A proposal for self-correcting stabilizer quantum memories in 3 dimensions (or slightly less)

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We propose a family of local CSS stabilizer codes in 3D as candidates for self-correcting quantum memories. The construction is inspired by the classical Ising model on a Sierpinski carpet fractal, which acts as a classical self-correcting memory. Our models are naturally defined on fractal subsets of a 4D hypercubic lattice which can be embedded into \( \mathbb{R}^3 \) with finite distortion. The \( X \) and \( Z \) sectors of the code are dual to one another, and we show that there exists a finite temperature phase transition associated with each of these sectors, providing evidence that the system may robustly store quantum information at finite temperature.

I. INTRODUCTION

There is significant interest from both an abstract and practical perspective as to if and how self-correcting quantum memories might be realised. A practical self-correcting memory would allow for arbitrarily long storage of quantum information at finite temperature without the need for constant active error-correction techniques. The 4D toric code is a simple, exactly solvable example of a system with local interactions in 4 spatial dimensions that is known to have self-correcting properties [1, 2]. In 2D, the toric code is known to be unstable at finite temperature [3], and there are numerous no-go theorems that rule out broad classes of models for self-correction [4–6]. Despite this, some attempts have been made to engineer self-correcting behaviour in 2D systems [7–9]. Many approaches towards realising some aspects of self-correction in 3D have also been found, notably including the Haah code [10–13] among others [14–20], though no previously known local spin models in 2D or 3D are fully self-correcting. There are also several no-go results restricting possible self-correcting models in 3D [21–24]. For a comprehensive review of the field of memories at finite temperature, see Ref. [25].

We propose here a local spin model in 3D, and argue that it may act as a self-correcting quantum memory. Our approach is based on fractal geometries, and inspired by the classical self-correcting behaviour of an Ising model on a Sierpinski carpet graph. The Sierpinski carpets [26] are a family of fractal subsets of \( \mathbb{R}^2 \) with Hausdorff dimension between 1 and 2. We propose a family of quantum CSS codes that can be considered as 4D toric codes on discretizations of the product of two Sierpinski carpet fractals (with appropriate boundary conditions). Concretely, our codes are defined through the homological product construction [27, 28] applied to two toric codes on 2D Sierpinski carpet graphs, yielding a code family with extensive degeneracy. Though this system naturally embeds in \( \mathbb{R}^4 \), by choosing the Hausdorff dimension of the Sierpinski carpets small enough, we can ensure that the resulting system can be embedded in \( \mathbb{R}^3 \) with finite distortion. We call these codes embeddable fractal product codes (EFPCs).

Though we call them codes, EFPCs should more properly be considered Hamiltonian systems given by a (negative) sum of stabilizer generators, and we show that such systems have (at least) two phase transitions at finite temperature, one associated with each sector (\( X \) or \( Z \)) of the CSS code. The tools we use to show this are generalized duality transformations and correlation inequalities. Given these phase transitions, we argue that the EFPC system may act as a self-correcting quantum memory at sufficiently low temperatures. Though there is an extensive degeneracy, we expect the phase transitions we identify to correspond to the appearance of thermal stability for only one preferred encoded qubit, and the stability of the other encoded qubits is a little more subtle.

The use of the Sierpinski carpet fractals is not crucial for our construction, and so we also briefly discuss the more general family of EFPCs that could arise from alternative fractal structures. However, to the best of our knowledge no other well-studied fractals are known to have the properties required to build an EFPC.

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A. The Caltech rules

A practical quantum memory could in principle take any number of forms. In order to concretely discuss a self-correcting quantum memory (SCQM), it is convenient to set a series of criteria which such a system should satisfy. As such, we briefly review the so-called Caltech rules:

A model is a $D$-dimensional SCQM under the Caltech rules if:

1. (finite spins) It consists of finite dimensional spins embedded in $\mathbb{R}^D$ with finite density
2. (bounded local interactions) It evolves under a Hamiltonian comprised of a finite density of interactions of bounded strength and bounded range
3. (nontrivial codespace) It encodes at least one qubit in its degenerate ground space
4. (perturbative stability) The logical space associated with at least one encoded qubit must be perturbatively stable in the thermodynamic limit
5. (efficient decoding) This encoded qubit allows for a polynomial time decoding algorithm
6. (exponential lifetime) Under coupling to a thermal bath at some non-zero temperature in the weak-coupling Markovian limit, the lifetime of this encoded qubit scales exponentially in the number of spins

We purposely leave the precise definition of perturbative stability vague, and will discuss it further in Sec. V A. It is also often required that the Hamiltonian be gapped, but while this may be desirable it is not a necessary condition for self-correcting behaviour.

The 4D toric code is an example of a 4D Caltech SCQM. Other proposals in 2 or 3 dimension often make use of long-range interactions, bosonic modes in place of spins, or do not achieve asymptotically exponential memory lifetime. Our proposal may be the first example of a 3D Caltech SCQM, though we stress that we merely argue, and do not prove, that it satisfies all of these constraints, notably lacking a rigorous treatment of the memory lifetime, perturbative stability, and decoding algorithm.

II. FRACTALS AND DIMENSIONALITY

Key to our construction will be the notion of fractal geometry. Fractal objects have spatial dimension that interpolates between the familiar integral topological dimensions. This dimension can be quantified in several useful ways, such as the Hausdorff dimension or the box-counting dimension. We will not give details of many results familiar in fractal geometry, instead we refer the interested reader to a standard text such as Ref. [29].

We will consider only fractals that are particularly well-behaved, in that they are self-similar Borel sets satisfying the open set condition. For fractals with these properties, many different fractal dimensions coincide, and so we will simply denote the dimension of a fractal $F$ as $\dim F$ (for concreteness this can be taken as the Hausdorff dimension). The dimensions of these sets also satisfy $\dim(F_1 \times F_2) = \dim F_1 + \dim F_2$, which will be a useful identity.

A. Sierpinski carpets

Our construction is motivated by a particular family of fractals, the Sierpinski carpets [26]. These fractals have dimension between 1 and 2, and are naturally defined as self-similar subsets of $\mathbb{R}^2$. Although more general definitions of Sierpinski carpets are sometimes used, for our purposes it will be sufficient to define a Sierpinski carpet by two positive integers $b$ and $c$, with $(b-c)$ even and positive (following e.g. [30] or [31]). We denote the resulting fractals by $SC(b,c)$.

The fractals $SC(b,c)$ can be defined as the limit of a sequence $SC(b,c,l)$ as $l \to \infty$. We call the $SC(b,c,l)$ the $(b,c)$ Sierpinski carpets at level $l$, and they are constructed by dividing the unit square into $b^2$ smaller squares, deleting the central $c^2$ squares, and iterating this procedure $l$ times. An example is shown in Fig. 1. The dimension of a $(b,c)$ Sierpinski carpet is $\dim SC(b,c) = \frac{\ln(b^2-c^2)}{\ln b}$. For positive integral $b$ and $(b-c)$, it is clear that achievable dimensions are dense in the interval $1 < \dim SC(b,c) \leq 2$ (and empty outside).
FIG. 1: $SC(3, 1, l)$ for $l = 0$ to 4

FIG. 2: $SC'(3, 1, l)$ for $l = 0$ to 4, overlaid on $SC(3, 1, l)$

B. Sierpinski carpet graphs

At level $l$, a $(b, c)$ Sierpinski carpet is an arrangement of $(b^2 - c^2)^l$ out of a possible $b^2$ elementary squares. Drawing the borders of these squares yields the Sierpinski carpet graphs, i.e. with a vertex at each corner of occupied squares (note alternative conventions exist in the literature, see [30] or [31] for discussion). These graphs consist of

$$|V(b, c, l)| = (b^2 - c^2)^l + 2c \frac{b^l - (b^2 - c^2)^l}{b - (b^2 - c^2)} - \frac{1 - (b^2 - c^2)^l}{1 - (b^2 - c^2)} + 2b^l + 1 \text{ vertices, and}$$

$$|E(b, c, l)| = 2(b^2 - c^2)^l + 2c \frac{b^l - (b^2 - c^2)^l}{b - (b^2 - c^2)} + 2b^l \text{ edges.}$$

(1)

We will denote such graphs by $SC(b, c, l)$. Examples are shown in Fig. 2.

It will be convenient to distinguish between “interior” and “exterior” plaquettes of the Sierpinski carpet graphs. The interior plaquettes are those bounding occupied squares of the Sierpinski carpet fractal, while the exterior plaquettes are the minimal cycles not generated by interior ones (i.e. those cycles bounding “deleted” regions of the fractal, plus the outer boundary). $SC(b, c, l)$ can be shown to contain $|P_i(b, c, l)|$ interior plaquettes and $|P_e(b, c, l)|$ independent exterior plaquettes, with

$$|P_i(b, c, l)| = (b^2 - c^2)^l$$

(3)

$$|P_e(b, c, l)| = \frac{1 - (b^2 - c^2)^l}{1 - (b^2 - c^2)}$$

(4)

noting that the outer boundary can be generated by the product of all interior and exterior plaquettes.

We will sometimes use the words fractal or graph interchangeably to refer to the Sierpinski carpet when it is clear which object is meant from context.

C. Sierpinski carpet Ising models

It is possible to define a classical ferromagnetic Ising model on a Sierpinski carpet graph, and study the thermodynamic properties for fixed $b$ and $c$ as $l \to \infty$. These models have 2-fold degenerate ground spaces, and thus can be considered as classical codes. General arguments suggest [32, 33], many numerical studies demonstrate (e.g. [30, 31]), and it can be rigorously proved [34–36], that such a family of Ising models has a phase transition at non-zero temperature. Intuitively, this is due to the fact that the
Sierpinski carpet graphs have infinite ramification order (i.e. in the limit $l \to \infty$, an infinite number of bonds must be cut to separate the graph into two infinite pieces). General arguments suggest that an Ising model defined on any family of fractal graphs with sufficiently large ramification (in our context, scaling fast enough with $l$) will have a finite temperature phase transition, while those with finite ramification order do not exhibit finite-temperature phase transitions. The Sierpinski triangle graphs (Fig. 3) are an example of such a family of fractal graphs with finite ramification, and the corresponding Ising model has a zero-temperature phase transition [37].

Another quantity of interest is the minimum energy barrier that must be overcome to transition between the two degenerate code states by a sequence of spin flips. While this quantity is not as important as the existence of a phase transition, we can nonetheless compute it. In order to transition between the two code states, every spin on the lattice must be flipped. The energy barrier must then be at least proportional to the number of bonds it takes to separate the graph into two comparably sized pieces (this can be thought of as the ramification). To calculate the energy barrier in this way, we choose a cut that minimizes the number of crossed bonds in the limit $l \to \infty$, and then compute the total number of bonds that would be frustrated if the spins on one side of the cut were flipped and those on the other not. An example of such a minimal cut for odd $b$ is a vertical line running down the centre of the graph, as shown in Fig. 4. It can directly be computed that the total number of bonds that such a minimal cut crosses is $\left(\frac{(b-c)^{r+1}-1}{(b-c)^{r}-1} + 1\right)$, providing a lower bound to the energy barrier $\Delta E$. Since the total number of spins in this Ising model is $n \equiv |V(b, c, l)|$, we see that $\Delta E \geq O\left(n^{\frac{1}{\log(b-c)(b^2-c^2)}}\right)$, i.e. $\Delta E$ is polynomial in $n$ (as the maximum possible energy is also polynomial in $n$, $\Delta E$ cannot be superpolynomial). Though a polynomial energy barrier is typical for models with finite temperature phase transitions such as the 2D Ising model or the 4D toric code, examples exist of systems with polynomial energy barrier but no finite temperature phase transition, such as the welded codes [18].

D. Embedding fractals in Euclidean space

Crucial to our work will be the ability to embed certain fractals into $\mathbb{R}^3$ with finite distortion (i.e. without increasing or decreasing the distance between any two points by more than a constant multiplicative factor). Such a finite distortion embedding $\mathcal{E}$ of a (possibly subset of a) metric space $(F_1, d_1)$ into a metric space $(F_2, d_2)$ is called bilipschitz and satisfies

$$c^{-1}d_1(x, y) \leq d_2(\mathcal{E}(x), \mathcal{E}(y)) \leq cd_1(x, y)$$

(5)
for some finite constant $c > 0$ and for all $x, y \in F_1$.

It is in general difficult to determine whether one fractal (for simplicity, considered as a subset of, and inheriting the metric structure of, some $\mathbb{R}^D$) can be bilipschitz embedded into another. However, for certain classes of fractals sufficient criteria for the construction of such an embedding are known. In particular, we will appeal to the dusty self-similar embedding theorem [38] that characterizes bilipschitz embeddings from fractals that are “dust-like”, or obey the so-called strong separation condition. To formulate this condition, it is convenient to consider a self-similar fractal $F$ as the fixed point of a set of similitudes $S_i$, i.e. $F = \bigcup_i S_i(F)$.

**Definition 1** (Strong separation [29]). A self-similar fractal $F = \bigcup_i S_i(F)$ is said to satisfy the strong separation condition if $S_i(F) \cap S_j(F) = \emptyset$ for $i \neq j$.

**Theorem 2** (Dusty self-similar embedding theorem [38]). Consider two self-similar subsets of metric spaces $E_1$ and $E_2$ such that $\dim E_1 < \dim E_2$. If $E_1$ satisfies the strong separation condition, then there exists a bilipschitz embedding of $E_1$ in $E_2$.

Notably, we could choose $E_2$ in Thm. 2 to be $\mathbb{R}^3$, and then the theorem shows that any self-similar strongly separated fractal of dimension $< 3$ can be bilipschitz embedded in $\mathbb{R}^3$.

The EFPCs that we define in Sec. III are naturally defined on $SC(b, c, l) \times SC(b, c, l)$, and so it would be convenient for us to embed $SC(b, c) \times SC(b, c)$ into $\mathbb{R}^3$. However, the Sierpinski carpets themselves do not satisfy the strong separation condition (nor does their product). Instead, we consider a related class of fractals that we call the dusty Sierpinski carpets. The dusty Sierpinski carpets $DSC_c(b, c)$ are labelled by a small parameter $0 < \epsilon \ll 1$, as well as the $b$ and $c$ parameters of the standard Sierpinski carpets $SC(b, c)$. They are also the limit of a sequence $DSC_c(b, c, l)$ as $l \to \infty$. This sequence is obtained by again dividing the unit square into $b^2$ smaller squares, and deleting the central $c^2$, but additionally deleting a boundary of width $\epsilon$ (as a fraction of the total square edge length) around each of the remaining $(b^2 - c^2)$ squares (see Fig. 5). For any $0 < \epsilon < 1$ this defines a fractal that is satisfies the strong separation condition, or is dust-like.

The product of two self-similar sets satisfying the strong separation condition will give a self-similar set satisfying the strong separation condition, and so it follows that for suitably chosen $b$ and $c$ such that $\dim DSC^2(b, c) < 3$, we can embed $DSC(b, c) \times DSC(b, c) \equiv DSC^2(b, c)$ in $\mathbb{R}^3$ with finite distortion. Since the dusty Sierpinski carpets are Borel sets satisfying the open set condition, $\dim DSC^2(b, c) = 2 \cdot \dim DSC(b, c)$, and since removing points from a set cannot increase its Hausdorff dimension, we conclude that $\dim DSC^2(b, c) \leq 2 \dim SC(b, c)$. It can easily be seen that there exist $b$ and $c$ such that this quantity is less than 3, for example choosing $c = b - 2$ and $b \geq 14$. The fractals $DSC^2(b, c)$ for these values can be embedded in $\mathbb{R}^3$ with finite distortion.

### III. EMBDEBBLE FRACTAL PRODUCT CODES

In this section we define the embeddable fractal product codes (EFPCs). Although we call them codes, they should be understood as either quantum codes (in the sense of a subspace of a larger Hilbert space) or as local commuting Hamiltonians (such that the frustration-free ground space is the corresponding codespace) depending on context. Since the codes we present are (CSS) stabilizer codes, the Hamiltonian formulation simply corresponds to a negative sum of generators of the stabilizer group. As such, the presentation of the codes will contain more information than is necessary to specify the codespace only, as we will also be interested in the particular (typically non-minimal) choice of generators of the stabilizer group. By setting the Hamiltonian, this choice of generators will set the energetics and thermodynamic

![FIG. 5: $DSC_c(3, 1, l)$ for $l = 0$ to 3](image)

properties of the system. Since the code is CSS, we can also consider the X sector and the Z sectors of the code separately as classical codes or classical Hamiltonian systems in the analogous way.

In order to define the EFPCs, it will be convenient to recall the homological product of two codes.

### A. Homological product codes

The homological (or hypercomplex) product \([27, 28]\) is a construction for building new CSS codes from existing ones, making use of tools from algebraic topology. We need not introduce the full generality of the homological product here and will simply sketch it as is appropriate for our needs; we refer the interested reader to Refs. \([27, 28]\), or a standard reference on algebraic topology, e.g. \([39]\).

In the homological product construction, each quantum CSS code \(C\) is represented by three vector spaces over \(\mathbb{Z}_2\) (or as convenient the corresponding abelian groups): \(C_0\), \(C_1\), and \(C_2\), and two maps \(\partial^C_0 : C_2 \rightarrow C_1\) and \(\partial^C_1 : C_1 \rightarrow C_0\) such that \(\partial^C_0 \partial^C_1 = 0\). The elements of \(C_1\) correspond to qubits, while the elements of \(C_0\) (\(C_2\)) correspond to X-type (Z-type) stabilizer generators (note that this need not be a minimal generating set, and in fact the choice of generators significantly affects the construction). The maps \(\partial^C_0\) and \((\partial^C_1)^T\) define the qubits on which each stabilizer generator has support, and the constraint \(\partial^C_1 = 0\) enforces that the stabilizer group is abelian. Often, as will be the case in our construction, the \(C_0\) (\(C_2\)) elements of the spaces \((\partial^C_1)^T\) correspond to qubits, while elements of the \(C_0\) (\(C_2\)) \(C\)-type \((C\)-type) stabilize generators.

Associated with each space \(C_i\) is a homology group \(H_i(C) = \ker (\partial^C_i) / \text{im} (\partial^C_{i-1})\) and a cohomology group \(H^i(C) = \ker (\partial^C_i)^T / \text{im} (\partial^C_{i-1})^T\) (with \(\partial^C_0\) and \(\partial^C_1\) maps from and to the zero space, respectively, such that \(\text{im} (\partial^C_0)^T = 0\)). \(H^1(C) = \ker (\partial^C_1)^T = C_0\), and \(\ker (\partial^C_1)^T = C_3\). In this language, the X-type and Z-type logical operators correspond to elements of \(H_1(C)\) and \(H^1(C)\) respectively. As such, the number of encoded qubits in such a code is given by \(k_C \equiv \dim H_1(C) = \dim(\ker (\partial^C_1)) - \dim(\text{im} (\partial^C_2))\) (or equivalently \(\dim H^1(C)\)).

The homological product of two codes \(C\) and \(C'\) yields a new object \(C \otimes C'\) with five spaces and four maps, given by

\[
H_i(C \otimes C') = \bigoplus_{j=0}^{j} \bigoplus_{j=0}^{j} (C_i \otimes C'_{i-j})
\]

\[
\partial^C_0(C' \otimes C') = (\partial^C_1 C_i) \otimes C'_{i-1} + C_i \otimes (\partial^{C'}_{i-j} C'_{i-j-1})
\]

for \(c_i \in C_i\) and \(c'_j \in C'_j\).

We could define three different codes from the general construction, but for our purposes we take the middle homological product code, denoted by \(\text{mid}(C \otimes C')\) and defined by the three spaces \((C \otimes C')_1\), \((C \otimes C')_2\), \((C \otimes C')_3\), and two maps \(\partial^C_2(C \otimes C')\) and \(\partial^C_3(C \otimes C')\). In \(\text{mid}(C \otimes C')\), elements of the space \((C \otimes C')_2\) correspond to qubits, while elements of the spaces \((C \otimes C')_1\) and \((C \otimes C')_3\) correspond to X- and Z-type stabilizer generators, respectively.

Properties of the homological product codes can be determined directly from those of their component codes. In particular, it will be useful to determine the logical operators of such codes. These can be calculated using the Künneth formulae

\[
H_i(C \otimes C') = \bigoplus_{j+k=i} H_j(C) \otimes H_k(C')
\]

\[
H^i(C \otimes C') = \bigoplus_{j+k=i} H_j(C) \otimes H^k(C')
\]

The logical operators of the middle homological product code correspond to the elements of the homology and cohomology groups \(H_2(C \otimes C')\) and \(H^2(C \otimes C')\), and correspondingly the number of encoded qubits is given by \(k_{\text{mid}(C \otimes C')} = \dim H_2(C \otimes C')\).
where, in the second last line, we note that $H_1(T) \cong \mathbb{Z}_2$ breaks into even and odd elements. In contrast, the second homology group $H_2(T)$ is given as the sets of plaquettes with no boundary, which is empty, giving $H_2(T) \cong 0$. The first homology group is given by the quotient of the cycles in the graph by the interior boundaries. Since there are $|P_e| = k_T$ independent exterior boundaries, we find $H_1(T) \cong \mathbb{Z}_2^k$ as expected.

We will also make use of the dual code $T^*$, where the $X$- and $Z$-type stabilizers have been exchanged. This is the toric code on the dual graph to $S\tilde{C}(b,c,l)$ (i.e. where plaquettes and vertices have been exchanged, and which we denote $S\tilde{C}^*(b,c,l)$, see Fig. 6), with the appropriate high-coordination vertices neglected for the purposes of defining the stabilizer group. In the homological language, a dual code has $C_0^* = C_2$, $C_1^* = C_1$, $\partial_1^* = (\partial_0^*)^T$ and $C^* = C$. From these duality properties, the homology groups of this code are immediate: $H_1(T^*) \cong \mathbb{Z}_2$, $H_0(T^*) \cong 0$, and $H_1(T) \cong \mathbb{Z}_2^k$.

We define the family of embeddable fractal product codes $EFPC_{\{b,c,l\}}$ as the middle homological product $\text{mid}(T \otimes T^*)$ for suitable $b$ and $c$. The physical qubits of this code correspond to elements of $(T \otimes T^*) = \mathbb{L}_0 \otimes \mathbb{L}_0^* \oplus \mathbb{L}_1 \otimes \mathbb{L}_1^* \oplus \mathbb{L}_2 \otimes \mathbb{L}_2^*$. There are thus $n_{EFPC_{\{b,c,l\}}} \equiv |V(b,c,l)|^2 + |E(b,c,l)|^2 + |P_e(b,c,l)|^2$ physical qubits in this code.

The degeneracy of these codes can easily be determined by the K"unneth formula as

$$k_{EFPC_{\{b,c,l\}}} = \dim (H_0(T) \otimes H_2(T^*)) + \dim (H_1(T) \otimes H_1(T^*)) + \dim (H_2(T) \otimes H_0(T^*))$$

$$= (1 \cdot 1) + (k_T)^2 + (0 \cdot 0)$$

$$= 1 + \left( \frac{1 - (b^2 - c^2)}{1 - (b^2 - c^2)} \right)^2$$

where, in the second last line, we note that $H_2(T^*) \cong \mathbb{Z}_2$ and $H_2(T) \cong \mathbb{Z}_2$. The number of qubits in the code $n_{EFPC_{\{b,c,l\}}}$ scales asymptotically as $\left( \frac{b^2 - c^2}{1 - (b^2 - c^2)} \right)^2$, as does the number of encoded qubits $k_{EFPC_{\{b,c,l\}}}$, suggesting a constant rate $r_{EFPC_{\{b,c,l\}}} \equiv \frac{k_{EFPC_{\{b,c,l\}}}}{n_{EFPC_{\{b,c,l\}}}}$.

The degeneracy of this system comes from two different sources. The $H_1(T) \otimes H_1(T^*)$ term gives

FIG. 6: $\tilde{SC}^*(3,1,l)$ for $l = 0$ to $3$, neglecting the exterior vertices.

B. Defining the EFPCs
rise to an extensive number of encoded qubits \( \left( \frac{1 - (b^2 - c^2)}{1 - (b - c)} \right)^2 \), and we will call these qubits “local”. By contrast, the \( H_0(T) \otimes H_2(T^*) \) term produces a single qubit of degeneracy, and we call this qubit “global”.

The two kinds of encoded qubits have quite different properties, and we will largely focus on the global encoded qubit, leaving discussion of the local encoded qubits to Sec. V D.

Some intuition for the properties of an EFPC can be gained by considering its relation to the 4D toric code. Noting that both encoded qubit, leaving the two kinds of encoded qubits have quite different properties, and we will largely focus on the global encoded qubit, leaving discussion of the local encoded qubits to Sec. V D.

The two kinds of encoded qubits have quite different properties, and we will largely focus on the global encoded qubit, leaving discussion of the local encoded qubits to Sec. V D.

\[
H_{EFPC}(b, c, l) = - \sum_{A \in \{T \otimes T^*\}} \prod_{j \in \{H_{T \otimes T^*}^A\}} X_j - \sum_{B \in \{T \otimes T^*\}} \prod_{k \in \{H_{T \otimes T^*}^B\}} Z_k
\]  

for \( X_i \) and \( Z_i \) the relevant Pauli matrices on qubit \( i \).

In order to study the thermodynamic properties of the quantum Hamiltonians corresponding to \( EFPC(b, c, l) \), it will be convenient to consider each of the two sectors \( X \) and \( Z \) individually. Since the \( X \)-type and \( Z \)-type stabilizers commute pairwise, for the purposes of considering thermalization processes we can consider each sector separately (see Ref. [2] for an analogous discussion). We denote
the corresponding classical codes $EFPC_X(b,c,l)$ and $EFPC_Z(b,c,l)$ with associated Hamiltonians

$$H_{EFPC_X}(b,c,l) = - \sum_{A \in (T \otimes T^*)_L} \prod_{j \in \partial A} X_j$$

$$H_{EFPC_Z}(b,c,l) = - \sum_{B \in (T \otimes T^*)_L} \prod_{k \in \partial B} Z_k$$

As noted, these two systems are related by a rotation and so can be considered dual to one another, meaning we need only study the properties of one of these classical codes. The thermodynamic limit corresponds to taking $l \to \infty$ for fixed $b$ and $c$.

The main technical result of this paper is the following:

**Theorem 3.** The classical Hamiltonians corresponding to each of the classical codes $EFPC_X(b,c)$ and $EFPC_Z(b,c)$ have finite temperature phase transitions.

The proof of this theorem, given in Sec. IV C, makes use of two main tools: duality transformations and correlation inequalities. Both of these results apply to the class of generalized Ising models, to which $EFPC_X$ and $EFPC_Z$ belong. Generalized Ising models consist of ferromagnetic interactions on sets of spins (for simplicity, spin-$\frac{1}{2}$). Such systems are specified by a lattice of spins $\Lambda$, and interaction strengths $J_R \geq 0$ for each $R \subset \Lambda$, with a Hamiltonian

$$H((\{J_R\}_R) = - \sum_{R \in \Lambda} J_R \prod_{j \in R} Z_j$$

It is clear that each of the sectors of the EFPC can be trivially written in this way.

We will be interested in the phase transitions of such a model at finite inverse temperature $\beta$, and so it will be convenient to define rescaled interaction strengths $K_R = \beta J_R$, the set of non-trivial interactions $K = \{K_R > 0\}$, and their supports $B = \{R \subseteq \Lambda | K_R \in K\}$. We will also abuse notation and treat suitable sets as groups when convenient, with multiplication given as the exclusive union $g_1g_2 = (g_1 \cup g_2) \setminus (g_1 \cap g_2)$ for $g_i$ in an appropriate set such as $B$. Also of interest is the symmetry set (or group) $S$, with elements $S = \{R|(-1)^{|R \cap B|} = 1 \ \forall B \in B\}$, and $N_S$, defined as the number of generators of the group $S$, so that $|S| = 2^{N_S}$. $S$ should be understood physically as the group of spin flips that commute with each interaction. Finally, it is convenient to construct the group of constraints $C = \{C \subseteq B | \prod_{c_i \in C} \prod_{j \in c_i} Z_j = 1\}$. Elements of this group are sets of interactions that are not independent.

The partition function of a generalized Ising model is

$$Z(\Lambda, \mathcal{K}) = \sum_{R \subseteq \Lambda} \exp \left( \sum_{B \in \mathcal{B}} K_B (-1)^{|R \cap B|} \right)$$

where the $R$ represents possible sets of spins pointing down, and so $\prod_{j \in B} Z_j$ acting on the corresponding state will accumulate a phase $(-1)^{|R \cap B|}$.

**A. Correlation inequalities**

The GKS inequalities [41, 42] (see also Ref. [43]) are simple correlation inequalities for ferromagnetic systems. These inequalities state that for a generalized Ising model,

$$\left\langle \left( \prod_{j \in A \cup \Lambda} X_j \right) \left( \prod_{j' \in B \cup \Lambda} X_{j'} \right) \right\rangle \geq \left\langle \left( \prod_{j \in A} X_j \right) \right\rangle \left\langle \left( \prod_{j' \in B} X_{j'} \right) \right\rangle$$

An immediate corollary of this inequality is that

$$\frac{\partial}{\partial J_l} \left\langle \left( \prod_{j \in A \cup \Lambda} X_j \right) \right\rangle = \left\langle \left( \prod_{j \in A \cup \Lambda} X_j \right) \left( \prod_{j' \in K_l} X_{j'} \right) \right\rangle - \left\langle \left( \prod_{j \in A \cup \Lambda} X_j \right) \right\rangle \left\langle \left( \prod_{j' \in K_l} X_{j'} \right) \right\rangle \geq 0$$

Intuitively, increasing the strength of, or adding more ferromagnetic interactions cannot decrease the correlations present in the system. In particular, if there exists a (possibly long-range) ordered phase of a system, it cannot be destroyed by adding extra ferromagnetic terms to the Hamiltonian. This is the crucial sense in which we will make use of the GKS inequalities.
B. Duality transformations on general Ising models

We also make use of a particular derivation of duality transformations on general Ising models due to Merlini and Gruber [44] that have a natural geometrical interpretation. Given a system of spins $\Lambda$ and set of interactions $K$, Merlini and Gruber give a prescription to construct a dual system $\Lambda^*$ with interactions $K^*$ and a (surjective) map $\varphi : B \to B^*$ such that

$$Z(\Lambda, K) = \sqrt{2}^{(|\Lambda| - |\Lambda^*| + N_2 - N_2^*)} \prod_{B \in B} \sqrt{\sinh 2K(B)} \cdot Z(\Lambda^*, K^*)$$

(20)

for $e^{-2K^*(B^*)} = \prod_{B \in e^{-1}B^*} \tanh K(B)$. This relation between the partition functions ensures that a non-analyticity in the free energy (i.e. a phase transition) in the $(\Lambda^*, K^*)$ system also corresponds to a phase transition in the $(\Lambda, K)$ system (at an appropriately rescaled temperature).

In order to construct such a dual system, we consider a generating set of the constraint group $C$. For each generator $C_i$, we assign a point $v_i^* \in \Lambda^*$. The interaction regions $B_j^*$ are labelled by interaction terms $B_j \in B$ such that $B_j^* = \{v_i^* \in \Lambda^* | B_j \subset C_i\}$.

C. Finite temperature phase transition

Given the Merlini-Gruber duality transformation and the GKS inequalities, we can now prove Thm. 3. Consider $EFPC_Z(b,c,l)$. The set of spins in this model $\Lambda$ are associated with (a subset of the) faces of the hypercubic lattice. Of course, this is according to their presence in $(T \otimes T)^{2}$ regions lattice, where it is understood that exterior boundaries as neglected as appropriate). The interaction terms $B$ each generator $C_i$, and we determine a point $w_i^* \in \Lambda^*$. The interaction regions $B_j^*$ are labelled by interaction terms $B_j \in B$ such that $B_j^* = \{v_i^* \in \Lambda^* | B_j \subset C_i\}$.

Thus, the Hamiltonian of the dual system will consist of an Ising model on $\Lambda^*$ and a (surjective) map $\varphi : B \to B^*$ such that

$$H(\Lambda^*, K^*) = H_{SC} + \ldots$$

(21)

for $H_{SC}$ the ferromagnetic Ising model on the relevant $(b,c,l)$ Sierpinski carpet graph as described in Sec. II C.

Now appealing to the GKS inequality, if we were to begin with the system $H_{SC}$, then by adding the additional ferromagnetic terms to give $H(\Lambda^*, K^*)$ we cannot decrease the magnetization of the system. Since it is known that the Ising model on a Sierpinski carpet $H_{SC}$ has a magnetically ordered phase for some range of finite inverse temperatures $\beta$ [34–36], this tells us that $H(\Lambda^*, K^*)$ will also have such an ordered phase. Since $H(\Lambda^*, K^*)$ acts with a bounded density of bounded strength operators, it is also clear that in the $\beta \to 0$ limit the system will be disordered. Thus there must be a finite temperature phase transition.

Since $H(\Lambda^*, K^*)$ is dual to the Hamiltonian $H_{EFPC_Z}$, we conclude that it too possesses a finite temperature phase transition. The symmetry between $EFPC_Z$ and $EFPC_X$ implies that the same is true for the $X$ sector of the theory, completing the proof of Thm. 3.

Though this is by no means a rigorous argument, we intuitively associate the identified phase transitions with the global encoded qubit, as the Hamiltonian $H_{SC}$ in Eq. (21) corresponds only to action on those qubits of $H_{EFPC_Z}$ in a cross-sectional slice of the lattice, as does the logical operators of the global...
encoded qubit. Similar arguments for phase transitions corresponding to local encoded qubits can also be made, and will be discussed briefly in Sec. V D. The existence of finite temperature phase transitions such as those identified in Thm. 3 is indicative that the system may be able to robustly store quantum information at finite temperature.

V. DISCUSSION

We have shown that EFPCs satisfy Caltech rules 1, 2 and 3 for a 3D SCQM (finite spins, bounded local interactions, and nontrivial codespace). We have not proved that they satisfy rule 6 (exponential lifetime), though we believe that the existence of a finite temperature phase transition is strong evidence that this rule may be satisfied. It remains to discuss rules 4 and 5 (perturbative stability and efficient decoding).

A. Perturbative stability

Typically, perturbative stability of the codespace of a system similar to an EFPC is shown by proving that the gap and ground space degeneracy is stable under arbitrary quasi-local perturbations, as in the topological stability theorems [45–47]. This guarantees that a quasi-adiabatic continuation between the ground spaces of the perturbed and unperturbed models exists [45, 48], ensuring that their properties are stable. Though $H_{\text{EFPC}}$ are gapped, the topological stability theorems do not apply to our system, as the ground space does not have macroscopic distance (due to the presence of the local encoded qubits). Under generic perturbation, naive perturbation theory suggests that the degeneracies associated with the local encoded qubits will be lifted, while that associated with the global encoded qubit will be exponentially suppressed in system size as desired (due to the local indistinguishability of the global qubit states).

In order to make the stability of the global encoded qubit more concrete, a finer notion of perturbative stability seems to be required. We do not necessarily expect that our entire ground space will be stable, nor a subspace of it. Instead, it seems likely that by factorizing the ground space into the global encoded qubit system and the local encoded qubit systems, the global subsystem will be stable in the sense that an appropriate continuation could be constructed from the unperturbed logical space to the perturbed logical space. Similar considerations would apply to other systems that might be of independent interest, such as a toric code with a small but non-zero density of punctures, or a nonabelian anyon model with a finite density of particles.

B. Decoding algorithm

Due to the similarity between the 4D toric code and the EFPCs, it seems likely that any decoding algorithm for the 4D toric code may be adapted to decode the EFPCs. A notable example of such a decoder that also applies to topological codes in general is the topological renormalization group decoder due to Bravyi and Haah [12, 13]. Unfortunately, for the same reasons that we could not apply the topological stability theorems (namely the presence of local encoded qubits), the proof of threshold for this decoder does not directly apply for the EFPCs. However, we anticipate that this algorithm and proof of threshold could be adapted to our setting. Given the self-similar nature of the EFPCs, alternative renormalization group based decoding methods [49–51] might also be effective decoders for these systems.

For the purposes of decoding the global encoded qubit, the EFPC should be treated as a subsystem code [52, 53], with the local encoded degrees of freedom playing the role of gauge qubits. Other natural candidates for an EFPC decoding algorithm are the 4D toric code heat-bath algorithm [1] or a variant of Toom’s rule [54]. Particularly, a concrete proof that the EFPC is indeed a self-correcting memory would likely also guarantee the existence of a polynomial decoding algorithm, as in Ref. [2] for the 4D toric code. For similar reasons, we also expect that the EFPCs may be single-shot fault-tolerant [55].

C. Rigorous proof of memory lifetime

We have not proven that the memory lifetime of and EFPC scales exponentially with system size for some finite temperatures, simply that there are phase transitions of the system at finite temperature.
This seems highly suggestive that the system may function as a self-correcting quantum memory below the critical temperatures, but the relation between thermodynamic phase transitions and memory lifetime is not fully understood [56, 57] and so a rigorous proof of this fact is desirable. We anticipate that it may be possible to construct such a proof by combining the techniques used to prove the exponential lifetime of the 4D toric code [2] (see also Ref. [58]) with those used in the Peierls-type proof of the finite-temperature phase transition in the Sierpinski carpet Ising model [35, 36].

D. Local encoded qubits

Though most of our discussion has centered around the stability of the global encoded qubit, it seems likely that the local encoded qubits might also enjoy stability in an appropriately defined sense. At any fixed system size, there are local qubits whose distance is a small constant, associated with the smallest punctures to appear in the Sierpinski carpet at that level. However, if we were to increase the system size (by moving to the higher levels of the Sierpinski carpet graph), then the distance of the local qubits associated with these same holes will increase (as will the energy barrier needed to transition between the corresponding logical states).

If we focus on any one local qubit (i.e. any particular puncture in the lattice) as we approach the thermodynamic limit, we can also demonstrate a finite temperature phase transition associated with the region on which the logical operators of this qubit act. By performing the duality transformation as in Sec. IV C, and then disregarding all terms not acting along a boundary of the puncture, we find a 2D Ising model of increasing size as we increase \( l \). The 2D Ising model is of course known to have a finite temperature phase transition. Just as in Sec. IV C, the GKS inequality and the Merlini-Gruber duality transformation then guarantee an analogous finite temperature phase transition in the EFPC Hamiltonian. Again, we intuitively associate this phase transition to the local qubit under consideration, though we have not demonstrated this concretely.

It thus seems that any fixed local qubit may also enjoy thermal stability in a similar way to the global qubit, though the argument is more subtle. In particular, depending on precisely how the thermodynamic limit is taken, one might conclude that all logical qubits in an EFPC system may be thermally stable in principle, though this seems a counterintuitive proposition given that for all finite-sized EFPCs there exist logical operators with distance upper-bounded by a constant.

E. More general EFPCs

The main ideas of this construction are clearly not limited to the Sierpinski carpet fractals. The majority of the results and methods apply directly to any pair of well-behaved fractals (self-similar Borel sets satisfying the open set condition), with combined dimension \( < 3 \) and corresponding Ising models with finite temperature phase transitions. For these fractals an EFPC can be constructed in an analogous way, and may also act as a self-correcting quantum memory. Similarly, by varying the relative dimension of the two fractals used, one can introduce an asymmetry between the critical temperatures of the \( X \) and \( Z \) sectors of the system. Though general results are known on the existence of phase transitions in fractal graphs [32, 33, 36], there do not seem to be many well-known fractals that have the required properties apart from the Sierpinski carpets studied here, and slight generalizations thereof.

It should also be noted that the particular discretization of the fractal used to define the fractal graph does not seem to play a critical role in our construction. An alternative convention for defining the Sierpinski carpet graphs, for example as discussed in Refs. [30, 31], would be equally amenable to our analysis.

Though our construction seems to manifestly break translation invariance, it might also be possible to restore it approximately. There is some evidence that variants of the Sierpinski carpet with low lacunarity (a measure of violation of translation invariance) behave like concrete geometric realizations of hypercubic lattices with fractional dimension in certain limits [59]. It would be interesting to see whether the use of these fractal graphs would allow for a translation-invariant EFPC as a limiting case. In order to answer such questions, the precise details of the embedding into \( \mathbb{R}^3 \) would need to be investigated in much more detail.
F. Relation to previous work

Several existing quantum codes have relations to fractal geometry. Two of the most prominent are the Haah codes \[10\] and Yoshida’s fractal spin liquids \[60\]. While one motivation for considering such codes has also been to engineer self-correcting behaviour, the relationship between our work and these models appears largely superficial. In these previous works, the systems were typically defined as local, translation-invariant Hamiltonians acting on a regular cubic lattice. These systems are engineered so that the support of their logical operators is a fractal subset of the lattice. Significantly, such fractals are of finite ramification. In contrast, the EFPC Hamiltonians we study here are themselves defined on a fractal lattice, breaking translation invariance and directly giving rise to the infinitely ramified fractal logical operators. The breaking of translation invariance, along with scale invariance, also directly allows us to escape no-go theorems such as that of Ref. \[22\].

Recently, a new code construction has appeared that produces local subsystem codes with properties inherited from an arbitrary base stabilizer code \[61\]. Using a concatenated base code, the resulting subsystem code appears to have fractal structure. Again, the relationship between these codes and our work is purely superficial, in particular noting that the EFPCs are commuting stabilizer code models while the codes of Ref. \[61\] are non-commuting subsystem codes.

G. Numerical simulation

In the absence of a rigorous proof of self-correction, an attractive strategy is to attempt some numerical simulation of thermalization for an EFPC system. However, this could prove prohibitively difficult since the EFPCs are only defined for certain (exponentially spaced) system sizes. The smallest EFPC family, corresponding to \(b = 14\) and \(c = 12\), for \(l = 0, 1, 2, 3\) requires \(3.8 \times 10^4, 8.3 \times 10^7,\) and \(2.1 \times 10^{11}\) qubits respectively. Since the \(l = 0\) case is simply a standard toric code, this leaves very little ability to reasonably simulate these systems. Even by dropping the requirement that the system be embeddable in \(\mathbb{R}^3\), the smallest code family (with \(b = 3, c = 1\)) still requires \(1.4 \times 10^8\) qubits for \(l = 4\). In order to realistically simulate these systems, it may be necessary to find some way to consistently interpolate between these system sizes.

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