Topological Defects in Gravitational Theories with Non Linear Lagrangians

J. Audretsch, A. Economou and C.O. Lousto

Fakultät für Physik der Universität Konstanz, Postfach 5560, D - 7750 Konstanz, Germany

The gravitational field of monopoles, cosmic strings and domain walls is studied in the quadratic gravitational theory $R + \alpha R^2$ with $\alpha |R| \ll 1$, and is compared with the result in Einstein’s theory. The metric acquires modifications which correspond to a short range ‘Newtonian’ potential for gauge cosmic strings, gauge monopoles and domain walls and to a long range one for global monopoles and global cosmic strings. In this theory the corrections turn out to be attractive for all the defects. We explain, however, that the sign of these corrections in general depends on the particular higher order derivative theory and topological defect under consideration. The possible relevance of our results to the study of the evolution of topological defects in the early universe is pointed out.

I. INTRODUCTION

After the paradigm of the Hilbert’s Lagrangian formulation of Einstein’s theory of gravity it was clear how one could consistently formulate other, higher derivative gravitational theories (that is theories in which the field equations have higher than second metric derivatives). And such theories where indeed proposed and used as alternatives to Einstein’s theory in attempts to unify other fields with gravity [1] and to remedy some of its seemingly undesirable consequences as, for example, at the classical level, the avoidance of cosmological singularities [2] and, at the quantum level, the non renormalizability of the quantized version of general relativity [3].

One of the main motivations for studying higher derivative theories comes from the semiclassical general relativity. There, it seems to be a matter of self-consistency to consider higher derivative terms in the gravitational Lagrangian since such terms arise generically in one-loop calculations [4]. Certainly this notion of self-consistency is a delicate issue and, as Simon has recently suggested [5], it needs to be reconsidered if one wants to avoid undesirable semiclassical predictions such as unstable Minkowski spacetime. Another recent motivation for considering higher derivative gravitational theories is that such theories have arisen as low energy limit of several superstring theories [6].

Higher derivative theories are of interest to cosmology mainly because, even vacuum theories admit cosmological models which give rise to the, so called, Starobinsky inflation [7] (see however Ref. [8] for a criticism on its consistency in the semiclassical limit), without fine tuning of the initial conditions [8].

In this paper we want to look at another topic of cosmological relevance namely, the effects of higher derivative theories on the gravitational field of topological defects as monopoles, cosmic strings and domain walls. These are objects that may have formed during phase transitions in the cooling down of the Early-Universe and may have played a key role in the formation of the large scale structure of the Universe mainly through their gravitational interactions [9]. Since their main interaction is gravitational, it is important to have an idea of what modifications one should expect in their gravitational field when the relevant gravitational theory has higher derivatives. Some work has recently been done in this direction, but only for gauge cosmic strings [10]. This work shows that in the weak field limit only short range corrections to the Einstein theory arise which are associated with the presence of additional massive fields in the spectrum of higher derivative theories. However, this is not expected to be in general true, especially for global topological defects which are extended field configurations and not localized as the gauge cosmic strings.

In this work we have in mind theories that can be separated in a part $\mathcal{L}_G$ for the gravitational field $g_{\mu\nu}$ constructed with geometrical scalars of the Ricci tensor $R_{ab}$, and another part $\mathcal{L}_M$ containing matter fields with standard coupling to the gravitational field $g_{\mu\nu}$.
Hereafter $\kappa := 8\pi G$ where $G$ is the gravitational constant. For theories of this type it has been noted that they can be recasted into an equivalent theory of Einstein gravity interacting with additional matter fields \[12, 14\]. However, as it was stressed by Brans \[15\] and we shall explain in the next section, this equivalence is in general only at a mathematical level and not at a physical one. Nevertheless, based on such an equivalent system, Whitt \[12\] was able to show that the black holes of general relativity are the only black hole solutions of $R + R^2$ theories (no hair theorem).

For the discussion of this paper we will deal with theories that have as gravitational part the following, often appearing in the literature, Lagrangians

$$L_G = \sqrt{-g}(R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu})$$

and

$$L_G = \sqrt{-g}F(R),$$

where $\alpha, \beta$ are some coupling constants, $g := \det g_{\mu\nu}$, and $R = g^{\mu\nu}R_{\mu\nu}$. Finally the $F$ in Eq. (3) is in principle an arbitrary function of the curvature scalar $R$. However, later on we will take $F$ to differ only slightly from the Einstein value $R$, that is $F = R + \alpha R^2$ with $\alpha|R| \ll 1$. See Ref. \[12\] for a treatment of the $F = R + \alpha R^2$ theory in vacuum and the Ref. \[16\] together with references therein for generalizations to arbitrary $F(R)$ in the presence of particular forms of matter.

The structure of the paper is as follows. Section II contains a brief review of higher derivative theories to the extent needed in this paper. The field equations for the theories in (1),(2) and (1),(3), are written down, and their spectrum is explained. With the procedure that enables the recasting of these field equations into Einstein type ones we obtain the basic result that is used in the Sec. III for the comparison of the gravitational field of global monopoles, cosmic strings and domain walls in Einstein’s theory, and in the quadratic $R + \alpha R^2$ theory with $\alpha|R| \ll 1$. Section III also contains at the beginning a brief introduction to the topological defects. Finally in Sec. IV we conclude with a brief summary and comments.

Throughout this paper we use the conventions $\hbar = c = 1$, metric signature $(-+++)$, Riemann tensor $R^a_{\ bcd} := -\partial_d \Gamma^a_{bc} + \ldots$, and Ricci tensor $R_{\mu\nu} := R^c_{\ aceb}$.

II. THEORIES WITH HIGHER DERIVATIVES

A. Field equations

We shall give now the gravitational field equations for the higher derivative theories given by the Eqs. (1), (2) and Eqs. (1), (3). The field equations for $g_{\mu\nu}$ are obtained by varying the action corresponding to Eq. (1) with respect to $g_{\mu\nu}$ and contain derivatives of the metric up to the fourth order. For the case of $L_G$ of Eq. (2) they read

$$ (1 + 2\alpha R)(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \frac{\alpha}{2}R^2 g_{\mu\nu} $$

$$ + (2\alpha + \beta)g_{\mu\nu}R_{p\nu}^\ p - (2\alpha + \beta)R_{\mu\nu} $$

$$ + \beta R_{\mu\nu;p}^\ p - \frac{\beta}{2}R_{pq}R^{pq}g_{\mu\nu} + 2\beta R_{pq}R_{\mu\nu}^\ p + g $$

$$ = \frac{-2\kappa}{\sqrt{-g}} \delta S_M := \kappa T^{(M)}. $$

Notice that the trace of this equation is an inhomogeneous massive Klein-Gordon equation for the curvature scalar $R$

$$ (6\alpha + 2\beta)R_{p\nu}^\ p - R = \kappa T^{(M)}. $$

Finally, the field equations for the theory (1) and (3) can be written as
\[ F'G_{\mu\nu} = \kappa T^{(M)}_{\mu\nu} + \frac{1}{2} g_{\mu\nu} (F - F' R - 2 F'_p) + F'_p, \] (6)

where \( F' = \partial F / \partial R \) and \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor. The trace of this equation is

\[ 3 F'_p + F' R - 2 F = \kappa T^{(M)}. \] (7)

B. Spectrum of quadratic theories -
Weak gravitational limit

We would like to stress here the fact that quadratic theories do not contain only the usual massless (long-range) spin-2 graviton field but also, in general, two massive (short-range) fields with spin-0 and spin-2.

This spectrum can be easily recognized in the case of \( L_G \) of Eq. (2) when one writes the field equations in the linearized weak field limit using a convenient gauge (coordinate system). Indeed following Teyssandier [17] we have that \( g_{\mu\nu} \) can be decomposed in the weak gravitational limit (where \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \) with \( |h_{\mu\nu}| << 1 \)) as

\[ g_{\mu\nu} = \eta_{\mu\nu} + h^{(E)}_{\mu\nu} + \chi \eta_{\mu\nu} + \psi_{\mu\nu}, \] (8)

with the field equations

\[ \Box h^{(E)}_{\mu\nu} = -2 \kappa (T_{\mu\nu} - \frac{1}{2} T \eta_{\mu\nu}), \]
\[ (\Box - m_0^2) \chi = -\frac{1}{3} \kappa T, \quad m_0^{-2} := 6 \alpha + 2 \beta, \]
\[ (\Box - m_1^2) \psi_{\mu\nu} = 2 \kappa (T_{\mu\nu} - \frac{1}{3} T \eta_{\mu\nu}), \quad m_1^{-2} := -\beta, \] (9)

and the gauge conditions

\[ \partial^a (h^{(E)}_{\mua} - \frac{1}{2} h^{(E)}_{\lambda\nu} \eta_{\mua}) = 0, \]
\[ (\psi^{ab} - \psi^{\lambda} \eta^{ab})_{,ab} = 0. \] (10)

Here indices are raised and lowered with the Minkowski metric tensor and the operator \( \Box \) is the Minkowskian one. One recognizes in Eqs. (8)-(13) the usual Einstein contribution \( h^{(E)}_{\mu\nu} \), that is the graviton field which has 2 degrees of freedom. Then, a scalar field \( \chi \) with mass \( m_0 \), which obviously has one degree of freedom and appears as an overall conformal factor (in the considered approximation). Finally the massive tensorial field \( \psi_{\mu\nu} \) with mass \( m_1 \) which turns out to have five degrees of freedom (note that in contrast to \( h^{(E)}_{\mu\nu} \) its components satisfy only one gauge condition) and thus it possesses the structure of a massive spin-2 field. In order to keep the “mass” parameters \( m_0, m_1 \) real we shall demand the no-tachyon constraint

\[ 3 \alpha + \beta \geq 0, \quad \beta \leq 0. \] (11)

We leave for the next subsection the case of the Lagrangian in Eq. (3) where we will go beyond the weak gravitational limit and we will see that the spectrum of this theory consists of a graviton and a massive interacting scalar field.

C. Reformulation of quadratic theories

Interestingly enough, besides \( g_{\mu\nu} \), there is an alternative candidate for the metric field of the spacetime \([13,14]\), namely the \( \gamma_{\mu\nu} \) which is the inverse of \( \gamma^{\mu\nu} \) where

\[ \sqrt{-\gamma} \gamma^{\mu\nu} := \frac{\partial L_G}{\partial R^{\mu\nu}}, \] (12)
and $\gamma := \det\gamma_{\mu\nu}$.

In particular for the $L_G$ of Eq. (3) we have
\[
\sqrt{-\gamma} \gamma^{\mu\nu} := \sqrt{-g} \left[ 1 + 2\alpha R g^{\mu\nu} + 2\beta R_{\alpha\beta} g^{\alpha\mu} g^{\beta\nu} \right].
\]

Expressing the theory in terms of $\gamma_{\mu\nu}$ via a Legendre transformation, one can reduce the order of the derivatives that appear in the field equations from fourth to second. But what is also important is, that the resulting theory takes the form of Einstein gravity for the metric $\gamma_{\mu\nu}$ plus some additional massive fields interpreted as matter fields. Thus the equation (13) can be considered as a non linear decomposition of $g^{\mu\nu}$ in the physical spectrum of the full theory: a spin-2 massless field $\gamma^{\mu\nu}$, a scalar field that appears as a conformal factor and is a linear function of $R$, and finally, a tensor field related to $R_{ab}$ with 5 degrees of freedom ($R_{ab}$ is symmetric satisfying the 4 contracted Bianchi equations and its trace is essentially the previously mentioned scalar field).

We can be more explicit in the interesting case of the theory in Eq. (1) where $L_G$ is given by Eq. (3).

Here, only an additional scalar field appears since $\gamma_{\mu\nu}$ and $g_{\mu\nu}$ are conformally related. Indeed Eq. (12) gives
\[
\sqrt{-\gamma} \gamma_{\mu\nu} := \sqrt{-g} F' g_{\mu\nu},
\]
which implies that
\[
\gamma_{\mu\nu} = F'^{-1} g_{\mu\nu}.
\]

Defining a scalar field $\psi$ (not to be confused with the tensorial field $\psi_{\mu\nu}$ of Eqs. (8), (9) via
\[
F' = \exp \left( \sqrt{\frac{2\kappa}{3}} \psi \right),
\]
the field equations (1) are written in the system $\gamma_{\mu\nu}$ as
\[
\hat{G}_{\mu\nu} = \frac{\kappa}{F'} T_{\mu\nu}^{(M)}(g_{ab}) + \kappa T_{\mu\nu}^{(\psi)}
\]
where, hereafter, hats denote quantities with respect to the metric $\gamma_{\mu\nu}$ and
\[
T_{\mu\nu}^{(\psi)} = \hat{\nabla}_\mu \psi \hat{\nabla}_\nu \psi - \frac{1}{2} \gamma_{\mu\nu} \hat{\nabla}_\lambda \psi \hat{\nabla}^\lambda \psi - \frac{1}{2} \gamma_{\mu\nu} U(\psi).
\]

The potential $U(\psi)$ is given by
\[
U(\psi) = \frac{1}{2\kappa} F'^{-2} (RF' - F),
\]
which can be written as a function of $\psi$ alone in regions where Eq. (14) is invertible. Finally the scalar field $\psi$ satisfies the equation
\[
\hat{\Box} \psi = \left( \frac{\kappa}{6} \right)^{1/2} F'^{-2} \left[ \frac{2F - RF'}{\kappa} + g^{\mu\nu} T_{\mu\nu}^{(M)}(g_{ab}) \right],
\]
which can be checked to be equivalent to the trace (1) of the initial field equations (1).

These field equations (17) and (20) follow from the Lagrangian
\[
L' = \frac{1}{2\kappa} \sqrt{-\gamma} \hat{R}(\gamma_{ab}) + \sqrt{-\gamma} \left[ -\frac{1}{2} \gamma^{\mu\nu} \psi_{,\mu} \psi_{,\nu} - U(\psi) \right] + L_M(g_{ab}),
\]
which shows that the quadratic theory (1) is, loosely speaking, equivalent to “Einstein’s” gravitational theory for the metric $\gamma_{ab}$ plus an interacting massive scalar field $\psi$ plus the “peculiar” (not anymore usual) matter fields of $L_M(g_{ab})$. They are indeed peculiar if $\gamma_{ab}$ is considered as the metric of spacetime since then the dependence of $L_M$ on $g_{ab}$ implies, via Eq. (13), a non standard interaction of the metric
field $\gamma_{ab}$ and of the field $\psi$ with the matter fields of $\mathcal{L}_M$. More on this issue of equivalence will be said at the end of this section.

Case: $F(R) \approx R + \alpha R^2$ with $\alpha |R| \ll 1$.

We will now consider the interesting case where $F(R)$ can be expanded as a Taylor series around $R = 0$ and deviates only slightly from Einstein’s theory

$$F(r) = R + \alpha R^2 + O(R^3), \quad \alpha |R| \ll 1, \quad \alpha := \frac{F''}{2} |_{R=0}. \quad (22)$$

Assuming that the $O(R^3)$ terms can be ignored in this expression then the field equations (17) and (20) simplify considerably. Indeed, in this approximation Eqs. (16), (22) imply

$$\psi \approx \left(\frac{6}{\kappa}\right)^{1/2} \alpha R, \quad (23)$$

while the leading term in the potential $U(\psi)$ of Eq. (19) is, assuming the no-tachyon constraint $\alpha > 0$, a mass term

$$U(\psi) = \frac{1}{2} m_0^2 \psi^2 + O(\psi^3), \quad m_0^2 = \frac{1}{6\alpha}. \quad (24)$$

The metric $\gamma_{\mu\nu}$ and the field $\psi$ are obtained from the field equations (17) and (20) which, to lowest order in the approximation (22), read

$$\hat{G}_{\mu\nu} \approx \kappa T^{(M)}_{\mu\nu}(\gamma_{ab}), \quad (\Box - m_0^2)\psi \approx \left(\frac{\kappa}{6}\right)^{1/2} T^{(M)}(\gamma_{ab}). \quad (25)$$

Making use of Eq. (13) one can finally obtain the metric $g_{\mu\nu}$. Notice that, to the considered approximation, it does not matter which metric we actually use in $T^{(M)}_{\mu\nu}$. It is more convenient, however, from the technical point of view to use $\gamma_{ab}$.

Concluding we arrive at the following result:

For a given matter source $T^{(M)}_{\mu\nu}(g_{ab})$, the metric $g_{\mu\nu}$ in the quadratic theory (1) and (3) with $F(R) \approx R + \alpha R^2$ and $\alpha |R| \ll 1$ is given by

$$g_{\mu\nu} = [1 + \chi] \gamma_{\mu\nu}, \quad (26)$$

where $\gamma_{\mu\nu}$ is the metric in the Einstein’s theory with source $T^{(M)}_{\mu\nu}(\gamma_{ab})$ while the field $\chi$ satisfies the equation

$$\Box - m_0^2)\chi = -\frac{\kappa}{3} T^{(M)}(\gamma_{ab}), \quad m_0^2 := \frac{1}{6\alpha}. \quad (27)$$

with the $\Box$ operator taken with respect to the $\gamma_{ab}$ metric.

This result follows directly from Eqs. (21), (15) using the variable $\chi$ related to the field $\psi$ via $\chi := -2(\kappa/6)^{1/2} \psi$. Note that from (23) follows $\alpha R = -\chi/2$ and therefore the condition $\alpha |R| \ll 1$ for our approximation is equivalent to $|\chi/2| \ll 1$. Finally, let us notice that in the weak gravitational limit the Eqs. (26) and (27) are consistent with the $\beta = 0$ limit of the linearized equations (18) and (19).

D. Some remarks

Based on the decomposition (12) several authors [13,14,15,16,17] have dealt with the question of whether quadratic theories are equivalent to Einstein’s theory plus some additional fields. It seems that this may well be true for vacuum theories. However, as was pointed out by Brans [15], (see also Refs. [20,21]), a subtlety appears in the case where usual matter is present. The problem is that the equivalence principle, a basic guide that one may use in constructing theories coupled to gravity and in particular to Einstein’s theory, cannot be valid in both the original and the reformulated theories. If it is valid in the original theory, then a test matter in $\mathcal{L}_M$ of the Lagrangian (1) will follow geodesics
of the spacetime with metric $g_{\mu\nu}$ but, in general, will fail to do the same in the spacetime with the metric $\gamma_{\mu\nu}$. In this sense we are not entitled to consider the reformulated theory as Einstein’s theory in the presence of some interacting fields. In the case that one is philosophically inclined to consider the $\gamma_{\mu\nu}$ as the physical metric, while the $g_{\mu\nu}$ as some sort of unifying field, then the equivalence principle should be implemented in the matter part $\mathcal{L}_M$ of Eq. (9) using the metric $\gamma_{\mu\nu}$ in the place of $g_{\mu\nu}$. Of course then, this $\mathcal{L}_M$ will be non standard with respect to $g_{\mu\nu}$.

Whether or not nature chooses to couple usual matter universally only to a spin-2 field (as the $\gamma_{\mu\nu}$) and not to a more composite one (as the $g_{\mu\nu}$) is far from being experimentally testable. Trying to find an answer one may, however, employ some criteria of principle, as positivity of energy [20]. In any case, the use of new variables, as those of Eq. (13) and Eq. (15), which turn out to simplify technically a physical problem, is undoubtedly very useful even if it is not clear whether one can attribute to these variables a foundational character.

### III. TOPOLOGICAL DEFECTS IN HIGHER DERIVATIVE THEORIES

Cosmological defects are formed during phase transitions in the evolving Early Universe whenever the symmetry group $G$ of the relevant field theory breaks down to a subgroup $H$ so that the vacuum manifold $M = G/H$ has some non trivial homotopy group [23]. Such a symmetry breakdown at an energy scale $\eta$ can be realized, e.g., with an $n$-component scalar field $\phi^{(i)}$ having a Mexican-hat type of potential

$$V(\phi) = -\frac{\lambda}{4} \sum_{i=1}^{n} (\phi^{(i)} - \eta)^2.$$  

(28)

The homotopic structure of the vacuum manifold depends on the number $n$ of components of the scalar field and, thus, we may have the formation of domain walls for $n = 1$, cosmic strings for $n = 2$, monopoles for $n = 3$. These defects are respectively surface-, line-, and point-like configurations. Sufficiently away from these configurations, at distances $d \gg \delta$, the scalar field $\phi^{(i)}$ approaches quickly its vacuum value $\sum_i \phi^{(i)}(\eta)^2 \approx \eta^2$. Here $\delta$ is the width of the core of these defects, of the order of $m_\phi^{-1}$ where $m_\phi = \eta\sqrt{\lambda}$ is the mass of the scalar field’s massive mode. Typically, for symmetry breaking at grand unification scale, $\delta \approx 10^{-30}\text{cm}$ and $\kappa \eta^2 \approx 10^{-6}$.

Depending on whether the symmetry that breaks down is a gauge (local) or a global one we have respectively the formation of gauge or global topological defects. In the case of gauge symmetry there exists a well defined core, with width $\delta$, where most of the energy of the topological defect configuration is localized. On the other hand, for global topological defects the components of the respective stress-energy-momentum tensor have, outside the “core”, a relatively slow fall off due to the gradients of the Goldstone modes of the scalar field $\phi^{(i)}$. Thus, global defects are extended configurations. The reason for this difference between gauge and global defects is that in the case of gauge symmetry the presence of gauge fields can compensate the gradients of the scalar field. Finally, in the case of discrete symmetry breaking, which gives rise to domain walls, there are no Goldstone modes and thus domain walls are localized configurations.

Based on the above properties, we will make in what follows the following approximations:

(i) Gauge topological defects and domain walls will be considered in the zero core-thickness approximation and thus their stress-energy-momentum tensors will have components with appropriate Dirac $\delta$-fuctions.

(ii) For global defects, we will make the $\sigma$-model approximation where the scalar field is fixed to its asymptotic vacuum value everywhere outside the defect. This is a sensible approximation at distances from the defect sufficiently larger than the “core” width $\delta$.

In the following subsections we will obtain the gravitational field of cosmic strings, monopoles and domain walls in the quadratic theory $R + \alpha R^2$ with $\alpha |R| \ll 1$. For this we will make use of the result of the previous section (see Eqs. (20), (21)), stating that the metric in the quadratic theory, $ds^2_{(Q)}$, is conformally related with the metric in Einstein’s theory, $ds^2_{(E)}$,

$$ds^2_{(Q)} = (1 + \chi)ds^2_{(E)}, \quad |\chi/2| \ll 1,$$

(29)
with $\chi$ satisfying the massive Klein-Gordon equation (27) in the $ds^2_{(E)}$ metric. A consequence of Eq. (29) is that there will be a modification of the "Newtonian" potential equal to $\chi/2$. We will have below the opportunity to study its nature and its range in the case of topological defects, be them localized or extended sources.

In general we shall restrict our attention to sufficiently large distances, $d$, away from the core, ($d \gg \delta$), but we will keep in mind that a proper treatment at short distances requires a proper model for the core of the defect itself. In this way we will be able to use the existing results in General Relativity for the gravitational field of cosmic strings, monopoles and domain walls which were obtained by making use of the above approximations in model Lagrangians with symmetry breaking potential of the form (28).

A. Global monopoles

The stress-energy-momentum tensor of a global monopole configuration, in regions far away from the core, can be approximated by $T^t_t\approx -\eta r^2$, $T^\theta_\theta\approx T^{\phi \phi}=0$, while the respective metric in Einstein’s theory of gravity is (approximately) given by $ds^2_{(E)} = -(1-\Delta)dt^2 + (1-\Delta)^{-1}dr^2 + d\Omega^2$, $\Delta := 8\pi G\eta^2 = \kappa\eta^2$.

This metric corresponds to a spacetime with a solid deficit angle: test particles are deflected by an angle $\pi\Delta/2$ irrespective of their velocity and their impact parameter. Here it should be added that a more careful treatment [24] that takes into account the actual behaviour of the field at the monopole core, shows that the metric (31) gets modified by terms which at distances $r \gg \delta = (\sqrt{\lambda}\eta)^{-1}$ correspond effectively to a negative mass term $M_{\text{eff}}$, that is e.g. $g_{tt} \approx (1-\Delta - 2GM_{\text{eff}}/r)$. According to numerical analysis [24] $M_{\text{eff}} \approx -6\pi\sqrt{\lambda}\eta$. Thus, besides the topological deflection caused by the solid deficit angle, test particles experience also a repulsive radial force $-GM_{\text{eff}}/r^2$ away from the monopole.

The metric in the quadratic theory is given by Eq. (29) with $\chi$ satisfying Eq. (27). Looking for spherically symmetric solutions we find that this equation for $\chi = \chi(r)$ reads\n
$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{d\chi}{dr} - \tilde{m}^2 \chi \right] = \frac{2\Delta}{3(1-\Delta)r^2},$$

$$\tilde{m}^2 := m_0^2/(1-\Delta).$$

Making use of the Green function for this equation,

$$G(r,r') = -\frac{1}{\tilde{m}r} [e^{-\tilde{m}r'} \sinh(\tilde{m}r) \Theta(r' - r) + e^{-\tilde{m}r} \sinh(\tilde{m}r') \Theta(r - r')],$$

where the step function $\Theta(z) := \{0,1,1/2\}$ for $\{z<0,z>0,z=0\}$ respectively, we can write down the solution for $\chi(r)$ in terms of the Exponential-Integral (Ei) and Hyperbolic-Sine-Integral (shi) functions [24] as

$$\chi(r) = \frac{2\Delta}{3(1-\Delta)} \frac{1}{\tilde{m}r} [\text{Ei}(-\tilde{m}r) \sinh(\tilde{m}r) - e^{-\tilde{m}r} \text{shi}(\tilde{m}r)].$$

Checking numerically the behavior of this function we find that its contribution to the 'Newtonian' potential $\chi/2$ is an attractive one. In particular, using the asymptotic behavior of the Ei and shi functions [24] we find that at large radial distances $r \to \infty$

$$\chi(r) \approx \frac{2}{3} \frac{\Delta}{(m_0 r)^2},$$

7
which implies a long range potential, exerting on test particles an attractive force $-(2\Delta/3m_0^2)r^{-3}$. Comparing this force to the repulsive force due to the core of the monopole we see that the former falls off faster by one power of $r$ and thus is negligible at very large distances. It overcomes, however, the effect of the latter at a distance $r \approx m_0^{-2}/(\lambda \delta)$ and, thus, it can be the dominant force within the region $\delta \ll r \ll m_0^{-2}/(\lambda \delta)$ which will exist provided that $m_0^{-1} \gg \delta$.

Finally, let us note that the expression (34) diverges as $r \to 0$. This is due to the form of the energy-momentum-tensor in Eq. (30) which is not valid at distances comparable to the core of the monopole.

**B. Gauge Monopoles**

A gauge monopole is a spherically symmetric configuration with mass $M$ and a magnetic charge $g$. Its stress energy momentum tensor can be approximated by

$$T^t_t = -\frac{M}{4\pi} \frac{\delta(r)}{r^2} - \frac{(g/4\pi)^2}{r^4},$$

$$T^\varphi_\varphi = -T^\theta_\theta = -\frac{(g/4\pi)^2}{r^4}.$$  \hspace{1cm} (36)

We consider the case where the metric outside the core of the monopole matches to a Reissner-Nordstrom one, (see Ref. [26] for a recent review and new results on the gravitational field of monopoles)

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{Gg^2}{4\pi r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{Gg^2}{4\pi r^2}\right)^{-1} dr^2 + d\Omega^2.$$ \hspace{1cm} (37)

Since the source for the $\chi$ field is the trace of the stress-energy-momentum tensor, only the mass term in Eq. (36) will contribute. Furthermore, if we consider distances sufficiently far from the monopole $r \gg \delta \gg GM$, the equation for $\chi$ approximately reads

$$\left\{ \frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{d\chi}{dr} \right] - m_0^2 \right\} \chi(r) = \frac{\kappa M}{12\pi} \frac{\delta(r)}{r^2}.$$ \hspace{1cm} (38)

Demanding finiteness at radial infinity, this equation has as solution the Yukawa fuction

$$\chi(r) = -\frac{\kappa M}{12\pi} \frac{e^{-m_0 r}}{r}.$$ \hspace{1cm} (39)

Notice that the Newtonian potential of the monopole will be modified by the amount $\chi/2$ corresponding to an attractive potential exponentially decreasing with an $e$-folding term characteristic of a massive scalar field with mass $m_0$.

It worths remarking that the short range corrections of Eq. (39) apply also to the external metric of spherically symmetric mass distributions \cite{27} such as neutron stars, giving thus rise to “fifth force” terms. However, when one deals with black holes, the no-hair theorem for $R + R^2$ theories \cite{12} implies that corrections of the type (39) are absent.

**C. Global Cosmic Strings**

As we explained in the introductory part of this section global cosmic strings are extended line configurations. The stress-energy-momentum tensor for a straight, static, cylindrically symmetric global string lying along the $z$-axis is approximately given for $r \gg \delta$ by

$$T^t_t = T^z_z = T^r_r = -T^\phi_\phi = -\frac{\eta^2}{2\pi^2}.$$ \hspace{1cm} (40)

The respective exact solution for the metric in Einstein’s theory has been found in \cite{27}. However, it is quite complicated for the purpose of solving the equation (27) for the field $\chi$. Furthermore, besides
this technical problem, the spacetime of a global string has true spacetime singularities \cite{28,29}, a fact that demands careful checking of the range of validity of the approximation ($\alpha |R| \ll 1$) on which our treatment is based. Instead, we prefer to work here in the weak field limit where the equation for the $\chi$ field is in Minkowski background metric.

In the weak field limit of general relativity the metric of the global string reads \cite{28}

$$ds_{(E)}^2 = \left[ 1 - 4G\mu \ln\left(\frac{r}{\delta}\right) \right] (-dt^2 + dz^2) + \frac{1}{1 - 4G\mu \ln\left(\frac{r}{\delta} + c\right)} d\theta^2,$$

(41)

Here $\mu := \pi \eta^2$, $\delta$ is the core width and $c$ is a constant of order unity that may partially take into account a global effect of the string core. Studying the motion of test particles it is seen that the static global string exerts a repulsive force $2G\mu/r$ \cite{28}. It is interesting to explore how this force is modified in the quadratic theory that we are currently considering.

The equation that $\chi$ satisfies in the weak field limit is

$$\left\{ \frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \right] - m_0^2 \right\} \chi(r) = \frac{\kappa \mu}{3\pi r^2},$$

(42)

The Green function for the homogeneous part of this differential equation, with the boundary conditions of finitness at the origin and at infinity, is easily found to be

$$G(r, r') = -K_0(m_0r)I_0(m_0r')\Theta(r - r') - I_0(m_0r)K_0(m_0r')\Theta(r' - r)$$

(43)

where $\Theta$ is the step function. Thus the solution of Eq. (42) can be written as

$$\chi(r) = -\frac{\kappa \mu}{3\pi m_0^2} \left[ K_0(m_0r) \int_{\delta}^{r} I_0(m_0r') \frac{dr'}{r'} \right]$$

$$+ I_0(m_0r) \int_{r}^{\infty} K_0(m_0r') \frac{dr'}{r'}],$$

(44)

where we have introduced a lower cutoff at $r = \delta$ to cope effectively with the divergence that appears in the first integral if we let $r \to 0$. This divergence is only due to the approximate form of the stress-energy-momentum tensor which as we have already stressed is not valid near the core of the string.

The leading term in an asymptotic expansion of Eq. (44) at large radial distances is

$$\chi(r) \approx -\frac{\kappa \mu}{3\pi m_0^2 r^2}, \quad r \to \infty.$$

(45)

From this expression we conclude that the additional ‘Newtonian’ potential in the quadratic theory implies at large distances an attractive force $-(\kappa \mu/3m_0^2) r^{-3}$. Due to the slower fall off of the original repulsive force, the total force on test particles remains repulsive at large distances from the string. At around $r \sim m_0^{-1}$ the total force is expected to change sign.

D. Gauge Cosmic Strings

Gauge cosmic strings are, in contrast to global ones, localized line configurations. The stress-energy-momentum tensor for a static, straight along the $z$-axis, gauge cosmic string with line energy density $\mu$ is

$$T^t_t = T^z_z = -\frac{\mu}{2\pi(1 - \kappa \mu/2)} \frac{\delta(r)}{r}, \quad T^r_r = T^\theta_\theta = 0$$

(46)

with corresponding metric in Einstein’s theory \cite{33}

$$ds_{(E)}^2 = -dt^2 + dz^2 + dr^2 + (1 - \kappa \mu/2)^2 r^2 d\theta^2.$$

(47)
Here the polar coordinates \( r, \theta \) have the usual range. This spacetime is everywhere flat except along the \( z \)-axis where the string is located. As one goes around the string one notices an angle deficit. This topological property has the consequence that test particles which locally do not feel any gravitational forces are, however, deflected by the string.

Let us now turn our attention to the field \( \chi \). It satisfies the equation

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{d}{dr} \right] \chi(r) = \frac{\kappa \mu}{3\pi(1 - \kappa \mu/2)} \frac{\delta(r)}{r}.
\]

which can be easily solved by demanding for the field \( \chi \) finiteness at infinity and correct discontinuity at the origin. The solution reads

\[
\chi(r) = -\frac{\kappa \mu}{3\pi(1 - \kappa \mu/2)} K_0(m_0 r),
\]

which can be easily checked that satisfies Eq. (48) using the small argument asymptotic behavior of the modified Bessel function \( K_0(z) \approx -\ln(z/2) \). Finally notice that the field \( \chi \) decays exponentially fast since at large distances \( K_0(m_0 r) \approx (\pi/2m_0 r)^{1/2} \exp(-m_0 r) \). Very close to the string the expression \( (49) \) diverges logarithmically. Again, as the physical cosmic string has a finite core this divergence should not appear in a more proper treatment near the core.

From Eq. (49) follows that the ‘Newtonian’ potential of the cosmic string spacetime in an attractive, short range one. The respective force that the string will exert on test particles is \(-|\kappa \mu m_0/6\pi(1 - \kappa \mu/2)| K_1(m_0 r)\). Because of the large distance exponential fall off behavior of \( K_1(m_0 r) \propto (m_0 r)^{-1/2} \exp(-m_0 r)\) it follows that this force is significant only close to the string up to distances \( r \sim m_0^{-1} \).

In closing this subsection let us remark that the result obtained here is in agreement with the recent result of Linet and Teyssandier in the weak field limit where \( \kappa \mu \ll 1 \). These authors also obtained the cosmic string metric in the weak field limit of the quadratic theory and which contains also the massive tensorial field \( \psi_{\mu\nu} \) of Eqs. (3)–(4). In particular they find that the effect of this field on the Newtonian potential is a repulsive one with a range set by the inverse mass \( m_1 \) of the field \( \psi_{\mu\nu} \).

### E. Domain Walls

The energy content and the gravitational field of domain wall configurations in Einstein’s theory has been studied extensively in the literature, see Ref. [29] and references therein. We will consider here Vilenkin’s vacuum plain domain wall solution discussed in [31]. For a domain wall with surface energy density \( \sigma \), lying on the \( |z| = 0 \) plane, the stress-energy-momentum tensor is

\[
T_t^t = T_x^x = T_y^y = -\sigma \delta(z), \quad T_z^z = 0,
\]

while the respective domain wall spacetime is described by the metric

\[
dx^2 = (1 - \nu|z|)^2 \left[-dt^2 + e^{2\nu t}(dx^2 + dy^2)\right] + dz^2,
\]

\[
\nu := 2\pi G \sigma = \kappa \sigma/4.
\]

Note that some of the metric components are time dependent (no static solutions can be found). Test particles in this spacetime are repelled with a proper acceleration \( \nu \) away from the wall, a property that we may deduce just by looking at the ‘Newtonian’ potential term in the \( g_{tt} \) component of the metric. Finally we should mention that at \( |z| = \nu^{-1} \) an event horizon appears. In what follows we restrict our attention to spacetime regions with \( |z| \leq \nu^{-1} \).

Although the metric in (51) is time dependent we can find, however, static solutions to the equation (27) for the field \( \chi \) depending only on \( |z| \). For such solutions, equation (27) reduces to the ordinary differential equation

\[
\left(1 - \nu|z|\right)^{-3} \frac{d}{dz} \left[ (1 - \nu|z|)^3 \frac{d}{dz} \right] - m_0^2 \chi(z) = 4\nu \delta(z).
\]
The homogeneous part of this equation can be easily transformed into a Bessel differential equation for $\hat{\chi}$ where $\chi(|z|) := \hat{\chi}(\hat{z})/\hat{z}$ using the new variable $\hat{z} = \nu^{-1} - |z|$. In this way we find that the solution for $\chi(z)$ is, in regions with $z \neq 0$, a linear combination of the terms $I_1(m_0\hat{z})/\hat{z}$ and $K_1(m_0\hat{z})/\hat{z}$ where $K_1, I_1$ denote modified Bessel functions. The coefficients of this solution are determined by demanding finiteness at the horizon $\hat{z} = 0$, while on the domain wall, $z = 0$, the field $\chi$ should be continuous and have the appropriate discontinuity in its first derivative which, according to Eq. (52), is $[\frac{d}{dz} \chi]|_{z=0} = 4\nu$. Thus we finally obtain

$$\chi(z) = -2\left[\frac{m_0}{\nu} I_2(m_0)\right]^{-1} I_1(\frac{m_0}{\nu}[1 - \nu|z|]) \frac{1}{1 - \nu|z|}.$$  \hspace{1cm} (53)

It is easy to check that this implies an attractive and short range contribution to the Newtonian potential. This cannot overwhelm the original repulsive potential of the domain wall except very close to the wall for $|z| \lesssim m_0^{-1}$. This should be obvious in writing Eq. (53) in the sensible approximation $m_0/\nu \gg 1$ and near the domain wall where $\chi$ takes its largest value

$$\chi(z) \approx -\frac{2\nu}{m_0} \exp(-m_0|z|).$$ \hspace{1cm} (54)

Away from the wall the field $|\chi|$ decreases exponentially and at the horizon $\chi$ attains the small value $-1/I_2(m_0\nu)$.

IV. CONCLUSIONS AND REMARKS

We have dealt in this paper mainly with the higher derivative theory [1], [3] with $F = R + \alpha R^2$ in the approximation $|\alpha| R \ll 1$. We showed that one can simplify the problem of solving the corresponding fourth order field equations using the conformal transformation [17] which leaves us with the system of field equations [17], [20] having only second order derivatives. This is formally a system of Einstein type equations plus the field equations for a massive scalar field with mass $m_0^2 = 1/(6\alpha)$ interacting non minimally with gravity. We then found that in the approximation $|\alpha| R \ll 1$ the gravitational fields in the $R + \alpha R^2$ theory and in the Einstein theory are conformally related according to the Eqs. [21], [27]. Using this result we looked in Sec. III for solutions representing the gravitational field of monopoles, cosmic strings and domain walls.

For localized topological defect configurations as gauge strings, gauge monopoles and domain walls we have found short range, attractive corrections. Their range $\sim 1/m_0$ is characteristic of the presence of the massive field.

For extended sources as global monopoles and global strings we have again found attractive corrections but with a long range. Their fall off rate depends on the corresponding stress-energy-momentum of these defect configurations. In particular, for distances $r \gg m_0^{-1}$ is found that the attractive correction to the Newtonian potential is $\approx \kappa T/(6m_0^2)$ where $T$ is the trace of the corresponding stress-energy-momentum tensor.

A more detailed investigation of the gravitational effects of topological defects in more general higher order derivative theories is in progress and we hope to present the results elsewhere. We can however already here reestablish the above given conclusions in the $R + \alpha R^2$ theory and at the same time extend them to the case of the theory [1], [2] which, for $\beta \neq 0$, contains also a massive tensorial field. This will be done by making use of the linearized field equations [3], [3]. First observe that the Newtonian potential $\Phi_N$ will have, besides the Einstein term $\Phi_N^{(E)}$, also the contributions $\Phi_N^{(x)}$ and $\Phi_N^{(\psi \psi)}$ from the scalar field $\chi$ and massive field $\psi_{\mu \nu}$ respectively

$$\Phi_N = \Phi_N^{(E)} + \Phi_N^{(x)} + \Phi_N^{(\psi \psi)} = \frac{1}{2}[\nu^{(E)} - \psi_{\mu \nu}].$$ \hspace{1cm} (55)

For static spacetimes we have from Eqs. (8) and (3) that

$$\nabla^2 \Phi_N^{(E)} = \frac{\kappa}{2}(\rho + P_1 + P_2 + P_3),$$

$$\nabla^2 - m_0^2 \Phi_N^{(x)} = \frac{\kappa}{6}(\rho - P_1 - P_2 - P_3).$$
\[
(\nabla^2 - m_1^2)\Phi_N^{(\psi_{\mu\nu})} = -\frac{\kappa}{3}(2\rho + P_1 + P_2 + P_3)
\]
\[m_0^{-2} := 6\alpha + 2\beta, \quad m_1^{-2} := -\beta,
\]

where \(\rho\) denotes the mass density and \(P_1, P_2, P_3\) the principal pressures of the matter.

**Sign of forces:** Look at the r.h.s of these equations. For the topological defect configurations discussed in the present paper \(\rho + \sum_i P_i > 0\) for gauge monopoles, 0 for gauge strings and global monopoles, and \(< 0\) for global strings and domain walls. On the other hand \(\rho - \sum_i P_i > 0\), while \(-(2\rho + \sum_i P_i) \leq 0\) with equality holding for domain walls. Thus the Einstein contribution in Eq. (55) is attractive for gauge monopoles, zero for gauge strings and global monopoles, and repulsive for global strings and domain walls. The \(\chi\)-contribution is always attractive, while the \(\psi_{\mu\nu}\)-contribution is in general repulsive except for domain walls where it is zero.

**Range of forces:** From Eqs. (56) it is clear that the Einstein term provides in general a long range interaction due to an effective mass density \(\rho + \sum_i P_i\). It gives, however, a zero effect for gauge cosmic strings and global monopoles. For the contributions of \(\chi\) and \(\psi_{\mu\nu}\) terms we have that:

(i) If the stress energy momentum tensor vanishes (or falls off sufficiently rapidly) outside a localized source then at distances \(d\) from the source, these contributions are of short range \(\propto \exp(-md)/d^p\) where \(m\) stands for the mass \(m_0\) (or \(m_1\)) and \(p\) is a parameter depending on the symmetry of the spacetime and is equal to 0, \(\frac{1}{2}\), and 1 for plain domain walls, strings and monopoles respectively.

(ii) If the sources are not localized then for the \(\chi\), \(\psi_{\mu\nu}\) contributions there are two characteristic regimes:

(a) At distances \(d \gg \max(m_0^{-1}, m_1^{-1})\) the mass terms dominate over the derivative terms in the two last equations of (56). Thus, asymptotically at large distances we have long range contributions \(\Phi_N^{(\chi)} \approx -\kappa(p - \sum_i P_i)/(6m_0^3) = \kappa T/(6m_0^3)\), as we found in this paper, and \(\Phi_N^{(\psi_{\mu\nu})} \approx \kappa(2\rho + \sum_i P_i)/(3m_1^2)\).

(b) At distances \(d \ll \min(m_0^{-1}, m_1^{-1})\) the derivative terms dominate over the mass terms. The interesting thing to note here is that at such distances the total ‘Newtonian’ potential in Eq. (53) satisfies \(\nabla^2 \Phi_N \approx 0\). This has implications for the differentiability of the spacetime metric and implies drastic changes in the singularity structure of gravity at short distances. For example, the gravitational potential of a point massive particle is finite at the origin in contrast to the \(1/r\) Coulomb behavior in the Newtonian theory.

These considerations are in agreement with the results of the previous sections and the results of [1] for gauge cosmic strings. They may be particularly relevant to the study of the evolution of topological defects in the very early universe: (a) for structure formation scenarios based on global defects where the long range modifications of the quadratic theories may play an important role; (b) for collisions of cosmic strings where the drastic short range modifications may change significantly the predictions of these simulations for the evolution parameters of a string network. Thus it is interesting to study further topological defects and collisions of cosmic strings in quadratic gravitational theories and implement appropriate modifications in future numerical simulations. The outcome of such an investigation confronted with observation, may, among other things, allow one to put constraints on the \(m_0, m_1\) parameters of quadratic gravitational theories.

**ACKNOWLEDGMENTS**

This work was supported by the European Community DG XII. C.O.L was also supported by the Alexander von Humboldt foundation.

[1] H. Weyl, Raum-Zeit-Materie, (1921), 4th edition, (Springer-Verlag, Berlin).
[2] R. Kerner, Gen. Rel. Grav. 14, 453 (1982).
[3] K.S. Stelle, Phys. Rev. D 16, 953 (1977); K. S. Stelle, Gen. Rel. Grav. 9, 353 (1978).
[4] N. D. Birrell and P.C.W. Davies, Quantum Fields in Curved Space (Cambridge University Press, Cambridge, 1982).
[5] J. Z. Simon, Phys. Rev. D 45, 1953 (1992).
[6] B. Zwiebach, Phys. Lett. 156B, 315 (1985); A. Tseytlin, Phys. Lett. 176B, 92 (1986).
[7] A. Starobinsky, Phys. Lett. 91B, 99 (1980).
[8] H. Feldman, Phys. Lett. 249B, 200 (1990).
[9] A. Vilenkin, Phys. Rep. 121, 263 (1985).
[10] R. Brandenberger, J. Phys. G 15, 1 (1989).
[11] B. Linet and P. Teyssandier, Class. Quantum. Grav. 9, 159, (1992).
[12] B. Whitt, Phys. Lett. 145B, 176 (1984).
[13] G. Magnano, M. Ferraris and M. Francaviglia, Gen. Rel. Grav. 19, 465, (1987).
[14] A. Jakubié and J. Kijowski, Phys. Rev. D 37, 1406 (1988).
[15] C. Brans, Class. Quantum. Grav. 5, L197, (1988).
[16] K. Maeda, Phys. Rev. D 39, 3159 (1989).
[17] P. Teyssandier, Class. Quantum. Grav. 6, 219, (1989).
[18] M. Ferraris, M. Francaviglia and G. Magnano, Class. Quantum. Grav. 5, L95, (1988).
[19] H.J. Schmidt, Astron. Nach. 308, 183 (1987),
[20] L. Sokolowski, Class. Quantum. Grav. 6, 2045, (1989).
[21] M. Ferraris, M. Francaviglia and G. Magnano, Class. Quantum. Grav. 7, 261, (1990).
[22] T.W.B. Kibble, J. Phys. A 9, 1387 (1976).
[23] M. Barbiola and A. Vilenkin, Phys. Rev. Lett. 63, 341 (1989).
[24] D. Harari and C. Lousto, Phys. Rev. D 42, 2626 (1990).
[25] I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series and Products (Academic, New York, 1980).
[26] G.W. Gibbons, Self-gravitating magnetic monopoles, global monopoles and black holes in The Physical Universe: The Interface between Cosmology, Astrophysics and Particle Physics, Proceedings of the XII Autumn School of Physics, Lisbon, 1990, edited by J. Barrow, A. Henriques, M. Lago and M. Longair (Springer-Verlag, 1991).
[27] A. Cohen and D. Kaplan, Phys. Lett. 215B, 67 (1988).
[28] D. Harari and P. Sikivie, Phys. Rev. D 37, 3438 (1988).
[29] R. Gregory, Phys. Lett. 215B, 663 (1988).
[30] A. Wang, Dynamics of plane-symmetric thin walls in general relativity, Ioannina report, (1992).
[31] A. Vilenkin, Phys. Lett. 133B, 177 (1983).