Collective modes in the anisotropic unitary Fermi gas and the inclusion of a backflow term

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We study the collective modes of the confined unitary Fermi gas under anisotropic harmonic confinement as a function of the number of atoms. We use the equations of extended superfluid hydrodynamics, which take into account a dispersive von Weizsäcker-like term in the equation of state. Finally, we discuss the inclusion of a backflow term in the extended superfluid Lagrangian and the effects of this anomalous term on sound waves and Beliaev damping of phonons.

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I. INTRODUCTION

In this paper we calculate the collective monopole and quadrupole modes of the unitary Fermi gas (characterized by an infinite s-wave scattering length) under axially-symmetric anisotropic harmonic confinement by using the extended Lagrangian density of superfluids which we proposed a few years ago [1], to study the unitary Fermi gas [1–8]. The internal energy density of our extended Lagrangian density contains a term proportional to the kinetic energy of a uniform non-interacting gas of fermions, plus a gradient correction of the von-Weizsacker form

\[ L \approx \frac{\hbar^2}{8m} \left( \nabla n \right)^2 . \]

which takes into account density variations. Thus, the local internal energy depends not only on the local density \( n(r,t) \) but also on its space gradient, namely

\[ \mathcal{E}(n,\nabla n) = \mathcal{E}_0(n) + \lambda \frac{\hbar^2}{8m} \left( \nabla n \right)^2 , \]

where, as previously mentioned, \( \mathcal{E}_0(n) \) is the internal energy of a uniform unitary Fermi gas with density \( n \). The parameter \( \lambda \) giving the gradient correction must be determined from microscopic calculations or from comparison with experimental data.

By using the Lagrangian density \( L \) the Euler-Lagrange equation for \( \theta \) gives

\[ \frac{\partial n}{\partial t} + \frac{\hbar}{m} \nabla \cdot (n \nabla \theta) = 0 , \]

while the Euler-Lagrange equation for \( n \) leads to

\[ \hbar \dot{\theta} + \frac{\hbar^2}{2m} (\nabla \theta)^2 + U(r) + X(n,\nabla n) = 0 , \]

which describes how the internal energy varies as the local density and its gradient vary, may be considered a local chemical potential. The local velocity field \( \mathbf{v}(r,t) \) of the superfluid is related to \( \theta(r,t) \) by

\[ \mathbf{v}(r,t) = \frac{\hbar}{m} \nabla \theta(r,t) . \]

This definition ensures that the velocity is irrotational, i.e. \( \nabla \times \mathbf{v} = 0 \). By using the definition in both Eqs.
and applying the gradient operator $\nabla$ to Eq. one finds the extended hydrodynamic equations of superfluids

$$\frac{\partial n}{\partial t} + \nabla \cdot \left( n \mathbf{v} \right) = 0 . \quad (9)$$

$$m \frac{\partial \mathbf{v}}{\partial t} + \nabla \left[ \frac{1}{2} m \mathbf{v}^2 + U(\mathbf{r}) + X(n, \nabla n) \right] = 0 . \quad (10)$$

We stress that in the presence of an external confinement $U(\mathbf{r})$ the chemical potential $\mu$ of the system does not coincide with the local chemical potential $X(n, \nabla n)$. In the presence of an external potential the relation between the equilibrium (ground state) density $n_0(\mathbf{r})$ and the chemical potential $\mu$ can be obtained from Eq. by setting $\theta(\mathbf{r}, t) = -\mu t / \hbar$ and $\mathbf{v}(\mathbf{r}, t) = 0$, so that

$$U(\mathbf{r}) + X(n_0, \nabla n_0) = \mu . \quad (11)$$

**III. COLLECTIVE MODES OF THE ANISOTROPIC UNITARY FERMII GAS**

In the case of the unitary Fermi gas the bulk internal energy can be written as

$$E_0(n) = \frac{3 \xi}{2} \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{5/3} , \quad (12)$$

where $\xi \approx 0.4$ is a universal parameter and various approaches [1, 2, 16, 31] suggest that $\lambda \simeq 0.25$. The local chemical potential is then:

$$X(n, \nabla n) = \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n^{1/3} - \lambda \frac{\hbar^2}{2m} \nabla^2 \sqrt{n} . \quad (13)$$

with the above mentioned values of $\xi$ and $\lambda$.

In this section we consider the unitary Fermi gas under the anisotropic axially-symmetric harmonic confinement

$$U(\mathbf{r}) = \frac{m}{2} \omega_0^2 (x^2 + y^2) + \frac{m}{2} \omega_z^2 z^2 , \quad (14)$$

where $\omega_0$ is the cylindrical radial frequency while $\omega_z$ is the axial frequency. In this case, Eq. for the ground-state density profile $n_0(\mathbf{r})$ becomes

$$\frac{m}{2} \omega_0^2 (x^2 + y^2) + \frac{m}{2} \omega_z^2 z^2 + \frac{\hbar^2}{2m} (3\pi^2)^{2/3} n_0(x, y, z)^{2/3} - \lambda \frac{\hbar^2}{2m} \nabla^2 \sqrt{n_0(x, y, z)} = \mu . \quad (15)$$

We have solved numerically this 3D partial differential equation, by using a finite-difference predictor-corrector Crank-Nicolson method with imaginary time after choosing $\xi = 0.42$ and $\lambda = 0.25$. In the case of isotropic trap ($\omega_0 = \omega_z$) the fermionic cloud is spherically symmetric and consequently axial and radial density profiles coincide. Instead, as expected, by increasing the trap anisotropy also the fermionic cloud becomes more anisotropic.

![Figure 1](image-url)  
**FIG. 1:** (Color online). Unitary Fermi gas under isotropic $(\omega_0 = \omega_z)$ harmonic confinement. In the two panels there are the monopole frequency $\Omega_0$ (upper panel) and the quadrupole frequency $\Omega_2$ (lower panel) as a function of the number $N$ of atoms. Filled circles with error bars: numerical results obtained solving Eqs. and with Eq. and $\lambda = 0.25$. Dashed lines: analytical results, i.e. exact Eq. and Thomas-Fermi Eq. Universal parameter of the unitary Fermi gas: $\xi = 0.42$.

We are interested in calculating the frequencies of low-lying collective oscillations of the anisotropic unitary Fermi gas. Exact scaling solutions for the unitary Fermi gas have been considered by Castin [26] and also by Hou, Pitaevskii, and Stringari [27]. Unfortunately, in the presence of anisotropic trapping potential and including the gradient term in the hydrodynamic equations, these scaling solutions are no more exact.

For this reason we solve numerically the extended hydrodynamic equations and . In particular, by using our finite-difference predictor-corrector Crank-Nicolson code in real time [25], we integrate a time-dependent nonlinear Schrödinger equation, which is fully equivalent (see [1, 5]) to Eqs. and .

Fig. refers to the unitary Fermi gas under isotropic $(\omega_0 = \omega_z)$ harmonic confinement. In the two panels we plot the monopole frequency $\Omega_0$ (upper panel) and the quadrupole frequency $\Omega_2$ (lower panel) as a function of the number $N$ of atoms. As expected [26], the frequency $\Omega_0$ of the monopole mode does not depend on the number $N$ of particles and it is given by

$$\Omega_0 = 2 \omega_0 . \quad (16)$$

On the contrary, the figure shows that the frequency $\Omega_2$
frequencies reduce to the results without gradient term.

\[ \Omega_{0,2}^{(a), (b)} = \sqrt{\frac{5}{3} \omega_{\rho}^2 + 4 \omega_{z}^2} \pm \frac{1}{3} \sqrt{25 \omega_{\rho}^4 + 16 \omega_{z}^4 - 32 \omega_{\rho}^2 \omega_{z}^2} , \]

which correspond to the dashed lines. Our calculations show that the frequency \( \Omega_{2} \) of Fig. 2 and the frequencies \( \Omega_{0,2}^{(a)} \) and \( \Omega_{0,2}^{(b)} \) of Figs. 2 give a clear signature of the presence of the von-Weizsacker gradient term.

We stress that current experiments with ultracold atoms at unitarity can detect deviations from the Thomas-Fermi approximation, as done some years ago for Bose-Einstein condensates [29].

**IV. INCLUSION OF A BACKFLOW TERM**

Inspired by the papers of Son and Wingate [30] and Manes and Valle [31] in this section we consider the inclusion of a backflow term in the extended superfluid Lagrangian. This backflow term depends on the velocity strain, as suggested for superfluid 4He many years ago by Thouless [21] and more recently by Dalfovo and collaborators [22]. In particular, we consider the Lagrangian density

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_W + \mathcal{L}_B , \]

where \( \mathcal{L}_0 \) and \( \mathcal{L}_W \) are given by Eqs. (8) and (19) respectively, and the backflow term \( \mathcal{L}_B \) reads

\[ \mathcal{L}_B = -\frac{\hbar^2}{m n^{1/3}}[\gamma_1 (\nabla^2 \theta)^2 + \gamma_2 (\partial_i \partial_j \theta)^2] . \]

Notice that \( i, j = x, y, z \) and summations over repeated indices are implied. Again, for a generic superfluid the parameters \( \gamma_1 \) and \( \gamma_2 \) of the backflow term must be determined from microscopic calculations or from comparison with experimental data.

The Lagrangian density [19] depends on the dynamical variables \( \theta(r, t) \) and \( n(r, t) \). The conjugate momenta of these dynamical variables are then given by

\[ \pi_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = -\hbar n , \]

\[ \pi_n = \frac{\partial \mathcal{L}}{\partial \dot{n}} = 0 , \]

and the corresponding Hamiltonian density reads

\[ \mathcal{H} = \pi_\theta \dot{\theta} + \pi_n \dot{n} - \mathcal{L} = -\hbar n \dot{\theta} - \mathcal{L} , \]

namely

\[ \mathcal{H} = \frac{\hbar^2}{2m} (\nabla \theta)^2 n + U(r) n + \mathcal{E}_0(n) \]

\[ + \lambda \frac{\hbar^2}{8m} (\nabla n)^2 + \frac{\hbar^2}{m} n^{1/3}[\gamma_1 (\nabla^2 \theta)^2 + \gamma_2 (\partial_i \partial_j \theta)^2] , \]
which is the sum of the flow kinetic energy density $h^2(\nabla \theta)^2 n/(2m) = (1/2)m v^2 n$, the external energy density $U(\mathbf{r})n$, the internal energy density $\mathcal{E}_0(n)$ without out the gradient correction, the gradient correction $\lambda(h^2/8m)((\nabla n)^2/n$ to the internal energy, and the backflow energy density $[h^2/m n^{1/3}\gamma_1(\nabla^2 \theta)^2 + \gamma_2(\partial_i \partial_j \theta)^2] = m n^{1/3}[\gamma_1(\nabla \cdot \mathbf{v})^2 + \gamma_2(\partial_i v_j)^2].$

The Hamiltonian density $[24]$ is nothing else than the energy density recently found by Manes and Valle $[31]$ with a derivative expansion from their effective field theory of the the Goldstone field $[30,31]$. The effective field theory of Manes and Valle $[31]$ traces back to the old hydrodynamic results of Popov $[24]$ and generalizes the one derived by Son and Wingate $[30]$ for the unitary Fermi gas from general coordinate invariance and conformal invariance. Actually, at next-to-leading order Son and Wingate $[30]$ found an additional term proportional to $\nabla^2 U(\mathbf{r})$, which has been questioned by Manes and Valle $[31]$ and which is absent in our approach. In addition, Manes and Valle $[31]$ have stressed that the conformal invariance displayed by the unitary Fermi gas implies

$$\gamma_2 = -3\gamma_1. \tag{25}$$

Note that a paper of Schakel $[32]$ confirms the results of Manes and Valle.

We are interested on the propagation of sound waves in superfluids. For simplicity we set $U(\mathbf{r}) = 0$, and consider a small fluctuation $\phi(\mathbf{r}, t)$ of the phase $\theta(\mathbf{r}, t)$ around the stationary phase $\theta_0(t) = -(\mu/h)t$, namely

$$\phi(\mathbf{r}, t) = \theta(\mathbf{r}, t) - \theta_0(t), \tag{26}$$

and a small fluctuation $\rho(\mathbf{r}, t)$ of the density $n(\mathbf{r}, t)$ around the constant and uniform density $n_0$, namely

$$\rho(\mathbf{r}, t) = n(\mathbf{r}, t) - n_0. \tag{27}$$

From the full Lagrangian density $[19]$ it is then quite easy to find the quadratic Lagrangian density $\mathcal{L}^{(2)}$ of the fluctuating fields $\phi(\mathbf{r}, t)$ and $\rho(\mathbf{r}, t)$:

$$\mathcal{L}^{(2)} = -\hbar \dot{\phi} \rho - \frac{\hbar^2 n_0}{2m} (\nabla \phi)^2 - \frac{mc^2}{2m_0} \rho^2 - \lambda \frac{\hbar^2}{8mn_0} (\nabla \rho)^2 - \gamma \frac{h^2 n_0^{1/3}}{m} (\nabla^2 \phi)^2, \tag{28}$$

where $c_s$ is the sound velocity of the generic superfluid, given by

$$c_s^2 = \frac{n_0}{m} \frac{\partial^2 \mathcal{E}_0(n_0)}{\partial n^2}, \tag{29}$$

and $\gamma = \gamma_1 + \gamma_2$. In fact, $(\nabla^2 \theta)^2$ and $(\partial_i \partial_j \theta)^2$ differ by a total derivative $[41]$ and consequently, since at the quadratic order the coefficients in front of them are constants, one derives Eq. $[28]$ with $\gamma = \gamma_1 + \gamma_2$. The linear equations of motion associated to the quadratic Lagrangian $\mathcal{L}^{(2)}$ read

$$\frac{\partial}{\partial t} \dot{\phi} + n_0 \nabla \cdot \mathbf{v} - 2n_0^{1/3} \gamma \nabla^2 (\nabla \cdot \mathbf{v}) = 0, \tag{30}$$

$$\frac{\partial}{\partial t} \mathbf{v} + \frac{c^2}{n_0} \nabla \rho - \frac{\lambda \hbar^2}{4mn_0} \nabla (\nabla^2 \rho) = 0, \tag{31}$$

with $\mathbf{v} = (\hbar/m) \nabla \phi$. These equations can be arranged in the form of the following wave equation

$$\left[ \frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2 + \left( \frac{\hbar^2}{4m_0} + \gamma \frac{2c^2}{n_0^{1/3}} \right) \nabla^4 - \lambda \frac{\hbar^2}{2m^2 n_0^{1/3}} \nabla^6 \right] \rho(\mathbf{r}, t) = 0. \tag{32}$$

This wave equation admits monochromatic plane-wave solutions, where the frequency $\omega$ and the wave vector $\mathbf{q}$ are related by the dispersion formula $\omega = \omega(q)$ given by

$$h \omega(q) = \sqrt{\left( \frac{\hbar^2 q^2}{2m} + \gamma \frac{\hbar^4 q^4}{mn_0^{2/3}} \right) \left( \frac{\hbar^2 q^2}{2m} + 2mc^2_s \right)}. \tag{33}$$

Notice that a negative value of $\gamma$ implies that the frequency $\omega(q)$ becomes imaginary for $q > n_0^{1/3}/\sqrt{2} \gamma$. However, $\gamma$ is expected to be very small and the hydrodynamics is no longer valid for these large values of $q$.

It is instead useful to expand $\omega(q)$ for small values of $q$ (long-wavelength hydrodynamic regime), finding

$$h \omega(q) = c_s \hbar q + \frac{h}{2} \left( \frac{\hbar^2}{4m^2 c_s^2} + \gamma \frac{2c_s^4}{n_0^{2/3}} \right) q^4 + \ldots. \tag{34}$$

The dispersion relation is linear in $q$ only for small values of the wavenumber $q$ and the coefficient of cubic correction depends on a combination of the gradient parameter $\lambda$ and backflow parameter $\gamma$. For $\gamma = 0$ one recovers the dispersion relation we have proposed some years ago $[3]$, while setting also $\lambda = 0$ one gets the familiar linear dispersion relation $\omega = c_s q$ of phonons. In the case of the unitary Fermi gas one has

$$c_s^2 = \frac{\hbar^2}{m^2} \frac{\xi}{3} (3\pi^2)^{2/3} n_0^{2/3}. \tag{35}$$

Moreover, we have seen that the backflow parameters are related by the formula $[20]$, which means

$$\gamma = \gamma_1 + \gamma_2 = -2\gamma_1. \tag{36}$$

Consequently, at the cubic order in $q$ Eq. $[33]$ gives

$$\frac{\omega(q)}{c_s k_F} = \frac{q}{k_F} + 1 \frac{q^3}{k_F^3}, \tag{37}$$

where $k_F = (3\pi^2 n_0)^{2/3}$ is the Fermi wavenumber and

$$\Gamma = \frac{3\lambda}{8\xi} - 2(3\pi^2)^{2/3} \gamma_1. \tag{38}$$
Within a mean-field approximation Manes and Valle \cite{31} have found $\gamma_1 \simeq 0.006$, which implies $\gamma \simeq -0.012$ and $\Gamma \simeq 0.12$, using $\xi = 0.4$ and $\lambda = 0.25$. As recently discussed by Mannarelli, Manuel and Tolos \cite{33}, the sign of $\Gamma$ has a dramatic effect on the possible phonon interaction channels: the three-phonon Beliaev process, i.e. the decay of a phonon into two phonons \cite{22}, is only allowed for positive values of $\Gamma$. Under this condition ($\Gamma > 0$) the phonon has a finite life-time and the frequency $\omega(q)$ possesses an imaginary part $\text{Im}[\omega(q)]$ due to this three-phonon decay \cite{22,32}. In particular, we find

$$\text{Im}[\omega(q)] = -\frac{\hbar q^5}{270 \pi m n_0}. \quad (39)$$

This formula of Beliaev damping is easily derived from Beliaev theory \cite{23} taking into account Eq. \cite{35}.

It is important to point out that the sign of $\Gamma$ in Eq. \cite{35} was debated also without the backflow term. In 1998 Marini, Pistolesi and Strinati \cite{36} found $\Gamma > 0$ at unitarity by including Gaussian fluctuations to the mean-field BCS-BEC crossover. In 2005 Combescot, Kagan and Stringari \cite{37} derived Eq. \cite{35} with a negative $\Gamma$ at unitarity on the basis of a dynamical BCS model. In 2011 Schakel \cite{32} obtained a positive $\Gamma$ at unitarity by using a derivative expansion technique, finding exactly the values of $\Gamma$ predicted by Ref. \cite{36} in the full BCS-BEC crossover.

To conclude this section, we observe that, for a generic many-body system, the dispersion relation can be written as

$$\hbar \omega(q) = \sqrt{m_n(q) \cdot m_{-1}(q)}, \quad (40)$$

where $m_n(q)$ is the $n$ moment of the dynamic structure function $S(q, \omega)$ of the many-body system under investigation, namely

$$m_n(q) = \int_0^\infty d\omega \ S(q, \omega) \ (\hbar \omega)^n. \quad (41)$$

In our problem, Eq. \cite{32}, it is straightforward to recognize (see also \cite{22}) that

$$m_1(q) = \frac{\hbar^2 q^2}{2 m} + \gamma \frac{\hbar^2 q^4}{m n_0^{2/3}} \quad (42)$$

and

$$m_{-1}(q) = \frac{1}{\lambda \frac{\hbar^2 q^2}{2 m} + 2m c_s^2}. \quad (43)$$

In general, the static response function $\chi(q)$ is defined as \cite{35}

$$\chi(q) = -2 \ m_{-1}(q) ; \quad (44)$$

in our problem it reads:

$$\chi(q) = -\frac{2}{\lambda \frac{\hbar^2 q^2}{2 m} + 2 m c_s^2}, \quad (45)$$

which satisfies the exact sum rule $\chi(0) = -1/mc_s^2$ \cite{35}.

The static structure factor $S(q)$, defined as \cite{35}

$$S(q) = m_0(q) = \int_0^\infty d\omega \ S(q, \omega), \quad (46)$$

can be approximated by the expression

$$S(q) = \sqrt{m_1(q) m_{-1}(q)} = \sqrt{\frac{\hbar^2 q^2}{2 m} + \gamma \frac{\hbar^2 q^4}{m n_0^{2/3}}} \sqrt{\frac{\lambda \hbar^2 q^2}{2 m} + 2m c_s^2}, \quad (47)$$

which gives an upper bound of $S(q)$ \cite{35} and reduces to $S(q) = \hbar q/(2mc_s)$ for small $q$.

Finally, we remark that one can also calculate the frequencies $\Omega$ of collective oscillations of the unitary Fermi gas under the action of the trapping potential given by Eq. \cite{14} taking into account the backflow. We have verified that in the case of spherically-symmetric harmonic confinement ($\omega_r = \omega_z$) the monopole mode $\Omega_0$ is not affected by the backflow term, i.e. $\Omega_0 = 2\omega_p$. Moreover, for large values of $N$ the contribution due to the backflow becomes negligible, similarly to the von Weizsäcker one.

V. CONCLUSIONS

We have calculated collective modes of the anisotropic unitary Fermi gas by using the equations of extended superfluid hydrodynamics. In particular, we have shown that a gradient correction of the von-Weizsacker form in the hydrodynamic equations strongly affects the frequencies of collective modes of the system under axially-symmetric anisotropic harmonic confinement. We have found that, for both monopole and quadrupole modes, this effect becomes negligible only in the regime of a large number of fermions, where one recovers the predictions of superfluid hydrodynamics \cite{28}. In the last part of the paper we have considered the inclusion of a backflow term in the extended hydrodynamics of superfluids.

We believe our results can trigger the interest of experimentalists. Some years ago beyond-Thomas-Fermi effects due to the dispersive gradient term have been observed by measuring the frequencies of collective modes in trapped Bose-Einstein condensates \cite{29}. Moreover, the spectrum of phonon excitations and Beliaev decay have been observed in a quasi-uniform Bose-Einstein condensate with Bragg pulses \cite{38}. Performing similar measurements in the unitary Fermi gas can shed light on the role played by gradient and backflow corrections in the superfluid hydrodynamics.

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