Piecewise planar method of bicubic interpolation

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Abstract

The article on the example of finite elements of bicubic interpolation presents the key ideas and technology of the piecewise planar method (PPM) for the construction of basic functions. The efficient use of triangular plane fragments characterizes PPM as one of the simple and visual methods of constructive theory of serendipity approximations. A model series (7 models) of Q12 elements with special "portraits" of zero level lines has been built. Appropriate influence functions (Lagrange coefficients) for angular and intermediate nodes were obtained. The first attempt to use PPM based on triangular simplexes is to construct triangular finite elements of higher order (complexes). As is known, in these cases, higher order Lagrangian polynomials are obtained by directly multiplying first order polynomials. Generalizing the basic idea of Courant about linear bases was a decisive step in the finite elements technique. The authors will show how PPM works based on the example of bicubic interpolation. A series of computer experiments were carried out to test different formulas for determining the spectrum of nodal loads on Q12 from a single uniform mass force. Typically, such a spectrum is determined by the double integration of serendipity polynomials (basis functions). Authors suggest instead of double integration to use stratified averaging of the surface applicators (rule of 9 applicators). Analysis of the stratified sample of the applicators revealed a simple relationship between the average surface applicator and the applicator in the barycenter base. This frees up the double integration and immediately gives the total body volume between the base and the surface.

Keywords: piecewise planar method; finite element; bicubic interpolation; alternative bases; stratified sample; spectrum of nodal loads.

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1. INTRODUCTION

The year of birth of the finite element method (FEM) is considered to be 1943 (R. Courant), and in 1968 there appeared serendipian (isoparametric) finite element (Ergatoudis, Irons, Zienkiewicz). Thus, FEM supporters in 2018 celebrated the double anniversary. Standard serendipity elements of higher order belong to "hard" models (V.I. Arnold term). The disadvantages of "hard" models are well known, so the actual task is to design alternative ("soft") models. Piecewise planar method (PPM) is the best example of the effective use of the famous Courant triangle in constructive theory of serendipity approximations. The paper shows how to form interpolation basis-finite surfaces of higher orders from plane fragments. The most common models of bicubic interpolation are considered as an example.

1.1. Problem statement

To illustrate the key ideas and technologies of PPM, authors consider the problem of bicubic interpolation, which is relevant in FEM (Q12 element). Let the interpolation nodes \((x_i, y_i)\), \(i = 1, 12\) be evenly spaced at the boundary of the element \((x = \pm 1, y = \pm 1)\). Denote the nodal values of the function \(f(x, y)\) by \(f_i\). These values can be obtained from the experiment or determined using complex calculations. There is a problem of approximate functions \(f(x, y)\) restoration at any point of the element. Specialists prefer polynomial interpolation for known reasons. In this case, the interpolation polynomial is constructed as a linear combination of basis functions \(N_i(x, y)\):

\[
 f(x, y) \doteq \sum_{i=1}^{12} N_i(x, y) \cdot f_i
\]

within the framework of Lagrange hypothesis

\[
 N_i(x_k, y_k) = \delta_{ik} \sum_{i=1}^{12} N_i(x, y) = 1
\]

where \(\delta_{ik}\) is the Kronecker symbol; \(i\) is the number of the basis function; \(k\) is the number of the interpolation node.

It should be emphasized that every basic function \(N_i(x, y)\) (influence function) within the limits of "its" area on the boundary of the square changes according to the law of the cubic parabola. This guarantees inter-elemental continuity of the surface \(f(x, y)\). It is clear that the necessary monomials can be conveniently selected from Pascal’s algebraic scheme. In this case, alternative models do not differ from the standard only at the boundary of the element; inside the base, the difference can be significant. This fact is reflected in the "portraits" of the zero level lines.

1.2. Literature review

The key ideas of FEM were outlined by Courant in [1]. The cell of 6 triangles and the pyramid of Courant can be read in [2]. The first serendipian finite elements (SFE) were selected and reported in [3]. These are known isoparametric models, which are called standard. It is about standard SFE and (unfortunately) only these are discussed in [4, 5, 6, 7, 8, 9]. Non-standard (alternative) SFE appeared in [10]. It was the first successful attempt to use PPM in serendipity approximations. The concept of "soft" mathematical modeling is outlined in [11].
Some examples of "soft" modeling of SFE are given in [12]. It is interesting to create a model line of alternative bicubic SFE that the "portraits" of zero level lines do not have a single curve line. In these cases, the simple and visual PPM is most effective.

Note that the "portrait" of the zero level lines of the standard bicubic FSE [3, 4, 5, 9] contains a second-order curve. The fractional-rational basis is discussed in [13].

2. METHODS AND TECHNOLOGIES

As is known [1, 2] modern FEM began with the problem of approximation of the function of two arguments by means of piecewise linear (finite) functions. In Fig. 1a depicts a Courant cell consisting of 6 rectangular triangles and is a base of Courant function. This function is linear in each triangle $\Delta_i$, equal to 1 in the control node (center of the cell) and 0 in the rest of the interpolation nodes. In canonical coordinates $(-1 \leq s, t \leq 1)$, the Courant function looks like:

$$
\varphi(s, t) = \begin{cases} 
1 - s \text{ in } \Delta_1, & \text{if } \Delta_2, \text{ } 1 + s - t \text{ in } \Delta_3, \\
1 + s \text{ in } \Delta_4, & \text{if } \Delta_5, \text{ } 1 - s + t \text{ in } \Delta_6 
\end{cases}
$$

$$
(3)
$$

Fig. 1. Courant hexagonal cell and zero level lines of bicubic interpolation surfaces: solid for "angular" surfaces $N_1(x, y)$, dotted for "intermediate" surfaces $N_5(x, y)$. 

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Fig. 2. Zero level lines of bicubic interpolation surfaces: solid for "angular" surfaces $N_1(x, y)$, dotted for "intermediate" surfaces $N_5(x, y)$.

From a geometric point of view $\varphi(s, t)$ is a pyramid formed of 6 triangular fragments of planes that intersect at a point $(0, 0, 1)$. In fact, Courant inventively used the barycentric coordinates of the two-dimensional simplex, which were discovered in 1827. It can be considered that the result of Courant [1] was the first successful attempt of piecewise planar design of stress surface in problems of rotation of prismatic rods. It should be noted that, the first alternative SFE models based on the probabilistic content of the barycentric coordinates of the triangles were obtained [10]. These are the models in Fig. 2c and 2d. At the beginning of FEM development, the triangular elements were independently investigated by Turner and other authors. Therefore, in some sources they are called the Courant-Turner triangles. These triangles are successfully applied to higher order triangular finite elements (FE), which is quite natural. Authors extend this approach to higher order square FE.

Classic PPM assumes that all traces of the intersection of the basis surface $N_i(x, y)$ with the base of the plane are straight. That is, the polynomial $N_i(x, y)$ a priori decomposes into linear factors. From a geometric point of view, the surface $N_i(x, y)$ is restored by multiplying the equations of the planes passing through the point $(x_i, y_i, 1)$. The number of required planes for bicubic interpolation ranges from 4 to 6. By the way, as author’s research shows, the "portrait" of zero level lines can be quite complicated, and $N_i(x, y)$ not always decompose into linear factors. A prime example is the standard model of bicubic SFE, among the alternative models there are many such examples [12].
The authors consider a series of 7 third-order SFE models (Fig. 1 and Fig. 2). The PPM algorithm will be performed in detail on the example of two models: Fig. 1b and Fig. 2c. The models are arranged in order to reduce the volume contained between the top part of the “intermediate” surface, for example, \( Z = N_5(x, y) \) and the plane of the base \( Z = 0 \). It is clear that the volume under the ”angular” surface increases, for example, \( N_1(x, y) \). The balance governs the Lagrange hypothesis.

Let’s start with the \( g \) model, which was the first among the alternative models of the SFE [10]. The interpolant (1) is constructed using the 12 influence functions (Lagrange coefficients) and the nodal values of the function \( f \). To obtain a complete set of polynomials \( N_i(x, y) \), it is sufficient to construct only two polynomials: ”angular”, for example, \( N_1(x, y) \) and ”intermediate”, for example \( N_5(x, y) \). Let’s start with \( N_1(x, y) \). The chosen variant of arrangement of zero level lines (Fig. 2c) is carried out by use of 4 fragments of planes passing through the corresponding pair of boundary nodes and a point \( P(-1, -1, 1) \). For a triangle \( 2 - 3 - P \) have \( Z_1 = \frac{1}{2}(1 - x) \), for a triangle \( 3 - 4 - P \) is \( Z_2 = \frac{1}{2}(1 - y) \), for a triangle \( 6 - 11 - P \) is \( Z_3 = -\frac{1}{4}(3x + 3y + 2) \), for \( 5 - 12 - P \) is \( Z_4 = -\frac{1}{2}(3x + 3y + 4) \). Superposition of fragments gives:

\[
N_1(x, y) = \frac{1}{32}(1 - x)(1 - y)(3x + 3y + 2)(2x + 3y + 4). \tag{4}
\]

Similarly, \( N_i(x, y) \) for \( i = 2, 3, 4 \). The ”intermediate” function \( N_5(x, y) \) is also constructed with the help of the PPM, taking into account the requirements of the interpolation hypothesis (2). Using only the lines in the plane of the zero level, write the ”intermediate” function in the general form:

\[
N_5(x, y) = \mu(1 - x)(1 + x)(1 - y)(Ax + By + 1). \tag{5}
\]

The normalizing factor \( \mu \) is determined by the second condition (2), and the coefficients \( A \) and \( B \) are determined by the first condition (2). This is how the ”intermediate” function is defined in the general case. For the model \( g \), the construction algorithm \( N_5(x, y) \) is greatly simplified. Given that, \( N_1(0, 0) = \frac{1}{4} \), it is easy to understand that \( N_5(0, 0) = 0 \). This means that the line, exiting node 6 passes through (0; 0). Thus, all equations of the lines at zero level \( N_5(x, y) \) are known. Therefore,

\[
N_5(x, y) = \frac{9}{32}(1 - x^2)(1 - y)(-3x + y). \tag{6}
\]

Similarly, \( N_i(x, y) \) for \( i = 6, ..., 12 \).

Now construct the “angular” function \( N_1(x, y) \) of model \( b \). Here have a more complex surface relief due to the superposition of 5 triangular fragments. From the previous model, three fragments remain, two triangles are added: \( 5 - 10 - P \) and \( 7 - 12 - P \). So,

\[
N_1(x, y) = \frac{1}{64}(1 - x)(1 - y)(1 + 3x)(1 + 3y)(-3x - 3y - 2). \tag{7}
\]

Similarly, \( N_i(x, y) \) for \( i = 2, 3, 4 \).

Formula (5) in this case looks like:

\[
N_5(x, y) = \frac{9}{64}(1 - x^2)(1 - y)(Ax + By + 1).
\]
From the first condition (2): $A = -6$ and $B = -1$. Therefore,

$$N_5(x, y) = \frac{9}{64} (1 - x^2)(1 - y)(Ax + By + 1). \tag{8}$$

Similarly, $N_i(x, y)$ for $i = 6, \ldots, 12$.

Let’s show polynomials $N_i(x, y)$ for the rest of the models (Fig. 1 and Fig. 2).

**Model c):**

$$N_1(x, y) = \frac{1}{128} (1 - x)(1 - y)(1 - 3x)(1 - 3y)(-3x - 3y - 4);$$

$$N_5(x, y) = \frac{9}{64} (1 - x^2)(1 - y)(-6x - y + 1).$$

**Model d):**

$$N_1(x, y) = \frac{1}{256} (1 - x)(1 - y)(1 - 9x^2)(1 - 9y^2);$$

$$N_5(x, y) = \frac{9}{512} (1 - x^2)(1 - y)(-48x - 9y + 7).$$

**Model e):**

$$N_1(x, y) = \frac{1}{128} (1 - x)(1 - y)(1 - 3x)(1 - 3y)(6x + 3y + 5);$$

$$N_5(x, y) = \frac{9}{256} (1 - x^2)(1 - y)(-24x - 5y + 3).$$

**Model f):**

$$N_1(x, y) = \frac{1}{128} (1 - x)(1 - y)(1 + 3x)(1 - 3y)(3x + 6y + 5);$$

$$N_5(x, y) = \frac{9}{256} (1 - x^2)(1 - y)(-24x - 5y + 3).$$

**Model h):**

$$N_1(x, y) = \frac{1}{64} (1 - x)(1 - y)(3x + 6y + 5)(6x + 3y + 5);$$

$$N_5(x, y) = \frac{9}{128} (1 - x^2)(1 - y)(-12x - 5y - 1).$$

3. **Experiments**

A series of computer experiments on visualization of "portraits" of zero level lines of constructed surfaces was performed in the work. Different formulas for determining the spectrum of nodal loads on FE Q12 from a single uniform mass force was tested. The load is an integral characteristic of the function $f(x, y) = N_i(x, y)$ and is usually determined by the double integral:

$$\bar{f} = \frac{1}{S} \int \int_D N_i(x, y) dxdy \tag{9}$$

As you can see, from a geometric point of view, the load is the average value of the surface applicate $f = (x, y)$. Authors propose instead of (9) to use cubatures of Newton-Cotes or
Gauss-Legendre. The results of cubic testing have unexpectedly shown that Newton-Cotes cubature performs better on the SFE of bicubic interpolation. Below are the centered computational template (Fig. 3) and the corresponding cubature of Newton-Cotes (rule of 9 applicates).

$$\bar{f} = \frac{4}{9} \cdot f_0 \cdot + \frac{1}{36}\sum_{i=1}^{4} f_i + \frac{1}{9}\sum_{i=5}^{12} f_i$$ (10)

Formula (10) incorporates the idea of the Monte Carlo method, but uses stratified instead of simple sample. The weighting coefficients (10) are consistent with the parabolic trapezoidal rule.

As an example, let’s determine the average value of the surface applicate for model h):

$$f_0 = \frac{25}{64}; f_1 = 1; f_2 = f_3 = f_4 = 0; f_5 = f_8 = -\frac{1}{16}; f_6 = f_7 = 0;$$

$$\bar{f} = \frac{4}{9} \cdot \frac{25}{64} \cdot + \frac{1}{36} \cdot 1 + \frac{1}{9}(-\frac{1}{16} - \frac{1}{16}) = \frac{3}{16}.$$ 

4. Results

The main result, of course, is that PPM is distributed to higher-ranking SFE without any interference. Note that there are only two models (standard and alternative) of the second-order SFE with the corresponding ”portraits” of the zero level lines. Among the third-order SFE, as you can see, there are seven such models. It seems to us that fourth-order SFE and above have not been considered from this point of view. The results of the calculation of the integral characteristics of the ”angular” and ”intermediate” surfaces of SFE Q12 are shown in Table 1.

Note that the results of double integration (9) and stratified averaging (10) coincide.

5. Conclusions

The list of methods of constructive theory of serendipity approximations is constantly expanding. In this list, the vast majority today are non-matrix methods. Simple and convenient PPM is one of such methods. Unfortunately, in the theory of FEM, there is still a belief that higher order FE are poorly formalized [14]. PPM overrules this misconception. The procedure of weighted averaging of two models of bicubic interpolation is capable of generating many alternative models of bicubic interpolation. This applies to all higher order FE. Thus, PPM eliminates
Table 1. Average surfaces applicates SFE Q12.

| Model Surface | Average surface applicate $f(x, y) = N_i(x, y)$, $(i = 1, \ldots, 4)$ | Average surface applicate $f(x, y) = N_i(x, y)$, $(i = 5, \ldots, 12)$ |
|---------------|-------------------------------------------------|-------------------------------------------------|
| standard [4, 9] | $\bar{f} = -\frac{1}{8}$ | $\bar{f} = \frac{3}{16}$ |
| b) | $\bar{f} = 0$ | $\bar{f} = \frac{1}{8}$ |
| c) | $\bar{f} = 0$ | $\bar{f} = \frac{1}{8}$ |
| d) | $\bar{f} = \frac{1}{64}$ | $\bar{f} = \frac{15}{128}$ |
| e) | $\bar{f} = \frac{1}{32}$ | $\bar{f} = \frac{7}{64}$ |
| f) | $\bar{f} = \frac{1}{32}$ | $\bar{f} = \frac{7}{64}$ |
| g) | $\bar{f} = \frac{1}{8}$ | $\bar{f} = \frac{1}{16}$ |
| h) | $\bar{f} = \frac{3}{16}$ | $\bar{f} = \frac{1}{32}$ |

the disadvantages of standard models. Now you can create the model with "custom" spectrum using weighted averaging. Analysis of the results obtained by stratified averaging of the applicates revealed a simple relationship between the average surface applicate $\bar{f}$ and the barycentric applicate $f_0$. This dependence for bisquare surfaces is $\bar{f} = \frac{4}{9} \cdot f_0 + \frac{1}{36}$ and for bicubic surfaces is $\bar{f} = \frac{4}{9} \cdot f_0 + \frac{1}{72}$ double integration and immediately gives the total body volume between the base and the surface.

6. CONFLICT OF INTEREST STATEMENT

None declared.

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