Multi-loop integrals made simple: applications to QCD processes

Johannes M. Henn
Institute for Advanced Study

based on [JMH, Phys. Rev. Lett. 110 (2013) 25]

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Marvin L. Goldberger Member

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Towards automating NNLO calculations

• status scattering amplitudes & cross sections at NLO
  - efficient tools for generating & organizing integrands
  - one-loop integrals known
  - subtraction methods available
  - many automated programs

• at NNLO: bottleneck often missing analytic expressions for loop integrals

• this talk: new method for computing loop integrals  
  [JMH, PRL 110 (2013) 25]
  - based on better understanding Feynman integrands and integrals
  - identifies appropriate class of functions (and computes them!)
  - makes analytic properties manifest (e.g. singularities)
  - especially useful for integrals that depend on several scales
  - for massive/massless, planar/non-planar integrals
  - uses differential equations  
  [Kotikov, Phys. Lett. B254 (1991) 158
   Remiddi, Nuovo Cimento A110 (1997) 1435
   Gehrmann and Remiddi, NPB 580 (2000) 485]
Sample applications: two-scale problems

- massless 2-2 scattering to 3 loops
  
  non-planar integrals work in the same way

  physics motivation: scattering amplitudes in Yang-Mills & supergravity

- heavy quark effective theory
  
  all 3-loop cusp integrals, e.g.

  physics motivation: infrared divergences of massive scattering amplitudes

\[
s = (p_1 + p_2)^2 \quad t = (p_2 + p_3)^2 \quad x = t/s
\]

\[
\cos \phi = \frac{v_1 \cdot v_2}{\sqrt{v_1^2} \sqrt{v_2^2}}, \quad x = e^{i\phi}
\]
Sample applications: multi-scale problems

- integrals for Bhabha scattering
  
  \[ J.M.H., V. Smirnov, JHEP 1311 (2013) 041 \]

- scattering amplitudes & cross sections in massive toy model

- vector boson production \( pp \rightarrow VV \)

\[ J.M.H., S. Caron-Huot, to appear \]

similar integrals in QCD for finite top quark mass

- equal mass case: \[ Gehrmann, Tancredi, Weihs, JHEP 1308 (2013) 070 \]
Key points of the method

• differential equations for master integrals $\vec{f}$
• crucial: choose convenient basis (systematic procedure) → makes solution trivial to obtain
• elegant description: Feynman integrals specified by:
  (1) set of ‘letters’ (related to singularities $x_k$)
  (2) set of constant matrices $A_k$

Example: one dimensionless variable $x$; $D = 4 - 2\epsilon$

$$\partial_x \vec{f}(x; \epsilon) = \epsilon \sum_k \frac{A_k}{x - x_k} \vec{f}(x; \epsilon)$$

• expansion to any order in $\epsilon$ is linear algebra answer: multiple polylogarithms of uniform weight (‘transcendentality’)
• asymptotic behavior $\vec{f}(x; \epsilon) \sim (x - x_k)^{\epsilon A_k} \vec{f}_0(\epsilon)$
• natural extension to multi-variable case
vector boson production \[ pp \to VV \]

- planar integral families

\begin{align*}
\frac{S}{M_3^2} &= (1+x)(1+xy), \quad \frac{T}{M_3^2} = -xz, \quad \frac{M_4^2}{M_3^2} = x^2y \\
\end{align*}

physical region \[ 0 < x, \quad 0 < y < z < 1 \]

- differential equations

\[
\frac{df(x,y,z;\epsilon)}{d\epsilon} = \epsilon d\tilde{A}(x,y,z) f(x,y,z;\epsilon)
\]

\[
\tilde{A} = \sum_{i=1}^{15} \tilde{A}_{\alpha_i} \log(\alpha_i)
\]

- alphabet \[ \alpha = \{x, y, z, 1+x, 1-y, 1-z, 1+xy, z-y, 1+y(1+x)-z, xy+z, \]
  \[ 1+x(1+y-z), 1+xz, 1+y-z, z+x(z-y)+xyz, z-y+yz+xyz\} \]

- solution in terms of multiple polylogarithms
Conclusions and outlook

• families of Feynman integrals described by iterated integrals

  analytic answer specified by:

  (1) “alphabet” for iterated integrals

  (2) constant matrices provide rules to form words

• many applications for LHC physics, e.g.

  - amplitudes involving top quarks
  - vector boson production
  - ... (insert your wish here)
  - outlook: library for NNLO Feynman integrals

• interesting mathematics

  - Fuchsian differential equations; monodromies, asymptotic limits
  - extension to elliptic functions
Curious?

Lecture notes for this method available in chapter 3.8 of

Scattering Amplitudes in Gauge Theories