HOMEOMORPHISM CLASSIFICATION OF 
POSITIVELY CURVED MANIFOLDS WITH 
ALMOST MAXIMAL SYMMETRY RANK

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Abstract. We show that a closed simply connected 8-manifold (9-manifold) of positive sectional curvature on which a 3-torus (4-torus) acts isometrically is homeomorphic to a sphere, a complex projective space or a quaternionic projective plane (sphere). We show that a closed simply connected 2m-manifold ($m \geq 5$) of positive sectional curvature on which a ($m - 1$)-torus acts isometrically is homeomorphic to a complex projective space if and only if its Euler characteristic is not $2 \times 2$. By [Wi], these results imply a homeomorphism classification for positively curved $n$-manifolds ($n \geq 8$) of almost maximal symmetry rank $\left\lfloor \frac{n+1}{2} \right\rfloor$.

0. Introduction

An interesting problem in Riemannian geometry is the classification of the positively curved manifolds with large symmetry rank i.e. the rank of a maximal torus of the isometry group (cf. [Gro]). This study has gained considerable progress in the recent years (cf. [FMR], [FR2,3], [GS], [Hi], [HK], [PS], [Ro1], [Wi], [Ya], etc).

Grove-Searle proved that any closed positively curved $n$-manifold $M$ has symmetry rank $\leq \left\lfloor \frac{n+1}{2} \right\rfloor$ (the integer part) and “=” implies that $M$ is diffeomorphic to a sphere, a lens space or a complex projective space ([GS]).

Very recently, Wilking made a remarkable discovery that for a closed $n$-manifold of positive sectional curvature $M$, its a closed totally geodesic $m$-submanifold $N$ (if any) may capture the homotopy information i.e. $\pi_i(M, N) = 0$ for $i \leq 2m - n + 1$ ([Wi], cf. [FMR]). Using this result, he proved the following almost $1/2$-maximal rank theorem: For $n \geq 10$, if a closed simply connected positively curved $n$-manifold $M$ with symmetry rank at least $(\frac{2n}{2} + 1)$, then $M$ is homeomorphic to a sphere or a quaternionic projective space or homotopically equivalent to a complex projective space (for non-simply connected case, see [Ro1], [Wi]).

Subsequently, an analog of the above results for positively curved manifolds which admit isometric actions by elementary $p$-groups of large rank has been obtained in [FR2].

We now state the main results of this paper.

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Theorem A.

A closed 8-manifold (9-manifold) of positive sectional curvature on which a 3-torus (4-torus) acts isometrically is homeomorphic to a sphere or a complex projective space or a quaternionic projective plane $\mathbb{H}P^2$ (a sphere).

Theorem A implies an extension of the Wilking’s almost 1/2-maximal rank theorem to dimensions 8 and 9. It also arises a natural question: Is there a critical symmetry rank between $\frac{n}{4} + 1$ and $\left[\frac{n+1}{2}\right]$ so that above which there is a homeomorphism classification?

The following provides a partial positive answer.

Theorem B.

A closed simply connected $2n$-manifold ($n \geq 5$) $M$ of positive sectional curvature on which a $(n-1)$-torus acts isometrically is homeomorphic to a complex projective space if and only if the Euler characteristic of $M$ is not equal to 2.

As a consequence of Theorems A, B and the Wilking’s almost 1/2-maximal rank theorem (also, [FR2]), we obtain the following homeomorphism classification for positively curved manifolds with almost maximal rank.

Corollary C (Almost Maximal Rank).

For $n \geq 8$, a closed simply connected $n$-manifold $M$ of positive sectional curvature and almost maximal symmetry rank $\left[\frac{n-1}{2}\right]$ is homeomorphic to a sphere, a complex projective space or the quaternionic projective plane $\mathbb{H}P^2$.

Corollary C fits nicely between the maximal rank theorem of Grove-Searle and the almost 1/2-maximal rank theorem of Wilking mentioned earlier; in terms of the largeness on symmetry ranks and different types of classifications: diffeomorphism, homeomorphism and homotopy. Note that Corollary C is also sharp in the sense that a sphere, a complex projective space and a quaternionic projection plane all admit metrics satisfying Corollary C.

Corollary C is indeed valid for $n = 4$ ([HK]), and $n = 5$ ([Ro1]) where manifolds in Corollary C were studied. Hence, Corollary C also generalizes these early results.

To the contrast, the statement of Corollary C no longer holds in the remaining dimensions $n = 6, 7$. There are closed simply connected 6- and 7-manifolds of positive curvature and almost maximal symmetry rank which are not homeomorphic to any of rank one symmetric spaces ([AW], [Es]).

Due to the subtleness in dimensions $n = 6, 7$, one may like to take the approach by solving two problems below:

Problem 0.1: Show that if $M$ is a positively curved 6- (resp. 7-) manifold of almost maximal rank, then the second Betti number is at most two (resp. one).

Problem 0.2: Classify positively curved 6- (resp. 7-) manifolds of almost maximal rank with $b_2(M) \leq 2$ (resp. $\leq 1$).

Our last result will provide with a partial solution to Problem 0.2 (for a further development, see [FR3]).
Theorem D.

A closed simply connected 6-manifold (7-manifold) of positive sectional curvature on which a 2-torus (3-torus) acts isometrically is homeomorphic to a sphere if and only if its second Betti number vanishes.

Obviously, Theorem D is false if one replaces “positive curvature” by “non-negative curvature”; e.g. the metric product of the unit spheres, $S^3 \times S^3$ and $S^3 \times S^4$. By [GZ], each $S^3$-bundle over $S^4$ admits infinitely many non-negatively curved metrics. An interesting open question is to determine if there is a $S^3$-bundle over $S^4$ which is not diffeomorphic to a standard example and which admits a metric of positive sectional curvature. Theorem D implies that such a positively curved metric, if exists, has symmetry rank at most 2 (cf. [GKS]).

We now give an indication for the proof of Theorem B.

First, by [Ro1] (see Theorem 1.11) and the almost 1/2-maximal rank theorem of Wilking, one concludes that $M$ is homotopy equivalent to $\mathbb{C}P^n$. Then, from the homeomorphism classification of homotopy complex projective spaces by Sullivan ([Su]), it is easy to see that $M$ is homeomorphic to $\mathbb{C}P^n$ if $M$ has a submanifold $N$ which is homeomorphic to $\mathbb{C}P^{n-1}$ and the inclusion, $N \hookrightarrow M$, is at least 3-connected (see Corollary 1.4).

The proof is to construct a codimension 2 submanifold $N$ that meets the above requirements. Observe that combining the maximal rank theorem and the almost 1/2-maximal rank theorem mentioned earlier, it is not hard to see that any invariant closed totally geodesic submanifold of codimension 2 can serve as the desired $N$ (see Lemma 2.1). Unfortunately, in our circumstance $M$ may not have any invariant totally geodesic submanifold of codimension 2 (see Example 1.8 in [Ro1]). Hence, the submanifold $N$ we constructed may not be a totally geodesic submanifold (see Theorem 5.1).

Our strategy is to classify the singular structure of an isometric $T^{n-1}$-action. In spirit, this is similar to [HK] and [Ro1]. The motivation is that the singular set which is the union of the fixed point sets of all isotropy groups may provide more topological information than that from the fixed point set of any single isotropy group. The two main ingredients in our construction of $N$ are:

(0.3.1) A classification of the singular set $S$ (as a stratified space) of the $T^{n-1}$-action (see Theorem 3.6).

(0.3.2) The orbit space $M^* = M/T^{n-1}$ is homeomorphic to a sphere (see Theorem 4.1).

The singular set $S$ is stratified, and a stratum is a component of the singular orbits whose isotropy groups are the same. We will call the closure of the orbit projection of a singular stratum a simplex of $S^*$ (see Section 2), and call the union of the simplex of dimension $i$ the $i$-skeleton of $S^*$.

Perhaps it is easier to explain the idea of our construction if one pretends that the $T^{n-1}$-action is sitting in some isometric $T^n$-action on $M$ that is invisible to an observer. Then $S^*$ should be the codimension one ‘skeleton’ of the singular set $(S')^*$ of the $T^n$-action. Observe that a codimension 2 invariant totally geodesic submanifold (which is from the $T^n$-action and which is invisible) in $M$ corresponds to a top simplex of $(S')^*$. Here (0.3.1) and (0.3.2) enter to guarantee that one may fill in a ‘topological’ simplex to replace the top simplex that is invisible because the codimension one skeleton of the top simplex is visible in $S^*$ and because whose total space is an embedded sphere in $M^*$ of codimension 3. Since the embedded
sphere is unknotted ([St]), it must bound a standard topological ball in $M^*$ (see (0.3.2)). Then, it is not hard to check that the preimage $N$ of the ball by the orbit projection has the desired property (Lemma 5.2).

The ingredients in (0.3.1) is the rigidity of the local singular structure and the connectedness of $S$ (the Frankel’s theorem). The local rigidity is due to the almost maximality of symmetry rank and Berger’s vanishing theorem (cf. [GS], [Ro1]). The proof of (0.3.2) is based on (0.3.1) and uses, among others, the connectedness result of Wilking mentioned earlier.

It is worth to mention that (0.3.1) and (0.3.2) are also the main ingredients in the proof of Theorems A and D; using which we may either determine the homology structure or present $M$ as a union of two standard pieces from which the homeomorphism type of $M$ is obvious.

For $n = 6$ and 7, $S$ may not be connected and this is the reason we do not know a classification for $S$ in the general situation. However, such a classification is obtained in the case $b_2(M) = 0$.

The rest of the paper is organized as follows:

In Section 1, we collect results that are used in the proofs.

In Section 2, we prove Theorems A and B for the case that there is a non-trivial isotropy group with a codimension 2 fixed point set. Hence, we may assume that in the rest sections, any isotropy group has fixed point set codimension at least 4.

In Section 3, we classify the singular structure.

In Section 4, we prove that the orbit space $M^*$ is homeomorphic to a sphere.

In Section 5, we prove Theorem B.

In Section 6, we prove Theorem A.

In Sections 7 and 8, we prove Theorem D.

1. Preliminaries

In this section, we will collect results that will be served as tools in the rest of the paper.

a. Homeomorphic classification of a homotopy type.

The generalized Poincaré conjecture says that any homotopy $n$-sphere is homeomorphic to a sphere $S^n$. By the famous works of Smale and Freedman, only the case $n = 3$ (the original Poincaré conjecture) remains unsolved.

**Theorem 1.1 ([Fr]).**

*Any homotopy 4-sphere is homeomorphic to $S^4$.***

**Theorem 1.2 ([Sm]).**

*For $n \geq 5$, any homotopy $n$-sphere is homeomorphic to $S^n$.***

Contrast to being a sphere, the homotopy type of a closed manifold may contain many homeomorphism types. In [Su], Sullivan classified the homeomorphism types of a homotopy complex projective space.

Let $h : M \to \mathbb{C}P^n$ be a homotopy equivalence. For any $\mathbb{C}P^i \subset \mathbb{C}P^n$, let $h^{-1}(\mathbb{C}P^i) \subset M$ denote the transverse submanifold (the preimage of a map homotopic to $h$ and transverse to $\mathbb{C}P^n$.) Let $\sigma_h(i)$ be the signature (an integer) of
Theorem 1.3 ([Su]).

Let $hS(\mathbb{C}P^n)$ denote the set of homeomorphism classes of closed manifolds homotopy equivalent to $\mathbb{C}P^n$. For $i = 1, \cdots, [n/2]$, $\sigma_h(i)$ defines an one-to-one correspondence between $hS(\mathbb{C}P^n)$ and the set $\prod_{i=1}^{[n/4]}(\mathbb{Z} \times \mathbb{Z}_2)$.

Recall that the Kervaire invariant is zero if a manifold of dimension $(4i+2)$ has no middle dimensional homology, e.g. manifolds homotopy equivalent to $\mathbb{C}P^{2i+1}$. On the other hand, the signature is a homotopy invariant. As an immediate corollary of Theorem 1.3, we conclude

Corollary 1.4.

Assume that $M \in hS(\mathbb{C}P^n)$ has a codimension 2 submanifold $N$ which is homeomorphic to $\mathbb{C}P^{n-1}$. Then $M$ is homeomorphic to $\mathbb{C}P^n$ under one of the following conditions:

(1.4.1) $N$ represents a generator for $H_{2n-2}(M)$.
(1.4.2) The inclusion, $N \hookrightarrow M$, is at least 3-connected.

b. $G$-spaces.

Let $G$ denote a compact Lie group. The following two results, which concern a $G$-space, will be frequently quoted in our proofs. The first one is the so-called homotopy lifting property (cf. [Br]).

Lemma 1.5 (Homotopy Lifting).

Let $G$ be a compact Lie group, and let $M$ be a connected $G$-manifold. If there is a connected $G$-orbit, then $p_* : \pi_1(M) \rightarrow \pi_1(M/G)$ is surjective.

Let $F(G, M)$ denote the $G$-fixed point set. It is well known that if $G$ is abelian, then topology of $F(G, M)$ is closely related to the topology of $M$ (cf. [Hs]). Note that given an $G$-invariant metric, a component of $F(G, M)$ is a totally geodesic submanifold of even codimension.

Theorem 1.6 (Fixed point).

Let $M$ be a closed $G$-space.

(1.6.1) If $G = T^k$ a torus, then the Euler characteristic $\chi(M) = \chi(F(T^k, M))$.
(1.6.2) If $G = \mathbb{Z}_p^k$ ($p$ is a prime), then $\chi(M) = \chi(F(\mathbb{Z}_p^k, M)) \pmod{p}$

c. Positive curvature with large symmetry rank.

The rank of a Lie group is the rank of a maximal torus. Hence, Symrank($M$) = $k$ implies that $M$ admits an isometry torus $T^k$-action. When a $T^k$-invariant metric has positive sectional curvature, the basic fact is the following Berger's vanishing theorem ([Ko], [Ro2]):
Theorem 1.7.

Let $M$ be a closed $n$-manifold of positive sectional curvature. Assume that $M$ admits an isometric $T^k$-action.

(1.7.1) If $n$ is even, then the fixed point set is not empty.
(1.7.2) If $n$ is odd, then there is a circle orbit.

A consequence of Theorem 1.7 is that a large symmetry rank implies a totally geodesic submanifold of small codimension. Recently, Wilking [Wi] discovered that a totally geodesic submanifold of small codimension may capture a lot homotopy information of $M$ (see [FMR] for a further development).

A map $f : N \to M$ is called $(i + 1)$-connected, if $f_* : \pi_q(N) \to \pi_q(M)$ is an isomorphism for $q \leq i$ and onto for $q = i + 1$.

Theorem 1.8 ([Wi]).

Let $M$ be a closed $n$-manifold of positive sectional curvature, and let $N$ be a closed totally geodesic $k$-submanifold. If there is a Lie group $G$ that acts isometrically on $M$ and fixes $N$ pointwisely, then the inclusion map is $(2k - n + 1 + C(G))$-connected, where $C(G)$ is the dimension of a principal orbit of $G$.

In the extremal case when a $T^k$-fixed point set has the maximal possible dimension (e.g. a circle group has a fixed point set codimension 2 or a torus has a fixed point set codimension 4), one has the Grove-Searle diffeomorphism classification for the maximal symmetry rank.

Theorem 1.9 ([GS]).

Let $M$ be a closed $n$-manifold of positive sectional curvature.

(1.9.1) If $M$ admits an isometric circle action with a fixed point set of codimension 2, then $M$ is diffeomorphic to a lens space $S^n/Z_p$ ($p$ may be 1) or a complex projective space.
(1.9.2) If $M$ admits an isometric $T^k$-action, then $k \leq \lfloor \frac{n+1}{2} \rfloor$ and “=” implies a circle subgroup whose fixed point set has codimension 2.

The above Theorems 1.8 and 1.9 play an important role in the proof of the following almost 1/2-maximal rank theorem of Wilking.

Theorem 1.10 ([Wi]).

For $n \geq 10$, let $M$ be a closed simply connected $n$-manifold of positive sectional curvature. If $\text{Symrank}(M) \geq \frac{n}{4} + 1$, then $M$ is homeomorphic to a sphere or a quaternionic projective space or $M$ is homotopically equivalent to a complex projective space.

The following theorem in [Ro1] on the singular structure of a positively curved manifold with almost maximal symmetry rank will be used.

Theorem 1.11 ([Ro1]).

For $n \geq 8$, let $M$ be a closed simply connected $n$-manifold of positive sectional curvature. Assume that $M$ admits an isometric $T^{|\frac{n+1}{2}|}$-action such that there is no circle subgroup with fixed point set of codimension 2.

(1.11.1) If $n = 8$, then the fixed points are isolated whose number is 2, 3 or 5.
(1.11.2) If $n = 2m > 8$, then the fixed points are isolated whose number is either 2 or $(n + 1)$. 

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If \( n = 2m + 1 \), then the circle orbits are isolated whose number is \((m + 1)\).

A consequence of Theorems 1.11 and 1.6 is

**Corollary 1.12.**

For \( n \geq 4 \), let \( M \) be a closed 2n-manifold of positive sectional curvature and almost maximal symmetry rank \((n - 1)\). Then the Euler characteristic of \( M \), \( \chi(M) = 2 \) or \((n + 1)\) or 3 when \( n = 4 \). In particular, \( M \) is not homotopically equivalent to \( \mathbb{H}P^n \) for \( n \neq 2 \).

2. A Special Case of Theorems A and B

In this section, we prove the case of Theorems A and B that there is a non-trivial isotropy group with codimension 2 fixed point set (Lemma 2.1). We then show that in the complementary situation, all isotropy groups are connected (Lemma 2.3). This condition is crucial to obtain a complete classification of the singular set (see Section 3).

**Lemma 2.1.**

Let \( M \) be a closed simply connected \( n \)-manifold of positive sectional curvature. Assume that \( M \) admits an isometric \( T^{[n-1]/2} \)-action. If there is a non-trivial isotropy group with codimension 2 fixed point set, then \( M \) is homeomorphic to a sphere or a complex projective space.

**Proof.** First, we may assume \( n \geq 6 \) since the cases of \( n = 4 \) and \( n = 5 \) are covered by \([HK]\) and \([Ro1]\).

Let \( H \) denote an isotropy group with a codimension 2 fixed point component \( F_0 \subset M \). Since \( H \) acts effectively on a normal 2-disk of \( F_0 \), \( H = S^1 \) or \( H \) is a finite cyclic group. By Theorem 1.9, we may assume that \( H \) is finite. Since \( T^{[n-1]/2}/H \) acts effectively on \( F_0 \), \( F_0 \) has the maximal symmetric rank. Since \( M \) is simply connected, \( F_0 \) is simply connected (Theorem 1.8), and since the induced metric has positive sectional curvature, \( F_0 \) is diffeomorphic to \( S^{n-2} \) or \( \mathbb{C}P^{n-2} \) (Theorem 1.9).

Case 1. If \( F_0 \) \( \simeq \) \( S^{n-2} \), since \( F_0 \) \( \hookrightarrow \) \( M \) is \((n - 3)\)-connected, by the Hurewicz theorem and the Poincaré duality it is easy to see that \( M \) is a homotopy sphere. By Theorem 1.2, \( M \) is homeomorphic to \( S^n \).

Case 2. If \( F_0 \) \( \simeq \) \( \mathbb{C}P^{n-2} \). Since \( F_0 \) \( \hookrightarrow \) \( M \) is at least \( 3 \)-connected, by Corollary 1.4 \( M \) is homeomorphic to \( \mathbb{C}P^\frac{n}{2} \). \( \Box \)

The following are cases where Lemma 2.1 may apply.

**Lemma 2.2.**

Let \( M \) be a closed \( n \)-manifold of positive sectional curvature. Assume that \( M \) admits an isometric \( T^{[n-1]/2} \)-action. Then there is an isotropy group with fixed point set of codimension 2 if one of the following conditions holds:

(2.2.1) For \( n = 2m \), the fixed point set \( F(T^{[n-1]/2}, M) \) has dimension \( > 0 \).
(2.2.2) For \( n = 2m + 1 \), there are non-isolated circle orbits (e.g. if the fixed point set is not empty).
(2.2.3) There is a non-trivial finite isotropy group.
Proof. (2.2.1) Let \( F_0 \) denote a component of \( F(T^{\frac{n-1}{2}}, M) \) of positive dimension. Since \( T^{\frac{n-1}{2}} \) acts effectively on the normal space \( T_x^\perp F_0 \) via differentials, \( T_{\ell}^\perp F_0 \simeq \mathbb{R}^{n-2} \) and thus \( T^{\frac{n-1}{2}} \) has a circle subgroup \( S^1 \) with fixed point set codimension 2 in \( T_x^\perp F_0 \). Consequently, \( \dim(F(S^1, M)) = n - 2 \).

(2.2.2) Let \( H \subset T^{\frac{n-1}{2}} \) be an isotropy group of a non-isolated circle orbit, and let \( F_0 \) be a component of the fixed point set \( F(H, M) \) of dimension \( 2i + 1 > 1 \). Then \( H \) has rank \( \frac{n-2}{2} \). Since \( H \) acts effectively on the normal space of \( F_0 \), as in (2.2.1) we conclude the desired result.

(2.2.3) Let \( x \in M \) such that the isotropy group \( H \) at \( x \) is finite. Let \( F_0 \) denote a \( H \)-fixed point component containing \( x \). Then \( T^{\frac{n-1}{2}} \simeq T^{\frac{n-1}{2}}/H \) acts effectively on \( F_0 \). Since \( F_0 \) is a closed totally geodesic submanifold of even codimension, by Theorem 1.9 it follows that \( \dim(F_0) = n - 2 \). □

If (2.2.3) does not occur, then we have

**Lemma 2.3.**

Let \( M \) be a closed \( n \)-manifold with a \( T^k \)-action. Then there is no finite isotropy group if and only if every isotropy group is connected.

**Proof.** It suffices to prove the necessity. We argue by contradiction. Assume that \( T^\ell \times A \) is an isotropy group of some \( x \in M \), where \( \ell > 0 \) is minimal and\( A \neq 1 \) is a finite abelian group. By the slice theorem ([Br]), the orbit \( T^k(x) \) has a tubular neighborhood \( U = T^k \times T^\ell \times A D^\perp_\epsilon \), and the isotropy groups of the \( T^k \)-action in \( U \) are in one-to-one correspondence to the isotropy subgroups of \( T^\ell \times A \)-action in \( D^\perp_\epsilon \), where \( D^\perp_\epsilon \) is the \( \epsilon \)-ball in the normal slice of \( T^k(x) \). Hence, it suffices to prove that the linear \( T^\ell \times A \)-action on \( \partial D^\perp_\epsilon \) has a non-trivial finite isotropy group.

Assume that \( \mathbb{Z}_p \) is a cyclic subgroup of \( A \) of prime order. Consider the action of \( \mathbb{Z}_p^\ell \times \mathbb{Z}_p \) on \( \partial D^\perp_\epsilon \), where \( \mathbb{Z}_p^\ell \) is the subgroup of \( T^\ell \). By [Br] III. Theorem 10.12 (and its remark) one sees that, there are at least two isotropy groups of \( p \)-rank \( \ell \) in \( \mathbb{Z}_p^\ell \times \mathbb{Z}_p \). Therefore, there is at least one \( p \)-rank \( \ell \) subgroup of \( T^\ell \times A \) but not contained in \( T^\ell \), which acts on \( \partial D^\perp_\epsilon \) with non-empty fixed point set. Note that every isotropy group of this fixed point set has the form \( T^s \times B \), where \( s \leq \ell - 1 \) and \( B \cong \mathbb{Z}_p^{s-n} \). A contradiction to the minimality of \( \ell \). □

3. **Singular Structures in The Reduced Case**

By Lemma 2.1, the proofs of Theorems A, B and D may be reduced to the case where there is no non-trivial isotropy group with codimension 2 fixed point set. By Lemmas 2.2 and 2.3, we may assume the action of \( T^{\frac{n-1}{2}} \) satisfies that

(3.1.1) All isotropy groups are connected.

(3.1.2) The union of singular orbits is of codimension 4 (cf. [Ko]).

Note that by Lemma 2.2, (3.1.2) implies the following:

(3.1.2a) If \( n = 2m \), then all fixed points are isolated.

(3.1.2b) If \( n = 2m + 1 \), then all circle orbits are isolated.

For an example of a torus action satisfying the above, see Example 1.8 in [Ro1].

The goal of this section is to classify the singular set of an isometric \( T^{\frac{n-1}{2}} \)-action on a positively curved \( n \)-manifold \( M \) which satisfies (3.1.1) and (3.1.2) (see
Then S, A, B and D (see the discussion following (0.3.1)).

Let S denote the singular set i.e. the union of non-principle orbits. For an isotropy group \( H \), let \( S(H) \) denote the union of orbits whose isotropy group is \( H \). Then \( S \) is stratified by singular strata i.e. components of \( S(H) \) for all isotropy group \( H \neq 1 \). The \( i \)-th skeleton of \( S \), \( S^{(i)} \), is the union of singular strata whose dimensions are at most \( i \). Let \( p : M \to M^* \) denote the orbit projection. We call \( S^* = p(S) \) the singular set of \( M^* \). Clearly, the stratification of \( S \) descends to a stratification on \( S^* \).

Note that the closure of a singular stratum is a fixed point component of \( H \). We will call a component of the closure \( \overline{S(H)} \) a simplex.

Let \( \Delta_s \) denote the standard \( s \)-simplex in \( \mathbb{R}^{s+1} \) whose vertices are \((1, 0, \cdots, 0, 0)\), \((0, 1, 0, \cdots, 0)\), \cdots, \((0, 0, \cdots, 0, 1)\). Then \( \Delta_s \) is a stratified space with the natural stratification: a vertex is a 0-stratum, an open 1-simplex (edge) is a 1-stratum, ..., an open \( s \)-simplex is an \( s \)-stratum. Let \( \Delta^k_s \) denote the \( k \)-skeleton of \( \Delta_s \).

Two stratified spaces \( S_1^* \) and \( S_2^* \) are isomorphic, if there is a homeomorphism \( f : S_1^* \to S_2^* \) which maps a stratum onto a stratum.

**a. The singular structure of the maximal rank.**

We will see that the singular set \( S^* \) of an almost maximal symmetry rank action is the codimension one skeleton of a singular set in some standard example of the maximal symmetry rank; for which we will now describe its singular structure.

It is an easy exercise to verify the following three model cases (Lemmas 3.2-3.4).

**Lemma 3.2.**

Let \( T^{m+1} \) denote a maximal torus of \( O(2m+2) \). Then \( T^{m+1} \) acts isometrically on the unit sphere \( S^{2m+1} \) with the orbit space \( S^{2m+1}/T^{m+1} \) homeomorphic to \( \Delta_m \) and the singular set \( S^* \) is isomorphic to \( \partial \Delta_m = \Delta_m^{m-1} \) as stratified space. Moreover, a simplex of \( S \) is a totally geodesic sphere and its orbit projection is a simplex of \( \Delta_m \).

**Lemma 3.3.**

Let \((\mathbb{C}P^m, T^m)\) denote the complex projective space with the Fubini-Study metric and \( T^m \) be a maximal torus of \( \text{Isom}(\mathbb{C}P^m) \). Then the orbit space \( \mathbb{C}P^m/T^m \) is homeomorphic to \( \Delta_m \) and the singular set \( S^* \) is isomorphic to \( \partial \Delta_m = \Delta_m^{m-1} \) as stratified space. Moreover, a simplex of \( S \) is a totally geodesic complex projective space and its orbit projection is a simplex of \( \Delta_m \).

We now consider the standard \((S^{2m}, T^m)\). Observe that since there are only two isolated \( T^m \)-fixed points, \( S^* \) with its simplices defined in the above does not form a simplicial complex. Because of this, one may like to refine the stratification of \( S^* \) to a simplicial complex structure as follows.

Let \( S^{2m-1} \) denote the subsphere in \( S^{2m} \) consisting of points whose distance from a \( T^{2m} \)-fixed point is \( \pi/2 \). Then the \( T^m \)-action preserves \( S^{2m-1} \). Let \( S^*_i \) denote the singular set of \( S^{2m-1}/T^m \). Note that each \( i \)-simplex \( B_i \) of \( S^*_i \) is contained in a unique \((i + 1)\)-simplex \( A_{i+1} \) in \( S^* \). We then divide \( A_{i+1} \) into two new \((i + 1)\)-simplices whose intersection is \( B_i \). These new simplices also form a stratification of \( S^* \), denoted by \( \text{rf}(S^*) \).
Let $\Sigma(\Delta_{m-1}^{n-2})$ denote the two sides suspension of $\Delta_{m-1}^{n-2}$ obtained by identify the two ends of $\Delta_{m-1}^{n-2} \times [-1,1]$ with two points. Then $\Sigma(\Delta_{m-1}^{n-2})$ has a natural stratification which also forms a simplicial complex.

**Lemma 3.4.**

Let $T^m \subset O(2m+1)$ denote a maximal torus. Then $S^{2m}/T^m$ is homeomorphic to $\Delta_m$ and $\text{rf}(S^*)$ is isomorphic to $\Sigma(\Delta_{m-1}^{n-2})$ as stratified space. Moreover, a simplex of $S$ is a totally geodesic sphere and the orbit projection preserves vertices and maps a simplex of $S$ to the suspension of a simplex of $\Sigma(\Delta_{m-1}^{n-2})$.

**Lemma 3.5.**

Let $M$ be a simply connected positively curved closed $n$-manifold with an isometric $T[\frac{n+1}{2}]$-action. Then the orbit space $M^*$ is homeomorphic to $\Delta_m$, where $n = 2m$ or $2m + 1$. Moreover,

(3.5.1) If $n = 2m + 1$, then $S \cong S(S^{2m+1}, T^{m+1})$ and $S^* \cong \partial \Delta_m$.

(3.5.2) If $n = 2m$ and $\chi(M) = 2$, then $S \cong S(S^{2m}, T^m)$ and $S^* \cong \Sigma(\Delta_{m-1}^{n-2})$.

(3.5.3) If $n = 2m$ and $\chi(M) = m + 1$, then $S \cong S(\mathbb{C}P^m, T^m)$ and $S^* \cong \partial \Delta_m$.

**Proof.** Since the proofs are similar, we will only present a proof for (3.5.1). First, there are $(m + 1)$ isolated circle orbits or equivalently, $S^*$ has $(m + 1)$-vertices (if $n \geq 8$, see Theorem 1.11; and the case for $n \leq 7$ can be easily verified). By Lemma 3.8 below, around each fixed vertex $v$ there are $m$ 1-simplices whose end ($\neq v$) are all distinct. The maximality implies that every $(i + 1)$-vertices determines an $i$-simplex of $S^*$ and this clearly implies the desired result. \(\square\)

**b. The singular structure of almost maximal rank.**

The goal of this subsection is to prove the following classification result:

**Theorem 3.6 (Classification of singularities).**

Let $M$ be a closed simply connected $n$-manifold of positive sectional curvature.

Assume that $M$ admits an isometric $T[\frac{n+1}{2}]$-action satisfying (3.1.1) and (3.1.2).

(3.6.1) If $n = 2m \geq 8$ and $\chi(M) = m + 1$, then $S^* \cong \Delta_{m-2}^n$ and $S \cong S(\mathbb{C}P^m, T^m)$.

(3.6.2) If $n = 2m + 1 \geq 9$, then $S^* \cong \Delta_{m-2}^n$ and $S \cong S(\mathbb{C}P^m, T^m)$.

(3.6.3) If $n = 2m \geq 8$ and $\chi(M) = 2$, then $\text{rf}(S^*) \cong \Sigma(\Delta_{m-1}^{n-2})$ and $S \cong S(S^{2m}, T^m)$.

In particular, a simplex of $S$ is a totally geodesic sphere in (3.6.1) and (3.6.2) and a complex projective space in (3.6.1) and (3.6.3) in (3.6.2). In any case, the orbit space of a simplex is contractible.

Roughly speaking, Theorem 3.6 says that the singular set $S$ (resp. $S^*$) is a codimension 2 (resp. codimension one) skeleton of some corresponding model space of maximal rank (see Lemmas 3.2-3.4).

**Remark 3.7.** Consider a positively curved 6- or 7-manifold $M$ with (almost maximal) symmetry rank 2 and 3 respectively. Note that unlike the case of $n \geq 8$, the singular set $S$ of $M$ may not be connected. This is a problem to classify the global singular structure of $M$ (cf. [FR3]).

Similar to the situation of the maximal rank, the local singular structure in our circumstance is rigid.
Lemma 3.8 (Local singular structure).
Let the assumptions be as in Theorem 3.6.

(3.8.1) If \( n = 2m \), then around each fixed point there are \( \frac{m!}{i!(m-i)!} \)-many 2\(i\)-dimensional singular strata whose isotropy groups have rank \((m-1-i)\), \(1 \leq i \leq m-2\).

(3.8.2) If \( n = 2m + 1 \), then around each isolated circle orbit there are \( \frac{m!}{i!(m-i)!} \)-many \((2i+1)\)-dimensional strata whose isotropy groups have rank \((m-1-i)\), \(1 \leq i \leq m-2\).

(3.8.3) If \( H \) is an isotropy group of rank \( r \), then \( \dim(S(H)) = n - 2(r + 1) \).

(3.8.4) If \( n \geq 8 \) then the singular set \( S \) is connected.

Proof. (3.8.1) Consider the isotropy representation of \( T^{m-1} \) at an isolated fixed point. This is clearly a sub-representation of the standard linear \( T^m \)-action on \( D^{2m} \) (cf. (3.2.1)). Since (3.1.2), \( T^{m-1} \) must transversely intersect every torus subgroup \( \{(u_1, \cdots, u_m) \in T^m \subset \mathbb{C}^m|\ \text{where only one } u_i = 1 \text{ for } 1 \leq i \leq m\} \). Therefore, by the standard linear algebra, around the fixed point there are exactly \( \binom{m}{i} \)-many 2\(i\)-dimensional strata whose isotropy groups have rank \( m - 1 - i \), \(1 \leq i \leq m-2\).

(3.8.2) Let \( O_x \) be an isolated circle orbit with isotropy group \( H \cong T^{m-1} \). Consider the isotropy representation of \( H \) at the normal slice \( D^{2m} \) of \( O_x \). Since the isotropy groups of the linear \( H \)-action on the slice are in one-to-one correspondence with the isotropy groups of the \( T^m \)-action on a slice neighborhood of \( O_x \). The desired result follows from (3.8.1).

(3.8.3) If \( n = 2m \) (resp. \( n = 2m + 1 \)), then \( S(H) \) contains a fixed point (resp. an isolated circle orbit). The desired result follows from (3.8.1) and (3.8.2).

(3.8.4) Since \( m \geq 4 \), by (3.8.1) over each vertex of \( S^* \), \( S \) has a codimension 4 simplex and any two codimension 4 simplices have a non-empty intersection (Frankel theorem). This shows that \( S \) is connected.

\( \square \)

Proof of Theorem 3.6.
Since each simplex of \( S \) is a closed totally geodesic submanifold, each codimension 4 simplex is simply connected (Theorem 1.8). Since a codimension 4 simplex has the maximal symmetry rank, it is diffeomorphic to a sphere or a complex projective space.

(3.6.1) Note that \( \Delta_m^{m-2} \) is isomorphic to a simplicial complex with \((m+1)\)-vertices such that every \((m-1)\)-vertices spans a simplex isomorphic to \( \Delta_{m-2} \).

By Theorem 1.11, \( S^* \) has \((m+1)\) vertices. By Lemma 3.8, at each (fixed) vertex \( v \) there are \((m+1)\) 1-simplices containing \( v \) and any \((m-1)\) of these 1-simplices are contained in a (top) \((m-2)\) simplex. By the above discussion, the corresponding top simplex of \( S \) is a sphere (\( n \) is odd) with the maximal symmetry rank. This implies that a top simplex of \( S^* \) is isomorphic to \( \Delta_{m-2} \). Since at \( v \), each 1-simplex contains another vertex (the other end point), we conclude that every \((m-1)\) vertices spans a simplex \( \Delta_{m-2} \). We then conclude the desired result with the observation that \( S^* \) completely determines \( S \) when specifying the preimage of every top simplex of \( S^* \).

(3.6.2) We omit the proof since it is exactly the same argument if replacing the assertion that a codimension 4 simplex of \( S \) is diffeomorphic to a sphere by that a top simplex is diffeomorphic to a complex projective space.

(3.6.3). In this case, \( S \) (and \( S^* \)) has exactly two vertices (Theorem 1.11). The vertices correspond to the two vertices of the suspension \( \Sigma(\Delta_{m-1}) \). \( \square \)
Lemma 3.9.

Let the assumptions be as in Theorem 3.6. If \( n = 8 \) and \( \chi(M) = 3 \), then \( S \cong S(\mathbb{H}P^2, T^3) \) and \( S^* \cong S^*(\mathbb{H}P^2, T^3) \), where \( \mathbb{H}P^2 \) is the quaternionic projective space with the standard metric and \( T^3 \) is a maximal torus in Isom(\( \mathbb{H}P^2 \)).

Proof. It is an easy exercise to see that \( S^*(\mathbb{H}P^2, T^3) \) is isomorphic to a 2-dimensional singular polyhedron with three vertices \( v_1, v_2, v_3 \), six segments \( \alpha_1, \alpha_2 \) (joining the vertices \( v_1, v_2 \)), \( \beta_1, \beta_2 \) (joining the vertices \( v_2 \) and \( v_3 \)), \( \gamma_1, \gamma_2 \) (joining the vertices \( v_3 \) and \( v_1 \)) and seven faces with boundaries \( \alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2, \alpha_1\beta_1\gamma_1, \alpha_1\beta_2\gamma_2, \alpha_2\beta_2\gamma_1, \alpha_2\beta_1\gamma_2 \). The preimage of a face by the orbit projection, which is a codimension 4 simplex, is diffeomorphic to a sphere or a complex projective space depending the number of the vertices in the simplex.

We now consider \((M^8, T^3)\). By Theorem 1.11, \( S^* \) has exact three vertices. Together with the fact that \( S^* \) is connected and that at each vertex there are 4 1-simplex imply that the 1-skeleton of \( S^* \) is isomorphic to the 1-skeleton of \( S^*(\mathbb{H}P^2, T^3) \). Since at a fixed vertex \( v \) of \( S \) there are six 4-simplices, this implies that \( S^* \) is isomorphic to \( S^*(\mathbb{H}P^2, T^3) \). Since each 4-simplex is diffeomorphic to either \( S^4 \) or \( \mathbb{C}P^2 \), we then conclude the desired result. \( \square \)

4. Orbit Spaces of The Reduced Case

The goal of this section is to prove the following theorem (see the discussion following (0.3.2)).

Theorem 4.1.

Let \( M \) be a closed simply connected \( n \)-manifold of positive sectional curvature. Assume that \( M \) admits an isometric \( T^{(\frac{n-1}{2})} \)-action satisfying (3.1.1) and (3.1.2). Then \( M^* \) is a closed topological manifold and for \( n \geq 8 \), \( M^* \) is homeomorphic to a sphere.

We first verify a special case of Theorem 4.1.

Lemma 4.2.

Let the assumptions be as in Theorem 4.1. Assume that \( M = S^n \) is the standard unit sphere. Then \( M^* \) is homeomorphic to a sphere for all \( n \).

Proof. We will only present a proof for the case \( n = 2m+1 \) since the proof for \( n = 2m \) is similar.

Note that \( T^m \subset \text{Isom}(S^{2m+1}) = O(2m+2) \) and thus we may assume that \( T^m \) is contained in some maximal torus \( T^{m+1} \subset O(2m+2) \). By Lemma 3.2, \( S^{2m+1}/T^{m+1} \) is homeomorphic to \( \Delta_m \). Let \( X = S^{2m+1}/T^m \). Then \( S^1 = T^{m+1}/T^m \) acts on \( X \) such that \( X/S^1 \cong \Delta_m \).

Note that a singular \( T^m \)-orbit is also a singular \( T^{m+1} \)-orbit and any singular \( T^{m+1} \)-orbit projects to a point in \( X \) which is a \( S^1 \)-fixed point. Hence, the \( S^1 \)-action on \( X \) is semi-free with fixed point set the boundary of \( \Delta_m \). i.e. every orbit over the interior of \( \Delta_m \) is principal. Therefore, \( X \) is homeomorphic to \((\partial \Delta_m \times D^2) \cup_\partial (\Delta_m \times S^1) \cong S^{m+1}. \) \( \square \)
Lemma 4.3.

Let the assumptions be as in Theorem 4.1. Then $M^*$ is a closed topological manifold.

Proof. It suffices to prove that any maximal singular point $x \in S^*$ has an open neighborhood homeomorphic to an open ball in $\mathbb{R}^{[\frac{n+1}{2}]}$, since all other type of singular points appear in any such an open neighborhood, here a maximal singular point is either a fixed point or a circle orbit.

Let $H$ denote the isotropy group of $x$. Assume that $n = 2m$ or $2m + 1$. By the slice theorem, the orbit $x$ has an open neighborhood of the type, $T^{[\frac{n-1}{2}]} \times_H D^{2m}/T^{[\frac{n-1}{2}]} = D^{2m}/H$, where $D^{2m}$ is the open unit disk in $\mathbb{R}^{2m}$, and $H \cong T^{m-1}$. Since $H$ acts linearly on $D^{2m}$ which also satisfies (3.1.1) and (3.1.2), by Lemma 4.2 it follows that $\partial D^{2m}(r)/H$ is a sphere for all $0 < r < 1$ and thus $D^{2m}/H$ is homeomorphic to a disk. \hfill \Box

To prove that $M^*$ is homeomorphic to a sphere, we also need the following two lemmas.

For a compact Lie group $G$, let $EG$ denote the universal principle $G$-bundle. For a $G$-space $X$, the $i$-th equivariant cohomology group $H^i_G(X) := H^i(X_G)$, where $X_G = EG \times_G X$ (cf. [Br]).

Lemma 4.4.

Let $X$ be a precompact $G$-space with $G = T^{[\frac{n-1}{2}]}$, and let $S$ denote the singular set. If all isotropy groups are connected, then

(4.4.1) For all $i \geq 0$, there is an isomorphism, $H^i_G(X, S) \cong H^i(X^*, S^*)$.

(4.4.2) $H^i(X, S) = 0$ (resp. $H_i(X, S) = 0$) for all $i \leq l$ if and only if $H^i_G(X, S) = 0$ (resp. $H_i(G, X, S) = 0$) for all $i \leq l$.

Proof. (4.4.1) Consider the natural map, $f : X_G \to X^*$. Note that for $x \in (X^* - S^*)$, the preimage of $f^{-1}(x)$ is the classifying space of the isotropy group at $x$ and thus contractible since the isotropy group is trivial. Then (4.4.1) follows from the proof of Proposition VII.1.1 of [Br] and the Vietoris-Begle mapping Theorem.

(4.4.2) By a direct calculation. \hfill \Box

Proof of Theorem 4.1.

Since $M^*$ is a topological manifold of dimension at least 5 (Lemma 4.3), by Theorem 1.2 it suffices to show that $M^*$ is a homotopy sphere. Since $M^*$ is simply connected (Lemma 1.5), $M^*$ is a homotopy sphere if and only if $M^*$ is a homology sphere (Hurewicz theorem). By the Poincaré duality, $M^*$ is a homology sphere if $H_i(M^*) = 0$ for all $i \leq [\frac{n+1}{2}]$, where $n = 2m$ or $(2m + 1)$.

We now assume that $n \geq 8$. Note that by Theorem 3.6, there is a unique 1-simplex $\Delta^*_{i}$ of $S^*$ such that $\Delta^*_{i} \cap N^* = \emptyset$ (note that $\Delta^*_{i}$ is spanned by the two vertices of $S^*$ which are not in $N^*$). By the transversality, $H_i(M^*) \cong H_i(M^* - \Delta^*_i) \cong H_i((M^* - \Delta^*_i), N^*)$ for $1 \leq i \leq m - 2$ since $N^*$ is contractible. Observe that the inclusion $N^* \subset (S^* - \Delta^*_i)$ (resp. $N \subset (S - p^{-1}(\Delta^*_i))$) is a homotopy equivalence. Therefore, by Lemma 4.4, $H_i((M^* - \Delta^*_i), N^*) = 0$ if and only if $H_i((M - p^{-1}(\Delta^*_i)), N) = 0$ for $i \leq n - 6$. Again by the transversality and Theorem 1.8, $H_i((M - p^{-1}(\Delta^*_i)), N) \cong H_i(M, N) = 0$ for $i \leq n - 6$. Since $n \geq 8$, $n - 6 \geq [\frac{n+1}{2}]$ and thus the desired result follows. \hfill \Box

We conclude the section with the following application of Theorem 4.1.
Corollary 4.5.
Let $M$ be as in Theorem 4.1 of dimension $n \geq 8$. Assume that $n$ is odd or $n$ is even and $\chi(M) = 2$ (see Theorem 1.11). Then there is a totally geodesic sphere $S^{n-4} \subset M$ and an embedded 3-sphere $S^3$ in $M$ such that $(M - S^3)$ is homotopy equivalent to $S^{n-4}$.

Proof. Let $N$ and $\Delta_1^*$ be as in the proof of Theorem 4.1. By Theorem 3.6 $N \simeq S^{n-4}$ is totally geodesic and $N^*$ is contractible. Observe that $p^{-1}(\Delta_1^*)$ is homeomorphic to $S^3$. Indeed, $p^{-1}(\Delta_1^*)$ is a totally geodesic 3-sphere if $n$ is odd, and may not be totally geodesic but contained in a totally geodesic 4-sphere $p^{-1}(\Sigma(\Delta_1^*))$ if $n$ is even. By Theorem 4.1, $(M^* - \Delta_1^*)$ is contractible, and thus by Lemma 4.4

$$H_i((M - p^{-1}(\Delta_1^*)), (S - p^{-1}(\Delta_1^*)) \cong H_i((M^* - \Delta_1^*), (S^* - \Delta_1^*)) \cong H_i((M^* - \Delta_1^*), N^*) = 0$$

for all $i$, since $(S^* - \Delta_1^*)$ homotopy retracts to $N^*$. Therefore the inclusion $j : (S - p^{-1}(\Delta_1^*)) \rightarrow (M - p^{-1}(\Delta_1^*))$ is a homology equivalence. By transversality the complement $(M - p^{-1}(\Delta_1^*))$ is simply connected. Hence, by the Hurewicz theorem, $j$ is a homotopy equivalence. The desired result follows. □

5. Proof of Theorem B

As explained in the introduction, in the proof of Theorem B the main technical result is the following.

Theorem 5.1.
Let the assumptions be as in Theorem B. Then $M$ has a submanifold $N$ such that

(5.1.1) $N$ is homeomorphic to $\mathbb{CP}^{n-1}$.
(5.1.2) The inclusion $N \hookrightarrow M$ is $(2n - 6)$-connected.

Proof of Theorem B by assuming Theorem 5.1.
First, the Euler characteristic number of $M$ is $(n + 1)$ (Corollary 1.12) and thus $M$ is homotopically equivalent to $\mathbb{CP}^n$ (Theorem 1.10). By now Theorem B follows from Theorem 5.1 and Corollary 1.4. □

Lemma 5.2.
Let $M$ be a closed $2n$-manifold. Assume that $M$ admits an effective $T^n$-action with $(n + 1)$ isolated fixed points and $(M^*, S^*) \cong (\Delta_n, \partial \Delta_n)$ as stratified spaces. Let $p : M \rightarrow M^*$ be the orbit projection. If every face $\Delta_{n-1} \subset \partial \Delta_n$, $p^{-1}(\Delta_{n-1})$ is diffeomorphic to $\mathbb{CP}^{n-1}$, then $M$ is homeomorphic to $\mathbb{CP}^n$.

Note that no curvature assumption is required in Lemma 5.2.

Proof. First note that $M$ is simply connected, since $M^*$ is contractible and $T^n$ acts on $M$ with non-empty fixed point set.

We identify $M^*$ with $\Delta_n$. Choose a face $\Delta_{n-1} \subset \Delta_n$ and identify $\mathbb{CP}^{n-1}$ with $p^{-1}(\Delta_{n-1})$. Let $D(\eta)$ be a tubular neighborhood of $\mathbb{CP}^{n-1}$, which is a closed $D^2$-bundle on $\mathbb{CP}^{n-1}$. Let $W$ be the closure of the complement $M - D(\eta)$. Note that $M = D(\eta) \cup_{\partial W} W$. Clearly, $W$ is homotopy equivalent to $(M - \mathbb{CP}^{n-1})$. 

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We claim that $W$ is homologous equivalent to a point. Assuming this, by duality the boundary $\partial W$ is a homology sphere. Since $\partial W = S(\eta)$, the circle bundle of $D(\eta)$, by Gysin exact sequence the Euler class of $\eta$ has to be a generator of $\mathbb{CP}^{n-1}$. Therefore $\partial W = S^{2n-1}$. By the Van-Kampen theorem $\pi_1(W) = 0$ since $\pi_1(M) = 0$. Thus $W$ is diffeomorphic to the ball $D^{2n}$ and so $M = D(\eta) \cup_{S^{2n-1}} D^{2n}$ is unique up to homeomorphism.

It remains to prove $(M - \mathbb{CP}^{n-1})$ is homologous to a point. By the assumption $S \cong S(\mathbb{CP}^n, T^n)$. In other words, it is union of $n$ copies of $\mathbb{CP}^{n-1}$ such that every two copies intersects in a $\mathbb{CP}^{n-2}$, etc. Observe that $(S - \mathbb{CP}^{n-1})$ is contractible (it obviously deformation retracts to the unique fixed point outside $\mathbb{CP}^{n-1}$). Applying Lemma 4.4 to $(M - \mathbb{CP}^{n-1})$ we get

$$H_i((M - \mathbb{CP}^{n-1}), (S - \mathbb{CP}^{n-1})) \cong H_i((\Delta_n - \Delta_{n-1}), (\partial \Delta_n - \Delta_{n-1})) = 0$$

for all $i$. Therefore $H_i(M - \mathbb{CP}^{n-1}) = 0$ for all $i \geq 1$. The desired result follows. □

**Proof of Theorem 5.1.**

(5.1.1) By the discussion in Section 2, we may assume (3.1.1) and (3.1.2). The singular set $S^*$ is isomorphic to $\Delta_n^{n-2}$ (Theorem 3.6) and $M^*$ is homeomorphic to $S^{n+1}$ (Theorem 4.1). These two facts are crucial for the following construction of the desired $N$.

Let $v_1, \ldots, v_n, v_{n+1}$ denote the vertices of $S^*$, and let $A$ denote the simplicial subcomplex of $S^*$ which consists of $(n-2)$-simplices which do not contain $v_{n+1}$. Then the total space $|A|/A$ is homeomorphic to $S^{n-2}$ (Theorem 3.6). A priori, $|A|$ is a codimension 3 knot in $M^*$. By [St], $|A|$ must be a topological trivial knot and thus bounds a $(n-1)$-ball $D \subset M^*$. We may assume that $D \cap S^* = A$ (since $(M^* - (S^* - |A|))$ is homeomorphic to $\mathbb{R}^{n+1}$). Put $N = p^{-1}(D)$. Note that $N$ is a closed invariant $(2n-2)$-manifold. Since every top simplex of $S$ in $N$ is a totally geodesic $\mathbb{CP}^{n-2}$, by Lemma 5.2 $N$ is homeomorphic to $\mathbb{CP}^{n-1}$.

(5.1.2) Note that $\mathbb{CP}^{n-2} \to N \approx \mathbb{CP}^{n-1}$ is $(2n - 4)$-connected. Hence, by Theorem 1.8 the inclusion $i : \mathbb{CP}^{n-2} \to N \hookrightarrow M$ is $(2n - 6)$-connected. □

6. Proof of Theorem A.

First, by the discussion in Section 2, we may reduce the proof of Theorem A to the case in (3.1.1) and (3.1.2).

The proof of Theorem A for $n = 8$ and $9$ follows from Lemmas 6.1 and 6.2.

**Lemma 6.1.**

Let $M$ be as in Theorem A. If $n = 8$, then $M$ is homeomorphic to $S^8$, $\mathbb{CP}^4$ or $\mathbb{H}P^2$.

**Proof.** We will show that $M$ is homeomorphic to $S^8$, $\mathbb{H}P^2$ or $\mathbb{CP}^4$ corresponding respectively to $\chi(M) = 2, 3$ or $5$ (Corollary 1.12).

Case 1. Assume that $\chi(M) = 2$.

We only need to show that $M$ is a homotopy sphere (Theorem 1.2). Since $\chi(M) = 2$, by Poincaré duality and the Hurewicz theorem it is easy to see that $M$ is a homotopy sphere if $M$ is 3-connected, which follows trivially from Corollary 4.5 and the Alexander duality.
Case 2. Assume that $\chi(M) = 3$.

Let $S^4$ denote a totally geodesic sphere which is a 4-simplex of $S$. By the argument similar to the above, one may verify that $(M - S^4)$ is contractible and so homeomorphic to the open ball $\text{int}(D^8)$ (by the h-cobordism theorem), and thus $M = D(\nu) \cup \partial D^8$, where $D(\nu)$ is the normal disk bundle over $S^4$ with fiber $D^4$. Since $S^4$ is the fixed point component of a circle isotropy group of $T^3$-action without any finite order isotropy group, the normal bundle $\nu$ must be a complex 2-bundle with normal sphere bundle $S^7$. This shows that the second Chern class of the bundle is a generator of $H^4(S^4, \mathbb{Z})$ and $\nu$ is equivalent to the normal bundle of $\mathbb{C}P^1 \subset \mathbb{C}P^2$. In particular, this gives a diffeomorphism $f: (M - \text{int}(D^8)) \to (\mathbb{C}P^2 - \text{int}(D^8))$.

Clearly the restriction of $f$ on the boundary may be extended to a homeomorphism of $D^8$ by radical extension (indeed one may prove the diffeomorphism type is unique using some deep topology in this case). The desired result follows.

Case 3. Assume $\chi(M) = 5$.

By Theorem B, it suffices to prove that $M$ is homotopy equivalent to $\mathbb{C}P^4$. To achieve this, we only need to prove that the cohomology ring of $M$ is the same as that of $\mathbb{C}P^4$. Let $v_1, v_2, v_3, v_4, v_5$ be the vertices of $S^* \cong \Delta^4_2$ (Theorem 3.6). Recall that a face $S^*$, saying $[v_3v_4v_5]$, corresponds to a totally geodesic $\mathbb{C}P^2$ in $M$. By removing a face spanned by $v_3, v_4, v_5$ in $\Delta^4_2$, we claim that the complement $M - \mathbb{C}P^2$ is homotopy equivalent to $S^2$, which corresponds to a segment $[v_1v_2]$ with vertices $v_1, v_2$. We omit the details since the argument is similar to the proof of Corollary 4.5. Therefore, by Alexander duality Theorem we get that $H_i(M) \cong H_i(\mathbb{C}P^2)$ for $i \leq 5$. By Poincaré duality it suffices to prove that the self intersection number of the cycle $[\mathbb{C}P^2]$ is $\pm 1$. This is because that the simplex of $S$ represented by the face $[v_1v_2v_3]$ intersects with $\mathbb{C}P^2$ at a single point $v_3$. The desired result follows.

Lemma 6.2.

Let $M$ be as in Theorem A. If $n = 9$, then $M$ is homeomorphic to a sphere.

Proof. Since $S$ has a 5-simplex (Theorem 3.6), $M$ is 3-connected (Theorem 1.8). By Poincaré duality, it suffices to prove that $H_4(M) = 0$. This follows easily from Corollary 4.5 and the Alexander duality.

7. Proof of Theorem D For $n = 6$

The goal of this section is to establish the following result.

Proposition 7.1.

Let $M$ be a closed simply connected 6-manifold. Assume $M$ admits a $T^2$-action satisfying following conditions:

(7.1.1) The $T^2$-fixed point set is non-empty and there is no circle subgroup with fixed point set codimension 2.

(7.1.2) All isotropy groups are connected.

(7.1.3) Each fixed point component of any isotropy group is either a 2-sphere or a point.

Then $M$ is diffeomorphic to $S^6$ if and only if $b_2(M) = 0$.

Remark 7.2. Proposition 7.1 will be true without the restriction (7.1.2). A reason we do not include a proof for this more general situation is to avoid some unnec-
essary technical complexity (compare to [Ro1]); in our circumstance we do have (7.1.2) (see (3.1.1)).

An interesting consequence of Proposition 7.1 is that any $T^2$-action on $S^3 \times S^3$ does not satisfy (7.1.2) or (7.1.3).

Proof of Theorem D for $n = 6$. By Lemma 2.1, we may assume (7.1.1) and (7.1.2). Since a fixed point component is an orientable totally geodesic submanifold of even-codimension, (7.1.3) follows from the positive curvature assumption. By now Theorem D for $n = 6$ follows from Proposition 7.1. □

Lemma 7.3.

Let $M$ be as in Proposition 7.1. If $b_2(M) = 0$, then the Euler characteristic $\chi(M) = 2$.

Proof. Since $b_2(M) = 0$, $\chi(M) = 2 - b_3(M) \leq 2$. By (7.1.1), from the isotropy representation of the $T^2$-action at an isolated fixed point one concludes that there is a circle subgroup whose fixed point set contains a sphere $S^2$ (see (7.1.3)). This implies that $\chi(M) \geq 2$ (see Theorem 1.6) and therefore $\chi(M) = 2$ and $b_3(M) = 0$. □

Lemma 7.4.

Let $M$ be as in Proposition 7.1. Then $H_2(M)$ has no torsion.

Proof. Note that $H_2(M) \cong \pi_2(M)$ (Hurewicz theorem). Since the singular set $S$ is a finite union of at most 2-dimensional submanifolds, by transversality $\pi_2(M) \cong \pi_2(M - S)$. Thus it suffices to show that $\pi_2(M - S)$ has no torsion.

By the homotopy exact sequence of the fibration, $T^2 \to (M - S) \to (M^* - S^*)$,

$$0 \to \pi_2(M - S) \to \pi_2(M^* - S^*) \to \mathbb{Z}^2 \to 0$$

it reduces to show that $\pi_2(M^* - S^*)$ has no torsion. By (7.1.1) and (7.1.2), it is easy to see that $M^*$ is a topological 4-manifold. Since $M^*$ is simply connected (see Lemma 1.5), by transversality $(M^* - S^*)$ is simply connected and thus $\pi_2(M^* - S^*) \cong H_2(M^* - S^*)$ (Hurewicz theorem). By Alexander duality, $H_2(M^* - S^*) \cong \tilde{H}^2(M^*, S^*)$. Since both $H^1(S^*)$ and $H^2(M^*)$ are torsion free, the desired result follows by the long exact sequence of the pair $(M^*, S^*)$. □

Proof of Proposition 7.1.

If $b_2(M) = 0$, by Lemma 7.4 we know that $\pi_2(M) = 0$ and by Lemma 7.3 $b_3(M) = 0$. By duality and Hurewicz theorem this implies that $M$ is a homotopy sphere. By Theorem 1.2 $M$ is homeomorphic and thus diffeomorphic to $S^6$. □

8. Proof of Theorem D For $n = 7$

The goal of this section is to establish the following topological result.
Proposition 8.1.
Let $M$ be a closed simply connected 7-manifold. Assume that $M$ admits a $T^3$-action such that
(8.1.1) There is no circle subgroup with fixed point set of codimension 2.
(8.1.2) All isotropy groups are connected.
(8.1.3) Each fixed point component of any isotropy group is either a circle or a lens space of dimension 3.
Then $M$ is homeomorphic to $S^7$ if and only if $b_2(M) = 0$.

Note that Proposition 8.1 will still be true without (8.1.2); compare to Remark 7.2.

Proof of Theorem D for $n = 7$. By Lemma 2.1, we may assume (8.1.1) and (8.1.2). Since any fixed point component of an isotropy group is a closed totally geodesic manifold with the maximal symmetry rank, (8.1.3) holds (see Theorem 1.9). By now Theorem D for $n = 7$ follows from Proposition 8.1. □

We first determine the singular structure and the homeomorphic type of the orbit space.

Lemma 8.2.
Let the assumptions be as in Proposition 8.1. Then $S^*$ is isomorphic to $\Delta^1_3$ and $M^*$ is homeomorphic to $S^4$.

Proof. By Theorem 4.1, $M^*$ is a closed topological 4-manifold. By Lemma 1.5 $M^*$ is simply connected. We further claim that $M^*$ is a homotopy 4-sphere. In fact, by [FR1] Lemma 5.2 we see that $H_2(M/H; \mathbb{Q}) = 0$, where $H \cong T^2$ is an isotropy group. Observe that $T^3/H$ acts on $M/H$ with non-empty fixed point set. Since $M^* = (M/H)/(T^3/H)$, by [FR1] Lemma 5.2 once again we get that $H_2(M^*; \mathbb{Q}) = 0$. By Poincare duality this implies that $M^*$ is a homotopy 4-sphere.

Consider the homotopy exact sequence of $T^3 \to (M - S) \to (M^* - S^*)$,

$$0 \to \pi_2(M - S) \to \pi_2(M^* - S^*) \to \mathbb{Z}^3 \to 0.$$ 

By the transversality (see (8.1.1)), $\pi_2(M - S) \cong \pi_2(M) = 0$ and thus $\pi_2(M^* - S^*) \cong \mathbb{Z}^3$. By Alexander duality $H_1(S^*) \cong H_2(M^*, S^*) \cong H^2(M^* - S^*) \cong \mathbb{Z}^3$.

Note that the local singular structure described in Lemma 3.8 remains valid in our situation: around each isolated circle orbit, there are exactly three 3-dimensional strata sharing the circle orbit. Hence, at each vertex of $S^*$ there are exactly three edges going out from the vertex. Since $S^*$ is a graph satisfying $b_1(S^*) = 3$, it is easy to see that $S^* \cong \Delta^1_3$. □

Proof of Proposition 8.1.
We claim that any 3-simplex of $S$ is homeomorphic to $S^3$ and $(M - S^3)$ is homology equivalent to a three sphere. By Alexander duality, it follows easily that $M$ is a homology sphere and thus a homotopy sphere since $M$ is simply connected. By Theorem 1.2 $M$ is homeomorphic to a sphere.

Let $L$ denote a 3-simplex of $S$. By (8.1.3), $L$ is a lens space. By Lemma 8.2, there is a unique 3-simplex $L'$ of $S$ such that $L^* \cap L'^* = \emptyset$. By (8.1.2) and Lemma 4.4, $H_i((M - L'), (S - L')) \cong H_i((M^* - L^*), (S^* - L'^*)) = 0$ for all $i$ (see Lemma
8.2). Consequently, \((M - L')\) is homology equivalent to \((S - L')\) which, in turn, is homology equivalent to \(L\) (see Lemma 8.2). On the other hand, by (8.1.1) and the transversality \(\pi_1(M - L') \cong \pi_1(M) = 1\). Hence, \(H_1(L) = 0\) and so \(L\) is simply connected and therefore homeomorphic to \(S^3\). Likewise, \(L'\) is also homeomorphic to \(S^3\). The desired result follows. \(\Box\)

References

[AW] S. Aloff; N. R. Wallach, An infinite family of 7-manifolds admitting positive curved Riemannian structures, Bull. Amer. Math. Soc. 81 (1975), 93-97.
[Bre] G. Bredon, Introduction to compact transformation groups, Academic Press 48 (1972).
[Es] J.-H Eschenburg, New examples of manifolds with strictly positive curvature, Invent. Math 66 (1982), 469-480.
[FMR] F. Fang; S. Mendonca; X. Rong, A connectedness principle in the geometry of positive curvature, Preprint (2001).
[FR1] F. Fang; X. Rong, Positive pinching, volume and second Betti number, Geom. Func. Anal. 9 (1999), 641-674.
[FR2] F. Fang; X. Rong, Positively curved manifolds of maximal discrete symmetry rank, Amer. J. Math (To appear).
[FR3] F. Fang; X. Rong, Positively curved manifolds of almost maximal symmetry rank in dimensions 6 and 7, In preparation.
[Fr] M. Freedman, Topology of Four Manifolds, J. of Diff. Geom. 28 (1982), 357-453.
[Gro] K. Grove, Geometry of, and via symmetries, Univ. Lecture Ser., Amer. Math. Soc., Providence, RT 27 (2002), 31-53.
[GKS] S. Goette; N. Kitchloo; K. Shankar, Diffeomorphism type of the Berger space SO(5)/SO(3), Preprint.
[GS] K. Grove, C. Searle, Positively curved manifolds with maximal symmetry-rank, J. Pure Appl. Alg 91 (1994), 137-142.
[GZ] K. Grove; W. Ziller, Curvature and symmetry of Milnor spheres, Ann. of Math 152 (2000), 301-367.
[Hs] W. Hsiang, Cohomology theory of topological transformation groups, Ergebnisse der Mathematik und inere Grenzegebiete 85 (1975).
[HK] W. Hsiang, B. Kleiner, On the topology of positively curved 4-manifolds with symmetry, J. Diff. Geom 30 (1989), 615-621.
[Ko] S. Kobayashi, Transformation groups in differential geometry, Springer-Verlag Berlin Heidelberg New York (1972).
[PS] T. Püttmann; C. Searle, The Hopf conjecture for manifolds with low cohomogeneity or high symmetry rank, Proc. Amer. Math. Soc. 130 (2002), 163-166.
[Ro1] X. Rong, Positively curved manifolds with almost maximal symmetry rank, Geometriae Dedicata 59 (2002), 157-182.
[Ro2] X. Rong, On the fundamental groups of compact manifolds of positive sectional curvature, Ann. of Math 143 (1996), 397-411.
[Sm] S. Smale, Generalized Poincaré conjecture in dimension \(\geq 4\), Ann. of Math 74 (1961), 391-466.
[St] J. Stallings, On topologically unknotted spheres, Ann. of Math 77 (1963), 490-503.
[Su] D. Sullivan, Triangulating homotopy equivalences and homeomorphisms, Geometric Topology Seminar Notes, in “The Hauptvermutung Book”, A collection of papers on the topology of manifolds, K-monographs in Math. 1 (edited by A. Ranicki, Kluwer Academic Publishers, 1995).
[Wi] B. Wilking, Torus actions on manifolds of positive sectional curvature, preprint (August 2002).
[Ya] D. Yang, On the topology of nonnegatively curved simply connected 4-manifolds with discrete symmetry, Duke Math. J. 74 (1994), 531-545.
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