Asymptotic flatness at spatial infinity in higher dimensions

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A definition of asymptotic flatness at spatial infinity in $d$ dimensions ($d \geq 4$) is given using the conformal completion approach. Then we discuss asymptotic symmetry and conserved quantities. As in four dimensions, in $d$ dimensions we should impose a condition at spatial infinity that the “magnetic” part of the $d$-dimensional Weyl tensor vanishes at faster rate than the “electric” part does, in order to realize the Poincare symmetry as asymptotic symmetry and construct the conserved angular momentum. However, we found that an additional condition should be imposed in $d > 4$ dimensions.

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I. INTRODUCTION

If one considers an “isolated” system in general relativity, one should impose some asymptotic boundary conditions on gravitational fields. As one of such conditions, there is the asymptotically flat condition, which states that the metric should approach to Minkowski metric at “far away” place from gravitational sources. In order to define the notion of this “far away” covariantly, one often uses the conformal completion method introduced by Penrose [1]. In this method, physical space-time $M$ is conformally embedded to unphysical space-time $\hat{M}$ with boundary, and this boundary is constituted of spatial infinity and null infinity. Hence, one can define asymptotic flatness, imposing some proper boundary conditions at this spatial infinity or null infinity.

In four dimensions, asymptotic flatness at spatial infinity was investigated using the conformal completion method by Ashtekar and Hansen [2]. They revealed that asymptotic symmetry at spatial infinity can be reduced to the Poincare symmetry which is a symmetry associated with “background” flat metric, and constructed 4-momentum and angular momentum. On the other hand, in higher dimensions, there is only a few works about asymptotic structure at spatial infinity [3] or null infinity [4] though recently the importance of higher dimensional black holes is increasing in string theory and TeV gravity scenario [5, 6].

While in four dimensions, uniqueness theorem was obtained [7], we cannot prove the uniqueness for stationary black holes (counterexamples are Myers-Perry black hole [8] and black ring [9] with the same mass and angular momentum) in higher dimensions (although uniqueness was shown in [10] for static black holes). If one would like to classify these higher dimensional black holes using some parameters, the investigation on asymptotic structure at spatial infinity could play a key role.

The purpose of this paper is to define asymptotic flatness and investigate asymptotic structure at spatial infinity in higher dimensions, following Ashtekar and Hansen [2]. (The reference [3] investigates into asymptotic flatness in higher dimensions following Ashtekar and Romano [11]. This analysis is useful when one is interested only in spatial infinity. For full understanding of asymptotic structures, however, Ashtekar and Hansen’s work is appropriate.)

The rest of this paper is organized as follows. In the section II we define asymptotic flatness at spatial infinity following Ashtekar and Hansen [2]. In the section III we investigate asymptotic structure: asymptotic symmetry and conserved quantities. Finally, we give a summary and discussion in the section IV. In the appendix A we introduce some important concepts in this literature such as directional dependence, and in the appendix B we summarize basic features of conformal completion taking Minkowski space-time for an example. Some important equations in this literature are derived in the appendix C and in the appendix D we prove the equivalence of our expressions for conserved quantities with the ADM formulae.

II. DEFINITION

We define asymptotic flatness at spatial infinity ($i^0$) in $d$ dimensions using the conformal completion method developed by Ashtekar and Hansen in four dimensions [2].

In this paper, for simplicity we assume physical space-time ($M, g_{ab}$) satisfies the vacuum Einstein equation $R_{ab} = 0$. It is easy to extend our current work to more general non-vacuum cases as long as one focuses on the asymptotically flat space-time.

Definition: $d$-dimensional physical space-time ($M, g_{ab}$) will be said to be asymptotically flat at spatial infinity $i^0$ if there exists ($\hat{M}, \hat{g}_{ab}$), where $\hat{g}_{ab}$ is $C^{d-4}$ at $i^0$ (see Appendix A for the definition of $C^{\geq n}$), and embedding of $M$ into $\hat{M}$ satisfying the following conditions:

1. $\bar{J}(i^0) = \hat{M} - M$, where $\bar{J}(i^0)$ is the closure of the union of chronological future and past of $i^0$. 


2. There exists a function $\Omega$ on $\hat{M}$ that is $C^2$ at $i^0$ such that $\hat{g}_{ab} = \Omega^2 g_{ab}$ on $M$ and $\nabla_a \nabla_b \Omega = 2g_{ab}$, $\Omega \equiv 0$ and $\hat{\nabla}_a \Omega \equiv 0$ at $i^0$ on $\hat{M}$.

Here, and $\hat{\nabla}$ is the connection for $g_{ab}$, and $\equiv$ implies the evaluation on $i^0$ (i.e. $\equiv \lim_{\omega \to \infty}^{}$ is equivalent to $\equiv \infty$). The first condition requires that, in $\hat{M}$, $\Omega$ is connected to the points on $M$ only via spacelike curves. The second condition says that $\Omega$ behaves $\sim 1/r^2$ near $i^0$. This is the same asymptotic behavior as in the Minkowski spacetime (see Appendix D).

Since we assume $g_{ab}$ is $C^{\infty}$ at $i^0$, $\hat{\theta}_a \cdots \hat{\theta}_{a(d-3)} \hat{g}_{bc}$ has directional dependent limit at $i^0$ (where $\hat{\theta}$ is flat connection on $\hat{M}$). This condition is equivalent to one such that $\Omega^{(5-d)/2} \hat{R}_{abcd}$ has directional dependent limit at $i^0$. When we discuss asymptotic structure, we often use the Weyl tensor $\hat{C}_{abcd}$ as asymptotic gravitational fields. Thus, it is convenient to use the latter condition on $\hat{R}_{abcd}$ for the discussions hereafter.

III. ASYMPTOTIC STRUCTURE

In this section, we show how to derive the asymptotic structure from the asymptotic flatness definition. Firstly, we discuss asymptotic symmetry in the section III A. We show that the asymptotic symmetry is constituted of the Lorentz group and supertranslation group (infinite group of angular-dependent translation) in higher dimensions. In the section III B we define asymptotic fields and study their transformation behavior under supertranslation. We find that supertranslation group reduces to the Poincare group if we impose an additional asymptotic condition $\hat{B}_{\alpha_1 \alpha_2 \cdots \alpha_{d-2}} \equiv 0$ in the definition of asymptotic flatness. We define conserved quantities ($d$-momentum and angular momentum) associated to this Poincare symmetry in the section III C. We confirm that the conserved quantities we define agree with the ADM formulae in this section and the appendix D.

A. Asymptotic symmetry

The asymptotic symmetry is a group of mappings which conserve asymptotic structure. Here, by asymptotic structure we mean $\hat{g}_{ab}(\Omega^{(4-d)/2} \hat{\nabla} \hat{g}_{bc})$ at $i^0$, since we impose $C^{\infty}$ condition on the behavior of $g_{ab}$ at $i^0$. In order to investigate this asymptotic symmetry, we consider the generator $\hat{\xi}$ of the asymptotic symmetry on $\hat{M}$. This generator $\hat{\xi}$ should be an extension of $\xi$, which is a generator of diffeomorphism on $M$. This extension $\hat{\xi}$ of $\xi$ to $i^0$ should satisfy

1. $\hat{\xi} \equiv 0$,
2. $\hat{\nabla}_{(a} \hat{\xi}_{b)} \equiv 0$,
3. $\hat{\nabla}_{(a} \hat{\xi}_{b)}$ is a $C^{\infty}$ tensor.

Roughly speaking, these conditions set the behavior of components of $\xi$ near $i^0$ as

$$\hat{\xi}^a \sim \frac{1}{r} + \frac{1}{r^{d-2}}. \quad (1)$$

The first condition says that a generator $\hat{\xi}$ does not touch $i^0$. The second condition implies that $\xi$ is asymptotically a Killing vector, i.e. $\hat{g}_{ab}$ at $i^0$ is not changed. Before explaining the meaning of the third condition, let us consider the gauge freedom of the conformal completion. First, let $\omega$ be a function on $\hat{M}$, $C^{\infty}$ at $i^0$ and $\omega \equiv 1$. Then, a conformal completion such that $\hat{g}_{ab} = \Omega^2 g_{ab}$ is equivalent to $\hat{g}_{ab} = \hat{g}_{ab}$, because $\delta \Omega$ satisfies

$$\omega \Omega \equiv 0, \; \hat{\nabla}_a (\omega \Omega) \equiv 0, \; \hat{\nabla}_a \hat{\nabla}_b (\omega \Omega) \equiv 2g_{ab}. \quad (2)$$

Then, we cannot distinguish these two conformal completions under the asymptotic flatness definition in section III. This gauge freedom $\omega$ of the conformal completion reshuffles the value $\Omega^{(4-d)/2} \hat{\nabla} \hat{g}_{bc}$ in the asymptotic structure as

$$\Omega^{(4-d)/2} \left( \hat{\nabla}_a - \hat{\nabla}_a \hat{\xi}^b \right) \hat{\nabla}_b \hat{\xi}^c \epsilon_{bc} \approx \Omega^{(4-d)/2} \hat{\nabla} \hat{\nabla} \hat{\xi}$$

$$\approx \frac{1}{\omega} \left[ \delta_c^b \Omega^{(4-d)/2} \hat{\nabla}_b \omega + \delta^c_\omega \Omega^{(4-d)/2} \hat{\nabla}_c \omega ight] \hat{\xi} \equiv 0. \quad (3)$$

where $\hat{\nabla}$ is the connection for $g_{ab}$ and $\epsilon_{bc}$ is any vector. This equation can also be written as

$$\Omega^{(4-d)/2} \hat{\nabla}_a (\omega \hat{\xi}) \equiv 2\Omega^{(4-d)/2} (\hat{\nabla}_a \omega) \hat{g}_{bc}. \quad (4)$$

Thus, asymptotic structure $\Omega^{(4-d)/2} \hat{\nabla} \hat{g}_{bc}$ has an ambiguity coming from gauge freedom $\omega$, and this ambiguity is reshuffled by order $1/r^{d-2}$ part of $\hat{\xi}$. Hence, asymptotic symmetry is the group of transformations which does not change the asymptotic structure except for this gauge ambiguity. Then, we call this asymptotic symmetry transformation, which is induced by order $1/r^{d-2}$ component of $\hat{\xi}$, supertranslation group. As any two generators $\hat{\xi}^1, \hat{\xi}^2$ of supertranslation group commute:

$$\left[ \hat{\xi}^1, \hat{\xi}^2 \right] \sim \frac{1}{r^{d-2}} \frac{\partial}{\partial U} r^{d-2} \sim O \left( \frac{1}{r^{2d-5}} \right), \quad (5)$$

supertranslation group is abelian (where we use the fact that the contribution to $\Omega^{(4-d)/2} \hat{\nabla} \hat{g}_{bc}$ from $O(1/r^{2d-5})$ part of $\hat{\xi}$ is only $O(1/r^{d-3})$, which is regarded as zero at $i^0$, and thus that part cannot transform the asymptotic structure). Because of angular dependence of $\omega$, however, supertranslation group has infinite translational directions. In this stage, asymptotic symmetry is not expected to be the Poincare symmetry.

B. Asymptotic fields

In order to construct conserved quantities associated with the asymptotic symmetry, we define asymptotic
gravitational fields using the Weyl tensor \( \hat{C}_{ambi} \) as\(^1\)
\[
\hat{E}_{ab} \equiv \Omega^{(5-d)/2}\hat{C}_{ambi}\hat{\eta}^m\hat{\eta}^n,
\]
\[
\hat{B}_{a_1\cdots a_{d-3}b} \equiv \Omega^{(9-2d)/2}\epsilon_{a_1\cdots a_{d-3}mpq}\hat{\eta}^m_b\hat{\eta}^p_a\hat{\eta}^q_b,
\]
where \( \epsilon_{a_1\cdots a_{d-3}mpq} \equiv \sqrt{-g}E_{a_1\cdots a_{d-3}mpq} \) is a totally antisymmetric tensor in \( M \), and we take the convention that \( E_{012\cdots d-1} = 1 \). \( \hat{\eta}^a = \nabla^a\Omega^{1/2} \) is a normal vector to \( \Omega = \text{constant surface which becomes a unit vector at } i^0 \). We call these asymptotic fields \( (6) \) and \( 7 \) electric and magnetic parts of the Weyl tensor respectively. As these fields do not have components parallel to \( \hat{\eta}^a \), we can regard them as fields on a timelike hypersurface \( S \) normal to \( \hat{\eta}^a \).

Firstly, let us derive asymptotic field equations. Using the Bianchi identity in the physical vacuum space-time \( \nabla_{[m}C_{ab]cd} = 0 \), we obtain the following equation in terms of the unphysical space-time quantities:
\[
\nabla_{[m}\hat{C}_{ab]cd} = \Omega^{-1}\left( \hat{g}_{c[m}\hat{C}_{ab]pd}\nabla^p\Omega + \hat{g}_{d[m}\hat{C}_{ab]cp}\nabla^p\Omega \right).
\]
(8)
It is better to rewrite the left-hand side as
\[
\nabla_{[m}\hat{C}_{ab]cd} = \Omega^{-1}\left[ \frac{5-d}{2}\left( \nabla_{[m}^2\hat{C}_{abcd} \right) \right],
\]
(9)
where \( \Omega^{(5-d)/2}\hat{C}_{abcd} \) has directional dependent limit at \( i^0 \). We project these equations into the timelike hypersurface \( S \), and contract with \( \hat{\eta}^a \). Then, we get the equations for the electric part
\[
\hat{D}_a\hat{E}_{bc} - \hat{D}_b\hat{E}_{ac} \equiv (4-d)\hat{h}_a^\hat{\eta}_b^\hat{\eta}_c^\hat{\eta}^r\Omega^{(5-d)/2}\hat{C}_{pqr}m\hat{\eta}^m
\]
(10)
and for the magnetic part
\[
\hat{D}_b\hat{B}_{a_1\cdots a_{d-3}c} - \hat{D}_c\hat{B}_{a_1\cdots a_{d-3}b} \equiv -(d-3)\Omega^{(9-2d)/2}\epsilon_{a_1\cdots a_{d-3}mpq}\hat{\eta}^m_b\hat{\eta}^p_a\hat{\eta}^q_b,
\]
(11)
where \( \hat{h}_{ab} \) is the induced metric on \( S \), and
\[
\hat{D}_a\hat{v}_b \equiv \Omega^{1/2}\hat{h}^p_a\hat{v}_b^q\nabla_p\hat{\eta}_q
\]
(12)
is a regular differentiation with respect to \( \hat{h}_{ab} \) on \( S \), \((d-1)\hat{C}_{abcd} \) is the \((d-1)\)-dimensional Weyl tensor with respect to \( \hat{h}_{ab} \), and \( \Omega^{(5-d)/2}\hat{C}_{abcd} \) have a directional dependent limit at \( i^0 \). (For detailed derivations of Eqs. (10) and (11), see Appendix C\(2\).

Next, in order to see how these fields transform under the supertranslational, we introduce potentials of the Weyl tensor. To do so, we will use the Bianchi identity in the unphysical space-time
\[
\nabla_m\hat{C}_{abc} + \frac{2(d-3)}{d-2}\nabla_{[a}\hat{S}_{b]c} = 0 ,
\]
(13)
where
\[
\hat{S}_{ab} \equiv \frac{\hat{R}_{ab} - \hat{\hat{R}}}{2(d-1)}\hat{g}_{ab} .
\]
(14)
Since we assume \( \hat{g}_{ab} \) to be \( C^{d-4} \), \( \Omega^{(5-d)/2}\hat{S}_{ab} \) admits directional dependent limit at \( i^0 \). Then, we define potentials as
\[
\hat{\hat{E}} \equiv \Omega^{(5-d)/2}\hat{S}_{pq}\hat{\eta}^p\hat{\eta}^q ,
\]
\[
\hat{\hat{Q}}_a \equiv \Omega^{(5-d)/2}\hat{S}_{ap}\hat{\eta}^p\hat{\eta}^a ,
\]
\[
\hat{\hat{U}}_{ab} \equiv \Omega^{(5-d)/2}\hat{S}_{ap}\hat{\eta}^p\hat{\eta}^q ,
\]
(15
(16
(17)
Using Eqs. (5) and (13), we can write down the electric and the magnetic part in terms of potentials as
\[
\hat{E}_{ab} \equiv \frac{-1}{2(d-2)}\left[ \frac{1}{d-3}\hat{D}_a\hat{D}_b\hat{E} + \hat{\hat{E}}\hat{h}_{ab} + (4-d)\hat{U}_{ab} \right] ,
\]
(18)
\[
\hat{B}_{a_1\cdots a_{d-3}b} \equiv \frac{-1}{d-2}\epsilon_{a_1\cdots a_{d-3}mpq}\hat{\eta}^m\Omega^{(4-d)/2} \times \nabla^p \left( \hat{\eta}^q_b - \frac{1}{d-3}\hat{\hat{E}}\hat{\eta}^q_b \right)
\]
\[
\equiv \frac{-1}{d-2}\epsilon_{a_1\cdots a_{d-3}mpq}\Omega^{(4-d)/2}\hat{\eta}^m\nabla^p\hat{k}_{qb} ,
\]
(19)
where we define a tensor \( \hat{k}_{qb} \) by Eq. (13). (Eqs. (15) and (16) are derived in Appendix C\(2\).

Now, we observe transformation behaviors of the asymptotic fields under the supertranslational. In a supertranslational transformation \( \hat{g}_{ab} \rightarrow \hat{g}'_{ab} = \omega^2\hat{g}_{ab} \), where \( \omega \) is a \( C^{d-4} \) function (\( \omega = 1 \)), \( \hat{S}_{ab} \) transforms as
\[
\hat{S}_{ab}' = \hat{S}_{ab} - (d-2)\omega^{-1}\nabla_a\hat{\omega}\nabla_b\omega
\]
\[
+ 2(d-2)\omega^{-2}(\nabla_c\hat{\omega})(\nabla_b\hat{\omega})
\]
\[
+ \frac{2}{d-2}\omega^{-2}\hat{g}_{ab}(\nabla_m\hat{\omega})(\nabla^m\hat{\omega}) .
\]
(20)
Since \( \omega \) is \( C^{d-4} \) and \( \omega = 1 \), it can be written as
\[
\omega = 1 + \Omega^{(d-3)/2}a ,
\]
(21)

---

\(^1\) In the definition of the magnetic part of the Weyl tensor \( \hat{C}_{ambi} \), the power of \( \Omega \) is determined by the following evaluation. Since \( a_1, \cdots, a_{d-3} \) are indices for angular coordinates and \( m \) is for the radial coordinate in polar coordinates, one of \( p \) and \( q \) has to be for the coordinate time \( t \) and the other one has to be for an angular coordinate \( \varphi \). Each parts in the magnetic part behaves near \( i^0 \) as \( \epsilon_{a_1\cdots a_{d-3}mpq} = C^0(\sqrt{-g}) = C^0(\varphi^2-d), \)
\( C^0_{ab} = C^0(\varphi^2-d), \)
\( \hat{g}_{ab} = C^0(\varphi^2-d), \)
\( \hat{\eta}^a = C^0(1) \), \( \hat{\eta}^b = C^0(1) \) and \( \hat{\eta}^a = C^0(1) \). Thus, \( \epsilon_{a_1\cdots a_{d-3}mpq}\hat{\eta}^m\hat{\eta}^p\hat{\eta}^q = C^0(\varphi^2-d) \sim \Omega^{(2d-9)/2} \) and we have to multiply an inverse of this factor to a regular quantity.

\( \hat{E}_{ab} \) is a symmetric traceless tensor since the Weyl tensor is traceless. \( \hat{B}_{a_1\cdots a_{d-3}b} \) is also a traceless tensor; \( \hat{B}_{a_1\cdots a_{d-3}b}g^{a_1b_1} = 0 \) due to antisymmetry of \( \epsilon \) in Eq. (4). \( \hat{B}_{a_1\cdots a_{d-3}b}g^{a_1b_1} = 0 \) since it contains \( C^0(\varphi^2-d) = 0 \). This \( B_{a_1\cdots a_{d-3}b} \) is antisymmetric on the first \( d-3 \) indices \( a_i \) (\( i = 1, \cdots, d-3 \)). There are no symmetry between the last index \( b \) and the other indices \( a_i \) in general, though in the four-dimensional case the magnetic part \( \hat{B}_{ab} \) is symmetric.
where \( \alpha \) is a function which has directional dependent limit at \( i^0 \). Then, the potentials \( E \) and \( U_{ab} \) transform under the supertranslational transformation as

\[
\tilde{E}' = \tilde{E} - (d-2)(d-3)(d-4)\alpha, \quad \tilde{U}_{ab}' = \tilde{U}_{ab} - (d-2)\left( \tilde{D}_a\tilde{D}_b\alpha + (d-3)\alpha \tilde{h}_{ab} \right). \tag{22}
\]

To show these equations, we use a relation

\[
\Omega^{(d-4)/2} = \Omega^{1/2} \eta^a \nabla_a \alpha + (d-3)\alpha = (d-3)\alpha. \tag{24}
\]

The second equality in this relation holds since \( \alpha \) has directional dependent limit at \( i^0 \) and \( \eta^a \nabla_a \alpha = 0 \). We note that only \( \nabla_a \eta_b \) term of Eq. (20) contributes to the variation of \( E \) and \( U_{ab} \).

It is easy to check that the electric part does not change in this transformation. On the other hand, the potential of the magnetic part \( \tilde{K}_a \) transforms as

\[
\tilde{K}_a' = \tilde{K}_a - (d-2)(\tilde{D}_a\tilde{D}_b\alpha + \alpha \tilde{h}_{ab}). \tag{25}
\]

Hence, the magnetic part \( \tilde{B}_{a_1 \cdots a_{d-3} b} \) does change under the supertranslational transformation.

### C. Conserved quantities and Poincare symmetry

Let us construct conserved quantities and the asymptotic symmetry in this section. First, as in four dimensions, we impose an additional condition

\[
\tilde{B}_{a_1 \cdots a_{d-3} b} = 0. \tag{26}
\]

This condition implies that the Taub-NUT charge is zero. Although it is of course possible to consider asymptotically locally Minkowski space-time with \( \tilde{B}_{a_1 \cdots a_{d-3} b} \neq 0 \), we focus only on asymptotically globally Minkowski space-time in this paper. In order to impose the condition (26), consistently with Eq. (11), we must require a further additional condition

\[
\Omega^{(5-d)/2} \tilde{C}_{abcd} = 0 \tag{27}
\]

as one of the conditions in the definition of asymptotic flatness. Note that \( \Omega^{(4-d)/2} \tilde{C}_{abcd} \) vanishes automatically in four dimensions. By the way, the condition (26) is not preserved under the supertranslation. To preserve the condition (26), we realise that one must impose

\[
\tilde{D}_a\tilde{D}_b\alpha + \alpha \tilde{h}_{ab} = 0. \tag{28}
\]

As in four dimensions, we can write down the solution to Eq. (28) as \( \alpha = \tilde{\omega}_a \tilde{\eta}^a \), where \( \tilde{\omega}_a \) is a fixed vector at \( i^0 \). The number of independent solutions is the number of dimensions. Thus, we can regard the transformation generated by \( \alpha \) satisfying Eq. (28) as translation. Then, the asymptotic symmetry reduces to the Poincare group which is constituted of the Lorentz group and the translation group, and we can define conserved quantities associated with this Poincare symmetry.

Now, it is ready to define conserved quantities. First, we define \( d \)-momentum \( P_a \) for translation \( \tilde{\omega}^a \) as

\[
P_a \tilde{\omega}^a = -\frac{1}{8\pi G_d (d-3)} \int_{S^{d-2}} \tilde{E}_{ab} \tilde{\omega}^a \tilde{\epsilon}_{c_1 \cdots c_{d-2} m} \tilde{\eta}^m dS^{c_1 \cdots c_{d-2}}, \tag{29}
\]

where \( dS^{c_1 \cdots c_{d-2}} \) is the volume element on \( (d-2) \)-dimensional unit sphere \( S^{d-2} \) on \( i^0 \). From Eq. (10), we get \( \tilde{D}_a \tilde{E}_{ab} = 0 \) since \( \tilde{E}_{ab} \) is traceless. Then, the integral of Eq. (29) is independent of the choice of time slice at \( i^0 \), and thus \( P_a \tilde{\omega}^a \) is conserved. After tedious calculations, we can show that Eq. (29) agrees with the ADM formula (see Appendix D1 and D2).

Next, in order to define angular momentum using the magnetic part of the Weyl tensor, we consider the next-to-leading order part of \( \tilde{B}_{a_1 \cdots a_{d-3} b} \):

\[
\tilde{\beta}_{a_1 \cdots a_{d-3} b} = \Omega^{(d-4)/2} \tilde{\hat{\epsilon}}_{a_1 \cdots a_{d-3} m p q} \tilde{\hat{\eta}}_{b n} \tilde{\hat{\eta}}^m \tilde{\hat{\eta}}^n. \tag{30}
\]

Since \( \tilde{\beta}_{a_1 \cdots a_{d-3} b} \) satisfies \( \tilde{D}_b \tilde{\beta}_{a_1 \cdots a_{d-3} c} = 0 \) due to Eq. (11) and the traceless property of \( \tilde{B}_{a_1 \cdots a_{d-3} b} \), we can define conserved quantity \( M_{ab} \) which is regarded as angular momentum:

\[
M_{ab} = -\frac{1}{8\pi G_d (d-2)!} \int_{S^{d-2}} \tilde{\beta}_{a_1 \cdots a_{d-3} b} \tilde{\xi}_{a_1 \cdots a_{d-3}} \tilde{\epsilon}_{c_1 \cdots c_{d-2} m} \tilde{\eta}_{b n} \tilde{\eta}^m dS^{c_1 \cdots c_{d-2}}, \tag{31}
\]

where

\[
\tilde{\xi}_{a_1 \cdots a_{d-3}} \equiv \tilde{\hat{\epsilon}}_{a_1 \cdots a_{d-3} m p q} \tilde{\hat{\eta}}_{b n} \tilde{\hat{\eta}}^m \tilde{\hat{\eta}}^n F_{pq}. \tag{32}
\]

and \( F_{ab} \) is any skew tensor in \( \mathcal{S} \). The coefficients in (31) so that angular momentum \( M_{ab} \) transforms properly under translation \( \tilde{\omega}_a \), including the coefficient:

\[
M_{ab} \to M'_{ab} = M_{ab} + 2P_a \tilde{\omega}_b. \tag{33}
\]

See Appendix D3 for details of the coefficient determination.

### IV. SUMMARY AND DISCUSSION

In this paper, we gave a definition of asymptotic flatness, and constructed conserved quantities, \( d \)-momentum and angular momentum in \( d \) dimensions. As in four dimensions, by imposing an additional constraints on the behavior of the “magnetic” part of the Weyl tensor, we can remove the supertranslational ambiguity. Then, the asymptotic symmetry of the space-time reduces to the Poincare symmetry, which is a symmetry of “background” flat metric, and we can construct conserved quantities associated with this Poincare symmetry. It can be shown that the expressions of these conserved quantities agree with the ADM formulae.
In four dimensions, the additional constraint is only $B_{ab} = 0$ to realize the Poincare symmetry as the asymptotic symmetry, and it is satisfied if there is a Killing vector in $M$, such as timelike Killing vector $(\partial/\partial t)$ or rotational Killing vector $(\partial/\partial \varphi)$ \cite{11}. On the other hand, in higher dimensions, due to the evolution equation \cite{11} of $B_{a_1 \ldots a_d \cdots a_b} = 0$, we need to impose a further condition $\Omega^{(5-d)/2}(d-1)\hat{C}_{bcd} = 0$ to remove the supertranslational ambiguity and realize the Poincare symmetry. As in four dimensions, $\hat{C}_{a} = 0$ to remove the supertranslation at null infinity, and investigate its connection to the supertranslation at spatial infinity.

Another future issue is the preparation for the uniqueness theorem in stationary black hole space-times. As mentioned in the introduction, at first glance, the uniqueness theorem does not hold in higher dimensions, although there are some partial achievements \cite{13, 14, 15, 16}. However, we would guess that the reason why we fail to prove it is due to lack of asymptotic boundary conditions. If we can specify the boundary condition appropriately, we will be able to prove the uniqueness theorem. The mass, charge and angular momentum are not enough to specify the black hole space-time uniquely. The additional information for the uniqueness may be higher multipole moments. Therefore, the study on higher multipole moments in stationary space-time will be useful.

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APPENDIX A: DIRECTIONAL DEPENDENCE

In the conformal completion method, spatial infinity which has a non-zero size in the physical space-time $\hat{M}$ contracts to a point $i^0$ in the unphysical space-time $M$. Hence, the definition of differentiability and continuity of physical fields (e.g. electromagnetic fields or gravitational fields) on $i^0$ is more subtle. In this appendix, we give the notion of directional dependent limit and $C^{n,n}$ class.

First, the tensor $\hat{T}_{c^{a \cdots b}}$ is said to have directional dependent limit at $i^0$ if $\hat{T}_{c^{a \cdots b}}$ satisfies the following conditions:

1. \[
\lim_{\tilde{\eta} \to i^0} \hat{T}_{c^{a \cdots b}} = \hat{T}_{c^{a \cdots b}}(\tilde{\eta}) ,
\]
   where $\tilde{\eta}$ is a vector on tangential space at $i^0$, which is tangent to the curve arriving at $i^0$.

2. The derivative coefficients at $i^0$ defined by \[
\left(\Omega^{1/2}\hat{\nabla}_{e_1} \right) \cdots \left(\Omega^{1/2}\hat{\nabla}_{e_n} \right) \hat{T}_{c^{a \cdots b}}
\]
   are regular.

The first condition says that, since $i^0$ has a non-zero size ($S^{d-2}$) in $M$, $\hat{T}_{c^{a \cdots b}}$ may have an angular dependence even in the limit $r \to \infty$. The operator $\Omega^{1/2}\hat{\nabla}_a$ in the second condition gives regular derivative coefficients, since an application of a derivative operator $\hat{\nabla}_a$ in $\hat{M}$ corresponds to a multiplication of $r$ near $i^0$ (see Appendix B). The second condition says that these regular derivative coefficients should be finite and regular.

Next, we define $C^{n,n}$ class. A tensor $\hat{T}_{c^{a \cdots b}}$ is $C^{n,n}$ at $i^0$ if $i^0$ have directional dependent limit at $i^0$. For example, when we set $\hat{T}_{c^{a \cdots b}}$ to be $C^{n,n}$ at $i^0$, the behavior of $\hat{g}_{ab}$ near $i^0$ is

$$\hat{g}_{ab} \sim \text{const.} + \frac{f(\theta, \varphi, \cdots)}{r^{n+1}} , \quad \text{(A1)}$$

where the dots stand for other angular coordinates.

APPENDIX B: CONFORMAL COMPLETION FOR MINKOWSKI SPACE-TIME

In this appendix, we discuss conformal completion for Minkowski space-time. This analysis tells us how we can define asymptotically flat space-time in general. First, we introduce coordinates $(U, V)$ such that

\[
ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2
= -dudv + \frac{(u - v)^2}{4} d\Omega_{d-2}^2
= - \frac{dUdV}{\cos^2 U \cos^2 V} + \frac{\sin^2 (U - V)}{4 \cos^2 U \cos^2 V} d\Omega_{d-2}^2 , \quad \text{(B1)}
\]

where

\[
u = t - r = \tan U , \quad v = t + r = \tan V , \quad \text{(B2)}
\]

and $d\Omega_{d-2}^2$ is a metric on unit $S^{d-2}$. Let us take $\Omega \equiv \cos U \cos V$ as a conformal factor. In this case, we can see that

\[
\hat{\nabla}_a \hat{\nabla}_b \Omega \equiv 2\hat{g}_{ab} \quad \text{(B3)}
\]
holds at \( i^0 \). The unit normal vector \( \hat{\eta}_a \) to \( \Omega = \text{constant surface} \) becomes

\[
\hat{\eta}_a = \hat{\nabla}_a \Omega^{1/2}
\]

on \( i^0 \).

It will be useful for discussions in the main text to look how the differential operators behave:

\[
\hat{\nabla}_U \sim \frac{\partial}{\partial U} \sim r^2 \frac{\partial}{\partial r},
\]

i.e. an application of \( \hat{\nabla}_a \) corresponds to a multiplication of \( r \). When we say that \( \hat{\eta}_ab \) is \( C^{\infty} \) at \( i^0 \), by the way, we should take differentiation in the coordinates \((U, V)\), and so this condition implies that the metric in the unphysical space-time is given by

\[
\hat{g}_{ab} = \hat{\eta}_{ab} \left( 1 + \frac{f(\theta, \varphi, \cdots)}{r^{n+1}} \right),
\]

where

\[
\hat{\eta}_{ab} dx^a dx^b = -dU dV + \frac{\sin^2 (U - V)}{4} d\Omega_{d-2}
\]

is the unphysical space-time metric corresponding to the flat metric \([B1]\) in the physical space-time.

APPENDIX C: DERIVATIONS OF EQUATIONS (10), (11), (18) AND (19)

In this appendix, we give detailed derivations of Eqs. (10), (11), (18) and (19). Since we compute quantities only at spatial infinity, we omit “hat” and limit \( \lim_{r \to 0} \) throughout this appendix for convenience.

1. Derivation of Eqs. (10) and (11)

First, from Eqs. (8) and (9), we obtain

\[
\Omega^{1/2} \nabla_{[m} \chi_{ab]cd} = \Omega^{-1/2} \left( g_{[m} \chi_{ab]cd} \nabla^\rho \Omega + g_{[m} \chi_{ab]cp} \nabla^\rho \Omega + \frac{5-d}{2} (\nabla_{(m} \Omega) \chi_{ab]cd} \right),
\]

where \( \chi_{abcd} = \Omega^{(5-d)/2} C_{abcd} \).

Multiplying \( \eta^b \eta^d \eta^e h^m_h^f_h^g \) to the above, the left-hand side becomes

\[
\Omega^{1/2} \eta^b \eta^d \eta^e h^m_h^f_h^g \nabla_{[m} \chi_{ab]cd} = \frac{1}{3} (D_a E_{fg} - D_f E_{eg}) - \frac{1}{3} (2W_{fg} + W_{gef} - W_{gef}),
\]

where we used the fact that

\[
\Omega^{1/2} \nabla_a \eta_b = g_{ab} - \eta_a \eta_b = h_{ab},
\]

and the definition

\[
W_{abc} = h^a_x h^f_x h^g_x \chi_{fgd} h^{d}.
\]

In addition, we used the fact that \( \chi_{abcd} \) has directional dependent limit and thus \( \hat{\eta}^a \nabla_a \chi_{abcd} \) vanishes. In the right-hand side of Eq. (11), the second and third terms become

\[
\Omega^{-1/2} \eta^b \eta^d \eta^e h^m_h^f_h^g \nabla_{[m} \Omega \chi_{ab]cd} = \frac{5-d}{3} W_{efg},
\]

The first term vanishes since \( \nabla^\rho \Omega = 2\Omega^{1/2} \eta^\rho \). Finally, we obtain Eq. (10) from Eq. (11), that is

\[
D_a E_{fg} - D_f E_{eg} = (d - 5) W_{fg} + W_{gef} + W_{gec} = (d - 4) W_{fg},
\]

where we used \( W_{[abc]} = 0 \) in the second line.

Next, we multiply \( \Omega^{(4-d)/2} \hat{\chi}_{a_1 \cdots a_d} \cdot fcd \eta^a \eta^b h^m_h^f h^g \) to Eq. (11), where \( \hat{\chi}_{a_1 \cdots a_d} \cdot fcd \equiv h^a_{a_2} \cdots h_{ab_d} h^b_{c_1} \cdots h_{c_d} \cdot fcd \), and then obtain

\[
\frac{1}{3} (D_a B_{a_1 \cdots a_d-3} - D_{a_1} B_{a_2 \cdots a_d-3})
- \frac{1}{3} \hat{\chi}_{a_1 \cdots a_d-3} \cdot fcd (\Omega^{(4-d)})
\times \left( h_{fg} h_{fg} \eta^a + h_{gf} h_{gf} \eta^b - h_{fg} h_{gf} \eta^b - h_{gf} h_{gf} \eta^a \right) \chi_{abcd}
\times \Omega^{(4-d)/2} \hat{\chi}_{a_1 \cdots a_d-3} \cdot fcd \eta^f.
\]

From this equation, we obtain Eq. (11):

\[
D_a B_{a_1 \cdots a_d-3} - D_{a_1} B_{a_2 \cdots a_d-3}
= -(d-3) \Omega^{(5-d)/2} h_{fg} h^f h^g \eta^a \hat{\chi}_{a_1 \cdots a_d-3} \cdot fcd (d-1) C_{pabcd},
\]

where \( (d-1) C_{abcd} \) is the \((d-1)\)-dimensional Weyl tensor on \( \Omega = \text{constant surface} \) at \( i^0 \). To transform Eq. (11) to Eq. (C2), we used the following relations

\[
\hat{\chi}_{a_1 \cdots a_d-3} \cdot fcd \eta^a \eta^b \chi_{abcd}
= \hat{\chi}_{a_1 \cdots a_d-3} \cdot fcd h_{fg} h^f h^g (h^a \chi_{a_1 \cdots a_d-3} \cdot fcd (d-1) C_{pabcd},
\]

and

\[
\chi_{abcd} h^a h^f h^g \eta^d = \Omega^{(5-d)/2 (d-1) C_{wyz}}
- \frac{2}{d-3} (E_{w[y} h^z]x - E_{x[y} h^z]|w).
\]

(C11)
In Eqs. (C10) and (C11), we used the fact that the extrinsic curvature of \( \Omega = \) constant surface at \( i^0 \) is

\[
\pi_{ab} \equiv (1/2)\mathcal{E}_{\eta} h_{ab} = \frac{1}{2} (\eta^c \nabla_c h_{ab} + h_{ac} \nabla_b \eta^c + h_{bc} \nabla_a \eta^c) = \Omega^{-1/2} h_{ab}.
\] (C12)

For the derivation of Eq. (C11), see Eq. (A6) in [18]. (Note that the magnetic part defined there is different from ours.)

2. Derivation of Eqs. (18) and (19)

Hereafter in this appendix, we derive Eqs. (18) and (19). Firstly, to facilitate the derivation, we derive the following relation:

\[
X_{abcm} \eta^m = \frac{1}{d-2} \Omega^{1/2} \nabla_{[b} T_{a]c} + (d-5) \eta_{[b} T_{a]c} \right],
\] (C13)

where \( T_{ab} = \Omega^{5-d/2} S_{ab} \) is a tensor which has directional dependent limit at \( i^0 \), and \( S_{ab} \) is defined in Eq. (14). The manipulation of \( g^{mn} \times \) Eq. (9) implies

\[
\nabla_m C_{abcm} \eta^m = (d-3) \Omega^{-1} C_{abcp} \eta^p \Omega.
\] (C14)

Note that Eq. (9) was derived from the Bianchi identity in the physical vacuum spacetime (\( \nabla_m C_{abcm} = 0 \)). On the other hand, from the Bianchi identity in the unphysical spacetime (\( \nabla_m \tilde{R}_{abcm} = 0 \)), we can derive

\[
\nabla_m C_{abcm} \eta^m + \frac{2(d-3)}{d-2} \nabla_{a} S_{bc} = 0.
\] (C15)

From these two equations, we can see that

\[
C_{abcm} \eta^m = - \frac{1}{d-2} \Omega^{1/2} \nabla_{a} S_{bc} \right).
\] (C16)

It is easy to see that Eq. (C13) holds from this equation.

Next, let us apply \( \Omega^{(d/2)} E_{a_1...a_d-3} f_{ab} \eta_f \) to Eq. (C13). Then we obtain Eq. (19):

\[
B_{a_1...a_d-3} c = \frac{1}{d-2} \mathcal{E}_{a_1...a_d-3} f_{ab} \eta_f \Omega^{(d/2)} (D_b U_{ac} + h_{bc} Q_a) = \frac{1}{d-2} \mathcal{E}_{a_1...a_d-3} f_{ab} \eta_f \Omega^{(d/2)} D_a (U_{bc} - \frac{h_{bc} E}{d-3}).
\] (C19)

APPENDIX D: \( (d-1) + 1 \) DECOMPOSITION

In this appendix, we show that \( d \)-momentum defined in Eq. (29) agrees with the ADM formulae for energy and momentum:

\[
E = \frac{1}{16 \pi G_d} \lim_{r_1 \to \infty} \int_{S_{d-2}} \left( \partial^a h_{ab} - \partial_b h^a \right) \eta^b \right|_{r=r_0},
\] (D1)

\[
Q_{Na} = - \frac{1}{8 \pi G_d} \lim_{r_1 \to \infty} \int_{S_{d-2}} \left( K_{ab} - K^m_{ab} h_{ab} \right) N^a \eta^b \right|_{r=r_0},
\] (D2)

where \( h_{ab} \equiv g_{ab} + t^a t_b \) and \( K_{ab} \equiv h^c h_{bd} \nabla_c t^d \) are the induced metric and the extrinsic curvature of a \( t = \) constant surface whose unit normal is \( t^a \), and \( \partial_a \) is a coordinate derivative with respect to asymptotic Cartesian coordinates. \( N^a \) is an asymptotic spacelike translational Killing vector such that \( D_a N_b \to 0 \) as \( r \to \infty \), where \( D_a \) is the connection for \( h_{ab} \).

We also show in this appendix that the angular momentum defined in Eq. (31) transforms in translational transformation as Eq. (33). This appendix may be regarded as an extension of the work by Ashtekar and Magnon in four dimensions [17]. We will describe in much detail because it is very hard to check their result.

1. Energy

First, let us consider the energy. Let \( \Sigma \) be a spacelike hypersurface in \( M \) on \( i^0 \) which has unit timelike vector \( t^a \) as its normal. Then, the energy defined by Eq. (29) becomes

\[
-P_a t^a = - \frac{1}{8 \pi G_d (d-3)} \int_{S_{d-2}} \tilde{E}_{ab} t^a \eta^b \eta_c \right|
\] (D3)

where \( dS \) is the volume element of a \( (d-2) \)-dimensional unit sphere \( S^{d-2} \). In order to compare the above with the ADM formula, we must write it down in terms of quantities in physical space-time \( M \). To do so, we introduce a spacelike hypersurface \( \Sigma \) in \( M \), unit timelike vector \( t^a \) normal to \( \Sigma \), and a unit radial vector \( \eta^a = \partial^a r \). \( t^a \) and \( \eta^a \) are related to \( t^\alpha \) and the unit radial vector in the unphysical space-time \( \eta^a \) as \( \lim_{r_0 \to \infty} \Omega^{-1} \eta^a = t^\alpha \).
\[ -P_a \hat{\mathbf{t}}^a = -\frac{1}{8\pi G_d(d-3)} \lim_{r_0 \to \infty} \int_{r=r_0}^{r_1} \tau_{abcd}^a \eta \eta d^4 \mathbf{r} dS, \] 

\[ \text{(D4)} \]

where we used the fact that \( \Omega \simeq 1/r^2 \) near \( \vec{r}^0 \).

Now, we define the usual electric part of the Weyl tensor \( e_{ab} \equiv C_{abcd}^{\mu\nu} \eta^{\mu\nu} \) in the physical space-time \( M \). This electric part can be decomposed as

\[ e_{ab} = (d-1)R_{ab} - K_a^m K_{bm} + KK_{ab} - \frac{1}{d-2}((d-3)h_a^m h_b^n + h_{ab}h^{mn})S_{mn}. \] 

\[ \text{(D5)} \]

Taking into account of asymptotic behaviors \( K_{ab} = O(1/r^{d-2}) \) and \( (d-1)R_{ab} = O(1/r^{d-1}) \) for \( r \to \infty \), and the vacuum Einstein equation \( R_{ab} = 0 \), we obtain

\[ -P_a \hat{\mathbf{t}}^a = -\frac{1}{8\pi G_d(d-3)} \lim_{r_0 \to \infty} \int_{r=r_0}^{r_1} \tau_{abcd}^a \eta \eta d^4 \mathbf{r} dS. \]

\[ \text{(D6)} \]

In order to integrate by parts in direction \( r \), we rewrite the integral into the following form:

\[ -P_a \hat{\mathbf{t}}^a = -\frac{1}{8\pi G_d(d-3)} \lim_{r_0 \to \infty} \int_{r=r_0}^{r_1} \tau_{abcd}^a \eta \eta d^4 \mathbf{r} dS \]

\[ \times \frac{1}{\Delta r} \int_{r=r_0}^{r_0+\Delta r} \int_{S_{d-2}}^{r_{d-1} \tau_{abcd}^a \eta \eta d^4 \mathbf{r} dr dS, \]

where we used the fact that the integrand in Eq. \( \text{(D6)} \) is independent of \( r \) at large \( r \). In this expression, the part which contribute to the integral is

\[ (d-1)R_{ab} \sim \frac{1}{2} (\partial^\nu \partial_b h_{ac} + \partial_a \partial^\nu h_{bc} - \partial^\nu \partial c h_{ab} - \partial_a \partial_b h_{ac}). \]

\[ \text{(D8)} \]

Substituting \( \text{(D8)} \) into \( \text{(D7)} \) and integrating by parts, we can get the desired result. Since this calculation is a little difficult, we describe carefully. First, we integrate the first part in \( \text{(D8)} \) by parts:

\[ \frac{1}{\Delta r} \int_{S_{d-2} \times \Delta r} \tau (\partial^\nu \partial_b h_{ac}) \eta \eta d^4 \mathbf{r} dV \]

\[ = \frac{1}{\Delta r} \int_{S_{d-2} \times \Delta r} [\partial^\nu (r (\partial_b h_{ac}) \eta \eta) - (\partial_b h_{ac}) \partial^\nu (r \eta \eta)] d^4 \mathbf{r} dV, \]

where \( dV \equiv r^{d-2} dr dS. \) The first term in the right-hand side becomes

\[ \frac{1}{\Delta r} \int_{S_{d-2} \times \Delta r} \partial^\nu (r (\partial_b h_{ac}) \eta \eta) dV \]

\[ = \frac{1}{\Delta r} \int_{S_{d-2}} r (\partial_b h_{ac}) \eta \eta d^4 \mathbf{r} \bigg|_{r=r_0+\Delta r} - \frac{1}{\Delta r} \int_{S_{d-2}} r (\partial_b h_{ac}) \eta \eta d^4 \mathbf{r} \bigg|_{r=r_0} \]

\[ = \int_{S_{d-2}} (\partial_b h_{ac}) \eta \eta d^4 \mathbf{r} \bigg|_{r=r_0}, \] 

\[ \text{(D10)} \]

where \( d^4 \mathbf{r} \equiv \eta \eta r^{d-2} dr dS. \) In the first and the second equalities, we used the Gauss theorem, and the fact that \((\partial_b h_{ac}) \eta \eta \) is independent of \( r \) in the limit of \( r_0 \to \infty \). The second term in the right-hand side of Eq. \( \text{(D9)} \) becomes

\[ \frac{1}{\Delta r} \int_{S_{d-2} \times \Delta r} (\partial_b h_{ac}) \partial^\nu (r \eta \eta) dV \]

\[ = \frac{1}{\Delta r} \int_{S_{d-2} \times \Delta r} (\partial_b h_{ac}) (\eta \eta \eta \eta + \eta \eta \eta \eta + q^{bc} \eta \eta) r^{d-2} dV \]

\[ = \int_{S_{d-2}} (\partial_b h_{ac}) (\eta \eta \eta \eta + \eta \eta \eta \eta + q^{bc} \eta \eta) r^{d-2} dS. \] 

\[ \text{(D11)} \]

To transform the second into the third line, we used the fact that the integrand in the second line does not depend on \( r \). Then, we obtain

\[ \int_{S_{d-2}} r^{d-1} (\partial^\nu \partial_b h_{ac}) \eta \eta \eta \eta \eta dS \]

\[ = - \int_{S_{d-2}} (\partial_b h_{ac}) (\eta \eta \eta \eta + \eta \eta \eta \eta + q^{bc} \eta \eta) r^{d-2} dS. \] 

\[ \text{(D12)} \]

Here, we defined a metric \( q_{ab} \) on \( r = \text{constant} \) surface such that \( \partial_a \eta_b = (h_{ab} - \eta_{ab})/r = q_{ab}/r \). In the same way, the other terms in \( \text{(D8)} \) are transformed as

\[ \int_{S_{d-2}} r^{d-1} (\partial_b \partial^\nu h_{ac}) \eta \eta \eta \eta \eta dS = -(d-2) \int_{S_{d-2}} \partial^\nu h_{ac} d^4 \mathbf{r}, \]

\[ \int_{S_{d-2}} r^{d-1} (\partial_b \partial^\nu h_{c}) \eta \eta \eta \eta \eta dS \]

\[ = - \int_{S_{d-2}} (\partial_c h_{ab}) (q^{ac} \eta \eta + q^{bc} \eta \eta) r^{d-2} dS, \]

\[ \int_{S_{d-2}} r^{d-1} (\partial_a \partial_b h_{c}) \eta \eta \eta \eta \eta dS = -(d-2) \int_{S_{d-2}} \partial_a h^c d^4 \mathbf{r}. \]

Finally, we obtain the desired result:

\[ -P_a \hat{\mathbf{t}}^a = \frac{1}{16\pi G_d} \lim_{r_0 \to \infty} \int_{r=r_0} r^{d-1} (\partial^\nu h_{ab} - \partial_b h^a) dS^b. \] 

\[ \text{(D13)} \]

2. 

2. Momentum

Next, let us consider momentum. The components of \((d-1)\)-momentum along a spacelike vector \( N^a \) at \( \vec{r}^0 \) can be written as

\[ P_a N^a = \frac{1}{8\pi G_d(d-3)} \int_{S_{d-2}} \hat{E}_{ab} N^a \hat{\mathbf{t}}^b dS. \] 

\[ \text{(D14)} \]

In terms of quantities of physical space-time, this equation becomes

\[ P_a \hat{N}^a = \frac{1}{8\pi G_d(d-3)} \lim_{r_0 \to \infty} \int_{r=r_0} r^{d-1} C_{abcd} \eta \eta d^4 \mathbf{r}, \] 

\[ \text{(D15)} \]
where \( N^a = \lim_{\rho \to 0} \Omega \hat{N}^a \). Using the Codacci equation and the vacuum Einstein equation, this expression becomes

\[
P_a \hat{N}^a \equiv \frac{1}{8\pi G_d (d-3)} \lim_{r \to r_0} \int_{r=r_0}^{r_0} r^{d-1} \left( (D_d K_{ab} - D_a K_{db}) \eta^b \eta^d N^a \right) dS.
\]

Using the fact that the leading part of \( r^{d-1} D_d K_{ab} \) does not depend on \( r \), the first term in the right-hand side is reexpressed as volume integral as

\[
\int_{r=r_0}^{r_0} r^{d-1} (D_d K_{ab}) \eta^b \eta^d N^a dS = \frac{1}{\Delta r} \int_{S^{d-2} \times \Delta r} r (D_d K_{ab}) \eta^b \eta^d N^a dV
\]

Using the fact that \( K_{ab} N^a r^{d-2} \) does not depend on \( r \). The second term in Eq. (D17) can be rearranged as

\[
-\frac{1}{\Delta r} \int_{S^{d-2} \times \Delta r} K_{ab} D_a (r \eta^b \eta^d N^a) dV = -(d-1) \int_{S^{d-2}} K_{ab} N^a dS_b
\]

In the same way, the second term of Eq. (D16) is rearranged as

\[
\int_{r=r_0}^{r_0} r^{d-1} (D_d K_{ab}) \eta^b \eta^d N^a dS = 2 \int_{S^{d-2}} K_{ab} \eta^b N^c dS_c - 2 \int_{S^{d-2}} K_{ab} N^a dS_b
\]

From this equation, we can obtain a relation

\[
2 \lim_{r_0 \to \infty} \int_{r=r_0}^{r_0} r^{d-2} K_{ab} N^a \eta^b \eta^d dS = \lim_{r_0 \to \infty} \int_{r=r_0}^{r_0} \left( K_{ab} - (d-3) Kh_{ab} \right) N^a dS_b.
\]

Derivation of this relation is a little non-trivial, so we describe it in detail. Note that the Gauss theorem makes the surface integral into the volume integral as

\[
\int_{r=r_0}^{r_0} r^{d-2} K_{ab} N^a \eta^b \eta^d dS = \frac{1}{\Delta r} \int_{S^{d-2} \times \Delta r} \partial a \left( r K_{ab} N^a \eta^b \eta^c \right) dV
\]

where we used the momentum constraint equation for the vacuum Einstein equation, \( D_a K^a - D_b K = 0 \), in the last line. The first and the second terms are rearranged respectively as

\[
\frac{1}{\Delta r} \int_{S^{d-2} \times \Delta r} r (D_b K) \eta^b N^c \eta_c dV
\]

and

\[
\frac{1}{\Delta r} \int_{S^{d-2} \times \Delta r} K_{ab} D_a (r \eta^b N^c \eta_c) dV
\]

Then, we proceed as

\[
\int_{r=r_0}^{r_0} r^{d-2} K_{ab} N^a \eta^b \eta^d dS = \int_{S^{d-2}} \left( K_{ab} N^b - (d-3) KN_a \right) dS^a \bigg|_{r=r_0}
\]

The last term in the right-hand side is the same with the left-hand side except for the signature. Therefore, we have the relation of Eq. (D21) into Eq. (D20), we can show

\[
\int_{r=r_0}^{r_0} r^{d-1} (D_d K_{ab} - D_a K_{db}) \eta^b \eta^d N^a dS = -(d-3) \int_{S^{d-2}} (K_{ab} - Kh_{ab}) N^a dS^b \bigg|_{r=r_0}
\]

and then, combining this equation with Eq. (D19), we see that our formula (29) for (\( d-1 \))-momentum becomes the ADM formula, that is

\[
P_a \hat{N}^a = -\frac{1}{8\pi G_d} \lim_{r_0 \to \infty} \int_{r=r_0}^{r_0} (K_{ab} - Kh_{ab}) N^a dS^b.
\]
3. Angular momentum

Finally, we consider translational transformation of angular momentum $M_{ab}$. We consider translation $\omega_a$ which is a fixed vector at $i^0$, and relate it with $\alpha = \omega_a \eta^a$. This translation transforms $\eta^a$ as

$$\hat{\eta}_a = \hat{\eta}_a + \frac{1}{2}\Omega^{(d-3)/2} \left( (d-2)\alpha \hat{\eta}_a + \Omega^{1/2} \nabla_a \alpha \right). \quad (D28)$$

Then, the magnetic part of the Weyl tensor $\hat{\beta}_{a_1 \cdots a_{d-3} b}$ transforms as

$$\hat{\beta}'_{a_1 \cdots a_{d-3} b} = \hat{\beta}_{a_1 \cdots a_{d-3} b} + \frac{d-2}{d-3} \hat{\xi}_{a_1 \cdots a_{d-3} m p} \hat{\eta}^m \hat{E}_{b}^{\alpha} \hat{D}_{\alpha} \hat{\eta}^\beta \hat{\xi}_{a_1 \cdots a_{d-3} m p} \hat{\eta}^m \hat{E}_{\beta}^{\nu} \hat{D}_{\nu} \hat{\xi}_{a_1 \cdots a_{d-3} b}. \quad (D29)$$

We used the projection formulae of the Weyl tensor (C10) and (C11) to derive this equation. Substituting (D29) into (C11), and noting that $F_{ab} \hat{e}^{c_1 \cdots c_{d-2} m} \hat{\eta}^m dS_{c_1 \cdots c_{d-2}}$ vanishes since $d$ and $1$-indices of $\hat{\xi}$ are projected onto $(d-2)$-dimensional surface, we find that usual translational transformation

$$M'_{ab} = M_{ab} + 2P_{a} \hat{\omega}_{b}, \quad (D30)$$

where $D_{\alpha} \hat{\xi}_{a_1 \cdots a_{d-3} b} = \hat{\omega}_a$, is correctly reproduced including the angular momentum as $\hat{\xi}_{a_1 \cdots a_{d-3} b}$. 

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