An Optimal Linear Coding for Index Coding Problem

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Abstract
An optimal linear coding solution for index coding problem is established. Instead of network coding approach by focus on graph theoretic and algebraic methods a linear coding program for solving both unicast and groupcast index coding problem is presented. The coding is proved to be the optimal solution from the linear perspective and can be easily utilize for any number of messages. The importance of this work is lying mostly on the usage of the presented coding in the groupcast index coding problem.

1 INTRODUCTION
We consider the problem of Index Coding when the sender wish to communicate $n$ messages $w_i, i \epsilon \{1, \ldots, n\}$ to $m$ receivers $r_j, j \epsilon \{1, \ldots, m\}$ with desire to recover a subset of messages $R_j$, with prior knowledge of $S_j$, as a side information which are subset of messages.

If we consider the encoder as a function $f(w_1, \ldots, w_n)$ which role is to satisfy the desire of ever single one of the receivers, the index coding problem’s goal is to minimize the size of $f$’s range.

To do so, we characterize index coding problem as a bipartite graph where the first set of nodes represent indices of the messages and the other one represent indices of the receivers. The edges in this graph represent the knowledge of the receiver node from the message node as a side information. By using the same definition as [4], we represent the desire of each receiver as $r_j(R_j \mid S_j)$.

As an example, consider an index coding problem with 3-message index and 2 receiver with $S_1 = \{2, 3\}, S_2 = \{1\}, R_1 = \{1\}$ and $R_2 = \{2, 3\}$. As a result this problem is represented as:

$$r_1(1 \mid 2, 3), r_2(2, 3 \mid 1),$$

(1.1)
The remainder of this paper is organized as follows. In Section 2, the coding solution for unicast cases and an algorithm to simplify its performance is provided. The generalization of this coding program for groupcast cases is provided in Section 3. Section 4 concludes the paper.

2 THE CODING PROGRAM FOR UNICAST CASES

Let’s consider a general unicast index coding problem as following:

\[ r_1(1 \mid S_1), r_2(2 \mid S_2), \ldots, r_n(n \mid S_n), \]  

(2.1)

By considering the graph characterization for this problem, we call two receiver nodes a cross neighbor if the desire message in the first one be part of the other one’s side information and vice versa. As a result, a cross neighbor set is a set of receiver nodes which any two of them are a cross neighbor. It is obvious that we can satisfy the desire of every nodes in a cross neighbor set by sending the summation of all the desired messages in the set.

The key to the optimal coding solution is to find the minimum number of cross neighbor sets in the graph. As a result, our coding program is to partition the graph to the minimum number of cross neighbor sets and then send the summation of all the desired nodes in the each set to satisfy their desire and separately send every desired messages which are not in any cross neighbor set.

**Theorem 1.** The presented coding program is an optimal solution for the index coding problem from linear perspective.

**Proof.** As it is mentioned before, we can satisfy the desire of every nodes in a cross neighbor set by sending one message through the channel. As a result, to get to the optimal program we must send this message for every single one of the cross neighbor sets. Since, we considered the minimum number of existing cross
neighbor sets, it is clear that this approach will satisfy the maximum number of receivers with the minimum number of outputs. Now that we satisfy the cross neighbor sets we can omit them from our original graph without effecting the final result. The remaining graph has no cross neighbor. As a result, by a single transmission we can at most satisfy one receiver node, so it is optimal to separately send all the desired messages in the remaining nodes through the channel.

2.1 AN ALGORITHM FOR UTILIZING PRESENTED CODING PROGRAM

The only Achilles heel in this program, is the fact that partitioning the graph to the minimum number of cross neighbor sets is not a simple task. To do so, we first need to specify all the cross neighbors in our graph. Then, by defining a new graph which its nodes are all the cross neighbor nodes and its edges are between every two nodes which were cross neighbor, we can transform our cross neighbor sets to complete subgraphs. Therefore, instead of finding the minimum number of cross neighbor sets we must find the minimum number of complete subgraphs which previously considered thoroughly in graph theory.

Example. Lets consider the following example:

\[
r_1(1 | 2, 3, 4), r_2(2 | 5), r_3(3 | 1, 4), r_4(4 | 1, 3), r_5(5 | 2, 6), r_6(6 | 4),
\]

This instance can be illustrate as Fig. 2. To perform the presented coding program we must partitioning the graph to the minimum number of cross neighbor sets, and since this case is not very complicated we can obtain this goal without using provided algorithm. The partitioning of the graph to the cross neighbor sets is provided in Fig. 3. As a result, we can solve this index coding problem by sending \(w_1 + w_3 + w_4, w_2 + w_5, w_6\) through the channel.
Fig. 3. The partitioning of the graphical characterization of the example to the cross neighbor sets.

3 THE CODING PROGRAM FOR GROUPCAST CASES

Let's consider a general groupcast index coding problem as following:

\[ r_1(R_1 | S_1), r_2(R_2 | S_2), ..., r_n(R_n | S_n), \]

(3.1)

To perform the coding program presented in the last section, we first need to characterize the problem to the same format as considered in the previous section. To do so, we need to treat every single desired messages as a separate receiver node, it means that we must break down the \( r_i \) to \( r_{i1}, ..., r_{i|R_i|} \) which have the same side information as \( r_i \). As a result, the total number of receiver nodes would be equal to \( \sum_{i=1}^{n} | R_i | \). The reason behind this characterization, is the fact that to get to the optimal solution we need to consider all the desired messages in every receivers in the same time.

Now that we transform the problem to a unicast case, By doing the same method as the previous section, and probably use the presented algorithm to finalize the coding, since the number of nodes will grow much larger, we can easily get to the optimal solution.

Remark. It is obvious that we cannot satisfy two or more desire of a receiver node with a single transmission. As a result, breaking down the receiver nodes will not affect the optimal solution.

Remark 2. Note that although in this case we will have different nodes with the same desire, since we build our coding program on finding the minimum number of cross neighbor sets, considering these same desires as separate receiver nodes will not affect the fact that our coding program is optimal.
4 CONCLUSION

In this work by considering the linear concept of index coding problem, an optimal coding program for unicast and groupcast cases provided. The most important privilege of this coding is the fact that it can be used in the general case of index coding problem. Although performing this method may be difficult in cases with lots of messages, but presented algorithm will simplify the usage of this coding solution.

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