A decomposition of the bifractional Brownian motion and some applications

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Abstract

In this paper we show a decomposition of the bifractional Brownian motion with parameters $H, K$ into the sum of a fractional Brownian motion with Hurst parameter $HK$ plus a stochastic process with absolutely continuous trajectories. Some applications of this decomposition are discussed.

1 Introduction

The bifractional Brownian motion is a generalization of the fractional Brownian motion, defined as a centered Gaussian process $B^{H,K}(t) = (B_t^{H,K}, t \geq 0)$, with covariance

$$R^{H,K}(t,s) = 2^{-K}((t^{2H} + s^{2H})^K - |t-s|^{2HK}),$$

(1.1)

where $H \in (0,1)$ and $K \in (0,1]$. Note that, if $K = 1$ then $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$, and we denote this process by $B^H$. Some properties of the bifractional Brownian motion have been studied in [3], and [13]. In particular, in [13] the authors show that the bifractional Brownian motion behaves as a fractional Brownian motion with Hurst parameter $HK$. The stochastic calculus with respect to the

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bifractional Brownian motion has been recently developed in the references [7] and [2].

The purpose of this note is to show a decomposition of the bifractional Brownian motion as the sum of a fractional Brownian motion with Hurst parameter $HK$ plus a process with absolutely continuous trajectories. This decomposition leads to a better understanding, and simple proofs of some of the properties of the bifractional Brownian motion that have been obtained in the literature.

## 2 Preliminaries

Suppose that $B^{H,K}$ is a bifractional Brownian motion with covariance $\langle 1.1 \rangle$. The following properties have been proved in [3] and summarized in [13].

(i) The bifractional Brownian motion with parameters $(H, K)$ is $HK$-self-similar, that is, for any $a > 0$, the processes $(a^{-HK}B^{H,K}_{at}, t \geq 0)$ and $(B^{H,K}_{t}, t \geq 0)$ have the same distribution. This is an immediate consequence of the fact that the covariance function is homogeneous of order $2HK$.

(ii) For every $s, t \in [0, \infty)$, we have

$$2^{-K}|t - s|^{2HK} \leq \mathbb{E}\left((B^{H,K}_{t} - B^{H,K}_{s})^2\right) \leq 2^{1-K}|t - s|^{2HK}. \quad (2.1)$$

This inequality shows that the process $B^{H,K}$ is a quasi-helix in the sense of J.P. Kahane [4, 5]. Applying Kolmogorov’s continuity criterion, it follows that $B^{H,K}$ has a version with Hölder continuous trajectories of order $\delta$ for any $\delta < HK$.

Note that the bifractional Brownian motion does not have stationary increments, except in the case $K = 1$.

It turns out that the bifractional Brownian motion is related to some stochastic partial differential equations. For example, suppose that $(u(t, x), t \geq 0, x \in \mathbb{R})$ is the solution of the one-dimensional stochastic heat equation on $\mathbb{R}$ with initial condition $u(0, x) = 0$

$$\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 W}{\partial t \partial x},$$
where $W = \{W(t, x), t \geq 0, x \in \mathbb{R}\}$ is a two-parameter Wiener process. In other words, $W$ is a centered Gaussian process with covariance
\[
E(W(t, x)W(s, y)) = (t \wedge s)(|x| \wedge |y|).
\]
Then, for any $x \in \mathbb{R}$, the process $(u(t, x), t \geq 0)$ is a bifractional Brownian motion with parameters $H = K = \frac{1}{2}$, multiplied by the constant $(2\pi)^{\frac{1}{4}}2^{-\frac{1}{8}}$. In fact,
\[
u(t, x) = \int_0^t \int_\mathbb{R} p_{t-s}(x-y)W(ds, dy),
\]
where $p_t(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, and the covariance of $u(t, x)$ is given by
\[
E(u(t, x)u(s, x)) = \int_0^{t \wedge s} \int_\mathbb{R} p_{t-r}(x-y)p_{s-r}(x-y)dydr
= \int_0^{t \wedge s} p_t(x-y)dy
= \frac{1}{\sqrt{2\pi}}(\sqrt{t+s} - \sqrt{|t-s|}).
\]

3 A decomposition of the bifractional Brownian motion

Consider the following decomposition of the covariance function of the bifractional Brownian motion:
\[
R^{H,K}(t, s) = \frac{1}{2K}[(t^{2H} + s^{2H}) - t^{2HK} - s^{2HK}]
+ \frac{1}{2K}[t^{2HK} + s^{2HK} - |t - s|^{2HK}].
\]
The second summand in the above equation is the covariance of a fractional Brownian motion with Hurst parameter $HK$. The first summand turns out to be non-positive definite and with a change of sign it will be the covariance of a Gaussian process. In order to define this process, consider a standard
Brownian motion \((W_{\theta}, \theta \geq 0)\). For any \(0 < K < 1\), define the process \(X^K = (X^K_t, t \geq 0)\) by
\[
X^K_t = \int_0^\infty (1 - e^{-\theta t})\theta^{-\frac{1+K}{2}}dW_{\theta}.
\]
(3.2)

Then, \(X^K\) is a centered Gaussian process with covariance:
\[
\gamma^K(t, s) = \mathbb{E}[X^K_t X^K_s] = \int_0^\infty (1 - e^{-\theta t})(1 - e^{-\theta s})\theta^{-1-K}d\theta = \frac{\Gamma(1-K)}{K}[t^K + s^K - (t+s)^K].
\]
(3.3)

In this way we obtain the following result.

**Proposition 1** Let \(B^{H,K}\) be a bifractional Brownian motion, and suppose that \((W_{\theta}, \theta \geq 0)\) is a Brownian motion independent of \(B^{H,K}\). Let \(X^K\) be the process defined in (3.2). Set \(X^K_t = X^K_{2t}^{H,K}\). Then, the processes \((C_1X^K_t + B^K_t, t \geq 0)\) and \((C_2B^K_t, t \geq 0)\) have the same distribution, where \(C_1 = \sqrt{\frac{2^{-K}K}{\Gamma(1-K)}}\) and \(C_2 = 2^{1-K}\).

**Proof.** Let \(Y_t = C_1X^K_t + B^K_t\). Then, from (3.1) and (3.3) for \(s, t \geq 0\) we have
\[
\mathbb{E}(Y_sY_t) = C_1^2\mathbb{E}(X^K_s X^K_t) + \mathbb{E}(B^K_s B^K_t)
\]
\[
= \frac{1}{2K}(t^{2HK} + s^{2HK} - (t^{2H} + s^{2H})K) + \frac{1}{2K}((t^{2H} + s^{2H})K - |t-s|^{2HK})
\]
\[
= \frac{1}{2K}(t^{2HK} + s^{2HK} - |t-s|^{2HK}),
\]
which completes the proof. 

The next result provides some regularity properties for the process \(X^K\).

**Theorem 1** The process \(X^K\) has a version with trajectories which are infinitely differentiable trajectories on \((0, \infty)\) and absolutely continuous on \([0, \infty)\).

**Proof.** Note that \(\mathbb{E}[(X^K_t)^2] = C_3t^K\), where \(C_3 = \frac{\Gamma(1-K)}{K}(2 - 2^K)\). For any \(t > 0\), define \(Y_t = \int_0^\infty \theta^{1-K}e^{-\theta t}dW_{\theta}\). This integral exists because
\[
\mathbb{E}[Y^2_t] = \int_0^\infty \theta^{1-K}e^{-2\theta t}d\theta = \Gamma(K)2^{K-2}t^{K-2}.
\]
Applying Fubini’s theorem and Cauchy-Schwartz inequality, we have:

$$E\left( \int_0^t |Y_s| ds \right) = \sqrt{\frac{2}{\pi}} \int_0^t \sqrt{E[|Y_s|^2]} ds = \sqrt{\Gamma(K) \frac{2}{\pi}} 2^{\frac{K}{2} - 1} \int_0^t s^{-\frac{K}{2}} ds < \infty.$$ 

On the other hand, applying stochastic Fubini’s theorem, we have:

$$\int_0^t Y_s ds = \int_0^t \left( \int_0^\infty \theta^{1-K} e^{-\theta s} dW_\theta \right) ds$$

$$= \int_0^\infty \theta^{\frac{1-K}{2}} \left( \int_0^t e^{-\theta s} ds \right) dW_\theta$$

$$= \int_0^\infty \theta^{-\frac{1+K}{4}} (1 - e^{-\theta t}) dW_\theta$$

$$= X^K(t).$$

This implies that $X^K$ is absolutely continuous and $Y_t = (X^K)'$ on $(0, \infty)$. Similarly, the $n^{th}$ derivative of $X^K$ exists on $(0, \infty)$ and it is given by

$$(X^K_t)^{(n)} = \int_0^\infty (-1)^{n-1} (\theta)^{n-\frac{1}{4} - \frac{K}{4}} e^{-\theta t} dW_\theta.$$ 

Proposition 2: There exists a nonnegative random variable $G(\omega)$ such that for all $t \in [0, 1]$

$$|X^K_t| \leq G(\omega) \sqrt{t^K \log \log(t^{-1}).}$$

Proof. Applying an integration by parts yields

$$X^K_t = \int_0^\infty \varphi(\theta, t) W_\theta d\theta,$$

where

$$\varphi(\theta, t) = te^{-\theta t} \theta^{-\frac{1+K}{2}} - \frac{1}{2}e^{-\theta t} \frac{1}{\theta^{\frac{3+K}{2}}} (1 - e^{-\theta t}).$$

By the law of iterated logarithm for the Brownian motion, given $c > 1$ we can find two random points $0 < t_0 < t_1$, with $t_0 < \frac{1}{e}, t_1 > e$, such that almost surely, for all $\theta \leq t_0$

$$|W_\theta| \leq c \sqrt{2 \theta \log \log \theta^{-1}}.$$
and for all $\theta \geq t_1$

$$|W_\theta| \leq c\sqrt{2\theta \log \log \theta}.$$ 

Then we make the decomposition

$$X^K_t = \int_0^{t_0} W_\theta \varphi(\theta, t)d\theta + \int_{t_0}^{t_1} W_\theta \varphi(\theta, t)d\theta + \int_{t_1}^{\infty} W_\theta \varphi(\theta, t)d\theta.$$ 

For the first term we obtain

$$\left| \int_0^a W_\theta \varphi(\theta, t)d\theta \right| \leq c \int_0^{t_0} \sqrt{2\theta \log \log \theta} |\varphi(\theta, t)|d\theta \leq G_1 t,$$

for some nonnegative random variable $G_1$. Similarly, we can show that

$$\left| \int_{t_0}^{t_1} W_\theta \varphi(\theta, t)d\theta \right| \leq G_2 t.$$ 

With the change of variables $\eta = \theta t$ the third term can be bounded as follows

$$\left| \int_{t_1}^{\infty} W_\theta \varphi(\theta, t)d\theta \right| \leq c \int_{t_1}^{\infty} \sqrt{2\theta \log \log \theta} |\varphi(\theta, t)|d\theta$$

$$\leq 2\sqrt{2ct^2} \int_{t_1}^{\infty} \sqrt{\log \log \frac{\eta}{t} (\eta^{-1 - \frac{K}{2}} + \eta^{-2 - \frac{K}{2}})}d\eta.$$ 

Applying the inequality

$$\log(\log |\eta| + \log t^{-1}) \leq \log 2 + \log |\log \eta| + \log \log t^{-1}$$

we obtain

$$\left| \int_{t_1}^{\infty} W_\theta \varphi(\theta, t)d\theta \right| \leq G_3 t^K \sqrt{\log \log t^{-1}}.$$ 

This completes the proof. ■

4 Applications

We first describe the space of integrable functions with respect to the bifractional Brownian motion.
Suppose that \( X = (X_t, t \in [0, T]) \) is a continuous zero mean Gaussian process. Denote by \( \mathcal{E} \) the set of step functions on \([0, T]\). Let \( \mathcal{H}_X \) be the Hilbert space defined as the closure of \( \mathcal{E} \) with respect to the scalar product
\[
\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = \mathbb{E}(X_t X_s).
\]
The mapping \( 1_{[0,t]} \to X_t \) can be extended to a linear isometry between \( \mathcal{H}_X \) and the Gaussian space \( H_1(X) \) associated with \( X \). We will denote this isometry by \( \varphi \to X(\varphi) \). The problem is to find \( \mathcal{H}_X \) for a particular process \( X \).

In the case of the standard Brownian motion \( B \), the space \( \mathcal{H}_B \) is \( L^2([0, T]) \). For the fractional Brownian motion \( B^H \) with Hurst parameter \( H \in (0, \frac{1}{2}) \) it is known (see [1]) that \( \mathcal{H}_B^H \) coincides with the fractional Sobolev space \( H^{1/2-H}(L^2([0, T])) \). In the case \( H > \frac{1}{2} \), the space \( \mathcal{H}_B^H \) contains distributions, according to the work by Pipiras and Taqqu [11]. In a recent work, Jolis [4] has proved that if \( H > \frac{1}{2} \), the space \( \mathcal{H}_B^H \) is the set of restrictions to the space of smooth functions \( D(0, T) \) of the distributions of \( W^{1/2-H,2}(\mathbb{R}) \) with support contained in \([0, T]\).

For the bifractional Brownian motion we can prove the following result.

**Proposition 3** For \( H \in (0, 1) \) and \( K \in (0, 1] \), the equality \( \mathcal{H}_{X,H,K} \cap \mathcal{H}_{B,H,K} = \mathcal{H}_{B,H,K} \) holds.

**Proof.** For any step function \( \varphi \in \mathcal{E} \) the following equality is a consequence of the decomposition proved in Proposition 1 and the independence of \( X^K \) and \( B^{H,K} \):
\[
C_1 \mathbb{E}(\int_0^T \varphi(t) dX^{H,K}_t)^2 + \mathbb{E}(\int_0^T \varphi(t) dB^{H,K}_t)^2 = C_2 \mathbb{E}(\int_0^T \varphi(t) dI^{H,K}_t)^2
\]
where \( C_1 \) and \( C_2 \) are positive constants. The equality \( \mathcal{H}_{X,H,K} \cap \mathcal{H}_{B,H,K} = \mathcal{H}_{B,H,K} \) follows immediately.

On the other hand, for any step function \( \varphi \in \mathcal{E} \) it holds that
\[
\mathbb{E}(X^{H,K}(\varphi)^2) \leq C_{H,K} \left( \int_0^T |\varphi(t)| t^{H,K-1} dt \right)^2,
\]
where \( C_{H,K} \) is a constant depending only on \( H \) and \( K \). As a consequence, \( L^1([0, T]; t^{H,K-1} dt) \subset \mathcal{H}_{X,H,K} \). The proof is sketched as follows. By taking partial derivative of the covariance function \( \gamma^K \) given in (3.3) it follows that
\[
\frac{\partial^2 \gamma^K(s^{2H}, t^{2H})}{\partial s \partial t} = C_{H,K}(t^{2H} + s^{2H})^{K-2} t^{2H-1} s^{2H-1}.
\]
for some constant $C_{H,K}$. Then, for any $\varphi \in \mathcal{E}$ it holds that

$$
\int_0^T \int_0^T |\varphi(s)\varphi(t)|(st)^{2H-1}(t^{2H} + s^{2H})^{K-1}dsdt \\
\leq \int_0^T \int_0^T |\varphi(s)\varphi(t)|(st)^{2H-1}(s^{2H}t^{2H})^{\frac{K-2}{2}}dsdt \\
= \left(\int_0^T |\varphi(t)|^{2HK-1}dt\right)^2.
$$

By Hölder’s inequality this implies that $L^p([0,T]) \subset \mathcal{H}_{X_{H,K}}$ for any $p > \frac{1}{HK}$. As a consequence,

$$
L^1([0,T];t^{HK-1}dt) \cap \mathcal{H}_{B_{H,K}} \subset \mathcal{H}_{X_{H,K}} \cap \mathcal{H}_{B_{H,K}} = \mathcal{H}_{B_{H,K}}.
$$

In the case $HK < \frac{1}{2}$ this implies that a function in $\mathcal{H}_{B_{H,K}}$ which is in $L^1([0,T];t^{HK-1}dt)$ must belong to the Sobolev space $L^{\frac{2}{HK}}(L^2([0,T]))$.

Consider now the notion of $\alpha$-variations for a continuous process $X = (X_t, t \geq 0)$. The process $X$ admits an $\alpha$-variation if

$$
V_t^{n,\alpha}(X) = \sum_{i=0}^{n-1} |\Delta X_{t_i}|^\alpha.
$$

(4.1)

converges in probability as $n$ tends to infinity for all $t \geq 0$, where $t_i = \frac{i}{n}$ and $\Delta X_{t_i} = X_{t_{i+1}} - X_{t_i}$.

As a consequence of the Ergodic Theorem and the scaling property of the fractional Brownian motion, it is easy to show (see, for instance, [12]) that the fractional Brownian motion with Hurst parameter $H \in (0,1)$ has an $\frac{1}{H}$-variation equals to $C_H t$, where $C_H = \mathbb{E}(|\xi|^H)$ and $\xi$ is a standard normal random variable. Then, Proposition 1 allows us to obtain the $\frac{1}{HK}$-variation of bifractional Brownian motion. This provides a simple proof of a similar result in [13].

**Proposition 4** The bifractional Brownian motion with parameters $H$ and $K$ has a $\frac{1}{HK}$-variation equals to $C_2^{\frac{1}{HK}}C_{HK}t$, where $C_{HK} = \mathbb{E}(|\xi|^{HK})$ and $\xi$ is a standard normal random variable.

**Proof.** Proposition 1 implies $B_{t}^{H,K} = C_2 B_{t}^{H,K} - C_1 X_{t}^{H,K}$.

Applying Minkowski’s inequality,

$$
\left(\sum_{i=0}^{n-1} |\Delta B_{t_i}^{H,K}|^{\frac{1}{HK}}\right)^{HK} \leq C_2 \left(\sum_{i=0}^{n-1} |\Delta B_{t_i}^{H,K}|^{\frac{1}{HK}}\right)^{HK} + C_1 \left(\sum_{i=0}^{n-1} |\Delta X_{t_i}^{H,K}|^{\frac{1}{HK}}\right)^{HK}.
$$

8
On the other hand,
\[
C_2 \left( \sum_{i=0}^{n-1} |\Delta B_{t_i}^{HK}| \frac{1}{n^{1/\alpha}} \right)^{HK} - C_1 \left( \sum_{i=0}^{n-1} |\Delta X_{t_i}^{H,K}| \frac{1}{n^{1/\alpha}} \right)^{HK} \leq \left( \sum_{i=0}^{n-1} |\Delta B_{t_i}^{H,K}| \frac{1}{n^{1/\alpha}} \right)^{HK}.
\]

From the results for the fractional Brownian motion we know that
\[
\lim_{n \to \infty} V_n^{n^{-1/\alpha}} = C_{HK} t
\]
almost surely and in \( L^1 \). To complete the proof, it is enough to show that
\[
\sum_{i=0}^{n-1} |\Delta X_{t_i}^{H,K}| \frac{1}{n^{1/\alpha}} \text{ converges to zero.}
\]
We can write
\[
\sum_{i=0}^{n-1} |\Delta X_{t_i}^{H,K}| \frac{1}{n^{1/\alpha}} \leq \sup_i |\Delta X_{t_i}^{H,K}| \frac{1}{n^{1/\alpha}} \sum_{i=0}^{n-1} |\Delta X_{t_i}^{H,K}|.
\]
The first factor in the above expression converges to zero by continuity, and the second factor is bounded by the total variation of \( X^{H,K} \) on \([0, T] \) since \( X^{H,K} \) is absolutely continuous on \([0, T] \) by Theorem 1. The proof is complete.

Similarly, for the \( \frac{1}{n^{1/\alpha}} \)-strong variation of the process \( B^{H,K} \) we can show that
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t |B^{H,K}_{s+\varepsilon} - B^{H,K}_s| \frac{1}{n^{1/\alpha}} ds = C_{HK}^{1/\alpha} C_{HK} t.
\]

In [14] the authors have proved the Chung’s law of the iterated logarithm for the bifractional Brownian motion:
\[
\liminf_{r \to 0} \max_{t \in [0,r]} \frac{|B^{H,K}_{t+t_0} - B^{H,K}_{t_0}|}{r^{HK}/((\log \log(1/r))^{HK}} = C_0(HK), \tag{4.2}
\]
where \( C_0 \) is a positive and finite constant depending on HK, for all \( t_0 \geq 0 \). A similar result for the fractional Brownian motion was obtained by Monrad and Rootzen in [8]. The decomposition obtained in this paper allows us to deduce Chung’s law of iterated logarithm for \( t_0 > 0 \) for the bifractional Brownian motion, from the same result for the fractional Brownian motion with Hurst parameter \( HK \), with the same constant.

Let us finally remark that the decomposition established in this paper permits to develop a stochastic calculus for the bifractional Brownian motion using the well-known results in the literature on the stochastic integration with respect to the fractional Brownian motion, and taking into account that the process \( X^{H,K} \) has absolutely continuous trajectories.
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