Arithmetic on curves
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We intend to give a brief account of what is known or conjectured about the set \( \mathbb{C}(\mathbb{Q}) \) of rational points on a smooth projective absolutely connected curve \( \mathbb{C} \) of genus \( g \) over \( \mathbb{Q} \). The idea is to show how the arithmetic properties of algebraic curves are governed by the familiar trichotomy: \( g = 0 \), \( g = 1 \), \( g \geq 2 \). Only incidentally shall we mention fields other than \( \mathbb{Q} \) and varieties other than curves.

We shall not give even the definitions of all the concepts which appear below, let alone the proofs of the theorems we state. Our wish is merely to provide some references — for the most part to survey articles and books — which the interested reader is advised to look up for the precise definitions, statements and — where they are known — proofs.

If I were to summarise in a few lines the work of more than a hundred mathematicians on this topic over the last century, I’d say: if \( g = 0 \), then \( \mathbb{C}(\mathbb{Q}) \) is either empty or infinite; we know how to distinguish between the two cases and, in the latter case, how to describe it completely. If \( g = 1 \), we don’t know when \( \mathbb{C}(\mathbb{Q}) \) can be empty; if it is not empty, it is a torsor under a finitely generated commutative group \( T \) attached to \( \mathbb{C} \); we know how to compute the torsion subgroup of \( T \) but not its rank; we know which groups can occur as the torsion but not which integers as ranks. Finally, if \( g \geq 2 \), we know that \( \mathbb{C}(\mathbb{Q}) \) is finite but not how “large” its points can be, nor whether the number of points remains bounded as \( \mathbb{C} \) varies among curves of genus \( g \). These cryptic remarks will now be slightly amplified.

A smooth projective curve of genus 0 can be realised as a plane conic \( \mathbb{C} \) of the form
\[
aX^2 + bY^2 + cZ^2 = 0 \quad (a, b, c \in \mathbb{Q}^\times).
\]
A theorem of Legendre [33] can be rephrased as saying that for \( \mathbb{C} \) to have a rational point it is sufficient for it to have a real point — which amounts to saying that \( a, b, c \) are not of the same sign — and, for every odd prime \( p \), a point over \( \mathbb{Q}_p \). As a matter of fact, it is sufficient to check solvability at the real place and — after rescaling \( X, Y, Z \) so that the coefficients \( a, b, c \) are in \( \mathbb{Z} \) — to check that the congruence \( aX^2 + bY^2 + cZ^2 \equiv 0 \) mod. \( M \) has a primitive solution for a certain integer \( M \) determined by the coefficients. Thus, checking for solvability is a finite amount of computation; so is finding a solution if there is one. If \( \mathbb{C}(\mathbb{Q}) = \emptyset \), one can write down a quadratic extension \( \mathbb{K} \) of \( \mathbb{Q} \) such that \( \mathbb{C}(\mathbb{K}) \neq \emptyset \). If \( \mathbb{C} \) has a rational point, then it is isomorphic to the projective line \( \mathbb{P}_1 \), so that we have an explicit description of the set \( \mathbb{C}(\mathbb{Q}) \) of all its rational points. Similar results continue to hold over finite extensions of \( \mathbb{Q} \).

The set \( \mathbb{P}_1(\mathbb{Q}) \) comes equipped with a natural “height” function \( H \) which measures
the size of a rational point \( P \): one defines \( H(P) = \sup_i |x_i| \) for \( P = (x_0 : x_1) \), where \( x_0, x_1 \) are rational integers with no common factor (determined by \( P \) up to sign). The number of points of height \( \leq B \) grows as \( \frac{2}{\zeta(2)} B^2 \) when \( B \to +\infty \). There is also a good expression for the “error term” [27].

Curves of genus 0 are the 1-dimensional case of quadrics — smooth projective varieties defined by a quadratic form. Hasse’s theorem asserts that the local to global principle continues to hold for quadrics: if a smooth quadric has a point over every completion of \( \mathbb{Q} \), then it has a rational point [24]. Curves of genus 0 are also the 1-dimensional case of varieties which are potentially isomorphic to the projective space \( \mathbb{P}_n \): those varieties which become isomorphic to \( \mathbb{P}_n \) over \( \overline{\mathbb{Q}} \) — one also says that they are twisted forms of \( \mathbb{P}_n \). The local to global principle holds for them as well. These results about smooth quadrics and about twisted forms of projective spaces remain valid over any number field \( K \) [25].

The asymptotic growth of the number of points of bounded height on \( \mathbb{P}_n(K) \) has been established by Schanuel, although the best possible “error term” seems to be known only in the case of the projective line over \( \mathbb{Q} \) [27].

In contrast to the genus 0 case, the local to global principle fails for curves of genus 1. The best-known example is the curve \( 3X^3 + 4Y^3 + 5Z^3 = 0 \) which has points in every completion of \( \mathbb{Q} \) but no rational points [13]. There is no proven algorithm to decide, given a curve \( C \) of genus 1, whether \( C(\mathbb{Q}) \) is empty or not. If \( C(\mathbb{Q}) = \emptyset \), there appears to be no characterisation of the finite extensions \( K \) of \( \mathbb{Q} \) such that \( C(K) \neq \emptyset \) or of the least integer \( n \geq 0 \) which can be the degree of a 0-cycle on \( C \) [30].

If \( C(\mathbb{Q}) \) is not empty and if we fix a point \( O \in C(\mathbb{Q}) \), there is a unique (commutative) group law \( C \times C \to C \) for which \( O \) is the neutral element; a curve with such a group law is called an elliptic curve [28] — it would have been more appropriate to call them abelian curves since they are 1-dimensional abelian varieties. A theorem of Mordell asserts that the commutative group \( E(\mathbb{Q}) \) is finitely generated ([28], [29]) for any elliptic curve \( E \) over \( \mathbb{Q} \). There is no proven algorithm to decide, given an elliptic curve \( E \) over \( \mathbb{Q} \), whether the group \( E(\mathbb{Q}) \) is finite or not. If \( E(\mathbb{Q}) \) is finite, there seems to be no characterisation of those finite extensions \( K \) of \( \mathbb{Q} \) for which \( E(K) \) is infinite.

For a given elliptic curve \( E \) over \( \mathbb{Q} \), a fairly easy theorem of Élisabeth Lutz and Nagell determines the torsion subgroup of \( E(\mathbb{Q}) \) [28]. A deep theorem of Mazur [26] gives the list of all the (finite commutative) groups which can occur as the torsion subgroup of an elliptic curve over \( \mathbb{Q} \); this list is finite and explicitly given; given a group \( G \) on the list, one knows the elliptic curves of which \( G \) is the rational torsion. On the other hand, given an elliptic curve \( E \) over \( \mathbb{Q} \), it is not easy to compute the rank of the group \( E(\mathbb{Q}) \) and there is no proven algorithm which is guaranteed to do so. It is not known which numbers can occur as the rank of \( E(\mathbb{Q}) \) when \( E \) varies among all elliptic curves defined over \( \mathbb{Q} \) [22].
In any attempt at computing the group $E(\mathbb{Q})$, one comes across the group $\Theta(E, \mathbb{Q})$ of $E$-torsors over $\mathbb{Q}$ which are trivial over every completion of $\mathbb{Q}$; it is widely conjectured to be finite. Manin has proved that its finiteness for every elliptic curve would allow us to give an algorithm to test the existence of a rational point on a curve of genus 1 \cite{10}.

The most important open problem in the arithmetic theory of elliptic curves is the conjecture of Birch and Swinnerton-Dyer \cite{1}. It involves the $L$-function $L(E, s)$ of $E$ — a function of a complex variable $s$ defined using the number of points of $E$ over the various finite fields $\mathbb{F}_p$ (this can be made precise) \cite{28}. It was widely conjectured that $L(E, s)$ admits an analytic continuation to the whole of $\mathbb{C}$. This is now known as a consequence of the seminal work of Wiles, continued by Breuil, Conrad, Diamond and Taylor \cite{4}, \cite{5}. Birch and Swinnerton-Dyer conjecture that the rank of $E(\mathbb{Q})$ equals the order of vanishing $r = \text{ord}_{s=1} L(E, s)$ of $L(E, s)$ at $s = 1$. Assuming the finiteness of the group $\Theta(E, \mathbb{Q})$, a refined version of this conjecture gives an expression for the “special value” $\lim_{s \to 1} \frac{L(E, s)}{(s - 1)^r}$ in terms of $\text{Card} \, \Theta(E, \mathbb{Q})$ and certain other arithmetic invariants of $E$. Manin has proved that the truth of this conjecture would give an effective method for computing the order of $\Theta(E, \mathbb{Q})$ and a system of generators for $E(\mathbb{Q})$ \cite{11}.

Results of Gross & Zagier and Kolyvagin, among others, imply that if $L(E, s)$ has at most a simple zero at $s = 1$, then $\Theta(E, \mathbb{Q})$ is finite, $\text{rk} \, E(\mathbb{Q}) = \text{ord}_{s=1} L(E, s)$ and the conjectured formula for the special value of $L(E, s)$ at $s = 1$ is true — to within small powers of small primes \cite{13}, \cite{21}. Practically nothing is known about the rank of $E(\mathbb{Q})$ when the order of vanishing of $L(E, s)$ is $\geq 2$ (in fact, no elliptic curve is known whose $L$-function can be shown to vanish to order $\geq 4$). In the other direction, it is not known that if $E(\mathbb{Q})$ is finite, then $L(E, 1) \neq 0$ or that if $E(\mathbb{Q})$ is of rank 1, then $L(E, s)$ has a simple zero at $s = 1$. An important recent result of Nekovář \cite{17} says that if $\Theta(E, \mathbb{Q})$ is finite, then $\text{ord}_{s=1} L(E, s)$ and $\text{rk} \, E(\mathbb{Q})$ have the same parity.

Which integers $d \geq 1$ can be the area of a right triangle with rational sides? This ancient problem comes down to determining the $d$ for which the group $E_d(\mathbb{Q})$ is infinite, $E_d$ being the elliptic curve $dy^2 = x^3 - x$. Tunnell has given a characterisation under the assumption that $E_d(\mathbb{Q})$ is infinite if $L(E_d, 1) = 0$ — a very special case of the Birch and Swinnerton-Dyer conjecture. Even this particular case is not yet known \cite{9}.

Mazur’s theorem about the possible torsion subgroups has been extended by Merel \cite{15} to all number fields, although the explicit list of all the possibilities is known only in a few cases \cite{6}.

Elliptic curves are the 1-dimensional case of abelian varieties. The theorem of Mordell has been generalised to abelian varieties over number fields by Weil \cite{16} and indeed to abelian varieties over fields finitely generated over their prime subfields by Néron \cite{18}. He
has also given an expression for the asymptotic growth of the number of points of bounded height on a projectively embedded abelian variety over a number field [27].

There is a version [31] of the conjecture of Birch and Swinnerton-Dyer for abelian varieties \( A \) over global fields \( K \) which predicts — assuming the analytic continuation of the L-function \( L(A, s) \) to the whole of \( \mathbb{C} \) — that the rank of the (finitely generated commutative) group \( A(K) \) of \( K \)-rational points of \( A \) is equal to \( r = \text{ord}_{s=1} L(A, s) \). There is also a conjectural expression for the “special value” \( \lim_{s \to 1} \frac{L(A, s)}{(s-1)^r} \) in terms of certain arithmetic invariants of \( A \) over \( K \), among them the order of the conjecturally finite group \( \text{III}(A, K) \) of \( A \)-torsors over \( K \) which are trivial over every completion of \( K \).

Elliptic curves \( E \) defined over the function field \( K \) of a curve over a finite field \( k \) have been used by Elkies and Shioda to give the best known examples of sphere packings in certain dimensions [20]. Ulmer [32] has given examples of \( E | K \) which have arbitrarily high ranks and such that \( E_K \) is not definable over \( \overline{K} \). In both these results, the known cases of the Birch and Swinnerton-Dyer conjecture play a prominent role.

An important recent result of Kato and Trihan [8] says that if the group \( \text{III}(A, K) \) is finite for an abelian variety \( A \) over a function field \( K \) over a finite field, then the conjecture of Birch and Swinnerton-Dyer is true for \( A \): the rank of \( A(K) \) equals the order \( r \) of vanishing of \( L(A, s) \) at \( s = 1 \) and equality holds in the conjectured expression for \( \lim_{s \to 1} \frac{L(A, s)}{(s-1)^r} \).

Let us now come to a curve \( C \) of genus \( \geq 2 \) over a number field \( K \). Shuji Saito, extending Manin’s result for the case \( g = 1 \), has proved that the failure of the local to global principle for the existence on \( C \) of a 0-cycle of degree 1 can be accounted for by an obstruction introduced by Manin if the group \( \text{III}(J, K) \) — where \( J \) is the jacobian of \( C \) — is finite [10].

The most striking result about the set \( C(K) \) is Faltings’ proof [3] that it is finite, as had been conjectured by Mordell; the function field case was treated earlier by Grauert, Manin and Samuel [23]. Let \( f \in \mathbb{Q}[X, Y] \) be an absolutely irreducible polynomial defining a (smooth projective) curve \( C \) of genus \( \geq 2 \). An important open problem asks for a bound on the size or height a point in \( C(\mathbb{Q}) \) in terms of the coefficients of \( f \). It has been proved [7] that a version of the celebrated abc conjecture ([14], [19]) implies such a bound.

Lang has conjectured [10] that for a (smooth projective) variety \( V \) of general type over a number field \( K \), the set \( V(K) \) is never dense in \( V \); in the case \( \dim V = 1 \), this is a restatement of Mordell’s conjecture as proved by Faltings. This conjecture has been shown to imply the boundedness of \( \text{Card} C(K) \) as \( C \) runs through smooth projective curves of a given genus \( g \geq 2 \) over the number field \( K \) [2].

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