SPECTRUM OF WEIGHTED COMPOSITION OPERATORS
PART VIII
LOWER SEMI-FREDHOLM SPECTRUM OF WEIGHTED COMPOSITION OPERATORS ON $C(K)$.
THE CASE OF NON-INVERTIBLE SURJECTIONS.

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1. Introduction

In this paper we continue the study of essential spectra of weighted composition operators. In [5] the first named author obtained a description of essential spectra of weighted composition operators on $C(K)$ in the case when the corresponding map is a homeomorphism of the compact space $K$ onto itself. That allowed to provide a similar description of essential spectra of operators of the form $wU$ acting on Banach lattices in the case when $w$ is a central operator, $U$ is a $d$-isomorphism, and the spectrum of $U$ lies on the unit circle.

In [7] the authors described essential spectra of weighted composition operators on $C(K)$ in the case when the corresponding map is a non-invertible homeomorphism of $K$ into itself.

In [6] we obtained some general results concerning essential spectra of $d$-homomorphisms of Banach $C(K)$-modules. From this results follows as a very special case that the upper semi-Fredholm spectrum of a weighted composition operator on $C(K)$ is rotation invariant provided that the set of eventually periodic points of the corresponding map is of first category in $K$.

Despite these partial results, the problem of describing essential spectra of general weighted composition operators on $C(K)$ and other Banach lattices remains, to the best of our knowledge, unsolved. The current paper represents another step toward the solution of this problem: namely, we describe the lower semi-Fredholm and the Fredholm spectrum.
spectrum of a weighted composition operator on \( C(K) \) in the case when the corresponding map is a surjection.

The paper is organized as follows. In section 2 we introduce the notations we use throughout the paper and state some known results needed in the sequel. In section 3 we obtain a criterion for the operator \( \lambda I - T \) to be Fredholm or lower semi-Fredholm, providing that \( |\lambda| < \rho_{\text{min}}(T) \). We refer the reader to formula (2) below for the definition of \( \rho_{\text{min}}(T) \). In section 4 we obtain a criterion for the operator \( \lambda I - T \) to be Fredholm or lower semi-Fredholm, providing that \( \rho_{\text{min}}(T) < |\lambda| < \rho(T) \). To obtain the corresponding results, and at the same time to avoid making our statements too cumbersome, we had to impose the following additional conditions.

1. The surjection \( \varphi \) is open.
2. The weight \( w \) is an invertible element of the algebra \( C(K) \).
3. The set of all eventually \( \varphi \)-periodic points is of first category in \( K \).

It might be worth noticing that the criterions we obtained involve the notion of almost homeomorphisms of compact Hausdorff spaces that was introduced and studied by Louis Friedler and the first named author in [3]. In particular, Theorem 4.14 states that if the compact Hausdorff space belongs to the class AH (see Definition 4.13) then the spectrum \( \sigma(T) \) coincides with the Fredholm spectrum \( \sigma_f(T) \).

2. Notations and preliminaries

Throughout the paper the following notations are used. \( K \) is a compact Hausdorff space. \( C(K) \) is the Banach algebra of all continuous complex-valued functions on \( K \) endowed with the supremum norm. \( \varphi \) is a continuous map of \( K \) onto itself. \( w \in C(K) \) is the weight. \( \mathbb{N} \) is the semigroup of all positive integers. \( \mathbb{Z} \) is the group of all integers. \( \mathbb{R} \) is the field of all real numbers. \( \mathbb{C} \) is the field of all complex numbers. \( T \) is the unit circle. \( D \) is the open unit disk. \( U \) is the closed unit disk. For any \( n \in \mathbb{N} \) we denote by \( \varphi^n \) the \( n^{\text{th}} \) iteration of the map \( \varphi \). \( \varphi^0 \) is the identical map: \( \varphi^0(k) = k, k \in K \).
If the map $\varphi$ is invertible and $n \in \mathbb{N}$ then $\varphi^{-n}$ is the $n^{th}$ iteration of the inverse map $\varphi^{-1}$.

For any subset $F$ of $K$ and for any $n \in \mathbb{N}$ we denote by $\varphi^{(-n)}(F)$ the full $n^{th}$ preimage of $F$, i.e.

$$\varphi^{(-n)}(F) = \{k \in K : \varphi^n(k) \in F\}$$

For any $n \geq 1$ we denote by $w_n$ the function $w(w \circ \varphi) \ldots (w \circ \varphi^{n-1})$. Thus, $w_1 = w$.

A point $k \in K$ is called eventually $\varphi$-periodic if for some $n \geq 0$ the point $\varphi^n(k)$ is $\varphi$-periodic. When it does not cause any ambiguity we will write eventually periodic or periodic instead of eventually $\varphi$-periodic or $\varphi$-periodic, respectively.

For a given $w \in C(K)$ and a continuous surjection $\varphi$ we define the weighted composition operator $T = wT_\varphi$ as

$$(Tf)(k) = w(k)f(\varphi(k)), \ f \in C(K), \ k \in K.$$
and Φ instead of Φ−(X), Φ+(X), and Φ(X) when it will not cause any ambiguity.

We consider the following subsets of σ(T).

\[ \sigma_{a.p.}(T) = \{ \lambda \in \mathbb{C} : \exists x_n \in X, \|x_n\| = 1, Tx_n - \lambda x_n \to 0 \}. \]

Thus, \( \sigma_{a.p.}(T) \) is the union of the point spectrum and the approximate point spectrum of T.

\[ \sigma_r(T) = \sigma(T) \setminus \sigma_{a.p.}(T); \text{ i.e., } \lambda \in \sigma_r(T) \text{ if and only if the operator } \lambda I - T \text{ is not invertible but bounded from below.} \]

\[ \sigma_{usf}(T) \text{ is the upper semi-Fredholm spectrum of an operator } T \in L(X). \]

It is defined as

\[ \sigma_{usf}(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \notin \Phi^+ \}. \]

\[ \sigma_{lsf}(T) \text{ is the lower semi-Fredholm spectrum of an operator } T \in L(X). \]

It is defined as

\[ \sigma_{lsf}(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \notin \Phi^- \}. \]

\[ \sigma_f(T) \text{ is the Fredholm spectrum of an operator } T \in L(X). \]

It is defined as

\[ \sigma_f(T) = \{ \lambda \in \mathbb{C} : \lambda I - T \notin \Phi \}. \]

We will need the following well known characterizations of \( \sigma_{usf}(T) \) and \( \sigma_{lsf}(T) \) (see e.g. [2]).

**Proposition 2.1.** Let X be a Banach space and \( T \in L(X) \). Let \( \lambda \in \mathbb{C} \).

The following conditions are equivalent.

1. \( \lambda \in \sigma_{usf}(T) \)
2. There is a sequence \( x_n, x_n \in X, n \in \mathbb{N} \), such that \( \|x_n\| = 1, Tx_n - \lambda x_n \to 0 \), and the sequence \( x_n \) is singular, i.e. it does not contain any norm convergent subsequence.

**Proposition 2.2.** Let X be a Banach space and \( T \in L(X) \). Let \( \lambda \in \mathbb{C} \).

The following conditions are equivalent.

1. \( \lambda \in \sigma_{lsf}(T) \)
2. There is a sequence \( x'_n, x'_n \in X', n \in \mathbb{N} \), such that \( \|x'_n\| = 1, T'x'_n - \lambda x'_n \to 0 \), and the sequence \( x'_n \) is singular, i.e. it does not contain any norm convergent subsequence.

**Definition 2.3.** A subset \( \{k_n, n \in \mathbb{Z}\} \) of K is called a \( \varphi \)-string if \( \varphi(k_n) = k_{n+1}, n \in \mathbb{Z} \).

Consider the space \( \prod_{n=-\infty}^{\infty} K_n \) endowed with the Tikhonov topology, where each \( K_n \) is a copy of K. The set of all \( \varphi \)-strings is a closed subset of \( \prod_{n=-\infty}^{\infty} K_n \). We will denote the compact space of all \( \varphi \)-strings by \( K \).
We define the homeomorphism $\Phi$ of $K$ onto itself as follows. If $k \in K$ and $k = (k_n)_{n=-\infty}^{\infty}$ then $\Phi(k) = (k_{n+1})_{n=-\infty}^{\infty}$.

We define $w \in C(K)$ by the equality

$$w(k) = w(k_0).$$

**Definition 2.4.** Let $K$ be a compact Hausdorff space, $\varphi : K \to K$ be a continuous surjection, and $w \in C(K)$. Let $K$, $\Phi$, and $w$ be the objects introduced above. We will say that $\Phi$ is the homeomorphism associated with $\varphi$ and that $T = wT_\varphi$ is the operator associated with $T = wT_\varphi$.

We will need the following two facts about the lower semi-Fredholm spectrum of weighted automorphisms of $C(K)$ (see [5]).

**Theorem 2.5.** Let $\varphi$ be a homeomorphism of $K$ onto itself, $w \in C(K)$, and $T = wT_\varphi$. Assume that the set of all $\varphi$-periodic points is of first category in $K$. Let $\lambda \in \mathbb{C}\{0\}$. The following conditions are equivalent

1. $\text{def}(\lambda I - T) \neq 0$.
2. $\exists k \in K$ such that
   $$|w_n(k)| \leq |\lambda|^n \text{ and } |w_n(\varphi^{-n}(k))| \geq |\lambda|^n, \ n \in \mathbb{N}.$$  

**Theorem 2.6.** Let $\varphi$ be a homeomorphism of $K$ onto itself, $w \in C(K)$, and $T = wT_\varphi$. Assume that the set of all $\varphi$-periodic points is of first category in $K$. Let $\lambda \in \sigma(T) \setminus \{0\}$. The following conditions are equivalent

1. $(\lambda I - T)C(K) = C(K)$.
2. $K = K_1 \cup K_2 \cup O$, where $K_1$, $K_2$, and $O$ are pairwise disjoint nonempty subsets of $K$ such that
   (a) $K_1$ and $K_2$ are closed $\varphi$-invariant subsets of $K$.
   (b) $\rho(T, C(K_1)) < |\lambda|$.
   (c) The operator $T$ is invertible on $C(K_2)$ and $\rho(T^{-1}, C(K_2)) < |\lambda|^{-1}$.
   (d) For any closed subset $F$ of $O$ we have
   $$\bigcap_{n=1}^{\infty} \text{cl} \bigcup_{j=n}^{\infty} \varphi^j(F) \subseteq K_2$$
   and
   $$\bigcap_{n=1}^{\infty} \text{cl} \bigcup_{j=n}^{\infty} \varphi^{-j}(F) \subseteq K_1.$$  

We will end this section with the following definition.
Definition 2.7. Let $K$ be a compact Hausdorff space and $\varphi : K \to K$ be a surjection. We will say that the point $k$ is **perfectly periodic** if $k$ is periodic and $\varphi^{(-1)}(\{k, \varphi(k), \ldots, \varphi^{p-1}(k)\}) = \{k, \varphi(k), \ldots, \varphi^{p-1}(k)\}$, where $p$ is the period of $k$.

3. Conditions for $\lambda I - T$ to be lower semi-Fredholm. The case $\lambda < \rho_{\min}(T)$.

In this section we will obtain a criterion for the operator $T = wT_\varphi$ to be lower semi-Fredholm, providing that $\varphi : K \to K$ is a non-invertible surjection. The answer involves the notion of **almost homeomorphism** of a compact Hausdorff space introduced in [3]

Definition 3.1. Let $K$ be a compact Hausdorff space and $\varphi : K \to K$ be a continuous surjection. The map $\varphi$ is called an almost homeomorphism of $K$ if there is a finite subset $S$ of $K$ such that the map $\varphi : (K \setminus S) \to (K \setminus \varphi(S))$ is a homeomorphism.

Definition 3.2. We say that a compact Hausdorff space $K$ belongs to the class $\mathcal{AH}$ if every almost homeomorphism of $K$ is a homeomorphism.

Theorem 3.3. Let $\varphi$ be a continuous non-invertible map of a compact Hausdorff space $K$ onto itself and let $w \in C(K)$. Assume that $|w| > 0$ on $K$. Let $T = wT_\varphi$ be the corresponding weighted composition operator on $C(K)$. The following conditions are equivalent

1. The operator $\lambda I - T$ is lower semi-Fredholm for every $\lambda \in \rho_{\min}(T)\mathbb{D}$.
2. The operator $\lambda I - T$ is Fredholm for every $\lambda \in \rho_{\min}(T)\mathbb{D}$.
3. The map $\varphi$ is an almost homeomorphism of $K$.

Moreover, for every $\lambda \in \rho_{\min}(T)\mathbb{D}$

\[
\text{ind}(\lambda I - T) = \text{def}(\lambda I - T) = \text{card}\{(p, q) : p, q \in K, p \neq q, \varphi(p) = \varphi(q)\}.
\]

Proof. (2) \Rightarrow (1). This implication is trivial.

(1) \Rightarrow (2). It is sufficient to prove that $\ker(\lambda I - T) = 0$ if $|\lambda| < \rho_{\min}(T)$. Moreover, we will prove that $\rho_{\min}(T)\mathbb{D} \subseteq \sigma_r(T)$. Indeed, if $\lambda \in \sigma_{a.p.}(T)$ then (see [4]) there is $k \in K$ such that for any $u \in \varphi^{(-n)}(k)$ and for any $n \in \mathbb{N}$ we have

\[
|w_n(u)| \leq |\lambda|^n.
\]

Because the case when $\varphi$ is a homeomorphism of $K$ onto itself was considered in [5], in all the statements in the current paper we assume that the surjection $\varphi$ is not invertible.
It is immediate to see from (7) and (2) that \(|\lambda| \geq \rho_{\min}(T)\).

(2) \(\Rightarrow\) (3). This implication is also trivial.

(3) \(\Rightarrow\) (2). First notice that \(T_{\varphi} C(K)\) is a closed unital \(C^*\) subalgebra of \(C(K)\) and therefore (3) implies that the operator \(T_{\varphi}\) is Fredholm and that

\[
\text{ind}(T_{\varphi}) = \text{card}\{ (p,q) : p,q \in K, p \neq q, \varphi(p) = \varphi(q) \}.
\]

Because the operator of multiplication by \(w\) is invertible in \(C(K)\) we see that \(T\) is Fredholm and \(\text{ind}(T) = \text{ind}(T_{\varphi})\).

To prove the last statement of the theorem recall that the index of a semi-Fredholm operator is stable under small norm perturbations. Therefore the set \(A = \{ \lambda \in \rho_{\min}(T) : \text{ind}(\lambda I - T) = \text{ind}(T) \}\) is open in \(C\). Assume, contrary to our claim, that there is \(\alpha \in \rho_{\min}(T) \cap \partial A\). The operator \(\alpha I - T\) is upper semi-Fredholm (because \(\alpha \in \sigma_r(T)\)) and \(\text{ind}(\alpha I - T) \neq \text{ind}(T)\). But it contradicts the stability of index under small norm perturbations. \(\square\)

If we do not assume that the weight \(w\) is invertible in \(C(K)\) the corresponding statement becomes more complicated. Let us denote by \(Z(w)\) the set of all zeros of \(w\) in \(K\), by \(P\) the subset of \(Z(w)\) that consists of \(\varphi\)-periodic points, and by \(S\) the smallest \(\varphi^{(-1)}\)-invariant subset of \(K\) that contains \(P\). Let \(K_1 = K \setminus S\).

**Theorem 3.4.** Let \(\varphi\) be a continuous non-invertible map of a compact Hausdorff space \(K\) onto itself and let \(w \in C(K)\). Let \(T = wT_{\varphi}\) be the corresponding weighted composition operator on \(C(K)\). The following conditions are equivalent

(1) \(T\) is lower semi-Fredholm.

(2) \(T\) is Fredholm.

(3) \(\varphi\) is an almost homeomorphism of \(K\) and every point of the set \(Z(w)\) is isolated in \(K\).

Moreover, if \(T\) is Fredholm then \(\lambda I - T\) is Fredholm for any \(\lambda \in \rho_{\min}(T, C(K))\) and

\[
\text{ind}(\lambda I - T) = \text{def}(\lambda I - T) = \text{card}\{ (p,q) : p,q \in K, p \neq q, \varphi(p) = \varphi(q) \}.
\]

Proof. (3) \(\Rightarrow\) (2). Let \(f_n \in C(K)\) be such that \(\|f_n\| = 1\) and \(T_{\varphi} f_n \to 0\).

Condition (3) guarantees that \(|w| > c > 0\) on \(K \setminus Z(w)\). Therefore \(f_n \to 0\) uniformly on the set \(\varphi(K \setminus Z(w))\). Indeed, \(|f_n(\varphi(k))| \leq c^{-1}\|T_{\varphi} f_n\|, k \in K \setminus Z(w)\). It follows from the equality \(\varphi(K \setminus Z(w)) \cup \varphi(Z(w)) = K\) that \(K \setminus \varphi(Z(w)) \subseteq \varphi(K \setminus Z(w))\).

It follows from (3) that \(K_1\) is a compact subspace of \(K\).
Let $B$ be the subset of $\varphi(z(w))$ that consists of points isolated in $K$. Notice that the set $B$ is either finite or empty. Let $A = \varphi(z(w)) \setminus B$. We claim that $|f_n(t)| \leq c^{-1} \|Tf_n\|$, $t \in A$. Indeed, let us fix $t \in A$. Because the point $t$ is not isolated in $K$ and the set $\varphi(z(w))$ is finite there is a net $t_\alpha$ in $K$ that converges to $t$ and such that $t_\alpha \not\in \varphi(z(w))$. Thus, $|f_n(t_\alpha)| \leq c^{-1} \|Tf_n\|$ and therefore, $|f_n(t)| \leq c^{-1} \|Tf_n\|, t \in A$. The sequence $f_n$ converges uniformly to zero on the set $K \setminus B$. Because the set $B$ is either empty or finite and consists of points isolated in $K$, there is a subsequence of the sequence $f_n$ that converges in $C(K)$. By Proposition 2.2 $T$ is upper semi-Fredholm.

Assume now that there is a sequence $\mu_n$ in $C(K)'$ such that $\|\mu_n\| = 1$ and $\mu_n \to 0$. Let $\nu_n = \mu_n |Z(w)$ and $\tau_n = \mu_n - \nu_n$. Then $T'\nu_n = 0$, and therefore $T'\tau_n \to 0$. Let

$$w_1(k) = \begin{cases} w(k), & \text{if } k \in K \setminus Z(w); \\ 1, & \text{if } k \in Z(w), \end{cases}$$

and let $T_1 = w_1 T_\varphi$. Then $T_1'\tau_n = T'\tau_n \to 0$ and by Theorem 3.3 the sequence $\tau_n$ contains a convergent subsequence. Because the sequence $\nu_n$ obviously has a convergent subsequence it follows that $\mu_n$ contains a convergent subsequence. By Proposition 2.2 $T$ is lower semi-Fredholm and therefore, Fredholm.

(2) $\Rightarrow$ (1). This implication is trivial.

(1) $\Rightarrow$ (3). Assume (1). If $k \in Z(w)$ and $k$ is not isolated in $K$ we can find pairwise distinct points $k_n \in K$ such that $|w(k_n)| \leq \frac{1}{n}, n \in \mathbb{N}$. The sequence $\delta_{k_n} \in C(K)'$ is singular and $T'\delta_{k_n} \to 0$, a contradiction. Let us fix an $\varepsilon > 0$ and let

$$w_\varepsilon(k) = \begin{cases} w(k), & \text{if } k \in K \setminus Z(w); \\ \varepsilon, & \text{if } k \in Z(w), \end{cases} \quad (9)$$

For any sufficiently small $\varepsilon$ the operator $T_\varepsilon = w_\varepsilon T_\varphi$ is lower semi-Fredholm, and by Theorem 3.3 $\varphi$ is an almost homeomorphism of $K$.

Having proved the equivalence of (1), (2), and (3) we will prove now the last statement of the theorem.

Assume one of equivalent conditions (1) - (3). Let $S = \bigcup_{n=1}^{\infty} \varphi(-n)(P)$.

Because $\varphi$ is an almost homeomorphism and the points of $P$ are isolated in $K$, the set $S$ is an at most countable open subset of $K$. Hence, the set $K_1 = K \setminus S$ is a compact subset of $K$ and $\varphi(K_1) = \varphi(-1)(K_1) = K_1$.

Let $\varepsilon > 0$ and let $T_\varepsilon$ be as above. By Theorem 3.3 for every $\lambda \in \rho_{\min}(T_\varepsilon) \mathbb{D}$ the operator $\lambda I - T_\varepsilon$ is Fredholm. But $\rho_{\min}(T(C(K_1))) \leq \rho_{\min}(T_\varepsilon)$ and the operator $T - T_\varepsilon$ is finite dimensional. Hence the last statement of the theorem follows. $\square$
4. Conditions for $\lambda I - T$ to be lower semi-Fredholm. The case $|\lambda| > \rho_{\text{min}}(T)$.

We start with the following lemma.

**Lemma 4.1.** Let $K$ be a compact Hausdorff space and $\varphi: K \to K$ be a continuous non-invertible surjection. Let $\Phi$ be the homeomorphism of $K$ associated with $\varphi$ (see Definition [2.4]). The following conditions are equivalent

1. The set of perfectly $\varphi$-periodic points (see Definition [2.7]) is of first category in $K$.
2. The set of $\Phi$-periodic points is of first category in $K$.

**Proof.** Denote by $P_n$ (respectively, $P_n^\Phi$) the set of all $\varphi$-periodic (respectively, $\Phi$-periodic) points of period $n$.

(1) $\Rightarrow$ (2). Assume (1) and assume to the contrary that for some $n \in \mathbb{N}$ we have $\text{Int } P_n \neq \emptyset$. Considering, if necessary, the map $\Phi^n$ instead of $\Phi$ we can assume without loss of generality that $n = 1$. The set $E = \{k_0 : k \in \text{Int } P_1\}$ is an open subset of $P_1 = \{s \in K : \varphi(s) = s\}$ and $\varphi(E) = E$. Therefore, by (1) there is an $s_1 \in K$ such that $s_1$ is not a $\varphi$-periodic point and $\varphi(s_1) = k \in E$. Let $k \in K$ be such that $(k_n) = k, n \in \mathbb{Z}$. Let $s_n : n \geq 2$ be a sequence of points in $K$ such that $\varphi(s_{n+1}) = s_n, n \in \mathbb{N}$. For any $m \in \mathbb{N}$ consider the point $k^m \in K$ such that

$$k^m_n = \begin{cases} k, & \text{if } n \geq -m; \\ s_{-(n+m)}, & \text{if } n < -m. \end{cases}$$

The points $k^m, m \in \mathbb{N}$, are not $\Phi$-periodic and they converge to $k$ in $K$, a contradiction.

(2) $\Rightarrow$ (1). Assume (2) and assume to the contrary that for some $n \in \mathbb{N}$ there is an open in $K$ nonempty subset $E$ of $P_n$ such that $E = \varphi^{-1}(E)$. Let $E = \{k \in K : k_0 \in E\}$. Then $E$ is an open nonempty subset of $K$ and $E \subseteq P_n$, a contradiction. \(\square\)

**Corollary 4.2.** Let $K$ be a compact Hausdorff space and $\varphi: K \to K$ be a continuous non-invertible surjection. Let $\Phi$ be the homeomorphism of $K$ associated with $\varphi$ (see Definition [2.4]).

If the set of all eventually $\varphi$-periodic points is of first category in $K$, then the set of $\Phi$-periodic points is of first category in $K$.

We will proceed with proving a series of lemmas needed for our main result in this section, Theorem [4.7].

**Lemma 4.3.** Let $K$ be a compact Hausdorff space, $\varphi: K \to K$ be a continuous non-invertible surjection, $w \in C(K)$, and $T = wT_\varphi$. Assume that...
(1) \( \lambda \in \sigma(T) \setminus \sigma(T) \) and \( |\lambda| > \rho_{\text{min}}(T) \).
(2) The operator \( \lambda I - T \) is lower semi-Fredholm.
(3) The set of eventually \( \varphi \)-periodic points is of first category in \( K \). Then \( K = K_1 \cup K_2 \) where \( K_1 \) and \( K_2 \) are nonempty closed disjoint subsets of \( K \) such that
- \( \varphi(K_i) = K_i, i = 1, 2 \).
- \( K_2 \neq \emptyset \) and \( \rho_{\text{min}}(T, C(K_2)) > |\lambda| \).
- The restriction of \( \varphi \) on \( K_2 \) is an almost homeomorphism, but not a homeomorphism of \( K_2 \).
- \( \rho(T, C(K_1)) < |\lambda| \).

Proof. By (3), Lemma 4.3 and Theorem 3.7 in [4] the spectrum \( \sigma(T) \) is rotation invariant and therefore \( \lambda T \cap \sigma(T) = \emptyset \). By Theorem 3.10 in [4] we have \( K = K_1 \cup K_2 \) where \( K_1 \) and \( K_2 \) are disjoint \( \Phi \)-invariant closed subsets of \( K \), \( \rho(T, C(K_1)) < |\lambda| \), the operator \( T \) is invertible on \( C(K_2) \) and \( \rho(T^{-1}, C(K_2)) < |\lambda| \). Notice that the sets \( K_1 \) and \( K_2 \) cannot be empty because on the one hand, \( \rho_{\text{min}}(T) = \rho_{\text{min}}(T) > |\lambda| \), while on the other hand, \( \rho(T) = \rho(T) > |\lambda| \). Let \( p : K \to K \) be the map defined as \( p(K) = k_0 \) and let \( K_1 = p(K_1) \) and \( K_2 = p(K_2) \). It is immediate to see that \( K_1 \) and \( K_2 \) are disjoint \( \varphi \)-invariant closed subsets of \( K \) and that \( \rho(T, C(K_1)) < |\lambda| \) while \( \rho_{\text{min}}(T, C(K_2)) > |\lambda| \). The map \( \varphi \) cannot be a homeomorphism of \( K_2 \) onto itself because by theorem 3.10 in [4] it would imply that \( \lambda \notin \sigma(T) \), and by Theorem 3.3 it must be an almost homeomorphism of \( K_2 \).

Lemma 4.4. Let \( K \) be a compact Hausdorff space, \( \varphi : K \to K \) be a continuous non-invertible surjection, \( w \in C(K) \), and \( T = wT_\varphi \). Assume that
(1) \( w \) is invertible in \( C(K) \).
(2) The map \( \varphi \) is open.
(3) \( \lambda \in \sigma(T) \cap \sigma(T) \) and \( |\lambda| > \rho_{\text{min}}(T) \).
(4) The operator \( \lambda I - T \) is lower semi-Fredholm.
(5) \( (\lambda I - T)C(K) = C(K) \).
(6) The set of eventually \( \varphi \)-periodic points is of first category in \( K \).

Then there are subsets \( K_1 \), \( K_2 \), and \( Q \) of \( K \) such that
(a) The set \( K_1 \) is closed in \( K \) and \( \varphi(K_1) = \varphi^{-1}(K_1) = K_1 \).
(b) \( \rho(T, C(K_1)) < |\lambda| \).
(c) The set \( K_2 \) is closed in \( K \), \( \varphi(K_2) = K_2 \), \( \rho_{\text{min}}(T, C(K_2)) > |\lambda| \) and \( \varphi \) is an almost homeomorphism of \( K_2 \).
(d) If
\[
K = K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2)
\]
then the set $K_2$ is open in $K$.

(e) The set $Q$

$$Q = K \setminus \left( K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2) \right)$$

is open in $K$ and $\varphi(Q) = \varphi^{-1}(Q) = Q$.

(f) The sets $K_1, K_2,$ and $Q$ are pairwise disjoint and

$$K = K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2) \cup Q.$$

(g) Assume that $Q \neq \emptyset$ and let $E$ be a closed in $K$ subset of $Q$ and $V_1, V_2$ be open neighborhoods of $K_1$ and $K_2$, respectively. Then there is an $m \in \mathbb{N}$ such that for any $n \geq m$ we have $\varphi^n(E) \subset V_2$ and $\varphi^{-n}(E) \subset V_1$.

(h) Assume that $Q \neq \emptyset$. Then there is an open neighborhood $V$ of $K_2$ such that $\varphi(V \cap Q) \subset V \cap Q$ and the restriction of $\varphi$ on $V \cap Q$ is one-to-one.

Proof. Conditions (3) and (5) combined with Lemma 4.1 and Theorem 2.6 provide that $K$ is the union of three disjoint nonempty $\Phi$ and $\Phi^{-1}$-invariant subsets $K_1, K_2,$ and $O$ with the properties

(I) The sets $K_1$ and $K_2$ are closed in $K$.

(II) $T$ is invertible on $C(K_2)$ and $\rho(T^{-1}, C(K_2)) < 1$.

(III) $\rho(T, C(K_1)) < 1$.

(IV) If $V_1$ and $V_2$ are open neighborhoods of $K_1$ and $K_2$, respectively, and $E$ is a closed subset of $O$ than there is an $m \in \mathbb{N}$ such that for any $n \geq m$ we have $\Phi^n(E) \subset V_2$ and $\Phi^{-n}(E) \subset V_1$.

Let $p : K \to K$ be the map $p(k) = k_0$ introduced in the proof of Lemma 4.3 and let $K_1 = p(K_1), K_2 = p(K_2),$ and $O = p(O)$. We proceed with proving the statements (a) - (i) of the lemma.

(a). Obviously the set $K_1$ is closed. It follows from the equalities $\Phi(K_1) = K_1, \Phi(O) = O,$ and $\Phi(K_2) = K_2$ and the definition of the map $p$ that $\varphi(K_1) = K_1, \varphi(O) = O,$ and $\varphi(K_2) = K_2$. Moreover, it follows from properties (II) - (IV) that $K_1$ does not intersect with $K_2 \cup O$ and therefore, $\varphi(K_1) = K_1 = \varphi^{-1}(K_1)$.

(b). Follows from (III) and the definition of $p$.

(c). The inequality $\rho(T^{-1}, C(K_2)) < 1$ follows from (II). It follows from the fact that the operator $\lambda I - T$ is lower semi-Fredholm and Theorem 3.3 that $\varphi$ is an almost homeomorphism of $K_2$. 

(d) It follows from Baire Category Theorem and the fact that $K_1 \cap F = \emptyset$ that there is an $n \in \mathbb{N}$ such that $\text{Int} \varphi^{(-n)}(K_2) \neq \emptyset$. Then, because $\varphi$ is open, $\text{Int} K_2 \neq \emptyset$. Assume, contrary to our claim, that $\text{Int} K_2 \nsubseteq K_2$. Let $\tilde{K} = K \setminus \bigcup_{n=0}^{\infty} \varphi^{(-n)}(\text{Int} K_2)$. Notice that $\tilde{K} = K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{(-n)}(E)$ where $E = K_2 \setminus \text{Int} K_2$. Applying again Baire category theorem we see that there is an $n \in \mathbb{N}$ such that $\text{Int}_{\tilde{K}} \varphi^{(-n)}(E) \neq \emptyset$. Therefore, there is an open subset $V$ of $K$ such that $V \cap \varphi^{(-n)} \subset \varphi^{(-n)}$. Then $\varphi^{n}(V) \cap E \subset E \neq \emptyset$. The set $\text{Int} K_2 \cup \varphi^{n}(V) \cap E \cup \text{Int} K_2 \cup (\varphi^{n}(V) \cap E)$ is open in $K$ and $\text{Int} K_2 \nsubseteq \text{Int} K_2 \cup (\varphi^{n}(V) \cap E)$, a contradiction.

(e) - (h). For brevity let $F = \bigcup_{n=0}^{\infty} \varphi^{(-n)}(K_2)$. We assume that $Q \neq \emptyset$, otherwise the statement is trivial. Let $k \in Q$ and let $E = p^{-1}(\{k\})$. $E$ is a compact subset of $O$ and therefore by Theorem 2.6 for arbitrary open neighborhoods $V_1$ and $V_2$ of, respectively, $K_1$ and $K_2$ there is an $N \in \mathbb{N}$ such that $\Phi^n(E) \subseteq V_2$ and $\Phi^{-n}(E) \subseteq V_1$ for any $n \geq N$. Define a function $f$ on $K$ as follows,

$$f(x) = \begin{cases} 
1, & \text{if } x = k, \\
\lambda^i/w_i(k), & \text{if } x = \varphi^{(i)}(k), i \in \mathbb{N}, \\
\lambda^{-i}w_i(x), & \text{if } x \in \varphi^{-i}(\{k\}), i \in \mathbb{N}, \\
0, & \text{otherwise}, 
\end{cases}$$

It is immediate to see that $f$ is an element of $C(K)^n$ and $T^\mu f = \lambda f$. Because we assumed that $\lambda I - T$ is a semi-Fredholm operator it follows that (see e.g. [1], Theorem 4.42) $\ker(\lambda I - T) \neq \emptyset$. Let $f \in C(K)$, $f \neq 0$, and $Tf = \lambda f$. Let $U = \{k \in K : f(k) \neq 0\}$. It follows from $Tf = \lambda f$, $\lambda \neq 0$, and $w \in C(K)^{-1}$ that

$$\varphi(U) = U = \varphi^{(-1)}(U) \text{ and } U \cap (K_1 \cup F) = \emptyset. \tag{10}$$

By Zorn’s lemma there is a maximal by inclusion open subset $V$ of $K$ with properties (10). We claim that

$$V \cup K_1 \cup F = K. \tag{11}$$

To prove it consider the compact space $G = K \setminus V$. It follows from (10) that $\varphi(G) = G$ and therefore the operator $T$ is defined on $C(G)$. Moreover, the operator $\lambda I - T$ is lower semi-Fredholm. Assume that $K_1 \cup F \nsubseteq G$. Then our previous reasoning shows that there is $g \in C(G)$ such that $g \neq 0$ and $Tg = \lambda g$. Let $H = \{k \in G : g(k) \neq 0\}$. Then the set $V \cup H$ is open in $K$ and satisfies (10) in contradiction with the maximality of $V$. 


We claim that for any compact subset \( W \) of \( V \) and for any open neighborhoods \( U_1 \) and \( U_2 \) of \( K_1 \) and \( K_2 \), respectively, there is an \( N \in \mathbb{N} \) such that for any \( n > N \) we have

\[
\varphi^n(W) \subset U_2 \text{ and } \varphi^{-n}(W) \subset U_1.
\] (12)

Indeed, (12) follows from the fact that \( p^{(-1)}(W) \) is a compact subset of \( O \).

Notice that it follows from (12) that \( clV \cap K_2 \neq \emptyset \).

Next we claim that there is an open neighborhood \( Q \) of \( K_2 \) in \( K \) such that \( \varphi(Q \cap V) \subseteq Q \cap V \) and the restriction of \( \varphi \) on \( Q \cap V \) is a homeomorphism of \( Q \cap V \) onto its image. First notice that ker \((\lambda I - T') = \text{ker} \((\lambda I - T'), C(K_2)) \). Indeed, let \( \mu \in C(K)' \) be such that \( T'\mu = \mu \).

Let \( U \) be an open neighborhood of \( K_2 \) and let \( f \in C(K) \) be such that \( supp f \subseteq K \setminus U \). Then \( \int f d\mu = \lambda^{-n} \int T^n f d\mu \to 0 \), because \( supp T^n f \subseteq K \setminus \varphi^{(-n)}(U), \bigcap_{n=1}^{\infty} (K \setminus \varphi^{(-n)}(U)) = K_1 \), and \( \rho(T, C(K_1)) < |\lambda| \). It follows that \( supp \mu \subseteq K_2 \).

Assume, contrary to our claim, that for any open neighborhood \( Q \) of \( K_2 \) there are \( p, q \in Q \cap V \) such that \( \varphi(p) = \varphi(q) \). The proof of Lemma 5.9 in [5] shows that there is a sequence \( \mu_n \in (C(K)')' \) such that \( \|\mu_n\| = 1, T'\mu_n - \lambda \mu_n \to 0 \), and for each \( n \) the measure \( \mu_n \) is a finite linear combination of point measures \( \delta_{S_i} \) where \( S_i \in V \). Because the operator \( \lambda I - T' \) is lower semi-Fredholm the sequence \( \mu_n \) must contain a norm convergent subsequence. Let the limit of such a subsequence be \( \nu \) then \( T'\nu = \nu \) and by our previous step \( supp \nu \subseteq K_2 \). That means that for every \( n \) we have \( |\nu| \wedge |\mu_n| = 0 \), a contradiction.

Thus, there is an open neighborhood \( Q \) of \( K_2 \) such that the restriction of \( \varphi \) on \( Q \cap V \) is one-to-one. The proofs of Lemma 5.10 and Corollary 5.11 in [5] show that we can choose \( Q \) in such a way that \( \varphi(V \cap Q) \subset V \cap Q \). □

We will now consider what happens when the operator \( \lambda I - T \) is lower semi-Fredholm and \( \lambda \in \sigma_{a.p.}(T') \). In this case by Theorem 2.5 there is \( k \in K \) such that

\[
|w_n(k)| \leq |\lambda|^n, \quad |w_n(\Phi^{-n}(k))| \geq |\lambda|^n, n \in \mathbb{N}.
\] (13)

**Lemma 4.5.** Assume conditions of Lemma 4.4. Let the operator \( \lambda I - T \) be lower semi-Fredholm and \( \lambda \in \sigma_{a.p.}(T') \). Let \( S \) be the set of \( \Phi \)-strings
defined as
\[ S = \{ s : s = \{ \Phi^n(k) : n \in \mathbb{Z} \} \} \]
\[ |w_n(k)| \leq |\lambda|^n, \quad |w_n(\Phi^{-n}(k))| \geq |\lambda|^n, \quad n \in \mathbb{N} \} . \]

Then the set \( S \) is finite and for every \( s = \{ \Phi^n(k) : n \in \mathbb{Z} \} \in S \) the point \( k \) is isolated in \( K \).

Proof. Let \( k \) be a point in \( K \) such that inequalities (13) are satisfied. Let \( \{ k_n : n \in \mathbb{Z} \} \) be the corresponding \( \varphi \)-string. We have to consider several cases.

Case 1. The set \( \{ k_n : n \in \mathbb{N} \} \) is infinite, i.e. the point \( k_0 \) is not eventually periodic. We claim that
\[ \sum_{n=1}^{\infty} \frac{|\lambda|^n}{|w_n(k-n)|} + \sum_{n=0}^{\infty} \frac{|w_n(k_0)|}{|\lambda|^n} < \infty. \tag{14} \]
To prove (14) assume first to the contrary that
\[ \sum_{n=1}^{\infty} \frac{|\lambda|^n}{|w_n(k-n)|} = \infty. \tag{15} \]
For every \( m \in \mathbb{N} \) consider the discrete measure \( \mu_m = \sum_{n=-m}^{0} \frac{\lambda^n}{w_n(k-n)} \delta_{k_n} \). Then
\[ \| \mu_m \| = \sum_{n=0}^{m} \frac{|\lambda|^n}{|w_n(k-n)|} \rightarrow \infty \quad \text{as} \quad m \rightarrow \infty. \tag{16} \]
and
\[ \| T' \mu_m - \lambda \mu_m \| = \| \frac{\lambda^m}{w_m(k-m)} \delta_{k-m} + w(k_0)\delta_{k_1} \| \leq 1 + \| w \|_{C(K)}. \tag{17} \]
Let \( \nu_m = \mu_m/\| \mu_m \| \). It follows from (16) and (17) that we can find positive integers \( m_l \) such that \( \| T' \nu_{m_l} - \lambda \nu_{m_l} \| \rightarrow 0 \) and the sequence \( \nu_{m_l} \) is singular, in contradiction with our assumption that the operator \( \lambda I - T \) is lower semi-Fredholm.

Similarly we can prove that
\[ \sum_{n=0}^{\infty} \frac{|w_n(k_0)|}{|\lambda|^n} < \infty. \]
Let
\[ \mu = \sum_{n=1}^{\infty} \frac{\lambda^n}{w_n(k-n)} \delta_{k-n} + \sum_{n=0}^{\infty} \frac{w_n(k_0)}{\lambda^n} \delta_{k_n} . \]
Then $T'\mu = \lambda \mu$. We claim that the point $k_0$ is isolated in $K$. Indeed, otherwise using [14] and the fact that the map $\varphi$ is open we can find points $k_{i,m}, m \in \mathbb{N}, -m \leq i \leq m$ such that
(a) $\varphi(k_{i,m} = k_{i+1,m}, m \in \mathbb{N}, -m \leq i \leq m - 1$.
(b) The points $k_{i,m}$ are pairwise distinct.
(c) $\mu_m - \mu_{m+\infty} \to 0$, where

$$
\mu_m = \sum_{i=-m}^{-1} \frac{|\lambda|^i}{w_i(k_{m,i})} \delta_{k_{m,i}} + \sum_{i=0}^{m} \frac{w_i(k_{m,i})}{\lambda^i} \delta_{k_{m,i}}.
$$

Let $\nu_m = \mu_m/\|\mu_m\|$. It follows from (a) - (c) that $T'\nu_m - \lambda \nu_m \to 0$ and the sequence $\nu_m$ is singular in contradiction with $\lambda I - T$ assumed to be lower semi-Fredholm.

Case 2. The set $\{k_n : n \in \mathbb{Z}\}$ is infinite but the set $\{k_n : n \in \mathbb{N}\}$ is finite. In other words, there is $s \in \mathbb{Z}$ such that the point $k_s$ is $\varphi$-periodic, but the point $k_{s-1}$ is not. Let $p$ be the period of the point $k_s$.

There are two possibilities.
(1) $w_p(k_s) = \lambda^p$. Let

$$
\mu = \sum_{i=0}^{p-1} \lambda^{p-i-1}(T')^i \delta_s.
$$

Then $T'\mu = \lambda \mu$.
(2) $w_p(k_s) \neq \lambda^p$. Like in case 1 we can prove that

$$
\sum_{n=1}^{\infty} \frac{|\lambda|^n}{w_n(k_{s-n})} < \infty.
$$

Let

$$
\mu = (\lambda^p - w_p(k_s) \sum_{n=1}^{\infty} \frac{\lambda^n}{w_n(k_{s-n})} \delta_{k_{s-n}} + \sum_{i=0}^{p-1} \lambda^{p-i-1}(T')^i \delta_s.
$$

Then $T'\mu = \lambda \mu$.

In both cases we can prove that the point $k_s$ is isolated in $K$ using the same reasoning as in Case 1.

Case 3. The set $\{k_n : n \in \mathbb{Z}\}$ is finite. Assume that the point $k_0$ is not isolated in $K$. Then there are the following possibilities.

1. For every open neighborhood $V$ of $k_0$ and for every $n \in \mathbb{N}$ the neighborhood $V$ contains either a point that is not $\varphi$-periodic or a $\varphi$-periodic point of period at least $n$. In this case we can construct a singular sequence $\mu_n$ such that $T'\mu_n - \lambda \mu_n \to 0$.

   Indeed we can find points $t_{n,i} \in K, n \in \mathbb{N} - n \leq i \leq n$ such that
\( \varphi(t_{n,i}) = t_{n,i+1}, -n \leq i < n, \)

(II) The points \( t_{n,i}, n \in \mathbb{N}, -n \leq i \leq n \) are pairwise distinct,

(III) \( |w_j(t_{n,0})| \leq 2|\lambda|^n \) and \( |w_j(t_{n,-j})| \geq 1/2|\lambda|^n, j = 1, \ldots, n. \)

Let

\[
\mu_n = \sum_{j=0}^{n-1} \left( 1 - \frac{1}{\sqrt{n}} \right)^j \lambda^{-j} w_j(t_{j,0}) \delta_{t_{j,n}} + \sum_{j=1}^{n-1} \left( 1 - \frac{\lambda^j}{\sqrt{n}} \right)^j \frac{1}{w_j(t_{n,-j})} \delta_{t_{n,-j}}. \tag{18}
\]

It is not difficult to see from (III) and (18) that

\[
\|T' \mu_n - \lambda \mu_n\| = o(\|\mu_n\|), n \to \infty. \tag{19}
\]

But the measures \( \mu_n \) are pairwise disjoint in virtue of II and therefore the sequence \( \nu_n = |\mu_n| \) is singular, a contradiction.

2. There are an open neighborhood \( V \) of \( k_0 \) and an \( n \in \mathbb{N} \) such that every point in \( V \) is \( \varphi \)-periodic and has period less or equal to \( n \). That obviously contradicts our assumption that the set of eventually \( \varphi \)-periodic points is of first category in \( K \).

Thus it follows from our assumption that the set of all eventually \( \varphi \)-periodic points is of first category that only the first case is possible. Moreover, because in this case to each \( s \in S \) there is a unique up to a constant factor discrete measure \( \mu \) on the set \( p(s) \) such that \( T' \mu = \lambda \mu \), we see that the set \( S \) is finite. \( \square \)

**Corollary 4.6.** Assume conditions of Lemma 4.4. Let the operator \( \lambda I - T \) be lower semi-Fredholm and let the set \( S \) be not empty. Then there is a countable open subset \( S \) of \( K \) such that \( S \) is the union of finite number of strings, \( \varphi(K \setminus S) = K \setminus S \) and the operator \( \lambda I - T \) is lower semi-Fredholm on \( C(K \setminus S) \).

**Proof.** We take \( S = p(S) \) and apply Lemma 4.5. \( \square \)

Finally we can state our main result.

**Theorem 4.7.** Let \( K \) be a compact Hausdorff space and \( \varphi \) be an open continuous non-invertible map of \( K \) onto itself. Let \( w \) be an invertible element of the algebra \( C(K) \). Assume that the set of all eventually \( \varphi \)-periodic points is of first category in \( K \). Let

\[
(Tf)(k) = w(k)f(\varphi(k)), f \in C(K), k \in K.
\]

Let \( \lambda \in \mathbb{C} \) be such that \( \lambda \in \sigma(T) \) and \( |\lambda| > \rho_{\min}(T) \). The following conditions are equivalent.

(1) The operator \( \lambda I - T \) is lower semi-Fredholm.

(2) One of the following conditions is satisfied.
(I) The compact space $K$ is the union of two disjoint $\varphi$-invariant clopen subsets $K_1$ and $K_2$ such that $\rho(T, C(K_1)) < |\lambda|$, $\rho_{\min}(T, C(K_2)) > |\lambda|$, and $\varphi$ is an almost homeomorphism but not a homeomorphism of $K$ onto itself.

(II) There are closed subsets $K_1$ and $K_2$ of $K$ such that $\varphi^{-1}(K_1) = \varphi(K_1) = K_1$, $\rho(T, C(K_1)) < |\lambda|$, $\varphi(K_2) = K_2$, $\rho_{\min}(T, C(K_2)) > |\lambda|$, and $\varphi$ is an almost homeomorphism but not a homeomorphism of $K$ onto itself.

Moreover, the set $K_2$ is a clopen subset of $K$, $K_2 \subseteq \varphi^{-1}(K_2)$, and

$K = K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2)$.

(III)

$K = K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2) \cup Q$,

where the sets $K_1$, $K_2$, and $Q$ have properties described in the statement of Lemma 4.4.

(IV)

$K = K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2) \cup Q \cup S$,

where the sets $K_1$, $K_2$, and $Q$ have properties described in the statement of Lemma 4.4 and $S$ is an open countable subset of $K$ disjoint with the set $K_1 \cup \bigcup_{n=0}^{\infty} \varphi^{-n}(K_2) \cup Q$. Moreover, the set $S$ is the union of finite number of $\varphi$-strings and for every $s \in S$ we have

$|w_n(s_0)| \leq |\lambda|^n, |w_n(s_{-n})| \geq |\lambda|^n, n \in \mathbb{N}$

proof. The implication (1) \Rightarrow (2) follows from Lemmas 4.3 - 4.5.

To prove the implication (2) \Rightarrow (1) we will prove that every of conditions (I) - (IV) implies (2).

(I) \Rightarrow (2). This implication follows from Theorem 3.3.

(II) \Rightarrow (2). First we notice that (II) implies that $\lambda \in \sigma_r(T)$, and therefore the image $(\lambda I - T)C(K)$ is closed in $C(K)$. Indeed, assume to the contrary that $\lambda \in \sigma_{a.p.}(T)$. Then (see [4]) there is $k \in K$ such that

$|w_n(k)| \geq |\lambda|^n$ and $\forall t \in \varphi^{-n}(k)$, $|w_n(t)| \leq |\lambda|^n, n \in \mathbb{N}$. \hfill (20)

But it is immediate to see that the existence of a point $k$ satisfying (20) contradicts condition (II).
Assume that \( \mu \in C(K)', \mu \neq 0 \), and \( T'\mu = \lambda \mu \). The proof of Theorem 5.14 in [5] shows that \( \text{supp} \mu \subseteq K_2 \). Because \( \varphi \) is an almost homeomorphism of \( K_2 \) we see that \( \text{def}(\lambda I - T) < \infty \).

\((III) \Rightarrow (2)\). Assume (III) and assume to the contrary that there is a singular sequence \( \mu_n \in C(K)' \) such that \( T'\mu_n - \lambda \mu_n \to 0 \). From the previous step we conclude that without loss of generality we can assume that \( \text{supp} \mu_n \subseteq Q \). Assume first that the compact space \( K \) is extremally disconnected. Then the restriction of \( \varphi \) on \( \text{cl}V \) is one-to-one. Then we come to contradiction as in the proof of Theorem 5.14 in [5].

If \( K \) is not extremally disconnected we can consider the operator \( T'' \) on the second dual \( C(K)'' \cong C(Q) \) where the compact space \( Q \) is extremally disconnected. Notice that \( T'' = w''T_\psi \) where \( \psi \) is a continuous map of \( Q \) onto itself. It is not difficult to see that \( \psi \) satisfies condition (III).

\((IV) \Rightarrow (2)\). Assume (IV) and assume to the contrary that there is a singular sequence \( \mu_n \in C(K)' \) such that \( T'\mu_n - \lambda \mu_n \to 0 \). Without loss of generality we can assume that \( \text{supp}\mu_n \subseteq S \) Let \( S^{**} \) be the set \( j^{-1}(S) \) where \( j : Q \to K \) is the surjection corresponding to the isometric embedding of \( C(K) \) into \( C(Q) \). The set \( S^{**} \) is a finite union of \( \psi \)-strings and points of \( S^{**} \) are isolated in \( Q \). The map \( \psi \) extends to a homeomorphism of \( \text{cl}S^{**} \) onto itself. It follows from Theorem 2.11 in [5] that the sequence \( \mu_n \) contains a convergent subsequence, a contradiction.

\[ \square \]

**Corollary 4.8.** Assuming one of conditions (I) - (IV) from Theorem 4.7 is satisfied, the defect of \( \lambda I - T \) can be computed as

\[ \text{def}(\lambda I - T) = \text{card}(\{(p, q) : p, q \in K_2, p \neq q, \varphi(p) = \varphi(q)\}) + \text{card}(S), \]

where \( S \) is the set introduced in the statement of Lemma 4.5.

In particular, \( \text{def}(\lambda I - T) = 0 \), i.e. \( (\lambda I - T)C(K) = C(K) \) if the following two conditions are satisfied

- The set \( S \) is empty,
- The map \( \varphi \) is a homeomorphism of \( K_2 \) onto itself.

From Theorem 4.7 easily follows the following criterion for the operator \( \lambda I - T \) (where \( \lambda > \rho_{\text{min}}(T) \)) to be Fredholm.

**Theorem 4.9.** Let \( K \) be a compact Hausdorff space, \( \varphi \) be a continuous open non-invertible map of \( K \) onto itself. Let \( w \in C(K)^{-1} \) and

\[ (Tf)(k) = w(k)f(\varphi(k)), \quad f \in C(K), k \in K. \]

\[ ^3\text{See also Theorem 5.14 in [5]} \]
Assume that the set of all eventually $\varphi$-periodic points is of first category in $K$. Let $\lambda \in \sigma(T)$ and $\lambda > \rho_{\min}(T)$. The following conditions are equivalent.

1. The operator $\lambda I - T$ is Fredholm.
2. One of conditions (I) - (IV) is satisfied. Moreover, if the set $Q$ is not empty, then every point of $Q$ is isolated in $K$ and there is a finite subset $\{k_1, \ldots, k_p\}$ of $Q$ such that the sets $A_1, \ldots, A_p$ of pairwise disjoint and $Q = \bigcup_{j=1}^{p} A_j$, where $A_j$ is the smallest $\varphi$ and $\varphi^{(-1)}$ invariant subset of $Q$ that contains $k_j$, i.e. $A_j = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{\infty} \varphi^{(-m)}(\varphi^{n}(k_j))$.

**Corollary 4.10.** Assume condition (2) of Theorem 4.9. Then
\[ \text{ind}(\lambda I - T) = p - \text{def}(\lambda I - T), \]
where $p$ is from the statement of Theorem 4.9 and $\text{def}(\lambda I - T)$ is defined by (21).

**Corollary 4.11.** Assume either conditions of Theorem 4.7, or of Theorem 4.9. Then the spectrum $\sigma_{sfl}(T)$ or, respectively, the Fredholm spectrum $\sigma_f(T)$ is rotation invariant.

**Corollary 4.12.** Let $K$ be a compact Hausdorff space, $\varphi$ be a continuous open map of $K$ onto itself. Let $w \in C(K)^{-1}$ and
\[ (Tf)(k) = w(k)f(\varphi(k)), \quad f \in C(K), k \in K. \]
Assume that the set of all eventually $\varphi$-periodic points is of first category in $K$. Assume that either $|\lambda| = \rho(T)$ or $|\lambda| = \rho_{\min}(T)$. Then the operator $\lambda I - T$ cannot be lower semi-Fredholm.

**Proof.** Let $|\lambda| = \rho(T)$. If the operator $\lambda I - T$ is semi-Fredholm then by the punctured neighborhood theorem (see [9]) $\lambda$ is an isolated point in $\sigma(T)$. On the other hand, it follows from our assumption that the set of all eventually $\varphi$-periodic points is of first category in $K$ that $\sigma(T)$ is rotation invariant (see [4]). As $\rho(T) > 0$, we have a contradiction.

Next, assume that $|\lambda| = \rho_{\min}(T)$ and the operator $\lambda I - T$ is lower semi-Fredholm. Then there are $\lambda_n \in \mathbb{C}, n \in \mathbb{N}$ such that $|\lambda_n| \downarrow |\lambda|$ and either $\lambda_n \notin \sigma(T), n \in \mathbb{N}$, or $\lambda_n \in \sigma(T)$ and the operator $\lambda_n I - T$ is lower semi-Fredholm for any $n \in \mathbb{N}$.

Applying in the first case Theorem 3.10 from [4] and in the second case Theorem 4.7 we can see that there is a closed subset $K_\infty$ of $K$ such that $\varphi(K_\infty) = K_\infty$ and $\rho(T, C(K_\infty)) = |\lambda| = \rho_{\min}(T)$.

We claim that the operator $\lambda I - T$ considered on the space $C(K_\infty)$ is lower semi-Fredholm. Indeed, otherwise there is a singular sequence...
\( \mu_n \in C(K_\infty)' \) such that \( \|\mu_n\| = 1 \) and \( T'\mu_n - \lambda \mu_n \to 0 \). Considering the measures \( \mu_n \) as elements of \( C(K)' \) we come to a contradiction.

Applying the punctured neighborhood theorem to the operator \( \lambda I - T \) on \( C(K_\infty) \) we see that the set \( \sigma(T, C(K_\infty)) \) is not rotation invariant. By Theorem 3.12 from [4] there is a \( \varphi \)-periodic point \( k \in K_\infty \) such that \( |w_\varphi(k)| < |\lambda|^p \), where \( p \) is the period of \( k \). But the last inequality contradicts the definition of \( \rho_{\text{min}}(T) \). \( \square \)

To state our next result we have to recall the following definition introduced in [3]

**Definition 4.13.** Let \( K \) be a compact Hausdorff space. We say that \( K \in AH \) if every almost homeomorphism of \( K \) onto itself is a homeomorphism.

**Theorem 4.14.** Let \( K \) be a compact Hausdorff space, \( \varphi \) be a continuous non-invertible map of \( K \) onto itself, and \( w \in C(K)^{-1} \). Assume that the set of all eventually \( \varphi \)-periodic points is of first category in \( K \).

Assume that \( K \in AH \). Then, \( \sigma_f(T) = \sigma(T) \).

**Proof.** Consider \( \lambda \in \sigma(T) \). we have to consider three cases.

1. \( |\lambda| < \rho_{\text{min}}(T) \). The operator \( \lambda I - T \) cannot be Fredholm by Theorem 3.3
2. \( |\lambda| = \rho_{\text{min}}(T) \) or \( |\lambda| = \rho(T) \). The operator \( \lambda I - T \) cannot be Fredholm by Corollary 4.12
3. \( \rho_{\text{min}}(T) < |\lambda| < \rho(T) \). The operator \( \lambda I - T \) cannot be Fredholm by Theorem 4.9

\( \square \)

While a complete characterization of the class \( AH \) remains unknown, numerous sufficient conditions guaranteeing that \( K \in AH \) were obtained by Friedler and Kitover in [3] and by Vermeer in [10]. We list some of this conditions in Corollary 4.15.

**Corollary 4.15.** Let \( K \) be a compact Hausdorff space, \( \varphi \) be a continuous non-invertible map of \( K \) onto itself, and \( w \in C(K)^{-1} \). Assume that the set of all \( \varphi \)-periodic points is of first category in \( K \).

Assume one of the following conditions.

- The compact space \( K \) is extremally disconnected and has no isolated points.
- The compact space \( K \) is an \( F \)-space without isolated points that satisfies the countable chain condition.
• The compact space $K$ is arcwise connected and one of the following conditions is satisfied
  (a) The fundamental group $\Pi_1(X)$ is finite.
  (b) The fundamental group $\Pi_1(X)$ is abelian.
  (c) The fundamental group $\Pi_1(X)$ is finitely generated.
• $K$ is a convex, compact subset of a linear topological space.
• $K$ is a compact, arcwise connected subset of the plane $\mathbb{R}^2$ such that $\mathbb{R}^2 \setminus K$ consists of a finite number of components.
• $K$ is a locally simply connected compact subset of $\mathbb{R}^2$.
• $K = X \times Y$ where $X$ and $Y$ are locally connected compact spaces that have no isolated points.

Then $\sigma_f(T) = \sigma(T)$.

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