Coherent states for a system of an electron moving in a plane: case of discrete spectrum

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Abstract
In this work, we construct different classes of coherent states related to a quantum system, recently studied in [1], of an electron moving in a plane in uniform external magnetic and electric fields which possesses both discrete and continuous spectra. The eigenfunctions are realized as an orthonormal basis of a suitable Hilbert space appropriate for building the related coherent states. These latter are achieved in the context where we consider both spectra purely discrete obeying the criteria that a family of coherent states must satisfy.

1. Introduction
From a generalization of the definition of canonical coherent states, Gazeau and Klauder proposed a method to construct temporally stable coherent states for a quantum system with one degree of freedom [2]. Then, in the literature, the method has been explored for different kinds of quantum systems with several degrees of freedom. See for example [3–5] and references therein. Also, in some previous works, motivated by these developments, multidimensional vector coherent states have been performed for Hamiltonians describing the nanoparticle dynamics in terms of a system of interacting bosons and fermions [6]; from a matrix (operator) formulation of the Landau problem and the corresponding Hilbert space, an analysis of various multi-matrix vector coherent states extended to diagonal matrix domains has been performed on the basis of Landau levels [7]. Moreover, the motion of an electron in a noncommutative (x,y) plane, in a constant magnetic field background coupled with a harmonic potential has been examined with the relevant vector coherent states constructed and discussed [8].

Following the method developed in [2, 4], we investigate in a recent work [1] by considering Landau levels, various classes of coherent states as in [5, 9, 10] arising from a physical Hamiltonian describing a charged particle in an electromagnetic field, by introducing additional parameters useful for handling discrete and continuous spectra of the Hamiltonian. In this work, we consider an electron moving in a plane (x,y) in the uniform external electric field $E^y = -\nabla \Phi(x, y)$ and the uniform external magnetic field $B^z$ which is perpendicular to the plane described by the Hamiltonian [1]

$$H = \frac{1}{2m} \left( \vec{p}^2 + \frac{e}{c} \vec{A} \right)^2 - e\Phi.$$  (1)

We briefly recall here a summary of results where the details are given in [1]. In the symmetric gauge $\vec{A} = \left( \frac{B}{c} x, -\frac{B}{c} y \right)$ with the scalar potential given by $\Phi(x, y) = -Ey$, the corresponding classical Hamiltonian, obtained from (1), denoted by $H_1$, reads
The Hamiltonian $\hat{H}_1$ can be then re-expressed as follows:

$$\hat{H}_1 = \frac{1}{4m}(b^\dagger b + bb^\dagger) - \frac{\lambda}{2m}(d^\dagger + d) - \frac{\lambda^2}{2m}$$

(3)

where the two sets of annihilation and creation operators $b, b^\dagger$ and $d, d^\dagger$ are given by

$$b = 2P_z - i\frac{eB}{2c}Z + \lambda, \quad b^\dagger = 2P_z + i\frac{eB}{2c}Z + \lambda,$$

(4)

$$d = 2P_z + i\frac{eB}{2c}Z, \quad d^\dagger = 2P_z - i\frac{eB}{2c}Z,$$

(5)

with $\lambda = \frac{mcE_B}{\hbar}$. The Hamiltonian $\hat{H}_1$ splits into two commuting parts in the following manner:

$$\hat{H}_1 = \hat{H}_{1_{osc}} - \hat{\mathcal{T}}_i,$$

(6)

where $\hat{H}_{1_{osc}}$ denotes the harmonic oscillator part

$$\hat{H}_{1_{osc}} = \frac{1}{4m}(b^\dagger b + bb^\dagger),$$

(7)

while the part linear in $d$ and $d^\dagger$ is given by

$$\hat{\mathcal{T}}_i = \frac{\lambda}{2m}(d^\dagger + d) + \frac{\lambda^2}{2m}.$$  

(8)

The eigenvalue equation $\hat{\mathcal{T}}_i \phi = \mathcal{E}\phi$ can be reduced to

$$-i\frac{\partial}{\partial x} - \frac{m\omega_y}{2\hbar}\phi - m\mathcal{E}\phi = 0.$$  

(9)

Setting $\alpha = \frac{mcE_B}{\hbar\lambda}$, it becomes

$$-i\frac{\partial}{\partial x} - \left(\frac{m\omega_y}{2\hbar} + \alpha\right)\phi,$$

(10)

whose solution is readily found to be

$$\phi_n \equiv \phi_n(x, y) = e^{i(\alpha x + \frac{m\omega_y}{2\hbar} y)}, \quad \alpha \in \mathbb{R}.$$  

(11)

Then, the eigenvalues of the operator $\hat{\mathcal{T}}_i$, corresponding to eigenfunctions (11), are given by

$$\mathcal{E}_n = \frac{\hbar\lambda}{m}\alpha + \frac{\lambda^2}{2m}, \quad \alpha \in \mathbb{R},$$

(12)

indicating that this spectrum, labeled by $\alpha$, is continuous. Therefore, the eigenvectors and the energy spectrum of the Hamiltonian $\hat{H}_1$ are determined by the following formulas:

$$\Psi_{n,\alpha} = \Phi_n \otimes \phi_n \equiv |n, \alpha\rangle,$$

(13)

$$\mathcal{E}_{(n,\alpha)} = \frac{\hbar\omega_y}{2}(2n + 1) - \frac{\hbar\lambda}{m}\alpha - \frac{\lambda^2}{2m}, \quad n = 0, 1, 2, ....$$

In the symmetric gauge $A^x = \left(\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right)$ with the scalar potential given by $\Phi(x, y) = -Ex$, the classical Hamiltonian $H$ in equation (1) becomes

$$H_2(x, y, P_x, P_y) = \frac{1}{2m}\left[\left(P_x + \frac{eB}{2c}y\right)^2 + \left(P_y + \frac{eB}{2c}x\right)^2\right] + eEx.$$  

(14)

The Hamiltonian operator $\hat{H}_2$ can be then written as

$$\hat{H}_2 = \frac{1}{4m}(b^\dagger b + bb^\dagger) - \frac{\lambda}{2m}(\mathcal{D}^\dagger + \mathcal{D}) - \frac{\lambda^2}{2m},$$

(15)

where

$$b^\dagger = -2iP_x + \frac{eB}{2c}Z + \lambda, \quad b = 2iP_x + \frac{eB}{2c}Z + \lambda,$$

(16)

$$\mathcal{D} = 2iP_x - \frac{eB}{2c}Z, \quad \mathcal{D}^\dagger = -2iP_x - \frac{eB}{2c}Z,$$

(17)

with $\lambda$ defined as in (4) and (5).
The harmonic oscillator part is given by
\[ \hat{H}_{\text{osc}} = \frac{1}{4m}(\hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger) \]  
(18)
and the linear part by
\[ \hat{T}_z = \frac{\lambda}{2m}(\hat{b}^\dagger \hat{b} + \hat{b} \hat{b}^\dagger) + \frac{\lambda^2}{2m}. \]  
(19)

Then, the eigenvalue equation \( \hat{T}_z \phi = E\phi \) is equivalent in this case to
\[ \hbar \lambda \left( -i \frac{\partial}{\partial y} - \frac{m\omega_1}{2\hbar} x \right) \phi = E\phi, \]  
(20)
which leads to
\[ \left( -i \frac{\partial}{\partial y} - \frac{m\omega_1}{2\hbar} x \right) \phi - \frac{mE}{\hbar \lambda} \phi = 0. \]  
(21)

Taking again \( \alpha = \frac{mc}{\hbar \lambda} \), it follows the equation
\[ -i \frac{\partial}{\partial y} \phi = \left( \frac{m\omega_1}{2\hbar} x + \alpha \right) \phi \]  
(22)
which can be solved to give the eigenfunctions
\[ \phi_\alpha \equiv \phi_\alpha(x,y) = e^{i(\alpha y + \frac{m\omega_1}{\hbar \lambda} xy)} \quad \alpha \in \mathbb{R}, \]  
(23)
of the operator \( \hat{T}_z \) corresponding to eigenvalues expressed as in (12). The eigenvectors and the eigenvalues of the Hamiltonian \( \hat{H}_2 \), as previously determined for \( \hat{H}_1 \), are obtained as
\[ \Psi_{l,\alpha} = \Psi_l \otimes \phi_\alpha \equiv \ket{l, \alpha}, \]  
\[ E_{(l,\alpha)} = \frac{\hbar \omega_1}{2}(2l + 1) - \frac{\hbar \lambda}{m} \alpha - \frac{\lambda^2}{2m} \quad l = 0, 1, 2, \ldots. \]  
(24)

The eigenvectors denoted \( \ket{\Psi_{nl}} = \ket{n, l} \otimes \ket{l} \) of \( \hat{H}_{\text{osc}} \) can be so chosen that they are also the eigenvectors of \( \hat{H}_{\text{osc}} \), since \([\hat{H}_{\text{osc}}, \hat{H}_{\text{osc}}] = 0\), as follows:
\[ \hat{H}_{\text{osc}} \ket{\Psi_{nl}} = \hbar \omega_1 \left( n + \frac{1}{2} \right) \ket{\Psi_{nl}}, \quad \hat{H}_{\text{osc}} \ket{\Psi_{n l}} = \hbar \omega_1 \left( l + \frac{1}{2} \right) \ket{\Psi_{n l}}, \quad n, l = 0, 1, 2, \ldots \]  
(25)
so that \( \hat{H}_{\text{osc}} \) lifts the degeneracy of \( \hat{H}_{\text{osc}} \), and vice versa.

The present paper is a direct continuation of our work in [1], where we construct different classes of coherent states corresponding to the case of discrete spectrum.

The paper is organized as follows. Section 2 is devoted to the construction of coherent states for the quantum Hamiltonian possessing purely discrete spectrum by following the method developed in [2, 4]. Section 3 is about the coherent states of the unshifted Hamiltonians \( \hat{H}_1 \) and \( \hat{H}_2 \) defined through multiple summations. In section 4 we construct coherent states related to the Hamiltonian \( \hat{H}_{\text{osc}} - \hat{H}_{\text{osc}} \). An outlook is given in section 5.

2. Coherent states for shifted Hamiltonians with more than one degree of freedom

In this section, we construct various classes of coherent states for Hamiltonian operators that admit discrete eigenvalues and eigenfunctions in appropriate separable Hilbert spaces as elaborated in [4, 5, 9]. Let \( \mathcal{H}_D \) be spanned by the eigenvectors \( \ket{\Psi_{nl}} \equiv \ket{n, l} \) provided by (25) of \( \hat{H}_{\text{osc}} \) and \( \hat{H}_{\text{osc}} \), supplied by equation (7) and (18), respectively. Let us consider \( I_{\mathcal{H}_D}, I_{\mathcal{H}_D}^\dagger \), the identity operators on the subspaces \( \mathcal{H}_D, \mathcal{H}_D^\dagger \) of \( \mathcal{H}_D \) such that
\[ \sum_{n=0}^{\infty} \ket{\Psi_{nl}} \bra{\Psi_{nl}} = I_{\mathcal{H}_D}, \quad \sum_{l=0}^{\infty} \ket{\Psi_{nl}} \bra{\Psi_{nl}} = I_{\mathcal{H}_D^\dagger}. \]  
(26)

Since the Hamiltonians \( \hat{H}_1 \) and \( \hat{H}_2 \), given by (3) and (15), respectively, are formed by self-adjoint operators that act in infinite dimensional Hilbert spaces, from the equations (13) and (24), we set
\[ \mathcal{E}_{n, \alpha_k} = \frac{\hbar \omega_c}{2} (2n + 1) - \frac{\hbar \lambda}{m} \alpha_k - \frac{\lambda^2}{2m}, \quad n, k = 0, 1, 2, \ldots \]
\[ \mathcal{E}_{l, \alpha_k} = \frac{\hbar \omega_c}{2} (2l + 1) - \frac{\hbar \lambda}{m} \alpha_k - \frac{\lambda^2}{2m}, \quad l, k = 0, 1, 2, \ldots \]

(27)

that define families of discrete eigenvalues associated with the eigenvectors \{ |\psi_m\rangle \otimes |\alpha_k\rangle \}_{n,L,k=0}^{\infty}, forming an orthonormal basis of the separable Hilbert space \( \tilde{\mathcal{H}} = \mathcal{H}_0 \otimes \mathcal{H}_1 \), with \( \tilde{\mathcal{H}} \) spanned by the states \{ |\alpha_k\rangle \}_{k=0}^{\infty}, with

\[ \sum_{k=0}^{\infty} |\alpha_k\rangle \langle \alpha_k| = I_\mathcal{H}. \]

(28)

2.1. Coherent states of the shifted Hamiltonians

From (27), using

\[ \mathcal{E}_{n, \alpha_k}' = \frac{\hbar \omega_c}{2} (2n + 1) \quad \text{and} \quad \mathcal{E}_{\alpha_k} = \frac{\hbar \lambda}{m} \alpha_k + \frac{\lambda^2}{2m} \]

(29)

with \( \mathcal{E}_{\alpha_k} \) eigenvalues of the operator \( \hat{T} \) through the eigenvalue equation

\[ \left( \hat{T} - \frac{\lambda^2}{2m} I_\mathcal{H} \right) |\alpha_k\rangle = \frac{\hbar \lambda}{m} |\alpha_k\rangle |\alpha_k\rangle, \]

(30)

where the states \{ |\alpha_k\rangle \}_{k=0}^{\infty} satisfy (28), the eigenenergies of the shifted Hamiltonian \( \hat{H}_{n,m} = \hat{T}_1 - \frac{\hbar \omega_c}{2} - \frac{\lambda^2}{2m} I_\mathcal{H} \)

can be written as

\[ \mathcal{E}_{n, \alpha_k}' = \mathcal{E}_{n, \alpha_k} - \frac{\hbar \omega_c}{2} - \frac{\lambda^2}{2m} = \hbar \omega_c n - \frac{\hbar \lambda}{m} \alpha_k \]

(31)

with the following relation

\[ \left[ \hat{H}_{n,m} - \hat{T}_1 - \frac{\hbar \omega_c}{2} - \frac{\lambda^2}{2m} I_\mathcal{H} \right] |\psi_n\rangle \otimes |\alpha_k\rangle = \mathcal{E}_{n, \alpha_k}' |\psi_n\rangle \otimes |\alpha_k\rangle. \]

(32)

satisfied. Then, the condition \( \mathcal{E}_{n, \alpha_k}' \geq 0 \), for all \( (k, n) \in \mathbb{N} \times \mathbb{N}^* \), requires

\[ 0 \leq \alpha_k \leq \frac{m \omega_c}{\lambda}. \]

(33)

Let us suppose \( \alpha_k \) fixed, then set

\[ \mathcal{E}_{n, \alpha_k}' = \hbar \omega_c \left( n - \frac{\lambda}{m \omega_c} \alpha_k \right), \quad \kappa = \hbar \omega_c. \]

(34)

We define

\[ \rho(n) = \mathcal{E}_{1, \alpha_k}' \mathcal{E}_{2, \alpha_k}' \cdots \mathcal{E}_{n, \alpha_k}', \]

(35)

with the increasing order \( \mathcal{E}_{1, \alpha_k}' < \mathcal{E}_{2, \alpha_k}' < \cdots < \mathcal{E}_{n, \alpha_k}' \), such that

\[ \rho(n) = \prod_{q=1}^{n} \hbar \omega_c \left( q - \frac{\lambda}{m \omega_c} \alpha_k \right) = \kappa^n (\gamma)_n, \quad \gamma = 1 - \frac{\lambda}{m \omega_c} \alpha_k, \]

(36)

where \((\gamma)_n\) is the Pochhammer symbol, with \( (\gamma)_n = \gamma (\gamma + 1) (\gamma + 2) \cdots (\gamma + n - 1) \).

Let \( n \) be fixed and set

\[ \mathcal{E}_{n, \alpha_k}' = \frac{\hbar \lambda}{m} \left( \frac{m \omega_c}{\lambda} n - \alpha_k \right), \quad \xi = \frac{\hbar \lambda}{m} \]

(37)

allowing the definition of the quantity

\[ \rho(\alpha_k) = \mathcal{E}_{n, \alpha_1}' \mathcal{E}_{n, \alpha_2}' \cdots \mathcal{E}_{n, \alpha_k}' \]

(38)

with \( \mathcal{E}_{n, \alpha_1}' < \mathcal{E}_{n, \alpha_2}' < \cdots < \mathcal{E}_{n, \alpha_k}' \) and \( \epsilon_k = \epsilon_n \epsilon_{n-1} \cdots \epsilon_1 \), such that

\[ \rho(k) = \prod_{q=1}^{k} \frac{\hbar \lambda}{m} \left( \frac{m \omega_c}{\lambda} n - \alpha_q \right) = \epsilon_k \xi^{k-1}, \quad \epsilon_1 = \frac{m \omega_c}{\lambda} n - \alpha_1. \]

(39)
2.1.1. Coherent states with one degree of freedom

Let $l$ and $n$ be fixed. Define the coherent states for the Hamiltonian $\hat{H}_{\text{osc}} - \hat{\omega} - \frac{\chi}{2m} \hat{J}_y$, where $\alpha_k$ satisfies (33), with one degree of freedom. Denoting them with the fixed index $l$, they are given from (39), with $K \geq 0$ and $0 \leq \Delta < 2\pi$, by

$$|K, \delta; l\rangle = \mathcal{N}(K, \delta; l) \sum_{k=0}^{\infty} K^k e^{-i\Delta E_{\text{osc}}l} \sqrt{\rho(k)} |\Psi_k \rangle \otimes |\alpha_k \rangle. \quad (40)$$

With the normalization condition

$$\langle K, \delta; l|K, \delta; l\rangle = 1, \quad (41)$$

the normalization constant is determined such that we must have

$$\langle K, \delta; l|K, \delta; l\rangle = \mathcal{N}(K, \delta; l) \sum_{k=0}^{\infty} K^k e^{-i\Delta E_{\text{osc}}l} < \infty. \quad (42)$$

Thus if $\lim_{k \to \infty} \epsilon_k = \epsilon$, we need to restrict $K$ to $0 \leq K < L = \sqrt{\tau}$ for the convergence of the above series. In this case we have

$$\mathcal{N}(K, \delta; l) = \sum_{k=0}^{\infty} K^k e^{-i\Delta E_{\text{osc}}l}. \quad (43)$$

Proposition 2.1. For fixed $l$ and $n$, let us write the following measures

$$d\mu(K, \delta) = d\nu(K) d\delta(\delta) = \mathcal{N}(K, \delta; n) \varphi(K) dK \frac{d\delta}{2\pi}. \quad (44)$$

Then, on the Hilbert subspace $\mathcal{H}^D_{\text{osc}}$ of $\hat{\mathcal{H}}_0$, the coherent states satisfy the resolution of the identity given by

$$\int_0^L \int_0^{2\pi} |K, \delta; l\rangle \langle K, \delta; l| d\mu(K, \delta) = \hat{I}_{\mathcal{H}^D} \otimes \hat{I}_{\mathcal{H}^D}. \quad (45)$$

Proof. See in the Appendix. \qed

2.1.2. Coherent states with two degrees of freedom

Let $\alpha_k$ be fixed. We obtain, from equation (25), infinite component vector coherent states [9], with two degrees of freedom, where each component $n, l = 0, 1, 2,$... labelling these states, counts the infinite degeneracy of the energy level of the harmonic oscillators $\hat{H}_{\text{osc}}$ and $\hat{H}_{\text{osc}}^{\prime}$ shifted eigenvalues $E_{n}^{\prime} = \hbar \omega n$ and $E_{l}^{\prime} = \hbar \omega l$, respectively. Taking $J \geq 0$, $J' \geq 0$ and $0 \leq \theta, \theta' < 2\pi$, the coherent states are given by

$$|J, \theta; J', \theta'; l\rangle = \mathcal{N}(J, \alpha_k) e^{-i\Delta E_{\text{osc}}l} |J, \theta; J', \theta'; l\rangle \sum_{n=0}^{\infty} J^{n/2} e^{-i\Delta E_{\text{osc}}l} \sqrt{\rho(n)} |\Psi_n \rangle \otimes |\alpha_k \rangle, \quad (46)$$

where

$$\mathcal{N}(J, \alpha_k) = \sum_{n=0}^{\infty} \frac{J^n}{(\gamma)_{n}} = F_1 \left( 1; \gamma; \frac{J}{\kappa} \right). \quad (47)$$

Proposition 2.2. Provided the measures

$$d\gamma(J, \theta) = d\rho(J) d\mu(\theta) = \frac{1}{\kappa^{1/2}(\gamma)} F_1 \left( 1; \gamma; \frac{J}{\kappa} \right) e^{-i\Delta E_{\text{osc}}l} d\theta \frac{d\theta}{2\pi} \quad (48)$$

with fixed $\alpha_k$, the resolution of the identity can be expressed as follows:

$$\int_0^\infty \int_0^\infty \int_0^{2\pi} |J, \theta; J', \theta'; l\rangle \langle J, \theta; J', \theta'; l| d\gamma(J, \theta) = \hat{I}_{\mathcal{H}^D} \otimes \hat{I}_{\mathcal{H}^D}, \quad (49)$$

where $\hat{I}_{\mathcal{H}^D}$ is defined previously in (26) and $\mathcal{H}_{\text{osc}}$ the subspace of $\mathcal{H}^D$ being spanned by $\{|\alpha_k \rangle\}_{k=0}^{\infty}$ with

$$\sum_{k=0}^{\infty} |\alpha_k \rangle \langle \alpha_k | = \sum_{k=0}^{\infty} \hat{I}_{\mathcal{H}^D} = \hat{I}_{\mathcal{H}^D}. \quad (50)$$
Proof. See in the Appendix.

**Proposition 2.3.** The coherent states defined in (46) and (40) satisfy the temporal stability property given as follows:

\[ e^{-i\theta_1} [J_1, \theta; J_1', \theta'] = [J, \theta + t; J', \theta' + t], \]

\[ e^{-i\theta_2} [K, \delta; l] = [K, \delta + t; l], \]

with \( \dot{H}_1 = \hat{H}_{1\text{osc}} - \hat{T}_1 - \left( \frac{\hbar \omega}{2} - \frac{\hbar}{2m} \right) \hat{I}_\beta. \)

Proof. See in the Appendix.

**Remark 2.4.** Note that since in the equation (31), we have \( \mathcal{E}_{0,\alpha_k}^t = 0, \) the coherent states \((40)\) and \((46)\) cannot satisfy the action identity. In this case, we phrase the resulting coherent states as 'temporally stable coherent states'. Indeed, the eigenenergy of the shifted Hamiltonian \( H_{1\text{osc}} - T_1 - \left( \frac{\hbar \omega}{2} - \frac{\hbar}{2m} \right) \hat{I}_\beta, \) see (13), can be written as in (31). Then using the definition of \( \lambda \) see after (5), and from (31) and (33) together, we obtain for \( n = 0, \) the corresponding shifted eigenvalue as

\[ \mathcal{E}^t_{0,\alpha_k} = -\frac{\hbar \lambda}{m} \alpha_k < 0 \text{ since } \lambda = \frac{mcE}{B} > 0 \text{ and } \alpha_k \geq 0. \]

From the fact that \( \mathcal{E}^t_{0,\alpha_k} = 0, \) let us shift the spectrum of the Hamiltonian \( H_{1\text{osc}} - T_1 - \left( \frac{\hbar \omega}{2} - \frac{\hbar}{2m} \right) \hat{I}_\beta \) backward by \( \mathcal{E}^t_{0,\alpha_k}, \) i.e, we set

\[ \epsilon_{n,\alpha_k} = \mathcal{E}^t_{n,\alpha_k} - \mathcal{E}^t_{0,\alpha_k} = \left( nh\omega - \frac{\hbar \lambda}{m} \alpha_k \right) + \frac{\hbar \lambda}{m} \alpha_k \]

implying for \( n = 0 \)

\[ \epsilon_{0,\alpha_k} = \mathcal{E}^t_{0,\alpha_k} - \mathcal{E}^t_{0,\alpha_k} = 0. \]

Then, dealing with the shifted spectrum \( \{ \epsilon_{n,\alpha_k} \}_{n,\alpha_k = 0}^\infty = \{ n h\omega \}_{n=0}^\infty \) corresponding to the spectrum of the shifted quantum harmonic oscillator Hamiltonian \( \left( \hat{H}_{1\text{osc}} - \frac{\hbar \omega}{2} \hat{I}_\beta \right), \) the constructed coherent states given in \( (40) \) and \( (46) \) correspond to the Gazeau-Klauder coherent states of the shifted quantum harmonic oscillator coinciding with the standard coherent standard states, with \( z = \hat{J}^{1/2} e^{-\hat{w}}, \dot{z} = \hat{J}^{1/2} e^{\hat{w}}. \) Thereby, the action identity property will be satisfied for both coherent states \((40)\) and \((46)\).

**3. Coherent states for unshifted Hamiltonians \( \hat{H}_1 \) and \( \hat{H}_2 \) defined through multiple summations**

In this paragraph, two types of temporally stable coherent states are constructed in line with the general scheme developed in [4], (see also [5]). The first type is defined as a tensor product of two classes of coherent states with one and two degrees of freedom by setting \( \rho_1(n), \mathcal{E}_n, \mathcal{E}_\alpha, \) and \( \rho(k) \) as independent quantities. The second type, which cannot be considered as a tensor product of vectors with one and two degrees of freedom, is defined by letting that one sum depends on the other through the same quantities.

1- When the summations are independent

Let us set \( \mathcal{E}_{n,\alpha_k} = \mathcal{E}_n + \mathcal{E}_{\alpha_k}, \)

\[ \mathcal{E}_n = n h\omega_n, \quad \mathcal{E}_{\alpha_k} = -\frac{\hbar \lambda}{m} \alpha_k = \mathcal{E}_k. \]

Then, the conditions of positivity required for the eigenvalues \( \mathcal{E}_n^t \) imposes \( \alpha_k \leq 0. \) Setting

\[ \rho_1(n) = \mathcal{E}_1 \mathcal{E}_2 \cdots \mathcal{E}_n, \rho(k) = \mathcal{E}_1^t \mathcal{E}_2^t \cdots \mathcal{E}_k^t \]

and \( \epsilon_k = \rho_2(k)/\rho_2(k - 1), \) where \( \epsilon_k = -\alpha_k, \) for \( k = 1, 2, 3, \ldots \) leads to

\[ \rho_2(k) = \epsilon_k \epsilon_{k-1} \cdots \epsilon_1 = \epsilon_k \text{ with, by convention, } \epsilon_0 = 1. \]

Given the relations \( \mathcal{E}_1^t < \mathcal{E}_2^t < \cdots < \mathcal{E}_n^t \) and \( \mathcal{E}_1^t < \mathcal{E}_2^t < \cdots < \mathcal{E}_k^t, \) we can rewrite

\[ \rho_1(n) = \prod_{k=1}^n h\omega_k^\kappa = n! \kappa^n, \quad \kappa = h\omega, \]

\[ \tilde{\rho}(k) = \prod_{q=1}^k \frac{\hbar \lambda}{m} \epsilon_q = v^k \rho_2(k) = \epsilon_k v^k, \quad v = \frac{\hbar \lambda}{m}. \]

Under these considerations, the coherent states for the Hamiltonians \( \hat{H}_{1\text{osc}} - \hat{T}_1 \) and \( \hat{H}_{1\text{osc}} - \hat{T}_2, \) when taking into account the degeneracies of the Landau levels as before, are defined with three degrees of freedom as a tensor product of two coherent states defined with one and two degrees of freedom, respectively, as follows:
\[ |J, \theta; J', \theta'; K, \delta; l \rangle = [\mathcal{N}(J)\mathcal{N}(J')]^{-1/2}\mathcal{N}(K)^{-1/2}J^{n/2}\epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{\rho_{1}(l)}}\frac{e^{-i\epsilon_{\alpha}\theta}}{\sqrt{\rho_{1}(l)}} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle \]
\[ |J, \theta; J', \theta'; K, \delta; n \rangle = [\mathcal{N}(J)\mathcal{N}(J')]^{-1/2}\mathcal{N}(K)^{-1/2}J^{n/2}\epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} \sum_{l=0}^{\infty} \frac{J^{n/2}}{\sqrt{\rho_{1}(l)}}\frac{e^{-i\epsilon_{\alpha}\theta}}{\sqrt{\rho_{1}(l)}} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle \].

**Proposition 3.1.** The normalization requirements

\[ \sum_{l=0}^{\infty} \sum_{n=0}^{\infty} \frac{J^{n}}{\rho_{1}(n)} = 1 \]

are such that we must have the relation (43). Then, we obtain

\[ \mathcal{N}(J) = \sum_{k=0}^{\infty} K^{k} \epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle. \]

\[ \mathcal{N}(K) = \sum_{k=0}^{\infty} K^{k} \epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle. \]

**Proposition 3.2.** The coherent states (54), satisfy, on the Hilbert space \( \tilde{H} \otimes \tilde{H} \), the following resolutions of the identity:

\[ \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} \int_{0}^{\infty} |J, \theta; J', \theta'; K, \delta; l \rangle |J, \theta; J', \theta'; K, \delta; l \rangle \rho_{1}(J, \theta) \rho_{1}(J', \theta') d\mu(K, \delta) = I_{\tilde{H}^{2}} \otimes I_{\tilde{H}^{2}}. \]

**Proof.** See in the Appendix.

**Proposition 3.3.** The coherent states (54) are temporally stable, i.e.,

\[ e^{-iH_{\text{osc}}t} |J, \theta; J', \theta'; K, \delta; l \rangle = |J, \theta + t; J', \theta'; K, \delta + t; l \rangle. \]

**Proof.** See that of proposition 4.2

2- When the summations depend one on the other

For fixed \( \alpha_{k} \), with \( \epsilon_{n,\alpha}^\prime \) given in (34), let us set

\[ \rho(n, \alpha_{k}) = \epsilon_{n,\alpha}^\prime \epsilon_{n,\alpha} \epsilon_{n,\alpha} \ldots \epsilon_{n,\alpha}. \]

From (36), one has

\[ \rho(n, \alpha_{k}) = \kappa^{n}(\gamma)_{n}. \]

The definition (53) gives

\[ \rho_{1}(l) = l!n^{n}. \]

Under the condition \( \alpha_{k} \leq 0 \), one can define the following coherent states:

\[ |J, \theta; J', \theta'; K, \delta; l \rangle = \mathcal{N}(J)^{-1/2}\mathcal{N}(J')^{-1/2}J^{n/2}\epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{\rho_{1}(l)}}\frac{e^{-i\epsilon_{\alpha}\theta}}{\sqrt{\rho_{1}(l)}} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle. \]

In order to obtain the normalization constant, let us compute the norm of the coherent states

\[ |J, \theta; J', \theta'; K, \delta; l \rangle, \] with the conditions given in (55) leading to

\[ |J, \theta; J', \theta'; K, \delta; l \rangle = \mathcal{N}(J)^{-1/2}\mathcal{N}(J')^{-1/2}J^{n/2}\epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} \sum_{k=0}^{\infty} \frac{J^{n/2}}{\sqrt{\rho_{1}(l)}}\frac{e^{-i\epsilon_{\alpha}\theta}}{\sqrt{\rho_{1}(l)}} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle. \]

\[ \mathcal{N}(J) = \sum_{k=0}^{\infty} K^{k} \epsilon^{l/2}e^{-i\epsilon_{\alpha}\theta} |\Psi_{\alpha n} \rangle \otimes |\alpha_{k} \rangle. \]
if, as done in (43) and (56),
\[ \mathcal{N}(J, \alpha_k) = \mathcal{N}' \left( 1; \gamma; \frac{J}{\kappa} \right) \geq 1, \quad \mathcal{N}(J) = e^{-p/J} \geq 1, \] (64)
and
\[ \mathcal{N}(K, J) = \sum_{n=0}^{\infty} \frac{K^k}{\rho(k)\mathcal{N}(J)'\mathcal{N}(J, \alpha_k)}. \] (65)

Then, we get
\[ \mathcal{N}(K, J) = \sum_{k=0}^{\infty} a_k\xi_k e^{-p/J} \mathcal{N}' \left( 1; \gamma; \frac{J}{\kappa} \right) \leq \sum_{k=0}^{\infty} K^k \]
which converges for all \( 0 \leq K \leq L = \sqrt{\varepsilon} \).

**Proposition 3.4.** The coherent states (62) satisfy, on \( \tilde{\mathcal{H}}_D \otimes \tilde{\mathcal{H}}_\theta \), the resolutions of the identity given by
\[
\int_0^L \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(J, \theta; J', \theta'; K, \delta; l) |J, \theta; J', \theta'; K, \delta; l| \, df(J, \theta) \times d\eta(J', \theta') d\mu(K, n; \delta) = I_{\tilde{\mathcal{H}}_0} \otimes I_{\tilde{\mathcal{H}}_\theta}.
\] (67)

**Proof.** See in the Appendix. \( \square \)

Note that, as previously mentioned, the coherent states (62) are temporally stable.

### 4. Coherent states related to the Hamiltonian \( \hat{H}_{\text{osc}} - \hat{H}_{\text{osc}} \)

- When \( \alpha_k \) is fixed: the coherent states are defined on \( \mathcal{H}_D \otimes \tilde{\mathcal{H}}_\theta \), in an analogous way, by using the 'bi-coherent states' (BCS) [9] as follows:

\[
|J, \theta; J', \theta'\rangle = |J, \theta; J', \theta'\rangle^{\text{BCS}} \otimes |\alpha_k\rangle
= \frac{1}{\mathcal{N}(J, \alpha_k)\mathcal{N}(J')\mathcal{N}(J)\mathcal{N}(J')} \sum_{n,l=0}^{\infty} \frac{J^{n/2}J'^{l/2}e^{-i(n\theta-l\theta')}}{\sqrt{n!l!}} |\Psi_{nl}\rangle \otimes |\alpha_k\rangle
\] (68)
or, by using the complex labels, as
\[
|z, \tilde{z}\rangle = |z, \tilde{z}\rangle^{\text{BCS}} \otimes |\alpha_k\rangle
= e^{-|z|^2-|\tilde{z}|^2} \sum_{n,l=0}^{\infty} \frac{z^n\tilde{z}^l}{\sqrt{n!l!}} |\Psi_{nl}\rangle \otimes |\alpha_k\rangle.
\] (69)

They correspond to the multidimensional coherent states [4] of the Hamiltonian \( \hat{H}_{\text{osc}} - \hat{H}_{\text{osc}} \).

The normalization condition is given by
\[
\langle J, \theta; J', \theta'| J, \theta; J', \theta' \rangle = 1.
\] (70)

**Proposition 4.1.** They satisfy, on the separable Hilbert space \( \tilde{\mathcal{H}}_D \), the following resolution of the identity:
\[
\int_0^L \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(J, \theta; J', \theta') |J, \theta; J', \theta'\rangle \langle J, \theta; J', \theta'| d\mu(\theta) d\mu(\theta')
\times N(J, \mathcal{N}(J') d\nu(J) d\nu(J') = I_{\mathcal{H}_D}.
\] (71)

- When \( \alpha_k \) is not fixed: the coherent states are denoted by \( |J, \theta; J', \theta'; \alpha_k\rangle \) or \( |z, \tilde{z}; \alpha_k\rangle \) and given by the same equations (68) and (69).

Here, the normalization condition is given by
\[
\sum_{k=0}^{\infty} \langle J, \theta; J', \theta'; \alpha_k | J, \theta; J', \theta'; \alpha_k \rangle = 1.
\] (72)
Proposition 4.2. They satisfy, on the separable Hilbert space \( \hat{\mathcal{H}} \), the following resolution of the identity:
\[
\int_0^\infty \int_0^\infty \int_0^{2\pi} |J, \theta; J', \theta'; \alpha_k\rangle \langle J, \theta; J', \theta'; \alpha_k| d\mu(\theta) d\mu(\theta') \times \mathcal{N}(J)\mathcal{N}(J') d\nu(J) d\nu(J') = \mathbf{1}_{\mathcal{H}_0} \otimes \mathbf{1}_{\hat{\mathcal{H}}}. \tag{73}
\]

Proof. See in the Appendix.

Proposition 4.3. The coherent states (68) also satisfy the properties of temporal stability and action identity as stated in [9]. In the situation of the coherent states \(|J, \theta; J', \theta'; \alpha_k\rangle\), these properties are given as below:
\[
e^{-i\hbar t(J, \theta; J', \theta'; \alpha_k)} = |J, \theta + \omega t; J', \theta' + \omega t; \alpha_k\rangle \tag{74}
\]
\[
(J, \theta; J', \theta'; \alpha_k)\hat{H}|J, \theta; J', \theta'; \alpha_k\rangle = \omega(J - J') \tag{75}
\]
with \( \hat{H} = \hat{H}_{\text{osc}} - \hat{H}_{\text{osc}} \).

Proof. See in the Appendix.

5. Outlook

The behaviour of an electron moving in a plane in an electromagnetic field background, arising in the quantum Hall effect, has been studied with the related Hamiltonian spectra having both discrete and continuous parts provided. Also, a Hamiltonian in the case of an electric field depending simultaneously on both x and y directions has been discussed, with its corresponding eigenvalues and eigenvectors provided. The eigenfunctions have been obtained as a countable set realizing an infinite dimensional appropriate Hilbert space. Various coherent states have been constructed by considering shifted and unshifted spectra, respectively. Two kinds of coherent states classes have been obtained. The first kind, with one degree of freedom, is achieved by fixing each index counting the energy levels. The second kind is realized by taking tensor product of two classes of coherent states with one and two degrees of freedom.

The discussion can be extended for the case of the potential \( V = E_1x + E_2y \). Here, the uniform electric field is defined as \( E_i = (E_{i1}, E_{i2}, 0) \) with the scalar potential \( \Phi(x, y) = E_1x + E_2y = E \cdot r \), and the magnetic field given by \( A = \left( -\frac{B}{2}y, \frac{B}{2}x \right) \). Then, the Hamiltonian writes as
\[
H(x, y, p_x, p_y) = \frac{1}{2M} \left( p_x - \frac{eB}{2c}y \right)^2 + \left( p_y + \frac{eB}{2c}x \right)^2 - eE_1x - eE_2y. \tag{76}
\]

Let us introduce the following pairs of annihilation and creation operators defined by
\[
\hat{b}^\dagger = -2iP_x + \frac{eB}{2c}Z - 2\lambda_0, \quad \hat{b} = 2iP_x + \frac{eB}{2c}Z - 2\lambda_1, \tag{77}
\]
\[
d = 2iP_x - \frac{eB}{2c}Z, \quad d^\dagger = -2iP_x - \frac{eB}{2c}Z, \tag{78}
\]
and
\[
\hat{l} = 2P_x + i\frac{eB}{2c}Z + 2\lambda_2, \quad \hat{l}^\dagger = 2P_x - i\frac{eB}{2c}Z + 2\lambda_3, \tag{79}
\]
\[
\hat{k}^\dagger = 2P_x - i\frac{eB}{2c}Z, \quad \hat{k} = 2P_x + i\frac{eB}{2c}Z, \tag{80}
\]
where \( \lambda_0 = \frac{Me}{\hbar} \) and \( \lambda_2 = \frac{Me}{\hbar} \). They satisfy the following commutation relations
\[
[b, b^\dagger] = 2M\hbar\omega, \quad [d, d^\dagger] = 2M\hbar\omega, \quad [b, d] = 0, \quad [b^\dagger, d^\dagger] = 0, \quad [b, d^\dagger] = 0, \quad [b^\dagger, d] = 0, \tag{81}
\]
\[
[\hat{l}, \hat{l}^\dagger] = 2M\hbar\omega, \quad [\hat{k}^\dagger, \hat{k}] = 2M\hbar\omega, \quad [\hat{l}, \hat{l}^\dagger] = 0, \quad [\hat{k}, \hat{l}^\dagger] = 0, \quad [\hat{k}^\dagger, \hat{l}] = 0, \tag{82}
\]
and
\[
[b^\dagger, \hat{k}] = 0 = [b, \hat{k}^\dagger], \quad [\hat{l}^\dagger, d] = 0 = [\hat{l}, d^\dagger]. \tag{83}
\]

The operator Hamiltonian \( \hat{H} \) is delivered as follows:
\[
\hat{H} = \frac{1}{4M}(\hat{l}^\dagger \hat{l} + \hat{l} \hat{l}^\dagger) + \frac{\hbar^2}{M} + \frac{1}{2M}(\xi^\dagger \xi + \xi \xi^\dagger) + \frac{\lambda_1}{2M}(d^\dagger + d) + \frac{\lambda_2}{2M}(\hat{k}^\dagger + \hat{k}) - \frac{\lambda_3}{2M}. \tag{84}
\]
where we use the following relations
\[ b^i = -i(\hat{\mathcal{L}}^1 + 2\xi), \quad b = i(\hat{\mathcal{L}} + 2\zeta), \quad \xi = i\lambda_1 - \lambda_2, \quad a = i\hat{k}, \quad d^i = -i\hat{k}^i. \]  

The eigenvalue equation \( \hat{H}\Psi = E\Psi, \Psi(r, \theta) = \varphi(r)e^{i\theta}, \) provides the radial equation
\[ \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{l^2}{r^2} \right) \varphi(r) - (2Br + 2C r^2 - 2E) \varphi(r) = 0 \]  

where
\[ B = -\frac{M_0\omega}{\hbar^2} f_i(\theta), \quad C = \left( \frac{M_0\omega}{2\sqrt{2} \hbar} \right)^2, \quad E = \frac{M}{\hbar^2} \left( E - \frac{\hbar}{2} \omega_i \right) \]

with \( f_i(\theta) = \lambda_1 \cos \theta + \lambda_2 \sin \theta. \) The corresponding radial eigenfunctions are obtained in terms of Heun functions as follows
\[ \varphi(r) = A_1 r^{l+1/2} e^{i\lambda(r)} \text{HeunB}\{l_i, l_b, l_a, 0, 0\} + A_2 r^{l-1/2} e^{i\lambda(r)} \text{HeunB}\{-l_i, l_b, l_a, 0, 0\} \]

with
\[ l_i = 2l, \quad l_b(r) = r \sqrt{\frac{2}{C}} \left( \sqrt{C} r + \frac{B}{\sqrt{C}} \right), \quad l_a = iB \left( \frac{2}{C} \right)^{1/2}, \quad l_d(r) = -4EC + B^2 / \sqrt{2}C^2, \quad l_s(r) = i(2C)^{1/2}, \]

where \( A_1 \) and \( A_2 \) are constants. Thus, the solutions of the Schrödinger equation \( \hat{H}\Psi = E\Psi \) are obtained as the product of (88) by \( e^{i\theta}. \)

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

Appendix

Proof of proposition 2.1. From the definition (40), we have
\[ |K, \delta; l\rangle \langle K, \delta; l| = |\Psi_0\rangle \langle \Psi_0| \otimes N(K, \delta; l)^{-1} \sum_{k,q=0}^\infty \frac{K_{k,q} e^{i\eta_{k,q}} K_{l,q}}{\sqrt{\hat{\rho}(k) \rho(q)}} |\alpha_k\rangle \langle \alpha_q|, \]

which allows to write
\[ \int_0^L \int_0^{2\pi} |K, \delta; l\rangle \langle K, \delta; l| d\mu(K, \delta) = |\Psi_0\rangle \langle \Psi_0| \otimes \int_0^L \sum_{k,q=0}^\infty \frac{K_{k,q}}{\sqrt{\hat{\rho}(k) \rho(q)}} \int_0^{2\pi} e^{i(\eta_{k,q} - \eta_{l,q})} \frac{d\theta}{2\pi} \varpi(K) dK |\alpha_k\rangle \langle \alpha_q|. \]

Assuming that the density \( \varpi(K) \) satisfies the relation
\[ \int_0^L K^l \varpi(K) dK = \epsilon_{l\xi} \xi^k, \]

we get
\[ \int_0^L \int_0^{2\pi} |K, \delta; l\rangle \langle K, \delta; l| d\mu(K, \delta) = |\Psi_0\rangle \langle \Psi_0| \otimes \sum_{k=0}^\infty |\alpha_k\rangle \langle \alpha_q| = I_{\Psi_0} \otimes I_\delta. \]
Proof of proposition 2.2. Using the definition (46) leads to

\[
\int_0^{\infty} \int_0^{\infty} \int_0^{2\pi} \int_0^{2\pi} |l, \theta; J', \theta'; l \rangle \langle J, \theta; J', \theta'; l | d\eta(J, \theta) d\eta(J', \theta')
\]

\[
= \int_0^{\infty} \int_0^{\infty} \frac{J^l}{\rho(l)} e^{-l/\kappa} \int_{\Gamma(\gamma)} \Gamma(-l) \frac{J^{\frac{1}{2}}}{\sqrt{\rho(n)\rho(q)}} e^{-l/\kappa} \int_0^{2\pi} e^{-iE_{\alpha}(\theta) \theta} d\theta \frac{d\theta'}{2\pi} dJ' d|\Psi_{\alpha}\rangle \otimes |\alpha_k\rangle \langle \alpha_k|
\]

(94)

The following relations

\[
\int_0^{\infty} J^{l/2} e^{-l/\kappa} dJ = \kappa^{l/2} \Gamma(l + \gamma), \quad \int_0^{\infty} J^{l/2} e^{-l/\kappa} dJ' = \kappa^{l/2} \Gamma(l + \gamma),
\]

are satisfied, where we have used the inverse Mellin transform [11, 12]

\[
\int_0^{\infty} e^{-at} d\tau = a^{-\Gamma}(s),
\]

with \( s = n + \gamma \) and \( a = \frac{1}{\kappa} \). Thereby

\[
\int_0^{\infty} \int_0^{\infty} \int_0^{2\pi} \int_0^{2\pi} |l, \theta; J', \theta'; l \rangle \langle J, \theta; J', \theta'; l | d\eta(J, \theta) d\eta(J', \theta')
\]

\[
= \sum_{n=0}^{\infty} |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}| \otimes |\alpha_k\rangle \langle \alpha_k| = I_{\alpha} \otimes I_{\delta}.
\]

(97)

Proof of proposition 2.3. From the definition (46), it follows that

\[
e^{-i\frac{J^l}{\kappa} |l, \theta; J', \theta'; l \rangle = \mathcal{N} |l, \theta; \alpha_k\rangle^{-1/2} \mathcal{N}^*(J', \theta'; \alpha_k)^{-1/2} J^{l/2} e^{iE_{\alpha}(\theta)}
\]

\[
\times \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-iE_{\alpha}(\theta)}}{\sqrt{\rho(n)\rho(l)}} e^{-i\frac{J^l}{\kappa} |\Psi_{\alpha}\rangle \otimes |\alpha_k\rangle
\]

\[
= \mathcal{N} |l, \theta; \alpha_k\rangle^{-1/2} \mathcal{N}^*(J', \theta'; \alpha_k)^{-1/2} J^{l/2} e^{iE_{\alpha}(\theta)}
\]

\[
\times \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-iE_{\alpha}(\theta)}}{\sqrt{\rho(n)\rho(l)}} |\Psi_{\alpha}\rangle \otimes |\alpha_k\rangle
\]

\[
= |l, \theta + t; J', \theta'; l \rangle.
\]

(98)

Proof of proposition 3.2. Starting from (54), with the measures given by

\[
d\eta(J, \theta) = e^{-i/\kappa} e^{-l/\kappa} \frac{d\theta}{2\pi}, \quad d\eta(J', \theta') = e^{-i/\kappa} e^{-l/\kappa} \frac{d\theta'}{2\pi}
\]

and (44), we get

\[
\int_0^{\infty} \int_0^{\infty} \int_0^{L} \int_0^{2\pi} \int_0^{2\pi} |l, \theta; J', \theta'; K, \delta; l \rangle \langle J, \theta; J', \theta'; K, \delta; l | d\eta(J, \theta)
\]

\[
\times d\eta(J', \theta') d\mu(K, \delta)
\]

\[
= \int_0^{\infty} \int_0^{\infty} \frac{J^l}{\rho(l)} e^{l/\kappa} \sum_{n,m=0}^{\infty} \frac{J^{n/2}}{\sqrt{\rho(n)\rho(m)}} e^{-l/\kappa} \int_0^{2\pi} \int_0^{2\pi} e^{-iE_{\alpha}(\theta) \theta} d\theta \frac{d\theta'}{2\pi} e^{iE_{\alpha}(\theta) \theta'}
\]

\[
\times d\eta(J' d|\Psi_{\alpha}\rangle \otimes \sum_{k=0}^{\infty} \frac{1}{\sqrt{\rho(k)\rho(q)}} e^{iE_{\alpha}(\theta) \theta} dK \int_0^{\infty} \int_0^{L} \int_0^{2\pi} e^{-iE_{\alpha}(\theta) \theta} d\theta \frac{d\theta'}{2\pi} dK |\alpha_k\rangle \langle \alpha_k|
\]

\[
= \sum_{n=0}^{\infty} \int_0^{\infty} \frac{J^l}{l!} dJ' \int_0^{\infty} \frac{J^m}{m!} dJ d|\Psi_{\alpha}\rangle \otimes \sum_{k=0}^{\infty} \frac{1}{\sqrt{\rho(k)\rho(q)}} e^{iE_{\alpha}(\theta) \theta} dK |\alpha_k\rangle \langle \alpha_k|
\]

(100)

Using the definition of the Gamma function, we obtain, on the Hilbert space \( \mathcal{H}_0 \otimes \mathcal{H} \), the following resolution of the identity:

\[
\int_0^{\infty} \int_0^{\infty} \int_0^{L} \int_0^{2\pi} \int_0^{2\pi} |l, \theta; J', \theta'; K, \delta; l \rangle \langle J, \theta; J', \theta'; K, \delta; l | d\eta(J, \theta)
\]

\[
\times d\eta(J', \theta') d\mu(K, \delta) = \sum_{n=0}^{\infty} |\Psi_{\alpha}\rangle \langle \Psi_{\alpha}| \otimes \sum_{k=0}^{\infty} |\alpha_k\rangle \langle \alpha_k| = I_{\alpha} \otimes I_{\delta},
\]

(101)

providing that the relation (57) is satisfied.

\[\square\]
Proof of proposition 3.4. From (62), and the measures given by \(d\eta(J', \theta') = dv(J')d\mu(\theta')\) as in (48), and \(d\sigma(J, \alpha_k, \delta) = \varrho_1(J, \alpha_k)dJd\mu(\theta)\), \(d\mu(K, \delta) = \varrho_2(K)dKd\mu(\delta)\), we have

\[
\int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{\infty} \int_0^{2\pi} \int_0^{2\pi} |J, \theta; J', \theta' ; K, \delta; l| |J, \theta; J', \theta' ; K, \delta; l| \ d\sigma(J, \alpha_k, \theta) \\
\times d\eta(J', \theta') \ d\mu(\delta) = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty J_n(J, \alpha_k) \int_0^{2\pi} \int_0^\infty e^{-i(\varepsilon_{\alpha_k} - \varepsilon_{\alpha_k})} \\
\times \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} d\mu(\delta) |\Psi_n\rangle \langle \Psi_n| \otimes \sum_{k=0}^\infty \frac{K^k}{\hat{p}(k)} N(J, K) \int_0^{2\pi} e^{i(\varepsilon_{\alpha_k} - \varepsilon_{\alpha_k})} \\
\times \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi} d\mu(\delta) = \sum_{n=0}^\infty |\Psi_n\rangle \langle \Psi_n| \otimes \sum_{k=0}^\infty |\alpha_k\rangle \langle \alpha_k| = I_{\Psi_0} \otimes I_{\eta_0},
\]

(102)

Thereby we obtain the following relations,

\[
\int_0^\infty \int_0^\infty \int_0^{L} \int_0^{2\pi} \int_0^{2\pi} \int_0^{2\pi} |J, \theta; J', \theta' ; K, \delta; l| |J, \theta; J', \theta' ; K, \delta; l| \ d\sigma(J, \alpha_k, \theta) \\
\times d\eta(J', \theta') \ d\mu(\delta) = \sum_{n=0}^\infty |\Psi_n\rangle \langle \Psi_n| \otimes \sum_{k=0}^\infty |\alpha_k\rangle \langle \alpha_k| = I_{\Psi_0} \otimes I_{\eta_0},
\]

(103)

if there exist the densities \(\varrho_1(J, \alpha_k)\) and \(\varrho_2(K)\) such that

\[
\int_0^\infty \frac{J_n}{\rho_1(n, \alpha_k)N(J, \alpha_k)} \varrho_1(J, \alpha_k) \ dJ \int_0^{L} \frac{K^k}{\hat{p}(k)N(K, J)} \varrho_2(K) \ dK = 1,
\]

(104)

where

\[
\varrho_1(J, \alpha_k) = N(J, \alpha_k) \varpi_1(J, \alpha_k), \quad \varrho_2(K) = N(K, J) \varpi_2(K)
\]

(105)

supplying that the measures \(\varpi_1(J, \alpha_k)\) and \(\varpi_2(K)\) satisfy

\[
\int_0^\infty \frac{J_n}{\rho_1(n, \alpha_k)} \varpi_1(J, \alpha_k) \ dJ \int_0^{L} \frac{K^k}{\hat{p}(k)} \varpi_2(K) \ dK = 1.
\]

(106)

Proof of proposition 4.2. From the definition of the coherent states (68), we get

\[
|J, \theta; J', \theta' ; \alpha_k \rangle = \frac{1}{\sqrt{N(J)N(J')}} \sum_{n,l=0}^\infty \sum_{q,m=0}^\infty J_n^{(l-q)} f_{n,l}^{(q,m)} e^{i(l-q)\theta + (l' - m)\theta'} \\
|\Psi_n\rangle \langle \Psi_{qm}| \otimes |\alpha_k\rangle \langle \alpha_k|,
\]

(107)

such that

\[
\int_0^{2\pi} \int_0^{2\pi} |J, \theta; J', \theta' ; \alpha_k \rangle \langle J, \theta; J', \theta' ; \alpha_k| d\mu(\theta) d\mu(\theta')N(J)N(J') \\
= \sum_{n,l=0}^\infty \sum_{q,m=0}^\infty \frac{1}{\sqrt{n!!q!!m!!}} \epsilon_{qm} \epsilon_{lm} f_{n,l}^{(q,m)} \langle \Psi_n\rangle \langle \Psi_{qm}| \otimes |\alpha_k\rangle \langle \alpha_k|
\]

(108)

Thus the resolution of the identity is obtained as

\[
\int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} |J, \theta; J', \theta' ; \alpha_k \rangle \langle J, \theta; J', \theta' ; \alpha_k| d\mu(\theta) d\mu(\theta')N(J)N(J') \\
\times d\eta(J', \theta') = \sum_{n,l=0}^\infty |\Psi_n\rangle \langle \Psi_n| \otimes |\alpha_k\rangle \langle \alpha_k| = I_{\eta_0} \otimes I_{\eta_0},
\]

(109)

Proof of proposition 4.3. Let us set

\[
\hat{H} = \hat{H}_{\text{osc}} - \hat{H}_{\text{osc}} = \sum_{n,l=0}^\infty \omega_n (n - l) |\Psi_n\rangle \langle \Psi_n| \otimes |\alpha_k\rangle \langle \alpha_k|,
\]

(110)
with $|\Psi_{nl}; \alpha_k\rangle := |\Psi_{nl}\rangle \otimes |\alpha_k\rangle$. Then, we have

$$
\hat{H}|J, \theta; J', \theta'; \alpha_k\rangle = \frac{1}{[N(J)N'(J)]^{1/2}} \sum_{n_{ij}=0}^{\infty} \sum_{m_{ij}=0}^{\infty} \omega_c(n - l)f^{n/2}f^{m/2}e^{-i(q\theta - m\theta')} \sqrt{q!m!} \times \langle \Psi_{nl}|\Psi_{qm}\rangle |\Psi_{nl}\rangle \otimes \langle \alpha_k|\alpha_k\rangle |\alpha_k\rangle
$$

$$
= \frac{1}{[N(J)N'(J)]^{1/2}} \sum_{n_{ij}=0}^{\infty} \sum_{m_{ij}=0}^{\infty} \omega_c(n - l)f^{n/2}f^{m/2}e^{-i(q\theta - m\theta')} \sqrt{q!m!} |\Psi_{nl}\rangle \otimes |\alpha_k\rangle.
$$

Thus,

$$
\langle J, \theta; J', \theta'; \alpha_k|\hat{H}|J, \theta; J', \theta'; \alpha_k\rangle
$$

$$
= \frac{1}{[N(J)N'(J)]^{1/2}} \sum_{n_{ij}=0}^{\infty} \sum_{m_{ij}=0}^{\infty} \omega_c(n - l)f^{n/2}f^{m/2}e^{-i(q\theta - m\theta')} \sqrt{q!m!} \delta_{q\theta \delta_{m\theta}}
$$

$$
= \omega_c \left[ \frac{1}{N(J)} \sum_{n=0}^{\infty} \frac{f^n}{n!} - J' \frac{1}{N'(J')} \sum_{n=0}^{\infty} \frac{f^n}{n!} \right] = \omega_c(J - J').
$$

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