Axially symmetric static scalar solitons 
and black holes with scalar hair

Burkhard Kleihaus, Jutta Kunz, Eugen Radu and Bintoro Subagyo
Institut für Physik, Universität Oldenburg, Postfach 2503 D-26111 Oldenburg, Germany

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Abstract

We construct static, asymptotically flat black hole solutions with scalar hair. They evade the no-hair theorems by having a scalar potential which is not strictly positive. By including an azimuthal winding number in the scalar field ansatz, we find hairy black hole solutions which are static but axially symmetric only. These solutions possess a globally regular limit, describing scalar solitons. A branch of axially symmetric black holes is found to possess a positive specific heat.

Introduction.– The energy conditions are an important ingredient of various significant results in general relativity [1]. Essentially, they imply that some linear combinations of the energy-momentum tensor of the matter fields should be positive, or at least non-negative. However, over the last decades, it has become increasingly obvious that these conditions can be violated, even at the classical level. Remarkably enough, the violation may occur also for the simplest case of a scalar field (see e.g. [2] for a discussion of these aspects).

Once we give up the energy conditions (and in particular the weak one), a number of results in the literature show that the asymptotically flat black holes may possess scalar hair\(^1\), which otherwise is forbidden by a number of well-known theorems [4]. Restricting to the simplest case of a minimally coupled scalar field with a scalar potential which is not strictly positive, this includes both analytical [5], [6], [7], [8], [9] and numerical [10], [11] results.

Interestingly, in the limit of zero event horizon radius, some of these hairy black holes describe globally regular, particle-like objects, the so-called ‘scalarons’ [10]. At the same time, a complex scalar field is known for long time to possess non-topological solitonic solutions [12], even in the absence of gravity. These are the Q-balls introduced by Coleman in [13]. Such configuration owe their existence to a harmonic time dependence of the scalar field and possess a positive energy density.

However, as argued below, the Q-balls can be reinterpreted as non-gravitating scalarons. The scalar field is static in this case and has a potential which takes negative values as well. As expected, the scalarons possess gravitating generalizations. However, different from the standard Q-ball case [14], their regular origin can be replaced with an event horizon. In this work we study such solutions for the simple case of a massive complex scalar field with a negative quartic self-interaction term in the potential. Apart from spherically symmetric configurations, we construct solitons and hairy black hole solutions which are static but axially symmetric only.

The model.– Let us consider the action of a self-interacting complex scalar field $\Phi$ coupled to Einstein gravity in four spacetime dimensions,

$$ S = \int \! d^4 x \sqrt{-g} \left[ \frac{1}{16\pi G} R - \frac{1}{2} g^{\mu\nu} (\Phi^*, \Phi, \Phi, \Phi) + U \right]. $$

\(^1\)One has to remark that the existence of black holes with scalar hair is perhaps the mildest consequence of giving up the energy conditions, see e.g. the discussion in [3].
where $R$ is the curvature scalar, $G$ is Newton’s constant and the asterisk denotes complex conjugation. Using the principle of variation, one finds the coupled Einstein–Klein-Gordon equations

$$E_{\mu \nu} = R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R - 8 \pi G T_{\mu \nu} = 0, \quad \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \Phi) = \frac{\partial U}{\partial |\Phi|^2} \Phi, \quad (2)$$

where $T_{\mu \nu}$ is the stress-energy tensor of the scalar field

$$T_{\mu \nu} = (\Phi^* \mu, \Phi + \Phi^* \nu, \Phi, \Phi) - g_{\mu \nu} \left[ \frac{1}{2} g^{\alpha \beta} (\Phi^* \alpha, \Phi \beta + \Phi^* \beta, \Phi, \alpha) + U \right]. \quad (3)$$

In the above relations $U$ denotes the scalar field potential, which, in order to retain the $U(1)$ symmetry of the whole Lagrangian, must be a function of $|\Phi|^2$. In what follows, we assume that $U$ can be written as

$$U = \sum_{k \geq 1} c_k |\Phi|^{2k}, \quad (4)$$

the $k > 1$ terms taking effectively into account various interactions. Of interest here is the case of a potential which is not strictly positive definite. Then the polynomial $F(x) = \sum_{k \geq 1} c_k x^k$ is negative for some range of $x > 0$, which implies that at least one of coefficients $c_k$ is smaller than zero. Since we assume $2 \Phi \rightarrow 0$ asymptotically, the requirement to obtain a bound state imposes $c_1 = \mu^2 > 0$, with $\mu$ the scalar field mass.

**Flat space solitons: Q-balls as scalarons.**– Let us start our discussion with the simple observation that when ignoring the gravity effects, a class of solutions of the model (1) is already known. We recall that in a flat spacetime background, the Klein-Gordon equation possesses non-topological soliton solutions—the so-called Q-balls, in which case the scalar has a harmonic time dependence, $\Phi = \phi(x) e^{-i \omega t}$ [13] (with $x^\mu = (x^a, t)$). As a result, the solutions possess a nonvanishing conserved Noether charge, $Q = 2w \int d^3 x |\phi|^2$.

Then, even though $\Phi$ depends on time, the energy-momentum tensor $T_{\mu \nu}$ is time independent and the effective action of this model reads

$$S_Q = -\int d^3 x dt \left[ \phi^* a \phi \phi^a - w^2 |\phi|^2 + U \right]. \quad (5)$$

The Q-balls have been extensively discussed in the literature (see the review work [12], [15]) and they have found a variety of physically interesting applications. If one assumes a potential of the form (4), then $U$ necessarily contains powers of $|\Phi|^2$ higher than two, the usual choice in the literature being $U = \mu^2 |\Phi|^2 - |\lambda| |\Phi|^4 + \nu |\Phi|^6$, with $\lambda > 0$, $\nu > 0$ and $\lambda^2 < 4 \mu^2 \nu$ for a positive potential.

However, one can see from (5) that $w^2$ acts as an effective tachyonic contribution to the mass term, and thus it can be absorbed into $\mu^2$. The scalar field is static in this case, $\Phi = \phi(x^a)$ and thus the Noether charge vanishes. Therefore all Q-ball solutions in a flat spacetime background can be interpreted as static scalar solitons, i.e. they become scalarons in a model with a shifted scalar field mass, for a new potential $U = U_Q - w^2 |\phi|^2$. Note that although $\phi$ satisfies the same equation as before, the energy-momentum tensor and the total mass of the scalarons are different. Also, as implied by the Derick-type virial identity

$$\int d^3 x \left[ \phi^* a \phi \phi^a + 3U \right] = 0, \quad (6)$$

the redefined potential $U$ is necessarily negative for some range of $|\phi|^2$ which is realised by the solutions. However, the scalarons’ total mass is strictly positive,

$$M = \int d^3 x \left[ \phi^* a \phi \phi^a + U \right] = \frac{2}{3} \int d^3 x \phi^* a \phi \phi^a. \quad (7)$$

Finally, we mention also that the mass of the scalarons is fixed by the mass $M_Q$ and the Noether charge $Q$ of the Q-balls, $M = M_Q - wQ$.

\[This can always be realized via a redefinition of the scalar field.\]

\[For example, the Q–ball solutions appear in supersymmetric generalizations of the standard model [16]. Also, they may be responsible for the generation of baryon number or may even be regarded as candidates for dark matter [17].\]
Spherically symmetric, gravitating solutions.— However, the curved spacetime scalarons cannot be interpreted as boson stars and thus require a separate study. For example, following [14], one can show that, even in the absence of backreaction, one cannot add a black hole horizon inside a Q-ball\(^4\) (this follows essentially because a Q-ball possesses a \(e^{-iut}\) time dependence and \(t \to \infty\) at the horizon of a black hole). However, this obstruction does not apply to scalarons, which possess finite energy, regular generalizations also for a static black hole background.

Let us start with a discussion of the spherically symmetric gravitating solutions of the model (1). These configurations are easier to study and some of their properties seem to be generic. A sufficiently general metric ansatz in this case reads

\[
ds^2 = g_{rr}dt^2 + g_{\Omega\Omega}d\Omega^2_2 + g_{tt}dt^2,
\]

(with \(d\Omega^2_2 = d\theta^2 + \sin^2\theta d\varphi^2\)), and the scalar field is a function of \(r\) only, \(\Phi = Z(r)\). One possible direction here is to choose a metric gauge with \(-g_{tt} = 1/g_{rr} = V(r), g_{\Omega\Omega} = P^2(r)\). Then the Einstein equations imply the relation \(P'' + 8\pi GZ'^2 = 0\) (where the prime denotes a derivative with respect to \(r\)). The approach taken in [5], [6] (see also [7]) is to postulate an expression for the scalar field and to use this relation to derive \(P\).

In the next step, the remaining Einstein equations are used to reconstruct the scalar potential \(U\) and the metric function \(V\) compatible with \(Z\) and \(P\). This approach has the advantage to lead to partially closed form solutions, but the resulting expressions are very complicated; also the potential cannot be written in the form (4).

In what follows we solve the field equations numerically for a given potential. In this case it is convenient to work in Schwarzschild-like coordinates with

\[
g_{rr} = \frac{1}{N(r)}, \quad g_{\Omega\Omega} = r^2, \quad g_{tt} = -N(r)\sigma^2(r), \quad \text{with} \quad N(r) = 1 - \frac{2m(r)}{r},
\]

where \(m(r)\) may be interpreted as the total mass-energy within the radius \(r\); its derivative \(m'\) is proportional to the energy density \(\rho = -T^t_t\). Then the field equations (2) reduce to

\[
m' = 4\pi Gr^3(NZ'^2 + U), \quad \sigma' = 8\pi Gr\sigma Z'^2, \quad Z'' + \left(\frac{\sigma'}{\sigma} + \frac{N'}{N} + \frac{2}{r}\right)Z' - \frac{1}{N}\frac{\partial U}{\partial Z^2}Z = 0.
\]

For a generic \(U\), it is possible to write an approximate form of the solutions close to the horizon (or at the origin) and also for large \(r\). These asymptotics are connected by constructing numerically the solutions, which requires to specify the expression for the scalar field potential.

The horizon of the black holes is located at \(r = r_H > 0\), where the solutions have a power-series expansion

\[
m(r) = \frac{r_H}{2} + m_1(r - r_H) + \ldots, \quad \sigma(r) = \sigma_0 + 8\pi G\sigma_0 r_H z^2_1(r - r_H) + \ldots, \quad Z(r) = z_0 + z_1(r - r_H) + \ldots
\]

in terms of two arbitrary parameters \(Z(r_H) = z_0\) and \(\sigma(r_H) = \sigma_0\) (with \(m_1 = 4\pi Gr^2_HU(z_0)\), and \(z_1 = \frac{r_H}{1-2m_1}\frac{\partial U}{\partial Z^2}|_{z_0}z_0\)). One can write an approximate form of the solutions also for \(r \to \infty\), with

\[
m(r) = GM - 4\pi G\mu z^2_1 e^{-2\mu r} + \ldots, \quad \log \sigma(r) = -8\pi Gz^2_1 \mu \left(\frac{e^{-2\mu r}}{r} + \mu Ei(-2\mu r)\right) + \ldots, \quad Z(r) = z_1 e^{-\mu r} + \ldots,
\]

with \(Ei(x)\) the exponential integral function [19]; \(M, z_1\) are two parameters fixed by the numerical calculations, \(M\) corresponding to the total mass of the solutions.

The Hawking temperature and event horizon area of a spherically symmetric black hole are

\[
T_H = \frac{\sigma'(r_H)}{4\pi r_H^3} (1 - 2m'(r_H)), \quad A_H = 4\pi r_H^2,
\]

the entropy of the solutions being \(S = A_H/4G\).

\(^4\)Note, however, the boson shells harbouring black holes in [18]. These solutions require a \(V\)-shaped scalar potential which is not of the form (4).
For any symmetry, the static black hole solutions satisfy the Smarr relation [20]
\[ M = 2T_H S + M^{(ext)}, \]
where
\[ M^{(ext)} = - \int_{r>r_H} d^3x \sqrt{-g} (2T^t_t - T^\mu_\mu), \]
is the contribution to the total mass of the matter outside the event horizon (with \( d^3x = 4\pi \int_{r_H}^{\infty} dr \) for spherically symmetric configurations), and the first law of thermodynamics [21],
\[ dM = T_H dS. \]
Also, by using the approach in [22], one can prove the following virial identity:
\[ \int_{r_H}^{\infty} dr \sigma r^2 \left[ Z'^2 \left( 1 - \frac{r_H}{r} (1 + N) \right) + U \left( 3 - \frac{2r_H}{r} \right) \right] = 0, \]
\((r_H = 0 \text{ gives the corresponding relation for the solitonic case}).

For a quantitative study of the solutions, we need to specify the expression for \( U \). The results reported in this work correspond to the simplest potential allowing for \( U < 0 \), with
\[ U = \mu^2 |\Phi|^2 - \lambda |\Phi|^4, \]
where \( \lambda \) is a strictly positive parameter. The existence of hairy black hole solutions for this choice of the potential has been noticed in [23]; a possible physical justification for this expression of \( U \) can also be found there.

In this case, the system possesses two scaling symmetries (these symmetries are independent of any specific ansatz and hold also for the axially symmetric solutions below):
\[(i) \ x^a \rightarrow x^a c, \ \mu \rightarrow \mu/c, \ \lambda \rightarrow \lambda/c, \ \text{and} \ (ii) \ \Phi \rightarrow \Phi c, \ \lambda \rightarrow \lambda/c^2, \ G \rightarrow G/c^2, \]
(also with \( m \rightarrow mc \) for (i); note the invariant parameters are not shown here) which are used to define a dimensionless radial variable \( r \rightarrow r_H \) and a scaled scalar field, \( \Phi \rightarrow \Phi \sqrt{4\pi/M_{Pl}} \) (with \( M_{Pl} = 1/\sqrt{G} \) the Planck mass for the units employed in this work). Then, for a given \( r_H \), families of solutions can be parametrized by the single dimensionless quantity
\[ \Lambda = \frac{\lambda M^2_{Pl}}{4\pi \mu^2}. \]
Also, all quantities in this work are given in natural units set by μ and G.

The properties of the spherically symmetric solutions can be summarized as follows\(^5\). First, the model possesses soliton configurations \((r_H = 0)\), which are the gravitating scalarons. These globally regular configurations possess at \(r = 0\) a power-series expansion with (here we employ dimensionless variables) \(m(r) = \frac{1}{6}b^2(1 - b^2\Lambda)r^3 + O(r^5)\), \(\sigma(r) = \sigma_0 + 2\sigma_0 z_2 r^4 + O(r^6)\), \(Z(r) = b + z_2 r^2 + O(r^4)\) (with \(z_2 = b(1 - 2b^2\Lambda)/6\)), while for large \(r\) the expressions (12) are still valid.

The only input parameter in this case is \(\Lambda\). Rather unexpected, we have found that the scalarons do not exist for an arbitrarily small coefficient of the quartic term in the potential. That is, we could find solutions around the origin, both the scalar field and the metric functions possessing a nontrivial dependence on \(\sigma\). Thus, as expected, the se solutions possess a negative specific heat.

Furthermore, it turns out that the free energy \(F\) decreases (see Figure 2). These equations imply that both \(\delta \sigma(r, t)\) and \(\delta N(r, t)\) are determined by \(\delta Z(r, t)\). For a harmonic time dependence \(e^{-\Omega^2t}\), the linearized scalar field equation reduces to a standard Schrödinger equation

\[
\left\{-\frac{d^2}{d\rho^2} + V_{\text{eff}}\right\} \Psi(\rho) = \Omega^2 \Psi(\rho),
\]

\(^5\)The solutions reported in this work have a nodeless scalar field. However, solutions with \(Z\) taking both positive and negative values do also exist, but we did not attempt to study them systematically.
where $\Psi = \delta Z e^{i\Omega t}/r$ and a new radial coordinate is introduced, $d/d\rho = N d/dr$. The effective potential in (21) is given by $V_{\rho \rho} = \sigma^2 N \left[ \frac{2}{r} \left( \frac{1}{2r} + \frac{1}{\bar{r}} \right) - 4rN \left( \frac{1}{2r} + \frac{1}{\bar{r}} + 1 \right) Z^2 + 8r^2 \frac{d^2 Z}{\bar{r} d\rho} + \frac{1}{2} \frac{d^2 \bar{r}}{d\rho^2} \right]$. By solving numerically the above Schrödinger equation with suitable boundary conditions (namely $\Psi(r_H) = 0$ and $\Psi(r) \to 0$ as $r \to \infty$), we have found that $\Omega^2 < 0$ in all cases considered. Thus we conclude that these scalar hairy black holes are unstable against linear fluctuations. This result holds also when considering instead the soliton case.

**Static axially symmetric black holes.**—All known static black hole solutions with scalar hair are spherically symmetric. However, a Q-ball model possesses also solutions with a spinning phase [25], [26], [27], [15]. As discussed above, when ignoring the gravity effects, these Q-balls can be interpreted as static, axially symmetric scalarons in a flat spacetime background. The scalar field in this case is complex, with a phase depending on the azimuthal angle $\varphi$,

$$\Phi = Z(r, \theta) e^{in\varphi},$$

(22)

with $n$ a winding number, $n = \pm 1, \pm 2, \ldots$ (the value $n = 0$ corresponds to the spherically symmetric case discussed above). These solutions should survive when considering the full model (1); therefore we expect to find black hole solutions as well, which are static but axially symmetric only$^6$.

In the numerical construction of such solutions, we have found it convenient to use a metric ansatz with three independent functions$^7$

$$ds^2 = -e^{2F_0(r, \theta)} \Delta(r) dt^2 + e^{2F_1(r, \theta)} \left( \frac{dr^2}{\Delta(r)} + r^2 d\theta^2 \right) + e^{2F_2(r, \theta)} r^2 \sin^2 \theta d\varphi^2,$$

and a scalar field given by (22). $r, \theta$ and $\varphi$ are spherical coordinates; however, the coordinate range for $r$ is $r_{H} \leq r < \infty$, with $r = r_{H}$ corresponding to an event horizon (thus we do not consider the behaviour of the solutions inside the horizon).

A straightforward computation leads to the following expressions for the horizon area, Hawking temperature and mass of the solutions:

$$A_{H} = 4\pi \bar{r}_{H}^2 \int_{0}^{\pi} d\theta \sin \theta e^{F_1(r_{H}, \theta) + F_2(r_{H}, \theta)}, \quad T_{H} = \frac{1}{4\pi \bar{r}_{H}} e^{F_0(r_{H}, \theta) - F_1(r_{H}, \theta)}, \quad M = \frac{\bar{r}_{H}}{2} - c,$$

(24)

where $c$ is a constant which enters the leading order terms in the large $r$ expansion of the metric functions, $F_1 = -\frac{\varphi}{\theta} + \ldots$, $F_2 = -\frac{\varphi}{\theta} + \ldots$, $F_0 = \frac{\varphi}{\theta} + \ldots$.

The equations for the metric functions $F = (F_0, F_1, F_2)$ employed in the numerical calculations, are found by using a suitable combination of the Einstein equations $E^t_t = 0$, $E^i_i + E^\theta_\theta = 0$ and $E^\varphi_\varphi = 0$, which diagonalizes them with respect to $\nabla^2 F$ (where $\nabla^2 = \partial_r^2 + \frac{\varphi}{\theta} \partial_r + \frac{1}{\sin \theta} \partial_\theta$). In the numerical calculations, it turns out to be convenient to introduce a new radial variable $\bar{r} = \sqrt{r^2 - r_{H}^2}$, such that the event horizon is located at $\bar{r} = 0$. Then the Einstein-Klein-Gordon equations are solved with the following boundary conditions:

$$F|_{\bar{r}=\infty} = Z|_{\bar{r}=\infty} = 0, \quad \partial_{\bar{r}} F|_{\bar{r}=0} = \partial_{\bar{r}} Z|_{\bar{r}=0} = 0, \quad \partial_\theta F|_{\theta=0,\pi} = Z|_{\theta=0,\pi} = 0,$$

which follow from a study of the approximate form of the solutions, similar to (11), (12). The absence of conical singularities imposes on the symmetry axis the supplementary condition $F_1|_{\theta=0,\pi} = F_2|_{\theta=0,\pi}$, which is used to verify the accuracy of the solutions. Other numerical tests were provided by the Smarr relation (14) and the first law (16). Based on that, we estimate a typical relative error around $10^{-3}$ for the solutions reported here$^8$. Also, all solutions here reach a reflection symmetry with respect to the equatorial plane, which is used to restrict the domain of integration for $\theta$ to $[0, \pi/2]$, with $\partial_\theta F|_{\theta=\pi/2} = \partial_\theta Z|_{\theta=\pi/2} = 0$.

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$^6$All known solutions with this property exist in models with gravitating non-Abelian fields [28].

$^7$Some of the solutions have been recovered by using a Lewis-Papapetrou-type metric Ansatz instead of (23), with $ds^2 = -fdr^2 + \frac{2}{r} (dr^2 + r^2 d\theta^2) + \frac{1}{r^2} \sin^2 \theta d\varphi^2$, the functions $f, l, m$ depending on $r$ and $\theta$ (note that the radial coordinate here differs from the one used in (23)).

$^8$The resulting set of four coupled non-linear partial differential equations is solved numerically by employing a finite difference solver, based on the Newton-Raphson method. To decrease the errors, a new compactified radial coordinate $x = \bar{r}/(1 + \bar{r})$ is introduced. Then the equations are discretized on a non-equidistant grid in $x$ and $\theta$. Typical grids used have sizes $250 \times 40$, covering the integration region $0 \leq x \leq 1$ and $0 \leq \theta \leq \pi/2$. 

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Figure 3. Some properties of static axially symmetric black holes with scalar hair are shown as functions of the Hawking temperature for several values of the dimensionless self-coupling constant $\Lambda$.

The picture in the axially symmetric case is more complicated than the one found for spherically symmetric solutions. The solutions reported here have a winding number $n = 1$; however, we have also constructed configurations with $n = 2, 3$.

As expected, the limit $r_H = 0$ still corresponds to static solitons (the boundary conditions at the origin are $\partial_r F \big|_{r=0} = Z \big|_{r=0} = 0$). These axially symmetric scalarons have an intrinsic toroidal shape, as one can see e.g. by plotting surfaces of constant energy density. Also in this case $\rho = -T_t^t$ becomes negative in a region close to the origin (however, for both solitons and black holes, the total mass is still positive).

Again, for any soliton, the origin can be replaced by a black hole with a small radius. However, although it resides at a constant value of $r$, the horizon is not a round sphere. This can be seen by evaluating the circumference of the horizon along the equator, $L_e = 2 \pi r_H e^{F_2(r_H, \pi/2)}$, and the circumference of the horizon along the poles, $L_p = 2 r_H \int_0^{\pi} d\theta e^{F_2(r_H, \theta)}$. We have found that the ratio $L_e/L_p$ is always slightly larger than one.

Some thermodynamical properties of these black hole solutions are shown in Figure 3 as a function of the temperature (with $T_H \to \infty$ corresponding again to the scalaron limit), for several values of the self-coupling constant $\Lambda$. One can see that the horizon size cannot be arbitrarily large, while the mass takes smaller values than in the spherically symmetric case (note the existence, for some range of $T_H$, of two different solutions with the same mass). Another puzzling feature there is the existence of a branch of configurations with a positive specific heat.

Also, any family of axially symmetric black holes with a fixed $\Lambda$, emerges smoothly from the respective scalaron and appears to end in a critical configuration. Extrapolating the numerical results suggests that the mass and the horizon area of these critical configurations remain finite, while the temperature tends to zero. Constructing such solutions explicitly, and thus clarifying this limiting behaviour of axially symmetric black holes, however, remains a numerical challenge beyond the purposes of this paper.

Further remarks.— All known asymptotically flat scalarons and scalar hairy black holes in the literature are spherically symmetric and were found for a rather complicated potential. In this work we have shown that such solutions exist already in a simple model which possesses only a negative quartic term in the scalar field potential in addition to the usual mass term. Furthermore, black hole solutions with a regular event horizon which are static and possess only axial symmetry, do also exist. Rather unexpected, some of these black holes have a positive specific heat.

In fact, we predict a variety of more complicated solutions to exist. For example, based on the analogy with Q-balls [27], odd-parity static axially symmetric scalarons and black holes should also exist, with the scalar field vanishing in the equatorial plane. Moreover, it is likely that the model possesses solitons and hairy black holes with discrete crystal-like symmetries only.

Although the analysis in this work was restricted to the (simplest) case of the potential (18), we expect...
some basic features of the solutions in this work to be generic. This conjecture is based mainly on the analogy with Q-balls and their gravitating generalizations—boson stars, which are known to possess a certain degree of universality of the properties, for any potential choice (see e.g. [29]). Apart from that, we have verified that the qualitative features of the solutions in this work remain unchanged when adding a positive sextic term to the potential (18), provided that $U$ is negative for a range of $|\Phi|^2$.

We hope to return elsewhere with a systematic study of these solutions, including an existence proof for the spherically symmetric case. Finally, let us mention that we have verified that the solitons and black holes in this work possess five dimensional generalizations with rather similar properties. It is likely that they can be generalized for any spacetime dimension $d > 4$.

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References

[1] S. W. Hawking and G. F. R. Ellis, *The Large scale structure of space-time*, Cambridge University Press, Cambridge, 1973

[2] C. Barcelo and M. Visser, Int. J. Mod. Phys. D 11 (2002) 1553 [gr-qc/0205066].

[3] T. Hertog, G. T. Horowitz and K. Maeda, JHEP 0305 (2003) 060 [hep-th/0304199].

[4] J. D. Bekenstein, *Black hole hair: 25 - years after*, [gr-qc/9605059].

[5] E. Radu and M. S. Volkov, Phys. Rept. 221 (1992) 251.

[10] M. Heusler and N. Straumann, Class. Quant. Grav. 10 (1993) 1299.

[21] F. E. Schunck and D. F. Torres, Int. J. Mod. Phys. D 9 (2000) 601 [gr-qc/9911038].