The degenerate parametric oscillator and Ince’s equation

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Abstract

We construct Green’s function for the quantum degenerate parametric oscillator in the coordinate representation in terms of standard solutions of Ince’s equation in a framework of a general approach to variable quadratic Hamiltonians. Exact time-dependent wavefunctions and their connections with dynamical invariants and SU(1,1) group are also discussed. An extension to the degenerate parametric oscillator with time-dependent amplitude and phase is also mentioned.

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1. The degenerate parametric amplifier Hamiltonian

We use standard annihilation and creation operators in the coordinate representation given by

\[ \hat{a} = \sqrt{\frac{m}{2\hbar\omega}} \left( \omega x + \frac{ip}{m} \right), \quad \hat{a}^\dagger = \sqrt{\frac{m}{2\hbar\omega}} \left( \omega x - \frac{ip}{m} \right) \] (1.1)

with \( p = -i\hbar \partial/\partial x \) and \( \hat{a}^\dagger \hat{a} - \hat{a} \hat{a}^\dagger = 1 \) throughout this paper. The Hamiltonian

\[ H(t) = \frac{\hbar\omega}{2} (\hat{a}^\dagger \hat{a}^\dagger + \hat{a} \hat{a}) - \frac{\hbar\lambda}{2} \left( e^{2i\omega t} \hat{a}^3 + e^{-2i\omega t} (\hat{a}^\dagger)^2 \right) \] (1.2)

of the time-dependent Schrödinger equation

\[ i\hbar \frac{\partial \psi}{\partial t} = H(t) \psi \] (1.3)

describes the process of degenerate parametric amplification in quantum optics. The first term corresponds to the self-energy of the oscillator representing the mode of interest, and the second

1 http://hahn.la.asu.edu/~suslov/index.html

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term describes the coupling of the classical pump to that mode (with the phenomenological constant $\lambda$), giving rise to the parametric amplification process (the phase of the pump is taken to be zero at $t = 0$ for convenience). Takahasi [41] wrote a detailed paper on this subject way back in 1965 (see also papers [18, 19, 22, 32, 34, 35, 38], books [21, 30, 31, 33, 36, 42] and references therein). The same Hamiltonian had also been considered by Angelow and Trifonov in the mid-1990s in order to describe the light propagation in the nonlinear Ti:LiNbO$_3$ anisotropic waveguide (details can be found in [1–3]). Another form of this variable quadratic Hamiltonian is given by

$$H = \frac{1}{2m} \left( 1 + \frac{\lambda}{\omega} \cos (2\omega t) \right) p^2 + \frac{m\omega^2}{2} \left( 1 - \frac{\lambda}{\omega} \cos (2\omega t) \right) x^2 + \frac{\lambda}{2} \sin (2\omega t) (px + xp),$$

where $p = \hbar \frac{\partial}{\partial x}$. (1.4)

We refer to (1.2) and (1.3) and/or (1.4) as the degenerate parametric oscillator [30]. It can be recognized as a special case of the so-called generalized harmonic oscillators that had attracted considerable attention over many years in view of their great importance to several advanced quantum problems including coherent states and uncertainty relations, Berry’s phase, asymptotic and numerical methods, quantization of mechanical systems, Hamiltonian cosmology, charged particle traps and motion in uniform magnetic fields, molecular spectroscopy and polyatomic molecules in varying external fields. Quadratic Hamiltonians have particular applications in quantum electrodynamics because the electromagnetic field can be represented as a set of forced harmonic oscillators. See, for example, [4, 7, 10–14, 24, 40, 43, 44] and references therein. Nonlinear oscillators play a central role in the theory of Bose–Einstein condensation [9] because the dynamics of gases of cooled atoms in a magnetic trap at very low temperatures can be described by an effective equation for the condensate wavefunction known as the Gross–Pitaevskii (or nonlinear Schrödinger) equation (see also [15, 16, 39] and references therein).

A goal of this paper is to construct Green’s function of the Hamiltonian (1.4) in the coordinate representation in terms of solutions of Ince’s equation studied in [23] and [27] (see also references therein). Exact time-dependent wavefunctions and connections with linear and quadratic dynamical invariants and the $SU(1, 1)$ group are also briefly discussed.

2. Generalized harmonic oscillators

The degenerate parametric oscillator Hamiltonian (1.4) belongs to quantum systems described by the one-dimensional time-dependent Schrödinger equation

$$i \frac{\partial \psi}{\partial t} = H \psi$$

with variable quadratic Hamiltonians of the form

$$H = a(t) p^2 + b(t) x^2 + c(t) px + d(t) xp, \quad p = -i \frac{\partial}{\partial x},$$

where $a(t)$, $b(t)$, $c(t)$ and $d(t)$ are real-valued functions of time, $t$, only (see, for example, [5–8, 10, 12, 19, 20, 26, 43] and [44] for a general approach and known elementary solutions; a case related to Airy functions is discussed in [17] and our paper deals with another special case of transcendental solutions).

The corresponding Green’s function, or Feynman’s propagator, can be found as follows [5]:

$$\psi = G(x, y, t) = \frac{1}{\sqrt{2\pi \mu_0(t)}} e^{i(\alpha_0(t)x^2 + \rho_0(t)xy + \beta_0(t)y^2)}.$$

(2.3)
where
\[ \alpha_0(t) = \frac{1}{4a(t)} \frac{\mu_0'(t)}{\mu_0(t)} - \frac{c(t)}{2a(t)}, \]  
(2.4)
\[ \beta_0(t) = -\frac{\dot{\lambda}(t)}{\mu_0(t)}, \quad \lambda(t) = \exp \left( \int_0^t (c(s) - d(s)) \, ds \right), \]  
(2.5)
\[ \gamma_0(t) = \frac{1}{2\mu_1(0)} \frac{\mu_1(t)}{\mu_0(t)} + \frac{c(0)}{2a(0)}, \]  
(2.6)
and \( \mu_0(t) \) and \( \mu_1(t) \) are two linearly independent solutions of the characteristic equation
\[ \mu'' - \tau(t)\mu' + 4\sigma(t)\mu = 0 \]  
(2.7)

with
\[ \tau(t) = a' + 2c - 2d, \quad \sigma(t) = ab - cd + \frac{c}{2} \left( \frac{a'}{a} - \frac{c'}{c} \right) \]  
(2.8)
subject to initial conditions
\[ \mu_0(0) = 0, \quad \mu_0'(0) = 2a(0) \neq 0, \]  
(2.9)
\[ \mu_1(0) \neq 0, \quad \mu_1'(0) = 0. \]  
(2.10)

In this paper, we present the time-dependent coefficient \( \gamma_0(t) \) in terms of two standard solutions \( \mu_0 \) and \( \mu_1 \) of the characteristic equations (2.7) and (2.8) with initial data (2.9) and (2.10), respectively, instead of a previously used formula in terms of only one solution \( \mu_0 \) (see, for example, [5, 7] and [40]). Our expression (2.6) can be verified by a direct differentiation:
\[ \left( \frac{\mu_1}{\mu_0} \right)' = \frac{\mu_1'\mu_0 - \mu_1\mu_0'}{\mu_0^2} = \frac{W(\mu_0, \mu_1)}{\mu_0^2}, \]  
(2.11)
where the Wronskian can be found with the help of Abel’s theorem as follows:
\[ W(\mu_0, \mu_1) = \text{constant} \, \lambda^2(t) a(t). \]  
(2.12)

Then, as required,
\[ \frac{dy_0}{dt} + a(t)\beta_0^2(t) = 0 \]  
(2.13)
and the constant term in (2.6) gives a correct asymptotic as \( t \to 0^+ \) (see references in [8] and [7]). The Green’s function (2.3) is an eigenfunction of the linear dynamical invariant of Dodonov, Malkin, Man’ko and Trifonov [10, 11, 24, 25] (also see theorem 1 of [40]).

By the superposition principle, the solution of the Cauchy initial value problem can be presented in an integral form
\[ \psi(x, t) = \int_{-\infty}^{\infty} G(x, y, t) \chi(y) \, dy, \quad \lim_{t \to 0^+} \psi(x, t) = \chi(x) \]  
(2.14)
for a suitable initial function \( \chi \) on \( \mathbb{R} \) (a rigorous proof will be discussed elsewhere and uniqueness is analyzed in [7]; another form of solution is provided by an eigenfunction expansion [40]; generalized coherent states and transition amplitudes will be discussed elsewhere). In the next sections, we apply these general results to the case of the degenerate parametric oscillator (1.4).
3. The Green’s function and Ince’s equation

For the Hamiltonian (1.4), variable coefficients are

\[ a = \frac{\hbar}{2m} \left( 1 + \frac{\lambda}{\omega} \cos (2\omega t) \right), \]

\[ b = \frac{m\omega^2}{2\hbar} \left( 1 - \frac{\lambda}{\omega} \cos (2\omega t) \right), \]

\[ c = d = \frac{\lambda}{2} \sin (2\omega t) \]

and general expressions for Green’s function (2.3)-(2.6) can be simplified by letting \( \lambda(t) \equiv 1 \) and \( c(0) = 0 \).

The characteristic equation is given by

\[ \mu'' + \frac{2\lambda \omega \sin (2\omega t)}{\omega + \lambda \cos (2\omega t)} \mu' + \frac{\omega (\omega^2 - 3\lambda^2) - \lambda (\omega^2 + \lambda^2) \cos (2\omega t)}{\omega + \lambda \cos (2\omega t)} \mu = 0, \]

which can be identified as a special case of the Ince equation [23, 27]:

\[ (1 + a_0 \cos 2s) y''(s) + b_0 \sin 2s y'(s) + (c_0 + d_0 \cos 2s) y(s) = 0, \]

when \( s = \omega t \) and parameters are given by

\[ a_0 = \frac{\lambda}{\omega}, \quad b_0 = 2a_0 = 2\frac{\lambda}{\omega}, \quad c_0 = 1 - 3\frac{\lambda^2}{\omega^2}, \quad d_0 = -\frac{\lambda}{\omega} \left( 1 + \frac{\lambda^2}{\omega^2} \right). \]

Traditionally, a special question which arises in the theory of Ince’s equation is the problem of the existence of periodic solutions. By theorem 7.1 on p 93 of [23], if Ince’s equation (3.5) has two linearly independent solutions of period \( \pi \), then the polynomial

\[ P(\xi) = 2a_0 \xi^2 - b_0 \xi - d_0/2 \]

has a zero at one of the points \( \xi = 0, \pm 1, \pm 2, \ldots \). If (3.5) has two linearly independent solutions of period \( 2\pi \), then

\[ Q(\xi) = 2P(\xi - 1/2) \]

vanishes for one of the values \( \xi = 0, \pm 1, \pm 2, \ldots \).

In the case of the degenerate parametric oscillator Hamiltonian (3.6), both of these polynomials are strictly positive:

\[ P(\xi) = \frac{2\lambda}{\omega} \left( \left( \xi - \frac{1}{2} \right)^2 + \frac{\lambda^2}{4\omega^2} \right) > 0 \]

and

\[ Q(\xi) = \frac{4\lambda}{\omega} \left( (\xi - 1)^2 + \frac{\lambda^2}{4\omega^2} \right) > 0. \]

Thus the corresponding Ince’s equation (3.4) may not have two periodic solutions of period \( \pi/\omega \) or \( 2\pi/\omega \) and our standard solutions \( \mu_0 \) and \( \mu_1 \) that satisfy initial conditions (2.9) and (2.10) cannot be constructed in terms of simple variants of the Fourier sine and cosine series:

\[ \mu_1(t) = \sum_{n=0}^{\infty} A_{2n} \cos (2\omega nt), \quad \mu_0(t) = \sum_{n=1}^{\infty} B_{2n} \sin (2\omega nt) \]

or

\[ \mu_1(t) = \sum_{n=0}^{\infty} A_{2n+1} \cos (\omega (2n+1)t), \quad \mu_0(t) = \sum_{n=0}^{\infty} B_{2n+1} \sin (\omega (2n+1)t), \]

when coefficients \( A_m \) and \( B_m \) satisfy certain three-term recurrence relations (more details can
be found in [23] and [27]). Therefore, the degenerate parametric oscillator (1.4) motivates a detailed study of non-periodic solutions of Ince’s equation.

4. The dynamical invariant and time-dependent wavefunctions

Exact time-dependent wavefunctions of generalized harmonic oscillators (2.1) and (2.2) can also be obtained as eigenfunctions of the quadratic invariant in terms of Hermite polynomials and certain solutions of Ermakov-type equation (see [40] for a modern introduction and references therein). The dynamical invariant of the Hamiltonian (1.4) can be derived from a general form presented in [7, 40] as follows:

\[ E(t) = \left( \mu \frac{p}{\mu} + \frac{m}{\hbar} \frac{\lambda \sin(2\omega t) \mu}{1 + (\lambda/\omega) \cos(2\omega t)} \right)^2 + \frac{C_0}{\mu^2} x^2, \quad p = -i \frac{\partial}{\partial x}, \quad (4.1) \]

where \( C_0 > 0 \) is a constant and function \( \mu(t) \) satisfies the nonlinear auxiliary equation:

\[ \mu'' + \frac{2\omega \sin(2\omega t)}{\omega + \lambda \cos(2\omega t)} \mu' + \frac{\omega(\omega^2 - 3\lambda^2) - \lambda(\omega^2 + \lambda^2) \cos(2\omega t)}{\omega + \lambda \cos(2\omega t)} \mu = \frac{C_0}{\mu} \frac{(\hbar/m)^2}{1 + (\lambda/\omega) \cos(2\omega t)^2} \mu^3. \quad (4.2) \]

A general solution of this nonlinear equation can be found by the following ‘law of cosines’:

\[ \mu^2(t) = A u^2(t) + B v^2(t) + 2 C u(t) v(t) \quad (4.3) \]

in terms of two linearly independent solutions \( u \) and \( v \) of the homogeneous equation. The constant \( C_0 \) is related to the Wronskian of two linearly independent solutions \( u \) and \( v \):

\[ AB - C^2 = C_0 \frac{(2a)^2}{W^2(u, v)}, \quad W(u, v) = uv - u'v \quad (4.4) \]

(more details can be found in [40] and [13]; see also references therein regarding this so-called Pinney’s solution). Thus solution of the corresponding initial value problem is given by

\[ \mu^2(t) = \left( \frac{\mu'(0)}{2a(0)} \mu_0(t) + \frac{\mu(0)}{\mu_1(0)} \mu_1(t) \right)^2 + \frac{C_0}{\mu^2(0)} \mu_0^2(t), \quad \mu(0) \neq 0 \quad (4.5) \]

in terms of standard solutions \( \mu_0 \) and \( \mu_1 \) corresponding to initial data (2.9) and (2.10) of the homogeneous equation (4.2).

Time-dependent wavefunctions can be presented in the form

\[ \psi_n(x, t) = e^{-i(n+1/2)\phi(t)} \Psi_n(x, t) \quad (4.6) \]

(see, for example, [40, 44] and references therein), where

\[ \frac{d\psi}{dt} = \sqrt{C_0} \frac{\hbar}{m} \frac{1 + (\lambda/\omega) \cos(2\omega t)}{\mu^2}, \quad \psi(0) = 0 \quad (4.7) \]

and orthonormal time-dependent eigenfunctions \( \Psi_n(x, t) \) of the quadratic invariant (4.1) are expressed in terms of Hermite polynomials [29]:

\[ \Psi_n(x, t) = D_n \exp \left( i x^2 \frac{m}{2\hbar} \left( \frac{\mu' \mu}{\mu} - \frac{\lambda \sin(2\omega t)}{2} \right) \right) \times e^{-x^2/\sqrt{C_0}/(2a^2)} H_n \left( \frac{C_0^{1/4} x}{\mu} \right), \quad |D_n|^2 = \frac{C_0^{1/4}}{\sqrt{\pi} \sqrt{2^n n!} \mu}. \quad (4.8) \]
\[ i \frac{\partial \psi_n}{\partial t} = H \psi_n, \quad E \psi_n = 2 \sqrt{C_0} \left( n + \frac{1}{2} \right) \psi_n. \quad (4.9) \]

If the orthonormal initial wavefunction is given by
\[ \psi_n(x,0) = \frac{e^{(i\delta - \epsilon^2/2)x^2}}{\sqrt{\pi n!}} H_n(\epsilon x), \quad (4.10) \]
where \( \epsilon \) and \( \delta \) are constants, then initial conditions for the auxiliary equation (4.2) are given by
\[ \mu(0) = \frac{C_0^{1/4}}{\epsilon}, \quad \mu'(0) = 2C_0^{1/4} \left( 1 + \frac{\lambda}{\omega} \right) \frac{\delta}{\epsilon} \quad (4.11) \]
(the integral \( C_0 > 0 \) can be chosen at reader’s convenience). Then integration of Ermakov’s equation (4.2), say with the help of Pinney’s solution (4.5), determines complete dynamics of the oscillator-type wavefunction via the explicit formula (4.8). Further relations of these particular solutions of the Schrödinger equation with the Cauchy initial value problem are discussed in [40] (see theorem 2).

5. The SU(1, 1) symmetry

The Hamiltonian (1.2) can also be rewritten as follows:
\[ H(t) = \hbar \omega J_0 - \hbar \lambda (e^{2i\omega t} J_+ + e^{-2i\omega t} J_-), \quad (5.1) \]
where generators of a non-compact SU(1, 1) group are given by
\[ J_+ = \frac{1}{2}(a^\dagger)^2, \quad J_- = \frac{1}{2}a^2, \quad J_0 = \frac{1}{4}(aa^\dagger + a^\dagger a). \quad (5.2) \]

Then the group properties of the corresponding discrete positive series \( D'_n \) can be used for further investigation of the degenerate parametric oscillator. This is a ‘standard procedure’—more details can be found in [19, 24, 26, 29] and [37] and/or elsewhere.

6. A special case

An integrable special case \( \lambda = \omega = m = \hbar = 1 \) of the degenerate parametric oscillator has been recently considered by Meiler, Cordero-Soto and Suslov [8, 26] (this Hamiltonian is a simplest time-dependent quadratic integral of motion for the linear harmonic oscillator [7]). Here, the Ince equation (3.4) simplifies to
\[ \mu'' + 2 \tan t \mu' - 2 \mu = 0. \quad (6.1) \]
It has two elementary non-periodic standard solutions:
\[ \mu_0 = \cos t \sinh t + \sin t \cosh t, \quad \mu_0(0) = 0, \quad \mu_0'(0) = 2, \quad (6.2) \]
\[ \mu_1 = \cos t \cosh t + \sin t \sinh t, \quad \mu_1(0) = 1, \quad \mu_1'(0) = 0 \quad (6.3) \]
and Green’s function is given in terms of trigonometric and hyperbolic functions as follows:
\[ G(x, y, t) = \frac{1}{\sqrt{2\pi i} (\cos t \sinh t + \sin t \cosh t)} \times \exp \left( \frac{(x^2 - y^2) \sin t \sinh t + 2xy - (x^2 + y^2) \cos t \cosh t}{2i(\cos t \sinh t + \sin t \cosh t)} \right) \quad (6.4) \]
as a simple consequence of expressions (2.3)–(2.6). It is worth noting that formula (6.4) has been obtained in [26] by a totally different approach using the SU(1, 1)-symmetry of \( n \)-dimensional oscillator wavefunctions and properties of the Meixner–Pollaczek polynomials. More details can be found in [8, 26].
7. An extension

The degenerate parametric amplification with time-dependent amplitude and phase had been considered by Raiford [35]. The corresponding Hamiltonian, without damping and neglecting high-frequency terms, has the form

$$H(t) = \frac{\hbar \omega}{2} (\hat{a} \hat{a}^\dagger + \hat{a}^\dagger \hat{a}) - \frac{\hbar \lambda(t)}{2} (e^{i(2\omega t + \delta(t))} \hat{a}^2 + e^{-i(2\omega t + \delta(t))} (\hat{a}^\dagger)^2).$$

(7.1)

In this model, the phenomenological coupling parameter $\lambda(t)$, which describes the strength of the interaction between the quantized signal of frequency $\omega$ and the classical pump of frequency $2\omega$, and the pump phase $\delta(t)$ are in general functions of time as indicated. It includes the special case of the pump and signal being off-resonance by a given amount $\epsilon$, i.e. the pump frequency being $2\omega + \epsilon$, by letting $\delta(t) = \epsilon t$ and $\lambda(t) = \lambda$, a constant. More details, including Heisenberg’s equation of motion for annihilation and creation operators for signal photons, can be found in the original paper [35].

In the coordinate representation, the Hamiltonian (7.1) is rewritten as

$$H = \frac{1}{2m} \left( 1 + \frac{\lambda(t)}{\omega} \cos (2\omega t + \delta(t)) \right) p^2 + \frac{m\omega^2}{2} \left( 1 - \frac{\lambda(t)}{\omega} \cos (2\omega t + \delta(t)) \right) \chi^2$$

$$+ \frac{\lambda}{2} \sin (2\omega t + \delta(t)) (px + xp), \quad p = \frac{\hbar}{i} \frac{\partial}{\partial x},$$

(7.2)

and the Green’s function can be found by using the method described in section 2. The corresponding characteristic equation takes the form

$$\mu'' + \frac{\lambda \sin (2\omega t + \delta)(2\omega + \delta') - \lambda' \cos (2\omega t + \delta)}{\omega + \lambda \cos (2\omega t + \delta)} \mu'$$

$$+ \frac{\omega (\omega^2 - 3\lambda^2) - \lambda \omega (\omega^2 + \lambda^2 + \omega' \delta') \cos (2\omega t + \delta) - \lambda' \omega \sin (2\omega t + \delta)}{\omega + \lambda \cos (2\omega t + \delta)} \mu = 0,$$

(7.3)

which can be thought of as an extension of Ince’s equation (3.4). Further details are left to the reader.

8. Conclusion

In this paper, we have constructed Green’s function of the degenerate parametric oscillator (1.4) in terms of standard non-periodic solutions of Ince’s equation. To the best of our knowledge, this oscillator was introduced by Takahasi [41] in order to describe the process of degenerate parametric amplification in quantum optics (see also [18, 19, 21, 22, 30–36, 38, 42] and references therein). The corresponding Hamiltonian had also been considered later by Angelow and Trifonov [1, 3] in order to describe the light propagation in a nonlinear anisotropic waveguide. Our observation motivates further investigation of properties of the degenerate parametric oscillator including a systematic study of corresponding non-periodic solutions of Ince’s equation that seems to be missing in mathematical literature. Moreover, the Hamiltonian (7.1), which was considered by Raiford [35] in the case of the degenerate parametric amplification with time-dependent amplitude and phase, requires a generalization of Ince’s equation given by (7.3). An application of a Green’s function approach to quantum optical correlations is discussed in [28].
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