Symmetric Set Coloring of Signed Graphs

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Abstract. There are many concepts of signed graph coloring which are defined by assigning colors to the vertices of the graphs. These concepts usually differ in the number of self-inverse colors used. We introduce a unifying concept for this kind of coloring by assigning elements from symmetric sets to the vertices of the signed graphs. In the first part of the paper, we study colorings with elements from symmetric sets where the number of self-inverse elements is fixed. We prove a Brooks’-type theorem and upper bounds for the corresponding chromatic numbers in terms of the chromatic number of the underlying graph. These results are used in the second part where we introduce the symset-chromatic number $\chi_{\text{sym}}(G, \sigma)$ of a signed graph $(G, \sigma)$. We show that the symset-chromatic number gives the minimum partition of a signed graph into independent sets and non-bipartite antibalanced subgraphs. In particular, $\chi_{\text{sym}}(G, \sigma) \leq \chi(G)$. In the final section we show that these colorings can also be formalized as DP-colorings.

1. Introduction

A signed graph $(G, \sigma)$ is a multigraph $G$ together with a function $\sigma : E(G) \rightarrow \{\pm\}$, where $\{\pm\}$ is seen as a multiplicative group. The function $\sigma$ is called a signature of $G$ and $\sigma(e)$ is called the sign of $e$. An edge $e$ is negative if $\sigma(e) = -$ and it is positive otherwise. The set of negative edges is denoted by $E^-_\sigma$, and $E(G) - E^-_\sigma$ is the set of positive edges. A multigraph $G$ is sometimes called the underlying graph of the signed graph $(G, \sigma)$.

Let $(G', \sigma|_{E(G')})$ be a subgraph of $(G, \sigma)$. The sign of $(G', \sigma|_{E(G')})$ is the product of the signs of its edges. A circuit is a connected 2-regular graph. It is positive if its sign is + and negative otherwise. A subgraph $(G', \sigma|_{E(G')})$ is balanced if all circuits in $(G', \sigma|_{E(G')})$ are positive, otherwise it is unbalanced. Furthermore, negative (positive) circuits are also often called unbalanced (balanced) circuits. If $\sigma(e) = +$ for all $e \in E(G)$, then $\sigma$ is the all-positive signature.
and it is denoted by $+$, and if $\sigma(e) = -$ for all $e \in E(G)$, then $\sigma$ is the all-negative signature and it is denoted by $-$. 

A switching of a signed graph $(G, \sigma)$ at a set of vertices $X$ defines a signed graph $(G, \sigma')$ which is obtained from $(G, \sigma)$ by reversing the sign of each edge of the edge cut $\partial_G(X)$, where $\partial_G(X)$ denotes the set of edges having exactly one end in $X$, i.e. $\sigma'(e) = -\sigma(e)$ if $e \in \partial_G(X)$ and $\sigma'(e) = \sigma(e)$ otherwise.

If $X = \{v\}$, then we also say that $(G, \sigma')$ is obtained from $(G, \sigma)$ by switching at $v$. Switching defines an equivalence relation on the set of all signed graphs on $G$. We say that $(G, \sigma_1)$ and $(G, \sigma_2)$ are equivalent if they can be obtained from each other by a switching at a vertex set $X$. We also say that $\sigma_1$ and $\sigma_2$ are equivalent signatures of $G$. From Harary’s [4] characterization of balanced signed graphs it follows that a signed graph $(G, \sigma)$ is balanced if and only if it is equivalent to $(G, +)$. A signed graph $(G, \sigma)$ is antibalanced if it is equivalent to $(G, -)$. The signed extension $\pm H$ of a graph $H$ is obtained from $H$ by replacing every edge by two edges, one positive and one negative.

1.1. Motivation

We consider colorings of signed graphs which are defined by assigning colors to its vertices.

Let $(G, \sigma)$ be a signed graph and $S$ be a set of colors. A function $c : V(G) \rightarrow S$ is a coloring of $(G, \sigma)$. A coloring $c$ is proper if $c(v) \neq \sigma(e)c(w)$ for each edge $e = vw$. If $(G, \sigma)$ admits a proper coloring with elements from $S$, we say that $(G, \sigma)$ is $S$-colorable.

While the coloring-condition for positive edges remains unchanged with respect to the unsigned case, the condition on a negative edge $e = vw$ requires that $c(v) \neq -c(w)$. It implies that $-s \in S$ for each $s \in S$. At this point, the choice of the elements of $S$ has strong consequences on the colorings. Indeed, two cases have to be distinguished: when $s$ is a non-self-inverse element, that is $s \neq -s$, and when $s$ is a self-inverse element, and so $s = -s$.

The objective of this paper is to define a coloring and a corresponding chromatic number for a signed graph $(G, \sigma)$ which gives a minimum color-coded partition of the vertex set and which does not depend on the number of self-inverse colors which are admitted for coloring. To achieve this we will first discuss colorings where the number of self-inverse elements is fixed. This somewhat technical part is performed in Sect. 2, where we determine the chromatic spectrum of signed graphs and prove a Brooks’-type theorem for these kinds of coloring. These results are used in Sect. 3, where we show that the symset chromatic number (which will be defined later in this section) describes the minimum partition of the signed graph into independent sets and antibalanced non-bipartite subgraphs. It follows that this parameter gives a lower bound on the number of pairwise vertex-disjoint negative circuits of a signed graph. We further give an upper bound for the symset-chromatic number for a specific class of signed graphs. In the concluding section we show that circular coloring of signed graphs [7,10] is also covered by our approach and that all these colorings can also be formalized as $DP$-coloring.
1.2. Colorings and Basic Results

If a vertex \( v \) of \((G, \Sigma)\) is incident to a positive loop, then \((G, \sigma)\) does not have a proper coloring. If \( v \) is incident to a negative loop, then it has to be colored with a non-self-inverse color, which somehow counteracts our aforementioned objective. For these reasons, we will consider multigraphs without loops.

Next, we will introduce coloring of signed graphs which covers all approaches of signed graph coloring which are defined by assigning colors to the vertices of the graph. The sets of colors will be symmetric sets.

**Definition 1.1.** A set \( S \) together with a sign “−” is a symmetric set if it satisfies the following conditions:

1. \( s \in S \) if and only if \( -s \in S \).
2. If \( s = s' \), then \( -s = -s' \).
3. \( s = -(−s) \).

An element \( s \) of a symmetric set \( S \) is self-inverse if \( s = -s \). A symmetric set with self-inverse elements \( 0, 1, \ldots, 0 \) and non-self-inverse elements \( ±s_1, \ldots, ±s_k \) is denoted by \( S^t_{2k} \). Clearly, \( |S^t_{2k}| = t + 2k \).

A further natural requirement on signed graph coloring is that equivalent signed graphs should have the same coloring properties. Let \((G, \sigma')\) be obtained from \((G, \sigma)\) by switching at a vertex \( u \). If \((G, \sigma)\) admits a proper coloring \( c \) with elements from \( S^t_{2k} \), then \( c' \) with \( c'(u) = -c(u) \) and \( c'(v) = c(v) \) for \( v \neq u \) is a proper coloring of \((G, \sigma')\) with elements from \( S^t_{2k} \).

**Proposition 1.2.** Let \((G, \sigma)\) and \((G, \sigma')\) be equivalent signed graphs. Then \((G, \sigma)\) admits a proper \( S^t_{2k} \)-coloring if and only if \((G, \sigma')\) admits a proper \( S^t_{2k} \)-coloring.

Schweser and Stiebitz [11] used the term symmetric set for subsets \( Z \subseteq \mathbb{Z} \) with the property that \( Z = -Z \), where \( -Z = \{ -z : z \in Z \} \). In case of finite sets this gives symmetric sets with \( t \) self-inverse elements for \( t \in \{0, 1\} \). Examples for symmetric sets with more than one self-inverse element are subsets \( Z' \) of \( \mathbb{Z}^k_{2n} \), with \( Z' = -Z' \). Here the vectors whose entries are either 0 or \( n \) are self-inverse.

Self-inverse elements somehow annul the effect of the sign, so the following proposition naturally holds.

**Proposition 1.3.** Every signed graph \((G, \sigma)\) has a proper \( S^\chi(G)_0 \)-coloring, where \( \chi(G) \) denotes the chromatic number of the underlying graph \( G \).

The main problem in coloring with symmetric sets is that their cardinality and the number self-inverse elements have the same parity. This has some surprising consequences as it can be that the set of colors has more elements than the vertex set of the graph. This issue has been addressed in several ways (see e.g. [13]).

Zaslavsky [14,15] first considered two different sets for coloring signed graphs, \( M_{2k} = \{±1, \ldots, ±k\} \) and \( M_{2k+1} = \{0, ±1, \ldots, ±k\} \). He worked on the chromatic polynomial by distinguishing the two cases, the 0-free coloring with elements from \( M_{2k} \) and the coloring with elements from \( M_{2k+1} \), i.e. coloring where 0 is allowed.
Based on this coloring, Máčajová, Raspaud, and Škoviera [9] introduced the signed chromatic number \( \chi_{\pm}(G, \sigma) \) to be the smallest integer \( n \) for which \((G, \sigma)\) admits a proper \( M_n \)-coloring.

Kang and Steffen [7] introduced cyclic coloring of signed graphs and they used cyclic groups \( \mathbb{Z}_n \) as the set of colors. The cyclic chromatic number, denoted by \( \chi_{\text{mod}}(G, \sigma) \), is the smallest integer \( n \) such that \((G, \sigma)\) admits a proper coloring with elements of \( \mathbb{Z}_n \).

Interestingly, these approaches may not only provide different chromatic numbers for the same signed graphs, but they even have different general bounds. On one side an antibalanced triangle is colorable with \( M_2 \) by assigning 1 to all its vertices, but it is not \( Z_2 \)-colorable. On the other side, the signed extension of the complete graph on 4 vertices, \( \pm K_4 \), has a \( Z_6 \)-coloring but no \( M_6 \)-coloring. Note that here, in both types of coloring the set of colors contains more elements than the vertex set of the graph. The reason for this is given by the different number of self-inverse elements allowed: indeed, in the coloring defined by Zaslavsky we can have either 0 or 1 self-inverse element, while cyclic coloring uses 1 or 2 self-inverse elements.

Let \((G, \sigma)\) be a signed graph and \( t \in \{0, \ldots, \chi(G)\} \) be fixed. The \textit{sym}-set \( t \)-chromatic number (or \( t \)-chromatic number for short) of \((G, \sigma)\) is the minimum \( \lambda_t = t + 2k \) for which \((G, \sigma)\) admits an \( S_{2k}^t \)-coloring, and it is denoted by \( \chi_{\text{sym}}^t(G, \sigma) \). By Proposition 1.2, if \((G, \sigma)\) and \((G, \sigma')\) are equivalent, then \( \chi_{\text{sym}}^t(G, \sigma) = \chi_{\text{sym}}^t(G, \sigma') \). If a graph has symset \( t \)-chromatic number \( \chi_{\text{sym}}^t(G, \sigma) = \lambda_t \), we say that \((G, \sigma)\) is \( \lambda_t \)-chromatic.

Clearly, Zaslavsky’s coloring is equivalent to coloring with elements from \( S_{2k}^0 \) or \( S_{2k}^1 \) and cyclic coloring is equivalent to coloring with elements from \( S_{2k}^1 \) and \( S_{2k}^2 \). Consequently, the chromatic numbers studied in [7,9] can be defined as the minimum between two specific \( t \)-chromatic numbers of signed graphs.

\textbf{Proposition 1.4.} \textit{If \((G, \sigma)\) is a signed graph, then \( \chi_{\pm}(G, \sigma) = \min\{\chi_{\text{sym}}^0(G, \sigma), \chi_{\text{sym}}^1(G, \sigma)\} \) and \( \chi_{\text{mod}}(G, \sigma) = \min\{\chi_{\text{sym}}^1(G, \sigma), \chi_{\text{sym}}^2(G, \sigma)\} \).}

Thus, for a signed graph \((G, \sigma)\), \( \chi_{\pm}(G, \sigma) \) and \( \chi_{\text{mod}}(G, \sigma) \) depend on the number of self-inverse colors which can be used for coloring. In general, fixing the number of self-inverse elements (instead of choosing from two different cases) causes some issues due to parity. For instance, the all-positive complete graph on \( 2n \) vertices has a proper \( S_{2j}^{2(n-j)} \)-coloring for each \( j \in \{0, \ldots, n\} \), and in particular \( \chi_{\text{sym}}^{2j}(K_{2n}, +) = 2n \). However, for each \( j \in \{0, \ldots, n\} \) it also holds \( \chi_{\text{sym}}^{2j+1}(K_{2n}, +) = 2n + 1 \).

\textbf{The Symset-Chromatic Number}

Since \( S_{2k}^t \subseteq S_{2k}^t \) for all \( t \geq \chi(G) \), it follows by Proposition 1.3 that every signed graph \((G, \sigma)\) has a proper \( S_{2k}^t \)-coloring for all \( t \geq \chi(G) \). For this reason, we assume \( t \leq \chi(G) \) in the following.

If an antibalanced subgraph of \((G, \sigma)\) which is induced by a non-self-inverse color is bipartite, then the non-self-inverse color can be replaced by two self-inverse colors. In that case, \((G, \sigma)\) has an \( S_{2k}^t \)- and an \( S_{2(k-1)}^{t+2} \)-coloring.
Let \( N = \min \{ \chi^t_{sym}(G, \sigma) : 0 \leq t \leq \chi(G) \} \). The above examples show that \( N \) is not necessarily associated with a unique symmetric set \( S \) for which \((G, \sigma)\) admits a minimum proper \( S \)-coloring. To overcome this problem we define \( \max_t \min \{ \chi^t_{sym}(G, \sigma) : 0 \leq t \leq \chi(G) \} \) to be the symset chromatic number of \((G, \sigma)\), which is denoted by \( \chi_{sym}(G, \sigma) \). Furthermore, we say that an \( S^t_{2k} \)-coloring is minimum if \( \chi_{sym}(G, \sigma) = t + 2k \). By Proposition 1.2 it follows that equivalent signed graphs have the same symset chromatic number.

**Proposition 1.5.** For every signed graph \((G, \sigma)\) it holds \( \chi_{sym}(G, \sigma) \leq \chi(G) \). Furthermore, if \((G, \sigma)\) and \((G, \sigma')\) are equivalent, then \( \chi_{sym}(G, \sigma) = \chi_{sym}(G, \sigma') \). In particular, if \((G, \sigma)\) is equivalent to \((G, +)\), then \( \chi_{sym}(G, \sigma) = \chi(G) \).

In preparation for the study of the symset-chromatic number in Sect. 3 we study the symset-\( t \)-chromatic number in the next section.

## 2. The Symset \( t \)-Chromatic Number

An \( S^t_{2k} \)-coloring of \((G, \sigma)\) provides some information on the structure of \((G, \sigma)\).

**Proposition 2.1.** If a signed graph \((G, \sigma)\) admits a proper \( S^t_{2k} \)-coloring \( c \), then \( c \) induces a partition of \( V(G) \) such that \( c^{-1}(0) \) is an independent set in \( G \) for every self-inverse color \( 0 \), and \((G[c^{-1}(\pm s)], \sigma_{|G[c^{-1}(\pm s)]})\) is an antibalanced subgraph of \((G, \sigma)\) for every non-self-inverse color \( \pm s \).

We first prove upper bounds for the symset \( t \)-chromatic number in terms of the chromatic number of the underlying graph. The cases \( t \leq 1 \) and \( t = 2 \) had been proved in [5,9], respectively.

**Theorem 2.2.** Let \( G \) be a graph with chromatic number \( k \). Then for every \( t \in \{0, \ldots, k\} \), \( \chi^t_{sym}(G, \sigma) \leq 2k - t \). Furthermore, \( \chi^t_{sym}(G) = 2k - t \) and there are simple signed graphs \((H, \sigma_H)\) such that \( \chi(H) = k \) and \( \chi^t_{sym}(H, \sigma_H) = 2k - t \).

**Proof.** Let \( c \) be a \( k \)-coloring of \( G \) with colors from \( \{0_1, \ldots, 0_t, s_{t+1}, \ldots, s_k\} \). This coloring is a coloring of \( \pm G \) with colors from \( \{0_1, \ldots, 0_t, \pm s_{t+1}, \ldots, \pm s_k\} \). Hence, \( \chi^t_{sym}(G, \sigma) \leq 2k - t \).

If \( t = k \), then \( \chi^t_{sym}(\pm G) = 2k - t \), since \( \chi(G) = k \).

Let \( t < k \) and suppose to the contrary, that \( \chi^t_{sym}(\pm G) < 2k - t \). Then, there is a coloring with elements \( \{0_1, \ldots, 0_t, \pm s_{t+1}, \ldots, \pm s_l\} \) and \( l < k \). If necessary by switching there is a \( 2l - t \) coloring of \((G, \sigma)\) which only uses colors \( \{0_1, \ldots, 0_t, s_{t+1}, \ldots, s_l\} \). This is also an \( l \)-coloring of \( G \), a contradiction. Hence, \( \chi^t_{sym}(G) = 2k - t \) and \( \chi^t_{sym}(G, \sigma) \leq 2k - t \), since \((G, \sigma)\) is a subgraph of \( \pm G \).

Let \( G_k \) be the Turan graph on \( k(k - t + 1) \) vertices which is the complete \( k \)-partite graph with \( k \) independent sets of cardinality \( k - t + 1 \). Thus, \( G_k \) contains \( k - t + 1 \) pairwise disjoint copies \( H_1, \ldots, H_{k-t+1} \) of \( K_k \). Let \( \sigma \) be a signature on \( G_k \) with \( E^- = \bigcup_{i=2}^{k-t+1} E(H_i) \). Clearly, \( \chi(G_k) = k \) and therefore, \( \chi^t_{sym}(G_k, \sigma) \leq 2k - t \).

If \( t = k \), then \( \chi^t_{sym}(G_k, \sigma) = \chi(G_k) = k (= 2k - k) \). Let \( t \in \{0, \ldots, k - 1\} \) and suppose to the contrary that \( \chi^t_{sym}(G_k, \sigma) < 2k - t \), say \((G, \sigma)\) is colored with colors from \( \{0_1, \ldots, 0_t, \pm s_{t+1} \cdots \pm s_l\} \) with \( l < k \).
Then, at least \( k - t \) vertices of \( H_1 \) are colored with pairwise different colors from \( \{\pm s_{t+1} \cdots \pm s_t\} \). Furthermore, for each \( i \in \{2, \ldots, k - t\} \) each all-negative copy \( H_i \) of \( K_k \) contains at least two vertices of the same color of \( \{\pm s_{t+1} \cdots \pm s_t\} \). Since for all \( 2 \leq i < j \leq k - t + 1 \) each vertex of \( H_i \) is connected by a positive edge to every vertex of \( H_j \), it follows that for all \( i \neq j \), the multiple used colors in \( H_i \) are different from the multiple used colors in \( H_j \). Thus, at least \( 2(k - t) \) pairwise different non-self-inverse colors are needed; \( k - t \) for the all-negative copies of \( K_k \) and \( k - t \) for the coloring of \( H_1 \). But we have only \( 2(l - t) \) non-self-inverse colors and \( l < k \), a contradiction. Thus, \( \chi^t_{\text{sym}}(G_k, \sigma) = 2k - t. \) \( \square \)

2.1. Brooks’ Type Theorem for the Symset \( t \)-Chromatic Number

We are going to prove a Brooks’ type theorem for the symset \( t \)-chromatic number, which implies Brooks’ Theorem for unsigned graphs.

The symset \( t \)-chromatic number has the same parity as \( t \). Observe that, if \( t = \chi(G) - l \), then Theorem 2.2 can be reformulated as \( \chi^t_{\text{sym}}(G, \sigma) \leq \chi(G) + l. \) By parity we obtain equality in the following statement.

**Proposition 2.3.** Let \((G, \sigma)\) be a signed graph. If \( t = \chi(G) - 1 \), then \( \chi^t_{\text{sym}}(G, \sigma) = \chi(G) + 1. \)

If \( G \) is a graph with \( \chi(G) = \Delta(G) = t + 1 \), then \( \chi^t_{\text{sym}}(G, \sigma) = \Delta(G) + 1 \) by Proposition 2.3. The following Brooks’ type statement is the main result of this section.

**Theorem 2.4.** Let \( G \) be a connected graph and \( t \in \{0, \ldots, \chi(G)\} \). If \( \Delta(G) - t \) is odd, then \( \chi^t_{\text{sym}}(G, \sigma) \leq \Delta(G) + 1. \)

If \( \Delta(G) - t \) is even, then \( \chi^t_{\text{sym}}(G, \sigma) = \Delta(G) + 2 \) or \( \chi^t_{\text{sym}}(G, \sigma) \leq \Delta(G) \). Furthermore, \( \chi^t_{\text{sym}}(G, \sigma) = \Delta(G) + 2 \) if and only if

- \( G \) is a complete graph and \( t = \chi(G) - 1(= \Delta(G)) \) or
- \( (G, \sigma) \) is a balanced complete graph or
- \( (G, \sigma) \) is a balanced odd circuit or
- \( (G, \sigma) \) is an unbalanced even circuit and \( t = 0 \) or
- \( (G, \sigma) \) is an unbalanced odd circuit and \( t = 2 \).

We will prove the statement by formulating some propositions, some of which might be of own interest.

**Proposition 2.5.** Let \( K_n \) be the complete graph on \( n \geq 3 \) vertices and let \( t \in \{0, \ldots, n\} \).

If \( \Delta(K_n) - t \) is odd, then \( \chi^t_{\text{sym}}(K_n, \sigma) \leq \Delta(K_n) + 1. \)

If \( \Delta(K_n) - t \) is even, then \( \chi^t_{\text{sym}}(K_n, \sigma) = \Delta(K_n) + 2 \) or \( \chi^t_{\text{sym}}(K_n, \sigma) \leq \Delta(K_n) \). Furthermore, \( \chi^t_{\text{sym}}(K_n, \sigma) = \Delta(K_n) + 2 \) if and only if \( (K_n, \sigma) \) is equivalent to \( (K_n, +) \) or \( t = n - 1. \)

**Proof.** If \( t = n \) or \( t = n - 2 \), then \( \chi^t_{\text{sym}}(K_n, \sigma) = \chi(K_n) = n = \Delta(K_n) + 1. \)

If \( t = n - 1 \), then by Proposition 2.2, \( \chi^t_{\text{sym}}(K_n, \sigma) = \chi(K_n) + 1 = n + 1 = \Delta(K_n) + 2. \)
Let \( t \leq n-3 \) and \( \chi^t_{\text{sym}}(K_n, \sigma) = t+2k \). Hence, \( k \geq 1 \). First we consider the case when \((K_n, \sigma)\) is not balanced. Then it contains an induced antibalanced circuit \( C_3 \) of length 3, which can be colored with one pair of non-self-inverse colors. Thus, \((K_n - V(C_3), \sigma|_{K_n - V(C_3)})\) can be colored with at most \( n-3 \) pairwise different colors. Taking the parity into account it follows that if \( \Delta(K_n) - t \) is even, then \( \chi^t_{\text{sym}}(K_n, \sigma) \leq n - 1 = \Delta(K_n) \), and if \( \Delta(K_n) - t \) is odd, then \( \chi^t_{\text{sym}}(K_n, \sigma) \leq n = \Delta(K_n) + 1 \).

If \( \sigma \) is equivalent to +, then any coloring of \((K_n, +)\) needs \( n \) pairwise different colors. Thus, \( \Delta(K_n) - t \) is even if and only if \( \chi^t_{\text{sym}}(K_n, +) = n + 1 = \Delta(K_n) + 2 \).

**Proposition 2.6.** For each circuit \( C_n \) on \( n \) vertices:

- If \( t \in \{1, 3\} \), then \( \chi^t_{\text{sym}}(C_n, \sigma) = 3 \).
- If \( t \in \{0, 2\} \), then \( \chi^t_{\text{sym}}(C_n, \sigma) \in \{2, 4\} \), and \( \chi^t_{\text{sym}}(C_n, \sigma) = 4 \) if and only if
  - \( (C_n, \sigma) \) is a balanced odd circuit or
  - \( (C_n, \sigma) \) is an unbalanced even circuit and \( t = 0 \) or
  - \( (C_n, \sigma) \) is an unbalanced odd circuit and \( t = 2 \).

**Proof.** Since we assume that \( t \leq \chi(C_n) \), it follows that \( t \in \{0, 1, 2, 3\} \), where \( t = 3 \) only applies if \( n \) is odd. In this case, we have \( \chi^3_{\text{sym}}(C_n, \sigma) = \chi(C_n) = 3 \). Furthermore, \( \chi^2_{\text{sym}}(C_n, \sigma) = 3 \) is easy to check. The statements for \( t = 2 \) follow with Proposition 2.3. It is easy to see that \( \chi^0_{\text{sym}}(C_n, \sigma) \leq 4 \) and \( \chi^0_{\text{sym}}(C_n, \sigma) = 2 \) if and only if \( n \) is even and \( C_n \) is balanced or \( n \) is odd and \( C_n \) is unbalanced.

The following statement is a standard lemma for coloring.

**Lemma 2.7.** The vertices of a connected graph \( G \) can be ordered in a sequence \( x_1, x_2, \ldots, x_n \) so that \( x_n \) is any preassigned vertex of \( G \) and for each \( i < n \) the vertex \( x_i \) has a neighbor among \( x_{i+1}, \ldots, x_n \).

**Lemma 2.8.** Let \((G, \sigma)\) be a simple connected signed graph. If \( G \) is not regular, then

\[
\chi^t_{\text{sym}}(G, \sigma) \leq \begin{cases} 
\Delta(G) + 1, & \text{if } \Delta(G) - t \text{ is odd} \\
\Delta(G), & \text{if } \Delta(G) - t \text{ is even}.
\end{cases}
\]

**Proof.** Let \( v \) be a vertex having degree \( d_G(v) \leq \Delta - 1 \). By Lemma 2.7, there exists an ordering of the vertices \( x_1, \ldots, x_n \) such that \( x_n = v \) and for each \( i < n \) the vertex \( x_i \) has neighbors among \( x_{i+1}, \ldots, x_n \). We follow this order to color the vertices by using the greedy algorithm. We can first use the \( t \) self-inverse colors, and then add pairs of non-self-inverse colors when it is necessary. If \( \Delta - t = 2n \) is even, then we use exactly \( n \) non-self-inverse colors \( \pm s \). Each vertex \( x_i \), \( i < n \), has at most \( \Delta - 1 \) neighbors which have been colored previously. Since it also holds \( d(x_n) \leq \Delta - 1 \), the graph has an \( S^t_{2k} \)-coloring, with \( t + 2k = \Delta \).

If \( \Delta - t \) is odd, then the result follows similarly.

For the proof of Theorem 2.4 we also use the following lemma.
Lemma 2.9. [8] Let $G$ be a 2-connected graph with $\Delta(G) \geq 3$ other than a complete graph. Then $G$ contains a pair of vertices $a$ and $b$ at distance 2 such that the graph $G - \{a, b\}$ is connected.

2.1.1. Proof of Theorem 2.4.

Proof. Propositions 2.5 and 2.6 imply that the statement is true for complete graphs and circuits. By Lemma 2.8 it suffices to prove it for non-complete regular graphs with a maximum vertex degree of at least 3. We can also assume that the graph is connected.

Let $(G, \sigma)$ be a signed graph of order $n$ and $0 \leq t \leq \chi(G)$. If $\Delta(G) - t$ is odd, then $(G, \sigma)$ can be colored greedily with $\Delta(G) + 1$ colors. Hence, we focus on the case where $\Delta(G) - t$ is even. We show that the graph has an $S_{2k}^t$-coloring with $t + 2k \leq \Delta(G)$.

Assume now that $(G, \sigma)$ is not 2-connected, that is, there exists a cut vertex $v$.

Let $H_1, H_2, \ldots, H_k$ be the components of $G - v$. For each $i \in \{1, \ldots, k\}$, the subgraph $H_i' = H_i \cup v$ is not regular and $d_{H_i'}(v) < \Delta(H_i)$. Thus, it can be colored by $\Delta(G)$ colors by Lemma 2.8. By relabeling we can always suppose that $v$ is colored with the same element in each graph, so the entire graph is also $S_{2k}^t$-colorable, with $t + 2k = \Delta(G)$. \hfill \Box

Corollary 2.10 (Brooks’ Theorem [2]). Let $G$ be a connected graph. If $G$ is neither complete nor an odd circuit, then $\chi(G) \leq \Delta(G)$.

Proof. By induction we get $\chi(G) \leq \Delta(G) + 1$. Assume that $\chi(G) = \Delta(G) + 1$. For $t = \Delta(G)$ it follows by Proposition 2.3 that $\chi_{\text{sym}}^t(G, +) = \Delta(G) + 2$. Hence, by Theorem 2.4, $G$ is a complete graph or it is an odd circuit. \hfill \Box

As a simple consequence of Theorem 2.2 and Corollary 2.10 we obtain the following statement on the signed extension of a graph.

Corollary 2.11. Let $G$ be a connected graph. If $G$ is a complete graph or an odd circuit, then $\chi_{\text{sym}}^t(\pm G) = \Delta(\pm G) + 2 - t$. Otherwise $\chi_{\text{sym}}^t(\pm G) \leq \Delta(\pm G) - t$.

2.2. Symset $t$-Chromatic Spectrum

Let $G$ be a graph and $\Sigma(G)$ be the set of its non-equivalent signatures. The symset $t$-chromatic spectrum of $G$ is the set $\Sigma_{\chi_{\text{sym}}^t}(G) := \{\chi_{\text{sym}}^t(G, \sigma) : \sigma \in \Sigma(G)\}$. We define $m_{\chi_{\text{sym}}^t}(G) = \min_{\chi_{\text{sym}}^t}(G, \sigma)$, and $M_{\chi_{\text{sym}}^t}(G) = \max_{\chi_{\text{sym}}^t}(G)$.
Since \(|S_{2k}^t|\) has the same parity as \(t\), it follows that the \(t\)-chromatic spectrum contains only values of the same parity.

The question of the \(t\)-chromatic spectrum of a signed graph was studied for \(t \in \{0, 1, 2\}\) in [6] first. There, it is shown that \(\Sigma_{\chi_{\text{sym}}^0}(G) \cup \Sigma_{\chi_{\text{sym}}^1}(G)\) and \(\Sigma_{\chi_{\text{sym}}^t}(G) \cup \Sigma_{\chi_{\text{sym}}^{t+2}}(G)\) are intervals of integers.

Observe that, if \(t = \chi(G)\), then it follows that \(\Sigma_{\chi_{\text{sym}}^t}(G) = \{t\}\). Hence, we assume \(t \leq \chi(G) - 1\).

**Proposition 2.12.** Let \(G\) be a graph and \(t\) a positive integer. Then \(m_{\chi_{\text{sym}}^t}(G) = t + 2\).

**Proof.** Consider the signed graph \((G, \cdot)\) and the coloring \(c : V(G) \to S_2^t\) with \(c(v) = 1 \forall v \in V(G)\). This coloring is proper and uses \(t + 2\) colors. Since \(t \leq \chi(G)\), there exists no signature \(\sigma'\) such that \(\chi_{\text{sym}}^t(G, \sigma') = t\), so \(m_{\chi_{\text{sym}}^t}(G) = t + 2\).

A \(\lambda_t\)-chromatic signed graph \((G, \sigma)\) is \(\lambda_t\)-critical if \(\chi_{\text{sym}}^t(G - v, \sigma|_{G - v}) < \lambda_t\) for every \(v \in V(G)\).

**Lemma 2.13.** If \((G, \sigma)\) is a \(\lambda_t\)-chromatic signed graph with \(\lambda_t = t + 2k\), then \(\chi_{\text{sym}}^t(G - v, \sigma|_{G - v}) \in \{t + 2k, t + 2k - 2\}\).

**Proof.** Suppose that there exists a vertex \(v\) such that \(\chi_{\text{sym}}^t(G - v, \sigma|_{G - v}) \leq t + 2k - 4\). The coloring can be easily extended to \((G, \sigma')\) by adding at most two colors, so \(\chi_{\text{sym}}^t(G, \sigma) \leq t + 2k - 2\), which is a contradiction.

As a consequence, if a signed graph \((G, \sigma)\) is \(\lambda_t\)-critical, then \(\chi_{\text{sym}}^t(G - v, \sigma|_{G - v}) = \lambda_t - 2\) for each \(v \in V(G)\). In particular, the following statement holds:

**Theorem 2.14.** If \((G, \sigma)\) is a \(\lambda_t\)-chromatic signed graph with \(\lambda_t = t + 2k\), then \((G, \sigma)\) has a critical \(\lambda_i^t\)-chromatic subgraph for each \(\lambda_i^t = t + 2i\), \(i \in \{1, \ldots, k\}\).

**Proof.** First, we stepwise remove vertices \(v\) such that the removal of \(v\) does not decrease the symset \(t\)-chromatic number. The remaining subgraph \((G', \sigma|_{G'})\) is \(\lambda_t\)-critical. Secondly, we remove another vertex \(w\) from \(G'\). Lemma 2.13 implies that this graph has \(t\)-chromatic number \(t + 2k - 2\). By proceeding as before, we find a critical subgraph with the same \(t\)-chromatic number. This process can be iterated until we obtain a \(\lambda_i^t\)-critical graph, for each \(i \in \{1, \ldots, k\}\).

**Theorem 2.15.** Let \(G\) be a graph, then \(\Sigma_{\chi_{\text{sym}}^t}(G) = \{m_{\chi_{\text{sym}}^t}(G) = t + 2, t + 4, \ldots, t + 2k = M_{\chi_{\text{sym}}^t}(G)\}\).

**Proof.** Let \((G, \sigma)\) be a signature such that \(\chi_{\text{sym}}^t(G, \sigma) = M_{\chi_{\text{sym}}^t}(G) = t + 2k\). By Theorem 2.14, we know that for each value of \(\lambda_i^t = t + 2i\), where \(i \in \{1, \ldots, k\}\), \((G, \sigma)\) has a \(\lambda_i^t\)-chromatic subgraph \((H, \tau)\). Our aim is to prove that the signature \(\tau\) can be extended to a signature \(\tau'\) in \(G\) such that \(\chi_{\text{sym}}^t(G, \tau') = t + 2i\).
Let \( c : V(H) \to S_{2i}^t \) the \( \lambda_i^t \)-coloring of \((H, \tau)\). For each edge \( uv \in E(G) \) we define \( \tau' \) in the following way:

- If \( u, v \in V(H) \), \( \tau(uv) = \tau'(uv) \).
- If \( u, v \notin V(H) \) or \( u \notin V(H) \) and \( v \) is colored with 1, \( \tau'(uv) = -\).
- If \( v \in V(H) \) and \( u \notin V(H) \) and \( v \) is not colored with 1, \( \tau'(uv) = +\).

By defining \( c' : V(G) \to S_{2i}^t \) as \( c'(v) = c(v) \) if \( v \in V(H) \) and \( c'(v) = 1 \) if \( v \notin V(H) \) we obtain a proper \( S_{2i}^t \)-coloring, so the statement follows. \( \square \)

### 3. The Symset Chromatic Number

Next we skip the constraint on the set of colors given by fixing the value of \( t \) and we focus on the symset chromatic number. If \( c \) is an \( S_{2k}^t \)-coloring, then \( c^{-1}(0_j) \) is also called a self-inverse color class for each \( j \in \{1, \ldots, t\} \). Similarly, \( c^{-1}(\pm s_i) \) is called a non-self-inverse color class for \( i \in \{1, \ldots, k\} \). Clearly, a self-inverse color class is an independent set of the graph, while a non-self-inverse color class induces an antibalanced subgraph. If the color classes are induced by a \( \lambda_i \)-coloring of \((G, \sigma)\) and \( \lambda_i = \chi_{\text{sym}}(G, \sigma) \), then any non-self-inverse color class induces a non-bipartite subgraph of \( G \), see Proposition 2.1.

#### 3.1. The Chromatic Spectrum and Structural Implications

The symset chromatic number gives some information on circuits in the underlying graph \( G \) and on the frustration index \( l(G, \sigma) \), which is defined as the minimum number of edges which have to be removed from \((G, \sigma)\) to make the graph balanced [4].

**Theorem 3.1.** Let \((G, \sigma)\) be a signed graph and \( t, k \geq 0 \). If \( \chi_{\text{sym}}(G, \sigma) = t + 2k \), then \( G \) has at least \( k \) pairwise vertex-disjoint odd circuits, which are unbalanced in \((G, \sigma)\). In particular, \( k \leq l(G, \sigma) \).

**Proof.** For \( k = 0 \) there is nothing to prove. So assume \( k \geq 1 \). Let \( c \) be an \( S_{2k}^t \)-coloring of \((G, \sigma)\) and let \( S \) be a non-self-inverse color class. Since \( t \) is maximum, it follows that \( \chi(G[S]) > 2 \). Hence, \( G[S] \) is not bipartite. Thus, it contains an odd and therefore unbalanced circuit. Since this is true for every subgraph induced by a non-self-inverse color class, the statement follows. \( \square \)

Furthermore, the bound regarding the frustration index is sharp: the graph in Fig. 1 has frustration index 2 and it can be easily seen that a minimum coloring requires two non-self-inverse colors.

Next, we will prove that the symset chromatic spectrum is an interval of integers.

**Theorem 3.2.** The symset chromatic spectrum of a graph \( G \) is the interval \( \Sigma_{\chi_{\text{sym}}}(G) = \{2, \ldots, \chi(G)\} \).

**Proof.** We proceed by induction on the order of the graph. If \( G \) is the \( K_1 \), then the statement is trivial. Let us remark that it obviously true for bipartite graphs.

Let \( v \in V(G) \) and \( G' = G - v \). By induction hypothesis \( \Sigma_{\chi_{\text{sym}}}(G') = \{2, \ldots, \chi(G')\} \). Let \( i \in \{2, \ldots, \chi(G') - 1\} \) and \( \sigma_i' \) be a signature of \( G' \) such
that $\chi_{\text{sym}}(G', \sigma'_i) = i$. Since $i < \chi(G')$ it follows that $i = t + 2k$ and $k \geq 1$. Let $c'$ be an $S^t_{2k}$-coloring of $(G', \sigma'_i)$. We also assume, by switching, that $c'$ does not use the negative colors. It implies that all the edges connecting vertices in the same non-self-inverse color class are negative.

Extend $\sigma'_i$ to a signature $\sigma_i$ of $G$ as follows. Let $vw \in E(G)$. If $c'(w)$ is self-inverse, then let $\sigma_i(vw) = +$ and $\sigma_i(vw) = -$ for otherwise. Since $v$ is connected to a non-self-inverse color class by negative edges only, it can be colored with the same color. Thus, $\chi_{\text{sym}}(G, \sigma_i) \leq i$. It cannot be smaller, since for otherwise we would get a symset-coloring of $(G', \sigma'_i)$ with less than $i$ colors. Thus, $\chi_{\text{sym}}(G, \sigma_i) = i$ and therefore, $\{2, \ldots, \chi(G') - 1\} \subseteq \Sigma_{\chi_{\text{sym}}}(G)$. Since $\chi_{\text{sym}}(G, +) = \chi(G)$, if $\chi(G') = \chi(G)$ the statement follows.

Assume now that $\chi(G') = \chi(G) - 1$. We define now a signature $\sigma$ of $G$ such that $\chi_{\text{sym}}(G, \sigma) = \chi(G) - 1$.

Let $S_1$ and $S_2$ be two of the $\chi(G')$ self-inverse color classes induced by the all-positive signature of $G'$. Define $\sigma'$ in the following way: for each $e = wz$, $\sigma'(e) = -$ if $\{w, z\} \subseteq S_1 \cup S_2$ and $\sigma'(e) = +$ otherwise. $(G', \sigma')$ can be colored with a pair of non-self-inverse colors instead of two self-inverse colors. By extending $\sigma'$ to $\sigma$ as before, we obtain a $S^\chi(G)-3$-coloring, so the statement follows. \qed

Let $(G, \sigma)$ be a signed graph with $\chi_{\text{sym}}(G, \sigma) = \lambda_t = t + 2k$ ($t$ maximum) and let $c$ be a $\lambda_t$-coloring of $(G, \sigma)$. Let $0_1, \ldots, 0_t$ be the self-inverse colors and $\pm s_1, \ldots, \pm s_k$ be the non-self-inverse colors. Let $I_p = \bigcup_{j=1}^p c^{-1}(0_{i_j})$ be the union of $p$ self-inverse color classes and $S_q = \bigcup_{j=1}^q c^{-1}(\pm s_{i_j})$ be the union of $q$ non-self-inverse color classes, and $(H_{p, q}, \sigma_{p, q}) = (G[I_p \cup S_q], \sigma_{|G[I_p \cup S_q]|})$.

**Theorem 3.3.** Let $(G, \sigma)$ be a signed graph with $\chi_{\text{sym}}(G, \sigma) = \lambda_t = t + 2k$ and let $c$ be a $\lambda_t$-coloring of $(G, \sigma)$. Then $\chi_{\text{sym}}(H_{p, q}, \sigma_{p, q})) = \chi_{\text{sym}}(H_{p, q}, \sigma_{p, q})) = p + 2q$, for each $p \in \{0, \ldots, t\}$ and $q \in \{0, \ldots, k\}$.

**Proof.** By the coloring $c$ of $(G, \sigma)$ we have that $\chi_{\text{sym}}(H_{p, q}, \sigma_{p, q})) \leq p + 2q$. However, if there would be a better coloring with less colors or one with the same number of colors but more self-inverse colors, then there would be a better coloring for $(G, \sigma)$, a contradiction. \qed
For \((p, q) = (t, 0)\) and \((p, q) = (0, k)\) we obtain the following corollary.

**Corollary 3.4.** Let \((G, \sigma)\) be a signed graph with \(\chi_{\text{sym}}(G, \sigma) = \lambda_t = t + 2k\) and let \(c\) be a \(\lambda_t\)-coloring of \((G, \sigma)\). Then \((G, \sigma)\) can be partitioned into two induced subgraph \((H_1, \sigma_1)\) and \((H_2, \sigma_2)\), such that \(\chi_{\text{sym}}(H_1, \sigma_1) = t = \chi(H_1)\) and \(\chi_{\text{sym}}(H_2, \sigma_2) = 2k = \chi_{\text{sym}}(H_2, \sigma_2)\).

We conclude with the following statement.

**Theorem 3.5.** Let \((G, \sigma)\) be a signed graph. Then \(\chi_{\text{sym}}(G, \sigma) = \lambda_t = t + 2k\) if and only if \((G, \sigma)\) can be partitioned into \(t\) independent sets and \(k\) non-bipartite antibalanced subgraphs such that \(t + 2k\) is minimum and \(t\) maximum for such a partition.

**Proof.** Clearly, each self-inverse color class induces an independent set in \((G, \sigma)\) and each non-self-inverse color class an antibalanced subgraph \((H, \gamma)\). Since \(t\) is maximum it follows that \((H, \gamma)\) is not bipartite.

On the other side, assume that \((G, \sigma)\) has a partition into \(t'\) independent sets and \(k'\) antibalanced subgraphs with \(t' + 2k'\) minimum. Then there is a partition \(P\) with the maximum number of independent sets. Let \(t\) be this number. Then \(P\) has \(k = \frac{1}{2}(t' + 2k' - t)\) non-bipartite antibalanced subgraphs. Thus, \((G, \sigma)\) has an \(S_{2k}\)-coloring and therefore, \(\chi_{\text{sym}}(G, \sigma) = t + 2k\). \(\square\)

**Upper Bounds for the Symset Chromatic Number.** By definition, \(\chi_{\text{sym}}(G, \sigma) \leq \chi(G)\) and, therefore, Brooks’ Theorem can easily be extended to the symset chromatic number. Indeed, if \(\chi_{\text{sym}}(G, \sigma) \neq \chi(G)\), then \(\chi_{\text{sym}}(G, \sigma) \leq \Delta(G) - 1\) unless \(G\) is complete or an odd circuit. This statement can be further improved if there is a small non-self-inverse color class.

**Theorem 3.6.** Let \((G, \sigma)\) be a signed graph and \(\chi_{\text{sym}}(G, \sigma) = \lambda_t = t + 2k < \chi(G)\). If there exists a \(\lambda_t\)-coloring with a non-self-inverse color class of cardinality 3, then \(\chi_{\text{sym}}(G, \sigma) \leq \Delta(G) - k + 1\).

**Proof.** Among all \(\lambda_t\)-colorings of \((G, \sigma)\) which have a non-self-inverse color class with precisely three vertices choose coloring \(c\) with a maximum number of vertices in the union of the self-inverse color classes and then choose the non-self-inverse \(S_1, \ldots, S_k\), such that \(|S_1|\) is maximum, according to the choice of \(S_1, \ldots, S_i\) choose \(S_{i+1}\) such that \(|S_{i+1}|\) is maximum. Since every non-self-inverse color class has at least three vertices we can assume that \(S_k = T\).

Clearly, \(G[T]\) is a triangle. Note that by the choice of \(c\) every vertex of \(T\) is connected to each self-inverse color class by an edge and to each non-self-inverse color class by a positive and a negative edge. It implies that each vertex \(v \in T\) has \(d_G(v) \geq t + 2k\).

We will show that there is a vertex \(v \in T\) with \(d_G(v) \geq t + 3k - 1\).

Let \(S = c^{-1}(\pm s)\) be a non-self-inverse color class. Then \(c\) induces an \(S^0\)-coloring of \((G[S \cup T], \sigma_{|G[S \cup T]}),\) and we can assume that all edges of \(G[S]\) and \(G[T]\) are negative. Furthermore, each vertex of \(T\) has degree at least 4 in \(G[T \cup S]\) and \(d_{G[S \cup T]}(T) \geq 6\). We will show that there are at least nine edges between \(T\) and \(S\). Let \(V(T) = \{v_1, v_2, v_3\}\) and we assume that \(d_{G[T \cup S]}(v_1) \leq d_{G[T \cup S]}(v_2) \leq d_{G[T \cup S]}(v_3)\).
Claim. \( d_{G[S \cup T]}(T) \geq 9 \).

Proof of the claim. Suppose to the contrary that the claim is not true. Then \( d_{G[T \cup S]}(v_1) = 4 \) and \( 8 \leq d_{G[T \cup S]}(v_2) + d_{G[T \cup S]}(v_3) \leq 10 \). Thus, \( d_{G[T \cup S]}(v_2) \leq 5 \). Let \( \{w_1, w_2\} \) be the neighbors of \( v_1 \) in \( S \). We assume that \( v_1w_1 \) is negative and \( v_1w_2 \) is positive.

Suppose that \( w_2 \) is not a neighbor of \( v_i, i \in \{2, 3\} \). Then \( w_2 \) and \( v_1 \) can be colored with one self-inverse color, \( v_j (j \neq 1, i) \) can be colored with another self-inverse color and \( v_1 \) with color \( s \), since it is connected by a negative edge to its second neighbor in \( S \). Thus, \( w_2 \) is also neighbor of \( v_2 \) and \( v_3 \). By switching at \( T \) we deduce that \( w_1, w_2 \) are both neighbors of \( v_2 \) and of \( v_3 \).

Let \( d_{G[T \cup S]}(v_2) = 5 \). Hence, \( d_{G[T \cup S]}(v_3) = 5 \). Let \( w_3 \) be the third neighbor of \( v_2 \) in \( S \). By possible switching at \( T \) we can assume that \( v_2w_3 \) is negative. If \( G(\{v_3, w_1, w_2\}) \) is bipartite, then color it with two (new) self-inverse colors and \( v_1, v_2 \) with color \( s \) to obtain an \( S^2_{2} \)-coloring of \( G[T \cup S] \), a contradiction.

Thus, \( w_1w_2 \in E(G) \) (indeed in \( E_{\sigma} \)) and \( G(\{v_3, w_1, w_2\}) \) is a triangle. Furthermore, \( G(\{v_1, w_1, w_2\}) \) is a balanced triangle. Suppose that \( G(\{v_2, v_1, w_2\}) \) is anti-balanced, then \( v_2 \) can be colored with \( \pm s \) and \( v_1, v_3 \) with two self-inverse colors to obtain an \( S^2_{2} \)-coloring of \( G[T \cup S] \), a contradiction. By possible switching we analogously argue for \( G(\{v_3, w_1, w_2\}) \) and hence, \( G(\{v_1, w_1, w_2\}) \) is a balanced triangle for each \( i \in \{1, 2, 3\} \). Since \( w_1w_2 \) is negative, precisely one of the remaining two edges is positive. If one of \( w_1, w_2 \), say \( w_1 \) is incident to three positive edges \( v_1w_1, v_2w_1, v_3w_1 \), then color \( v_1 \) and \( v_4 \) -the third neighbor of \( v_3 \) in \( S \)- with two self-inverse colors and the remaining vertices with color \( s \) to obtain an \( S^2_{2} \)-coloring of \( G[T \cup S] \), a contradiction.

Hence, \( G(T \cup \{w_1, w_2\}) \) is a complete signed subgraph \((H_5, \sigma_5)\) of \((G, \sigma)\). Clearly, all edges within two vertices of \( T \) are negative and all edges between two vertices of \( S \) are negative. We will discuss the following distribution of positive and negative edges: \( v_1w_2, v_2w_2, v_3w_3, v_1w_1, v_2w_1, v_3w_1 \) are positive and all other edges in \((H_5, \sigma_5)\) are negative. Furthermore, we can assume that \( v_2w_3 \) is negative (see Figure 2). The argumentation for other distributions is similar.

If \( w_3 = w_4 \) and \( v_3w_3 \) is negative or \( w_3 \neq w_4 \), then color \( v_3 \) and \( w_2 \) with two self-inverse colors and the remaining vertices with color \( s \) to obtain the desired contradiction.

(*) If \( w_3 = w_4 \) and \( v_3w_3 \) is positive, then color \( v_3, w_2 \) with two self-inverse colors and the remaining vertices with color \( s \) to obtain an \( S^2_{2} \)-coloring of \( G[T \cup S] \), which is the desired contradiction and finishes the proof of this case.

It remains to consider the case when \( d_{G[T \cup S]}(v_2) = 4 \). We analogously deduce that \((G, \sigma)\) contains \((H_5, \sigma_5)\). If \( d_{G[T \cup S]}(v_3) = 4 \), then \( w_1, w_2 \) is a bipartite cut in \( G[T \cup S] \) and we easily get an \( S^2_{2} \)-coloring of \( G[T \cup S] \). If \( d_{G[T \cup S]}(v_3) = 5 \), we similarly argue as above by discussing the edge \( v_3w_3 \) instead of \( v_2w_3 \). If \( d_{G[T \cup S]}(v_3) = 6 \), then we may assume that \( v_3w_3 \) is negative. However, the coloring given in (*) works here as well and the proof of the claim is finished.
Figure 2. The graph \((G[T \cup S], \sigma|_{G[T \cup S]})\), with dotted edges negative and small dotted edges undefined.

Since \((G[T], \sigma|_{G[T]})\) is connected to \(t\) self-inverse color classes and \(k - 1\) non-self-inverse color classes, it holds that \(d_G(T) \geq 3t + 9(k - 1)\). Seeing that \(T\) only contains three vertices, each of degree 2 in \(G[T]\), it follows that there exists \(v \in T\) such that \(d_G(v) \geq t + 3(k - 1) + 2 = t + 3k - 1\).

4. Concluding Remarks on Variants of Coloring Parameters of Signed Graphs

4.1. Circular Coloring

Circular coloring is a well studied refinement of ordinary coloring of graphs. Here the set of colors is provided with a (circular) metric. Kang and Steffen [7] used elements of cyclic groups as colors for their definition of \((k,d)\)-coloring of a signed graph \((G, \sigma)\). For positive integers \(k, d\) with \(k \geq 2d\), a \((k,d)\)-coloring of a signed graph \((G, \sigma)\) is a map \(c: V(G) \rightarrow \mathbb{Z}_k\) such that for each edge \(e = vw\), \(|c(v) - \sigma(e)c(w)| \geq d \mod k\). Hence, this coloring is a specific \(S_{1k'}\)-coloring if \(k = 2k' + 1\), and a specific \(S_{2(k'-1)}\)-coloring if \(k = 2k'\).

Naserasr, Wang and Zhu [10] generalized circular coloring of graphs to signed graphs as follows. For \(i, j \in \{0, 1, \ldots, p-1\}\), the modulo-\(p\) distance between \(i\) and \(j\) is \(d_{(\mod p)}(i, j) = \min\{|i - j|, p - |i - j|\}\). For an even integer \(p\), the antipodal color of \(x \in \{0, 1, \ldots, p-1\}\) is \(\bar{x} = x + \frac{p}{2} \mod p\).

Let \(p\) be an even integer and \(q \leq \frac{p}{2}\) be a positive integer. A \((p,q)\)-coloring of a signed graph \((G, \sigma)\) is a mapping \(f: V(G) \rightarrow \{0,1,\ldots,p-1\}\) such that for each positive edge \(xy\), \(d_{(\mod p)}(f(x), f(y)) \geq q\), and for each negative edge \(xy\), \(d_{(\mod p)}(\overline{f(x)}, f(y)) \geq q\). Now it is easy to see that this defines a specific \(S_{p^{0}}\)-coloring of \((G, \sigma)\).
4.2. DP-Coloring

In this subsection, we show that coloring of signed graphs with elements from a symmetric set can be described as special DP-coloring. The DP-coloring was introduced for graphs by Dvořák and Postle [3] under the name correspondence coloring. We follow Bernshtein, Kostochka, and Pron [1] and consider multigraphs.

Let $G$ be a multigraph. A cover of $G$ is a pair $(L, H)$, where $L$ is an assignment of pairwise disjoint sets to the vertices of $G$ and $H$ is the graph with vertex set $\bigcup_{v \in V(G)} L(v)$ satisfying the following conditions:

1. $H[L(v)]$ is an independent set for each $v \in V(G)$.
2. For any two distinct vertices $v, w$ of $G$ the set of edges between $L(v)$ and $L(w)$ is the union of $\mu_G(v, w)$ (possible empty) matchings, where $\mu_G(v, w)$ denotes the number of edges between $v$ and $w$ in $G$.

An $(L, H)$-coloring of $G$ is an independent transversal $T$ of cardinality $|V(G)|$ in $H$, i.e. for each vertex $v \in V(G)$ exactly one vertex of $L(v)$ belongs to $T$ and $H[T]$ is edgeless. We also say that $G$ is $(L, H)$-colorable.

Let $t, k$ be positive integers and $L_{2^k}^t = \{s_1, \ldots, s_t, r_0, \ldots, r_{2^k-1}\}$. A $S_{2^k}^t$-cover of a signed multigraph is a cover of $(G, \sigma)$ with $L(v) = L_{2^k}^t$ for each vertex $v \in V(G)$ and $H_{2^k}^t$ satisfies the following conditions:

1. $H_{2^k}^t[L(v)]$ is an independent set for each $v \in V(G)$.
2. If there is no edge between $u$ and $w$, then $E_{H_{2^k}^t}(L(u), L(w)) = \emptyset$.
3. For each edge $e$ between $u$ and $w$ we associate a perfect matching $M_e$ of $E_{H_{2^k}^t}(L(u), L(w))$ with the property that, if $e$ is a positive edge, then $M_e = \{(q, u)(q, w) : q \in L_{2^k}^t\}$ and if $e$ is a negative edge, then $M_e$ is a perfect matching of $E_{H_{2^k}^t}(L(u), L(w))$ which consists of the edges $(s_i, u)(s_i, w)$ for each $i \in \{1, \ldots, t\}$ and $(r_j, u)(r_j+k, w)$ for each $j \in \{0, \ldots, 2^k - 1\}$, where the indices are added mod $2^k$.

It is easy to see that a signed graph is $S_{2^k}^t$-colorable if and only if it $(L, H_{2^k}^t)$-colorable. The associated chromatic numbers are to be defined accordingly.

If we consider coloring of signed graphs we can restrict to multigraphs with edge multiplicity at most 2, since more than one positive and one negative edge between two vertices do not have any effect on the coloring properties of the multigraph. That is, if we consider $(L, H_{2^k}^t)$-cover of a signed extension $\pm G$ of a graph $G$, then $H_{2^k}^t[E_H(L(u), L(w))]$ is a 2-regular multigraph whose components are digons and circuits of lengths 4, for any two adjacent vertices $u, v$ of $G$.

However, the DP-coloring approach allows further flexibility and generalizations. For instance, DP-coloring is considered in the more general context of
gain graphs in a short note of Slilaty [12], where the corresponding chromatic polynomials are defined.

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**declarations**

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