REVIEW

Minimal model holography

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Abstract

We review the duality relating 2D $W_N$ minimal model conformal field theories, in a large-$N$ 't Hooft like limit, to higher spin gravitational theories on AdS\textsubscript{3}.

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The search for simple examples of holography is important in the effort to penetrate the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence. It involves seeking a hard-to-achieve balance between analytic tractability and intrinsic complexity. One wants to be able to capture enough physics of holography, especially of the aspects relevant to the puzzles of quantum gravity, with quantitative precision so as to be able to transfer the resulting understanding to more ‘realistic examples’.

In this paper, we review one such attempt in this search which appears to have a number of promising features. It is a particular instance of the general class of examples involving Vasiliev higher spin gauge theories on AdS with dual vector-like CFTs (in a large-$N$ limit). The papers in this issue discuss various aspects as well as examples of higher spin holography. Here, we focus on the specific case of a class of interacting vector-like two-dimensional (2D) (generically non-supersymmetric) CFTs and their AdS$_3$ duals in terms of a higher spin gauge theory coupled to matter fields.

Two-dimensional CFTs are among the best understood non-trivial quantum field theories [18] and, moreover, have wide applications in diverse areas of physics. Since one has a high degree of analytic control over these theories, they can potentially provide a rich source of CFTs with interesting bulk AdS$_3$ duals. Of course, an essential ingredient in having a classical bulk dual is to have a large number of degrees of freedom as in a large-$N$ vector or matrix theory. It is in such a family of theories that one can recover classical gravitational physics (not necessarily described by an Einstein Lagrangian) in a parametrically controlled manner from the finite-$N$ quantum regime.

However, systematic studies of the large-$N$ limit of families of 2D CFTs have not been carried out until recently. One can imagine at least two categories of such theories: these are the vector-like and the gauge-like models whose number of degrees of freedom (i.e. the central charge) scales as $N$ or $N^2$, respectively; here, $N$ is the rank of some underlying gauge group. In complexity, the former are obviously simpler, as familiar from the usual large-$N$ vector models. Nevertheless, even these are quite intricate in their detailed structure as we will see in this review. Thus, these theories may strike a good balance between complexity and tractability. We will only briefly mention the case of the matrix-like families, which have just begun to be analysed; see, e.g., [71], at the end of this review.

More specifically, the family of theories we will be considering is the so-called coset CFTs of the form

$$\frac{\text{SU}(N)_k \otimes \text{SU}(N)_1}{\text{SU}(N)_{k+1}}.$$  \hspace{1cm} (1)

They have the central charge

$$c_{N,k} = (N-1) \left[ 1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right] \leq (N-1),$$  \hspace{1cm} (2)

and hence are vector-like. We will review many of the already known properties of these CFTs in section 2.2. In our context, the most important characteristic is that they have conserved
higher spin currents $W^s(\tau)$ with $s = 3, \ldots, N$; their symmetry algebra is therefore a $\mathcal{W}$-algebra, and the models (1) are usually referred to as the $\mathcal{W}_N$ minimal models. It is an important feature of 2D quantum field theories (and CFTs in particular) that higher spin conserved currents are compatible with interactions—this is for example not the case in 3D [94]. We will review some of the salient facts about the $\mathcal{W}_N$-algebras in section 3.1; as we will explain there, these algebras are all special cases of an extended symmetry algebra known as $\mathcal{W}_\infty[\mu]$ which typically has all integer spins $s \geq 2$, and which can be truncated to $\mathcal{W}_N$ for $\mu = N$.

We will be interested, as mentioned, in the large-$N$ limit of these theories. We shall consider a ’t Hooft-like limit, where we take $N, k \to \infty$ while keeping the ’t Hooft coupling

$$0 \leq \lambda = \frac{N}{N + k} \leq 1$$

(3)

fixed. Note that in this limit, the central charge in (2) behaves as $c = N(1 - \lambda^2)$. We will describe, as we go along, some of the evidence that this limit is well behaved; for instance, in sections 5.2–5.4, we will study the spectrum of operators in this limit, while in section 6.1, we will review some of the results from studies of correlation functions. We will see that an appropriate part of the spectrum will organize itself, at large $N$, into a Fock space of multi-particle states. The correlation functions, in turn, will exhibit, rather non-trivially, the factorization required for a good large-$N$ limit.

Let us now turn to the bulk AdS theories that are believed to be dual to these minimal models. They are gravitational theories in AdS$_3$, containing one additional higher spin $s > 2$ gauge field (for each $s$) together with some scalar fields. Theories of this kind were constructed by Vasiliev first in AdS$_4$ [112], and then generalized to other dimensions including AdS$_5$ [113, 114]. In 3D, they are labelled by a single parameter $\mu$ and based on a higher spin gauge group known as hs$[\mu]$ [103, 104]; we summarize some of the relevant facts about these theories and their symmetries in section 2.1. As familiar from the classic calculation of Brown and Henneaux [27], partial information about the dual CFT comes from the analysis of the asymptotic symmetry algebra. For the case of the hs$[\mu]$ theory, this symmetry algebra was determined in [76, 29, 61] and shown to define a classical Poisson algebra which agrees, in the classical ($c \to \infty$) limit, with $\mathcal{W}_\infty[\mu]$; this will be reviewed in section 3.2.

Based on this observation, it was proposed in [57] that the hs$[\mu]$ higher spin theory in AdS$_3$ is dual to the above ’t Hooft limit of the $\mathcal{W}_N$ minimal models, where the ’t Hooft coupling $\lambda$ agrees with $\mu = \frac{\lambda}{1 - \lambda}$, $\mu$. Furthermore, in order to account for the full spectrum of the minimal model CFTs, it was proposed that the higher spin theory is coupled to two complex scalar fields. Unlike the higher dimensional case, the scalar field is, in 3D, not part of the higher spin multiplet, and hence does not need to be included from the start. However, in order to couple it consistently to the higher spin theory based on hs$[\mu]$, its mass is fixed to equal $M^2 = -1 + \mu^2$ [103, 104]. For $0 < \mu < 1$, this is the case of relevance since the ’t Hooft coupling is by construction between $0 < \lambda < 1$—the mass therefore lies in the window where two quantizations are possible [91]. The proposal in [57] was then that one of the scalars is quantized in the standard way ($+$), whereas the other is quantized in the alternate way ($-$). The corresponding primary fields in the dual CFT then have conformal dimensions equal to $h_{\pm} = 1/2(1 \pm \lambda)$; these are precisely the conformal dimensions of the ’primitive’ representations of the minimal model CFT in the ’t Hooft limit.

The symmetry algebras of the hs$[\mu]$ higher spin theory on AdS$_3$ and the ’t Hooft limit of the minimal model CFTs are both $\mathcal{W}_\infty$-algebras, but a priori, it is not at all obvious whether they are the same $\mathcal{W}_\infty$-algebra. This issue was first raised in [61], see also [59], and then finally resolved in [58]: there is a unique way of ‘quantizing’ the asymptotic symmetry algebra of the higher spin theory (that is initially a commutative Poisson algebra). The resulting quantum algebra $\mathcal{W}_\infty[\mu]$ exhibits a non-trivial equivalence which implies, among other things, that
\( W^\infty [\lambda] \) agrees indeed with the 't Hooft limit of the \( W_N \)-algebras. In fact, the equivalence holds also for finite \( N \) and \( k \) (and hence finite \( c \)): the \( W^N_{N,k} \) minimal model algebra at the central charge \( c = c_{N,k} \), see (2), is equivalent to the \( W^\infty [\lambda] \)-algebra at the same value of the central charge and with \( \lambda \) given by (3); this will be reviewed in section 4.

Given the detailed understanding of the \( W^\infty [\mu] \)-algebra for arbitrary \( \mu \) and \( c \), it is then also possible to analyse the semi-classical (large \( c \)) behaviour of its representations at fixed \( \mu \). In particular, one can study the two 'primitive' coset representations (that correspond to the two quantizations of the massive scalar field, from above) for fixed \( N \) and large \( c \). As it turns out, the two representations behave rather differently in this limit: while the conformal dimension \( h_+ \) remains finite, \( h_- \) is proportional to \( c \). This suggests that the AdS dual of the \( h_- \) primary should not be thought of as a perturbative massive scalar field with alternate boundary conditions, but rather as a non-perturbative state [58]. This point of view also ties in nicely with the fact that the higher spin theory possesses a large number of semi-classical 'conical defect' solutions [33] that are in one-to-one correspondence with the closely related 'light' states of the coset CFT. The picture that emerges from these considerations [58, 100] is that the bulk AdS theory should be thought of as an \( \text{hs}[\lambda] \) theory coupled to one complex scalar field (dual to \( h_+ \)). Other states, including those dual to \( h_- \) and the 'light' states, are to be viewed as conical defects (and their generalizations) bound with perturbative quanta [100]; all of this will be discussed in section 5.

There are various aspects of this proposal that can be checked in some detail. In particular, one can show that the perturbative spectrum of the higher spin AdS theory matches exactly with the 'perturbative' part of the CFT spectrum, i.e. with those states that appear in multiple OPEs of the \( h_+ \) primary (and its conjugate). This calculation represents a highly non-trivial consistency check on the proposal, and will be explained, in some detail, in section 5.3. Further checks, including the comparison of correlation functions as well as the calculation of the black hole entropy in [92] from the dual CFT point of view [62]—for a review about the construction of black hole solutions for these theories, see [8]—are discussed in section 6. In section 7, we summarize the generalizations of the duality conjecture to the orthogonal groups, as well as to the case with \( \mathcal{N} = 2 \) supersymmetry. Finally, section 8 outlines some of the possible lines of future development of this fruitful subject.

2. The ingredients

In this section, we briefly review the basic ingredients that go into the duality, namely higher spin theories on AdS3 on the one hand (see section 2.1) and the coset conformal field theories in two dimensions on the other (see section 2.2).

2.1. The higher spin theory

Higher spin gauge fields in AdS3 are relatively simple compared to their higher dimensional counterparts. (The general Vasiliev approach to constructing higher spin theories in diverse dimensions and its relevance for the AdS/CFT correspondence is reviewed elsewhere in this volume, for instance in the articles of Giombi and Yin [69] and Vasiliev [115].) The basic reason is that these fields, just like gravity, do not contain propagating degrees of freedom in three dimensions. Thus, their bulk dynamics is topological and the only states come from boundary degrees of freedom generalizing the Brown–Henneaux states of pure AdS3 gravity. The precise higher spin theory that will be dual to the \( W_N \) minimal models will, however, have bulk propagating degrees of freedom coming from a scalar. The mass and couplings of this scalar are determined by the higher spin symmetry algebra.
Below, we will first review the Chern–Simons construction for pure gravity in AdS$_3$, and then explain how it can be generalized to higher spin \[21\]. After a discussion of the higher spin symmetry algebra, we will also mention how the scalar field can be coupled.

2.1.1. Review of pure gravity. Recall that the Einstein equations of pure gravity in AdS$_3$ can be written in the Chern–Simons form \[1, 117\]. In order to see this, let us work with the vielbein formalism, where the basic variables are the dreibein $e^a_\mu$ and the spin connection $\omega^a_{\mu}$.

Dualizing the spin connection as $\omega^a_\mu = -\frac{1}{2} \epsilon^a_{bc} \omega^b_\mu$, the Einstein equations take the form (in the following, we work in the form language, and hence drop the explicit spacetime indices)

$$\hat{R}^a_\mu = d\omega^a_\mu + \frac{1}{2} \epsilon^a_{bc} \omega^b_\mu \wedge \omega^c_\mu = \frac{1}{2\ell^2} \epsilon^a_{bc} e^b_\mu \wedge e^c_\mu,$$

(4)

where $\ell$ is the AdS radius (which will often be set equal to 1). In addition, we have the condition that the torsion vanishes

$$T^a_\mu = de^a_\mu + \epsilon^a_{bc} e^b_\mu \wedge e^c_\mu = 0.$$  

(5)

We now want to obtain these two equations from a Chern–Simons point of view. To see how this goes, we recall that the isometry group of AdS$_3$ is $\text{SO}(2, 2) \cong \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$. Let us introduce the fields

$$A^a_\mu = \omega^a_\mu + \frac{1}{\ell} e^a_\mu, \quad \bar{A}^a_\mu = \omega^a_\mu - \frac{1}{\ell} e^a_\mu,$$

(6)

which transform in the adjoint representation with respect to the two $\text{SL}(2, \mathbb{R})$ factors. Thus, both $A^a_\mu$ and $\bar{A}^b_\mu$ take values in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, and we can consider the Chern–Simons action

$$S = S_{\text{CS}}[A] - S_{\text{CS}}[\bar{A}] \quad \text{with} \quad S_{\text{CS}}[A] = \frac{\hat{k}}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

(7)

It was observed in \[1\] that the flatness conditions $F^a_\mu \equiv dA^a_\mu + \epsilon^a_{bc} A^b_\mu \wedge A^c_\mu = 0$ and $\bar{F}^a_\mu = 0$ that arise as equations of motion from (7) are in fact equivalent to the Einstein equations of pure gravity (4) and (5). In a similar vein, it was shown in \[117\] that the Chern–Simons action (7) reduces, up to some boundary terms, to the Einstein–Hilbert action (with a negative cosmological constant) provided we identify

$$\hat{k} = \frac{\ell}{4G},$$

(8)

where $G$ is Newton’s constant. We should stress that this identification requires that we choose appropriate boundary conditions for the gauge fields\(^3\). The precise form of the boundary conditions will be explained in section 3.2.

2.1.2. Spin 3 and higher. Next, we want to discuss the generalization of the above analysis to higher spin theories. In three dimensions, it is actually possible to define consistent higher spin theories containing only a finite number of spin fields; the simplest example is the theory that contains, in addition to the graviton, a single field of spin $s = 3$. It is simply obtained from the above description by replacing $\mathfrak{sl}(2)$ by $\mathfrak{sl}(3)$. This is to say, we consider the Chern–Simons theory of the form (7), where now the gauge fields $A$ and $\bar{A}$ take values in the Lie algebra $\mathfrak{sl}(3)$ \[76, 29\]. In order to relate this Chern–Simons theory to a higher spin theory, we need to

\(^3\) Indeed, without imposing any additional boundary conditions, we would conclude that the field theory living on the boundary would be a WZW model based on $\mathfrak{sl}(2)$, and this is clearly not the conformal field theory dual to pure gravity in AdS$_3$. 

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identify the ‘gravitational’ subalgebra $\mathfrak{sl}(2) \subset \mathfrak{sl}(3)$. The most natural choice is to take $\mathfrak{sl}(2)$ to be the principal embedding. This essentially means that (the adjoint of) $\mathfrak{sl}(3)$ decomposes as

$$\mathfrak{sl}(3) = \mathfrak{sl}(2) \oplus \mathfrak{5},$$

where 5 denotes the five-dimensional $j = 2$ representation of $\mathfrak{sl}(2)$. These components of the two $\mathfrak{sl}(3)$ gauge fields correspond to generalized vielbein and connection 1-forms $e^{ab}_{\nu}$ and $\omega^{ab}_{\nu}$, respectively, that are symmetric and traceless in the $a, b$ indices and generalize (6).

In this case, it was shown in [29] that the resulting equations of motion of the Chern–Simons theory reduce, at the linearized level, to the Fronsdal equations [56], characterizing a massless spin $s = 3$ gauge field on AdS$_3$. Indeed, at the linearized level, the generalized vielbeins $e^a_{\mu}$ are related to the symmetric rank-3 tensor field $\phi_{\mu\nu\rho}$ in the Fronsdal formulation as

$$\phi_{\mu\nu\rho} \sim \text{Tr}(e^{ab}_{\mu} e_{\nu a} e_{\rho b}),$$

where $e_{\nu a}$ are the background vielbeins for the AdS metric. This demonstrates that the Chern–Simons theory based on $\mathfrak{sl}(3)$ indeed describes spin-3 gravity on AdS$_3$.

The above construction can be generalized by replacing the gauge group in the Chern–Simons theory by $\mathfrak{sl}(N)$ (where the gravitational $\mathfrak{sl}(2)$ is principally embedded). The analogue of (9) is now

$$\mathfrak{sl}(N) = \mathfrak{sl}(2) \oplus \mathfrak{5} \oplus \mathfrak{7} \oplus \cdots \oplus (\mathfrak{2N} - \mathfrak{1}),$$

where the representation of dimension $(2s - 1)$ corresponds to the spin $s$ field, which is described by generalized vielbein and connection 1-forms $e^{a_1 \cdots a_{s-1}}_{\nu}$, $\omega^{a_1 \cdots a_{s-1}}_{\nu}$ (whose $a_i$ indices are symmetric and traceless), respectively; thus, the resulting higher spin gauge theory has spin fields of spin $s = 2, 3, \ldots, N$. At the linearized level, we again have a generalization of (10) relating these generalized vielbeins to the Fronsdal fields. For more details, we refer the reader to [29].

In all of these cases, the higher spin theory is the sum of two Chern–Simons terms as in (7) with equal and opposite levels. One can also consider a parity violating version of the theory, where the two levels are different [39, 11]. One then needs to impose the zero torsion condition additionally through a Lagrange multiplier term. As a consequence, this theory turns out to have propagating modes [11, 12, 40].

2.1.3. The underlying algebra of the higher spin theory. The higher spin theories we are primarily interested in are a generalization of the above $\mathfrak{sl}(N)$ theories. They have one massless higher spin field for each spin $s = 3, 4, 5, \ldots$. These generalizations can be constructed by considering the Chern–Simons theory [21] based on the infinite-dimensional Lie algebra $\mathfrak{hs}[\mu]$. Let us first describe the structure of this Lie algebra in some detail, following [53, 23, 19, 55].

Consider the quotient of the universal enveloping algebra $U(\mathfrak{sl}(2))$ by the ideal generated by $(C^{\text{sl}} - \frac{1}{2}(\mu^2 - 1) \mathbf{1})$,

$$B[\mu] = \frac{U(\mathfrak{sl}(2))}{C^{\text{sl}} - \frac{1}{2}(\mu^2 - 1) \mathbf{1}}.$$

Here, $C^{\text{sl}}$ is the quadratic Casimir of $\mathfrak{sl}(2)$; if we denote the generators of $\mathfrak{sl}(2)$ by $J_0, J_\pm$ with commutation relations

$$[J_+, J_-] = 2J_0, \quad [J_\pm, J_0] = \pm J_\pm,$$

then $C^{\text{sl}}$ is given by

$$C^{\text{sl}} \equiv J_0^2 - \frac{1}{2}(J_+ J_- + J_- J_+).$$
A basis for $B[\mu]$ as a vector space can be described as follows. There is one zero-letter word, namely the identity generator $1 \equiv V^1_0$ of the universal enveloping algebra. Then, there are three one-letter words, namely

$$V^1_2 = J_+, \quad V^2_0 = J_0, \quad V^2_{-1} = J_-, \quad (15)$$

and five two-letter words since the linear combination described by the Casimir (14) is proportional to $1$ in $B[\mu]$; we may denote them by

$$V^3_2 = J_+ J_+, \quad V^3_0 = J_0 J_0 + \frac{1}{2} J_+, \quad V^3_0 = \frac{1}{2} (J_- J_+ + J_+ J_- + 2 J_0 J_0) \equiv J_0 J_0 - \frac{1}{12} (\mu^2 - 1) \quad (16)$$

Continuing in this manner, one finds that there are $2s + 1$ different $s - 1$ letter words, which we may define to be

$$V^s_n = (-1)^{s-1-n} \frac{(n + s - 1)!}{(2s - 2)!} [J_s, \ldots, [J_m, J_{s-1}]], \quad (17)$$

where $|n| \leq s - 1$. Thus, we have a basis for the full vector space $B[\mu]$ given by $V^s_n$ with $s = 1, 2, \ldots$ and $|n| \leq s - 1$.

The vector space $B[\mu]$ actually defines an associative algebra, where the product $\ast$ is the one inherited from the universal enveloping algebra, i.e. defined by concatenation; this is what is sometimes called the ‘lone-star product’ in the literature. We can thus turn $B[\mu]$ into a Lie algebra by defining the commutator of two generators $X, Y \in B[\mu]$ to be

$$[X, Y] = X \ast Y - Y \ast X. \quad (18)$$

On $B[\mu]$, we can define an invariant bilinear trace [112] via

$$\text{tr}(X \ast Y) = X \ast Y |_{x=0}. \quad (19)$$

i.e. by retaining only the term proportional to $1 \equiv V^1_0$ (after rewriting the product in terms of the generators $V^s_n$). One easily checks that this trace is symmetric. Thus, the commutator of two elements in $B[\mu]$ does not involve $1$, and hence, as a Lie algebra, $B[\mu]$ decomposes as

$$B[\mu] = \mathbb{C} \oplus \text{hs}[\mu], \quad (20)$$

where the vector corresponding to $\mathbb{C}$ in (20) is the identity generator $1$ of the universal enveloping algebra, and a basis of the Lie algebra $\text{hs}[\mu]$, thus defined, is given by $V^s_n$ with $s = 2, \ldots$ and $|n| \leq s - 1$. The generators with $s = 2$ define an $sl(2)$ subalgebra, with respect to which the generators $V^s_n$ transform in the $(2s - 1)$-dimensional representation

$$[V^2_m, V^2_n] = (-m + m(s - 1))V^s_{m+n} \quad (21)$$

We thus conclude that the bulk fields associated with $V^s_n$ have spacetime spin $s$. The Chern–Simons theory based on $\text{hs}[\mu]$ therefore describes a higher spin theory with massless spin fields of spin $s = 2, 3, 4, \ldots$.

Let us analyse the structure of the Lie algebra $\text{hs}[\mu]$ in a little more detail. Using (16), the first few commutators are, for example,

$$[V^2_2, V^2_1] = 4V^4_2 \quad [V^3_2, V^0_0] = 4V^4_2 \quad (22)$$

A closed formula for all commutation relations is known [101]; see, e.g., equation (A.1) in [61]. Note that the commutators (22) suggest that, for $\mu = 2$, the Lie algebra generated by $V^s_n$
with \( s \geq 3 \) forms a proper subalgebra of \( \text{hs}[\mu] \). In fact, this is a special case of a more general phenomenon. If \( \mu = N \) with integer \( N \geq 2 \), then the quadratic form (19) degenerates \([112, 55]\)

\[
\text{tr} \left( V^*_m V^*_n \right) = 0 \quad \text{for} \quad s > N.
\] (23)

This implies that an ideal \( \chi_N \) appears, consisting of all generators \( V^*_s \) with \( s > N \). Factoring over this ideal truncates to the finite-dimensional Lie algebra \( \mathfrak{sl}(N) \), \( \text{hs}[\mu] = N \big/ \chi_N \cong \mathfrak{sl}(N) \quad (N \geq 2) \). (24)

Thus, we can think of \( \text{hs}[\mu] \) as being the continuation of \( \mathfrak{sl}(N) \) to non-integer \( N \). This relation will be important in the following.

In summary, we therefore have a one-parameter family of higher spin theories on AdS_3 that are described by a Chern–Simons theory based on the Lie algebra \( \text{hs}[\mu] \times \text{hs}[\mu] \). The classical theory reduces to a higher spin theory with a finite number of spins only when we take the parameter \( \mu \) to equal a positive integer greater than or equal to 2; in fact, if \( \mu = N \), then the theory becomes the \( \mathfrak{sl}(N) \times \mathfrak{sl}(N) \) higher spin theory described in the previous subsection.

2.1.4. Coupling to scalar fields. Unlike in higher dimensions, in three dimensions, the scalar field is not part of the higher spin multiplet and its presence in the theory is optional. The theory with a scalar field becomes considerably more complicated than the pure higher spin theory since the scalar field carries propagating degrees of freedom.

The full set of interactions of the scalar with the higher spin fields is difficult to write out explicitly \([103, 104]\). However, the interactions at the linearized level are relatively simple (see, for instance, \([9]\)). The scalar field \( C_0(x) \) is the part proportional to the identity of a field \( C(x) \) which takes values in the Lie algebra \( B[\mu] \) (see (20)). The latter obeys the linearized field equation

\[
dC + A \star C - C \star \bar{A} = 0,
\] (25)

where \( A, \bar{A} \) are the \( \text{hs}[\mu] \) gauge fields introduced in the previous section. When expanded around the AdS vacuum, these field equations imply that the scalar obeys the Klein–Gordon equation with mass \( M^2 = -1 + \mu^2 \). (Here, we have set the AdS radius \( \ell = 1 \).) Note that for any real value of \( \mu \), this is above the Breitenlohner–Freedman bound \([26]\) \( M^2_{\text{BF}} = -1 \). One can also work out the cubic and higher couplings of the scalar field, see \([103, 104, 36, 9, 37]\), but we will not go into the details here.

2.2. The \( \mathcal{W}_N \) minimal model conformal field theories

The CFTs we are interested in are the so-called \( \mathcal{W}_N \) minimal models \([52]\). They have higher spin conserved currents whose charges form an extended global symmetry of the CFT—in contrast to the higher spin gauge symmetry of the bulk AdS theory described in the previous subsection. This is, of course, to be expected from the point of view of the AdS/CFT correspondence where gauge fields in the bulk AdS couple to conserved currents in the boundary theory.

Interacting 2D conformal field theories with conserved currents \( W^{(s)}(z) \) with spin \( s \geq 3 \) were first constructed by Zamolodchikov \([119]\) and called \( W \)-algebras. They define a new class of chiral algebras beyond the more familiar cases of (super-)Virasoro/Kac–Moody algebras. In the following, we shall describe one route towards these theories, namely by explaining the construction of the \( \mathcal{W}_N \) minimal models via the coset construction. We shall also review their spectrum of primary operators and sketch the structure of the associated partition function.
2.2.1. The coset construction. The $\mathcal{W}_N$ minimal models are most easily described in terms of a coset [14]

$$\frac{\text{SU}(N)_k \otimes \text{SU}(N)_1}{\text{SU}(N)_{k+1}},$$

(26)

which is a special instance of the general $G/H$ coset construction [70]. In our case, this means that we consider a WZW theory based on the group $G = \text{SU}(N) \otimes \text{SU}(N)$ in which we gauge the diagonal subgroup $H = \text{SU}(N)$. The stress tensor of the coset theory equals

$$T_{G/H} = T_G - T_H,$$

(27)

where the individual stress tensors $T_G$ and $T_H$ are given by the usual Sugawara construction, i.e. in terms of bilinears of the currents. The stress tensors $T_{G/H}$ and $T_H$ have non-singular OPEs with each other by construction. We can therefore decompose the Hilbert space $H_G$ (or more particularly, the affine representation space $H_G^{(\Lambda_1)}$ corresponding to a highest weight representation (hwr) $\Lambda_1$) into representations of $H$ as

$$H_G^{(\Lambda_1)} = \bigoplus_{\Lambda'} (H_{G/H}^{(\Lambda,\Lambda')} \otimes H_{H}^{(\Lambda')}).$$

(28)

The multiplicity spaces $H_{G/H}^{(\Lambda,\Lambda')}$ then define the Hilbert space of the coset theory, and the corresponding operators commute with the $H$ currents (i.e. have a non-singular OPE with them).

It follows from (27) that the central charge of the coset stress tensor $T_{G/H}$ equals

$$c_{G/H} = c_G - c_H.$$

(29)

For our particular coset (26), this leads to

$$c_{N,k} = (N^2 - 1) \left[ \frac{k}{N+k} + \frac{1}{N+1} - \frac{k+1}{N+k+1} \right],$$

(30)

$$= (N-1) \left[ 1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right] \leq (N-1).$$

We will at times also use the notation $p = N+k \geq (N+1)$. Note that for $N = 2$, (26) agrees exactly with the original coset construction in [70], which describes the familiar unitary series of the Virasoro minimal models with

$$c_{2,k} = 1 - \frac{6}{p(p+1)}, \quad p = k+2.$$

(31)

For general $N$, the coset theory (26) with the smallest value of $k = 1$, i.e. $p = N+1$, has the central charge $c = \frac{2(N-1)}{N+2}$, and can alternatively be realized in terms of $\mathbb{Z}_N$ parafermions [51]. Another extreme case corresponds to $p \to \infty$ (taking $k \to \infty$ while keeping $N$ finite), where $c = (N-1)$, and the symmetry algebra is equivalent to the Casimir algebra of the $\mathfrak{su}(N)$ affine algebra at level $k = 1$ [13, 14]. The Casimir algebra consists of all $\mathfrak{su}(N)$ singlets in the affine vacuum representation of the affine algebra. Since the affine algebra is at level 1, it can be realized in terms of $(N-1)$ free bosons; thus in this limit, the coset model can be described as a singlet sector of a free (boson) theory [63].

2.2.2. Higher spin currents. The cosets (26) are the simplest examples of interacting CFTs which have (for $N \geq 3$) conserved currents of spin $s > 2$. We now describe an explicit method for constructing these higher spin currents. Actually, this procedure applies to the more general cosets of the form

$$\frac{G}{H} = \frac{\text{SU}(N)_k \otimes \text{SU}(N)_l}{\text{SU}(N)_{k+l}}.$$
Let us consider the cubic combination of currents

\[ W^3(z) \propto d_{abc}(a_1(J^a_{(1)}J^b_{(1)}J^c_{(1)})(z) + a_2(J^a_{(2)}J^b_{(1)}J^c_{(1)})(z) + a_3(J^a_{(3)}J^b_{(2)}J^c_{(1)})(z) + a_4(J^a_{(2)}J^b_{(2)}J^c_{(2)})(z)), \]

where \( d_{abc} \) is the totally symmetric cubic invariant of \( \mathfrak{su}(N) \) which is present for \( N \geq 3 \), while \( a_i \) are initially free parameters. The currents \( J^a_{(i)} \), \( J^b_{(j)} \) refer to the \( \mathfrak{su}(N) \) currents in the two factors in the numerator of the coset. The OPE of any of the four independent terms on the right-hand side (rhs) with the diagonal current \( (J^a_{(1)} + J^a_{(2)}) \) will generate singular terms of the kind \( d_{abc}J^a_{(i)}J^b_{(j)} \) with \( i, j \in \{1, 2\} \). Since there are only three such terms (since \( d_{abc} \) is symmetric), we can choose \( a_i \) such that the resulting \( W^3(z) \) has a non-singular OPE with \( (J^a_{(1)} + J^a_{(2)}) \). Thus, it defines a chiral current of weight and spin 3 in the coset theory. The explicit expressions for the coefficients can be found, for instance, in equations (7.42) and (7.43) of [24].

Since \( \mathfrak{su}(N) \) has independent invariant symmetric tensors for each rank \( s \) with \( s \leq N \)—these are the independent Casimirs of \( \mathfrak{su}(N) \)—a similar construction exists for each spin \( s \leq N \). Indeed, the analogue of the ansatz (33) contains now \( (s + 1) \) independent terms, and the OPE with the diagonal \( (J^a_{(1)} + J^a_{(2)}) \) current generates a singular term involving \( s \) distinct combinations of currents. By choosing the \( (s + 1) \) coefficients suitably, we can then arrange to have one combination which has a non-singular OPE with the diagonal current. Thus, we obtain one such field \( W^s(z) \) for every spin \( s \leq N \).

This construction works for general \( l \) and \( k \) in (32). What is special about taking one of the level, say \( l = 1 \), is that the OPEs of the \( W^s \) close among themselves. The additional fields that are generated in these OPEs for general \( l \) become null for \( l = 1 \) and hence decouple; see, e.g., [24].) The resulting algebra of \( W^l(z) \) defines the \( \mathcal{W}_N \)-algebra that is of primary interest to us here.

2.2.3. Minimal model primaries. The above higher spin currents are in the vacuum sector \((\Lambda = \Lambda' = 0 \text{ in the notation of (28)})\) of the coset Hilbert space since they are built purely from the currents \( J^a_{(1)} \) and \( J^b_{(2)} \), and are singlets with respect to the diagonal zero mode action. The other states of the theory (26) fall into non-trivial hwr\(^s\) of the coset algebra. As clear from (28), a general representation is parametrized by taking \( \Lambda = \rho \oplus \mu \), where \( \rho \) is an hwr of \( \mathfrak{su}(N)_k \), while \( \mu \) is an hwr of \( \mathfrak{su}(N)_1 \), and \( \Lambda' = \nu \), where \( \nu \) is an hwr of \( \mathfrak{su}(N)_{k+1} \); thus, the most general coset representations are labelled by \((\rho, \mu; \nu)\). Actually, only those combinations are allowed for which \( \nu \) appears in the decomposition of \((\rho \oplus \mu)\) under the action of \( \mathfrak{su}(N)_{k+1} \). The relevant selection rule is simply that

\[ \rho + \mu - \nu \in \Lambda_R, \]

where \( \rho, \mu \) and \( \nu \) are thought of as weights of the finite-dimensional Lie algebra \( \mathfrak{su}(N) \), and \( \Lambda_R \) is the corresponding root lattice. In addition, there are field identifications: the two triplets

\[ (\rho, \mu; \nu) \cong (A\rho, A\mu; Av) \]

define the same hwr of the coset algebra, provided that \( A \) is an outer automorphism of the affine algebra \( \mathfrak{su}(N)_{k} \). The group of outer automorphisms of \( \mathfrak{su}(N)_{k} \) is \( \mathbb{Z}_N \) (independent of \( k \)), and it is generated by the cyclic rotation of the affine Dynkin labels \( l_j \), i.e. the map

\[ [l_0; l_1, \ldots, l_{N-1}] \mapsto [l_1; l_2, \ldots, l_{N-1}, l_0], \]

It is important to note though that the states in the coset do not transform under any non-trivial representations of \( \mathfrak{su}(N) \).
where the first entry is the affine Dynkin label. In this notation, the allowed hwrs of $\mathfrak{su}(N)$ at level $k$ are labelled by

\[ P^+_k(\mathfrak{su}(N)) = \left\{ [l_0; l_1, \ldots, l_{N-1}] : l_j \in \mathbb{N}_0, \quad \sum_{j=0}^{N-1} l_j = k \right\}. \tag{37} \]

Note that the field identification (35) does not have any fixed points since $\mathbb{Z}_N$ acts transitively on the hwrs of $\mathfrak{su}(N)$ at level $k = 1$.

### 2.2.4. The spectrum of primaries

It is easy to see that for any choice of hwrs $(\rho; v)$, there always exists a unique $\mu \in P^+_1(\mathfrak{su}(N))$, such that $\rho + \mu - v \in A_R$. Thus, we may label the hwrs of the coset algebra in terms of unconstrained pairs $(\rho; v)$ and suppress the $\mu$ label since it is completely determined by the other two. The labels are still subject to the field identifications

\[ (\rho; v) \cong (\Lambda; \Lambda v). \tag{38} \]

Since the coset theory has a stress tensor which is the difference of the two stress tensors of the mother and daughter theories, the conformal weight of the corresponding hwr has the form

\[ h(\rho; v) = \frac{C_2(\rho)}{N + k} + \frac{C_2(\mu)}{N + 1} - \frac{C_2(v)}{N + k + 1} + n, \tag{39} \]

where $C_2(\sigma)$ is the eigenvalue of the quadratic Casimir operator of $\mathfrak{su}(N)$. Furthermore, $n$ is a non-negative integer, describing the ‘height’ (i.e. the conformal weight above the ground state) at which the $\mathfrak{su}(N)_{k+1}$ primary $v$ appears in the representation $(\rho \oplus \mu)$. Unfortunately, an explicit formula for $n$ is not available, but it is not difficult to work out $n$ for simple examples.

Alternatively, one may use the Drinfel’d–Sokolov (DS) description of these models; see, e.g., [24] for more details. In that language, the hwrs are labelled by $(\Lambda_+, \Lambda_-) \cong (\rho; v)$, and the conformal weights equal

\[ h(\Lambda_+, \Lambda_-) = \frac{1}{2p(p+1)} \left((p+1)(\Lambda_+ + \tilde{\rho}) - p(\Lambda_- + \tilde{\rho})\right)^2 - \tilde{\rho}^2, \tag{40} \]

where $\tilde{\rho}$ is the Weyl vector of $\mathfrak{su}(N)$. For $N = 2$ (the Virasoro minimal models), (40) reduces to the familiar formula

\[ h(r, s) = \frac{(r(p + 1) - sp)^2 - 1}{4p(p+1)} = h(p-r, p+1-s), \tag{41} \]

with $1 \leq r \leq p - 1$, $1 \leq s \leq p$. Here, we have identified $\Lambda_+ = \frac{(r - 1)}{2} \tilde{\alpha}$ and $\Lambda_- = \frac{(r - 1)}{2} \bar{\alpha}$ (with $|\tilde{\alpha}|^2 = 2$).

In the following, the primary where $v = [1, 0^{N-2}] = f$ is the fundamental representation\(^6\) with $\rho = [0^{N-1}] = 0$, the trivial representation will play an important role. Then, either (40) or (39) gives in the latter case $\mu = f$ with $n = 0$

\[ h(0; f) = \frac{C_2(f)}{N+1} - \frac{C_2(f)}{N+k+1} = \frac{(N-1)}{2N} \left( 1 - \frac{N+1}{N+k+1} \right), \tag{42} \]

where we have used that $C_2(f) = \frac{1}{2} (\Lambda_0, \Lambda_0 + 2\tilde{\rho}) = \frac{N^2-1}{2N}$. On the other hand, for the coset representation with $\rho = f$ and $v = 0$, $\mu$ is the anti-fundamental representation, $\mu = \bar{f}$, and we obtain (again with $n = 0$)

\[ h(f; 0) = \frac{C_2(f)}{N+k} + \frac{C_2(f)}{N+1} = \frac{(N-1)}{2N} \left( 1 + \frac{N+1}{N+k} \right). \tag{43} \]

\(^6\) Note that the representation of the affine $\mathfrak{su}(N)$ algebra has $N$ entries as in (36). Here and below, we will mostly drop the affine Dynkin label and use a description in terms of the usual $(N-1)$ Dynkin labels for representations of the finite-dimensional Lie algebra $\mathfrak{su}(N)$.\]
An example with \( n = 1 \) arises for the case where \( \rho = 0 \) and \( \nu = \text{adj} \), the adjoint representation. Then, \( \mu = 0 \) but \( n = 1 \), and we obtain

\[
h(0; \text{adj}) = 1 - \frac{C_2(\text{adj})}{N + k + 1} = 1 - \frac{N}{N + k + 1},
\]

(44)

where we have used that \( C_2(\text{adj}) = N \). Finally, the representation with \( \rho = \text{adj} \) and \( \nu = 0 \) also has \( \mu = 0 \) and \( n = 1 \), and the conformal weight is

\[
h(\text{adj}; 0) = 1 + \frac{C_2(\text{adj})}{N + k} = 1 + \frac{N}{N + k}.
\]

(45)

2.2.5. Partition functions. To determine the complete partition function of the theory, we need to know the full tower of descendants for each of the allowed primaries. These descendant states are generated from the ground states by the action of the negative Virasoro and higher spin modes, modulo the null states that decouple. The most efficient way to calculate the corresponding character is by using (28) since we know the affine characters \( \chi^{(p \nu \mu)}_N \) and \( \chi^{(\nu)}_N \), and hence can read off the character of the coset as a branching function.

For the coset we are considering, the branching functions \( b_{(\Lambda_+; \Lambda_-)} \) are known explicitly, see, e.g., equation (7.51) of [24], and given by

\[
b_{(\Lambda_+; \Lambda_-)}(q) = \frac{1}{\eta(q)^{N-1}} \sum_{w \in \hat{W}} \epsilon(w) q^{\frac{1}{2} \sum_{m \geq 1} (\rho + \hat{\nu} + \hat{\mu}) - p(\Lambda_+ + \hat{\nu} - \Lambda_-)}.
\]

(46)

where \( \hat{\rho} \) is the Weyl vector of \( \mathfrak{su}(N) \), \( p = k + N \) and the sum is over the full affine Weyl group \( \hat{W} \).

The full partition function is then obtained by putting together the holomorphic and anti-holomorphic branching functions in a modular-invariant manner. There are many non-equivalent ways of doing so. However, we will be concentrating here on the simplest choice—the so-called charge conjugation invariant. Its partition function is given by

\[
Z_{cc} = \sum_{\Lambda_+; \Lambda_-} |b_{(\Lambda_+; \Lambda_-)}(q)|^2.
\]

(47)

2.2.6. Fusion rules and characters. The fusion rules of the coset theory follow directly from the mother and daughter theories. Indeed, in terms of the triplets \( (\rho, \mu; \nu) \), the fusion rules are simply

\[
N_{(\rho_1; \mu_1; \nu_1)}(\rho_2, \mu_2; \nu_2) = N^{(k)}_{\rho_1 \rho_2} N^{(1)}_{\mu_1 \mu_2} N^{(k+1)}_{\nu_1 \nu_2},
\]

(48)

where the fusion rules on the rhs are those of \( g_0, g_1 \) and \( g_{k+1} \), respectively. Note that the fusion rules are invariant under the field identification (35). Since the fusion rules of the level-1 factor are just a permutation matrix, we can also directly give the fusion rules for the representatives \( (\rho; \nu) \) as

\[
N_{(g_1; \nu_1)}(g_2; \nu_2) = N^{(k)}_{g_1 g_2} N^{(1)}_{\nu_1 \nu_2}.
\]

(49)

Note that the fusion rules on the rhs for the affine theories are strongly constrained by the \( \mathfrak{su}(N) \) symmetry—they are essentially Clebsch–Gordan coefficients. This will play an important role when we consider the large-\( N \) limit of correlation functions.

This completes our brief review of the \( \mathcal{W}_N \) minimal models; further details about coset theories in general can be found in [48], and various aspects of \( \mathcal{W} \)-algebras are explained in the review [24].

12
3. $\mathcal{W}_\infty$-symmetries in the boundary and the bulk

In this section, we explain the emergence of $\mathcal{W}_\infty$-symmetries in our context. First, in section 3.1 (see in particular section 3.1.1), we describe the $\mathcal{W}_\infty$-algebras that appear in the 2D CFTs of free bosons and free fermions. Then, in section 3.2, we show that closely related $\mathcal{W}_\infty$-algebras appear as the asymptotic symmetry algebra of higher spin theories in AdS$_3$. The precise relationship between the two constructions will be discussed in section 4.

3.1. $\mathcal{W}$-algebras in 2D conformal field theories

Unlike in higher dimensions, it is possible to have non-trivial interacting quantum field theories in two dimensions which possess conserved currents of spin $s > 2$. The Coleman–Mandula argument [41] does not rule out their existence and indeed there is a rich collection of 2D (massive) integrable quantum field theories which have higher spin conserved charges.

The Coleman–Mandula theorem itself applies to theories with an $S$-matrix and there is an assumption about the spectrum having a mass gap [41]. Therefore, it does not strictly apply to conformal field theories. An analogue of the Coleman–Mandula theorem (with some mild assumptions) was recently proven for conformal field theories in 3D [94]. This theorem shows that (in a theory with a finite number of fields) the correlation functions of higher spin currents are necessarily given by either those in a theory of free fermions or in one of free bosons. Thus, at least the sector of higher spin currents behaves like a free theory.

In two dimensions, this theorem does not hold. Indeed, as we have seen in section 2.2, the coset conformal theories (that are typically interacting) have conserved higher spin currents. The resulting $\mathcal{W}$-algebras are generically nonlinear (in contradistinction to the more familiar Kac–Moody or Virasoro algebras). This is to say that quadratic terms in the current modes appear on the rhs of current commutators. The OPE of the currents is nevertheless associative and hence Jacobi identities are obeyed. This nonlinear structure is directly responsible for the fact that the algebra undergoes a non-trivial deformation at the quantum level (as we shall explain in some detail below, see section 4). It is also the reason why these theories are much harder to analyse. In fact, the complete commutation relations have only been written down explicitly for a few $\mathcal{W}$-algebras involving fields of small spin.

While a large number of different $\mathcal{W}$-algebras have been studied (and there are probably many more yet to be discovered), we will restrict our attention in this review mostly to a special class of $\mathcal{W}$-algebras. We will consider the so-called $\mathcal{W}_N$-algebras which contain exactly one conserved current $W_s(z)$ of integer spin $s = 2, 3, \ldots, N$, with the spin-2 current being the stress tensor $W_2(z) \equiv T(z)$. For fixed $N$, these algebras are parametrized by the central charge $c$, and for $c = c_{W_N}$, see (30), the algebras coincide with those arising in the coset construction of section 2.2. For $c \geq N - 1$, the algebras also appear in a family of (generically non-unitary) CFTs known as the $\mathcal{A}_N$ Toda theories of which the Liouville theory is the simplest member (for $N = 2$).

The $\mathcal{W}_N$-algebras are in turn a special case of an even more general family of $\mathcal{W}$-algebras which will play a central role in our analysis and which we will denote by $\mathcal{W}_\infty[\mu]$. These algebras are parametrized by two labels: the central charge $c$, as well as the parameter $\mu$. Generically, the algebras are generated by the currents $W_s(z)$, where $s = 2, 3, \ldots$ without any bound on $s$. For special values of $\mu$, however, e.g., for $\mu = N \geq 2$, they reduce to the $\mathcal{W}_N$-algebras discussed above. There are also simplifications for $\mu = 0$ and $\mu = 1$, where the algebras are closely related to those of free fermions and free bosons, respectively. 

7 The proof may be generalizable to higher dimensions; see [107, 120] for the first steps in this direction.
The higher spin algebra corresponding to free bosons and fermions is an analogue of a similar algebra in the higher dimensional theories (though extended in two dimensions to chiral and anti-chiral currents); the $\mathcal{W}_\infty[\mu]$-algebras for $\mu \neq 0, 1$, on the other hand, do not seem to have an analogue in higher dimensions. In the following, we shall sketch the construction of the $\mathcal{W}_\infty[\mu]$-algebras for the case of free bosons and free fermions; we shall come back to the $\mathcal{W}_\infty[\mu]$-algebra for general $\mu$ in section 4.

3.1.1. Free bosons and free fermions. As in higher dimensions, we can write down conserved currents of spin higher than 2 for a system of free bosons or free fermions. The new feature in two dimensions is the enlargement to separate holomorphic and anti-holomorphic currents as in the case of the stress tensor.

Consider, for instance, a complex free boson. We can write down conserved currents with $s = 1, 2, \ldots \infty$:

$$W_s^B(z) \propto s - 2 \sum_{k=0}^{s-2} (-1)^k \binom{s-1}{k} \binom{s-1}{k+1} \partial^{s-k-1} \bar{\phi} \partial^k \phi.$$ (50)

For $s = 1, 2$, this reduces to the familiar charge and energy momentum currents, respectively. The combinatorial coefficients are chosen so that $W_s^B$ transforms as a quasi-primary under the global conformal transformations. The conservation follows from the equation of motion for the free theory $\bar{\partial} \partial \phi = 0$. It is straightforward to generalize this construction to an $N$ component boson $\phi_i$—the above currents are singlet bilinears under the resulting global $SU(N)$.

Using the OPE of free bosons, it is not difficult to work out the OPE of the currents $W_s^B(z)$. Schematically, one finds

$$W_s \cdot W_{s'} \sim W_{s+s'-2} + W_{s+s'-4} + \cdots + c_s \delta_{s,s'},$$ (51)

with a central term $c_s \propto c = N$ for the general case of $N$ free bosons. Note that the algebra has no nonlinear terms. Explicit expressions for the commutation relations of the modes of these currents can, for example, be found in section 3 of [102]. The resulting Lie algebra is related by a nonlinear change of basis to the general $\mathcal{W}_\infty[\mu]$-algebra at the special value $\mu = 1$ (and $c = N$) [61].

Similarly for a free Dirac fermion, we can define [19, 20, 46, 102]

$$W_s^F(z) \propto \sum_{k=0}^{s-1} (-1)^k \binom{s-1}{k} \partial^{s-k-1} \bar{\psi} \partial^k \psi,$$ (52)

with the $s = 1, 2$ expressions being the more familiar conserved currents. Again, these combinations are quasi-primary, and the OPE also has the schematic form as in (51) though the (suppressed) coefficients in front of the individual terms are different; see [102] for explicit expressions. This algebra is believed to be related to the $\mathcal{W}_\infty[\mu]$-algebra at $\mu = 0$ after an appropriate truncation to the sector without the spin-1 current [61].

3.2. Asymptotic symmetries of higher spin theories

Next, we want to explain how similar $\mathcal{W}_\infty$-algebras also appear as asymptotic symmetry algebras of higher spin gauge theories on AdS$_3$. Recall from section 2.1 that these higher spin gauge theories can be described in terms of a Chern–Simons theory. In this section, we pay close attention to the boundary conditions that need to be imposed in this description, first in the case of pure gravity following closely [16] as reviewed in [29]. Then, we explain how to generalize this analysis to the case of spin-3 gravity, and how to obtain the asymptotic symmetry algebra in the manner of Brown and Henneaux [27]. (Again, this follows closely [29], see
3.2.1. Asymptotic symmetry analysis for gravity. In order to describe the boundary conditions in the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ Chern–Simons formulation of gravity, let us introduce a basis for $\mathfrak{sl}(2)$ consisting of $L_0$, $L_{\pm 1}$ with $[L_m, L_n] = (m - n)L_{m+n}$. Furthermore, we parametrize the solid cylinder on which the Chern–Simons theory is defined by $(t, \rho, \phi)$, where $(\rho, \phi)$ are 2D polar coordinates on the disc, while $t$ is the variable along the length of the cylinder. Introducing light-cone variables as $x^\pm = \frac{t}{\ell} \pm \phi$, (53) the 1-form $A^\mu$ from (6) takes the form

$$A^\mu = A^\rho d\rho + A^+ dx^+ + A^- dx^-.$$ (54)

The solid torus has a boundary, and hence the variation of the Chern–Simons action includes the boundary term

$$\delta S|_{\text{bdy}} = -\frac{\hat{k}}{4\pi} \int_{\mathbb{R} \times S^1} dx^+ dx^- \text{Tr}(A_+ \delta A_- - A_- \delta A_+).$$ (55)

A natural boundary condition that guarantees that this boundary term vanishes is then, for example,

$$A_- = 0 \quad \text{at the boundary.}$$ (56)

Note that this is necessary in order to really reproduce the equations of motion of Einstein gravity from the Chern–Simons point of view.

Next, we want to characterize the physically inequivalent solutions of the Chern–Simons theory that are asymptotically AdS$_3$. We can partially fix the gauge by setting

$$A_\rho = b^{-1}(\rho) \partial_\rho b(\rho),$$ (57)

where $b(\rho)$ is an arbitrary function with values in $\text{SL}(2, \mathbb{R})$. Solving the equations of motion ($F = 0$) then leads to

$$A_+ = b^{-1}(\rho) a(x^+) b(\rho), \quad A_- = 0.$$ (58)

The analysis can be done similarly for $\bar{A}$, leading to

$$\bar{A}_\rho = b(\rho) \partial_\rho b^{-1}(\rho), \quad \bar{A}_+ = 0, \quad \bar{A}_- = b(\rho) \bar{a}(x^-) b(\rho)^{-1},$$ (59)

where $b(\rho)$ is the same function as above—this is necessary for the solution to be asymptotically AdS$_3$. In fact, AdS$_3$ is described in this framework by the solution

$$A_{\text{AdS}} = b^{-1} \left( L_1 + \frac{1}{4} L_{-1} \right) b \, dx^+ + b^{-1} \partial_\rho b \, d\rho$$ (60)

$$\bar{A}_{\text{AdS}} = -b \left( \frac{1}{4} L_1 + L_{-1} \right) b^{-1} dx^- + b \partial_\rho b \partial_\rho^{-1} d\rho,$$ (61)

where

$$b(\rho) = e^{\rho L_0}.$$ (62)

Actually, the condition that (59) takes the above form is not quite sufficient to obtain an asymptotically AdS solution (in the sense of Brown and Henneaux [27]), as discussed in [16, 29]. In addition, we have to require that

$$(A - A_{\text{AdS}})|_{\text{bdy}} = (\bar{A} - \bar{A}_{\text{AdS}})|_{\text{bdy}} = O(1).$$ (63)
In particular, this implies that the functions \( a(x^+) \) and \( \bar{a}(x^-) \) that appear in (58) and (59) are of the form

\[
a(\phi) = L_1 + \bar{l}^0(\phi)L_0 + l^{-1}(\phi)L_{-1}, \quad \bar{a}(\phi) = L_{-1} + \bar{p}^0(\phi)L_0 + \bar{p}^{-1}(\phi)L_1,
\]

where \( \bar{l}^0(\phi) \) and \( l^{-1}(\phi) \) (as well as their barred cousins) are arbitrary functions of \( \phi \), and we have set (for simplicity) \( t = 0 \).

Among the asymptotically AdS solutions, we should now identify those as physically equivalent that can be related by a gauge transformation that vanishes at the boundary\(^8\). Using only gauge transformations of this type, we can set \( \bar{l}^0(\phi) = \bar{p}^0(\phi) = 0 \), but we cannot change \( l^{-1}(\phi) \) and \( \bar{p}^{-1}(\phi) \). Thus, we conclude that the space of physically inequivalent asymptotically AdS solutions is parametrized by the functions \( l^{-1}(\phi) \) and \( \bar{p}^{-1}(\phi) \). This space then carries naturally an action of \( \text{Diff}(S^1) \times \text{Diff}(S^3) \), corresponding to the two commuting Virasoro actions predicted by the analysis of Brown and Henneaux [27]. (The asymptotic symmetry analysis can also be carried out in this framework, see [16]—we shall sketch this for the case of spin-3 gravity in the following section.)

3.2.2. Asymptotic symmetry analysis for spin-3 gravity. Now we generalize the analysis to the pure higher spin theory containing in addition to the spin-2 graviton a massless spin-3 field. As reviewed in section 2.1.2, there is a Chern–Simons description in terms of \( sl(3) \times sl(3) \) gauge fields. In this framework, we need to discuss how asymptotically AdS solutions can be characterized. To be concrete, let us denote the basis elements of the five-dimensional subspace in (9) by \( V_n^3 \) with \( n = -2, -1, 0, 1, 2 \); their commutators are then given by

\[
[L_n, V_m^3] = (2m - n)V_{m+n}^3
\]

(65)

\[
[V_m^3, V_n^3] = (m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n}.
\]

(66)

The most general ansatz for the function \( a(\phi) \) in equation (58) is then (analogous statements hold also for \( \bar{a}(\phi) \))

\[
a(\phi) = \sum_{m=-1}^{1} l^m(\phi)L_m + \sum_{n=-2}^{2} u^n(\phi)V_n^3.
\]

(67)

We can characterize the asymptotic boundary condition as in (63), where \( A_{\text{AdS}} \) is the solution for which \( u^0(\phi) \equiv 0 \), and \( l^m(\phi) \) is given as in (64):

\[
l^1(\phi) = 1, \quad u^0(\phi) = u^1(\phi) = 0.
\]

(68)

By means of gauge transformations that vanish at the boundary, we can also set

\[
l^0(\phi) = w^0(\phi) = w^{-1}(\phi) = 0,
\]

(69)

and hence the space of physically inequivalent asymptotically AdS solutions is parametrized by the functions \( l^{-1}(\phi) \) and \( u^{-2}(\phi) \) (as well as their right-moving analogues).

The next step is now to determine the (classical) asymptotic symmetry algebra of this higher spin theory. Concentrating on the left-moving fields—the analysis for the right movers is analogous—the most general gauge transformation that preserves the gauge (57) and (58) is

\[
\Gamma(x^+) = e^{-\delta a} \gamma(x^+) e^{\delta a},
\]

(70)

where \( \gamma(x^+) \) is an arbitrary Lie algebra valued function; here, we have used that its action on the gauge field is of the form

\[
\delta a = \gamma' + [a, \gamma].
\]

(71)

\(^{8}\) Indeed, since 3D gravity is topological, any two solutions are gauge equivalent, and physical degrees of freedom only arise if we are careful about boundary conditions.
Let us parametrize $\gamma(\phi)$ as
\[
\gamma(\phi) = \sum_{s=2}^{3} \sum_{|n|<s} \gamma_{s,n}(\phi) V_{s}^{n}, \tag{72}
\]
where $V_{s}^{n} \equiv L_{n}$ with $n = 0, \pm 1$. Demanding that, after the gauge transformation (72), the gauge connection $A$ is still of the form (67) with (68) and (69) then leads to the recursion equations (from the conditions that the coefficients of $L_{1} \equiv V_{1}^{2}$ and $L_{0} \equiv V_{2}^{2}$ are unchanged):
\[
\gamma_{2,0} = -\gamma_{2,1}^{\prime}, \tag{73}
\]
\[
\gamma_{2,-1} = \frac{1}{2} \gamma_{2,1}^{\prime} + \frac{2\pi}{k} \gamma_{2,1} L + \frac{4\pi}{k} \gamma_{3,2} W, \tag{74}
\]
where
\[
L(\phi) = \frac{k}{2\pi} f^{-1}(\phi), \quad W(\phi) = \frac{6\hat{k}}{\pi} w^{-2}(\phi). \tag{75}
\]
Similarly, from the requirement that the coefficients of $V_{2}^{3}$, $V_{1}^{3}$, $V_{0}^{3}$ and $V_{1}^{-3}$ continue to vanish, we obtain
\[
\gamma_{3,1} = -\gamma_{3,2}', \tag{76}
\]
\[
\gamma_{3,0} = \frac{1}{2} \gamma_{3,2}^{\prime} + \frac{4\pi}{k} \gamma_{3,2} L \tag{77}
\]
\[
\gamma_{3,-1} = -\frac{1}{6} \gamma_{3,2}^{\prime'} - \frac{10\pi}{3k} \gamma_{3,2} L - \frac{4\pi}{3k} \gamma_{3,2} L' \tag{78}
\]
\[
\gamma_{3,-2} = \frac{1}{24} \gamma_{3,2}^{\prime''} + \frac{4\pi}{3k} \gamma_{3,2} L + \frac{7\pi}{6k} \gamma_{3,2} L' + \frac{\pi}{3k} \gamma_{3,2} L'' + \frac{4\pi^{2}}{k^{2}} \gamma_{3,2} L^{2} + \frac{\pi}{6k} \gamma_{2,1} W. \tag{79}
\]
Writing $\epsilon \equiv \gamma_{2,1}^{\prime}$ and $\chi \equiv \gamma_{3,2}$, we then obtain altogether the variations
\[
\delta_{\epsilon} L = \epsilon L' + 2\epsilon' L + \frac{k}{4\pi} \epsilon'' \tag{80}
\]
\[
\delta_{\chi} W = \epsilon W' + 3\epsilon' W \tag{81}
\]
as well as
\[
\delta_{\epsilon} L = 2\chi W' + 3\chi' W \tag{82}
\]
\[
\delta_{\chi} W = 2\chi L'' + 9\chi' L' + 15\chi'' L + 10\chi''' L + \frac{k}{4\pi} \chi^{(5)} + \frac{64\pi}{k} (\chi L L' + \chi' L'^{2}). \tag{83}
\]
Interpreting these variations in terms of charges, we can read off the Poisson brackets of the associated currents; see [29] for details. In particular, it follows from equation (80) that $L$ plays the role of the stress–energy tensor, i.e. the associated modes satisfy the Virasoro algebra
\[
i[L_{m}, L_{n}] = (m - n) L_{m+n} + \frac{c}{12} m(m^{2} - 1) \delta_{m,-n}, \quad c = 6\hat{k}. \tag{84}
\]
Furthermore, equation (81) means that $W$ is a primary field of conformal weight $h = 3$ since we have the Poisson brackets
\[
i[L_{m}, W_{n}] = (2m - n) W_{m+n}. \tag{85}
\]
Finally, the Poisson bracket of the $W$ modes with themselves contains bilinear terms that originate from equation (83)

$$
i[W_m, W_n] = -\left[(m-n)(2m^2+2n^2-mn-8)L_{m+n} + \frac{96}{c}(m-n) \Lambda_{m+n}^{(4)}
+ \frac{c}{12}m(m^2-1)(m^2-4)\delta_{m-n}\right],$$

(86)

where $\Lambda_{m+n}^{(4)} \equiv \sum_{n \in \mathbb{Z}} L_n L_{m-n}$. The Poisson algebra defined by (84)–(86) is the classical $W_4$-algebra, which is a well-defined Poisson algebra (in particular satisfying the Jacobi identity) for any value of $c$. Because of the nonlinear term, the ‘quantization’ of this algebra, where we replace Poisson brackets by commutators, is not straightforward since we will have to worry about normal ordering terms. We will come back to this issue in section 4.

3.2.3. Asymptotic symmetry algebra of $\text{hs}[\mu]$ Chern–Simons theory. Next, we want to study the asymptotic symmetry algebra of the Chern–Simons theory based on $\text{hs}[\mu] \times \text{hs}[\mu]$; this can be done in close analogy to the case of $\text{sl}(3)$ in section 3.2.2. The asymptotic boundary condition (63) together with the gauge transformations that vanish at the boundary now allows one to set the coefficients of all Lie algebra generators $V_n^s$ to zero, except for $V_1^s$ (whose coefficient equals 1), as well as $V_{s+1}^s$ (whose coefficients $\gamma_{s,-s+1}$ are the analogues of the functions $t^{-1}(\phi)$ and $w^{-2}(\phi$ from above)). The requirement that the gauge transformation (70) leaves this form of the solution invariant leads then again to recursion relations analogous to (73)–(79). This allows one to determine the variations, i.e. the analogues of (80)–(83), and from them the Poisson brackets of the fields $W^{(s)} \sim \gamma_{s,-s+1}$. For the first few cases, this was explicitly worked out in [61], and it was observed that the answer agrees precisely with the classical $W^{cl}_{\infty}[\mu]$-algebra that had been obtained before in [54, 88, 89].

Explicit closed-form expressions for the Poisson brackets (albeit in a non-primary basis) are known for $\mathcal{W}^{cl}_{\infty}[\mu]$; see [54] or the appendix of [61]. Recursion relations for the algebra in a primary basis were later given in [28]. The $\mathcal{W}^{cl}_{\infty}[\mu]$-algebra is generated by the elements $W^{(s)}_n$, where $s = 2, 3, \ldots$ and $n \in \mathbb{Z}$. Because of the nonlinear terms (i.e. the analogue of the $\Lambda^{(4)}$ term in (86)), it is not immediately clear how to turn the Poisson brackets into commutators—we shall come back to this point in section 4. However, these difficulties go away for $c \to \infty$ (since the nonlinear terms are suppressed by inverse powers of $c$) [25]. In this limit, the generators $W^{(s)}_n$ with $|n| \leq s-1$—we shall sometimes refer to the corresponding algebra as the ‘wedge algebra’—reduce to those of $\text{hs}[\mu]$. Thus, we can think of $\mathcal{W}^{cl}_{\infty}[\mu]$ as an extension of the wedge algebra $\text{hs}[\mu]$, ‘beyond the wedge’. However, at finite $c$ (and with the exception of $\mu = 1$), even the commutation relations of the wedge generators acquire nonlinear correction terms and thus do not agree with those of $\text{hs}[\mu]$. Thus, we expect that $\text{hs}[\mu]$ will not be a subalgebra of the quantum $\mathcal{W}^{cl}_{\infty}[\mu]$-algebra. As we have mentioned before, $\mu = 1$ corresponds to a free boson theory, and thus the fact that $\text{hs}[\mu] \times \text{hs}[\mu]$ with $\mu \neq 1$ is not a genuine symmetry of the theory at finite $c$ is the 2D incarnation of the result of [94, 95].

4. Matching the symmetries

Next, we want to understand the precise relation between the asymptotic symmetry algebra $\mathcal{W}^{cl}_{\infty}[\mu]$ of the higher spin theory on AdS that we have just derived, and the limit algebra of

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9 Actually, one can argue on general grounds [29] that, at least formally, the asymptotic symmetry algebra is the DS reduction [50] (see [47] for a review) of the affine algebra based on $\text{hs}[\mu]$. Then, the identification of the asymptotic symmetry algebra with $\mathcal{W}^{cl}_{\infty}[\mu]$ can also be deduced from the work of [87].
the $\mathcal{W}_{N,k}$ minimal models. In order to do so, it is important to understand how we can turn the classical Poisson algebra $\mathcal{W}_\infty^d[\mu]$ into a consistent quantum algebra $\mathcal{W}_\infty[\mu]$; the following analysis follows closely [58].

4.1. The quantum algebra $\mathcal{W}_\infty[\mu]$ 

As we have mentioned before, the main difficulty in replacing the Poisson brackets by commutators comes from the nonlinear terms in the commutation relations. For example, naively ‘quantizing’ the Poisson brackets of $c$ leads to the commutators

$$[W_m, W_n^3] = 2(m - n)W_{m+n}^4 + \frac{N_3}{12}(m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n} + \frac{8N_3}{c}(m - n)\Lambda_{m+n}^{(4)} + \frac{N_3}{444}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}$$

(87)

$$[W_m, W_n^2] = (3m - 2n)W_{m+n}^5 + \frac{208N_4}{25N_3^2c}(3m - 2n)\Lambda_{m+n}^{(5)} + \frac{84N_4}{25N_3^2}\Theta_{m+n}^{(6)} - \frac{N_4}{15N_3}n^3 - 3mn^2 + 5m^2n - 9n + 17m)W_{m+n}^3$$

(88)

where $\Lambda_{m}^{(4)} \sim LL$, $\Lambda_{m}^{(5)} \sim W_m^3L$, and $\Theta_{m}^{(6)} \sim \frac{2}{9}L(W_m^2)^L - L'W_m^3$, and we have denoted the Virasoro generators by $W_n^2 \equiv L_n$. Furthermore, the structure constants take the form

$$N_3 = \frac{10}{9} q^2 (\mu^2 - 4)$$

(89)

$$N_4 = \frac{384}{125} q^4 (\mu^2 - 4) (\mu^2 - 9)$$

(90)

where $q$ is a normalization parameter, i.e. different values of $q$ describe the same algebra. As written, these commutation relations do not satisfy the Jacobi identities

$$[L_m, [L_n, W_l^3]] + \text{cycl.} = [L_m, [W_n^3, W_l^3]] + \text{cycl.} = [W_m^4, [W_n^3, L_l]] + \text{cycl.} = 0,$$

(91)

except to leading order in $1/c$. However, we can satisfy the Jacobi identities exactly, i.e. for arbitrary finite $c$, (i) by defining carefully what we mean by $\Lambda_{m}^{(4)}$, $\Lambda_{m}^{(5)}$, $\Theta_{m}^{(6)}$, i.e. by specifying the correct ‘normal ordering prescription’; (ii) by modifying the above commutation relations by $1/c$ corrections. Explicitly, the correct normal ordered expressions are

$$\Lambda_{m}^{(4)} = \sum_p \langle L_{m-p}L_p : \frac{1}{5}x_nL_n \rangle$$

(92)

$$\Lambda_{m}^{(5)} = \sum_p \langle L_{m-p}W_p^3 : \frac{1}{14}y_nW_n^3 \rangle$$

(93)

$$\Theta_{m}^{(6)} = \sum_p \langle \frac{5}{3}p - n \rangle : L_{m-p}W_p^3 : \frac{1}{6}z_nW_n^3 \rangle$$

(94)

where

$$x_l = (l + 1)(1 - l), \quad x_{l-1} = (l + 1)(2 - l),$$

(95)

$$y_l = (l + 2)(3 - 5l), \quad y_{l-1} = 5(l + 1)(2 - l),$$

(96)

$$z_l = l(l + 2), \quad z_{l-1} = 0,$$

(97)
and the modified form of the above commutation relations reads as

\[
\left[ W^3_m, W^3_n \right] = 2(m - n)W^4_{m+n} + \frac{N_3}{12}(m - n)(2m^2 + 2n^2 - mn - 8)L_{m+n}
\]

\[+ \frac{8N_3}{(c + \frac{1}{2})} (m - n) \Lambda_{m+n}^{(4)} + \frac{N_3c}{144}m(m^2 - 1)(m^2 - 4)\delta_{m,-n}
\]  

(98)

\[
\left[ W^3_m, W^4_n \right] = (3m - 2n)W^5_{m+n} + \frac{208N_4}{25N_3(c + \frac{1}{2})} (3m - 2n) \Lambda_{m+n}^{(5)} + \frac{84N_4}{25N_3(c + 2)} \Theta_{m+n}^{(6)}
\]

\[- \frac{N_4}{15N_3}(n^3 - 5m^3 - 3mn^2 + 5m^2n - 9n + 17m)W^3_{m+n},
\]

(99)

where the \(1/c\) corrections have been indicated in red. Similar corrections appear at higher order; see [32].

For the low-lying commutation relations given above, this is sufficient to solve the constraints coming from the Jacobi identities. However, for the higher commutators, we also obtain conditions on the structure constants, i.e. on the analogues of \(N_3, N_4\). In order to describe this succinctly, it is convenient to rescale \(W^3\) such that \(N_3 = \frac{2}{5}\), i.e. to choose \(q^2 = \frac{1}{8(c^2-1)}\), and to redefine \(W^4\) by

\[ W^4 = \beta^{-1} W^4 \quad \text{with} \quad \beta^2 = \frac{56 N_4}{75 N_3^2} = \frac{4 \mu^2 - 9}{5 \mu^2 - 4}.
\]  

(100)

(This redefinition has been chosen for convenience and the apparent singularities thus induced at \(\mu^2 = 4\) in the expressions below are spurious and have no significance.)

As a result, the OPEs are of the form

\[ W^3 \cdot W^3 \sim \frac{c}{3} \cdot \mathbf{1} + 2 \cdot L + 8 \sqrt{\frac{1}{5} \frac{\mu^2 - 9}{\mu^2 - 4}} \cdot W^3 + \ldots
\]  

(101)

\[ W^3 \cdot W^4 \sim + 6 \sqrt{\frac{1}{5} \frac{\mu^2 - 9}{\mu^2 - 4}} \cdot W^3 + \ldots,
\]

and thus in the conventions of [79], the structure constant \(C^4_{33}\) satisfies

\[
\left( C^4_{33} \right)^2 = \frac{64 \mu^2 - 9}{5 \mu^2 - 4} + O \left( \frac{1}{c} \right).
\]  

(103)

Note that we have included the possibility of an \(1/c\) correction, given that we now know that the algebra has to be corrected at that order.

The Jacobi identities now imply that at least some of the higher structure constants are uniquely determined in terms of \(C^4_{33}\) and \(c\). For example, for the structure constants that were calculated explicitly in [78, 80, 22, 81], one finds [58]

\[
C^4_{44} = \frac{9(c + 3)}{4(c + 2)} \gamma - \frac{96(c + 10)}{(5c + 22)} \gamma^{-1}
\]  

(104)

\[
\left( C^5_{34} \right)^2 = \frac{75(c + 7)(5c + 22)}{16(c + 2)(7c + 114)} \gamma^2 = 25
\]

(105)

\[
C^5_{45} = \frac{15(17c + 126)(c + 7)}{8(7c + 114)(c + 2)} \gamma - \frac{240(c + 10)}{(5c + 22)} \gamma^{-1},
\]

(106)

where

\[
\gamma^2 \equiv \left( C^4_{33} \right)^2.
\]  

(107)
This suggests that at least these structure constants are fixed by the Jacobi identities, and this was subsequently confirmed by an explicit analysis [32] where, in addition, the next 40 or so structure constants were found to be determined uniquely in this manner. Note that there is a sign ambiguity in the definition of \( C_{33}^i \), \( C_{34}^i \), etc; this is a consequence of the normalization convention of [79] which is defined by fixing the OPE of the spin field \( W^s \) with itself

\[
W^s \cdot W^t \sim \frac{c}{s} \cdot I + \ldots,
\]

and hence only determines the normalization of each field up to a sign. We should also stress that these relations modify the value of the structure constants in \( \mathcal{W}_\infty[\mu] \) relative to those in \( \mathcal{W}_\infty[\mu] \) by \( 1/c \) corrections; this justifies a posteriori why we also included a \( 1/c \) correction in (103).

Assuming that the Jacobi identities continue to determine all of these higher structure constants, it then follows that the quantum \( \mathcal{W}_\infty[\mu] \)-algebra is completely characterized by the two parameters

\[
\gamma^2 \equiv (C_{33}^i)^2 \quad \text{and} \quad c.
\]

Furthermore, we know that to leading order in \( 1/c \), the parameter \( \gamma^2 \) is determined by the classical Poisson algebra \( \mathcal{W}_\infty[\mu] \) equal to (103), i.e. \( \gamma^2 \) captures essentially the \( \mu \)-dependence of \( \mathcal{W}_\infty[\mu] \). The fact that we find a consistent two-parameter family of \( \mathcal{W}_\infty[\mu] \)-algebras characterized by (109) is therefore what one should have expected: it simply means that every classical \( \mathcal{W}_\infty[\mu] \)-Poisson algebra can be quantized in a unique manner.

The final step of the argument is to determine the exact \( \mu \)-dependence of \( \gamma \); this can be done by employing the following trick. We know that, for \( \mu = N \), \( \text{hs}[N] \) can be truncated to \( \mathfrak{s}(N) \), and we similarly expect that \( \mathcal{W}_\infty[N] \) can be truncated to \( \mathcal{W}_N \). Thus, the representation theory of \( \mathcal{W}_\infty[N] \) must be compatible with the known representation theory of \( \mathcal{W}_N \). Using this constraint, the exact \( (c, \mu) \)-dependence of \( \gamma^2 \) can be determined [58] to be (see also [78, 80, 22] for earlier work using essentially the same idea)

\[
(C_{33}^i)^2 \equiv \gamma^2 = \frac{64(c + 2)((\mu - 3)(c(\mu + 3) + 2(4\mu + 3)(\mu - 1)))}{(5c + 22)(\mu - 2)(c(\mu + 2) + (3\mu + 2)(\mu - 1))}.
\]

Note that (110) is indeed of the form (103). The resulting algebra \( \mathcal{W}_\infty[\mu] \) is now a well-defined \( \mathcal{W} \)-algebra for all values of \( c \) and \( \mu \).

4.2. The triality relation

The fact that \( \mathcal{W}_\infty[\mu] \) actually only depends on \( \gamma^2 \) (rather than directly on \( \mu \)) has a very important consequence. It means that the algebras \( \mathcal{W}_\infty[\mu] \) are equivalent for generically three different values of \( \mu \). Indeed, for given \( c \) and \( \gamma \), it follows directly from (110) that the three values are the roots of the cubic equation

\[
(3\tilde{\gamma}^2 - 8)\mu^3 + (\tilde{\gamma}^2(c - 7) + (26 - c))\mu^2 - (4\tilde{\gamma}^2(c - 1) - 9(c - 2)) = 0,
\]

where we have defined \( \tilde{\gamma}^2 = \gamma^2 \). Thus, we have shown that

\[
\mathcal{W}_\infty[\mu_1] \cong \mathcal{W}_\infty[\mu_2] \cong \mathcal{W}_\infty[\mu_3] \quad \text{at fixed } c,
\]

where \( \mu_{1,2,3} \) are the roots of the cubic equation (111), evaluated for a given \( \gamma \). Note that the cubic equation does not have a linear term in \( \mu \); thus, the three solutions satisfy

\[
\mu_1 \mu_2 + \mu_2 \mu_3 + \mu_3 \mu_1 = 0,
\]

which is equivalent to \( \sum_{i=1}^{3} \frac{1}{\mu_i} = 0 \) provided that all \( \mu_j \neq 0 \).

These algebras look very different from the point of view of \( \text{hs}[\mu] \) or even at the classical level. In fact, at very large \( c \), equation (111) reduces to a linear equation in \( \mu^2 \), and hence reduces
to the familiar equivalence between the classical $\mathcal{W}_\infty[\mu]$-algebras for $\pm \mu$—this property is directly inherited from $\text{hs}[\mu]$. The statement in (112) is a very non-trivial generalization to the quantum level (finite $c$), where the equivalence is a triality between the three values $\mu_{1,2,3}$. There are three special cases where the cubic equation (111) degenerates: for $\mu = 0$, we have $\tilde{\gamma} = \frac{\sqrt{4(c-2)}}{16\sqrt{7}}$, and the constant term in (111) vanishes. Then, $\mu = 0$ is a double zero, and another solution simply becomes

$$
\mathcal{W}_\infty[\mu = 0] \cong \mathcal{W}_\infty[\mu = c + 1].
$$

(114)

For $\mu = 1$, on the other hand, we have $\tilde{\gamma}^2 = \frac{8}{3}$, and the cubic power vanishes; then, we have the equivalences

$$
\mathcal{W}_\infty[\mu = 1] \cong \mathcal{W}_\infty[\mu = -1] \cong \mathcal{W}_\infty[\mu = \infty].
$$

(115)

The fact that for $\mu = 1$ the symmetry $\mu \mapsto -\mu$ survives at the quantum level is a direct consequence of the fact that for this value of $\mu$, $\mathcal{W}_\infty[\mu]$ is a linear $\mathcal{W}$-algebra whose structure constants are simply the (analytic continuation of the) $\text{hs}[\mu]$ structure constants.

Finally, the coefficient in front of the $\mu^2$ term in (111) vanishes for $\tilde{\gamma}^2 = \frac{16 - 26k}{16\sqrt{7}}$, when the equation becomes $\mu^3 = (c + 1)$. Thus, the three cubic roots of $(c + 1)$ define equivalent $\mathcal{W}_\infty[\mu]$-algebras.

### 4.3. Triality in minimal model holography

The above triality relation now allows us to prove that the asymptotic quantum symmetry of the higher spin gauge theory on AdS agrees exactly with the $\mathcal{W}_{N,k}$-symmetry in the ’t Hooft limit. In order to see this, we take $\mu = N$, and hence determine $\gamma = \gamma(\mu = N, c)$. Then, it follows from (111) that the other two roots $\mu_{2,3}$ satisfy the quadratic equation

$$
\mu^2(N^2 - 1) - \mu(N - 1 - c) - N(N - 1 - c) = 0,
$$

(116)

whose solutions are

$$
\mu_{2,3}(N, c) = \frac{1}{2(N^2 - 1)} \left[(N - 1 - c) \pm \sqrt{(N - 1 - c)(4N^3 - 3N - c - 1)}\right].
$$

(117)

For the particular value $c = c_{N,k}$ defined in (30), we then find

$$
\mu_2(N, c_{N,k}) = \frac{N}{N + k} \quad \text{and} \quad \mu_3(N, c_{N,k}) = -\frac{N}{N + k + 1}.
$$

(118)

Thus, we conclude, in particular, that the minimal model algebra $\mathcal{W}_{N,k}$ is isomorphic to

$$
\mathcal{W}_{N,k} \cong \mathcal{W}_\infty[\lambda] \quad \text{for} \quad \lambda = \frac{N}{N + k} \quad \text{and} \quad c = c_{N,k}.
$$

(119)

This therefore proves that the $\mathcal{W}$-algebra of the dual 2D CFT agrees indeed with the quantization of the classical symmetry algebra of higher spin gravity based on $\text{hs}[\lambda]$. This correspondence is not at all obvious at the classical level, and is a very non-trivial confirmation of the minimal model holography conjecture. We should also stress that (119) actually holds for finite $N, k$, not just in the ’t Hooft limit. This implies that the finite $N, k$ version of the duality should be constrained by this exact quantum symmetry.

We should also mention in passing that another value of $\mu$, namely $\mu_3 = -\frac{N}{N + k + 1}$, becomes in the large-$N$ ’t Hooft limit $\mu_3 = -\mu_2$. This just recovers the by-now familiar statement about the classical equivalence of the $\text{hs}[\pm \mu]$ theories. The relation between the
different algebras can thus be summarized as
\[ \mathcal{W}_{\infty}[N] \xrightarrow{\text{Hoch} N} \mathcal{W}_{\infty}[\lambda] \]

where
\[ \lambda = \frac{N}{N+k} \]

4.4. Relation to coset level-rank duality

The above triality relation is in some sense an analytic continuation of the conjectured level-rank duality of coset models \([93, 6]\)

\[ \mathcal{W}_{N,k} \equiv \mathcal{W}_{\infty}[N] \]

where the relation between the parameters is
\[ k = \frac{N}{M} - N, \quad l = \frac{M}{N} - M. \]  \(121\)

Here, \(M\) and \(N\) are taken to be positive integers, whereas \(k\) and \(l\) are fractional (real) numbers, and the central charges of both sides are equal to

\[ c_{N,k} = (N-1) \left( 1 - \frac{N(N+1)}{(N+k)(N+k+1)} \right) \]

\[ = (M-1) \left( 1 - \frac{M(M+1)}{(M+l)(M+l+1)} \right) \equiv c_{M,l}. \]  \(122\)

If we assume that this level-rank duality will also hold if instead of integer \(N, M\), we consider the situation where \(N\) and \(k\) are integers, then we can solve (121) for \(M\) to obtain

\[ M \equiv \lambda = \frac{N}{N+k}. \]  \(123\)

while \(l\) is determined by the condition that both sides have the same central charge. Next, we observe that we have also quite generically that

\[ \frac{su(M)_1 \oplus su(M)_1}{su(M)_{1+1}} \cong \text{DS reduction of } su(M) \text{ at level } \hat{l}, \]  \(124\)

where again \(\hat{l}\) is determined so as to have the same central charge as the left-hand side. For non-integer \(M\), we can think of

\[ su(\lambda) \cong hs[\lambda]. \]  \(125\)

and the DS reduction of \(hs[\lambda]\) equals \(W_\infty[\lambda]\). Combining these statements then leads to the claim that we have an isomorphism of algebras

\[ \mathcal{W}_{N,k} \equiv \frac{su(N)_1 \oplus su(N)_1}{su(N)_{k+1}} \cong \mathcal{W}_{\infty}[\lambda] \quad \text{with } \lambda = \frac{N}{N+k}. \]  \(126\)

Here, the central charge of \(\mathcal{W}_{\infty}[\lambda]\) is taken to agree with that of \(\mathcal{W}_{N,k}\), i.e. with \(c_{N,k}\) defined in (122). This then reproduces (119).

Actually, there is a second variant of this relation. The \(\mathcal{W}_N\)-algebra at level \(k\) is identical to the \(\mathcal{W}_N\)-algebra at level

\[ k' = -2N - k - 1 \]  \(127\)
since the central charges of the two algebras agree, i.e. \( c_{N,k} = c_{N,k'} \). Incidentally, this identification has a natural interpretation from the DS point of view. Recall that the cosets \( \mathcal{W}_{N,k} \) in (120) are equivalent to the DS reduction of \( su(N) \) at level \( \hat{k} \equiv \delta_{DS} \), where the two levels are related as (see, e.g., [24] for a review of these matters)

\[
\frac{1}{\hat{k} + N} = \frac{1}{k + N} - 1.
\]

(128)

From the DS point of view, replacing \( k \mapsto k' \) as in (127) is equivalent to replacing \( \hat{k} \) by \( \hat{k}' \) with

\[
\hat{k}' + N = \frac{1}{k + N}.
\]

(129)

In terms of the underlying free-field description, this corresponds to exchanging (see, e.g., [24] or [59, section 6.2.2]) the roles of \( \alpha_{\pm} \), i.e. to define \((\hat{\alpha}_+, \hat{\alpha}_-) = (-\alpha_-, -\alpha_+)\). This is an obvious symmetry of the DS reduction under which the representations are related as \( \Lambda_+ \leftrightarrow \Lambda^* \). Thus, we can repeat the above analysis with \( k' \) in place of \( k \) to conclude that \( \mathcal{W}_{N,k'} \) is also equivalent to \( \mathcal{W}_\infty[\mu] \) with \( \mu = \frac{-N}{N+k+1} \); this then reproduces also the third root \( \mu_3 \) in (118).

5. Matching the spectrum

In the previous section, we have shown that the symmetries of the higher spin theory on AdS\(_3\) and the proposed dual 2D CFT match in a rather intriguing manner. Now we want to check that the full spectrum of the two theories also agrees. We only know how to calculate the spectrum of the higher spin theory in the semi-classical regime, i.e. for \( c \to \infty \); thus, we can only compare it to the CFT prediction in the ’t Hooft limit.

We begin by studying the spectrum of the higher spin fields which, given the results of the previous section, must agree with the vacuum representation of the CFT in the ’t Hooft limit. From the 2D CFT point of view, modular invariance requires that the CFT also has other representations in its spectrum. By studying the finite-\(N, c \to \infty \) behaviour of these representations, we argue that some of them correspond to non-perturbative and some to perturbative states. We then explain that the contribution of the perturbative states is precisely reproduced by adding to the higher spin theory a complex massive scalar field. Finally, we review a proposal for the interpretation of the remaining non-perturbative states as analytic continuations in \( c \) of classical conical defect solutions.

5.1. Higher spin fields

The contribution of the massless higher spin fields to the one-loop partition function on thermal AdS\(_3\) only requires knowledge of their kinetic term. This can be most easily calculated using the Fronsdal description of higher spin fields [56]. Taking carefully the various gauge transformations into account, it was shown in [60] that the contribution of a massless spin \( s \) field to the one-loop partition function equals

\[
Z_{(s)}^{1-\text{loop}} = \left[ \det \left( -\Delta + \frac{s(s-3)}{\ell^2} \right)_{(s)}^{\text{TT}} \right]^{-\frac{1}{2}} \left[ \det \left( -\Delta + \frac{s(s-1)}{\ell^2} \right)_{(s-1)}^{\text{TT}} \right]^{\frac{1}{2}},
\]

(130)

where ‘TT’ means that only the transverse traceless part of the determinant is considered, and the index \((s)\) refers to the spin. (As before, \( \ell \) is the AdS radius.) Determinants of this form were explicitly evaluated in [44] using group-theoretic techniques; applying these results to the
present context, one finds that the one-loop answer factorizes nicely into left- and right-moving pieces

\[
Z_{\text{1-loop}}^{(s)} = \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2},
\]

where \( q = e^{i\tau} \) is the modular parameter of the boundary \( T^2 \) of the thermal background. This generalizes the expression for the case of pure gravity \((s = 2) [96]\), as explicitly checked in [66]. Putting together the contributions of the fields of arbitrary spin \( s = 2, 3, \ldots \), the total one-loop contribution of the massless higher spin fields equals

\[
Z_{\text{1-loop}}^{\text{hs}} = \prod_{s=2}^{\infty} \prod_{n=1}^{\infty} \frac{1}{|1-q^n|^2} = \prod_{n=1}^{\infty} |1-q^n|^2 \equiv |M(q)|^2,
\]

where \( M(q) \) is the MacMahon function, and \( \tilde{M}(q) \) is defined by

\[
M(q) = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^{s-1}},
\]

\[
\tilde{M}(q) = \prod_{n=2}^{\infty} \frac{1}{(1-q^n)^{s-1}}.
\]

The partition function \( Z_{\text{hs}}^{\text{1-loop}} \) in (132) now matches exactly the one-loop contribution of the vacuum representation \( |\chi(0,0); 0\rangle(q)\) of the \( \mathcal{W}_N \), \( k \) CFTs in the 't Hooft limit. Indeed, by the usual Poincaré–Birkhoff–Witt theorem (see, for example, [116]), a basis for the vacuum representation of \( \mathcal{W}_\infty[\lambda] \) is given by

\[
W_{s_1}^{n_1} \cdots W_{s_j}^{n_j} \Omega,
\]

where \( s_1 > s_2 > \cdots > s_j \geq 2 \) and

\[
n_1 \geq n_2 \geq \cdots \geq n_j \geq 0.
\]

Here, we have used that \( W_n^s \Omega = 0 \) for \( n \geq -s + 1 \)—this is the reason for the lower bound in (135)—but we have assumed that there are no other null vectors in the vacuum representation, which is true in the 't Hooft limit. (Note that we have denoted the Virasoro modes by \( W_n^s \equiv L_n \).) Thus, the character of the vacuum representation equals

\[
\chi(0,0) = q^{-\frac{c}{12}} \prod_{n=2}^{\infty} \prod_{s=1}^{\infty} \frac{1}{(1-q^n)}.
\]

The contribution of \( |q|^{-c/12} \) in \( |\chi(0,0); 0\rangle(q)\) corresponds to the tree-level part of the higher spin gravity calculation, and the remaining terms in (136) then reproduce precisely (132).

5.2. Other states in the conformal field theory

As we have reviewed in section 2.2, the minimal model CFTs also have other representations (apart from the vacuum representation). As familiar from rational CFTs, these representations have to be present in the spectrum for a consistent (modular-invariant) CFT\[11\]. Note that modular invariance is really a crucial ingredient in our analysis since the boundary of thermal AdS3 is in fact a torus, and hence the possibility to go to finite temperature in AdS requires that the dual 2D CFT must be modular-invariant (i.e. consistent on a torus).

\[10\] A similar one-loop calculation in the parity violating topologically massive higher spin theory is suggestive of the vacuum character of a logarithmic \( \mathcal{W}_\infty \) CFT [12].

\[11\] Typically, there will be more than one modular-invariant combination of characters, and therefore more than one consistent CFT. In the following, we shall concentrate on the simplest modular invariant, the ‘charge conjugation’ theory, which exists for every rational CFT.
Recall that the most general representation of the $W_{N,k}$ minimal model is described by $(\Lambda_+; \Lambda_-)$, where $\Lambda_\pm$ are integrable hwrs of the affine algebra $\mathfrak{su}(N)$ at level $k$ and level $k+1$, respectively. (Thus, $\Lambda_\pm$ are Young diagrams of at most $N$ rows, and at most $k$ and $k+1$ columns, respectively.) The simplest representations (that generate all representations upon taking successive fusions) are $(f; 0)$ and $(0; f)$, as well as their conjugates, where $f$ denotes the fundamental representation of $\mathfrak{su}(N)$. Their conformal dimension equals (see equations (42) and (43))

$$h(f; 0) = \frac{N-1}{2N} \left(1 + \frac{N+1}{N+k}\right), \quad h(0; f) = \frac{N-1}{2N} \left(1 - \frac{N+1}{N+k+1}\right).$$

(137)

In the 't Hooft limit, they therefore become

$$'t \text{ Hooft limit: } h(f; 0) = \frac{1}{2}(1 + \lambda), \quad h(0; f) = \frac{1}{2}(1 - \lambda).$$

(138)

However, in order to understand the nature of their duals in the higher spin theory, one should instead consider the limit where $N$ is being kept fixed, while $c \to \infty$ (the semi-classical limit) [58]. In that limit, the two states behave rather differently, as one finds

$$\text{semi-classical: } h(f; 0) \sim -\frac{(N-1)}{2}, \quad h(0; f) \sim -\frac{c}{2N^2}.$$ 

(139)

In particular, the conformal dimension of $(0; f)$ is proportional to $c$, thus suggesting that this state should correspond to a non-perturbative (classical solution), rather than to a perturbative excitation of the higher spin theory. Actually, a similar consideration applies to any state for which $\Lambda_-$ is non-trivial. Thus, one is led to propose that only the states of the form $(\Lambda_+; 0)$ should have a perturbative origin in the higher spin theory [58]. We shall come back to the description of the remaining states (i.e. those with $\Lambda_- \neq 0$) in section 5.4, but for the moment, we now concentrate on these perturbative states.

5.3. Perturbative states

It was proposed in [57, 58] that all CFT representations of the form $(\Lambda_+; 0)$ are accounted for by adding to the higher spin theory a complex massive scalar of mass

$$M^2 = -(1 - \lambda^2).$$

(140)

Recall from section 2.1.4 that in the 3D higher spin theory of [103, 104] (see also [114]), it is consistent to add a scalar multiplet to the higher spin theory, but the mass of the scalar is then determined by the $\lambda$-parameter of the underlying hs$[\lambda]$ algebra as in (140). By the usual AdS/CFT dictionary, the mass of the scalar field is related to the conformal dimension $\Delta$ of the corresponding conformal field; in 3D, the relation takes the form

$$M^2 = \Delta(\Delta - 2).$$

(141)

Since $0 \leq \lambda \leq 1$, $M^2$ in (140) lies in the window $-1 \leq M^2 \leq 0$; there are two real solutions for $\Delta$, namely

$$\Delta = (1 \pm \lambda).$$

(142)

They correspond to the two different quantizations of the scalar field (since they characterize two different types of asymptotic behaviour of the scalar field) [91]. In the following, we shall concentrate on the ‘usual’ quantization with $\Delta = 1 + \lambda$, for which $h = \bar{h} = \frac{1}{2}(1 + \lambda)$. Note that this agrees precisely with the conformal dimension of the ‘fundamental’ field $(f; 0)$ or its conjugate; see equation (138).
The main evidence in favour of the above proposal comes from the comparison of partition functions \([57, 59]\). A real scalar field with a boundary conformal dimension \(h = \bar{h} = \frac{1}{2} \Delta\) contributes to the one-loop partition function on thermal AdS as \([66]\)

\[
Z_{\text{scal}}^{1-\text{loop}}(h) = \prod_{j,j'=0}^{\infty} \frac{1}{1 - q^{h+j} \bar{q}^{j'}}
\]

and hence the contribution of a complex scalar is the square of (143). Note that the form of (143) can be understood intuitively: a local operator of dimension \(h\) has descendants which are obtained by acting on it with derivatives. Thus, the ‘single particle’ contribution to the partition function is given by

\[
Z_{\text{sing par}}(h, q, \bar{q}) = q^h \bar{q}^\bar{h} (1 - q)(1 - \bar{q}).
\]

In the non-interacting limit, where we can neglect the anomalous dimensions of composite operators, we can obtain the ‘multi-particle’ partition function by using the standard formula for Bose statistics, leading to

\[
Z_{\text{scal}}^{1-\text{loop}}(h) = \exp \left[ \sum_{n=1}^{\infty} \frac{Z_{\text{sing par}}(h, q^n, \bar{q}^n)}{n} \right] = \prod_{j,j'=0}^{\infty} \frac{1}{1 - q^{h+j} \bar{q}^{j'}}
\]

thus reproducing (143). For the comparison with the CFT calculation, it is useful to rewrite \(Z_{\text{scal}}^{1-\text{loop}}(h)\) in terms of \(U(\infty)\) characters following \([59]\). Recall that characters of \(u(N)\) in a representation \(R\) are given by Schur polynomials in \(N\) variables,

\[
\chi_{u(N)}^R(z_i) = \prod_{i=1}^{N} \frac{1}{1 - q^{z_i}}
\]

Taking the large-\(N\) limit and evaluating on the Weyl vector, we can define the specialized Schur functions

\[
P_R(q) \equiv \chi_{u(N)}^R(z_i) (z_i = q^{-\frac{1}{2}}),
\]

where \(B(R)\) is the number of boxes in the Young diagram \(R\); explicit formulae for the Schur functions can be found in the appendix of \([59]\). In terms of \(U(\infty)\) characters, the scalar determinant (143) equals then

\[
Z_{\text{scal}}^{1-\text{loop}}(h) = \sum_R |P_R^+(q)|^2
\]

Here, the sum is over all Young diagrams of \(U(\infty)\), i.e. without any restrictions on the lengths of rows or columns. Combining the contribution of two real (i.e. one complex) scalars then leads to

\[
Z_{\text{pert}}^{\text{bulk}} = (q \bar{q})^{-c/24} \cdot |\tilde{M}(q)|^2 \cdot \sum_{R,S} |P_R^+(q) P_S^+(q)|^2,
\]

where the sum runs over two sets of Young diagrams.

### 5.3.1. Comparison to CFT

This partition function should now be compared to the ‘perturbative part’ of the CFT partition function, i.e. to

\[
Z_{\text{CFT}}^{\text{pert}}(N, k) = \sum_{A} |b_{A,0}(q)|^2
\]

where \(A\) runs over all allowed representations of \(su(N)_k\), and \(b_{A,0}(q)\) is the branching function (i.e. the character) of the corresponding \(W_{N,k}\) representation; see equation (46).
Since we can only calculate the gravity answer in the semi-classical limit, we need to take the $N \to \infty$ 't Hooft limit, and hence have to be careful about which representations $\Lambda$ we should include. As familiar from similar situations, see, e.g., [72], a natural prescription is to consider those representations $\Lambda$ that are contained in finite tensor powers of the fundamental and anti-fundamental (where the number of tensor powers does not scale with $N$); note that the conformal dimension of $(\Lambda; 0)$ is essentially proportional to the number of tensor powers in $\Lambda$, and hence this prescription takes account of all the low-lying representations of this type. As in [72], the corresponding Young diagrams can then be viewed as two Young diagrams placed side by side:

$$\Lambda = (\bar{R}, S),$$

where $\bar{R}$ is a tensor power of anti-fundamentals (anti-boxes) and $S$ is a tensor power of fundamentals (boxes) as in figure 1.

We should also mention that the field identification (38) becomes trivial in this limit since it does not lead to identifications among representations for which $R$ and $S$ are finite Young diagrams.

In order to calculate (150), we next observe that the branching functions $b_{(\Lambda; 0)}$ from section 2.2.4, see equation (46), simplify considerably in the 't Hooft limit [59]. In particular, we can restrict the sum over the affine Weyl group to the finite Weyl group $W$, and we can simplify the exponent to arrive at

$$b_{(\Lambda; 0)}(q) \cong q^{\frac{N-1}{N} \frac{2}{N-1}} \frac{q^{\frac{2}{N-1}} B}{q(q)^{N-1}} q^{-\frac{2}{N-1}} \sum_{w \in W} \epsilon(w) q^{-\langle w(\hat{\rho}) + \hat{\rho}, \hat{\rho} \rangle},$$

where $\cong$ denotes identities that are true up to terms that tend to zero as $N \to \infty$, and we have specialized to the case $\Lambda_- = 0$ and written $\Lambda_+ \equiv \Lambda = (\bar{R}, S)$. Furthermore, $B = B(R) + B(S)$ is the total number of boxes in the Young diagrams corresponding to $R$ and $S$. Following again [59], we can use the Weyl denominator formula for $\mathfrak{su}(N)$

$$\sum_{w \in W} \epsilon(w) q^{-\langle w(\hat{\rho}), \hat{\rho} \rangle} = q^{-\hat{\rho}^2} \prod_{n=1}^{N-1} (1 - q^n)^{N-n},$$

which is equivalent to the identity (153) for $N 

Figure 1. A Young diagram of $SU(N)$ in the large-$N$ limit. The full representation $\Lambda = (\bar{R}, S)$ has a finite number of “boxes” $S$ and “anti-boxes” $\bar{R}$. 

28
which we solve for $q^{\frac{c}{\beta}}$, to obtain

$$b_{(\Lambda; 0)}(q) \equiv q^{-\frac{c}{\beta}} q^{C_{3}(\Lambda)} \frac{\sum_{w \in W} \epsilon(w) q^{-\langle u(\Lambda + \hat{\rho}), \hat{\rho} \rangle}}{\sum_{w \in W} \epsilon(w) q^{-\langle u(\hat{\rho}), \hat{\rho} \rangle}} \hat{M}(q), \quad (154)$$

where we have used that $c = (N - 1)(1 - \lambda^2)$ and $\hat{M}(q)$ is as defined in (133). The ratio of sums in (154) actually equals the so-called quantum dimension of $\Lambda$:

$$\frac{S_{\Lambda 0}}{S_{00}} = \text{dim}_q(\Lambda) = \frac{\sum_{w \in W} \epsilon(w) q^{-\langle u(\Lambda + \hat{\rho}), \hat{\rho} \rangle}}{\sum_{w \in W} \epsilon(w) q^{-\langle u(\hat{\rho}), \hat{\rho} \rangle}}. \quad (155)$$

(Here, $S_{ab}$ are the matrix elements of the $S$ modular transformation matrix of the affine algebra.) Using results from [2] and [72], one can show that the quantum dimension of $\Lambda$ factorizes as

$$q^{C_{3}(\Lambda)} \text{dim}_q(\Lambda) \cong q^{C_{3}(R)} \text{dim}_q(R) \cdot q^{C_{3}(S)} \text{dim}_q(S), \quad (156)$$

and for each finite Young diagram $L = R, S$, we have

$$\text{dim}_q(L) = x^u_{\Lambda}(\tilde{z}_i) = x^u_{\Lambda}(\tilde{z}_i) = q^{-\frac{c}{\beta} A(L)} x^u_{\Lambda}(\tilde{z}_i), \quad (157)$$

where $B(L)$ is the number of boxes of $L$, and

$$\tilde{z}_i = q^{-\frac{2}{\beta}} z_i, \quad z_i = q^{\frac{1}{2}}. \quad (158)$$

Finally, using the large-$N$ expansion of the quadratic Casimir (see [59] for details), it follows that

$$q^{C_{3}(L)} \text{dim}_q(L) \cong x^u_{\Lambda}(\tilde{z}_i) = P_{L^T}(q), \quad (159)$$

where $L^T$ is the representation whose Young diagram has been flipped relative to $L$, and we have used the notation introduced in equation (147). Inserting this relation into (154), we finally obtain

$$b_{(\Lambda; 0)}(q) \cong q^{-\frac{c}{\beta}} P_{R^T}(q) P_{S^T}(q) \hat{M}(q). \quad (160)$$

Summing over all $R, S$ independently, it is then obvious that $Z_{\text{pert}}^{\text{CFT}}$ in (150) reproduces exactly $Z_{\text{pert}}^{\text{bulk}}$; see equation (149). This is a highly non-trivial check on the duality conjecture.

As an aside, we should mention that in the original analysis of [59], the above calculation was performed both for the representations of the form $(\Lambda; 0)$ and for those of the form $(0; \Lambda)$. Furthermore, it was shown that the ‘light states’, see section 5.4, decouple in the ’t Hooft limit, and therefore that the full CFT partition function (after removing the null states that appear in the ’t Hooft limit) is exactly reproduced by adding to the higher spin theory two complex scalar fields. However, this agreement only works in the strict $N \to \infty$ limit; if we are interested in understanding the theory at finite $N$, we need to treat the states of the form $(0; \Lambda)$ differently.

5.4. Non-perturbative states

As described in section 5.2, only states in the CFT of the form $(\Lambda; 0)$ have dimensions of order 1 in the semi-classical ($c \to \infty$, $N$ fixed) limit. Therefore, we would like to interpret all states $(\Lambda_+; \Lambda_-)$ with $\Lambda_- \neq 0$ as non-perturbative states in the bulk theory.

To understand what these excitations might be, first focus on a class of states in the CFT of the form $(\Lambda_-; \Lambda_-)$, the so-called light states. The reason for this terminology is that in the ’t Hooft limit (as opposed to the semi-classical limit), these states are very light. Indeed, the dimension formula (40) gives

$$h(\Lambda_+; \Lambda_-) = \frac{1}{2p(p + 1)} (\Lambda_- + 2\hat{\rho}, \Lambda_-) = \frac{C_{2}(\Lambda_-)}{(N + k)(N + k + 1)}, \quad (161)$$

29
which reduces in the \('t\) Hooft limit to (for \(\Lambda\) having a finite number of boxes \((S)\) or anti-boxes \((R)\), in the notation explained below figure 1)

\[
\text{'t Hooft limit: } h(\Lambda_+; \Lambda_-) = \frac{\lambda^2}{N^2} C_2(\Lambda_-) = \frac{\lambda^2}{2N} (B(R) + B(S)).
\]

(162)

Thus for finite \(B(R), B(S)\), these dimensions tend to zero and form a continuum of light states near the vacuum. However, in the semi-classical limit, they behave as

\[
\text{semi-classical: } h(\Lambda_+; \Lambda_-) \sim -\frac{c}{N(N^2 - 1)} C_2(\Lambda_-) + O(1),
\]

(163)

and thus are candidates for non-perturbative states. Here, we have used the fact that

\[
a_0^2 \equiv \frac{1}{(N + k)(N + k + 1)} = \frac{(N - 1 - c)}{N(N^2 - 1)} \to -\frac{c}{N(N^2 - 1)}.
\]

(164)

In fact, in the semi-classical limit, it turns out that all states of the form \((\Lambda_+; \Lambda_-)\) have the same dimension, i.e.

\[
\text{semi-classical: } h(\Lambda_+; \Lambda_-) \sim -\frac{c}{N(N^2 - 1)} C_2(\Lambda_-) + O(1),
\]

(165)

with only the \(O(1)\) terms depending on the representation \(\Lambda_+\).

5.4.1. Conical defects. We will now outline how all these states \((\Lambda_+; \Lambda_-)\) (with \(\Lambda_- \neq 0\)) can be exactly accounted for, in the semi-classical limit (with \(N\) fixed), by a class of solutions to the bulk equations of motion \([33, 100]\). We first describe the solutions without scalar fields turned on. We can describe this sector by the Chern–Simons theory reviewed in section 2.1. There are some important differences, which we will mention later, between the Lorentz signature theory, which has the gauge group \(\text{SL}(N, \mathbb{R}) \times \text{SL}(N, \mathbb{R})\), and the Euclidean theory with the gauge group \(\text{SL}(N, \mathbb{C})\). For the moment, we will consider the Lorentzian case and then mention the extension to the Euclidean setting later.

The equations of motion of the Chern–Simons theory are simply those for flat connections, \(F(A) = F(\bar{A}) = 0\). Therefore, the only gauge-invariant observables to characterize solutions are the holonomies of the gauge field. We will consider geometries which have the boundary topology of a torus. We will further look for solutions in which the topology is such that the spatial circle of the torus is contractible in the bulk while the time circle is not. This is therefore the same topology as global \(\text{AdS}_3\). Note that for black holes, the role of the two circles is interchanged; see \([8]\).

We now address the question of what the admissible (or smooth) classical solutions of the higher spin theory are. The geometric notion of smoothness is somewhat subtle in a higher spin theory since the usual curvature invariants (which one uses to characterize smoothness) are actually not invariant under higher spin gauge transformations; see also \([8]\) for a discussion of this issue. However, in the present case, the higher spin gauge fields are simply \(\text{SL}(N)\) gauge fields, and we can use our experience from gauge theory to rephrase the question. It is therefore natural to take the criterion to be that the gauge field configuration should not be singular. This is ensured if the holonomy along a contractible curve is trivial (i.e. gauge equivalent to the identity element). Otherwise, the gauge connection would be singular somewhere in the interior of the curve.

To see what this implies, let us fix a gauge and solve the equations of motion via the choice (57), (58). Then, the holonomy

\[
\text{Hol}_b(A) = \mathcal{P} \exp \left( \oint_{S^1} A \right) = b^{-1} \exp(2\pi a) b
\]

(166)
has to be trivial, i.e. an element of the centre of the gauge group since the gauge fields are in the adjoint representation of the gauge group. This can be achieved if \( \exp(2\pi a) \) is diagonalizable to an appropriate multiple of the identity matrix.

We can arrange this by choosing the \( \mathfrak{sl}(N) \) gauge field to be of the form
\[
a = \sum_{j=1}^{[N/2]} \mathcal{B}_{2j-1}^{(1)}(n_j, n_j),
\]
(167)
where the band \( \mathfrak{sl}(N) \) matrices \( \mathcal{B}_k^{(1)}(a, b) \) are defined via
\[
[\mathcal{B}_k^{(1)}(a, b)]_{lj} = a \delta_{l,k} \delta_{j,k+1} - b \delta_{l,k+1} \delta_{j,k}.
\]
(168)
Since \( a \) in (167) has eigenvalues \( \pm in_j \) \( (j = 1 \ldots \lfloor N/2 \rfloor) \), the holonomy in (166) will be an element of the centre if we choose
\[
n_j \in \mathbb{Z} \quad \text{for } a \in \mathfrak{sl}(N, \mathbb{R}) \quad (N \text{ odd})
\]
(169)
\[
n_j \in \mathbb{Z} \quad \text{or} \quad n_j \in \mathbb{Z} + \frac{1}{2} \quad \text{for } a \in \mathfrak{sl}(N, \mathbb{R}) \quad (N \text{ even})
\]
(170)
\[
n_j \in \mathbb{Z} - \frac{m}{N} \quad \text{for } a \in \mathfrak{sl}(N, \mathbb{C}).
\]
(171)
This corresponds to the fact that the centre of \( \text{SL}(N, \mathbb{R}) \) is \( \mathbb{Z}_2 \) for \( N \) even while being trivial for \( N \) odd. On the other hand for \( \text{SL}(N, \mathbb{C}) \), the centre is \( \mathbb{Z}_N \), and thus \( m \in \{0, \ldots, N-1\} \) (independent of \( j \)).

On the other hand, not all of these solutions satisfy the Brown and Henneaux boundary conditions (63) which we needed for the asymptotic symmetry analysis. It can be shown [33] that the above solution can be brought to the highest weight gauge used in section 3.2 if and only if \( n_j \) are all distinct.

In the highest weight gauge, we can easily read off the quantum numbers of the solution (mass, higher spin charges). Indeed, in that gauge, the gauge field \( a \) takes the form, generalizing (67) and the considerations that follow,
\[
a = L_1 + \sum_{s=2}^{N} a_s w_0^{(s)} V_{s+1}^s,
\]
(172)
where \( w_0^{(s)} \) are the spin \( s \) charges and \( a_s \) is a suitable normalization constant (see [33]). One can therefore express the charges \( w_0^{(s)} \) in terms of traces of powers of \( a \). Given the form of the solution (167) with eigenvalues \( \pm in_j \), we have
\[
\frac{(-i)^s}{s} \text{tr}(a)^s = \frac{1}{s} \sum_{i=1}^{N} (n_i)^s \equiv C_s(n), \quad s = 2, \ldots, N.
\]
(173)
This then leads to [33]
\[
w_0^{(2)} = - \frac{c}{N(N^2 - 1)} C_2(n),
\]
\[
w_0^{(3)} = - \frac{c}{N(N^2 - 1)} \left( \frac{3}{2} \right)^{3/2} C_3(n),
\]
\[
w_0^{(4)} = \left( \frac{c}{N(N^2 - 1)} \right)^2 \left( C_4(n) - \frac{C_3(\hat{\rho})}{C_2(\hat{\rho})^{3/2}} C_2(n)^2 \right),
\]
(174)
where \( \hat{\rho} \) is the Weyl vector with components \( \hat{\rho}_i = \frac{N+1}{2} - i \). Note that in our conventions, the vacuum AdS has \( n_i = \hat{\rho}_i \) so that it has \( w_0^{(2)} = L_0 = -\frac{N+1}{2} \) and vanishing spin 3 and higher spin charges.
5.4.3. Interpretation.

Given the above identification of the conical surpluses with the condition (63), we can also find a discrete spectrum of conical deficit metrics as solutions. They do not, however, obey the boundary condition (64). We will loosely refer to the generic solutions as conical surpluses though not all of them can be thus viewed. One can also find a discrete spectrum of conical deficit metrics as solutions. They do not, however, obey the boundary condition (63).

5.4.2. Comparison.

We can now compare this class of solutions with the non-perturbative states of the CFT. The key fact that we need is that the conical surplus geometries have quantum numbers which are exactly those of the states of the CFT. The key fact that we need is that while the presence of a scalar field, leading to a rich spectrum of bound states of perturbative scalar quanta with the conical surpluses.

\[ C_2(\Lambda) = \frac{1}{2} \sum_i n_i^2 - \frac{N(N^2 - 1)}{24} = C_2(\tilde{n}) - \frac{N(N^2 - 1)}{24}, \]  

where \( n_i \) are distinct numbers given in terms of the row lengths \( r_i \) of the corresponding Young diagrams

\[ \tilde{n}_i = \Lambda \psi + \rho_i = r_i + \frac{N + 1}{2} - i - \frac{B(\Lambda)}{N}, \]  

and \( B(\Lambda) = \sum_i r_i \) is the total number of boxes. We have also used the definition of \( C_2(n) \) given in (173). With the identification \( n_i = \tilde{n}_i \), the first line of (174) agrees now, up to the constant shift by the vacuum energy \( \frac{c}{2\pi} \), exactly with the spectrum of the states in (165), to leading order in \( c \). Note that both \( n_i \) and \( \tilde{n}_i \) are individually required to be distinct, and that \( \tilde{n}_i \) in (176) are indeed of the form (171), which is the appropriate condition in Euclidean signature.

One can similarly work out the higher spin charges of the \( (\Lambda_+; \Lambda_-) \) states, at least in the semi-classical limit, and compare them to the other expressions in (174). As shown in [33], there is exact agreement in the semi-classical large \( c \) limit (with fixed \( N \)). In [100], the comparison was carried further to include the \( \mathcal{O}(1) \) terms and it was found that the pure conical surplus geometries have quantum numbers which are exactly those of the \( (0; \Lambda_-) \) states (rather than the \( (\Lambda_-, \Lambda_-) \) states as originally proposed in [33]).

5.4.3. Interpretation.

Given the above identification of the conical surpluses with the \( (0; \Lambda_-) \) states, we can revisit the other states discussed in section 5.2. As an illustration, consider the states \( (\Lambda; f) \). We see from (139) that \( h(0; f) \sim -\frac{c}{2\pi f} \sim h(f; f) \) in the large \( c \) limit. Indeed, we have, in this limit,

\[ h(\Lambda; f) \sim h(0; f) - (\Lambda; f + \hat{\rho}) \sim h(0; f) + h(\Lambda; 0) - (\Lambda, f), \]  

where we have kept the term of order 1 but dropped terms of order \( 1/c \). Thus, as mentioned earlier, all states \( (\Lambda; f) \) (with \( \Lambda \) having a finite number of boxes and anti-boxes) have approximately the same energy as the conical surplus \( (0, f) \) in this limit. It can be shown that the sum of the last two terms on the rhs of (177) is always negative [100]. Thus, the state \( (0; f) \) is at the top of a band of states with energy spacings of order 1. The proposal in [58] (modified suitably by [100], as we describe below) is to interpret all the non-perturbative states as bound states of the conical surplus with perturbative scalar excitations, i.e. the states \( (\Lambda; 0) \). Specifically, Perlmutter et al [100] propose to identify the general state \( (\Lambda_+; \Lambda_-) \) with \( \Lambda_+ \neq 0 \) as a bound state of perturbative scalars \( (\Lambda_+; 0) \) with a pure geometric surplus state \( (0; \Lambda_-) \). Several pieces of evidence, including a matching of the quantum numbers to order 1 as well as the structure of null states, were provided in [100].

Thus, we now have fairly persuasive evidence for a candidate bulk dual for all states of the CFT, albeit in a semi-classical regime. This regime is related by analytic continuation in
c (keeping N fixed) to the regime of the $W_N$ minimal models ($c < (N - 1)$). However, the primaries continue smoothly as we change $c$ and so we have evidence that the bulk $h_\lambda$ Vasiliev dual to the minimal models does capture all the states of the CFT.

6. Further checks

In this section, we briefly review a number of additional consistency checks that have been performed: in section 6.1, we discuss the matching of correlation functions, while in section 6.2, we explain the recent construction of black holes and the calculation of their entropy.

6.1. Correlation functions

While the spectrum is an important check of the duality, more dynamical information is encoded in correlation functions. In particular, in a 2D CFT, the three-point function on the sphere is an important independent ingredient which then determines higher point functions via factorization. We would like to match the CFT answer with the predictions from the bulk Vasiliev theory. Recall that this was the compelling piece of evidence [67, 68] for the Klebanov–Polyakov proposal for $\text{AdS}_4/\text{CFT}_3$ [90] and its generalizations [106]; see also [109, 118, 97, 105] for earlier work. Below, we will review the calculations [36, 4, 9, 37] that perform the analogous checks in the present case.

Another reason to study correlation functions has to do with the large-$N$ limit. In gauge theories (or vector models), ’t Hooft’s diagrammatic argument shows that the large-$N$ limit is well defined (when we keep the ’t Hooft coupling fixed). In particular, if we normalize the two-point functions to be of order 1, higher point functions of single trace operators are suppressed by inverse powers of $\sqrt{N}$. Furthermore, double trace operators behave like two particle states and thus their correlators can be factorized, to leading order in $N$, into those of the single particle states.

While our coset CFTs seem to behave like a vector model, we do not have any general argument that the ’t Hooft limit defined in section 1 leads to a familiar large-$N$ expansion. For instance, the presence of a large number of light states (whose energy is proportional to $1/N$, see section 5.4) could indicate that the $N \to \infty$ limit is not well behaved. In particular, even if every three-point function is suppressed by $1/N$, this may not be sufficient to deduce a similar suppression for the four-point functions since the large degeneracy of intermediate light states could potentially overcome the individual $1/N$ suppression factors. It is therefore also important to check that the four-point functions are well behaved in the ’t Hooft limit. We shall review below (see section 6.1.2) the non-trivial checks on the factorization of the three- and four-point functions that have been performed [99, 37].

6.1.1. Three-point functions. The simplest class of three-point functions involve two scalar primaries with one higher spin current, $\langle \bar{O} O J^{(s)} \rangle$. Here, $\bar{O}$ denotes the scalar primary $(0; f)$ which is dual to the perturbative scalar in the bulk (and $\bar{O}$ is its complex conjugate)\(^{13}\). This correlator was first computed for small values of the spin $s$ and compared with the bulk calculation at $\lambda = 1/2$ in [36, 4], and later generalized to arbitrary spin and $\lambda$ in [9]; the answer is

$$\langle \bar{O}(z_1) O(z_2) J^{(s)}(z_3) \rangle = \frac{(-1)^{s-1}}{2\pi} \frac{\Gamma(s)^2}{\Gamma(2s - 1)} \frac{\Gamma(s + \lambda)}{\Gamma(1 + \lambda)} \times \left( \frac{z_{12} z_{23} z_{13}}{z_{23} z_{13}} \right)^s \langle \bar{O}(z_1) O(z_2) \rangle. \quad (178)$$

\(^{13}\)The calculation can also be carried out analogously for the scalar primary $(0; 0)$ which was later identified with a non-perturbative scalar [58].
The CFT calculation in [9] assumes that the theory has $\mathcal{W}_\infty[\lambda]$-symmetry, and it follows from the triality described in section 4 that this is indeed the case for the 't Hooft limit of the $\mathcal{W}_N$ theories. On the bulk side, one uses the coupling of the scalar field to the higher spin gauge fields (25) to compute the three-point function, and finds exactly the same formula as the CFT answer from above. The computation makes clever use of the higher spin gauge symmetry to generate the solutions for the scalar field in the presence of the gauge fields.

6.1.2. Factorization. The issue of large-$N$ factorization of correlation functions of the CFT was studied in [99, 37]. Through explicit computation of a large number of correlators in the coset CFT using Coulomb gas and related techniques and then taking the large-$N$ 't Hooft limit, the following conclusions can be drawn.

- Perturbative primaries built from multiple tensor powers of fundamental/anti-fundamental fields behave as multi-particle states. Thus, a primary such as $(\text{adj}; 0)$ behaves in three-point functions like a double trace operator—the answer factorizes, at leading order in $N$, into two two-point functions.
- Four-point functions of perturbative primaries also factorize at large $N$, and the light states do not appear in the intermediate channel at large $N$. They have a well-defined large-$N$ limit.
- Four-point functions of perturbative primaries such as $(f; 0)$ with non-perturbative primaries such as $(0; f)$ also factorize even though there are light states such as $(f; f)$ in the intermediate channel. The important point here is that the fusion rules of the CFT guarantee that of the very large number of light states, only a finite number propagates in the intermediate channel. Furthermore, the non-zero couplings are of order $1/N$.

Thus, the perturbative primaries $(\Lambda; 0)$ form a closed consistent subsector (at large $N$) for sphere amplitudes. Furthermore, all of these states can be viewed as multi-particle states of a single complex scalar. Some of the non-perturbative states such as $(0; f)$ (and an infinite number of others at higher levels [37]) behave in much the same way as perturbative single particle states as far as their large-$N$ behaviour is concerned. Their correlation functions also have a well-behaved 't Hooft limit. However, because they essentially do not appear in any correlation function of perturbative states (unless there are order $N$ operators), we can view them as a decoupled sector. As observed earlier, in the semi-classical limit, these non-perturbative states indeed have $h \propto c$ justifying their name, even though in the 't Hooft limit, their dimensions are of order 1.

6.1.3. Torus two-point function. Let us also mention that in [37], the torus two-point function of $(f; 0)$ and its conjugate were calculated. This could potentially answer the question whether thermalization occurs in these theories at large but finite $N$ at timescales small compared to the Poincaré recurrence time which is $\sim N^4$. However, the explicit answer is not in a form which is easily amenable to a large-$N$ expansion, and so more work needs to be done in order to be able to extract interesting physics from it. A numerical study of the $N = 2$ case does show encouraging signs of thermalization occurring at intermediate timescales before recurrence sets in.

\[\text{From the factorization of correlators in the CFT, we know that we have a sum of terms like } q^h n, \text{ where } h \text{ are the conformal dimensions of various primaries and } n \text{ is an integer. From the form of } h \text{ given in equation (39), we see that it is a rational number with a denominator which goes like } N^4 \text{ (the quadratic Casimir has a piece like } 1/N^4). \text{ Therefore, the Poincaré recurrence time, i.e. the periodicity of the Euclidean correlator in imaginary time, behaves as } N^4.\]
6.2. Black hole entropy

As implicit from the discussion in section 5.4.1, it is not immediately obvious how to construct black hole solutions in higher spin gravity. Indeed, the usual definition—a spacetime singularity hidden behind a horizon—is difficult to apply because neither the Riemann tensor nor the causal structure of the metric is gauge-invariant. However, in Euclidean signature, the problem is simpler because a black hole is simply a smooth classical solution with torus boundary conditions. This definition has been used to construct explicit black hole solutions carrying higher spin charge [73]; see also [7, 35, 110, 92] as well as the review [8] in this volume.

The original construction in [73] was done for spin-3 gravity, but this was later generalized to the case of the $\text{hs}[\lambda]$ higher spin theories in [92]. The mass, angular momentum and charges of the black hole were computed and used to infer the free energy [92],

$$
\log Z_{\text{BH}}(\hat{\tau}, \alpha) = \frac{\pi e}{12} \left[ 1 - \frac{4}{3} \frac{\lambda^2}{\hat{\tau}^2} + \frac{400}{27} \frac{\lambda^2 - 7}{\hat{\tau}^8} - \frac{1600}{27} \frac{5\lambda^4 - 85\lambda^2 + 377}{(\lambda^2 - 4)^2} \frac{\alpha^8}{\hat{\tau}^{12}} + \cdots \right],
$$

(179)

where $\alpha$ is the chemical potential for the spin-3 charge, and $\hat{\tau}$ is the complex structure of the torus, related to the black hole temperature $T_H$ and (imaginary) angular potential $\Omega_H$ by

$$
\hat{\tau} = \frac{i}{2\pi T_H} (1 + \Omega_H).
$$

(180)

Furthermore, the central charge equals $c = \frac{3\ell}{2G}$ with $\ell$ being the AdS radius and $G$ Newton’s constant. Note that (179) only exhibits the holomorphic part of the full partition function; the right-moving sector gives a similar contribution.

By the usual AdS/CFT dictionary, one expects (179) to agree with the CFT partition function

$$
Z_{\text{CFT}}(\hat{\tau}, \alpha) = \text{Tr}(\hat{q}^{L_0} - \frac{c}{24}), \quad \hat{q} = e^{2\pi i \hat{\tau}}, \quad y = e^{2\pi i \alpha},
$$

(181)

in the high-temperature regime, i.e. for $\hat{\tau} \to 0$, and to leading order in the central charge $c$. Here, $W_0$ is the zero mode of the spin-3 current of $W_{\infty}[\lambda]$. Since (179) is an expansion in powers of the central charge $c$, it should be compared to the CFT expansion

$$
Z_{\text{CFT}}(\hat{\tau}, \alpha) = \text{Tr}(\hat{q}^{L_0} - \frac{c}{24}) + \frac{(2\pi i \alpha)^2}{2!} \text{Tr}((W_0)^2 \hat{q}^{L_0} - \frac{c}{24}) + \frac{(2\pi i \alpha)^4}{4!} \text{Tr}((W_0)^4 \hat{q}^{L_0} - \frac{c}{24}) + \cdots.
$$

(182)

At high temperatures, the $\hat{\tau}$-dependence of each term in the expansion is fixed by conformal invariance, which requires that [62]

$$
\log Z_{\text{CFT}}(\hat{\tau}, \alpha) \approx \frac{1}{\hat{\tau}} f \left( \frac{\alpha}{\hat{\tau}^2} \right)
$$

(183)

for some function $f$. As familiar from entropy calculations [108], the standard method to obtain the partition function from a dual conformal field theory point of view is to perform the $S$-modular transformation

$$
\tau = -\frac{1}{\hat{\tau}}, \quad q = e^{2\pi i \tau}.
$$

(184)

In the high-temperature regime, i.e. for $\hat{\tau} \to 0$, we have $q \to 0$. The answer for the trace is then dominated by the contribution from the vacuum state. This argument can be directly applied to the first term in the expansion (182):

$$
\text{Tr}(\hat{q}^{L_0} - \frac{c}{24}) = \sum_{s,r} S_{sr} \text{Tr}_s(\hat{q}^{L_0} - \frac{c}{24}) \sim \left( \sum_s S_{s0} \right) q^{-\frac{c}{24}} + \cdots,
$$

(185)
where the sum runs over all primaries labelled by \( r, s \) (with \( r = 0 \) being the vacuum representation), \( S_{rs} \) is the modular \( S \)-matrix (not to be confused with the black hole entropy), and the dots indicate terms exponentially suppressed at high temperature. The leading behaviour of the logarithm is then

\[
\log \text{Tr}(\hat{q}^{\alpha} \hat{\pi}) = -\frac{i\pi c}{12} \tau + \cdots,
\]

and this reproduces precisely the \( \alpha \)-independent term in (179), using the relation \( \tau = -\frac{1}{\tau} \).

This is equivalent to the Cardy formula for the entropy.

In order to reproduce the subleading terms in (179) from a CFT point of view, one therefore has to understand the modular behaviour of traces with the insertion of \( W_0 \) modes

\[
\text{Tr}(\hat{q}^{\alpha} \hat{\pi} (W_0)^n)
\]

for \( n = 1, 2, \ldots \)—it is relatively easy to see that odd powers of \( W_0 \) will not contribute at leading order in the high-temperature expansion (183). Using the general transformation formula for torus correlation functions of conformal primary fields under modular transformations [121], the leading high-temperature behaviour of (187) was determined for \( n = 1, 2, 3 \) in [62], thereby reproducing exactly (179) from (182). As for (185), the calculation effectively only depends on the vacuum representation of the CFT, and hence does not probe the detailed spectrum of the conjectured dual. However, at least for \( n = 3 \), various nonlinear terms of \( \mathcal{W}_\infty[\lambda] \) contributed to leading order, and hence the agreement is a pretty non-trivial test of the structure of \( \mathcal{W}_\infty[\lambda] \). The result is also in agreement with a direct free-field calculation [92] of (181) that is available for \( \lambda = 0, 1 \), where we have a realization of the CFT in terms of free fermions and free bosons, respectively.

The agreement between the two calculations demonstrates that the black hole solutions of [73, 92] dominate the bulk thermodynamics for \( T \to \infty \). However, it is currently not known whether Vasiliev gravity in three dimensions has a Hawking–Page transition, or whether the black hole dominates the bulk thermodynamics anywhere besides \( T \to \infty \). If there is indeed a phase transition above which the black hole dominates, then the dual CFT should have a gap large enough so that (179) applies above the transition temperature. The microscopic CFT proposed in [57] has a large number of light states with the dimension \( h + \bar{h} < 1 \), so it presumably obeys (179) only at asymptotically high temperatures. This is mirrored by the fact that the Vasiliev gravity theory has other saddle point solutions [33] (see section 5.4.1) which would contribute to the bulk thermodynamics.

7. Generalizations

In this section, we sketch a number of relatively straightforward generalizations of the above duality conjecture.

7.1. The orthogonal algebras

The most obvious generalization is the one that is analogous to the \( O(N) \) vector model in one dimension higher [3, 64]: it consists of replacing the \( SU(N) \) groups by \( SO(2N) \), i.e. it involves instead of (26) the cosets

\[
\frac{SO(2N)_k \otimes SO(2N)_{k+1}}{SO(2N)_{k+1}}.
\]

The \( SO(2N) \) groups have independent Casimir operators of even degree \( 2, 4, \ldots, 2N - 2 \), as well as a Casimir operator of degree \( N \), and thus the corresponding \( \mathcal{W} \)-algebra is generated.
by currents of the corresponding spin 2, 4, ... 2N − 2, as well as N. The algebra possesses a \( \mathbb{Z}_2 \)-symmetry under which the spin-\( N \) field is odd, and the even subalgebra is then generated by the fields of even spin 2, 4, ... 2N, together with the normal ordered product of the spin \( N \) field with itself and its higher derivatives, see [22] for details. In the large-\( N \) limit, we therefore obtain a \( \mathcal{W} \)-algebra with one current for every even spin.

The central charge of the coset (188) equals
\[ c = N \left( 1 - \frac{(2N - 1)(2N - 2)}{p(p + 1)} \right), \]
where \( p \equiv k + 2N - 2 \). The hwrs of the coset are labelled by triplets \( (\Lambda_+, \mu; \Lambda_-) \), where \( \Lambda_+ \) and \( \Lambda_- \) are the integrable hwr of \( \mathfrak{so}(2N)_k \) and \( \mathfrak{so}(2N)_{k+1} \), respectively, while \( \mu \) is an \( \mathfrak{so}(2N)_1 \) hwr. The triplets have to satisfy the selection rule that \( \Lambda_+ + \mu - \Lambda_- \) (interpreted as a weight of the finite-dimensional Lie algebra \( \mathfrak{so}(2N) \)) lies in the root lattice of \( \mathfrak{so}(2N) \). Modulo the root lattice, the weight lattice of \( \mathfrak{so}(2N) \) has four conjugacy classes, and there is precisely one level-1 representation in each conjugacy class; thus, the selection rule determines \( \mu \) uniquely, and we can label our coset representations by the pairs \( (\Lambda_+, \Lambda_-) \). In addition, there is the field identification \( (\Lambda_+; \Lambda_-) \cong (A\Lambda_+; A\Lambda_-) \), where \( A \) is the outer automorphism of the affine algebra \( \mathfrak{so}(2N)_k \) and \( \mathfrak{so}(2N)_{k+1} \), respectively. \( A \) permutes the four roots of the extended Dynkin diagram with the \( \mathfrak{sl}(N) \) case, the field identification becomes irrelevant in the 't Hooft limit.

We are again interested in the 't Hooft limit, where we take \( N \) and \( k \) to infinity, keeping the ratio
\[ \lambda = \frac{2N}{k + 2N - 2} = \frac{2N}{p} \]
fixed. In this limit, the conformal weight of the representations \( (\Lambda; 0) \) or \( (0; \Lambda) \) that involve spinor labels is proportional to \( N \), and the corresponding states decouple; for example, for the two spinor representations \( s = [0^{N-2}, 1, 0] \) and \( c = [0^{N-1}, 1] \), one finds [64]
\[ h_{(s;0)} = h_{(0;c)} = \frac{N}{8} \left( 1 + \frac{2N - 1}{p} \right), \quad h_{(0,s)} = h_{(c;0)} = \frac{N}{8} \left( 1 - \frac{2N - 1}{p + 1} \right). \]
Thus, only the non-spinor representations survive. These are contained in tensor products of the vector representations and they have a small conformal dimension in the 't Hooft limit; for example, for the vector representation \( v = [1, 0^{N-1}] \) itself, we have
\[ h_{(c;v)} = \frac{1}{2} \left( 1 + \frac{2N - 1}{p} \right) \cong \frac{1}{2} (1 + \lambda), \quad h_{(0,v)} = \frac{1}{2} \left( 1 - \frac{2N - 1}{p + 1} \right) \cong \frac{1}{2} (1 - \lambda), \]
where we have denoted by \( \cong \) the value in the 't Hooft limit. The tensor products of the vector representation can be labelled by Young diagrams, and thus the situation is very similar to what was discussed above. There is only one small difference: the vector representation \( (v; 0) \) (and similarly for \( (0; v) \)) is its own conjugate representation, and thus there is no analogue of \( (\bar{v}; 0) \) in the current context.

Based on these observations, one expects the dual higher spin theory to have higher spin gauge fields of every even spin \( s = 2, 4, 6, \ldots \). In addition, one may guess that the contribution of the representations that are contained in the tensor products of \( (v; 0) \) corresponds to adding to the topological higher spin theory a real massive scalar field of mass [3, 64]
\[ M^2 = -(1 - \lambda^2) \]
that is again quantized in the usual manner, i.e. leading to \( \hbar = \tilde{\hbar} = \frac{1}{2} (1 + \lambda) \). It was shown in [64] that this proposal satisfies one important consistency check: the spectrum of the higher spin theory together with this scalar field agrees exactly with the contribution of the perturbative \( (\Lambda; 0) \) states of the coset (188) in the large-\( N \) 't Hooft limit.
Unfortunately, the comparison of the partition functions does not directly determine the underlying higher spin symmetry of the AdS theory (since the calculation of the higher spin partition function only depends on the quadratic part of the action). However, there is a proposal for what should replace \( \text{hs}[\mu] \) in this context, namely the subalgebra
\[
\text{hs}[\mu]^{(s)} = \text{span}\{ V^s_m \in \text{hs}[\mu] : s \text{ even} \}.
\] (194)
In particular, the algebra \( \text{hs}[\mu]^{(s)} \) contains the ‘gravity’ \( \mathfrak{sl}(2) \) algebra generated by \( V^2_{0,\pm 1} \), and the Chern–Simons theory based on it will lead to spin fields of all even spacetime spins. Recently, the quantum \( \mathcal{W}_\infty^{(s)}[\mu] \)-algebra consisting of one conserved current for every even spin was studied in some detail [32]. It was found that it is again characterized in terms of two parameters: the central charge \( c \) as well as the self-coupling constant of the spin \( s = 4 \) field. The analogues of the triality relations of section 4 were also derived, thereby proving the equivalence of the quantum symmetries. It was furthermore shown in [32] that the wedge algebra of \( \mathcal{W}_\infty^{(s)}[\mu] \) becomes in the \( c \to \infty \) limit precisely \( \text{hs}[\mu]^{(s)} \), thereby proving that the higher spin theory is indeed the one based on (194).

7.2. The \( \mathcal{N} = 2 \) supersymmetric models

The bosonic higher spin theories we have discussed so far arise most naturally from truncations of the \( \mathcal{N} = 2 \) supersymmetric higher spin theories [103, 104]. These supersymmetric higher spin theories have two (real) bosonic gauge fields of each spin \( s = 2, 3, \ldots \) together with a single current of spin \( s = 1 \). In addition, there are two (real) fermionic gauge fields for each spin \( s = \frac{1}{2}, \frac{3}{2}, \ldots \). As in the bosonic case above, the structure of the theory depends on a real parameter \( \mu \) that characterizes the underlying Lie algebra symmetry in the Chern–Simons formulation. For the supersymmetric case, the relevant algebra is \( \text{shs}[\mu] \), which can be defined in close analogy to \( \text{hs}[\mu] \) in (20). To this end, consider
\[
\text{sB}[\mu] = \frac{U(\mathfrak{o}(1|2))}{C^{\text{op}} - \frac{1}{2}\mu(\mu - 1)1},
\] (195)
where \( \mathfrak{o}(1|2) \) is the Lie algebra generated by \( L_m, \ m = 0, \pm 1 \) and \( G_r, \ r = \pm \frac{1}{2} \), with commutation relations
\[
[L_m, L_n] = (m - n) L_{m+n},
\]
\[
[L_m, G_r] = \left( \frac{m}{2} - r \right) G_{m+r},
\] (196)
\[
[G_r, G_s] = 2L_{r+s},
\] (197)
and the Casimir operator \( C^{\text{op}} \) takes the form
\[
C^{\text{op}} = C^{\text{bos}} + \frac{1}{4} C^{\text{fer}} \equiv L_0^2 - \frac{1}{2}(L_1 L_{-1} + L_{-1} L_1) + \frac{1}{4}(G_2^2 G_{-2}^2 - G_{-2}^2 G_2^2).
\] (198)
By construction, \( \text{sB}[\mu] \) is an associative superalgebra with product \( \ast \), and we can make it into a Lie superalgebra by defining \([A, B]_\ast = A \ast B \pm B \ast A\). As before, the resulting Lie superalgebra contains an Abelian subalgebra generated by the identity \( 1 \), and we define \( \text{shs}[\mu] \) by
\[
\text{sB}[\mu] = \text{shs}[\mu] \oplus \mathbb{C},
\] (199)
in close analogy to (20). By a straightforward calculation, one shows that \( C^{\text{fer}} \), defined by (twice) the second term in (198), satisfies
\[
(C^{\text{fer}})^2 = C^{\text{bos}} + C^{\text{fer}} = C^{\text{op}} + \frac{1}{4} C^{\text{fer}},
\] (200)
and hence we can define orthogonal projection operators

\[ P_{\pm} = \frac{1}{2} \left[ 1 \pm \frac{2}{(\mu - \frac{1}{2})} \left( C^{\text{ferm}} - \frac{1}{4} \cdot 1 \right) \right], \quad P_{\pm}^2 = P_{\pm}, \quad P_{\pm} P_{-} = 0 \]  

(201)

that commute with the bosonic subalgebra of \( \text{shs}[\lambda] \). Thus, the bosonic subalgebra of \( \text{shs}[\lambda] \) actually decomposes as a direct sum into

\[ \text{shs}[\mu]^{\text{bos}} \cong \text{hs}[\mu] \oplus \text{hs}[1 - \mu], \]  

(202)

since on the image of \( P_{\pm} \), the eigenvalue of \( C^{\text{bos}} \) equals

\[ C^{\text{bos}} = C^{\text{supp}} - \frac{1}{2} C^{\text{ferm}} = \frac{1}{4} \mu (\mu - 1) - \frac{1}{4} \left[ \mp \frac{1}{2} (\mu - \frac{1}{2}) + \frac{1}{2} \right], \]  

(203)
i.e. either \( C^{\text{bos}} = \frac{1}{4} (\mu^2 - 1) \) or \( C^{\text{bos}} = \frac{1}{4} (\mu^2 - 2\mu) = \frac{1}{4} ((1 - \mu)^2 - 1) \). Finally, the analogue of (24) is now

\[ \text{shs}[\mu = -N]/\chi = \mathfrak{s}(N + 1|N). \]  

(204)

The above formulation is manifestly \( \mathcal{N} = 1 \) supersymmetric—(196) is the wedge algebra of the \( \mathcal{N} = 1 \) superconformal algebra—but actually the theory has \( \mathcal{N} = 2 \) supersymmetry. In particular, the massless gauge fields organize themselves into \( \mathcal{N} = 2 \) multiplets as

\[ (1 \frac{1}{2} 1 \frac{3}{2} 2) \quad (2 \frac{3}{2} 2 \frac{5}{2} 3) \quad (3 \frac{5}{2} 3 \frac{7}{2} 4), \quad \text{etc.} \]  

(205)

By analogy with the bosonic case, one expects that a massive scalar multiplet has to be added to the higher spin theory. In the supersymmetric case, each matter multiplet consists of a complex scalar field of mass

\[ M_{\mu}^2 = -1 + \mu^2, \]  

(206)
a Dirac fermion of mass \( m_{\mu} \) with

\[ m_{\mu}^2 = m_{1-\mu}^2 = (\mu - \frac{1}{2})^2 \]  

(207)
as well as a complex scalar and Dirac fermion of mass \( M_{1-\mu} \) and \( m_{1-\mu} \), respectively. These fields must be quantized so that the corresponding conformal dimensions fit also into an \( \mathcal{N} = 2 \) multiplet, i.e. as

\[ \left[ \frac{1}{2} (1 + \mu), \frac{1}{2} (1 + \mu) \right] \quad \left[ \frac{1}{2} (1 + \mu), \frac{\mu}{2} \right] \quad \left[ \frac{1}{2} (1 + \mu), \frac{\mu}{2} \right] \quad \left[ \frac{\mu}{2}, \frac{\mu}{2} \right], \]  

(208)

where the \( \left[ \frac{1}{2}, \frac{1}{2} \right] \) field corresponds to the massive scalar field with mass \( M_{1-\mu} \), quantized in non-standard fashion, i.e. with \( h = \bar{h} = \frac{1}{2} (1 - (1 - \mu)) = \frac{h}{2} \).

7.2.1. The dual Kazama–Suzuki models. It was proposed in [42] that the above higher spin theory with \( \mu = \lambda \) is dual to the ’t Hooft-like limit of a family of minimal \( \mathcal{N} = 2 \) superconformal coset theories based on

\[ sW_{\kappa,k} = \frac{\mathfrak{s}(N + 1)_{k+N+1}^{(1)}}{\mathfrak{su}(N)_{k+N+1}^{(1)} \oplus \mathfrak{u}(1)_{k}^{(1)}}, \]  

(209)

where \( \kappa = N(N + 1)(k + N + 1) \) is the ‘level’ of the \( \mathfrak{u}(1) \) algebra, and the superscript ‘(1)’ indicates that the relevant algebras are \( \mathcal{N} = 1 \) supersymmetric. (The \( \mathcal{N} = 1 \) affine algebras are actually isomorphic to a direct sum of the corresponding bosonic algebra (at a shifted level), together with \( \text{dim}(\mathfrak{su}(N)) \) free fermions.) The ’t Hooft limit consists again of taking \( N, k \) to infinity, with \( \lambda = \frac{N}{N+1} \) fixed.
These cosets are manifestly $\mathcal{N} = 1$ supersymmetric, but according to Kazama and Suzuki [84, 85], the actual chiral algebra contains the $\mathcal{N} = 2$ superconformal algebra. Geometrically, this is a consequence of the fact that the coset (209) is associated with the homogeneous space

$$\mathbb{CP}^N = \frac{\text{U}(N+1)}{\text{U}(N) \times \text{U}(1)},$$

which is actually a Hermitian symmetric space, i.e. possesses a complex structure. We should also mention in passing that (209) coincides with the DS reduction of the affine superalgebra $\mathfrak{sl}(N+1|N)_{\text{ext}}$ at level [82]

$$k_{\text{DS}} = -1 + \frac{1}{k + N + 1}.$$  

Given that the $\mathcal{N} = 1$ superconformal algebras are actually isomorphic to direct sums of the corresponding bosonic subalgebras and free Majorana fermions, we can reformulate the bosonic subalgebra of $\mathcal{W}_{N,k}$ in (209) as

$$s\mathcal{W}^{(0)}_{N,k} = \frac{\text{su}(N+1)_k \oplus \text{so}(2N)_1}{\text{su}(N)_{k+1} \oplus \text{u}(1)_c},$$

where $\text{so}(2N)_1$ is the bosonic algebra associated with the $2N$ free Majorana fermions that survive after subtracting from the $N^2 + 2N$ free fermions of the numerator in (209) the $N^2$ free fermions of the denominator. The central charge of the coset algebra $s\mathcal{W}_{N,k}$ is therefore

$$c = (N - 1) + \frac{kN(N+2)}{k + N + 1} - \frac{(k + 1)(N^2 - 1)}{k + N + 1} = \frac{3kN}{k + N + 1}.$$  

As reviewed in detail in [30], the supersymmetric representations of the coset $s\mathcal{W}_{N,k}$ are labelled by $(\Lambda; \Xi; l)$, where $\Lambda$ and $\Xi$ denote hwrs of $\text{su}(N+1)_k$ and $\text{su}(N)_{k+1}$, respectively, while $l$ is an integer defined modulo $\kappa$. The selection rule takes the form

$$\frac{B(\Lambda)}{N + 1} - \frac{B(\Xi)}{N} - \frac{l}{N(N+1)} \in \mathbb{Z},$$

where $B(\Lambda)$ denotes the number of boxes in the Young diagram corresponding to $\Lambda$, and similarly for $\Xi$, there are also field identifications (that are worked out in [65]), but they are again irrelevant in the ’t Hooft limit. The analogue of the $(f; 0)$ representation of the bosonic theory is now the representation with $\Lambda = f$, $\Xi = 0$, with $l = N$ because of (214); its conformal dimension equals the ’t Hooft limit; see, e.g., equation (3.63) of [30]

$$h(f; 0, N) = \frac{N(N + 2)}{2(N+1)(N+k+1)} - \frac{N^2}{2(N+1)(N+k+1)} \approx \frac{\lambda}{2}.$$  

This reproduces the lowest conformal dimension of the scalar multiplet (208) with $\mu = \lambda$. It was shown in [30] that the one-loop partition function of the supersymmetric higher spin theory, together with the massive scalar multiplet (208), is reproduced exactly by the perturbative states (i.e. the states with $\Xi = 0$) of the above Kazama–Suzuki model in the ’t Hooft limit. It was also shown in [75, 74, 5] that the symmetries match at least partially, and the analogue of the quantum symmetry analysis of section 4 was recently performed in [31]. More recently, the smooth supersymmetric conical defect geometries in the bulk were studied in [111, 43, 77], and it was suggested in [77] that these classical solutions may account for all primaries of the dual CFT, as suggested by the analysis of [31].
8. Questions and future directions

In the preceding sections, we have outlined many of the features of the $\mathcal{W}_N$ minimal models and the evidence accumulated thus far, for a dual description, at large $N$, in terms of a classical higher spin theory on AdS$_3$. In the process, we have also exhibited the tractability as well as complexity of the CFT:

(1) The spectrum and partition function of the $\mathcal{W}_N$ minimal models are explicitly known for any $N$ (and $k$). Nevertheless, analysing the spectrum in the large-$N$ 't Hooft limit is quite subtle. We see the presence of a large number of light states $\Delta \sim O\left(\frac{1}{N}\right)$—a feature not seen thus far in other examples of the AdS/CFT correspondence. While we have concentrated on the states with $\Delta \sim O(1)$, there is also a rich structure of primaries of dimension $N$ and higher which we have not touched upon.

(2) Three- and four-point sphere correlation functions in the CFT can also be explicitly calculated using conventional CFT techniques [99, 36]. It is non-trivial that they have a sensible large-$N$ limit which is consistent with a classical theory in the bulk. The two-point torus correlator has also been computed for finite $N$, $k$ and clearly exhibits intricate structure [36].

The boundary theory therefore appears to be rich enough to serve as an insightful example of the AdS/CFT correspondence. In particular, unlike most studies of the AdS/CFT correspondence thus far, one may hope to use the CFT to learn about aspects of stringy/quantum gravity in AdS. Clearly, a first task is to build on existing studies of the spectrum and correlation functions to extract quantitative information about bulk physics. Specifically, one may envisage:

(1) Obtaining a more refined understanding of the spectrum of states from the bulk point of view. We have identified the $(\Lambda, 0)$ primaries (with a finite number of boxes and anti-boxes) with perturbative multi-particle states of the complex scalar in the bulk. The $(\Lambda_+, \Lambda_-)$ primaries (with $\Lambda_- \neq 0$), on the other hand, behave as non-perturbative states in the semi-classical (large $c$, finite-$N$) limit, i.e. have $\Delta \propto c$. There is a class of non-trivial classical solutions in the bulk (the conical defects) whose quantum numbers match with those of the $(\star, \Lambda)$ primaries to leading order in $c$. It will be interesting to quantitatively check whether all the $(\Lambda_+, \Lambda_-)$ primaries can indeed be viewed in the semi-classical Vasiliev theory as bound states of these defects with the perturbative scalar excitations.

(2) Understanding the significance of the light states in the bulk $\text{hs}[\lambda]$ theory. The identification of light states as conical defects is in the semi-classical $SL(N)$ theory which is related by an analytic continuation in the central charge to the $\text{hs}[\lambda]$ theory. Is there a way to understand these directly through some kind of quantization of semi-classical solutions in the $\text{hs}[\lambda]$ higher spin theory? Can one give a more geometrical interpretation for them?

(3) Studying the interactions between perturbative and non-perturbative sectors. The sector of non-perturbative primaries contains states which behave like single or multi-particle excitations in correlation functions [99, 36] with each other and with perturbative primaries. What is the meaning of this from the bulk?

(4) Understanding the primaries in the CFT whose dimensions grow like $N$ or higher. As mentioned before, the CFT also has primaries whose dimension grows at least like $N$. Can these states be identified with micro states of black-hole-like solutions? Is there a

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15 See, though, [17] for a similar phenomena in 3D Chern–Simons vector models on $T^3$.

16 A precise proposal for this has recently been put forward in [100], together with supporting evidence.
phase transition at temperatures of order 1 where such states dominate the spectrum? Note
that at asymptotically high temperatures (and thus very high energies), we have seen, in
section 6.2, a match of the states in the CFT with those of black holes in the bulk [92, 62].

(5) Extracting thermal behaviour from torus two-point function. We need to put the two-point
function computed in [36] into a form amenable to taking the large-$N$
limit. Then one
may hope to see whether it exhibits exponential thermal decay for intermediate times
much smaller than the Poincaré recurrence time. This is related to the previous question
of whether we have black holes dominating the phase diagram at any finite temperature.

Symmetry is playing a very active role in this duality. Again, unlike other examples of
AdS/CFT duality, here the matching of the global and gauge symmetries between the boundary
and the bulk is a non-trivial dynamical fact. Specifically, from the bulk point of view, we have
an $\text{hs}[\lambda]$ classical gauge symmetry which is enhanced to the classical $\mathcal{W}_\infty[\lambda]$ asymptotic
symmetry algebra. As we saw in section 4, this is non-trivially equivalent to the large-$N$
‘t Hooft of the $\mathcal{W}_N$-algebra of the boundary CFT. We believe that this equivalence is pointing
to directions worth exploring further:

1) Quantum deformation of the bulk symmetry algebra. At finite $N$ when we need to go
to a quantum version of the Vasiliev bulk theory, the prediction is that the symmetry
algebra is deformed to $\mathcal{W}_N[\lambda = \frac{Nk}{N^2}] \cong \mathcal{W}_N$. This requires a non-perturbative truncation
of the symmetry currents to a maximal spin $s_{\text{max}} = N$. This is reminiscent of the stringy
exclusion principle that arises in other (stringy) AdS/CFT examples.

2) Integrability at the quantum level. The above truncation immediately leads to the fact that
instead of an infinite number of commuting conserved charges at the classical level, there
are only finitely many at finite $N$. What does this mean for the integrability of the theory?
Does it affect the physics of black holes in the theory?

3) Quantization of the Vasiliev theory. What kind of quantization of the bulk can produce
a truncation like the above which would not be visible in the $\frac{1}{N}$ expansion? Is there a
naive first quantization like those of strings which is adequate for the $\frac{1}{N}$ expansion but not
beyond? Is there a more geometric formulation of the quantization in which the $\mathcal{W}_\infty[\lambda]$-symmetry plays a central role? The $\mathcal{W}_\infty[\lambda]$-algebra makes definite predictions for the
exact $c$-dependence of, for example, the mass of the scalar as well as the structure of the
higher spin algebra. Can one derive these corrections, at least to lowest order in $\frac{1}{N}$, directly
from the higher spin theory point of view?

4) Proving the duality. Could the quantum $\mathcal{W}_\infty[\lambda]$-symmetry perhaps be powerful enough
to prove the duality? One is looking for unitary representations of this algebra as well as
modular invariance of the thermal partition function. Perhaps, this constrains the matter
primaries to be those of the $\mathcal{W}_N$ minimal models (up to the discrete choices of modular
invariants). Alternatively, could one generalize the ideas in [45, 49, 83] to the interacting
CFTs considered here?

We have discussed in section 7 some of the generalizations of the original duality to
orthogonal gauge groups as well as $\mathcal{N} = 2$ supersymmetric cosets. There are many other
avenues here as well:

1) Other modular invariants. Up to now, we have focused on the diagonal modular invariant
while constructing the $\mathcal{W}_N$ CFT from its chiral sectors. There is a large class of other
modular invariants as well which are also consistent CFTs, and it is natural to wonder
whether large-$N$ families of these admit higher spin AdS3 duals.

2) Massive deformations and RG flows. The $\mathcal{W}_N$ minimal models have many relevant
operators and it is possible to deform the CFTs by turning these on. Some of these
RG flows, especially between nearby minimal models, have been studied, see, e.g., [57], being in some cases even integrable deformations. It would be of obvious interest to have nice examples of holographic duals to such massive non-supersymmetric theories and their RG flows.

(3) ‘Stringy cosets’. We can consider the general family of cosets

\[
\frac{\text{SU}(N)_k \otimes \text{SU}(N)_l}{\text{SU}(N)_{k+l}}.
\]

(216)

If we define the ’t Hooft limit in this case with \( k, l, N \to \infty \) with relative ratios held finite as in [86], then we find that the central charge grows like \( N^2 \). This is like in a gauge theory and it is natural to expect a stringy dual\(^{17}\). Indeed, the special case of \( k = l = N \) recently studied in [71] does arise as the low-energy limit of a 2D gauge theory coupled to adjoint fermions. It would be very interesting to understand the string duals for these generically non-supersymmetric theories. These would also provide an embedding of the vector-like cosets into a larger string theory, perhaps along the lines of [38].

(4) de Sitter analogue. Vasiliev higher spin theories can also be defined on dS spacetimes. A dS\(_4/\text{CFT}^3\) correspondence has been advanced for 4D Vasiliev theories [10]. A similar attempt for the case of dS\(_3/\text{CFT}^2\) seems to require an imaginary central charge for the CFT and other such undesirable features [98]. Are there perhaps ways around this?

We have not described the features of black holes and other classical bulk solutions in this theory, in any detail. There are tantalizing hints here of a stringy generalization of geometry and what it has to say about fundamental issues of singularities, existence of horizons, etc. Some of these issues will be addressed in the accompanying article in this issue [8].

To summarize, we expect various fruitful insights to emerge in the coming years from the study of minimal models and their holographic duals.

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