EXTINCTION PROFILE OF THE LOGARITHMIC DIFFUSION EQUATION

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ABSTRACT. Let $u$ be the solution of $u_t = \Delta \log u$ in $\mathbb{R}^N \times (0,T)$, $N = 3$ or $N \geq 5$, with initial value $u_0$ satisfying $B_{k_0}(x,0) \leq u_0 \leq B_k(x,0)$ for some constants $k_1 > k_2 > 0$ where $B_k(x,t) = 2(N-2)(T-t)^{N+2}/(k + (T-t)^{N-2}|x|^2)$ is the Barenblatt solution for the equation. We give a new different proof on the uniform convergence of the rescaled function $\tilde{u}(x,s) = (T-t)^{-N/2}v(x/(T-t)^{1/2}, t)$, $s = -\log(T-t)$, on $\mathbb{R}^N$ to the rescaled Barenblatt solution $B_{k_0}(x) = 2(N-2)/(k_0 + |x|^2)$ for some $k_0 > 0$ as $s \to \infty$. We also obtain convergence of the rescaled solution $\tilde{u}(x,s)$ as $s \to \infty$ when the initial data satisfies $0 \leq u_0 \leq B_{k_0}(x,0)$ in $\mathbb{R}^N$ and $|u_0(x) - B_{k_0}(x,0)| \leq f(|x|) \in L^1(\mathbb{R}^N)$ for some constant $k_0 > 0$ and some radially symmetric function $f$.

1. INTRODUCTION

The equation

$$u_t = \Delta \phi_m(u) \quad \text{in} \quad \mathbb{R}^N \times (0,T) \quad (1.1)$$

where $\phi_m(u) = u^m/m$ for $m \neq 0$ and $\phi_m(u) = \Delta \log u$ for $m = 0$ arises in many physical models such as the flow of gases through porous media [A], [P]. When $m = 1$, $(1.1)$ is the heat equation. When $m = 0$ and $N = 1$, the equation $(1.1)$ arises as the limiting density distribution of two gases moving against each other and obeying the Boltzmann equation [K], as the diffusive limit for finite velocity Boltzmann kinetic models [LT], and in the model of viscous liquid film lying on a solid surface and subjecting to long range Van der Waals interactions with the fourth order term being neglected [G], [WD]. When $m = 0$ and $N = 2$, $(1.1)$ arises as the Ricci flow on the complete surface $\mathbb{R}^2$ [W1], [W2]. We refer the reader to the book [V3] by J.L. Vazquez for the basics of the above equation and the books [DK], [V2], by P. Daskalopoulos, C.E. Kenig, and J.L. Vazquez for the recent research results on $(1.1)$.

As observed by J.L. Vazquez [V1] there is a great difference in the behaviour of the solutions of $(1.1)$ for $m > (N-2)_+/N$ and for $m \leq (N-2)_+/N$. For example for $m > (N-2)_+/N$ there exists global $L^1(\mathbb{R}^N)$ solution of $(1.1)$ while for $0 < m \leq (N-2)_+/N$ and $N \geq 3$ the $L^1(\mathbb{R}^N)$ solutions of $(1.1)$ vanish in a finite time. For $m \leq -1$ and $N = 1$ there exists no finite mass solution of $(1.1)$.

In [DS1] P. Daskalopoulos and N. Sesum proved the convergence of the rescaled solution of $(1.1)$ to the rescaled Barenblatt solution of $(1.1)$ near the extinction time for the case $0 < m \leq (N-2)_+/N$, $N > 2$, with initial data that behaves like $O(|x|^{-2/(1-m)})$ as $|x| \to \infty$. Extinction behaviour of the solution of

$$\begin{cases}
    u_t = \Delta \log u & \text{in} \mathbb{R}^N \times (0,T), \\
    u(x,0) = u_0(x) & \text{in} \mathbb{R}^N
\end{cases} \quad (1.2)$$

for the case $N = 2$ was studied by S.Y. Hsu [Hs2], [Hs3], P. Daskalopoulos, M.A. del Pino and N. Sesum [DP2], [DS2] and K.M. Hui [Hu3].

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In [Hu2] K.M. Hui proved that any solution of (1.2) with $N \geq 3$ and initial value satisfying the condition $0 \leq u_0(x) \leq C/|x|^2$ for all $|x| \geq R_0$ and some constants $R_0 > 0$, $C > 0$, will vanish in a finite time. It would be interesting to find the extinction behaviour of the solution of (1.2) for the case $N \geq 3$. In this paper we will study the asymptotic behaviour of solutions of (1.2) for $N = 3$ and $N \geq 5$ near its extinction time under the assumption that the initial value $u_0$ is non-negative, locally integrable, and

$$u_0(x) \approx \frac{C}{|x|^2} \quad \text{as } |x| \to \infty.$$  

Note that the self-similar Barenblatt solutions of (1.2) for $N \geq 3$ are given explicitly by

$$B_k(x, t) = \frac{2(N-2)(T-t)^{N-2}}{k + (T-t)^{N-2}|x|^2}, \quad k > 0,$$

which satisfy the growth condition (1.3).

Note that the asymptotics of the solutions of the fast diffusion equation (1.1) for the case $0 < m < 1$ and the case $m = (N-4)/(N-2)$ is studied by A. Blanchet, M. Bonfort, J. Dolbeault, G. Grillo and J.L. Vaquez in [BBDGV] and [BGV]. Sharp decay rate of the solutions of (1.1) for the case $0 < m < 1$ and $m \neq (N-4)/(N-2)$ is proved in [BDGV]. A sketch that their proofs extended to the case $m = 0$ is also given in appendix B of [BBDGV]. The proof in [BBDGV], [BGV] and [BDGV] used Lyapunov functional technique. On the other hand in this paper we will give a totally different proof of the asymptotics of the solutions of the fast diffusion equation (1.1) for the case $m = 0$ and $N = 3$ or $N > 5$ near the extinction time using a modification of the potential technique of P. Daskalopoulos and N. Sesum [DS1].

We will assume $N \geq 3$ for the rest of the paper. We will also assume in the first part of this paper that the initial condition $u_0$ is trapped in between two Barenblatt solutions, i.e.,

$$B_{k_1}(x, 0) \leq u_0(x) \leq B_{k_2}(x, 0)$$

for some constants $k_1 > k_2 > 0$. We will consider first solutions of (1.2) which satisfy the condition

$$B_{k_1}(x, t) \leq u(x, t) \leq B_{k_2}(x, t) \quad \text{in } \mathbb{R}^N \times (0, T).$$

Note that if $u$ is the maximal solution of (1.2) for $N \geq 3$ with initial value satisfying (1.5), then by the result of [Hu2] $u$ satisfies (1.6).

Consider the rescaled function

$$\tilde{u}(x, s) = \frac{1}{(T-t)^{N/2}} u \left( \frac{x}{(T-t)^{N/2}}, t \right), \quad s = -\log(T-t).$$

By direct computation $\tilde{u}$ satisfies

$$\tilde{u}_s = \Delta \log \tilde{u} + \frac{1}{N-2} \text{div}(x \cdot \tilde{u}) \quad \text{in } \mathbb{R}^N \times (-\log T, \infty).$$

By (1.6) and (1.7),

$$\tilde{B}_{k_1}(x) \leq \tilde{u}(x, s) \leq \tilde{B}_{k_2}(x)$$

holds in $\mathbb{R}^N \times (-\log T, \infty)$ where

$$\tilde{B}_k(x) = \frac{2(N-2)}{k + |x|^2}.$$

The main convergence results that we will prove in this paper are the following.
Theorem 1.1. Let $N = 3$ and let $u_0$ satisfy (1.5) for some constant $k_1 > k_2 > 0$. Suppose $u$ is a solution of (1.2) with initial value $u_0$ which satisfies (1.6). Then the rescaled function $\tilde{u}$ given by (1.4) converges uniformly on $\mathbb{R}^3$ and also in $L^1(\mathbb{R}^3)$ as $s \to \infty$ to the rescaled Barenblatt solution $\tilde{B}_{k_0}$ for some constant $k_0 > 0$ uniquely determined by

$$\int_{\mathbb{R}^N} (u_0(x) - B_{k_0}(x, 0)) \, dx = 0. \quad (1.11)$$

Theorem 1.2. Let $N \geq 5$ and let $u$ be a solution of (1.2) with initial value $u_0$ satisfying (1.5) and

$$u_0 = B_{k_0} + f \quad (1.12)$$

for some constants $k_1 > k_2 > 0$, $k_0 > 0$ and $f \in L^1(\mathbb{R}^N)$ where $B_{k_0}$ is the Barenblatt solution. Suppose $u$ satisfies (1.6). Let $\tilde{u}$ be the rescaled function given by (1.7). Then $\tilde{u}$ converges uniformly on $\mathbb{R}^N$ and in the weighted space $L^1(\tilde{B}_{N-1}^{N-2}, \mathbb{R}^N)$ as $s \to \infty$ to the rescaled Barenblatt solution $\tilde{B}_{k_0}$.

The plan of the paper is as follows. In section 2 we will establish some a priori estimates for the solutions of (1.2). We will prove Theorem 1.1 and Theorem 1.2 in sections three and four respectively. In section five we will improve Theorem 1.2 by removing the condition (1.5) on the initial data.

We start with some definitions. We say that $u$ is a solution of (1.2) in $\mathbb{R}^N \times (0, T)$ if $u > 0$ in $\mathbb{R}^N \times (0, T)$ and $u$ satisfies (1.2) in the classical sense in $\mathbb{R}^N \times (0, T)$ with

$$u(\cdot, t) \to u_0 \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{as } t \to 0.$$

We say that $u$ is a maximal solution of (1.2) in $\mathbb{R}^N \times (0, T)$ if $u$ is a solution of (1.2) in $\mathbb{R}^N \times (0, T)$ and $u \geq v$ for any solution $v$ of (1.2). For any $R > 0$ and $x_0 \in \mathbb{R}^N$, let $B_R(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < R\}$. Let $\omega_N$ be the surface area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$. For any $a \in \mathbb{R}$, let $a_{\pm} = \max(\pm a, 0)$. We will assume $N \geq 3$ for the rest of the paper.

For any $\alpha > 0$, we define the weighted $L^1$-space with weight $\tilde{B}^\alpha(x) := \left(\frac{2(N-2)}{k_2 + |x|^2}\right)^{\alpha}$ as

$$L^1(\tilde{B}^\alpha, \mathbb{R}^N) := \left\{ f \left| \int_{\mathbb{R}^N} f(x) \tilde{B}^\alpha(x) \, dx < \infty \right. \right\}.$$

2. Preliminary Estimates

In this section we will establish some a priori estimates for the solutions of (1.2).

Lemma 2.1. Let $u, v$ be two solutions of (1.2) with initial values $u_0, v_0$ respectively. Assume in addition that $u, v \geq B$, for some Barenblatt solution $B = B_k$ given by (1.4). Then there exists a constant $C > 0$ such that

(i) \( \left( \int_{B_R(x)} (u - v)_+(y, t) \, dy \right)^{\frac{1}{2}} \leq \left( \int_{B_{2R}(x)} (u_0 - v_0)_+(y) \, dy \right)^{\frac{1}{2}} + CR^{\frac{N+1}{N-2}} \sqrt{T} \)

and

(ii) \( \left( \int_{B_R(x)} |u - v|(y, t) \, dy \right)^{\frac{1}{2}} \leq \left( \int_{B_{2R}(x)} |u_0 - v_0|(y) \, dy \right)^{\frac{1}{2}} + CR^{\frac{N-1}{N-2}} \sqrt{T} \)

holds for any $R \geq |x| + \sqrt{k\delta - \frac{1}{2N-2}}$, $x \in \mathbb{R}^N$, $0 < t \leq T - \delta$, and $0 < \delta < T$.\[\]
Proof. We will use a modification of the argument of [12] to prove the lemma. Without loss of
generality we may assume that \( x = 0 \). Let \( \eta \in C_0^\infty(\mathbb{R}^N) \), \( 0 \leq \eta \leq 1 \), be such that \( \eta(x) = 1 \) for
\( |x| \leq 1 \), \( \eta = 0 \) for \( |x| \geq 2 \) and \( \eta_R(x) = \eta(x/R) \) for any \( R > 0 \). Then \( |\nabla \eta_R| \leq C_\delta / R \) and \( |\nabla \eta_R| \leq C_\delta / R \)
for some constant \( C_\delta > 0 \). By the Kato inequality \([K]\),
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^N} (u - v)^+ (x, t) \eta^4_R(x) \, dx \leq \int_{\mathbb{R}^N} \left( \log u - \log v \right)^+ (x, t) \triangle \eta^4_R(x) \, dx \quad \forall 0 < t < T.
\]
(2.1)
Since \( v \geq B_k \) for some Barenblatt solution \( B_k \),
\[
(\log u - \log v)^+ = \left( \log \left( \frac{u}{v} \right) \right)^+ \leq C \left( \frac{u - v}{v} \right)^+ \leq CB_k^{-\frac{1}{2}} (u - v)^{\frac{1}{2}}
\]
for some generic constant \( C > 0 \). By (2.1), (2.2), and the Hölder inequality,
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^N} (u - v)^+ (x, t) \eta^4_R(x) \, dx \leq C \int_{\mathbb{R}^N} (u - v)^{\frac{1}{2}} (x, t) B_k^{-\frac{1}{2}}(x) \triangle \eta^4_R(x) \, dx
\]
\[
\leq C \left( \int_{\mathbb{R}^N} (u - v)^+ (x, t) \eta^4_R(x) \, dx \right)^{\frac{1}{2}} \left( \int_{R \leq |x| \leq 2R} \eta_R^{-4} B_k^{-1} |\triangle \eta_R|^2 \, dx \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \int_{\mathbb{R}^N} (u - v)^+ (x, t) \eta^4_R(x) \, dx \right)^{\frac{1}{2}} \left( \int_{R \leq |x| \leq 2R} B_k^{-1} \left( 32 \eta_R^2 |\triangle \eta_R|^2 + 288 |\nabla \eta_R|^4 \right) \, dx \right)^{\frac{1}{2}}.
\]
(2.3)
Since
\[
(B_k(x, t))^{-1} \leq \frac{C|x|^2}{T-t} \quad \forall |x| \geq \sqrt{k \delta^{-\frac{3}{N-2}}}, 0 \leq t \leq T - \delta, 0 < \delta < T,
\]
by (2.3) we have
\[
\frac{\partial}{\partial t} \int_{\mathbb{R}^N} (u - v)^+ (x, t) \eta^4_R(x) \, dx \leq \frac{C R^{\frac{N-2}{2}}}{(T-t)^{\frac{1}{2}}} \left( \int_{\mathbb{R}^N} (u - v)^+ (x, t) \eta^4_R(x) \, dx \right)^{\frac{1}{2}}
\]
for any \( R^2 \geq k \delta^{-\frac{2}{N-2}} \), \( 0 < t \leq T - \delta \), and \( 0 < \delta < T \). By integrating the above differential
inequality with respect to \( t \), we get (i). Similarly,
\[
\left( \int_{B_R(x)} (u - v)^- (x, t) \, dx \right)^{\frac{1}{2}} \leq \left( \int_{B_{2R}(x)} (u_0 - v_0)^- (x, t) \, dx \right)^{\frac{1}{2}} + CR^{\frac{N-2}{2}} \sqrt{T}
\]
holds for any \( R^2 \geq k \delta^{-\frac{2}{N-2}} \), \( 0 < t \leq T - \delta \), and \( 0 < \delta < T \). (ii) then follows by adding the above
inequality with (i).
\[\square\]

Lemma 2.2. Let \( u, v \) be two solutions of (1.2) with initial values \( u_0, v_0 \), respectively. Assume in
addition that \( u, v \geq B \), for some Barenblatt solution \( B = B_k \) given by (1.3). If \( f = u_0 - v_0 \in L^1(\mathbb{R}^N) \), then \( u(\cdot, t) - v(\cdot, t) \in L^1(\mathbb{R}^N) \) for all \( t \in [0, T) \).

Proof. We will use a modification of the proof of Lemma 2.1 of [DS1] to prove the lemma. We
introduce the potential function
\[
w(x, t) = \int_0^t |(\log u - \log v)(x, s)| \, ds \quad \forall 0 < t \leq T - \delta.
\]
By the Kato inequality [K],
\[ \triangle |\log u - \log v| \geq \text{sign}(u - v)\triangle (\log u - \log v), \]
and so from equation (1.2), we obtain
\[ \frac{\partial}{\partial t} |u - v| \leq \triangle |\log u - \log v|. \quad (2.5) \]
Integrating the above inequality in time, and using that
\[ |f| = |u_0 - v_0|, \]
we obtain
\[ \Delta w \geq -|f| \quad \text{in } \mathbb{R}^N \quad \forall 0 < t < T. \quad (2.6) \]
Let
\[ Z(x) = \frac{1}{(N - 2)\omega_N} \int_{\mathbb{R}^N} \frac{|f(y)|}{|x - y|^{N-2}} \, dy \]
denote the Newtonian potential of $|f|$ where $\omega_N$ is the surface area of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$. Then by (2.6),
\[ \Delta (w(\cdot, t) - Z) \geq 0 \quad (2.7) \]
in the sense of distributions in $\mathbb{R}^N$ for any $0 < t < T$. Next we would like to show that
\[ \int_{R \leq |x| \leq 2R} w(x, t) \, dx \leq \int_{R \leq |x| \leq 2R} Z(x) \, dx \quad \forall R > 0. \quad (2.8) \]
In order to prove this estimate we first suppose that $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By (2.7) and the mean value property for subharmonic functions,
\[ w(x, t) \leq Z(x) + \frac{N}{\omega_N \rho^N} \int_{B_\rho(x)} (w(y, t) - Z(y)) \, dy \]
\[ \leq Z(x) + \frac{N}{\omega_N \rho^N} \int_{B_\rho(x)} w(y, t) \, dy \quad (2.9) \]
holds for any $x \in \mathbb{R}^N$, $0 < t < T$, and $\rho > 0$. We claim that
\[ \lim_{\rho \to \infty} \frac{1}{\rho^N} \int_{B_\rho(x)} w(y, t) \, dy = 0 \quad \forall x \in \mathbb{R}^N, 0 < t < T. \quad (2.10) \]
In order to prove (2.10) it suffices to prove that
\[ \lim_{\rho \to \infty} \frac{1}{\rho^N} I_1(\rho, t) = 0 \quad \text{and} \quad \lim_{\rho \to \infty} \frac{1}{\rho^N} I_2(\rho, t) = 0 \quad \forall 0 < t < T \]
where
\[ \begin{align*}
I_1(\rho, t) &= \int_0^t \int_{B_\rho(x)} (\log u - \log v)_+ (y, s) \, dy \, ds \\
I_2(\rho, t) &= \int_0^t \int_{B_\rho(x)} (\log u - \log v)_- (y, s) \, dy \, ds.
\end{align*} \]
Since $u$ and $v$ are the solutions of (1.2), by the Green Theorem \((\text{GT})\) and an approximation argument,

\[
\frac{\partial}{\partial s} \int_{B_\rho(x)} (u - v)_+ (y, s)(\rho^2 - |x - y|^2) \, dy
\]

\[
= \int_{B_\rho(x) \cap \{u > v\}} \triangle (\log u - \log v) (y, s)(\rho^2 - |x - y|^2) \, dy
\]

\[
\leq \int_{B_\rho(x) \cap \{u > v\}} (\log u - \log v) (y, s)\frac{\partial}{\partial \nu}(\rho^2 - |x - y|^2) \, d\sigma_y
\]

\[
\leq -2N \int_{B_\rho(x)} (\log u - \log v)_+ (y, s) \, dy
\]

\[
+ 2\rho \int_{\partial B_\rho(x)} (\log u - \log v)_+ (y, s) \, d\sigma_y \quad \forall 0 < s < T
\]  

(2.11)

where $\frac{\partial}{\partial \nu}$ is the derivative with respect to the unit outer normal $\nu$ on $\partial \{B_\rho(x) \cap \{u > v\}\}$. Integrating (2.11) with respect to $s$ over $(0, \tau)$, we have

\[
\int_{B_\rho(x)} (u - v)_+ (y, \tau)(\rho^2 - |x - y|^2) \, dy 
\]

\[
\leq \int_{B_\rho(x)} (u_0 - v_0)_+ (\rho^2 - |x - y|^2) \, dy
\]

\[
- 2N \int_0^\tau \int_{B_\rho(x)} (\log u - \log v)_+ (y, s) \, dy \, ds
\]

\[
+ 2\rho \int_0^\tau \int_{\partial B_\rho(x)} (\log u - \log v)_+ (y, s) \, d\sigma_y \, ds \quad \forall 0 < \tau < T.
\]  

(2.12)

Integrating (2.12) with respect to $\tau$ over $(0, t)$,

\[
\int_0^t \int_{B_\rho(x)} (u - v)_+ (y, \tau)(\rho^2 - |x - y|^2) \, dy \, d\tau
\]

\[
\leq T \int_{B_\rho(x)} (u_0 - v_0)_+ (\rho^2 - |x - y|^2) \, dy
\]

\[
- 2N \int_0^t \int_0^\tau \int_{B_\rho(x)} (\log u - \log v)_+ (y, s) \, dy \, ds \, d\tau
\]

\[
+ 2\rho \int_0^t \int_0^\tau \int_{\partial B_\rho(x)} (\log u - \log v)_+ (y, s) \, d\sigma_y \, ds \, d\tau \quad \forall 0 < t < T.
\]  

(2.13)

Let $0 < t_0 < T$ and $\delta = T - t_0$. Now we divide the proof into two cases depending on whether

\[
\int_0^{t_0} \int_0^\tau \int_{\mathbb{R}^N} (\log u - \log v)_+ (y, s) \, dy \, ds \, d\tau < \infty
\]  

(2.14)

or

\[
\int_0^{t_0} \int_0^\tau \int_{\mathbb{R}^N} (\log u - \log v)_+ (y, s) \, dy \, ds \, d\tau = \infty.
\]  

(2.15)

Case 1: (2.14) holds.
Then for any $0 < \delta' < t_0$,

$$\begin{align*}
\infty &> \int_{t_0-\delta'}^{t_0} \int_0^\tau \int_{\mathbb{R}^N} (\log u - \log v)_+ (y, s) \, dy \, ds \, d\tau \\
&\geq \delta' \int_0^t \int_0^\tau \int_{\mathbb{R}^N} (\log u - \log v)_+ (y, s) \, dy \, ds \quad \forall 0 < t < t_0 - \delta'.
\end{align*}$$

(2.16)

Hence

$$\lim_{\rho \to \infty} \frac{1}{\rho^N} I_1(\rho, t) = 0 \quad \forall 0 < t < t_0 - \delta'. \quad (2.17)$$

Since $\delta'$ is arbitrary, (2.17) holds for any $0 < t < t_0$.

**Case 2:** (2.15) holds.

By the l'Hospital rule,

$$\lim_{\rho \to \infty} \frac{1}{\rho^N} \int_0^t \int_0^\tau \int_{B_\rho(x)} (\log u - \log v)_+ \, dy \, ds \, d\tau = \lim_{\rho \to \infty} \frac{1}{\rho^N} \int_0^t \int_0^\tau \int_{\partial B_\rho(x)} (\log u - \log v)_+ \, d\sigma_y \, ds \, d\tau. \quad (2.18)$$

Let $r_1 = |x| + k\delta^{-\frac{N}{N-2}}$. By (2.2), (2.4), Lemma 2.1 and the Hölder inequality, for any $\rho > |x| + k\delta^{-\frac{N}{N-2}}$ and $0 < t \leq T - \delta$, we have

$$\begin{align*}
\frac{1}{\rho^N} \int_0^t \int_0^\tau \int_{B_\rho(x)} (\log u - \log v)_+ \, dy \, ds \, d\tau
&= \frac{1}{\rho^N} \int_0^t \int_0^\tau \int_{B_\rho(x) \cap B_1(0)} (\log u - \log v)_+ \, dy \, ds \, d\tau \\
&\quad + \frac{1}{\rho^N} \int_0^t \int_0^\tau \int_{B_\rho(x) \setminus B_1(0)} (\log u - \log v)_+ \, dy \, ds \, d\tau \\
&\leq \frac{C_1}{\rho^N} \int_0^t \int_0^\tau \int_{B_1(0)} (u - v)_+^\frac{1}{2} (y, s) \, dy \, ds \, d\tau \\
&\quad + \frac{C_1}{\rho^N} \int_0^t \int_0^\tau \int_{B_\rho(x) \setminus B_1(0)} \rho \left( \int_{B_\rho(x)} (u - v)_+^\frac{1}{2} (y, s) \, dy \right) \, ds \, d\tau \\
&\leq \frac{C'}{\rho^N} \int_0^t \int_0^\tau \left( \int_{B_1(0)} (u - v)_+ (y, s) \, dy \right)^\frac{1}{2} \, ds \, d\tau + C' T^\frac{3}{2} \left( \frac{||f||_{L^1(\mathbb{R}^N)}}{\rho^{\frac{N}{2}-1}} + \sqrt{T} \right)
\end{align*}$$

(2.19)

for some constant $C_1 > 0$, $C' > 0$, depending on $\delta$ and $k$. By (2.19) the limit in (2.18) is finite. Since $u_0 - v_0 \in L^1(\mathbb{R}^N)$,

$$\lim_{\rho \to \infty} \frac{1}{\rho^{N-2}} \int_{B_\rho(x)} (u_0 - v_0)_+ (y) \, dy = 0. \quad (2.20)$$

Dividing (2.13) by $\rho^N$ and letting $t = t_0$ and $\rho \to \infty$ as $i \to \infty$, by (2.18) and (2.20),

$$\lim_{i \to \infty} \frac{1}{\rho^N} \int_0^{t_0} \int_{B_\rho(x)} (u - v)_+ (y, \tau) \left( \rho^2 - |x - y|^2 \right) \, dy \, d\tau = 0. \quad (2.21)$$
Let $0 < \varepsilon < 1/2$. Since

$$\rho^2 \leq \frac{\rho^2 - |x - y|^2}{1 - (1 - \varepsilon)^2} \quad \forall y \in B_{(1-\varepsilon)\rho}(x),$$

by (2.2), (2.4), Lemma 2.1 and the Hölder inequality, for any $\rho > |x|$ we have

$$\int_0^t \int_{B_{(1-\varepsilon)\rho}(x)} (\log u - \log v)_{+} (y, s) \, dy \, ds$$

$$= \int_0^t \int_{B_{(1-\varepsilon)\rho}(x) \cap B_{r_1}(0)} (\log u - \log v)_{+} (y, s) \, dy \, ds$$

$$+ \int_0^t \int_{B_{(1-\varepsilon)\rho}(x) \cap B_{r_1}(0)} (\log u - \log v)_{+} (y, s) \, dy \, ds$$

$$\leq C_2 \int_0^t \int_{B_{r_1}(0)} (u - v)^{\frac{1}{2}} (y, s) \, dy \, ds + C_2 \int_0^t \frac{1}{(T - s)^{\frac{1}{2}}} \left( \int_{B_{(1-\varepsilon)\rho}(x)} \rho \cdot (u - v)^{\frac{1}{2}} \, dy \right) \, ds$$

$$\leq C_3 T + \frac{C_3 \rho \frac{N}{2} \sqrt{T}}{\delta^{\frac{1}{2}} (1 - (1 - \varepsilon)^2)^{\frac{1}{2}}} \left( \int_0^t \int_{B_{\frac{x}{2}}(x)} (u - v)_{+} (y, s) (\rho^2 - |x - y|^2) \, dy \, ds \right)^{\frac{1}{2}} (2.22)$$

for some constants $C_2, C_3 > 0$. Thus by (2.21) and (2.22),

$$\lim_{\rho \to \infty} \frac{1}{\rho^N} \int_0^t \int_{B_{(1-\varepsilon)\rho}(x)} (\log u - \log v)_{+} (y, s) \, dy \, ds = 0. \quad (2.23)$$

Now for any $y \in B_{\rho}(x) \setminus B_{(1-\varepsilon)\rho}(x)$ and $\rho > 2(|x| + r_1)$ we have

$$\frac{3}{2} \rho \geq |x| + \rho \geq |y| \geq |x - y| - |x| \geq (1 - \varepsilon)\rho - |x| \geq r_1.$$

Hence by (2.2), (2.4), Lemma 2.1 and the Hölder inequality, for any $\rho > 2(|x| + r_1),$

$$\int_0^t \int_{B_{\rho}(x) \setminus B_{(1-\varepsilon)\rho}(x)} (\log u - \log v)_{+} (y, s) \, dy \, ds$$

$$\leq C \int_0^t \int_{B_{\rho}(x) \setminus B_{(1-\varepsilon)\rho}(x)} v^{-\frac{1}{2}} (u - v)^{\frac{1}{2}} \, dy \, ds$$

$$\leq C \rho \int_0^t \frac{1}{(T - s)^{\frac{1}{2}}} \int_{B_{\rho}(x) \setminus B_{(1-\varepsilon)\rho}(x)} (u - v)^{\frac{1}{2}} \, dy \, ds$$

$$\leq C' (1 - (1 - \varepsilon)^N)^{\frac{1}{2}} \sqrt{T} \rho^\frac{N}{2} + (\|f\|_{L^1(\mathbb{R}^N)} + \rho^{N-2}T)^{\frac{1}{2}}$$

for some constants $C > 0, C' > 0$. Hence

$$\limsup_{\rho \to \infty} \frac{1}{\rho^N} \int_0^t \int_{B_{\rho}(x) \setminus B_{(1-\varepsilon)\rho}(x)} (\log u - \log v)_{+} (y, s) \, dy \, dsds \leq C' \left(1 - (1 - \varepsilon)^N\right)^{\frac{1}{2}} T. \quad (2.24)$$

By (2.23) and (2.24),

$$\limsup_{\rho \to \infty} \frac{1}{\rho^N} I_1(\rho, t_0) \leq C' \left(1 - (1 - \varepsilon)^N\right)^{\frac{1}{2}} T. \quad (2.25)$$

Since $0 < \varepsilon < 1/2$ is arbitrary, letting $\varepsilon \to 0$ in (2.25) we get that

$$\lim_{\rho \to \infty} \frac{1}{\rho^N} I_1(\rho, t) = 0 \quad (2.26)$$
holds for any $0 < t < t_0$. By Case 1 and Case 2, (2.26) holds for any $0 < t < t_0$. Since $0 < t_0 < T$ is arbitrary, (2.26) holds for any $0 < t < T$. Similarly,

$$\lim_{\rho \to \infty} \frac{1}{\rho^N} I_2(\rho, t) = 0 \quad \forall 0 < t < T$$

and (2.10) follows. Letting $\rho \to \infty$ in (2.9), by (2.10),

$$w(x, t) \leq Z(x) \quad \forall x \in \mathbb{R}^N, 0 < t < T.$$  \hspace{1cm} (2.27)

By (2.27), we get that (2.8) holds for any $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

For general $f \in L^1(\mathbb{R}^N)$. Let $\varphi \in C^\infty_0(\mathbb{R}^N)$ be such that $0 \leq \varphi \leq 1$ and $\int_{\mathbb{R}^N} \varphi = 1$. Let

$$\varphi_\epsilon(y) = \epsilon^{-N} \varphi \left( \frac{y}{\epsilon} \right)$$

and

$$g_\epsilon(x) = g * \varphi_\epsilon(x) = \int_{\mathbb{R}^N} g(x - y) \varphi_\epsilon(y) \, dy$$

for any $0 < \epsilon < 1$ and $g \in L^1(\mathbb{R}^N)$. Then by (2.7),

$$\triangle (w_\epsilon - Z_\epsilon) \geq 0 \quad \text{in } \mathbb{R}^N \quad \forall 0 < t < T.$$  \hspace{1cm} (2.28)

By (2.10),

$$\lim_{\rho \to \infty} \frac{1}{\rho^N} \int_{B_{\rho+2R}(0)} w(y, t) \, dy = 0.$$  \hspace{1cm} (2.29)

Hence by letting $\rho \to \infty$ in (2.28), (2.8) follows. Let $\eta_R$ be as in the proof of Lemma 2.1. By (2.5),

$$\int_{\mathbb{R}^N} |u - v|(\cdot, t) \eta_R \, dx \leq \int_{\mathbb{R}^N} |f| \, dx + \int_0^t \int_{R \leq |x| \leq 2R} |\log u - \log v| \, |\triangle \eta_R| \, dx \, ds$$

$$\leq \|f\|_{L^1(\mathbb{R}^N)} + \frac{C}{R^2} \int_{R \leq |x| \leq 2R} w(x) \, dx.$$  \hspace{1cm} (2.29)
By (2.8),

\[
\int_{R \leq |x| \leq 2R} w(x, t) \, dx \leq \frac{1}{N(N-2)2^N} \int_{R \leq |x| \leq 2R} \left( \int_{R^N} \frac{|f(y)|}{|x-y|^{N-2}} \, dy \right) \, dx
\]

\[
\leq C \int_{R^N} |f(y)| \left( \int_{R \leq |x| \leq 2R} \frac{dx}{|x-y|^{N-2}} \right) \, dy
\]

\[
\leq C \int_{R^N} |f(y)| J_R(y) \, dy.
\]

where \( J_R(y) = \int_{R \leq |x| \leq 2R} \frac{dx}{|x-y|^{N-2}} \). Let \( R \leq |x| \leq 2R \). Then for \(|y| \leq \frac{R}{2}\) we have \(|x-y| \geq |x|/2\).

Hence

\[
J_R(y) \leq \int_{R \leq |x| \leq 2R} \frac{dx}{|x|^{N-2}} \leq CR^2 \quad \forall |y| \leq \frac{R}{2}.
\]

(2.31)

For \(|y| \geq 4R\), we have \(|x-y| \geq |y|/2 \geq 2R\). Thus

\[
J_R(y) \leq \int_{R \leq |x| \leq 2R} \frac{dx}{(2R)^{N-2}} \leq CR^2 \quad \forall |y| \leq 4R.
\]

(2.32)

Finally for \( \frac{R}{2} < |y| < 4R\), we have \(|x-y| < 6R\). Therefore

\[
J_R(y) \leq \int_{|x-y| < 6R} \frac{dx}{|x-y|^{N-2}} \leq CR^2 \quad \forall \frac{R}{2} < |y| < 4R.
\]

(2.33)

By (2.30), (2.31), (2.32) and (2.33),

\[
\int_{R \leq |x| \leq 2R} w(x, t) \, dx \leq C'R^2 \|f\|_{L^1} \quad \forall 0 < t < T
\]

(2.34)

for some constant \( C' > 0 \). By (2.29) and (2.34),

\[
\int_{R^N} |u - v|(x, t) \eta_R(x) \, dx \leq C \|f\|_{L^1(R^N)} \quad \forall R > 0, 0 < t < T
\]

for some constant \( C > 0 \). Letting \( R \to \infty \), we get

\[
\int_{R^N} |u - v|(x, t) \, dx \leq C \|f\|_{L^1(R^N)} \quad \forall 0 < t < T
\]

and the lemma follows.

By an argument similar to the proof of Corollary 2.2 of [DS1] but with Lemma 2.2 replacing Lemma 2.1 of [DS1] in the proof, we have the following \( L^1 \)-contraction principle for the solutions of (1.2) that are bounded below by some Barenblatt solution \( B \).

**Lemma 2.3.** Let \( u, v \) be two solutions of (1.2) with initial values \( u_0, v_0 \) respectively and \( f = u_0 - v_0 \in L^1(R^N) \). Assume in addition that \( u, v \geq B \), for some Barenblatt solution \( B = B_k \) given by (1.4). Then

\[
\int_{R^N} |u(\cdot, t) - v(\cdot, t)| \, dx \leq \int_{R^N} |u_0 - v_0| \, dx, \quad \forall t \in [0, T).
\]

As a consequence of Lemma 2.3, we have the following result concerning the rescaling solutions \( \widetilde{u} \) and \( \widetilde{v} \) of solutions \( u \) and \( v \) of (1.2).
Corollary 2.4. Let \( u, v, u_0, v_0 \), be as in Lemma 2.3. If \( u_0 - v_0 \in L^1(\mathbb{R}^N) \), then
\[
\int_{\mathbb{R}^N} |\bar{u}(x, s) - \bar{v}(x, s)| \, dx \leq \int_{\mathbb{R}^N} |u_0 - v_0| \, dx, \quad \forall s > -\log T.
\]

3. The integrable case (\( N = 3 \))

This section will be devoted to the proof of Theorem 1.1. Note that when \( N = 3 \), the difference of two solutions \( u, v \), satisfying (1.6) is integrable. We will use a modification of the technique of [HS1] to prove Theorem 1.1. We begin this section with the following technical lemma, which constitutes the main step in the proof of Theorem 1.1.

Lemma 3.1. Let \( N \geq 3, s_0 > 0, 0 \leq f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) and \( 0 \leq g, \hat{g} \in C(\mathbb{R}^N \times (0, s_0]) \cap L^1(\mathbb{R}^N \times [0, s_0]) \) such that \( 0 \leq g \leq \hat{g} \) on \( \mathbb{R}^N \times (0, s_0) \). Let \( \tilde{a}(x, s) \in C^\infty(\mathbb{R}^N \times (0, s_0]) \) satisfy
\[
C_1(1 + |x|^2) \leq \tilde{a}(x, s) \leq C_2(1 + |x|^2) \quad \forall x \in \mathbb{R}^N, 0 \leq s \leq s_0,
\]
for some constants \( C_1 > 0, C_2 > 0 \). For any \( R > 1 \), let \( p_R(x, s) \) be a solution of
\[
\begin{align*}
p_a(x, s) &= \Delta(\tilde{a}(x, s)p(x, s)) + \frac{1}{N-2} \text{div}(x \cdot p(x, s)) \quad \text{in } B_R(0) \times (0, s_0) \\
p(x, s) &= g(x, s) \quad \text{on } \partial B_R(0) \times (0, s_0) \\
p(x, 0) &= f(x) \quad \text{in } B_R(0)
\end{align*}
\]
Then there exists a sequence of positive numbers \( \{R_i\}_{i=1}^\infty \), \( R_i \to \infty \) as \( i \to \infty \), depending on \( \hat{g} \) and independent of \( g \) such that \( p_{R_i} \) converges uniformly on every compact subsets of \( \mathbb{R}^N \times (0, s_0] \) as \( i \to \infty \) to a solution \( p \) of
\[
q_s = \Delta(\tilde{a}(x, s)q) + \frac{1}{N-2} \text{div}(x \cdot q)
\]
in \( \mathbb{R}^N \times (0, s_0] \) which satisfies
\[
\int_{\mathbb{R}^N} p(x, s) \, dx \leq \int_{\mathbb{R}^N} f \, dx \quad \forall 0 < s \leq s_0.
\]

Proof. Since \( \hat{g} \in L^1(\mathbb{R}^N \times [0, s_0]) \),
\[
\int_{\mathbb{R}^N} \int_{|x|=R} \hat{g}(y, s) \, d\sigma_R \, ds \, dR \to 0 \quad \text{as } i \to \infty
\]
where \( d\sigma_R \) is the surface measure on \( \partial B_R(0) \). For each \( i \in \mathbb{N} \), there exists \( R_i \in [i/2, i] \) such that
\[
\int_{0}^{s_0} \int_{|x|=R_i} \hat{g}(y, s) \, d\sigma_{R_i} \, ds = \min_{\frac{i}{2} \leq R \leq i} \left\{ \int_{0}^{s_0} \int_{|x|=R} \hat{g}(y, s) \, d\sigma_{R} \, ds \right\}.
\]
Then by (3.5),
\[
\frac{i}{2} \int_{0}^{s_0} \int_{|x|=R_i} \hat{g}(y, s) \, d\sigma_{R_i} \, ds \to 0 \quad \text{as } i \to \infty
\]
\[
\Rightarrow R_i \int_{0}^{s_0} \int_{|x|=R_i} \hat{g}(y, s) \, d\sigma_{R_i} \, ds \to 0 \quad \text{as } i \to \infty.
\]
By choosing a subsequence if necessary we may assume without loss of generality that \( R_{i+1} > R_i \) for any \( i \in \mathbb{N} \). By (3.4) and the Schauder estimates for parabolic equations [LSU], the sequence \( \{p_{R_i}\}_{i=1}^\infty \) is equi-Hölder continuous in \( C^{2,1} \) on every compact subsets of \( \mathbb{R}^N \times (0, s_0] \). Hence by the Ascoli Theorem and a diagonalization argument there exists a subsequence, which we will still
denote by $\{p_{R_i}\}_{i=1}^{\infty}$, that converges uniformly on every compact subsets of $\mathbb{R}^N \times (0, s_0]$ to a solution $\{\tilde{p}\}$ in $\mathbb{R}^N \times (0, s_0]$ as $i \to \infty$.

It remains to prove (3.4). We fix $s_1 \in (0, s_0]$ and define the operator $L$ by

$$L[\psi] = \psi_s + \bar{a} \Delta \psi - \frac{1}{N-2} x \cdot \nabla \psi.$$  

For any $R > 1$ and $h \in C_0^\infty(B_R(0))$, $0 \leq h \leq 1$ on $B_R(0)$, such that

$$h(x) = 0 \quad \text{on} \quad B_R(0) \setminus B^{\frac{R}{2}}(0),$$  

let $\psi_R(x, s)$ be the solution of

$$
\begin{cases}
L[\psi] = 0 & \text{in} \ B_R(0) \times (0, s_1) \\
\psi(x, s) = 0 & \text{on} \ \partial B_R(0) \times (0, s_1) \\
\psi(x, s_1) = h(x) & \text{in} \ B_R(0).
\end{cases}
$$

By the maximum principle $0 \leq \psi_R \leq 1$ in $B_R(0) \times (0, s_1)$. Let

$$H_k(x) = \frac{2^{2k}}{2^{2k} - 1} \left( 1 - \frac{|x|^{2k}}{R^{2k}} \right)$$

for some $k > 0$ to be determined later. By direct computation for any $k \in \mathbb{N}$

$$L[H_k(x)] \leq -\left( \frac{2^{2k}}{2^{2k} - 1} \right) \frac{2^k|x|^{2k - 2}}{R^{2k}} \left[ C_1(N + 2k - 2) + \left( C_1(N + 2k - 2) - \frac{1}{N-2} \right) |x|^2 \right]$$

on $B_R(0) \setminus B^{\frac{R}{2}}(0)$ and

$$
\begin{cases}
H_k(x) = 0 & \forall |x| = R \\
H_k(R/2) = 1 & \geq \psi_R(x, s) & \forall |x| = R/2, 0 < s < s_1.
\end{cases}
$$

We now choose $k > \frac{1}{2C_1(N-2)} + 1$. Then by (3.10),

$$L[H_k(x)] < 0 \quad \text{in} \ B_R(0) \setminus B^{\frac{R}{2}}(0).$$

Hence $H_k(x)$ is a super-solution of $L(\xi) = 0$ in $(B_R(0) \setminus B^{R/2}(0)) \times (0, s_1)$. By (3.8), (3.9), (3.11), (3.12), and the maximum principle in $(B_R(0) \setminus B^{R/2}(0)) \times (0, s_1)$,

$$\psi_R(x, s) \leq H_k(x) = \frac{2^{2k}}{2^{2k} - 1} \left( 1 - \frac{|x|^{2k}}{R^{2k}} \right) \quad \text{on} \ (B_R(0) \setminus B^{R/2}(0)) \times (0, s_1).$$

Then by (3.11) and (3.13),

$$\left| \frac{\partial \psi_R}{\partial \nu} \right| \leq \left| \frac{\partial}{\partial \nu} \left( \frac{2^{2k}}{2^{2k} - 1} \left( 1 - \frac{|x|^{2k}}{R^{2k}} \right) \right) \right| \leq \frac{C}{R} \quad \text{on} \ \partial B_R(0) \times (0, s_1)$$

for some constant $C > 0$ depending on $k$ where $\frac{\partial}{\partial \nu}$ is the derivative with respect to the unit outer normal $\nu$ on the boundary $\partial B_R(0)$. 
Lemma 3.1 of [DS1]). Let

\[
\frac{\partial}{\partial s} \int_{|x| \leq R} p_R \psi_R \, dx = \int_{|x| \leq R} \left[ \psi_{Rs} + \tilde{a} \Delta \psi_R - \frac{1}{N-2} x \cdot \nabla \psi_R \right] p_R \, dx - \int_{|x|=R} \tilde{a} g \frac{\partial \psi_R}{\partial \nu} \, d\sigma_R
\]

\[
= - \int_{|x|=R} \frac{\partial \psi_R}{\partial \nu} \, d\sigma_R
\]

\[
\leq CR \int_{|x|=R} \hat{g} \, d\sigma_R
\]

\[\forall 0 < s < s_1, \quad R > 1. \quad (3.14)\]

Hence

\[
\int_{|x| \leq R} p_R(x, s_1) h(x) \, dx \leq \int_{|x| \leq R} f(x) \psi_R(x, 0) \, dx + CR \int_0^{s_1} \int_{|x|=R} \hat{g} \, d\sigma_R \, ds. \quad (3.15)
\]

We now choose \( h(x) = \eta_{R/4}(x) \) where \( \eta_{R/4}(x) \) is as in the proof of Lemma 2.1. By the maximum principle \( p_R \geq 0 \) in \( B_R \times (0, \infty) \). Then putting \( R = R_i \) in (3.15) and letting \( i \to \infty \), by (3.7),

\[
\int_{|x|=R} p(x, s_1) \, dx \leq \int_{|x|=R} f \, dx.
\]

Since \( 0 < s_1 \leq s_0 \) is arbitrary, (3.4) follows. \( \square \)

Lemma 3.2. Let \( N \geq 3 \). Let \( u, v \), be two solutions of (1.2) with initial values \( u_0, v_0 \), satisfying (1.3) for some constants \( k_1 > k_2 > 0 \) and let \( \tilde{u}, \tilde{v} \), be given by (1.7) with \( u = u, v \), respectively. Let \( \tilde{u}_0(x) = \tilde{u}(x, -\log T) \) and \( \tilde{v}_0(x) = \tilde{v}(x, -\log T) \). Suppose \( u, v \), satisfy (1.6) and

\[
\min(\|u_0 - \tilde{u}_0\|_{L^\infty(\mathbb{R}^N)}, \|v_0 - \tilde{v}_0\|_{L^\infty(\mathbb{R}^N)}) > 0.
\]

Then

\[
\|\tilde{u} - \tilde{v}\|(\cdot, s)\|_{L^1(\mathbb{R}^N)} \leq \|u_0 - \tilde{u}_0\|_{L^1(\mathbb{R}^N)} \forall s > -\log T. \quad (3.16)
\]

Proof. We will use a modification of the proof of Lemma 2.1 of [Hs1] to prove the lemma (cf. Lemma 3.1 of [DS1]). Let \( q = \tilde{u} - \tilde{v} \). Then \( q \) satisfies (3.3) in \( \mathbb{R}^N \times (-\log T, \infty) \) with

\[
\tilde{a}(x, s) = \int_0^1 \frac{d\theta}{\partial \tilde{u} + (1 - \theta)\tilde{v}}.
\]

Since both \( \tilde{u} \) and \( \tilde{v} \) satisfy (1.9), \( \tilde{a}(x, s) \) satisfies the growth estimate

\[
\frac{k_2 + |x|^2}{2(N-2)} \leq \tilde{a}(x, s) \leq \frac{k_1 + |x|^2}{2(N-2)}. \quad (3.18)
\]

Hence (3.3) is uniformly parabolic on any compact subset of \( \mathbb{R}^N \times (-\log T, \infty) \).

For any \( R > 0 \), by standard parabolic theory there exist solutions \( q_1^R, q_2^R \) of (3.3) in \( Q_R = B_R(0) \times (-\log T, \infty) \) with initial values \( q_+(\cdot, -\log T), q_-(\cdot, -\log T) \) and boundary value \( q_+ \), \( q_- \) on \( \partial B_R(0) \times (-\log T, \infty) \), respectively. Notice that \( q_1^R - q_2^R \) is a solution of (3.3) in \( Q_R \) with initial value \( q(\cdot, -\log T) \) and boundary values \( q \). By the maximum principle \( q = q_1^R - q_2^R \) on \( Q_R \). Similarly there are solutions \( \tilde{q}_1^R, \tilde{q}_2^R \) of (3.3) in \( Q_R \) with initial values \( q_+(\cdot, -\log T), q_-(\cdot, -\log T) \) and zero lateral boundary value. By the maximum principle

\[
\begin{cases}
0 \leq q_1^R \leq \tilde{q}_1^R \quad \text{and} \quad \tilde{q}_1^R \leq q_2^R \\
q_1^R \leq \tilde{q}_1^R' \quad \text{and} \quad \tilde{q}_1^R' \leq q_2^R' \quad \text{in} \quad Q_R \quad \forall R' \geq R > 0.
\end{cases}
\]

(3.19)
Since both $\tilde{u}$ and $\tilde{v}$ satisfy (1.9),
\begin{equation}
|q| \leq \tilde{B}_{k_2} - \tilde{B}_{k_1} \quad \text{in } \mathbb{R}^N \times (-\log T, \infty). \tag{3.20}
\end{equation}

By (3.19) and (3.20) the families of solutions $\overline{q}_1^R(x,s)$ and $\overline{q}_2^R(x,s)$ are monotone increasing in $R$ and uniformly bounded above by $\tilde{B}_{k_2} - \tilde{B}_{k_1}$, which implies
\[
\overline{q}_1 = \lim_{R \to \infty} \overline{q}_1^R \quad \text{and} \quad \overline{q}_2 = \lim_{R \to \infty} \overline{q}_2^R
\]
exist and are both solutions of (3.3) in $\mathbb{R}^N \times (0, \infty)$.

Let $\eta_{R'} \in C_0^\infty(\mathbb{R}^N)$ be as in the proof of Lemma 2.1. By Lemma 2.2 and the same computation as the proof of Lemma 2.1 of [Hs1],
\[
\int_{\mathbb{R}^N} |q(x,s)| \eta_{R'}(x) \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0|(x) \eta_{R'}(x) \, dx
\]
\[
= \int_{-\log T}^s \int_{|x| \leq R} \left( \tilde{a}(x, \tau) q_1^R(x, \tau) \triangle \eta_{R'} - \frac{1}{N-2} q_1^R(x, \tau) \cdot \nabla \eta_{R'} \right) \, dx \, d\tau
\]
\[
+ \int_{-\log T}^s \int_{|x| \leq R} \left( \tilde{a}(x, \tau) q_2^R(x, \tau) \triangle \eta_{R'} - \frac{1}{N-2} q_2^R(x, \tau) \cdot \nabla \eta_{R'} \right) \, dx \, d\tau
\]
\[
- 2 \int_{|x| \leq R} \min(q_1^R(x,s), q_2^R(x,s)) \eta_{R'}(x) \, dx \quad \forall R \geq 2R' > 0, s > -\log T.
\]

Hence
\[
\int_{\mathbb{R}^N} |q(x,s)| \eta_{R'}(x) \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0|(x) \eta_{R'}(x) \, dx
\]
\[
\leq \frac{C}{R'^2} \int_{-\log T}^s \int_{R' \leq |x| \leq 2R'} \tilde{a}(x, \tau) q_1^R(x, \tau) \, dx \, d\tau + C \int_{-\log T}^s \int_{R' \leq |x| \leq 2R'} q_1^R(x, \tau) \, dx \, d\tau
\]
\[
+ \frac{C}{R'^2} \int_{-\log T}^s \int_{R' \leq |x| \leq 2R'} \tilde{a}(x, \tau) q_2^R(x, \tau) \, dx \, d\tau + C \int_{-\log T}^s \int_{R' \leq |x| \leq 2R'} q_2^R(x, \tau) \, dx \, d\tau
\]
\[
- 2 \int_{|x| \leq R_0} \min(|\overline{q}_1^R(x,s)|, |\overline{q}_2^R(x,s)|) \eta_{R'}(x) \, dx \quad \forall R \geq 2R' > 0, R \geq R_0 > 0, s > -\log T. \tag{3.21}
\]

By Corollary 2.4
\begin{equation}
0 \leq q_+ , q_- \leq |q| \in L^1(\mathbb{R}^N \times (-\log T, s)) \quad \forall s > -\log T. \tag{3.22}
\end{equation}

Let $s > -\log T$ be fixed. Then by (3.22) and Lemma 3.1 there exists a sequence of positive numbers $\{R_i\}_{i=1}^\infty, R_i \to \infty$ as $i \to \infty$, such that $\overline{q}_1^{R_i}$, $\overline{q}_2^{R_i}$, converges uniformly on every compact subset of $\mathbb{R}^N \times (-\log T, s)$ to some solutions $\overline{q}_1, \overline{q}_2$, of (3.3) respectively as $i \to \infty$. Moreover
\[
\begin{aligned}
\int_{\mathbb{R}^N} \overline{q}_1(x, \tau) \, dx &\leq \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0| \, dx \quad \forall -\log T \leq \tau \leq s \\
\int_{\mathbb{R}^N} \overline{q}_2(x, \tau) \, dx &\leq \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0| \, dx \quad \forall -\log T \leq \tau \leq s.
\end{aligned}
\]

Hence
\begin{equation}
\overline{q}_1, \overline{q}_2 \in L^1(\mathbb{R} \times [0, s)). \tag{3.23}
\end{equation}
Putting $R = R_i$ in \eqref{3.21} and letting $i \to \infty$, by \eqref{3.18},
\[
\int_{\mathbb{R}^N} |q(x, s)| \eta_R(x) \, dx - \int_{\mathbb{R}^N} |\bar{u}_0 - \bar{v}_0|(x) \eta_R(x) \, dx \\
\leq C \int_{-\log T}^s \int_{R' \leq |x| \leq 2R'} \bar{q}_1(x, \tau) \, dx \, d\tau + C \int_{\frac{-\log T}{2}}^s \int_{R' \leq |x| \leq 2R'} \bar{q}_2(x, \tau) \, dx \, d\tau \\
- 2 \int_{|x| \leq R_0} \min(\bar{q}_1(x, s), \bar{q}_2(x, s)) \eta_R(x) \, dx \quad \forall R' \geq 1, R_0 > 0.
\]
By \eqref{3.22},
\[
\int_{-\log T}^s \int_{R' \leq |x| \leq 2R'} \bar{q}_j(x, \tau) \, dx \, d\tau \to 0 \quad \text{as } R' \to \infty, \ j = 1, 2.
\]
Hence letting $R' \to \infty$ in \eqref{3.24},
\[
\int_{\mathbb{R}^N} |q(x, s)| \, dx \leq \int_{\mathbb{R}^N} |\bar{u}_0 - \bar{v}_0|(x) \, dx - 2 \int_{|x| \leq R_0} \min(\bar{q}_1(x, s), \bar{q}_2(x, s)) \, dx
\]
holds for any $R_0 > 0$, $s > -\log T$. We now choose $R_0 > 0$ such that
\[
\min(\|\bar{u}_0 - \bar{v}_0\|_{L^\infty(B_{R_0}(0))}, \|(\bar{u}_0 - \bar{v}_0)\|_{L^\infty(B_{R_0}(0))}) > 0.
\]
Since $\bar{q}_1 \geq \bar{q}_{R_0}^2$ and $\bar{q}_2 \geq \bar{q}_{R_0}^2$, by \eqref{3.25},
\[
\int_{\mathbb{R}^N} |q(x, s)| \, dx - \int_{\mathbb{R}^N} |\bar{u}_0 - \bar{v}_0|(x) \, dx \leq -2 \int_{|x| \leq R_0} \min(\bar{q}_{R_0}^2(x, s), \bar{q}_{R_0}^2(x, s)) \, dx \quad \forall s > -\log T.
\]
Since $\bar{q}_{R_0}^2(x, s)$ and $\bar{q}_{R_0}^2(x, s)$ are the solutions of \eqref{3.3} in $Q_{2R_0}$ with zero boundary value and initial values $q_+(-, - \log T)$, $q_+(-, - \log T)$, respectively, by the Green function representation for solutions, for any $s > -\log T$, there exists a constant $c(s)$ such that
\[
\min_{|x| \leq R_0} \bar{q}_{R_0}^2 \geq c(s) > 0 \quad \text{and} \quad \min_{|x| \leq R_0} \bar{q}_{R_0}^2 \geq c(s) > 0
\]
and the lemma follows. \hfill \Box

We next note that $\tilde{B}_k$ given by \eqref{1.10} is a stationary solution of \eqref{1.8} for any $k > 0$. By an argument similar to the proof of Lemma 1 of [OR] we have the following lemma.

**Lemma 3.3.** (cf. Lemma 1 of [OR]) **Suppose** \(\|\bar{u}(\cdot, s_i) - \bar{u}_0\|_{L^1(\mathbb{R}^N)} \to 0\) **as** \(i \to \infty\). **If** $\tilde{w}$ **is a solution of** \eqref{1.8} **in** $\mathbb{R}^N \times [0, \infty)$ **with initial value** $w(x, 0) = \bar{u}_0(x)$, **then**
\[
\|\tilde{w}(\cdot, s) - \tilde{B}_k\|_{L^1(\mathbb{R}^N)} \to 0 \quad \forall s > 0, k > 0,
\]
**where** $\tilde{B}_k$ **is given by** \eqref{1.10}.

**Proof of Theorem 1.1** Since the proof of the case $N = 3$ is similar to that of [HS1] and section 3 of [DS1], we will only sketch the argument here. Let
\[
f(k) = \int_{\mathbb{R}^N} (u_0(x) - B_k(x)) \, dx.
\]
Then $f(k)$ is a continuous monotone increasing function of $k > 0$. By \eqref{1.5}, $f(k_1) \geq 0 \geq f(k_2)$. Hence by the intermediate value theorem there exists a unique $k_0$ such that $f(k_0) = 0$. By \eqref{1.9}, Lemma 3.2, Lemma 3.3 and an argument similar to the proof of Theorem 2.3 in [HS1], one gets that the rescaled function $\tilde{u}(\cdot, s)$ converges uniformly on $\mathbb{R}^3$, and also in $L^1(\mathbb{R}^3)$, to the rescaled Barenblatt solution $\tilde{B}_{k_0}$ as $s \to \infty$. \hfill \Box
4. The non-integrable case \((N \geq 5)\)

In this section we will prove Theorem 1.2. Since the difference of any two solutions \(u, v\) of (1.2) that satisfies (1.6) may not be integrable when \(N \geq 4\), for any solution \(u\) that satisfies (1.6) we cannot ensure the existence of a constant \(k_0 > 0\) such that (1.11) holds from the condition (1.6) alone. Thus we need additional conditions on the initial data to ensure convergence. We will assume that \(u_0\) also satisfies (1.12) for some constant \(k_0 > 0\) and function \(f \in L^1(\mathbb{R}^N)\) in this section. Unless stated otherwise in this section we will assume that \(u\) is a solution of (1.2) which satisfies the bound (1.6), \(\tilde{u}\) will denote the rescaled solution defined by (1.7), and \(B_k\) will be the rescaled Barenblatt solution given by (1.10).

We will use a modification of the technique of [DS1] to find the asymptotic behaviour of the solution of (1.2) near its extinction time \(T\). The following simple convergence result will be used in the proof of Theorem 1.2.

**Lemma 4.1.** Let \(u_0\) satisfy (1.5) for some constants \(k_2 > k_1 > 0\) and \(u\) be a solution of (1.2) that satisfies (1.6). Let \(\tilde{u}\) be given by (1.7). Let \(\{s_i\}_{i=1}^{\infty}\) be a sequence of positive numbers such that \(s_i \to \infty\) as \(i \to \infty\) and \(\tilde{u}_i(. , s) = \tilde{u}(., s_i + s)\). Then the sequence \(\{\tilde{u}_i\}_{i=1}^{\infty}\) has a subsequence \(\{\tilde{u}_{ik}\}_{k=1}^{\infty}\) that converges uniformly on every compact subsets of \(\mathbb{R}^N \times (-\infty, \infty)\) to a solution \(\tilde{w}(x, s)\) of (1.8) in \(\mathbb{R}^N \times (-\infty, \infty)\) which satisfies (1.9) in \(\mathbb{R}^N \times (-\infty, \infty)\) as \(k \to \infty\).

**Proof.** Since \(\tilde{u}\) satisfies (1.9) in \(\mathbb{R}^N \times (-\log T, \infty)\), equation (1.8) is uniformly parabolic on \(\mathbb{R}^N \times [-\log T, \infty)\), for any \(R > 0\). By the Schauder estimates for parabolic parabolic equation [LSU] the sequence \(\tilde{u}_i\) is equi-Hölder continuous in \(C^2\) on every compact subsets of \(\mathbb{R}^N \times (-\infty, \infty)\). Hence by the Arzela-Ascoli theorem and a diagonalization argument the sequence \(\{\tilde{u}_i\}_{i=1}^{\infty}\) has a convergent subsequence \(\{\tilde{u}_{ik}\}_{k=1}^{\infty}\) that converges uniformly in \(C^2\) on every compact subsets of \(\mathbb{R}^N \times (-\infty, \infty)\) to a solution \(\tilde{w}\) of (1.8) in \(\mathbb{R}^N \times (-\infty, \infty)\) which satisfies (1.9) in \(\mathbb{R}^N \times (-\infty, \infty)\) as \(k \to \infty\). \(\square\)

**Lemma 4.2.** Let \(N \geq 5\) and let \(\tilde{u}, \tilde{v}\), be two solutions of (1.8) with initial values \(\tilde{u}_0, \tilde{v}_0\), respectively which satisfy (1.9). Let \(\tilde{B} = \tilde{B}_{k_2}\). Suppose \(\tilde{u}_0 - \tilde{v}_0 \in L^1(\mathbb{B}^\alpha, \mathbb{R}^N)\) with \(\alpha = \frac{N-4}{2}\). Then there exists a constant \(C > 0\) such that

\[
\int_{\mathbb{R}^N} |\tilde{u} - \tilde{v}|(x, s)\tilde{B}^\alpha(x) \, dx \leq \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0|\tilde{B}^\alpha(x) \, dx + Cs \quad \forall s > -\log T. \tag{4.1}
\]

**Proof.** Let \(\eta_R \in C^\infty_0(\mathbb{R}^N)\) be as in the proof of Lemma 2.4 and let \(q = \tilde{u} - \tilde{v}\). By the Kato inequality [K] \(q\) satisfies

\[
|q|, \leq \Delta(|\log \tilde{u} - \log \tilde{v}|) + \frac{1}{N-2} \nabla(x \cdot |q|) \quad \text{in } \mathbb{R}^N \times (-\log T, \infty)
\]

in the distribution sense. Then

\[
\frac{d}{ds} \int_{\mathbb{R}^N} |\tilde{u} - \tilde{v}|(x, s)\tilde{B}^\alpha(x) \eta_R(x) \, dx \\
\leq \int_{\mathbb{R}^N} |\log \tilde{u} - \log \tilde{v}|(x, s) \left( \tilde{B}^\alpha(x)\Delta\eta_R(x) + \eta_R(x)\Delta\tilde{B}^\alpha(x) + 2\nabla\tilde{B}^\alpha(x) \cdot \nabla\eta_R(x) \right) \, dx \\
- \frac{1}{N-2} \int_{\mathbb{R}^N} |\tilde{u} - \tilde{v}|(x, s) x \cdot \left\{ \eta_R(x)\nabla\tilde{B}^\alpha(x) + \tilde{B}^\alpha(x)\nabla\eta_R(x) \right\} \, dx.
\]
Hence
\[
\frac{d}{ds} \int_{\mathbb{R}^N} |\tilde{u} - \tilde{v}|(x, s) \tilde{B}^\alpha(x) \eta_R(x) \, dx \\
\leq \int_{\mathbb{R}^N} \left| \log \tilde{u} - \log \tilde{v} \right|(x, s) \left( \tilde{B}^\alpha(x) \Delta \eta_R(x) + 2 \nabla \tilde{B}^\alpha(x) \cdot \nabla \eta_R(x) \right) \, dx \\
- \frac{1}{N-2} \int_{\mathbb{R}^N} |\tilde{u} - \tilde{v}|(x, s) \tilde{B}^\alpha(x) \cdot x \cdot \nabla \eta_R(x) \, dx \\
+ \int_{\mathbb{R}^N} \left\{ \tilde{a}(x, s) \Delta \tilde{B}^\alpha(x) - \frac{1}{N-2} x \cdot \nabla \tilde{B}^\alpha(x) \right\} |\tilde{u} - \tilde{v}|(x, s) \eta_R(x) \, dx \\
= I_{1,R} + I_{2,R} + I_{3,R} \quad \forall s > -\log T. \tag{4.2}
\]
where \( \tilde{a}(x, s) \) is given by (3.17). By direct computation,
\[
\Delta \tilde{B}^\alpha(x) = -\frac{(N-4)(2|x|^2 + k_2 N)}{(k_2 + |x|^2)^2} \tilde{B}^\alpha < 0 \quad \text{in } \mathbb{R}^N. \tag{4.3}
\]
Since \( \tilde{u}, \tilde{v} \), satisfies (1.9), by (3.17) \( \tilde{a}(x, s) \) satisfies
\[
\frac{k_2 + |x|^2}{2(N-2)} \leq \tilde{a}(x, s) \leq \frac{k_1 + |x|^2}{2(N-2)} \quad \text{in } \mathbb{R}^N \times (-\log T, \infty). \tag{4.4}
\]
Then by (4.3) and (4.4),
\[
\tilde{a}(x, s) \Delta \tilde{B}^\alpha(x) - \frac{1}{N-2} x \cdot \nabla \tilde{B}^\alpha(x) \leq \frac{k_2 + |x|^2}{2(N-2)} \Delta \tilde{B}^\alpha - \frac{1}{N-2} x \cdot \nabla \tilde{B}^\alpha(x) \\
= -\frac{k_2(N-4)N}{2(N-2)(k_2 + |x|^2)} \tilde{B}^\alpha(x) < 0 \tag{4.5}
\]
in \( \mathbb{R}^N \times (-\log T, \infty) \). Hence
\[
I_{3,R} \leq 0. \tag{4.6}
\]
Since \( \tilde{u}, \tilde{v} \geq \overline{B} \),
\[
|\log \tilde{u} - \log \tilde{v}| = \left| \log \left( \frac{\tilde{u}}{\tilde{v}} \right) \right| \leq \left\{ \begin{array}{ll}
C \left| (\tilde{u}/\tilde{v}) - 1 \right| & \text{if } \tilde{u} \geq \tilde{v} \\
C \left| (\tilde{v}/\tilde{u}) - 1 \right| & \text{if } \tilde{v} \geq \tilde{u}
\end{array} \right.
\leq C \bar{B}^{-1} |\tilde{u} - \tilde{v}|
\]
for some constant \( C > 0 \). Then
\[
|I_{1,R}| \leq C_1 \int_{B_{2R} \setminus B_R} |\tilde{u} - \tilde{v}|(x, s) \tilde{B}^{-1}(x) \left| \tilde{B}^\alpha(x) \Delta \eta_R(x) + 2 \nabla \tilde{B}^\alpha(x) \cdot \nabla \eta_R(x) \right| \, dx \tag{4.7}
\]
Since
\[
|\tilde{B}| \leq \frac{C}{R^2}, \quad |\tilde{B}^{-1}| \leq CR^2, \quad |\nabla \tilde{B}| \leq \frac{C}{R^3}, \quad |\Delta \tilde{B}| \leq \frac{C}{R^4}, \quad |\nabla \eta_R| \leq \frac{C}{R}, \quad |\Delta \eta_R| \leq \frac{C}{R^2} \tag{4.8}
\]
and
\[
|\tilde{u} - \tilde{v}| \leq \left| \tilde{B}_{k_1} - \tilde{B}_{k_2} \right| \leq \frac{C}{R^3}
\]
in \( B_{2R}(0) \setminus B_R(0) \) for any \( R > 1 \) and some constant \( C > 0 \), by (4.7)
\[
|I_{1,R}| \leq C' \quad \forall R > 1, s > -\log T. \tag{4.9}
\]
Similarly there exists a constant \( C > 0 \) such that
\[
|I_{2,R}| \leq C \quad \forall R > 1, s > -\log T \tag{4.10}
\]
By \((4.2), (4.6), (4.9),\) and \((4.10),\)
\[
\frac{d}{ds} \int_{\mathbb{R}^N} |\tilde{u} - \tilde{v}| \eta_R(x) \, dx \leq C \quad \forall R > 1, s > -\log T
\]
for some constant \(C > 0.\) Integrating the above differential inequality and letting \(R \to \infty\) we get \((4.1)\) and the lemma follows. \(\square\)

**Lemma 4.3.** Let \(N \geq 5\) and let \(\tilde{u}, \tilde{v}\), be two solutions of \((4.8)\) with initial values \(\tilde{u}_0, \tilde{v}_0,\) satisfying \((4.9)\) and \(\tilde{u}_0 - \tilde{v}_0 \in L^1(\tilde{B}^\alpha, \mathbb{R}^N)\) with \(\alpha = \frac{N-1}{2}.\) Let \(\tilde{B} = \tilde{B}_{k_2}.\) If
\[
\max_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0| \neq 0, \tag{4.11}
\]
then for any \(s > -\log T\) there exist constants \(C(s) > 0\) and \(R_0 > 1\) such that
\[
\left\| (\tilde{u} - \tilde{v}) (\cdot, s) \tilde{B}^\alpha \eta_R \right\|_{L^1(\mathbb{R}^N)} < \left\| (\tilde{u}_0 - \tilde{v}_0)(\cdot, s) \tilde{B}^\alpha \eta_R \right\|_{L^1(\mathbb{R}^N)} - C(s) \quad \forall R \geq R_0 \tag{4.12}
\]
where \(\eta_R\) is as in the proof of Lemma 2.1.

**Proof.** We will use a modification of the proof of Lemma 4.1 of [DS1] to prove the lemma. Let \(\eta_R \in C_0^\infty(\mathbb{R}^N)\) be as in the proof of Lemma 2.1. Let \(q = \tilde{u} - \tilde{v}\) and \(\bar{a}(x, s)\) be given by \((3.17)\). By the proof of Lemma 4.2 \((4.2)\) holds. Integrating \((4.2),\)
\[
\int_{\mathbb{R}^N} q(x, s)|\tilde{B}^\alpha(x)\eta_R(x) \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0| |\tilde{B}^\alpha(x)\eta_R(x) \, dx
\]
\[
\leq \int_{-\log T}^{s} \int_{\mathbb{R}^N} \bar{a}(x, \tau)|q|(x, \tau) \tilde{B}^\alpha \Delta \eta_R + \eta_R \tilde{B}^\alpha \Delta \tilde{B}^\alpha + 2\tilde{B}^\alpha \nabla \eta_R \cdot \nabla \tilde{B}^\alpha \, dx \, d\tau
\]
\[
- \frac{1}{N-2} \int_{-\log T}^{s} \int_{\mathbb{R}^N} |q|(x, \tau) \tilde{B}^\alpha \nabla \eta_R + \eta_R \tilde{B}^\alpha \nabla \tilde{B}^\alpha \, dx \, d\tau.
\]
Hence
\[
\int_{\mathbb{R}^N} q(x, s)|\tilde{B}^\alpha(x)\eta_R(x) \, dx - \int_{\mathbb{R}^N} |\tilde{u}_0 - \tilde{v}_0| |\tilde{B}^\alpha(x)\eta_R(x) \, dx
\]
\[
\leq \frac{C}{R^2} \int_{-\log T}^{s} \int_{R \leq |x| \leq 2R} \bar{a}(x, \tau)|q|(x, \tau) \tilde{B}^\alpha \, dx \, d\tau
\]
\[
+ \frac{C}{R} \int_{-\log T}^{s} \int_{R \leq |x| \leq 2R} \bar{a}(x, \tau)|q|(x, \tau) \left| \nabla \tilde{B}^\alpha \right| \, dx \, d\tau
\]
\[
+ C \int_{-\log T}^{s} \int_{R \leq |x| \leq 2R} \left| q(x, \tau) \tilde{B}^\alpha \right| \, dx \, d\tau
\]
\[
+ \int_{-\log T}^{s} \int_{R \leq |x| \leq 2R} \left\{ \alpha(x, \tau) \Delta \tilde{B}^\alpha - \frac{1}{N-2} x \cdot \nabla \tilde{B}^\alpha \right\} \left| q(x, \tau) \eta_R \right| \, dx \, d\tau
\]
\[
= I_{1,R} + I_{2,R} + I_{3,R} + I_{4,R} \quad \forall R > 0, s > -\log T. \tag{4.13}
\]
Now by \((4.8),\)
\[
|\nabla \tilde{B}^\alpha| \leq C R^{-1} \tilde{B}^\alpha \quad \forall R \leq |x| \leq 2R, R > 1 \tag{4.14}
\]
for some constant \(C > 0.\) Then by \((4.4)\) and \((4.14),\)
\[
0 \leq I_{2,R} \leq C I_{1,R} \leq C' I_{3,R} \quad \forall R > 0. \tag{4.15}
\]
Since by Lemma 4.2
\[
\int_{-\log T}^{s} \int_{\mathbb{R}^N} |q|(x, \tau) \tilde{B}^\alpha \, dx \, d\tau < \infty \quad \forall s > -\log T,
\]
we have
\[
\lim_{R \to \infty} I_{3,R} = \lim_{R \to \infty} \int_{-\log T}^{s} \int_{R \leq |x| \leq 2R} |q|(x,\tau) \tilde{B}^\alpha(x) \, dx \, d\tau = 0. \tag{4.16}
\]
By (4.15) and (4.16),
\[
\lim_{R \to \infty} I_{1,R} = \lim_{R \to \infty} I_{2,R} = 0. \tag{4.17}
\]
By (4.5) and (4.11) for any \( s > -\log T \) there exist constants \( C(s) > 0 \) and \( R_1 > 1 \) such that
\[
I_{4,R} < -C(s) \quad \forall R \geq R_1. \tag{4.18}
\]
By Lemma 4.3 and Corollary 2.2 in the proof there we have the following result.

Lemma 4.4. (cf. Lemma 1 of [OR]) Let \( N \geq 5 \), \( \alpha = (N-4)/2 \), and \( \tilde{B}_{k_0} \) be the rescaled Barenblatt solution. Suppose \( \| \tilde{w}(\cdot, s_i) - \tilde{w}_0 \|_{L^1(\tilde{B}_{k_0}^{\alpha},\mathbb{R}^N)} \to 0 \) as \( i \to \infty \). If \( \tilde{w} \) is a solution of (1.8) in \( \mathbb{R}^N \times [0,\infty) \) with initial value \( \tilde{w}(x,0) = \tilde{w}_0(x) \), then
\[
\| \tilde{w}(\cdot, s) - \tilde{B}_{k_0} \|_{L^1(\tilde{B}_{k_0}^{\alpha},\mathbb{R}^N)} = \| \tilde{w}_0 - \tilde{B}_{k_0} \|_{L^1(\tilde{B}_{k_0}^{\alpha},\mathbb{R}^N)} \quad \forall s > 0.
\]

By an argument similar to the proof of Claim 4.4 of [DS1] but with Lemma 4.3 and Corollary 2.2 replacing Lemma 4.1 and Corollary 2.2 in the proof there we have the following result.

Lemma 4.5. Let \( N \geq 5 \) and let \( u_0, u, \tilde{u}, \tilde{u}_i, \tilde{u}_k \) and \( \tilde{w} \) be as in Lemma 4.4. Then the sequence \( \tilde{u}_k(x,s) \) converges to \( \tilde{u}(x,s) \) in \( L^1(\tilde{B}_{k_0}^{\alpha},\mathbb{R}^N) \)-norm as \( k \to \infty \).

Then by the same argument as the proof of Theorem 1.2 of [DS1] on P.110 but with Lemma 4.2, Lemma 4.3, and Claim 4.4 there being replaced by Lemma 4.3, Lemma 4.1 and Lemma 4.5 we get Theorem 1.2. This completes the proof of Theorem 1.2.

5. Improvement

In this section we will improve Theorem 1.2 by removing the assumption \( u_0 \geq B_{k_1}(x,0) \) where \( B_{k_1}(x,t) \) is given by (1.4) for some \( T > 0 \). Let \( T > 0 \) and \( k_0 > 0 \) be fixed constants. Denoting by
\[
B_{k_0}(x,t) = \frac{2(N-2)(T-t)^{\frac{N-4}{2}}}{k_0 + (T-t)^{\frac{N-4}{2}} |x|^2},
\]
we will prove the following result.

Theorem 5.1. Let \( N \geq 5 \). Suppose
\[
0 \leq u_0 \leq B_{k_0}(x,0) \quad \text{in} \ \mathbb{R}^N \tag{5.1}
\]
and
\[
|u_0(x) - B_{k_0}(x,0)| \leq f(|x|) \in L^1(\mathbb{R}^N) \tag{5.2}
\]
for some nonnegative radially symmetric function \( f \). Then the maximal solution \( u \) of (1.2) vanishes at the same time \( T \) as \( B_{k_0}(x,t) \) and the rescaled solution \( \tilde{u}(x,s) \) given by (1.7) converges uniformly on \( \mathbb{R}^N \) and in \( L^1(\tilde{B}_{k_0}^{\alpha},\mathbb{R}^N) \) as \( s \to \infty \) to the rescaled Barenblatt solution \( \tilde{B}_{k_0} \).

We will first prove that condition (5.2) implies the \( L^1 \)-contraction principle.
Lemma 5.2. Let \( N \geq 3 \) and \( 0 \leq u_0 \) satisfy (5.2) for some function \( 0 \leq f \in L^1(\mathbb{R}^N) \). Suppose \( u \) is the maximal solution of (1.2) in \( \mathbb{R}^N \times (0, T_0) \) for some \( T_0 > 0 \). Then
\[
\int_{\mathbb{R}^N} |u(\cdot, t) - B_{k_0}(\cdot, t)| \, dx \leq \|f\|_{L^1(\mathbb{R}^N)} \quad \forall 0 < t < \min(T, T_0).
\]

Proof. For any \( k \geq k_0 \), let \( u_k \) be the maximal solutions of (1.2) (cf. [Hu2]) in \( \mathbb{R}^N \times (0, T_k) \) with initial values
\[
u_{0,k}(x) = \max(B_k(x, 0), u_0(x)), \quad \forall k \geq k_0
\]
where \( T_k \) is the maximal time of existence of the solution \( u_k \). Since
\[
u_0(x) \leq \nu_{0,k}(x) \leq 2(N - 2)k_0^{-1}T_0^{N-2} \quad \text{in } \mathbb{R}^N \quad \forall k' \geq k \geq k_0
\]
and \( u, u_k \) are the maximal solutions of (1.2) with initial values \( u_0, u_{0,k} \) respectively, by the result of [Hu2],
\[
u(x, t) \leq \nu_{k'}(x, t) \leq \nu_k(x, t) \leq 2(N - 2)k_0^{-1}T_0^{N-2} < \infty \quad \text{in } \mathbb{R}^N \times (0, T_0)
\]
for any \( k' \geq k \geq k_0 \). Then \( T_k \geq T_{k'} \geq T_0 \) for all \( k' \geq k \geq k_0 \). Hence the equation (1.2) for the sequence \( \{u_k\}_{k \geq k_0} \) is uniformly parabolic on any compact subset of \( \mathbb{R}^N \times (0, T_0) \). By the Schauder estimates [LSU], \( \{u_k\}_{k \geq k_0} \) is equi-Hölder continuous on any compact subset of \( \mathbb{R}^N \times (0, T_0) \). Since the sequence of solution \( \{u_k\}_{k \geq k_0} \) is decreasing as \( k \to \infty \) and bounded below by \( u, u_k \) converges uniformly to a solution \( v \) of (1.2) on every compact subset of \( \mathbb{R}^N \times (0, T_0) \) as \( k \to \infty \). By an argument similar to the proof of Theorem 2.4 in [Hu2], \( v \) has initial value \( u_0 \). Letting \( k \to \infty \) in (5.3),
\[
\nu(x, t) \geq \nu(x, t) \quad \text{in } \mathbb{R}^N \times (0, T_0).
\]
On the other hand since \( u \) is the maximal solution of with initial value \( u_0 \),
\[
u(x, t) \geq \nu(x, t) \quad \text{in } \mathbb{R}^N \times (0, T_0).
\]
Hence \( u = v \) on \( \mathbb{R}^N \times (0, T_0) \). Since both \( B_{k_0} \geq B_k \) and \( u_k \geq B_k \) for any \( k \geq k_0 \), by Lemma 2.3
\[
\int_{\mathbb{R}^N} |B_{k_0} - u_k|(x, t) \, dx \leq \int_{\mathbb{R}^N} |B_{k_0} - u_k|(x, 0) \, dx \leq \|f\|_{L^1(\mathbb{R}^N)}, \quad \forall k \geq k_0, \quad \forall 0 < t < \min(T, T_0).
\]
Letting \( k \to \infty \) in (5.3), we get (5.3) and the lemma follows. \( \square \)

Note that if \( 0 \leq u_0 \in L^\infty(\mathbb{R}^N) \) satisfies (5.2) for some function \( 0 \leq f \in L^1(\mathbb{R}^N) \), then the maximal solution \( u \) of (1.2) and \( B_{k_0} \) have the same vanishing time. The reason is as follows. Let \( T_0 > 0 \) be the maximal time of existence of the solution \( u \) of (1.2). We first suppose that \( T_0 < T \), then by (5.3)
\[
\int_{\mathbb{R}^N} |B_{k_0}(x, T_0)| \, dx \leq \|f\|_{L^1(\mathbb{R}^N)}.
\]
On the other hand, since the dimension \( N \geq 3 \), \( B_{k_0}(x, T_0) \notin L^1(\mathbb{R}^N) \). Contradiction arises. Hence \( T_0 \geq T \). We now assume that \( T_0 > T \). Letting \( t \nearrow T \) in (5.3),
\[
\int_{\mathbb{R}^N} |u(x, T)| \, dx \leq \|f\|_{L^1(\mathbb{R}^N)}.
\]
This contradicts the result of Vazquez [V1] which said that (1.2) has no solution that is in \( L^1(\mathbb{R}^N) \). Hence \( T = T_0 \) and the maximal solution \( u \) vanishes at the same time as \( B_{k_0}(x, t) \).

We next prove a lemma on the existence of maximal solutions of (1.2).
Then there exist positive constants $C$ and $\omega$ such that the rescaled function $\tilde{u}(\cdot,t)$ is radially symmetric in $\mathbb{R}^N \times (0,T)$ for some radially symmetric initial value $u_0$.

**Lemma 5.3** (cf. Corollary 2.8 in [Hu2]). Let $N \geq 3$ and let $g(x) = B_{k_0}(x,0) - h(x)$ for some radially symmetric function $0 \leq h \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ be such that $g(x) \geq 0$ on $\mathbb{R}^N$. Then there exists a unique maximal solution $u$ of (1.2) in $\mathbb{R}^N \times (0,T)$ with initial value $g$.

**Proof.** Since the proof is similar to that of Corollary 2.8 of [Hu2], we will only give a sketch of the proof here. For any $R > 0$ and any function $\psi \in L^1(B_R(0))$, let

$$\tilde{G}_R(\psi)(x) = \int_{B_R(0)} (G_R(0,y) - G_R(x,y)) \psi(y) \, dy \quad \forall |x| \leq R$$

where $G_R$ is the Green function for the Laplacian on $B_R(0)$. Since $u_0$ is radially symmetric and

$$B_{k_0}(x,0) \geq \frac{C_1}{|x|^2} \quad \forall |x| \geq 1$$

for some constant $C_1 > 0$, for any $R > 1$, we have (cf. [Hu2])

$$\tilde{G}_R(g)(x) = \int_0^{|x|} \frac{1}{\omega_{N-1} \omega_{N-1}} \left( \int_{|y| \leq r} g(y) \, dy \right) \, dr \geq 0, \quad \forall |x| \leq R$$

and

$$\tilde{G}_R(g)(x) \geq \int_1^{|x|} \frac{1}{\omega_{N-1}} \left( \int_{|y| \leq r} g(y) \, dy \right) \, dr$$

$$\geq \frac{C_1}{N-2} \log |x| - \frac{C_1}{(N-2)^2 + \frac{1}{(N-2)\omega_N}} \left( 1 - |x|^{2-N} \right) \quad \forall 1 \leq |x| \leq R.$$

Hence there exist constants $R_1 > 1$ and $C_2 > 0$ such that

$$\tilde{G}_R(g)(x) \geq C_2 \log |x| \quad \forall R_1 \leq |x| \leq R. \quad (5.6)$$

Then by (5.4) and the result of [Hu2], (1.2) has a unique maximal solution $u$ with initial value $g$ in $\mathbb{R}^N \times (0,T_1)$ for some constant $T_1 > 0$. Since the solution $u$ is unique and $g$ is radially symmetric, $u(\cdot,t)$ is radially symmetric in $\mathbb{R}^N \times (0,T_1)$. Let $T_2 > 0$ be the maximal time of existence of the solution $u$. By the discussion just before the lemma we have $T_2 = T$ and the lemma follows. \qed

By Lemma 1.8 of [Hu2], Lemma 5.4 and an argument similar to the proof of Corollary 2.8 of [Hu2] we have the following corollary.

**Corollary 5.4.** Let $N \geq 3$ and let $B_{k_0}(x,0) - h(x) \leq u_0(x) \leq B_{k_0}(x,0)$ for some radially symmetric function $h \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ satisfying $0 \leq h(x) \leq B_{k_0}(x,0)$ on $\mathbb{R}^N$. Then there exists a unique maximal solution $u$ of (1.2) in $\mathbb{R}^N \times (0,T)$ satisfying $0 \leq u(x,t) \leq B_{k_0}(x,t)$ in $\mathbb{R}^N \times (0,T)$ with initial value $u_0$.

**Lemma 5.5.** Let $N \geq 3$. Suppose $u_0$ satisfies (5.1), (5.2), and $u$ is the maximal solution of (1.2).

Then there exist positive constants $C_1, C_2, C_3, r_0, s_0$ such that the rescaled function $\tilde{u}$ given by (1.7) satisfies

$$C_1 \frac{e^{-C_3\|f\|_{L^1}}}{1 + r^2} \leq \tilde{u}(r,s) \leq C_2 \frac{e^{C_3\|f\|_{L^1}}}{1 + r^2} \quad \forall r \geq r_0, \ s \geq s_0. \quad (5.7)$$

**Proof.** We will use a modification of the proof of Proposition 6.2 of [DS1] to prove the lemma. We will first prove (5.7) under the assumption that $u_0$ is radially symmetric. Let $\{u_k\}_{k \geq k_0}$ be the sequence constructed in the proof of Lemma 5.2 As observed in the proof of Lemma 2.2 the function

$$w_k(x) = \int_l^t |\log u_k - \log B_{k_0}|(x,\tau) \, d\tau \quad \forall 0 < l < t < T$$
satisfies
\[ \Delta w_k(x) \geq -|u_k - B_{k_0}|(x, l) \quad \text{in } \mathbb{R}^N \quad \forall 0 < l < t < T \]
and
\[ \Delta (w_k - Z_k(\cdot, l)) \geq 0 \quad \text{in } \mathbb{R}^N \quad \forall 0 < l < t < T \]
where \( Z_k(x, l) \) is given by
\[ Z_k(x, l) = \int_r^\infty \frac{1}{\omega_N \rho^{N-1}} \int_{|y| \leq \rho} |u_k - B_{k_0}|(y, l) \, dy \, d\rho, \quad r = |x|. \]
Note that \( Z_k \) satisfies \( \Delta Z_k(x, l) = -|u_k - B_{k_0}|(x, l) \) in \( \mathbb{R}^N \). Then, as in the proof of Lemma 2.2
\[ w_k(x) \leq Z_k(x, l) \quad \text{in } \mathbb{R}^N. \]
Thus
\[ w_k(x) \leq C_3 \frac{\|(u_k - B_{k_0})(l)\|_{L^1(\mathbb{R}^N)}}{r^{N-2}}, \quad r = |x| \geq 1, \quad (5.8) \]
for some constant \( C_3 > 0 \). By \( (5.2), (5.5) \) and \( (5.8) \),
\[ \int_t^\infty \log B_{k_0}(x, \tau) \, d\tau - C_3 \frac{\|f\|_{L^1(\mathbb{R}^N)}}{r^{N-2}} \leq \int_t^\infty \log u_k(x, \tau) \, d\tau \leq \int_t^\infty \log B_{k_0}(x, \tau) \, d\tau + C_3 \frac{\|f\|_{L^1(\mathbb{R}^N)}}{r^{N-2}} \quad (5.9) \]
holds for any \( r = |x| \geq 1 \) and \( 0 < l < t < T \). We now let \( t \in [3T/4, T) \) and choose \( l \in [T/2, T) \) such that \( T - t = t - l \). For any \( l \leq \tau \leq t \),
\[ B_{k_0}(x, \tau) \leq \frac{2(N - 2)(T - l)^{N/2}}{k_0 + (T - t)^{N/2} |x|^2} = \frac{2(N - 2)[2(T - t)^{N/2}]}{k_0 + (T - t)^{N/2} |x|^2} = 2^{N/2} B_{k_0}(x, t) \]
and similarly
\[ B_{k_0}(x, \tau) \geq 2^{-N/2} B_{k_0}(x, l) \quad \forall \, l < \tau < T. \]
Hence
\[ (t - l) \left\{ \log B_{k_0}(x, l) - \log 2^{N/2} \right\} \leq \int_t^\infty \log B_{k_0}(x, \tau) \, d\tau \leq (t - l) \left\{ \log B_{k_0}(x, t) + \log 2^{N/2} \right\}. \quad (5.10) \]
By \( (5.9) \) and \( (5.10) \),
\[ \log \left( \frac{B_{k_0}(x, l)}{C_4} \right) \leq \frac{1}{t - l} \int_t^\infty \log u_k(x, \tau) \, d\tau \leq \log (C_4 B_{k_0}(x, t)) \quad \forall |x| = r \geq 1 \quad (5.11) \]
where \( C_4 = C_3 \frac{\|f\|_{L^1(\mathbb{R}^N)}}{2^{N/2}} \). Since \( u_k \) satisfies the Aronson-Benilan inequality (cf. [Hu2]),
\[ u_t \leq \frac{u}{t} \quad \text{in } \mathbb{R}^N \times (0, T), \]
we have
\[ \frac{r}{t} u_k(x, t) \leq u_k(x, \tau) \leq \frac{r}{t} u_k(x, l) \quad \forall x \in \mathbb{R}^N, l \leq \tau \leq t \]
\[ \Rightarrow \quad \log \left( \frac{1}{t} u_k(x, t) \right) \leq \frac{1}{t - l} \int_t^\infty \log u_k(x, \tau) \, d\tau \leq \log \left( \frac{1}{t} u_k(x, l) \right) \quad \forall x \in \mathbb{R}^N, l \leq \tau \leq t. \quad (5.12) \]
Now by our choice for \( l \) we have \( t/l \leq 2 \). Then by \( (5.11) \) and \( (5.12) \),
\[ u_k(x, t) \leq C_5 e^{C_3 \frac{\|f\|_{L^1(\mathbb{R}^N)}}{t^{N/2}}} B_{k_0}(x, t) \quad \forall \, |x| \geq 1, 3T/4 \leq t < T \quad (5.13) \]
and
\[ u_k(x, l) \geq C_6 e^{-C_3 \frac{\|f\|_{L^1(\mathbb{R}^N)}}{t^{N/2}}} B_{k_0}(x, l) \quad \forall \, |x| \geq 1, 3T/4 \leq l < T \quad (5.14) \]
for some constants $C_5, C_6 > 0$. Letting $k \to \infty$ in (5.13) and (5.14),

$$C_6 e^{-C_3 \|f\|_{L^1}} B_{k_0}(x,t) \leq u(x,t) \leq C_5 e^{-C_3 \|f\|_{L^1}} B_{k_0}(x,t) \quad \forall \, |x| \geq 1, 3T/4 \leq t < T. \quad (5.15)$$

By (5.15) we conclude after rescaling,

$$C_1 e^{-C_3 e^s \|f\|_{L^1}} \leq \tilde{u}(r,s) \leq C_2 e^{-C_3 e^s \|f\|_{L^1}} \quad \forall \, r \geq T^{1/(N-2)}, s \geq -\log(T/4)$$

for some constants $C_1 > 0, C_2 > 0$.

When $u_0(x)$ is nonradial and satisfies (5.2), by the above result for the radially symmetric initial data case and an argument similar to the last step of the proof of Proposition 6.2 of [DS1] on p.118 of [DS1] the lemma follows. □

Proof of Theorem 5.1. By Lemma 5.5 there exist positive constants $C_1, C_2, C_3, s_0$ and $r_0$ such that (5.7) holds. Let $s_1 > s_0 > -\log T$ and $Q_{r_0}^{s_1} = B_{r_0}(0) \times (s_0, s_1)$. Then there exist constants $C_4 > 0, C_5 > 0$ such that

$$\frac{C_4}{1 + r_0^2} \leq \tilde{u}(x,s) \leq \frac{C_5}{1 + r_0^2} \quad \text{in} \, Q_{r_0}^{s_1}. \quad (5.16)$$

on the parabolic boundary $\partial_p Q_{r_0}^{s_1} = (\overline{B_{r_0}(0) \times \{s_0\}}) \cup (\partial B_{r_0}(0) \times (s_0, s_1))$. By the maximum principle,

$$\frac{C_4}{1 + r_0^2} \leq \tilde{u}(x,s) \leq \frac{C_5}{1 + r_0^2} \quad \text{in} \, Q_{r_0}^{s_1}. \quad (5.17)$$

By (5.16) and (5.17),

$$\frac{C_4'}{1 + |x|^2} \leq \tilde{u}(x,s) \leq \frac{C_5'}{1 + |x|^2} \quad \text{on} \, \mathbb{R}^N \times [s_0, s_1)$$

for some constants $C_4' > 0, C_5' > 0$. Hence $\tilde{u} - \tilde{B}_{k_0}$ satisfies (3.3) in $\mathbb{R}^N \times [s_0, \infty)$ with

$$\tilde{a}(x,s) = \int_0^1 \frac{d\theta}{\theta \tilde{u} + (1 - \theta) \tilde{B}_{k_0}}$$

and $\tilde{a}$ satisfies (3.1) for some constants $C_1 > 0$ and $C_2 > 0$. By (5.1),

$$\tilde{u}(x,s) \leq \frac{2(N - 2)}{k_0 + |x|^2} \quad \text{in} \, \mathbb{R}^N \times [s_0, \infty).$$

Hence

$$\tilde{a}(x,s) \geq \frac{k_0 + |x|^2}{2(N - 2)} \quad \text{in} \, \mathbb{R}^N \times [s_0, \infty).$$

Then by an argument similar to the the proof of Theorem 1.2 in section 4 the theorem follows. □

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