). The group carried by \( f_k \) is equal to the group carried by its initial vertex \( v_k \). A given lift \( \tilde{v}_k \) of \( v_k \) to \( T_k \) is therefore adjacent to only one lift of \( f_k \) (but to several lifts of \( e_k \)). On any translation axis in \( T_k \), every occurrence of a lift of \( f_k \) is immediately preceded by an occurrence of a lift of \( e_k \). The length function of the prime factor of \( T_k \) and \( \overline{T}_{k-1} \) corresponding to \( f_k \) is therefore bounded from above by that of the prime factor of \( T_k \) and \( \overline{T}_k \) corresponding to \( e_k \). Since this is true for every segment, we get \( \ell_{k-1} \leq \ell_k \) as required.

Let \( S'_k = S \cup \overline{T}_k \). It collapses to \( S \), belongs to the same deformation space as \( S \) (because it is a collapse of \( S_k \)), and the number of edges of \( S'_k / G \) is bounded. By an observation due to Forester (see [GL07, p. 169]), this implies an inequality \( \ell(S'_k) \leq C \ell(S) \), with \( C \) independent of \( k \). Since \( \ell_k \leq \ell(S'_k) \), we get convergence. \( \square \)

We call \( \ell \) the limit of \( \ell_k \). It is the length function of a tree \( T \) because the set of length functions of trees is closed [CMS87]. This tree is simplicial because \( \ell \) takes values in \( \mathbb{Z} \), and irreducible because \( \ell_k \) is non-decreasing. It is universally compatible as a limit of universally compatible trees, by Corollary 3.9. Since \( \ell \geq \ell_k \), every \( g \in G \) elliptic in \( T \) is elliptic in \( T_k \), and \( T \) dominates \( T_k \) by Lemma 3.6 of [GL09].

There remains to prove that every edge stabilizer \( G_e \) of \( T \) belongs to \( \mathcal{A} \). If \( G_e \) is finitely generated, there is a simple argument using the Gromov topology. In general, we argue as follows. We may find hyperbolic elements \( g, h \) such that \( G_e \) is the stabilizer of the bridge between \( A(g) \) and \( A(h) \) (the bridge might be \( e \cup e' \) as in the proof of Lemma 3.18 if an endpoint of \( e \) is a valence 2 vertex). Choose \( k \) so that the values of \( \ell_k \) and \( \ell \) coincide on \( g, h, gh \). In particular, the axes of \( g \) and \( h \) in \( \overline{T}_k \) are disjoint.

Any \( s \in G_e \) is elliptic in \( \overline{T}_k \) since \( \ell_k \leq \ell \). Moreover, \( \ell_k(gs) \leq \ell(gs) \leq \ell(g) = \ell_k(g) \).

The fixed point set of \( s \) in \( \overline{T}_k \) must intersect the axis of \( g \), since otherwise \( \ell_k(gs) > \ell_k(g) \), a contradiction. Similarly, it intersects the axis of \( h \). It follows that \( G_e \) fixes the bridge between the axes of \( g \) and \( h \) in \( \overline{T}_k \), so \( G_e \in \mathcal{A} \). This concludes the proof of Proposition 4.3. \( \square \)
4.2 The compatibility JSJ tree $T_{co}$

When $D_{co}$ is irreducible, we show that it contains a preferred tree $T_{co}$.

Given a deformation space $D$, we say that a tree $T$ is $D$-compatible if $T$ is compatible with every tree in $D$.

**Lemma 4.5.** An irreducible deformation space $D$ can only contain finitely many reduced $D$-compatible trees.

Recall (see Subsection 2.4) that $T$ is reduced if no proper collapse of $T$ lies in the same deformation space as $T$. If $T$ is not reduced, one may perform collapses so as to obtain a reduced tree $T'$ in the same deformation space (note that $T'$ is $D$-compatible if $T$ is).

**Proof.** This follows from results in [GL07]. We refer to [GL07] for definitions not given here. Suppose there are infinitely many reduced $D$-compatible trees $T_1, T_2, \ldots$. Let $S_k = T_1 \lor T_2 \lor \cdots \lor T_k$. It belongs to $D$ by Assertion 3 of Proposition 3.22.

As pointed out on page 172 of [GL07], the tree $S_k$ is BF-reduced, i.e. reduced in the sense of [BF91], because all its edges are surviving edges (they survive in one of the $T_i$’s), and the space $D$ is non-ascending by Assertion 4 of Proposition 7.1 of [GL07]. Now there is a bound $C_D$ for the number of orbits of edges of a BF-reduced tree in $D$ ([GL07, Proposition 4.2]; this is an easy form of accessibility, which requires no smallness or finite presentability hypothesis). It follows that the sequence $S_k$ is eventually constant.

**Corollary 4.6.** If $D$ is irreducible and contains a $D$-compatible tree, it has a preferred element: the lcm of its reduced $D$-compatible trees.

**Definition 4.7.** If the compatibility JSJ deformation space $D_{co}$ exists and is irreducible, its preferred element is called the compatibility JSJ tree $T_{co}$ of $G$ (over $A$). If $D_{co}$ is trivial, we define $T_{co}$ as the trivial tree (a point).

It may happen that $D_{co}$ is neither trivial nor irreducible. It then follows from Remark 3.19 that it is the only deformation space of $A$-trees. If there is a unique reduced tree $T$ in $D_{co}$ (in particular, if $D_{co}$ consists of actions on a line), we define $T_{co} = T$. Otherwise we do not define $T_{co}$. See Subsection 5.5 for an example where $D_{co}$ consists of trees with exactly one fixed end.

4.3 Relative splittings

Besides $A$, the set of allowed edge stabilizers, we now fix an arbitrary set $H$ of subgroups of $G$ and we only consider $(A, H)$-trees, i.e. $A$-trees in which every element of $H$ is elliptic. We then define the compatibility deformation space $D_{co}$ as in the non-relative case (with universal compatibility defined with respect to $(A, H)$-trees).

**Theorem 4.8.** If $G$ is finitely presented, and $H$ is a finite family of finitely generated subgroups, the compatibility JSJ space $D_{co}$ of $G$ (over $A$ relative to $H$) exists. If $D_{co}$ is irreducible, it has a preferred element $T_{co}$.

Existence is proved as in the non-relative case, using relative accessibility [GL09, Section 5.1]. The trees $S, S_k, T_k, \bar{T}_k$ are relative to $H$ (note that an lcm of $(A, H)$-trees is an $(A, H)$-tree). Since the groups in $H$ are finitely generated, they are elliptic in $T$ because all their elements are elliptic, so $T$ is an $(A, H)$-tree.

**Remark 4.9.** The theorem remains true if $G$ is only finitely presented relative to $H$. 

5 Examples

5.1 Free groups

When \( \mathcal{A} \) is \( \text{Aut}(G) \)-invariant, the compatibility JSJ tree \( T_{\text{co}} \) is \( \text{Out}(G) \)-invariant. This sometimes forces it to be trivial. Suppose for instance that \( G \) has a finite generating set \( a_i \) such that all elements \( a_i \) and \( a_i a_j^{\pm 1} \) \((i \neq j)\) belong to the same \( \text{Aut}(G) \)-orbit. Then the only \( \text{Out}(G) \)-invariant length function \( \ell \) is the trivial one. This follows from Serre’s lemma (Lemma 2.1) if the generators are elliptic, from the inequality \( \max(\ell(a_i a_j), \ell(a_i a_j^{-1})) \geq \ell(a_i) + \ell(a_j) \) (see Lemma 3.1) if they are hyperbolic. In particular:

**Proposition 5.1.** If \( G \) is a free group and \( \mathcal{A} \) is \( \text{Aut}(G) \)-invariant, then \( T_{\text{co}} \) is trivial. \( \Box \)

5.2 Algebraic rigidity

The following result provides simple examples with \( T_{\text{co}} \) non-trivial.

**Proposition 5.2.** Assume that there is only one reduced JSJ tree \( T_J \in \mathcal{D}_{JSJ} \), and that \( G \) does not split over a subgroup contained with infinite index in a group of \( \mathcal{A} \). Then \( T_{\text{co}} \) exists and equals \( T_J \).

**Proof.** Let \( T \) be any \( \mathcal{A} \)-tree. The second assumption implies that \( T \) is elliptic with respect to \( T_J \) by Remark 2.3 of [FP06] or Corollary 7.11 of [GL09], so there is a tree \( \hat{T} \) refining \( T \) and dominating \( T_J \) as in Lemma 3.2 of [GL09]. Collapse all edges of \( \hat{T} \) whose stabilizer is not universally elliptic. Arguing as in the proof of [GL09, Proposition 8.1], one sees that the collapsed tree \( T' \) also dominates \( T_J \), hence is a JSJ tree because it is universally elliptic. Since \( T_J \) is the unique reduced JSJ tree, \( T' \) is a refinement of \( T_J \), so \( T_J \) is compatible with \( T \). This shows that \( T_J \) is universally compatible. Thus \( T_{\text{co}} = T_J \). \( \Box \)

A necessary and sufficient condition for a tree to be the unique reduced tree in its deformation space is given in [Lev05] (see also [CF09]).

The proposition applies for instance to free splittings and splittings over finite groups, whenever there is a JSJ tree with only one orbit of edges. This provides examples of virtually free groups with \( T_{\text{co}} \) non-trivial: any amalgam \( F_1 \ast_F F_2 \) with \( F_1, F_2 \) finite and \( F \neq F_1, F_2 \) has this property (with \( \mathcal{A} \) the set of finite subgroups).

5.3 Free products

Let \( \mathcal{A} \) consist only of the trivial group. Let \( G = G_1 \ast \cdots \ast G_p \ast F_q \) be a Grushko decomposition \((G_i \) is non-trivial, not \( \mathbb{Z} \), and freely indecomposable, \( F_q \) is free of rank \( q \)). If \( p = 2 \) and \( q = 0 \), or \( p = q = 1 \), there is a JSJ tree with one orbit of edges and \( T_{\text{co}} \) is a one-edge splitting as explained above. We now show that \( T_{\text{co}} \) is trivial if \( p + q \geq 3 \) (of course it is trivial also if \( G \) is freely indecomposable or free of rank \( \geq 2 \)).

Assuming \( p + q \geq 3 \), we actually show that there is no non-trivial tree \( T \) with trivial edge stabilizers which is invariant under a finite index subgroup of \( \text{Out}(G) \). By collapsing edges, we may assume that \( T \) only has one orbit of edges. Since \( p + q \geq 3 \), we can write \( G = A \ast B \ast C \) where \( A, B, C \) are non-trivial and \( A \ast B \) is a vertex stabilizer of \( T \). Given a non-trivial \( c \in C \) and \( n \neq 0 \), the subgroup \( c^n Ac^{-n} \ast B \) is the image of \( A \ast B \) by an automorphism but is not conjugate to \( A \ast B \). This contradicts the invariance of \( T \).

5.4 Hyperbolic groups

Let \( G \) be a one-ended hyperbolic group, and let \( \mathcal{A} \) consist of all virtually cyclic subgroups. It follows from [Gui00b] that the tree \( T_{\text{Bow}} \) constructed by Bowditch [Bow98] using the topology of \( \partial G \) is universally compatible, and refines any reduced tree in \( \mathcal{D}_{\text{co}} \). In particular,
such that $G$ is a maximal virtually cyclic subgroup fixing no other edge, and by removing redundant vertices.

5.5 (Generalized) Baumslag-Solitar groups

We consider cyclic splittings of generalized Baumslag-Solitar groups. These are groups which act on a tree with all edge and vertex stabilizers infinite cyclic. Unless $G$ is isomorphic to $\mathbb{Z}$, $\mathbb{Z}^2$, or the Klein bottle group, all such trees belong to the cyclic JSJ deformation space (see [For03] or Subsection 6.6 of [GL09]).

First consider a solvable Baumslag-Solitar group $BS(1, s)$, with $|s| \geq 2$. In this case $D_{\text{co}}$ is trivial if $s$ is not a prime power. If $s$ is a prime power, $D_{\text{co}}$ is the JSJ deformation space (it is reducible).

When $G = BS(r, s)$ with none of $r, s$ dividing the other, Proposition 5.2 applies by [Lev05]. In particular, $D_{\text{co}}$ is non-trivial. This holds, more generally, when $G$ is a generalized Baumslag-Solitar group defined by a labelled graph with no label dividing another label at the same vertex. See [Bee] for a study of $D_{\text{co}}$ for generalized Baumslag-Solitar groups.

5.6 The canonical decomposition of Scott and Swarup

Recall that a group is VPC if it is virtually polycyclic of Hirsch length $n$ (resp. $\leq n$). Let $G$ be a finitely presented group, and $n \geq 1$. Assume that $G$ does not split over a VPC subgroup, and that $G$ is not VPC. Let $A$ consist of all subgroups of VPC subgroups. Then $D_{\text{co}}$ dominates the Bass-Serre tree $T_{SS}$ of the regular neighbourhood $T_n = \Gamma(F_n : G)$ constructed by Scott-Swarup in Theorem 12.3 of [SS03]. This follows directly from the fact that $T_{SS}$ is universally compatible ([SS03, Definition 6.1(1)], or [GL08, Corollary 8.4] and [GL10, Theorem 4.1]).

The domination may be strict: if $G = BS(r, s)$, the tree $T_{SS}$ is always trivial but, as pointed out above, $D_{\text{co}}$ is non-trivial when none of $r, s$ divides the other.

5.7 Poincaré Duality groups

Let $G$ be a Poincaré duality group of dimension $n$ (see also work by Kropholler on this subject [Kro90]). Although such a group is not necessarily finitely presented, it is almost finitely presented [Wal04, Proposition 1.1], which is sufficient for Dunwoody's accessibility, so the JSJ deformation space and the compatibility JSJ deformation space exist. By [KR89b, Theorem A], if $G$ splits over a virtually solvable subgroup $H$, then $H$ is VPC. We therefore consider the family $A$ consisting of VPC subgroups.

By [KR89a, Corollary 4.3], for all VPC subgroups $H$, the number of coends $\tilde{c}(G, H)$ is 2 (see [SS03, p. 33] for a discussion of the relation between $\tilde{c}(G, H)$ and the number of coends, [Geo08, Section 14.5]). By [KR89a, Theorem 1.3], if $G$ is not virtually polycyclic, then $H$ has finite index in its commensurizer. By Corollary 8.4(2) of [GL08], this implies that the JSJ deformation space contains a universally compatible tree (namely its tree of cylinders, see Subsection 9.1 or [GL08]), so equals $D_{\text{co}}$.

But one has more in this context: any universally elliptic tree is universally compatible. Indeed, since VPC subgroups of $G$ have precisely 2 coends, Proposition 7.4 of [SS03] implies that any two one-edge splittings $T_1, T_2$ of $G$ over $A$ with edge stabilizers of $T_1$ elliptic in $T_2$ are compatible. Indeed, strong crossing of almost invariant subsets corresponding to $T_1$ and $T_2$ occurs if and only edge stabilizers of $T_1$ are not elliptic in $T_2$ ([Gui05, Lemme 11.3]), and the absence of (weak or strong) crossing is equivalent to compatibility of $T_1$ and $T_2$ [SS00]. To sum up, we have:
Corollary 5.3. Let $G$ be a Poincaré duality group of dimension $n$, with $G$ not virtually polycyclic. Let $\mathcal{A}$ the family of $\text{VPC}_{\leq n-1}$-subgroups. Then $T_{\text{co}}$ exists and lies in the JSJ deformation space of $G$ over $\mathcal{A}$. □

In particular, $G$ has a canonical JSJ tree over $\mathcal{A}$.

5.8 Trees of cylinders
We will see additional examples based on the tree of cylinders in Part III.

Part II
Acylindrical JSJ deformation spaces

6 Introduction
In this part of the paper, we prove existence of the usual JSJ deformation space $D_{JSJ}$, under some acylindricity assumptions. We will return to the compatibility JSJ in Subsection 9.3.

As above, we fix a finitely generated group $G$ and a class $\mathcal{A}$ of subgroups of $G$ which is stable under conjugating and passing to subgroups. We will work in a relative situation, both for applications and because the proof uses relative trees (see Section 8).

We therefore fix an arbitrary set $\mathcal{H}$ of subgroups of $G$ (possibly empty), and we consider $(\mathcal{A}, \mathcal{H})$-trees, i.e. minimal actions of $G$ on simplicial trees with edge stabilizers in $\mathcal{A}$, in which every subgroup occuring in $\mathcal{H}$ is elliptic.

Using acylindrical accessibility, we first show (Sections 7 and 8) that one may construct the JSJ deformation space of $G$ over $\mathcal{A}$ relative to $\mathcal{H}$, and describe its flexible subgroups, provided that every tree $T$ may be mapped to an acylindrical tree $T^*$ whose deformation space is not too different from that of $T$. We then explain (Section 9) how the tree of cylinders introduced in [GL08] yields such acylindrical trees $T^*$. But the tree of cylinders also has strong compatibility properties, which are used in Subsection 9.3 to construct and describe the JSJ compatibility tree.

Applications of these constructions will be given in Part III. For concreteness, we describe the example studied in Section 11 right away. See also the example given in the introduction.

Example 6.1. $G$ is a torsion-free CSA group (for instance a limit group, or a torsion-free hyperbolic group), $\mathcal{A}$ is the family of its abelian subgroups, $\mathcal{H}$ is arbitrary. Later we will introduce $C, k, \mathcal{S}, \mathcal{S}_{\text{nc}}, T^*, \mathcal{E}, \sim; \mathcal{E}$ is the family of non-trivial abelian subgroups, $\sim$ is the commutation relation on $\mathcal{E}$, and $T^* = T_{\text{c}}$ will be the tree of cylinders of $T$ for $\sim$. The tree $T^*$ is $(k, C)$-acylindrical with $k = 2$ and $C = 1$: stabilizers of arcs of length $> 2$ have cardinality $\leq 1$. Finally, $\mathcal{S}$ will be the family of abelian subgroups, $\mathcal{S}_{\text{nc}}$ the family of non-cyclic abelian subgroups. Subgroups becoming elliptic when passing from $T$ to $T^*$ belong to $\mathcal{S}$.

All trees considered in Sections 7 through 14 are simplicial, except for the limit tree $T_{\infty}$ introduced in Subsection 7.1.

7 Uniform acylindricity
In this section, we assume that for each $(\mathcal{A}, \mathcal{H})$-tree $T$ there is an acylindrical tree $T^*$ (with uniform constants) in the same deformation space, and we deduce the existence of the JSJ deformation space. In the setting of Example 6.1, this applies if all abelian subgroups of
are cyclic (for instance if $G$ is a torsion-free hyperbolic group), and more generally if $H$
contains all non-cyclic abelian groups. This is a first step towards the proof of Theorem
8.3, where we allow $T^*$ not to lie in the same deformation space as $T$.
An infinite group $H$ is virtually cyclic if some finite index subgroup is infinite cyclic.
Such a group maps onto $\mathbb{Z}$ or $D_\infty$ (the infinite dihedral group $\mathbb{Z}/2 \ast \mathbb{Z}/2$) with finite kernel.

**Definition 7.1.** Given $C \geq 1$, we say that $H$ is **C-virtually cyclic** if it maps onto $\mathbb{Z}$ or $D_\infty$ with kernel of order at most $C$. Equivalently, $H$ acts non-trivially on a line with edge stabilizers of order $\leq C$.

An infinite virtually cyclic group is **C-virtually cyclic** if the order of its maximal finite normal subgroup is $\leq C$. Note that finite subgroups of a $C$-virtually cyclic group have cardinality bounded by $2C$.

**Definition 7.2.** A tree $T$ is **$(k, C)$-acylindrical** if all arcs of length $\geq k + 1$ have stabilizer of cardinality $\leq C$.

If $T'$ belongs to the same deformation space as $T$, it is $(k', C)$-acylindrical (see [GL07]), but in general there is no control on $k'$.

**Theorem 7.3.** Given $A$ and $H$, suppose that there exist numbers $C$ and $k$ such that:

- $A$ contains all $C$-virtually cyclic subgroups of $G$, and all subgroups of order $\leq 2C$;
- for any $(A, H)$-tree $T$, there is an $(A, H)$-tree $T^*$ in the same deformation space which is $(k, C)$-acylindrical.

Then the JSJ deformation space of $G$ over $A$ relative to $H$ exists.

Moreover, if all groups in $A$ are small in $(A, H)$-trees, then the flexible vertices are relative QH vertices with fiber of cardinality at most $C$, and all boundary components are used.

Recall (Subsection 2.3) that $H$ is small in $T$ (resp. in $(A, H)$-trees) if it fixes a point, or an end, or leaves a line invariant in $T$ (resp. in all $(A, H)$-trees on which $G$ acts). See Subsection 2.5 for the definitions of flexible, QH, and used boundary components.

**Remark 7.4.** Theorem 7.3 holds, with the same proof, if $G$ is only finitely generated relative to a finite collection of subgroups, and these subgroups are in $H$.

Theorem 7.3 will be proved in Subsections 7.1 and 7.2. We start with two general lemmas.

**Lemma 7.5.** If a group $H$ acts non-trivially on a $(k, C)$-acylindrical tree $T$, and $H$ is small in $T$, then $H$ is $C$-virtually cyclic.

**Proof.** By smallness, $H$ preserves a line or fixes an end. If it acts on a line, it is $C$-virtually cyclic. If it fixes an end, the set of its elliptic elements is the kernel of a homomorphism to $\mathbb{Z}$. Every finitely generated subgroup of this kernel is elliptic. It fixes a ray, so has order $\leq C$ by acylindricity. It follows that $H$ is $C$-virtually cyclic.

**Lemma 7.6.** If a finitely generated group $G$ does not split over subgroups of order $\leq C$ relative to a family $H$, there exists a finite family $H' = (H_1, \ldots, H_p)$, with $H_i$ a finitely generated group contained in a group of $H$, such that $G$ does not split over subgroups of order $\leq C$ relative to $H'$.

A special case of this lemma is proved in [Per09].
Proof. All trees in this proof will have stabilizers in the family $A(C)$ consisting of all subgroups of order $\leq C$. Note that over $A(C)$ all trees are universally elliptic, so having no splitting is equivalent to the JSJ deformation space being trivial.

Let $H_1, \ldots, H_n, \ldots$ be an enumeration of all finitely generated subgroups of $G$ contained in a group of $\mathcal{H}$. We have pointed out ([GL09, Remark 4.6]) that $G$ admits JSJ decompositions over $A(C)$ (this follows from Linnell’s accessibility [Lin83]). This is also true in the relative setting, so let $T_n$ be a JSJ tree relative to $\mathcal{H}_n = (H_1, \ldots, H_n)$. We show the lemma by proving that $T_n$ is trivial for $n$ large.

By Lemma 4.8 of [GL09], the tree $T_n$, which is relative to $\mathcal{H}_{n-1}$, may be refined to a JSJ tree relative to $\mathcal{H}_{n-1}$. If we fix $n$, we may therefore find trees $S_1(n), \ldots, S_n(n)$ such that $S_i(n)$ is a JSJ tree relative to $\mathcal{H}_i$ and $S_i(n)$ refines $S_{i+1}(n)$. By Linnell’s accessibility theorem, there is a uniform bound for the number of orbits of edges of $S_1(n)$ (assumed to have no redundant vertices). This number is an upper bound for the number of $n$’s such that $T_n$ and $T_{n+1}$ belong to different deformation spaces, so for $n$ large the trees $T_n$ all belong to the same deformation space $D$ (they have the same elliptic subgroups).

Since every $H_n$ is elliptic in $D$, so is every $H \in \mathcal{H}$ (otherwise $H$ would fix a unique end, and edge stabilizers would increase infinitely many times along a ray going to that end). The non-splitting hypothesis made on $G$ then implies that $D$ is trivial. □

We also note:

Lemma 7.7. It suffices to prove Theorem 7.3 under the additional hypothesis that $G$ does not split over subgroups of order $\leq 2C$ relative to $\mathcal{H}$ (“one-endedness condition”).

Proof. Let $A(2C) \subset A$ denote the family of subgroups of order $\leq 2C$. As mentioned in the previous proof, Linnell’s accessibility implies the existence of a JSJ tree over $A(2C)$ relative to $\mathcal{H}$. We can now apply Subsection 8.1 of [GL09]. □

7.1 Existence of the JSJ deformation space

Because of Lemma 7.7, we assume from now on that $G$ does not split over subgroups of order $\leq 2C$ relative to $\mathcal{H}$.

In this subsection, we prove the first assertion of Theorem 7.3. We have to construct a universally elliptic tree $T_J$ which dominates every universally elliptic tree (of course, all trees are $(A, \mathcal{H})$-trees and universal ellipticity is defined with respect to $(A, \mathcal{H})$-trees). Countability of $G$ allows us to choose a sequence of universally elliptic trees $U_i$ such that, if $g \in G$ is elliptic in every $U_i$, then it is elliptic in every universally elliptic tree. By Corollary 3.6 of [GL09], it suffices that $T_J$ dominates every $U_i$. Refining $U_i$ if necessary, as in the proof of [GL09, Theorem 4.3], we may assume that $U_{i+1}$ dominates $U_i$. In particular, we are free to replace $U_i$ by a subsequence when needed.

Let $T_i$ be a $(k, C)$-acylindrical tree in the same deformation space as $U_i$. Let $\ell_i$ be the length function of $T_i$. The proof has two main steps. First we assume that, for all $g$, the sequence $\ell_i(g)$ is bounded, and we construct a universally elliptic $(A, \mathcal{H})$-tree $T_J$ which dominates every $U_i$. Such a tree is a JSJ tree. In the second step, we deduce a contradiction from the assumption that the sequence $\ell_i$ is unbounded.

• If $\ell_i(g)$ is bounded for all $g$, we pass to a subsequence so that $\ell_i$ has a limit $\ell$ (possibly 0). Since the set of length functions of trees is closed [CM87, Theorem 4.5], $\ell$ is the length function associated to the action of $G$ on an $\mathbb{R}$-tree $T$. Since $\ell$ takes values in $\mathbb{Z}$, the tree $T$ is simplicial (but in general it is not an $A$-tree).

We can assume that all trees $T_i$ are non-trivial. By Lemma 7.5, $T_i$ is irreducible except if $G$ is virtually cyclic, in which case the theorem is clear. If $g \in G$ is hyperbolic in some $T_{i_0}$, then $\ell_i(g) \geq 1$ for all $i \geq i_0$ (because $T_i$ dominates $T_{i_0}$), so $g$ is hyperbolic in $T$. Since every $T_i$ is irreducible, there exist $g, h \in G$ such that $g, h$, and $[g, h]$ are hyperbolic in $T_i$.

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(for some $i$), hence in $T$, so $T$ is also irreducible. By [Pau89], $T_i$ converges to $T$ in the Gromov topology (see Subsection 3.1).

We will not claim anything about the edge stabilizers of $T$, but we study its vertex stabilizers. We claim that a subgroup $H \subset G$ is elliptic in $T$ if and only if it is elliptic in every $T_i$; in particular, every $H \in \mathcal{H}$ is elliptic in $T$, and $T$ dominates every $T_i$.

Note that the claim is true if $H$ is finitely generated, since $g \in G$ is elliptic in $T$ if and only if it is elliptic in every $T_i$. If $H$ is infinitely generated, let $H'$ be a finitely generated subgroup of cardinality $> C$. If $H$ is elliptic in $T$, it is elliptic in $T_i$ because otherwise it fixes a unique end and $H'$ fixes an infinite ray of $T_i$, contradicting acylindricity of $T_i$. Conversely, if $H$ is elliptic in every $T_i$ but not in $T$, the group $H'$ fixes an infinite ray in $T$. By Gromov convergence, $H'$ fixes a segment of length $k + 1$ in $T_i$ for $i$ large. This contradicts acylindricity of $T_i$, thus proving the claim.

We now return to the trees $U_i$. They are dominated by $T$ since the trees $T_i$ are. Their edge stabilizers are in $A_{ell}$, the family of groups in $A$ which are universally elliptic. Since $A_{ell}$ is stable under taking subgroups, any equivariant map $f : T \rightarrow U_i$ factors through the tree $T_j$ obtained from $T$ by collapsing all edges with stabilizer not in $A_{ell}$. The tree $T_j$ is a universally elliptic $(A, H)$-tree dominating every $U_i$, hence every universally elliptic tree. It is a JSJ tree.

- We now suppose that $\ell_i(g)$ is unbounded for some $g \in G$, and we work towards a contradiction. Since the set of projectivized non-zero length functions is compact [CM87, Theorem 4.5], we may assume that $\ell_i/\lambda_i$ converges to the length function $\ell$ of a non-trivial $\mathbb{R}$-tree $T_\infty$, for some sequence $\lambda_i \rightarrow +\infty$. By Theorem 3.5, convergence also takes place in the Gromov topology (all $T_i$‘s are irreducible, and we take $T_\infty$ to be a line if it is not irreducible).

**Lemma 7.8.** 1. Any subgroup $H < G$ of order $> C$ which is elliptic in every $T_i$ fixes a unique point in $T_\infty$. In particular, elements of $\mathcal{H}$ are elliptic in $T_\infty$.

2. Tripod stabilizers of $T_\infty$ have cardinality $\leq C$.

3. If $I \subset T_\infty$ is a non-degenerate arc, then its stabilizer $G_I$ has order $\leq C$ or is $C$-virtually cyclic.

4. Let $J \subset I \subset T_\infty$ be two non-degenerate arcs. If $G_I$ is $C$-virtually cyclic, then $G_I = G_J$.

**Proof.** Recall that $T_{i+1}$ dominates $T_i$, so a subgroup acting non-trivially on $T_i$ also acts non-trivially on $T_j$ for $j > i$.

To prove 1, we may assume that $H$ is finitely generated. It is elliptic in $T_\infty$ because all of its elements are. But it cannot fix an arc in $T_\infty$: otherwise, since $T_i/\lambda_i$ converges to $T_\infty$ in the Gromov topology, $H$ would fix a long segment in $T_i$ for $i$ large, contradicting acylindricity.

Using the Gromov topology, one sees that a finitely generated subgroup fixing a tripod of $T_\infty$ fixes a long tripod of $T_i$ for $i$ large, so has cardinality at most $C$ by acylindricity. This proves 2.

To prove 3, consider a group $H$ fixing a non-degenerate arc $I = [a, b]$ in $T_\infty$. It suffices to show that $H$ does not contain $F_2$: depending on whether $H$ is elliptic in every $T_i$ or not, Assertion 1 or Lemma 7.5 then gives the required conclusion.

If $H$ contains a non-abelian free group, choose elements $\{h_1, \ldots, h_n\}$ generating a free subgroup $F_n \subset H$ of rank $n \gg C$, and choose $\varepsilon > 0$ with $\varepsilon \ll |I|$. For $i$ large, there exist two points $a, b \in T_i$ at distance at least $(|I| - \varepsilon)\lambda_i$ from each other, and contained in the characteristic set (axis or fixed point set) of each $h_j$. Additionally, the translation length of every $h_j$ in $T_i$ is at most $\varepsilon\lambda_i$ if $i$ is large enough. Then all commutators $[h_{j_1}, h_{j_2}]$ fix most of the segment $[a, b]$, contradicting acylindricity of $T_i$ if $n(n-1)/2 > C$. 

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We now prove 4. By Assertion 1, we know that $G_I$ acts non-trivially on $T_i$, for $i$ large, so we can choose a hyperbolic element $h \in G_I$. We suppose that $g \in G_I$ does not fix an endpoint, say $a$, of $I = [a, b]$, and we argue towards a contradiction. Let $a_i, b_i \in T_i$ be in the axis of $h$ as above, with $d(a_i, ga_i) \geq \delta \lambda_i$ for some $\delta > 0$. For $i$ large the translation lengths of $g$ and $h$ in $T_i$ are small compared to $\frac{\|I\|}{\lambda_i}$, and the elements $[g, h], [g, h^2], \ldots, [g, h^{C+1}]$ all fix a common long arc in $T_i$. By acylindricity, there exist $j_1 \neq j_2$ such that $[g, h^{j_1}] = [g, h^{j_2}]$, so $g$ commutes with $h^{j_1-j_2}$. It follows that $g$ preserves the axis of $h$, and therefore moves $a_i$ by $\ell_i(g)$, a contradiction since $\ell_i(g) = o(\lambda_i)$ as $i \to \infty$.

Sela’s proof of acylindrical accessibility now comes into play to describe the structure of $T_\infty$. We use the generalization given by Theorem 5.1 of [Gui08], which allows non-trivial tripod stabilizers. Lemma 7.8 shows that stabilizers of unstable arcs and tripods have cardinality at most $C$, and we have assumed that $G$ does not split over a subgroup of cardinality at most $C$ relative to $H$, hence also relative to a finite $H'$ by Lemma 7.6. It follows that $T_\infty$ is a graph of actions as in Theorem 5.1 of [Gui08]. In order to reach the desired contradiction, we have to rule out several possibilities.

First consider a vertex action $G_v \ltimes Y_v$ of the decomposition of $T_\infty$ given by [Gui08]. If $Y_v$ is a line on which $G_v$ acts with dense orbits through a finitely generated group, then, by Assertion 3 of Lemma 7.8, $G_v$ contains a finitely generated subgroup $H$ mapping onto $\mathbb{Z}^2$ with finite or virtually cyclic kernel, and acting non-trivially. The group $H$ acts non-trivially on $T_i$ for $i$ large, contradicting Lemma 7.5.

Now suppose that $G_v \ltimes Y_v$ has kernel $N_v$, and the action of $G_v/N_v$ is dual to a measured foliation on a 2-orbifold $\Sigma$ (with conical singularities). Then $N_v$ has order $\leq C$ since it fixes a tripod. Consider a one-edge splitting $S$ of $G$ (relative to $H$) dual to a simple closed curve on $\Sigma$. This splitting is over a $C$-virtually cyclic group $G_e$. In particular, $S$ is an $(A, H)$-tree. Since $G_e$ is hyperbolic in $T_\infty$, it is also hyperbolic in $T_i$, hence in $U_i$, for $i$ large enough. On the other hand, being universally elliptic, $U_i$ is elliptic with respect to $S$. By Remark 2.3 of [FP06] (see [GL09, Lemma 7.10]), $G$ splits relative to $H$ over an infinite index subgroup of $G_e$, i.e. over a group of order $\leq 2C$, contradicting our assumptions.

By Theorem 5.1 of [Gui08], the only remaining possibility is that $T_\infty$ itself is a simplicial tree, and all edge stabilizers are $C$-virtually cyclic. Then $T_\infty$ is an $(A, H)$-tree, and its edge stabilizers are hyperbolic in $T_i$ for $i$ large. This leads to a contradiction as in the previous case.

### 7.2 Description of flexible vertices

We now prove the second assertion of Theorem 7.3: if all groups in $A$ are small in $(A, H)$-trees, then the flexible vertices are relatively QH with fiber of cardinality at most $C$.

Let $G_v$ be a flexible vertex group of a JSJ tree $T_J$. We consider splittings of $G_v$ (over groups in $A$) relative to the incident edge groups and the subgroups of $G_v$ conjugate to some $H \in H$, as studied in Subsection 5.2 of [GL09]. These splittings extend to splittings of $G$ relative to $H$. We claim that every edge stabilizer $A \subset G_v$ of such a splitting is $C$-virtually cyclic.

The group $A$ cannot be $(A, H)$-universally elliptic, since otherwise the splitting could be used to refine $T_J$, contradicting maximality of the JSJ tree $T_J$. Being small in $(A, H)$-trees, $A$ is $C$-virtually cyclic: apply Lemma 7.5 to the action of $A$ on $T^*$, where $T$ is some $(A, H)$-tree in which $A$ is not elliptic. This proves the claim.

We assume that $G$ does not split (relative to $H$) over a subgroup of order $\leq 2C$, so it does not split over a subgroup of infinite index of $A$. It follows that all splittings of $G_v$ are minimal in the sense of Fujiwara-Papasoglu, by Remark 3.2 of [FP06]. Moreover, any pair of splittings is either hyperbolic-hyperbolic or elliptic-elliptic [FP06, Proposition 2.2]. Since $G_v$ is flexible, there are hyperbolic-hyperbolic splittings, and we consider a maximal set $I$ of hyperbolic-hyperbolic splittings, in the sense of Definition 4.4 of [FP06].
We now construct an enclosing graph of groups decomposition for $I$, as in the proof of Proposition 5.4 in [FP06]. Associated to a finite set of splittings $I_i \subset I$ (consisting of splittings $T_1, \ldots, T_i$ where each $T_k$ is hyperbolic-hyperbolic with respect to some $T_j, j < k$, if $k > 1$) is a splitting of $G_v$ with a QH vertex stabilizer $S_i$ fitting in an exact sequence $1 \to F_i \to S_i \to \pi_1(\Sigma_i) \to 1$. One can check from the construction in [FP06] that this decomposition is relative to $\mathcal{H}$, and that any non-peripheral element in $S_i$ is hyperbolic in some splitting $T_j$. In particular, any conjugate of a group in $\mathcal{H}$ intersects $S_i$ in an extended boundary subgroup, so $S_i$ is a relative QH group.

As in [FP06], the point is to show that the complexity of the 2-orbifold $\Sigma_i$ is bounded. If not, one uses non-peripheral simple closed 1-suborbifolds on $\Sigma_i$ to construct splittings $\Delta_i$ of $G_v$ with arbitrarily many orbits of edges. In [FP06] this is ruled out by Bestvina-Feighn’s accessibility theorem, but we have to use a different argument because we do not assume that $G$ is finitely presented.

Since $S_i$ was constructed using splittings over $\mathbb{C}$-virtually cyclic groups, the fibers $F_i$ have order bounded by $C$, so $\Delta_i$ is $(2,C)$-acylindrical. Refining $T_j$ at the vertices in the orbit of $v$ using $\Delta_i$, and collapsing the original edges, we obtain splittings of $G$ with arbitrarily many orbits of edges. These splittings are $(2,C)$-acylindrical since the $\Delta_i$’s are over non-peripheral 1-suborbifolds. This contradicts the main result of [Wei07], which generalizes Sela’s acylindrical accessibility [Sel97] by allowing groups of order $\leq C$ to have arbitrary fixed point sets.

We now have an enclosing graph of groups decomposition of $G_v$ for $I$, that is relative to $\mathcal{H}$, and with a relative QH vertex stabilizer $S_i$. Its edge stabilizers are $C$-virtually cyclic, hence in $\mathcal{A}$.

We complete the proof by showing $S = G_v$. Assume otherwise. Then some boundary component of the underlying orbifold yields a non-trivial splitting $\Delta$ of $G_v$. Maximality of the JSJ tree $T_j$ therefore implies that $\Delta$ is hyperbolic-hyperbolic with respect to some other splitting. As in Lemma 5.5 of [FP06], the “rigidity” property of enclosing groups contradicts the maximality of $I$.

This proves that flexible vertices of $T_j$ are relative QH with fiber of cardinality at most $C$. By Proposition 7.5 and Remark 8.19 of [GL09], all boundary components of the underlying orbifold are used. This completes the proof of the second assertion of Theorem 7.3.

8 Acylindricity up to small groups

In this section, we generalize Theorem 7.3. Instead of requiring that every deformation space contains a $(k,C)$-acylindrical tree, we require that one can make every tree acylindrical without changing its deformation space too much: the new elliptic subgroups should be small in $(\mathcal{A}, \mathcal{H})$-trees.

**Definition 8.1.** Besides $\mathcal{A}$ and $\mathcal{H}$, we fix a family $\mathcal{S}$ of subgroups of $G$. It should be closed under conjugation and under taking subgroups, and every group in $\mathcal{S}$ should be small in $(\mathcal{A}, \mathcal{H})$-trees. If $C \geq 1$ is fixed, we denote by $\mathcal{S}_{\text{nvC}}$ the family of groups in $\mathcal{S}$ which are not $C$-virtually cyclic.

In applications $\mathcal{S}$ will contain $\mathcal{A}$, sometimes strictly.

**Definition 8.2.** Let $T, T^*$ be $(\mathcal{A}, \mathcal{H})$-trees. Given $\mathcal{S}$, we say that $T$ smally dominates $T^*$ (with respect to $\mathcal{S}$) if:

- $T$ dominates $T^*$;
- any group which is elliptic in $T^*$ but not in $T$ belongs to $\mathcal{S}$ (in particular, it is small in $(\mathcal{A}, \mathcal{H})$-trees);
Theorem 8.3. Given $\mathcal{A}$, $\mathcal{H}$, and $\mathcal{S}$ as in Definition 8.1, suppose that there exist numbers $C$ and $k$ such that:

- $\mathcal{A}$ contains all $C$-virtually cyclic subgroups, and all subgroups of cardinal $\leq 2C$;
- every $(\mathcal{A}, \mathcal{H})$-tree $T$ smally dominates some $(k,C)$-acylindrical $(\mathcal{A}, \mathcal{H})$-tree $T^*$.

Then the JSJ deformation space of $G$ over $\mathcal{A}$ relative to $\mathcal{H}$ exists.

Moreover, if all groups in $\mathcal{A}$ are small in $(\mathcal{A}, \mathcal{H})$-trees, then the flexible vertex groups that do not belong to $\mathcal{S}$ are relatively QH with fiber of cardinality at most $C$. All the boundary components of the underlying orbifold are used.

The rest of this section is devoted to the proof. As above, Lemma 7.7 lets us assume that $G$ does not split over groups of order $\leq 2C$ relative to $\mathcal{H}$ ("one-endedness" assumption).

Lemma 8.4. Assume that $G$ does not split over groups of order $\leq 2C$ relative to $\mathcal{H}$. Suppose that $T$ smally dominates a $(k,C)$-acylindrical tree $T^*$.

1. If a vertex stabilizer $G_v$ of $T^*$ is not elliptic in $T$, it belongs to $\mathcal{S}_{\text{vwc}}$ (in particular, it is not $C$-virtually cyclic).

2. Let $H$ be a subgroup. It is elliptic in $T^*$ if and only if it is elliptic in $T$ or contained in a group $K \in \mathcal{S}_{\text{vwc}}$. In particular, all $(k,C)$-acylindrical trees smally dominated by $T$ belong to the same deformation space.

3. Assume that $T^*$ is reduced. Then every edge stabilizer $G_e$ of $T^*$ has a subgroup of index at most 2 fixing an edge in $T$. In particular, if $T$ is universally elliptic, so is $T^*$.

Proof. Clearly $G_v \in \mathcal{S}$ because $T$ smally dominates $T^*$. Assume that it is $C$-virtually cyclic. Stabilizers of edges incident on $v$ have infinite index in $G_v$ since they are elliptic in $T$ and $G_v$ is not, so they have order $\leq 2C$. This contradicts the "one-endedness" assumption (note that $T^*$ is not trivial because then $G = G_v$ would be $C$-virtually cyclic, also contradicting one-endedness).

The "if" direction of Assertion 2 follows from Lemma 7.5. Conversely, assume that $H$ is elliptic in $T^*$ but not in $T$. If $v$ is a vertex of $T^*$ fixed by $H$, we have $H \subset G_v$ and $G_v \in \mathcal{S}_{\text{vwc}}$ by Assertion 1.

For Assertion 3, let $u$ and $v$ be the endpoints of an edge $e$ of $T^*$. First suppose that $G_u$ is not elliptic in $T$. It is small in $T$, so it preserves a line or fixes an end. Since $G_e$ is elliptic in $T$, some subgroup of index at most 2 fixes an edge.

If $G_u$ fixes two distinct points of $T$, then $G_e$ fixes an edge. We may therefore assume that $G_u$ and $G_v$ each fix a unique point in $T$. If these fixed points are different, $G_e = G_u \cap G_v$ fixes an edge of $T$. Otherwise, $(G_u, G_v)$ fixes a point $x$ in $T$, and is therefore elliptic in $T^*$. Since $T^*$ is reduced, some $g \in G$ acting hyperbolically on $T^*$ maps $u$ to $v$ and conjugates $G_u$ to $G_v$ (unless there is such a $g$, collapsing the edge $uv$ yields a tree in the same deformation space as $T^*$). This element $g$ fixes $x$, a contradiction. \[\square\]

Corollary 8.5. 1. $T^*$ is an $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{\text{vwc}})$-tree.

2. If $T_1$ dominates $T_2$, then $T_1^*$ dominates $T_2^*$.

3. If $T$ is an $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{\text{vwc}})$-tree, then $T^*$ lies in the same deformation space as $T$. In particular, Theorem 7.3 applies to $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{\text{vwc}})$-trees. \[\square\]
Applying Theorem 7.3, we get the existence of a JSJ tree \( T_r \) of \( G \) over \( A \) relative to \( H \cup S_{nvc} \). We can assume that \( T_r \) is reduced. We think of \( T_r \) as relative, as it is relative to \( S_{nvc} \) (not just to \( H \)). In the setting of Example 6.1, \( S_{nvc} \) is the class of all non-cyclic abelian subgroups and \( T_r \) is a JSJ tree over abelian groups relative to all non-cyclic abelian subgroups.

**Lemma 8.6.** If \( T_r \) is reduced, then it is \((A, H)\)-universally elliptic.

**Proof.** We let \( e \) be an edge of \( T_r \) such that \( G_e \) is not elliptic in some \((A, H)\)-tree \( T \), and we argue towards a contradiction. We may assume that \( T \) only has one orbit of edges. The first step is to show that \( T_r \) dominates \( T^* \).

Since \( T^* \) is relative to \( H \cup S_{nvc} \), the group \( G_e \) fixes a vertex \( u \in T^* \) by \((A, H \cup S_{nvc})\)-universal ellipticity of \( T_r \). This \( u \) is unique because edge stabilizers of \( T^* \) are elliptic in \( T \). Also note that \( G_u \in S_{nvc} \) by Assertion 1 of Lemma 8.4.

There is an equivariant map \( T \to T^* \), so \( G_u \) contains the stabilizer of some edge \( f \subset T \), and \( G_f \subset G_u \in S_{nvc} \) is (trivially) \((A, H \cup S_{nvc})\)-universally elliptic. Since \( T \) has a single orbit of edges, it is \((A, H \cup S_{nvc})\)-universally elliptic (but it is not relative to \( H \cup S_{nvc} \)). On the other hand, \( T^* \) is an \((A, H \cup S_{nvc})\)-tree, and it is \((A, H \cup S_{nvc})\)-universally elliptic by Assertion 3 of Lemma 8.4 (replace \( T^* \) by a reduced tree in the same deformation space if needed). By maximality of the JSJ, \( T_r \) dominates \( T^* \).

Recall that \( u \) is the unique fixed point of \( G_e \) in \( T^* \). Denote by \( a, b \) the endpoints of \( e \) in \( T_r \). Since \( T_r \) dominates \( T^* \), the groups \( G_a \) and \( G_b \) fix \( u \), so \( G_a, G_b \subset G_u \in S_{nvc} \) is elliptic in \( T_r \). As in the previous proof, some \( g \in G \) acting hyperbolically on \( T_r \) maps \( a \) to \( b \) (because \( T_r \) is reduced). This \( g \) fixes \( u \), so belongs to \( G_u \in S_{nvc} \), a contradiction since \( T_r \) is relative to \( H \cup S_{nvc} \).

We shall now construct a JSJ tree \( T_a \) relative to \( H \) by refining \( T_r \) at vertices with small stabilizer. This JSJ tree \( T_a \) is thought of as absolute as it is not relative to \( S_{nvc} \).

**Lemma 8.7.** There exists a JSJ tree \( T_a \) over \( A \) relative to \( H \). It may be obtained by refining \( T_r \) at vertices with stabilizer in \( S \) (in particular, the set of vertex stabilizers not belonging to \( S \) is the same for \( T_a \) as for \( T_r \)). Moreover, \( T_a^* \) lies in the same deformation space as \( T_r^* \).

**Proof.** Let \( v \) be a vertex of \( T_r \). We shall prove the existence of a JSJ tree \( T_v \) for \( G_v \) relative to the family \( P_v \) consisting of incident edge groups and subgroups conjugate to a group of \( H \) (see Subsection 5.2 of [GL09]). By Lemma 5.3 of [GL09], which applies because \( T_r \) is \((A, H)\)-universally elliptic (Lemma 8.6), one then obtains a JSJ tree \( T_a \) for \( G \) relative to \( H \) by refining \( T_v \) using the trees \( T_r \).

If \( G_v \) is elliptic in every \((A, H)\)-universally elliptic tree \( T \), its JSJ is trivial and no refinement is needed at \( v \). Assume therefore that \( G_v \) is not elliptic in such a \( T \). Consider the \((A, H \cup S_{nvc})\)-tree \( T^* \). It is \((A, H)\)-universally elliptic by Assertion 3 of Lemma 8.4, hence \((A, H \cup S_{nvc})\)-universally elliptic, so it is dominated by \( T_r \). In particular, \( G_v \) is elliptic in \( T_v \), so belongs to \( S \).

Being small in \((A, H)\)-trees, \( G_v \) has at most one non-trivial deformation space containing a universally elliptic tree [GL09, Proposition 6.1]. Applying this to splittings of \( G_v \) relative to \( P_v \), we deduce that \( T \) is a (non-minimal) JSJ tree of \( G_v \) relative to \( P_v \). This shows the first two assertions of the lemma.

We now show the “moreover”. Since \( T_v^* \) is an \((A, H \cup S_{nvc})\)-universally elliptic \((A, H \cup S_{nvc})\)-tree, it is dominated by \( T_r \). Conversely, \( T_a \) dominates \( T_r \) and therefore \( T_a^* \) dominates \( T_r^* \) by Corollary 8.5(2), so \( T_a^* \) dominates \( T_r \) since \( T_r^* \) lie in the same deformation space by Corollary 8.5(3).
We have just proved that the JSJ deformation space relative to $\mathcal{H}$ exists. To complete the proof of Theorem 8.3, there remains to describe flexible vertex groups under the assumption that all groups in $\mathcal{A}$ are small in $(\mathcal{A}, \mathcal{H})$-trees.

Let $G_v$ be a flexible vertex stabilizer of $T_a$ which does not belong to $S$. Because of the way $T_a$ was constructed by refining $T$, at vertices with stabilizer in $S$, the group $G_v$ is also a vertex stabilizer of $T_v$, with the same incident edge groups. Being flexible in $T_a$, the group $G_v$ is non-elliptic in some $(\mathcal{A}, \mathcal{H})$-tree $T$. By Lemma 8.4, $G_v$ is non-elliptic in the $(\mathcal{A}, \mathcal{H} \cup S_{\text{nvc}})$-tree $T^*$. This means that $G_v$ is not $(\mathcal{A}, \mathcal{H} \cup S_{\text{nvc}})$-universally elliptic, so it is a flexible vertex stabilizer of $T_v$. Since groups of $\mathcal{A}$ are small in $(\mathcal{A}, \mathcal{H} \cup S_{\text{nvc}})$-trees, Theorem 7.3 implies that $G_v$ is relatively QH with fiber of order $\leq C$, and all boundary components are used.

9 The tree of cylinders

In order to apply Theorem 8.3, one has to be able to construct an acylindrical tree $T^*$ smally dominated by a given tree $T$. We show how the (collapsed) tree of cylinders $T^*$ may be used for that purpose.

9.1 Definitions

We first recall the definition and some basic properties of the tree of cylinders (see [GL08] for details).

Besides $\mathcal{A}$ and $\mathcal{H}$, we have to fix a conjugacy-invariant subfamily $E \subset \mathcal{A}$. For the purposes of this paper, we always let $E$ be the family of infinite groups in $\mathcal{A}$. We assume that $G$ is one-ended relative to $\mathcal{H}$, so that all $(\mathcal{A}, \mathcal{H})$-trees have edge stabilizers in $E$.

As in [GL08], an equivalence relation $\sim$ on $E$ is admissible (relative to $\mathcal{H}$) if the following axioms hold for any $A, B \in E$:

1. If $A \sim B$, and $g \in G$, then $gAg^{-1} \sim gBg^{-1}$.
2. If $A \subset B$, then $A \sim B$.
3. Let $T$ be an $(\mathcal{A}, \mathcal{H})$-tree. If $A \sim B$, and $A, B$ fix $a, b \in T$ respectively, then for each edge $e \subset [a, b]$ one has $G_e \sim A \sim B$.

Fix an admissible equivalence relation $\sim$. Let $T$ be an $(\mathcal{A}, \mathcal{H})$-tree. We declare two edges $e, f$ to be equivalent if $G_e \sim G_f$. The union of all edges in an equivalence class is a subtree $Y$, called a cylinder of $T$. Two distinct cylinders meet in at most one point. The tree of cylinders $T_c$ of $T$ is the bipartite tree such that $V_0(T_c)$ is the set of vertices $v$ of $T$ which belong to at least two cylinders, $V_1(T_c)$ is the set of cylinders $Y$ of $T$, and there is an edge $\varepsilon = (v, Y)$ between $v$ and $Y$ in $T_c$ if and only if $v \in Y$. The tree $T_c$ is dominated by $T$ (in particular, it is relative to $\mathcal{H}$). It only depends on the deformation space $D$ containing $T$ (we sometimes say that it is the tree of cylinders of $D$).

The stabilizer of a vertex $v \in V_0(T_c)$ is the stabilizer of $v$, viewed as a vertex of $T$. The stabilizer $G_Y$ of a vertex $Y \in V_1(T_c)$ is the stabilizer $G_C$ (for the action of $G$ on $E$ by conjugation) of the equivalence class $C \in E / \sim$ containing stabilizers of edges in $Y$. If $A \in E$, then $A \subset G_c$ where $C$ is the equivalence class of $A$. The stabilizer of an edge $\varepsilon = (v, Y)$ of $T_c$ is $G_\varepsilon = G_v \cap G_Y$; it is elliptic in $T$.

It often happens that edge stabilizers of $T_c$ belong to $\mathcal{A}$. But this is not always the case, so we have to consider the collapsed tree of cylinders $T^*_c$ obtained from $T_c$ by collapsing all edges whose stabilizer does not belong to $\mathcal{A}$ (see Section 5.2 of [GL08]). It is an $(\mathcal{A}, \mathcal{H})$-tree and is equal to its collapsed tree of cylinders.
9.2 Trees of cylinders and acylindricity up to small groups

We assume that $G$ is one-ended relative to $H$ and we fix an admissible equivalence relation $\sim$ on $E$. The following lemma gives conditions ensuring that the collapsed tree of cylinders $T^*_e$ of $T$ is smally dominated by $T$, and $(2,C)$-acylindrical.

**Lemma 9.1.** Let $S$ be a class of subgroups which are small in $(A,H)$-trees, as in Definition 8.1. Suppose that, for every equivalence class $C \in E/\sim$, the stabilizer $G_C$ belongs to $S$. Also assume that one of the following holds:

1. For all $C \in E/\sim$, the group $G_C$ belongs to $A$.
2. $A$ is stable by extension of index 2: if $A$ has index 2 in $A'$, and $A' \in A$, then $A' \in A$.

If $T$ is any $(A,H)$-tree, then $T^*_e$ is an $(A,H)$-tree smally dominated by $T$. Any vertex stabilizer of $T^*_e$ which is not elliptic in $T$ is a $G_C$.

Assume furthermore that there exists $C$ with the following property: if two groups of $E$ are inequivalent, their intersection has order $\leq C$. Then $T^*_e$ is $(2,C)$-acylindrical.

**Proof.** It is a general fact that edge stabilizers of $T_e$ are elliptic in $T$, and that $T$ dominates $T_e$. A vertex stabilizer of $T_e$ which is not elliptic in $T$ is a $G_C$, so belongs to $S$ by assumption. This proves that $T$ smally dominates $T_e$ (but in general $T_e$ may have edge stabilizers not in $A$).

Under the first hypothesis ($G_C \in A$), the stabilizer of an edge $\varepsilon = (v,Y)$ of $T_e$ belongs to $A$ because $G_\varepsilon$ does, so $T_e = T^*_e$ is an $(A,H)$-tree. Under the second hypothesis, Remark 5.11 of [GL08] guarantees that $T^*_e$ belongs to the same deformation space as $T_e$, so $T^*_e$ also is smally dominated by $T$. Its vertex stabilizers are vertex stabilizers of $T_e$.

Acyldricity of $T^*_e$ follows from the fact that any segment of length 3 in $T^*_e$ contains edges with inequivalent stabilizers (see [GL08]).

**Corollary 9.2.** Let $G$ be one-ended relative to $H$, and let $S$ be as in Definition 8.1. Suppose that:

- $A$ contains all finite and virtually cyclic subgroups, and is stable by extension of index 2.
- $\sim$ is an admissible equivalence relation on the set $E$ of infinite groups in $A$, with all groups $G_C$ belonging to $S$.
- There exists $C$ such that, if two groups in $E$ are inequivalent, their intersection has order $\leq C$.

Then the JSJ deformation space $\mathcal{D}_{JSJ}$ over $A$ relative to $H$ exists. Its flexible subgroups belong to $S$ or are relatively QH with finite fiber. The collapsed tree of cylinders $(T_a)^*_e$ of $\mathcal{D}_{JSJ}$ is a JSJ tree relative to $H \cup S_{nvc}$. It has the same vertex stabilizers not contained in $S$ as trees in $\mathcal{D}_{JSJ}$.

**Proof.** Lemma 9.1 allows us to apply Theorem 8.3 (with $T^*_e = T^*_e$ and $k = 2$). Every $A \in A$ is finite or contained in a $G_C$, hence is small in $(A,H)$-trees because $G_C \in S$, so we can apply the “moreover” of the theorem. Thus $\mathcal{D}_{JSJ}$ exists, and flexible groups are as described.

We consider the JSJ tree $T_a$ relative to $H$ and the JSJ tree $T_r$ relative to $H \cup S_{nvc}$ as in Subsection 8. By Lemma 8.7, the collapsed tree of cylinders $(T_a)^*_e$ of $T_a$ lies in the same deformation space as $T_r$. If $(T_a)^*_e$ is reduced, its edge stabilizers fix an edge in $T_r$ (see Section 4 of [GL07]), so $(T_a)^*_e$ is $(A,H \cup S_{nvc})$-universally elliptic and therefore a JSJ tree relative to $H \cup S_{nvc}$.
In general, we have to prove that $(T_a)_e^*$ is $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{nvc})$-universally elliptic: if $e = (x, Y)$ is an edge of $(T_a)_e$ with $G_e \in \mathcal{A}$, then $G_e$ is $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{nvc})$-universally elliptic. Let $T$ be an $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{nvc})$-tree. If $G_Y$ is not $C$-virtually cyclic, then $G_Y \in \mathcal{S}_{nvc}$, and therefore $G_e \subset G_Y$ is elliptic in $T$. If $G_Y$ is $C$-virtually cyclic, consider an edge $e$ of $T_a$ with $G_e \subset G_e \subset G_Y$. By one-endedness of $G$, the group $G_e$ is infinite, so has finite index in $G_e$. Since $G_e$ is $(\mathcal{A}, \mathcal{H})$-universally elliptic, hence $(\mathcal{A}, \mathcal{H} \cup \mathcal{S}_{nvc})$-universally elliptic, so is $G_e$.

The last assertion of the theorem follows from Lemma 8.7 (recall that two trees belonging to the same deformation space have the same vertex stabilizers not in $\mathcal{A}$ [GL07, Corollary 4.4]).

### 9.3 Trees of cylinders and compatibility

The tree of cylinders may also be used to construct and describe the compatibility JSJ deformation space $\mathcal{D}_{co}$ (see Section 4). This is based on the following observation.

**Lemma 9.3.** Let $T_a$ be a universally elliptic $(\mathcal{A}, \mathcal{H})$-tree. Assume that, to each $(\mathcal{A}, \mathcal{H})$-tree $T$, one can associate an $(\mathcal{A}, \mathcal{H})$-tree $T^*$ in such a way that:

1. $T^*$ is compatible with any $(\mathcal{A}, \mathcal{H})$-tree dominated by $T$;
2. If an $(\mathcal{A}, \mathcal{H})$-tree $T$ is a refinement of $T_a$, then $T^*$ is a refinement of $T_a^*$.

Then $T_a^*$ is universally compatible (i.e. compatible with every $(\mathcal{A}, \mathcal{H})$-tree).

**Proof.** Let $T$ be any $(\mathcal{A}, \mathcal{H})$-tree. Since $T_a$ is universally elliptic, there exists a refinement $S$ of $T_a$ dominating $T$ [GL09, Lemma 3.2]. By the first assumption, $S^*$ and $T$ have a common refinement $R$. Since $S^*$ is a refinement of $T_a^*$, the tree $R$ is a common refinement of $T$ and $T_a^*$.

In applications, $T_a$ is a JSJ tree and $T^*$ is the collapsed tree of cylinders $T_a^*$, so the lemma says that $(T_a)_e^*$ is universally compatible. The compatibility JSJ deformation space, if it exists, is dominated by $T_a$ and dominates $(T_a)_e^*$.

The first assumption of Lemma 9.3 is a general property of trees of cylinders (Proposition 8.1 of [GL08]). For the second one, we invoke the following lemma.

**Lemma 9.4.** Let $G$ be one-ended relative to $\mathcal{H}$. Let $\sim$ be an admissible equivalence relation. Suppose that each group $G_c$ which contains $F_2$ is $(\mathcal{A}, \mathcal{H})$-universally elliptic, and that one of the two assumptions of Lemma 9.1 holds.

Let $S, T$ be $(\mathcal{A}, \mathcal{H})$-trees, with $S$ refining $T$. Assume that each vertex stabilizer $G_v$ of $T$ is small in $S$ or relatively QH with finite fiber. Then $S^*$ refines $T^*$.

Since $S$ dominates $T$, there is a cellular map from $S^*_e$ to $T^*_e$ (it maps a vertex to a vertex, an edge to a vertex or an edge) [GL08, Lemma 5.6]. The point of the lemma is that this is a collapse map.

**Proof.** It suffices to show that $S_e$ refines $T_e$. First suppose that $T$ is obtained from $S$ by collapsing the orbit of a single edge $e$ (this is the key step). We consider two cases, depending on the image $v$ of $e$ in $T$. We assume that $G_e$ is not elliptic in $S$, since otherwise $S$ and $T$ belong to the same deformation space and therefore $S_e = T_e$.

First suppose that $G_e$ is small in $S$. In this case, we claim that $v$ belongs to only one cylinder of $T$. Indeed, the preimage $S_v$ of $v$ in $S$ is a line or a subtree with a $G_v$-fixed end. All edges in $S_v$ belong to the same cylinder $Y$ of $S$. Edges $f$ of $S$ with one endpoint $x$ in $S_v$ also belong to $Y$. This is clear under the first assumption of Lemma 9.1 ($G_C \in \mathcal{A}$) since $G_f \subset G_v \subset G_Y \in \mathcal{A}$, so $G_f \sim G_Y$; under the second assumption, we note that $G_x$ contains the stabilizer of an edge in $Y$ with index at most 2, so belongs to $\mathcal{A}$, and $G_f \sim G_x \sim G_e$. Thus $v$ belongs to only one cylinder of $T$. This implies $S_e = T_e$. 

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Now suppose that $v$ is relatively QH with finite fiber $F$. We claim that a cylinder of $S$ containing an edge of the preimage $S_v$ of $v$ is entirely contained in $S_v$. This implies that $S_v$ refines $T_v$ by Remark 4.13 of [GL08].

The action of $G_v$ on $S_v$ is minimal because it has only one orbit of edges. Standard arguments (see [MS84, III.2.6]) show that this action is dominated by an action dual to a non-peripheral simple closed 1-suborbifold of the underlying orbifold of $G_v$. In particular, its edge stabilizers are not universally elliptic (this is clear if the orbifold is a surface, see [Gui00b, Lemma 5.3] for the general case).

Now consider an edge $f$ of $S$ with one endpoint in $S_v$. Since $G$ is one-ended relative to $H$, and $F$ is finite, the image of $G_f$ in the orbifold group $G_v/F$ is infinite so contains a finite index subgroup of a boundary subgroup. If $f'$ is an edge of $S_v$, then $G_f$ and $G_{f'}$ generate a group which contains $F_2$ and is not universally elliptic. Our assumption on the groups $G_c$ implies that $G_f$ and $G_{f'}$ cannot be equivalent. The claim follows since $f$ and $f'$ cannot be in the same cylinder.

If several orbits of edges of $S$ are collapsed in $T$, one iterates the previous argument, noting that all vertex stabilizers of intermediate trees are relatively QH with finite fiber or small in $S$.

**Corollary 9.5.** Let $G$ be one-ended relative to $H$. Suppose that:

1. $A$ contains all finite and virtually cyclic subgroups, and is stable by extension of index 2;

2. $\sim$ is an admissible equivalence relation on the set $E$ of infinite groups in $A$; if $G_C$ contains $F_2$, it is $(A, H)$-universally elliptic.

3. There exists $C$ such that, if two groups in $E$ are inequivalent, their intersection has order $\leq C$.

Then the collapsed tree of cylinders $(T_a)_c^*$ of the JSJ deformation space (over $A$ relative to $H$) is universally compatible.

**Proof.** Assumption 2 implies that the groups $G_C$ and the elements of $A$ are small in $(A, H)$-trees. We can apply Corollary 9.2 with $S$ consisting of all groups $G_C$ and their subgroups. It follows that a JSJ tree $T_a$ exists, and its flexible subgroups belong to $S$ or are relatively QH with finite fiber. By Lemmas 9.3 and 9.4 (applied with $T^* = T^*_c$), its collapsed tree of cylinders $(T_a)_c^*$ is universally compatible.

**Proposition 9.6.** Further assume that each group $G_C$ belongs to $A$, and $G \notin A$. Then the compatibility JSJ deformation space $D_{co}$ exists and contains $(T_a)_c$. It is trivial or irreducible, so the JSJ compatibility tree $T_{co}$ is defined.

The assumption $G_C \in A$ guarantees that $(T_a)_c$ is an $(A, H)$-tree, so no collapsing is necessary. Also note that $S$ (as defined in the previous proof) and $A$ contain the same infinite groups, so Corollary 9.2 implies that the flexible subgroups of $(T_a)_c$ belong to $A$ or are relative QH subgroups with finite fiber.

**Proof.** The point is to show that $(T_a)_c$ is maximal (for domination) among universally compatible trees. This will prove that $D_{co}$ exists and contains $(T_a)_c$. It is irreducible or trivial because $T_c$ is equal to its tree of cylinders [GL08, Corollary 5.8], and the tree of cylinders of a reducible tree is trivial.

Consider a universally compatible tree $T$. We have to show that each vertex stabilizer $G_v$ of $(T_a)_c$ is elliptic in $T$. If $G_v$ is not contained in any $G_C$, then it is elliptic in $T_a$, hence in $T$ because $T$, being universally elliptic, is dominated by $T_a$. We can therefore assume that $G_v \in A$, and also that $G_v$ does not contain $F_2$.
Since \( G \notin \mathcal{A} \), the quotient graph \((T_a)_c/G\) is not a point. There are two cases. If the image of \( v \) in \((T_a)_c/G\) has valence at least 2, we can refine \((T_a)_c\) to a minimal tree \( T' \) (in the same deformation space) having \( G_v \) as an edge stabilizer. Since \( G_v \in \mathcal{A} \), this is an \((\mathcal{A}, \mathcal{H})\)-tree. Its edge group \( G_e \) is elliptic in \( T \) because \( T \) is universally compatible.

The remaining case is when the image of \( v \) in \((T_a)_c/G\) has valence 1. In this case we assume that \( G_v \) is not elliptic in \( T \), and we argue towards a contradiction. Let \( e \) be an edge of \((T_a)_c\), containing \( v \). We are going to prove that \( G_v \) contains a subgroup \( G_0 \) of index 2 with \( G_e \subset G_0 \). Assuming this fact, we can refine \((T_a)_c\) to a minimal \((\mathcal{A}, \mathcal{H})\)-tree \( T' \) in which \( G_0 \) is an edge stabilizer. As above, \( G_0 \) is elliptic in \( T \), and so is \( G_v \). This is the required contradiction.

We now construct \( G_0 \). Replacing \( T \) by \( T \lor (T_a)_c \) (which is universally compatible by Assertion 2 of Proposition 3.22), and then collapsing, we can assume that \((T_a)_c\) is obtained from \( T \) by collapsing the orbit of an edge. We consider the action of \( G_v \) on \( T \). By Lemma 3.6 of [GL09], \( G_v \) contains a hyperbolic element. Since \( G_v \) does not contain \( F_2 \), it is small in \( T \), and the action defines a non-trivial morphism \( \varphi : G_v \to \mathbb{Z} \) (if there is a fixed end) or \( \varphi : G_v \to \mathbb{Z}/2 \ast \mathbb{Z}/2 \) (if the action is dihedral). Since \( G_e \) is elliptic in \( T \), its image under \( \varphi \) is trivial, or contained in a conjugate of a \( \mathbb{Z}/2 \) factor. It is now easy to construct \( G_0 \), as the preimage of a suitable index 2 subgroup of the image of \( \varphi \).

\( \square \)

Part III

Applications

10 Introduction

In Part I, we gave a general definition and proved the existence of the compatibility JSJ tree \( T_{\text{co}} \) for \( G \) finitely presented. In Part II, we gave a general existence theorem for the JSJ decomposition of a finitely generated group under acylindricity hypotheses. In the following sections, we are going to describe examples where these results apply. They will mostly be based on the tree of cylinders construction.

As we saw in Subsection 9.3, the tree of cylinders enjoys strong compatibility properties. Because of this, we will see that in many examples, the (collapsed) tree of cylinders \( T_e^* \) of the JSJ deformation space satisfies the same compatibility properties, and lives in the same deformation space, as the compatibility JSJ tree \( T_{\text{co}} \).

Although defined in less generality, \( T_e^* \) is usually easier to describe than \( T_{\text{co}} \) and, for this reason, is more useful than \( T_{\text{co}} \) in concrete situations (see Corollary 13.2). Note that any automorphism of \( G \) leaving \( \mathcal{A} \) and \( \mathcal{H} \) invariant leaves \( D_{JSJ} \) and \( D_{\text{co}} \) invariant, hence fixes both \( T_e^* \) and \( T_{\text{co}} \).

We first treat the case of abelian splittings of CSA groups. To allow torsion, we introduce \( K\)-CSA groups in Section 12.2, and describe their JSJ decomposition over virtually abelian groups. We then consider elementary splittings of relatively hyperbolic groups. Finally, we consider splittings of a finitely generated group over virtually cyclic groups under the assumption that their commensurizers are small. Then we relate these JSJ decompositions to actions on \( \mathbb{R} \)-trees.

11 CSA groups

In our first application, \( G \) is a torsion-free CSA group, and we consider splittings over abelian, or cyclic, groups. Recall that \( G \) is CSA if the commutation relation is transitive on \( G \setminus \{1\} \), and maximal abelian subgroups are malnormal. Toral relatively hyperbolic groups, in particular limit groups and torsion-free hyperbolic groups, are CSA.
If \( G \) is freely indecomposable relative to \( \mathcal{H} \), commutation is an admissible equivalence relation on the set of infinite abelian (resp. infinite cyclic) subgroups (see [GL08]). The groups \( G_C \) are maximal abelian subgroups, so are small in all trees. Over abelian groups, all edge stabilizers of \( T_c \) are abelian since \( G_C \in \mathcal{A} \), so \( T_c^* = T_c \). Over cyclic groups, \( T_c \) may have non-cyclic edge stabilizers, so we have to use \( T_c^* \).

**Theorem 11.1.** Let \( G \) be a finitely generated torsion-free CSA group, and \( \mathcal{H} \) any family of subgroups. Assume that \( G \) is freely indecomposable relative to \( \mathcal{H} \).

1. The abelian (resp. cyclic) JSJ deformation space \( D_{JSJ} \) of \( G \) relative to \( \mathcal{H} \) exists. Its non-abelian flexible subgroups are relative QH surface groups, with every boundary component used.

2. The collapsed tree of cylinders \( T_c^* \) of \( D_{JSJ} \) is an abelian (resp. cyclic) JSJ tree relative to \( \mathcal{H} \) and all non-cyclic abelian subgroups. It is universally compatible.

3. If \( G \) is not abelian, the abelian compatibility JSJ tree \( T_c^0 \) (relative to \( \mathcal{H} \)) exists and belongs to the same deformation space as \( T_c^* \). Trees in \( D_{JSJ} \) and \( D_c^0 \) have the same non-abelian vertex stabilizers (they are universally elliptic or surface groups).

See Subsection 2.5 for the definitions of flexible, QH, and used boundary components.

**Remark 11.2.** By Corollary 8.4 of [GL09], the first assertion holds even if \( G \) is not freely indecomposable.

**Proof.** We apply Corollaries 9.2 and 9.5 with \( \mathcal{A} \) the family of abelian (resp. cyclic) subgroups, \( C = 1 \), and \( S \) the family of abelian subgroups. The CSA property guarantees that \( \mathcal{A} \) is stable by extension of index 2. As mentioned above, \( \sim \) is commutation. Non-abelian flexible subgroups are relatively QH with finite fiber, hence surface groups because \( G \) is torsion-free. Assertion 3 follows from Proposition 9.6 since the condition \( G_C \in \mathcal{A} \) holds over abelian groups. \( \square \)

The last assertion of the theorem does not apply to cyclic splittings, because the condition \( G_C \in \mathcal{A} \) does not hold. In this setting \( T_c^* \) is universally compatible by Corollary 9.5, but as the following example shows it may happen that \( D_c^0 \) strictly dominates \( T_c^* \). As illustrated by this example, we will show in Subsection 14 that one obtains \( D_c^0 \) by possibly refining \( T_c^* \) at vertices with group \( \mathbb{Z}^2 \).

**Example.** Let \( H \) be a torsion-free hyperbolic group with property FA, and \( \langle a \rangle \) a maximal cyclic subgroup. Consider the HNN extension \( G = \langle H, t \mid tat^{-1} = a \rangle \), a one-ended torsion-free CSA group. The Bass-Serre tree \( T_0 \) is a JSJ tree over abelian groups. Its tree of cylinders \( T_1 \) is the Bass-Serre tree of the amalgam \( G = H \ast_{\langle a \rangle} \langle a, t \rangle \), it is also the compatibility JSJ tree over abelian groups by Assertion 3 of Theorem 11.1. Over cyclic groups, \( T_0 \) is a JSJ tree, its tree of cylinders is \( T_1 \), but the compatibility JSJ tree is \( T_0 \) (this follows from Proposition 5.2 and [Lev05]; the non-splitting assumption of Proposition 5.2 holds over cyclic groups, but not over abelian groups).

**12 \( \Gamma \)-limit groups and \( K \)-CSA groups**

The notion of CSA groups is not well-adapted to groups with torsion. This is why we introduce \( K \)-CSA groups, where \( K \) is an integer. Every hyperbolic group \( \Gamma \) is \( K \)-CSA for some \( K \). Being \( K \)-CSA is a universal property; in particular, all \( \Gamma \)-limit groups are \( K \)-CSA.

We say that a group is \( K \)-virtually abelian if it contains an abelian subgroup of index \( \leq K \) (note that the infinite dihedral group is 1-virtually cyclic, in the sense of Definition 7.1, but only 2-virtually abelian).
**Lemma 12.1.** If a countable group $G$ is locally $K$-virtually abelian, then $G$ is $K$-virtually abelian.

**Proof.** Let $g_1, \ldots, g_n, \ldots$ be a numbering of the elements of $G$. Let $A_n \subset \langle g_1, \ldots, g_n \rangle$ be an abelian subgroup of index $\leq K$. For a given $k$, there are only finitely many subgroups of index $\leq K$ in $\langle g_1, \ldots, g_k \rangle$, so there is a subsequence $A_{n_i}$ such that $A_{n_i} \cap \langle g_1, \ldots, g_k \rangle$ is independent of $i$. By a diagonal argument, one produces an abelian subgroup $A$ of $G$ whose intersection with each $\langle g_1, \ldots, g_n \rangle$ has index $\leq K$, so $A$ has index $\leq K$ in $G$. \qed

**Definition 12.2.** Say that $G$ is $K$-CSA for some $K > 0$ if:

1. Any finite subgroup has cardinality at most $K$ (in particular, any element of order $\leq K$ has infinite order).
2. Any element $g \in G$ of infinite order is contained in a unique maximal virtually abelian group $M(g)$, and $M(g)$ is $K$-virtually abelian.
3. $M(g)$ is its own normalizer.

A 1-CSA group is just a torsion-free CSA group. The Klein bottle group is 2-CSA but not 1-CSA. Any hyperbolic group $\Gamma$ is $K$-CSA for some $K$ since finite subgroups of $\Gamma$ have bounded order, and there are only finitely many isomorphism classes of virtually cyclic groups whose finite subgroups have bounded order. Corollary 12.5 will say that $\Gamma$-limit groups also are $K$-CSA.

**Lemma 12.3.** Let $G$ be a $K$-CSA group.

1. If $g$ and $h$ have infinite order, the following conditions are equivalent:
   
   (a) $g$ and $h$ have non-trivial commuting powers;
   (b) $g^{K^1}$ and $h^{K^1}$ commute;
   (c) $M(g) = M(h)$;
   (d) $\langle g, h \rangle$ is virtually abelian.

2. Any infinite virtually abelian subgroup $H$ is contained in a unique maximal virtually abelian group $M(H)$. The group $M(H)$ is $K$-virtually abelian and almost malnormal: if $M(H) \cap M(H)^9$ is infinite, then $g \in M(H)$.

**Proof.** $(c) \Rightarrow (b) \Rightarrow (a)$ in Assertion 1 is clear since $g^{K^1} \in A$ if $A \subset M(g)$ has index $\leq K$. We prove $(a) \Rightarrow (c)$. If $g^n$ commutes with $h^n$, then $g^n$ normalizes $M(h^n)$, so $M(g^n) = M(h^n)$ and $M(g) = M(g^n) = M(h^n) = M(h)$. Clearly $(c) \Rightarrow (d) \Rightarrow (a)$. This proves Assertion 1.

Being virtually abelian, $H$ contains an element $g$ of infinite order, and we define $M(H) = M(g)$. We have to show $h \in M(H)$ for every $h \in H$. This follows from Assertion 1 if $h$ has infinite order since $M(h) = M(g)$. If $h$ has finite order, we write $M(g) = M(hgh^{-1}) = hM(g)h^{-1}$, so $h \in M(g)$ because $M(g)$ equals its normalizer. A similar argument shows almost malnormality. \qed

We now prove that, for fixed $K$, the class of $K$-CSA groups is closed in the space of marked groups ($K$-CSA is a universal property). We refer to [CG05] for the topological space of marked groups, and the relation with universal theory.

**Proposition 12.4.** For any fixed $K > 0$, the class of $K$-CSA groups is defined by a set of universal sentences. In particular, the class of $K$-CSA groups is stable under taking subgroups, and closed in the space of marked groups.
Proof. For any finite group \( F = \{a_1, \ldots, a_n\} \), the fact that \( G \) does not contain a subgroup isomorphic to \( F \) is equivalent to a universal sentence saying that for any \( n \)-tuple \( (x_1, \ldots, x_n) \) satisfying the multiplication table of \( F \), not all \( x_i \)'s are distinct. Thus, the first property of \( K \)-CSA groups is defined by (infinitely many) universal sentences.

Now consider the second property. We claim that, given \( \ell \) and \( n \), the fact that \( (g_1, \ldots, g_n) \) is \( \ell \)-virtually abelian may be expressed by the disjunction \( \text{VA}_{\ell,n} \) of finitely many systems of equations in the elements \( g_1, \ldots, g_n \). To see this, let \( \pi : F_n \to G \) be the homomorphism sending the \( i \)-th generator \( x_i \) of \( F_n \) to \( g_i \). If \( A \subset \{g_1, \ldots, g_n\} \) has index \( \leq \ell \), so does \( \pi^{-1}(A) \) in \( F_n \). Conversely, if \( B \subset F_n \) has index \( \leq \ell \), so does \( \pi(B) \) in \( \langle g_1, \ldots, g_n \rangle \). To define \( \text{VA}_{\ell,n} \), we then enumerate the subgroups of index \( \leq \ell \) of \( F_n \). For each subgroup, we choose a finite set of generators \( w_i(x_1, \ldots, x_n) \) and we write the system of equations \( \{w_i(g_1, \ldots, g_n), w_j(g_1, \ldots, g_n)\} = 1 \). This proves the claim.

By Lemma 12.1 (and Zorn’s lemma), any \( g \) is contained in a maximal \( K \)-virtually abelian subgroup. The second property of Definition 12.2 is equivalent to the fact that, if \( \langle g, h \rangle \) is virtually abelian, \( \langle g, g_1, \ldots, g_n \rangle \) is \( K \)-virtually abelian, and \( g \) has order \( > K \), then \( \langle g, h, g_1, \ldots, g_n \rangle \) is \( K \)-virtually abelian. This is defined by a set of universal sentences constructed using the \( \text{VA}_{\ell,n} \)'s.

If the first two properties of the definition hold, the third one is expressed by saying that, if \( g \) has order \( > K \) and \( \langle g, hgh^{-1} \rangle \) is \( K \)-virtually abelian, so is \( \langle g, h \rangle \). This is a set of universal sentences as well.

Corollary 12.5. Let \( \Gamma \) be a hyperbolic group. There exists \( K \) such that any \( \Gamma \)-limit group is \( K \)-CSA.

Moreover, any subgroup of a \( \Gamma \)-limit group \( G \) contains a non-abelian free subgroup or is \( K \)-virtually abelian.

Remark 12.6. We will not use the “moreover”. There are additional restrictions on the virtually abelian subgroups. For instance, there exists \( N \geq 1 \) such that, if \( hgh^{-1} = g^{-1} \) for some \( g \) of infinite order, then \( hg^N h^{-1} = g'^{-N} \) for all \( g' \) of infinite order in \( M(g) \).

Proof. The first assertion is immediate from Proposition 12.4.

Now let \( H \) be an infinite subgroup of \( G \) not containing \( F_2 \). By [Kou98, Proposition 3.2], there exists a number \( M \) such that, if \( x_1, \ldots, x_M \) are distinct elements of \( \Gamma \), some element of the form \( x_i \) or \( x_i x_j \) has infinite order (i.e. order \( > K \)). This universal statement also holds in \( G \), so \( H \) contains an element \( g \) of infinite order. Recall that there exists a number \( N \) such that, if \( x, y \in \Gamma \), then \( x^N \) and \( y^N \) commute or generate \( F_2 \) (see [Del96]). The same statement holds in \( G \) since, for each non-trivial word \( w \), the universal statement \( \{x^N, y^N\} \neq 1 \Rightarrow w(x^N, y^N) \neq 1 \) holds in \( \Gamma \) hence in \( G \). Thus, for all \( h \in H \), the elements \( g^N \) and \( hg^N h^{-1} \) commute. By Lemma 12.3 \( H \) normalizes \( M(g) \), so \( H \subset M(g) \) and \( H \) is \( K \)-virtually abelian.

Let \( G \) be a \( K \)-CSA group. We now show how to define a tree of cylinders for virtually abelian splittings of \( G \) (hence also for virtually cyclic splittings). Let \( \mathcal{A} \) be the family of all virtually abelian subgroups of \( G \), and \( \mathcal{E} \) the family of infinite subgroups in \( \mathcal{A} \). Given \( H, H' \in \mathcal{E} \), define \( H \sim H' \) if \( M(H) = M(H') \). Equivalently, \( H \sim H' \) if and only if \( \langle H, H' \rangle \) is virtually abelian. The stabilizer of the equivalence class \( C \) of any \( H \in \mathcal{E} \) is \( M(H) \).

Lemma 12.7. If \( G \) is one-ended relative to \( \mathcal{H} \), the equivalence relation \( \sim \) is admissible (see Subsection 9.1).

Proof. By one-endedness, all \( (\mathcal{A}, \mathcal{H}) \)-trees have edge stabilizers in \( \mathcal{E} \). The first two properties of admissibility are obvious. Consider \( H, H' \in \mathcal{E} \) with \( H \sim H' \), and an \( (\mathcal{A}, \mathcal{H}) \)-tree \( T \) in which \( H \) fixes some \( a \) and \( H' \) fixes some \( b \). Since the group generated by two commuting elliptic groups is elliptic, there are finite index subgroups \( H_0 \subset H \) and \( H'_0 \subset H' \).
such that $\langle H_0, H'_0 \rangle$ fixes a point $c \in T$. Any edge $e \subset [a, b]$ is contained in, say, $[a, c]$, so $G_e \sim H_0 \sim H$ as required.

**Theorem 12.8.** Let $G$ be a $K$-CSA group, and $\mathcal{H}$ any family of subgroups. Assume that $G$ is one-ended relative to $\mathcal{H}$. Then:

1. The virtually abelian (resp. virtually cyclic) JSJ deformation space $D_{JSJ}$ of $G$ relative to $\mathcal{H}$ exists. Its flexible subgroups are virtually abelian, or relatively QH with finite fiber and every boundary component used.

2. The collapsed tree of cylinders $T^*_c$ of $D_{JSJ}$ is a virtually abelian (resp. virtually cyclic) JSJ tree relative to $\mathcal{H}$ and to all virtually abelian subgroups which are not virtually cyclic. It is universally compatible.

3. If $G$ is not virtually abelian, the virtually abelian compatibility JSJ tree $T_{co}$ (relative to $\mathcal{H}$) exists and $T^*_c \in D_{co}$. Trees in $D_{JSJ}$ and $D_{co}$ have the same non-virtually abelian vertex stabilizers.

The proof is similar to that of Theorem 11.1, with $C = K$. The first assertion still holds if $G$ is not one-ended relative to $\mathcal{H}$, using Linnell’s accessibility as in Lemma 7.7.

### 13 Relatively hyperbolic groups

**Theorem 13.1.** Let $G$ be hyperbolic relative to finitely generated subgroups $H_1, \ldots, H_p$. Let $\mathcal{H}$ be any family of subgroups. If $G$ is one-ended relative to $\mathcal{H}$, and every $H_i$ which contains $F_2$ is contained in a group of $\mathcal{H}$, then:

1. The elementary (resp. virtually cyclic) JSJ deformation space $D_{JSJ}$ of $G$ relative to $\mathcal{H}$ exists. Its non-elementary flexible subgroups are relatively QH with finite fiber, and every boundary component is used.

2. The collapsed tree of cylinders $T^*_c$ of $D_{JSJ}$ for co-elementarity is an elementary (resp. virtually cyclic) JSJ tree relative to $\mathcal{H}$ and to all elementary subgroups which are not virtually cyclic. It is universally compatible.

3. If $G$ is neither parabolic nor virtually cyclic, the elementary compatibility JSJ tree $T_{co}$ (relative to $\mathcal{H}$) exists and $T^*_c \in D_{co}$. Trees in $D_{JSJ}$ and $D_{co}$ have the same non-elementary vertex stabilizers (they are universally elliptic or QH).

We make a few comments before giving the proof.

Recall that a subgroup is **parabolic** if it is conjugate to a subgroup of some $H_i$, elementary if it is virtually cyclic (possibly finite) or parabolic. Co-elementarity, defined by $A \sim B$ if $\langle A, B \rangle$ is elementary, is an admissible equivalence relation on the set of infinite elementary subgroups (see [GL08], Examples 3.3 and 3.4; if $H_i$ is not contained in a group of $\mathcal{H}$, it has finitely many ends, so Lemma 3.5 of [GL08] applies). The stabilizer $G_E$ of an equivalence class is a maximal elementary subgroup, so over elementary groups no collapsing is necessary (i.e. $T_E = T^*_c$).

If $\mathcal{H} = \{H_1, \ldots, H_p\}$, then $G$ is finitely presented relative to $\mathcal{H}$, and existence of the JSJ deformation space follows from [GL09, Section 5.1].

The first assertion of the theorem remains true if $G$ is not one-ended relative to $\mathcal{H}$, provided that $G$ is accessible relative to $\mathcal{H}$. As before, the last assertion may fail over virtually cyclic groups.

**Proof.** We apply Corollary 9.2 and Proposition 9.6, with $\mathcal{A}$ consisting of all elementary (resp. virtually cyclic) subgroups, and $S$ consisting of all elementary subgroups. Elementary subgroups are small in $(\mathcal{A}, \mathcal{H})$-trees by our assumption on the $H_i$’s, and every $G_E$
is elementary. We choose \( C \) bigger than the order of any finite subgroup which is non-parabolic or contained in a virtually cyclic \( H_i \), to ensure that \( S_{\text{vvc}} \) contains no virtually cyclic subgroup.

Remark. For the first two assertions of the theorem, the hypothesis that every \( H_i \) which contains \( F_2 \) is contained in a group of \( \mathcal{H} \) may be weakened to: every \( H_i \) is small in \((\mathcal{A}, \mathcal{H})\)-trees. This requires a slight generalization of Lemma 3.5 of [GL08].

**Corollary 13.2.** Let \( G \) be hyperbolic relative to a finite family of finitely generated subgroups \( \mathcal{H} = \{H_1, \ldots, H_p\} \). Let \( \mathcal{A} \) be the family of elementary subgroups of \( G \). If \( G \) is one-ended relative to \( \mathcal{H} \), there is a JSJ tree \( T \) over \( \mathcal{A} \) relative to \( \mathcal{H} \) which is equal to its tree of cylinders, invariant under automorphisms of \( G \) preserving \( \mathcal{H} \), and compatible with every \((\mathcal{A}, \mathcal{H})\)-tree. The vertex stabilizers of \( T \) are elementary, universally elliptic, or relatively \( \text{QH} \) with finite fiber.

If \( G \) is one-ended, and no \( H_i \) is 2-ended or contains \( F_2 \), then \( T \) is also the tree of cylinders of the non-relative JSJ deformation space over \( \mathcal{A} \). It is compatible with every tree with elementary edge stabilizers.

**Proof.** We define \( T \) as the tree of cylinders of the JSJ deformation space over \( \mathcal{A} \) relative to \( \mathcal{H} \). It is equal to its tree of cylinders by Corollary 5.8 of [GL08]. Applying Theorem 13.1 with \( \mathcal{H} \) empty, we see that the tree of cylinders of the non-relative JSJ deformation space is a JSJ tree relative to all elementary subgroups which are not virtually cyclic, hence relative to \( \mathcal{H} \) because no \( H_i \) is 2-ended.

### 14 Virtually cyclic splittings

In this subsection we consider splittings of \( G \) over virtually cyclic groups, assuming smallness of their commensurizers.

Let \( \mathcal{A} \) be the family of virtually cyclic (possibly finite) subgroups of \( G \), and \( \mathcal{E} \) the family of all infinite virtually cyclic subgroups of \( G \). Recall that two subgroups \( A \) and \( B \) of \( G \) are commensurable if \( A \cap B \) has finite index in \( A \) and \( B \). The commensurability relation \( \sim \) is an admissible relation on \( \mathcal{E} \) (see [GL08]), so one can define a tree of cylinders.

The stabilizer \( G_C \) of the commensurability class \( C \) of a group \( A \in \mathcal{E} \) is its commensurizer \( \text{Comm}(A) \), consisting of elements \( g \) such that \( gAg^{-1} \) is commensurable with \( A \). The condition \( G_C \in \mathcal{A} \) does not hold, so Proposition 9.6 does not apply.

**Theorem 14.1.** Let \( \mathcal{A} \) be the family of virtually cyclic (possibly finite) subgroups, and let \( \mathcal{H} \) be any set of subgroups of \( G \), with \( G \) one-ended relative to \( \mathcal{H} \). Let \( \mathcal{S} \) be the set of subgroups of commensurizers of infinite virtually cyclic subgroups. Assume that there is a bound \( C \) for the order of finite subgroups of \( G \), and that all groups of \( \mathcal{S} \) which contain \( F_2 \) are contained in a group of \( \mathcal{H} \). Then:

1. The virtually cyclic JSJ deformation space \( \mathcal{D}_{\text{JSJ}} \) relative to \( \mathcal{H} \) exists. Its flexible subgroups either commensurize some infinite virtually cyclic subgroup, or are relative \( \text{QH} \)-subgroups with finite fiber and every boundary component used.

2. The collapsed tree of cylinders \( T^*_c \) of \( \mathcal{D}_{\text{JSJ}} \) (for commensurability) is a virtually cyclic JSJ tree relative to \( \mathcal{H} \) and the groups of \( \mathcal{S} \) which are not virtually cyclic. It is universally compatible.

3. If \( G \) contains \( F_2 \), then the virtually cyclic JSJ tree \( T_{\text{co}} \) relative to \( \mathcal{H} \) exists, and \( \mathcal{D}_{\text{co}} \) can be obtained from \( T^*_c \) by (possibly) refining at vertices \( v \) with \( G_v \) virtually \( \mathbb{Z}^2 \). Trees in \( \mathcal{D}_{\text{JSJ}} \) and \( \mathcal{D}_{\text{co}} \) have the same vertex stabilizers \( G_v \notin \mathcal{S} \).
Remark 14.2. The assumptions are satisfied if $G$ is a torsion-free CSA group, or a $K$-CSA group, or any relatively hyperbolic group whose finite subgroups have bounded order as long as all parabolic subgroups containing $F_2$ are contained in a group of $\mathcal{H}$. If $G$ is $K$-CSA, the trees of cylinders of a given $T$ for commutation and for commensurability belong to the same deformation space (this follows from Lemma 12.3).

See Subsection 11 for an example with $T^*_c \not\in D_{co}$.

Proof. The first two assertions follow from Corollary 9.2 as in the previous examples. For the third assertion, let $T$ be a JSJ tree (relative to $\mathcal{H}$). We know that $T^*_c$ is universally compatible by Corollary 9.5. By [GL08, Remark 5.11], $T_c$ and $T^*_c$ are in the same deformation space. This implies that any group elliptic in $T^*_c$ but not in $T$ belongs to $\mathcal{S}$, hence does not contain $F_2$. Existence of $D_{co}$ follows as there are only finitely many deformation spaces between $T$ and $T^*_c$ by Proposition 6.1 of [GL09]. The assumption $F_2 \subset G$ guarantees that $D_{co}$ is trivial or irreducible, so $T_{co}$ is defined.

One may obtain a tree $T' \in D_{co}$ by refining $T^*_c$ at vertices $v$ with $G_v$ not elliptic in $T$. Such a $G_v$ does not contain $F_2$, so is small in $T'$. If it fixes exactly one end, then $D_{co}$ is an ascending deformation space, contradicting [GL07, Prop.7.1(4)]. Thus $G_v$ acts on a line, hence is virtually $\mathbb{Z}^2$ because edge stabilizers are virtually cyclic.

We also have:

**Theorem 14.3.** Let $G$ be torsion-free and commutative transitive. Let $\mathcal{H}$ be any family of subgroups. If $G$ is freely indecomposable relative to $\mathcal{H}$, then:

1. The cyclic JSJ deformation space $D_{JSJ}$ of $G$ relative to $\mathcal{H}$ exists. Its flexible subgroups are QH surface groups, with every boundary component used.

2. Its collapsed tree of cylinders $T^*_c$ (for commensurability) is a JSJ tree relative to $\mathcal{H}$ and all subgroups isomorphic to a solvable Baumslag-Solitar group $BS(1, s)$. It has the same non-solvable vertex stabilizers as trees in $D_{JSJ}$. It is universally compatible.

3. If $G$ is not a solvable Baumslag-Solitar group, the cyclic compatibility JSJ deformation space $D_{co}$ relative to $\mathcal{H}$ exists and may be obtained by (possibly) refining $T^*_c$ at vertices with stabilizer isomorphic to $\mathbb{Z}^2$.

Recall that $G$ is *commutative transitive* if commutation is a transitive relation on $G \setminus \{1\}$.

Proof. Here we do not apply Corollary 9.2 directly, because we do not have enough control on the groups $G_C$ (though they may be shown to be metabelian).

By Proposition 6.5 of [GL08], if $T$ is any tree with cyclic edge stabilizers, a vertex stabilizer of its collapsed tree of cylinders which is not elliptic in $T$ is a solvable Baumslag-Solitar group $BS(1, s)$ (with $s \neq -1$ because of commutative transitivity). Arguing as in the proof of Lemma 9.1, we deduce that Theorem 8.3 applies, with $T^*$ the collapsed tree of cylinders and $\mathcal{S}$ consisting of all solvable Baumslag-Solitar subgroups and their subgroups. No subgroup of $BS(1, s)$ can be flexible, except if $G_v = G \simeq \mathbb{Z}^2$, so all flexible groups are surface groups.

Assertion (2) of the theorem follows from Lemma 8.7. For Assertion (3), one argues as in the proof of Theorem 14.1.

15 Reading actions on $\mathbb{R}$-trees

Rips theory gives a way to understand stable actions on $\mathbb{R}$-trees, by relating them to actions on simplicial trees. Therefore, they are closely related to JSJ decompositions. We consider first the JSJ deformation space, then the compatibility JSJ tree.
15.1 Reading $\mathbb{R}$-trees from the JSJ deformation space

**Proposition 15.1.** Let $G$ be finitely presented. Let $T_J$ be a JSJ tree over the family of slender subgroups. If $T$ is an $\mathbb{R}$-tree with a stable action of $G$ whose arc stabilizers are slender, then edge stabilizers of $T_J$ are elliptic in $T$.

**Proof.** By [Gui98], $T$ is a limit of simplicial trees $T_k$ with slender edge stabilizers. Since $T_J$ is universally elliptic, edge stabilizers of $T_J$ are elliptic in $T_k$. They are elliptic in $T$ because they are finitely generated and all their elements are elliptic.

**Remark 15.2.** More generally, suppose that $G$ is finitely presented and $A$ is stable under extension by finitely generated free abelian groups: if $H < G$ is such that $1 \to A \to H \to \mathbb{Z}^k \to 1$, with $A \in A$, then $H \in A$. Let $T_J$ be a JSJ tree over $A$ with finitely generated edge stabilizers (this exists by Theorem 4.3 of [GL09]). If $T$ is a stable $\mathbb{R}$-tree with arc stabilizers in $A$, then edge stabilizers of $T_J$ are elliptic in $T$.

Recall [FP06] that, when $G$ is finitely presented, flexible vertices of the slender JSJ deformation space are either slender or QH (with slender fiber).

**Proposition 15.3.** Let $G, T_J, T$ be as in the previous proposition. There exists an $\mathbb{R}$-tree $\hat{T}$ obtained by blowing up each flexible vertex $v$ of $T_J$ into

1. an action by isometries on a line if $G_v$ is slender,
2. an action dual to a measured foliation on the underlying 2-orbifold of $G_v$ if $v$ is QH,

which resolves (or dominates) $T$ in the following sense: there exists a $G$-equivariant map $f : \hat{T} \to T$ which is piecewise linear.

**Proof.** Using ellipticity of $T_J$ with respect to $T$, we argue as in the proof of [GL09, Lemma 3.2], with $T_1 = T_J$ and $T_2 = T$. If $v \in V(T_J)$ and $G_v$ is elliptic in $T$, we let $Y_v \subset T$ be a fixed point. If $G_v$ is not elliptic in $T$, we let $Y_v$ be its minimal subtree. It is a line if $G_v$ is slender. If $v$ is a QH vertex, then $Y_v$ is dual to a measured foliation of the underlying orbifold by Skora’s theorem [Sk09] (applied to a covering surface $\Sigma_0$).

**Remark 15.4.** The arguments given above may be applied in more general situations. For instance, assume that $G$ is finitely generated, that all subgroups of $G$ not containing $F_2$ are slender, and that $G$ has a JSJ tree $T_J$ over slender subgroups whose flexible subgroups are QH. Let $T$ be an $\mathbb{R}$-tree with slender arc stabilizers such that $G$ does not split over a subgroup of the stabilizer of an unstable arc or of a tripod in $T$. Then, applying [Gui08] and the techniques of [Gui98], we see that $T$ is a limit of slender trees, so Propositions 15.1 and 15.3 apply.

15.2 Reading $\mathbb{R}$-trees from the compatibility JSJ tree

In [Gui00b], the first author explained how to obtain all small actions of a one-ended hyperbolic group $G$ on $\mathbb{R}$-trees from a JSJ tree. The proof was based on Bowditch’s construction of a JSJ tree from the topology of $\partial G$. Here we give a different, more general, approach, based on Corollary 3.9 (saying that compatibility is a closed condition) and results of Part III (describing the compatibility JSJ space). Being universally compatible, $T_{co}$ is compatible with any $\mathbb{R}$-tree which is a limit of simplicial $A$-trees. We illustrate this idea in a simple case.

Let $G$ be a one-ended finitely presented torsion-free CSA group. Assume that $G$ is not abelian, and let $T_{co}$ be its compatibility JSJ tree over the class $A$ of abelian groups (see Definition 4.7 and Theorem 11.1). Let $G \circ T$ be an action on an $\mathbb{R}$-tree with trivial tripod stabilizers, and abelian arc stabilizers. By [Gui98], $T$ is a limit of simplicial $A$-trees. Since $T_{co}$ is compatible with all $A$-trees, it is compatible with $T$ by Corollary 3.9. Let
\( \hat{T} \) be the standard common refinement of \( T \) and \( T_{co} \) with length function \( \ell_T + \ell_{T_{co}} \) (see Subsection 3.2). Let \( f_{co} : \hat{T} \to T_{co} \) and \( f : \hat{T} \to T \) be maps preserving alignment such that \( d_f(x, y) = d_{T_{co}}(f_{co}(x), f_{co}(y)) + d_T(f(x), f(y)) \).

To each vertex \( v \) and each edge \( e \) of \( T_{co} \) there correspond closed subtrees \( \hat{T}_v = f_{co}^{-1}(v) \) and \( T_e = \hat{T}^{-1}(e) \) of \( T \). By minimality, \( \hat{T}_v \) is a segment of \( \hat{T} \) containing no branch point except maybe at its endpoints. The relation between \( d_{\hat{T}}, d_{T_{co}} \), and \( d_T \) shows that the restriction of \( f \) to \( \hat{T}_v \) is an isometric embedding. In particular, \( T \) can be obtained from \( \hat{T} \) by changing the length of the segments \( \hat{T}_e \) (possibly making the length 0).

We shall now describe the action of \( G_v \) on \( \hat{T}_v \). Note that \( G_v \) is infinite. Its action on \( \hat{T}_v \) need not be minimal, but it is finitely supported, see [Gui08]. Given an edge \( e \) of \( T_{co} \) containing \( v \), we denote by \( x_e \) the endpoint of \( \hat{T}_v \) belonging to \( \hat{T}_v \). The tree \( \hat{T}_v \) is the convex hull of the points \( x_e \).

First suppose that \( G_v \) fixes some \( x \in \hat{T}_v \). Note that, if \( x \neq x_e \), the stabilizer of the arc \([x, x_e] \) contains \( G_v \), so is infinite. We claim that, if \( e \) and \( f \) are edges of \( T_{co} \) containing \( v \) with \( x_e \neq x_f \), then \([x, x_e] \cup [x, x_f] \) contains no tripod. Assume otherwise. Then the intersection \([x, x_e] \cap [x, x_f] \) is an arc \([x, y] \). The stabilizer of \([x, y] \) is abelian and must also fix \( x_e \) and \( x_f \) by triviality of tripod stabilizers. This proves that \( \text{Stab}[x, y] \) fixes \([x, x_e] \cup [x, x_f] \), contradicting triviality of tripod stabilizers. We have thus proved that \( \hat{T}_v \) is a cone on a finite number of orbits of points.

If \( G_v \) does not fix a point in \( \hat{T}_v \), then it is flexible. If it is abelian, triviality of tripod stabilizers implies that \( \hat{T}_v \) is a line. If \( G_v \) is not abelian, it is a surface group by Theorem 11.1. Skora’s theorem [Sk096] asserts that the minimal subtree \( (T_v)_{min} \) of \( G_v \) is dual to a measured lamination on a compact surface. By triviality of tripod stabilizers, \( T_v \ \setminus (T_v)_{min} \) is a disjoint union of segments, and the pointwise stabilizer of each such segment has index at most 2 in a boundary subgroup of \( G_v \).

It follows from particularity from this analysis that \( \hat{T} \) and \( T \) are geometric (see [LP97]).

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Vincent Guirardel
Institut de Recherche Mathématique de Rennes
Université de Rennes 1 et CNRS (UMR 6625)
263 avenue du Général Leclerc, CS 74205
F-35042 RENNES Cedex
e-mail:vincent.guirardel@univ-rennes1.fr

Gilbert Levitt
Laboratoire de Mathématiques Nicolas Oresme
Université de Caen et CNRS (UMR 6139)
BP 5186
F-14032 Caen Cedex
France
e-mail:levitt@math.unicaen.fr