\textbf{L}^p\text{-Poincaré inequalities on nested fractals}

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\textbf{Abstract}

We prove on some nested fractals scale invariant \textit{L}^p\text{-Poincaré inequalities on metric balls in the range } 1 \leq p \leq 2. Our proof is based on the development of the local \textit{L}^p\text{-theory of Korevaar-Schoen-Sobolev spaces on fractals using heat kernel methods. Applications to scale invariant Sobolev inequalities and to the study of maximal functions and Hajlasz-Sobolev spaces on fractals are given. Results are illustrated and further developed in the case of the Vicsek set.}

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1 Introduction

Scale invariant \textit{L}^p\text{-Poincaré inequalities on metric balls play a fundamental role in the analytic local and global theory of metric measure spaces, see [21, 22]. In this theory, such Poincaré inequalities are stated using upper gradients, a notion that depends on the availability of rectifiable curves. In the context of nested fractals, there may not be such curves at all and therefore the theory has to be modified.}

The interest in analysis on fractals arose from mathematical physics, and dates back at least to the 1980’s, see for instance [27]. Since then, the literature has been extensive and fractals have been studied from different but complementary viewpoints using harmonic analysis, Dirichlet forms, and probability theory; see for instance [7, 28, 37] and the references therein.

In the present paper we are interested in the study of certain \textit{L}^p\text{-Poincaré inequalities in nested fractals in the range } 1 \leq p \leq 2. While the case \( p = 2 \) has already extensively been studied in the literature using Dirichlet form theory (see for instance [33], [10], [11], [4], [19], [31]), the Poincaré inequalities we

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are interested in are unexplored in the range $1 \leq p < 2$. A difficulty with the local $L^p$ theory on fractals, $p \neq 2$, is to find analogues of the energy measures since energy measures are specific to the $L^2$ theory of Dirichlet forms, see [17]. In this paper, as analogue of the energy measures, we will work with $L^p$ variations of Korevaar-Schoen type [29] and the $L^p$-Poincaré inequalities we aim to prove write

$$
\int_{B(x_0,R)} |f(x) - \int_{B(x_0,R)} f \, d\mu|^p \, d\mu(x) \leq CR^{(p-1)d_w + (2-p)d_h} \mathbf{Var}_{B(x_0,AR),p}(f)^p,
$$

where $\mu$ is the Hausdorff measure, $d_w$ and $d_h$ are respectively the walk dimension and the Hausdorff dimension of the fractal and where we define the $L^p$ Korevaar-Schoen variation of a Borel set $F$ by

$$\mathbf{Var}_{F,p}(f)^p := \liminf_{r \rightarrow 0^+} \frac{1}{r^{(p-1)d_w + (2-p)d_h}} \int_F \int_{B(x,r) \cap F} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} \, d\mu(x) \, d\mu(y).$$

Let us note that the exponent $(p-1)d_w + (2-p)d_h$ in (1) appears as a critical exponent in the theory of Besov spaces on fractals, see [2], and coincides when $p = 2$ with the walk dimension $d_w$, which is the known exponent for scale invariant $L^2$-Poincaré inequalities. We also remark that for the case of the Vicsek set, which is an example of a nested fractal for which all of our results apply, this exponent reduces to $d_h + p - 1$ and coincides with a known optimal exponent for some Poincaré inequalities on Vicsek graphs, see [14].

The paper is organized as follows. In Section 2, we give reminders about nested fractals and Dirichlet forms and heat kernels on them. We also introduce the Korevaar-Schoen-Sobolev classes of functions for which the Poincaré inequalities will be proved. Such classes belong to the more general family of heat semigroup based Besov classes introduced in [2, 3], see also [1]. In Section 3, we prove the $L^p$-Poincaré inequalities in the range $1 < p \leq 2$. We will actually prove a stronger statement, namely the uniform estimates

$$|f(x) - f(y)| \leq CR^{(d_w - d_h)(1 - \frac{1}{p})} \mathbf{Var}_{B(x_0,AR),p}(f), \quad x, y \in B(x_0, R).$$

The proof is rather long and intricate, but the overall strategy is to first prove $L^p$-Poincaré inequalities on simplices using pseudo-Poincaré inequalities for the Neumann heat semigroup, and to use then the Hölder regularity of Sobolev functions together with a covering argument on nested fractals in the spirit of [33]. This will yield the estimate (2) with a sub-Gaussian type variation on the right hand side. The final and key step in Section 3.3 is to compare this sub-Gaussian type variation to the $L^p$ Korevaar-Schoen variation which is a priori sharper.

In Section 4, we study the case of $L^1$-Poincaré inequalities. A difficulty with the case $p = 1$ is that it corresponds to the class of BV functions and BV functions might be discontinuous (see [2]) so that the covering argument used when $p > 1$ fails. Instead, we will be using a coarea formula and a topological argument which is based on the geometry of the fractal. For this reason we will restrict ourselves to the case where the underlying fractal is the Vicsek set. In Section 5, we discuss some applications of the $L^p$-Poincaré inequalities. The first application is to the study of scale invariant Sobolev inequalities on balls. Using methods of [5, 35] we prove that for $L^p$ Sobolev functions

$$\|f\|_{L_\infty(B(x_0,R))} \leq C \left( R^{\frac{d_h}{p}} \|f\|_{L^p(B(x_0,R))} + R^{(1 - \frac{1}{p})(d_w - d_h)} \mathbf{Var}_{B(x_0,C2R),p}(f) \right),$$

When $p = 1$ and $R \to \infty$, this recovers an oscillation inequality for BV functions on the Vicsek set first proved in [2]. Then, as a second application, we introduce a fractal version of the Hardy-Littlewood maximal function

$$g(x) := \sup_{r > 0} \frac{1}{\mu(B(x,r))^{1/p}} \mathbf{Var}_{B(x,r),p}(f)$$

and prove the Lusin-Hölder type estimate

$$|f(x) - f(y)| \leq C d(x,y)^{(d_w - d_h)(1 - \frac{1}{p}) + \frac{d_h}{p}}(g(x) + g(y)).$$

While it is easy to check that the maximal function is weak $L^p$-bounded, the study of its strong $L^p$ boundedness is left to further investigation. The Lusin-Hölder estimate yields a natural connection between the Korevaar-Schoen-Sobolev spaces and the Hajłasz-Sobolev spaces on fractals (see [24]) which is discussed. Finally, to conclude the paper and give an idea of the scope of our results we give, for the Vicsek set, a useful description of the Sobolev space of functions for which the Korevaar-Schoen $p$-variations are finite.
Notations: Throughout the paper, we denote by \( c, C \) non important positive constants which may vary from line to line. For any Borel set \( F \) and any measurable function \( f \), we write the average of \( f \) on the set \( F \) as

\[
\int_F f(x) \, d\mu(x) := \frac{1}{\mu(F)} \int_F f(x) \, d\mu(x).
\]

If \( \Lambda_1 \) and \( \Lambda_2 \) are two non-negative functionals defined on a space of functions \( f \in \mathcal{A} \), we will write \( \Lambda_1(f) \simeq \Lambda_2(f) \) if there exists a constant \( C > 0 \) such that for every \( f \in \mathcal{A} \), \( \frac{1}{C} \Lambda_1(f) \leq \Lambda_2(f) \leq C \Lambda_1(f) \).

2 Preliminaries on nested fractals

In this section, we collect some preliminaries about nested fractals, their associated Dirichlet forms, heat kernels and functional spaces.

2.1 Compact nested fractals

Nested fractals were introduced by Lindstrom [32]. We briefly recall their construction below, see also [7, 16, 33] for the general definition. Let \( L > 1 \), then an \( L \)-similitude is a map \( \psi : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
\psi(x) = L^{-1} U(x) + a
\]

where \( U \) is a unitary linear map and \( a \in \mathbb{R}^d \). We call \( L \) the contraction factor of \( \psi \).

Consider a collection of similitudes \( \{\psi_i\}_{i=1}^M \) in \( \mathbb{R}^d \) with a common contraction factor \( L > 1 \). There exists a unique nonempty compact set \( K \subseteq \mathbb{R}^d \) such that

\[
K = \bigcup_{i=1}^M \psi_i(K) =: \Psi(K).
\]

Each map \( \psi_i \) has a unique fixed point \( q_i \). Denote the set of all fixed points by \( V \). We say that \( x \in V \) is an essential fixed point if there exist \( y \in V \) and \( i \neq j \) such that \( \psi_i(x) = \psi_j(y) \). The set of all essential fixed points will be denoted by \( V^{(0)} \). For any \( n \in \mathbb{N} \), set \( V^{(n)} = \Psi^n(V^{(0)}) \) and

\[
V^{(\infty)} = \bigcup_n V^{(n)}.
\]

Denote \( W_n := \{1, 2, \ldots, M\}^n \) and \( W_\infty := \bigcup_n W_n \). For any \( w = (i_1, \ldots, i_n) \in W_n \), write \( \psi_w = \psi_{i_n} \circ \cdots \circ \psi_{i_1} \) and \( A_w = \Psi_w(A) \), for any set \( A \subseteq K \). In particular, we call \( K_w \) an \( n \)-simplex and \( V_w^{(0)} := \psi_w(V^{(0)}) \) an \( n \)-cell.

Definition 2.1. The self-similar structure \( (K, \psi_1, \ldots, \psi_M) \) described above is called a nested fractal if the following conditions are satisfied

1. \( \#(V^{(0)}) \geq 2 \);
2. \( \text{(Connectivity)} \) For any \( i, j \in W \), there exists a sequence of \( 1 \)-cells \( V_w^{(0)}, \ldots, V_i^{(0)} \) such that \( i_0 = i, i_k = j \) and \( V_{i_{r-1}}^{(0)} \cap V_{i_r}^{(0)} \neq \emptyset \), for \( 1 \leq r \leq k \).
3. \( \text{(Symmetry)} \) If \( x, y \in V^{(0)} \), then reflection in the hyperplane \( H_{xy} = \{z : |x - z| = |y - z|\} \) maps \( n \)-cells to \( n \)-cells.
4. \( \text{(Nesting)} \) If \( w, v \in W_n \) and \( w \neq v \), then \( K_w \cap K_v = V_w^{(0)} \cap V_v^{(0)} \).
5. \( \text{(Open set condition)} \) There exists a non-empty bounded open set \( U \) such that \( \psi_i(U), 1 \leq i \leq M \), are disjoint and \( \Psi(U) \subseteq U \).

We call \( M \) the mass scaling factor of \( K \) and also call \( L \) the length scaling factor. For any \( x, y \in K \), let \( d(x, y) \) be the Euclidean distance. We observe that for \( w \in W_n \),

\[
d(\psi_w(x), \psi_w(y)) = L^{-n}d(x, y).
\]
Let $\mu$ be the normalized Hausdorff measure on $K$ such that for any $w \in W_n$,\
\[
\mu(K_w) = \mu(\psi_w(K)) = M^{-n}.
\]
Then the \textit{Hausdorff dimension} of $K$ is\
\[
d_h = \frac{\log M}{\log L},
\]
see for instance [15, Theorem 9.3].\
We now need to introduce more notations and definitions which can also be found in [33].

**Definition 2.2.** Let $n \geq 1$.
- The collection of all $n$-simplices is denoted by $\mathcal{T}_n$ and $K$ itself is a 0-simplex.
- For $K_w \in \mathcal{T}_n$, we denote by $K^*_w$ the union of $K_w$ and all the adjacent $n$-simplices of $K_w$, and by $K^*_w$ the union of $K^*_w$ and all the $n$-simplices of $K^*_w$.
- For any $x \in K \setminus V^{(\infty)}$, we use $K_n(x)$ to denote the unique $n$-simplex which contains $x$.

Since the similitudes $\{\psi_i\}_{i=1}^{M}$ have the same unitary part $U$, it was proved in [33, Lemma 5.2] that there exists $\beta \in (0,1)$ such that for two disjoint $(n+1)$-simplices $K_{w_1}$ and $K_{w_2}$ located in two adjacent $n$-simplices,\
\[
d(K_{w_1}, K_{w_2}) \geq \beta L^{-n} \tag{3}
\]
Moreover (see [33, Proposition 5.1]), for any $n \in \mathbb{N}$ and $x, y \in K \setminus V^{(\infty)}$ such that $y \in K^*_n(x) \setminus K^*_{n+1}(x)$\
\[
d(x, y) \geq \beta L^{-n}.
\]
As a consequence, for every $x \in K \setminus V^{(\infty)}$ and $n \geq 1$ we have\
\[
B(x, \beta L^{-n}) \subset K^*_n(x) \subset B(x, 2L^{-n}). \tag{4}
\]
Sierpiński gasket and Vicsek sets (also snowflakes etc) are typical examples of nested fractals whose similitudes have the same unitary parts and that satisfy all of the above assumptions.

**Example 2.3.** [Sierpiński gasket]\
We recall the definition of Sierpiński gasket $K_{SG}$. Let $q_1 = 0, q_2 = 1, q_3 = e^{\pi i}$ be three vertices on $\mathbb{R}^2 = \mathbb{C}$. Define $\psi_i(z) = \frac{1}{4}(z - q_i) + q_i$ for $i = 1, 2, 3$. Then the Sierpiński gasket $K_{SG}$ is the unique non-empty compact set such that\
\[
K = \bigcup_{i=1}^{3} \psi_i(K).
\]
The measure $\mu$ is a normalized Hausdorff measure on $K_{SG}$ such that for any $i_1, \cdots, i_n \in \{1, 2, 3\}$\
\[
\mu(\psi_{i_1} \circ \cdots \circ \psi_{i_n}(K_{SG})) = 3^{-n}.
\]

**Example 2.4.** [Vicsek sets]\
Let $\{q_1, q_2, q_3, q_4\}$ be the 4 corners of the unit square and let $q_5 = (1/2, 1/2)$. Define $\psi_i(z) = \frac{1}{4}(z - q_i) + q_i$ for $1 \leq i \leq 5$. Then the Vicsek set $K_{VS}$ is the unique non-empty compact set such that\
\[
K = \bigcup_{i=1}^{5} \psi_i(K).
\]
The measure $\mu$ is a normalized Hausdorff measure on $K_{VS}$ such that $i_1, \cdots, i_n \in \{1, 2, 3, 4, 5\}$\
\[
\mu(\psi_{i_1} \circ \cdots \circ \psi_{i_n}(K_{VS})) = 5^{-n}.
\]
More generally, let $\{q_1, \cdots, q_{2^N}\}$ be the corners of the unit cube $[0,1]^N$ on $\mathbb{R}^N$ $(N \geq 2)$ and let $q_0 = (1/2, \cdots, 1/2)$ be the center of the unite cube. Define $\psi_i(z) = \frac{1}{4}(z - q_i) + q_i$ for $0 \leq i \leq 2^N$. Then the $N$-dimensional version of Vicsek set $K_{VS_N}$ is the unique non-empty compact set such that\
\[
K = \bigcup_{i=0}^{2^N} \psi_i(K).
\]
The measure $\mu$ is a normalized Hausdorff measure on $K_{VS_N}$ such that $i_1, \cdots, i_n \in \{1, \cdots, 2^N + 1\}$\
\[
\mu(\psi_{i_1} \circ \cdots \circ \psi_{i_n}(K_{VS_N})) = (2^N + 1)^{-n}.
\]
2.2 Unbounded nested fractals

By unbounded nested fractals we mean blow-ups of compact nested fractals, see [16, 9, 26] and also [36] for different constructions. Without loss of generality, we assume that $\psi_1 = L^{-1} x$ and consider the unbounded nested fractal $X$ defined by

$$X = \bigcup_{n=1}^{\infty} K^{(n)},$$

where $K^{(n)} = L^n K$. Set $V_n = L^n V^{(n)}$. Then the set of essential fixed points is defined by $V^{(0)} = \bigcup_{n=0}^{\infty} V_n$ and $V^{(n)} = L^{-n} V^{(0)}$. We may still denote $V^{(0)}$ and $V^{(n)}$ by $V^{(0)}$ and $V^{(n)}$ if there is no confusion.

The Hausdorff measure $\mu$ on $X$ is such that $\mu(K^{(n)}) = M^n$ and $\mu$ is $d_h$-Ahlfors regular on $X$, that is, for $x \in X$, $r \geq 0$,

$$cr^{d_h} \leq \mu(B(x, r)) \leq C r^{d_h}. \quad (5)$$

2.3 Dirichlet forms, Heat kernels

We now introduce the canonical Dirichlet forms on $K$ and $X$ and recall some of the basic properties of their associated heat kernels.

Let $f \in C(V^{(\infty)}) = \{f : V^{(\infty)} \to \mathbb{R}\}$ and define

$$\mathcal{E}_n(f, f) = \rho^n \sum_{u \in V_n} \sum_{x, y \in V^{(0)}} (f \circ \psi_w(x) - f \circ \psi_w(y))^2,$$

where $\rho > 1$ is the resistance scale factor of $K$. The Dirichlet form on $K$, denoted by $\mathcal{E}_K$, is given by

$$\mathcal{E}_K(f, f) = \lim_{n \to \infty} \mathcal{E}_n(f, f),$$

where $f \in \mathcal{F}_K = \{f \in C(K) : \sup \mathcal{E}_n(f, f) < \infty\}$. For more details, we refer to, for instance, [16, Section 2] and [7, Corollary 6.28, Section 7]. Then $(\mathcal{E}_K, \mathcal{F}_K)$ is a strongly local regular Dirichlet form on $L^2(K, \mu)$. Let $\Delta_K$ be the generator of $(\mathcal{E}_K, \mathcal{F}_K)$ on $L^2(K, \mu)$. The associated heat semigroup $(P^K_t)_{t \geq 0}$ admits a heat kernel that we denote by $p^K_t(x, y)$.

Next we introduce the Dirichlet form on $X$. Define $\sigma_n : C(K^{(n)}) \to C(K)$ by

$$\sigma_n f(x) = f(L^n x) = f \circ \psi_1^{-n}(x), \quad \forall x \in K.$$

Set $\mathcal{F}_{K^{(n)}} = \sigma_n \mathcal{F}_K$ and

$$\mathcal{E}_{K^{(n)}}(f, f) = \rho^{-n} \mathcal{E}_K(\sigma_n f, \sigma_n f), \quad \forall f \in \mathcal{F}_{K^{(n)}}.$$

Let

$$\mathcal{F} = \{f : |K^{(n)}| \in \mathcal{F}_{K^{(n)}} \text{ for every } n, \lim_{n \to \infty} \mathcal{E}_{K^{(n)}}(f |_{K^{(n)}}, f |_{K^{(n)}}) < \infty\}$$

and let $\mathcal{F}_X = \mathcal{F} \cap L^2(X, \mu)$. Then the Dirichlet form $\mathcal{E}_X$ is then defined by

$$\mathcal{E}_X(f, f) = \lim_{n \to \infty} \mathcal{E}_{K^{(n)}}(f |_{K^{(n)}}, f |_{K^{(n)}}), \quad \forall f \in \mathcal{F}_X.$$

Then $(\mathcal{E}_X, \mathcal{F}_X)$ is a strongly local regular Dirichlet form on $L^2(X, \mu)$. The associated heat semigroup $(P^X_t)_{t \geq 0}$ admits a heat kernel that we denote by $p^X_t(x, y)$.

Set

$$d_w = \frac{\log M \rho}{\log L}.$$

The parameter $d_w$ is the so-called walk dimension of the nested fractal and we have $\rho = L^{d_w - d_h}$. We note that $p^X_t(x, y)$ is the Neumann heat kernel of $K$ if $K$ is seen as a subset of $X$. The heat kernels $p^K_t(x, y)$ and $p^X_t(x, y)$ satisfy sub-Gaussian estimates, see [7, Theorem 8.18] and [16, Theorem 1]. More precisely, for every $(x, y) \in K \times K$ and $t \in (0, 1)$

$$c_1 t^{-d_w/d_w} \exp\left(-c_2 \left(\frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{d_w}}\right) \leq p^K_t(x, y) \leq c_3 t^{-d_h/d_w} \exp\left(-c_4 \left(\frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{d_w}}\right), \quad (6)$$

5
and for every \((x, y) \in X \times X\) and \(t > 0\)
\[
c_{5}t^{-d_{h}/d_{w}} \exp\left( -c_{6}\left( \frac{d(x, y)^{d_{w}}}{t} \right)^{\tau_{w}^{-1}} \right) \leq p_{t}^{X}(x, y) \leq c_{7}t^{-d_{h}/d_{w}} \exp\left( -c_{8}\left( \frac{d(x, y)^{d_{w}}}{t} \right)^{\tau_{w}^{-1}} \right). \tag{7}
\]
Those sub-Gaussian estimates easily imply (see Lemma 2.3 in [2]) that for every \(\kappa \geq 0\), there are constants \(C, c > 0\) so that for every \(x, y \in X\), and \(t > 0\)
\[
d(x, y)^{\kappa}p_{t}^{X}(x, y) \leq Ct^{-d_{h}/d_{w}}p_{t}^{X}(x, y). \tag{8}
\]
Furthermore, the weak Bakry-Émery non-negative curvature condition holds true on both \(K\) and \(X\) (see [2, Theorem 3.7]), that is, there exists a constant \(C > 0\) such that for every \(g \in L^{\infty}(X, \mu)\) and every \(t > 0\)
\[
|P_{t}^{K}g(x) - P_{t}^{K}g(y)| \leq C\frac{d(x, y)^{d_{w}-d_{h}}}{t^{1-d_{h}/d_{w}}}\|g\|_{L^{\infty}(K, \mu)}, \tag{9}
\]
and
\[
|P_{t}^{X}g(x) - P_{t}^{X}g(y)| \leq C\frac{d(x, y)^{d_{w}-d_{h}}}{t^{1-d_{h}/d_{w}}}\|g\|_{L^{\infty}(X, \mu)}. \tag{10}
\]

**Example 2.5.** [Sierpiński gasket] The Sierpiński gasket \(K_{SG}\) satisfies the properties (5)–(9) with scaling factors \(L_{SG} = 2, M_{SG} = 3\) and \(\rho_{SG} = 5/3\). In particular \(d_{h} = \log 3\) and \(d_{w} = \frac{\log 5}{\log 2}\). For further details about the heat kernel on the Sierpiński gasket we refer to [12].

**Example 2.6.** [Vicsek sets] The Vicsek set \(K_{VS}\) satisfies the properties (5)–(9) with scaling factors \(L_{VS} = 3, M_{VS} = 5\) and \(\rho_{VS} = 3\). In particular \(d_{h} = \frac{\log 5}{\log 3}\) and \(d_{w} = \frac{\log 15}{\log 3}\). For the \(N\)-dimensional version of the Vicsek set \(K_{VS_{N}}\), the properties (5)–(9) are satisfied with \(d_{h} = \frac{\log(2^{N}+1)}{\log 3}\) and \(d_{w} = \frac{\log 3(2^{N}+1)}{\log 3}\).

We note that on Vicsek sets we therefore have \(d_{w} - d_{h} = 1\) so that from (9), the heat semigroup \(P_{t}^{K}\) therefore transforms bounded Borel functions into Lipschitz functions. For further details about the heat kernel on Vicsek sets we refer to [7, 6, 8, 11].

### 2.4 Korevaar-Schoen-Sobolev and BV spaces on fractals

Following [3], we introduce the definitions below.

**Definition 2.7.** For any \(p \geq 1\) and \(\alpha > 0\), define the heat semigroup based Besov class
\[
\mathcal{B}^{p, \alpha}(K) := \left\{ f \in L^{p}(K, \mu), \|f\|_{p, \alpha} := \sup_{t > 0} t^{-\alpha} \left( \int_{K} \int_{K} p_{t}^{K}(x, y)|f(x) - f(y)|^{p}d\mu(x)d\mu(y) \right)^{1/p} < \infty \right\},
\]
where \(\| \cdot \|_{p, \alpha}\) is called the Besov seminorm.

As was proved in [3], \((\mathcal{B}^{p, \alpha}(K), \| \cdot \|_{p, \alpha} + \| \cdot \|_{L^{p}(K, \mu)})\) is a complete Banach space. The definition of \(\mathcal{B}^{p, \alpha}(X)\) is identical, replacing the integrals over \(K\) by integrals over \(X\). Korevaar-Schoen-Bovle and BV spaces appear at the critical exponents of the spaces \(\mathcal{B}^{p, \alpha}(X)\), see [1, 2], and in the present paper we shall use the following definitions.

**Definition 2.8.** For any \(1 < p \leq 2\), the Korevaar-Schoen-Sobolev spaces on the compact nested fractal \(K\) or its blowup \(X\) are defined by using the \(L^{p}\) Korevaar-Schoen energy:
\[
W^{1, p}(K) = \left\{ f \in L^{p}(K, \mu), \limsup_{r \to 0^{+}} \frac{1}{r^{d_{w}}} \left( \int_{K} \int_{B(x, r) \cap K} \frac{|f(y) - f(x)|^{p}}{\mu(B(x, r))} d\mu(y) d\mu(x) \right)^{1/p} < +\infty \right\},
\]
\[
W^{1, p}(X) = \left\{ f \in L^{p}(X, \mu), \limsup_{r \to 0^{+}} \frac{1}{r^{\alpha_{p}} d_{w}} \left( \int_{X} \int_{B(x, r)} \frac{|f(y) - f(x)|^{p}}{\mu(B(x, r))} d\mu(y) d\mu(x) \right)^{1/p} < +\infty \right\},
\]
where
\[
\alpha_{p} = \left(1 - \frac{2}{p}\right) \left(1 - \frac{d_{h}}{d_{w}}\right) + \frac{1}{p}.
\]

**Remark 2.9.**
1. From [2], the sub-Gaussian estimates for the heat kernel imply that $W^{1,p}(K) = B^{p,p}(K)$ and $W^{1,p}(X) = B^{p,p}(X)$.

2. From [2], one can also see that $W^{1,2}(K) = \mathcal{F}_K$ is the domain of the Dirichlet form $\mathcal{E}_K$. The similar result holds with $X$ instead of $K$.

3. It follows from Theorem 5.1 in [1] that for $p > 1$, $W^{1,p}(K) \subset C^0(K)$, meaning that any function $f \in W^{1,p}(K)$ admits a continuous version. We will always work with such continuous version without further comments. The same holds for $X$ instead of $K$.

4. As we will prove in Theorem 5.8, for the Vicsek set the space $W^{1,p}(K), 1 < p \leq 2$ is dense in $L^p(K,\mu)$. A similar proof shows that this result also holds true on the $N$-dimensional version of Vicsek set. For the Sierpiński gasket, we do not know if the spaces $W^{1,p}(K)$ are trivial or not for $1 < p < 2$, however combining the results from [25] and the numerical simulations from [23] suggests that the spaces $W^{1,p}(K)$ are trivial for $1 < p < 2$.

**Definition 2.10.** For $p = 1$, the definitions are the same as in Definition 2.8 but we then speak of BV type spaces and will use the notation $BV(K)$ and $BV(X)$. Note that

$$\alpha_1 = \frac{d_h}{d_w}.$$  

**Remark 2.11.** From Theorem 5.1 in [2], if $K_w \subset T_n$, then $1_{K_w} \in BV(K)$. More generally, if $F$ is a set with finite boundary then $1_F \in BV(K)$. As a consequence $BV(K)$ is dense in $L^1(K,\mu)$ for any nested fractal.

Throughout the paper, we will use two types of (inner) variations for functions in the Sobolev or BV space. The Korevaar-Schoen type (inner) variation is the one that shall be the most important for us and the other one is used as a comparison tool.

**Definition 2.12 (p-Variations).** Let $1 < p \leq 2$. The sub-Gaussian $p$-variation of a function $f \in W^{1,p}(X)$ along a Borel set $F \subset X$ is defined by

$$\var^*_F(f) := \liminf_{t \to 0^+} \frac{1}{t^{\alpha_p + \frac{1}{p}}} \left( \int_F \int_F \exp \left( -\frac{(d(x,y)^d_w)}{t} \right) \left( f(x) - f(y) \right)^p d\mu(x)d\mu(y) \right)^{1/p}. \quad (11)$$

The Korevaar-Schoen $p$-variation is defined by

$$\var^*_F(f) := \liminf_{r \to 0^+} \frac{1}{r^{\alpha_p}} \left( \int_F \int_{B(x,r)\cap F} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} d\mu(x)d\mu(y) \right)^{1/p}. \quad (12)$$

**Remark 2.13.** From [1], on $W^{1,2}(X)$,

$$\mathcal{E}_X(f) \simeq \var^*_{X,2}(f) \simeq \var^*_{X,2}(f),$$

where $\mathcal{E}_X(f) \simeq \var^*_{X,2}(f)$ means that there exist constants $c, C > 0$ such that for every $f \in W^{1,2}(X)$, $c\mathcal{E}_X(f) \leq \var^*_{X,2}(f) \leq C\mathcal{E}_X(f)$.

**Remark 2.14.** It is easy to prove (see for instance [2] for further details) that if $f \in W^{1,p}(X)$ then $\var^*_{F,p}(f) < +\infty$ for every Borel set $F$. Indeed, $f \in W^{1,p}(X)$ implies that

$$\sup_{r > 0} \frac{1}{r^{\alpha_p}} \left( \int_X \int_{B(x,r)} \frac{|f(y) - f(x)|^p}{\mu(B(x,r))} d\mu(y) d\mu(x) \right)^{1/p} < +\infty$$

from which, using a dyadic annuli decomposition, we deduce that

$$\sup_{r > 0} \frac{1}{t^{\alpha_p + \frac{1}{p}}} \left( \int_X \int_X \exp \left( -\frac{(d(x,y)^d_w)}{t} \right) \left( f(x) - f(y) \right)^p d\mu(x)d\mu(y) \right)^{1/p}.$$
Remark 2.15. It is clear that $\text{Var}_{F,p}(f) \leq C\text{Var}_{F,p}^\ast(f)$ because
\[
\frac{1}{t^{\alpha'}} \frac{1}{p} \left( \int_F \int_F \exp \left( - \frac{(d(x,y)^{d_w})}{t} \right) |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} \\
\geq \frac{1}{t^{\alpha'}} \frac{1}{p} \left( \int_F \int_F \exp \left( - \frac{(d(x,y)^{d_w})}{t} \right) |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} \\
\geq \frac{1}{t^{\alpha'}} \frac{1}{p} \left( \int_F \int_F |f(x) - f(y)|^p d\mu(x) d\mu(y) \right)^{1/p}.
\]

It will be proved later that on the simplices $K_w$ one actually has $\text{Var}_{K_w,p}(f) \sim \text{Var}_{K_w,p}^\ast(f)$. Similar definitions will be used for the space $BV(X)$.

Definition 2.16 (1-Variations). The sub-Gaussian 1-variation of a function $f \in BV(X)$ along a Borel set $F \subset X$ is defined by
\[
\text{Var}_F^\ast(f) := \liminf_{t \to 0^+} \frac{1}{t^{d_h/d_w}} \int_F \int_F \exp \left( - \frac{(d(x,y)^{d_w})}{t} \right) |f(x) - f(y)| d\mu(x) d\mu(y). \tag{13}
\]
The Korevaar-Schoen type (inner) 1-variation is defined by
\[
\text{Var}_F(f) := \liminf_{t \to 0^+} \frac{1}{t^{d_h/d_w}} \int_F \int_{B(x,r) \subset F} \frac{|f(y) - f(x)|}{\mu(B(x,r))} d\mu(y) d\mu(x). \tag{14}
\]

It is also clear for the same reason as above that $\text{Var}_F(f) \leq C\text{Var}_F^\ast(f)$.

Remark 2.17. Let $1 < p \leq 2$ and $f \in W^{1,p}(X)$. The following two direct consequences from the definition of the Korevaar-Schoen type $p$-variation are useful in later proofs. The same also holds for the sub-Gaussian type $p$-variation and the above two 1-Variations.

- If $E$ is a subset of $F$, one has $\text{Var}_{F,p}(f) \leq \text{Var}_{E,p}(f)$.
- If $\{F_i\}$ is a bounded overlapping family of subsets on $X$, that is, there exists $N \in \mathbb{N}$ such that every $x \in \bigcup_i F_i$ belongs to at most $N$ number of $F_i$’s, then one has $\sum_i \text{Var}_{F_i,p}(f)^p \leq C \text{Var}_{\bigcup_i F_i,p}(f)^p$.

3 \textbf{L}^p \textbf{P}oincaré inequality for $1 < p \leq 2$

Throughout the section we assume that $1 < p \leq 2$ and work on the unbounded nested fractal $X$. Our goal is to prove the following uniform Morrey estimates on balls.

Theorem 3.1. Let $1 < p \leq 2$. Then there exist constants $C > 0$ and $A > 1$ such that for every $f \in W^{1,p}(X)$, $x_0 \in X$, $R > 0$, $x, y \in B(x_0, R)$
\[
|f(x) - f(y)| \leq CR^{(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{B(x_0, AR), p}(f). \tag{15}
\]

We note that the $L^p$-Poincaré inequality
\[
\left\| f - \int_{B(x_0, R)} f \, d\mu \right\|_{L^p(B(x_0, R), \mu)} \leq CR^{(d_w - d_h)(1 - \frac{1}{p})} + \frac{d_w}{p} \text{Var}_{B(x_0, AR), p}(f). \tag{16}
\]
of course easily follows from Theorem 3.1 since from Hölder’s inequality one has
\[
\int_{B(x_0, R)} |f(x) - \int_{B(x_0, R)} f \, d\mu|^p \, d\mu(x) \leq \int_{B(x_0, R)} \int_{B(x_0, R)} |f(x) - f(y)|^p \, d\mu(x) d\mu(y).
\]

and $\mu(B(x_0, R)) \sim R^{d_w}$.

Remark 3.2. Notice that the exponent $(d_w - d_h)(1 - \frac{1}{p}) + \frac{d_w}{p}$ in (16) is exactly $\alpha_p d_w$, where $\alpha_p$ is from Definition 2.8.
The proof of Theorem 3.1 is rather long and we divide it into three parts. Section 3.1 will be to obtain first, using heat kernel methods, a family of scale-invariant $L^p$-Poincaré inequalities on the simplices $K_w$ for the sub-Gaussian $p$-variation. Then in Section 3.2, we will prove a uniform modulus of continuity estimate for functions in $f \in W^{1,p}(X)$, $p > 1$, by using a covering argument that is based on the geometry of nested fractals. The final step of the proof, see Section 3.3, will be to use a self-improvement property of the $L^p$-Poincaré inequalities together with some heat kernel estimates to replace the sub-Gaussian $p$-variation by the a priori sharper Korevaar-Schoen $p$-variation.

### 3.1 $L^p$-Poincaré inequality on simplices

To prove Theorem 3.1, we will first prove the following $L^p$-Poincaré inequality on simplices.

#### Theorem 3.3

There exists a constant $C > 0$ such that for every $n$-simplex $K_w \subset K$ and $f \in W^{1,p}(K)$

$$\left\| f - \int_{K_w} f \, dp \right\|_{L^p(K_w,\mu)} \leq Cr(K_w)^{(d_w-d_h)(1-\frac{1}{p})} \frac{d_w}{d_h} \mathsf{Var}^{*}_{K_w,p}(f).$$

Here, $r(K_w)$ denotes the diameter of $K_w$. The proof uses heat semigroups and heat kernels, and relies on several lemmas as follows.

#### Lemma 3.4

There exist $\lambda, C > 0$ such that for every $t > 0$, $g \in L^\infty(K,\mu)$

$$|P_t^K g(x) - P_t^K g(y)| \leq C \frac{d(x,y)^{d_w-d_h}}{t^{d-d_h/d_w}} e^{-\lambda t} \|g\|_{L^\infty(K,\mu)}.$$

**Proof.** In view of the weak Bakry-Émery condition (9), we assume $t \geq 2$ without loss of generality. Some ideas used here are similar to the proof of [1, Lemma 2.14]. First note that $-\Delta_K$ has eigenvalues $(\lambda_j)_{j \geq 0}$ with finite multiplicity satisfying

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \wedge \infty,$$

see for instance [13, Theorem 2.4]. Let $(\phi_j)_{j \geq 0}$ be the corresponding eigenfunctions in $C(K)$ such that $P_t^K \phi_j = e^{-\lambda_j t} \phi_j$. From spectral theory, one has for any $g \in L^2(K,\mu)$

$$P_t^K g(x) - \int_K g \, d\mu = \sum_{j=1}^{\infty} e^{-\lambda_j t} \phi_j(x) \int_K \phi_j(z) g(z) \, d\mu(z).$$

For any $0 < t_0 < 1$, using the Cauchy-Schwartz inequality and (6), we obtain for every $x \in K$,

$$|\phi_j(x)| = e^{\lambda_j t_0} |P_{t_0}^K \phi_j(x)| \leq e^{\lambda_j t_0} \left( \int_K P_{t_0}^{K}(x,y)^2 \, d\mu(y) \right)^{1/2} \leq C t_0^{-d_h/d_w} e^{\lambda_j t_0}.$$

In particular, taking $t_0 = 1/2$ leads to

$$|\phi_j(x)| \leq C e^{\lambda_j/2}, \quad x \in K.$$

Now we write $e^{-\lambda_j t} \phi_j = e^{-\lambda_j t/2} P_{t/2}^{K} \phi_j$ and apply (9), then

$$|P_t^K g(x) - P_t^K g(y)| \leq \sum_{j=1}^{\infty} \left| e^{-\lambda_j t} \phi_j(x) - e^{-\lambda_j t} \phi_j(y) \right| \int_K \phi_j(z) g(z) \, d\mu(z)$$

$$\leq C \sum_{j=1}^{\infty} e^{-\lambda_j t/2} \left| P_{t/2}^K \phi_j(x) - P_{t/2}^K \phi_j(y) \right| \int_K \phi_j(z) \, d\mu(z)$$

$$\leq C \sum_{j=1}^{\infty} e^{-\lambda_j t/2} \frac{d(x,y)^{d_w-d_h}}{(t/2)^{d_h/d_w}} \|g\|_{L^\infty(K,\mu)}$$

$$\leq C \frac{d(x,y)^{d_w-d_h}}{t^{d/d_w}} \|g\|_{L^\infty(K,\mu)} \sum_{j=1}^{\infty} e^{-\lambda_j (t-1)/2}$$

$$\leq C \frac{d(x,y)^{d_w-d_h}}{t^{d/d_w}} e^{-\lambda t} \|g\|_{L^\infty(K,\mu)},$$

where in the last inequality we can choose $\lambda = \lambda_1/4$. \qed
Lemma 3.5. For any $p \geq 2$, there exists a constant $C > 0$ and $\lambda > 0$ such that for every $t > 0$ and $g \in L^p(K, \mu)$,

$$
\sup_{0 < s < 1} \frac{1}{s^{\alpha_p}} \left( \int_K \int_K p^K_s(x, y)|P^K_t f(x) - P^K_t f(y)|^p d\mu(x) d\mu(y) \right)^{1/p} \leq \frac{C}{t^{\alpha_p}} e^{-\lambda t} \|f\|_{L^p(K, \mu)}.
$$

Proof. The proof is adapted from that of [2, Proposition 3.9]. Consider the map $P^K_t$ defined by $P^K_t f(x, y) = P^K_t f(x) - P^K_t f(y)$. It was proved in [3, Theorem 5.1] that $\|P^K_t f\|_{L^2(K, \mu)} \leq C t^{-1/2} \|f\|_{L^2(K, \mu)}$. Equivalently, for all $s > 0$:

$$
\int_K \int_K p^K_s(x, y)|P^K_t f(x) - P^K_t f(y)|^2 d\mu(x) d\mu(y) \leq C \|f\|_{L^2(K, \mu)}^2.
$$

Hence $P^K_t : L^2(K, \mu) \to L^2(K \times K, p^K_s \mu \otimes \mu)$ is bounded by $C \left( \frac{t}{s} \right)^{1/2}$.

We now consider the case where $f \in L^\infty(K, \mu)$, $t > 0$, $0 < s < 1$, and $q > 1$. Recall that $\mu(K) = 1$. It follows from Lemma 3.4 and (8) that

$$
\left( \int_K \int_K p^K_s(x, y)|P^K_t f(x) - P^K_t f(y)|^q d\mu(x) d\mu(y) \right)^{1/q} \leq C \|f\|_{L^\infty(K, \mu)} e^{-\lambda t} \left( \int_K \int_K p^K_s(x, y)d(x, y) d\mu(x) d\mu(y) \right)^{1/q} = C \|f\|_{L^\infty(K, \mu)} e^{-\lambda t}.
$$

Since $(K \times K, p^K_s \mu \otimes \mu)$ is a finite measure space we conclude on sending $q \to \infty$ that $P^K_t : L^\infty(K, \mu) \to L^\infty(K \times K, p^K_s \mu \otimes \mu)$ is a bounded operator with bound $C \left( \frac{t}{s} \right)^{1-\alpha_d/d_a}$ on its operator norm. By the Riesz-Thorin interpolation theorem it follows that $P^K_t : L^p(K, \mu) \to L^p(K \times K, p^K_s \mu \otimes \mu)$ is a bounded operator whose operator norm is bounded by $C \left( \frac{t}{s} \right)^{\alpha_p}$. So dividing by $s^{\alpha_p}$ and taking the supremum over $0 < s < 1$ give the result. \hfill \Box

Lemma 3.6 (Pseudo-Poincaré inequality). Let $1 < p \leq 2$. There exists $C > 0$ such that for every $t > 0$, $f \in W^{1, p}(K)$

$$
\|f - P^K_t f\|_{L^p(K, \mu)} \leq C \text{Var} \left( P^K_t f, \nu \right).
$$

Proof. We use ideas from [2, Proposition 3.10]. Denote

$$
\mathcal{E}_t^K(u, v) := \frac{1}{t} \int_K v(x - P^K_t u) d\mu = \frac{1}{2t} \int_K \int_K p^K_t(x, y)(u(x) - u(y))(v(x) - v(y)) d\mu(x) d\mu(y).
$$

Let $q$ be the conjugate of $p$ and let $g \in L^q(K, \mu)$, then $\alpha_q = 1 - \alpha_p$. For $f \in W^{1, p}(K)$, one writes

$$
\int_K (f - P^K_t f) g d\mu = \lim_{\tau \to 0^+} \int_0^\tau \mathcal{E}_t^K(P^K_s f, g) ds.
$$

From the symmetry of $\mathcal{E}_t^K$, Hölder’s inequality and Lemma 3.5,

$$
2|\mathcal{E}_t^K(P^K_s f, g)| = \frac{1}{t} \int_K \int_K p^K_s(x, y)|P^K_s f(x) - P^K_s f(y)|^p d\mu(x) d\mu(y)
$$

$$
\leq \frac{1}{t^{1-\alpha_p}} \left( \int_K \int_K p^K_s(x, y)^q d\mu(x) d\mu(y) \right)^{1/q} \left( \int_K \int_K f(x)^p d\mu(x) d\mu(y) \right)^{1/p}
$$

$$
\leq C \frac{1}{t^{1-\alpha_p}} \left( \int_K \int_K p^K_s(x, y)f(x) - f(y))^p d\mu(x) d\mu(y) \right)^{1/p}.
$$

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Integrating over $s \in (0, t)$ and taking $\liminf_{\tau \to 0^+}$, we obtain the expected inequality by using duality and the following sub-Gaussian upper bound valid for $\tau \in (0, 1)$, $x, y \in K$

$$p^K_t(x,y) \leq c_3 \tau^{-d_w/d_w} \exp\left(-c_4 \left(\frac{d(x,y)^{d_w}}{\tau}\right)^{\frac{1}{d_v-1}}\right).$$

□

**Lemma 3.7.** Let $1 \leq p \leq \infty$ and let $f \in L^p(K, \mu)$. Then as $t \to \infty$ one has in $L^p(K, \mu)$

$$P^K_t f \to \int_K f \, d\mu.$$

**Proof.** Since $K$ is compact and $\mu(K) = 1$, it suffices to prove the result for $p = \infty$. One can apply the proof of [13, Proposition 2.6], which is given here for the sake of completeness.

From spectral theory, $P^K_t f$ converges in $L^2(K, \mu)$ to a constant function that is denoted by $P^K_\infty f$. This convergence is also uniform. Indeed, for any $s, t > 0$ and $0 < r < 1$,

$$\left|P^K_{t+r} f(x) - P^K_s f(x)\right| = \sup_{x \in K} \left|P^K_{t+r} (P^K_t f - P^K_s f)(x)\right|$$

$$= \sup_{x \in K} \left|\int_K p^K_t(x,y)(P^K_t f - P^K_s f)(y) \, d\mu(y)\right|$$

$$\leq \left(\sup_{x \in K} \int_K p^K_t(x,y)^2 \, d\mu(y)\right) \|P^K_t f - P^K_s f\|_{L^2(K, \mu)}$$

$$\leq C \|P^K_t f - P^K_s f\|_{L^2(K, \mu)},$$

where we use the upper bound of the heat kernel for $0 < r < 1$: $p^K_t(x,y) \leq C r^{-d_w/d_w}$.

On the other hand, for every $t > 0$, $\int_K P^K_t f \, d\mu = \int_K f \, d\mu$. Therefore $\int_K P^K_\infty f \, d\mu = \int_K f \, d\mu$. Since $P^K_\infty f$ is a constant and $\mu(K) = 1$, we deduce that $P^K_\infty f = \int_K f \, d\mu$. □

**Lemma 3.8 (L^p Poincaré inequality on K).** Let $1 < p \leq 2$. Then, there exists a constant $C > 0$ such that for every $f \in W^{1,p}(K)$

$$\left\|f - \int_K f \, d\mu\right\|_{L^p(K, \mu)} \leq C \var^*_{K,p}(f).$$

**Proof.** This is a consequence of Lemma 3.6 and Lemma 3.7. □

Now we are ready to prove the $L^p$ Poincaré inequality on simplices by using the scaling properties of nested fractals.

**Proof of Theorem 3.3.** Let $f \in L^p(K, \mu)$. Then $f \circ \psi_w \in L^p(K, \mu)$ and from Lemma 3.8, one has

$$\left\|f \circ \psi_w - \int_K f \circ \psi_w \, d\mu\right\|_{L^p(K, \mu)} \leq C \var^*_{K,p}(f \circ \psi_w).$$

Observe that $\int_K f \circ \psi_w \, d\mu = \int_K f \circ \psi_w \, d\mu = \int_{K_w} f \, d\mu$. Hence the left hand side above becomes

$$M^{n/p} \left\|f - \int_{K_w} f \, d\mu\right\|_{L^p(K_w, \mu)}.$$

On the other hand, one has

$$\var^*_{K,p}(f \circ \psi_w)$$

$$= \liminf_{t \to 0^+} \frac{1}{t^{d_w} + \frac{d_w}{p}} \left(\int_K \int_K \exp\left(-\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_v-1}}\right) |f \circ \psi_w(x) - f \circ \psi_w(y)| \, d\mu(x) \, d\mu(y)\right)^{1/p}$$

$$\leq \liminf_{t \to 0^+} \frac{M^{2n/p}}{t^{d_w} + \frac{d_w}{p}} \left(\int_{K_w} \int_{K_w} \exp\left(-\left(\frac{d(x,y)^{d_w}}{t\mu(K_w)^{d_w} \mu(K_w)}\right)^{\frac{1}{d_v-1}}\right) |f(x) - f(y)| \, d\mu(x) \, d\mu(y)\right)^{1/p}$$

$$= r(K_w)^{d_w/d_v + d_v/p} \var^*_{K_w, p}(f).$$

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As a consequence, since $M^{-n} \leq C r(K_w)^{d_w}$ we obtain

$$\left\| f - \int_{K_w} f \, d\mu \right\|_{L^p(K_w, \mu)} \leq C r(K_w)^{n_d_w} \text{Var}_{K_w, p}(f).$$

\[\blacksquare\]

### 3.2 $L^p$-Poincaré inequality on balls and inner sub-Gaussian variation

Our next goal will be to go from the simplices to the metric balls using chaining arguments. We begin with the following Morrey type result on simplices.

**Lemma 3.9.** Let $f \in W^{1,p}(X)$. Let $K_w \subset K$ be an $m$-simplex. Then for $x,y \in K_w$,

$$|f(x) - f(y)| \leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_m(x), p}(f),$$

where we denote $f_m(x) := \frac{1}{\mu_m} \int_{K_m(x)} f(z) \, d\mu(z)$ and $\mu_m := \mu(K_m(x))$.

Indeed, an elementary argument gives

$$|f_{m+1}(x) - f_m(x)| = \left\| \frac{1}{\mu_m} \int_{K_m(x)} f(z) \, d\mu(z) - \frac{1}{\mu_{m+1}} \int_{K_{m+1}(x)} f(z) \, d\mu(z) \right\|_{L^p(K_m(x), \mu)}$$

$$= \left\| \frac{1}{\mu_m \mu_{m+1}} \int_{K_{m+1}(x)} \int_{K_{m}(x)} f(z') \, d\mu(z) \, d\mu(z') - \frac{1}{\mu_{m+1}} \int_{K_{m+1}(x)} f(z) \, d\mu(z) \right\|_{L^p(K_m(x), \mu)}$$

$$\leq \frac{1}{\mu_{m+1}} \int_{K_{m+1}(x)} \left| f(z') - f_m(x) \right| \, d\mu(z) \leq \frac{1}{\mu_{m+1}} \int_{K_m(x)} |f(z) - f_m(x)| \, d\mu(z).$$

In view of Hölder’s inequality and Theorem 3.3, we thus have

$$|f_{m+1}(x) - f_m(x)| \leq C \mu_m^{-1/p} |f - f_m(x)|_{L^p(K_m(x), \mu)} \leq C M^{\frac{1}{p+1}} L^{-m_\alpha d_w} \text{Var}_{K_m(x), p}(f).$$

Observe that $L^{-m_\alpha d_w} = L^{-m((1 - \frac{1}{p})d_w - d_h) + \frac{m_\alpha}{p+1}} = M^{-\frac{m_\alpha}{p+1}} L^{-m(d_w - d_h)(1 - \frac{1}{p})}$, hence

$$|f_{m+1}(x) - f_m(x)| \leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_m(x), p}(f).$$

By a telescopic type argument, we have

$$|f(x) - f_m(x)| \leq \sum_{j=m}^{\infty} |f_j(x) - f_{j+1}(x)| \leq C \sum_{j=m}^{\infty} L^{-j(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_j(x), p}(f)$$

$$\leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_m(x), p}(f).$$

Now consider $x,y \in K_w \setminus V^{(\infty)}$. We note that $K_m(x) = K_m(y) = K_w$ and hence $f_m(x) = f_m(y)$. Then from (18), we have

$$|f(x) - f(y)| \leq |f(x) - f_m(x)| + |f(y) - f_m(y)| \leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_w, p}(f).$$

Since it is known that $W^{1,p}(X) \subset C(X)$ for $p > 1$, the Hölder estimate also holds for every $x,y \in K_w$. \[\blacksquare\]

**Lemma 3.10.** Let $f \in W^{1,p}(X)$. Let $K_w \subset K$ be an $m$-simplex. Then for $x,y \in K_w^*$,

$$|f(x) - f(y)| \leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_w^*, p}(f).$$
Proof. Recall that $K_w^{**}$ is the union of $K_w^*$ and all its adjacent $m$-simplices. For any $x, y \in K_w^*$, assume that $x \in K_{w_x}$ and $y \in K_{w_y}$, where $K_{w_x}, K_{w_y}$ are two $m$-simplices in $K_w^{**}$. Then there are four cases:

1. $K_{w_x} = K_{w_y}$;
2. $K_{w_x}$ and $K_{w_y}$ are neighboring $m$-simplices.
3. $K_{w_x}$ and $K_{w_y}$ are disjoint and have a common neighboring $m$-simplex.
4. $K_{w_x}$ and $K_{w_y}$ are disjoint and don’t have a common neighboring $m$-simplex, but there exist two neighboring $m$-simplices which are respectively the neighbors of $K_{w_x}$ and $K_{w_y}$.

Case 1 is a direct subsequence of Lemma 3.9 since $\text{Var}_{K_{w_x}, p}(f) \leq \text{Var}_{K_{w_y}, p}(f)$.

Next we work for Case 2. The other two cases can be treated in a similar way and the details are left to the interested reader. Indeed, we pick $z \in K_{w_z} \cap V^{(n)}$ and $w \in K_{w_w} \cap V^{(n)}$ such that $w, z$ are in the boundary of the same $m$-simplex in $K_w^{**}$, denoted by $K_{w_{wz}}$. By Lemma 3.9, we have

$$|f(x) - f(y)| \leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_{w_{wz}}, p}(f) + \text{Var}_{K_{w_{wz}}, p}(f) + \text{Var}_{K_{w_{wz}}, p}(f)$$

$$\leq C L^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_{w_{wz}}, p}(f).$$

□

Using (4), the previous Morrey’s type estimate also applies to balls.

**Proposition 3.11.** Let $1 < p \leq 2$. Then, there exists a constant $C > 0$ such that for every $f \in W^{1,p}(X)$, $x_0 \in X, R > 0$ and $x, y \in B(x_0, R)$,

$$|f(x) - f(y)| \leq CR^{(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{B(x_0, AR), p}(f).$$

**Proof.** By using dilation and translation, it is enough to prove the result when $x_0 \in K$ and $B(x_0, AR) \subset K$ where $A$ is a fixed number large enough ($A = \frac{4L}{\beta}$ will do).

We first consider the case that $x_0$ is not a vertex point, i.e., $x_0 \in K \setminus V^{(\infty)}$. Note that there exists a unique $n_0$ such that

$$L^{-(n_0 + 1)} < R/\beta \leq L^{-n_0}.$$

Hence one has

$$B(x_0, R) \subset B(x_0, \beta L^{-n_0}) \subset K_{n_0}^*(x_0) \subset B\left(x_0, \frac{2L}{\beta} R\right).$$

where $K_{n_0}^*(x_0) = \bigcup_{i=1}^l K_i$ and $K_i$ is either $K_{n_0}(x_0)$ or its adjacent $n_0$-simplices. Observe that $l$ is a uniformly bounded integer. Consider now any $x, y \in B(x_0, R)$. Then there exists $1 \leq i \leq l$ such that $x \in K_i$ and $y \in K_{i}^{**}$. On the other hand, for any $i = 1, \cdots, l$, one has $K_{i}^{**} \subset B(x_0, AR)$. Therefore by Lemma 3.10,

$$|f(x) - f(y)| \leq CL^{-m(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{K_{i}^{**}, p}(f) \leq CR^{(d_w - d_h)(1 - \frac{1}{p})} \text{Var}_{B(x_0, AR), p}(f).$$

This completes the proof.

Next assume that $x_0 \in V^{(m)}$ for some fixed $m \in \mathbb{N}$. For $R > 0$, there exists a unique $n$ such that $L^{-(n+1)} < R \leq L^{-n}$. We consider two different cases.

Case $m \leq n$: we denote by $K_n(x_0)$ the union of all adjacent simplices which meet at $x_0$. Then $B(x_0, R) \subset K_n(x_0) \subset B(x_0, LR)$. The above proof also applies.

Case $m > n$: we then repeat the proof for the case $x_0 \in K \setminus V^{(\infty)}$.

□

3.3 $L^p$-Poincaré inequality on balls and inner Korevaar-Schoen variation

We prove in this section the following proposition that will conclude the proof of Theorem 3.1.

**Proposition 3.12.** There exists a constant $C$ such that for every $f \in W^{1,p}(X)$,

$$\text{Var}_{i,p}(f) \leq C \text{Var}_{K,p}(f).$$
Remark 3.13. As a consequence, similar dilation argument as in the proof of Theorem 3.3 also yields that for any simplex $K_w \subset X,$
\[
\Var_{K_w,p}(f) \leq C \Var_{K_w,p}(f).
\]

The proof of Proposition 3.12 is divided in several lemmas.

Lemma 3.14. There exists a constant $C > 0$ such that for every $f \in W^{1,p}(X)$ and $0 < R < \beta$
\[
\int_K \int_{B(x,R) \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \leq CR^{d_w - d_h}(p-1) \Var_{K_p}(f)^p.
\]

Proof. There exists a unique $n$ such that
\[
L^{-(n+1)} < R/\beta \leq L^{-n}.
\]

Consider the covering of $K$ by the $M^n$ $n$-simplices $\{K_{w_i}\}_{1 \leq i \leq M^n}$. For any $x \in K_w$, we claim that $B(x,R) \subset K_w$. Indeed, let $K_{w_i} \subset K_w$ be an $(n+1)$-simplex containing $x$. Then by (3), one deduces that
\[
d(x, K_w \setminus (K_{w_i} \cup K_{w_i}^\ast)) \geq d(K_{w_i}^\ast \setminus (K_{w_i} \cup K_{w_i}^\ast)) \geq \beta L^{-n},
\]
and therefore $B(x,R) \subset B(x,\beta L^{-n}) \subset K_{w_i}^\ast$.

We have then
\[
\int_K \int_{B(x,R) \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) = \sum_i \int_{K_{w_i}} \int_{B(x,R) \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x)
\]
\[
\leq \sum_i \int_{K_{w_i}^\ast \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x).
\]

From Lemma 3.10 for $x, y \in K_{w_i}^\ast \cap K$, one has
\[
|f(x) - f(y)| \leq C L^{-n(d_w - d_h)(1 - \frac{2}{p})} \Var_{K_w^\ast \cap K_p}(f)
\]
\[
\leq CR^{d_w - d_h}(p-1) \Var_{K_p}(f).
\]

One concludes
\[
\int_K \int_{B(x,R) \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \leq CR^{d_w - d_h}(p-1) \sum_i \Var_{K_w^\ast \cap K_p}(f)^p
\]
\[
\leq CR^{d_w - d_h}(p-1) \Var_{K_p}(f)^p.
\]

Lemma 3.15. There exists a constant $C$ such that for every $f \in W^{1,p}(X)$
\[
\limsup_{t \to 0^+} \frac{1}{t^{d_w + d_h}} \int_K \int_{B(x,tR) \cap K} \exp\left(-\frac{R(x,y)^{d_w}}{t}\right) |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x)
\]
\[
\leq C \limsup_{t \to 0^+} \frac{1}{t^{d_w + d_h}} \int_K \int_{B(x,tR) \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x).
\]

Proof. Fix $\delta > 0$. For $d(x,y) > \delta t^{1/d_w}$, we see that
\[
\exp\left(-\frac{R(x,y)^{d_w}}{t}\right) = \exp\left(-\frac{1}{2} \frac{R(x,y)^{d_w}}{t} \frac{1}{\delta t^{\frac{d_w}{\alpha}}} \right) \leq \frac{1}{2} \frac{R(x,y)^{d_w}}{t} \frac{1}{\delta t^{\frac{d_w}{\alpha}}}.
\]

\[14\]
Therefore,
\[
\int_K \int_K \exp \left( -\left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq \int_K \int_{B(y, \delta t^{1/d_w}) \cap K} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
+ \int_K \int_{K \setminus B(y, \delta t^{1/d_w}) \cap K} \exp \left( -\left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq \int_K \int_{B(y, \delta t^{1/d_w}) \cap K} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
+ \exp \left( -\frac{1}{2} \delta^{\frac{d_w}{\alpha}} \right) \int_K \int_K \exp \left( -\frac{1}{2} \left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y).
\]

This yields
\[
\limsup_{t \to 0^+} \frac{1}{1 \times \alpha} \int_K \int_K \exp \left( -\left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq \limsup_{t \to 0^+} \frac{1}{1 \times \alpha} \int_K \int_{B(y, \delta t^{1/d_w}) \cap K} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
+ \exp \left( -\frac{1}{2} \delta^{\frac{d_w}{\alpha}} \right) \limsup_{t \to 0^+} \frac{1}{1 \times \alpha} \int_K \int_K \exp \left( -\frac{1}{2} \left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y).
\]

Choosing then \( \delta \) large enough gives the result.

**Proof of Proposition 3.12.** From the proof of Lemma 3.15, one has for every \( \delta > 0 \)
\[
\frac{1}{1 \times \alpha} \int_K \int_K \exp \left( -\left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
\leq \frac{1}{1 \times \alpha} \int_K \int_{B(y, \delta t^{1/d_w}) \cap K} |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y)
\]
\[
+ \exp \left( -\frac{1}{2} \delta^{\frac{d_w}{\alpha}} \right) \limsup_{t \to 0^+} \frac{1}{1 \times \alpha} \int_K \int_K \exp \left( -\frac{1}{2} \left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y).
\]

However, combining Lemmas 3.14 and 3.15 gives
\[
\limsup_{t \to 0^+} \frac{1}{1 \times \alpha} \int_K \int_K \exp \left( -\left( \frac{d(x, y)^{d_w}}{t} \right)^{\frac{1}{\alpha}} \right) |f(x) - f(y)|^p \, d\mu(x) \, d\mu(y) \leq C \text{Var}_{K, p}^\alpha(f).
\]

Thus, taking \( \liminf_{t \to 0^+} \) in the inequality (19) and choosing \( \delta \) large enough give the result.

We are now finally ready for the proof of Theorem 3.1.

**Proof of Theorem 3.1.** Thanks to Proposition 3.12 we obtain the pseudo-Poincaré inequality in Theorem 3.3 with \( \text{Var}_{K, p}^\alpha(f) \) instead of \( \text{Var}_{K, p}^\alpha(f) \). More precisely, there exists a constant \( C > 0 \) such that for every \( n \)-simplex \( K_w \subset K \) and \( f \in L^p(K, \mu) \)
\[
\left\| f - \int_{K_w} f \, d\mu \right\|_{L^p(K_w, \mu)} \leq C \text{Cr}(K_w)^{d_w \alpha} \text{Var}_{K, p}^\alpha(f).
\]

The same arguments then yield the conclusion of Proposition 3.11 with \( \text{Var}_{B(x_0, AR), p}^\alpha(f) \) instead of \( \text{Var}_{B(x_0, AR), p}^\alpha(f) \).

## 4 \( L^1 \)-Poincaré Inequality

Our goal in this section is to prove the \( L^1 \)-Poincaré inequality. In the first part of the section we work under the general assumptions of the paper and in the second for topological reasons we will have to assume that \( K \) is the Vicsek set (or its \( N \)-dimensional version). The argument in the second part of the section might possibly be generalized to other treelike structures, but for the sake of clarity we restrict ourselves to the Vicsek set.
4.1 $L^1$-Poincaré inequality on simplices

Our first goal will be to prove Lemma 4.5, i.e. the uniform estimate

$$\int_{K_w} \int_{B(x,r) \cap K_w} |f(x) - f(y)| d\mu(x) d\mu(y) \leq C r^{2d_w} \var_K(f), \quad f \in BV(X).$$

The methods of the previous section do not work anymore since functions in $BV$ might not be continuous. Interestingly, this difficulty can be overcome using a co-area formula type argument which is specific to the $L^1$ case, see the proof of Lemma 4.2. We start with a small-time $L^1$-pseudo-Poincaré inequality for the heat semigroup $P_t^K$.

**Lemma 4.1** ($L^1$ Pseudo-Poincaré inequality). There exists a constant $C > 0$ such that for every $f \in BV(K)$ and $t \in (0, r(K)^{d_w}]$

$$\|f - P_K^t f\|_{L^1(K, \mu)} \leq C t^{d_w/d_w} \var_K(f).$$

**Proof.** As in the proof of Lemma 3.6, we denote

$$\mathcal{E}^K_t(u, v) = \frac{1}{\tau} \int_K v(I - P^K_t)d\mu = \frac{1}{2\tau} \int_K P^K_t(x, y) (u(x) - u(y))(v(x) - v(y)) d\mu(x) d\mu(y).$$

Then for $f \in BV(K)$ and $g \in L^\infty(K, \mu)$ one writes

$$\int_K (f - P^K_t f)g d\mu = \lim_{\tau \to 0^+} \int_0^\tau \mathcal{E}^K_t(P^K_t f, g) ds.$$

From the symmetry of $\mathcal{E}^K_t$ and the weak Bakry-Émery estimate (9), there holds for $0 < \tau < 1$,

$$2|\mathcal{E}^K_t(P^K_t f, g)| = \frac{1}{\tau} \int_K \int_K P^K_t(x, y) |P^K_t g(x) - P^K_t g(y)||f(x) - f(y)| d\mu(x) d\mu(y)$$

$$\leq C \|g\|_{L^\infty} \frac{1}{\tau^{s(d_w - d_n)/d_w}} \int_K \int_K d(x, y)^{d_w - d_n} P^K_t(x, y)|f(x) - f(y)| d\mu(x) d\mu(y)$$

$$\leq C \|g\|_{L^\infty} \frac{1}{\tau^{d_w/d_w}} \int_K \int_K P^K_t(x, y)|f(x) - f(y)| d\mu(x) d\mu(y).$$

Integrating (20) over $s \in (0, t)$ and taking $\liminf_{\tau \to 0^+}$, we obtain the expected inequality by duality and the sub-Gaussian upper bound for $p^K_t(x, y)$.

**Lemma 4.2.** There exists a constant $C > 0$ such that for every $f \in BV(K)$ and $t \in (0, r(K)^{d_w}]$

$$\int_K \int_K P^K_t(x, y)|f(x) - f(y)| d\mu(x) d\mu(y) \leq C t^{d_w/d_w} \var_K^*(f).$$

**Proof.** The proof is similar as [2, Lemma 4.12]. Without loss of generality, we assume $f \geq 0$. In general, it suffices to work for $f_n = (f + n)_+$. For any $s > 0$, denote $E_s = \{x \in K : f(x) > s\}$. By [2, Lemmas 4.10 and 4.11] and Lemma 4.1, we have

$$\int_K \int_K p^K_t(x, y)|f(x) - f(y)| d\mu(x) d\mu(y) \leq \int_0^\infty \|P^K_t(1_{E_s}) - 1_{E_s}\|_{L^1(K, \mu)} ds$$

$$\leq C t^{d_w/d_w} \int_0^\infty \var_K^*(1_{E_s}) ds \leq C t^{d_w/d_w} \var_K^*(f).$$

**Lemma 4.3.** There exists a constant $C > 0$ such that for every $f \in BV(K)$ and $r > 0$

$$\int_K \int_{B(x,r) \cap K} |f(x) - f(y)| d\mu(x) d\mu(y) \leq C r^{2d_w} \var_K^*(f).$$
Proof. It is enough to prove the inequality for $0 < r \leq r(K)$. For $0 < r \leq r(K)$ and $d(x, y) \leq r$, the sub-Gaussian lower bound in (6) gives

$$p^K_{r \Delta K}(x, y) \geq C r^{-d_h}.$$ 

Applying Lemma 4.2, one has

$$\int_K \int_{B(x, r) \cap K} |f(x) - f(y)| d\mu(x) d\mu(y) \leq C r^{d_h} \int_K \int_K p^K_{r \Delta K}(x, y) |f(x) - f(y)| d\mu(x) d\mu(y) \leq C r^{2d_h} \mathbf{Var}_K(f).$$

\[\square\]

**Lemma 4.4.** There exists a constant $C > 0$ such that for every $f \in \mathbf{BV}(K)$ and $r \geq 0$

$$\int_K \int_{B(x, r) \cap K} |f(x) - f(y)| d\mu(x) d\mu(y) \leq C r^{2d_h} \mathbf{Var}_K(f).$$

**Proof.** By Lemma 4.3, it suffices to show that $\mathbf{Var}_K^*(f) \leq C \mathbf{Var}_K(f)$. We apply the method developed in [2, Lemma 4.13]. Write

$$\Theta(t) = \frac{1}{t^{2d_h/d_u}} \int_K \int_K \exp \left( - \left( \frac{d(x, y)^{d_u}}{t} \right)^{\frac{1}{d_u-1}} \right) |f(x) - f(y)| d\mu(x) d\mu(y).$$

Let $\delta > 0$ and let $r = \delta t^{1/d_u}$. Then for any $d(x, y) \leq r$, one has

$$\frac{1}{t^{2d_h/d_u}} \int_K \int_{K \cap B(x, r)} \exp \left( - \left( \frac{d(x, y)^{d_u}}{t} \right)^{\frac{1}{d_u-1}} \right) |f(x) - f(y)| d\mu(x) d\mu(y) \leq C \delta^{2d_h} \frac{1}{t^{2d_h}} \int_K \int_{K \cap B(x, r)} |f(x) - f(y)| d\mu(x) d\mu(y) =: \Phi(t).$$

On the other hand, for $d(x, y) > r$,

$$\exp \left( - \left( \frac{d(x, y)^{d_u}}{t} \right)^{\frac{1}{d_u-1}} \right) \leq \exp \left( - \frac{1}{2} \left( \frac{d(x, y)^{d_u}}{t} \right)^{\frac{1}{d_u-1}} \right).$$

Therefore

$$\Theta(t) \leq \Phi(t) + \exp \left( \frac{1}{2} \delta^{d_u} \right) \frac{1}{t^{2d_h/d_u}} \int_K \int_{K \setminus B(x, r)} \exp \left( \frac{1}{2} \left( \frac{d(x, y)^{d_u}}{t} \right)^{\frac{1}{d_u-1}} \right) |f(x) - f(y)| d\mu(x) d\mu(y) \leq \Phi(t) + C \exp \left( - \frac{1}{2} \delta^{d_u} \right) \Theta(ct) = \Phi(t) + D \Theta(ct),$$

where $c = 2^{d_u-1} > 1$ and we choose $\delta$ large enough such that $D < \frac{1}{2}$. Adapting an iteration strategy used in the proof of [2, Lemma 4.13], we obtain that $\mathbf{Var}_K^*(f) \leq C \mathbf{Var}_K(f)$ and thus conclude the proof. \[\square\]

**Lemma 4.5.** There exists a constant $C > 0$ such that for every $n$-simplex $K_w \subset K$, $r \geq 0$ and $f \in \mathbf{BV}(K)$

$$\int_{K_w} \int_{B(x, r) \cap K_w} |f(x) - f(y)| d\mu(x) d\mu(y) \leq C r^{2d_h} \mathbf{Var}_{K_w}(f).$$

**Proof.** Recall that $K_w = \psi_w(K)$ and $d(\psi_w(x), \psi_w(y)) = L^{-n} d(x, y)$, for any $x, y \in K$. By scaling,

$$\frac{1}{L^{2d_h}} \int_K \int_{B(x, r) \cap K} |f \circ \psi_w(x) - f \circ \psi_w(y)| d\mu(x) d\mu(y) = \frac{M^{2n}}{L^{2d_h} (L^{-n})^{2d_h}} \int_{K_w} \int_{B(x, L^{-n}r) \cap K_w} |f(x) - f(y)| d\mu(x) d\mu(y) = \frac{1}{(L^{-n})^{2d_h}} \int_{K_w} \int_{B(x, L^{-n}r) \cap K_w} |f(x) - f(y)| d\mu(x) d\mu(y),$$

where the last inequality is due to the fact that $d_h = -\log \frac{M}{\log L}$. In view of Lemma 4.4, we thus conclude the proof following the definition of $\mathbf{Var}_{K_w}(f)$. \[\square\]
Note that, as a corollary, we of course obtain in particular the $L^1$-Poincaré on simplices:

**Corollary 4.6.** There exists a constant $C > 0$ such that for every $n$-simplex $K_w \subset K$ and $f \in BV(K)$

$$\left\| f - \int_{K_w} f \, d\mu \right\|_{L^1(K_w, \mu)} \leq C r(K_w)^{d_B} \mathbf{Var}_{K_w}^*(f).$$

### 4.2 $L^1$-Poincaré inequality on balls for the Vicsek set

A problem to go from the $L^1$-Poincaré on simplices to the $L^1$-Poincaré on balls is to control $f(x) - f(y)$ when $x, y$ are in adjacent simplices. When $p > 1$, functions in the space $W^{1, p}(X)$ are Hölder continuous and a chaining argument was possible, see Lemma 3.10. In the case $p = 1$, functions in $BV(X)$ do not need to be continuous and we need to come up with new methods by passing this chaining argument.

The method we use, a cutoff argument, relies on the topology of $X$ and requires that $X$ has a treelike structure. For simplicity of presentation we restrict ourselves to the case of the Vicsek set. So, throughout the section, we assume that $X$ is the unbounded Vicsek set and $K$ the bounded one.

**Lemma 4.7.** For any $m \geq 0$, let $\{F_i\}_{1 \leq i \leq \ell}$ be a sequence of adjacent $m$-simplices in $K$. There exists a constant $C > 0$ depending only on $\ell$ such that for every $f \in BV(K)$,

$$\int_{\bigcup_{i=1}^{\ell} F_i} \int_{\bigcup_{i=1}^{\ell} F_i} |f(x) - f(y)| \, d\mu(x) \, d\mu(y) \leq C L^{-2md_B} \mathbf{Var}_{\bigcup_{i=1}^{\ell} F_i}^*(f).$$

**Proof.** We first prove the theorem when $f = 1_G \in BV(K)$. In the following, we work with the intrinsic topology of $K$. Consider the finite set of vertices

$$V := \{v_1, \ldots, v_{m, \ell}\} = \partial \left( \bigcup_{i=1}^{\ell} F_i \right) \cap \text{int}(K).$$

Here $n_{m, \ell}$ is a finite number depending on $m$ and $\ell$ (at most $3\ell$ when $\ell$ is very large, comparable to $5^m$).

Due to the topology of $K$, the set $K \setminus \text{int}(\bigcup_{i=1}^{\ell} F_i)$ has $n_{m, \ell}$ connected components $C_1, \ldots, C_{n_{m, \ell}}$ such that $v_i \in C_i$. Consider then a family of sets $S_1, \ldots, S_{n_{m, \ell}}$ such that

$$S_i = \begin{cases} C_i, & \text{if } v_i \in G \\ \emptyset, & \text{if } v_i \notin G \end{cases}$$

Finally, define

$$\Omega = \left( G \cap \left( \bigcup_{i=1}^{\ell} F_i \right) \right) \cup \left( \bigcup_{i=1}^{n_{m, \ell}} S_i \right).$$

Note that by construction

$$\partial \Omega = \partial G \cap \text{int} \left( \bigcup_{i=1}^{\ell} F_i \right)$$

For any $x \in \bigcup_{i=1}^{\ell} F_i$, there exists a constant $C_\ell > 0$ (only depending on $\ell$) such that $\bigcup_{i=1}^{\ell} F_i \subset B(x, C_\ell L^{-m})$. Hence from Lemma 4.4, we have

$$\int_{\bigcup_{i=1}^{\ell} F_i} \int_{\bigcup_{i=1}^{\ell} F_i} |1_G(x) - 1_G(y)| \, d\mu(x) \, d\mu(y) = \int_{\bigcup_{i=1}^{\ell} F_i} \int_{\bigcup_{i=1}^{\ell} F_i} |1_\Omega(x) - 1_\Omega(y)| \, d\mu(x) \, d\mu(y)$$

$$\leq \int_{K} \int_{B(x, C_\ell L^{-m})} |1_\Omega(x) - 1_\Omega(y)| \, d\mu(x) \, d\mu(y)$$

$$\leq C L^{-2md_B} \mathbf{Var}_K^*(1_\Omega).$$

We now note that

$$\mathbf{Var}_K^*(1_\Omega) = \liminf_{r \to 0^+} \frac{1}{r^{2md_B}} \int_{K} \int_{B(x, r) \cap K} |1_\Omega(y) - 1_\Omega(x)| \, d\mu(y) \, d\mu(x).$$
The integral \( \int_K \int_{B(x,r) \cap K} \) can be divided into four integrals according to \( x, y \in \bigcup_{i=1}^f F_i \) or not. The first integral is

\[
\int_{\bigcup_{i=1}^f F_i} \int_{B(x,r) \cap (\bigcup_{i=1}^f F_i)} |1_{\Omega}(x) - 1_{\Omega}(y)|d\mu(y)d\mu(x) = \int_{\bigcup_{i=1}^f F_i} \int_{B(x,r) \cap (\bigcup_{i=1}^f F_i)} |1_G(x) - 1_G(y)|d\mu(y)d\mu(x).
\]

The second integral is

\[
\int_{\bigcup_{i=1}^f F_i} \int_{B(x,r) \cap (\bigcup_{i=1}^f F_i) \setminus K(\bigcup_{i=1}^f F_i))} |1_{\Omega}(x) - 1_{\Omega}(y)|d\mu(y)d\mu(x)
= \int_{\bigcup_{i=1}^f F_i} \int_{B(x,r) \cap (\bigcup_{i=1}^f F_i \setminus F_i)} |1_G(x) - 1_{B(x,r) \cap (\bigcup_{i=1}^f F_i)}|d\mu(y)d\mu(x).
\]

By construction, the function \( 1_G(x) - 1_{\cup_{i=1}^F S_i(x)} \) is zero on a neighborhood of the boundary of \( \bigcup_{i=1}^f F_i \).

Thus, for small \( r \) the second integral is zero. For the same reason, the third integral is zero as well for \( r \) small enough. We conclude

\[
\text{Var}_K(1_{\Omega}) = \text{Var}_{\bigcup_{i=1}^f F_i}(1_G).
\]

Thus, if \( f = 1_G \in BV(K) \), we obtain

\[
\int_{\bigcup_{i=1}^f F_i} \int_{\bigcup_{i=1}^f F_i} |1_G(x) - 1_G(y)|d\mu(y)d\mu(y) \leq C L^{-2nd_h} \text{Var}_{\bigcup_{i=1}^f F_i}(1_G).
\]

Now, for general \( f \in BV(K) \), we can assume \( f \) to be nonnegative and use the representation

\[
|f(y) - f(x)| = \int_0^{+\infty} |1_{E_t(f)}(x) - 1_{E_t(f)}(y)|dt,
\]

where for almost every \( t \geq 0 \) we define the set \( E_t(f) = \{ x \in K : f(x) > t \} \). We observe that the following co-area formula estimate holds

\[
\int_0^{+\infty} \text{Var}_{\bigcup_{i=1}^f F_i}(1_{E_t(f)})dt \leq C \text{Var}_{\bigcup_{i=1}^f F_i}(f).
\]

In fact, for any Borel set \( F \subset K \) and nonnegative \( f \in BV(K) \), set

\[
A_r = \{ (x, y) \in F \times F : d(x, y) < r, f(x) < f(y) \}.
\]

It follows from Fatou’s Lemma that

\[
\int_0^{+\infty} \text{Var}_{\bigcup_{i=1}^f F_i}(1_{E_t(f)})dt = \int_0^{+\infty} \liminf_{r \to 0^+} \frac{1}{r^{2d_h}} \int_F \int_{B(x,r) \cap F} |1_{E_t(f)}(y) - 1_{E_t(f)}(x)|d\mu(x)d\mu(y)dt
\]

\[
\leq 2 \liminf_{r \to 0^+} \frac{1}{r^{2d_h}} \int_F \int_{B(x,r) \cap F} |1_{E_t(f)}(y) - 1_{E_t(f)}(x)|d\mu(x)d\mu(y)
\]

\[
\leq 2 \liminf_{r \to 0^+} \frac{1}{r^{2d_h}} \int_{A_r} (f(y) - f(x))d\mu(x)d\mu(y)
\]

\[
\leq \liminf_{r \to 0^+} \frac{1}{r^{2d_h}} \int_F \int_{B(x,r) \cap F} |f(y) - f(x)|d\mu(x)d\mu(y) = \text{Var}_F(f).
\]

Finally, applying Fubini’s theorem and equations (21), (22), we conclude that

\[
\int_{\bigcup_{i=1}^f F_i} \int_{\bigcup_{i=1}^f F_i} |f(y) - f(x)|d\mu(x)d\mu(y) = \int_0^{+\infty} \int_{\bigcup_{i=1}^f F_i} \int_{\bigcup_{i=1}^f F_i} |1_{E_t(f)}(x) - 1_{E_t(f)}(y)|d\mu(x)d\mu(y)dt
\]

\[
\leq C L^{-2nd_h} \int_0^{+\infty} \text{Var}_{\bigcup_{i=1}^f F_i}(1_{E_t(f)})dt
\]

\[
\leq C L^{-2nd_h} \text{Var}_{\bigcup_{i=1}^f F_i}(f).
\]

\[\square\]
Now we can prove the main theorem of this section.

**Theorem 4.8.** For any $B(x_0, R) \subset K$, there exist constants $C > 0$ and $A > 1$ such that if $B(x_0, AR) \subset K$, then for every $f \in BV(K)$,

$$\left\| f - \int_{B(x_0, R)} f \, d\mu \right\|_{L^1(B(x_0, R), \mu)} \leq CR^{d_h} \text{Var}_{B(x_0, AR)}(f).$$

**Proof.** We analyze as in Proposition 3.11. There are two situations.

If $x_0 \not\in V(\infty)$, then there exists a unique $n_0$ such that $L^{-n_0} < R/\beta \leq L^{-n_0}$, and

$$B(x_0, R) \subset B(x_0, \beta^{-n_0}) \subset K^*_n(x_0) \subset B \left(x_0, \frac{2L}{\beta} R \right).$$

where $K^*_n(x_0) = \bigcup_{i=1}^l K_i$ and $K_i$ is either $K_n(x_0)$ or its adjacent $n_0$-simplices. Observe that $l$ is a uniform bounded integer and $\mu(K^*_n(x_0)) \approx R^{d_h}$. Then from Lemma 4.7 one has

$$\int_{B(x_0, R)} \left| f(x) - \int_{B(x_0, R)} f \, d\mu(x) \right| d\mu(x) \leq \frac{1}{\mu(B(x_0, R))} \int_{B(x_0, R)} \int_{B(x_0, R)} |f(x) - f(y)| \, d\mu(x) \, d\mu(y)$$

$$\leq \frac{1}{\mu(B(x_0, R))} \int_{K^*_n(x_0)} \int_{K^*_n(x_0)} |f(x) - f(y)| \, d\mu(x) \, d\mu(y)$$

$$\leq \frac{C}{\mu(B(x_0, R))} L^{-2n d_h} \text{Var}_{K^*_n(x_0)}(f)$$

$$\leq CR^{d_h} \text{Var}_{B(x_0, AR)}(f),$$

where we take $A = \frac{2L}{\beta}$.

If $x_0 \in V^{(m)}$ for some fixed $m \in \mathbb{N}$, then for $R > 0$ there exists a unique $n$ such that $L^{-n+1} < R \leq L^{-n}$. We consider two different cases. When $m \leq n$, we denote by $K_n(x_0)$ the union of all adjacent simplices which meet at $x_0$. Then $B(x_0, R) \subset K_n(x_0) \subset B(x_0, LR)$. The above proof also applies. When $m > n$, we then repeat the proof for the case $x \not\in V(\infty)$.

Finally, to go from the $L^1$ Poincaré inequalities on $K$ to the $L^1$ Poincaré inequalities on $X$, one can use a scaling argument.

**Theorem 4.9.** For any $x_0 \in X$ and $R > 0$, there exist constants $C > 0$ and $A > 1$ such that for every $f \in BV(X)$,

$$\left\| f - \int_{B(x_0, R)} f \, d\mu \right\|_{L^1(B(x_0, R), \mu)} \leq CR^{d_h} \text{Var}_{B(x_0, AR)}(f).$$

**Proof.** Recall that from the construction of $X$, there exists a minimal integer $n$ such that $B(x_0, AR) \subset K^{(n)}$, where $A$ is the constant from Theorem 4.8. That is, $B(x_0, AR) \subset L^n K = \varphi^n(K)$.

By scaling and the proof of Theorem 4.8, we have

$$\frac{1}{\mu(B(x_0, R))} \int_{B(x_0, R)} \int_{B(x_0, R)} |f(x) - f(y)| \, d\mu(x) \, d\mu(y)$$

$$= M^{2n} \frac{1}{L^{nd_h} \mu(B(x_0, L^{-n} R))} \int_{B(\tilde{x}_0, L^{-n} R)} \int_{B(\tilde{x}_0, L^{-n} R)} |f(L^n x) - f(L^n y)| \, d\mu(x) \, d\mu(y)$$

$$\leq C M^{2n} \frac{1}{L^{nd_h} (L^{-n} R)^{d_h}} \text{Var}_{B(\tilde{x}_0, AL^{-n} R)}(f \circ L^n) \leq CR^{d_h} \text{Var}_{B(\tilde{x}_0, AL^{-n} R)}(f \circ L^n),$$

where $\tilde{x}_0 = L^{-n} x_0$ and we use the fact $d_h = \log M / \log L$.

On the other hand,

$$\text{Var}_{B(\tilde{x}_0, AL^{-n} R)}(f \circ L^n)$$

$$= \liminf_{r \to 0^+} \frac{1}{r^{2d_h}} \int_{B(\tilde{x}_0, AL^{-n} R)} \int_{B(x, R) \cap B(\tilde{x}_0, AL^{-n} R)} |f(L^n y) - f(L^n x)| \, d\mu(x) \, d\mu(y)$$

$$= \liminf_{r \to 0^+} \frac{M^{-2n}}{L^{-2d_h} (L^n R)^{2d_h}} \int_{B(x_0, AR)} \int_{B(x, L^n R) \cap B(x_0, AR)} |f(y) - f(x)| \, d\mu(x) \, d\mu(y)$$

$$= \text{Var}_{B(x_0, AR)}(f).$$

We conclude the proof by combining the above two equations. \(\square\)
5 Applications

In this section we point out three applications of the $L^p$-Poincaré inequalities. The first one concerns scale invariant Sobolev type inequalities on balls. The second one introduces a fractal version of the Hardy-Littlewood maximal function and studies its relation to the Hajlasz-Sobolev spaces. Last but not least we give a characterization of the Sobolev spaces $W^{1,p}(K)$, $1 < p \leq 2$ when $K$ is the Vicsek set.

5.1 Sobolev inequalities on balls

As a first application of the $L^p$-Poincaré inequalities on balls, we prove scale invariant Sobolev inequalities on balls. As before, let $K$ be a compact nested fractal and let $X$ be its blowup. Throughout this section we assume $1 \leq p \leq 2$. The method to prove the Sobolev inequalities on balls will again be to use pseudo-Poincaré inequalities, but this time for moving averages instead of the heat semigroup, and then apply the general theory of [5]. For $f \in W^{1,p}(X)$ (or $BV(X)$ for $p = 1$), we will denote

$$f_s(x) = \int_{B(x,s)} f(y)d\mu(y), \ \forall x \in X, s > 0,$$

and $f_B = \int_{B} f(y)d\mu(y)$ for any ball $B \subset X$.

We start working on the case $1 < p \leq 2$.

**Lemma 5.1.** There exist constants $C_1, C_2 > 0$ such that for every $f \in W^{1,p}(X)$, $x_0 \in X$, $R > 0$ and $0 < s < R$,

$$\|f - f_s\|_{L^p(B(x_0,R))} \leq C_1s^{(d_\alpha - d_h)(1 - \frac{2}{p})} \frac{d \mu}{\mathbf{Var}_{B(x_0,C_2R)^p}(f)}.$$

**Proof.** To prove this pseudo-Poincaré inequality, we use similar arguments as in [34, Lemma 2.4]. Since $X$ is $d_h$-Ahlfors regular, there exists a collection of balls $\{B_i = B(x_i,s)\}_{i \in I}$ such that

$$\{x_i\}_{i \in I} \subset B(x_0,R), \quad B_i^{s/2} \cap B_j^{s/2} = \emptyset \text{ if } i \neq j, \quad B(x_0,R) \subset \bigcup_{i \in I} B_i^s.$$

Moreover, let $A$ be the constant appeared in Theorem 3.1, then there exists an integer $P$ such that each point $x \in X$ is contained in at most $P$ balls from the family $\{B_i^{2s} = B(x_i, 2As)\}_{i \in I}$. In other words, the bounded overlapping number $N(x) = \# \{i \in I : x \in B_i^{2As}\}$ is bounded by $P$. Now we can write

$$\|f - f_s\|_{L^p(B(x_0,R))}^p \leq \sum_{i \in I} \int_{B_i^s} |f(x) - f_s(x)|^p d\mu(x) \leq C \sum_{i \in I} \int_{B_i^s} \left(|f(x) - f_{B_i^{2s}}|^p + |f_{B_i^{2s}} - f_s(x)|^p\right) d\mu(x).$$

Observe that the Poincaré inequality in Theorem 3.1 implies

$$\int_{B_i^{s}} |f(x) - f_{B_i^{2s}}|^p d\mu(x) \leq C_{s^{p\alpha_d}d_\alpha} \mathbf{Var}_{B_i^{2As},p}(f)^p.$$

Moreover, note that for any $x \in B_i^{s}$, one has $B(x,s) \subset B_i^{2s}$. Recall also $\mu(B(x,s)) \simeq s^{d_h}$. Then applying Hölder’s inequality and again the Poincaré inequality yields

$$\int_{B_i^{s}} |f_s(x) - f_{B_i^{2s}}|^p d\mu(x) \leq C_{s^{-d_h}} \int_{B_i^{s}} \int_{B(x,s)} |f(y) - f_{B_i^{2s}}|^p d\mu(y) d\mu(x)$$

$$\leq C_{s^{-d_h}} \int_{B_i^{s}} \int_{B_i^{2s}} |f(y) - f_{B_i^{2s}}|^p d\mu(y) d\mu(x) \leq C_{s^{p\alpha_d}d_\alpha} \mathbf{Var}_{B_i^{2As},p}(f)^p.$$

Combining the above two estimates, we obtain

$$\|f - f_s\|_{L^p(B(x_0,R))}^p \leq C_{s^{p\alpha_d}d_\alpha} \sum_{i \in I} \mathbf{Var}_{B_i^{2As},p}(f)^p.$$
Moreover, since $0 < s < R$, there exists a constant $C_2 > 1$ such that $\bigcup_{x \in I} B_{2s}^x \subset B(x_0, C_2R)$. Hence from the bounded overlapping property,

$$\sum_{x \in I} \mathbf{Var}_{B_{2s}^x, p}(f)^p \leq C \mathbf{Var}_{B(x_0, C_2R), p}(f)^p.$$

This completes the proof of Lemma 5.1.

The following lemma will play a crucial role in proving the Sobolev inequalities.

**Lemma 5.2.** Let $F \subset X$ be a Borel set and let $1 < p \leq 2$. For any nonnegative $f \in W^{1,p}(X)$, it holds that

$$\left( \sum_{k \in \mathbb{Z}} \mathbf{Var}_{F, p}(f_k)^p \right)^{1/p} \leq C \mathbf{Var}_{F, p}(f),$$

where $f_k = (f - 2^k)_+ \wedge 2^k$, $k \in \mathbb{Z}$.

**Proof.** For any $f \in W^{1,p}(X)$, set

$$W_{F, r, p}(f) = \int_F \int_F |f(x) - f(y)|^p 1_{B(x, r)}(y) d\mu(y) d\mu(x).$$

Then applying the method in [5, Lemma 7.1] (see also [1, Lemma 2.6]) for $K(x, dy) = 1_F(x) 1_{B(x, r) \cap F}(y) d\mu(y)$ and $a = p$, we obtain

$$\sum_{k \in \mathbb{Z}} W_{F, r, p}(f_k) \leq 2(p + 1) W_{F, r, p}(f).$$

Now dividing by $r^{p\alpha_d - d_k}$ and taking the lim inf as $r \to 0^+$, we get

$$\liminf_{r \to 0^+} \frac{1}{r^{p\alpha_d - d_k}} \int_F \int_{F \cap B(x, r)} |f_k(x) - f_k(y)|^p d\mu(y) d\mu(x) \leq 2(p + 1) \liminf_{r \to 0^+} \frac{1}{r^{p\alpha_d - d_k}} \int_F \int_{F \cap B(x, r)} |f(x) - f(y)|^p d\mu(y) d\mu(x)$$

The superadditivity of the lim inf then gives

$$\sum_{k \in \mathbb{Z}} \liminf_{r \to 0^+} \frac{1}{r^{p\alpha_d - d_k}} \int_F \int_{F \cap B(x, r)} |f_k(x) - f_k(y)|^p d\mu(y) d\mu(x) \leq 2(p + 1) \liminf_{r \to 0^+} \frac{1}{r^{p\alpha_d - d_k}} \int_F \int_{F \cap B(x, r)} |f(x) - f(y)|^p d\mu(y) d\mu(x)$$

and we conclude the proof.

**Proposition 5.3.** Let $1 < p \leq 2$. There exists a constant $C > 0$ such that for every $f \in W^{1,p}(X)$, $x_0 \in X$ and $R > 0$,

$$\|f\|_{L^\infty(B(x_0, R))} \leq C \left( R^{-\frac{1}{p'}} \|f\|_{L^p(B(x_0, R))} + R^{(1-\frac{1}{p})(d_w - d_h)} \mathbf{Var}_{B(x_0, C_2R), p}(f) \right).$$

**Proof.** We recall that for any $s > 0$,

$$|f_s(x)| \leq C s^{-d_h} \|f\|_{L^1(X, \mu)}$$

and for $0 < s < R$ (see Lemma 5.1),

$$\|f - f_s\|_{L^p(B(x_0, R))} \leq C s^{\alpha_d - d_h} \mathbf{Var}_{B(x_0, C_2R), p}(f).$$

In view of Lemma 5.2, we can then apply [5, Theorem 9.1] for $r = \infty$, $s = 1$ and $q \neq 0$ such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_d - d_h}{d_w}$. It follows that

$$\|f\|_{L^\infty(B(x_0, R))} \leq C \left( R^{-\alpha_d} \|f\|_{L^p(B(x_0, R))} + \mathbf{Var}_{B(x_0, C_2R), p}(f)^p \right)^{\frac{q}{p(q - 1)}} \|f\|_{L^1(B(x_0, R), \mu)}^{\frac{1}{q - 1}} \leq CR^{\frac{d_w}{d_h}} \left( R^{-\alpha_d} \|f\|_{L^p(B(x_0, R))} + \mathbf{Var}_{B(x_0, C_2R), p}(f)^p \right)^{\frac{q}{p(q - 1)}} \|f\|_{L^1(B(x_0, R), \mu)}^{\frac{1}{q - 1}}.$$
This implies
\[ \|f\|_{L^\infty(B(x_0,R))} \leq CR^{-\frac{d_p}{p}} \left( R^{-\alpha_{p,d_w}} \|f\|_{L^p(B(x_0,R))} + \var B(x_0,C_2 R),p(f) \right) \]
\[ = CR^{-\alpha_{p,d_w}} \frac{d_p}{p} \left( R^{-\alpha_{p,d_w}} \|f\|_{L^p(B(x_0,R))} + \var B(x_0,C_2 R),p(f) \right) \]
and we finish the proof by plugging \( \alpha_p = \left( 1 - \frac{2}{p} \right) \left( 1 - \frac{d_w}{d_p} \right) + \frac{1}{p}. \)

Now we consider the case \( p = 1 \) on the infinite Vicsek set.

**Proposition 5.4.** Let \( X \) be the infinite Vicsek set. There exists a constant \( C > 0 \) such that for every \( f \in BV(X), x_0 \in X \) and \( R > 0, \)
\[ \|f\|_{L^\infty(B(x_0,R))} \leq C \left( R^{-d_h} \|f\|_{L^1(B(x_0,R))} + \var B(x_0,C_2 R),p(f) \right). \]

**Remark 5.5.** When \( R = \infty \), we recover the oscillation result in [2, Proposition 4.17] on the Vicsek set.

**Proof.** By Theorem 4.9, the same argument as in Lemma 5.1 gives the \( L^1 \) pseudo-Poincaré inequality:
for every \( f \in BV(X), x_0 \in X, R > 0 \) and \( 0 < s < R, \)
\[ \|f - f_s\|_{L^1(B(x_0,R))} \leq C_1 s^{d_h} \var B(x_0,C_2 R),p(f). \]
Moreover, Lemma 5.2 also holds for \( p = 1 \) and \( f \in BV(X) \). Thus we are able to apply [5, Theorem 9.1] for \( r = \infty, s = 1 \) and \( q = \infty \), which concludes the proof. \( \square \)

### 5.2 Maximal function and Hajlasz-Sobolev spaces

Let \( 1 \leq p \leq 2 \). For \( f \in W^{1,p}(X) \) (or \( BV(X) \) for \( p = 1 \)), we introduce the following fractal version of the Hardy-Littlewood maximal function
\[ g(x) := \sup_{r > 0} \frac{1}{\mu(B(x,r))^{1/p}} \var B(x,r),p(f). \tag{23} \]

It is easy to see that the maximal function \( g \) is weak \( L^p \) bounded. Indeed, for any \( t > 0 \), let \( E_t = \{ x : g(x) > t \} \). Then for each \( x \in E_t \), one can find an \( r_x > 0 \) such that
\[ \mu(B(x,r_x)) \leq \frac{1}{tp} \var B(x,r_x),p(f)^p. \]

Thus we get a family of balls \( \{ B(x,r_x) \}_{x \in E_t} \), which covers \( E_t \). By the \( 5 \)-covering theorem (Vitali covering lemma), there exists a disjoint countable subfamily of balls \( \{ B(x_i,r_i) \}_{i \in I} \) such that \( E_t \subset \bigcup_{i \in I} B(x_i,5r_i) \).

Hence by the \( d_h \)-Ahlfors regularity,
\[ \mu(E_t) \leq \sum_{i \in I} \mu(B(x_i,5r_i)) \leq C \mu(B(x_i,r_i)) \leq \frac{C}{p} \sum_{i \in I} \var B(x_i,r_i),p(f)^p \leq \frac{C}{p} \var X,f(p). \]

The \( L^p \) boundedness of \( g \) is an open question for future investigation.

In this section, our main result is the following Lusin-Hölder estimate for \( f \in W^{1,p}(X) \) in terms of the maximal function \( g \) defined as above.

**Proposition 5.6.** Let \( 1 < p \leq 2 \). Then there exist a constant \( C \) such that for every \( f \in W^{1,p}(X) \) and \( x,y \in X, \)
\[ |f(x) - f(y)| \leq Cd(x,y)^{(d_u - d_p)} \left( 1 - \frac{2}{p} \right) + \frac{d_u}{p} (g(x) + g(y)). \]

**Proof.** We will use a telescopic argument. Denote \( d(x,y) = R \) and note that
\[ |f(x) - f(y)| \leq |f(x) - f_{B(x,R)}| + |f_{B(x,R)} - f_{B(y,R)}| + |f(y) - f_{B(y,R)}|. \]
where \( f_B := \frac{1}{\mu(B)} \int_B f(z) d\mu(z) \) for any ball \( B \subset X \). Applying the Poincaré inequality in Remark 3.2, we have

\[
|f(x) - f_{B(x, R)}| \leq \sum_{m=0}^{\infty} |f_{B(x, 2^{-m} R)} - f_{B(x, 2^{-(m+1)} R)}| \\
\leq \sum_{m=0}^{\infty} \int_{B(x, 2^{-m} R)} |f(z) - f_{B(x, 2^{-m} R)}| d\mu(z) \\
\leq C \sum_{m=0}^{\infty} (2^{-m} R)^{2d(u)} \frac{1}{\mu(B(x, 2^{-m} R))^{1/p}} \text{Var}_{B(x, 2^{-m} R), p}(f) \\
\leq CR^{2d(u)} g(x).
\]

Similarly,

\[
|f(y) - f_{B(y, R)}| \leq CR^{2d(u)} g(y).
\]

It remains to estimate \( |f_{B(x, R)} - f_{B(y, R)}| \). Observe that \( B(y, R) \subset B(x, 2R) \), then the \( d_h \)-Ahlfors regularity and Theorem 3.1 deduce that

\[
|f_{B(x, R)} - f_{B(y, R)}| \leq |f_{B(x, R)} - f_{B(x, 2R)}| + |f_{B(x, 2R)} - f_{B(y, R)}| \\
\leq 2 \int_{B(x, 2R)} |f(z) - f_{B(x, 2R)}| d\mu(z) \\
\leq CR^{2d(u)} \frac{1}{\mu(B(x, 2R))^{1/p}} \text{Var}_{B(x, 2AR), p}(f) \leq CR^{2d(u)} g(x).
\]

We thus conclude the proof by combining the above three estimates.

Similarly, applying Theorem 4.8, we obtain

**Corollary 5.7.** Let \( X \) be the infinite Vicsek set. Then, there exist a constant \( C \) such that for every \( f \in BV(X) \),

\[
|f(x) - f(y)| \leq Cd(x, y)^{2d(u)} (g(x) + g(y)).
\]

Adapting to the fractal case the original definition by P. Hajłasz [20], we say that a function \( f \in L^p(X, \mu) \) is in the Hajłasz-Sobolev space \( HS^{1,p}(X) \) if there exists \( g \in L^p(X, \mu) \) such that for \( \mu \)-a.e. \( x, y \in X \),

\[
|f(x) - f(y)| \leq Cd(x, y)^{(d_u - d_h)(1 - \frac{1}{p}) + \frac{d_h}{p}} g(x) + g(y).
\]

On metric measure spaces, we refer to, for instance, [30, 18, 22] for the study of Hajłasz-Sobolev spaces and their relations with Poincaré inequalities and Korevaar-Schoen type Sobolev spaces. Those have also been explored on fractal spaces, see [24, 33].

In our setting, one always has \( HS^{1,p}(X) \subset W^{1,p}(X) (1 < p \leq 2) \) and \( HS^{1,1}(X) \subset BV(X) \), for which the proof can be found in [24, Theorem 1.1].

### 5.3 Further characterizations of the Korevaar-Schoen-Sobolev spaces in the Vicsek set

Finally, to conclude the paper, we provide in the Vicsek set a description of the Korevaar-Schoen-Sobolev spaces which is similar to the description of the Sobolev space \( W^{1,2}(K) \) as the domain of the Dirichlet form \( E_K \).

Let \( \{q_1, q_2, q_3, q_4\} \) be the 4 corners of the unit square and let \( q_5 = (1/2, 1/2) \) be the center of the square. Define \( \psi_i(z) = \frac{1}{2}(z - q_i) + q_i \) for \( 1 \leq i \leq 5 \). Recall that the Vicsek set \( K \) is the unique non-empty compact set such that

\[
K = \bigcup_{i=1}^{5} \psi_i(K).
\]

Let \( 1 < p \leq 2 \). For \( n \geq 0 \) and \( f \in C(K) \) we define the discrete \( p \)-Korevaar-Schoen energy as

\[
E_n^{(p)}(f, f) = \rho^{(p-1)n} \sum_{w \in W_n} \sum_{x, y \in V^{(n)}} |f \circ \psi_w(x) - f \circ \psi_w(y)|^p,
\]
where $\rho = L^{d_w-d_h}$ is as before the resistance scale factor of $K$. We note that since $K$ is assumed to be the Vicsek set in this section, one actually has $L = 3$ and $d_w - d_h = 1$. We have then the following theorem:

**Theorem 5.8.** Let $1 < p \leq 2$ and $f \in C(K)$. Then, $f \in W^{1,p}(K)$ if and only if $\lim inf_n \mathcal{E}_n^{(p)}(f,f) < +\infty$. Moreover, on $W^{1,p}(K)$

$$\text{Var}_{K,p}(f)^p \simeq \sup_n \mathcal{E}_n^{(p)}(f,f) \simeq \lim inf_n \mathcal{E}_n^{(p)}(f,f),$$

and $W^{1,p}(K)$ is dense in $L^p(K,\mu)$.

**Remark 5.9.** Since $W^{1,2}(K)$ is exactly the domain of the Dirichlet form $\mathcal{E}_K$, this theorem is well-known for $p = 2$, see Section 2.3 for further details.

**Proof.** Let $f \in W^{1,p}(K)$. From Lemma 3.9 and Remark 3.13, there exists a constant $C > 0$ such that for every $n$-simplex $K_w \subset K$

$$|f(x) - f(y)| \leq CL^{-n(1-\frac{1}{p})} \text{Var}_{K,w,p}(f), \quad x,y \in K_w.$$ 

This yields

$$\rho^{(p-1)n} \sum_{w \in W_n} \sum_{x,y \in V_n} |f \circ \psi_w(x) - f \circ \psi_w(y)|^p \leq C \rho^{(p-1)n} \sum_{w \in W_n} L^{-n(p-1)} \text{Var}_{K,w,p}(f)^p \leq C \rho^{(p-1)n} L^{-n(p-1)} \text{Var}_{K,p}(f)^p.$$ 

Since $\rho = L$, we obtain $\sup_n \mathcal{E}_n^{(p)}(f,f) \leq C \text{Var}_{K,p}(f)^p$.

We now prove that $\text{Var}_{K,p}(f)^p \leq C \lim inf_n \mathcal{E}_n^{(p)}(f,f)$. The idea is to use approximations by piecewise affine functions. Let $\bar{V}^{(0)}$ be the metric graph with vertices $\{q_1, q_2, q_3, q_4, q_5\}$ which consists of the union of the two diagonals of the unit square. We consider the sequence of metric graphs $\bar{V}^{(n)}$ inductively defined as follows. The first metric graph is $\bar{V}^{(0)}$ and then

$$\bar{V}^{(n+1)} = \bigcup_{i=1}^5 \psi_i(\bar{V}^{(n)}).$$

Note that $\bar{V}^{(n)} \subset K$. A continuous function $\Phi : K \to \mathbb{R}$ is called $n$-piecewise affine, if there exists $n \geq 0$ such that $\Phi$ is piecewise affine on the metric graph $\bar{V}^{(n)}$ (i.e linear between the vertices of $\bar{V}^{(n)}$) and constant on any connected component of $\bar{V}^{(m)} \setminus \bar{V}^{(n)}$ for every $m > n$.

If $\Phi$ is $n$-piecewise affine, then the basic convexity inequality

$$|x+y+z|^p \leq 3^{p-1}(|x|^p + |y|^p + |z|^p)$$

shows that $\mathcal{E}_0^{(p)}(\Phi,\Phi) \leq \cdots \leq \mathcal{E}_n^{(p)}(\Phi,\Phi)$. It is moreover seen that for $m \geq n$, $\mathcal{E}_m^{(p)}(\Phi,\Phi) = \mathcal{E}_n^{(p)}(\Phi,\Phi)$. Furthermore, for every $(x,y) \in K$,

$$|\Phi(x) - \Phi(y)|^p \leq \mathcal{E}_n^{(p)}(\Phi,\Phi).$$

We deduce that if $w \in \cup_n W_n$, then for every $x,y$ in $K_w$

$$|\Phi(x) - \Phi(y)|^p \leq \mathcal{E}_n^{(p)}(\Phi \circ \psi_w, \Phi \circ \psi_w).$$

Using then similar chaining arguments as before, we get that for every $(x,y)$ in $K_w \times K_w$

$$|\Phi(x) - \Phi(y)|^p \leq C \sum_{w^*} \mathcal{E}_n^{(p)}(\Phi \circ \psi_{w^*}, \Phi \circ \psi_{w^*})$$

where the summation is made over the set of $w^*$ such that $K_{w^*} \subset K_w$. Let now $0 < R < \beta$. As before, there exists a unique $k$ such that

$$L^{-(k+1)} < R/\beta \leq L^{-k}.$$
Consider the covering of $K$ by the $M^k$ $k$-simplices $\{K_w\}_{w \in W_k}$ (here $M = 5$). For any $x \in K_w$, we have that $B(x, R) \subset K_w^\ast$. We have then

$$\int_K \int_{B(x, R) \cap K} |\Phi(x) - \Phi(y)|^p \, d\mu(y) \, d\mu(x) = \sum_{w \in W_k} \int_{K_w} \int_{B(x, R) \cap K} |\Phi(x) - \Phi(y)|^p \, d\mu(y) \, d\mu(x)$$

$$\leq \sum_{w \in W_k} \int_{K_w} \int_{K_w^\ast \cap K} |\Phi(x) - \Phi(y)|^p \, d\mu(y) \, d\mu(x)$$

$$\leq C \sum_{w \in W_k} \int_{K_w^\ast \cap K} d\mu(y) \, d\mu(x) \sum_{w^\ast} \mathcal{E}_n^{(p)}(\Phi \circ \psi_{w^\ast}, \Phi \circ \psi_{w^\ast})$$

$$\leq CM^{-2k} \sum_{w \in W_k} \mathcal{E}_n^{(p)}(\Phi \circ \psi_w, \Phi \circ \psi_w).$$

We now observe that:

$$\sum_{w \in W_k} \mathcal{E}_n^{(p)}(\Phi \circ \psi_w, \Phi \circ \psi_w) = \rho^{-(p-1)k} \mathcal{E}_{n+k}^{(p)}(\Phi, \Phi) = \rho^{-(p-1)k} \mathcal{E}_n^{(p)}(\Phi, \Phi).$$

Since $M^{-2k} \rho^{-(p-1)k} \leq CR^{p \alpha \rho_d + d_h}$, we deduce that $n$-piecewise affine functions are in $W^{1,p}(K)$ and that for every $R < \beta$

$$\int_K \int_{B(x, R) \cap K} |\Phi(x) - \Phi(y)|^p \, d\mu(y) \, d\mu(x) \leq CR^{p \alpha \rho_d + d_h} \mathcal{E}_n^{(p)}(\Phi, \Phi).$$

Let now $f \in C(K)$ such that $\inf_{K_w} \mathcal{E}_n^{(p)}(f, f) < +\infty$. For every $n \geq 0$ we define $f_n$ to be the unique $n$-piecewise affine function on $K$ that coincides with $f$ on $V^{(n)}$. By construction of $f_n$, it is clear that for every $n \geq 0$ and $w \in W_n$, we have for every $x \in K_w$,

$$\inf_{K_w} f \leq f_n(x) \leq \sup_{K_w} f.$$

Since $f \in C(K)$, we deduce that $f_n$ converges to $f$ in the supremum norm. We have for every $n \geq 0$,

$$\int_K \int_{B(x, R) \cap K} |f_n(x) - f_n(y)|^p \, d\mu(y) \, d\mu(x) \leq CR^{p \alpha \rho_d + d_h} \mathcal{E}_n^{(p)}(f_n, f_n) = CR^{p \alpha \rho_d + d_h} \mathcal{E}_n^{(p)}(f, f).$$

Taking $\lim \inf_{n \to +\infty}$ yields

$$\int_K \int_{B(x, R) \cap K} |f(x) - f(y)|^p \, d\mu(y) \, d\mu(x) \leq CR^{p \alpha \rho_d + d_h} \lim \inf \mathcal{E}_n^{(p)}(f, f).$$

Thus $f \in W^{1,p}(K)$. Finally, in order to prove that $W^{1,p}(K)$ is dense in $L^p(K, \mu)$, it is enough to prove that the set of piecewise affine functions is dense in $C(K)$ which follows from the previous arguments.

**Remark 5.10.** The results of that section also hold for the $n$-dimensional analogue of the Vickers set with similar proofs.

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