BOUNDARY INTEGRAL OPERATOR FOR THE FRACTIONAL LAPLACIAN
IN THE BOUNDED SMOOTH DOMAIN

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Abstract. We study the boundary integral operator induced from the fractional Laplace equation in a bounded smooth domain. For \( \frac{1}{2} < \alpha < 1 \), we show the bijectivity of the boundary integral operator \( S_{2\alpha} : L^p(\partial \Omega) \to H_2^{2\alpha-1}(\partial \Omega) \), \( 1 < p < \infty \). As an application, we show the existence of the solution of the boundary value problem of the fractional Laplace equation.

1. Introduction

In this paper, we study a boundary integral operator defined on the boundary of a bounded domain in \( \mathbb{R}^n \), \( n \geq 3 \). Let \( \Omega \) be a bounded smooth domain and \( \Gamma_{2\alpha}(x) = c(n,2\alpha) \frac{1}{|x|^{n-2\alpha}} \) be the Riesz kernel of order \( 2\alpha \), \( 0 < 2\alpha < n \) in \( \mathbb{R}^n \), where \( c(n,2\alpha) \) is the normalized constant. The layer potential of a fractional Laplacian for \( \phi \in L^p(\partial \Omega) \), \( 1 < p < \infty \) is defined by

\[
S_{2\alpha}\phi(x) = \int_{\partial \Omega} \Gamma_{2\alpha}(x-Q)\phi(Q)dQ, \quad x \in \mathbb{R}^n \setminus \partial \Omega. \tag{1.1}
\]

The boundary integral operator

\[
S_{2\alpha} \phi = (S_{2\alpha}\phi)|_{\partial \Omega} \tag{1.2}
\]

is defined by the restriction of \( S_{2\alpha}\phi \) over \( \partial \Omega \).

M. Zähle studied the Riesz potentials in a general metric space \((X, \rho)\) with Ahlfors \( d \)-regular measure \( \mu \). She showed \( S_{2\alpha} : L^2(X,d\mu) \to L^2_{2\alpha}(X,d\mu) \), \( 0 < 2\alpha < n \) is invertible, where \( L^2(X,d\mu) \) is decomposed by the null space \( N(S_{2\alpha}) \) and the orthogonal compliment of \( N(S_{2\alpha}) \), that is, \( L^2(X,d\mu) = N(S_{2\alpha}) \otimes L^2_{2\alpha}(X,d\mu) \) (see [19] and [20]).

The author of [3] showed that the boundary integral operator \( S_{2\alpha} \) defined in (1.2) is extended to \( H_2^{-\alpha+\frac{1}{2}}(\partial \Omega) \), \( \frac{1}{2} < \alpha < 1 \) such that \( S_{2\alpha} : H_2^{-\alpha+\frac{1}{2}}(\partial \Omega) \to H_2^{\alpha-\frac{1}{2}}(\partial \Omega) \) is bijective operator and \( S_{2\alpha}\phi \in H_2^\gamma(\mathbb{R}^n) \) for \( \phi \in H_2^{-\alpha+\frac{1}{2}}(\partial \Omega) \) (see section 2 for the definition of the function spaces).

When \( 2\alpha = 2 \), \( \Gamma_2 \) is the fundamental solution of the Laplace equation in \( \mathbb{R}^n \) and (1.1) is a single layer potential of the Laplace equation. The single layer potential and boundary layer potential of the Laplace equation were studied by many mathematicians to show the existence of the solution of the boundary value problem of the Laplace equation in a bounded domain (see [9], [11], [13] and [18]).
The first result of this paper is the following Theorem.

**Theorem 1.1.** Let \( \Omega \) be a bounded \( C^2 \)-domain in \( \mathbb{R}^n \), \( 3 \leq n \). Let \( \frac{1}{2} < \alpha < 1 \) and \( 1 < p < \infty \). Then, \( S_{2\alpha} : L^p(\partial \Omega) \to H^{2\alpha-1}_p(\partial \Omega) \) is bijective.

The function space \( H^{2\alpha-1}_p(\partial \Omega) \) is defined in the section 2.

The layer potential for \( \phi \in B^s_p(\partial \Omega) \), \( s < 0 \), \( 1 < p < \infty \) is defined by

\[
\mathcal{S}_{2\alpha}\phi(x) = \langle \phi, \Gamma_{2\alpha}(x - \cdot) \rangle, \quad x \in \mathbb{R}^n \setminus \partial \Omega,
\]

(1.3)

where \( \langle \cdot, \cdot \rangle \) is the duality paring between \( B^s_p(\partial \Omega) \) and \( B^{-s}_p(\partial \Omega) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \). In particular, if \( \phi \in L^p(\partial \Omega) \), then \( \mathcal{S}_{2\alpha}\phi \) is defined by (1.4). The second result is the following Theorem.

**Theorem 1.2.** Let \( \frac{1}{2} < \alpha < 1 \) and \( 1 < p < \infty \). For \( \phi \in B^s_p(\partial \Omega) \), let \( u = \mathcal{S}_{2\alpha}\phi \) be a layer potential defined in (1.3). Let \(-2\alpha + 1 - \frac{1}{p} < s < 0\). Then, \( u \in B^{s+2\alpha-1+\frac{1}{p}}_{\text{loc},p}(\mathbb{R}^n) \) and

\[
\|u\|_{B^{s+2\alpha-1+\frac{1}{p}}_{\text{loc},p}(\mathbb{R}^n)} \leq c_R \|\phi\|_{B^s_p(\partial \Omega)},
\]

(1.4)

where \( B_R \) is open ball in \( \mathbb{R}^n \) whose radius is \( R \) and center is origin such that \( \Omega \subset B_R \). Moreover, if \( p > \frac{n-1}{n+s-1} \), the \( u \in B^{s+2\alpha-1+\frac{1}{p}}_p(\mathbb{R}^n) \) such that

\[
\|u\|_{B^{s+2\alpha-1+\frac{1}{p}}_p(\mathbb{R}^n)} \leq c \|\phi\|_{B^s_p(\partial \Omega)}.
\]

(1.5)

The function spaces \( B^{s+2\alpha-1+\frac{1}{p}}_{\text{loc},p}(\mathbb{R}^n) \) and \( B^{s+2\alpha-1+\frac{1}{p}}_p(\mathbb{R}^n) \) are defined in the section 2.

The boundary integral operators (the single layer potential and the double layer potential) have been studied by many mathematicians. The bijectivity of the layer potential has been used to show the existence of the solutions of the partial differential equations in a bounded domain or bounded cylinder (see [3], [4], [7], [10], [12], [15] and [16]).

As in the vast literature, we apply the bijectivity of the boundary integral operator to the boundary value problem of the fractional Laplace equation in the bounded smooth domain. The fractional Laplacian of order \( 0 < \alpha < 1 \) of a function \( v : \mathbb{R}^n \to \mathbb{R} \) is expressed by the formula

\[
\Delta^{\alpha} v(x) = C(n, \alpha) \int_{\mathbb{R}^n} \frac{v(x + y) - 2v(x) + v(x - y)}{|x - y|^{n+2\alpha}} dy,
\]

where \( C(n, \alpha) \) is some normalization constant. The fractional Laplacian can also be defined as a pseudo-differential operator

\[
(-\Delta)^\alpha \hat{v}(\xi) = (2\pi|\xi|)^{2\alpha} \hat{v}(\xi),
\]

where \( \hat{v}(\xi) := \int_{\mathbb{R}^n} v(x) e^{-2\pi i \xi \cdot x} dx, \) \( \xi \in \mathbb{R}^n \) is the Fourier transform of \( v \) in \( \mathbb{R}^n \). In particular, when \( 2\alpha = 2 \) it is the Laplacian \( \Delta v(x) = \sum_{1 \leq i \leq n} \frac{\partial^2}{\partial x_i^2} v(x) \).
**Definition 1.3.** Let \( 0 < \alpha < 1 \). We say that \( v \) is a weak solution of \( \Delta^\alpha u = 0 \) in \( \mathbb{R}^n \setminus \partial \Omega \) if \( v \) satisfies

\[
\int_{\mathbb{R}^n} v(x) \Delta^\alpha \psi(x) dx = \int_{\mathbb{R}^n} (2\pi |\xi|)^{2\alpha} \hat{\psi}(\xi) \hat{\psi}(\xi) d\xi = 0
\]

for all \( \psi \in C_0^\infty(\mathbb{R}^n \setminus \partial \Omega) \).

In fact, if \( u \) is weak solution, then \( u \) is continuous function in \( \mathbb{R}^n \setminus \partial \Omega \) and satisfies

\[
\Delta^\alpha u(x) = 0 \quad \text{for} \quad x \in \mathbb{R}^n \setminus \partial \Omega.
\]

(See Theorem 3.9 in [2]).

For the application of Theorem 1.1 and Theorem 1.2, we show the existence of the solution of the boundary value problem of the fractional Laplace equation in the bounded smooth domain.

**Theorem 1.4.** Let \( \Omega \) be a bounded \( C^2 \) domain in \( \mathbb{R}^n \), \( 3 \leq n \) and \( \frac{1}{2} < \alpha < 1 \). Let \( 0 < t < 1 \) and \( 1 < p < \infty \). Then, for given \( g \in B_t^p(\partial \Omega) \), the following equation

\[
\begin{align*}
\Delta^\alpha u &= 0 \quad \text{in} \quad \mathbb{R}^n \setminus \partial \Omega, \\
u|_{\partial \Omega} &= g \in B_t^p(\partial \Omega), \\
u &= B_{t+\frac{1}{2}}^{t}p(\mathbb{R}^n), \\
|u(x)| &= O(|x|^{-n+2\alpha}) \quad \text{near} \quad \infty.
\end{align*}
\]

has a weak solution. And, if \( \frac{n-1}{n+1-2\alpha} < p < \infty \), then \( u \in B_{t+\frac{1}{2}}^{t}p(\mathbb{R}^n) \). Furthermore, there exists \( \phi \in B_{t+\frac{1}{2}}^{t}p(\mathbb{R}^n) \) such that

\[
u = S_{2\alpha}\phi.
\]

The rest of this paper is organized as follows. In section 2 we introduce the several function spaces. In section 3 we introduce the several property of the layer potential. In section 4 we prove the Theorem 1.1. In section 5 we prove the Theorem 1.2. In section 6 we prove the Theorem 1.4.

### 2. Function spaces

#### 2.1. Function spaces in \( \mathbb{R}^n \)

In this section, we introduce the several function spaces, Sobolev space and Besov space. For \( s \in \mathbb{R} \), we consider a distribution \( G_s \) whose Fourier transform in \( \mathbb{R}^n \) is defined by

\[
\hat{G}_s(\xi) = (1 + 4\pi^2|\xi|^2)^{-\frac{s}{2}}.
\]

For \( s \in \mathbb{R} \), \( 1 \leq p \leq \infty \), we define the Sobolev space \( H_p^s(\mathbb{R}^n) \) by

\[
H_p^s(\mathbb{R}^n) := \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{H_p^s(\mathbb{R}^n)} := \| G_{-s} * f \|_{L_p(\mathbb{R}^n)} < \infty \}.
\]
where \( * \) is a convolution in \( \mathbb{R}^n \) and \( \mathcal{S}'(\mathbb{R}^n) \) is the dual space of the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \). In particular, when \( s = k \in \mathbb{N} \cup \{0\} \) and \( 1 < p < \infty \),

\[
H^k_p(\mathbb{R}^n) = \{ f \mid D^\beta f \in L^p(\mathbb{R}^n), \quad |\beta| \leq k \},
\]

where \( \beta \in (\mathbb{N} \cup \{0\})^n \) and \( |\beta| = \beta_1 + \beta_2 + \cdots + \beta_n \) for \( \beta = (\beta_1, \beta_2, \ldots, \beta_n) \).

For \( k < s < k + 1 \), \( k \in \mathbb{N} \), we define the Besov space \( B^s_p(\mathbb{R}^n) \) and homogeneous Besov space \( \dot{B}^s_p(\mathbb{R}^n) \) by

\[
B^s_p(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{B^s_p} < \infty \}, \quad \dot{B}^s_p(\mathbb{R}^n) = \{ f \in \mathcal{S}'(\mathbb{R}^n) \mid \| f \|_{\dot{B}^s_p} < \infty \}
\]

with the norms

\[
\| f \|_{B^s_p} := \| f \|_{H^s_p} + \left( \sum_{|\beta|=k} \frac{\int \int_{\mathbb{R}^n \times \mathbb{R}^n} |D^\beta f(x) - D^\beta f(y)|^p |x-y|^{n+ps-k} \, dydx}{|x-y|^{n+ps-k}} \right)^{\frac{1}{p}},
\]

\[
\| f \|_{\dot{B}^s_p} := \left( \sum_{|\beta|=k} \frac{\int \int_{\mathbb{R}^n \times \mathbb{R}^n} |D^\beta f(x) - D^\beta f(y)|^p |x-y|^{n+ps-k} \, dydx}{|x-y|^{n+ps-k}} \right)^{\frac{1}{p}}.
\]

If \( s \in \mathbb{R} \) is negative, then we define \( B^s_p \) and \( \dot{B}^s_p \) as the dual spaces of \( B^{-s}_{p'} \) and \( \dot{B}^{-s}_{p'} \), respectively, where \( \frac{1}{p} + \frac{1}{p'} = 1 \). The real interpolation method and complex interpolation method give

\[
(H^s_{p_0}, H^s_{p_1})_{\theta,p} = B^s_p, \quad [H^s_{p_0}, H^s_{p_1}]_\theta = H^s_p
\]

for \( s = (1 - \theta)s_0 + \theta s_1, \quad s_0, s_1 \in \mathbb{R}, \quad 0 < \theta < 1 \) (see theorem 6.4.5 in [1]).

2.2. Function spaces in \( \Omega \). Let \( \Omega \) be bounded \( C^2 \)-domain in \( \mathbb{R}^n \). Let \( R_\Omega f \) be a restriction over \( \Omega \) of the function \( f \) defined in \( \mathbb{R}^n \). For \( s \geq 0 \), we define the function spaces

\[
H^s_p(\Omega) := \{ R_\Omega f \mid f \in H^s_p(\mathbb{R}^n) \}, \quad B^s_p(\mathbb{R}^n) := \{ R_\Omega f \mid f \in B^s_p(\mathbb{R}^n) \}
\]

with norms

\[
\| f \|_{H^s_p(\Omega)} := \inf \| F \|_{H^s_p(\mathbb{R}^n)}, \quad \| f \|_{B^s_p(\Omega)} := \inf \| F \|_{B^s_p(\mathbb{R}^n)},
\]

where infimums are taken in \( F \in H^s_p(\mathbb{R}^n) \) and \( F \in B^s_p(\mathbb{R}^n) \), respectively, such that \( R_\Omega F = f \).

Note that for non-negative integer \( k \) and \( 1 < p < \infty \),

\[
H^k_p(\Omega) = \{ f \in L^p(\Omega) \mid \sum_{0 \leq l \leq k} \| D^l f \|_{L^p(\Omega)} < \infty \}
\]

and for \( 0 < \theta < 1 \),

\[
(H^{k_1}_p(\Omega), H^{k_2}_p(\Omega))_{\theta,p} = B^s_p(\Omega), \quad [H^{k_1}_p(\Omega), H^{k_2}_p(\Omega)]_{\theta} = H^s_p(\Omega),
\]

where \( s = \theta k_1 + (1 - \theta) k_2 \) (see chapter 2 in [13]). In particular, for \( k < s < k + 1 \), we have

\[
\| f \|_{H^s_p(\Omega)} \approx \| f \|_{H^k_p(\Omega)} + \left( \sum_{|\beta|=k} \int_{\Omega} \int_{\Omega} \frac{|D^\beta f(x) - D^\beta f(y)|^p}{|x-y|^{n+ps-k}} \, dxdy \right)^{\frac{1}{p}}.
\]
For negative \( s \in \mathbb{R} \), we define \( B_{p}^{s}(\Omega) \) and \( H_{p}^{s}(\Omega) \) as the dual spaces of \( B_{p}^{-s}(\Omega) \) and \( H_{p}^{-s}(\Omega) \), respectively, where \( \frac{1}{p} + \frac{1}{p'} = 1 \).

For \( s > 0 \), we define the spaces \( H_{p0}^{s}(\Omega) \) and \( B_{p0}^{s}(\Omega) \) as the closures of \( C_{c}^{\infty}(\Omega) \) in \( H_{p}^{s}(\Omega) \) and \( B_{p}^{s}(\Omega) \), respectively.

### 2.3. Function spaces in \( \partial \Omega \)

Let \( \Omega \) be a bounded \( C^{2} \)-domain in \( \mathbb{R}^{n} \). Let us \( \Delta(P,r) = B(P,r) \cap \partial \Omega \) for \( P \in \partial \Omega \). Then, there is \( r_{0} > 0 \) such that for all \( P \in \partial \Omega \), there is bijective \( C^{2} \)-function \( \Psi : B'(0,r_{0}) \to \Delta(P,r_{0}) \), where \( B'(0,r_{0}) \) is open balls in \( \mathbb{R}^{n-1} \) whose radius is \( R \) and center is origin. Since \( \Omega \) is bounded domain, there are \( P_{1}, P_{2}, \ldots, P_{N} \) such that \( \partial \Omega \subset \cup \Delta(P_{i}, r_{0}) \). Moreover, there are bijective \( C^{2} \)-functions \( \Psi_{i} : B'(0,r_{0}) \to \Delta(P_{i}, r_{0}) \). Now, we say that \( \phi \) is in function space \( H_{p}^{s}(\partial \Omega) \), \( -2 \leq s \leq 2 \) if \( \phi \circ \Psi_{i} \in H_{p}^{s}(B'(0,r_{0})) \) for all \( 1 \leq i \leq N \). And the norm is

\[
\| \phi \|_{H_{p}^{s}(\partial \Omega)} = \sum_{1 \leq i \leq N} \| \phi \circ \Psi_{i} \|_{H_{p}^{s}(B'(0,r_{0}))}.
\]

Similarly, we define the function space \( B_{p}^{s}(\partial \Omega) \). Clearly, for \( 0 < s < 2 \), \( H_{p}^{-s}(\partial \Omega) \) and \( B_{p}^{-s}(\partial \Omega) \) are dual spaces of \( H_{p}^{s}(\partial \Omega) \) and \( B_{p}^{s}(\partial \Omega) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), respectively.

For \( 0 < \theta < 1 \),

\[
(H_{p}^{k_{1}}(\partial \Omega), H_{p}^{k_{2}}(\partial \Omega))_{\theta,p} = B_{p}^{\theta}(\partial \Omega), \quad [H_{p}^{k_{1}}(\partial \Omega), H_{p}^{k_{2}}(\partial \Omega)]_{\theta} = H_{p}^{\theta}(\partial \Omega),
\]

where \( s = (1 - \theta)k_{1} + \theta k_{2} \) (see chapter 2 in \[13\]).

### 3. Boundary layer potential

Let \( S_{2\alpha} \) be a boundary layer potential defined by \[13\]. Because of \( \partial \Omega \) is \( C^{2} \), we get the following Proposition:

**Proposition 3.1.** Let \( \Omega \) be \( C^{2} \) bounded domain. For \( -2 \leq s \leq 3 - 2\alpha \) and \( 1 < p < \infty \),

\[ S_{2\alpha} : H_{p}^{s}(\partial \Omega) \to H_{p}^{s+2\alpha-1}(\partial \Omega) \]

is bounded operator. Moreover, for \( -1 \leq s \leq 0 \),

\[ S_{2} : H_{p}^{s}(\partial \Omega) \to H_{p}^{1+s}(\partial \Omega) \]

is bijective.

(See \[13\]).

Let \( \psi, \phi \in C^{2}(\partial \Omega) \) and \( S_{2\alpha} : H_{p}^{-s-2\alpha+1}(\partial \Omega) \to H_{p}^{-s}(\partial \Omega) \) is a dual operator of \[3.1\], where \( \frac{1}{p} + \frac{1}{p'} = 1 \). Using \[1.1\], we have

\[
\langle \langle S_{2\alpha}^{*} \psi, \phi \rangle \rangle = \langle \psi, S_{2\alpha} \phi \rangle = \int_{\partial \Omega} \psi(P) S_{2\alpha} \phi(P) dP
\]

\[
= \int_{\partial \Omega} \phi(P) S_{2\alpha} \psi(P) dP = \langle \phi, S_{2\alpha} \psi \rangle,
\]

(3.2)
Lemma 4.1. Let \( \kappa \) be a smooth function. Given sufficiently small \( \epsilon > 0 \), there are bounded linear operators \( T^1 : L^p(\partial \Omega) \to H^1_p(\partial \Omega) \) with \( \|T^1\|_{L^p(\partial \Omega) \to H^1_p(\partial \Omega)} < c\epsilon \) and \( T^2 : H^{-1}_p(\partial \Omega) \to H^1_p(\partial \Omega) \) such that
\[
S_{2\alpha}S_{3-2\alpha} = S_2 + T^1 + T^2. \tag{4.1}
\]

Remark 4.2.

1. Since \( S_2 : L^p(\partial \Omega) \to H^1_p(\partial \Omega) \) is bijective, for sufficiently small \( \epsilon > 0 \), \( S_2 + T^1 : L^p(\partial \Omega) \to H^1_p(\partial \Omega) \) is bijective.

2. Since \( S_2 \), \( S_{2\alpha}S_{3-2\alpha} \) and \( T^2 \) are bounded operators from \( H^{-1}_p(\partial \Omega) \) to \( L^p(\partial \Omega) \), \( T^1 : H^{-1}_p(\partial \Omega) \to L^p(\partial \Omega) \) is also bounded operator. Then, by the complex interpolation property \( \mathbb{L} \), we get that for \(-1 < s < 0\),
\[
\|T^1\|_{H^{-1}_p(\partial \Omega) \to H^{1+s}_p(\partial \Omega)} \leq c\epsilon^{1+s}. \tag{4.2}
\]

3. By the argument of (1), (2) and by Proposition \( \mathbb{L} \) we get \( S_2 + T^1 : H^s_p(\partial \Omega) \to H^{1+s}_p(\partial \Omega) \), \(-1 < s < 0\) is bijective.

Proof of Lemma 4.1. Let \( 0 < 9\epsilon < r_0 \), where \( r_0 > 0 \) is defined in a section \( \mathbb{L} \). Let \( P_1, P_2, \ldots, P_m \in \partial \Omega \) such that \( |P_i - P_j| > \epsilon \) and \( \partial \Omega \subset \cup_{1 \leq i \leq m} B(P_i, \frac{3}{2}\epsilon) \). Let \( \{\eta_i\}_{1 \leq i \leq m} \) be a partition of unity of \( \{B(P_i, 2\epsilon)\}_{1 \leq i \leq m} \) such that \( \text{supp} \eta_i \subset B(P_i, 2\epsilon), \eta_i \equiv 1 \) in \( B(P_i, \frac{3}{2}\epsilon) \). Let \( \kappa_i = \sum_{|P_i - P_j| \leq 5\epsilon} \eta_j \) and \( \lambda_i = \sum_{|P_i - P_j| \leq 7\epsilon} \eta_j \) such that \( \text{supp} \kappa_i \subset B(P_i, 7\epsilon) \) and \( \text{supp} \lambda_i \subset B(P_i, 9\epsilon) \).

Then, for \( \phi \in L^p(\partial \Omega) \), we have
\[
S_{2\alpha}S_{3-2\alpha}\phi = \sum_i \eta_i S_{2\alpha} \kappa_i S_{3-2\alpha}\phi + \sum_i \eta_i S_{2\alpha} (1 - \kappa_i) S_{3-2\alpha}\phi
= I_1\phi + I_2\phi.
\]

Note that for \( P \in \Delta(P_i, 2\epsilon), \kappa_i(\cdot) \Gamma_{2\alpha}(P - \cdot) \) has no singularity in \( \partial \Omega \setminus \Delta(P_i, 3\epsilon) \) and so \( I_2\phi \) is a smooth function.
For $I_1\phi$, we have

$$I_1\phi(P) = \sum_i \eta_i S_{2a} \kappa_i S_{3-2a} \lambda_i \phi(P) + \sum_i \eta_i S_{2a} \kappa_i S_{3-2a} (1 - \lambda_i) \phi(P)$$

$$= \sum_i \eta_i(P) \int_{\partial \Omega} \Gamma_{2a}(P - Z) \kappa_i(Z) \int_{\Delta(P, 9\epsilon)} \lambda_i(Q) \Gamma_{3-2a}(Z - Q) \phi(Q) dQ dZ$$

$$+ \sum_i \eta_i(P) \int_{\partial \Omega} \Gamma_{2a}(P - Z) \kappa_i(Z) \int_{\partial \Omega \setminus \Delta(P, 9\epsilon)} (1 - \lambda_i(Q)) \Gamma_{3-2a}(Z - Q) \phi(Q) dQ dZ$$

$$:= \sum_i I_{11}^i \phi(P) + \sum_i I_{12}^i \phi(P)$$

$$:= I_{11} \phi(P) + I_{12} \phi(P).$$

For fixed $Z \in \Delta(P_1, 7\epsilon)$, $\Gamma_{3-2a}(Z - \cdot)$ has no singularity in $\partial \Omega \setminus \Delta(P_1, 9\epsilon)$ and so $\kappa_i(Z) \int_{\partial \Omega \setminus \Delta(P_1, 9\epsilon)} (1 - \lambda_i(Q)) \Gamma_{3-2a}(Z - Q) \phi(Q) dQ$ is a smooth function in $\partial \Omega$. Hence, by Proposition 3.1, $I_{12} \phi$ is a smooth function.

Similarly, we decompose $S_2 \phi$ as

$$S_2 \phi(P) = J_{11} \phi(P) + J_{12} \phi(P) + J_2 \phi(P),$$

where $J_{12} \phi$, $J_2 \phi$ are smooth functions and

$$J_{11} \phi(P) = \sum_i J_{11}^i \phi(P)$$

with $J_{11}^i \phi(P) := \eta_i(P) \int_{\Delta(P, 9\epsilon)} \Gamma_2(P - Q) \lambda_i(Q) \phi(Q) dQ.$

For $I_{11} \phi$, $J_{11} \phi$, we fix $i$. Using the translation and rotation, we assume that $P_i = 0$ and there is $\Psi : B'(0, 9\epsilon) \to \mathbb{R}$ with

$$|\Psi_i(x')| < c|x'|^2 < \epsilon c^2, \quad |\nabla \Psi_i(x')| < c|x'| < \epsilon c, \quad \text{for } x' \in B'(0, 9\epsilon) \quad (4.3)$$

such that for $Q \in \Delta(P_i, 9\epsilon) := \Delta_{9\epsilon}$, $Q$ is represented by $Q = (y', \Psi(y'))$ for some $y' \in B'(0, 9\epsilon) := B'(9\epsilon)$. Let $P = (x', \Psi(x'))$, $x' \in B'(0, 2\epsilon)$. Then, we have

$$I_{11}^i \phi(P) = \eta_i(P) \int_{\Delta_{9\epsilon}} \lambda_i(Q) \phi(Q) \int_{\partial \Omega} \kappa_i(Z) \Gamma_{2a}(P - Z) \Gamma_{3-2a}(Q - Z) dZ dQ$$

$$= \eta_i(P) \int_{B'(9\epsilon)} \lambda_i(y', \Psi(y')) \phi(y', \Psi(y')) \sqrt{1 + |\nabla \Psi(y')|^2} \times$$

$$\int_{B'(7\epsilon)} \kappa_i(z', \Psi(z')) \Gamma_{2a}(x' - z', \Psi(x') - \Psi(z')) \Gamma_{3-2a}(y' - z', \Psi(y') - \Psi(z')) \sqrt{1 + |\nabla \Psi(z')|^2} dz' dy'$$

and

$$J_{11}^i \phi(P) := \eta_i(P) \int_{B'(9\epsilon)} \Gamma_2(x' - y', \Psi(x') - \Psi(y')) \lambda_i(y', \Psi(y')) \phi(y', \Psi(y')) \sqrt{1 + |\nabla \Psi(y')|^2} dy'.$$
Let
\[
I^{i}_{111}\phi(P) := \eta(P) \int_{B'(\gamma)} \lambda_i(y', 0) \phi(y', \Psi(y')) \int_{B'(\gamma)} \kappa_i(z', 0) \Gamma_{2\alpha}(x' - z', 0) \Gamma_{3 - 2\alpha}(y' - z', 0) dz' dy',
\]
\[
J^{i}_{111}\phi(P) := \eta(P) \int_{B'(\gamma)} \lambda_i(y', 0) \Gamma_2(x' - y', 0) \phi(y', \Psi(y')) dy'.
\]
Note that by (4.3), we have
\[
\|I^{i}_{111} - I^{i}_{111}\|_{L^p(\Delta_\alpha)} \rightarrow H^2_\alpha(\Delta_\alpha), \quad \|J^{i}_{111} - J^{i}_{111}\|_{L^p(\Delta_\alpha)} \rightarrow H^2_\alpha(\Delta_\alpha) \leq \epsilon.
\]
(See [6]).

It is well-known
\[
\int_{\mathbb{R}^{n-1}} \Gamma_2(x' - y', 0) \Gamma_{3 - 2\alpha}(y' - z', 0) dz' = \Gamma_2(x' - y', 0).
\]
(See Section 5.1 in [17]). Hence, we have
\[
\Gamma_2(x' - y', 0) = \int_{\mathbb{R}^{-1}} \kappa_i(z', 0) \Gamma_{2\alpha}(x' - z', 0) \Gamma_{3 - 2\alpha}(y' - z', 0) dz' + \int_{\mathbb{R}^{n-1}} (1 - \kappa_i(z', 0)) \Gamma_{2\alpha}(x' - z', 0) \Gamma_{3 - 2\alpha}(y' - z', 0) dz' = \int_{\mathbb{R}^{n-1}} \kappa_i(z', 0) \Gamma_2(x' - z', 0) \Gamma_{3 - 2\alpha}(y' - z', 0) dz' + k_i(x', y').
\]
Hence, we have that
\[
I^{i}_{111}\phi(P) - J^{i}(111)\phi(P) = \eta(P) \int_{B'(\gamma)} \lambda_i(y', 0) \phi(y', \Psi(y')) k_i(x', y') dy'
\]
is a smooth function. Let \( T^1 = \sum_i (I^{i}_{111} - I^{i}_{111}) + \sum_i (J^{i}_{111} - J^{i}_{111}) \) and \( T_2 = I_2 + J_2 + I_{12} + J_{12} + \sum_i (I^{i}_{111} - J^{i}(111)) \). Then, \( S_{2\alpha}S_{3-2\alpha} = S_2 + T^1 + T^2 \) such that \( T^2 \) is a smooth function and
\[
\|T^1\|_{H^2_\alpha(\partial\Omega)} \leq \epsilon \sum_i \|I^{i}_{111} - J^{i}_{111}\|_{L^p(\Delta_\alpha)} + \|J^{i}_{111} - J^{i}_{111}\|_{L^p(\Delta_\alpha)} \leq \epsilon \sum_i \|\phi\|_{L^p(\Delta_\alpha)}
\]
Hence, we complete the proof of Lemma 4.1. □

**Proof Theorem 1.1**

(1). In the case of \( p \geq p_0 := \frac{2(n-1)}{n-2+2\alpha} \) (\( p_0 < 2 \)). To show the injectivity, suppose that \( S_{2\alpha} = 0 \) for \( \phi \in L^p(\partial\Omega) \). By the Hölder inequality and Sobolev imbedding, we have \( L^p(\partial\Omega) \subset L^{p_0}(\partial\Omega) \subset H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \). Hence, by the bijectivity of \( S_{2\alpha} : H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \rightarrow H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \) (see [5]), we have \( \phi = 0 \). Hence, \( S_{2\alpha} : L^p(\partial\Omega) \rightarrow H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \) is bijective for \( p \geq p_0 \).

To show that \( S_{2\alpha} : L^p(\partial\Omega) \rightarrow H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \) is surjective, let \( f \in H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \). By the Sobolev imbedding, the Hölder inequality, and the bijectivity, we have \( H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \subset H^{2\alpha-1}_p(\partial\Omega) \subset H^{2\alpha-1}_p(\partial\Omega) \subset H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \). By the bijectivity of \( S_{2\alpha} : H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \rightarrow H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \), there is \( \phi \in H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \) such that \( S_{2\alpha}\phi = f \). Note that by the Lemma 4.1, we get that \( S_{3-2\alpha}S_{2\alpha} = S_2 + T^1 + T^2 \), where \( \|T^1\|_{L^p(\partial\Omega)} \rightarrow H^{-\alpha+\frac{1}{2}}_2(\partial\Omega) \) is bounded. Then, by the Lemma 4.1, we obtain that \( (S_2 + T^1)\phi = \).
$S_{3-2a}S_{2a}\phi - T^{2}\phi \in H_{p}^{1}(\partial \Omega)$. Taking $\epsilon > 0$ sufficiently small such that $S_{2} + T^{1} : L^{p}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is bijective (see (1) of the Remark 4.2), we obtain that $\phi \in L^{p}(\partial \Omega)$. This implies that $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2-2a}(\partial \Omega)$ is surjective. Hence, we complete the proof of the bijectivity of $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2-2a}(\partial \Omega)$ for $p \geq \frac{2(n-1)}{n-2+2a}$.

\[\text{Remark 4.3.}\]
(1) Note that the dual operator $S_{2a}^{*} : H_{p}^{-2a+1}(\partial \Omega) \to L^{p}(\partial \Omega)$ of $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$ are same with the operator $S_{2} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$, where $\frac{1}{p} + \frac{1}{p^{*}} = 1$ by the section 3. Hence, by the property of the dual operator, we get that $S_{2} : L^{p}(\partial \Omega) \to L^{p}(\partial \Omega)$ is bijective. This implies that $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$ is bijective for $1 < p \leq \frac{2(n-1)}{n-2+2a}$.

(2) In the Lemma 4.1, $S_{3-2a}S_{2a}$ is sum of a bijective operator $S_{2} + T^{1}$ and a compact operator $T^{2}$ and so $S_{3-2a}S_{2a}$ is the Fredholm operator with index zero. Since $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$, $S_{3-2a} : H_{p}^{2a-1}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is injective, $S_{3-2a}S_{2a}$ is injective and so by the Fredholm operator theorem, $S_{3-2a}S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is bijective. This implies that $S_{3-2a} : H_{p}^{2a-1}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is bijective for $p \geq \frac{2(n-1)}{n-2+2a}$.

(2). In the case of $1 < p < 2$. Now, we will show that $S_{2a} : H_{q}^{-2a+1}(\partial \Omega) \to L^{q}(\partial \Omega)$ is surjective for $q = \frac{p}{p-1} > 2$. Let $f \in L^{q}(\partial \Omega)$. By the Hölder inequality, $L^{q}(\partial \Omega) \subset L^{2}(\partial \Omega)$ and by the bijectivity of $S_{2a} : H_{2}^{2a+1}(\partial \Omega) \to L^{3}(\partial \Omega)$ (see (1) of Remark 4.2), there is $\phi \in H_{2}^{2a+1}(\partial \Omega)$ such that $S_{2a}\phi = f$. Then,

$$S_{3-2a}S_{2a}\phi = S_{3-2a}f \in H_{q}^{2-2a}(\partial \Omega).$$

By (3) of Remark 4.2 we obtain that $(S_{2} + T^{1})\phi \in H_{q}^{2-2a}(\partial \Omega)$. Since $S_{2} + T^{1} : H_{q}^{-2a+1}(\partial \Omega) \to H_{q}^{2-2a}(\partial \Omega)$ is bijective (see (3) of Remark 4.2), we obtain that $\phi \in H_{q}^{-2a+1}(\partial \Omega)$. This implies that $S_{2a} : H_{q}^{-2a+1}(\partial \Omega) \to L^{q}(\partial \Omega)$ is surjective.

By the property of the dual operator, we have that $S_{2a}^{*} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$, $1 < p < 2$ are injective. Since $S_{2a} = S_{2a}^{*}$, $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$ is injective and so $S_{2a} : H_{p}^{2a-1}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is injective. Hence, we get $S_{2a}S_{3-2a} : L^{p}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$, $1 < p < 2$ are injective.

Note that in (3) of Remark 4.2, $S_{2a}S_{3-2a}$ is the sum of a bijective operator and a compact operator. Hence, by the Fredholm theorem, $S_{2a}S_{3-2a} : L^{p}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is bijective.

To show that $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$ is surjective for $1 < p < 2$, let us $f \in H_{p}^{2a-1}(\partial \Omega)$. Then, we have $S_{3-2a}f \in H_{p}^{1}(\partial \Omega)$. Since, $S_{3-2a}S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{1}(\partial \Omega)$ is bijective, there is $\phi \in L^{p}(\partial \Omega)$ such that $S_{3-2a}S_{2a}\phi = S_{3-2a}f$. Since $S_{3-2a}$ is injective, we get $S_{2a}\phi = f$ and so $S_{2a} : L^{p}(\partial \Omega) \to H_{p}^{2a-1}(\partial \Omega)$ is bijective. □
Corollary 4.4. Let $\frac{1}{2} < \alpha < 1$ and $1 < p < \infty$. For $-2\alpha + 1 \leq s \leq 2 - 2\alpha$,
\[
S_{2\alpha} : H^s_p(\partial \Omega) \to H^{s+2\alpha-1}_p(\partial \Omega),
\]
\[
S_{2\alpha} : B^s_p(\partial \Omega) \to B^{s+2\alpha-1}_p(\partial \Omega)
\]
are bijective.

Proof. In the proof of the Theorem 1.1, we have
\[
S_{3-2\alpha} : L^p(\partial \Omega) \to H^{2-2\alpha}_p(\partial \Omega), \quad S_{2\alpha} : H^{2-2\alpha}_p(\partial \Omega) \to H^1_p(\partial \Omega)
\]
are injective and so $S_{2\alpha}S_{3-2\alpha} : L^p(\partial \Omega) \to H^1_p(\partial \Omega)$ is injective. Since $S_{2\alpha}S_{3-2\alpha} : L^p(\partial \Omega) \to H^1_p(\partial \Omega)$ is Fredholm operator with index zero, we get $S_{2\alpha}S_{3-2\alpha} : L^p(\partial \Omega) \to H^1_p(\partial \Omega)$ is bijective. (4.4)

By the dual operator property and the fact of $S_{2\alpha}^* = S_{2\alpha}$, we have
\[
S_{2\alpha} : H^{-1}_p(\partial \Omega) \to H^{-2+2\alpha}_p(\partial \Omega)
\]
is bijective. (4.5)

Using (4.4), (4.5) and the properties of the real interpolation and complex interpolation, we obtain the corollary. □

5. Proof of Theorem 1.2

We introduce a Riesz potential $I_{2\alpha}$, $0 < 2\alpha < n$, by
\[
I_{2\alpha}f(x) = c(n,\alpha) \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2\alpha}} f(y) dy \quad \text{for} \quad \psi \in C_0^\infty(\mathbb{R}^n),
\]
where $c(n,\alpha) := \frac{(2\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2} - \alpha\right)}{\pi^{\frac{n}{2}} \Gamma(\alpha)}$.

The following proposition is well known fact and will be useful in the subsequent estimates (see chapter 5 of [17]).

Proposition 5.1. 1). Let $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{n}$. Then
\[
I_{2\alpha} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)
\]
is bounded.

2). Let $1 < p < \infty$ and $s \in \mathbb{R}$. Then, the following operators are bounded.
\[
I_{2\alpha} : H^s_p(\mathbb{R}^n) \to H^{s+2\alpha}_p(\mathbb{R}^n), \quad I_{2\alpha} : B^s_p(\mathbb{R}^n) \to B^{s+2\alpha}_p(\mathbb{R}^n).
\]

Remark 5.2. Let $B_R$ be the open ball in $\mathbb{R}^n$ centered at the origin with radius $R$. Then, by proposition 5.1, the following operator is bounded.
\[
\tilde{I}_{2\alpha} : B^s_p(\partial \Omega) \to B^{s+2\alpha}_p(\partial \Omega), \quad \tilde{I}_{2\alpha} f(x) = \int_{\mathbb{R}^n} \Gamma_{2\alpha}(x-y)f(y) dy, \quad f \in B^s_p(\partial \Omega), \quad s \in \mathbb{R}.
\]
Proof of (1.3) Let $-2\alpha + 1 - \frac{1}{p} < s < 0$. Let $\phi \in C^1(\partial \Omega)$ and $f \in C^\infty_c(B_R)$. Then, we have

$$\int_{\mathbb{R}^n} f(x) S_{2\alpha} \phi(x) dx = \int_{\partial \Omega} \phi(P) \mathcal{I}_{2\alpha} f(P) dP.$$

Since $C^1(\partial \Omega)$ is dense subspace of $B^s_p(\partial \Omega)$ and $C^\infty_c(B_R)$ is dense subspace of $B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R)$, we have

$$< f, S_{2\alpha} \phi > (B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R), B^s_{p} + \frac{2\alpha+1}{p} (B_R)) = < \phi, \mathcal{I}_{2\alpha} f > (B^s_{p} + \frac{2\alpha+1}{p} (B_R), B^{-s+\frac{1}{p}}_q (B_R)),$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$< f, S_{2\alpha} \phi > (B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R), B^s_{p} + \frac{2\alpha+1}{p} (B_R)) \leq \| \phi \|_{B^s_{p}(\partial \Omega)} \| \mathcal{I}_{2\alpha} f \|_{B^{-s+\frac{1}{p}}_q (B_R)} \leq c \| \phi \|_{B^s_{p}(\partial \Omega)} \| f \|_{B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R)} \leq c \| \phi \|_{B^s_{p}(\partial \Omega)} \| f \|_{B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R)}.$$

Hence, we have

$$\| S_{2\alpha} \phi \|_{B^s_{p} + \frac{2\alpha+1}{p} (B_R)} = \sup_{B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R)} \| f \|_{B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R)} = 1 | < f, S_{2\alpha} \phi > (B^{-s-2\alpha+\frac{1}{p}}_{q_0}(B_R), B^s_{p} + \frac{2\alpha+1}{p} (B_R)) | \leq c \| \phi \|_{B^s_{p}(\partial \Omega)}.$$

We complete the proof of (1.3). $\square$

Proof of (1.5). For $\phi \in B^s_{p}(\partial \Omega)$, $-2\alpha + 1 - \frac{1}{p} < s < 0$, let us $u$ be a the layer potential of $\phi$ defined by (1.3). Note that $u$ is in $C^\infty(\mathbb{R}^n \setminus \partial \Omega)$ and for large $|x|$, we have

$$|D^\beta u(x)| \leq \| \phi \|_{B^s_{p}(\partial \Omega)} \| D^\beta \Gamma_{2\alpha}(x - \cdot) \|_{B^{-s+\frac{1}{p}}_q(\partial \Omega)} \leq c \| \phi \|_{B^s_{p}(\partial \Omega)} \frac{1}{|x|^{n-2\alpha+s}} ,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Let $B_R$ be an open ball whose center is origin and radius is $R \geq 2$ such that $\Omega \subset B_\frac{1}{2} R$. We divide the left-hand side of (1.5) into three parts

$$A_1 = \int_{|x| \leq R} \int_{|y| \leq R} \frac{|D^\beta u(x) - D^\beta u(y)|^p}{|x-y|^{n+p(x+2\alpha+1-k)\frac{1}{p}}} dy dx,$$

$$A_2 = 2 \int_{|x| \leq R} \int_{|y| \geq R} \frac{|D^\beta u(x) - D^\beta u(y)|^p}{|x-y|^{n+p(x+2\alpha+1-k)\frac{1}{p}}} dy dx,$$

$$A_3 = \int_{|x| \geq R} \int_{|y| \geq R} \frac{|D^\beta u(x) - D^\beta u(y)|^p}{|x-y|^{n+p(x+2\alpha+1-k)\frac{1}{p}}} dy dx.$$

By (1.4), $A_1$ is dominated by $\| \phi \|^p_{B^s_{p}(\partial \Omega)}$. For $|x| \leq R$ and $|y| \geq 2R$, we get that $|x-y| \geq |y| - |x| \geq |y| - R \geq \frac{1}{2} |y|$. Note that by (1.1), for $|y| \geq 2R$, we have that $|D^\beta u(y)| \leq c |y|^{-n+2\alpha-k} \| \phi \|^2_{B^s_{p}(\partial \Omega)}$.
Hence, by (1.4), we have
\[
A_2 \leq 2 \int_{|x| \leq R} \int_{|y| \leq 2R} \frac{|D^k_x u(x) - D^k_y u(y)|^p}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx \\
+ 2^{n+2\alpha+2} \int_{|x| \leq R} \int_{|y| \geq 2R} \frac{|D^k_x u(x)|^p + |D^k_y u(y)|^p}{|y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx \\
\leq c_R \|u\|_{B^p_{\ell}(\partial \Omega)}^p + c\|\phi\|_{B^p_{\ell}(\partial \Omega)}^p \int_{|x| \leq R} \int_{|y| \geq 2R} \frac{1}{|y|^{n+p(s-1+\frac{k}{p}+n)}} dy dx \\
\leq c_R \|\phi\|_{B^p_{\ell}(\partial \Omega)}^p.
\]

We divide $A_3$ into two parts:
\[
\int_{|x| \geq R} \int_{|y| \geq R, |x-y| \leq \frac{1}{2}|x|} \frac{|D^k_x u(x) - D^k_y u(y)|^p}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx \\
+ \int_{|x| \geq R} \int_{|y| \geq R, |x-y| \geq \frac{1}{2}|x|} \frac{|D^k_x u(x) - D^k_y u(y)|^p}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx.
\]

Applying the mean-value theorem, for $|x| \geq R$, $|x-y| \leq \frac{1}{2}|x|$, there is a $\xi$ between $x$ and $y$ such that $D^k_x u(x) - D^k_y u(y) = D^{k-1} u(\xi) \cdot (x-y)$. Note that $|x-\xi| \leq \frac{1}{2}|x|$ and hence $|\xi| \geq \frac{1}{2}|x| \geq \frac{1}{2}R$.

Since $s + 2\alpha - k - 2 + \frac{1}{p} < 0$ and $p > \frac{n-1}{n+s-1}$, by (5.1), the first term of (5.3) is dominated by
\[
\int_{|x| \geq R} \int_{|y| \geq R, |x-y| \leq \frac{1}{2}|x|} \frac{|D^k_x u(x)|^p}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})-p}} dy dx \\
\leq c\|\phi\|_{B^p_{\ell}(\partial \Omega)}^p \int_{|x| \geq R} \frac{1}{|x|^{n-2p\alpha+(k+1)p}} \int_{|x-y| \leq \frac{1}{2}|x|} \frac{1}{|x-y|^{n+p(s+2\alpha-k-2+\frac{k}{p})}} dy dx \\
\leq c\|\phi\|_{B^p_{\ell}(\partial \Omega)}^p \int_{|x| \geq R} \frac{1}{|x|^{p(n+s-1)+1}} dx \\
= cR^{-p(n+s-1)-1+n} \|\phi\|_{B^p_{\ell}(\partial \Omega)}^p.
\]

Since $|x|, |y| \geq R$, by (5.1), the second term of (5.3) is dominated by
\[
\int_{|x| \geq R} \int_{|y| \geq R, |x-y| \geq \frac{1}{2}|x|} \frac{|D^k_x u(x)|^p + |D^k_y u(y)|^p}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx \\
\leq \|\phi\|_{B^p_{\ell}(\partial \Omega)}^p \int_{|x| \geq R} \frac{1}{|x|^{p(n+2\alpha-k)}} \int_{|y| \geq R, |x-y| \geq \frac{1}{2}|x|} \frac{1}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx \\
+ \|\phi\|_{B^p_{\ell}(\partial \Omega)}^p \int_{|x| \geq R} \int_{|y| \geq R, |x-y| \leq \frac{1}{2}|x|} \frac{1}{|x-y|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dy dx.
\]

Since $p > \frac{n-1}{n+s-1}$, the second term of right-hand side of (5.4) is dominated by $R^{-p(n+s-1)-1+n} \|\phi\|_{B^p_{\ell}(\partial \Omega)}^p$.

Note that
\[
\int_{|x| \geq R} \int_{|y| \leq 2|x|} \frac{1}{|x|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} \frac{1}{|y|^{p-2p\alpha+kp}} dy dx \\
\leq c \begin{cases} 
R^{-p(n+2\alpha+n)} \int_{|x| \geq R} \frac{1}{|x|^{n+p(s+2\alpha-k-1+\frac{k}{p})}} dx, & \text{if } pn - 2p\alpha + kp > n, \\
\ln |x| \int_{|x| \geq R} \frac{1}{|x|^{p(n+2\alpha-k-1+\frac{k}{p})}} dx, & \text{if } pn - 2p\alpha + kp = n, \\
\int_{|x| \geq R} \frac{1}{|x|^{p(n+2\alpha-k-1+\frac{k}{p})}} dx, & \text{if } pn - 2p\alpha + kp < n
\end{cases}
\leq cR^{-p(n-k-1+s)-1+n} \ln R.
\]
Then, since $p > \frac{n-1}{n+s-1}$, the first term of right-hand side of (5.4) is dominated by
\[
\|\phi\|_{B^p_s(\partial \Omega)}^p \int_{|x| \geq R} \left( \int_{|y| \geq R} \frac{1}{|x-y|^{n+p-2\alpha+1}} \frac{1}{|y|^{p(n-2\alpha+k_p)}} \right) dydx
\]
\[
\leq c \|\phi\|_{B^p_s(\partial \Omega)}^p \int_{|x| \geq R} \left( \int_{|y| \geq 2|x|} \frac{1}{|x|^{n+p-2\alpha+1}} \frac{1}{|y|^{p(n-2\alpha+k_p)}} \right) dydx
\]
\[
+ \int_{|x| \geq R} \left( \int_{|y| \geq 2|x|} \frac{1}{|y|^{n+p(n+s)-1}} \right) dydx
\]
\[
\leq c R^{-(n-1+s)-1+n} \ln R \|\phi\|_{B^p_s(\partial \Omega)}^p.
\]
Therefore, we showed that $A_1 + A_2 + A_3 \leq c R \|\phi\|_{B^p_s(\partial \Omega)}^p$ and hence showed (1.5). \(\square\)

6. PROOF OF THEOREM 1.4

**Theorem 6.1.** Let $1 - 2\alpha - \frac{1}{p} < s < 0$. For $\phi \in B^p_s(\partial \Omega)$, let us $u = S_{2\alpha}\phi$ be a layer potential defined in (1.3). The, we have

1. $\hat{u}(\xi) = |\xi|^{-2\alpha} \phi, e^{2\pi i \xi \cdot \cdot} > . \quad (6.1)$
2. $u$ is a weak solution of
   \[
   \Delta^\alpha u = 0, \quad \text{in} \quad \mathbb{R}^n \setminus \partial \Omega. \quad (6.2)
   \]

**Proof.** For the proof of (6.1), let us $\phi \in C^2(\partial \Omega)$ and $\psi \in C^\infty(\mathbb{R}^n)$. Then, we have

\[
\int_{\mathbb{R}^n} u(x) \psi(x) dx = c(n, s) \int_{\partial \Omega} \phi(Q) \int_{\mathbb{R}^n} \frac{1}{|x-Q|^{n-2\alpha}} \psi(x) dx dQ
\]
\[
= \int_{\partial \Omega} \phi(Q) \int_{\mathbb{R}^n} |\xi|^{-2\alpha} e^{2\pi i \xi \cdot Q} \hat{\psi}(\xi) d\xi dQ
\]
\[
= \int_{\mathbb{R}^n} \hat{\psi}(\xi) |\xi|^{-2\alpha} \int_{\partial \Omega} \phi(Q) e^{2\pi i \xi \cdot Q} dQ d\xi.
\]

Hence, we get

\[
\hat{u}(\xi) = |\xi|^{-2\alpha} \int_{\partial \Omega} \phi(Q) e^{2\pi i \xi \cdot Q} dQ.
\]

Since $C^2(\partial \Omega)$ is dense in $B^p_s(\partial \Omega)$, we get (6.1) for all $\phi \in B^p_s(\partial \Omega)$.

For the proof (6.2), suppose that $\phi \in C^2(\partial \Omega)$ and $\psi \in C^\infty(\mathbb{R}^n \setminus \partial \Omega)$, then, by (6.1),

\[
\int_{\mathbb{R}^n} u(x) \Delta^\alpha \psi(x) dx = \int_{\mathbb{R}^n} |\xi|^{2\alpha} \hat{u}(\xi) \hat{\psi}(\xi) d\xi
\]
\[
= \int_{\mathbb{R}^n} \hat{\psi}(\xi) \int_{\partial \Omega} e^{-2\pi i \xi \cdot Q} \phi(Q) dQ d\xi
\]
\[
= \int_{\partial \Omega} \phi(Q) \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot Q} \hat{\psi}(\xi) d\xi dQ
\]
\[
= \int_{\partial \Omega} \phi(Q) \psi(Q) dQ
\]
\[
= 0 \quad (6.3)
\]
Since $\Delta^t : \dot{B}^s_p(\mathbb{R}^n) \to \dot{B}^{s-2t}_p(\mathbb{R}^n)$ is isomorphism, we have
\[
| \int_{\mathbb{R}^n} u(x)\Delta^a \psi(x) dx | = | \int_{\mathbb{R}^n} \Delta^\frac{1}{2}(s+2\alpha -\frac{1}{p})u(x)\Delta^\frac{1}{2}(-s+\frac{1}{p})\psi(x) dx |
\leq \| \Delta^\frac{1}{2}(s+2\alpha -\frac{1}{p})u \|_{\dot{B}^0_p(\mathbb{R}^n)} \| \Delta^\frac{1}{2}(-s+\frac{1}{p})\psi \|_{\dot{B}^0_{p'}(\mathbb{R}^n)}
\leq \|u\|_{\dot{B}^{s+2\alpha -\frac{1}{p}}_p(\mathbb{R}^n)} \|\psi\|_{\dot{B}^{-s+\frac{1}{p}}_{p'}(\mathbb{R}^n)}
\leq c \|\phi\|_{\dot{B}_p^s(\partial \Omega)} \|\psi\|_{\dot{B}^{-s+\frac{1}{p}}_{p'}(\mathbb{R}^n)}.
\]
Let $\phi_k \in C^2(\partial \Omega)$ such that $\phi_k \to \phi$ in $\dot{B}^s_p(\partial \Omega)$ and $u_k = S\phi_k$. Then, we get
\[
| \int_{\mathbb{R}^n} (u_k(x) - u(x))\Delta^a \psi(x) dx | \leq c \|\phi_k - \phi\|_{\dot{B}_p^s(\partial \Omega)} \|\psi\|_{\dot{B}^{-s+\frac{1}{p}}_{p'}(\mathbb{R}^n)} \to 0, \quad k \to \infty.
\]
Hence, since $C^2(\partial \Omega)$ is the dense subspace of $\dot{B}_p^s(\partial \Omega)$, (6.3) holds for $\phi \in \dot{B}_p^s(\partial \Omega)$ and so we get (6.2) for all $\phi \in \dot{B}_p^s(\partial \Omega)$. □

**Proof of Theorem 1.4.** By the Corollary 4.3, we get that $S_{2\alpha} : B^{t-2\alpha+1}_p(\partial \Omega) \to B^t_p(\partial \Omega), \ 0 < t < 1, \ 1 < p < \infty$ are bijective.

To show the existence of solution, let $g \in B^t_p(\partial \Omega)$. By the bijectivity of $S_{2\alpha} : B^{t-2\alpha+1}_p(\partial \Omega) \to B^t_p(\partial \Omega)$, there is a $\phi \in B^{t-2\alpha+1}_p(\partial \Omega)$ such that $S_{2\alpha}\phi = g$. Let $u = S_{2\alpha}\phi$ defined by (1.3). Then, by (2) of the Theorem 6.1 u is a weak solution of (6.2) and by the Theorem 1.2 u satisfies the equation (1.7). Hence, we complete the proof of the Theorem 1.4. □

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