Screening for Sparse Online Learning

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Abstract. Sparsity promoting regularizers are widely used to impose low-complexity structure (e.g. \(\ell_1\)-norm for sparsity) to the regression coefficients of supervised learning. In the realm of deterministic optimization, the sequence generated by iterative algorithms (such as proximal gradient descent) exhibit “finite activity identification”, namely, they can identify the low-complexity structure in a finite number of iterations. However, most online algorithms (such as proximal stochastic gradient descent) do not have the property owing to the vanishing step-size and non-vanishing variance. In this paper, by combining with a screening rule, we show how to eliminate useless features of the iterates generated by online algorithms, and thereby enforce finite activity identification. One consequence is that when combined with any convergent online algorithm, sparsity properties imposed by the regularizer can be exploited for computational gains. Numerically, significant acceleration can be obtained.

1 Introduction

1.1 Background

Regression problems play a fundamental role in various fields including machine learning, data science and statistics. A typical example of regression problem has the following form

\[
\min_{\beta \in \mathbb{R}^n} \{ F(\beta) \overset{\text{def}}{=} \mathbb{E}_{(x,y)}[f(x^\top \beta; y)] \}. \tag{P}
\]

The expectation is taken over random variable \((x, y)\) whose probability distribution \(\Lambda\) is supported on some compact domain \(\mathcal{X} \times \mathcal{Y} \subset \mathbb{R}^n \times \mathbb{R}\). For each \(y \in \mathcal{Y}\), \(f(\cdot; y) : \mathbb{R} \to \mathbb{R}\) is a real-valued loss function, and representative examples of \(f\) include

- Squared loss: \(y \in \mathbb{R}\) and
  \[f(z; y) = \frac{1}{2}(z - y)^2.\]

- Squared hinge loss [9, 18]: \(y \in \{-1, +1\}\) and
  \[f(z; y) = (\max\{0, 1 - yz\})^2.\]

- Logistic regression [14]: \(y \in \{-1, 1\}\) and
  \[f(z; y) = \log(1 + \exp(-yz)).\]

To improve the robustness of regression models and avoid over-fitting problems, a common approach to strengthen \((P)\) is to add a regularization term to the regression coefficient vector \(\beta\). As a result, one considers the following regularized regression problem

\[
\min_{\beta \in \mathbb{R}^n} F(\beta) + \lambda \Omega(\beta), \tag{P_\lambda}
\]

Here, \(\Omega(\cdot)\) is a regularization term and \(\lambda > 0\) is a trade-off parameter to balance the two components. Popular choices of \(\Omega\) are sparsity promoting functionals, which include

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• \ell_1\)-norm for enforcing sparsity [36]:
\[ \Omega(\beta) = ||\beta||_1 = \sum_{i=1}^{n} |\beta_i|. \]

• \ell_{1,2}\)-norm for enforcing group sparsity [43]: Let \( G \) be a non-overlap partition of the index of \( \beta \) such that \( \beta = (\beta_g)_{g \in G} \),
\[ \Omega(\beta) = \sum_{g \in G} \|\beta_g\|_2. \]

• \ell_1 + \ell_{1,2}\)-norms for enforcing sparsity within groups [34]:
\[ \Omega(\beta) = (1 - \epsilon)||\beta||_1 + (1 - \epsilon)\sum_{g \in G} \|\beta_g\|_2 \]
where \( \epsilon > 0 \) is a weight parameter.

Throughout the paper, we impose the following assumptions to the regularized regression model \((P_\lambda)\):

\( \text{(H.1)} \) The loss function \( f \) is convex, differentiable, and for each \( y \), \( f_y \equiv f(\cdot; y) \) has L-Lipschitz continuous gradient for some \( L > 0 \);

\( \text{(H.2)} \) The regularization function \( \Omega \) is a convex and group decomposable norm, that is, given \( \beta \in \mathbb{R}^n \) and a partition \( G \) on \( \{1, \ldots, n\} \) such that \( \beta = (\beta_g)_{g \in G} \), we have
\[ \Omega(\beta) \equiv \sum_{g \in G} \Omega_g(\beta_g) \]
where \( \Omega_g \) is a norm on \( \mathbb{R}^{n_g} \) with \( n_g \) being the cardinality of \( \beta_g \).

**Empirical loss minimization** In practice, it is often unrealistic to minimize the loss function \( F \) over the distribution \( \Lambda \). Consequently, a practical approach is to draw samples from \( \Lambda \) which results in empirical loss
\[ F_\eta(\beta) \equiv \sum_{i=1}^{m} \eta_i f(x_i^\top \beta, y_i) \]
where \( m \) samples \( \{x_i, y_i\}_{i=1}^{m} \in (\mathbb{R}^n)^m \times \mathbb{R}^m \) are drawn from \( \Lambda \), with positive weight \( \eta_i \), for each sample, which sum to 1. A popular choice of \( (\eta_i) \) is uniform weights, i.e. \( \eta_i = \frac{1}{m} \). Correspondingly, \((P_\lambda)\) becomes the following regularized empirical loss
\[ \min_{\beta \in \mathbb{R}^n} F_\eta(\beta) + \lambda \Omega(\beta). \] (1.1)

### 1.2 Dimension reduction of sparsity promoting regularization

The purpose of using sparsity promoting regularizers is so that the solution of the optimization problem (1.1) has as few non-zeros coefficients as possible. If the position of the non-zeros coefficient can be known in advance, then one simply needs to solve (1.1) restricted to the support of the solution which can greatly reduce the dimension of the problem. In general, finding the support of the solution can be achieved via two ways

- Automatic support identification: the iterates generated by the numerical algorithms themselves can automatically find the support of the solution.
- Artificial support identification: besides the numerical schemes, extra operations are evaluated which enforce support identification.

In what follows, we provide a short discussion about these two approaches.

Let \( \beta^* \) be a global minimizer of the regularized empirical loss minimization (1.1), the first-order optimality condition is
\[ 0 \in \nabla F_\eta(\beta^*) + \lambda \partial \Omega(\beta^*). \] (1.2)
1.2.1 Support identification of proximal gradient descent

One of the most popular algorithms, to solve (1.1), is proximal gradient descent (PGD) [23] (a.k.a Forward–Backward splitting (FBS)) which dates back to late 1970s. It is well known that for sparsity promoting regularizers, deterministic proximal gradient descent has the support identification property. In [21, 20, 22], it is shown that under the following non-degeneracy condition

\[ 0 \in \text{ri}(\nabla F_{\eta}(\beta^*) + \lambda \partial \Omega(\beta^*)) \]  

(1.3)

where \( \text{ri}(\cdot) \) stands for relative interior, after a finite number of iterations, the iterate \( \beta_t \) has the same sparsity pattern as the minimizer \( \beta^* \) that \( \beta_t \) converges to. In the case of \( \Omega(u) = \|u\|_1 \), this means that \( \text{supp}(\beta_t) = \text{supp}(\beta^*) \) holds for all \( t \geq T \) for some finite-valued \( T > 0 \).

Such a result implies that, along the course of iteration, proximal gradient descent can automatically reduce the dimension of the problem, hence reducing computational cost per iteration. Other popular deterministic variants of proximal gradient descent, including the inertial variants [22] and FISTA [2, 6], also have this property [22].

It should be noted that, most results on support identification only mention the existence of such a constant \( T \) and limited result are available regarding the estimate of \( T \). There are some lower bound estimates of \( T \), such as [22, Proposition 10], however such an estimate requires the knowledge of \( \beta^* \) which makes it impractical.

Prox-SGD cannot identify sparsity For large-scale problems, proximal gradient descent may be impractical, mainly because of the calculation of \( \nabla F_{\eta} \) or \( \nabla F \), which needs to be evaluated over the distribution \( \Lambda \). It is therefore preferable to use online algorithms [5]: at each iteration \( t \), draw a sample \((x_t, y_t)\) randomly from the distribution \( \Lambda \) and perform the update

\[ \beta_{t+1} = \beta_t - \gamma_t \left( f_{y_t}(x_t^\top \beta_t)x_t + \lambda Z_t \right) \quad (1.4) \]

where \( Z_t \in \partial \Omega(\beta_t) \) is a subgradient (see Eq. (A.1)). This is a special instance of stochastic gradient descent, which can be traced back to [32, 16].

To deal with the non-smoothness imposed by the regularization term \( \Omega \), various stochastic algorithms have been proposed in the literature, such as truncated gradient [17] or stochastic versions of proximal gradient descent [11] (Prox-SGD)

\[ \beta_{t+1} = \text{prox}_{\lambda \gamma_t \Omega}(\beta_t - \gamma_t f_{y_t}(x_t^\top \beta_t)x_t), \]

where \( \text{prox}_{\lambda \gamma_t \Omega}(\cdot) \) is called the proximal operator which is defined by \( \text{prox}_{\gamma_t \Omega}(\cdot) = \text{argmin}_\beta \gamma_t \Omega(\beta) + \frac{1}{2}\|\beta - \cdot\|^2 \), and has closed form expression for the aforementioned sparsity promoting regularization terms [8]. However, different from the deterministic proximal gradient type methods, the support identification property is not always preserved. For example, for the standard Prox-SGD, due to the vanishing step-size \( \gamma_t \) and non-vanishing variance in the stochastic gradient estimates [42, 19], the generated sequence \( \{\beta_t\}_{t \in \mathbb{N}} \) does not have support identification property. Moreover, \( \beta_t \) tends to have full support for all \( t \); see [29] for an explicit example. As a result, one cannot easily exploit the sparsity promoting structure of \( \Omega \) for computational gains.

We do however mention that some stochastic algorithms have been shown to have this identification property: The first support identification result appears in [19], where the authors established the support identification property of the dual averaging method proposed in [42]. Then in [29], identification property are established for variance reduced stochastic gradient methods as these methods allow constant step-size, and hence, asymptotically behave like deterministic methods.
1.2.2 Dimension reduction via (safe) screening

In high dimensional statistics, (safe) screening techniques are popular approaches for filtering out features whose corresponding coefficients are 0, hence achieving dimension reduction; See \[13, 37, 28\] and the references therein. Safe feature elimination was first proposed by El Ghaoui et al. \[13\] for $\ell_1$ regularization problems. The rules introduced were static rules, where features are screened out as a preprocessing step, and sequential rules where one solves a sequence of optimization problems with a decreasing list of parameters $\{\lambda_k\}_k$, so solutions of an optimization problem with $\lambda_{k+1}$ are used to screen out features when solving with $\lambda_k$. The core idea behind these rules also rely on the optimality condition (1.2), which has a nice geometric interpretation with the construction of the safe regions driven by properties of the dual problem. Since this work, several extensions of these rules have been proposed \[24, 38, 41\].

Dynamic screening rules were later proposed by \[4\], where the safe screening region are updated along the iterates of a solver. Another work in this direction are so-called gap-safe rules \[26, 28\] where the calculation of the safe regions along the iterates are done via primal-dual gap computations. The present article is largely inspired by \[28\], where we dynamically construct safe regions by computing an ‘online’ primal-dual gap.

For algorithms that already have support identification property, screening can speed up the identification speed \[28\]. While for algorithms which do not have identification property, screening can enforce identification property. For example in a recent work \[35\], the authors developed a gap safe rule for conditional gradient descent. One highlight of their work is that through safe screening, identification is achieved whereas simply running conditional gradient descent will never achieve identification. The present article is largely inspired by \[35\] and \[28\], where we dynamically construct safe regions by computing an “online” primal-dual gap.

1.3 Our contributions

Motivated by the popularity of online optimization algorithms, in this paper, we address the non-sparseness problem (i.e. no support identification) of online optimization algorithms by combining them with the idea of safe screening rules \[13, 28\]. More precisely, our contributions in this paper are,

- By adapting gap safe screening rules of \[28\] to online algorithms, we propose an online screening rule. The proposed rule only needs to evaluate function values at the sampled data, hence has very low per iteration complexity. In particular, we show how to construct a “dual certificate” along the iterations which allows us to apply gap safe rules to screen out certain features. Moreover, this certificate can be built alongside any convergent online algorithms.

- The consequence of screening rules for online optimization is support identification of $\beta_t$, i.e. dimension reduction. It allows us to locate exactly which features are of interests. More importantly, significant computational gains can be obtained since the per iteration complexity scales from $n$ the dimension of the variable to $\kappa$ the sparsity of the solution.

It should also be noted that, the support identification indicates the global non-smooth optimization problem (in $\mathbb{R}^n$) locally becomes smooth (restricted to the support of $\beta^\star$). It can be even strongly convex if the Hessian of $F_\eta$ along supp($\beta^\star$) is positive definite. As a result, higher-order optimization methods such as BFGS and other quasi-Newton methods can be applied, which in turn provides even faster performance.

Remark 1.1. To remedy the problem of no activity identification, Regularized Dual Averaging (RDA) \[42\] was proposed as an alternative online algorithm, and it is known that RDA has activity identification in finite time \[19\]. The work \[19\] furthermore proposed to exploit this identification property by changing to higher order methods after identification. One consequence of our present
work is that screening can also be used alongside RDA and thereby provide savings in memory cost, even before identification is achieved.

**Paper organization** The rest of the paper is organized as follows. We recall the basic derivation of screening rules for sparsity regularized regression problem in Section 2. Theoretical analysis of our online screening rule is presented in Section 3. Numerical experiments on LASSO and sparse logistic regression problems are provided in Section 4. Finally, in the appendix we collect some basic concepts of convex analysis and the proofs of main theorems.

## 2 Safe screening

In this section, we provide a compact derivation of screening rules and focus on the original screening rule from [13] and gap safe rule from [28]. For the regularized empirical loss (1.1), the optimality condition states that $\beta^*$ is a minimizer if and only if

\[
Z^* \overset{\text{def}}{=} \frac{1}{\lambda} \sum_{i=1}^{m} \eta_i \theta^*_i x_i \in \partial \Omega(\beta^*), \quad \text{where} \quad \theta^*_i = \nabla f_i(x_i \beta^*), \quad i = 1, \ldots, m. \tag{2.1}
\]

Given a sparsity promoting norm $\Omega$, its dual norm is defined by

\[
\Omega^D(x) \overset{\text{def}}{=} \sup_{\Omega(z) \leq 1} \langle z, x \rangle
\]

and its sub-differential can be expressed as

\[
\partial \Omega(\beta) = \{ Z : \langle Z, \beta \rangle = \Omega(\beta), \quad \Omega^D(Z) \leq 1 \}.
\]

Note that if $\Omega$ is group decomposable, then we have

\[
\Omega^D(x) = \sup_{g \in G} \Omega^D_g(x).
\]

### 2.1 Safe screening

The fundamental of deriving screening rule is the optimality condition of the minimizers. For generality, suppose the regularization function $\Omega$ is group decomposable, then give any group $g \in G$,

\[
\Omega^D_g(Z^*_g) < 1 \iff \beta^*_g = 0. \tag{2.2}
\]

In general, the converse is not true, and an example of LASSO problem can be found in [29] where $\beta^*_g = 0$ but $\Omega^D_g(Z^*_g) = 1$. We also refer to [12, 7, 30] for more general theoretical studies. However, if the non degeneracy condition (1.3) holds, then one has the equivalence

\[
\Omega^D_g(Z^*_g) < 1 \iff \beta^*_g = 0.
\]

As can be seen here, the vector $Z^*$ certifies the support of $\beta^*$, and hence, it is called a *dual certificate*. Moreover, one can relate $Z^*$ corresponds to the solution $\theta^*$ of the *dual problem* of (1.1)

\[
\theta^* \in \operatorname{argmax}_{\theta \in K_{\lambda,\eta}} D_g(\theta) \overset{\text{def}}{=} - \sum_{i=1}^{m} \eta_i f^*_i(\theta_i) \tag{2.3}
\]

where

\[
K_{\lambda,\eta} \overset{\text{def}}{=} \{ \theta : \Omega^D(\sum_{i=1}^{m} \theta_i \eta_i x_i) \leq \lambda \} \subset \mathbb{R}^m
\]

is the dual constraint set.
The above message implies that, if $Z^*$ is known, we identify an index set which includes the support of the solution
\[\mathcal{I} \overset{\text{def}}{=} \{ g \in \mathcal{G} : \Omega^D_g(Z^*) = 1 \} \supseteq \text{supp}(\beta^*).\]
Consequently, one can restrict to optimization over $\beta_I \in \mathbb{R}^{|I|}$ instead. This can lead to huge computation gains if $\mathcal{I}$ tightly estimates the true support $\text{supp}(\beta^*)$ which very often is much smaller than the dimension of the problem.

Unfortunately, computing $Z^*$ is generally as difficult as finding $\beta^*$. However, the positions where $Z^*$ saturates can be estimated more readily. This is exactly the idea of safe screening, which constructs a “safe region” $Z$ such that $Z^* \in Z$. Then instead of using (2.2) to determine the zero entries of $\beta^*$, one can consider the relaxed criteria: $\beta^*_g = 0$ if
\[
\sup_{Z \in Z} \Omega^D(Z) < 1.
\]
The following result, which can be found in [35], illustrates how to perform screening rules based on a safe region $Z$ with center $c$.

**Proposition 2.1 (Safe screen rule).** Let $\beta^*$ be a minimizer to (1.1) and suppose that
\[Z^* \in Z \overset{\text{def}}{=} \{ Z : \Omega^D_g(Z - c) \leq r_g, \ g \in \mathcal{G} \}.
\]
Then, $\beta^*_g = 0$ if $1 - \Omega^D_g(c) > r_g$.

The proof of the result is rather simple, hence we provide below.

**Proof.** We know that for each $g \in \mathcal{G}$, $\beta^*_g = 0$ if $Z^*$ satisfies $\Omega^D_g(Z^*) < 1$. So, if $1 - \Omega^D_g(c) > r_g$ then by the triangle inequality,
\[
\Omega^D_g(Z^*) \leq \Omega^D_g(Z^* - c) + \Omega^D_g(c) \leq r_g + \Omega^D_g(c) < 1
\]
which implies that $\beta^*_g = 0$. \qed

There are several ground rules for constructing a safe region $Z$, which are

- The supreme of dual norm over the region $\sup_{Z \in Z} \Omega^D_g(Z)$ is easy to compute;
- The size of the safe region should be as small as possible: as the most trivial safe region is the whole space which screens out nothing, while the best one is $Z = \{Z^*\}$ which screens out all useless features.

In the literature, various safe regions are proposed: the simplest safe region is the safe sphere, proposed in the very first safe screening work [13] and several others [24, 26, 28]. Other safe regions include dual polytope projection safe sphere [39] and safe dome [40].

There are also different approaches to apply safe screening: static screening [13], sequential screening [13] and dynamic screening [26, 28]. For static safe screening, screening is only implemented as a pre-processing of data, so the amount of discarded features are fixed, and it is very crucial to construct a good safe region such that the amount of discarded features is as many as possible. If we have a finite sequence of regularization parameter $\lambda_j$ for $j = 0, ..., J$ such that
\[
\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_J = \lambda.
\]

Then static screening can be applied to each $\lambda_j$ which results in sequential screening. For both static and sequential screening, the volume of the safe regions is always bounded way from 0 which limits the potential of screening. Lastly, dynamic screening combines both screening rule and numerical methods such that the constructed safe regions are parameterized by the sequence generated by the numerical scheme. As a result, the safe region can eventually converge to the dual certificate and screen out all useless features.
2.2 Gap safe screening

In a series of work [26, 27, 28], the authors develop a gap safe rule for screening, where the “gap” here refers to the primal and dual function value gap. The key of gap screening rule is constructing the safe region via duality gap. Recall the primal optimization problem (1.1)

$$\min_{\beta \in \mathbb{R}^n} \left\{ P_{\lambda,\eta}(\beta) = \lambda \Omega(\beta) + \sum_{i=1}^{m} \eta_i f(x_i^\top \beta, y_i) \right\},$$

whose dual problem reads

$$\max_{\theta \in K_{\lambda,\eta}} \left\{ D_{\lambda,\eta}(\theta) \right\} \text{def} = -\sum_{i=1}^{m} \eta_i f^*_i(\theta_i)$$

with dual feasible set $K_{\lambda,\eta} \text{def} = \{ \theta : \Omega^D(\sum_{i=1}^{m} \theta_i \eta_i x_i) \leq \lambda \}$. For any $\beta \in \mathbb{R}^n$ and $\theta \in K_{\lambda,\eta}$, the duality gap is defined by

$$G_{\lambda,\eta}(\beta,\theta) = P_{\lambda,\eta}(\beta) - D_{\lambda,\eta}(\theta).$$

Let $\beta^*$ and $\theta^*$ be primal and dual solution respectively, then strong duality holds

$$D_{\lambda,\eta}(\theta) \leq D_{\lambda,\eta}(\theta^*) = P_{\lambda,\eta}(\beta^*) \leq P_{\lambda,\eta}(\beta).$$

As a result, the duality gap $G_{\lambda,\eta}(\beta,\theta)$ is always non-negative.

Since the loss function is differentiable with gradient being $L$-Lipschitz continuous, the dual problem $D_{\lambda,\eta}(\theta)$ is then $\mu$-strongly concave with $\mu = 1/L$, and one has

$$\frac{\mu^2}{2} \|\theta - \theta^*\|^2 \leq D_{\lambda,\eta}(\theta^*) - D_{\lambda,\eta}(\theta) + \langle \nabla D_{\lambda,\eta}(\theta^*), \theta - \theta^* \rangle$$

$$\leq P_{\lambda,\eta}(\beta) - D_{\lambda,\eta}(\theta).$$

Now given a numerical scheme, at each iteration, $\beta_t$ is explicitly available and one can compute a dual feasible variable $\theta_t$ by projecting $(f'_{y_i}(x_i^\top \beta_t, y_i))_{i=1}^m$ on to the dual feasible set $K_{\lambda,\eta}$. Finally, let $r_t = \sqrt{2G_{\lambda,\eta}(\beta_t,\theta_t)} / \mu$, one obtains the following safe sphere:

$$Z \text{def} = \left\{ -\frac{1}{\lambda} \sum_{i=1}^{m} \eta_i \theta_i^* x_i, \forall \theta \in Z_\theta \right\} \text{ with } Z_\theta \text{def} = \{ \theta \in \mathbb{R}^m : \|\theta - \theta_t\| \leq r_t \}.$$

2.3 Gap safe screening for Prox-SGD

For deterministic optimization (e.g. regularized empirical loss minimization (1.1)), screening rules are algorithm agnostic. That is to say, given an algorithm which outputs $\beta_t$, one can always compute a safe region $Z$ for screening. As a result, we can incorporate screening to proximal stochastic gradient descent when the problem to solve has a finite sum empirical loss.

For the finite sum problem, proximal stochastic gradient descent takes the following general form: for each $t$, sample $(x_t, y_t)$ uniformly at random from the finite data

$$\theta_t = f'_{y_t}(x_t^\top \beta_t)$$

$$\beta_{t+1} = T(\beta_t, \theta_t x_t, \gamma_t)$$

(2.4)

In the case of SGD, $T(\beta_t, \phi_t, \gamma_t) = \beta_t - \gamma_t(\theta_t + \lambda Z_t)$, while for Prox-SGD we have $T(\beta_t, \theta_t, \gamma_t) = \text{prox}_{\gamma \tau \Omega}(\beta_t - \gamma_t \phi_t)$. We have the following algorithm which combines (2.4) with safe screening rules.
Algorithm 1: Safe screening for finite sum problem

1. Given: $T > 0$, step-size $\{\gamma_t\}_{t \in \mathbb{N}}$;
2. initialization $t = 1$, $\bar{\beta}_0 \in \mathbb{R}^n$;
3. while not terminate do
   4. $\beta_0 = \bar{\beta}_{t-1}$; // set the anchor point
   5. for $j = 0, \ldots, T - 1$ do
      6. Sample $(x_j, y_j)$ uniformly at random; // random sampling
      7. $\theta_j = f'_{y_j}(x_j^T \bar{\beta}_j)$; $\beta_j = T(\beta_{j-1}, \theta_j, x_j, \gamma_j)$; // standard gradient update
      8. $j = j + 1$;
   9. end
   10. $\bar{\beta}_t = \beta_T$, $\bar{\theta}_t = (f'_{y_t}(x_t^T \bar{\beta}_t))_{i=1,\ldots,m}$; // primal and dual variables
   11. $Z = Z(\bar{\beta}_t, \bar{\theta}_t)$; // constructing safe region via $\bar{\beta}_t, \bar{\theta}_t$
   12. $S = \{ g \in \mathcal{G} : \Omega^\sigma_g(Z) < 1 - r_g \}$; // screening set
   13. $(\bar{\beta}_t)_S = 0$; // pruning the primal point
   14. $t = t + 1$;
15. end

Remark 2.2. The above algorithm has two loops of iteration: the inner loop is standard stochastic gradient methods, while for the outer loop, screening with certain safe rules is applied. The key of outer loop is the calculation of $\theta_t$ which requires evaluation over all data. Such a setting is similar to that of SVRG algorithm [15], where the full gradient of the loss function at an anchor point needs to be computed. Likewise the choice of steps for inner loop, the value of $T$ in Algorithm 1 should be the order of $m$, such that in average the overhead of computing $\theta_t$ is not significant.

Remark 2.3. For Algorithm 1, all the aforementioned safe screening rules can be applied; see [13, 24, 26, 28] and the references therein. However, for online learning, this is no longer true, since for online learning it is impossible to obtain $\theta_t$, let alone constructing the safe region $Z$.

3 Screening for online algorithms

For very large scale problems and online optimization methods, it is unrealistic or impossible to compute the dual variable $\theta_t$. Consequently, one cannot construct the safe region $Z$ for screening. However, for gap safe screening rule, since its safe region is built on function duality gap, therefore it is possible to generalize the rule to online setting via stochastic approximation. The purpose of this section is to build this generalization. The roadmap of this section reads

1. We first describe how to construct online dual certificates and primal/dual objectives, which consist of the following aspects: a) the dual problem of the online problem $(P_\lambda)$; b) online duality gap via stochastic approximation; c) online dual certificate; d) convergence guarantees.
2. With the online duality gap and dual certificate obtained in the first stage, we then can extend the gap-safe screening rule to the online setting. This extension is described in Section 3.2.
3. Finally in Section 3.3, we summarize our online screening scheme for proximal stochastic gradient descent in Algorithm 2.

3.1 Online dual certificates and objectives

Given an online optimization method, at each iteration, we sample $(x_t, y_t)$ from the distribution $\Lambda$ and evaluate

$$\theta_t \overset{\text{def}}{=} f'_{y_t}(x_t^T \bar{\beta}_t)$$

(3.1)
In what follows, we define an online dual point $\tilde{Z}_t$ which is constructed as weighted average of the past evaluated points $\{\theta_s\}_{s\leq t}$ and define online primal and dual objectives which are again weighted averages of the past selected functions $\{f_{y_s}\}_{s\leq t}$ and $\{f_{y_s}^*\}_{s\leq t}$, where $f^*$ denotes the convex conjugate of loss function $f_y$.

**Dual problem** We first consider the dual problem of the primal problem $(\mathcal{P}_\lambda)$, which reads

$$\max_v \{ \mathcal{D}(v) \overset{\text{def}}{=} -\mathbb{E}_{(x,y)}[f_{y}(v(x,y))] : \Omega^D(\mathbb{E}_{(x,y)}[v(x,y)x]) \leq \lambda \} \quad (\mathcal{D}_\lambda)$$

where we maximize over $\Lambda$-measurable functions $v$. Note that it admits a unique maximizer, since $f_y$ is $L$-Lipschitz smooth which implies that $f_y^*$ is $\frac{1}{L}$-strongly convex. The problems $(\mathcal{P}_\lambda)$ and $(\mathcal{D}_\lambda)$ are related as primal and dual problems and their solutions are related: Any minimizer $\beta^*$ of $(\mathcal{P}_\lambda)$ is related to the optimal solution $v^*$ of $(\mathcal{D}_\lambda)$ by $v^*(x,y) = f^T(x)\beta^*, y)$ and

$$Z^* \overset{\text{def}}{=} -\frac{1}{\lambda} \mathbb{E}_{(x,y)}[v^*(x,y)x] \in \partial \Omega(\beta^*).$$

**Online duality gap** Observe that the primal and dual objective functions are expectations. We now discuss their online ergodic estimations over the sampled data generated. For each iteration step $t \in \mathbb{N}$, let $\mu_t \in (0,1)$. Given a primal variable $\beta \in \mathbb{R}^n$ and a dual variable $\zeta \in \mathbb{R}^t$, define the online primal objective

$$\tilde{P}^{(t)}(\beta) \overset{\text{def}}{=} \tilde{F}^{(t)}(\beta) + \lambda \Omega(\beta) \quad (3.2)$$

where

$$\tilde{F}^{(t)}(\beta) \overset{\text{def}}{=} \mu_t f_{y_t}(x_t^\top \beta) + (1-\mu_t)\tilde{F}^{(t-1)}(\beta), \quad \tilde{F}^{(1)}(\beta) = f_{y_1}(x_1^\top \beta)$$

and the online dual objective

$$\tilde{D}^{(t)}(\zeta) \overset{\text{def}}{=} -\mu_t f_{y_t}^*(\zeta_t) + (1-\mu_t)\tilde{D}^{(t-1)}((\zeta_s)_{s\leq t-1}), \quad \tilde{D}^{(1)}(\zeta_1) = -f_{y_1}^*(\zeta_1)$$

**Remark 3.1.** Note that the primal variable has a fixed dimension which is $n$, however the dimension of the dual variable $\zeta$ grows over iteration.

**Assumption 3.2 ([32]).** We assume that $\mu_t \in (0,1)$ and

$$\sum_{t} \mu_t = +\infty \quad \text{and} \quad \sum_{t} \mu_t^2 < \infty \quad (3.3)$$

It is straightforward to check (see Lemma A.4 (i)) that there exists a decreasing sequence $\eta^{(t)}_s > 0$ such that

$$\sum_{s\leq t} \eta^{(t)}_s = 1 \quad \text{and} \quad \eta^{(t)}_s \overset{\text{def}}{=} \mu_s \prod_{i=s+1}^{t} (1-\mu_i) \quad (3.4)$$

so that in our previous notation of (1.1) and (2.3), we have $\tilde{P}^{(t)} = P^{(t)}$ and $\tilde{D}^{(t)} = D^{(t)}$, and they are related by

$$\min_{\beta} \tilde{P}^{(t)}(\beta) = \max_{\xi \in \mathcal{K}_{\lambda,\eta^{(t)}}} \tilde{D}^{(t)}(\xi)$$

**Definition 3.1 (Online duality gap).** We define the online duality gap at $\beta \in \mathbb{R}^n$ as

$$\text{Gap}_t(\beta) \overset{\text{def}}{=} \tilde{P}^{(t)}(\beta) - \tilde{D}^{(t)}((\theta_s)_{s\leq t}). \quad (3.5)$$

where we recall the definition of $\theta_s$ from (3.1).

**Remark 3.3.** Since $\theta_s$ is not necessarily a dual feasible point, the online duality gap $\text{Gap}_t(\beta)$ is not guaranteed to be non-negative. As we shall see in the next paragraph, while the gap $\text{Gap}_t$ can be computed in an online fashion, the feasible point $\theta_s$, in $\tilde{P}^{(t)}(\beta) - \tilde{D}^{(t)}((\theta_s)_{s\leq t})$, which is the projection of $\theta_s$ onto the constraint set cannot be computed online.
**An online estimate of the dual certificate** With the online duality gap, now we need to construct a dual certificate from $\beta_t$ and $\theta_t$. For the primal variable, it is clear that $\beta_t$ converges to $\beta^*$. It is therefore natural to define a candidate point, for $\mu_t > 0$, as

$$
\bar{Z}_t \overset{\text{def}}{=} -\frac{1}{\lambda} \mu_t \theta tx_t + (1 - \mu_t) \bar{Z}_{t-1}, \quad \text{and} \quad \bar{Z}_1 \overset{\text{def}}{=} -\frac{1}{\lambda} \theta_1 x_1.
$$

(3.6)

In the notation introduced in (3.4), we can write

$$
\bar{Z}_t = -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_{s}^{(t)} \theta_s x_s = -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_{s}^{(t)} f_{y_s}^t (x_s^\top \beta_s) x_s.
$$

(3.7)

**Convergence results** Before presenting our online screening rule, we provide some theoretical convergence analysis of the above online estimates. We first establish uniform convergence of the online objective $\bar{P}^{(t)}$ to its expectation $P$, and uniform convergence of the corresponding dual certificate $Z^{*,(t)}$ to $Z^*$.

**Proposition 3.4 (Convergence of online objectives).** The following result holds

(i) let $O$ be a compact set of $\mathbb{R}^n$,

$$
\sup_{\beta \in O} |\bar{P}^{(t)}(\beta) - P(\beta)| \to 0, \quad t \to +\infty
$$

almost surely.

(ii) Let $Z^{*,(t)}$ be the dual certificate associated to $\bar{P}^{(t)}$, we have uniform convergence to $Z^*$:

$$
\|Z^{*,(t)} - Z^*\|_\infty \to 0, \quad t \to +\infty.
$$

We also have convergence of the online certificate $\bar{Z}_t$ to $Z^*$, and the online gap evaluated at converging points also converges to zero.

**Proposition 3.5 (Convergence of the online estimate).** Let $\beta^*$ be a minimizer of $P$ and assume that $\beta_t \to \beta^*$ almost surely. Then with probability 1,

(i) $\bar{Z}_t$ converge to $Z^*$ as $t \to \infty$.

(ii) If $\bar{Z}_t \to \beta^*$, then $\text{Gap}_t(\bar{Z}_t) \to 0$ as $t \to \infty$.

### 3.2 Online screening

In this section, we derive a screening rule for solutions to the online objective $\bar{P}^{(t)}$ based on the certificate $\bar{Z}_t$ and $\text{Gap}_t$. In the following, let $\bar{Z}_t$ be as defined in (3.7), $\theta_t$ be as in (3.1) and let

$$
\bar{Z}_t^* = -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_{s}^{(t)} \theta_s^{(t),*} x_s
$$

where $\theta_s^{(t),*}$ is the maximizer of $D(t)$.

**Lemma 3.6 (Screen gap).** Let $\bar{\beta} \in \mathbb{R}^n$, then there holds

$$
\frac{1}{2L} \sum_{s=1}^{t} \eta_{s}^{(t)} |\theta_s - \theta_s^{(t),*}|^2 \leq \text{Gap}_t(\bar{\beta}) + \sum_{s=1}^{t} \eta_{s}^{(t)} f_{y_s}^t (0) (\Omega^D(\bar{Z}_t) - 1) +
$$

Moreover, for all $g \in \mathcal{G}$,

$$
\Omega^D_g(\bar{Z}_t - \bar{Z}_t^*) \leq r_g^{(t)}(\bar{\beta}, \bar{Z}_t)
$$

where

$$
r_g^{(t)}(\bar{\beta}, \bar{Z}_t) \overset{\text{def}}{=} \frac{2LN_g}{\lambda} \sqrt{\text{Gap}_t(\bar{\beta}) + \sum_{s=1}^{t} \eta_{s}^{(t)} f_{y_s}^t (0) (\Omega^D(\bar{Z}_t) - 1) +}
$$

with $N_g \overset{\text{def}}{=} \sum_{s=1}^{t} \eta_{s}^{(t)} \Omega^D_g(x_s)^2$. 

10
By combining Lemma 3.6 with Proposition 2.1, we obtain the following online screening rule.

**Corollary 3.7 (Screen rule).** Let $\beta^{(t),*} \in \arg\min_\beta \tilde{P}^{(t)}(\beta)$. Then, given any $\bar{\beta} \in \mathbb{R}^n$, $\bar{\beta}^{(t),*} = 0$ if

$$1 - \Omega^D(\bar{Z}_t) > \gamma^{(t)}_g(\bar{\beta}, \bar{Z}_t).$$

**Remark 3.8.**

(i) Compared with the gap safe rules of [26, 27, 28], we do not project $\bar{Z}_t$ onto the dual feasible set $\mathcal{K}$ and hence have an additional term $(\Omega^D(\bar{Z}_t) - 1)_+$ in $\gamma^{(t)}_g(\bar{\beta}, \bar{Z}_t)$.

(ii) The above screening is safe for the online problem $\tilde{P}^{(t)}$ in the sense that screening will not falsely remove features which are in the solution $\beta^{(t),*}$ of $\tilde{P}^{(t)}$, however, the support of $\beta^{(t),*}$ may not necessarily coincide with that of the global minimizer $\beta^*$ of $(P_\lambda)$ and hence, our rule is not necessarily safe for the expectation $(P_\lambda)$. Further discussions on the safety of our rule can be found in Section 4.4.

We can directly apply Corollary 3.7 to screen out variables while running SGD. However, the effectiveness of this rule will depend on the proximity of $\bar{\beta}$ to the optimal point $\beta^*$. We therefore propose to progressively update this anchor point $\bar{\beta}$.

Let $0 = t_0 < t_1 < t_2 < \cdots < t_k = T$ and denote $[t_{j-1}, t_j] \overset{\text{def}}{=} \{t_{j-1} + 1, \ldots, t_j\}$. Let $\eta^{(T)} = (0, 1)$ for $s \in [0, T]$ be such that $\sum_{s \in [0, T]} \eta^{(T)} = 1$. Given $j \in \{1, \ldots, k\}$, let $(\theta^*_s)_{s \in [t_{j-1}, t_j]}$ be the optimal dual solution to

$$\max_{\gamma} \sum_{s \in [t_{j-1}, t_j]} - \eta^{(T)} f_{y_s}(\gamma_s)$$

where

$$\Omega^D(\sum_{s \in [t_{j-1}, t_j]} \eta^{(T)} x_s \gamma_s) \leq \lambda \sum_{s \in [t_{j-1}, t_j]} \eta^{(T)}.$$ 

Note that (3.8) is dual to the primal problem

$$\min_{\beta} \sum_{s \in [t_{j-1}, t_j]} \eta^{(T)} (f_{y_s}(x_s^\top \beta) + \lambda \Omega(\beta)).$$

The corresponding dual certificate is

$$Z^{*}_{j} \overset{\text{def}}{=} -\frac{1}{\lambda} \sum_{s \in [t_{j-1}, t_j]} \frac{\eta^{(T)}_s}{\gamma_j} \theta^*_s, \quad \text{where} \quad \gamma_j = \sum_{s \in [t_{j-1}, t_j]} \eta^{(T)}_s.$$ 

For each $j$, by applying Lemma 3.6 to $\{\theta_s\}_{s \in [t_{j-1}, t_j]}$, $\bar{\beta} \overset{\text{def}}{=} \beta_{t_{j-1}}$ and $\{\theta^*_s\}_{s \in [t_{j-1}, t_j]}$, we obtain

$$\sum_{s \in [t_{j-1}, t_j]} \eta^{(T)}_s |\theta_s - \theta^*_s|^2 \leq \sum_{s \in [t_{j-1}, t_j]} \eta^{(T)}_s \left(f_{y_s}(x_s^\top \beta_{t_{j-1}}) + \lambda \Omega(\beta_{t_{j-1}}) - f_{y_s}(\theta_s) + f_{y_s}(0)(\Omega^D(Y^*_j) - 1)_+\right)$$

where

$$Y^*_j \overset{\text{def}}{=} \frac{1}{\sum_{s \in [t_{j-1}, t_j]} \eta^{(T)}_s} \sum_{n \in [t_{j-1}, t_j]} \eta^{(T)}_s \theta_s x_s.$$ 

Summing (3.9) over $j = 1, \ldots, k$ and denoting $\bar{\beta}_s \overset{\text{def}}{=} \beta_{t_{j-1}}$ for $s \in [t_{j-1}, t_j]$, we obtain

$$\sum_{s=1}^{T} \eta^{(T)}_s |\theta_s - \theta^*_s|^2 \leq R_T$$
where

\[ R_t \overset{\text{def}}{=} \sum_{s=1}^{T} \eta_s^{(T)} \left( f_{y_s}(x_s^\top \bar{\beta}_s) + \lambda \Omega(\bar{\beta}_s) - f_{y_s}^*(\theta_s) \right) + \sum_{j=1}^{k} V_j (\Omega^D(Y_j) - 1) + \]

with \( V_j \overset{\text{def}}{=} \sum_{s\in[t_{j-1},t_j]} \eta_s^{(T)} f_{y_s}(0) \).

Define

\[ \bar{Z}_T = \sum_{j=1}^{k} \gamma_j Z_j^* = -\frac{1}{\lambda} \sum_{j} \sum_{s\in[t_{j-1},t_j]} \eta_s^{(T)} f_{y_s}^*(x_s^\top \bar{\beta}_s^*) x_s, \]

and we have \( \bar{Z}_T \overset{\text{def}}{=} \sum_{s=1}^{T} \eta_s^{(T)} \theta_s x_s \) satisfy

\[ \Omega^D \left( \bar{Z}_T - \bar{Z}^* \right) \leq \frac{\sqrt{2LR_T N_{T,g}}}{\lambda}, \tag{3.11} \]

where \( N_{T,g} \overset{\text{def}}{=} \sum_{s=1}^{T} \eta_s^{(T)} \Omega^D(x_s)^2 \). Note that the residual term \( R_T \) is now dependent on the sequence \( \beta_{t_j} \) which converges to \( \beta^* \) as \( j \to \infty \).

### 3.3 Screening procedure

Finally, we are able to present our online screening rule for online optimization algorithms. Consider an online algorithm of the following form

\[ \theta_t = f_{y_t}'(x_t^\top \beta_t) \]
\[ \beta_{t+1} = \mathcal{T}(\beta_t, \theta_t x_t, \gamma_t) \tag{3.12} \]

where \( \mathcal{T} \) is a fixed point operator. In the case of SGD, \( \mathcal{T}(\beta_t, \phi_t, \gamma_t) = \beta_t - \gamma_t(\theta_t + \lambda Z_t) \), while for Prox-SGD we have \( \mathcal{T}(\beta_t, \theta_t, \gamma_t) = \text{prox}_{\gamma_t \tau \Omega}(\beta_t - \gamma_t \phi_t) \).

We state our screening framework for online optimization methods in Algorithm 2 below.
Algorithm 2: Online optimization algorithm with screening

1. Given: step-size $\{\gamma_t\}_{t \in \mathbb{N}}$, exponent $w$, $\mu_t \overset{\text{def}}{=} 1/t^w$, initial point $\bar{\beta} \in \mathbb{R}^n$; 
2. initialization $t = 1$; $p_0 = d_0 = N_{0,g} = 0$; 
3. $u_0 = 0$; $S = 0$; $Z = 0$; 
4. while not terminate do 
   5.   $\beta_0 = \bar{\beta}$; // set the anchor point 
   6.   $v_0 = 0$; $X_0 = 0$; 
   7.   for $t = 1, \ldots, T$ do 
      8.     $(x_t, y_t) \sim \Lambda$; // random sampling 
      9.     $\theta_t = f_{y_t'}(x_t^T \beta_{t-1})$; $\beta_t = T(\beta_{t-1}, \theta_t x_t, \gamma_t)$; // standard gradient update 
     10.    $X_t = -\frac{1}{2} \mu_t \theta_t x_t + (1 - \mu_t) X_{t-1}$; // certificate update 
     11.    $p_t = \mu_t (f_{y_t'}(x_t^T \beta) + \lambda \Omega(\beta)) + (1 - \mu_t) p_{t-1}$; // primal value 
     12.    $d_t = -\mu_t f_{y_t'}(\theta_t) + (1 - \mu_t) d_{t-1}$; // dual value 
     13.    $\forall g \in \mathcal{G}$, $N_{t,g} = \mu_s \Omega_{y_t}^D(x_t^2) + (1 - \mu_s) N_{t-1,g}$; 
     14.    $v_t = v_t f_{y_t'}(0) + (1 - \mu_v) v_{t-1}$; 
     15.    $u_t = (1 - \mu_u) u_{t-1}$; 
   16.  end 
   17.  $\bar{\beta} = \beta_{t-1}$; // update anchor point 
   18.  $Z = u_t Z + X_t$; // estimated certificate 
   19.  $S = u_t S + v_t \Omega^D(X_t/(1 - u_t) - 1)_+$; 
   20.  $R = p_t - d_t + S$; 
   21.  $S = \{g \in \mathcal{G} : \Omega_{y_t}^D(Z) < 1 - \frac{2LN_{0,g}}{\lambda}\}$; // screening set 
   22.  $(\beta_0) S = 0$; // pruning the primal point 
   23.  $u_0 = 1$; 
   24. end 

Next we provide some discussions on how to compute some key values of the algorithm, for instance the terms described in (3.10) and (3.11).

- It is straightforward to compute $\bar{Z}_T$ and $N_{T,g}$, as we have

  \[ \bar{Z}_s \overset{\text{def}}{=} \mu_s \theta_s x_s + (1 - \mu_s) \bar{Z}_{s-1}, \quad \text{where} \quad \bar{Z}_0 = 0, \]

  \[ N_{s,g} \overset{\text{def}}{=} \mu_s \Omega_{y_t}^D(x_s^2) + (1 - \mu_s) N_{s-1,g}, \quad \text{where} \quad N_{0,g} = 0. \]

- Next is the computation of

  \[ R_T \overset{\text{def}}{=} \sum_{s=1}^{T} \eta_{s}^{(T)}(f_{y_s}(x_s^T \beta_s) + \lambda \Omega(\beta_s)) - \sum_{s=1}^{T} -\eta_{s}^{(T)} f_{y_s}(\theta_s) + \sum_{j=1}^{k} V_j(\Omega^D(Y_j) - 1)_+ \]

  The first two terms are straightforward: define $\bar{\beta}_s = \beta_{t_j-1}$ for all $s \in [t_{j-1}, t_j]$ and repeat this over $s = 1, \ldots, t$:

  \[ p_s \overset{\text{def}}{=} \mu_s (f_{y_s}(x_s^T \beta_s) + \lambda \Omega(\beta_s)) + (1 - \mu_s) p_{s-1}, \quad \text{where} \quad p_0 = 0, \]

  \[ d_s \overset{\text{def}}{=} -\mu_s f_{y_s}(\theta_s) + (1 - \mu_s) d_{s-1}, \quad \text{where} \quad d_0 = 0. \]
In comparison, it is a bit more complicated to compute $S_T$: during the first $s = 1, \ldots, t_1$ iterations, define
\[ X_s^{(1)} \equiv \mu_s \theta_s x_s + (1 - \mu_s) X_{s-1}^{(1)}, \quad \text{where} \quad X_0^{(1)} = 0, \]
\[ v_s^{(1)} \equiv \mu_s f_{y_s}(0) + (1 - \mu_s) v_{s-1}^{(1)}, \quad \text{where} \quad v_0^{(1)} = 0. \]
and that
\[ Y_1 \equiv X_{t_1}, \quad V_1 \equiv v_{t_1}, \quad S_{t_1} \equiv V_1 (\Omega^D (Y_1) - 1)_+. \]

Then for iteration $s \in [t_{j-1}, t_j], j = 2, \ldots, k$, we have
\[ X_s^{(j)} \equiv \mu_s \theta_s x_s + (1 - \mu_s) X_{s-1}^{(j)}, \quad \text{where} \quad X_{t_j}^{(j)} = 0, \]
\[ v_s^{(j)} \equiv \mu_s f_{y_s}(0) + (1 - \mu_s) v_{s-1}^{(j)}, \quad \text{where} \quad v_{t_j}^{(j)} = 0. \]

At iteration $t_j$, define $\gamma_j \equiv \prod_{s \in [t_{j-1}, t_j]} (1 - \mu_s)$ and
\[ Y_j \equiv \frac{1}{1 - \gamma_j} X_{t_j}^{(j)}, \quad V_j^{(j)} \equiv v_{t_j}^{(j)} \quad \text{and} \quad S_{t_j} \equiv \gamma_j S_{t_{j-1}} + V_j (\Omega^D (Y_j) - 1)_+. \]

Note that we in fact have $\bar{Z}_{t_j} = X_{t_j}^{(j)} + \gamma_j \bar{Z}_{t_{j-1}}$.

We conclude this section by few remarks.

**Remark 3.9 (Computational pains and gains).** Our screening rule adds several computational overheads to the original online optimization problem, however, all of them are of $O(n)$ complexity. Denote by $n_t$ the dimension of the problem at current iteration

- For the *inner loop* of Algorithm 2, line 10–12 computing the dual certificate and primal/dual function values are of $O(n_t)$ complexity.
- For the *outer loop* of Algorithm 2, all computations are at most $O(n_t)$.

Overall, the computational overheads added by screening is $O(n_t)$ per iteration where $n_t$ is the dimension of $\beta_t$ at iteration step $t$.

On the other hand, since our screening rule can effectively remove useless features along iteration. Suppose the sparsity of $\beta^*$ is $\kappa$ which is much smaller than $n$ and our screening rule manages to screen out all useless features, then we have eventually $n_t = \kappa$ for all $t$ large enough, which in turn means the computational overheads are negligible.

**Remark 3.10 (Effect of the exponent w).** For Algorithm 2, the weight parameter $\mu_t$, essentially determined by the exponent $w$, determines how important the latest iterate is. As a result, $w$ is crucial to the screening behavior of Algorithm 2. In general the value of $w$ lies in $[0.5, 1]$. As we shall see later in the numerical experiments, that the smaller the value of $w$, the more aggressive the screening rule which make Algorithm 2 not safe. While for larger choice of $\gamma$, the screening is much more passive, hence safer.

**Remark 3.11 (Choices of T).** For Algorithm 2, the inner loop iteration number is controlled by $T$. The choice of $T$ should balance the number of screening and the total number of iteration of the method. Similar to Algorithm 1, in practice, choices like $\ell m$ with $\ell$ being small integers demonstrate good performance, and we refer to Section 4 the numerical experiments for more detailed discussions.

**Remark 3.12 (Online screening is not safe).** Though our screening rule is adapted from gap safe rule, which is guaranteed to be safe, i.e. only removes useless features and keeps all the active ones. Algorithm 2 is not guaranteed to be safe. This is due to the fact that the rule we derive is with respect to the online objective $P(t)$ which is not the original objective $(P_A)$. As a result, potentially our screening rule can falsely remove useful features, which makes it not safe. However, this can be avoided by incorporating safe guard step, for instance, we can combine Algorithm 2 with the strong rules developed in [37] to avoid false removal.
4 Numerical results

In this section, we provide numerical experiments on LASSO and Sparse Logistic Regression (SLR) problems to compare the performances of the proposed online screening (Algorithm 2) and the full screening (Algorithm 1). For both problems, datasets from LIBSVM\footnote{https://www.csie.ntu.edu.tw/cjlin/libsvm/} are considered, whose details are listed below, including number of samples ($m$) and dimension of the problem ($n$).

| Name        | $m$ | $n$  |
|-------------|-----|------|
| colon-cancer| 62  | 2,000|
| leukemia    | 38  | 7,129|
| breast-cancer| 44 | 7,129|
| gisette     | 6,000| 5,000|
| arcene      | 200 | 10,000|
| dexter      | 600 | 20,000|
| dorothea    | 1,150| 100,000|
| rcv1        | 20,242| 47,236|

As seen from the table, four relatively small scale datasets and four bigger scale datasets are considered. Three algorithms are compared: the standard proximal gradient descent (Prox-SGD), Algorithm 1 with Prox-SGD (FS-Prox-SGD), Algorithm 2 with Prox-SGD (OS-Prox-SGD). Below are the details of settings of our experiments

- SAGA \cite{10} is used for computing the solution of the problems.
- Step-sizes are also set the same for them $\gamma_t = \frac{1}{mL_{\text{grad}}}.$
- The exponent $w$ is set as $0.51$ for all the tests.
- Regularization parameter: for LASSO problem we have $\lambda = \frac{\lambda_{\text{max}}}{2}$, while for SLR problem, various choices are chosen and provided below.
- For both screening schemes, we set $T = 4m$, i.e. screening is applied every $4m$ steps.

4.1 Dimension reduction of screening schemes

We first compare the support identification properties of Prox-SGD, FS-Prox-SGD and OS-Prox-SGD. The obtained results are shown in Figure 1 (LASSO) and Figure 2 (SLR), respectively.

For each figure, two quantities are provide: size of support over number of epochs of $\beta_t$ for \textit{solid lines} and elapsed time over number of epochs for \textit{dashed lines}. For LASSO problem, we obtain the following observations,

- Prox-SGD, black lines in all figures, indeed does not have support identification property, as the size of support is oscillating and does not decrease.
- Except for rcv1, online screening can effectively remove redundant features, while full screening mainly works for smaller datasets. It should be noted that, limited by the number of iterations, support identification is not exactly reached.
- Between FS-Prox-SGD and OS-Prox-SGD, overall the latter demonstrates a better screening outcomes. In particular, OS-Prox-SGD can significantly reduce the dimension of the problem at the very early state of the iteration.

For SLR problem, the choice of regularization parameter $\lambda$ is provided under each subfigurs of Figure 2. The advantages of online screening is similar to those of LASSO problem for the first six datasets. However, for dorothea and rcv1 datasets, the behaviors are different.
we have $\lambda = \frac{\lambda_{\max}}{2}$.

- For **dorothea** dataset, both screening schemes are slower than the pure Prox-SGD, while for LASSO problem, online screening achieve dimension reduction.
- For **rcv1** dataset, online screening eventually provides dimension reduction while for LASSO case, both schemes are not working.

Figure 2: Comparison of support reduction and wall clock time for SLR problems.

We observe from above that, for both problems, when online screening works, it can achieve dimension at the very early stage of the iteration, which means practically it is more attractive than the full screening scheme, since in practice stochastic algorithms are run for limited number of epochs.
4.2 LASSO

In this part, we present absolute error $\|\beta_t - \beta^*\|$ and solution quality comparisons for the LASSO problem. We start with error comparisons in Figure 3. Similarly to the wall-clock time comparison in Figure 1, that faster algorithm yields faster error decays.

![Graphs showing error comparison for LASSO problems.](image)

Figure 3: Comparison of errors $\|\beta_t - \beta^*\|$ for LASSO problems.

In Figure 4, we provide comparisons of the solutions obtained by the algorithms. Recall that, we use SAGA to compute the solution, which is verified via optimality condition, for reference.

- It can be observed that for Prox-SGD, non-identification can be observed by the large number of tiny values around order $10^{-6}$.
- In general, there are discrepancies between the outputs of Prox-SGD schemes and the solution by SAGA, which is caused by the vanishing step-size of SGD schemes.
- Screening can be effective in screening out these tiny values, with online screening being better than full screening.

Note that for the dexter dataset, online screening over-screens the features which results in only 2-sparse output while the solution obtained by SAGA is 6-sparse. As we mentioned in Remark 3.10, the aggressiveness of online screening is controlled by the exponent parameter $w$, in the last part of this section we provide a detailed discussion on this parameter.

4.3 Sparse logistic regression

For SLR problems, the comparisons of error-time and solution quality are provided in Figure 5 and Figure 6, respectively. Similar to the LASSO problem, the error comparison is in consistent with time comparison of Figure 2. For solution comparison, this time, for the dorothea dataset, the outputs of all Prox-SGD schemes are far from close to the solution obtained by SAGA.

It is worth noting that, this time for dexter dataset, the behavior of online screening is not as aggressive as that of LASSO case (see Figure 6(f)), yet still the outputs of Prox-SGD schemes are quite far away from the solution obtained by SAGA. Moreover, for rcv1 data set, this time our online screening is not safe where screen out the 3rd feature of the output of SAGA.
Regarding the non safe behaviors of online screening for LASSO on dexter dataset and SLR on rcv1 dataset, in this part we discuss the effect of the exponent parameter $w$ in Algorithm 2. For comparison purpose, three choices of the exponent $w$ are tested, which are $w = 0.51, 0.6$ and $0.75$. In the meantime, three quantities are considered: dimension reduction, solution and error decay. Since SAGA has support identification property [29], we also include it for reference.

In the experiments, we only consider LASSO on dexter dataset and SLR on rcv1 dataset. Though limited, these two examples are demonstrative of the situation where online screening correctly identifies the solution support and where small values of $w$ results in false removal of solution support.
Similar results can be observed for the other datasets described in the previous section.

**LASSO problem** The results for LASSO and **dexter** dataset are provided below in Figure 7, from which we observe the followings

- **Dimension reduction** For all choices of the exponent \( w \), there is a sharp dimension reduction at the beginning stage of the iteration. Eventually, the smaller the value of \( w \), the sparser the output of Prox-SGD. Moreover, the support identification of SAGA, see magenta line, indicates the non-safe behavior of \( w = 0.51 \).

- **Solution** As we have seen previously, \( w = 0.51 \) is not safe as it screens out true support\(^2\). For \( w = 0.6, 0.75 \), both choices retain the support solution, with \( w = 0.6 \) being the better one.

- **Error decay** Since \( w = 0.51 \) is not safe, the fast decay of the black line is meaningless. Both \( w = 0.6 \) and \( w = 0.75 \) produce almost the same error decay.

Finally, it is worth mentioning that the wall clock time for all three choices of \( w \) are very close and around 10 seconds.

The main conclusion is that smaller values of \( w \) will lead to a more aggressive screening rule. This can however (in 2 of our tested datasets) lead to false removal of the solution support. To guarantee its output, certain safeguards need to be considered. For example, we can include the rules developed in [37], so that once false dimension reduction happens, the screening will be reset and the value of \( w \) will be increased by a certain margin until an upper bound is reached.

**SLR problem** To conclude this section, we present numerical results for SLR on **rcv1** dataset, which are shown in Figure 8. The observations are similar to those we obtain from Figure 7, except that

- Online screening only starts to be effective after certain number of epochs, which is already seen in Figure 2 (h).

- Only \( w = 0.75 \) is safe for this case, though the error decay of \( w = 0.6 \) is better than that of \( w = 0.75 \).

\(^2\)We ran the method until convergence and checked its optimality condition, which showed that the output obtained by \( w = 0.51 \) is not a solution of the problem
The wall clock time for these three choices of $w$ are 325, 371 and 297 seconds, respectively. The reason that $w = 0.75$ being better than other two, though eventually provides largest support size, is that it has slightly faster dimension reduction that the other two after 50’th epoch.

Figure 7: Comparison of exponent parameter $w$ for LASSO problem and dexter dataset.

Figure 8: Comparison of exponent parameter $w$ for SLR problem and rcv1 dataset.

5 Conclusion

Online optimization methods are widely used for solving large scale problems arising from machine learning, data science and statistics. However, when encountered with sparsity promoting regularizers, online methods break the support identification property of these regularizers. In this paper, we combined the well established safe screening technique with online optimization methods which allows online methods to discard useless features along the iteration, hence achieving dimension reduction. Numerical result demonstrated that dramatic wall time gain can be achieved for classic regression tasks over real datasets.

A Appendix

A.1 Convex analysis

The sub-differential of a proper convex and lower semi-continuous function $\Omega : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a set-valued mapping defined by

$$\partial \Omega : \mathbb{R}^n \to \mathbb{R}^n, \beta \mapsto \{Z \in \mathbb{R}^n \mid \Omega(\beta') \geq \Omega(\beta) + \langle Z, \beta' - \beta \rangle, \forall \beta' \in \mathbb{R}^n\}. \quad (A.1)$$
Lemma A.1 (Descent lemma [3]). Suppose that $F : \mathbb{R}^n \to \mathbb{R}$ is convex continuously differentiable and $\nabla F$ is $L$-Lipschitz continuous. Then, given any $\beta, \beta' \in \mathbb{R}^n$,

$$F(\beta) \leq F(\beta') + \langle \nabla F(\beta'), \beta - \beta' \rangle + \frac{L}{2} \|\beta - \beta'\|^2.$$ 

Lemma A.2. Fenchel-Young inequality Let $F : \mathbb{R}^n \to \mathbb{R}$ be convex, lower semicontinuous and proper, then for all $p, x \in \mathbb{R}^n$,

$$F(x) + F^*(p) \geq \langle p, x \rangle$$

with equality if $p \in \partial F(x)$.

A.2 Derivation of the dual problem ($D_\lambda$)

Write $f_y(z) \overset{\text{def}}{=} f(z; y)$ and $v_\beta(x, y) = f_y'(x^\top \beta)$, then applying the Fenchel-Young (in)equality twice,

$$P(\beta) = \mathbb{E}_{(x,y)}[v_\beta(x, y)x^\top \beta] - \mathbb{E}_{(x,y)}[f_y^*(v_\beta(x, y))] + \lambda \Omega(\beta)$$

$$= -\mathbb{E}_{(x,y)}[f_y^*(v_\beta(x, y))] - \lambda \left( -\frac{1}{\lambda} \mathbb{E}_{(x,y)}[v_\beta(x, y)], \beta \right) - \Omega(\beta)$$

$$\geq -\mathbb{E}_{(x,y)}[f_y^*(v_\beta(x, y))] - \lambda \Omega^\ast( -\frac{1}{\lambda} \mathbb{E}_{(x,y)}[v_\beta(x, y)])$$

where the final inequality is an equality if $-\frac{1}{\lambda} \mathbb{E}_{(x,y)}[v_\beta(x, y)] \in \partial \Omega(\beta)$, which is the case at the optimal point $\beta^\ast$. Therefore, it follows that

$$\min_{\beta \in \mathbb{R}^n} P(\beta) = -\mathbb{E}_{(x,y)}[f_y^*(v_\beta^\ast(x, y))] - \lambda \Omega^\ast( -\frac{1}{\lambda} \mathbb{E}_{(x,y)}[v_\beta^\ast(x, y)])$$

On the other hand, again by the Fenchel-Young inequality,

$$\min_{\beta} P(\beta) \geq \max_v D(v) \overset{\text{def}}{=} -\mathbb{E}_{(x,y)}[f_y^*(v(x, y))] - \lambda \Omega^\ast( -\frac{1}{\lambda} \mathbb{E}_{(x,y)}[v(x, y)])$$

where we maximize over $\Lambda$-measurable functions $v$. Therefore, the dual problem is $\max_v D(v)$ and strong duality holds. Finally, note that $\Omega^\ast$ is the indicator function on the dual constraint set $K_{\lambda, \eta}$.

A.3 Proofs of Section 3.1

We prove Propositions 3.4 and 3.5 in this section. The proofs are provided for completeness although they use standard techniques, see for example [19]. Similar results can be found in [25] (although we relax the condition of $\sum_t \mu_t^2 \sqrt{I} < +\infty$ to simply $\sum_t \mu_t^2 < +\infty$). We make use of the following lemma.

Lemma A.3 (Super-martingale convergence [33]). Let $\mathcal{F}_k$ be a set of random variables with $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for all $k \in \mathbb{N}$. Let $Y_k, Z_k, W_k$ be non-negative random variables which are functions of random variables in $\mathcal{F}_k$, such that

- $\mathbb{E}[Y_{k+1} | \mathcal{F}_k] \leq Y_k + W_k - Z_k$
- $\sum_k W_k < +\infty$ with probability 1

Then, $\sum Z_k < +\infty$ and $Y_k$ converges to a non-negative random variable $Y$ with probability 1.

Lemma A.4. For some $\{\mu_n\}_n \subset (0, 1)$ and let $\{f_n\}_n \subset \mathbb{R}^d$ be random variables. Suppose $\sum_j \mu_j = +\infty$ and $\sum_j \mu_j^2 < +\infty$. Define $\bar{f}_n \overset{\text{def}}{=} \mu_n f_n + (1 - \mu_n) \bar{f}_{n-1}$ with $\bar{f}_1 = f_1$. Let $\eta_j^{(n)} \overset{\text{def}}{=} \mu_j \prod_{i=j+1}^n (1 - \mu_i)$.

(i) $\bar{f}_n = \sum_{j=1}^n \eta_j^{(n)} f_j$
(ii) $\sum_j \eta_j^{(n)} = 1$
(iii) $\lim_{n \to +\infty} \sum_{j=1}^n (\eta_j^{(n)})^2 = 0$
(iv) Suppose that \( \mathbb{E}[f_j] = \mathbb{E}[f_1], \{f_j\} \) are iid random variables and \( \|f_j\|_{\infty} \leq B \). Then, \( \lim_{n \to +\infty} \tilde{f}_n = \mathbb{E}[f_1] \) with probability 1.

**Proof.** The first two statements are a simple computation. For the third statement,

\[
\sum_{j=1}^{n} (\eta_{j}^{(n)})^2 = \sum_{j=1}^{n} \mu_j^2 \prod_{i=j+1}^{n} (1 - \mu_i)^2 \leq \sum_{j=m}^{n} \mu_j^2 + \prod_{i=m}^{n} (1 - \mu_i)^2 \sum_{j=1}^{m-1} \mu_j^2
\]

Since \( \sum_j \mu_j^2 < +\infty \), we have \( \sum_{j=m}^{n} \mu_j^2 \to 0 \) as \( m, n \to +\infty \). Moreover, since \( \sum_{i=m}^{n} \mu_i \to +\infty \) as \( n \to +\infty \), we have

\[
\prod_{i=m}^{n} (1 - \mu_i)^2 \leq \prod_{i=m}^{n} \exp(-2\mu_i) = \exp \left(-2 \sum_{i=m}^{n} \mu_i \right) \to 0
\]
as \( n \to +\infty \).

Finally, fix \( j \in [d] \) and \( n \in \mathbb{N} \). For \( m \leq n \), define \( Y_m \defeq \sum_{j=1}^{m} \eta_{j}^{(n)} (f_j - (\mathbb{E}[f_1])_j) \). Then, \( \{Y_m\}_{m \leq n} \) is a Martingale and \( |Y_m - Y_{m-1}| \leq 2\eta_{m}^{(n)} B \). By Azuma-Hoeffding inequality, given any \( v > 0 \),

\[
\mathbb{P} \left( |Y_n| \geq v \right) \leq 2 \exp \left( - \frac{v^2}{8B^2 \sum_{j=1}^{n} (\eta_{j}^{(n)})^2} \right)
\]

and hence, by the union bound,

\[
\mathbb{P} \left( \|\tilde{f}_n - \mathbb{E}[f_1]\|_{\infty} \geq v \right) \leq 2d \exp \left( - \frac{v^2}{8B^2 \sum_{j=1}^{n} (\eta_{j}^{(n)})^2} \right) \to 0, \quad n \to +\infty,
\]

so \( \tilde{f}_n \) converges to \( \mathbb{E}[f_1] \) in probability. To show that it converges almost surely, we make use of Lemma A.3: Let \( Y_{n} \defeq |\tilde{f}_{n} - \mathbb{E}[f_1]|^2 \). Note that

\[
Y_{n} = |(1 - \mu_n)(\tilde{f}_{n-1} - \mathbb{E}[f_1]) + \mu_n (f_n - \mathbb{E}[f_1])|^2 \\
= (1 - \mu_n)^2 Y_{n-1} + \mu_n^2 (f_n - \mathbb{E}[f_1])^2 + (1 - \mu_n)\mu_n(\tilde{f}_{n-1} - \mathbb{E}[f_1])(f_n - \mathbb{E}[f_1]).
\]

So, taking expectation with respect to \( \{f_j\}_{j=1}^{n-1} \), it follows that

\[
\mathbb{E}_{n-1}[Y_{n}] = (1 - \mu_n)^2 Y_{n-1} + \mu_n^2 [E_{n-1}[(f_n - \mathbb{E}[f_1])^2]]
\]

\[
\leq Y_{n-1} + 4B^2 \mu_n^2.
\]

Since \( \sum_n \mu_n^2 < +\infty \), it follows from Lemma A.3 that \( Y_n \) converges almost surely, and this converges almost surely to 0 since \( Y_n \) converges to 0 in probability. In particular, \( f_n \) converges to \( \mathbb{E}[f_1] \) almost surely.

Now we are ready to prove the result of Proposition 3.4.

**Proof of Proposition 3.4.** To prove (i), first note that for each \( \beta \), with probability 1, \( |\bar{P}(t)(\beta) - P(\beta)| \to 0 \) as \( t \to +\infty \), by (iv) of Lemma A.4.

Note that \( \{\bar{P}(t)\}_t \) is equi-continuous: By the mean value theorem there exists \( \xi_{s}^{t} \subset \mathcal{B}_{R} \), where we denote by \( \mathcal{B}_{R} \) the ball of radius \( R \) with \( R \defeq \sup_{x \in \mathcal{X}, \beta \in \mathcal{C}} |\langle x, \beta \rangle| \), such that

\[
|\bar{P}(t)(\beta) - \bar{P}(t)(\beta')| = |\sum_{s=1}^{t} \eta_{s}^{(t)} f_{y_{s}}^{(t)}(\xi_{s}^{t}) x_{s}^{T}(\beta - \beta')|
\]

22
For some fixed point $\xi_0 \in D$, 
\[
|\bar{P}(t)(\beta) - \bar{P}(t)(\beta')| \leq |\sum_{s=1}^{t} \eta_s(t)(f'_{y_s}(\xi_s) - f'_{y_s}(\xi_0))x_s^\top(\beta - \beta')| \\
+ |\sum_{s=1}^{t} \eta_s(t)f'_{y_s}(\xi_0)x_s^\top(\beta - \beta')|
\]
\[
\leq L \sum_{s=1}^{t} \eta_s(t)|\xi_s - \xi_0||x_s||\beta - \beta'|
\]
\[
+ |(\sum_{s=1}^{t} \eta_s(t)f'_{y_s}(\xi_0)x_s)^\top(\beta - \beta')|
\]
where we have used that fact that $f'_{y_s}$ is $L$-Lipschitz. By boundedness of $B_R$ and $\mathcal{X}$, there exists $B$ such that 
\[
L \sum_{s=1}^{t} \eta_s(t)|\xi_s - \xi_0||x_s||\beta - \beta'| \leq B\|\beta - \beta'\| \sum_{s=1}^{t} \eta_s(t) = B\|\beta - \beta'\|.
\]
By Lemma A.4 (iv), we know that with probability 1, 
\[
\sum_{s=1}^{t} \eta_s(t)f'_{y_s}(\xi_0)x_s \rightarrow E[f'(\xi_0)x], \quad t \rightarrow +\infty.
\]
Therefore, $\|\sum_{s=1}^{t} \eta_s(t)f'_{y_s}(\xi_0)x_s\|$ is uniformly bounded, and there exists a constant $C > 0$ such that, with probability 1, 
\[
|\bar{P}(t)(\beta) - \bar{P}(t)(\beta')| \leq C\|\beta - \beta'\|
\]
Hence, by Arzela-Ascoli (see [31, Theorem 11.3.2] or [1]), it follows that $\bar{P}(t)$ converges uniformly to $P$ on compact sets.

To prove (ii), let $\beta^*$ be a minimiser of $P$, then 
\[
P(\beta(t),*) - P(\beta^*) = P(\beta(t),*) - \bar{P}(t)(\beta^*) + \bar{P}(t)(\beta^*) - P(\beta^*)
\]
\[
\leq P(\beta(t),*) - \bar{P}(t)(\beta(t),*) + \bar{P}(t)(\beta^*) - P(\beta^*)
\]
By (i), the right hand side converges to zero as $t \rightarrow +\infty$. Therefore, $\{\beta(t),*\}$ is a minimising sequence. By compactness of the sublevel sets of $\Omega$, there exists a convergent subsequence, and the limit of this subsequence is a minimiser since $\Omega$ is lower-semicontinuous and $f(\cdot,y)$ is continuous. 

To prove (iii), denote $Z(t)(\beta) \overset{\text{def}}{=} -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_s(t)f'(x_s^\top\beta,y_s)x_s$. Note that given $\beta(t) \in \text{argmin}_\beta \bar{P}(t)(\beta)$, $\beta^*(t) = Z(t)(\beta(t))$. By the triangle inequality, 
\[
\|Z(t)(\beta(t)) - E[-\frac{1}{\lambda} f'(x^\top \beta^*, y)x]\|_\infty \leq \|Z(t)(\beta(t)) - Z(t)(\beta^*)\|_\infty
\]
\[
+ \|Z(t)(\beta^*) - E[-\frac{1}{\lambda} f'(x^\top \beta^*, y)x]\|_\infty
\]
By Lemma A.4 (iv), we have that $\|Z(t)(\beta^*) - E[-\frac{1}{\lambda} f'(x^\top \beta^*, y)x]\|_\infty \rightarrow 0$ as $t \rightarrow +\infty$. To bound the first term on the RHS, let $\theta_s(t) = f'(x_s^\top \beta(t), y_s)$ and $\theta^*_s = f'(x_s^\top \beta^*, y_s)$. Note that by strong convexity of $D(t)$ (c.f. the proof of Lemma 3.6), we have 
\[
\frac{1}{2\ell} \sum_{s=1}^{t} \eta_s(t)|\theta_s(t) - \theta^*_s|^2 \leq \bar{D}(t)(\theta(t)) - \bar{D}(t)(\theta^*) \leq \bar{P}(t)(\beta^*) - \bar{D}(t)(\beta^*),
\]
which again converges to zero by Lemma A.4 (iv) and optimality of $\beta^*$. 

\[
\square
\]

23
Proof of Proposition 3.5. Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $\{(x_s, y_s)\}_{s \leq t}$. Observe that

$$-\frac{1}{\lambda} \sum_{s=1}^{t} \eta_s^{(t)} \theta_s x_s - Z^*$$

$$= -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_s^{(t)} (f'_{y_s}(x_s^\top \beta_s)x_s - \mathbb{E}[f'_{y_s}(x_s^\top \beta_s)x_s | \mathcal{F}_{s-1}])$$

$$+ \sum_{s=1}^{t} \eta_s^{(t)} (\frac{-1}{\lambda} \mathbb{E}[f'_{y_s}(x_s^\top \beta_s)x_s | \mathcal{F}_{s-1}] - Z^*)$$

$$= \frac{1}{\lambda} \sum_{s=1}^{t} \eta_s^{(t)} (f'_{y_s}(x_s^\top \beta_s)x_s - \mathbb{E}[f'_{y_s}(x_s^\top \beta_s)x_s | \mathcal{F}_{s-1}])$$

$$+ \sum_{s=1}^{t} \eta_s^{(t)} (\frac{-1}{\lambda} \mathbb{E}[f'_y(x^\top \beta_s)x] - Z^*).$$

Therefore,

$$\| -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_s^{(t)} \theta_s x_s - Z^* \|_{\infty} \leq \frac{1}{\lambda} \| \sum_{s=1}^{t} \eta_s^{(t)} z_s \|_{\infty} + \frac{CL}{\lambda} \sum_{s=1}^{t} \eta_s^{(t)} \| \beta_s - \beta^* \|$$

where $z_s = f'_{y_s}(x_s^\top \beta_s)x_s - \mathbb{E}[f'_{y_s}(x_s^\top \beta_s)x_s | \mathcal{F}_{s-1}]$, and the constant $C > 0$ comes from the assumption that $x \in \mathcal{X}$ which is compact.

Fix $k \in [n]$, and define

$$Y^k_n \overset{d}{=} \sum_{s=1}^{n} \eta_s^{(t)} (z_s)_k$$

This is a martingale with bounded difference $|Y^k_n - Y^k_{n-1}| = \eta_n^{(t)}|(z_n)_k| \leq 2B\eta_n^{(t)}$. By Azuma-Hoeffding inequality,

$$\mathbb{P}(|Y^k_n| \geq v) \leq 2 \exp \left( -\frac{v^2}{8B^2 \sum_{s=1}^{t} (\eta_s^{(t)})^2} \right)$$

Therefore, by the union bound,

$$\mathbb{P}(\max_k |Y^k_n| \geq v) \leq 2n \exp \left( -\frac{v^2}{8B^2 \sum_{s=1}^{t} (\eta_s^{(t)})^2} \right).$$

The RHS converges to 0 as $t \to +\infty$ by (iii) of Lemma A.4.

Finally, $\sum_{s=1}^{t} \eta_s^{(t)} \| \beta_s - \beta^* \| \to 0$ provided that $\| \beta_s - \beta^* \| \to 0$: For any $m \leq t$:

$$\sum_{s=1}^{m} \eta_s^{(t)} \| \beta_s - \beta^* \| = \sum_{s=1}^{m} \eta_s^{(t)} \| \beta_s - \beta^* \| + \sum_{s=m+1}^{t} \eta_s^{(t)} \| \beta_s - \beta^* \|.$$

For any $\epsilon > 0$, there exists $m$ such that the second term on the RHS is at most $\epsilon/2$ since $\sum_{s=1}^{t} \eta_s^{(t)} = 1$ and $\| \beta_s - \beta^* \| \to 0$ as $s$ increases. For the first term on the RHS, since $\| \beta_s - \beta^* \|$ is uniformly bounded and for any fixed $m$, $\sum_{s=1}^{m} \eta_s^{(t)} \to 0$ as $t \to +\infty$, it is clear that for any $\epsilon > 0$, there exists $t_0$ such that for all $t \geq t_0$, $\sum_{s=1}^{t} \eta_s^{(t)} \| \beta_s - \beta^* \| \to 0$. It therefore follows that $Z_t - Z^*$ converges to 0 in probability.

To conclude almost sure convergence, we apply Lemma A.3. Let $Y_t = |Z_t - Z^*|^2$, then,

$$\mathbb{E}_{t-1}[Y_t] = (1 - \mu_t)Y_{t-1} + \mu_t^2 \mathbb{E}_{t-1} \left(-\frac{1}{\lambda} \theta_t x_t - Z^* \right)^2$$

$$+ (1 - \mu_t) \mu_t Y_{t-1} \mathbb{E}_{t-1} \left(-\frac{1}{\lambda} \theta_t x_t - Z^* \right).$$
Recall that \( Z^* = -\frac{1}{\lambda}E_{(x,y)}[f'(x^\top \beta^*, y)x] \) and \( \|\beta_s\| \) is uniformly bounded, so there exists \( B, B' > 0 \) such that
\[
E_{t-1}[Y_t] \leq (1 - \mu_t)^2 Y_{t-1} + \mu_t^2 B \\
+ (1 - \mu_t)\mu_t Y_{t-1} - E_{(x,y)}[(f'(x^\top \beta_t, y) - f'(x^\top \beta^*, y))x] \\
\leq (1 - \mu_t)^2 Y_{t-1} + \mu_t^2 B + (1 - \mu_t)\mu_t Y_{t-1} - \beta_t - \beta^* \\
= (1 - \mu_t)(1 - \mu_t + \mu_t\|\beta_t - \beta^*\|)Y_{t-1} + \mu_t^2 B.
\]
Since \( \beta_t \to \beta^* \), for \( t \) sufficiently large, \( 1 - \mu_t + \mu_t\|\beta_t - \beta^*\| \in (0, 1) \), so we can apply Lemma A.3 to conclude that \( Y_t \) converges almost surely to 0.

For (ii), let \( \beta^* \) be a minimizer of \( P \). We can establish in the same way as above that
\[
\sum_{s=1}^{t} \eta_s f^*_y(\theta_s) \to E[f^*_y(f'(x^\top \beta^*))], \quad t \to +\infty,
\]
and
\[
|\tilde{P}^{(t)}(\beta_t) - P(\beta^*)| \leq |\tilde{P}^{(t)}(\beta_t) - \tilde{P}(\beta_t)| + |\tilde{P}(\beta_t) - P(\beta^*)|
\]
which converges to 0 as \( t \to +\infty \) since we have uniform convergence of \( \tilde{P}^{(t)} \) to \( P \) by Proposition 3.4 and \( \beta_t \to \beta^* \). Finally, since \( -\frac{1}{\lambda} \sum_{s=1}^{t} \eta_s(\theta_s x_s) \to Z^* \), and \( \Omega^* \) is lower semicontinuous, we have that
\[
\liminf_{t \to +\infty} \Omega^*\left(-\frac{1}{\lambda} \sum_{s=1}^{t} \eta_s(\theta_s x_s)\right) \geq \Omega^*(Z^*).
\]
Therefore,
\[
\lim_{t \to +\infty} \tilde{P}^{(t)}(\beta_t) - \tilde{D}^{(t)}((\theta_s)_{s \leq t}) \leq P(\beta^*) - E[f^*_y(f'(x^\top \beta^*))] - \Omega^*(Z^*) = 0,
\]
and we conclude the proof. \( \square \)

A.4 Proofs for Section 3.2

Proof of Lemma 3.6. Since \( f_{y_s} \) is \( L \)-Lipschitz smooth, it follows that \( f^*_y \) is \( 1/L \)- strongly convex,
\[
\frac{1}{2L}||\theta_s - \theta_s^*||^2 \leq -f^*_y(\theta_s^*) + f^*_y(\theta_s) - (f^*_y)'(\theta_s^*)(\theta_s - \theta_s^*) \\
\leq -f^*_y(\theta_s) + f^*_y(\theta_s) - (f^*_y)'(\theta_s^*)(a_t\theta_s - \theta_s^*) \\
- (1-a_t)(f^*_y)'(\theta_s^*)\theta_s
\] \hspace{1cm} (A.2)
where \( a_t = \min(1, 1/\Omega^D(\tilde{Z}_t)) \) is such that \( a_t(\theta_s)_{s \leq t} \in \mathcal{K}_{\lambda,y(t)} \), the dual constraint set defined in (2.3). Since \( \theta_s^* \) is a dual optimal point, by Fermat’s rule
\[
-\sum_{s \leq t} \eta_s(\theta_s^*) = (a_t\theta_s - \theta_s^*) \leq 0.
\]
Therefore, multiplying \( (A.2) \) by \( \eta_s(\theta_s) \) and summing from \( s = 1, \ldots, t \), we obtain
\[
\frac{1}{2L} \sum_{s=1}^{t} \eta_s(\theta_s - \theta_s^*) \leq \sum_{s=1}^{t} \eta_s\left(-f^*_y(\theta_s^*) + f^*_y(\theta_s)\right) \\
- (1-a_t)\sum_{s=1}^{t} \eta_s(\theta_s^*) = 0.
\] \hspace{1cm} (A.3)
Note that by optimality of $\theta^{(t),*}$, $\sum_{s=1}^{t} \eta_{s}^{(t)} (-f_{y_{s}}^{*}(\theta_{s}^{(t),*})) \leq \bar{P}^{(t)}(\beta)$ for all $\beta \in \mathbb{R}^{d}$. Moreover, letting $\beta^{(t),*} \in \text{argmin}_{\beta} \bar{P}^{(t)}(\beta)$, we have $\theta_{s}^{(t),*} = f_{y_{s}}^{*}(x_{s}^{\top} \beta^{(t),*})$, and using the fact that $(f_{y_{s}}^{*})' \circ f_{y_{s}}^{*} = \text{Id}$, we have
\[
\sum_{s=1}^{t} \eta_{s}^{(t)} (f_{y_{s}}^{*})'((\theta_{s}^{(t),*})) \theta_{s} = \sum_{s=1}^{t} \eta_{s}^{(t)} (f_{y_{s}}^{*})'((f_{y_{s}}^{*})(x_{s}^{\top} \beta^{(t),*}))) \theta_{s} = \sum_{s=1}^{t} \eta_{s}^{(t)} \theta_{s} x_{s}^{\top} \beta^{(t),*}.
\]

By optimality of $\beta^{(t),*}$, we have
\[
\Omega(\beta^{(t),*}) \leq \frac{1}{\lambda} \sum_{s=1}^{t} \eta_{s}^{(t)} f_{y_{s}}(0).
\]

Plugging these estimates back into (A.3) yields, for any $\beta \in \mathbb{R}^{n}$,
\[
\frac{1}{2E} \sum_{s=1}^{t} \eta_{s}^{(t)} |\theta_{s} - \theta_{s}^{(t),*}|^2 \leq \text{Gap}_{t}(\beta) + \sum_{s=1}^{t} \eta_{s}^{(t)} f_{y_{s}}(0) (\Omega_{g}(\bar{Z}_{t}) - 1)_{+} \quad (A.4)
\]
since $(1 - a_{t}) \Omega_{g}(\bar{Z}_{t}) = (\Omega_{g}(\bar{Z}_{t}) - 1)_{+}$. Finally, by the Cauchy-Schwarz inequality,
\[
\Omega_{g}^{D} \left( \frac{1}{\lambda} \sum_{s=1}^{t} \eta_{s}^{(t)} (\theta_{s} x_{s} - \theta_{s}^{(t),*} x_{s}) \right)
= \frac{1}{\lambda} \sup_{\Omega_{g}(z) \leq 1} \langle \sum_{s=1}^{t} \eta_{s}^{(t)} (\theta_{s} x_{s} - \theta_{s}^{(t),*} x_{s}), z \rangle
\leq \frac{1}{\lambda} \sqrt{\sum_{s=1}^{t} \eta_{s}^{(t)} |\theta_{s} - \theta_{s}^{(t),*}|^2} \sup_{\Omega_{g}(z) \leq 1} \sqrt{\sum_{s=1}^{t} \eta_{s}^{(t)} |\langle x_{s}, z \rangle|^2}
\leq \frac{1}{\lambda} \sqrt{\sum_{s=1}^{t} \eta_{s}^{(t)} |\theta_{s} - \theta_{s}^{(t),*}|^2} \sqrt{\sum_{s=1}^{t} \eta_{s}^{(t)} \Omega_{g}^{D}(x_{s})^2}
\]
and the result follows by combining this with (A.4).

\[\square\]

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