Relationship between the distinguishing index, minimum
degree and maximum degree of graphs

Saeid Alikhani ∗ Samaneh Soltani

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Department of Mathematics, Yazd University, 89195-741, Yazd, Iran
alikhani@yazd.ac.ir, s.soltani1979@gmail.com

Abstract

Let \( \delta \) and \( \Delta \) be the minimum and the maximum degree of the vertices of a simple
connected graph \( G \), respectively. The distinguishing index of a graph \( G \), denoted
by \( D'(G) \), is the least number of labels in an edge labeling of \( G \) not preserved by
any non-trivial automorphism. Motivated by a conjecture by Pilániak (2017) that
implies that for any 2-connected graph \( D'(G) \leq \lceil \sqrt{\Delta(G)} \rceil + 1 \), we prove that for
any graph \( G \) with \( \delta \geq 2 \), \( D'(G) \leq \lceil \delta \sqrt{\Delta} \rceil + 1 \). Also, we show that the distinguishing
index of \( k \)-regular graphs is at most 2, for any \( k \geq 5 \).

Keywords: distinguishing index; edge colourings; bound

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1 Introduction

Let \( G = (V, E) \) be a simple connected graph. We use the standard graph notation. In
particular, \( \text{Aut}(G) \) denotes the automorphism group of \( G \). For simple connected graph
\( G \), and \( v \in V \), the neighborhood of a vertex \( v \) is the set \( N_G(v) = \{ u \in V(G) : uv \in E(G) \} \). The degree of a vertex \( v \) in a graph \( G \), denoted by \( \text{deg}_G(v) \), is the number of
edges of \( G \) incident with \( v \). In particular, \( \text{deg}_G(v) \) is the number of neighbours of \( v \) in
\( G \). We denote by \( \delta \) and \( \Delta \) the minimum and maximum degrees of the vertices of \( G \),
respectively. A graph \( G \) is \( k \)-regular if \( \text{deg}_G(v) = k \) for all \( v \in V \). The diameter of a
graph \( G \) is the greatest distance between two vertices of \( G \), and denoted by \( \text{diam}(G) \).

The distinguishing index \( D'(G) \) of a graph \( G \) is the least number \( d \) such that \( G \)
has an edge labeling with \( d \) labels that is preserved only by the identity automorphism
of \( G \). The distinguishing edge labeling was first defined by Kalinowski and Pilániak \[6\] for graphs (was inspired by the well-known distinguishing number \( D(G) \) which was
defined for general vertex labelings by Albertson and Collins \[1\]). The distinguishing
index of some examples of graphs was exhibited in [6]. For instance, \( D'(P_n) = 2 \) for every \( n \geq 3 \), and \( D'(C_n) = 3 \) for \( n = 3, 4, 5 \), \( D'(C_n) = 2 \) for \( n \geq 6 \). They showed that if \( G \) is a connected graph of order \( n \geq 3 \), then \( D'(G) \leq \Delta \), unless \( G \) is \( C_3, C_4 \) or \( C_5 \). It follows for connected graphs that \( D'(G) \geq \Delta \) if and only if \( D'(G) = \Delta + 1 \) and \( G \) is a cycle of length at most five. The equality \( D'(G) = \Delta \) holds for all paths, for cycles of length at least 6, for \( K_4, K_{3,3} \) and for symmetric or bisymmetric trees. Also, Pilśniak showed that \( D'(G) < \Delta \) for all other connected graphs. Pilśniak put forward the following conjecture.

**Conjecture 1.1** [7] If \( G \) is a 2-connected graph, then \( D'(G) \leq 1 + \lceil \sqrt{\Delta(G)} \rceil \).

In [3], we proved that if \( \delta \geq 2 \), then \( D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1 \), which proves the conjecture.

Motivated by Conjecture 1.1, in the next section, we prove that for any connected graph \( G \), if \( \delta \geq 2 \), then \( D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1 \). Also, in Section 3, we show that the distinguishing index of \( k \)-regular graphs is at most 2, for any \( k \geq 5 \).

## 2 An upper bound for \( D'(G) \) in terms of \( \delta \) and \( \Delta \)

In this section, we shall obtain an upper bound for the distinguishing index of graph \( G \) in terms of its maximum and minimum degree. For this purpose, we need some preliminaries. The friendship graph \( F_n \) \((n \geq 2)\) can be constructed by joining \( n \) copies of the cycle graph \( C_3 \) with a common vertex. The distinguishing index of \( F_n \) can be computed by the following result.

**Theorem 2.1** [2] Let \( a_n = 1 + 27n + 3\sqrt{81n^2 + 6n} \). For every \( n \geq 2 \),

\[
D'(F_n) = \left\lceil \frac{1}{3} (a_n)^{\frac{1}{3}} + \frac{1}{3(a_n)^{\frac{1}{3}}} + \frac{1}{3} \right\rceil.
\]

Also we need the following theorem:

**Theorem 2.2** [7] Let \( G \) be a connected graph that is neither a symmetric nor an asymmetric tree. If the maximum degree of \( G \) is at least 3, then \( D'(G) \leq \Delta(G) - 1 \) unless \( G \) is \( K_4 \) or \( K_{3,3} \).

**Theorem 2.3** For any connected graph \( G \), if \( \delta \geq 2 \), then \( D'(G) \leq \lceil \sqrt{\Delta} \rceil + 1 \).

**Proof.** If \( \Delta \leq 5 \), then the result follows from Theorem 2.2. So, we suppose that \( \Delta \geq 6 \). Let \( v \) be a vertex of \( G \) with the maximum degree \( \Delta \). By Theorem 2.1, we can label the pendant friendship graph (a subgraph is pendant if it has only one vertex in common with the rest of a graph) in common with \( G \) at \( v \) for which \( v \) is the central point of the friendship graph, with at most \( \lceil \sqrt{\Delta} \rceil \) labels from label set \( \{0, 1, \ldots, \lceil \sqrt{\Delta} \rceil \} \), distinguishingly. If there exists one pendant triangle in common with \( G \) at \( v \), then we label the two its incident edges to \( v \) with 0 and 1, and another edges of the pendant triangle with label 2.
Let \( N^{(1)}(v) = \{v_1, \ldots, v_{|N^{(1)}(v)|}\} \) be the vertices of \( G \) at distance one from \( v \), except the vertices of pendant friendship or triangle graph in common with \( G \) at \( v \). Suppose that \( d := \lceil \sqrt[3]{\Delta^3-1} \rceil - 1 \) and we continue our labeling by the following steps:

Step 1) Since \( |N^{(1)}(v)| \leq \Delta \), so we can label the edges \( v_vd+j \) with label \( i \), for \( 0 \leq i \leq \lceil \sqrt[3]{\Delta} \rceil \) and \( 1 \leq j \leq d \), and we do not use label 0 any more. With respect to the number of incident edges to \( v \) with label 0, we conclude that the vertex \( v \) is fixed under each automorphism of \( G \) preserving the labeling. Also, since the pendant friendship or triangle graph in common with \( G \) at \( v \) has been labeled distinguishingly, so the vertices of pendant graph are fixed under each automorphism of \( G \) preserving the labeling. Hence, every automorphism of \( G \) preserving the labeling must map the set of vertices of \( G \) at distance \( i \) from \( v \) to itself setwise, for any \( 1 \leq i \leq \text{diam}(G) \). We denote the set of vertices of \( G \) at distance \( i \) from \( v \) for any \( 2 \leq i \leq \text{diam}(G) \), by \( N^{(i)}(v) \).

If \( N^{(i)}(v) = \emptyset \), for any \( i \geq 2 \), then we suppose that \( E_k(v_{j}d+k) \) is the set of unlabeled edges of \( G \) incident to the vertex \( v_{j}d+k \). For every \( 0 \leq j \leq \lceil \sqrt[3]{\Delta} \rceil \), we can label the elements of each \( E_k(v_{j}d+k) \) with labels \( \{1, \ldots, \lceil \sqrt[3]{\Delta} \rceil \} \) such that for every pair of \( (E_k(v_{j}d+k), E'_k(v_{j}d+k')) \), where \( k \neq k' \), there exist a label \( l, 1 \leq l \leq \lceil \sqrt[3]{\Delta} \rceil \), such that the number of label \( l \) used for labeling of elements of \( E_k(v_{j}d+k) \) and \( E'_k(v_{j}d+k') \) is distinct. Therefore all elements of \( N^{(1)}(v) \) is fixed under each automorphism of \( G \) preserving the labeling. Thus we suppose that \( N^{(i)}(v) \neq \emptyset \), for some \( i \geq 2 \).

Now we partition the vertices \( N^{(1)}(v) \) to two sets \( M^{(1)}_1 \) and \( M^{(1)}_2 \) as follows:

\[
M^{(1)}_1 = \{x \in N^{(1)}(v) : N(x) \subseteq N(v)\}, \quad M^{(1)}_2 = \{x \in N^{(1)}(v) : N(x) \not\subseteq N(v)\}.
\]

Thus the sets \( M^{(1)}_1 \) and \( M^{(1)}_2 \) are mapped to \( M^{(1)}_1 \) and \( M^{(1)}_2 \), respectively, setwise, under each automorphism of \( G \) preserving the labeling. For \( 0 \leq i \leq \lceil \sqrt[3]{\Delta} \rceil \), we set \( L_i = \{v_{i}d+j : 1 \leq j \leq d\} \). By this notation, we get that for \( 0 \leq i \leq \lceil \sqrt[3]{\Delta} \rceil \), the set \( L_i \) is mapped to \( L_i \) under each automorphism of \( G \) preserving the labeling, setwise. Let the sets \( M^{(1)}_{1i} \) and \( M^{(1)}_{2i} \) for \( 0 \leq i \leq \lceil \sqrt[3]{\Delta} \rceil \) are as follows:

\[
M^{(1)}_{1i} = M^{(1)}_1 \cap L_i, \quad M^{(1)}_{2i} = M^{(1)}_2 \cap L_i.
\]

It is clear that the sets \( M^{(1)}_{1i} \) and \( M^{(1)}_{2i} \) are mapped to \( M^{(1)}_{1i} \) and \( M^{(1)}_{2i} \), respectively, setwise, under each automorphism of \( G \) preserving the labeling. Since for any \( 0 \leq i \leq \lceil \sqrt[3]{\Delta} \rceil \), we have \( |M^{(1)}_{1i}| \leq d \), so we can label all incident edges to each element of \( M^{(1)}_{1i} \) with labels \( \{1, 2, \ldots, \lceil \sqrt[3]{\Delta} \rceil \} \), such that for any two vertices of \( M^{(1)}_{1i} \), say \( x \) and \( y \), there exists a label \( k, 1 \leq k \leq \lceil \sqrt[3]{\Delta} \rceil \), such that the number of label \( k \) for the incident edges to \( x \) is different from the number of label \( k \) for the incident edges to \( y \). Hence, it can be deduce that each vertex of \( M^{(1)}_{1i} \) is fixed under each automorphism of \( G \) preserving the labeling, where \( 0 \leq i \leq \lceil \sqrt[3]{\Delta} \rceil \). Thus every vertices of \( M^{(1)}_1 \) is fixed under each automorphism of \( G \) preserving the labeling. In sequel, we want to label the edges incident to vertices of \( M^{(1)}_2 \) such that \( M^{(1)}_2 \) is fixed under each automorphism of \( G \) preserving the labeling, pointwise. For this purpose, we partition the vertices of
$M_{2i}^{(1)}$ to the sets $M_{2ij}^{(1)}$, where $1 \leq j \leq \Delta - 1$ as follows:

$$M_{2ij}^{(1)} = \{x \in M_{2i}^{(1)} : |N(x) \cap N^{(2)}(v)| = j\}.$$ 

Since the set $N^{(i)}(v)$, for any $i$, is mapped to itself, it can be concluded that $M_{2ij}$ is mapped to itself under each automorphism of $G$ preserving the labeling, for any $i$ and $j$. Let $M_{2ij}^{(1)} = \{x_{j1}, x_{j2}, \ldots, x_{j_{s_j}}\}$. It is clear that $|M_{2ij}^{(1)}| \leq |M_{2i}^{(1)}| \leq d$. Now we consider the two following cases for each $0 \leq i \leq \lceil \sqrt[3]{\Delta}\rceil$:

Case 1) Let $j < \delta - 1$ and $\delta \geq 3$. Since $|M_{2ij}^{(1)}| \leq d$, so we can label all incident edges to each element of $M_{2ij}^{(1)}$ with labels $\{1, 2, \ldots, \lceil \sqrt[3]{\Delta}\rceil\}$, such that for any two vertices of $M_{2ij}^{(1)}$, say $x$ and $y$, there exists a label $k$, $1 \leq k \leq \lceil \sqrt[3]{\Delta}\rceil$, such that the number of label $k$ for the incident edges to $x$ is different from the number of label $k$ for the incident edges to $y$. Hence, it can be deduced that each vertex of $M_{2ij}^{(1)}$ is fixed under each automorphism of $G$ preserving the labeling, where $1 \leq j < \delta - 1$.

Case 2) Let $j \geq \delta - 1$. Let $x_{jk} \in M_{2i}^{(1)}$, and $N(x_{jk}) \cap N^{(2)}(v) = \{x'_{jk1}, x'_{jk2}, \ldots, x'_{jk_{s_j}}\}$. We assign to the $j$-tuple $(x_{jk}x'_{jk1}, \ldots, x_{jk}x'_{jk_{s_j}})$ of edges, a $j$-tuple of labels such that for every $x_{jk}$ and $x_{jk'}$, $1 \leq k, k' \leq s_j$, there exists a label $l$ in their corresponding $j$-tuples of labels with different number of label $l$ in their coordinates. For constructing $|M_{2ij}^{(1)}|$ numbers of such $j$-tuples we need, $\min\{r: \binom{j+r-1}{r-1} \geq |M_{2ij}^{(1)}|\}$ distinct labels. Since for any $\delta - 1 \leq j \leq \Delta - 1$, we have

$$\min\left\{r: \binom{j+r-1}{r-1} \geq |M_{2ij}^{(1)}|\right\} \leq \min\left\{r: \binom{j+r-1}{r-1} \geq d\right\} \leq \lceil \sqrt[3]{\Delta}\rceil,$$

so we need at most $\lceil \sqrt[3]{\Delta}\rceil$ distinct labels from label set $\{1, 2, \ldots, \lceil \sqrt[3]{\Delta}\rceil\}$ for constructing such $j$-tuples. Hence, the vertices of $M_{2ij}^{(1)}$, for any $\delta - 1 \leq j \leq \Delta - 1$, are fixed under each automorphism of $G$ preserving the labeling.

Therefore, the vertices of $M_{2i}^{(1)}$ for any $0 \leq i \leq \lceil \sqrt[3]{\Delta}\rceil$, and so the vertices of $M_2^{(1)}$ are fixed under each automorphism of $G$ preserving the labeling. Now, we can get that all vertices of $N^{(1)}(v)$ are fixed. If there exist unlabeled edges of $G$ with the two endpoints in $N^{(1)}(v)$, then we assign them an arbitrary label, say 1.

Step 2) Now we consider $N^{(2)}(v)$. We partition this set such that the vertices of $N^{(2)}(v)$ with the same neighbours in $M_2^{(1)}$, lie in a set. In other words, we can write $N^{(2)}(v) = \bigcup_i A_i$, such that $A_i$ contains that elements of $N^{(2)}(v)$ having the same neighbours in $M_2^{(1)}$, for any $i$. Since all vertices in $M_2^{(1)}$ are fixed, so the set $A_i$ is mapped to $A_i$ setwise, under each automorphism of $G$ preserving the labeling. Let $A_i = \{w_{i1}, \ldots, w_{it_i}\}$, and we have

$$N(w_{i1}) \cap M_2^{(1)} = \cdots = N(w_{it_i}) \cap M_2^{(1)} = \{v_{i1}, \ldots, v_{ip_i}\}.$$ 

We consider the two following cases:

Case 1) If for every $w_{ij}$ and $w_{ij'}$ in $A_i$, where $1 \leq j, j' \leq t_i$, there exists a $k$, $1 \leq k \leq p_i$, for which the label of edges $w_{ij}v_{ik}$ is different from label of edge $w_{ij'}v_{ik}$,
then all vertices of $G$ in $A_i$ are fixed under each automorphism of $G$ preserving the labeling.

Case 2) If there exist $w_{ij}$ and $w_{ij'}$ in $A_i$, where $1 \leq j, j' \leq t_i$, such that for every $k$, $1 \leq k \leq p_i$, the label of edges $w_{ij}v_{ik}$ and $w_{ij'}v_{ik}$ are the same, then we can make a labeling such that the vertices in $A_i$ have the same property as Case 1, and so are fixed under each automorphism of $G$ preserving the labeling, by using at least one of the following actions:

- By permuting the components of the $j$-tuple of labels assigned to the incident edges to $v_{ik}$ with an end point in $N^2(v)$,
- By using a new $j$-tuple of labels, with labels $\{1, 2, \ldots, \lceil \frac{\sqrt{\Delta}}{\log r} \rceil \}$, for incident edges to $v_{ik}$ with an end point in $N^2(v)$, such that the vertices in $M_2^{(1)}$ are fixed under each automorphism of $G$ preserving the labeling,
- By labeling the unlabeled edges of $G$ with the two end points in $N^2(v)$ which are incident to the vertices in $A_i$,
- By labeling the unlabeled edges of $G$ which are incident to the vertices in $A_i$, and another their endpoint is $N^3(v)$,
- By labeling the unlabeled edges of $G$ with the two end points in $N^3(v)$ for which the end points in $N^3(v)$ are adjacent to some of vertices in $A_i$.

Using at least one of above actions, it can be seen that every two vertices $w_{ij}$ and $w_{ij'}$ in $A_i$ have the property as Case (1). Thus we conclude that all vertices in $A_i$, for any $i$, and so all vertices in $N^2(v)$, are fixed under each automorphism of $G$ preserving the labeling. If there exist unlabeled edges of $G$ with the two endpoints in $N^2(v)$, then we assign them an arbitrary label, say 1.

By following this method, in the next step we partition $N^3(v)$ exactly by the same method as partition of $N^2(v)$ to the sets $A_i$s in Step 2, we can make a labeling such that $N^3(v)$ is fixed pointwise, under each automorphism of $G$ preserving the labeling, for any $3 \leq i \leq \text{diam}(G)$.

By the result obtained by Fisher and Isaak [4] and independently by Imrich, Jerebic and Klavzar [5] the distinguishing index of complete bipartite graphs is as follows. By using Theorem 2.4, we can see that the upper bound of Theorem 2.3 is sharp for some complete bipartite graphs.

**Theorem 2.4** [4, 5] Let $p, q, r$ be integers such that $r \geq 2$ and $(r-1)^p < q \leq r^p$. Then

$$D'(K_{p,q}) = \left\{ \begin{array}{ll} r & \text{if } q \leq r^p - \lceil \log r \rceil - 1, \\ r + 1 & \text{if } q \geq r^p - \lceil \log r \rceil + 1. \end{array} \right.$$  

If $q = r^p - \lceil \log r \rceil$ then the distinguishing index $D'(K_{p,q})$ is either $r$ or $r + 1$ and can be computed recursively in $O(\log(q))$ time.
3  Distinguishing index of regular graphs

By Theorem 2.3, we can conclude that the distinguishing index of a $k$-regular graph is at most 3. In the following we improve this upper bound to 2. A palette of a vertex is the set of labels of edges incident to it. We need the following result to obtain the main result of this section.

**Theorem 3.1** [7] If $G$ is a graph of order $n \geq 7$ such that $G$ has a Hamiltonian path, then $D'(G) \leq 2$.

**Theorem 3.2** Let $G$ be a connected $k$-regular graph of order $n$ with $k \geq 5$. Then $D'(G) \leq 2$.

**Proof.** If $k \geq \frac{n-1}{2}$, then it is known that $G$ has a Hamiltonian path, and so $D'(G) \leq 2$, by Theorem 3.1. Then, we suppose that $5 \leq k < \frac{n-1}{2}$.

Let $v$ be an arbitrary vertex of $G$, and $N^{(1)}(v) = \{v_1, \ldots, v_k\}$ be the vertices of $G$ at distance one from $v$. We state our labeling by the following steps:

Step 1) We label all incident edges to $v$ with 1. In our edge labeling of the graph $G$, the vertex $v$ will be the unique vertex with the monochromatic palette $\{1\}$. Hence, the vertex $v$ is fixed under each automorphism of $G$ preserving the labeling. Thus, every automorphism of $G$ preserving the labeling must map the set of vertices of $G$ at distance $i$ from $v$ to itself setwise, for any $1 \leq i \leq \text{diam}(G)$. We denote the set of vertices of $G$ at distance $i$ from $v$ for any $2 \leq i \leq \text{diam}(G)$, by $N^{(i)}(v)$. If $N^{(i)}(v) = \emptyset$ for any $i \geq 2$, then $k \geq \frac{n-1}{2}$, which is a contradiction. Thus we suppose that $N^{(i)}(v) \neq \emptyset$ for some $i \geq 2$.

We can label all incident edges to each element of $N^{(1)}(v) \setminus \{v_1\}$ with labels 1 and 2, such that for any two vertices of $N^{(1)}(v) \setminus \{v_1\}$, say $x$ and $y$, there exists a label $k$, $k = 1, 2$, such that the number of label $k$ for the incident edges to $x$ is different from the number of label $k$ for the incident edges to $y$, and also the number of label 2 for the incident edges to each element of $N^{(1)}(v) \setminus \{v_1\}$ is at least one. Next we label the incident edges to $v_1$ exactly the same as labeling of the incident edges of one of the vertices in $N^{(1)}(v) \setminus \{v_1\}$, say $v_2$. Therefore all vertices in $N^{(1)}(v)$ will also be fixed, except, possibly $v_1$ and $v_2$. To distinguish $v_1$ and $v_2$, we label the incident edges to $v_1$ and $v_2$ which are incident to a vertex in $N^{(2)}(v)$, such that there exists a label $k = 1, 2$, for which the number of label $k$ for the incident edges to $v_1$ and $v_2$ are distinct. Thus, all vertices in $N^{(1)}(v)$ will be also fixed.

Step 2) Now we consider $N^{(2)}(v)$. We partition this set such that the vertices of $N^{(2)}(v)$ with the same neighbours in $N^{(1)}(v)$, lie in a set. In other words, we can write $N^{(2)}(v) = \bigcup_i A_i$, such that $A_i$ contains that elements of $N^{(2)}(v)$ having the same neighbours in $N^{(1)}(v)$, for any $i$. Since all vertices in $N^{(1)}(v)$ are fixed, so the set $A_i$ is mapped to $A_i$ setwise, under each automorphism of $G$ preserving the labeling. Let $A_i = \{w_{i1}, \ldots, w_{ik}\}$, and we have

$$N(w_{i1}) \cap N^{(1)}(v) = \cdots = N(w_{ik}) \cap N^{(1)}(v) = \{v_{i1}, \ldots, v_{ip_i}\}.$$
We consider the two following cases:

Case 1) If for every \(w_{ij}\) and \(w_{ij'}\) in \(A_i\), where \(1 \leq j, j' \leq t_i\), there exists a \(k, 1 \leq k \leq p_i\), for which the label of edges \(w_{ij} v_{ik}\) is different from label of edge \(w_{ij'} v_{ik}\), then all vertices of \(G\) in \(A_i\) are fixed under each automorphism of \(G\) preserving the labeling.

Case 2) If there exist \(w_{ij}\) and \(w_{ij'}\) in \(A_i\), where \(1 \leq j, j' \leq t_i\), such that for every \(k, 1 \leq k \leq p_i\), the label of edges \(w_{ij} v_{ik}\) and \(w_{ij'} v_{ik}\) are the same, then we can make a labeling such that the vertices in \(A_i\) have the same property as Case 1, and so are fixed under each automorphism of \(G\) preserving the labeling, by using at least one of the following actions:

- By permuting the labels assigned to the incident edges to \(v_{ik}\) with an end point in \(N^{(2)}(v)\),
- By using a new labeling for incident edges to \(v_{ik}\) with an end point in \(N^{(2)}(v)\), such that the vertices in \(N^{(1)}(v)\) are fixed under each automorphism of \(G\) preserving the labeling,
- By labeling the unlabeled edges of \(G\) with the two end points in \(N^{(2)}(v)\) which are incident to the vertices in \(A_i\),
- By labeling the unlabeled edges of \(G\) which are incident to the vertices in \(A_i\), and another their endpoint is \(N^{(3)}(v)\),
- By labeling the unlabeled edges of \(G\) with the two end points in \(N^{(3)}(v)\) for which the end points in \(N^{(3)}(v)\) are adjacent to some of vertices in \(A_i\).

Using at least one of above actions, it can be concluded that all vertices in \(A_i\), for any \(i\), and so all vertices in \(N^{(2)}(v)\), are fixed under each automorphism of \(G\) preserving the labeling. If there exist unlabeled edges of \(G\) with the two endpoints in \(N^{(2)}(v)\), then we assign them an arbitrary label, say 2.

By following this method, in the next step we partition \(N^{(3)}(v)\) exactly by the same method as partition of \(N^{(2)}(v)\) to the sets \(A_i\)s in Step 2, we can make a labeling such that \(N^{(3)}(v)\) is fixed pointwise, under each automorphism of \(G\) preserving the labeling, for any \(3 \leq i \leq \text{diam}(G)\).

□

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