The Exact Superconformal R-Symmetry Extremizes $Z$

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Abstract

The three sphere partition function, $Z$, of three dimensional theories with four supercharges and an $R$-symmetry is computed using localization, resulting in a matrix integral over the Cartan of the gauge group. There is a family of couplings to the curved background, parameterized by a choice of $R$-charge, such that supersymmetry is preserved; $Z$ is a function of those parameters. The magnitude of the result is shown to be extremized for the superconformal $R$-charge of the infrared conformal field theory, in the absence of mixing of the $R$-symmetry with accidental symmetries. This exactly determines the IR superconformal $R$-charge.


1 Introduction

This paper contains evidence for two results. The first is that the $S^3$ partition function of a theory with four supercharges and an $R$-symmetry can be computed using localization. The partition function depends on supersymmetry preserving couplings to curvature, which are parameterized by a choice of $R$-charge. These differ by abelian non-$R$ flavor symmetries. The second result is that the magnitude of that partition function is extremized for the superconformal $R$-charge of the 3d infrared conformal field theory.

Under the assumption that the $R$-symmetry does not mix with accidental flavor symmetries, the partition function on the sphere can be computed in a weakly coupled UV theory. The results of this paper then lead to explicit exact formulas for the superconformal $R$-charge in the IR.

There has recently been considerable interest in the partition function of supersymmetric three dimensional theories on $S^3$. This quantity is extensive in the number of degrees of freedom in the system, yet it can be calculated exactly using supersymmetry. Moreover the resulting answers can be written in closed form, unlike the perhaps richer invariants given by the superconformal index on $S^2 \times S^1$ studied in [23].

The technique of localization of supersymmetric partition functions involves the addition of a $Q$-exact operator to the action, which does not affect the path integral, but renders the 1-loop approximation exact. This was applied to the infrared-finite problem of gauge theories on spheres by Pestun [28], who obtained exact results for $\mathcal{N} = 2$ 4d theories on $S^4$. The study of the three dimensional case was initiated by Kapustin, Willett, and Yaakov [21], for theories with $\mathcal{N} = 2$ supersymmetry and no anomalous dimensions for chiral operators.

In practice, this means that it has been used in 3d theories with $\mathcal{N} = 3$ or more supersymmetry, since their nonabelian $R$-charges cannot receive quantum corrections. In this work, I will generalize the localization argument to any $\mathcal{N} = 2$ three dimensional field theory with an $R$-symmetry.

In particular, I will explain how the $S^3$ partition function of an $\mathcal{N} = 2$ superconformal field theory can be computed exactly at 1-loop from a localized UV Lagrangian, given the input of the $R$-charge that controls the dimensions of operators in the infrared CFT.

The full path integral on the sphere localizes to a matrix model, which may be solved in the ‘t Hooft limit using large $N$ techniques. This method has been put to impressive use by Drukker, Marino, and Putrov [11] (see also [18, 30]), who found the famous $N^{3/2}$ scaling of the entropy of multiple M2 branes directly in the IR $\mathcal{N} = 6$ conformal field
theory \[1\] on \( \mathcal{N} \) M2 branes.

Recall that in three dimensions, there are no anomalies in the trace of the stress energy tensor, and thus no obvious analogues of the \( c \) and \( a \) anomaly coefficients of four dimensional theories. The quantity \( a \) is the coefficient of the Euler density, and \( c \) is the coefficient of the square of the Weyl tensor.

The \( a \) function can be expressed simply \[3, 4\] in terms of the superconformal \( R \)-charge in 4d \( \mathcal{N} = 1 \) theories. Moreover, it was shown by Intriligator and Wecht that \( a \) is maximized as a function of a trial \( R \)-charge, thus determining the exact superconformal one \[20\]. In this work, evidence will be given that, in 3d theories with \( \mathcal{N} = 2 \) supersymmetry, the sphere partition function, \( Z \), plays a very similar role. An intriguing and seemingly different proposal for a quantity that behaves monotonically along renormalization group flow is the entanglement entropy defined and holographically studied by \[26, 27\].

The partition function of a four dimensional theory on \( S^4 \) suffers from logarithmic divergences. Thus the exact results of Pestun \[28\] have usually been applied to ratios of such partition functions, or VEVs of BPS operators. The coefficient of the logarithmic divergence in the free energy is proportional to the \( a \) function multiplying the Euler density in the trace anomaly (the Weyl curvature vanishes on \( S^4 \), so \( c \) does not appear). This was one of the original motivations of Cardy \[9\] in proposing that \( a \) was should be decrease in the IR.

In contrast, for three dimensional conformal theories, the partition function on the sphere has only linear and quadratic UV divergences, thus the finite piece, after appropriate regularization, itself provides a measure of the number of degrees of freedom.

Moreover, I will show that for \( \mathcal{N} = 2 \) SCFTs, there is an explicit formula for \( Z \) given the IR superconformal \( R \)-charge. The function is extremized for the exact superconformal \( R \)-charge. In examples, it appears that \( Z \) is in fact always minimized as a function of the \( R \)-charge associated to the curvature couplings on the sphere.

The decrease in of the number of degrees of freedom of a field theory along a renormalization group trajectory has been made precise in two and four dimensions. The beautiful theorem of Zamolodchikov \[36\] proves that in two dimensions the trace anomaly, \( c \), is strictly decreasing, and the rg flow is the gradient flow of \( c \).

In four dimensions, it is conjectured \[9\] that the coefficient, \( a \), of the Euler density appearing in the conformal anomaly decreases along rg flows. Strong evidence for this was found in presence of \( \mathcal{N} = 1 \) supersymmetry \[9\]. Moreover, an apparent counter-example

\[1\]In \[38\], which appeared after the first version of this paper, it is shown that these two quantities are in fact equivalent.
has recently been removed [14].

In theories with a gravity dual, the $S^3$ partition function is identified with the exponential of the euclidean Einstein action in AdS$_4$. It diverges, and is regulated using a boundary counter term [16, 17] (see also [12, 5, 25, 35]) leading to a finite result for even dimensional AdS. The regularized action turns out to be negative.

This gives some holographic evidence for a $Z$-theorem that $Z$ is greater at the endpoint of a renormalization group flow [13], since the volume of the dual AdS must be less. If one could prove that $Z$ was always minimized in determining the exact $R$-symmetry, then the loss of flavor symmetries along rg flows would give evidence for such a $Z$-theorem for $\mathcal{N} = 2$ theories.

Many exact results involve the superconformal $R$-charge of $\mathcal{N} = 2$ conformal field theories in three dimensions. Just as in 4d $\mathcal{N} = 1$ superconformal theories, it determines the dimensions of chiral operators. Furthermore, as I will show, the partition function on the sphere may also be determined given the exact superconformal $R$-charge.

Until now, there was no known method to determine the superconformal $R$-charge in three dimensions, beyond considerations of symmetry. It can, of course, be calculated in perturbation theory.

There is a further result known about the $R$-charges in 3d. Barnes, Gorbatov, Intriligator and Wright showed [7] that the two point function of an $R$-current is minimized for the superconformal one. This result gives another derivation of $a$-maximization when applied to four dimensions. However, this two point function receives quantum corrections, and cannot be computed exactly in three dimensions.

The $S^3$ partition function that I define and calculate in this paper is an explicit function of an $R$-charge. Extremizing it gives an exact formula for the superconformal $R$-charge.

The recipe is as follows. Consider an $\mathcal{N} = 2$ theory with $f$ abelian flavor symmetries. If $R_0$ is an $R$-charge, then so is

$$R = R_0 + \sum_{j=1}^f a_j F_j,$$

where $F_j$ are the flavor charges.

I will show that the partition function of the theory on $S^3$, as a function of the $R$-multiplet used to couple to the background geometry, is given by

$$Z(R) = \int \prod_{\text{Cartan}} du \ e^{i\pi \text{Tr} u^2} \det_{Ad}(\sinh(\pi u)) \prod_{\text{Chirals in rep } R_i} \det_{R_i}(e^{\ell(1-\Delta_i+iu)}),$$

(1.2)
where the $\text{Tr}$ is the Chern-Simons form (normalized such that for $U(N)$ at level $k$, it is $k$ times the ordinary trace), $\Delta_i$ is the $R$-charge of the chiral multiplet under the $R$-symmetry, and the function $\ell$ is defined below.

Suppose the $R$-symmetry can mix with a baryonic flavor symmetry associated to the conserved current $\star \text{Tr} F$. The $R$-charge may be expressed as $R = R_{\text{matter}} + \Delta_B B$, where $B$ is the topological (ie. monopole) charge. A factor of $\exp(2\pi \Delta_B \text{Tr} u)$ should then be included in (1.2), which is now a function of $\Delta_B$ as well.

The function (1.2) is computed using localization, and the resulting 1-loop determinants involve the function

$$ \ell(z) = -z \log (1 - e^{2\pi iz}) + \frac{i}{2} \left( \pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz}) \right) - \frac{i\pi}{12}, \quad (1.3) $$

which satisfies $\partial_z \ell(z) = -\pi z \cot(\pi z)$.

Setting $\partial_{a_j}|Z|^2 = 0$ gives $f$ real equations for $f$ unknowns, determining the exact superconformal $R$-charge in the infrared, up to possible discrete degeneracy.

As will be explained in section 2, an $\mathcal{N} = 2$ field theory in three dimensions with an $R$-symmetry can be coupled to curvature such that the 4 supercharges contained in $OSp(2|2) \times SU(2)$ are preserved. That supergroup contains the $SO(4) \cong SU(2) \times SU(2)$ isometries of $S^3$, but no conformal transformations. The action which preserves this symmetry is very similar to those appearing in [31, 32, 33, 29] in the related context of $\mathcal{N} = 1$ four dimensional theories on $S^3 \times S^1$, and generalizes discussions for Yang-Mills theories in [8]. In general, there is a family of such supersymmetric curvature couplings, parameterized by a choice of $R$-charge. The linear combination of an $R$-charge with an abelian flavor charge is another $R$-charge.

The partition function may then be calculated using localization, and is independent of the radius of the $S^3$. Thus the partition function of the IR CFT, conformally coupled to curvature, can be computed using localization of a UV theory. The latter is a function of the supersymmetric curvature couplings that are parameterized by a choice of $R$-charge. It is equal to the partition function of the IR CFT conformally coupled to curvature for the special choice of $R$-multiplet which sits in the superconformal algebra in the IR. The result is calculated explicitly in section 3, as a function of yet unknown IR $R$-charges. Here it must be assumed that the $R$-symmetry does not mix with accidental $U(1)$’s.

In section 4, I consider deforming the theory by real mass parameters, which preserve the same supersymmetry algebra. The dependence of the supersymmetry transformations on the real masses and the choice of $R$-multiplet used to couple the theory to gravity is shown to be holomorphic. The structure of the localization argument then implies that
the partition function is also a holomorphic function.

This relates the derivative with respect to the $R$-charge to the one point function of an operator. As explained in section 5, this operator may mix with the identity in the IR, but parity implies that its VEV must be purely real in the conformal field theory. I conclude that $|Z|^2$ is extremized when the $R$-symmetry is taken to be the exact IR value. This determines the dimensions of all chiral operators in the IR CFT.

In the last section, I check the proposal in several examples.

2 Supersymmetric theories on $S^3$

I begin by explaining how to put quantum field theories with 4 supercharges and an $R$-symmetry on $S^3$, while preserving supersymmetry.

There is a canonical way of coupling conformal field theories to curvature, defined by requiring Weyl invariance. Our ultimate aim is to compute the partition function of an $\mathcal{N} = 2$ SCFT on $S^3$, conformally coupled to curvature.

One main result will be that this partition function can be calculated exactly using localization in a UV Lagrangian description. A prerequisite is the existence and uniqueness, in the more general non-conformal context, of curvature couplings on a round $S^3$ that preserve particular supersymmetries.

2.1 Superconformal symmetries on $S^3$

Recall that in three Euclidean dimensions, $\mathcal{N} = 2$ supersymmetry implies two complex spinors.

In Lorentz signature, the superconformal group is $OSp(2|4)$, containing the $SO(2)$ $R$-symmetry and $USp(4) \cong SO(3,2)$ conformal group. In Euclidean signature, the conformal group, in the notation of [10], is $USp(2,2)$, the real form of $USp(4)$ that is equivalent to $SO(4,1)$. Thus the superconformal group in $\mathbb{R}^3$ is $OSp(2|2,2)$.

Denote the super(conformal) charges by $Q^i_A$, where $A = 1, \ldots, 4$ is an $SO(4,1)$ spinor index and $i = 1, 2$ is an $SO(2)$ $R$-symmetry index. The anti-commutator of the supersymmetries is given by

$$\{Q^i_A, Q^j_B\} = \delta^{ij} M_{AB} + i \omega_{AB} \epsilon_{ij} R,$$

where $M_{AB} \in USp(2,2)$ and $R$ is the $U(1)$ superconformal $R$-symmetry. One has that $[R, Q^i_A] = \epsilon^{ij} Q^j_A$. 

Here, $\omega$ is the symplectic form of $USp(2, 2)$,

$$\omega = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix},$$

where $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, is the anti-symmetric symbol on each $SU(2)$ factor.

On the sphere, the superconformal group is the same as in flat space, however the interpretation of the bosonic $USp(2, 2)$ is different. In particular, the full $SU(2)_L \times SU(2)_r \cong SO(4)$ subgroup\footnote{I reserve capital $R$ for an $R$-symmetry. $S^3$ is isomorphic to $SU(2)$, and these are the left and right group actions.} is realized as isometries of $S^3$, while only the diagonal $SU(2)$ are rotations of $\mathbb{R}^3$ in the flat space limit.

Localization of the path integral requires only a single global supercharge. On $S^3$ there are four Killing spinors, satisfying the Killing equation $\nabla_\mu \varepsilon = \gamma \varepsilon'$. There is a basis of Killing spinors\footnote{My conventions for spinors are collected in the appendix.} which further satisfy $\varepsilon' \propto \varepsilon$. In particular, on $S^3$, one has $\nabla_\mu \varepsilon = \pm \frac{1}{2r} \gamma_\mu \varepsilon$, where $r$ is the radius of the $S^3$. Following [21], choose such an $\varepsilon$, with the plus sign, normalized such that $\varepsilon^\dagger \varepsilon = 1$.

Take the holomorphic supersymmetry with spinor parameter $\varepsilon$ that satisfies the homogeneous Killing spinor equation to be $\delta = \frac{1}{\sqrt{2}} (Q_1^1 + iQ_1^2)$, with $R$-charge $+1$. If $\delta$ is a symmetry of any Euclidean theory on the sphere whose flat space limit is the analytic continuation of a unitary Lorentzian theory, then $\delta^\dagger = \frac{1}{\sqrt{2}} (Q_2^1 - iQ_2^2)$ must also be a symmetry. In Euclidean signature, $\delta^\dagger$ now denotes not the Hermitian conjugate, but an independent supercharge, defined by $(\tilde{Q})^j_A = \omega_{AB} \varepsilon^i Q^j_B$. It is easy to check that $\delta^2 = (\delta^\dagger)^2 = 0$.

Thus one must preserve

$$\{\delta, \delta^\dagger\} = M_{12} + R,$$

where $M_{12}$ is a rotation of $S^3$, and $R$ is the $R$-symmetry. The particular rotation which appears is in the direction of the vector field $v_\mu = \varepsilon^i \gamma_\mu \varepsilon$, as defined in [21]. It can be viewed as translation along the Hopf fiber of $S^1 \hookrightarrow S^3 \to S^2$.

More abstractly, there is an $OSp(2|2)_r \times SU(2)_L$ subgroup of $OSp(2|2, 2)$, containing the $SO(4) \cong SU(2)_L \times SU(2)_r$ rotations of $S^3$, the $R$-symmetry, and 4 supercharges. The supersymmetry $\delta$ is part of a doublet under the $SU(2)_r$; it is a singlet of $SU(2)_L$.

The subgroup $OSp(2|2)$ does not contain any of the conformal transformations. It is somewhat novel that the $R$ charge appears on the right hand side of a supersymmetry
algebra without conformal generators. This also occurs for the supersymmetry group $OSp(2|4)$ in Pestun’s work [28] on $\mathcal{N} = 2$ theories on $S^4$.

Regarding $S^3$ as a Hopf fibration of $S^1$ over $S^2$, one $SU(2) \subset SO(4)$ acts as rotations on the $S^2$ (together with the induced action on the fiber), while the other contains a $U(1)$ subgroup of the translations along the $S^1$ fiber. In the flat space limit, the diagonal $SU(2)$ becomes the $SO(3)$ rotation group that fixes a given point.

The $\delta$ used in localization is one of the fermionic generators in $OSp(2|2)$. Note that this subgroup is preserved under conjugation by $SO(4)$ rotations of the $S^3$. Thus $\delta$ and $\delta^\dagger$, together with the $SO(4)$ isometries, do not generate the entire superconformal group.

Some of the theories considered in this work will be invariant under parity. On $\mathbb{R}^3$, parity acts by reflection of one coordinate, or, equivalently (up to an $SO(3)$ rotation) by inversion through the origin.

This becomes a reflection through an equatorial $S^2$ on the sphere, or, after an $SO(4)$ rotation, the antipodal map. Thus parity acts by exchanging the two $SU(2)$ isometries.

Therefore, any theory invariant under $OSp(2|2)$ and this $\mathbb{Z}_2$ inversion must in fact have the full superconformal symmetry. The converse is of course false: there are superconformal theories (invariant under $OSp(2|2,2)$ on the sphere) that are not parity invariant. This includes most theories with Chern-Simons terms.

### 2.2 Supersymmetric curvature couplings

I will now show explicitly how to couple an $\mathcal{N} = 2$ Lagrangian field theory with an $R$ symmetry to the curvature of a round $S^3$ such that $OSp(2|2)$ is preserved. In addition to ordinary curvature couplings, proportional to $1/r^2$, there will be terms in the action that depend on $1/r$, so calling them curvature couplings is a slight misnomer.

In general, there will be a whole family of actions, differing only in terms that vanish in the flat space limit, which preserve $OSp(2|2)$. They will be parameterized by a choice of $R$-symmetry. Recall that an $R$-symmetry together with any combination of abelian flavor symmetries is again an $R$ symmetry.

The $R$-charges that appears on the right hand side of the algebra will differ by abelian flavor charges. The couplings to curvature are uniquely determined given that $R$-multiplet.

Consider a single chiral multiplet, coupled to an abelian gauge field. Let $\Delta$ be the
$R$-charge of the lowest component. The supersymmetry transformations are

$$
\delta \phi = 0 \hspace{1cm} (2.1)
$$

$$
\delta \phi^\dagger = \psi^\dagger \varepsilon \hspace{1cm} (2.2)
$$

$$
\delta \psi = (-i \not\partial \phi - i \sigma \phi + \frac{\Delta}{r} \phi) \varepsilon \hspace{1cm} (2.3)
$$

$$
\delta \psi^\dagger = \varepsilon^T F^\dagger \hspace{1cm} (2.4)
$$

$$
\delta F = \varepsilon^T (-i \not\partial \psi + i \sigma \psi + \frac{1}{r} \left( \frac{1}{2} - \Delta \right) \psi + i \lambda \phi) \hspace{1cm} (2.5)
$$

$$
\delta F^\dagger = 0, \hspace{1cm} (2.6)
$$

and

$$
\delta A_\mu = -\frac{i}{2} \lambda^\dagger \gamma_\mu \varepsilon \hspace{1cm} (2.7)
$$

$$
\delta \sigma = -\frac{i}{2} \lambda^\dagger \varepsilon \hspace{1cm} (2.8)
$$

$$
\delta \lambda = \left( -\frac{1}{2} \gamma^{\mu \nu} F_{\mu \nu} - D + i \gamma^\mu \partial_\mu \sigma - \frac{1}{r} \sigma \right) \varepsilon \hspace{1cm} (2.9)
$$

$$
\delta \lambda^\dagger = 0 \hspace{1cm} (2.10)
$$

$$
\delta D = \left( -\frac{i}{2} (D_\mu \lambda^\dagger) \gamma^\mu + \frac{1}{4r} \lambda^\dagger \right) \varepsilon. \hspace{1cm} (2.11)
$$

Here $r$ is the radius of $S^3$, $\gamma^{\mu \nu} = \frac{1}{2} \left[ \gamma^\mu, \gamma^\nu \right]$, and the gauge covariant derivative is defined as $D_\mu = \partial_\mu + i [A_\mu, \cdot]$. When $\Delta = \frac{1}{2}$, this is a symmetry of the classical action of an $\mathcal{N} = 2$ chiral multiplet with canonical dimension, conformally coupled to curvature.

As before, $\varepsilon$ is a Killing spinor satisfying $\nabla_\mu \varepsilon = -\frac{i}{2} \gamma_\mu \varepsilon$ and $\varepsilon^\dagger \varepsilon = 1$. Define a vector field $v_\mu = \varepsilon^\dagger \gamma_\mu \varepsilon = -\varepsilon^T \gamma_\mu \varepsilon^*$, which satisfies $\nabla^\mu v_\mu = 0$, as shown in [21]. Note that $\varepsilon^\dagger$ is also a left invariant Killing spinor.

It is easy to check that these transformations satisfy the above algebra,

$$
\{ \delta, \delta^\dagger \} \phi = -i (v^\mu D_\mu + \sigma) \phi + \Delta \phi \hspace{1cm} (2.12)
$$

$$
\{ \delta, \delta^\dagger \} \psi = -i (v^\mu D_\mu + \sigma) \psi + (\Delta - 1) \psi \hspace{1cm} (2.13)
$$

$$
\{ \delta, \delta^\dagger \} F = -i (v^\mu D_\mu + \sigma) F + (\Delta - 2) F, \hspace{1cm} (2.14)
$$

and likewise for their conjugates. Note that the terms involving $\sigma$ are simply a gauge transformation of the fields, and do not give rise to a nontrivial charge in the supersymmetry algebra.
Suppose one takes a non-conformal theory. To put it on the sphere, one needs to specify how to couple it to curvature. If the theory were conformal, those couplings are uniquely determined by requiring Weyl invariance.

Instead, consider a theory which preserves $OSp(2|2)$. For any choice of the $R$-charge (and thus $\Delta$ in the above expressions), one can find a unique action on the sphere such that the theory is invariant under that supersymmetry.

In particular, on the sphere, the matter action that preserves $OSp(2|2)$ is

$$S = \int \sqrt{g} \left( \sum_{\mu} D_\mu \phi^\dagger D^\mu \phi + i \psi^\dagger \Gamma \psi + F^\dagger F + \phi^\dagger \sigma^2 \phi + i \phi^\dagger D \phi - i \psi^\dagger \sigma \psi + i \phi^\dagger \lambda^\dagger \psi - i \psi^\dagger \lambda \phi ight)$$

$$+ \frac{\Delta - \frac{1}{2}}{r} \psi^\dagger \psi + \frac{2i}{r} (\Delta - \frac{1}{2}) \phi^\dagger \sigma \phi + \frac{\Delta(2 - \Delta)}{r^2} \phi^\dagger \phi.$$

(2.15)

Similar actions have appeared before in the work of D. Sen [31, 32, 33] and, more recently, Römelberger [29], in the related context of the (superconformal) index of four dimensional $\mathcal{N} = 1$ theories on $S^3$.

Note that when $\Delta$ differs from $1/2$, the $\mathbb{Z}_2$ parity is broken by the new terms required to preserve supersymmetry on the sphere. This is not too surprising, considering that the choice of $OSp(2|2)_L$ already breaks parity, as does the choice of Killing spinor.

The careful reader will also note that the factor of $i$ in the term $\psi^\dagger \sigma \psi$ and the sign of $\phi^\dagger \sigma^2 \phi$ correct errors in the literature; moreover the auxiliary field, $D$, is defined with a slightly unusual factor of $i$.

The supersymmetry transformation of the vector multiplet is identical with that appearing in [21], since those fields cannot have anomalous dimensions ($* \text{Tr } F$ is a conserved current, and must have dimension 2 in the IR).

Therefore the unique supersymmetric Yang-Mills action on $S^3$ (preserving $OSp(2|2) \times SU(2)_L$) is given by

$$\frac{1}{g_{YM}^2} \int \sqrt{g} \text{Tr} \left( -i F^{\mu \nu} F_{\mu \nu} + D_\mu \sigma D^\mu \sigma + D^2 + i \lambda^\dagger \nabla \lambda + i [\lambda^\dagger, \sigma] \lambda + \frac{2}{r} D \sigma - \frac{1}{2r} \phi^\dagger \sigma \phi + \frac{1}{r^2} \sigma^2 \right)$$

(2.16)

Note that the curvature couplings of the scalars in the vector multiplet already break parity, since $\sigma$ is a pseudo-scalar, while $D$ is an ordinary scalar. There is a parity violating mass for the gauginos as well.

This avoids the following potential contradiction. QED with $N_f$ conjugate flavor pairs is parity invariant, yet has nontrivial anomalous dimensions. Given that $OSp(2|2)$ ×
$SU(2)_L$ and parity generate the entire superconformal algebra, if the curvature couplings
of pure YM had not broken parity, then neither would those QED with flavors for the
canonical choice of dimensions. But that theory is not a CFT, so it is impossible.

Another important point is that the YM action (2.16) is not only invariant under
supersymmetry, it is in fact $Q$-exact.

Therefore, by the standard localization argument, the partition function does not
depend on the dimensionless parameter $r g^2_{YM}$. One can easily check that the $S^3$ partition
function computed using localization in the next section will in fact always independent
of the radius of the sphere.

The addition of a superpotential does not change the action of the supercharges nor
the value of the action on the localized configuration space discussed in the next section.
Therefore it does not change the $S^3$ partition function.

Of course, superpotentials will typically break some flavor symmetries, and thus will
restrict the possible $R$-multiplets. In other words, only a subset of the possible curvature
couplings discussed above will still preserve supersymmetry - namely those associated to
$R$-symmetries that are unbroken by the superpotential.

Any theory with four supercharges and an $R$-symmetry possesses an $R$-multiplet [24].
If there are abelian flavor symmetries, then there is a family of $R$-multiplets related by
improvement terms. Such a theory may be put in curved space by first coupling it to
supergravity. Those supergravity theories do depend on which $R$-multiplet is gauged.

It would be very interesting to understand in that more abstract language precisely
what background of the fields in the supergravity multiplet must be turned on for the
round $S^3$, so that in addition to the $SO(4)$ isometries, the global $OSp(2|2)$ supersymmetry
is preserved.

This would explain the origin of the somewhat mysterious $1/r$ couplings, give a general
interpretation of the preservation of supersymmetry on this space with Killing spinors,
and would generalize the result to theories with an $R$-multiplet without a Lagrangian
description.

2.3 The essential logic

In the UV, there is a way of coupling the theory to curvature on a round $S^3$ such the
$OSp(2|2)$ algebra is preserved. There is a family of such theories on the sphere, parameter-
ized by the $R$-charge which appears in the algebra.

The partition function can then be computed using localization, which renders the
1-loop approximation exact, and the result does not depend on the radius of the sphere. In the next section, I will exactly calculate this partition function, generalizing [21] to situations with nontrivial anomalous dimensions. The path integral will be a function of the choice of $R$-charge that parameterizes the couplings to curvature on $S^3$.

One important subtlety is that the 1-loop determinants are UV divergent. I will regulate them using zeta function regularization, as in [21], [11]. This is justified $a$ posteriori by the various checks of the result.

Moreover, it is reasonable that the 1-loop determinant for a chiral multiplet charged under several gauge and flavor groups depends only on the natural combination of the real scalars, $\sum_a q_a \sigma_a$, where $q_a$ are the changes under the Cartan factors. Strictly speaking, this is not guaranteed since the chiral field is not gauge invariant.

In later sections, it will be shown that the dependence of the 1-loop determinant on the parameters $\Delta_j$ is related by analytic continuation to the dependence on background flavor gauge fields.

Therefore, assuming the regulator is such that the previous two paragraphs apply, the 1-loop determinant for a single free chiral multiplet would be enough to obtain the general result. It would be much more satisfactory to determine the principle which selects zeta function regularization.

The IR CFT can be conformally coupled to curvature. It preserves the whole $OSp(2|2, 2)$. Therefore, the UV theory “$OSp(2|2)$-coupled” to curvature on a large $S^3$ will be the same as the IR theory conformally coupled to curvature exactly if the $R$-charge is chosen to be the $R$-symmetry that sits in the IR superconformal algebra.

Therefore, the partition function of the CFT on $S^3$ can be computed using localization of a UV Lagrangian definition of the theory, which is coupled to curvature such that the $OSp(2|2)$ which is preserved contains the exact superconformal $R$-charge of the IR theory.

3 Localizing the partition function

Consider any theory which is invariant under the supersymmetry (2.1). Then the addition of $Q$-exact terms to the action will not change the partition function by the standard localization argument. Thus one can compute

$$Z = \lim_{t \to \infty} e^{-S - t\{\delta, P\}},$$

for any sufficiently regular, odd operator $P$ that is invariant under the bosonic symmetry $\delta^2$. In the case at hand, $\delta^2 = 0$, so the last is not a constraint.
The action for the vector multiplets, including their couplings to curvature, is identical to that appearing in [21], since those fields cannot have anomalous dimension. Thus the arguments of Kapustin-Willett-Yaakov may be carried over directly.

In brief, [21] takes \( P = \text{Tr} \left( (\delta \lambda)^\dagger \lambda \right) \), where the \( \delta \lambda \) is understood to be stripped of the Killing spinor \( \varepsilon \). The bosonic part of the localizing action \( S_{\text{loc}} = t \{ \delta, P \} \) is positive definite, and vanishes exactly when \( \delta \lambda = 0 \).

The only solutions are that \( D = -\sigma/r \) is a constant and all other fields are set to zero. The path integral reduces to a matrix model, that is, a finite dimensional integral over the constant VEVs of \( \sigma \).

The Yang-Mills action coupled to curvature such that \( OSp(2|2) \) is preserved is \( Q \)-exact, and indeed restricts to zero on the localized space of field configurations. In fact, the only non-vanishing piece of the action is from the supersymmetric Chern-Simons term,

\[
\frac{ik}{4\pi} \int_{S^3} 2\text{Tr} \left( D\sigma \right) = i\pi kr^2 \text{Tr} \left( \sigma^2 \right),
\]

where one uses the fact that the volume of \( S^3 \) is \( 2\pi^2 r^3 \).

Turning now to the matter sector, a particularly natural choice, as in [21], is to take

\[
P = (\delta \psi)^\dagger \psi + \psi^\dagger (\delta \psi)^\dagger, \]

where in this expression, \( \delta \psi \) is understood to not include the \( \varepsilon \). This ensures that the bosonic part of the \( Q \)-exact localizing term in action, \( S_{\text{loc}} \), is positive, and vanishes on supersymmetric configurations. On \( S^3 \), this implies that all of the fields are localized to zero with the exception of the scalars in the vector multiplets. Integrating over their constant VEVs results in a finite dimensional matrix integral.

The localizing Lagrangian, \( L_{\text{loc}} = \{ \delta, P \} \), is

\[
\partial_\mu \phi \partial^\mu \phi + \phi^\dagger \sigma_0^2 \phi + \frac{2i}{r} (1 - \Delta) \phi^\dagger \nu^\mu \partial_\mu \phi + \frac{\Delta^2}{r^2} \phi^\dagger \phi + F^\dagger F + \psi^\dagger \left( i \nabla - i \sigma_0 + \frac{1}{2} + (1 - \Delta) \psi \right) \psi,
\]

up to total derivatives. It breaks the right \( SU(2) \) isometry down to \( U(1) \).

The 1-loop determinant from the bosonic fields is \( \text{det}^{-1}(D_{\text{bos}}) \), where

\[
D_{\text{bos}} = -\nabla^2 + 2i \frac{1 - \Delta}{r} \nu^\mu \partial_\mu + \frac{\Delta^2}{r^2} + \sigma_0^2 = \frac{1}{r^2} \left( -\ell_j \ell^j + 2i(1 - \Delta) \ell_3 + \Delta^2 + \sigma_0^2 r^2 \right),
\]

in terms of the left invariant vector fields defined in the appendix. After expanding in

\[4\]I keep the radius of the sphere explicit; it is set to 1 in [21].
terms of angular momentum modes on the $S^3$, one finds that

$$r^{-2(\ell+1)}\det_{\ell/2}(D_{\text{bos}}) = \prod_{m=-\ell/2}^{\ell/2} \left( \ell(\ell+2) - 4m(1-\Delta) + \Delta^2 + \sigma_0^2 r^2 \right). \quad (3.2)$$

The fermionic action is also quadratic, and one needs to compute the determinant of the operator

$$rD_{\text{ferm}} = i\gamma^j \ell_j - 1 - ir\sigma_0 + (1-\Delta)\psi = -4\vec{S} \cdot \vec{L} + 2(1-\Delta)S_3 - 1 - ir\sigma_0.$$

Up to a change in the coefficients, this is identical to the 1-loop determinant in the case of fields with canonical dimensions studied by [21]. They found that

$$\det_{\ell/2} \left( 2\alpha \vec{L} \cdot \vec{S} + 2\beta S_3 + \gamma \right) = \left( \alpha \frac{\ell}{2} + \beta + \gamma \right) \left( \alpha \frac{\ell}{2} - \beta + \gamma \right) \times \prod_{m=-\ell/2}^{\ell/2-1} \left( -\frac{\ell}{2} + 1 \right) \alpha^2 - (2m+1)\alpha \beta - \alpha \gamma - \beta^2 + \gamma^2. \quad (3.3)$$

Therefore, in this case,

$$\det_{\ell/2}(D_{\text{ferm}}) = r^{-2(\ell+1)}(-\ell-\Delta + i\sigma_0 r)(-\ell-2+\Delta - i\sigma_0 r) \prod_{m=-\ell/2}^{\ell/2-1} (-\ell(\ell+2)+4(1-\Delta)m-\Delta^2-\sigma_0^2 r^2).$$

The matter determinants mostly cancel between bosons and fermions, leaving the following infinite product.\footnote{See also [37] that appeared shortly after the first version of this paper, which derived the same result also using localization of non-conformal $R$-symmetric $\mathcal{N} = 2$ theories on $S^3$.}

$$Z_{1\text{-loop}} = \prod_{n=1}^{\infty} \left( \frac{n+1 + iu - \Delta}{n-1 - iu + \Delta} \right)^n, \quad (3.4)$$

where I defined $u = \sigma_0 r$.

Define $z = 1 - \Delta + iu$, and let $\ell(z) = \log Z_{1\text{-loop}}$. Then

$$\partial_z \ell(z) = \sum_{n=1}^{\infty} \left( \frac{n}{n+z} + \frac{n}{n-z} \right),$$

which has a linear divergence. Regulating using zeta functions, one finds that

$$\partial_z \ell(z) = \frac{\partial}{\partial s} \bigg|_{s=0} \left( \zeta_H(s-1, -z) + z\zeta_H(s, -z) - \zeta_H(s-1, z) + z\zeta_H(s, z) \right) ,$$
where $\zeta_H$ is the Hurwitz zeta function.

This results in

$$\partial_z \ell(z) = -\pi z \cot(\pi z),$$

which can be integrated to give

$$\ell(z) = -z \log \left(1 - e^{2\pi iz}\right) + \frac{i}{2} \left(\pi z^2 + \frac{1}{\pi} \text{Li}_2(e^{2\pi iz})\right) - \frac{i\pi}{12}. \tag{3.5}$$

This is not manifestly real when $z$ is real, but it is clear from the original definition that it will be.

\section{Deformation by real masses and a mysterious holomorphicity}

One initially surprising feature of the matter one loop determinant (3.4) is that it is identical to the result for $\Delta = 1/2$ found in [21] up to the replacement $u \rightarrow u + i(\Delta - 1/2)$. In particular, the dependence on the variable $z$ is holomorphic.

This can be explained simply by the fact that the supersymmetry transformation, $\delta$, itself depends holomorphically on $z$. This holomorphy was not manifest in the above calculation of the 1-loop determinant, since the operator $P$ depended anti-holomorphically on $z$.

However, the final answer does not depend on the choice of $P$, and only depends on $z$ due to the dependence of $\delta$. Therefore it must be holomorphic. Note that a fixed $P$, independent of $z$ could have been chosen in this case since $\delta^2 = 0$, so there is no $z$ dependent constraint on $P$.

So far, this looks like a curious feature of the matter 1-loop determinant of a chiral multiplet charged under an abelian gauge field. However, it has more general implications.

Recall that the supersymmetry preserving couplings of the QFT to the gravity background of $S^3$ are parameterized by the space of abelian flavor symmetries. Coupling all of those flavor symmetries to background $\mathcal{N} = 2$ vector multiplets, one may turn on real mass parameters, $m_j$, that are constant VEVs for the real scalars in those background multiplets.

I will now explain that the $S^3$ partition function depends holomorphically on the parameters $z_j = -a_j + irm_j$, where $j$ runs over the Cartan of the flavor symmetry group and the chosen $R$-charge is $R_0 + \sum a_j F_j$. Here $F_j$ denote the flavor charges, and $R_0$ is some
$R$-charge. It would be extremely interesting to understand the origin of this holomorphy - it will remain an empirical observation in this paper.

In flat space, the real mass deformation is defined by turning on a constant $\theta \bar{\theta}$ component (denoted by $\sigma$) of a background abelian vector field. On $S^3$, the supersymmetry transformations are modified, as we saw above in the matter sector. In the case of the vector multiplet, the supersymmetry preserving configuration on $S^3$ is given by a constant $D = -\sigma/r$, where $r$ is the radius of $S^3$, and $D$ is the auxiliary, $\theta^2 \bar{\theta}^2$ component [21]. Obviously, this reproduces the standard real mass in the flat space limit.

Alternatively, one can use the standard definition of a real mass deformation, and include extra terms in the curvature couplings needed to supersymmetrize the theory on the sphere.

As observed in [22], the supersymmetry transformation written above is still a symmetry of the theory after such a real mass has been turned on, simply by setting $\sigma$ to be the real mass parameter, and taking the vector multiplet to be non-dynamical. This modifies the supersymmetries relative to those of the theory without real masses.

Moreover, the real mass parameter appears in the anti-commutator of supercharges; it is a central charge of the algebra,

$$\{\delta, \delta^\dagger\} = M_{12} + \frac{1}{r} R_{UV} + (\frac{a_j}{r} - im_j)F_j.$$

Further note that the supersymmetry transformation itself depends holomorphically on $a - im$.

The real mass must break conformal invariance, however it preserves $OSp(2|2)$ if one also turns on $D = -\sigma/r$. I will refer to that susy preserving deformation as the real mass on $S^3$. In the flat space limit, it is just the usual real mass.

For example, consider a chiral multiplet with canonical dimension $1/2$. Then invariance under supersymmetry requires that the action on the sphere be given by

$$S = \int \sqrt{g} \left( g^{\mu \nu} \partial_\mu \phi^\dagger \partial_\nu \phi + i \psi^\dagger \nabla \psi + F^\dagger F + im \psi^\dagger \psi + (m^2 - \frac{im}{r} + \frac{3}{4r^2})\phi^\dagger \phi \right).$$

The terms $im \psi^\dagger \psi - m^2 \phi^\dagger \phi$ are the usual real mass terms in Euclidean signature. The term $\frac{3}{4r^2} \phi^\dagger \phi$ is the standard conformal coupling of a free scalar. The term $-i \frac{m}{r} \phi^\dagger \phi$ can be understood as arising because a constant value of $D = -\frac{m}{r}$ must be turned on in the background vector multiplet to preserve supersymmetry on $S^3$.

The partition function on the sphere can be computed using localization. It depends on the real masses, as well as the $R$-multiplet used to couple the theory to curvature. The
mechanism of localization implies that the only dependence on $m_j$ and $\Delta_j$ is through the supersymmetry transformation, $\delta$, and the value of the action on the localized space of field configurations.

The terms $\int D\sigma$ appearing in $\mathcal{N} = 2$ Chern-Simons are the only ones in a YM-CS-matter action that are non-zero on the space of configurations for which the fermion variations vanish. These have no dependence on $m_j$ or $\Delta_j$.

Therefore the full $S^3$ partition function depends on $m_j$ and $\Delta_j$ only through the supercharge, and it inherits the holomorphy, $Z(m_j, \Delta_j) = Z(\Delta_j - irm_j)$.

It is natural to assume that this holomorphy also applies to baryonic flavor symmetries. Their associated real masses are Fayet-Iliopoulos parameters, appearing in the action as $-\frac{i}{\pi}\xi \text{Tr}(D)$.

After localization, this results in a term $-\frac{i}{\pi} \int_{S^3} \xi \text{Tr}(-\sigma/r) = 2\pi i \xi \text{Tr}(\sigma)r^2$, as in [22]. Therefore, I conjecture that if the $R$-symmetry mixes with a baryonic $U(1)$, then the partition function (1.2) is modified by the inclusion of a factor

$$e^{2\pi \Delta_B \text{Tr}(u)}$$

in the integrand. This is consistent with the equivalence of the baryonic $U(1)$ and the gauged matter current in abelian gauge theories with a non-trivial Chern-Simons level (so that it is really $j_{\text{matter}} + k \star F$ that is gauged).

5 $Z$-extremization

The holomorphy of the partition function implies that $\partial_\Delta Z = \frac{i}{r} \partial_m Z$. At the conformal point, $\frac{1}{Z} \partial_m Z \big|_{m=0, \Delta=\Delta_{\text{IR}}}$ is the 1-point function of an operator in a CFT on $S^3$.

Thus, by conformal invariance, it vanishes unless the operator contains the identity. In general, the operator associated to a real mass, as defined in the UV theory, may indeed mix with the identity.

First consider the case that the CFT is parity symmetric, and it is not spontaneously broken. The real mass term is parity odd.

Therefore, its VEV must vanish, since parity isn’t spontaneously broken, by assumption. This argument does not appear to utilize conformal invariance, but recall that parity, together with $OSp(2|2) \times SU(2)_L$, generates the entire superconformal algebra.

In parity invariant theories, $Z$ is a real number, and the requirements that $\partial_{\Delta_j} Z(\Delta_j) = 0$ are exactly the right number of equations to determine all of the $\Delta_j$ (up to a possible
discrete degeneracy).

In general, the CFT will not preserve parity. Then the $S^3$ partition function will be complex, and parity acts on it by complex conjugation.

In a non-conformal field theory without parity, the VEVs of parity violating operators may be non-zero. However in a CFT, only the identity operator has a non-vanishing VEV.

The identity operator is parity invariant, thus its VEV must be a real number, $\text{Im} \left( \frac{1}{2} \partial_{m_j} Z \right) = 0$, where the partition function is evaluated at the conformal point.

Therefore,

$$\partial_{\Delta_j} |Z|^2 = 0.$$  \hspace{1cm} (5.1)

This is the main result. Given the calculation of section 3, it is an explicit formula, in terms of the UV data of gauge groups, matter representations and Chern-Simons levels, that determines the exact $R$-charge in the superconformal algebra of the IR theory.

It may seem somewhat unusual that the operator which couples to the real mass at linear order mixes with the identity operator in parity violating theories. The effect appears already at 1-loop in perturbation theory, when computing the VEV of $\psi^\dagger \psi$ in the UV theory.

For example, in a Chern-Simons theory at level $k$ with some charged matter, there is a 1-loop diagram involving $\psi^\dagger \psi$, the two fermion-two boson vertex, and a $\phi$ loop, at order $1/k$. It is quadratically divergent, and in flat space one usually throws it away.

However, on $S^3$, the zeta function regularization instructs one to keep a finite piece.

It would also be interesting to look at the second derivative, which is related to the two point function. Its positivity would imply that $|Z|^2$ was always minimized, which is indeed seen in examples.

The two point function is integrated over the entire $S^3$, so contact terms must be taken into account. Similarly, there may be explicit $m^2$ couplings in the action. I leave this question and that of the interpretation of multiple extrema of $|Z|^2$ for future work.

6 Examples

I will now check the above prescription in several examples, which for simplicity involve only a single integral after localization.
6.1 $\mathcal{N} = 4$ vector multiplet

The $\mathcal{N} = 4$ vector multiplet contains, in $\mathcal{N} = 2$ language, an adjoint vector superfield and an chiral superfield, $\Phi$. The latter has dimension 1, rather than the canonical dimension $1/2$.

Such theories were studied in [22]. To obtain answers in agreement with 3d mirror symmetry, those authors set the 1-loop determinant arising from that chiral field in the $\mathcal{N} = 4$ vector multiplet to 1, a constant independent of the variables in the matrix integral.

This was justified by noting that an F-term mass for that adjoint chiral, $\int d^2\Phi^2$, is in fact Q-exact. Thus $\Phi$ may be localized to zero, independently of the rest of the theory, and its contribution to the 1-loop determinant must be trivial.

The formula (3.4) implies that 1-loop determinant of an adjoint chiral multiplet with dimension 1 is

$$\prod_{i<j} \prod_{n=1}^{\infty} \left( \frac{n + i(u_i - u_j)}{n - i(u_i - u_j)} \right)^n \left( \frac{n + i(u_j - u_i)}{n - i(u_j - u_i)} \right)^n = 1,$$

as predicted.

6.2 SQED

Consider $\mathcal{N} = 2$ QED with $N_f$ conjugate flavor pairs, $Q$ and $\tilde{Q}$. This theory has a $U(1)$ flavor symmetry under which the flavors all have the same charge. There is also a topological $U(1)_B$, which cannot mix with the $R$-charge in this charge conjugation invariant theory, and an $SU(N_f)$ flavor symmetry, which cannot mix with the abelian $R$-symmetry.

The partition function of the theory coupled to curvature on $S^3$ using the $R$-multiplet under which the flavors have charge $\Delta$ is given by

$$Z = \int_{-\infty}^{\infty} du \exp \left( N_f (\ell(1 - \Delta + iu) + \ell(1 - \Delta - iu)) \right). \quad (6.1)$$

SQED is parity invariant, and this is a real valued function.

Setting $\partial_\Delta Z = 0$ determines the superconformal value of $\Delta$. In particular, for $N_f = 1$, one can check numerically that $\Delta = 1/3$. An analytic proof will be given below.

This is precisely the prediction of 3d mirror symmetry, which relates SQED with one flavor to the $XYZ$ model [2]. This Wess-Zumino model has three chiral fields, and a superpotential, $W = XYZ$ with an $S_3$ discrete symmetry.
The gauge invariant operator $\tilde{Q}Q$ of SQED therefore has dimension $2/3$, since it corresponds to the chiral operator $X$.

The three dimensional mirror symmetry relating $\mathcal{N} = 4$ SQED with one hypermultiplet flavor and the free theory of a twisted hyper implies the equality of their sphere partition functions. This was confirmed in [22], and generalized to the case of $\mathcal{N} = 2$ preserving real masses in [19]. Using the holomorphy, the $\mathcal{N} = 2$ relation with the $XYZ$ model can also be seen analytically as follows.

The partition function of $\mathcal{N} = 4$ SQED with one flavor deformed by the $\mathcal{N} = 2$ real mass is given by

$$Z_{\mathcal{N}=4}^{SQED}(m) = \int_{-\infty}^{\infty} du \ e^{\ell(1/2+im+iu)+\ell(1/2-im-\ell(-2im)}.$$  

The first two terms are from the conjugate pair of fundamental chirals with dimension $1/2$, and the third term from the adjoint chiral with dimension $1$ in the $\mathcal{N} = 4$ vector multiplet. The last term was neglected in the analysis of [19], resulting in a slight mismatch that they remark on. There is a superpotential $\mathcal{W} = \tilde{Q}\Phi Q$, so the real mass is associated to a flavor $U(1)$ under which $\Phi$ has charge $-2$ when $Q$ and $\tilde{Q}$ have charge $1$.

The statement of 3d mirror symmetry is then $Z_{\mathcal{N}=4}^{SQED}(m) = e^{2\ell(1/2-im)}$, which corrects the result of [19] by the factor of $e^{\ell(1-2im)}$ on the left hand side. To relate this to $\mathcal{N} = 2$ SQED, which does not have the adjoint chiral multiplet, one must divide the whole equation by its contribution.

Therefore,

$$\int_{-\infty}^{\infty} du \ e^{\ell(1-\Delta+iu)+\ell(1-\Delta-\ell(2\Delta-1)},$$

using the holomorphy to set $m = i(\Delta - 1/2)$. The expression on the right hand side is exactly the sphere partition function of the $XYZ$ model with $R$-charges taken to be $1 - \Delta$, $2\Delta$ and $2\Delta$. It is trivial to check that it is minimized by the symmetric choice, $\Delta = 1/3$.

### 6.3 Abelian Chern-Simons

Adding an $\mathcal{N} = 2$ Chern-Simons term to the previous example breaks parity. A term $e^{i\pi ku^2}$ is introduced into the matrix integral. The superconformal $R$-charge is determined by setting $\partial_\Delta |Z|^2$ to zero.

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6I would like to thank Davide Gaiotto for suggesting this argument.
7See also version 2 of [19] that appeared shortly after this paper.
The partition function can be evaluated in the large $k$ limit to give
\[
\int_{-\infty}^{\infty} du \ e^{i\pi ku^2} e^{N_f(\ell(1-\Delta+iu)+\ell(1-\Delta-iu))} = \frac{1}{2} \sqrt{\frac{i}{k}} \left[ \frac{N_f \pi^2 a^2}{2} + \frac{i\pi N_f}{4k} \left( 1 + 4a + \pi^2 a^2 + \frac{\pi^2 N_f a^2}{2} \right) + \frac{\pi^2 N_f}{16k^2} \left( 1 - 8a - \frac{3}{2} N_f - 12a N_f - 24a^2 N_f - 4a^2 \pi^2 - \frac{7}{2} a^2 N_f \pi^2 - \frac{3}{4} a^2 N_f^2 \pi^2 \right) + O\left( \frac{1}{k^3} \right) \right],
\]
where $\Delta = \frac{1}{2} + a$.

The dimensions of the fundamentals in the IR were calculated perturbatively in the Chern-Simons-matter theory by Gaiotto and Yin \cite{Gaiotto:2017yup},
\[
\Delta = \frac{1}{2} - \frac{b_0}{4k^2} + O\left( \frac{1}{k^4} \right),
\]
where $b_0 = \frac{2}{\dim R} \left( \Tr_R(T^a T^b) \Tr_R(T^a T^b) + \Tr_R(T^a T^b T^a T^b) \right)$, for matter in a representation, $R \oplus \bar{R}$, of the gauge group with Lie algebra generated by $T^a$, normalized such that $\Tr_{fund}(T^a T^b) = \delta^{ab}$.

The $R$-charges determined by minimizing $|Z|^2$ are given by
\[
\Delta = \frac{1}{2} - \frac{N_f + 1}{2k^2} + O\left( \frac{1}{k^4} \right),
\]
in perfect agreement with the perturbative field theory calculation, which gives $b_0 = 2(N_f + 1)$.

### 6.4 $SU(2)$ Chern-Simons

The integral over the Cartan of $SU(2)$ again involves only a single integration variable. The Vandermonde and 1-loop determinant for the gauge sector are now non-trivial.

Consider $SU(2)$ Chern-Simons theory at level $k$ and $N_f$ chiral multiplets in the self-conjugate 2 representation. The $R$-symmetry may mix with the $U(1)$ flavor symmetry under which all of the matter has the same charge.

The partition function on the sphere is given by
\[
Z = \int_{-\infty}^{\infty} du \ \sinh^2(2\pi u) e^{2i\pi ku^2} e^{N_f(\ell(1-\Delta+iu)+\ell(1-\Delta-iu))},
\]
up to an overall $\Delta$ independent constant.

\footnote{This differs slightly from the normalization in \cite{Gaiotto:2017yup}.}
Calculating the large $k$ asymptotics of $Z$, and solving the equation $\text{Re} \left( \partial_\Delta \log Z \right) = 0$, determines the infrared dimensions to be

$$\Delta = 1/2 - \frac{3}{8k^2}(N_f - 1) + O\left(\frac{1}{k^4}\right),$$

again agreeing with the perturbative computation, for which $b_0 = \frac{3}{2}(N_f - 1)$. In this case, the representation $R$ is $\frac{N_f}{2}$ copies of the fundamental of $SU(2)$, in the notation of [15] in which the matter is in the $R \oplus \bar{R}$.

For $g$ adjoint chiral multiplets, the method of $Z$-extremization gives

$$\Delta = \frac{1}{2} - \frac{4}{k^2}(g + 1).$$

This exactly matches the two-loop calculation in appendix (D.1) of [15].

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A Conventions for spinors and derivatives on $S^3$

I use the following conventions, based on [21].

On $S^3$, spinors have 2 complex components, and transform in the fundamental of $SU(2) = \text{Spin}(3)$.

Regarding $S^3$ as the $SU(2)$ group manifold, one can find an orthonormal triplet of vector fields that are invariant under the left (resp. right) action of $SU(2)$, denoted by $\ell^i_\mu$ (resp. $\xi^i_\mu$), where $i = 1, 2, 3$ is a frame index.

The Laplacian on the sphere of radius $r$ can be expanded as $r^2\nabla^2 = \sum (\varphi')^2 = \sum (\ell^i)^2$. Defining $L_i = -\frac{i}{2}\ell_i$, one can check that $[L_i, L_j] = i\epsilon_{ijk}L_k$.

One can use the left invariant vector fields to define a vielbein as the dual 1-forms, $e^i_\mu$. The gamma matrices can be written using frame indices, $\gamma_\mu = e^k_\mu \gamma_k$, where the $\gamma_k$ are just the Pauli matrices, satisfying $[\gamma_i, \gamma_j] = 2i\epsilon_{ijk}\gamma^k$. Let $S^i = \frac{1}{2}\gamma^i$.

In the left invariant frame, the spin connection is given by $\omega_{ij} = e_{ijk}e^k$, and the spinor covariant derivative is $\nabla_\mu = \partial_\mu + \frac{i}{2}e^k_\mu \gamma_k$. 21
Half of the 4 Killing spinors on the sphere are constant in this frame, satisfying
\[ \nabla_\mu \varepsilon_a = \frac{i}{2} \epsilon_\mu^k \gamma_k \varepsilon_a = \frac{i}{2} \gamma_\mu \varepsilon_a. \]
These 2 Killing spinors are invariant under the left SU(2) by construction, and transform in a doublet of the right SU(2) (that is the \( a \) index).

Pick one of the left invariant Killing spinors, and define a left invariant vector field
\[ v_\mu = \varepsilon^\dagger \gamma_\mu \varepsilon, \]
which can be identified as \( \ell^3 \). Therefore \( \phi = \gamma^3 = 2S^3 \).

The other pair of Killing spinors satisfy \( \nabla_\mu \eta_b = -\frac{i}{2} \gamma_\mu \eta_b \), and transform in a doublet of the left SU(2). They are constant in the right invariant frame constructed from the \( i_\mu \).

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