Chemical factors in canonical statistical models for relativistic heavy ion collisions

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We study the effect of enforcing exact conservation of charges in statistical models of particle production for systems as large as those relevant to relativistic heavy ion collisions. By using a numerical method developed for small systems, we have been able to approach the large volume limit keeping the exact canonical treatment of all relevant charges, namely baryon number, strangeness and electric charge. Hence, we hereby give the information needed in a hadron gas model whether the canonical treatment is necessary or not in actual cases. Comparison between calculations and experimental particle multiplicities is shown. Also, a discussion on relative strangeness chemical equilibrium is given.

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I. INTRODUCTION

Over the era of relativistic nuclear collisions, the statistical-thermal models have been widely used in the analysis of particle production. During the past few years, these models have been successful in describing particle multiplicities in high energy nuclear reactions at the stage where inelastic collisions between hadrons cease (chemical freeze-out), see e.g. \cite{1, 2, 3, 4, 5}.

The easiest to handle, thence the most used, statistical model calculational framework is based on the grand canonical (GC) ensemble. In the GC approach entropy is maximized using the constraints of conserved ensemble averages of energy and charges. This allows the net charges to fluctuate from sample to sample even though the actual charges brought into the physical reactions were exactly same every time. In the large volume and energy limit, the error in such an assumption due to charge fluctuations is negligible.

On the other hand, when the involved volumes are small, like in $e^+e^-$, $p\bar{p}$, $pA$ reactions or in peripheral $AA$ reactions, the charge fluctuations allowed in GC formalism give a large theoretical error. So, the GC ensemble turns out to be inadequate for the analysis of experimental results and the statistical ensemble to be used is rather the canonical one, enforcing exact conservation of relevant charges. In this case entropy is maximized using the constraint of conserved ensemble average of energy, but charges are not allowed to fluctuate.

Statistical models based on the canonical ensemble have been able to reproduce particle multiplicities even in elementary collisions, although in those there is clearly no room for kinetic thermalization \cite{6, 7}. Also, the strangeness enhancement with increasing system size in $p-A$ reactions can be explained by a canonical effect \cite{8}.

In this work, we investigate whether and to what extent the canonical treatment is relevant for actual heavy ion collisions. The main difficulty of canonical calculations is related to the involved large values of baryon number and electric charge, which make the numerical computation of canonical partition functions quite hard. We present here an efficient numerical method to carry out such calculations which enabled us to implement exact charge conservation up to baryon number $O(100)$ whereas with old methods it was only possible to reach $B \sim 20$ \cite{8} with very long computation times.

After deriving the needed expressions of mean particle numbers, we present a systematical study of the chemical factors appearing in the canonical statistical model. Along with some comparison between grand-canonical and canonical calculations, we show a comparison with experimental results for peripheral to central $A-A$ reactions at AGS and SPS energies.

In section IV, we briefly address the role of the so-called relative strangeness chemical equilibrium, usually parametrized with $\gamma_S$, in canonical statistical-thermal models.

II. THE ANALYTICAL DEVELOPMENT

In nuclear physics, the problem of calculating the relativistic canonical partition function has been handled by several different methods, see e.g. \cite{9, 10, 11, 12}. Here, we employ a very general group theoretical method – applicable to any internal symmetry represented by a semi-simple Lie algebra – first introduced by Cerulius \cite{13, 14, 15}.

By denoting the set of conserved quantum numbers by $\{C_i\}$, the canonical partition function $Z_{\{C_i\}}$ can be obtained from the usual grand-canonical partition function

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oscillating for large net quantum numbers, so it is worth
approach, which has been applied in elementary collisions
than
functions
rotated fugacity factor
U
Z
where \( \phi_i \in [0, 2\pi] \) is a \( U(1) \) group parameter and a Wick-
rotated fugacity factor \( \lambda_{C_i} = e^{i\phi_i} \) is introduced, for every
charge \( C_i \).
In order to calculate \( Z_{GC} \) in the above integral, we will
assume the Boltzmann statistics for all hadron species.
The error involved in such an approximation for final par-
ticle multiplicities, when resonance decays is taken into
account, is some percent in case of pions and much less for
all other hadrons. Quantum statistics may be used, but
requires somewhat heavier calculations and disregarding
it does not essentially affect any forthcoming argument.

In heavy ion reactions, the relevant set of conserved charges is \( \{C_i\} = B, S, Q \), namely baryon number, strangeness and electric charge. Denoting
the partition function of hadrons carrying none of
the previous charges by \( Z_0 \), and writing \( Z_{GC} \) by using the one-particle partition function
\( z_i^1 = [(2J_i + 1)V/(2\pi)^3] \int d^3p \exp[-\sqrt{p^2 + m_i^2}/T] \) for each hadron \( i \),
we find

\[
Z_{B,S,Q}(T, V) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi_B e^{-iB\phi_B} \int_0^{2\pi} d\phi_S e^{-iS\phi_S} \int_0^{2\pi} d\phi_Q e^{-iQ\phi_Q} \times \exp \left\{ \sum_i z_i^1 \left[ e^{i(B,\phi_B+S,\phi_S+Q,\phi_Q)} + e^{-i(B,\phi_B+S,\phi_S+Q,\phi_Q)} \right] \right\},
\]

(2)

The direct numerical computation of the triple integral
above is a formidable task for \( B \geq 5 \). The integral can be
turned into a sum over many indices of modified Bessel
functions \( I_n(x) \), yet its evaluation is very time consum-
ing and becomes impractical with baryon number larger
than \( \sim 10 \). Therefore, we have chosen another ap-
proach, which has been applied in elementary collisions
\[ \text{Eq. \ref{Bb}} \text{ through Eq. \ref{Bb+1}} \]. From \ref{Bb+1} one finds the integrand to be violently
oscillating for large net quantum numbers, so it is worth
trying to eliminate analytically the source of strongest
oscillation; in heavy ion reactions, the baryon number is
always the largest of the set \( B, S, Q \), so it is certainly the
most beneficial to eliminate the integration over \( B \). In
fact, this can be done by taking advantage of a special
feature of baryon number, that is no elementary hadron
exists with \( |B_i| > 1 \). We first rewrite equation \ref{Bb} in the
form

\[
Z_{B,S,Q}(T, V) = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi_B e^{-iB\phi_B} \int_0^{2\pi} d\phi_S e^{-iS\phi_S} \int_0^{2\pi} d\phi_Q e^{-iQ\phi_Q} \times \exp \left[ \sum_{M,\overline{M}} z_i^1 e^{i(M,\phi_M+S,\phi_S+Q,\phi_Q)} \right],
\]

(3)

where the two summations run over baryons and mesons,
respectively. As \( |B_i| = 1 \) for all of the baryons, we can
write the baryon summation as:

\[
\sum_{B,\overline{B}} z_i^1 e^{i(B,\phi_B+S,\phi_S+Q,\phi_Q)} = \sum_B e^{iB\phi_B} \sum_{\overline{B}} z_i^1 e^{i(S,\phi_S+Q,\phi_Q)} + e^{-iB\phi_B} \sum_{\overline{B}} z_i^1 e^{-i(S,\phi_S+Q,\phi_Q)},
\]

(4)

where baryonic and antibaryonic terms have been sepa-
rated. Introducing the notation \( \sum_B z_i^1 e^{i(S,\phi_S+Q,\phi_Q)} = \omega \), the above equation can be rewritten as:

\[
\sum_{B,\overline{B}} z_i^1 e^{i(B,\phi_B+S,\phi_S+Q,\phi_Q)} = e^{iB\phi_B} \omega + e^{-iB\phi_B} \omega^* = e^{i(\phi_B + \text{arg} \omega)} |\omega| + e^{-i(\phi_B + \text{arg} \omega)} |\omega|.
\]

(5)

Substituting this expression in Eq. \ref{Bb+1} and changing the
baryon variable according to \( \phi_B \rightarrow \phi_B - \text{arg} \omega \) allows
us to perform analytically the integration in \( \phi_B \), which
yields a modified Bessel function $I_B$:

$$Z_{B,S,Q}(T,V) = \frac{Z_0}{(2\pi)^2} \int_0^{2\pi} d\phi_S \int_0^{2\pi} d\phi_Q \times \cos (S\phi_S + Q\phi_Q - B \arg(\omega(\phi_S, \phi_Q))) \times I_B(2|\omega(\phi_S, \phi_Q)|) \times \exp \left[ 2 \sum_i z_i^1 \cos (S_i \phi_S + Q_i \phi_Q) \right].$$  

(6)

Thus, we are left with a double integration only which can be then performed numerically with no major problem provided that, $B$ being very large, the uniform asymptotic expansion of Bessel functions for large orders is used to compute $I_B$.

It is worth noting, that the form $\langle N_i \rangle$ does not involve a saddle-point approximation used in $\langle N_i \rangle$ and similarly for other kinds of multistrange hadrons, while $\langle N_j \rangle$ stands for the sum $\sum_{S=3} \lambda_B^{B_s} \lambda_Q^{Q_s} z_i^1$. In this case, mean particle numbers read:

$$\langle N_i \rangle = \frac{Z_{S=3}(T,V,\lambda_B,\lambda_Q)}{Z_{S=3}(T,V,\lambda_B,\lambda_Q)} \lambda_B^{B_s} \lambda_Q^{Q_s} z_i^1.$$  

(11)

III. NUMERICAL RESULTS

First of all, we point out that we do not perform fits to experimental results in this paper. Indeed, as our main motivation is to test the statistical canonical formalism in the heavy ion collision regime, we only take some suggestive values for freeze-out temperatures and baryon densities from our recent work on the analysis of particle multiplicities in heavy ion systems in the laboratory momentum range from GSI SIS 1.7 A GeV to CERN SPS 158 A GeV and make comparisons between canonical and grand-canonical calculations for those values.

The peculiar feature of canonical formalism is a nonlinear volume behaviour of mean particle numbers because of the dependence of chemical factors on the volume according to eqs. (11). To show this effect, we first fix the baryon density $n_B$ so that the dependence of the chemical factors on the volume turns into a dependence on net baryon number. As the temperature is fixed too, then the grand-canonical baryon chemical potential can be calculated and the difference with the corresponding chemical factor can be studied. Furthermore, net strangeness $S$ and electric charge $Q$ are fixed by the initial conditions of the collision (namely zero and $(Z/A) \times B$ respectively).

The canonical effect is the most significant at comparatively low energies, such as 1.7 GeV per nucleon in SIS Au–Au collisions. In our recent analysis we find the
best fit to experimental results with thermal parameters $T \sim 50$ MeV and $\mu_B \sim 800$ MeV. These values yield, in the pointlike hadron gas framework, a baryon density lower than 0.1 fm$^{-3}$, which is used as a reference value in order to study the canonical effect in heavy ion collisions as a function of the number of participants - i.e. net baryon number - in the collision. In figure 1 the chemical factor $C_B$ is shown in comparison with its GC limit $\lambda_B$. Under these circumstances, the canonical effect related to the baryon chemical factor is found to decrease from 2.1% at $B = 2$ down to 0.7% at $B = 10$, and to further decrease rather slowly towards the GC limit with increasing baryon number or volume. For central Au–Au reactions, the number of participating nucleons is more than 300, so we can conclude that the canonical baryon chemical effect is negligible. However, this is not the case for strangeness. In figure 2 the theoretical production ratio $K^+$ over number of participating nucleons is shown to point out the canonical strangeness effect. Note that we have been able to compute the full chemical factor even for $B = 400$. The ratio increases very slowly towards the GC limit, thus strangeness must always be handled canonically when analysing SIS results. The relative difference between full canonical and strangeness-canonical results is 13% for $B = 2$, decreasing to 4% for $B = 10$ and to 1% for $B = 40$. It must be pointed out that the above results are calculated using an isospin symmetric initial configuration whereas $Z/A \approx 0.4$ for gold nucleus. However, this variation essentially gives no change on the relative differences above, although the ratio $K^+ / N_{\text{part}}$ decreases naturally due to the initial neutron excess.

In Au–Au collisions at AGS energies (11.6 A MeV momentum), the chemical freeze-out temperature is found to be around $T = 120$ MeV and the baryon density close to the normal nuclear density [1]. Using these values, we show the canonical baryon chemical factor in comparison with the GC limit in figure 3 keeping a symmetric isospin initial condition. Relative difference between $C_B$ and $\lambda_B$ decreases rather quickly from 23% at $B = 2$ to 1% at $B = 40$. A more realistic comparison of results in Au–Au with pp collisions requires different initial isospin configuration to be into account. By using temperature and baryon density quoted above, $C_B$ at $B = Q = 2$ turns out to be 81.9 and $\lambda_B = 118$ with $Q/B = 0.4$.

Recent experimental results on AGS $K/N_{\text{part}}$ ratios with increasing centrality [2] serve as a basis for a further study of strangeness production behaviour at $T = 120$ MeV. In figure 4 we plot the calculated $K^+ / N_{\text{part}}$ with results obtained in peripheral to central Au–Au, Si–Au and Si–Al collisions. It can be seen that all the way up from $B = 2$ to $B = 60$ the full canonical and strangeness-canonical results are essentially the same. All central reaction results lie on the region where the canonical effects are negligible and the GC formalism applies. Strangeness enhancement in Au–Au system does not definitely look like being of canonical origin, whereas Si–Au and Si–Al follow roughly the curve at the normal nuclear density.
squares are full canonical results. The curves are strangeness canonical while crosses are full canonical results.

FIG. 5: Theoretical $K^+/N_{\text{part}}$ curves at fixed temperature for two different baryon densities shown along with AGS experimental results. Curves are strangeness canonical while squares are full canonical results.

FIG. 6: Strange baryon enhancement in the conditions relevant to the SPS energies. Hadron multiplicities are normalized to full canonical results at $B = 2$. Curves are strangeness canonical while crosses are full canonical results.

reasonable canonical effect. All results in figure 6 are normalized to the full canonical baryon multiplicities per participant at the point $N_{\text{part}} = 2$. This choice reveals the slight difference between the results obtained using the $S$-canonical approximation and the $(B,S,Q)$-canonical calculation, and forces both canonical methods to reach for the same GC limit. An interesting feature here is the fact, that the $S$-canonical method leads to an overestimation of the canonical effect. Corresponding curves have also been calculated in ref. [21] within the strangeness canonical ensemble with some numerical approximations not used here. This enhancement picture has been calculated again with the assumption of an initial symmetric isospin configuration. Changing the freeze-out conditions to $T = 160$ MeV and $n_B = 0.17$ fm$^{-3}$, the total relative enhancement of $\langle N \rangle / N_{\text{part}}$ from canonical $B = Q = 2$ to GC $Q/B = 0.4$ raises to 30% for $\Lambda$'s, 241% for $\Xi$'s and 918% for $\Omega$'s.

SPS NA49 collaboration has also measured the participant dependence of $K^+/\pi^+$ ratio in pp, C–C, Si–Si, S–S and Pb–Pb reactions. In fig. we show that the roughly linear enhancement pattern is far away from the purely canonical shape. In the same figure, one can see the weak dependence of the $K^+/\pi^+$ ratio on thermal parameters. Indeed, theoretical curves always lie well above the measured points, which is a strong indication that strangeness is not in complete chemical equilibrium. Usually, this lack of equilibrium is taken into account by introducing a parameter $\gamma_S$ (see discussion in section 4), which, in the Boltzmann approximation, appears as a linear coefficient in front of the $K^+/\pi^+$ ratio. Taking our best fit value $\gamma_S \sim 0.8$ brings the theoretical curves asymptotically to the experimental central Pb–Pb reaction points. Curves with $T = 160$ MeV (lowest two) assume initial isospin symmetry, whilst the $S$ canonical curve with $T = 150$ MeV is corrected for the condition $Q/B = 0.4$. It is worth quoting our result for pp to cen-
where the chemical potentials for all the relevant charges have been grouped under the symbol $\mu$. The fugacity factor for the constrained number of strange quarks $e^{\beta n_s n_s}$ is usually called $\gamma_s$.

In the full canonical picture, where charges are not allowed to fluctuate from sample to sample, the constraint of the average number of strange and antistrange quarks can be turned into the enforcement of a fixed exact number of $s, \bar{s}$ quarks. Corresponding $U(1)_n$ symmetry yields another integration variable and another phase factor, $\exp(i n_s \phi_{n_s})$, into the projection integral [20] [17]. Following the procedure described in section [11] the average number of hadrons $h_i$ can be written as:

$$< h_i > = \frac{Z_{B,S,Q,n_s}}{Z_{B,S,Q,n_s}^1}. \quad (13)$$

In the grand canonical limit, the corresponding number is

$$\lim_{V \to \infty} < h_i > = \lambda_B^B \lambda_S^S \lambda_Q^Q \gamma_s^{n_s} \gamma_s^{n_s} \gamma_s^{n_s} \gamma_s^{n_s} \gamma_s^{n_s}. \quad (14)$$

As an illustrative example, take the average number of $\phi$ mesons. It does not carry any relevant charge, but the valence quark content is $s\bar{s}$. In the GC limit $< \phi > = \gamma_s^{2\phi}$. This leads us to the identification

$$\lim_{V \to \infty} \frac{Z_{B,S,Q,n_s}^{n_s}}{Z_{B,S,Q,n_s}^1} = \gamma_s^n. \quad (15)$$

There is one major caveat in those formulae and the use of $\gamma_s$. Indeed, the canonical formalism for the net conserved charges in the system is to be applied in any reaction where they are known from the initial state; if the system is large enough, the fluctuations allowed in GC ensemble give negligible deviation from canonical results and one is allowed to use formulae such as eq. [14]. However, the case of the number of strange quarks $N_s$ is intrinsically different. In fact, this number is not known from the initial state and may undergo large dynamical fluctuations from event to event, well beyond those (small) predicted by the GC formalism and its related fugacity $\gamma_s$. If this was the case, $\gamma_s$ in eq. [14] must be understood as a sort of average fugacity rather than a proper fugacity. On the other hand, when the expected mean number of $s\bar{s}$ pairs is small, the canonical formalism should be used (i.e. eq. [13]) but the probability distribution of generating a given number of $s\bar{s}$ pairs should be known in advance. As a reasonable ansatz, one can assume those pairs are independently produced so that their distribution is a Poissonian [17].

V. CONCLUSIONS

We have presented a method to calculate particle densities in the canonical framework of the statistical model, which can be effectively used in heavy ion reactions at very large values of baryon number. This method has

FIG. 7: Theoretical $K^+/\pi^+$ curves at fixed temperature compared with SPS experimental data [22]. Curves are strangeness canonical while crosses are full canonical results. Curves from top to bottom are indicated in the lower right corner. Crosses correspond to the curves with $T = 160$ MeV.
allowed us to study the applicability of different approximations in chemical analysis, namely the grand-canonical and the strangeness-canonical ensembles. It is found that the full \((B, S, Q)\) canonical formalism is only needed for elementary and very small nuclear systems up to baryon number \(\sim 10\). For larger systems, the baryon number and the electric charge can be safely handled in grand-canonical manner. For central Au–Au and Pb–Pb reactions, exact strangeness conservation is needed at GSI SIS energies \((p_{lab} \leq 2\ \text{GeV A})\), while at AGS and SPS even strangeness can be handled grand-canonical. Certainly, this also applies to much higher energy RHIC and LHC collisions.

The observed strangeness enhancement from peripheral to central nuclear collisions \([20, 22]\) is generally not reproduced by the canonical curves with fixed temperature and baryon density whilst the AGS p–p to p–A to A–A central reaction measurements are found to follow the canonical results \([5]\). This is an indication of a possible change in reaction systematics going from peripheral to central collisions. The strange hadron enhancement in SPS from p–p to Pb–Pb reactions is only qualitatively reproduced by canonical model.

Whilst \(\gamma_S\) was not used in the derivation of our numerical results, the weak dependence of \(K/\pi\) ratio on the thermal parameters (shown in fig. 7) serves as a further compelling piece of evidence in favour of incomplete strangeness chemical equilibration in CERN SPS heavy ion reactions, confirming previous findings \([1, 10]\).

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