HOMOGENIZATION OF VARIATIONAL PROBLEMS 
IN MANIFOLD VALUED BV-SPACES

Jean-François BABADJIAN*
Laboratoire Jean Kuntzmann
Université Joseph Fourier
BP 53
38041 Grenoble Cedex 9, France.
babadjia@imag.fr

Vincent MILLOT
Université Paris Diderot - Paris 7
CNRS, UMR 7598 Laboratoire Jacques-Louis Lions
F-75005 Paris, France.
millot@math.jussieu.fr

Abstract. This paper extends the result of [9] on the homogenization of integral functionals with linear growth defined for Sobolev maps taking values in a given manifold. Through a Γ-convergence analysis, we identify the homogenized energy in the space of functions of bounded variation. It turns out to be finite for BV-maps with values in the manifold. The bulk and Cantor parts of the energy involve the tangential homogenized density introduced in [9], while the jump part involves an homogenized surface density given by a geodesic type problem on the manifold.

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1. Introduction

In this paper we extend our previous result [9] concerning the homogenization of integral functionals with linear growth involving manifold valued mappings. More precisely, we are interested in energies of the form

\[ \int_{\Omega} f \left( \frac{x}{\varepsilon}, \nabla u \right) \, dx, \quad u : \Omega \to M \subset \mathbb{R}^d, \tag{1.1} \]

where \( \Omega \subset \mathbb{R}^N \) is a bounded open set, \( f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty) \) is a periodic integrand in the first variable with linear growth in the second one, and \( M \) is a smooth submanifold. Our main goal is to find an effective description of such energies as \( \varepsilon \to 0 \). To this aim we perform a Γ-convergence analysis which is an appropriate approach to study asymptotics in variational problems (see [21] for a detailed description of this subject). For energies with superlinear growth, the most general homogenization result has been obtained independently in [15,38] in the nonconstrained case, and in [9] in the setting of manifold valued maps.

The functional in (1.1) is naturally defined for maps in the Sobolev class \( W^{1,1} \). However if one wants to apply the Direct Method in the Calculus of Variations, it becomes necessary to extend the original energy to a larger class of functions (possibly singular) in which the existence of minimizers is ensured. In the nonconstrained case, this class is exactly the space of functions of bounded variation and the problem of finding an integral representation for the extension, the so-called “relaxed functional”, has been widely investigated, see e.g., [33,29,20,7,8,28,5,26,27,14] and [13,22] concerning homogenization in BV-spaces.

Many models from material science involve vector fields taking their values into a manifold. This is for example the case in the study of equilibria for liquid crystals, in ferromagnetism or

* Current adress: CMAP, Ecole Polytechnique, 91128 Palaiseau, France. E-mail: babadjian@cmap.polytechnique.fr
for magnetostrictive materials. It then became necessary to understand the behaviour of integral functionals of the type (1.1) under this additional constraint. In the framework of Sobolev spaces, it was the object of [19,3,9]. For $\varepsilon$ fixed, the complete analysis in the linear growth case has been performed in [2] assuming that the manifold is the unit sphere of $\mathbb{R}^d$. Using a different approach, the arbitrary manifold case has been recently treated in [37] where a further isotropy assumption on the integrand is made. We will present in the Appendix the analogue result to [2] for a general integrand and a general manifold.

We finally mention that the topology of $\mathcal{M}$ does not play an important role here. This is in contrast with a slightly different problem originally introduced in [18,11], where the starting energy is assumed to be finite only for smooth maps. In this direction, some recent results in the linear growth case can be found in [31,32] where the study is performed within the framework of Cartesian Currents [30]. When the manifold $\mathcal{M}$ is topologically nontrivial, it shows the emergence in the relaxation process of non local effects essentially related to the non density of smooth maps (see [10,12]).

Throughout this paper we consider a compact and connected smooth submanifold $\mathcal{M}$ of $\mathbb{R}^d$ without boundary. The classes of maps we are interested in are defined as

$$BV(\Omega; \mathcal{M}) := \{ u \in BV(\Omega; \mathbb{R}^d) : u(x) \in \mathcal{M} \text{ for } L^N\text{-a.e. } x \in \Omega \},$$

and $W^{1,1}(\Omega; \mathcal{M}) = BV(\Omega; \mathcal{M}) \cap W^{1,1}(\Omega; \mathbb{R}^d)$. For a smooth $\mathcal{M}$-valued map, it is well known that first order derivatives belong to the tangent space of $\mathcal{M}$, and this property has a natural extension to $BV$-maps with values in $\mathcal{M}$, see Lemma 2.1.

The function $f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)$ is assumed to be a Carathéodory integrand satisfying

(H$_1$) for every $\xi \in \mathbb{R}^{d \times N}$ the function $f(\cdot, \xi)$ is 1-periodic, i.e. if $\{e_1, \ldots, e_N\}$ denotes the canonical basis of $\mathbb{R}^N$, one has $f(y + e_i, \xi) = f(y, \xi)$ for every $i = 1, \ldots, N$ and $y \in \mathbb{R}^N$;

(H$_2$) there exist $0 < \alpha \leq \beta < +\infty$ such that

$$\alpha |\xi| \leq f(y, \xi) \leq \beta (1 + |\xi|) \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } \xi \in \mathbb{R}^{d \times N};$$

(H$_3$) there exists $L > 0$ such that

$$|f(y, \xi) - f(y, \xi')| \leq L |\xi - \xi'| \quad \text{for a.e. } y \in \mathbb{R}^N \text{ and all } \xi, \xi' \in \mathbb{R}^{d \times N}.$$

For $\varepsilon > 0$, we define the functionals $\mathcal{F}_\varepsilon : L^1(\partial^1; \mathbb{R}^d) \to [0, +\infty]$ by

$$\mathcal{F}_\varepsilon(u) := \begin{cases} \int_{\Omega} f \left( \frac{x}{\varepsilon}, \nabla u \right) \, dx & \text{if } u \in W^{1,1}(\Omega; \mathcal{M}), \\ +\infty & \text{otherwise.} \end{cases}$$

We have proved in [9] the following representation result on $W^{1,1}(\Omega; \mathcal{M})$.

**Theorem 1.1 ([9]).** Let $\mathcal{M}$ be a compact and connected smooth submanifold of $\mathbb{R}^d$ without boundary, and $f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a Carathéodory function satisfying (H$_1$) to (H$_3$). Then the family $\{\mathcal{F}_\varepsilon\}_{\varepsilon > 0}$ $\Gamma$-converges for the strong $L^1$-topology at every $u \in W^{1,1}(\Omega; \mathcal{M})$ to $\mathcal{F}_\text{hom} : W^{1,1}(\Omega; \mathcal{M}) \to [0, +\infty)$, where

$$\mathcal{F}_\text{hom}(u) := \int_{\Omega} T_{\text{hom}}(u, \nabla u) \, dx,$$

and $T_{\text{hom}}$ is the tangentially homogenized energy density defined for every $s \in \mathcal{M}$ and $\xi \in [T_s(\mathcal{M})]_N$ by

$$T_{\text{hom}}(s, \xi) = \lim_{t \to +\infty} \inf_{\varphi} \left( \int_{(0,t)^N} f(y, \xi + \nabla \varphi(y)) \, dy : \varphi \in W^{1,\infty}_0((0,t)^N; T_s(\mathcal{M})) \right). \quad (1.2)$$

Note that the previous theorem is not really satisfactory since the domain of the $\Gamma$-limit is obviously larger than the Sobolev space $W^{1,1}(\Omega; \mathcal{M})$. In view of the studies performed in [31,37], the domain is exactly given by $BV(\Omega; \mathcal{M})$. Under the additional (standard) assumption,
Let $\Omega$ be a generic bounded open subset of $\mathbb{R}^N$. We write $\mathcal{A}(\Omega)$ for the family of all open subsets of $\Omega$, and $\mathcal{B}(\Omega)$ for the $\sigma$-algebra of all Borel subsets of $\Omega$. We also consider a countable subfamily $\mathcal{R}(\Omega)$ of $\mathcal{A}(\Omega)$ made of all finite unions of cubes with rational edge length centered at rational points of $\mathbb{R}^N$. Given $\nu \in \mathbb{S}^{N-1}$, $Q_\nu$ stands for an open unit cube in $\mathbb{R}^N$ centered at the origin with two of its faces orthogonal to $\nu$. $Q_\nu(x_0, \rho) := x_0 + \rho Q_\nu$. Similarly $Q := (-1/2,1/2)^N$ is the unit cube in $\mathbb{R}^N$ and $Q(x_0, \rho) := x_0 + \rho Q$. We denote by $h^\infty$ the recession function of a generic scalar function $h$, i.e.,

$$h^\infty(\xi) := \limsup_{t \to +\infty} \frac{h(t\xi)}{t}.$$
The space of vector valued Radon measures in $\Omega$ with finite total variation is denoted by $\mathcal{M}(\Omega; \mathbb{R}^m)$. We shall follow [6] for the standard notation on functions of bounded variation. We only recall Alberti Rank One Theorem which states that for $|D^c u|$-a.e. $x \in \Omega$,

$$A(x) := \frac{dD^c u}{d|D^c u|}(x)$$

is a rank one matrix.

In this paper, we are interested in Sobolev and $BV$ maps taking their values into a given manifold. We consider a connected smooth submanifold $M$ of $\mathbb{R}^d$ without boundary. The tangent space of $M$ at $s \in M$ is denoted by $T_s(M)$, $\text{co}(M)$ stands for the convex hull of $M$, and $\pi_1(M)$ is the fundamental group of $M$.

It is well known that if $u \in W^{1,1}(\Omega; M)$, then $\nabla u(x) \in [T_{u(x)}(M)]^N$ for $\mathcal{L}^N$-a.e. $x \in \Omega$. The analogue statement for $BV$-maps is given in Lemma 2.1 below.

**Lemma 2.1.** For every $u \in BV(\Omega; M)$,

$$\tilde{u}(x) \in M \text{ for every } x \in \Omega \setminus S_u; \tag{2.1}$$

$$u^\pm(x) \in M \text{ for every } x \in J_u; \tag{2.2}$$

$$\nabla u(x) \in [T_{u(x)}(M)]^N \text{ for } \mathcal{L}^N\text{-a.e. } x \in \Omega; \tag{2.3}$$

$$A(x) := \frac{dD^c u}{d|D^c u|}(x) \in [T_{\tilde{u}(x)}(M)]^N \text{ for } |D^c u|\text{-a.e. } x \in \Omega. \tag{2.4}$$

**Proof.** We first show (2.1). By definition of the space $BV(\Omega; M)$, $u(y) \in M$ for a.e. $y \in \Omega$. Therefore for any $x \in \Omega \setminus S_u$, we have $|u(y) - \tilde{u}(x)| \geq \text{dist}(\tilde{u}(x), M)$ for a.e. $y \in \Omega$. By definition of $S_u$, this yields $\text{dist}(\tilde{u}(x), M) = 0$, i.e., $\tilde{u}(x) \in M$. Arguing as for the approximate limit points, one obtains (2.2).

Now it remains to prove (2.3) and (2.4). We introduce the function $\Phi : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\Phi(s) = \chi(\delta^{-1}\text{dist}(s, M)^2) \text{dist}(s, M)^2,$$

where $\chi \in C^\infty(\mathbb{R}; [0, 1])$ with $\chi(t) = 1$ for $|t| \leq 1$, $\chi(t) = 0$ for $|t| \geq 2$, and $\delta > 0$ is small enough so that $\Phi \in C^1(\mathbb{R}^d)$. Note that for every $s \in M$, $\Phi(s) = 0$ and

$$\text{Ker} \nabla \Phi(s) = T_s(M). \tag{2.5}$$

By the Chain Rule formula in $BV$ (see, e.g., [6, Theorem 3.96]), $\Phi \circ u \in BV(\Omega)$ and

$$D(\Phi \circ u) = \nabla \Phi(u) \nabla u \mathcal{L}^d \nabla \Phi(u) \nabla u \mathcal{L}^d \Omega + \nabla \Phi(u) D^c u + (\Phi(u^+) - \Phi(u^-)) \otimes \nu_u \mathcal{H}^{N-1} J_u$$

$$= \nabla \Phi(u) \nabla u \mathcal{L}^d \nabla \Phi(u) \nabla u \mathcal{L}^d \Omega + \nabla \Phi(u) A|D^c u|,$$

thanks to (2.2). On the other hand, $\Phi \circ u = 0$ a.e. in $\Omega$ since $u(x) \in M$ for a.e. $x \in \Omega$. Therefore we have that $D(\Phi \circ u) \equiv 0$. Since $\mathcal{L}^d \Omega$ and $|D^c u|$ are mutually singular measures, we infer that $\nabla \Phi(u(x)) \nabla u(x) = 0$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and $\nabla \Phi(u(x)) A(x) = 0$ for $|D^c u|$-a.e. $x \in \Omega$. Hence (2.3) and (2.4) follow from (2.5) together with (2.1).

In [10,12], density results of smooth functions between manifolds into Sobolev spaces have been established. In the following theorem, we summarize these results only in $W^{1,1}$. Let $\mathcal{S}$ be the family of all finite unions of subsets contained in a $(N-2)$-dimensional submanifold of $\mathbb{R}^N$.

**Theorem 2.1.** Let $\mathcal{D}(\Omega; M) \subset W^{1,1}(\Omega; M)$ be defined by

$$\mathcal{D}(\Omega; M) := \begin{cases} W^{1,1}(\Omega; M) \cap C^\infty(\Omega; M) & \text{if } \pi_1(M) = 0, \\ \{u \in W^{1,1}(\Omega; M) \cap C^\infty(\Omega \setminus \Sigma; M) \text{ for some } \Sigma \in \mathcal{S} & \text{otherwise}. \end{cases}$$
Then $\mathcal{D}(\Omega; \mathcal{M})$ is dense in $W^{1,1}(\Omega; \mathcal{M})$ for the strong $W^{1,1}(\Omega; \mathbb{R}^d)$-topology.

We now present a useful projection technique (taken from [23] for $\mathcal{M} = \mathbb{S}^{d-1}$). It was first introduced in [34,35], and makes use of an averaging device going back to [25]. We sketch the proof for the convenience of the reader.

**Proposition 2.1.** Let $\mathcal{M}$ be a compact connected $m$-dimensional smooth submanifold of $\mathbb{R}^d$ without boundary, and let $v \in W^{1,1}(\Omega; \mathbb{R}^d) \cap C^\infty(\Omega \setminus \Sigma; \mathbb{R}^d)$ for some $\Sigma \in \mathcal{S}$ such that $v(x) \in \text{co}(\mathcal{M})$ for a.e. $x \in \Omega$. Then there exists $w \in W^{1,1}(\Omega; \mathcal{M})$ satisfying $w = v$ a.e. in \( \{ x \in \Omega : v(x) \in \mathcal{M} \} \) and

\[
\int_\Omega |\nabla w| \, dx \leq C \int_\Omega |\nabla v| \, dx,
\]

for some constant $C > 0$ which only depends on $d$ and $\mathcal{M}$.

**Proof.** According to [35, Lemma 6.1] (which holds for $p = 1$), there exist a compact Lipschitz polyhedral set $X \subset \mathbb{R}^d$ of codimension greater or equal to 2, and a locally Lipschitz map $\pi : \mathbb{R}^d \setminus X \to \mathcal{M}$ such that

\[
\int_{B_R(0)} |\nabla \pi(s)| \, ds < +\infty \quad \text{for every } R < +\infty.
\]

Moreover, in a neighborhood of $\mathcal{M}$ the mapping $\pi$ is smooth of constant rank equal to $m$.

We argue as in the proof of [35, Theorem 6.2]. Let $B$ be an open ball in $\mathbb{R}^d$ containing $\mathcal{M} \cup X$, and let $\delta > 0$ small enough so that the nearest point projection on $\mathcal{M}$ is a well defined smooth mapping in the $\delta$-neighborhood of $\mathcal{M}$. Fix $\sigma < \inf\{ \delta \, \text{dist}(\text{co(\mathcal{M})}, \partial B) \}$ small enough, and for $a \in B^d(0, \sigma)$ we define the translates $B_a := a + B$ and $X_a := a + X$, and the projection $\pi_a : B_a \setminus X_a \to \mathcal{M}$ by $\pi_a(s) := \pi(s - a)$. Since $\pi$ has full rank and is smooth in a neighborhood of $\mathcal{M}$, by the Inverse Function Theorem the number

\[
\Lambda := \sup_{a \in B^d(0, \sigma)} \text{Lip}(\pi_a|_\mathcal{M})^{-1}
\]

is finite and only depends on $\mathcal{M}$. Using Sard's lemma, one can show that $\pi_a \circ v \in W^{1,1}(\Omega; \mathcal{M})$ for $\mathcal{L}^d$-a.e. $a \in B^d(0, \sigma)$. Then Fubini's theorem together with the Chain Rule formula yields

\[
\int_{B_R(0)} \int_\Omega |\nabla (\pi_a \circ v)(x)| \, d\mathcal{L}^N(x) \, d\mathcal{L}^d(a) \leq \int_\Omega |\nabla v(x)| \left( \int_{B_R(0)} |\nabla \pi(v(x) - a)| \, d\mathcal{L}^d(a) \right) \, d\mathcal{L}^N(x) \leq \left( \int_B |\nabla \pi(s)| \, d\mathcal{L}^d(s) \right) \left( \int_\Omega |\nabla v(x)| \, d\mathcal{L}^N(x) \right).
\]

Therefore we can find $a \in B_R(0, \sigma)$ such that

\[
\int_\Omega |\nabla (\pi_a \circ v)| \, dx \leq C \mathcal{L}^d \left( B(0, \sigma) \right)^{-1} \int_\Omega |\nabla v| \, dx,
\]

where we used (2.7). To conclude, it suffices to set $w := (\pi_a|_\mathcal{M})^{-1} \circ \pi_a \circ v$, and (2.6) arises as a consequence of (2.8) and (2.9).

\[\square\]

### 3. Properties of homogenized energy densities

In this section we present the main properties of the energy densities $Tf_{\text{hom}}$ and $\vartheta_{\text{hom}}$, defined in (1.2) and (1.3). In particular we will prove that $\vartheta_{\text{hom}}$ is well defined in the sense that the limit in (1.3) exists.
3.1. The tangentially homogenized bulk energy

We start by considering the bulk energy density $T_{f_{\text{hom}}}$ defined in (1.2). As in [9] we first construct a new energy density $g : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ satisfying

$$g(\cdot, s, \xi) = f(\cdot, \xi) \quad \text{and} \quad g_{\text{hom}}(s, \xi) = T_{f_{\text{hom}}}(s, \xi) \quad \text{for} \ s \in \mathcal{M} \ \text{and} \ \xi \in [T_s(\mathcal{M})]^N.$$  

Hence upon extending $T_{f_{\text{hom}}}$ by $g_{\text{hom}}$ outside the set $\{(s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N} : s \in \mathcal{M}, \xi \in [T_s(\mathcal{M})]^N\}$, we will tacitly assume $T_{f_{\text{hom}}}$ to be defined over the whole $\mathbb{R}^d \times \mathbb{R}^{d \times N}$. We proceed as follows.

For $s \in \mathcal{M}$ we denote by $P_s : \mathbb{R}^d \to T_s(\mathcal{M})$ the orthogonal projection from $\mathbb{R}^d$ into $T_s(\mathcal{M})$, and we set

$$P_s(\xi) := (P_s(\xi_1), \ldots, P_s(\xi_N)) \quad \text{for} \ \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^{d \times N}.$$  

For $\delta_0 > 0$ fixed, let $U := \{s \in \mathbb{R}^d : \text{dist}(s, \mathcal{M}) < \delta_0\}$ be the $\delta_0$-neighborhood of $\mathcal{M}$. Choosing $\delta_0 > 0$ small enough, we may assume that the nearest point projection $\Pi : U \to \mathcal{M}$ is a well defined Lipschitz mapping. Then the map $s \in U \mapsto P_{\Pi(s)}$ is Lipschitz. Now we introduce a cut-off function $\chi \in C^\infty_c(\mathbb{R}^d; [0, 1])$ such that $\chi(t) = 1$ if $\text{dist}(s, \mathcal{M}) \leq \delta_0/2$, and $\chi(s) = 0$ if $\text{dist}(s, \mathcal{M}) \geq 3\delta_0/4$, and we define

$$P_s(\xi) := \chi(s)P_{\Pi(s)}(\xi) \quad \text{for} \ (s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}.$$  

Given the Carathéodory integrand $f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)$ satisfying assumptions $(H_1)$ to $(H_3)$, we construct the new integrand $g : \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty)$ as

$$g(y, s, \xi) := f(y, P_s(\xi)) + |\xi - P_s(\xi)|.$$  

Lemma 3.1. The integrand $g$ as defined in (3.1) is a Carathéodory function satisfying

$$g(y, s, \xi) = f(y, \xi) \quad \text{and} \quad g^{\infty}(y, s, \xi) = f^{\infty}(y, \xi) \quad \text{for} \ s \in \mathcal{M} \ \text{and} \ \xi \in [T_s(\mathcal{M})]^N,$$  

and

(i) $g$ is 1-periodic in the first variable;
(ii) there exist $0 < \alpha' \leq \beta'$ such that

$$\alpha' |\xi| \leq g(y, s, \xi) \leq \beta' (1 + |\xi|) \quad \text{for every} \ (s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N} \ \text{and a.e.} \ y \in \mathbb{R}^N;$$  

(iii) there exist $C > 0$ and $C' > 0$ such that

$$|g(y, s, \xi) - g(y, s', \xi)| \leq C |s - s'| |\xi|,$$  

$$|g(y, s, \xi) - g(y, s, \xi')| \leq C' |\xi - \xi'|$$  

for every $s, s' \in \mathbb{R}^d$, every $\xi \in \mathbb{R}^{d \times N}$ and a.e. $y \in \mathbb{R}^N$;
(iv) if in addition $(H_4)$ holds, there exists $0 < q < 1$ and $C'' > 0$ such that

$$|g(y, s, \xi) - g^{\infty}(y, s, \xi)| \leq C'' (1 + |\xi|^{1-q})$$  

for every $(s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N}$ and a.e. $y \in \mathbb{R}^N$.

We can now state the properties of $T_{f_{\text{hom}}}$ and the relation between $T_{f_{\text{hom}}}$ and $g_{\text{hom}}$ through the homogenization procedure.

Proposition 3.1. Let $f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a Carathéodory integrand satisfying $(H_1)$ to $(H_3)$. Then the following properties hold:
Proof of Proposition 3.1.

(i) for every \( s \in \mathcal{M} \) and \( \xi \in [T_s(\mathcal{M})]^N \),
\[
T_{\text{hom}}(s, \xi) = g_{\text{hom}}(s, \xi),
\]
where
\[
g_{\text{hom}}(s, \xi) := \lim_{t \to +\infty} \inf \varphi \left\{ \int_{(0,t)^N} g(y, s, \xi + \nabla \varphi(y)) \, dy : \varphi \in W^{1,\infty}_0((0,t)^N; \mathbb{R}^d) \right\}
\]
is the usual homogenized energy density of \( g \) (see, e.g., [16, Chapter 14]);

(ii) the function \( T_{\text{hom}} \) is tangentially quasiconvex, i.e., for all \( s \in \mathcal{M} \) and all \( \xi \in [T_s(\mathcal{M})]^N \),
\[
T_{\text{hom}}(s, \xi) \leq \int_Q T_{\text{hom}}(s, \xi + \nabla \varphi(y)) \, dy
\]
for every \( \varphi \in W^{1,\infty}_0(Q; T_s(\mathcal{M})) \). In particular \( T_{\text{hom}}(s, \cdot) \) is rank one convex;

(iii) there exists \( C > 0 \) such that
\[
\alpha|\xi| \leq T_{\text{hom}}(s, \xi) \leq \beta(1 + |\xi|),
\]
and
\[
|T_{\text{hom}}(s, \xi) - T_{\text{hom}}(s, \xi')| \leq C|\xi - \xi'|
\]
for every \( s \in \mathcal{M} \) and \( \xi, \xi' \in [T_s(\mathcal{M})]^N \);

(iv) there exists \( C_1 > 0 \) such that
\[
|T_{\text{hom}}(s, \xi) - T_{\text{hom}}(s', \xi)| \leq C_1|s - s'|(1 + |\xi|),
\]
for every \( s, s' \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^{d \times N} \). In particular \( T_{\text{hom}} \) is continuous;

(v) if in addition \((H_4)\) holds, there exist \( C_2 > 0 \) and \( 0 < q < 1 \) such that
\[
|T_{\text{hom}}^\infty(s, \xi) - T_{\text{hom}}(s, \xi)| \leq C_2(1 + |\xi|^{1-q}),
\]
for every \( (s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N} \).

Remark 3.1. Observe that, if \( f \) satisfies assumption \((H_3)\), then \( f^\infty \) satisfies \((H_3)\) as well. In particular the function \( f^\infty \) is Carathéodory, 1-periodic in the first variable, and positively 1-homogeneous with respect to the second variable. In view of the growth and coercivity condition \((H_2)\), one gets that
\[
\alpha|\xi| \leq f^\infty(y, \xi) \leq \beta|\xi| \quad \text{for all } \xi \in \mathbb{R}^{d \times N} \text{ and a.e. } y \in \mathbb{R}^N.
\]
Then, as for \( f^\infty \), the function \( g^\infty \) is Carathéodory, 1-periodic in the first variable, and positively 1-homogeneous with respect to the second variable. Moreover,
\[
\alpha'|\xi| \leq g^\infty(y, s, \xi) \leq \beta'|\xi| \quad \text{for every } (s, \xi) \in \mathbb{R}^d \times \mathbb{R}^{d \times N} \text{ and a.e. } y \in \mathbb{R}^N,
\]
and \( g^\infty \) satisfies estimates analogue to (3.4) and (3.5). Hence we may apply classical homogenization results to \( g^\infty \). In addition, in view of (3.2), claim(i) in Proposition 3.1 holds for \( f^\infty \) and \( g^\infty \), and we have
\[
T(f^\infty)_{\text{hom}}(s, \xi) = (g^\infty)_{\text{hom}}(s, \xi) \quad \text{for every } s \in \mathcal{M} \text{ and } \xi \in [T_s(\mathcal{M})]^N.
\]
In particular \( T(f^\infty)_{\text{hom}} \) will be tacitely extended by \( (g^\infty)_{\text{hom}} \).

Proof of Proposition 3.1. The proofs of claims (i)-(iii) can be obtained exactly as in [9, Proposition 2.1] and we shall omit it. It remains to prove (iv) and (v).

Fix \( s, s' \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^{d \times N} \). For any \( \eta > 0 \), we may find \( k \in \mathbb{N} \) and \( \varphi \in W^{1,\infty}_0((0,k)^N; \mathbb{R}^d) \) such that
\[
\int_{(0,k)^N} g(y, s, \xi + \nabla \varphi) \, dy \leq g_{\text{hom}}(s, \xi) + \eta.
\]
We infer from (3.3) that \( \alpha' |\xi| \leq g_{\text{hom}}(s, \xi) \leq \beta'(1 + |\xi|) \) and consequently

\[
\int_{(0,k)^N} |\nabla \varphi| \, dy \leq C(1 + |\xi|), \tag{3.13}
\]

for some constant \( C > 0 \) depending only on \( \alpha' \) and \( \beta' \). Then from (3.7) and (3.4) it follows that

\[
T_{f,\text{hom}}(s', \xi) - T_{f,\text{hom}}(s, \xi) = g_{\text{hom}}(s', \xi) - g_{\text{hom}}(s, \xi) \leq \\
\leq \int_{(0,k)^N} (g(y, s', \xi + \nabla \varphi) - g(y, s, \xi + \nabla \varphi)) \, dy + \eta \leq \\
\leq C|s - s'|\int_{(0,k)^N} |\xi + \nabla \varphi| \, dy + \eta \leq C|s - s'|(1 + |\xi|) + \eta.
\]

We deduce relation (3.10) inverting the roles of \( s \) and \( s' \), and sending \( \eta \) to zero. In particular, we obtain that \( T_{f,\text{hom}} \) is continuous as a consequence of (3.10) and (3.9).

To show (3.11), let us consider sequences \( t_n \nearrow +\infty \), \( k_n \in \mathbb{N} \) and \( \varphi_n \in W_0^{1,\infty}(\mathbb{R}^N; T_s(M)) \) such that

\[
T_{f,\text{hom}}^\infty(s, \xi) = \lim_{n \rightarrow +\infty} \frac{T_{f,\text{hom}}(s, t_n \xi)}{t_n}, \tag{3.14}
\]

and

\[
\int_{(0,k_n)^N} f(y, t_n \xi + t_n \nabla \varphi_n) \, dy \leq T_{f,\text{hom}}(s, t_n \xi) + \frac{1}{n}.
\]

Then (\( H_2 \)) and (3.8) yield

\[
\int_{(0,k_n)^N} |\nabla \varphi_n| \, dy \leq C(1 + |\xi|), \tag{3.15}
\]

for some constant \( C > 0 \) depending only on \( \alpha \) and \( \beta \). Using (\( H_4 \)) and (3.14), we derive that

\[
T_{f,\text{hom}}(s, \xi) - T_{f,\text{hom}}^\infty(s, \xi) \leq \liminf_{n \rightarrow +\infty} \left\{ \int_{(0,k_n)^N} \left| f(y, \xi + \nabla \varphi_n) - f^\infty(y, \xi + \nabla \varphi_n) \right| \, dy + \\
+ \int_{(0,k_n)^N} \left| f^\infty(y, \xi + \nabla \varphi_n) - \frac{f(y, t_n \xi + t_n \nabla \varphi_n)}{t_n} \right| \, dy \right\} \leq \\
\leq \liminf_{n \rightarrow +\infty} \left\{ C \int_{(0,k_n)^N} (1 + |\xi + \nabla \varphi_n|^{1-q}) \, dy + \\
+ \frac{C}{t_n} \int_{(0,k_n)^N} (1 + t_n^{1-q} |\xi + \nabla \varphi_n|^{1-q}) \, dy \right\},
\]

where we have also used the fact that \( f^\infty(y, \cdot) \) is positively homogeneous of degree one in the last inequality. Then (3.15) and Hölder’s inequality lead to

\[
T_{f,\text{hom}}(s, \xi) - T_{f,\text{hom}}^\infty(s, \xi) \leq C(1 + |\xi|^{1-q}). \tag{3.16}
\]

Conversely, given \( k \in \mathbb{N} \) and \( \varphi \in W_0^{1,\infty}(\mathbb{R}^N; T_s(M)) \), we deduce from (\( H_2 \)) that

\[
\frac{f(\cdot, t(\xi + \nabla \varphi(\cdot)))}{t} \leq \beta(1 + |\xi + \nabla \varphi|) \in L^1((0,k)^N)
\]

whenever \( t > 1 \). Then Fatou’s lemma implies

\[
T_{f,\text{hom}}^\infty(s, \xi) \leq \limsup_{t \rightarrow +\infty} \int_{(0,k)^N} \frac{f(y, t \xi + t \nabla \varphi)}{t} \, dy \leq \int_{(0,k)^N} f^\infty(y, \xi + \nabla \varphi) \, dy.
\]

Taking the infimum over all admissible \( \varphi \)'s and letting \( k \rightarrow +\infty \), we infer

\[
T_{f,\text{hom}}^\infty(s, \xi) \leq T(f^\infty)_{\text{hom}}(s, \xi). \tag{3.17}
\]
For $\eta > 0$ arbitrary small, consider $k \in \mathbb{N}$ and $\varphi \in W^{1,\infty}_0((0,k)^N; T_s(M))$ such that

$$
\int_{(0,k)^N} f(y, \xi + \nabla \varphi) \, dy \leq T_{f_{\text{hom}}}(s, \xi) + \eta.
$$

In view of $(H_2)$ and $(3.8)$, it turns out that $(3.13)$ holds with constant $C > 0$ only depending on $\alpha$ and $\beta$. Then it follows from $(3.17)$ that

$$
T_{f_{\text{hom}}}(s, \xi) - T_{f_{\text{hom}}}(s, \xi) \leq (f)_{\text{hom}}(s, \xi) - T_{f_{\text{hom}}}(s, \xi) \leq
$$

$$
\leq \int_{(0,k)^N} |f^\infty(y, \xi + \nabla \varphi) - f(y, \xi + \nabla \varphi)| \, dy + \eta \leq C \int_{(0,k)^N} (1 + |\xi + \nabla \varphi|^{-\eta}) \, dy + \eta,
$$

where we have used $(H_4)$ in the last inequality. Using Hölder’s inequality, relation $(3.13)$ together with the arbitrariness of $\eta$ yields

$$
T_{f_{\text{hom}}}(s, \xi) - T_{f_{\text{hom}}}(s, \xi) \leq C(1 + |\xi|^{1-\eta}).
$$

(3.18)

Gathering $(3.16)$ and $(3.18)$ we conclude the proof of $(3.11)$. \qed

3.2. The homogenized surface energy

We now present the homogenized surface energy density $\vartheta_{\text{hom}}$. We start by introducing some useful notations.

Given $\nu = (\nu_1, \ldots, \nu_N)$ an orthonormal basis of $\mathbb{R}^N$ and $(a, b) \in M \times M$, we denote by

$$Q_\nu := \{a_1 \nu_1 + \ldots + a_N \nu_N : a_1, \ldots, a_N \in (-1/2, 1/2)\},$$

and for $x \in \mathbb{R}^N$, we set $\|x\|_{\nu, \infty} := \sup_{i \in \{1, \ldots, N\}} |x \cdot \nu_i|$, $x_\nu := x \cdot \nu_1$ and $x' := (x \cdot \nu_2) \nu_2 + \ldots + (x \cdot \nu_N) \nu_N$ so that $x$ can be identified to the pair $(x', x_\nu)$. Let $u_{a, b, \nu} : Q_\nu \to M$ be the function defined by

$$u_{a, b, \nu}(x) := \begin{cases} a & \text{if } x_\nu > 0, \\ b & \text{if } x_\nu \leq 0. \end{cases}$$

We introduce the class of functions

$$A_t(a, b, \nu) := \{ \varphi \in W^{1,1}(tQ_\nu; M) : \varphi = u_{a, b, \nu} \text{ on } \partial(tQ_\nu) \}.$$

We have the following result.

**Proposition 3.2.** For every $(a, b, \nu_1) \in M \times M \times S^{N-1}$, there exists

$$
\vartheta_{\text{hom}}(a, b, \nu_1) := \lim_{t \to +\infty} \inf_{\varphi} \left\{ \frac{1}{t^{N-1}} \int_{tQ_\nu} f^\infty(y, \nabla \varphi(y)) \, dy : \varphi \in A_t(a, b, \nu) \right\},
$$

where $\nu = (\nu_1, \ldots, \nu_N)$ is any orthonormal basis of $\mathbb{R}^N$ with first element equal to $\nu_1$ (the limit being independent of such a choice).

The proof of Proposition 3.2 is quite indirect and is based on an analogous result for a similar surface energy density $\vartheta_{\text{hom}}$ (see (3.19) below). We will prove in Proposition 3.3 that the two densities coincide.

Given $a$ and $b \in M$, we introduce the family of geodesic curves between $a$ and $b$ by

$$G(a, b) := \{ \gamma \in C^\infty(\mathbb{R}; M) : \gamma(t) = a \text{ if } t \geq 1/2, \gamma(t) = b \text{ if } t \leq -1/2, \int_{\mathbb{R}} |\dot{\gamma}| \, dt = d_M(a, b) \},$$

where $d_M$ denotes the geodesic distance on $M$. We define for $\varepsilon > 0$ and $\nu = (\nu_1, \ldots, \nu_N)$ an orthonormal basis of $\mathbb{R}^N$,

$$B_\varepsilon(a, b, \nu) := \{ u \in W^{1,1}(Q_\nu; M) : u(x) = \gamma(x_\nu/\varepsilon) \text{ on } \partial \nu \text{ for some } \gamma \in G(a, b) \}.$$
Proposition 3.3. For every \((a, b) \in \mathcal{M} \times \mathcal{M}\) and every orthonormal basis \(\nu = (\nu_1, \ldots, \nu_N)\) of \(\mathbb{R}^N\), there exists the limit

\[
\tilde{\partial}_\text{hom}(a, b, \nu) := \lim_{\varepsilon \to 0} \inf_{u} \left\{ \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon}, \nabla u \right) dx : u \in B_\varepsilon(a, b, \nu) \right\}.
\] (3.19)

Moreover \(\tilde{\partial}_\text{hom}(a, b, \nu)\) only depends on \(a, b\) and \(\nu_1\).

**Proof.** The proof follows the scheme of the one in [17, Proposition 2.2]. We fix \(a, b \in \mathcal{M}\). For every \(\varepsilon > 0\) and every orthonormal basis \(\nu = (\nu_1, \ldots, \nu_N)\) of \(\mathbb{R}^N\), we set

\[
I_\varepsilon(\nu) := I_\varepsilon(a, b, \nu) := \inf \left\{ \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon}, \nabla u \right) dx : u \in B_\varepsilon(a, b, \nu) \right\}.
\]

We divide the proof into several steps.

**Step 1.** Let \(\nu\) and \(\nu'\) be two orthonormal bases of \(\mathbb{R}^N\) with equal first vector, i.e., \(\nu_1 = \nu'_1\). Suppose that \(\nu\) is a rational basis, i.e., for all \(i \in \{1, \ldots, N\}\) there exists \(\gamma_i \in \mathbb{R} \setminus \{0\}\) such that \(v_i := \gamma_i \nu_i \in \mathbb{Z}^N\). Similarly to Step 1 of the proof of [17, Proposition 2.2], we readily obtain that

\[
\limsup_{\varepsilon \to 0} I_\varepsilon(\nu') \leq \liminf_{\varepsilon \to 0} I_\varepsilon(\nu).
\] (3.20)

**Step 2.** Let \(\nu\) and \(\nu'\) be two orthonormal rational bases of \(\mathbb{R}^N\) with equal first vector. By Step 1 we immediately obtain that the limits \(\lim_{\varepsilon \to 0} I_\varepsilon(\nu)\) and \(\lim_{\varepsilon \to 0} I_\varepsilon(\nu')\) exist and are equal.

**Step 3.** We claim that for every \(\sigma > 0\) there exists \(\delta > 0\) (independent of \(a\) and \(b\)) such that if \(\nu\) and \(\nu'\) are two orthonormal bases of \(\mathbb{R}^N\) with \(|\nu_i - \nu'_i| < \delta\) for every \(i = 1, \ldots, N\), then

\[
\liminf_{\varepsilon \to 0} I_\varepsilon(\nu) - K\sigma \leq \liminf_{\varepsilon \to 0} I_\varepsilon(\nu') \leq \limsup_{\varepsilon \to 0} I_\varepsilon(\nu') \leq \limsup_{\varepsilon \to 0} I_\varepsilon(\nu) + K\sigma
\]

where \(K\) is a positive constant which only depends on \(\mathcal{M}, \beta\) and \(N\).

We use the notation \(Q_{\nu, \eta} := (1 - \eta)Q_{\nu}\) where \(0 < \eta < 1\). Let \(\sigma > 0\) be fixed and let \(0 < \eta < 1\) be such that

\[
\eta < \frac{1}{34} \quad \text{and} \quad \max \left\{ 1 - (1 - \eta)^{N-1}, \frac{(1 - \eta)^{N-1}(1 - 2\eta)^{N-1}}{(1 - 3\eta)^{N-1}} - (1 - 2\eta)^{N-1} \right\} < \sigma. \] (3.21)

Consider \(\delta_0 > 0\) (that may be chosen so that \(\delta_0 \leq \eta/(2\sqrt{N})\)) such that for every \(0 < \delta \leq \delta_0\) and every pair \(\nu\) and \(\nu'\) of orthonormal basis of \(\mathbb{R}^N\) satisfying \(|\nu_i - \nu'_i| \leq \delta\) for \(i = 1, \ldots, N\), one has

\[
Q_{\nu, 3\eta} \subset \overline{Q_{\nu', 2\eta}}, \quad Q_{\nu, \eta}, \quad Q_{\nu', 2\eta} \subset Q_{\nu, \eta}, \] (3.22)

and \(\{x \cdot \nu'_1 = 0\} \cap \partial Q_{\nu, \eta} \subset \{x \cdot \nu_1 \leq 1/8\}\).

Given \(\varepsilon > 0\) small, we consider \(u_\varepsilon \in B_\varepsilon(a, b, \nu')\) such that

\[
\int_{Q_{\nu'}} f^\infty \left( \frac{x}{\varepsilon}, \nabla u_\varepsilon \right) dx \leq I_\varepsilon(\nu') + \sigma,
\]

where \(u_\varepsilon(x) = \gamma_\varepsilon(x_{\nu'}/\varepsilon)\) for \(x \in \partial Q_{\nu'}\). Now we construct \(v_\varepsilon \in B_{(1 - 2\eta)\varepsilon}(a, b, \nu)\) satisfying the boundary condition \(v_\varepsilon(x) = \gamma_\varepsilon(x_{\nu'}/(1 - 2\eta)\varepsilon)\) for \(x \in \partial Q_{\nu'}\). Consider \(F_\eta: \mathbb{R}^N \to \mathbb{R}\),

\[
F_\eta(x) := \left( \frac{1 - 2\|x'\|_{\nu'\infty}}{\eta} \right) \frac{x_{\nu'}}{1 - 2\eta} + \left( \frac{\eta - 1 + 2\|x'\|_{\nu'\infty}}{\eta} \right) \frac{x_{\nu'}}{1 - 2\eta},
\]
and define

\[ v_\varepsilon(x) := \begin{cases} 
  u_\varepsilon \left( \frac{x}{1 - 2\eta} \right) & \text{if } x \in Q_{\nu', 2\eta}, \\
  \gamma_\varepsilon \left( \frac{x_{\nu'}}{(1 - 2\eta)\varepsilon} \right) & \text{if } x \in Q_{\nu, \eta} \setminus Q_{\nu', 2\eta}, \\
  a & \text{if } x \in Q_{\nu} \setminus Q_{\nu, \eta} \text{ and } x_{\nu} \geq \frac{1}{4}, \\
  \gamma_\varepsilon \left( \frac{F_\eta(x)}{\varepsilon} \right) & \text{if } x \in A_\eta := \{ x : |x_{\nu}| \leq 1/4 \} \cap (Q_{\nu} \setminus Q_{\nu, \eta}), \\
  b & \text{if } x \in Q_{\nu} \setminus Q_{\nu, \eta} \text{ and } x_{\nu} \leq -1/4.
\]

Hence, thanks the growth condition (3.12), (3.26) and (3.27), we get that

We now estimate these three integrals. First, we easily get that

\[ Q_{\nu, \eta}(1-2\eta) \varepsilon (\nu) \leq \int_{Q_{\nu, \eta}} f^\infty \left( \frac{x}{(1 - 2\eta)\varepsilon}, \nabla v_\varepsilon \right) dx \\
= \int_{Q_{\nu, \eta}} f^\infty \left( \frac{x}{(1 - 2\eta)\varepsilon}, \nabla v_\varepsilon \right) dx + \int_{Q_{\nu, \eta} \setminus Q_{\nu', 2\eta}} f^\infty \left( \frac{x}{(1 - 2\eta)\varepsilon}, \nabla v_\varepsilon \right) dx \\
+ \int_{A_\eta} f^\infty \left( \frac{x}{(1 - 2\eta)\varepsilon}, \nabla v_\varepsilon \right) dx =: I_1 + I_2 + I_3. \tag{3.23} \]

We can check that \( v_\varepsilon \) is well defined for \( \varepsilon \) small enough and that \( v_\varepsilon \in B_{(1-2\eta)\varepsilon}(a, b, \nu) \). Therefore

\[ I_1 = (1 - 2\eta)^{N-1} \int_{Q_{\nu'}} f^\infty \left( \frac{\nu}{\varepsilon}, \nabla u_\varepsilon \right) dy \leq I_\varepsilon(\nu') + \sigma. \tag{3.24} \]

In view of (3.22) we have \( Q_{\nu, \eta} \subset (1 - \eta)(1 - 2\eta)(1 - 3\eta)^{-1} Q_{\nu'} =: D_\eta \). Then we infer from the growth condition (3.12) together with Fubini’s theorem that

\[ I_2 \leq \beta \int_{D_\eta} |\nabla v_\varepsilon| dx = \frac{\beta}{(1 - 2\eta)\varepsilon} \int_{(D_\eta \setminus Q_{\nu, 2\eta} \cap \{ |x_{\nu'}| \leq (1 - 2\eta)\varepsilon / 2 \}} \left| \frac{\gamma_\varepsilon \left( \frac{x_{\nu'}}{(1 - 2\eta)\varepsilon} \right)}{(1 - 2\eta)\varepsilon} \right| dx \\
= \beta \mathcal{H}^{N-1} ((D_\eta \setminus Q_{\nu', 2\eta}) \cap \{ x_{\nu'} = 0 \}) \frac{1}{(1 - 2\eta)\varepsilon} \int_{(1 - 2\eta)\varepsilon / 2}^{1} \left| \frac{t}{(1 - 2\eta)\varepsilon} \right| dt \\
= \beta d_{\mathcal{M}}(a, b) \left( \frac{(1 - \eta)^{N-1}(1 - 2\eta)^{N-1}}{(1 - 3\eta)^{N-1}} - (1 - 2\eta)^{-1} \right). \tag{3.25} \]

Now it remains to estimate \( I_3 \). To this purpose we first observe that (3.22) yields

\[ \| \nabla F_\eta \|_{L^\infty(A_\eta; \mathbb{R}^N)} \leq C, \tag{3.26} \]

for some absolute constant \( C > 0 \), and

\[ |\nabla F_\eta(x) \cdot \nu_1| \geq 1 \quad \text{for a.e. } x \in A_\eta. \tag{3.27} \]

Hence, thanks the growth condition (3.12), (3.26) and (3.27), we get that

\[ I_3 \leq \beta \int_{A_\eta} |\nabla v_\varepsilon| dx \leq \frac{C \beta}{\varepsilon} \int_{A_\eta} \left| \gamma_\varepsilon \left( \frac{F_\eta(x)}{\varepsilon} \right) \right| dx \leq \frac{C \beta}{\varepsilon} \int_{A_\eta} \left| \gamma_\varepsilon \left( \frac{F_\eta(x)}{\varepsilon} \right) \right| |\nabla F_\eta(x) \cdot \nu_1| dx = \]

\[ = \frac{C \beta}{\varepsilon} \int_{A_\eta'} \left| \frac{1}{\varepsilon} \int_{-1/4}^{1/4} \left| \gamma_\varepsilon \left( \frac{F_\eta(t\nu_1 + x')}{\varepsilon} \right) \right| |\nabla F_\eta(t\nu_1 + x') \cdot \nu_1| dt \right| d\mathcal{H}^{N-1}(x'), \]

where we have set \( A_\eta' := A_\eta \cap \{ x_{\nu} = 0 \} \), and used Fubini’s theorem in the last equality. Changing variables \( s = (1/\varepsilon)F_\eta(t\nu_1 + x') \), we obtain that for \( \mathcal{H}^{N-1}\text{-a.e. } x' \in A_\eta' \),

\[ \frac{1}{\varepsilon} \int_{-1/4}^{1/4} \left| \gamma_\varepsilon \left( \frac{F_\eta(t\nu_1 + x')}{\varepsilon} \right) \right| |\nabla F_\eta(t\nu_1 + x') \cdot \nu_1| dt \leq \int_{\mathbb{R}} |\gamma_\varepsilon(s)| ds = d_{\mathcal{M}}(a, b). \]
Consequently,
\[ I_3 \leq C\beta \mathcal{H}^{N-1}(A_0) \mathbf{d}_M(a, b) = C\beta (1 - (1 - \eta)^{N-1}) \mathbf{d}_M(a, b). \] (3.28)

In view of (3.23), (3.21) and estimates (3.24), (3.25) and (3.28), we conclude that
\[ I_{(1-\eta)\varepsilon}(\nu) \leq I_\varepsilon(\nu') + K\sigma, \]
where \( K = 1 + \beta \Delta (1 + C), \Delta \) is the diameter of \( \mathcal{M} \) and \( C \) is the constant given by (3.26). Finally, letting \( \varepsilon \to 0 \) we derive
\[ \liminf_{\varepsilon \to 0} I_\varepsilon(\nu) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(\nu') + K\sigma, \]
and \( \limsup_{\varepsilon \to 0} I_\varepsilon(\nu) \leq \limsup_{\varepsilon \to 0} I_\varepsilon(\nu') + K\sigma. \)

The symmetry of the roles of \( \nu \) and \( \nu' \) allows us to invert them, thus concluding the proof of Step 3.

**Step 4.** Let \( \nu \) and \( \nu' \) be two orthonormal bases of \( \mathbb{R}^N \) with equal first vector. Similarly to Step 4 of the proof of [17, Proposition 2.2], by Steps 2 and 3 we readily obtain that the limits \( \lim_{\varepsilon \to 0} I_\varepsilon(\nu) \) and \( \lim_{\varepsilon \to 0} I_\varepsilon(\nu') \) exist and are equal.

**Proof of Proposition 3.2.** We use the notation of the previous proof. Given \( \varepsilon > 0 \) and an orthonormal basis \( \nu = (\nu_1, \ldots, \nu_N) \) of \( \mathbb{R}^N \), we set
\[ J_\varepsilon(\nu) = J_\varepsilon(a, b, \nu) := \inf \left\{ \int_{Q_\nu} f^\infty \left( \frac{x}{\varepsilon}, \nabla u \right) dx : u \in \mathcal{A}_1(a, b, \nu) \right\} \]
\[ = \inf \left\{ \varepsilon^{N-1} \int_{4Q_\nu} f^\infty(y, \nabla \varphi) dy : \varphi \in \mathcal{A}_1/\varepsilon(a, b, \nu) \right\}. \]

We claim that
\[ \lim_{\varepsilon \to 0} J_\varepsilon(\nu) = \lim_{\varepsilon \to 0} I_\varepsilon(\nu). \] (3.29)

For \( 0 < \varepsilon < 1 \) we set \( \tilde{\varepsilon} = \varepsilon/(1 - \varepsilon) \), and we consider \( u_\varepsilon \in \mathcal{B}_\varepsilon(a, b, \nu) \) satisfying
\[ \int_{Q_\nu} f^\infty \left( \frac{x}{\tilde{\varepsilon}}, \nabla u_\varepsilon \right) dx \leq I_\varepsilon(\nu) + \varepsilon, \]
where \( u_\varepsilon(x) = \gamma_\varepsilon(x_{\nu}/\tilde{\varepsilon}) \) if \( x \in \partial Q_\nu \), for some \( \gamma_\varepsilon \in \mathcal{G}(a, b) \). We define for every \( x \in Q_\nu \),
\[ v_\varepsilon(x) := \begin{cases} u_\varepsilon \left( \frac{x}{1 - \varepsilon} \right) & \text{if } x \in Q_{\nu, \varepsilon}, \\ \gamma_\varepsilon \left( \frac{x_{\nu}}{1 - 2 \|x'\|_{\nu, \infty}} \right) & \text{otherwise}. \end{cases} \]

One may check that \( v_\varepsilon \in \mathcal{A}_1(a, b, \nu) \), and hence
\[ J_\varepsilon(\nu) \leq \int_{Q_\nu} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx = \int_{Q_{\nu, \varepsilon}} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx + \int_{Q_\nu \setminus Q_{\nu, \varepsilon}} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx = I_1 + I_2. \]

We now estimate these two integrals. First, we have
\[ I_1 = (1 - \varepsilon)^{N-1} \int_{Q_\nu} f^\infty \left( \frac{y}{\varepsilon}, \nabla u_\varepsilon \right) dy \leq (1 - \varepsilon)^{N-1} (I_\varepsilon(\nu) + \varepsilon). \] (3.30)

In view of the growth condition (3.12),
\[ I_2 \leq \beta \int_{Q_\nu \setminus Q_{\nu, \varepsilon}} \left| \gamma_\varepsilon \left( \frac{x_{\nu}}{1 - 2 \|x'\|_{\nu, \infty}} \right) \right| \left( \frac{1}{1 - 2 \|x'\|_{\nu, \infty}} + \frac{|x_{\nu}| \|\nabla (\|x'\|_{\nu, \infty})\|}{(1 - 2 \|x'\|_{\nu, \infty})^2} \right) dx \]
\[ \leq 2\beta \int_{(Q_\nu \setminus Q_{\nu, \varepsilon}) \cap \{ \|x_{\nu}| \leq (1 - 2 \|x'\|_{\nu, \infty})/2 \}} \left| \gamma_\varepsilon \left( \frac{x_{\nu}}{1 - 2 \|x'\|_{\nu, \infty}} \right) \right| \left( \frac{1}{1 - 2 \|x'\|_{\nu, \infty}} \right) dx. \]
where we have used the facts that \( \dot{\gamma}(x_{\nu}/(1 - 2\|x\|_{\infty})) = 0 \) in the set \( \{x_{\nu} > (1 - 2\|x\|_{\infty})/2\} \) and \( \|\nabla(x\|_{\infty})\|_{L^\infty(Q_{\nu};\mathbb{R}^N)} \leq 1 \). Setting \( Q'_{\nu} = Q_{\nu} \cap \{x_{\nu} = 0\} \) and \( Q'_{\nu,\epsilon} = Q_{\nu,\epsilon} \cap \{x_{\nu} = 0\} \), we infer from Fubini's theorem that

\[
I_2 \leq 2\beta \int_{Q'_{\nu}} \left( \int_{(1 - 2\|x\|_{\infty})/2}^{(1 - 2\|x\|_{\infty})/2} \dot{\gamma}(t) \left( \frac{1}{1 - 2\|x\|_{\infty}} \right)^{\epsilon} dt \right) d\mathcal{H}^{N-1}(x') \leq 2\beta \mathcal{H}^{N-1}(Q'_{\nu}) \tau_{\mathcal{M}}(a, b) \leq 2\beta \tau_{\mathcal{M}}(a, b)(1 - (1 - \epsilon)^{N-1}).
\]

(3.31)

In view of the estimates (3.30) and (3.31) obtained for \( I_1 \) and \( I_2 \), we derive that

\[
\lim_{\epsilon \to 0} J_{\epsilon}(\nu) = \lim_{\epsilon \to 0} I_{\epsilon}(\nu).
\]

(3.32)

Conversely, given \( 0 < \epsilon < 1 \), we consider \( \tilde{u}_{\epsilon} \in A_1(a, b, \nu) \) such that

\[
\int_{Q_{\nu}} f_{\infty} \left( \frac{x}{\epsilon}, \nabla \tilde{u}_{\epsilon} \right) dx \leq J_{\epsilon}(\nu) + \epsilon,
\]

and \( \gamma \in \mathcal{G}(a, b) \) fixed. We define for \( x \in Q_{\nu}, \)

\[
w_{\epsilon}(x) := \begin{cases} \tilde{u}_{\epsilon} \left( \frac{x}{1 - \epsilon} \right) & \text{if } x \in Q_{\nu,\epsilon}, \\ \gamma \left( \frac{x_{\nu}}{(1 - \epsilon)(2\|x\|_{\infty} - 1 + \epsilon)} \right) & \text{otherwise}. \end{cases}
\]

We can check that \( w_{\epsilon} \in B_{(1 - \epsilon)x\epsilon}(a, b, \nu) \), and arguing as previously we infer that

\[
I_{(1 - \epsilon)x\epsilon}(\nu) \leq \int_{Q_{\nu,\epsilon}} f_{\infty} \left( \frac{x}{(1 - \epsilon)x\epsilon}, \nabla w_{\epsilon} \right) dx + \int_{Q_{\nu}\setminus Q_{\nu,\epsilon}} f_{\infty} \left( \frac{x}{(1 - \epsilon)x\epsilon}, \nabla w_{\epsilon} \right) dx \\
\leq (1 - \epsilon)^{N-1} (J_{\epsilon}(\nu) + \epsilon) + 2\beta \tau_{\mathcal{M}}(a, b)(1 - (1 - \epsilon)^{N-1}).
\]

Consequently, \( \lim_{\epsilon \to 0} I_{\epsilon}(\nu) \leq \liminf_{\epsilon \to 0} J_{\epsilon}(\nu) \), which, together with (3.32), completes the proof of Proposition 3.2.

We now state the following properties of the surface energy density.

**Proposition 3.4.** The function \( \vartheta_{\text{hom}} \) is continuous on \( \mathcal{M} \times \mathcal{M} \times S^{N-1} \) and there exist constants \( C_1 > 0 \) and \( C_2 > 0 \) such that

\[
|\vartheta_{\text{hom}}(a_1, b_1, \nu_1) - \vartheta_{\text{hom}}(a_2, b_2, \nu_1)| \leq C_1(|a_1 - a_2| + |b_1 - b_2|),
\]

(3.33)

and

\[
\vartheta_{\text{hom}}(a_1, b_1, \nu_1) \leq C_2|a_1 - b_1|
\]

(3.34)

for every \( a_1, b_1, a_2, b_2 \in \mathcal{M} \) and \( \nu_1 \in S^{N-1} \).

**Proof.** We use the notation of the previous proof. By Proposition 3.2 together with steps 3 and 4 of the proof of Proposition 3.3, we get that \( \vartheta_{\text{hom}}(a, b, \cdot) \) is continuous on \( S^{N-1} \) uniformly with respect to \( a \) and \( b \). Hence it is enough to show that (3.33) holds to get the continuity of \( \vartheta_{\text{hom}} \).

**Step 1.** We start with the proof of (3.33). Fix \( \nu_1 \in S^{N-1} \) and let \( \nu = (\nu_1, \nu_2, \ldots, \nu_N) \) be any orthonormal basis of \( \mathbb{R}^N \). For every \( \epsilon > 0 \), let \( \bar{\epsilon} := \epsilon/(1 - \epsilon) \) and consider \( \gamma_{\bar{\epsilon}} \in \mathcal{G}(a_1, b_1) \) and \( u_{\bar{\epsilon}} \in B_{\bar{\epsilon}}(a_1, b_1, \nu) \) such that \( u_{\bar{\epsilon}}(x) = \gamma_{\bar{\epsilon}}(x_{\nu}/\bar{\epsilon}) \) for \( x \in \partial Q_{\nu} \) and

\[
\int_{Q_{\nu}} f_{\infty} \left( \frac{x}{\bar{\epsilon}}, \nabla u_{\bar{\epsilon}} \right) dx \leq I_{\bar{\epsilon}}(a_1, b_1, \nu) + \epsilon.
\]
We shall now carefully modify \( u_\varepsilon \) in order to get another function \( v_\varepsilon \in A_1(a_2, b_2, \nu) \). We will proceed as in the proofs of Propositions 3.2 and 3.3. Let \( \gamma_a \in \mathcal{G}(a_2, a_1) \) and \( \gamma_b \in \mathcal{G}(b_2, b_1) \), and define

\[
v_\varepsilon(x) := \begin{cases} 
\frac{x}{1 - \varepsilon} & \text{if } x \in Q_{\nu, \varepsilon}, \\
\frac{x - x_\nu}{1 \varepsilon - \nu} & \text{if } x \in A_1, \\
\frac{2||x||_{\nu, \infty} - 1}{\varepsilon} + \frac{1}{2} & \text{if } x \in A_2 := (Q_{\nu} \setminus Q_{\nu, \varepsilon}) \cap \{x_\nu \geq \varepsilon / 2\}, \\
\frac{2||x||_{\nu, \infty} - 1}{\varepsilon} + \frac{1}{2} & \text{if } x \in A_3 := (Q_{\nu} \setminus Q_{\nu, \varepsilon}) \cap \{x_\nu \leq -\varepsilon / 2\}, \\
\frac{2||x'||_{\nu, \infty} - 1}{2x_\nu} + \frac{1}{2} & \text{if } x \in A_4 := \left\{ 0 < x_\nu \leq \frac{\varepsilon}{2}, \frac{1}{2} - x_\nu \leq ||x'||_{\nu, \infty} < \frac{1}{2} \right\}, \\
\frac{1 - 2||x'||_{\nu, \infty} + \frac{1}{2}}{2x_\nu} & \text{if } x \in A_5 := \left\{ -\frac{\varepsilon}{2} < x_\nu \leq 0, \frac{1}{2} + x_\nu \leq ||x'||_{\nu, \infty} < \frac{1}{2} \right\},
\end{cases}
\]

with

\[
A_1 := \left\{ \frac{1 - \varepsilon}{2} \leq ||x'||_{\nu, \infty} < \frac{1}{2} \text{ and } |x_\nu| \leq -||x'||_{\nu, \infty} + \frac{1}{2} \right\}.
\]

One may check that the function \( v_\varepsilon \) has been constructed in such a way that \( v_\varepsilon \in A_1(a_2, b_2, \nu) \), and thus

\[
J_\varepsilon(a_2, b_2, \nu) \leq \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx.
\]

Arguing exactly as in the proof of Proposition 3.3, one can show that

\[
\int_{Q_{\nu, \varepsilon}} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx \leq I_\varepsilon(a_1, b_1, \nu) + \varepsilon,
\]

and

\[
\int_{Q_{\nu, \varepsilon}} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx \leq C_1 d_M(a_1, b_1)(1 - (1 - \varepsilon)^{N - 1}).
\]

Now we only estimate the integrals over \( A_2 \) and \( A_4 \), the ones over \( A_3 \) and \( A_5 \) being very similar. Define the Lipschitz function \( F_\varepsilon : \mathbb{R}^N \to \mathbb{R} \) by

\[
F_\varepsilon(x) := \frac{2||x||_{\nu, \infty} - 1}{\varepsilon} + \frac{1}{2}.
\]

Using the growth condition (3.12) together with Fubini’s theorem, and the fact that \( A_2 \subset F_\varepsilon^{-1}((-1/2, 1/2)) \), we derive

\[
\int_{A_2} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx \leq \int_{A_2} \left| \gamma_a(F_\varepsilon(x)) \right| \left| \nabla F_\varepsilon(x) \right| dx \leq \beta \int_{F_\varepsilon^{-1}((-1/2, 1/2))} \left| \gamma_a(F_\varepsilon(x)) \right| \left| \nabla F_\varepsilon(x) \right| dx \leq \beta \int_{F_\varepsilon^{-1}((-1/2, 1/2))} \left| \gamma_a(t) \right| H^{N - 1}(F_\varepsilon^{-1}\{t\}) dt,
\]

where we used the Coarea formula in the last inequality. We observe that for every \( t \in (-1/2, 1/2) \), \( F_\varepsilon^{-1}\{t\} = \partial Q_{\nu, \varepsilon(1/2)} \) so that \( H^{N - 1}(F_\varepsilon^{-1}\{t\}) \leq H^{N - 1}(\partial Q) \). Therefore

\[
\int_{A_2} f^\infty \left( \frac{x}{\varepsilon}, \nabla v_\varepsilon \right) dx \leq \beta H^{N - 1}(\partial Q) d_M(a_1, a_2).
\]

Define now \( G : \mathbb{R}^N \setminus \{x_\nu = 0\} \to \mathbb{R} \) by

\[
G(x) := \frac{2||x'||_{\nu, \infty} - 1}{2x_\nu} + \frac{1}{2}.
\]
The growth condition (3.12) and Fubini’s theorem yield
\[
\int_{A_4} f^\infty \left( \frac{x}{\varepsilon}, \nabla u_x \right) \, dx \leq \\
\leq \beta \int_0^{\varepsilon/2} \left( \int_{G(\cdot,x_{\nu})^{-1}((-1/2,1/2))} |\gamma_\alpha(G(x',x_{\nu}))| \, |\nabla G(x',x_{\nu})| \, d\mathcal{H}^{N-1}(x') \right) \, dx_{\nu}.
\]
As \( |\nabla_x G(x)| = 1/x_{\nu} \) and \( |\nabla_{x_{\nu}} G(x)| \leq 1/x_{\nu} \) for a.e. \( x \in A_4 \), it follows that \( |\nabla G(x)| \leq 2|\nabla_{x_{\nu}} G(x)| \) for a.e. \( x \in A_4 \). Hence
\[
\int_{A_4} f^\infty \left( \frac{x}{\varepsilon}, \nabla u_x \right) \, dx \leq \\
\leq 2\beta \int_0^{\varepsilon/2} \left( \int_{G(\cdot,x_{\nu})^{-1}((-1/2,1/2))} |\gamma_\alpha(G(x',x_{\nu}))| \, |\nabla_x G(x',x_{\nu})| \, d\mathcal{H}^{N-1}(x') \right) \, dx_{\nu}.
\]
For every \( x_{\nu} \in (0,\varepsilon/2) \) the function \( G(\cdot,x_{\nu}) : \mathbb{R}^{N-1} \to \mathbb{R} \) is Lipschitz, and thus the Coarea formula implies
\[
\int_{A_4} f^\infty \left( \frac{x}{\varepsilon}, \nabla u_x \right) \, dx \leq 2\beta \int_0^{\varepsilon/2} \left( \int_{-1/2}^{1/2} |\gamma_\alpha(t)| \, \mathcal{H}^{N-2}\{x' : G(x',x_{\nu}) = t\} \, dt \right) \, dx_{\nu} \\
\leq C\varepsilon \, d_M(a_1,a_2), \quad (3.39)
\]
where we used as previously the estimate \( \mathcal{H}^{N-2}\{x' : G(x',x_{\nu}) = t\} \leq \mathcal{H}^{N-2}(\partial(\varepsilon^{1/2} \cdot \frac{1}{\varepsilon})^{N-1}) \). Gathering (3.35) to (3.39) and considering the analogous estimates for the integrals over \( A_3 \) and \( A_5 \) (with \( b_1 \) and \( b_2 \) instead of \( a_1 \) and \( a_2 \)), we infer that
\[
J_\varepsilon(a_2,b_2,\nu) \leq \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon}, \nabla u_x \right) \, dx \leq I_\varepsilon(a_1,b_1,\nu) + C(\varepsilon + d_M(a_1,a_2) + d_M(b_1,b_2)).
\]
Taking the limit as \( \varepsilon \to 0 \), we get in light of Propositions 3.2 and 3.3 that
\[
\vartheta_{\text{hom}}(a_2,b_2,\nu) \leq \vartheta_{\text{hom}}(a_1,b_1,\nu) + C(d_M(b_1,b_2) + d_M(a_1,a_2)).
\]
Since the geodesic distance on \( \mathcal{M} \) is equivalent to the Euclidian distance, we conclude, possibly exchanging the roles of \( (a_1,b_1) \) and \( (a_2,b_2) \), that (3.33) holds.

**Step 2.** We now prove (3.34). Given an arbitrary orthonormal basis \( \nu = (\nu_1,\ldots,\nu_N) \) of \( \mathbb{R}^N \), let \( \gamma \in G(a_1,b_1) \) and define \( u_x(x) := \gamma(x_{\nu}/\varepsilon) \). Obviously \( u_x \in B_\varepsilon(a_1,b_1,\nu) \). Using (3.29) together with the growth condition (3.12) satisfied by \( f^\infty \), we derive that
\[
\vartheta_{\text{hom}}(a_1,b_1,\nu_1) \leq \liminf_{\varepsilon \to 0} \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon}, \nabla u_x \right) \, dx \leq \liminf_{\varepsilon \to 0} \beta \int_{Q_{\nu}} |\gamma \left( \frac{x \cdot \nu_1}{\varepsilon} \right) | \, dx = \beta d_M(a_1,b_1).
\]
Then (3.34) follows from the equivalence between \( d_M \) and the Euclidian distance.

4. Localization and integral representation on partitions

In this section we first show that the \( \Gamma \)-limit defines a measure. Then we prove an abstract representation on partitions in sets of finite perimeter. This two facts will allow us to obtain the upper bound on the \( \Gamma \)-limit in the next section.

4.1. Localization

We consider an arbitrary given sequence \( \{\varepsilon_n\} \searrow 0^+ \) and we localize the functionals \( \{F_{\varepsilon_n}\}_{n \in \mathbb{N}} \) on the family \( \mathcal{A}(\Omega) \), i.e., for every \( u \in L^1(\Omega;\mathbb{R}^d) \) and every \( A \in \mathcal{A}(\Omega) \), we set
\[
F_{\varepsilon_n}(u,A) := \begin{cases} 
\int_A f \left( \frac{x}{\varepsilon_n}, \nabla u \right) \, dx & \text{if } u \in W^{1,1}(A;\mathcal{M}), \\
+\infty & \text{otherwise}.
\end{cases}
\]
Next we define for \( u \in L^1(\Omega; \mathbb{R}^d) \) and \( A \in \mathcal{A}(\Omega) \),
\[
\mathcal{F}(u, A) := \inf_{\{u_n\}} \left\{ \liminf_{n \to +\infty} \mathcal{F}_{\varepsilon_n}(u_n, A) : u_n \to u \text{ in } L^1(A; \mathbb{R}^d) \right\}.
\]

Note that \( \mathcal{F}(u, \cdot) \) is an increasing set function for every \( u \in L^1(\Omega; \mathbb{R}^d) \) and that \( \mathcal{F}(\cdot, A) \) is lower semicontinuous with respect to the strong \( L^1(A; \mathbb{R}^d) \)-convergence for every \( A \in \mathcal{A}(\Omega) \).

Since \( L^1(A; \mathbb{R}^d) \) is separable, [21, Theorem 8.5] and a diagonalization argument bring the existence of a subsequence (still denoted \( \{\varepsilon_n\} \)) such that \( \mathcal{F}(\cdot, A) \) is the \( \Gamma \)-limit of \( \mathcal{F}_{\varepsilon_n}(\cdot, A) \) for the strong \( L^1(A; \mathbb{R}^d) \)-topology for every \( A \in \mathcal{R}(\Omega) \) (or \( A = \Omega \)).

We have the following locality property of the \( \Gamma \)-limit which, in the \( BV \) setting, parallels [9, Lemma 3.1].

**Lemma 4.1.** For every \( u \in BV(\Omega; \mathcal{M}) \), the set function \( \mathcal{F}(u, \cdot) \) is the restriction to \( \mathcal{A}(\Omega) \) of a Radon measure absolutely continuous with respect to \( \mathcal{L}^N + |Du| \).

**Proof.** Let \( u \in BV(\Omega; \mathcal{M}) \) and \( A \in \mathcal{A}(\Omega) \). By Theorem 3.9 in [6], there exists a sequence \( \{u_n\} \subset W^{1,1}(A; \mathbb{R}^d) \cap C^\infty_c(A; \mathbb{R}^d) \) such that \( u_n \to u \text{ in } L^1(A; \mathbb{R}^d) \) and \( \int_A |\nabla u_n| \, dx \to |Du|(A) \). Moreover, \( u_n(x) \in co(\mathcal{M}) \) for a.e. \( x \in A \) and every \( n \in \mathbb{N} \). Applying Proposition 2.1 to \( u_n \), we obtain a new sequence \( \{w_n\} \subset W^{1,1}(A; \mathcal{M}) \) satisfying
\[
\int_A |\nabla w_n| \, dx \leq C_* \int_A |\nabla u_n| \, dx,
\]
for some constant \( C_* > 0 \) depending only on \( \mathcal{M} \) and \( d \). From construction of \( w_n \), we have that \( w_n \to u \) in \( L^1(A; \mathbb{R}^d) \). Taking \( \{w_n\} \) as admissible sequence, we deduce in light of the growth condition \( (H_2) \) that
\[
\mathcal{F}(u, A) \leq \beta \left( \mathcal{L}^N(A) + C_* |Du|(A) \right).
\]

We now prove that
\[
\mathcal{F}(u, A) \leq \mathcal{F}(u, B) + \mathcal{F}(u, A \setminus \overline{C})
\]
for every \( A, B \) and \( C \in \mathcal{A}(\Omega) \) satisfying \( \overline{C} \subset B \subset A \). Then the measure property of \( \mathcal{F}(u, \cdot) \) can be obtained as in the proof of [9, Lemma 3.1] with minor modifications. For this reason, we shall omit it.

Let \( R \in \mathcal{R}(\Omega) \) such that \( C \subset R \subset B \) and consider \( \{u_n\} \subset W^{1,1}(R; \mathcal{M}) \) satisfying \( u_n \to u \) in \( L^1(R; \mathbb{R}^d) \) and
\[
\lim_{n \to +\infty} \mathcal{F}_{\varepsilon_n}(u_n, R) = \mathcal{F}(u, R).
\]
Given \( \eta > 0 \) arbitrary, there exists a sequence \( \{v_n\} \subset W^{1,1}(A \setminus \overline{C}; \mathcal{M}) \) such that \( v_n \to u \) in \( L^1(A \setminus \overline{C}; \mathbb{R}^d) \) and
\[
\liminf_{n \to +\infty} \mathcal{F}_{\varepsilon_n}(v_n, A \setminus \overline{C}) \leq \mathcal{F}(u, A \setminus \overline{C}) + \eta.
\]
By Theorem 2.1, we can assume without loss of generality that \( u_n \in \mathcal{D}(R; \mathcal{M}) \) and \( v_n \in \mathcal{D}(A \setminus \overline{C}; \mathcal{M}) \). Let \( L := \text{dist}(C, \partial R) \) and define for every \( i \in \{0, \ldots, n\} \),
\[
R_i := \left\{ x \in R : \text{dist}(x, \partial R) > \frac{1 + L}{n} \right\}.
\]
Given \( i \in \{0, \ldots, n - 1\} \), let \( S_i := R_i \setminus R_{i+1} \) and consider a cut-off function \( \zeta_i \in C^\infty_c(\Omega; [0, 1]) \) satisfying \( \zeta_i(x) = 1 \) for \( x \in R_{i+1} \), \( \zeta_i(x) = 0 \) for \( x \in \Omega \setminus R_i \) and \( |\nabla \zeta_i| \leq 2n/L \). Define
\[
z_{n,i} := \zeta_i u_n + (1 - \zeta_i) v_n \in W^{1,1}(A; \mathbb{R}^d).
\]
If \( \pi_1(\mathcal{M}) \neq 0 \), \( z_{n,i} \) is smooth in \( A \setminus \Sigma_{n,i} \) with \( \Sigma_{n,i} \in \mathcal{S} \), while \( z_{n,i} \) is smooth in \( A \) if \( \pi_1(\mathcal{M}) = 0 \). Observe that \( z_{n,i}(x) \in co(\mathcal{M}) \) for a.e. \( x \in A \) and actually, \( z_{n,i} \) fails to be \( \mathcal{M} \)-valued exactly in
the set $S_i$. To get an admissible sequence, we project $z_{n,i}$ on $\mathcal{M}$ using Proposition 2.1. It yields a sequence $\{w_{n,i}\} \subset W^{1,1}(A; \mathcal{M})$ satisfying $w_{n,i} = z_{n,i}$ a.e. in $A \setminus S_i$, 
$$
\int_A |w_{n,i} - u| \, dx \leq \int_A |z_{n,i} - u| \, dx + C \mathcal{L}^N(S_i), 
$$
for some constant $C > 0$ depending only on the diameter of $\text{co}(\mathcal{M})$, and 
$$
\int_{S_i} |\nabla w_{n,i}| \, dx \leq C_* \int_{S_i} |\nabla z_{n,i}| \, dx \leq C_* \int_{S_i} \left( |\nabla u_n| + |\nabla v_n| + \frac{n}{2L} |u_n - v_n| \right) \, dx.
$$
Arguing exactly as in the proof of [9, Lemma 3.1], we now find an index $i_n \in \{0, \ldots, n - 1\}$ such that 
$$
\mathcal{F}_{\varepsilon_n}(w_{n,i_n}, A) \leq \mathcal{F}_{\varepsilon_n}(u_n, B) + \mathcal{F}_{\varepsilon_n}(v_n, A \setminus \overline{C}) + C_0 \int_{R \setminus \overline{C}} |u_n - v_n| \, dx + \frac{C_0}{n} \sup_{k \in \mathbb{N}} \int_{R \setminus \overline{C}} (1 + |\nabla u_k| + |\nabla v_k|) \, dx, 
$$
for some constant $C_0$ independent of $n$.

A well known consequence of the Coarea formula yields (see, e.g., [24, Lemma 3.2.34]),
$$
\mathcal{L}^N(S_n) = \int_{i_nL/n}^{(i_n+1)L/n} \mathcal{H}^{N-1} \left( \{ x \in R : \text{dist}(x, \partial R) = t \} \right) \, dt \to 0 \quad \text{as} \quad n \to +\infty. 
$$
As a consequence of (4.3) and (4.5), $w_{n,i_n} \rightharpoonup u$ in $L^1(\mathcal{A}; \mathbb{R}^d)$. Taking the lim inf in (4.4) and using (4.1) together with (4.2), we derive 
$$
\mathcal{F}(u, A) \leq \mathcal{F}(u, B) + \mathcal{F}(u, A \setminus \overline{C}) + \eta \leq \mathcal{F}(u, B) + \mathcal{F}(u, A \setminus \overline{C}) + \eta.
$$
The conclusion follows from the arbitrariness of $\eta$.

Remark 4.1. In view of Lemma 4.1, for every $u \in BV(\Omega; \mathcal{M})$, the set function $\mathcal{F}(u, \cdot)$ can be uniquely extended to a Radon measure on $\Omega$. Such a measure is given by 
$$
\mathcal{F}(u, B) := \inf \{ \mathcal{F}(u, A) : A \in \mathcal{A}(\Omega), B \subset A \},
$$
for every $B \in \mathcal{B}(\Omega)$ (see, e.g., [6, Theorem 1.53]).

4.2. Integral representation on partitions

Besides the locality of $\mathcal{F}(u, \cdot)$, another key point of the analysis is to prove an abstract integral representation on partitions. Similarly to e.g. [17, Lemma 3.7], using $(H_1)$ we easily obtain the translation invariance property of the $\Gamma$-limit, the proof of which is omitted.

Lemma 4.2. For every $u \in BV(\Omega; \mathcal{M})$, every $A \in \mathcal{A}(\Omega)$ and every $y \in \mathbb{R}^N$ such that $y + A \subset \Omega$, we have 
$$
\mathcal{F}(\tau_y u, y + A) = \mathcal{F}(u, A),
$$
where $(\tau_y u)(x) := u(x - y)$.

We are now in position to prove the integral representation of the $\Gamma$-limit on partitions.

Proposition 4.1. There exists a unique function $K : \mathcal{M} \times \mathcal{M} \times S^{N-1} \to [0, +\infty)$ continuous in the last variable and such that 
(i) $K(a, b, \nu) = K(b, a, -\nu)$ for every $(a, b, \nu) \in \mathcal{M} \times \mathcal{M} \times S^{N-1}$,
(ii) for every finite subset $T$ of $\mathcal{M}$,
$$
\mathcal{F}(u, S) = \int_S K(u^+, u^-, \nu) \, d\mathcal{H}^{N-1}, 
$$
for every $u \in BV(\Omega; T)$ and every Borel subset $S$ of $\Omega \cap S_u$.

Proof. It follows the argument of [17, Proposition 4.2] that is based on the general result [4, Theorem 3.1], on account to Lemmas 4.1, 4.2 and Remark 4.1. We omit any further details. \qed
5. The upper bound

We now address the $\Gamma$-lim sup inequality. The upper bound on the diffuse part will be obtained using an extension of the relaxation result of [2] (see Theorem 7.1 in the Appendix) together with the partial representation of the $\Gamma$-limit already established in $W^{1,1}$ (see Theorem 1.1). The estimate of the jump part relies on the integral representation on partitions in sets of finite perimeter stated in Proposition 4.1.

In view of the measure property of the $\Gamma$-limit, we may write for every $u \in BV(\Omega; \mathcal{M})$,
\[ F(u, \Omega) = F(u, \Omega \setminus S_u) + F(u, \Omega \cap S_u). \]
(5.1)
Hence the desired upper bound $F(u, \Omega) \leq F_{\text{hom}}(u)$ will follow estimating separately the two terms in the right handside of (5.1).

Lemma 5.1. For every $u \in BV(\Omega; \mathcal{M})$, we have
\[ F(u, \Omega \setminus S_u) \leq \int_\Omega T_{f_{\text{hom}}}(u, \nabla u) \, dx + \int_\Omega T_{f_{\text{hom}}}(\tilde{u}, \frac{dD^c u}{d|D^c u|}) \, d|D^c u|. \]

Proof. Let $A \in A(\Omega)$ and $\{u_n\} \subset W^{1,1}(A; \mathcal{M})$ be such that $u_n \to u$ in $L^1(A; \mathbb{R}^d)$. Since $F(\cdot, A)$ is sequentially lower semicontinuous for the strong $L^1(A; \mathbb{R}^d)$ convergence, it follows from Theorem 1.1 that
\[ F(u, A) \leq \liminf_{n \to +\infty} F(u_n, A) = \liminf_{n \to +\infty} \int_A T_{f_{\text{hom}}}(u_n, \nabla u_n) \, dx. \]

Since the sequence $\{u_n\}$ is arbitrary, we deduce
\[ F(u, A) \leq \inf \left\{ \liminf_{n \to +\infty} \int_A T_{f_{\text{hom}}}(u_n, \nabla u_n) : \{u_n\} \subset W^{1,1}(A; \mathcal{M}), u_n \to u \text{ in } L^1(A; \mathbb{R}^d) \right\}. \]

According to Proposition 3.1, the energy density $T_{f_{\text{hom}}}$ is a continuous and tangentially quasiconvex function which fulfills the assumptions of Theorem 7.1. Hence
\[ F(u, A) \leq \int_A T_{f_{\text{hom}}}(u, \nabla u) \, dx + \int_A T_{f_{\text{hom}}}(\tilde{u}, \frac{dD^c u}{d|D^c u|}) \, d|D^c u| + \int_{S_u \cap A} H(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1}. \]
(5.2)
for some function $H : \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{N-1} \to [0, +\infty)$. By outer regularity, (5.2) holds for every $A \in \mathcal{B}(\Omega)$. Taking $A = \Omega \setminus S_u$, we obtain
\[ F(u, \Omega \setminus S_u) \leq \int_\Omega T_{f_{\text{hom}}}(u, \nabla u) \, dx + \int_\Omega T_{f_{\text{hom}}}(\tilde{u}, \frac{dD^c u}{d|D^c u|}) \, d|D^c u|, \]
and the proof is complete.

To prove the upper bound of the jump part, we first need to compare the energy density $K$ obtained in Proposition 4.1 with the expected density $\vartheta_{\text{hom}}$.

Lemma 5.2. We have $K(a, b, \nu_1) \leq \vartheta_{\text{hom}}(a, b, \nu_1)$ for every $(a, b, \nu_1) \in \mathcal{M} \times \mathcal{M} \times \mathbb{S}^{N-1}$.

Proof. We will partially proceed as in the proof of Proposition 3.3 and we refer to it for the notation. Consider $\nu = (\nu_1, \ldots, \nu_N)$ an orthonormal basis of $\mathbb{R}^N$. We shall prove that $K(a, b, \nu_1) \leq \vartheta_{\text{hom}}(a, b, \nu_1)$. Since $K$ and $\vartheta_{\text{hom}}$ are continuous in the last variable, we may assume that $\nu$ is a rational basis, i.e., for all $i \in \{1, \ldots, N\}$, there exists $\gamma_i \in \mathbb{R} \setminus \{0\}$ such that $\nu_i := \gamma_i \nu_i \in \mathbb{Z}^N$, and the general case follows by density.

Given $0 < \eta < 1$ arbitrary, by Proposition 3.2 and (3.29) we can find $\varepsilon_0 > 0$, $u_0 \in B_{\varepsilon_0}(a, b, \nu)$ and $\gamma_{e_0} \in G(a, b)$ such that $u_0(x) = \gamma_{e_0}(x \cdot \nu_1 / \varepsilon_0)$ and
\[ \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon_0}, \nabla u_0 \right) \, dx \leq \vartheta_{\text{hom}}(a, b, \nu_1) + \eta. \]
For every \( \lambda = (\lambda_2, \ldots, \lambda_N) \in \mathbb{Z}^{N-1} \), we set \( x^{(\lambda)}_n := \varepsilon_n \sum_{i=2}^N \lambda_i v_i \) and \( Q^{(\lambda)}_{\nu,n} := x^{(\lambda)}_n + (\varepsilon_n / \varepsilon_0) Q_{\nu} \). We define the set \( \Lambda_n \) by

\[
\Lambda_n := \left\{ \lambda \in \mathbb{Z}^{N-1} : Q^{(\lambda)}_{\nu,n} \subset Q_{\nu} \text{ and } x^{(\lambda)}_n \in \sum_{i=2}^N \frac{l_i}{\varepsilon_0} \varepsilon_n \gamma_i \nu_i + \varepsilon_n P \right\}
\]

for some \((l_2, \ldots, l_N) \in \mathbb{Z}^{N-1}\).

where

\[
P := \left\{ \alpha_2 v_2 + \ldots + \alpha_N v_N : \alpha_2, \ldots, \alpha_N \in [-1/2, 1/2] \right\}.
\]

Next consider

\[
u_n(x) = \begin{cases} u_0 \left( \frac{\varepsilon_0(x - x^{(\lambda)}_n) \cdot \nu_i}{\varepsilon_n} \right) & \text{if } x \in Q^{(\lambda)}_{\nu,n} \text{ for some } \lambda \in \Lambda_n, \\ \gamma \varepsilon_0 \left( \frac{x \cdot \nu_i}{\varepsilon_n} \right) & \text{otherwise}. \end{cases}
\]

Note that \( u_n \in W^{1,1}(Q_{\nu}; \mathcal{M}) \), \( \{\nabla u_n\} \) is bounded in \( L^1(Q_{\nu}; \mathbb{R}^{d \times N}) \), and \( u_n \rightarrow u^{a,b}_{\nu_i} \) in \( L^1(Q_{\nu}; \mathbb{R}^d) \) as \( n \rightarrow +\infty \) with \( u^{a,b}_{\nu_i} \) given by

\[
u^{a,b}_{\nu_i}(x) := \begin{cases} a & \text{if } x \cdot \nu_i \geq 0, \\ b & \text{if } x \cdot \nu_i < 0, \end{cases} \quad \Pi_{\nu_i} := \{x \in \mathbb{R}^N : x \cdot \nu_i = 0\}
\]

Arguing as in Step 1 of the proof of [17, Proposition 2.2], we obtain that

\[
\limsup_{n \to +\infty} \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx \leq \liminf_{n \to +\infty} \int_{A_{\nu}} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx \leq \beta L^N(A_{\nu}) + \limsup_{n \to +\infty} \int_{A_{\nu}} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx \leq \beta L^N(A_{\nu}) + \limsup_{n \to +\infty} \int_{A_{\nu}} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx + \beta \eta.
\]

where we have used \((H_2)\) and the fact that \( \nabla u_n = 0 \) outside \( A_{\varepsilon_n} \). On the other hand, Proposition 4.1 yields

\[
\mathcal{F}(u^{a,b}_{\nu_i}, A_{\eta} \cap \Pi_{\nu_i}) = \int_{A_{\eta} \cap \Pi_{\nu_i}} K(a, b, \nu_i) d\mathcal{H}^{N-1} = K(a, b, \nu_i).
\]

Using \((H_4)\), the boundedness of \( \{\nabla u_n\} \) in \( L^1(Q_{\nu}; \mathbb{R}^{d \times N}) \), the fact that \( f^\infty(\cdot, 0) \equiv 0 \), and Hölder’s inequality, we derive

\[
\left| \int_{A_{\varepsilon_n}} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx - \int_{Q_{\nu}} f^\infty \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) dx \right| \leq C \int_{A_{\varepsilon_n}} (1 + |\nabla u_n|^{1-q}) dx
\]

\[
\leq C \varepsilon_n + \varepsilon_n \|\nabla u_n\|_{L^1(Q_{\nu}; \mathbb{R}^{d \times N})} \leq 0 (5.6)
\]

as \( n \to \infty \). Gathering (5.3), (5.4), (5.5) and (5.6), we obtain \( K(a, b, \nu_i) \leq \vartheta_{\text{hom}}(a, b, \nu_i) + (\beta + 1) \eta \) and the conclusion follows from the arbitrariness of \( \eta \).}

We are now in position to prove the upper bound on the jump part of the energy. The argument is based on Lemma 5.2 together with an approximation procedure of [7]. In view of Lemma 5.1 and (5.1), this will complete the proof of the upper bound \( \mathcal{F}(u, \Omega) \leq \mathcal{F}_{\text{hom}}(u) \).
Corollary 5.1. For every $u \in BV(\Omega; \mathcal{M})$, we have
\[
\mathcal{F}(u, \Omega \cap S_u) \leq \int_{\Omega \cap S_u} \vartheta_{\text{hom}}(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1}.
\]

Proof. First assume that $u$ takes a finite number of values, i.e., $u \in BV(\Omega; T)$ for some finite subset $T \subset \mathcal{M}$. Then the conclusion directly follows from Proposition 4.1 together with Lemma 5.2.

Fix an arbitrary function $u \in BV(\Omega; \mathcal{M})$ and an open set $A \in \mathcal{A}(\Omega)$. For $\delta_0 > 0$ small enough, let $\mathcal{U} := \{s \in \mathbb{R}^d : \text{dist}(s, \mathcal{M}) < \delta_0\}$ be the $\delta_0$-neighborhood of $\mathcal{M}$ on which the nearest point projection $\Pi : \mathcal{U} \to \mathcal{M}$ is a well defined Lipschitz mapping. We extend $\vartheta_{\text{hom}}$ to a function $\tilde{\vartheta}_{\text{hom}}$ defined in $\mathbb{R}^d \times \mathbb{R}^d \times S^{N-1}$ by setting
\[
\tilde{\vartheta}_{\text{hom}}(a, b, \nu) := \chi(a)\chi(b)\vartheta_{\text{hom}}\left(\Pi(a), \Pi(b), \nu\right),
\]
for a cut-off function $\chi \in C^\infty(\mathbb{R}^d; [0, 1])$ satisfying $\chi(t) = 1$ if dist$(s, \mathcal{M}) \leq \delta_0/2$, and $\chi(s) = 0$ if dist$(s, \mathcal{M}) \geq 3\delta_0/4$. In view of Proposition 3.4, we infer that $\tilde{\vartheta}_{\text{hom}}$ is continuous and satisfies
\[
|\tilde{\vartheta}_{\text{hom}}(a_1, b_1, \nu) - \tilde{\vartheta}_{\text{hom}}(a_2, b_2, \nu)| \leq C(|a_1 - a_2| + |b_1 - b_2|),
\]
and
\[
\tilde{\vartheta}_{\text{hom}}(a_1, b_1, \nu) \leq C|a_1 - b_1|,
\]
for every $a_1, b_1, a_2, b_2 \in \mathbb{R}^d, \nu \in S^{N-1}$, and some constant $C > 0$. Therefore we can apply Step 2 in the proof of [7, Proposition 4.8] to obtain a sequence $\{v_n\} \subset BV(\Omega; \mathbb{R}^d)$ such that, for every $n \in \mathbb{N}, v_n \in BV(\Omega; T_n)$ for some finite set $T_n \subset \mathbb{R}^d$, $v_n \to u$ in $L^\infty(\Omega; \mathbb{R}^d)$ and
\[
\limsup_{n \to +\infty} \int_{A \cap S_{v_n}} \tilde{\vartheta}_{\text{hom}}(v_n^+, v_n^-, \nu_{v_n}) \, d\mathcal{H}^{N-1} \leq C|Du|\left(A \setminus S_u\right) + \int_{A \cap S_u} \tilde{\vartheta}_{\text{hom}}(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1}
\]
\[
\leq C|Du|\left(A \setminus S_u\right) + \int_{A \cap S_u} \tilde{\vartheta}_{\text{hom}}(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1}.
\]
Hence we may assume without loss of generality that for each $n \in \mathbb{N}, \|v_n - u\|_{L^\infty(\Omega; \mathbb{R}^d)} < \delta_0/2$, and thus dist$(v_n^+(x), \mathcal{M}) \leq |v_n^+(x) - u^+(x)| < \delta_0/2$ for $\mathcal{H}^{N-1}$-a.e. $x \in S_{v_n}$. In particular, we can define
\[
u_{v_n}(x) = v_n^+(x) - u^+(x)
\]
and then $u_n := \Pi(v_n)$, and $u_n \in BV(\Omega; \mathcal{M}), u_n \to u$ in $L^1(\Omega; \mathbb{R}^d)$. Moreover, one may check that for each $n \in \mathbb{N}, S_{v_n} \subset S_u$, so that $\mathcal{H}^{N-1}(S_{v_n} \setminus (J_{u_n} \cap J_u)) \leq \mathcal{H}^{N-1}(S_{v_n} \setminus J_{u_n}) + \mathcal{H}^{N-1}(S_u \setminus J_{u_n}) = 0$, and
\[
u_{u_n}(x) = \nu_{v_n}(x)
\]
for every $x \in J_{u_n} \cap J_u$.

Consequently,
\[
\limsup_{n \to +\infty} \int_{A \cap S_{u_n}} \tilde{\vartheta}_{\text{hom}}(u_n^+, u_n^-, \nu_{u_n}) \, d\mathcal{H}^{N-1} \leq C|Du|\left(A \setminus S_u\right) + \int_{A \cap S_u} \tilde{\vartheta}_{\text{hom}}(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1}. \tag{5.7}
\]
Since $u_n$ takes a finite number of values, Proposition 4.1 and Lemma 5.2 yield
\[
\mathcal{F}(u_n, A \cap S_{u_n}) \leq \int_{A \cap S_{u_n}} \tilde{\vartheta}_{\text{hom}}(u_n^+, u_n^-, \nu_{u_n}) \, d\mathcal{H}^{N-1}, \tag{5.8}
\]
and, in view of Lemma 4.1,
\[
\mathcal{F}(u_n, A \setminus S_{u_n}) \leq C\mathcal{L}^N(A). \tag{5.9}
\]
Combining (5.7), (5.8) and (5.9), we deduce
\[ \limsup_{n \to +\infty} F(u_n, A) = \limsup_{n \to +\infty} \left( F(u_n, A \setminus S_{u_n}) + F(u_n, A \cap S_{u_n}) \right) \]
\[ \leq \int_{A \cap S_u} \varrho_{\text{hom}}(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1} + C \left( \mathcal{L}^N(A) + |Du|(A \setminus S_u) \right). \]

On the other hand, \( F(\cdot, A) \) is lower semicontinuous with respect to the strong \( L^1(A; \mathbb{R}^d) \)-convergence, and thus \( F(u, A) \leq \liminf_{n \to +\infty} F(u_n, A) \) which leads to
\[ F(u, A) \leq \int_{A \cap S_u} \varrho_{\text{hom}}(u^+, u^-, \nu_u) \, d\mathcal{H}^{N-1} + C \left( \mathcal{L}^N(A) + |Du|(A \setminus S_u) \right). \]
Since \( A \) is arbitrary, the above inequality holds for any open set \( A \in \mathcal{A}(\Omega) \) and, by Remark 4.1, it also holds if \( A \) is any Borel subset of \( \Omega \). Then taking \( A = \Omega \cap S_u \) yields the desired inequality. \( \square \)

### 6. The lower bound

We address in this section with the \( \Gamma \)-lim inf inequality. Using the blow-up method, we follow the approach of [27], estimating separately the Cantor part and the jump part, while the bulk part is obtained exactly as in the \( W^{1,1} \) analysis, see [9, Lemma 5.2].

**Lemma 6.1.** For every \( u \in BV(\Omega; \mathcal{M}) \), we have \( F(u, \Omega) \geq F_{\text{hom}}(u) \).

**Proof.** Let \( u \in BV(\Omega; \mathcal{M}) \) and \( \{u_n\} \subset W^{1,1}(\Omega; \mathcal{M}) \) be such that
\[ F(u, \Omega) = \lim_{n \to +\infty} \int_{\Omega} f \left( \frac{x}{\varepsilon_n}, \nabla u_n \right) \, dx. \]
Define the sequence of nonnegative Radon measures
\[ \mu_n := f \left( \frac{\cdot}{\varepsilon_n}, \nabla u_n \right) \mathcal{L}^N \mathbb{L} \Omega. \]

Up to the extraction of a subsequence, we can assume that there exists a nonnegative Radon measure \( \mu \in \mathcal{M}(\Omega) \) such that \( \mu_n \rightharpoonup \mu \) in \( \mathcal{M}(\Omega) \). By the Besicovitch Differentiation Theorem, we can split \( \mu \) into the sum of four mutually singular nonnegative measures \( \mu = \mu^a + \mu^j + \mu^c + \mu^s \) where \( \mu^a \ll \mathcal{L}^N, \mu^j \ll \mathcal{H}^{N-1}_u S_u \) and \( \mu^c \ll |D^s u| \). Since we have \( \mu(\Omega) \leq F(u, \Omega) \), it is enough to check that
\[ \frac{d\mu}{d\mathcal{L}^N}(x_0) \geq T_{\text{hom}}(u(x_0), \nabla u(x_0)) \quad \text{for } \mathcal{L}^N \text{-a.e. } x_0 \in \Omega, \tag{6.1} \]
\[ \frac{d\mu}{d|D^s u|}(x_0) \geq T_{\text{hom}}(\tilde{u}(x_0), \frac{D^s u(x_0)}{|D^s u|}(x_0)) \quad \text{for } |D^s u| \text{-a.e. } x_0 \in \Omega, \tag{6.2} \]
and
\[ \frac{d\mu}{d\mathcal{H}^{N-1}_u S_u}(x_0) \geq \varrho_{\text{hom}}(u^+(x_0), u^-(x_0), \nu_u(x_0)) \quad \text{for } \mathcal{H}^{N-1} \text{-a.e. } x_0 \in S_u. \tag{6.3} \]
The proof of (6.1) follows the one in [9, Lemma 5.2] and we shall omit it. The proofs of (6.2) and (6.3) are postponed to the remaining of this subsection. \( \square \)

**Proof of (6.2),** The lower bound on the density of the Cantor part will be achieved in three steps. We shall use the blow-up method to reduce the study to constant limits, and then a truncation argument as in the proof of [9, Lemma 5.2], to replace the starting sequence by a uniformly converging one.
Step 1. Choose a point \( x_0 \in \Omega \) such that
\[
\lim_{\rho \to 0^+} \int_{Q(x_0, \rho)} |u(x) - \tilde{u}(x_0)| \, dx = 0, \tag{6.4}
\]
\[
A(x_0) := \lim_{\rho \to 0^+} \frac{Du(Q(x_0, \rho))}{|Du(Q(x_0, \rho))|} \in |T\tilde{u}(x_0)(\mathcal{M})|^N \text{ is a rank one matrix with } |A(x_0)| = 1, \tag{6.5}
\]
\[
\frac{d\mu}{d|Du|}(x_0) \text{ exists and is finite and } \frac{|Du|(Q(x_0, \rho))}{\rho^N} \leq 1 - \tau^N \text{ for every } 0 < \tau < 1. \tag{6.6}
\]
\[
\liminf_{\rho \to 0^+} \frac{|Du|(Q(x_0, \rho))}{|Du|(Q(x_0, \rho))} < 1. \tag{6.7}
\]

It turns out that \( |Du|\)-a.e. \( x_0 \in \Omega \) satisfy these properties. Indeed (6.6) is immediate while (6.4) is a consequence of the fact that \( S_\nu \) is \( |Du|\)-negligible. Property (6.5) comes from Alberti Rank One Theorem together with Lemma 2.1, (6.7) from [6, Proposition 3.92 (a), (c)] and (6.8) from [27, Lemma 2.13]. Write \( A(x_0) = a \otimes \nu \) for some \( a \in \mathcal{M} \) and \( \nu \in S^{N-1} \). Upon rotating the coordinate axis, one may assume without loss of generality that \( \nu = e_N \). To simplify the notations, we set \( s_0 := \tilde{u}(x_0) \) and \( A_0 := A(x_0) \).

Fix \( t \in (0, 1) \) arbitrarily close to 1, and in view of (6.8), find a sequence \( \rho_k \searrow 0^+ \) such that
\[
\limsup_{k \to +\infty} \frac{|Du|(Q(x_0, \rho_k) \setminus Q(x_0, t\rho_k))}{|Du|(Q(x_0, \rho_k))} \leq 1 - t^N. \tag{6.9}
\]

Now fix \( t < \gamma < 1 \) and set \( \gamma' := (1 + \gamma)/2 \). Using (6.6), we derive
\[
\frac{d\mu}{d|Du|}(x_0) = \lim_{k \to +\infty} \frac{\mu(Q(x_0, \rho_k))}{|Du|(Q(x_0, \rho_k))} \geq \limsup_{k \to +\infty} \frac{\mu(Q(x_0, \gamma' \rho_k))}{|Du|(Q(x_0, \rho_k))} \geq \limsup_{k \to +\infty} \frac{1}{|Du|(Q(x_0, \rho_k))} \int_{Q(x_0, \gamma' \rho_k)} f \left( \frac{x}{\rho_k}, \nabla \tilde{u}_n \right) \, dx. \tag{6.10}
\]

Arguing as in the proof of [9, Lemma 5.2] with minor modifications, we construct a sequence \( \{\tilde{v}_n\} \subset W^{1,\infty}(Q(0, \rho_k); \mathbb{R}^d) \) satisfying \( \tilde{v}_n \to u(x_0 + \cdot) \) in \( L^1(Q(0, \rho_k); \mathbb{R}^d) \) and
\[
\limsup_{n \to +\infty} \int_{Q(x_0, \gamma' \rho_k)} f \left( \frac{x}{\rho_k}, \nabla \tilde{u}_n \right) \, dx \leq \limsup_{n \to +\infty} \int_{Q(0, \gamma' \rho_k)} g \left( \frac{x}{\rho_k}, \tilde{v}_n, \nabla \tilde{v}_n \right) \, dx, \tag{6.11}
\]
where \( g \) is given by (3.1). Setting \( w_{n,k}(x) := \tilde{v}_n(\rho_k x) \), a change of variable together with (6.10) and (6.11) yields
\[
\frac{d\mu}{d|Du|}(x_0) \geq \limsup_{k \to +\infty} \limsup_{n \to +\infty} \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \int_{Q} g \left( \frac{\rho_k x}{\rho_k}, w_{n,k}, \frac{1}{\rho_k} \nabla w_{n,k} \right) \, dx. \tag{6.12}
\]

Then we infer from (6.4) that
\[
\lim_{k \to +\infty} \lim_{n \to +\infty} \int_{Q} |w_{n,k} - s_0| \, dx = 0, \tag{6.13}
\]
and
\[
\lim_{k \to +\infty} \lim_{n \to +\infty} \int_{Q} \left| w_{n,k}(x) - u(x_0 + \rho_k x) - \int_{Q} (w_{n,k}(y) - u(x_0 + \rho_k y)) \, dy \right| \, dx = 0. \tag{6.14}
\]
By (6.12), (6.13) and (6.14), we can extract a diagonal sequence \( n_k \to +\infty \) such that \( \delta_k := \varepsilon_{n_k}/\rho_k \to 0 \), \( w_k := w_{n_k,k} \to s_0 \) in \( L^1(Q; \mathbb{R}^d) \),

\[
\frac{d\mu}{d|Du|}(x_0) \geq \limsup_{k \to +\infty} \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \int_{\gamma_Q} g \left( \frac{x}{\delta_k}, w_k, \frac{1}{\rho_k} \nabla w_k \right) dx,
\]

and

\[
\lim_{k \to +\infty} \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \int_Q \left| w_k(x) - u(x_0 + \rho_k x) - \int_Q (w_k(y) - u(x_0 + \rho_k y)) dy \right| dx = 0. \tag{6.15}
\]

**Step 2.** Now we reproduce the truncation argument used in Step 2 of the proof of [9, Lemma 5.2] with minor modifications (make use of (6.7) and [27, Lemma 2.12] instead of [26, Lemma 2.6], see [27] for details). Setting \( a_k := \int_Q w_k(y) dy \), it yields a sequence of cut-off functions \( \{\zeta_k\} \subset C_c^\infty(\mathbb{R}; [0, 1]) \) such that \( \zeta_k(\tau) = 1 \) if \( |\tau| \leq s_k \), \( \zeta_k(\tau) = 0 \) is \( |\tau| \geq t_k \) for some

\[
\|w_k - a_k\|_{L^1(Q; \mathbb{R}^d)}^{1/2} < s_k < t_k < \|w_k - a_k\|_{L^1(Q; \mathbb{R}^d)}^{1/3},
\]

for which \( \tilde{w}_k := a_k + \zeta_k((w_k - a_k)(w_k - a_k)) \in W^{1,1}(Q; \mathbb{R}^d) \) satisfies \( \tilde{w}_k \to s_0 \) in \( L^\infty(Q; \mathbb{R}^d) \) and

\[
\frac{d\mu}{d|Du|}(x_0) \geq \limsup_{k \to +\infty} \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \int_{\gamma_Q} g \left( \frac{x}{\delta_k}, \tilde{w}_k, \frac{1}{\rho_k} \nabla \tilde{w}_k \right) dx. \tag{6.16}
\]

In view of the coercivity condition (3.3), (6.6) and (6.16),

\[
\sup_{k \in \mathbb{N}} \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \int_{\gamma_Q} |\nabla \tilde{w}_k| dx < +\infty.
\]

Therefore, (3.4), (6.16) and \( \|\tilde{w}_k - s_0\|_{L^\infty(Q; \mathbb{R}^d)} \to 0 \) lead to

\[
\frac{d\mu}{d|Du|}(x_0) \geq \limsup_{k \to +\infty} \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \int_{\gamma_Q} g \left( \frac{x}{\delta_k}, s_0, \frac{1}{\rho_k} \nabla s_0 \right) dx.
\]

Next we define the three following sequences for every \( x \in Q \),

\[
\begin{align*}
\bar{u}_k(x) &:= \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} \left( u(x_0 + \rho_k x) - \int_Q u(x_0 + \rho_k y) dy \right), \\
z_k(x) &:= \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} (w_k(x) - a_k), \\
\bar{z}_k(x) &:= \frac{\rho_k^N}{|Du|(Q(x_0, \rho_k))} (\tilde{w}_k(x) - a_k).
\end{align*}
\]

As a consequence of (6.15) we have \( \|z_k - \bar{z}_k\|_{L^1(Q; \mathbb{R}^d)} \to 0 \), and since

\[
\int_Q \bar{z}_k(x) dx = 0 \quad \text{and} \quad |D\bar{z}_k|(Q) = 1,
\]

it follows that the sequence \( \{\bar{z}_k\} \) is bounded in \( BV(Q; \mathbb{R}^d) \) and thus relatively compact in \( L^1(Q; \mathbb{R}^d) \). Hence \( \{\bar{z}_k\} \) is equi-integrable, and consequently so is \( \{z_k\} \). Up to a subsequence, \( \bar{z}_k \) converges in \( L^1(Q; \mathbb{R}^d) \) to some function \( v \in BV(Q; \mathbb{R}^d) \), and then \( z_k \to v \) in \( L^1(Q; \mathbb{R}^d) \). By [6, Theorem 3.95] the limit \( v \) is representable by

\[
v(x) = a \theta(x_N)
\]

for some increasing function \( \theta \in BV((-1/2, 1/2); \mathbb{R}) \) (recall that we assume \( A_0 = a \otimes e_N \)).
By construction, \(\overline{w}_k\) coincides with \(w_k\) in the set \(\{|w_k - a_k| \leq s_k\}\). Hence
\[
\|\tau_k - z_k\|_{L^1(Q; \mathbb{R}^d)} = \frac{\rho_k^{N-1}}{|Du(Q(x_0, \rho_k))|} \int_{\{|w_k - a_k| > s_k\}} |w_k(x) - \overline{w}_k(x)| \, dx \leq \frac{\rho_k^{N-1} - a_k}{|Du(Q(x_0, \rho_k))|} \int_{\{|w_k - a_k| > s_k\}} |w_k(x) - a_k| \, dx = \int_{\{|w_k - a_k| > s_k\}} |z_k(x)| \, dx .
\]
(6.17)

By Chebyshev inequality, we have
\[
\mathcal{L}^N(\{ |w_k - a_k| > s_k \}) \leq \frac{1}{s_k} \int_Q |w_k(x) - a_k| \, dx \leq \|w_k - a_k\|_{L^1(Q; \mathbb{R}^d)}^{1/2} \to 0 ,
\]
and thus (6.17), (6.18) and the equi-integrability of \(\{z_k\}\) imply \(\|\tau_k - z_k\|_{L^1(Q; \mathbb{R}^d)} \to 0\). Therefore \(\tau_k \to v\) in \(L^1(Q; \mathbb{R}^d)\), and setting \(\alpha_k := |Du(Q(x_0, \rho_k))|/\rho_k^N \to +\infty\),
\[
\frac{d\mu}{|D^2u|}(x_0) \geq \limsup_{k \to +\infty} \frac{1}{\alpha_k} \int_{\gamma Q} g \left( \frac{x}{\delta_k}, s_0, \alpha_k \nabla \tau_k \right) \, dx .
\]
(6.19)

Using (3.6) and the positive 1-homogeneity of the recession function \(g^\infty(y, s, \cdot)\), we infer that
\[
\int_{\gamma Q} \left| \frac{1}{\alpha_k} g \left( \frac{x}{\delta_k}, s_0, \alpha_k \nabla \tau_k \right) - g^\infty \left( \frac{x}{\delta_k}, s_0, \nabla \tau_k \right) \right| \, dx \leq \frac{C}{\alpha_k} \int_{\gamma Q} (1 + \alpha_k^{-q} |\nabla \tau_k|^{1-q}) \, dx \leq C (\alpha_k^{-1} + \alpha_k^{-q} \|\nabla \tau_k\|_{L^1(\gamma Q; \mathbb{R}^d; \mu, N)}) \to 0 ,
\]
where we have used Hölder’s inequality and the boundedness of \(|\nabla \tau_k|\) in \(L^1(\gamma Q; \mathbb{R}^d)\) (which follows from (3.3) and (6.19)). Consequently,
\[
\frac{d\mu}{|D^2u|}(x_0) \geq \limsup_{k \to +\infty} \int_{\gamma Q} g^\infty \left( \frac{x}{\delta_k}, s_0, \nabla \tau_k \right) \, dx .
\]

**Step 3.** Extend \(\theta\) continuously to \(\mathbb{R}\) by the values of its traces at \(\pm 1/2\). Define \(v_k(x) = v_k(x_N) := a \theta + \phi_k(x_N)\) where \(\phi_k\) is a sequence of (one dimensional) mollifiers. Then \(v_k \to v\) in \(L^1(Q; \mathbb{R}^d)\) and thus, since \(\overline{v}_k - v_k \to 0\) in \(L^1(Q; \mathbb{R}^d)\), it follows that (up to a subsequence)
\[
D\overline{v}_k(\tau Q) - Dv_k(\tau Q) \to 0
\]
(6.20)

for \(L^1\text{-a.e. } \tau \in (0, 1)\). Fix \(\tau \in (t, \gamma)\) for which (6.20) holds. Since \(\|\overline{z}_k - v_k\|_{L^1(Q; \mathbb{R}^d)} \to 0\), one can use a standard cut-off function argument (see [27, p. 29–30]) to modify the sequence \(\{\overline{\tau}_k\}\) and produce a new sequence \(\{\overline{\tau}_k\}\) satisfying \(\overline{\tau}_k \to v\) in \(L^1(\tau Q; \mathbb{R}^d)\), \(\overline{v}_k = v_k\) on a neighborhood of \(\partial(\tau Q)\) and
\[
\frac{d\mu}{|D^2u|}(x_0) \geq \limsup_{k \to +\infty} \int_{\tau Q} g^\infty \left( \frac{x}{\delta_k}, s_0, \nabla \overline{\tau}_k \right) \, dx .
\]
(6.21)

A simple computation shows that
\[
D\overline{v}_k(\tau Q) = \frac{Du(Q(x_0, \tau \rho_k))}{|Du(Q(x_0, \rho_k))|} \quad \text{and} \quad Dv_k(\tau Q) = \tau^N A_k ,
\]
where \(A_k \in \mathbb{R}^{d \times N}\) is the matrix given by
\[
A_k := a \otimes e_N \frac{\theta \ast \phi_k(\tau/2) - \theta \ast \phi_k(-\tau/2)}{\tau} .
\]

We observe that \(A_k\) is bounded in \(k\) since \(\theta\) has bounded variation.

Let \(m_k := \lfloor \tau/\delta_k \rfloor + 1 \in \mathbb{N}\), and define for \(x = (x', x_N) \in \delta_k m_k Q\),
\[
\varphi_k(x) := \begin{cases} \overline{v}_k(x) - A_k x & \text{if } x \in \tau Q , \\
v_k(x_N) - A_k x & \text{if } |x_N| \leq \tau/2 \text{ and } |x'| \geq \tau/2 , \\
v_k(\tau/2) - A_k (x', \tau/2) & \text{if } x_N \geq \tau/2 , \\
v_k(-\tau/2) - A_k (x', -\tau/2) & \text{if } x_N \leq -\tau/2 . 
\end{cases}
\]
One may check that \( \varphi_k \in W^{1,\infty}(\delta_k m_kQ; \mathbb{R}^d) \), \( \varphi_k \) is \( \delta_k m_k \)-periodic, and that
\[
\limsup_{k \to +\infty} \int_{\tau_Q} g^\infty \left( \frac{x}{\delta_k}, s_0, \nabla \varphi_k \right) dx = \limsup_{k \to +\infty} \int_{\delta_k m_kQ} g^\infty \left( \frac{x}{\delta_k}, s_0, A_k + \nabla \varphi_k \right) dx.
\] (6.23)

Setting \( \phi_k(y) := \tau^N \delta_k^{-1} \varphi_k(\delta_k y) \) for \( y \in m_kQ \), we have \( \phi_k \in W^{1,\infty}(m_kQ; \mathbb{R}^d) \), and a change of variables yields
\[
\int_{\delta_k m_kQ} g^\infty \left( \frac{x}{\delta_k}, s_0, A_k + \nabla \varphi_k \right) dx = \tau^{-N} \delta_k^N \int_{m_kQ} g^\infty \left( y, s_0, \tau^N A_k + \nabla \phi_k \right) dy
\]
\[
\geq \tau^{-N} \delta_k^N m_k^N (g^\infty)_{\text{hom}}(s_0, \tau^N A_k),
\] (6.24)
since \( (g^\infty)_{\text{hom}} \) can be computed as follows (see Remark 3.1 and e.g., [16, Remark 14.6]),
\[
(g^\infty)_{\text{hom}}(s, \xi) = \inf \left\{ \int_{(0,m)} g^\infty(y, s, \xi + \nabla \phi(y)) dy : m \in \mathbb{N}, \phi \in W^{1,\infty}(m); \mathbb{R}^d \right\}.
\]

Gathering (6.21), (6.23) and (6.24), we derive
\[
\frac{d\mu}{d|D^\nu u|}(x_0) \geq \limsup_{k \to +\infty} (g^\infty)_{\text{hom}}(s_0, \tau^N A_k).
\]

In view of (6.20), (6.22), (6.9) and (6.5), we have
\[
\limsup_{k \to +\infty} |\tau^N A_k - A_0| = \limsup_{k \to +\infty} |Dv_k(\tau Q) - A_0| = \limsup_{k \to +\infty} |D\bar{\nu}_k(\tau Q) - A_0| =
\]
\[
= \limsup_{k \to +\infty} \left| \frac{D u(Q(x_0, \tau \rho_k))}{D u(Q(x_0, \tau \rho_k))} - A_0 \right| = \limsup_{k \to +\infty} \frac{|D u(Q(x_0, \rho_k) \setminus Q(x_0, \tau \rho_k))|}{|D u(Q(x_0, \rho_k))|} \leq 1 - t^N.
\]

By Remark 3.1, \( (g^\infty)_{\text{hom}}(s_0, \cdot) \) is Lipschitz continuous, and consequently
\[
\frac{d\mu}{d|D^\nu u|}(x_0) \geq (g^\infty)_{\text{hom}}(s_0, A_0) - C(1 - t^N).
\]

From the arbitrariness of \( t \), we finally infer that
\[
\frac{d\mu}{d|D^\nu u|}(x_0) \geq (g^\infty)_{\text{hom}}(s_0, A_0).
\]

Since \( s_0 \in \mathcal{M} \) and \( A_0 \in [T_{s_0}(\mathcal{M})]^N \), Remark 3.1 and (3.17) yield \( (g^\infty)_{\text{hom}}(s_0, A_0) = T(f^\infty)_{\text{hom}}(s_0, A_0) \geq T f^\infty_{\text{hom}}(s_0, A_0) \), and the proof is complete. \( \square \)

**Proof of (6.3).** The strategy used in that part follows the one already used for the bulk and Cantor parts. It still rests on the blow up method together with the projection argument in Proposition 2.1. 

**Step 1.** Let \( x_0 \in S_u \) be such that
\[
\lim_{\rho \to 0^+} \int_{Q^\pm_{\nu u}(x_0 \cup \rho)} |u(x) - u^\pm(x_0)| dx = 0,
\] (6.25)

where \( u^\pm(x_0) \in \mathcal{M} \),
\[
\lim_{\rho \to 0^+} \frac{\mathcal{H}^{N-1}(S_u \cap Q_{\nu u}(x_0, \rho))}{\rho^{N-1}} = 1,
\] (6.26)

and such that the Radon-Nikodým derivative of \( \mu \) with respect to \( \mathcal{H}^{N-1} \) \( \mathcal{L} S_u \) exists and is finite. By Lemma 2.1, Theorem 3.78 and Theorem 2.83 (i) in [6] (with cubes instead of balls), it turns out that \( \mathcal{H}^{N-1} \)-a.e. \( x_0 \in S_u \) satisfy these properties. Set \( s_0^\pm := u^\pm(x_0), \nu_0 := \nu_u(x_0) \).
By a standard diagonal argument, we find a sequence \((n \to \infty)\) of nonnegative Radon measure \(\lambda \in \mathcal{M}(\Omega)\) for some \(\lambda \in \mathcal{M}(\Omega)\). Consider a sequence \(\rho_k \neq 0^+\) satisfying \(\mu(\partial Q_{\rho_k}(x_0, \rho_k)) = 0\) for each \(k \in \mathbb{N}\). Using (6.26) we derive

\[
\frac{d\mu}{d\mathcal{H}^{N-1}S_u}(x_0) \lim_{k \to \infty} \frac{\mu(Q_{\rho_k}(x_0, \rho_k))}{\rho_k^{N-1}} = \lim_{k \to \infty} \int_{Q_{\rho_k}(x_0, \rho_k)} f \left( \frac{x}{\rho_k}, \nabla u \right) dx.
\]

Thanks to Theorem 2.1, one can assume without loss of generality that \(u_n \in \mathcal{D}(\Omega; \mathcal{M})\) for each \(n \in \mathbb{N}\). Arguing exactly as in Step 1 of the proof of [9, Lemma 5.2] (with \(Q_{\rho_k}(x_0, \rho_k)\) instead of \(Q(x_0, \rho_k)\)) we obtain a sequence \(\{v_n\} \subset \mathcal{D}(Q_{\rho_k}(0, \rho_k); \mathcal{M})\) such that \(v_n \to u(x + \cdot)\) in \(L^1(Q_{\rho_k}(0, \rho_k); \mathbb{R}^d)\) as \(n \to \infty\), and

\[
\frac{d\mu}{d\mathcal{H}^{N-1}S_u}(x_0) \geq \limsup_{k \to \infty} \limsup_{n \to \infty} \frac{1}{\rho_k^N} \int_{Q_{\rho_k}(0, \rho_k)} f \left( \frac{x}{\rho_k}, \nabla v_n \right) dx
\]

(note that the construction process to obtain \(v_n\) from \(u_n\) does not affect the manifold constraint). Changing variables and setting \(w_{n,k}(x) = v_n(\rho_k x)\) lead to

\[
\frac{d\mu}{d\mathcal{H}^{N-1}S_u}(x_0) \geq \limsup_{k \to \infty} \limsup_{n \to \infty} \rho_k \int_{Q_{\rho_k}} f \left( \frac{\rho_k x}{\rho_k}, \frac{1}{\rho_k} \nabla w_{n,k} \right) dx.
\]

Defining

\[
u_0(x) := \begin{cases} s_0^+ & \text{if } x \cdot v_0 > 0, \\ s_0^- & \text{if } x \cdot v_0 \leq 0, \end{cases}
\]

we infer from (6.25) that

\[
\lim_{k \to \infty} \lim_{n \to \infty} \int_{Q_{\rho_k}} |w_{n,k} - \nu_0| dx = 0.
\]

By a standard diagonal argument, we find a sequence \(n_k \nearrow \infty\) such that \(\delta_k := \varepsilon_{n_k}/\rho_k \to 0\), \(w_k := w_{n_k,k} \in \mathcal{D}(Q_{\rho_k}; \mathcal{M})\) converges to \(\nu_0\) in \(L^1(Q_{\rho_k}; \mathbb{R}^d)\), and

\[
\frac{d\mu}{d\mathcal{H}^{N-1}S_u}(x_0) \geq \limsup_{k \to \infty} \rho_k \int_{Q_{\rho_k}} f \left( \frac{x}{\delta_k}, \frac{1}{\rho_k} \nabla w_k \right) dx. \tag{6.27}
\]

According to \((H_4)\) and the positive 1-homogeneity of \(f^{\infty}(y, \cdot)\), we have

\[
\int_{Q_{\rho_k}} \left| \rho_k f \left( \frac{x}{\delta_k}, \frac{1}{\rho_k} \nabla w_k \right) - f^{\infty} \left( \frac{x}{\delta_k}, \nabla w_k \right) \right| dx \leq C \rho_k \int_{Q_{\rho_k}} (1 + \rho_k^{q-1} |\nabla w_k|^{1-q}) dx
\]

\[
\leq C \left( \rho_k + \rho_k^q \|\nabla w_k\|_{L^1(Q_{\rho_k}; \mathbb{R}^d)}^{1-q} \right), \tag{6.28}
\]

where we have used Hölder’s inequality and \(0 < q < 1\). From (6.27) and the coercivity condition \((H_2)\), it follows that \(\{\nabla w_k\}\) is uniformly bounded in \(L^1(Q_{\rho_k}; \mathbb{R}^{d \times N})\). Gathering (6.27) and (6.28) yields

\[
\frac{d\mu}{d\mathcal{H}^{N-1}S_u}(x_0) \geq \limsup_{k \to \infty} \int_{Q_{\rho_k}} f^{\infty} \left( \frac{x}{\delta_k}, \nabla w_k \right) dx. \tag{6.29}
\]

**Step 2.** Now it remains to modify the value of \(w_k\) on a neighborhood of \(\partial Q_{\rho_k}\) in order to get an admissible test function for the surface energy density. We argue as in [2, Lemma 5.2]. Using the notations of Subsection 3.2, we consider \(\gamma \in \mathcal{G}(s_0^+, s_0^-)\), and set

\[
\phi_k(x) := \gamma \left( \frac{x \cdot v_0}{\delta_k} \right).
\]
Using a De Giorgi type slicing argument, we shall modify \( w_k \) in order to get a function which matches \( \psi_k \) on \( \partial Q_{\nu_0} \). To this end, define

\[
r_k := \|w_k - \psi_k\|_{L^1(Q_{\nu_0};\mathbb{R}^d)}^{1/2}, \quad M_k := k[1 + \|w_k\|_{W^{1,1}(Q_{\nu_0};\mathbb{R}^d)} + \|\psi_k\|_{W^{1,1}(Q_{\nu_0};\mathbb{R}^d)}], \quad \ell_k := \frac{r_k}{M_k}.
\]

Since \( \psi_k \) and \( w_k \) converge to \( u_0 \) in \( L^1(Q_{\nu_0}%;\mathbb{R}^d) \), we have \( r_k \to 0 \), and one may assume that \( 0 < r_k < 1 \). Set

\[
Q_k^{(i)} := (1 - r_k + i \ell_k)Q_{\nu_0} \quad \text{for } i = 0, \ldots, M_k.
\]

For every \( i \in \{1, \ldots, M_k\} \), consider a cut-off function \( \varphi_k^{(i)} \in C_c^\infty(Q_k^{(i)};[0,1]) \) satisfying \( \varphi_k^{(i)} = 1 \) on \( Q_k^{(i-1)} \) and \( |\nabla \varphi_k^{(i)}| \leq c/\ell_k \). Define

\[
z_k^{(i)} := \varphi_k w_k + (1 - \varphi_k^{(i)})\psi_k \in W^{1,1}(Q_{\nu_0};\mathbb{R}^d),
\]

so that \( z_k^{(i)} = w_k \) in \( Q_k^{(i-1)} \), and \( z_k^{(i)} = \psi_k \) in \( Q_{\nu_0} \setminus Q_k^{(i)} \). Since \( z_k^{(i)} \) is smooth outside a finite union of sets contained in some \( (N - 2) \)-dimensional submanifolds and \( z_k^{(i)}(x) \in \operatorname{co}(M) \) for a.e. \( x \in Q_{\nu_0} \), one can apply Proposition \( 2.1 \) to obtain new functions \( \hat{z}_k^{(i)} \in W^{1,1}(Q_{\nu_0};M) \) such that \( \hat{z}_k^{(i)} = z_k^{(i)} \) on \( (Q_{\nu_0} \setminus Q_k^{(i)}) \cup Q_k^{(i-1)} \), and

\[
\int_{Q_k^{(i)} \setminus Q_k^{(i-1)}} |\nabla \hat{z}_k^{(i)}| \, dx \leq C \int_{Q_k^{(i)} \setminus Q_k^{(i-1)}} |\nabla z_k^{(i)}| \, dx
\]

\[
\leq C \int_{Q_k^{(i)} \setminus Q_k^{(i-1)}} \left( |\nabla w_k| + |\nabla \psi_k| + \frac{1}{\ell_k}|w_k - \psi_k| \right) \, dx.
\]

In particular \( \hat{z}_k^{(i)} \in B_{\delta_k}(s_0^+, s_0^-, \nu_0) \), and by the growth condition (3.12),

\[
\int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla \hat{z}_k^{(i)} \right) \, dx \leq \int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla w_k \right) \, dx + C \int_{Q_{\nu_0} \setminus Q_k^{(i)}} |\nabla \psi_k| \, dx +
\]

\[
+ C \int_{Q_k^{(i)} \setminus Q_k^{(i-1)}} \left( |\nabla w_k| + |\nabla \psi_k| + \frac{1}{\ell_k}|w_k - \psi_k| \right) \, dx.
\]

Summing up over all \( i = 1, \ldots, M_k \) and dividing by \( M_k \), we get that

\[
\frac{1}{M_k} \sum_{i=1}^{M_k} \int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla \hat{z}_k^{(i)} \right) \, dx \leq \int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla w_k \right) \, dx +
\]

\[
+ C \int_{Q_{\nu_0} \setminus Q_k^{(0)}} |\nabla \psi_k| \, dx + \frac{C}{k} + C \|w_k - \psi_k\|_{L^1(Q_{\nu_0};\mathbb{R}^d)}^{1/2}.
\]

Since

\[
\int_{Q_{\nu_0} \setminus Q_k^{(0)}} |\nabla \psi_k| \, dx \leq d_M(s_0^+, s_0^-)H^{N-1}((Q_{\nu_0} \setminus Q_k^{(0)}) \cap \{x : \nu_0 = 0\}) \to 0
\]

as \( k \to +\infty \), there exists a sequence \( \eta_k \to 0^+ \) such that

\[
\frac{1}{M_k} \sum_{i=1}^{M_k} \int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla \hat{z}_k^{(i)} \right) \, dx \leq \int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla w_k \right) \, dx + \eta_k.
\]

Hence, for each \( k \in \mathbb{N} \) we can find some index \( i_k \in \{1, \ldots, M_k\} \) satisfying

\[
\int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla \hat{z}_k^{(i_k)} \right) \, dx \leq \int_{Q_{\nu_0}} f^\infty \left( \frac{x}{\delta_k}, \nabla w_k \right) \, dx + \eta_k.
\]
Gathering (6.29) and (6.30), we obtain that
\[
\frac{d\mu}{dH^{N-1}}(x_0) \geq \limsup_{k \to +\infty} \int_{Q_{x_0}} f^\infty \left( \frac{x}{\delta_z^{(ik)}}, \nabla z_k^{(ik)} \right) dx.
\]
Since \( z_k^{(ik)} \in B_{\delta_k}(s_0^+, s_0^-, \nu_0) \), we infer from Proposition 3.2, Proposition 3.3 and (3.29) that
\[
\frac{d\mu}{dH^{N-1}}(x_0) \geq \vartheta_{hom}(s_0^+, s_0^-, \nu_0),
\]
which completes the proof.

\[\square\]

6.1. \textbf{Proof of Theorem 1.2}

\textbf{Proof of Theorem 1.2.} In view of (H2) and the closure of the pointwise constraint under strong \(L^1\)-convergence, \( \mathcal{F}(u) < +\infty \) implies \( u \in BV(\Omega; \mathcal{M}) \). In view of (5.1), Lemma 5.1, Corollary 5.1 and Lemma 6.1, the subsequence \( \{ \mathcal{F}_{\varepsilon_n} \} \) \( \Gamma \)-converges to \( \mathcal{F}_{hom} \) in \( L^1(\Omega; \mathbb{R}^d) \). Since the \( \Gamma \)-limit does not depend on the particular choice of the subsequence, we get in light of [21, Proposition 8.3] that the whole sequence \( \Gamma \)-converges.

\[\square\]

7. Appendix

We present in this appendix a relaxation result already proved in [2] for \( \mathcal{M} = \mathbb{S}^{d-1} \), and in [37] for isotropic integrands. The proof can be obtained following the one of [2, Theorem 3.1] replacing the standard projection on the sphere (used in Lemma 5.2, Proposition 6.2 and Lemma 6.4 of [2]) by the projection on \( \mathcal{M} \) of [35] as in Proposition 2.1. Since we only make use of the upper bound on the diffuse part, we will just enlight the differences in the main steps leading to it.

Assume that \( \mathcal{M} \) is a smooth, compact and connected submanifold of \( \mathbb{R}^d \) without boundary, and let \( f : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty) \) be a continuous function satisfying:

\( (H'_1) \) \( f \) is tangentially quasiconvex, \textit{i.e.}, for all \( x \in \Omega \), all \( s \in \mathcal{M} \) and all \( \xi \in [T_s(\mathcal{M})]^N \),
\[
f(x, s, \xi) \leq \int_Q f(x, s, \xi + \nabla \varphi(y)) dy \quad \text{for every } \varphi \in W_{0, \infty}^1(Q; T_s(\mathcal{M}));
\]

\( (H'_2) \) there exist \( \alpha > 0 \) and \( \beta > 0 \) such that
\[
\alpha |\xi| \leq f(x, s, \xi) \leq \beta (1 + |\xi|) \quad \text{for every } (x, s, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N};
\]

\( (H'_3) \) for every compact set \( K \subset \Omega \), there exists a continuous function \( \omega : [0, +\infty) \to [0, +\infty) \) satisfying \( \omega(0) = 0 \) and
\[
|f(x, s, \xi) - f(x', s', \xi)| \leq \omega(|x - x'| + |s - s'|)(1 + |\xi|)
\]
for every \( x, x' \in \Omega, s, s' \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^{d \times N} \);

\( (H'_4) \) there exist \( C > 0 \) and \( q \in (0, 1) \) such that
\[
|f(x, s, \xi) - f^\infty(x, s, \xi)| \leq C(1 + |\xi|^{1-q}), \quad \text{for every } (x, s, \xi) \in \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N},
\]
where \( f^\infty : \Omega \times \mathbb{R}^d \times \mathbb{R}^{d \times N} \to [0, +\infty) \) is the recession function of \( f \) defined by
\[
f^\infty(x, s, \xi) := \limsup_{t \to +\infty} \frac{f(x, s, t\xi)}{t}.
\]

Consider the functional \( F : L^1(\Omega; \mathbb{R}^d) \to [0, +\infty] \) given by
\[
F(u) := \begin{cases} 
\int_{\Omega} f(x, u, \nabla u) dx & \text{if } u \in W^{1,1}(\Omega; \mathcal{M}), \\
+\infty & \text{otherwise},
\end{cases}
\]
and its relaxation for the strong $L^1(\Omega;\mathbb{R}^d)$-topology $\overline{F}: L^1(\Omega;\mathbb{R}^d) \to [0, +\infty]$ defined by

$$\overline{F}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n \to +\infty} F(u_n) : u_n \to u \text{ in } L^1(\Omega;\mathbb{R}^d) \right\}. $$

Then the following integral representation result holds:

**Theorem 7.1.** Let $\mathcal{M}$ be a smooth compact and connected submanifold of $\mathbb{R}^d$ without boundary, and let $f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function satisfying $(H'_1)$ to $(H'_4)$. Then for every $u \in L^1(\Omega;\mathbb{R}^d)$,

$$\overline{F}(u) = \begin{cases} \int_{\Omega} f(x, u, \nabla u) dx + \int_{\Omega \cap S_u} K(x, u^+, u^-, \nu_u) d\mathcal{H}^{N-1} + \\
\quad + \int_{\Omega} f^\infty(x, \tilde{u}, \frac{dD^c u}{|D^c u|}) d|D^c u| & \text{if } u \in BV(\Omega; \mathcal{M}), \\
+\infty & \text{otherwise}, \end{cases}$$

where for every $(x, a, b, \nu) \in \Omega \times \mathcal{M} \times \mathcal{M} \times S^{N-1}$,

$$K(x, a, b, \nu) := \inf_{\varphi} \left\{ \int_{Q_{\nu}} f^\infty(x, \varphi(y), \nabla \varphi(y)) dy : \varphi \in W^{1,1}(Q_{\nu}; \mathcal{M}), \varphi = a \text{ on } \{ x \cdot \nu = 1/2 \}, \right.$$

$$\varphi = b \text{ on } \{ x \cdot \nu = -1/2 \} \text{ and } \varphi \text{ is 1-periodic in the } \nu_2, \ldots, \nu_N \text{ directions}\},$$

$\{\nu, \nu_2, \ldots, \nu_N\}$ forms any orthonormal basis of $\mathbb{R}^N$, and $Q_{\nu}$ stands for the open unit cube in $\mathbb{R}^N$ centered at the origin associated to this basis.

**Sketch of the Proof.** The proof of the lower bound “≥” in (7.1) can be obtained as in [9, Lemma 5.2] and Lemma 6.1 using standard techniques to handle with the dependence on the space variable. The lower bounds for the bulk and Cantor parts rely on the construction of a suitable function $f$ on $\mathcal{M}$ defined by

$$f(x, u, \nabla u) := \begin{cases} 0 & \text{if } u \in W^{1,1}(\mathcal{M}), \\
+\infty & \text{otherwise}, \end{cases}$$

and let $f : \mathbb{R}^N \times \mathbb{R}^{d \times N} \to [0, +\infty)$ be a continuous function satisfying $(H'_1)$ to $(H'_4)$. Then for every $u \in L^1(\Omega;\mathbb{R}^d)$,

$$F(u, A) := \begin{cases} \int_A f(x, u, \nabla u) dx & \text{if } u \in W^{1,1}(A; \mathcal{M}), \\
+\infty & \text{otherwise}, \end{cases}$$

Arguing as in the proof of Lemma 4.1, we obtain that for every $u \in BV(\Omega; \mathcal{M})$, the set function $\overline{F}(u, \cdot)$ is the restriction to $A(\Omega)$ of a Radon measure absolutely continuous with respect to $\mathcal{L}^N + |Du|$. Hence it uniquely extends into a Radon measure on $\Omega$ (see Remark 4.1), and it suffices to prove that for any $u \in BV(\Omega; \mathcal{M})$,

$$\overline{F}(u, \Omega \cap S_u) \leq \int_{\Omega \cap S_u} K(x, u^+, u^-, \nu_u) d\mathcal{H}^{N-1},$$

$$\frac{d\overline{F}(u, \cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0)) \text{ for } \mathcal{L}^N\text{-a.e. } x_0 \in \Omega,$$

$$\frac{d\overline{F}(u, \cdot)}{|D^c u|}(x_0) \leq f^\infty(x_0, \tilde{u}(x_0), \frac{dD^c u}{|D^c u|}(x_0)) \text{ for } |D^c u|\text{-a.e. } x_0 \in \Omega,$$
Proof of (7.2). Concerning the jump part, one can proceed as in [2, Lemma 6.5]. A slight difference lies in the third step of its proof where one needs to approximate in energy an arbitrary \( u \in BV(\Omega; \mathcal{M}) \) by a sequence \( \{ u_n \} \subset BV(\Omega; \mathcal{M}) \) such that for each \( n \), \( u_n \) assumes a finite number of values. This can be performed as in the proof of Corollary 5.1 using the regularity properties of \( K \) stated in [2, Lemma 4.1] for \( \mathcal{M} = \mathbb{R}^{d-1} \).

Proof of (7.3). Let \( x_0 \in \Omega \) be a Lebesgue point for \( u \) and \( \nabla u \) such that \( u(x_0) \in \mathcal{M} \), \( \nabla u(x_0) \in [T_u(x_0) (\mathcal{M})]^{\eta} \),

\[
\lim_{\rho \to 0^+} \int_{Q(x_0, \rho)} |u(x) - u(x_0)|(1 + |\nabla u(x)|) \, dx = 0, \quad \lim_{\rho \to 0^+} \frac{|D^s u(Q(x_0, \rho))|}{\rho^N} = 0,
\]

and

\[
\frac{d|Du|}{d\mathcal{L}^N}(x_0) \quad \text{and} \quad \frac{d\mathcal{F}(u, \cdot)}{d\mathcal{L}^N}(x_0)
\]

exist and are finite. Note that \( \mathcal{L}^N \)-a.e. \( x_0 \in \Omega \) satisfy these properties. We select a sequence \( \rho_k \searrow 0^+ \) such that \( Q(x_0, 2\rho_k) \subset \Omega \) and \( |Du|(|\partial Q(x_0, \rho_k)) = 0 \) for each \( k \in \mathbb{N} \). Next consider a sequence of standard mollifiers \( \{ \varrho_n \} \), and define \( u_n := \varrho_n * u \in W^{1,1}(Q(x_0, \rho_k); \mathbb{R}^d) \cap C^\infty(Q(x_0, \rho_k); \mathbb{R}^d) \). In the sequel, we shall assume as in the proof of Proposition 2.1 and we refer to it for the notation. Fix \( \delta > 0 \) small enough such that \( \pi : \mathbb{R}^d \setminus \mathcal{M} \) is smooth in the \( \delta \)-neighborhood of \( \mathcal{M} \).

Since \( u_n \) takes its values in \( \text{co}(\mathcal{M}) \), we can reproduce the proof of Proposition 2.1 to find \( a_n^k \in \mathbb{R}^d \) with \( |a_n^k| < \delta/4 \) such that setting \( p_n^k := (\pi a_n^k | \mathcal{M})^{-1} \circ \pi a_n^k \), \( w_n^k := p_n^k \circ u_n \in W^{1,1}(Q(x_0, \rho_k); \mathcal{M}) \) and

\[
\int_{A_n^k} |\nabla w_n^k| \, dx \leq C \int_{A_n^k} |\nabla u_n| \, dx,
\]

where \( A_n^k \) denotes the open set \( A_n^k := \{ x \in Q(x_0, \rho_k) : \text{dist}(u_n(x), \mathcal{M}) > \delta/2 \} \). Furthermore, since \( \pi \) is smooth in the \( \delta \)-neighborhood of \( \mathcal{M} \) and \( |a_n^k| < \delta/4 \), there exists a constant \( C_\delta > 0 \) independent of \( n \) and \( k \) such that

\[
|\nabla^2 p_n^k(s)| + |\nabla p_n^k(s)| \leq C_\delta \quad \text{for every } s \in \mathbb{R}^d \text{ satisfying } \text{dist}(s, \mathcal{M}) \leq \delta/2,
\]

and consequently,

\[
|\nabla w_n^k| \leq C_\delta |\nabla u_n| \quad \mathcal{L}^N \text{-a.e. in } Q(x_0, \rho_k) \setminus A_n^k.
\]

Since \( u(x) \in \mathcal{M} \) for \( \mathcal{L}^N \)-a.e. \( x \in \Omega \), it follows that

\[
\mathcal{L}^N(A_n^k) \leq \frac{2}{\delta} \int_{Q(x_0, \rho_k)} \text{dist}(u_n, \mathcal{M}) \, dx \leq \frac{2}{\delta} \int_{Q(x_0, \rho_k)} |u_n - u| \, dx \xrightarrow{n \to +\infty} 0,
\]

and then (6) yields

\[
\int_{Q(x_0, \rho_k)} |w_n^k - u| \, dx = \int_{A_n^k} |w_n^k - u| \, dx + \int_{Q(x_0, \rho_k) \setminus A_n^k} |p_n^k(u_n) - p_n^k(u)| \, dx \leq \text{diam}(\mathcal{M}) \mathcal{L}^N(A_n^k) + C_\delta \int_{Q(x_0, \rho_k)} |u_n - u| \, dx \xrightarrow{n \to +\infty} 0.
\]

Hence \( w_n^k \to u \) in \( L^1(Q(x_0, \rho_k); \mathbb{R}^d) \) as \( n \to +\infty \) so that we are allowed to take \( w_n^k \) as competitor, i.e.,

\[
\mathcal{F}(u, Q(x_0, \rho_k)) \leq \liminf_{n \to +\infty} \int_{Q(x_0, \rho_k)} f(x, u_n^k, \nabla u_n^k) \, dx.
\]

At this stage we can argue exactly as in [2, Lemma 6.4] to prove that for any \( \eta > 0 \) there exists \( \lambda = \lambda(\eta) > 0 \) such that

\[
\mathcal{F}(u, Q(x_0, \rho_k)) \leq \liminf_{n \to +\infty} \left\{ \int_{Q(x_0, \rho_k)} f(x_0, u(x_0), \nabla u_n) \, dx + C \int_{Q(x_0, \rho_k)} |\nabla u_n - \nabla w_n^k| \, dx + C(\eta + \lambda \rho_k) \int_{Q(x_0, \rho_k)} (1 + |\nabla u_n|) \, dx + C\lambda \int_{Q(x_0, \rho_k)} |w_n^k - u(x_0)|(1 + |\nabla u_n|) \, dx \right\}.
\]
The first and third term in the right-hand side of (7.8) can be treated as in the proof of [27, Theorem 2.16]. Concerning the remaining terms, we proceed as follows. Using (7.5), (7.6) and (7.7), we get that
\[
\int_{Q(x_0, \rho_k)} |u_n^k - u(x_0)||\nabla u_n^k| \, dx \leq \text{diam}(\mathcal{M}) \int_{A_n^k} |\nabla u_n^k| \, dx +
\int_{Q(x_0, \rho_k) \setminus A_n^k} |p_n^k(u_n) - p_n^k(u(x_0))||\nabla u_n^k| \, dx \leq C \int_{A_n^k} |\nabla u_n| \, dx +
+C \int_{Q(x_0, \rho_k) \setminus A_n^k} |u_n - u(x_0)||\nabla u_n| \, dx = C_\delta \int_{Q(x_0, \rho_k)} |u_n - u(x_0)||\nabla u_n| \, dx,
\]
where \(C_\delta > 0\) still denotes some constant depending on \(\delta\) but independent of \(k\) and \(n\). Arguing in a similar way, we also derive
\[
\int_{Q(x_0, \rho_k)} |\nabla u_n - \nabla u_n^k| \, dx \leq C_\delta \int_{Q(x_0, \rho_k)} |u_n - u(x_0)||\nabla u_n| \, dx + \int_{Q(x_0, \rho_k) \setminus A_n^k} |L_n^k \nabla u_n| \, dx,
\]
where \(L_n^k := \text{Id} - \nabla p_n^k(u(x_0)) \in \text{Lin}(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})\). Gathering (7.8), (7.9) and (7.10) we finally obtain that
\[
F(u, Q(x_0, \rho_k)) \leq \liminf_{n \to +\infty} \left\{ \int_{Q(x_0, \rho_k)} f(x_0, u(x_0), \nabla u_n) \, dx + C \int_{Q(x_0, \rho_k) \setminus A_n^k} |L_n^k \nabla u_n| \, dx +
+C(\eta + \lambda \rho_k) \int_{Q(x_0, \rho_k)} (1 + |\nabla u_n|) \, dx + C_\delta \lambda \int_{Q(x_0, \rho_k)} |u_n - u(x_0)|(1 + |\nabla u_n|) \, dx \right\}.
\]
Now we can follow the argument in [2, Lemma 6.4] to conclude that
\[
\frac{dF(u, \cdot)}{d\mathcal{L}^N}(x_0) \leq f(x_0, u(x_0), \nabla u(x_0)),
\]
which completes the proof of (7.3).

Proof of (7.4). Once again the proof parallels the one in [2, Lemma 6.4]. We first proceed as in the previous reasoning leading to (7.11). Then we can exactly follow the argument of [2, Lemma 6.4] to obtain (7.4). \(\square\)

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References
[1] G. ALBERTI: Rank-one property for derivatives of functions with bounded variation, Proc. Royal Soc. Edinburgh Sect. A 123 (1993), 239–274.
[2] R. ALICANDRO, A. C. ESPOSITO & C. LEONE: Relaxation in \(BV\) of integral functionals defined on Sobolev functions with values in the unit sphere, J. Conv. Anal. 14 (2007), 69–98.
[3] R. ALICANDRO & C. LEONE: 3D-2D asymptotic analysis for micromagnetic energies, ESAIM Cont. Optim. Calc. Var. 6 (2001), 489–498.
[4] L. AMBROSIO & A. BRAIDES: Functionals defined on partitions in sets of finite perimeter I: integral representation and \(\Gamma\)-convergence, J. Math. Pures Appl. 69 (1990), 285–306.
[5] L. AMBROSIO & G. DAL MASO: On the relaxation in \(BV(\Omega; \mathbb{R}^m)\) of quasiconvex integrals, J. Funct. Anal. 109 (1992), 76–97.
[6] L. AMBROSIO, N. FUSCO & D. PALLARA: Functions of bounded variation and free discontinuity problems, Oxford University Press (2000).
[7] L. Ambrosio, S. Mortola & V.M. Tortorelli: Functionals with linear growth defined on vector valued $BV$ functions, *J. Math. Pures Appl.* **70** (1991), 269–323.

[8] L. Ambrosio & D. Pallara: Integral representation of relaxed functionals on $BV(R^n; R^k)$ and polyhedral approximation, *Indiana Univ. Math. J.* **42** (1993), 295–321.

[9] J.-F. Babadjian & V. Millot: Homogenization of variational problems in manifold valued Sobolev spaces, preprint (2008).

[10] F. Béthuel: The approximation problem for Sobolev maps between two manifolds, *Acta Math.* **167** (1991), 153–206.

[11] F. Béthuel, H. Brézis & J.M. Coron: Relaxed energies for harmonic maps, in *Variational methods* (Paris, 1988), 37–52. Progress in Nonlinear Differential Equations and Their Applications 4, Birkhäuser, 1990.

[12] F. Béthuel & X. Zheng: Density of smooth functions between two manifolds in Sobolev spaces, *J. Funct. Anal.* **80** (1988), 60–75.

[13] G. Bouchitté: Convergence et relaxation de fonctionnelles du calcul des variations à croissance linéaire. Application à l’homogénéisation en plasticité, *Ann. Fac. Sci. Univ. Toulouse* **8** (1986), 7–36.

[14] G. Bouchitté, I. Fonseca & L. Mascarenhas: A global method for relaxation, *Arch. Rational Mech. Anal.* **145** (1998), 51–98.

[15] A. Braides: Homogenization of some almost periodic coercive functional, *Rend. Accad. Naz. Sci. XL.* **103** (1985), 313–322.

[16] A. Braides & A. Defranceschi: Homogenization of multiple integrals, Oxford Lecture Series in Mathematics and its Applications **12**, Oxford University Press, New York (1998).

[17] A. Braides, A. Defranceschi & E. Vitali: Homogenization of free discontinuity problems, *Arch. Rational Mech. Anal.* **135** (1996), 297–356.

[18] H. Brézis, J.M. Coron & E.H. Lieb: Harmonic maps with defects, *Comm. Math. Phys.* **107** (1986), 649–705.

[19] B. Dacorogna, I. Fonseca, J. Malý & K. Trivisa: Manifold constrained variational problems, *Calc. Var. Part. Diff. Eq.* **9** (1999), 185–206.

[20] G. Dal Maso: Integral representation on $BV(\Omega)$ of $\Gamma$-limits of variational integrals, *Manuscripta Math.* **30** (1980), 387–410.

[21] G. Dal Maso: *An Introduction to $\Gamma$-convergence*, Birkhäuser, Boston (1993).

[22] R. De Arcangelis & G. Gargiulo: Homogenization of integral functionals with linear growth defined on vector-valued functions, *NoDEA* **2** (1995), 371–416.

[23] F. Demengel: On some spaces of functions with bounded derivatives between manifolds, *Differential Integral Equations* **9** (1996), 173–185.

[24] H. Federer: Geometric measure theory, Springer-Verlag (1969).

[25] H. Federer & W.H. Fleming: Normal and integral currents, *Ann. Math.* **72** (1960), 458–520.

[26] I. Fonseca & S. Müller: Quasiconvex integrands and lower semicontinuity in $L^p$, *SIAM J. Math. Anal.* **23** (1992), 1081–1098.

[27] I. Fonseca & S. Müller: Relaxation of quasi-convex functionals in $BV(\Omega; R^p)$ for integrands $f(x, u, \nabla u)$, *Arch. Rational Mech. Anal.* **123** (1993), 1–49.

[28] I. Fonseca & P. Ryska: Relaxation of multiple integrals in space $BV(\Omega; R^p)$, *Proceedings Roy. Soc. Ed.* **121A** (1992), 321–348.

[29] M. Giaquinta, L. Modica & J. Souček: Functionals with linear growth in the calculus of variations, *Comment. Math. Univ. Carolinae* **20** (1979), 143–172.

[30] M. Giaquinta, L. Modica & J. Souček: Cartesian currents in the calculus of variations, Modern surveys in Mathematics **37–38**, Springer-Verlag, Berlin (1998).

[31] M. Giaquinta & D. Muçć: The $BV$-energy of maps into a manifold: relaxation and density results, *Ann. Scuola Norm. Sup. Pisa. Cl. (5) 5* (2006), 483–548.

[32] M. Giaquinta & D. Muçć: Relaxation results for a class of functionals with linear growth defined on manifold constrained mappings, to appear in *J. Convex Anal.*

[33] C. Goffman & J. Serrin: Sublinear functions of measures and variational integrals, *Duke Math. J.* **31** (1964), 159–178.

[34] R. Hardt, R. Kinderlehrer & F. H. Lin: Stable defects of minimizers of constrained variational principles, *Ann. Inst. Henri Poincaré Anal. Non Linéaire* **5** (1986), 297–322.

[35] R. Hardt & F. H. Lin: Mappings minimizing the $L^p$ norm of the gradient, *Comm. Pure Appl. Math.* **40** (1987), 555–588.

[36] P. Marcellini: Periodic solutions and homogenization of nonlinear variational problems, *Ann. Mat. Pura Appl.* (4) **117** (1978), 139–152.

[37] D. Muçć: Relaxation of isotropic functionals with linear growth defined on manifold values constrained Sobolev mappings, to appear in *ESAIM Cont. Optim. Calc. Var.*

[38] S. Müller: Homogenization of nonconvex integral functionals and cellular elastic materials, *Arch. Rational Mech. Anal.* **99** (1987), 189–212.