KNOTS AND CLASSICAL 3-GEOMETRIES

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Abstract. It has been conjectured by Rovelli that there is a correspondence between the space of link classes of a Riemannian 3-manifold and the space of 3-geometries (on the same manifold). An exact statement of his conjecture will be established and then verified for the case when the 3-manifold is compact, orientable and closed.

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1. Introduction.

In [6, p. 1661], Rovelli sketched a proof showing how a certain collection of n-loops, which he called weaves, are related to the flat 3-metric. He then conjectured that perhaps there exists a relationship between n-loops, for n < ∞, and 3-metrics. The relationship between n-loops, for n < ∞, and 3-metrics will not be answered in this paper (and so, it still remains an open question); however, what will be shown in this article is that there exists a precise relationship between 3-geometries and a subset of ℵ₀-knots, where an n-knot is defined to be an equivalence class of n-loops

\[ n \text{-loop } \gamma \overset{\text{def}}{=} \{ \gamma^1, \ldots, \gamma^n \} \text{ is just a subset of the loop space consisting of } n \text{ loops; i.e., } \gamma^i, \text{ for each } i = 1, \ldots, n, \text{ are (distinct) closed curves in } \Sigma. \]
under (smooth) ambient isotopies (cf. §3). The approach given here is entirely different to that outlined by Rovelli in [6]: tersely, Rovelli introduced a lattice spacing on the 3-manifold—the distance between parallel non-intersecting curves which defines a weave in 3-space—to obtain his conclusion regarding weaves and flat metrics; this, in turn, motivated his conjecture. Here, no such assumptions will be made and the results are purely ‘topological’.

The attention here will be focused on compact, Riemannian 3-manifolds. In this paper, the term Riemannian metric means a smooth, non-degenerate, symmetric, covariant 2-tensor that is positive-definite on the base manifold. The fact that the 3-manifold is separable is crucial in the construction: this, at least, explains why \( \aleph_0 \)-loops are used rather than \( n \)-loops for \( n < \infty \). The main interest in Rovelli’s conjecture is that it will provide a tentative physical interpretation of the loop representation of quantum gravity [7]: it yields a possible insight into the interweaving of topology and geometry at the quantum level. More will be said in section 5.

In all that follows, the spatial (Riemannian) 3-manifold, denoted by \( \Sigma \), is assumed to be smooth, orientable, closed and compact; \( \mathbb{R}_+ \equiv \{ s \in \mathbb{R} \mid s \geq 0 \} \) and \( I \equiv [0, 1] \). Lastly, let \( \text{Diff}^+(\Sigma) \) denote the group of smooth, orientation-preserving, diffeomorphisms on \( \Sigma \). An overview of this paper runs as follows: section 2 introduces the required notations and definitions, whilst in section 3, the property of the space of \( \aleph_0 \)-knots of a subset of \( \aleph_0 \)-loops will be examined. This space will establish the sought for correspondence between topology and geometry. In section 4, a variant of the Rovelli conjecture will be formulated precisely and then verified; then, in section 5, some interesting speculations regarding the results established in §4 will be outlined.

2. Preliminary Definitions and Notations.

Let \( \Omega_\Sigma = \{ \gamma : I \rightarrow \Sigma \mid \gamma(0) = \gamma(1), \gamma \text{ continuous} \} \) be the space of loops in \( \Sigma \) and equip it with the compact-open topology. Fix a Riemannian metric \( \hat{q} \) on \( \Sigma \) and let \( \hat{d} : \Sigma \times \Sigma \rightarrow \mathbb{R}_+ \) be the distance function on \( \Sigma \) induced by \( \hat{q} \). Then, the metric \( d_\Omega : \Omega_\Sigma \times \Omega_\Sigma \rightarrow \mathbb{R}_+ \) defined by

\[
d_\Omega(\gamma, \eta) \equiv \sup_{t \in I} \hat{d}(\gamma(t), \eta(t))
\]

induces a metric topology on \( \Omega_\Sigma \) which is compatible with its compact-open topology [1, p. 263, theorem 4.2.17].
2.1. **Remark.** Since for each (admissible) Riemannian metric \( q \) on \( \Sigma \),\(^2\) the \( d_q \)-topology—where \( d_q \) is the distance function induced on \( \Sigma \) by \( q \)—coincides with the manifold topology, it follows that all the metrics \( d_q \) are equivalent to one another. Hence, all the metrics \( d_\Omega \) are also equivalent.

Let \( \tilde{L}_\Sigma \subset \Omega_\Sigma \) be the space of piecewise smooth loops in \( \Sigma \) endowed with the subspace topology. Next, quotient away the constant loops—i.e., \( \gamma(I) = \{ x_\gamma \} \), some \( x_\gamma \in \Sigma \)—in \( \tilde{L}_\Sigma \) as follows. Define an equivalence relation \( \tilde{R} \subset \tilde{L}_\Sigma \times \tilde{L}_\Sigma \) such that \( \forall (\gamma, \eta) \in \tilde{R}, \gamma \) and \( \eta \) are constant loops in \( \tilde{L}_\Sigma \). Let \( L_\Sigma \overset{\text{def}}{=} \tilde{L}_\Sigma / \tilde{R} \) be the quotient space and \( \tilde{\pi} : \tilde{L}_\Sigma \to L_\Sigma \) the natural map. It is clear that \( \tilde{R} \) is closed in \( \tilde{L}_\Sigma \times \tilde{L}_\Sigma \). Furthermore, observe from the construction that if \( L_0 = \{ \gamma \in \tilde{L}_\Sigma \mid \gamma \) is a constant loop \}, then \( \tilde{\pi}(\tilde{L}_\Sigma - L_0) \) coincides with the inclusion map \( \tilde{L}_\Sigma - L_0 \hookrightarrow \tilde{L}_\Sigma \); that is, \( \tilde{\pi}(\tilde{L}_\Sigma - L_0) = \text{id}_{\tilde{L}_\Sigma}(\tilde{L}_\Sigma - L_0) \).

2.2. **Lemma.** \( L_0 \) is closed and nowhere dense in \( \tilde{L}_\Sigma \).

**Proof.** Let \( \{ \gamma_n \}_n \) be a sequence in \( L_0 \) which converges to \( \gamma_0 \in \tilde{L}_\Sigma \). By definition, \( \gamma_n(I) = x_n \in \Sigma \forall n \) and \( \Sigma \) compact Hausdorff imply that \( \exists x_0 \in \Sigma \) and a subsequence \( \{ x_{n_k} \}_k \subset \{ x_n \}_n \) such that \( x_{n_k} \to x_0 \). Since \( \forall \varepsilon > 0, \exists N_\varepsilon > 0 \) such that \( d_\Omega(\gamma_n, \gamma_0) = \sup_{t \in I} d(\gamma_n(t), \gamma_0(t)) = \sup_{t \in I} \hat{d}(x_n, \gamma_0(t)) < \varepsilon \forall n > N_\varepsilon \), it follows at once that \( \gamma_0(t) \equiv x_0 \) on \( I \) and \( L_0 \) is thus closed, where \( d_\Omega | \tilde{L}_\Sigma \times \tilde{L}_\Sigma \) is denoted by \( d_\Omega \) for simplicity.

Finally, to complete the proof, suppose that the interior \( L_0^\circ \neq \emptyset \). Then, for any fixed \( \eta \in L_0^\circ \), there exists a neighbourhood \( N_\eta = \bigcap_{i=1}^n M(K_i, O_i) \), for some \( n < \infty \)—where \( M(K_i, O_i) = \{ \gamma \in \tilde{L}_\Sigma \mid K_i \subset I \) is compact, \( \gamma(K_i) \subset O_i, O_i \subset \Sigma \) is open \)—such that \( N_\eta \subset L_0^\circ \). However, \( \eta \in N_\eta \) and \( \eta(I) = x_\eta \) for some \( x_\eta \in \Sigma \Rightarrow O \equiv \bigcap_{i=1}^n O_i \neq \emptyset \). Let \( L_\eta = \{ \gamma \in \tilde{L}_\Sigma \mid \gamma(I) \subset O \} - L_0 \). Evidently, \( L_\eta \neq \emptyset \), and in particular, \( L_\eta \cap N_\eta \neq \emptyset \), which is a contradiction. Hence, \( L_0^\circ \equiv \emptyset \), as required. \( \square \)

The neighbourhood base of \( 0_\Sigma \equiv \tilde{\pi}(\gamma) \forall \gamma \in L_0 \) can now be constructed. It follows from the quotient topology and lemma 2.2 that for each neighbourhood \( N_{0_\Sigma} \) of \( 0_\Sigma \), \( \tilde{\pi}^{-1}(N_{0_\Sigma}) \) must be a neighbourhood of \( L_0 \). Hence, from the definition of \( \tilde{\pi} \), the neighbourhood base of \( 0_\Sigma \) consists precisely of subsets \( \tilde{\pi}(N) \), where \( N \) is a neighbourhood of \( L_0 \) in \( \tilde{L}_\Sigma \). Explicitly, a neighbourhood of \( 0_\Sigma \) is of the form

\(^2\)The topology on \( \Sigma \) and its differentiable structure are of course assumed fixed throughout the discussion.
\[ \bigcup_{\eta \in \mathcal{L}_0} \tilde{\pi}(N_\eta), \] where \( N_\eta \) is a neighbourhood of \( \eta \) in \( \tilde{L}_\Sigma \). Notice however, that \( \tilde{\pi} \) is not an open map. For given any neighbourhood \( N_\eta \) of \( \eta \in \mathcal{L}_0 \), \( \tilde{\pi}^{-1} \circ \tilde{\pi}(N_\eta) = N_\eta \cup \mathcal{L}_0 \) which is neither closed nor open (by lemma 2.2) if \( \mathcal{L}_0 \not\subset N_\eta \). Nevertheless, for each neighbourhood \( N \) of \( \mathcal{L}_0 \), \( \tilde{\pi}(N) \) is a neighbourhood of \( 0_{\Sigma} \).

**2.3. Lemma.** \( \tilde{\pi} : \tilde{L}_\Sigma \to L_\Sigma \) is closed.

**Proof.** To establish this claim, it is enough to show that \( \tilde{\pi} \) maps closed neighbourhoods \( C_\eta \) of \( \eta \in \mathcal{L}_0 \) into closed neighbourhoods of \( 0_{\Sigma} \). Invoking the quotient topology, it will suffice to verify that \( \tilde{\pi}^{-1} \circ \tilde{\pi}(C_\eta) \) is closed in \( \tilde{L}_\Sigma \). Since \( \tilde{\pi}^{-1} \circ \tilde{\pi}(C_\eta) = C_\eta \cup \mathcal{L}_0 \) by definition, lemma 2.2 yields the desired result. \( \square \)

**2.4. Theorem.** \( L_\Sigma \) is metrizable.

**Proof.** By corollary 2.3, it will suffice to show that \( L_\Sigma \) is first countable [1, p. 285, theorem 4.4.17], and from the definition of \( \tilde{\pi} \), it is enough to verify that \( 0_{\Sigma} \) has a countable neighbourhood base, since each \( \gamma \in L_\Sigma - \{0_{\Sigma}\} \) has a countable neighbourhood base (by definition). Let \( \mathcal{B}_\eta = \{ B_\frac{1}{n}(\eta) \mid n \in \mathbb{N} \} \) be a countable neighbourhood base of \( \eta \in \mathcal{L}_0 \) in \( \tilde{L}_\Sigma \). Since, by definition, each subset of the form \( \bigcup_{\eta \in \mathcal{L}_0} \tilde{\pi}(B_\frac{1}{n}(\eta)) \) defines a neighbourhood of \( 0_{\Sigma} \), it follows that the collection \( \mathcal{B}_{0_{\Sigma}} \) defined by

\[
\mathcal{B}_{0_{\Sigma}} = \left\{ \bigcup_{\eta \in \mathcal{L}_0} \tilde{\pi}(B_\frac{1}{n}(\eta)) \mid n \in \mathbb{N} \right\}
\]

is a countable neighbourhood base of \( 0_{\Sigma} \). Hence, \( L_\Sigma \) is metrizable, as claimed. \( \square \)

In the following account, call a curve in \( \Sigma \) a \textit{q-geodesic} if it is a geodesic in \( \Sigma \) relative to the Riemannian metric \( q \). Also, if \( \gamma, \eta \) are curves such that \( \gamma(1) = \eta(0) \), then define \( \gamma \ast \eta \) by

\[
\gamma \ast \eta(t) = \begin{cases} 
\gamma(2t) & \text{for } 0 \leq t \leq \frac{1}{2}, \\
\eta(2t-1) & \text{for } \frac{1}{2} \leq t \leq 1.
\end{cases}
\]

**2.5. Definition.** Let \( \gamma \in L_\Sigma \). Then, \( \gamma \) is said to be a \textit{piecewise geodesic loop} if there exists a Riemannian metric \( q \) on \( \Sigma \) and \( n \) smooth \( q \)-geodesics \( \gamma_1, \ldots, \gamma_n : I \to \Sigma \), \( 1 \leq n < \infty \), such that \( \gamma = \gamma_1 \ast \cdots \ast \gamma_n \).

Let \( \Gamma^+_2 \) denote the space of (smooth) Riemannian metrics on \( \Sigma \) (endowed with the compact \( C^\infty \) topology\(^4\)) and \( D_\Sigma \subset \Sigma \) a countably dense subset of \( \Sigma \). Now,

\(^3\)Note that each \( \gamma_i \) in \( \gamma \) is still a q-geodesic with respect to its new parametrization \([\frac{i-1}{n}, \frac{i}{n}]\), as the reparametrization \( \gamma_i(t) \to \gamma_i(nt - i + 1) \equiv \gamma_i[\frac{i-1}{n}, \frac{i}{n}] \) is clearly an affine transformation.

\(^4\)The compact \( C^\infty \) topology is defined in the appendix.
define $\mathcal{M}_\infty[q]$, for each $q \in \Gamma_2^+$, to be the set of all countably infinite multi-loops $\gamma = \{ \gamma^i | i \in \mathbb{N} \}$ satisfying the following two properties:

1. for each $i$, $\gamma^i \in L_\Sigma$ is a piecewise (affinely parametrized) $q$-geodesic loop in $\Sigma$,
2. the subset $\gamma$ is in bijective correspondence with $D_\Sigma$ under the map $\gamma^i \mapsto \gamma^i(0)$.

Finally, set $\mathcal{M}_\infty[\Gamma_2^+] = \bigcup_{q \in \Gamma_2^+} \mathcal{M}_\infty[q]$. An immediate consequence of the definition is the following two observations. Suppose $\gamma \in \mathcal{M}_\infty[q] \cap \mathcal{M}_\infty[q']$. Let $\Gamma(q)$ and $\Gamma(q')$ be the Riemannian connections of $q$ and $q'$ respectively. Fix an admissible atlas $\{(U_\alpha, \psi_\alpha)\}_\alpha$ on $\Sigma$. Then, with respect to each chart $U_\alpha$,

$$\left(\ddot{\gamma}_\alpha^i\right)^{\ell} + \Gamma_\alpha(q_{k_j}^\ell)\left(\dot{\gamma}_\alpha^i\right)^{k\ell} \overset{a.e.}{=} 0 \quad \text{and} \quad \left(\ddot{\gamma}_\alpha^i\right)^{\ell} + \Gamma_\alpha(q_{k_j}^\ell)\left(\dot{\gamma}_\alpha^i\right)^{k\ell} \overset{a.e.}{=} 0$$

on $\gamma^i(I) \cap U_\alpha$ for each $i$ (no summation over $\alpha$, obviously), where $F(t) = 0$ on $I$ apart from a finite number of points in $I$. Hence, $(\Gamma_\alpha(q_{k_j}^\ell) - \Gamma_\alpha(q_{k_j}^\ell))\left(\dot{\gamma}_\alpha^i\right)^{k\ell} \overset{a.e.}{=} 0 \ \forall \gamma^i \in \gamma$ and $\alpha$. Thus, by property (2), $\Gamma(q_{k_j}^\ell)(x) \equiv \Gamma(q_{k_j}^\ell)(x)$ for each $i \in \mathbb{N}$ since $\gamma^i \cap U_\alpha$ is a dense subset of $\Sigma$ as $\bigcup_{\gamma \in \gamma} \gamma^i(0)$ by (2). Hence, invoking the continuity of $\Gamma(h)$ for $h = q, q'$, it follows at once that $\Gamma(q) \equiv \Gamma(q')$ on $\Sigma$. Now, with respect to local coordinate basis, $\Gamma(q)_{k_j}^i = \frac{1}{2}q^{ih}(\partial_k q_{hj} + \partial_j q_{hk} - \partial_h q_{kj})$ (and likewise for $q'$); consequently, $q$ and $q'$ are related homothetically; that is, $\exists c > 0$ constant such that $q' = cq$. More generally, $q, q'$ are related by some coordinate transformation, as is shown below.

Let $f : \Sigma \to \Sigma$ be a smooth diffeomorphism, where $\Sigma = (\Sigma, q)$ and set $\Sigma_f = f(\Sigma) \overset{\text{def}}{=} (\Sigma, (f^{-1})^*q)$. Clearly, if $\gamma : I \to \Sigma$ is a $q$-geodesic, then $\gamma : I \to \Sigma_f$ is an $(f^{-1})^*q$-geodesic in $\Sigma_f$ and conversely, by symmetry (as isometries map geodesics into geodesics). Hence, in view of these two observations, each $\gamma \in \mathcal{M}_\infty[\Gamma_2^+]$ is assigned to a unique 3-geometry of $\Sigma$, where the space of 3-geometries is defined to be the quotient space $\mathcal{Q} = \Gamma_2^+/\text{Diff}^+(\Sigma)$. Recall that each element $[q] \in \mathcal{Q}$ is defined by $[q] = \{ f^*q | f \in \text{Diff}^+(\Sigma) \}$. Let $\pi_+ : \Gamma_2^+ \to \mathcal{Q}$ denote the natural projection. Then, $\pi_+$ is open [2, p. 317, §3.1] and $\mathcal{Q}$ is a second countable, metrizable space [2, p. 318, theorem 1].

As a converse remark, notice that if $\Sigma$ were not separable or that $\gamma_q = \{ \gamma_q^i | i \in \mathbb{N} \}$ were not chosen to satisfy (2), $\gamma_q$ need not uniquely determine $[q] \in \mathcal{Q}$. For want of a better term, call $\mathcal{M}_\infty[\Gamma_2^+]$ the space of piecewise geodesic $\aleph_0$-loops.

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5 Note trivially that as $q, q'$ are positive-definite, $c < 0$ is not an admissible solution.
Now, a suitable topology can be defined on this space. To do this, let $L_\infty$ be the set of affinely parametrized, piecewise geodesic loops in $\Sigma$ and let $L_\infty^\Sigma$ denote the countably infinite set-theoretic product of $L_\Sigma$. Define an equivalence relation $R_\Sigma \subset L_\infty^\Sigma \times L_\infty^\Sigma$ by

$$R_\Sigma \overset{\text{def}}{=} \{ (\gamma, \gamma') \subset L_\infty^\Sigma \times L_\infty^\Sigma : [\gamma] = [\gamma'] \},$$

where $[\eta] \overset{\text{def}}{=} \{ \eta^i \mid i \in \mathbb{N} \}$ is just the set of components of $\eta \overset{\text{def}}{=} (\eta^i)_{i=1}^\infty$. Let $\pi_\Sigma : L_\infty^\Sigma \rightarrow M_\Sigma \overset{\text{def}}{=} L_\infty^\Sigma / R_\Sigma$ denote the natural map. Then clearly, as a subset, $\mathcal{M}_\infty[\Gamma^+_2] \subset M_\Sigma$.

Now, let $M_\infty \subset L_\infty^\Sigma$ be a subset satisfying

(i) for each $\gamma = (\gamma^i)_{i=1}^\infty$, $\gamma^i \neq \gamma^j \ \forall i \neq j$,

(ii) $\pi_\Sigma(M_\infty) = \mathcal{M}_\infty[\Gamma^+_2]$.

It is clear from the definition of $M_\infty$ that there exists a family of subsets $M_\sigma \subset M_\infty$ such that

(a) $M_\infty = \bigcup_\sigma M_\sigma$,

(b) $M_\sigma \cap M_{\sigma'} = \emptyset \ \forall \sigma \neq \sigma'$,

(c) $\pi_\Sigma|M_\sigma : M_\sigma \rightarrow \mathcal{M}_\infty[\Gamma^+_2]$ is a (set-theoretic) bijection.

Let $h_\sigma = \overset{\pi_\Sigma}{\overset{\text{def}}{\mid}} M_\sigma$ and for each $\gamma \in \mathcal{M}_\infty[\Gamma^+_2]$, set $\gamma_\sigma = h_\sigma^{-1}(\gamma) \in M_\sigma$. The subsets $M_\sigma$ admit suitable metrics to be constructed below. Firstly, fix a finite atlas $\mathfrak{A}$ on $\Sigma$ and let $\tilde{d}_\Omega$ be a metric on $L_\Sigma$ compatible with its quotient topology. Then, for any pair $\gamma, \eta \in M_\sigma$, let $d_\sigma(\gamma, \eta) \overset{\text{def}}{=} \sup_I \tilde{d}_\Omega(\gamma^i, \eta^i) + \sup_I \tilde{d}'_\Omega(\gamma^i, \eta^i)$, where

$$\tilde{d}_\Omega(\gamma^i, \eta^i) \overset{\text{def}}{=} \overset{\text{ess sup}}{\text{sup}} \{ \| D^k \gamma^i(t) - D^k \eta^i(t) \| : t \in I, \ k \geq 1 \}$$

with sup running over all relevant (finite) charts $(U, \varphi) \in \mathfrak{A}$, ess denoting that the expression $\| D^k \gamma^i(t) - D^k \eta^i(t) \|$ is not defined only on a finite (possibly zero) set of points in $I$ wherein $\gamma^i$ and $\eta^i$ are not differentiable, and $D^k \gamma^i(t)$ denotes the $k$th differential of $\gamma^i$ at $t$ in abused notation. It is routine to verify that $d_\sigma$ is indeed a metric. In all that follows, $M_\sigma$ will be endowed with the $d_\sigma$-topology.

2.6. Remark. Let $\overline{\mathfrak{A}}$ denote the maximal atlas of $\Sigma$ and define a topology on $M_\sigma$ to be generated by subbasic open sets $N_i(\gamma; (U_{\alpha(i)}, \varphi_{\alpha(i)})_{i=1}^\infty, K)$ in $M_\sigma$ to be constructed below, where $K \subset I$ is compact, $\gamma^i(K) \subset U_{\alpha(i)}$ and $(U_{\alpha(i)}, \varphi_{\alpha(i)}) \in \mathfrak{A}$.

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6The subscript $\sigma$ on $\gamma_\sigma$ will be omitted should no confusion arise from the context.
\[ ∃ \forall i. \text{Denote } \{ \alpha(i) \mid 1 \leq i \leq \infty \} \text{ by } \alpha \text{ and } (U_{\alpha(i)}, \varphi_{\alpha(i)})_i \text{ by } (U, \varphi)_\alpha \text{ for notational simplicity, and let} \]

\[ \tilde{d}_{\sigma_\alpha K}(\gamma^i, \eta^i) \defeq \{ \| D^k \varphi_{\alpha(i)} \circ \gamma^i(t) - D^k \varphi_{\alpha(i)} \circ \eta^i(t) \| : t \in K, k \geq 1 \} \]

whenever \( \gamma^i(K), \eta^i(K) \subset U_{\alpha(i)} \ \forall i \). Then, for a fixed \( \gamma \in M_\sigma \) such that \( \gamma^i(K) \subset U_{\alpha(i)} \ \forall i \), let \( N_\epsilon(\gamma; (U, \varphi)_\alpha, K) \defeq \{ \eta \in M_\sigma \mid \tilde{d}_{\sigma_\alpha K}(\gamma, \eta) < \epsilon, \eta^i(K) \subset U_{\alpha(i)} \ \forall i \} \), where

\[ \tilde{d}_{\sigma_\alpha K}(\gamma, \eta) \defeq \sup_i \tilde{d}_{\Omega}(\gamma^i, \eta^i) + \sup_i \tilde{d}_{\sigma_\alpha K}(\gamma^i, \eta^i). \]

It can be shown that this topology is equivalent to the \( \tilde{d}_\sigma \)-topology on \( M_\sigma \). In particular, the \( \tilde{d}_\sigma \)-topologies on \( M_\sigma \) defined relative to any two finite atlases of \( \Sigma \) are equivalent. Hence, in this sense, the \( \tilde{d}_\sigma \)-topology is well-defined as it does not depend on the choice of finite atlas \( \mathfrak{A} \) on \( \Sigma \).

A topology on \( \mathcal{M}_\infty[\Gamma_2^+] \) can now be constructed. Firstly, notice that the spaces \( M_\sigma \) and \( M_{\sigma'} \) are homeomorphic for each pair \( \sigma, \sigma' \)—define \( h_{\sigma_\sigma'} : M_\sigma \to M_{\sigma'} \) by \( \gamma_\sigma \mapsto \gamma_{\sigma'} \), where \( h_\sigma(\gamma_\sigma) = \gamma = h_{\sigma'}(\gamma_{\sigma'}) \). The existence of \( h_{\sigma_\sigma'} \) follows immediately from conditions (i) and (c). Hence, it is possible to endow \( \mathcal{M}_\infty[\Gamma_2^+] \) with a topology such that each \( h_\sigma : M_\sigma \to \mathcal{M}_\infty[\Gamma_2^+] \) defines a homeomorphism. This will be the topology imposed on \( \mathcal{M}_\infty[\Gamma_2^+] \). As an aside, if \( M_\infty \) is given the sum topology, \( M_\infty \defeq \bigoplus_\sigma M_\sigma \), then \( h : M_\infty \to \mathcal{M}_\infty[\Gamma_2^+] \) given by \( h|M_\sigma \defeq h_\sigma \) defines a continuous open surjection.

3. The Space of \( \mathcal{N}_0 \)-Knots of \( \mathcal{M}_\infty[\Gamma_2^+] \).

First of all, some notations and elementary properties of the space of equivalence \( \mathcal{N}_0 \)-loop classes will be established. Let \( \mathcal{G}_a^+ \) be the set of (smooth) orientation-preserving, ambient isotopies on \( \Sigma \). That is, \( \mathcal{G}_a^+ \subset C^\infty(\Sigma \times I, \Sigma \times I) \) is the following set:

\[ \{ F : \Sigma \times I \to \Sigma \times I \mid F(x, t) \defeq (F_t(x), t), F_0 = \text{id}_\Sigma, F_t \in \text{Diff}^+(\Sigma) \ \forall t \in I \} \]

and define composition \( \circ \) on \( \mathcal{G}_a^+ \) by

\[ (F' \circ F) : (x, t) \mapsto (F'_t \circ F_t(x), t). \]

Then, clearly, \( F' \circ F \in \mathcal{G}_a^+ \) and \( 1_{\Sigma \times I} \defeq \text{id}_\Sigma \times \text{id}_I \in \mathcal{G}_a^+ \). It is straight forward to check that \( \langle \mathcal{G}_a^+, \circ \rangle \) forms a group under \( \circ \), where the inverse \( F^{-1} \) of \( F = (F_t, \text{id}_I) \)
is defined to be \((F_t^{-1}, \text{id}_I)\). In particular, \(\circ\) is compatible with the compact \(C^\infty\)-topology on \(G^+_a\) — cf. [3, p. 64, ex. 9]. Moreover, since \(\text{Diff}^+(\Sigma)\) is closed in the group \(\text{Diff}(\Sigma)\) of smooth diffeomorphisms endowed with the compact \(C^\infty\)-topology (as it is a subgroup of \(\text{Diff}(\Sigma)\)), \(G^+_a\) is also closed in \(C^\infty(\Sigma \times I, \Sigma \times I)\) (with respect to the compact \(C^\infty\)-topology).

If \(\gamma, \eta \in \mathcal{L}_\Sigma\) are any pair of loops and \(\gamma\) is ambiently isotopic to \(\eta\) under some \(F \in G^+_a\), denote this by \(F: \gamma \simeq \eta\). Now, given any pair of \(\aleph_0\)-loops \(\gamma, \eta \in M_\infty[\Gamma^+_2]\), define an equivalence relation \(R\) generated by \(\simeq\) on \(M_\infty[\Gamma^+_2]\) as follows:

\[
\gamma \simeq \eta \iff \exists F \in G^+_a \text{ such that } F \cdot \gamma = \eta,
\]

where \(F \cdot \gamma \stackrel{\text{def}}{=} \{ F_1 \circ \gamma^1, F_1 \circ \gamma^2, \ldots \}\) and \(F : \gamma^i \simeq \eta^i \ \forall \ i\). Then, the space \(K[\Gamma^+_2]\) of equivalence classes of \(\aleph_0\)-loops in \(M_\infty[\Gamma^+_2]\) is defined to be the quotient space \(M_\infty[\Gamma^+_2]/G^+_a\). Henceforth, for simplicity, the term \((\text{piecewise geodesic}) \aleph_0\)-knot will mean an element of the quotient space \(K[\Gamma^+_2]\); that is, an \(\aleph_0\)-knot denotes an equivalence class of \(\aleph_0\)-loops under a smooth, orientation-preserving, ambient isotopy. The space \(K[\Gamma^+_2]\) will be called the \((\aleph_0, \Gamma^+_2)\)-knot space of \(M_\infty[\Gamma^+_2]\). Let \(\kappa_\infty : M_\infty[\Gamma^+_2] \to K[\Gamma^+_2]\) denote the natural map, where \(K[\Gamma^+_2]\) is endowed with the quotient topology.

**3.1. Lemma.** The natural projection \(\kappa_\infty : M_\infty[\Gamma^+_2] \to K[\Gamma^+_2]\) is open.

**Proof.** A sketch of the proof will be given. To see that \(\kappa_\infty\) is an open mapping, it is enough to note that for each open subset \(N \subset M_\infty[\Gamma^+_2]\),

\[
\kappa_\infty^{-1} \circ \kappa_\infty(N) = \bigcup_{F \in G^+_a} F \cdot N,
\]

where \(F \cdot N = \{ F \cdot \gamma \mid \gamma \in N \}\). Since \(F \cdot N\) is open in \(M_\infty[\Gamma^+_2]\), as \(F\) defines a homeomorphism from \(M_\infty[\Gamma^+_2]\) onto itself, the quotient topology implies that \(\kappa_\infty^{-1} \circ \kappa_\infty(N)\), and hence \(\kappa_\infty\), must also be open. \(\square\)

**3.2. Proposition.** \(K[\Gamma^+_2]\) is Hausdorff.

**Proof.** By lemma 3.1, it will suffice to show that the equivalence relation \(R\) generated by \(\simeq\) is closed in \(M_\infty[\Gamma^+_2] \times M_\infty[\Gamma^+_2]\) [4, p. 98, theorem 11]. Let \(\{(\gamma_n, \eta_n)\}_n\) be a sequence in \(R\) which converges in \(M_\infty[\Gamma^+_2] \times M_\infty[\Gamma^+_2]\) to \((\gamma_0, \eta_0)\). By definition, \(\exists\) a sequence \(\{F_n\}_n\) in \(G^+_a\) such that \(F_n : \gamma_n \simeq \eta_n\) for each \(n\). So, \((\gamma_n, F_n \cdot \gamma_n) \to (\gamma_0, \eta_0)\) \(\Rightarrow F_n \cdot \gamma_n \to \eta_0\) and \(\gamma_n \to \gamma_0\), and hence implying that
\{F_n\}_n is a convergent sequence in \(G^+_a\). Consequently, \(G^+_a\) is closed implies that \(F_n \to F_0 \in G^+_a\) for some \(F_0\). Whence, \(\eta_0 \equiv F_0 \cdot \gamma_0\) and \(R\) is thus closed, as desired. \(\square\)

In the interest of simplicity, call \(\gamma \in M_\infty[\Gamma^+_2]\) a piecewise \((N_0, q)\)-geodesic loop whenever the 3-metric \(q\) is required to be specified.

**3.3. Lemma.** Let \(\gamma, \tilde{\gamma} \in M_\infty[\Gamma^+_2]\) be piecewise \((N_0, q)\)- and \((N_0, \tilde{q})\)-geodesic loops respectively. If \(\gamma \simeq \tilde{\gamma}\), then \(\exists f \in \text{Diff}^+(\Sigma)\) such that \(q = f^* \tilde{q}\).

**Proof.** Let \(F \in G^+_a\) be an ambient isotopy of \(\gamma\) and \(\gamma\): \(F \cdot \gamma = \tilde{\gamma}\). Then, evidently, \(\tilde{\gamma}\) is a piecewise \((N_0, (F_1^{-1})^*q)\)-geodesic. However, \(\tilde{\gamma}\) is also a piecewise \((N_0, \tilde{q})\)-geodesic; hence, by \(\S 2\) (2), \(\exists f \in \text{Diff}^+(\Sigma)\) such that \(\tilde{q} = f^* q\), as required. \(\square\)

**4. \(N_0\)-Knots and Classical Geometry.**

In this section, the relationship between the equivalence classes of \(N_0\)-loops in \(\Sigma\) and the (classical) geometries admissible on \(\Sigma\) will be studied. This correspondence can be easily sought simply by noting that each element in \(M_\infty[\Gamma^+_2]\) corresponds to a unique 3-geometry \([q]\) of \(\Sigma\) by construction. The modified form of Rovelli’s Conjecture can now be formulated.

**4.1. Theorem.** There exists a continuous, open surjection \(\check{\chi} : M_\infty[\Gamma^+_2] \to \mathcal{Q}\) given by \(\gamma_q \mapsto [q]\), where \(\gamma_q\) is a (piecewise) \((N_0, q)\)-geodesic loop and \(q \in [q]\).

**Proof (Sketch).** Firstly, \(\check{\chi}\) is well-defined from the definition of \(M_\infty[\Gamma^+_2]\). Secondly, the surjective property of \(\check{\chi}\) is also clear. Thirdly, in this proof, \(\Gamma^+_2\) will be identified with its image under the (topological) imbedding \(j^\infty : \Gamma^+_2 \hookrightarrow C(\Sigma, J^\infty[p_\Sigma])\).\(^7\) So, \(\mathcal{Q} \equiv j^\infty \Gamma^+_2 / \text{Diff}^+(\Sigma)\) and \(\pi_+ : j^\infty \Gamma^+_2 \to \mathcal{Q}\).

Now, fix some \(\gamma_0 \in M_\infty[\Gamma^+_2]\) and let \(N(q_0) = \bigcap_{i=1}^n M(K_i, (\pi^+_\Sigma)^{-1}(U^n))\) be a neighbourhood of \(q_0\) in \(\Gamma^+_2\), where \(n_i \in \mathbb{N}, n < \infty\) and \(q_0 \in \check{\chi}(\gamma_0) = [q_0]\) is a representative of the \(q_0\)-equivalence class. Set \(N([q_0]) = \pi_+(N(q_0))\). Then, \(\check{N}([q_0]) = \pi_+^{-1}(N([q_0])) = \bigcup\{ f^* \circ N(q_0) \mid f \in \text{Diff}^+(\Sigma) \}\), where \(f^* \circ N(q_0) \overset{\text{def}}{=} \{ f^* q \mid q \in N(q_0) \}\). Let \(D_\varepsilon(\gamma_0)\) be an \(\varepsilon\)-neighbourhood of \(\gamma_0\) defined by \(B_\varepsilon(h^{-1}_\sigma(\gamma_0)) = h^{-1}_\sigma(D_\varepsilon(\gamma_0)) \forall \sigma\). Then, \(\forall \eta \in D_\varepsilon(\gamma_0), \tilde{d}_\Omega(\gamma_0^i, \eta^i_\sigma) + \tilde{d}_\Omega(\gamma_0^i, \eta^i_\sigma) < \varepsilon \forall i \text{ and } \sigma\), where \(h^{-1}_\sigma(\gamma) \overset{\text{def}}{=} \gamma_\sigma\).

\(^7\)The notations used here—the \(C^\infty\)-jets and compact \(C^\infty\)-topology—can be found in the appendix.
Next, observe from the definition that

\[(\dot{\gamma}^i)_\ell + \Gamma(q)_{kj}^\ell (\dot{\gamma}^i)^k (\dot{\gamma}^j)^j \overset{\text{a.e.}}{=} 0 \quad \forall i \in \mathbb{N} \text{ and } \ell = 1, 2, 3,\]

where \(\Gamma(q)\) is a Riemannian connection determined by the 3-metric \(q\) (with the connection coefficients written with respect to the natural frame for simplicity).

So, by choosing \(\varepsilon > 0\) to be sufficiently small, and by fixing any \(\sigma\)—and setting \(\gamma_0^i = \gamma_0^\sigma(i), \eta^i = \eta^\sigma(i)\)—it follows that \(|\bar{\eta}^i - \dot{\gamma}_0^i| < \varepsilon\) and \(|\ddot{\eta}^i - \ddot{\gamma}_0^i| < \varepsilon\) (almost everywhere), and in particular, using (*)

\[
\left| (\dot{\gamma}_0^i)^\ell + \Gamma(q_\eta)_{kj}^\ell (\dot{\gamma}_0^i)^k (\ddot{\gamma}_0^i)^j \right| \overset{\text{a.e.}}{=} \left| (\ddot{\gamma}_0^i)^\ell + \Gamma(q_\eta)_{kj}^\ell (\ddot{\gamma}_0^i)^k (\ddot{\gamma}_0^i)^j \right| = \left| (\Gamma(q_0) - \Gamma(q_\eta))_{kj}^\ell (\dot{\gamma}_0^i)^k (\ddot{\gamma}_0^i)^j \right| \sim \mathcal{O}(\varepsilon) \text{ a.e. on } I,
\]

where \(q_\eta \in \hat{\chi}(\eta)\). Whence, \(|(\dot{\gamma}_0^i)^\ell + \Gamma(q_\eta)_{kj}^\ell (\dot{\gamma}_0^i)^k (\ddot{\gamma}_0^i)^j| \overset{\text{a.e.}}{=} |(\dot{\gamma}_0^i)^\ell + \Gamma(q_0)_{kj}^\ell (\dot{\gamma}_0^i)^k (\ddot{\gamma}_0^i)^j| - |(\Gamma(q_0) - \Gamma(q_\eta))_{kj}^\ell (\dot{\gamma}_0^i)^k (\ddot{\gamma}_0^i)^j| \sim \mathcal{O}(\varepsilon) \text{ a.e. (from above)} \forall i \in \mathbb{N} \Rightarrow |\Gamma(q_\eta)_{kj} - \Gamma(q_0)_{kj}|\) is small on \(\Sigma\) for each fixed \(\ell, k, j\) whenever \(\varepsilon > 0\) is small enough by appealing to §2 (2) and the continuity of \(\Gamma\). Thus, from \(\Gamma(q)_{kj}^\ell \overset{\text{def}}{=} \frac{1}{2} q^{\ell h} (\partial_k q_{ij} + \partial_j q_{hk} - \partial_h q_{kj})\) (in the natural frame), it follows that \(\exists f \in \text{Diff}^+(\Sigma)\) such that \(f^* q_\eta\) and \(q_0\), together with their \(k\)th derivatives, must be close to one another: \(f^* q_\eta(K_i) \subset (\pi^*_{\Sigma})^{-1}(U_{n_i})\) \(\forall i = 1, \ldots, n\). So, \(f^* q_\eta\) and hence \(q_\eta\) must both belong to \(\tilde{N}([q_0])\) for \(\varepsilon > 0\) sufficiently small. Whence, \(\hat{\chi}(D_\varepsilon(\gamma_0)) \subset N([q_0])\), and the continuity of \(\hat{\chi}\) follows.

Finally, to conclude this proof, observe that for any \(\gamma \in \mathcal{M}_\infty[\Gamma_2^+], \hat{\chi}^{-1} \circ \hat{\chi}(\gamma) = \{ f \circ \gamma \mid f \in \text{Diff}^+(\Sigma) \}\), where \(f \circ \gamma \overset{\text{def}}{=} \{ f \circ \gamma^1, f \circ \gamma^2, \ldots \}\). Hence, for any \(\varepsilon\)-neighbourhood \(D_\varepsilon(\gamma)\),

\[
\hat{\chi}^{-1} \circ \hat{\chi}(D_\varepsilon(\gamma)) = \bigcup_{f \in \text{Diff}^+(\Sigma)} f \circ D_\varepsilon(\gamma),
\]

and \(\hat{\chi}\) is thus open, as desired. \(\square\)

In spite of the divergent approach given here with Rovelli’s original idea, the following corollary could perhaps be christened as the weak Rovelli conjecture inasmuch as the notion of relating knots with geometry originated from Rovelli [6].
4.2. Corollary (Weak Rovelli Conjecture). The map \( \hat{\chi} \) induces a continuous, open surjection \( \chi : \mathcal{K}[\Gamma^+] \to \mathcal{Q} \) given by \( [\gamma_q] \mapsto \hat{\chi}(\gamma_q) \), where \( \gamma_q \in \kappa^{-1}_\infty([\gamma_q]) \) is any fixed representative.

Proof. This map \( \chi \) is well-defined by lemma 3.3. The result now follows immediately from theorem 4.1, lemma 3.1 and the commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathcal{M}_\infty[\Gamma^+] & \xrightarrow{\hat{\chi}} & \mathcal{Q} \\
\kappa_\infty \downarrow & & \downarrow \text{id} \\
\mathcal{K}[\Gamma^+] & \xrightarrow{\chi} & \mathcal{Q}.
\end{array}
\]

\( \square \)

Two comments regarding theorem 4.1 and its corollary are now in order. Firstly, it is certainly evident that if \( \Sigma \) be separable (which, here, it is in any case!), then it is sufficient to characterized its 3-geometries by the \( \aleph_0 \)-loops in \( \mathcal{M}_\infty[\Gamma^+] \) since, by construction, \( \{ \gamma^i(0) \mid i \in \mathbb{N} \} \equiv \Sigma \), whereas \( n \)-loops, for \( n < \infty \) (using this construction), are not sufficient to determine the 3-geometry uniquely (as might well be expected): cf. §2.5 for a detailed account.

Secondly, it has been established elsewhere—cf. for example, [7, p. 132, §5.1] using the diffeomorphism constraints of general relativity (in the loop representation)—that functionals on \( \mathcal{L}_\Sigma \) which describe gravitational states are constant on the \( G^+_a \)-orbits of \( \mathcal{L}_\Sigma \): \( \psi[\gamma] = \psi[\gamma'] \ \forall \gamma, \gamma' \in [\gamma] \), where \( \psi : \mathcal{L}_\Sigma \to \mathbb{C} \) is a loop functional. However, surprisingly, this condition follows immediately from corollary 4.2. This can be easily seen as follows. Functionals on \( \Gamma^+_2 \) that describe gravitational states are those which are invariant under \( \text{Diff}^+(\Sigma) \): i.e., they are essentially functionals on \( \mathcal{Q} \). Let \( C(\mathcal{Q}, \mathbb{C}) \) be the set of continuous functionals on \( \mathcal{Q} \) and let \( C(\mathcal{K}[\Gamma^+_2], \mathbb{C}) \) be the set of functionals on \( \mathcal{K}[\Gamma^+_2] \). Then, \( \forall \tilde{\Psi} \in C(\mathcal{Q}, \mathbb{C}), \tilde{\Psi} \circ \chi \in C(\mathcal{K}[\Gamma^+_2], \mathbb{C}) \); that is, \( \chi^*(C(\mathcal{Q}, \mathbb{C})) \subset C(\mathcal{K}[\Gamma^+_2], \mathbb{C}) \), and the assertion thus follows.

This concludes the classical description of \( \aleph_0 \)-knots and their relationship with 3-geometries.

5. Discussion.

In this final section, a possible physical interpretation—albeit a highly speculative one!—regarding knots and gravity will be sketched. As was pointed out before,
the separability of $\Sigma$ guarantees that $\hat{\chi}$ in theorem 4.1 remains well-defined. Furthermore, as classically, gravity—or equivalently, the 4-metric—of space-time is determined by the distribution of matter in the universe via Einstein’s field equations, gravity is a ‘global’ concept. In this sense, if $n$-loops can describe gravity in any way, then, provided that space-time be separable, loops that will best describe it are $\aleph_0$-loops. Indeed, a judicious choice of $\aleph_0$-loops—such as those given in the preceding sections—enables one to recover the underlying Riemannian 3-manifold $\Sigma$ simply because $\{ \gamma_q^i(0) | i \in \mathbb{N} \} = \Sigma$, and $\hat{\chi}(\gamma_q) = [q]$. In the light of this observation, it is not unreasonable to conclude that gravity is the result of the way 3-space (and hence, space-time) is knotted, where $(\Sigma, q)$ is said to be $[\gamma]$-knotted if $\chi([\gamma]) = [q]$. And since $\chi$ is not one-one, $\Sigma$ can be knotted in two $G_\aleph^+$-inequivalent ways and yet give rise to the same gravitational configuration (determined by $\chi$).

In short, having determined $\mathcal{M}_\infty[\Gamma_2^+]$ from $\Sigma$, each element in $\mathcal{M}_\infty[\Gamma_2^+]$ contains the necessarily information to reconstruct $\Sigma$.

To conclude with a speculative note on the quantum aspect of a knot $[\gamma]$, one might heuristically interpretate a knot state $[[\gamma]]$ to correspond to the pair $[(\Sigma, q)]$, where $[(\Sigma, q)] \overset{\text{def}}{=} \{ (\Sigma, q) | q \in \chi([\gamma]) \}$. In particular, $[[\gamma]]$ is associated with a particular 3-geometry $\chi([\gamma])$. Thus, $[[\gamma]]$ corresponds to the global degrees of freedom of gravity: and since gravitons are associated with the local degrees of freedom of gravity, it has no direct relationship with a knot state. In the full quantum theory, it is quite reasonable to expect that $[[\gamma]]$ will not span a Hilbert space due to the highly non-linear nature of gravity and the violation of the asymptotic completeness condition. Hence, a knot state most probably cannot be interpreted in the usual quantum field theoretic sense in that it lies in some Hilbert space, although it is tempting to conjecture that the knot states lie in some $\aleph_0$-dimensional smooth Kähler manifold.

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Appendix

A. Compact $C^\infty$-Topology.

The definition of a compact $C^\infty$-topology will be reviewed [5, pp. 32–33, §§4.1–4.3]. Let $J^n[\Sigma]$ be the space of $C^n$-jets from $\Sigma$ into $\Sigma$ and denote an element
in $J^n[\Sigma]$ by either $j^n(f(x))$ or $[f, x]_n$ (which ever proves more convenient). Fix an atlas $\mathfrak{A}_\Sigma = \{ (U_{\alpha}, \psi_{\alpha}) \}_{\alpha \in \Lambda}$ on $\Sigma$ and set $\mathfrak{A}_\Sigma(U_{\alpha}) = \{ U \subset U_{\alpha} \mid U \text{ open} \}$. Then, $\mathfrak{B}_\Sigma = \bigcup_{\alpha} \mathfrak{A}_\Sigma(U_{\alpha})$ forms a base for $\Sigma$. Let $J^0[\Sigma] = \Sigma \times \Sigma$ and let $\pi_1^0 : J^1[\Sigma] \to J^0[\Sigma]$ by $j^1(\phi(p)) \mapsto (p, \phi(p))$. Set $U^1_{\alpha\alpha'} \equiv (\pi_1^0)^{-1}(U_{\alpha} \times U_{\alpha'})$ and define $p^\pm_1 : J^1[\Sigma] \to \Sigma$ by $p^+_0 : j^1(\phi(p)) \mapsto \phi(p)$ and $p^-_0 : j^1(\phi(p)) \mapsto p$. Then, it is clear that $U^1_{\alpha\alpha'} = (p^+_0)^{-1}(U_{\alpha}) \cap (p^-_0)^{-1}(U_{\alpha'})$. Finally, let $\mathfrak{A}^1_{\alpha\alpha'} = \{ (\pi_1^0)^{-1}(U \times U') \mid U \times U' \subset U_{\alpha} \times U_{\alpha'} \text{ open} \}$. Then, $\mathfrak{B}^1 = \bigcup_{\alpha, \alpha'} \mathfrak{A}^1_{\alpha\alpha'}$ forms a base for $J^1[\Sigma]$. Following [8, p. 94, definition 4.1.5], define $\Psi^1_{\alpha\alpha'} : U^1_{\alpha\alpha'} \equiv 3B_{\varepsilon_1}(x_{\alpha}) \times 3B_{\varepsilon_1}(x_{\alpha'}) \times N_1 B_{\varepsilon_1}(x_{\alpha'})$

\[ \{ \phi, p \} \mapsto (\psi_{\alpha}(p), \psi_{\alpha'}(\phi(p)), D_{\alpha}(j^1(\phi(p)))) \],

where $nB_{\varepsilon}(x)$ is an open $\varepsilon$-ball in $\mathbb{R}^n$ and $D_{\alpha}j^1(\phi(p)) \equiv \left\{ \frac{\partial}{\partial x_{\alpha}} \phi_{\alpha\alpha'}(\psi_{\alpha}(p)) \right\}_{i \leq i}$. For some $N_1 \in \mathbb{N}$ such that $D_{\alpha\alpha'} : U^1_{\alpha\alpha'} \equiv N_1 B_{\varepsilon_1}(x_{\alpha})$ and $\phi_{\alpha\alpha'} \equiv \psi_{\alpha} \circ \phi \circ \psi_{\alpha}^{-1}$. The pair $(U^1_{\alpha\alpha'}, \Psi^1_{\alpha\alpha'})$ defines a chart on $J^1[\Sigma]$. Denote $\Psi^1_{\alpha\alpha}$ symbolically by $\psi_{\alpha} \times \psi_{\alpha'} \times D_{\alpha}$.

Now, define $p^+_2 : J^2[\Sigma] \to \Sigma$ by $p^+_2 : j^2(\phi(p)) \mapsto p$ and $p^-_2 : j^2(\phi(p)) \mapsto \phi(p)$. Furthermore, define $\pi_2^0 : J^2[\Sigma] \to J^0[\Sigma]$ by $\pi_2^0(j^2(\phi(p)) = (p, \phi(p))$ and let $U^2_{\alpha\alpha'} \equiv (\pi_2^0)^{-1}(U_{\alpha} \times U_{\alpha'})$. Then, $U^2_{\alpha\alpha'} \equiv (p^-_2)^{-1}(U_{\alpha}) \cap (p^+_2)^{-1}(U_{\alpha'})$. And as with the case for $J^1[\Sigma]$, the pair $(U^2_{\alpha\alpha'}, \Psi^2_{\alpha\alpha'})$ defines a chart in $J^2[\Sigma]$, where $\Psi^2_{\alpha\alpha} \equiv \psi_{\alpha} \times \psi_{\alpha'} \times D_{\alpha} \times D_{\alpha'}$ and $U^2_{\alpha\alpha'} \equiv N_2 B_{\varepsilon_2}(x_{\alpha})$. For some $\varepsilon_2 > 0$ and some $N_2 \in \mathbb{N}$, is defined by $D^2(\phi_{\alpha\alpha'}(p)) \equiv \left\{ \frac{\partial}{\partial x_{\alpha}^i} \phi_{\alpha\alpha'}(\psi_{\alpha}(p)) \right\}_{i \leq i}$. Also, define $\pi_2^1 : J^2[\Sigma] \to J^1[\Sigma]$ by $[\phi, p] \mapsto [\phi, p]$. Then, by definition, $\pi_2^0 = \pi_1^0 \circ \pi_2^1$ and $\pi_1^1(U^1_{\alpha\alpha'}) = U^1_{\alpha\alpha'}$. Finally, let $\mathfrak{A}^2_{\alpha\alpha'} = \{ (\pi_2^0)^{-1}(U \times U') \mid U \times U' \subset U_{\alpha} \times U_{\alpha'} \text{ open} \};$ then, $\mathfrak{B}^2 = \bigcup_{\alpha, \alpha'} \mathfrak{A}^2_{\alpha\alpha'}$, forms a base for $J^2[\Sigma]$.

By induction, given $J^n[\Sigma]$, $(\pi_n^0)^{-1}(U_{\alpha} \times U_{\alpha'}) = (p^n_+)^{-1}(U_{\alpha}) \cap (p^n_-)^{-1}(U_{\alpha'})$ and $\pi_{n-1}^n(U^1_{\alpha\alpha'}) = U^{n-1}_{\alpha\alpha'}$. Furthermore, the pair $(U^n_{\alpha\alpha'}, \Psi^n_{\alpha\alpha'})$ forms a chart on $J^n[\Sigma]$ as follows: $\Psi^n_{\alpha\alpha'} \equiv \psi_{\alpha} \times \psi_{\alpha'} \times \prod_{i=1}^n D^i_{\alpha}$, where

\[ D^i_{\alpha} : [\phi, p] \mapsto \left\{ \frac{\partial^i \phi_{\alpha\alpha'} \circ \psi_{\alpha}(p)}{\partial x_{\alpha}^i} \right\}_{i \leq i} \in \mathbb{R}^n_{\ell} \]

with some $N_\ell \in \mathbb{N}$ such that $D^i_{\alpha}(U^\ell_{\alpha\alpha'}) = N_\ell B_{\varepsilon_\ell}(x_{\ell})$. Tersely, $\Psi^n_{\alpha\alpha'} : U^n_{\alpha\alpha'} \equiv 3B_{\varepsilon_1}(x_{\alpha}) \times 3B_{\varepsilon_1}(x_{\alpha'}) \times N_\ell B_{\varepsilon_\ell}(x_{\ell})$. The topology on $J^n[\Sigma]$ is generated by the base $\mathfrak{B}^n = \bigcup_{\alpha, \alpha'} \mathfrak{A}^n_{\alpha\alpha'}$, where $\mathfrak{A}^n_{\alpha\alpha'} = \{ (\pi_n^0)^{-1}(U \times U') \mid U \times U' \subset U_{\alpha} \times U_{\alpha'} \text{ open} \}$. 

---

\[ ^8 \text{For } j^1(\phi(p)) \in U^1_{\alpha\alpha'} \Rightarrow \pi_1^0(j^1(\phi(p))) = (p, \phi(p)) \in U_{\alpha} \times U_{\alpha'} \Rightarrow j^2(\phi(p)) \in U^2_{\alpha\alpha'} \text{ and so, } U^1_{\alpha\alpha'} \subseteq \pi_1^1(U^2_{\alpha\alpha'}). \text{ Conversely, } j^1(\phi'(p')) \in U^1_{\alpha\alpha'} \Rightarrow j^2(\phi'(p')) \in U^2_{\alpha\alpha'} \Rightarrow (p', \phi'(p')) \in U_{\alpha} \times U_{\alpha'} \Rightarrow \text{ the converse set-inequality, as required.} \]
It follows from the construction that \( \{ J^n[\Sigma], \pi_n^{n-1}, N \} \) forms an inverse sequence. Let \( J^\infty[\Sigma] \ define \lim J^n[\Sigma] \) denote the limit of the inverse sequence. Then, \( \mathcal{B}^\infty = \{ (\pi^n)^{-1}(U) \mid U \in \mathcal{B}^n \ \forall \ n \} \) defines a base of \( J^\infty[\Sigma] \), where \( \pi^n \ define p^n|J^\infty[\Sigma] \) and \( p^n : \prod_{i \in \mathbb{N}} J^i[\Sigma] \rightarrow J^n[\Sigma] \) is the nth projection. Observe from [1, p. 98, proposition 2.5.1] that \( J^\infty[\Sigma] \) is closed in the Cartesian product \( \prod_{i \in \mathbb{N}} J^i[\Sigma] \).

The compact (or weak) \( C^\infty \)-topology on \( C^\infty(\Sigma, \Sigma) \) is the topology induced by the map \( j^\infty : C^\infty(\Sigma, \Sigma) \rightarrow C(\Sigma, J^\infty[\Sigma]) \) defined by \( f \mapsto j^\infty f \ define [f, \cdot]_\infty \) such that it is a topological imbedding. Let \( \text{Diff}(\Sigma) \subset C^\infty(\Sigma, \Sigma) \) denote the set of \( C^\infty \)-diffeomorphisms on \( \Sigma \). The composition mapping \( \circ : \text{Diff}(\Sigma) \times \text{Diff}(\Sigma) \rightarrow \text{Diff}(\Sigma) \) given by \( (f, g) \mapsto f \circ g \) defines a group structure on \( \text{Diff}(\Sigma) \). Indeed, the group structure is compatible with the compact \( C^\infty \)-topology on \( \text{Diff}(\Sigma) \) [3, p. 64, ex. 9].

Lastly, observe from [3, p. 38, theorem 1.6] that \( \text{Diff}(\Sigma) \) is open in \( C^\infty(\Sigma, \Sigma) \) (as \( \Sigma \) is compact implies that the weak and strong \( C^\infty \)-topology coincide).

This appendix will conclude with a brief sketch of the compact \( C^\infty \)-topology on the space \( \Gamma^+_2 \) of (admissible) Riemannian metrics on \( \Sigma \). Let \( p_\Sigma : S^+_2 \Sigma \rightarrow \Sigma \) be the symmetric covariant 2-tensor bundle over \( \Sigma \) and \( p_{\Sigma n} : J^n[p_\Sigma] \rightarrow \Sigma \) be the \( C^n \)-jet bundle of the cross-sections of \( S^+_2 \Sigma \). Then, defining \( \pi^0_{\Sigma 1} : J^1[p_\Sigma] \rightarrow \Sigma \times S^+_2 \Sigma \) as above by \( j^1 q(x) \mapsto (x, q(x)) \) and \( \pi^m_{\Sigma n} : J^n[p_\Sigma] \rightarrow J^m[p_\Sigma] \) by \( j^n q(x) \mapsto j^m q(x) \) whenever \( m \leq n \), one again obtains an inverse sequence \( \{ J^n[p_\Sigma], \pi_n^{n-1}, N \} \), where \( J^0[p_\Sigma] \ define \Sigma \times S^+_2 \Sigma \). Finally, let \( J^\infty[p_\Sigma] \) denote the inverse limit of the sequence and set \( \pi_\Sigma \ define p_\Sigma^\infty|J^\infty[p_\Sigma] \), where \( p_\Sigma^\infty : \prod_{i \in \mathbb{N}} J^i[p_\Sigma] \rightarrow J^n[p_\Sigma] \) is the nth projection. The topology of \( \Gamma^+_2 \) is then defined by the (topological) imbedding \( j^\infty : \Gamma^+_2 \mapsto C(\Sigma, J^\infty[p_\Sigma]) \).

References

1. Engelking, R., General Topology, SSPM 6, Heldermann Verlag-Berlin, 1989.
2. Fischer, A. E., Theory of Superspace, Relativity, ed. Carmeli, Fickler, Witten.
3. Hirsch, M., General Topology, 4. Kelley, J., General Topology.
4. Fischer, A. E., Theory of Superspace, vol. 8, Shiva Maths Series, Kent, 1980.
5. Michor, P., Manifolds of Differentiable Mappings, vol. 8, LMS 142, Cambridge University Press, 1989.