On Dobrushin Ergodicity Coefficient and weak ergodicity of Markov Chains on Jordan Algebras

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Abstract. In this paper we study certain properties of Dobrushin’s ergodicity coefficient for stochastic operators defined on non-associative $L^1$-spaces associated with semi-finite $JBW$-algebras. Such results extends the well-known classical ones to a non-associative setting. This allows us to investigate the weak ergodicity of nonhomogeneous discrete Markov processes (NDMP) by means of the ergodicity coefficient. We provide a necessary and sufficient conditions for such processes to satisfy the weak ergodicity.

1. Introduction
It is known (see [23]) that the investigations of asymptotical behavior of iterations of Markov operators on commutative $L^1$-spaces are very important. On the other hand, these investigations are related with several notions of ergodicity of $L^1$-contractions of measure spaces. To the investigation of such ergodic properties of Markov operators were devoted lots of papers (see for example, [7, 23]). On the other hand, such kind of operators were studied in noncommutative settings. Since, the study of quantum dynamical systems has had an impetuous growth in the last years, in view of natural applications to various field of mathematics and physics. It is then of interest to understand among the various ergodic properties, which ones survive and are meaningful by passing from the classical to the quantum case. Due to noncommutativity, the latter situation is much more complicated than the former. The reader is referred e.g. to [2, 15, 16, 18, 29] for further details relative to some differences between the classical and the quantum situations. It is therefore natural to study the possible generalizations to quantum case of the various ergodic properties known for classical dynamical systems. One of the generalizations of noncommutative algebras is Jordan algebras. Note that Jordan Banach algebras [3, 6] are a non-associative real analogue of von Neumann algebras. The existence of exceptional $JBW$-algebras does not allow one to use the ideas and methods from von Neumann algebras. The motivation of these investigations arose in quantum statistical mechanics and quantum field theory (see, [9]). Mostly, in those investigations homogeneous Markov processes were considered. Many ergodic type theorems have been proved for Markov operators acting in nonassociative and noncommutative $L^p$-spaces (see for example, [4, 5, 20, 21, 8, 18]).

On the other hand, nonhomogeneous Markov processes with general state space have become a subject of interest due to their applications in many branches of mathematics and natural sciences. In many papers (see for example, [24, 17, 31]) the weak ergodicity of nonhomogeneous Markov process are given in terms of Dobrushin’s ergodicity coefficient [12]. In [33] some...
sufficient conditions for weak and strong ergodicity of nonhomogeneous Markov processes are given and estimates of the rate of convergence are proved. Lots of papers were devoted to the investigation of ergodicity of nonhomogeneous Markov chains (see, for example [12]-[19],[30]).

Until now a limited number of investigations are devoted to the ergodic properties of nonhomogeneous Markov processes defined on nonassociative and noncommutative spaces (see for example, [1, 25]). In this paper we are going to study ergodic properties of nonhomogeneous discrete Markov processes defined on nonassociative $L^1$-spaces. Note that in the context of inhomogeneous Markov chains, ergodicity refers to the asymptotic behavior of products of stochastic operators where the number of factors grows unbounded. In the simplest case, when all factors in the products are identical to the same stochastic operator $T$, ergodicity corresponds to the investigation of iterations of $T$. The Dobrushin’s ergodicity coefficient is one of the effective tools to study a behavior of such products (see [17] for review). Therefore, we will define such an ergodicity coefficient of a positive mapping defined on nonassociative $L^1$-space, and study its properties. In this direction we extend the results of [24] to a noncommutative setting. This allows us to investigate the weak ergodicity of nonhomogeneous discrete Markov processes by means of such ergodicity coefficient. We shall provide sufficient conditions for such processes to satisfy the weak ergodicity. Note that in [13] similar conditions were found for classical ones to satisfy weak ergodicity.

2. Preliminaries

In this section recall some well known facts concerning Jordan algebras. Let $A$ be a Jordan algebra with unity $I$ and a Banach space over the reals. Multiplication operation on $A$ we will denote by $\circ$. If the norm on $A$ respects multiplication so that $\|a^2\| = \|a\|^2$ and $\|a^2\| \leq \|a^2 + b^2\|$ for all $a, b \in A$, then $A$ is called a JB-algebra (see [3],[6]). A Jordan subalgebra $A_0$ of $A$ is said to be strongly associative if $(a \circ b) \circ c = a \circ (c \circ b)$ for all $a, b \in A_0$ and $c \in A$. A family $M$ of elements of $A$ is said to be consistent if it is contained in a strongly associative subalgebra.

**Definition 2.1.** A partial ordering $\geq$ of a Jordan algebra $A$ is said to be compatible with the algebraic operations if

1. $a \geq b \Rightarrow a + c \geq b + c$ for each $c \in A$;
2. $a \geq b \Rightarrow \lambda a \geq \lambda b$ for all $\lambda \in \mathbb{R}$, $\lambda \geq 0$;
3. $a \geq 0$, $b \geq 0$, $a$ and $b$ are consistent $\Rightarrow a \circ b \geq 0$;
4. $a^2 \geq 0$ for each $a \in A$.

Note that in each JB-algebra $A$ the set $A^+ = \{a^2 : a \in A\}$ is regular convex cone and defines in $A$ a partial ordering compatible with the algebraic operations. A JB-algebra $A$ is called a JBW-algebra if there exists a Banach space $N$, which is said to be pre-dual to $A$, such that $A$ is isometrically isomorphic to the space $N^*$ of continuous linear functionals on $N$. On the JBW-algebra $A$ one can introduce the $\sigma(A, N)$-weak topology. A linear functional $f$ is continuous in the $\sigma(A, N)$-weak topology. It is known that the pre-dual space $N$ of a JBW-algebra $A$ can be identified with the space of continuous linear functionals $A_*$ on $A$.

An important role in the theory of Jordan algebras is played by the operator $U_a$ defined for each $a \in A$ by the formula $U_a x = 2a \circ (a \circ x) - a^2 \circ x$. If $A$ is a JBW-algebra, then $U_a$ has the following properties:

1. $U_a$ is positive, that is, $U_a A^+ \subset A^+$;
2. $U_a$ is normal, that is, for each increasing net $x_a$ bounded above in $A$ one has $U_a(\sup x_a) = \sup U_a(x_a)$;
3. if $a$ and $b$ are consistent elements, then $U_{a \circ b} = U_a U_b$.

**Definition 2.2.** A trace on a JBW-algebra is a map $\tau : A^+ \to [0, \infty]$ such that
Besides, if \( T \) is increasing net for every or asymptotically stable (NDMC), if each \( A \), \( p \) forms a logic. For nonhomogeneous discrete Markov process (NDMP) \( T \) is a contraction. For given Markov process.

Moreover, if \( \tau \) is faithful if \( \tau(a) > 0 \) for all \( a \in A^+ \), \( a \neq 0 \); it is normal if for each increasing net \( x_\alpha \) in \( A^+ \) that is bounded above one has \( \tau(\sup x_\alpha) = \sup \tau(x_\alpha) \); it is semifinite if there exists a net \( \{b_\alpha\} \subset A^+ \) increasing to 1 such that \( \tau(b_\alpha) < \infty \) for all \( \alpha \), and it is finite if \( \tau(1) < \infty \).

Throughout the paper we will consider a JBW-algebra \( A \) with a faithful finite normal trace \( \tau \). Therefore we omit this condition from the formulation of theorems.

An element \( p \in A \) is called an idempotent if \( p^2 = p \). Let \( \nabla \) be the set of idempotents: \( \nabla \) forms a logic. For \( p \in \nabla \) we set \( p^\perp = 1 - p \).

The map \( || \cdot ||_1 : A \rightarrow [0, \infty) \) defined by the formula \( || \cdot ||_1 = \tau(||a||) \) is a norm (see [6]). The completion of \( A \) in the norm \( || \cdot ||_1 \) will be denoted by \( L^1(A, \tau) \). It is shown [7] that the spaces \( L^1(A, \tau) \) and \( A, \tau \) isometrically isomorphic, therefore they can be indentified. Further we will use this fact without noting.

**Theorem 2.1.** [6] The space \( L^1(A, \tau) \) coincides with the set

\[
L^1 = \{ x = \int_{-\infty}^{\infty} \lambda d\nu \in A : \int_{-\infty}^{\infty} |\lambda| d\tau(\nu) < \infty \}.
\]

Moreover,

\[
||x||_1 = \int_{-\infty}^{\infty} |\lambda| d\tau(\nu).
\]

Besides, if \( x, y \in L^1(A, \tau) \) such that \( x \geq 0, y \geq 0 \) and \( x \circ y = 0 \) then \( ||x + y||_1 = ||x||_1 + ||y||_1 \).

For more information about Jordan algebras we refer a reader to [3, 6].

Let \( T : L^1(A, \tau) \rightarrow L^1(A, \tau) \) be linear bounded operator. We say that a linear operator \( T \) is positive is \( T x \geq 0 \) whenever \( x \geq 0 \). A linear operator \( T \) is said to be a contraction if \( ||T|| \leq 1 \). A positive operator \( T \) is called stochastic if \( \tau(Tx) = \tau(x), x \geq 0 \). It is clear that any stochastic operator is a contraction. For given \( y \in L^1(A, \tau) \) and \( z \in A \) define a linear operator \( T_{y,z} : L^1(A, \tau) \rightarrow L^1(A, \tau) \) as follows

\[
T_{y,z}(x) = \tau(x \circ z)y.
\]

Put \( T_y := T_y 1 \).

Recall that a family of contractions \( \{T^{m,n} : L^1(A, \tau) \rightarrow L^1(A, \tau)\} (m \leq n, m, n \in N) \) is called a nonhomogeneous discrete Markov process (NDMP) if one satisfies

\[
T^{m,n} = T^{k,n}T^{m,k}
\]

for every \( m \leq k \leq n \). A NDMP \( \{T^{m,n}\} \) is called nonhomogeneous discrete Markov chain (NDMC), if each \( T^{m,n} \) is a stochastic operator. A NDMP \( \{T^{m,n}\} \) is called uniformly asymptotically stable or uniformly ergodic if there exist an element \( y \in L^1(A, \tau) \) such that

\[
\lim_{n \rightarrow \infty} ||T^{m,n} - T_y|| = 0
\]

for any \( m \geq 0 \).

Recall that if for a NDMP \( \{T^{k,m}\} \) one has \( T^{k,m} = (T^{0,1})^{m-k} \), then such a process becomes homogeneous, and therefore, in what follows, by \( T \), where \( T := T^{0,1} \), we denote the homogeneous Markov process.

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(1) \( \tau(a + \lambda b) = \tau(a) + \lambda \tau(b) \) for all \( a, b \in A^+ \) and \( \lambda \in \mathbb{R}_+ \), provided that \( 0.(\infty) = 0 \),

(2) \( \tau(Us a) = \tau(a) \) for all \( a \in A^+ \) and \( s \in A, s^2 = 1 \).

Note that [6] the last condition is equivalent to \( \tau(x \circ a) \geq 0 \) for every \( a, x \in A^+ \).
3. Dobrushin ergodicity coefficient

Let $A$ be a $JBW$-algebra with faithful normal semifinite trace $\tau$. Let $L^1(A, \tau)$ be a $L^1$-space. In the sequel by $\| \cdot \|$ we mean the norm $\| \cdot \|_1$. Let $T : L^1(A, \tau) \to L^1(A, \tau)$ be a linear bounded operator. Define

$$X = \{ x \in L^1(A, \tau) : \tau(x) = 0 \},$$

$$\delta(T) = \sup_{x \in X, x \neq 0} \frac{\|Tx\|_1}{\|x\|_1}, \quad \alpha(T) = \|T\| - \delta(T).$$

(1)

The magnitude $\delta(T)$ is called the Dobrushin ergodicity coefficient of $T$.

**Remark 3.1.** We note that in a commutative case, the notion of the Dobrushin ergodicity coefficient was studied in [11,12,32].

We have the following theorem which extends the results of [11,32,27].

**Theorem 3.1.** Let $T : L^1(A, \tau) \to L^1(A, \tau)$ be a linear bounded operator. Then the following inequality holds

$$\|Tx\|_1 \leq \delta(T)\|x\|_1 + \alpha(T)\|\tau(x)\|$$

(2)

for every $x \in L^1(A, \tau)$.

Note that the proved theorem extends the results of [11,32,27]. Now before formulating a main result of this section we need an auxiliary result. Next lemma has been proved in [27].

First denote

$$D = \{ x \in L^1(A, \tau) : x \geq 0, \|x\|_1 = 1 \}.$$

**Lemma 3.2.** For every $x, y \in L^1(A, \tau)$ such that $x - y \in X$ there exist $u, v \in D$, such that

$$x - y = \frac{\|x - y\|_1}{2} (u - v).$$

The next result establishes several properties of the Dobrushin ergodicity coefficient in a noncommutative setting. Note that when $M$ is commutative and $\tau$ is finite, similar properties were studied in [24, 17].

**Theorem 3.3.** Let $T, S : L^1(A, \tau) \to L^1(A, \tau)$ be stochastic operators. Then the following assertions hold true:

(i) $0 \leq \delta(T) \leq 1$;

(ii) $|\delta(T) - \delta(S)| \leq \delta(T - S) \leq \|T - S\|$

(iii) $\delta(TS) \leq \delta(T)\delta(S)$;

(iv) if $K : L^1(A, \tau) \to L^1(A, \tau)$ is a linear bounded operator with $\tau(Kx) = 0$ for all $x \in A$, then $\|TK\| \leq \|K\|\delta(T)$.

(v) one has

$$\delta(T) = \sup \left\{ \frac{\|Tu - Tv\|_1}{2} : u, v \in D \right\}. \quad (3)$$

(vi) if $\delta(T) = 0$, then there is $y \in L^1(A, \tau)$, $y \geq 0$ such that $T = Ty$.

**Proof** (i) is obvious. Let us prove (ii). From (1) we immediately find that $\delta(T - S) \leq \|T - S\|$. Now let us establish the first inequality. Without loss of generality, we may assume that $\delta(T) \geq \delta(S)$. For an arbitrary $\varepsilon > 0$ from (1) one can find $x_\varepsilon \in X$ with $\|x_\varepsilon\|_1 = 1$ such that

$$\delta(T) \leq \|Tx_\varepsilon\|_1 + \varepsilon.$$
Then we have
\[\delta(T) - \delta(S) \leq \|Tx\|_1 + \|S\|_1 - \sup_{x \in X, \|x\|_1 = 1} \|Sx\|_1 \]
\[\leq \|Tx\|_1 - \|Sx\|_1 + \|S\|_1 \]
\[\leq \|(T - S)x\|_1 + \epsilon \]
\[\leq \sup_{x \in X, \|x\|_1 = 1} \|T - S\|_1 + \epsilon \]
and the arbitrariness of \(\epsilon\) implies the assertion.

(iii). Let \(x \in X\), then the stochasticity of \(S\) implies \(\tau(Sx) = 0\), hence due to (2) one finds
\[\|TSx\|_1 \leq \delta(T)\|Sx\|_1 + \alpha(T)\|\tau(Sx)\| \]
which yields \(\delta(TS) \leq \delta(T)\delta(S)\).

(iv). Using the same argument as (iii) one gets the desired assertion.

(v). For \(x \in X, x \neq 0\) using Lemma 3.2 we have
\[\frac{\|Tx\|_1}{\|x\|_1} = \frac{\|T(x^+ - x^-)\|_1}{\|x^+ - x^-\|_1} \]
\[= \frac{\|x^+ - x^-\|_1}{2} \frac{\|T(u - v)\|_1}{\|x^+ - x^-\|_1} \]
\[= \frac{\|Tu - Tv\|_1}{2}.\]

The equality (1) with the last one implies (3).

(vi). Let \(\delta(T) = 0\), then from (3) one gets \(Tu = Tv\) for all \(u, v \in D\). Therefore, denote \(y := Tu\). It is clear that \(y \in D\). Moreover, \(Ty = y\). Let \(x \in L^1(A,\tau), x \geq 0\), then noting \(\|x\|_1 = \tau(x)\) we find
\[Tx = \|x\|_1 T \left( \frac{x}{\|x\|_1} \right) = \tau(x)y.\]

If \(z \in L^1(A,\tau), \) then \(z = z_+ - z_-\), where \(z_+, z_- \geq 0\). Therefore
\[T(z) = T(z_+) - T(z_-) = \tau(z_+)y - \tau(z_-)y = \tau(z)y.\]

This completes the proof.

4. Uniform ergodicity

In this section, as an application of Theorem 3.3 we are going to prove the uniform ergodicity of homogeneous Markov chain.

**Theorem 4.1.** Let \(\{T^n\}\) be a homogeneous Markov chain generated by a stochastic operator \(T : L^1(A,\tau) \rightarrow L^1(A,\tau)\). The following assertions are equivalent:

(i) there exists \(\rho \in [0,1)\) and \(n_0 \in \mathbb{N}\) such that \(\delta(T^{n_0}) \leq \rho;\)

(ii) \(T\) is uniformly ergodic.
Proof (i) ⇒ (ii). Let $\rho \in [0,1)$ and $n_0 \in \mathbb{N}$ such that $\delta(T^{n_0}) \leq \rho$. Now from (iii) and (i) of Theorem 3.3 one gets

$$\delta(T^n) \leq \rho^{n/n_0} \to 0 \quad \text{as} \quad n \to \infty,$$

where $[a]$ stands for the integer part of $a$.

Let us show that $\{T^n\}$ is a Cauchy sequence w.r.t. to the norm. Indeed, using (iv) of Theorem 3.3 and (4) we have

$$\|T^n - T^{n+m}\| = \|T^{n-1}(T - T^{m+1})\| \leq \delta(T^{n-1})\|T - T^{m+1}\| \to 0 \quad \text{as} \quad n \to \infty.$$ 

Hence, there is a stochastic operator $Q$ such that $\|T^n - Q\| \to 0$. Let us show that $Q = T_y$, for some $y \in L^1(M, \tau)$. To do so, due to (vi) of Theorem 3.3 it is enough to establish $\delta(Q) = 0$.

So, using (ii) of Theorem 3.3 we have

$$|\delta(T^n) - \delta(Q)| \leq \|T^n - Q\|.$$ 

Now passing to the limit $n \to \infty$ at the last inequality and taking into account (4), we obtain $\delta(Q) = 0$, which is the desired assertion.

(ii)⇒ (i). Let $y$ be the element of $L^1(M_{sa}, \tau)$ defined at (ii). Let $\eta \in (0,1/4)$ be given a fixed number. Then (ii) implies that there is a number $n_0 \in \mathbb{N}$ such that $\|T^n - T_y\| < \eta$ for every $n \geq n_0$. Since $T_y = y$ we get

$$\|T^{n_0}u - T^{n_0}v\|_1 \leq \|T^{n_0}u - y\|_1 + \|T^{n_0}v - y\|_1 < 2\eta,$$

for every $u, v \in D$.

Hence, using (v) of Theorem 3.3 (see (3)) we obtain $\delta(T^{n_0}) \leq 2\eta$ which yields the desired assertion. The proof is complete.

Remark 4.1. Note that the proved theorem is a non-associative version Bartoszek’s result [7]. A similar result has been obtained in [8, 27] without using Dobrushin ergodicity coefficient, when $A$ is a self-adjoint part of von Neumnan algebra with a finite trace.

By $\Sigma(A)_{ue}$ we denote the set of all uniformly ergodic stochastic operators. Again using Theorem 3.3 we can prove the following

Theorem 4.2. The set $\Sigma(A)_{ue}$ is a norm dense and open subset of $\Sigma(M)$.

5. Weak ergodicity of nonhomogeneous Markov chains

In this section we study weak ergodicity of nonhomogeneous discrete Markov chains defined on $L^1(A, \tau)$.

Let us recall that NDMP $\{T^{k,n}\}$ defined on $L^1(A, \tau)$ is weakly ergodic if for every $k \in \mathbb{N} \cup \{0\}$ one has

$$\lim_{n \to \infty} \delta(T^{k,n}) = 0.$$ 

Theorem 5.1. Let $\{T^{k,n}\}$ be a NDMC defined on $L^1(A, \tau)$. If for each $k \in \mathbb{N} \cup \{0\}$ there exist $\lambda_k \in [0,1]$, a number $n_k \in \mathbb{N}$ such that $\delta(T^{k,k+n_k}) \leq \lambda_k$ with

$$\sum_{j' \geq 0} (1 - \lambda_{j'}) = \infty$$

for every subsequence $\{j'\}$ of $\{j\}_{j \in \mathbb{N}}$. Then the process $\{T^{k,n}\}$ is weak ergodic.
Proof Take any \( k \in \mathbb{N} \cup \{0\} \). Then due to the condition of Theorem there exist \( \lambda_k \in [0,1] \), a number \( n_1 \in \mathbb{N} \) such that \( \delta(T^{k,n_1}) \leq \lambda_k \). For \( \ell_1 := k + n_k \) we again apply the given condition, then one can find \( \lambda_{\ell_1}, n_{\ell_1} \) such that \( \delta(T^{\ell_1,\ell_1+n_{\ell_1}}) \leq \lambda_{\ell_1} \). Now continuing this procedure one finds sequences \( \{\ell_j\} \) and \( \{\lambda_{\ell_j}\} \) such that

\[
\ell_0 = k, \quad \ell_1 = \ell_0 + n_k, \quad \ell_2 = \ell_1 + n_{\ell_1}, \ldots, \ell_m = \ell_{m-1} + n_{\ell_{m-1}}, \ldots
\]

and \( \delta(T^{\ell_j,\ell_j+1}) \leq \lambda_{\ell_j} \).

Now for large enough \( n \) one can find \( M \) such that

\[
M = \max\{j : \ell_j + n_j \leq n\}.
\]

Then due to (iii) of Theorem 3.3 we get

\[
\delta(T^{k,n}) = \delta(T^{n,\ell_M} T^{\ell_{M-1},\ell M-1} \ldots T^{\ell_0,\ell_1}) \\
\leq \prod_{j=0}^{M-1} \delta(T^{\ell_{M-j},\ell_{M-j+1}}) \\
\leq \prod_{j=0}^{M-1} \lambda_{\ell_j}.
\]

Now taking into account (7), the last inequality implies the weak ergodicity of \( \{T^{k,n}\} \).

It is well-known [30] that one of the most significant conditions for weak ergodicity is the Doeblin’s Condition. Now we are going to define some non-associative analogous of such a condition.

We say that a NDMP \( \{T^{k,n}\} \) defied on \( L^1(A,\tau) \) satisfies condition \( \mathcal{D} \) if there exists \( \mu \in \mathcal{D} \) and for each \( k \) there exist a constant \( \lambda_k \in [0,1] \), an integer \( n_k \in \mathbb{N} \), and for every \( \varphi \in \mathcal{D} \), one can find \( \sigma_{k,\varphi} \in L^1(A_+,\tau) \) with \( \sup_{\varphi} \|\sigma_{k,\varphi}\|_1 \leq \lambda_k^\frac{1}{4} \) such that

\[
T^{k,n_k}\varphi + \sigma_{k,\varphi} \geq \lambda_k \mu,
\]

and

\[
\sum_{j' \geq 0} \lambda_{j'} = \infty
\]

for every subsequence \( \{j'\} \) of \( \{j\}_{j \in \mathbb{N}} \).

Using Theorem 5.1 one can proves the following

**Theorem 5.2.** Assume that a NDMC \( \{T^{k,n}\} \) defined on \( L^1(A,\tau) \) satisfies eak ergodic if and only if the condition \( \mathcal{D} \) is satisfied.

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**References**

[1] Accardi L and Fidaleo F 2003 *J. Funct. Anal.* **200** 324
[2] Albeverio S and Høegh-Krohn R 1978 *Comm. Math. Phys.* **64** 83
[3] Alfsen E M, Shultz F W and Stormer E 1978 *Adv. in Math.* **128** 11
[4] Ayupov Sh A 1982 *Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat.* **5** 7 (Russian)
[5] Ayupov Sh A 1981 *Russian Math. Surveys* **36** 201
[6] Ayupov Sh, Rakhimov A and Usmanov Sh. 1997 *Jordan, Real and Lie Structures in Operator Algebras* (Dordrecht: Kluwer Acad. Publ.)
[7] Bartoszek W 1990 *Anal. Polon. Math.* **52** 165
[8] Bartoszek W and Kuna B 2006 *Colloq. Math.* **105** 311
[9] Bratteli O and Robertson D W 1979 *Operator algebras and quantum statistical mechanics* (Berlin: Springer)
[10] Bożejko M, Kummerer B and Speicher R 1997 *Commun. Math. Phys.* **185** 129
[11] Cohen J E, Iwasa Y, Rautu G, Ruskai M B, Seneta E and Zbaganu G 1993 *Linear Algebra Appl.* **179** 211
[12] Dobrushin R L 1956 *Theor. Probab. Appl.* **1** 65, 329
[13] Dorea C C Y and Pereira A G C 2006 *Acta Math. Hungar.* **110** 287
[14] Emel’yanov E Yu and Wolff M P H 2003 *Positivity* **7** 3
[15] Fagnola F and Rebolledo R 2001 *Jour. Math. Phys.* **42** 1296
[16] Fagnola F and Rebolledo R 2003 *Theory Relat. Fields* **126** 289
[17] Ipsen I C F and Salee T M 2011 *SIAM J. Matrix Anal. Appl.* **32** 153
[18] Jajte R 1984 *Strong limit theorems in non-commutative probability* (Berlin: Springer)
[19] Johnson J and Isaacson D 1988 *J. Appl. Probab.* **25** 34
[20] Karimov A K and Mukhamedov F M 2003 *Sbornik Math.* **194** 237
[21] Karimov A and Mukhamedov F 2010 *Probab. Math. Statist.* **30** 153
[22] Komorowski T and Tycha J 1989 *Bull. Polish Acad. Sci. Math.* **37** 220
[23] Krengel U 1985 *Ergodic Theorems* (Berlin: Walter de Gruyter)
[24] Madsen R W and Isaacson D L 1973 *Ann. Probab.* **1** 329
[25] Mukhamedov F M 2004 *Izvestiya Math.* **68** 1009
[26] Mukhamedov F M (in press) *Rev. Mat. Compult.*
[27] Mukhamedov F, Temir S and Akin H 2005 *Siberian Adv. Math.* **15**(3) 28
[28] Nelson E 1974 *J. Funct. Anal.* **15** 103
[29] Parthasarathy K R 1992 *An introduction to quantum stochastic calculus* (Basel: Brirkäuser)
[30] Seneta E 1973 *Proc. Cambridge Philos. Soc.* **74** 507
[31] Tan Ch P 1996 *Statis. & Probab. Lett.* **26** 293
[32] Zaharopol R and Zbaganu G 1999 *Jour. Theor. Probab.* **99** 885
[33] Zeifman A I and Isaacson D L 1994 *Stochast. Process. Appl.* **50** 263