DECOMPOSABILITY OF THE HIGSON CORONAE OF FINITELY GENERATED GROUPS WITH ONE END

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ABSTRACT. We characterize a space which is coarsely equivalent to the space of natural numbers using the indecomposability of its Higson corona. This leads to the characterization that a finitely generated group has exactly one end if and only if its Higson corona is a decomposable continuum. In contrast, in the case of a finitely generated group having exactly two ends, we characterize it as a group whose Higson corona is a topological sum of two indecomposable continua.

1. INTRODUCTION

The notion of Higson compactification was introduced by Higson for non-compact complete Riemannian manifolds. Then Roe defined the Higson compactification for more general spaces (cf. [10], [17]). It has been used in large scale geometry to capture the global properties of Riemannian manifolds and geometric groups and others. In particular, the Higson corona of a space is important in large scale geometry because most of the global information is thought to be condensed in it.

In the first part of this paper, we consider spaces whose Higson coronae are indecomposable continua. For example, the ray $\mathbb{R}_+ = [0, \infty)$ is known as such a space, that is, the Higson corona $\nu \mathbb{R}_+$ of the ray $\mathbb{R}_+$ is known to be a non-metrizable indecomposable continuum [12]. This is closely related to the fact that the Stone-Čech remainder $\beta \mathbb{R}_+ \setminus \mathbb{R}_+$ of the ray is a non-metrizable indecomposable continuum [3], where $\beta \mathbb{R}_+$ denotes the Stone-Čech compactification of $\mathbb{R}_+$. For a non-compact locally connected generalized continuum $X$, Dickman [5] gave a characterization that the Stone-Čech remainder $\beta X \setminus X$ is an indecomposable continuum if and only if $X$ has the strong complementation property. The ray $\mathbb{R}_+$ is coarsely equivalent to the subspace $\mathbb{N} \subset \mathbb{R}_+$ of natural numbers. Hence, the Higson corona $\nu \mathbb{N}$ of $\mathbb{N}$ is homeomorphic to $\nu \mathbb{R}_+$ since coarsely equivalent spaces have homeomorphic Higson coronae. Thus $\nu \mathbb{N}$ is an indecomposable continuum. As this example shows, when trying to characterize a space whose Higson corona is an indecomposable continuum, not only the local properties of the space but also the connectivity of itself does not make sense. Therefore, we try to characterize spaces whose Higson coronae are indecomposable continua not by intrinsic but by extrinsic. Proceeding the study, we found that for some class of non-compact proper metric spaces, the Higson corona $\nu X$ of $X$ is an indecomposable continuum if and only if $X$ is coarsely equivalent to $\mathbb{N}$. As a result, we characterize a space that is coarsely equivalent to either the space of natural numbers $\mathbb{N}$ or the space of integers $\mathbb{Z}$, using the indecomposability of the components of its Higson corona (Theorems 3.10 and 3.12).

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We apply these characterizations to finitely generated groups. It is known that a finitely generated group \( G \) has 0, 1, 2 or infinitely many ends, and the group structure is determined when it has two or infinitely many ends (cf. [4], [8], [10]). Also, the number of the ends of \( G \) is zero if and only if \( G \) is a finite group. In that sense, finitely generated groups having exactly one end are fascinating. We characterize a finitely generated group having exactly one end as a group whose Higson corona is a decomposable continuum (Theorem 4.3). In contrast, in the case of a group having exactly two ends, we characterize it as a group whose Higson corona is a topological sum of two indecomposable continua, each of which is homeomorphic to \( \nu \mathbb{N} \) (Theorem 4.5).

2. Preliminaries

Throughout this paper, \( \mathbb{R}_+ \) denotes the ray \([0, \infty)\) with the metric
\[
d(x, y) = |x - y|
\]
and \( \mathbb{N} \subset \mathbb{R}_+ \) denotes the space of natural numbers with the induced metric. In what follows, a metric space \((X, d_X)\) is assumed to have a base point \( x_0 \).

2.1. Basic properties of the Higson compactification. In general, the Higson compactification \( X^\nu \) of a proper metric space \( X \) is defined as the Gelfand dual of an unital commutative \( C^* \)-algebra of the bounded complex-valued continuous Higson functions on \( X \) [17]. On the other hand, there is another way to define the Higson compactification of \( X \) using an evaluation map of \( X \) into Tychonoff cube. It is known that the compactifications obtained by these two definitions are equivalent for the metric coarse structure of \( X \). The latter was first introduced by Keesling [14] and we adopt here the latter definition.

Let \((X, d_X)\) be a metric space and let \( B_{d_X}(x, r) \) be the closed ball of radius \( r \) centered at \( x \in X \). A metric \( d_X \) on \( X \) is called proper if \( B_{d_X}(x, r) \) is compact for every \( x \in X \) and \( r > 0 \). For a subset \( A \) of \( X \), the diameter of \( A \) is denoted by \( \text{diam}_{d_X} A \), that is,
\[
\text{diam}_{d_X} A = \sup \{ d_X(x, y) \mid x, y \in A \}.
\]

Let \((X, d_X)\) and \((Y, d_Y)\) be proper metric spaces. A map \( f : X \to Y \) is a Higson function provided that
\[
(\ast)_f \quad \lim_{d_X(x_0, x) \to \infty} \text{diam}_{d_Y} f(B_{d_X}(x, r)) = 0
\]
for each \( r > 0 \), that is, \( f : X \to Y \) is a Higson function if and only if, given \( r > 0 \) and \( \varepsilon > 0 \), there exists a compact subset \( K \subset X \) such that \( \text{diam}_{d_Y} f(B_{d_X}(x, r)) < \varepsilon \) whenever \( x \in X \setminus K \).

Let \( C_b(X) \) be the set of all bounded real-valued continuous functions on \( X \). For each \( f \in C_b(X) \), let \( I_f \) denote the closed interval \([\inf f, \sup f] \subset \mathbb{R} \). For a subset \( F \) of \( C_b(X) \), let
\[
e_F : X \to \prod_{f \in F} I_f
\]
be the evaluation map of \( F \), that is, \( (e_F(x))_f = f(x) \) for every \( x \in X \). It is known that if \( F \) separates points from closed sets, then \( e_F \) is a topological embedding [9 2.3.20]. Identifying \( X \) with \( e_F(X) \), the closure \( e_F(X) \) of \( e_F(X) \) in \( \prod_{f \in F} I_f \) gives a compactification of \( X \).
For a proper metric space $X$, we consider the following subsets of $C_b(X)$:

$$C_H(X) = \{ f \in C_b(X) \mid f \text{ satisfies } (*)_r \text{ for every } r > 0 \}.$$  

Then $C_H(X)$ is a closed subring of $C_b(X)$ with respect to the sup-metric. Also, it contains all constant maps and separates points from closed sets. Hence, the subring $C_H(X)$ uniquely determines a compactification of $X$ (see [9, 3.12.22 (e)]) which is called the Higson compactification $X^\nu$ of $X$. Then the compact subset $\nu X = X^\nu \setminus X$ is called the Higson corona of $X$.

The following proposition is a fundamental property of the Higson compactification.

**Proposition 2.1** ([14], Proposition 1). Let $X$ be a proper metric space. Then the Higson compactification is the unique compactification of $X$ such that if $Y$ is a compact metric space and $f : X \to Y$ is a continuous map, then $f$ has a continuous extension $f^\nu : X^\nu \to Y$ if and only if $f$ satisfies $(*)_r$ for any $r > 0$.

A finite collection $E_1, \ldots, E_n$ of subsets of a metric space $(X, d_X)$ diverges coarsely in $X$ if for any $r > 0$ the intersection of $r$-neighborhoods of them is bounded, that is, there exists $R > 0$ such that

$$\bigcap_{i=1}^n N_{d_X}(E_i, r) \cap (X \setminus B_{d_X}(x_0, R)) = \emptyset,$$

where $N_{d_X}(E_i, r)$ is the $r$-neighborhood of $E_i$ in $X$.

For each subset $A$ of a proper metric space $X$, $A^*$ denotes the space $\overline{A} \setminus A$, where $\overline{A}$ is the closure of $A$ in the Higson compactification $X^\nu$.

**Proposition 2.2** ([7], Proposition 2.3). Let $X$ be a non-compact proper metric space. For a finite collection $E_1, \ldots, E_n$ of subsets of $X$, the following are equivalent:

1. $\bigcap_{k=1}^n E_k^* = \emptyset$;
2. the collection $E_1, \ldots, E_n$ diverges coarsely in $X$.

Let $X^\xi$ and $X^\zeta$ be compactifications of $X$. We say $X^\xi \succeq X^\zeta$ provided that there is a continuous map $f : X^\xi \to X^\zeta$ such that $f|_X = \text{id}_X$. We note that a continuous map $f : X^\xi \to X^\zeta$ with $f|_X = \text{id}_X$ is unique and surjective by the density of $X$. If $X^\xi \succeq X^\zeta$ and $X^\xi \succeq X^\zeta$ then we say that $X^\xi$ and $X^\zeta$ are equivalent. Of course, two equivalent compactifications of $X$ are homeomorphic.

**Proposition 2.3** ([7], Theorem 1.4). Let $X$ be a non-compact proper metric space and let $Y$ be a closed subspace of $X$ with the induced metric. Then the closure $\overline{Y}$ of $Y$ in $X^\nu$ is a compactification of $Y$ which is equivalent to the Higson compactification $Y^\nu$. In particular, $Y^\nu$ is homeomorphic to the Higson corona $\nu Y$ of $Y$.

### 2.2. Maps in large scale geometry and Weighill’s property ($\mathcal{C}$).

Here, we summarize the definitions of maps used in large scale geometry. Then we state Weighill’s property ($\mathcal{C}$) and related results.

A (not necessarily continuous) function $f : X \to Y$ is called a coarse map if $f$ satisfies the following two conditions:

1. $f$ is uniformly expansive, that is, there exists a non-decreasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_Y(f(x), f(y)) \leq \sigma(d_X(x, y))$$

for every $x, y \in X$ (in this case, $f$ is called $\sigma$-uniformly expansive); and
(2) $f$ is metrically proper, that is, for any bounded set $B \subseteq Y$, $f^{-1}(B)$ is bounded in $X$ with respect to the metric $d_X$.

Two maps $f : X \to Y$ and $g : X \to Y$ are said to be close if there exists $r > 0$ such that $d_Y(f(x), g(x)) < r$ for every $x \in X$. A coarse map $f : X \to Y$ is called a coarse equivalence if there exists a coarse map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to $\text{id}_Y$ and $\text{id}_X$ respectively. If there exists a coarse equivalence between $X$ and $Y$ then $X$ and $Y$ are called coarsely equivalent.

A (not necessarily continuous) function $f : X \to Y$ is said to be uniformly metrically proper if the following condition is satisfied:

(3) there exist non-decreasing functions $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_Y(x, y) \leq \tau(d_Y(f(x), f(y)))$$

for every $x, y \in X$ (in this case, $f$ is called $\tau$-uniformly metrically proper).

A map $f$ is called a rough map if it is uniformly expansive and uniformly metrically proper. If a rough map $f$ is $\sigma$-uniformly expansive and $\tau$-uniformly metrically proper then $f$ is called $(\sigma, \tau)$-rough map. In case $\sigma = \tau$, a $(\sigma, \tau)$-rough map $f$ is called a $\sigma$-rough map. Note that if we define $\lambda = \max\{\sigma, \tau\}$ then any $(\sigma, \tau)$-rough map is a $\lambda$-rough map. A rough map $f : X \to Y$ is called a rough equivalence if there exists a rough map $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to $\text{id}_X$ and $\text{id}_Y$ respectively. A rough equivalence which is a $\sigma$-rough map is called a $\sigma$-rough equivalence.

It is known that a map $f : X \to Y$ is a coarse equivalence if and only if it is a rough equivalence. In particular, it is known that $f$ is a coarse equivalence if and only if $f$ satisfies the following two conditions:

(4) $f(X)$ is a net in $Y$, that is, there exists $r > 0$ such that, for every $y \in Y$, there is $z \in f(X)$ such that $d_Y(y, z) < r$; and

(5) there exist non-decreasing functions $\rho_+, \rho_- : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\lim_{t \to \infty} \rho_-(t) = \infty$$

and the inequality

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

holds for every $x, y \in X$.

Clearly, a net $A$ in a space $X$ is coarsely equivalent to $X$. Refer to [2, §2.6] for more information on the relation between coarse maps and rough maps.

For proper metric spaces $X$ and $Y$, it is known that a coarse map $f : X \to Y$ induces a continuous map $\nu f : \nu X \to \nu Y$ between their Higson coronae. In particular, if $f$ is a coarse equivalence then $\nu f$ is a homeomorphism between their Higson coronae (cf. [17], [2]).

Let $(Y, d_Y)$ and $(Z, d_Z)$ be metric spaces with base points $y_0 \in Y$ and $z_0 \in Z$ respectively. The coarse coproduct of $(Y, d_Y)$ and $(Z, d_Z)$ is the metric space $(Y + Z, d_{Y+Z})$ whose underlying set is the disjoint union of $Y$ and $Z$ and the distance function $d_{Y+Z}$ is defined as follows:

$$d_{Y+Z}(a, b) = \begin{cases} d_Y(a, b) & \text{if } a, b \in Y; \\ d_Z(a, b) & \text{if } a, b \in Z; \\ d_Y(y_0, a) + 1 + d_Z(z_0, b) & \text{if } a \in Y, b \in Z; \\ d_Z(z_0, a) + 1 + d_Y(y_0, b) & \text{if } b \in Y, a \in Z. \end{cases}$$
Although the Higson compactification is defined for proper metric spaces, the Higson corona can be defined for arbitrary metric spaces. For a metric space $X$, we consider the property $(\mathcal{C})$ defined by Weighill [18]:

$$(\mathcal{C})$$

for every coarse map $f : X \to Y + Z$, we can take $A \in \{ Y, Z \}$ and a coarse map $g_A : X \to A$ such that $i_A \circ g_A : X \to Y + Z$ is close to $f$, where $i_A : A \to Y + Z$ is the inclusion.

**Theorem 2.4** ([18], Theorem 4.6). Let $X$ be a non-compact metric space. The Higson corona $\nu X$ of $X$ is topologically connected if and only if $X$ satisfies the property $(\mathcal{C})$.

The following is a consequence of [18, Theorem 4.1], [18, Proposition 4.5], and Proposition 2.3.

**Proposition 2.5** ([18]). If a metric space $X$ does not satisfy $(\mathcal{C})$ then there are two unbounded closed sets $A$ and $B$ of $X$ such that

1. $X = A \cup B$;
2. $A$ and $B$ diverges coarsely in $X$; and
3. the Higson corona $\nu X$ is homeomorphic to the topological sum of the Higson coronae of $A$ and $B$, i.e., $\nu X \approx \nu A \oplus \nu B$.

3. Indecomposable continua as Higson coronae

In this section, we give characterizations of a space that is coarsely equivalent to either the space of natural numbers $\mathbb{N}$ or the space of integers $\mathbb{Z}$ using the indecomposability of the components of its Higson corona.

A continuum is a non-empty, compact and topologically connected Hausdorff space. A subcontinuum is a continuum which is a subset of a continuum. A proper subset of $X$ is a subset of $X$ which is not equal to $X$. A continuum is called decomposable if it can be represented as the union of two of its proper subcontinua. A continuum which is not decomposable is said to be indecomposable.

An example of a space whose Higson corona is an indecomposable continuum is the following:

**Theorem 3.1** ([12], Theorem 1.6). The Higson corona $\nu \mathbb{R}_+$ is a non-metrizable indecomposable continuum.

Since the inclusion $\mathbb{N} \to \mathbb{R}_+$ is a coarse equivalence, $\nu \mathbb{N}$ is homeomorphic to $\nu \mathbb{R}_+$. Hence, we have the following:

**Corollary 3.2.** The Higson corona $\nu \mathbb{N}$ is a non-metrizable indecomposable continuum which is homeomorphic to $\nu \mathbb{R}_+$.

**Remark 3.3.** The subpower Higson compactification was introduced in [15] as a variant of the Higson compactification. It is known that the subpower Higson corona of the ray $\mathbb{R}_+$ is also a non-metrizable indecomposable continuum [13].

Let $\mu > 0$ and let $X$ be a metric space. A subset $Y$ of $X$ is said to be $\mu$-connected if, for every two points $x$ and $y$ of $Y$, there exists a finite sequence $\{p_i\}_{i=1}^n$ in $Y$ such that $p_1 = x$, $p_n = y$ and $d_X(p_i, p_{i+1}) \leq \mu$ for every $i$.

**Lemma 3.4.** Let $(X, d_X)$ be a non-compact proper metric space. If there exists a coarse equivalence $f : M \to X$ from a geodesic metric space $M$ then there exist positive number $\mu > 0$ and a non-decreasing function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to \infty} \tau(t) = \infty$. 
such that, for each unbounded sequence \( \{y_n\}_{n=1}^{\infty} \subset f(M) \), there exist a subsequence \( \{x_n\}_{n=1}^{\infty} \subset \{y_n\}_{n=1}^{\infty} \) and a family \( \{B_n\}_{n=1}^{\infty} \) of subsets of \( X \) satisfying the following:

(A) \( \text{diam } B_n \leq \tau(n) \);
(B) \( B_n \) is \( \mu \)-connected;
(C) \( B_n \) contains an \( n \)-ball \( B_{d_M}(x_n, n) \); and

(D) \( \lim_{n \to \infty} d_X(x_0, B_n) = \infty. \)

Proof. Let \( f : M \to X \) be a coarse equivalence from a geodesic metric space \( (M, d_M) \). Since any coarse equivalence is a rough equivalence, we may assume that \( f \) is a \( \sigma \)-rough equivalence for some non-decreasing function \( \sigma : \mathbb{R}_+ \to \mathbb{R}_+ \), that is,

(1) \( d_M(x, y) \leq \sigma(d_X(f(x), f(y))) \);
(2) \( d_X(f(x), f(y)) \leq \sigma(d_M(x, y)) \)

for every \( x, y \in M \). Note that \( \lim_{t \to \infty} \sigma(t) = \infty \) since \( X \) is a non-compact proper metric space and \( f(M) \) is a net in \( X \). Put \( A = f(M) \) and say that \( A \) is an \( r \)-net in \( X \) for some \( r > 0 \). Let \( \{y_n\}_{n=1}^{\infty} \subset A \) be an unbounded sequence. Then we can take a subsequence \( \{x_n\}_{n=1}^{\infty} \) of \( \{y_n\}_{n=1}^{\infty} \) such that

(3) \( d_X(x_n, x_0) > n + \sigma^2(n + r) \)

for each \( n \), where \( \sigma^2(n + r) = \sigma(\sigma(n + r)) \). Let \( \{v_n\}_{n=1}^{\infty} \) be a sequence in \( M \) such that \( f(v_n) = x_n \) for every \( n \). Put

\[
D_n = B_{d_M}(v_n, \sigma(n + r))
\]

for each \( n \in \mathbb{N} \). Then we define \( B_n \) as the \( r \)-neighborhood of \( f(D_n) \), that is,

\[
B_n = N_{d_X}(f(D_n), r).
\]

Define \( \tau : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\tau(t) = 2(\sigma^2(t + r) + r).
\]

Thus the condition (A) is satisfied.

Put \( \mu = \sigma(1) + r. \) Since \( M \) is a geodesic metric space, each ball \( D_n \) is 1-connected. So the image \( f(D_n) \) is \( \sigma(1) \)-connected. Thus, \( B_n \) is \( \mu \)-connected, the condition (B) is satisfied.

Let \( x \in B_{d_X}(x_n, n) \). We take \( y = f(m) \in A \) such that \( d_X(x, y) < r \). Then \( d_X(f(v_n), f(m)) = d_X(x_n, y) \leq n + r \). So we have \( d_M(v_n, m) \leq \sigma(n + r) \) by (1), that is, \( m \in B_{d_M}(v_n, \sigma(n + r)) = D_n \). Hence, \( y \in f(D_n) \), i.e., \( x \in B_n \). Thus the condition (C) is satisfied.

Finally, we shall check the condition (D). Let \( x \in B_n \). Then we can take \( y = f(m) \in f(D_n) \) such that \( d(x, y) < r \). So we have

\[
d_X(x, x_n) \leq d_X(x, y) + d_X(y, x_n) = d_X(x, y) + d_X(f(m), f(v_n)) \leq r + \sigma(d_M(m, v_n)) \leq r + \sigma(\sigma(n + r)) = r + \sigma^2(n + r).
\]

By (3), we have

\[
d_X(x, x_0) \geq d_X(x_n, x_0) - d_X(x, x_n) \geq n + \sigma^2(n + r) - (r + \sigma^2(n + r)) = n - r.
\]
Therefore, \( \lim_{n \to \infty} d_X(x_0, B_n) = \infty \), the condition (D) is satisfied. \( \square \)

**Lemma 3.5.** Let \( \mu > 0 \). Let \( (X, d_X) \) be a non-compact proper metric space and let \( N \subset X \) be a subset such that \( N^* \) is connected. Suppose that \( \{B_n\}_{n=1}^\infty \) is a family of \( \mu \)-connected bounded subsets of \( X \) such that

1. \( N \cap B_n \neq \emptyset \) for every \( n \); and
2. \( \lim_{n \to \infty} d_X(x_0, B_n) = \infty \).

If we define \( E = N \cup (\cup_{n=1}^\infty B_n) \) then \( E^* \) is a continuum.

**Proof.** To see this, it suffices to show that \( E \) satisfies the condition (\( \mathcal{C} \)) by Proposition 2.3. Let \( f : E \to Y + Z \) be a coarse map into a coarse coproduct space \( Y + Z \). We assume that \( f \) is \( \sigma \)-uniformly expansive for some non-decreasing function \( \sigma : \mathbb{R}^+ \to \mathbb{R}^+ \), i.e.,
\[
d_{Y+Z}(f(x), f(y)) \leq \sigma(d_X(x, y))
\]
for every \( x, y \in E \). Since \( N^* \) is connected, using Theorem 2.4, we may assume without loss of generality that there is a coarse map
\[
g_N : N \to Y
\]
such that \( i_Y \circ g_N \) and \( f|_N \) are \( r_0 \)-close for some \( r_0 > 0 \), where \( i_Y : Y \to Y + Z \) is the inclusion. Since \( g_N \) is metrically proper, there exists \( r_1 > 0 \) such that
\[
g_N^{-1}(B_{d_Y}(y_0, r_0)) \subset B_{d_X}(x_0, r_1).
\]
Then \( d_Y(g_N(x), y_0) > r_0 \) for every \( x \in N \setminus B_{d_X}(x_0, r_1) \). So we have
\[
(1) \quad f(x) \in Y \text{ for every } x \in N \setminus B_{d_X}(x_0, r_1).
\]
Indeed, if there exists a point \( x \in N \setminus B_{d_X}(x_0, r_1) \) such that \( f(x) \in Z \) then
\[
d_{Y+Z}(i_Y \circ g_N(x), f(x)) = d_Y(g_N(x), y_0) + 1 + d_Z(f(x), x_0)
\]
\[
> r_0 + 1 > r_0.
\]
This contradicts the fact that \( i_Y \circ g_N \) and \( f|_N \) are \( r_0 \)-close.

Since \( f \) is metrically proper, we can take \( r_2 > r_1 \) so that
\[
(2) \quad f^{-1}(B_{d_{Y+Z}}(y_0, \sigma(\mu))) \subset B_{d_X}(x_0, r_2).
\]
Put
\[
\Lambda = \{k \in \mathbb{N} \mid B_k \cap B_{d_X}(x_0, r_2) \neq \emptyset \}.
\]
Then \( \Lambda \) is a finite set since \( \lim_{n \to \infty} d_X(x_0, B_n) = \infty \). Put
\[
B = (E \cap B_{d_X}(x_0, r_2)) \cup (\cup_{k \in \Lambda} B_k).
\]
Note that \( B \) is a bounded set in \( E \) since \( \Lambda \) is finite. Then it follows that
\[
(3) \quad f(E \setminus B) \subset Y.
\]
Indeed, if \( x \in N \setminus B \) then \( f(x) \in Y \) by (1). Suppose \( x \in E \setminus (N \cup B) \). Then there is \( B_k \subset E \setminus B_{d_X}(x_0, r_2) \) such that \( x \in B_k \). Since \( N \cap B_k \neq \emptyset \), we can take \( x_k \in N \cap B_k \). Note that \( f(x_k) \in Y \) by (1). Since \( B_k \) is \( \mu \)-connected, there is a finite sequence \( \{p_i\}_{i=1}^n \subset B_k \) such that \( x_k = p_1, \ldots, p_n = x \) and \( d_X(p_{i-1}, p_i) \leq \mu \) for every \( i \). Then
\[
d_{Y+Z}(f(p_i), f(p_{i+1})) < \sigma(\mu)
\]
for every $i$. If $f(x)$ is contained in $Z$ then there is $j \leq n$ such that $f(p_j) \in Z$ and $f(p_{j-1}) \in Y$ since $f(p_1) = f(x_k) \in Y$. Since $p_j \not\in B_{d_X}(x_0, r_2)$, $d_Y(f(p_{j-1}), y_0) > \sigma(\mu)$ by (2). So we have

$$d_{Y^+Z}(f(p_{j-1}), f(p_j)) = d_Y(f(p_{j-1}), y_0) + 1 + d_Z(f(p_j), z_0) > \sigma(\mu) + 1 > \sigma(\mu) \geq \sigma(d_X(p_{j-1}, p_j)),$$

which contradicts to the $\sigma$-uniformly expansiveness of $f$. Hence, $f(x) \in Y$.

Now we define $g : E \to Y$ by $g|_{E \setminus B} = f|_{E \setminus B}$ and $g(x) = y_0$ for every $x \in B$. The map $g$ is well-defined by (3). Since $B$ is a bounded set, $g$ is metrically proper and close to $f$. To see that $g$ is uniformly expansive, we fix $b_0 \in B$. Then for $x \in B$ and $x' \in E \setminus B$, we have

$$d_Y(g(x), g(x')) = d_Y(y_0, f(x')) \leq d_Y(y_0, f(b_0)) + d_Y(f(b_0), f(x)) + d_Y(f(x), f(x')) \leq d_Y(y_0, f(b_0)) + \sigma(\text{diam } B) + \sigma(d_X(x, x')),$$

that is, $g$ is uniformly expansive. Thus $g$ is a coarse map close to $f$. Hence, $E$ satisfies the property $(C)$. \hfill \Box

The next lemma follows from [1, Theorem 3.11] since any geodesic metric space is coarsely quasi-convex and any bijective bi-Lipschitz map into a net is a coarse equivalence (cf. [2, Proposition 3.25]).

**Lemma 3.6** ([1]). If $X$ is a geodesic metric space then there exist a coarse equivalence $f : V(G) \to X$ from the vertices $V(G)$ of a connected graph $G$, where the metric on $V(G)$ is induced by the path metric on $G$.

A space $X$ has **coarse bounded geometry** if it has a net $A$ of $X$ and a function $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\#(A \cap B(x, t)) \leq \lambda(t)$ for every $x \in X$ and $t > 0$, where $\#(A \cap B(x, t))$ denotes the number of elements of $A \cap B(x, t)$.

It is known that if $X$ is the metric space of vertices of a connected graph $G$ then $X$ has coarse bounded geometry if and only if $G$ is of **finite type**, that is, there is a positive integer $K$ such that every vertex of $G$ has degree bounded by $K$ (cf. [2, Example 3.7]).

**Proposition 3.7** ([2], Proposition 3.10). Let $X$ and $Y$ be metric space. If $X$ has coarsely bounded geometry and there exists a coarse equivalence $f : X \to Y$ then $Y$ has coarsely bounded geometry.

A metric space $X$ is said to be **coarsely geodesic** if $X$ is coarsely equivalent to a geodesic metric space.

**Lemma 3.8.** If $X$ is coarsely geodesic and has coarse bounded geometry then there exists a rough map $\alpha : \mathbb{N} \to X$.

**Proof.** Since $X$ is coarsely geodesic, there exists a coarse equivalence $f : M \to X$ from a geodesic metric space $M$. By Lemma 3.6 there exists a coarse equivalence $f : V(G) \to M$ from the vertices of $V(G)$ of a connected graph $G$. By Proposition 3.7 $G$ must be of finite type since $M$ has coarse bounded geometry. It is known [6, 8.2.1] that every infinite connected graph has a vertex of infinite degree or contains a ray. Since $G$ is of finite type, there exists a ray $R \subset G$. Note that the vertices

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1 The coarse structure of a coarsely geodesic space is called monogenic in [17].
$V(R)$ of $R$ is isometric to $\mathbb{N}$, so let $g : \mathbb{N} \to V(R) \subset V(G)$ be the isometry. Then the composition $\alpha = f \circ g : \mathbb{N} \to X$ is a required rough map. 

The following proposition is useful to verify the decomposability of a continuum.

**Proposition 3.9** (\cite{11}, Theorem 3.41). A continuum $K$ is decomposable if and only if there is a proper subcontinuum $C \subset K$ with non-empty interior in $K$.

Now we shall give a characterization of a space which is coarsely equivalent to $\mathbb{N}$ using the indecomposability of its Higson corona.

**Theorem 3.10.** Let $X$ be a non-compact proper metric space. Then $X$ is coarsely equivalent to $\mathbb{N}$ if and only if

1. $X$ is coarsely geodesic and has coarse bounded geometry; and
2. the Higson corona $\nu X$ is an indecomposable continuum.

**Proof.** If $X$ is coarsely equivalent to $\mathbb{N}$ then $X$ satisfies the condition (i) since $\mathbb{N}$ has coarse bounded geometry and is coarsely equivalent to $\mathbb{R}_+$. Also, $\nu X$ is an indecomposable continuum by Theorem 3.1.

Let $X$ be coarsely geodesic and has coarse bounded geometry. Suppose that $\nu X$ is a continuum but $X$ is not coarsely equivalent to $\mathbb{N}$. We shall show that $\nu X$ is a decomposable continuum. Let $\alpha : \mathbb{N} \to X$ be a rough map assured by Lemma 3.8. Put $N = \alpha(\mathbb{N})$ and let $\overline{N}$ denote the closure of $N$ in $X^\nu$. By Proposition 2.3, $N^*$ is homeomorphic to $\nu N$ which is homeomorphic to $\nu \mathbb{N}$ by $\alpha$. Hence, $N^*$ is a continuum (in fact, an indecomposable continuum) by Theorem 3.1.

Since $X$ is coarsely geodesic, there exists a coarse equivalence $f : M \to X$ from a geodesic metric space $M$. We take a positive number $\mu > 0$ and a non-decreasing function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ as in Lemma 3.4 for $f : M \to X$. Since $X$ is not coarsely equivalent to $\mathbb{N}$, the rough map $\alpha : \mathbb{N} \to X$ cannot be a coarse equivalence. Hence, $N$ cannot be a net in $X$. So we can take a sequence $\{z_i\}_{i=1}^\infty \subset X$ such that

1. $d_X(z_n, N) > 3\tau(n)$ for each $n \in \mathbb{N}$.

Put $Z = \{z_i\}_{i=1}^\infty$. Note that $Z^* \neq \emptyset$ since $Z$ is an unbounded set. Since $\alpha$ is metrically proper, we can take a sequence $\{y_i\}_{i=1}^\infty \subset N \subset A$, such that

2. $d_X(z_k, y_n) > 3\tau(n)$ for every $1 \leq k \leq n$.

By Lemma 3.4 there exist a subsequence $\{x_n\}_{n=1}^\infty \subset \{y_n\}_{n=1}^\infty$ and a family $\{B_n\}_{n=1}^\infty$ of subsets of $X$ satisfying the following:

- (A) $\text{diam} B_n < \tau(n)$;
- (B) $B_n$ is $\mu$-connected;
- (C) $B_n$ contains an $n$-ball $B(x_n, n)$; and
- (D) $\lim_{n \to \infty} d_X(x_0, B_n) = \infty$.

By (2), we may assume that

3. $d_X(z_k, x_n) > 3\tau(n)$ for every $1 \leq k \leq n$.

Put

$$E = N \cup (\cup_{n=1}^\infty B_n).$$

Then $E^*$ is a continuum by Lemma 3.5.

Now we shall show that $Z$ and $E$ diverge coarsely in $X$. Given $r > 0$, we can take $n_0 > 0$ such that $\tau(n_0) > r$ since $\lim_{t \to \infty} \tau(t) = \infty$. Fix $m \geq n_0$. Then
$d_X(z_m, N) > 3\tau(m) > 3r$ by (1). So we have $B(z_m, r) \cap B(N, r) = \emptyset$. Recall that $x_n \in B_n \cap N$ for each $n \in \mathbb{N}$. So, if $n < m$ then

$$d_X(z_m, B_n) \geq d_X(z_m, N) - \text{diam } B_n$$

$$> 3\tau(m) - \tau(n) > 2\tau(m) > 2r$$

by (1) and (A). If $m \leq n$ then

$$d_X(z_m, B_n) \geq d_X(z_m, x_n) - \text{diam } B_n$$

$$> 3\tau(n) - \tau(n) = 2\tau(n) > 2r$$

by (3) and (A). As a consequence, $B(z_m, n) \cap N(B_n, r) = \emptyset$ for every $n$. Thus the intersection $N(Z, r) \cap N(E, r)$ is contained in the bounded set $\bigcup_{i=1}^{n-1} B(z_i, r)$, that is, the collection $Z$ and $E$ diverge coarsely in $X$. Hence, $Z^* \cap E^* = \emptyset$ by Proposition 2.2. Since $Z^* \neq \emptyset$, this means that $E^*$ is a proper subcontinuum of $\nu X$.

Recall that $E$ contains $B(x_n, n)$ by (C) for every $n \in \mathbb{N}$, that is, $E$ contains metric balls of arbitrary large radius. Hence, $E^*$ has non-empty interior in $\nu X$ [11]. Proposition 4.12]. We have shown that $E^*$ is a proper subcontinuum of $\nu X$ with non-empty interior. Thus $\nu X$ is a decomposable continuum by Proposition 3.9.

**Corollary 3.11.** Suppose that $X$ is a non-compact proper metric space that is coarsely geodesic and has coarse bounded geometry. Then $\nu X$ is an indecomposable continuum if and only if $\nu X$ is homeomorphic to $\nu \mathbb{N}$.

Next we shall give a characterization of a space which is coarsely equivalent to $B$ using the indecomposability of the components of its Higson corona.

**Theorem 3.12.** Let $X$ be a non-compact proper metric space. Then $X$ is coarsely equivalent to $B$ if and only if

(i) $X$ is coarsely geodesic and has coarse bounded geometry; and

(ii) the Higson corona $\nu X$ is a topological sum of two indecomposable continua, each of which is homeomorphic to $\nu \mathbb{N}$.

**Proof.** Suppose that $X$ is coarsely equivalent to $B$. Then $X$ satisfies the condition (i) since $B$ has coarse bounded geometry and is coarsely equivalent to $\mathbb{R}$. Since $Z_{>0}$ and $Z_{<0}$ diverge in $B$, $\nu B$ is a topological sum of $(Z_{>0})^*$ and $(Z_{<0})^*$. Clearly, both of $Z_{>0}$ and $Z_{<0}$ are coarsely equivalent to $B$. Thus both of $(Z_{>0})^*$ and $(Z_{<0})^*$ are homeomorphic to $\nu \mathbb{N}$ by Proposition 2.3. Thus, $\nu X \approx (Z_{>0})^* \oplus (Z_{<0})^* \approx \nu \mathbb{N} \oplus \nu \mathbb{N}$, the condition (ii) is satisfied.

Now suppose that $X$ satisfies the conditions (i) and (ii). Then $X$ cannot satisfy the condition (C) by Theorem 2.4. By Proposition 2.5 there are unbounded closed subspaces $A_1$, $A_2 \subset X$ such that

1. $X = A_1 \cup A_2$;
2. $A_1$ and $A_2$ diverges coarsely in $X$; and
3. $\nu X \approx \nu A_1 \oplus \nu A_2$.

Then $\nu A_i$ must be indecomposable continuum by the condition (ii) for each $i = 1, 2$.

For $i = 1, 2$, we shall transform $A_i$ into $A_i'$ that is coarsely geodesic and have coarse bounded geometry. By Lemma 3.14 there exists a coarse equivalence $f$:

In fact, one can construct a Higson function $h : X \to [0, 1]$ such that $\supp h \subset \bigcup_{n=1}^{\infty} B(x_n, n) \subset E$ and $h(x_n) = 1$ for every $n$. By Proposition 2.4 there is the extension $h : X^* \to [0, 1]$ of $h$. Then $h^{-1}((0, 1]) \subset \overline{E}$ is open in $X^*$ and $h^{-1}((0, 1]) \cap E^* \neq \emptyset$ by (D).
$V(G) \to X$ from the vertices $V(G)$ of a connected graph $G$. We may assume that $f$ is a $\sigma$-rough equivalence for some non-decreasing function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$. Put

$$\Gamma = f(V(G))$$

and say that $\Gamma$ is an $r$-net in $X$ for some $r > 0$. Take $\delta > \sigma(1)$. Since $A_1$ and $A_2$ diverge coarsely, there exists $R > 0$ such that

$$(3.1) \quad N(A_1, \delta) \cap N(A_2, \delta) \cap (X \setminus B(x_0, R)) = \emptyset.$$  

Note that $G$ is of finite type by Proposition 3.7. Thus $f^{-1}(B(x_0, R) \cap \Gamma)$ is a finite set since $f$ is metrically proper. Since $G$ is a connected graph, there exists a connected subgraph $G_0$ of $G$ such that $f^{-1}(B(x_0, R) \cap \Gamma) \subset G_0$. We may assume that $G_0$ is a finite graph since $f^{-1}(B(x_0, R) \cap \Gamma)$ is a finite set. Let $V(G_0)$ be the vertices of $G_0$. Put

$$\Gamma_i = \Gamma \cap A_i$$

for each $i = 1, 2$. Let $H_i$ be the subgraph of $G$ such that each edge of which contains a vertex in $f^{-1}(\Gamma_i) \setminus V(G_0)$, $i = 1, 2$. Put

$$G_i = H_i \cup G_0$$

for each $i = 1, 2$. Note that $H_1 \cap H_2 \subset G_0$ by (3.1) since each edge of $G$ has diameter 1 and $\delta > \sigma(1)$. Thus we have $G = G_1 \cup G_2$ and $G_1 \cap G_2 = (H_1 \cap H_2) \cup G_0 = G_0$. Since both of $G$ and $G_0 = G_1 \cap G_2$ are connected, both $G_1$ and $G_2$ are connected graphs. For each $i = 1, 2$, put

$$A'_i = N(f(V(G_i)), r),$$

where $V(G_i)$ is the vertices of $G_i$. Then both $A'_1$ and $A'_2$ are coarsely geodesic and have coarsely bounded geometry since $A'_i$ and $G_i$ are coarsely equivalent by $f|_{V(G_i)}$ for each $i = 1, 2$. Note that $f(V(G_1)) \cup f(V(G_2)) = f(V(G)) = \Gamma$. Thus $X = A'_1 \cup A'_2$. Since $A'_i \setminus B(x_0, R') = A_i \setminus B(x_0, R')$ for sufficiently large $R' > 0$, we have $A'_i = (A'_i)^r = \nu A_i$, $i = 1, 2$. Thus each $\nu A'_i$ is an indecomposable continuum for $i = 1, 2$.

By Theorem 3.10 there exists a coarse equivalence $f_i : \mathbb{N} \to A'_i$ for $i = 1, 2$. Define $f : Z \to X$ by $f(0) = x_0$, $f(k) = f_1(k)$ if $k > 0$, and $f(k) = f_2(|k|)$ if $k < 0$. Then $f$ is a coarse equivalence of $Z$ into $X$. \hfill \Box

4. Decomposability of the Higson Coronea of Finitely Generated Groups

Let $X$ be a metric space. A proper ray in $X$ is a proper map $r : \mathbb{R}_+ \to X$. Two proper rays $r_1$ and $r_2$ in $X$ are said to define the same end if for every compact $K \subset X$ there exists $N \in \mathbb{N}$ such that $r_1([N, \infty))$ and $r_2([N, \infty))$ are contained in the same path component of $X \setminus K$. This is an equivalence relation on continuous proper rays. An end of $X$ is an equivalence class of proper rays in $X$.

For a finitely generated group $G$, let $\text{Cay}(G, S)$ denote its Cayley graph with respect to a finite generating set $S$. The number of ends of $G$ is defined as the number of ends of $\text{Cay}(G, S)$ and it does not depend on the choice of finite generating sets. It is known that $G$ can have 0, 1, 2 or infinitely many ends, and the group structure is determined when it has two or infinitely many ends. Also, the number of the ends of $G$ is zero if and only if $G$ is a finite group. In that sense, finitely generated groups having exactly one end are fascinating. Refer to [4], [8], or [10] for more information on ends of groups.
In this section, we give characterizations of finitely generated groups that have one or two ends by decomposability/indecomposability of the components of their Higson corona.

**Lemma 4.1.** Let $G$ be a finitely generated infinite group with the word length metric. Then $G$ cannot be coarsely equivalent to $\mathbb{N}$.

**Proof.** Let $G$ be a finitely generated infinite group. Let $\text{Cay}(G)$ be a Cayley graph of $G$ with respect to some finite generating set. Suppose that $G$ is coarsely equivalent to $\mathbb{N}$. Then there exists a coarse equivalence $f : \mathbb{N} \to \text{Cay}(G)$, such that

$$\rho_-(d(x, y)) \leq d_G(f(x), f(y)) \leq \rho_+(d(x, y))$$

for every $x, y \in \mathbb{N}$, where $\rho_-$ is a non-decreasing function with $\lim_{t \to \infty} \rho_-(t) = \infty$. Note that $f(\mathbb{N})$ is a net in $G$, say an $r$-net in $G$ for some $r > 0$. Since $G$ is a finitely generated infinite group, we can take an isometry $\alpha : \mathbb{R} \to \text{Cay}(G)$ from the real line $(\mathbb{R}, | \cdot |)$ [8, Exercise 7.84]. Let $\lambda = \max\{\rho_+(1), r\}$ and put $g_i = \alpha(3\lambda i + 2r)$ for each $i \in \mathbb{N}$. Since $\alpha$ is an isometry, $d_G(g_i, g_{i+1}) = 3\lambda$ for every $i \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $n_i \in \mathbb{N}$ be a point such that $d_G(f(n_i), g_i) < r$. Note that $n_i \neq n_j$ whenever $i \neq j$. For each $i \in \mathbb{N}$, let $H(i) = \max\{n_i, n_{i+1}\}$ and $L(i) = \min\{n_i, n_{i+1}\}$. Since $\lim_{t \to \infty} \rho_-(t) = \infty$, there exists $m_0 \in \mathbb{N}$ such that $\rho_-(m_0) > 5\lambda$. Put

$$A = \{n \in \mathbb{N} \mid d_G(f(n), \alpha((-\infty, 0])) < r\}.$$

Since $A$ is an infinite set, there exists $i \geq m_0$ such that $A \cap [L(i), H(i)] \neq \emptyset$. Let $a \in A \cap [L(i), H(i)]$. Then

$$d_G(f(n_i), f(a)) \geq d_G(\alpha(i), \alpha(0)) - 2r = d_G(\alpha(3\lambda i + 2r), \alpha(0)) - 2r \\
\geq 3\lambda i + 2r - 2r = 3\lambda i.$$

So we have $d(n_i, a) \geq 3i$ since $\lambda \geq \rho_+(1)$. Hence, $d(n_i, n_{i+1}) \geq d(n_i, a) \geq 3i \geq m_0$. On the other hand, we have

$$d_G(f(n_i), f(n_{i+1})) \leq d_G(g_i, g_{i+1}) + 2r \leq 5\lambda \\
< \rho_-(m_0) \leq \rho_-(d(n_i, n_{i+1})).$$

a contradiction. Thus $G$ cannot be coarsely equivalent to $\mathbb{N}$. \hfill $\square$

If $G$ has exactly one end then the following is known:

**Proposition 4.2** ([13], Corollary 7.3). A finitely generated group $G$ with the word length metric has a connected Higson corona if and only if $G$ has exactly one end.

Now we can refine this result as follows:

**Theorem 4.3.** Let $G$ be a finitely generated group with the word length metric. Then $G$ has exactly one end if and only if the Higson corona $\nu G$ is a decomposable continuum.

**Proof.** Suppose that $G$ has exactly one end. Then the Higson corona $\nu G$ is a continuum by Proposition 4.2. Since $G$ is a finitely generated group with the word length metric, it is coarsely geodesic and has coarse bounded geometry. Recall that $G$ cannot be coarsely equivalent to $\mathbb{N}$ by Lemma 4.1. Thus $\nu G$ cannot be an indecomposable continuum by Theorem 3.10, that is, $\nu G$ is a decomposable continuum. The inverse implication follows from Proposition 4.2. \hfill $\square$
Lemma 4.4. Let $G$ be a finitely generated group with the word length metric. Then $G$ has exactly two ends if and only if $G$ is coarsely equivalent to $\mathbb{Z}$.

Proof. Suppose that $G$ has exactly two ends. Then $G$ is virtually infinite cyclic, that is, $G$ contains $\mathbb{Z}$ as a subgroup of finite index [4, Chap. I, 8.32]. Thus $G$ is coarsely equivalent to $\mathbb{Z}$ [17, Corollary 1.19].

Now let $G$ be a finitely generated group which is coarsely equivalent to $\mathbb{Z}$. Then the Higson corona $\nu G$ of $G$ is disconnected by Theorem 3.12. Thus $G$ has at least two ends by Proposition 4.2. Let $\text{Cay}(G)$ be the Cayley graph of $G$ with respect to some finite generating set. Then there exists a coarse equivalence $f : \mathbb{Z} \to \text{Cay}(G)$ such that $d_G(f(x), f(y)) \leq \rho_+(d(x, y))$ for every $x, y \in \mathbb{Z}$ and $f(\mathbb{Z})$ is an $r$-net in $\text{Cay}(G)$ for some $r > 0$. Suppose that there are three proper rays $r_1, r_2, r_3 : \mathbb{R}_+ \to \text{Cay}(G)$ which correspond to distinct three ends. Then there exist $R > 0$ and $T \in \mathbb{N}$ such that $r_i(t)$ and $r_j(t')$ cannot be connected by a $(\rho_+(1) + 2r)$-path in $\text{Cay}(G) \setminus \text{Ball}(1, R)$ for every $t, t' \geq T$, $i \neq j$ (cf. [4, Lemma 8.28]). Since $f(\mathbb{Z})$ is an $r$-net in $\text{Cay}(G)$, either $f(\mathbb{Z}_{>0})$ or $f(\mathbb{Z}_{<0})$ must have infinitely many common elements with two of three sets $N(r_i([T, \infty)), r)$, $i = 1, 2, 3$. We may assume without loss of generality that

$$\# (f(\mathbb{Z}_{>0}) \cap N(r_i([T, \infty)), r)) = \infty$$

for $i = 1, 2$. Then we can take a sequence $\{k_n\}_{n=1}^{\infty}$ such that, for each $m \in \mathbb{N}$,

1. $k_{2m} < k_{2m+1} < k_{2(m+1)}$; and
2. $f(k_{2m+i}) \in N(r_i([T, \infty)), r)$ for $i = 1, 2$.

Since $r_1(t)$ and $r_2(t')$ cannot be connected by a $(\rho_+(1)+2r)$-path in $\text{Cay}(G) \setminus \text{Ball}(1, R)$ for every $t, t' > T$, $f(k_{2m})$ cannot be connected to $f(k_{2m+1})$ by a $\rho_+(1)$-path in $\text{Cay}(G) \setminus \text{Ball}(1, R)$. Thus for each $n \in \mathbb{N}$, there exists $x_n \in \mathbb{Z}$ such that $k_n < x_n < k_{n+1}$ and $f(x_n) \in \text{Ball}(1, R)$. As a consequence, $f^{-1}(\text{Ball}(1, R))$ contains the infinite set $\{x_i\}_{i=1}^{\infty}$, so $f$ cannot be metrically proper, a contradiction. Hence, $G$ has exactly two ends. \hfill $\square$

For a finitely generated group having exactly two ends, we obtain the following characterization by the indecomposability of the component of its Higson corona.

Theorem 4.5. Let $G$ be a finitely generated group with the word length metric. Then $G$ has exactly two ends if and only if the Higson corona $\nu G$ is a topological sum of two non-metrizable indecomposable continua, each of which is homeomorphic to $\nu \mathbb{N}$.

Proof. This is a consequence of Theorem 3.12 and Lemma 4.3 \hfill $\square$

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REFERENCES

1. J. A. Álvarez López and A. Candel, Algebraic characterization of quasi-isometric spaces via the Higson compactification, Topology Appl. 158(13), 1679–1694 (2011).
2. J. A. Álvarez López and A. Candel, Generic coarse geometry of leaves, Lecture Notes in Mathematics, 2223. Springer, Cham, 2018.
3. D. Bellamy, A non-metric indecomposable continuum, Duke Math. J., 38 (1971), 15–20.
4. M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften, 319. Springer-Verlag, Berlin, 1999.
5. R. F. Dickman, Jr, *A necessary and sufficient condition for $\beta X \setminus X$ to be an indecomposable continuum*, Proc. Amer. Math. Soc. 33 (1972), 191-194.
6. R. Diestel, *Graph theory*, Fifth edition, Graduate Texts in Mathematics, 173. Springer, Berlin, 2017.
7. A. N. Dranishnikov, J. Keesling and V. V. Uspenskij, *On the Higson corona of uniformly contractible spaces*, Topology 37(4) (1999), 791–803.
8. C. Drută-Romaniuc and M. Kapovich, *Geometric group theory* With an appendix by Bogdan Nica, American Mathematical Society Colloquium Publications, 63. American Mathematical Society, Providence, RI, 2018.
9. R. Engelking, *General topology*, Heldermann Verlag, Berlin, Revised and completed ed., 1989.
10. R. Geoghegan, *Topological methods in group theory*, Graduate Texts in Mathematics, 243, Springer, New York, 2008.
11. J. G. Hocking and G. S. Young, *Topology*, Second edition. Dover Publications, Inc., New York, 1988.
12. Y. Iwamoto and K. Tomoyasu, *Higson compactifications obtained by expanding and contracting the half-open interval*, Tsukuba J. Math., 25 (2001), No. 1, 179–186.
13. Y. Iwamoto, *An indecomposable continuum as subpower Higson corona*, Tsukuba J. Math. 42 (2018), no. 2, 173-190.
14. J. Keesling, *The one-dimensional Čech cohomology of the Higson compactification and its corona*, Topology Proc., 19 (1994), 129–148.
15. J. Kucab and M. Zarichnyi, *Subpower Higson corona of a metric space*, Algebra and Discrete Mathematics, 17 (2014), No. 2, 280–287.
16. J. Roe, *Coarse cohomology and index theory on complete Riemannian manifolds*, Mem. Amer. Math. Soc. 104 (1993), no. 497.
17. J. Roe, *Lectures on Coarse Geometry*, University Lecture Series, Vol. 31, AMS, 2003.
18. T. Weighill, *On spaces with connected Higson coronas*, Topology Appl. 209 (2016), 301-315.

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