DISCRETE REFLECTION GROUPS AND INDUCED REPRESENTATIONS OF POINCARÉ GROUP ON THE LATTICE

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Abstract

We continue the program, presented in previous Symposia, of discretizing physical models. In particular we calculate the integral Lorentz transformations with the help of discrete reflection groups, and use them for the covariance of Klein-Gordon and Dirac wave equation on the lattice. Finally we define the unitary representation of Poincaré group on discrete momentum and configuration space, induced by integral representations of its closed subgroup.

INTRODUCTION

The assumption of a physical model on discrete space and time leads to some changes of the mathematical equations. In particular, the symmetries of the model has to be restricted to integral transformations acting on some vector space defined over the integers. The space-time groups are subgroups of the set of non singular integral matrices, and the classical and quantum laws are written with the help of difference operators. In [1] we have developped some properties of non-compact groups acting on lattice with non-euclidean metric. In [2] we have modify some of the postulates of quantum mechanics in order to incorporate the hypothesis of discrete space and time. In [3] we have proposed a new scheme for the Klein-Gordon and Dirac wave equation. The Quantum fields on the lattice is the subject of very extensive litterature [4] and the theoretical difficulties of it have not yet been solved.

In section 2, we review some results of discrete reflection groups that give raise to a complete classification of n-simplex reflection groups of the spherical, euclidean or hyperbolic type, and of the generalized hyperbolic reflection groups whose connection with integral space-time groups is made via Coxeter groups.

In section 3 and 4 we introduce the definitions and main properties of a discrete differential geometry in order to construct covariant expression for the Maxwell, Dirac and Klein-Gordon difference equations.
In section 5 we apply the standard formulation of induced representations to calculate the transformation properties of the Mackey, Wigner and covariant state functions that belong to the carrier space for the unitary representations of Poincaré group on discrete momentum space.

In section 6 we construct the unitary representations of Poincaré group in the discrete coordinate space. We recover the formulas for the momentum space by the use of a discrete Fourier transform for non-periodic functions.

**REFLECTION GROUPS**

Let \( X \) be a metric space of dimension \( n \) of the type spherical \( S^n \), euclidean \( E^n \) or hyperbolic \( H^n \). Let \( S \) be a side of an \( n \)-dimensional convex polyhedron \( P \) in \( X \). The reflection of \( X \) in the side \( S \) of \( P \) is the reflection of \( X \) in the hyperplane spanned by \( S \). The reflections of \( X \) in the sides of a finite sided \( n \)-dimensional convex polyhedron \( P \) in \( X \) of finite volume generate a reflection group.

Let \( P \) be a finite-sided \( n \)-dimensional convex polyhedron in \( X \) of finite volume all of whose dihedral angles are submultiple of \( \pi \). Then the group generated by the reflections of \( X \) in the sides of \( P \) is a discrete reflection group \( \Gamma \) with respect to the polyhedron \( P \).

In order to construct a presentation for a discrete reflection group we take all the sides \( \{S_i\} \) of \( P \) and for each pair of indices, \( i, j \), let \( k_{ij} \equiv \frac{\pi}{\vartheta(S_i, S_j)} \) where \( \vartheta(S_i, S_j) \) is the angle between \( S_i \) and \( S_j \). Let \( F \) be the group generated by the sides \( S_i \) and \( \Gamma \) the group generated by the reflections on \( S_i \). When \( P \) has finitely many sides and finite volume the map \( \psi : G \to \Gamma \) is an isomorphism where \( G \) is the quotient of \( F \) by the normal closure of the words \( (S_i S_j)^{k_{ij}} \). We call

\[
\{ S_i, (S_i S_j)^{k_{ij}} \}
\]

a presentation of the discrete reflection group \( \Gamma \).

A discrete reflection group \( \Gamma \) with respect to a finite-sided polyhedron \( P \) of finite volume is isomorphic to a Coxeter group \( G \), that is, an abstract groups defined by a group presentation of the form \( \{ S_i, (S_i S_j)^{k_{ij}} \} \), where:

i) the indices \( i, j \) vary over some countable indexing set \( J \);

ii) the exponent \( k_{ij} \) is either a positive integer or \( \infty \) for each \( i, j \);

iii) \( k_{ij} = k_{ji} \);

iv) \( k_{ii} = 1 \), for each \( i, j \);

v) \( k_{ij} > 1 \) if \( i \neq j \); and

vi) if \( k_{ij} = \infty \), the term \( (S_i S_j)^{\infty} \) is omitted.

The Coxeter graph of \( G \) is the labeled graph with vertices \( J \) and edges

\[
\{(i, j) : k_{ij} > 2\}
\]

Each edge \( (i, j) \) is labeled by \( k_{ij} \).

For simplicity, the edges with \( k_{ij} = 3 \) are usually not labeled in a Coxeter graph.

Only some particular types of Coxeter groups have been classified. We present here some of the most fundamental and useful Coxeter groups: the simple reflection groups and the generalized simplex reflection groups.
Let $\Delta$ be an $n$-simplex in $X (= S^n, E^n, \text{or} H^n)$ all of whose dihedral angles are submultiple of $\pi$ (by $n$-simplex we define a convex $n$-dimensional polyhedron in $X$ with exactly $n + 1$ vertices). The group $\Gamma$ generated by the reflections of $X$ in the sides of $\Delta$ is an $n$-simplex reflection group.

If $n = 1$ then $\Delta$ is a geodesic segment in $X$, and $\Gamma$ is the dihedral group of order $2k$ with $k > 1$, where $\frac{\pi}{k}$ is the angle of $\Delta$. The Coxeter graph is

$$
\begin{array}{c}
\bullet \\
\bullet \\
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\end{array}
$$

For $X = S^1$, $k$ is finite; for $X = E^1$ or $H^1$, $k = \infty$.

Assume that $n = 2$. Then $\Delta$ is a triangle in $X$ whose angles $\frac{\pi}{a}, \frac{\pi}{b}, \frac{\pi}{c}$ are submultiple of $\pi$. If we call $T(a, b, c)$ the triangle determined by the integer numbers $(a, b, c)$, then the group $\Gamma$ generated by the reflections in the sides of $T(a, b, c)$ is denoted by $G(a, b, c)$ and it is called a triangle reflection group.

If $X = S^2$ the only spherical triangle reflection group have the following Coxeter graph:

$$
G(2,2,2) \quad G(2,2,c) \quad G(2,3,3)
$$

$$
G(2,3,4) \quad G(2,3,5)
$$

If $X = E^3$ we have the euclidean triangle reflection groups with Coxeter diagram:

$$
G(3,3,3) \quad G(2,4,4) \quad G(2,3,6)
$$

If $X = H^3$ we have the hyperbolic triangle reflection groups with Coxeter diagram:

$$
G(2,b,c) \quad G(a,b,c)
$$

Let $\Gamma$ be the group generated by the reflections of $X$ in the sides of an $n$-simplex $\Delta$ all of whose dihedral angles are submultiples of $\pi$. The group $\Gamma$ said to be irreducible if and only if its Coxeter graph is connected.

The classification of all the irreducible $n$-simplex (spherical, euclidean and hyperbolic) reflection groups is complete. Spherical and euclidean $n$-simplex reflections groups exist in all dimensions; however, hyperbolic $n$-simplex reflections groups exist only for dimension $n \leq 5$. [5].

Another type of Coxeter groups that have been classified are the generalized simplex reflection groups, which are defined only in $H^n$. A generalized $n$-simplex in $H^n$ is a $n$-dimensional polyhedron with $n + 1$ generalized vertices (a generalized vertex of a convex polyhedron $P$ is either a vertex of $P$ or an ideal vertex of $P$).

The generalized hyperbolic triangle reflection groups have the following Coxeter graphs:

$$
\begin{array}{c}
\infty \\
\infty \\
\infty \\
\end{array} \\
\begin{array}{c}
\infty \\
\infty \\
a \\
\end{array} \\
\begin{array}{c}
a \\
b \\
\infty \\
\end{array}
$$

\[ a \geq b \geq 3 \]
The generalized hyperbolic n-simplex reflections groups exist only for \( n \leq 10 \). [6]

Some particular very interesting cases of the last hyperbolic generalized reflection groups are the following:

\[
\Gamma_{2,1} : \quad \bullet \quad 4 \quad \bullet \quad \Gamma_{3,1} : \quad \bullet \quad 4 \quad 4
\]

\[
\Gamma_{4,1} : \quad \bullet \quad 4 \quad \Gamma_{5,1} - \Gamma_{9,1} : \quad \bullet \quad \cdots \quad \bullet \quad 4
\]

If we take a cartesian basis \( \{e_i\} \, i = 0, 1, \cdots, 9 \) a realization of the reflections corresponding to these Coxeter groups can be given in matricial form:

\[
\Gamma_{2,1} : \quad S_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

\[
S_3 = \begin{pmatrix} 3 & 2 & 2 \\ -2 & -1 & -2 \\ -2 & -2 & -1 \end{pmatrix}
\]

\[
\Gamma_{3,1} : \quad S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}
\]

\[
S_4 = \begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}
\]

\[
\Gamma_{n,1} : \quad S_i = x_i \leftrightarrow x_{i+1} \quad i = 1, 2, \cdots, n-1; \quad n = 4, 5, \cdots, 9.
\]

\[
S_n = x_n \leftrightarrow -x_n
\]

\[
S_{n+1} = \begin{pmatrix} 2 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & -1 & 0 \\ -1 & -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 \end{pmatrix}
\]

It can be checked that these reflections satisfy the presentation of the Coxeter groups

\[ \{S_i, (S_iS_j)^{k_{ij}}\} \]

corresponding to the Coxeter graphs \( \Gamma_{n,1}, (n = 2, 3, \cdots, 9) \)

The groups \( \Gamma_{n,1} \) have a very important connection with the Lorentian lattices \( \Lambda_n \) or the set of points with integral components with respect to a Cartesian basis \( \{e_0, e_1, \cdots, e_n\} \) equipped with the bilinear form

\[-x_0^2 + x_1^2 + \cdots + x_n^2\]
The set of all proper Lorentz transformations that leave the Lorentian lattice \( \Lambda_n \) invariant is generated by the elements of the Coxeter group \( \Gamma_{n,1} \), \((n = 2, \cdots, 9)\), the proof is given by Kac [7]. Similarly Coxeter has proved [8] that all integral Lorentz transformations (including reflections) are obtained combining the operation of permuting the spacial coordinates \( x_1, x_2, x_3 \) and changing the signs of the coordinates \( x_0, x_1, x_2, x_3 \), together with the transformation

\[
\begin{pmatrix}
  2 & -1 & -1 & -1 \\
  -1 & 0 & -1 & -1 \\
  -1 & -1 & 0 & -1 \\
  -1 & -1 & -1 & 0
\end{pmatrix}
\]

We have arrived at the same result in the case of \( \Gamma_{2,1} \) using the isomorphism between \( \text{SO}(2,1) \cap \text{GL}(3, \mathbb{Z}) \) and \( \text{GL}(2, \mathbb{Z}) \) [9].

### A DIFFERENCE CALCULUS OF SEVERAL INDEPENDENT VARIABLES

Given a function of one independent variable the forward and backward differences are defined as

\[
\Delta f(x) \equiv f(x + \Delta x) - f(x) \quad , \quad \nabla f(x) \equiv f(x) - f(x - \Delta x)
\]

Similarly, we can define the forward and backward promediate operator

\[
\tilde{\Delta} f(x) \equiv \frac{1}{2} \{f(x + \Delta x) + f(x)\} \quad , \quad \tilde{\nabla} f(x) \equiv \frac{1}{2} \{f(x - \Delta x) + f(x)\}
\]

Hence the difference or promediate of the product of two functions follows:

\[
\begin{align*}
\Delta \{f(x)g(x)\} & = \Delta f(x)\tilde{\Delta}g(x) + \tilde{\Delta} f(x)\Delta g(x) \quad \text{(3.1)} \\
\tilde{\Delta} \{f(x)g(x)\} & = \tilde{\Delta} f(x)\tilde{\Delta}g(x) + \frac{1}{4}\Delta f(x)\Delta g(x) \quad \text{(3.2)}
\end{align*}
\]

This calculus can be enlarged to functions of several independent variables. We use the following definitions:

\[
\begin{align*}
\Delta_x f(x, y) & \equiv f(x + \Delta x, y) - f(x, y) \\
\Delta_y f(x, y) & \equiv f(x, y + \Delta y) - f(x, y) \\
\tilde{\Delta}_x f(x, y) & \equiv \frac{1}{2} \{f(x + \Delta x, y) + f(x, y)\} \\
\tilde{\Delta}_y f(x, y) & \equiv \frac{1}{2} \{f(x, y + \Delta y) + f(x, y)\} \\
\Delta f(x, y) & \equiv f(x + \Delta x, y + \Delta y) - f(x, y) \\
\tilde{\Delta} f(x, y) & \equiv \frac{1}{2} \{f(x + \Delta x, y + \Delta y) + f(x, y)\}
\end{align*}
\]

These definitions can be easily generalized to more independent variables but for the sake of brevity we restrict ourselves to two independent variables. From the last definitions it can be proved the following identities:

\[
\begin{align*}
\Delta f(x, y) & = \Delta_x \tilde{\Delta}_y f(x, y) + \tilde{\Delta}_x \Delta_y f(x, y) \quad \text{(3.3)} \\
\tilde{\Delta} f(x, y) & = \tilde{\Delta}_x \tilde{\Delta}_y f(x, y) + \frac{1}{4}\Delta_x \Delta_y f(x, y) \quad \text{(3.4)}
\end{align*}
\]

Given a vectorial space \( V^n \) over \( \mathbb{Z} \) we can define a real-valued linear function over \( \mathbb{Z} \)

\[
f(u) \equiv \langle \omega, u \rangle \quad u \in V^n \quad \text{(3.5)}
\]
The forms \( \omega \) constitute a vectorial linear space (dual space) \( {}^*V^n \), and can be expanded in terms of a basis \( \omega^\alpha \)

\[
\omega = \sigma_\alpha \omega^\alpha
\]

The basis \( e_\beta \) of \( V^n \) and \( \omega^\alpha \) of \( {}^*V^n \) can be contracted in the following way

\[
\langle \omega^\alpha, e_\beta \rangle \delta^\alpha_\beta \quad (3.6)
\]

hence

\[
\langle \omega, e_\alpha \rangle = \sigma_\alpha, \quad \langle \omega^\beta, u \rangle = \sigma_\alpha u^\alpha, \quad \langle u, \omega \rangle = \sigma_\alpha u_\alpha \quad (3.7)
\]

with \( u = u^\alpha e_\alpha \).

If we take \( \omega^\beta = \Delta x^\beta \) as coordinate basis for the linear forms we can construct discrete differential forms (a discrete version of the continuous differential forms) \([10]\)

A particular example of this discrete form is the total difference operator \((3.3)\) of a function of several discrete variables written in the following way:

\[
\Delta f(x, y) = \left( \frac{\Delta f}{\Delta x} \right) \Delta x + \left( \frac{\Delta f}{\Delta y} \right) \Delta y \quad (3.8)
\]

For these discrete forms we can define the exterior product of two form \( \sigma \) and \( \rho \)

\[
\rho \wedge \sigma = -\sigma \wedge \rho
\]

which is linear in both arguments.

For the coordinate basis we also have

\[
\Delta x \wedge \Delta y = -\Delta y \wedge \Delta x
\]

With the help of this exterior product we can construct a second order discrete differential form or 2-form, namely

\[
\rho \wedge \sigma = \rho_\alpha \Delta x^\alpha \wedge \sigma_\beta \Delta x^\beta = \frac{1}{2} (\rho_\alpha \sigma_\beta - \rho_\beta \sigma_\alpha) \Delta x^\alpha \wedge \Delta x^\beta \equiv \sigma_{\alpha\rho} \Delta x^\alpha \wedge \Delta x^\beta \quad (3.9)
\]

where \( \sigma_{\alpha\rho} \) is an antisymmetric tensor. Similarly we can define a discrete p-form in a \( n \)-dimensional space \((p < n)\)

\[
\sigma = \frac{1}{p!} \sigma_{i_1i_2 \ldots i_p} \Delta x^{i_1} \wedge \Delta x^{i_2} \ldots \wedge \Delta x^{i_p}
\]

where \( \sigma_{i_1i_2 \ldots i_p} \) is a totally antisymmetric tensor.

The dual of a p-form in a \( n \)-dimensional space is the \((n-p)\)-form \(*\alpha\) with components

\[
(*\alpha)_{k_1k_2 \ldots k_{n-p}} = \frac{1}{p!} \alpha_{i_1i_2 \ldots i_p} \varepsilon_{i_1i_2 \ldots i_p k_1 \ldots k_{n-p}}
\]

where \( \varepsilon \) is the \( n \)-dimensional Levy-Civittá totally antisymmetric tensor \((\varepsilon_{123 \ldots} \equiv 1)\)

**EXTERIOR CALCULUS AND LORENTZ TRANSFORMATIONS**

Given a 1-form in a two-dimensional space

\[
\omega = a(x, y) \Delta x + b(x, y) \Delta y
\]

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we can define the exterior difference, in the similar way as the exterior derivative, namely,

\[
\Delta \omega \equiv \Delta a \wedge \Delta x + \Delta b \wedge \Delta y \\
= \left( \frac{\Delta z \tilde{\Delta}_y a}{\Delta x} \Delta x + \frac{\tilde{\Delta}_y \Delta_z a}{\Delta y} \Delta y \right) \wedge \Delta x + \left( \frac{\Delta z \tilde{\Delta}_y b}{\Delta x} \Delta x + \frac{\tilde{\Delta}_y \Delta_z b}{\Delta y} \Delta y \right) \wedge \Delta y \\
= \left( \frac{\Delta z \tilde{\Delta}_y b}{\Delta x} - \frac{\tilde{\Delta}_y \Delta_z a}{\Delta y} \right) \Delta x \wedge \Delta y
\]  

(4.1)

where in the last expression we have used the properties of the exterior product.

This definition of exterior difference can be easily written for 1-form in \(n\)-dimensional space.

Given a 2-form in a 3-dimensional space, 
\[
\omega = a(x, y, z) \Delta y \wedge \Delta z + b(x, y, z) \Delta z \wedge \Delta x + c(x, y, z) \Delta x \wedge \Delta y
\]

(4.2)

we can also define the exterior difference as:

\[
\Delta \omega = \Delta a \wedge \Delta y \wedge \Delta z + \Delta b \wedge \Delta z \wedge \Delta x + \Delta c \wedge \Delta x \wedge \Delta y
\]

(4.3)

The exterior derivative applied to the product of a 0-form (scalar function \(f\)) and a 1-form \(\omega = a \Delta x + b \Delta y\) is

\[
\Delta (f \omega) = \tilde{\Delta} f \Delta \omega + \Delta f \wedge \tilde{\Delta} \omega
\]

(4.4)

where \(\tilde{\Delta} f\) is expressed in (3.4) and \(\tilde{\Delta} \omega = \tilde{\Delta} a \Delta x + \tilde{\Delta} b \Delta y\).

The exterior difference of the product of two 1-forms is easily obtained

\[
\Delta \{\omega_1 \wedge \omega_2\} = \Delta \omega_1 \wedge \tilde{\Delta} \omega_2 - \tilde{\Delta} \omega_1 \wedge \Delta \omega_2
\]

(4.5)

The exterior difference of the product of a p-form \(\rho\) and a q-form \(\sigma\) is

\[
\Delta \{\rho \wedge \sigma\} = \Delta \rho \wedge \tilde{\Delta} \sigma + (-1)^p \tilde{\Delta} \rho \wedge \Delta \sigma
\]

(4.6)

Finally for any p-form \(\omega\) we have

\[
\Delta^2 \omega = \Delta (\Delta \omega) = 0
\]

(4.7)

Some examples:

From the Faraday 2-form \(F = \frac{1}{2} F_{\mu\nu} \Delta x^\mu \wedge \Delta x^\nu\) we write down one set of Maxwell difference equations

\[
\Delta F = \Delta (\Delta A) = 0
\]

\[
\left( \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_x}{\Delta x} + \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_y}{\Delta y} + \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_y}{\Delta z} \right) \Delta x \wedge \Delta y \wedge \Delta z \\
+ \left( \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_x}{\Delta t} + \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_y}{\Delta y} - \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_y}{\Delta z} \right) \Delta t \wedge \Delta y \wedge \Delta z \\
+ \left( \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_x}{\Delta t} + \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_y}{\Delta z} - \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_x}{\Delta y} \right) \Delta t \wedge \Delta z \wedge \Delta x \\
+ \left( \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z B_y}{\Delta t} + \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z E_y}{\Delta x} - \frac{\Delta_x \tilde{\Delta}_y \tilde{\Delta}_z E_y}{\Delta y} \right) \Delta t \wedge \Delta x \wedge \Delta y
\]

(4.8)
from the Maxwell 2-dual form $\star \mathbf{F}$ we get the other set of Maxwell equations:

$$\Delta^* \mathbf{F} = 4\pi \mathbf{J}$$

where $\mathbf{J} = J_\mu \Delta x^\mu$ is the charge-current 1-form.

Taking the exterior derivative of the last equation we get an other example of $\Delta^2 = 0$.

$$\left( \frac{\Delta x \Delta y \Delta z \Delta t \rho}{\Delta t} + \frac{\Delta x \Delta y \Delta z \Delta t J_x}{\Delta x} + \frac{\Delta x \Delta y \Delta z \Delta t J_y}{\Delta y} + \frac{\Delta x \Delta y \Delta z \Delta t J_z}{\Delta z} \right) \cdot \Delta t \wedge \Delta x \wedge \Delta y \wedge \Delta z = 0$$

Note that the coefficient of the difference form is the discrete version of the continuity equation.

From a scalar function we get the wave equations in terms of difference operators, namely,

$$- \star \Delta^* \Delta \phi \equiv \square \phi$$

where $\square$ is the discrete d’Alambertian operator:

\[
\left\{ -\widetilde{\nabla}_x \widetilde{\nabla}_y \widetilde{\nabla}_z \widetilde{\nabla}_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \right) + \nabla_x \nabla_y \nabla_z \nabla_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \right) + \nabla_x \nabla_y \nabla_z \nabla_t \left( \Delta_x \Delta_y \Delta_z \Delta_t \right) \right\} \phi (xyzt) = 0
\]

In order to compute the Lorentz transformation of the discrete differential forms we start with the coordinate-independent nature of 1-form

$$\omega = \omega_\mu \Delta x^\mu$$

where the $\Delta x^\mu$ are the space-time intervals in Minkowski space-time. From

$$\Delta x^\mu = \Lambda^\mu_\nu \Delta x^\nu$$

where $\Lambda^\mu_\nu$ is a global Lorentz transformation, and from the coordinate-free expresion for $\omega$ we get

$$\omega_{\mu} = \omega_\nu \Lambda^\nu_\mu$$

Recall that $\Lambda^\nu_\mu \Lambda^\mu_\rho = \delta^\nu_\rho$

From the total difference of a function of several variables $f(x, y, z, t)$

$$\Delta f = \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta x} \Delta x + \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta y} \Delta y + \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta z} \Delta z + \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta t} \Delta t$$

it follows that the coefficients of the 1-forms, namely,

$$\left( \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta x}, \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta y}, \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta z}, \frac{\Delta x \Delta y \Delta z \Delta t f}{\Delta t} \right)$$

transform covariantly like the coefficients $\omega_\mu$ of (4.1).

The same technic can be applied to components of discrete p-forms.

**UNITARY REPRESENTATIONS OF THE DISCRETE POINCARE GROUP IN MOMENTUM SPACE**

Let $\mathcal{P}_+\subset$ be the integral proper inhomogeneous Lorentz group or discrete Poincaré group, which is homomorphic to the semidirect product of the integral subgroups of $SL(2, \mathbb{C})$
and $T_4$ the transformation group on the Minkowski lattice. The multiplication law for the Poincaré group is

$$\{a_2, \Lambda_2\} \{a_1, \Lambda_1\} = \{a_2 + \Lambda_2 a_1, \Lambda_2 \Lambda_1\} \quad (5.1)$$

where $\Lambda_1, \Lambda_2$ are two integral Lorentz transformations described in section 2 and $a_1, a_2$ are two set of four integer numbers that define the parameters of the translation groups $T_4$ in Minkowski space.

In order to construct the unitary representation of the Poincaré group we use the standard method of induced representation [11]. All the properties of these representations can be translated into the language of integral representations. It is well known that the irreducible representations of the inhomogeneous space-time groups can be realized on different kind of states such as Mackey, Wigner or covariant states.

Consider now the representations acting on the Wigner states, corresponding to massive particles. Following standard procedure we need:

i) the unitary representation of the translation group

$$U(a) = \prod_{\mu=0}^{3} \left( 1 + \frac{i}{2} \epsilon P_\mu \right)^{a_\mu}$$

$k_\mu$ being the discrete momentum, $a = (a_0, a_1, a_2, a_3)$ the parameters of the group.

ii) The unitary representation of the Wigner little group (in our case $SU(2)$) corresponding to integral rotations in 3-dimensional space. They are only 24 different elements of this type.

iii) The representative momentum is $(m, 0, 0, 0) \equiv P_\mu$ and the orbit generated by this vector is given by the boost

$$P_\mu = \Lambda_\mu^{\nu} P_\nu \quad (5.3)$$

$P_\nu$ is defined on the Minkowski lattice:

$$P_\mu = m \left( \frac{\Delta t}{\sqrt{\Delta t^2 - (\Delta x^2)^2}}, \frac{\Delta \bar{x}}{\sqrt{\Delta t^2 - (\Delta x^2)^2}} \right) \quad (5.4)$$

the points $(t, \bar{x})$ lying in the integral hyperboloid generated by the reflection $S_4$ of $\Gamma_{3,1}$ (see section 2).

With these ingredients we can write down the transformation properties of the Wigner functions in momentum space under the transformations of the restricted Poincaré group, namely [Ref. 11, formula 16.2]

$$U(a, \Lambda) \psi(P) = \prod_{\mu=0}^{3} \left( 1 + \frac{i}{2} \epsilon P_\mu \right)^{a_\mu} D(SU(2)) \psi(\Lambda^{-1}P)$$

The integral representation of the $SU(2)$ group can be given by the use of Cayley parametrization

$$A = \frac{1}{\det} \left( \begin{array}{ccc} n_0 + in_3 & n_2 + in_1 & n_1 + in_0 \\ -n_2 + in_1 & n_0 - in_3 & n_2 + in_1 \\ -n_1 + in_0 & -n_2 + in_1 & n_0 + in_3 \end{array} \right) = \frac{1}{\det} (n_01 + in_1 \sigma_1 + in_2 \sigma_2 + in_3 \sigma_3)$$

with $\det = n_0^2 + n_1^2 + n_2^2 + n_3^2 = 1$ or 2.

For representation of higher dimension we have to substitute the generators $\tilde{\sigma}$ by the corresponding representations of $\tilde{\sigma}$. 

9
For the covariant states the unitary representations of the Poincaré group restricted to integral transformations is given by the following transformations [see Ref. 11, formula 17.1]

\[
U (a, \Lambda) \psi (P) = \prod_{\mu=0}^{3} \left( \frac{1 + \frac{1}{2} i \varepsilon P_{\mu}}{1 - \frac{1}{2} i \varepsilon P_{\mu}} \right)^{a_{\mu}} D (\Lambda) \psi (\Lambda^{-1} P)
\]  

(5.6)

where \( D (\Lambda) \) is an integral representation of the restricted Lorentz group.

In our case it is a representation of the subgroup \( SL (2, C) \cap GL (2, Z) \) which is homomorphic to the integral Lorentz group as described in section 2.

In order to construct covariant functions that transform under irreducible representation of the Poincaré group we need a subsidiary condition on the functions:

\[
Q \psi (\hat{P}) = \psi (\hat{P})
\]

(5.7)

where \( Q \) is a projection operator that restricts the components of the vector \( \psi (\hat{P}) \) to one representation of unique spin.

For example for the Dirac representation of \( SL (2, C) \) which contains \( D^{1/2} (SU (2)) \) twice, we take

\[
Q = \frac{1}{2} (1 \pm \beta), \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}
\]

(5.8)

Taking \( \hat{P} = (m_0, 0, 0, 0) \) this equation leads to the wave equation in the rest system

\[
\beta \psi^{(\pm)} (\hat{P}) = \pm \psi^{(\pm)} (\hat{P})
\]

(5.9)

If we transform this equation to an arbitrary inertial system we calculate

\[
Q (P) = F (P) Q F^{-1} (P)
\]

\[
\psi (P) = F (P) \psi (\hat{P})
\]

with

\[
F (P) = \frac{\left( \tilde{P}^2 + m^2 \right)^{1/2} + |m| - \beta \tilde{\alpha} \cdot \tilde{P}}{\left\{ 2 \left[ \left( \tilde{P}^2 + m^2 \right)^{1/2} + |m| \right] \right\}^{1/2}}
\]

and \( \tilde{\alpha} = \begin{pmatrix} 0 & \tilde{\alpha} \\ \tilde{\alpha} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \),

we obtain

\[
(\tilde{\alpha} \cdot \tilde{\beta} + |m| \beta) \psi^{(\pm)} (P) = \pm P_0 \psi^{(\pm)} (P)
\]

(5.10)

The transformation properties of the Dirac vectors are

\[
U (a, \Lambda) \psi^{(\pm)} (P) = \prod_{\mu=0}^{3} \left( \frac{1 + \frac{1}{2} i \varepsilon P_{\mu}}{1 - \frac{1}{2} i \varepsilon P_{\mu}} \right)^{a_{\mu}} \left( \frac{1 + \frac{1}{2} i \tilde{\omega} \cdot \tilde{\sum}}{1 - \frac{1}{2} i \tilde{\omega} \cdot \tilde{\sum}} \right) \psi^{(\pm)} (\Lambda^{-1} P)
\]

(5.11)

for spatial rotations, and
\[ U(a, \Lambda) \psi^+(P) = \prod_{\mu=0}^{3} \left( 1 + \frac{1}{2} i \varepsilon P_{\mu} \right)^{\alpha_{\mu}} \left( \frac{1 - \frac{1}{2} i \vec{u} \cdot \vec{\alpha}}{1 + \frac{1}{2} i \vec{u} \cdot \vec{\alpha}} \right) \psi^+(\Lambda^{-1} P) \] (5.12)

for pure Lorentz transformations.

The negative energy covariant vectors carry the same representation.

**TRANSFORMATION OF THE COVARIANT FUNCTIONS IN CONFIGURATION SPACE**

In order to construct the wave functions in configuration space we solve the Klein-Gordon and Dirac wave equation on the lattice. This procedure can be easily generalized to higher spin representations.

We define the scalar function on the \((3 + 1)\) dimensional cubic lattice

\[ \phi(j_1 \epsilon_1, j_2 \epsilon_2, j_3 \epsilon_3, n \tau) \equiv \phi(\vec{j}, n) \]

where \(\epsilon_1, \epsilon_2, \epsilon_3, \tau\) are small quantities in the space-time directions and \(j_1, j_2, j_3, n\) are integer numbers.

We define the difference operators

\[ \delta^+_{\mu} \equiv \frac{1}{\epsilon_{\mu}} \Delta_{\mu} \prod_{\nu \neq \mu} \tilde{\Delta}_\nu, \mu = 0, 1, 2, 3 \quad \delta^-_{\mu} \equiv \frac{1}{\epsilon_{\mu}} \nabla_{\mu} \prod_{\nu \neq \mu} \tilde{\nabla}_\nu \]

\[ \eta^+ \equiv \prod_{\mu=0}^{3} \tilde{\Delta}_{\mu} \quad \eta^- \equiv \prod_{\mu=0}^{3} \tilde{\nabla}_{\mu} \]

Then the Klein-Gordon wave equations defined on the grid points of the lattice can be read off

\[ \left( \delta^+_{1} \delta^-_{1} + \delta^+_{2} \delta^-_{2} + \delta^+_{3} \delta^-_{3} - \delta^+_{0} \delta^-_{0} - M^2 \eta^+ \eta^- \right) \phi(\vec{j}, n) = 0 \] (6.1)

It can be verified by direct substitution that the plane wave solution satisfy the difference equation

\[ f(\vec{j}, n) = \left( \frac{1 + \frac{1}{2} i \varepsilon_1 k_1}{1 - \frac{1}{2} i \varepsilon_1 k_1} \right)^{j_1} \left( \frac{1 + \frac{1}{2} i \varepsilon_2 k_2}{1 - \frac{1}{2} i \varepsilon_2 k_2} \right)^{j_2} \left( \frac{1 + \frac{1}{2} i \varepsilon_3 k_3}{1 - \frac{1}{2} i \varepsilon_3 k_3} \right)^{j_3} \left( \frac{1 - \frac{1}{2} i \tau \omega}{1 + \frac{1}{2} i \tau \omega} \right)^n \] (6.2)

provided the dispersion relation is satisfied

\[ \omega^2 - k_1^2 - k_2^2 - k_3^2 = M^2 \] (6.3)

From section 5 the Klein-Gordon equation is invariant under finite Lorentz transformations.

The discrete version of the Dirac wave equation can be written as

\[ \left( \gamma_1 \delta^+_1 + \gamma_2 \delta^+_2 + \gamma_3 \delta^+_3 - i \gamma_0 \delta^+_0 + M \eta^+ \right) \psi(\vec{j}, n) = 0 \] (6.4)

where \(\gamma_\mu, \gamma = 0, 1, 2, 3\) are the usual Dirac matrices. Applying the operator

\[ \left( \gamma_1 \delta^-_1 + \gamma_2 \delta^-_2 + \gamma_3 \delta^-_3 - i \gamma_0 \delta^-_0 - M \eta^- \right) \]
from the left on both sides of (6.4) we recover the Klein-Gordon equation (6.1). Let now construct solutions to (6.4) of the form
\[
\psi (j, n) = \omega (k, E) \, f (j, n)
\]
where the \( f (j, n) \) are given in (6.2).

The four-component spinors \( \omega (k, E) \), with spatial momentum \( k \equiv (k_1, k_2, k_3) \), must satisfy
\[
\left( i \vec{\gamma} \cdot \vec{k} - \gamma_0 E + M \right) \omega (k, E) = 0 \quad (6.5)
\]
as in the continuous case. Multiplying this equation from the left by
\[
\left( i \vec{\gamma} \cdot \vec{k} - \gamma_0 E - M \right)
\]
we obtain the dispersion relation
\[
E^2 - \vec{k}^2 = M^2 \quad (6.6)
\]

The transformation properties of the wave functions in the configuration space are given, as in the continuous case, as follows:
\[
U (a, \wedge) \psi (\epsilon j) = \left( \frac{1 + i \vec{\omega} \cdot \vec{\Sigma}}{1 - i \vec{\omega} \cdot \vec{\Sigma}} \right) \psi (\wedge^{-1} (\epsilon j - a)) \quad (6.7)
\]
for spatial rotations of angle \( \vec{\omega} \) and
\[
U (a, \wedge) \psi (\epsilon j) = \left( \frac{1 - \frac{i}{2} \vec{u} \cdot \vec{\alpha}}{1 + \frac{i}{2} \vec{u} \cdot \vec{\alpha}} \right) \psi (\wedge^{-1} (\epsilon j - a)) \quad (6.8)
\]
for special Lorentz transformation of relative velocity \( \vec{u} \equiv \vec{v} / v \).

The connection of these transformations in the configuration space and the transformation in momentum space is given via the Fourier transform in the lattice, for non-periodic functions, namely,
\[
\psi (P) = \sum_{j=-\infty}^{\infty} \left( \frac{1 + \frac{i}{2} \epsilon P}{1 - \frac{i}{2} \epsilon P} \right)^j \psi (\epsilon j) \quad (6.9)
\]
where \( \psi (\epsilon j) \) satisfy boundary conditions \( \psi (\epsilon j) \to 0 \), when \( j \to \infty \) and
\[
\sum_{j=-\infty}^{\infty} \psi (\epsilon j) < \infty \quad [12]
\]

Using summation by parts and the boundary conditions we derive that (5.11) is the Fourier transform of (6.7) and (5.12) is the Fourier transform of (6.8), and that (6.5) is the Fourier transform of (6.4).

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[10] See M. LORENTE, “Discrete Differential Geometry and Lattice Field Theory” in VII International Conference on Symmetry Methods in Physics (Dubna 1995).

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[12] The consistency of this new discrete Fourier transform is proved with the help of the $\delta$ sequence:

$$s_L(j) = \sum_{k=-L}^{L} \left( \frac{1 + \frac{i}{2} \epsilon k}{1 - \frac{i}{2} \epsilon k} \right)^j$$

that satisfies $\lim_{L \to \infty} s_L(0) = \infty$ and

$$\lim_{N \to \infty} \frac{1}{2N} \sum_{j=-N}^{N} s_L(j) = 1$$

for all L.