Nonlinear waves and related nonintegrable and integrable systems

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Abstract
Spectral method related to Lamé equation with finite-gap potential is used to study the optical cascading equations. These equations are known not to be integrable by inverse scattering method. Due to ”partial integrability” two-gap solutions are obtained in terms of products of elliptic functions and are classified in five different families related to eigenvalues of appropriate spectral problem. In special cases, when periodic solutions reduce to localized solitary waves, previously known phase-locked solutions are recovered, and additional one solution is obtained. For vector nonlinear Schrödinger equation \( n = 3 \) we present exact solutions in a form of multicomponent cnoidal waves.

1 Introduction
The aim of the present paper is devoted to the rather old but still open problem how to construct exact periodic solutions of integrable and nonintegrable soliton systems. This problem is important from the physical point of view to study nonlinear waves. These solutions are expressed in terms of Hermite and Lamé polynomials. Results are presented both for integrable and nonintegrable dynamical systems. As examples optical cascading equations and vector nonlinear Schrödinger equation are considered.

2 Optical cascading equations and phase-locked solutions
We consider \( \chi^{(2)} : \chi^{(2)} \) cascading equations in the normalized form

\[
\begin{align*}
ia_{1t} - \frac{r}{2}a_{1xx} + a_1^*a_2 &= 0, \\
la_{2t} - \beta a_2 - i\delta a_{2x} - \frac{\alpha}{2}a_{2xx} + a_1^2 &= 0,
\end{align*}
\]  

(1)

where \( a_1 \) and \( a_2 \) are the normalized complex envelopes of FW and SHW, respectively, \( t \) is the normalized distance along the wave guide, and \( x \) is the normalized...
transverse coordinate. The real constant $\alpha$ is given by minus ratio of the wave numbers of the FW and SHW, the quantity $\beta$ corresponds to the normalized wave number mismatch and the parameter $\delta$ corresponds to normalized walkoff coefficient, $r = \pm 1$.

We seek solution of (1) in the following form

$$ a_1 = q_1(\xi)e^{i\phi_1(\xi)e^{i(k_1t-x_1)}}, \quad (2) $$

$$ a_2 = q_2(\xi)e^{i\phi_2(\xi)e^{i2(k_1t-x_1)}}, \quad (3) $$

with $\xi = \omega x - vt$ and we define $\phi = \phi_2 - 2\phi_1$. We will consider phase-locked solutions ($\phi = k\pi$, $k \in \mathbb{Z}$), then the system (1) is reduced to two ordinary differential equations,

$$ q_{1\xi\xi} + A_0 q_1 + B_0 q_1 q_2 = 0, \quad (4) $$

$$ q_{2\xi\xi} + C_0 q_2 + D_0 q_2^2 = 0, \quad (5) $$

where we have

$$ A_0 = A^2 - D, \quad B_0 = -\frac{2\epsilon}{r\omega^2}, \quad C_0 = B^2 - C, \quad D_0 = -\frac{2\epsilon}{\alpha\omega^2}, \quad \epsilon^2 = 1, \quad (6) $$

and

$$ A = \frac{r\omega\omega_1 - v}{r\omega^2}, \quad B = \frac{2\alpha\omega_1 - v - \delta\omega}{\alpha\omega^2}, \quad C = 2\frac{2\alpha\omega_1^2 - 2k_1 - 2\delta\omega - \beta}{\alpha\omega^2}, \quad D = \frac{r\omega_1^2 - 2k_1}{r\omega^2}. \quad (7) $$

Introducing new variable

$$ q_1^2 = \frac{4F}{B_0 D_0}, \quad F = \lambda^2 - 3\psi\lambda + 9\psi^2 - \frac{9}{4}g_2 \quad (8) $$

where $F$ is Hermite polynomial $[9]$, $g_2, g_3$ are elliptic invariants defined in $[9]$. $\psi = \varphi(\xi + \omega')$ is Weierstrass function shifted by half period $\omega'$ is related to sn Jacobian elliptic function with modulus $k$

$$ \varphi(\xi + \omega'; g_2, g_3) = \alpha^2 k^2 sn^2(\alpha\xi, k) - (1 + k^2), \quad (9) $$

where $\alpha = \sqrt{e_1 - e_3}$ and $e_i, i = 1, 2, 3$, $e_3 \leq e_2 \leq e_1$ are the real roots of the cubic equation

$$ 4\lambda^3 - g_2\lambda - g_3 = 0. \quad (10) $$

Using wave height $\alpha$ and modulus $k = \sqrt{\frac{e_2 - e_3}{e_1 - e_2}}$, we have the following relations

$$ e_1 = \frac{1}{3}(2 - k^2)\alpha^2, \quad e_2 = \frac{1}{3}(2k^2 - 1)\alpha^2, \quad e_3 = -\frac{1}{3}(1 + k^2)\alpha^2, $$

$$ g_2 = -4(e_1 e_2 + e_1 e_3 + e_2 e_3) = \frac{4}{3}\alpha^2(1 - k^2 + k^4), $$

$$ g_3 = 4e_1 e_2 e_3 = \frac{4}{27}\alpha^6(1 + (2 - k^2)(1 - 2k^2)). \quad (11) $$
Inserting this expression in (11) we have the following nonlinear differential equation with spectral parameter $\lambda = -C_0/2$

$$\frac{1}{2} FF'_{\xi} - \frac{1}{4} F^2_{\xi} - (u(\xi) + \lambda) F^2 + \frac{1}{4} R(\lambda) = 0,$$  

(12)

with eigenvalue equations

$$R(\lambda) = 4\lambda^5 - 21\lambda^3 g_2 + 27\lambda^2 g_2^2 + 27\lambda^2 g_3 - 81g_2 g_3 = 0,$$

$$u(\xi) = -(B_0 q_2 + \lambda + A_0) = 6\wp(\xi + \omega'),$$  

(13)

or in factorized form

$$R(\lambda) = 4 \prod_{i=1}^{5} (\lambda - \lambda_i) = 0, \quad \lambda_1 = -\sqrt{3}g_2, \quad \lambda_2 = 3e_3$$

$$\lambda_3 = 3e_2, \quad \lambda_4 = 3e_1, \quad \lambda_5 = \sqrt{3}g_2.$$  

(14)

It is well known that equation (12) is reduced to linear spectral problem of one dimensional Schrödinger equation with two gap potential $u(x) = 6\wp(\xi + \omega')$ and with five normalized eigenfunctions $q_1^{(i)}$, $(i) = 1, \ldots, 5$:

$$\frac{d^2 q_1^{(i)}}{d\xi^2} - u(\xi) q_1^{(i)} = \lambda q_1^{(i)}, \quad (i) = 1, \ldots, 5.$$  

(15)

Under these conditions the second equation (11) is automatically satisfied. Second equation can be considered as "self-consistent" equation for potential $u(\xi)$. Finally the five spectral families of periodic solutions can be written in the following Table 1

| (I) | $q_1 = \frac{\alpha^2 k^2}{\sqrt{B_0 D_0}} E_2^{(ac)}$ | $q_2 = -\frac{1}{B_0} (u(\xi) + \frac{2g_2}{\lambda_1} - 2\lambda_1)$ | $(i) = 1$ |
| (II) | $q_1 = \frac{\alpha^2 k^2}{\sqrt{B_0 D_0}} E_2^{(sd)}$ | $q_2 = -\frac{1}{B_0} (u(\xi) + \frac{2g_2}{\lambda_2} - 2\lambda_2)$ | $(i) = 2$ |
| (III) | $q_1 = \frac{\alpha^2 k^2}{\sqrt{B_0 D_0}} E_2^{(cd)}$ | $q_2 = -\frac{1}{B_0} (u(\xi) + \frac{2g_2}{\lambda_3} - 2\lambda_3)$ | $(i) = 3$ |
| (IV) | $q_1 = \frac{\alpha^2 k^2}{\sqrt{B_0 D_0}} E_2^{(sc)}$ | $q_2 = -\frac{1}{B_0} (u(\xi) + \frac{2g_2}{\lambda_4} - 2\lambda_4)$ | $(i) = 4$ |
| (V) | $q_1 = \frac{\alpha^2 k^2}{\sqrt{B_0 D_0}} E_2^{(uc)}$ | $q_2 = -\frac{1}{B_0} (u(\xi) + \frac{2g_2}{\lambda_5} - 2\lambda_5)$ | $(i) = 5$ |

where

$$E_2^{(ac)} = \text{sn}(\alpha \xi, k) \text{cn}(\alpha \xi, k),$$

$$E_2^{(sd)} = \text{sn}(\alpha \xi, k) \text{dn}(\alpha \xi, k),$$

$$E_2^{(cd)} = \text{cn}(\alpha \xi, k) \text{dn}(\alpha \xi, k),$$

$$E_2^{(uc)} = \text{sn}^2(\alpha \xi, k) - \frac{1 + k^2 \pm \sqrt{1 - k^2 + k^4}}{3k^2},$$

(16)
are normalized two-gap Lamé functions [9], $cn$, $dn$ are Jacobian elliptic functions and potential $u(\xi)$ have the form

$$u(\xi) = 6\alpha^2 k^2 \text{sn}^2(\alpha\xi, k) - 2(1 + k^2)\alpha^2.$$  

(17)

If, however, we restrict ourselves to using solitary waves that corresponds to limit $k \to 1$, for FW $q_1$ we have the following forms (case I): \(\text{sech}^2(\alpha\xi) - \frac{2}{3}\); case (II) and (V): \(\text{sech}^2(\alpha\xi)\); case (III) and (IV): \(\tanh(\alpha\xi)\text{sech}(\alpha\xi)\), the red, white and blue solitary waves respectively. These wavetrains, so-called order two solitary waves are well known phenomena only in the case of coupled nonlinear Schrödinger equations [11], [12].

The theory of optical cascading in materials with a pure quadratic (or $\chi^{(2)}$) nonlinearity became the subject of many theoretical and experimental investigations. In the case of $\chi^{(2)}$ materials it has been possible to produce solitary waves and periodic solutions as background through $\chi^{(2)}$ cascading. This effect occurs in parametrically coupled fields with quadratic nonlinearities and interacting fundamental (FW) and second harmonic (SHW) waves. Several particular wave solutions of the system describing this phenomenon have been obtained, e.g. with the aid of the Hamiltonian formalism [1], direct substitution [2, 3, 4], and Lie group analysis [5]. General families of periodic waves are reported recently in [6, 7].

### 3 Vector nonlinear Schrödinger equation

We consider the system of coupled nonlinear Schrödinger equations

$$i \frac{\partial}{\partial t} Q_j + s \frac{\partial^2}{\partial x^2} Q_j + \sigma \left( \sum_{k=1}^{n} |Q_k|^2 \right) Q_j = 0, \quad j = 1, \ldots, n,$$  

(18)

where $s = \pm 1$, $\sigma = \pm 1$. These equations are important for a number of physical applications. For example, for photorefractive media with a drift mechanism of nonlinear response, a good approximation describing the propagation of $n$ self-trapped mutually incoherent wave packets is the set of equations for a Kerr-type nonlinearity [13]

$$i \frac{\partial}{\partial z'} \tilde{Q}_j + \frac{1}{2} \frac{\partial^2}{\partial x'^2} \tilde{Q}_j + \alpha \delta \eta \tilde{Q}_j = 0, \quad j = 1, \ldots, n,$$  

(19)

where $\tilde{Q}_j$ denotes the $j$th component of the beam, $\alpha$ is a coefficient representing the strength of nonlinearity, $z'$ and $x'$ are the coordinate along the direction of propagation and transverse coordinate respectively. The change in refractive index profile $\eta$ created by all the incoherent components in the light beam is defined by

$$\delta \eta = \sum_{k=1}^{n} |\tilde{Q}_k|^2.$$  

(20)
Inserting (20) in (19) and renormalising the variables as $\tilde{Q}_j = Q_j/\sqrt{2\alpha}$, $z' = 2t, x' = x$ we obtain the vector nonlinear Schrödinger equation (18). Stability, localization, and soliton asymptotics of multicomponent photorefractive cnoidal waves are discussed in [14]. New solutions are presented next for the case $n = 3, 4$.

We seek solution of (18) in the following form [15]

$$Q_j = q_j(z) e^{i\Theta_j}, \quad j = 1, \ldots, n,$$

where $z = x - ct, \Theta_j = \Theta_j(z, t)$, with $q_j, \Theta_j$ real. Substituting (21) into (18) and separating real and imaginary parts by supposing that the functions $\Theta_j, j = 1, \ldots, n$ behave as

$$\Theta_j = \frac{1}{2} scx + (a_j - \frac{1}{4} sc^2)t - s C_j \int_0^z \frac{dz'}{q_j(z')^2} + \Theta_{j0},$$

we obtain the system ($\sigma = s = \pm 1$)

$$\frac{d^2}{dz^2}q_j + \left( \sum_{k=1}^n \frac{\sigma}{s} \delta_{jk} - \frac{a_j}{s} \right) q_j - \frac{C_j^2}{q_j^2} = 0, \quad k, j = 1, \ldots, n,$$

where $C_j, j = 1, \ldots, n$ are free parameters and $\Theta_{j0}$ are constants. These equations describe the integrable case of motion of a particle in a quartic potential perturbed with inverse squared potential, which is separable in ellipsoidal coordinates. The solutions of the system (22) are then given as

$$q_i^2(z) = 2 \frac{F(z, a_i - \Delta)}{\prod_{k\neq i} (a_i - a_k)}, \quad i = 1, \ldots, n,$$

where $F(z, \lambda)$ is Hermite polynomial associated with Lamé potential. The final formula for the solutions of the system (18) then reads

$$Q_i(x, t) = \sqrt{2 \frac{F(z, a_i - \Delta)}{\prod_{k\neq i} (a_i - a_k)}} \exp(\Theta_i),$$

where

$$\Theta_i = \left\{ \frac{1}{2} icx + i(a_j - \frac{1}{4} sc^2)t - \frac{1}{2} \nu(a_j - \Delta) \int_0^z \frac{dz'}{F(z', a_j - \Delta)} \right\},$$

and $i = 1, \ldots, n$ and we have made use of (23) and (21). To obtain the special class of periodic solution of (22) we introduce the following ansätze

$$q_i(\zeta) = \sqrt{A_i \psi(\zeta + \omega') + B_i}, \quad i = 1, 2, 3, \text{ or } i = 1, \ldots, 4.$$  

As a result we obtain:

$$\sum_{k=1}^m A_k = -2, \quad a_i = \sum_{k=1}^m B_k - \frac{B_i}{A_i}, \quad m = 3 \text{ or } 4,$$

$$-\frac{4C_i^2}{A_i^2} = (4\lambda^3 - \lambda g_2 - g_3)|_{\lambda = -B_i/A_i}, \quad i = 1, \ldots, 3 \text{ or } 4$$  

(25)
and using the well known relations
\[ \int_0^z \frac{dz'}{\wp(z') - \wp(\tilde{a}_j)} = \frac{1}{\wp'(\tilde{a}_j)} \left( 2z\zeta(\tilde{a}_j) + \ln \frac{\sigma(z - \tilde{a}_j)}{\sigma(z + \tilde{a}_j)} \right), \]  
(28)
and
\[ \wp(z + \omega') - \wp(\tilde{a}_j) = -\frac{\sigma(z + \omega' + \tilde{a}_j)\sigma(z + \omega' - \tilde{a}_j)}{\sigma(z + \omega')^2\sigma(\tilde{a}_j)^2}. \]  
(29)

We derive the following result
\[ Q_j = \sqrt{-A_j} \sigma(z + \omega' + \tilde{a}_j) \times \]  
\[ \exp \left( \frac{i}{2}cx + i(a_j - \frac{1}{4}c^2)t - (z + \omega')\zeta(\tilde{a}_j) \right), \]  
(30)

where
\[ \sum_{j=1}^{\epsilon_1} A_j = -2, \quad a_j = \sum_{k=1}^{\epsilon_1} B_k - \frac{B_j}{A_j}, \]  
\[ C_j = \frac{i}{2} \sqrt{4\lambda^3 - \lambda^2g_2 - g_3} |_{\lambda = -\frac{\tilde{a}_j}{A_j}}, \]  
\[ \wp(\tilde{a}_j) = -\frac{B_j}{A_j} = \hat{a}_j, \quad j = 1 \ldots \epsilon_1, \quad \epsilon_1 = 3, 4 \]  
(31)

One special solution is written by
\[ q_j = C_j \text{cn}(\alpha z, k), \quad j = 1, 2, 3, \]  
(32)

where
\[ \alpha^2 = \frac{a_1}{2k_2 - 1}, \quad \sum_{j=1}^{3} C_j^2 = 2\alpha^2 k_2, \quad a_1 = a_2 = a_3 = a, \]  
(33)
in the limit \( k \to 1 \) we obtain soliton solution
\[ Q_j = \sqrt{2a\epsilon_j} \exp \left\{ i \frac{1}{2}c(x - x_0) + \frac{1}{4}c^2 t \right\} \frac{\text{ch}(\sqrt{a}(x - x_0 - ct))}{\text{ch}(\sqrt{a}(x - x_0 - ct))}, \quad j = 1, 2, 3, \]  
(34)

where we introduce the following notations
\[ \sum_{k=1}^{3} |\epsilon_k|^2 = 1, \quad \zeta_1 = \frac{1}{2}c + i\sqrt{a} = \xi + i\eta, \]  
(35)

where \( x_0 \) is the position of soliton, \( \epsilon_j, j = 1, 2, 3 \) are the components of polarization vector. One notes that the real part of \( \zeta_1 \) i.e. \( c/2 \) gives us the soliton
velocity while the imaginary part of $\zeta_1$ i.e. $\sqrt{2a}$ gives the soliton amplitude and width. Another special solution is written by

$$q_j = C_j \text{sn}(\alpha z, k), \quad j = 1, 2, \quad q_2 = C_2 \text{dn}(\alpha z, k),$$  \hspace{1cm} (36)

where

$$\alpha^2 = a_3 - a_1, \quad C_2^2 = 2a_3 - a_1 - \alpha^2(1 - k^2),$$

$$\frac{1}{k^2} \sum_{j=1}^{2} C_j^2 = a_1 - \alpha^2(1 - k^2),$$

in the limit $k \to 1$ we obtain the following soliton solution

$$Q_j = \sqrt{a \epsilon_j} \exp \left\{ i \left( \frac{1}{2} c(x - x_0) + (a - \frac{1}{4} c^2)t \right) \right\} \times \text{th}(\alpha(x - x_0 - ct)), \quad Q_3 = \sqrt{a + 2\alpha^2 \epsilon_3} \exp \left\{ i \left( \frac{1}{2} c(x - x_0) + (a + \alpha^2 - \frac{1}{4} c^2)t \right) \right\} \times \text{ch}(\alpha(x - x_0 - ct)),$$

where we introduce the following notations

$$\sum_{j=1}^{2} |\epsilon_j|^2 = 1, \quad |\epsilon_3|^2 = 1, \quad a_1 = a_2 = a = \sum_{j=1}^{2} C_j^2, \quad a_3 = a + \alpha^2.$$

To obtain the class of periodic solutions of system (22) for $n = 3, 4$ we introduce the following two ansatizes in terms of the Weierstrass function $\wp(\zeta + \omega')$

$$q_i(\zeta) = \sqrt{A_i \wp(\zeta + \omega')^3 + B_i \wp(\zeta + \omega')^2 + C_i \wp(\zeta + \omega') + D_i},$$  \hspace{1cm} (37)

where $i = 1, \ldots, 3$. Next for conciseness we denote $\wp = \wp(\zeta + \omega')$, then the second ansatz have the form

$$q_i(\zeta) = \sqrt{A_i \wp^4 + B_i \wp^3 + C_i \wp^2 + D_i \wp + E_i}, \quad i = 1, \ldots, 4$$  \hspace{1cm} (38)

with the constants $A_i, B_i, C_i, D_i, E_i$ defined from the compatibility condition of the ansatz with the equations of motion (22). Inserting (37) and (38) into Eqs. (22), using the basic equations for Weierstrass $\wp$ function [9]

$$\left( \frac{d}{d\zeta} \wp(\zeta) \right)^2 = 4\wp(\zeta)^3 - g_2\wp(\zeta) - g_3, \quad \frac{d^2}{d\zeta^2} \wp(\zeta) = 6\wp(\zeta) - \frac{g_2}{2},$$  \hspace{1cm} (39)

and equating to zero the coefficients at different powers of $\wp$ we obtain the following algebraic equations for the parameters of the solutions $A_i, B_i, C_i, D_i, i =
\[A_1 + A_2 + A_3 = 0, \quad B_1 + B_2 + B_3 = 0, \quad C_1 + C_2 + C_3 = -12, \quad C_i = \frac{2 B_i^2}{3 A_i} - \frac{1}{4} A_i g_2,\]

\[a_i = \sum_{i=1}^{3} D_i - 5 \frac{B_i}{A_i}, \quad D_i = \frac{5 B_i^3}{9 A_i} - \frac{1}{3} B_i g_2 - \frac{1}{4} A_i g_3.\]

The analogical algebraic system for \(n = 4\) is as follows

\[A_1 + A_2 + A_3 + A_4 = 0, \quad B_1 + B_2 + B_3 + B_4 = 0, \quad C_1 + C_2 + C_3 + C_4 = 0, \quad D_1 + D_2 + D_3 + D_4 = -20, \]

\[C_i = \frac{3 B_i^2}{5 A_i} - \frac{3}{10} A_i g_2, \quad D_i = \frac{14 B_i^3}{45 A_i^2} - \frac{53}{180} B_i g_2 - \frac{2}{9} A_i g_3,\]

\[E_i = \frac{49 B_i^4}{225 A_i^3} - \frac{113 B_i^2}{450 A_i} g_2 - \frac{11}{36} B_i g_3 + \frac{9}{400} A_i g_2^2,\]

\[a_i = \sum_{i=1}^{4} E_i - 7 \frac{B_i}{A_i}.\]

Another result from the algebraic systems is the expression for constants \(C_i\) which parametrize our solutions. For them we obtain

\[C_i^2 = -\frac{\nu (a_i - \Delta)^2}{\prod_{k \neq i} (a_i - a_k)},\]

where \(i, k = 3 \text{ or } 4\) and parameters \(\nu\) are defined by (for \(n = 3\))

\[\nu^2 = \frac{4185}{18225} \frac{g_2}{g_3} \lambda^2 + \frac{297}{9125} \frac{g_3}{g_2} \lambda^4 + \frac{63129}{3375} \frac{g_2^3 g_3}{g_2^2 g_3^2} - \frac{16 \lambda^2}{3375} g_2^2 \lambda^3 - \frac{2}{8} g_2 g_3 \lambda^5 + \frac{63}{2} g_2 \lambda^3 - \frac{1}{2} g_3 \lambda^6 - \frac{518505}{16} g_2 g_3 \lambda^4 - \frac{231}{2} \lambda^2 g_2,
\]

and (for \(n = 4\))

\[\nu^2 = \lambda^9 - \frac{231}{2} \lambda^4 g_2^2 + \frac{2145}{2} \lambda^6 g_3 + \frac{63129}{16} \lambda^5 g_2^2 - \frac{518505}{8} g_2 g_3 \lambda^4 + \left( - \frac{563227}{16} g_2^3 + \frac{4549125}{16} g_3^2 \right) \lambda^3 + \frac{991515}{2} g_3 g_2 \lambda^2 - \frac{361179}{4} g_2^4 - \frac{5273625}{4} g_2 g_3^2 \lambda - 9724050 g_2 g_3^3 - 1500625 g_3^3.
\]

Using the general formulae, we will consider below the physically important cases of \(n = 3, 4\) which are associated with the three-gap \(12\wp(\zeta + \omega')\), and four-gap elliptic potentials \(20\wp(\zeta + \omega')\).
The Hermite polynomial $\mathcal{F}(\wp(x), \lambda)$ associated to the Lamé potential $12\wp(\zeta)$ has the form

$$
\mathcal{F}(\wp(\zeta), \lambda) = \lambda^3 - 6\wp(\zeta + \omega')\lambda^2 - 3 \cdot 5(-3\wp(\zeta + \omega')^2 + g_2)\lambda
$$

$$
- \frac{3^2 \cdot 5^2}{4}(4\wp(\zeta + \omega')^3 - g_2\wp(\zeta + \omega') - g_3).
$$

(49)

The solution is real under the choice of the arbitrary constants $a_i, i = 1, \ldots, n$ in such a way that the constants $a_i - \Delta, i = 1, \ldots, n$ lie in different lacunae.

Comparing (37) and (49) and using (23) the solutions of polynomial equations (40), (41), (42) can be given by

$$
A_i = \frac{2 \cdot 5^2 \cdot 3^2}{\prod_{k \neq i}(a_i - a_k)},
$$

(50)

$$
B_i = -\frac{2 \cdot 3^2 \cdot 5(a_i - \Delta)}{\prod_{k \neq i}(a_i - a_k)},
$$

(51)

$$
\Delta = \frac{2}{5} \sum_{i=1}^{3} a_i.
$$

(52)

The Hermite polynomial $\mathcal{F}(\wp(\zeta), \lambda)$ associated to the Lamé potential $20\wp(\zeta)$ can be written as

$$
\mathcal{F}(\wp(\zeta), \lambda) = 11025\wp(\zeta + \omega')^4 - 1575\wp(\zeta + \omega')^3\lambda +
$$

$$
(135\lambda^2 - \frac{22615}{2}g_2)\wp(\zeta + \omega')^2 +
$$

$$
(-10\lambda^3 + \frac{1855}{4}g_2 - 2450g_3)\wp(\zeta + \omega') +
$$

$$
\lambda^4 - \frac{113}{2}\lambda^2g_2 + \frac{3969}{16}g_2^2 + \frac{195}{4}\lambda g_3.
$$

(53)

Comparing (48) and (53) and using (26) the solutions of polynomial equations (43-46) can be given by

$$
A_i = \frac{11025 \cdot 2}{\prod_{k \neq i}(a_i - a_k)},
$$

(54)

$$
B_i = -\frac{1575 \cdot 2(a_i - \Delta)}{\prod_{k \neq i}(a_i - a_k)}
$$

$$
\Delta = \frac{2}{7} \sum_{i=1}^{4} a_i.
$$

Next solution of system (22, $n = 3$) we obtain using the following ansatz

$$
q_i(\zeta) = \sqrt{A_i\wp(\zeta + \omega')^2 + B_i\wp(\zeta + \omega') + C_i}, \quad i = 1, 2, 3,
$$

(55)
then we have
\[ \sum_{i=1}^{3} A_i = 0, \quad \sum_{i=1}^{3} B_i = -6, \quad (56) \]
\[ a_i = \sum_{k=1}^{3} C_k - 3 \frac{B_i}{A_i}, \quad C_i = \frac{B_i^2}{A_i} - \frac{1}{4} A_i g_2, \quad (57) \]
\[ \frac{C_i^2 \cdot 3^3 \cdot 4}{A_i^2} = (4\lambda^5 + 27\lambda^2 g_3 + 27\lambda g_2^2 - 21\lambda^3 g_2 - 81g_2g_3), \quad (58) \]

where \( \lambda = -3B_i/A_i \). More complicated solution we can obtain as a special solution of last polynomial system
\[ q_1 = C_1 \alpha \left( \frac{1}{3} C_- - k^2 \text{sn}^2(\alpha z, k) \right), \]
\[ q_2 = C_2 \alpha \text{sn}(\alpha z, k) \text{cn}(\alpha z, k), \]
\[ q_3 = C_3 \alpha \text{cn}(\alpha z, k) \text{dn}(\alpha z, k), \]

where
\[ C_\pm = 1 + k^2 \pm \sqrt{1 - k^2 + k^4}, \]
\[ k^2 = \frac{1}{3} \left( 2 - \gamma + 2\sqrt{\gamma^2 - \gamma - 2} \right), \]
\[ \gamma = \frac{2\alpha_3 + \alpha_2 - 3\alpha_1}{\alpha_3 - \alpha_2}, \]
\[ \alpha^2 = \frac{1}{3} (\alpha_3 - \alpha_2), \quad C_3^2 = (k^2 - \frac{2}{3} C_-) C_1^2 + 6, \]
\[ C_2^2 = k^2 (C_1^2 + C_3^2). \]

Another solution is written by
\[ q_1 = C_1 \alpha \left( \frac{1}{3} C_- - k^2 \text{sn}^2(\alpha z, k) \right), \]
\[ q_2 = C_2 \alpha \text{sn}(\alpha z, k) \text{cn}(\alpha z, k), \]
\[ q_3 = C_3 \alpha \left( \frac{1}{3} C_+ - k^2 \text{sn}^2(\alpha z, k) \right), \quad (59) \]

where
\[ C_\pm = 1 + k^2 \pm \sqrt{1 - k^2 + k^4}, \]
\[ \alpha^2 = \frac{(\alpha_3 - \alpha_2)}{4 + k^2 - 2C_-}, \quad C_3^2 = -\frac{(6 + (k^2 - \frac{2}{3} C_-) C_1^2)}{k^2 - \frac{2}{3} C_+}, \]
\[ C_2^2 = k^2 (C_1^2 + C_3^2). \]

Next solution is given by
\[ q_1 = C_1 \alpha \left( \frac{1}{3} C_- - k^2 \text{sn}^2(\alpha z, k) \right), \]

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\[q_2 = C_2 \alpha \text{sn}(\alpha z, k) \text{dn}(\alpha z, k),\]
\[q_3 = C_3 \alpha \left( \frac{1}{3} C_+ - k^2 \text{sn}^2(\alpha z, k) \right),\]  
(60)

where
\[C_\pm = 1 + k^2 \pm \sqrt{1 - k^2 + k^4},\]
\[\alpha^2 = \frac{(a_3 - a_2)}{1 + 4k^2 - 2C_-}, \quad C_3^2 = \frac{-6 + (1 - \frac{2}{3}C_-)C_1^2}{1 - \frac{2}{3}C_+},\]
\[C_2^2 = C_1^2 + C_3^2.\]

From last three solutions in the limit \(k \to 1\) we obtain the following soliton solutions
\[Q_1 = \sqrt{\frac{3a}{4}} \epsilon_1 \exp \left\{ i \left( \frac{1}{2} c(x - x_0) + (a - \frac{1}{4} c^2) t \right) \right\} \times \left( -\frac{2}{3} + \text{sech}^2(\alpha(x - x_0 - ct)) \right),\]
\[Q_2 = \sqrt{3(2\alpha^2 - a)} \epsilon_2 \exp \left\{ i \left( \frac{1}{2} c(x - x_0) + (\alpha^2 + a - \frac{1}{4} c^2) t \right) \right\} \times \text{th}(\alpha(x - x_0 - ct)) \text{sech}(\alpha(x - x_0 - ct)),\]
\[Q_3 = \sqrt{\frac{3(2\alpha^2 - 4a)}{4}} \epsilon_3 \exp \left\{ i \left( \frac{1}{2} c(x - x_0) + (4\alpha^2 + a - \frac{1}{4} c^2) t \right) \right\},\]

where we introduce the following notations
\[|\epsilon_1|^2 = |\epsilon_2|^2 = |\epsilon_3|^2 = 1, \quad a_1 = a, \quad a_2 = a^2 + a, \quad a_3 = 4a^2 + a.\]

Another type solution in the limit \(k \to 1\) have the following form
\[Q_1 = C_1 \epsilon_1 \exp \left\{ i \left( \frac{1}{2} c(x - x_0) - (a + \frac{1}{4} c^2) t \right) \right\} \times \left( -\frac{2}{3} + \text{sech}^2(\sqrt{\frac{a}{8}}(x - x_0 - ct)) \right),\]
\[Q_2 = C_2 \epsilon_2 \exp \left\{ i \left( \frac{1}{2} c(x - x_0) - (\frac{7}{8} a + \frac{1}{4} c^2) t \right) \right\} \times \text{th}(\sqrt{\frac{a}{8}}(x - x_0 - ct)) \text{sech}(\sqrt{\frac{a}{8}}(x - x_0 - ct)),\]
\[Q_3 = \frac{\sqrt{2} \alpha \epsilon_3 \exp \left\{ i \left( \frac{1}{2} c(x - x_0) + (\frac{7}{8} a - \frac{1}{4} c^2) t \right) \right\}}{\text{ch}^2(\sqrt{\frac{7a}{8}}(x - x_0 - ct))},\]

where we introduce the following notations
\[|\epsilon_1|^2 = 1, |\epsilon_2|^2 + |\epsilon_3|^2 = 1,\]
\[a_1 = -a, \quad a_2 = a_3 = -\frac{7}{8} a, \quad C_2^2 = C_1^2 + C_3^2.\]
4 Lamé polynomials and multicomponent cnoidal waves

Next we skip the details of derivation of exact solutions and give only the final results for special case \( n = 3 \) and all \( C_j = 0 \) using methods presented in [15]. The \((2n + 1)\) Lamé polynomials of order \( n \) are solutions of

\[
d^2 E_i \over dz^2 + \left( \lambda_i - n(n + 1)k^2 \sin^2(\alpha z) \right) E_i = 0.
\] (61)

For \( n = 3 \) we introduce the following eigenfunctions functions \( E_i \), \( i = 1, \ldots, 7 \) and eigenvalues \( \lambda_i \) given in table 2, \( \lambda_1 < \lambda_2 < \ldots < \lambda_7 \) and the constants \( C_i^{(3)} \) given in table 3.

Table 2:

| \( i \) | \( E_i^{(3)} \) | \( \lambda_i^{(3)} \) |
|---|---|---|
| 1 | \( \text{sn}(\alpha z, k) \left( \text{dn}^2(\alpha z, k) + C_1^{(3)} \right) \) | \( 7 - 5k^2 - 2\sqrt{4 - 7k^2 + 4k^4} \) |
| 2 | \( \text{cn}(\alpha z, k) \left( \text{dn}^2(\alpha z, k) + C_2^{(3)} \right) \) | \( 7 - 2k^2 + 2\sqrt{4 - k^2 + k^4} \) |
| 3 | \( \text{dn}(\alpha z, k) \left( \text{dn}^2(\alpha z, k) + C_3^{(3)} \right) \) | \( 5(2 - k^2) - 2\sqrt{1 - k^2 + 4k^4} \) |
| 4 | \( \text{sn}(\alpha z, k) \text{cn}(\alpha z, k) \text{dn}(\alpha z, k) \) | \( 4(2 - k^2) \) |
| 5 | \( \text{sn}(\alpha z, k) \left( \text{dn}^2(\alpha z, k) + C_5^{(3)} \right) \) | \( 7 - 5k^2 + 2\sqrt{4 - 7k^2 + 4k^4} \) |
| 6 | \( \text{cn}(\alpha z, k) \left( \text{dn}^2(\alpha z, k) + C_6^{(3)} \right) \) | \( 7 - 2k^2 + 2\sqrt{4 - k^2 + k^4} \) |
| 7 | \( \text{dn}(\alpha z, k) \left( \text{dn}^2(\alpha z, k) + C_7^{(3)} \right) \) | \( 5(2 - k^2) + 2\sqrt{1 - k^2 + 4k^4} \) |

Table 3:

| \( i \) | \( C_i^{(3)} \) |
|---|---|
| 1 | \( \frac{1}{5}(2k^2 - 3 - \sqrt{4 - 7k^2 + 4k^4}) \) |
| 2 | \( \frac{1}{5}(k^2 - 3 - \sqrt{4 - k^4 + k^4}) \) |
| 3 | \( \frac{1}{5}(2k^2 - 4 - \sqrt{1 - k^2 + 4k^4}) \) |
| 4 | \( \frac{1}{5}(2k^2 - 3 + \sqrt{4 - 7k^2 + 4k^4}) \) |
| 5 | \( \frac{1}{5}(k^2 - 3 + \sqrt{4 - k^4 + k^4}) \) |
| 6 | \( \frac{1}{5}(2k^2 - 4 + \sqrt{1 - k^2 + 4k^4}) \) |

Next we enumerate the periodic solutions of the system (62) \( \sigma = 1, s = 1, n = 3 \). The results are collected in Table 4. For convenience we present solutions in the following form

\[
q_i = C_i \alpha E_i^{(3)}, \quad q_j = C_j \alpha k^2 E_j^{(3)}, \quad q_k = C_k \alpha k E_k^{(3)},
\] (62)
and

\[ C_i^2 = -\Lambda/\Lambda_{i;j,k}, \quad C_j^2 = \Lambda/\Lambda_{j;i,k}, \quad C_k^2 = -\Lambda/\Lambda_{k;i,j}, \]  

(63)

where \( i \neq j \neq k \) and

\[ \Lambda = 2 \cdot 3^2 \cdot 5^2, \quad \Lambda_{i;j;k} = (\lambda_i^{(3)} - \lambda_j^{(3)}) (\lambda_i^{(3)} - \lambda_k^{(3)}). \]

Table 4:

| A1  | \( q_1 = C_1 \alpha E_3^{(3)} \), \( q_2 = C_2 \alpha k^2 E_4^{(3)} \), \( q_3 = C_3 \alpha k E_5^{(3)} \) | \{3, 4, 6\} |
|-----|-------------------------------------------------|-------------|
|     | \( C_1^2 = -\Lambda/\Lambda_{2,4,5} \), \( C_2^2 = \Lambda/\Lambda_{4,3,6} \), \( C_3^2 = -\Lambda/\Lambda_{6,3,4} \) |             |
| A2  | \( q_1 = C_1 \alpha E_3^{(3)} \), \( q_2 = C_2 \alpha k E_5^{(3)} \), \( q_3 = C_3 \alpha E_7 \) | \{3, 5, 7\} |
|     | \( C_1^2 = \Lambda/\Lambda_{3,5,7} \), \( C_2^2 = -\Lambda/\Lambda_{5,3,7} \), \( C_3^2 = \Lambda/\Lambda_{7,3,5} \) |             |
| A3  | \( q_1 = C_1 \alpha E_3^{(3)} \), \( q_2 = C_2 \alpha k E_5^{(3)} \), \( q_3 = C_3 \alpha k E_6^{(3)} \) | \{3, 5, 6\} |
|     | \( C_1^2 = \Lambda/\Lambda_{3,5,6} \), \( C_2^2 = -\Lambda/\Lambda_{5,3,6} \), \( C_3^2 = \Lambda/\Lambda_{6,3,5} \) |             |
| A4  | \( q_1 = C_1 \alpha E_3^{(3)} \), \( q_2 = C_2 \alpha k^2 E_4^{(3)} \), \( q_3 = C_3 \alpha k E_6^{(3)} \) | \{3, 4, 7\} |
|     | \( C_1^2 = \Lambda/\Lambda_{3,4,7} \), \( C_2^2 = -\Lambda/\Lambda_{4,3,7} \), \( C_3^2 = \Lambda/\Lambda_{7,3,4} \) |             |
| A5  | \( q_1 = C_1 \alpha k E_2^{(3)} \), \( q_2 = C_2 \alpha k E_5^{(3)} \), \( q_3 = C_3 \alpha k E_6^{(3)} \) | \{2, 5, 6\} |
|     | \( C_1^2 = \Lambda/\Lambda_{2,5,6} \), \( C_2^2 = -\Lambda/\Lambda_{5,2,6} \), \( C_3^2 = \Lambda/\Lambda_{6,2,5} \) |             |
| A6  | \( q_1 = C_1 \alpha k E_2^{(3)} \), \( q_2 = C_2 \alpha k E_5^{(3)} \), \( q_3 = C_3 \alpha E_6^{(3)} \) | \{2, 5, 7\} |
|     | \( C_1^2 = \Lambda/\Lambda_{2,5,7} \), \( C_2^2 = -\Lambda/\Lambda_{5,2,7} \), \( C_3^2 = \Lambda/\Lambda_{7,2,5} \) |             |
| A7  | \( q_1 = C_1 \alpha k E_2^{(3)} \), \( q_2 = C_2 \alpha k^2 E_4^{(3)} \), \( q_3 = C_3 \alpha E_6^{(3)} \) | \{2, 4, 7\} |
|     | \( C_1^2 = \Lambda/\Lambda_{2,4,7} \), \( C_2^2 = -\Lambda/\Lambda_{4,2,7} \), \( C_3^2 = \Lambda/\Lambda_{7,2,4} \) |             |
| A8  | \( q_1 = C_1 \alpha k E_2^{(3)} \), \( q_2 = C_2 \alpha k^2 E_4^{(3)} \), \( q_3 = C_3 \alpha E_6^{(3)} \) | \{2, 4, 6\} |
|     | \( C_1^2 = \Lambda/\Lambda_{2,4,6} \), \( C_2^2 = -\Lambda/\Lambda_{4,2,6} \), \( C_3^2 = \Lambda/\Lambda_{6,2,4} \) |             |

For defocusing case \( \sigma = -1 \), \( s = 1 \), \( n = 3 \) we have presented solutions in Table 5.

Table 5:

| B1  | \( q_1 = C_1 \alpha k E_1^{(3)} \), \( q_2 = C_2 \alpha E_3^{(3)} \), \( q_3 = C_3 \alpha k E_5^{(3)} \) | \{1, 3, 5\} |
|-----|-------------------------------------------------|-------------|
|     | \( C_1^2 = -\Lambda/\Lambda_{1,3,5} \), \( C_2^2 = -\Lambda/\Lambda_{3,1,5} \), \( C_3^2 = \Lambda/\Lambda_{5,1,3} \) |             |
| B2  | \( q_1 = C_1 \alpha k E_1^{(3)} \), \( q_2 = C_2 \alpha k E_3^{(3)} \), \( q_3 = C_3 \alpha k E_5^{(3)} \) | \{1, 2, 5\} |
|     | \( C_1^2 = \Lambda/\Lambda_{1,2,5} \), \( C_2^2 = -\Lambda/\Lambda_{2,1,5} \), \( C_3^2 = \Lambda/\Lambda_{5,1,2} \) |             |
| B3  | \( q_1 = C_1 \alpha k E_1^{(3)} \), \( q_2 = C_2 \alpha k E_3^{(3)} \), \( q_3 = C_3 \alpha E_5^{(3)} \) | \{1, 2, 4\} |
|     | \( C_1^2 = \Lambda/\Lambda_{1,2,4} \), \( C_2^2 = -\Lambda/\Lambda_{2,1,4} \), \( C_3^2 = \Lambda/\Lambda_{4,1,2} \) |             |
| B4  | \( q_1 = C_1 \alpha k E_1^{(3)} \), \( q_2 = C_2 \alpha E_3^{(3)} \), \( q_3 = C_3 \alpha k^2 E_4^{(3)} \) | \{1, 3, 4\} |
|     | \( C_1^2 = \Lambda/\Lambda_{1,3,4} \), \( C_2^2 = -\Lambda/\Lambda_{3,1,4} \), \( C_3^2 = \Lambda/\Lambda_{4,1,3} \) |             |

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5 Conclusions

In this paper we find five different families of periodic solutions of optical cascading equations related to eigenvalues of Lamé equation with two-gap potential. We investigate also localized solitary wave solutions as a special limit of periodic solutions.

The existence of two-gap solutions expressed as product of two elliptic functions can be viewed as manifestation of "partial integrability" of these equations. We expected that both group theoretical method [5] and spectral method will lead us to new understanding of "partial integrability" of optical cascading equations.

For vector nonlinear Schrödinger equation $n = 3, 4$ we present exact solutions in a form of multicomponent cnoidal waves. Multicomponent cnoidal waves are previously discussed in [14].

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