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ON EVOLUTION QUASI-VARIATIONAL INEQUALITIES AND IMPLICIT STATE-DEPENDENT SWEEPING PROCESSES.

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Abstract. In this paper, we study a variant of the state-dependent sweeping process with velocity constraint. The constraint $C(\cdot, u)$ depends upon the unknown state $u$, which causes one of the main difficulties in the mathematical treatment of quasi-variational inequalities. Our aim is to show how a fixed point approach can lead to an existence theorem for this implicit differential inclusion. By using Schauder’s fixed point theorem combined with a recent existence and uniqueness theorem in the case where the moving set $C$ does not depend explicitly on the state $u$ (i.e. $C := C(t)$) given in [3], we prove a new existence result of solutions of the quasi-variational sweeping process in the infinite dimensional Hilbert spaces with a velocity constraint. Contrary to the classical state-dependent sweeping process, no conditions on the size of the Lipschitz constant of the moving set, with respect to the state, is required.

1. Introduction. In [11], J.J Moreau introduced and studied thoroughly the following differential inclusion

$$\dot{u}(t) \in -N_{C(t)}(u(t)) \quad \text{a.e. on } [0,T], \quad u(0) = u_0 \in C(0),$$

(1)

where $C : [0,T] \to H$ is a set-valued mapping defined from $[0,T]$ ($T > 0$) to a Hilbert space $H$ and takes convex and closed values, and $N_{C(t)}(u(t))$ denotes the outward normal cone, in the sense of convex analysis, to the set $C(t)$ at $u(t)$.

Translating the dynamic (1) to a mechanical language, we obtain the following interpretation:
- if the position $u(t)$ of a material point lies in the interior of the moving set $C(t)$, then $\dot{u}(t) = 0$, which means that the material point remains at rest;
- when the boundary of the moving set $C(t)$ catches up the material point, then this latter is pushed in an inward normal direction by the boundary of $C(t)$ to stay inside the moving set and satisfies the constraint. This mechanical visualization

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leads J.J. Moreau to call this problem the sweeping process: the material point is swept by the moving set.

The original concrete motivations of the sweeping process by J.J. Moreau are the study of quasi-static evolution in elastoplasticity [11]. Nowadays, many models in nonsmooth mechanics, mathematical economics, simulation of switched electrical circuits and crowd motion can be formulated by the sweeping process approach (for more details, we refer to [1, 2, 3, 4, 7, 8, 10, 12] and references therein).

Problem (1) is equivalent to the following evolution variational inequality with a moving constraint subset:

Find \( u(t) \in C(t) \) for all \( t \in [0, T] \) such that

\[
\langle \dot{u}(t), v-u(t) \rangle \geq 0, \quad \text{for all } v \in C(t) \text{ and for a.e. } t \in [0, T].
\]

As a special case, the sweeping process includes the following time-dependent variational inequality:

Find \( u(t) \in K \) for all \( t \in [0, T] \) such that

\[
\langle \dot{u}(t), v-u(t) \rangle \geq (f(t), v-u(t)), \quad \text{for all } v \in K \text{ and for a.e. } t \in [0, T],
\]

where \( K \) is a fixed (time-independent) closed convex subset of a Hilbert space \( H \), \( u : [0, T] \rightarrow H \) and \( f \in L^2([0, T], H) \).

Several extensions of the sweeping process in diverse ways have been studied in the literature (see for instance [2, 3, 9] and references therein).

In this paper, we are interested in a new variant of the state-dependent sweeping process with velocity constraint

\[
\begin{aligned}
&\begin{cases}
A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -N_{C(t,u(t))}(\dot{u}(t)) & \text{a.e. } t \in [0, T], \\
u(0) = u_0 \in H,
\end{cases}
\end{aligned}
\]  

(2)

with the following assumptions:

\( (H_1) \) \( A_0 : H \rightarrow H \) is a bounded symmetric linear operator which is coercive, that is, for all \( x \in H \),

\[
\langle A_0 x, x \rangle \geq \alpha \|x\|^2,
\]

for some real constant \( \alpha > 0 \); and \( A_1 : H \rightarrow H \) is a bounded symmetric linear operator which is positive semi-definite in the standard sense, that is,

\[
\langle A_1 x, x \rangle \geq 0, \quad \text{for all } x \in H;
\]

\( (H_2) \) For every \( (t, x) \in [0, T] \times H \), \( C(t, x) \subset H \) is a nonempty closed and convex set, \( C(t, x) \subset K \subset rB_H \) for some compact set \( K \) in \( H \) and some \( r > 0 \);

\( (H_3) \) The set \( C(t, x) \) varies in a Lipschitz continuous way with respect to the Hausdorff metric \( d_H \), that is, there exist a constants \( L_1, L_2 \geq 0 \) such that

\[
d_H(C(t, x), C(s, y)) \leq L_1 |t-s| + L_2 \|x-y\|, \quad \text{for all } t, s \in [0, T], \quad x, y \in H; \tag{3}
\]

\( (H_4) \) \( f : [0, T] \rightarrow H \) is a continuous mapping such that

\[
\|f(t)\| \leq \beta, \quad \text{for all } t \in [0, T] \text{ with } \beta > 0.
\]

The nonsmooth dynamic (2) can be rewritten as the following evolution quasi-variational inequality:

\[
\begin{aligned}
&\begin{cases}
\text{Find } u : [0, T] \rightarrow H, \text{ with } u(0) = u_0 \in H, \\
such that \dot{u}(t) \in C(t, u(t)) \text{ a.e. } t \in [0, T] \text{ and}
\end{cases}
\end{aligned}
\]  

\[
\begin{aligned}
&\begin{cases}
a_1(\dot{u}(t), v - \dot{u}(t)) + a_0(u(t), v - \dot{u}(t)) \geq (f(t), v - \dot{u}(t)), \quad \forall v \in C(t, u(t)).
\end{cases}
\end{aligned}
\]
Here \( a_0(\cdot, \cdot) \) and \( a_1(\cdot, \cdot) \) are real bilinear, bounded and symmetric forms associated to the operators \( A_1 \) and \( A_0 \) defined in \((H_1)\).

The above type of evolution quasi-variational inequalities are widely used in applied mathematics, unilateral mechanics and various fields of sciences and engineering such as for instance traffic networks, energy market, transportation, elastoplasticity etc . . . (see e.g. [6]).

In [9], the authors proved the existence of Lipschitz continuous solutions to the classical state dependent sweeping process

\[
\begin{cases}
-\dot{u}(t) \in N_{C(t,u(t))}(u(t)) & \text{a.e. } t \in [0, T], \\
u(0) = u_0 \in C(0, u_0),
\end{cases}
\]

where the moving set \( C(t,u) \) is supposed to move in a Lipschitz continuous way with the constants \( 0 \leq L_1 \) and \( 0 \leq L_2 < 1 \). In the case \( L_2 > 1 \), the differential inclusion (4) may have no absolutely continuous solutions. In fact, a counter-example is given in [9] (see Example 3.1 in [9]). Now, unlike problem (4), the following simple example shows that the existence of solutions of problem (2) can be expected even for \( L_2 > 1 \).

**Example 1.** Let \( H = \mathbb{R}, C(u) = [2 \cos(u), 2] \) for all \( u \in \mathbb{R} \). Define \( u_0 = 0 \) and \( f(t) = 2, t \in [0, 1] \).

Then \( u \mapsto C(u) \) is \( d_H \)-Lipschitz with constant \( L_2 = 2 \) (also \( C \) is bounded).

Consider the mapping \( u : [0, 1] \to \mathbb{R} \) defined by: \( u(t) = 2t \).

We have, \( \dot{u}(t) = 2 \in C(u(t)) = [2 \cos(2t), 2] \), for all \( t \in [0, 1] \), \( N_{C(u(t))}(\dot{u}(t)) = N_{[2 \cos(2t), 2]}(2) = [0, +\infty[, \) for all \( t \in [0, 1] \), and \( N_{C(u(0))}(\dot{u}(0)) = N_{(2)}(2) = \mathbb{R} \) at \( t = 0 \).

Hence, \( -\dot{u}(t) + f(t) = 2(1-t) \in N_{C(u(t))}(\dot{u}(t)), \) for all \( t \in [0, 1] \).

Consequently, \( u \) is a solution of (2) with the initial value \( u(0) = 0 \).

**Figure 1.** The moving set \( C(u) \) of Example 1.
particular case of a shifted moving set is discussed in section 4. Section 5 is devoted to some concluding remarks and open questions.

2. Notation and preliminaries. Let $H$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. For any $x \in H$ and $r \geq 0$, the closed ball centered at $x$ with radius $r$ will be denoted by $B(x, r)$. For $x = 0$ and $r = 1$, we set $B$ instead of $B(0, 1)$. For a closed convex subset $C$ of $H$ the set

$$N_C(x) = \{ y \in H : \langle y, v - x \rangle \leq 0, \forall v \in C \},$$

$x \in C$, denotes the normal cone to $C$ at $x$. Let $d_H(C_1, C_2)$ denotes the Hausdorff distance between two subsets $C_1$ and $C_2$ of $H$ defined as follows

$$d_H(C_1, C_2) = \sup_{x \in C_1} \inf_{y \in C_2} d(x, y),$$

where $d(x, y)$ denotes the distance between two subsets $x \in C_1$ and $y \in C_2$.

Let us recall the following classical Schauder’s fixed point lemmas that will be used in the proof of our main result, for a reference we refer e.g. to [13].

Lemma 2.1. Let $K$ be a closed convex subset of the Banach space $X$. Suppose $f : K \rightarrow K$ a compact map (i.e. bounded sets in $K$ are mapped into relatively compact sets). Then, $f$ has a fixed point in $K$.

We recall the following existence and uniqueness theorem proved in [3].

Lemma 2.2. Suppose that $C : [0, T] \rightarrow H$, $t \mapsto C(t)$ depends only on time. Assume that (H1) and (H2) are satisfied. Assume that $C(0)$ is bounded and the nonempty closed convex sets $C(t)$ of $H$ have a continuous variation in the sense that there is some nondecreasing continuous function $v(\cdot) : [0, T] \rightarrow \mathbb{R}$ with $v(0) = 0$ and such that

$$|d(y, C(t)) - d(y, C(s))| \leq |v(t) - v(s)|$$

for all $y \in H$ and $s, t \in [0, T]$.

Then, for any initial point $u_0 \in H$, the evolution variational inequality with velocity constraint

$$\begin{cases}
A_1 \dot{u}(t) + A_2 u(t) - f(t) \in -N_{C(t)}(\dot{u}(t)) & \text{a.e. } t \in [0, T], \\
u(0) = u_0 \in H,
\end{cases}$$

admits unique Lipschitz continuous solution $u : [0, T] \rightarrow H$.

Remark 1. The proof of Lemma 2.2 is based on a modified version of the so called catching-up algorithm, initially introduced by J.J. Moreau and is performed into 5 steps:

- **Step 1.** Construction of approximants $u^n_i$: for $n \in \mathbb{N}$, let

$$0 = t^n_0 < t^n_1 < \ldots < t^n_i < \ldots < t^n_n = T,$$

be a finite partition of the interval $[0, T]$. We denote by $\eta^n = t^n_{i+1} - t^n_i = \frac{T}{n}$.

Algorithm.

Fix $n \geq 2$ and set $\eta^n = \frac{T}{n}$, $u^n_0 = u_0$ and $f^n_0 = f(t^n_0)$.

For $i = 0, 1, \ldots, n - 1$

- Compute $f^n_{i+1} = f(t^n_{i+1})$

- Solve for $z^n_{i+1}$ the following variational inequalities
Assume that Theorem 3.1.

Main result. A by assuming that 3.

Remark 2. The coercivity assumption on $A_0$ in Lemma 2.2 was relaxed in [5] by assuming that $A_0$ is a semi-coercive operator. An existence result is proved in [5], however the uniqueness of solution, which is an important issue for applying Schauder’s fixed point Theorem, is not guaranteed and the technique used in this paper can not be adapted.

Step 1. Let us apply Schauder’s fixed point argument (Lemma 2.1) to deduce the existence result.

Let us fix an arbitrary element $u_0 \in H$, there exists at least one Lipschitz continuous mapping $u : [0, T] \rightarrow H$ satisfying (2) i.e.

$$\begin{cases}
A_1 \dot{u}(t) + A_0 u(t) - f(t) \in -N C_{C(t, u(t))} (\dot{u}(t)) & \text{a.e. } t \in [0, T], \\
u(0) = u_0 \in H,
\end{cases}$$

Proof. Let us apply Schauder’s fixed point argument (Lemma 2.1) to deduce the existence result.

Step 1. Let $u_0 \in H$. Set

$$\mathcal{K} := \left\{ v \in C([0, T], H) : v(t) = u_0 + \int_0^t \dot{v}(s) \, ds \, \forall t \in [0, T], \|\dot{v}(t)\| \leq r \, \text{a.e. } t \in [0, T] \right\}.$$ 

It is easy to check that $\mathcal{K}$ is a closed convex subset of $C([0, T], H)$.

Step 2. Let us fix an arbitrary element $v \in \mathcal{K}$. We consider the problem: find $w : [0, T] \rightarrow H$, with $w(0) = u_0 \in H$ such that

\[
(P_v) \begin{cases}
\dot{w}(t) \in C(t, v(t)), \text{ a.e. } t \in [0, T], \text{ and; } \\
-A_1 \dot{w}(t) - A_0 w(t) + f(t) \in N C_{C(t, v(t))}(\dot{w}(t)) \text{ a.e. on } [0, T]; \\
w(0) = u_0.
\end{cases}
\]
For each \( t \in [0, T] \) and each \( v \in K \) let us set
\[
D(t) = C(t, v(t)).
\]
Obviously, for each \( t \in [0, T] \), the set valued map \( D(\cdot) \) is convex and closed.
Let now \( x \in H \) and \( t, s \in [0, T] \) such that \( s < t \).
Using \((H_3)\) one has,
\[
d_H(D(t), D(s)) = d_H\left(C(t, v(t)), C(s, v(s)) \right)
\leq L_1(t - s) + L_2\|v(t) - v(s)\|
\leq L_1(t - s) + L_2\left| \int_s^t \dot{v}(\tau) d\tau \right|
\leq L_1(t - s) + L_2\left| \int_s^t \|\dot{v}(\tau)\| d\tau \right|
\leq L_1(t - s) + L_2 r(t - s).
\]
Therefore, for \( x \in H \) and \( t, s \in [0, T] \) one has
\[
d_H(D(t), D(s)) \leq (L_1 + L_2 r)|t - s|.
\]
Since, \( d_H(D(t), D(s)) := \sup_{y \in H} |d(y, C(t)) - d(y, C(s))| \), by Lemma 2.2, for every \( u_0 \in H \) and for each \( v \in K \), problem \((P_v)\) has one and only one Lipschitz continuous solution denoted \( w(\cdot) \) satisfying
\[
\|\dot{w}(t)\| \leq r \tag{5}
\]
and
\[
\|w(t)\| \leq \|u_0\| + rT. \tag{6}
\]
**Step 3.** Let us define the mapping \( \Psi : K \to K \) by, \( \Psi(v) = w \) such that \( w \) is the unique solution for the problem \((P_v)\).
It’s clear that \( \Psi(K) \subset K \). To apply the classical Schauder’s fixed point theorem (see Lemma 2.1), it is sufficient to show that \( \Psi \) is continuous with respect to \( \|\cdot\|_{C([0, T], H)} \) and that \( \Psi(K) \) is relatively compact in the Banach space \( C([0, T], H) \). We prove the latter by the classical Ascoli-Arzelà theorem.
Indeed, let \( v \in K \) and \( t, s \in [0, T] \)
\[
\|\Psi(v)(t) - \Psi(v)(s)\| = \|w(t) - w(s)\|
\leq \| \int_s^t \dot{w}(\tau) d\tau \|
\leq r|t - s|.
\]
This implies that \( \{\Psi(v)\}_{v \in K} \) is equi-Lipschitz and therefore equicontinuous.
Let us now show that it is bounded. Observe that, since \( \dot{w}(t) \in C(t, v(t)) \subset K \), we have
\[
\Psi(v)(t) = w(t) \in u_0 + tK, \text{ for all } t \in [0, T].
\]
We set \( S = \overline{w}(K \cup \{0\}), \) then \( S \) is convex, compact and \( 0 \in S \).
We have
\[
\Psi(v)(t) = w(t) \in u_0 + TS,
\]
where \( T \) is a positive constant.
and hence \( \{ \Psi(v) \}_{v \in K} \) is bounded in \( C([0,T], H) \) with respect to the uniform norm. Therefore by the Ascoli-Arzela theorem, the set \( \Psi(K) \) is relatively compact with respect to the uniform norm \( \| \cdot \|_{C([0,T], H)} \).

Now let us show that \( \Psi \) is continuous. Let \( (v_n) \subset K \) be a sequence of continuous functions converging uniformly to \( v \in K \), i.e.
\[
\|v_n - v\|_{C([0,T], H)} \to 0, \quad n \to \infty.
\]

Let \( w_n = \Psi(v_n) \in K \) and \( w = \Psi(v) \in K \). Since \( \dot{w}_n(t) \in C(t,v_n(t)) \), We have
\[
\langle -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t), x - \dot{w}_n(t) \rangle \leq 0, \quad \forall x \in C(t,v_n(t)) \tag{7}
\]
and for \( \dot{w}(t) \in C(t,v(t)) \), We have
\[
\langle -A_1 \dot{w}(t) - A_0 w(t) + f(t), x - \dot{w}(t) \rangle \leq 0, \quad \forall x \in C(t,v(t)). \tag{8}
\]
Taking \( x := P_{C(t,v_n(t))}(\dot{w}(t)) \in C(t,v_n(t)) \) in (7), we get
\[
\langle -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t), P_{C(t,v_n(t))}(\dot{w}(t)) - \dot{w}_n(t) \rangle \leq 0. \tag{9}
\]
Using (5),(6) and (9), we obtain
\[
(A) = \langle -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t), \dot{w}(t) - \dot{w}_n(t) \rangle
= \langle -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t), \dot{w}(t) - P_{C(t,v_n(t))}(\dot{w}(t)) \rangle
+ \langle -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t), P_{C(t,v_n(t))}(\dot{w}(t)) - \dot{w}_n(t) \rangle
\leq \| -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t) \| \| \dot{w}(t) - P_{C(t,v_n(t))}(\dot{w}(t)) \|
\leq \|[A_1] r + \| A_0 \| \|u_0\| + rT + \beta \| d_{C(t,v_n(t))}(\dot{w}(t)) \|
\leq \|[A_1] r + \| A_0 \| \|u_0\| + rT + \beta \| d_{H}(C(t,v_n(t)), C(t,v(t))) \|.
\]

Using (H3), we get
\[
(B) = \langle -A_1 \dot{w}_n(t) - A_0 w_n(t) + f(t), \dot{w}(t) - \dot{w}_n(t) \rangle
\leq L_2 \| A_1 \| r + \| A_0 \| \|u_0\| + rT + \beta \| v_n(t) - v(t) \|.
\]

Hence,
\[
(C) = (A_1 \dot{w}_n(t) + A_0 w_n(t) - f(t), \dot{w}(t) - \dot{w}_n(t))
\leq L_2 \| A_1 \| r + \| A_0 \| \|u_0\| + rT + \beta \| v_n(t) - v(t) \|. \tag{10}
\]
By setting \( x := P_{C(t,v(t))}(\dot{w}_n(t)) \in C(t,v(t)) \) in (8), we get
\[
\langle -A_1 \dot{w}(t) - A_0 w(t) + f(t), P_{C(t,v(t))}(\dot{w}_n(t)) - \dot{w}(t) \rangle \leq 0. \tag{11}
\]
Using (5),(6) and (11), one has
\[
(D) = \langle -A_1 \dot{w}(t) - A_0 w(t) + f(t), \dot{w}_n(t) - \dot{w}(t) \rangle
= \langle -A_1 \dot{w}(t) - A_0 w(t) + f(t), \dot{w}_n(t) - P_{C(t,v(t))}(\dot{w}_n(t)) \rangle
+ \langle -A_1 \dot{w}(t) - A_0 w(t) + f(t), P_{C(t,v(t))}(\dot{w}_n(t)) - \dot{w}(t) \rangle
\leq \| -A_1 \dot{w}(t) - A_0 w(t) + f(t), \dot{w}_n(t) - P_{C(t,v(t))}(\dot{w}_n(t)) \|
\leq \| -A_1 \dot{w}(t) - A_0 w(t) + f(t) \| \| \dot{w}_n(t) - P_{C(t,v(t))}(\dot{w}_n(t)) \| \|
\leq \|[A_1] r + \| A_0 \| \|u_0\| + rT + \beta \| d_{C(t,v(t))}(\dot{w}(t)) \|
\leq \|[A_1] r + \| A_0 \| \|u_0\| + rT + \beta \| d_{H}(C(t,v_n(t)), C(t,v(t))) \|.\]
By \((\mathcal{H}_3)\), we have
\[
\langle A_1 \dot{w}(t) - A_0 w(t) + f(t), \dot{w}_n(t) - \dot{w}(t) \rangle \\
\leq L_2 \|A_1\| r + \|A_0\| (\|u_0\| + rT) + \beta \|v_n(t) - v(t)\|. \tag{12}
\]
By adding both inequalities (10) and (12) we obtain
\[
\langle A_1 \dot{w}_n(t) - A_1 \dot{w}(t) + A_0 w_n(t) - A_0 w(t), \dot{w}_n(t) - \dot{w}(t) \rangle \leq 2L_2 \|A_1\| r + \|A_0\| (\|u_0\| + rT) + \beta \|v_n(t) - v(t)\|, \text{ a.e. } t \in [0, T]. \tag{13}
\]
Since \(A_1\) is monotone, we deduce
\[
(\mathcal{F}) = \langle A_0 w_n(t) - A_0 w(t), \dot{w}_n(t) - \dot{w}(t) \rangle \\
\leq 2L_2 \|A_1\| \|r + \|A_0\| (\|u_0\| + rT) + \beta \|v_n(t) - v(t)\|, \text{ a.e. } t \in [0, T]. \tag{14}
\]
Consequently, for almost every \(t \in [0, T]\), (14) implies
\[
(\mathcal{G}) = \frac{d}{dt} \langle A_0 w_n(t) - A_0 w(t), \dot{w}_n(t) - \dot{w}(t) \rangle \\
= 2 \langle A_0 w_n(t) - A_0 w(t), \dot{w}_n(t) - \dot{w}(t) \rangle \\
\leq 4L_2 \|A_1\| \|r + \|A_0\| (\|u_0\| + rT) + \beta \|v_n(t) - v(t)\|.
\]
By integrating, using the equalities \(w_n(0) = w(0) = u_0\) and the coerciveness of \(A_0\), we have for every \(t \in [0, T]\)
\[
\alpha \|w_n(t) - w(t)\|^2 \leq \langle A_0 w_n(t) - A_0 w(t), \dot{w}_n(t) - \dot{w}(t) \rangle \\
\leq 4L_2 \|A_1\| \|r + \|A_0\| (\|u_0\| + rT) + \beta \|v_n(t) - v(t)\|.
\]
The last inequality gives
\[
\lim_{n \to \infty} \|w_n - w\|_{C([0, T], H)} = 0, \quad \text{when } \lim_{n \to \infty} \|v_n - v\|_{C([0, T], H)} = 0.
\]
Therefore, \(\Psi\) is continuous. This completes the proof. \(\square\)

4. **Uniqueness in the case of a shifted moving set.** Due to the dependence of the constraint \(C(\cdot, u)\) upon the unknown state \(u\), the uniqueness of a solution to problem (2) can not be expected in general. However, the uniqueness of a solution could be proved in the particular case where the operator \(A_1 \equiv 0\) and the state-dependent moving set \(C(\cdot, u)\) has the following structure:
\[
C(t, u) = K(t) + m(u), \tag{15}
\]
where \(K(t) \subset K_0\) is a fixed compact convex set of \(H\) for every \(t \in [0, T]\) and \(m : H \to H\) is a single-valued and \(L_2\)-Lipschitz operator such that \(m(u) \in K_1\) for some fixed compact convex set of \(H\) and all \(u \in H\).

Let us consider the following problem
\[
\begin{aligned}
\{ & A_0 u(t) - f(t) \in -N_{C(t, u(t))}(\tilde{u}(t)) & \text{ a.e. } t \in [0, T], \\
& u(0) = u_0 \in H, \}
\end{aligned} \tag{16}
\]
with the following assumptions:
\((\mathcal{H}_1)\) \(A_0 : H \to H\) is a bounded symmetric linear operator which is coercive, that is, for all \(u \in H\),
\[
\langle A_0 u, u \rangle \geq \alpha \|u\|^2,
\]
for some real constant \(\alpha > 0\);
\((\mathcal{H}_2)\) For every \((t, u) \in [0, T] \times H\), \(C(t, u) = K(t) + m(u) \subset H\) where \(K(t) \subset K_0\) is a
fixed compact convex set of $H$ and $m : H \to H$ is a single-valued and $L_2$-Lipschitz operator such that $m(u) \in K_1$ for some fixed compact convex set of $H$.

$(\tilde{H}_4)$ We assume that $K(t)$ moves in a Lipschitz way, i.e. there exist a constants $L_1 \geq 0$ such that

$$|d(u, K(t)) - d(v, K(s))| \leq \|u - v\| + L_1|t - s|$$

for all $u, v \in H$ and $s, t \in [0, T]$.

$(\tilde{H}_4)$ $f : [0, T] \to H$ is a continuous mapping such that

$$\|f(t)\| \leq \beta, \text{ for all } t \in [0, T] \text{ with } \beta > 0.$$

**Proposition 1.** Assume that $(\tilde{H}_1), (\tilde{H}_2), (\tilde{H}_3)$ and $(\tilde{H}_4)$ are satisfied. Then, for any initial point $u_0 \in H$, there exists one and only one Lipschitz continuous solution of (16) i.e.

$$\begin{cases} A_0 u(t) - f(t) \in -N_{K(t)+m(u(t))}(\dot{u}(t)) \quad \text{a.e. } t \in [0, T], \\ u(0) = u_0 \in H, \end{cases}$$

with $K(t)$ and $m(u)$ defined in $(\tilde{H}_2)$ and $(\tilde{H}_3)$.

It is clear that in this case the Lipschitz condition (3) is satisfied. Indeed, by $(\tilde{H}_3)$ for every $0 \leq s < t \leq T$, $x, u, v \in H$, we have

$$|d(x, C(t, u)) - d(x, C(s, v))| = |d(x - m(u), K(t)) - d(x - m(v), K(s))|$$

$$\leq \|m(u) - m(v)\| + L_1|t - s|$$

$$\leq L_1|t - s| + L_2\|u - v\|.$$ 

Hence,

$$d_H(C(t, u), C(s, v)) \leq L_1|t - s| + L_2\|u - v\|,$$

for all $t, s \in [0, T], u, v \in H$.

All assumptions of Theorem 3.1 are satisfied. Hence, problem (16) has at least one solution.

Suppose that $(u_1, u_2)$ are two solutions for (16) such that $u_1(0) = u_2(0) = u_0$. Then, for almost every $t \in [0, T]$, we have

$$\begin{cases} \langle -A_0 u_1(t) + f(t), x - \dot{u}_1(t) \rangle \leq 0, \text{ for all } x \in C(t, u_1(t)), \\ \dot{u}_1(t) \in C(t, u_1(t)), \end{cases}$$

and

$$\begin{cases} \langle -A_0 u_2(t) + f(t), x - \dot{u}_2(t) \rangle \leq 0, \text{ for all } x \in C(t, u_2(t)), \\ \dot{u}_2(t) \in C(t, u_2(t)). \end{cases}$$

Using the fact that $\dot{u}_2(t) \in C(t, u_2(t)) = K(t) + m(u_2(t))$, we replace in (17) $x := \dot{u}_2(t) - m(u_2(t)) + m(u_1(t)) \in C(t, u_1(t))$, we obtain for a.e. $t \in [0, T],$

$$\langle -A_0 u_1(t) + f(t), \dot{u}_2(t) - \dot{u}_1(t) - m(u_2(t)) + m(u_1(t)) \rangle \leq 0,$$

or equivalently

$$\langle A_0 u_1(t) - f(t), \dot{u}_1(t) - \dot{u}_2(t) + m(u_2(t)) - m(u_1(t)) \rangle \leq 0.$$  

(19)

On the other hand, as

$$\dot{u}_1(t) \in C(t, u_1(t)) = K(t) + m(u_1(t)),$$

by taking in (18)

$$x := \dot{u}_1(t) - m(u_1(t)) + m(u_2(t)) \in C(t, u_2(t)),$$

(17) and (18) yield
we get for a.e. $t \in [0, T],$

$$\langle -A_0 u_2(t) + f(t), \dot{u}_1(t) - \dot{u}_2(t) - m(u_1(t)) + m(u_2(t)) \rangle \leq 0.$$  \hspace{1cm} (20)

Adding the inequalities (19) and (20), we obtain

$$\langle A_0 u_1(t) - A_0 u_2(t), \dot{u}_1(t) - \dot{u}_2(t) - m(u_1(t)) + m(u_2(t)) \rangle \leq 0, \text{ a. e. } t \in [0, T].$$  \hspace{1cm} (21)

Since $m(\cdot)$ is $L_2$ Lipschitz, we deduce

$$\langle A_0 u_1(t) - A_0 u_2(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq L_2 \|A_0\| \|u_1(t) - u_2(t)\|^2, \text{ a. e. } t \in [0, T].$$  \hspace{1cm} (22)

Consequently, for almost every $t \in [0, T],$

$$\frac{d}{dt} \langle A_0 u_1(t) - A_0 u_2(t), u_1(t) - u_2(t) \rangle = 2 \langle A_0 u_1(t) - A_0 u_2(t), \dot{u}_1(t) - \dot{u}_2(t) \rangle \leq 2L_2 \|A_0\| \|u_1(t) - u_2(t)\|^2.$$

By integrating, using the equalities $u_1(0) = u_2(0) = u_0$ and the coerciveness of $A_0,$ we have

$$\alpha_0 \|u_1(t) - u_2(t)\|^2 \leq \langle A_0 u_1(t) - A_0 u_2(t), u_1(t) - u_2(t) \rangle \leq 2L_2 \|A_0\| \int_0^t \|u_1(s) - u_2(s)\|^2 ds.$$

Gronwall’s Lemma permits us to deduce that $\|u_1(t) - u_2(t)\|^2 = 0.$ Therefore $u_1 = u_2.$ Which completes the proof. \hfill \Box

5. Concluding remarks. In this note, using a fixed point argument, we give a new existence result for a variant of the state-dependent sweeping process with velocity constraint. The fact that the constraint $C(\cdot, u)$ depends upon the unknown state $u,$ makes the study of the evolution quasi-variational inequalities more complicate. The smallness condition of the Lipschitz constant of the moving set with respect to the state, used in the classical state-dependent sweeping process (4), is not required here.

We handled the situation using the Schauder fixed point Theorem which requires a compactness assumption of the moving set. Many issues remain open and need further investigations. It includes the relaxation of the compactness assumption or the use of another technique. The extension of this result beyond the convex setting could be also of great interest. In fact, the class of prox-regular sets play an important role in optimization and control theory with concrete applications and constitutes an alternative assumption to the convexity used in this paper. This out of the scope of the current paper and will be the subject of a future research project.

REFERENCES

1. V. Acary, O. Bonnefon, B. Brogliato, Nonsmooth Modeling and Simulation for Switched Circuits, Springer, Lecture notes in electrical engineering (2011).
2. S. Adly, T. Haddad. An implicit sweeping process approach to quasistatic evolution variational inequalities. SIAM J. Math. Anal. 50 (2018), no. 1, 761-778.
3. S. Adly, T. Haddad, L. Thibault, Convex sweeping process in the framework of measure differential inclusions and evolution variational inequalities, Math. Program. Ser. B 148 (2014), 5-47.
4. K. Addi, S. Adly, B. Brogliato, D. Goeleven, A method using the approach of Moreau and Panagiotopoulos for the mathematical formulation of non-regular circuits in electronics, Nonlinear Anal. Hybrid Syst. 1 (2007), no. 1, 30-43.
5. S. Adly, B.K. Le. On semicoercive sweeping process with velocity constraint. Optimization letters. DOI: 10.1007/s11590-017-1149-2
6. D. Duvaut and J. L. Lions, Inequalities in mechanics and physics. Springer-Verlag, Berlin (1976)
7. C. Henry, An existence theorem for a class of differential equations with multivalued right-hand side, J. Math. Anal. Appl. 41 (1973), 179-186.
8. B. Maury and J. Venel, A mathematical framework for a crowd motion model, C. R. Math. Acad. Sci. Paris 346 (2008), 1245-1250.
9. M. Kunze and M. D. P. Monteiro Marques, On parabolic quasi-variational inequalities and state-dependent sweeping processes. Topol. Methods Nonlinear Anal. 12, 179-191 (1998)
10. M. D. P. Monteiro Marques, Differential inclusions in nonsmooths mechanical problems, Shockcs and dry Friction, Progress in Nonlinear Differential Equations an Their Applications, Birkhauser. 9 (1993)
11. J. J. Moreau, Evolution problem associated with a moving convex set in a Hilbert space. J. Differential Equations 26, 347-374 (1977).
12. J.J. Moreau, Numerical aspects of the sweeping process, Comput. Methods Appl. Mech. Engrg. 177 (1999), 329-349.
13. E. Zeidler, Nonlinear functional analysis and its applications, I: Fixed Point Theorems, Springer-Verlag, New York, 1986.

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