MARKED NON-ORIENTABLE SURFACES AND CLUSTER CATEGORIES VIA SYMMETRIC REPRESENTATIONS

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ABSTRACT. We initiate the investigation of representation theory of non-orientable surfaces. As a first step towards finding an additive categorification of Dupont and Palesi’s quasi-cluster algebras associated marked non-orientable surfaces, we study a certain modification on the objects of the cluster category associated to the orientable double covers in the unpunctured case. More precisely, we consider symmetric representation theory studied by Derksen-Weyman and Boos-Cerulli Irelli, and lift it to the cluster category. This gives a way to consider ‘indecomposable orbits of objects’ under a contravariant duality functor. Hence, we can assign curves on a non-orientable surface $(S, M)$ to indecomposable symmetric objects. Moreover, we define a new notion of symmetric extension, and show that the arcs and quasi-arcs on $(S, M)$ correspond to the indecomposable symmetric objects without symmetric self-extension. Consequently, we show that quasi-triangulations of $(S, M)$ correspond to a symmetric analogue of cluster tilting objects.

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1. Introduction

1.1. Background. Cluster algebras are certain commutative algebras, first axiomatized by Fomin and Zelevinsky \cite{FZ}, that comes with a distinguished set of generators called cluster variables organised in overlapping sets called clusters. Since the birth of cluster algebras, connections between cluster theory and many mathematical fields have been discovered, such as combinatorics, dynamical systems, knot theory, mirror symmetry, etc. The motivation of this article originates from the connection between cluster algebras arising from surface topology and representation theory \cite{MRT, FZT, FM, FZ1, FZ2, FZ3, FZ4}.

More precisely, the cluster algebras in this setting - usually called surface cluster algebras - comes from the coordinate ring of a certain Teichmüller space associated to an orientable marked surface with triangulation. The cluster variables are in correspondence with arcs on the surface, i.e. self non-crossing curves that connect marked points; clusters are in correspondence with maximal sets of pairwise non-crossing arcs, i.e. triangulations. The combinatorial nature of these correspondence have been fruitful to reveal the algebraic structure of surface cluster algebras; see, for example, been studied in \cite{TZ, FTZ}.

It is natural to ask for an extension to non-orientable surfaces since one still have a coordinate ring for a Teichmüller space associated to the surface. Their idea is to enlarge the set of arcs to include what they call quasi-arcs. A quasi-arc is a self-non-intersecting 1-sided closed curves, meaning that its cylindrical neighbourhood forms a Möbius strip. The analogue of clusters is now given by quasi-triangulations, which are maximal sets of pairwise non-crossing arcs and quasi-arcs. Thus far, a few classical results from Fomin-Shapiro’s surface cluster algebras have been extended; see \cite{TZ, FTZ, FZT}.

We are interested in the role of representation theory in the quasi-cluster structure associated to marked non-orientable surfaces, in particular the additive categorification \cite{AM1, AM2, AM3} of quasi-cluster algebras. In the classical case, this is given by a triangulated category $\mathcal{C}$ called Amiot’s cluster category. The category $\mathcal{C} = C_{Q,W}$ is determined by a quiver with potential $(QP) \ (Q,W)$ associated to a chosen initial cluster – or triangulation in the surface case \cite{AM1, AM2}. It turns out that the indecomposable rigid objects – i.e. those without self-extensions – in $C_{Q,W}$
correspond to cluster variables, and the so-called cluster tilting objects correspond to clusters; see, for example, [17]. In practice, one can analyse \( C_{Q,W} \) through the module category \( \text{mod} \ J_{Q,W} \) of the Jacobian algebra associated to \((Q,W)\).

In the surface cluster algebra case, it was shown in [6] that \( J_{Q,W} \) falls under a prominent class of algebras in representation theory called gentle algebras [19]. The modules over gentle algebras are well-understood [20] and can be calculated combinatorially. In particular, an indecomposable module is either a string module, which is described by (Dynkin) 'type \( A \)' combinatorics, or a band module, which is described by (extended Dynkin) type \( \tilde{A} \) combinatorics along with a parameter \( \lambda \in k^\times \) from the underlying field \( k \). Brüstle and Zhang [7] used this to study the cluster category \( C_{Q,W} \) by showing the following dictionary (see Section 3 for details). Note that by [21] \( C_{Q,W} \) is independent of the choice of initial triangulation, and so we can denote it by \( C_{(S,M)} \) where \((S,M)\) is the marked surface of interest.

| Topology                  | Cluster category         |
|---------------------------|--------------------------|
| (A) curve \( \gamma \) connecting marked points | indecomposable string object \( \gamma \) |
| (B) closed curve \( \omega \) with a parameter \( \lambda \in k^\times \) | indecomposable band objects \( (\omega, \lambda) \) |
| (C) arcs                   | rigid indecomposable objects |
| (D) triangulations         | cluster tilting objects   |

1.2. Our goal. We aim to extend the above dictionary to the case of non-orientable surfaces. To do this, we rely on the fact that any non-orientable surface always has an orientable double cover. By lifting a triangulation \( T \) of a non-orientable marked surface \((S,M)\) to a triangulation \( \tilde{T} \) on the orientable double cover \((\tilde{S},\tilde{M})\), we can then canonically associate a QP \((Q,W)\) and hence a cluster category \( \mathcal{C}_{\tilde{(S,M)}} = C_{Q,W} \). The associated Deck transformation group is generated by an orientation-reversing automorphism \( \sigma_S \), which gives rise to an (arrow-reversing) involution \( \sigma \) on the QP. We show that this gives rise to a contravariant duality \( \nabla \) on both \( \mathcal{C}_{(S,M)} \) and \( \text{mod} \ J_{Q,W} \) (Proposition 4.12).

Next, we want a categorical way - such as an ‘orbit category’ \( \mathcal{C}_{(S,M)}/\nabla \) - to treat a \( \nabla \)-orbit of objects as indecomposable. We have to address two problems here. Firstly, the classical orbit construction requires a covariant autoequivalence; however, \( \nabla \) is contravariant. As a consequence, we need a different theory to deal with \( \nabla \)-orbits. Secondly, a primitive closed curve \( \tilde{\omega} \) on \((\tilde{S},\tilde{M})\) with \( \sigma_S(\tilde{\omega}) = \tilde{\omega} \) is a lift of two different closed curves, say \( \omega, \omega' \), on \((S,M)\). Indeed, one of \( \omega, \omega' \) is 1-sided (has a unique lift) and the other is 2-sided (admits a 2-sheeted cover); see discussion at the end of Section 2. In view of the dictionary [B], a good orbit construction should induce a partition of \( \{ (\tilde{\omega}, \lambda) \in \mathcal{C}_{\tilde{(S,M)}} \mid \lambda \in k^\times \} \) into two subsets, with one corresponds to \( \omega \) and the other to \( \omega' \).
To this end, we employ the symmetric representation theory, developed by Derksen and Weyman [22] as well as Boos and Cerulli Irelli [23], associated to algebras with duality – such as \((J_Q,W,\sigma)\) in our setting. A symmetric representation is an ordinary representation equipped with some extra data that forces each dual pair \((\alpha,\sigma(\alpha))\) of arrows of \(Q\) to act adjointly, see Section 5. Although symmetric representations do not form an additive category, there is still a natural notion indecomposability. It was shown in [22, 23] (see Proposition 5.4) that every indecomposable symmetric representation \(X\) is uniquely determined by the \(\nabla\)-orbit of an indecomposable (ordinary) module \(M\) in one of the following forms.

1. Split type: \(X = M \oplus \nabla M\) for \(M \not\cong \nabla M\).
2. Ramified type: \(X = M \oplus \nabla M\) for \(M \cong \nabla M\).
3. 1-sided type: \(X = M\) for \(M \cong \nabla M\).

Conversely, every indecomposable module \(M\) give rise to exactly one of these indecomposable symmetric representations. This trichotomy closely resembles the behaviour of curves on \((S,M)\), with split and ramified type corresponds to curves admitting a 2-sheeted cover whereas 1-sided type corresponds to 1-sided (closed) curves. Indeed, by fully classifying indecomposable symmetric representations over \(J_{Q,W}\) (Theorem 5.15), we have that for a closed curve \(\tilde{\omega}\) on \((S,M)\) with \(\sigma_{S}(\tilde{\omega}) = \tilde{\omega}\), there is a unique parameter \(\lambda \in k^\times\) so that \((\tilde{\omega},\lambda)\) gives rise to a 1-sided indecomposable symmetric representation. In other words, the notion of indecomposable symmetric representations does satisfy our desired criteria of good \(\nabla\)-orbit on the level of \(\text{mod} J_{Q,W}\).

We next define the notion of symmetric objects (respectively indecomposable symmetric objects) (Definition 6.1) in the category \(C\) \((S,M)\), which allows us to lift symmetric (respectively indecomposable symmetric) representations from \(\text{mod} J_{Q,W}\) to \(C\) \((S,M)\). Combining with the dictionary \([A]\) and \([B]\) from [7], this allows us to write down the following correspondences.

**Theorem 1.1.** (Theorem 6.4) There are the following bijections between curves on \((S,M)\) and indecomposable symmetric objects of the cluster category \(C\) \((S,M)\):

\[
\begin{align*}
\{ \text{curves connecting marked points} \} & \quad \overset{1:1}{\longrightarrow} \quad \{ \text{indecomposable symmetric objects of split string type} \} \\
\{ \text{primitive one-sided closed curves} \} & \quad \overset{1:1}{\longrightarrow} \quad \{ \text{indecomposable symmetric objects of one-sided primitive band type} \}
\end{align*}
\]

We note that the set on the right in the second row is discrete, as opposed to a continuous family in \([B]\). We omit the statement for the 2-sided closed curves due to technicalities; see Theorem 6.4 for details.

Our next goal is to find a categorical criteria for indecomposable symmetric objects that characterises self-non-crossing property of a curve on \((S,M)\) - these are encoded by vanishing of extensions of objects in the classical case. As an analogue, we define symmetric extensions (Definition 6.5) for symmetric objects.
Using this, we further define the notion of symmetric rigid objects and symmetric cluster tilting objects, which yields the following correspondences extending \((C)\) and \((D)\).

**Theorem 1.2.** (Theorem 6.22) For a marked non-orientable unpunctured surface \((S, M)\) with orientable double cover \((\tilde{S}, \tilde{M})\), the correspondences in Theorem 1.1 restrict to

\[
\{ \text{arcs and quasi-arcs of } (S, M) \} \leftrightarrow \{ \text{indecomposable symmetric rigid objects of } \mathcal{C}_{(S, M)} \}.
\]

This induces the following correspondence

\[
\{ \text{quasi-triangulations of } (S, M) \} \leftrightarrow \{ \text{symmetric cluster tiling objects of } \mathcal{C}_{(S, M)} \}.
\]

**1.3. Future Direction.** A more precise additive categorification of a cluster algebra \(A\) asks not just correspondences between rigid/cluster tilting objects of a category \(\mathcal{C}\) and cluster variables/clusters of \(A\), but also existence of a cluster character \(\chi: \text{ob } \mathcal{C} \rightarrow A\) (a.k.a. Caldero-Chapoton map) that satisfies additional properties. Having the correspondence is only a first step; in a future project, we will look at analogue of cluster characters for quasi-cluster algebras associated to non-orientable surfaces.

**1.4. Structure of the paper.** Our paper is structured as follows. In Section 2, we review the necessary topological theory we use in this article; namely, non-orientable surfaces, quasi-arcs, quasi-triangulations, double cover. In Section 3, we review the algebraic prerequisites. This includes QP’s associated to orientable triangulated surfaces, the arising gentle Jacobian algebra, and cluster category. In Section 4, we describe a certain type of involutions on the QP’s we use. This gives rise to a contravariant duality on the cluster categories and the module categories. In subsection 4.2, we justify that such a duality categorifies the orientation-reversing automorphism on the double cover defining the non-orientable surface of interest. In Section 5, by thoroughly analysing the possible symmetric structure on the modules of the arising Jacobian algebras, we prove our first main result (Theorem 5.15) — the classification of indecomposable symmetric representations. In the final Section 6, we lift symmetric representation theory from the module category to the cluster category, which yields the correspondence (Theorem 6.4) between curves and symmetric objects. Moreover, we define symmetric analogue of rigid objects and cluster tilting objects, which lead us to the final main result (Theorem 6.22), namely, the categorification of quasi-triangulations of non-orientable surfaces.

**1.4.1. Conventions.** Throughout, we assume any underlying field \(k\) is algebraically closed. For any \((k-)\)algebra \(\Lambda\) is assumed to be finite-dimensional unless otherwise specified. The category of (finitely generated) \(\Lambda\)-modules is denoted by \(\text{mod } \Lambda\). The
full subcategory of finitely generated projective \( \Lambda \)-modules is denoted by \( \text{proj} \Lambda \).

For an additive category \( \mathcal{C} \) and any object \( X \in \mathcal{C} \), denote by \( \text{add}_\mathcal{C}(C) \), or simply \( \text{add}(C) \) if there is no confusion, the additive closure of \( C \) in \( \mathcal{C} \).

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2. Marked surfaces

In this section, we review the necessary topological notions that will be used throughout the paper, namely, the nuances of arcs and triangulations of non-orientable surfaces.

By a surface, we mean a compact, connected, real 2-dimensional, (possibly non-orientable) surface \( S \) with non-empty boundary \( \partial S \). Let \( M \) be a set of marked points, that is, a finite discrete set of points in \( \partial S \) such that \( M \cap \partial \neq \emptyset \) for each connected component \( \partial \) of \( \partial S \). Such a pair \( (S, M) \) is called a marked surface. We will assume marked surfaces can be non-trivially triangulated, i.e. \( (S, M) \) is not a monogon, digon, or a triangle.

Suppose that \( (S, M) \) is a marked non-orientable surface. Up to homeomorphism, \( S \) is the connected sum of \( k \) projective planes \( \mathbb{RP}^2 \). The number \( k \) is called the non-orientable genus, or simply just the genus if no confusion arises. Note that any non-orientable surface \( N_k \) of genus \( k \) admits an orientable double cover \( \tilde{N}_k \) of genus \( k - 1 \).

In order to visualize these surfaces, recall that \( \mathbb{RP}^2 \) is the quotient of the 2-sphere \( S^2 \) under the antipodal map, which leads to a practical representation by drawing a crosscap \( \bigotimes \). Figure 1 shows how attaching a crosscap to a disk gives a surface homeomorphic to the Möbius strip. More generally, \( N_k \) can be drawn as a disk with \( k \) crosscaps attached, or equivalently, an orientable genus \( g \geq 0 \) surface with \( k - 2g \geq 1 \) crosscaps attached.

![Figure 1. Identifying crosscap-attached disk with the Möbius strip.](image-url)
Unless otherwise specified, a curve in \((S,M)\) is (the image of) a continuous function \(\gamma : I \to S\) such that

- either \(\gamma\) is a curve with endpoints in \(M\), i.e. \(I \simeq [0,1]\) with \(\gamma \cap \partial S = \{I(0), I(1)\} \subseteq M\) and \(\gamma\) does not cut out a monogon containing no marked point when \(I(0) = I(1)\),
- or \(\gamma\) is a closed curve, i.e. \(I \simeq S^1\) with \(\gamma \cap \partial S = \emptyset\) and \(\gamma\) non-contractible.

We always work with curves up to isotopy relative its endpoints (whenever 'endpoints' make sense). In particular, two curves bounding a marked point are considered distinct. We will use the following notation:

\[
\begin{align*}
C_{\text{nc}}(S,M) &:= \{\text{(isotopy classes of) curves with endpoints in } M\} \\
C_{\text{cc}}(S,M) &:= \{\text{(isotopy classes of) closed curves in } (S,M)\} \\
C(S,M) &:= C_{\text{nc}}(S,M) \sqcup C_{\text{cc}}(S,M)
\end{align*}
\]

A boundary arc is a curve that is isotopic to an interval on \(\partial S\) with endpoints in \(M\). A regular arc \(\gamma\) is a curve in \((S,M)\) with endpoints in \(M\) that is not a boundary arc and has no self-intersections (i.e. simple), except possibly at its endpoints. Denote by \(\mathbf{A}(S,M)\) the set of regular arcs in \((S,M)\).

Every curve on \((S,M)\) can be drawn as a curve on an orientable surface with crosscaps, such that when it hit a crosscap, it needs to come out from the antipodal point of this crosscap. See, for example, how the 'arc' given by the (concatenation of) red-and-blue (dashed, then dash-dotted) line in Figure 1 represents the vertical (green) oriented curve) on the Möbius strip on the far-right.

**Example 2.1.** Consider the curves \(\alpha, \beta, \gamma\) in Figure 2. We have \(\alpha, \beta \in C_{\text{nc}}(S,M)\) and \(\omega, \omega' \in C_{\text{cc}}(S,M)\). Note that \(\alpha\) is not a regular arc as it contains a self-intersection, whereas \(\beta\) is a regular arc as the only self-intersection is at its endpoints.

**Example 2.2.** There are five regular arcs in the Möbius strip with two marked points. See Figure 5.

Recall that a closed curve on \((S,M)\) is 2-sided if local orientation is preserved when traversing along itself; 1-sided, otherwise. This is equivalent to say that it has a 2-sheeted cover or a unique lift on the orientable double cover; see the discussion at the end of this section. A simple closed curve is 1-sided if and only
if the number of crosscaps it goes through is odd \[27\]. Denote by \(C_{1\text{si}}(S, M)\) the set of 1-sided closed curves and by \(C_{2\text{si}}(S, M)\) the set of 2-sided closed curves. So we have \(C_{cc}(S, M) = C_{1\text{si}}(S, M) \cup C_{2\text{si}}(S, M)\).

The dotted curve in the far-right of Figure 1 shows the core of a Möbius strip, i.e. a 1-sided simple closed curve whose neighbourhood is homeomorphic to a Möbius strip. We call such a curve a quasi-arc. The simplest pictorial representation of a quasi-arc is given by drawing a closed curve through a crosscap once; see Figure 3.

![Figure 3. Quasi-arc on the Möbius strip](image)

Quasi-arcs can take more complicating forms. For example, Figure 4 shows a quasi-arc that goes through three crosscaps on \((\mathbb{RP}^2)^#3\) with a disk removed. This can be turned back into the more familiar form by using a different representation of the same surface, as a sphere attached with 3 crosscaps is homeomorphic to a torus attached with 1 crosscap.

![Figure 4. Quasi-arc that goes through three crosscaps](image)

From now on, by an arc, or internal arc if we want to emphasis its nature, we mean a regular arc or a quasi-arc. Denote by \(\mathcal{A}(S, M)\) the set of all arcs. Next, we would like to talk about triangulations (and their extended notion) formed by curves of \((S, M)\). This requires the notion of non-crossing.

Recall that two curves (in particular, two arcs) \(\gamma, \gamma'\) in a (orientable or non-orientable) marked surface \((S, M)\) are non-crossing if, up to isotopy, they do not intersect one another except at their endpoints. More formally, define

\[
\text{int}(\gamma, \gamma') := \min\{\alpha \cap \alpha' \mid \alpha \simeq \gamma \text{ and } \alpha' \simeq \gamma'\}
\]
where $\alpha$ (respectively $\alpha'$) ranges over curves isotopic to $\gamma$ (respectively $\gamma'$), and define also

$$\text{cross}(\gamma, \gamma') := \begin{cases} \text{int}(\gamma, \gamma') \setminus \{\gamma(0), \gamma(1)\}, & \text{if both } \gamma, \gamma' \text{ are not closed;} \\ \text{int}(\gamma, \gamma'), & \text{otherwise.} \end{cases}$$

Then, $\gamma$ and $\gamma'$ being non-crossing is the same as saying $\text{cross}(\gamma, \gamma') = \emptyset$.

With the crosscap representation of non-orientable surfaces, two arcs may appear to have a crossing at the crosscap, but they actually do not cross in reality.

**Definition 2.3.** A quasi-triangulation (respectively triangulation) of $(S, M)$ is a maximal collection $T \subset A^\oplus(S, M)$ (respectively $T \subset A(S, M)$) of pairwise non-crossing arcs (respectively regular arcs).

See Figure 5 for all the triangulations and quasi-triangulations of the Möbius strip with 2 marked points $M_2$.

**Definition 2.4.** Let $(S, M)$ be a marked unpunctured (not necessarily orientable) surface. A (orientable) double cover of $(S, M)$ is a double cover $p : \tilde{S} \to S$ of surfaces with orientable $\tilde{S}$ and Deck transformation group $\{1, \sigma\}$ for some orientation-reversing automorphism $\sigma : \tilde{S} \to \tilde{S}$, such that $p^{-1}(\partial S) = \partial \tilde{S}$ and $(\tilde{S}, \tilde{M}) = (\tilde{S}, M := p^{-1}(M))$ is an orientable marked (unpunctured) surface. In such a case, if $T$ is a triangulation of $(S, M)$, then we call the preimage $\tilde{T} := p^{-1}(T)$ of $T$ the double cover of $T$.

We will omit the covering map $p$ and just say that $(\tilde{(S, M)}, \sigma)$, or even just $(\tilde{S, M})$, is an orientable double cover of $S$. In Figure 6 we have $S = M_{2n}$, the
Möbius strip with $2n$ marked points on the left-hand side, and its double cover – an annulus with $2n$ marked points on each boundary component – on the right-hand side. We show also a triangulation $T = \{1, 2, \ldots, n\}$ on the Möbius strip and its double cover $\tilde{T} = \{1, 1', 2, 2', \ldots, n, n'\}$. We call this the fan triangulation of $\mathcal{M}_{2n}$.

![Figure 6. Fan triangulation of Möbius strip and its double cover](image)

Recall that a closed curve $\omega \in C_{cc}(S, M)$ is primitive if for it cannot be written as $\kappa^w$ for some $\kappa \in C_{cc}(S, M)$ with $r > 1$. Therefore, we have a partition $C_{cc}(S, M) = \bigsqcup_{r \geq 1} C_{cc}^r(S, M)$ where $C_{cc}^r(S, M)$ are closed curves of the form $\omega^r$ for a primitive $\omega \in C_{cc}^1(S, M)$. Likewise, we write $C_{1si}^1(S, M)$ and $C_{2si}^1(S, M)$ for the sets of primitive 1-sided and 2-sided closed curves respectively. We can characterise the whether a closed curve $\omega \in C_{cc}^1(S, M)$ is 1- or 2-sided from its lift in $\tilde{(S, M)}$ as follows:

$$\omega \in C_{1si}^1(S, M) \iff p^{-1}(\omega) \text{ is a single closed curve on } \tilde{(S, M)},$$

$$\omega \in C_{2si}^1(S, M) \iff p^{-1}(\omega) = \tilde{\omega} \sqcup \sigma(\tilde{\omega}) \text{ for some lift } \tilde{\omega} \in C_{cc}(\tilde{S}, \tilde{M}).$$

In particular, if $\tilde{\omega} := p^{-1}(\omega)$ is the unique lift of $\omega \in C_{1si}^1(S, M)$, then there is also a 2-sided closed curve $\kappa$ such that $p^{-1}(\kappa)$ is (isotopic to) the disjoint union of two copies of $\tilde{\omega}$. An example of such a pair $(\omega, \kappa)$ is already shown in Example 2.

3. Quiver with potential, Jacobian algebra, and cluster category

In this section, we review the notion of quivers with potential associated to an orientable surface. We discuss two algebraic objects one can associate to such a quiver with potential: the cluster category and the Jacobian algebra. In this, we review string and band modules arising from arcs and closed curves on a surface.

3.1. QP and Jacobian algebra. Suppose $(S, M)$ is a marked orientable surface and $T$ is a triangulation on $(S, M)$. Recall from \cite{5} that one can associate to $T$ is quiver with potential (QP) $(Q_T, W_T)$ given by

- The quiver $Q_T$ has vertices being the internal arcs of $T$ and arrows clockwise rotation of arcs around marked points.
Each internal triangle \( \triangle \) yields an oriented cycle \( a_{\triangle}b_{\triangle}c_{\triangle} \) in \( QT \) that is unique up to cyclic permutation. Then the potential \( W_T \) is given by the sum of all these oriented cycles over all internal triangles of \( T \).

**Example 3.1.** Consider the triangulation of \( M_2 \) in the middle of the top row of Figure 5. The double cover is given by the triangulation \( T \) of the annulus with four arcs as shown in Figure 7a.

\[
\begin{align*}
Q_T &= \begin{tikzpicture}[scale=0.7]
    \node (1) at (0,0) {1};
    \node (2) at (1,0) {2};
    \node (3) at (1,-1) {3};
    \node (4) at (0,-1) {4};
    \draw (1) to (2);
    \draw (2) to (3);
    \draw (3) to (4);
    \draw (4) to (1);
    \end{tikzpicture}
\end{align*}
\]

\[
W_T = \alpha_1\alpha_2\alpha_3 + \beta_3\beta_2\beta_1
\]

**Figure 7.** A triangulation of annulus and its associated quiver with potential.

A QP gives rise to an algebra \( J_{Q,W} \) called *Jacobian algebra*. In the case when \( (Q,W) = (Q_T, W_T) \) for some triangulation \( T \), then [6] showed that the arising Jacobian algebra \( J_T := J_{Q_T, W_T} \) belongs to a special class of finite-dimensional (basic) algebras called *gentle algebras* (see Definition 4.5); note that having \( M \subset \partial S \) is crucial. Its quiver-and-relations, i.e. the pair \( (Q, R) \) of a quiver \( Q \) and a set \( R \) of linear combinations of paths in \( Q \) such that \( J_T \cong kQ/(R) \), can be written in a more practical form:

\[
Q = Q_T, \quad \text{and} \quad R = \{ \text{length 2 paths in internal triangles of} \ T \}.
\]

**Example 3.2.** Using the QP from Example 3.1 we obtain the following ideal

\[
R = \langle \alpha_1\alpha_2, \alpha_2\alpha_3, \alpha_3\alpha_1, \beta_2\beta_3, \beta_3\beta_2, \beta_1\beta_3 \rangle.
\]

This gives the Jacobian algebra \( J_T = kQ/R \) which has a basis given by

\[
\{ \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \alpha_1\beta_2, \alpha_2\beta_1, \beta_3\alpha_2, \beta_2\alpha_3 \}.
\]

### 3.2. Cluster category associated to a QP.

By a result of Amiot [28, Thm 3.6], for any QP \( (Q,W) \) with a finite-dimensional Jacobian algebra \( J_{Q,W} \), then one can associate to it a Krull-Schmidt Hom-finite 2-Calabi-Yau k-linear triangulated category, denoted by \( \mathcal{C}_{(Q,W)} \), called the (Amiot’s) *cluster category*. For an introductory exposition on the cluster category, see [29]. When \( (Q,W) = (Q_T, W_T) \) for some triangulation \( T \), we will use a simpler notation \( \mathcal{C}_T := \mathcal{C}_{(Q_T, W_T)} \).

Let us now focus on the case when \( (Q,W) = (Q_T, W_T) \) for some triangulation \( T \) of marked orientable surface \( (S,M) \). Denote by \( \text{ind} \mathcal{C} \) the set of isomorphism
classes of indecomposable objects in $\mathcal{C}$, whose relation with surface combinatorics is first explained in [6] and [7]. Then we have a bijection
\begin{equation}
(C_{cc}(S, M) \times k^\times) \cup C_{nc}(S, M) \leftrightarrow \text{ind} \mathcal{C}
\end{equation}

**Example 3.3.** Let $\omega$ be the unique closed curve on the triangulation of the annulus given in Figure 7a. For some $\lambda \in k \times$, this curve corresponds to the indecomposable module $V_\lambda = k \xrightarrow{1} k \xrightarrow{\lambda} k \xrightarrow{1} k$.

Henceforth, we use the same notation for curves with endpoints and the corresponding indecomposable objects in $\mathcal{C}_T$; likewise for the coloured closed curves $(\gamma, \lambda) \in C_{cc}(S, M) \times k \times$.

**Definition 3.4.** Let $\mathcal{C}$ be a triangulated category with shift functor [1]. Denote by $\text{Ext}^1_C(X, Y) := \text{Hom}_C(X, Y[1])$ for any $X, Y \in \mathcal{C}$.

An object $T \in \mathcal{C}$ is called **cluster-tilting** if
\begin{itemize}
  \item it is rigid, i.e. $\text{Ext}^1_C(T, T) = 0$,
  \item and $T^\perp := \text{Ker} \text{Ext}^1_C(T, -)$ and $^\perp T := \text{Ker} \text{Ext}^1_C(-, T)$ coincides with $\text{add}(T) \subset \mathcal{C}$.
\end{itemize}

A cluster-tilting object is said to be **basic** if its indecomposable direct summands are pairwise distinct. Denote by $\text{c-tilt} (\mathcal{C})$ the set of (isomorphism classes of) basic cluster-tilting objects in $\mathcal{C}$.

Suppose $T$ is a triangulation of an orientable marked surface $(S, M)$. It turns out that $\Gamma := \bigoplus_{\gamma \in T} \gamma[-1] \in \mathcal{C}_T$ (and hence also $T$ itself) is a cluster-tilting object. In fact, more generally, we have the following commutative diagram
\begin{align*}
\text{A}(S, M) \xrightarrow{1:1} & \{\text{indecomposable rigid objects of } \mathcal{C}\} \\
\cup & \{\text{arcs of } T\} \xrightarrow{1:1} \{\text{indecomposable direct summands of } \Gamma[1]\}.
\end{align*}

By Koenig and Zhu’s result [18], the functor $\text{Hom}_{\mathcal{C}(Q, W)}(\Gamma, -) : \mathcal{C}(Q, W) \to \mod J_{Q, W}$ induces an equivalence
\begin{equation}
M(-) := \text{Hom}_{\mathcal{C}(Q, W)}(\Gamma, -) : \mathcal{C}(Q, W)/[\Gamma[1]] \to \mod J_{Q, W}
\end{equation}
where $\mathcal{C}(Q, W)/[\Gamma[1]]$ denotes the additive quotient of $\mathcal{C}$ by the ideal of morphisms factoring through $\Gamma[1]$. Practically, this means that indecomposable objects of $\mathcal{C}(Q, W)$ are ‘given’ by those of $\mod J_{Q, W}$ along with the indecomposable direct summands of $\Gamma[1]$.

Let us now describe the explicit structure of the indecomposable $J_T$-modules for a triangulation $T$ of a marked orientable unpunctured surface. There are two types of such modules, called **strings** and **bands**.
3.3. String modules vs curves with endpoints. Consider first the case of a curve with endpoints $γ : [0, 1] \to S$ that is not an arc of $T$. We have the following set of crossings between $γ$ and arcs of $T$

cross($γ, T$) = \{ $γ(t_0), γ(t_1), \ldots, γ(t_c)$ \}, arranged so that $0 < t_0 < t_1 < \cdots t_c < 1$.

Let $γ_i := γ|_{(t_{i-1}, t_i)}$ for $i = 1, 2, \ldots, c$. Since each $γ_i$ is an angle of an (internal) triangle of $T$, this can be identified with an arrow $α_i ∈ Q = QT$. Note that $α_i$ may not have the same orientation as the (oriented) segment $γ_i$, in which case, we write $α_i ≃ γ_i^{-1}$; otherwise, $α_i ≃ γ_i$. The underlying vector space of the module $M(γ)$ is given by

$$M(γ) \cong SP \bigoplus_{i=0}^c \mathbb{K}γ(t_i) \text{ with } α_i \text{ action: } \begin{cases} kγ(t_{i-1}) \rightarrow kγ(t_i), & \text{if } α_i ≃ γ_i; \\ kγ(t_i) \rightarrow kγ(t_{i-1}), & \text{if } α_i ≃ γ_i^{-1}; \end{cases}$$

and with $e_τ$-action, for primitive idempotent $e_τ ∈ JT$ corresponding to the arc $τ$, on $γ(t_i)$ is given by identity if $γ(t_i) ∈ τ$; by zero otherwise.

As we can see, a string module $M(γ)$ can be encoded purely by combinatorial means, namely, the sequence $γ_i$'s. This is what people call string combinatorics in the representation theory of gentle (or generally, special biserial) algebras, which we will describe more properly in the following.

We consider the elements of $Q_0 \sqcup Q_1 \sqcup Q_1^{-1}$ as letters, where $Q_1^{-1}$ is the set of formal inverses $α^{-1}$ of arrows $α ∈ Q_1$. We call a letter $α$ a directed arrow if $α ∈ Q_1$, an inverse arrow if $α ∈ Q_1^{-1}$, and trivial if $α ∈ Q_0$. It is customary to use the trivial path $e_τ$ as letter for $x ∈ Q_0$. For a directed arrow $α ∈ Q_1$, its inverse $α^{-1}$ has source $s(α^{-1}) := t(α)$ and target $t(α^{-1}) = s(α)$. The inverse $(α^{-1})^{-1}$ of an inverse arrow $α^{-1}$ is just $α$, and the inverse of a trivial letter is itself. A word $w$ is called a walk if it is either trivial, i.e. $w = e_i ∈ Q_0$, or $w = w_1 \cdots w_L$ with letters $w_i$'s such that the following hold:

- $t(w_i) = s(w_{i+1})$ for all $i$;
- if both $w_i, w_{i+1}$ are directed (respectively inverse), then $w_i w_{i+1} ∈ R$ (respectively $w_i^{-1} w_{i+1}^{-1} ∈ R$);
- if $w_i$ and $w_{i+1}$ are in different direction, then $w_i \neq w_{i+1}^{-1}$.

Inverting a letter extends to a reflection operation on the set of walks, and the induced equivalence classes are called string.

Curves with endpoints can be identified with strings. Indeed, for such a curve $γ$, the segment $γ_i = γ|_{(t_i, t_{i+1})}$ defines a letter $w_i$ in the corresponding walk; note that $γ_1, \ldots, γ_c$ suffices to determine $γ$ as the remaining starting interval of $γ$ are uniquely determine by going from the arc containing $γ(t_0)$ to the opposite marked point of the triangle, and likewise for the ending interval. Note that the trivial strings $e_τ$ for $x ∈ Q_0$ correspond to the curve that crosses $T$ only once at arc $x ∈ T$. Now, the module $M(γ)$ for a string $γ$ (equivalently, curve with endpoints) is called a string module.

**Example 3.5.** Consider the triangulation $T$ of the surface $(S, M)$ in Example 3.1. In Figure S we show a (non-closed) curve $γ$ on $(S, M)$. Orient $γ$ so it starts with
the top marked point. Then it crosses $T$ in the order of 1, 4, 2, 4, 3 before reaching its other endpoint. The string correspond to this is $\alpha_3^{-1}\beta_2^{-1}\alpha_2\beta_1$ which gives the string module $M(\gamma)$ shown on the right of Figure 8.

3.4. Band modules vs closed curves. Suppose now $\gamma : S^1 \to S$ is a (non-contractible) closed curve. Write $\gamma = \omega^n$ for some primitive closed curve $\omega$ on $S$, i.e. $\gamma$ is homotopic the concatenation of $n$ copies of a ‘shorter’ closed curve $\omega$, and $\omega$ itself cannot be written as concatenation of a shorter closed curve. Similar to the previous case, write

$$\text{cross}(\omega, T) = \{\omega(z_0), \omega(z_1), \ldots, \omega(z_{c-1})\}$$

with $z_i = \exp(2t_i + \pi \sqrt{-1})$ and $0 = t_1 < \cdots < t_c < 1$.

Removing the intersections yields intervals $\omega_1, \ldots, \omega_c$ of $\omega$ which can be identified with arrows $\alpha_1, \ldots, \alpha_c \in Q$ By rotating the pieces if necessary, we assume that $\alpha_1 \simeq \omega_1$ and $\alpha_c \not\simeq \omega_c$. For $\lambda \in k^\times$, denote by $J_n(\lambda)$ the Jordan block of size $n$ with eigenvalue $\lambda$. Then we can define the indecomposable $J_T$-module $M_\lambda(\gamma) := M((\gamma, \lambda))$ associated to the indecomposable object $(\gamma = \omega^n, \lambda) \in C_{cc}(S, M) \times k^\times \subset \text{ind} \mathcal{C}$ by

$$M_\lambda(\omega^n) = \bigoplus_{i=0}^{c-1} V_i,$$

with $V_i = \bigoplus_{j=1}^n \mathbb{C} \omega(z_i)^{(j)}$, defining the $\alpha_i$ action by

$$\begin{align*}
V_{i-1} &\xrightarrow{1} V_i, & \text{if } \alpha_i \simeq \gamma_i; \\
V_i &\xrightarrow{1} V_{i-1}, & \text{if } \alpha_i \not\simeq \gamma_i \text{ and } i \neq c; \\
V_0 &\xrightarrow{J_n(\lambda)} V_{c-1}, & \text{if } i = c.
\end{align*}$$

and having primitive idempotent $e_\tau \in J_T$ corresponding to $\tau \in T$ acts by identity on $V_i$ if $\omega(z_i) \in \tau$; by zero otherwise.

These indecomposable modules are called band modules. Like string modules, they can be encoded completely by string combinatorics. Consider a walk $w = w_1 \cdots w_c$ with $s(w_1) = t(w_c)$, we can rotate it to form a new word $w_2 \cdots w_c w_1$. If this new word is also a walk, then the equivalence class of $w$ under compositions of reflections and rotations is called a band. Similar to strings, bands correspond
to closed curves on $(S, M)$. Unless otherwise specified, we will assume the representative $w = w_1 \cdots w_c$ we take from the equivalence class has $w_1$ directed and $w_c$ inverse, which matches our convention of indexing the segments of closed curves in the previous paragraph. Concatenation of strings is the natural operation inherited from concatenation of words; in particular, notations $w^n$ mean self-concatenating $n$ times. A band $w$ is primitive if $w \neq u^n$ for any subword $u$ of $w$ that is also a band; hence, primitive bands correspond to a primitive closed curve.

**Example 3.6.** Consider the fan triangulation of the Möbius strip with two marked points (left of the top row of Figure 5). It has an orientable double cover (see Definition 2.4) by an annulus as shown in Figure 9 (this is a special case of Figure 6), where the preimage $\tilde{T} := \{1, 2, 1', 2'\}$ of the triangulation $\{T, \overline{T}\}$ and the associated QP $(Q_{\tilde{T}}, \emptyset)$ are as shown.

![Figure 9](image)

**Figure 9.** A primitive closed curve $\omega = \omega_1 \omega_2 \omega_3 \omega_4$ in the double cover of the fan triangulation of the Möbius strip with two marked points and the quiver with (zero) potential associated to the orientable double cover.

There is a unique primitive closed curve $\omega = \omega_1 \omega_2 \omega_3 \omega_4 = b_2 b_1 a_2^{-1} a_1^{-1}$ on the double cover that represent the preimage of the unique quasi-arc $\overline{\omega}$. We have a band module over $J_{\tilde{T}}$ given by $M_{\lambda}(\omega^n)$ for each $n \geq 1$ and $\lambda \in k^\times$. Explicitly, the case when $n \in \{1, 2\}$ can be written as follows.

$$M_{\lambda}(\omega) = \begin{cases} k & 1 \\ k & 1 \\ k & 1 \end{cases}, \quad M_{\lambda}(\omega^2) = \begin{cases} k^2 & 1 \\ k^2 & 1 \\ [\lambda \ 0] & 1 \end{cases}.$$ 

We recall the following description of Hom-spaces between indecomposable objects.

**Proposition 3.7.** [24 Lem 3.3], [27 Cor 5.4], [30 Prop 4.3] Let $C$ be a Hom-finite Krull-Schmidt 2-CY triangulated category with cluster-tilting object $T$ with $\Lambda := \text{End}_C(T)$, and $X, Y$ be objects of $C$. Write $X = X' \oplus U[1]$ and $Y = Y' \oplus V[1]$
so that $U, V \in \text{add}(T)$ and $X', Y'$ has no direct summand in $\text{add}(T[1])$. Then the following hold.

(a) $M(X'[1]) \cong \tau M(X')$ (and likewise for $Y'$).
(b) There is an exact sequence
\[ 0 \to D \text{Hom}_\Lambda(M(Y'), \tau M(X')) \to \text{Ext}^1_C(X', Y') \to \text{Hom}_\Lambda(M(X'), \tau M(Y')) \to 0. \]
(c) There is a bifunctorial isomorphism
\[ \text{Ext}^1_C(X, Y) \cong \text{Ext}^1_C(X', Y') \oplus \text{Hom}_\Lambda(M(U), M(Y')) \oplus \text{Hom}_\Lambda(M(V), M(X')). \]

**Proposition 3.8.** For the cluster category $\mathcal{C} = \mathcal{C}_T$ associated to surface triangulation, the following hold.

(a) For any $\gamma \in \mathcal{C}_{\text{nc}}(S, M)$, $\gamma$ is rigid if and only if $\gamma \in A(S, M)$.
(b) For any curves $\gamma, \delta$ that are self-non-crossing, any $X \in \mathcal{C}$ with underlying curve $\gamma$ and any $Y \in \mathcal{C}$ with underlying curve $\delta$, $\text{Ext}^1_C(X, Y) = 0$ if and only if $\text{cross}(\gamma, \delta) = \emptyset$.

**Proof.** (b) is [7, Prop 5.3] and (a) follows from (b). \qed

4. INVOLUTION AND DUALITY

In this section, we associate to the orientable double cover of a surface to a QP with involution. We then further enhance this to a contravariant duality functor $\nabla$ on the cluster category. This provides the setup in the following sections, where we utilise the symmetric representation theory first studied in [22, 23] to write down a dictionary between the surface combinatorics and phenomena in the cluster categories.

4.1. Symmetric QP and relation to non-orientable surfaces. From now on, we assume $\Bbbk$ is an algebraically closed field with characteristic different from 2.

**Definition 4.1.** A symmetric quiver is a pair $(Q, \sigma)$ of a quiver $Q$ equipped with an involutive anti-automorphism (or simply involution) $\sigma$, i.e. $\sigma(i \xrightarrow{a} j) = \sigma(j) \xleftarrow{\sigma(a)} \sigma(i)$ for all arrows $a \in Q_1$ with $\sigma^2 = \text{id}$. This is equivalent to saying that $\sigma$ defines an algebra isomorphism, which we denote by $\sigma$ again by abusing notation, $\sigma : kQ^{op} \to kQ$. A symmetric QP is a tuple $(Q, W, \sigma)$ such that $(Q, \sigma)$ is a symmetric quiver, $(Q, W)$ is a QP, and $\sigma(W) = W$.

**Example 4.2.**
1. The type $A_n$ quiver $1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n$ with $\sigma(i) = n + 1 - i$ and $\sigma(a_i) = a_{n+1-i}$.
2. The Kronecker quiver $1 \xrightarrow{a} a' 1'$ has two choices of involutions - one of them fixes the arrows while the other swap $a$ and $a'$.
3. Consider the following QP $(Q, W)$ in Example 3.1. Take $\sigma(1) = 3$ and $\sigma(2) = 4$, then this defines an involution $\sigma$ on $(Q, W)$. This induces an algebra isomorphism on the Jacobian algebras as $\sigma(W) = \sigma(a_3)\sigma(a_2)\sigma(a_1) + \sigma(\beta_1)\sigma(\beta_2)\sigma(\beta_3) = \beta_3\beta_2\beta_1 + \alpha_1\alpha_2\alpha_3 = W$. 

(4) Suppose \((Q, W)\) is a QP. Take \(\tilde{Q} := Q \sqcup Q^{\text{op}}\) and \(\tilde{W} := W + W^{\text{op}}\), where \(W^{\text{op}}\) is the linear combination of cycles on \(Q^{\text{op}}\) given by reversing the cycles in \(W\). Let \(\sigma\) be the automorphism on \(Q\) that swaps \(i \in Q_0\) with the corresponding \(i \in Q^{\text{op}}_0\), and \(\alpha \in Q_1\) with \(\alpha^{\text{op}} \in Q^{\text{op}}_1\). Then \((\tilde{Q}, \tilde{W}, \sigma)\) is a symmetric QP.

**Definition 4.3.** A fixed-point-free symmetric QP, or FF-symmetric QP for short, is a symmetric QP \((Q, W, \sigma)\) such that both \(\sigma|_{Q_0}\) and \(\sigma|_{Q_1}\) are fixed-point-free.

**Example 4.4.** The smallest example of a FF-symmetric QP is the Kronecker quiver (Example 4.2 (2)) with \(\sigma\) swapping both vertices and swapping both arrows. More generally, for the quiver

![Diagram](image)

we have \((\tilde{A}_{n,n}, 0, \sigma)\), where \(\sigma\) swaps the unprimed and the primed, defines a FF-symmetric QP. This quiver corresponds to the fan triangulations of the Möbius strip shown in Figure 6.

Both Example 4.2 (3) and (4) are examples of FF-symmetric QP.

Let us now review the characterisation of QPs arising from orientable marked unpunctured surfaces.

**Definition 4.5.** Let \((Q, W)\) be a QP, and \(R\) be the set of monomials that appear in the cyclic derivative \(\partial_\alpha W\) with respect to some \(\alpha \in Q_1\). We say that \((Q, W)\) is gentle if

1. \((Q, W)\) is Jacobi-finite, i.e. \(J_{Q,W}\) is finite-dimensional;
2. for each \(v \in Q_0\), there are at most 2 in-coming arrows and at most 2 out-going arrows;
3. For all \(\alpha \in Q_1\), there is at most one \(\beta \in Q_1\) such that \(\beta \alpha \in R\) and at most one \(\beta' \in Q_1\) such that \(\beta' \alpha \notin R\);
4. For all \(\alpha \in Q_1\), there is at most one \(\gamma \in Q_1\) such that \(\alpha \gamma \in R\) and at most one \(\gamma' \in Q_1\) such that \(\alpha \gamma' \notin R\).

This is, by definition, equivalent to saying that the Jacobian algebra \(J_{Q,W}\) is a (finite-dimensional) gentle algebra.

**Proposition 4.6.**[6] For any marked unpunctured orientable surface \((S, M)\) equipped with a triangulation \(T\), \((Q_T, W_T)\) is a gentle QP.

Keeping in the mind the relation between non-orientable surface and its double covering, we can extend Proposition 4.6 as follows.

**Proposition 4.7.** For a marked unpunctured (not necessarily orientable) surface \((S, M)\) equipped with a triangulation \(T\). Let \((\tilde{S}, \tilde{M})\) be an orientable double
cover of \((S, M)\), and \(\tilde{T}\) be the associated double cover of \(T\). Then we have a fixed-
point-free symmetric gentle QP \((Q_{\tilde{T}}, W_{\tilde{T}}, \sigma)\) where \(\sigma\) is induced by the restriction
of \(\sigma_S\) to \(\tilde{T}\).

Remark 4.8. One can consider relaxing surfaces equipped with ‘marked unpunc-
tured points with triangulations’ to surfaces equipped with ‘a pair of dual cellular
dissections’ in the sense of [31]. Then ‘symmetric gentle QP’ can be replaced by
locally gentle algebra with involution (equivalently, locally gentle quiver with sym-
metric structure in the obvious sense). The proof of this generalisation is analogous
to the one presented below and we omit them for simplicity as these settings is
beyond the scope of this script.

\begin{proof}
Since \(\sigma : \tilde{S} \rightarrow \tilde{S}\) only fixes some (or none if \(S\) is already orientable) of
the closed curves on \(\tilde{(S, M)}\), it acts transitively on the set \(A_{\tilde{(S, M)}}\) of arcs, which
means that the induced \(\sigma\) on \(Q_{\tilde{T}}\) is a fixed-point free involution. The claim then
follows from Proposition 4.6. \(\square\)
\end{proof}

Example 4.9. (1) Let \((S, M)\) be the Möbius strip with 1 marked point. This is
a unique triangulation whose double cover \(\tilde{T}\) defines the Kronecker quiver with
fixed-point free involution as described in Example 4.2 (2).

(2) The \(A_{n, n}\) symmetric QP in Example 4.4 is associated to the orientable
double cover of ‘fan triangulation’ on the Möbius strip as shown in Figure 6.

4.2. Duality functor associated to involution. Suppose \((Q, W, \sigma)\) is a sym-
metric QP. Let \(\Lambda := J_{Q, W}\) be the associated Jacobian algebra. We denote by \(-\sigma\)
the algebra map \(\Lambda^{op} \rightarrow \Lambda\) given by \(\alpha_1 \alpha_2 \cdots \alpha_\ell \mapsto (-1)^\ell \sigma(\alpha_\ell) \sigma(\alpha_{\ell-1}) \cdots \sigma(\alpha_1)\) for
all paths \(\alpha_1 \cdots \alpha_\ell\) of length \(\ell\). Consider the \(\Lambda-\Lambda^{op}\)-bimodule \(1_{\Lambda-\sigma}\) whose underly-
ing space is \(\Lambda\) with the natural left \(\Lambda\)-action and with the right \(\Lambda^{op}\)-action given
by the \(m \cdot a := -\sigma(a)m\) for all \(a \in \Lambda^{op}, m \in 1_{\Lambda-\sigma}\). Note that subtle choice of
putting the minus twist on \(\sigma\) comes from the use of ‘symmetric representations’
(see next section) of these algebras. Anyway, the bimodule is invertible of order 2
and defines an equivalence \(- \otimes_{\Lambda} 1_{\Lambda-\sigma} : \text{mod} \Lambda \rightarrow \text{mod} \Lambda^{op}\). Composing with the
\(k\)-linear dual \((\cdot)^* := \text{Hom}_k(\cdot, k)\) yields a contravariant equivalence
\(\nabla : \text{mod} \Lambda \cong \text{mod} \Lambda\) such that \(\nabla^2 \cong \text{Id}\).

We call this the \textit{duality} (associated to \(\sigma\)). Note that if we denote by \(M_\alpha\) the
transformation on \(M \in \text{mod} \Lambda\) representing \(\alpha \in Q_1\), then \((\nabla M)_{\sigma(\alpha)} = -M_\alpha^*\).

Example 4.10. Consider the module \(M = M_\Lambda(\omega)\) from Example 3.6. Then,

\[
\nabla M = \begin{array}{c}
\k 1 & \k 1 \\
\k \omega & \k \omega \\
\k -1 & \k -1 \\
\end{array}
\]

We have the following observation.
Proposition 4.11. The following hold.

(a) We have an equivalence $\nabla: \text{proj} \Lambda \xrightarrow{\sim} \text{inj} \Lambda$ sends $e\Lambda$ to $D(\Lambda \sigma(e))$. In particular, there is a natural isomorphism $\nabla \circ \nu^\pm \cong \nu^\mp \circ \nabla$, where $\nu := - \otimes_A DA$ is the Nakayama functor.

(b) For any $M \in \text{mod} \Lambda$, denote by $P_M^*$ the minimal projective presentation of $M$, and $I_M^*$ the minimal injective copresentation of $M$. Then we have $P_{\nabla M}^* = \nabla(I_M^*)$ and $I_{\nabla M}^* = \nabla(P_M^*)$.

(c) [22, Prop 3.4] There is a natural isomorphism $\nabla \circ \tau^\pm \cong \tau^\mp \circ \nabla$, where $\tau$ denotes the Auslander-Reiten translation ($\text{AR}$-translation for short).

Proof. (a) is just stating the algebra isomorphism $\sigma: \Lambda^{op} \to \Lambda$ categorically. (b) follows from (a). (c) follows by combining (a) and (b), as $\tau M$ is the kernel of $\nu(P_M^*)$ and $\tau^\mp M$ is the cokernel of $\nu(I_M^*)$.

We can lift this duality to the cluster category. This is the first justification of $\nabla$ being a categorification of the defining involution of a non-orientable surface from its double cover. Recall from Section 3 that there is a canonical projection $M(-): C_{Q,W} \to \text{mod} J_{Q,W}$.

Proposition 4.12. Suppose $(Q,W,\sigma)$ is a Jacobi-finite symmetric QP. Then $\sigma$ induces a contravariant exact duality $\nabla$ on $C = \overline{C_{Q,W}}$ so that we have a commutative diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\nabla} & C \\
\downarrow{M(-)} & & \downarrow{M(-)} \\
\text{mod } J & \xrightarrow{\nabla} & \text{mod } J.
\end{array}
\]

Proof. Recall that $C$ is defined as the quotient $\text{per}(\Gamma)/D_{id}(\Gamma)$, where $\Gamma$ is the Ginzburg dg-algebra associated to a certain quiver with potential $(Q,W)$, $\text{per}(\Gamma)$ the perfect derived category, and $D_{id}(\Gamma)$ the full subcategory of the derived category supported at totally finite-dimensional (dg) $\Gamma$-modules. Here, $W$ is the sum of the form $\sum_{i,j} (i,j) = (i,j)$, one for each internal triangle of $\tilde{T}$ where $\alpha \beta \gamma$ is (a choice of) a length 3 path given by bouncing inside the triangle. Recall also that the underlying graded algebra structure of $\Gamma$ is given by $kQ$ where $Q$ is the quiver with the same set of vertices as $Q$, and the set of arrows is $Q_1 \cup Q_1^* \cup Q_0$ with $Q_0$ representing the set of loops $t_i$ of degree $-2$ for $i \in Q_0$ and $Q_1^* := Q_1^{op}$ is the set of ‘dual arrows’ $\alpha^*$ of degree $-1$ (for each $\alpha \in Q_1$). The differential of $\Gamma$ is given by $d(\alpha) = 0$ and $d(\alpha^*) = \partial_\alpha W$ for all $\alpha \in Q_1$, and $d(t_i) = e_i(\sum_{\alpha^*}[\alpha, \alpha^*])e_i$ for all $i \in Q_0$.

By abuse of notation, define $\sigma: \Gamma \to \Gamma^{op}$ by extending that of the original one $\sigma: kQ \to kQ^{op}$. So we have $\sigma(\alpha^*) = \sigma(\alpha)^*$ for all $\alpha^* \in Q_1^*$ and $\sigma(t_i) = -t_{\sigma(i)}$. This clearly defines an isomorphism of graded algebra. We claim that $\sigma$ is also a chain map (and so $\sigma$ is a dga isomorphism). Indeed, it is clear that $\sigma d(Q_1) = 0 = d(\sigma(Q_1))$. For $\alpha^* \in Q_1^*$, we have

\[
d(\sigma(\alpha^*)) = d(\sigma(\alpha)^*) = \partial_{\sigma(\alpha)}(W) = \sigma(\partial_\alpha(W)) = \sigma d(\alpha^*)
\]
by the assumption of \((Q, W, \sigma)\) being symmetric. For \(t_i\)'s, we have
\[
d(\sigma(t_i)) = -d(t_{\sigma(i)})
\]
\[
= -e_{\sigma(i)} \left( \sum_{a} [a, a^*] \right) e_{\sigma(i)}
\]
\[
= -\sigma(e_i) \left( -\sum_{a} [a^*, a] \right) \sigma(e_i) = \sigma d(t_i).
\]

Define \(\nabla := D(- \otimes \Gamma \Gamma^\op - \sigma)\), where \(\Gamma^\op \Gamma\) is the \(\Gamma\)-\(\Gamma\) op-dg-bimodule given by \(\Gamma\) equipped with the natural left \(\Gamma\)-action and the right \(\Gamma\) op-action is given by the dga isomorphism \(\Gamma^\op \to \Gamma\) sending \(a\) to \(-\sigma(a)\). Both \((- \otimes \Gamma \Gamma^\op - \sigma)\) and \(D(-)\) are exact equivalence that restricts to equivalence on the respective bounded derived categories. Hence, this induces the require equivalence on \(\mathcal{C} = \text{per}(\Gamma)/\mathcal{D}_{\text{id}}(\Gamma)\).

Note also that restricting \(\nabla\) to \(\text{add}(\Gamma[1]) \subset \text{per}(\Gamma)\) yields an equivalence \(\nabla : \text{add}(\Gamma[1]) \xrightarrow{\sim} \text{add}(\Gamma[-1])\). Since \(\mathcal{C}\) is 2-CY, \(\nabla\) is an equivalence on the full subcategory \(\text{add}(\Gamma[1]) \subset \mathcal{C}\), where, by abusing notation, \(\Gamma\) here is the image of \(\Gamma \in \text{per}(\Gamma)\) in \(\mathcal{C}\). Thus, we have an induced duality \(\nabla : \mathcal{C}/[\Gamma[1]] \xrightarrow{\sim} \mathcal{C}/[\Gamma[1]]\). Note that \(\Gamma\) is the cluster-tilting object of \(\mathcal{C}\) with endomorphism ring \(J\), so it remains to see that the induced duality coincide with the one naturally defined on \(\text{mod } J\) via \(\sigma_J : \text{mod } J \to \text{mod } J\).

\[\square\]

Remark 4.13. Note that the exactness here means that \(\nabla \circ [\pm 1] \simeq [\mp 1] \circ \nabla\) and a triangle \((X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]) \in \mathcal{C}\) is sent to \((\nabla(X[1]) \xrightarrow{\nabla(h)} \nabla(Z) \xrightarrow{\nabla(g)} \nabla(Y) \xrightarrow{\nabla(f)} \nabla(X))\).

5. Symmetric representations vs curves

We view the duality \(\nabla\) as the categorification of orientation-reversing automorphism \(\sigma = \sigma_\sigma\) on a triangulated surface \(S\). To categorify curves - in particular, quasi-arcs - of a non-orientable surface arising as \(S/\sigma\), we need to makes sense of indecomposability for the \(\nabla\)-orbit of an indecomposable object. Classical orbit category construction does not work well in this setting as \(\nabla\) is a contravariant duality; fortunately, a resolution called symmetric representation has recently been suggested in the literature [22, 23]. As in the previous section, we assume the underlying field \(k\) is of characteristic not equal to 2.

5.1. Symmetric representations.

Definition 5.1. Suppose now that the characteristic of the underlying field \(k\) is not 2 and \(\Lambda \cong kQ/I\) is an algebra equipped with an involution \(\sigma\) such that \(\sigma(I) = I\) (such as the Jacobian algebra of a symmetric QP). Let \(\varepsilon \in \{+1, -1\}\). By an \(\varepsilon\)-form on a \(k\)-vector space \(V\) we mean a bilinear form \(\langle -, - \rangle\) that is symmetric
when $\varepsilon = +1$, and skew-symmetric when $\varepsilon = -1$. An $\varepsilon$-representation $(M, \langle -, - \rangle)$ of $(\Lambda, \sigma)$ is a $\Lambda$-module equipped with a bilinear form on $M$ such that

1. $\langle -, - \rangle$ is a non-degenerate $\varepsilon$-form;
2. $\langle -, - \rangle|_{M_i \times M_j} \neq 0$ implies $j = \sigma(i)$, where $M_i := Me_i$ for the primitive idempotent $e_i$ corresponding to $i \in Q_0$;
3. $\langle v, w \rangle + \langle v, w \sigma(\alpha) \rangle = 0$ for all arrow $(\alpha : i \to j) \in Q_1$ and all $v \in M_i, w \in M_{\sigma(j)}$.

We also consider $\varepsilon$-representations to be symmetric representation if we do not want to emphasise the parity of $\varepsilon$.

**Example 5.2.** Consider the quiver from Example 3.1 and the module $M$ given by

$$M = \begin{bmatrix} 0 & k^2 \\ \kappa & 0 \end{bmatrix} \cong \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \oplus \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \cong \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}.$$

Define a symmetric form $\langle -, - \rangle$ on $M$ as follows.

$$\langle x, y \rangle |_{M_i \times M_j} = \begin{cases} xy, & \text{if } (i, j) = (2, 4) \text{ or } (4, 2); \\ ad + bc, & \text{if } (i, j) = (1, 3) \text{ or } (3, 1) \text{ with } x = \begin{pmatrix} a \\ b \end{pmatrix}, y = \begin{pmatrix} c \\ d \end{pmatrix}; \\ 0, & \text{otherwise}. \end{cases}$$

It is routine to check that $(M, \langle -, - \rangle)$ is a $\varepsilon$-representation for $\varepsilon = +1$.

**Remark 5.3.** Instead of the more compact terminology ‘$\varepsilon$-representations’, [22] calls them orthogonal (when $\varepsilon = +1$) and symplectic (when $\varepsilon = -1$) representations since the underlying vector space equipped with the form is an orthogonal/symplectic vector space. We follow [23] practice. We also remark that [23] uses only the complex number instead of arbitrary algebraically closed field of non-2 characteristic; the results we need from them can be argued in the latter more general setting.

The notion of direct sum of $\varepsilon$-representations is well-defined by naturally extending that of ordinary representations, i.e. the collection of matrices $(M_\alpha)_{\alpha \in Q_1}$ cannot be block-decomposed in a uniform way. This allows one to talk about the notion of $\varepsilon$-indecomposability. Notably, the Krull-Schmidt theorem applies in this context too; that is, every $\varepsilon$-representation can be written as direct sum of the $\varepsilon$-indecomposables in a unique way. As far as the application within this article is concerned, the following result from [22, 2.7], [23, 2.10] suffices to act as a substitute of the proper definition of indecomposable $\varepsilon$-representation.

**Proposition 5.4.** Let $M$ be an indecomposable $\varepsilon$-representation. Then precisely one of the following three cases occur:
(a) $M$ is 1-sided, i.e. $M$ is indecomposable as a $\Lambda$-module.
(b) $M$ is ramified, i.e. $M \cong L \oplus \nabla L$ as $\Lambda$-module for some indecomposable $\Lambda$-module $L \cong \nabla L$.
(c) $M$ is split, i.e. $M \cong L \oplus \nabla L$ as $\Lambda$-module for some indecomposable $\Lambda$-module $L \cong \nabla L$.

Moreover, for an indecomposable $\Lambda$-module $L$, it gives rise to exactly one of the three types of indecomposable $\varepsilon$-representation of the form $M$ above.

Remark 5.5. Note that what is called ‘1-sided’ here is called ‘type I’ in [22, 23].

From now on, we will often omit $\langle -, - \rangle$ from the notation as Proposition 5.4 implies that the $\varepsilon$-representation structure is determined by underlying module structure.

If we already know the classification of indecomposable ordinary modules, then Proposition 5.4 gives us a way to classify all indecomposable $\varepsilon$-representations. Namely, for each indecomposable $L \in \text{mod} \Lambda$, we first check whether or not that module is self-dual. If it is not self-dual, then we have a split $\varepsilon$-indecomposable $M = L \oplus \nabla L$. Otherwise, we check whether one can equip an $\varepsilon$-form $\langle -, - \rangle$ so that $(L, \langle -, - \rangle)$ defines a $\varepsilon$-indecomposable. If this is the case, then we have a 1-sided $\varepsilon$-indecomposable on the spot; otherwise, $L \oplus \nabla L$ can be given a structure of a ramified $\varepsilon$-indecomposable.

Example 5.6. Let us take $\varepsilon = 1$. Consider the quiver from Example 3.1 and the $\varepsilon$-representation $M$ from Example 5.2. As ordinary module we have $M = M' \oplus \nabla M'$ with the ordinary indecomposable module.

\[
M' = \begin{array}{ccc}
1 & \downarrow & 0 \\
\downarrow & \leftarrow & \leftarrow \\
1 & \downarrow & 0 \\
\end{array}
\]

Example 5.7. Consider the quiver from Example 3.1 again and also the band $\omega := \alpha_2 \beta_2^{-1}$, which defines a family of indecomposable band $\Lambda$-module $M_\lambda(\omega)$ with $\lambda \in \mathbb{K}^\times$. We have

\[
M_\lambda(\omega) = \begin{array}{ccc}
0 & \downarrow & \downarrow \\
\leftarrow & \leftarrow & \leftarrow \\
\downarrow & \downarrow & 0 \\
\end{array}
\]

and

\[
\nabla M_\lambda(\omega) = \begin{array}{ccc}
0 & \downarrow & \downarrow \\
\leftarrow & \leftarrow & \leftarrow \\
\downarrow & \downarrow & 0 \\
\end{array} \cong \begin{array}{ccc}
0 & \downarrow & \downarrow \\
\leftarrow & \leftarrow & \leftarrow \\
\downarrow & \downarrow & 0 \\
\end{array} = M_{\lambda^{-1}}(\omega),
\]
so 1-sided and ramified $\varepsilon$-indecomposables appear only when $\lambda \in \{+1, -1\}$. Define a bilinear form $\langle -, - \rangle$ on the underlying vector space given by $\begin{pmatrix} 0 & 1 \\ -\lambda & 0 \end{pmatrix}$. Then this satisfies Definition 5.1 (3) only if $\varepsilon = -\lambda \in \{+1, -1\}$.

Let $N = M_{-\varepsilon}(\omega)$, $L' = M_\varepsilon(\omega)$, and $L = L' \oplus \nabla L$, i.e.

$$
N = \begin{array}{c}
\kappa & 0 \\
0 & \left[ \begin{array}{c} 1 \\ \varepsilon \end{array} \right]
\end{array} \rightarrow \kappa, \quad L' = \begin{array}{c}
\kappa & 0 \\
0 & \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]
\end{array} \rightarrow \kappa \quad \text{and} \quad L = \begin{array}{c}
\kappa & 0 \\
0 & \left[ \begin{array}{c} 1 \\ 0 \end{array} \right]
\end{array} \rightarrow \kappa^2.
$$

Then one can find an appropriate bilinear form $\langle -, - \rangle$ so that $N$ defines a 1-sided $\varepsilon$-indecomposable, whereas $L$ defines a ramified $\varepsilon$-indecomposable.

### 5.2. Curves as indecomposable symmetric representations

Throughout this subsection, we fix the following notation. Let $\pi : (\widetilde{S}, \widetilde{M}) := (\overline{S}, \overline{M}) \xrightarrow{\sim} (S, M)$ denote the orientable double cover of a non-orientable marked unpunctured surface $(\overline{S}, \overline{M})$, and let $\sigma_S$ denote the associated orientation-reversing automorphism of $\overline{S}$. Fix a triangulation $T$ on $(S, M)$ and let $\widetilde{T}$ be the triangulation on $(\widetilde{S}, \widetilde{M})$ so that $\pi(\widetilde{T}) = T$. Let $(Q, W, \sigma)$ be the gentle symmetric QP associated to $T$, $J = J_{Q, W}$ the associated Jacobian algebra, and $C = C_{\widetilde{T}}$ the associated cluster category.

**Lemma 5.8.** Let $\gamma \in C_{\text{nc}}(\widetilde{S}, \widetilde{M})$ be a non-closed curve on $(\widetilde{S}, \widetilde{M})$. Then $\nabla(\gamma) \cong \sigma_S(\gamma)$ as object on $C$. In particular, $M(\gamma) \oplus M(\sigma_S(\gamma))$ has a structure of an indecomposable split $\varepsilon$-representation.

**Proof.** Suppose $\gamma \in \widetilde{T}$. Then $\gamma = \pi(e\Gamma)$, where $\Gamma$ is the Ginzburg dga associated to $(Q, W)$, $e$ is the primitive idempotent corresponding to $\gamma \in Q_0$, and $\pi : \text{per}(\Gamma) \rightarrow C$ is the canonical projection. It follows from the definition of $\nabla$ and $\sigma_S$ that $\nabla(e\Gamma) \cong \sigma(e\Gamma)$ and $\sigma(e)$ is the primitive idempotent corresponding to $\sigma_S(e) \in Q_0$. This immediate implies that $\nabla(\gamma) = \pi(\nabla(e\Gamma)) = \sigma_S(\gamma)$.

For any non-closed $\gamma \notin \widetilde{T}$, we show that $\nabla(M(\gamma)) \cong M(\sigma_S(\gamma))$ and the claim follows by using Proposition 4.12. Indeed, first recall that the underlying vector space of $M(\gamma)$ is $\bigoplus_{i=0}^c k\gamma(t_i)$ where $\gamma(t_i)$'s are the crossings of $\gamma$ with $\widetilde{T}$. Let $\tau_i \in \widetilde{T}$ be the arc containing $\gamma(t_i)$. By definition of $\sigma_S$, the underlying vector space of $M(\sigma_S(\gamma))$ is given by $\bigoplus_{i=0}^c k\sigma_S(\gamma)(1 - t_i)$ with $\sigma_S(\gamma)(1 - t_i) \in \sigma_S(\tau_i)$. Hence, as vector spaces $M(\sigma_S(\gamma))$ agrees with $\nabla(M(\gamma))$. Consider now the arrow $\alpha_i$ determined by $\gamma|_{(t_{i-1}, t_i)}$ and denote by $M_{\alpha_i}$ the action of $\alpha_i$ on $k\gamma(t_j) \subset M(\gamma)$ with an appropriate $j \in \{i, i - 1\}$. Then $\sigma(\alpha_i)$-action on $\nabla(M(\gamma))$ is given by $-M_{\alpha_i}^\ast$, which is just the negative of the identity map. Hence $\nabla(M(\gamma)) \cong M(\sigma_S(\gamma))$ via the map that multiplies all basis vector by $-1$. \qed
We will now determine the $\varepsilon$-indecomposables arising from $M_\lambda(\omega)$. Let us fix some notations and terminologies first. From now on until further notice, $\omega$ will always be a primitive closed curve on $(S, \mathcal{M})$. We will identify $\omega$ with its band form $\omega = \omega_1\omega_2 \cdots \omega_c$. As in Section 3, we will always assume without loss of generality that $\omega_1 = \alpha_1 \in Q_1$ is directed and $\omega_c = \alpha_c^{-1}$ is inverse; otherwise, one can rotate the indices until this criteria is met. For convenience, we will call the number $c$ the length of $\omega$. In picture, we can display $\omega$ and $M_\lambda(\omega^n)$ as follows.

\begin{equation}
\omega = (0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{c-1}} c \xleftarrow{\alpha_c} 0)
\end{equation}

\begin{equation}
M_\lambda(\omega^n) = (V_0 \xrightarrow{1} V_1 \xrightarrow{1} \cdots \xrightarrow{1} V_{c-1} \xleftarrow{\lambda_0(\lambda)} V_0).
\end{equation}

For simplicity, we let $v_i^1$ be the basis vector $\omega(z_i)^{(j)}$ of $V_i$, and take also $v_i^0 := 0$, so that $\alpha_c$-action on $V_i$ is given by $v_i^0\alpha_c = v_{i-1}^j + \lambda v_i^j$ for all $i \in \{0, 1, \ldots, c\}$. In the case when $n = 1$, we will further omit the superscript index as long as there is no confusion. All arithmetic operations on the subscript index $v_i^j$ will be taken modulo $c$.

**Lemma 5.9.** If $\omega = \sigma_S(\omega)$, then the length $c$ of $\omega$ is even, say, $c = 2r$, and for all $i \in \{1, 2, \ldots, c\}$, we have $\sigma(\alpha_i) = \alpha_{i+r}$ and $\omega_{i+r} = \sigma(\alpha_{i})^{-1}$.

**Proof.** Viewing the picture (5.1) as a $c$-gon by forgetting the orientation, $\sigma$ acts as a non-identity element of the dihedral group $\langle \rho, \beta \mid \rho^2 = 1, \beta^2 = 1, \beta \rho \beta = \rho^{-1} \rangle$. Note that $\sigma$ cannot act as the reflection $\beta$ as it will fix at least a vertex of the $c$-gon.

Suppose on the contrary that $c$ is odd, then $\sigma$ acts as $\beta\rho^k$ for some $k$ as it is of order 2. But then $\sigma$ will fix an edge of the $c$-gon, contradicting the fixed-point-free property of $\sigma$.

It remains to show that $\sigma$ acts cyclically on the $c$-gon. Suppose the contrary, i.e. the source and target of the arrow $\alpha_i := \sigma(\alpha_c)$ are $i$ and $i - 1$ respectively. Since the source of $\sigma(\alpha_{c-1})$ is given by applying $\sigma$ on the target of $\alpha_{c-1}$ (and vice versa), this means that $\sigma(\alpha_{c-j}) = \alpha_{i-j}$ for all $j$. Hence, there is some $1 \leq k \leq i$ such that $\sigma(\alpha_k) = \alpha_k$, which contradicts the fixed-point-free property of $\sigma$. \hfill \square

**Lemma 5.10.** Let $\omega$ be a primitive closed curve on $(S, \mathcal{M})$ and $n \geq 1$ be a positive integer. Then $\nabla M_\lambda(\omega^n) \cong M_\mu(\sigma_S(\omega^n))$ for some $\mu \in \{\lambda, \lambda^{-1}\}$; the same holds for the corresponding object $(\omega^n, \lambda) \in \mathcal{C}$. Moreover, the $J$-module $M_\lambda(\omega^n)$ (respectively the coloured closed curve object $(\omega^n, \lambda) \in \mathcal{C}$) is self-dual if and only if $\sigma_S(\omega) = \omega$ and $\lambda \in \{\pm 1\}$.

**Proof.** By Proposition 4.12, we only need to show for the $J$-module case. Moreover, we only need to argue the case $n = 1$ as there are exact sequences

\[0 \to M_\lambda(\omega) \to M_\lambda(\omega^{n+1}) \to M_\lambda(\omega^n) \to 0\]

for all $n$ that allows us to iteratively apply the exact equivalence $\nabla$ to get the desired result for the case when $n > 1$.
Similar to the construction of $M_\lambda(\omega)$. Let us consider a module $N_\lambda^j(\omega)$ of the form

$$u_0 \xrightarrow{1} u_1 \xrightarrow{1} \cdots \xrightarrow{1} u_{j-1} \xrightarrow{\lambda} u_j \xrightarrow{1} \cdots \xrightarrow{1} u_{c-1} \xrightarrow{1} u_0$$

where $\alpha_j$ acts by multiplying $\lambda$. Note that $M_\lambda(\omega) = N_\lambda^j(\omega)$.

We claim that, if both $\alpha_j$ and $\alpha_{j-1}$ points in the same direction (i.e. either both $\omega_j$ and $\omega_{j-1}$ are arrows or both are inverses of arrows), then $N_\lambda^j(\omega) \cong N_\lambda^{j-1}(\omega)$; otherwise, $N_\lambda^j(\omega) \cong N_\lambda^{j-1}(\omega)$. Indeed, it is enough to see locally that the commutative diagrams

$$
\begin{array}{ccc}
k & \xrightarrow{1} & k \\
\downarrow \alpha_j & & \downarrow \alpha_j \\
k & \xrightarrow{\lambda} & k
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
k & \xrightarrow{1} & k \\
\downarrow \alpha_j & & \downarrow \alpha_j \\
k & \xrightarrow{\lambda^{-1}} & k
\end{array}
$$

induce the required isomorphism.

Following the same argument in the proof of Lemma 5.8 (for the $J$-module case), we have $M_\lambda(\omega) = N_\lambda^j(\omega) \cong N_\lambda^j(\sigma_S(\omega))$ for some $j \in \{1, 2, \ldots, c-1\}$. Note that $j \neq c$ as $\sigma$ is fixed-point-free on arrows. By the claim in the previous paragraph, we have $N_\lambda^j(\sigma_S(\omega)) \cong M_\mu(\sigma_S(\omega))$ for some $\mu \in \{\lambda, \lambda^{-1}\}$. This finishes the proof of the first assertion.

To show the second part, it remains to show that $M_{\lambda,1}(\omega)$ is self-dual when $\sigma_S(\omega) = \omega$. By the argument of translating $\lambda$ to a neighbouring arrow above, we only need to show that $\sigma(\alpha_c)$ and $\alpha_c$ points in opposite direction, which is already shown in Lemma 5.9.

We are going to determine the exact values of $n$ and $\lambda$ so that $M_\lambda(\omega^n)$ has a structure of an $\varepsilon$-representation; hence, a 1-sided indecomposable $\varepsilon$-representation. Note that by Proposition 5.3 this implies that any other values of $n, \lambda$, the module $M_\lambda(\omega^n)^{\oplus 2}$ has a structure of a ramified indecomposable $\varepsilon$-representation.

From now on until the end of the section, unless otherwise specified, we will always assume the primitive closed $\omega = \sigma(\omega)$ and $\lambda \in \{\pm 1\}$; otherwise, as $M_\lambda(\omega)$ is not self-dual and so it must give rise to split $\varepsilon$-indecomposable. We keep the symbol $c$ for the length of $\omega$, and let $r$ be $c/2$, which is an integer as guaranteed by Lemma 5.9.

We will achieve our goal in two steps: Lemma 5.12 and Lemma 5.14. The first one will tell us which $M_\lambda(\omega^n)$ is an $\varepsilon$-representation and the second one tells us that all other cases cannot have $\varepsilon$-representation structure. Both steps will follow same strategy - we start by taking an $\varepsilon$-form $(\varepsilon, -)$ (for arbitrary $\varepsilon \in \{\pm 1\}$) and use conditions (2) and (3) of Definition 5.1 to obtain a list of equations on the values $b_{i,j}^{l,k}$, then we use these equations to determine the value of $\varepsilon$ and whether the form is non-degenerate form. For convenience, we also define $b_{i,j}^{l,k} := 0$ whenever one of $j, k$ is zero.
First, condition (2) of Definition 5.1 says that it is sufficient to consider only \( b_{i,r+i}^{j,k} \). Let us now write out the list of equations that condition (3) requires. Recall from Lemma 5.9 that \( \sigma(\alpha_i) = \alpha_{r+i} \) for all \( i \in \{1, 2, \ldots, c\} \). If we consider an arrow \( \alpha_i = \omega_i \), then \( \sigma(\alpha_i) = \omega_{r+i}^{-1} = \alpha_{r+i} \), and we have \( \langle v_{i-1}^j \alpha_i, v_{r+i}^k \rangle = -(v_{i-1}^j, v_{r+i}^k) \). Writing down similar equations for the cases when \( \alpha_i \neq \omega_i \), and evaluating both sides of all these equations yields the following set of conditions indexed over \( i \in \{1, 2, \ldots, c\} \) and \( j, k \in \{1, 2, \ldots, n\} \):

\[
\text{(Eq: } i, j, k \text{)} \quad \langle v_i^j, v_{r+i}^k \rangle = -\langle v_{i-1}^j, v_{r+i-1}^k \rangle \quad \forall i \in \{1, 2, \ldots, c-1\} \setminus \{r\}
\]

\[
\text{(Eq: } r, j, k \text{)} \quad \langle v_r^j, v_0^k \rangle = -\langle v_{r-1}^j, v_{r-1}^k \rangle - \lambda(v_{r-1}^j, v_{r-1}^k)
\]

\[
\text{(Eq: } c, j, k \text{)} \quad \langle v_c^j, v_0^k \rangle = -\langle v_{c-1}^j, v_{c-1}^k \rangle - \lambda(v_{c-1}^j, v_{c-1}^k)
\]

For each \( j, k \), going through (Eq: \( i, j, k \)) from \( i = 1 \) to \( i = c-1 \) says that \( (b_{i,r+i}^{j,k})_{i \in \mathbb{Z}/c\mathbb{Z}} \) is dependent only on \( b_{i,r}^{j,k} := b_{0,r}^{j,k} \), namely,

\[
(5.3) \quad b_{i,r+i}^{j,k} = \begin{cases} (-1)^i b_{i,r}^{j,k}, & \text{if } 0 \leq i < r; \\ (-1)^i (b_{i,r}^{j,k} + b_{i,r-k}^{j,k}), & \text{if } r \leq i < c. \end{cases}
\]

Putting this into (Eq: \( c, j, k \)) and rearranging the equation (also using \( \lambda^2 = 1 \) and \( c \in 2\mathbb{Z} \)) yields

\[
\text{(Eq: } j, k \text{)} \quad b_{r-1,k-1}^{j,-1} = \lambda(b_{r-1,k}^{j,-1} + b_{r,k-1}^{j,-1}).
\]

Note that this is a null condition in the case when \( j = k = 1 \), as (Eq: \( c, 1, 1 \)) says only \( b_{0,r}^{1,1} = -\lambda b_{r-1,r-1}^{1,1} \), which is already guaranteed by (5.3).

Summarising what we have so far:

**Lemma 5.11.** \( M_\lambda(\omega^n) \) is a 1-sided indecomposable \( \epsilon \)-representation if and only if there is a non-degenerate \( \epsilon \)-form \( \langle -,- \rangle \) such that \( b_{i}^{l,k} := \langle v_i^j, v_k^l \rangle = 0 \) for all \( l \neq r+i \), and the equations (5.3) and (Eq: \( j, k \)) hold for all \( i, j, k \).

**Lemma 5.12.** The following hold.

(a) The indecomposable module \( M_\lambda(\omega) \) is an \( \epsilon \)-representation if and only if \( \epsilon = (-1)^r\lambda \).

(b) The indecomposable module \( M_\lambda(\omega^2) \) is an \( \epsilon \)-representation if and only if \( \epsilon = (-1)^{r+1}\lambda \).

**Proof.** (a) We define a bilinear form so that its values on the basis is given by

\[
\langle v_i, v_j \rangle := \begin{cases} (-1)^i, & \text{if } 0 \leq i < r \text{ and } j = r+i; \\ (-1)^i \lambda, & \text{if } r \leq i < c \text{ and } j = r+i; \\ 0, & \text{otherwise}. \end{cases}
\]

It is straightforward to check that this is a non-degenerate \( \epsilon \)-form for \( \epsilon = (-1)^r\lambda \). Every bilinear form satisfying (3.2) will be a scalar multiple of the one here (which has \( b^{1,1} = 1 \)). The assertion follows.
(b) We first show the only-if direction. By (5.3), we have $b_{r+1,i}^{2,1} = (-1)^{r+1} \lambda b_{r+2,1}^{2,1}$ and $b_{r+1,i}^{1,2} = (-1)^{r} b_{r+2,1}^{1,2}$ for all $0 \leq i < r$. If $j \neq k$, then (Eq. 5.3) says that $\lambda b_{1,1}^{1,2} = 0$. Now substituting $b_{1,1}^{1,2} = 0$ into (Eq. 5.3) for the case when $j = k = 2$, we get that $b_{r+1,i}^{2,1} = (-1)^{r} b_{r+2,1}^{1,2}$. Hence, we have $b_{r+1,i}^{2,1} = (-1)^{r+i+1} \lambda b_{r+2,1}^{1,2}$ for $0 \leq i < r$.

By definition, $b_{r+1,i}^{k,j}$ defines an $\varepsilon$-form if and only if $b_{r+1,i}^{1,2} = \varepsilon b_{r+2,1}^{2,1}$. Substituting the equations from the previous paragraph yields

$$(-1)^{r} b_{1,r+1}^{1,2} = \varepsilon b_{r+2,1}^{2,1} = \varepsilon (-1)^{r+i+1} \lambda b_{r+2,1}^{1,2},$$

i.e. $\varepsilon = (-1)^{r+1} \lambda$ as claimed.

For the converse, define a bilinear form given by

$$\langle v_i^j, v_i^k \rangle = b_{r+1,i}^{k,j} :=
$$

$$
\begin{cases}
(-1)^{i+j}, & \text{if } j \neq k \text{ and } 1 \leq i \leq r \text{ and } l = r + i;
(-1)^{i+j} \lambda, & \text{if } j \neq k \text{ and } r < i \leq c \text{ and } l = r + i;
(-1)^{i+1} \lambda / 2, & \text{if } j = k = 2 \text{ and } 1 \leq i \leq r \text{ and } l = r + i;
(-1)^{i+1} / 2, & \text{if } j = k = 2 \text{ and } r < i \leq c \text{ and } l = r + i;
0, & \text{otherwise}.
\end{cases}
$$

Note that here we have $b_{1,1}^{1,2} = 0, b_{2,1}^{1,2} = 1, b_{2,1}^{2,1} = -1$, and $b_{2,2}^{2,1} = \lambda / 2$.

Again, it is straightforward to check that (5.3) holds. As argued before for the only-if direction, (Eq. 5.3) yields $b_{1,r+1}^{1,2} = 0$ and $b_{2,2}^{1,2} = -b_{2,1}^{1,2}$ - both of which are satisfied in our case.

Finally, as $\varepsilon = (-1)^{r+1} \lambda$, we have $b_{1,r+1}^{2,2} = (-1)^{r+1} \lambda b_{r+2,1}^{2,2}$ for all $0 \leq i < r$. Apply (5.3) to both sides yields

$$(-1)^{r} b_{2,2}^{2,2} = b_{1,r+1}^{2,2} = (-1)^{r+1} \lambda b_{r+2,1}^{2,2} = (-1)^{r+1+r+i} \lambda (\lambda b_{2,2}^{2,2} + b_{2,1}^{2,1}).$$

Since $\lambda^2 = 1$ and $b_{2,2}^{2,1} = -b_{2,1}^{1,2}$, this equation rearranges to $b_{2,2}^{2,2} = b_{2,1}^{2,1} \lambda / 2$. Hence, our bilinear form defines an $\varepsilon$-representation on $M_\lambda(\omega^2)$.

Now, we look at higher $n$.

**Lemma 5.13.** The equations (5.4) forces $b_{r+1,i}^j = 0$ for all $j + k \leq n$, and $b_{r+1,i}^j = (-1)^{j+1} b_{r+1,i}^j$ for all $1 \leq j \leq n$.

**Proof.** For $j = 1$, the equation (Eq. 5.4) says that $b_{r+1,i}^j = 0$ for all $1 \leq k \leq n$; likewise, taking $k = 1$ in equation (Eq. 5.4) yields $b_{r+1,i}^j = 0$ for all $1 \leq j \leq n$. Now we go through the equations (Eq. 5.4) starting from the $j + k = 4$ (the equation for $j + k = 2$ is null and the equations for $j + k = 3$ are included in the previous sentence) to $j + k = n + 1$. For each fixed $\ell := j + k \in \{4, \ldots, n + 1\}$, if we iterate the equations (Eq. 5.4) from $j = 1$ to $j = \ell - 1$, then in each iteration, we obtain $b_{r+1,i}^j = 0$. Hence we have the first part of the assertion. In a similar way, taking $j + k = n + 2$ in (Eq. 5.4) yields $b_{r+1,i}^{j+1} = -b_{r+1,i}^{j-1}$, and the second part of the assertion follows.

**Lemma 5.14.** If $M_\lambda(\omega^n)$ is an $\varepsilon$-representation, then the following hold.

(a) $b_{r+1,i}^j \neq 0$. 
(b) \( \varepsilon = \lambda(-1)^{n+r+1} \).
(c) \( n \leq 2 \).

In particular, \( M_\lambda(\omega^n) \oplus \nabla(M_\lambda(\omega^n)) \) is a ramified \( \varepsilon \)-indecomposable for all \( n \geq 3 \) and \( \lambda \in \{\pm 1\} \).

Proof. (a) Suppose the contrary. Then combining Lemma 5.13 yields \( b^{1,k} = 0 \) for all \( k \in \{1, \ldots, n\} \). By \((5.3)\), we then have \( \langle v_1^{(1)}, - \rangle = 0 \) for all \( i \). This contradicts the non-degeneracy of \( \langle -,- \rangle \).

(b) Since \( \langle -,- \rangle \) is an \( \varepsilon \)-form, we have \( b^{j,k} = \varepsilon b^{k,j} \). By \((5.3)\), the case \( 0 \leq i < r \) yields
\[
(-1)^i b^{j,k} = (-1)^{r+i}(\lambda b^{k,j} + b^{k,j+1}),
\]
then \( b^{j,k} = (-1)^r(\lambda b^{k,j} + b^{k,j+1}) \).

Note that if \( j + k = n+1 \), then \( b^{j,k} = 0 \) in these cases become \( (-1)^{j+k+1} b^{1,n} = \varepsilon(-1)^{r}(\lambda(-1)^{n-j} b^{1,n}) \), rearranging yields
\[
b^{1,n} = \varepsilon \lambda(-1)^{n+r+1} b^{1,n}.
\]
Hence, by (a) and the assumption that \( \varepsilon, \lambda \in \{\pm 1\} \) we get \( \varepsilon = \lambda(-1)^{n+r+1} \) as required.

(c) By rearranging equation \((5.4)\) after substituting \( \varepsilon = \lambda(-1)^{n+r+1} \) from (b), we obtain
\[
(-1)^{n+1} b^{j,k} + b^{k,j} + \lambda b^{k,j+1} = 0.
\]
Now consider the equation with \((j,k) = (2,n)\) and with \((j,k) = (n,2)\).

When \( n > 2 \), we can multiply the latter equation by \((-1)^{n+1} \) and subtract it from the first equation, which yields
\[
(-1)^n b^{2,n-1} + b^{n,1} = 0.
\]
On the other hand, by Lemma 5.13 we can rewrite \( b^{2,n-1} = -b^{1,n} \) and \( b^{n,1} = (-1)^{n+1} b^{1,n} \) in terms of \( b^{1,n} \), which means that we have \( 2(-1)^{n+1} b^{1,n} = 0 \). Hence, we deduce that \( b^{1,n} = 0 \) - a contradiction. \( \square \)

Let us summarise our investigation so far. Recall (from the discussion after Lemma 5.8) that the length \( \text{len}(\omega) \) of a closed curve \( \omega \) is the number of intersections of \( \omega \) with the initial triangulation.

**Theorem 5.15.** Let \((J = J_{Q,W}, \sigma)\) be a FF-symmetric gentle Jacobian algebra. If \( M \) is an indecomposable \( \varepsilon \)-representation over \((J,\sigma)\), then exactly one of the following hold.

- \( M \) is a split \( \varepsilon \)-indecomposable with underlying \( J \)-module being either one of the following:
  1. **(SS)** \( M(\gamma) \oplus M(\sigma(\gamma)) \) for some curve \( \gamma \) with endpoints.

\(^1\)Note that when \( n = 2 \) the two equations are the same, which is why the remaining of the proof does not work in this case.
(SB) $M_\lambda(\delta)$ for some (not necessarily primitive) closed curve $\delta$ with $\sigma(\omega) \neq \omega$ or $\lambda \neq \pm 1$.
- $M$ is a 1-sided $\epsilon$-indecomposable with underlying $J$-module being $M_\lambda(\omega^n)$ with $\omega = \sigma(\omega)$ a primitive closed curve, $\lambda \in \{\pm 1\}$, $n \leq 2$, and $\epsilon = (-1)^{\text{len}(\omega)/2+n-1}\lambda$.
- $M$ is a ramified $\epsilon$-indecomposable.

Proof. Proposition 5.4 that $M$ is a split $\epsilon$-indecomposable if and only if $M = N \oplus \nabla N$ for some indecomposable non-self-dual $J$-module $N \cong \nabla N$. Thus, the first case follows from Lemma 5.8 and Lemma 5.10 as they combine to say that non-self-dual indecomposable $J$-modules are precisely those that satisfy the condition (SS) or (SB).

On the other hand, Proposition 5.4 says that 1-sided $\epsilon$-indecomposable are given by self-dual indecomposable $J$-module that can be equipped with an $\epsilon$-representation structure. By Lemma 5.14, such an indecomposable $J$-module must be of the form given in Lemma 5.12. Now the rest of the claim follows. \hfill \square

6. Categorifying quasi-triangulations

In this section, we lift our work from Section 5 to the cluster category. Recall that when $(S, M)$ is orientable with triangulation $T$, triangulations are categorified by cluster-tilting objects in the cluster category $C_T$. We will formulate a symmetric representation theoretic analogue of cluster-tilting to categorify quasi-triangulations for the case when $(S, M)$ is non-orientable.

We will use the same notations as in the previous section, that is:
- $(S, M)$ is a non-orientable unpunctured marked surface with $T$ a triangulation on it.
- $(S, M)$ is the double cover of $(S, M)$ with Deck transformation group $\langle \sigma_S \rangle$ and $\tilde{T}$ be the induced double cover of $T$.
- $(Q, W, \sigma)$ is the symmetric QP associated to $(\tilde{T}, \sigma_S)$, $C := C_{(Q, W)}$ is the associated cluster category, and $M(\cdot) : C \to \mod J$ be the projection on the module category of $J = J_{Q, W} \cong \End_C(\Gamma)$, where $\Gamma$ is the cluster-tilting object of $C$ given by $\tilde{T}[-1] := \bigoplus_{\gamma \in \tilde{T}} \gamma[-1]$.
- $\epsilon \in \{\pm 1\}$ and by $\epsilon$-representation we always mean $\epsilon$-representation of $J$ with respect to the involution $\sigma$.

6.1. Lifting $\epsilon$-representation theory to the cluster category. Before going into the definitions, note that an $\epsilon$-representation $M$ comes with a canonical isomorphism $\psi_M : \nabla M \xrightarrow{\sim} M$ such that $\nabla(\psi_M) = \epsilon\psi_M$, and conversely, specifying such an isomorphism on an ordinary representation $M$ is equivalent to specifying an $\epsilon$-representation structure; see [23, Sec 2.3]. We can use this idea to lift symmetric representations to the cluster category $C$ as follows.

Definition 6.1. Let $X \in C = C_{(Q, W)}$ and write $X = T'[1] \oplus Y$ a decomposition with $T'[1]$ the maximal direct summand of $X$ in $\add(\tilde{T}[1])$. 
(i) We say that $X$ is an $\varepsilon$-object if there is an isomorphism $\psi_X : \nabla X \xrightarrow{\sim} X$ such that $\nabla \psi_X = \varepsilon \psi_X$. If, moreover, either one of the following cases occur:
- $T' = 0$ and $M(Y) \in \text{mod } J$ is an indecomposable $\varepsilon$-representation of $J$,
- $Y = 0$ and $T' = \nabla(\alpha) \oplus \alpha$ for some arc $\alpha \in \tilde{T}$,
then we call $X$ an indecomposable $\varepsilon$-object (or $\varepsilon$-indecomposable for short). To distinguish the two cases, we will call them non-initial and initial respectively.

(ii) Suppose $X$ is an indecomposable $\varepsilon$-object. The notion split, 1-sided, ramified in Proposition 5.4 extends naturally to $\varepsilon$-indecomposable objects by regarding the case $X = T' = \nabla(\alpha) \oplus \alpha$ as split.

(iii) For an $\varepsilon$-indecomposable $X$, we define $\varepsilon$-factors of $X$ as follows.
- If $X = \gamma \oplus \nabla(\gamma)$ is split or ramified, then an $\varepsilon$-factor of $X$ is either $\gamma$ or $\nabla(\gamma)$.
- If $X$ is one-sided with $X = (\omega, \lambda)$ for some primitive closed curve $\omega = \omega_1 \cdots \omega_c$, then an $\varepsilon$-factor of $X$ is $(\omega, \lambda)$.
- If $X$ is one-sided with $X = (\omega, \lambda)$ for some primitive closed curve $\omega = \omega_1 \cdots \omega_{2r}$, then an $\varepsilon$-factor of $X$ is a non-closed curve $\alpha = \omega_{i+1} \cdots \omega_{i+r-1}$ so that $\omega_i$ is an inverse arrow (or equivalently $\omega_{i+r}$ is a direct arrow).

**Example 6.2.** Consider an initial arc $\alpha \in \tilde{T}$. Under the correspondence (3.1), we have an object $X := \alpha \oplus \sigma_S(\alpha) \in \mathcal{C}$, and the involution $\sigma$ on $(Q, W)$ induces an isomorphism $\psi_{\alpha} : \nabla(\alpha) \rightarrow \sigma_S(\alpha)$. Then we have

$$\psi_X := \begin{pmatrix} 0 & \psi_{\alpha} \\ \varepsilon \psi_{\sigma\alpha} & 0 \end{pmatrix} : \alpha \oplus \nabla(\alpha) \rightarrow \sigma_S(\alpha) \oplus \alpha$$

that satisfies $\nabla(\psi_X) = \varepsilon \psi_X$. Every initial $\varepsilon$-object arise this way.

Let us give a more concrete example as well.

**Example 6.3.** Consider the quiver from Example 3.1. Let $S_x$ denotes the simple module corresponding to vertex $x \in Q_0$. Since $\nabla(S_1) \cong S_{1'}$, $S_1 \oplus S_{1'}$ is a split $\varepsilon$-indecomposable representation, and we have an isomorphism $\psi_{M(X)} : S_1 \oplus S_{1'} \rightarrow S_1 \oplus S_{1'}$ that satisfies $\nabla(\psi_{M(X)}) = \varepsilon \psi_{M(X)}$.

Let $X \in \mathcal{C}$ be the lift of $S_1 \oplus S_{1'}$, i.e. $M(X) = S_1 \oplus S_{1'}$, and $\psi_X \in \mathcal{C}$ be the lift of $\psi_{M(X)}$. Then $(X, \psi_X)$ defines a split $\varepsilon$-indecomposable of string type.

Consider the band module $M_\lambda(\omega)$ in Example 3.6 associated to the primitive band $\omega$. By Theorem 6.13 this is a 1-sided $\varepsilon$-indecomposable if $\lambda = \varepsilon$; ramified if $\lambda = -\varepsilon$. Hence, $(\omega, \varepsilon)$ is a 1-sided $\varepsilon$-indecomposable object in $\mathcal{C}$ and $(\omega, -\varepsilon) \oplus (\omega, -\varepsilon)$ is a ramified $\varepsilon$-indecomposable object in $\mathcal{C}$.

As an application Theorem 5.18 we have the following correspondences between curves on $(S, M)$ with $\varepsilon$-indecomposables in $\mathcal{C}$.
Theorem 6.4. For every curve $\gamma \in C(S, M)$, fix a lift $\overline{\gamma} \in \tilde{C}(S, M)$.

(1) If $\gamma \in C_{nc}(S, M)$, then correspondence (3.1) induces the following bijection

\[
\begin{align*}
\{ \text{curves connecting marked points} \} & \leftrightarrow \{ \varepsilon\text{-indecomposable objects of split string type} \} \\
\gamma & \mapsto \overline{\gamma} \oplus \nabla \overline{\gamma}.
\end{align*}
\]

(2) If $\omega \in C_{cc}^1(S, M) = C_{1st}(S, M) \sqcup C_{2st}(S, M)$ is a primitive closed curve, then there are the following bijections

\[
\begin{align*}
\{ \text{primitive 1-sided closed curves} \} & \leftrightarrow \{ \varepsilon\text{-indecomposable objects of 1-sided primitive band type} \} \\
\omega & \mapsto (\overline{\omega}, \varepsilon(-1)^{\text{len}(\overline{\omega})/2}), \\
\{ \text{primitive 2-sided closed curves} \} & \leftrightarrow \{ \varepsilon\text{-indecomposable objects of ramified or split primitive band type} \}/\sim \\
\omega & \mapsto [(\overline{\omega}, \lambda) \oplus \nabla(\overline{\omega}, \lambda)],
\end{align*}
\]

where the equivalence $\sim$ in the last row is defined by $(\omega, \lambda) \oplus \nabla(\omega, \lambda) \sim (\omega', \lambda') \oplus \nabla(\omega', \lambda')$ if $\omega'$ is one of $\omega$ or $\nabla(\omega)$.

Note that the equivalence is effectively forgetting the colouring on the closed curves.

Proof. Suppose $\overline{X}$ is an indecomposable object in $\mathcal{C}$. If $\overline{X}$ is initial, then the $\overline{X} \oplus \nabla(\overline{X})$ is an $\varepsilon$-object as discussed in Example 6.2. If $\overline{X}$ is non-initial, then $M(\overline{X})$ is an indecomposable $J$-module. This gives rise to an indecomposable $\varepsilon$-representation $M(X)$ for some $X \in \mathcal{C}$, and hence an isomorphism $\psi_{M(\overline{X})}$ in $\text{mod} J$ satisfying $\nabla(\psi_{M(\overline{X})}) = \varepsilon\psi_{M(\overline{X})}$. Thus, by Proposition 4.12, this lifts to an isomorphism $\psi_X$ in $\mathcal{C}$ satisfying $\nabla(\psi_X) = \varepsilon\psi_X$. Hence, non-initial $\varepsilon$-indecomposable objects in $\mathcal{C}$ are in bijection with indecomposable $\varepsilon$-representations of $J$.

For indecomposable $\varepsilon$-representations of $J$ that come from string modules, Theorem 5.15 says that they are always of the form $M(\gamma) \oplus M(\sigma(\gamma))$ for some non-initial $\gamma \in C_{nc}(S, M) \setminus \tilde{T}$. Since $\sigma$-orbits of $C_{nc}(S, M) \setminus \tilde{T}$ is just $C_{nc}(S, M) \setminus T$, we get the bijection in (1).

Using the classification of one-sided indecomposable $\varepsilon$-representations in Theorem 5.15 and restricts to primitive bands, and the fact that $C_{1st}(S, M)$ is the set of fixed points under $\sigma$ in $C_{2st}(S, M)$, the bijection for $C_{1st}(S, M)$ follows.

Finally, since $C_{1st}(S, M)/\sigma = C_{1st}(S, M) \sqcup C_{2st}(S, M)$, the correspondence of the claim follows from the classification of ramified and split indecomposable $\varepsilon$-representations of the form $M_\lambda(\omega)$ for primitive $\omega$ in Theorem 5.15. □
Recall that $\text{Ext}_C^1(X,Y)$ is defined as $\text{Hom}_C(X,Y[1])$. Therefore, for each $f \in \text{Hom}_C(X,Y[1])$, we have a triangle

$$Y \to C_f \to X \xrightarrow{f} Y[1]$$

in $C$. So far as context is clear, we call the morphism $f$, the cocone $C_f$, as well as any isomorphic triangle an \textit{extension from $X$ to $Y$}. Recall also that triangulations are in bijection with cluster-tilting (i.e. maximal rigid) objects of $C$, and rigid objects are those with only split self-extension.

In the following, we give a ‘symmetric analogue’ of these notions.

**Definition 6.5.** Let $X, Y$ be $\varepsilon$-objects in $C$ and $Y'$ be an $\varepsilon$-factor of $Y$. We call $f \in \text{Hom}_C(X,Y'[1])$ an \textit{$\varepsilon$-extension} from $X$ to $Y$ if $f \circ \psi_X \circ \nabla(f) = 0$. The case when $f = 0$ is called a \textit{trivial $\varepsilon$-extension}.

We call $X$ \textit{$\varepsilon$-rigid} if every $\varepsilon$-extension from $X$ to itself is trivial, i.e. for any $\varepsilon$-factor $X'$ of $X$, a morphism $f : X \to X'[1]$ satisfies $f \circ \psi_X \circ \nabla(f) = 0$ implies that $f = 0$.

**Remark 6.6.** $\varepsilon$-extensions are closed under scalar multiple, but unlike ordinary morphisms, they are not necessarily closed under addition.

Note that, if we consider the triangle $C_f \to X \xrightarrow{f} Y'[1] \to$ with $C_f$ being the cocone of $f$, and consider also the long exact sequence induced by applying $\text{Hom}_C(\nabla(Y')[−1], −)$ to this triangle, then we can see that $f \circ \psi_X \circ \nabla(f) = 0$ is equivalent to saying that $\psi_X \circ \nabla(f)$ factors through $C_f$. In particular, we have the following commutative diagram from the octahedral axiom of triangulated categories

where every row and every column is a triangle in $C$. We call this diagram the $\varepsilon$-\textit{extension diagram associated to $f$}, and $E$ the $\varepsilon$-\textit{extension from $X$ to $Y$} by abusing terminology (as in the classical case).
Since $M(-) : \mathcal{C} \to \text{mod} J$ is a cohomological functor, to find $\varepsilon$-extensions from $X$ to $Y$ in $\mathcal{C}$, it suffices to find a commutative diagram

\[
\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M(Y') & \rightarrow & M(C_f) & \rightarrow & M(X) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M(Y') & \rightarrow & M(E) & \rightarrow & M(D_f) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M(\nabla(Y')) & \rightarrow & M(\nabla(Y')) & & & & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

in $\text{mod} J$ where all rows and columns are exact, with the first row being equivalent (as a short exact sequence) to the $\nabla$-dual of the right hand column. This is the strategy we will use to determine (non-)$\varepsilon$-rigidity throughout.

6.2. $\varepsilon$-rigidity for split and ramified $\varepsilon$-indecomposables.

Lemma 6.7. Consider an $\varepsilon$-indecomposable object $X \in \mathcal{C}$.

(i) In the case when $X$ is a split $\varepsilon$-indecomposable, we have that $X$ is $\varepsilon$-rigid if and only if it is rigid.

(ii) If $X$ is a ramified $\varepsilon$-indecomposable, then $X$ is never $\varepsilon$-rigid.

Proof. (1) Write $X = \gamma \oplus \nabla(\gamma)$ for an indecomposable object $\gamma$ and so $\text{Ext}^1_{\mathcal{C}}(X, X)$ is a direct sum of $\text{Ext}^1_{\mathcal{C}}(\delta, \delta')$ over all $\delta, \delta' \in \{\gamma, \nabla(\gamma)\}$. If $X$ is rigid, then all of these individual spaces is necessarily zero, so $X$ is clearly $\varepsilon$-rigid. Conversely, we have a non-zero morphism $f : \delta \rightarrow \delta'[1]$ for some $\delta, \delta' \in \{\gamma, \nabla(\gamma)\}$ which yields $(f \pi)\psi_X \nabla(f \pi) = 0$ for $\pi : X \rightarrow \delta$ the natural projection, and so $X$ cannot be $\varepsilon$-rigid.

(2) If $X$ is ramified, then $X \cong (\omega, \lambda)^{\oplus 2}$ for some closed curve $\omega$ and some $\lambda \in k^\times$. But there is no rigid band object $(\omega, \lambda)$, so we must have a non-zero $f : (\omega, \lambda) \rightarrow (\omega, \lambda)[1]$, and apply the same argument as the converse part of (1) to see that $X$ is not $\varepsilon$-rigid. \qed

Lemma 6.8. Suppose $\omega = \sigma(\omega)$ is a primitive closed curve such that $(\omega^2, \lambda) \in \mathcal{C}$ is a 1-sided $\varepsilon$-indecomposable object. Then $X$ is not $\varepsilon$-rigid.

Proof. Recall that there is a functor $G_\omega : \text{mod} k[x, x^{-1}] \rightarrow \text{mod} J$ so that the full subcategory of modules of the form $M_\lambda(\omega^\alpha)$ can be understood from the representation theory of $k[x, x^{-1}]$-module; see for example from [32 II.3, II.4]. We have $M_\lambda(\omega^\alpha) = G_\omega(V_\lambda^\alpha)$ corresponds to the indecomposable $k[x, x^{-1}]$-module $V_\lambda^\alpha$. 

where \( x \) acts as the \( \lambda \)-Jordan block of size \( n \). For any \( n \geq 1 \), let \( v_n \) be the generator of \( V^\lambda_n \), and so \( V^\lambda_n \) have basis \( \{(x - \lambda)^k v_n \}_{1 \leq k \leq n} \). Define \( \iota_n : V^\lambda_n \to V^\lambda_{n+1} \) to be the map \( v_n \mapsto (x - \lambda)v_{n+1} \) and \( \pi_n : V^\lambda_{n+1} \to V^\lambda_n \) to be the map \( v_{n+1} \mapsto v_n \). Let \( \iota_{n,1} := \iota_n \iota_{n-1} \cdots \iota_1 \) and \( \pi_{1,n} := \pi_1 \pi_2 \cdots \pi_n \). Then we have short exact sequences

\[
\xi_n : 0 \to V^\lambda_1 \to V^\lambda_{n+1} \to V^\lambda_n \to 0,
\]

\[
\zeta_n : 0 \to V^\lambda_n \to V^\lambda_{n+1} \to V^\lambda_1 \to 0,
\]

\[
\rho_n : 0 \to V^\lambda_n \to V^\lambda_{n-1} \oplus V^\lambda_{n+1} \to V^\lambda_1 \to 0;
\]

see, for example, [32, Lemma II.4.2]. This induces the following commutative diagram

where the first row is \( \xi_2 \), second row is \( \xi_3 \), third right-hand column is \( \zeta_2 \), and left-hand column is \( \zeta_3 \). Note that the commutation on the top-right square comes from the sequence \( \rho_3 \).

Consider applying \( G_\omega \) to \( \xi_1 = \zeta_1 = \rho_1 \), we have a non-split short exact sequence

\[
G(\xi_1) : \quad 0 \to M_\lambda(\omega) \to M_\lambda(\omega^2) \to M_\lambda(\omega) \to 0,
\]

which says that we can take \((\omega, \lambda)\) to be an \( \varepsilon \)-factor of \((\omega^2, \lambda)\).
By applying $G_\omega$ to the commutative diagram of $\mathbb{k}[x,x^{-1}]$-modules above yields a commutative diagram

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & M_\lambda(\omega) & M_\lambda(\omega^3) & M_\lambda(\omega^2) \\
0 & M_\lambda(\omega) & M_\lambda(\omega^4) & M_\lambda(\omega^3) \\
0 & M_\lambda(\omega) & M_\lambda(\omega) \\
0 & 0
\end{array}
\]

with all rows and columns exact, and the commutation of the top-right square comes from $G_\omega(\rho_3)$. Therefore, we can lift this diagram to an $\varepsilon$-extension diagram (as in Definition 6.5) with $C_f = (\omega^3, \lambda) \oplus t$, $D_f = (\omega^3, \lambda) \oplus t'$, $E = (\omega^6, \lambda) \oplus t''$ for some (possibly zero) $t, t', t'' \in \text{add}(\widetilde{T})$, we have that $(\omega^2, \lambda)$ is not $\varepsilon$-rigid. \[\square\]

**Example 6.9.** Let $J$ be the Kronecker algebra and $\omega$ be the unique closed curve of the annulus. Then $M_\lambda(\omega^n) = U_n \xrightarrow{J_n(\lambda)} V_n$, where $U_n = \langle u_1, \ldots, u_n \rangle$ and $V_n = \langle v_1, \ldots, v_n \rangle$ over $\mathbb{k}$. We demonstrate that $M_\lambda(\omega^2)$ is not $\varepsilon$-rigid by showing the maps involved in the commutative diagram. The first row $0 \to M_\lambda(\omega) \to M_\lambda(\omega^3) \to M_\lambda(\omega^2) \to 0$ is given by

\[
\begin{array}{cccc}
0 & U_1 & [0] & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & V_1 & [0] & 1 \\
\end{array}
\quad
\begin{array}{cccc}
U_3 & [0] & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
V_3 & [0] & 1 \\
\end{array}
\quad
\begin{array}{cccc}
U_2 & [0] & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
V_2 & [0] & 0 \\
\end{array}
\quad
\begin{array}{cccc}
0 & 0 \\
\end{array}
\]

The second row is similar with the one-column matrices having an extra zero entry on top and the 2-by-3 matrices is enlarged to 3-by-4 matrices of the form $[I, 0]$ for a 3x3 identity matrix $I$.

The rightmost column of the commutative diagram is given by the short exact sequence $0 \to M_\lambda(\omega^2) \to M_\lambda(\omega^3) \to M_\lambda(\omega) \to 0$ which can be written explicitly
The left column can be described similarly by enlarging the matrices similar to the previous paragraph. Then one can check the commutative of the symmetric extension diagram explicitly.

**Definition 6.10.** An (oriented) almost crossing from $\gamma$ to $\delta$ is an overlap (underlined and encapsulated by the $|$-separators) of the form

$$
\begin{align*}
\gamma &= [\gamma_L a_L^- L] \\
\delta' &= [\delta_L b_L] \quad \kappa = [a_R\gamma R] \\
\end{align*}
$$

where $\delta' \in \{\delta, \delta^-\}$, each of the bracketed parts can possibly be empty, $\kappa$ can be a trivial string, and $a_L, a_R, b_L, b_R$ are arrows. We denote by $ac(\gamma, \delta)$ the set of almost positive crossings from $\gamma$ to $\delta$.

If all $a_L, a_R, b_L, b_R$ exists, then we get a genuine topological crossing between the corresponding curves; see Figure 10. If, for example, $a_L$ (hence $\gamma_L$ as well) does not exist in the almost crossing, then we should modify Figure 10 so that the curve $\gamma$ starts from the marked point $m$ instead. In particular, if both $a_L$ and $b_L$ do not exist, then we get only an intersection of the curves at the marked point $m$.

**Figure 10.** An oriented (almost) crossing $d$ from $\gamma$ to $\delta$

We remark that not every intersection (up to isotopy) between $\gamma, \delta$ is encoded in $ac(\gamma, \delta) \cup ac(\delta, \gamma)$; the missing ones are called ‘crossing in a an arrow’ and ‘crossing in a 3-cycle’ in [33].

6.2.1. Homomorphisms between string modules. Let $\{x_i\}_{i=0,1,\ldots,c}$ be the canonical basis of $M(\gamma)$, i.e. $x_i = \gamma(t_i)$ in the notation of Section 3.

Likewise, let $\{y_j\}_{j \in J}$ (for some finite indexing set $J$) be the basis elements of $M(\delta)$. 

If there is an overlap $\kappa = \gamma_a \cdots \gamma_{a+t} = \delta_b \cdots \delta_{b+t}$ between the (string form of) $\gamma$ and $\delta$, then we have $x_{a-1}, y_{b-1} \in s(\gamma_a) = s(\delta_b), x_{a+t}, y_{b+t} \in t(\gamma_{a+t}) = t(\delta_{b+t})$. If $\kappa$ is the overlap specified by $d \in ac(\gamma, \delta)$, we write $x_i \sim_d y_j$ when $x_i = x_{a+k}$ and $y_j = y_{b+k}$ for some $k \in \{-1, 0, \ldots, t\}$. In this case, we have a module homomorphism

$$f_d : M(\gamma) \to M(\delta)$$

$$x_i \mapsto \begin{cases} y_j & \text{if } x_i \sim_d y_j; \\ 0 & \text{otherwise.} \end{cases}$$

We can regard $f_d$ as a counter-clockwise turning of subinterval $\omega$ of the curve $\gamma$; see Figure [10].

The following result of Crawley-Boevey says that $f_d$ defines a canonical basis of Hom-space between string modules.

**Theorem 6.11.** [34] For two (possibly the same) strings $\gamma$ and $\delta$, the set $\{f_d \mid d \in ac(\gamma, \delta)\}$ is a basis of $\text{Hom}_J(M(\gamma), M(\delta))$.

### 6.2.2. Homomorphism between a string and a band module

A similar result exists for Hom-spaces involving band modules. In the scope of this article, we only need maps involving primitive band $\omega$, where the associated Hom-spaces have much simpler descriptions than the non-primitive ones. We refer the interested reader to the original paper of Krause [35], or to [36] for which the following formulation is based on.

Suppose $\omega = \omega_1 \omega_2 \cdots \omega_c$ is a primitive band. Let $\{x_i\}_{i=0,2,\ldots,c-1}$ be the basis of $M_\lambda(\omega)$ where $x_i := \omega(t_i+1) \in s(\omega_{i+1})$ in the notation of Section [3] We repeatedly self-concatenate $\omega$ to form a bi-infinite string $\omega^\infty := \cdots \omega \omega \omega \cdots$. This yields an infinite-dimension string module $M(\infty \omega^\infty) = \bigoplus_{i \in \mathbb{Z}} k \hat{x}_i$ and a surjection

$$\pi_\omega : M(\infty \omega^\infty) \to M_\lambda(\omega)$$

$$\hat{x}_{i-cm} \mapsto \lambda^m x_i \quad \text{for all } 0 \leq i < c \text{ and } m \in \mathbb{Z}.$$

The map $p : \omega_i \mapsto \omega_{i-c}$ defines a $\langle p \rangle \cong \mathbb{Z}$-action called period-shifting on $\infty \omega^\infty$. The definition of almost crossing extends naturally to bi-infinite string, in which case, the $\mathbb{Z}$-action on $\infty \omega^\infty$ induces a $\mathbb{Z}$-action on $ac(\infty \omega^\infty, \delta)$ and on $\text{ac}(\delta, \infty \omega^\infty)$ for any (finite) string $\delta$.

As before, we take $\{y_j\}_{j \in J}$ to be the basis of $M(\delta)$ for a (finite) string $\delta$. For an almost crossings $d \in ac(\delta, \infty \omega^\infty)$, we have a homomorphism of modules

$$g_d^{bs} : M(\delta) \to M_\lambda(\omega)$$

$$y_j \mapsto \pi_\omega f_d(y_j) = \lambda^m x_i \text{ where } y_j \sim_d \hat{x}_{i-cm} \text{ with } 1 \leq i \leq c \text{ and } m \in \mathbb{Z}.$$

We can write the $g_d^{bs}$ in matrix form with respect to the canonical bases:

$$(6.1) \quad g_d^{bs} = \begin{pmatrix} 0 & \lambda^d f_{d,0} & \lambda^{d-1} f_{d,1} & \cdots & \lambda^{d-l} f_{d,l} & 0 \end{pmatrix}$$
where each $f_{a,m}$ for $0 \leq m \leq l$ is (the $K$-linear map corresponding to) a $c$-by-$c$ matrix and $f_{a,m}$ for $0 < m < l$ are the identity matrices. Here the exponent $a$ is determined by the smallest index $i - dc$ with $0 \leq i < c$ and $x_{i-de} \in \text{Im}(g_{d}^{bs})$.

One checks that $g_{p(\delta)}^{bs} = \lambda g_{d}^{bs}$ for all $k \in \mathbb{Z}$. This means that the $\mathbb{Z}$-orbit $[d] \in ac(\delta, \infty, \omega^\infty)/\mathbb{Z}$ determines the same map up to scalar multiple. Define $g_{d}^{bs} := g_{d}^{bs}$ where $d'$ is determined by $y_{j} \sim_{d'} x_{i}$ for some $0 \leq i < c$.

Dually, for an almost crossings $d \in ac(\infty, \omega^\infty, \delta)$, we have a homomorphism $g_{d}^{ab} : M_{\delta}(\omega) \rightarrow M(\delta)$ of modules so that $g_{d}^{ab} \pi_{\omega} = \sum_{m \in \mathbb{Z}} \lambda^{m} f_{p^{m}(d)}$, i.e. explicitly $g_{d}^{ab}$ is given by

$$
g_{d}^{ab} : M_{\delta}(\omega) \rightarrow M(\delta)
$$

$$
x_{i} \mapsto \sum_{m \in \mathbb{Z}} \lambda^{m} f_{d}(\tilde{x}_{i+cm}) = \sum_{m \in \mathbb{Z}} \lambda^{m} y_{j}.
$$

Note that this is a finite sum as the overlap in $d$ is of finite length. We can also write $g_{d}^{ab}$ in matrix form:

$$
g_{d}^{ab} = (0 \quad \lambda^{0} f_{d,0} \quad \lambda^{1} f_{d,1} \quad \cdots \quad \lambda^{l} f_{d,l})^{T},
$$

where each $f_{a,m}$ for $0 \leq m \leq l$ are $c$-by-$c$ matrix and $f_{a,m}$ for $0 < m < l$ are the identity matrices.

It is straightforward to check that $g_{p(\delta)}^{ab} = \lambda^{-1} g_{d}^{ab}$, so for a $\mathbb{Z}$-orbit $[d] \in ac(\omega^\infty, \omega^\infty, \delta)/\mathbb{Z}$, define $g_{d}^{ab} := g_{d}^{ab}$ where $d'$ is determined by $y_{j} \sim_{d'} x_{i}$ for some $0 \leq i < c$.

**Theorem 6.12.** [35] For a string $\delta$, a band $\omega$, and $\lambda \in \mathbb{k}^\times$, we have

(a) $\{g_{d}^{ab} \mid [d] \in ac(\omega^\infty, \omega^\infty, \delta)/\mathbb{Z}\}$ is a basis of $\text{Hom}_{J}(M_{\delta}(\omega), M(\delta))$;

(b) $\{g_{d}^{bs} \mid [d] \in ac(\delta, \infty, \omega^\infty)/\mathbb{Z}\}$ is a basis of $\text{Hom}_{J}(M(\delta), M_{\delta}(\omega))$.

### 6.2.3. Composition of homomorphism.

Suppose we have three strings $\gamma, \delta, \eta$ and two almost crossings $d \in ac(\gamma, \delta), e \in ac(\delta, \eta)$. If the intersection of the overlaps of $d$ and $e$ is non-empty, then we have an almost crossing $e \cdot d \in ac(\gamma, \delta)$; otherwise, we define $e \cdot d := \emptyset$. It is straightforward from the definition of the canonical maps that $f_{e} f_{d} = f_{e \cdot d}$, where $f_{\emptyset} := 0$.

Consider $g_{e}^{ab} : M_{\delta}(\omega) \rightarrow M(\delta)$ and $g_{d}^{bs} : M(\gamma) \rightarrow M_{\delta}(\omega)$. Then we have

$$
(6.2) \quad g_{e}^{ab} g_{d}^{bs} = g_{e}^{ab} \pi_{\omega} f_{d} = (\sum_{m \in \mathbb{Z}} \lambda^{m} f_{p^{m}(e)}) f_{d} = \sum_{m \in \mathbb{Z}} \lambda^{m} f_{p^{m}(e)} \cdot d.
$$

### 6.2.4. Interaction with duality.

We explain the effect of applying the duality $\nabla$ to the canonical maps. We will use the following convention. For a curve $\gamma$ with string form $\gamma = a_{1}^{\varepsilon_{1}} a_{2}^{\varepsilon_{2}} \cdots a_{l}^{\varepsilon_{l}}$ where $a_{i}$'s are all arrows and $\varepsilon_{i} \in \{+, -\}$ for all $i$, we define

$$
\nabla(\gamma) := \sigma(a_{1})^{-\varepsilon_{1}} \cdots \sigma(a_{l})^{-\varepsilon_{l}}.
$$

Let us first describe the isomorphism from $\nabla(M)$ of an indecomposable string-or-band module $M$ to the canonical basis of the resulting string-or-band
module. For a string module $M(\gamma)$, we have defined the combinatorial operation $\nabla$ on the string form of $\gamma$ in a way so that there is an isomorphism $\psi_{M(\gamma)} : \nabla(M(\gamma)) \xrightarrow{\sim} M(\nabla(\gamma))$. Recall from subsection 3.3 that the canonical basis of $M(\gamma)$ is given by $\{u_i := \gamma(t_i)\}_{0 \leq i \leq c}$. Therefore, the module $\nabla(M(\gamma))$ has a natural basis $\{u_i^*\}_{0 \leq i \leq c}$ dual to $\{u_i\}_i$. On the other hand, by design of $\nabla$ (on strings), the canonical basis of $M(\nabla(\gamma))$ is $\{v_i := \sigma(\gamma(t_i))\}$. Under this setup, we have

$$\psi_{M(\gamma)} : \nabla(M(\gamma)) \xrightarrow{\sim} M(\nabla(\gamma))$$

given by $u_i^* \mapsto (-1)^i v_i \forall 0 \leq i \leq c$.

Let $E(\gamma) := M(\gamma) \oplus M(\nabla(\gamma))$, then we have an isomorphism

$$\psi_{E(\gamma)} = \begin{bmatrix} 0 & \psi_{M(\nabla(\gamma))} \\ \varepsilon_{\psi_{M(\nabla(\gamma))}} & 0 \end{bmatrix} : \nabla(M(\gamma)) \oplus \nabla(M(\nabla(\gamma))) \xrightarrow{\sim} M(\gamma) \oplus M(\nabla(\gamma)).$$

This isomorphism is associated to the bilinear map defining the $\varepsilon$-representation structure on $E(\gamma)$ (see discussion in [23 Sec. 2.3]), that is, $\nabla(\psi_{E(\gamma)}) = \psi_{E(\gamma^*)}$.

For a 1-sided indecomposable $\varepsilon$-representation $M_{\lambda}(\omega)$ where $\omega = \sigma(\omega)$ is a primitive closed curve (note that $\lambda = (-1)^{\text{len}(\omega)/2} \varepsilon$ by Lemma 5.12), we have a similar isomorphism $\psi_{M_{\lambda}(\omega)} : \nabla(M_{\lambda}(\omega)) \rightarrow M_{\lambda^{-1}}(\omega)$ that satisfies $\nabla(\psi_{M_{\lambda}(\omega)}) = \varepsilon \psi_{M_{\lambda}(\omega)}$ given as follows.

**Lemma 6.13.** Suppose that $M_{\lambda}(\omega)$ is a 1-sided indecomposable $\varepsilon$-representation for some primitive $\omega = \sigma(\omega)$. Let $\{u_i\}_{0 \leq i \leq 2r}$ and $\{v_i\}_{0 \leq i \leq 2r}$ be the canonical basis of $M_{\lambda^{-1}}(\omega)$ and $M_{\lambda}(\omega)$ respectively, i.e. $u_i, v_i$ are symbols given by $\omega(z_i)$ in the notation of subsection 3.4. Let $\{v_i^\prime\}_{0 \leq i \leq 2r}$ be the standard basis of $\nabla(M_{\lambda}(\omega))$ (whose underlying space is the $k$-linear dual of $M_{\lambda}(\omega)$) dual to the canonical basis $\{v_i\}_{0 \leq i \leq 2r}$. Then we have an isomorphism

$$\psi_{M_{\lambda}(\omega)} : \nabla(M_{\lambda}(\omega)) \xrightarrow{\sim} M_{\lambda^{-1}}(\omega)$$

given by $v_i^\prime \mapsto (-1)^i u_{r+i}$ for all $0 \leq i \leq r$

such that $\nabla(\psi_{M_{\lambda}(\omega)}) = \varepsilon \psi_{M_{\lambda}(\omega)}$.

**Proof.** The isomorphism $\psi_{M_{\lambda}(\omega)}$ is the one naturally induced by the bilinear form defining the $\varepsilon$-representation on $M_{\lambda}(\omega)$, using the data from the proof of Lemma 5.12. The fact that it satisfies $\nabla(\psi_{M_{\lambda}(\omega)}) = \varepsilon \psi_{M_{\lambda}(\omega)}$ can be found in [23 Sec. 2.3].

For any almost crossing $d \in ac(\gamma, \eta)$, say

$$\begin{cases} \gamma = [\gamma_L \alpha_L] \\ \eta^\pm = [\eta_L b_L] \end{cases} \in \begin{bmatrix} \sigma(a_R \gamma R) \\ \sigma(b_R \eta R) \end{bmatrix},$$

its dual $\nabla(d) \in ac(\nabla(\gamma), \nabla(\eta))$ is just the natural

$$\begin{cases} \nabla(\eta) \mp = \begin{bmatrix} \nabla(\eta_L) \sigma(\alpha_L)^- \\ \nabla(\alpha_L) \sigma(\eta_L)^- \end{bmatrix} \\ \nabla(\gamma) = \begin{bmatrix} \sigma(a_R) \nabla(\eta_R) \\ \sigma(b_R) \nabla(\gamma_R) \end{bmatrix} \end{cases}.$$

In the case when $\eta = \infty \omega \infty = \cdots \omega \omega_i \omega_{i+1} \cdots$ for a $\sigma$-stable closed curve $\sigma(\omega) = \omega$, we identify $\nabla(\infty \omega \infty)$ with the ‘half-period shift’ of $\infty \omega \infty$, i.e. if $\kappa \subset \infty \omega \infty$ starts at the $i$-th letter, then $\nabla(\kappa) \subset \nabla(\infty \omega \infty)$ starts with the $(\text{len}(\omega)/2 + i)$-th letter.
Fix a \( d \in ac(\gamma, \eta) \) as above. Let \( a, b, \ell \) be integers such that
\[
\kappa = \gamma_{a+1} \gamma_{a+2} \cdots \gamma_{a+\ell} = \eta_{b+1} \eta_{b+2} \cdots \eta_{b+\ell}.
\]
This means that, if we let \( \{x_i\}_i \) and \( \{y_j\}_j \) be the canonical basis of the indecomposable modules of \( \gamma, \eta \) respectively, then the canonical map sends \( x_i \) for \( a \leq i \leq a+\ell \) to a linear combination of \( y_j \) with \( b \leq j \leq b+\ell \); here the indices are taken modulo \( c := \text{len}(\omega) \) when any one of \( \gamma, \eta \) is \( \infty \omega \infty \).

**Proposition 6.14.** For strings \( \gamma, \delta \), and primitive band whose underlying closed curve satisfies \( \omega = \sigma(\omega) \), the following hold.

(a) For any \( d \in ac(\gamma, \delta) \), we have \( \nabla(f_d) = (-1)^{a+b} \psi^{-1}_{M(\delta)} f_{\nabla(\delta)} \psi_{M(\gamma)} \).

(b) For any \( d \in ac(\gamma, \infty \omega \infty) \), we have \( \nabla(g_{d}^{ab}) = (-1)^{a+b} \psi^{-1}_{M(\delta)} g_{\nabla(\delta)} \psi_{M(\omega)} \).

(c) For any \( d \in ac(\infty \omega \infty, \delta) \), we have \( \nabla(g_{d}^{ab}) = (-1)^{a+b} \psi^{-1}_{M(\omega)} \psi_{M(\delta)} \).

**Proof.** (a) This is straightforward to check from the definition of the \( \psi \) maps and \( \nabla(d) \).

(b) Consider \( g_{d}^{ab} \) in its matrix form as in (6.11) but with each submatrix \( f_{d,i} \) decomposed further into a 2-by-2 block \( \begin{pmatrix} f_{d,i}^L & 0 \\ 0 & f_{d,i}^R \end{pmatrix} \), each of size \( r \)-by-\( r \) for \( r := \text{len}(\omega)/2 \), that is,
\[
g_{d}^{ab} = \begin{pmatrix} 0 & \lambda^d f_{d,0}^L & 0 & \lambda^d f_{d,1}^L & 0 & \cdots & \lambda^{d-1} f_{d,1}^L & 0 & 0 \\ 0 & \lambda^d f_{d,0}^L & 0 & \lambda^d f_{d,1}^L & 0 & \cdots & \lambda^{d-1} f_{d,1}^L & 0 & 0 \\ \end{pmatrix}.
\]

Since the overlap of \( \nabla(d) \) starts \( r \) places later than that of \( d \) in \( \infty \omega \infty \), the matrix form of \( g_{\nabla(d)}^{ab} \) will be of the form:
\[
g_{\nabla(d)}^{ab} = \begin{pmatrix} \lambda^d f_{\nabla(d),0}^L & 0 \\ 0 & \lambda^d f_{\nabla(d),1}^L \\ \lambda^{d-1} f_{\nabla(d),0}^L & 0 \\ \vdots & \vdots \\ \lambda^{d-1} f_{\nabla(d),1}^L & 0 \\ 0 & \lambda^{d-1} f_{\nabla(d),1}^R \\ 0 & 0 \\ \end{pmatrix}.
\]

Note that here, for each \( 0 \leq i \leq l \), \( f_{\nabla(d),i}^R \) is the transpose of \( f_{d,i}^L \) and \( f_{\nabla(d),i+1}^L \) is the transpose of \( f_{d,i}^R \). Also, the domain of \( g_{d}^{ab} \) is \( M_{-1}(\omega) \), and so the exponent on \( \lambda \) decreases as we go down the rows.

Now we compare the above matrix with \( \psi_{M(\gamma)} (g_{d}^{ab}) \psi_{M(\omega)}^{-1} \). As maps of vector spaces, \( \nabla(g_{d}^{ab}) \) is just \( k \)-linear dual of \( g_{d}^{ab} \), so the corresponding matrix is just taking the transpose of \( g_{d}^{ab} \). By the definition of \( \psi_{M(\omega)}^{-1} \), it swaps the two (block-)columns of the transposed matrix, and then multiply \( \lambda^{-1} \) to the first column, and multiply a further \((-1)^b\) to the resulting matrix. The effect of \( \psi_{M(\gamma)} \)
further multiplies a factor of \((-1)^a\) but does not permute any entries of the matrix. This results in the same matrix as \((-1)^{a+b}g_{\psi(d)}\), as claimed.

(c) Follows from (b) by applying \(\nabla\) on both sides.

\[\text{Corollary 6.15. For any } d \in ac(\omega^\infty, \delta), \text{ we have } g_{\psi(W(\omega))} \nabla(g_{\psi(W(\omega))}) = 0 \text{ if } p^m(d) - \nabla(d) \neq 0 \text{ for some } m \in \mathbb{Z}. \]

\[\text{Proof. This follows from combining Proposition 6.14 with (6.2).} \]

\[\text{Corollary 6.16. For any strings } \gamma, \delta, \eta, \text{ and almost crossings } d \in ac(\gamma, \delta), e \in ac(\delta, \eta), \text{ we have } \nabla(e \cdot d) = \nabla(d) - \nabla(e). \text{ In particular, we have } \nabla(f_a \circ f_d) = \pm \psi^{-1}(M(\omega))f_a f_d \psi(M(\gamma)). \]

\[\text{Proof. First part is straightforward from definition of composition and } \nabla\text{-operation on almost crossings. The second part then follows by applying Proposition 6.14 (a).} \]

6.3. \(\varepsilon\)-rigidity for \(1\)-sided \(\varepsilon\)-indecomposables.

\[\text{Lemma 6.17. Let } \omega = \sigma(\omega) \text{ be a primitive closed curve and } (\omega, \lambda) \text{ a } 1\text{-sided } \varepsilon\text{-indecomposable object. If } \omega \text{ is simple, then is } \varepsilon\text{-rigid.} \]

\[\text{Proof. We need to show that every non-zero } f \in Ext^1_\omega((\omega, \lambda), \rho) \text{ has } f \psi(\omega, \lambda) \nabla(f) \text{ for all } \varepsilon\text{-factor } \rho \text{ of } (\omega, \lambda). \text{ Note that by Proposition 3.7 and } \tau\text{-invariance of band modules, } Ext^1_\omega((\omega, \lambda), \rho) \cong Hom_J(M_\lambda(\omega), \tau M(\rho)) \oplus Hom_J(M(\rho), M_\lambda(\omega)) \text{ as vector space.} \]

Write the underlying walk of \(\omega\) as \(\omega_1 \cdots \omega_r \cdots \omega_{2r}\) and \(\omega_i = \alpha_i \epsilon_i\) with \(\alpha_i \in Q_1\) and \(\epsilon_i \in \{\pm 1\}\) for all \(1 \leq i \leq 2r\). For the indices of letters or arrows in \(\omega\), we will also use arithmetic modulo \(2r\) taking values in \(\{1, \ldots, 2r\}\). Recall from Lemma 5.9 that \(\sigma(\alpha_i) = \alpha_{i+r}\) for all \(1 \leq i \leq 2r\). Let us fix an \(i \in \{1, \ldots, r\}\) so that \(\rho = \omega_{i+1} \omega_{i+2} \cdots \omega_{i+2r-1}\). The form of the curves \(\rho, \sigma(\rho), \omega\) are shown in Figure 11 where the dashed lines are supposed to be identified, the shaded parts represent the ‘outside’ of \(\partial S\), and two triangles are part of the triangulation.

The first thing we claim is that \(\dim_k Ext^1_\omega((\omega, \lambda), \rho) = 1\). It is a folklore that dimension of \(Ext^1_\omega\) counts geometric intersection number (see [33] for related result for the case of intersections between two curves with endpoints), so the claim can be easily seen from picture (Figure 11). Since the exact statement is not shown in the literature, we will do it explicitly here.

\[\text{Claim 1: } \dim_k Hom_J(M(\rho), M_\lambda(\omega)) = 0 \text{ and } \dim_k Hom_J(M_\lambda(\omega), \tau M(\rho)) = 1. \]

\[\text{Proof of Claim: By Theorem 6.14, it suffices to show } ac(\rho, \infty^\omega) = 0 \text{ and } |ac(\infty^\omega, \tau(\rho))/\mathbb{Z}| = 1, \text{ where } \tau(\rho) \text{ is the string so that } \tau M(\rho) \cong M(\tau(\rho)). \]

Since \(\alpha_i \rho \alpha_{i+r}\) is a subwalk of \(\infty^\omega\), any almost crossing from \(\rho\) to \(\infty^\omega\) will induce an almost crossing from \(\infty^\omega\) to itself, which then means that the closed curve \(\omega\) has a self-crossing. Thus, the set \(ac(\rho, \infty^\omega)\) is empty.

For \(ac(\infty^\omega, \tau(\rho))\), first recall from [7] Sec 3 that \(\tau(\rho)\) can be described by moving the endpoints of \(\rho\) to the ‘next’ marked point on the boundary; see Figure
Figure 11. The curves $\omega, \rho, \sigma(\rho), \tau(\rho)$ on $(S, M)$

In terms of strings, $\tau(\rho) = p_L \alpha_i^{-1} \rho \alpha_{r+j} p_R$ for some (maximal or trivial) paths $p_L, p_R$: see Figure 12. Hence, there are some $j, k \in \{1, \ldots, 2r\}$ so that we have an almost crossing

$$d = \{ \infty \omega \infty = \omega L \alpha_j, \tau(\rho) = p_L \alpha_i^{-1} \rho \alpha_{i+r} p_R \} \alpha_k \omega R$$

in $ac(\infty \omega \infty, \tau(\delta))$. This is the only almost crossing (up to $\mathbb{Z}$-action) as otherwise we will have a almost self-crossing in $\omega$, which contradicts the simplicity assumption.

Figure 12. Understanding the string form of $\tau(\rho)$

By Claim 1 and Proposition 3.7, $\text{Ext}^1_C(\omega, \lambda, \rho)$ is one-dimensional, and since the space of $\epsilon$-extensions is closed under scalar multiple, it is enough to show for an arbitrary non-zero morphism $f \in \text{Ext}^1_C(\omega, \lambda, \rho)$ satisfies $f \psi(\omega, \lambda) \nabla(f)$. Since there is an equivalence $M(-) : C/[\tilde{T}] \xrightarrow{\sim} \text{mod} J$, it suffices to show that the canonical
basis element \( g_d \in \text{Hom}_J(M_\lambda(\omega), \tau M(\rho)) \) for \( d \) the unique (equivalence class of) almost crossing in \( \text{ac}(\omega^\infty, \tau(\rho))/\mathbb{Z} \) that \( g_d \psi_{M_\lambda(\omega)} \nabla(g_d) \neq 0 \). By Corollary 6.15 it suffices to show that for the unique almost crossing \( d \in \text{ac}(\omega^\infty, \tau(\rho)) \), there is some \( m \in \mathbb{Z} \) so that \( d \cdot \nabla(d) \neq 0 \).

If we index \( \omega = (\omega_t)_{t \in \mathbb{Z}} \) so that \( \omega_t = \alpha_T^r \), where \( T \in \{1, \ldots, 2r\} \) is \( t \) modulo \( 2r \), then the effect of applying \( \nabla \) to \( \omega^\infty \) takes \( \omega_t = \alpha_T^r \) to \( \omega_{t+i} = \alpha_{T+i}^r \). Now the dual almost crossing is:

\[
\nabla(d) = \left\{ \begin{array}{ll}
\nabla(\tau(\rho)) = \\
\infty, \omega = \\
\nabla(\omega_L)\alpha_{r+j} \rightarrow \nabla(p_L)\alpha_{r+i} \alpha_i \alpha_j \nabla(p_R) \rightarrow \alpha_{r+k} \nabla(\omega_R). 
\end{array} \right.
\]

Note that the \( \alpha_i \) appearing in \( \omega^\infty \) in the second line is \( \omega_{2r+i} \), i.e. this \( \alpha_i \) appears in the ‘next copy of \( \omega \) in \( \omega^\infty \) relative to the \( \alpha_i \) in \( d \). On the hand, \( \alpha_{r+i} \) in both \( d \) and \( \nabla(d) \) are the same letter in \( \omega^\infty \) (namely, \( \omega_{r+i} \)). Also, observe from Figure 12 that \( p_R \) is a prefix of \( \nabla(\rho) \), and so \( \nabla(\rho) = p_R q \) for some walk \( q \).

Dually, we have \( \rho = q' \alpha_{r+i} \nabla(p_R) \).

Now combine all the information we have

\[
d \cdot \nabla(d) = \left\{ \begin{array}{ll}
\nabla(\tau(\rho)) = \\
\tau(\rho) = p_L \alpha_i q' \alpha_{r+j} \rightarrow \nabla(p_L)\alpha_{r+i} \alpha_i \alpha_j \nabla(p_R) \rightarrow \alpha_k q \alpha_i \nabla(p_R), 
\end{array} \right.
\]
a well-defined almost crossing as required. \( \square \)

The rest of this subsection is to show the following converse of Lemma 6.17:

**Lemma 6.18.** Let \( \omega = \sigma(\omega) \) be a primitive closed curve and \( (\omega, \lambda) \) be a 1-sided \( \varepsilon \)-indecomposable object. If \( \omega \) has self-intersection, then \( (\omega, \lambda) \) is not \( \varepsilon \)-rigid.

We write the \( \omega = \omega_1 \omega_2 \cdots \omega_{2r} \) where \( \omega_i \) are arrows or formal inverse of arrows. The fact that the length \( 2r \) of \( \omega \) is even comes from Lemma 5.9 which also says that \( \nabla(\omega_i) = \omega_{i+r} \). Note that throughout, indices appearing in \( \omega \) (and its rotations and reflection) are always taken modulo \( 2r \).

An self-intersection of \( \omega \) is given by \( a(n\text{ almost}) \) crossing:

\[
(6.3) \quad \begin{cases}
\omega' = \xi'_L b'_L \\
\omega = \xi_L a_L \infty \rightarrow b_R \xi_R
\end{cases}
\]

where \( \omega' \) is a primitive band obtained from some rotation of \( \omega^\kappa \) for some \( \kappa \in \{+1, -1\} \), the symbols \( a_L, a_R, b_L, b_R \) are arrows, and all other symbols are (possibly trivial) subwords. Note that we do not need to work \( \infty \omega^\infty \) and only focus on the primitive band since we are looking at self-intersection.

For all \( k \in \mathbb{Z} \), write \( \omega_k = a_k^\omega \) for an arrow \( a_i \) and a sign \( \varepsilon_k \in \{+1,-1\} \). Let \( i, i', l \) be integers so that

\[
\begin{align*}
b^+_L \omega^\kappa b^+_R &= \omega_1 \omega_{i'+\kappa} \cdots \omega_{i'+\kappa} = a^\varepsilon_{i'+\kappa} a^\varepsilon_{i'+\kappa} \cdots a^\varepsilon_{i'+\kappa} \\
&= a^\varepsilon_{i'+\kappa} a^\varepsilon_{i'+1} \cdots a^\varepsilon_{i+l}
\end{align*}
\]

In particular, we have \( \varepsilon_i = +1 = \varepsilon_{i'+\kappa} \), and \( \varepsilon_{i'} = -1 = \varepsilon_{i+l} \).

The following result says that we can always assume the overlap of the crossing comes from two different copies of the subinterval \( \omega_C \) in \( \omega \).
Lemma 6.19. Suppose that \( \omega_0 \) is of minimal possible length.

Then \( I' := \{ i' + \kappa, \ldots, i' + \kappa l - 1 \} \) and \( I := \{ i + 1, i + 2, \ldots, i + l - 1 \} \) are distinct intervals of \( \mathbb{Z}/2\mathbb{Z} \).

Proof. Clear the two intervals cannot be identical; otherwise, \( \omega_0 \) is not an overlap of a crossing. This means that it is sufficient to show that \( i', i' + \kappa l \notin I \) and \( i, i + l \notin I' \). We can further reduce to showing \( i' \notin I \) and \( i \notin I' \), as the other case follows by inverting both \( \omega \) and \( \omega' \). We present the argument for \( i \notin I' \); the argument for \( i \notin I' \) is analogous.

Consider first the case when \( \kappa = 1 \). We have that \( \omega' \) is a substring in \( \omega \) aligned with \( a_{2i'-i} \) in \( \omega' \), and \( a_{i+l} \) is a substring in \( \omega' \) aligned with \( a_{i+1-l} \) in \( \omega \).

This means that the crossing is either of the form
\[
\begin{align*}
\omega' &= \cdots a_{i' + l} \cdots a_{2i' - i} \cdots a_{i - l} \cdots a_{i + 1} \cdots a_{i + l - 1} \\
\omega &= \cdots a_i \cdots a_{i + l - \nu} \cdots a_{i + \nu} \cdots a_{i + l - \nu} \cdots,
\end{align*}
\]
or of a similar form where the column indexed by \((i + l, i + l - \nu)\) appears on the right of the column indexed by \((2i' - i, i')\).

This implies that the substring \( a_{i' + l} \cdots a_{i + l - \nu} \cdots a_{i + \nu} \cdots a_{i + l - \nu} \cdots \) of \( \omega \) coincides with the substring \( a_{i' + l} \cdots a_{i + l - \nu} \cdots \), and so we can find a new crossing:
\[
\begin{align*}
\omega' &= \cdots a_{2i' - i} \cdots a_{i - l} \cdots a_{i + 1} \cdots a_{i + l - 1} \\
\omega &= \cdots a_i \cdots a_{i + 1} \cdots a_{i + l - 1} \cdots a_{i + l - \nu} \cdots a_{i + l - \nu} \cdots,
\end{align*}
\]
where the overlap is of short length than the original one. Hence, this contradicts the minimality assumption on \( \omega_0 \).

Consider now the case when \( \kappa = -1 \). If on the contrary that \( i' \in I \), then as indices decrease as we go right on \( \omega' \), the substring \( a_{i' - \nu} \cdots a_{i} \cdots \) of \( \omega' \) will align with \( a_{i - \nu} \subset \omega_C \subset \omega \). But this means that \( \varepsilon_i = \varepsilon_{i'}, \) a contradiction.\( \square \)

Lemma 6.19 says that, possibly after inverting \( \omega \), we can write
\begin{equation}
\omega = b_U^\varepsilon \omega_C^\nu b_V \theta a_L \omega_C a_R^\nu \theta', \tag{6.4}
\end{equation}
for some substring \( \theta, \theta' \) such that \((U, V) = (L, R)\) if \( \kappa = 1 \); otherwise, \((U, V) = (R, L)\). Note that the part \( b_V \theta a_L \) can possibly contract to a single letter \( b_V = a_L \); likewise \( a_R^\nu \theta' b_U \) may contract to a single letter, i.e. \( \omega = b_U^\varepsilon \omega_C^\nu b_V \theta a_L \omega_C \).

Since \( \omega \) is self-dual, applying \( \nabla \) to \( \omega \) yields a new self-crossing
\begin{equation}
\begin{align*}
\omega &= \nabla(\xi) \sigma(a_L)^- \nabla(\phi_L) \sigma(b_L) \left[ \nabla(\omega_C) \sigma(a_R) \nabla(\xi_R) \sigma(b_R) \right] \nabla(\phi_R).
\end{align*}
\end{equation}
From this, one should expect that we can arrange the two copies of \( \omega_C \) in only half of \( \omega \) (and the other two copies of \( \nabla(\omega_C) \)) in the other half; in particular, we should only have at most one of \( b_V \theta a_L \) and \( a_R^\nu \theta' b_U \) contracting to a single letter.

Lemma 6.20. There is a walk \( \omega'' \) that is equivalent to \( \omega \) as a band, so that we can arrange both copies of \( \omega_C \) in \( a_L \omega_C a_R \) and in \( b_U^\varepsilon \omega_C^\nu b_V \) to lie in an \( \varepsilon \)-factor \( \rho \).
of \( \omega \), namely, that
\[
\omega'' = b_U^\omega \rho \sigma(b_U) \nabla(\rho)
\]
(6.6)
\[
= b_U^\omega \omega_C^b \theta \rho a_L \omega_C \omega \varphi a_R \theta' \rho \sigma(b_U) \nabla(\omega_C^b) \nabla(\theta) \sigma(a_L) \nabla(\omega_C) \sigma(a_R) \nabla(\theta'),
\]
(\( \nabla(\rho) \))

for some substring \( \theta, \theta' \) with the possibility that \( b_V \rho a_L \) contracts to \( b_V = a_L \).

**Proof.** Assume \( \omega \) is in the form of (6.4). Recall that \( a_L \) is positioned at the \( i \)-th letter in the walk \( \omega \). If \( i < r \), then we can just take \( \omega'' = \omega \).

Suppose that \( i \geq r \), by Lemma 5.9 we have \( \sigma(b_U) \nabla(\omega_C^b) \sigma(b_V) = \omega_r \cdots \omega_{r+1} \) and also \( \sigma(a_L) \nabla(\omega_C) \sigma(a_R) = \omega_{i-r} \cdots \omega_{i-r+1} \). Note that by Lemma 6.19 \( a_R \) must lie before the end of the walk \( \omega \) (i.e. \( i + l < 2r \)).

For ease of reading, let \( \beta := b_U^\omega \omega_C b_V \) and \( \alpha := a_L \omega_C a_R \), so \( \omega \) is of the form
\[
\omega = \beta \phi \nabla(\alpha) \phi' \nabla(\beta) \nabla(\phi) \nabla(\phi')
\]
for some substring \( \phi, \phi' \). By rotate \( \omega^- \) to a new primitive band that starts with \( \beta^- \), we obtain
\[
\omega'' = \beta^- \nabla(\phi')^- \alpha^- \nabla(\phi)^- \nabla(\beta^-)^- \nabla(\phi^-) \nabla(\phi^-).
\]
The resulting band is of the form as claimed by swapping the role of \( b_U, \omega_C, a_L \) with \( b_V, \omega_C^b, a_R \) respectively.

Let us rotate \( \omega \) again so that it takes the form
(6.7)
\[
\omega = \begin{cases} 
\omega_C a_R \theta' \nabla(b_L^\omega \rho) b_L^\omega \omega_C b_R \theta \rho a_L, & \text{if } \kappa = +1; \\
\alpha_R \theta' \nabla(b_R^\omega \rho) b_R^\omega \omega_C b_L \theta \rho a_L \omega_C, & \text{if } \kappa = -1.
\end{cases}
\]

Consider the following string
(6.8)
\[
\eta := \begin{cases} 
\omega \rho, & \text{if } \kappa = +1; \\
\rho^- \omega & \text{if } \kappa = -1.
\end{cases}
\]

We define another string \( \delta \) from \( \nabla(\eta) \) as follows
(6.9)
\[
\nabla(\eta) = \begin{cases} 
\nabla(\omega_C a_R \theta' \nabla(b_L^\omega \rho) b_L^\omega \omega_C b_R \theta \rho a_L \omega_C a_R \theta' \nabla(b_L^\omega \omega_C b_R \theta \rho a_L \omega_C) \theta' \nabla(b_L^\omega \omega_C b_R \theta \rho a_L \omega_C) & \text{if } \kappa = +1; \\
\nabla(\rho^- a_R \theta' \nabla(b_R^\omega \omega_C b_L \theta \rho a_L \omega_C) a_R \theta' \nabla(b_R^\omega \omega_C b_L \theta \rho a_L \omega_C) & \text{if } \kappa = -1.
\end{cases}
\]

We have now completed the setup needed.

**Proof of Lemma 6.18.** We prove the claim for the case when \( \kappa = +1 \) by constructing the following commutative diagram where all rows and columns are exact sequences; for the case when \( \kappa = -1 \) one just needs to modify the crossings
\[ a, b, c, d, h. \]

\[
\begin{array}{cccccc}
0 & \xrightarrow{M(\rho)} & f_s - \lambda f_s & \xrightarrow{M(\eta)} & \frac{\partial}{\partial s} & \xrightarrow{M_{\lambda}(\omega)} & 0 \\
0 & \xrightarrow{M(\rho)} & M(\delta) & \xrightarrow{\lambda f_{\tau s} f_{\tau c}} & (\lambda f_{\tau s} f_{\tau c}) & \xrightarrow{M(\nabla(\eta))} & 0 \\
0 & \xrightarrow{M(\rho)} & \frac{\lambda f_{\tau s} f_{\tau c}}{f_{\tau s} - \lambda f_{\tau c}} & \xrightarrow{M(\nabla(\delta))} & 0 & \xrightarrow{0} & 0 \\
0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 \\
\end{array}
\]

For ease of reading, we use \( \cdots \) instead of writing the full walks as long as there is no confusion, and use a cross \( \times \) to denote an empty entry. To further reduce complication, we take

\[ \phi_A := \omega_C a_R \theta' \quad \text{and} \quad \phi_B := \omega_C b_R \theta a_L, \]

which means that

\[ \rho = \phi_B \phi_A, \quad \omega = \phi_A \sigma(b_L) \nabla(\phi_B \phi_A) b_{\Lambda} \phi_B, \quad \text{and} \quad \delta = \phi_A \sigma(b_L) \nabla(\phi_B \rho). \]

The almost crossings we needed are

\[ a := \begin{cases} 
\rho = & \times \\
\eta = & \cdots b_R \theta a_L \\
\phi_A := & \cdots \rho \\
\end{cases}, \]

\[ b := \begin{cases} 
\rho = & \times \\
\eta = & \times \omega_C \\
\phi_A := & b_R \theta a_L \phi_A \\
\end{cases}, \]

\[ c := \begin{cases} 
\eta = & \times \\
\delta = & \times \left( \phi_A \sigma(b_L) \nabla(\phi_B) \nabla(\omega_C) \sigma(a_R) \cdots \right) \\
\phi_A := & \sigma(b_R) \cdots \right) \\
\end{cases}, \]

\[ d := \begin{cases} 
\nabla(\delta) = & \times \\
\phi_A := & \cdots \sigma(a_L) \cdots \\
\end{cases}, \]

\[ h := \begin{cases} 
\eta = & \times \\
\phi_A := & b_R \cdots \\
\phi_A := & a_R \cdots a_L \omega \infty, \end{cases} \]
These yield the following compositions:

\[ c \cdot a = 0 = d \cdot b, \]
\[ c \cdot b =: x = \left\{ \begin{array}{l}
\rho = \times \\
\delta = \times \end{array} \right| b_R \cdots a_R \cdots, \]
\[ d \cdot a =: y = \left\{ \begin{array}{l}
\rho = \\
\nabla(\delta) = \times a_L \end{array} \right| \times x, \]
\[ \nabla c \cdot y =: w = \left\{ \begin{array}{l}
\rho = \\
\nabla(\eta) = \nabla(\phi_A) b_L^{-1} \omega_C b_R \theta a_L \end{array} \right| \times \frac{b_R \cdots}{a_R} \omega \nabla(b_L^{-1} \phi_B \rho), \]
\[ h \cdot a = \left\{ \begin{array}{l}
\infty \omega = \\
\infty \omega_C \cdots a_L \end{array} \right| \times \frac{b_R \theta a_L \phi_A}{a_R} \omega \cdots, \]
\[ = p^{-1}(h \cdot b), \]
\[ \nabla h \cdot h =: u = \left\{ \begin{array}{l}
\eta = \\
\nabla(\eta) = \nabla(\phi_A) b_L^{-1} \omega_C \nabla(\phi_B) \phi_A \cdots \nabla \phi_A \end{array} \right| \frac{b_R \cdots}{a_R} \omega \cdots. \]
\[ v = p^{-1}(\nabla h) \cdot h = \left\{ \begin{array}{l}
\nabla(\eta) = \\
\phi_A \sigma(b_L) \nabla(\phi_B) \nabla(\omega_C) \phi_A \cdots \nabla \phi_A \sigma(\theta_B) \sigma(\phi_B) \nabla(\phi_A) \sigma(\theta_B) \sigma(\phi_B) \nabla(\omega_C) \phi_A \cdots \nabla \phi_A \end{array} \right| \frac{\sigma(a_R) \cdots}{\sigma(a_R) \cdots} \frac{\sigma(b_R) \nabla(\theta_B) \nabla(\phi_A)}{\sigma(b_R) \nabla(\theta_B) \nabla(\phi_A)} = \nabla d \cdot c. \]

From these datum (and that \( \lambda = \lambda^{-1} \)), it is easy to verify the commutation of the left-hand square and the bottom square, as well as the exactness of the second row and the left-hand column.

For the exactness of the first row, since \( p(h \cdot a) = h \cdot b \), we have

\[ \eta = \phi_A \sigma(b_L) \nabla(\phi_B) \nabla(\omega_C) \phi_A \cdots \nabla \phi_A \sigma(\theta_B) \sigma(\phi_B) \nabla(\phi_A) \sigma(\theta_B) \sigma(\phi_B) \nabla(\omega_C) \phi_A \cdots \nabla \phi_A = \nabla d \cdot c, \]

which yields the required exactness. The exactness of the right-hand column then follows by applying \( \nabla \).

Finally, for the commutation of the upper right squares, by (6.2) we have

\[ g_{h_b} f_b = g_{h_b} = \lambda^{-1} f_r + f_u \]

whereas the composition through \( M(\delta) \oplus M(\nabla(\delta)) \) yields \( \lambda f_r + f_u \). This completes the proof. \( \square \)

6.4. \( \varepsilon \)-cluster-tilting object.

**Definition 6.21.** Let \( X \in \mathcal{C} \) be an \( \varepsilon \)-object with \( \varepsilon \)-indecomposable decomposition \( X = \bigoplus_{i=1}^n X_i \). We say that \( X \) is an \( \varepsilon \)-cluster-tilting object in \( \mathcal{C} \) if

- each \( X_i \) is an \( \varepsilon \)-rigid object,
- \( \text{Ext}^1_{\mathcal{C}}(X_i, X_j) = 0 \),
- \( X \) is maximal with respect to the above properties, i.e. if \( Y \) is an indecomposable \( \varepsilon \)-rigid object \( Y \) satisfying \( \text{Ext}^1_{\mathcal{C}}(Y, X_i) = 0 = \text{Ext}^1_{\mathcal{C}}(X_i, Y) \) for all \( X_i \not\sim Y \), then there must be some \( j \in \{1, \ldots, n\} \) so that \( Y \cong X_j \).
Now we can collect everything we have in the section to obtain our main result.

**Theorem 6.22.** The bijections in Theorem 6.4 restricts to a correspondence
\[ A \otimes (S, M) \leftrightarrow \{ \varepsilon\text{-rigid objects of } C_{\hat{(S,M)}} \}. \]
This induces a correspondence
\[ \{ \text{quasi-triangulations of } (S, M) \} \leftrightarrow \{ \varepsilon\text{-cluster-tilting objects of } C_{\hat{(S,M)}} \}, \]
which restricts to a correspondence
\[ \{ \text{\sigma-stable triangulations of } \hat{(S,M)} \} \leftrightarrow \{ \text{triangulations of } (S, M) \} \leftrightarrow \{ \nabla\text{-stable cluster-tilting objects of } C_{\hat{(S,M)}} \}. \]

**Proof.** The first bijection follows from combining Lemma 6.7, Lemma 6.8, Lemma 6.18, and Lemma 6.17. Now the second one follows from Proposition 3.8. In particular, when restricting to triangulations, we only get split \( \varepsilon \)-indecomposables appearing and so \( \varepsilon \)-rigid is just usual rigidity and \( \varepsilon \)-cluster-tilting is the usual cluster-tilting; these yields the last correspondence. \( \square \)

**Example 6.23.** Consider the fan triangulation \( T \) of \( M_2 \) from Example 3.6 which is associated to \( \hat{A}_{2,2} \)-quiver. Fix \( \varepsilon \in \{ \pm 1 \} \). It is well-known that \( C_{\hat{T}} \) has infinitely many rigid objects (corresponding to arcs on \( \hat{(S,M)} \)) and cluster tilting objects (corresponding to triangulations on \( (S,M) \)), but not many of them are give rise to indecomposable \( \varepsilon \)-rigid object. For example, if we consider the arc \( \alpha \in A(\hat{S,M}) \) corresponding to the string \( b_2 \), then \( \alpha \oplus \nabla(\alpha) \) non-rigid in \( C_{\hat{T}} \). Note that \( \alpha, \sigma(\alpha) \) are the lifts of the self-intersecting curve shown in Figure 2; see Figure 13 where we also display the structure of \( M(\alpha) \) and \( M(\sigma \alpha) \).

![Figure 13. Self-crossing arc, its lift, and corresponding non-\( \varepsilon \)-rigid object](image-url)

There are only finitely many indecomposable object \( X \in C_{\hat{T}} \) such that \( X \oplus \nabla(X) \) is rigid, which means that there are only finitely many \( \varepsilon \)-rigid objects of split string type. The two obvious one are the initial arcs in \( \hat{T} \subset \hat{(S,M)} \), which correspond to the initial arcs in \( T \subset (S,M) \). The remaining ones are given in Figure 14 where we display from top to bottom their structures as indecomposable
\( \varepsilon \)-representations over \( J \), the corresponding arcs in \((S, M)\), and the corresponding arcs in \((S, \bar{M})\).

\[
\begin{align*}
&\begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{array} \\
&\begin{array}{c}
\delta
\end{array}
\end{align*}
\]

\textbf{Figure 14.} Non-initial \( \varepsilon \)-rigid objects of split string types on \( \mathcal{M}_2 \)

There is one more indecomposable \( \varepsilon \)-rigid object, namely the unique 1-sided \( \varepsilon \)-indecomposable \((\omega, \varepsilon)\) corresponding to the quasi-arc; see Figure 9.

\textbf{Example 6.24.} We consider \((S, M)\) a genus 2 non-orientable surface with 1 boundary component and 1 marked point. Let \( T = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \) be a triangulation on \((S, M)\) as shown in the middle of Figure 15. We display the gentle algebra as a quiver with relation on the left. Here \( i, i' \) for each \( 1 \leq i \leq 4 \) are
the vertices corresponding to the lifts $\tilde{\alpha}_i, \sigma(\tilde{\alpha}_i)$ of $\alpha_i$ to $(\widetilde{S,M})$, and the dotted line connecting two arrows represents a monomial quadratic relation given by the composition of the two arrows.

Consider the curves $\gamma, \delta$ on $(S,M)$ as shown on the right. Let $\tilde{\gamma}, \tilde{\delta}$ be a lift to $(\widetilde{S,M})$. Then

$$M(\tilde{\gamma}) \oplus \nabla M(\tilde{\gamma}) = 1 \oplus 1', \text{ and } M(\tilde{\delta}) \oplus \nabla M(\tilde{\gamma}) = \frac{1}{3} \oplus 3' 1'. $$

The set $\{\gamma, \delta, \alpha_2, \alpha_4\}$ form a triangulation of $(S,M)$, and so the object

$$\tilde{\gamma} \oplus \tilde{\delta} \oplus \tilde{\alpha}_2 \oplus \tilde{\alpha}_4 \oplus \nabla (\tilde{\gamma} \oplus \tilde{\delta} \oplus \tilde{\alpha}_2 \oplus \tilde{\alpha}_4) $$

is an $\varepsilon$-cluster tilting object in $C_{(S,M)}$.

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