On the bi-Hamiltonian structures of the Camassa-Holm and Harry Dym equations

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Abstract

We show that the bi-Hamiltonian structures of the Camassa-Holm and Harry Dym hierarchies can be obtained by applying a reduction process to a simple Poisson pair defined on the loop algebra of $\mathfrak{sl}(2,\mathbb{R})$. The reduction process is a bi-Hamiltonian reduction, that can be canonically performed on every bi-Hamiltonian manifold.
1 Introduction

In recent years a lot of papers have been devoted to the Camassa-Holm equation (CH)

\[ u_t - u_{txx} = -3uu_x + 2u_xu_xx + uu_{xxx} \]  \hspace{1cm} (1)

or, putting \( m = u - u_{xx} \),

\[ m_t = -2mu_x - mxu, \] \hspace{1cm} (2)

introduced in [3] as a model of shallow water waves. Part of them [1, 2, 12, 21] have investigated the connections with the Korteweg-de Vries (KdV) equation

\[ u_t = -3uu_x + u_{xxx}, \] \hspace{1cm} (3)

the Hunter-Saxton (HS) equation [10]

\[ u_{txx} = -2u_xu_xx - uu_{xxx}, \] \hspace{1cm} (4)

and the Harry Dym (HD) equation

\[ u_t = \left( u^{-\frac{1}{2}} \right)_{xxx}, \] \hspace{1cm} (5)

attributed in [13] to Harry Dym. In particular, Khesin and Misiolek [12], motivated by [1] and [2], have explained the connections between KdV, CH with linear dispersion,

\[ u_t - u_{txx} = -3uu_x + 2u_xu_xx + uu_{xxx} + cu_{xxx}, \] \hspace{1cm} (6)

and HS in terms of their common symmetry group, the Virasoro group. Indeed, these equations can be interpreted as Euler equations describing the geodesic flow (with respect to different metrics): on the Virasoro group in the KdV and CH \( (c \neq 0) \) case [23, 22], on the diffeomorphism group of the circle in the CH \( (c = 0) \) case [22, 6], and on a suitable homogeneous space in the HS case [12]. Moreover, to any (codimension 2) coadjoint orbit there corresponds a bi-Hamiltonian structure: the first Poisson bracket is just the Lie-Poisson bracket, while the second one is a constant bracket depending on the choice of a point in the dual of the Virasoro algebra. Points on the same orbit give rise to equivalent choices. There are three types of orbit and
three different associated bi-Hamiltonian structures: the KdV, the CH and the HS bi-Hamiltonian structures.

As regards the connections between CH and HD, they are clear in the framework of the inverse scattering techniques \[1\]. Indeed, both the equations are associated to the scattering problems for the family of operators

\[ L_k = \partial_x^2 + k^2 \rho^2 - q. \]

A different choice of the boundary conditions for the function \( \rho(x) \) and of the value of the constant \( q \) selects the associated equation: the case \( q = 0 \) and \( \rho \to 1 \) at infinity corresponds to HD, while the case \( q = \frac{1}{4} \) and \( \rho \to 0 \) at infinity corresponds to CH.

In this paper we investigate the connections between CH and HD from a different point of view. More precisely, we show that the CH and HD bi-Hamiltonian structures can be obtained by a bi-Hamiltonian reduction procedure from the Poisson pencil

\[ P(\lambda) = P_2 + \lambda P_1 = \partial_x + [\cdot, A] + \lambda [\cdot, S], \]

defined on the space \( \mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2, \mathbb{R})) \) of \( C^\infty \) maps from the unit circle to the Lie algebra of \( 2 \times 2 \) traceless matrices. \( S \) is a point of \( \mathcal{M} \) and \( A \) is an arbitrary constant traceless matrix. The reduction procedure depends on the choice of the conjugacy class of \( A \).

It turns out that there are three different reduced bi-Hamiltonian structures: one is the CH bi-Hamiltonian structure (\( \text{rank}(A) = 2 \)), one is the HD bi-Hamiltonian structure (\( \text{rank}(A) = 1 \)) and one, up to a change of coordinates, is still the HD bi-Hamiltonian structure (\( A = 0 \)).

Since the HD bi-Hamiltonian structure can be obtained from HS bi-Hamiltonian structure just by a change of variables \([\Pi]\), and taking into account the correspondence between coadjoint Virasoro orbits and Jordan normal forms in \( SL(2, \mathbb{R}) \), we observe that this result seems to be strictly related to those of Khesin and Misiolek.

The paper is organized as follows: In section 2 we summarize some useful techniques in the theory of the bi-Hamiltonian reduction. In section 3 we formalize and prove the above mentioned results.

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2 The bi-Hamiltonian reduction

In this section we recall a reduction process of the Marsden-Ratiu type \[20\], that can be performed on every bi-Hamiltonian manifold. It has been presented in \[4\] and then applied to the Drinfeld-Sokolov hierarchies \[8\] in \[5, 24\] and to the stationary reductions of KdV in \[9\].

Let \((M, P_1, P_2)\) be a bi-Hamiltonian manifold, i.e., a manifold \(M\) endowed with two Poisson tensors \(P_1\) and \(P_2\) that are compatible, in the sense that their sum (and hence any linear combination) is still a Poisson tensor (see, e.g., \[13, 17\]). Let us fix a symplectic leaf \(S\) of \(P_1\) and consider the distribution \(D = P_2(\text{Ker}P_1)\) on \(M\).

**Theorem 1** The distribution \(D\) is integrable. If \(E = D \cap TS\) is the distribution induced by \(D\) on \(S\) and the quotient space \(N = S/E\) is a manifold, then it is a bi-Hamiltonian manifold.

The reduced Poisson tensors \(P_1^{\text{red}}\) and \(P_2^{\text{red}}\) on \(N\) are constructed as follows. For any point \(p \in S\) and any covector \(\alpha \in T^*_p(N)\), where \(\pi : S \to N\) is the canonical projection, there is a covector \(\tilde{\alpha} \in T^*_p(M)\) such that

\[
\tilde{\alpha}|_{D_p} = 0, \quad \tilde{\alpha}|_{T_p S} = \pi_p^* \alpha ,
\]

where \(\pi_p^*: T^*_p(N) \to T^*_p S\) is the codifferential of \(\pi\) at \(p\). Then

\[
(P_i^{\text{red}})_{\pi(p)} \alpha = \pi_p^*((P_i)_p \tilde{\alpha}) , \quad i = 1, 2 .
\]

Whenever an explicit description of the quotient manifold \(N\) is not available, the following technique to compute the reduced bi-Hamiltonian structure (already employed in \[5\] for the Drinfeld-Sokolov case) is very useful.

**Theorem 2** Suppose \(Q\) to be a submanifold of \(S\) which is transversal to the distribution \(E\), in the sense that

\[
T_p Q \oplus E_p = T_p S \quad \text{for all } p \in Q .
\]
Then $Q$ (which is locally diffeomorphic to $N$) also inherits a bi-Hamiltonian structure from $M$. The reduced Poisson pair on $Q$ is given by

$$(P^i_{rd})_p \alpha = \Pi_p ((P_i)_p \tilde{\alpha}), \quad i = 1, 2, \quad (8)$$

where $p \in Q$, $\alpha \in T^*_p Q$, $\Pi_p : T_p S \to T_p Q$ is the projection relative to $\mathcal{J}$, and $\tilde{\alpha} \in T^*_p M$ satisfies

$$\tilde{\alpha}|_{D_p} = 0, \quad \tilde{\alpha}|_{T^*_p M} = \alpha. \quad (9)$$

In the next section we will follow this procedure to construct the bi-Hamiltonian structures of Camassa-Holm and Harry Dym as suitable bi-Hamiltonian reduced structures.

3 The bi-Hamiltonian structure of CH and HD

The aim of this section is to obtain the Poisson pair of Camassa-Holm and that of Harry Dym by applying the reduction procedure we have just described to a simple class of bi-Hamiltonian structures on the loop algebra of $\mathfrak{sl}(2, \mathbb{R})$.

Let $\mathcal{M} = C^\infty(S^1, \mathfrak{sl}(2, \mathbb{R}))$ be the space of $C^\infty$-maps from the unit circle to the Lie algebra of traceless $2 \times 2$ real matrices. The tangent space $T_S \mathcal{M}$ at $S \in \mathcal{M}$ is obviously identified with $\mathcal{M}$ itself. As far as the cotangent space is concerned, we will assume that $T^*_S \mathcal{M} \simeq T_S \mathcal{M}$ by means of the nondegenerate form

$$\langle V_1, V_2 \rangle = \int_{S^1} \text{tr} (V_1(x), V_2(x)) \, dx, \quad V_1, V_2 \in \mathcal{M}.$$ 

As well-known (see, e.g., [12]), the manifold $\mathcal{M}$ admits a 3-parameter family of compatible Poisson tensors given by

$$V \mapsto (P_{(a,b,c)})_S V = a \partial_x V + b[V, S] + c[V, A], \quad S \in \mathcal{M}, \quad V \in T^*_S \mathcal{M}, \quad (10)$$

where $a, b, c \in \mathbb{R}$ and $A$ is any matrix in $\mathfrak{sl}(2, \mathbb{R})$. We have the following theorems.
**Theorem 3** The bi-Hamiltonian reduction process applied to the pair \((P_1 = P_{(1,1,0)}, P_2 = P_{(0,0,1)})\) gives rise:

- to the Poisson pair of the KdV hierarchy if \(A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\);

- to the Poisson pair of the AKNS hierarchy if \(A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\),

for suitable choices of the symplectic leaf.

**Proof:** See [4] and [19]. The reduction used in the latter paper is not the one presented in Theorem 1, but it is easily shown to be equivalent. \(\square\)

**Theorem 4** The bi-Hamiltonian reduction process applied to the pair \((P_1 = P_{(0,1,0)}, P_2 = P_{(1,0,1)})\) gives rise:

- to the Poisson pair of the CH hierarchy if \(A = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\);

- to the Poisson pair of the HD hierarchy if \(A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\),

for suitable choices of the symplectic leaf.

**Proof:** Let

\[
S = \begin{pmatrix} p & q \\ r & -p \end{pmatrix}, \quad A = \begin{pmatrix} P & Q \\ R & -P \end{pmatrix},
\]

with \(p, q, r \in C^\infty(S^1, \mathbb{R})\) and \(P, Q, R \in \mathbb{R}\). Since \(\text{Ker}(P_1)_S\) is spanned by \(S\), the symplectic leaves of \(P_1\) are the level submanifolds of \(\det S = -p^2 - qr\). Moreover, we have that

\[
D_S = (P_2)_S (\text{Ker}(P_1)_S) = \{(\mu S)_x + [\mu S, A] | \mu \in C^\infty(S^1, \mathbb{R})\}.
\]

Explicitly,

\[
D_S = \left\{ \left( \frac{\mu p}{r} + \frac{(Rq - Qr)\mu}{\mu r} \right) \right\}.
\]

The distribution \(D\) is not tangent to the generic symplectic leaf of \(P_1\), but it is easily shown to be tangent to the symplectic leaf

\[
S = \left\{ \left( \frac{p}{r} \quad \frac{q}{-p} \right) | p^2 + qr = 0, (p, q, r) \neq (0, 0, 0) \right\}, \quad (11)
\]
so that $E_p = D_p \cap T_p \mathcal{S}$ coincides with $D_p$ for all $p \in \mathcal{S}$. In order to determine the reduced bi-Hamiltonian structure we first show that, under the assumption that $R \neq 0$, the submanifold
\[ \mathcal{Q} = \left\{ \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \mid q \in C^\infty(S^1, \mathbb{R}), q(x) \neq 0 \ \forall x \in S^1 \right\} \] (12)
of $\mathcal{S}$ is transversal to the distribution $E$. Indeed, if $S(q) = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$, then
\[ T_{S(q)} \mathcal{S} = \left\{ \begin{pmatrix} \dot{p} \\ q \end{pmatrix} \mid \dot{p}, q \in C^\infty(S^1, \mathbb{R}) \right\} \approx C^\infty(S^1, \mathbb{R}) \oplus C^\infty(S^1, \mathbb{R}) , \]
and every tangent vector in $T_{S(q)} \mathcal{S}$ admits the unique decomposition
\[ (\dot{p}, \dot{q}) = (\dot{p}, \frac{1}{R}(\dot{p} + 2P\dot{q})) + (0, \dot{q} - \frac{1}{R}(\dot{p} - 2P\dot{p})) , \]
where the first summand belongs to $E_{S(q)}$ and the second one to $T_{S(q)} \mathcal{Q}$. This also shows that $\Pi_{S(q)} : T_{S(q)} \mathcal{S} \to T_{S(q)} \mathcal{Q}$ is given by
\[ \Pi_{S(q)} : (\dot{p}, \dot{q}) \mapsto (0, \dot{q} - \frac{1}{R}(\dot{p} - 2P\dot{p})) . \] (13)

At this point we can compute the reduced Poisson pair on $\mathcal{Q}$. For the sake of simplicity we will deal simultaneously with the Poisson pencil $P(\lambda) = P_2 + \lambda P_1$. Given $\alpha \in T^*_S \mathcal{Q} \approx C^\infty(S^1, \mathbb{R})$, we look for a covector
\[ \tilde{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ -\alpha_1 \end{pmatrix} \in T^*_S \mathcal{M} \]
such that $\tilde{\alpha}|_{D_{S(q)}} = 0$ and $\tilde{\alpha}|_{T_{S(q)} \mathcal{Q}} = \alpha$. We easily find that
\[ \tilde{\alpha} = \begin{pmatrix} \frac{1}{2R}(\alpha_x + 2P\alpha) \\ \alpha \\ -\frac{1}{2R}(\alpha_x + 2P\alpha) \end{pmatrix} , \]
where $\alpha_2$ is arbitrary. Then we have that $(P(\lambda))_{S(q)} \tilde{\alpha} \in T_S \mathcal{Q}$ is given by
\[ \dot{p} = \frac{1}{2R}(\alpha_x + 2P\alpha_x) + R\alpha_2 - Q\alpha - \lambda q \]
\[ \dot{q} = \alpha_2 + Q \frac{R}{P}(\alpha_x + 2P\alpha) - 2P\alpha_2 + \lambda q \frac{R}{P}(\alpha_x + 2P\alpha) . \]
Thus the reduced Poisson pencil is

\[
(P_{\lambda})_q^r \alpha = \Pi_{S(q)} \left( (P_{\lambda})_{S(q)}^r \alpha \right) = \left[ -\frac{1}{2R^2} \partial_x^3 + 2 \frac{QR + P^2}{R^2} \partial_x + \frac{\lambda}{R} (2q \partial_x + q_x) \right] \alpha,
\]

that is to say

\[
(P_{1d}^r)_q = \frac{1}{R} (2q \partial_x + q_x) \]

\[
(P_{2d}^r)_q = -\frac{1}{2R^2} \partial_x^3 + 2 \frac{QR + P^2}{R^2} \partial_x.
\]

The case \(P = 0, Q = R = \frac{1}{2}, \) i.e., \(A = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), corresponds to the Poisson pair

\[
(P_{1d}^r)_q = 2(2q \partial_x + q_x) \]

\[
(P_{2d}^r)_q = 2(-\partial_x^3 + \partial_x).
\]

It is well-known that it is the CH bi-Hamiltonian structure \[3\]. Indeed, if we put \(q = m = u - u_{xx}\):

\[
m_t = -2mu_x - m_x u = P_{1d}^r \delta H_1 = P_{2d}^r \delta H_2,
\]

where

\[
H_1 = -\frac{1}{4} \int (u^2 + u_x^2) dx,
\]

\[
H_2 = -\frac{1}{4} \int (u^3 + uu_x^2) dx.
\]

The case \(P = Q = 0, R = 1, \) i.e., \(A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\), gives rise to the Poisson pair

\[
(P_{1d}^r)_q = 2q \partial_x + q_x
\]

\[
(P_{2d}^r)_q = -\frac{1}{2} \partial_x^3.
\]
It is well-known that it is the HD bi-Hamiltonian structure (see [7, 16, 25]). Indeed, if we put \( q = u \):

\[
\frac{\partial}{\partial t} = \left( u^{-\frac{1}{2}} \right)_{xxx} = P_1^{rd} \frac{\delta H_1}{\delta u} = P_2^{rd} \frac{\delta H_2}{\delta u},
\]

where

\[
H_1 = \frac{1}{8} \int u^{-\frac{5}{2}} u_x^2 dx,
\]

\[
H_2 = -4 \int u^2 dx.
\]

\[\square\]

**Theorem 5** The bi-Hamiltonian reduction process applied to the pair \( P_1 = P_{(0,1,0)}, P_2 = P_{(1,0,1)} \) with \( A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), in the space \( C_0^\infty(\mathbb{R}, \mathfrak{sl}(2, \mathbb{R})) \) of rapidly decreasing \( C^\infty \)-maps from the real line to \( \mathfrak{sl}(2, \mathbb{R}) \), gives rise to the Poisson pair

\[
(-2 \left( \partial_x^{-1} p_x + p_x \partial_x^{-1} \right), \partial_x),
\]

where \( p(x) \) is a smooth function vanishing at infinity and

\[
\partial_x^{-1} = \frac{1}{2} \left( \int_{-\infty}^{x} - \int_{x}^{+\infty} \right).
\]

**Proof:** The distribution

\[
D_S = \left\{ \begin{pmatrix} (\mu p)_x & (\mu q)_x \\ (\mu p)_x & (\mu r)_x \end{pmatrix} \, | \, \mu \in C_0^\infty(\mathbb{R}, \mathbb{R}) \right\}
\]

is still tangent to \( S \). In this case the transversal submanifold is

\[
Q = \left\{ \begin{pmatrix} p & 1 \\ -p^2 & \mu \end{pmatrix} \, | \, p \in C_0^\infty(\mathbb{R}, \mathbb{R}), p(x) \neq 0 \, \forall x \in \mathbb{R} \right\}
\]

and the projection \( \Pi_{S(q)} : T_{S(q)}S \to T_{S(q)}Q \) is given by

\[
\Pi_{S(q)} : (\dot{p}, \dot{q}) \mapsto (\dot{p} - p\dot{q} - p_x \partial_x^{-1} \dot{q}, 0).
\]
Following the same procedure used above it is easy to see that

\[
(P_{1d})_p = -2(\partial_x^{-1}p_x + p_x\partial_x^{-1})
\]

\[
(P_{2d})_p = \partial_x.
\]

Moreover, taking into account that, after a change of coordinates \(u' = u'(u)\), a bivector \(P\) transforms as

\[
P' = \left( \sum_{s \geq 0} \frac{\partial u'}{\partial u^{(s)}} \partial_x^s \right) P \left( \sum_{t \geq 0} (-\partial_x)^t \frac{\partial u'}{\partial u^{(t)}} \right), \tag{19}
\]

we obtain that, in the variable \(u = p_x\), the Poisson pair \((P_{1d}, P_{2d})\) coincides with the Harry Dym bi-Hamiltonian structure.

\[\square\]

4 Conclusions

In this paper we have shown that the bi-Hamiltonian structures of CH and HD can be seen as reductions of suitable structures on \(C^\infty(S^1, sl(2, \mathbb{R}))\). In the KdV case this is a well-known result, that can be interpreted both from the Drinfeld-Sokolov point of view and in the bi-Hamiltonian reduction scheme \[24\].

The results of this paper could be used to construct 2-field extensions of the CH and HD hierarchies, in the same way as Drinfeld and Sokolov construct a 2-component generalization of the KdV hierarchy. This extended hierarchy lives on a symplectic leaf of \(P_{(0,0,1)}\) and projects on the usual scalar KdV hierarchy (see \[8\] and \[17\]). Analogously, we plan to define a 2-field hierarchy on the symplectic leaf \(S\) of CH, whose projection on the transversal submanifold \(Q\) is the CH hierarchy. The “unprojected” CH equation should be compared with the 2-component generalization of CH recently introduced by Liu and Zhang \[15\].

A crucial point in Theorem 4 is the choice of the symplectic leaf \(S\). As we have already said, this leaf is special in that the distribution \(D\) is tangent to it. Thus, for a different leaf the reduction process is more complicated and turns out to be similar to the AKNS case. It would be interesting to construct the equations corresponding to these alternative choices.
References

[1] R. Beals, D. Sattinger and J. Szmigielski, *Acoustic scattering and the extended Koteweg de Vries hierarchy*, Adv. in Math., 140 (1998), 190-206.

[2] R. Beals, D. Sattinger and J. Szmigielski, *Inverse scattering solutions of the Hunter-Saxton equations*, Appl. Anal. 78 (2001), 255-269.

[3] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Phys. Lett. Rev. 71 (1993), 1661-1664.

[4] P. Casati, F. Magri, M. Pedroni, *Bi-Hamiltonian Manifolds and τ–function*, in: Mathematical Aspects of Classical Field Theory 1991 (M. J. Gotay et al. eds.), Contemporary Mathematics vol. 132, American Mathematical Society, Providence, R.I., 1992, pp. 213–234.

[5] P. Casati and M. Pedroni, *Drinfeld–Sokolov Reduction on a Simple Lie Algebra from the Bi-Hamiltonian Point of View*, Lett. Math. Phys. 25 (1992), 89-101.

[6] A. Constantin and B. Kolev, *On the geometric approach to the motion of inertial mechanical systems*, J. Phys. A: Math. Gen. 35 (2002), R51-R79.

[7] I. Dorfman, *Dirac structures and integrability of nonlinear evolution equations*, John Wiley and Sons Ltd, Chichester, 1993.

[8] V. G. Drinfeld, V. V. Sokolov, *Lie Algebras and Equations of Korteweg–de Vries Type*, J. Sov. Math. 30 (1985), 1975–2036.

[9] G. Falqui, F. Magri, M. Pedroni, J.P. Zubelli, *A Bi-Hamiltonian Theory for Stationary KdV Flows and their Separability*, Regul. Chaotic Dyn. 5 (2000), 33-52.

[10] J. Hunter and R. Saxton, *Dynamics of director fields*, SIAM J. Appl. Math. 51 (1991), 1498-1521.

[11] J. Hunter and Y. Zheng, *On a completely integrable nonlinear variational equation*, Phys. D 79 (1994) 361-386

[12] B. Khesin and G. Misiolek, *Euler equations on homogeneous spaces and Virasoro orbits*, Adv. Math. 176 (2003), 116-144.

[13] M. Kruskal, *Nonlinear wave equations*, Dynamical Systems, Theory and Applications (J. Moser ed.), Lecture Notes in Phys. 38, Springer, Heidelberg, 1975, pp. 310-354.

[14] P. Libermann and C. M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.

[15] S. Liu and Y. Zhang, *Deformations of semisimple bi-Hamiltonian structures of hydrodynamic type*, math.DG/0405146
[16] F. Magri, *A geometrical approach to the nonlinear solvable equations*, Nonlinear evolution equations and dynamical systems (Proc. Meeting, Univ. Lecce, Lecce, 1979), Lecture Notes in Phys. **120**, 1980, pp. 233-263.

[17] F. Magri, P. Casati, G. Falqui and M. Pedroni, *Eight lectures on Integrable Systems*, in: Integrability of Nonlinear Systems (Y. Kosmann-Schwarzbach et al. eds.), Lecture Notes in Physics **495** (2nd edition), 2004, pp. 209–250.

[18] F. Magri, G. Falqui, M. Pedroni, *The method of Poisson pairs in the theory of nonlinear PDEs*, in: Direct and Inverse Methods in Nonlinear Evolution Equations, Lectures Given at the C.I.M.E. Summer School Held in Cetraro, Italy, 1999, Lecture Notes in Physics **632**, 2003; [nlmi.SI/0002009](http://arxiv.org/abs/nlmi.SI/0002009).

[19] F. Magri, C. Morosi and O. Ragnisco, *Reduction techniques for infinite-dimensional Hamiltonian systems: some ideas and applications*, Commun. Math. Phys. **99** (1985), 115-140.

[20] J.E. Marsden, T. Ratiu, *Reduction of Poisson Manifolds*, Lett. Math. Phys. **11** (1986), 161–169.

[21] H. McKean, *The Liouville correspondence between the Korteweg-de Vries and the Camassa-Holm hierarchies*, Comm. Pure Appl. Math. **56** (2003), 998-1015.

[22] G. Misiolek, *A shallow water equation as a geodesic flow on the Bott-Virasoro group*, J. Geom. Phys. **24** (1998), 203-208.

[23] V. Ovsienko and B. Khesin, *The (super) KdV equation as an Euler equation*, Funct. Anal. Appl. **21** (1987), 81-82.

[24] M. Pedroni *Equivalence of the Drinfeld-Sokolov reduction to a bihamiltonian reduction*, Lett. Math. Phys. **35** (1995), 291-302.

[25] M. Pedroni, V. Sciacca, J.P. Zubelli, *On the Bi-Hamiltonian Theory for the Harry Dym Equation*, Theor. Math. Phys. **133** (2002), 1583-1595.