Sigma clique covering of graphs

Akbar Davoodi Ramin Javadi Behnaz Omoomi
Department of Mathematical Sciences
Isfahan University of Technology
84156-83111, Isfahan, Iran

Abstract

The sigma clique cover number (resp. sigma clique partition number) of graph $G$, denoted by $scc(G)$ (resp. $scp(G)$), is defined as the smallest integer $k$ for which there exists a collection of cliques of $G$, covering (resp. partitioning) all edges of $G$ such that the sum of sizes of the cliques is at most $k$. In this paper, among some results we provide some tight bounds for $scc$ and $scp$.

Keywords: clique covering; clique partition; sigma clique covering; sigma clique partition; set intersection representation; set system.

1 Introduction

Throughout the paper, all graphs are simple and undirected. By a clique of a graph $G$, we mean a subset of mutually adjacent vertices of $G$ as well as its corresponding complete subgraph. The size of a clique is the number of its vertices. Also, a biclique of $G$ is a complete bipartite subgraph of $G$. A clique covering (resp. biclique covering) of $G$ is defined as a family of cliques (resp. bicliques) of $G$ such that every edge of $G$ lies in at least one of the cliques (resp. bicliques) comprising this family. A clique (resp. biclique) covering in which each edge belongs to exactly one clique (resp. biclique), is called a clique (resp. biclique) partition. The minimum size of a clique covering, a biclique covering, a clique partition and a biclique partition of $G$ are called clique cover number, biclique cover number, clique partition number and biclique partition number of $G$ and are denoted by $cc(G)$, $bc(G)$, $cp(G)$ and $bp(G)$, respectively.

The subject of clique covering has been widely studied in recent decades. First time, Erdős et al. in [6] presented a close relationship between the clique covering and the set intersection representation. Also, they proved that the clique partition number of a graph on $n$ vertices cannot exceed $n^2/4$ (known as Erdős-Goodman-Pónsa theorem). The connections of clique covering and other combinatorial objects have been explored (see e.g. [14, 16]). For a survey of the classical results on the clique and biclique coverings, see [11, 13].
Chung et al. in [4] and independently Tuza in [15] considered a weighted version of the biclique covering. In fact, given a graph $G$, they were concerned with minimizing $\sum_{B \in \mathcal{B}} |V(B)|$ among all biclique coverings $\mathcal{B}$ of $G$. They proved that every graph on $n$ vertices has a biclique covering such that the sum of number of vertices of these bicliques is $O(n^2 / \log n)$ [4, 15]. Furthermore, a clique counterpart of weighted biclique cover number has been studied. Following a conjecture by Katona and Tarjan, Chung [3], Győri and Kostochka [7] and Kahn [10], independently, proved that every graph on $n$ vertices has a clique partition such that the sum of number of vertices in these cliques is at most $n^2/2$. This can be considered as a generalization of Erdős-Goodman-Pósa theorem.

In this paper, we are concerned with a weighted version of the clique cover number. Let $G$ be a graph. The sigma clique cover number of $G$, denoted by $scc(G)$, is defined as the minimum integer $k$ for which there exists a clique covering $\mathcal{C}$ of $G$, such that the sum of its clique sizes is at most $k$. For a clique covering $\mathcal{C}$ of a graph $G$ and a vertex $u \in V(G)$, let the valency of $u$ (with respect to $\mathcal{C}$), denoted by $V_C(u)$, be the number of cliques in $\mathcal{C}$ containing $u$. In fact,

$$scc(G) = \min \sum_{C \in \mathcal{C}} |C| = \min \sum_{u \in V(G)} V_C(u),$$

where the minimum is taken over all clique coverings of $G$. Analogously, one can define sigma clique partition number of $G$, denoted by $scp(G)$. As a matter of fact, the above-mentioned result in [3, 7, 10] states that for every graph $G$ on $n$ vertices, $scp(G) \leq n^2/2$.

In order to reveal inherent difference between $cc(G)$ and $scc(G)$, we introduce a similar parameter $scc'(G)$ which is defined as the minimum of the sum of clique sizes in a clique covering $\mathcal{C}$ achieving $cc(G)$, i.e.

$$scc'(G) := \min \left\{ \sum_{C \in \mathcal{C}} |C| : \mathcal{C} \text{ is a clique covering of } G \text{ and } |\mathcal{C}| = cc(G) \right\}.$$

It is evident that $scc(G) \leq scc'(G)$. In Section 2, first in Theorem 1, we will see that for some classes of graphs $G$, the quotient $scc'(G)/scc(G)$ can be arbitrary large. Then, we give some general bounds on the sigma clique cover number and the sigma clique partition number. In particular, we prove that if $G$ is a graph on $n$ vertices with no isolated vertex and the maximum degree of the complement of $G$ is $d - 1$, for some integer $d$, then $scc(G) \leq cnd[\log ((n - 1)/(d - 1))]$, where $c$ is a constant. We conjecture that this upper bound is best up to a constant factor for large enough $n$. In Section 3, using a well-known result by Bollobás, we prove the correctness of this conjecture for $d = 2$. In other words, we show that for every even integer $n$, if $G$ is the complement of an induced matching on $n$ vertices, then $scc(G) \sim n \log n$. Finally, in Section 4 we give an interpretation of this conjecture as an interesting set system problem.
2 Some Bounds

In this section, first we present a class of graphs for which the family of clique coverings achieving $cc(G)$ is disjoint from the family of clique coverings achieving $scc(G)$. Then, we provide several inequalities relating the introduced clique covering parameters. Moreover, we present an upper bound for $scc(G)$ in terms of the number of vertices and the maximum degree of the complement of $G$.

Theorem 1. There exists a sequence of graphs $\{G_n\}$ such that $scc'(G_n)/scc(G_n)$ tends to infinity as $n$ tends to infinity.

Proof. Let $n$ be a positive integer and $G_n$ be a graph on $3n+2$ vertices, such that $V(G_n) = \{x_0, y_0\} \cup X \cup Y \cup Z$, where $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and $Z = \{z_1, \ldots, z_n\}$ and adjacency is as follows. The sets $X \cup \{x_0\}$, $Y \cup \{y_0\}$ and $Z$ are three cliques and every vertex in $Z$ is adjacent to every vertex in $X \cup Y$. Moreover, for all $i, j \in \{1, \ldots, n\}$, $x_i$ is adjacent to $y_j$ if and only if $i = j$ (see Figure 1).

First, note that each clique of $G_n$ covers at most one edge from the set $\{x_iy_i : 1 \leq i \leq n\} \cup \{x_0x_1, y_0y_1\}$. This yields $cc(G_n) \geq n + 2$. Now, we show that $G_n$ has a unique clique covering containing exactly $n + 2$ cliques. Let $\mathcal{C}$ be a clique covering of $G_n$ consisting of $n + 2$ cliques. Assume that the clique $C_i \in \mathcal{C}$ covers the edge $x_iy_i$, for $1 \leq i \leq n$, and the cliques $C_{n+1} \in \mathcal{C}$ and $C_{n+2} \in \mathcal{C}$ cover the edges $y_0y_1$ and $x_0x_1$, respectively. Note that $C_{n+2} \subseteq \{x_0\} \cup X$ and $x_0 \notin \bigcup_{i=1}^{n+1} C_i$. Therefore, $C_{n+2} = \{x_0\} \cup X$. Similarly, $C_{n+1} = \{y_0\} \cup Y$. Also, we have $x_j, y_j \notin C_i$, for every $1 \leq i \neq j \leq n$. Thus, $C_i = \{x_i, y_i\} \cup Z$, $1 \leq i \leq n$. Hence, the clique covering $\mathcal{C} = \{C_i : 1 \leq i \leq n+2\}$ is the unique clique covering of $G_n$ with $n + 2$ cliques and then $cc(G_n) = n + 2$. Consequently,

$$soc'(G_n) = \sum_{C \in \mathcal{C}} |C| = n(n+2) + 2(n+1) = n^2 + 4n + 2.$$ 

On the other hand, the $n + 4$ cliques $\{x_0\} \cup X$, $\{y_0\} \cup Y$, $X \cup Z$, $Y \cup Z$ and $\{x_i, y_i\}$,
1 \leq i \leq n$, form a clique covering $C'$ and thus,
\[ \text{scc}(G_n) \leq \sum_{C \in C'} |C| = 2(n + 1) + 2(2n) + 2n = 8n + 2. \]

Hence, the families of the optimum clique coverings achieving $\text{cc}(G_n)$ and $\text{scc}(G_n)$ are disjoint and $\text{scc}'(G_n)/\text{scc}(G_n)$ tends to infinity.

In the following, we prove some relations between $\text{scc}(G)$, $\text{scp}(G)$ and $\text{cp}(G)$.

**Theorem 2.** If $G$ is a graph with $m$ edges and $\omega(G)$ is the clique number of $G$, then

i) \[ \frac{2m}{\omega(G) - 1} \leq \text{scc}(G) \leq \text{scp}(G) \leq 2m, \]

ii) \[ \frac{\text{scp}^2(G)}{2m + \text{scp}(G)} \leq \text{cp}(G). \]

Also, in all relations, the equalities hold for the triangle-free graphs.

**Proof.** i) Since the collection of all edges of $G$ is a clique partition for $G$, we have $\text{scc}(G) \leq \text{scp}(G) \leq 2m$. Now, suppose that $C$ is a clique covering of $G$ such that $\sum_{C \in C} |C| = \text{scc}(G)$. Clearly $m \leq \sum_{C \in C} \binom{|C|}{2}$. Hence,
\[ 2m \leq \sum_{C \in C} |C|^2 - \text{scc}(G) \leq (\omega(G) - 1) \text{scc}(G). \]

ii) Let $\text{cp}(G) = t$ and $\{C_1, \ldots, C_t\}$ be a clique partition of $G$. Then, $m = \sum_{i=1}^t \binom{|C_i|}{2}$. Thus,
\[
2m = \sum_{i=1}^t |C_i|^2 - \sum_{i=1}^t |C_i| \\
\geq \frac{1}{t} \left( \sum_{i=1}^t |C_i| \right)^2 - \sum_{i=1}^t |C_i| \\
\geq \frac{1}{t} \text{scp}^2(G) - \text{scp}(G),
\]
where the second inequality is due to Cauchy-Schwarz inequality and the last inequality holds because the function $f(x) = \frac{1}{t} x^2 - x$ is increasing for $x \geq \frac{t}{2}$ and clearly $\text{scp}(G) \geq \text{cp}(G) = t$.

For a vertex $u \in V(G)$, let $N_G(u)$ denotes the set of all neighbours of $u$ in $G$ and let $\overline{G}$ stand for the complement of $G$. Moreover, let $\Delta(G)$ be the maximum degree of $G$. Alon in [1] proved that if $G$ is a graph on $n$ vertices and $\Delta(\overline{G}) = d$, then $\text{cc}(G) = O(d^2 \log n)$. In the following, modifying the idea of Alon, we establish an upper bound for $\text{scc}(G)$.
Theorem 3. If $G$ is a graph on $n$ vertices with no isolated vertex and $\Delta(G) = d - 1$, then
\[ \text{scc}(G) \leq (e^2 + 1)nd \left[ \ln \left( \frac{n-1}{d-1} \right) \right]. \] (1)

Proof. Let $0 < p < 1$ be a fixed number and let $S$ be a random subset of $V(G)$ defined by choosing every vertex $u$ independently with probability $p$. For every vertex $u \in S$, if there exists a non-neighbour of $u$ in $S$, then remove $u$ from $S$. The resulting set is a clique of $G$. Repeat this procedure $t$ times, independently, to get $t$ cliques $C_1, C_2, \ldots, C_t$ of $G$.

Let $F$ be the set of all the edges which are not covered by the cliques $C_1, \ldots, C_t$. For every edge $uv$, using inequality $(1 - \alpha) \leq e^{-\alpha}$, we have
\[ \Pr(uv \in F) = \left( 1 - p^2(1-p)^{|N_G(u) \cup N_G(v)|} \right)^t \leq (1 - p^2(1-p)^{2(d-1)})^t \leq e^{-tp^2(1-p)^{2(d-1)}}. \]

The cliques $C_1, \ldots, C_t$ along with all edges in $F$ comprise a clique covering of $G$. Hence,
\[ \text{scc}(G) \leq \mathbb{E} \left( \sum_{i=1}^{t} |C_i| + 2|F| \right) \leq npt + 2 \left( \frac{n}{2} \right) e^{-tp^2(1-p)^{2(d-1)}}. \]

Now, set $p := 1/d$. Since $(1 - 1/d)^{d-1} \geq 1/e$, we have
\[ \text{scc}(G) \leq \frac{nt}{d} + n(n-1)e^{-td^2e^{-2}}. \]

Finally, by setting $t := \lceil e^2d^2 \ln(\frac{n-1}{d-1}) \rceil > 0$, we have
\[ \text{scc}(G) \leq \frac{n(e^2d^2 \ln(\frac{n-1}{d-1}) + 1)}{d} + n(d-1) \]
\[ \leq nd \left[ \ln \left( \frac{n-1}{d-1} \right) \right] \left( e^2 + \frac{1}{\ln(\frac{n-1}{d-1})} \right) \]
\[ \leq nd \left[ \ln \left( \frac{n-1}{d-1} \right) \right] (e^2 + 1). \]

The upper bound in $(1)$ gives rise to the question that for positive integers $n, d$, how large can be the sigma clique cover number of an $n-$vertex graph where the maximum degree of its complement is $d - 1$. A first candidate for graphs with large scc is the family of complete multipartite graphs.

For positive integers $n, k$, an orthogonal array $OA(n,k)$ is an $n^2 \times k$ array of elements in \{1, \ldots, n\}, such that in every two columns each ordered pair $(i, j)$, $1 \leq i, j \leq n$, appears exactly once.
Theorem 4. For positive integers \( n, d \) with \( n \geq 2d \), let \( G \) be a complete multipartite graph on \( n \) vertices with at least two parts of size \( d \) and the other parts of size at most \( d \). Then, \( \Delta(G) = d - 1 \) and \( scc(G) \geq nd \). Moreover, if \( d \) is a prime power and \( n \leq d(d+1) \), then \( scc(G) = scp(G) = nd \).

Proof. Let \( C \) be a clique covering for \( G \). For every vertex \( u \), \( N_G(u) \) contains a stable set (a set of pairwise nonadjacent vertices) of size \( d \). Therefore, \( u \) is contained in at least \( d \) cliques of \( C \), i.e. the valency of \( u \), \( \nu_C(u) \) is at least \( d \). Thus, \( scc(G) \geq nd \).

Now, let \( d \) be a prime power. It is known that there exists an orthogonal array \( OA(d, d+1) \). Let \( k = d + 1 \) and denote the \( i \)th row of the orthogonal array by \( a_{i1}, a_{i2}, \ldots, a_{ik} \). Also, let \( H \) be a complete \( k \)-partite graph on \( d(d+1) \) vertices with the parts \( V_1, \ldots, V_k \), where \( V_j = \{v_{j1}, \ldots, v_{jd}\} \), for \( 1 \leq j \leq k \). For each \( i \in \{1, \ldots, d^2\} \), the set \( C_i := \{v_{i1a_1}, v_{i2a_2}, \ldots, v_{ika_k}\} \) is a clique of \( H \). Since in every two columns of \( OA \), each ordered pair \((i, j)\), \( 1 \leq i, j \leq d \), appears exactly once, the collection \( C := \{C_i : 1 \leq i \leq d^2\} \) forms a clique partition for \( H \). Moreover, for every vertex \( u \in V(H) \), \( \nu_C(u) = d \). On the other hand, \( G \) is an induced subgraph of \( H \). Thus, the collection \( C' := \{C_i \cap V(G) : 1 \leq i \leq d^2\} \) is a clique partition of \( G \) and for every vertex \( u \in V(G) \), \( \nu_{C'}(u) \) is at most \( d \). Hence, \( scc(G) \leq scp(G) \leq nd \). \( \square \)

For positive integers \( t, d \), let us denote the complete \( t \)-partite graph with each part of size \( d \) by \( K_t(d) \). Theorem 3 asserts that \( scc(K_t(d)) \leq cd^2t \log t \), for some constant \( c \). Although Theorem 4 says that \( scc(K_t(d)) = d^2t \) when \( t \leq (d+1) \) and \( d \) is a prime power, we believe that \( scc(K_t(d)) \) is much larger when \( t \) is sufficiently large. This leads us to the following conjecture.

Conjecture 5. There exists a function \( f \) and a constant \( c \), such that for every positive integers \( t \) and \( d \), if \( t \geq f(d) \), then \( scc(K_t(d)) \geq cd^2t \log t \).

In fact, if Conjecture 5 is correct, then the upper bound in (1) is best possible up to a constant factor, at least for sufficiently large \( n \). In the following section, we will prove that Conjecture 5 is true for \( d = 2 \).

3 Cocktail Party Graphs

In this section, we investigate the sigma clique cover number of the Cocktail party graph \( K_t(2) \). Given a positive integer \( t \), the Cocktail party graph \( K_t(2) \) is obtained from the complete graph \( K_{2t} \) with the vertex set \( \{x_1, \ldots, x_t\} \cup \{y_1, \ldots, y_t\} \) by removing all the edges \( x_iy_i, 1 \leq i \leq t \).

Various clique covering parameters of the Cocktail party graphs have been studied in the literature. In 1977, Orlin [12] asked about asymptotic behaviour of \( cc(K_t(2)) \), with this motivation that it arises in an optimization problem in Boolean functions theory. He also conjectured that \( cp(K_t(2)) \sim t \). Gregory et al. [8] proved that for \( t \geq 4 \), \( cp(K_t(2)) \geq 2t \) and for large enough \( t \), \( cp(K_t(2)) \leq 2t \log \log 2t \). The problem that \( cp(K_t(2)) \sim 2t \)
is still an open problem. Moreover, Gregory and Pullman [9], by applying a Sperner-type theorem of Bollobás and Schönheim on set systems, proved that for every integer $t$, $cc(K_t(2)) = \sigma(t)$, where

$$\sigma(t) = \min \left\{ k : t \leq \left( \frac{k - 1}{\lceil k/2 \rceil} \right) \right\}.$$  

Furthermore, the authors in [5], using the pairwise balanced designs, have proved that $scp(K_t(2)) \sim (2t)^{3/2}$.

Here, using the following well-known theorem by Bollobás, we prove a lower bound for the sigma clique cover number of $K_t(2)$ which determines the asymptotic behaviour of $scc(K_t(2))$ and implies that Conjecture 5 is true for $d = 2$.

**Bollobás’ Theorem.** [2] Let $A_1, \ldots, A_t$ be some sets of size $a_1, \ldots, a_t$, respectively and $B_1, \ldots, B_t$ be some sets of size $b_1, \ldots, b_t$, respectively, such that $A_i \cap B_j = \emptyset$ if and only if $i = j$. Then

$$\sum_{i=1}^{t} \left( \frac{a_i + b_i}{a_i} \right)^{-1} \leq 1.$$  

**Theorem 6.** Let $K_t(2)$ be the Cocktail party graph on $2t$ vertices. Then

$$t\delta(t) \leq scc(K_t(2)) \leq t\sigma(t),$$

where $\sigma(t)$ is defined as above and $\delta(t) = \min \left\{ k - 1 : t \leq \left( \frac{k}{\lceil k/2 \rceil} \right) \right\}$.

**Proof.** Since $cc(K_t(2)) = \sigma(t)$ and every clique in $K_t(2)$ is of size at most $t$, we have $scc(K_t(2)) \leq t\sigma(t)$.

For the lower bound, assume that $\{C_1, \ldots, C_k\}$ is an arbitrary clique covering for $K_t(2)$. For every $i \in \{1, \ldots, t\}$, define

$$A_i = \{a : x_i \in C_a\}, \quad B_i = \{a : y_i \in C_a\}.$$  

Also, let $a_i = |A_i|, b_i = |B_i|$ and $c_i = a_i + b_i$. Then for every $i \neq j$, there exists a clique containing the edge $x_iy_j$. Hence, $A_i \cap B_j \neq \emptyset$. Moreover, since no clique contains both vertices $x_i$ and $y_i$, we have $A_i \cap B_i = \emptyset$.

Therefore, by Bollobás’ theorem, we have

$$\sum_{i=1}^{t} \left( \frac{a_i + b_i}{a_i} \right)^{-1} \leq 1.$$  

For every integer $m$, let $f(m) = \left( \frac{m}{\lceil m/2 \rceil} \right)^{-1}$ and $f(x)$ be the linear extension of $f(m)$ in $\mathbb{R}^+$. Since $f$ is non-increasing and convex, by Jensen inequality, we have

$$f \left( \frac{1}{t} \sum_{i=1}^{t} c_i \right) \leq f \left( \frac{1}{t} \sum_{i=1}^{t} c_i \right) \leq \frac{1}{t} \sum_{i=1}^{t} \left( \frac{c_i}{\lceil c_i/2 \rceil} \right)^{-1} \leq \frac{1}{t} \sum_{i=1}^{t} \left( \frac{a_i + b_i}{a_i} \right)^{-1} \leq \frac{1}{t}.$$
Thus, \( \left\lceil \frac{1}{t} \sum_{i=1}^{t} c_i \right\rceil \geq t \). Therefore,

\[
\delta(t) \leq \left\lceil \frac{1}{t} \sum_{i=1}^{t} c_i \right\rceil - 1 \leq \frac{1}{t} \sum_{i=1}^{t} c_i = \frac{1}{t} \sum_{a=1}^{k} |C_a|.
\]

Consequently, \( t\delta(t) \leq \text{scc}(K_t(2)) \). \( \square \)

Theorem 6 along with the approximation \( \binom{2n}{n} \sim 2^{2n}/\sqrt{\pi n} \) yields the following corollary which proves Conjecture 5 for \( d = 2 \).

Corollary 7. For every integer \( t \), \( \text{scc}(K_t(2)) \sim t \log t \).

4 Concluding Remarks

In previous section, by considering a clique covering as a set system and applying Bollobás’ theorem, we proved Conjecture 5 for \( d = 2 \). In this point of view, this conjecture can be restated as an interesting set system problem and thus it can be viewed as a generalization of Bollobás’ theorem, as follows.

Conjecture 8. Let \( d \geq 2 \), \( t \geq 1 \) and \( \mathcal{F} = \{(A_1^i, A_2^i, \ldots, A_d^i) : 1 \leq i \leq t\} \) such that \( A_j^i \) is a set of size \( k_{ij} \) and \( A_j^i \cap A_{j'}^{i'} = \emptyset \) if and only if \( i = i' \) and \( j \neq j' \). Then, there exists a function \( f \) and a constant \( c \), such that for every \( t \geq f(d) \),

\[
\sum_{i,j} k_{ij} \geq cd^2 t \log t.
\]

Note that Conjecture 8 is true for \( d = 2 \), due to Bollobás’ theorem.

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