Multi-step Fermi normal coordinates

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Abstract

We generalize the concept of Fermi normal coordinates adapted to a geodesic to the case where the tangent space to the manifold at the base point is decomposed into a direct product of an arbitrary number of subspaces, so that we follow several geodesics in turn to find the point with given coordinates. We compute the connection and the metric as integrals of the Riemann tensor. We calculate explicitly an example of a five-dimensional brane inflation spacetime. In the case of one subspace (Riemann normal coordinates) or two subspaces, we recover some results previously found by Nesterov, using somewhat different techniques.

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1. Introduction

The construction of Riemann normal coordinates is well known. For any point $p$ of a Riemannian or Lorentzian manifold $(M, g)$ and any vector $V_p$ at $p$ there exists a maximal geodesic $\gamma_V(\lambda)$ with starting point $p$ and initial direction $V_p$. We define the exponential map $\exp_p$ that takes a subset of $T_p p$, the tangent space to $M$ at $p$, into $M$, such that $\exp_p(V)$ is the point $q$ a unit parameter distance along the geodesic $\gamma_V$ from $p$.

We can choose an orthonormal tetrad basis $\{E^{(\alpha)}\}$ at $p$ and then define the coordinates at $q$ by the relation $q = \exp(x^{(\alpha)}E^{(\alpha)})$. Such coordinates are called Riemann normal coordinates.

The Fermi normal coordinate construction [1] is also well known. We start with a timelike geodesic $\gamma_K(\lambda)$ with tangent vector $K$ at $p$. (We will consider only geodesics, not arbitrary timelike curves.) Given any vector $V \in T_p(M)$, we can write $V = A + B$, where $A$ is in the direction of $K$ and $B$ perpendicular to $K$. We then let $q = \exp_p(A)$ and define a map $\text{Fermi}_p(V) = \exp_p(B)$. That is to say, $\text{Fermi}_p(V)$ is found by first moving unit distance along the geodesic $\gamma_A$ from $p$ to $q$. We parallel transport $B$ from $p$ to $q$ and then move unit distance along the geodesic whose tangent vector at $q$ is $B$.

We can define an orthonormal basis $\{E^{(\alpha)}\}$ at $p$ such that $E^{(0)}$ is parallel to $K$. Then $A = x^{(0)}E^{(0)}$, $B = x^{(\alpha)}E^{(\alpha)}$, giving the usual construction of Fermi normal coordinates [1].
In this paper, we generalize this construction to allow an arbitrary number of arbitrary subspaces (and an arbitrary number of dimensions $d$), rather than just a timelike geodesic and the perpendicular space. In section 2 we construct the generalized coordinate system, in section 3 we compute the connection, and in section 4 we compute the metric in the generalized Fermi coordinates. In section 5, we briefly discuss the question of when these coordinates are well defined, and in section 6 we explicitly construct multi-step Fermi coordinates for a particular brane-world model. We conclude in section 7.

We use the sign convention $(+,-,+)$ in the classification of Misner, Thorne and Wheeler [2].

2. Multi-step Fermi coordinates

Consider a $d$-dimensional Riemannian or Lorentzian manifold $(\mathcal{M}, g)$. We will start our construction by choosing a base point $p \in \mathcal{M}$. We decompose the tangent space $T_p$ into $n$ subspaces, $T_p = A_p^{(1)} \times A_p^{(2)} \times A_p^{(3)} \times \cdots \times A_p^{(n)}$ so that any $V \in T_p$ can be uniquely written as $V = V^{(1)} + V^{(2)} + V^{(3)} + \cdots + V^{(n)}$. We choose, as a basis for $T_p$, $d$ linearly independent vectors $\{E^{(\alpha)}\}$ adapted to the decomposition of $T_p$, so that for each $m = 1, \ldots, n$, $\{E^{(\alpha)}|\alpha \in c_m\}$ is a basis for $A_p^{(m)}$, where $c_1, c_2, \ldots, c_n$ is an ordered partition of $\{1, \ldots, d\}$. Thus each $V^{(m)} = \sum_{\alpha \in c_m} x^\alpha E^{(\alpha)}$. The vectors $\{E^{(\alpha)}\}$ need not be normalized or orthogonal.

The point corresponding to coordinates $x^\alpha$ is then found by starting from $p$ and going along the geodesic whose tangent vector is $V^{(1)}$, parallel transporting the rest of the vectors, then along the geodesic whose tangent vector is $V^{(2)}$, and so on. An example is shown in figure 1.

With that general construction of multi-step Fermi coordinates we can define a general Fermi mapping $q = \text{Fermi}_p(V)$ given by

\[
q(0) = p \\
q(1) = \exp_p(V^{(1)}) \\
q(2) = \exp_{q(1)}(V^{(2)}) \\
\vdots \\
q = q(n) = \exp_{q(n-1)}(V^{(n)}). \tag{1}
\]

From that general construction we can return to the original Fermi case by choosing $c_1 = \{t\}$ and $c_2 = \{x, y, z\}$. In the Lorentzian case, we could also choose a pseudo-orthonormal tetrad $E_u, E_v, E_x, E_y$, with $E_u$ and $E_v$ null, $E_u \cdot E_v = -1$, and other inner products vanishing, and $c_0 = \{u\}$ and $c_1 = \{v, x, y\}$.
For later use we will define

\[ V_{\leq m} = \sum_{\alpha \in c_1 \cup \cdots \cup c_m} x^{\alpha} E_\alpha = \sum_{l=1}^{m} V_l \]

Then we can write

\[ V_{\leq (m-1)} = V_{\leq m} - V_m. \]  

An example of a spacetime where these multi-step Fermi coordinates might be used comes from brane-world models. A general brane-world metric with one extra dimension can be written

\[ ds^2 = b(w)^2 [-dt^2 + a(t)(dx^2 + dy^2 + dz^2)] + dw^2. \]

In this kind of spacetime it might be useful to introduce three-step Fermi coordinates with \( c_1 = \{ w \}, c_2 = \{ t \} \) and \( c_3 = \{ x, y, z \} \). We will work out an example like this in section 6.

Another application arises when one is interested in a congruence of geodesics. If one uses usual Fermi coordinates, points of the form \( (t, x, y, z) \) with \( x, y, \) and \( z \) constant will not in general lie on a geodesic, even if the generating line \((t, 0, 0, 0)\) is a geodesic. But if one uses coordinates as described here with the last step being just \( \{ z \} \), then lines with other coordinates fixed and \( z \) varying will be geodesics. In [3] we use this technique to analyze a congruence of null geodesics using pseudo-orthonormal Fermi-like coordinates with one null and two spacelike directions in the first step and a single null direction in the second step.

3. Connection

We will parallel transport our orthonormal basis vectors \( E_\alpha \) along the geodesics that generate the coordinates, and use them as a basis for vectors and tensors throughout the region of \( M \) covered by our coordinates. Components in this basis will be denoted by Greek indices. We will use Latin letters from the beginning of the alphabet to denote indices in the Fermi coordinate basis. Of course at \( p \), there is no difference between these bases.

Latin letters from the middle of the alphabet will denote the subspaces of \( T_p \), or equivalently the steps of the Fermi mapping process.

We would like to calculate the covariant derivatives of the basis vectors, \( \nabla_\beta E_\alpha \), which are related to connection 1-forms (see for example [4]) by

\[ \omega_{\beta\alpha\delta} = \eta_{\beta\gamma} \nabla_\beta E_\gamma, \]

because we are using a orthonormal tetrad basis.

We can then calculate the covariant derivative of any vector field \( V = V_\beta E_\beta \) along a curve \( f(\lambda) \) as

\[ \frac{DV_\beta}{d\lambda} = \frac{dV_\beta}{d\lambda} + V_\gamma \left( \frac{\partial}{\partial \lambda} \right)^{\alpha} \nabla_\alpha E_\beta. \]

To evaluate \( \nabla_\beta E_\alpha \) at some point \( q_1 = \exp_p(X) \), consider an infinitesimally separated point \( q_2 = \exp_p(X + E_\beta dx) \). The covariant derivative of \( E_\alpha \) at \( q_1 \) is the difference between \( E_\alpha(q_2) \) parallel transported to \( q_1 \) and the actual \( E_\alpha(q_1) \), divided by \( dx \). That difference is the same as the change in \( E_\alpha \) by parallel transport around a loop following the geodesics from \( q_1 \) backward to \( p \), the infinitesimally different geodesics forward from \( p \) to \( q_2 \), and the infinitesimal distance back to \( q_1 \). We can write this loop parallel transport as an integral over the Riemann tensor.
Let us first consider the Riemannian case, as shown in figure 2. The total parallel transport can be written as the sum of parallel transport around a succession of small trapezoidal regions whose sides are $\lambda E_{(\beta)}dx$ and $Xd\lambda$. By using the definition of the Riemann tensor we have

$$\nabla_{\beta}E_{(\alpha)} = \int_0^1 d\lambda R^\gamma_{\alpha\delta\beta}(\lambda X)\lambda X^\delta. \quad (7)$$

Here $R$ is evaluated at the point $\exp_p(\lambda X)$, which we have denoted merely $\lambda X$ for compactness.

Equation (7) reproduces equation (13) of [5]. Note, however, that equation (7) is exact and does not require $R$ to be smooth, whereas that of [5] was given as first order in $R$ and was derived by means of a Taylor series.

We see immediately that the covariant derivative of any $E_{(\alpha)}$ at $X$ in the direction of $X$ vanishes. This happens simply because changes with $dX$ in the direction of $X$ correspond to additional parallel transport of $E_{(\alpha)}$.

Let us now consider the general case where there are $n$ steps, and compute $\nabla_{\beta}E_{(\alpha)}$. Since the coordinates are adapted to our construction, the index $\beta$ must be in some specific set $c_m$, which is to say that the direction of the covariant derivative, $E_{(\beta)}$, is part of step $m$ in the Fermi coordinate process. We will write the function that gives that $m$ as $m(\beta)$. Some particular cases are shown in figure 3.
Figure 4. Part of the calculation of $\nabla_\beta E^{(\alpha)}$ in the case $n = 2$, $m(\beta) = 1$. The geodesic of the first step has been modified, causing the geodesic in the second step to be displaced. The parallel transport integrates the Riemann tensor over a series of rectangular regions between the two second-step geodesics.

If $m = n$ (leftmost in figure 3), only the last step is modified. The integration is exactly as shown in figure 2, except that it covers only the final geodesic from $X_{(n-1)}$ to $X_{(n)}$.

$$\nabla_\beta E^{(\alpha)} = \int_0^1 d\lambda R^{(\alpha)}_{\gamma\delta\beta}(X_{(n-1)} + \lambda X_{(n)}) \lambda X^{(n)}.$$  \hfill (8)

If $m < n$, then we are modifying some intermediate step, and the path followed at later steps is displaced parallel to itself. In that case we get an integral over rectangular rather than trapezoidal regions, as shown in figure 4. For general $m$ there is a contribution for each step $j \geq m$. The $j = m$ contribution integrates over trapezoids that grow with $\lambda$, while the $j > m$ contributions integrate over rectangles with fixed width $dx$. We can write the complete result

$$\nabla_\beta E^{(\alpha)} = \sum_{j=m}^n \int_0^1 d\lambda a_{jm}(\lambda) R^{(\alpha)}_{\gamma\delta\beta}(X_{(n-j)} + \lambda X_{(j)}) X^{(n)}.$$  \hfill (9)

where $m = m(\beta)$ and

$$a_{jm}(\lambda) = \begin{cases} 1 & j \neq m \\ \lambda & j = m. \end{cases}$$  \hfill (10)

Equation (9) is exact and includes equations (7), (8) as special cases.

Consider the case where $c_1$ consists only of one index. If $m > 1$, there is no $j = 1$ term in equation (9). If $m = 1$, then $\beta$ is the single index in $c_1$, and $X^{(1)}_{(1)} = 0$ unless $\delta = \beta$, so the $j = 1$ term in vanishes because $R_{\gamma\delta\beta\lambda}$ is antisymmetric under $\delta \leftrightarrow \beta$. Thus there is never a $j = 1$ contribution to equation (9) when there is only one index in $c_1$.

Now suppose $X$ lies on the first generating geodesic, so $X_{(j)} = 0$ for $j > 1$. Then all $j > 1$ terms vanish in equation (9). So if $c_1$ consists only of one index, all Christoffel symbols vanish at $X$. This is well known in the case of the usual Fermi coordinates.

4. Metric

Now we would like to compute the metric $g$ at some point $X$. Specifically, we would like to compute the metric component $g_{ab}$ in our generalized Fermi coordinates.
We will start by considering the vectors $Z_{(a)} = \partial/\partial x^a$. These are the basis vectors of the Fermi coordinate basis for the tangent space, so the metric is given by $g_{ab} = Z_{(a)} \cdot Z_{(b)}$. Thus if we compute the orthonormal basis components $Z_{(a)}^\beta$ we can write $g_{ab} = \eta_{\alpha\beta} Z_{(a)}^\alpha Z_{(b)}^\beta$.

Again we will start with the case of Riemann normal coordinates. Let $W(t, s)$ be the point $\exp_p s(X + tE_{(a)})$. Define $Y = \partial W/\partial t$ and $V = \partial W/\partial s$. Then $Y(X) = Z_{(a)}$ and $V^\beta = X^\beta + t d_{\beta}^\alpha$. The components of $Z_{(a)}$ at $X$ can be calculated by integration,

$$Z_{(a)}^\beta(X) = Y^\beta(X) = \int_0^1 ds \frac{\partial Y^\beta(sX)}{ds}. \quad (11)$$

Because the orthonormal basis is parallel transported we can write

$$\frac{dY^\beta}{ds} = \frac{DY^\beta}{ds}. \quad (12)$$

By construction, the Lie derivative $L_Y Y = 0$ and thus [6, chapter 4]

$$\frac{DY}{ds} = \frac{DV}{dt} \quad (13)$$

From equation (6) we have

$$\frac{DV^\beta}{dt} = \frac{dV^\beta}{dt} + V^\gamma V_a \nabla_a E_{(\gamma)}^\beta = \delta^\beta_{\alpha} + \delta^\beta_a V^\alpha \nabla_a E_{(\gamma)}^\beta + O(R^2), \quad (14)$$

where we have retained $\delta^\beta_a$ instead of writing $\nabla_a E_{(\gamma)}^\beta$ to make it clear that the covariant derivative is with respect to the orthonormal basis.

From equation (7) we have

$$\nabla_a E_{(\gamma)}(sX) = \int_0^s ds \int_0^1 d\lambda \lambda R^a_{\gamma\lambda}(\lambda sX)sX^\lambda = \frac{1}{s} \int_0^s d\lambda \lambda R^a_{\gamma\lambda}(\lambda X)X^\lambda. \quad (15)$$

Taking $t = 0$, $V$ is just $X$. Combining equations (11)–(15), we find

$$Z_{(a)}^\beta(X) = \int_0^1 ds \left[ \delta^\beta_{\alpha} + \delta^\beta_a \int_0^s \int_0^1 d\lambda \lambda R^a_{\gamma\lambda}(\lambda X)X^\lambda X^\gamma \right] + O(R^2)$$

$$= \delta^\beta_{\alpha} + \delta^\beta_a \int_0^1 d\lambda \lambda (1 - \lambda) R^a_{\gamma\lambda}(\lambda X)X^\lambda X^\gamma + O(R^2). \quad (16)$$

From equation (16), the metric is given by

$$g_{ab} = \eta_{ab} + 2 \delta^\alpha_a \delta^\beta_b \int_0^1 d\lambda \lambda (1 - \lambda) R^a_{\gamma\lambda}(\lambda X)X^\lambda X^\gamma + O(R^2). \quad (17)$$

Equation (17) reproduces equation (14) of [5] \(^1\).

Next let us consider the case where there are $n$ steps in our procedure. We will define a set of functions $W_j$ as

$$W_j(s) = \text{Fermi}_p(X_0 + sX_{(j)}). \quad (18)$$

The path $W_j(s)$, $j = 1, \ldots, n$, $s = 0, \ldots, 1$ traces the geodesics generating the Fermi coordinates for the point $X$. Now consider $Z_{(a)} = \partial/\partial x^a$. Let $m = m(a)$, so $Z_{(a)}(p) \in \Lambda^{[m]}_p$. Then let

$$W_j(s, t) = \text{Fermi}_p \begin{cases} 
X_{(j)} + s(X_{(j)} + tE_{(a)}) & j < m \\
X_{(j)} + sE_{(a)} + sX_{(j)} & j = m \\
X_{(j)} + tE_{(a)} + sX_{(j)} & j > m.
\end{cases} \quad (19)$$

\(^1\) Reference [5] uses the same sign convention for $R^\rho_{\mu\nu\beta}$ as the present paper, but the opposite convention for $g_{ab}$ and consequently also for $R_{a\beta\gamma}\delta$. 
The generalized version of equation (14) is

\[ F = \sum_{j=1}^{n} \int_{0}^{1} \frac{dY^j}{ds}(W_j(s)) ds. \]  

(20)

The generalized version of equation (14) is

\[ \frac{DV^\beta}{dt} = \frac{dV^\beta}{dt} + V^\gamma Y^a \nabla_a E^\beta_{(\gamma)} = \begin{cases} 0 & j < m \\ \delta^a_b V^\gamma \nabla_a E^\beta_{(\gamma)} + O(R^2) & j = m \\ \delta^a_b V^\gamma \nabla_a E^\beta_{(\gamma)} + O(R^2) & j > m. \end{cases} \]

(21)

Now

\[ \nabla_a E^\beta_{(\gamma)}(W_j(s)) = \sum_{k=m}^{j} \frac{1}{s \kappa_j(s)} \int_{0}^{t_j(s)} d\lambda \ a_{km}(\lambda) R^\beta_{\gamma \delta \alpha}(X_{(c k)}) + \lambda X_{(k)} X_{(j)} \]

where

\[ s_{kj}(s) = \begin{cases} 1 & k \neq j \\ s & k = j. \end{cases} \]

(23)

The \( k = j \) term is analogous to equation (15), while the others have no dependence on \( s \).

Combining equations (12),(13),(20)-(22) we get

\[ Z^{\beta}_{(a)}(X) = \delta^\alpha_b + F^\beta_a + O(R^2), \]

(24)

where

\[ F^\beta_a = \sum_{j=m}^{n} \sum_{k=m}^{j} \int_{0}^{1} ds \int_{0}^{s_{kj}(s)} d\lambda \ a_{km}(\lambda) R^\beta_{\gamma \delta \alpha}(X_{(c k)}) + \lambda X_{(k)} X_{(j)} \]

\[ = \sum_{j=m}^{n} \sum_{k=m}^{j} \int_{0}^{1} d\lambda \ a_{km}(\lambda) b_{kj}(\lambda) R^\beta_{\gamma \delta \alpha}(X_{(c k)}) + \lambda X_{(k)} X_{(j)} \]

(25)

where \( m = m(\alpha) \) and

\[ b_{kj}(\lambda) = \begin{cases} 1 & k \neq j \\ 1 - \lambda & k = j. \end{cases} \]

(26)

Thus the metric is

\[ g_{ab} = \eta_{ab} Z^{\alpha}_{(a)} Z^{\beta}_{(b)} = \eta_{ab} + F_{ab} + O(R^2), \]

(27)

where

\[ F_{ab} = \sum_{j=m}^{n} \sum_{k=m}^{j} \int_{0}^{1} d\lambda \ a_{km}(\lambda) b_{kj}(\lambda) R_{\gamma \delta \beta}(X_{(c k)}) + \lambda X_{(k)} X_{(j)} \]

(28)

where \( m = m(\beta) \).

Florides and Synge [7] construct coordinates by taking geodesics perpendicular to an embedded submanifold. Our construction for \( n = 2 \) is of this kind in the case where all basis vectors with indices in \( c_1 \) lie tangent to the surface generated by all first-step geodesics. This will be so if \( R_{\gamma \delta \beta} = 0 \) everywhere on this surface whenever \( \gamma, \delta, \beta \in c_1 \) and \( \alpha \in c_2 \). In that case, equation (28) agrees with theorem I of [7].

Now, consider again the case where \( c_1 \) contains only one index. As discussed with respect to equation (9), if \( \beta \in c_1 \), there is no non-vanishing \( k = 1 \) term in equation (28). Thus \( g_{ab} = \eta_{ab} \) everywhere on the first generating geodesic. This is also well known in the usual Fermi case.
Now suppose $c_1$ consists only of one index and furthermore $n = 2$. The only possible term in equation (28) is then $j = k = 2$, so

$$F_{a\beta} = \int_0^1 d\lambda a_{2m}(\lambda)(1 - \lambda)R_{a\gamma\beta\delta}(X_1 + \lambda X_2)X^\epsilon_\gamma X^\delta_\epsilon$$

(29)

where $m = m(\beta)$. Equation (29) is equivalent to equation (28) in [5] in the case where the generating curve of the Fermi coordinates is a geodesic.

5. Regularity of the coordinates

Riemann normal coordinates cannot in general be defined over the entirety of a manifold, because there might be points conjugate to the base point $p$. At such a point, some infinitesimal change to the coordinates would yield no change to the resulting point, and the metric would be singular.

Similar considerations apply to Fermi normal coordinates [1]. No trouble can occur in the first step, because that consists merely of traveling down a geodesic. Along the geodesic, and thus by continuity in some neighborhood surrounding the geodesic, Fermi normal coordinates are regular. If we attempt to extend beyond this neighborhood, we may find points that are conjugate to the generating geodesic. In such places the metric will become singular.

The situation here is more complicated. The metric will be singular whenever an infinitesimal change in coordinates fails to yield a change in the location of the resulting point. But when there are more than two steps, the result can no longer be described in terms of conjugate points. Nevertheless, it is easy to see that one if one chooses a sufficiently small neighborhood around $p$, all multi-step Fermi coordinates will be well defined, since any such coordinates approach Riemann normal coordinates when all coordinate values are sufficiently small.

In case $c_1$ contains only one index, multi-step Fermi coordinates will be well defined in a neighborhood of the initial geodesic, because when all coordinate values except for the first are small, the multi-step Fermi coordinates approach the usual Fermi coordinates. There is no particular advantage to having a single index in any later $c_m$.

One can get a simple condition sufficient for the existence of multi-step Fermi coordinates in a small region by looking at equations (27), (28). As long as $|F_{ab}| \ll 1$, the metric $g_{ab}$ cannot degenerate. Thus the coordinates will be well defined if [5]

$$|R_{a\gamma\beta\delta}(X^\epsilon)|^2 \ll 1$$

(30)

throughout the region of interest, for all $\alpha, \gamma, \beta, \delta, \epsilon$. In the case where there is only one index in $c_1$, there is no contribution to $F_{ab}$ from $X^\epsilon_\gamma$. Then it is sufficient for equation (30) to hold for $\epsilon > 1$. Thus if the first step is one-dimensional, it can be arbitrarily long [1], as discussed above.

6. Examples

We now consider two examples of the use of Fermi coordinates. First we check our formula for the metric in terms of the curvature using the known solution for de Sitter space. Then we calculate three-step Fermi coordinates for a five-dimensional brane inflation metric and again verify our approximation of the metric as an integral over the Riemann tensor.
6.1. de Sitter space

Let us consider the case of de Sitter space in Robertson–Walker coordinates,
\[ ds^2 = -dt^2 + e^{2Ht} [dr^2 + r^2 d\Omega^2]. \] (31)

We start from the point \( r = t = 0 \) and proceed first in the \( t \) direction for distance \( \bar{t} \) and then along a geodesic pointing initially in the radial direction for distance \( \bar{r} \). Thus there is one coordinate in the first step and three in the second step, but the angular coordinates \( \theta \) and \( \phi \) are fixed at the beginning and not changed as we travel along the geodesics.

The conversion between Fermi coordinates (which we will denote here with overbars) and the original coordinates, as well as the resulting metric, are known [8, 9]:
\[ t = \bar{t} + \ln \cos H \bar{r}, \] (32)
\[ r = e^{-H \bar{t}} \tan H \bar{r}, \] (33)
\[ ds^2 = -\cos^2 H \bar{r} d\bar{t}^2 + d\bar{r}^2 + \sin^2 H \bar{r} \frac{dr^2 + r^2 d\Omega^2}{H^2}. \] (34)

We now check this metric against our prediction above. The curvature components are all proportional to \( H^2 \), so we compute only to that order.

In the present case, Equation (29) becomes
\[ F_{\alpha \beta} = \int_0^1 d\lambda \ a_{2m}(\lambda)(1 - \lambda) R_{\alpha \beta \gamma \delta} \bar{r}^2. \] (35)

The Riemann tensor component here is the component in the tetrad basis, which differs from the coordinate-basis component in two ways. First, components other than \( t \) are rescaled by \( e^{Ht} \), but since the components are already \( O(H^2) \), we can take this as 1. The second effect is that as we move along a geodesic, the relationship between tetrad and coordinate-basis vectors changes. However, this is again at a higher order in \( H \).

The only non-vanishing Riemann tensor components of the form \( R_{\alpha \beta \gamma \delta} \) have \( \alpha = \beta \). We start with
\[ R_{ttrt} = H^2 e^{2Ht} = H^2 e^{2Ht} \cos^2 H \bar{r} \] (36)
so
\[ F_{tt} = \frac{1}{2} H^2 r^2 e^{2Ht} \cos^2 H \bar{r} = \frac{1}{2} H^2 r^2 + O(H^4) \] (37)
and thus
\[ g_{tt} = -1 + 2F_{tt} = -(1 - H^2 r^2). \] (38)

We also have
\[ R_{rrtt} = H^2 r^2 e^{2Ht} = H^2 r^2 + O(H^4). \] (39)
In this case, \( m = 1, a_{2m} = \lambda \), and we find
\[ F_{\theta \theta} = \frac{1}{2} H^2 H^2 r^2 + O(H^4) \] (40)
and
\[ g_{\theta \theta} = \bar{r}^2 (1 - H^2 \bar{r}^2 / 3). \] (41)

Equations (38), (40) agree with equation (34) at order \( H^2 \). The calculation for \( g_{\phi \phi} \) is essentially the same.
6.2. Brane-world spacetime

A specific example of the general brane-world metric of equation (4) is
\[ ds^2 = e^{2Lt} \left(-dt^2 + e^{2Ht} \left(dr^2 + r^2 d\Omega^2\right)\right) + dw^2. \]  
(42)

This just the metric of [10] with a change of coordinates. We will use three-step Fermi coordinates with \( c_1 = \{w\}, c_2 = \{t\} \) and \( c_3 = \{r, \theta, \phi\} \).

The only non-zero Christoffel symbols are
\[ \Gamma^w_{tt} = L e^{2wL}, \quad \Gamma^w_{tw} = L, \quad \Gamma^w_{rr} = -L e^{2Ht} e^{2wL}, \quad \Gamma^w_{wt} = 0, \quad \Gamma^w_{wr} = 0, \quad \Gamma^w_{rr} = H e^{2Ht} \]  
(43)

and those formed from equation (43) by exchange of lower indices. The equations for a geodesic are thus
\[ \frac{d^2 w}{ds^2} = -e^{2wL} \left(\frac{dr}{ds}\right)^2 + e^{2Ht} e^{2wL} \left(\frac{dr}{ds}\right)^2 \]  
(44a)
\[ \frac{d^2 t}{ds^2} = -2L \frac{dr}{ds} \frac{dw}{ds} - H e^{2Ht} \left(\frac{dr}{ds}\right)^2 \]  
(44b)
\[ \frac{d^2 r}{ds^2} = -2H \frac{dr}{ds} \frac{dw}{ds} - 2L \frac{dr}{ds} \frac{dw}{ds}, \]  
(44c)

which keep constant the squared length of the tangent vector,
\[ \ell^2 = -e^{2wL} \left(\frac{dr}{ds}\right)^2 + e^{2Ht} e^{2wL} \left(\frac{dr}{ds}\right)^2 + \left(\frac{dw}{ds}\right)^2. \]  
(45)

We start at the point \( w = t = r = 0 \). The first step is to start in the \( w \) direction and go length \( \bar{w} \). Since \( g_{ww} = 1 \), and \( \Gamma^w_{ww} = \Gamma^w_{ww} = 0 \) for all \( \alpha \), this is trivial, and we arrive at \( w = \bar{w} \) with \( t = r = 0 \). In this step we parallel transport vectors pointing in the \( t \) and \( r \) directions. But since \( \Gamma^w_{\beta w} = 0 \) unless \( \alpha = \beta \), the directions of these vectors do not change.

The second step is to travel along a geodesic initially directed in the \( t \) direction. Since \( \Gamma^w_{tt} = \Gamma^w_{tw} = \Gamma^w_{ww} = 0 \), we will always have \( r = 0 \) in this step. The squared length of the tangent vector will be the constant \( \ell^2 = -\bar{t}^2 \). Using equation (45) in equation (44a), we find
\[ \frac{d^2 w}{ds^2} + L \left(\frac{dw}{ds}\right)^2 + L\bar{t}^2 = 0 \]  
(46)

with initial conditions
\[ w(0) = \bar{w}, \quad \frac{dw}{ds}\bigg|_0 = 0. \]  
(47)

The solution is
\[ w(s) = \bar{w} + \frac{1}{L} \ln(\cos s\bar{t}L). \]  
(48)

We can put this form for \( w \) into equation (45) to get
\[ \frac{dr}{ds} = \bar{t} e^{-s\bar{t}L} \sec^2 s\bar{t}L. \]  
(49)

With initial condition \( t(0) = 0 \), we find
\[ t(s) = \frac{1}{L} e^{-s\bar{t}L} \tan s\bar{t}L. \]  
(50)
Going distance $s = 1$ on the geodesic we can find the position after the second step,

$$w_2 = \bar{w} + \frac{1}{L} \ln(\cosh \bar{t} L)$$  \hspace{1cm} (51a)

$$t_2 = \frac{1}{L} \tan \bar{t} L e^{-\bar{w} L}$$  \hspace{1cm} (51b)

and $r = 0$.

In this step we parallel transport a vector pointing in the $r$ direction. But since $\Gamma^\rho_{\mu\nu} = 0$ unless $\alpha = r$, the directions of this vector does not change.

The third step is to travel on a geodesic initially in the $r$ direction with some fixed $\theta$ and $\phi$. The squared length of the tangent vector will now be $\ell^2 = \bar{r}^2$. Using equation (45) in equation (44), we find

$$\frac{d^2 w}{ds^2} + L \left( \frac{dw}{ds} \right)^2 - L \bar{r}^2 = 0$$  \hspace{1cm} (52)

with initial conditions

$$w(0) = w_2, \quad \frac{dw}{ds} \bigg|_0 = 0.$$  \hspace{1cm} (53)

The solution is

$$w(s) = w_2 + \frac{1}{L} \ln(\cosh s \bar{r} L).$$  \hspace{1cm} (54)

Putting this form for $w$, together with equation (45), into equation (44), we get

$$\frac{d^2 t}{ds^2} + H \left( \frac{dt}{ds} \right)^2 + 2\bar{r} \tanh s \bar{r} L \frac{dt}{ds} + H \bar{r}^2 e^{-2w_2 L} \text{sech}^4 s \bar{r} L = 0$$  \hspace{1cm} (55)

with initial conditions

$$t(0) = t_2, \quad \frac{dr}{ds} \bigg|_0 = 0.$$  \hspace{1cm} (56)

The solution is

$$t(s) = t_2 + \frac{1}{H} \ln(\cos u)$$  \hspace{1cm} (57)

with

$$u = e^{-w_2 L} \frac{H}{L} \tanh \bar{r}s L.$$  \hspace{1cm} (58)

Putting equations (54), (57) into equation (45) gives

$$\frac{dr}{ds} = \bar{r} e^{-Ht_2 - Lw_2} \text{sech}^2 \bar{r}s L \sec^2 u.$$  \hspace{1cm} (59)

With initial condition $r(0) = 0$, we find

$$r(s) = \frac{1}{H} e^{-Ht_2} \tan u.$$  \hspace{1cm} (60)

Finally, we put $s = 1$ in equations (54), (57), (60) and use equation (49) to find the location of a point with given Fermi coordinates,

$$w = \bar{w} + \frac{1}{L} \ln(\cosh \bar{t} L) + \frac{1}{L} \ln(\cosh \bar{r} L)$$  \hspace{1cm} (61a)

$$t = \frac{1}{L} e^{-\bar{w} L} \tan \bar{t} L + \frac{1}{H} \ln(\cos u)$$  \hspace{1cm} (61b)

$$r = \frac{1}{H} e^{-u} \tan u.$$  \hspace{1cm} (61c)
with
\[ u = \frac{H}{L} e^{-\tilde{u}L} \sec \tilde{t}L \tanh \tilde{r}L \] (62a)
\[ v = \frac{H}{L} e^{-\tilde{u}L} \tan \tilde{t}L. \] (62b)

The non-vanishing components of the metric are then
\[ g_{\tilde{r}\tilde{r}} = 1 \] (63a)
\[ g_{\tilde{w}\tilde{w}} = \frac{1}{4} \cosh^2 \tilde{r}L (3 + \cos 2\tilde{t}L - 2 \cos 2\tilde{u} \sin^2 \tilde{t}L) \] (63b)
\[ g_{\tilde{t}\tilde{t}} = -\frac{1}{2} \left( \cos 2\tilde{t}L + \cos 2\tilde{u} \right) \cosh^2 \tilde{r}L \sec^2 \tilde{t}L \] (63c)
\[ g_{\tilde{t}\tilde{w}} = -2 \cosh^2 \tilde{r}L \sin^2 \tilde{t}L \tan \tilde{t}L. \] (63d)
\[ g_{\tilde{u}\tilde{u}} = e^{2\tilde{L}w+2\tilde{H}t} r^2 = \frac{1}{H^2} e^{2\tilde{L}w} \cos^2 \tilde{r}L \cosh^2 \tilde{r}L \sin^2 \tilde{u} \] (63e)
\[ g_{\phi\phi} = g_{\theta\theta} \sin^2 \theta. \] (63f)

We will now check against our previous formulas. All curvature components are proportional to either \( H^2 \) or \( L^2 \), so we will include only up to this order in all calculations below. Just as in the previous subsection, this enables us to ignore the distinction between the components of the Riemann tensor in the original basis and in the orthonormal basis.

We start by making the first-order expansions
\[ u = H\tilde{r} \] (64a)
\[ v = H\tilde{t} \] (64b)
which are sufficient to write
\[ g_{\tilde{w}\tilde{w}} = 1 + \tilde{r}^2 L^2 - \tilde{t}^2 L^2 \] (65a)
\[ g_{\tilde{t}\tilde{t}} = -(1 - \tilde{r}^2 H^2 + \tilde{r}^2 L^2) \] (65b)
\[ g_{\tilde{w}\tilde{t}} = g_{\tilde{t}\tilde{w}} = 0 \] (65c)
\[ g_{\phi\phi} = g_{\theta\theta} \sin^2 \theta. \] (65d)

Because of the presence of \( 1/H^2 \) in \( g_{\theta\theta} \), we need a further expansion of \( u \). The easiest plan is to first approximate \( \sin^2 \tilde{u} = \tilde{u}^2 - \tilde{u}^4/3 \). For the \( \tilde{u}^4 \) term, equation (64a) is sufficient. Using equation (62a), the \( \tilde{u}^2 \) contribution to \( g_{\theta\theta} \) is \( (1/L^2) \sinh^2 \tilde{r}L = \tilde{r}(1 + \tilde{r}^2 L^2/3) \), so \( g_{\theta\theta} = \tilde{r}^2 (1 + \tilde{r}^2 (L^2 - H^2)/3) \). (66)

Since there is only one index, there is no \( k = 1 \) term in equation (28). We never move in \( \theta \) or \( \phi \), so the only possible values for \( \delta \) and \( \gamma \) in equation (28) are \( t \) and \( r \). The Riemann tensor components with those indices are
\[ R_{\tilde{w}\tilde{w}\tilde{w}} = -L^2 e^{2\tilde{L}w} \] (67a)
\[ R_{\tilde{w}\tilde{w}t\tilde{t}} = L^2 e^{2\tilde{L}w+2\tilde{H}t} \] (67b)
\[ R_{\tilde{t}\tilde{t}\tilde{w}\tilde{w}} = e^{2\tilde{L}w+2\tilde{H}t} (\tilde{H}^2 - L^2 e^{2\tilde{L}w}) \] (67c)
\[ R_{\phi\phi\theta\theta} = e^{2\tilde{L}w+2\tilde{H}t} (\tilde{H}^2 - L^2 e^{2\tilde{L}w}) r^2 \] (67d)
\[ R_{rr\theta} = e^{2Lw+4Ht}(L^2E^{2Lw} - H^2)r^2 \] (67e)
\[ R_{\theta\phi r} = R_{dr\theta r} \sin^2 \theta \] (67f)
\[ R_{\phi rr} = R_{dr\phi r} \sin^2 \theta \] (67g)

and those that can be formed from equation (62a) by permuting indices.

First consider \( g_{ab} \) with \( a \neq b \). Rotation and reflection symmetry prevent any such component with \( a \) or \( b \) an angular coordinate. So \( a \) and \( b \) must be in different steps. Since there are no Riemann components with more than two different indices, we must have \( \gamma = \beta \) and \( \delta = \alpha \) in equation (28). But then \( X(j)^\gamma = 0 \) unless \( j = m \), which implies \( k = m \), and thus \( X(k)^3 = 0 \). Thus there are no off-diagonal components in the metric.

Now we compute \( F_{rr} \). This makes \( m = 3 \), but the only component of \( X^3(j) \) is the \( r \) component, for which the Riemann tensor vanishes, so \( g_{rr} = 1 \) in agreement with equation (63a).

Similarly in \( F_{tt} \) there are no \( k = 2 \) terms, but there is the \( k = 3 \) contribution just as in equation (37),
\[ F_{tt} = \frac{1}{2}R_{rrtt} = (H^2 - L^2)r^2 \] (68)
In \( F_{ww} \) the second and third steps contribute,
\[ F_{ww} = \frac{1}{2}[R_{wttw} + R_{wrrw}r^2] = \frac{1}{2}L^2(\bar{r}^2 - \bar{t}^2) \] (70)
so
\[ g_{ww} = 1 + L^2(\bar{r}^2 - \bar{t}^2) \] (71)
in agreement with equation (65b).

Finally, for \( F_{\theta\theta} \) we have only the third step, analogous to equation (40),
\[ F_{\theta\theta} = \frac{1}{2}R_{\theta\rr\theta} = \frac{1}{5}(L^2 - H^2)r^4 \] (72)
so
\[ g_{\theta\theta} = r^2 + (L^2 - H^2)r^4/3 \] (73)
in agreement with equation (66). Again the case of \( g_{\phi\phi} \) is similar. This completes the check of the metric computation.

7. Conclusion

It is often convenient in general relativity calculations to choose coordinates that are adapted to the problem to be solved. For example, to calculate what one observer sees, one can use Fermi coordinates adapted to that observer’s path. For other purposes, other coordinates may be useful. For example, for a three-dimensional congruence of paths (as in [3]), it may be useful to use first use three coordinates to specify the path and then one to specify the position on the path. To describe the experiences of a one-dimensional congruence of observers, one might want to use the first coordinate to specify the choice of observer, the second to specify the position on the observer’s world line, and the remaining two to describe the space near that position.
This motivates us to generalize the usual Fermi normal coordinates in the case where the generating curve is a geodesic to allow for any number of steps and for a subspace of any dimension at each step. In that context, we have derived the connection (exactly) and the metric (to first order in the curvature) as integrals over the Riemann tensor. This approach differs from techniques involving Taylor series around a base point, in that derivatives of the curvature do not appear. Thus it is applicable to situations where Taylor series do not converge, and in particular where the curvature is not smooth.

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