A GEOMETRIC MODEL FOR THE DERIVED CATEGORY OF GENTLE ALGEBRAS

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Abstract. In this paper we construct a geometric model for the bounded derived category of a gentle algebra. The construction is based on the ribbon graph associated to a gentle algebra in [54]. This ribbon graph gives rise to an oriented surface with boundary and marked points in the boundary. We show that the homotopy classes of curves connecting marked points and of closed curves are in bijection with the isomorphism classes of indecomposable objects in the bounded derived category of the gentle algebra up to the action of the shift functor. Intersections of curves correspond to morphisms and resolving the crossings of curves gives rise to mapping cones. The Auslander-Reiten translate corresponds to rotating endpoints of curves along the boundary. Furthermore, we show that the surface encodes the derived invariant of Avella-Alaminos and Geiss.

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Introduction

Giving a description of the (bounded) derived category of a finite dimensional algebra is a difficult undertaking in general. This problem is best approached by restricting to special classes of examples; for instance, in [11] the notion of derived-tameness of a bounded derived category of finite dimensional modules over a finite dimensional algebra was introduced and in [14] it was shown that for a gentle algebra its bounded derived category is derived-tame (a result which also follows from [53] for gentle algebras of finite global dimension).

Gentle algebras first appeared in the form of iterated tilted algebras of type $A$ [6, 7] and type $\tilde{A}$ [8]. It has transpired since that they naturally appear in many different contexts, these include dimer models [20, 21], enveloping algebras of Lie algebras [46], and cluster theory, where they appear as $m$-cluster tilted and $m$-Calabi–Yau tilted algebras as well as Jacobian algebras associated to surfaces with marked points in the boundary [5, 37, 48]. Of particular interest in relation to our construction is the appearance of the derived category of graded gentle algebras in the context of partially wrapped Fukaya categories [43].

The bounded derived categories of gentle algebras have been extensively studied. Their indecomposable objects were completely classified in terms of homotopy strings and bands in [14] and using different matrix reduction techniques in [25, 27, 26]. A basis for their morphism spaces was given in [4], and the cones of these morphisms were studied in [31]. The almost-split triangles of these categories were described in [17], see also [4]. The introduction of a combinatorial derived invariant for gentle algebras in [10] has also sparked a lot of research on the question of when two gentle algebras are derived equivalent, see for instance [19, 20, 35, 34, 1, 2, 45, 18, 50] (other invariants had also been introduced in [15]). This study was also extended to unbounded homotopy categories in [29]. The derived category of related classes of algebras have also been studied in some of the references mentioned above, see also [13, 12, 28, 11].

In this paper, given a gentle algebra $A$, we construct a geometric model of $D^b(A - \text{mod})$ in form of a lamination of an oriented surface with boundary and marked points in the boundary. In [54], see also [55], for every gentle algebra, a ribbon graph was given. Our model is based on the embedding of the ribbon graph into its ribbon surface where the marked points correspond to the vertices of the ribbon graph embedded in the boundary of the surface. The lamination then corresponds to a form of dual of the ribbon graph within the surface. Furthermore, we show that the fundamental group of the surface is isomorphic to the fundamental group of the quiver considered as a graph.

We give an explicit description of the correspondence of homotopy classes of (infinite) curves in the surface with the indecomposable objects (up to shift) in the derived category of a gentle algebra based on the homotopy strings and bands of [14] (Theorem 2.5). Based on the basis of homomorphism in $D^b(A - \text{mod})$ given in [4], we show that these basis elements correspond to crossings of curves (Theorem 3.3). Using the graphical mapping cone calculus given in [31], we show that the mapping cone of a map corresponding to a crossing of curves is given by the resolution of the crossing (Theorem 4.3). The Auslander–Reiten translate of a perfect object in $D^b(A - \text{mod})$ then corresponds to the rotation of the endpoints along the boundary of the corresponding curve in the surface (Corollary 5.2). Finally, we show that the surface encodes the derived invariant of Avella-Alaminos and Geiss [10] in terms
of the number of boundary components, the number of marked points on each boundary component and the number of laminates starting and ending on each boundary component (Theorem 6.1).

As we were finalising our paper, an independent construction of a geometric model of the bounded derived category of homologically smooth gentle algebras has come to our attention [50]. It is establishing, together with the results in [43], an equivalence with the partially wrapped Fukaya category associated to the surface. While it would seem that our geometric models coincide, our focus is on the explicit description of the connection of the surface combinatorics and the representation theory related to $D^b(A - \text{mod})$.

In the context of a classification of thick subcategories of discrete derived categories, a geometric model was given in [22]. Discrete derived categories were classified in [58], where it is shown that they correspond to bounded derived categories of a class of gentle algebras. The geometric model constructed in [22] coincides with our model for the class of discrete derived algebras.

Jacobian algebras of (ideal) triangulations of marked surfaces with all marked points in the boundary are gentle algebras [48, 5]. We note that the ribbon graph of such a gentle algebra corresponds exactly to the triangulation of the surface. In this context, the indecomposable objects of the associated cluster category were classified in [24] in terms of arcs and closed curves on the surface, and the Auslander-Reiten translation was described in [23]. Bases for the extension spaces were described in terms of crossings of arcs in [22]. These results were then extended to the case where the surface has punctures (that is, marked points in its interior) to objects associated to arcs in [52], and a complete description of indecomposable objects using arcs and closed curves was given in [3]. Furthermore, for gentle algebras associated to triangulations of surfaces with marked points in the boundary, the geometric description of the Auslander-Reiten translation is the same in both the associated module category [24], the cluster category [24] and we show in this paper, that it is the same again in the bounded derived category. We also note that the triangulated marked bounded surface $S$ corresponding to a gentle Jacobian algebra $A$ and the marked bounded surface underlying our geometric model of the bounded derived category $D^b(A - \text{mod})$ correspond if we add a boundary component with no marked points in the interior of each internal triangle in $S$. However, we also note that the sets of marked points do not necessarily coincide.

The layout of the paper is as follows. In Section 1 we construct the marked bounded surface $S_A$ of a gentle algebra $A$ from its ribbon graph as well as a laminating of $S_A$. The correspondence of homotopy classes of curves with the objects in the bounded derived category $D^b(A - \text{mod})$ (up to shift) is given in Section 2. In Section 3 we establish a correspondence between the basis of homomorphism in $D^b(A - \text{mod})$ given in [4] and the crossing of curves (in minimal position) in $S_A$. The mapping cones of the basis of homomorphism in terms of resolutions of crossings is given in Section 4 and it is shown in Section 5 that the Auslander-Reiten translate corresponds to a rotation of both endpoints of the homotopy class of curves corresponding to an indecomposable object in $D^b(A - \text{mod})$. Finally, in Section 6 a description of the derived invariant of Avella-Alaminos and Geiss in terms of the surface is given.
Conventions

In this paper, all algebras will be assumed to be finite-dimensional over a base field $k$. All modules over such algebras will be assumed to be finite-dimensional. Arrows in a quiver are composed from left to right.

1. Surfaces with boundaries for gentle algebras

In this section, we recall the construction of a surface with boundary associated to a gentle algebra. Our main references in this section are [55] and [49].

1.1. Ribbon graphs and ribbon surfaces. A ribbon graph is an unoriented graph with a cyclic ordering of the edges around each vertex. In order to give a precise definition, it is useful to define a graph as a collection of vertices and half-edges, each of which is attached to a vertex and another half-edge. More precisely:

**Definition 1.1.** A graph is a quadruple $\Gamma = (V, E, s, \iota)$, where

- $V$ is a finite set, whose elements are called vertices;
- $E$ is a finite set, whose elements are called half-edges;
- $s : E \to V$ is a function;
- $\iota : E \to E$ is an involution without fixed points.

We think of $s$ as a function sending each half-edge to the vertex it is attached to, and of $\iota$ as sending each half-edge to the other half-edge it is glued to. This definition is equivalent to the usual definition of a graph, and in practice we will draw graphs in the usual way.

**Definition 1.2.** A ribbon graph is a graph $\Gamma$ endowed with a permutation $\sigma : E \to E$ whose orbits correspond to the sets $s^{-1}(v)$, for all $v \in V$. A ribbon graph is a graph $\Gamma$ endowed with a permutation $\sigma : E \to E$ such that the cycles of $\sigma$ correspond to the sets $s^{-1}(v)$, for all $v \in V$.

In other words, a ribbon graph is a graph endowed with a cyclic ordering of the half-edges attached to each vertex.

Any ribbon graph can be embedded in the interior of a canonical oriented surface with boundary, called the ribbon surface, in such a way that the orientation of the surface is induced by the cyclic orderings of the ribbon graph. Whenever we deal with oriented surfaces in this paper, we will call clockwise orientation the orientation of the surface, and anti-clockwise orientation the opposite orientation. When drawing surfaces or graphs in the plane, we will do so that locally, the orientation of the surface or graph becomes the clockwise orientation of the plane.

**Definition 1.3.** Let $\Gamma$ be a connected ribbon graph. The ribbon surface $S_\Gamma$ is constructed by gluing polygons as follows.

- For any vertex $v \in V$ with valency $d(v) \geq 1$, let $P_v$ be an oriented $2d(v)$-gon.
- Following the cyclic orientation, label every other side of $P_v$ with the half-edges $e \in E$ such that $s(e) = v$.
- For any half-edge $e$ of $\Gamma$, identify the side of $P_v$ labelled $e$ with the side of the polygon $P_{s(\iota(e))}$ labelled $\iota(e)$, respecting the orientations of the polygons.

In this definition, we exclude the degenerate case where $\Gamma$ has only one vertex and no half-edges.
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Figure 1. Example of a ribbon graph $\Gamma$ with orientation given by the planar embedding and with half edge labelling on the left and on the right the associated ribbon surface $S_{\Gamma}$ obtained by gluing the two polygons $P_1$ and $P_2$ corresponding to vertices 1 and 2 of the ribbon graph.

Note that $S_{\Gamma}$ is oriented, and that we can embed $\Gamma$ in $S_{\Gamma}$ as follows: the vertices of $\Gamma$ are the centers of the polygons $P_v$, and the half edges of $\Gamma$ are arcs joining the center of each $P_v$ to the middle of the side with the same label. By [49 Corollary 2.2.11], $S_{\Gamma}$ is, up to homeomorphism, the only oriented surface $S$ in which we can embed $\Gamma$, preserving the cyclic ordering around each vertex, and such that the complement of the embedding of $\Gamma$ in $S$ is a disjoint union of discs (we say that $\Gamma$ is filling for $S$). Moreover, by [49 Proposition 2.2.7], the number of boundary components of $S_{\Gamma}$ is equal to the number of faces of $\Gamma$, according to the following definition.

**Definition 1.4.** Let $\Gamma$ be a ribbon graph. A face of $\Gamma$ is an equivalence class, up to cyclic orientation, of tuples of half-edges $(e_1, \ldots, e_n)$ such that

1. $e_{p+1} = \begin{cases} \ell(e_p) & \text{if } s(e_p) = s(e_{p-1}), \\ \sigma(e_p) & \text{otherwise}, \end{cases}$ where the indices are taken modulo $n$;
2. the tuple is non-repeating, in the sense that if $p \neq q$ and $e_p = e_q$, then $e_{p+1} \neq e_{q+1}$.

**1.2. Marked ribbon graphs.** When we study gentle algebras in Section 1.3, we will obtain ribbon graphs endowed with one additional piece of information. We will call these marked ribbon graph, and we define them as follows.

**Definition 1.5.** A marked ribbon graph is a ribbon graph $\Gamma$ together with a map $m : V \to E$ such that for every vertex $v \in V$, $m(v) \in s^{-1}(v)$.

In other words, a marked ribbon graph is a ribbon graph in which we have chosen one half-edge $m(v)$ around each vertex $v$.

If $\Gamma$ is a marked ribbon graph, we can construct its ribbon surface $S_{\Gamma}$ like in Definition 1.3. Moreover, with the additional information given by the map $m$, we can do the following:

**Proposition 1.6.** There is an orientation-preserving embedding of $\Gamma$ in $S_{\Gamma}$ which sends all vertices of $\Gamma$ to boundary components of $S_{\Gamma}$ such that for each vertex...
v ∈ V, the boundary component lies between m(v) and σ(m(v)) in the clockwise orientation. This embedding is unique up to homotopy relative to ∂SΓ.

Proof. With the notations of Definition 1.3 to prove the existence of the embedding, it suffices to move v to the unlabelled side of Pν that lies between the sides labeled with m(v) and σ(m(v)). Uniqueness follows from the fact that there is precisely one boundary component inside every face of Γ. □

We call an embedding as in Proposition 1.6 a marked embedding of Γ in SΓ. We usually denote by M the set of marked points on SΓ corresponding to the vertices of Γ.

1.3. The marked ribbon graph of a gentle algebra. Here, we follow [54], see also [55, Section 3]. Gentle algebras are finite-dimensional algebras having a particularly nice description in terms of generators and relations. Their representation theory is well understood and their study goes back to [42, 56, 59, 50, 50]. Let us recall their definition:

Definition 1.7. An algebra A is gentle if it is isomorphic to an algebra of the form kQ/I, where

(1) Q is a finite quiver;
(2) I is an admissible ideal of Q (that is, if R is the ideal generated by the arrows of Q, then there exists an integer m ≥ 2 such that Rm ⊂ I ⊂ R2);
(3) I is generated by paths of length 2;
(4) for every arrow α of Q, there is at most one arrow β such that αβ ∈ I; at most one arrow γ such that γα ∈ I; at most one arrow β′ such that αβ′ /∈ I; and at most one arrow γ′ such that γ′α /∈ I.

Definition 1.8. For a gentle algebra A = kQ/I, let

• M be the set of maximal paths in (Q, I), that is, paths w /∈ I such that for any arrow α, αw ∈ I and wα ∈ I;
• M0 be the set of trivial paths ev such that either v is the source or target of only one arrow, or v is the target of exactly one arrow α and the source of exactly one arrow β, and αβ /∈ I;
• M = M ∪ M0.

We call M the augmented set of maximal paths of A.

Then the marked ribbon graph ΓA of A is defined as follows.

(1) The set of vertices of ΓA is M.
(2) For every vertex of ΓA corresponding to a path ω, there is a half-edge attached to ω and labeled by i for every vertex i of Q through which ω passes. Note that this includes the vertices at which ω starts and ends. Furthermore, if ω passes through i multiple times (at most 2), then there is one half-edge labeled by i for every such passage.
(3) For every vertex i of Q, there are exactly two half-edges labeled with i. The involution i sends each one to the other.
(4) For each vertex ω of ΓA, the vertices through which the path ω passes are ordered from starting point to ending point. The permutation σ sends each vertex in this ordering to the next, with the additional property that it sends the ending point of ω to its starting point.
(5) The map m takes every ω to the half-edge labeled by its ending point.
Remark 1.9. In a dual construction, the marked ribbon graph of a gentle algebra can also be defined using instead of $\overline{M}$, the augmented set of all paths in $Q$ such that any subpaths of length 2 is in $I$. This is the set of forbidden threads as defined in [10].

Using Section 1.2 we can now define a surface with boundary and marked points for every gentle algebra.

Definition 1.10. Let $A = kQ/I$ be a gentle algebra. Its ribbon surface $S_A$ is the ribbon surface of $\Gamma_A$. It has marked points on the boundary and collection of arcs joining them given by the embedding of $\Gamma_A$ as in Proposition 1.6.

Thus the marked points of $S_A$ are in bijection with the vertices of $\Gamma_A$, and the edges of $\Gamma_A$ are in bijection with the vertices of $Q$.

Example 1.11. (1) Let $A$ be the algebra defined by the quiver

$$
1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4
$$

with no relations. The ribbon graph $\Gamma_A$ of this algebra is

![Diagram of the ribbon graph](image)

and its ribbon surface $S_A$ is a disc.

(2) Let $A$ be the algebra defined by the quiver

$$
1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4
$$

with relations $\alpha_1\alpha_2$ and $\alpha_2\alpha_3$. The ribbon graph $\Gamma_A$ of this algebra is

![Diagram of the ribbon graph](image)

and its ribbon surface $S_A$ is, again, a disc.

For any gentle algebra $A$, the edges of $\Gamma_A$ cut $S_A$ into polygons as follows.

Proposition 1.12. Let $A = kQ/I$ be a gentle algebra, and let $\Gamma_A$ and $S_A$ be as in Definitions 1.8 and 1.10. Then $S_A$ is divided into two types of pieces glued together by their edges:
(1) polygons whose edges are edges of $\Gamma_A$, except for exactly one boundary edge, and whose interior contains no boundary component of $S_A$;
(2) polygons whose edges are edges of $\Gamma_A$ and whose interior contains exactly one boundary component of $S_A$ with no marked points.

Proof. Take any point $X$ in the interior of $S_A$ which does not belong to any edge of $\Gamma_A$. Then this point belongs to a polygon $P_v$ as in Definition 1.3. This polygon has $2d$ sides (for a certain integer $d$) and contains exactly one marked point on one of its boundary segments, from which emanate $d$ edges of $\Gamma_A$. Below is the local picture if $P_v$ is an octogon:

We see that $X$ belongs to a region of $P_v$ (grayed on the picture) that is partly bounded by a segment of a boundary component $B$ of $S_A$. Around this boundary component are other polygons $P_{v_1}, P_{v_2}, \ldots, P_{v_r}$, each containing exactly one marked point on one of its boundary segments. The picture around this boundary component $B$ is as follows (in the example, $B$ is a square).

Two cases arise.

Case 1: There is at least one marked point on $B$. In this case, the point $X$ belongs to a polygon cut out by edges of $\Gamma_A$ and by exactly one boundary edge on $B$, as illustrated in the following picture.
Case 2: There are no marked points on $B$. In this case, the point $X$ belongs to a polygon on $S_A$ cut out by edges of $\Gamma_A$ and which contains the boundary component $B$, as illustrated below.

This finishes the proof. □

Remark 1.13. Let $A$ be a gentle algebra with ribbon graph $\Gamma_A$ and associated ribbon surface $S_A$. Suppose that $\Gamma_A$ admits $v$ vertices, $2e$ half-edges and $f$ faces.
1) The complement of $S_A$ is a disjoint union of open discs.
2) The Euler characteristic $\chi(\Gamma_A) = v - e + f$ of the ribbon graph $\Gamma$ is equal to the Euler characteristic of $\hat{S}_A$, where $\hat{S}_A$ is the surface without boundary obtained from $S_A$ by gluing an open disc to each of the boundary components of $S_A$.
3) The genus of $S_A$ (as well as the genus of $\hat{S}_A$) is equal to $1 - \chi(\Gamma_A)/2$ that is the genus of $S_A$ is $(e - v - f + 2)/2$.

1.4. A lamination on the surface of a gentle algebra. On any surface with boundary and marked points on the boundary, the notion of lamination is defined in [39 Definiton 12.1]. We will need to modify the definition slightly for what follows.

Definition 1.14. Let $S$ be a surface with boundary and a finite set $M$ of marked points on its boundary. A lamination on $S$ is a finite collection of non-selfintersecting and pairwise non-intersecting curves on $S$, considered up to isotopy relative to $M$. Each of these curves is one of the following:
- a closed curve not homotopic to a point; or
a curve from one non-marked point to another non-marked point, both on the boundary of \( S \). We exclude such curves that are isotopic to a part of the boundary of \( S \) containing no marked points.

A curve that is part of a lamination is called a laminate.

**Remark 1.15.** In [39, Definition 12.1], the case of a curve from a non-marked point to another on the boundary that is isotopic to a part of the boundary containing exactly one marked point is also excluded. For our purposes, we need to allow such curves in our laminations.

Let \( A = kQ/I \) be a gentle algebra, and let \( S_A \) be its ribbon surface as in Definition 1.10. We will now define a canonical lamination of the ribbon surface of a gentle algebra.

**Proposition 1.16.** Let \( A = kQ/I \) be a gentle algebra, and let \( S_A \) be its ribbon surface as in Definition 1.10. There exists a unique lamination \( L \) of \( S_A \) such that

1. \( L \) contains no closed loops;
2. for every vertex \( i \) of \( Q \) (that is, every edge of \( \Gamma_A \)), there is a unique curve \( \gamma_i \in L \) such that \( \gamma_i \) crosses the edge labeled by \( i \) of the embedding of \( \Gamma_A \) once, and crosses no other edges;
3. \( L \) contains no other curves than those described in (2).

**Proof.** Every edge \( E \) of \( \Gamma_A \) is part of two (not necessarily distinct) faces, in the sense of Definition 1.4, and each of these faces encloses a boundary component in \( S_A \). Therefore, if a curve \( \gamma \) in a lamination crosses \( E \), then either it starts and ends on these two boundary components, or it has to cross at least another edge. Moreover, there is a unique curve starting on one of these two boundary components and ending on the other that crosses \( E \) once and no other edges of \( \Gamma_A \). \( \square \)

**Example 1.17.** We give the laminations for the two gentle algebras in Example 1.11

1. \[ \begin{array}{c}
1 \quad \alpha_1 \quad 2 \\
\alpha_2 \quad 3 \\
\alpha_3 \quad 4
\end{array} \]

**Figure 2.** On the right side is the ribbon graph embedded in the ribbon surface as well as the lamination of the hereditary gentle algebra on the left.
**Definition 1.18.** Let $A = kQ/I$ be a gentle algebra. Then we denote by $L_A$ the lamination described in Proposition 1.16 and we call it the *lamination of $A$*. 

**Definition 1.19.** Let $A$ be a gentle algebra with marked ribbon graph $\Gamma_A$ and associated ribbon surface $S_A$ together with a marked embedding of $\Gamma_A$ into $S_A$. Denote by $M$ the set of marked points (these are the vertices of $\Gamma_A$) in $S_A$.

An arc $\gamma$ in $S_A$ is a homotopy class of non-contractible curves whose endpoints coincide with marked points on the boundary. A *closed curve* in $S_A$ is a free homotopy class of non-contractible curves whose starting points and ending points are equal and lie in the interior of $S_A$. A closed curve is *primitive* if it is not a non-trivial power of a different closed curve in the fundamental group of $S_A$.

In addition to the arcs with endpoints in the marked points, we also consider particular classes of rays and lines in $S_A$. Recall that a *ray* is a map $r : (0, 1] \to S_A$ or a map $r : [0, 1) \to S_A$ and that a *line* is a map $l : [0, 1) \to S_A$. In what follows all rays will be such they start or end in a marked point and such that the other end wraps infinitely many times around a single unmarked boundary component with no marked points. All lines will be such that on each end they wrap infinitely many times around a single boundary component with no marked points.

An *infinite arc* is given by homotopy classes associated to rays or lines in $S_A$ as follows: Let $B$ and $B'$ be boundary components in $S_A$ such that $B \cap M = B' \cap M = \emptyset$. We call such boundary components *unmarked*. We say two rays $r : (0, 1] \to S_A$ and $r' : (0, 1] \to S_A$ wrapping infinitely many times around the same unmarked boundary component $B$ are equivalent if $r(1) = r'(1) \in M$ and if for every closed neighbourhood $N$ of $B$ the induced maps $r, r' : [0, 1] \to S_A/N$ are homotopy equivalent relative to their endpoints. Similarly, we say two lines $l : (0, 1) \to S_A$ and $l' : (0, 1) \to S_A$ are equivalent if they wrap infinitely many times around the same unmarked boundary components $B$ and $B'$ on each end and if for every closed neighbourhood $N$ of $B$ and $N'$ of $B'$ the induced maps $l, l' : [0, 1] \to S_A/(N \cup N')$ are homotopy equivalent relative to their endpoints.

**Remark 1.20.** It will sometimes be useful to think of boundary components with no marked points as punctures in the surface. Infinite arcs wrapping around such a boundary component can then be viewed as arcs going to the puncture.
1.5. **Recovering the gentle algebra from its lamination.** The surface $S_A$ and lamination $L_A$ of a gentle algebra $A$ contain, by construction, enough information to recover the algebra $A$. We record the procedure in the following proposition.

**Proposition 1.21.** Let $A = kQ/I$ be a gentle algebra, and let $L_A$ be the associated lamination (see Definition 1.18). Define a quiver $Q_L$ as follows:

- its vertices correspond to curves in $L_A$;
- whenever two curves $i$ and $j$ in $L_A$ both have an endpoint on the same boundary segment of $S_A$, so that no other curve has an endpoint in between, then there is an arrow from $i$ to $j$ if the endpoint of $j$ follows that of $i$ on the boundary in the clockwise order.

Let $I_L$ be the ideal of $kQ_L$ defined by the following relations: whenever there are curves $i, j$ and $k$ in $L_A$ that have an endpoint on the same boundary segment of $S_A$, so that the endpoint of $k$ follows that of $j$, which itself follows that of $i$, and if $\alpha : i \to j$ and $\beta : j \to k$ are the corresponding arrows, then $\beta \alpha$ is a relation. Then $A \cong kQ_L/I_L$.

1.6. **The fundamental group of the surface of a gentle algebra.** We show that the fundamental group of the surface $S_A$ of a gentle algebra $A = kQ/I$ is isomorphic to $\pi_1(Q)$, the fundamental group of the graph underlying its quiver $Q$.

**Proposition 1.22.** Let $A = kQ/I$ be a gentle algebra. Let $\pi_1(Q)$ be the fundamental group of its underlying graph. There exists an isomorphism $\pi_1(S_A) \cong \pi_1(Q)$.

**Corollary 1.23.** Let $A = KQ/I$ be a gentle algebra. Then following are equivalent

(i) the graph underlying $Q$ is a tree
(ii) $S_A$ is a disc
(iii) $A$ is derived equivalent to a path algebra of Dynkin type $\mathbb{A}$.

**Proof.** The equivalence of (i) and (ii) directly follows from Proposition 1.22, the equivalence of (i) and (iii) is due to [6].

**Corollary 1.24.** Let $A = KQ/I$ be a gentle algebra. Then $S_A$ is an annulus if and only if $A$ has precisely one cycle.

**Proof.** Follows directly from Proposition 1.22 and the fact that the annulus is the only (compact oriented) surface with fundamental group $\mathbb{Z}$. 

By [6], the algebras appearing in Corollary 1.23 are precisely the algebras which are derived equivalent to a path algebra of type $\mathbb{A}$; by [10], those appearing in Corollary 1.24 are determined, up to derived equivalence, by their AG-invariant (for more on the AG-invariant, see Section 6).

**Proof of Proposition 1.22.** It follows from Proposition 1.21 that there exists an embedding of $Q$ into $S_A$ such that each vertex is mapped to an interior point on the corresponding laminate and such that each arrow is mapped to a path with no intersection with the boundary and no intersection with any of the laminates apart from its endpoints. For our assertions it is sufficient to prove that this embedding is a strong deformation retract of the surface. We do this by gluing deformation retractions of the individual polygons cut out by the lamination.

For each polygon $P_v, v \in \mathcal{M}$, denote by $Q(v)$ the subquiver of $Q$, which contains all arrows of the path $v$, if $v \in \mathcal{M}$, and, in case $v \in \mathcal{M}_0$, let $Q(v)$ be the subquiver
with a single vertex corresponding to \( v \in \mathcal{M}_0 \). We define a strong deformation retraction of \( P_v \) onto the embedding of \( Q(v) \), which contracts each laminate to a single point and projects each boundary segment onto an arrow.

This is done in two steps. Every arrow \( \alpha \) of \( Q(v) \) singles out a square in \( P_v \) bounded by the edge \( \alpha \), a boundary segment and segments of the laminates crossed by \( \alpha \). \( P_v \) is glued from these squares and another polygon \( P'_v \), which contains the marked point. For each \( L \in \mathcal{L} \) and for \( a_L \in [0, 1] \), such that \( L(a_L) = p_L \), denote \( H_L \) the homotopy from \( \text{Id}_L \) to the constant map \( p_L \) corresponding to \( t \to a_L + (1-t) \cdot (t-a_L) \). Convex linear combinations enable us to extend any homotopy, which is constant on \( \{0, 1\} \times \{0\} \), from \( \text{Id}_{(0,1) \times \{0\}} \) to the map \( (a, t) \mapsto (a, 0) \) to a homotopy, which is constant on \( \{0, 1\} \times [0, 1] \), from \( \text{Id}_{[0,1]^2} \) to the map \( (a, t) \mapsto (a, 0) \). In particular, we find a homotopy from the identity of each square to a map, which projects the square onto the corresponding arrow of \( Q(v) \) and which extends the contractions of the segments of laminates \( L \) to the point \( p_L \) (as restrictions of \( H_L \)). We finally find a homotopy from \( \text{Id}_{P'_v} \), which is constant on all arrows of \( Q(v) \), to a map, which projects each point of \( P'_v \) to a point of the embedding of \( Q(v) \). By construction, we can glue all the homotopies showing that \( P_v \) strongly deformation retracts onto the embedding of \( Q(v) \). All such homotopies can be glued at the laminates, which finishes the proof. \( \square \)

2. Indecomposable objects in the derived category of a gentle algebra

Throughout this section let \( A = KQ/I \) be a gentle algebra. In this section, we prove that the indecomposable objects of the bounded derived category \( D^b(A \mod) \) are in bijection, up to shift, with certain curves on the surface \( S_A \).

2.1. Homotopy strings and bands. There are several approaches to the description of indecomposable objects in the bounded derived category of a gentle algebra. One approach makes use of combinatorial objects called homotopy strings and bands \([14, 13]\) and this is the approach that we will use in this paper.

In this section, we briefly recall the classification of the indecomposable objects in the bounded derived category of a gentle algebra in terms of homotopy string and band complexes \([14]\). Throughout this section let \( A = KQ/I \) be a gentle algebra. We recall that there is a triangle equivalence \( D^b(A \mod) \simeq K^{-b}(A \text{ proj}) \), where \( A \text{ proj} \) is the full subcategory of \( A \mod \) given by the finitely generated projective \( A \)-modules, \( K^{-b}(A \text{ proj}) \) is the homotopy category of complexes of objects in \( A \text{ proj} \) which are bounded on the right and have bounded homology, and \( D^b(A \mod) \) is the bounded derived category of \( A \mod \). The definition
of the homotopy string and band complexes that we will use is that introduced in [14].

For every \( a \in Q_1 \), we define a formal inverse \( \overline{a} \) where \( s(\overline{a}) = t(a) \) and \( t(\overline{a}) = s(a) \). We denote by \( \overline{Q_1} \) the set of formal inverses of the elements in \( Q_1 \), and we extend the operation \((-\)\) to an involution of \( Q_1 \cup \overline{Q_1} \) by setting \( \overline{a} = a \).

A walk is a sequence \( w_1 \ldots w_n \), where \( w_i \in Q_1 \cup \overline{Q_1} \) is such that \( s(w_{i+1}) = t(w_i) \). We also allow trivial walks \( e_u \) for every vertex \( u \) of \( Q \). A string is a walk \( w \) such that \( w_{i+1} \neq \overline{w_i} \) and such that for all substrings \( w' = w_i w_{i+1} \ldots w_j \) of \( w \) with the \( w_i, \ldots, w_j \) all in \( Q_1 \) (or all in \( \overline{Q_1} \)), we have that \( w' \notin I \) (or \( \overline{w'} \notin I \), respectively).

We say that \( w = w_1 \ldots w_n \) is a direct (resp. inverse) string if for all \( 1 \leq i \leq n \), we have \( w_i \in Q_1 \) (resp. \( w_i \in \overline{Q_1} \)).

A generalized walk is a sequence \( \sigma_1 \ldots \sigma_m \) such that each \( \sigma_i \) is a string such that \( s(\sigma_{i+1}) = t(\sigma_i) \).

**Definition 2.1.** Let \( A = KQ/I \) be a gentle algebra. A finite homotopy string \( \sigma = w_1 \ldots w_n \), where \( w_i \in Q_1 \cup \overline{Q_1} \), is a (possibly trivial) walk in \( (Q, I) \) consisting of subwalks \( \sigma_1, \ldots, \sigma_r \) with \( \sigma = \sigma_1 \ldots \sigma_r \) and such that

1. \( \sigma_k \) is a direct or inverse string;
2. If \( \sigma_k, \sigma_{k-1} \) are both direct strings then \( \sigma_{k-1} \sigma_k \in I \) (resp. if both \( \overline{\sigma_k}, \overline{\sigma_{k-1}} \) are inverse strings then \( \overline{\sigma_{k-1}} \overline{\sigma_k} \in I \)).

If \( \sigma_k \) is a direct string it is called a direct homotopy letter, otherwise it is called an inverse homotopy letter.

A homotopy band is a finite homotopy string \( \sigma = \sigma_1 \ldots \sigma_r \) with an equal number of direct and inverse homotopy letters \( \sigma_i \) such that \( t(\sigma_r) = s(\sigma_1) \) and \( \sigma_1 \neq \overline{\sigma_r} \) and \( \sigma \neq \tau^m \) for some homotopy string \( \tau \) and \( m > 1 \).

A homotopy string or band \( \sigma = \sigma_1 \ldots \sigma_r \) is reduced if \( \sigma_i \neq \overline{\sigma_{i+1}} \) for all \( i \in \{1, \ldots, r-1\} \).

A generalized walk is called a direct (resp. inverse) antipath if each homotopy letter is a direct (resp. inverse) homotopy letter.

**Definition 2.2.** A left (resp. right) infinite generalized walk \( \sigma = \ldots \sigma_{-2} \sigma_{-1} \) (resp. \( \sigma = \sigma_1 \sigma_2 \ldots \)) is called a left (resp. right) infinite homotopy string if there exists \( k \geq 1 \) such that \( \ldots \sigma_k \sigma_{k+1} \) (resp. \( \sigma_{-k+1} \sigma_{-k} \ldots \)) is a direct (resp. inverse) antipath which is eventually periodic and eventually involves only homotopy letters of length 1.

A two-sided infinite generalized walk \( \sigma = \ldots \sigma_{-1} \sigma_0 \sigma_1 \ldots \) is called a two-sided infinite homotopy string if \( \ldots \sigma_{-1} \sigma_0 \) is a left infinite homotopy string and \( \sigma_0 \sigma_1 \ldots \) is a right infinite homotopy string.

To each homotopy string and homotopy band \( \sigma \) as described above is associated a (possibly infinite) complex of projective modules \( P^{\bullet}_{\sigma} \) [14]. We now recall this construction.

**Definition 2.3 (14).** (1) Let \( \sigma = \sigma_1 \ldots \sigma_r \) be a finite reduced homotopy string. Define \( v_0 = s(\sigma_1) \) and \( v_i = t(\sigma_i) \) for all \( i \in \{1, \ldots, r\} \). Define further \( \mu_0 = 0 \) and \( \mu_{i+1} = \begin{cases} \mu_i + 1 & \text{if } \sigma_i \text{ is a direct homotopy letter;} \\ \mu_i - 1 & \text{if } \sigma_i \text{ is an inverse homotopy letter,} \end{cases} \) and let \( \mu(\sigma) := \min_{i \in \{0, 1, \ldots, r\}} (\mu_i) \). Then the complex

\[
P^{\bullet}_{\sigma} : \ldots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \rightarrow \ldots
\]
is given by
  • for all \( j \in \mathbb{Z} \),
    \[
P^j = \bigoplus_{0 \leq i \leq r, \mu_i = j} P_{v_i},
    \]
    where each \( P_{v_i} \) is the indecomposable projective module associated to
    the vertex \( v_i \);
  • each direct (resp. inverse) homotopy letter \( \sigma_i \) defines a morphism
    \( P_{v_{i-1}} \xrightarrow{\sigma_i} P_{v_i} \) (resp. \( P_{v_i} \xrightarrow{\overline{\sigma}_i} P_{v_{i-1}} \)). These form the components of the
differentials \( d^j \) in the natural way. We call \( P_\sigma \) a string object.

(2) The definition of \( P_\sigma \) when \( \sigma \) is an infinite reduced homotopy string is sim-
ilar, and we again call \( P_\sigma \) a string object.

(3) Let \( \sigma = \sigma_1 \cdots \sigma_r \) be a reduced homotopy band. Let \( M \) be a finite-dimensional
indecomposable \( K[X] \)-module, and let \( m = \dim_K M \). Let \( F \) be the matrix of the
multiplication by \( X \) for a given basis of \( M \). Define \( v_0, \ldots, v_r, \mu_0, \ldots, \mu_r \)
as for homotopy strings.

Then the complex \( P_{\sigma,F} \) is defined by
  • for all \( j \in \mathbb{Z} \),
    \[
P^j = \bigoplus_{0 \leq i \leq r-1, \mu_i = j} P_{v_i}^\oplus m,
    \]
    where \( \operatorname{Id}^m \) is the \( m \times m \) identity matrix. These form the components
of the differentials \( d^j \) in the natural way.
  • The homotopy letter \( \sigma_r \) defines a final component of the differential.

If it is a direct letter, then the morphism used is \( P_{v_{r-1}}^\oplus m \xrightarrow{\sigma_r F} P_{v_0}^\oplus m \),
otherwise, the morphism is \( P_{v_0}^\oplus m \xrightarrow{\overline{\sigma}_r F} P_{v_{r-1}}^\oplus m \).
In this case, \( P_{\sigma,F} \) is called a band object.

Furthermore, it is shown in [14] that the isomorphism classes of indecomposable
objects in \( D^b(A - \text{mod}) \) up to shift are in bijection with homotopy strings and
bands up to inverse. More precisely, the equivalence is modulo the equivalence
relation \( \sigma \sim \overline{\sigma} \) for a homotopy string \( \sigma \), infinite homotopy strings up to inverse,
and pairs consisting of a homotopy band up to inverse and up to permutation and
of an isomorphism class of indecomposable \( K[X] \)-modules. This bijection is the
one described in Definition 2.3.

Remark 2.4. If the field \( K \) is algebraically closed, then the matrix \( F \) of Defini-
tion 2.3 (3) can always be chosen to be a Jordan block \( J_m(\lambda) \) of size \( m \)
corresponding to a scalar \( \lambda \in K \). Note that for \( m = 1 \), we have \( P_{\sigma,J_1(\lambda)} = P_{\sigma,\lambda} \).
In the text, if the result or proof does not depend on the scalar \( \lambda \), we will sometimes omit it in our
notation and we will write \( P_\sigma \) instead of \( P_{\sigma,\lambda} \).

2.2. Main result on indecomposable objects of the derived category. We
are now ready to prove our classification of indecomposable objects (up to shift) in
\( D^b(A - \text{mod}) \) using curves on the surface \( S_A \). Before stating our main result, we
recall, see for example [33, 47], the definition of the orbit category \( D^b(A - \text{mod})/1 \)
where \([1]\) is the shift functor. Namely the objects of \(D^b(A - \text{mod})/[1]\) are the same as the objects of \(D^b(A - \text{mod})\) and

\[
\text{Hom}_{D^b(A - \text{mod})/[1]}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(A - \text{mod})}(X, Y[n]).
\]

**Theorem 2.5.** Let \(A = KQ/I\) be a gentle algebra with marked ribbon graph \(\Gamma_A\) and a marked embedding in the associated ribbon surface \(S_A\). Let \([1]\) be the shift functor in \(D^b(A - \text{mod})\). Then

1. the isomorphism classes of the indecomposable string objects in \(D^b(A - \text{mod})/[1]\) are in bijection with the finite arcs on \(S_A\) and infinite arcs on \(S_A\) whose infinite rays circle around a boundary component in counter-clockwise orientation;

2. the isomorphism classes of the indecomposable band objects in \(D^b(A - \text{mod})/[1]\) are in bijection with the pairs \((\gamma, M)\), where \(\gamma\) is a closed curve on \(S_A\) satisfying condition (3) of Lemma 2.11 below and \(M\) is an isomorphism class of indecomposable \(K[X]\)-modules.

More precisely, arcs correspond to homotopy string complexes, in finite arcs correspond to infinite homotopy string complexes and closed arcs correspond to homotopy bands.

Before proving the theorem, we need some results on the geometry of the lamination.

**Lemma 2.6.**

1. The lamination \(L_A\) subdivides \(S_A\) into polygons whose sides are laminates and boundary segments. The laminates of \(L_A\) can be chosen to be the “glued edges” of Definition 1.3.

2. Each polygon contains exactly one marked point.

3. Every boundary segment of \(S_A\) contains the endpoint of at least one laminate of \(L_A\).

**Proof.** It suffices to observe that the “glued edges” of Definition 1.3 cut the surface \(S_A\) into the polygons \(P_v\) of Definition 1.3 which contain exactly one marked point each by definition. Moreover, every boundary segment of these polygons is adjacent to at least one laminate.

Once we know that a surface is cut into polygons, then any arc is determined by the order in which it crosses the edges of the polygons. Note that the edges crossed correspond exactly to the laminates.

Whenever we shall be dealing with collections of arcs and curves on a surface, we will make the following assumption.

**Assumption 2.7.** Any finite collection of curves or arcs is in minimal position, that is, the number of intersections of each pair of (not necessarily distinct) curves in this set is minimal in their respective homotopy class.

As pointed out in [57], it follows from [40] and [51] that, up to homotopy, this assumption is always satisfied.

**Lemma 2.8.** Let \(\gamma\) be a possibly infinite arc or a closed curve on \(S_A\), and assume that every laminate of \(L_A\) that \(\gamma\) crosses, it crosses transversally (we can assume this, up to homotopy).

1. If \(\gamma\) is an arc, then it is completely determined by the (possibly infinite) sequence of the laminates that it crosses.
(2) If \( \gamma \) is a closed curve, then it is completely determined by the sequence of the laminates that it crosses, up to cyclic ordering.

The order in which an arc or a closed curve crosses the laminates gives rise to a homotopy string or band, as we will see in Lemma 2.11

**Definition 2.9.** Let \( P_v \) be a polygon on the surface \( S_A \), as per Lemma 2.6, and let \( M_v \) be the unique marked point in \( P_v \). Let \( \delta \) be a curve in \( P_v \) starting and ending on edges \( \ell_1 \) and \( \ell_2 \) of \( P_v \) which are laminates.

- If \( M_v \) lies between \( \ell_2 \) and \( \ell_1 \) in the clockwise order, then let \( w_1, \ldots, w_r \) be the laminates between \( \ell_1 = w_1 \) and \( \ell_2 = w_r \) in clockwise order. By Proposition 1.21, these correspond to vertices of the quiver \( Q \) of \( A \) which are joined by arrows \( \alpha_1, \ldots, \alpha_{r-1} \).
  
  Then define \( \sigma(\delta) := \alpha_1 \cdots \alpha_{r-1} \).

- If \( M_v \) lies between \( \ell_1 \) and \( \ell_2 \) in the clockwise order, then let \( w_1, \ldots, w_r \) be the laminates between \( \ell_2 = w_1 \) and \( \ell_1 = w_r \) in clockwise order. By Proposition 1.21, these correspond to vertices of the quiver \( Q \) of \( A \) which are joined by arrows \( \alpha_1, \ldots, \alpha_{r-1} \).
  
  Then define \( \sigma(\delta) := (\alpha_1 \cdots \alpha_{r-1})^{-1} \).

**Lemma 2.10.** Let \( P_v \) and \( \delta \) be as in Definition 2.9. Then \( \sigma(\delta) \) is a homotopy letter.

**Proof.** By Proposition 1.21, the compositions of the arrows of \( \sigma(\delta) \) are not in the ideal of relations of \( A \).

**Lemma 2.11.**

1. Let \( \gamma \) be a finite arc on \( S_A \). Let \( \ell_1, \ell_2, \ldots, \ell_r \) be the laminates crossed (in that order) by \( \gamma \), as per Lemma 2.8. For every \( i \in \{1, 2, \ldots, r-1\} \), let \( \gamma_i \) be the part of \( \gamma \) between its crossing of \( \ell_i \) and of \( \ell_{i+1} \). Let

\[
\sigma(\gamma) := \begin{cases} 
\prod_{i=1}^{r-1} \sigma(\gamma_i) & \text{if } r > 1; \\
\ell_{\ell_i} & \text{if } r = 1.
\end{cases}
\]

Then \( \sigma(\gamma) \) is a homotopy string.

2. Let \( \gamma \) be an infinite arc. Assume that on any infinite end of \( \gamma \), the arc cycles infinitely many times around a boundary component in counter-clockwise orientation. Let \( (\ell_i) \) be the sequence of laminates crossed by \( \gamma \) (this sequence can be infinite on either side). For every \( i \), let \( \gamma_i \) be the part of \( \gamma \) between its crossing of \( \ell_i \) and of \( \ell_{i+1} \). Let

\[
\sigma(\gamma) := \prod_i \sigma(\gamma_i).
\]
Then $\sigma(\gamma)$ is an infinite homotopy string.

(3) Let $\gamma$ be a primitive closed curve on $S_A$. Let $\ell_1, \ell_2, \ldots, \ell_r$ be the laminates crossed (in that order) by $\gamma$. For every $i \in \{1, 2, \ldots, r-1\}$, let $\gamma_i$ be the path of $\gamma$ between its crossing of $\ell_i$ and of $\ell_{i+1}$, and let $\gamma_r$ be the path of $\gamma$ between its crossing of $\ell_r$ and of $\ell_1$. Let

$$\sigma(\gamma) := \prod_{i=1}^r \sigma(\gamma_i).$$

If there is an equal number of inverse and direct homotopy letters among the $\sigma(\gamma_i)$, then $\sigma(\gamma)$ is a homotopy band.

**Proof.** In all three cases, for any index $i$, if $\sigma(\gamma_i)$ and $\sigma(\gamma_{i+1})$ are both direct homotopy letters, then by Proposition 1.21 composition of the last arrow of $\sigma(\gamma_i)$ and of the first of $\sigma(\gamma_{i+1})$ form a relation. The argument for consecutive inverse homotopy letters is similar. This proves (1).

To prove (2), assume that $\gamma$ is an infinite arc. Then $\gamma$ eventually wraps around one of the boundary components without marked points. By Lemma 2.6 there is at least one laminate with one endpoint on this boundary component. Thus, by Proposition 1.21 every full turn of $\gamma$ around the boundary component induces a subword of $\sigma(\gamma)$ of the form $\alpha_1 \cdots \alpha_r$, where the $\alpha_i$ form an oriented cycle of $Q$ such that every composition is a relation. Thus $\sigma(\gamma)$ is eventually periodic, with homotopy letters of length one. Since the infinite ends of $\gamma$ cycle around a boundary component in counter-clockwise direction, we get that the start (or the end) of $\sigma(\gamma)$, if infinite, is a direct (resp. inverse) antipath. This proves (2).

To prove (3), assume that $\gamma$ is a primitive closed curve. Write $\sigma(\gamma) := \prod_{i=1}^r \sigma(\gamma_i)$ as in the statement of the Lemma. Clearly, $s(\sigma(\gamma_1)) = t(\sigma(\gamma_r))$ and $\sigma(\gamma_1) \neq \sigma(\gamma_r)$. The condition on the number of inverse and direct homotopy letters among the $\sigma(\gamma_i)$ ensures that $\sigma(\gamma)$ is a homotopy band. \(\square\)

**Remark 2.12.** In Lemma 2.11 (3), it should be stressed that the condition on the number of inverse and direct homotopy letters among the $\sigma(\gamma_i)$ is not satisfied by all closed curves. It would be interesting to find a geometric characterisation of the closed curves that do satisfy the condition.

Conversely, any homotopy string or band defines an arc or a closed curve on $S_A$.

**Lemma 2.13.**

1. For any finite homotopy string $\tau$, there exists a unique finite arc $\gamma$ on $S_A$ (up to homotopy) such that $\tau = \sigma(\gamma)$.

2. For any infinite homotopy string $\tau$, there exists a unique infinite arc $\gamma$ on $S_A$ (up to homotopy) such that $\tau = \sigma(\gamma)$.

3. For any homotopy band $b$, there exists a unique closed curve $\gamma$ on $S_A$ (up to homotopy) such that $b = \sigma(\gamma)$.

**Proof.** We only prove (1); the proofs of (2) and (3) are similar. Write $\tau = \tau_1 \cdots \tau_r$, where each $\tau_i$ is a homotopy letter. Write $\tau_i = \alpha_1^i \cdots \alpha_{s_i}^i$, where the $\alpha_i^j$ are either all arrows or all inverse arrows. By Proposition 1.21 since there are no relations in the (possibly inverse) path $\alpha_1^1 \cdots \alpha_{s_1}^1$, then there are laminates $\ell_1^1, \ldots, \ell_{s_1}^{1+1}$ inside a unique polygon $P_1$ such that $\ell_1^1$ and $\ell_{s_1}^{1+1}$ have an endpoint on the same boundary segment of $P_1$, and $\ell_{j}^{1+1}$ follows $\ell_j^1$ in the clockwise order if $\tau_i$ is a direct homotopy letter, and counter-clockwise order if $\tau_i$ is an inverse homotopy letter.
Define $\gamma_i$ to be a segment in $P_i$ going from $\ell_i^1$ to $\ell_i^{i+1}$ if $\tau_i$ is a direct homotopy letter, or the other way around if $\tau_i$ is an inverse homotopy letter. We can assume that the endpoint of $\gamma_i$ is the starting point of $\gamma_{i+1}$.

If we define $\gamma(\tau)$ to be the concatenation of $\gamma_1, \ldots, \gamma_r$, then $\sigma(\gamma(\tau)) = \tau_1 \cdots \tau_r = \tau$. This proves the existence result.

To prove uniqueness, assume that $\gamma$ and $\gamma'$ are such that $\sigma(\gamma) = \sigma(\gamma')$. Let $\tau$ be the (unique) reduced expression of the homotopy string $\sigma(\gamma) = \sigma(\gamma')$. Then $\gamma(\tau)$ is homotopic to $\gamma$ and $\gamma'$. Indeed, if $\sigma(\gamma)$ is reduced, then $\tau = \sigma(\gamma)$ and we are done. Otherwise, it means that in the expression $\sigma(\gamma)_1 \cdots \sigma(\gamma)_r$ of $\sigma(\gamma)$ as a product of homotopy letters, there are two adjacent letters $\sigma(\gamma)_i$ and $\sigma(\gamma)_{i+1}$ that are inverse to each other. Then the corresponding segments in the polygon $P_i$ described above are the same path going in opposite directions; their concatenation is thus homotopic to a trivial path. Thus if we cancel the two inverse homotopy letters, we get that $\gamma\left(\sigma(\gamma)_1 \cdots \sigma(\gamma)_{i-1} \sigma(\gamma)_{i+2} \cdots \sigma(\gamma)_r\right)$ is homotopic to $\gamma(\sigma(\gamma))$.

By induction on the number of reduction steps to get from $\sigma(\gamma)$ to $\tau$, we get that $\gamma(\sigma(\gamma)) = \gamma(\tau)$.

The same applies if we replace $\gamma$ by $\gamma'$. This proves the uniqueness, and finishes the proof of the Lemma.

With this, we can prove Theorem 2.5.

Proof of Theorem 2.5. It follows from the results of [14] that indecomposable objects of $D^b(A - \text{mod})$ are in bijection with homotopy strings (finite and infinite) and homotopy bands paired with an isomorphism class of indecomposable $K[X]$-modules. By Lemma 2.11 we can associate a homotopy string or band to each of the curves listed in the statement of Theorem 2.5. Then Lemma 2.13 ensures that if the endpoint of $\gamma_{i+1}$ is algebraically closed, then $\tau_1 \cdots \tau_r = \tau$. This proves the existence result.

By induction on the number of reduction steps to get from $\sigma(\gamma)$ to $\tau$, we get that $\gamma(\sigma(\gamma)) = \gamma(\tau)$.

The same applies if we replace $\gamma$ by $\gamma'$. This proves the uniqueness, and finishes the proof of the Lemma.

3. Homomorphisms in the derived category of a gentle algebra

The morphism spaces in the bounded derived category $D^b(A - \text{mod})$ of a gentle algebra $A$ were completely described in [4]. Our aim in this section is to describe a basis of the morphism spaces in the orbit category $D^b(A - \text{mod})/\{1\}$ in terms of curves on the surface $S_A$ that was associated to $A$ in Section 1.

3.1. Bases for morphism spaces in the derived category. We now briefly recall the results of [4]. These results are proved in the case where the base field $K$ is algebraically closed; for the rest of this section, we will assume that we are in this situation. Also, their results deal with morphisms in $D^b(A - \text{mod})$, but we will immediately translate them to the setting of the orbit category $D^b(A - \text{mod})/\{1\}$. Let $\sigma$ and $\tau$ be two homotopy strings or bands. Let $P^*_{\sigma, \lambda}$ and $P^*_{\tau, \lambda}$ be the associated indecomposable objects in $D^b(A - \text{mod})/\{1\}$ (if $\sigma$ is a homotopy band and $\lambda \in K^*$, then we write $P^*_{\sigma, \lambda}$ instead of $P^*_{\sigma, \lambda}$). In all that follows, we consider $\sigma$ and $\tau$ only up to the action of the inverse operation $\overline{\cdot}$; this means that whenever we are comparing $\sigma$ and $\tau$, we also need to compare $\overline{\sigma}$ and $\overline{\tau}$ in order to get all morphisms.

3.1.1. Graph maps. Assume that $\sigma$ and $\tau$ have a maximal subword in common, say $\sigma_i \sigma_{i+1} \cdots \sigma_j$ and $\tau_i \tau_{i+1} \cdots \tau_j$, with each $\sigma_i$ equal to $\tau_i$. We also allow this subword to be a trivial homotopy string.

Consider the following conditions.
LG1: Either the homotopy letters $\sigma_{i-1}$ and $\tau_{i-1}$ are both direct and there exists a path $p$ in $Q$ such that $p\tau_{i-1} = \sigma_{i-1}$, or they are both inverse letters and there exists a path $p$ in $Q$ such that $\tau_{i-1} = \sigma_{i-1}p$.

LG2: The homotopy letter $\sigma_{i-1}$ is either zero or inverse, and $\tau_{i-1}$ is either zero or direct.

RG1: Dual of (LG1).

RG2: Dual of (LG2).

If one of (LG1) and (LG2) holds, and one of (RG1) and (RG2) holds, then one can construct a morphism from $P_{\sigma}^{*}$ to $P_{\tau}^{*}$ called a graph map. Note that if $\sigma$ and $\tau$ are infinite homotopy strings, then the definition above extends to the case where the strings have an infinite subword in common: for instance, if this subword is on the left, then one simply drops conditions (LG1) or (LG2).

3.1.2. Quasi-graph maps. Keep the above notations. If none of the conditions (LG1), (LG2), (RG1) and (RG2) hold, then one can construct a morphism in $\mathcal{D}^{b}(A-\text{mod})/\mathbb{Z}$ from $P_{\sigma}^{*}$ to $P_{\tau}^{*}$, called a quasi-graph map. Again, this definition extends to infinite homotopy strings in the natural way.

Note that a quasi-graph map gives rise to a homotopy class of single and double maps, defined in the next section. In fact, all single and double maps that are not singleton maps arise in this way, see [4].

3.1.3. Single maps. Assume that there are direct homotopy letters $\sigma_{i}$ and $\tau_{j}$ and a non-trivial path $p$ such that $s(p) = t(\sigma_{i})$ and $t(p) = t(\tau_{j})$. (What follows also works if $\sigma_{i}$ and $\tau_{j}$ are both inverse letters by working with $\overline{\tau}$ and $\overline{\sigma}$ instead).

Consider the following conditions:

L1: If $\sigma_{i}$ is direct, then $\sigma_{i}p \in I$.

L2: If $\tau_{i}$ is inverse, then $p\tau_{i} \in I$.

R1: If $\sigma_{i+1}$ is inverse, then $\overline{\tau}_{i+1}p \in I$.

R2: If $\tau_{i+1}$ is direct, then $p\tau_{i+1} \in I$.

If conditions (L1), (L2), (R1) and (R2) are satisfied, then $p$ induces a morphism of complexes from $P_{\sigma}^{*}$ to $P_{\tau}^{*}$ called a single map.

Assume, moreover, that

- $\sigma_{i+1}$ is zero or is a direct homotopy letter of the form $pa'_{i+1}$, where $\sigma'_{i+1}$ is a direct homotopy letter;
- $\tau_{i}$ is zero or is a direct homotopy letter of the form $\tau'_{i}p$, where $\tau'_{i}$ is a direct homotopy letter;
- $p$ is neither a subword of $\sigma_{i}$ nor a subword of $\tau_{i+1}$.

If that is the case, then $p$ induces a morphism from $P_{\sigma}^{*}$ to $P_{\tau}^{*}$ in $\mathcal{D}^{b}(A-\text{mod})/\mathbb{Z}$ called a singleton single map.

3.1.4. Double maps. Keeping the above notations, assume now that there are non-trivial paths $p$ and $q$ such that $s(p) = s(\sigma_{i})$, $t(p) = s(\tau_{j})$, $s(q) = t(\sigma_{i})$ and $t(q) = t(\tau_{j})$, and such that $\sigma_{i}q = p\tau_{j}$.

If conditions (L1) and (L2) above are satisfied for $p$ and conditions (R1) and (R2) are satisfied for $q$, then $p$ and $q$ induce a morphism of complexes from $P_{\sigma}^{*}$ to (a shift of) $P_{\tau}^{*}$ called a double map.

If, moreover, there exists a non-trivial path $r$ such that $\sigma_{i} = \sigma'_{i}r$ and $\tau_{j} = r\tau'_{j}$, with $\sigma'_{i}$ and $\tau'_{j}$ direct homotopy letters, then $p$ and $q$ induce a morphism from $P_{\sigma}^{*}$ to $P_{\tau}^{*}$ in $\mathcal{D}^{b}(A-\text{mod})/\mathbb{Z}$ called a singleton double map.
The basis. We can now state the main result of \cite{4}.

**Theorem 3.1** (Theorem 3.15 of \cite{4}). A basis of the space of morphisms from $P^\bullet_\sigma$ to $P^\bullet_\tau$ in $\mathcal{D}^b(A-\text{mod})/\langle 1 \rangle$ is given by all graph maps, quasi-graph maps, singleton single maps and singleton double maps in $\mathcal{D}^b(A-\text{mod})/\langle 1 \rangle$.

**Definition 3.2.** The basis described in Theorem 3.1 will be called the standard basis.

3.2. Morphisms as Intersections. As before, let $A = kQ/I$ denote a fixed gentle algebra and let $S_A$ denote its surface (see Section 1).

Let $\gamma_1$ and $\gamma_2$ be two arcs or closed curves on $S_A$. The main result of this section (Theorem 3.3) is that the set of intersection points of $\gamma_1$ and $\gamma_2$ gives rise to the standard basis of the vector space of morphisms from $P^\bullet_{\sigma(\gamma_1)}$ to $P^\bullet_{\sigma(\gamma_2)}$ in $\mathcal{D}^b(A-\text{mod})/\langle 1 \rangle$.

The precise statement of this result requires some notation. Throughout this section, we will replace all boundary components of $S_A$ without marked points by punctures, and consider infinite arcs wrapping around such a boundary component as a finite arc going to the puncture (see Remark 1.20). We can do this, since according to our conventions every infinite arcs wraps around such a boundary component only in one direction, namely the counter-clockwise direction.

Let $\gamma_1 \cap \gamma_2$ be the set of intersection points of $\gamma_1$ and $\gamma_2$ (including intersections on the boundary of $S_A$). To be very precise, we need to view the curves $\gamma_1$ and $\gamma_2$ as maps from the interval $[0, 1]$ to the surface $S_A$; an intersection point is then a point $(s_1, s_2) \in [0, 1]^2$ such that $\gamma_1(s_1) = \gamma_2(s_2)$. By abuse of language and notation, we will nevertheless speak of intersection points on $S_A$.

Let $\gamma_1 \cap \gamma_2$ be the subset of $\gamma_1 \cap \gamma_2$ of all interior intersections and of those boundary intersections of $\gamma_1$ and $\gamma_2$ such that locally around the intersection, $\gamma_1$ 'lies before' $\gamma_2$ in the counter-clockwise orientation as shown in the following picture.

Let us state the main result of this section.

**Theorem 3.3.** Let $\gamma_1$ and $\gamma_2$ be arcs or closed curves on $S_A$, and let $\mathcal{B}$ be the standard basis of $\text{Hom}_{\mathcal{D}^b(A-\text{mod})/\langle 1 \rangle}(P^\bullet_{\sigma(\gamma_1)}, P^\bullet_{\sigma(\gamma_2)})$. Then there exists an explicit injection

$$\mathcal{B} : \gamma_1 \cap \gamma_2 \to \mathcal{B}.$$ 

Moreover, the following hold true.

i) The map $\mathcal{B}$ is a bijection, unless $\gamma_1$ and $\gamma_2$ are the same closed curve and $P^\bullet_{\sigma(\gamma_1)}, \lambda_1$ and $P^\bullet_{\sigma(\gamma_2)}, \lambda_2$ are isomorphic.

ii) If $\gamma_1$ and $\gamma_2$ are the same closed curve and $P^\bullet_{\sigma(\gamma_1)}, \lambda_1$ and $P^\bullet_{\sigma(\gamma_2)}, \lambda_2$ are isomorphic, then $\mathcal{B}$ is not surjective, and the missing elements in its image
are an invertible graph map and the quasi graph map $\xi$ that appears in an Auslander-Reiten triangle

$$\tau P^{\bullet}(\gamma_1) \to E \to P^{\bullet}(\gamma_1) \xrightarrow{\xi} \tau P^{\bullet}(\gamma_1)[1]$$

(keeping in mind that in this case, $\tau P^{\bullet}(\gamma_1) = P^{\bullet}(\gamma_1)[1] = P^{\bullet}(\gamma_1)$ in $D^b(A - \text{mod})[1]$).

The proof of Theorem 3.3 occupies the rest of this section.

The Walk of an Intersection. It will be useful to lift intersection points of arcs and curves to a universal cover of $S_\Lambda$. Let $\pi : \hat{S}_\Lambda \to S_\Lambda$ be a fixed universal covering map, and let $\hat{L}_\Lambda$ be the set of all lifts of laminates $\ell \in L_\Lambda$. Note that $\hat{S}_\Lambda$ is a union of polygons whose edges are either boundary segments or laminates in $\hat{L}_\Lambda$. We lift arcs on $S_\Lambda$ to arcs on $\hat{S}_\Lambda$ and closed curves on $S_\Lambda$ to infinite lines on $\hat{S}_\Lambda$.

Let $\gamma_1$ and $\gamma_2$ be two arcs or closed curves on $S_\Lambda$. Let $q \in \gamma_1 \cap \gamma_2$, and let $\hat{q}$ be any lift of $q$ on $\hat{S}_\Lambda$. There are unique lifts $\hat{\gamma}_1$ and $\hat{\gamma}_2$ on $\hat{S}_\Lambda$ such that $\hat{\gamma}_1$ and $\hat{\gamma}_2$ intersect at $\hat{q}$.

**Lemma 3.4.** The curves $\hat{\gamma}_1$ and $\hat{\gamma}_2$ intersect only at $\hat{q}$.

**Proof.** It follows from [56] that $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are simple. Assume that they intersect twice in succession, say at $\hat{r}_1$ and $\hat{r}_2$. Then $\hat{r}_1$ and $\hat{r}_2$ are not two lifts of the same intersection point of $\gamma_1$ and $\gamma_2$. Indeed, the sections of $\gamma_1$ and $\gamma_2$ between $\hat{r}_1$ and $\hat{r}_2$ form a disc. Around the boundary of this disc, we can assume without loss of generality that $\hat{\gamma}_1$ comes before $\hat{\gamma}_2$ at $\hat{r}_1$ in the orientation of the surface. But then $\hat{\gamma}_2$ comes before $\hat{\gamma}_1$ at $\hat{r}_2$, which is impossible if $\hat{r}_1$ and $\hat{r}_2$ are lifts of the same intersection point of $\gamma_1$ and $\gamma_2$.

Therefore, by the bigon criterion (see [38], Proposition 1.7), we can find a homotopy of $\hat{\gamma}_1$ which descends to a homotopy of $\gamma_1$ that reduces the number of intersections with $\gamma_2$ - a contradiction with the assumption that the two are in minimal position.

Next, we define a region $S_{\hat{q}}$ of $\hat{S}_\Lambda$. Let $P_0$ be the polygon of $\hat{S}_\Lambda$ containing $\hat{q}$. Define a set of polygons $P_n$ recursively by setting $P_0 = \{P_0\}$ and by letting $P_{n+1}$ contain all polygons of $P_n$ and all polygons $P_n$ adjacent to a polygon of $P_n$ such that both $\hat{\gamma}_1$ and $\hat{\gamma}_2$ go through $P_n$. Then $S_{\hat{q}}$ is defined to be the union of all polygons belonging to one of the $P_n$. In other words, $S_{\hat{q}}$ is the region of $\hat{S}_\Lambda$ containing the laminates intersected by both $\hat{\gamma}_1$ and $\hat{\gamma}_2$ as well as $\hat{q}$.

**Lemma 3.5.** The surface $S_{\hat{q}}$ is a union of finitely many polygons.

**Proof.** The result is trivial if $\gamma_1$ or $\gamma_2$ is an arc. Assume that $\gamma_1$ and $\gamma_2$ are closed curves and suppose that $S_{\hat{q}}$ contains an infinite number of polygons. Since a fundamental domain of $S_\Lambda$ in $\hat{S}_\Lambda$ contains only finitely many polygons, one of the polygons in $S_{\hat{q}}$ will contain another lift of $q$, say $\hat{q}'$. At this lift, $\hat{\gamma}_1$ and $\hat{\gamma}_2$ must intersect. This contradicts Lemma 3.4.

Let $\delta_1$ and $\delta_2$ be the parts of $\gamma_1$ and $\gamma_2$ contained in $S_{\hat{q}}$. In the interior of $S_{\hat{q}}$, the curves $\delta_1$ and $\delta_2$ cross the same laminates of $\hat{L}_\Lambda$ in the same order. These crossings define a (possibly empty) homotopy string $\sigma(q)$. If $\sigma(q)$ is non-empty, it is a subwalk of $\sigma(\gamma_1)$ and $\sigma(\gamma_2)$ in a canonical way, where in case of a homotopy
The surface $S_{\tilde{q}}$: Dashed curves belong to $\tilde{L}_A$, whereas the blue and the red curve show $\delta_1$ and $\delta_2$, respectively, and solid black lines belong to $\partial \tilde{S}_A$. In this example $\sigma(q)$ has 3 homotopy letters.

Remark 3.6. Since we have replaced boundary components without marked points by punctures, one should note that if $\gamma_1$ and $\gamma_2$ meet at a puncture, then they cross infinitely many laminates. In this case, the walk $\sigma(q)$ is infinite on one side.

Lemma 3.7. If $\sigma(q)$ is empty, then the unique marked point in $P$ as before is contained in $\mathcal{M}$ (see Definition 1.8).

Proof. Suppose the marked point in $P$ is an element in $\mathcal{M}_0$. Then the only laminate on the boundary of $S_{\tilde{q}}$ is a lift of the laminate $\ell$ associated with a vertex $v \in Q_0$. But by definition of $S_{\tilde{q}}$, there exists $i \in \{1, 2\}$, such that $\gamma_i$ does not cross $\ell$. Thus, $\gamma_i$ is contained in $P$ and homotopic to a constant path - a contradiction. $\square$

We now explain how an intersection gives rise to a morphism. For $j \in \{1, 2\}$, denote $p_{j, 1}, \ldots, p_{j, m}$ the ordered sequence of intersections of the curve $\delta_j$ with the boundary or the laminates of $L_A$. We may assume that if $\sigma(q)$ is non-empty, then for each $i \in (1, m)$, $p_{1, i}$ and $p_{2, i}$ lie on the same laminate.

Then $\delta_1$ and $\delta_2$ in $S_{\tilde{q}}$ are homeomorphic to two arcs crossing in a closed disc; their endpoints alternate on the boundary of $S_{\tilde{q}}$. We distinguish two cases:

a) if $\sigma(q)$ is non-empty, then either $p_{1, 3}$ comes immediately before $p_{1, 1}$ in the counter-clockwise orientation of $\partial S_{\tilde{q}}$, or vice versa;

b) if $\sigma(q)$ is empty (i.e. $m = 2$), let $X$ be the unique marked point on the boundary of the polygon containing $\delta_1$ and $\delta_2$. Then either $X$ is after an endpoint of $\delta_2$ and before an endpoint of $\delta_1$ in the counter-clockwise orientation of $\partial S_{\tilde{q}}$, or vice versa.

The subsequent lemmas prove that $\delta_1$ and $\delta_2$ encode a basis element of the corresponding homomorphism space in a natural way.
Lemma 3.8. Let $B$ be the basis of $\text{Hom}_{\text{D}^b(A-\text{mod})/[1]}(P^{\sigma(\gamma_1)}, P^{\sigma(\gamma_2)})$ described in Theorem 3.1, and let $q \in \gamma_1 \cap \gamma_2$. Then $q$ gives rise to an element $B(q) \in B$. Furthermore,

i) If $\sigma(q)$ is non-empty, then $B(q)$ is a graph map if $p_2$ comes immediately before $p_1$ in the counter-clockwise orientation, and a quasi graph map otherwise.

ii) If $\sigma(q)$ is empty, then $B(q)$ is a singleton single or singleton double map.

Proof. Suppose first that $\sigma(q)$ is non-empty. Assume for the moment that $q$ is not on the boundary of $S_A$ and is not a puncture. Let $P_v$ be the polygon in which $p_1, p_2, p_3, p_4$ are found, and let $X$ be the marked point on its boundary. The position of $X$ with respect to $p_1$ and $p_2$ decides whether one of the conditions (LG1) to (RG2) of Section 3.1 is satisfied, as illustrated below. Note that this includes the case that $q$ is a boundary intersection, that is $p_1$ and $p_2$ both coincide with the marked point.

As the analogous statements hold for $p_{m-1}, p_{m-1}, p_m$ and $p_m$, it follows that the intersection point $q$ defines the data of a graph map if $p_2$ comes immediately before $p_1$ in the counter-clockwise orientation and the data of a quasi graph map otherwise.

Next, assume that $q$ is on the boundary of $S_A$, and that $\delta_1 \neq \delta_2$. Then (LG2) is automatically satisfied for $\sigma(q)$, and we need only repeat the above argument for $p_{m-1}, p_{m-1}, p_m$ and $p_m$.

Now, assume that $q$ is on the boundary of $S_A$, and that $\delta_1$ and $\delta_2$ are equal. This implies that $\gamma_1$ and $\gamma_2$ are the same arc, and that $q$ is on the boundary of $S_A$. Then $B(q)$ will be an identity map from one of $P^{\sigma(\gamma_1)}$ and $P^{\sigma(\gamma_2)}$ to the other. Indeed, we can choose representatives of the arcs $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ such that they only cross at their endpoints. In this case, one of the endpoints will be in $\gamma_1 \cap \gamma_2$ and the other in $\gamma_2 \cap \gamma_1$. The image of these points by $B$ will be the identity graph maps $P^{\sigma(\gamma_1)} \rightarrow P^{\sigma(\gamma_2)}$ and $P^{\sigma(\gamma_2)} \rightarrow P^{\sigma(\gamma_1)}$.

Finally, we note that in the case where $\delta_1$ and $\delta_2$ meet in one puncture, then $\sigma(q)$ is infinite on one side, and we need to look at the conditions (LG1) to (RG2) on one side only. If they meet at two punctures, then $\gamma_1$ and $\gamma_2$ are equal, and $B(q)$ will be an identity morphism, as above.

This finishes the proof for case i).
To prove case ii), suppose that $\sigma(q)$ is empty and that $q \notin \partial S_A$. Consequently, $\delta_1$ and $\delta_2$ are contained in a single polygon $P_v$. Depending on the position of the marked point in $P_v$, $q$ gives rise to different types of singleton maps. If the marked point lies between $p^j_2$ and $p^j_1$ in counter-clockwise order for some $j \in \{1, 2\}$, then $q$ defines the data of a singleton single map. In the other situation, i.e. the marked point lies between $p^j_1$ and $p^j_2$ in counter-clockwise order, $q$ gives rise to a singleton double map. In the previous picture this is the situation we obtain by interchanging the labels $\delta_1$ and $\delta_2$.

Finally, if $\delta_1$ or $\delta_2$ have a marked endpoint it can be seen that $q$ gives rise to a singleton single map.

Remark 3.9. The graph and quasi graph maps which occur in the previous Lemma as $\mathcal{B}(q)$ cannot be invertible graph maps or maps of the form $\xi$ occurring in Auslander-Reiten triangles as described in Theorem 3.3 (2).

Remark 3.10. The precise definition of $\mathcal{B}$ depends on the homotopy representatives of the curves $\gamma_1$ and $\gamma_2$.

Indeed, suppose that $\gamma_1$ and $\gamma_2$ are the same arc, and that $q$ is on the boundary of $S_A$. Recall from the proof of Lemma 3.8 that the identity morphisms between $P^\bullet_{\sigma(\gamma_1)}$ and $P^\bullet_{\sigma(\gamma_2)}$ are obtained as follows. First, choose representatives of the arcs
\( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) which only cross at their endpoints. Then one of the endpoints will be in \( \gamma_1 \cap \gamma_2 \) and the other in \( \gamma_2 \cap \gamma_1 \). The image of these points by \( \mathcal{B} \) will be the identity graph maps \( P_{\sigma(\gamma_1)}^* \to P_{\sigma(\gamma_2)}^* \) and \( P_{\sigma(\gamma_2)}^* \to P_{\sigma(\gamma_1)}^* \).

Choosing different representatives of \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) could lead to the first intersection point belonging to \( \gamma_2 \cap \gamma_1 \) and the second one belonging to \( \gamma_1 \cap \gamma_2 \). Then the images of these two points by \( \mathcal{B} \) would be permuted.

**Lemma 3.11.** Let \( f \in \text{Hom}_{D^b(A-\text{mod})/\langle 1 \rangle}(P_{\sigma(\gamma_1)}^*, P_{\sigma(\gamma_2)}^*) \) be an element of \( \mathcal{B} \) which is neither an invertible graph map nor a quasi graph map of the form \( \xi \) occurring inAuslander-Reiten triangles as described in Theorem 3.3 (2). (see also Remark 3.9). Then there exists a unique \( q \in \gamma_1 \cap \gamma_2 \) such that \( f = \mathcal{B}(q) \).

**Proof.** Let \( \tilde{\gamma}_1 \) be a lift of \( \gamma_1 \). We distinguish two cases.

First, assume that \( f \) is a graph or quasi graph map and let \( \sigma \) be the maximal common subword associated with \( f \) as in Section 3.1. Our assumptions imply that \( \sigma \) is finite. The subword \( \sigma \) of \( \sigma(\gamma_1) \) corresponds to a section \( \delta_1 \) of \( \tilde{\gamma}_1 \). Let \( \tilde{\gamma}_2 \) be the unique lift of \( \gamma_2 \) such that the section \( \delta_2 \) corresponding to the subword \( \sigma \) passes through the same polygons as \( \delta_1 \).

As we have seen in the proof of Lemma 3.5 the conditions (LG1), (LG2), (RG1) and (RG2) are equivalent to certain configurations of \( \delta_1 \), \( \delta_2 \) and of the marked point in the first and last polygons that \( \delta_1 \) and \( \delta_2 \) cross (see Figure 6). These conditions force \( \delta_1 \) and \( \delta_2 \) to intersect in a (unique) point \( \tilde{q} \). By construction, \( f = \mathcal{B}(\pi(\tilde{q})) \).

Next, assume that \( f \) is a singleton single or singleton double map. If \( f \) is a single map, denote by \( p \) the non-trivial path which appears in the definition of single maps, see Section 3.1.4. Otherwise, let \( p \) denote the non-trivial path which was denoted by \( r \) in the definition of singleton double maps, see Section 3.1.4. There exists a polygon \( P \) of the surface \( \tilde{S}_A \), which corresponds to \( p \) and is crossed by \( \tilde{\gamma}_1 \). We write \( \tilde{\gamma}_2 \) for the unique lift of \( \gamma_2 \) which crosses \( P \), and denote \( \delta_i \) the restriction of \( \tilde{\gamma}_i \) to \( P \). The combinatorial conditions in the definition of singleton single and singleton double maps are then equivalent to certain configurations of the marked point in \( P \) and the endpoints of \( \delta_1 \) and \( \delta_2 \) as shown in Figure 6 and Figure 7. As above, this proves that \( \delta_1 \) and \( \delta_2 \) intersect in a (unique) point \( \tilde{q} \), such that \( \mathcal{B}(\pi(\tilde{q})) = f \).

The previous lemma finishes the proof of Theorem 3.3.

4. **Mapping Cones in the Derived Category of a Gentle Algebra**

In this Section we will show that the mapping cone of a map in \( D^b(A-\text{mod})/\langle 1 \rangle \) is given by the homotopy strings of the two curves resolving the corresponding crossing.

**Theorem 4.1.** Let \( A \) be a gentle algebra and let \( P_\sigma^* \) and \( P_\tau^* \) be indecomposable objects in \( D^b(A-\text{mod})/\langle 1 \rangle \) with homotopy strings or bands \( \sigma \) and \( \tau \). Let \( f^* \in \text{Hom}_{D^b(A)}(P_\sigma^*, P_\tau^*) \) be an standard basis element. Then the indecomposable summands of the mapping cone \( M_{f^*}^* \) are given by the homotopy strings and bands occurring in the green and red boxes resulting from the following graphical calculus.

1. Let \( \sigma = \ldots \sigma_{i-2} \sigma_{i-1} \sigma_i \ldots \sigma_j \sigma_{j+1} \sigma_{j+2} \ldots \) and \( \tau = \ldots \tau_{i-2} \tau_{i-1} \tau_i \ldots \tau_j \tau_{j+1} \tau_{j+2} \ldots \) and suppose \( f^* \) is a graph map with common homotopy substring \( \sigma_i \ldots \sigma_j = \tau_i \ldots \tau_j \). Then \( M_{f^*}^* = P^*_{\sigma_1} \oplus P^*_{\sigma_2} \) with homotopy strings \( c_1 = \ldots \sigma_{i-2} \sigma_{i-1} \tau_{i-1} \tau_{i-2} \ldots \) and \( c_2 = \ldots \tau_{j+2} \tau_{j+1} \tau_{j+2} \ldots \).
is given by:

\( \sigma_i, \tau_i, \sigma_j, \tau_j \)

(2) Let \( \sigma = \ldots \sigma_i \sigma_{i+1} \ldots \) and \( \tau = \ldots \tau_j \tau_{j+1} \ldots \) and suppose \( f^\bullet \) is a single map. Then \( M_{f^\bullet} = P_{c_1}^\bullet \oplus P_{c_2}^\bullet \) with homotopy strings \( c_1 = \ldots \sigma_i-1 \sigma_i \tau_j \tau_{j+1} \ldots \) and \( c_2 = \ldots \tau_{j+1} \tau_{j+2} \) is given by:

(3) Let \( \sigma = \ldots \sigma_i-2 \sigma_{i-1} \sigma_i \sigma_{i+1} \sigma_{i+2} \ldots \) and \( \tau = \ldots \tau_{j-2} \tau_j \tau_{j+1} \tau_{j+2} \ldots \) and suppose \( f^\bullet \) is a double map. Then \( M_{f^\bullet} = P_{c_1}^\bullet \oplus P_{c_2}^\bullet \) with homotopy strings \( c_1 = \ldots \sigma_i-2 \sigma_{i-1} \tau_j \tau_{j+1} \ldots \) and \( c_2 = \ldots \tau_{j+1} \tau_{j+2} \ldots \) is given by:

Remark 4.2. Note that quasi-graph maps are implicitly treated in the previous theorem, since they give rise to homotopy classes of single and double maps.

Theorem 4.3. Let \( \sigma \) and \( \tau \) be homotopy strings or bands and \( f^\bullet : P_{\sigma}^\bullet \to P_{\tau}^\bullet \) be a map in \( D^b(A - \text{mod})/\mathbb{Z} \) associated to a crossing point \( \gamma(\sigma) \cap \gamma(\tau) \) of the corresponding arcs \( \gamma(\sigma) \) and \( \gamma(\tau) \). Suppose further that \( f^\bullet \) is different from the identity and if \( \sigma \) is a band, different from the map \( P_{\sigma}^\bullet \to \tau P_{\sigma}^\bullet \mathbb{Z} \). Let \( M_{f^\bullet} = P_{c_1}^\bullet \oplus P_{c_2}^\bullet \) be its mapping cone (as described in Theorem 4.1). Then the arcs \( \gamma(c_1) \) and \( \gamma(c_2) \) corresponding to \( P_{c_1}^\bullet \) and \( P_{c_2}^\bullet \) are given by the following resolution of the crossing \( \gamma(\sigma) \cap \gamma(\tau) \).
Figure 8. Curves associated to the mapping cone $M_f^* = P_{c_1}^* \oplus P_{c_2}^*$ of a map $f^*: P_\sigma^* \to P_\tau^*$.

Proof. We first consider the case when $f^*$ is a graph map. Locally in the surface this corresponds to the following configuration.

Figure 9. Curves associated to the mapping cone $M_f^* = P_{c_1}^* \oplus P_{c_2}^*$ of a map $f^*: P_\sigma^* \to P_\tau^*$ when $f^*$ is a graph map. In this picture we have decomposed $\gamma(\sigma)$ into segments $\gamma(\sigma_{i-2}\sigma_{i-1})$, $\gamma(\sigma_i\ldots\sigma_j)$ and $\gamma(\sigma_{j+1}\sigma_{j+2}\ldots)$ and $\gamma(\tau)$ into segments $\gamma(\tau_{i-2}\tau_{i-1})$, $\gamma(\tau_i\ldots\tau_j)$ and $\gamma(\tau_{j+1}\tau_{j+2}\ldots)$.

The blue dotted region corresponds to the topological disc $S_q$ of Section 3.2. Thus we see that the curve $\gamma_{c_1}$ at the top is split into two subcurves, so that $\gamma(c_1) = \gamma(\sigma_{i-2}\sigma_{i-1}) \gamma(\sigma_i\ldots\sigma_j) \gamma(\sigma_{j+1}\sigma_{j+2}\ldots)$. This proves that $P_{c_1}^*$ has the form in the statement of the theorem.

A similar argument at the bottom of the picture proves the result for $P_{c_2}^*$.

Next, we treat the case of single maps. In that case, $\gamma(\sigma)$ and $\gamma(\tau)$ meet in a polygon which forms the whole of $S_q$. 

We see that $\gamma(c_2)$ is obtained by $\gamma(\tau_{i+1}\tau_{i+2}\ldots)^{-1}\gamma(p)^{-1}\gamma(\sigma_{i+1}\sigma_{i+2}\ldots)$, as in the previous case. We also see that $\gamma(c_1)$ is obtained in a similar fashion, by noticing that $\sigma_i\tau_i$ contains one copy of $p$, since $\sigma_i$ ends in (and $\tau_i$ starts in) $p^{-1}$.

The remaining cases of a double map or of a single map arising from an intersection on the boundary of $S_A$ are treated in a similar fashion. \hfill \Box

5. Auslander-Reiten triangles

It is shown in [44] that in the bounded derived category of a module category of a finite dimensional algebra, Auslander-Reiten triangles ending in an indecomposable object $X$ exist if and only if $X$ is perfect. For a gentle algebra $A$, the indecomposable perfect objects in $D^b(A\text{-mod})$ are given by the string objects with finite homotopy string and the band objects. The purpose of this section is to show that an Auslander-Reiten triangle of a perfect string object $P^\bullet$ is determined by the corresponding arc $\gamma(\sigma)$ on $S_A$.

For this, we make the convention that given a finite homotopy string $\sigma$ with corresponding arc $\gamma(\sigma)$, the start of $\gamma(\sigma)$ corresponds to $s(\sigma)$ and the end of $\gamma(\sigma)$ corresponds to $t(\sigma)$ and we define the following:

1. Let $s_\sigma$ be the homotopy string corresponding the arc $s_\gamma(\sigma)$ obtained from $\gamma(\sigma)$ by rotating its start clockwise to the next marked point on the boundary.

2. Let $e_\sigma$ be the homotopy string corresponding the arc $e_\gamma(\sigma)$ obtained from $\gamma(\sigma)$ by rotating its end clockwise to the next marked point on the boundary.

3. Let $s_{e_\sigma}$ be the homotopy string corresponding the arc $s_{e_\gamma(\sigma)}$ obtained from $\gamma(\sigma)$ by rotating its end and its start clockwise to the next marked point on the boundary.

\begin{center}
\begin{figure}
\includegraphics[width=\textwidth]{figure10.png}
\caption{Figure 10}
\end{figure}
\end{center}
It follows from the above that $s \gamma(\sigma) = \gamma(s \sigma)$, $\gamma(\sigma)_e = \gamma(\sigma_e)$ and $s \gamma(\sigma)_e = s(\gamma(\sigma))_e$.

We can now state the main result of this Section.

**Theorem 5.1.** Let $P^\bullet_\sigma \in D^b(A - \text{mod})/[1]$ be an indecomposable object with finite homotopy string $\sigma$. Then the Auslander-Reiten triangle starting in $P^\bullet_\sigma$ is given by

$$
P^\bullet_\sigma \xrightarrow{\gamma(\sigma)} P^\bullet_{s \sigma} \oplus P^\bullet_{s \sigma} \xrightarrow{h} P^\bullet_{\sigma}[1]
$$

Furthermore, every morphism in the above triangle can be given by a standard basis element corresponding to the associated intersection.

It directly follows that the Auslander-Reiten translate $\tau P^\bullet_\sigma$ of $P^\bullet_\sigma$ corresponds to rotating both endpoints of the corresponding arc $\gamma(\sigma)$.

**Corollary 5.2.** Let $P^\bullet_\sigma \in D^b(A - \text{mod})/[1]$ be an indecomposable object with finite homotopy string. Let $\tau^{-1} \gamma(\sigma)$ be the arc corresponding to $\tau^{-1} P^\bullet_\sigma$. Then

$$
\tau^{-1} \gamma(\sigma) = \gamma(s \sigma_e)
$$

In Figure 12 we give an example of the geometric realisation of the Auslander-Reiten translate of $P^\bullet_\sigma$.

**Remark 5.3.** A version of Theorem 5.1 holds for string complexes of homotopy strings which are infinite. Indeed, with a similar proof, one can show that these irreducible maps [4] are represented by intersections of arcs $\gamma(\sigma)$ and $s(\gamma(\sigma))_e$ (resp. $\gamma(\sigma)_e$), where $s(-)$ (resp. $(-)_e$) is extended to arcs which end (resp. start) at a puncture. In this case, the corresponding intersection is at the puncture and the associated map is a graph map given by an infinite subword.
We now recall some general facts on Auslander-Reiten triangles in $K^b(A - \text{proj})$. The first explicit description of such triangles for gentle algebras was given in [17].

For every indecomposable object $P^\bullet_\sigma \in K^b(A - \text{proj})$, where $\sigma$ is a finite homotopy string or a homotopy band, there exists a distinguished triangle

$$P^\bullet_\sigma \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

in $K^b(A - \text{proj})$ such that the composition $h \circ u$ vanishes for every non-split morphism $u : U \rightarrow Z$. Such a triangle is unique up to (non-unique) isomorphism. Furthermore, $Z$ is indecomposable and is denoted by $\tau^{-1}P^\bullet_\sigma$. It defines a bijection $\tau$, called the Auslander-Reiten translate, on the set of isomorphism classes of indecomposable objects.

Moreover,

- $Y$ is the direct sum of at most two indecomposable objects and up to isomorphism the entries in $f, g, h$ are standard basis elements [4].
- If $P^\bullet_\sigma$ is a band complex associated with a $m$-dimensional $K[X]$-module, then $Z \cong P^\bullet_\sigma$, that is $\tau^{-1}P^\bullet_\sigma = P^\bullet_\sigma$, and $Y$ is (isomorphic to) a direct sum of band complexes associated with $m \pm 1$-dimensional $K[X]$-modules.

It can be shown that if $P^\bullet_\sigma$ is a band complex associated with a 1-dimensional $K[X]$-module, then $h$ as above is not represented by an intersection. In particular, if the representing closed curve of $P_\sigma$ is simple, then none of the maps in an Auslander-Reiten triangle are represented by an intersection.

Let $\sigma$ be a finite homotopy string. The arc $\gamma(\sigma)$ shares its endpoint with $s\gamma(\sigma)$ and its starting point with $e\gamma(\sigma)$. Therefore we have distinguished boundary intersections $\gamma(\sigma) \cap s\gamma(\sigma)$ and $\gamma(\sigma) \cap e\gamma(\sigma)$.

**Proof of Theorem 5.1.** We first note that since $s\gamma(\sigma) = \gamma(\sigma)_e$, we have $s\sigma = (\sigma_e)$. Therefore it is enough to prove the result for $f_e$, the proof for $s\sigma$ then follows. Furthermore, the proof for $sg$ and $g_e$ then also follows noting that $s\sigma_e = (\sigma_e) = (s\sigma)_e$.

To prove that $f_e$ is an irreducible map of the required form we follow Algorithm 6.3 in [4] step by step. Algorithm 6.3 in [4] breaks down into five cases. In each case, it suffices to prove that $\sigma_e$ is the homotopy string of the resolution of the boundary intersection of $\gamma(\sigma)$ with the arc $\gamma_B$ containing in its homotopy class the boundary arc connecting the end of $\gamma(\sigma)$ and the end of $\gamma(\sigma)_e$. Let $\rho$ be the homotopy string of $\gamma_B$, that is $\gamma_B = \gamma(\rho)$. Locally in the surface we have the following configuration

For what follows, write $\sigma = \sigma_1 \cdots \sigma_n$ with homotopy letters $\sigma_i$. 
Case 1: Suppose that there exists a maximal path \( q \) in \( Q \) (which then corresponds to a single homotopy letter) and a maximal inverse antipath \( \theta = \theta_1 \cdots \theta_m \), such that \( \sigma q \theta \) is a homotopy string. Note that by Remark 6.5 (4) in \([4]\), \( \theta \) is finite. Furthermore, \( \gamma(\sigma) \) ends on the marked point corresponding to the start of \( \gamma(q) \). Since \( t(q) = s(\theta) \) and \( s(q) = t(\sigma) \), \( \sigma q \theta \) is obtained from \( \sigma \) as the homotopy string of the mapping cone of the singleton single map \( \phi^* : P_{\theta_1} \cdots \theta_m \rightarrow P_{\sigma}^* \) induced by \( q \) and thus \( \sigma_e = \sigma q \theta \). By maximality of \( \theta \), it follows that \( \gamma(\theta) \sim \gamma(\rho) \) and the single map \( \phi^* \) corresponds to the boundary intersection \( \gamma(\rho) \overline{\gamma(\sigma)} \). Then \( f_e \) is a graph map induced by the subword \( \sigma \) and hence is represented by the distinguished intersection \( \gamma(\sigma) \overline{\gamma(\sigma)}_e \) as claimed.

The other cases are treated in a similar way and we only give an outline for each.

Case 2: Suppose that \( \sigma_{r+1} \cdots \sigma_n \) is a direct antipath and that \( \sigma_r \) is an inverse homotopy letter. Suppose further that there exists \( \alpha \in Q_1 \) such that \( \alpha \sigma_r \not\in I \). Assume also that there exists a maximal inverse antipath \( \theta \), such that \( \overline{\sigma \theta} \) is a homotopy string. Then we have \( \rho = \overline{\sigma \theta} \) and \( \sigma_e = \alpha \cdots \sigma_r \sigma \theta \), which is the homotopy string of the mapping cone of a graph map \( P_{\rho}^* \rightarrow P_{\sigma}^* \) associated with the common subword \( \sigma_{r+1} \cdots \sigma_n \) and \( f_e \) is a graph map associated with the common subword \( \sigma_1 \cdots \sigma_r \) of \( \sigma \) and \( \sigma_e \).

Case 3: Suppose that \( \sigma_{r+1} \cdots \sigma_n \) is a direct antipath and that \( \sigma_r \) is an inverse homotopy letter. Suppose further that there exists \( \alpha \in Q_1 \) such that \( \alpha \sigma_r \not\in I \). In this case, \( \sigma_e = \alpha \cdots \sigma_{r-1} \) and \( \rho = \sigma_{r+1} \cdots \sigma_n \) where \( \sigma_e \) is the homotopy string of the mapping cone of the graph map \( P_{\rho} \rightarrow P_{\sigma} \) associated to the common subword \( \sigma_{r+1} \cdots \sigma_n \) and \( f_e \) is a graph map associated with the common subword \( \sigma_e \).

Case 4 Suppose that \( \sigma_{r+1} \cdots \sigma_n \) is a direct antipath and that \( \sigma_r \) is a direct homotopy letter and write \( \sigma_r = q \alpha \) where \( \alpha \in Q_1 \). Let \( \theta \) be a maximal inverse antipath such that \( \sigma_1 \cdots \sigma_{r-1} \overline{q \theta} \) is a homotopy string. Then one verifies that \( \sigma_e = \alpha \cdots \sigma_{r+1} q \theta \) and \( \rho = \overline{\theta \alpha} \sigma_{r+1} \cdots \sigma_n \), where \( \sigma_e \) is the homotopy string of the mapping cone of the graph map \( P_{\rho}^* \rightarrow P_{\sigma}^* \) associated to the common subword \( \sigma_{r+1} \cdots \sigma_n \) and \( f_e \) is given by the graph map determined by the common subword \( \sigma_1 \cdots \sigma_{r-1} \).

Case 5: Suppose that \( \sigma \) is a direct antipath and suppose that there exists \( \alpha \in Q_1 \) such that \( \alpha \sigma_1 \in I \). Let \( \theta \) be a maximal inverse antipath starting at \( s(\alpha) \). Then \( \sigma_e = \theta \), which is the mapping cone of the graph map \( P_{\rho}^* \rightarrow P_{\sigma}^* \) associated to the subword \( \sigma \), where \( \rho = \overline{\theta \alpha} \sigma \). In that case, \( f_e \) is a singleton single map.

If none of the above cases hold, then \( \sigma_e \) is empty and \( P_{\rho}^* = 0 \), so there is nothing to show. \( \Box \)

6. Avella-Alaminos–Geiss invariants in the surface

In \([10]\) Avella-Alaminos and Geiss define invariants for derived equivalence classes of gentle algebras. We will refer to these invariants as AG-invariants. In this Section we show that these derived invariants are encoded in the ribbon surface of a gentle algebra. In their paper Avella-Alaminos and Geiss show that two gentle
algebras that are derived equivalent have the same AG-invariant but they also give an example of two gentle algebras that are not derived equivalent yet have the same AG-invariant. Since then, many other examples of non-derived equivalent gentle algebras with the same AG-invariants have appeared in the literature, see for example [15, 11].

6.1. The Avella-Alaminos–Geiss invariants. We begin by briefly recalling the definition of the AG-invariants. Let \( A = KQ/I \) be a gentle algebra with augmented set of maximal paths \( \overline{M} = M \cup M_0 \) (see Definition 1.8). Let \( \mathcal{F} \) be the set of paths \( w \) in \((Q, I)\) such that if \( w = a_1 \ldots a_n \) then \( a_i a_{i+1} \in I \) for all \( i \in \{1, \ldots, n-1\} \), and such that \( w \) is maximal for this property, that is for all \( \beta \in Q_I \), if \( t(\beta) = s(\alpha) \) then \( \beta \alpha \notin I \) and if \( t(\alpha_i) = s(\beta) \) then \( \alpha_i \beta \notin I \). Let \( \mathcal{F}_0 = \{e_v \mid v \in W_0\} \) where \( W_0 \) is the subset of \( Q_0 \) containing all vertices that are either the source or target of only one arrow and those vertices that are the target of exactly one arrow \( \alpha \) and the source of exactly one arrow \( \beta \) and \( \alpha \beta \in I \).

Let \( H_0 = m_0 \) with \( m_0 \in \overline{M} \). Set \( F_0 = f_0 \) where \( f_0 \) is the unique element in \( \mathcal{F} \), if it exists, such that \( t(f_0) = t(m_0) \) and such that if \( m_0 = p \alpha \) is non-trivial with \( \alpha \in Q_I \) then \( f_0 = q \beta \) with \( \beta \neq \alpha \) and \( \beta \in Q_1 \). If no such \( f_0 \in \mathcal{F} \) exists then we set \( f_0 = \epsilon_t(m_0) \). Note that in this case \( \epsilon_t(m_0) \in \mathcal{F}_0 \).

Now define \( H_1 = m_1 \) where \( m_1 \) is the unique element in \( \mathcal{M} \), if it exists, such that \( s(m_1) = t(f_0) \) and such that if \( f_0 = q \beta \) is non-trivial with \( \gamma \in Q_1 \) then \( m_1 = \delta \beta \) with \( \delta \neq \gamma \) and \( \delta \in Q_1 \). If no such \( m_1 \) exists then we set \( m_1 = \epsilon_s(f_0) \) and we note that \( \epsilon_s(f_0) \in \mathcal{M}_0 \).

Define \( F_{i-1} \) and \( H_i \) for \( i \geq 2 \) in an analogous way to the above. The algorithm stops as soon as \( H_i = H_0 \) and we set \( k = i \). Set \( l \) equal to the number of arrows in \( F_0, \ldots, F_{k-1} \).

We repeat this process until every element of \( \overline{M} \) has appeared once as one of the \( H_i \). This gives rise to a set of tuples \((k, l)\). We add to this a pair \((0, n)\) for each full cycle of relations of length \( n \).

The AG-invariant of \( A \) is the function \( \phi_A : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) defined by sending \((i, j)\) to the number of pairs corresponding to these entries in the above algorithm. the boundary consists of as many boundary segments as there are marked points.

**Theorem 6.1.** Let \( A \) be a gentle algebra with associated ribbon surface \( S_A \) and lamination \( L_A \). Let \( B_1, \ldots, B_n \) be the boundary components in \( S_A \). Then the AG-invariant of \( A \) is given by the set of pairs \((b_i, c_i)\) for \( 1 \leq i \leq n \) where

- \( b_i \) is given by the number of marked points on \( B_i \),
- \( c_i = l_i - b_i \) where \( l_i \) is equal to the number of laminates starting or ending on \( B_i \).

Furthermore, if \( b_i \neq 0 \), we also have \( c_i = \sum_j k_j - 2 \) where \( j \) runs over all \( k_j \)-gons which have at least one side isotopic with a boundary segment of \( B_i \).

Note that in Theorem 6.1 if a laminate ends and starts on the same boundary component, then it is counted twice.

**Proof of Theorem 6.1.** First suppose that \( B \) is a boundary component with no marked points. Then, by Proposition 1.12 \( B \) lies in the interior of an \( n \)-gon \( P \) which corresponds to an \( n \)-cycle with full relations in \((Q, I)\). Therefore it corresponds to a pair \((0, n)\) in the algorithm of the AG-invariant. Furthermore, by construction each side of \( P \) corresponds to exactly one laminate incident with \( B \).
Now let $B$ be a boundary component with marked points $m_1, \ldots, m_r$ ordered in counter-clockwise occurrence on $B$. Then set $H_0$ to be the maximal path associated to the fan at $m_1$ or if $i_1$ is the only edge of $\Gamma_A$ incident with $m_1$ set $H_0 = e_{i_1}$. Let $F_0$ be the inverse path corresponding to the arrows inscribed in the polygon $P_1$ with boundary segment between $m_1$ and $m_2$. Clearly if $P_1$ has $k_1$ edges (exactly one of which is a boundary segment by Proposition 1.12) then there are $k_1 - 2$ arrows inscribed in that polygon giving an element in $\mathcal{F}$ except when $k_1 = 2$ in which case we set $F_0 = e_{i_1}$, where $j_1$ is the only internal edge of $P_1$. Now let $H_1$ be the path corresponding to the maximal fan at $m_2$ or if this fan consists of a single edge $i_2$ then set $H_1 = e_{i_2}$. We set $F_1$ to be equal to the inverse path consisting of $k_2 - 2$ inverse arrows inscribed in the $k_2$-gon $P_1$ with (unique) boundary segment between $m_2$ and $m_3$ where $F_1 = e_{j_2}$ with $j_2$ the only internal edge of $P_1$ if $k_2 = 2$.

We continue in a similar fashion along the boundary component $B$ in a counter-clockwise direction until we return to the fan at $m_1$. At this point the algorithm repeats and therefore stops and we move on to the next boundary component. The number of steps in each part of the algorithm is given by the number of fans on the boundary component which is equal to the number of marked points on $B$. The total number of arrows in the inverse paths at $B$ corresponds to the sum of the arrows in the $k_j$-gons $P_j$ incident with $B$, that is it is equal to $\sum_{j=1}^r k_j - 2$ as claimed. We repeat this for every boundary component, thus covering every element in $\mathcal{M}$ exactly once.

Given a $k_j$-gon $P_j$ with one side isotopic to a boundary component $B_i$, it follows from the construction of the lamination $L_A$ that there are exactly $k_j - 1$ laminates incident with the only boundary edge of $P_j$ and since there are as many marked points on a boundary component as there are boundary segments, we have $c_i = l_i - b_i$ as claimed.

\[ \square \]

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