ACCELERATING PLANAR ORNSTEIN-UHLENBECK DIFFUSION WITH SUITABLE DRIFT

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ABSTRACT. The principal aim of this paper is to construct an explicit sequence of weighted divergence free vector fields which accelerates the rate of convergence of planar Ornstein-Uhlenbeck diffusion to its equilibrium state. The rate of convergence is expressed in terms of the spectral gap of the diffusion generator. We construct an explicit sequence of vector fields which pushes the spectral gap to infinity. The acceleration of the diffusion results from the strong oscillation of the flow lines generated by the vector field.

1. Introduction. The main objective of this paper is to study the influence of some oscillatory measure preserving drift on the rate of convergence of some diffusion process to its equilibrium state. Let us informally introduce our topic in some general setting. We assume $V$ to be a smooth function on $\mathbb{R}^d$ such that $\varphi(x) = \frac{1}{Z} e^{-V(x)}$ is a probability density with respect to the Lebesgue measure on $\mathbb{R}^d$, where $Z$ is a suitable normalizing constant. The resulting probability measure on $\mathbb{R}^d$ will be denoted by $\gamma_\varphi$. We are interested in the diffusions $X^u$ which are solution of the stochastic differential equation

$$dX(t) = -\nabla V(X(t))dt + \sqrt{2}dB(t) + u(X(t))dt,$$

where $B$ is a Brownian motion and $u$ is a vector field satisfying $\text{div}(ue^{-V}) = 0$. Together with some suitable growth condition on the function $V$ this last property ensures that the process $X^u$ has $\gamma_\varphi$ as its unique equilibrium measure. Note that $X^u$ is reversible if and only if $u = 0$.

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A classical result on the asymptotic behavior of the diffusion $X^u$ is given by the following $L^2$-inequality which holds for all $t > 0$ (see [12]):

$$\left\| \mathbb{E}(f(X^u(t))) - \int_{\mathbb{R}^d} f \, d\gamma_\varphi \right\| \leq K_u e^{-t \rho_u} \left\| f - \int_{\mathbb{R}^d} f \, d\gamma_\varphi \right\|.$$ 

In this inequality $K_u$ is a suitable positive constant and

$$\rho_u := -\sup \left\{ \text{Re}(z) ; z \in \text{sp}(A_u) \setminus \{0\} \right\}$$

is the spectral gap of the infinitesimal generator of the diffusion $X^u$ which acts on smooth functions with compact support as

$$A_u f = \Delta f - \nabla V \cdot \nabla f + u \cdot \nabla f.$$ 

It follows from this that the spectral gap is an important indicator for understanding the speed at which the diffusion generated by $A_u$ converges to its equilibrium distribution. We want to understand which specific feature of the vector field forces this spectral gap to become large. This is interesting to know, since the spectral gap is often used to measure the performance of Markov chain Monte Carlo methods. Those algorithms use processes of the type $X$ to approximate integrals with respect to $\gamma_\varphi$ through iterated simulations (see [9]). It was shown by Hwang, Hwang-Ma and Sheu in [11] and [12] that the optimal algorithms are to be found in the class of non-reversible diffusions, where the vector field $u$ is non-zero and the generator $A_u$ is not self-adjoint. In other words, for $u \neq 0$, the process $X^u(t)$ converges to the equilibrium faster than $X^0(t)$ as $t$ goes to infinity. In fact they showed that $\rho_u \geq \rho_0$ for all $u$ satisfying $\text{div}(ue^{-V}) = 0$. Moreover, they proved that the inequality is strict whenever the flow generated from the vector field $u$ does not preserve a subspace of the eigenspace associated to the first non-zero eigenvalue of the operator $A_0$. An explicit asymptotic expression for the spectral gap resulting from the multiplication of the vector field $u$ with a large constant is found in Franke, Hwang, Pai and Sheu [6]. In the same context Constantin, Kiselev, Ryzhik and Zlatos proved in [4] that the operator norm of the resulting semi group converges toward zero if and only if the operator $u \cdot \nabla$ don’t have eigen-functions in the Sobolev space $H^1$. This implies the convergence of the spectral gap to infinity. For example vector fields $u$ generating weak mixing flows satisfy the above condition. However, the existence of such flows is not always guaranteed and in some cases, like for example on two dimensional spaces, those flows do not exist. Moreover, the existence of mixing flows is usually proved by non-constructive methods and thus those flows are not explicit, which makes their application in numerical simulations difficult. This leads to the question, whether one can find a sequence of explicit vector fields $u_n$ such that $\rho_{u_n}$ diverges to infinity. The problem was studied on $d$ dimensional torus by Hwang and Pai in [14] and by Franke and Yaakoubi for various compact two-dimensional Riemannian manifolds in [7] and [8]. In those papers the authors prove that the spectral gap denoted by $\rho_u$ can be arbitrarily large with a suitable choice of $u$. In this paper we will study the case of the two dimensional Ornstein-Uhlenbeck diffusion (i.e.: $V(x) = x^2$), which is of some importance in probability theory, where it is used to modelize mean reverting processes. It grew from the PhD thesis of N. Yaakoubi which was directed by B. Franke and M. Damak (see [15]) and completes some of the arguments given in this thesis.

The proof is based on the construction of strongly oscillating periodic flow-lines in the plane. Those flow-lines are used to define a weighted divergence free vector
field $u$ in $\mathbb{R}^2$. This vector field $u$ can be modified in such a way that the differential operator $u \cdot \nabla$ has no $H^1$-eigenfunctions associated to non-zero eigenvalues. Only the zero eigenvalue has $H^1$-eigenfunctions, which then are essentially constant along the flow-lines. In this context the result from [6] yields an explicit asymptotic expression for the spectral gap $\rho_{cu}$ as $c \to \infty$ in terms of some Rayleigh quotient over functions from the $H^1$-eigenspaces of the operator $u \cdot \nabla$. Note that by the special construction of $u$, only functions which are invariant along the flow-lines have to be considered in this quotient. Increasing the oscillation of the flow-lines implies longer level sets for those functions which results in large gradients and thus a large Rayleigh quotient. To prove this we use the isoperimetric structure of the flow-invariant sets in $\mathbb{R}^2$ to introduce some suitable comparison measure and prove some Faber-Krahn type rearrangement inequality for flow-invariant $H^1$-functions.

This general scheme was used in [7] and [8] for compact two dimensional spaces. The Ornstein Uhlenbeck process being a process on an unbounded domain, a considerable amount of effort has to be spent to understand the asymptotic behavior of the isoperimetric structure of the flow-invariant sets as the oscillation increases.

It is also worthy to note that the explicit construction of the flow lines has the advantage to bypass the difficult task of integrating the vector field which is necessary to run the Monte Carlo simulations.

An interesting question is whether the method presented in this manuscript can be generalized to higher dimensional Ornstein Uhlenbeck diffusion. While most arguments can be carried over to higher dimensions, the behavior of the isoperimetric problem as oscillation grows is quite difficult to estimate in higher dimensions.

1.1. Formal definitions and abbreviations. On the plane $\mathbb{R}^2$, let

$$\varphi(x) = \frac{1}{2\pi} e^{-|x|^2/2}$$

be the density of the normalized 2-dimensional Gaussian measure $\gamma_\varphi = \varphi \gamma$ where $\gamma$ denotes the Lebesgue measure on $\mathbb{R}^2$. The weighted arc length measure with respect to the gaussian measure $\gamma_\varphi$ will be denoted $\ell_\varphi$. This means that $\ell_\varphi = \varphi \ell$, where $\ell$ is the euclidean arc length measure on $\mathbb{R}^2$. The spectra of the operators we consider are complex valued and the eigenfunctions will be complex valued functions as well. We therefore introduce the following Hilbert space of mean-zero complex-valued functions:

$$H(\mathbb{R}^2, \gamma_\varphi) = \left\{ f = f_1 + if_2; f_1, f_2 \in L^2(\mathbb{R}^2, \gamma_\varphi) : \int_{\mathbb{R}^2} f_1 d\gamma_\varphi = \int_{\mathbb{R}^2} f_2 d\gamma_\varphi = 0 \right\}$$

with scalar product and norm

$$\langle f, g \rangle = \int_{\mathbb{R}^2} \overline{f} g d\gamma_\varphi \quad \text{and} \quad ||f||^2 = \langle f, f \rangle.$$

We define the following Sobolev space of mean-zero functions:

$$H^1 = \left\{ f = f_1 + if_2 \in H(\mathbb{R}^2, \gamma_\varphi) : \int_{\mathbb{R}^2} |\nabla f_1|^2 d\gamma_\varphi + \int_{\mathbb{R}^2} |\nabla f_2|^2 d\gamma_\varphi < \infty \right\}$$

with scalar product and norm

$$\langle f, g \rangle_1 = \int_{\mathbb{R}^2} \nabla f \overline{\nabla g} d\gamma_\varphi \quad \text{and} \quad ||f||^2_1 = \langle f, f \rangle_1.$$

Note that

$$A_0 f = \Delta f - x_1 \partial_1 f - x_2 \partial_2 f$$
is the generator of the ordinary two dimensional Ornstein-Uhlenbeck process, which has $\gamma_\varphi$ as its invariant probability measure. This process is reversible and converges in law toward $\gamma_\varphi$.

**Remark 1.** The operator $A_0$ has only discrete spectrum as an operator on $L^2(\mathbb{R}^2, \gamma_\varphi)$ (see [13]).

For $f \in C^\infty(\mathbb{R}^2)$, we define the operator

$$A_u f = \Delta f - x_1 \partial_1 f - x_2 \partial_2 f + u \cdot \nabla f,$$

where $u$ is a vector field satisfying $\text{div}(u_\varphi) = 0$.

### 1.2. Main result.

The notion of the spectral gap

$$\rho_u = -\sup \left\{ \text{Re}(z); z \in \sigma(A_u) \setminus \{0\} \right\}$$

is often used to quantify the acceleration of the diffusion introduced above. The following is our main result:

**Theorem 1.1.** There exists a sequence of vector fields $(u_n)_{n \in \mathbb{N}}$ with the property $\text{div}(u_n \varphi) = 0$ such that one has

$$\lim_{n \to \infty} \rho_{u_n} = \infty.$$

### 2. Construction of the flow lines.

In this section we will construct some strongly oscillating flow in the plane. Each flow line is obtained from a rotation around the origin with some simultaneous oscillation in radial direction. Later in Proposition 1 the isoperimetric behavior of those flow lines will be studied as the number of oscillations grows to infinity. Consider the $C^1$-function $\eta$:

$$\eta(r) := \begin{cases} r & \text{for } r \in ]0, \frac{1}{3}[; \\ \frac{1}{3} + \frac{1}{2} \sin (\pi(3r - 1)) & \text{for } r \in ]\frac{1}{3}, \frac{1}{2}[; \\ \frac{1}{3} + \frac{1}{2} & \text{for } r \in ]\frac{1}{2}, \infty[. \end{cases}$$

For $n \in \mathbb{N}$, let us consider the $C^1$- function $h_n : \mathbb{R}^+ \times [0, 2\pi[ \to \mathbb{R}^2$ defined by:

$$h_n(r, \theta) := \left( r \left( 1 + \frac{1}{2} \eta(r) \sin(n\theta) \right) \cos(\theta), r \left( 1 + \frac{1}{2} \eta(r) \sin(n\theta) \right) \sin(\theta) \right).$$

For all $r > 0$, we note the graph of the function $\theta \mapsto h_n(r, \theta)$ by:

$$\Gamma_r^{(n)} := \left\{ h_n(r, \theta); \theta \in [0, 2\pi[ \right\}.$$

**Remark 2.** For every $r > 0$, the set $\Gamma_r^{(n)}$ is a closed curve. Further, since

$$\frac{d}{dr} \left| h_n(r, \theta) \right| = \frac{d}{dr} \left( r \left( 1 + \frac{1}{2} \eta(r) \sin(n\theta) \right) \right) > 0$$

for all $\theta \in [0, 2\pi[ \text{ and } r > 0$, it follows that the graphs $\Gamma_r^{(n)}$ are disjoint. Note $\hat{\Gamma}_r^{(n)}$ the open domain enclosed by the curve $\Gamma_r^{(n)}$.

**Lemma 2.1.** As $r \to 0$, one has that $\ell_\varphi(\Gamma_r^{(n)}) \sim \ell(\partial B_r(0))$ and $\gamma_\varphi(\hat{\Gamma}_r^{(n)}) \sim \gamma(B_r(0))$ where $B_r(0)$ denotes the ball of radius $r$ in $\mathbb{R}^2$. 

Proof. At first, note that, for \( r < \frac{1}{3} \), the curve \( \Gamma_r^{(n)} \) oscillates \( n \) times between two concentric circle of radius \( (r - \frac{\pi^2}{r^3}) \) and of radius \( (r + \frac{\pi^2}{r^3}) \). Also, note that the gaussian density \( \varphi \) decays with \( r \). Thus,

\[
2\pi\left(r - \frac{\pi^2}{2}\right)\varphi\left(r + \frac{\pi^2}{2}, 0\right) \leq \ell_\varphi(\Gamma_r^{(n)}) \leq \left(2\pi\left(r + \frac{\pi^2}{2}\right) + 2nr^2\right)\varphi\left(r - \frac{\pi^2}{2}, 0\right).
\]

Thanks to those considerations, there exist two constants \( \beta_1 < 0 \) and \( \beta_2 < 0 \) such that:

\[
2\pi\left(r - \frac{\pi^2}{2}\right)\left(1 + \beta_1\left(r + \frac{\pi^2}{2}\right)^2\right) \leq \ell_\varphi(\Gamma_r^{(n)}) \leq \left(2\pi\left(r + \frac{\pi^2}{2}\right) + 2nr^2\right)\left(1 + \beta_2\left(r - \frac{\pi^2}{2}\right)^2\right).
\]

After dividing the 3 sides of those inequalities by the arc length of \( \partial B_r(0) \), we obtain

\[
\frac{2\pi(r - \frac{\pi^2}{2})(1 + \beta_1(r + \frac{\pi^2}{2})^2)}{2\pi r} \leq \frac{\ell_\varphi(\Gamma_r^{(n)})}{\ell(\partial B_r(0))} \leq \frac{(2\pi(r + \frac{\pi^2}{2}) + nr^2)(1 + \beta_2(r - \frac{\pi^2}{2})^2)}{2\pi r}.
\]

We obtain the first assertion when taking the limit \( r \to 0 \). The second assertion follows from the first by integration over \( r \).

**Remark 3.** For \( r > \frac{1}{2} \), the function

\[
r \mapsto re^{-\frac{\pi^2}{2}(1 + \frac{1}{2}\frac{1 + \frac{1}{2}}{2\pi^2})\sin(n\theta)^2}\sqrt{(1 + \frac{1}{2}\frac{1 + \frac{1}{2}}{2\pi^2})\sin(n\theta)^2 + n^2\frac{1}{4}\left(1 + \frac{1}{2\pi^2}\right)^2\cos^2(n\theta)}
\]

determines the weight of an infinitesimal arc length element of the curve \( \Gamma_r^{(n)} \) at \( \theta \). It is maximal in \( r_{max} = \frac{\pi r}{\pi + \frac{1}{2}}\sin(n\theta) \). We note that the arc length element at \( \theta \) of \( \Gamma_r^{(n)} \) is increasing in \( \left[\frac{1}{2}, r_{max}\right] \) and decreasing in \( [r_{max}, \infty[ \). However, \( r_{max} \) depends on \( \theta \) and \( n \). The biggest possible value of \( r_{max} \) is equal to \( \frac{6\pi}{\pi + 1} \) and the smallest equals \( \frac{6\pi}{\pi + 1} \). As a conclusion, \( r \mapsto \ell_\varphi(\Gamma_r^{(n)}) \) is increasing in \( \left[\frac{1}{2}, \frac{6\pi}{\pi + 1}\right] \), and decreasing in \( \left[\frac{6\pi}{\pi + 1}, \infty[ \).

**Remark 4.** Since for all \( \alpha \in [-1, 1] \) and all \( n \in \mathbb{N} \) the sets \( \{\theta \in [0, 2\pi]; \sin(\theta) < \alpha\} \) and \( \{\theta \in [0, 2\pi]; \sin(n\theta) < \alpha\} \) have the same Lebesgue measure, it follows that for all \( r > 0 \) and \( n \in \mathbb{N} \) one has

\[
\gamma_\varphi(\Gamma_r^{(n)}) = \gamma_\varphi(\hat{\Gamma}_r^{(1)}).
\]

**Remark 5.** For \( c \in [0, 1] \), let \( r_c \) denote the radius such that

\[
\gamma_\varphi(\hat{\Gamma}_r^{(n)}) = c.
\]

It follows from the previous remark that \( r_c \) does not depend on \( n \). When \( r_c \geq \frac{1}{2} \), let \( D_{int} \) and \( D_{ext} \) be the open disks around zero in \( \mathbb{R}^2 \) of radius respectively \( \frac{(5\pi - 1)r_c}{6\pi} \) and \( \frac{(7\pi - 1)r_c}{6\pi} \). From the expression of \( h_n \), \( D_{int} \) is the biggest open disk strictly contained in \( \hat{\Gamma}_r^{(n)} \) and \( D_{ext} \) is the smallest open disc which contains \( \hat{\Gamma}_r^{(n)} \), i.e.:

\[
D_{int} \subset \hat{\Gamma}_r^{(n)} \subset D_{ext}.
\]

**Remark 6.** For \( c \in [0, 1] \), the expression \( R_c := \sqrt{-2\ln(1 - c)} \) satisfies that

\[
\gamma_\varphi(B_{R_c}(0)) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{R_c} re^{-\frac{\pi^2}{2}} d\theta = c.
\]

Moreover, since we have that

\[
\gamma_\varphi(D_{int}) \leq \gamma_\varphi(B_{R_c}(0)) = c \leq \gamma_\varphi(D_{ext}),
\]
it follows that
\[
\frac{(5\pi - 1)r_e}{6\pi} \leq R_e \leq \frac{(7\pi + 1)r_e}{6\pi}.
\]

**Lemma 2.2.** There exist four positive constants \(C_1, C_2, C_3\) and \(C_4\) such that for all \(r > 0\), one has that
\[
\ell_\varphi(\Gamma_r^{(n)}) \geq C_1 r_e e^{-C_2 r^2} \sqrt{C_3 + n^2 C_4 \eta^2(r)}.
\]

**Proof.** Since for all \(r > 0\), the infinitesimal arc length element of the curve \(\Gamma_r^{(n)}\) in polar coordinates equals
\[
\sqrt{\left(1 + \frac{1}{2}\eta(r) \sin(n\theta)\right)^2 + \left(n\frac{1}{2}\eta(r) \cos(n\theta)\right)^2} d\theta,
\]
it follows that
\[
\ell_\varphi(\Gamma_r^{(n)}) \geq \frac{1}{2\pi} \int_0^{2\pi} \left(1 + \frac{1}{2}\eta(r) \sin(n\theta)\right) r e^{-\frac{1}{2}(1+\frac{1}{2}\eta(r))^2} \sin(n\theta) d\theta.
\]

Proposition 1. We have
\[
\inf_{0 < c < 1} \frac{\ell_\varphi(\Gamma_r^{(n)})}{c} \to \infty \quad \text{as} \quad n \to \infty
\]

and
\[
\inf_{0 < c < 1} \frac{\ell_\varphi(\Gamma_{r_1-\epsilon}^{(n)})}{c} \to \infty \quad \text{as} \quad n \to \infty.
\]

**Proof.** The proof will address the following three cases separately. The first case is \(\frac{1}{3} \leq r_e \leq \frac{6\pi}{5\pi - 1}\), the second is \(\frac{6\pi}{5\pi - 1} < r_e\) and the last is \(r_e < \frac{1}{3}\).

1) For the first case, i.e. \(\frac{1}{3} \leq r_e \leq \frac{6\pi}{5\pi - 1}\), we use Lemma 2.2 to see that
\[
\inf_{0 < c < \frac{1}{2}} \frac{\ell_\varphi(\Gamma_r^{(n)})}{c} = \inf_{\frac{1}{3} \leq r_e \leq \frac{6\pi}{5\pi - 1}} \frac{C_1 r_e e^{-C_2 r^2} \sqrt{C_3 + n^2 C_4 \eta^2(r)}}{c} \to \infty.
\]
In exactly the same way, we can show that:

\[
\inf_{0 < c < \frac{1}{2}, \frac{1}{3} \leq r_1 - c \leq \frac{6\pi}{5\pi - 1}} \ell_{\phi}(\Gamma_{r_1 - c}) \geq \frac{2}{3} C_1 e^{-C_2 \left(\frac{6\pi}{5\pi - 1}\right)^2} \sqrt{C_3 + n^2 \frac{C_4}{9}} \quad n \to \infty.
\]

2) When \( r_c > \frac{6\pi}{5\pi - 1} \), we use inequality (1) to see that

\[
R'_c := \frac{6\pi}{5\pi - 1} R_c \geq r_c.
\]

Using the fact that \( r \mapsto \ell_{\phi}(\Gamma_r) \) is decreasing for \( r > \frac{6\pi}{5\pi - 1} \), we obtain that

\[
\ell_{\phi}(\Gamma_{R'_c}) \geq \ell_{\phi}(\Gamma_{r_c}).
\]

It follows then that

\[
\inf_{0 < c < \frac{1}{2}, \frac{6\pi}{5\pi - 1} < r_c} \frac{\ell_{\phi}(\Gamma_{r_c})}{c} \geq \inf_{0 < c < \frac{1}{2}, \frac{6\pi}{5\pi - 1} < r_c} \frac{\ell_{\phi}(\Gamma_{R'_c})}{c}.
\]

From Lemma 2.2, it follows that

\[
\ell_{\phi}(\Gamma_{R'_c}) \geq C_1 R'_c e^{-C_2(R'_c)^2} \sqrt{C_3 + n^2 C_4} \eta^2(R'_c)
\]

\[= C_1 R'_c e^{-C_2(R'_c)^2} \sqrt{C_3 + n^2 \frac{C_4}{9}}.
\]

Replacing \( R'_c = \frac{6\pi}{5\pi - 1} \sqrt{-2 \ln(1 - c)} \) in the previous expression yields that

\[
\ell_{\phi}(\Gamma_{R'_c}) \geq C_1 \frac{6\pi}{5\pi - 1} \sqrt{-2 \ln(1 - c)} e^{2C_2 \left(\frac{6\pi}{5\pi - 1}\right)^2 \ln(1 - c)} \sqrt{C_3 + n^2 \frac{C_4}{9}}
\]

with \( C_2 = \left(\frac{1 + \frac{1}{2} \left(\frac{1}{2} + \frac{1}{\pi}\right)^2}{2}\right) \) (see proof of Lemma 2.2). It follows that

\[
\inf_{0 < c < \frac{1}{2}, \frac{6\pi}{5\pi - 1} < r_c} \frac{\ell_{\phi}(\Gamma_{R'_c})}{c} \geq C_1 \frac{6\pi}{5\pi - 1} \sqrt{C_3 + n^2 \frac{C_4}{9}} \inf_{0 < c < \frac{1}{2}} \sqrt{-2 \ln(1 - c)(1 - c)}^{\left(\frac{7\pi - 1}{\pi}\right)^2} \left(\frac{6\pi}{5\pi - 1}\right)^2 \frac{C_2}{c}
\]

However, the function

\[
c \to \frac{1}{c} \sqrt{-2 \ln(1 - c)(1 - c)}^{\left(\frac{7\pi - 1}{\pi}\right)^2} \left(\frac{6\pi}{5\pi - 1}\right)^2 \frac{C_2}{c}
\]

is bounded away from zero on the interval \([0, \frac{1}{3}]\). From this follows that

\[
\inf_{0 < c < \frac{1}{2}, \frac{6\pi}{5\pi - 1} < r_c} \frac{\ell_{\phi}(\Gamma_{r_c})}{c} \quad n \to \infty.
\]
Note that for \( a_n := \sqrt{\frac{1}{c^2} \ln \sqrt{n}} \) we have
\[
\inf_{\frac{a}{n^{\frac{1}{2}}} < r < \frac{1}{3}} \frac{\ell_r(\Gamma_{r_{1-n}})}{c} = \inf_{\frac{a}{n^{\frac{1}{2}}} < r < \frac{1}{3}} \frac{\ell_r(\Gamma_{r_{1-n}})}{\gamma_r((\widehat{\Gamma}_r^{(n)})^c)} \geq \inf_{\frac{a}{n^{\frac{1}{2}}} < r} \frac{\ell_r(\Gamma_r^{(n)})}{\gamma_r((\widehat{\Gamma}_r^{(n)})^c)} = \min \left\{ \inf_{a_n < r} \frac{\ell_r(\Gamma_r^{(n)})}{\gamma_r((\widehat{\Gamma}_r^{(n)})^c)}, \inf_{\frac{a}{n^{\frac{1}{2}}} < r < a_n} \gamma_r((\widehat{\Gamma}_r^{(n)})^c) \right\}.
\]
We will show that both expressions in the minimum go to infinity. First, note that for \( r > \frac{1}{2} \) we have
\[
\gamma_r((\widehat{\Gamma}_r^{n})^c) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^\infty se^{-\frac{r^2}{2}} ds d\theta
= \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{r^2}{2}} \left(1 + \frac{1}{2} \eta(r) \sin(n \theta)\right)^2 d\theta
\]
and
\[
\ell_r(\Gamma_r^{(n)}) \geq \frac{1}{2\pi} \int_0^{2\pi} \left(1 + \frac{1}{2} \eta(r) \sin(n \theta)\right) re^{-\frac{r^2}{2}} \left(1 + \frac{1}{2} \eta(r) \sin(n \theta)\right)^2 \times \sqrt{\frac{2}{3}} + \frac{n^2}{4} \eta^2(r) \cos^2(n \theta) d\theta
\geq \frac{\sqrt{2}}{3} \frac{r}{2\pi} \int_0^{2\pi} \left(1 + \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3\pi}\right) \sin(n \theta)\right) e^{-\frac{r^2}{2}} \left(1 + \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3\pi}\right) \sin(n \theta)\right)^2 d\theta
\geq r \sqrt{\frac{2}{3}} \left(1 - \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3\pi}\right)\right) \gamma_r((\widehat{\Gamma}_r^{n})^c).
\]
It thus follows that uniformly in \( n \in \mathbb{N} \)
\[
\frac{\ell_r(\Gamma_r^{(n)})}{\gamma_r((\widehat{\Gamma}_r^{n})^c)} \to \infty.
\]
Since \( a_n \to \infty \) this implies that
\[
\inf_{a_n < r} \frac{\ell_r(\Gamma_r^{(n)})}{\gamma_r((\widehat{\Gamma}_r^{n})^c)} \to \infty.
\]
For \( \frac{6\pi}{5n-1} < r \leq a_n \leq \) we have that
\[
\gamma_r((\widehat{\Gamma}_r^{n})^c) = \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{r^2}{2}} \left(1 + \frac{1}{2} \eta(r) \sin(n \theta)\right)^2 d\theta
\leq \frac{1}{2\pi} \int_0^{2\pi} e^{-\frac{r^2}{2}} \left(1 + \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3\pi}\right)\right)^2 d\theta
\leq e^{-\frac{r^2}{2}} \left(1 - \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3\pi}\right)^2\right)
\leq e^{-\frac{r^2}{2}} \left(\frac{6\pi}{5n-1}\right)^2 \left(1 - \frac{1}{2} \left(\frac{1}{3} + \frac{1}{3\pi}\right)^2\right) =: C_5.
It then follows from Lemma 2.2 and the choice of $a_n$ that

$$\inf_{0 < r < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} \geq \frac{1}{C_5} \inf_{0 < r < a_n} \frac{C_1 r e^{-C_2 r^2}}{C_3 + n^2 C_4 r^2} \geq \frac{1}{C_5} \frac{6\pi}{5\pi - 1} \frac{C_1}{C_3} \sqrt{C_3 + n^2 \frac{C_4}{9}} = C_1 \frac{6\pi}{5\pi - 1} \frac{1}{\sqrt{n}} \sqrt{C_3 + n^2 \frac{C_4}{9}} \to \infty. n \to \infty.$$ 

Thus we have proved that

$$\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} \to \infty, n \to \infty.$$ 

3) For the case when $r_c \leq \frac{1}{2}$, Lemma 2.2 yields

$$\ell_{\varphi}(\Gamma_r^{(n)}) \geq C_1 r_c e^{-C_2 r_c^2} \sqrt{C_3 + n^2 C_4 r_c^2} \geq C_1 r_c e^{-C_2 r_c^2} \sqrt{C_3 + n^2 C_4 r_c^2},$$

We have that

$$\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} = \min\left\{ \inf_{r \leq \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c}, \inf_{\frac{1}{2} < r < \frac{3}{4}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} \right\}.$$ 

By the above consideration, the second expression in the minimum is bigger than

$$2C_1 e^{-C_2 r_c^2} \frac{1}{\sqrt{n}} \sqrt{C_3 + n^2 C_4 \frac{1}{\sqrt{n}}} = 2C_1 e^{-C_2 r_c^2} \sqrt{C_3 \sqrt{n} + C_4 n^4} \to \infty.$$ 

By Lemma 2.1, $c = \gamma_{\varphi}(\Gamma_r^{(n)}) \sim \pi r_c^2$. This implies that

$$\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} \geq \inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} \to \infty, n \to \infty.$$ 

This shows that

$$\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_r^{(n)})}{c} \to \infty, n \to \infty.$$
On the other hand we have from Lemma 2.2 and Remark 6 that
\[
\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_{r_1, r_2})}{c} \geq \inf_{0 < c < \frac{1}{3}} \frac{C_1 r_1 - \frac{1}{c}}{c} \sqrt{C_3 + n^2 C_4 \eta^2 (r_1 - c)}
\]
\[
\geq C_1 e^{-C_2 \frac{1}{c}} \inf_{0 < c < \frac{1}{3}} \frac{r_1 - \frac{1}{c}}{c} \sqrt{C_3 + n^2 C_4 \eta^2 (r_1 - c)}
\]
\[
\geq \frac{6 \pi C_1 e^{-C_2 \frac{1}{c}}}{7 \pi + 1} \inf_{0 < c < \frac{1}{3}} \frac{r_1 - \frac{1}{c}}{c} \sqrt{C_3 + n^2 C_4 \left(\frac{6 \pi}{1 + 1}\right)^2 (R_1 - c)^2}
\]
\[
= \frac{6 \pi C_1 e^{-C_2 \frac{1}{c}}}{7 \pi + 1} \inf_{0 < c < \frac{1}{3}} \frac{\sqrt{-2 \ln c}}{c} \sqrt{C_3 + n^2 C_4 \left(\frac{6 \pi}{1 + 1}\right)^2 (R_1 - c)^2}
\]
\[
\geq C_1 e^{-C_2 \frac{1}{c}} \frac{6 \pi}{7 \pi + 1} \sqrt{-2 \ln \frac{1}{2} c} \sqrt{C_3 + n^2 C_4 \left(\frac{6 \pi}{1 + 1}\right)^2 (R_1 - c)^2}
\]
\[
= C_6 \sqrt{C_3 + n^2 C_4} \rightarrow \infty.
\]

As a conclusion, we see that
\[
\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_{r_1, r_2})}{c} \geq \min \left\{ \inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_{r_1, r_2})}{c} \right\}
\]
\[
\text{goes to infinity as } n \rightarrow \infty. \text{ Furthermore, we also have that}
\]
\[
\inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_{r_1, r_2})}{c} \geq \min \left\{ \inf_{0 < c < \frac{1}{2}} \frac{\ell_{\varphi}(\Gamma_{r_1, r_2})}{c} \right\}
\]
\[
\text{goes to infinity as } n \rightarrow \infty. \text{ This proves our proposition.}
\]

3. Construction of the vector field. In this section we construct a sequence of vector fields $u_n$ which will have the property to generate the sequence of flow lines $\Gamma_{n}^{(n)}$ which was introduced in the previous section. In Proposition 4 we will see that $u_n$ can be chosen in such a way that the operator $u_n \cdot \nabla$ has no $H^1$-eigenfunctions except the ones associated to the zero eigenvalue. In order to achieve this, we first use the implicit value theorem to construct a $C^1$-function $\alpha_n^*$ which is constant on the orbits $\Gamma_{r_1}^{(n)}$. From this function, we obtain a vector field $u_0$ by applying some symplectic gradient $\nabla^+$ to the function $\alpha_n^*$ and multiplying with $\varphi^{-1}$. This vector field is tangential to the orbit $\Gamma_{r_1}^{(n)}$ and generates a flow which follows the curves $\Gamma_{r_1}^{(n)}$ and also satisfies our requirement $\text{div}(\varphi u_0) = 0$. However, we have to modify this vector field by a multiplication with a suitable function $S$ to ensure that the resulting vector field $u_n$ leads to a well defined anti-symmetric operator $u_n \cdot \nabla$ with domain containing $H^1$ and without $H^1$ eigenfunctions to non-zero eigenvalues.

For a $C^1$-function $F$ we introduce its gradient
\[
\nabla F = \left( \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2} \right)^t
\]

In the following we will use polar coordinates \((r, \theta)\). Obviously we have the two following properties:

- \((\nabla F)\cdot (\nabla F) = 0\);
- \(\text{div}(\nabla F) = 0\).

In the following we will use polar coordinates \((r, \theta)\) to parametrize \(\mathbb{R}^2\). For \(n \in \mathbb{N}\) we use the implicit function theorem to define a \(C^1\)-function \(\alpha_n^* : \mathbb{R}^+ \times [0, 2\pi] \rightarrow \mathbb{R}^+\) which satisfies for all \((r, \theta) \in \mathbb{R}^+ \times [0, 2\pi]\)

\[
\alpha_n^*(r, \theta) = \left(1 + \frac{1}{2} \eta(\alpha_n^*(r, \theta)) \sin(n\theta)\right) = r.
\]

It follows from this construction, that the function \(\alpha_n\) is constant on the sets \(\Gamma_r^{(n)}\) for all \(r > 0\). We denote by \(\alpha_n\) the representation of \(\alpha_n^*\) with respect to cartesian coordinates in the plane; i.e.: for all \((r, \theta) \in \mathbb{R}^+ \times [0, 2\pi]\) we have

\[
\alpha_n(r, \varphi) = \alpha_n(r \cos \varphi, r \sin \varphi).
\]

Note that we put \(\alpha_n\) to be zero in the origin to obtain a well defined differentiable function on the plane. From this family of functions, we define a sequence of vector fields \(u_0^n\) on \(\mathbb{R}^2\) by:

\[
u_n^0 := \frac{1}{\varphi} \nabla \perp \alpha_n.
\]

By construction \(u_0^n\) is a vector field which satisfies \(\text{div}(\varphi u_0^n) = 0\). Moreover, it generates a flow following the curves \(\Gamma_r^{(n)}\) which are also the level sets of the function \(\alpha_n\). Since the vector field \(u_0^n\) is not bounded it might not lead to a well defined operator \(u_0^n \cdot \nabla\) on \(H^1\). This point will be addressed in Proposition 2.

**Remark 7.** In order to obtain some bounds on the return times of the flow generated by the vector field \(u_0^n\), we introduce polar coordinates in the plane; i.e.: \((x_1, x_2) = (r \cos \theta, r \sin \theta)\) for \(r > 0\) and \(\theta \in [0, 2\pi]\). For a given \(C^1\)-function \(f : \mathbb{R}^2 \setminus \{0\} \rightarrow \mathbb{R}\) we can represent its gradient \(\nabla f\) with respect to the local basis

\[
\partial_r = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \text{and} \quad \partial_\theta = \begin{pmatrix} -\sin \theta \\ \cos \theta \end{pmatrix}
\]

of the tangent space of \(\mathbb{R}^2 \setminus \{0\}\) in the point \((r, \theta)\). The gradient then can be expressed as

\[
\nabla f = \frac{\partial f}{\partial r} \partial_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \partial_\theta.
\]

It then also follows that

\[
\nabla \perp f = -\frac{1}{r} \frac{\partial f}{\partial \theta} \partial_r + \frac{\partial f}{\partial r} \partial_\theta.
\]

**Lemma 3.1.** For every \(n \in \mathbb{N}\) there exist constants \(M_1, M_2, M_3, M_4, M_5, M > 0\), \(0 < L_1 < L_2 < 1\), \(0 < K_1 < K_2 < 1\) and \(0 < K < 1\) such that the radial and angular components of the vector field \(u_0^n\) satisfy the following bounds:

- for \(r > \frac{1}{2}\) one has \(M_1 e^{K_1 r^2} \leq \langle u_0^n, \partial_r \rangle \leq M_2 e^{K_2 r^2}\);
- for \(r < \frac{1}{3}\) holds \(M_3 e^{L_1 r^2} \leq \langle u_0^n, \partial_\theta \rangle \leq M_4 e^{L_2 r^2}\);
- for \(r > \frac{1}{2}\) one has \(\langle u_0^n, \partial_r \rangle \leq M_5 e^{Kr^2}\);
- for \(r < \frac{1}{3}\) one has that \(\langle u_0^n, \partial_\theta \rangle \leq M\).
Proof. We can partly express $\alpha_n^*$ explicitly as follows:

$$
\alpha_n^*(r, \theta) = \begin{cases} 
\frac{r}{1+\frac{1}{2}(\frac{1}{3} + \frac{1}{3\pi})} & \text{for } r > \frac{1}{2}; \\
\frac{1}{\sqrt{1+2r\sin(n\theta)}} & \text{for } \frac{1}{3} \leq r \leq \frac{1}{2}; \\
\frac{1}{\sqrt{1+2r\sin(n\theta)}} & \text{for } r < \frac{1}{3}.
\end{cases}
$$

From this one can compute the partial derivatives

$$
\partial_r\alpha_n^*(r, \theta) = \begin{cases} 
\frac{1}{1+\frac{1}{2}(\frac{1}{3} + \frac{1}{3\pi})} \sin(n\theta) & \text{for } r > \frac{1}{2}; \\
\frac{1}{\sqrt{1+2r\sin(n\theta)}} & \text{for } \frac{1}{3} \leq r \leq \frac{1}{2}; \\
\frac{1}{\sqrt{1+2r\sin(n\theta)}} & \text{for } r < \frac{1}{3}.
\end{cases}
$$

This implies that the angular component of $\nabla^\perp \alpha_n^*$ satisfies the following bounds when $r > \frac{1}{2}$:

$$
\left(1 + \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3\pi}\right)\right)^{-1} \leq \langle \nabla^\perp \alpha_n^*, \partial_r \rangle \leq \left(1 - \frac{1}{2}\left(\frac{1}{3} + \frac{1}{3\pi}\right)\right)^{-1}.
$$

From this follows the following bounds for the angular component of the vector field $u_n^0$ along the level set $\Gamma_r^{(n)}$:

$$
2\pi e^{\frac{2}{7}(1+\frac{1}{2})(\frac{1}{3} + \frac{1}{3\pi})^2} \left(1 + \frac{1}{3}\right)^{-1} \leq \langle u_n^0, \partial_r \rangle \leq 2\pi e^{\frac{2}{7}(1+\frac{1}{2})(\frac{1}{3} + \frac{1}{3\pi})^2} \left(1 - \frac{1}{3}\right)^{-1}.
$$

This implies:

$$
\frac{3}{2} \pi e^{\frac{1}{2}(1-\frac{1}{2})^2} \leq \langle u_n^0, \partial_r \rangle \leq 3\pi e^{\frac{1}{2}(1+\frac{1}{2})^2}.
$$

This is the bound given in the lemma for $r > \frac{1}{2}$.

We now turn to the case $r < \frac{1}{2}$. The above computation of $\partial_r\alpha_n^*$ yields the following bounds for the angular component of the vector field $\nabla^\perp \alpha_n^*$ when $r < \frac{1}{3}$:

$$
(\sqrt{1+2r})^{-1} \leq \langle \nabla^\perp \alpha_n^*, \partial_r \rangle \leq (\sqrt{1-2r})^{-1}.
$$

This implies the following bounds for the angular component for $u_n^0$ along the orbit $\Gamma_r^{(n)}$ when $r < \frac{1}{3}$:

$$
2\pi e^{\frac{2}{7}(1-\frac{1}{2})^2} \left(1 + 2r(1+\frac{1}{2})\right)^{-1} \leq \langle u_n^0, \partial_r \rangle \leq 2\pi e^{\frac{2}{7}(1+\frac{1}{2})^2} \left(1 - 2r(1+\frac{1}{2})\right)^{-1}.
$$

Using the fact $r < \frac{1}{3}$ one obtains:

$$
2\pi e^{\frac{2}{7}(1-\frac{1}{2})^2} \left(1 + \frac{1}{3}(1 + \frac{1}{6})\right)^{-1} \leq \langle u_n^0, \partial_r \rangle \leq 2\pi e^{\frac{2}{7}(1+\frac{1}{2})^2} \left(1 - \frac{3}{2}(1 + \frac{1}{6})\right)^{-1}.
$$

This leads to the bounds given in the statement of the lemma.

When computing the derivative of $\alpha_n^*$ with respect to $\theta$ one obtains

$$
\partial_\theta\alpha_n^*(r, \theta) = \begin{cases} 
\frac{r}{(1+\frac{1}{2}+\frac{1}{3\pi})\sin(n\theta)}\left(\frac{1}{3} + \frac{1}{3\pi}\right)n\cos(n\theta) & \text{for } r > \frac{1}{2}; \\
\frac{n\cot(n\theta)}{\sqrt{1+2r\sin(n\theta)}} & \text{for } \frac{1}{3} \leq r \leq \frac{1}{2}; \\
\frac{n\cot(n\theta)}{\sqrt{1+2r\sin(n\theta)}} & \text{for } r < \frac{1}{3}.
\end{cases}
$$

Note, that in the case $r < \frac{1}{3}$ an asymptotic expansion up to second order of the square root in the second term of the above expression yields

$$
\partial_\theta\alpha_n^*(r, \theta) = \frac{n}{2}r^2
$$
when \( \sin(n\theta) \to 0 \). From those computations we obtain the following bounds for the radial part of the vector field \( \nabla^\perp \alpha_n \) when \( r > \frac{1}{3} \):

\[
- \frac{n}{3(1 - \frac{1}{2}(\frac{1}{3} + \frac{1}{3\pi}))^2} \leq \langle \nabla^\perp \alpha_n, \partial_r \rangle \leq \frac{n}{3(1 - \frac{1}{2}(\frac{1}{3} + \frac{1}{3\pi}))^2}.
\]

This yields the following bounds for the radial part of the vector field \( u_n^0 \) along the orbit \( \Gamma_r^{(n)} \):

\[
|\langle u_n^0, \partial_r \rangle| \leq \frac{2\pi ne^{\frac{2}{3}(1 + \frac{1}{3} + \frac{1}{3\pi})^2}}{3(1 - \frac{1}{2}(\frac{1}{3} + \frac{1}{3\pi}))^2}.
\]

From this follows:

\[
|\langle u_n^0, \partial_r \rangle| \leq \frac{2\pi ne^{\frac{2}{3}(1 + \frac{1}{3})^2}}{3(1 - \frac{1}{3})^2}.
\]

This implies the bounds given in the statement.

We finally have to analyze \( \langle u_n^0, \partial_r \rangle \) for \( r < \frac{1}{3} \). In this case we have on \( \Gamma_r^{(n)} \)

\[
\langle u_n^0, \partial_r \rangle = \frac{n \cot(n\theta)}{r(1 + \frac{1}{2} \sin(n\theta))} \left( \frac{r}{\sqrt{1 + 2r \sin(n\theta)}} - \frac{\sqrt{1 + 2r \sin(n\theta)} - 1}{\sin(n\theta)} \right).
\]

The statement of the lemma follows from the observation that the function

\[
(r, \theta) \mapsto \begin{cases} 
\frac{1}{r \sin(n\theta)} \left( \frac{r}{\sqrt{1 + 2r \sin(n\theta)}} - \frac{\sqrt{1 + 2r \sin(n\theta)} - 1}{\sin(n\theta)} \right) & \text{for } \sin(n\theta) \neq 0, r > 0; \\
\frac{1}{\frac{1}{2}r} & \text{for } \sin(n\theta) = 0, r > 0; \\
0 & \text{for } r = 0
\end{cases}
\]

is continuous on the compact set \([0, \frac{1}{2}] \times [0, 2\pi]\). To see this, one can use an asymptotic expansion in zero of the function \( y \mapsto \sqrt{1 + 2r y} \). \(\square\)

**Remark 8.** Note that for a function \( S \in \ker(u_n^0 \cdot \nabla) \cap C^1(\mathbb{R}^2) \) we have that \( S \) is constant along the level sets of the function \( \alpha_n \). Further, we have

\[
\text{div}(\varphi Su_n^0) = u_n^0 \cdot \nabla S + S\text{div}(u_n^0) = 0.
\]

Moreover, the vector field \( u_n = Su_n^0 \) has the same trajectories as \( u_n^0 \). Those trajectories are the level sets of the function \( \alpha_n \).

**Proposition 2.** For all \( n \in \mathbb{N} \), there exists a function \( S \in \ker(u_n^0 \cdot \nabla) \cap C^1(\mathbb{R}^2) \) such that the two following properties hold:

- the flow \( (\phi_t)_{t \in \mathbb{R}} \) generated by the vector field \( u_n := Su_n^0 \) satisfies for all \( r > 0 \) that \( \phi_{t\rightarrow n}(x) = x \) for all \( x \in \Gamma_r^{(n)} \);
- the vector field \( u_n \) satisfies \( \text{div}(\varphi u_n) = 0 \) and is bounded;
- there exists a constant \( D > 0 \) such that \( |u_n(x)| \leq D|x| \) for all \( x \) from a neighbourhood of the origine.

**Proof.** For a function \( S \in \ker(u_n^0 \cdot \nabla) \cap C^1(\mathbb{R}^2) \) we introduce the flows \( (\phi_t)_{t \in \mathbb{R}} \) and \( (\phi_t^0)_{t \in \mathbb{R}} \) generated by \( u_n = Su_n^0 \) resp. \( u_n^0 \) through the equations \( \dot{\phi}_t^0 = u_n^0(\phi_t^0) \) resp. \( \dot{\phi}_t = u_n(\phi_t) \). This implies for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^2 \) the relation

\[
\phi_t(x) = \phi_t^0 S(x).
\]
We want to use this relation to determine a suitable function $S$, which satisfies the properties stated in the lemma. For $x \in \mathbb{R}^2 \setminus \{0\}$ let $\tau^0(x) > 0$ denote the first return time of the flow $(\phi^t_0)_{t \geq 0}$. It follows that for all $x \in \mathbb{R}^2 \setminus \{0\}$ one has 
\[ \phi^0_{\tau^0(x)}(x) = x. \]

Note that it follows from the implicit function theorem, that the function $\tau^0$ is a $C^1$-function. Moreover, it is constant along the trajectories of the flow $(\phi^t_0)_{t \in \mathbb{R}}$. Since for $r > \frac{1}{2}$ the orbit $\Gamma^{(n)}_r$ stays between the two centred circles of radius $r \frac{2}{3}$ and $r \frac{4}{3}$, it follows from the bounds on the angular speed of the flow given in Lemma 3.1 that $\tau^0(x)$ satisfies the following bounds for $x \in \Gamma^{(n)}_r$ and $r < \frac{1}{3}$ 
\[ \frac{2}{3} \pi r M_2^{-1} e^{-K_2 r^2} \leq \tau^0(x) \leq \frac{4}{3} \pi r M_1^{-1} e^{-K_1 r^2}. \]

Further, by a similar argument using the fact, that for $r < \frac{1}{4}$ the orbit $\Gamma^{(n)}_r$ stays between the two circles of radius $r - \frac{r^2}{2}$ and $r + \frac{r^2}{2}$ it follows that $\tau^0(x)$ is bounded as follows for $x \in \Gamma^{(n)}_r$ 
\[ \pi (r - \frac{r^2}{2}) M_1^{-1} e^{-L_2 r^2} \leq \tau^0(x) \leq 4 \pi (r + \frac{r^2}{2}) M_3^{-1} e^{-L_1 r^2}. \]

Since $r < \frac{1}{3}$, this yields the following bounds 
\[ \frac{5}{6} \pi r M_4^{-1} e^{-L_2 r^2} \leq \tau^0(x) \leq \frac{6}{7} \pi r M_5^{-1} e^{-L_1 r^2}. \]

Note that it is sufficient to prove the statement of the lemma for the unique point $x_r = (r, 0)$ in the intersection of $\Gamma_r^{(n)}$ with the positive real line, since the return time is constant on the orbit $\Gamma_r^{(n)}$. It is then sufficient to find a function $S \in \ker (u_n^0, \nabla) \cap C^1(\mathbb{R}^2)$ such that 
\[ \phi^0_{\tau^0(x_r)}(x_r) = \phi^0_{\tau^0(x_r)}(x_r) = x_r = \phi_{e^{-r^2}}(x_r). \]

This implies that the function $S(x)$ has to be equal to $\tau^0(x_r) e^{-r^2}$ for $x \in \Gamma_r^{(n)}$.

We now have to prove that the resulting vector field is bounded. We treat the angular component first. By Lemma 3.1 we have for $r > \frac{1}{2}$ and $x \in \Gamma_r^{(n)}$ that 
\[ |\langle u_n, \partial \theta \rangle| \leq |\langle u_n^0, \partial \theta \rangle| \tau^0(x_r) e^{-r^2} \leq M_2 e^{K r^2} \frac{4}{3} \pi r M_1^{-1} e^{-K_1 r^2} e^{-r^2}. \]

This expression is bounded for $r > \frac{1}{2}$. For the radial component of the vector field $u_n$ we see from Lemma 3.1 for $r > \frac{5}{2}$ 
\[ |\langle u_n, \partial r \rangle| \leq |\langle u_n^0, \partial r \rangle| \tau^0(x_r) e^{-r^2} \leq M_5 e^{K r^2} \frac{4}{3} \pi r M_1^{-1} e^{-K_1 r^2} e^{-r^2}, \]

which also stays bounded for $r > \frac{1}{2}$.

For $r < \frac{1}{4}$ one has on the orbit $\Gamma_r^{(n)}$ by Lemma 3.1 
\[ |\langle u_n, \partial \theta \rangle| \leq |\langle u_n^0, \partial \theta \rangle| \tau^0(x_r) e^{-r^2} \leq M_4 e^{L_2 r^2} \frac{6}{7} \pi r M_3^{-1} e^{-L_1 r^2} e^{-r^2} = O(r). \]

Further, one also has from Lemma 3.1 that 
\[ |\langle u_n, \partial r \rangle| \leq |\langle u_n^0, \partial r \rangle| \tau^0(x_r) e^{-r^2} \leq M_6 \pi r M_2^{-1} e^{-L_2 r^2} e^{-r^2} = O(r). \]

The last statement of the lemma follows since we have for $x \in \Gamma_r^{(n)}$ and $r < \frac{1}{3}$ that 
\[ |x| \geq r (1 - \frac{r^2}{2}) \geq r (1 - \frac{1}{6}) = \frac{r^2}{6}, \]

which shows that $|\langle u_n, \partial r \rangle| = O(|x|)$.
\[ \square \]
Remark 9. Note that since the vector field $u_n$ is bounded, it follows that the operator $u_n \cdot \nabla$ has a domain, which contains $H^1$. In order to study the spectral behavior of the operators $u_n \cdot \nabla$, we define the weak eigenspace of the operator $u_n \cdot \nabla$ by

$$H^1_\lambda = \left\{ f \in H^1; \int_{\mathbb{R}^2} f(x)u \cdot \nabla \psi(x)d\gamma_\varphi(x) = i\lambda \int_{\mathbb{R}^2} f(x)\psi(x)d\gamma_\varphi(x), \forall \psi \in C^1_b(\mathbb{R}^2) \right\}.$$ 

Proposition 3. Let $\tau(x)$ be the return time to the point $x$ of the flow $(\phi_t)_{t \in \mathbb{R}}$ generated by the vector field $u_n$. If $f \in H^1_\lambda$, then for every $t \in \mathbb{R}$ holds:

$$f \circ \phi_{-t\tau} = e^{i\lambda t\tau}f \quad \gamma_\varphi\text{-a.e.}$$

Proof. First we note that the flow $(\phi_{t\tau})_{t \in \mathbb{R}}$ is also preserving the measure $\gamma_\varphi$, since $\tau$ is invariant along the orbit sets $\Gamma^{(n)}$. For $\psi \in C^1_b(\mathbb{R}^2)$, we then have

$$\frac{d}{dt} \bigg|_{t=0} \int_{\mathbb{R}^2} f \circ \phi_{-t\tau} \psi d\gamma_\varphi = \int_{\mathbb{R}^2} f \circ \phi_{t\tau} \psi d\gamma_\varphi = \int_{\mathbb{R}^2} \frac{d}{dt} \bigg|_{t=0} (\psi \circ \phi_{t\tau}) d\gamma_\varphi$$

$$= \int_{\mathbb{R}^2} f \tau u_n \cdot \nabla \psi d\gamma_\varphi = i\lambda \int_{\mathbb{R}^2} \tau \psi d\gamma_\varphi.$$ 

For $t = t_0$, since $\psi \circ \phi_{t_0\tau}$ is in $C^1_b(\mathbb{R}^2)$, we have that

$$\frac{d}{dt} \bigg|_{t=t_0} \int_{\mathbb{R}^2} f \circ \phi_{-t\tau} \psi d\gamma_\varphi = \int_{\mathbb{R}^2} f \circ \phi_{t\tau} \circ \phi_{t_0\tau} \psi d\gamma_\varphi$$

$$= i\lambda \int_{\mathbb{R}^2} f \tau \psi \circ \phi_{t_0\tau} d\gamma_\varphi$$

$$= i\lambda \int_{\mathbb{R}^2} f \circ \phi_{-t_0\tau} \tau \psi d\gamma_\varphi.$$ 

Note that in the last line we used that the return time is invariant with respect to the flow. We also have that

$$\frac{d}{dt} \bigg|_{t=t_0} \int_{\mathbb{R}^2} e^{i\lambda t\tau} f \psi d\gamma_\varphi = \int_{\mathbb{R}^2} \frac{d}{dt} \bigg|_{t=t_0} e^{i\lambda t\tau} f \psi d\gamma_\varphi = \int_{\mathbb{R}^2} \tau e^{i\lambda t\tau} f \psi d\gamma_\varphi.$$ 

It follows that the function

$$g(t) = \int_{\mathbb{R}^2} (f \circ \phi_{-t\tau} - e^{i\lambda t\tau} f) \psi d\gamma_\varphi$$

satisfies the differential equation

$$\frac{d}{dt} g(t) = 0$$

with the intial condition

$$g(0) = \int_{\mathbb{R}^2} (f \circ \phi_0 - e^{i\lambda 0} f) \psi d\gamma_\varphi = 0.$$ 

This implies for any $t \in \mathbb{R}$ and for all $\psi \in C^1_b(\mathbb{R}^2)$ that

$$\int_{\mathbb{R}^2} (f \circ \phi_{-t\tau} - e^{i\lambda t\tau} f) \psi d\gamma_\varphi = g(t) = 0.$$ 

Finally, one obtains $f \circ \phi_{-t\tau}(x) = e^{i\lambda t\tau} f(x)$ for $\gamma_\varphi$-almost all $x \in \mathbb{R}^2$. 

$\square$
Proposition 4. The sequence of divergence free vector fields $u_n$ constructed in Lemma 2 satisfies that $H^1_\lambda = \{0\}$ for all $\lambda \neq 0$.

Proof. Let $\tau(x)$ be the return time to the point $x$ of the flow $(\phi_t)_{t \in \mathbb{R}}$ generated by the vector field $u_n$. Then it follows from Proposition 3 that

$$e^{i\lambda \tau} f = f \circ \phi_{\tau} \quad \gamma_\varphi\text{-a.e.}$$

For the choice $t = 1$ we obtain from this

$$e^{i\lambda \tau} f = f \circ \phi_{\tau} = f \quad \gamma_\varphi\text{-a.e.}$$

If we put

$$A := \{x \in \mathbb{R}^2 : e^{i\lambda \tau} f(x) = f(x)\},$$

then it follows from the coarea formula applied to the function $C^1$-function $\alpha_n$, which was used to introduce the vector field $u_0$, that

$$0 = \gamma_\varphi(A) = \int_{\mathbb{R}^2} 1_A \varphi d\gamma = \int_0^\infty \int_{\alpha_n^{-1}(r)} 1_A \frac{\varphi}{|\nabla \alpha_n|} d\gamma dr = \int_0^\infty \int_{\Gamma_r^{(n)}} 1_A \frac{1}{|\nabla \alpha_n|} d\gamma dr.$$

This means that for almost all $r > 0$ the equation

$$e^{i\lambda \tau} f(x) = f(x)$$

must be true for $\ell_\varphi$-almost all $x \in \Gamma_r^{(n)}$. Since by Proposition 2 on $\Gamma_r^{(n)}$ we have $\tau = e^{-r^2}$ it follows that one has either

$$\exists k \in \mathbb{Z} : e^{-r^2} \lambda = 2\pi k$$

or

$$f(x) = 0 \quad \text{for } \ell_\varphi\text{-almost all } x \in \Gamma_r^{(n)}.$$

The first possibility can only be true for a set of real values $r > 0$ forming a set of Lebesgue measure zero. It then follows that

$$f = 0 \quad \gamma_\varphi\text{-a.e.}$$

This implies $H^1_\lambda = \{0\}$. \qed

4. Comparison arguments. The proof of our result is based on some lower bound for the gradients of functions which are invariant with respect to the flow we constructed in the previous sections. We will use the information gained in Proposition 1 to construct a suitable comparison measure having some large Cheeger constant. Further, we will prove some Faber-Krahn type inequality relating flow invariant functions on the plane with some symmetrized functions on this comparison space.

4.1. Some suitable symmetric comparison measure. In this part, we will construct some comparison probability density $\hat{\varphi}_n$ on $\mathbb{R}^2$ based on the function $c \mapsto \ell_\varphi(\Gamma_r^{(n)})$.

Proposition 5. There exists a $C^1$-probability density $\hat{\varphi}_n$ on $\mathbb{R}^2$ which is constant on every circle around zero and which satisfies the equality:

$$\ell_{\hat{\varphi}_n}(\partial B_{R_n(c)}(0)) = \ell_\varphi(\Gamma_r^{(n)}),$$

where the function $R_n(c)$ satisfies $\gamma_{\hat{\varphi}_n}(B_{R_n(c)}(0)) = c$ for all $c \in [0,1]$. 
Proof. The following proof is motivated from the analogous construction of comparison manifolds in [2]. See also [5] for an alternative construction. Let $\psi_n(r)$ be the value of the density $\hat{\varphi}_n$ on the circle $\partial B_r(0)$. In order to satisfy the stated properties, the function $\psi_n$ has to satisfy

$$2\pi \int_0^{R_n(c)} r\psi_n(r)dr = c.$$ 

Differentiating both sides of this equality with respect to $c$ yields

$$\left(\frac{d}{dc}R_n(c)\right)2\pi R_n(c)\psi_n(R_n(c)) = 1.$$ 

Due to the requirements on the function $\hat{\varphi}$ stated in the proposition the function $\psi$ has to satisfy $2\pi R_n(c)\psi_n(R_n(c)) = \ell_\varphi(\Gamma^{(n)}_{r_c})$. This implies

$$\left(\frac{d}{dc}R_n(c)\right)\ell_\varphi(\Gamma^{(n)}_{r_c}) = 1.$$ 

This gives that

$$\frac{d}{dc}R_n(c) = \frac{1}{\ell_\varphi(\Gamma^{(n)}_{r_c})}.$$ 

Thus we obtain the following explicit formula for the function $R(c)$

$$R_n(c) = \int_0^c \frac{1}{\ell_\varphi(\Gamma^{(n)}_{r_c})}d\sigma.$$ 

As a conclusion, the suitable density is given by

$$\hat{\varphi}_n : \mathbb{R}^2 \rightarrow \mathbb{R}; x \mapsto \frac{\ell_\varphi(\Gamma^{(n)}_{r_c} \setminus \{x\})}{2\pi |x|}.$$ 

This ends the proof. 

4.2. The rearrangement method and Faber-Krahn type inequalities. Let $A$ be a measurable set in $\mathbb{R}^2$. Its symmetric rearrangement $A_n^*$ is the open ball centered around zero in $\mathbb{R}^2$ satisfying

$$\gamma \hat{\varphi}_n(A_n^*) = \gamma \varphi(A).$$ 

For all non negative measurable function $f$, the following representation holds almost every where with respect to $\gamma$

$$f(x) = \int_0^\infty \chi_{\{y : f(y) > t\}}(x) dt.$$ 

We define its rearrangement function by:

$$f_n^*(x) = \int_0^\infty \chi_{\{y : f(y) > t\}_n}(x) dt.$$ 

Here we used the notation $\chi_A$ for the indicator function over the set $A$ which is one for $x \in A$ and zero for $x \notin A$.

Then $f_n^*$ is lower semi-continuous (since its level sets are open), and is uniquely determined by the distribution function $\gamma \varphi(\{x : f(x) \geq s\})$ of $f$. 

Remark 10. The function \( f_n^* \) constructed above verifies, for all \( s \in \mathbb{R} \)
\[ \gamma_\varphi \{ x; f(x) \geq s \} = \gamma_{\tilde{\varphi}_n} \{ x; f_n^*(x) \geq s \} \].
This means that the distribution function of \( f \) with respect to the measure \( \gamma_\varphi \)
and the distribution function of \( f_n^* \) with respect to the measure \( \gamma_{\tilde{\varphi}_n} \) are equal. It then follows that \( \|f\|_{L^2(\mathbb{R}^2, \gamma_\varphi)} = \|f_n^*\|_{L^2(\mathbb{R}^2, \gamma_{\tilde{\varphi}_n})} \).

For general bounded function \( f \), we define \( f_n^* := (f - \inf f)_n^* + \inf f \).

Remark 11. Note that \( C^1(\mathbb{R}^2) \cap \ker(u \cdot \nabla) \) is dense in \( H^1 \cap \ker(u \cdot \nabla) \) with respect to the \( H^1 \)-norm.

Proposition 6. For all \( f \in C^1(\mathbb{R}^2) \cap \ker(u \cdot \nabla) \) we have that
\[ \int_{\mathbb{R}^2} |\nabla f|^2 d\gamma_\varphi \geq \int_{\mathbb{R}^2} |\nabla f_n^*|^2 d\gamma_{\tilde{\varphi}_n}. \]

The following proof follows the line of arguments for proving Faber-Krahn type inequalities as given in [3] and adopted in [7].

Proof. We start the proof by citing the co-area formula for the integral of some integrable function \( h \) (see [3])
\[ \int_{\mathbb{R}^2} h d\gamma = \int_{\inf f}^{\sup f} \left( \int_{f^{-1}(t)} h \left| \nabla f \right| dt \right) dt. \]

Applying the co-area formula to \( h = |\nabla f|^2 \varphi \), one obtains
\[ \int_{\mathbb{R}^2} |\nabla f|^2 d\gamma_\varphi = \int_{\inf f}^{\sup f} \left( \int_{f^{-1}(t)} |\nabla f| d\ell_\varphi \right) dt. \]

Cauchy-Schwarz inequality implies that the right side of the above equation is larger or equal than
\[ \int_{\inf f}^{\sup f} \left( \ell_\varphi(f^{-1}(t)) \right)^2 \left( \int_{f^{-1}(t)} \frac{1}{|\nabla f|} d\ell_\varphi \right)^{-1} dt. \]

On the other hand, \( |\nabla f_n^*| \) is constant along the level sets \( (f_n^*)^{-1}(t) \). It follows from this property that
\[ \int_{\mathbb{R}^2} |\nabla f_n^*|^2 d\gamma_\varphi = \int_{\inf f}^{\sup f} \left( \ell_{\tilde{\varphi}_n}((f_n^*)^{-1}(t)) \right)^2 \left( \int_{(f_n^*)^{-1}(t)} \frac{1}{|\nabla f_n^*|} d\ell_{\tilde{\varphi}_n} \right)^{-1} dt. \]

Note that \( f_n^* \) is Lipschitz continuous, and by the co-area formula we have
\[ \int_{(f_n^*)^{-1}(t)} \frac{1}{|\nabla f_n^*|} d\ell_{\tilde{\varphi}_n} = - \frac{1}{d} \frac{d}{dt} \gamma_{\tilde{\varphi}_n} \{ x; f_n^*(x) \geq t \} \]
which, by the construction of \( f_n^* \), equals
\[ - \frac{d}{dt} \gamma_\varphi \{ x; f(x) \geq t \} = \int_{f^{-1}(t)} \frac{1}{|\nabla f|} d\ell_\varphi. \]

It follows that
\[ \int_{f^{-1}(t)} \frac{1}{|\nabla f|} d\ell_\varphi = \int_{(f_n^*)^{-1}(t)} \frac{1}{|\nabla f_n^*|} d\ell_{\tilde{\varphi}_n}. \]

However, the construction of \( f^* \) and Proposition 5 implies that:
\[ \ell_\varphi(f^{-1}(t)) \geq \ell_{\tilde{\varphi}_n}((f_n^*)^{-1}(t)). \]
As a conclusion

\[
\int_{\mathbb{R}^2} |\nabla f|^2 d\gamma \geq \int_{\mathbb{R}^2} \left( \inf_f \left( \int_{f^{-1}(t)} |\nabla f| dt \right) \right) dt \geq \int_{\mathbb{R}^2} \sup_f \left( \ell_{\tilde{\phi}_n}((f^*_n)^{-1}(t)) \right)^2 \left( \int_{(f^*_n)^{-1}(t)} \frac{1}{|\nabla f^*_n|} d\tilde{\phi}_n \right)^{-1} dt \geq \int_{\mathbb{R}^2} |\nabla f^*_n|^2 d\tilde{\phi}_n.
\]

5. **Proof of the main result.** In the next paragraph, we will need some Cheeger constant for the space of rotationally invariant functions on \(\mathbb{R}^2\). Let \(F^*\) be the set of all the closed subsets in \(\mathbb{R}^2\) which are invariant with respect to rotations around the origin in \(\mathbb{R}^2\). The appropriate Cheeger constant is given by

\[
\hat{C}_n = \inf_{A \in F^*: \gamma_{\tilde{\phi}_n}(A) < \frac{1}{2}} \frac{\ell_{\tilde{\phi}_n}(\partial A)}{\gamma_{\tilde{\phi}_n}(A)}.
\]

Since \(\hat{C}\) depends on the vector field \(u_n\) used to construct \(\tilde{\phi}_n\), we will denote \(\hat{C}\) as \(\hat{C}_n\) in the following.

**Proposition 7.** For the sequence of vector fields constructed in Lemma 2, holds

\[
\lim_{n \to \infty} \hat{C}_n = \infty.
\]

**Proof.** By the same arguments as in [7], it follows from Proposition 1 that

\[
\hat{C}_n \geq \min \left\{ \inf_{0 < c < \frac{1}{2}} \frac{\ell_{\tilde{\phi}_n}(\Gamma_{r_c})}{c}, \inf_{0 < c < \frac{1}{2}} \frac{\ell_{\tilde{\phi}_n}(\Gamma_{r_{1-c}})}{c} \right\} \to \infty.
\]

**Proposition 8.** For all function \(f \in H^1(\mathbb{R}^2, \gamma_{\tilde{\phi}_n})\) which are invariant by rotation around the origin in \(\mathbb{R}^2\) and satisfy \(||f|| = 1\) we have

\[
\int_{\mathbb{R}^2} |\nabla f|^2 d\gamma_{\tilde{\phi}_n} \geq \frac{\hat{C}_n^2}{4}.
\]

**Proof.** This is a modification of Cheeger inequality (see [3]) to our setting. Its proof follows the same arguments like in [7].

**Remark 12.** An expression for the limit of the spectral gap of the operator \(A_{cu}\) is given in [6]:

\[
\lim_{c \to \infty} \rho_{cu} = \inf_{\lambda \in \mathbb{R}} \left\{ \frac{\int_{\mathbb{R}^2} |\nabla f|^2 d\gamma_{\tilde{\phi}_n}}{\int_{\mathbb{R}^2} |f|^2 d\gamma_{\tilde{\phi}_n}} : f \in H^1_{\lambda} \right\},
\]

with

\[
H^1_{\lambda} = \left\{ f \in H^1 : \int_{\mathbb{R}^2} f(x) u \cdot \nabla \psi(x) d\gamma_{\tilde{\phi}_n} = i\lambda \int_{\mathbb{R}^2} f(x) \psi(x) d\gamma_{\tilde{\phi}_n}, \forall \psi \in C^1(\mathbb{R}^2) \right\}.
\]

Note that the above results was proved in [6] for diffusions on compact manifolds \(M\) in the context of \(L^2(M, vol)\). The proof presented there is essentially based on the Rellich Lemma and the fact that the Laplace operator has compact resolvent. It carries over to our situation since the operator \(A_0\) has compact resolvent as was noted in Remark 1.
5.1. Proof of Theorem 1.1. The main argument follows the arguments given in [7]. Let $\gamma_{\hat{\varphi}}$ be the comparison probability density associated to the vector-field $u_n$. For a given $K > 0$, Proposition 8 and Proposition 7 tell us that there exists $N \in \mathbb{N}$ sufficiently large such that

$$\inf \left\{ \int_{\mathbb{R}^2} |\nabla f|^2 d\gamma_{\hat{\varphi}N}; f \in H^1(\mathbb{R}^2, \gamma_{\hat{\varphi}N}) \text{ with } \|f\| = 1 \right\} \geq \frac{\tilde{C}_N^2}{4} \geq 2K.$$ 

By Proposition 6 and Remark 10 one has that:

$$\inf \left\{ \int_{\mathbb{R}^2} |\nabla f|^2 d\gamma_{\varphi}; f \in H^1_0 \text{ with } \|f\| = 1 \right\} \geq 2K.$$ 

We now can use Remark 12 and Proposition 4 to see that

$$\lim_{a \uparrow \infty} \rho_{au} \geq 2K.$$ 

This implies $\rho_{au_n} > K$ for some suitable choice of $a > 0$. \hfill \square

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