Influence of the boundary effect, viscosity factor and heat conductivity mechanism on the shock front evolution

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Abstract. In this paper a new economical discrete-analytical approach is applied to the problem penetrating of a shock wave in the shear layer of a viscous heat-conducting gas at high Reynolds numbers.

1. Introduction
Mathematical modeling of shock flows of an inviscid as well as viscous heat-conducting gas at high Reynolds numbers (Re∞), based on the approaches combining the advantages of numerical and analytical methods is obviously promising. If constructed correctly, a numeric and analytical methods-symbiosis can have much greater resolving power and/or efficiency in calculating gasdynamic flows characterized by shock waves, shear layers, as well as their interactions than some of those belonging to the classes of numerical and analytical methods. The methods based on the above mentioned symbiosis are able to give new information concerning the details of the inner structure of complex two dimensional, in particular, stationary gasdynamic shock flows; these methods can also reveal both individual and combined impact of such factors as the non-uniformity of an undisturbed flow in front of the shock wave, the boundary effect behind it which is formed by disturbances catching up with it, and viscosity-heat conductivity on the physical process under study. Here, the term “boundary effect” is understood as being a derivative from the generally recognized term “boundary condition” when applied to the back surface of the shock wave; the boundary effect is associated with a continuous system of the corresponding conservation laws. (The analytical apparatus in such numerical analytical methods is applied due to the schematization (isolation) of the shock front by the flat curvilinear surface of a strong gasdynamic discontinuity; thus, along with the known relations in the case of the inclined shock wave (or, even, instead of them), use is made of the continuous-generalized differential relations asymptotically taking into account the viscosity factor and mechanism of heat conductivity (VHC factor) [1–5]. Here, it is worth mentioning the contribution of V.N. Uskov who paid great attention to the analytical apparatus in the research of shock wave flows. This creative process was continued by his students, in particular, by the author of the present study who has attempted to combine the advantages of the previously distinct analytical and numerical methods.

2. Discrete-analytical approach
In the case of numerical modeling of shock flows of a viscous heat-conducting gas with large Re∞ numbers without the shock wave schematization, performing the “through” calculations it is almost impossible to distinguish the contribution into the gasdynamic parameters realized behind it due to the
action of the external (!) (from the side of the macro-flow on both sides of the shock wave outside the shock transition), viscous stresses and heat flow [1–3], in other words, to correctly switch off the action of the external VHC factor only on the shock wave, thereby revealing the contribution of this factor in the shock process. Here and in what follows it is not schematic (numeric) but physical external viscosity factor and conductivity mechanism that are implied, which do not have to be taken into account within the method of the through calculations of shock flows. The main advantages of the shock capturing method (SCM) include its versatility, relative simplicity of constructing a calculation algorithm (its software implementation) and, as a consequence, some reliability. The actual accuracy (without taking into account the unavoidable approximate losses on the shocks), obtained using SCM in most cases is satisfactory for practical applications; thus, this direction is intensively being developed.

In all fairness, it is worth noting that good fulfillment of the standard conditions on the inclined (straight) shock wave often means that at high Re_∞ the VHC factor on the strong gasdynamic discontinuity can be neglected; in the integral formulas of the schematized shock transition this factor is used with the small parameter; the gradients on both sides of the shock wave are assumed to be limited [1]. In the case of a uniform flow on both sides of the shock wave the contribution of the above mentioned factor into the parameters of the shock transition is generally absent at any Re_∞, since the external gradients of the values are absent and the shock itself is smooth.

If the shock wave is schematized, then, it is obvious that in the general case of the non-uniform macroflow outside its curvilinear surface, the influence of the VHC factor on the parameters of the shock transition must correctly be taken into account by the mathematical model even at the average values of (~ 10^3) Re_∞, which results into the necessity to use generalized conditions on such a shock wave. However, in the case of schematization of the shock wave, in terms of the boundary problem statement (!) it appears to be impossible to vary all the possible boundary effects in a wide range, to reveal the degree of their influence on the shape of the shock itself, as well as on the distribution of the parameters and their derivatives along its back surface and the action of the VHC factor. Besides, the numerical methods for solving the systems of Euler equations and Navier-Stokes equations of viscous heat conducting gas (NSEVHC) integrated into a particular commercial software, still require considerable calculation resources, which cannot be neglected despite the existence of supercomputers with petaflop performance.

The above mentioned problems which appear in the numerical modeling of shock flows turn out to be surmountable by using the numerical-analytical methods, and, in particular, the “discrete-analytical approach” [1].

Let us specify what is implied by the compound term numerical-analytical method (as well as by the terms difference-analytical method, discrete-analytical approach). In the general case, this is not to be understood as a method constructed based on direct (independent) combining of the numeric (discrete) and analytical apparatus which allows constructing some “total solution” of a particular problem. However, in some cases an analytical (in the extended sense [6]) solution for the shock process is constructed by its “submersion” into another solution-source which can be determined by analytical dependences or have a numerical origin – the gasdynamic field obtained from the preliminary calculation and tabularized using smooth interpolation. A possibility of obtaining analytical solutions in the classical sense is often connected with the assumptions simplifying the problem statement, consideration of particular cases etc. There are numerous studies dealing with such methods since in the pre-computer era and even with the appearance of the first low-performance PCs these methods and the physical experiment were the main tools for studying complex aero gas dynamic processes. These methods have been used and improved until now since they prove to be efficient in the mathematical description of particular high-gradient shockwave and shock-free gasdynamic processes. In contrast, in the studies being carried out only some minimum assumptions are made which allow, as a maximum, the reduction of the boundary problem for the calculated shock flow in terms of the Euler equations or NSEVHC to the Cauchy problem for a nonlinear ODE system which then, is solved numerically. The final working equations of this system are obtained by complex mathematical calculations by means of computer algebra [7]: they are exact only in terms of gas dynamics; however, taking into account the viscosity - heat conductivity factor, they are approximate
asymptotic equations containing the elements of difference approximations. Unfortunately, such an ODE system requires the development of non-standard approaches for its numerical integration. [1, 3]. In any case, analytical or exact solutions can be discussed only in the extended sense [6].

To conclude, the numerical-analytical method in the present study is understood to be a method symbiosis constructed using the synthesis of the apparatus of difference schemes and analytical apparatus applied locally (discretely) to particular characteristics. In this case, the shock wave appears to be either distinguished, or schematized by the flat curvilinear surface of the strong gasdynamic discontinuity, depending on the applied mathematical model of the compressible gas flow: inviscid (ideal) or viscous with heat conductivity at high Re, correspondingly. The main motivation for the smooth representation of the shock wave with the adequate local application of the analytical apparatus is the natural (infinite (!) in the absence of other disturbances, and, moreover, taking into account the VHC factor) order of the smoothness of the solution in the direction tangent to the shock, which is to be used. Here, if on the schematized curved shock wave in a non-uniform flow of viscous or inviscid heat-conducting gas, the relations of the 0-th order hold, it is required that the corresponding “continuous relations”, relations of the 1-st order (in other words, differential relations in the shock wave [1–2]) should be fulfilled. Using the mathematical language of spline apparatus (spline functions) one can express this idea in a different manner: when used locally, the analytical apparatus allows passing to the spline of a higher order in the smooth representation of the shock wave front. It is important that in the case of such a smooth representation of the shock wave there is no necessity of point-to-point statement of the 0-th order relations: in any of the calculation points they will be fulfilled automatically (!), and, thus, can be employed in the course of the calculation only for the purpose of additional point-to-point control of calculation errors. The analytical apparatus of differential relations on the shock wave employed in connection with the smoothness of the process allows one to analyze the influence (in particular, the combined one) of the main physical parameters on the evolution of the shock front itself and to control its behavior. These factors are the following: gradients of gasdynamic values in an undisturbed flow in front of the shock wave, boundary effect behind it, as well as VHC factor when it is asymptotically taken into account in the present relations. Note that based only on the standard (of the Rankine-Hugoniot type) relations of the 0-th order and, correspondingly, on the apparatus of the shock or shock wave polars [1] (traditionally used by experimentalists and other specialists in calculations at present) such an analysis and control of the behavior of the shock wave front cannot be made.

The motivation for constructing effective numerical-analytical methods for the calculation of shock flows of inviscid and, in particular, of viscous heat-conducting gas at high Re, has a more interesting mathematical background. The shock wave when it has a pronounced (discontinuous upon schematization) change of gasdynamic values in its small neighborhood, and has “the margin of smoothness” in the tangential direction which, undoubtedly (!), would be relevant for the calculation algorithm of the problem solving. Otherwise, as is often the case when, for example, difference SCM are used, apart from the usual local loss of accuracy for the shock wave and other approximative defects, the so-called defect of saturation of the calculation method will appear. [8]. The essence of this phenomenon which is negative but less pronounced than the approximation error, is in the fact that beginning from some critical point the next gradual rank increase of the solution smoothness no longer results in the adequate decrease of the calculation error of the algorithm, i.e. it is no longer meaningful. In particular, this brings about the necessity of the local application of the analytical apparatus in the calculations of gasdynamic flows containing shock waves or other special features. Moreover, if a shock wave is significantly isolated from the oncoming and catching up neighboring features or the problem is solved in a more general case with respect to the small VHC factor automatically providing a greater smoothness of the solution (!), the role of the above mentioned shock increases. For example, in schematizing a single shock wave penetrating in the shear layer (layer) of a viscous heat-conducting gas at high Re, the initial initial-boundary problem for NSEVHVC and moreover, in terms of its inviscid approximation for Euler equations when the layer is considered to be a vortex layer with a non-degenerate profile of entropy, velocity and enthalpy, the reduction to the Cauchy problem for the ODE system becomes possible.
The above mentioned reduction of NSEVHC to ODE in terms of the statement of the viscous problem in the process of the mathematical modeling within a unified computational algorithm, on the one hand, allows one to pass “through” from the gasdynamic to the diffusion stage of the shock wave evolution in the layer, given different boundary effects, while on the other hand, it enables one to significantly save computational power: the calculation on PC takes several seconds rather than numerous hours (NSEVHC). Here, as indicated, in such problem statement the natural smoothness of the solution (especially when the VHC factor is taken into account) in the direction tangent to the shock wave is retained, which was the main motivation for the development and application of the discrete-analytical approach [1].

3. Generalized differential relations
Consider the given problem of the shock wave penetration into the layer of the inviscid or viscous heat-conducting gas at high Reynolds numbers. On the surface of the schematized curved shock wave either ordinary or generalized relations [1–3] of the 0-th and 1-st (differential) order asymptotically taking into account the effect of the viscosity - heat conductivity factor hold; they are exact in terms of gas dynamics. In deriving generalized differential relations (GDR) a number of assumptions is used [1–3], including the assumption on the scale of derivatives of the main values in the layer. The asymptotic order of the VHC terms preserved in GDR corresponds to $O(1)$ at $Re \to \infty$, which appears to be quite sufficient for the VHC effect to be estimated. The compact matrix record of GDR [1, 2] fulfilled in such a shock wave can be written as:

$$a_{ij}(\Phi_j)_n + b_{ij}Kw + c_{ij}(\Phi_j)_n + c_{ij}\delta + \varepsilon^2\left[e_{ij}(\Phi_j)_{nn} + f_{ij}Kw' + g_{ij}(\Phi_j)_{nn} + N_i\right] = 0,$$

$$\Phi = \left(\vec{w}, \theta, \rho, h\right)^T; \quad i, j = 1, ..., 4,$$

where $\delta = 0 \, (\delta = 1); \quad \varepsilon^2 = \left(Re\right)^{-1}; \quad N_i = N_i\left(\left(\Phi_j\right)_n, \left(\Phi_j\right)_n, \left(\Phi_j\right)_n, K_w\right)$.

In (1) the summation over the repeated index $j$ is applied; the values $K_w$ and $K_w' = \frac{c^2\Omega}{\partial w^2}$, where $\Omega = \theta + \sigma \equiv \hat{\theta} + \hat{\sigma}$, represent the longitudinal curvature of the shock wave (as opposed to radial curvature $1/\gamma$ in the axisymmetric case) and its derivative; the vector $N$ explicitly combines nonlinear terms (products of lower derivatives and curvature). All the values are assumed to be dimensionless. In (1) in the notations $K_w$ and $K_w' \equiv \xi_{s} - \xi$ is the coordinate in the direction which is longitudinal to the shock wave (in contrast to the velocity module $W$ it is denoted by the small $w$); $p$ is the pressure; $\rho$ is the density; $h$ is the enthalpy; $\theta$ and $\hat{\theta}$ are the angles of inclination of the streamline to the axis $OX$ of the Cartesian (cylindrical) system of coordinates $XOY$ in front of and behind the shock wave, respectively; all the derivatives are presented in the local system of natural (eigen-) coordinates $(s, n)$ in front of the shock wave, $(s, n)$ – behind the shock wave; the derivatives in front of the shock wave are assumed to be known. The acute angle $\sigma$ between the velocity vector $\vec{v}$ and the shock wave (between $s$ and $\xi_{s}$) determines the shock wave intensity $(J = \hat{\rho}/\hat{p} = (1 + \varepsilon)\beta^2\sin^2\sigma - \varepsilon)$; $\varepsilon = (\gamma - 1)/(\gamma + 1); \quad \gamma = \frac{c_p}{c_v}$: $|\sigma| \geq \alpha_{m}^\circ$, where $\alpha_{m} = \arcsin(1/\beta)$ is the Mach angle at a given point in front of the shock wave and the sign $\chi$ of this angle determines the family to which the shock wave belongs (the characteristic, at $\sigma = \chi\alpha_{m}^\circ$) in the inviscid case. The functional matrices (vectors) of the coefficients $A, ..., G$ in (1) in a complex way depend on the gasdynamic values on both sides of the shock wave; in the inviscid case ($\varepsilon^2 = 0 \, \psi (1)$) $A, ..., D$ after the corresponding
admissible transformations exactly results in analogous coefficients in [4, 5], where Uskov V.N. used another, more convenient in this case, group of dependent variables.

Complex analytical machine calculations were performed in the symbol transformation system «REDUCE» on PC in order to obtain the symbolic products to be used later in calculation software modules, namely generalized relations of the 0-th and 1-st (GDR (1)) order on the curved shock wave, and those following from (1), after their closure and difference approximation of the upper derivatives behind the shock wave, differential-difference working equations, their Jacobians etc. [7].

The application of GDR (1) implies their closure; for this it is required to set any derivative behind the shock wave or their combination, thereby determining the boundary effect behind it. As such a combination use can be made of an additional differential constraint, in particular, that excluding the boundary effect [1]. In the case of setting the boundary effect behind the shock wave, i.e. closing the system (1) which is nonlinear (at $\varepsilon \neq 0$) in the first derivatives behind the shock wave, it becomes uniquely solvable with respect to these derivatives and shock wave curvature in any of its calculation points and can be numerically integrated along the shock wave front.

Having generalized all possible boundary effects behind the shock wave, one has the extended differential constraint, which allows excluding $p^\wedge_n$ from (1), in the following form

$$\begin{align*}
 p^\wedge_n + b \cdot \frac{\wedge \chi}{m} \cdot p^\wedge_n &= c, \\
 \frac{\wedge \chi}{m} &= \left( \frac{\chi}{M} \right)^{\frac{1}{2}} = -\chi \tan(\alpha^\wedge_M),
\end{align*}$$

where $\alpha^\wedge_M$ is the Mach angle behind the shock wave, and $b$ and $c$ are the constraint parameters.

Substituting (2) in (1) it is assumed that the tangent derivative of pressure $p^\wedge_n$, is substituted through the Euler equations of the normal derivative of the inclination angle $\theta^\wedge_n$. Varying the constraint coefficients (2) allows one to set strong and weak boundary effects behind the shock wave of the front compression type (intensification), rarefaction (attenuation), as well as elimination of the boundary effect (neutral boundary effect). Among the boundary effects are the forbidden ones which are defined by the Uskov angle [1], which in a wide range of the shock wave intensities for the fixed M turned out to be close to the known Mach angle (being actually «its double») behind the shock wave, but as opposed to the latter it is related only to the problem of the first order (differential one).

4. Results, discussion and conclusion

Based on (1)–(2), the shock wave penetrating into the shear layer is calculated with the initial intensity $J_n(t_0 = 63.7 \%)$, which is taken in $t_0$ percents of the logarithm of the sound intensity $J_n$($M$,$J_n(t_0) = J_n(t_0)/100\%$. As a model of undisturbed layer the supersonic part of the boundary layer on a heat insulated plate ($\delta = 0$) is taken; Pr=$1$; the power coefficient in the viscosity law is $\omega = 1$; $\partial p/\partial n = 0$; $W|_{x=0} = \partial p/\partial n|_{x=0} = \partial h/\partial n|_{x=0} = 0$.

Figure 1 shows the characteristic corridor for viscous and inviscid solutions with different boundary effects and without them. The characteristic corridor presents in each calculation points of the shock wave the values $|\wedge \chi \alpha^\wedge_n|$ (in front of the shock wave), $|p|$ (on the shock wave), $|\beta + \wedge \chi \alpha^\wedge_n|$ (behind the shock wave), which are the modules of the shock wave inclination angles and characteristics of the same family on both sides of the streamline in the undisturbed flow.

Obviously, with the complete degeneration of the shock wave due to the boundary effect or/and viscosity - heat conductivity all the three curves of the same type from different families (see figure 1), corresponding to a particular solution would intersect at one point below: the corridor «collapses». As is seen in figure 1, regardless of whether the boundary effect weakens or enhances the shock wave,
there will be a partial narrowing of the *characteristic corridor* owing to the weakening effect of the VHC factor on the shock wave. Thus, if there is a boundary effect weakening the shock wave, as a result of the total shockwave-weakening effect of both factors, the narrowing of the *corridor* turns out to be more significant.

Figure 1. *Characteristic corridor*: on the left — in front of the shock wave; in the center — on the shock wave; on the right — behind the shock wave; 1 – viscous solution with the boundary effect, weakening the shockwave; 2 – inviscid solution with the boundary effect weakening the shockwave; 3 – viscous solution without the boundary effect; 4 – inviscid solution without the boundary effect; 5 – viscous solution with the boundary effect, enhancing the shockwave; 6 – inviscid solution with the boundary effect, enhancing the shockwave.

Note that the “inviscid solution” with different boundary effects presented in figure 1 means that the VHC factor was switched off ($\varepsilon^2 = 0$ in (1)) only in the problem of the shockwave interaction with the *layer*, i.e. the latter was considered in this case only as a *vortex layer*, though it had been formed with account of the *viscosity* and *heat conductivity* factor.

Additionally, during the computational experiment, the degree of the influence of individual parameters of the *discrete analytical model* [1], such as the order of asymptotic expansions, approximation order of difference representations, etc. on the solution has also been revealed.

Thus, an important conclusion can be made. From the studies on the shock wave penetration into an ordinary *shear layer* it follows that even with significant boundary effects of one sign or another and the fixed average ($\sim 10^3$) number of $Re_\infty$, taking into account the *external* viscosity - heat conductivity factor does not (!) significantly reduce the intensity of the shock wave as compared with the inviscid formulation of the problem. This is confirmed by numerous physical experiments in which these factors are always present.
In the general case, all the incoming disturbances, including strong and weak (in an inviscid formulation) discontinuities, usually occur “spontaneously”, determining the further evolution of all the outcoming (initial) elements of the resulting shock wave structure, which can be optimized by solving a specific problem [5] (Uskov V.N.) [9–12].

This study considers only smooth secondary disturbances (such as compression-rarefaction) arriving at a single schematized front of the shock wave; due to the use of the viscosity factor and heat conductivity mechanism, weak discontinuities are also excluded. It is worth noting that the formation of the shock front, as well as its subsequent dynamics are always determined from two sides! Therefore, if, in addition to one or another boundary effect, the secondary smooth disturbances in front of the shock wave are also added to the problem, then the goal of possible optimization may be to maximize the process of degradation of the shock front itself. It is extremely difficult to select the necessary set of secondary disturbances, matching (synchronizing) them on both sides of the front, even in terms of a single gas model, boundary formulation of the problem, having powerful computing facilities (experimental equipment). However, in the framework of the discrete-analytical approach [1] such studies are actually being carried out.

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References

[1] Adrianov A L 2016 Mathematical modeling of shock flows of ideal and viscous heat conducting gas based on the discrete-analytical approach: monograph. Krasnoyarsk: Siberian Federal University 216 p
[2] Adrianov A L 2009 Generalized differential relations on a shock wave. Problems of Nuclear Science and Technology. (Issue Mathematical modeling of physical processes) 4 pp 22–30
[3] Adrianov A L 2010 Mathematical modeling of shock flows of viscous heat conducting gas based on the asymptotic model. Problems of Nuclear Science and Technology (Issue Mathematical modeling of physical processes) 4 pp 10–26
[4] Uskov V N 1983 Interaction of stationary gasdynamic discontinuities. Supersonic gas jets: Collected papers (Novosibirsk: Nauka. Siberian Department) pp 22–46
[5] Adrianov A L, Starykh A L and Uskov V N 1995 Interaction of stationary gasdynamic discontinuities. Novosibirsk: Nauka. Siberian Publishing house 180 p
[6] Sidorov A F, Shapeev V P and Yanenko N N 1984 Method of differential constraints and its applications in gas dynamics (Novosibirsk: Nauka) 272 p

[7] Valiullin A N, Ganzha V G, Ilyin V P, Shapeev V P and Yanenko N N 1984 The problem of automatic construction and research of analytical difference schemes on computers Doklady AS USSR 275 (3) pp 528–532

[8] Babenko K I 1986 Fundamentals of numerical analysis (M.: Science) Ch. ed. Phys.-Math. lit. 744 p

[9] Omelchenko A V and Uskov V N 1995 Optimal shock-wave systems Bulletin of the Russian Academy of Sciences. Mechanics of fluid and gas. 6 pp 118-126

[10] Uskov V N and Chernyshov M V 2006 Special and extreme triple configurations of shock waves Applied Mechanics and Technical Physics 47(4) pp 39-53

[11] Uskov V N and Chernyshov M V 2014 Extreme shock wave systems in external aerodynamic problems. Thermophysics and Aeromechanics 21 (1) pp 15-31

[12] Silnikov M V and Chernyshov M V 2018 Supersonic flow at an overexpanded nozzle lip. Shock Waves 28 pp 765-784