A CRITERION FOR TALAGRAND’S QUADRATIC TRANSPORTATION COST INEQUALITY.

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Abstract. We show that the quadratic transportation cost inequality $T_2$ is equivalent to both a Poincaré inequality and a strong form of the Gaussian concentration property. The main ingredient in the proof is a new family of inequalities, called modified quadratic transportation cost inequalities in the spirit of the modified logarithmic-Sobolev inequalities by Bobkov and Ledoux [6], that are shown to hold as soon as a Poincaré inequality is satisfied.

Key words: Transportation inequalities, spectral gap, Gaussian concentration.

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1. Introduction, framework and main results.

Transportation inequalities recently deserved a lot of interest, especially in connection with the concentration of measure phenomenon (see [17], [18]). Links with others renowned functional inequalities, in particular logarithmic-Sobolev inequalities, were also particularly studied (see [5], [21], [4], [18] ...), as no direct or tractable criteria were available for this kind of inequalities.

Given a metric space $(E,d)$ equipped with its Borel $\sigma$ field, the $L^p$ Wasserstein distance between two probability measures $\mu$ and $\nu$ on $E$ is defined as

$$W_p(\mu, \nu) = \left( \inf_{\pi} \int_{E \times E} d^p(x, y) \pi(dx, dy) \right)^{1/p},$$

where $\pi$ describes the set of all coupling of $(\mu, \nu)$, i.e. the set of all probability measures on the product space with marginal distributions $\mu$ and $\nu$.

A probability measure $\mu$ is said to satisfy the $T_p(C)$ transportation cost inequality if for all probability measure $\nu$,

$$W_p(\mu, \nu) \leq \sqrt{2C H(\nu, \mu)},$$

where $H(\nu, \mu)$ stands for the Kullback-Leibler information (or relative entropy), i.e.

$$H(\nu, \mu) = \int \log \frac{d\nu}{d\mu} d\nu \quad \text{if } \nu \ll \mu \quad ; \quad +\infty \quad \text{otherwise.}$$
As shown by K. Marton ([19]), \( T_1 \) implies a Gaussian type concentration for \( \mu \).
Let us briefly recall the general argument, we shall use later.
For any Borel set \( A \) with measure \( \mu(A) \geq 1/2 \) introduce \( A_r^c = \{x, d(x, A) \geq r\} \) and \( d\mu_A = \frac{1}{\mu(A)} d\mu \). Set \( B \) for \( A_r^c \) and assume that \( W_1(\nu, \mu) \leq \varphi(H(\nu, \mu)) \) for all \( \nu \). Then
\[
(1.3) \quad r \leq W_1(\mu_B, \mu_A) \leq W_1(\mu_B, \mu) + W_1(\mu, \mu_A) \leq \varphi(H(\mu_A, \mu)) + \varphi(H(\mu_B, \mu)) = \varphi \left( \log \frac{1}{\mu(A)} \right) + \varphi \left( \log \frac{1}{\mu(A_r^c)} \right).
\]
When \( \varphi(u) = \sqrt{2Cu} \) we immediately obtain
\[
\mu(A_r^c) \leq \exp \left( -1/2C \left( r - \sqrt{2C \log(\frac{1}{\mu(A)})} \right)^2 \right).
\]
Hence criteria for \( T_1 \) to hold are very useful. Such a criterion was first obtained by Bobkov and Götze ([5] Theorem 3.1) and recently discussed by Djellout, Guillin and Wu ([12] Theorem 2.3) where the following is proved

**Theorem 1.4.** ([12]) \( \mu \) satisfies \( T_1 \) if and only if there exist \( \varepsilon > 0 \) and \( x_0 \in E \) such that
\[
\int_E e^{\varepsilon d^2(x,x_0)} \mu(dx) < +\infty.
\]

Unfortunately \( T_1 \) is not well adapted to dimension free bounds, while \( T_2 \) is, as shown by Talagrand ([25]). The first example of measure satisfying \( T_2 \) is the standard Gaussian measure ([25]), for which \( C = 1 \). When \( E \) is a complete smooth Riemannian manifold of finite dimension, with \( d \) the geodesic distance and \( dx \) the volume measure, Otto and Villani ([21]) have studied the \( T_2 \) property for absolutely continuous probability measures (Boltzmann measures)

\[
(B.M) \quad \mu(dx) = e^{-V(x)} dx,
\]
for \( V \in C^2(E) \) in connection with the logarithmic-Sobolev inequality. Their method was recently improved by Wang ([32]) in order to skip the curvature assumption made in [21].

In the sequel we shall assume that \( \mu \) is a Boltzmann measure with \( V \in C^3 \), and that the diffusion process built on \( E \) with generator \( L = 1/2 \text{div}(\nabla) - 1/2 \nabla V.\nabla \) is non explosive.
This is assumption (A) in [32]. Conditions for non explosion are known. Here are some among others when \( E = \mathbb{R}^d \):

- \( V(x) \to +\infty \) as \( |x| \to +\infty \) and \( |\nabla V|^2 - \Delta V \) is bounded from below,
- \( x.\nabla V(x) \geq -a|x|^2 - b \) for some \( a \) and \( b \) in \( \mathbb{R} \),
- \( \int |\nabla V|^2 \, d\mu < +\infty \).

For the first two see e.g. [21] p.26, for the third one see e.g. [9].

The first result is thus
Theorem 1.5. \cite{21,4,32}, (also see \cite{11}) Let \( \mu \) be as above with finite moment of order 2. If \( \mu \) satisfies the logarithmic-Sobolev inequality (L.S.I)

\[
\int g^2 \log(g^2) \, d\mu - \left( \int g^2 \, d\mu \right) \log \left( \int g^2 \, d\mu \right) \leq 2C \int |\nabla g|^2 \, d\mu,
\]

for all smooth \( g \), then \( \mu \) satisfies \( T_2(C) \).

A partial converse of Theorem 1.5 is also shown in \cite{21} (Corollary 3.1), namely

Theorem 1.6. \cite{21,4} Let \( \mu \) be as above with finite moment of order 2, and \( E = \mathbb{R}^n \). If \( \mu \) satisfies \( T_2(C) \) and the curvature assumption

\[ \text{Hess}(V) \geq K Id_n \]

for some \( K \in \mathbb{R} \), then \( \mu \) satisfies a logarithmic-Sobolev inequality (with some new constant \( \bar{C} \)), provided

\[ 1 + KC > 0. \]

The latter restriction is very important and has to be compared with Wang's results (\cite{28} and \cite{31}) telling that a logarithmic-Sobolev inequality holds provided the curvature assumption above and the integrability condition \( EI_x(2) \) in Theorem 1.4 hold with

\[ \varepsilon + K > 0. \]

In other words, according to Theorem 1.4 and Theorem 1.6 under the curvature assumption, log-Sobolev, \( T_1(C_1) \), \( T_2(C_2) \) are all equivalent for appropriate constants \( C_1 \) and \( C_2 \). Whether this equivalence holds without restrictions on the constants or not was left open by these authors.

Let us recall that another approach of Theorems 1.5 and 1.6 was introduced by Bobkov, Gentil and Ledoux (\cite{4}). First of all the general Monge-Kantorovich duality theory indicates that for \( p \geq 1 \),

\[
W_p^p(\nu, \mu) = \sup \left( \int g \, d\nu - \int f \, d\mu \right),
\]

where the supremum is running over all pairs \( (f, g) \) of measurable and bounded functions satisfying

\[
g(x) \leq f(y) + d^p(x, y),
\]

for every \( x, y \in E \). In the infimum-convolution notation of Maurey (\cite{20}),

\[ Qf(x) = \inf_{y \in E} \left( f(y) + d^p(x, y) \right) \]

achieves the optimal choice. Defining

\[ Q_t f(x) = \inf_{y \in E} \left( f(y) + \frac{1}{t} d^2(x, y) \right) \]

one thus introduces a semi-group satisfying the Hamilton-Jacobi initial value problem. Relying some kind of hypercontractivity of this semi-group to the logarithmic-Sobolev inequality, these authors obtain both Theorems 1.5 and 1.6 (without any curvature assumption for 1.6 improving Otto and Villani result as and before Wang’s result, also see \cite{22}). In particular, the following originally due to Otto and Villani is elementary shown in \cite{4} subsection 4.1
Theorem 1.9. Let $\mu$ be as above. If $\mu$ satisfies $T_2(C)$ then $\mu$ satisfies the Poincaré (or spectral gap) inequality (S.G.I) i.e. for all smooth $f$ ,
\[
Var_\mu(f) \leq C \int |\nabla f|^2 \, d\mu.
\]
This result gives us a first hint on what should be the difference between $T_1$ and $T_2$ as $T_1$ is well known to hold when (S.G.I.) fails (see [12], Remark 2.4).

One aim of the present paper is to show that actually

Theorem 1.10. Let $\mu$ be as above. Then $\mu$ satisfies $T_2$ if and only if $\mu$ satisfies some Poincaré inequality, the integrability condition $EI_\varepsilon(2)$ of Theorem 1.4 and the following property:

there exists some $a > e^{3/2}$ and some constant $c(a)$ such that for all $\nu = h \mu$ with $H(\nu, \mu) \leq 1/2$
\[
\text{(Tronc)} \quad W_2^2(\nu_a, \nu) \leq c(a) H(\nu, \mu),
\]
where $\nu_a = (1/\nu(h \leq a)) h \mathbf{1}_{h \leq a} \mu$.

An explicit upper bound of the constant of this $T_2$ inequality in terms of the constants arising in the Poincaré’s inequality, $EI_\varepsilon(2)$, choice of $a$ and $c(a)$ can be computed (and $c(a)$ being given optimized in $a$). We shall see that, furthermore, if $EI_\varepsilon(2)$ holds, (Tronc) is implied by the following Variance-Entropy property
\[
(\text{Var - Ent}) \quad \int d^2(x, x_0) \mathbf{1}_{h > a} \, d\nu \leq D(a) H(\nu, \mu),
\]
for $\nu$ as before.

The proof of Theorem 1.10 lies on the recent work by Wang [32]. The limitation to the finite dimensional setting is due to the fact we want to use Otto-Villani coupling technique as in section 2 of [32]. However, one expects that Theorem 1.10 extends to infinite dimensional settings, as path spaces. Indeed Theorem 1.10 is extended to this setting in [32] section 5 by using finite dimensional approximation (also see the final section in [12] for an approach using Girsanov transform), and Monge-Ampère theory was extended to this setting by Feyel and Ustunel (13 and 14). This will not be studied here.

The proof of Theorem 1.10 splits into two parts. In section 2 we shall show that (S.G.I) implies some transportation inequality for measures $\nu$ with a bounded density. Actually we prove an interpolation result between (S.G.I) and (L.S.I) through a family of inequalities $I(\alpha)$ introduced by Latala and Oleszkiewicz (see [16]) for $0 \leq \alpha \leq 1$,
\[
I(\alpha) \sup_{p \in [1, 2)} \frac{\int f^2 \, d\mu - (\int f^p \, d\mu)^{2/p}}{(2 - p)^\alpha} \leq C(\alpha) \int |\nabla f|^2 \, d\mu.
\]
It is easily seen that $I(0)$ is the Poincaré inequality and $I(1)$ reduces to the logarithmic-Sobolev inequality. Our first result is the following
Theorem 1.12. Let \( \mu \) be as above. If \( I(\alpha) \) holds then for all \( \nu \) such that \( \| \frac{d\nu}{d\mu} \|_\infty \leq K \) the following modified transportation inequality holds
\[
W_2(\nu, \mu) \leq D(\alpha) \left( \log K \right)^{\frac{1-\alpha}{2}} \sqrt{C(\alpha) H(\nu, \mu)},
\]
where
\[
D(\alpha) = 16 \exp \left( \frac{1-\alpha}{2} (1 - \log(1-\alpha)) \right).
\]
Remark that the previous Theorem and Marton’s trick allow to recover the concentration property shown in [16]. Indeed, recall (1.3) and remark that the interesting \( K \) is given by
\[
K = \frac{1}{\mu(A_c^c)}.
\]
We immediately see that if \( I(\alpha) \) holds, \( \mu(A_c^c) \) behaves like \( \exp(-Cr^2\alpha^2) \).

Another characterization of \( I(0) \) (i.e. (S.G.I)) is obtained in [4] section 5.3 in terms of a mixed transportation cost \( W_L \). It is almost immediate that for some constants \( C \) and \( D \),
\[
CW_L \leq W_1(\nu, \mu) \leq D(H(\nu, \mu) + H_2^\frac{1}{2}(\nu, \mu)).
\]
But the behavior of Wasserstein metrics for large entropy is easily related to exponential integrability thanks to the following elementary lemma proved in section 3.

Lemma 1.13. Assume that \( \mu \) satisfies \( EI_\varepsilon(p) \) for some \( \varepsilon > 0 \). There exists a constant \( C(\varepsilon) \) such that for all \( \nu \) satisfying \( H(\nu, \mu) \geq 1 \), \( W_p(\nu, \mu) \leq C(\varepsilon) H(\nu, \mu) \).

The first consequence of Lemma 1.13 combined with Theorem 1.4 is that the transportation inequalities \( T_2 \) and \( T_1 \) are “equivalent” for large entropy. Since Marton’s method is essentially concerned with large entropy, \( T_2 \) cannot furnish a better concentration result than \( T_1 \).

The second consequence is that \( T_2 \) is mainly (and surprisingly) concerned with small entropy. That is why one can expect that the modified transportation inequality 1.12 together with a small entropy (so that the density cannot be too big except on a small set) will yield the statement in Theorem 1.10. The proof will be given in section 3.

At this point we shall mention that the proof of Lemma 1.13 is using the trivial independent coupling. We learned from F. Bolley and C. Villani [7] that, using a less trivial coupling in [26], this statement can be greatly improved, in particular

Proposition 1.14. Bolley and Villani
\[
EI_\varepsilon(p) \quad \Rightarrow \quad W_p(\nu, \mu) \leq C(\varepsilon) (H(\nu, \mu) + H_2^\frac{1}{2}(\nu, \mu)).
\]

Since (S.G.I) implies \( EI_\varepsilon(1) \), this result for \( p = 1 \) is stronger than the one we already recalled. Bolley and Villani are then able to get back Theorem 1.4 i.e. \( EI_\varepsilon(2) \) is equivalent to the transportation inequality \( T_1 \), but with some better constant than in 1.12.

Section 2 mainly contains the proof of Theorem 1.12. Section 3 contains the proofs of Lemma 1.13, Theorem 1.10 and related topics. In particular, going back to the proof of Theorem
one can see that the main term to be controlled is either $W^2_2(\nu_a, \nu)$ (using (Tronc)) or the left hand side in (Var-Ent). Elementary computations allow to control the later and show

**Theorem 1.15.** Let $\mu$ be as above.

1. If $EI_\varepsilon(2)$ holds and $a > c^{\frac{3}{2}}$ there exists some constant $c(a)$ such that for all $\nu$ with $H(\nu, \mu) \leq 1/2$

   $W^2_2(\nu, \mu) \leq W^2_2(\nu_a, \mu) + c(a) H(\nu, \mu) \log(1/H(\nu, \mu))$.

2. If $EI_\varepsilon(2)$ and Poincaré are satisfied, there exists some constant $C$ such that

   $W^2_2(\nu, \mu) \leq C \left( 1 + \sqrt{\log^+(1/H(\nu, \mu))} \right) H(\nu, \mu)$.

Even if this last inequality is not dimension free, one may use the concavity of $x \to x \sqrt{\log^+ x}$ to get some tensorization over the dimension for $\mu^\otimes n$ which thus verifies the preceding inequality with constant $C(n) = C \sqrt{\log n}$ (see [19] or [12, Th.2.5] for dependent sequences) to be compared to $C.n$ obtained with the sole $T_1$ inequality.

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## 2. Modified transportation inequalities.

**Proof. of Theorem 1.15.**
Let $\nu$ be a probability measure such that $h = \frac{dh}{d\mu}$ satisfies $0 < \beta \leq h(x) \leq K$. We assume first that $h \in \mathbb{D}$ i.e. is the sum of a constant and a $C^\infty$ function with compact support.

Let $P_t$ denotes the $\mu$ symmetric semigroup with generator $L = 1/2 \div(\nabla) - 1/2 \nabla V.\nabla$, and define $\mu_t = (P_t h) \mu$.

Our method relies on Otto-Villani’s coupling [21], refined by Wang [32], whose idea is the following: to provide a coupling between $\mu_t$ and $\mu_{t+s}$ as $\pi_s(dx, dy) = \mu_t(dx) \delta_{\varphi_s(x)}(dy)$ where $\varphi_s$ is the well defined unique (under our assumptions) solution of the p.d.e.

$$\frac{d}{ds}\varphi_s = -\xi_{t+s} \circ \varphi_s, \quad \varphi_0 = Id, S \geq 0$$

with $\xi_{t+s}(x) = \nabla \log P_{t+s} h(x)$.

Then, according to Otto and Villani [21], Lemma 2 (or more exactly its proof), or Wang [32] section 3,

$$A = \frac{d^+}{dt} (-W_2(\mu_t, \mu)) \leq \limsup_{s \to 0^+} \frac{1}{s} W_2(\mu_t, \mu_{t+s}) \leq 2 \left( \int |\nabla \sqrt{P_t h}|^2 d\mu \right)^{\frac{1}{2}}.$$
Using $I(\alpha)$ we obtain for all $1 \leq p < 2$,
\begin{equation}
A \leq \frac{2\sqrt{C(\alpha)}(2-p)^\alpha}{\sqrt{1 - \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}}} \int |\nabla \sqrt{P_t h}|^2 \, d\mu.
\end{equation}

Now using a similar argument as in Lemma 3.1 in [32] or simply the fact that $D$ is a nice core for the diffusion semigroup, the following computation is rigorous
\begin{equation}
\frac{d}{dt} \left(1 - \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}\right) = -\frac{1}{2} \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p} - 1} \int (P_t h)^{\frac{p}{2} - 1} \, d\mu
\end{equation}
\begin{align*}
&= \frac{1}{2} \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p} - 1} \int \left(\frac{p}{2} - 1\right) (P_t h)^{\frac{p}{2} - 2} |\nabla P_t h|^2 \, d\mu \\
&\leq 0.
\end{align*}

Here we have used $\int (\varphi'(g) L g + \varphi''(g) |\nabla g|^2) \, d\mu = 0$, with $\varphi(g) = g^{\frac{p}{2} - 1}$.

But since $h \leq K$, $P_t h \leq K$ hence according to \[(2.2)\] and \[(2.3)\]
\begin{equation}
A \leq \frac{2\sqrt{C(\alpha)}(2-p)^\alpha}{\sqrt{1 - \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}}} \frac{K^{1 - \frac{p}{2}}}{\left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}} \int |\nabla \sqrt{P_t h}|^2 \, d\mu
\end{equation}
\begin{align*}
&\leq -\frac{4\sqrt{C(\alpha)}(2-p)^\alpha}{\sqrt{1 - \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}}} \frac{K^{1 - \frac{p}{2}}}{\left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}} \frac{d}{dt} \left(1 - \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}\right) \\
&\leq 16 \sqrt{C(\alpha)}(2-p)^{\frac{p}{2} - 1} K^{1 - \frac{p}{2}} \left(\int \frac{d}{dt} \left(1 - \left(\int (P_t h)^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}\right)\right).
\end{align*}

For the latter inequality we have used $\int (P_t h)^{\frac{p}{2}} \, d\mu \leq 1$.

It remains to integrate in $t$. Since $I(\alpha)$ implies (S.G.I), we know that $P_t h$ goes to 1 in $L^2(\mu)$ as $t$ goes to infinity. Arguing as in [32] p.10, one can show that $W_2(\mu_t, \mu)$ goes to 0 as $t$ goes to $\infty$, so that we have obtained
\begin{equation}
W_2(\nu, \mu) \leq 16 \sqrt{C(\alpha)}(2-p)^{\frac{p}{2} - 1} K^{1 - \frac{p}{2}} \sqrt{\left(1 - \left(\int h^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}\right)}
\end{equation}
\begin{align*}
&\leq 16 \sqrt{C(\alpha)}(2-p)^{\frac{p}{2} - 1} K^{1 - \frac{p}{2}} \sqrt{\left(1 - \left(\int h^{\frac{p}{2}} \, d\mu\right)^{\frac{1}{p}}\right)}.
\end{align*}
Now we shall use the two following elementary inequalities for $p \in [1, 2)$:

1. $1 - u^2 \leq \frac{2}{p} (1 - u)$ for $u \in [0, 1]$;
2. $\xi \log \xi + 1 - \xi \geq 0$ for $\xi > 0$.

The former yields log $\xi^k \geq 1 - \xi^{-k}$, hence $\xi \log \xi^k \geq \xi - \xi^{1-k}$ and finally for $k = 1 - \frac{p}{2}$, $(1 - \frac{p}{2}) \xi \log \xi \geq \xi - \frac{p}{2}$. We apply this with $h(x) = \xi$, integrate with respect to $\mu$ and use the former inequality in order to get

$$1 - (\int h^p \, d\mu)^{\frac{1}{p}} \leq \frac{2}{p} \left(1 - \frac{p}{2}\right) H(\nu, \mu).$$

Plugging (2.6) into (2.5) furnishes (using $p \geq 1$)

$$W_2(\nu, \mu) \leq 16 \sqrt{C(\alpha)} (2 - p)^{\frac{p-1}{2}} K^{1 - \frac{p}{2}} \sqrt{H(\nu, \mu)}.$$

It is now enough to optimize in $p$. The optimal value is obtained for $2 - p = \frac{1-\alpha}{\log K}$, and a simple calculation yields the exact bound in Theorem 1.12.

It remains to extend the result to densities $h$ that are no more bounded away from 0, by using standard tools.

One may ask whether this modified transportation inequality is dimension free. It does not seem so. Actually the only kind of modified inequalities we are able to tensorize (following the induction method in [25]) are the ones where we replace $(\log K)^{1-\frac{p}{2}}$ by $K^\theta$ for $\theta > 1/2$.

For the concentration property, such a bound furnishes a polynomial tail estimate for $\mu(A^c)$, precisely $(1/r)^{\frac{1}{2}}$ which is not really exciting.

3. Exponential integrability and the proof of Theorem 1.10

We start this section by the proof of the elementary Lemma 1.13 showing that the obstruction for $T_2$ to hold is in a neighborhood of $\mu$. Notice that except for the conclusion (i.e. Theorem 1.10 itself) all the intermediate results are available in a general metric space.

**Proof of Lemma 1.13.**

Introduce the Young function

$$\tau(u) = u \log^+(u),$$

and its Legendre conjugate function $\tau^*(v) = v \mathbb{1}_{v<1} + e^{v-1} \mathbb{1}_{v \geq 1}$.

Among all possible coupling of $(\mu, \nu)$, the simplest one is the independent one i.e. if we denote $h = \frac{d\nu}{d\mu}$,

$$\pi(dx, dy) = h(x) \mu(dx) \mu(dy).$$

Accordingly

$$W_p^p(\nu, \mu) \leq \int d^p(x, y) h(x) \mu(dx) \mu(dy) \leq 2 N_r(h) N_r(d^p),$$

where $N_r(h) N_r(d^p)$ is the metric entropy of $\nu$ with respect to $d^p$.
where $N_\tau$ and $N_{\tau^*}$ are the gauge norms in the corresponding Orlicz spaces, the second inequality being the classical Hölder-Orlicz inequality (see e.g. [23] for all concerned with Orlicz spaces). Recall that the gauge norm for a general Young function $\psi$ is defined as

$$N_\psi(g) = \inf \{ \lambda > 0, \int \psi(g/\lambda)(x,y) \, \mu(dx) \, \mu(dy) \leq 1 \},$$

such that an easy convexity argument yields

$$N_\psi(g) \leq \max \{ 1, \int \psi(g) \, d\mu \otimes d\mu \}.$$  

(3.2)

In addition remark that

$$\int h \log^+(h) = \int h \log(h) - \int_{h<1} h \log(h) \leq \int h \log(h) + 1/e.$$  

Hence if $H(\nu,\mu) \geq 1$,

$$1 \leq \int h \log^+(h) \leq (1 + 1/e) H(\nu,\mu),$$

and according to (3.2) and what precedes

$$W^p_\nu(\nu,\mu) \leq 2(1 + 1/e) N_{\tau^*}(d^p) H(\nu,\mu).$$

Finally, thanks to $I_\varepsilon(p)$, $N_{\tau^*}(d^p) < +\infty$ and the result follows.

One can improve the preceding result by showing that (up to the constant) it holds for $H(\nu,\mu)$ bounded away from 0. But as quoted in Proposition 1.14 one can also get a precise bound for the behavior of the Wasserstein distances when entropy goes to 0.

**Proof. of Theorem 1.10.** We now proceed with the proof of Theorem 1.10. It breaks into several lemmata. According to Lemma 1.13 and (3.2) we may and will assume that $H(\nu,\mu)$ is small enough.

**Lemma 3.3.** Let $\nu = h\mu$ be a probability measure. If $a > e$, then

1. $H(\nu,\mu) \geq (1 - 1/\log a) \int_{h>a} h \log h \, d\mu$,
2. $\nu(h > a) \leq \left( 1/ (\log a - 1) \right) H(\nu,\mu)$.

**Proof.** Again we start with $u \log u + 1 - u \geq 0$ which yields

$$\int_{h\leq a} h \log h \, d\mu + 1 - \int_{h\leq a} h \, d\mu \geq 0,$$

hence

$$H(\nu,\mu) \geq \int_{h>a} h \log h \, d\mu - \nu(h > a).$$

(2) follows immediately since $\log h > \log a$ on $\{h > a\}$. For (1) we have

$$\nu(h > a) \leq \int_{h>a} \frac{\log h}{\log a} h \, d\mu = (1/\log a) \int_{h>a} h \log h \, d\mu.$$
Now we introduce a cut-off for $\nu$ i.e. if $a > 0$ we define

\begin{equation}
\nu_a = \left(1/\nu(h \leq a)\right) h \mathbb{1}_{h \leq a} \mu.
\end{equation}

**Lemma 3.5.** Let $\nu = h \mu$ be a probability measure such that $H(\nu, \mu) \leq 1/2$. If $a > e^{\frac{3}{2}}$ and $\nu_a$ is given by (3.4), then

\[H(\nu_a, \mu) \leq \left(1 + \frac{1}{2(\log a - 3/2)} + \frac{2}{\log a - 1}\right) H(\nu, \mu).\]

**Proof.**

\[
H(\nu_a, \mu) = \int \frac{h \mathbb{1}_{h \leq a}}{\nu(h \leq a)} \log \left(\frac{h}{\nu(h \leq a)}\right) d\mu \\
\leq H(\nu, \mu) + \left((1/\nu(h \leq a)) - 1\right) \int_{h \leq a} h \log h d\mu \\
- \log(\nu(h \leq a)) - \int_{h > a} h \log h d\mu \\
\leq H(\nu, \mu) + \frac{\nu(h > a)}{\nu(h \leq a)} H(\nu, \mu) - \log(1 - \nu(h > a)).
\]

But if $0 \leq x \leq 1/2$, $-\log(1 - x) \leq 2x$, hence according to (3.3)\((2)\), if $H(\nu, \mu) \leq 1/2$, $\log(1 - \nu(h > a)) \leq (2/(\log a - 1)) H(\nu, \mu)$ and

\[
\frac{\nu(h > a)}{\nu(h \leq a)} \leq \frac{H(\nu, \mu)}{\log a - 1 - H(\nu, \mu)}
\]

and we get the desired result. \hfill \Box

We shall now proceed with the proof of an intermediate result: Poincaré, $EI_{\epsilon}(2)$ and (Var-Ent) imply $T_2$.

Recall the dual formulation of $W_2$ in (1.8) and (1.9) i.e.

\[W_2^2(\nu_a, \mu) = \sup \left( \int g d\nu_a - \int f d\mu \right)\]

for $f$ and $g$ such that for all $x$ and $y$ $g(x) \leq f(y) + d^2(x, y)$.

Remark that in the above formula we may add the same constant to both $f$ and $g$ so that we may assume that $\int f d\mu = 0$. In this case, integrating with respect to $\mu(dy)$ the condition \((1.9)\) we have

\[
g(x) \leq \int f d\mu + \int d^2(x, y) \mu(dy) \\
\leq 2d^2(x, x_0) + 2 \int d^2(y, x_0) \mu(dy) = q_2(x).
\]
Now
\[ \int g \, d\nu_a = \int g \, \frac{h \mathbb{1}_{h \leq a}}{\nu(h \leq a)} \, d\mu \]
\[ = \frac{1}{\nu(h \leq a)} \int g \, d\nu - \frac{1}{\nu(h \leq a)} \int g \, h \mathbb{1}_{h > a} \, d\mu \]
\[ \geq \frac{1}{\nu(h \leq a)} \int g \, d\nu - \frac{1}{\nu(h \leq a)} \int q_2 \, h \mathbb{1}_{h > a} \, d\mu. \]

Hence, since \( \nu(h \leq a) \leq 1 \),
\[ (3.6) \quad W^2_2(\nu, \mu) \leq W^2_2(\nu_a, \mu) + \int q_2 \, h \mathbb{1}_{h > a} \, d\mu. \]

Recall that \( q_2 \) is the sum of a constant term and \( 2d^2(x, x_0) \). So we have to control
\[ (3.7) \quad \int h \mathbb{1}_{h > a} \, d\mu = \nu(h > a), \]
and
\[ (3.8) \quad \int d^2(x, x_0) \, h \mathbb{1}_{h > a} \, d\mu, \]
by some constant times \( H(\nu, \mu) \). For (3.7) we may just use (3.3) (2), and for (3.8) we may just use the hypothesis (Var-Ent) in Theorem 1.10. So applying successively (3.6), (3.7), (3.8), Theorem 1.12 and Lemma 3.5, if \( H(\nu, \mu) \leq 1/2 \)
\[ W^2_2(\nu, \mu) \leq W^2_2(\nu_a, \mu) + c(a) H(\nu, \mu) \]
\[ \leq D^2(0) C(0) \log a H(\nu_a, \mu) + c(a) H(\nu, \mu) \]
\[ \leq C(\alpha, a) H(\nu, \mu). \]

For \( H(\nu, \mu) \geq 1 \) we may use Lemma 1.13 and for \( H(\nu, \mu) \in [1/2, 1] \) we may use (3.1) and (3.2) and get
\[ W^2_2(\nu, \mu) \leq 2N_{\tau^*}(d^2) \leq 4N_{\tau^*}(d^2) H(\nu, \mu). \]

Hence we have proved that Poincaré, \( EI_\epsilon(2) \) and (Var-Ent) imply \( T_2 \), that is a consequence of Theorem 1.10.

We did so because we shall use this method later to evaluate (3.8) when (Var-Ent) property fails to hold. Furthermore, (Var-Ent) is well suited to study (Tronc).

Indeed, according to a well known result in mass transportation theory (see Proposition 7.10) if (Var-Ent) is satisfied, if \( a > e^{3/2} \) and \( H(\nu, \mu) \leq 1/2 \),
\[ W^2_2(\nu, \nu_a) \leq C \int d^2(x, x_0) \left| 1 - \mathbb{1}_{h \leq a} \right| \, d\nu, \]
\[ \leq C \left( \frac{\nu(h > a)}{\nu(h \leq a)} \right) \int_{h \leq a} d^2(x, x_0) \, d\nu + \int_{h > a} d^2(x, x_0) \, d\nu \]
\[ \leq C' H(\nu, \mu), \]
according to Lemma 3.3 the Hölder-Orlicz inequality, \( EI_\epsilon(2) \) and (Var-Ent). Hence (Tronc) is a consequence of (Var-Ent), provided \( EI_\epsilon(2) \) is satisfied.
To finish, we proceed with the end of the proof of Theorem 1.10. For one way, it is enough to write for $H(\nu, \mu) \leq 1/2$

$$W_2^2(\nu, \mu) \leq 2 W_2^2(\nu_a, \mu) + 2 W_2^2(\nu_a, \nu) \leq C' H(\nu, \mu)$$

according to the distance property of $W_2$, Theorem 1.12, Lemma 3.5 and the (Tronc) property. Conversely we already know that $T_2$ implies both a Poincaré inequality and $EI_\varepsilon(2)$. It remains to show that it also implies (Tronc). But if $H(\nu, \mu) \leq 1/2$,

$$W_2^2(\nu, \nu_a) \leq 2 W_2^2(\nu_a, \mu) + 2 W_2^2(\nu_a, \nu) \leq 2 C H(\nu, \mu) + H(\nu_a, \mu) \leq C' H(\nu, \mu)$$

according to the distance property, $T_2$ and Lemma 3.5.

To conclude this section we shall proceed with the proof of Theorem 1.15.

**Proof. of Theorem 1.15**

**Part (1).** According to (3.6)-(3.8) all we have to do is to estimate

$$\int_{h > a} d^2(x, x_0) d\nu.$$ 

Applying again the Hölder-Orlicz inequality and $EI_\varepsilon(2)$ what we have to do is to estimate the Orlicz norm

$$N_\tau(h \mathbb{1}_{h > a}),$$

i.e. we have to estimate $\lambda$ such that

$$\int_{h > a} \frac{h}{\lambda} \log \left( \frac{h}{\lambda} \right) d\mu \leq 1.$$

According to Lemma 3.3 it is enough to have

$$\frac{1}{\lambda} H(\nu, \mu) + \frac{1}{\lambda} \log \left( \frac{1}{\lambda} \right) \nu(h > a) \leq 1,$$

and it is easily seen that $\lambda \leq C(a) H(\nu, \mu) \log(1/H(\nu, \mu))$.

**Part (2).** We shall be more accurate with the previous estimate. Indeed if $EI_\varepsilon(2)$ and Poincaré are satisfied, it holds

$$W_2^2(\nu, \mu) \leq W_2^2(\nu_K, \mu) + \int_{h > K} d^2(x, x_0) d\nu \leq C_1 \log \left( K/\nu(h \leq K) \right) H(\nu_K, \mu) + \int_{h > K} d^2(x, x_0) d\nu \leq C_2 \log(K) H(\nu, \mu) + \int_{h > K} d^2(x, x_0) d\nu,$$

where we used successively (3.6)-(3.8), Theorem 1.12, Lemma 3.5 and previous estimates (for small entropy, and large $K$).
Now we choose $K = 1/H^q(\mu, \nu)$ for some $q > 0$ and we assume that $H(\nu, \mu)$ is small enough (we already saw it is not a restriction). Lemma 3.3(2) furnishes
\begin{equation}
\nu(h > K) \leq H(\nu, \mu)/q \log(1/H(\nu, \mu)),
\end{equation}
so that the computation of $N_\tau(h \mathbf{1}_{h > K})$ as in (3.9)-(3.10) yields this time $N_\tau(h \mathbf{1}_{h > K}) = C q^{-1} H(\nu, \mu)$. Plugging this estimate into (3.11) yields
\begin{equation}
W^2_2(\nu, \mu) \leq (C_2 q \log(1/H(\nu, \mu)) + C_3 q^{-1}) H(\nu, \mu),
\end{equation}
and the result follows optimizing in $q$ and using the same arguments as before for large entropy. 
\[\square\]

**Remark 3.14.** We hardly tried to improve the above estimates. For instance one can reduce the problem to estimate
\[\int_{1/H^q \geq e^{h^p} \geq h^p \geq K^p} d^2(x, x_0) \, d\nu\]
for some fixed $K, p > 0$ large, $q > 0$ small (this is left to the reader). Unfortunately we did not succeed in removing the extra $\log(1/H(\nu, \mu))$ in this estimate, hence in Theorem 1.15. Actually we do not know whether this is possible or not, only assuming Poincaré and the exponential integrability. However we shall see in the next section that for some less general potentials $V$ one can do the job.

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