THE MAXIMUM NUMBER OF MINIMAL CODEWORDS IN AN $[n,k]$–CODE

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Abstract. Upper and lower bounds are derived for the quantity in the title, which is tabulated for modest values of $n$ and $k$. An application to graphs with many cycles is given.

1. Introduction

Consider a binary linear code $C$. A codeword of $C$ is called minimal if its support does not contain properly the support of another nonzero codeword. This concept was discovered independently in code-based secret sharing schemes [6] and also in the study of the Voronoi domain of a code in the context of decoding [11]. What is the maximum number $M(C)$ of minimal codewords a code $C$ of given length and dimension might have? Formally, denoting by $C[n,k]$ the set of all $[n,k]$ codes, we define, following the companion paper [4], the function

$$M(n,k) = \max \{ M(C) : C \in C[n,k] \},$$

as the maximum of $M(C)$ over that set of codes. While the concern of [4] was asymptotics, we will consider in this note only bounds on or exact values of that function for finite values of $n$ and $k$. We will consider three upper bounds. The so-called trivial bound, the matroid bound as in [4] and the Agrell bound [1]. We derive a recursive inequality on $M(n,k)$ which gives an alternative proof of the matroid bound, independent of matroid theory as a special case. The connection with intersecting codes shows that the trivial bound is sharp when $k$ is small compared to $n$. The Agrell bound which is asymptotically equivalent to the trivial bound can be sharper for finite values of $n$ and $k$ when $k$ is close to $n$. In particular in the special case of the cycle code of graphs this bound is a sharpening of the $\frac{15}{16}$ bound of [3] in a special case. For lower bounds, neither the random coding bound of [6] nor the combinatorial bound of [2] matches explicit constructions.

The material is organized as follows. Section 2 is dedicated to upper bounds. Section 3 considers lower bounds. Section 4 builds a table of values of and bounds on $M(n,k)$ for $k \leq 13$ and $n \leq 15$.

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2. Upper bounds

An immediate upper bound is $M(n, k) \leq 2^k - 1$. We call this the trivial bound. Another upper bound derived in [4] by use of matroid theory is $M(C) \leq \binom{n}{k} - 1$, which is sharper than the trivial bound at high rates. We give a recurrence relation that implies the matroid bound.

**Theorem 1.** For all $1 \leq k \leq n$ we have

$$M(n, k) \leq M(n - 1, k - 1) + \binom{n - 1}{k - 1}.$$

**Proof:** Let $H$ be the parity check of $C$ that realizes $M(n, k)$, and $H'$ the matrix with column $n$ removed. Assume, up to column reordering, that there is a basis of the column space not containing column $n$, or equivalently that the rank of $H'$ is $n - k$. This is always possible if $k \geq 1$. Let $x$ be a nonzero minimal codeword in $C$ and discuss according to the value of $x_n$.

If $x_n = 0$ then the projection $x'$ on the first $n - 1$ coordinates is a minimal codeword in $Ker(H')$ an $[n - 1, k - 1]$ code. Therefore there are at most $M(n - 1, k - 1)$ such vectors.

If $x_n = 1$ then the set of columns where the projection $x'$ on the first $n - 1$ coordinates is nonzero form an independent set of the column space of $H'$, because of the minimality property of $x$. There are at most

$$\binom{n - 1}{n - k} = \binom{n - 1}{k - 1}$$

possible such $x'$.

The matroid bound now follows as a Corollary of the above Theorem.

**Corollary 1.** For all $1 \leq k \leq n$ we have $M(n, k) \leq \binom{n}{k} - 1$.

**Proof:** We reason by induction on $k$. Clearly, the bound is true for $k = 1$, since $M(n, 1) = 1 = \binom{n}{0}$. Assuming $M(n - 1, k - 1) \leq \binom{n - 1}{k - 2}$, by Pascal’s triangle, using the above theorem, we are done. $\square$

Another upper bound is given in [1, Theorem 5].

**Theorem 2.** For $\frac{k - 1}{n} > \frac{1}{2}$ we have

$$M(n, k) \leq \frac{2^k}{4n(\frac{k - 1}{n} - \frac{1}{2})^2}.$$

A difficult problem in graph theory is to bound above the maximum number of cycles a connected graph on $p$ vertices and with $q$ edges can have [8]. The analogue of the trivial bound in that context is $2^{q - p + 1}$. The first bound significantly below that was [3]. The next result is a strengthening for graphs of average degree $> 4$ of that result.
Corollary 2. If $\Gamma$ is a connected graph on $p$ vertices and with $q$ edges satisfying $q > 2p$ then its number of elementary cycles is at most
\[
\frac{q2^{q-p+1}}{(q-2p)^2}.
\]

Proof: Recall that with every connected graph $\Gamma$ on $p$ vertices and with $q$ edges is attached a binary $[q,q-p+1]$ code $C(\Gamma)$ called the cycle code of the graph. Its codewords are indicator vectors of either elementary cycles or edge disjoint unions of such. The minimal codewords of $C(\Gamma)$ are the indicator vectors of the (elementary) cycles of the graph. The result follows by applying Theorem 2 to that code, after some algebra.

The bound in [3] was $\frac{15\cdot 2^{q-p+1}}{16}$. The last result is sharper for $p \geq 1$ as can be seen by computing the discriminant of a quadratic equation in $q$.

3. Lower bounds

As in [4] there is a random coding lower bound from [6].
\[
M(n,k)2^{n-k} \geq \sum_{j=0}^{n-k+1} \binom{n}{j} \prod_{i=0}^{j-2} (1 - 2^{-(n-k-i)}).
\]

Another existence bound is as follows. Denote by $d(n,k)$ the largest minimum distance of an $[n,k]$ code. The following Proposition is a direct consequence of [2, Prop. 2.1].

Proposition 1. For all $n \geq k \geq 1$, we have
\[
\sum_{i=1}^{\left\lfloor \frac{n}{d(n,k)} \right\rfloor} \binom{M(n,k)}{i} \geq 2^k - 1.
\]

Proof:

Let $C$ be an $[n,k,d]$ code. By induction it can be seen that every nonzero codeword can be written as a sum of at most $\left\lfloor \frac{n}{d(n,k)} \right\rfloor$ support disjoint minimal codewords. Hence, enumeration of such sums yields
\[
\sum_{i=1}^{\left\lfloor \frac{n}{d} \right\rfloor} \binom{M(C)}{i} \geq 2^k - 1.
\]

The result follows by choosing $C$ to be optimal for $d$.

This result shows that good codes cannot have too few minimum codewords. It is not very sharp. We only get $M(8,4) \geq 5$, when the example of the extended Hamming code shows that $M(8,4) \geq 14$.

4. Tabulating $M(n,k)$

4.1. Monotonicity properties. It is easy to show that $M(n,k) \leq M(n+1,k)$ by adding a zero column to a code realizing $M(n,k)$. This innocent remark provides better bounds that the random coding bound for $k = 4$ and $n = 7,8$ as well as $k = 5$ and $n = 7,8,9$ and so on. It is false that $M(n,k) \leq M(n,k+1)$ as the values of $M(4,k)$ already show. We conjecture, but cannot prove, that $M(n,k)$ is an unimodal function of $k$ for fixed $n$. 

Proposition 2. For binary codes $C, D$ we have $M(C \oplus D) = M(C) + M(D)$.

Proof: If $c$ and $d$ are minimal codewords of respectively $C$ and $D$ then $(c, 0)$ and $(0, d)$ are minimal codewords of $C \oplus D$. Conversely, we claim that all minimal codewords of the latter code arise in that way. Indeed if $(c, d)$ is a minimal codeword of $C \oplus D$, with both $c$ and $d$ nonzero, then $(c, d) = (c, 0) + (0, d)$ contradicting minimality.

Using the above Proposition, we see that $M(n, k)$ is super additive

$$M(n + m, k + j) \geq M(n, k) + M(m, j).$$

4.2. Exact values. Trivial values are $M(n, 1) = 1$, and $M(n, n) = n$ for all $n \geq 1$. Already $M(n, n - 1)$ is known but requires a proof.

Proposition 3. $M(n, n - 1) = \binom{n}{2}$ for $3 \leq n$.

Proof: We claim that

$$M(n, n - 1) = \max\{\left(\frac{x}{2}\right) + (n - x) | 2 \leq x \leq n\}.$$ 

Indeed, denoting by $P_x$ the parity-check code of length $x$ and by $U_y$ the universe code of length $y$, we see that, by Proposition 2, $M(P_x \oplus U_{n-x}) = \binom{x}{2} + (n-x) =: f(x)$. Studying the variation of the quadratic $f(x)$ shows that it is increasing for $x \in [2, n]$. Since $f(n) = \binom{n}{2}$, we are done.

4.3. Intersecting codes. Recall that a code is intersecting [9] if the respective supports of any two nonzero codewords intersect. As observed in [4] a linear binary code meets the trivial bound with equality iff it is intersecting. Following [9], denote by $f(k)$ the shortest length of a binary linear intersecting code. Equivalently, there is a function $g(n)$ such that if $k \leq g(n)$ then there is an intersecting $[n, k]$ code; and there is not if $k > n$; thus if $k \leq g(n)$ then $M(n, k) = 2^k - 1$, and if $k > g(n)$ then $M(n, k) \leq 2^k - 2$. The function $g(n)$, the inverse of $f$, is known exactly for $1 \leq n \leq 15$ [9] and given in Table 1.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|----|----|----|----|----|----|
| $g(n)$ | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 4 | 4 | 5 | 5 | 6 |

Table 1. $g(n)$ for $3 \leq n \leq 15$

4.4. Table of $M(n, k)$. The exponents in Table 2 are as follows.

- $t$ - Trivial bound
- $m$ - Matroid bound
- $a$ - Agrell bound

When the trivial bound is met with equality the exponent $t$ is omitted. Empty entries correspond to $k > n$ when $M(n, k)$ is undefined. The lower bounds are derived by explicit constructions of codes $C$ derived from A. Betten list of indecomposable codes [5], followed by application of rule G or rule H of [1] to derive $M(C)$. The codes realizing $M(n, k)$ are not in general optimal for the minimum distance, but they are in general indecomposable.
5. Conclusion and open problems

In this note we have considered the function $M(n, k)$ maximum number of minimal codewords of a binary linear code of parameters $[n, k]$. Three upper bounds have been considered in turn: trivial, Agrell and matroid bound. From Table 2 we see that they are most relevant respectively at low rate, high rate and very high rate. Lower bounds have been derived by selecting suitable indecomposable codes. It seems possible but computationally heavy to derive the exact values of $M(n, k)$ by combining Proposition 2 with a database of indecomposable codes that would be more comprehensive than that of [5] where only indecomposable codes with large minimum distance are listed. We conjecture that the lower bounds in Table 2 are in fact exact values, and that they are obtained for indecomposable codes.

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References

[1] E. Agrell, On the Voronoi neighbor ratio for binary linear codes, IEEE Transactions on Information Theory (1998) 3064–3072.
[2] A. Ashikhmin, A. Barg, G. Cohen, L. Huguet, Variations on minimial codewords in linear codes, Springer LNCS 948 (1995) 96-105.
[3] R.E.L. Aldred, C. Thomassen On the maximum number of cycles in a planar graph, J. Graph Theory 53 (2008) 255–264.
[4] A. Alamadhi, R.E.L. Aldred, R. de la Cruz, P. Solé, C. Thomassen, The maximum number of minimal codewords in long codes, submitted.
[5] http://www.math.colostate.edu/~betten/research/codes/GF2/codes_GF2.html
[6] A. Ashikhmin, A. Barg, Minimal vectors in linear codes, IEEE Transactions in Information Theory (1998) 2010–2017.
[7] Gy. Dosa, I. Szalkai, C. Laflamme, The maximum and minimum number of circuits and bases of matroids, PU. M. A. 15 (2004) 383–392.
[8] R.C. Entringer, P.J. Slater, On the maximum number of cycles in a graph, Ars Combinatoria 11 (1981) 289–294.
[9] N. J. A. Sloane, Covering Arrays and Intersecting Codes, J. Combinatorial Designs, 1 (1993), 51–63.
