Higgs instability in gapless superfluidity/superconductivity

Ioannis Giannakis¹, Defu Hou², Mei Huang³,⁴, Hai-cang Ren¹,²

¹ Physics Department, The Rockefeller University, 1230 York Avenue, New York, NY 10021-6399
² Institute of Particle Physics, Huazhong Normal University, Wuhan 430079, China
³ Physics Department, University of Tokyo, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan and
⁴ Institute of High Energy Physics, Chinese Academy of Sciences, Beijing 100039, China

In this letter we explore the Higgs instability in the gapless superfluid/superconducting phase. This is in addition to the (chromo)magnetic instability that is related to the fluctuations of the Nambu-Goldstone bosonic fields. While the latter may induce a single-plane-wave LOFF state, the Higgs instability favors spatial inhomogeneity and cannot be removed without a long range force. In the case of the g2SC state the Higgs instability can only be partially removed by the electric Coulomb energy. But this does not exclude the possibility that it can be completely removed in other exotic states such as the gCFL state.

PACS numbers: 12.38.-t, 12.38.Aw, 26.60.+c

Superfluidity or superconductivity with mismatched Fermi momenta appears in many systems such as the charge neutral dense quark matter, the asymmetric nuclear matter, and in imbalanced cold atomic gases. The mismatch plays the role of breaking the Cooper pairing. But it is not understood how a BCS superconductor is destroyed as the mismatch is increased. It was proposed in the 1960s that the competition between the pair breaking and the pair condensation would induce an unconventional superconducting phase, the Larkin-Ovchinnikov-Fulde-Ferrel (LOFF) state. Recently it was found that if the mismatch is larger than the gap magnitude, a gapless superconducting phase can be formed in a charge neutral quark matter or in a constrained imbalanced cold atom system, respectively. Another scenario for the ground state at moderate mismatch is the phase separation between superfluid/superconducting and normal phases.

Both gapless superconducting and BP phases exhibit instabilities. The former exhibits a chromomagnetic instability while the latter a superfluid density instability. Recent studies have demonstrated that this type of instability is related to the local phase fluctuation of the superconducting order parameter, i.e., the Nambu-Goldstone boson fields. This type of instability induces the formation of the single-plane-wave FF-like state. The imbalanced cold atom systems such as the ⁴⁰K and ⁶Li offer an intriguing experimental opportunity to understand the pair breaking states. However, recent experiments on these systems did not show explicit evidence of the BP state or the LOFF state, rather they produced strong evidence of the phase separation at moderate mismatch. We explain in this letter that the BP state or the FF state maybe prevented to form there because of the Higgs instability induced by the mismatch. The Higgs instability is related to the magnitude fluctuation of the superconducting order parameter, i.e., the Higgs field. It causes spatial inhomogeneity that may lead to phase separation. The Higgs instability was also considered in [20], where it was named "amplitude instability".

The Higgs instability is an inhomogeneous extension of the Sarma instability against a homogeneous variation of the order parameter. Consider a system described by the Hamiltonian $H$ with a spontaneous symmetry breaking, its thermodynamic potential density is given by

$$\Omega = - \lim_{V \to \infty} \frac{T}{V} \ln \text{Tr} e^{-\beta H + \nu N},$$

where $N$ is a conserved charge with $\nu$ the corresponding chemical potential and $V$ is the volume of the system. The trace extends to a subset of the Hilbert space specified by an order parameter $\phi$, which is invariant under the broken symmetry group. The eigenvalue condition reads $(\partial \Omega / \partial \phi)_\nu = 0$. A homogeneous variation of $\phi$ around the equilibrium leads to $\delta \Omega = \frac{1}{2} (\partial^2 \Omega / \partial \phi^2)_\nu \delta \phi^2 + ...$. The Sarma instability

$$\left( \frac{\partial^2 \Omega}{\partial \phi^2} \right)_\nu < 0 \quad (2)$$

renders the equilibrium unstable. In an ensemble of a fixed density of the $N$-charge,

$$n = - \left( \frac{\partial \Omega}{\partial \nu} \right)_\phi \quad (3)$$

however, the thermodynamic quantity to be minimized is the Legendre transformed free energy density, $F = \Omega + \nu n$. While the equilibrium condition $(\partial F / \partial \phi)_n = 0$ is equivalent to $(\partial \Omega / \partial \phi)_\nu = 0$, the second order derivative reads

$$\left( \frac{\partial^2 F}{\partial \phi^2} \right)_n = \left( \frac{\partial^2 \Omega}{\partial \phi^2} \right)_\nu + \left( \frac{\partial n}{\partial \nu} \right)_\phi^2. \quad (4)$$

Because $(\partial n / \partial \nu)_\phi = - (\partial^2 \Omega / \partial \nu^2)_\phi > 0$ as can be verified from the definition, $(\partial^2 F / \partial \phi^2)_n > (\partial^2 \Omega / \partial \phi^2)_\nu$.,
The removal of the Sarma instability by the constraint \( \delta \phi \) amounts to \( \langle \partial^2 F / \partial \phi^2 \rangle_n > 0 \), leaving the 2nd order variation of the free energy
\[
\delta F = \frac{1}{2} \left( \frac{\partial^2 F}{\partial \phi^2} \right)_n \delta \phi^2 > 0
\]
at the equilibrium. This is, however, not enough. The inclusion of inhomogeneous variations \( \phi \) extends \( \delta \phi \) to
\[
\delta F = \frac{1}{2} \left( \frac{\partial^2 F}{\partial \phi^2} \right)_n \delta \phi^2 + \frac{1}{2} \sum_{k \neq 0} \left( \frac{\partial^2 F}{\partial \phi_k^* \partial \phi_k} \right)_n \delta \phi_k^* \delta \phi_k,
\]
with \( \left( \frac{\partial^2 F}{\partial \phi_k^* \partial \phi_k} \right)_n = \left( \frac{\partial^2 \Omega}{\partial \phi_k^* \partial \phi_k} \right)_\nu \). As long as
\[
\lim_{k \to 0} \lim_{V \to \infty} \left( \frac{\partial^2 \Omega}{\partial \phi_k^* \partial \phi_k} \right)_\nu = \left( \frac{\partial^2 \Omega}{\partial \phi^2} \right)_\nu < 0,
\]
the Sarma instability will return with respect to \( \delta \phi \) of sufficiently low but nonzero \( k \) without offsetting the globally imposed constraint \( \delta \mu \).

The Higgs instability can also be understood intuitively. Let us divide the whole system being considered into subsystems. As long as the size of each subsystem is much larger than all interaction length scales, its thermodynamics is identical to that of the master system but with the constraint relaxed. The Sarma instability will develop through different variations of the order parameters of each subsystem while maintaining the constraint conditions of the master system.

To analyze the Higgs instability of the g2SC, we start from the bosonized 2-flavor Nambu–Jona-Lasinio (NJL) model, the Lagrangian density has the form of
\[
L_{2SC} = \bar{q} (i \partial_\nu + \mu \gamma^0) q - \frac{1}{4G_D} \Delta^\rho \Delta^\rho + \frac{i \Delta^\rho}{2} [\bar{q} i \gamma^5 \tau_2 e^\rho q] - \frac{i \Delta^\rho}{2} [\bar{q} i \gamma^5 \tau_2 e^\rho q^C].
\]
The requirement of \( \beta \)-equilibrium induces the mismatch between the Fermi surfaces of the pairing quarks. The matrix of chemical potentials in the flavor-space color \( \mu \) is given in terms of the quark number chemical potential \( \mu \), the color chemical potential \( \mu_a \), and the electric chemical potential \( \mu_e = 2 \delta \mu \). Because \( \mu_a \simeq O(\Delta^2 / \mu) \ll \Delta \), we safely take \( \mu_a = 0 \) in our weak coupling calculations. Then \( \mu_a = \mu - \delta \mu \) and \( \mu_2 = \mu + \delta \mu \) with \( \delta \mu = \mu - \delta \mu / 3 \).

The conserved charge \( N \) and its chemical potential \( \nu \) of the general discussion above correspond to the electric charge \( Q \) and the electric chemical potential \( \mu_e \) of the model. The 4th anti-triplet composite diquark field reads \( \Delta^\rho = 2iG_D(q^C \gamma^5 \tau_2 e^\rho q) \), where \( e^\rho \) is an anti-symmetric matrix in the color space with \( \rho = 1, 2, 3 \) indicating respectively the anti-red, anti-green, and anti-blue colors of the diquark. In the 2SC phase, the color symmetry \( SU(3)_c \) is spontaneously broken to \( SU(2)_c \) and diquark field obtains a nonzero expectation value. Without loss of generality, one can always assume that diquark condenses in the anti-blue direction, the ground state of the 2SC phase is characterized then by \( \langle \Delta^3 \rangle = \Delta > 0 \), and \( \langle \Delta^1 \rangle = \langle \Delta^2 \rangle = 0 \).

The most general spatial fluctuation of the order parameter can be parameterized as
\[
\begin{pmatrix} \Delta^1(\vec{r}) \\ \Delta^2(\vec{r}) \\ \Delta^3(\vec{r}) \end{pmatrix} = \exp \left[ i \sum_{\alpha = 4}^8 \phi_\alpha(\vec{r}) T_\alpha \right] \begin{pmatrix} 0 \\ 0 \\ \Delta + H(\vec{r}) \end{pmatrix},
\]
where \( \phi_\alpha \)'s are five Nambu-Goldstone bosons, \( T_\alpha \)'s are the corresponding generator of \( SU(3)_c \) and \( H \) is the real Higgs field. The Fourier components of \( H \) correspond to \( \delta \phi \) and \( \delta \phi \) of Eqs. (3) - (7). For the rest of the letter, we focus only on the Higgs mode by setting \( \varphi_a = 0 \), since \( H \) and \( \varphi \) do not mix quadratically.

The thermodynamic potential of the system including Higgs fluctuation reads \( \Omega = \Omega_M + \Omega_H \). The mean-field free-energy at equilibrium is given by
\[
\Omega_M = -\frac{T}{2} \sum_n \int \frac{d^3 \vec{p}}{(2\pi)^3} \text{Tr} \ln \left( S_M(P) \right) - \Delta^2 / 4G_D, \tag{10}
\]
where the inverse propagator \( S_M^{-1} \) takes the form of
\[
\left( \frac{\partial^2 \Omega}{\partial \Delta^\rho \partial \Delta^\rho} \right)_\mu = \frac{4\bar{\mu}^2}{\mu^2} \left[ 1 - \frac{\delta \mu}{\sqrt{(\delta \mu)^2 - \Delta^2}} \right] < 0 \tag{12}
\]
in weak coupling, i.e. \( \delta \mu \ll \mu \).

The thermodynamic potential of the Higgs field reads
\[
\Omega_H = \frac{1}{2} \int \frac{d^3 \vec{k}}{(2\pi)^3} H^*(\vec{k}) \Pi(k) H(\vec{k}), \tag{13}
\]
with the self-energy function
\[
\Pi(k) = \frac{1}{4G_D} - \frac{T}{4} \sum_n \int \frac{d^3 \vec{p}}{(2\pi)^3} \text{Tr} \left[ S_M(P) \gamma_5 \gamma_3 \tau_2 \rho_2 S_M(P - K) \gamma_5 \gamma_3 \tau_2 \rho_2 \right], \tag{14}
\]
where the trace extends to Dirac, color, flavor and NG indexes and \( K = (0, \vec{k}) \). At \( T = 0 \), we obtain the result of \( \Pi(k) = A_H + B_H k^2 \) for \( k \ll \Delta \) with \( A_H = \langle \partial^2 \Omega / \partial \Delta^2 \rangle_{\mu_e} \) given in Eq. (12) and
\[
B_H = \frac{2 \bar{\mu}^2}{9 \pi^2 \Delta^2 \left[ 1 - \frac{(\delta \mu)^3}{((\delta \mu)^2 - \Delta^2)^2} \right]}. \tag{15}
\]
The negative $A_H$ signals the Higgs instability and the negative $B_H$ indicates that the instability gets stronger at nonzero $k$. Note that our coefficient $B_H$ is more negative than that of Ref. 21. Numerically, we found that the function $\Pi(k)$ reaches a minimum at some intermediate $k$ before increasing with $k$. For $k \gg \Delta$, we find $\Pi(k) \simeq (\bar{\mu}^2/\pi^2)(\ln(\bar{\mu}^2/\Delta) + \text{const.}) > 0$.

Unless there exists a competing mechanism that results in a positive contribution to the Higgs self-energy function $\Pi(k)$ for $k \neq 0$. The Higgs instability will prevent the exotic states of gapless superfluidity/superconductivity from being implemented in nature. This explains the failure of observing BP state in cold neutral atoms. In case of quark matter, however, the long range Coulomb interaction of the electric charge fluctuation induced by the inhomogeneous Higgs field has to be examined before reaching a conclusion.

The inhomogeneity generated Coulomb energy density that should be added to RHS of $\Omega_H$ of Eq. (13) reads

$$E_{\text{coul}} = \frac{1}{2V} \sum_{\bar{k} \neq 0} \frac{\delta \rho(\bar{k})^* \delta \rho(\bar{k})}{k^2 + m_D^2(\bar{k})}$$

where $\delta \rho(\bar{k})$ is the Fourier component of the Higgs-induced charge density and $m_D$ is the Coulomb polarization function (Debye mass at $\bar{k} = 0$). We have $\delta \rho(\bar{k}) = \kappa(\bar{k})H(\bar{k})$, with

$$\kappa(k) = -\frac{eT}{2V} \sum_P \text{tr} \gamma_0 Q S_M(P) \gamma_5 \tau_3 \rho_3 S_M(P - K),$$

and

$$m_D^2(k) = -\frac{e^2 T}{2V} \sum_P \text{tr} \gamma_0 Q S_M(P) \gamma_0 Q S_M(P - K),$$

where the charge matrix $Q = \rho_3(a + b\gamma_3)$ with $a = 1/6$ and $b = 1/2$ for the quark matter consisting of $u$ and $d$ flavors. It can be shown that the Goldstone fields do not generate charge fluctuations. The Legendre transformed free energy of Higgs field is obtained from Eq. (13) with $\Pi(k)$ replaced by

$$\tilde{\Pi}(k) \equiv \left(\frac{\partial^2 F}{\partial H^* \partial H(k)}\right)_{n_Q} = \Pi(k) + \frac{\kappa(k)\kappa(k)}{k^2 + m_D^2(k)}.$$  

(19)

Notice that $m_D^2(0) = e^2(\partial n_Q/\partial \mu)\Delta$ and $\kappa(0) = e(\partial n_Q/\partial \Delta)\mu$ with $n_Q$ the charge density. The second term on RHS of Eq. (13) is included in the second term of $\tilde{\Pi}(k)$ in the infinite volume limit, invalidating the equality $\Pi$. We have

$$\kappa(0) = \frac{4eb\bar{\mu}^2}{\pi^2} \frac{\Delta}{\sqrt{\Delta^2 - \Delta^2}},$$

(20)

$$m_D^2(0) = \frac{2e^2b\bar{\mu}^2}{\pi^2} (1 + \frac{2\delta \mu}{2\sqrt{\Delta^2 - \Delta^2}}).$$

(21)

On writing $\tilde{A}_H \equiv \tilde{\Pi}(0)$, we obtain that

$$\tilde{A}_H = \frac{4(b^2 - a^2)\bar{\mu}^2(\delta \mu - \sqrt{\delta \mu^2 - \Delta^2})}{\pi^2(3a^2\sqrt{\delta \mu^2 - \Delta^2} + b^2(2\delta \mu + \sqrt{\delta \mu^2 - \Delta^2}^2))},$$

(22)

for the whole range of g2SC state. This quantity together with that without Coulomb term are plotted in Fig. 1.

FIG. 1: $A_H$ (red dashed-dotted line) and $\tilde{A}_H$ (green solid line) are plotted as functions of $\Delta/\delta \mu$ in the g2SC phase, and the black dashed line shows the corresponding quantity in the 2SC phase.

The detailed calculation of the form factor $\kappa(k)$, function $m_D(k)$ as well as the Coulomb improved Higgs self-energy in the whole momentum $k$ space will be reported in another paper 21. We find that there is always some domain of $k$ where $\tilde{\Pi}(k) < 0$ throughout the entire region of g2SC state ($\delta \mu > \Delta$) for an arbitrary ratio.
m_D(0)/\Delta. Therefore the electric Coulomb energy is not strong enough to cure the Higgs instability of g2SC for all momenta. See Fig. 2 for an example. It is noticed that II(k) reaches its minimum at a rather large momentum, i.e., $k \approx 4\Delta$, which indicates the formation of phase separation.

In this letter, we have explored the Higgs instability against local fluctuations of the magnitude of the order parameter that are not prohibited by global constraints. In the case of the imbalanced neutral atom systems, the instability extends from arbitrarily low momenta—albeit not zero-to momenta much higher than the inverse coherence length. Without a long range force a phase separation is likely to be the structure of the ground state since it minimizes the gradient energy of the inhomogeneity. For the case of two flavor quark matter that is globally neutral, the instability can be removed by the electric Coulomb energy for low and high momenta. But it remains in a window of intermediate momenta. Nevertheless, the long range Coulomb interactions might favor a ground state with crystalline structure. Possible candidates include the heterotic mixture of normal and superconducting phases and the multi-plane-wave LOFF state. Although our calculations are limited to the case of g2SC, the existence of the Higgs instability in the absence of Coulomb interactions is universal for all gapless superfluidity/superconductivity that are subject to the Sarma instability. Our general formulation from to (1) to (7) can be trivially generalized to the system with several invariant order parameters and constraints. It would be extremely interesting to examine whether the electric and color Coulomb energies are capable of eliminating the Higgs instability completely in the gCFL phase.

Acknowledgments: We thank M. Forbes, K. Fukushima, K. Iida, M. Hashimoto, T. Hatsuda, D.K. Hong, I. Shovkovy and P.F. Zhuang for stimulating discussions. The work of I.G. and H.C.R is supported in part by US Department of Energy under grants DE-FG02-91ER40651-TASKB. The work of D.F.H is supported in part by Educational Committee under grants NCET-05-0675 and 704035. The work of D.F.H and H.C.R is also supported in part by NSFC under grant No. 10575043. The work of M.H. is supported in part by the Japan Society for the Promotion of Science Fellowship Program and the Institute of High Energy Physics, Chinese Academy of Sciences, she also would like to thank the APCTP for its hospitality.

[1] P. Fulde and R. A. Ferrell, Phys. Rev. 135, A550 (1964); A. I. Larkin and Yu. N. Ovchinnikov, Sov. Phys. JETP 20, 762 (1965).
[2] M. Huang, P. F. Zhuang and W. Q. Chao, Phys. Rev. D 67, 065015 (2003); I. Shovkovy and M. Huang, Phys. Lett. B 564, 205 (2003); M. Huang and I. Shovkovy, Nucl. Phys. A729, 835 (2003).
[3] M. Alford, C. Kouvaris and K. Rajagopal, Phys. Rev. Lett. 92, 222001 (2004); Phys. Rev. D 71, 054009 (2005).
[4] W. V. Liu and F. Wilczek, Phys. Rev. Lett. 90, 047002 (2003); E. Gubankova, W.V. Liu and F. Wilczek, Phys. Rev. Lett. 91, 032001 (2003).
[5] M. G. Alford, K. Rajagopal, S. Reddy and F. Wilczek, Phys. Rev. D 64, 074017 (2001) F. Neumann, M. Buballa, and M. Oertel, Nucl. Phys. A714, 481 (2003); I. Shovkovy, M. Hanauske and M. Huang, Phys. Rev. D 67, 103004 (2003); H. Grigorian, D. Blaschke and D. N. Aguilera, Phys. Rev. C 69, 065802 (2004); S. Reddy and G. Rupak, Phys. Rev. C 71, 025201 (2005); P. F. Bedaque, H. Caldas and G. Rupak, Phys. Rev. Lett. 91, 247002 (2003); H. Caldas, cond-mat/0605005.
[6] M. Huang and I. A. Shovkovy, Phys. Rev. D 70, 051501 (2004); M. Huang and I. A. Shovkovy, Phys. Rev. D 70, 094030 (2004).
[7] R. Casalbuoni, R. Gatto, M. Mannarelli, G. Nardulli and M. Ruggieri, Phys. Lett. B 605, 362 (2005); M. Alford and Q. li. Wang, hep-ph/0501078.
[8] S.-T. Wu and S. Vip. Phys. Rev. A 67, 053603 (2003).
[9] D. K. Hong, hep-ph/0506007.
[10] Mei Huang, Int. J. Mod. Phys. A 21, 910 (2006); Mei Huang, Phys. Rev. D 73, 045007 (2006).
[11] M. Hashimoto, hep-ph/0605323.
[12] A. Kryjevski, hep-ph/0508180; T. Schafer, Phys. Rev. Lett. 96, 012005 (2006); A. Gerhold and T. Schafer, hep-ph/0603257.
[13] I. Giannakis and H. C. Ren, Phys. Lett. B 611, 137 (2005); I. Giannakis and H. C. Ren, Nucl. Phys. B 723, 255 (2005); I. Giannakis, D. f. Hou and H. C. Ren, Phys. Lett. B 631, 16 (2005).
[14] E. V. Gorbar, M. Hashimoto and V. A. Miransky, hep-ph/0507303.
[15] K. Fukushima, hep-ph/0603216.
[16] R. Casalbuoni, R. Gatto, N. Ippolito, G. Nardulli and M. Ruggieri, Phys. Lett. B 627, 89 (2005); [Erratum-ibid. B 634, 565 (2006)]; M. Ciminale, G. Nardulli, M. Ruggieri and R. Gatto, Phys. Lett. B 636, 317 (2006); M. Mannarelli, K. Rajagopal and R. Sharma, hep-ph/0603076; K. Rajagopal and R. Sharma, hep-ph/0605316.
[17] D. T. Son and M. A. Stephanov, cond-mat/0507586; L. He, M. Jin and P. f. Zhuang, cond-mat/0601147; H. Hu and X.J. Liu, cond-mat/0603332; K. Machida, T. Mizushima, and M. Ichikawa, cond-mat/0604339.
[18] M. W. Zwierlein, A. Schirotzek, C. H. Schunck, and W. Ketterle, Science 311 (5760), 492-496 (2006); M.W. Zwierlein, C. H. Schunck, A. Schirotzek, and W. Ketterle, cond-mat/0605258; G. B. Partridge, W. Li, R. I. Kamar, Y. Lia, and R. G. Hulet, cond-mat/0517152; M. W. Zwierlein, and W. Ketterle, cond-mat/0603434.
[19] G. Sarma, J. Phys. Chem. Solids 24, 1029 (1963).
[20] K. Iida and K. Fukushima, hep-ph/0603179.
[21] I. Giannakis, D. Hou, M. Huang and H. C. Ren, in preparation.
[22] We thank I. Shovkovy for pointing this out for us.
[23] J. A. Bowers and K. Rajagopal, Phys. Rev. D 66, 065002(2002); K. Rajagopal and R. Sharma, hep-ph/0605316.