Asymptotic behavior of second-order dissipative evolution equations combining potential with non-potential effects

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Dedicated to Alain Haraux on the occasion of his 60th birthday

Abstract We investigate, in the setting of a real Hilbert space \(\mathcal{H}\), the asymptotic behavior, as time \(t\) goes to infinity, of trajectories of second-order evolution equations

\[
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A(u(t)) = 0,
\]

where \(\nabla \phi\) is the gradient operator of a convex differentiable potential function \(\phi : \mathcal{H} \to \mathbb{R}\), \(A : \mathcal{H} \to \mathcal{H}\) is a maximal monotone operator which is assumed to be \(\lambda\)-cocoercive, and \(\gamma > 0\) is a damping parameter. Potential and non-potential effects are associated respectively to \(\nabla \phi\) and \(A\). We prove that, under condition \(\lambda \gamma^2 > 1\), each trajectory asymptotically weakly converges to a zero of \(\nabla \phi + A\). This condition, which only involves the non-potential operator and the damping parameter, is sharp and consistent with time rescaling. Passing from weak to strong convergence of the trajectories is obtained by introducing an asymptotically vanishing Tikhonov-like regularizing term. As special cases, we recover the asymptotic analysis of the heavy ball with friction dynamic attached to a convex potential, the second-order gradient-projection dynamic, and the second-order dynamic governed by the Yosida approximation of a general maximal monotone operator. The breadth and flexibility of the proposed framework is illustrated through applications in the areas of constrained optimization, dynamical approach to Nash equilibria for noncooperative games, and asymptotic stabilization in the case of a continuum of equilibria.

Key words: Second-order evolution equations, asymptotic behavior, dissipative systems, maximal monotone operators, potential and non-potential operators, cocoercive operators, Tikhonov regularization, heavy ball with friction dynamic system, constrained optimization, coupled systems, dynamical games, Nash equilibria.

AMS Subject Classifications (2000): 34C35, 34D05, 65C25, 90C25, 90C30.

1 Introduction

Throughout this paper, \(\mathcal{H}\) is a real Hilbert space. Its scalar product is denoted by \(\langle \cdot, \cdot \rangle\) and the associated norm by \(|\cdot|\). We wish to investigate the asymptotic behavior, as the time variable \(t\) goes to \(+\infty\), of second order evolution equations

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\[ \ddot{u}(t) + \gamma \dot{u}(t) + \mathcal{A}(u(t)) = 0 \] (1)

where \( \mathcal{A} : \mathcal{H} \to \mathcal{H} \) is a maximal monotone operator and \( \gamma > 0 \) is a positive damping parameter. Here and henceforth, a dot (\( \dot{\cdot} \)) denotes first-order differentiation with respect to time \( t \), and a double dot (\( \ddot{\cdot} \)) denotes second-order differentiation. In order to grasp the respective influence of potential and non-potential operators in such inertial systems, and their effects on the asymptotic behavior of trajectories, we consider the class of maximal monotone operators which can be splitted up into the sum of two terms \( \mathcal{A} = \nabla \phi + A \) where

- \( \nabla \phi : \mathcal{H} \to \mathcal{H} \) is the gradient operator of a convex and continuously differentiable function \( \phi : \mathcal{H} \to \mathbb{R} \);
- \( A : \mathcal{H} \to \mathcal{H} \) is a maximal monotone operator that is assumed to be cocoercive, which means that there exists some constant \( \lambda > 0 \) such that
  \[ \forall (v, w) \in \mathcal{H}^2 \quad \langle Av - Aw, v - w \rangle \geq \lambda |Av - Aw|^2. \] (2)

An operator \( A : \mathcal{H} \to \mathcal{H} \) which satisfies (2) is said to be \( \lambda \)-cocoercive (the relevancy of this hypothesis with respect to applications is examined below). Note that \( \lambda \)-cocoercive implies that \( A \) is \((1/\lambda)\)-Lipschitz continuous.

An outline of the present work is as follows:

1. In section 2, we study the asymptotic behavior of second-order autonomous evolution systems governed by such operators \( \mathcal{A} = \nabla \phi + A \), namely
  \[ \ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A(u(t)) = 0, \] (3)
  with \( \phi \) convex and continuously differentiable, \( A \) \( \lambda \)-cocoercive for some \( \lambda > 0 \), and \( \gamma > 0 \) as a damping parameter. Let us denote by
  \[ S := \{ v \in \mathcal{H} \mid \nabla \phi(v) + Av = 0 \} \] (4)
  the set of equilibria and suppose that \( S \neq \emptyset \). In Theorem 2.1 we establish that, under the sole assumption
  \[ \lambda \gamma^2 > 1, \] (5)
  each trajectory \( t \to u(t) \) of (3) weakly converges in \( \mathcal{H} \), as \( t \to +\infty \), to an element of \( S \). This condition, which only involves the non-potential part of the maximal monotone operator governing the equation, is proved to be sharp, in the sense that one can exhibit situations where condition \( \lambda \gamma^2 < 1 \) does not ensure convergence of all trajectories.

  When the operator \( \nabla \phi + A \) is strongly monotone we prove strong asymptotic convergence of the trajectories.

  In the potential case, \( \mathcal{A} = \nabla \phi \) (corresponding to \( A = 0 \)), taking advantage of the fact that, in the above result, no restrictive assumption is made on the potential operator \( \nabla \phi \), we recover the asymptotic convergence result for the so-called heavy ball with friction dynamical system (Alvarez [2]).
\[ \ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) = 0. \] 

(6)

In recent years, because of its rich connections with mechanics and optimization, this system has been the object of active research, see \cite{2}, \cite{8}, \cite{11}, \cite{14}, and the references therein.

In the non-potential case, \( \nabla \phi = 0 \), (3) becomes

\[ \ddot{u}(t) + \gamma \dot{u}(t) + A(u(t)) = 0 \] 

(7)

with \( A \) a cocoercive operator. Equation (7) covers several situations of practical interest:

- \( A = I - T \) where \( T : \mathcal{H} \rightarrow \mathcal{H} \) is a contraction. It can be easily checked that \( A \) is \((1/2)\)-cocoercive. Condition (5) gives \( \gamma > \sqrt{2} \). When specialized to this situation, Theorem 2.1 yields convergence results for the second-order gradient-projection dynamical system, first established in \cite{5} and \cite{8}.

- \( A = B_\lambda \) where \( B_\lambda \) (with parameter \( \lambda > 0 \)) is the Yosida approximation of a general maximal monotone operator \( B : \mathcal{H} \rightarrow 2^{\mathcal{H}} \), (see \cite{19}). One can easily verify that \( B_\lambda \) is \( \lambda \)-cocoercive. Noticing that \( B \) and \( B_\lambda \) have the same zeroes, we shall derive new inertial second-order dynamical approach to the set of zeroes \( B^{-1}(0) \), for \( B \) a general maximal monotone operator.

2. In section 3, we introduce a Tikhonov-like regularizing term \( \nabla \Theta(u(t)) \) with vanishing coefficient \( \epsilon(t) \) in the above dynamic, and consider the nonautonomous system

\[ \ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A(u(t)) + \epsilon(t) \nabla \Theta(u(t)) = 0, \] 

(8)

where the function \( \Theta : \mathcal{H} \rightarrow \mathbb{R} \) is supposed to be convex, differentiable, with \( \nabla \Theta \) Lipschitz continuous and strongly monotone, while \( \epsilon : [0, +\infty) \rightarrow [0, +\infty) \) is a function of class \( C^1 \) such that \( \epsilon(t) \rightarrow 0^+ \) as \( t \rightarrow +\infty \).

In Theorem 3.1 we establish that, under condition (5), and the slow vanishing condition on \( \epsilon(\cdot) \)

\[ \int_0^{+\infty} \epsilon(s)ds = +\infty, \] 

(9)

each trajectory of (8) strongly converges as \( t \rightarrow +\infty \) to \( u_* \), which is the unique minimizer of \( \Theta \) over the set \( S := (\nabla \phi + A)^{-1}(0) \). Equivalently, \( u_* \) solves the following variational inequality problem: find \( u_* \in S \) such that

\[ \langle \nabla \Theta(u_*), v - u_* \rangle \geq 0 \quad \forall v \in S. \] 

(10)

Note that the above regularization technique allows both to obtain strong convergence of the trajectories, together with a limit which no longer depends on the initial data. As a consequence of Theorem 3.1 we recover various results which have been devoted to the Tikhonov dynamics, see \cite{13}, \cite{23} and the references therein.
In the final section, we briefly outline situations where our results can be applied. Indeed, the convergence results obtained in Theorems 2.1 and 3.1 offer promising views on numerical optimization and algorithms (by time discretization as in [2], [4], [12], [22]), on the modeling of inertial dynamical approach to Nash equilibria in decision sciences and game theory (see [15], [23]), and on dissipative dynamical systems and PDE’s (as in [6] for the damped wave equation).

2 Asymptotic convergence results

This section is devoted to the study of the asymptotic behavior, as time variable $t$ goes to infinity, of trajectories of (3).

2.1 Weak asymptotic convergence results

Throughout this section we make the following assumptions:

(H1) $\phi : H \rightarrow \mathbb{R}$ is a convex differentiable function whose gradient $\nabla \phi$ is Lipschitz continuous on the bounded subsets of $H$;

(H2) $A : H \rightarrow H$ is maximal monotone and $\lambda$-cocoercive for some $\lambda > 0$.

Note that the cocoerciveness of $A$ implies that $A$ is Lipschitz continuous, so that $\nabla \phi + A$ is Lipschitz continuous on bounded sets. By Cauchy-Lipschitz theorem, for any initial data $u(0) = u_0$, $\dot{u}(0) = v_0$ with $(u_0, v_0)$ in $H^2$ there exists a unique local solution to the Cauchy problem

$$
\begin{cases}
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A(u(t)) = 0, \\
u(0) = u_0, \quad \dot{u}(0) = v_0.
\end{cases}
$$

The fact that $u$ is infinitely extendible to the right follows from a uniform bound on $|\dot{u}(t)|$ as given in the proof of Theorem 2.1, which provides a unique classical global solution $u \in C^2([0, +\infty); H)$ of (11).

The following theorem establishes the weak asymptotic convergence property of $u(t)$, solution of (11), under the sole assumption $\lambda \gamma^2 > 1$.

**Theorem 2.1** Let us suppose that (H1)-(H2) hold with $S := (\nabla \phi + A)^{-1}(0) \neq \emptyset$, and that the cocoercive parameter $\lambda$ and the damping parameter $\gamma$ satisfy

$$
\lambda \gamma^2 > 1.
$$

Then, for each initial data $u_0$ and $v_0$ in $H$, the unique solution $u \in C^2([0, +\infty); H)$ of (11) satisfies:

i1) There exists $u_\infty \in S$ such that $u(t) \rightharpoonup u_\infty$ weakly in $H$ as $t \rightarrow +\infty$.

Moreover,

i2) $\dot{u} \in L^2(0, +\infty; H)$; $\lim_{t \rightarrow +\infty} |\dot{u}(t)| = 0$;

i3) $\ddot{u} + \nabla \phi(u) + Ap \in L^2(0, +\infty; H)$ whenever $p \in S$;

i4) for every $p \in S$, $\lim_{t \rightarrow +\infty} |u(t) - p|$ exists.
Before proceeding with the proof of Theorem 2.1, we establish three technical lemmas. The following lemma plays a crucial role in the proof of optimality of the weak cluster points of a given trajectory of (11). Because of its independent interest, it is stated in a general setting, the potential \( \Phi \) being allowed to be non-smooth. We use the classical notation \( \partial \Phi \) for the subdifferential operator of \( \Phi \) (it coincides with the gradient operator in the smooth case).

**Lemma 2.1** Let \( \Phi : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) be a convex and lower semicontinuous function, \((u_n)\) a bounded sequence in \( \mathcal{H} \), and \( p \in \mathcal{H} \). If there exist \( \eta_n \in \partial \Phi(u_n) \) and \( \bar{\eta} \in \partial \Phi(p) \) such that
\[
\lim_{n \to +\infty} \langle u_n - p, \eta_n - \bar{\eta} \rangle = 0, \tag{13}
\]
then any weak cluster point \( \bar{u} \) of \((u_n)\) satisfies
\[
\bar{\eta} \in \partial \Phi(\bar{u}). \tag{14}
\]

**Proof:** Set \( w_n = \langle u_n - p, \eta_n - \bar{\eta} \rangle \) and introduce the functional \( F : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) defined for any \( v \in \mathcal{H} \) by
\[
F(v) = \Phi(v) - \Phi(p) - \langle v - p, \bar{\eta} \rangle. \tag{15}
\]
By convexity of \( \Phi \) and \( \bar{\eta} \in \partial \Phi(p) \), we observe that \( F \) is a convex and nonnegative function, and it can be easily checked that
\[
F(u_n) = \Phi(u_n) - \Phi(p) - \langle u_n - p, \eta_n \rangle + w_n.
\]
Therefore, from convexity of \( \Phi \) and \( \eta_n \in \partial \Phi(u_n) \), we have
\[
0 \leq F(u_n) \leq w_n,
\]
which by (13) (that is \( w_n \to 0 \) as \( n \to +\infty \)) gives
\[
\lim_{n \to +\infty} F(u_n) = 0. \tag{16}
\]
Now consider any weak cluster point \( \bar{u} \) of \((u_n)\), namely there exists a subsequence \((u_{n_k})\) of \((u_n)\) such that \((u_{n_k})\) weakly converges to \( \bar{u} \) as \( k \to +\infty \). Then invoking the weak lower semicontinuity of \( F \) (as it is convex and lower semicontinuous) and using (16), we obtain
\[
0 \leq F(\bar{u}) \leq \liminf_{k \to +\infty} F(u_{n_k}) = \lim_{n \to +\infty} F(u_n) = 0,
\]
which entails \( F(\bar{u}) = 0 \). This implies that \( \bar{u} \) is a minimizer of the convex and nonnegative function \( F \), so that \( 0 \in \partial F(\bar{u}) \), which from (15) is equivalent to (14).

**Lemma 2.2** Let \( A : \mathcal{H} \rightarrow \mathcal{H} \) be a maximal monotone single-valued operator and \( \phi : \mathcal{H} \rightarrow \mathbb{R} \) a convex differentiable function such that \( S := (\nabla \phi + A)^{-1}(0) \neq \emptyset \). Suppose that \( p \in S \) and \((u_n)\) is a bounded sequence in \( \mathcal{H} \) verifying
\[
\begin{align*}
(c1) & \quad \lim_{n \to \infty} \langle \nabla \phi(u_n) - \nabla \phi(p), u_n - p \rangle = 0, \\
(c2) & \quad \lim_{n \to \infty} |Au_n - Ap| = 0.
\end{align*}
\]
Then, any weak cluster point of \((u_n)\) belongs to \( S \).
Proof: Let \( \bar{u} \) be a weak cluster point of \( (u_n) \). From (c1) and invoking Lemma 2.3, we have \( \nabla \phi(p) = \nabla \phi(\bar{u}) \). Moreover, from (c2) and recalling that the maximal monotone operator \( A \) has a graph which is closed in \( w - \mathcal{H} \times s - \mathcal{H} \) (see [19] for example), we have \( Au = Ap \). As a straightforward consequence, \( \nabla \phi(\bar{u}) + Au = \nabla \phi(p) + Ap = 0 \), so that \( \bar{u} \in S \), which completes the proof. ●

The following lemma will be used in the proof of Theorem 2.1 in order to obtain the asymptotic convergence of the mapping \( t \to |u(t) - p| \), whenever \( p \in S \) and \( u \) is solution of (11). This lemma appears implicitly in [2], its proof is given for the sake of completeness.

Lemma 2.3 If \( w \in C^2([0, +\infty); \mathbb{R}) \) is bounded from below and satisfies the following inequality
\[
\dot{w}(t) + \gamma \ddot{w}(t) \leq g(t),
\] (17)
where \( \gamma \) is a positive constant and \( g \in L^1([0, +\infty); \mathbb{R}) \), then \( w(t) \) converges as \( t \to +\infty \).

Proof: From (17) and denoting \( [\dot{w}]_+ = \max\{\dot{w}(t), 0\} \), we classically have
\[
[\dot{w}(t)]_+ \leq e^{-\gamma t}[\dot{w}(0)]_+ + \int_0^t e^{-\gamma(t-\tau)}|g(\tau)|d\tau,
\]
while Fubini’s theorem gives us
\[
\int_0^{+\infty} \int_0^t e^{-\gamma(t-\tau)}|g(\tau)|d\tau dt = \frac{1}{\gamma} \int_0^{+\infty} |g(\tau)|d\tau < +\infty.
\]
This shows that \( [\dot{w}]_+ \in L^1([0, +\infty); \mathbb{R}) \). Now setting \( z(t) = w(t) - \int_0^t [\dot{w}(\tau)]_+ d\tau \), we observe that \( z(.) \) is bounded from below (as \( w(.) \) is assumed to be bounded from below) with \( \dot{z}(t) = \dot{w}(t) - [\dot{w}(t)]_+ \leq 0 \), hence \( z \) converges as \( t \to +\infty \), and so does \( w(t) \). ●

We are now in position to prove the main result of this section.

Proof of Theorem 2.1 Take \( p \in S = (\nabla \phi + A)^{-1}(0) \) and set \( h(t) = (1/2)|u(t) - p|^2 \). From \( \dot{h}(t) = \langle u(t) - p, \dot{u}(t) \rangle \) and \( \ddot{h}(t) = \langle u(t) - p, \ddot{u}(t) \rangle + |\dot{u}(t)|^2 \) we obtain
\[
\ddot{h}(t) + \gamma \dot{h}(t) = \langle u(t) - p, \ddot{u}(t) \rangle + |\dot{u}(t)|^2, \tag{18}
\]
which, by using (11) yields
\[
\ddot{h}(t) + \gamma \dot{h}(t) + \langle \nabla \phi(u(t)) + Au(t), u(t) - p \rangle = |\dot{u}(t)|^2. \tag{19}
\]
Recalling that \( \nabla \phi(p) + Ap = 0 \), we deduce that
\[
\ddot{h}(t) + \gamma \dot{h}(t) + \langle \nabla \phi(u(t)) - \nabla \phi(p), u(t) - p \rangle + \langle Au(t) - Ap, u(t) - p \rangle = |\dot{u}(t)|^2. \tag{20}
\]
Then, by using the \( \lambda \)-cocoercive property (2) of \( A \), together with the monotonicity of \( \nabla \phi \), we obtain
\[
\ddot{h}(t) + \gamma \dot{h}(t) + \lambda |Au(t) - Ap|^2 \leq |\dot{u}(t)|^2. \tag{21}
\]
Regarding the third term in the left side of (21), by using (11) again, replacing \( Au(t) \) by \(- (\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t))) \), and setting \( D(t) := |\dot{u}(t) + \nabla \phi(u(t)) + Ap|^2 \), we obtain

\[
|Au(t) - Ap|^2 = |(\ddot{u}(t) + \nabla \phi(u(t)) + Ap) + \gamma \dot{u}(t)|^2
= D(t) + \gamma^2 |\ddot{u}(t)|^2 + 2\gamma \langle \ddot{u}(t), \ddot{u}(t) + \nabla \phi(u(t)) + Ap \rangle
= D(t) + \gamma^2 |\ddot{u}(t)|^2 + \gamma \frac{d}{dt} \left( |\dot{u}(t)|^2 + 2\phi(u(t)) + 2\langle u(t) - p, Ap \rangle \right).
\]

Recalling that \( Ap = -\nabla \phi(p) \) and by using the above equality, let us rewrite (21) as

\[
\ddot{h}(t) + \gamma \dot{h}(t) + (\lambda \gamma^2 - 1) |\dot{u}(t)|^2 + \lambda D(t)
+ \gamma \frac{d}{dt} \left( |\dot{u}(t)|^2 + 2\phi(u(t)) - 2\langle u(t) - p, \nabla \phi(p) \rangle \right) \leq 0.
\]

By using assumption \( \lambda \gamma^2 \geq 1 \), with \( D(t) \geq 0 \) and (22), we then get

\[
\dot{h}(t) + \gamma \dot{h}(t) + \lambda \frac{d}{dt} \left( |\dot{u}(t)|^2 + 2\phi(u(t)) - 2\langle u(t) - p, \nabla \phi(p) \rangle \right) \leq 0,
\]

which expresses that the function

\[
\Gamma_0(t) := \dot{h}(t) + \gamma h(t) + \lambda \gamma \left( |\dot{u}(t)|^2 + 2\phi(u(t)) - 2\langle u(t) - p, \nabla \phi(p) \rangle \right)
\]

is nonincreasing on \([0, +\infty)\). Indeed, \( \Gamma_0(\cdot) \) will serve us as a Lyapunov function in the asymptotic analysis of (11). Let us first show the boundedness of trajectories. Note that by convexity of \( \phi \), we obviously have

\[
\phi(p) \leq \phi(u(t)) - \langle u(t) - p, \nabla \phi(p) \rangle,
\]

hence the nonincreasing property of \( \Gamma_0(\cdot) \) leads to

\[
\dot{h}(t) + \gamma h(t) + \lambda \gamma \left( |\dot{u}(t)|^2 + 2\phi(p) \right) \leq \Gamma_0(t) \leq \Gamma_0(0).
\]

Set

\[
C_0 := |u_0 - p||v_0| + \frac{\gamma}{2} |u_0 - p|^2 + \lambda \gamma |v_0|^2 + 2\lambda \gamma (\phi(u_0) - \phi(p) - \langle u_0 - p, \nabla \phi(p) \rangle),
\]

which clearly satisfies

\[
C_0 \geq \Gamma_0(0) - 2\lambda \gamma \phi(p).
\]

From (26) and (27) we deduce that \( \dot{h}(t) + \gamma h(t) \leq C_0 \), so that, after integration, we obtain

\[
\dot{h}(t) = \frac{1}{2} |u(t) - p|^2 \leq \frac{1}{2} |u_0 - p|^2 + \frac{C_0}{\gamma},
\]

which shows that the trajectory \( t \to u(t) \) remains bounded on \([0, +\infty)\):

\[
\sup_{t \in [0, +\infty)} |u(t)| < +\infty.
\]
Let us now establish estimates on \( \dot{u} \) and \( \ddot{u} \). Integrating (22) from 0 to \( t \), together with (25) yields
\[
\dot{h}(t) + \gamma h(t) + \lambda \gamma |\dot{u}(t)|^2 + (\lambda^2 - 1) \int_0^t |\dddot{u}(s)|^2 ds + \lambda \int_0^t |\dddot{u}(s) + \nabla \phi(u(s)) + Ap|^2 ds \leq C_0,
\]
which implies \( \dot{h}(t) + \lambda \gamma |\dot{u}(t)|^2 \leq C_0 \). Equivalently, by definition of \( h(t) \), we have
\[
\langle u(t) - p, \dot{u}(t) \rangle + \lambda \gamma |\dot{u}(t)|^2 \leq C_0. \tag{30}
\]
Since \( u(t) \) remains bounded on \([0, +\infty)\), inequality (30) implies that \( \dot{u}(t) \) also remains bounded on \([0, +\infty)\):
\[
\sup_{t \in [0, +\infty)} |\dot{u}(t)| < +\infty. \tag{31}
\]
From (23), (31) and \( \dot{h}(t) = \langle u(t) - p, \dot{u}(t) \rangle \), we observe that \( \dot{h}(t) \) is bounded on \([0, +\infty)\). Returning to (29) we deduce that
\[
\int_0^{+\infty} |\dot{u}(s)|^2 ds < +\infty \quad \text{and} \quad \int_0^{+\infty} |\dddot{u}(s) + \nabla \phi(u(s)) + Ap|^2 ds < +\infty. \tag{32}
\]
Using that \( u \) and \( \dot{u} \) are bounded on \([0, +\infty)\), together with the Lipschitz continuity property of \( A \) and equation (3), we deduce that \( \ddot{u} \) is also bounded:
\[
\sup_{t \in [0, +\infty)} |\dddot{u}(t)| < +\infty. \tag{33}
\]
Properties (32) and (33) express that the function \( g := \dot{u} \) satisfies both
\[
g \in L^2(0, +\infty; \mathcal{H}) \quad \text{and} \quad \dot{g} \in L^\infty(0, +\infty; \mathcal{H}).
\]
By a classical result, this implies \( \lim_{t \to +\infty} g(t) = 0 \), that is,
\[
\lim_{t \to +\infty} |\dot{u}(t)| = 0. \tag{34}
\]
Returning to (3), by using that functions \( u \) and \( \dot{u} \) are Lipschitz continuous on \([0, +\infty)\) (as their derivatives are uniformly bounded), that \( A \) is Lipschitz continuous and \( \nabla \phi \) is Lipschitz continuous on bounded sets, we deduce that \( \ddot{u} \) is Lipschitz continuous on \([0, +\infty)\). Once again, this property together with (32) implies that \( \lim_{t \to +\infty} |\ddot{u}(t) + \nabla \phi(u(t)) + Ap| = 0 \), which in light of (3) and (31) leads to
\[
\lim_{t \to +\infty} |Au(t) - Ap| = 0. \tag{35}
\]
We have all the ingredients to conclude thanks to Opial’s lemma, see (32). To that end, we need to prove that properties (a) and (b) are fulfilled:

(a) for every \( p \in S \), \( \lim_{t \to +\infty} |u(t) - p| \) exists;

(b) for every \( t_n \to +\infty \) with \( u(t_n) \to \overline{u} \) weakly in \( \mathcal{H} \), we have \( \overline{u} \in S \).

Opial’s lemma asserts that, under the above two properties, \( u(t) \) weakly converges as \( t \to +\infty \) to an element \( u_\infty \in S \).
Let us first prove (a). From \( (20) \) and the monotonicity of \( A \), we have
\[
\dot{h}(t) + \gamma \dot{h}(t) + w(t) \leq |\dot{u}(t)|^2, \tag{36}
\]
where
\[
w(t) = \langle \nabla \phi(u(t)) - \nabla \phi(p), u(t) - p \rangle.
\]
It is obviously seen that \( w(t) \) is a nonnegative term (thanks to the monotonicity of \( \nabla \phi \)), hence, from \( (36) \), we immediately derive
\[
\dot{h}(t) + \gamma \dot{h}(t) \leq |\dot{u}(t)|^2. \tag{37}
\]
By \( (32) \) we know that \( |\dot{u}|^2 \) belongs to \( L^1(0, +\infty; \mathcal{H}) \). Noticing that \( h \) is nonnegative, Lemma \( 2.3 \) shows that property (a) holds.

Let us now prove (b). By \( (36) \), after integration we obtain
\[
f_0^{+\infty} w(s)ds < +\infty.
\]
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Owing to the Lipschitz continuity property of \( w(\cdot) \) we deduce that \( \lim_{t \to +\infty} w(t) = 0 \).

On the other hand, by \( (35) \) we have \( \lim_{t \to +\infty} |Au(t) - Ap| = 0 \). We can now apply Lemma \( 2.2 \) to get property (b). 

\[ 
\]

### 2.2 The strongly monotone case

Let us recall that the operator \( \nabla \phi + A \) is called **strongly monotone over bounded sets** if, for all positive real number \( R \), there exists a continuous function \( w_R : \mathbb{R}_+ \to \mathbb{R}_+ \) verifying
\[
w_R(t_n) \to 0 \Rightarrow t_n \to 0,
\]
and such that \( (u, v) \in \mathcal{H}_2 \) with \( |u| \leq R \) and \( |v| \leq R \) yields
\[
\langle (\nabla \phi + A)u - (\nabla \phi + A)v, u - v \rangle \geq w_R(|u - v|). \tag{38}
\]
Clearly, this property implies that \( S := (\nabla \phi + A)^{-1}(0) \) contains at most one element and it holds if, for instance, \( \nabla \phi \) or \( A \) is strongly monotone over bounded sets (with the other operator being assumed to be monotone).

**Proposition 2.1** Under the assumptions of Theorem \( 2.1 \), assuming moreover that \( \nabla \phi + A \) is strongly monotone over bounded sets, then the solution \( u(t) \) of \( (14) \) strongly converges as \( t \) goes to \( +\infty \) towards the unique element of \( S := (\nabla \phi + A)^{-1}(0) \).

**Proof:** By Theorem \( 2.1 \), we have \( u, \dot{u} \in L^\infty([0, +\infty); \mathcal{H}) \). Set \( R := \sup_{t \geq 0} |u(t)| + |p| \), with \( p \in S \), and combine \( (20) \) and \( (38) \), to obtain
\[
\dot{h}(t) + w_R(|u(t) - p|) \leq |\dot{u}(t)|^2, \tag{39}
\]
where \( h(t) := (1/2)|u(t) - p|^2 \). After integration, observing that \( h \in L^\infty([0, +\infty); \mathcal{H}) \) and recalling that \( u \in L^2([0, +\infty); \mathcal{H}) \) (Theorem \( 2.1 \) (2)), we deduce that
\[
f_0^{+\infty} w_R(|u(s) - p|)ds < +\infty.
\]
On the other hand, by Theorem \( 2.1 \) (i4), \( \lim_{t \to +\infty} |u(t) - p| \) exists, which by continuity of \( w_R \) implies that \( \lim_{t \to +\infty} w_R(|u(t) - p|) \) exists. Hence, \( \lim_{t \to +\infty} w_R(|u(t) - p|) = 0 \). Then use the property of \( w_R \) \( (w_R(t_n) \to 0 \Rightarrow t_n \to 0) \) to obtain \( \lim_{t \to +\infty} |u(t) - p| = 0 \), which ends the proof. 

\[ 
\]
2.3 Some particular cases

Let us specialize Theorem 2.1 and Proposition 2.1 to some important particular situations.

a) Let us first take the non-potential operator \( A \) equal to zero. Then notice that the null operator is \( \lambda \)-cocoercive for any \( \lambda > 0 \). By taking \( \lambda > (1/\gamma^2) \), we can apply Theorem 2.1. Owing to the fact that no restrictive assumption is made on the potential part, we obtain the Alvarez theorem [2]:

**Corollary 2.1** Let \( \phi : \mathcal{H} \to \mathbb{R} \) be a convex differentiable function whose gradient \( \nabla \phi \) is Lipschitz continuous on the bounded subsets of \( \mathcal{H} \), and let \( \gamma > 0 \). Let us assume that \( S := \text{argmin}_{u} \phi \) is nonempty. Then, for each initial data \( u_0 \) and \( v_0 \) in \( \mathcal{H} \), the unique solution \( u \in C^2([0, +\infty); \mathcal{H}) \) of

\[
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) = 0,
\]

with initial conditions \( u(0) = u_0 \) and \( \dot{u}(0) = v_0 \), satisfies:

i) there exists \( u_{\infty} \in S \) such that \( u(t) \to u_{\infty} \) weakly in \( \mathcal{H} \) as \( t \to +\infty \);

ii) \( \dot{u} \in L^2(0, +\infty; \mathcal{H}) \); \( \lim_{t \to +\infty} |\dot{u}(t)| = 0 \).

b) Operators of the form \( A = I - T \), where \( T : \mathcal{H} \to \mathcal{H} \) is a contraction, play a central role in the realm of fixed point theory and constrained optimization. Let us verify that \( A \) is \((1/2)\)-cocoercive. Indeed, by setting

\[
E := \langle (u - Tu) - (v - Tv), u - v \rangle - \frac{1}{2} |(u - Tu) - (v - Tv)|^2,
\]

we obviously have

\[
E = \langle (u - v) - (Tu - Tv), u - v \rangle - \frac{1}{2} |(u - v) - (Tu - Tv)|^2,
\]

\[
= |u - v|^2 - \langle Tu - Tv, u - v \rangle - \frac{1}{2} |u - v|^2 - \frac{1}{2} |Tu - Tv|^2 + \langle (u - v), Tu - Tv \rangle,
\]

which, by contraction property of \( T \), is nonnegative. Thus we can take \( \lambda = 1/2 \), and condition (5) boils down to \( \gamma > \sqrt{2} \). Applying Theorem 2.1 to this situation, we obtain the following result (see [5], Theorem 3.2):

**Corollary 2.2** Let \( T : \mathcal{H} \to \mathcal{H} \) be a contraction and \( \gamma > \sqrt{2} \). Let us assume that \( S := \text{Fix} T = \{ v \in \mathcal{H} : Tv = v \} \) is nonempty. Then, for each initial data \( u_0 \) and \( v_0 \) in \( \mathcal{H} \), the unique solution \( u \in C^2([0, +\infty); \mathcal{H}) \) of

\[
\ddot{u}(t) + \gamma \dot{u}(t) + u(t) - T(u(t)) = 0,
\]

with initial data \( u(0) = u_0 \) and \( \dot{u}(0) = v_0 \), satisfies:

i) there exists \( u_{\infty} \in \text{Fix} T \) such that \( u(t) \to u_{\infty} \) weakly in \( \mathcal{H} \) as \( t \to +\infty \);

ii) \( \dot{u} \in L^2(0, +\infty; \mathcal{H}) \); \( \lim_{t \to +\infty} |\dot{u}(t)| = \lim_{t \to +\infty} |\ddot{u}(t)| = 0 \).

c) When working with a general maximal monotone operator \( A : \mathcal{H} \to \mathcal{H} \), let us observe that, for every \( \lambda > 0 \), its Yosida approximation \( A_\lambda \) is \( \lambda \)-cocoercive. We recall that \( A_\lambda = \frac{1}{\lambda}(I - J_\lambda^A) \), where \( J_\lambda^A := (I + \lambda A)^{-1} \), is the resolvent operator of index \( \lambda \) of \( A \). The operator \( J_\lambda^A \) is everywhere defined, single-valued (it is a
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contraction) and \( A_\lambda v \in (A \circ J_\lambda^1)v \) for any \( v \in \mathcal{H} \) (see [19]). Let \( (u,v) \in \mathcal{H}^2 \) and set \( E := \langle A_\lambda u - A_\lambda v, u - v \rangle \). Noticing that \( u = J_\lambda^A u + \lambda A_\lambda u \) and \( v = J_\lambda^A v + \lambda A_\lambda v \), and recalling that \( A_\lambda u \in A(J_\lambda^A u) \) and \( A_\lambda v \in A(J_\lambda^A v) \), we immediately obtain

\[
E = \langle A_\lambda u - A_\lambda v, (J_\lambda^A u - J_\lambda^A v) + \lambda(A_\lambda u - A_\lambda v) \rangle \geq \lambda |A_\lambda u - A_\lambda v|^2,
\]

which proves that \( A_\lambda \) is \( \lambda \)-cocoercive. Observing that \( A \) and \( A_\lambda \) have the same zeroes, and as a straight consequence of Theorem 2.1, we reach the following result:

**Corollary 2.3** Let \( A : \mathcal{H} \to \mathcal{H} \) be a general maximal monotone operator and let \( \gamma > 0 \) and \( \lambda > 0 \) be such that \( \lambda \gamma^2 > 1 \). Let us assume that \( S := A^{-1}(0) \), the set of zeroes of \( A \), is nonempty. Then, for each initial data \( u_0 \) and \( v_0 \) in \( \mathcal{H} \), the unique solution \( u \in C^2([0, +\infty); \mathcal{H}) \) of

\[
\ddot{u}(t) + \gamma \dot{u}(t) + A_\lambda(u(t)) = 0,
\]

with initial data \( u(0) = u_0 \) and \( \dot{u}(0) = v_0 \), \( A_\lambda \) being the Yosida approximation of index \( \lambda \) of \( A \), satisfies:

1) there exists \( u_\infty \in S \) such that \( u(t) \rightharpoonup u_\infty \) weakly in \( \mathcal{H} \) as \( t \to +\infty \);
2) \( \dot{u} \in L^2(0, +\infty; \mathcal{H}) \); \( \lim_{t \to +\infty} |\dot{u}(t)| = 0 \).

Let us examine an interesting consequence of Corollary 2.3 regarding numerical schemes. From resolvent equation (see [19], Proposition 2.6), we have \( (A_\lambda^\mu) = A_\lambda + \mu \) whenever \( \lambda > 0 \) and \( \mu > 0 \). Then it can be easily derived that, for any \( v \in \mathcal{H} \),

\[
J_\lambda^A v = \frac{\lambda}{\lambda + \mu} v + \frac{\mu}{\lambda + \mu} J_\lambda^A v.
\]

Implicit time discretization of (41) naturally leads to a second order relaxed proximal algorithm, whose trajectories (sequences) converge to the set of zeroes \( A^{-1}(0) \), for \( A \) a general maximal monotone operator, see [3] and references herein.

**2.4 Condition (5) is sharp**

Let \( B : \mathbb{R}^2 \to \mathbb{R}^2 \) be the \( \pi/2 \) rotation of center \((0,0)\), namely \( B \) is the linear operator whose matrix in the canonical basis is given by

\[
B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Consider the dynamical system

\[
\ddot{X}(t) + \gamma \dot{X}(t) + B_\lambda X(t) = 0,
\]

where \( \gamma > 0 \) and \( B_\lambda \) is the Yosida approximation of \( B \) of parameter \( \lambda > 0 \). An easy computation shows that

\[
B_\lambda = \frac{1}{1 + \lambda^2} H, \quad \text{where } H = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}.
\]
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The operator $B_\lambda$ is $\lambda$-cocoercive, as follows from general properties of Yosida approximation, or by a direct elementary computation. Let us compute explicitly the solutions of (43). The matrix $H$ admits two complex conjugate eigenvalues. One of them is $\rho_1 = \lambda - i$, and it is associated to the eigenvector $W_1 = (1, i)^T$. Subsequently, searching for a solution of (43) of the form $X(t) = e^{rt}W_1$ (with $r \in \mathbb{C}$), we see that $r$ satisfies the quadratic equation

$$r^2 + \gamma r + \frac{\lambda - i}{1 + \lambda^2} = 0.$$  \hspace{1cm} (44)

Equation (44) has a complex discriminant $\Delta = \gamma^2 - \frac{4\lambda}{1 + \lambda^2} + \frac{4i}{1 + \lambda^2} = (x + iy)^2$, where $x$ and $y$ are real values given by

$$x = \frac{1}{\sqrt{2}} \left( (\gamma^2 - \frac{4\lambda}{1 + \lambda^2}) + \sqrt{\left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right)^2 + \frac{16}{(1 + \lambda^2)^2}} \right)^{1/2},$$

$$y = \frac{1}{\sqrt{2}} \left( -(\gamma^2 - \frac{4\lambda}{1 + \lambda^2}) + \sqrt{\left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right)^2 + \frac{16}{(1 + \lambda^2)^2}} \right)^{1/2}.$$  

Hence (44) has two complex solutions

$$r_1 = (1/2)[(-\gamma - x) - iy] = a_1 - ib,$$

$$r_2 = (1/2)[(-\gamma + x) + iy] = a_2 + ib,$$

where $a_1, a_2$ and $b$ are real values defined by

$$a_1 = \frac{1}{2} \left( -\gamma - \frac{1}{\sqrt{2}} \left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right) + \sqrt{\left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right)^2 + \frac{16}{(1 + \lambda^2)^2}} \right)^{1/2},$$

$$a_2 = \frac{1}{2} \left( -\gamma + \frac{1}{\sqrt{2}} \left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right) + \sqrt{\left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right)^2 + \frac{16}{(1 + \lambda^2)^2}} \right)^{1/2},$$

$$b = \frac{1}{2\sqrt{2}} \left( -\gamma^2 + \frac{4\lambda}{1 + \lambda^2} + \sqrt{\left( \gamma^2 - \frac{4\lambda}{1 + \lambda^2} \right)^2 + \frac{16}{(1 + \lambda^2)^2}} \right)^{1/2}.$$  

As a straightforward consequence, we deduce that a family of complex solutions to (43) is given by the functions \{V_1, V_2\}, where $V_1(.)$ and $V_2(.)$ are defined for any $t \in \mathbb{R}$ by

$$V_1(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(a_1 - ib)t} = e^{a_1t} \begin{pmatrix} \cos bt - i \sin bt \\ i \cos bt + \sin bt \end{pmatrix} = e^{a_1t} \begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix} + i \begin{pmatrix} -\sin bt \\ \cos bt \end{pmatrix},$$

$$V_2(t) = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(a_2 + ib)t} = e^{a_2t} \begin{pmatrix} \cos bt + i \sin bt \\ i \cos bt - \sin bt \end{pmatrix} = e^{a_2t} \begin{pmatrix} \cos bt \\ -\sin bt \end{pmatrix} + i \begin{pmatrix} \sin bt \\ \cos bt \end{pmatrix}.$$
Note also that the complex conjugate of any solution to (43) is also a solution to (43). It is then immediate that a family of real solutions to (43) is given by the functions \( \{ U_1, U_2, U_3, U_4 \} \) defined for any \( t \in \mathbb{R} \) by
\[
U_1(t) = e^{a_1t} \begin{pmatrix} \cos bt \\ \sin bt \end{pmatrix}, \quad U_2(t) = e^{a_1t} \begin{pmatrix} -\sin bt \\ \cos bt \end{pmatrix},
\]
\[
U_3(t) = e^{a_2t} \begin{pmatrix} \cos bt \\ -\sin bt \end{pmatrix}, \quad U_4(t) = e^{a_2t} \begin{pmatrix} \sin bt \\ \cos bt \end{pmatrix}.
\]
In light of these last results, we deduce that (43) will have non-convergent solutions if \( a_2 \geq 0 \), or equivalently if
\[
\sqrt{\gamma^4 - \frac{8\gamma^2\lambda}{1 + \lambda^2} + \frac{16}{1 + \lambda^2}} \geq (\gamma^2 + \frac{4\lambda}{1 + \lambda^2}). \tag{45}
\]
In addition, setting \( \lambda = \theta / \gamma^2 \) (with \( \theta \geq 0 \)), and after an easy computation, we obtain that (45) is equivalent to \( \gamma^4(1 - \theta) \geq \theta^3 \). Hence (43) has non-convergent solution trajectories when \( \theta \in [0, 1) \), which means that \( \lambda \gamma^2 < 1 \) does not ensure convergence of dynamical system (11).

2.5 Invariance of condition (5) with respect to time rescaling

Given some positive real parameter \( k \), let us consider the time rescaling \( t = ks \). For any trajectory \( u(.) \) of (11), the rescaled trajectory \( v(s) = u(ks) \) satisfies
\[
\ddot{v}(s) + \gamma k \dot{v}(s) + k^2 \nabla \phi(v(s)) + A(v(s)) = 0. \tag{46}
\]
Let us suppose that \( A \) is a \( \lambda \)-cocoercive maximal monotone operator. One can easily verify that \( k^2A \) is a \( (\lambda/k^2) \)-cocoercive maximal monotone operator. Clearly, this time rescaling does not change the asymptotic convergence properties of the trajectories. Indeed, the condition insuring the convergence of the trajectories, namely \( \lambda \gamma^2 > 1 \), remains invariant under this time rescaling, as shown by the following relation: \( (\lambda/k^2) \times (\gamma k)^2 = \lambda \gamma^2 \). This elementary observation tells us that the condition \( \lambda \gamma^2 > 1 \) makes sense from a physical point of view.

3 Asymptotic stabilization by Tikhonov regularization methods

In order to correct some drawbacks of system (3), namely the weak (and possibly not strong) asymptotic convergence property of its trajectories, and the dependence of the limit equilibrium on the initial data, we introduce in the equation a Tikhonov-like regularization term with a vanishing coefficient. To be more precise, we are going to establish strong asymptotic convergence results regarding the solution of the non-autonomous system
\[
\begin{aligned}
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A(u(t)) + \epsilon(t) \nabla \Theta(u(t)) &= 0, \\
u(0) = u_0, \quad \dot{u}(0) = v_0,
\end{aligned} \tag{47}
\]
where $\gamma > 0$ and $(u_0, v_0)$ in $H^2$ are arbitrary given initial data. We make the following assumptions:

(H1) $\phi : H \to \mathbb{R}$ is a convex differentiable function whose gradient $\nabla \phi$ is Lipschitz continuous on bounded sets;

(H2) $A : H \to H$ is maximal monotone and $\lambda$-cocoercive for some $\lambda > 0$;

(H3) $\epsilon : [0, +\infty) \to (0, +\infty)$ is of class $C^1$ and tends to zero as $t \to +\infty$;

(H4) $\Theta : H \to \mathbb{R}$ is differentiable, convex and bounded below on $H$, and its derivative $\nabla \Theta$ is $\delta$-Lipschitz continuous (LC) and $\eta$-strongly monotone (SM), with $\delta > 0$ and $\eta > 0$, i.e.,

\[
\begin{align*}
1. & \quad \nabla \Theta(u) - \nabla \Theta(v) \leq \delta |u - v| \quad \forall u, v \in H, \\
2. & \quad \langle \nabla \Theta(u), u - v \rangle \geq \eta |u - v|^2 \quad \forall u, v \in H.
\end{align*}
\]

At once, we claim the main result of this section.

**Theorem 3.1** Let us suppose that (H1)-(H4) hold with $S := (A + \nabla \phi)^{-1}(0) \neq \emptyset$ and with $\lambda \gamma^2 > 1$. Let us assume moreover that $\epsilon(t)$ is decreasing and converges slowly to zero as $t \to +\infty$ in the following sense:

\[
\int_{0}^{+\infty} \epsilon(s) ds = +\infty.
\]

Then, for any $u_0$ and $v_0$ in $H$, there exists a unique solution $u \in C^2([0, +\infty); H)$ of (47) which satisfies $u(t) \to u^*_s$ strongly in $H$ as $t \to +\infty$, where $u^*_s$ is the unique solution of the variational inequality: find $u^*_s \in S$ such that

\[
\langle \nabla \Theta(u^*_s), v - u^*_s \rangle \geq 0 \quad \forall v \in S.
\]

Before proving Theorem 3.1 we introduce a series of preliminary results. To begin with, we establish a key estimate on the trajectories of (47).

**Lemma 3.1** Under conditions (H1)-(H4), the solution $u$ of (47) satisfies

\[
\lambda G(t) + w(t) + (\lambda \gamma^2 - 1)|\dot{u}(t)|^2 + \dot{\Gamma}_1(t) \leq -\epsilon(t)\langle \nabla \Theta(u(t)), u(t) - p \rangle + 2\lambda \gamma \epsilon(t) \Theta(u(t)),
\]

where $p$ is any element in $S := (A + \nabla \phi)^{-1}(0)$, $w(.)$ and $G(.)$ are defined by

\[
w(t) = \langle \nabla \phi(u(t)) - \nabla \phi(p), u(t) - p \rangle, \quad G(t) := |A(u(t)) + \gamma \dot{u}(t) - Ap|^2,
\]

$\Gamma_1(t)$ denotes the first derivative of the mapping $\Gamma_1(\cdot)$ given by

\[
\Gamma_1(t) = h(t) + \gamma h(t) + \lambda \gamma (||\dot{u}(t)||^2 + 2 \langle \phi(u(t)) - \nabla \phi(p) \rangle) + 2 \epsilon(t) \Theta(u(t)),
\]

with $h(t) = (1/2)||u(t) - p||^2$, $\Theta(v) := \Theta(v) - \inf_A \Theta$.

**Proof:** Without ambiguity, in order to get simplified notations, we omit to write the variable $t$. Hence $u$ stands for $u(t)$, and so on. Taking $p \in S$ and using (47), we have

\[
\langle \nabla \phi(u) - \nabla \phi(p), u - p \rangle + \langle Au - Ap, u - p \rangle = -\langle \ddot{u} + \gamma \dot{u} + \epsilon(t)\nabla \Theta(u), u - p \rangle,
\]

which by $\lambda$-cocoerciveness of $A$ and by definition of $w(t)$ amounts to

\[
\lambda ||Au - Ap||^2 + w(t) \leq -\langle \ddot{u} + \gamma \dot{u}, u - p \rangle - \epsilon(t)\langle \nabla \Theta(u), u - p \rangle.
\]
Furthermore, from $h = (1/2)|u - p|^2$, we have $\dot{h} = \langle \ddot{u}, u - p \rangle$ and $\ddot{h} = \langle \dddot{u}, u - p \rangle + |\ddot{u}|^2$, so that $\dot{h} + \gamma \dot{h} = \langle \dddot{u} + \gamma \ddot{u}, u - p \rangle + |\ddot{u}|^2$. This combined with (51) gives

$$\lambda|Au - Ap|^2 + w(t) + (\dot{h} + \gamma \dot{h}) \leq |\ddot{u}|^2 - \epsilon(t)\langle \nabla \Theta(u), u - p \rangle. \quad (52)$$

Let us reexpress $|Au - Ap|^2$ with the help of (17):

$$|Au - Ap|^2 = |\dot{u} + \nabla \phi(u) + \epsilon(t)\nabla \Theta(u) + Ap + \gamma \dot{u}|^2$$

$$= G(t) + \gamma^2 |\dot{u}|^2 + 2\gamma \langle \dot{u}, \ddot{u} + \nabla \phi(u) + \epsilon(t)\nabla \Theta(u) + Ap \rangle,$$ 

where $G(t) := |\dot{u} + \nabla \phi(u) + \epsilon(t)\nabla \Theta(u) + Ap|^2$ (hence $G(t) = |Au(t) + \gamma \ddot{u}(t) - Ap|^2$).

Set $\bar{\Theta}(\cdot) = \Theta(\cdot) - \inf_{H} \Theta, \ Q(t) = (1/2)|\dot{u}|^2 + \phi(u) + \langle u - p, Ap \rangle$, and rewrite (53) as

$$|Au - Ap|^2 = G(t) + \gamma^2 |\dot{u}|^2 + 2\gamma \frac{d}{dt} (Q(t) + \epsilon(t)\bar{\Theta}(u)) - 2\gamma \epsilon(t)\bar{\Theta}(u).$$

Replacing this last expression in (52) we obtain

$$w(t) + (\dot{h} + \gamma \dot{h}) + \lambda \left( G(t) + \gamma^2 |\dot{u}|^2 + 2\gamma \frac{d}{dt} (Q(t) + \epsilon(t)\bar{\Theta}(u)) - 2\gamma \epsilon(t)\bar{\Theta}(u) \right)$$

$$\leq |\ddot{u}|^2 - \epsilon(t)\langle \nabla \Theta(u), u - p \rangle + 2\lambda \gamma \epsilon(t)\bar{\Theta}(u),$$

or equivalently

$$w(t) + \lambda G(t) + (\lambda \gamma^2 - 1)|\dot{u}|^2 + \bar{\Gamma}_1(t) \leq -\epsilon(t)\langle \nabla \Theta(u), u - p \rangle + 2\lambda \gamma \epsilon(t)\bar{\Theta}(u),$$

where the function $\bar{\Gamma}_1(\cdot)$ is defined by

$$\bar{\Gamma}_1(t) = \dot{h}(t) + 2\gamma \epsilon(t)\bar{\Theta}(u(t))$$

$$= \dot{h}(t) + \gamma h(t) + 2\gamma \epsilon(t)\bar{\Theta}(u(t)) + 2\lambda \gamma \epsilon(t)\bar{\Theta}(u(t)),$$

which by $Ap = -\nabla \phi(p)$ leads to (49). •

We also need the following variant of Gronwall’s inequality, see (23) (Lemma 1).

**Lemma 3.2** Let $\psi : [0, +\infty) \to \mathbb{R}$ be absolutely continuous with

$$\dot{\psi}(t) + \epsilon(t)\psi(t) \leq \epsilon(t)g(t), \quad a.e.,$$

where $g(t)$ is bounded and $\epsilon(t) \geq 0$ with $\epsilon \in L^1_{loc}(\mathbb{R})$. Then the function $\psi(t)$ is bounded and if $\int_0^{+\infty} \epsilon(\tau)d\tau = +\infty$ we have $\lim sup_{t \to +\infty} \psi(t) \leq \lim sup_{t \to +\infty} g(t)$.

**Proof:** Let $\kappa_s := \sup\{g(t) : t \geq s\}$ so that $\dot{\psi}(t) + \epsilon(t)\psi(t) - \kappa_s \leq 0$ for $t \geq s$.

Multiplying by $\exp \left( \int_0^{t} \epsilon(\tau)d\tau \right)$ and integrating over $[s, t]$ we get $|\psi(t) - \kappa_s| \leq |\psi(s) - \kappa_s| \exp \left( - \int_0^{t} \epsilon(\tau)d\tau \right)$. It follows that $\psi(t)$ is bounded and, if $\int_0^{+\infty} \epsilon(\tau)d\tau = +\infty$, by letting $t \to +\infty$ in the above estimate we obtain $\lim sup_{t \to +\infty} \psi(t) \leq \kappa_s$. Letting $s \to +\infty$ in this last inequality yields $\lim sup_{t \to +\infty} \psi(t) \leq \lim sup_{t \to +\infty} g(t)$. •

At once we prove the boundedness of the trajectory given by (17).

**Lemma 3.3** Let conditions (H1)-(H4) be satisfied with parameters such that
Then for any solution \( u \) of (47) it holds that

\[
u \in L^\infty([0, +\infty), \mathcal{H}) \text{ and } \dot{v} \in L^\infty([0, +\infty), \mathcal{H}).
\]

If in addition \( \int_0^\infty \epsilon(s)ds = +\infty \), then the following properties are equivalent:

(i) any weak cluster point of \( u(t) \) for \( t \to +\infty \) belongs to \( S := (\nabla \phi + A)^{-1}(0) \),

(ii) \( \liminf_{t \to +\infty} \langle \nabla \phi(u), u(t) - u* \rangle \geq 0 \),

(iii) \( u(t) \to u* \) strongly,

where \( u* \) is the unique solution of (10).

**Proof:** Let us take \( p = u* \) in Lemma 3.1 use the same notations and set

\[ z(\cdot) := \langle \nabla \phi(u), u - u* \rangle, \quad K(\cdot) = \phi(u) - \langle u - u*, \nabla \phi(u*) \rangle, \]

so that \( \Gamma_1 = \dot{h} + \gamma h + \lambda_2 (|\dot{h}|^2 + 2K + 2\epsilon(t)\Theta(u)) \). With these notations, by using \( \dot{e} \leq 0 \) and \( \dot{\Theta}(u) \geq 0 \), Lemma 3.1 gives

\[
\lambda G + w + (\lambda^2 - 1)|\dot{u}|^2 + \dot{\Gamma}_1 \leq -\epsilon z. \tag{54}
\]

For any real values \( \nu > 0 \) and \( \rho \in (0, 1) \), by definition of \( \Gamma_1 \) we also have

\[
\nu \Gamma_1 - z = (-z + \rho \eta \nu + \nu 2\lambda \gamma e \Theta(u)) + (\nu \gamma - \rho \eta) h + \nu \lambda \gamma |\dot{u}|^2 + \nu \dot{h} + 2(\lambda \gamma \nu)K,
\]

which by (54) amounts to

\[
\lambda G + w + (\lambda^2 - 1)|\dot{u}|^2 + \dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \\
\leq \epsilon (-z + \rho \eta \nu + 2\lambda \gamma e \Theta(u)) + \epsilon (\nu \gamma - \rho \eta) h + \nu \epsilon \lambda \gamma |\dot{u}|^2 + \nu \epsilon \dot{h} + 2(\lambda \gamma \nu)K. \tag{55}
\]

By convexity of \( \phi \) and definition of \( K \) we have \( K(t) \leq \phi(u) + w(t) \), while by Young's inequality \( \dot{h} \leq (1/2)|\dot{u}|^2 + h \). As a result, combining these last two inequalities with (55), we deduce that

\[
\lambda G + (1 - 2\lambda \gamma \nu)w + [\lambda^2 - 1 - \nu \epsilon (\lambda \gamma + 1/2)]|\dot{u}|^2 + \dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \\
\leq \epsilon (-z + \rho \eta \nu + 2\lambda \gamma e \Theta(u)) + \epsilon (\nu \gamma - \rho \eta) h + 2\lambda \gamma \nu e \phi(u*). \tag{56}
\]

Hence, for \( \nu = (\rho \eta)/(\gamma + 1) \), we equivalently obtain

\[
\lambda G + (1 - 2\lambda \gamma \nu)w + (\lambda^2 - 1 - \nu \epsilon (\lambda \gamma + 1/2))|\dot{u}|^2 + \dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \\
\leq \epsilon (-z + \rho \eta \nu + 2\nu \lambda \gamma e \Theta(u)) + 2\lambda \gamma \nu e \phi(u*). \tag{57}
\]

Moreover, since \( \lambda \gamma^2 - 1 > 0 \) and \( \epsilon(t) \to 0 \) as \( t \to +\infty \), for \( t \) large enough, say \( t \geq t_0 \), we have \( \lambda \gamma^2 - 1 - \nu \epsilon(t) (\lambda \gamma + 1/2) \geq 0, 1 - 2\lambda \gamma \nu e(t) \geq 0 \), which by (57) and positivity of \( G(\cdot) \) (and omitting the variable \( t \)) leads to

\[
\dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \leq \epsilon (-z + \rho \eta \nu + 2\lambda \gamma \nu e \Theta(u)) + 2\lambda \gamma \nu e \phi(u*). \tag{58}
\]

By convexity of \( \Theta \), we know that

\[
z \geq \bar{\Theta}(u) - \bar{\Theta}(u*), \tag{59}
\]

while by \( \eta \)-strong-monotononicity of \( \nabla \Theta \) we have

\[
z \geq \langle \nabla \Theta(u*), u - u* \rangle + 2\eta h. \tag{60}
\]
From Young’s inequality this latter inequality yields
\[ z \geq -(1/2\eta)|\nabla \Theta(u_*)|^2 + \eta h. \] (61)
Recalling that \( \rho \in (0,1) \), and using (59) and (61), we obtain
\[ z \geq (1 - \rho)(\Theta(u) - \Theta(u_*)) - \rho(1/2\eta)|\nabla \Theta(u_*)|^2 + \rho \eta h, \] (62)
which by (58) amounts to
\[ \dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \leq \epsilon \left( (1 - \rho)\Theta(u_*) + \rho(1/2\eta)|\nabla \Theta(u_*)|^2 \right) - \epsilon ((1 - \rho) - 2\gamma \nu \epsilon) \Theta(u) + 2\lambda \gamma \nu \epsilon \phi(u_*). \] (63)
Observing that \((1 - \rho) - 2\lambda \gamma \nu \epsilon(t) \geq 0\) for \( t \geq t_1 \) large enough, we obtain
\[ \dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \leq \epsilon \left( (1 - \rho)\Theta(u_*) + \rho(1/2\eta)|\nabla \Theta(u_*)|^2 \right) + 2\lambda \gamma \nu \epsilon \phi(u_*). \] (64)
Applying Lemma 3.2 we deduce that \( \Gamma_1 \) is bounded. By definition of \( \Gamma_1 \) and convexity of \( \phi \) (note that \( \phi(u_*) \leq K(t) \)) we have
\[ \dot{h}(t) + \gamma h(t) + \lambda \gamma (|\dot{u}(t)|^2 + 2\phi(u_*)) \leq \Gamma_1(t). \] (65)
Hence there exists a positive constant \( C \) such that \( t \) large enough, say \( t \geq t_2 \), yields
\[ \dot{h}(t) + \gamma h(t) + \lambda \gamma |\dot{u}(t)|^2 \leq C, \] (66)
which obviously implies that \( h \) is bounded on \([0, +\infty[\), and so is \( u \):
\[ \sup_{t \in [0, +\infty)} |u(t)| < +\infty. \] (67)
From \( \dot{h} = \langle \dot{u}, u - u_* \rangle \), (66) and (67) we also derive the boundedness of \( \dot{u} \):
\[ \sup_{t \in [0, +\infty)} |\dot{u}(t)| < +\infty. \] (68)
Let us now prove the equivalences. Note that (i1) \( \Rightarrow \) (i2) follows from (10), while (i3) \( \Rightarrow \) (i1) is obvious. It remains to prove (i2) \( \Rightarrow \) (i3). Let us assume that (i2) holds. Again from (58) and (60) we immediately have
\[ \dot{\Gamma}_1 + \nu \epsilon \Gamma_1 \leq \epsilon (-\langle \nabla \Theta(u_*), u - u_* \rangle + 2\lambda \gamma \nu \epsilon \Theta(u)) + 2\lambda \gamma \nu \epsilon \phi(u_*), \] (69)
which in light of Lemma 3.2 entails
\[ \limsup_{t \to +\infty} \Gamma_1(t) \leq (1/\nu) \limsup_{t \to +\infty} [-\langle \nabla \Theta(u_*), u(t) - u_* \rangle + 2\lambda \gamma \nu \epsilon(t) \Theta(u(t)) + 2\lambda \gamma \nu \epsilon \phi(u_*)] \]
\[ = (1/\nu)[2\lambda \gamma \nu \phi(u_*)] - \liminf_{t \to +\infty} \langle \nabla \Theta(u_*), u(t) - u_* \rangle. \]
It is then immediate that (i2) leads to \( \limsup_{t \to +\infty} \Gamma_1(t) \leq 2\lambda \gamma \phi(u_*), \) which in light of (65) entails \( \limsup_{t \to +\infty} \left( \dot{h}(t) + \gamma h(t) \right) \leq 0. \) Then, it is easily checked (one can apply Lemma 3.2) that this inequality implies \( \limsup_{t \to +\infty} \dot{h}(t) = 0, \) namely (i3). \( \bullet \)
We are now in position to prove convergence in norm of the trajectory of the solution \( u \) of \((47)\) towards a special zero of \( \nabla \phi + A \).

**Proof of Theorem 3.1.** From Lemma 3.3 we know that
\[
    u \in L^\infty([0, +\infty); \mathcal{H}) \quad \text{and} \quad \dot{u} \in L^\infty([0, +\infty); \mathcal{H}).
\] (70)

By choosing \( p = u_s \) in Lemma 3.1 keeping the same notations, and setting \( K(t) = \phi(u(t)) - \langle u(t) - u_s, \nabla \phi(u_s) \rangle \), we recall that
\[
    \Gamma_1(t) = \dot{h}(t) + \gamma h(t) + \lambda \gamma (|\dot{u}(t)|^2 + 2K(t) + 2\epsilon(t)\Theta(u(t))),
\]
and we also have
\[
    \lambda G(t) + w(t) + (\lambda \gamma^2 - 1)|\dot{u}(t)|^2 + \dot{\Gamma}_1(t) \leq -\epsilon(t)\langle \nabla \Theta(u(t)), u(t) - u_s \rangle. \tag{71}
\]

The remainder of the proof will be divided into two cases:

Case 1. Assume that there exists a time \( t_0 \) such that the function \( \Gamma_1(\cdot) \) is nonincreasing on \([t_0, +\infty)\). Then recalling that \( \Gamma_1 \) is lower bounded (thanks to its definition and using \((70)\)) we deduce that \( \Gamma_1(t) \) converges as \( t \to +\infty \) to some real value \( \alpha \). This leads to \( \lim_{t \to +\infty} \dot{\Gamma}_1(t) = 0 \). Indeed, for any reals \( t \) and \( s \) verifying \( t \geq s \geq t_0 \), we have \( \int_s^t \Gamma_1(\tau)d\tau = -\int_s^t \dot{\Gamma}_1(\tau)d\tau = \Gamma_1(t) - \Gamma_1(s) \), hence, passing to the limit as \( t \to +\infty \), we get \( \int_s^{+\infty} |\dot{\Gamma}_1(\tau)|d\tau = \alpha - \Gamma_1(s) < +\infty \), and, by Lipschitz continuity of \( |\dot{\Gamma}_1(\cdot)| \), we conclude that \( \dot{\Gamma}_1(t) \to 0 \) as \( t \to +\infty \). Therefore, by \((71)\), the boundedness of \( u(\cdot) \), together with \( \lambda \gamma^2 - 1 > 0 \) and \( \epsilon(t) \to 0 \) as \( t \to +\infty \), we deduce that \( \lim_{t \to +\infty} w(t) = \lim_{t \to +\infty} G(t) = \lim_{t \to +\infty} |\dot{u}(t)| = 0 \). It follows that \( \lim_{t \to +\infty} w(t) = 0 \) and \( \lim_{t \to +\infty} |Au(t) - Au_s| = 0 \). Hence, in light of Lemma 2.2 we observe that any weak cluster point of \( u \) belongs to \( S \), which by Lemma 3.3 shows that \( u(t) \to u_s \) strongly.

Case 2. Assume that there exists an increasing sequence \( (l_k) \subset [0, +\infty) \) such that \( \lim_{k \to \infty} l_k = +\infty \) and \( \dot{\Gamma}_1(l_k) > 0 \) for all \( k \geq 0 \). This allows us to introduce the mapping \( s : [0, +\infty) \to [0, +\infty) \) defined for \( t \) large enough by
\[
    s(t) = \sup \left\{ \beta \leq t \quad | \quad \dot{\Gamma}_1(\beta) > 0 \right\},
\]
and we establish that \( \limsup_{t \to +\infty} \Gamma_1(t) \leq 0 \). Let \( (t_n) \subset [0, +\infty) \) be a sequence such that \( \lim_{n \to +\infty} t_n = +\infty \) and set \( s_n = s(t_n) \). It is obviously seen that \( (s_n) \) is a nonincreasing sequence such that \( s_n \to +\infty \) as \( n \to +\infty \), \( (\dot{\Gamma}_1(s_n)) \subset [0, +\infty) \), while by \((71)\) we have
\[
    \lambda G(s_n) + w(s_n) + (\lambda \gamma^2 - 1)|\dot{u}(s_n)|^2 \leq -\epsilon(s_n)\langle \nabla \Theta(u(s_n)), u(s_n) - u_s \rangle. \tag{72}
\]

From this last inequality, keeping in mind that \( \lambda \gamma^2 - 1 > 0 \) and \( \epsilon(s_n) \to 0 \) as \( n \to +\infty \), we obviously deduce that
\[
    \lim_{n \to +\infty} |\dot{u}(s_n)| = \lim_{n \to +\infty} w(s_n) = \lim_{n \to +\infty} |Au(s_n) - Au_s| = 0. \tag{73}
\]

By using \( \epsilon(s_n) \in (0, +\infty) \), we additionally obtain
\[
    \langle \nabla \Theta(u(s_n)), u(s_n) - u_s \rangle \leq 0. \tag{74}
\]
Asymptotic behavior of second-order dissipative evolution equations

Clearly, (73) entails that any weak-cluster point of \( u(s_n) \) belongs to \( S \) (according to Lemma 2.2), while by (71) and using the \( \eta \)-strong monotonicity of \( \nabla \Theta \), we obtain

\[
\eta|u(s_n) - u_*|^2 \leq (\nabla \Theta(u(s_n)) - \nabla \Theta(u_*), u(s_n) - u_*) \leq -(\nabla \Theta(u_*), u(s_n) - u_*).
\]

Hence

\[
\limsup_{n \to +\infty} \eta|u(s_n) - u_*|^2 \leq \liminf_{n \to +\infty} (\nabla \Theta(u_*), u(s_n) - u_*) \leq 0,
\]

this last inequality being obtained by passing to the limit on a weak convergent subsequence of \( u(s_n) \), using the fact that any weak-cluster point of \( u(s_n) \) belongs to \( S \) and the definition (10) of \( u_* \). Hence \( u(s_n) \to u_* \) strongly, which together with (73) and the definition of \( \Gamma_1 \) implies that

\[
\lim_{n \to +\infty} \Gamma_1(s_n) = 2\lambda \gamma \phi(u_*). \tag{75}
\]

In addition, using the definition of \( s(t_n) \) we clearly have \( s_n \leq t_n \) and \( \Gamma_1(t) \leq 0 \) for \( t \in [s(t_n), t_n] \) when \( s(t_n) \neq t_n \). Let us write \( \Gamma_1(t_n) = \Gamma_1(s_n) + \int_{s_n}^{t_n} \dot{\Gamma}_1(t) dt \leq \Gamma_1(s_n) \), which by (75) yields \( \limsup_{n \to +\infty} \Gamma_1(t_n) \leq 2\lambda \gamma \phi(u_*) \). By convexity of \( \phi \)

\[
\dot{h}(t) + \gamma h(t) + 2\lambda \gamma \phi(u_*) \leq \Gamma_1(t),
\]

which yields \( \limsup_{n \to +\infty} (\dot{h}(t_n) + \gamma h(t_n)) \leq 0 \). This being true for any sequence \( (t_n) \) that converges to infinity, we conclude that \( \limsup_{t \to +\infty} (\dot{h}(t) + \gamma h(t)) \leq 0 \), which immediately leads to \( h(t) \to u_* \) strongly. \( \bullet \)

The case \( \Theta(v) = \frac{1}{2}|v|^2 \), which corresponds to the classical Tikhonov regularization method, has been the object of active research, especially in the case of first order dynamical systems governed by maximal monotone operators, see [23] for some advanced results and references. Concerning second order dynamics, the potential case has been considered in [13]. Let us now state an extension of this last result to the non-potential case.

**Corollary 3.1** Let us suppose that (H1)-(H3) hold with \( S := (\nabla \phi + A)^{-1}(0) \neq \emptyset \) and with \( \lambda \gamma^2 > 1 \). Let us assume moreover that \( \epsilon(t) \) is decreasing and converges slowly to zero as \( t \to +\infty \) in the following sense:

\[
\int_0^{+\infty} \epsilon(s) ds = +\infty. \tag{76}
\]

Then, for any \( u_0 \) and \( v_0 \) in \( \mathcal{H} \), there exists a unique solution \( u \in C^2([0, +\infty); \mathcal{H}) \) of

\[
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A(u(t)) + \epsilon(t)u(t) = 0, \tag{77}
\]

with \( u(0) = u_0 \) and \( \dot{u}(0) = v_0 \), which satisfies \( u(t) \to u_* \) strongly in \( \mathcal{H} \) as \( t \to +\infty \), where \( u_* \) is the element of minimal norm of the closed convex set \( S \).

4 Examples

The following examples aim at illustrating our abstract results. Each of them requires further studies which are out of the scope of the present article.
4.1 Constrained optimization

Let $C$ be a closed convex subset of $\mathcal{H}$, and $g : \mathcal{H} \to \mathbb{R}$ a convex continuously differentiable function, whose gradient $\nabla g$ is Lipschitz continuous over $\mathcal{H}$ with Lipschitz constant $L$. Let us consider the constrained optimization problem

$$\min \{ g(v) : v \in C \}.$$  

Because of their direct linking with numerical gradient methods, we are interested in the study of smooth dissipative dynamical systems whose trajectories asymptotically converge towards minimizers of (78). Considering second-order dynamics aims at making the algorithms faster.

Let us denote by $S$ the set of solutions of (78) and suppose that $S \neq \emptyset$. A first-order optimality condition for (78) is given by the following inclusion

$$\nabla g(u) + N_C(u) \ni 0,$$  

where $N_C(u)$ is the outward normal cone to $C$ at $u \in C$. Observe that (79) is equivalent to

$$u - P_C[u - \mu \nabla g(u)] = 0,$$  

where $P_C$ is the projection operator on $C$ and $\mu$ is a positive parameter. Whence we considerer the following gradient-projection dynamic

$$\ddot{u}(t) + \gamma \dot{u}(t) + u(t) - P_C[u(t) - \mu \nabla g(u(t))] = 0.$$  

This continuous dynamical system has been first introduced by Antipin [8] and further studied by Attouch-Alvarez [5]. An extended study of the corresponding first-order system has been achieved in [17], [18]. Let us show that the operator

$$Av := v - P_C(v - \mu \nabla g(v))$$  

is $(1/2)$-cocoercive for $0 < \mu < (2/L)$. To that end, let us notice that $A = I - T$ where $Tv = P_C(v - \mu \nabla g(v))$. According to Corollary 2.2 and noticing that $P_C$ is a contraction, we just need to verify that the mapping $v \mapsto v - \mu \nabla g(v)$ is a contraction too. Given arbitrary $u \in \mathcal{H}$, $v \in \mathcal{H}$, we have

$$\left| (u - \mu \nabla g(u)) - (v - \mu \nabla g(v)) \right|^2 = |u - v|^2 - 2\mu \langle \nabla g(u) - \nabla g(v), u - v \rangle + \mu^2 |\nabla g(u) - \nabla g(v)|^2.$$  

Thus proving that $v \mapsto v - \mu \nabla g(v)$ is a contraction, is equivalent to prove that $\langle \nabla g(u) - \nabla g(v), u - v \rangle \geq \frac{\mu}{2}|\nabla g(u) - \nabla g(v)|^2$, which amounts to say that $\nabla g$ is $(\mu/2)$-cocoercive. Clearly, this property forces $\nabla g$ to be $(2/\mu)$-Lipschitz continuous. Indeed, as a striking property, the converse statement holds true for the gradient of a convex function, that’s the Baillon-Haddad theorem, see [16]. Hence $\nabla g$ being Lipschitz continuous with Lipschitz constant $L \leq (2/\mu)$ forces $\nabla g$ to be $(\mu/2)$-cocoercive.

Hence (81) falls in our setting. Applying Theorem 2.1 with $\lambda = 1/2$, we deduce that, for $\gamma > \sqrt{2}$ and $0 < \mu \leq (2/L)$, every trajectory of (81) weakly converges to an element of $S$, which is a minimizer of problem (78).
Moreover, when considering the Tikhonov-like regularized dynamic with $\epsilon(t) \to 0$ as $t$ goes to $+\infty$,

$$
\ddot{u}(t) + \gamma \dot{u}(t) + u(t) - P_C[u(t) - \mu \nabla g(u(t))] + \epsilon(t) \nabla \Theta(u(t)) = 0, \tag{83}
$$

by a direct application of Theorem 3.1, under the assumption $\int_0^{+\infty} \epsilon(s) ds = +\infty$, one obtains that the trajectories of (83) strongly converge to the unique solution $\bar{u} \in S$ verifying

$$
\Theta(\bar{u}) = \inf_{v \in S} \Theta(v). \tag{84}
$$

**Remark** Inertial dynamical systems, like the "heavy ball with friction dynamical system" have been first introduced by B. Polyak [33] in the realm of optimization. Since then, an abundant literature has been devoted to this subject. Rich connections between second order dissipative dynamical systems in their respective continuous and discretized forms have been put to the fore, and so provide new algorithms together with a deeper insight into their convergence analysis, see [2], [4], [5], [6], [7], [11], [13], [20], [25], [30].

### 4.2 Coupled systems and dynamical games

Throughout this section we make the following standing assumptions:

- $\mathcal{H} = \mathcal{X}_1 \times \mathcal{X}_2$ is the cartesian product of two Hilbert spaces equipped with norms $|.|_{\mathcal{X}_1}$ and $|.|_{\mathcal{X}_2}$, while $x = (x_1, x_2)$, with $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$, stands for any element in $\mathcal{H}$;
- $\phi(x) = f_1(x_1) + f_2(x_2) + \Phi(x_1, x_2)$, where $f_1 : \mathcal{X}_1 \to \mathbb{R}$, $f_2 : \mathcal{X}_2 \to \mathbb{R}$ are smooth convex functions, $\Phi : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$ is a smooth convex coupling function;
- $A = (\nabla_x L, -\nabla_x L)$ is the maximal monotone operator which is attached to a smooth convex-concave function $L : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$. The operator $A$ is assumed to be $\lambda$-cocoercive for some $\lambda > 0$.

A typical example of coupling function $\Phi$ (see [10]) is given by

$$
\Phi(x) = \frac{1}{2} |L_1 x_1 - L_2 x_2|^2_Z,
$$

where $L_1 \in L(\mathcal{X}_1, \mathcal{Z})$ and $L_2 \in L(\mathcal{X}_2, \mathcal{Z})$ are linear continuous operators acting respectively from $\mathcal{X}_1$ and $\mathcal{X}_2$ into a third Hilbert space $\mathcal{Z}$ with norm $|.|_\mathcal{Z}$.

A smooth convex-concave function $L$, such that the associated maximal monotone operator $A$ is $\lambda$-cocoercive, can be obtained in a systematic way by using epi-hypo Moreau-Yosida regularization, as described below. Given a general closed convex-concave function $L : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$ (see [33]) and a positive parameter $\lambda$, we consider $L_\lambda : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathbb{R}$ defined for any $(x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2$ by

$$
L_\lambda(x_1, x_2) = \min_{u_1 \in \mathcal{X}_1} \max_{u_2 \in \mathcal{X}_2} \left\{ L(u_1, u_2) + \frac{1}{2\lambda} |x_1 - u_1|^2_{\mathcal{X}_1} - \frac{1}{2\lambda} |x_2 - u_2|^2_{\mathcal{X}_2} \right\}.
$$
Then \( \mathcal{L}_\lambda \) is a smooth convex-concave function whose associated maximal monotone operator \( (\nabla_{x_1} \mathcal{L}_\lambda, -\nabla_{x_2} \mathcal{L}_\lambda) \) is precisely the Yosida approximation \( A_\lambda \) of the operator \( A = (\partial_{x_1} \mathcal{L}, -\partial_{x_2} \mathcal{L}) \), i.e., \( A_\lambda = (\nabla_{x_1} \mathcal{L}_\lambda, -\nabla_{x_2} \mathcal{L}_\lambda) \) (see [31] for further details).

In this setting, with \( u(t) = (x_1(t), x_2(t)) \) system (3) becomes

\[
\begin{align*}
\dot{x}_1(t) + \gamma \dot{x}_1(t) + \nabla f_1(x_1(t), x_2(t)) + \nabla_x \nabla_x \mathcal{L}(x_1(t), x_2(t)) &= 0, \\
\dot{x}_2(t) + \gamma \dot{x}_2(t) + \nabla f_2(x_2(t)) + \nabla_x \nabla_x \mathcal{L}(x_1(t), x_2(t)) &= 0.
\end{align*}
\]

(85)

As a straight application of Theorem 2.1, assuming relation \( \lambda \gamma^2 > 1 \) holds, one obtains \( x(t) = (x_1(t), x_2(t)) \to x_\infty = (x_{1,\infty}, x_{2,\infty}) \) weakly in \( \mathcal{H} \) as \( t \) goes to \( +\infty \), where \( (x_{1,\infty}, x_{2,\infty}) \) is solution of the coupled system

\[
\begin{align*}
\nabla f_1(x_1) + \nabla_x \Phi(x_1, x_2) + \nabla_x \mathcal{L}(x_1, x_2) &= 0, \\
\nabla f_2(x_2) + \nabla_x \Phi(x_1, x_2) - \nabla_x \mathcal{L}(x_1, x_2) &= 0.
\end{align*}
\]

(86)

Structured systems such as (86) contain both potential and non-potential terms whose antagonistic effects are often present in decision sciences and physics.

In game theory, (80) describes Nash equilibria of the normal form game with two players 1, 2 whose static loss functions are respectively given by

\[
\begin{align*}
F_1 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 &\to F_1(x_1, x_2) = f_1(x_1) + \Phi(x_1, x_2) + \mathcal{L}(x_1, x_2), \\
F_2 : (x_1, x_2) \in \mathcal{X}_1 \times \mathcal{X}_2 &\to F_2(x_1, x_2) = f_2(x_2) + \Phi(x_1, x_2) - \mathcal{L}(x_1, x_2),
\end{align*}
\]

(87)

where the \( f_i(.) \) represent the individual convex payoffs of the players, \( \Phi(.,.) \) is their joint convex payoff, and \( \mathcal{L} \) is a convex-concave payoff with zero-sum rule. The case \( \mathcal{L} = 0 \) corresponds to a potential team game (see [31]), while case \( \Phi = 0 \) corresponds to a non cooperative zero-sum game. Game (87) involves both cooperative and non cooperative aspects.

A central question in game theory, decision sciences and economics is to describe realistic dynamics which converge to Nash equilibria, see [29] and the references herein. Indeed, implicit time discretization of dynamical system (85) leads to the following best response dynamics (players 1 and 2 play alternatively) with inertia and “costs to change”

\[
(x_{1,k+1}, x_{2,k+1}) \to (x_{1,k+2}, x_{2,k+1}) \to (x_{1,k+2}, x_{2,k+2});
\]

\[
\begin{align*}
x_{1,k+2} &= \arg \min_{\xi \in \mathcal{X}_1} \left\{ f_1(\xi) + \Phi(\xi, x_{2,k+1}) + \mathcal{L}(\xi, x_{2,k+1}) + \frac{1}{2\alpha_k} \| \xi - (x_{1,k+1} + \beta_k(x_{1,k+1} - x_{1,k})) \|^2_{\mathcal{X}_1} \right\}, \\
x_{2,k+2} &= \arg \min_{\eta \in \mathcal{X}_2} \left\{ f_2(\eta) + \Phi(x_{1,k+2}, \eta) - \mathcal{L}(x_{1,k+2}, \eta) + \frac{1}{2\alpha_k} \| \eta - (x_{2,k+1} + \beta_k(x_{2,k+1} - x_{2,k})) \|^2_{\mathcal{X}_2} \right\}.
\end{align*}
\]

(88)

The terms \( \| \xi - x_{1,k+1} - \beta_k(x_{1,k+1} - x_{1,k}) \|^2 \) and \( \| \eta - x_{2,k+1} - \beta_k(x_{2,k+1} - x_{2,k}) \|^2 \) naturally come into play when discretizing first and second order time derivatives, see [2], [3]. In decision sciences, they reflect some aspects of agents behaviors like anchoring and inertia, and can be interpreted as “low local costs to change”, see
The various parameters $\alpha_k, \nu_k, \beta_k$ model adaptive, learning abilities of the agents as well as their reactivity or resistance to change.

Continuous dynamical system (85) together with its convergence properties offers a valuable guideline to study discrete dynamic (88). Clearly, this requires further study beyond the limit of the present paper.

4.3 Further perspectives

It is natural to relax the regularity assumptions on the potential operator $\nabla \phi$ and the maximal monotone operator $A$, and to wonder whether our asymptotic convergence results remain valid in respect of the more general differential inclusion

$$\ddot{u}(t) + \gamma \dot{u}(t) + Au(t) + \partial \phi(u(t)) \ni 0. \quad (89)$$

Let us give some precisions (regarding cases when either $A$ or $\partial \phi$ is single-valued):

1. The non-smooth potential case (obtained by taking $A = 0$), that is

$$\ddot{u}(t) + \gamma \dot{u}(t) + \partial \phi(u(t)) \ni 0, \quad (90)$$

has been considered in the finite dimensional case in [11]. Trajectories of (90) naturally exhibit elastic shocks, which makes this system play an important role in unilateral mechanics, see [35]. Taking $\phi$ equal to the indicator function of a closed convex set $C$ yields (damped) billiard dynamics. Given a non-smooth closed convex potential $\phi: \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ and a $\lambda$-cocoercive operator $A$, a natural approach to (89) (in the same lines as [11], [35]) would consist in approximating $\phi$ by smooth convex functions $\phi_n$ (this is always possible for example via Moreau-Yosida approximation), which leads to study

$$\ddot{u}_n(t) + \gamma \dot{u}_n(t) + Au_n(t) + \nabla \phi_n(u_n(t)) = 0. \quad (91)$$

Equation (92) enters our setting (as $A_\lambda$ is $\lambda$-cocoercive) and it can be equivalently rewritten as

$$\begin{cases} 
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + \frac{1}{\lambda}(u(t) - v(t)) = 0, \\
Av(t) + \frac{1}{\lambda}(v(t) - u(t)) \ni 0.
\end{cases} \quad (93)$$

Note that our basic assumption $\lambda \gamma^2 > 1$, which does not concern the potential part of the operator, is not affected by this operation. Thus, (91) falls in our setting. Then, one has to establish estimations on the sequence $(u_n)$, pass to the limit on these estimations as $n$ goes to infinity, and derive conclusions on the asymptotic behavior of trajectories of (89). As it has been done in [6] and [11], this program would allow to consider the asymptotic behavior of damped hyperbolic equations (with non-isolated equilibria) combining potential with non potential effects.

2. The study of (89) with a general maximal monotone operator $A$ leads to considerable difficulties. In that case, another type of regularization or relaxation method can be used. It relies on the following remark. Let $\lambda > 0$ and $\gamma > 0$ (being fixed) such that $\lambda \gamma^2 > 1$, $A_\lambda$ the Yosida approximation of $A$, and consider the equation

$$\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + A_\lambda(u(t)) = 0. \quad (92)$$

Equation (92) enters our setting (as $A_\lambda$ is $\lambda$-cocoercive) and it can be equivalently rewritten as

$$\begin{cases} 
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + \frac{1}{\lambda}(u(t) - v(t)) = 0, \\
Av(t) + \frac{1}{\lambda}(v(t) - u(t)) \ni 0.
\end{cases} \quad (93)$$
This naturally suggests that some of our results can be extended to coupled systems of the following type

\[
\begin{aligned}
\ddot{u}(t) + \gamma \dot{u}(t) + \nabla \phi(u(t)) + \frac{1}{\lambda} (u(t) - v(t)) &= 0, \\
\dot{v}(t) + Av(t) + \frac{1}{\lambda} (v(t) - u(t)) &\ni 0.
\end{aligned}
\]  

(94)

Our approach, which is based on Liapunov methods, ([28], [21], [27]), can be adapted to such nonlinear systems. In the case of linear hyperbolic PDE’s, it would be worthwhile to compare it with spectral analysis methods, see for example [24], [26].

3. In view of further applications, it would be interesting to consider other types of dissipation phenomena: For example, dry friction most likely produces stabilization within finite time, a desirable feature of human decision processes, see [1]. Geometrical damping is of great importance for numerical optimization, and modeling of non elastic shocks in mechanics, see [7].

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