A NEW APPROACH TO POINTWISE HEAT KERNEL UPPER BOUNDS ON DOUBLING METRIC MEASURE SPACES

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ABSTRACT. On doubling metric measure spaces endowed with a strongly local regular Dirichlet form, we show some characterisations of pointwise upper bounds of the heat kernel in terms of global scale-invariant inequalities that correspond respectively to the Nash inequality and to a Gagliardo-Nirenberg type inequality when the volume growth is polynomial. This yields a new proof and a generalisation of the well-known equivalence between classical heat kernel upper bounds and relative Faber-Krahn inequalities or localized Sobolev or Nash inequalities. We are able to treat more general pointwise estimates, where the heat kernel rate of decay is not necessarily governed by the volume growth. A crucial role is played by the finite propagation speed property for the associated wave equation, and our main result holds for an abstract semigroup of operators satisfying the Davies-Gaffney estimates.

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1. Introduction

1.1. Background and motivation. Let $M$ be a complete non-compact connected Riemannian manifold, $d$ is the geodesic distance and $\mu$ the Riemannian measure on $M$. Denote by $V(x, r) := \mu(B(x, r))$ the volume of the ball $B(x, r)$ of center $x \in M$ and radius $r > 0$ with respect to $d$.

Let $\Delta$ be the non-negative Laplace-Beltrami operator and $p_t$ be the heat kernel of $M$, that is by definition the smallest positive fundamental solution of the heat equation:

$$\frac{\partial u}{\partial t} + \Delta u = 0,$$

or the kernel of the heat semigroup $e^{-t\Delta}$, i.e.

$$e^{-t\Delta} f(x) = \int_M p_t(x, y) f(y) d\mu(y), \ f \in L^2(M, \mu), \ \mu - \text{a.e.} \ x \in M.$$ 

It is well-known that in this situation, contrary to more general ones, $p_t(x, y)$ is smooth in $t > 0$, $x, y \in M$ and everywhere positive (see for instance [36]).

In the Euclidean space $\mathbb{R}^n$, $p_t$ is given by the classical Gauss-Weierstrass kernel:

$$p_t(x, y) = \frac{1}{(4\pi t)^{n/2}} \exp \left( -\frac{|x - y|^2}{4t} \right), \ t > 0, \ x, y \in \mathbb{R}^n.$$ 

On general manifolds, where of course no such formula is available, the subject of upper estimates of the heat kernel has led to an intense activity in the last three decades (see for instance [17, 35, 36, 51, 52] for references and background).

One says that $M$ satisfies the volume doubling property if there exists $C > 0$ such that:

$$(VD) \quad V(x, 2r) \leq CV(x, r), \ \forall x \in M, \ r > 0.$$ 

For such manifolds, a typical upper estimate for $p_t(x, x)$ (so-called diagonal upper estimate) is

$$(DUE) \quad p_t(x, x) \leq \frac{C}{V(x, \sqrt{t})}, \forall t > 0, \ x \in M.$$ 

This estimate holds in particular on manifolds with non-negative Ricci curvature, see [17].

Under $(VD)$, $(DUE)$ is equivalent to the apparently stronger Gaussian upper estimate

$$(UE) \quad p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{Ct} \right), \forall t > 0, \ x, y \in M,$$ 

see [34] Theorem 1.1, also [21, Corollary 4.6] and the references therein.

A fundamental characterisation of $(UE)$ or $(DUE)$ was found by Grigor’yan. One says that $M$ admits the relative Faber-Krahn inequality if there exists $c > 0$ such that, for any ball $B(x, r)$ in $M$ and any open set $\Omega \subset B(x, r)$:

$$(FK) \quad \lambda_1(\Omega) \geq \frac{c}{r^2} \left( \frac{V(x, r)}{\mu(\Omega)} \right)^{\alpha},$$
where \( c \) and \( \alpha \) are some positive constants and \( \lambda_1(\Omega) \) is the smallest Dirichlet eigenvalue of \( \Delta \) in \( \Omega \). It was proved in [33] that \((FK)\) is equivalent to the upper bound \((DUE)\) together with \((VD)\). The proof that \((FK)\) implies \((DUE)\) is difficult. It goes through a mean value inequality for solutions of the heat equation which is proved via a non-trivial Moser type iteration. One then deduces \((DUE)\) from this mean value inequality by using either the integrated maximum principle (see [36, chapter 15]) or the Davies-Gaffney estimates which will play an important role in the present article (see [19, Section 5] and [1, Theorem 4.4]).

It turns out that the relative Faber-Krahn inequality is equivalent to the following family of localised Sobolev inequalities introduced by Saloff-Coste (see [49] and also [52, section 2.3]): there exists \( C > 0 \) and \( q > 2 \) such that, for every ball \( B = B(x, r) \) in \( M \) and for every \( f \in C^\infty_0(B) \),

\[
(LS_q) \quad \left( \int_B |f|^q \, d\mu \right)^{\frac{1}{q}} \leq \frac{C r^2}{V^{1 - \frac{q}{2}}(x, r)} \int_B (|\nabla f|^2 + r^{-2}|f|^2) \, d\mu.
\]

This family of inequalities implies in turn by Hölder’s inequality a family of localised Nash inequalities

\[
(LN_\alpha) \quad \left( \int_B |f|^2 \, d\mu \right)^{1+\alpha} \leq \frac{C r^2}{V^\alpha(x, r)} \left( \int_B |f| \, d\mu \right)^{2\alpha} \int_B (|\nabla f|^2 + r^{-2}|f|^2) \, d\mu,
\]

where \( \alpha = 1 - \frac{2}{q} > 0 \), and in fact the methods of [7] show that they are equivalent. One can prove directly that \((LS_q)\) implies \((DUE)\) (see [51, Section 5.2] and the references therein; this has been extended in [56] to a more general Dirichlet form setting), but the proof again relies on the Moser iteration process. A direct proof of the implication from \((DUE)\) to \((LS_q)\) is implicit in [50, Theorem 10.3] and can be found also in [42, Theorem 2.6, proof of Lemma 4.4], but it is not straightforward either.

In the case where there exists \( C, n > 0 \) such that

\[
C^{-1} r^n \leq V(x, r) \leq C r^n,
\]

one says that the volume growth is polynomial with exponent \( n \). This is a much more restrictive and less natural condition than \((VD)\), but in that situation the characterisation of heat kernels upper bounds turns out to be much easier. Indeed, the upper bound \((DUE)\) then reads

\[
(1.1) \quad p_t(x, x) \leq C t^{-n/2}, \forall t > 0, x \in M,
\]

and the fact that this estimate is uniform (meaning that the RHS is independent of \( x \in M \)) allows one to make use of purely functional analytic methods, which yield many characterisations of \((1.1)\) in terms of Sobolev type inequalities. First, a necessary and sufficient condition for this upper bound is the Sobolev inequality:

\[
\|f\|_p \leq C \|\Delta^{\alpha/2} f\|_p, \forall f \in C^\infty_0(M),
\]

for \( p > 1 \) and \( 0 < \alpha p < n \), see [58] and [14]. Note especially, when \( n > 2 \), the particular case \( \alpha = 1, p = 2 \) of this inequality:

\[
(1.2) \quad \|f\|_{2n/p}^2 \leq C \mathcal{E}(f), \forall f \in C^\infty_0(M),
\]
where

\[ \mathcal{E}(f) := \|\Delta^{1/2} f\|^2_2 = \langle \Delta f, f \rangle = \|\nabla f\|^2_2 \]

is the Dirichlet form associated with the Laplace-Beltrami operator. Also equivalent to (1.1) is the Nash inequality:

\[ \|f\|^2_2 + \frac{4}{n} \|f\|^4/n \mathcal{E}(f) \leq C \langle f \rangle^{4/n} \|f\|^4/n, \quad \forall f \in C_0^\infty(M) \]

(for this equivalence, see [11], and for generalisations see [16]). Yet another characterisation of (1.1) is given by the Gagliardo-Nirenberg type inequalities:

\[ \|f\|^2_q \leq C \|f\|^{2 - \frac{2}{n}q} \mathcal{E}(f)^{\frac{q}{2^n}}, \quad \forall f \in C_0^\infty(M), \]

for \( q > 2 \) such that \( \frac{q - 2}{n} < 2 \), see [15] for such inequalities and other extensions. For instance, if one takes \( q = 2 + \frac{4}{n} \) (in which case the above conditions on \( q \) are satisfied), then (1.4) is the well-known Moser inequality

\[ \|f\|^2_2 + \frac{4}{n} \|f\|^4/n \mathcal{E}(f)^{\frac{2}{n}}, \quad \forall f \in C_0^\infty(M). \]

Note also that in (1.4) the limit case \( \frac{q - 2}{n} = 2 \), that is \( q = \frac{2n}{n - 2} \), is nothing but (1.2). Let us insist on the fact that the proofs of the above equivalences work in the general setting of a measure space endowed with a Dirichlet form. For more on this, see [17], and for a summary in book form, see [48, Section 6.1].

The equivalences between (1.1) on the one hand, (1.2), (1.3), and (1.4) on the other hand do not use the fact that the majorizing function \( Ct^{-n/2} \) is tied to the volume growth via \( V(x, r) \approx r^n \). This raises the question of characterising estimates of the type

\[ p_t(x, x) \leq m(x, t), \forall t > 0, x \in M, \]

where \( m \) is not necessarily linked to the volume function.

The aim of the present paper is, assuming the volume doubling property \((VD)\) instead of the more restrictive polynomial volume growth property, to establish new characterisations of the upper bound \((DUE)\) in terms of two types of one-parameter weighted inequalities, which coincide respectively with the Nash inequality (1.3) and with the Gagliardo-Nirenberg type inequalities (1.4) when the volume growth happens to be polynomial of exponent \( n \). We will provide a proof of these characterisations that does not rely on Grigor’yan’s theorem on the equivalence between relative Faber-Krahn inequalities \((FK)\) and the heat kernel upper bound \((DUE)\). As a matter of fact, we will obtain as a by-product a new proof of this equivalence, also of the one with families of localised Sobolev inequalities. All this will rely (as in the uniform case) on functional analytic methods as opposed to PDE methods such as the Moser iteration process.

Further interesting features of our approach are the following: instead of considering a family of inequalities indexed by all balls, we deal with global inequalities (with a scale parameter though); the fact that \((DUE)\) implies such inequalities is rather straightforward; the converse relies on a new argument with respect to the preceding proofs, namely the finite propagation speed of the wave equation (note that the latter, in its equivalent form of Davies-Gaffney estimate, is also an underlying principle of the equivalence between \((DUE)\) and \((UE)\), see [21]); more
importantly, we shall consider a more general form \((DUE^v)\) of \((DUE)\), where the volume function \(V\) is replaced by a more general doubling function \(v\), and we shall prove the equivalence between \((DUE^v)\) in the one hand, and matching versions \((N^v)\) and \((GN_q^v)\) of Nash and Gagliardo-Nirenberg inequalities on the other hand. We shall also show that the latter inequalities are equivalent to their localised Sobolev and Nash counterparts and also to a more general version of the relative Faber-Krahn inequality. Finally, we shall work in a much more general setting than the Riemannian one.

### 1.2. Framework and main results.

Let \((M, \mu)\) be a \(\sigma\)-finite measure space. Denote by \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) with dense domain \(\mathcal{D}\). The quadratic form \(\mathcal{E}\) associated with \(L\) is defined, for \(f, g \in \mathcal{D}\), by

\[
\mathcal{E}(f, g) := \langle Lf, g \rangle,
\]

where \(\langle \cdot, \cdot \rangle\) is the scalar product in \(L^2(M, \mu)\). We shall abbreviate

\[
\mathcal{E}(f) := \mathcal{E}(f, f) = \|L^{1/2}f\|^2_2.
\]

Let \(\mathcal{F}\) the domain of \(\mathcal{E}\), which is usually larger than \(\mathcal{D}\). The form \(\mathcal{E}\) is closed, symmetric, non-negative. By spectral theory, the operator \(-L\) generates an analytic contraction semigroup \((e^{-tL})_{t \geq 0}\) on \(L^2(M, \mu)\). For \(1 \leq p \leq +\infty\) we denote the norm of a function \(f\) in \(L^p(M, \mu)\) by \(\|f\|_p\) and, if \(T\) is a bounded linear operator from \(L^p(M, \mu)\) to \(L^q(M, \mu)\), \(1 \leq p \leq q \leq +\infty\), we denote by \(\|T\|_{p \rightarrow q}\) the operator norm of \(T\). If \(A\) is an unbounded operator acting on \(L^p(M, \mu)\), \(\mathcal{D}_p(A)\) will denote its domain.

Let \(v(x, r)\) be a function of \(x \in M\) and \(r > 0\), measurable in \(x\), a.e. finite and positive, and non-decreasing in \(r\) for a.e. fixed \(x\). These will be standing assumptions that we will call \((A)\).

We shall often have to assume also that \(v\) is doubling, in the sense that there exists \(C > 0\) such that

\[ (D_v) \quad v(x, 2r) \leq Cv(x, r), \forall r > 0, \text{ for } \mu - \text{a.e. } x \in M. \]

As a consequence of \((D_v)\), there exist positive constants \(C\) and \(\kappa_v\) such that

\[ (D_v^{\kappa_v}) \quad v(x, r) \leq C \left( \frac{r}{s} \right)^{\kappa_v} v(x, s), \quad \forall r \geq s > 0, \text{ for } \mu - \text{a.e. } x \in M. \]

From Section 3 on, we shall consider the case where \(M\) is endowed with a metric \(d\) (and \(\mu\) is Borel). In that situation, we may need to assume in addition to \((D_v)\):

\[ (D_v') \quad v(y, r) \leq Cv(x, r), \forall x, y \in M, r > 0, d(x, y) \leq r. \]

One should compare the above definitions with the notion of doubling gauge in [45]. Note that we do not need to assume \(\inf_{x \in M} v(x, r) > 0\) for some \(r > 0\).

Let again \(B(x, r) := \{y \in M, d(x, y) < r\}\) be the open ball in \(M\) for the distance \(d\), of center \(x \in M\) and radius \(r > 0\). Assume that \(V(x, r) := \mu(B(x, r))\) is finite and positive for all \(x \in M, r > 0\). Exactly as in the case of Riemannian manifolds, define property \((VD)\), which may or may not be satisfied by \((M, d, \mu)\):

\[ V(x, 2r) \leq CV(x, r), \quad \forall x \in M, r > 0. \]
and, if it is the case, let $\kappa > 0$ be such that:

\[(V D_\kappa) \quad V(x, r) \leq C \left( \frac{r}{s} \right)^\kappa V(x, s), \quad \forall \, r \geq s > 0, \, x \in M.\]

In other terms, $(V D)$ is nothing but $(D_V)$, $\kappa = \kappa_V$, and in that case, it is easy to check that $(D_V)$ always holds. We shall sometimes say in short that $(M, d, \mu)$ is a doubling metric measure space meaning that it satisfies $(V D)$.

When $(V D)$ is satisfied, a typical example of function $v$ satisfying $(D_v)$ is $v(x, r) := V^\alpha(x, r^\beta)$, $\alpha, \beta > 0$; if $\beta = 1$ then $(D'_v)$ is satisfied in addition. Another interesting example is $v(x, r) := V(x, \min(r, r_0))$, which satisfies $(D_v)$ and $(D'_v)$ as soon as $(M, d, \mu)$ satisfies $(V D_{loc})$, that is $(V D)$ for $0 < r \leq r_0$. As a consequence, the family of general pointwise heat kernel upper estimates $(DUE^v)$ defined below encompasses $(DUE_{loc})$, that is $(DUE)$ for $0 < t \leq t_0$.

Note by the way that, except in Section 3.4, we treat finite as well as infinite measure spaces, and compact as well as non-compact metric measure spaces (recall that according to an observation by Martell, under $(V D)$ compactness is equivalent to finiteness of the measure if balls in $M$ are precompact, see [37, Corollary 5.3]).

Let us finally record a consequence of $(V D)$, which we will use several times in the sequel, even in the presence of a function $v$ instead of $V$, namely the bounded covering principle $(BCP)$: for every $r > 0$, there exists a sequence $x_i \in M$ such that $d(x_i, x_j) > r$ for $i \neq j$ and $\sup_{x \in M} \inf_i d(x, x_i) \leq r$. The balls $B(x_i, r/2)$ are pairwise disjoint and, for all $\theta > 1/2$, there exists $K_0$ only depending on $\theta$ and on the constant in $(V D)$ such that

\[K(x) := \# \{ i \in I, \, x \in B(x_i, \theta r) \} \leq K_0, \quad \forall \, x \in M.\]

Let us turn now towards the heat kernel pointwise upper estimates. Even in the case $v = V$, the definition of the heat kernel upper bound $(DUE^v)$ requires some adaptation from the Riemannian setting to the metric measure space setting. First, in our general setting the existence of a measurable heat kernel is not granted, and it will be part of the definition of $(DUE^v)$ that $(e^{-tL})_{t > 0}$ has a measurable kernel $p_t$, that is

\[e^{-tL} f(x) = \int_M p_t(x, y) f(y) d\mu(y), \quad t > 0, \quad f \in L^2(M, \mu), \quad \text{for } \mu - \text{a.e. } x \in M.\]

This being said, even if $p_t$ exists as a measurable function of $(x, y) \in M \times M$, the expression $p_t(x, x)$ is not properly defined in general, since the diagonal is a set of measure zero in $M \times M$. The following definition overcomes this difficulty: we shall say that $(M, \mu, L, v)$ satisfies $(DUE^v)$ if $(e^{-tL})_{t > 0}$ has a measurable kernel $p_t$ and there exist $C, c > 0$ such that

\[|p_t(x, y)| \leq \frac{C}{\sqrt{v(x, c\sqrt{t})v(y, c\sqrt{t})}}, \quad \text{for all } t > 0, \quad \text{for } \mu - \text{a.e. } x, y \in M.\]

If $v$ satisfies $(D_v)$, one can obviously take $c = 1$ in the above estimate.

If $p_t$ happens to be continuous in $x, y$, the above inequality holds for all $x, y$. Taking $x = y$ and observing that $p_t(x, x) \geq 0$ yields the original form of the estimate

\[p_t(x, x) \leq \frac{C}{v(x, c\sqrt{t})}, \quad \text{for all } t > 0, \quad x \in M,\]
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with \( c = 1 \) if \( v \) satisfies \((Dv)\). Conversely, using the symmetry of the kernel \( p_t(x, y) \) and the semigroup property, it is easy to prove that

\[
|p_t(x, y)| \leq \sqrt{p_t(x, x)p_t(y, y)},
\]

for all \( t > 0, x, y \in M \), hence the two forms of \((DUE^v)\) are equivalent as soon as they both have a meaning.

The reader may wonder why, since we make almost no assumptions on \( v \), we do not write the upper estimate under consideration in a more compact form like

\[
p_t(x, x) \leq m(x, t)
\]

when \( p_t \) is continuous, or

\[
|p_t(x, y)| \leq \sqrt{m(x, t)m(y, t)}
\]

in general. The advantage of our choice is that it makes the comparison with the classical case \( v = V \) easier. Our notation is adapted to a classical time-space scaling \( t = r^2 \). One can of course easily change this notation in order to treat the sub-Gaussian case, but this raises other questions, to which we will come back in a future work. Finally, we keep the apparently useless constant \( C \) in the definition of \((DUE^v)\) in order to stress the fact that the equivalences we will show between \((DUE^v)\) and the functional inequalities we are going to consider are up to a multiplicative constant.

Note that there are a posteriori some limitations on \( v \): our results in Section 3.4 will rely on the assumption that \( v \) is bounded from below by \( V \). In the opposite direction, in many situations, \( v \) cannot be substantially larger than \( V \). This follows from the fact which we are about to explain that \((DUE^v)\) implies a Gaussian upper bound \((UE^v)\); then, if \( v \) is too large, \( \int_M p_t(x, y) \, d\mu(y) \) cannot be uniformly bounded from below, therefore \((e^{-tL})_{t>0}\) cannot preserve the constant function 1 (the so-called conservativeness property). More precisely, it follows from [18, Theorem 6.1, see also beginning of Section 7] that, at least in a Riemannian situation, \((DUE^v)\) can only hold if \( v(x, r) \leq CV(x, r \log r), r \geq 2 \); but this relies strongly on the conservativeness property (also called stochastic completeness), and there are many interesting situations where this property does not hold, for instance Dirichlet boundary conditions. In any case, there is no reason to tie a priori \( v \) to \( V \).

If \((M, \mu)\) is endowed with a metric \( d \), the full Gaussian estimate \((UE^v)\) can be formulated essentially in the same way as in the Riemannian setting:

\[
(UE^v) \quad |p_t(x, y)| \leq \frac{C}{\sqrt{v(x, \sqrt{t})v(y, \sqrt{t})}} \exp \left( -\frac{d^2(x, y)}{Ct} \right),
\]

\( \forall t > 0, \) for \( \mu - \text{a.e.} \ x, y \in M \). If \( v \) satisfies \((Dv)\) and \((Dv')\), \((UE^v)\) can be rewritten in the simpler form

\[
|p_t(x, y)| \leq \frac{C}{v(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{Ct} \right), \forall t > 0, \text{ for } \mu - \text{a.e.} \ x, y \in M.
\]

Let us now briefly introduce a major but very general assumption, namely the Davies-Gaffney estimate. Let \((M, d, \mu)\) be a metric measure space and \( L \) a non-negative self-adjoint operator on \( L^2(M, \mu) \) with dense domain. For \( U_1, U_2 \) open
subsets of $M$, let $d(U_1, U_2) = \inf_{x \in U_1, y \in U_2} d(x, y)$. One says that $(M, d, \mu, L)$, or in short $L$, satisfies the Davies-Gaffney estimate if

$$(DG) \quad |\langle e^{-tL} f_1, f_2 \rangle| \leq \exp \left( -\frac{r^2}{4t} \right) \|f_1\|_2 \|f_2\|_2,$$

for all $t > 0$, $U_i \subset M$, $f_i \in L^2(U_i, \mu)$, $i = 1, 2$ and $r = d(U_1, U_2)$. Davies-Gaffney estimate holds for essentially all self-adjoint, elliptic or subelliptic second-order differential operators including Laplace-Beltrami operators on complete Riemannian manifolds, Schrödinger operators with real-valued potentials and electromagnetic fields; as we already said, it is equivalent to the finite propagation speed of the wave equation. For more information, see for instance [21] and the beginning of Section 4. Davies-Gaffney estimate also holds for measure spaces endowed with a strongly local Dirichlet form and the associated intrinsic metric, see [56], and also [44], [3] for the same estimate with an optimal metric. Note however that this intrinsic metric can degenerate, or be discontinuous, and in such instances our approach cannot be used. For example in the case of fractals, such metrics degenerate. Indeed in this case $d(x, y) = 0$ for all $x, y$ from the ambient space and of course it is impossible to use this intrinsic metric in a meaningful way. Recall finally that $(DUE^v)$ and $(UE^v)$ are equivalent under $(DG)$ and $(D_v)$ (see [21, Section 4.2]).

Let us finally introduce the functional inequalities that are going to generalise (1.3) and (1.4). Denote

$$v_r(x) := v(x, r), \quad r > 0, \quad x \in M.$$ 

For $v$ as above (not necessarily satisfying $(D_v)$ or $(D_v^0)$), consider the inequality

$$(N^v) \quad \|f\|_2^2 \leq C(\|fv^{-1/2}\|_1^2 + r^2 \mathcal{E}(f)), \quad \forall r > 0, \quad f \in \mathcal{F}.$$ 

Of course, our understanding is that if the RHS is infinite (here because $\|fv^{-1/2}\|_1$ is infinite) then $(N^v)$ holds. The same applies to the inequalities we will consider in the sequel. When $v(x, r) \simeq r^m$ for some $n > 0$, for instance when $M$ is endowed with a metric $d$, $v = V$, and $(M, d, \mu)$ has polynomial volume growth of exponent $n$, $(N^v)$ yields

$$\|f\|_2^2 \leq C'(r^{-n}) \|f\|_1^2 + r^2 \mathcal{E}(f), \quad \forall r > 0, \quad f \in \mathcal{F},$$

which, as one sees by minimising in $r$, has exactly the same form as (1.3). This why we shall call $(N^v)$ a $v$-Nash inequality, or in short a Nash inequality.

The following inequality was introduced in [45]:

$$(KN^v) \quad \|f\|_2^2 \leq C \left( \frac{\|f\|_1^2}{\inf_{z \in \text{supp}(f)} v_r(z)} + r^2 \mathcal{E}(f) \right), \quad \forall r > 0, \quad f \in \mathcal{F}.$$ 

Obviously, $(N^v)$ implies $(KN^v)$. Kigami shows that $(DUE^v)$ implies $(Kn^v)$ and that, under an exit time estimate, $(Kn^v)$ implies $(DUE^v)$. We shall see that if one replaces the exit time assumption by a Davies-Gaffney estimate which holds in great generality, $(Kn^v)$, $(N^v)$, and $(DUE^v)$ are in fact equivalent. Kigami also considered a version of $(Kn^v)$ that is adapted to the case of so-called sub-Gaussian estimates. We will not pursue this direction in the present article (see however the remarks at
the very end). The unpublished note [20] contains further generalisations of \((KN^v)\) type inequalities.

In the case where \((M,d,\mu)\) is a metric space, we can also introduce the family of localised \(v\)-Nash inequalities: there exist \(\alpha, C > 0\) such that for every ball \(B = B(x,r)\), for every \(f \in \mathcal{F}_c(B)\),

\[
\left( LN^v_\alpha \right) \quad \|f\|_2^{2(1+\alpha)} \leq \frac{C}{v_r^\alpha(x)} \|f\|_1^{2\alpha} \left( \|f\|_2^2 + r^2 \mathcal{E}(f) \right).\]

Here \(\mathcal{F}_c(\Omega)\) is the set of functions in \(\mathcal{F}\) that are compactly supported in the open set \(\Omega\). The positive parameter \(\alpha\) plays a minor role here. For instance, if \(v = V\) one can check easily that \((LN^v_\alpha) \Rightarrow (LN^v_{\alpha'})\) for all \(0 < \alpha' < \alpha\). We will often drop \(\alpha\) in \((LN^v_\alpha)\), and then \((LN^v)\) will mean: \((LN^v_\alpha)\) for some \(\alpha > 0\) . We shall see in Proposition 3.1.4 below that \((KN^v) \Rightarrow (LN^v)\) if \(v\) satisfies \((D_v)\) and \((D'_v)\), and in Proposition 3.4.1 that \((LN^v)\) is equivalent with some form of relative Faber-Krahn inequality, which coincides with \((FK)\) if \(v = V\) and \((M,d,\mu)\) satisfies the doubling and reverse doubling volume conditions.

Introduce now, for \(2 < q \leq +\infty\), the inequality:

\[
\left( GN^v_q \right) \quad \|f v_r^{\frac{1}{2} - \frac{1}{q}}\|_q^2 \leq C (\|f\|_2^2 + r^2 \mathcal{E}(f)), \quad \forall r > 0, f \in \mathcal{F}.
\]

When \(v(x,r) \simeq r^n\) for some \(n > 0\), for instance when \(M\) is endowed with a metric space, \(v = V\), and \((M,d,\mu)\) has polynomial volume growth of exponent \(n\), \((GN^v_q)\) yields

\[
\|f\|_q^2 \leq C r^{-\frac{q-2}{2}n} (\|f\|_2^2 + r^2 \mathcal{E}(f)), \quad \forall r > 0, f \in \mathcal{F},
\]

which, as one sees again by minimising in \(r\), is equivalent to \((1.4)\) if \(\frac{q-2}{2}n < 2\) and to \((1.2)\) if \(\frac{q-2}{2}n = 2\). This is why we shall call \((GN^v_q)\) a \(v\)-Gagliardo-Nirenberg inequality or in short a Gagliardo-Nirenberg inequality. Note at once that \((GN^v_q)\) is nothing but a resolvent estimate:

\[
\sup_{r > 0} \|v_r^{\frac{1}{2} - \frac{1}{q}} (I + r^2 L)^{-1/2}\|_{2 \rightarrow q} < +\infty.
\]

Similarly as for \((N^v)\), one can formally weaken \((GN^v_q)\) in the spirit of [43], by introducing a \(v\)-Kigami-Gagliardi-Nirenberg inequality

\[
\left( KGN^v_q \right) \quad \left( \inf_{z \in \text{supp}(f)} v_r^{1-\frac{2}{q}}(z) \right) \|f\|_q^2 \leq C (\|f\|_2^2 + r^2 \mathcal{E}(f)), \quad \forall r > 0, f \in \mathcal{F}.
\]

In the case where \((M,d,\mu)\) is a metric space and if \(v\) satisfies \((D'_v)\), by restricting oneself to functions supported in \(B(x,r)\), one sees that \((KGN^v_q)\) implies the following version of the family of localised Sobolev inequalities \((LS^v_q)\): there exists \(C > 0\) such that for every ball \(B = B(x,r)\), for every \(f \in \mathcal{F}_c(B)\),

\[
\left( LS^v_q \right) \quad \|f\|_q^2 \leq \frac{C}{v_r^{1-\frac{2}{q}}(x)} (\|f\|_2^2 + r^2 \mathcal{E}(f)).
\]

Note that in [49], [51], and [52] such inequalities are considered (in the case \(v = V\)) and called localised or scale-invariant Sobolev inequalities. For the sake of coherence with the notation \((N^v)\), \((KN^v)\), \((LN^v)\) on the one hand, \((GN^v_q)\), \((KGN^v_q)\) on the
other hand, we could also have denoted this inequality by $(LN_v^v)$, but we rather chose the name $(LS_v^v)$ to emphasise the connection with Sobolev type inequalities.

It is an easy exercise to check that $(GN_q^v)$, $(KN_q^v)$, $(LS_q^v)$ respectively imply $(GN_{q_1}^v)$, $(KN_{q_1}^v)$, $(LS_{q_1}^v)$ for $2 < q_1 < q$. We leave this to the reader. Inequalities $(GN_{2}^v)$, $(KN_{2}^v)$, $(LS_{2}^v)$ are all trivial.

It will be understood that if $M$ is endowed with a metric and $v = V$ we omit the superscript $v$ in all the inequalities considered above.

There are several reasons why we consider two families of inequalities, namely $(N^v)$ and its variants on the one hand, and $(GN_q^v)$ and its variants on the other hand, instead of one, even though the $(GN_q^v)$ family is easily seen to imply the $(N^v)$ one (Proposition 3.1.1 below) and a more involved converse happens to hold under additional assumptions (Proposition 4.2.3). First, we want to bridge as much as possible the polynomial theory and the existing doubling theory, and they both involve analogues of these two families. Second, and more importantly, the two families display different features. We shall see that $(N^v)$ admits a variant which is adapted to the case where $v$ is not doubling, whereas $(GN_q^v)$, being in essence a resolvent estimate, is more directly related to the matching pointwise heat kernel upper estimate $(DUE_v)$ to be defined below. As a matter of fact, we shall have to go through $(GN_q^v)$ in order to show, under additional assumptions, that $(N^v)$ implies $(DUE_v)$.

We are now ready to state our main result.

**Theorem 1.2.1.** Let $(M,d,µ)$ be a doubling metric measure space, $v : M \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (A), $(D_v)$, and $(D'_v)$, and $L$ a non-negative self-adjoint operator on $L^2(M,µ)$. Assume that $(M,d,µ,L)$ satisfies the Davies-Gaffney estimate $(DG_v)$ and that the semigroup $(e^{-tL})_{t>0}$ is uniformly bounded on $L^1(M,µ)$. Then the upper bound $(DUE_v)$ is equivalent to $(GN_q^v)$, for any $q$ such that $2 < q \leq +\infty$ and $\frac{q-2}{q} \kappa_v < 2$, where $\kappa_v$ is as in $(L^p_v)$; if in addition $(e^{-tL})_{t>0}$ is positivity preserving, $(DUE_v)$ is also equivalent to each of the following conditions: $(N^v)$, $(KN^v)$, $(LN^v)$, as well as $(KN_{q_1}^v)$, $(LS_{q_1}^v)$, under the same condition on $q$.

Note that $\frac{q-2}{q} \kappa < 2$ together with $2 < q \leq +\infty$ means that either $\kappa < 2$ and $q \in (2, +\infty]$, or $\kappa \geq 2$ and $q \in (2, \frac{2\kappa}{\kappa-2}]$. In the latter case, for $q > \frac{2\kappa}{\kappa-2}$, $(GN_q^v)$ may happen to be trivially false, as the polynomial case shows, even though $(DUE_v)$ is true.

We obtain also a characterization of $(DUE_v)$ in terms of Faber-Krahn inequalities, but in that case there are additional subtleties, so that we refer the reader directly to Section 3.3.

Recall that, already 15 years ago, Carron showed in [12] that $(DUE_v)$, with $v$ not necessarily doubling, implies a non-uniform Sobolev-Orlicz inequality involving $\mathcal{E}(f)$; interestingly enough, he also claimed that a converse should rely on the finite propagation speed of the wave equation (that is, on the Davies-Gaffney estimate).

The present paper proves that he was right. It would be interesting to check directly that, under our assumptions, $(N^v)$ and $(GN_q^v)$ imply this Sobolev-Orlicz inequality, and to investigate whether they are equivalent or not.
Here is the plan of this article:

In Section 2, we will use purely functional analytic techniques. We will work with a quadruple \((M, \mu, L, v)\), with \((M, \mu)\) a measure space, \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\), and \(v\) a function on \(M \times \mathbb{R}_+\), satisfying \((A)\). Sometimes, but not always, we will also assume that \(v\) satisfies some doubling properties. In Section 2.1 we introduce the weighted \(L^p - L^q\) estimates that will be our main technical tool, in Section 2.2 we prove that \((\text{DUE}_v)\) implies \((N_v^q)\), in Section 2.3 that \((\text{DUE}_v)\) implies \((\text{GN}_q^v)\) for \(q > 2\) small enough, in Section 2.4 that \((\text{GN}_q^v)\) implies \((\text{DUE}_v)\) if one assumes that \(v\) does not depend on \(x \in M\); the fact that \((N_v^q)\) implies \((\text{DUE}_v)\) is already known in that case, see [16], but we elaborate on that result.

In Section 3, we introduce a metric \(d\) on \(M\), and from Section 3.2 on we assume that the quadratic form \(E\) associated with \(L\) is a strongly local and regular Dirichlet form. In Section 3.1 we observe that \((\text{GN}_q^v)\) implies \((N_v^q)\), that \((\text{KG}_q^v)\) implies \((\text{K}^N_v)\), and that \((L_{S_q}^v)\) implies \((L_{N}^v)\), in Section 3.2 that \((\text{GN}_q^v)\), \((\text{KG}_q^v)\) and \((L_{S_q}^v)\) (resp. \((N_v)\), \((K^N_v)\) and \((L_{N}^v)\)) are equivalent, in Section 3.3 we establish the connection with Faber-Krahn inequalities, in Section 3.4 we study the effect of the so-called reverse doubling condition on Faber-Krahn inequalities and on localised Nash inequalities.

In Section 4, we assume that \((M, d, \mu)\) is a doubling metric measure space and that \(L\) satisfies the Davies-Gaffney estimate. In Section 4.1 we prove that \((\text{GN}_q^v)\) implies \((\text{DUE}_v)\) and in Section 4.2 that \((N_v^q)\) implies \((\text{GN}_q^v)\) under an \(L^1\) assumption on \((e^{-tL})_{t>0}\). This section finishes the proof of Theorem 1.2.1.

As we have just seen, the setting in which we work may vary from section to section. All our results are however valid in the setting of a doubling metric measure space endowed with a strongly local regular Dirichlet form compatible with the distance (see Section 3.2 for details). Therefore they not only cover the Laplace-Beltrami operators on Riemannian manifolds, but a significantly larger class of self-adjoint differential operators acting on more general spaces. Such Dirichlet forms include restrictions of the Laplace operator to open subsets with Dirichlet or Neumann boundary conditions, see for example [11]. This setting also includes degenerate elliptic operators, a class which was studied for instance in [27, 28, 30, 57] or subelliptic operators as in [29]. In some instances we will also consider Schrödinger type operators with positive or negative potentials.

Note that Theorem 1.2.1 itself holds in a even more general setting, that is, in principle, beyond differential operators or even Dirichlet forms: the only assumption for the first assertion is Davies-Gaffney and uniform boundedness on \(L^1\) (in particular, one could treat semigroups acting on vector bundles). In the second assertion, some positivity is required.

Let us finally point out that a very preliminary version of parts of the present work, with the same authors, appeared in [10].

**Remark on notation:** In the sequel, letters \(c\), \(C\) and \(C'\) will denote positive constants, whose value may change at each occurrence.
2. Functional analysis

2.1. Weighted  $L^p - L^q$ estimates. In this section, $(M, \mu)$ will be a $\sigma$-finite measure space, $L$ a non-negative self-adjoint operator on $L^2(M, \mu)$, and $v$ a function from $M \times \mathbb{R}_+$ to $\mathbb{R}_+$ satisfying (A).

For a function $W : M \to \mathbb{R}$, let $M_W$ the operator of multiplication by $W$, that is

$$ (M_W f)(x) = W(x)f(x). $$

In the sequel, we shall identify the function $W$ and the operator $M_W$. That is, if $T$ is a linear operator, we shall denote by $W_1TW_2$ the operator $M_{W_1}TM_{W_2}$. In other words

$$ W_1TW_2f(x) := W_1(x)T(W_2f)(x). $$

Let $1 \leq p \leq q \leq +\infty$. Let $\gamma$, $\delta$ be real numbers such that $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$. We shall denote

$$ (vE_{p,q,\gamma}) \sup_{t>0} ||v^{\gamma}e^{-tL}v^{\delta}||_{p\to q} < +\infty. $$

Of course, this condition may or may not hold, and requires in the first place that the operator $v^{\gamma}e^{-tL}v^{\delta}$ is bounded from $L^p$ to $L^q$ for all $t > 0$, which is certainly not always true. When $\gamma = \frac{1}{p} - \frac{1}{q}$ (that is $\delta = 0$), we shall abbreviate $(vE_{p,q,\gamma})$ by

$$ (vE_{p,q}) \sup_{t>0} ||v^{\frac{1}{p} - \frac{1}{q}}e^{-tL}||_{p\to q} < +\infty, $$

and when $\gamma = 0$, by

$$ (Ev_{p,q}) \sup_{t>0} ||e^{-tL}v^{\frac{1}{p} - \frac{1}{q}}||_{p\to q} < +\infty. $$

Finally, we shall abbreviate $(vE_{1,\infty,1/2})$, that is

$$ \sup_{t>0} ||v^{1/2}e^{-tL}v^{1/2}||_{1\to \infty} < +\infty, $$

by $(vEv)$. Another noteworthy particular case is $(vE_{p,p,0})$, which is nothing but the uniform boundedness of $(e^{-tL})_{t>0}$ on $L^p(M, \mu)$.

In the case where we take $v = V$, we shall of course use the notation $(VE_{p,q,\gamma})$, $(VE_{p,q})$, $(EV_{p,q})$, $(VEV)$. Note that the above estimates are on-diagonal versions of the generalised Gaussian estimates introduced by Blunck and Kunstmann (see for instance [9]).

Observe that, by duality, $(vE_{p,q,\gamma})$ is equivalent to $(vEv_{q',p',\gamma})$, where $p'$, $q'$ are the conjugate exponents to $p$, $q$ and $\gamma + \delta = \frac{1}{p} - \frac{1}{q}$. In particular, $(vE_{p,q})$ and $(Ev_{q',p'})$ are equivalent. The following statement does not use any doubling assumption on $v$. To this purpose, we introduce slightly modified versions of $(Ev_{1,2})$ and $(vE_{2,\infty})$, which under $(D_v)$ are equivalent to their counterparts.

**Proposition 2.1.1.** The estimates (DUE$^v$), $(vEv)$,

$$ (\tilde{E}v_{1,2}) \sup_{t>0} ||e^{-(t/2)L}v^{1/2}||_{1\to 2} < +\infty $$

and

$$ (v\tilde{E}_{2,\infty}) \sup_{t>0} ||v^{1/2}e^{-(t/2)L}||_{2\to \infty} < +\infty $$
are equivalent.

Proof. According to Dunford-Pettis theorem (for a nice account see [37, Proposition 3.1]), \((vEv)\) implies that the operator \(v^{1/2} e^{-tL} v^{1/2}\) has a bounded measurable kernel. It follows that \((e^{-tL})_{t>0}\) also has a measurable kernel \(p_t(x,y)\) and that

\[
\text{ess sup}_{x,y \in M} v_{\sqrt{t}}^{1/2}(x) |p_t(x,y)| v_{\sqrt{t}}^{1/2}(y) = \|v_{\sqrt{t}}^{1/2} e^{-tL} v_{\sqrt{t}}^{1/2}\|_{1 \to \infty} < +\infty,
\]

hence \((DUE^v)\) holds. The converse from \((DUE^v)\) to \((vEv)\) also follows from the above equality.

Similarly to what we already observed, \([Ev_{1,2}]\) and \([vE_{2,\infty}]\) are equivalent by duality. Furthermore, it is well-known that, for an operator \(T\) mapping \(L^1\) to \(L^2\),

\[
\|T^*T\|_{1 \to \infty} = \|T\|_{2 \to \infty}^2 = \|T\|_{1 \to 2}^2.
\]

By taking \(T = e^{-(t/2)L} v_{\sqrt{t}}^{1/2}\), we obtain

\[(1.1) \quad \|v_{\sqrt{t}}^{1/2} e^{-tL} v_{\sqrt{t}}^{1/2}\|_{1 \to \infty} = \|v_{\sqrt{t}}^{1/2} e^{-(t/2)L}\|_{2 \to \infty} = \|e^{-(t/2)L} v_{\sqrt{t}}^{1/2}\|_{1 \to 2},\]

which shows the equivalence of \((DUE^v)\) with the two other conditions.

The consequence if one does assume doubling is now obvious.

**Corollary 2.1.2.** Assume that \(v\) satisfies \((D_v)\). The estimates \((DUE^v)\), \((vEv)\), \((Ev_{1,2})\), and \((vE_{2,\infty})\) are equivalent.

**Remark 2.1.3.** The above so-called \(T^*T\)-argument yields in the same way

\[(Ev_{p,2}) \Leftrightarrow (vE_{p,\gamma}) \Leftrightarrow (vE_{p,p'},\gamma)\]

for all \(1 \leq p \leq 2, \gamma = \frac{1}{p} - \frac{1}{2},\)

**Remark 2.1.4.** The equivalence between \((vEv)\) and \((Ev_{1,2})\) means that an equivalent definition for \((DUE^v)\) is the following:

\[(2.2) \quad \|p_t(x,.)\|_2^2 \leq \frac{C'}{v(x,\sqrt{t})}, \text{ for all } t > 0, \text{ for } \mu - \text{a.e. } x \in M\]

(this also holds with a slight modification if \(v\) is not doubling). Also, it is worth emphasising the difference between \((Ev_{1,2})\) and \((vE_{1,2})\): \((Ev_{1,2})\) (or \((vE_{2,\infty})\) is equivalent to

\[
\text{ess sup}_{x \in M, t>0} v_{\sqrt{t}}(x) \int_M p_t^2(x,y) d\mu(y) < +\infty.
\]

whereas \((vE_{1,2})\) (or \((Ev_{2,\infty})\)) is equivalent to

\[
\text{ess sup}_{x \in M, t>0} \int_M p_t^2(x,y) v_{\sqrt{t}}(y) d\mu(y) < +\infty.
\]

The cornerstone of our main results, Proposition 4.1.6 below, yields in particular that, under the so-called Davies-Gaffney estimate and additional assumptions on \((M,\mu)\) and \(v, (Ev_{1,2})\) and \((vE_{1,2})\) are actually equivalent.

By applying complex interpolation to the family of operators

\[T_z := v_{\sqrt{t}}^{\gamma_1 t + (1-z)\gamma_2} e^{-tL} v_{\sqrt{t}}^{\delta_1 t + (1-z)\delta_2}, \ 0 \leq \Re z \leq 1,\]

one obtains easily:
Proposition 2.1.5 (Interpolation). Let \( 1 \leq p_1 \leq q_1 \leq +\infty, 1 \leq p_2 \leq q_2 \leq +\infty, \gamma_1, \gamma_2 \in \mathbb{R} \). Then \((vEv_{p_1,q_1,\gamma_1})\) and \((vEv_{p_2,q_2,\gamma_2})\) imply \((vEv_{p,q,\gamma})\), where \(1 = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}\), \(\frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}\), \(\gamma = \theta \gamma_1 + (1 - \theta) \gamma_2\).

Of particular interest will be the case \(p_2 = q_2, \gamma_2 = 0\), which amounts to the uniform boundedness of \((e^{-tL})_{t>0}\) on \(L^p(M, \mu)\). Note that the main technical point of the present paper will be an extrapolation counterpart to this consequence of Proposition 2.1.5 namely Proposition 2.1.9 below.

Next for any pair \((p, q)\) such that \(1 \leq p \leq q \leq +\infty\) we define exponents

\[ 0 \leq \gamma_-(p, q) \leq \gamma_+(p, q) \]

by the formulae

\[ \gamma_-(p, q) = \max \left\{ \frac{1}{2p} - \frac{1}{q}, 0 \right\} \]

and

\[ \gamma_+(p, q) = \min \left\{ \frac{1}{p} - \frac{1}{q}, \frac{1}{2} - \frac{1}{2q} \right\} \]

Using this notation, we can state a consequence of Corollary 2.1.2 and Proposition 2.1.5 that only relies on duality and interpolation.

Corollary 2.1.6. Assume that \(v\) satisfies \((D_v)\). The pointwise heat kernel upper bound \((DUE^v)\) implies \((Ev_{p,2})\) for all \(p\) such that \(1 \leq p \leq 2\), \((vEv_{2,q})\) for all \(q\) such that \(2 \leq q \leq +\infty\), and \((vEv_{p,q,\gamma})\) for all \(p, q\) such that \(1 \leq p \leq 2 \leq q \leq +\infty\) and \(\gamma = \frac{1}{2} - \frac{1}{q}\). If in addition \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\), one obtains also \((vEv_{p,q,\gamma})\) for all \(p, q\) such that \(1 \leq p \leq q \leq +\infty\) and all \(\gamma_-(p, q) \leq \gamma \leq \gamma_+(p, q)\).

Proof. Corollary 2.1.2 says that, under \((D_v)\), \((DUE^v)\) implies \((vEv_{1,\infty,1/2})\), \((vEv_{1,2,0})\) and \((vEv_{2,\infty,1/2})\). Since the semigroup \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^2(M, \mu)\), we have in addition \((vEv_{2,2,0})\). Proposition 2.1.5 applied to \((vEv_{1,2,0})\) and \((vEv_{2,2,0})\) yields \((vEv_{p,2,0})\), that is \((Ev_{p,2})\), for all \(1 \leq p \leq 2\), and applied to \((vEv_{1,\infty,1/2})\) and \((vEv_{2,\infty,1/2})\) it yields \((vEv_{p,\infty,1/2})\) for all \(1 \leq p \leq 2\). Now interpolating again between \((vEv_{p,2,0})\) and \((vEv_{p,\infty,1/2})\) yields the first part of the statement. Next if \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\) then by duality and interpolation \((vEv_{p,0})\) holds for all \(1 \leq p \leq \infty\). One checks easily that interpolation between \((vEv_{1,1,0})\), \((vEv_{\infty,\infty,0})\), and \((vEv_{1,2,0})\) yields \((vEv_{p,q,\gamma})\) for all \(1 \leq p \leq q \leq \infty\) such that \(1/p \leq 2/q\), that is \((vEv_{p,q,\gamma_-(p,q)})\) for this range of \(p, q\). To obtain \((vEv_{p,q,\gamma_-(p,q)})\) for \(1/p > 2/q\), one interpolates between \((vEv_{\infty,\infty,0})\), \((vEv_{1,2,0})\), and \((vEv_{1,\infty,1/2})\). One then obtains \((vEv_{p,q,\gamma_-(p,q)})\) by duality and interpolating between \((vEv_{p,q,\gamma_-(p,q)})\) and \((vEv_{p,q,\gamma_+(p,q)})\) yields \((vEv_{p,q,\gamma})\) for all \(\gamma_-(p, q) \leq \gamma \leq \gamma_+(p, q)\).

Note that one can drop assumption \((D_v)\) in Corollary 2.1.6 at the expense of modifying conditions \((vEv_{p,q,\gamma})\) in the spirit of Proposition 2.1.1. If one is prepared to assume the full Gaussian upper estimate \((UE^v)\) instead of \((DUE^v)\), one can obtain the same conclusion as in Corollary 2.1.6 in a more straightforward way and without any restriction on the exponent \(\gamma\).
Corollary 2.1.7. Let \((M, d, \mu)\) be a metric measure space, let \(v : M \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\) satisfy \((A), (D_v),\) and \((D_u^p)\), and let \(L\) be a non-negative self-adjoint operator on \(L^2(M, \mu)\). Assume that \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^p_{0}(M, \mu)\) for some \(p_0 \in [1, 2)\). The full Gaussian upper bound \((UE^v)\) implies \((vEv_{p,q,\gamma})\), for all \(p, q\) such that \(p_0 \leq p < q \leq p_0'\) and all \(\gamma \in \mathbb{R}\).

Proof. It is easy to see using \((D_v)\) and \((D_u^p)\) that \((UE^v)\) implies \((vEv_{1,\infty,\gamma})\) for all \(\gamma \in \mathbb{R}\). On the other hand, \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^r(M, \mu)\) for all \(p_0 \leq r \leq p_0'\) by duality and interpolation. Applying Proposition 2.1.5 with \(p_1 = 1, q_1 = \infty\), every \(\gamma_1 \in \mathbb{R}, p_2 = q_2 = r\), for all \(r\) such that \(p \leq r \leq q\), and \(\gamma_2 = 0\) yields the claim. 

Note that if \((M, d, \mu)\) is a doubling metric measure space and if \(v \geq \varepsilon V\) for some \(\varepsilon > 0\), one need not assume the uniform boundedness of \((e^{-tL})_{t>0}\) on \(L^p_{0}(M, \mu)\) in Corollary 2.1.7, since it then follows for \(p_0 = 1\) from \((UE^v)\).

Now recall that \((UE^v)\) and \((DUE^v)\) coincide if the Davies-Gaffney estimate holds (see section 2.1.2). We can therefore state the following.

Theorem 2.1.8. Let \((M, d, \mu)\) be a metric measure space, \(v : M \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\) satisfy \((A), (D_v),\) and \((D_u^p)\), and let \(L\) be a non-negative self-adjoint operator on \(L^2(M, \mu)\). Assume that \((M, d, \mu, L)\) satisfies \((DG)\) and that \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^p_{0}(M, \mu)\) for some \(p_0 \in [1, 2)\). Then the pointwise heat kernel upper bound \((DUE^v)\) implies \((vEv_{p,q,\gamma})\), for all \(p, q\) such that \(p_0 \leq p < q \leq p_0'\) and all \(\gamma \in \mathbb{R}\).

Theorem 2.1.8 indicates that a sensible step towards \((DUE^v)\), at least if the Davies-Gaffney estimate holds, is to show that all estimates \((vEv_{p,q,\gamma})\) are equivalent. Indeed, this will be proved for \(q = 2\) in Proposition 4.1.6 below.

2.2. The heat kernel upper bound implies Nash. Our main result here is the following.

Proposition 2.2.1. Let \((M, \mu)\) be a measure space, \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\), and \(v\) a function from \(M \times \mathbb{R}_+\) to \(\mathbb{R}_+\) satisfying \((A)\). The heat kernel upper bound \((DUE^v)\) implies the inequality \((N^v)\).

In the case where \(v\) satisfies \((D_v)\), this result will also follow from Propositions 2.3.1 and 3.1.1 below, but even in that case, it is nice to have the following simple and direct proof. According to Corollary 2.1.2, it is enough to prove that \((Ev_{1,2})\) implies \((N^v)\). For the sake of simplicity and for future record, we prefer to state the implication from \((Ev_{1,2})\) to \((N^v)\), but the proof is similar.

Proposition 2.2.2. The estimate:

\[(Ev_{1,2})\quad \sup_{t>0} \| e^{-tL} v^{1/2} \|_{1 \rightarrow 2} < +\infty\]

implies the inequality \((N^v)\).

Proof. One can write, for \(f \in \mathcal{D}\) and \(t > 0\),

\[(2.3)\quad f = e^{-tL} f + \int_0^t Le^{-sL} f \, ds.\]
This formula is also valid in $L^2(\mu)$ because $(e^{-sL})_{s>0}$ is analytic on $L^2(\mu)$. Thus, for $f \in \mathcal{F} = \mathcal{D}_2(L^{1/2})$,
\[
\|f\|_2 \leq \|e^{-tL}f\|_2 + \int_0^t \|Le^{-sL}f\|_2 ds \\
\leq \|e^{-tL}v^{1/2}_t v^{-1/2}_t f\|_2 + \int_0^t \|L^{1/2}e^{-sL}f\|_2 ds \\
\leq \|e^{-tL}v^{1/2}_t|_{1 \to 2} f v^{-1/2}_t\|_1 + C \int_0^t s^{-1/2}\|L^{1/2}f\|_2 ds.
\]
In the last inequality, we have again used the analyticity of $(e^{-sL})_{s>0}$ on $L^2(\mu)$, which yields
\[
\|L^{1/2}e^{-sL}f\|_2 \leq C s^{-1/2}\|f\|_2, \quad \forall f \in L^2(\mu), \quad s > 0.
\]
Hence
\[
\|f\|_2 \leq \|e^{-tL}v^{1/2}_t|_{1 \to 2} f v^{-1/2}_t\|_1 + C \sqrt{2t}\|L^{1/2}f\|_2, \quad \forall f \in \mathcal{F}, \quad t > 0,
\]
therefore using (Ev$_{1,2}$),
\[
\|f\|_2 \leq C'(\|f v^{-1/2}_t\|_1 + \sqrt{t}\|L^{1/2}f\|_2), \quad \forall f \in \mathcal{F}, \quad t > 0,
\]
that is, setting $r = \sqrt{t}$, $(N^v)$.

Note that one could also adapt the proof of Kigami in [45, pp.528-529] to get a proof of Proposition 2.2.1 at least if one assumes a priori the existence of measurable $p_t$. We leave this to the interested reader. Our more functional analytic approach enables one to treat other $L^p$ spaces than $L^2$.

**Remark 2.2.3.** One can prove in a similar way that if (Ev$_{p,q}$) holds for some $p, q$ such that $1 \leq p < q < +\infty$ and, if $(e^{-tL})_{t>0}$ is bounded analytic on $L^q(\mu)$,
\[
(N_{p,q}) \quad \|f\|_q \leq C(\|fv^{1/2}_t\|_p + r\|L^{1/2}f\|_q), \quad \forall r > 0, \quad f \in \mathcal{D}_q(L^{1/2}),
\]
follows. More generally, for $\beta > 0$,
\[
(N_{p,q,\beta}) \quad \|f\|_q \leq C_\beta(\|fv^{1/2}_t\|_p + r^\beta\|L^{\beta/2}f\|_q), \quad \forall r > 0, \quad f \in \mathcal{D}_q(L^{\beta/2}).
\]
For $\beta \geq 2$, one uses a higher order Taylor formula instead of (2.3).

**Theorem 2.2.4.** Let $(M, d, \mu)$ be a metric measure space, $L$ a non-negative self-adjoint operator on $L^2(\mu)$, and $v$ a function from $M \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (A), (D$_v$) and (D$_\mu$). Assume that $(M, d, \mu, L)$ satisfies (DG) and that $(e^{-tL})_{t>0}$ is uniformly bounded on $L^{p_0}(\mu)$ for some $p_0 \in [1, 2]$. Then the heat kernel upper bound (DUE$^v$) implies $(N^v_{p,q,\beta})$ for all $p_0 \leq p < q < p_0'$ and $\beta > 0$.

In the proof of the above statement, we make use of the fact that, since $(e^{-tL})_{t>0}$ is bounded analytic on $L^2(\mu)$, if in addition it is uniformly bounded on $L^p(\mu)$, for some $p \in [1, +\infty]$, then by interpolation it is bounded analytic on $L^q(\mu)$, for all $q$ strictly between 2 and $p$.

Although it will come under some additional assumptions as a by-product of our results in Section 4, we do not know how to prove in a direct way, that is, without
going through $(GN^v_r)$, that conversely $(N^v)$ implies $(E^v_{1,2})$ under these assumptions. More generally, one may wonder whether $(N^v_{p,q})$ and $(E^v_{p,q})$ coincide or not (we will be able to answer positively a similar question for $(GN^v_r)$ and its variants, see Proposition 2.3.3). The following observation together with Proposition 2.1.5 shows that, if it were to be the case, all $(N^v_{p,q})$ (and $(E^v_{p,q})$) would be equivalent for fixed $q$ and all $p \in [1, q]$ as long as $(e^{-tL})_{t>0}$ is uniformly bounded on $L^1(M, \mu)$. This is unlikely to hold without further assumptions.

**Proposition 2.2.5.** $(N^v_{p,q})$ implies $(N^v_{p_0,q})$ for $1 \leq p_0 < p < q < +\infty$. In particular, $(N^v_{p,2})$ for $1 < p < 2$ implies $(N^v)$.

**Proof.** Assume $(N^v_{p,q})$ and write

$$
\|fv^r_{\theta} \|_p \leq \|f\|_q \|fv^r_{\theta} \|_{p_0}^{1-\theta},
$$

where $\theta$ is such that $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{p_0}$. It follows that

$$
\|f\|_q \leq C(\|f\|_q \|fv^r_{\theta} \|_{p_0}^{1-\theta} + r \|L^{1/2}f\|_q)
\leq C(\varepsilon \|f\|_q + \varepsilon^{-\frac{\theta}{1-\theta}} \|fv^r_{\theta} \|_{p_0}^{1-\theta} + r \|L^{1/2}f\|_q),
$$

for all $r, \varepsilon > 0$, $f \in D_q(L^{1/2})$. One obtains $(N^v_{p_0,q})$ by choosing $\varepsilon = \frac{1}{2C}$. □

It is known, already in the case where $v$ does not depend on $x$ but decays more quickly than a negative power (in particular $v$ is not doubling), that Proposition 2.2.2 is not optimal: in that case, under mild conditions on $v$, $(DUE^v)$ is equivalent to a so-called generalised Nash inequality, which is strictly stronger than $(\tilde{N}^v)$ (see [16, Theorem II.5]). Using the technique introduced in [16], one can indeed obtain in a simple way a stronger version of Proposition 2.2.2.

**Proposition 2.2.6.** The estimate $(\tilde{E}^v_{1,2})$ implies the inequality

$$(\tilde{N}^v) \quad \|f\|_2 \log \frac{c \|f\|_2^2}{\|fv^r_{L^{-1/2}}\|_2^2} \leq r^2 \mathcal{E}(f), \quad \forall r > 0, f \in \mathcal{F},$$

where $c$ is the inverse of the supremum in $(\tilde{E}^v_{1,2})$ squared.

**Proof.** Start with the inequality from [16] Proposition II.2,

$$
\exp \left( - \frac{\mathcal{E}(f)}{\|f\|_2^2} \right) \leq \frac{\|e^{-(t/2)L}f\|_2^2}{\|f\|_2^2},
$$

which follows from Jensen’s inequality applied to the spectral resolution of $L$.

Then $(\tilde{E}^v_{1,2})$ with constant $C$ yields

$$
\exp \left( - \frac{\mathcal{E}(f)}{\|f\|_2^2} \right) \leq \frac{\|e^{-(t/2)L}f\|_2^2}{\|f\|_2^2} \leq \frac{C^2 \|fv^r_{\sqrt{7}}\|_1^2}{\|f\|_2^2},
$$

which is obviously equivalent to $(\tilde{N}^v)$ by changing $t$ to $r^2$ and taking logarithms. □

Propositions 2.2.6 and 2.1.1 yield the following corollary, where one still does not assume $v$ to be doubling. Note that [6, Theorem 3.9] corresponds to the particular case where $v(x, r)$ is a product of a function of $x$ and a function of $r$.
Corollary 2.2.7. The heat kernel upper bound \((DUE^v)\) implies the inequality
\[
(\tilde{N}^v) \quad \|f\|_2^2 \log \frac{c\|f\|_2^2}{\|f v_r^{-1/2}\|_1} \leq r^2 \mathcal{E}(f), \quad \forall r > 0, \ f \in \mathcal{F};
\]
for some \(c > 0\).

Rewriting \((N^v)\) as
\[
0 < \frac{1}{C} \leq \frac{\|f v_r^{-1/2}\|_1^2}{\|f\|_2^2} + \frac{r^2 \mathcal{E}(f)}{\|f\|_2^2}
\]
for all \(f \in \mathcal{F}\setminus\{0\}\) and all \(r > 0\) and using the elementary fact that \(A, B, c > 0, \ \log c \leq B \Rightarrow A + B \geq \min\left(\frac{c}{2}, \log 2\right)\), one sees that \((N^v)\) implies \((N^v)\) with \(C = \frac{1}{\min(\frac{1}{2}, \log 2)}\), and as we already said the converse is false even in the case where \(v\) does not depend on \(x\) (see [16]).

A posteriori, if \(v\) is doubling and if \((M, d, \mu, L)\) satisfies the additional assumptions of Proposition 4.2.4 below, one can see that \((N^v)\) does imply \((N^v)\); indeed, Proposition 4.2.4 states that in that situation \((N^v)\) implies \((DUE^v)\), which implies back \((N^v)\) by Corollary 2.2.7. One can see this directly.

Proposition 2.2.8. If \(v\) satisfies \((D_v)\), \((N^v)\) and \((N^v)\) are equivalent.

Proof. We have already seen that \((N^v)\) always implies \((N^v)\). Now for the converse. For \(f \in \mathcal{F}\setminus\{0\}\) and \(r > 0\), denote
\[
A(r, f) := \frac{\|f v_r^{-1/2}\|_1^2}{\|f\|_2^2}
\]
and
\[
B(f) := \frac{\mathcal{E}(f)}{\|f\|_2^2}.
\]
Since \(v\) is assumed to be non-decreasing in \(r\), the function \(r \mapsto A(r, f)\) is non-increasing, and since \(v\) satisfies \((D_v)\),
\[
\frac{A(s, f)}{A(r, f)} \leq C \left(\frac{r}{s}\right)^{\nu_v}, \text{ for } r \geq s > 0,
\]
where \(C > 0\) being the doubling constant is independent of \(f\).

The validity of \((N^v)\) means that
\[
(2.5) \quad \inf_{r > 0, f \in \mathcal{F}\setminus\{0\}} A(r, f) + r^2 B(f) = c > 0.
\]

Assume first that \(B(f) \neq 0\).

Define \(r_0 = r_0(f) := \inf\{r > 0; r^2 B(f) \geq A(r, f)\}\).

Note that \(r_0 = 0\) would not be compatible with \((2.5)\), hence \(r_0 > 0\). Then \((r_0/2)^2 B(f) < A(r_0/2, f)\) since \(r_0/2 < r_0\). Also, there exists \(\varepsilon > 0\) arbitrarily small such that \((r_0 + \varepsilon)^2 B(f) \geq A(r_0 + \varepsilon, f)\), hence, for \(\varepsilon \leq r_0\), \((2r_0)^2 B(f) \geq A(2r_0, f)\).

But by doubling \(A(2r_0, f) \geq C^{-1} A(r_0, f)\) and \(A(r_0/2, f) \leq CA(r_0, f)\), hence
\[
\frac{1}{4C} A(r_0, f) \leq r_0^2 B(f) \geq 4CA(r_0, f).
\]
It follows that
\[ \min \{ A(r_0, f), r_0^2B(f) \} \geq \frac{c}{4C + 1}. \]
If \( r \leq r_0 \) then
\[ A(r, f)e^{r^2B(f)} \geq A(r, f) \geq A(r_0, f) \geq \frac{c}{4C + 1} := c' > 0. \]
Now for \( r \geq r_0 \). Using (2.4),
\[ A(r, f)e^{r^2B(f)} \geq C^{-1}A(r_0, f)\left(\frac{r_0}{r}\right)^{\kappa_v}e^{r^2B(f)} = C^{-1}A(r_0, f)\left(\frac{r_0}{r}\right)^{\kappa_v}e^{(r/r_0)^{2}\kappa_vB(f)} \geq C^{-1}c'\left(\frac{r_0}{r}\right)^{\kappa_v}e^{c'(r/r_0)^{2}} \]
Set \( b = \inf_{x \geq 1} x^{\kappa_v}e^{c'x^2} \). Note that \( b > 0 \) only depends on \( \kappa_v, c' \). Finally
\[ A(r, f)e^{r^2B(f)} \geq C^{-1}c'b > 0, \]
which is nothing but \( [N^v] \).

The argument for the case \( B(f) = 0 \) is straightforward so we skip it.

\[ \square \]

Remark 2.2.9. Similarly as Carron in [12], one can observe that the best upper bound for \( p_t \) is... \( p_t \) itself, and obtain a universal Nash inequality \( (\tilde{N}) \) by taking \( v(x, r) := \frac{1}{p_t(x, r)} \), or more generally \( v(x, r) := \frac{1}{\|p_t(x, \cdot)\|_2} \) if \( p_t \) is not assumed or known to be continuous.

Remark 2.2.10. One may conjecture that [16, Theorem II.5], see also Section 2.4 below, generalises to the case where \( v \) does depend on \( x \), that is \( [N^v] \) implies back \( (DUE^v) \). The difficulty is related to the fact that we do not know so far how to prove directly, even when \( v \) is doubling, that \( (N^v) \) implies \( (DUE^v) \). We have to go through \( (GN_q^v) \), hence the next section. The article [6] does contain a partial converse to Corollary 2.2.7 in the case where \( v(x, r) \) is a product of a function of \( x \) and a function of \( r \), and the function of \( x \) satisfies a Lyapunov type condition.

2.3. The heat kernel upper bound implies Gagliardo-Nirenberg. In this section, we will prove:

Proposition 2.3.1. Let \( (M, \mu) \) be a measure space, \( L \) a non-negative self-adjoint operator on \( L^2(M, \mu) \) and let \( v : M \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) satisfy \( (A) \) and \( (D_v) \). Then the heat kernel upper bound \( (DUE^v) \) implies the inequality \( (GN_q^v) \) for all \( q \) such that \( 2 < q \leq +\infty \) and \( \frac{q-2}{q}\kappa_v < 2 \), where \( \kappa_v \) is as in \( (D_q^v) \).

The above statement does not cover the limit case \( \frac{q-2}{q}\kappa_v = 2 \); we suspect the latter might be obtained by using the self-improvement of \( (DUE^v) \) into \( (UE^v) \).

Recall that the constraint on \( q \) can be reformulated in the following way: either \( \kappa_v < 2 \) and \( q \in (2, +\infty] \), or \( \kappa_v \geq 2 \) and \( q \in (2, \frac{2\kappa_v}{\kappa_v - 2}) \). For specific considerations on the case \( q = +\infty \), see Corollary 2.3.6 below.
A remark similar to Remark 2.2.9 is in order, except that the universal inequality \((GN_q^v)\) one obtains in this way only holds under the condition that

\[ p_t(x, x) \leq C p_{2t}(x, x), \forall t > 0, x \in M \]

(similar formulation if needed with \(\|p_t(x, .)\|_{\frac{3}{2}}\)).

According to Corollary 2.1.6 it is enough to prove that \((vE_{2,q})\) implies \((GN_q^v)\). In fact, these two conditions happen to be equivalent. For the converse, we shall use the fact that \((GN_q^v)\) can be reformulated as a resolvent estimate:

\[
\sup_{t > 0} \|v^{\frac{t}{\sqrt{q}}} (I + t L)^{-1/2}\|_{2 \to q} < +\infty.
\]

We shall develop further this point of view in Proposition 2.3.4 below, and it will also be instrumental in Section 4.1. Let us start by adopting a point of view more similar to the one in Proposition 2.2.2.

**Proposition 2.3.2.** Assume \(v\) satisfies \((D_v)\). Let \(q\) be such that \(2 < q \leq +\infty\) and \(\frac{q-2}{q} \kappa_v < 2\), where \(\kappa_v\) is as in \((D_{v}^{0})\). Then the estimate:

\[
(vE_{2,q}) \quad \sup_{t > 0} \|v^{\frac{t}{\sqrt{q}}} e^{-t L}\|_{2 \to q} < +\infty
\]

implies \((GN_q^v)\). Conversely, \((GN_q^v)\) implies \((vE_{2,q})\) for all \(q\) such that \(2 < q \leq +\infty\). The latter assertion does not require \(v\) to be doubling.

**Proof.** Assume \((vE_{2,q})\) and \(\frac{q-2}{q} \kappa_v < 2\). Set \(\alpha = \frac{1}{2} - \frac{1}{q}\). Again, for \(f \in L^2(M, \mu)\), write

\[ f = e^{-t L} f + \int_0^t e^{-s L} f ds, \quad \forall t > 0, \]

hence

\[ v^{\alpha t} f = v^{\alpha t} e^{-t L} f + \int_0^t v^{\alpha t} e^{-(s/2) L} e^{-(s/2) L} f ds. \]

Then

\[
\|v^{\alpha t} f\|_q \leq \|v^{\alpha t} e^{-t L} f\|_q + \int_0^t \|v^{\alpha t} e^{-(s/2) L}\|_{2 \to q} \|e^{-(s/2) L} f\|_2 ds
\]

\[
\leq C \|v^{\alpha t} e^{-t L}\|_{2 \to q} \|f\|_2 + \int_0^t \|v^{\alpha t} e^{-(s/2) L}\|_{\infty} \|v^{\alpha t} e^{-(s/2) L}\|_{2 \to q} \|e^{-(s/2) L} f\|_2 ds.
\]

Using \((vE_{2,q})\) and \((D_{v}^{0})\), we obtain, for \(f \in \mathcal{F}\),

\[
\|v^{\alpha t} f\|_q \leq C \|f\|_2 + C' \int_0^t \left( \frac{t}{s} \right)^{\alpha \kappa_v/2} \|L^{1/2} e^{-(s/2) L} (L^{1/2} f)\|_2 ds
\]

\[
\leq C \|f\|_2 + C' t^{\alpha \kappa_v/2} \left( \int_0^t s^{-\alpha \kappa_v/2 - \frac{1}{2}} ds \right) \|L^{1/2} f\|_2
\]

\[
\leq C(\|f\|_2 + \sqrt{t} \|L^{1/2} f\|_2),
\]

that is, setting \(r = \sqrt{t}\), \((GN_q^v)\). In the second inequality, we have used the analyticity of \((e^{-tL})_{t>0}\) on \(L^2(M, \mu)\), and in the last one the fact that \(\alpha \kappa_v < 1\).
Now for the converse. Assume that
\[(GN_q^v) \quad \| f v_{\sqrt{t}}^{\frac{1}{2} - \frac{1}{q}} \|_q \leq C(\| f \|_2 + \sqrt{t} \| L^{1/2} f \|_2), \quad \forall \ t > 0, \ \forall \ f \in D.\]

This can be rewritten as
\[
\| v_{\sqrt{t}}^{\frac{1}{2} - \frac{1}{q}} f \|_q^2 \leq C'(\| f \|_2^2 + t < L f, f >)
= C' < (I + tL) f, f >
= C' \| (I + tL)^{1/2} f \|_2^2,
\]
thus, replacing \( f \) by \( e^{-tL} f \),
\[
\| v_{\sqrt{t}}^{\frac{1}{2} - \frac{1}{q}} e^{-tL} f \|_q \leq C'' \| (I + tL)^{1/2} e^{-tL} f \|_2
\leq C'' \| (I + tL)^{1/2} e^{-tL} \|_{2 \rightarrow 2} \| f \|_2
= C'' \left( \sup_{\lambda > 0} (1 + t\lambda)^{1/2} e^{-t\lambda} \right) \| f \|_2
= C'' \| f \|_2.
\]

\[\square\]

As we already said, Proposition 2.3.1 yields Proposition 2.2.1 as a by-product in the case where \( v \) satisfies \( (D_v) \), because, according to Proposition 3.1.1 below, its conclusion is stronger. On the other hand, the converse part of Proposition 2.3.2 can be used to give a (rather indirect) proof of the implication from \( (GN_q^v) \) to \( (N^v) \) which will see in a straightforward way in Proposition 3.1.1 below.

**Proposition 2.3.3.** For any \( q > 2 \), \( (GN_q^v) \) implies \( (N_{p,2}^v) \) for all \( p, 1 \leq p \leq q' \), and in particular \( (N^v) \).

**Proof.** By Proposition 2.3.2, \( (GN_q^v) \) implies \( (vE_{2,q}) \). By duality, \( (vE_{2,q}) \) is equivalent to \( (Ev_{q',2}) \). On the other hand, as noticed in Remark 2.2.3, \( (Ev_{q',2}) \) implies \( (N_{q',2}^v) \). Now, according to Proposition 2.2.5, \( (N_{q',2}^v) \) implies \( (N_{p,2}^v) \) for all \( p, 1 \leq p \leq q' \). The case \( p = 1 \) yields \( (N^v) \). \[\square\]

Next we show a variation on (and generalisation of) Proposition 2.3.2 which yields characterisations of \( (vE_{p,q}) \) in terms of some resolvent type conditions and of some generalised forms of \( (GN_q^v) \).

**Proposition 2.3.4.** Let \( 1 \leq p < q \leq +\infty \) and \( \beta > (\frac{1}{p} - \frac{1}{q})\kappa_v \), where \( \kappa_v \) is as in \( (D_v^\kappa) \). Assume that \( (e^{-tL})_{t>0} \) is bounded analytic on \( L^p(M, \mu) \). Then the following conditions are equivalent:

\[(vE_{p,q}) \quad \sup_{t>0} \| v_{\sqrt{t}}^{\frac{1}{2} - \frac{1}{q}} e^{-tL} \|_{p \rightarrow q} < +\infty \]

\[(vR_{p,q,\beta}) \quad \sup_{t>0} \| v_{\sqrt{t}}^{\frac{1}{2} - \frac{1}{q}} (I + tL)^{-\beta/2} \|_{p \rightarrow q} < +\infty, \]

\[(GN_{p,q,\beta}^v) \quad \| f v_{\sqrt{t}}^{\frac{1}{2} - \frac{1}{q}} \|_q \leq C(\| f \|_p + r^\beta \| L^\beta f \|_p), \quad \forall \ r > 0, \ f \in D_p(L^\beta). \]
Note that the condition $\beta > \left(\frac{1}{p} - \frac{1}{q}\right)\kappa_v$, together with $p < q \leq +\infty$, means that either $\beta > \frac{\kappa_v}{p}$ and $q \in (p, +\infty]$, or $\beta \leq \frac{\kappa_v}{p}$ and $q \in (p, \frac{p\kappa_v}{\kappa_v - p\beta})$. Note also that $(GN_{\|v\|}^{\|v\|})$ is nothing but $(GN_{\|v\|}^{\|v\|})$. In particular, taking $p = 2$ and $\beta = 1$ in the proof below yields an interesting alternative proof to the implication from $(vE_{2,q})$ to $(GN_{\|v\|}^{\|v\|})$ in Proposition 2.3.2.

**Proof.** Note first that $(\mathbb{R}_{p,q,\beta})$ and $(\mathbb{V}_{p,q,\beta})$ can be rewritten respectively as

$$\|f v^{\frac{1}{p} - \frac{1}{q}}\|_q \leq C \| (I + tL)^{\beta/2} f \|_p,$$

uniformly in $t > 0$ and $f \in D_p(L^{\beta/2})$ and

$$\|f v^{\frac{1}{p} - \frac{1}{q}}\|_q \leq C (\|f\|_p + t^{\beta/2} \|L^{\beta/2} f\|_p),$$

uniformly in $t > 0$ and $f \in D_p(L^{\beta/2})$. The equivalence between $(\mathbb{V}_{p,q,\beta})$ and $(\mathbb{R}_{p,q,\beta})$ follows therefore from

$$(2.6) \quad \|(I + tL)^{\beta/2} f\|_p \simeq \|(I + (tL)^{\beta/2}) f\|_p \simeq \|f\|_p + t^{\beta/2} \|L^{\beta/2} f\|_p,$$

uniformly in $t > 0$ and $f \in D_p(L^{\beta/2})$.

The norm equivalence (2.6) is classical (see for instance [5, Proposition 3.1]). Let us sketch a proof for the sake of completeness. In order to prove (2.6), it is clearly enough to prove

$$(2.7) \quad \sup_{t > 0} \|(I + tL)^{\beta/2}(I + (tL)^{\beta/2})^{-1}\|_{p \to p} < +\infty,$$

$$(2.8) \quad \sup_{t > 0} \|(I + (tL)^{\beta/2})(I + tL)^{-\beta/2}\|_{p \to p} < +\infty,$$

$$(2.9) \quad \sup_{t > 0} \|(I + tL)^{-\beta/2}\|_{p \to p} < +\infty$$

and

$$(2.10) \quad \sup_{t > 0} \|(tL)^{\beta/2}(I + tL)^{-\beta/2}\|_{p \to p} < +\infty.$$ 

Note that (2.8) obviously follows from (2.9) and (2.10). An equivalent formulation of (2.7) is

$$\sup_{t > 0} \|(I + tL)^{\beta/2}(I + (tL)^{\beta/2})^{-1} - I\|_{p \to p} < +\infty.$$ 

Set $F(z) = (1 + z)^{\beta/2}(1 + z^{\beta/2})^{-1} - 1$, which can be defined as an analytic function on $\mathbb{C} \setminus (-\infty, 0]$. One checks easily that

$$|F(z)| \leq C \min(|z|^b, |z|^{-b}),$$

where $b = \min(1, \beta/2)$. On the other hand, by Hille-Yosida,

$$(2.11) \quad \|(tL + zI)^{-1}\|_{p \to p} \leq C(-Rez)^{-1},$$

for all $t > 0$ and $z \in \mathbb{C}$ such that $Rez < 0$.

Hence

$$F(tL) = \int_{\Gamma} F(z)(tL + zI)^{-1} \, dz,$$
where the curve $\Gamma$ consists of two half-lines $re^{i\theta_i}, \, r > 0,$ and $\theta_1, \theta_2$ chosen so that $\pi/2 < \theta_1 < \pi$ and $\pi < \theta_2 < 3\pi/2$. Finally, using (2.11),
\[
\|F(tL)\| \leq C \sum_{i=1,2} \int_0^\infty \min(r^b, r^{-b}) \|((tL + re^{i\theta_i})^{-1})_{p\to q} dr
\]
\[
\leq 2C \int_0^\infty \frac{\min(r^b, r^{-b})}{r} dr = C'.
\]
This proves (2.7), and (2.9), (2.10) can be proved in the same way.

Now for the equivalence between $(vE_{p,q})$ and $(vR_{p,q,\beta})$.
\[
(vE_{p,q}) \Rightarrow (vR_{p,q,\beta}). \text{ Note that}
\]
(2.12)  
\[
(I + tL)^{-\beta/2} = \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} e^{-s}\sqrt{s}^{\beta/2-1} e^{-s(tL)} ds,
\]
so that
\[
v_{\sqrt{t}}^{\beta/2} (I + tL)^{-\beta/2} = \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} e^{-s}\sqrt{s}^{\beta/2-1} v_{\sqrt{t}}^{\beta/2} e^{-stL} ds.
\]
Hence
\[
\|v_{\sqrt{t}}^{\beta/2} (I + tL)^{-\beta/2}\|_{p\to q} \leq \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} e^{-s}\sqrt{s}^{\beta/2-1} \|v_{\sqrt{t}}^{\beta/2} e^{-stL}\|_{p\to q} ds
\]
\[
\leq \frac{1}{\Gamma(\beta/2)} \int_0^{+\infty} e^{-s}\sqrt{s}^{\beta/2-1} \|v_{\sqrt{t}}^{\beta/2}\|_{\infty} \|v_{\sqrt{t}}^{\beta/2} e^{-stL}\|_{p\to q} ds.
\]
Using $(D^\infty)$ and assumption $(vE_{p,q})$, we obtain
\[
\|v_{\sqrt{t}}^{\beta/2} (I + tL)^{-\beta/2}\|_{p\to q} \leq \frac{C}{\Gamma(\beta/2)} \int_0^{+\infty} e^{-s}\sqrt{s}^{\beta/2-1} \max \left(1, \frac{1}{s}\right) \right(\frac{p}{p} - \frac{q}{q}\right) ds,
\]
which is finite since $\beta > \kappa_v (\frac{1}{p} - \frac{1}{q})$.

$(vR_{p,q,\beta}) \Rightarrow (vE_{p,q})$. Observe that
\[
\|v_{\sqrt{t}}^{\beta/2} e^{-tL}\|_{p\to q} \leq \|v_{\sqrt{t}}^{\beta/2} (I + tL)^{-\beta/2}\|_{p\to q} \|I + tL\|_{\beta/2} \|e^{-tL}\|_{p\to p}.
\]
Now, according to (2.9),
\[
\|I + tL\|_{\beta/2} \|e^{-tL}\|_{p\to p} \leq C \left(\|e^{-tL}\|_{p\to p} + \|tL\|_{\beta/2} \|e^{-tL}\|_{p\to p}\right),
\]
and the RHS is bounded uniformly in $t > 0$ by bounded analyticity on $L^p(M,\mu)$ of $(e^{-tL})_{t>0}$. This yields the claim.

\[\Box\]

**Theorem 2.3.5.** Let $(M,d,\mu)$ be a metric measure space, $L$ a non-negative self-adjoint operator on $L^2(M,\mu)$, and $v$ a function from $M \times \mathbb{R}_+$ to $\mathbb{R}_+$ satisfying (A), $(D_v)$ and $(D_{v}^\infty)$. Assume that $(M,d,\mu, L)$ satisfies $(DG)$ and that $(e^{-tL})_{t>0}$ is bounded analytic on $L^{p_0}(M,\mu)$ for some $p_0 \in [1,2)$. Then $(DUE_v)$ implies $(GN_{p,q,\beta}^v)$ for all $p,q$ such that $p_0 \leq p < q \leq p_0$ and $\beta$ such that $\beta > (\frac{1}{p} - \frac{1}{q})\kappa_v$, where $\kappa_v$ is as in $(D_{v}^\infty)$. 

Let us now emphasise a consequence of the particular case \( p = 2, q = +\infty \), and \( \beta > \kappa_v/2 \) of Proposition 2.3.4 where we take advantage of the fact that, according to Corollary 2.1.2, \((vE_{2,\infty})\) is equivalent to \((D_{vE})\). This yields a direct characterisation of \((D_{vE})\) in terms of a Gagliardo-Nirenberg inequality (compare with [15] where, in the case where \( v = V \) does not depend on \( x \) and is polynomial in \( r \), no extrapolation is needed for the case \( q = +\infty \)). This result is much easier to obtain than Theorem 1.2.1.

**Corollary 2.3.6.** Let \( \beta > \kappa_v/2 \), where \( \kappa_v \) is as in \((D_{vE})\). Then \((D_{vE})\) is equivalent to

\[
(GN_{2,\infty,\beta}) \quad \| f \sqrt{v_r} \|_\infty \leq C(\| f \|_2 + r^\beta \| L^{\beta/2} f \|_2), \quad \forall r > 0, f \in D_2(L^{\beta/2}).
\]

The drawback of the above result is that it involves a high power of \( L \) in the expression \( \| L^{\beta/2} f \|_2 \), instead of the Dirichlet form \( \mathcal{E} \), which is much easier to handle in applications. For instance, unless \( \kappa_v < 2 \), in which case one can choose \( \beta = 1 \), it is not clear how to see from the Corollary 2.3.6 that \((D_{vE})\) is invariant under quasi-isometry. If one insists, as one should, on taking \( \beta = 1 \), one cannot in general take \( p = 2, q = +\infty \). This is why the implication from \((GN_q)\) to \((D_{vE})\) will require an extrapolation argument.

### 2.4. Converse in the uniform case

Again, let \((M, \mu)\) be a measure space and \( L \) a non-negative self-adjoint operator on \( L^2(M, \mu) \). In this section, we shall study the case where \( v \) does not depend on \( x \in M \), but only on \( r > 0 \). We shall see that in this particular case, if in addition \((e^{-tL})_{t \geq 0}\) is uniformly bounded on \( L^1(M, \mu) \), one can prove the converse of Propositions 2.2.1 and 2.3.1 by using existing arguments, and conclude that \((D_{vE})\), \((N^v)\) and \((GN_q^v)\) for \( q > 2 \) small enough are equivalent. The general case will require new arguments and more assumptions. It will be treated in Sections 4.1 and 4.2.

Let us start with Nash type inequalities. Since they are \( L^1 - L^2 \) inequalities, one can derive \((D_{vE})\) from them without any interpolation argument. Unfortunately, we do not see so far how to implement the argument of Lemma 2.4.2 below in a non-uniform situation. We will consider \((N^v)\), but also \((\tilde{N}^v)\) introduced in Section 2.2 which will enable us to go beyond condition \((D_v)\). The following statement elaborates on [16, Proposition II.1]. Assume for simplicity that \( v \) is one-to-one from \( \mathbb{R}_+ \) onto itself and \( \mathcal{C}^1 \). This excludes for instance the case where \( v = V \) and \( M \) has finite measure, which can probably be also treated with similar methods; we leave the details to the reader. Say that \( v \) satisfies \((*_v)\) if \( U(r) = \log \nu(r) \) is such that

\[
U''(s) \geq \sigma U'(r), \quad \forall r > 0, \forall s \in [r, 2r],
\]

for some \( \sigma > 0 \). Functions \( v(r) = \exp(r^\alpha), r^\alpha, \alpha > 0 \), and many others satisfy \((*_v)\).

**Proposition 2.4.1.** Assume that \((e^{-tL})_{t \geq 0}\) is uniformly bounded on \( L^1(M, \mu) \) and that \( v \) satisfies \((A)\) but does not depend on \( x \in M \). Then, if \( v \) satisfies \((D_v)\), \((N^v)\) implies \((D_{vE})\) and if \( v \) satisfies \((*_v)\), \((\tilde{N}^v)\) implies \((D_{vE})\).

When \( v \) does not depend on \( x \), the \( v \)-Nash inequality \((N^v)\) reads

\[
\| f \|_2^2 \leq C \left( \frac{\| f \|_2^2}{v(r)} + r^2 \mathcal{E}(f) \right), \quad \forall f \in \mathcal{F}, \forall r > 0.
\]
Choosing \( \frac{1}{v(r)} = \frac{\|f\|_2^2}{2C\|f\|_1^2} \), that is \( r = v^{-1} \left( \frac{2C\|f\|_1^2}{\|f\|_2^2} \right) \) yields

\[
\|f\|_2^2 \log \frac{cv(r)}{\|f\|_1^2} \leq r^2 \mathcal{E}(f), \quad \forall r > 0, \ f \in \mathcal{F} \setminus \{0\},
\]

where \( \theta_1(\tau) = \frac{\tau}{2C[v^{-1}(\frac{2C}{\tau})]^2} \). Note that it follows from our assumptions on \( v \) that \( \tau \to \frac{\theta_1(\tau)}{\tau} \) is non-decreasing and continuous.

Similarly, when \( v \) does not depend on \( x \), \((\tilde{N}^v)\) reads

\[
\|f\|_2^2 \log \frac{cv(r)}{\|f\|_1^2} \leq r^2 \mathcal{E}(f), \quad \forall r > 0, \ f \in \mathcal{F} \setminus \{0\},
\]

and can be rewritten as

\[
\|f\|_1^2 \theta_2 \left( \frac{\|f\|_2^2}{\|f\|_1^2} \right) \leq \mathcal{E}(f), \quad \forall f \in \mathcal{F} \setminus \{0\},
\]

where

\[
\theta_2(\tau) = \tau \sup_{r > 0} \log \left( \frac{cv(r)}{r^2} \right).
\]

If \((\tilde{N}^v)\) holds, this supremum has to be finite (this is certainly the case if \( v \) has at most exponential growth in addition to the above assumptions), and under our assumptions on \( v \) it is always positive. Again, note that \( \tau \to \frac{\theta_2(\tau)}{\tau} \) is non-decreasing and continuous. To show continuity at \( \tau_1 > 0 \), let \( r_1 \) be such that \( cv(r_1)\tau_1 = 1/2 \). Then \( \frac{\theta_1(\tau_1)}{\tau_1} = \sup_{r > r_1} \frac{\log(cv(r)\tau_1)}{r^2} \) and, for \( \tau_2 \leq 2\tau_1 \), \( \frac{\theta_2(\tau_2)}{\tau_2} = \sup_{r > r_1} \frac{\log(cv(r)\tau_2)}{r^2} \). Now

\[
\left| \frac{\theta_2(\tau_1)}{\tau_1} - \frac{\theta_2(\tau_2)}{\tau_2} \right| \leq \sup_{r > r_1} \left| \frac{\log(cv(r)\tau_1)}{r^2} - \frac{\log(cv(r)\tau_2)}{r^2} \right| \leq \tau_1^2 \log(\tau_1/\tau_2).
\]

According to \([33, p.414]\), see also \([8, Section 3.3]\), if \( v \) satisfies \((\ast_v)\), then

\[
\theta_2(\tau) \geq \tilde{\theta}_2(\tau) = \frac{\tilde{c} \tau^2 v'(v^{-1}(\frac{1}{\tau}))}{v^{-1}(\frac{1}{\tau})},
\]

where \( \tilde{c} \) depends on \( c \) in \((\tilde{N}^v)\) and on \( \sigma \) in \((\ast_v)\).

Then Nash’s argument as adapted in \([16]\) shows that if \((e^{-tL})_{t > 0}\) is uniformly bounded on \( L^1(M, \mu) \), then inequalities like \((2.13)\) or \((2.14)\) imply a heat kernel upper bound:

**Lemma 2.4.2.** Assume that \((e^{-tL})_{t > 0}\) is uniformly bounded on \( L^1(M, \mu) \) and that

\[
\|f\|_1^2 \theta \left( \frac{\|f\|_2^2}{\|f\|_1^2} \right) \leq \mathcal{E}(f), \quad \forall f \in \mathcal{F} \setminus \{0\},
\]

where \( \theta : \mathbb{R}_+ \to \mathbb{R}_+ \) is continuous and \( \tau \to \frac{\theta(\tau)}{\tau} \) is non-decreasing. Assume that

\[
\int_{0}^{+\infty} \frac{d\tau}{\theta(\tau)} < +\infty.
\]

Then \((DUE^w)\) holds, for \( w \) defined by

\[
w(r) = \frac{1}{A^2 m(r^2/2)}
\]
and

\[ \int_{m(t)}^{+\infty} \frac{d\tau}{\theta(\tau)} = 2t, \]

where \( A = \sup_{t>0} \|e^{-tL}\|_{1\to1} < +\infty. \)

**Proof.** Substitute \( e^{-tL}f \) to \( f \) in (2.15). Use the fact that

\[ \frac{\|e^{-tL}f\|_2^2}{\|e^{-tL}f\|_1^2} \geq \frac{\|e^{-tL}f\|_2^2}{A^2\|f\|_1^2}, \]

and that the function \( \tau \to \frac{\theta(\tau)}{\tau} \) is non-decreasing. It follows that

\[ A^2\|f\|_1^2 \theta \left( \frac{\|e^{-tL}f\|_2^2}{A^2\|f\|_1^2} \right) \leq E(e^{-tL}f), \quad \forall f \in F \setminus \{0\}, \ t > 0. \]

Set \( u(t) = \frac{\|e^{-tL}f\|_2^2}{A^2\|f\|_1^2} \). Since \( \frac{d}{dt}\|e^{-tL}f\|_2^2 = -2E(e^{-tL}f) \), (2.18) becomes

\[ \theta(u(t)) \leq -\frac{u'(t)}{2}, \ t > 0, \]

hence

\[ \int_{u(t)}^{u(0)} \frac{d\tau}{\theta(\tau)} \geq 2t, \ t > 0. \]

Define \( m(t) \) by

\[ \int_{m(t)}^{+\infty} \frac{d\tau}{\theta(\tau)} = 2t. \]

It follows from (2.19) and (2.20) that \( u(t) \leq m(t) \), that is

\[ \|e^{-tL}f\|_2^2 \leq A^2m(t)\|f\|_1^2, \]

in other words

\[ \|e^{-tL}\|_{1\to2} \leq A\sqrt{m(t)}. \]

By duality \( \|e^{-tL}\|_{2\to\infty} \leq A\sqrt{m(t)} \), hence by writing

\[ \|e^{-tL}f\|_{1\to\infty} \leq \|e^{-(t/2)L}\|_{2\to\infty}\|e^{-(t/2)L}\|_{1\to2} \]

it follows that

\[ \|e^{-tL}\|_{1\to\infty} \leq A^2m(t/2) = \frac{1}{w(\sqrt{t})}. \]

\[ \square \]

**Lemma 2.4.3.** Functions \( \theta_1 \) and \( \theta_2 \) satisfy the assumptions of Lemma 2.4.2. If \( w_1 \) and \( w_2 \) are the associated functions defined via (2.16) and (2.17), then, if \( v \) is doubling, there exists \( C > 0 \) such that \( w_1(r) \geq Cv(r), \ \forall r > 0 \), and, if \( v \) satisfies \((\ast_v)\), there exist \( C, c > 0 \) such that \( w_2(r) \geq Cv(cr), \ \forall r > 0 \). In particular, under these assumptions, \((DUE^{w_1})\), resp. \((DUE^{w_2})\), implies \((DUE^v)\).
Proof. We have already observed that the functions $\tau \to \frac{\theta_i(\tau)}{\tau}$, $i = 1, 2$, are nondecreasing and continuous. The computations below will prove that $\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_i(\tau)} < +\infty$, $i = 1, 2$. This gives the first assertion of the lemma.

As for the second assertion, let us start with $\theta_2$ which is simpler to treat. Changing variables $\tau = \frac{1}{v(r)}$ in the expression of $\tilde{\theta}_2$ yields

$$\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_2(\tau)} = \frac{1}{c} \int_0^{\sqrt{t}} r \, dr = \frac{t}{2c}$$

therefore

$$\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_2(\tau)} \leq \int_{+\infty}^{+\infty} \frac{d\tau}{\theta_2(\tau)} \leq 2t = \int_{m_2(t)}^{+\infty} \frac{d\tau}{\theta_2(\tau)}.$$

The first inequality follows from the comparison between $\theta_2$ and $\tilde{\theta}_2$, the second one from (2.21), and the equality is the definition of $m_2$ (with obvious notation). From

$$\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_2(\tau)} \leq \int_{m_2(t)}^{+\infty} \frac{d\tau}{\theta_2(\tau)}$$

it follows that

$$m_2(t) \leq \frac{1}{v(\sqrt{4ct})},$$

thus

$$\frac{1}{w_2(\sqrt{t})} = A^2 m_2(t/2) \leq \frac{A^2}{v(\sqrt{2ct})}.$$

Now for $\theta_1$. We are going to use a trick from [8]. Consider $\tilde{v}(r) = \frac{2}{r} \int_{r/2}^{r} v(s) \, ds$. Clearly,

$$v(r/2) \leq \tilde{v}(r) \leq v(r),$$

and by $(D_v)$, $\tilde{v}(r)$ and $v(r)$ are within multiplicative constants. One can check by a simple calculation (see [8, Lemma 2.1]) that $\tilde{v}$ is also one-to-one. Define $\tilde{\theta}_1(\tau) := \frac{\tau}{[\tilde{v}(\tau/4)]^{1/2}}$. Again, $\tilde{\theta}_1$ and $\theta_1$ are uniformly comparable.

By the change of variables $\tau = \frac{1}{v(r)}$,

$$\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_1(\tau)} = \int_0^{\sqrt{t}} \frac{\tilde{v}'(r) r^2}{\tilde{v}(r)} \, dr.$$

But again by calculations similar to the ones in [8, Lemma 2.1],

$$\frac{\tilde{v}'(r)}{\tilde{v}(r)} \leq \frac{C}{r},$$

therefore

$$\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_1(\tau)} \leq \frac{Ct}{2},$$

hence

$$\int_{+\infty}^{+\infty} \frac{d\tau}{\theta_1(\tau)} \leq C't.$$
\[ \int_{\tau}^{+\infty} \frac{d\tau}{\theta_1(\tau)} \leq 2t = \int_{m_1(t)}^{+\infty} \frac{d\tau}{\theta_1(\tau)}, \]

thus

\[ m_1(t) \leq \frac{1}{v(\sqrt{2t/C'})}, \]

and

\[ \frac{1}{w_1(\sqrt{t})} = A^2 m_1(t/2) \leq \frac{A^2}{v(\sqrt{t/C'})} \leq \frac{A'}{v(\sqrt{t})}, \]

where we use \((D_v)\) in the last inequality.

\[ \square \]

Lemmas \ref{lem:1} and \ref{lem:2} together yield Proposition \ref{prop:1}.

Consider now \((GN^v_q)\) for some \(q > 2\) and assume \((D_v)\). When \(v\) does not depend on \(x\), \((GN^v_q)\) reads:

\[ v^{1-\frac{2}{q}}(r) \|f\|_q^2 \leq C \left( \|f\|_2^2 + r^2 \mathcal{E}(f) \right), \quad \forall f \in \mathcal{F}, \quad \forall r > 0, \]

that is

\[ \|f\|_q^2 \leq \frac{1}{\|f\|_2^2} \left( \frac{1}{v^{1-\frac{2}{q}}(r)} + \frac{r^2 \mathcal{E}(f)}{v^{1-\frac{2}{q}}(r) \|f\|_2^2} \right), \quad \forall f \in \mathcal{F} \setminus \{0\}, \quad \forall r > 0, \]

or

\[ \frac{\|f\|_q^2}{\|f\|_2^2} \leq K \left( \mathcal{E}(f) \|f\|_2^2 \right), \quad \forall f \in \mathcal{F} \setminus \{0\}, \]

where

\[ K(s) = C \inf_{r > 0} \frac{1 + r^2 s}{v^{1-\frac{2}{q}}(r)}. \]

Note that if \(v\) satisfies \((D_{\kappa_v}^{\infty})\) then \(v(r) \leq C \sqrt{v(1)r^{\kappa_v}}\), \(r \geq 1\), and if \(q\) is such that \(q-2\kappa_v < 2\), \(\frac{r^2}{v^{1-\frac{2}{q}}(r)} \to +\infty\) as \(r \to +\infty\), and \(K(s)\) is positive and finite for every \(s > 0\). One checks easily that \(K\) is one-to-one from \(\mathbb{R}_+\) into itself. Finally \((GN^v_q)\) can be written in the more concise form

\[ \|f\|_2 \eta \left( \frac{\|f\|_q^2}{\|f\|_2^2} \right) \leq \mathcal{E}(f), \quad \forall f \in \mathcal{F} \setminus \{0\}, \]

where \(\eta(\tau) := K^{-1}(\tau)\). If \(v\) is doubling, then choosing \(\tau = r^2\) yields

\[ \eta(\tau) \geq \frac{c}{(\tau^{-1/2}(\tau^{-C}/2)^{\frac{1}{2}}).} \]

This yields a more general version of the inequalities in \cite{15}.

To go back from \((GN^v_q)\) to \((D_{\kappa_v}^{\infty})\), we will use the equivalence between \((GN^v_q)\) and \((vE_{2,q})\), which does not require the above optimisation, and we will extrapolate from \((vE_{2,q})\) to \((vE_{2,\infty})\), which will require a uniform boundedness assumption on \(L^1\). Indeed, the extrapolation lemma \cite[Lemma 1]{14} can be extended to the situation where the decay of the semigroup is governed by a doubling function of time instead of a power function.
Proposition 2.4.4 ([23], Lemma 1.3). Assume that \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\). Let \(w\) be a non-decreasing positive function on \((0, +\infty)\) satisfying the doubling condition \((D_w)\). If there exist \(1 \leq p < q \leq +\infty\) such that:

\[
(2.22) \quad \|e^{-tL}\|_{p \to q} \leq \frac{1}{w(t)}, \forall t > 0,
\]

then there exists a constant \(C\) such that

\[
\|e^{-tL}\|_{1 \to \infty} \leq \frac{C}{w^\alpha(t)}, \forall t > 0, \quad \text{with} \quad \alpha = 1/(1/p - 1/q).
\]

Proposition 2.4.5. Assume that \(v : \mathbb{R}_+ \to \mathbb{R}_+\) satisfies \((A)\) and \((D_v)\) but does not depend on \(x \in M\) and that \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\). Then, for all \(q\) such that \(2 < q \leq +\infty\), \((GN^v_q)\) implies \((DUE^v)\).

Proof. By Proposition 2.3.2, \((GN^v_q)\) implies \((vE_{2,q})\). Now, since \(v\) does not depend on \(x\), \((vE_{2,q})\) is equivalent to \((2.22)\) with \(p = 2\) and \(w(t) = cv^{1/2 - 1/4(\sqrt{t})}, c > 0\). Note that \(w\) satisfies \((D_w)\) since \(v\) satisfies \((D_v)\). Proposition 2.4.4 then yields

\[
\|e^{-tL}\|_{1 \to \infty} \leq \frac{C''}{v(\sqrt{t})}, \forall t > 0,
\]

which is obviously equivalent to \((DUE^v)\). \(\square\)

In Proposition 4.1.11 below, we shall be able to drop the assumption of independence on \(x\). And this will rely on an adapted extrapolation result, namely Proposition 4.1.9 below, which will require the use of new ingredients.

3. Local and Global Inequalities

The section will be devoted to a closer study of the relationship between on the one hand global inequalities such as \((GN^v_q)\) and \((N^v)\) and on the other hand local inequalities like \((KN^v_q)\), \((KN^v)\), and \((LN^v)\), and also various forms of relative Faber-Krahn inequalities. More precisely, in Sections 3.1 and 3.2, we are going to see that conditions \((GN^v_q)\), \((KN^v_q)\), and \((LS^v_q)\), (resp. \((N^v)\), \((KN^v)\), and \((LN^v)\)) are equivalent and that the first group implies the second one. In Section 3.3 we shall establish the link with various versions of Faber-Krahn inequalities. In Section 3.4 we shall see in a systematic way that in the case where \((M, d, \mu)\) is non-compact and connected, the so-called reverse doubling property enables one to get rid of (or not to introduce) of a certain local term in Nash and Faber-Krahn type inequalities.

From Section 3.2 on, we shall work in the setting of a metric measure space endowed with a strongly local regular Dirichlet form together with a proper distance, as described for instance in [41] or [56].

3.1. Gagliardo-Nirenberg implies Nash and global implies local. We will start by showing that the implications in the following diagram hold:

\[
\begin{array}{c}
(GN^v_q) \iff (KN^v_q) \iff (LS^v_q) \\
\iff \iff \\
(N^v) \iff (KN^v) \iff (LN^v)
\end{array}
\]
The inequalities in the two first columns can be formulated on any measure space \((M, \mu)\) endowed with a non-negative self-adjoint operator \(L\). The ones in the last column require in addition \(M\) to be endowed with a distance \(d\). Let us first consider the vertical implications. Note that they do not require \((D_v)\) or \([D_v']\).

**Proposition 3.1.1.** Let \((M, \mu)\) be a measure space, \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) and let \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfy \((A)\). For any \(q > 2\), \((GN_q^v)\) implies \((N^v)\) and \((KGN_q^v)\) implies \((KN^v)\). If in addition \(M\) is endowed with a metric, \((LS_q^v)\) implies \((LN^v)\).

**Proof.** Let \(q > 2\). Let \(\theta \in [0, 1]\) be such that \(\frac{1}{2} = \frac{\theta}{q} + (1 - \theta)\). By Hölder’s inequality,

\[
\|f\|_2 \leq \|fv_r^{-\frac{1}{r}}\|^\theta \|fv_r^{-1/2}\|^{1-\theta}.
\]

Hence \((GN_q^v)\) yields

\[
\|f\|_2^2 \leq C \left(\|f\|_2 + r^2 \mathcal{E}(f)\right)^\theta \|fv_r^{-1/2}\|_1^{2(1-\theta)}
\]

\[
\leq C \varepsilon \left(\|f\|_2^2 + r^2 \mathcal{E}(f)\right) + C \varepsilon^{-\frac{\theta}{1-\theta}} \|fv_r^{-1/2}\|_1^2,
\]

for all \(r, \varepsilon > 0\), \(f \in \mathcal{F}\). Choosing \(\varepsilon = \frac{1}{2r^2}\) proves the first assertion of the proposition. The second one can be proved in a similar way. The last one again follows directly from Hölder’s inequality. 

**Remark 3.1.2.** According to [7], \((LS_q^v)\) and \((LN^v)\) are actually equivalent if the quadratic form associated with \(L\) is a Dirichlet form. It will follow from Propositions 3.2.2 and 3.2.3 that \((GN_q^v)\) and \((N^v)\) are also equivalent, at least in the setting of metric measure spaces endowed with a strongly local regular Dirichlet form and a proper intrinsic distance. See also Section 4.2 for a slightly more general setting where all these inequalities happen to be equivalent.

**Remark 3.1.3.** The same argument as in Proposition 3.1.1 shows more generally that, for all \(1 \leq \tilde{p} < p < q < +\infty\) and \(\beta > 0\), \((GN_{p,q}^v)\) (see Proposition 2.3.4) implies \((N_{p,p,\beta}^v)\) (see Remark 2.2.3).

Now for the horizontal implications in the above diagram. On the top line, we have already noticed that both implications were obvious, and on the bottom line, that the first was obvious. From the first column to the second one, we need not assume \((D_v)\) or \([D_v']\). Again, \((M, \mu)\) need not be endowed with a metric. To formulate the localised inequalities \((LS_q^v)\) and \((LN^v)\), one does need a metric \(d\). Then the implication from \((KGN_q^v)\) to \((LS_q^v)\) is obvious if one assumes \([D_v']\). To complete the above diagram, it remains to prove:

**Proposition 3.1.4.** Let \((M, d, \mu)\) be a metric measure space, \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) and let \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfy \((A)\), \((D_v)\), and \([D_v']\). Then \((KN^v)\) implies \((LN_{2/\kappa_v}^v)\), where \(\kappa_v\) is as in \([D_v']\).

**Proof.** Write \((KN^v)\):

\[
(3.1) \quad \|f\|_2^2 \leq C \left( \inf_{z \in \text{supp}(f)} \frac{\|f\|_2^2}{v_s(z)} + s^2 \mathcal{E}(f) \right), \quad \forall s > 0, f \in \mathcal{F}.
\]
Now consider a ball $B = B(x, r)$ and let $f \in \mathcal{F}_c(B)$. Since $\text{supp}(f) \subset B$, \(\mathcal{D}_v\) implies

\[v_r(x) \leq C \inf_{z \in \text{supp}(f)} v_r(z), \quad \forall \ r > 0.\]

For $r \geq s$, \(\mathcal{D}_v\) implies

\[
\inf_{z \in \text{supp}(f)} v_r(z) \leq C \left( \frac{r}{s} \right)^{\kappa_v} \inf_{z \in \text{supp}(f)} v_s(z).
\]

Gathering these two estimates yields, for $x \in M$, $r \geq s > 0$, $f \in \mathcal{F}_c(B(x, r))$,

\[
\frac{1}{\inf_{z \in \text{supp}(f)} v_s(z)} \leq C \left( \frac{r}{s} \right)^{\kappa_v} v_r(x).
\]

Thus (3.1) implies

\[
\int_B |f|^2 d\mu \leq C \left( \left( \frac{r}{s} \right)^{\kappa_v} \left( \int_B |f| d\mu \right)^2 + s^2 \mathcal{E}(f) \right)
\]

if $r \geq s > 0$. In order to obtain an inequality which is also valid for $s \geq r > 0$, it enough to add a term $\frac{s^2}{r^2} \int_B |f|^2 d\mu$ in the RHS:

\[
\int_B |f|^2 d\mu \leq C \left( \frac{r}{s}^{\kappa_v} \left( \int_B |f| d\mu \right)^2 + s^2 \left( \mathcal{E}(f) + r^{-2} \int_B |f|^2 d\mu \right) \right).
\]

Now taking the infimum in $s > 0$ yields

\[
\left( \int_B |f|^2 d\mu \right)^{\frac{2}{\kappa_v} + 1} \leq \frac{C r^2}{v_r^2(x)} \left( \int_B |f| d\mu \right)^{4/\kappa_v} \left( \mathcal{E}(f) + r^{-2} \int_B |f|^2 d\mu \right),
\]

for all $x \in M$, $r > 0$, $f \in \mathcal{F}_c(B(x, r))$, that is, $(LN^\alpha)$ with $\alpha = 2/\kappa_v$. \(\square\)

One may wonder why the implication from $(KN^v_q)$ to $(LS^v_q)$ is direct, as we have seen already in Section 1.2, whereas the one from $(KN^v)$ to $(LN^\alpha)$ requires first the consideration of two different values $r$ and $s$ respectively for the radius of the ball and the parameter in the inequality, then the use of $(D_v)$ and an optimisation. There are two answers to this question and both are interesting.

The first one is that we could perform a similar optimisation on $(KN^v_q)$. With the notation of Section 3.4, assume $(RD_v)$ and write

\[
v_{r-\frac{2}{q}}(x) w(r, s)^{\kappa_v (\frac{2}{q} - 1)} \left( \int_B |f|^q d\mu \right)^{\frac{2}{q}} \leq C \left( \int_B |f|^2 d\mu + s^2 \mathcal{E}(f) \right),
\]

for all $x \in M$, $r, s > 0$, $f \in \mathcal{F}_c(B)$, $B = B(x, r)$. Then choose $s$ such that

\[s^2 \mathcal{E}(f) = \int_B |f|^2 d\mu.
\]

One obtains a formally stronger form of $(LS^v_q)$ which should be in fact equivalent to $(LS^v_q)$ by the methods of \[7\]. We leave the details to the reader.
The second answer is that one could also perform a similar optimisation already at the level of \((N^v)\) by writing
\[
\|f_{v_s}^{-1/2}\|^2_1 + s^2\mathcal{E}(f) \leq C \left( \left( \frac{L}{s} \right)^{\kappa_v} \|f_{v_r}^{-1/2}\|^2_1 + s^2 \left( \mathcal{E}(f) + r^{-2}\|f\|^2_2 \right) \right),
\]
and improve this inequality into
\[
\|f\|_2^{2(1+\alpha)} \leq C \|f_{v_r}^{-1/2}\|^2_1 \left( \|f\|^2_2 + r^2\mathcal{E}(f) \right), \quad \forall r > 0, f \in \mathcal{F}
\]
for some \(\alpha > 0\) depending on the constant in \((VD)\). The implication from the latter inequality to \((LN^v)\) is then obvious by restricting oneself to functions supported in balls and using \((D\mu)\).

3.2. From local to global. In this section we show that, in the setting of a doubling measure space endowed with a strongly local and regular Dirichlet form and a proper distance, one can go back from the family of localised inequalities \((LN^v)\) (resp. \((LS^v_q)\)) to the global inequality \((N^v)\) (resp. \((GN^v_q)\)). We are grateful to Gilles Carron ([12]) for this observation.

Recall that in this framework, it is well-known that \((LN)\), or \((LS_q)\), implies \((DUE)\) (see [56] or [51, Section 5.2]). Together with Proposition 2.3.1, the results in the present section therefore give a short-cut to Theorem 1.2.1 in the case \(v = V\). Remember however that one of our main goals is precisely to give an alternative and more general approach to the above equivalences from [56] or [51].

Indeed, later in Section 4, we will see, in a slightly more general setting, first that the strongest global inequality \((GN^v_q)\) is equivalent to \((DUE^v)\), second that the weakest of the local inequalities, namely \((LN^v)\), implies back \((GN^v_q)\). In particular, the local and the global inequalities are all equivalent. The current section is nevertheless important for clarity, since we will see this equivalence directly without going through the machinery of Section 4.

We will use the setting introduced for instance in [11, 2.2] (for more information see also [55, 56, 41, 3], and [52, Section 3]). We shall only recall the basic notions and notations. Let \((M, \mu)\) be a locally compact separable measure space endowed with a Borel measure \(\mu\) which is finite on compact sets and strictly positive on non-empty open sets. Let \(L\) be a non-negative self-adjoint operator on \(L^2(M, \mu)\) and \(\mathcal{E}\) the associated quadratic form with domain \(\mathcal{F}\). Assume that \(\mathcal{E}\) is a strongly local and regular Dirichlet form (see [31] for definitions). In particular, \((e^{-tL})_{t>0}\) is a submarkovian semigroup, that is \(0 \leq e^{-tL}f \leq 1\) if \(0 \leq f \leq 1\). Let \(d\Gamma\) be the energy measure associated with \(\mathcal{E}\), that is
\[
\mathcal{E}(f, g) = \int_M d\Gamma(f, g)
\]
for all \(f, g \in \mathcal{F}\).

Then \(d\Gamma\) satisfies a Leibniz rule (see [31, Lemma 3.2.5] or [1, p.?]), which yields the following inequality between measures
\[
d\Gamma(fg, fg) \leq 2 \left( f^2d\Gamma(g, g) + g^2d\Gamma(f, f) \right)
\]
for all \(f, g \in \mathcal{F} \cap L^\infty(M, \mu)\), or rather their quasi-continuous versions (see for instance [11, Lemma 2.5]).
Define now the intrinsic quasi-metric:

\[(3.3) \quad d(x, y) = \sup \{ f(x) - f(y); f \in \mathcal{F} \cap C_0(M) \text{ s.t. } d\Gamma(f, f) \leq d\mu \}.\]

Here \( C_0(M) \) denotes the space of continuous functions on \( M \) which vanish at infinity. Assume that \( d \) is finite everywhere, separates points, and defines the original topology on \( M \); assume also that the metric space \((M, d)\) is complete. According to [55, Lemma 1'] (see also [41, Theorem 2.11]), \( \forall x \in M, d_x \in \mathcal{F}_{loc} \), where \( d_x(y) := d(x, y) \), (see [41, Definition 2.3] for a precise definition of \( \mathcal{F}_{loc} \)) and

\[d\Gamma(d_x, d_x) \leq d\mu.\]

In fact, definition (3.3) is not essential, as long as one has the latter properties. In other words, we could consider any distance \( d \) on \((M, \mu)\) defining the original topology, such that \( (M, d) \) is complete and

\[(3.4) \quad d_x \in \mathcal{F}_{loc}, \quad d\Gamma(d_x, d_x) \leq d\mu, \quad \forall x \in M.\]

It follows that the balls in \( M \) are relatively compact (see [39, footnote 4, p. 1215]).

In the sequel, let us say that \((M, d, \mu, L)\) satisfies \((H)\) if \((M, d, \mu)\) is a locally compact separable and complete metric measure space endowed with a Borel measure \( \mu \) which is finite on compact sets and strictly positive on non-empty open sets, \( L \) is a non-negative self-adjoint operator on \( L^2(M, \mu) \) and the associated quadratic form \( \mathcal{E} \) with domain \( \mathcal{F} \) is Dirichlet, strongly local and regular, and if \( d \) satisfies (3.4), where \( d\Gamma \) is the energy measure associated with \( \mathcal{E} \).

**Lemma 3.2.1.** Assume that \((M, d, \mu, L)\) satisfies \((H)\). For \( x \in M \) and \( r > 0 \), define

\[
\rho(y) = \rho_x^{r, \varepsilon}(y) := \left( 1 - \frac{d(y, B(x, r - 2\varepsilon))}{\varepsilon} \right)_+.\]

Then for all \( g \in \mathcal{F}, g\rho \in \mathcal{F}_c(B(x, r)) \) and

\[
\mathcal{E}(g\rho) \leq 2 \varepsilon^{2} \int_{B(x,r)} g^2 \, d\mu + 2 \int_{B(x,r)} d\Gamma(g, g).\]

**Proof.** Let us first observe that \( \rho \) is supported in \( B(x, r - \varepsilon) \) and that \( \rho \equiv 1 \) on \( B(x, r - 2\varepsilon) \). According to [56, Lemma 1.9] (see also [41, Theorem 2.11]), \( \rho \in \mathcal{F} \) and

\[d\Gamma(\rho, \rho) \leq \frac{1}{\varepsilon^2} d\mu,\]

and in fact, due to the local character of \( \mathcal{E} \) (see [31, Corollary 3.2.1, p.115]),

\[d\Gamma(\rho, \rho) \leq \frac{1}{\varepsilon^2} \chi_{B(x,r)} \, d\mu.\]

Using (3.2), \( g\rho \in \mathcal{F} \) and

\[
\mathcal{E}(g\rho) = \int_M d\Gamma(g\rho, g\rho) \leq 2 \left( \int_M g^2 d\Gamma(\rho, \rho) + \int_M \rho^2 d\Gamma(g, g) \right) \leq 2 \left( \frac{1}{\varepsilon^2} \int_{B(x,r)} g^2 \, d\mu + \int_{B(x,r)} d\Gamma(g, g) \right).
\]

\(\blacksquare\)
Proposition 3.2.2. Assume that $(M, d, \mu, L)$ satisfies $(H)$ and $(VD)$ and that $v$ satisfies $(D_i)$. Then conditions $(N^v)$, $(KN^v)$, and $(LN^v)$ are equivalent.

Proof. Given the considerations in Section 3.1, it only remains to prove that $(LN^v)$ implies $(N^v)$. Assume $(LN^v)$, that is

$$\|f\|_{2}^{2(1+\alpha)} \leq \frac{C}{\nu_0^2(x)} \|f\|_{1}^{2\alpha} (r^2 \mathcal{E}(f) + \|f\|_{2}^2),$$

for every ball $B = B(x, r)$, for every $f \in \mathcal{F}(B)$, and for some $\alpha, C > 0$. Using $(D_i)$, this can be rewritten as

$$\|f\|_{2}^{2(1+\alpha)} \leq C \|f v_r^{-1/2}\|_{1}^{2\alpha} (r^2 \mathcal{E}(f) + \|f\|_{2}^2),$$

hence, for all $\varepsilon > 0$,

$$\|f\|_{2}^2 \leq C\varepsilon^{-1/\alpha} \|f v_r^{-1/2}\|_{1}^2 + \varepsilon (r^2 \mathcal{E}(f) + \|f\|_{2}^2).$$

Invoking $(BCP)$, consider a covering of $M$ by balls $B(x_i, r/2)$, $i \in I$, such that the balls $B(x_i, r/4)$ are pairwise disjoint. Recall that $K(x) = \# \{i \in I, x \in B(x_i, r)\} \leq K_0$, where $K_0$ only depends on the constant in $(VD)$. Define cut-off functions $\rho_i$ by

$$\rho_i(x) := \left(1 - \frac{4d(x, B(x_i, r/2))}{r}\right)_+,$$

that is, in the notation of Lemma 3.2.1, $\rho_i = \rho_{x_i,r/4}^r$. Let $g \in \mathcal{F}$. Since $\rho_i \equiv 1$ on $B(x_i, r/2)$, one can write

$$\|g\|_{2}^2 \leq \sum_i \|g\rho_i\|_{2}^2.$$

Since $g\rho_i \in \mathcal{F}(B(x_i, r))$ by Lemma 3.2.1, one can apply (3.5) to each $g\rho_i$ in $B(x_i, r)$. It follows

$$\|g\|_{2}^2 \leq C\varepsilon^{-1/\alpha} \sum_i \|g\rho_i v_r^{-1/2}\|_{1}^2 + \varepsilon r^2 \sum_i \mathcal{E}(g\rho_i) + \varepsilon \sum_i \|g\rho_i\|_{2}^2. \tag{3.6}$$

Now

$$\sum_i \|g\rho_i v_r^{-1/2}\|_{1}^2 \leq \left( \sum_i \|g\rho_i v_r^{-1/2}\|_{1} \right)^2 \leq K_0^2 \|g v_r^{-1/2}\|_{1}^2. \tag{3.7}$$

On the other hand, by Lemma 3.2.1 $g\rho_i \in \mathcal{F}$ and

$$\mathcal{E}(g\rho_i) \leq \frac{32}{r^2} \int_{B(x_i, r)} g^2 d\mu + 2 \int_{B(x_i, r)} d\Gamma(g, g),$$

hence

$$\sum_i \mathcal{E}(g\rho_i) \leq \frac{32K_0}{r^2} \|g\|_{2}^2 + 2K_0 \mathcal{E}(g). \tag{3.8}$$

and finally

$$\sum_i \|g\rho_i\|_{2}^2 \leq K_0 \|g\|_{2}^2. \tag{3.9}$$

Gathering (3.6), (3.7), (3.8) and (3.9), one obtains

$$\|g\|_{2}^2 \leq C\varepsilon^{-1/\alpha} K_0^2 \|g v_r^{-1/2}\|_{1}^2 + 33K_0\varepsilon \|g\|_{2}^2 + 2K_0\varepsilon r^2 \mathcal{E}(g).$$
Choosing $\varepsilon = \frac{1}{66K_0}$ yields $(N^w)$.

\( \square \)

**Proposition 3.2.3.** Assume that $(M, d, \mu, L)$ satisfies $(H)$ and $(VD)$ and that $v$ satisfies $(D^L_v)$. For all $q > 2$, conditions $(GN^w_q)$, $(KGN^w_q)$, and $(LS^w_q)$ are equivalent.

**Proof.** It only remains to prove that $(LS^w_q)$ implies $(GN^w_q)$. Assume $(LS^w_q)$, that is

$$\|f\|_q^2 \leq C \left( \|f\|_2^2 + r^2 \mathcal{E}(f) \right)$$

for every ball $B = B(x, r)$, for every $f \in \mathcal{F}_c(B)$, and for some $C > 0$. Using $(D^L_v)$, this can be rewritten as

\[(3.10) \quad \left\| \frac{1}{r^{1-q}} f \right\|_q^2 \leq C \left( \|f\|_2^2 + r^2 \mathcal{E}(f) \right).\]

Consider the $x_i$ and $\rho_i$ as before. Let $g \in \mathcal{F}$. Write

$$\left\| \frac{1}{r^{1-q}} g \right\|_q^q \leq \sum_i \left\| \frac{1}{r^{1-q}} g \rho_i \right\|_q^q$$

and apply $(3.10)$ to each $g \rho_i$ in $B(x_i, r)$. It follows

$$\left\| \frac{1}{r^{1-q}} g \right\|_q^q \leq C \left( \sum_i \|g \rho_i\|_2^2 + r^2 \sum_i \mathcal{E}(g \rho_i) \right),$$

and one concludes by using $(3.9)$ and $(3.8)$.

\( \square \)

A similar argument as in Propositions 3.2.2 and 3.2.3 can also be applied if one substracts from an operator $L$ satisfying the above assumptions a strongly positive potential $V$. Let $(M, \mu, L)$ be as above, $\mathcal{E}$ the associated Dirichlet form, and let $V$ be a positive function on $M$. Following [22], we shall say that $L - V$ is strongly positive (or strongly subcritical in the terminology of [24]) if there exists $0 < \varepsilon < 1$ such that

\[(3.11) \quad (1 - \varepsilon) \mathcal{E}(f) \geq \|V^{1/2} f\|_2^2.\]

It follows that $L - V$ is well-defined as an operator with dense domain on $L^2(M, \mu)$. Indeed, according to $(3.11)$, the quadratic form

$$\mathcal{E}_V(f) := \langle Lf - Vf, f \rangle = \mathcal{E}(f) - \|V^{1/2} f\|_2^2$$

satisfies

\[(3.12) \quad \varepsilon \mathcal{E} \leq \mathcal{E}_V \leq \mathcal{E}\]

and is defined on the domain $\mathcal{F}$ of $\mathcal{E}$.

The semigroup generated by $L - V$ is not necessarily submarkovian and possibly does not act on the whole range of $L^p$ spaces. As a matter of fact, $\mathcal{E}_V$ is no more a Dirichlet form in general and even when it is one, it is not strongly local but only local. In any case, we cannot apply directly Propositions 3.2.2 and 3.2.3. However,
one can consider again the family \( \{ \rho_i \}_{i=1}^\infty \) introduced in the proof of Proposition 3.2.2. Then

\[
\sum_i \mathcal{E}_V(g\rho_i) = \sum_i \mathcal{E}(g\rho_i) - \sum_i \| \nabla^{1/2} g\rho_i \|_2^2 \\
\leq \sum_i \mathcal{E}(g\rho_i) \\
\leq \frac{32K_0}{r^2} \| g \|_2^2 + 2K_0 \mathcal{E}(g) \\
\leq \frac{32K_0}{r^2} \| g \|_2^2 + \frac{2}{\varepsilon} K_0 \mathcal{E}_V(g).
\]

The one before last inequality is (3.8) and the last one follows from the first inequality in (3.12). Again the same argument as above can then be used to show that conditions \( (GN_q^v), (KGN_q^v), \) and \( (LS_q^v) \) associated with \( \mathcal{E}_V \) are equivalent.

A cousin of inequality \( (LS_q^v) \) was introduced in [4, Proposition 2.1 (ii)] in the setting of a metric measure space \( (M, d, \mu) \) endowed with a non-negative self-adjoint operator \( L \), namely

\[
(3.13) \quad \| \chi_{B(x,r)} f \|_q^2 \leq \frac{C}{V_r^{1-2/q}} (\| f \|_2^2 + r^2 \mathcal{E}(f)),
\]

for all \( x \in M, r > 0, f \in \mathcal{F} \). It is shown there that (3.13) implies a localised version of \( (VE_{2,q}) \). Note that restricting (3.13) to \( \mathcal{F}_c(B(x,r)) \) yields \( (LS_q) \). Now Proposition 3.2.3 says that if \( \mathcal{E} \) is strongly local and regular and \( (M, d, \mu) \) satisfies \( (VD) \), \( (LS_q) \) implies \( (GN_q) \) which implies the full \( (VE_{2,q}) \) by Proposition 2.3.2. This yields an improvement of [4, Proposition 2.1] in that situation, as well as an extension to the case where \( v \neq V \).

### 3.3. Nash and Faber-Krahn

In the setting of Riemannian manifolds, \( (DUE) \) has been characterised by Grigor’yan in [33] in terms of relative Faber-Krahn inequalities. These methods also work in the setting of strongly local and regular Dirichlet forms as in Section 3.2. Our aim in this section is to extend this characterisation to \( (DUE_v) \) for \( v \neq V \). More precisely, we are going to make the connection between suitable versions of relative Faber-Krahn inequalities and localised \( v \)-Nash inequalities. Together with Proposition 4.2.6 below, this will establish the connection between \( (DUE_v) \) and these relative Faber-Krahn inequalities. For any open subset \( \Omega \) of a topological measure space \( M \) endowed with a closed non-negative quadratic form \( \mathcal{E} \), set

\[
(3.14) \quad \lambda_1(\Omega) := \inf \left\{ \frac{\mathcal{E}(f)}{\| f \|_2^2}, f \in \mathcal{F}_c(\Omega) \text{ and } f \neq 0 \right\}.
\]

Of course one could replace \( \mathcal{F}_c(\Omega) \) with its closure in \( \mathcal{F} \) for the norm \( \mathcal{E}(f) + \| f \|_2^2 \) without changing anything. This definition is the one used for instance in [11]. More interestingly, if \( (M, d, \mu, L) \) satisfies \( (H) \), the above definition is also equivalent to the one in [37], namely

\[
(3.15) \quad \lambda_1(\Omega) := \inf \left\{ \frac{\mathcal{E}(f)}{\| f \|_2^2}, f \in \mathcal{F} \cap C_c(\Omega) \text{ and } f \neq 0 \right\},
\]
where $C_c(\Omega)$ is the space of continuous functions that are compactly supported in $\Omega$ (see Lemma 3.3.1 below). As above, one can replace $\mathcal{F} \cap C_c(\Omega)$ with its closure in $\mathcal{F}$.

The classical Faber-Krahn theorem says that, for any open set $\Omega$ of $\mathbb{R}^n$, we have

$$\lambda_1(\Omega) \geq c_n \mu(\Omega)^{-2/n}.$$ We shall say that $M$ admits the relative $v$-Faber-Krahn inequality if, for any ball $B(x, r) \subset M$ and any relatively compact open set $\Omega \subset B(x, r)$:

$$(FK^v_\alpha) \quad \lambda_1(\Omega) \geq \frac{c}{r^2} \left( \frac{v_r(x)}{\mu(\Omega)} \right)^\alpha,$$

where $c$ and $\alpha$ are some positive constants. As usual, we abbreviate $(FK^v_\alpha)$ into $(FK_\alpha)$, $(FK^v)$ means $(FK^v_\alpha)$ for some $\alpha > 0$, and $(FK)$ means $(FK_\alpha)$, that is $(FK^v_\alpha)$, for some $\alpha > 0$. Note that in general, contrary to the case $v = V$, the inclusion $\Omega \subset B(x, r)$ does not guarantee any more that $\frac{v_r(x)}{\mu(\Omega)} \geq 1$.

It is known (see [33]) that Nash and Faber-Krahn inequalities are equivalent in the setting of Riemannian manifolds. Let us make this more precise in the present generality. Consider the following stronger version of $(LN^v_\alpha)$, homogeneous in the sense that it does not display the local term $\|f\|_2^2$ in the RHS: there exist $\alpha, C > 0$ such that for every ball $B = B(x, r)$, for every $f \in \mathcal{F}_c(B)$,

$$(HLN^v_\alpha) \quad \|f\|_2^{2(1+\alpha)} \leq \frac{C r^2}{v_r^\alpha(x)} \|f\|_1^{2\alpha} \mathcal{E}(f).$$

Conversely, let us introduce a weaker, inhomogeneous, form of the relative $v$-Faber-Krahn inequality, namely

$$(\tilde{FK}^v_\alpha) \quad r^2 \lambda_1(\Omega) + 1 \geq c \left( \frac{v_r(x)}{\mu(\Omega)} \right)^\alpha.$$

The following lemma is probably well-known. We will use it in the proof of Proposition 3.3.2 below.

**Lemma 3.3.1.** Assume that $(M, d, \mu, L)$ satisfies (H). Let $\Omega$ be an open subset of $M$. Then $\mathcal{F}_c(\Omega) \subset \overline{\mathcal{F} \cap C_c(\Omega)}^\mathcal{F}$.

**Proof.** Let $f \in \mathcal{F}_c(\Omega)$. Choose $\varepsilon > 0$ so that $f \in \mathcal{F}_c(\Omega_{4\varepsilon})$, where

$$\Omega_\varepsilon := \{ x \in \Omega, d(x, \Omega^c) > \varepsilon \}.$$

Select a finite family of points $x_1, \ldots, x_k$ in $M$ such that the balls $B(x_i, \varepsilon)$ cover the support of $f$. Consider next the functions $\eta_i = \rho_{x_i, \varepsilon/2}^2$ defined in Lemma 3.2.1. Note that $\eta_i \equiv 1$ on $B(x_i, \varepsilon)$ and $\eta_i$ is compactly supported in $B(x_i, 2\varepsilon)$. Set

$$g_0 = f \quad \text{and} \quad g_i = \prod_{j=1}^i (1 - \eta_j) f.$$

Note that $g_k = 0$ so if we put $h_i = g_{i-1} - g_i$ then $f = \sum_{i=1}^k h_i$. Note also that $h_i = \eta_i g_{i-1}$ so, by applying several times Lemma 3.2.1 one sees that $h_i \in \mathcal{F}_c(B(x_i, 2\varepsilon))$. Hence to prove Lemma 3.3.1 it is enough to show that if $h \in \mathcal{F}_c(B(x, 2\varepsilon))$ there exists a sequence of continuous functions $h_n \in \mathcal{F}_c(B(x, 3\varepsilon))$ which converges to $h$ in $\mathcal{F}$. 
Indeed, since $\mathcal{E}$ is regular, we may approximate $h \in \mathcal{F}_c(B(x,2\varepsilon))$ by a sequence of continuous functions $\tilde{h}_n \in \mathcal{F}$ such that $\mathcal{E}(h - \tilde{h}_n) \to 0$ and $\|\tilde{h}_n\|_2 \to 0$. Then, again by Lemma 3.2.1,

$$\mathcal{E}(h - \tilde{h}_n\rho_x^{3\varepsilon,\varepsilon/2}) = \mathcal{E}((h - \tilde{h}_n)\rho_x^{3\varepsilon,\varepsilon/2}) \leq \frac{8}{\varepsilon^2}\|h - \tilde{h}_n\|_2^2 + 2\mathcal{E}(h - \tilde{h}_n).$$

It follows that $\overline{\tilde{h}_n} = \tilde{h}_n\rho_x^{3\varepsilon,\varepsilon/2}$ converges to $h$ in $\mathcal{F}$ as $n \to +\infty$. Moreover, the functions $\tilde{h}_n$ are continuous and compactly supported in $B(x,3\varepsilon)$.

Note that the following statement does not require any doubling assumption on function $v$.

**Proposition 3.3.2.** Assume that $(M, d, \mu, L)$ satisfies (H) and let $v : M \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (A). Then $(FK^v_\alpha)$ is equivalent to $(HLN^v_\alpha)$ and $(\widehat{FK}^v_\alpha)$ is equivalent to $(LN^v_\alpha)$ for all $\alpha > 0$.

**Proof.** Inequality $(HLN^v_\alpha)$ can be rewritten as

$$\left(\frac{\|f\|_2^2}{\|f\|_1}\right)^\alpha \leq \frac{Cr^2}{v^\alpha(x)\|f\|_2^2} \mathcal{E}(f), \quad \forall x \in M, r > 0, \forall f \in \mathcal{F}_c(B(x,r)) \setminus \{0\}.$$

Let $\Omega$ be an open subset of $B(x,r)$. We can restrict the above inequality to $f \in \mathcal{F}_c(\Omega) \subset \mathcal{F}_c(B(x,r))$. By Hölder’s inequality, the LHS is larger than $\frac{1}{\mu^\alpha(\Omega)}$. Then taking the infimum in $f \in \mathcal{F}_c(\Omega)$ in the RHS and using definition $(3.14)$ of $\lambda_1(\Omega)$ yields

$$\frac{1}{\mu^\alpha(\Omega)} \leq \frac{Cr^2}{v^\alpha(x)}\lambda_1(\Omega), \quad \forall x \in M, r > 0, \forall \Omega \subset B(x,r),$$

that is, $(FK^v_\alpha)$. Similarly, $(LN^v_\alpha)$ can be rewritten as

$$\left(\frac{\|f\|_2^2}{\|f\|_1}\right)^\alpha \leq \frac{C}{v^\alpha(x)} \left(\frac{r^2\mathcal{E}(f)}{\|f\|_2^2} + 1\right), \quad \forall x \in M, r > 0, \forall f \in \mathcal{F}_c(B(x,r)) \setminus \{0\},$$

and the same argument yields $(\widehat{FK}^v_\alpha)$.

For the converse, we use a trick introduced by Grigor’yan in [33]. First observe that by definition $(3.15)$ of $\lambda_1(\Omega)$, $(FK^v_\alpha)$ can be rewritten as

$$(3.16) \quad \|g\|_2^2 \leq C \left(\frac{\mu(\Omega)}{v^\alpha(x)}\right)^\alpha r^2\mathcal{E}(g), \quad \forall x \in M, r > 0, \forall g \in \mathcal{F} \cap C_c(\Omega).$$

Now, for $f \in L^2(M,\mu)$, $g \geq 0$, and $\lambda > 0$, write

$$\|f\|_2^2 = \int_{f \geq 2\lambda} f^2 \, d\mu + \int_{f < 2\lambda} f^2 \, d\mu \leq 4 \int_{f \geq 2\lambda} (f - \lambda)^2 \, d\mu + 2\lambda \int_{f < 2\lambda} f \, d\mu,$$

hence

$$(3.17) \quad \|f\|_2^2 \leq 4 \int f^2 \, d\mu + 2\lambda \|f\|_1.$$
Let $x \in M$, $r > 0$, $f \in \mathcal{F} \cap C_c(B(x, r))$. Assume in addition that $f \geq 0$. Obviously $\Omega_\lambda = \{ f > \lambda \}$ is an open set. Since the semigroup $(e^{-tL})_{t>0}$ is submarkovian, $(f - \lambda)_+ = f - \min(f, \lambda) \in \mathcal{F}$. Now
\[ \sqrt{\mathcal{E}((f - \lambda)_+)} \leq \sqrt{\mathcal{E}(f)} + \sqrt{\mathcal{E}(\min(f, \lambda))} \]
and since by the submarkovian property $\mathcal{E}(\min(f, \lambda)) \leq \mathcal{E}(f)$,
\[ (3.18) \quad \mathcal{E}((f - \lambda)_+) \leq 4 \mathcal{E}(f). \]

Apply now (3.16) to $\Omega = \Omega_\lambda/2$ and $g = (f - \lambda)_+ \in \mathcal{F} \cap C_c(\Omega_\lambda/2)$. This yields, using Bienaymé-Tchebycheff and (3.18),
\[ \| (f - \lambda)_+ \|_2 \leq C' \left( \frac{\| f \|_1}{\lambda v_r(x)} \right)^\alpha r^2 \mathcal{E}(f), \]
therefore, together with (3.17),
\[ (3.19) \quad \| f \|_2 \leq 4 C' \left( \frac{\| f \|_1}{\lambda v_r(x)} \right)^\alpha r^2 \mathcal{E}(f) + 2 \lambda \| f \|_1. \]

The same inequality holds, with a different constant $C''$, for all $f \in \mathcal{F} \cap C_c(B(x, r))$ by applying (3.19) to $f_+$ and $f_-$, using the fact that $f_+, f_- \in \mathcal{F} \cap C_c(B(x, r))$ because $(e^{-tL})_{t>0}$ is submarkovian,
\[ \mathcal{E}(f) = \mathcal{E}(f_+) + \mathcal{E}(f_-) \]
since $\mathcal{E}$ is local,
\[ \| f \|_2^2 = \| f_+ \|_2^2 + \| f_- \|_2^2 \]
and
\[ \| f_+ \|_1, \| f_- \|_1 \leq \| f \|_1. \]

Taking $\lambda = \frac{\| f \|_2^2}{4 \| f \|_1}$ then yields
\[ (3.20) \quad \| f \|_2^2 \leq 2 C'' \left( \frac{4 \| f \|_1^2}{\lambda v_r(x) \| f \|_2^2} \right)^\alpha r^2 \mathcal{E}(f), \]
for all $x \in M$, $r > 0$, $f \in \mathcal{F} \cap C_c(B(x, r))$. According to Lemma 3.3.1, this is nothing but $\mathbb{HLN}^\omega$. If one assumes $(F\tilde{K}_\alpha^v)$, one starts with
\[ \| g \|_2^2 \leq C \left( \frac{\mu(\Omega)}{\lambda v_r(x)} \right)^\alpha \left( r^2 \mathcal{E}(g) + \| g \|_2^2 \right), \]
and the argument is similar. \[ \square \]

Putting together Proposition 4.2.6, which will be proved in Section 4.2, and Proposition 3.3.2, we can establish the link between $(DUE^v)$ and $(F\tilde{K}^v)$. We will see in Section 3.4 under which conditions one can replace $(F\tilde{K}_\alpha^v)$ with the more classical $(FK^\alpha)$. Note that the following statement does apply to doubling compact spaces, in particular to compact Riemannian manifolds, in the case $v = V$. In other
terms, considering \( \overline{FK} \) instead of \( FK \) solves the difficulty raised in [37, comment 5, p.9].

**Theorem 3.3.3.** Assume that \((M, d, \mu, L)\) satisfies \((H)\) and that \((M, d, \mu)\) satisfies \((VD)\). Let \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfy \((A)\), \((D_v)\), and \(\overline{D_v'}\). Then the upper bound \(\overline{DUE_v}\) is equivalent to \(\overline{FK_v}\).

Of course, if \((M, d, \mu, L)\) satisfies \((H)\), \(\mathcal{E}\) is a Dirichlet form and \((e^{-tL})_{t>0}\) is submarkovian, hence in particular positivity preserving and uniformly bounded on \(L^1\). The Davies-Gaffney estimate is known as well for a strongly local and regular Dirichlet form (see [56, Corollary 1.11]). This is why we can use Proposition 4.2.6 towards the proof of Theorem 3.3.3. Moreover, under the assumptions of Theorem 3.3.3, \(\overline{FK_v}\) can be added to the string of equivalences of Theorem 1.2.1.

### 3.4. Killing the local term with reverse doubling

We will introduce the notion of reverse doubling for a general function \(v\). Let us first consider the case \(v = V\). In this case the notion originates for Riemannian manifolds in [32, Theorem 1.1]. Let \((M, d, \mu)\) be a metric measure space satisfying \((VD)\). It is known (see [37, Proposition 5.2]), that, if in addition \(M\) is unbounded and connected, one has a so-called reverse doubling volume property, namely there exist \(0 < \kappa' < \kappa\) and \(c > 0\) such that, for all \(r \geq s > 0\) and \(x, y \in M\) such that \(d(x, y) < r + s\),

\[
\tag{RD} c \left( \frac{r}{s} \right)^{\kappa'} \leq \frac{V_r(y)}{V_s(x)},
\]

Together with \((D_\kappa)\), this yields

\[
\tag{D_{\kappa,\kappa'}^{\kappa'}} c \left( \frac{r}{s} \right)^{\kappa'} \leq \frac{V_r(y)}{V_s(x)} \leq C \left( \frac{r}{s} \right)^{\kappa}, \quad \forall r \geq s > 0, \ d(x, y) < r + s,
\]

which can also be written:

\[
\tag{3.21} \frac{V_r(y)}{V_s(x)} \leq C' w(r, s), \quad \forall r, s > 0, \ x, y \in M \text{ such that } d(x, y) < r + s
\]

where \(w(r, s) := \max\{\left( \frac{r}{s} \right)^{\kappa}, \left( \frac{s}{r} \right)^{\kappa'}\}\).

Consider now a measure space \((M, \mu)\) endowed with a function \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfying \((A)\). We shall say that \((M, \mu, v)\) satisfies weak \((RD_v)\) if there exist \(\kappa', c > 0\) such that, for any \(x \in M\) and any \(r \geq s > 0\),

\[
c \left( \frac{r}{s} \right)^{\kappa'} \leq \frac{v_r(x)}{v_s(x)},
\]

and if in addition \((M, \mu)\) is endowed with a metric \(d\), that \((M, d, \mu, v)\) satisfies strong \((RD_v)\) or simply \((RD_v)\) if there exist \(\kappa', c > 0\) such that, for any \(r \geq s > 0\) and \(x, y \in M\) such that \(d(x, y) < r + s\),

\[
c \left( \frac{r}{s} \right)^{\kappa'} \leq \frac{v_r(y)}{v_s(x)}.
\]

One checks easily that, under \((D_v)\),

\[ (RD_v) \iff \text{weak } (RD_v) + (D_v') \]
As above, the conjunction of \((D_v)\) and \((RD_v)\) can be rewritten as

\[
(3.22) \quad \frac{v_r(y)}{v_s(x)} \leq C' w_v(r, s), \quad \forall r, s > 0, \ x, y \in M \text{ such that } d(x, y) < r + s,
\]

where \(w_v(r, s) := \max\{\left(\frac{r}{s}\right)^{\kappa_v}, \left(\frac{s}{r}\right)^{\kappa_v}\}\).

**Proposition 3.4.1.** Assume that \((M, d, \mu, \nu)\) satisfies weak \((RD_v)\) and that \(v \geq \varepsilon V\) for some \(\varepsilon > 0\). Let \(L\) be a nonnegative self-adjoint operator on \(L^2(M, \mu)\). Then \((\text{HN}_\alpha^v)\) is equivalent to \((\text{HLN}_\alpha^v)\) and \((\text{FK}_\alpha^v)\) is equivalent to \((\text{FK}_\alpha^v)\).

**Proof.** It is obvious that \((\text{HLN}_\alpha^v)\) implies \((\text{HN}_\alpha^v)\) and that \((\text{FK}_\alpha^v)\) implies \((\text{FK}_\alpha^v)\).

Now for the converses. Assume \((\text{HN}_\alpha^v)\), that is

\[
(3.23) \quad \|f\|^2_{2(\alpha+1)} \leq \frac{C}{v_r^\alpha(x)} \|f\|^2_1 (r^2 \mathcal{E}(f) + \|f\|_2^2),
\]

\(\forall B = B(x, r), \ f \in \mathcal{F}_c(B)\).

Now use a trick from [19, Proposition 2.3]. Let \(A > 1\) to be chosen later. Applying \((3.23)\) in the ball \(B(x, Ar)\) to \(f \in \mathcal{F}_c(B) \subset \mathcal{F}_c(B(x, Ar))\), one obtains

\[
(3.24) \quad \|f\|^2_{2(\alpha+1)} \leq \frac{C}{v^\alpha(x, Ar)} \|f\|^2_1 (A^2 r^2 \mathcal{E}(f) + \|f\|_2^2), \quad \forall B = B(x, r), \ f \in \mathcal{F}_c(B).
\]

Since

\[
\|f\|_1 \leq V^{1/2}(x, r) \|f\|_2 \leq C' V^{1/2}(x, r) \|f\|_2,
\]

the first inequality uses Cauchy-Schwarz inequality and the second one the assumption that \(v \geq \varepsilon V\), \((3.24)\) yields

\[
(3.25) \quad \|f\|^2_{2(\alpha+1)} \leq \frac{CA^2 r^2}{v^\alpha(x, Ar)} \|f\|^2_1 \mathcal{E}(f) + C'' \left(\frac{v_r(x)}{v_{Ar}(x)}\right)^\alpha \|f\|_2^2,
\]

Now, by weak \((RD_v)\), one has

\[
(3.26) \quad \frac{v_r(x)}{v_{Ar}(x)} \leq \frac{1}{cA^\kappa_v}.
\]

One can therefore choose \(A\) so large that \(C'' \left(\frac{v(x, r)}{v(x, Ar)}\right)^\alpha \leq 1/2\). Then \((3.25)\) implies

\[
\|f\|^2_{2(\alpha+1)} \leq \frac{2CA^2 r^2}{v^\alpha_{Ar}(x)} \|f\|^2_1 \mathcal{E}(f), \quad \forall B = B(x, r), \quad \forall f \in \mathcal{F}_c(B),
\]

that is, using \((3.26)\) once again, \((\text{HN}_\alpha^v)\).

The statement about the implication from \((\text{FK}_\alpha^v)\) to \((\text{FK}_\alpha^v)\) follows from the one we have just proved through Proposition [3.3.2] if \((M, d, \mu, \nu)\) satisfies \((H)\).

Alternatively, one can rewrite \((\text{FK}_\alpha^v)\) as

\[
c \left(\frac{v_r(x)}{\mu(\Omega)}\right)^\alpha \leq r^2 \lambda_1(\Omega) + 1,
\]

apply it in \(B(x, Ar), A \geq 1\), use \((RD_v)\), obtain

\[
cA^\alpha \left(\frac{v_r(x)}{\mu(\Omega)}\right)^\alpha \leq c \left(\frac{v_{Ar}(x)}{\mu(\Omega)}\right)^\alpha \leq A^2 r^2 \lambda_1(\Omega) + 1,
\]
and choose \( A \) so large that \( cA^{\alpha^\nu} \geq \frac{2}{\varepsilon^2} \). Since \( \frac{v_r(x)}{\mu(B)} \geq \varepsilon \frac{V_r(x)}{\mu(B)} \geq \varepsilon \), \((FK^\nu)\) follows. 

Note that the role of the local term \( \|f\|_2^2 \) is different in the case of \((LS^q)\): here it seems one cannot get rid of it except when \( v_r(x) \simeq r^n \) and one considers the limit case \( q = n \).

This time, putting together Theorem 3.3.3 and Proposition 3.4.1, we can establish the link between \((DUE^v)\) and \((FK^v)\) under doubling and reverse doubling for \( v \). In the case \( v = V \) and \( M \) a Riemannian manifold the following statement gives back Proposition 5.2 from [33], see also Theorem 15.21 in [36]; note the role played by reverse doubling in both instances.

**Theorem 3.4.2.** Assume that \((M, d, \mu, L)\) satisfies \((H)\) and that \((M, d, \mu)\) satisfies \((VD)\). Let \( v : M \times \mathbb{R}^+ \to \mathbb{R}^+ \) satisfy \((A)\), \((D_v)\), \((RD_v)\), and \( v \geq \varepsilon V \) for some \( \varepsilon > 0 \). Then the upper bound \((DUE^v)\) is equivalent to \((FK^v)\).

In fact, once one has \((RD_v)\), instead of killing the local term or non-homogeneous term in \((LN^\nu)\) or \((FK^\nu)\), one may as well avoid to introduce it in the first place. Let us show how this works by proving directly the following version of Proposition 3.4.1 (which can also be derived by using Proposition 3.3.2). One could also obtain directly \((FK^v)\) instead of \((HLN^v)\).

**Proposition 3.4.3.** Let \((M, d, \mu)\) be a metric measure space, \( L \) a non-negative self-adjoint operator on \( L^2(M, \mu) \) and let \( v : M \times \mathbb{R}^+ \to \mathbb{R}^+ \) satisfy \((A)\), \((D_v)\), \((RD_v)\), and \( v \geq \varepsilon V \) for some \( \varepsilon > 0 \). Then \((KN^v)\) implies \((HLN^v_{2/\kappa_v})\), where \( \kappa_v \) is as in \((D^\nu_v)\).

**Proof.** Let \( f \in \mathcal{F}_c(B) \), \( B = B(x, r) \). By (3.22), one has

\[
|v_r(x)| \leq C w(r, s)v_s(z), \quad \forall s > 0, \forall z \in \text{supp}(f).
\]

Thus \((KN^v)\) yields

\[
(3.27) \quad \int_B |f|^2 \, d\mu \leq C \left( \frac{w(r, s)}{v_r(x)} \left( \int_B |f| \, d\mu \right)^2 + s^2 \mathcal{E}(f) \right), \quad \forall f \in \mathcal{F}_c(B),
\]

for all \( s, r > 0 \) and \( x \in M \). Choose \( s_0 > 0 \) such that

\[
\frac{w(r, s_0)}{v_r(x)} \left( \int_B |f| \, d\mu \right)^2 = s_0^2 \mathcal{E}(f),
\]

which is possible since the function \( s \to \frac{w(r, s)}{s^2} \) is continuous, \( \lim_{s \to 0^+} \frac{w(r, s)}{s^2} = +\infty \) and \( \lim_{s \to +\infty} \frac{w(r, s)}{s^2} = 0 \).

Then (3.27) yields

\[
(i) \quad \int_B |f|^2 \, d\mu \leq 2Cs_0^2 \mathcal{E}(f) \quad \text{and} \quad (ii) \quad \int_B |f|^2 \, d\mu \leq \frac{2Cw(r, s_0)}{v_r(x)} \left( \int_B |f| \, d\mu \right)^2.
\]

If \( r \geq s_0 \), (ii) reads \( \int_B |f|^2 \, d\mu \leq \frac{2Cv_r}{v_r(x)s_0^2} \left( \int_B |f| \, d\mu \right)^2 \), that is

\[
\frac{c'}{r^2} \left( \frac{v_r(x)}{\int_B |f|^2 \, d\mu} \right)^{2/\kappa} \leq \frac{1}{s_0^2},
\]

where

\[
c' = \frac{2C}{s_0^2}.
\]
hence, together with (i),

\[(3.28) \quad \frac{c''}{r^2} \left( \frac{v_r(x) \int_B |f|^2 d\mu}{\left( \int_B |f| d\mu \right)^2} \right)^{2/\kappa} \leq \frac{\mathcal{E}(f)}{\int_B |f|^2 d\mu}.\]

If \( r \leq s_0 \), (ii) yields this time

\[ \frac{c'}{r^2} \left( \frac{v_r(x) \int_B |f|^2 d\mu}{\left( \int_B |f| d\mu \right)^2} \right)^{2/\kappa'} \leq \frac{1}{s_0^2}. \]

Since \( v \geq \varepsilon V \), Hölder’s inequality yields

\[ \frac{v_r(x) \int_B |f|^2 d\mu}{\left( \int_B |f| d\mu \right)^2} \geq \varepsilon. \]

Since \( \kappa' \leq \kappa \),

\[ \frac{c'' \varepsilon^{2/\kappa'} - 2/\kappa}{r^2} \left( \frac{v_r(x) \int_B |f|^2 d\mu}{\left( \int_B |f| d\mu \right)^2} \right)^{2/\kappa'} \leq \frac{c''}{r^2} \left( \frac{v_r(x) \int_B |f|^2 d\mu}{\left( \int_B |f| d\mu \right)^2} \right)^{2/\kappa'}, \]

hence again (3.28), and finally \((H_{LN_{2/\kappa'}})\). \( \square \)

4. **Converses under Davies-Gaffney estimate**

Let \((M, d, \mu)\) be a metric measure space and \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) with dense domain. We have already introduced the Davies-Gaffney estimate \((DG)\). In fact, contrary to what we did in [21], we will not use directly \((DG)\) in the present work (except for a technical argument in the end of the proof of Proposition 4.1.9), but rather an equivalent form, namely the finite propagation speed for the wave equation. By the way, it is known that one can also attack the problems treated in [21] (the implication from on-diagonal to off-diagonal bounds) by using the latter property instead of the former (see for instance [53]). It would be interesting to know whether conversely one could use exclusively \((DG)\) to prove the results in the present section. This would probably help to get \((DUE^v)\) and \((UE^v)\) in one go from \((N^v)\) or \((GN^v)\), instead of going through two steps: first the present article, then [21].

Following [21], we say that \((M, d, \mu, L)\), or in short \(L\), satisfies the finite propagation speed property for solutions of the corresponding wave equation if

\[(4.1) \quad \langle \cos(r\sqrt{L}) f_1, f_2 \rangle = 0 \]

for all \( f_i \in L^2(B_i, \mu), i = 1, 2 \), where \( B_i \) are open balls in \( M \) such that \( d(B_1, B_2) > r > 0 \). Here and in the sequel, if \( \Omega \) is a measurable subset of \( M \), \( L^2(\Omega, \mu) \) will mean \( L^2(\Omega, \mu|_{\Omega}) \).

We shall use the following notational convention. For \( r > 0 \), set

\[ D_r = \{(x, y) \in M \times M : d(x, y) \leq r\}. \]
Denote by $L^p_r(M, \mu)$, $p \geq 1$, the vector space of functions in $L^p(M, \mu)$ with support in a ball. Given a linear operator $T$ from $L^p_r(M, \mu)$ to $L^q_{loc}(M, \mu)$, for some $1 \leq p, q \leq +\infty$, write
\begin{equation}
\text{supp } T \subseteq D_r
\end{equation}
if $\langle Tf_1, f_2 \rangle = 0$ whenever $f_1 \in L^p(B_1, \mu)$, $f_2 \in L^q(B_2, \mu)$, and $B_1, B_2$ are balls such that $d(B_1, B_2) > r$. Note that if $T$ is an integral operator with kernel $K_T$, then
\begin{equation}
(4.2)
\end{equation}
coincides with the standard meaning of $\text{supp } K_T \subseteq D_r$, that is $K_T(x, y) = 0$ for all $(x, y) \notin D_r$. Now we can state the finite propagation speed property \((4.1)\) in the following way
\begin{equation}
\text{supp } \cos(r\sqrt{L}) \subseteq D_r, \quad \forall r > 0.
\end{equation}

The proof of the above-mentioned equivalence can be found in [54] or [21].

**Theorem 4.0.1.** Let $(M, d, \mu)$ be a metric measure space and $L$ be a non-negative self-adjoint operator acting on $L^2(M, \mu)$. Then the finite propagation speed property \((4.1)\) and the Davies-Gaffney estimate \((DG)\) are equivalent properties for $(M, d, \mu, L)$.

4.1. From Gagliardo-Nirenberg to heat kernel upper bounds. Our starting point will be the fact that, on a doubling metric space, support properties like \((4.2)\) enable one to commute the operator with multiplication by doubling weights in $L^p - L^q$ estimates.

**Proposition 4.1.1.** Let $(M, d, \mu)$ be a doubling metric measure space and let $v : M \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy \((A)\), \((D_v)\), and \((D'_v)\). For all $\gamma \in \mathbb{R}$, there exists $C_\gamma > 0$ only depending on the constants in \((VD)\), \((D_v)\), and \((D'_v)\) such that, for all $p, q, 1 \leq p \leq q \leq \infty$, and for every family of operators $T_r$, $r > 0$, mapping continuously $L^p(M, \mu)$ to $L^q(M, \mu)$ and satisfying
\begin{equation}
\text{supp } T_r \subseteq D_r,
\end{equation}
one has
\begin{equation}
\|v_r^\gamma T_r v_r^{-\gamma}\|_{p \to q} \leq C_\gamma \|T_r\|_{p \to q},
\end{equation}
uniformly in $r > 0$.

Proposition 4.1.1 relies on ideas that already appeared in [21] [40]. If $\Omega$ is a subset of $M$, denote by $\chi_\Omega$ its characteristic function. We will deduce Proposition 4.1.1 from the following statement which does not involve $v$.

**Lemma 4.1.2.** Let $(M, d, \mu)$ be a doubling metric measure space. There exists $C > 0$ only depending on the doubling constant such that, for all $p, q, 1 \leq p \leq q \leq \infty$, and for every family of operators $T_r$, $r > 0$ from $L^p_r(M, \mu)$ to $L^q_r(M, \mu)$ satisfying \((4.3)\), one has
\begin{equation}
\|T_r\|_{p \to q} \leq C \sup_{x \in M} \|T_r \chi_{B(x, r)}\|_{p \to q},
\end{equation}
uniformly in $r > 0$.

Note that the reverse inequality is obvious. Let us check that Proposition 4.1.1 follows from Lemma 4.1.2. Indeed, the operator $S_r := v_r^\gamma T_r v_r^{-\gamma}$ clearly also satisfies \((4.3)\). Lemma 4.1.2 applied to $S_r$ thus yields
\begin{equation}
\|v_r^\gamma T_r v_r^{-\gamma}\|_{p \to q} \leq C \sup_{x \in M} \|v_r^\gamma T_r v_r^{-\gamma} \chi_{B(x, r)}\|_{p \to q},
\end{equation}
uniformly in $r > 0$. Now, for $f \in L^p(M, \mu)$,
\[
\|v_r^\gamma T_r v_r^{-\gamma} \chi_{B(x,r)} f\|_q \leq \|v_r^\gamma T_r \chi_{B(x,r)}\|_{p \to q} \|v_r^{-\gamma} \chi_{B(x,r)} f\|_p
\leq \|v_r^\gamma T_r \chi_{B(x,r)}\|_{p \to q} \|v_r^{-\gamma} \chi_{B(x,r)}\|_\infty \|f\|_p.
\]
Since $v$ satisfies $\left[\text{D}_v\right]$, the function $v_r^{-\gamma} \chi_{B(x,r)}$ is dominated by $C|\gamma| v_r^{-\gamma}(x) \chi_{B(x,r)}$, where $C$ is the constant in $\left[\text{D}_v\right]$, hence
\[
\|v_r^\gamma T_r v_r^{-\gamma} \chi_{B(x,r)} f\|_q \leq C|\gamma| v_r^{-\gamma}(x) \|v_r^\gamma T_r \chi_{B(x,r)}\|_{p \to q} \|f\|_p.
\]
Now, since $T_r$ satisfies $\left[\text{L}_\gamma\right]$
\[
T_r \chi_{B(x,r)} = \chi_{B(x, 2r)} T_r \chi_{B(x,r)},
\]
and
\[
v_r^{-\gamma}(x) \|v_r^\gamma T_r \chi_{B(x,r)}\|_{p \to q} = v_r^{-\gamma}(x) \|v_r^\gamma \chi_{B(x,2r)} T_r \chi_{B(x,r)}\|_{p \to q}
\leq v_r^{-\gamma}(x) \|v_r^\gamma \chi_{B(x,2r)}\|_\infty \|T_r \chi_{B(x,r)}\|_{p \to q}
\leq C' \|T_r \chi_{B(x,r)}\|_{p \to q},
\]
where $C'_\gamma$ only depends on the constants in $\left[\text{D}_v\right]$, $\left[\text{D}_T\right]$ and on $\gamma$. Finally
\[
\|v_r^\gamma T_r v_r^{-\gamma} \chi_{B(x,r)}\|_{p \to q} \leq C|\gamma| C'_\gamma \|T_r \chi_{B(x,r)}\|_{p \to q}
\]
and
\[
\|v_r^\gamma T_r v_r^{-\gamma}\|_{p \to q} \leq CC'_\gamma |\gamma| \sup_{x \in M} \|T_r \chi_{B(x,r)}\|_{p \to q} \leq CC'_\gamma |\gamma| \|T_r\|_{p \to q},
\]
which proves Proposition $\left[\text{1.1.1}\right]$.

Proof of Lemma $\left[\text{1.1.2}\right]$. Fix $r > 0$. Apply $\left(\text{BCP}\right)$: there exists a sequence $x_i \in M$ such that $d(x_i, x_j) > r/2$ for $i \neq j$ and $\sup_{x \in M} \inf_{i} d(x, x_i) \leq r/2$. Define $\tilde{B}_i$ by the formula
\[
(4.4) \quad \tilde{B}_i = B(x_i, r) \setminus \left( \bigcup_{j \neq i} B(x_j, r) \right),
\]
so that $\left(\tilde{B}_i\right)_i$ is a denumerable partition of $M$. For $f \in L^p(M, \mu)$, write
\[
T_r f = \sum_{i,j} \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f.
\]
Now, if $d(x_i, x_j) > 3r$, $d(\tilde{B}_i, \tilde{B}_j) > r$, hence, using $\left[\text{1.3}\right]$,
\[
\sum_{i,j : d(x_i, x_j) > 3r} \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f = 0,
\]
thus
\[
T_r f = \sum_{i,j : d(x_i, x_j) \leq 3r} \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f.
\]
Assume $q < +\infty$. Obvious modifications yield the case $q = +\infty$. Write
\[
\|T_r f\|_q^q = \left\| \sum_{i,j : d(x_i, x_j) \leq 3r} \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f \right\|_q^q = \sum_{i} \left\| \sum_{j : d(x_i, x_j) \leq 3r} \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f \right\|_q^q.
\]
the last equality using the fact that the $\tilde{B}_i$’s are disjoint. According to (BCP),
\[ K_0 = \sup_i \#\{j: d(x_i, x_j) \leq 3r\} \]
is a finite integer which only depends on the doubling constant of $(M, d, \mu)$. It follows that
\[
\| \sum_{j: d(x_i, x_j) \leq 3r} \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f \|_q^q \leq \left( \sum_{j: d(x_i, x_j) \leq 3r} \| \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f \|_q^q \right)^{\frac{q}{q-1}} \leq K_0^{q-1} \sum_{j: d(x_i, x_j) \leq 3r} \| \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f \|_q^q .
\]
The last line uses convexity together with the fact that there are at most $K_0$ terms in the summation.

Gathering the above inequalities, one obtains
\[
\| T_r f \|_q^q \leq K_0^{q-1} \sum_i \sum_{j: d(x_i, x_j) \leq 3r} \left( \| \chi_{\tilde{B}_i} T_r \chi_{\tilde{B}_j} f \|_p \right)^{\frac{q}{q-1}} \| \chi_{\tilde{B}_j} f \|_p ^q \leq K_0^{q-1} \sum_i \sum_{j: d(x_i, x_j) \leq 3r} \| T_r \chi_{\tilde{B}_j} f \|_p \| \chi_{\tilde{B}_j} f \|_p ^q \leq K_0 \sum_{j} \| T_r \chi_{\tilde{B}_j} f \|_p \| \chi_{\tilde{B}_j} f \|_p ^q \leq K_0 \sup_\ell \| T_r \chi_{\tilde{B}_\ell} f \|_p \| f \|_p ^\frac{q}{p} \leq K_0 \sup_\ell \| T_r \chi_{\tilde{B}_\ell} f \|_p \| f \|_p ^\frac{q}{p} .
\]
The one before last inequality uses the fact that $p \leq q$. Finally,
\[
\| T_r \|_{p \to q} \leq K_0 \sup_\ell \| T_r \chi_{\tilde{B}_\ell} f \|_{p \to q},
\]
hence the claim. \qed

In the sequel we will use the following straightforward observation in order to build functions of $L$ that satisfy support properties of the type (4.2). Next our task will be to relate these operators to $(e^{-tL})_{t>0}$. 
Lemma 4.1.3. Assume that \((M, d, \mu, L)\) satisfies \((DG)\). Let \(\Phi \in L^1(\mathbb{R})\) be even and such that
\[
\text{supp } \hat{\Phi} \subset [-1, 1].
\]
Then
\[
\text{supp } \Phi(r \sqrt{L}) \subseteq D_r
\]
for all \(r > 0\).

Proof. The claim follows from Theorem 4.0.1 (4.1) and the formula
\[
(4.6) \quad \Phi(r \sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\Phi}(s) \cos(rs \sqrt{L}) \, ds.
\]
\(\square\)

Lemma 4.1.4. Let \((M, d, \mu, v)\) be as in Proposition 4.1.1. Assume that \((M, d, \mu, L)\) satisfies \((DG)\). Let \(1 \leq p \leq 2\), \(\gamma, \delta \in \mathbb{R}\). Suppose that the function \(\Phi\) on \(\mathbb{R}\) is even and satisfies \(\text{supp } \hat{\Phi} \subset [-1, 1]\) as well as
\[
\sup_{\lambda} |(1 + \lambda^2)^{N+1} \Phi(\lambda)| < \infty,
\]
for some \(N \in \mathbb{N}\). Then
\[
\left\| v_{\sqrt{t}}^\gamma \Phi(\sqrt{tL}) v_{\sqrt{t}}^\delta \right\|_{p \to 2} \leq C \left\| v_{\sqrt{t}}^\gamma (I + tL)^{-N} v_{\sqrt{t}}^\delta \right\|_{p \to 2}
\]
uniformly in \(t > 0\).

Proof. Set \(\Psi(\lambda) = (1 + \lambda^2)^N \Phi(\lambda)\). Function \(\Psi\) is even, bounded, belongs to \(L^1(\mathbb{R})\), and satisfies \(\text{supp } \hat{\Psi} \subset [-1, 1]\). By Lemma 4.1.3
\[
\text{supp } \Psi(\sqrt{tL}) \subseteq D_{\sqrt{t}}.
\]
Thus by Proposition 4.1.1
\[
\left\| v_{\sqrt{t}}^\gamma \Psi(\sqrt{tL}) v_{\sqrt{t}}^{-\gamma} \right\|_{2 \to 2} \leq C_{\gamma} \left\| \Psi(\sqrt{tL}) \right\|_{2 \to 2}
\]
and by spectral theory
\[
\left\| \Psi(\sqrt{tL}) \right\|_{2 \to 2} \leq C'_{\gamma}, \forall t > 0.
\]
Hence
\[
\left\| v_{\sqrt{t}}^\gamma \Phi(\sqrt{tL}) v_{\sqrt{t}}^\delta \right\|_{p \to 2} \leq \left\| v_{\sqrt{t}}^\gamma \Psi(\sqrt{tL}) v_{\sqrt{t}}^{-\gamma} \right\|_{2 \to 2} \left\| v_{\sqrt{t}}^\gamma (I + tL)^{-N} v_{\sqrt{t}}^\delta \right\|_{p \to 2} \leq C_{\gamma} C'_{\gamma} \left\| v_{\sqrt{t}}^\gamma (I + tL)^{-N} v_{\sqrt{t}}^\delta \right\|_{p \to 2}.
\]
\(\square\)

Lemma 4.1.5. Let \((M, \mu, L)\) be a measure space endowed with a non-negative self-adjoint operator and let \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfy \((A)\) and \((D_v)\). Let \(1 \leq p \leq q \leq \infty, \gamma, \delta \in \mathbb{R}\), and \(N > \kappa_v(\|\delta| + |\gamma|)/2\), where \(\kappa_v\) is the exponent in \((D_v^{\kappa_v})\). Then there exists \(C > 0\) such that
\[
\sup_{t > 0} \left\| v_{\sqrt{t}}^\gamma (I + tL)^{-N} v_{\sqrt{t}}^\delta \right\|_{p \to q} \leq C \sup_{t > 0} \left\| v_{\sqrt{t}}^\gamma e^{-tL} v_{\sqrt{t}}^\delta \right\|_{p \to q}.
\]
Proof. Recall the following standard integral representation
\[
(I + tL)^{-N} = \frac{1}{\Gamma(N)} \int_0^\infty e^{-s} s^{N-1} \exp(-s(tL)) \, ds.
\]
It yields
\[
\|v^\gamma_{\sqrt{t}}(I + tL)^{-N} v^\delta_{\sqrt{t}}\|_{p \to q} \leq C \int_0^\infty e^{-s} s^{N-1} \|v^\gamma_{\sqrt{s}} \exp(-s(tL))v^\delta_{\sqrt{s}}\|_{p \to q} \, ds,
\]
hence, using \((D_v)\),
\[
\|v^\gamma_{\sqrt{t}}(I + tL)^{-N} v^\delta_{\sqrt{t}}\|_{p \to q} \\
\leq C \int_0^\infty e^{-s} s^{N-1} \left(\sqrt{s} + \frac{1}{\sqrt{s}}\right)^{\kappa_v(|\delta| + |\gamma|)} \|v^\gamma_{\sqrt{s}} \exp(-s(tL))v^\delta_{\sqrt{s}}\|_{p \to q} \, ds \\
\leq C \left(\int_0^\infty e^{-s} s^{N-1} \left(\sqrt{s} + \frac{1}{\sqrt{s}}\right)^{\kappa_v(|\delta| + |\gamma|)} \, ds\right) \sup_{t > 0} \|v^\gamma_{\sqrt{t}} \exp(-tL)v^\delta_{\sqrt{t}}\|_{p \to q},
\]
which proves the claim. □

Proposition 4.1.6. Let \((M, d, \mu, v)\) be as in Proposition 4.1.1. Assume in addition that \((M, d, \mu, L)\) satisfies \((D_G)\). Let \(1 \leq p \leq 2\). Then for any \(\gamma_1, \gamma_2, \delta_1, \delta_2 \in \mathbb{R}\) such that \(\gamma_1 + \delta_1 = \gamma_2 + \delta_2 = 1/p - 1/2\), there exists a constant \(C\) such that
\[
\sup_t \|v^\gamma_{\sqrt{t}} e^{-tL} v^\delta_{\sqrt{t}}\|_{p \to 2} \leq C \sup_t \|v^\gamma_{\sqrt{t}} e^{-tL} v^\delta_{\sqrt{t}}\|_{p \to 2}.
\]
As a consequence, for fixed \(p\), \(1 \leq p < 2\), all the conditions \((vEv_{\gamma,2})\) for \(\gamma \in \mathbb{R}\) are equivalent.

Proof. A direct calculation shows that for all \(a > 0\), \(x \in \mathbb{R}\),
\[
\frac{1}{\Gamma(a)} \int_0^\infty (s - x^2)^a_+ e^{-s} \, ds = e^{-x^2},
\]
where
\[
(t)_+ = t \quad \text{if} \quad t \geq 0 \quad \text{and} \quad (t)_+ = 0 \quad \text{if} \quad t < 0.
\]
Hence
\[
C_{a} \int_0^\infty \left(1 - \frac{x^2}{s}\right)_+^a e^{-s/4} s^a \, ds = e^{-x^2/4},
\]
for some suitable \(C_a > 0\). Taking the Fourier transform of both sides of the above inequality yields
\[
\int_0^\infty F_a(\sqrt{s} \lambda)s^{a + \frac{1}{2}} e^{-s/4} ds = e^{-\lambda^2},
\]
where \(F_a\) is the Fourier transform of the function \(t \to (1 - t^2)_+^a\) multiplied by the appropriate constant. Hence, by spectral theory,
\[
\int_0^\infty F_a(\sqrt{s L} \lambda)s^{a + \frac{1}{2}} e^{-s/4} ds = e^{-tL}
\]
(this is a version of the well-known transmutation formula). Write now
\[ \|v^\gamma_1 t^{-L} v^\delta_1\|_{p \to 2} \leq \int_0^\infty \|v^\gamma_1(\sqrt{tsL})v^\delta_1\|_{p \to 2} s^{a+\frac{1}{2}} e^{-s/4} ds \]
\[ \leq C \int_0^\infty \|v^\gamma_1(\sqrt{tsL})v^\delta_1\|_{p \to 2} (\sqrt{s} + \frac{1}{\sqrt{s}})^{\kappa_v(|\delta_1|+|\gamma_1|)} s^{a+\frac{1}{2}} e^{-s/4} ds, \]
hence, for a large enough,
\[ \sup_{t>0} \|v^\gamma_1 e^{-tL} v^\delta_1\|_{p \to 2} \leq C' \sup_{t>0} \|v^\gamma_1(\sqrt{tL})v^\delta_1\|_{p \to 2}. \]

On the other hand, \( \Phi = F_a \) satisfies the assumptions of Lemma 4.1.3 thus
(4.7) \[ \text{supp } F_a(r \sqrt{L}) \subseteq D_r, \forall r > 0. \]
Setting \( T_r = v^\gamma_2 F_a(r \sqrt{L})v^\delta_2 \), it follows that
(4.8) \[ \text{supp } T_r \subseteq D_r, \forall r > 0. \]
By Proposition 4.1.1 with \( \gamma = \gamma_1 - \gamma_2 = \delta_2 - \delta_1 \),
\[ \sup_{t>0} \|v^\gamma_1(\sqrt{tL})v^\delta_1\|_{p \to 2} \leq C \sup_{t>0} \|v^\gamma_2(\sqrt{tL})v^\delta_2\|_{p \to 2}. \]
We can now apply Lemma 4.1.4 with \( \Phi = F_a \). One checks easily that the assumptions are satisfied as soon as \( 2N + 1 \leq a \), in which case
\[ \sup_{t>0} \|v^\gamma_2(\sqrt{tL})v^\delta_2\|_{p \to 2} \leq C' \sup_{t>0} \|v^\gamma_2(I + tL)^{-N}v^\delta_2\|_{p \to 2}. \]
By Lemma 4.1.5 since \( a \) hence \( N \) can be chosen arbitrarily large,
\[ \sup_{t>0} \|v^\gamma_2(I + tL)^{-N}v^\delta_2\|_{p \to 2} \leq C'' \sup_{t>0} \|v^\gamma_1 e^{-tL} v^\delta_1\|_{p \to 2}. \]
This finishes the proof of the proposition. \( \square \)

**Remark 4.1.7.** In [9, Proposition 2.1] and [46, Proposition 2.16], p.40, see also Remark 2.17, p.42, it is proved that a commutation phenomenon similar to the one in Proposition 4.1.6 holds in presence of generalised Gaussian estimates. In [21, Theorem 4.15 and Remarks a) and b)], it is shown that such estimates follow from (DG) and (Ev) estimates. This provides an alternative approach to Proposition 4.1.6, at least in the case \( \delta_2 = 0 \).

A first consequence of Proposition 4.1.6 is the following result, which is in some sense dual to Corollary 2.3.6.

**Proposition 4.1.8.** Let \( (M, d, \mu, v) \) be as in Proposition 4.1.1. Assume in addition that \( (M, d, \mu, L) \) satisfies (DG) and that \( (e^{-tL})_{t>0} \) is bounded analytic on \( L^1(M, \mu) \).
Then \( (DUE^v) \) is equivalent to
\[ (GN_{1,2,\beta}^v) \quad \| f \sqrt{v} \|_2 \leq C_\beta (\| f \|_1 + r^\beta \| L^{\beta/2} f \|_1), \quad \forall r > 0, \forall f \in D_1(L^{\beta/2}), \]
for all \( (or \ some) \ \beta > \kappa_v/2. \)

**Proof.** That \( (DUE^v) \) implies \( (GN_{1,2,\beta}^v) \) is a particular case of Theorem 2.3.5. Conversely, substituting \( e^{-rL} f \) to \( f \) in \( (GN_{1,2,\beta}^v) \) yields \( (vE_{1,2}) \), hence \( (Ev_{1,2}) \) by Proposition 4.1.6 hence \( (DUE^v) \) by Corollary 2.1.2. \( \square \)
Now we can extend the extrapolation lemma from [14] Lemma 1 or [23] Lemma 1.3 to our setting.

**Proposition 4.1.9.** Let $(M, d, \mu, v)$ be as in Proposition 4.1.4. Assume in addition that $(M, d, \mu, L)$ satisfies (DC) and that $(e^{-tL})_{t>0}$ is uniformly bounded on $L^{p_0}(M, \mu)$ for some $1 \leq p_0 < 2$. Then $(Ev_{p,2})$ for some $p_1$, $p_0 \leq p_1 < 2$, implies $(vEv_{p,2,\gamma})$, for all $\gamma \in \mathbb{R}$, and in particular $(Ev_{p,2})$, for all $p$, $p_0 \leq p \leq 2$.

**Proof.** Observe first that by Proposition 2.1.5, $(Ev_{p,2})$ implies $(Ev_{p,2})$ for $p_1 \leq p < 2$. Then under our assumptions Proposition 4.1.6 yields $(vEv_{p,2,\gamma})$, for all $\gamma \in \mathbb{R}$ and $p_1 \leq p < 2$. It remains to treat the case where $p_0 \leq p < p_1$. Now by interpolation $(e^{-tL})_{t>0}$ is also uniformly bounded on $L^p(M, \mu)$ if $p_0 \leq p < p_1 < 2$. We can therefore assume without loss of generality that $p = p_0$ and assume $(Ev_{p,2})$.

Fix $x \in M$ and $r > 0$, and let $f \in L^2(B(x, r), \mu) \cap L^{p_0}(B(x, r), \mu)$. Then

$$\left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-tL} f \right\|_2 \leq \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-\frac{t}{2}L} \right\|_{p_1 \to 2} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_1}} e^{-\frac{t}{2}L} f \right\|_{p_1}.$$ 

Proposition 4.1.6 yields in particular $(vEv_{p_1,2,\gamma})$ with $\gamma = \frac{1}{p_0} - \frac{1}{2}$, hence, by $(D_v)$,

$$\sup_{t>0} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-tL} v_{\sqrt{t}}^{1 - \frac{1}{p_1}} \right\|_{p_1 \to 2} \leq C,$$

therefore

$$(4.9) \quad \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-tL} f \right\|_2 \leq C \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_1}} e^{-\frac{t}{2}L} f \right\|_{p_1}.$$ 

Next, we choose $\theta$ such that

$$\frac{1}{p_1} = \frac{\theta}{p_0} + \frac{1 - \theta}{2}.$$ 

H"{o}lder’s inequality yields

$$\left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} g \right\|_{p_1} \leq \left\| g \right\|_{p_0} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} g \right\|_2^{1 - \theta}.$$ 

Taking $g = e^{-\frac{t}{2}L} f$, we obtain

$$\left\| v_{\sqrt{t}}^{1 - \frac{1}{p_1}} e^{-\frac{t}{2}L} f \right\|_{p_1} \leq \left\| e^{-\frac{t}{2}L} f \right\|_{p_0} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_1}} e^{-\frac{t}{2}L} f \right\|_2^{1 - \theta},$$

therefore, since $(e^{-tL})_{t>0}$ is uniformly bounded on $L^{p_0}(M, \mu)$,

$$(4.10) \quad \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-\frac{t}{2}L} f \right\|_{p_1} \leq C' \left\| f \right\|_{p_0} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-\frac{t}{2}L} f \right\|_2^{1 - \theta}.$$ 

It follows from $(D_v)$ that

$$\left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-\frac{t}{2}L} f \right\|_2 \leq C \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-\frac{t}{2}L} f \right\|_{p_0}.$$ 

Thus (4.9) and (4.10) yield

$$(4.11) \quad \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-tL} f \right\|_2 \leq C \left\| f \right\|_{p_0} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-\frac{t}{2}L} f \right\|_2^{1 - \theta}.$$ 

For $T > 0$, define

$$K(f, T) := \sup_{0 < t \leq T} \left\| v_{\sqrt{t}}^{1 - \frac{1}{p_0}} e^{-tL} f \right\|_2.$$
which is a finite quantity. Indeed, write
\[
K^2(f, T) = \sup_{0 \leq t \leq T} \left\| v \frac{1}{\sqrt{t}} e^{-tL} f \right\|_2^2
\]
\[
= \sup_{0 \leq t \leq T} \sum_{k=0}^{\infty} \left\| v \frac{1}{\sqrt{t}} e^{-tL} f \right\|_{L^2(B(x,(k+1)r),B(x,kr),\mu)}^2
\]
\[
\leq \sup_{0 \leq t \leq T} \sum_{k=0}^{\infty} \left\| v \frac{1}{\sqrt{T}} e^{-tL} f \right\|_{L^2(B(x,(k+1)r),B(x,kr),\mu)}^2.
\]

Now note that
\[
(4.12) \quad v(y, r) \leq C \left( 1 + \frac{d(x, y)}{r} \right)^{\kappa_v} v(x, r), \quad \forall r > 0, \quad \text{for } \mu - \text{a.e. } x, y \in M.
\]

Indeed, since \( v \) is non-decreasing in \( r \),
\[
v(y, r) \leq v(y, r + d(x, y)).
\]

By (\( D_{v}^{D} \))
\[
v(y, r + d(x, y)) \leq C v(x, r + d(x, y))
\]
and by (\( D_{v}^{\kappa_{v}} \))
\[
v(x, r + d(x, y)) \leq C \left( 1 + \frac{d(x, y)}{r} \right)^{\kappa_v} v(x, r).
\]

Therefore, using (\( D_{v}^{D_{v}} \)) and the fact that \( f \in L^2(B(x, r), \mu) \),
\[
K^2(f, T) \leq C \left[ v(x, \sqrt{T}) \right]^{2\gamma_0^{-1}} \left\| f \right\|_2^2
\]
\[
\sup_{0 \leq t \leq T} \left( \sum_{k=1}^{\infty} \left( 1 + \frac{r(k+1)}{\sqrt{T}} \right)^{\kappa_v \left( \frac{1}{\gamma_0} - \frac{1}{2} \right)} \exp \left( -\frac{(k-1)^2r^2}{4t} \right) + \left( 1 + \frac{r}{\sqrt{T}} \right)^{\kappa_v \left( \frac{1}{\gamma_0} - \frac{1}{2} \right)} \right)
\]
\[
\leq C \left[ v(x, \sqrt{T}) \right]^{2\gamma_0^{-1}} \left\| f \right\|_2^2
\]
\[
\left( \sum_{k=1}^{\infty} \left( 1 + \frac{r(k+1)}{\sqrt{T}} \right)^{\kappa_v \left( \frac{1}{\gamma_0} - \frac{1}{2} \right)} \exp \left( -\frac{(k-1)^2r^2}{4t} \right) + \left( 1 + \frac{r}{\sqrt{T}} \right)^{\kappa_v \left( \frac{1}{\gamma_0} - \frac{1}{2} \right)} \right) < +\infty.
\]

Taking the supremum for \( t \in [0, T] \) in (4.11) yields
\[
K(f, T) \leq C \left\| f \right\|_r^\theta K(f, T)^{1-\theta}, \quad \forall T > 0,
\]
hence, since \( K(f, T) \) is finite,
\[
K(f, T) \leq C' \left\| f \right\|_r, \quad \forall T > 0.
\]

It follows that
\[
\sup_{t > 0} \left\| v \frac{1}{\sqrt{t}} e^{-tL} f \right\|_2 \leq C' \left\| f \right\|_{p_0},
\]
for all \( x \in M, \ r > 0 \), and \( f \in L^2(B(x, r), \mu) \cap L^{p_0}(B(x, r), \mu) \), but this estimate does not depend on \( x \) and \( r \). Therefore (\( v E_{p_0, 2} \)) holds and by Proposition 4.1.6 we obtain all estimates (\( v E_{p_0, 2, \gamma} \)), \( \gamma \in \mathbb{R} \), and in particular (\( E_{p_0, 2} \)).

□
Proposition 4.1.10. Let \((M, d, \mu, v)\) be as in Proposition 4.1.11. Assume in addition that \((M, d, \mu, L)\) satisfies (DG), and that \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\). Let \(q > 2\). Then \((vE_{2,q})\) implies \((DUE^v)\).

Proof. By duality, \((vE_{2,q})\) implies \((Ev_{q',2})\). Proposition 4.1.9 with \(p_0 = 1\) and \(p = q'\) yields \((Ev_{1,2})\), hence \((DUE^v)\) by Corollary 2.1.2.

As a consequence of Propositions 2.3.2 and 4.1.10, we can at last state a converse to Proposition 2.3.1. By using Proposition 2.3.4 instead of Proposition 2.3.2 one could replace in the following \((GN_v^q)\) by any \((GN_{2,q,\beta})\) for \(\beta > (\frac{1}{2} - \frac{1}{q})\kappa_v\). One can also combine with Proposition 2.1.5 to obtain more results.

Proposition 4.1.11. Let \((M, d, \mu)\) be a doubling metric measure space and let
\[v : M \times \mathbb{R}_+ \to \mathbb{R}_+\]
satisfy \((A), (D_v), \) and \((D'_v)\). Assume that \((M, d, \mu, L)\) satisfies (DG) and that \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\). Let \(q > 2\). Then \((GN_v^q)\) implies \((DUE^v)\).

The first assertion of Theorem 4.1.1 follows from Propositions 2.3.1 and 4.1.11.

In the case where \(v = V\), one can use Proposition 4.1.9 together with [21, Corollary 4.16] to obtain \(L^p\) uniform boundedness results for semigroups which are not necessary uniformly bounded on \(L^1(M, \mu)\) or possibly do not even act on this space.

Proposition 4.1.12. Let \((M, d, \mu)\) be a doubling metric measure space. Assume that \((M, d, \mu, L)\) satisfies (DG). Assume further that \((M, \mu, L)\) satisfies \((GN_v^q)\) for some \(q\) such that \(2 < q < +\infty\) and \(\frac{2}{q'} - \kappa < 2\), where \(\kappa\) is as in \((VD''_\alpha)\). Then \((e^{-tL})_{t>0}\) satisfies \((VE_{2,q})\) and is uniformly bounded on \(L^p(M, \mu)\) for \(q' \leq p \leq q\). Let \(p\) outside this interval, \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^p(M, \mu)\) if and only if it satisfies \((VE_{2,q'})\), where \(\bar{q} = \max(p, p')\).

Proof. According to Proposition 2.3.2, \((GN_v^q)\) implies \((VE_{2,q})\) and, by duality, \((VE_{2,q})\) implies \((EV_{q',2})\). Now [21, Corollary 4.16] yields in particular the uniform boundedness of \((e^{-tL})_{t>0}\) on \(L^p(M, \mu)\) for \(q' \leq p \leq q\). Next, for \(1 \leq p < q'\), if \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^p(M, \mu)\), then Proposition 4.1.9 yields \((Ev_{p,2})\), hence \((VE_{2,p'})\) and again [21, Corollary 4.16] yields the converse. The case \(q < p \leq +\infty\) is treated by duality.

As an application of the above, let us present a result on Schrödinger semigroups which applies in particular to the case of negative inverse square potentials (see for instance [21, p.539]). Compare with [24, Theorem 11].

Theorem 4.1.13. Let \((M, d, \mu, L)\) be as in Proposition 4.1.12. Assume in addition that \(L - V\) is strongly positive in the sense of (3.11). Then the Schrödinger semigroup \((e^{-t(L-V)})_{t>0}\) also satisfies estimates \((VE_{2,q'})\) and is uniformly bounded on \(L^p(M, \mu)\) for \(q' \leq p \leq q\). For \(p\) outside this interval, \((e^{-t(L-V)})_{t>0}\) is uniformly bounded on \(L^p(M, \mu)\) if and only if it satisfies estimates \((VE_{2,q'})\), where \(\bar{q} = \max(p, p')\).

Proof. Since \((M, d, \mu, L)\) satisfies \((GN_v^q)\), we can write, using the first inequality in (3.12),
\[\|fV_{r_1 - \frac{1}{2}}\|_q^2 \leq C(\|f\|_2^2 + r^2 \mathcal{E}(f)) \leq C\|f\|_2^2 + r^2 \mathcal{E}_V(f).\]
This shows that \((M, d, \mu, L - \mathcal{V})\) satisfies estimates \((GN_\alpha)\) too and Theorem 4.1.13 follows from Proposition 4.1.12 since \((e^{-t(L-V)})_{t>0}\) satisfies \([DC]\), see [21] Theorem 3.3].

Let us finish this section by an application of Theorem 4.1.13 to the Hodge Laplacian.

**Theorem 4.1.14.** Let \(M\) be a complete non-compact Riemannian manifold satisfying the doubling volume property \((VD)\) and the upper estimate \((DUE)\) for the heat kernel on functions. Let \(\mathcal{V}(x)\) be the negative part of a lower bound on the Ricci curvature at \(x \in M\). Assume that \(\Delta - \mathcal{V}\) is strongly positive. Then the heat semigroup on 1-forms \((e^{-t\Delta})_{t>0}\) is uniformly bounded on \(L^p(M, \mu)\) for all \(p \in (1, +\infty)\) if \(\kappa < 2\) and for \(p \in \left(\frac{2\kappa}{\kappa+2}, \frac{2\kappa}{\kappa-2}\right)\) if \(\kappa \geq 2\), where \(\kappa\) is as in \([VD]\).

**Proof.** This is a straightforward consequence of Proposition 2.3.1, Theorem 4.1.13, and of the well-known domination property 
\[
|\bar{p}_t(x, y)| \leq p_t^\mathcal{V}(x, y), \forall x, y \in M,
\]
where \(\bar{p}_t\) (resp. \(p_t^\mathcal{V}\)) is the kernel of \(e^{-t\Delta}\) (resp. \(e^{-t(\Delta - \mathcal{V})}\)) (see for instance [43]).

**Remark 4.1.15.** Compare Theorem 4.1.14 with the results in [25], where, under stronger assumptions, one obtains a stronger conclusion, namely Gaussian estimates on \(\bar{p}_t\), therefore the boundedness of the Riesz transform on \(L^p(M, \mu)\) for all \(p \in (1, +\infty)\). Note that, if \(\kappa < 2\), the above uniform boundedness of \((e^{-t\Delta})_{t>0}\) on \(L^p(M, \mu)\) yields such Gaussian estimates.

### 4.2. From local Nash to Gagliardo-Nirenberg

One can skip this section in a first reading. Indeed, the second assertion from Theorem 1.2.1 follows from the first one and Remark 3.1.2 in the case where \(E\) is a strongly local regular Dirichlet form. And we have seen in Section 3.2 that in the same setting all the inequalities \((N^v)\), \((KN^v)\), \((LN^v)\) and \((GN_\alpha^v)\), \((KGN_\alpha^v)\), \((LS_\alpha^v)\) are equivalent. Our aim here is to prove all this under the more general assumptions of Theorem 1.2.1, meaning that we replace some properties of Dirichlet forms by the finite propagation speed of the wave equation.

Let \((M, d, \mu)\) be a separable locally compact metric measure space and \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) with associated quadratic form \(E\). Let \(\Omega\) be an open subset of \(M\). There is a classical notion of a restriction of \(L\) to \(\Omega\) with Dirichlet boundary conditions in the case where \(E\) is a strongly local and regular Dirichlet form (see for instance [41] section 2.4.1]). We are going to start this section by showing that the latter assumption can be replaced with \((M, d, \mu, L)\) satisfying \([DC]\). We initially define the Dirichlet operator \(L_\Omega\) on the set \(D(L)(\Omega)\) of all functions \(f \in L^2(\Omega, \mu) \subset L^2(M, \mu)\) such that \(\text{supp}(f) \subset \Omega\) is compact and \(f\) is in the domain \(D\) of the operator \(L\). Thanks to the following Lemma we will be able to consider the Friedrichs extension of \(L_\Omega\). With some abuse of notation we will still denote the resulting self-adjoint operator by \(L_\Omega\) and name it the Dirichlet restriction of the operator \(L\) to the open set \(\Omega\).
Lemma 4.2.1. Assume that \((M, d, \mu, L)\) satisfies \((DG)\) and let \(\Omega\) be an open subset of \(M\). The set \(\mathcal{D}_c(\Omega)\) is dense in \(L^2(\Omega, \mu)\). In addition, the quadratic form \(\mathcal{E}\) restricted to \(\mathcal{D}_c(\Omega)\) is closable in \(L^2(\Omega, \mu)\) and the domain of its closure contains the set \(\mathcal{F}_c(\Omega)\).

Proof. The set of all functions \(f \in L^2(\Omega, \mu)\) with compact support is dense in \(L^2(\Omega, \mu)\), so that it is enough to show that any such function is in the closure of \(\mathcal{D}_c(\Omega)\).

To do this consider the family

\[
\frac{\sin^2 r \sqrt{L} f}{r^2 L}
\]

for \(r > 0\). By spectral theory, the above expression converges to \(f\) in \(L^2(M, \mu)\) when \(r\) goes to zero. Next, note that the Fourier transform of the function \(\lambda \mapsto \frac{\sin^2 \lambda}{\sqrt{\lambda}}\) is supported in the interval \([-2, 2]\). Hence, if \(d(\text{supp}(f), \Omega^c) = \varepsilon > 0\) and \(2r \leq \varepsilon\), it follows from Lemma 4.1.3 that

\[
\text{supp} \frac{\sin^2 r \sqrt{L} f}{r^2 L} \subset \text{supp}(f) \subset \Omega.
\]

In particular,

\[
\text{supp} \frac{\sin^2 r \sqrt{L} f}{r^2 L}
\]

is compact. Also, by spectral theory,

\[
L \frac{\sin^2 r \sqrt{L} f}{r^2 L}
\]

is well-defined as a function in \(L^2(M, \mu)\), that is \(\frac{\sin^2 r \sqrt{L} f}{r^2 L} \in \mathcal{D}\). Therefore \(\frac{\sin^2 r \sqrt{L} f}{r^2 L} \in \mathcal{D}_c(\Omega)\). This proves that \(\mathcal{D}_c(\Omega)\) is dense in \(L^2(\Omega)\). Now consider the operator \(L_\Omega\) which is the restriction of \(L\) to the set \(\mathcal{D}_c(\Omega)\). Note that \(L_\Omega\) is symmetric. By Friedrichs's theorem, the quadratic form corresponding to the operator \(L_\Omega\) is closable.

Let now \(f \in \mathcal{F}_c(\Omega)\). Since

\[
\mathcal{E}\left( \frac{\sin^2 r \sqrt{L} f}{r^2 L}\right) = \left< L \frac{\sin^2 r \sqrt{L} f}{r^2 L}, \frac{\sin^2 r \sqrt{L} f}{r^2 L} \right> = \left| \frac{\sin^2 r \sqrt{L} f}{r^2 L} L^{1/2} f \right|^2,
\]

one sees that

\[
\lim_{r \to 0^+} \mathcal{E}\left( \frac{\sin^2 r \sqrt{L} f}{r^2 L}\right) = \mathcal{E}(f)
\]

and the same argument which we used in the above paragraph to show that \(\mathcal{D}_c(\Omega)\) is dense in \(L^2(\Omega, \mu)\) can be used to prove that the closure of \(\mathcal{D}_c(\Omega)\) with respect to the norm corresponding to \(\mathcal{E}\) contains the set \(\mathcal{F}_c(\Omega)\). This shows that the closures of \(\mathcal{E}\) restricted to \(\mathcal{D}_c(\Omega)\) and \(\mathcal{F}_c(\Omega)\) coincide. \(\square\)

In the next lemma we discuss the relation between the wave propagators for the operator \(L\) and for its Dirichlet restriction \(L_{B(x, r_0)}\) to an open ball \(B(x, r_0)\).
Lemma 4.2.2. Assume that \((M, d, \mu, L)\) satisfies \([DG]\). Let \(x \in M\), \(r_0 > 0\), and \(0 < \varepsilon < r_0\). Then
\[
\cos(r\sqrt{L})\chi_{B(x, \varepsilon)} = \cos(r\sqrt{L_{B(x, r_0)}})\chi_{B(x, \varepsilon)}
\]
for all \(r\) such that \(0 < r < r_0 - \varepsilon\).

Proof. It is enough to prove that
\[
\cos(r\sqrt{L})f = \cos(r\sqrt{L_{B(x, r_0)}})f
\]
for \(f \in L^2(B(x, \varepsilon), \mu)\). According to Lemma 4.2.1 one can assume in addition \(f \in D\). Since \(\text{supp}(f) \subset B(x, \varepsilon)\), the finite propagation speed for the wave equation associated with \(L\) yields
\[
\text{supp } \cos(r\sqrt{L})f \subset B(x, r + \varepsilon) \subset B(x, r_0)
\]
for all \(r\) such that \(0 < r < r_0 - \varepsilon\). Hence, for \(0 < r < r_0 - \varepsilon\),
\[
L \cos(r\sqrt{L})f = L_{B(x, r_0)}\cos(r\sqrt{L})f = -\partial^2_r \cos(r\sqrt{L})f,
\]
i.e. the function \(F(\cdot, r) = \cos(r\sqrt{L})f\) is a solution of the wave equation
\[
L_{B(x, r_0)}F = -\partial_r^2 F.
\]
The standard argument shows that for any solution of the above equation the energy function \(E(r) = |F_r|^2 + \langle L_{B(x, r_0)}F, F \rangle\) is conserved. The energy conservation implies the uniqueness of the solutions of the wave equation which in turn implies the claim. \(\square\)

In the following result we show that under an \(L^1\) uniform boundedness assumption on the semigroup \((e^{-tL})_{t>0}\), \((LN^v_\alpha)\) implies \((GN^q_\alpha)\) for \(q > 2\) small enough in terms of \(\alpha\). Together with the results of Sections 3.1 and 3.2 this yields the full equivalence, under the following assumptions, of \((N^v)\), \((KN^v)\), \((LN^v)\), and, for \(q > 2\) small enough, of \((GN^q)\), \((KGN^q)\), \((LS^q)\).

Proposition 4.2.3. Let \((M, d, \mu)\) be a doubling metric measure space, \(v : M \times \mathbb{R}_+ \rightarrow \mathbb{R}_+\) satisfy \((A)\), \((D_v)\), and \([DG]\), and \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) such that \((M, d, \mu, L)\) satisfies \([DG]\). Assume that the semigroup \(\exp(-tL_{B(x, r)})\) is bounded on \(L^1(B(x, r), \mu)\) uniformly in \(t > 0\), \(x \in M\), and \(r > 0\). Then \((LN^v_\alpha)\) implies \((GN^q_\alpha)\) for all \(q\) such that \(2 < q < +\infty\) and \(\frac{q-2}{q} < \alpha\).

Proof. Condition \((LN^v_\alpha)\) can be stated in the following way
\begin{equation}
\|f\|_{L^2}^{2(1+\alpha)} \leq \frac{C_r^2}{v^\alpha_r(x)} \|f\|_1^2 E_{L_{B(x, 3r) + r-2I}}(f), \quad \forall x \in M, \ r > 0, \ f \in \mathcal{F}_c(B(x, 3r)).
\end{equation}

Now since by assumption the semigroup \((\exp(-sL_{B(x, 3r)}))_{s>0}\) is uniformly bounded on \(L^1(B(x, 3r), \mu)\), this also holds for \((\exp(-s(L_{B(x, 3r) + r-2I}))_{s>0}\). Then by Nash’s classical argument, (4.13) implies
\[
\|\exp(-s(L_{B(x, 3r) + r-2I}))\|_{1 \rightarrow \infty} \leq \frac{C_r^{2/\alpha}}{v^\alpha_r(x)} s^{-1/\alpha}, \quad \forall x \in M, \ r, s > 0,
\]
hence
\[ \| \exp(-s(L_{B(x,3r)} + r^{-2}I)) \|_{2 \to q} \leq \left( \frac{C''r^{2/\alpha}}{v_r(x)} \right)^\frac{1}{2} \| s^{-\frac{1}{\alpha}(\frac{1}{2} - \frac{1}{q})} \|, \forall x \in M, r, s > 0. \]

In the last two inequalities as well as in the sequel, the \( L^p \) norms have to be understood on \( L^p(B(x,3r), \mu) \). Let \( \lambda > 0 \). By integrating in \( s > 0 \) the function \( s \to e^{-s}e^{-s\lambda H} \), with
\[ H = L_{B(x,3r)} + r^{-2}I, \]
we obtain, for \( \frac{q-2}{q} < \alpha \),
\[ \| (I + \lambda(L_{B(x,3r)} + r^{-2}I))^{-1} \|_{2 \to q} \leq \left( \frac{C''r^{2/\alpha}}{v_r(x)} \right)^\frac{1}{2} \lambda^{-\frac{1}{\alpha}(\frac{1}{2} - \frac{1}{q})}, \forall \lambda > 0, \]
where \( C'' \) does not depend on \( \lambda \).
Taking \( \lambda = 2r^2 \) yields
\[ \| (I + r^2L_{B(x,3r)})^{-1} \|_{2 \to q} \leq \frac{C}{\sqrt{v_r^{\frac{1}{2} - \frac{1}{q}}(x)}}, \forall r > 0, x \in M. \]

Set \( \Psi(\lambda) = \frac{\lambda - \sin \lambda}{\lambda^{\frac{3}{2}}}. \) Note that by the spectral theorem, the function \( (1 + \lambda^2)\Psi(\lambda) \) and its inverse being bounded,
\[ \| (I + r^2L_{B(x,3r)})^{-1} \|_{2 \to q} \lesssim \| \Psi(r\sqrt{L_{B(x,3r)}}) \|_{2 \to q}, \]
uniformly in \( r > 0 \) and \( x \in M \). Therefore
\[ \| \Psi(r\sqrt{L_{B(x,3r)}}) \|_{2 \to q} \leq \frac{C}{\sqrt{v_r^{\frac{1}{2} - \frac{1}{q}}(x)}}, \forall r > 0, x \in M. \]

and also
\[ \| \Psi(r\sqrt{L_{B(x,3r)}}) \chi_{B(x,3r)} \|_{2 \to q} \leq \frac{C}{\sqrt{v_r^{\frac{1}{2} - \frac{1}{q}}(x)}}, \forall x \in M, t > 0. \]

Now Lemma [1.2.2] with \( \varepsilon = r, r_0 = 3r \) yields
\[ \cos(r\sqrt{L})\chi_{B(x,r)} = \cos(r\sqrt{L_{B(x,3r)}})\chi_{B(x,r)}. \]
Since the Fourier transform of \( \Psi \) is supported in the interval \([-1,1] \), one can use formula (4.6) and conclude
\[ \Psi(r\sqrt{L})\chi_{B(x,r)} = \Psi(r\sqrt{L_{B(x,3r)}})\chi_{B(x,r)}. \]
Hence
\[ \| \Psi(r\sqrt{L})\chi_{B(x,r)} \|_{2 \to q} \leq \frac{C}{\sqrt{v_r^{\frac{1}{2} - \frac{1}{q}}(x)}}, \forall x \in M, r > 0. \]

Since by Lemma [1.13]
\[ \text{supp} \Psi(r\sqrt{L}) \subseteq D_r, \]
one may write
\[ \Psi(r\sqrt{L})\chi_{B(x,r)} = \chi_{B(x,2r)}\Psi(r\sqrt{L})\chi_{B(x,r)}. \]
and, by \([D^c_v]\),
\[
\left\| v^{\frac{1}{2} - \frac{1}{2}} t L \Psi(r \sqrt{L}) \chi_{B(x,r)} \right\|_{2 \to q} = \left\| v^{\frac{1}{2} - \frac{1}{2}} t L \Psi(r \sqrt{L}) \chi_{B(x,r)} \right\|_{2 \to q} \\
\leq v^{\frac{1}{2} - \frac{1}{2}} t L \chi_{B(x,2r)} \Psi(r \sqrt{L}) \chi_{B(x,r)} \right\|_{2 \to q} \\
= v^{\frac{1}{2} - \frac{1}{2}} t L \chi_{B(x,r)} \right\|_{2 \to q}.
\]
Thus it follows from (4.15) that
\[
\left\| v^{\frac{1}{2} - \frac{1}{2}} t L \Psi(r \sqrt{L}) \chi_{B(x,r)} \right\|_{2 \to q} \leq C, \quad \forall x \in M, r > 0.
\]
Lemma 4.1.12 yields
\[
\left\| v^{\frac{1}{2} - \frac{1}{2}} t L \Psi(r \sqrt{L}) \right\|_{2 \to q} \leq C.
\]
Using the \(L^2\)-boundedness of \(\Psi^{-1}(r \sqrt{L}(I + r^2 L)^{-1})\), we obtain
\[
\left\| v^{\frac{1}{2} - \frac{1}{2}} t L \Psi(r \sqrt{L}) \right\|_{2 \to q} \leq C,
\]
that is, \([vF_{2,q}]\). Since \(v\) satisfies \((D_v)\), we can use Proposition 2.3.4 and \((GN_v^u)\) follows.

Propositions 3.1.4, 4.2.3 and 4.1.11 yield the following. Together with Proposition 3.1.4, this gives finally as a by-product a converse to Proposition 2.2.2.

**Proposition 4.2.4.** Let \((M, d, \mu)\) be a doubling metric measure space, \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfy \((A), (D_v)\), and \([D^c_v]\), and \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\) such that \((M, d, \mu, L)\) satisfies \((DG)\). Assume that the semigroup \(\exp(-tL)\) is bounded on \(L^1(B(x, r), \mu)\) uniformly in \(t > 0, x \in M, \) and \(r > 0\). Then \((LN^v)\) implies \((DUE^v)\).

Verifying that the semigroup \(\exp(-tL)\) is bounded on \(L^1(B(x, r), \mu)\) uniformly in \(t > 0, x \in M, \) and \(r > 0\) is possibly not always an easy task. The final result of this section shows that the situation simplifies if the semigroup \(\exp(-tL)\) is positivity preserving.

**Proposition 4.2.5.** Suppose that \((M, d, \mu, L)\) satisfies \((DG)\), that the semigroup \(\exp(-tL)\) is positivity preserving and uniformly bounded on \(L^p(M, \mu)\) for some \(1 \leq p \leq \infty\). Then the semigroups \(\exp(-tL_B(x, r))\) are bounded on \(L^p(B(x, r), \mu)\) uniformly in \(t > 0, x \in M, \) and \(r > 0\).

**Proof.** We observe that the semigroup \(\exp(-tL_B(x, r))\) is also positivity preserving and dominated by the original semigroup \(\exp(-tL)\). That is if \(f \in L^2(\Omega, \mu)\) and \(f \geq 0\) a.e. then
\[
0 \leq \exp(-tL_B(x, r))f \leq \exp(-tL)f, \; \mu - \text{a.e.}
\]
Indeed it is not difficult to check that the proof of the similar property described in [2], (4.6), Theorem 4.2.1 or [26], Proposition 2.1, can be adapted to our setting (see also [38], Proposition 4.23). Now the uniform boundedness of the semigroup \(\exp(-tL_B(x, r))\) is a straightforward consequence of the domination property described above, positivity and uniform boundedness on \(L^p(M, \mu)\) of the initial semigroup \(\exp(-tL)\).
We can finally state:

**Proposition 4.2.6.** Let \((M, d, \mu)\) be a doubling metric measure space, \(v : M \times \mathbb{R}_+ \to \mathbb{R}_+\) satisfy (A), \((D_v)\), and \((D'_v)\), and \(L\) a non-negative self-adjoint operator on \(L^2(M, \mu)\). Assume that \((M, d, \mu, L)\) satisfies the Davies-Gaffney estimate \((DG)\) and that the semigroup \((e^{-tL})_{t>0}\) is uniformly bounded on \(L^1(M, \mu)\) and positivity preserving. Then \((LN^v)\) implies the upper bound \((DUE^v)\).

The second assertion of Theorem 1.2.1 follows from Propositions 2.2.1 and 4.2.6.

Let us conclude by a remark on on-diagonal sub-Gaussian heat kernel estimates. They are typically of the type
\[
p_t(x, x) \leq \frac{C}{V(x, t^{1/m})},
\]
for \(m > 2\), and they can be characterised by Faber-Krahn type inequalities together with exit time estimates (see for instance [37], [45], and the references therein). One could certainly replace these Faber-Krahn inequalities by inequalities similar to the ones we have considered in this paper, simply by taking \(v(x, r) = V(x, r^{2/m})\). But if we try to get rid of the exit time estimates and make our theory fully work as in the Gaussian case, we encounter two obstacles: we lose \((D'_v)\), and more importantly our main tool, namely the finite speed propagation of the wave equation. The only hope would be to exploit the existence of specific cut-off functions related to the exponent \(m\) as in [11].

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