Characteristic matrix functions for delay differential equations with symmetry

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Abstract

A characteristic matrix function captures the spectral information of a bounded linear operator in a matrix-valued function. In this article, we consider a delay differential equation with one discrete time delay and assume this equation is equivariant with respect to a compact symmetry group. Under this assumption, the delay differential equation can have discrete wave solutions, i.e. periodic solutions that have a discrete group of spatio-temporal symmetries. We show that if a discrete wave solution has a period that is rationally related to the time delay, then we can determine its stability using a characteristic matrix function. The proof relies on equivariant Floquet theory and results by Kaashoek and Verduyn Lunel on characteristic matrix functions for classes of compact operators. We discuss applications of our result in the context of delayed feedback stabilization of periodic orbits.

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1 Introduction

For infinite dimensional dynamical systems, determining the stability of an invariant set often poses challenges due to the infinite dimensional nature of the problem. However, sometimes we are able to make a dimension reduction in the sense that the stability of the invariant set can be determined by computing zeros of a scalar valued function. This simplifies the stability analysis since we can now apply analytical and numerical techniques directly to the scalar valued function. Concrete examples of this approach appear in the context of partial differential equations, where the stability of certain travelling wave solutions can be computed using the Evans function [San02]. For delay differential equations (DDE), the stability of equilibria and certain classes of periodic orbits can be computed using so-called characteristic matrix functions, as introduced by Kaashoek and Verduyn Lunel in [KV92] and [KV21].

This article is concerned with DDE that have built-in symmetries; in this case, periodic solutions of the DDE can satisfy additional spatio-temporal relations. Although spatio-temporal patterns and their stability are well studied in the context of ordinary differential equations (see e.g. [Fie88] [LI99] [WS06]), they are much less explored in the setting of DDE. This article makes a next step in the stability analysis of spatio-temporal patterns in DDE by combining the concept of characteristic matrix functions with techniques from symmetric systems.

Specifically, we consider a DDE of the form

\[ \dot{x}(t) = f(x(t), x(t-\tau)), \quad t \geq 0 \]  

with \( f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \) a \( C^2 \) function and time delay \( \tau > 0 \). We assume that the DDE (1.1) is equivariant with respect to a compact subgroup \( \Gamma \subseteq GL(N, \mathbb{R}) \) of the general linear group. In this case, a periodic orbit \( x_\ast \) of (1.1) can satisfy spatio-temporal relations of the form \( hx_\ast(t) = x_\ast(t + r) \), with \( h \in \Gamma \) a spatial transformation and \( r > 0 \) a fraction of the period. We prove that if the time shift \( r \) is equal to the time delay \( \tau \), we can determine the stability of \( x_\ast \) by computing the zeroes of a scalar valued function.

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In the situation where the DDE (1.1) is not symmetric but does have a periodic orbit with period equal to the delay, we can determine the stability of this periodic orbit by computing the roots of a scalar valued function. This result was proven under an additional condition in [Ver92] and later in full generality in [KV21, Section 11.4]. The result presented in this article can be viewed as a refinement of the result in [Ver92] and [KV21, Section 11.4] in the sense that under the extra assumption that the DDE is symmetric, we are able to make more precise statements about periodic solutions with spatio-temporal patterns.

The result presented in this article is particularly relevant in the context of equivariant Pyragas control, a delayed feedback control scheme that aims to stabilize spatio-temporal patterns. If the ordinary differential equation
\[
\dot{x}(t) = F(x(t)), \quad F : \mathbb{R}^N \rightarrow \mathbb{R}^N \tag{1.2}
\]
has an unstable periodic solution \(x_*\) with spatio-temporal relation \(hx_*(t) = x_*(t + r)\), then this is also a solution of the delay differential equation
\[
\dot{x}(t) = F(x(t)) + K [x(t) - hx(t - r)], \quad K \in \mathbb{R}^{N \times N}, \tag{1.3}
\]
cf. [FFS10]. However, the overall dynamics of systems (1.2) and (1.3) are radically different, and we can try to choose the matrix \(K \in \mathbb{R}^{N \times N}\) in such a way that \(x_*\) is a stable solution of (1.3). Despite its many experimental applications (see e.g. [DNES+19, HGTS21]), mathematical results on the control scheme (1.3) are rare due to the periodic and infinite dimensional nature of the problem. In fact, most analytical results in the literature so far are either close to a bifurcation point [HKRH19, HBKR17, dWV17, FLR+20, FFG+07, Fie08] or concern periodic orbits that can be transformed to stationary solutions of autonomous systems [PPK14, Fie08, Sjb16, Fie10, Fie08]. The contents of this article are a first step towards further insights in equivariant Pyragas control, such as the results on stabilization of non-stationary periodic orbits far away from bifurcation point presented by the author in [dW21].

We start the rest of this article by formally stating its main result and introducing the necessary terminology in Section 2. Section 3 then reviews material from [KV92] and [KV21] on characteristic matrices. Section 4 contains the proof of the main result. In Section 5 we further discuss applications of the result to delayed feedback control of periodic orbits.

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2 Setting & statement of the main result

Throughout this section, we consider the DDE (1.1) and assume that this DDE has a periodic orbit and is symmetric with respect to a compact subgroup of the general linear group. We summarize this in the following hypothesis:

Hypothesis 1.

1. DDE (1.1) has a periodic orbit \(x_*\) with minimal period \(p > 0\);

2. DDE (1.1) is equivariant with respect to a compact subgroup \(\Gamma\) of the general linear group \(\text{GL}(N, \mathbb{R})\), i.e.
\[
f(\gamma x, \gamma y) = \gamma f(x, y) \quad \text{for all } x, y \in \mathbb{R}^N \text{ and } \gamma \in \Gamma. \tag{2.1}
\]
If $x(t)$ is a solution of the DDE \((1.1)\) and $\gamma$ is an element of $\Gamma$, then the equivariance relation \((2.1)\) implies that $\gamma x(t)$ is a solution of the DDE \((1.1)\) as well. So we can view $\Gamma$ as a group of symmetries of the solutions of \((1.1)\). Moreover, the equivariance relation \((2.1)\) naturally induces two symmetry groups on the periodic orbit $x_*$; we define the groups:

$$K_* = \{ \gamma \in \Gamma \mid \gamma x_*(t) = x_*(t) \text{ for all } t \in \mathbb{R} \} ;$$

$$H_* = \{ \gamma \in \Gamma \mid \text{there exists a } \Theta(\gamma) \in [0,1) \text{ such that } \gamma x_*(t) = x_*(t + \Theta(\gamma)p) \text{ for all } t \in \mathbb{R} \} .$$

The elements of group $K_*$ leave the periodic orbit fixed pointwise; therefore we refer to $K_*$ as the group of <strong>spatial symmetries</strong> of $x_*$. Since every element $h \in H_*$ induces a spatio-temporal relation of the form $hx_*(t) = x_*(t + \Theta(h)p)$ on the periodic solution, we refer to $H_*$ as the group of <strong>spatio-temporal symmetries</strong> of $x_*$. If $h_1, h_2 \in H_*$ are two spatio-temporal symmetries of $x_*$, then $h_1 h_2 x_*(t) = x_*(t + \Theta(h_1)p + \Theta(h_2)p)$ and hence $\Theta(h_1 h_2) = \Theta(h_1) + \Theta(h_2) \mod 1$. Thus the map

$$\Theta : H_* \to S^1 \simeq \mathbb{R}/\mathbb{Z} .$$

is a group homomorphism and $K_* = \ker \Theta$ is a normal subgroup of $H_*$. Therefore $H_* / K_* \simeq \text{im } \Theta$ and $\text{im } \Theta$ is a subgroup of $S^1$. This implies that

$$H_* / K_* \simeq \mathbb{Z}_m \quad \text{for some } m \in \mathbb{N},$$

$$H_* / K_* \simeq S^1 ,$$

where $\mathbb{Z}_m$ denotes the cyclic group of order $m$. If $H_* / K_* \simeq S^1$, we say that the periodic solution $x_*$ is a <strong>rotating wave</strong>; if $H_* / K_* \simeq \mathbb{Z}_m$ (i.e. the group of spatio-temporal symmetries modulo the group of purely spatial ones is a finite group), we say that the periodic solution $x_*$ is a <strong>discrete wave</strong>; cf. [PleSS].

To determine whether $x_*$ is a stable solution of the DDE \((1.1)\), we consider the linearized system

$$\dot{y}(t) = \partial_1 f(x_*(t), x_*(t-\tau))y(t) + \partial_2 f(x_*(t), x_*(t-\tau))y(t-\tau) .$$

If we supplement \((2.3a)\) with the initial condition

$$y(s+t) = \varphi(t) \quad \text{for } t \in [-\tau,0] \text{ and } \phi \in C([-\tau,0], \mathbb{R}^N)$$

then the system \((2.3a) - (2.3b)\) has a unique solution $y(t)$ for $t \geq s$. We define the <strong>history segment</strong> $y_* \in C([-\tau,0], \mathbb{R}^N)$ of this solution as $y_*(\theta) = y(t+\theta)$. We then associate to \((2.3a) - (2.3b)\) a two-parameter system of operators

$$U(t,s) : C([-\tau,0], \mathbb{R}^N) \to C([-\tau,0], \mathbb{R}^N), \quad t \geq s$$

defined via the relation $y_* = U(t,s) \phi$. We refer to \((2.3)\) as the family of solution operators of \((2.3)\), cf. [DvGVW95] Chapter 12.

Floquet theory for DDE implies that the non-zero spectrum of the <strong>monodromy operator</strong> $U(p,0)$ consists of isolated eigenvalues of finite algebraic multiplicity; these eigenvalues of $U(p,0)$ determine the stability of the periodic solution $x_*$. [DvGVW95] Chapter 13. The equivariance assumption in Hypothesis \([1]\) allows us to refine Floquet theory for discrete waves. We make this precise in the following proposition, which we cite without proof from [AW21]. The statement of the proposition is analogous to the formulation of equivariant Floquet theory of ODE (cf. [WS06]), but the proof now also involves compactness of the relevant operator.

Throughout, we let an element $\gamma \in \Gamma \subseteq \text{GL}(N, \mathbb{R})$ act on the state space $C([-\tau,0], \mathbb{R}^N)$ via $(\gamma \phi)(\theta) = \gamma \phi(\theta)$ for $\phi \in C([-\tau,0], \mathbb{R}^N)$ and $\theta \in [-\tau,0]$.

**Proposition 2.1** (Stability of discrete waves, [AW21] Proposition 6.3). *Consider the DDE \((1.1)\) satisfying Hypothesis \((1)\) and additionally assume that the periodic solution $x_*$ is a discrete wave. Let $U(t,s)$, $t \geq s$ be*
the family of solution operators of the linearized problem (2.3a). For \( h \in H \), a spatio-temporal symmetry of \( x_* \), define the operator

\[
U_h = h^{-1} U(\Theta(h)p, 0).
\]

(2.5)

Then the following statements hold:

1. The non-zero spectrum of \( U_h \) consists of isolated eigenvalues of finite algebraic multiplicity;
2. \( 1 \in \mathbb{C} \) is an eigenvalue of \( U_h \);
3. If \( U_h \) has an eigenvalue strictly outside the unit circle, then \( x_* \) is an unstable solution of (1.1). If the eigenvalue \( 1 \in \sigma_{pt}(U_h) \) is algebraically simple and all other eigenvalues of \( U_h \) lie strictly inside the unit circle, then \( x_* \) is a stable solution of (1.1).

We aim to determine the eigenvalues of the operator \( U_h \) in (2.5) using the concept of a characteristic matrix function for bounded linear operators, as introduced by Kaashoek and Verduyn Lunel in [KV21].

We denote by \( \mathcal{L}(X, X) \) the space of bounded linear operators on a complex Banach space \( X \). Moreover, we denote by \( I_X \) the identity operator on a Banach space \( X \), but suppress the subscript when the underlying space is clear.

**Definition 2.2** ([KV21, Definition 5.2.1]). Let \( X \) be a complex Banach space, \( T : X \to X \) be a bounded linear operator and \( \Delta : \mathbb{C} \to \mathbb{C}^{n \times n} \) be an analytic matrix-valued function. We say that \( \Delta \) is a characteristic matrix function for \( T \) if there exist analytic functions \( E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^n \oplus X, \mathbb{C}^n \oplus X) \) such that

\[
\begin{pmatrix}
\Delta(z) & 0 \\
0 & I_X
\end{pmatrix}
= F(z)
\begin{pmatrix}
I_{\mathbb{C}^n} & 0 \\
0 & I - zT
\end{pmatrix}
E(z)
\]

holds for all \( z \in \mathbb{C} \).

The idea of the above definition is to make a conjugation between the analytic function \( \mathbb{C} \ni z \mapsto I - zT \) and the analytic function \( z \mapsto \Delta(z) \in \mathbb{C}^{n \times n} \).

However, this cannot be done directly, since in general the dimensions of \( X \) and \( \mathbb{C}^{n \times n} \) are not the same. So we first (trivially) extend the functions \( z \mapsto I - zT \) and \( z \mapsto \Delta(z) \) to the functions

\[
\begin{pmatrix}
I_{\mathbb{C}^n} & 0 \\
0 & I - zT
\end{pmatrix} \in \mathcal{L}(\mathbb{C}^n \oplus X, \mathbb{C}^n \oplus X), \quad z \mapsto \begin{pmatrix}
\Delta(z) & 0 \\
0 & I_X
\end{pmatrix} \in \mathcal{L}(\mathbb{C}^n \oplus X, \mathbb{C}^n \oplus X)
\]

(2.7)

respectively. If now the functions in (2.7) are related via multiplication by analytic functions whose values are invertible operators, then this directly relates kernelvectors of \( I - zT \) to kernelvectors of \( \Delta(z) \). In particular, zeroes of the scalar valued function \( z \mapsto \det(\Delta(z)) \) give information on the non-zero spectrum of \( T \). We make this precise in Section 3. We first state the main result of this article, which gives an explicit characteristic matrix for the operator \( U_h \) defined in (2.5) in case the time shift \( \Theta(h)p \) of the spatio-temporal pattern is equal to the time delay \( \tau \).

**Theorem 2.3** (Main result). Consider the DDE

\[
\dot{x}(t) = f(x(t), x(t - \tau))
\]

(2.8)

with \( f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \) a \( C^2 \)-function and with time delay \( \tau > 0 \). Assume that
1. system \((2.8)\) has a periodic solution \(x_\ast\) with minimal period \(p > 0\);

2. system \((2.8)\) is equivariant with respect to a compact subgroup \(\Gamma\) of the general linear group \(\text{GL}(N, \mathbb{R})\);

3. the periodic solution \(x_\ast\) is a discrete wave and there exists a spatio-temporal symmetry \(h \in H_\ast\) with \(hx_\ast(t) = x_\ast(t + \tau)\).

Let \(U(t, s), t \geq s\) be the family of solution operators of the linearized DDE
\[
\dot{y}(t) = \partial_t f(x_\ast(t), x_\ast(t - \tau)) y(t) + \partial_2 f(x_\ast(t), x_\ast(t - \tau)) y(t - \tau).
\]  
(2.9)

For \(z \in \mathbb{C}\), let \(F(t, z)\) be the fundamental solution of the ODE
\[
\dot{y}(t) = \partial_t f(x_\ast(t), x_\ast(t - \tau)) y(t) + z \cdot \partial_2 f(x_\ast(t), x_\ast(t - \tau)) y(t)
\]
with \(F(0, z) = I_{\mathbb{C}^N}\). Then the analytic function
\[
\Delta(z) = I_{\mathbb{C}^N} - zh^{-1} F(\Theta(h)p, z)
\]
is a characteristic matrix function for the operator
\[
U_h = h^{-1} U(\Theta(h)p, 0).
\]

3 Characteristic matrices & Spectral information

This section reviews material from [KV92 Section 1] and [KV21 Chapter 5] on characteristic matrix functions. The first part of this section (page 5) discusses how characteristic matrix functions capture the spectrum of a bounded linear operator. We have included the contents of page 5 in this article to give context to Theorem 2.3 and to illustrate the implications of this theorem; we discuss its applications further in Section 4. In the second part of this section (page 5), we state a theorem from [KV21] that constructs a characteristic matrix function for a class of compact operators. This theorem is the cornerstone for the proof of Theorem 2.3 and is therefore crucial for the rest of this article.

We start by recalling the notion of Jordan chains for analytic operator-valued functions.

**Definition 3.1.** Let \(X\) be a complex Banach space and \(L : \mathbb{C} \rightarrow \mathcal{L}(X, X)\) an analytic operator-valued function. Given a complex number \(\mu \in \mathbb{C}\), we say that an ordered set \(x_0, \ldots, x_{k-1}\) of vectors in \(X\) is a Jordan chain of length \(k\) for \(L\) at \(\mu\) if \(x_0 \neq 0\) and
\[
L(z) [x_0 + (z - \mu)x_1 + \ldots + (z - \mu)^{k-1} x_{k-1}] = O((z - \mu)^k).
\]  
(3.1)

The maximal length of a Jordan chain starting with \(x_0\) is called the rank of \(x_0\); the rank is said to be infinite if no maximum exists.

**Example 3.2 (cf. [KV92 p. 485]).** Given a bounded linear operator \(T : X \rightarrow X\), the usual notion of a Jordan chain for \(T\) coincides with the notion of a Jordan chain for the analytic function
\[
L : \mathbb{C} \rightarrow \mathcal{L}(X, X), \quad L(z) = zI - T.
\]

Indeed, let \(\mu \in \mathbb{C}\) be an eigenvalue of \(T\) and let \(x_0, \ldots, x_{k-1}\) be an associated Jordan chain, i.e.
\[
Tx_0 = \mu x_0, \quad Tx_1 = \mu x_1 + x_0, \quad \ldots \quad Tx_{k-1} = \mu x_{k-1} + x_{k-2}.
\]  
(3.2)

Then
\[
(zI - T) [x_0 + (z - \mu)x_1 + \ldots + (z - \mu)^{k-1} x_{k-1}]
= (z - \mu)[x_0 + (z - \mu)[x_1 - x_0] + \ldots + (z - \mu)^{k-1} (x_{k-1} - x_{k-2})]
= (z - \mu)^k x_{k-1}.
\]
So
\[(zI - T) \left[ x_0 + (z - \mu)x_1 + \ldots + (z - \mu)^{k-1}x_{k-1} \right] = \mathcal{O}\left((z - \mu)^k\right) \tag{3.3} \]
and \(x_0, \ldots, x_{k-1}\) is a Jordan chain for the analytic function \(z \mapsto zI - T\). Vice versa, suppose the vectors \(x_0, \ldots, x_{k-1}\) satisfy (3.3). Then evaluating the derivatives of (3.3) at \(z = \mu\) yields the equalities (3.2) and thus \(x_0, \ldots, x_{k-1}\) is a Jordan chain for the bounded operator \(T\).

If \(x_0, \ldots, x_{k-1}\) is Jordan chain for \(L\) at \(\mu\), then (3.1) implies that \(L(\mu)x_0 = 0\). Vice versa, if \(x_0 \neq 0\) satisfies \(L(\mu)x_0\), then \(L(z)x_0 = \mathcal{O}(z - \mu)\) and hence \(L\) has a Jordan chain (of at least length 1) at \(\mu\) starting with \(x_0\). So \(L\) has a Jordan chain at \(\mu\) starting with \(x_0\) if and only \(x_0\) is a non-zero element of the space
\[
\ker L(\mu) = \{ x \in X \mid L(\mu)x = 0 \}.
\]

We now consider the case in which the space \(\ker L(\mu)\) is finite dimensional and all Jordan chains of \(L\) at \(\mu\) have finite rank. We pick a basis \(x^0, \ldots, x^n\) of \(\ker L(\mu)\) and for \(1 \leq j \leq n\), we let \(r_j\) be the rank of \(x^j\). Then, if \(x \neq 0\) is an element of \(L(\mu)\), its rank has to be equal to one of the \(r_j\). In particular, the set \(\{r_1, \ldots, r_n\}\) does not depend on the choice of basis. We define the \textbf{algebraic multiplicity} of \(\mu\) as the number
\[
r_1 + \ldots + r_n.
\]

The next lemma shows that the algebraic multiplicity is invariant under conjugation with analytic matrix-valued functions whose values are invertible operators.

\textbf{Lemma 3.3 (\cite{KV92} Proposition 1.2, \cite{KV21} Proposition 5.1.1).} \textit{Given a complex Banach space} \(X\), let \(L, M : \mathbb{C} \to \mathcal{L}(X, X)\) be analytic operator-valued functions, and let \(E, F : \mathbb{C} \to \mathcal{L}(X, X)\) be analytic operator-valued functions whose values are invertible operators. Suppose that \(M(z) = F(z)L(z)E(z)\) for all \(z \in \mathbb{C}\). Then, for \(\mu \in \mathbb{C}\), the algebraic multiplicity of \(L\) at \(\mu\) equals the algebraic multiplicity of \(M\) at \(\mu\).

\textbf{Proof.} We show that there is a one-to-one correspondence between Jordan chains for \(L\) at \(\mu\) and Jordan chains for \(M\) at \(\mu\); from there the claim follows.

Let \(x_0, \ldots, x_{k-1}\) be a Jordan chain for \(L\) at \(\mu\), i.e.
\[
L(z) \left[ x_0 + \ldots + (z - \mu)^{k-1}x_{k-1} \right] = \mathcal{O}\left((z - \mu)^k\right).
\]

For \(n \in \mathbb{N}\), let \(y_n \in X\) be such that
\[
E(z)^{-1} \left[ x_0 + \ldots + (z - \mu)^{k-1}x_{k-1} \right] = \sum_{n=0}^{\infty} y_n(z - \mu)^n.
\]

Then \(y_0 \neq 0\) and
\[
E(z) \left[ y_0 + \ldots + (z - \mu)^{k-1}y_{k-1} \right] = x_0 + \ldots + (z - \mu)^{k-1}x_{k-1} - E(z) \sum_{n=k}^{\infty} y_n(z - \mu)^n
\]

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so \(y_0, \ldots, y_{k-1}\) satisfy

\[
M(z)[y_0 + \ldots + (z - \mu)^{k-1}y_{k-1}] = F(z)L(z)E(z)[y_0 + \ldots + (z - \mu)^{k-1}y_{k-1}]
= F(z)L(z)[x_0 + \ldots + (z - \mu)^{k-1}x_{k-1}]
= \mathcal{O}((z - \mu)^k).
\]

So \(y_0, \ldots, y_{k-1}\) is a Jordan chain for \(M\) at \(\mu\). Vice versa, every Jordan chain \(y_0, \ldots, y_{k-1}\) of \(M\) at \(\mu\) induces a Jordan chain for \(L\) at \(\mu\). So there is a one-to-one correspondence between Jordan chains for \(M\) at \(\mu\) and Jordan chains for \(L\) at \(\mu\). In particular, the algebraic multiplicity of \(L\) at \(\mu\) equals the algebraic multiplicity of \(M\) at \(\mu\).

We are now ready to make precise how a characteristic matrix function, as defined in Definition 2.2, captures the spectral information of a bounded linear operator:

**Lemma 3.4.** ([1KV21] Theorem 5.2.2). Let \(X\) be a complex Banach space, \(T : X \to X\) a bounded linear operator and \(\Delta : \mathbb{C} \to \mathbb{C}^{n \times n}\) a characteristic matrix function for \(T\). Let \(\mu \in \mathbb{C}\backslash \{0\}\), then

1. \(\mu^{-1} \in \sigma_{pt}(T)\) if and only if \(\det \Delta(\mu) = 0\), i.e.

\[
\sigma_{pt}(T)\backslash \{0\} = \{\mu^{-1} \in \mathbb{C} \mid \det \Delta(\mu) = 0\}.
\]

2. If \(\mu^{-1} \in \sigma_{pt}(T)\), then the geometric multiplicity of \(\mu^{-1}\) as an eigenvalue of \(T\) equals the dimension of the space

\[
\ker \Delta(\mu) = \{x \in \mathbb{C}^n \mid \Delta(\mu)x = 0\}.
\]

3. If \(\mu^{-1} \in \sigma_{pt}(T)\), then the algebraic multiplicity of \(\mu^{-1}\) as an eigenvalue of \(T\) equals the order of \(\mu\) as a root of

\[
\det \Delta(z) = 0.
\]

**Proof.** Let \(\mu \neq 0\), then \(\mu^{-1}\) is an eigenvalue of \(T\) if and only if \(\mu^{-1}I - T = \mu^{-1}(I - \mu T)\) has a non-trivial kernel, i.e. if and only if \(I - \mu T\) has a non-trivial kernel. We first show that there is a one-to-one correspondence between kernel vectors of \(I - \mu T\) and kernel vectors of \(\Delta(\mu)\). This then implies the first two statements of the lemma.

We write

\[
M(z) = \begin{pmatrix} \Delta(z) & 0 \\ 0 & I_X \end{pmatrix}, \quad L(z) = \begin{pmatrix} I_{\mathbb{C}^n} & 0 \\ 0 & I - zT \end{pmatrix},
\]

then the kernels of the operators \(L(\mu)\), \(M(\mu)\) are given by

\[
\ker M(\mu) = \ker \Delta(\mu) \oplus \{0\}, \quad \ker L(\mu) = \{0\} \oplus \ker (I - \mu T).
\]

Since \(\Delta\) is a characteristic matrix function for \(T\), there exists analytic functions \(E, F : \mathbb{C} \to \mathcal{L}(\mathbb{C}^n \oplus X, \mathbb{C}^n \oplus X)\) so that \(E(z), F(z)\) are invertible operators for all \(z \in \mathbb{C}\) and such that \(M(z) = F(z)L(z)E(z)\) for all \(z \in \mathbb{C}\). In particular, the operator \(E(\mu)\) maps the space \(\ker M(\mu)\) in a one-to-one way to the space \(\ker L(\mu)\). This implies that the map

\[
\ker \Delta(\mu) \to \ker (I - \mu T), \quad c \mapsto (0, I_X)E(\mu)\begin{pmatrix} c \\ 0 \end{pmatrix}
\]

with inverse

\[
\ker (I - \mu T) \to \ker \Delta(\mu), \quad x \mapsto (I_{\mathbb{C}^n}, 0)E(\mu)^{-1}\begin{pmatrix} 0 \\ x \end{pmatrix}
\]

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is a bijection. So there is a one-to-one correspondence between elements of ker$(I - \mu T)$ and elements of ker$\Delta(\mu)$, which proves the first two statements of the lemma.

To prove the third statement of the lemma, we first show that for any number $\mu \in \mathbb{C}$, the algebraic multiplicity of $\Delta$ at $\mu$ equals the order of $\mu$ as a root of the equation $\det \Delta(z) = 0$. To do so, we bring $\Delta$ in local Smith form: given $\mu \in \mathbb{C}$, there exists analytic functions $G, H$, whose values are invertible matrices, and unique non-negative integers $r_1, \ldots, r_N$ such that

$$\Delta(z) = G(z)D(z)H(z)$$

with

$$D(z) = \begin{pmatrix}
(z - \mu)^{r_1} & 0 & \cdots & 0 \\
0 & (z - \mu)^{r_2} & \cdots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & \cdots & (z - \mu)^{r_N}
\end{pmatrix},$$

see [KV92, Section 1]. The algebraic multiplicity of $D$ at $\mu$ equals $r_1 + \ldots + r_N$; therefore, Lemma 3.3 implies that the algebraic multiplicity of $\Delta$ at $\mu$ equals $r_1 + \ldots + r_N$ as well. On the other hand, by (3.6a)–(3.6b) we can write $\det \Delta$ as

$$\det \Delta(z) = \det (G(z)) (z - \mu)^{r_1} \cdots (z - \mu)^{r_N} \det (H(z)).$$

Since $G(z), H(z)$ are invertible matrices, the order of $\mu$ as a root of $\det \Delta(z) = 0$ is also given by $r_1 + \ldots + r_N$. We conclude that the algebraic multiplicity of $\Delta$ at $\mu$ equals the order of $\mu$ as a root of $\det \Delta(z) = 0$.

Now let $\mu^{-1} \in \sigma_p(T)$. Then by equality (2.6) and Lemma 3.3 the algebraic multiplicity of $\mu$ as an eigenvalue of $T$ equals the algebraic multiplicity of $\Delta$ at $\mu$; by the previous step, this equals the order of $\mu$ as a root of $\det \Delta(z) = 0$. We conclude that the algebraic multiplicity of $\mu^{-1}$ as an eigenvalue of $T$ equals the order of $\mu$ as a root of $\det \Delta(z) = 0$, as claimed.

The next theorem from [KV21] gives a sufficient condition for a bounded linear operator to have a characteristic matrix function. The proof of this theorem is beyond the scope of this article and hence we state the theorem without proof.

**Theorem 3.5** ([KV21 Theorem 6.1.1]). Let $X$ be a complex Banach space and $T : X \to X$ a bounded linear operator. Assume that $T$ is of the form $T = V + R$ with

1. $V : X \to X$ a Volterra operator, i.e. $V$ is compact and $\sigma(V) \subseteq \{0\}$;
2. $R$ an operator of finite rank $n \in \mathbb{N}$.

Decompose $R$ as $R = DC$ where

$$C : X \to \mathbb{C}^n \quad \text{and} \quad D : \mathbb{C}^n \to X.$$

Then the matrix-valued function

$$\Delta(z) = I_{\mathbb{C}^n} - zC(I - zV)^{-1}D$$

is a characteristic matrix function for $T$.

Note that, since $V$ is Volterra, the resolvent map $z \mapsto (I - zV)^{-1}$ is analytic on $\mathbb{C}$ and hence $z \mapsto \Delta(z)$ is an analytic matrix-valued function.
4 Characteristic matrices for DDE with symmetries

In this section we prove this article’s main result Theorem 2.3. We prove this theorem by writing the operator (2.5) as the sum of a Volterra operator and a finite rank operator; we then apply Theorem 3.5 to obtain a characteristic matrix function.

In [KV21, Section 11.4], the authors consider a DDE with a periodic solution; they do not assume any symmetry relations on the DDE, but do assume that the period of the periodic solution is equal to the time delay. In this setting, they construct a characteristic matrix function for the monodromy operator. The difference between Theorem 2.3 presented here and the result in [KV21, Section 11.4] is the following: we realize that in a symmetric setting, the operator (2.5) has a characteristic matrix function if the time shift of the spatio-temporal symmetry is equal to the time delay, whereas in the non-symmetric case considered in [KV21, Section 11.4] the monodromy operator has a characteristic matrix function if the period of periodic solution is equal to the time delay. The proof of Theorem 2.3 is similar in spirit to the arguments in [KV21, Section 11.4], but we additionally exploit the equivariance of the considered system.

This section is structured as follows: in Subsection 4.1 we first consider a linear, time-dependent DDE whose coefficients satisfy a spatio-temporal relation; for this DDE we construct a characteristic matrix function. The proof of Theorem 2.3 then follows in Subsection 4.2.

4.1 Linear, time-dependent DDE with spatio-temporal symmetry

We consider the initial value problem
\[ \begin{aligned}
\dot{y}(t) &= A(t)y(t) + B(t)y(t-\tau), \quad \text{for } t \geq s; \\
y(s + t) &= \varphi(t), \quad \text{for } t \in [-\tau, 0].
\end{aligned} \tag{4.1a} \]

with time delay \( \tau > 0 \) and initial condition \( \varphi \in C([-\tau, 0], \mathbb{R}^N) \) at time \( s \in \mathbb{R} \). We make the following assumptions on system (4.1a):

**Hypothesis 2.**

1. the functions \( A, B : \mathbb{R} \to \mathbb{R}^{N \times N} \) are \( C^2 \);
2. there exists an invertible matrix \( h \in \mathbb{R}^{N \times N} \) such that
\[ hA(t)h^{-1} = A(t + \tau), \quad hB(t)h^{-1} = B(t + \tau) \tag{4.1b} \]

for all \( t \in \mathbb{R} \).

We stress that the time shift \( \tau \) in equation (4.1b) is the same as the time delay of the DDE (4.1a). So the coefficients \( A, B \) satisfy some spatio-temporal relation with time shift equal to the time delay of (4.1a).

Under this hypothesis, we construct a characteristic matrix function \( \Delta \) for the operator \( h^{-1}U(\tau, 0) \), where \( U(t, s), \ t \geq s \) is the family of solution operators of (4.1a). We give an explicit expression for \( \Delta \) in terms of solutions of the family of ODE
\[ \hat{y}(t) = [A(t) + z \cdot B(t)h^{-1}] y(t) \tag{4.2} \]

with \( z \in \mathbb{C} \). To arrive at this expression for \( \Delta \), we make the following intermediate steps:

1. We show that the symmetry relations (4.1b) imply symmetry relations on the fundamental solution of the ODE (4.2) (Lemma 4.1);
2. We give an explicit expression for \( h^{-1}U(\tau, 0) \) (Lemma 4.2) and write \( h^{-1}U(\tau, 0) = V + R \), with \( V \) an integral operator and \( R \) an operator of finite rank;
3. We show that the integral operator \( V \) is in fact a Volterra operator (Lemma 4.3);
4. We apply Theorem 3.5 to find a characteristic matrix for \( h^{-1}U(\tau, 0) \) (Proposition 4.4).
In the case where the coefficients $A, B$ of (4.2) are periodic, i.e. when $A(t + p) = A(t), B(t + p) = B(t)$ for some $p > 0$, the fundamental solution $F(t, z)$ of (4.2) satisfies the additional relation
\[
F(t + p, z) F(s + p, z)^{-1} = F(t, z) F(s, z)^{-1}. \tag{4.3}
\]
Indeed, the matrix-valued function $t \mapsto F(t + p, z) F(s + p, z)^{-1}$ satisfies the ODE
\[
\frac{d}{dt} F(t + p, z) F(s + p, z)^{-1} = [A(t + p) + B(t + p) h^{-1}] F(t + p, z) F(s + p, z)^{-1}
= [A(t) + B(t) h^{-1}] F(t + p, z) F(s + p, z)^{-1}
\]
with initial condition $F(t + p, z) F(s + p, z)^{-1} = I_{CN}$ for $t = s$. So uniqueness of solutions implies (4.3). Similarly, the symmetry relation (4.1b) on the coefficients $A, B$ induce symmetry relations on the fundamental solution $F(t, z)$, as we make precise in the following lemma:

**Lemma 4.1.** Consider functions $A, B : \mathbb{R} \to \mathbb{R}^{N \times N}$ satisfying Hypothesis 4. For $z \in \mathbb{C}$, let $F(t, z)$ be the fundamental solution of the ODE
\[
\dot{y}(t) = [A(t) + z \cdot B(t) h^{-1}] y(t)
\]
with $F(0, z) = I_{CN}$. Then it holds that
\[
hF(t, z) F(s, z)^{-1} h^{-1} = F(t + \tau, z) F(s + \tau, z)^{-1}
\tag{4.4}
\]
for all $t \geq s$.

In particular, if $Y_A(t)$ is the fundamental solution of the ODE
\[
\dot{y}(t) = A(t) y(t)
\]
with $Y(0) = I_{CN}$, then
\[
hY_A(t) Y_A(s)^{-1} h^{-1} = Y_A(t + \tau) Y_A(s + \tau)^{-1}. \tag{4.5}
\]
**Proof.** The matrix-valued function $t \mapsto hF(t, z) F(s, z) h^{-1}$ satisfies the ODE
\[
\frac{d}{dt} hF(t, z) F(s, z) h^{-1} = h \left[ A(t) + z \cdot B(t) h^{-1} \right] F(t, z) F(s, z) h^{-1}
= \left[ A(t + \tau) + z \cdot B(t + \tau) h^{-1} \right] [hF(t, z) F(s, z)^{-1} h^{-1}]
\]
with initial condition $hF(t, z) F(s, z)^{-1} = I_{CN}$ for $t = s$. Similarly, the matrix-valued function $t \mapsto F(t + \tau, z) F(s + \tau, z)^{-1}$ satisfies the ODE
\[
\frac{d}{dt} F(t + \tau, z) F(s + \tau, z)^{-1} = [A(t + \tau) + z \cdot B(t + \tau) h^{-1}] F(t + \tau, z) F(s + \tau, z)^{-1}
\]
with initial condition $F(t + \tau, z) F(s + \tau, z)^{-1} = I_{CN}$ for $t = s$. Uniqueness of solutions now implies the relation (4.4).

Since $F(t, z) = Y_A(t)$ when $B \equiv 0$, the equality (4.4) implies the equality (4.5).

We next give an explicit expression for the operator $h^{-1} U(\tau, 0)$.

**Lemma 4.2.** Consider the DDE (4.1a) satisfying Hypothesis 3 and let $U(t, s)$, $t \geq s$ be the family of solution operators of (4.1a). Moreover, let $Y_A(t)$ be the fundamental solution of the ODE
\[
\dot{y}(t) = A(t) y(t)
\]
with $Y_A(0) = I_{CN}$. Then the operator
\[
h^{-1} U(\tau, 0) : C \left( [-\tau, 0], \mathbb{R}^N \right) \to C \left( [-\tau, 0], \mathbb{R}^N \right)
\]
is given by
\[
(h^{-1} U(\tau, 0) \phi)(\theta) = h^{-1} Y_A(\tau + \theta) \phi(0) + \int_{-\tau}^{\theta} Y_A(\theta) Y_A(s)^{-1} B(s) h^{-1} \phi(s) ds. \tag{4.6}
\]
Proof. For \( s = 0 \) and \( t \in [0, \tau] \), the initial value problem (4.1a) becomes
\[
\dot{y}(t) = A(t)y(t) + B(t)\phi(t - \tau), \quad y(0) = \phi(0),
\]
which we solve by the Variation of Constants formula as
\[
y(t) = Y_A(t)\phi(0) + \int_0^t Y_A(t)Y_A(s)^{-1}B(s)\phi(s - \tau)ds.
\] (4.7)
With \( t = \tau + \theta \), \( \theta \in [-\tau, 0] \), (4.7) becomes
\[
y(\tau + \theta) = Y_A(\tau + \theta)\phi(0) + \int_0^{\tau+\theta} Y_A(\tau + \theta)Y_A(s)^{-1}B(s)\phi(s - \tau)ds
\]
\[
= Y_A(\tau + \theta)\phi(0) + \int_0^\theta Y_A(\tau + \theta)Y_A(\tau + s)^{-1}B(\tau + s)\phi(s)ds
\]
\[
= Y_A(\tau + \theta)\phi(0) + \int_{-\tau}^\theta hY_A(\theta)Y_A(s)^{-1}h^{-1}B(h)s\phi(s)ds
\]
where in the last step we used (4.1b) and (4.5). So \( (h^{-1}U(\tau, 0)\phi)(\theta) := h^{-1}y(\tau + \theta) \) is given by
\[
(h^{-1}U(\tau, 0)\phi)(\theta) = h^{-1}Y_A(\tau + \theta)\phi(0) + \int_{-\tau}^\theta Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds,
\]
which proves the lemma. \( \square \)

To apply Theorem 3.5, we first complexify the operator \( h^{-1}U(\tau, 0) \) in (4.6) via a canonical procedure as detailed in, for example, [DvGVW95, Chapter 3.7]. However, we do not make the complexification explicit in notation, i.e. we write \( h^{-1}U(\tau, 0) \) both for the real operator on the real Banach space \( C([-\tau, 0], \mathbb{R}^N) \) and the complexified operator on the complex Banach space \( C([-\tau, 0], \mathbb{C}^N) \). We then decompose the complex operator \( h^{-1}U(\tau, 0) \) as
\[
h^{-1}U(\tau, 0) = V + R
\] (4.8)
with the (suggestive) notation
\[
V : X \to X, \quad (V\phi)(\theta) = \int_{-\tau}^\theta Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds,
\] (4.9a)
\[
R : X \to X, \quad (R\phi)(\theta) = h^{-1}Y_A(\tau + \theta)\phi(0).
\] (4.9b)
and complex Banach space \( X \) given by
\[
X = C([-\tau, 0], \mathbb{C}^N).
\]
We next prove that the integral operator (4.9a) is in fact a Volterra operator.

**Lemma 4.3.** The operator \( V \) defined in (4.9a) is Volterra, i.e. \( V \) is compact and \( \sigma(V) \subseteq \{0\} \).

**Proof.** We first prove that \( \sigma_{pt}(V) \subseteq \{0\} \). To do so, fix \( z \in \mathbb{C}\setminus\{0\} \) and let \( \phi \in X \) be such that \( V\phi = z\phi \), i.e.
\[
\int_{-\tau}^\theta Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds = z\phi(\theta).
\] (4.10)
Equality (4.10) implies that \( \phi(-\tau) = 0 \). Moreover, since the left hand side of (4.10) is \( C^1 \), the right hand side is \( C^1 \) as well; differentiating both sides with respect to \( \theta \) gives
\[
A(\theta)\int_{-\tau}^\theta Y_A(\theta)Y_A(s)^{-1}B(s)h^{-1}\phi(s)ds + B(\theta)h^{-1}\phi(\theta) = z\phi'(\theta).
\] (4.11)
Equality (4.10) also implies that
\[ A(\theta) \int_{-\tau}^{\theta} Y_A(\theta) Y_A(s)^{-1} B(s) h^{-1} \phi(s) ds = z A(\theta) \phi(\theta) \]
and substituting this into (4.11) gives
\[ \mu A(\theta) \phi(\theta) + B(\theta) h^{-1} \phi(\theta) = z \phi'(\theta). \]

So \( \phi \) satisfies the initial value problem
\[
\begin{cases}
\phi'(\theta) = A(\theta) \phi(\theta) + z^{-1} B(\theta) h^{-1} \phi(\theta), & \theta \in [-\tau, 0], \\
\phi(-\tau) = 0,
\end{cases}
\]
which implies that \( \phi \equiv 0 \). We conclude that \( z \in \mathbb{C} \setminus \{0\} \) is not an eigenvalue of \( V \), and thus that \( \sigma_{pt}(V) \subseteq \{0\} \).

If \( \phi \in C([-\tau, 0], \mathbb{R}^N) \), then (4.9a) implies that \( V \phi \in C^1 \) and hence by the Arzelà-Ascoli theorem \( V \) is compact. This implies that the non-zero spectrum of \( V \) consists of eigenvalues. Since we already showed that \( \sigma_{pt}(V) \subseteq \{0\} \), we conclude that \( \sigma(V) \subseteq \{0\} \). So \( V \) is a compact operator and \( \sigma(V) \subseteq \{0\} \), which proves the claim.

We are now ready to use Theorem 3.5 and give an explicit characteristic matrix function for the operator \( h^{-1} U(\tau, 0) \):

**Proposition 4.4.** Consider the DDE (4.1a) satisfying Hypothesis 2 let \( U(t, s) \), \( t \geq s \) be the family of solution operators of (4.1a). Moreover, for \( z \in \mathbb{C} \), let \( F(t, z) \) be the fundamental solution of the ODE
\[
\dot{y}(t) = [A(t) + z \cdot B(t) h^{-1}] y(t)
\]
with \( F(0, z) = I_{\mathbb{C}^N} \). Then the matrix-valued function
\[
\Delta(z) = I_{\mathbb{C}^N} - z h^{-1} F(\tau, z)
\]
is a characteristic matrix for the operator
\[
h^{-1} U(\tau, 0).
\]

**Proof.** We divide the proof into two steps:

**Step 1:** The finite rank operator \( R \) in (4.9a) factorizes as \( R = DC \) with
\[
\begin{align*}
C : X &\to \mathbb{C}^N, \quad C \phi = \phi(0) \quad (4.14a) \\
D : \mathbb{C}^N &\to X, \quad (Du)(\theta) = h^{-1} Y_A(\tau + \theta) u. \quad (4.14b)
\end{align*}
\]

For \( V \) as in (4.9a) and \( D \) as in (4.14b), we now give an explicit expression for \( (I - z V)^{-1} D \). To that end, fix \( z \in \mathbb{C} \) and \( u \in \mathbb{C}^N \); let \( \phi \in X \) be the unique element such that \( (I - z V)^{-1} Du = \phi \), i.e. \( \phi \) satisfies
\[
\phi(\theta) = h^{-1} Y_A(\tau + \theta) u + z \int_{-\tau}^{\theta} Y_A(\theta) Y_A(s)^{-1} B(s) h^{-1} \phi(s) ds.
\]
Equality (4.10) implies that \( \phi(-\tau) = h^{-1} u \). Moreover, since the right hand side of (4.15) is \( C^1 \), the left hand side is \( C^1 \) as well; differentiating both sides with respect to \( \theta \) gives
\[
\phi'(\theta) = h^{-1} A(\tau + \theta) Y_A(\tau + \theta) u + z A(\theta) \int_{-\tau}^{\theta} Y_A(\theta) Y_A(s)^{-1} B(s) h^{-1} \phi(s) ds + z B(\theta) h^{-1} \phi(\theta) \quad (4.16)
\]
\[
= A(\theta) h^{-1} Y_A(\tau + \theta) u + z A(\theta) \int_{-\tau}^{\theta} Y_A(\theta) Y_A(s)^{-1} B(s) h^{-1} \phi(s) ds + z B(\theta) h^{-1} \phi(\theta) \quad (4.17)
\]
where in the last step we used (4.1b). Equality (4.15) also implies that
\[ zA(\theta) \int_{-\tau}^{\theta} Y_A(\theta) Y_A(s)^{-1} B(s) h^{-1} \phi(s) ds = A(\theta) \phi(\theta) - A(\theta) h^{-1} Y_A(\tau + \theta) u \]
and substituting this into (4.17) gives that
\[ \phi'(\theta) = A(\theta) \phi(\theta) + z \cdot B(\theta) h^{-1} \phi(\theta). \]
So \( \phi \) satisfies the initial value problem
\[
\begin{align*}
\phi'(\theta) &= A(\theta) \phi(\theta) + z \cdot B(\theta) h^{-1} \phi(\theta), & \theta \in [-\tau, 0],
\phi(-\tau) &= h^{-1} u
\end{align*}
\]
which implies that \( \phi(\theta) = F(\theta, z) F(-\tau, z)^{-1} h^{-1} u \). Equality (4.4) with \( t = \theta \) and \( s = -\tau \) implies that
\[ F(\theta, z) F(-\tau, z)^{-1} h^{-1} = h^{-1} F(\tau + \theta, z) \]
and hence
\[ \phi(\theta) = h^{-1} F(\tau + \theta, z) u. \]
So we conclude that
\[
\left( (I - zV)^{-1} D \right)(\theta) = h^{-1} F(\tau + \theta, z).
\]
\( \text{Step 2:} \) We now prove the statement of the proposition. The operator \( h^{-1} U(\tau, 0) \) decomposes as
\[ h^{-1} U(\tau, 0) = V + R, \]
with \( V \) defined in (4.9a) and \( R \) defined in (4.9b). The operator \( R \) is a finite rank operator; by Lemma 4.3 the operator \( V \) is a Volterra operator. Therefore, if we let \( D, C \) be as in (4.14a)–(4.14b), Theorem 3.5 implies that
\[ \Delta(z) = (I - zV)^{-1} D = I_C - zC \]
is a characteristic matrix for \( h^{-1} U(\tau, 0) \). Equality (4.18) implies that
\[ C(I - zV)^{-1} D = h^{-1} F(\tau, z) \]
and hence
\[ \Delta(z) = I_C - z h^{-1} F(\tau, z) \]
is a characteristic matrix for \( h^{-1} U(\tau, 0) \), as claimed.

4.2 Proof of Theorem 2.3

Theorem 2.3 now follows from Proposition 4.4.

Proof of Theorem 2.3 Define
\[ A(t) = \partial_1 f(x_*(t), x_*(t - \tau)), \quad B(t) = \partial_2 f(x_*(t), x_*(t - \tau)). \]
We show that the coefficients \( A, B \) satisfy Hypothesis 2. By assumption, the periodic solution \( x_* \) has a spatio-temporal symmetry \( h \in H_* \) with
\[ hx_*(t) = x_*(t + \tau). \]
Moreover, since \( f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \) satisfies the equivariance relation (2.4), it holds that
\[ \partial_i f(hx, \gamma y) h = h \partial_i f(x, y) \]
for all $x, y \in \mathbb{R}^N$ and $i = 1, 2$. So it in particular holds that

\[
A(t + \tau)h = \partial_1 f(x_*(t + \tau), x_*(t))h
= \partial_1 f(hx_*(t), hx_*(t - \tau))h
= h\partial_1 f(x_*(t), x_*(t - \tau)) = hA(t)
\]

and similarly

\[
B(t + \tau)h = \partial_2 f(x_*(t + \tau), x_*(t))h
= \partial_2 f(hx_*(t), hx_*(t - \tau))h
= h\partial_2 f(x_*(t), x_*(t - \tau)) = hB(t).
\]

So the coefficients $A, B$ satisfy Hypothesis 2; therefore Proposition 4.4 implies Theorem 2.3. 

We considered the system (4.1a)–(4.1b) with in the back of our mind the linearized DDE (2.9). However, the equations (4.1a)–(4.1b) also cover the special case $h = I$. In this case, the equation (4.1a) has periodic coefficients with period equal to the time delay, and the operator (4.13) is the monodromy operator. So in this case, an application Proposition 4.4 gives a characteristic matrix for the monodromy operator, and we recover the result from [KV21, Section 11.4]:

Theorem 4.5 (cf. [KV21, Section 11.4]). Consider the DDE

\[
\dot{x}(t) = f(x(t), x(t - \tau))
\]

with $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ a $C^2$ function and with time delay $\tau > 0$. Assume that system (4.1) has a periodic solution $x_*$ with period $\tau$, i.e.

\[x_*(t + \tau) = x_*(t) \]

Let $U(t, s), t \geq s$ be the family of solution operators associated to the linearized DDE

\[
\dot{y}(t) = \partial_1 f(x_*(t), x_*(t - \tau))y(t) + \partial_2 f(x_*(t), x_*(t - \tau))y(t - \tau).
\]

For $z \in \mathbb{C}$, let $F(t, z)$ be the fundamental solution of the ODE

\[
\dot{y}(t) = (\partial_1 f(x_*(t), x_*(t - \tau)) + z\partial_2 f(x_*(t), x_*(t - \tau))) \ y(t)
\]

with $F(0, z) = I_{\mathbb{C}^N}$. Then the analytic function

\[
\Delta(z) = I_{\mathbb{C}^N} - zF(\tau, z)
\]

is a characteristic matrix function for the monodromy operator

\[U(\tau, 0).\]

Proof. Define

\[A(t) := \partial_1 f(x_*(t), x_*(t - \tau)), \quad B(t) := \partial_2 f(x_*(t), x_*(t - \tau)),\]

then it holds that

\[A(t + \tau) = A(t), \quad B(t + \tau) = B(t).\]

So the coefficients $A, B$ satisfy Hypothesis 2 with $h = I_{\mathbb{C}^N}$. Therefore Proposition 4.4 implies the statement of the theorem.

\[\square\]
5 Applications to delayed feedback control

In [Pyr92], Pyragas introduced a delayed feedback method (now known as Pyragas control) that aims to stabilize periodic orbits of the ordinary differential equation

\[ \dot{x}(t) = F(x(t)), \quad x(t) \in \mathbb{R}^N. \] (5.1)

The feedback term introduced by Pyragas measures the difference between the current state and the state \( t \) ago, and feeds this difference (multiplied by a matrix) back into the system. Concretely the system with feedback control becomes

\[ \dot{x}(t) = F(x(t)) + K [x(t) - x(t - \tau)] \] (5.2)

with time delay \( \tau > 0 \) and matrix \( K \in \mathbb{R}^{N \times N} \). If now \( x_\ast(t) \) is a \( \tau \)-periodic solution of (5.1), then it is also a solution of (5.2). However, the overall dynamics of the systems with and without feedback are different, and it is possible that \( x_\ast \) is an unstable solution of (5.1) but a stable solution of (5.2).

We can determine whether \( x_\ast \) is a stable solution of (5.2) by computing the eigenvalues of the monodromy operator \( U(\tau, 0) \) where \( U(t, s), t \geq s \) is the family of solution operators of the DDE

\[ \dot{y}(t) = F'(x_\ast(t))y(t) + K [y(t) - y(t - \tau)]. \]

The results in [KV92], [KV21, Section 11.4] (cf. Theorem 4.5 in this article) give a characteristic matrix of the monodromy operator \( U(\tau, 0) \) in terms of solutions of the ODE

\[ \dot{y}(t) = F'(x_\ast(t))y(t) + K [1 - z] y(t). \] (5.4)

Equivariant Pyragas control [FFS10] adapts the Pyragas feedback scheme so that the feedback term vanishes on a periodic orbit with a specific spatio-temporal pattern. More precisely, suppose that

- (5.1) is equivariant with respect to a compact symmetry group \( \Gamma \subseteq \text{GL}(N, \mathbb{R}) \);
- (5.1) has a periodic solution \( x_\ast \) with minimal period \( p > 0 \);
- \( x_\ast \) is a discrete wave and \( h \in H \) is a spatio-temporal symmetry of \( x_\ast \), i.e.

\[ hx_\ast(t) = x_\ast(t + \Theta(h)p) \]

for some \( \Theta(h) \in [0, 1) \).

Then the periodic solution \( x_\ast \) is also a solution of the feedback system

\[ \dot{x}(t) = F(x(t)) + K [x(t) - x(t - \Theta(h)p)] \] (5.3)

with \( K \in \mathbb{R}^{N \times N} \). We additionally make the mild assumption that the matrix \( K \in \mathbb{R}^{N \times N} \) satisfies \( hK = Kh \), so that the system (5.3) is again equivariant with respect to the group generated by \( h \).

In system (5.3), the delay \( \Theta(h)p \) is strictly smaller than the minimal period of \( x_\ast \), and hence we are not in the setting of Theorem 4.5. However, Theorem 2.3 gives a characteristic matrix function \( z \mapsto \Delta(z) \) for the operator

\[ h^{-1}U(\Theta(h)p, 0) \]

where \( U(t, s), t \geq s \) is the family of solution operators of the DDE

\[ \dot{y}(t) = F'(x_\ast(t))y(t) + K [y(t) - hy(t - \Theta(h)p)]. \] (5.4)
The eigenvalues of the operator $h^{-1}U(\Theta(h)p,0)$ determine whether $x_*$ is stable as a solution of (5.3) (cf. Proposition 4.4); therefore, we can establish whether the control scheme (5.3) succeeds or fails to stabilize $x_*$ by computing the roots of the equation $\det \Delta(z) = 0$ (see also Lemma 3.4). This result contributes to the current literature on equivariant Pyragas control in two ways:

1. To prove that $x_*$ is an unstable solution of (5.3), it suffices to find (at least) one eigenvalue of the operator $h^{-1}U(\Theta(h)p,0)$ outside the unit circle; and in specific situations, it is indeed possible to do exactly that [HKR18]. However, if we want to establish that $x_*$ is a stable solution of (5.3), we have to ensure that we find all non-zero eigenvalues of $h^{-1}U(\Theta(h)p,0)$ and have to be careful about the multiplicity of the trivial eigenvalue $1 \in \sigma_{pt}(h^{-1}U(\Theta(h)p,0))$. Theorem 2.3 paves a way to do that, since the characteristic matrix function captures all non-zero eigenvalues of $h^{-1}U(\Theta(h)p,0)$ and also captures both their geometric and their algebraic multiplicity (cf. Lemma 3.4).

2. In the literature so far, most analytical results on succesful equivariant Pyragas control are either close to a bifurcation point [HKRH19, HBKR17, dWV17, FLR+20, FFG+07] or consider rotating waves, i.e. periodic solutions that can be transformed to stationary states of autonomous systems [PPK14, FFG+08, SB16, FFS10, Fie08]. Both these approaches simplify the stability analysis, but also work only in specific settings, i.e. they strongly depend on the form of the ODE (5.1). In the context of equivariant Pyragas control, Theorem 2.3 also simplifies the stability analysis by reducing the infinite dimensional problem to a finite dimensional one. However, this simplification is general in the sense that it does not depend on the specific form of the ODE (5.1). Therefore, we believe that Theorem 2.3 is a first step in proving new stabilization results (such as the stabilization results for non-stationary periodic solutions and far away from bifurcation point in [dW21]) and will generally be a helpful tool in further developments in equivariant Pyragas control.

6 Discussion

In [SGH06], Szalai, Stépán and Hogan discuss a delay equation of the form

$$\dot{x}(t) = f(x(t), x(t-\tau)),$$

that has a periodic solution of period $2\tau$. To find geometrically simple eigenvalues of this periodic orbit, they construct a characteristic matrix function that takes values in $\mathbb{C}^{4 \times 4}$. In general, if the delay equation

$$\dot{x}(t) = f(x(t), x(t-\tau)),$$

has a periodic orbit with period $\tau/m$, one expects that monodromy operator has a characteristic matrix function taking values in $\mathbb{C}^{(N \times m) \times (N \times m)}$, see also [SS11]. In Theorem 2.3, in contrast, the period of the periodic orbit of (2.8) is rationally related to the delay, but the constructed characteristic matrix function takes values in $\mathbb{C}^{N \times N}$. The difference here is that we do not construct a characteristic matrix function for the monodromy operator, but exploit the equivariance relations and construct a characteristic matrix function for the operator (2.5). So working with the operator (2.5) also has a computational advantage, since it yields a lower dimensional characteristic matrix function.

Throughout this article, we studied stability of periodic orbits of DDE using the principle of linearized stability, i.e. by studying the behaviour of the linearized system. The advantage of this is that for linear DDE of the form

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau)$$

one can very explicitly compute the time $\tau$-map, cf. Lemma 4.2. In contrast, a Poincaré map for periodic orbits of DDE can be constructed abstractly [DeGVW95, Section 14.3], but in general no explicit expression for the Poincaré map is available.
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