On the relation of the monomial group with other algebraic structures

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Summary: It is shown in what way monomial group connects Abelian group $\mathbb{Z}^n$ and total linear group $GL(n)$. It is shown that any subgroup of Abelian group $\mathbb{Z}^n$ induces subgroup of monomial group $S_2 \wr S_n$, which in its turn induces corresponding subgroup of $GL_n(\mathbb{R})$.

1 Monomial group as a generator of total linear group

In combinatorial constructions of this section we use language of set and map theory therefore, succeeding its notions and notations, let $I = \{1, \ldots, n\}$; $s : I \to I$ — is bijection, $s' s : I \to I : s' s(i) = s'(s(i))$ — is the composition of bijections. Then set of all $n!$ bijections $S = \{s\}$ with respect to product $S \times S \to S : (s', s) \to s' s$ put together permutation group (symmetric group) of the order $n$. At the same time, set of all even bijections put together alternating group $S^+$, which in general case is generated by 3–cycles, namely, $S^+_n := \langle \{(i, j, k)\}_{i} \rangle$ where $n \geq 3$ and $i \neq j \neq k$.

Let also $A = \{\pm 1, \ldots, \pm n\}$ and $p : I \to A$ — is such an injection that $|p| : I \to I$ — is bijection, moreover $p' p : I \to A : p' p(i) = \text{sgn} p(i) \cdot p'(\text{|p(i)|})$ — is the composition of these injections. Then set of all $2^n n!$ modulo bijective injections $P = \{p\}$ with respect to product $P \times P \to P : (p', p) \to p' p$ put together monomial group described by the wreath product $P_n := S_2 \wr S_n$ and intuitively imagined as group of arrowy permutations (arrowy group) of the order $n$. Next let arrowy permutation is called as arrowy transposition if it is a simple elementary permutation or inversion of one arrow, in other words, element of monomial group is transposition if it is an elementary permutation in $S_n$–component or a rearrangement in one from $S_2$–component of the wreath product. Then, like the simple permutations, parity of arrowy permutation $p$ is defined by parity of number of arrowy transpositions for passage from identity permutation to $p$, which is invariant relative to the choice of composition of transpositions for passage to $p$. In this connection, the set of all even arrowy permutations put together group $P^+$, which in general case is generated by
rearrangements with inversion, i.e. by generators \((j, -k) : j \rightarrow (-k), k \rightarrow j\) where \(j \neq k\), so we have generation \(P^+_{n} := \langle \{(j, -k)\}_{I} \rangle \).

However, along with the notion of parity of permutations (arrowy permutations) it should be considered the notion of parity of arrangements (arrowy arrangements), for which it is sufficient to accept that elementary (transpositional) arrangement onto the place \(I^*\) (where \(I^* \subset I\)) corresponds to either elementary rearrangement into the set \(I^*\) or one–element replacement from complementary to \(I^*\) set \(I \setminus I^*\) (the notion of elementary arrowy arrangement must be extended by addition of one–element inversion onto the place \(I^*\)). Then parity of arrangements \(s^* := s(I^*)\) and \(p^* := p(I^*)\) is defined by factored at \(Z_2\) numbers \(\sigma(s^*), \sigma(p^*)\) of elementary arrangements, that are required for passage to \(s^*\) and \(p^*\) respectively.

The notion of arrangement parity allows to form new combinatorial groups. In fact, let \(I_1 = \{1, \ldots, r\}\), \(I_2 = \{r + 1, \ldots, n\}\), \(s^1 := s(I_1)\), \(s^2 := s(I_2)\), then we can present \(s\) as a component permutation, consisting of arrangements \(s^1\), \(s^2\) and next define component parity given by number \(\sigma(s^1) + \sigma(s^2)\). Hence, set of all the even–component permutations are gathered in group \(S_{r,n}^+\), which in general case is generated by 3–cycles, acting into subsets \(I_1, I_2\) and by arbitrary 2–cycles, acting between them, namely,

\[ S_{r,n}^+ := \langle \{(i, j, k)\}_{I_1}, (l, m), \{(i, j, k)\}_{I_2} \rangle \]

where \(l \in I_1\) and \(m \in I_2\).

In its turn, set of all the even–component arrowy permutations put together group \(P_{r,n}^+\), which in general case is generated by generators \((i, -j)\), acting into subsets \(I_1, I_2\), and by arbitrary pair generator \(\pm(l, m)\), acting between them, where \(+ (l, m)\) — is a simple rearrangement and \(- (l, m)\) — is an inverse rearrangement, i.e. \(- (l, m) : l \rightarrow (- m), m \rightarrow (- l)\). Hence, we have generation:

\[ P_{r,n}^+ := \langle \{(i, -j)\}_{I_1}, \pm(l, m), \{(i, -j)\}_{I_2} \rangle \]

where \(l \in I_1\) and \(m \in I_2\).

Further, before to enter upon the matrix realization of monomial group we define the determinant of row subset of square matrix \(A = (a_{ij})_{i,j \in I}\) (i.e. of rectangular matrix \(A(I^*) := (a_{ij})_{i \in I^*, j \in I}\) as a sum of products of elements indexed by arrangements of columns on rows and multiplied at the sign of parity of this arrangement. In other words, we assume that

\[ \det A(I^*) := \sum_{s^*} \prod_{i \in I^*} (-1)^{\sigma(s^*)} \cdot a_{is^*(i)} \]

Let now \(P(I) := \{(p_{ij})_I\}\) — is an exhaustive set of such square matrices in which every column and every row have only one nonzero element, moreover if \(p_{ij} \neq 0\), then \(p_{ij} \in \{\pm 1\}\). Then we have linear presentation \(P(I)\) of arrowy group \(P_n\) given by formula \(p_{ij} := \text{sgn} p(i) \cdot 1\) for every \(j = |p(i)|\) and \(p_{ij} = 0\) for every \(j \neq |p(i)|\). Besides, matrix groups

\[ \{(p_{ij})_I \mid \det P(I) = +1\} \]
we obtain that statement is proved.

isomorphic to group $P_n$ of the order $n$. Let now $GL_n^+(2)$ where $\cup_{m} I_{j} = I$, $\cap_{m} I_{j} = \emptyset$, card $I_{j} = n_{j}$, $\sum_{m} n_{j} = n$, moreover subset $I_{j+1}$ is filled by sequential sampling from $I$ after filling $I_{j}$ and $I_{1} = \{1, \ldots, n_{1}\}$, then matrix group

$$\{(p_{i,j})_{I} \mid \det P(I^+) \cdot \det P(I \setminus I^+) = +1\}$$

gives linear presentation of arbitrary subgroup $P_{n_1, \ldots, n_m}^+$ of even–component arrowy permutations.

At last, we establish the connection of monomial group with total linear group. Let $RP_n$ — is algebra over $R$, which is generated as the linear span of group $P_n$ represented as a subset of matrix algebra $ML_n(R)$, i.e. $RP_n = R\langle P(I) \rangle$; $GP_n$ — is multiplicative group of algebra $RP_n$; $R^+_{\theta}$ — is multiplicative group of field $R$; Aut $RP_n := \text{Int} GP_n \simeq GP_n / R^+_{\theta}$ — is group of automorphisms of this algebra. Then we get underlying statement:

$$GP_n = GL_n(R)$$

In fact, since group $P_n$ is realized by set of transitional matrices $P(I)$ from which we always can choose $n^2$ linearly independent matrices, then it includes some basis of algebra $ML_n(R)$, hence the linear span of this group coincides with $ML_n(R)$, i.e. $RP_n = ML_n(R)$, and therefore $GP_n = GL_n(R)$. In consequence of underlying statement we have equality Aut $RP_n = PGL_n(R)$ where $PGL_n(R)$ is the total projective group.

Let now $GL_{ij} := \text{diag} [1, \ldots, GL(2)_{i,j}^{i,j}, \ldots, 1]_{n_{i,j}}$ where $i \neq j$, moreover upper indexes point to rows numbers and lower indexes point to columns numbers onto intersection of which is disposed group $GL(2)$. Then as additional source of procedure of generation of linear groups will be statement:

$$GL(n) = \langle \{GL_{i,j}\}_{I} \rangle = \langle \{GL_{i,i+1}\}_{I} \rangle$$

Really, since $P_n = \langle \{P_{i,j}^2\}_{I} \rangle = \langle \{P_{i,i+1}^2\}_{I} \rangle$ where $P_{i,j}^2$ is arrowy subgroup of the order $n$ presented by the arrowy permutations onto place $i,j$ and isomorphic to group $P_2$, then $GP_n = \langle \{GP_{i,j}^2\}_{I} \rangle$, but $GP_{i,j}^2 = GL_{i,j}$, hence the statement is proved.

Further let there be given linear presentations of two arrowy groups $P_{2}^+ = \langle \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \rangle$ and $P_{1,1}^+ = \langle \pm \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \rangle$, then after elementary calculations we obtain that

$$\text{Aut } RP_{2}^+ = \left\{ \left(\begin{array}{cc} \cos x & \sin x \\ -\sin x & \cos x \end{array}\right), x \in R \right\} = SO(2)$$

and

$$\text{Aut } RP_{1,1}^+ = \left\{ \pm \left(\begin{array}{cc} \cosh x & \sinh x \\ \sinh x & \cosh x \end{array}\right), x \in R \right\} = SO(1,1)$$
and denoting

\[ SO(2)_{i,j} := \text{diag}\left[1, \ldots, SO(2)_{i,j}^{(2)}, \ldots, 1\right]_n \]

and

\[ SO(1,1)_{i,j} := \text{diag}\left[1, \ldots, SO(1,1)_{i,j}^{(1,1)}, \ldots, 1\right]_n \]

obtain generation of special orthogonal group

\[ \text{Aut} R^+ = \langle \{SO(2)_{i,i+1}\}_I \rangle = SO(n) \]

and special pseudo-orthogonal group

\[ \text{Aut} R^+_{r,n-r} = \langle \{SO(2)_{i,i+1}\}_I, SO(1,1)_{r,r+1}, \{SO(2)_{i,i+1}\}_I \rangle = SO(r,n-r) \]

However, generalization of special orthogonal group is possible, including special case of pseudo-orthogonal group, namely,

\[ \text{Aut} R^+_{n_1, \ldots, n_m} = \text{SO}(I_I) := SO(n_1, \ldots, n_m) = \langle \{SO(2)_{i,i+1}\}_I, \{SO(1,1)_{n_i,n_{i+1}}\}_I \rangle \]

In the whole, it is clear that all possible subgroups of monomial group generate all possible linear groups (among them — unitary, symplectic, exeptional and so on), therefore the problem of classification and construction of linear groups coincides with the one for subgroups of monomial group.

## 2 Monomial group as group of automorphisms of \( Z \)-modulus \( Z^n \)

Let \( Z \) — is ring of integers; \( Z^n \) — is \( n \)-dimensional \( Z \)-modulus and \( e_I := (e_i)_I \) — is its basis, then \( \text{Aut} Z^n := \text{Int} \text{GL}_n(Z) \cong P_n \), since from the equality \( \text{GL}_n(Z)/\{\pm 1\} = P(I) \) it follows that \( P(I) \) — is group of inner automorphisms of total integer linear group \( \text{GL}_n(Z) \), i.e. of group of nonsingular integral–valued matrices. Further we will look for group of such automorphisms of modulus \( Z^n \), which correspond to topology of the quotient \( \{-e_I, o, e_I\}/\sim \), i.e. to partition of the set consisting of base elements, inverse base elements and the point of reference at equivalence classes. But, since the partition of this base set must be induced by factor group \( Z^n/H^n \), then it should be said on automorphisms of modulus \( Z^n \), corresponding to topology of factor group. For example, for the factor group \( Z/Z \) we have one class \([-e \sim e \sim o]\) and for the factor group \( Z/mZ \), where \( m \geq 2 \), we have partition on two classes \([-e \sim e], [o]\), so topological type of the generalized factor group \( Z/H \) may be presented by groups \( Z/Z \) and \( Z/2Z \). It is characteristic also that any factor group \( Z^n/H^n \) induces at least equivalence relation \(-e_i \sim e_i\) for every \( i \in I \).
Moreover, reversible maps:
\[-e_I \times e_I \rightarrow -e'_I \times e'_I : e'_I = e_{p(i)} := \text{sgn } p(i) \cdot e_{|p(i)|}\]

realize elementary reversible rearrangements of elements of set \([-e_I, e_I]\), coordinated with arrowy inversion, simple rearrangement and inverse rearrangement respectively. If we present now quotient \([-e_I, o, e_I] \sim 2n\)-link graph glued in equivalence points, then it will be clear that rebuilding of this factored graph may be not solved, namely, it is impossible to realize rearrangement of two links, the ends of which are glued with each other but not glued with the point of reference, where 2–link element of graph it should be considered as 2–dimensional, i.e. planar graph.

Thus, starting from topological arguments, we postulate the prohibition of reversible rearrangement of arbitrary pair elements of set \([-e_I, e_I]\) if they are equivalent in some class, different from the class of the point of reference. Hence, we have restriction of the group of automorphisms \(Z^n\) to some subgroup of monomial group. So if pairs \((-e_I, e_i)\) of base elements are grouped in \(m\) classes, none from which is not equivalent to the point of reference, then group of topological automorphisms \(Z^n\) corresponds to subgroup \(P_{n_1, \ldots, n_m}^+\). Really, only generators of even–component arrowy group (i.e. rearrangement with inversion) exclude permutations, corresponding to all the elementary reversible rearrangements in each set \([-e_{I_j}, e_{I_j}]\), but include simple and inverse rearrangements between each \(I_j\) and \(I_k\), that corresponds to choosen factorization of set \([-e_I, o, e_I]\).

We present also convenient method of construction of arbitrary factor group. Let \(|z| \mod 2 : Z \rightarrow Z/2Z : z' \equiv z \pmod{2} : |z| \mod 2 = r\), where \(r\) — is remainder from division \(z; 2\), taken without the sign of \(z\), then we have field \(Z_2 = \langle\{0, 1\}; \oplus, \cdot, 0, 1\rangle\), where \(z \oplus y = |x + y| \pmod{2}\). At the same time, if \(Z^n_2 := \prod_{i} Z_2\), then \(Z^n_2\) — is Abelian group with component–wise addition and moreover \(Z^n_2\) — is linear space over \(Z_2\) and \(e_I := (e_i)_I\), where \(e_i = (0_{|1|}, \ldots, 1_{(i)}, \ldots, 0_{|n|})\), — is its proper base. Let also \(H^n_2\) — is the notation of the generalized subgroup of group \(Z_2^n\) and \(H(I)\) — is the concrete subgroup, consisting of all possible even sum of elements of proper base, namely, \(H(I) = \langle\{e_j \oplus e_k\}\rangle\), which divides \(Z_2^n\) at two coset, so \(Z_2^n/H(I) \approx Z_2\). Let us assume \(H(I_j) := \prod_{m} H(I_j)\), then \(Z_2^n/H(I_j) \approx Z_2^n\). At the same time, since we have homomorphism \(z : H^n_2 \rightarrow H^n : (0 \rightarrow 2Z, 1 \rightarrow 2Z + 1)\), then any factor group \(Z_2^n/H_2^n\) is isomorphic to some factor group \(Z^n/H^n\), which in its turn induces the group of
topological automorphisms of \( Z^n \), in particular \( Aut Z^n/z \ast H(I_J) = P_{n_1,\ldots,n_m}^+ \) and \( Aut Z^n/z * Z_2^n = Aut Z^n/Z^n = P_n \).

Further, keeping in mind that any quotient \( \{ R^n/H^n \} \) may be identified with some compact space, topology of which is defined by factor group \( Z^n/H^n \), we obtain identification with \( n \)-sphere \( \{ R^n/z \ast H(I_J) \} \approx S^n \) or with more of generality with toroidal space \( \{ R^n/z \ast H(I_J) \} \approx S^{n_1} \times \cdots \times S^{n_m} \). At the same time, since \( RP_n = \text{End } R^n \), then \( \text{End } S^{n_1} \times \cdots \times S^{n_m} \approx RP_{n_1,\ldots,n_m}^+ \) and \( \text{Aut } S^{n_1} \times \cdots \times S^{n_m} \approx SO(n_1,\ldots,n_m) \), hence the rigidity of Euclidean spaces stipulated by certain type of functional of scalar product is the result of narrowing of algebra of endomorphisms and group of automorphisms of the space \( R^n \), arising from its factorization. In particular, geometry of proper Euclidean space is associated with factorization of \( n \)-dimensional arithmetical space into \( n \)-dimensional sphere. On the other hand, projective space \( RP^n \approx \{ R^n/Z^n \} \) does not possess topological rigidity and therefore \( End RP^n = RP_n \) and \( Aut RP^n = PGL_n(R) \).

Thus, monomial group by own subgroups provides conformity between subgroups of Abelian group \( Z^n \) and subgroups of total linear group \( GL_n(R) \), moreover any Abelian subgroup is associated with some topologically compact space, group of automorphisms of which coincides with corresponding linear subgroup.