\textbf{\textit{p}-adic multiple zeta values and \textit{p}-adic pro-unipotent harmonic actions : summary of parts I and II}

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\textit{Abstract}

This is a review on the two first parts of our work on \textit{p}-adic multiple zeta values at $N$-th roots of unity ($p\text{MZV}_N$'s), the \textit{p}-adic periods of the crystalline pro-unipotent fundamental groupoid of $\mathbb{P}^1 - \{0, \mu_N, \infty\}$ (where $N$ and $p$ are coprime). We restrict for simplicity the review to the case of $N = 1$, i.e. the case of \textit{p}-adic multiple zeta values ($p\text{MZV}$'s).

The main tools are new objects which we call \textit{p}-adic pro-unipotent harmonic actions. These are continuous group actions on a space containing the non-commutative generating series of weighted multiple harmonic sums, they are related to the motivic Galois action on $\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$ and to the Poisson-Ihara bracket, and interrelated by some maps. They are defined in [J2] and [J3] ; the definition relies on a simplification of the differential equation of the Frobenius, proved as a preliminary technical fact by [J1].

Part I ([J1], [J2], [J3]) is an explicit computation of the Frobenius of $\pi_1^{\text{un},\text{cr}}(\mathbb{P}^1 - \{0, 1, \infty\})$, and in particular of $p\text{MZV}$'s. We give formulas which keep a track of the motivic Galois action.

Part II ([J4], [J5], [J6]) is a study of the algebraic properties of $p\text{MZV}$'s brought together with the formulas of part I. We state an explicit elementary version of the Galois theory of $p\text{MZV}$'s.

In this text, we emphasize the ideas and the intuition of this work. We review the general context and the motivations (§1), a technical description of $\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$ (§2), and we explain our general strategy (§3). We state most of the main results of part I (§4) and of part II (§5), and we also summarize and motivate the methods of the proofs. We conclude on the main messages of this work (§6).

\textbf{Keywords.} periods, \textit{p}-adic periods, motivic Galois actions, pro-unipotent fundamental groupoids, iterated path integrals, Frobenius, unipotent $F$-isocrystals, Coleman integration, the projective line minus three points, multiple zeta values, \textit{p}-adic multiple zeta values, Goncharov coproduct, Poisson-Ihara bracket, twisted Magnus group, pro-unipotent harmonic actions, multiple harmonic sums, multiple harmonic values, finite multiple zeta values, double shuffle relations
1 Context and motivation

We introduce the two general notions that we will study: periods, and the pro-unipotent fundamental groupoid (§1.1, §1.2), then the particular example of \( \pi^\text{un}_1(\mathbb{P}^1 - \{0, 1, \infty\}) \) and multiple zeta values, and its \( p \)-adic aspects (§1.3, §1.4).

1.1 Periods

1.1.1 Definition

There are several possible ways to define periods.

**Definition 1.1.** (See [Gr1], [Hu-MS]) Let \( X \) be a smooth algebraic variety over \( \mathbb{Q} \), and \( D \) a normal crossings divisor. One can associate to them:

- The algebraic De Rham cohomology \( H^\text{DR}(X, D) \) of \( X \) relative to \( D \), which is a graded \( \mathbb{Q} \)-vector space of finite dimension
- The rational Betti cohomology \( H^B(X, D) \) of \( X(\mathbb{C}) \) relative to \( D(\mathbb{C}) \), which is a graded \( \mathbb{Q} \)-vector space of finite dimension
- The isomorphism of comparison \( \text{comp} : H^\text{DR}(X, D) \otimes_k \mathbb{C} \xrightarrow{\sim} H^B(X, D)^\vee \otimes_\mathbb{Q} \mathbb{C} \), defined by \((\omega, \Delta) \mapsto \int_\Delta \omega\).

The coefficients of any matrix of the isomorphism of comparison associated with a couple of bases of \( H^\text{DR}(X, D) \) and \( H^B(X, D)^\vee \) over \( \mathbb{Q} \) are called the periods of \((X, D)\).

**Definition 1.2.** (Kontsevich-Zagier, [KoZ], §1.1) A period is a complex number whose real and imaginary parts are values of absolutely convergent integrals of rational functions with rational coefficients, over domains in \( \mathbb{R}^n \) given by polynomial inequalities with rational coefficients.

**Example 1.3.** All algebraic numbers are periods, and one can replace above "rational" by "algebraic". \( 2i\pi = \int_\gamma \frac{dz}{z} \), where \( \gamma \) is a simple counterclockwise loop around 0 in \( \mathbb{C} \), is a period of \( \mathbb{A}^1 - \{0\} \). For \( r \in \mathbb{Q}^+ \), \( \log(r) = \int_1^r \frac{dt}{t} \) is a period. Conjecturally, \( e = \exp(1) \) is not a period.

The set of periods is countable. In particular, there are complex numbers that are not periods.

There exists similarly a notion of \( p \)-adic periods: it arises from the comparison between De Rham cohomology and étale cohomology, resp. crystalline cohomology and étale cohomology [Fa1], [Fa2]; \( p \)-adic periods are elements of Fontaine rings [Fo]. In this text, we will consider only the image in \( \mathbb{Q}_p \) of certain elements of Fontaine rings by the reduction maps (see §1.4.1) and study them through the concept of pro-unipotent fundamental groupoid (reviewed in §1.2) thus the notion of \( p \)-adic periods is not technically needed for our purposes.

\(^1\)This definition is not the most general one.
1.1.2 The conjecture of periods and motivic Galois actions

The theory of motives, initiated by Grothendieck, aims for a unification of the cohomology theories of algebraic geometry. Several different categories of motives have been constructed so far (see [An1], [An2] for reviews).

Let us assume that we have such a category of motives $M$, with the following properties. First, for a $(X,D)$ as above, the triple $(H^{DR}(X,D), H^B(X,D), \text{comp})$ lifts to an object $H^M(X,D)$ in $M$ to the category of graded finite dimensional vector spaces over $\mathbb{Q}$, which send $H^M(X,D) \mapsto H^{DR}(X,D)$ and $H^M(X,D) \mapsto H^B(X,D)$. Let us assume moreover that $\omega^{DR}$ and $\omega^B$ make $M$ into a neutral Tannakian over $\mathbb{Q}$ (see [De3], or [An1] §2, for generalities on Tannakian categories): $M$ is then equivalent to the category of representations of an affine algebraic group over $\mathbb{Q}$: $G = \text{Aut}_{\otimes}^\otimes(\omega^B)$, resp. $G^{DR} = \text{Aut}_{\otimes}^\otimes(\omega^{DR})$.

The scheme $P = \text{Isom}_{\otimes}(\omega^{DR}, \omega^B)$ is a torsor over $G_B$ and $G^{DR}$. $G_B$ and $G^{DR}$ are called motivic Galois groups, and their action on $P$ is called a motivic Galois action; the isomorphism comp is a point of $P$ over $\mathbb{C}$.

The important conjecture of periods of Grothendieck (unpublished), is that every polynomial equation in a $\mathbb{Q}$-algebra of periods "is of geometric origin" i.e. is reflected the framework above. More precisely,

**Conjecture 1.4.** (Grothendieck, see [An1], [An2], [Hu-MS] for precise statements) In a context as above, comp is a generic point of the torsor of periods.

**Variant:** the previous facts imply that the transcendence degree of the extension of $\mathbb{Q}$ generated by the periods of $(X,D)$ is bounded by the dimension of the motivic Galois group and :

**Conjecture 1.5.** (Grothendieck, see [An1], [An2], [Hu-MS] for precise statements) In a context as above, the transcendence degree of the extension of $\mathbb{Q}$ generated by the periods of $(X,D)$ is equal to the dimension of the motivic Galois group.

A different and more elementary formulation of the conjecture of periods is as follows (Kontsevich-Zagier, [KoZ], §4.1). Let $P_{\text{formal}}$, the algebra of "formal periods", be the algebra generated the equivalence classes of quadruples $(X,D,\omega,\gamma)$ where $(X,D)$ is as in Definition 1.1, $\omega \in \Omega^d(X)$, $\gamma \in H_d(X(\mathbb{C}), D(\mathbb{C}), \mathbb{Q})$, $d = \dim(X)$, where the equivalence is defined by the operations of linearity of integration, changes of variables and Stokes.

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2 A $\otimes$-category over a commutative ring $F$ is a category $\mathcal{T}$ equipped with a functor $\otimes: \mathcal{T} \times \mathcal{T} \to \mathcal{T}$, and a unit object 1, which is "associative, commutative, unitary"; a $\otimes$-category is said to be rigid when it has a functor $\mathcal{T} \to \mathcal{T}^{op}$ called autoduality which satisfies certain axioms ([An1], §2.2.2).

If $\mathcal{T}$ be an abelian rigid $\otimes$-category over a field $F$, a fiber functor over $\mathcal{T}$ is a faithful and exact $\otimes$-functor $\mathcal{T} \to \text{Vect}(K)$, where $K$ is a field extension of $F$. If a fiber functor exists, $\mathcal{T}$ is said to be Tannakian and the $K$-affine group scheme $G = \text{Aut}_{\otimes}^\otimes(\omega)$ is called the Tannakian group of $\mathcal{T}$ attached to $\omega$. It is compatible with base-change by field extensions of $K$.

If there exists a fiber functor with $K = F$, $\mathcal{T}$ is said to be neutral Tannakian, and $\omega$ defines an equivalence of $\otimes$-rigid categories $\mathcal{T} \to \text{Rep}_F(G)$, where $\text{Rep}_F(G)$ is the category of finite dimensional representations of $G$. ([An1], §2.3.2).
Conjecture 1.6. (Kontsevich-Zagier, KoZ, §4.1) The map from $P_{\text{formal}} \to P_{\text{num}}$ sending the equivalence class of $(X, D, \omega; \gamma)$ to $\int_\gamma \omega$ is injective.

1.2 The pro-unipotent fundamental groupoid

The pro-unipotent fundamental groupoid $\pi^\text{un}_1$ is the collection of several functors, which send certain algebraic varieties to certain groupoids in pro-affine schemes over them: $\pi^\text{un, Betti}_1$, $\pi^\text{un, DR}_1$ (De Rham), $\pi^\text{un, crys}_1$ (crystalline), $\pi^\text{un, l}_1$ ($l$-adic); some morphisms comparisons between them; and, finally, a functor $\pi^\text{un, mot}_1$ (motivic) whose realizations (in the sense of the realizations of a motive) are the previous ones. One also has the Hodge realization which englobes the Betti and De Rham realizations and the comparison between them. The notion of $\pi^\text{un}_1$ has been defined by Deligne [De2] and has been extended by Goncharov [Go2] and by Deligne and Goncharov [DeGo].

1.2.1 The Betti and De Rham realizations and the comparison

The concept of $\pi^\text{un}_1$ relies on the Tannakian point of view on the topological fundamental groupoid, and the Riemann-Hilbert correspondence (see for example [De1], and also [De2], §10.10-§10.13).

Definition 1.7. (Deligne, [De2], §13.5, §10.30 ii)) Let $X$ be a smooth algebraic variety over a field $K$ of characteristic zero; let us view $X$ as $\overline{X} - D$, where $\overline{X}$ is a projective smooth variety over $K$ and $D$ is a normal crossings divisor; then $\pi^\text{un, DR}_1(X)$ is the fundamental groupoid associated with the Tannakian category $C^\text{un,DR}(X)$ of vector bundles with integrable connection, with logarithmic singularities at $D$, and which are unipotent.

Definition 1.8. (Deligne, [De2], §13.6) Same hypothesis plus the existence of an embedding $k \hookrightarrow \mathbb{C}$. Then $\pi^\text{un, Betti}_1(X)$ (relative to this embedding) is the algebraic unipotent envelope of the topological fundamental groupoid of $X(\mathbb{C})$, in the sense of [De2], §10.24.

Theorem 1.9. (Deligne, [De2], equation (10.33.4) and §10.43) Under the previous hypothesis, the Riemann-Hilbert correspondence implies an isomorphism $\pi^\text{un, DR}_1(X) \otimes \mathbb{C} \simeq \pi^\text{un, Betti}_1(X) \otimes \mathbb{C}$.

1.2.2 The $p$-adic aspects

In the $p$-adic world, one has Tannakian definitions similar to Definition 1.7, where 'bundle with connection' is replaced by 'F-isocrystal' in the sense of [BerO], [O1], or [Kat]. Let $k$ be a perfect field of characteristic $p > 0$. By Shiho [Sh1], [Sh2], if $(X, M)$ is a fine log scheme over $k$ satisfying certain conditions ([Sh1], §4) one has a notion of (log-)crystalline

\[^3\pi^\text{un}_1\text{ is compatible with base-change ([De2], §10.36-§10.43); note that this is not true for its variant defined without the condition of unipotence of the bundles ([De2], §10.35).}]}
pro-unipotent fundamental groupoid $\pi_1^{\text{un,crys}}(X_0)$. Alternatively, by Chiarellotto and Le Stum [ChiL], if $X_0$ is separated scheme of finite type over $k$, one has a notion of rigid pro-unipotent fundamental groupoid $\pi_1^{\text{un,rig}}(X_0)$.

Finally, by Deligne ([De2], §11) one has a notion of crystalline Frobenius of the De Rham $\pi_1^{\text{un}}$. Let us review briefly [De2], §§11.9 - §11.12. Let $k$ be a perfect field of characteristic $p > 0$, $W(k)$ its ring of Witt vectors, $K$ the field of fractions of $W(k)$; let $S = \text{Spec}(W(k))$. Let $X$ be a smooth algebraic variety over $S$, $X_K = X \times_S \text{Spec}(K)$; assume that $X = \overline{X} - D$, with $\overline{X}$ smooth over $S$ and $D$ a relative normal crossings divisor, sum of smooth divisors. Let $\sigma$ be the Frobenius automorphism of $W(k)$, $F = \sigma^* : S \to S$, $X^{(p)}$ the pull-back of $X$ by $F$, and $X_K^{(p)} = X^{(p)} \times_S \text{Spec}(K)$. One has a notion of pull-back by Frobenius $F^*\mathcal{V}$ of certain bundles with integrable connection $\mathcal{V}$ over $X$ ([De2] Definition 11.10). It gives rise to a $K$-linear functor $F_{X/K}^* : C_1^{\text{un,DR}}(X^{(p)}_K) \to C_1^{\text{un,DR}}(X_K)$ ([De2], equation (11.11.1)), where $C_1^{\text{un,DR}}$ is in the sense of Definition 1.7. There exists an isomorphism of groupoids, which is horizontal with respect to the connexions ($\pi_1^{\text{un,DR}}$ is an initial object of the category of groupoids with integrable connection: [De2], §10.49, in the sense of [De2], §10.28): [De2], equation (11.11.2)

$$F_{X/K}^* : \pi_1^{\text{un,DR}}(X_K) \xrightarrow{\sim} F_{X/K}^*(\pi_1^{\text{un,DR}}(X_K^{(p)}))$$

**Definition 1.10.** (Deligne, [De2], §13.6) Under the assumptions above, the crystalline Frobenius of $\pi_1^{\text{un,DR}}(X_K)$ is $\phi = (F_{X/K})_*^{-1}$.

Thus, the crystalline $\pi_1^{\text{un}}$ of $X_0 = X \times_{\text{Spec}(W(k))} \text{Spec}(k)$ is $\pi_1^{\text{un,DR}}(X_K)$ equipped with $\phi$. We will follow this point of view in this text. By contrast, in the two other points of view, there is a theorem of comparison involving $\pi_1^{\text{un,DR}}(X_K)$ when $X_0$ is liftable.

### 1.2.3 Relation with cohomology and periods

A theorem of Beilinson (unpublished, reviewed in [Go1], §4, Theorem 4.1 and [DeGo], §3, Proposition 3.4) expresses $\pi_1^{\text{un,B}}(X)$ in terms of Betti cohomology groups of $X(\mathbb{C})^n$ relative to certain divisors for all $n \in \mathbb{N}^*$. This theorem has variants for the other realizations of the $\pi_1^{\text{un}}$, which are compatible with usual structures on cohomology ([Go1], §4). This is the starting point of the definition of the motivic $\pi_1^{\text{un}}$ ([Go1], §4 and [DeGo], §3-§4). Thus, the $\pi_1^{\text{un}}$ is actually a very specific part of the cohomology of algebraic varieties. When $X$ is defined over $k \subset \overline{\mathbb{Q}} \subset \mathbb{C}$, it thus makes sense, by §1.2.1, to talk about the periods of the Betti-De Rham comparison of $\pi_1^{\text{un}}(X)$.

### 1.3 $\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$ and multiple zeta values

#### 1.3.1 $\mathbb{P}^1 - \{0, 1, \infty\}$ and the motivation for the $\pi_1^{\text{un}}$

Despite the generality of the definition of the $\pi_1^{\text{un}}$, the paper [De2] is entitled *Le groupe fondamental de la droite projective moins trois points*, which can be explained as follows (see the introductions of [De2] and of [I]). The origin of the notion of $\pi_1^{\text{un}}$ comes from the
desire to understand $\text{Gal} (\overline{\mathbb{Q}} / \mathbb{Q})$ via its action on geometric objects; a theorem of Belyi [Bel] says that the natural map from $\text{Gal} (\overline{\mathbb{Q}} / \mathbb{Q})$ to the group $\text{Out} (\pi_1 (\mathbb{P}^1 - \{0, 1, \infty\}) (\mathbb{C}))$ of outer automorphisms of the pro-finite completion of $\pi_1 (\mathbb{P}^1 - \{0, 1, \infty\}) (\mathbb{C})$ is injective. Grothendieck has suggested in [Gr2] a way to understand the image of this map by studying more generally the outer action of $\text{Gal} (\overline{\mathbb{Q}} / \mathbb{Q})$ on the $\pi_1$ of moduli spaces $\mathcal{M}_{g,n}$ of curves of genus $g$ with $n$ marked points; and, in particular, $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 - \{0, 1, \infty\}$, $\mathcal{M}_{0,5}$, $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$. In [De2], Deligne has proposed to replace in this program the pro-finite completion of the $\pi_1$ by its pro-unipotent completion, which is still equipped with an action of $\text{Gal} (\overline{\mathbb{Q}} / \mathbb{Q})$. The advantage of $\pi_1^{\text{un}}$ comes from its proximity with cohomology, which follows from anterior works; whence also a second type of motivation for studying the $\pi_1^{\text{un}}$: to test the theory of motives and periods.

1.3.2 Multiple zeta values and their expressions as periods, including as Betti-De Rham periods of $\pi_1^{\text{un}} (\mathbb{P}^1 - \{0, 1, \infty\}, -\vec{I}_1, \vec{I}_0)$

Multiple zeta values were discovered, in one and two variables, by Euler, and were forgotten during more than two centuries. Their reappearance in the 1990’s is partly due to the role that they play in quantum field theory.

**Definition 1.11.** (Euler-Zagier) Multiple zeta values (MZV’s) are the following numbers, for $n_d, \ldots, n_1 \in \mathbb{N}^*$, such that $n_d \geq 2$:

$$\zeta (n_d, \ldots, n_1) = \sum_{0 < m_1 < \ldots < m_d} \frac{1}{m_1^{n_1} \ldots m_d^{n_d}} \in \mathbb{R} \quad (1)$$

One says that $n = n_d + \ldots + n_1$ is the weight of $(n_d, \ldots, n_1)$, and that $d$ is the depth of $(n_d, \ldots, n_1)$. One recognizes the values at positive integers of the Riemann zeta function as the $d = 1$ case. It has been noticed by Kontsevich that we have, for all $(n_d, \ldots, n_1)$,

$$\zeta (n_d, \ldots, n_1) = (-1)^d \int_0^1 dt_n \int_0^{t_n - \epsilon_n} dt_2 \ldots \int_0^{t_2 - \epsilon_2} \int_0^{t_1 - \epsilon_1} dt_1 \quad (2)$$

where $(\epsilon_n, \ldots, \epsilon_1) = \left( 0, \ldots, 0, 1, \ldots, 0, 0, 0, 1 \right)$. This shows that multiple zeta values as periods by Definition 1.2. What one wants, then, is to construct out of this observation a framework which enables to apply the tools of algebraic geometry to study multiple zeta values according to §1.1. There are two canonical ways to do it; only the second one will be used in this text.

- After a sequence of blowing-ups applied to the integral of (2), one obtains an expression of each multiple zeta value $\zeta (n_d, \ldots, n_1)$ as a period in the sense of Definition 1.1 of a relative cohomology group $H^n (\overline{\mathcal{M}}_{0,n+3} - \mathcal{A}_{(n_d, \ldots, n_1)}, B - \mathcal{A}_{(n_d, \ldots, n_1)} \cap B)$, where $n = n_d + \ldots + n_1$, $\overline{\mathcal{M}}_{0,n+3}$ is the Deligne-Mumford compactification of $\mathcal{M}_{0,n+3}$ and $\mathcal{A}_{(n_d, \ldots, n_1)}$ and $B$ are some unions of irreducible components of the normal crossings divisor $\partial \overline{\mathcal{M}}_{0,n+3} = \overline{\mathcal{M}}_{0,n+3} - \mathcal{M}_{0,n+3}$ [GoMan].
• An integral such as the one of (2) is called a *iterated path integral*, and is the typical expression of a period of a pro-unipotent fundamental groupoid. The work of Chen on (homotopy-invariant) iterated path integrals \[\text{Chen}\] is actually a sort of preliminary Betti - De Rham theory of the \(\pi_{1}^{un}\). Equation (2) shows, precisely, that multiple zeta values are Betti-De Rham periods of \(\pi_{1}^{un}(\mathbb{P}^1 - \{0, 1, \infty\})\) at the couple of tangential base-points \((-\bar{I}_1, \bar{I}_0)\) (see §2.3.3 for details.)

1.3.3 The question of studying multiple zeta values as periods

The question of understanding the polynomial equations in \(\mathbb{Q}\)-algebras of periods (§1.1.2) applies to multiple zeta values. The main objects involved are the motivic Galois action, or certain substitutes to it, and families of polynomial equations which conjecturally generate the whole of the ideal of polynomial equations of multiple zeta values. We will review these objects in §2.2 - §2.3. They are all described by fully explicit and simple formulas; this is a rare and advantageous situation, which enables an explicit theory of multiple zeta values as periods; in the present work on \(p\)-adic multiple zeta values, we will extend the area of explicitness of this theory.

The known results of the type ‘a given polynomial equation does not hold’ are the transcendence of \(\pi(\text{and thus the one of } \zeta(2n) = \frac{B_{2n}|\pi|^{2n}}{2(2n)!})\), Apéry’s result that \(\zeta(3) \not\in \mathbb{Q}\), and other results such as Rivoal-Ball’s theorem that the values \(\zeta(n)\) with \(n \in \mathbb{N}^*\) odd generate an infinite dimensional \(\mathbb{Q}\)-vector space. It is conjectured that all multiple zeta values are transcendental numbers; more precisely, the theory started out with the following conjecture:

**Conjecture 1.12.** (Zagier) For each \(n \in \mathbb{N}\), let \(\mathcal{Z}_n\) be the \(\mathbb{Q}\)-vector space generated by multiple zeta values of weight \(n\) (by convention \(\mathcal{Z}_0 = \mathbb{Q}\) is generated by \(\zeta(\emptyset) = 1\)). Then:

i) The \(\mathcal{Z}_n\) \((n \in \mathbb{N})\) are in direct sum.

ii) The dimensions of the \(\mathcal{Z}_n\) are given by the formula: \(\sum_{n=0}^{\infty} \dim(\mathcal{Z}_n)\Lambda^n = \frac{1}{1 - \Lambda - \Lambda^2}\).

Note that we have \(\mathcal{Z}_n \mathcal{Z}_{n'} \subset \mathcal{Z}_{n+n'}\) for all \(n, n'\) (by the two shuffle equations reviewed in §2.3.6). The framework of algebraic geometry has enabled to prove that we have \(\sum_{n=0}^{\infty} \dim(\mathcal{Z}_n)\Lambda^n \leq \frac{1}{1 - \Lambda^2 - \Lambda\Lambda^2}\) \([T], [Go1]\), which is a special case of the majoration reviewed in §1.1.2. However, the inequality \(\geq\) is out of reach at present.

1.4 Motivations for the explicit theory of \(\pi_{1}^{um,\text{crys}}(\mathbb{P}^1 - \{0, 1, \infty\})\)

The main subject of this text is \(\pi_{1}^{um,\text{crys}}(X_0)\) with \(X_0 = \mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{F}_p\), and more specifically, the families (one for each \(\alpha \in (\mathbb{Z} \cup \{-\infty\} - \{0\})\) of numbers \(\zeta_{p,\alpha}(n_d, \ldots, n_1) \in \mathbb{Q}_p\) called \(p\)-adic multiple zeta values defined through it.

\(^4\)The conjectural homogeneity of algebraic relations with respect to the weight expressed by i) of Conjecture 1.12 does not remain true if we replace weight by depth: we have \(\zeta(3) = \zeta(2, 1)\)

\(^5\)i.e. for all \(n \in \mathbb{N}\), \(\dim(\mathcal{Z}_n)\) \(\geq\) the coefficient of \(\Lambda^n\) in \(\sum_{n=0}^{\infty} \dim(\mathcal{Z}_n)\Lambda^n\).
1.4.1 Nature of $p$-adic multiple zeta values

Unlike real multiple zeta values, which express the comparison of two realizations of the \( \pi^{un} \), $p$-adic multiple zeta values are defined (see §2.4.2 for details) as the numbers in $\mathbb{Q}_p$ expressing canonically the crystalline Frobenius (Definition 1.10) iterated $\alpha \in (\mathbb{Z} \cup \{\pm \infty\}) - \{0\}$ times, applied to $X_0 = \mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{F}_p$, at the base-points $(1, -1)$. The questions on $p$-adic multiple zeta values are similar to those on multiple zeta values, and we have:

**Conjecture 1.13.** (for $\alpha = 1$ : Deligne-Goncharov, [DeGo], §5.28)

For each $n \in \mathbb{N}$, $p$ prime and $\alpha \in (\mathbb{Z} \cup \{\pm \infty\}) - \{0\}$, let $Z^{(p,\alpha)}_n$ be the $\mathbb{Q}$-vector space generated by numbers $\zeta^{(p,\alpha)}_n(w)$ with $w$ of weight $n$ (where $Z_0 = \mathbb{Q}$ generated by $\zeta^{(\alpha)}_0(\emptyset) = 1$).

i) The $Z^{(p,\alpha)}_n$ are in direct sum.

ii) We have $\sum_{n=0}^{\infty} \dim(Z^{(p,\alpha)}_n) \Lambda^n = \frac{1 - \Lambda^2}{1 - \Lambda^2 - \Lambda^3}$.

iii) The ideal of polynomial equations over $\mathbb{Q}$ satisfied by the numbers $\zeta^{(p,\alpha)}_n(w)$ is generated by the ideal of polynomial equations over $\mathbb{Q}$ satisfied by multiple zeta values and the relation $\zeta^{(p,\alpha)}(2) = 0$.

We have, as in the real framework, $Z^{(p,\alpha)}_n Z^{(p,\alpha)}_{n'} \subset Z^{(p,\alpha)}_{n+n'}$ for all $n, n'$ (by the two shuffle equations whose $p$-adic analogues are reviewed in §2.4.3).

By (unpublished) work of Yamashita, $p$-adic multiple zeta values in this sense are the images of some $p$-adic periods living in $B = \bigoplus_{n=0}^{\infty} (B_{cris} \cup Fil^0 B_{dR})^\varphi = p^n$, by the reduction map $B \to \mathbb{C}_p$, and $\sum_{n=0}^{\infty} \dim(Z^{(p,\alpha)}_n) \Lambda^n \leq \frac{1 - \Lambda^2}{1 - \Lambda^2 - \Lambda^3}$ (Yam). The inequality $\geq$ is out of reach at present like in the real case.

Aside from their intrinsic interest, a motivation for studying $p$-adic multiple zeta values is that this is believed to be necessary in the long-term to understand certain arithmetic aspects of real multiple zeta values. We now explain two more precise and shorter-term motivations; both of them are related to the fact that, unlike the case of multiple zeta values, there is a priori no straightforward way to compute explicitly $p$-adic multiple zeta values (see §2.4.3).

1.4.2 The quasi-shuffle relation and Deligne-Goncharov’s question

One of the standard families of algebraic equations of multiple zeta values, also proved for $p$-adic multiple zeta values, is the so-called quasi-shuffle (sometimes called *stuffle*) relation (see §2.3.6 §2.4.3), whose example in depth (1, 1) is

$$\zeta(n)\zeta(n') = \zeta(n + n') + \zeta(n, n') + \zeta(n', n)$$

For real multiple zeta values, the quasi-shuffle relation is a direct consequence of the formula of series of (1) combined to a canonical way of writing a product of two sets $\{m_1 < \ldots < m_d\} \subset (\mathbb{N}^*)^d$, $\{m'_1 < \ldots < m'_d\} \subset (\mathbb{N}^*)^d$ as a disjoint union of sets of the form

\[\left\{ \sum_{i=1}^d \mathbb{N} \right\} \times \left\{ \sum_{j=1}^d \mathbb{N} \right\} \times \left\{ \sum_{k=1}^d \mathbb{N} \right\} \subset \mathbb{N}^d \times \mathbb{N}^d \times \mathbb{N}^d\]

Whence the vanishing of the $p$-adic analogue of $\zeta(2)$: the analogue of $2i\pi$ in $B$ is sent to zero by the reduction map.
The fact that the quasi-shuffle relation holds for \( p \)-adic multiple zeta values can be proved without explicit formulas (see §2.4.3) but suggests the following question:

**Question 1.14.** (Deligne-Goncharov, [DeGo], §5.28)

Il serait aussi intéressant de disposer pour ces coefficients d’expressions \( p \)-adiques qui rendent clair qu’ils vérifient des identités du type [...]

\[
\sum_{k \neq l} \frac{1}{kn} \sum_{m} \frac{1}{lm} = \sum_{k \geq l} \frac{1}{kn} \frac{1}{lm} + \sum_{l > k} \frac{1}{kn} \frac{1}{lm} + \sum_{l=k} \frac{1}{kn+m}
\]

i.e. they ask to compute explicitly \( p \)-adic multiple zeta values in a way which enables a \( p \)-adic analogue of the above proof that multiple zeta values satisfy the quasi-shuffle relation.

### 1.4.3 Finite multiple zeta values and Kaneko-Zagier’s conjecture

Let \( \mathcal{P} \) be the set of prime numbers, and let \( \mathbb{Z}/p^\infty \) be the ring of 'integers modulo infinitely large primes' [Ko], which is of characteristic zero:

\[
\mathbb{Z}/p^\infty = \left( \prod_{p \in \mathcal{P}} \mathbb{Z}/p \mathbb{Z} \right) / \left( \bigoplus_{p \in \mathcal{P}} \mathbb{Z}/p \mathbb{Z} \right) = \left( \prod_{p \in \mathcal{P}} \mathbb{Z}/p \mathbb{Z} \right) / \left( \prod_{p \in \mathcal{P}} \mathbb{Z}/p \mathbb{Z} \right)_{\text{tors}} \simeq \left( \prod_{p \in \mathcal{P}} \mathbb{Z}/p \mathbb{Z} \right) \otimes \mathbb{Z} \mathbb{Q}
\]

where the subscript tors refers to the torsion subgroup. We have an embedding \( \mathbb{Q} \hookrightarrow \mathbb{Z}/p^\infty \).

**Definition 1.15.** (Zagier, unpublished) Finite multiple zeta values are the following numbers, for \((n_d, \ldots, n_1) \in (\mathbb{N}^*)^d, d \in \mathbb{N}^*\):

\[
\zeta_{\mathbb{Z}/p^\infty}(n_d, \ldots, n_1) = \text{the image of } \left( \sum_{0 < m_1 < \ldots < m_d < p} \frac{1}{m_1^{n_1} \ldots m_d^{n_d}} \mod p \right)_{p \in \mathcal{P}} \text{ in } \mathbb{Z}/p^\infty
\]

This notion is close to other notions and ideas that appeared earlier in the work of several authors, in particular Hoffman [Ho] and Zhao [Zh]. The explanation for this terminology is the following:

**Conjecture 1.16.** (Kaneko-Zagier) The following correspondence defines an isomorphism of \( \mathbb{Q} \)-algebras from the algebra generated by finite multiple zeta values to the algebra

---

7 Kaneko and Zagier as well as people working on finite multiple zeta values denote this ring by \( \mathcal{A} \); our notation \( \mathbb{Z}/p^\infty \) is not standard.

8 The usual notation is \( \zeta_{\mathcal{A}}(n_d, \ldots, n_1) \)
generated by multiple zeta values modulo $(\zeta(2))$:
\[ \zeta_{\mathbb{Z}/p\rightarrow\infty}(n_d, \ldots, n_1) \mapsto \sum_{d' = 0}^{d} (-1)^{n_{d'+1} + \cdots + n_d} \zeta(n_{d'+1}, \ldots, n_d) \zeta(n_d, \ldots, n_1) \mod \zeta(2) \tag{5} \]

This conjecture is striking, both because it involves the ring $\mathbb{Z}/p\rightarrow\infty$ and because of the explicit formula. Comparing it to Conjecture 1.13 and writing
\[ \mathbb{Z}/p\rightarrow\infty = \left\{ (x_p)_p \in \prod_{p \in \mathcal{P}} \mathbb{Q}_p \mid v_p(x_p) \geq 0 \text{ for } p \text{ large} \right\} \bigg/ \left\{ (x_p)_p \in \prod_{p \in \mathcal{P}} \mathbb{Q}_p \mid v_p(x_p) \geq 1 \text{ for } p \text{ large} \right\} \]

enables to imagine a relation between finite multiple zeta values and $p$-adic multiple zeta values, which could be an accessible part of the conjecture and partially explain it. This suggests, by extrapolating a little more, the existence of readable explicit formulas for $p$-adic multiple zeta values.

1.4.4 The initial technical question

By §1.4.1, §1.4.2 and §1.4.3, we are motivated to solve the problem of computing explicitly $p$-adic multiple zeta values, and we have in mind that a test of the success of the computation would be to see whether it sheds light in some way on the questions of §1.4.2 and §1.4.3.

It is granted at least that there is a certain way to compute $p$-adic multiple zeta values, by the overconvergence of the differential equation satisfied by the Frobenius (§2.4.3). To our knowledge, not much is granted regarding the form of the answer. We will see that there exists a primary form of answer which looks complicated, but that, by introducing certain particular principles of computation (§3) combined to certain ideas of $p$-adic analysis (§4.2), we can obtain different, simpler and more exploitable formulas; this will also lead us to enlarge our set of questions.

2 Review on $\pi_{1}^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$

We review some elements of explicit description of $\pi_{1}^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\})$, which make concrete the question of computing $p$-adic multiple zeta values. The pro-unipotent harmonic actions, which we will define in §4.2-§4.3, are connected to the Poisson-Ihara bracket which we review in §2.2.
2.1 Explicit description of $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\})$

In this paragraph, $X$ is $\mathbb{P}^1 - \{0, 1, \infty\}$ over any field $K$ of characteristic zero. The canonical base-point $\omega_{\text{DR}}$ is defined by [De2], §12.4; the tangential base-points are defined by [De2], §15 and for $u \in \{0, 1, \infty\}$, $T_u$ means the tangent space of $\mathbb{P}^1$ at $u$.

2.1.1 The groupoid $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\})$ and its canonical base-point $\omega_{\text{DR}}$

For each element $x$ of $\mathbb{P}^1 - \{0, 1, \infty\}(K) \cup \left( \bigcup_{u \in \{0, 1, \infty\}} (T_u - \{0\}) \right)(K)$, one has a tensor functor "fiber at $x$" $\pi_x^{\text{un,DR}} : \mathbb{C}^{\text{un,DR}} \rightarrow \text{Vect}(K)$, thus a base-point of the fundamental groupoid, which we will denote also by $x$.

For each couple $(x,y)$ of base-points, one has a pro-affine scheme $\pi_1^{\text{un,DR}}(X,y,x)$ over $\mathbb{Z}$: it is, by definition, the scheme of tensor automorphisms between the fiber functors $\pi_x^{\text{un,DR}}$ for each couple $(x,y)$. The points of $\pi_1^{\text{un,DR}}(X,y,x)$ are called De Rham paths from $x$ to $y$. For each triple $(x,y,z)$ of base-points one has a morphism of schemes $\pi_i^{\text{un,DR}}(X,z,y) \times \pi_i^{\text{un,DR}}(X,y,x) \rightarrow \pi_i^{\text{un,DR}}(X,z,x)$ called the groupoid multiplication.

When $x = y$, $\pi_1^{\text{un,DR}}(X,x,x) = \pi_1^{\text{un,DR}}(X,x,x)$ is a group scheme. The groupoid multiplication makes each $\pi_1^{\text{un,DR}}(X,y,x)$ into a bi-torsor under $(\pi_1^{\text{un,DR}}(X,x), \pi_1^{\text{un,DR}}(X,y))$.

The functor "global section" $\omega_{\text{DR}} : \mathbb{C}^{\text{un,DR}} \rightarrow \text{Vect}(k)$ is a tensor functor (this is because $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$), and for each base-point $x$, we have a canonical isomorphism $x \simeq \omega_{\text{DR}}$. As a consequence, we have a pro-affine scheme $\pi_1^{\text{un,DR}}(X,\omega_{\text{DR}})$; with, for each $(x,y)$, a canonical isomorphism $\pi_1^{\text{un,DR}}(X,\omega_{\text{DR}}) \simeq \pi_1^{\text{un,DR}}(X,y,x)$ and a canonical path $y_0x_1$ of $\pi_1^{\text{un,DR}}(X,y,x)$. They are compatible with the groupoid structure: $(z_1y)(y_1x) = z_1x$. Describing explicitly the groupoid $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\}, \omega_{\text{DR}})$ is thus reduced to describing explicitly the group scheme $\pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\}, \omega_{\text{DR}})$.

2.1.2 Explicit description of $\pi_1^{\text{un}}(\mathbb{P}^1 - \{0, 1, \infty\}, \omega_{\text{DR}})$

The shuffle Hopf algebra is a functor from the category of (finite) sets to the category of graded Hopf algebras over $\mathbb{Q}$. We need only the following example.

Proposition 2.1. The $\mathbb{Q}$-vector space $\mathcal{O}^m = \mathbb{Q}\langle e_0, e_1 \rangle$, freely generated by words on $\{e_0, e_1\}$ including the empty word, endowed with the following operations $\mathfrak{m}$, $\Delta_{\text{dec}}, \epsilon, S$, is a Hopf algebra over $\mathbb{Q}$, graded by the number of letter of words:

- the shuffle product $\mathfrak{m} : \mathcal{O}^m \otimes \mathcal{O}^m \rightarrow \mathcal{O}^m$ defined by $e_{i_{n+n'}} \cdots e_{i_0} e_{i_0} \cdots e_{i_n} = \sum_{\sigma} e_{\sigma(1)} \cdots e_{\sigma(n+n')}$, where the sum is over permutations $\sigma$ of $\{1, \ldots, n+n'\}$ such that $\sigma(l) < \ldots < \sigma(1)$ and $\sigma(n+n') < \ldots < \sigma(l+1)$

- the deconcatenation coproduct $\Delta_{\text{dec}} : \mathcal{O}^m \rightarrow \mathcal{O}^m \otimes \mathcal{O}^m$, defined by $\Delta_{\text{dec}}(e_{i_0} \cdots e_{i_1}) = \sum_{n=0}^{n'} e_{i_0} \cdots e_{i_{n+n'+1}} \otimes e_{i_{n'}} \cdots e_{i_1}$

- the counit $\epsilon : \mathcal{O}^m \rightarrow \mathbb{Q}$ equal to the augmentation map of $\mathcal{O}^m$,

- the antipode $S : \mathcal{O}^m \rightarrow \mathcal{O}^m$, defined by $S(e_{i_0} \cdots e_{i_1}) = (-1)^n e_{i_1} \cdots e_{i_n}$. 

11
Definition 2.2. $O^\mathrm{sh}$ is called the shuffle Hopf algebra over the alphabet $\{e_0, e_1\}$, and the number of letters of a word is called its weight.

Proposition 2.3. (follows from [De2], §12, Proposition 12.7, Corollaire 12.9) The group scheme $\text{Spec}(O^\mathrm{sh})$ is pro-unipotent and is canonically isomorphic to $\pi^\text{un,DR}_1(X, \omega_{\text{DR}})$.

We will denote in the same way $e_0, e_1 \in O^\mathrm{sh}$ and their duals below. Let $K\langle\langle e_0, e_1 \rangle\rangle$ be the non-commutative $K$-algebra of formal power series with variables $e_0, e_1$ with coefficients in $K$. An element $f$ of $K\langle\langle e_0, e_1 \rangle\rangle$ can be written in a unique way as $f = \sum_w \text{word over } \{e_0, e_1\} f[w] w$ with $f[w] \in K$ for all $w$. The notation $f[w]$ extends to linear combinations of words by linearity.

Proposition 2.4. The dual of the topological Hopf algebra $O^\mathrm{sh} \otimes Q K$ is $K\langle\langle e_0, e_1 \rangle\rangle$ viewed as the topological Hopf algebra associated with the universal enveloping algebra of the complete free Lie algebra over the two variables $e_0, e_1$.

The group $\text{Spec}(O^\mathrm{sh})(K)$ is the group of the elements of $K\langle\langle e_0, e_1 \rangle\rangle$ which satisfy the following property called the shuffle equation: for all words $w, w'$, $f[w] f[w'] = f[w \shuffle w']$.

2.1.3 The fundamental torsor of paths and its connection

The groupoid $\pi^\text{un,DR}_1(\mathbb{P}^1 - \{0, 1, \infty\})$ is an initial object of the category of groupoids with integrable connection (by [De2], §10.49), in the sense of [De2], §10.28. It follows from [De2], §12 that:

**Proposition 2.5.** The connection on $\pi^\text{un,DR}_1(\mathbb{P}^1 - \{0, 1, \infty\})$ is, in the sense of §7.30.2, $S$, the map

$$\nabla_{\text{KZ}} : f \mapsto f^{-1} \left( df - \left( e_0 \frac{dz}{z} + e_1 \frac{dz}{z-1} \right) f \right)$$

**Definition 2.6.** $\nabla_{\text{KZ}}$ is called the Khnizhnik-Zamolodchikov connection.

2.2 The motivic Galois action on $\pi^\text{un,DR}_1(\mathbb{P}^1 - \{0, 1, \infty\})$, $-\vec{1}_1, \vec{1}_0$)

Following [DeGo], we now review how the Tannakian framework of the motivic Galois theory of periods evoked in §1.1.2 applies to $\pi^\text{un}(\mathbb{P}^1 - \{0, 1, \infty\})$.

2.2.1 Mixed Tate motives and $\pi^\text{un, mot}(\mathbb{P}^1 - \{0, 1, \infty\})$

Let $k$ be a number field. By Levine [L], one has the Tannakian category $\text{MT}(k)$ of mixed Tate motives over $k$ and, for $S$ a set of finite places $k$, one has the Tannakian category $\text{MT}(O_S)$ of mixed Tate motives over $O_S$, which is a subcategory of $\text{MT}(k)$. They are described in [DeGo], §1,§2.

The De Rham realization functor $\text{DR} : \text{MT}(k) \to \text{Vect}_k$ is canonically isomorphic to the extension of scalars from $Q$ to $k$ of the 'canonical realization functor' $\omega$ ([DeGo], Proposition §2.10). The motivic Galois group $G^\omega = \text{Aut}^\otimes(\omega)$ associated with $\omega$ on $\text{MT}(O_S)$ with $S$ as above has a semi-direct product decomposition $G^\omega = \mathbb{G}_m \ltimes U^\omega$ where $U^\omega$ is a
The groupoid $\pi_{\text{un,mot}}^{\text{un}}(P^1 - \{0, 1, \infty\})$ is defined as a groupoid over $P^1 - \{0, 1, \infty\}$ in affine schemes in MT($Q$) ([DeGo], §3.12, §3.13, §4); the notion of affine schemes in a Tannakian category is reviewed in [DeGo], §2.6.

2.2.2 Description of the action of $G^\omega$ on $\pi^{\text{un,DR}}_1(P^1 - \{0, 1, \infty\}, -\vec{1}_1, \vec{1}_0)$; the Goncharov coproduct and the Ihara bracket

The motivic Galois group $G^\omega$ associated with MT($Z$) acts on $\pi^{\text{un,DR}}_1(P^1 - \{0, 1, \infty\}, -\vec{1}_1, \vec{1}_0)$. We review its description.

- The action of $G_m \subset G^\omega$ expresses the weight grading: it provides the motivic way to formulate the conjecture that all polynomial relations between multiple zeta values are homogeneous for the weight (Conjecture 1.12 i):

**Definition 2.7.** Let $\tau$ be the action of $G_m$ on $\pi^{\text{un,DR}}_1(P^1 - \{0, 1, \infty\}, -\vec{1}_1, \vec{1}_0)$ which sends $(\lambda, f = \sum_{\text{word}} f[w]w)\mapsto \sum_{\text{word}} \lambda^{\text{weight}(w)} f[w]w$.

- Since the functors of Hodge realization of categories of mixed Tate motives are fully faithful ([DeGo], Proposition 2.14), in order to compute the action of $U^\omega$, it is sufficient to compute its Hodge realization. This has been done by Goncharov: [Go1], Theorem 6.4, Theorem 6.5. The formula is usually written in terms Hopf algebras, is often called the Goncharov coaction or Goncharov coproduct. Here, what we are interested in is its De Rham realization. We will denote, as in [DeGo], §5, by $\Pi_{1,0} = \pi^{\text{un,DR}}_1(P^1 - \{0, 1, \infty\}, -\vec{1}_1, \vec{1}_0)$, and by $\Pi_{0,0}$, $\Pi_{0,1}$, $\Pi_{1,0}$ the other similar schemes, where we associate 1 with $-\vec{1}_1$ and 0 with $-\vec{1}_0$.

**Definition 2.8.** ([DeGo], §5.6 - §5.9) Let $V_\omega$ be the group of automorphisms of the four schemes $\Pi_{0,0}$, $\Pi_{1,0}$, $\Pi_{0,1}$, $\Pi_{1,1}$ which preserve the groupoid multiplication and fix $\exp(e_0)$ at $(0, 0)$ and $\exp(e_1)$ at $(1, 1)$.

**Theorem 2.9.** ([DeGo], equation (5.10.3)) There exists a morphism $G_m \times U_\omega \to G_m \times V_\omega$, compatible to the semi-direct product decompositions and the actions on $\Pi_{1,0}$.

**Theorem 2.10.** ([DeGo], Proposition 5.11)

i) $\Pi_{1,0}$ is a $V_\omega$-torsor, and we thus have an isomorphism of schemes $V_\omega \simeq \Pi_{1,0}$, $v \mapsto v(11_0)$ (where $11_0 = -\vec{1}_1 \vec{1}_0$ is the canonical path as in §2.1.1).

ii) More precisely, for $v$ a point of $V_\omega$ and $g = v(11_0)$, the action of $v$ on $\Pi_{0,0}$ is $g(e_0, e_1) \mapsto (f(e_0, e_1) \mapsto f(e_0, g^{-1}e_1g))$ and the action of $v$ on $\Pi_{1,0}$ is $g(e_0, e_1) \mapsto (f(e_0, e_1) \mapsto g(e_0, e_1).f(e_0, g^{-1}e_1g))$.

iii) The map $(g(e_0, e_1), f(e_0, e_1) \mapsto g(e_0, e_1).f(e_0, g^{-1}e_1g)$ defines a group law on $\Pi_{1,0}$.

**Notation 2.11.** We denote the group law above by $\circ_\omega$. (This notation is not standard.)
Some authors call Ihara action or Ihara product the above action of $V_\omega$ on $\Pi_{1,0}$, since it has been first written explicitly by Ihara [1]. The Lie bracket of Lie $V_\omega$ is sent by the isomorphism $V_\omega \simeq \Pi_{1,0}$ to a Lie bracket on Lie $\Pi_{1,0}$ called the Ihara bracket, or the Poisson-Ihara bracket, or the Poisson bracket on the free Lie algebra in two generators. Some other authors call $(\Pi_{1,0}, \circ_{\ell})$ the twisted Magnus group.

2.3 Betti-De Rham comparison of $\pi^{1un}_1(\mathbb{P}^1 - \{0, 1, \infty\})$ and multiple zeta values

2.3.1 Iterated path integrals

Definition 2.12. (Chen, [Che]) Let $\eta_1, \ldots, \eta_r$ be differential 1-forms on a manifold $M$, and let $\gamma: [0,1] \to M$ be a smooth path. Denote by $f_i(t)dt = \gamma^*(\eta_i)$, $i=1,\ldots,r$. The iterated path integral of $(\eta_1, \ldots, \eta_r)$ along $\gamma$ is

$$\int_\gamma \eta_r \cdots \eta_1 = \int_0^1 f_r(t)dt \int_0^{f_r} \cdots \int_0^{f_2} f_1(t)dt$$

If the differential forms have logarithmic singularities at infinity, the definition can be extended to the case where the extremities of $\gamma$ are tangential-base points; this sometimes requires to regularize a divergent iterated integral.

In all the rest of this text, we will choose paths in $\mathbb{P}^1 - \{0, 1, \infty\}$ starting at the tangential base-point $\vec{1}_0$.

2.3.2 Multiple polylogarithms and their series expansion at 0

Definition 2.13. (Goncharov, [Go1]) Multiple polylogarithms (MPL’s) on $\mathbb{P}^1 - \{0, 1, \infty\}$ are multivalued holomorphic functions on $\mathbb{P}^1 - \{0, 1, \infty\}(\mathbb{C})$ solutions to $\nabla_{KZ}$ of §2.1.3: for each $\gamma$ a path in $\mathbb{P}^1 - \{0, 1, \infty\}$ starting at $\vec{1}_0$, let

$$\text{Li}(\gamma) = 1 + \sum_{i_n,...,i_1 \in \{0,1\}^n} \left( \int_{\gamma} \frac{dz}{z - i_n} \cdots \frac{dz}{z - i_1} \right) e_{i_n} \cdots e_{i_1} \in \pi^{1un,DR}_1(\mathbb{P}^1 - \{0,1,\infty\}, z, \vec{1}_0)$$

Indeed, $\text{Li}(\gamma)$ depends only on the homotopy class of $\gamma$; it defines a function on the topological $\pi_1$, and then on the $\pi^{1un,-}_1$ via the Malcev completion map.

---

10For example in the case of $\mathbb{P}^1 - \{0,1,\infty\}$: let $\gamma$ be a smooth path $[0,1] \to \mathbb{P}^1(\mathbb{C})$ such that for $t \in [0,1[$, we have $\gamma(t) \in (\mathbb{P}^1 - \{0,1,\infty\})(\mathbb{C})$, and $\gamma(0) \in \{0,1,\infty\}$, then $\gamma$ is said to have extremity $\gamma(0),\gamma(1)$; analogous definition at $t=1$. The notions of homotopy of paths and of topological fundamental groupoid extend to such paths, as does the Malcev completion defining $\pi^{1un,-}_1(\mathbb{P}^1 - \{0,1,\infty\})$ from Definition.

11For example in the case of $\mathbb{P}^1 - \{0,1,\infty\}$: if $\gamma(0) \in \{0,1,\infty\}$, we consider the analogue iterated integral on $\gamma(\epsilon,1)$ with $0 < \epsilon < 1$, which has an asymptotic expansion in $\mathbb{C}[[\epsilon]]\log(\epsilon)$ when $\epsilon \to 0$ and define the regularized iterated integral on $\gamma$ as the constant coefficient of its asymptotic expansion.
2.3.3 Series expansion of multiple polylogarithms; multiple harmonic sums

Proposition 2.14. (Goncharov, [Go1]) For \( d \in \mathbb{N}^* \), and \( n_1, \ldots, n_d \in \mathbb{N}^* \), let \( w = e_0^{n_d-1} e_1 \cdots e_0^{n_1-1} e_1 \). For \( z \in \mathbb{C} \) such that \( |z| < 1 \), we have, denoting by \( \text{Li}[w](z) \) the value of \( \text{Li}[w] \) at the straight path from 0 to \( z \),

\[
\text{Li}[w](z) = \sum_{0 < m_1 < \cdots < m_d} \frac{z^{m_d}}{m_1^{n_1} \cdots m_d^{n_d}}
\]

We retrieve the formula (1) for multiple zeta values by taking the limit \( z \rightarrow 1 \); see §2.3.3 below. We are led to consider intrinsically the coefficients of this series expansion:

Definition 2.15. i) Let weighted multiple harmonic sums be the following numbers, where \( d \in \mathbb{N}^* \), and \( n_1, \ldots, n_d \in \mathbb{N}^* \),

\[
\text{har}_m(n_d, \ldots, n_1) = m_1^{n_d+\cdots+n_1} \sum_{0 < m_1 < \cdots < m_d < m} \frac{1}{m_1^{n_1} \cdots m_d^{n_d}}
\]

ii) Let prime weighted multiple harmonic sums be the weighted multiple harmonic sums of the form \( \text{har}_p^{\alpha}(n_d, \ldots, n_1) \) with \( p \) a prime number and \( \alpha \in \mathbb{N}^* \).

The adjective weighted refers to the factor \( m_1^{n_d+\cdots+n_1} \). By (6) we have, for all \( l \in \mathbb{N}^* \),

\[
\text{har}_m(w) = \tau(n) \text{Li}[e_0 c_1 w][z^m] = \tau(m) \sum_{m' = 1}^{m-1} \text{Li}[w][z^{m'}]
\]

where \([z^{m'}]\) means the coefficient of degree \( m' \) in the power series expansion.

2.3.4 Expression of multiple zeta values as Betti-De Rham periods of \( \pi_1^{un}(\mathbb{P}^1 - \{0, 1, \infty\}, -\bar{I}_1, \bar{I}_0) \)

Definition 2.16. ([Dr], §2, [DeGo], §5.18)

Let \( \text{dch} \in \pi_1^{un}(\mathbb{P}^1 - \{0, 1, \infty\}, -\bar{I}_1, \bar{I}_0)(\mathbb{C}) \) be the image, by the completion map of Definition 1.8 of the homotopy class of the straight path \( \gamma : [0, 1] \to [0, 1] \). Let \( \Phi_{KZ} \) be the image of \( \text{dch} \) by the Betti-De Rham comparison isomorphism of Theorem 1.9. \( \Phi_{KZ} \) is called the KZ Drinfeld associator (the origin of this terminology is in [Dr]).

Proposition 2.17. Equation (2) is equivalent to: for all \( (n_d, \ldots, n_1) \) such that \( n_d \geq 2 \),

\[
\zeta(n_d, \ldots, n_1) = (-1)^d \Phi_{KZ}[e_0^{n_d-1} e_1 \cdots e_0^{n_1-1} e_1]
\]

Note that \( \text{Li}[e_1](z) = -\log(1 - z) \). We take the convention that the values \( \text{Li}[w] \) for \( w \) a word of the form \( w' e_0 \) are determined by \( \text{Li}[e_0](z) = \log(z) \), the values of Proposition 2.14 and the shuffle equation of Proposition 2.4: namely, using that for all words \( w \), and \( x \in \{0, 1\} \), we have \( \text{Li}[e_1 w^x e_0](z) = -\text{Li}[w^x e_0](z) = -\text{Li}[w^x e_0](z) \).

The other coefficients of \( \Phi_{KZ} \) are some \( \mathbb{Q} \)-linear combinations of multiple zeta values, as it can be seen either by computing regularized iterated integrals, or by the shuffle equation of Proposition 2.4 applied to \( \Phi_{KZ} \). They are called regularized multiple zeta values. We have \( \Phi_{KZ}[e_0] = \Phi_{KZ}[e_1] = 0 \).
2.3.5 Conjecture of periods for multiple zeta values

The conjecture of periods in the case of multiple zeta values amounts to say that all polynomial equations of multiple zeta values are preserved by the motivic Galois action evoked in §1.3.2, i.e. are "motivic relations".

2.3.6 Explicit algebraic relations of multiple zeta values and product $\phi^d$; focus on double shuffle relations

One has three standard families of explicit algebraic relations over $\mathbb{Q}$ between multiple zeta values, each of them is conjectured to generate all their algebraic relations: they are called respectively the associator relations, the regularized double shuffle relations, and the Kashiwara-Vergne relations. (See [Fu3] for a review of these three notions and [Dr] for the definition of associators). They are known to be motivic (see [So], [Fu4], [Sch]). There are several known inclusions between the ideals respectively generated by these relations [AET], [AT], [Fu4], [Sch].

In this text we will focus on the regularized double shuffle relations, namely

- two notions of regularized multiple zeta values, extending the Definition 1.11 of multiple zeta values: $\zeta_m(w) = \Phi_{KZ}[w]$ for all words $w$ on $e_0, e_1$, and $\zeta_\ast(n_d, \ldots, n_1)$ for all words $(n_d, \ldots, n_1)$ including $n_d = 1$, and a formula relating $\zeta_m$ and $\zeta_\ast$ (see [Ra] for details)

- the shuffle relation, satisfied by all points of $\text{Spec}(\mathcal{O}^m)$ (Proposition 2.3): for all words $w, w'$, $\zeta_m(w)\zeta_m(w') = \zeta_m(w \shuffle w')$. This also follows from equation (2) combined to the canonical way to write a product $\{0 < t_1 < \ldots < t_n < 1\} \times \{0 < t'_1 < \ldots < t'_{n'} < 1\} \subset \mathbb{R}^{n+n'}$ as a disjoint union of simplices of $\mathbb{R}^{n+n'}$ up to sets of measure 0. (first example: $\int_{0<t_1<1} \int_{0<t'_1<1} = \int_{0<t_1<t'_1<1} + \int_{0<t_1<1}$)

- the quasi-shuffle (or stuffle) relation, already described in §1.4.3: it is expressed by the following formula: for all words $w, w'$ whose furthest to the right letter is $e_1$, $\zeta_\ast(w)\zeta_\ast(w') = \zeta_\ast(w * w')$ where $*$ is a bilinear map $\mathcal{O}^m \times \mathcal{O}^m \to \mathcal{O}^m$ which lifts the canonical expression of a product $\{0 < m_1 < \ldots < m_d\} \times \{0 < m'_1 < \ldots < m'_d\} \subset \mathbb{N}^d \times \mathbb{N}^d$ as a disjoint union of "simplices" of $\mathbb{N}^r$, $r \in \{\max(d, d'), \ldots, d + d'\}$ (first example: $\sum_{0<m_1} \times \sum_{0<m'_1} = \sum_{0<m_1=m'_1} + \sum_{0<m_1<m'_1} + \sum_{0<m'_1<m_1}$)

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15 Although this paper concerns the other definition of motivic multiple zeta values arising from [GoMan] evoked in §1.3.2, which is conjecturally equivalent to the notion of motivic multiple zeta values arising from $\pi^m_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$

16 $\zeta_m$ is defined by replacing an iterated integral from 0 to 1 by an iterated integral from $\epsilon$ to $1 - \epsilon'$, writing an asymptotic expansion when $\epsilon, \epsilon' \to 0$ (this depends on a choice of tangential base-points and one takes $I_0$ and $-I_1$) and considering its constant term; similarly, $\zeta_\ast$ is obtained by replacing the infinite sum over $0 < n_1 < \ldots < n_d$ by a truncated sum over $0 < n_1 < \ldots < n_d < n$, writing an asymptotic expansion when $n \to \infty$ and considering its constant term, in which, in addition, we suppress the terms where the Euler-Mascheroni constant appears; this gives for example that $\zeta_\ast(1) = 0$. The notions $\zeta, \zeta_m, \zeta_\ast$ agree on words $(n_d, \ldots, n_1)$ with $n_d \geq 2$.
For the three standard families of algebraic relations, the product $\phi^\mu$ on $\Pi_{1,0}$ reviewed in §2.2.2 plays the role of an elementary substitute to a motivic Galois action. More precisely, for each field $K$ of characteristic zero and $\mu \in K$, one has [Ra] a pro-affine scheme $DMR_\mu$ of solutions $\varphi$ to the regularized double shuffle equations and the condition $\varphi[\epsilon_0 \epsilon_1] = \mu$, such that:

**Theorem 2.18.** (Racinet, [Ra]) The product $\phi^\mu$ makes $DMR_0$ into a group scheme and the left multiplication by $\phi^\mu$ makes $DMR_\mu$ into a torsor under this group scheme.

One has similar results for the two other standard families of relations (see [Dr] for associators, and [Fu3] for a general review including a comparison with the motivic framework.)

### 2.4 The crystalline realization of $\pi^{\text{un}}_1(\mathbb{P}^1 - \{0, 1, \infty\})$

Let $p$ be a prime number. The definition of the crystalline Frobenius $\phi$ of the De Rham $\pi^{\text{un}}_1$ reviewed in §1.2.2 applies to $X_{\mathbb{F}_p} = \mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{F}_p$ and its lift $X_{\mathbb{Z}_p} = \mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{Z}_p$; we will also denote by $X_{\mathbb{Q}_p} = \mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{Q}_p$. When $k = \mathbb{F}_p$, the Frobenius of $k$ is $(x \mapsto x^p) = \text{id}_{\mathbb{F}_p}$, thus the Frobenius $\sigma$ of $W(\mathbb{F}_p) = \mathbb{Z}_p$, which is the unique automorphism lifting it, is $\text{id}_{\mathbb{Z}_p}$. Let $\alpha \in \mathbb{N}^*$, which will represent the number of iterations of $\phi$.

#### 2.4.1 The Frobenius of $\pi^{\text{un,DR}}_1(\mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{Q}_p)$ at $(-\overline{1}, \overline{0})$ and $p$-adic multiple zeta values

There are two different, but conjecturally arithmetically equivalent, notions of $p$-adic multiple zeta values. We have extended them in [J1] [J3] by considering the Frobenius iterated a certain number of times instead of the Frobenius itself. Introducing the number of iterations of the Frobenius also enables to formulate the two definitions in a unified way (see §4.3). The first definition is the following (where $\tau$ is defined in §2.2.2):

**Definition 2.19.** (For $\alpha = 1$, Deligne-Goncharov, [DeGo], §5.28; for any $\alpha$, [J1])

Let, for $\alpha \in \mathbb{N}^*$, $\Phi_{p,\alpha} = \tau(p^\alpha)\phi^\alpha(1_0) \in \pi^{\text{un,DR}}_1(\mathbb{P}^1 - \{0, 1, \infty\}, -\overline{1}, \overline{0})(\mathbb{Q}_p)$. The following numbers are called $p$-adic multiple zeta values ($p$MZV’s) \footnote{One sometimes encounters in the literature the variant of this definition without the factor $\tau(p^\alpha)$}:

$$\zeta_{p,\alpha}(n_d, \ldots, n_1) = (-1)^d \Phi_{p,\alpha} \left[ \epsilon_0^{n_d-1} \epsilon_1 \ldots \epsilon_0^{n_1-1} \epsilon_1 \right]$$

These numbers provide a natural way of expressing the Frobenius at $(-\overline{1}, \overline{0})$:

**Proposition 2.20.** (Deligne-Goncharov, [DeGo], §5.28)

$\tau(p^\mu)$ restricted to base points in the set $\{-\overline{1}, \overline{0}\}$ is a $\mathbb{Q}_p$-point of $V_\omega$ from Definition 2.8. Thus, by Proposition 2.10 for all $\alpha \in \mathbb{N}^*$, $\tau(p^\alpha)\phi^\alpha$ restricted to $\pi^{\text{un}}_1(\mathbb{P}^1 - \{0, 1, \infty\}, -\overline{1}, \overline{0})(\mathbb{Q}_p)$ is the map $f \mapsto \Phi_{p,\alpha} \phi^\mu f$.

The second definition is based on the theory of Coleman integration, initiated in [Co], and generalized and formulated in a Tannakian way by Besser [Bes] and Vologodsky [V]. One of its main result is the existence and uniqueness of Frobenius-invariant paths.
2.4.2 The Frobenius of $\pi_{1,\mathrm{an}}^{\mathrm{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\}, -I_0)(\mathbb{Q}_p)$ be the unique path invariant by the Frobenius. The following numbers are called $p$-adic multiple zeta values ($p$MZV’s)

$$\zeta_p^{KZ}(n_d, \ldots, n_1) = (-1)^d \Phi_p^{KZ}[e_0^{n_d-1} e_1 \cdots e_0^{n_1-1} e_1]$$

Conjecturally, all versions of $p$-adic multiple zeta values satisfy the same algebraic relations, which should be the ones of multiple zeta values ‘modulo $\zeta(2)$’, as reviewed in §1.4.1.

2.4.2 The Frobenius of $\pi_{1,\mathrm{an}}^{\mathrm{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\}/\mathbb{Q}_p)$ on the fundamental torsor of paths starting at $\overline{I}_0$ and $p$-adic multiple polylogarithms

In general, to write the Frobenius of $\pi_{1,\mathrm{an}}^{\mathrm{un,DR}}(X_K)$, one must consider an open affine covering of the rigid analytic space $X_K^{an}$ over $K$ and write the differential equation of Frobenius in coordinates on each open affine of the covering. Here, it is sufficient to consider the open affine $U^{an} = \mathbb{P}^{1,\mathrm{an}} - \{|z - 1|_p < 1\}$, as suggested first by [De2], §19.6. 18 Let the four following $\mathbb{Q}_p$-algebras of functions on $U^{an}$ : $A^1(U^{an})$, the algebra of overconvergent rigid analytic functions ; $A^{\mathrm{rig}}(U^{an})$, the algebra of rigid analytic functions ; $A^{\mathrm{Col}}(U^{an})$, the algebra of Coleman functions ; $A^{\mathrm{loc}}(U^{an})$, the algebra of locally analytic functions for the $p$-adic topology. We have $A^1(U^{an}) \subset A^{\mathrm{rig}}(U^{an}) \subset A^{\mathrm{Col}}(U^{an}) \subset A^{\mathrm{loc}}(U^{an})$.

Definition 2.22. (for $\alpha = 1$, [Fu2], Definition 2.13 ; for any $\alpha$, [JJ]) Let $L_{p,\alpha}^{\dagger}$ be the map $z \mapsto \tau(p^n)|_{\phi_\alpha}(z, 1_0)$ on $U^{an}$ (the fundamental torsors of paths starting at $\overline{I}_0$ being trivialized at $\overline{I}_0$, $L_{p,\alpha}^{\dagger}$ can be seen as an element of $\pi_{1,\mathrm{an}}^{\mathrm{un,DR}}(X_{\mathbb{Q}_p}, \overline{I}_0)(A^1(U^{an}))$). The overconvergent $p$-adic multiple polylogarithms ($p$MPL$\dagger$’s) are the functions $L_{p,\alpha}^{\dagger}[w] \in A^1(U^{an})$, for words $w$ on $\{e_0, e_1\}$.

We have:

Proposition 2.23. The Frobenius $\tau(p^n)|_{\phi^\alpha}$ on $U^{an}$ (after trivialization at $\overline{I}_0$, we can say the Frobenius of $\pi_{1,\mathrm{an}}^{\mathrm{un,DR}}(X_{\mathbb{Q}_p}, \overline{I}_0)(A^{\mathrm{Col}}(U^{an}))$) is

$$f(z)(e_0, e_1) \mapsto L_{p,\alpha}^{\dagger}(z)(e_0, e_1).f(z^{p^n})(e_0, \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha})$$

In particular, the Frobenius iterated $\alpha$ times is entirely characterized by the couple $(\zeta_{p,\alpha}, L_{p,\alpha}^{\dagger})$.

---

18$U^{an}$ contains 0 and $\infty$; it is chosen instead of $\mathbb{P}^1 - \{|z - \infty|_p < 1\} = \mathbb{Z}_p^{an}$ which contains 0 and 1 because the good lift of Frobenius on it, $z \mapsto z^p$ is simpler and more adapted to our purposes than the good lift of Frobenius on $\mathbb{Z}_p^{an}$, which is $z \mapsto z^p$ conjugated by $z \mapsto \frac{z-1}{z+1}$, the unique homography sending $(0, 1, \infty) \mapsto (0, \infty, 1)$

19In particular, the elements of these four algebras have convergent power series expansions on a neighbourhood of 0. Elements of $A^{\mathrm{Col}}(U^{an})$ are characterized uniquely by their series expansion at 0. The values of elements of $A^{\mathrm{rig}}(U^{an})$ at any point are described explicitly in terms of the series expansion at 0, see §4.1.3.
An essential theorem of Coleman integration is the existence and uniqueness of a primitive in algebras of Coleman functions up to an additive constant \(^{20}\).

**Definition 2.24.** (Furusho, [Fu1], Definition 2.9) Let \(L_{p}^{KZ}\) be the unique Coleman function on \(P^{1} - \{0,1,\infty\}\) characterized by \(\nabla_{KZ}(L_{p}^{KZ}) = 0, L_{p}^{KZ}[w](0) = 0\) for \(w\) a word of the form \(ue_{1}\), and \(L_{p}^{KZ}(z)[e_{0}] = \log_{p}(z)\).\(^{21}\) The \(p\)-adic multiple polylogarithms (\(p\)MPL’s) are the Coleman functions \(L_{p}^{KZ}[w]\) where \(w\) is a word on \(\{e_{0},e_{1}\}\).

This definition can be rewritten in terms of Frobenius-invariant paths on the \(\pi_{1}^{\text{et}}\) (Fu2, Theorem 2.3). By this definition, the power series expansion of \(L_{p}^{KZ}\) at 0 is also expressed in terms of multiple harmonic sums, as in Proposition 2.14 :

**Proposition 2.25.** (Furusho, [Fu1], Lemma 2.7 and Remark 2.10 (1)) For \(|z|_{p} < 1\), for \(n_{d}, \ldots, n_{1} \in \mathbb{N}^{*}\), the right-hand side below is absolutely convergent and we have :

\[
L_{p}^{KZ}(z)[e_{0}^{n_{d}-1}e_{1} \ldots e_{0}^{n_{1}-1}e_{1}] = \sum_{0 < m_{1} \cdots < m_{d}} \frac{z^{m_{d}}}{m_{1}^{n_{1}} \cdots m_{d}^{n_{d}}} \tag{8}
\]

The topological field \(\mathbb{C}_{p}\) is totally disconnected and the disks \(|z|_{p} < 1\) and \(|z-1|_{p} < 1\) are disjoint. It does not make sense to take the limit \(z \to 1\) in equation (8), and we see why an immediate computation of \(p\)-adic multiple zeta values from (8) is prevented. This contrasts with the complex setting where the limit \(z \to 1\) of the analogous equation (6) gives the formula (11) for multiple zeta values as sums of series.

The fact that the Frobenius commutes to the connection (§1.2.2) is equivalent to the following differential equation, which is sometimes referred to as the equation of horizontality of the Frobenius.

**Proposition 2.26.** (for \(\alpha = 1\), [De2], §19.6 ; [Fu2], Theorem 2.14 ; for any \(\alpha\), [J1]) We have the following equivalent equations :

\[
\phi^{\alpha}(L_{p}^{KZ}) = L_{p}^{KZ} \tag{9}
\]

\[
L_{p,\alpha}(z)(e_{0}, e_{1}), \ L_{p}^{KZ}(z^{p^{n}})(e_{0}, \Phi_{p,\alpha}^{-1}e_{1}, \Phi_{p,\alpha}) = L_{p}^{KZ}(z)(p^{n}e_{0}, p^{n}e_{1}) \tag{10}
\]

\[
dL_{p,\alpha}^{\dagger} = (p^{n}\omega_{0}(z)e_{0} + \omega_{1}(z)e_{1})L_{p,\alpha}^{\dagger} - L_{p,\alpha}^{\dagger}(\omega_{0}(z^{p^{n}})e_{0} + \omega_{1}(z^{p^{n}})\Phi_{p,\alpha}^{-1}e_{1}\Phi_{p,\alpha}) \tag{11}
\]

The formal properties of Frobenius imply finally :

**Proposition 2.27.** ([U2], §5.2) We have, with \(e_{0} + e_{1} + e_{\infty} = 0\),

\[
e_{0} + \Phi_{p,\alpha}^{-1}e_{1}\Phi_{p,\alpha} + L_{p,\alpha}^{\dagger}(\infty)^{-1}e_{\infty}L_{p,\alpha}^{\dagger}(\infty) = 0 \tag{12}
\]

\(^{20}\)which of course does not hold for algebras of locally analytic functions because the \(p\)-adic topology is totally disconnected.

\(^{21}\)Thus \(L_{p}^{KZ}\) is relative to the choice of a determination of \(\log_{p}\) (we also have \(L_{p}^{KZ}(z) = -\log_{p}(1-z)\)). However, \(p\)-adic multiple zeta values and overconvergent \(p\)-adic multiple polylogarithms do not depend on such a choice. The statements reviewed in this text are true for any branch of \(\log_{p}\).
Since $\mathrm{Li}_{p,\alpha}^+[w] \in A^+(U^\infty)$, it is possible to express $\mathrm{Li}_{p,\alpha}^+[w](\infty)$ in terms of the power series expansion of $\mathrm{Li}_{p,\alpha}^+[w]$ at 0. Through equation (12), this can lead to a computation of $\zeta_{p,\alpha}$. This is the strategy initiated very briefly in [De2], §19.6 in depth one, and followed in depth one and two in [U2].

2.4.3 Known results on the algebraic theory of $p$-adic multiple zeta values

The application of the Tannakian motivic framework to $p$-adic multiple zeta values, evoked in §1.4.1, follows from work of Yamashita (unpublished, but reviewed and used in [Yam]) who constructed a crystalline realization functor of the categories of mixed Tate motives (which was asked in [DeGo], §5.28). A byproduct of this construction is a morphism from the $\mathbb{Q}$-algebra of Goncharov’s motivic multiple zeta values defined in [Go1] to the one of $p$-adic multiple zeta values $\zeta_{KZ}^p(w)$, which factors through the ring of $p$-adic periods $B$ evoked in §1.4.1.

It is known that $p$-adic multiple zeta values satisfy the regularized double shuffle relations [BesF], [FuJ], and that they satisfy the associator relations [U1]. Thus, (by [AET], [AT], [Sch]) they also satisfy Kashiwara-Vergne relations.

3 Principles of the study and notations

We explain a few principles (§3.1) and facts, definitions and notations (§3.2, §3.3), about multiple harmonic sums and $\pi_{1,n}(P^1 - \{0,1,\infty\})$, on which our work will rely.

3.1 Three frameworks of computations : $\int_{0}^{1}$, $\int_{0}^{z<<1}$ and $\Sigma$ ; and the comparison between them

3.1.1 Two types of computations associated to relations between multiple polylogarithms

The Conjecture 1.16 on finite multiple zeta values leads to ask why, and how, certain sequences of multiple harmonic sums - perhaps not only the ones defining finite multiple zeta values - should be equivalent to certain periods. According to the spirit of Grothendieck’s period conjecture, we must ask ourselves how multiple harmonic sums are "related to geometry". A basic answer is given by §2.3.3 : multiple harmonic sums are essentially the coefficients of the power series expansions of multiple polylogarithms at 0, both in the complex case (Proposition 2.14) and in the $p$-adic case (Proposition 2.25), the connection $\nabla_{KZ}$ being of course the same in the two frameworks.

Most of the algebraic relations of multiple zeta values, which are conjecturally reflected in finite multiple zeta values by Conjecture 1.16, are actually consequences of equations satisfied by multiple polylogarithms : this is for example the case for double shuffle relations, and also for associator relations. Thus, translating these equations on the coefficients of the power series expansions of multiple polylogarithms at 0 should give certain equations
on multiple harmonic sums, and it would not be surprising that they explain at least partially the analogy between real and finite multiple zeta values.

**Principle 3.1.** With any equation of multiple polylogarithms, one can associate two kinds of computations, and they may not necessarily be trivially equivalent to each other

i) one by taking limits at tangential base-points (in our case, \( z \to 1 \)).

ii) one by considering coefficients of the series expansions at the origin of paths of integration (in our case, at 0).

This simple principle combined to ideas of \( p \)-adic analysis will actually guide us both in the parts I and II.

### 3.1.2 Computations on multiple harmonic sums without any reference to \( \pi_1^{un} \) and comparison

Multiple harmonic sums are a computational bridge between complex and \( p \)-adic multiple zeta values. Any result of an explicit computation of \( p \)-adic multiple zeta values based on the differential equation satisfied by the Frobenius (Proposition 2.26), should be necessarily expressed in terms of multiple harmonic sums, as it follows from the discussion in §2.4.2. This motivates to study multiple harmonic sums intrinsically, and not only with the perspective of finding algebraic equations evoked above. They make sense by themselves, independently from the \( \pi_1^{un} \); given a property of multiple harmonic sums arising from the \( \pi_1^{un} \), is there a natural way to retrieve it without making any reference to the \( \pi_1^{un} \)? And vice-versa?

We will keep these questions in mind, since it is regularly fruitful to compute an object in two different ways and to compare the results of the two computations, as soon as the comparison is not immediate. This general principle, of comparing two different incarnations of a mathematical object, is also at the center of the notion of periods.

### 3.1.3 \( f_0^1, f_0^{<<1} \) and \( \Sigma \)

The computations will lead us to define several objects, and we will add to the notations symbols reflecting the type of computation from which they are originated. We will add the superscript \( \Sigma \) for referring to multiple harmonic sums viewed as iterated sums and \( f \) for referring to iterated integrals. It follows from the discussions of §3.1.1 and §3.1.2 that we actually have three different frameworks of computations :

**Convention 3.2.** In the next parts, we will add to the notations for the new objects :

- the symbol \( f_0^1 \) for objects arising from computations on iterated integrals at the tangential base-points \((\vec{1}_0, -\vec{1}_1)\)

- the symbol \( f_0^{<<1} \) for objects arising from the computations involving power series expansion of multiple polylogarithms at 0, and thus multiple harmonic sums viewed in this way via equation (8)
the symbol $\Sigma$ for objects arising from the computations on multiple harmonic sums viewed as elementary finite iterated sums made without any reference to $\pi_1^m(P^1 - \{0, 1, \infty\})$.

We will compare the respective outcomes of these three types of computations to each other when it is possible.

3.2 Framework for weighted multiple harmonic sums

Convention 3.3. We will add the subscript $\text{har}$ to the notations for objects referring to multiple harmonic sums.

3.2.1 The variant $Q_p(\langle e_0, e_1 \rangle)_{\text{har}}$ of $Q_p(\langle e_0, e_1 \rangle)$; non-commutative generating series of multiple harmonic sums

Equation (7) indicates that the correspondence $(n_d, \ldots, n_1) \leftrightarrow e_0^{n_d-1}e_1 \ldots e_0^{n_1-1}e_1$, valid for the indices of multiple zeta values and multiple polylogarithms, must be replaced for any ring $\mathbb{F}$ by combining it with the fact that we have $Li_{p}^{KZ}(z)[e_0^{n_d-1}e_1 \ldots e_0^{n_1-1}e_1] \equiv \prod_{r_0+1 \leq \cdots \leq r_1}^{d} \left( -\frac{l_j}{r_d+1} \right) \prod_{i=1}^{d} \left( -\frac{n_i}{r_i} \right) \text{har}_m \left( n_d + r_d, \ldots, n_1 + r_1 \right) \mod \log_p(z)$ modulo $\log_p(z)$, which follows from the shuffle equation for $Li_{p}^{KZ}(z)$.

By further generalization of this equation we get

$$
\text{har}_m(n_d, \ldots, n_1; r) = \sum_{r_d+1 \leq \cdots \leq r_1}^{d} \prod_{i=1}^{d} \left( -\frac{n_i}{r_i} \right) \text{har}_m \left( n_d + r_d, \ldots, n_1 + r_1 \right)
$$

In the next definition, the i) will be motivated by §4.3 (part I-3):

Definition 3.4. For any ring $A$,

i) Let $A(\langle e_0, e_1 \rangle)_{\text{har}}^I = \{ f \in A(\langle e_0, e_1 \rangle) : Q_p(\langle e_0, e_1 \rangle)_{\text{har}} \}

ii) Let $A(\langle e_0, e_1 \rangle)_{\text{har}}^C = \{ f \in A(\langle e_0, e_1 \rangle) : Q_p(\langle e_0, e_1 \rangle)_{\text{har}} \}

iii) Let $A(\langle e_0, e_1 \rangle)_{\text{har}}^\Sigma = \prod_{r_0+1 \leq \cdots \leq r_1}^{d} A(\langle n_d, \ldots, n_1; r \rangle) \mod \log_p(z)$

These three modules are obviously isomorphic to each other, and we will use the notation $A(\langle e_0, e_1 \rangle)_{\text{har}}^I$ to refer to all of them at the same time.

A function $w \mapsto \text{har}_m(w)$ can thus be seen as an element of $Q_p(\langle e_0, e_1 \rangle)_{\text{har}}$; for most purposes, it is however sufficient replace it by its image, which is in $Q_p(\langle e_0, e_1 \rangle)_{\text{har}}$, by the weight-adically continuous linear map $Q_p(\langle e_0, e_1 \rangle)_{\text{har}} \rightarrow Q_p(\langle e_0, e_1 \rangle)_{\text{har}}$ which sends all words of the form $ue_1$ to themselves and all other words to 0. We adopt this point of view in the rest of this text.

\[22\]
Convention-Notation 3.5. Sequences of weighted multiple harmonic sums are viewed and denoted in the following way:

i) For any word $w$, and for any $I \subset \mathbb{N}$, we denote by $\text{har}_I(w) = (\text{har}_n(w))_{n \in I}$.

ii) For any $n \in \mathbb{N}^*$, $\text{har}_n$ is the sequence $(\text{har}_n(w))_{w \text{ word}}$ viewed in the sense of the discussion above as an element of $Q_p(\langle \langle e_0, e_1 \rangle \rangle)_{\text{har}}$.

iii) For any $I \subset \mathbb{N}$, we denote by $\text{har}_I$ the sequence $(\text{har}_n(w))_{n \in I, w \text{ word}}$, and view it as an element of $\text{Map}(I, Q_p(\langle \langle e_0, e_1 \rangle \rangle)_{\text{har}})$.

3.2.2 The first term of the $p$-adic expansion of prime weighted multiple harmonic sums

The prime weighted multiple harmonic sums (Definition 2.15) will play a special role. The lower bound on their valuations below will be used constantly.

Fact 3.6. Let $p$ a prime number, $\alpha \in \mathbb{N}^*$, and a word $w = (n_d, \ldots, n_1)$.

We have $v_p(\text{har}_p(\alpha)(w)) \geq \text{weight}(w)$ and the reduction of $p^{-\text{weight}(w)} \text{har}_p(\alpha)(w)$ modulo $p$ is independent of $\alpha$. In particular, the image of $(p^{-\text{weight}(w)} \text{har}_p(\alpha)(w))_{p \in \mathbb{P}} \in \prod_{p \in \mathbb{P}} \mathbb{Z}_p$ in $\mathbb{Z}_p$ by reduction modulo infinitely large primes is the finite multiple zeta value $\zeta_{\mathbb{Z}_p}(w)$.

By this remark, all weight-adically convergent sums of $\mathbb{Q}$-linear combinations of sequences $\text{har}_I(w)$ of prime weighted multiple harmonic sums ($I \subset \mathbb{P}^{\mathbb{N}^*}$) with reasonable rational coefficients are also $p$-adically convergent. Such objects will be omnipresent in this work.

3.3 Topologies on $\pi_{1,0}^{\text{un,DR}}(\mathbb{P}^1 - \{0, 1, \infty\}, -\overline{1}, \overline{0})(\mathbb{Q}_p)$

For any $n, d \in \mathbb{N}$, let $W(e_0, e_1)$, resp. $W_n(e_0, e_1)$, $W_{n,d}(e_0, e_1)$, $W_{n,d}(e_0, e_1)$ be the set of words over $\{e_0, e_1\}$, resp. words of depth $d$, of weight $n$, of weight $n$ and depth $d$. We can view $Q_p(\langle \langle e_0, e_1 \rangle \rangle)$ as the set $\text{Map}(W(e_0, e_1), Q_p)$. Let $\Lambda$ and $D$ be formal variables.

Definition 3.7.

i) For $f \in Q_p(\langle \langle e_0, e_1 \rangle \rangle)$, let $\mathcal{N}_{\Lambda,D}(f) = \sum_{s, d \geq 0} \sup_{w \in W_{n,d}(e_0, e_1)} |f[w]|_p \Lambda^s D^d \in \mathbb{R}_+[\Lambda, D]$.

ii) For $f \in Q_p(\langle \langle e_0, e_1 \rangle \rangle)$ whose restriction to each $W_{n,d}(e_0, e_1)$ is bounded, let $\mathcal{N}_D(f) = \sum_{d \geq 0} \sup_{w \in W_{n,d}(e_0, e_1)} |f[w]|_p D^d \in \mathbb{R}_+[D]$.

One can associate with $\mathcal{N}_{\Lambda,D}$, resp. $\mathcal{N}_D$ the topology on $\text{Map}(W(e_0, e_1), Q_p)$ defined by the simple convergence, resp. uniform convergence on each $W_{n,d}(e_0, e_1)$. This also induces topologies on $\Pi_{1,0}(Q_p)$. We will see that these definitions are compatible with most usual operations on $\Pi_{1,0}(Q_p)$.

\[ \text{Indeed, for any } \alpha, \text{ the iterated sum } \text{har}_p(\alpha)(n_d, \ldots, n_1) = (p^\alpha)^{n_d + \ldots + n_1} \sum_{0 < m_1 < \ldots < m_d < p^\alpha} \frac{1}{m_1 \ldots m_d} \text{ is the sum of the terms } (m_1, \ldots, m_d) \in (p^\alpha - 1\mathbb{N})^d, \text{ whose contribution is equal to } \text{har}_p(n_d, \ldots, n_1) \in p^{\alpha + \ldots + n_1}\mathbb{Z}_p, \text{ and the other terms, whose contribution is in } p^{\alpha + \ldots + n_1 + 1}\mathbb{Z}_p. \]
Let $p$ be a prime number. In this part, we compute the Frobenius of $\pi_{1}^{un,DR}((\mathbb{P}^1 - \{0,1,\infty\})/\mathbb{Q}_p)$ reviewed in §2.4. The three sections I-1, I-2, I-3 give three different computations, but depend on each other; I-2 relies on I-1 and I-3 relies on I-2.

4.1 (I-1) Direct solution to the equation of horizontality of the Frobenius $[J1]$

This first step is inspired from Ünver’s work in depth one and two $[U2]$, who has also generalized it to $\mathbb{P}^1 - \{0,\mu_N,\infty\}$, $p \nmid N$ in $[U3]$: we will solve the system of equations on $(\mathbb{L}^{\dagger}_p,\alpha,\zeta_p,\alpha)_{(p,\alpha)}$ formed by (11) and (12), initially by induction on the weight (in fine by induction on the depth). Because of the motivation coming from finite multiple zeta values ($§1.4.2$), we want moreover to express the result in terms of prime multiple harmonic sums.

4.1.1 A space of convergent series defined via prime weighted multiple harmonic sums

The main result will be expressed in terms of the following object.

Definition 4.1. i) Let

$$\widehat{\text{Har}}_{p^{n},\alpha} = \text{Vect}_Q\left\{ \left( \sum_{L \in \mathbb{N}} F_L \prod_{\eta = 1}^{\eta_0} \text{har}_{p^{n}}(w_{L,\eta})) \right)_{(p,\alpha)} \mid (*) \right\} \subset \prod_{(p,\alpha) \in \mathbb{P} \times \mathbb{N}^{*}} \mathbb{Q}_p$$

where $(\ast)$ means that $(w_{L,1})_{L \in \mathbb{N}}, \ldots, (w_{L,\eta_0})_{L \in \mathbb{N}}$ are sequences of words on $\{e_0, e_1\}$ satisfying $\sum_{\eta = 1}^{\eta_0} \text{weight}(w_{L,\eta}) \rightarrow l \rightarrow \infty \infty$ and $\lim \sup_{l \rightarrow \infty} \sum_{\eta = 1}^{\eta_0} \text{depth}(w_{L,\eta}) < \infty$, and $(F_L)_{L \in \mathbb{N}}$ is a sequence of rational numbers (independent of $(p,\alpha)$), such that the series is convergent (see $§3.2.2$).

ii) For all $(n,d) \in (\mathbb{N}^{*})^2$, let

$$\widehat{\text{Har}}_{p^{n},d} \subset \text{Har}_{p^{n},\alpha}$$

be the subspace generated by the elements as above such that, for all $L \in \mathbb{N}$, $\sum_{\eta = 1}^{\eta_0} \text{weight}(w_{L,\eta}) \geq n$ and $\sum_{\eta = 1}^{\eta_0} \text{depth}(w_{L,\eta}) \leq d$. We have $\widehat{\text{Har}}_{p^{n},d} = \bigcup_{(n,d) \in (\mathbb{N}^{*})^2} \text{Har}_{p^{n},d}$.

4.1.2 Main result

The main result of this part is the following:

**Theorem 1.** $p$-adic multiple zeta values at a word $w$ of weight $n$ and depth $d$, viewed as sequences $(\zeta_{p,\alpha}(w))_{(p,\alpha) \in \mathbb{P} \times \mathbb{N}^{*}}$, are elements of $\widehat{\text{Har}}_{p^{n},d}$.

\footnote{With a difference of convention: in those papers, the Frobenius is $\phi^{-1}$ whereas here the Frobenius is $\tau(p^\alpha) \circ \phi^\alpha$ for all $\alpha \in \mathbb{N}^*$}
Furthermore (see §6.2 of [J1] for a precise statement):

i) There is an explicit formula, inductive with respect to the depth, for \((\Li_{p,\alpha}^\dagger, \zeta_{p,\alpha})\) where the maps \(\Li_{p,\alpha}^\dagger(w)\) are viewed as the maps \(m \in \mathbb{N}^* \mapsto \Li_{p,\alpha}^\dagger[w][z^m] \in \mathcal{Q}_p\), \(^{26}\) themselves viewed as elements of a specific subalgebra (defined in terms of multiple harmonic sums) of the one of locally analytic functions \(\mathcal{Z}_p \to \mathcal{Q}_p\).

ii) There are explicit lower bounds for the \(p\)-adic valuations of the numbers \(\Li_{p,\alpha}^\dagger[w][z^m]\), and of \(p\)-adic multiple zeta values \(\zeta_{p,\alpha}(w)\), which depend only on \((p, \text{weight}(w), \text{depth}(w))\).

### 4.1.3 Strategy of proof and example of explicit formula

1) In order to solve the system of equations (11), (12), we must first express \(\Li_{p,\alpha}^\dagger(\infty)\) in terms of the coefficients \(\Li_{p,\alpha}^\dagger[z^m]\). A way to do it is to remark that \(U_{\mathbb{a}n} = \mathbb{P}^{1,\mathbb{a}n} - \{|z|_p < 1\}\) is the image by \(z \mapsto \frac{z}{1 - z}\) (the homography of \(\mathbb{P}^1\) sending \((0,1,\infty)\) to \((0,\infty,1)\)) of \(Z_{\mathbb{a}n} = \mathbb{P}^{1,\mathbb{a}n} - \{|z|_p < 1\};\) by definition, the algebra of rigid analytic functions on \(Z_{\mathbb{a}n}\) is \(A_{\rig}(Z_{\mathbb{a}n}) = \{\sum_{m=0}^{\infty} a_m z^m \mid |a_m|_p \to \infty\}\). Then, the characterization of the functions \(\mathbb{N} \to \mathcal{Q}_p\) which extend to continuous functions \(\mathcal{Z}_p \to \mathcal{Q}_p\) in terms of their Mahler coefficients \([\text{Mah}]\) can be reformulated as an explicit description of \(A_{\rig}(\mathbb{P}^{1,\mathbb{a}n} - [1])\), which implies that the maps \(m \in \mathbb{N}^* \subset \mathbb{Z}_p \mapsto \Li_{p,\alpha}^\dagger[w][z^m]\) are continuously interpolable on \(\mathcal{Z}_p\), and the formula \(\Li_{p,\alpha}^\dagger(\infty) = -\lim_{|m|_p \to 0} \Li_{p,\alpha}^\dagger(\infty)[z^m]\) (see Proposition 2 of [U2] for another version of this result, proved differently.)

2) We must also check that equation (12) gives an expression of \(\Phi_{p,\alpha}\) and \(\Li_{p,\alpha}^\dagger(\infty)\) in terms of each other which is compatible with the depth filtration. This extends some results of [U2], §5 in low depth. We prove that we can express \(\Phi_{p,\alpha}\) in depth \(\leq d\) and \(\Li_{p,\alpha}^\dagger(\infty)\) in depth \(\leq d + 1\) in terms of each other by polynomial expressions. These results do not alterate lower bounds of valuations since these polynomials have coefficients in \(\mathbb{Z}\).

3) Combining the two previous steps gives a solution to (11), (12); however, the recursive expression of \((\Li_{p,\alpha}^\dagger, \zeta_{p,\alpha})\) obtained in this way is expressed in terms of certain limits of \(\mathcal{Q}_p\)-linear combinations of finite iterated sums, and this is not enlightening enough. In order to make any progress, we must understand via the formulas why the maps \(m \in \mathbb{N}^* \mapsto \Li_{p,\alpha}^\dagger[w][z^m]\) are continuously interpolable on \(\mathcal{Z}_p\). For \(w\) of depth 1\(^{27}\) and for \(w = e_1 e_0^{n-1} e_1\), this is easy, and it gives back the following essentially known result, \(^{28}\)

**Example 4.2.** For all \(n \in \mathbb{N}^*\), we have \(\zeta_{p,\alpha}(n) = \frac{1}{n-1} \sum_{l \in \mathbb{N}} \binom{n}{l} B_l \har_{p,\alpha}(n + l - 1)\)

which also leads to a proof for other words \(w\) of depth 2, using a power series expansion of \(m \mapsto \Li_{p,\alpha}^\dagger[w][z^m]\) when \(|m|_p \to 0\). In [U2], §5.15, it is explained (for \(\alpha = 1\)) that each integration operator \(f \mapsto \int f \omega, \) with \(\omega \in \{\frac{d\alpha}{z}, \frac{\mu^\alpha}{z - 1}, \frac{d\alpha^\mu}{z - 1}\}\), appearing in the resolution of (11) in low depth, can be replaced by a regularized variant; i.e the linear combinations

\(^{26}\)where \([z^m]\) means the coefficient of degree \(m\) in the power series expansion at 0

\(^{27}\)in depth one we have: \(\Li_{p,\alpha}^\dagger[e_0^{n-1} e_1](z) = (p^n)^n \sum_{m \leq \frac{n}{\rho_1 m}} \frac{\alpha_m}{m!}\)

\(^{28}\)see [Co], equation (4), and [W]

25
of iterated integrals of these forms can be replaced by the same linear combinations of
regularized variants, in such a way that the evaluation at ∞ makes sense for each term of
the linear combination. Then, in [U2], §5.16–§5.19, for certain w of depth 2 and 3 (which
leads to compute certain c_{p,α} in depth 2), it is proven that certain terms of the maps
m ∈ pN^* ⊂ Z_p ↦ Li_{p,α}^[[w]][z^m], are analytic.29

What we prove, by induction, is that the maps
m ∈ N^∗ ↦ Li_{p,α}^[[w]][z^m], are analytic.

4) It remains to lift each step of the inductive process above to an explicit map: these
steps are the dualizations (expressed in terms of shuffle Hopf algebras) of equations [11]
and [12], and the computation of regularized p-adic iterated integrals associated with a
word in the differential forms \( \frac{p^αdz}{z} \), \( \frac{p^αdz}{z-1} \), and \( \frac{d(p^α)}{p^α-1} \). A fourth step is the use of the formula
for \( Li_{p,α}^[[∞]] \) in terms of \( Li_{p,α}^[[z^m]] \), but it adds no complexity. We can also upgrade the
induction on the weight into an induction on the depth, which is much more efficient
algorithmically.

4.2 (I-2) Indirect solution to the equation of horizontality of the Frobenius [J2]

We are going to see that there exists a way to compute the Frobenius which is entirely
different from I-1 but relies on I-1. Let us try to follow the ideas of §3, and to find an
application of the Principle [3.1] to the differential equation of Frobenius.

4.2.1 The Frobenius viewed as an operation on multiple harmonic sums

We are going to view the Frobenius \( τ(p^α)φ^α \), as well as the map \( z ↦ z^p^α \), lift to \( U^\text{an} \) of the
Frobenius of \( X_{F_p} \), as transformations of multiple harmonic sums, via their action on \( Li_{p}^{KZ} \)
and via the expression of multiple harmonic sums in terms of the power series expansions
of \( Li_{p}^{KZ} \) of equation [8].

The map \( \mathcal{F} : z ↦ z^p^α \) induces the map on power series \( \sum_{m∈N} c_m z^m ↦ \sum_{m∈N} c_m z^{p^α m} \), i.e.
the map on sequences of coefficients \( (c_m)_{m∈N} ↦ (1_p^{α m} c_m)_{m∈N} \). Let us consider its right
inverse \( (c_m)_{m∈N} ↦ (c_{p^α m})_{m∈N} \); let us say that \( \mathcal{F} \) "sends har_m to har_{p^α m}".

The invariance of \( Li_{p}^{KZ} \) by the Frobenius \( τ(p^α)φ^α \) is expressed by equation [10]; let us take, in that equation, the coefficient of the series expansion at 0 of degree \( p^α m \), with
\( m ∈ N^* \), then, the coefficient a word \( e_0^{l-1} e_1 e_0^{n-1} e_1 \ldots e_0^{n-1} e_1 \) ; and, finally, let us apply

\[ 29 \text{by an indirect method using an equation satisfied by the logarithm of the p-adic Gamma function;} \]
\[ \text{the formulas of the type of Example [4.2] infinite sums of prime weighted multiple harmonic sums, do not appear in [U2, U3]} \]
We define a condition of summability in 4.2.2 Main result. We show that equation (14) can be simplified a lot if we consider the limit of the right-hand side, which is actually a constant function of $\tau$. Let us say that equation (14) can be simplified a lot if we consider the limit of the right-hand side when $l \to \infty$.

### 4.2.2 Main result

We define a condition of summability in $Q_p\langle\langle e_0, e_1 \rangle\rangle$:

**Definition 4.3.**

For any $A \subset Q_p\langle\langle e_0, e_1 \rangle\rangle$, let $A_S = \{ f \in A \mid \text{for all } d \in \mathbb{N}^*, \sup_{w \in W_n,d} |f[w]|_p \to n \to \infty 0 \}$. Let also $\tilde{\Pi}_{1,0}(Q_p)_S \subset \Pi_{1,0}(Q_p)_S$ defined by the elements $f$ such that $f[e_0] = f[e_1] = 0$.

We can show that $\tilde{\Pi}_{1,0}(Q_p)_S$ is a subgroup of $\Pi_{1,0}(Q_p)$ both for the usual multiplication and $\circ_{\text{ad}}^l$, and that it is closed for both topologies of $\mathfrak{S}_3$. We equip it, as well as $Q_p\langle\langle e_0, e_1 \rangle\rangle_{\text{har}}$, with the $\mathcal{N}_{\text{p}}$-topology. The topological group $(\tilde{\Pi}_{1,0}(Q_p)_S, \circ_{\text{ad}}^l)$ is isomorphic via the map $\text{Ad}(e_1)$ to the topological group $(\text{Ad}_{\tilde{\Pi}_{1,0}(Q_p)_S}(e_1), \circ_{\text{ad}}^l)$ where $\circ_{\text{ad}}^l$ is defined as $g \circ_{\text{ad}}^l f = f(e_0, g)$.

**Theorem 2**

i) (framework $f_{0}^{<<1}$) There exists an explicit free continuous group action of $(\text{Ad}_{\tilde{\Pi}_{1,0}(Q_p)_S}(e_1), \circ_{\text{ad}}^l)$, the $p$-adic pro-unipotent $f_{0}^{<<1}$-harmonic action of $\mathbb{P}^1 - \{0, 1, \infty\}$

$$
\circ_{\text{har}}^{f_{0}^{<<1}} : \text{Ad}_{\tilde{\Pi}_{1,0}(Q_p)_S}(e_1) \times \text{Map}(\mathbb{N}, Q_p\langle\langle e_0, e_1 \rangle\rangle_{\text{har}}^{f_{0}^{<<1}}) \to \text{Map}(\mathbb{N}, Q_p\langle\langle e_0, e_1 \rangle\rangle_{\text{har}}^{f_{0}^{<<1}})
$$

such that we have

$$
\text{har}_{p^\alpha \mathbb{N}} = \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha} \circ_{\text{har}}^{f_{0}^{<<1}} \text{har}_{\mathbb{N}}
$$

(15)

ii) (framework $\Sigma$) There exists an explicit continuous map, the $p$-adic pro-unipotent $\Sigma$-harmonic action of $\mathbb{P}^1 - \{0, 1, \infty\}$

$$
\circ_{\text{har}}^{\Sigma} : (Q_p\langle\langle e_0, e_1 \rangle\rangle_{\text{har}}^{\Sigma})_S \times \text{Map}(\mathbb{N}, Q_p\langle\langle e_0, e_1 \rangle\rangle_{\text{har}}^{\Sigma}) \to \text{Map}(\mathbb{N}, Q_p\langle\langle e_0, e_1 \rangle\rangle_{\text{har}}^{\Sigma})
$$

such that we have

$$
\text{har}_{p^\alpha \mathbb{N}} = \text{har}_{p^\alpha} \circ_{\text{har}}^{\Sigma} \text{har}_{\mathbb{N}}
$$

(16)
iii) (comparison) By relating to each other the two formulas above, we get the definition of an explicit map \( \text{comp}^{f \leftarrow \Sigma} \) satisfying

\[
\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha} = \text{comp}^{f \leftarrow \Sigma} \text{har}_{p^\alpha}
\]  

(17)

We call \( f_0^{z<<1} \)-harmonic Frobenius the map \( (\phi^\alpha)^{f_0^{z<<1}}_{\text{har}} : h \mapsto \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha} \circ_{\text{har}}^{f_0^{z<<1}} h \) and \( \Sigma \)-harmonic Frobenius the map \( (\phi^\alpha)_{\text{har}}^{\Sigma} : h \mapsto \text{har}_{p^\alpha} \circ_{\text{har}}^{\Sigma} h \).

We note that, by contrast with the formulas of I-1 which involves the five objects \( \Phi_{p,\alpha}, \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha}, \text{Li}_{p,\alpha}(\infty), \text{Li}_{p,\alpha}^{-1}(e_0 + e_1) \text{Li}_{p,\alpha}^{i}(\infty), m \mapsto \text{Li}_{p,\alpha}[z^m] \), and relations between them, the formula (15) above involves only \( \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha} \), and similarly, the formula (16) above involves only \( \text{har}_{p^\alpha} \). Moreover, by contrast with the formulas of I-1 which are recursive, the three formulas (15), (16), (17) can be written without recursion and in a few lines (see [J2]). Instead of solving a system of equations, as we did in I-1, here, we only write two different incarnations of the Frobenius and compare them to each other. The proof will however rely on I-1.

4.2.3 The \( m = 1 \) case of the main result

Let us look at the \( m = 1 \) term of (15) ; since \( \text{har}_{1}(w) = 0 \) for all non-empty words \( w \), almost all the terms of the equation vanish and the explicit formula for \( \circ_{\text{har}}^{f_0^{z<<1}} \) gives directly :

Corollary 2

For all \((n_d, \ldots, n_1)\), we have :

\[
\text{har}_{p^\alpha}(n_d, \ldots, n_1) = (\Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha}) \left[ \frac{1}{1 - e_0} e_1 e_0^{n_d - 1} e_1 \ldots e_0^{n_1 - 1} e_1 \right] = \sum_{d'=0}^{d} \sum_{l_{d'+1}, \ldots, l_{d} \in \mathbb{N}} \prod_{i=d'}^{d} \left( -n_i \right) \zeta_{p,\alpha}(n_{d'+1} + l_{d'+1}, \ldots, n_d + l_d) \zeta_{p,\alpha}(n_{d'}, \ldots, n_1)
\]  

(18)

This is an inversion of the expansion of \( p \)-adic multiple zeta values in terms of prime weighted multiple harmonic sums arising from Theorems 1 and 2. The \( \alpha = 1 \) case of (18) solves a conjecture of Akagi, Hirose and Yasuda.\footnote{In depth one and \( \alpha = 1 \), equation (18) is equivalent to a result on Washington on the values of the Kubota-Leopoldt zeta function [W] via a result of Coleman [Co]. In depth two and for \( \alpha = 1 \), equation (18) had been proved by M. Hirose, a priori by a different method.} One can lift the formula of (18) to the definition of a map \( \text{comp}^{\Sigma \leftarrow f} \), which reformulates (18) as :

\[
\text{har}_{p^\alpha} = \text{comp}^{\Sigma \leftarrow f} \left( \Phi_{p,\alpha}^{-1} e_1 \Phi_{p,\alpha} \right)
\]  

(19)

This map satisfies the following property, which re-implies equation (18) through equation (17) :

\[
\text{comp}^{\Sigma \leftarrow f} \circ \text{comp}^{f \leftarrow \Sigma} = \text{id}
\]  

(20)
This last statement (20) is used in the proof of iii) of the Theorem 2, and is obtained as a "lift" of the proof of (16) restricted to \( m = 1 \).

### 4.2.4 Application to finite multiple zeta values according to Akagi, Hirose and Yasuda

We now review, following [AkHYas], some consequences on finite multiple zeta values of Corollary 2 brought together with other results; they answer to the question of §1.4.3. The following integrality property of Furusho’s version of \( p \)-adic multiple zeta values holds: for all words \( w \), we have \( \zeta^{KZ}_{p}(w) \in \sum n \geq \text{weight}(w) p^n n! \mathbb{Z}_p \); this is proved in [AkHYas], by the logarithmic generalization [O2] of Mazur’s theorem on crystalline cohomology [Maz], and in [Cha]. It implies easily the same result for \( \zeta_{p,1} \). These integrality of \( \zeta_{p,1} \) and our equation (18) brought together imply, for all \( p > n_d + \ldots + n_1 \), that below the right-hand side is in \( p^{n_d + \ldots + n_1} \mathbb{Z}_p \) and:

\[
\text{har}_{p}(n_d, \ldots, n_1) \equiv \sum_{d' = 0}^{d} (-1)^{n_{d'+1} + \ldots + n_{d_0}} \zeta_{p}(n_{d'+1}, \ldots, n_d) \zeta_{p}(n_d, \ldots, n_1) \mod p^{n_d + \ldots + n_1 + 1} \mathbb{Z}_p
\]

whence a formula for \( \zeta_{\mathbb{Z}/p^{\infty}}(n_d, \ldots, n_1) \) in terms of \( p \)-adic multiple zeta values; this formula coincides with the formula in Kaneko-Zagier’s Conjecture 1.16, and thus explains it.

This also enables to recast Kaneko-Zagier’s conjecture as the injectivity of the map of reduction of \( p \)-adic multiple zeta values "modulo infinitely large primes", which it is possible to define by the integrality property, and which is a surjection onto the \( \mathbb{Q} \)-algebra of finite multiple zeta values by (21).

Finally, the dimensions of the spaces of \( p \)-adic multiple zeta values being upper bounded by the dimensions of the motivic Galois group [Yam] - this is a particular case of a central fact of the Galois theory of periods, see §1.1.2 - the same bounds of dimensions for finite multiple zeta values are deduced by reduction modulo infinitely large primes:

\[
\sum_{n=0}^{\infty} \dim(\mathcal{Z}_{n}^{\text{fin}}) \Lambda^{n} \leq \frac{1 - \Lambda^{2}}{1 - \Lambda^{2} - \Lambda^{3}}
\]

where \( \mathcal{Z}_{n}^{\text{fin}} \) is the \( \mathbb{Q} \)-vector space generated by finite multiple zeta values of weight \( n \). This is conjecturally an equality by Kaneko-Zagier’s conjecture. In the end, the Galois theory of periods applies to finite multiple zeta values as if they were as periods, despite that they live in the unusual ring \( \mathbb{Z}_{/p^{\infty}} = \left( \prod_{p \in \mathbb{P}} \mathbb{Z}/p \mathbb{Z} \right)/\left( \bigoplus_{p \in \mathbb{P}} \mathbb{Z}/p \mathbb{Z} \right) \).

### 4.2.5 Strategy of proof of i) and definition of \( \text{comp}_{\text{har}}^{\Sigma_{+1}} \) and \( \text{comp}_{\text{har}}^{\Sigma_{+1}} \)

1) Let us go back to equation (14). Having made the observation of §4.2.1, we prove \( \text{comp}_{\text{har}}^{\Sigma_{+1}} \).
Key Lemma 4.4. ([12], §3) The terms of equation (4) containing a factor $\log_{n}^{\varepsilon}$ tend to zero when $\varepsilon \to \infty$.

This is a consequence of the bounds of valuations of $\log_{n}^{\varepsilon}$ in Theorem 1. By this lemma, taking the limit $\varepsilon \to \infty$ in (4) is a big simplification of (4). By expressing the limit of the other terms, we have a formula for $\log_{m}^{\varepsilon}$ in terms of $\log_{m}$ (instead of $(\log_{m'})_{1 \leq m' \leq m}$) and certain power series of $n$ whose coefficients are made out of $\zeta_{p,\alpha}$ (instead of $(\zeta_{p,\alpha}, \log_{n}^{\varepsilon})$): this is a preliminary version of equation (15):

Example 4.5. In depth one, equation (15) is for all $s \in \mathbb{N}^*$, 

$$\log_{n}^{\varepsilon}(n) = \log_{n}(n) + \sum_{p \mid \varepsilon} m^{b+n}(\Phi_{p,\alpha}^{-1} \varepsilon_{1} \Phi_{p,\alpha})[\varepsilon_{0}^{b} \varepsilon_{1}^{n-1} \varepsilon_{1}]$$

In depth two, equation (15) is for all $n_{1}, n_{2} \in \mathbb{N}^*$,

$$\log_{n}^{\varepsilon}(n_{2}, n_{1}) = \log_{n}(n_{2}, n_{1}) + \sum_{p \mid \varepsilon} m^{b+n_{2}+n_{1}}(\Phi_{p,\alpha}^{-1} \varepsilon_{1} \Phi_{p,\alpha})[\varepsilon_{0}^{b} \varepsilon_{1}^{n_{2}-1} \varepsilon_{1}^{n_{1}-1} \varepsilon_{1}]$$

We also observe that specializing this equation to $n = 1$ gives equation (18).

2) Let $Q_{p}((0,1)) = \lim \{ f \in Q_{p}((0,1)) \mid \text{for all words } w, \lim f[w] = \lim f[e_{1}^{l} w] \}$. We have a map $\log_{n}^{\varepsilon}(n_{2}, n_{1})^{\varepsilon < 1}_{\text{har}}$ defined by, for all words $w$, $(\lim f)[w] = \lim f[e_{1}^{l} w]$. We define $\log_{n}^{\varepsilon}(n_{2}, n_{1})^{\varepsilon < 1}_{\text{har}}$ by the equation

$$g^{\varepsilon < 1}_{\text{har}}(m \mapsto h_{m}) = \left( m \mapsto \lim \left( \text{Ad}_{r}(m)(g_{1}) \circ_{\text{Ad}}^{\varepsilon}_{1} h_{m} \right) \right)$$

We prove that it is well-defined, that it is a group action, that it is continuous, and that it is free. Finally, equation (18) lifts to the definition of a map $\text{comp}^{\varepsilon < 1}_{1}$.}

4.2.6 Strategy of proof of ii) and iii) and definition of $\log_{n}^{\varepsilon}_{\text{har}}$ and $\text{comp}^{\varepsilon < 1}_{1}$

1) We consider a multiple harmonic sum $\log_{n}^{\varepsilon}(n_{d}, \ldots, n_{1})$, and we partitionate the set its indices $\{ 0 < m_{1} < \ldots < m_{d} < p^{m} m \}$ into the $2^{d}$ subsets defined by $d$ conditions of the type $p^{m} m_{i} \mid p^{m} m$, where $i$ runs through $\{1, \ldots, d\}$. For each of these subsets, we write the Euclidean division of each $m_{i}$ by $p^{m}$: $m_{i} = p^{m} u_{i} + r_{i}$ with $r_{i} \in \{1, \ldots, p^{m} - 1\}$; in this second case, we write $m_{i} - n_{i} = r_{i}^{s_{i}} (1 + p^{m} u_{i}^{n_{i}}) = \sum_{l_{i} \geq 0} \left( -n_{i} r_{i}^{s_{i}} \right) \left( p^{m} u_{i}^{n_{i}} \right)$. We then sum over the values of $r_{i} \in \{1, \ldots, p^{m} - 1\}$ and $u_{i}$. The sums over $r_{i}$ give directly prime weighted multiple harmonic sums $\log_{n}^{\varepsilon}$. The sums over $u_{i}$ are of the type $\sum_{0 \leq u_{i} \leq \ldots \leq u_{i} \leq n-1} u_{i}^{l_{i}^{1}} \ldots u_{i}^{l_{i}^{t}}, l_{i} \in \mathbb{N}$. A subsum indexed by an $u_{i}$ with a positive
exponent, i.e. $+l_i$, can be expressed as a polynomial of the upper and lower bounds of its domain of summation, namely, $u_{i_j-1}$ and $u_{i_j+1}$; by iterating this result, we obtain that the sums $\sum_{0 \leq u_1 \leq \ldots \leq u_r \leq m-1} u_1^{\pm l_1} \ldots u_r^{\pm l_r}$ are linear combinations of weighted multiple harmonic sums $\text{har}_m$ with coefficients in the ring of $Q$-polynomial functions on $m$. After inverting an absolutely convergent double sum, we obtain an expression of $\text{har}_{p^m}$ of the type of equation (16):

Example 4.6. In depth one and two equation (16) is, respectively:

$$\text{har}_{p^m}(n) = \text{har}_m(n) + \sum_{b \geq 1} \sum_{l \geq b-1} \binom{-n}{l} B_b^l \text{har}_{p^m}(n + l)$$

(24)

$$\text{har}_{p^m}(n_2, n_1) = \text{har}_m(n_2, n_1) +$$

$$\sum_{l_1, l_2 \geq 0} \sum_{l_1 + l_2 \geq 2} \left[ \frac{(-n_1)}{l_1 + n_2} B_{l_1}^{l_2 + n_2, -n_2} - \frac{(-n_2)}{l + n_1} B_{l_1}^{l_2 + n_1, -n_1} \right] \text{har}_{p^m}(n_1 + n_2 + l) + \sum_{l_1, l_2 \geq 0} \sum_{l_1 + l_2 \geq 2} \left[ \frac{2}{l_1} \binom{-n_1}{l} \right] \text{har}_{p^m}(n_2 + l_2, n_1 + l_1)
$$

$$- m^{n_2 + n_1} \left[ \sum_{l_1 \geq n_2 - 1} B_{l_1}^{l_2 + n_2, -n_2} \text{har}_{p^m}(n_1 + l_1) - \sum_{l_2 \geq n_1 - 1} B_{l_1}^{l_2 + n_1, -n_1} \text{har}_{p^m}(n_2 + l_2) \right] +$$

$$\sum_{1 \leq l < n_2} \sum_{l \geq 2} \text{har}_m(n_2 - l) B_{l_1}^{l_2 + n_2, -n_2} \text{har}_{p^m}(n_1 + l_1) - \sum_{1 \leq l < n_2} \sum_{l \geq 2} \text{har}_m(n_2 - l) B_{l_1}^{l_2 + n_2, -n_2} \text{har}_{p^m}(n_1 + l_1)
$$

(25)

2) Let $Q_p(\langle e_0^{\pm 1}, e_1 \rangle)$ be the set of linear maps $Q(\langle e_0^{\pm 1}, e_1 \rangle) \to Q_p$ where $Q(\langle e_0^{\pm 1}, e_1 \rangle)$ is the localization of the non-commutative ring $Q(\langle e_0, e_1 \rangle)$ equipped with the concatenation product at the multiplicative part generated by $e_0$. We define through it a variant $Q_p(\langle e_0^{\pm 1}, e_1 \rangle)_{\text{har}}$ in the way of §3.2.1 which contains, for each $m \in \mathbb{N}$, the generating sequence $\text{har}_m^{\text{loc}}$ of multiple harmonic sums $\text{har}_m(\pm n_d, \ldots, \pm n_1)$ in the sense above.

The first step of 1), of writing a $p$-adic expansion of $\text{har}_{p^m}$ and summing over $r_i$’s and $u_i$’s, lifts in a natural way into a map, for which one can write an explicit formula, which we call a $p$-adic pro-unipotent $\Sigma$-harmonic action for $\mathbb{P}^1 - \{0, 1, \infty\}$ localized at the source:

$$\left(\otimes_{\text{har}} \right)^{\text{loc}} : \left( Q_p(\langle e_0, e_1 \rangle)_{\Sigma, \text{har}} \right) \times \text{Map}(\mathbb{N}, Q_p(\langle e_0^{\pm 1}, e_1 \rangle)_{\text{har}}) \to \text{Map}(\mathbb{N}, Q_p(\langle e_0, e_1 \rangle)_{\text{har}})$$

\footnote{With $B_{l_1}^{l_2, \ldots, l_1}$, defined by for all $n \in \mathbb{N}^*$, $\sum_{0 \leq n_1 < \ldots < n_r < n} n_1^{l_1} = \sum_{k=1}^{l_1 + \ldots + l_r} B_{l_1}^{l_r, \ldots, l_1} n^{l_1 + \ldots + l_r}$ ($r \geq 1$, $l_1, \ldots, l_r, b \geq 0$ and $1 \leq b \leq l_1 + \ldots + l_r$) \text{; example : $B_{l_1}^{l_2, \ldots, l_1} = \frac{1}{l_1} (l_1 + 1) B_{l_1 + l - 1}$ where $B$ denotes Bernoulli numbers.} The generalization $B_{l_1}^{l_2, \ldots, l_1}$ is defined similarly as certain coefficients arising from the map of localization above.
The second step of 1), of expressing the maps \( n \mapsto \sum_{0 \leq u_1 \leq \ldots \leq u_r \leq m-1} u_1^{\pm l_1} \ldots u_r^{\pm l_r} \) as \( Q[m] \)-linear combinations of \( m \mapsto \text{har}_m \) lifts to a map for which it is also possible to write an explicit formula

\[
\text{loc}^\vee : \text{Map}((N, Q_p \langle \langle e_0, e_1 \rangle \rangle \text{har}) \to \text{Map}((N, Q_p \langle \langle e_0^{\pm 1}, e_1 \rangle \rangle \text{har})
\]

Then we define

\[
\circ_{\text{har}} = (\circ_{\text{har}})_{\text{loc}} \circ (\text{id} \times \text{loc}^\vee)
\]

The explicit formulas are shown in [J2], §5. The formula for \( \text{loc}^\vee \) is indexed by the paths from the root to the leaves of a certain tree.

3) We define the map \( \text{comp} \Sigma \rightarrow \int \) as follows : \( \text{comp} \Sigma \rightarrow \int \text{har}_m \) is the coefficient of \( m^{b+n_1+\ldots+n_1} \text{har}_m(\emptyset) \) in equation (16), and similarly when we replace \( \text{har}_m \) by any sequence \( (h_m)_{m \in \mathbb{N}} \).

A first observation is that we have \( \text{comp} \Sigma \rightarrow \int \circ \text{comp} \int \rightarrow \Sigma = \text{id} \); this follows from that, by specializing the equation \( \text{har}_{\text{loc}}^\text{loc} = \text{loc}^\vee \text{har}_N \) to \( m = 1 \), we get relations between the coefficients \( B \) from the fact that \( \text{har}_{m=1}(w) = 0 \) for all non-empty words \( w \); this is hidden behind the fact that specializing equation (16) to \( m = 1 \) yields only the tautological equality \( \text{har}_p^p = \text{har}_p^p \).

A second observation is that, under certain conditions on \( g \) we have \( g \circ_{\text{har}}^\Sigma h = (\text{comp}^{\Sigma \rightarrow \Sigma} g) \circ_{\text{har}}^\Sigma \circ_{\text{har}}^\Sigma h \); this follows essentially from the fact that \( \circ_{\text{har}}^\Sigma \circ_{\text{har}}^\Sigma \) and \( \circ_{\text{har}}^\Sigma \) are built out of similar processes of taking subwords and quotient words.

4.3 (I-3) The number of iterations of the Frobenius viewed as a variable [J3]

In I-1 and I-2, the number \( \alpha \in \mathbb{N}^* \) of iterations of Frobenius was fixed. We now study the Frobenius iterated \( \alpha \) times as a function of \( \alpha \) viewed as a \( p \)-adic variable. More generally, for any \( (\alpha_0, \alpha) \in (\mathbb{N}^*)^2 \) such that \( \alpha_0 | \alpha \), we are going to consider \( \phi \) iterated \( \alpha \) times as \( \phi^{\alpha_0} \) iterated \( \frac{\alpha}{\alpha_0} \) times.

For prime weighted multiple harmonic sums \( \text{har}_{p^\alpha} \), the variable \( p^\alpha \) has two roles : it reflects the number of iterations of the Frobenius \( \alpha \) and it is the upper bound of the domain of summation of the underlying iterated sum. This relates these considerations to the study of the map \( n \mapsto \text{har}_n \) viewed as a function of the \( p \)-adic variable \( n \), whose certain aspects (the regularization) played a central role in I-1. By this relation, it will be useful in [J7] (part III-1) to reason in terms of \( (\alpha_0, \alpha) \) and not only in terms of \( \alpha \).

4.3.1 The Frobenius as a contraction mapping

We are going to use again, but this time in a more essential way, the topological framework of §3.3. One can prove that \( \mathcal{N}_{\Lambda,D} \), and thus \( \mathcal{N}_D \) as well, is sub-multiplicative with respect
to the product $\circ^\lambda$; for all $f, f' \in \Pi_{1,0}(Q_p)$, we have

$$N_{\lambda,D}(f' \circ^\lambda f) \leq N_{\lambda,D}(f') \times N_{\lambda,D}(f)$$

(26)

This inequation, together with the completeness of $\Pi_{1,0}(Q_p)$ authorizes the following definition: below, the exponent $\lambda$ means "inverse for the product $\circ^\lambda$:

**Definition 4.7.** Let $\kappa \in Q_p^*$ with $|\kappa|_p < 1$. A map $\psi : \Pi_{1,0}(Q_p) \to \Pi_{1,0}(Q_p)$ is said to be a $\kappa$-contraction if, for all $f, f' \in \Pi_{1,0}(Q_p)$, we have:

$$N_{\lambda,D}\left(\psi(f')^{-1(\circ^\lambda)} \circ^\lambda \psi(f)\right)(\Lambda, D) \leq N_{\lambda,D}\left(f'^{-1(\circ^\lambda)} \circ^\lambda f\right)(\kappa\Lambda, D)$$

(27)

Indeed, by (26) and the fact that $\Pi_{1,0}(Q_p)$ is complete, the contractions in this sense satisfy the usual properties of contractions of complete metric spaces regarding fixed points: they have a unique fixed point and any sequence $(\phi^\lambda(f))_{\lambda \in \mathbb{N}^*}$, with $f \in \Pi_{1,0}(Q_p)$ tends to the fixed point when $a \to \infty$.

One can see that, for all $(\lambda, f') \in G_n(Q_p) \times \Pi_{1,0}(Q_p)$, if $|\lambda|_p < 1$, the map $f \mapsto f'^{-1(\circ^\lambda)} \tau(\lambda)(f)$ is a contraction in this sense. This implies, by the formula for the Frobenius of $\Pi_{1,0}(Q_p)$ (Proposition 2.20) that the inverse of the Frobenius iterated $\alpha_0$ times, i.e. $\phi^{-\alpha_0}$, is a contraction in this sense with $\lambda = p^{\alpha_0}$. This was hidden behind the existence and uniqueness of a Frobenius-invariant path in $\Pi_{1,0}(Q_p)$ subjacent to Definition 2.21 (see also the proof of Proposition 3.1 in Fu2 for an implicit evocation of this fact). Denoting by $\Phi_{\alpha, -\alpha} = \Phi_{\alpha, -\alpha}^{-1(\circ^\lambda)}$ for $\alpha \in \mathbb{N}^*$, we thus have $\Phi_{\alpha}^{KZ} = \lim_{\alpha \to \infty} \Phi_{\alpha, -\alpha}$ and $(\Phi_{\alpha}^{KZ})^{-1(\circ^\lambda)} = \lim_{\alpha \to \infty} \Phi_{\alpha, -\alpha}$.

We thus have for each $\alpha \in \mathbb{Z} \cup \{\pm \infty\} - \{0\}$ a notion of $p$-adic multiple zeta values associated with the Frobenius iterated $\alpha$ times, in a sense given by the discussion above when $\alpha = \pm \infty$, and this formulation unifies all the definitions of $p$-adic multiple zeta values. We can use the notations $\Phi_{\alpha}^{KZ} = \Phi_{\alpha, -\alpha}$ and $(\Phi_{\alpha}^{KZ})^{-1(\circ^\lambda)} = \Phi_{\alpha, -\alpha}$.

### 4.3.2 Main result

Below, $\Lambda$ and $a$ are formal variables; we use the notations of §4.2.2; and, for $n \in \mathbb{N}^*$, $p_r : Q_p(\langle e_0, e_1 \rangle) \to Q_p(\langle e_0, e_1 \rangle)$ is the map of "projection onto the terms of weight $n$" i.e. the sequence $(p_r)_{r \in \mathbb{N}}$ is characterized by: for all $f \in Q_p(\langle e_0, e_1 \rangle)$, and $\lambda \in Q_p^*$, $\tau(\lambda)(f) = \sum_{n \in \mathbb{N}} p_r(f)\lambda^n$. The parts o) and i) below rely on Corollary 2.

**Theorem 3**

Let $(\alpha_0, \alpha) \in (\mathbb{N}^*)^2$ such that $\alpha_0 \alpha$.

o) (framework $f_0^\lambda$) There exists a continuous free group action of $(\text{Ad}_{\Pi_{1,0}(Q_p)})(e_1), c^\lambda_0$, the $p$-adic pro-unipotent $f_0^\lambda$-harmonic action of $\mathbb{P}^1 - \{0, 1, \infty\}$

$$c_0^\lambda_{\text{har}} : \text{Ad}_{\Pi_{1,0}(Q_p)}(e_1) \times Q_p(\langle e_0, e_1 \rangle)^{f_0^\lambda}_{\text{har}} \to Q_p(\langle e_0, e_1 \rangle)^{f_0^\lambda}_{\text{har}}$$

\(^{35}\text{and the inequality (27) is in this case an equality with } \kappa = \lambda\)

\(^{36}\phi^{-\alpha_0} \text{ is of the form above with } g = \Phi_{\alpha, -\alpha}^{-1(\circ^\lambda)} \text{ and } \lambda = p^{\alpha_0}.\)

33
which extends to a map $Q_p \langle \langle e_0, e_1 \rangle \rangle_s \times Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0} \to Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0}$ which we will denote in the same way, such that we have the analytic expansion (equality of functions of $\alpha \in \mathbb{N}^*$):

$$\text{har}_{p^s} = \sum_{s \geq 0} \left( \text{pr}_{s+1} \left( \Phi_{p,\infty}^{-1} e_1 \Phi_{p,\infty} \right) \circ \text{pr}_{f_0} \circ \text{comp}^{\rightarrow} \left( \Phi_{p,\infty}^{-1} e_1 \Phi_{p,\infty} \right) \right) (p^N)^s \quad (28)$$

i) (framework $f_0^1$) There exists an explicit map, the $f_0^1$-harmonic iteration of the Frobenius

$$(\text{iter}_{\text{har}}^{f_0^1})_{\alpha,\Lambda} : \text{Ad}_{\Pi_{1,0}(Q_p)}(e_1) \to Q_p[[\Lambda^a]][\alpha](\langle\langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0^1})$$

such that, the map $(\text{iter}_{\text{har}}^{f_0^1})_{\alpha,\Lambda} : \Pi_{1,0}(Q_p)s \to Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0^1}$ defined as the post-composition of $(\text{iter}_{\text{har}}^{f_0^1})_{\alpha,\Lambda}$ by the reduction modulo $(\alpha - \frac{\alpha}{\alpha_0}, \Lambda - p^{\alpha_0})$ satisfies, at words $w$

$$\text{har}_{p^s}(w) = (\text{iter}_{\text{har}}^{f_0^1})_{\alpha,\Lambda} (\Phi_{p,\alpha_0}^{-1} e_1 \Phi_{p,\alpha_0}) (w) \quad (29)$$

ii) (framework $\Sigma$) There exists an explicit map, the $\Sigma$-harmonic iteration of the Frobenius

$$(\text{iter}_{\text{har}}^{\Sigma})_{\alpha,\Lambda} : (Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{\Sigma}) s \to Q_p[[\Lambda^a]][\alpha](\langle\langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0})$$

such that, the map $(\text{iter}_{\text{har}}^{\Sigma})_{\alpha,\Lambda} : (Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{\Sigma}) s \to Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0}$ defined as the composition of $(\text{iter}_{\text{har}}^{\Sigma})_{\alpha,\Lambda}$ by the reduction modulo $(\alpha - \frac{\alpha}{\alpha_0}, \Lambda - p^{\alpha_0})$, satisfies

$$\text{har}_{p^s} = \text{iter}_{\text{har}}^{\Sigma} \circ \text{har}_{p^s} \quad (30)$$

iii) (comparison) Let us fix $\alpha_0$, and view $(28), (29), (30)$ as three expansions of a function $\mathbb{N}^* \to Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}$ of $\frac{\alpha_0}{\alpha_0}$ in the ring $Q_p[[[p^{\alpha_0}]],\langle\langle e_0, e_1 \rangle \rangle_{\text{har}}]$. Then, the coefficients of these expansions are identical.

We thus have a natural way to compute the Frobenius-invariant path $\Phi^KZ_p$ indirectly, i.e. by identifying the computations in the frameworks $f_0^1$ and $\Sigma$ of a same object.

We note that it is easy to check that we have an isomorphism of topological spaces with continuous group action between $\text{Map}(\mathbb{N}, Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0^1})$ equipped with $m \mapsto (\phi_{\alpha_0}^{f_0^1} \circ (\tau(m) \times \text{id})$ and $\text{Map}(\mathbb{N}, Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0^1})$ equipped with $\phi_{\alpha_0}^{f_0^1}$.

4.3.3 Strategy of proof and examples of explicit formulas

o) The formula for the Frobenius on $\Pi_{1,0}(Q_p)$ (Proposition $\text{2.20}$) combined to the definition of $\Phi^KZ_p$ as the Frobenius-invariant yields a formula expressing $\Phi_{p,\alpha}$ in terms of $\Phi^KZ_p$ (see also $\text{[Fu2]}, \text{Theorem 2.8}$). We apply to this formula, first, $\text{Ad}(e_1)$, which fits into a commutative diagram involving $\phi_{\alpha_0}^{f_0^1}$ and $\phi_{\alpha_0}^{f_0^1}$, and then $\text{comp}^{\Sigma-f} : \text{Ad}_{\Pi_{1,0}(Q_p)}(e_1) \to Q_p \langle \langle e_0, e_1 \rangle \rangle_{\text{har}}^{f_0^1}$, and we apply Corollary 2 (equation $\text{[19]}$). We get the result, where we view $\text{comp}^{\Sigma-f}$
as a surjective map onto $\mathbb{Q}_p\langle\langle e_0, e_1 \rangle\rangle^{\text{Ad}}_{\text{har}}$, where $\text{Ad}_{\text{har}}$ is extended as a map $\mathbb{Q}_p\langle\langle e_0, e_1 \rangle\rangle \times \mathbb{Q}_p\langle\langle e_0, e_1 \rangle\rangle \to \mathbb{Q}_p\langle\langle e_0, e_1 \rangle\rangle$ by the same formula with the one of §4.2.2 and where $\text{Ad}_{\text{har}}$ is defined by

$$g \circ \text{Ad}_{\text{har}} \circ \text{comp}^{\Sigma-f}h = \text{comp}^{\Sigma-f}(g \circ \text{Ad}_{\text{har}} h)$$

We check that $\text{Ad}_{\text{har}}$ is well-defined, and that is a continuous free group action.

**Example 4.8.** In depth one, this gives : for all $n \in \mathbb{N}^*$,

$$\text{har}_{p^n}(n) = (\Phi_{p^{-\infty}}^{-1} e_1 \Phi_{p^{-\infty}})^{-1} \left[ \frac{1}{1 - e_0} e_1 e_0^{n-1} e_1 \right] + \sum_{b \in \mathbb{N}} (p^n)^{k+n} (\Phi_{p,\infty}^{-1} e_1 \Phi_{p,\infty}) \left[ e_1 e_0^{n-1} e_1 \right]$$

i) For $a \in \mathbb{N}^*$, and $\lambda \in \mathbb{Q}_p^*$ which is not a root of unity, let us call "adjoint iteration $a$ times weighted by $\lambda$" the map $g \in \text{Ad}_{\text{har}}(\mathbb{Q}_p)_{\text{har}}(e_1) \mapsto g \circ \text{Ad}_{\text{har}} \tau(\lambda)(g) \circ \text{Ad}_{\text{har}} \tau(\lambda^{a-1})(g)$. It can be written in terms of the standard multiplication of $\mathbb{Q}_p\langle\langle e_0, e_1 \rangle\rangle$, which enables to write its dual. Then, we can analyze how it depends on $a$, provided we restrict to $a$ greater than the depth of the word under consideration. The map of the statement is then obtained by applying $\text{comp}^{\Sigma-f}$ and Corollary 2 to the result.

ii) We introduce in $(p^n)^{m_1+\ldots+m_1} \sum_{0< m_1<\ldots< m_d<p^n} \frac{1}{m_1 m_2 \ldots m_d}$, the following additional parameters : we write each $m_i$ as $p^{m_0} \alpha_i + r_i$, where $\alpha_i, u_i \in \mathbb{N}$ and $r_i \in \{1, \ldots, p^{m_0} - 1\}$. Then, we write $\frac{1}{m_i} = \frac{1}{p^{m_0} \alpha_i} \sum_{l_i>0} \frac{p^{m_0} \alpha_i}{r_i^{m_0} + r_i^{m_i}}$. We first have to translate the condition $0 < m_1 < \ldots < m_d < p^n$ in terms of the new parameters, $u_i, v_i, r_i$. Summing over $u_i$ and $r_i$ has some similarities with a step of the proof of Theorem 2 in §4.2.6. Summing over $v_i$ is different since they appear as exponents. The result is obtained after having summed over all additional parameters.

**Example 4.9.** In depth one, this gives : for all $n \in \mathbb{N}^*$,

$$\text{har}_{p^n}(n) = \sum_{b \geq 1} \frac{p^{a(n+b)}-1}{p^{a(n+b)}-1} \sum_{l \geq -1} B_{b}^{L+b} \text{har}_{p^{a}}(n+b+L)$$

iii) the uniqueness of the expansion follows from a standard elementary argument.

### 5 (Part II) Algebraic relations [J4], [J5], [J6]

In this part, we want to understand explicitly the algebraic relations of $p$-adic multiple zeta values, using the computations of part I. We will apply again the principles of §3. We keep in mind the concepts of the Galois theory of periods and, at the same time, the fact that the formulas exchanging $\zeta_{p,\alpha}$ and $\text{har}_{p^n}$ involve (weight-adically and $p$-adically convergent) infinite summations and the group $\Pi_{1,0}(\mathbb{Q}_p)_{\text{har}}$ defined by bounds of valuations ; our framework is thus not strictly speaking algebraic, although it comes from an algebraic framework.
5.1 (II-1) Standard algebraic relations of adjoint multiple zeta values and multiple harmonic values [J4]

We are led to make certain infinite sequences of numbers \( \text{har}_{\rho^\alpha}(w) \) into a notion of their own, studied in the three frameworks : \( \Sigma \) by their definition, \( f_{0}^{1} \) by equation (18) and \( j_{0}^{1} \) by equation (19) ; this will shed light on the notion of finite multiple zeta values, by §1.4.3, §4.2.4 and Fact 3.6. We are also led to consider intrinsically the coefficients of \( \Phi_{p,\alpha}^{-1}e_{1}\Phi_{p,\alpha} \) rather than those of \( \Phi_{p,\alpha} \), because they are the ones directly involved in the explicit formulas of part I such as equation (17) and (19) ; we will call them adjoint \( p \)-adic multiple zeta values. Both these notions are intermediates for the explicit study of algebraic relations of \( p \)-adic multiple zeta values, but we also develop them for their intrinsic interest.

5.1.1 Multiple harmonic values

We define several types of infinite sequences of \( \text{har}_{\rho^\alpha}(w) \) which we intend to view as "periods" comparable to multiple zeta values ; the notations are the one of §3.2.1 :

**Definition 5.1.** For any word \( w = (n_d, \ldots, n_1) \) :

i) for each \( p \in \mathcal{P} \), we call \( p \)-adic multiple harmonic value the sequence \( \text{har}_{p^{\rho^\alpha}}(w) \)

ii) for each \( \alpha \in \mathbb{N}^* \), we call adelic multiple harmonic value the sequence \( \text{har}_{\rho^\alpha}(w) \)

iii) we call \( \mathcal{P} \)-adic adelic multiple harmonic value the sequence \( \text{har}_{\rho^\alpha}(w) \)

We will view i) as an explicit substitute to \( p \)-adic multiple zeta values, ii) as the canonical lift of finite multiple zeta values (via Fact 3.6), and iii) as the "product" of i) by ii), which provides the natural way to state the properties of i) and ii) in a unified way.

5.1.2 Adjoint multiple zeta values

In the framework \( f_{0}^{1} \), the study of multiple harmonic values via equation (18) will be equivalent to the study of a slightly different object. Indeed, the known algebraic relations of \( (p\text{-adic}) \) multiple zeta values are homogeneous for the weight, and it is conjectured that it is true for all algebraic relations (Conjecture 1.12, Conjecture 1.13) ; separating the terms of different weights in equation (18) leads to :

**Definition 5.2.** Let \( (p, \alpha) \in \mathcal{P} \times \mathbb{N}^* \), \( (n_d, \ldots, n_1) \) and \( b \in \mathbb{N} \).

i) We call adjoint \( p \)-adic multiple zeta value (AdpMZV) the number

\[
\zeta_{p,\alpha}^{\text{Ad}}(b; n_d, \ldots, n_1) = (-1)^d(\Phi_{p,\alpha}^{-1}e_{1}\Phi_{p,\alpha})[e_{0}^b e_{1} e_{n_d}^{n_d-1} e_{1} \ldots e_{n_1}^{n_1-1} e_{1} ] \in \mathbb{Q}_p
\]

In [Ro], Rosen has considered the sequences \( (\text{har}_{p^{\rho^\alpha}}(w))_{p \in \mathcal{P}} \), and made them into a notion in the complete topological ring \( \lim_{\longleftarrow} \mathbb{Z}_{p} / p^n\mathbb{Z} \), where the bar refers to the adherence with respect to the uniform topology on \( \mathbb{Z}_p \) relative to the \( p \)-adic topologies. We denote this ring by \( \mathbb{Z}_{p \rightarrow \infty} = \lim_{\longleftarrow} \mathbb{Z}_{p} / (p \rightarrow \infty)^n \). The proofs in [Ro] do not use \( \pi^1_{\mathfrak{S}}(P^1 - \{0, 1, \infty\}) \) (they are in what we called the \( \Sigma \) setting in §3.1.3). We prove in [J4] that the "asymptotic reflexion theorem" and the "asymptotic duality theorem" of [Ro] are equivalent certain particular cases of our main theorems of [J4], which concern the standard families of algebraic relations : double shuffle relations, associator relations and Kashiwara-Vergne relations.
ii) Let $\Lambda$ be a formal variable. We call $\Lambda$-adic adjoint $p$-adic multiple zeta values ($\Lambda Ad_p MZV$) the power series

$$
\zeta_{\Lambda, Ad}^{p, \alpha}(n_d, \ldots, n_1) = (-1)^d (\Phi_{p, \alpha}^{-1} \Phi_{p, \alpha}) [\frac{\Lambda^{n_d + \ldots + n_1}}{1 - \Lambda e_0} e_1 e_0^{n_d - 1} e_1 \ldots e_0^{n_1 - 1} e_1] \in \mathbb{Q}_p[[\Lambda]]
$$

We can define in the same way complex and motivic analogues of these notions, with notations $\zeta^{Ad, \text{mot}}, \zeta_{\Lambda, Ad, \text{mot}}, \zeta_{\Lambda, Ad}, \zeta_{\Lambda, Ad, \text{mot}}$, respectively where $\zeta_{\text{mot}}$ refers to Goncharov’s motivic multiple zeta values [Go1]. The numbers $\zeta^{Ad}_{p, \alpha}(0; n_d, \ldots, n_1)$, which we will denote more simply by $\zeta^{Ad}_{p, \alpha}(n_d, \ldots, n_1)$, are conjecturally equivalent to finite multiple zeta values by Kaneko-Zagier’s Conjecture 1.16, and have additional properties of their own; their reduction modulo infinitely large primes is equal to finite multiple zeta values by Akagi-Hirose-Yasuda’s equation (21). The equation

$$
\zeta_{\Lambda, Ad}^{p, \alpha}(n_d, \ldots, n_1) \equiv \zeta^{Ad}(n_d, \ldots, n_1) \mod \Lambda^{n_d + \ldots + n_1 + 1}
$$

is the counterpart for adjoint multiple zeta values of the Fact 3.6 which can be rewritten in terms of multiple harmonic values.

5.1.3 From pro-affine schemes to affine formal schemes over $\mathbb{Q}[[\Lambda]]$

Whereas the polynomial equations of periods considered earlier are homogeneous for the weight, their analogues for multiple harmonic values which we will find will be in general convergent infinite sums of algebraic relations of $p$-adic multiple zeta values, involving sequences $(\text{har}_p(w_n))_{n \in \mathbb{N}}$, or $(\zeta^{Ad}_{p, \alpha}(w_n))_{n \in \mathbb{N}}$ with weight $\langle w_n \rangle \to_{n \to \infty} \infty$ which we will call completed algebraic relations. A family of completed algebraic relations defines an affine formal scheme over $\mathbb{Q}[[\Lambda]]$ equipped with the $\Lambda$-adic topology, $\Lambda$ being a formal variable; more precisely, a sequence $(T_n)_{n \in \mathbb{N}}$, such that, for all $n \in \mathbb{N}$, $T_n$ is an affine scheme over $\mathbb{Q}[[\Lambda]]/(\Lambda^n)$, with $T_n \equiv T_{n+1} \mod \Lambda^n$.

There is a faithful functor $\tau(\Lambda)$ sending projective limits of affine schemes over $\mathbb{Q}$ such as the fibers of $\pi^m_1(P^1 - \{0, 1, \infty\})$ or Racinet’s scheme $\text{DMR}_0$, to formal schemes over $\mathbb{Q}[[\Lambda]]$, defined by multiplying each equation of weight $n$ by $\Lambda^n$. We will view all the pro-affine schemes over $\mathbb{Q}$ and the morphisms between them as their images by $\tau(\Lambda)$. A functor in the converse direction can be defined by considering the lowest weight term of the equations of affine formal schemes over $\mathbb{Q}[[\Lambda]]$; let us call it $\text{gr}$; we have then $\text{gr} \circ \tau(\Lambda) = \text{id}$.

5.1.4 Main result

One of the main results is a description of an analogue the double shuffle relations for adjoint multiple zeta values and multiple harmonic values. We denote by $\text{DMR}^{p-sym} = \text{DMR}_{i_0 < i_1}^{p-sym}$ the $\mathbb{Z}[[\Lambda]]$-formal scheme of solutions to the double shuffle equations satisfied

\[\zeta_{\text{mot}}(2) = 0\] which is convenient for our purposes, contrarily to the other notions of motivic multiple zeta values defined later.

38 They satisfy $\zeta_{\text{mot}}(2) = 0$ which is convenient for our purposes, contrarily to the other notions of motivic multiple zeta values defined later.
by multiple polylogarithms, or equivalently by the functions $\text{har}_N$, and of a certain equation encoding a basic $p$-adic property of prime weighted multiple harmonic sums, which we call the "symmetry equation" (see [14]).

**Theorem 4**

i) (framework $f_0^1$) There exists an explicit pro-affine scheme $\text{DMR}_{0,\text{Ad}}$ over $\mathbb{Z}$, which is a group scheme for $\phi^{f_0^1}_{\text{Ad}}$, such that the map $\text{Ad}(e_1)$ defines a morphism of group schemes $(\text{DMR}_{0,\phi^{f_0^1}_{\text{Ad}}}, \phi^{f_0^1}_{\text{Ad}}) \rightarrow (\text{DMR}_{0,\text{Ad}}, \phi^{f_0^1}_{\text{Ad}}).$

There exists an explicit affine $\mathbb{Z}[[\Lambda]]$-formal scheme $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^1}$, such that $\tau(\Lambda)$ sends $\text{DMR}_{0,\text{Ad}}$ to $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^1}$, and which is stable by the action of $(\text{DMR}_{0,\text{Ad}}, \phi^{f_0^1}_{\text{Ad}})$ by $\phi^{f_0^1}_{\text{har}}$.

ii) (frameworks $f_0^{<<1}$ and $\Sigma$) There exists an explicit sequence $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<<1}} = \text{DMR}_{\text{har},p\mathbb{R}}^{\Sigma}$, indexed by $\mathbb{N}$, of affine $\mathbb{Z}[[\Lambda]]$-formal schemes, such that the map of "restriction to $\mathbb{P}^\mathbb{N}$", defines a morphism : $(\text{DMR}_{p\mathbb{R}}^{\Sigma} / f_0^{<<1}) = \text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<<1}} \rightarrow \text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<<1}} = \text{DMR}_{\text{har},p\mathbb{R}}^{\Sigma}$.

iii) (comparison) We have $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}} = \text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<<1}} = \text{DMR}_{\text{har},p\mathbb{R}}^{\Sigma}$.

By i), $\text{Ad}_{\text{har},p\mathbb{R}}(e_1)$ which is the non-commutative generating series of adjoint multiple zeta values (§5.1.2) is a point of $\text{DMR}_{0,\text{Ad}}$ and $\Lambda \text{Ad}_{\text{har},p\mathbb{R}}(e_1)$, the non-commutative generating series of $\Lambda$-adic adjoint multiple zeta values, is a point of $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}}$. By ii), $\text{har},p\mathbb{R}$, the non-commutative generating series of multiple harmonic values, is a point of $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<<1}} = \text{DMR}_{\text{har},p\mathbb{R}}^{\Sigma}$. We note that in this case, the results for $\Sigma$ and $f_0^{<<1}$ are not different from each other, whereas comparing the $f_0^{1}$ and $f_0^{<<1}$ results requires a proof.

We can also define a variant $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}}$ of $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}}$ which admits as a point the non-commutative generating series of the numbers $\text{har}_{p\mathbb{R}}^{f_0^{<1}}(w) = (p^\alpha)^{\text{weight}(w)} \text{Li}_{p\alpha}[w][z^{p\alpha}]$, which are equal to the remainders in the series [13].

**Corollary 4**

The pro-affine schemes $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}} = \text{gr}(\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}})$ and $\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<1}} = \text{gr}(\text{DMR}_{\text{har},p\mathbb{R}}^{f_0^{<<1}})$ are equal to each other and admit as points the non-commutative generating series of the numbers $\zeta^{\text{Ad}}(n_d, \ldots, n_1) \mod \zeta(2)^*$, their $p$-adic and motivic analogues (§5.1.2), and the non-commutative generating series of finite multiple zeta values.

This follows from Remark 3.5, equation (31) and equation (21).

**5.1.5 Strategy of proof and explicit formulas**

Let, for all $d \in \mathbb{N}^*$ and $n_1, \ldots, n_d \in \mathbb{N}^*,$ shft$_* (e_0^{n_d-1} e_1 \ldots e_0^{n_1-1} e_1) = e_0^{n_d-1} (1+e_0)^{n_d} e_1 \ldots e_0^{n_1-1} (1+e_0)^{n_1} e_1$ and $S_Y(e_0^{n_d-1} e_1 \ldots e_0^{n_1-1} e_1) = (-1)^{n_d+\ldots+n_1} e_0^{n_1-1} e_1 \ldots e_0^{n_d-1} e_1$

i) The quasi-shuffle part of the result is a purely technical checking of a commutation
between the quasi-shuffle product $*$ and $\tau(\Lambda)$ resp. $\text{Ad}(e_1)$. The shuffle part of the result requires to guess the equation what we must prove : what we find is a formula involving $\text{har}_p^\Lambda$, equivalent to saying that $\Phi_{p,\alpha}^{-1} \Phi_{p,\alpha} e_1$ satisfies the shuffle equation modulo products : for all words $w, w'$, $(\Phi_{p,\alpha}^{-1} \Phi_{p,\alpha})[w \shuffle w'] = 0$. \footnote{Let $\Delta_m$ the 'shuffle coproduct', defined as the continuous multiplicative map $Q_p(\langle e_0, e_1 \rangle) \to Q_p(\langle e_0, e_1 \rangle) \otimes Q_p(\langle e_0, e_1 \rangle)$ satisfying $\Delta_m(e_i) = 1 \otimes e_i + e_i \otimes 1$ for $i = 0, 1$ ; for $f \in Q_p(\langle e_0, e_1 \rangle)$, the shuffle equation amounts to $\Delta_m(f) = f \otimes f$, and the shuffle equation modulo products amounts to $\Delta_m(f) = 1 \otimes f + f \otimes 1$ ; whence $\Phi_{p,\alpha}^{-1} \Phi_{p,\alpha} e_1$ satisfies the shuffle equation modulo products \[32\]}

\( h \left( (e_0^{n-1} e_1 w) \shuffle w' \right) = h \left( w \shuffle (-1)^n \text{shift}_a(e_0^{n-1} e_1 w') \right) \) \( 32 \)

The formula comparing the two regularizations (integral and series) of $\Phi_{p,\alpha} \text{Ra}$ and the vanishing of the coefficients $\Phi_{p,\alpha}[e_0^{2n-1}]$ imply that the two regularizations of $\Phi_{p,\alpha}^{-1} \Phi_{p,\alpha}$ are equal to each other. One can read the equations of $\text{DMR}_{\text{Ad}}$ through the ones of $\text{DMR}_{\text{har}_p^\Lambda}$ and the definition of $\zeta_{p,\alpha}^A \text{Ad}$.

The compatibilities involving $\circ_{\text{Ad}}^0$, $\circ_{\text{Ad}}^1$ and $\circ_{\text{har}_p^\Lambda}$ are obtained by showing that all the maps and equations necessary for Racinet's proof in \text{Ra} can appear as lines of commutative diagrams involving the maps $\text{Ad}(e_1)$ and $\tau(\Lambda)$.

ii) is proved by the usual strategy of the framework $\int_0^\infty \otimes$ (explained in §3.1.1) applied to the case of the double shuffle relations of multiple polylogarithms, combined to the fact that, for $m \in \{1, \ldots, p^n\}$, we have $v_p(m) < v_p(p^n)$, thus $v_p(\frac{m}{p^n}) > 0$, thus, for all $n \in \mathbb{N}^*$, $(1 - \frac{m}{p^n})^{-n} = \sum_{i \in \mathbb{N}} \left( \frac{-n}{i} \right) \left( \frac{m}{p^n} \right)^n$. $\text{DMR}_{\text{har}_p^\Lambda}$ is defined by the quasi-shuffle equation and, for all words $w, w'$,

\[ h(w \shuffle w') = h(\text{shift}_a(S_Y(w)w')) \] \( 33 \)

iii) We prove by that \( 32 \) and \( 33 \) are equivalent by analyzing the parameters they depend on : we prove a triple equivalence involving these equations and the following one : for all words, $u, w, w'$,

\[ h(uw \shuffle w') = h(w \shuffle \text{shift}_a(S_Y(w)w')) \] \( 34 \)

\subsection*{5.2 (II-2) From properties of multiple harmonic sums to those of adjoint $p$-adic multiple zeta values \textbf{J5}}

We now want to show that algebraic properties of adjoint $p$-adic multiple zeta values can be understood via the formulas of part I, and find converse to some arrows of §5.1.

\subsubsection*{5.2.1 The implicit and explicit points of view}

The question can be tackled using two different parts of the formulas, corresponding to two different points of view :

1) (By the implicit formula) Adjoint $p$-adic multiple zeta values are characterized, by

\[39\text{Let } \Delta_m \text{ the 'shuffle coproduct', defined as the continuous multiplicative map } Q_p(\langle e_0, e_1 \rangle) \to Q_p(\langle e_0, e_1 \rangle) \otimes Q_p(\langle e_0, e_1 \rangle) \text{ satisfying } \Delta_m(e_i) = 1 \otimes e_i + e_i \otimes 1 \text{ for } i = 0, 1 \text{ ; for } f \in Q_p(\langle e_0, e_1 \rangle), \text{ the shuffle equation amounts to } \Delta_m(f) = f \otimes f, \text{ and the shuffle equation modulo products amounts to } \Delta_m(f) = 1 \otimes f + f \otimes 1 \text{ ; whence } \Phi_{p,\alpha}^{-1} \Phi_{p,\alpha} e_1 \text{ satisfies the shuffle equation modulo products} \]
equation (15), as the coefficients of the unique element of $\text{Ad}_{\Pi_{1,0}(Q_p),s}(e_1)$ whose action by $c_{\text{har}}^{f_{<1}}$ sends $\text{har}_{\mathbb{N}}$ to $\text{har}_{p^{<1} \mathbb{N}}$; i.e. they are characterized as the coefficients of the $f_{<1}^{\text{har}}$-adic multiple zeta values are given explicitly in terms of multiple harmonic values by the formula (17).

2) (By the explicit formula) Adjoint $p$-adic multiple zeta values are given explicitly in terms of multiple harmonic values by the formula (17).

The first point is natural because it keeps track of the fact that the Frobenius is an isomorphism $\pi_1^{\text{un,DR}}(X_{Q_p}) \sim \pi_1^{\text{un,DR}}(X_{Q_p})$, whereas the second point of view is natural because it is explicit.

### 5.2.2 Main results

Let $O_{\text{har}}^{f_{<1}}$ be the orbit of $\text{har}_{\mathbb{N}}$ for the action of $(\text{Ad}_{\Pi_{1,0}(Q_p),s}(e_1), c_{\text{Ad}}^{f_{<1}})$ by $c_{\text{har}}^{f_{<1}}$; since $c_{\text{har}}^{f_{<1}}$ is free (Theorem 2), $O_{\text{har}}^{f_{<1}}$ is a torsor under this group action. We call adjoint quasi-shuffle relation the variant of the quasi-shuffle relation which is part of the equations defining the scheme $\text{DMR}_{0,\text{Ad}}$ of II-1.

**Theorem 5**

i) (point of view of the implicit formula) Let $h \in O_{\text{har}}^{f_{<1}}$, and $g \in \text{Ad}_{\Pi_{1,0}(Q_p),s}(e_1)$ such that $h$ and $g c_{\text{har}}^{f_{<1}} h$ satisfy the quasi-shuffle relation. Then $g$ satisfies the adjoint quasi-shuffle relation.

ii) (point of view of the explicit formula) The map $\text{comp}^{f \leftarrow \Sigma}$ sends a solution to the quasi-shuffle relation to a solution to the adjoint quasi-shuffle relation.

Moreover, we have somewhat analogous statements for the integral shuffle relations.

Bringing together i) resp. ii) with equation (15), resp. equation (17), both from Theorem 2, and with the fact that multiple harmonic sums satisfy the quasi-shuffle relation, reproves in two different ways that $\Phi_{p^{<1} e_1 \Phi_{p,\alpha}}$ satisfies the adjoint quasi-shuffle relation. This gives an answer to an 'adjoint variant' of the question of Deligne and Goncharov of §1.4.2.

### 5.2.3 Strategy of proof

We write the strategy of proof for the quasi-shuffle part.

i) We write the quasi-shuffle relation for $\tilde{h} : \tilde{h}(w)\tilde{h}(w') = \tilde{h}(w * w')$, and we translate it in terms of $g$ and $h$, via the equation $\tilde{h} = g c_{\text{har}}^{f_{<1}} h$.

When we encounter a product $h'(z')h(z'')$, we linearize it by the quasi-shuffle relation of $h$, writing it as $h(z' * z'')$. It remains a linear relation between the maps $h(w) : m \in \mathbb{N} \mapsto h_m(w) \in Q_p$, with coefficients in a ring of power series of $m$, which are actually overconvergent over $Z_p$, by the bounds of valuations of $g$.

By a lemma of Ünver (114), Proposition 2.0.3) the maps $\text{har}_{\mathbb{N}}(w)$ are linearly independent over the ring of overconvergent analytic functions of $m \in Z_p$. For any $g \in \text{Ad}_{\Pi_{1,0}(Q_p),s}(e_1)$, the map $f \mapsto g c_{\text{har}}^{f_{<1}} f$ is a pro-unipotent linear invertible operator, and it thus preserves
Thus for any $h$ in the orbit of $\text{har}_N$ for $\phi_{\text{har}}^{0 << 1}$, all coefficients of the relation above are zero. The vanishing of the coefficients of $\text{har}_m(\emptyset) = 1$ is the adjoint quasi-shuffle relation for $\Phi_{p,a}^{-1} e_1 \Phi_{p,a}$.

ii) The quasi shuffle relation, which is true for multiple harmonic sums $\text{har}_m(n_d, \ldots, n_1)$, remains true for the localized multiple harmonic sums $\text{har}_m(\pm n_d, \ldots, \pm n_1)$ defined in §4.2.6; this combined with the formula for the localization map (§4.2.6) yields a version of the quasi-shuffle relation for the coefficients $B$ of the map of elimination of positive powers.

On the other hand, $c_{\text{har}}^\Sigma$ satisfies certain properties of symmetry with respect to the variables $(n_d, \ldots, n_1)$, and more generally, to certain combinatorial indices subjacent to the explicit formula for $c_{\text{har}}^\Sigma$ written in [J2].

Bringing together these two facts give the result.

**Example 5.3.** In depth $(1,1)$, we have the following incarnation of the quasi-shuffle relation for the coefficients of the elimination of positive powers: $B_0^{b_2+l_1} + B_0^{b_2+l_1} + B_0^{l_1,l_2} = \sum B_b^{l_1} B_{b'}^{l_2}$, for $b, l_1, l_2 \in \mathbb{N}$ such that $1 \leq b \leq l_1 + l_2 + 2$.

On the other hand, one can see on equation (25) of Example 4.6 which expresses the formula for $c_{\text{har}}^\Sigma$ in depth 2, that certain lines of the formula are unchanged, resp. multiplied by $-1$ when we exchange $n_2$ and $n_1$.

These two facts brought together, and combined with the other formulas of Example 4.5, Example 1.6 for $c_{\text{har}}^{0 << 1}$ and $c_{\text{har}}^\Sigma$ in depth one and two imply the adjoint quasi-shuffle relation in depth $(1,1)$ for $\Phi_{p,a}^{-1} e_1 \Phi_{p,a}$:

$$\begin{align*}
(\Phi_{p,a}^{-1} e_1 \Phi_{p,a})[e_0^b e_1 e_0^{n_2-1} e_1 e_1 e_0^{n_1-1} e_1 + e_0^b e_1 e_0^{n_2-1} e_1 e_1 e_0^{n_1-1} e_1 + e_0^b e_1 e_0^{n_2+n_1-1} e_1] = \\
\sum_{b', b'' \geq 0 \atop b' + b'' = b} (\Phi_{p,a}^{-1} e_1 \Phi_{p,a})[e_0^b e_1 e_0^{n_2-1} e_1] \times (\Phi_{p,a}^{-1} e_1 \Phi_{p,a})[e_0^b e_1 e_0^{n_1-1} e_1]
\end{align*}$$

5.3 (II-3) Multiple harmonic values viewed as periods [J6]

We now formalize the idea, implicit in §5.1 and §5.2, that the Galois theory of periods can be applied to multiple harmonic values and $\Lambda$-adic adjoint $p$-adic multiple zeta values as if they were periods. We distinguish again the three frameworks $f_0^1, f_0^{0 << 1}$ and $\Sigma$.

5.3.1 In the framework $f_0^1$: motivic multiple harmonic values and a period conjecture

The Galois theory of multiple zeta values is formulated by means of the notions of motivic multiple zeta values and period map and the conjecture of injectivity of the period map. We are going to see that these have analogues for multiple harmonic values.

Let $\mathcal{Z}_{\text{mot}}$ be the $\mathbb{Q}$-vector space generated by motivic multiple zeta values of [Go1]. For all $n \in \mathbb{N}^*$, let $\mathcal{Z}_{\text{mot}}^n$ be the subspace generated by elements of weight $n$. The weight is a
grading on $Z^{\text{mot}}$, namely we have $Z^{\text{mot}} = \oplus_{n \in \mathbb{N}} Z^{\text{mot}}_n$.

Let $\hat{Z}^{\text{mot}}$ be the completion of $Z^{\text{mot}}$ with respect to the decreasing weight filtration whose $n$-th term is $\oplus_{n' \geq n} Z^{\text{mot}}_{n'}$ for all $n \in \mathbb{N}$. Since the weight on $Z^{\text{mot}}$ is a grading, we have a canonical isomorphism $\hat{Z}^{\text{mot}} \simeq \prod_{n \in \mathbb{N}} Z^{\text{mot}}_n$. This enables to represent an element of $\hat{Z}^{\text{mot}}$ as a formal sum $\sum_{n \in \mathbb{N}} \zeta^{\text{mot}}(w_n)$, where $w_n$ is a $Q$-linear combination of words of weight $n$, in such a way that we have the equivalence $\sum_{n \in \mathbb{N}} \zeta^{\text{mot}}(w_n) = 0$ if and only if for all $n \in \mathbb{N}$, $\zeta^{\text{mot}}(w_n) = 0$. We will use implicitly this representation below. Let $\Phi^{\text{mot}} = \sum_{w \text{ word}} \Phi^{\text{mot}}[w] w$.

**Definition 5.4.** For any $n_d, \ldots, n_1 \in \mathbb{N}^*$, we call motivic multiple harmonic value and, at the same time, motivic $\Lambda$-adic adjoint multiple zeta value the following infinite sequence of motivic multiple zeta values (to be compared with equation (18))

$$\text{har}^{\text{mot}}_{n_d, \ldots, n_1} = (\Phi^{\text{mot}}_1 \Phi^{\text{mot}}_0) \left[ \frac{1}{1 - e_0 e_1^{n_d-1} e_1 \cdots e_1^{n_1-1} e_1} \right] \in \hat{Z}^{\text{mot}}$$

In the next statement, we assume that Yamashita’s (currently unavailable) work on the crystalline realization of mixed Tate motives implies, for any $(p, \alpha) \in \mathcal{P} \times (\mathbb{Z} \cup \{+\infty\} - \{0\})$, the existence of a period map sending $\zeta^{\text{mot}}(w) \mapsto \zeta_{p,\alpha}(w)$ for all words $w$ - to our knowledge, this is known at least for certain values of $\alpha$.

**Definition 5.5.**

i) Let $Z^{\text{mot}}_{\text{conv}} \subset \hat{Z}^{\text{mot}}$ be the set of elements $\sum_{n \in \mathbb{N}} \zeta^{\text{mot}}(w_n)$, such that, for all $(p, \alpha) \in \mathcal{P} \times \mathbb{N}^*$, $\sum_{n \in \mathbb{N}} \zeta_{p,\alpha}(w_n)$ converges in $Q_p$.

ii) For each $I \subset \mathcal{P} \times \mathbb{N}$, let $\text{per}_{\zeta,I} : \sum_{n \in \mathbb{N}} \zeta^{\text{mot}}(w_n) \in \hat{Z}^{\text{mot}} \mapsto (\sum_{n \in \mathbb{N}} \zeta_{p,\alpha}(w_n))_{(p,\alpha) \in I}$.

iii) For each $I \subset \mathcal{P} \times \mathbb{N}$, let $\text{per}_{\text{har},I}$ be the restriction of $\text{per}_{\zeta,I}$ to the subset of weight-adically convergent sums of motivic multiple harmonic values.

The maps of ii) are well-defined by the fact that the weight is a grading on $Z^{\text{mot}}$ and from the ensuing description of $\hat{Z}^{\text{mot}}$ explained above. The maps of iii) are well-defined since $\hat{Z}^{\text{mot}}_{\text{conv}}$ contains the motivic multiple harmonic values.

**Conjecture 5.6.** If $I$ contains a set of the form $p^n \mathbb{N}^*$ with $p$ a prime number, or $\mathcal{P}^\alpha$, with $\alpha \in \mathbb{N}^*$, then $\text{per}_{\text{har},I}$ is injective. In other terms, any relation expressing the vanishing of a weight-adically and $p$-adically convergent sum of multiple harmonic values lifts to an infinite collection of equalities among motivic multiple zeta values.

This setting can be essentialized by defining a notion of "summable subgroupoid of $\pi^{\text{un,DR}}_1(\mathbb{P}^1 - \{0, 1, \infty\})^*"$ : the details are written in [16].

**5.3.2 In the framework $f_0^{<1} :$ the "Taylor periods" of $\pi^{\text{un}}_1(\mathbb{P}^1 - \{0, 1, \infty\})"**

The framework of computation which involves power series expansion of multiple polylogarithms can also be essentialized by a conjecture ; however, we have to involve not only $\mathbb{P}^1 - \{0, 1, \infty\} = M_{0,4}$ but also at least the moduli space $M_{0,5}$ since, for instance, the quasi-shuffle relation is a property of multiple polylogarithms on $\pi^{\text{un}}_1(M_{0,5})$ ; it is actually convenient to involve all the spaces $M_{0,n}$.
**Conjecture 5.7.** Any relation expressing the vanishing of an $p$-adically and weight-adically convergent sum of multiple harmonic values is a consequence of equations obtained from equations on multiple polylogarithms on $\mathcal{M}_{0,n}$, $n \geq 4$, by taking power series expansions, and from the 'p-adic symmetry equation' of prime weighted multiple harmonic sums evoked in §5.2 and defined in II-1.

5.3.3 In the framework $\Sigma$ : a conjecture à la Kontsevich-Zagier for multiple harmonic values viewed as elementary iterated sums

Let us define a counterpart for multiple harmonic values viewed as finite iterated sums of the elementary framework for periods stated by Kontsevich and Zagier (Conjecture 1.6).

**Conjecture 5.8.** Let us consider
i) the subsets of $\Pi_{d\in\mathbb{N}}(\mathbb{N}^*)^{d+1}$ obtained by taking finite disjoint unions and finite products of subsets of the form $\{(m_1, \ldots, m_d, m = m_{d+1}) \in (\mathbb{N}^*)^{d+1} \mid 0 \ast_0 m_1 \ast_1 \ldots \ast_{d-1} m_d \ast_d m\}$, with $\ast_i \in \{=, <\}$ for all $i$,
ii) the $\mathbb{Q}$-algebras of functions $(\mathbb{N}^*)^{d+1} \to \mathbb{Q}$ generated by the maps sending $(m_1, \ldots, m_{d+1})$ to $m_i^t$ with $t_i \in \mathbb{Z}$, binomial coefficients of affine functions of the $m'_i$s (including of the form $\binom{u}{u'}$ with $u < 0$), factorials of the $m_i$'s.
iii) the $\mathbb{Q}$-algebra generated by numbers $\sum_{x \in D} f(x)$ with $f$ as in ii) and $D$ as in i), viewed as elements of $\mathbb{Q}_p$.

Then, any equation expressing the vanishing of an $p$-adically and weight-adically convergent sum of $\mathbb{Q}$-linear combinations of multiple harmonic values, can be obtained using i) ii) iii) and the properties of the following type :
iv) equalities among domains of summation, equations satisfied by summands, equations of changes of variables
v) operations arising from the structure of complete topological field of $\mathbb{Q}_p$ (such as expanding $\frac{1}{1-x} = \sum_{l \in \mathbb{N}} x^l$ in $\mathbb{Q}_p$ for $|x|_p < 1$).

5.3.4 Conclusion

In the end, what we can call 'explicit $p$-adic periods' is the couple formed by ($\Lambda$-adic) adjoint $p$-adic multiple zeta values and multiple harmonic values, with the usual conjecture of periods for $p$-adic multiple zeta values, the conjectures of this §5.3, and the equations relating adjoint $p$-adic multiple zeta values and multiple harmonic sums from part I.

6 Conclusion

Let us summarize what seem to us to be the two main outcomes about $p$-adic multiple zeta values of this work. The initial question was to compute $p$-adic multiple zeta values and apply it to understand explicitly algebraic relations (§1.4.4).
a - About the explicitness

The formulas for \( p \)-adic multiple zeta values provided by our indirect method for solving the differential equation of Frobenius are fully explicit, i.e. they can be written concisely provided the introduction of certain combinatorial tools (the full formulas, which we did not reproduce here, are written in the papers I-2 and I-3); they also enable to visualize explicitly algebraic relations (§5.1-§5.2).

What we have proved is that the Frobenius, because of bounds of the valuations of its values at the canonical paths, becomes much simpler computationally when one considers a certain limit of its parameters, and that the corresponding limit of the Frobenius, which we call harmonic Frobenius, is equivalent to a particular explicit structure on multiple harmonic sums. Moreover, this limit of the Frobenius is enough to reconstruct the whole of the Frobenius.

The explicitness is obtained by introducing and using the notion of adjoint \( p \)-adic multiple zeta values; namely, for each question on \( p \)MZVs which we want to tackle via explicit formulas, we find the appropriate variant of this question for AdpMZVs, we solve the variant, and the step of going back to \( p \)MZVs from adjoint \( p \)MZVs, if we do it, is non-\( p \)-adic, it amounts to a problem of algebra over \( \mathbb{Q} \). \( p \)MZVs and AdpMZVs are equally natural families of values which characterize the Frobenius, and it seems to us that AdpMZVs are actually the most natural from certain points of view.

b - About the formulation related to the motivic Galois theory of periods

We have formulated this work in a way which keeps in mind the concepts of the motivic Galois theory of periods, despite the presence of infinite summations in the formulas. Aside from the regularization of \( p \)-adic iterated integrals, the central objects are the \( p \)-adic pro-unipotent harmonic actions \( \circ_\text{har}^l, \circ_\text{har}^l<<1, \circ_\text{har}^\Sigma \), the maps \( \text{comp}^\Sigma \rightarrow \int, \text{comp}^{l \rightarrow \Sigma} \) expressing the relations between them, and the maps of iterations of the harmonic Frobenius \( \text{iter}_\text{har}^l, \text{iter}_\Sigma \) which we have defined in part I. The \( p \)-adic pro-unipotent harmonic actions keep in mind the motivic Galois theory: they are related to the product \( \circ_\text{Ihara}^l \) (the Ihara product or twisted Magnus product) on \( \Pi_{1,0} = \pi_1^{\text{un,DR}}(\mathbb{P}^1 - \{0,1,\infty\}, -\overrightarrow{1}, \overrightarrow{0}) \), which is itself related to the motivic Galois action on \( \Pi_{1,0} \). In part I, these maps were the way to express the formulas for the Frobenius. In part II, the central idea was that these maps 'reflect algebraic relations' as motivic Galois actions do for periods. More precisely, we have defined multiple harmonic values as certain infinite sequences of prime weighted multiple harmonic sums, and viewed them as periods, we have studied their (completed) algebraic relations, and compared them to the algebraic relations among adjoint \( p \)-adic multiple zeta values.
In a summary, what we have showed is that 1) there exists a computation of the Frobenius which keeps a track of the motivic Galois action, and 2) the motivic Galois theory of $p$-adic multiple zeta values can be extended to multiple harmonic values: we have an "explicit elementary version" of the motivic Galois theory of $p$-adic multiple zeta values, formulated as a comparison between adjoint $p$-adic multiple zeta values and multiple harmonic values via the $p$-adic pro-unipotent harmonic actions.

**c - Other perspectives**

The ideas summarized above, and some variants of them, also play a central role in the three other parts of this work (parts III, IV, V : [J7] to [J12]), and in other forthcoming works on pro-unipotent fundamental groupoids.

**Acknowledgments**

I warmly thank Hidekazu Furusho for having invited me to the conference *Various aspects of multiple zeta values* held in July of 2016 in Kyoto. This text was written for the proceedings of that conference. I thank Kyoto University for hospitality and financial support during the conference. I thank Pierre Cartier for his support in the end of my PhD at Université Paris Diderot. I thank Benjamin Enriquez for his support during my post-doctorate in Université de Strasbourg. This text has been written at Institut de Recherche Mathématique Avancée of Université de Strasbourg, supported by Labex IRMIA, and revised at Université de Genève, supported by NCCR SwissMAP.

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