Harmonious labeling of the union of $m$-cycle and $n$-path graph

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Abstract. A graph $G$ is called harmonious graph if there is an injective mapping $f$ from the set of vertices of $G$ to the set $\mathbb{Z}_q$ so that the mapping induced a bijection mapping $f^*$ from the set of edges of $G$ to $\mathbb{Z}_q$, which is defined as $f^*(xy) = f(x) + f(y)$ for each edge $xy \in E(G)$. The mapping $f$ is called harmonious labeling. Harmonious labeling concept has been developed since 1980. At that time, harmonious labeling is inspired by channel assignment problems. Nowadays, harmonious labeling has been known for various classes of graphs. However, there are still many certain classes of graphs that have not been determined yet whether they can be harmonious labeled or not. One of the class of graphs that have not been determined yet whether it can be harmonious labeled or not is graph $C_m \cup P_n$. In this paper, we prove that there is a harmonious labeling of graph $C_m \cup P_n$ for some values $m$ and $n$.

Keywords: Cycle, harmonious labeling, labeling, path, union of graph

1. Introduction

A graph labeling is an assignment of labels (usually represented by integers) to the vertices, edges, or both, subject to given conditions. Formally, given a graph $G = (V,E)$ then vertex labeling can be defined as a mapping which assigns all elements of $V$ to a set of labels; edge labeling can be defined as a mapping which assigns all elements of $E$ to a set of labels; and total labeling can be defined as a mapping which assigns all elements of $V$ and $E$ to a set of labels.

Calderbank [1] stated that in many applications, a vertex set and an edge set in a graph are labeled with certain meanings on the selected domain. As an example, any edges in a graph can be labeled by some positive real numbers which represent the distance between two vertices. In this case, the vertex set is considered to represent some places.

One of the interesting graph labelings among mathematicians is harmonious labeling. In 1980, the term of ‘harmonious labeling’ is introduced by Graham and Sloane. The idea of harmonious labeling is based on the study of channel assignment problems. For example, there are some frequency channels. The vertex set in the graph represents a communication station and the edge set represents a communication path between one station to another. By assigning different labels to each station, each communication path can obtain a different frequency channel by summing the labels of the two communication stations [2].

A graph $G$ of order $p$ and size $q$ is called harmonious if there is an injective mapping $f : V(G) \rightarrow \mathbb{Z}_q$, where $\mathbb{Z}_q$ is the group of integers modulo $q$, such that the induced mapping
The mapping \(f^*\) is called harmonious labeling [3].

Based on Gallian’s survey [2], since the concept of harmonious labeling had been introduced in 1980, many mathematicians tried to prove the existence of harmonious labeling of some classes of graphs. In 2011, Fang [4] used a hybrid algorithm that involved probabilistic backtracking, tabu searching, and constraint programming satisfaction to verify that every tree with at most 31 vertices is harmonious. Meanwhile, according to Gallian [2], Aldred and McKay used a computer to show that all trees with at most 26 vertices are harmonious. Next, Graham and Sloane [3] proved that every caterpillar graph is harmonious. In the same paper, they also showed that the cycle graph \(C_n\) is harmonious if and only if \(n\) is odd. Graham and Sloane [3] also had proved that the wheel graph \(W_n = C_n + K_1\) is harmonious. Consequently, a subgraph of a harmonious graph need not be harmonious.

Until now, research on harmonious labeling in many classes of graphs is still developed. On December 2016, Gallian [2] summarized all of mathematicians’ research on graph labeling, including harmonious labeling of many classes of graphs in tables. Based on Gallian [2], it is known that harmonious labeling on \(C_n \cup P_n\) graph is still an open problem. In this paper, we prove the existence of harmonious labeling on \(C_n \cup P_n\) graph, especially for \(C_n\) with \(m \geq 3\) and \(P_n\) with \(n = 1, 2, 3\).

2. Preliminaries

Here are some definitions that will be used to prove the main results.

Definition 2.1. A cycle is a sequence of vertices in a graph which start and end at the same vertex without repeating edges. An \(m\)-cycle graph is a cycle with length \(m\), denoted by \(C_m\), \(m \geq 3\). [5].

Definition 2.2. A path graph with \(n\) vertices, denoted by \(P_n\), is a sequence of vertices in a graph without repeating vertices. The graph \(P_1\), consisting of a single isolated vertex, and is also called the singleton graph [5].

Definition 2.3. Let \(\{G_i = (V_i, E_i) : i = 1, 2, \ldots, n\}\) be a set of \(n\) subgraph of \(G\) with mutually disjoints set of vertices. The graph \(G\) is called the union of these graphs, with notation \(G = \bigcup_{i=1}^{n} G_i\), if every vertex and every edge of \(G\) belong to exactly one of these subgraphs. [5].

Definition 2.4. A graph \(G\) with \(p\) vertices and \(q\) edges is said to be harmonious if there exists an injection \(f : V(G) \rightarrow \mathbb{Z}_q\) such that the induced \(f^* : E(G) \rightarrow \mathbb{Z}_q\), defined by \(f^*(xy) = f(x) + f(y)\) for each edge \(xy \in E(G)\), is a bijection. The mapping \(f\) is called harmonious labeling. For graphs with \(p = q + 1\), exactly one label may be used on two vertices [3].

Graham and Sloane [3] give the following theorem.

Theorem 2.5. If a harmonious graph has an even number \(q\) of edges and the degree of each vertex is divisible by \(2^\alpha (\alpha \geq 1)\), then \(q\) is divisible by \(2^{\alpha + 1}\).

The following claim will help in proving some parts of the theorems in the main results.
Claim 2.6. Let \( n \) be an odd number and define \( \phi_n : \mathbb{Z}_n \rightarrow \mathbb{Z}_n \) by \( \phi_n(k) = 2k \mod n \). Then

(i) \( \phi_n \) is a bijection;

(ii) \( \phi_n(k+1) = \phi_n(k) + 2 \mod n \);

(iii) \( \phi_n^{-1}(k) = \frac{n+1}{2}k \) and \( \phi_n^{-1}(k+2) = \phi_n^{-1}(k) + 1 \mod n \).

3. Results and discussion

In this section, we give some results related to harmonious labeling of the graph \( C_n \cup P_n \).

Theorem 3.1. The graph \( C_n \cup P_1 \) is harmonious if and only if \( m \neq 2 \mod 4 \).

Proof. 

(\( \Rightarrow \)) Suppose that \( m \equiv 2 \mod 4 \). Note that \( C_n \cup P_1 \) has an even number \( m \) of edges and \( 2 \mid d(v) \) for each vertex \( v \in C_n \cup P_1 \). By theorem 2.5, \( m \) is divisible by \( 2^2 = 4 \), a contradiction.

(\( \Leftarrow \)) For an odd \( m \), Graham Dan Sloane [3] gives a harmonious labeling for \( C_m \). Note that while working with the graph \( C_n \cup P_1 \), the arithmetic is also done in \( \mathbb{Z}_m \). Thus, harmonious labeling for \( C_n \cup P_1 \) can be obtained by labeling the graph \( C_n \) harmoniously and labeling the vertex in \( P_1 \) with an arbitrary number in \( \mathbb{Z}_m \).

For \( m = 4j \), label the graph \( C_n \cup P_1 \) as follows:

- Let the vertex in \( P_1 \) has the label \( 3j \).
- Perform a contraction on an arbitrary edge in \( C_n \) and let \( v_0 \) be the vertex resulting from the merging of the corresponding two adjacent vertices. Let \( v_0, v_1, \ldots, v_{m-2} \) be the vertices of \( C_{m-1} \) resulting from the contraction. Label the graph \( C_{m-1} \) using the mapping

\[
f(v_{\phi_n^{-1}(k)}) = \begin{cases} 
  k, & 0 \leq k < 3j \\
  k+1, & 3j \leq k \leq m-2
\end{cases}
\]

where \( \phi_{m-1} \) is defined as in claim 2.6 (Note that \( \phi_{m-1} \) uses \( \mathbb{Z}_m \) arithmetic, while \( f \) uses \( \mathbb{Z}_n \) arithmetic).

Note that \( \phi_{m-1}(3j) = 2j+1 \). The second property from claim 2.6 implies that if \( k \geq 3j \), then \( \phi_{m-1}(k) \geq 2j+1 \). So, it is reasonable to check the bijective property of \( f^* \) by observing \( f^*(v_{k+v_{k+1}}) \) for \( 0 \leq k < 2j \) and \( 2j \leq k < m-2 \).

For \( 0 \leq k < 2j \), note that

\[
f^*(v_{k+v_{k+1}}) - f^*(v_{k+1}) = f(v_k) + f(v_{k+1}) - f(v_{k+1}) - f(v_k) \\
= f(v_k) - f(v_{k+1}) \\
= \phi_{m-1}^{-1}(k+1) - \phi_{m-1}^{-1}(k+1-2) \\
= 1.
\]

Since \( f^*(v_0v_1) = f(v_0) + f(v_1) = \phi_{m-1}^{-1}(0) + \phi_{m-1}^{-1}(1) = 0 + \frac{m}{2} = \frac{m}{2} \), then the labels for the edges \( v_0v_1, v_1v_2, \ldots, v_{2j-1}v_j \) are the consecutive integers \( \frac{m}{2}, \frac{m}{2}+1, \ldots, \frac{m}{2}+2j-1 = m-1 \).
For $2j < k < m - 2$, it can be shown similarly that $f^*(v_{kj+1}) - f^*(v_{kj}) = 1$. Since 
\[ f^*(v_{kj+1}) = f(v_{kj}) + (v_{kj}) \]
\[ = \phi_{m-1}^{-1}(2j) + 3j + 1 \]
\[ = \left( \frac{m}{2}, 2j \text{ mod}(m-1) \right) + 3j + 1 \text{ mod } m \]
\[ = f + 3j + 1 \text{ mod } m \]
\[ = 1, \]

then the labels for the edges $v_{kj+1}v_{kj+1}, v_{kj+1}v_{kj+2}, ..., v_{m+1}v_0$ are consecutive integers $1, 2, ..., 2j - 1 = \frac{m}{2}$.

Note that every element in $\mathbb{Z}_m$, except $0$, is already been used as an edge label. Note also that the vertex $v_0$ has vertex label $0$. Now, undo the contraction and let the adjacent vertices merged before have the same vertex label $0$. Then the corresponding edge has edge label $0$. Thus, $C_m \cup P_1$ with $m = 4j$ has a harmonious labeling. □

**Theorem 3.2.** The graph $C_m \cup P_1$ is harmonious.

**Proof.**

For an odd number $m$, label $C_m \cup P_1$ as follows:

- Let $v_0, v_1, ..., v_{m-1}$ be the vertices of $C_m$. Let $\phi_n$ be defined as in claim 2.6 and let $f(v_{n(i)}) = k, 0 \leq k \leq m - 1$, be the label on vertex $v_{n(i)}$ (note that $\phi_n$ uses $\mathbb{Z}_m$ arithmetic, while $f$ uses $\mathbb{Z}_{m+1}$ arithmetic);

- Let one vertex of $P_1$ has the label $m$ and the other vertex has the label $\frac{m-1}{2}$.

For $0 \leq k \leq m - 2$, $f^*(v_{kj+1}) - f^*(v_{kj}) = 1$.

The edge $v_0v_1$ has the label
\[ f^*(v_0v_1) = f(v_0) + f(v_1) \]
\[ = \phi_m^{-1}(0) + \phi_m^{-1}(1) \]
\[ = 0 + \left( \frac{m+1}{2} \text{ mod } m \right) \text{ mod } (m+1). \]

Since $m \geq 3$, then $f^*(v_0v_1) = \frac{m+1}{2}$. The edge $v_{m-1}v_0$ has the label
\[ f^*(v_{m-1}v_0) = f(v_{m-1}) + f(v_0) \]
\[ = \phi_m^{-1}(m-1) + \phi_m^{-1}(0) \]
\[ = \left( \frac{m+1}{2} \text{ mod } m \right) \text{ mod } (m+1). \]

Since $m \geq 3$, then $f^*(v_{m-1}v_0) = \frac{m-1}{2} = \frac{m+1}{2} - 1 = f^*(v_0v_1) - 1$. Hence, the edge labels of $C_m$ are the consecutive integers $\frac{m-1}{2}, \frac{m-1}{2} + 1, \frac{m-1}{2} + 2, ..., \frac{m-1}{2} + m - 1 = \frac{m-1}{2} - 2$. Note that in $\mathbb{Z}_{m+1}$, these integers are distinct and the edge label which has not been used yet is $\frac{m-1}{2} - 1$. Note that the single
edge of $P_i$ has the label $m + \frac{m-1}{2} = m + \frac{m-1}{2} - m - 1 = \frac{m-1}{2} - 1$. Thus, $f$ is a harmonious labeling for the graph $C_n \cup P_i$ with $m$ an odd number.

For even number $m$, two cases will be considered: $m = 2j$ with $j$ an odd number and $m = 2j$ with $j$ an even number.

For $m = 2j$ with odd number $j$, let $v_0, v_1, \ldots, v_{m-1}$ be the vertices $C_n$ and let

$$f(v_k) = \begin{cases} 
  j - 2k - 1, & 0 \leq k \leq \frac{j-1}{2} \\
  -j + 2k - 2, & \frac{j+1}{2} \leq k \leq j - 1 \\
  -j + 2k + 1, & j \leq k \leq \frac{3j-1}{2} \\
  5j - 2k, & \frac{3j+1}{2} \leq k \leq m - 1
\end{cases}$$

be the labeling of the vertices. To simplify the notations, define $I_1 = \left\{ k \in \mathbb{Z} : 0 \leq k \leq \frac{j-1}{2} \right\}$,

$I_2 = \left\{ k \in \mathbb{Z} : \frac{j+1}{2} \leq k \leq j - 1 \right\}$, $I_3 = \left\{ k \in \mathbb{Z} : j \leq k \leq \frac{3j-1}{2} \right\}$, and $I_4 = \left\{ k \in \mathbb{Z} : \frac{3j+1}{2} \leq k \leq m - 1 \right\}$.

It is not difficult to see that

- $0 \leq j - 2k - 1 \leq j - 1$ for $k \in I_1$,
- $1 \leq j + 2k + 1 \leq j - 2$ for $k \in I_2$,
- $j + 1 \leq j + 2k + 2 \leq m$ for $k \in I_3$, and
- $j + 2 \leq j - 2k - 2 \leq m - 1$ for $k \in I_4$,

with the bounds given is also the minimum and maximum of the labeling mapping $f$ in the corresponding set $I$. The term $2k$ in the formula for $f$ implies $\text{im} f(v_k) = \{ 0, 2, 4, \ldots, j - 1 \}$, $\text{im} f(v_k) = \{ j + 1, j + 3, \ldots, m \}$, and $\text{im} f(v_k) = \{ j + 2, j + 4, \ldots, m - 1 \}$.

Finally, let the two vertices of $P_i$ has the label $j$.

Now $f^*$ has the form

$$f^*(v_k v_{k+1}) = \begin{cases} 
  -4k - 5, & 0 \leq k \leq \frac{j-3}{2} \\
  1, & k = \frac{j-1}{2} \\
  4k + 3, & \frac{j+1}{2} \leq k \leq j - 2 \\
  -2, & k = j - 1 \\
  4k + 5, & j \leq k \leq \frac{3j-3}{2} \\
  -3, & k = \frac{3j-1}{2} \\
  -4k - 7, & \frac{3j+1}{2} \leq k \leq m - 2 \\
  0, & k = m - 1
\end{cases}$$
It can be shown that $f^*$ is indeed injective and $m$ is not in $\text{im}_{k \in Z} f(v_k v_{k+1})$. But the label $m = j + j$ appears in the edge of $P_j$. Thus, the labeling for graph $C_j \cup P_j$ with odd number $j$, is harmonious.

For $m = 2j$ with even number $j$, let the notations for the vertices as above and

$$f(v_k) = \begin{cases} 
  j - 2k - 2, & 0 \leq k \leq \frac{j}{2} - 1 \\
  -j + 2k + 1, & \frac{j}{2} \leq k \leq j - 1 \\
  -j + 2k + 2, & j \leq k \leq \frac{3j}{2} - 1 \\
  5j - 2k - 1, & \frac{3j}{2} \leq k \leq m - 1
\end{cases}$$

To simplify the notations, define $J_1 = \left\{ k \in Z : 0 \leq k \leq \frac{j}{2} - 1 \right\}$, $J_2 = \left\{ k \in Z : \frac{j}{2} \leq k \leq j - 1 \right\}$, $J_3 = \left\{ k \in Z : j \leq k \leq \frac{3j}{2} - 1 \right\}$, and $J_4 = \left\{ k \in Z : \frac{3j}{2} \leq k \leq m - 1 \right\}$.

It is not difficult to see that

- $0 \leq j - 2k - 2 \leq j - 2$ for $k \in J_1$,
- $1 \leq -j + 2k + 1 \leq j - 1$ for $k \in J_1$,
- $0 \leq j - 2k + 2 \leq m$ for $k \in J_2$,
- $j + 1 \leq j - 2k - 3 \leq m - 1$ for $k \in J_4$.

with the bounds given is also the minimum and maximum of the labeling mapping $f$ in the corresponding set $J_i$. The term $2k$ in the formula of $f$ implies $\text{im}_{k \in J_i} f(v_k) = \{0, 2, 4, ..., j - 2\}$.

$\text{im}_{k \in J_i} f(v_k) = \{j + 1, j + 3, ..., m - 1\}$.

Also, let the two vertices of $P_j$ have the label $j$.

The mapping $f^*$ has the form

$$f^*(v_k v_{k+1}) = \begin{cases} 
  -4k - 7, & 0 \leq k \leq \frac{j}{2} - 2 \\
  1, & k = \frac{j}{2} - 1 \\
  4k + 5, & \frac{j}{2} \leq k \leq j - 2 \\
  0, & k = j - 1 \\
  4k + 7, & j \leq k \leq \frac{3j}{2} - 2 \\
  -3, & k = \frac{3j}{2} - 1 \\
  -4k - 9, & \frac{3j}{2} \leq k \leq m - 2 \\
  -2, & k = m - 1
\end{cases}$$

It can be shown that this mapping is injective and $m$ is never the image of this mapping. But $m$ appears as the edge label in $P_j$. Therefore, $C_{2j} \cup P_j$ with $j$ an even number is harmonious. \hfill $\Box$

**Theorem 3.3.** The graph $C_m \cup P_j$ is harmonious for an odd number $m \geq 3$. 


Proof.
First, label the cycle graph $C_m$. Let $v_0, v_1, ..., v_{m-1}$ be the vertices of $C_m$. Let $\phi_m$ be defined as in claim 2.6 and let $f(v_{k+1}) = k, 0 \leq k \leq m-1$ be the label on vertex $v_{k+1}$ (note that $\phi_m$ uses $\mathbb{Z}_m$ arithmetic, while $f$ uses $\mathbb{Z}_{m+1}$ arithmetic).

Note that: $f^*(v_{k+1}) - f^*(v_k) = 1$.

Thus, by the facts that $m < m + 2$, it will be got that every edge label in $C_m$ will be different in $\mathbb{Z}_m$, the numbers which have not been chosen as edge label is $r$ and $r + 1$ for any $r \in \mathbb{Z}_m$.

Based on our label choice for every vertex in $C_m$, vertex labels which have not been used is $m$ and $m + 1$. Let the vertices in $P_3$ are $w_0, w_1$, and $w_2$. We label these vertices with $f(w_0) = m, f(w_1) = r - m$, and $f(w_2) = m + 1$. These labels will bring in edge label $r$ and $r + 1$. Thus, the graph $C_m \cup P_3$ for an odd number $m \geq 3$ is harmonious.

**Theorem 3.4.** The graph $C_m \cup P_3$ is harmonious for an even number $m = 4, 6, 8, 10, 12, 14, 16, 18$.

The labels for these graphs can be seen on figure 1.

**Figure 1.** Harmonious labeling of graph: (a) $C_4 \cup P_3$, (b) $C_6 \cup P_3$, (c) $C_{10} \cup P_3$, and (d) $C_{14} \cup P_3$.

4. Conclusion
Based on this study, the graph $C_m \cup P_1$ with $m \neq 2 \mod 4$; the graph $C_m \cup P_2$; and the graph $C_m \cup P_3$, for an odd number $m \geq 3$ is proven to be harmonious graph. Furthermore, $C_m \cup P_3$ for an even number $m = 4, 6, 8, 10, 12, 14, 16, 18$ is also proven as harmonious graphs. Here, we conjecture that the graph $C_m \cup P_3$, for an even number $m$ is harmonious.

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