Computing K3 and CY n-fold Metrics

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Abstract

The derivative expansion in the context of IIB string scattering compactified on non-trivial K3 and other Calabi-Yau manifolds is formulated. The scattering data in terms of automorphic functions can be inverted to find the these metrics. The solutions are parameterized by the moduli information, and the metrics may be found to any desired accuracy in derivatives. Metric information to low orders in derivatives allows for a counting of curves inside the manifold; in addition, the coefficients of these exponential terms via D-brane wrappings are polynomials that may admit an invariant interpretation in cohomology. An interesting case pertaining to M-theory compactifications is the collection of seven-dimensional $G_2$ manifolds; they can also be obtained when the moduli space degenerates into cases, such as a toroidal one or other limit in which modular functions on the space are known.\footnote{This work was written two years ago; the recipe without the explicit form of the scattering and metrics is given.}
1 Introduction

Compact Ricci-flat manifolds were discovered in 1957, but to date there is moderate progress in computing their metric form away from the large volume region or in some degenerate limit. Knowing the explicit metric would allow for a variety of computations: holomorphic curve counting, gravitational instanton contributions to string backgrounds, number theory results, and more. In this work the derivative expansion applied to gravitational theories, together with S-duality, generate a recipe for computing the Kähler potential of the manifold to an arbitrary order in derivatives; formally the complete classical gravitational scattering in the K3 background generates the K3 metric. (Recent work in [1] generated numerically partial metrics on certain K3 spaces.)

First, the method presented to computing the metrics is described; model dependencies vary the procedure slightly. The bosonic truncation of the covariantized gravitational effective action has the form in Einstein frame,

\[
S = \frac{1}{G^2_N} \int d^4x \sqrt{g} \left[ R + \alpha_1 \ln \Box R^2 + \alpha_2 R^3 + \ldots \right],
\]

and whose variation generates the on-shell S-matrix evaluated in some background. The coefficients parameterizing the effective action, in general, are computable through perturbative and non-perturbative methods. In theories with non-perturbative duality structure such as IIB superstring theory with \( \tau \leftrightarrow -\frac{1}{\tau} \), the coefficients of the curvatures are easier to obtain. The coefficients are obtainable via pseudo free-field x-space diagrams in the derivative expansion. IIB compactified on K3, and other certain Calabi-Yau manifolds, allow the non-perturbative form to be deduced via perturbative methods. The non-perturbative terms correspond to various D-branes (i.e. D-instantons, D1 and D3 branes) wrapping the internal cycles to be computed; they are connected to the perturbative terms through S-duality, similar to \( \mathcal{N} = 4 \) supersymmetric gauge theory [11] as the gauge theory instantons are derivable from the perturbative terms.

The idea in obtaining the non-trivial metrics of K3, or other Calabi-Yau n-folds, involves first substituting \( g_{\bar{i}j} = \partial_i \partial_j \phi \) within the effective action. The vanishing of the action density

\[
S[\phi] = 0 \quad S = \int S[\phi]
\]

\(^2\)The work in the context of various theories is described in [2]-[17].
should generate the Kähler potential; the kinetic term $\int R$ is a total derivative, and the total action in the quantum regime should vanish as there is a minimum to the action on the Kähler manifold. Whether there is an integration by parts or not, there should be a minimum. This variation is non-trivial due to both the presence of the logarithmic terms in the effective actions and the effective vertices which depend on the modular forms commanded by S- and T-duality. This routine in principle generates the metric to an arbitrary order of accuracy, in addition to allowing the computation of all of the gravitational instantons wrapping the sub-cycles within the manifold (and also the number and type of cycles in the manifold).

2 Nonperturbative coefficients on CY $n$-folds

First, the IIB superstring derivative expansion in a flat-space background, as formulated in [17], is reviewed. The generating function of the scattering amplitude at two to four-point order is found from,

$$S = \frac{1}{\alpha^4} \int d^{10}x \sqrt{g} \left[ R + R^{\Box} R + \alpha^3 \sum_{k=1}^{\infty} f_k(\tau, \bar{\tau})(\alpha \Box)^k R^4 + \text{non-analytic} \right] , \quad (2.1)$$

with the derivatives acting generally within the components of the contracted $R^4$ terms (the tensor of the $R^4$ term is determined by maximal supersymmetry). The non-analytic terms have the general expansion of logarithms mingling within the curvature terms; they may be computed via unitarity - the imaginary parts of amplitudes should be found via Cutkowsky rules. The prefactors $f_k$ are modular functions and under S-duality transform under $\tau \rightarrow (a\tau + b)/(c\tau + d)$ with $ac - bd = -1$. The remaining terms in the action are found from a supersymmetric extension of the action (for example, with a lightcone maximal superspace of [18]).

The modular functions at a given order $k$ are spanned by the ring of functions,

$$\prod E_{s_i}^{(q_i)} , \quad \sum s_i = s, \quad \sum q_i = 0 , \quad (2.2)$$

and

$$E_{s}^{(q,-q)} = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^s}{(m + n\tau)^{s+q}(m + n\bar{\tau})^{s-q}} , \quad (2.3)$$
with the sum ranging over coprime pairs of integers and $s = 3/2 + k/2$. The ring element coefficients (including possible non-holomorphic $q_i \neq 0$ terms) are determined by sewing relations, after including the supersymmetric terms.

The formulation previously discussed is valid when the vacuum of the IIB string is described by the coset SL(2,R)/U(1), the keyhole region in the complex region. When the vacuum moduli space changes, a different set of automorphic functions enter into the description; for example, S-duality is expected for the IIB string compactified on a multi-torus, and the moduli space of the vacuum is described by the semi-direct product of the SL(2,R) action and the T-duality group of the torus $SO(10 - d, 10 - d)/SO(10 - d)/SO(10 - d)$, a member of the potentially affine duality group $E_{11-d,11-d}$.

Next we examine the general graviton scattering on the K3 manifolds. These manifolds have 57 moduli, with a rather simple moduli space, $SO(3,19)/SO(19)/SO(3)$ [231-3-171 =57]. The semi-direct product of S and T-duality forms the moduli (Teichmuller) space $SO(5,21)/SO(5)/SO(21)$; because the K3s moduli space is toroidal it is possible to formulate the automorphic functions taking values on it; we may also derive the string scattering and derive the explicit set of metrics.

Consider the scattering of IIB on $T^6 \times K3$, with the base space being K3. The number of supersymmetries is naively 32 components as found on the K3 base, with one covariantly conserved K3 spinor times the eight spinors arising from the $T^6$ torii; however, the theory has only $\mathcal{N} = 4$ in $d = 4$ with the grav multiplet arising from the K3 and the toroidal space generating a tower of $\mathcal{N} = 4$ multiplets. S-duality is expected to be present in theories containing $\mathcal{N} \geq 16$ supersymmetries. We will neglect all of the massless and massive modes entering from the higher dimensional space, i.e. the $T^6$, by treating the theory with only the K3 moduli space and neglecting these Kaluza-Klein modes in the sewing procedure; this allows us to examine the string on the reduced four-dimensional space-time. Furthermore, the massive string modes are to be decouple, leaving only the massless theory containing the couplings dictated by the moduli of the K3 background; the latter is the Lagrangian description of the theory that we examine in the following. Dropping the massive modes allows the integrals to be computed without detailed knowledge of the massive mode couplings; in principle, however, M-theory graviton scattering can be used to obtain the coefficients of the S-duality invariant graviton scattering without the massive mode technicalities, as described later.

The gravitational string scattering on the K3 part of the K3$\times T^6$ is described by
\[ S_{grav}^{n-pt} = \sum_{k=1}^{\infty} \int d^4x \sqrt{g} \, O_k \beta_k \]  

similar to the flat ten-dimensional background. The most general set of covariantized gravitational operators consistent with the symmetries of the theory (supersymmetry, presence of massless modes) is

\[ R, \quad R^1 R, \quad R^2, \quad \Box^n R^k \quad \text{(mixed tensors)} \]  

\[ \ln^{n_1}(\Box) \ldots \ln^{n_m}(\Box) \Box^n R^k \]  

with the latter terms coming from integrating out massless modes. The general derivative term is explained in detail in [2]-[17], and the derivatives are in general placed within the contractions of the curvature terms. The duality invariant moduli dependent coefficients of the gravitational terms are determined from the ring of functions (within the S-duality compliant scattering),

\[ \prod E_{K3,s_i}^{(q_i,-q_i)}, \quad \sum s_i = s = \frac{3}{2} + \frac{n}{2} \frac{(k - 4)}{2}, \quad \sum q_i = 0, \]  

and

\[ E_{K3,s}^{(g,-q)} = \sum_{(n^a,m^a)} \frac{1}{(n^a G_{ab} m^b)^{s+q}(n^a G_{ab} m^b)^{s-q}}, \]  

with the sum over the coprime invariants of the toroidal coset metric (the \( E_1 \) function is to be replaced with its regularized version). The moduli of the space in this description have absorbed a factor of the string scale so as to be dimensionless (e.g. Wilson lines in toroidal compactifications in the the lattice sum at genus one.) The on-shell supersymmetric completion is required to complete the terms in the generating function of the full scattering.

Next, the coefficients of the terms in the ring of the automorphic functions are described iteratively in the derivative expansion; subsequently, the metric on K3 is determined.
3 Iterative procedure to coefficients

In the iterative procedure to determining the relative coefficients of the ring of automorphic coefficients, a series of simple x-space integrals are required; the main complication is the tensor algebra and supersymmetric completion. Once done to a given order in derivatives, we can exploit the modular properties of the scattering to finding some very non-trivial structure of the Calabi-Yau metrics. In this section we only require the gravitational sector, but we need to eliminate the massive modes from the scattering in order to obtain the coefficients, without involving the complications of the explicit massive string to accomplish the same, but the matter couplings derived on the K3 are not known in this work.

To a given order in the genus expansion, via the coupling \( \tau^{3/2-2g} \) at genus \( g \), the massive modes begin to contribute at order \( \alpha^2 \) greater than the massless modes (at the four-point order). This mixes their roles in the derivative expansion, as opposed to in the coupling expansion. This is obviously true at loops 0 to 2 and due to unitarity true also at higher orders (an analysis is performed in [15] and [17]). In order for us to obtain the coefficients of the modular functions in the duality invariant scattering without these massive modes, these contributions are to be eliminated through an expansion of the K3 massive forms.

The expansion at derivative order \( 2(m+4) \) begins with a coefficient at \( \alpha^m \). A modular constrained form of string coupling expansion at this order in derivatives has the form in the Einstein frame [17],

\[
a_{m,0}\tau_2^{3/2+m/2} + a_{m,1}\tau_2^{-1/2+m/2} + \ldots + \tau_2^{3/2-2g_{\text{max}}+m/2},
\]

and maximum genus contribution, \( 3/2 + m/2 - 2g_{\text{max}} = -1/2 - m/2, \) dictated by the expansion of the modular functions. Furthermore, there are a series of instantonic terms coming from the wrapped membranes within the internal space. The truncation of the massive modes has been examined in [17] and corresponds to throwing away half the terms in (3.1). There are an even number of terms ranging with indices of \( 3/2 + m/2 \) to \(-1/2 - m/2\), a total of \( 1 + \lceil m/2 \rceil \) terms. The truncation corresponds to dishing the first half terms.

In the case of the K3 metric the expansion in \( \tau_2 \) is the same as in (3.1), but the coefficients \( a_{m,g_{\text{max}}} \) are dependent on the moduli non-trivially. Due to the multiplicative nature of the Eisenstein functions on K3, this truncation involves first multiplying the various E-functions contributing at a given derivative order \( 2m \).
The perturbative terms have integral or half-integral powers in $\tau_2$, and the non-perturbative terms have exponential factors; the latter corresponds to wrappings of D-branes on the 1 and 2-cycles of the K3 space. The contribution of these wrappings are computable via the expansions of the modular functions, similar to instantons in $\mathcal{N} = 4$ gauge theory \[11\].

Next, the sewing of the derivative terms (with the determined modular coefficients $a_{m,g}$) is performed to find the relative coefficients of the modular functions generating at each order in derivatives. This follows via some straightforward integrals, however, with some non-trivial tensor components. The implementation is very well suited to be both analytically computed and numerically computed.

The n-point gravitational vertices derived from the effective action are,

\[
v_g^{\mu_\nu} = \prod \frac{\delta}{\delta g_{\mu_\nu}} S[g + \hat{g}].
\] (3.2)

The fermionic and remaining bosonic vertices are similarly derived, or found by supersymmetrization. The recursive implementation at the four-point order involves the sewing relation and the complete set of operators in the derivative expansion. This sewing procedure is identical to determining higher loops from lower loops in the usual Feynman diagrams; this is clear by examining the low-energy theory of the usual loop expansion via a momentum expansion of the graphs. The sewing generates, in momentum space,

\[
\sum_{L=1}^{\infty} \int d^dq_j v_g^{2L+1} \prod_{i=1}^{L+1} \Delta_{\mu_\nu;i;\hat{\mu}_j;\hat{\nu}_j} \bar{v}_g^{2L+1} + \text{perms} = \frac{1}{2} v_g^4.
\] (3.3)

The complete set of derivatives has been included in the vertices; there are an infinite number of terms in $\nu$ that have to be expanded as in $v_4 = \sum v_{4,j}$. The graviton propagator is $\Delta_{\mu_\nu;\alpha,\beta}$, and the integrals are easiest to evaluate by Fourier transforming them to x-space followed by possibly transforming back to x-space. The diagram in Figure 1 illustrates the sewing relation. This iterative procedure, after including the fermions, generates all of the required coefficients to any given order in derivatives, in the massles mode approximation. (In order to obtain the full scattering the massive modes would have to be included; this is possible by decompactification, T-dualizing,
and comparing the result with the $d = 11$ supergravity coefficients. In principle the contributions of the virtual massive modes to the graviton multiplet scattering may be obtained.) This completes the method of deriving all of the coefficients of the gravitational scattering on the Calabi-Yau manifold.

The integrals required in the sewing procedure are easiest to perform in $x$-space, and the string regulator must be used in order to preserve duality; this is analyzed to multi-loops in [17]. The generating function in the classical limit does not require the supersymmetric completion in the internal legs. However, as the Eisenstein functions receive genus contributions from internal supersymmetric matter, the massless supersymmetric completion must be inserted in the loop; an on-shell extension of maximal supergravity is developed in [18]. The inclusion of the remaining modes, $\mathcal{N} = 4$ supergravity and $\mathcal{N} = 4$ supersymmetric gauge theory complicates the quantum description.
4 Determine of the Kähler potential

The K3, and any Calabi-Yau n-fold, metrics are described locally via the Kähler potential \( g_{ij} = \partial_i \partial_j \phi \) (or \( g_{ij} = \partial_i \partial_j \ln \phi \)) with a \( \mu_k \) moduli dependent scalar \( \phi(\mu_k) \). We may substitute this scalar in the full generating function of the graviton scattering amplitude in place of the four-dimensional metric. The general form of the graviton scattering function, classical and quantum, to all point orders is

\[
S_{\text{grav}}(\phi) = \int d^4 x S(\phi),
\]

with

\[
S(\phi) = \sum_{k=1}^{\infty} \mathcal{O}_k(\phi) \beta_k .
\]

The coefficients \( \beta_k \) are determined to any order via the method given in the previous sections. The operators \( \mathcal{O}_k \) can be complicated due to the logarithms and the derivatives; however, only the Kähler scalar is involved in the expansion.

With this description we take the condition,

\[
S(\phi) = 0,
\]

which corresponds to minimizing the classical (or quantum action); this is without using partial differentiation to enforce the vanishing of the action. The variation is similar to minimizing the scalar potential to find the true vacuum as in the Coleman-Weinberg mechanism; this generates an iterative means to obtaining the potential in terms of the modular functions \( E_8^{(q,-q)} \). The variation seems trivial, but only for the true Kähler potential, rather than some general \( \phi \), will the condition in (4.3) hold. Due to the specific set of automorphic functions pertaining to the K3, and given one metric (and the moduli spanning this metric) for any set of functions, the solution should be unique.

The classical generating function of \( \phi \), including the instantons, depends on the values of the moduli of the K3 due to the background field expansion of the \( R \) term about the metric; the terms coming from the multi-derivative curvatures generate an equation for \( \phi(\mu_k) \). The quantum terms involving \( \phi(\mu_k) \) generate a quantum extension for the metrics.
In general the solution will contain exponentials due to solving (4.3) in the presence of logarithms (in the quantum case) and also due to massless singularities (in the classical case). An example series in terms of the Kähler scalar is, and for simplicity we ignore the determinant $\sqrt{g}$,

$$\frac{1}{2} \phi \Box \phi + \frac{1}{2} g_1 \ln(\phi \Box \phi) \phi + \frac{1}{2} g_2 (\phi \Box \phi)^2 + \ldots ,$$

with $g(\mu_j)$ a sample modular function and $\mu_j$ the moduli. The $\alpha'$ in the expansion sets the dimensional scale in the derivative expansion; its factor scales the moduli to be dimensionless (the Eisenstein functions are dimensionless). Setting this function to be zero, and after a field redefinition $\phi = e^\sigma$ makes the equation look like a Liouville equation, which is common in the case of Monge-Ampère equations in the construction of hyperkähler metrics.

To higher orders in curvatures there are $\phi^n$ terms with $n$ an arbitrarily large integer. In the case of the classical metric only there are no logarithms, but rather an infinite number of these $\phi^n$ terms with many derivatives acting on them. The full solution to the minimization of the action generates the Kähler metric, without integrating by parts to force the vanishing of the action.

Because unitarity mixes graphs at different orders, the full loop expansion is required to obtain the quantum metric; there is no truncation of the expansion that generates the full metric in this approach. As long as the found potential $\phi$ generates a non-singular metric the solution would have to correspond to the general metric on K3 simply by symmetries (compact hyperkähler) and smoothness; the latter is a consistency check on S-duality and the computed expansion.

5 Generalization to other Calabi-Yau manifolds

The same procedure utilized on the K3 manifolds may be generalized to a variety of other compactifications. First, non-compact four-dimensional manifolds and their respective metrics may be examined with both the string scattering and metric information. This set includes a variety of 4-dimensional hyperkähler metrics such as the A,D, and E series.

The Calabi-Yau manifolds of complex dimension 3, 4, and 5 may be examined through the compactifications $M_{10} \to T^4 \times M_{3-CY}$, $M_{10} \to T^2 \times M_{4-CY}$, and $M_{10} \to M_{5-CY}$. These scenarios have $N = 16$ in $d = 6$, $d = 8$, and $d = 10$. A
similar procedure as described in the previous may be used to compute the metrics on all of the compact Calabi-Yau manifolds having a manageable moduli space, such as the toroidal ones of projective degenerations; this requirement comes from the requirement of explicit modular forms constructable on the manifolds’ moduli spaces. The set of CYs also includes the Gepner models. (Non-compact higher dimensional Ricci-flat manifolds are also in principle available, with the moduli spaces being the necessary ingredient.)

Furthermore, the seven dimensional $G_2$ Joyce manifolds may be analyzed and their metrics computed for classes in which the automorphic functions can be computed on the moduli spaces; the latter being useful for M-theory compactifications. The models considered are IIB on $T^3 \times M_{G_2}$.

6  Curves

In this section we examine the exponential terms arising from the expansion of the automorphic forms and the coefficients multiplying them. These terms correspond to wrapping of membranes on the internal cycles of the K3 (and CY folds), and explicit numbers may be found from the graviton scattering. Both the exponential factors and the rational functions multiplying them are of mathematical and physical interest. The factors are cohomological, potentially corresponding to invariants.

The modular functions have the instanton expansion,

$$g_k^{\text{inst}} = b_k(G) e^{W(G)},$$

for a general configuration of the moduli. The instanton series are computed via the coefficients of the modular functions and is formed physically via the wrapping of membranes on the compact cycles. The coefficients $b_k(G)$ of the exponential terms model the counting of these wrapped modes in the string coupling expansion, and include the moduli of the manifold.

7  Summary

The derivative expansion of IIB superstring theory on Calabi-Yau spaces is examined with the intent of finding detailed metric information of the background geometry. A
recipe for computing the metrics on K3 and higher dimensional Calabi-Yau manifolds, including Joyce manifolds, is presented, given information of S-duality and string scattering. To any amount of accuracy these Ricci-flat metrics may be computed via IIB string theory of M theory amplitudes. The formalism does not have complicated integrals, but rather somewhat complicated tensorial algebra at higher orders; as a result, the implementation of the recipe is suitable for a computer implementation.

Furthermore, the metric information found from low orders of derivatives in the target space-time theory leads to a natural means of computing numbers of holomorphic curves of varying orders and the moduli dependent coefficients. In principle this explicit curve and metric information is computable with the use of the $T \times S$ modular functions as coefficients in the gravitational scattering.
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