Canonical General Relativity on a Null Surface
with Coordinate and Gauge Fixing

J.N. Goldberg
Department of Physics, Syracuse University, Syracuse, NY 13244-1130
and
C. Soteriou
Department of Mathematics, King's College London, Strand, London WC2R 2LS

ABSTRACT

We use the canonical formalism developed together with David Robins on to study the Einstein equations on a null surface. Coordinate and gauge conditions are introduced to fix the triad and the coordinates on the null surface. Together with the previously found constraints, these form a sufficient number of second class constraints so that the phase space is reduced to one pair of canonically conjugate variables: \( A^3_2 \) and \( \Sigma^3_2 \). The formalism is related to both the Bondi-Sachs and the Newman-Penrose methods of studying the gravitational field at null infinity. Asymptotic solutions in the vicinity of null infinity which exclude logarithmic behavior require the connection to fall off like \( 1/r^3 \) after the Minkowski limit. This, of course, gives the previous results of Bondi-Sachs and Newman-Penrose. Introducing terms which fall off more slowly leads to logarithmic behavior which leaves null infinity intact, allows for meaningful gravitational radiation, but the peeling theorem does not extend to \( \Psi_1 \) in the terminology of Newman-Penrose. The conclusions are in agreement with those of Chrusciel, MacCallum, and Singleton. This work was begun as a preliminary study of a reduced phase space for quantization of general relativity.

1. Introduction.

The canonical approach to quantum gravity received a strong impetus from the introduction of the new variables by Abhay Ashtekar [1-4]. The use of a self-dual connection one-form and a vector density triad as canonical variables leads to a Hamiltonian which is a polynomial of degree four in the new variables. This structure suggests a simplification of the canonical formalism which may lead to significant easing of the problems of quantization. By now there is a considerable literature detailing the efforts of many people to understand how to make effective use of these variables [3,4].

As part of this overall effort, together with David Robinson, we have renewed the study of canonical general relativity on a null surface [5]. This program had previously been undertaken using the metric or a tetrad as the configuration space variables [6,7]. Although these efforts did not recover all the Einstein equations in a natural way, their principal drawback is that the resulting system of second class constraints is complicated by the non-polynomial structure so that there does not appear to be any hope that a successful Dirac quantization [8,9] can be carried out. Our hope was that use of the self-dual formalism on a null surface would retain the polynomial structure of the Hamiltonian and the constraints. If so, the second class constraints might not be as formidable as in the previous treatments.

While some of the second class constraints are indeed simpler, for the most part they are sufficiently complicated that they cannot easily be eliminated either directly or by means of the introduction of Dirac brackets [8-10]. Therefore, as a first step toward considering a reduced phase space quantization, we have repeated the analysis of the gravitational field in the vicinity of null infinity. The use of the Ashtekar variables for this analysis falls between the metric formulation of Bondi [11] and Sachs [12] and the spin coefficient method of Newman-Penrose [13,14]. Although the full complement of Dirac brackets cannot be obtained, a machine calculation did show that the bracket of one component of the connection (related to the expansion and shear) with its conjugate triad density is the same as the corresponding Poisson bracket. This identified for us which variables should serve as the true dynamical variables in terms of which to express the remaining components of the gravitational field.

By focussing attention on the connection, we were immediately struck by how deep in a \( 1/r \) expansion one must go in order to avoid the appearance of logarithmic behavior. Therefore, the question naturally arises whether such logarithmic behavior is consistent with asymptotic flatness and finite energy. While we were still analyzing this question, Chrusciel, MacCallum, and Singleton [15] showed that this is indeed the case. However, because in this work we use a different formalism in which the need to examine the logarithmic terms is glaring, it is worthwhile to present the results. Also, there is a difference in our solutions. We find that even with the appearance of logarithmic behavior, the coefficient of \( 1/r \) in the connection must vanish.
whereas they do not. A detailed discussion of our solution leading to this difference is left for the Appendix.

To see how far the connection must drop in $1/r$ in order to avoid logarithmic behavior, we shall first carry out the analysis requiring an expansion in $1/r$ without logarithms and then go back to indicate what changes would be introduced by the logarithmic terms. The simplification in the calculations by neglect of logarithms is enormous. In agreement with the earlier work of Novak and Goldberg [16, 17], the present results and those of Chrusciel, MacCallum, and Singleton show that null infinity can be defined and that energy-momentum and radiation of energy-momentum remain finite. In our case, logarithmic terms come in below those responsible for radiation. In the more general considerations of Chrusciel et al, the coefficients of the leading logarithmic terms are independent of the time. This point does not come up in our work. However, in both studies, the logarithmic terms come in before those needed to define angular momentum.

In the following section, we shall give a streamlined review of the construction of the Hamiltonian on a null surface [5]. In section 3, we introduce our coordinate and tetrad conditions and present our analysis of the equations, listing the order in which they are to be solved. The solution in the absence of logarithmic terms is given in section 4 and in section 5 we discuss the logarithmic terms. We close with a discussion of our results.
Section 2. The Hamiltonian.

To obtain the Hamiltonian, we started from a complex Lagrangian constructed from the self-dual part of the Riemann tensor, following similar work by Jacobson and Smolin [18] and by Samuel [19] on a space-like surface. We assume the space-time to be real, but consider complex solutions of the Einstein equations. After the calculation has been completed, we impose reality conditions on the variables and recover a real metric and curvature. Because a null surface is degenerate, we lose one of the field equations if we allow the initial surface to be null from the beginning. Therefore, in order to be sure that we recover all of the Einstein equations, we included an auxiliary variable $\alpha$. The surfaces $t=$constant are space-like, time-like, or null, when $\alpha < 0$, $> 0$, or $= 0$, respectively. To guarantee that the surfaces would be null, we adjoined $\alpha^2 = 0$ to the Lagrangian with a Lagrange multiplier. Moreover, because working on a null surface imposes constraints not present on a space-like surface, we did not a priori eliminate any variables as non-dynamical. Therefore, we started with a phase space of 40 variables. The 12 first and 14 second class constraints leave two dynamical degrees of freedom per hypersurface point as is appropriate on a null surface [20]. An examination of the resulting structure shows that we may limit the phase space variables to the nine components of the connection and of the densitized triad vectors on the hypersurface. We shall do so below.

We introduce the null basis of one forms and the dual tetrad basis
\begin{align}
\theta^0 &= (N + \alpha \nu^1 i N^1)dt + \alpha \nu^4 i dx^i, \\
\theta^i &= \nu^i i (N^4 dt + dx^4) \\
e_0 &= N^{-1}(\partial_t - N^i \partial_i), \\
e_i &= (\nu^i + \alpha_i N^{-1} N^i)\partial_i - \alpha_i N^{-1} \partial_t,
\end{align}

where $\alpha_i = \alpha \delta^1 i$ and $\nu^i i \nu^j = \delta^1 j$. All indices have the range 1-3 and repeated indices sum. Bold face indices refer to the one forms and tetrads. The signature is -2, with
\[
ds^2 = \theta^0 \otimes \theta^1 + \theta^1 \otimes \theta^0 - \theta^2 \otimes \theta^3 - \theta^3 \otimes \theta^2, \\
= \eta_{\alpha \beta} \theta^\alpha \otimes \theta^\beta.
\]

(Greek letters range and sum from zero to three.) It follows that the surfaces $t=$constant are null surfaces when $\alpha = 0$.

We choose the orientation so that the volume form is
\[
V = -i N \nu \theta^0 \wedge \theta^1 \wedge \theta^2 \wedge \theta^3.
\]

Thus, $-i N \nu$ is positive, where $\nu$ is the determinent of $\nu^i i$ and the components of the Levi-Civita tensor with respect to this basis, $\epsilon_{\alpha \beta \gamma \delta}$ and $\epsilon^{\alpha \beta \gamma \delta}$, satisfy
\[
\epsilon_{0123} = \epsilon^{0123} = -i.
\]

The connection coefficients are defined by
\[
d\theta^\alpha = \theta^\beta \wedge \omega^\alpha_{\beta}
\]
and the Riemann tensor by
\[
d\omega^\alpha_{\beta} + \omega^\alpha_{\gamma} \wedge \omega^\gamma_{\beta} - \frac{1}{2} R^\alpha_{\beta \gamma \delta} \theta^\gamma \wedge \theta^\delta =: R^\alpha_{\beta}.
\]

We take as a basis of self-dual two-forms
\begin{align}
S^1 &= \frac{1}{2}(\theta^1 \wedge \theta^0 + \theta^3 \wedge \theta^2), \\
S^2 &= \theta^1 \wedge \theta^2, \\
S^3 &= \theta^3 \wedge \theta^0.
\end{align}
These two-forms satisfy the equations $(A, B = 1, 2, 3)$

\[ S^A = iS^A \]
\[ S^A \wedge S^B = ig^{AB} \]

(2.7)

The $SO(3)$ invariant metric

\[ g_{AB} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \] (2.8a)

and its inverse,

\[ g^{AB} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \] (2.8b)

are used to raise and lower the uppercase Latin self-dual, triad indices.

The self-dual components of the connection are

\[ \omega^{\alpha \beta} = \frac{1}{2} \left( \omega^{\alpha \beta} - i \frac{1}{2} \epsilon^{\alpha \beta \mu \nu} \omega_{\mu \nu} \right) \]

and are represented by $\Gamma^A$;

\[ \Gamma^1 := \frac{1}{2} (\omega^{01} + \omega^{23}), \quad \Gamma^2 := \omega^{21}, \quad \Gamma^3 := \omega^{03}. \] (2.9)

From these we obtain the self-dual components of the Riemann tensor as

\[ \frac{1}{2} R^A = d\Gamma^a + \eta^{ABC} \Gamma^B \wedge \Gamma^C. \] (2.10)

In a $3 + 1$ decomposition, we have

\[ \Gamma^A = A^A_i dx^i + B^A dt, \] (2.11)

and

\[ R^A_{ij} = 2 A^A_{[i,j]} + 2 \eta^{ABC} A^B_j A^C_i, \]
\[ R^A_{0i} = D_i B^A - A^A_{i,0}. \] (2.12)

The derivative operator $D$ acts on the index $A$ as

\[ D_i f^A := \partial_i f^A + 2 \eta^{ABC} B^B_i f^C. \]

The complex action introduced by Jacobson and Smolin [18] and by Samuel [19] is

\[ S = \int R^A \wedge S^B g_{AB}. \] (2.13)

With the above decomposition, the Lagrangian density formed using the self-dual curvature tensor takes the form

\[ \mathcal{L} = (A^A_{i,0} \Sigma^i_A + B^A D_i \Sigma^i_A + R^A_{ij} \Sigma^j_A -
\quad - N v^i (R^1_{ij} \Sigma^j + R^2_{ij} \Sigma^j_1)) + \mu_i (\Sigma^i_2 - \alpha v^i) + \rho (15). \] (2.14)

In the above we have introduced $v^i := \tilde{v}^i_2$ and

\[ \Sigma^i_1 := -\tilde{v}^i_1, \quad \Sigma^i_2 := -\alpha \tilde{v}^i_2, \quad \Sigma^i_3 := -\tilde{v}^i_3. \] (2.15)
The over (under) tilde indicates a density of weight plus (minus) one. Thus, \( v^i \) and \( \Sigma A^i \) are densities of weight one while the other variables are not densitized except as indicated by the tilde.

It is evident from this form of the Lagrangian that only variation of \( A^A_i \) and \( \Sigma A^i \) lead to dynamical equations. The remaining variables lead to equations on the hypersurface \( N \) weight one while the other variables are not densitized except as indicated by the tilde.

\[ \text{needed to make} \]

\[ t \text{ is complex,} \]

\[ N \text{ is arbitrary. But, when the reality conditions are imposed,} \]

\[ v^i = \Sigma A^i \text{ and} \]

\[ \Sigma \text{ will no longer be undetermined. This structure in the Lagrangian foreshadows the fact that the scalar constraint is second class.} \]

Therefore, we take the phase space to be defined by \( (A^A_i, \Sigma A^i) \). It is easy to see that they form canonically conjugate pairs with the Poisson brackets

\[ \{A^A_i(x), \Sigma B^j(x')\} = \delta^A_B \delta^3(x-x'). \] (2.16)

Then the Hamiltonian takes the form

\[ H = \int d^3x \{N \mathcal{H}_0 + N^i \mathcal{H}_i - B^A \mathcal{G}_A - \mu_i \mathcal{C}^i - \rho(\alpha)^2\} \] (2.17)

where

\[ \mathcal{H}_0 := v^i (R^1_{ij} \Sigma A^j + R^2_{ij} \Sigma 1^j) = 0, \]

\[ \mathcal{H}_i := -R^A_{ij} \Sigma A^j = 0, \]

\[ \mathcal{G}_A := D_i \Sigma A^i = 0, \]

\[ \mathcal{C}^i := \Sigma 2^i + \alpha v^i = 0. \] (2.18a)

are constraints which arise from varying the action with respect to \( N, N^i, B^A, \) and \( \mu_i \), respectively. Other constraints come from varying \( \rho, \alpha, \) and \( v^i \):

\[ \alpha = 0, \quad v^i \mu_i = 0, \] (2.18b)

and

\[ \phi_i := R^1_{ij} \Sigma 3^j + R^2_{ij} \Sigma 1^j = 0. \] (2.18c)

Propagation of the constraint \( \mathcal{C}^i = 0 \) leads to

\[ \chi^i := 2B^B D_2 \left( N\nu^i \Sigma A^j Q^{A}_B \right) - 2A^A_j N^i \Sigma 1^j - B^A \Sigma 1^i = 0, \] (2.18d)

\[ Q^{A}_B := \delta^A_B \delta^1_3 + \delta^A_1 \delta^2_B. \]

These are all the constraints, but propagation of the constraint \( \mathcal{G}_3 = 0 \) leads to a condition on \( \mu_i \) which is important in obtaining the otherwise missing equation:

\[ \mu_i \Sigma 1^i = R^1_{ij} \Sigma 3^j - R^2_{ij} \Sigma 3^i N^j. \] (2.19)

Note that \( v^i \phi_i \equiv \mathcal{H}_0 \) and \( \Sigma 3^i \phi_i \equiv \Sigma 3^i \mathcal{H}_i. \)

Thus, there are 14 constraints among the dynamical variables and three conditions on the Lagrange multipliers \( \alpha \) and \( \mu_i \). Five of the constraints, \( \mathcal{H}_i, \mathcal{G}_1, \) and \( \mathcal{G}_2 \) are first class while the remaining constraints, including \( \mathcal{H}_0 \) are second class. Three of the second class constraints are conditions on \( v^i \), so there are 16 conditions on the 18 phase space variables per hypersurface point.

The Hamiltonian equations of motion for the dynamical variables are

\[ A^A_i,0 = \delta^A_A D_i B^A + N^j R^1_{ij} - N v^j R^2_{ij}, \] (2.20a)
\[ A^{2}_{i,0} = \delta_{A \Delta i} B^A + N^j R_{ij}^2 - \mu_i, \quad (2.20b) \]
\[ A^{3}_{i,0} = \delta_{A \Delta i} B^A + N^j R_{ij}^3 - N \nu^j R_{ij}^1, \quad (2.20c) \]
\[ \Sigma^i_{1,0} = 2 \delta_{B \Delta i} D_j (N^{v \vert i j A^j B}) - 2 \delta_{B \Delta i} D_j (N^{v \vert i j B^j}) - 2 B^3 \Sigma_j^i, \quad (2.20d) \]
\[ \Sigma^i_{3,0} = 2 \delta_{B \Delta i} D_j (N^{v \vert i j A^j B}) - 2 \delta_{B \Delta i} D_j (N^{v \vert i j B^j}) + 2 B^1 \Sigma_j^i + B^2 \Sigma_1^i, \quad (2.20e) \]

The quantities we have introduced are complex. To recover the Einstein theory we must impose reality conditions at some point. These conditions are that
\[ \bar{\Sigma}_1^i = -\Sigma_1^i, \quad \nu^i = \bar{\Sigma}_3^i, \]
and that
\[ \Gamma^1 = 2 \omega^{23}, \quad \Gamma^2 = \omega^{31}, \quad \text{and} \quad \Gamma^3 = \omega^{02} \]
form the anti-self-dual components of the connection.
3. Analysis of the Equations.

We assume that space-time is asymptotically Minkowskian and that outside of a timelike cylinder, the coordinate \( t \) defines a congruence of hypersurfaces. When \( \alpha = 0 \), these are null surfaces which have the topology of null cones extending to null infinity. These null cones in turn are foliated by closed two-surfaces that each null generator is labeled by the usual angular coordinates \((\theta, \phi)\) of the unit sphere. Following Bondi and Sachs \[11,12\], we choose the coordinate \( r \), that is, the area of each two surface \( r = \) constant is \( 4\pi r^2 \). The coordinates \( x^i \) of the previous section are then \((r, \theta, \phi)\). For the boundary conditions, we assume that \( \Sigma A^i \) and \( v^i \), thus the metric as well, take their Minkowski space behavior in limit of null infinity. In as far as it is possible to do so without losing the gravitational radiation, the same is true for the connection. It turns out that \( A^2_i \) contains radiative terms in this limit. These conditions, including the radiative terms, are unaffected by super-translations.

The five first class constraints allow us to make a convenient choice for the triad densities and the coordinates. One coordinate condition has been used to fix \( r \) as the luminosity distance:

\[
-ir^2 \sin \theta \eta_{jk} v^j \Sigma_3^k = \nu^2.
\] (3.1a)

With the other two available coordinate conditions, we can set

\[
\Sigma_1^i = -ir^2 \sin \theta \delta_1^i.
\] (3.1b)

Then the four real functions in the null rotations generated by \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) allow us to fix \( \Sigma_3^i \) tangent to the surfaces \( r = \)constant, that is, \( \Sigma_3^1 = 0 \), and then to set \( A^1_1 = 0 \). The latter is equivalent to setting \( \epsilon = 0 \) in the Newman-Penrose formalism \[14\].

As pointed at the end of the previous section, the Dirac brackets of the conjugate pair \((A^3_2, \Sigma_3^2)\) are equal to their Poisson brackets. Therefore, these can be identified as the dynamically independent degrees of freedom for the gravitational field. However, they are not observables because they are not diffeomorphism invariant. Nonetheless, they represent the initial data which can be specified on an initial null surface. The remaining variables and parameters are then determined by any remaining gauge freedom, the constraints, and relations from the propagation equations.

With the coordinate and gauge conditions given above, the constraints and propagation equations have a natural order for their solution. Below we will give the equations in the order in which they can be solved. Which variable is to be solved for is generally clear. (The indices \( a, b, .. \) range over 2 and 3.)

\[
\mathcal{G}_1 = 0 \Rightarrow \Sigma_1^i, i + 2A^3_i \Sigma_3^i = 0
\] (3.2a)

\[
\mathcal{G}_2 = 0 \Rightarrow A^3_i \Sigma_1^i = 0
\] (3.2b)

\[
\mathcal{H}_1 = 0 \Rightarrow A^3_i, i \Sigma_3^i = 0
\] (3.2c)

\[
\mathcal{G}_3 = 0 \Rightarrow \Sigma_3^i, i - 2A^1_i \Sigma_3^i - A^2_i \Sigma_1^i = 0
\] (3.2d1)

\[
\mathcal{H}_a = 0 \Rightarrow \Sigma_1^i - A^1_a, a - A^2_a A^3_j + A^3_a A^2_j) \Sigma_1^j + [A^3_a, j - A^3_j, a + A^3_a A^1_j - A^3_j A^1_a] \Sigma_3^j = 0
\] (3.2d2)

\[
\phi_a = 0 \Rightarrow A^2_a, j \Sigma_1^j - \Sigma_3^c A^3_c A^2_b A^b_a + G_a = 0
\] (3.2e)

\[
Z^b_a := \delta^b_a - \frac{\Sigma_3^b A^3_a}{\Sigma_3^c A^3_c},
\]

\[
G_a := (A^1_a, b - A^1_{b, a}) \Sigma_3^b - (A^2_j, a + 2A^2_j A^1_a) \Sigma_1^j
\]
\[ \chi^a = 0 \Rightarrow (V^a \Sigma_1^a)_{,j} + \Sigma_3^a A_3^a Z^a_{b} V^b = 0, \]  
\[ N^a v^a \equiv V^a \]  
\[ \chi^1 = 0 \Rightarrow B^3 \Sigma_1^1 + (V^b \Sigma_1^1)_{,b} + 2 A^1_b V^b \Sigma_1^{1} - A^3_{b} N^b \Sigma_1^1 = 0 \]  
\[ \Sigma_1^a = 0 \Rightarrow (N^a \Sigma_1^1)_{,j} - (V^a \Sigma_3^b - V^b \Sigma_3^a)_{,b} + 2 A^2_{j} \Sigma_1^1 
\quad + 2 A^2_{b} \Sigma_3^{b} N^a + 2 B^3 \Sigma_3^{a} = 0 \]  
\[ \dot{\Sigma}_1^1 = 0 \Rightarrow 2 N^1 A^3_{b} \Sigma_3^{b} - A^2_{j} V^j \Sigma_1^{1} - (N^a \Sigma_1^1)_{,a} = 0 \]  
\[ \dot{A}_1^1 = 0 \Rightarrow B^1_{,a} + A^2_{2} B^3 - N^a (A^1_{a,1} + A^3_{a} A^2_{1}) 
\quad - V^a (A^2_{1,a} - A^2_{a,1} - 2 A^1_{a} A^1_{1}) = 0 \]  
\[ \dot{\Sigma}_3^a = 0 \Rightarrow B^2 \Sigma_1^1 - (N^1 \Sigma_3^{a})_{,a} + 2 N^1 A^1_{a} \Sigma_3^{a} 
\quad - N^a A^2_{a} \Sigma_1^{1} = 0 \]  

This completes the set of equations which can be solved either by integration along the null generators of the surface \( t = \text{constant} \) or algebraically on that surface. The equation for \( \dot{A}_1^3 = 0 \) yields another equation for \( B^3 \) which is then trivially satisfied.

The remaining equations all contain time derivatives.

\[ \dot{A}_1^1 = B^1_{,a} + A^2_{2} B^3 - A^3_{a} B^2 - N^j [A^1_{a,j} - A^1_{j,a} + A^2_{j} A^3_{a} - A^3_{j} A^2_{a}] 
\quad - V^a [A^2_{a,b} - A^2_{b,a} + A^1_{b} A^2_{a} - A^2_{b} A^1_{a}] \]  
\[ \dot{A}_1^2 = B^2_{j} + 2 A^1_{a} B^2 + 2 A^2_{j} B^1 
\quad + N^j [A^2_{i,j} - A^2_{j,i} - 2 A^1_{j} A^2_{i} + 2 A^2_{j} A^1_{i}] - \mu_{i} \]  
\[ \dot{A}_1^3 = B^3_{a} - 2 A^3_{a} B^1 + 2 A^1_{a} A^3_{a} + N^j [A^3_{a,j} - A^3_{j,a} + 2 A^1_{j} A^3_{a} - 2 A^3_{j} A^1_{a}] 
\quad - V^a [A^1_{b,a} - A^1_{a,b} + A^2_{a} A^3_{b} - A^3_{a} A^2_{b}] \]  
\[ \dot{\Sigma}_3^3 = - 2 \partial_j (N_{\Sigma_3}^j) + 4 A^1_{j} N_{\Sigma_3}^j + 2 A^2_{j} N_{\Sigma_1}^j 
\quad - 2 B^1_{a} \Sigma_3^a + 2 B^2 \Sigma_4^a \]  

The fact that the Poisson brackets of the constraints with the Hamiltonian vanish modulo the constraints themselves tells us that the propagation of the variables is determined by the propagation of \( (A^2_{a}, \Sigma_3^a) \) alone. That is not quite true because, as in the case of Bondi-Sachs [11,12] and Newman-Penrose [13], integration along the null generators introduces arbitrary functions of \( (t, \theta, \phi) \). We find that propagation of \( A^1_{a} \) and \( A^2_{a} \) lead to the conservation equations for angular momentum and mass. The time derivative of \( A^3_{a} \) defines \( \mu_{i} \Sigma_4^i \) which when equated to (2.19) yields an identically satisfied field equation. On the other hand, the time derivative of \( A^2_{a} \) defines \( \mu_{i} \Sigma_3^i \) as the null component of the conformal tensor. This is in complete agreement with the previous work.

The integration along the null generators can be carried out from the time-like cylinder to null infinity without an expansion in \( 1/r \) in a manner similar to that of Tamburino and Winicour [21], but this formal result does not exhibit the presence or lack of logarithmic behavior. Therefore, in the next section we shall set up the calculation of the asymptotic behavior and then list the results.
4. The Asymptotic Solution.

In this section we shall first solve the complex equations for the triad and the self-dual connection. Then we shall apply the reality conditions and show the explicit relationship of our results to those of Sachs [12]. We shall look for solutions for which the triad differs from its Minkowski space value by factors with an expansion in \(1/r\). Although there are logarithmic terms consistent with the assumption of asymptotically Minkowskian behavior, in this section we shall choose the powers of \(1/r\) to avoid their occurrence. The appearance of terms in \((\ln r)^m/r^n\) will be discussed in the following section. The Minkowski space solutions are given in Appendix 1.

The Solution.

As noted previously, the initial data on the surface \(t = 0\) will be given by \((A^3_2, \Sigma^2_3)\). By our coordinate conditions we have

\[ \Sigma^i_1 = -ir^2 \sin \theta \delta^i_1. \] (4.1)

Then, \(\mathcal{G}_1\) and \(\mathcal{G}_2\), Eqs.(3.2a,b) tell us that

\[ \Sigma^a_3 A^a_3 = ir \sin \theta, \] (4.2a)

\[ A^a_3 = 0. \] (4.2b)

To proceed, we write

\[ A^3_2 = -\frac{1}{\sqrt{2}} \left\{ 1 + \frac{3A}{r^3} + \cdots \right\} \]

\[ \Sigma^2_3 = -\frac{ir \sin \theta}{\sqrt{2}} \left( 1 + \frac{1}{r} + \frac{3}{r^3} + \cdots \right) \] (4.3)

and find from (4.2a) and \(\mathcal{H}_1\) that

\[ A^a_3 = -a_a + \frac{3A}{r^3} b_a \]

\[ \Sigma^a_3 = -ir \sin \theta \left\{ s^a + \frac{\Sigma}{r} t^a + \frac{3}{r^3} t^a \right\}. \] (4.4)

The forms \((a_a, b_a)\) and the vectors \((s^a, t^a)\) are the eigen-forms and eigenvectors of the flat space part of the matrix \(Z^a_b\) as defined in Appendix 2. They satisfy the following algebraic relations:

\[ a_a s^a = b_a t^a = 1, \quad a_a t^a = b_a s^a = 0. \]

We assign a spin-weight of -1 to \(b_a\) and \(s^a\) and a spin-weight of +1 to \(a_a\) and \(t^a\) [22-24]. This allows us to express the results in terms of spin-weighted quantities which act as a check on the calculations and simplifies their appearance through the use of the edth operator which is also defined in the Appendix. In (4.4) and below, we exhibit only the terms of the solution we have calculated and omit the dots indicating further terms.

The radial integrations are not unique, but lead to the introduction of a number of \(r\)-independent functions, \(C^4_s = C^4_a s^a, C^4_t = C^4_a t^a, \mathcal{M}, \mathcal{A}, \mathcal{V}, \) and \(\mathcal{V}\). The first four of these functions are related to the angular momentum, mass, and radiation. The remaining two are fixed by the reality conditions. Below we give the results of these radial integrations in the order given in Eqs. (3.2):

\[ A^4_a = \frac{\cot \theta}{2\sqrt{2}} - \frac{1}{r^2} \gamma^1 \Sigma + \frac{1}{r^2} C^1 s - \frac{1}{r^3} \phi \Sigma a + \]

\[ \left[ \frac{-\cot \theta}{2\sqrt{2}} + \frac{1}{r^2} C^1 t - \frac{1}{r^3} \phi \Sigma \right] b_a \] (4.5a)

\[ A^2_1 = \frac{1}{r^2} \left[ \phi \Sigma - \frac{1}{r} C^1 s + \frac{1}{r^2} \left( \phi \Sigma - C^1 t \right) \right] \] (4.5b)
\[ A^2_a = [A + \frac{1}{r} \bar{\partial} \dot{\Psi} - \frac{1}{2r^2}(\mathcal{M} \bar{\partial} \Sigma + \bar{\partial} C^A) + 2(\bar{\partial} \dot{\Psi})^2 + \bar{\partial} \dot{\Psi}^2] a_a + \left[ - \frac{1}{2} + \frac{1}{r^2} \bar{\partial} C^A \right] b_a \] (4.5c)

\[ V^a = - \frac{1}{r} \left[ \frac{1}{r} \mathcal{V} + \frac{1}{2r^3} (\mathcal{V})^2 + (\mathcal{V} + \frac{1}{2r^2} \mathcal{V} \bar{\partial} \Sigma) a_a \right] \] (4.5d)

\[ B^3 = O \left( \frac{1}{r^3} \right) \] (4.5e)

\[ N^a = - \frac{1}{r^2} \left[ \bar{\partial} \mathcal{V} s^a + \mathcal{V} \bar{\partial} \bar{\partial} t^a \right] \] (4.5f)

\[ N^1 = - \frac{1}{2} \left[ \mathcal{V} - \frac{1}{r} \left( \partial \mathcal{V} - \frac{1}{2} \mathcal{V} \mathcal{V} + \bar{\partial} \mathcal{V} \bar{\partial} \Sigma + 2 \mathcal{V} \mathcal{A} \right) \right] \] (4.5g)

\[ B^1 = \frac{1}{2r^2} \mathcal{V} (\bar{\partial} \Sigma - \mathcal{M}) \] (4.5h)

\[ B^2 = \frac{1}{r^2} \bar{\partial} \left( (\mathcal{V} \mathcal{M}) + \frac{1}{2} \bar{\partial} \Sigma \mathcal{M} - \mathcal{V} \bar{\partial} \mathcal{A} \right) \] (4.5i)

The integration for \( N^a \) can introduce a function independent of \( r \). However, such a term can be removed by a coordinate transformation [11, 12]. Furthermore, \( N^1 \) will grow like \( r \) at null infinity unless we require that

\[ \mathcal{V} \mathcal{V} = \bar{\partial} \mathcal{V} = 0. \]

The above solutions have been written with this requirement so that at most, \( \mathcal{V} \) can be a function of \( t \) alone.

The fact that the constraints form a closed system shows that the propagation of these equations is consistent. This means that the propagation equations will determine the evolution of the arbitrary functions we have introduced, but there will be no further conditions. This argument is equivalent to the use of the Bianchi identities by Bondi and Sachs [11, 12] to obtain a similar result. \( A, \Sigma, \) and \( \Sigma \) are part of our initial data. In addition we have \( C^A, \mathcal{M}, \mathcal{A}, \mathcal{V}, \) and \( \mathcal{V} \). These are exactly the same quantities we would have had to introduce if we were to integrate the equations without the asymptotic expansion.

From the evolution equation (3.3a), we obtain the following relations:

\[ \mathcal{V} \mathcal{A} = i \Sigma \] (4.6a)

\[ \dot{C}_s = - \bar{\partial}^2 B^1 + 2B^2 + \mathcal{V} \left[ - \dot{\mathcal{V}} (\bar{\partial} \dot{\Sigma} - \frac{1}{2} \Sigma) + \bar{\partial} \mathcal{M} \right] \] (4.6b)

\[ \dot{C}_t = - \bar{\partial}^2 B^1 + \mathcal{V} \bar{\partial} \mathcal{A} \] (4.6c)

These latter equations are identified with the change in dipole aspect and, hence, are connected with the conservation of angular momentum. The equation for \( \dot{A}^2_a \) defines \( \Sigma \) which leads to an identity with Eq. (2.18). \( \mu_i \Sigma^a_i \) is the null part of the conformal tensor, \( \Psi_3 \) in the Newman-Penrose notation [13]. Thus, only the equation for \( \dot{A}^2_a V^a \) is dynamical. Remembering that \( \mu_i v^i = 0 \), the relevant relation can be written as

\[ - \frac{\partial}{\partial t} (\mathcal{V} \mathcal{M} - \mathcal{V} \dot{\Sigma}) - \mathcal{V} \dot{\Sigma} \] (4.7)

All that remains now are the equations to propagate \( \dot{A}^3_a \) and \( \Sigma^a_i \). This completes the solution of the complex equations. The further discussion of (4.7) will be delayed until after the reality conditions have been applied.

The Reality Conditions.
It is only necessary to apply the reality conditions to the tetrad because through the solution of the field equations the connection is expressed in terms of the tetrad. We shall see that in applying the reality conditions, the arbitrary functions in the connection will be expressed in terms of those in the triad. The reality conditions on the tetrad are

\[ N = \tilde{N}, \quad N^i = \tilde{N}^i, \quad -i\nu = i\tilde{\nu} > 0, \]

\[ \Sigma_1^i = -\tilde{\Sigma}_1^i, \quad \nu^i = \tilde{\nu}^i. \]  

From (3.1) we find that \( \Sigma_1^1 \) already satisfies the reality condition. Writing \( \nu^a = \bar{\Sigma}_3 \), we have

\[ \bar{\Sigma}_3 = -ir \sin \theta \left( s^a + \frac{1}{r} \Sigma s^a + \frac{1}{r^3} \bar{\Sigma} s^a \right), \]

\[ \nu^a = ir \sin \theta \left( t^a + \frac{1}{r} \Sigma s^a + \frac{1}{r^3} \bar{\Sigma} s^a \right). \]  

From this we find that

\[ \nu^2 = \eta_{ijk} \Sigma_1^i \nu^j \Sigma_3^k - r^2 \sin^2 \theta \left[ 1 - \frac{1}{r^2} \Sigma \bar{\Sigma} \right]. \]

The requirement \( -i\nu > 0 \) gives us

\[ \nu = i\nu^2 \sin \theta \sqrt{1 - \frac{1}{r^2} \Sigma \bar{\Sigma}}. \]  

Now from \( \bar{\Sigma} v^a = V^a \) and the boundary condition that \( N \) should be 1 at null infinity, we find

\[ \frac{N}{r} \left[ 1 + \frac{1}{2r^2} \Sigma \bar{\Sigma} \right] \left[ t^a + \frac{1}{r} \Sigma s^a + \frac{1}{r^3} \bar{\Sigma} s^a \right] \]

\[ = -\frac{1}{r} \left( \frac{\partial \nu}{\partial r} + \frac{1}{2r^2} \Sigma \bar{\Sigma} t^a \right) + \left( \frac{1}{r} \bar{\nu} + \frac{1}{2r^3} \partial \Sigma \bar{\Sigma} \right) s^a, \]

which implies \( N = 1 \) and

\[ \partial \nu = -1, \quad \bar{\partial} \nu = -\bar{\Sigma}, \quad \frac{1}{2} A = \bar{\Sigma} + \frac{1}{2} \Sigma \bar{\Sigma} \bar{\Sigma} \bar{\Sigma}. \]  

Comparison with the Bondi-Sachs form of the metric [12,22]

\[ ds^2 = \frac{V e^{2\nu}}{r} du^2 - 2 e^{2\nu} dudr - r^2 h_{ij} (dx^i - U^i du)(dx^j - U^j du), \]

give us

\[ \nu = -ie^{2
\nu} \sin \theta, \quad N = 1, \quad N^1 = \frac{V}{r}, \]

and for the principal spin coefficients

\[ \rho = -\frac{1}{\nu} A^3_a \Sigma^a = -\frac{1}{r} \frac{\Sigma \bar{\Sigma}}{r^3}, \]

\[ \sigma = \frac{1}{\nu} A^3_a v^a = -\frac{\Sigma \bar{\Sigma}}{r^2} + \frac{1}{r^3} (A - \frac{1}{2} \Sigma \bar{\Sigma} \bar{\Sigma} - \bar{\Sigma}), \]

\[ \mu = \frac{1}{\nu} A^2_a v^a = -\frac{1}{2r} + \frac{1}{r^2} (M - \frac{1}{2} \Sigma \bar{\Sigma}), \]

\[ \bar{\lambda} = -\frac{1}{\nu} A^2_a \Sigma^a = \frac{\Sigma \bar{\Sigma}}{r} + \frac{1}{r^2} (\bar{\Sigma} \bar{\Sigma} \bar{\Sigma} - \frac{1}{2} \Sigma). \]  

This shows that the Newman-Penrose asymptotic shear \( \sigma^0 = -\Sigma \) and

\[ -M + i \Sigma \bar{\Sigma} = \frac{1}{2} (\bar{\Sigma} \nu_2 + \nu_2 - \partial^2 \bar{\Sigma} + \partial^2 \nu_2 + \Sigma \bar{\Sigma} + \Sigma \bar{\Sigma} + i \Sigma \bar{\Sigma}). \]  

11
Note that the right hand side of (4.13) is real and comparison with (4.7) shows that it is just the negative of the Bondi-Sachs mass aspect.

It is perhaps worthwhile to exhibit the real mass loss explicitly by writing the integral over a sphere at null infinity as (\(dS\) is the area element of the unit sphere)

\[
\frac{dM}{dt} = -\frac{1}{16\pi} \oint \left[ \vec{\Sigma} \cdot \vec{\Sigma} \right] dS,
\]

(4.14)

\[
M := \frac{1}{16\pi} \oint \left[ \mathcal{M} + \mathcal{M} - \frac{1}{2} \Sigma \Sigma - \frac{1}{2} \Sigma \Sigma \right] dS
\]

(4.15)

This definition of the mass aspect agrees with that of Bondi-Sachs and Newman-Penrose. Thus, (4.14) describes the mass loss from gravitational radiation. It is interesting to note that addition of the surface integral

\[
\oint V^a A^2 \Sigma^1 d\theta d\phi
\]

(4.16)

is needed to assure the differentiability of the Hamiltonian. Together with the reality conditions, (4.16) defines the mass \(M\).

5. Logarithmic Behavior.

In this section we shall consider the terms in \(1/r\) which were omitted in the definition of \((A^3_a, \Sigma^a)\) in the previous section. These terms lead to logarithmic behavior which comes in below the leading terms previously found. Chrusciel, MacCallum, and Singleton [15] have studied the logarithmic behavior within the Bondi-Sachs formalism [11, 12]. They introduce polyhomogeneous functions in the metric and then see what powers of \((\ln r)^m/r^n\) lead to consistency in the solution of the constraint and propagation equations. Here we follow a somewhat different approach. We start with a power series in \(1/r\) in \((A^3_a, \Sigma^a)\) and examine the logarithmic terms which arise in the remaining terms. We then look at the propagation equations to see what logarithmic terms are introduced into \((A^3_a, \Sigma^a)\). Consistency means that these new terms should not interfere with what has been previously required. That is, a new calculation with these new terms should reproduce the already found behavior and add lower order logarithmic behavior. Apart from this difference in approach, our results are essentially in agreement with those of Chrusciel, MacCallum, and Singleton, but perhaps they are more perspicuous. Our approach is closer in spirit to that of Winicour [25]. However, in our results the coefficient of \(1/r\) in \(A^3_a\) is found to be zero, whereas that does not appear to be the case in their analysis. For possible clarification, the details of the calculation leading to that result is given in Appendix 3. Below we sketch our final results and then discuss the conservation equations and the conformal tensor.

From \(G_1, G_2,\) and \(H_1\) we find

\[
A^3_a = -a_a + \frac{2A}{r^2} b_a + \frac{3A}{r^2},
\]

\[
\Sigma^a = -ir \sin \theta \left[ s^a + \frac{3\Sigma}{r} t^a + \frac{2\Sigma}{r^2} t^a + \frac{(3\Sigma - \Sigma^2 A)}{r^3} \right].
\]

(5.1)

The later integration for \(V^a\) requires that a possible term in \(1/r\) in \(A^3_a\) be set equal to zero. Then the remaining integrations along the null generators give

\[
A^1_a = \left[ \frac{\cot \theta}{2\sqrt{2}} - \frac{1}{r} \theta^\Sigma + \theta^\Sigma \frac{\ln r}{r^2} + \frac{1}{r^2} C^1_s \right] a_a +
\]

\[
\left[ -\frac{\cot \theta}{2\sqrt{2}} + \frac{\theta A \ln r}{r^2} + \frac{1}{r^2} C^1_t \right] b_a
\]

(5.2a)

\[
A^2 = \frac{1}{r^2} \left[ \theta^\Sigma + 2\theta^\Sigma \frac{\ln r}{r} \right]
\]

(5.2b)
\[ A^2_a = \left[ -\frac{1}{2} \Sigma + \frac{1}{r} \bar{\theta}^2 \Sigma + \frac{1}{r} \bar{\Sigma}^2 \Sigma \ln \frac{r}{r^2} + \frac{1}{r^2} A \right] b_a \\
+ \left[ -\frac{1}{2} + \frac{1}{r} \mathcal{M} - \frac{\ln r}{r^2} (\bar{\theta}^2 A - \bar{\theta}^2 \Sigma) \right] b_a \] 

(5.2c)

To the order considered, the solution for \( V^a \) is the same as in the previous section. However, as noted above, it imposes the condition that \( \frac{1}{2} A = 0 \). This result is contained in the above expressions.

In the remaining variables, the logarithms appear in the order below the leading order given in the previous section. Except for \( N^1 \), they have no important consequences. For \( N^1 \), we find a term in \( \ln \frac{r}{r^2} \). It then follows from the propagation equations that \( A^2_a \) develops a term in \( \ln \frac{r}{r^3} \) and \( \Sigma^3_a \) a term in \( \ln \frac{r}{r^2} \) which follows the term in \( \bar{\Sigma} \). As a result, the conservation equation for mass is unchanged. This means that what appears as gravitational radiation at null infinity is unaffected by the inclusion of this logarithmic behavior. On the other hand, because \( A^4_a \) has a term in \( \ln \frac{r}{r^2} \), the conservation equation for angular momentum will be changed. This represents another problem for angular momentum which is yet to be understood adequately. This is exhibited below by the failure of the conformal tensor to peel:

\[ \Psi_4 = R^{2 \rho \sigma} e^0 e^3 \rightarrow \frac{1}{2} \Sigma \frac{1}{r} \]

\[ \Psi_3 = R^{1 \rho \sigma} e^3 e^0 \rightarrow -\frac{1}{2} \bar{\Sigma} \frac{1}{r^2} \]

\[ \Psi_2 = R^{2 \rho \sigma} e^1 e^2 \rightarrow \frac{\bar{\theta}^2 \Sigma - \mathcal{M}}{r^3} \]

\[ \Psi_1 = R^{1 \rho \sigma} e^1 e^2 \rightarrow 2 \bar{\theta}^2 A \frac{\ln r}{r^4} + \frac{2C_1 - \bar{\theta}^2 A}{r^4} \]

\[ \Psi_0 = R^{1 \rho \sigma} e^1 e^2 \rightarrow -\frac{2A}{r^4} - 3\alpha \frac{\ln r}{r^5} - 3A - \alpha \]

(5.3)

In the above, the \( \Psi_n \) are the components of the conformal tensor, \( \frac{1}{2} \Sigma = -\bar{\sigma}^0 \) of Newman-Penrose, and \( \alpha \) comes from the lowest order logarithm in \( A^3_a \).

The difference between our results and those of Chrusciel, MacCallum, and Singleton [15] comes from the different question which is asked. We ask for the logarithmic behavior which is forced on us by adding the additional terms in \( 1/r \) in both \( A^3_a \) and \( \Sigma^3_a \) while they ask for the most general logarithmic behavior they can introduce into the metric which is self-consistent. As a result, they find that the metric can have terms in \( \frac{(\ln r)^N}{r} \) whose coefficients are independent of \( u \). However, the main physical conclusions are the same. The definition of Bondi mass and the radiation of gravitational energy remains unchanged from the results without logarithmic behavior. Furthermore, considerations about angular momentum are affected by these new terms. On the other hand, the current results are in complete agreement with those of Novak and Goldberg [16, 17] who showed that the existence of null infinity was consistent with the logarithmic behavior found here.
6. Conclusions.

One of the main points of this paper has been to see whether the new variables introduced by Abhay Ashtekar [1-4] are useful in classical problems. Indeed, use of the self-dual connection and the densitized triad \((A^a_a, \Sigma_a^a)\) in the canonical formalism leads to a set of equations which are intermediate between the Bondi-Sachs [11, 12] and Newman-Penrose [13] equations in the vicinity of future null infinity, \(I^+\). In Bondi-Sachs, the calculation begins with specification of the metric on a two-surface foliation of an outgoing asymptotic null cone; in Newman-Penrose, with the specification of a component of the conformal tensor, \(\Psi_0 \Sigma^2 C_{ijkm} \Sigma^i \Sigma^j \Sigma^k \Sigma^m\); and in the present calculation we give as our initial data \((A^3_2, \Sigma^2_3)\) which is only half of the two-surface metric and that part of the connection which is related to \(\bar{\Psi}_0\). We have not evaluated whether our calculation is the most efficient. Both Newman-Penrose and we work with first order equations. Eventually they make use of the “metric” equations to determine the tetrad whereas they are part of our canonical equations, but we must then compute the conformal tensor.

The one advantage of the present approach is that it is derived from a Lagrangian and a canonical formalism which may yet be useful for quantum gravity. It also makes the study of the logarithmic behavior in the vicinity of null infinity somewhat more imperative and somewhat easier. Apart from the point mentioned earlier and elaborated on in Appendix 3, the important conclusions we have arrived at are not significantly different from those of Chrusciel et al [15], if less complete. We hope that this difference can be resolved in the near future. As noted in the previous section, we ask different questions. The important results here and there are that one can have an asymptotically Minkowskian metric, with a future null infinity, and a mass and radiation of gravitational energy which is well defined. That is, the logarithmic terms fall off faster than those terms which define the mass and radiation of gravitational energy. The same is not true for angular momentum. But, that concept is not sufficiently clear even in the absence of the logarithmic terms, although, in that case, there is an expression for angular momentum which transforms properly under the super translations as well as the Lorentz transformations [25].

There is, in the present approach, the additional need to apply the reality conditions. However, note that there are no propagation equations for \(v^i\). Therefore, once \(\Sigma^i_4\) is identified with \(\bar{\Sigma}^i_4\), it follows that its propagation is also specified. Given that \(\Sigma^i_4\) is real and independent of time, the metric will propagate as real. That guarantees that all the reality conditions will be fulfilled.

Our original hope was that on the null surface we could carry out a reduced phase space quantization of general relativity. While the identification of the dynamical degrees of freedom for the gravitational field is easy, either one has to express all the remaining variables in terms of these degrees of freedom or construct the Dirac brackets. At this time, it appears to be very difficult to carry out that task.

Acknowledgement.

The authors wish to express their appreciation to David Robinson for his readings of the several drafts of this work. They also thank Piotr Chrusciel for a critical reading of the paper and an extended exchange in an attempt to understand the one apparent difference in our results. CS thanks the Department of Physics and the Relativity Group of Syracuse University for its hospitality during the period when this work was begun. JNG thanks the faculty and staff of the Department of Mathematics, Kings College London for their hospitality during June to December 1993 while part of this work was carried out. He also thanks John Madore for discussions and the physics department for its hospitality during a brief stay at the Université de Paris-sud. This work was supported in part by the NSF under Grant No. PHY 9005790 and by SERC under Grant No. GR/H456472.
Appendix 1. Minkowski Space Tetrad and Connection.

In Minkowski space the metric takes the form

$$ds^2 = dt (dt + 2dr) - r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$

so that the four one-forms and tetrad vectors are

$$\begin{align*}
\theta^0 &= dt, & e_0 &= \partial_t - \frac{1}{2} \partial_r, \\
\theta^1 &= (dr + \frac{1}{2} dt), & e_1 &= \partial_r, \\
\theta^2 &= \frac{1}{\sqrt{2}} r (d\theta - i \sin \theta \, d\phi), & e_2 &= \frac{1}{\sqrt{2}} (\partial_\theta + \frac{i}{\sin \theta} \partial_\phi) \\
\theta^3 &= \frac{1}{\sqrt{2}} r (d\theta + i \sin \theta \, d\phi), & e_3 &= \frac{1}{\sqrt{2}} (\partial_\theta - \frac{i}{\sin \theta} \partial_\phi)
\end{align*}$$

Therefore, the triad densities and the self-dual connection one-forms are

$$\begin{align*}
\Sigma_1^i &= -ir^2 \sin \theta \, \delta^i_1; & A_1^i &= \frac{1}{2} \cos \theta \delta^i_3; \\
\Sigma_3^a &= -ir \sin \theta \, s^a; & A_3^a &= -a_a;
\end{align*}$$

$$s^a = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ -i/sin \theta \end{array} \right); \quad t^a = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 1 \\ i/sin \theta \end{array} \right);$$

$$\begin{align*}
a_a &= \frac{1}{\sqrt{2}} (1, \, i \sin \theta); \\
b_a &= \frac{1}{\sqrt{2}} (1, \, -i \sin \theta).
\end{align*}$$

The duality relations between $s^a$, $t^a$ and $a_a, b_a$, respectively, are unchanged by a phase change $e^{-i\phi} s^a, e^{i\phi} t^a$ and $e^{i\phi} a_a, e^{-i\phi} b_a$.

We assign a spin-weight of $-s$ to those quantities which transform as $e^{-i\phi}$ and a spin-weight of $+s$ to those which transform as $e^{i\phi}$ under this possible change of phase. Keeping track of the spin-weight is a help in controlling the calculations [22-24].

This also leads to the introduction of spin-weight raising and lowering operators $\vartheta$ and $\bar{\vartheta}$. If $\eta$ has spin-weight $s$, then

$$\vartheta \eta := -\frac{1}{\sqrt{2}} \sin^2 \theta \left( \frac{\partial}{\partial \theta} + i \frac{\partial}{\sin \theta \, \partial \phi} \right) (\sin^{-s} \theta \, \eta)$$

and

$$\bar{\vartheta} \eta := -\frac{1}{\sqrt{2}} \sin^{-s} \theta \left( \frac{\partial}{\partial \theta} + i \frac{\partial}{\sin \theta \, \partial \phi} \right) (\sin^s \theta \, \eta).$$

The action of these operators on the spin-weighted spherical harmonics can be found in Appendix A of [22] and further details can be found in [23,26].

Appendix 2. Properties of $Z^a_b$

In Eq. (3.2e), we defined $(a,b = 2,3)$

$$Z^a_b := \delta^a_b - f^a_b,$$  

$$f^a_b := \frac{\Sigma^a c A^3_b}{\Sigma^c A^3_c}$$  

(A2.1b)

It is easy to see that the determinant of $f^a_b$ is zero and there are two eigenvalues $(1,0)$ with the respective eigenvectors $\Sigma^a$ and $\epsilon^{ab} A_b^3$ and eigenforms $A^3_a$ and $\epsilon_{ab} \Sigma^b$ where $\epsilon_{ab}$ is the two-dimensional Levi-Civita tensor. The same is true of $Z^a_b$ with the eigenvalues $(0,1)$. Therefore, $Z^a_b$ is a projection operator. In the
1/r expansion of the variables, the zeroth order term will select eigenforms and eigenvectors of spinweight −1 and +1, respectively. More specifically, we have

\[ 0Z = \frac{1}{2} \begin{pmatrix} 1 & -i \sin \theta \\ i / \sin \theta & 1 \end{pmatrix} \]  

(A2.2)

so that

\[ 0Z^a_b s^b = a_a \]  

(A2.3)

The vectors and covectors \( s_t, a, b \) specify \( \Sigma \) and tetrad conditions which are described at the beginning of Section 3. We could proceed as in Section 4 specifying \( \Sigma \) and \( A \), but equivalently we can write

\[ \Sigma^a = -i r \sin \theta \{ s^a + \frac{1}{r} \Sigma^a + \frac{1}{r^2} A^a \} \]

(A3.1)

\[ A^a = -\{ a_a + \frac{1}{r} A_a + \frac{1}{r^2} A_a \} \]

The restrictions which are a result of the equations \( G_1, G_2, \) and \( H_1 \) lead to the same result. \( G_2 \) tells us that \( A^3 = 0 \), then \( G_1 \) and \( H_1 \) give us

\[ \Sigma^a A^a = i r \sin \theta \]

(A3.2)

Substituting (A3.1) into these equations yields the result

\[ \Sigma^a = -i r \sin \theta \{ s^a + \frac{1}{r} \Sigma^a + \frac{1}{r^2} (2 \Sigma t^a - \frac{1}{2} A^a \Sigma s^a) \} \]

(A3.3)

\[ A^a = -\{ a_a + \frac{1}{r} A a_a + \frac{1}{r^2} (2 A a_a - \frac{1}{2} A A a_a) \} \]

From this point on, one just puts this result into the succeeding equations and proceeds as before looking for the solutions. Now some logarithmic terms appear, but below the first couple of terms. However, the problem arises in the equations for \( V^a, \chi^a = 0 \), which do not depend on these solutions. We have \( (V^a := \Sigma^a) \):

\[ (V^a \Sigma^a \Sigma^a, 1) - \frac{1}{r} Z^a_b (V^b \Sigma^a) = 0. \]  

(A3.4)

\[ Z^a_b := \delta^a_b - \frac{\Sigma^a A^b}{\Sigma^a \Sigma^c A^c} \]

\[ := t^a b_a - \frac{1}{r} \{ s^a b_a A + t^a b_a \Sigma \} - \frac{1}{r^2} \{ (s^a b_a + t^a b_a) A^a \Sigma + \Sigma \} \]

Write

\[ V^a \Sigma^a = -i r \sin \theta \{ \psi^a + \frac{\psi^a}{r} + \frac{2 \psi^a}{r^2} \} \]

(A3.5)

and substitute into (A3.4). We obtain

\[ (\psi^a - \frac{2 \psi^a}{r^2} = [t^a b_a - \frac{1}{r} (s^a b_a A + t^a b_a \Sigma) - \frac{1}{r^2} (s^a b_a + t^a b_a) A^a \Sigma + \Sigma] \]

(A3.6)
which gives the relations

\[ 0 \nu^a = 0 \nu^b b_b t^a, \quad (\nu^a + 0 \nu^b a_b \Sigma) t^a - 0 \nu^b b_b \Sigma s^a = 0. \]

Thus,

\[ 0 \nu^a a_a = b^b = (0 \nu^a b_a)^1 A = 0. \quad (3.7) \]

The other expansion terms are given in terms of known quantities except for \( 0 \nu^a b_a \) and \( 1 \nu^a a_a \) which remains arbitrary. In order that \( v^a = \Sigma^a_3 \) when the reality conditions are imposed, we necessarily choose \( 1 A = 0 \) as noted in Section 5. Then we have

\[ V^a = - \frac{1}{r} [0 \nu + \frac{1}{2r^2} b^b \Sigma] + (\frac{1}{r} \nu + \frac{1}{2r^3} 0 \nu^3 A), \]

\[ A^a_3 = - a_a + \frac{2A}{r^2} b_a + \frac{3A}{r^3} b_a. \quad (A3.8) \]
References.
1. Ashtekar, A. (1986) *Phys. Rev. Lett.* **57**, 2244.
2. Ashtekar, A. (1987) *Phys. Rev. D* **36**, 1587.
3. Ashtekar, A. (with invited contributions) (1988) *New Perspectives in Canonical Gravity* (Naples, Bibliopolis).
4. Ashtekar, A. (1991) *Lectures on Non-perturbative Canonical Gravity* (Notes prepared in collaboration with R. Tate) (Singapore: World Scientific).
5. Goldberg, J.N., Robinson, D.C., and Soteriou, C. (1992) *Class. Quantum Grav.* **9** 1309.
6. Goldberg, J.N. (1985) *Found. Phys.* **15**, 439.
7. Torre, C. (1986) *Class. Quantum Grav.* **3**, 773.
8. Dirac, P.A.M. (1958) *Proc. R. Soc.* **246**, 333.
9. Dirac, P.A.M. (1959) *Phys. Rev.* **114**, 924.
10. Bergmann, P.G. and Komar, A. (1960) *Phys. Rev. Lett.* **4**, 432.
11. Bondi, H., Van Der Burg, M., and Metzner, A. (1962) *Proc. R. Soc. A* **269**, 21.
12. Sachs, R. (1962) *Proc. R. Soc. A* **270**, 103.
13. Newman, E.T. and Penrose, R. (1962) *J. Math. Phys.* **3**, 566.
14. Newman, E.T. and Unti, T. (1962) *J. Math. Phys.* **3**, 891.
15. Chruściel, P.T., MacCallum, M.A.H., and Singleton, D.B. (1994) *Phil. Trans. of the R. Soc. London*, to appear.
16. S. Novak and J.N. Goldberg (1981) *Gen. Rel. Grav.* **13**, 79.
17. S. Novak and J.N. Goldberg (1982) *Gen. Rel. Grav.* **14**, 655.
18. Jacobson, T. and Smolin, L (1987) *Phys. Lett.* **196 B**, 39.
19. Samuel, J. (1987) *Pramana J. Phys.* **28**, L429.
20. Sachs, R. (1962) *J. Math. Phys.* **3**, 908.
21. L. Tamburino and J. Winicour, (1966) *Phys. Rev.* **150**, 1039.
22. Glass, E.N. and Goldberg, J.N. (1970) *J. Math. Phys.* **11**, 3400.
23. Penrose, R. and Rindler, W. (1984) *Spinors and Space-time* (Cambridge University Press, Cambridge), vol. 1.
24. Winicour, J. (1985) *Found. Phys.* **15**, 605.
25. Dray, T. (1985) *Class. Quantum Grav.* **2**, L7.
26. Newman, E.T. and Penrose, R. (1966) *J. Math. Phys.* **7**, 863.