Locking entanglement measures with a single qubit

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We study the loss of entanglement of bipartite state subjected to discarding or measurement of one qubit. Examining the behavior of different entanglement measures, we find that entanglement of formation, entanglement cost, and logarithmic negativity are lockable measures in that it can decrease arbitrarily after measuring one qubit. We prove that any convex and asymptotically non-continuous measure is lockable. As a consequence, all the convex roof measures can be locked.

Relative entropy of entanglement is shown to be a non-lockable measure.

One of the main differences between quantum and classical information are various superadditivities. An example of superadditivity was found in [1], where with a single bit, one can lock (unlock) an arbitrary amount of classical correlations contained in a quantum state (according to a physically significant measure of classical correlations). One can ask if similar effects can be found for entanglement. The basic question is: how much can entanglement of any bi- or multipartite system change when one qubit is discarded? The answer clearly depends on the measure of entanglement. In this paper we show that the effect of locking holds for the entanglement of formation $E_F$ and cost $E_c$, as well as a computable measure of entanglement - the logarithmic negativity $E_N$ [2] (cf. [3]). More specifically, we will show that for some state, measuring (or dephasing) one qubit can change the entanglement from an arbitrary large value to zero. We analyze other entanglement measures. We argue that if a measure is convex but not too much, then it does not admit locking. We show for example, that relative entropy of entanglement can change at most by two upon discarding one qubit. Moreover we link the effect of locking with the postulate that is often adopted in the asymptotic regime - "asymptotic continuity". An entanglement measure is asymptotically continuous, if its density (entanglement per qubit) is continuous, uniformly with respect to dimension. The importance of asymptotically continuous measures is that they give rise to "macro-parameters" describing entanglement. I.e. entanglement would be a measure which changes little if the state changes little. The effect of locking is a form of discontinuity, since by removing just one qubit, many e-bits are destroyed. This raises the question of whether locking is connected to asymptotic continuity. We confirm this by proving that a convex measure that is not asymptotically continuous admits locking. Our proof is constructive: from the states on which a function is discontinuous, one can build a state exhibiting locking. Examples are entanglement measures built by the convex-roof method [4].

Entanglement cost and Logarithmic negativity - We shall show that an arbitrary large $E_c$, and $E_N$ of a given state can be reduced to zero by a measurement on a single qubit. Consider the state on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B \sim \mathbb{C}^{d+2} \otimes \mathbb{C}^{d+2}$

$$\rho_{AB} = \frac{1}{2} \begin{bmatrix} \sigma & 0 & 0 & \frac{1}{\sqrt{2}} U^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} U & 0 & 0 & \sigma \end{bmatrix}$$

(1)

Here $U = \sum_{i,j=0}^{d-1} u_{ij} |ii\rangle \langle jj|$ and $\sigma = \sum_i \frac{1}{d} |ii\rangle \langle ii|$ is a separable maximally correlated state, and both defined on $\mathbb{C}^d$. The matrix is written in the computational basis $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ of a pair of qubits each with one of two parties Alice and Bob. Clearly after one party measures in the computational basis, the state will decohere, and the off-diagonal elements will go to zero – thus the state will be separable. However, before the measurement, the state has arbitrarily large entanglement cost i.e. it requires an arbitrarily large number of singlets shared between Alice and Bob to create, even in the asymptotic limit. To see this, we take the purification of the state

$$\psi_{ABE} = \frac{1}{\sqrt{2}^d} \sum_{i=0}^{d-1} \{ |ii\rangle_A \langle ii| \}_{AB} U |ii\rangle_E$$

(2)

with the third subsystem denoted as $E$ for Eve (we call the state $\rho_{AE}$ dual to $\rho_{AB}$). One sees that Eve gets a bit string $X$ of length $\log d$ encoded in one of two basis. The basis are complementary if $U$ is taken to be $H^{\otimes \log d}$ with $H$ the Hadamard transform. This is precisely the situation for locking classical information between one party (here AB taken together) and another (here Eve).

In [1] it was shown that Eve can learn at most $\log d/2$ bits of $X$. Thus, for Eve’s optimal measurement, the entropy of Alice will be greater than $\log d/2$. But this is precisely a definition of $E_F$ i.e.

$$E_F(\rho_{AB}) = \inf_{A_i} \sum_i p_i S(\rho_i)$$

(3)

where the infimum is taken over all measurements with outcomes $A_i$ performed on the purification of $\rho_{AB}$ and resulting in states $\rho_i$ with Alice. Thus $E_F \geq \log d/2$
which can be arbitrarily large with $d$. Furthermore, since the information that Eve can obtain is additive in the number of copies of the state, $E_ε$ is also arbitrarily large. The log-negativity can also be calculated and it is $E_N = \log_2(\sqrt{d} + 1)$, thus it too can be locked.

Let us consider another example for locking $E_N$ which is motivated by the results of [3]. To this aim consider the state defined on the Hilbert space $\mathcal{H}_A^{(n)} \otimes \mathcal{H}_B^{(n)}$ in such a way that $\mathcal{H}_A^{(n)} \sim \mathcal{H}_B^{(n)} = C^2 \otimes (C^d) \otimes \cdots \otimes$ with natural parameter $n$ and range of parameter $\alpha$ specified subsequently

\begin{equation}
\varrho_{AB}^{(n)} = \frac{1}{2}[P_{00} \otimes P_{00} \otimes \tau_0^{\otimes n} + P_{11} \otimes P_{11} \otimes \tau_1^{\otimes n} + \alpha^n P_{01} \otimes P_{01} \otimes (\tau_1^{\otimes n} - \tau_0^{\otimes n}) + \alpha^n P_{10} \otimes P_{10} \otimes (\tau_0^{\otimes n} - \tau_1^{\otimes n})].
\end{equation}

Here we use the hiding states from [3] $\varrho_0 = \varrho^{(1)}$, $\varrho_1 = (\varrho^{(1)})^{\otimes n}$, where $\varrho_0$, $\varrho_n$ are fully symmetric and antisymmetric Werner states on $C^2 \otimes C^d$. We also use the notation $P_{ij} = |i\rangle \langle j|$. The whole matrix can be written as before in the form

\begin{equation}
\varrho_{AB}^{(n)} = \frac{1}{2} \begin{bmatrix}
\tau_0^{\otimes n} & 0 & 0 & 0 \\
0 & 0 & \alpha^n (\tau_1^{\otimes n} - \tau_0^{\otimes n}) & 0 \\
0 & \alpha^n (\tau_0^{\otimes n} - \tau_1^{\otimes n}) & 0 & 0 \\
0 & 0 & 0 & \tau_0^{\otimes n}
\end{bmatrix}
\end{equation}

This is a state for any $|\alpha| \leq 1$ since it can be reproduced by specific LOCC recurrence protocol [3] from $\varrho^{(1)}$ defined by the formula above. $\varrho^{(1)}$ can be easily checked to be a state. The log-negativity of $\varrho^{(n)}$ for given $n$ is

\begin{equation}
E_N(\varrho^{(n)}) = \log_2[1 + (\alpha(2 - 2^{-(n+1)})^n)]
\end{equation}

which goes to infinity with $n$ whenever $|\alpha| > (2 - 2^{-(n+1)})^{-1}$ (this is because of orthogonality of $\varrho_0$ and $\varrho_n$ one has $||\tau_0 - \tau_1|| = 2 - 2^{-(n+1)}$). On the other hand, measurement of Alice’s qubit in the $|i\rangle$ basis, leads to the state $\frac{1}{2} \sum_{i=0}^1 |i\rangle |i\rangle^2 \otimes (\tau_i)^n$ which is completely separable. Hence we have that measurement on a single qubit has locked completely an arbitrary high amount of entanglement.

Relative entropy of entanglement. Let us now examine relative entropy of entanglement ($E_r$) [4]. We will show that it is not local. More precisely, two solutions will be presented, exhibiting that after tracing out one qubit of the state $\rho_{AB}$, $E_r(\rho_{AB})$ can decrease at most by two, and after a complete von Neumann measurement on one qubit, $E_r$ can decrease at most by one.

**Proposition 1** For any bipartite state $\rho_{AA'B} \equiv \rho$ and any complete von Neumann measurement $\Lambda_A$ on the one qubit system $A$ there holds:

\begin{equation}
E_r(\rho) - E_r(\Lambda_A \otimes I_{A'B})(\rho) \leq 1
\end{equation}

\begin{equation}
E_r(\rho) - E_r(Tr_A(\rho)) \leq 2
\end{equation}

where $Tr_A$ denotes partial trace over system $A$.

**Proof.** Both statements of this theorem are consequence of the following property of relative entropy of entanglement $E_r$ (see [4] in this context):

\begin{equation}
\sum_i p_i E_r(\rho_i) - E_r(\sum_i p_i \rho_i) \leq S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i)
\end{equation}

where $S$ stands for the von Neumann entropy of the state.

For the first part of the proof, it suffices to notice that any complete measurement can be implemented as dephasing of the system. To dephase one qubit, one can add a local random ancilla $\tau = \frac{1}{2} |0\rangle \langle 0| + |1\rangle \langle 1|$ and perform the controlled unitary operation $U = \sum_{i=0}^1 |i\rangle \langle i| \otimes \sigma^{(i)}_A$ with $\sigma^{(0)} = I_A$ and $\sigma^{(1)} = \sigma_z$, a Pauli matrix. Indeed, this operation followed by tracing out the ancilla $\tau$ will have the desired effect. One can easily check that random unitaries put phases which zero the coherences of the state:

\begin{equation}
Tr_{anc}[U(\tau \otimes \rho)U^\dagger] = \Lambda_A \otimes I_{A'B}(\rho) \equiv \rho_{meas}
\end{equation}

Taking now in [10] $\rho_i = \sigma_i \otimes I_{A'B}(\rho)$ and $p_i = \frac{1}{2}$ one gets

\begin{equation}
E_r(\rho) - E_r(\sum_i p_i \rho_i) \leq S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i),
\end{equation}

since local unitary transformations do not change $E_r$. For such choice of $\rho_i$ and $p_i$, the state $\sum_i p_i \rho_i$ is equal to state $\rho$ after dephasing, and by [10] is the same as the one after a complete measurement, which gives us:

\begin{equation}
E_r(\rho) - E_r(\rho_{meas}) \leq S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i).
\end{equation}

It is known [10] that the right hand side does not exceed $H(\rho)$ i.e. the Shannon entropy of the ”mixing” distribution $\{p_i\}$. In our case this distribution is homogeneous, so $S(\sum_i p_i \rho_i) - \sum_i p_i S(\rho_i) \leq 1$ which leads us to the first part of the theorem.

The second part of the theorem can be proven in a similar vain. Instead of tracing out, we apply total dephasing, which is equivalent to substitution of a qubit by the maximally mixed one, uncorrelated with the rest of the state. To this end we a need bigger random ancilla system $\tau^{\otimes 2}$ and the controlled unitary composed from all four Pauli matrices: $U = \sum_{i=0}^3 |i\rangle \langle i| \otimes \sigma^{(i)}_A$. The unitaries $\sigma^{(i)}$ are well known examples of ones which when applied randomly change any state to the maximally mixed one (see for example, [11] [12]).

Now the state after the transformation $U$ and tracing out the ancilla $\tau^{\otimes 2}$ is the following: $\varphi = \frac{1}{2} \otimes Tr_{AA'B}$. The relative entropy of entanglement of this state is the same as for $Tr_{AA'B}$, because it cannot increase after tracing out $\frac{1}{2}$ for this is a local operation, and it cannot decrease, since this qubit is product with the rest of the
state. In this case the right hand side of the inequality \(12\) is bounded by \(H(p) = 2\) which completes the proof.

Although it seems to be intuitive, we are not able to show, that both complete measurement and tracing out of a qubit decrease \(E_r\) by the same amount. I.e. for tracing out, we were only able to prove a bound of 2 rather than 1 for the change of \(E_r\). Were this tighter bound to be proven, one would have an interesting complementarity relation between measuring and forgetting. Clearly, measuring a qubit can decrease the entanglement by one ebit. An example of the latter is the measurement result which tells one something about the state. In this case the right hand side of the inequality \(12\) is bounded by \(2\) which completes the proof.

**Remark.** I.e. it satisfies

\[
\gamma_1 \leq (1 + \delta)(x_2 - x_1)
\]

hence due to subextensivity we get

\[
[f(\rho_1) - f(\rho_2)] \leq \delta|\gamma_1 - \gamma_2| + (1 + \delta)|x_1| + |x_2| \leq 2\delta M \log d + 4c
\]

This ends the proof.

Now let us exhibit what happens when a function is subextensive, but is not asymptotically continuous. To this end consider a subextensive function \(f\), i.e. let \(f(\rho) \leq M \log d\), where \(\rho\) acts on a \(d\) dimensional Hilbert space. Let us assume that \(f\) is not asymptotically continuous. This means that we have a sequence of states \(\rho_1^{(n)}\) and \(\rho_2^{(n)}\) approaching each other in trace distance, and acting on a Hilbert space of increasing dimension \(d_n\), such that

\[
\frac{|f(\rho_1^{(n)}) - f(\rho_2^{(n)})|}{\log d_n} \geq \Delta
\]

where \(\Delta\) is some positive constant. We now consider states \(\sigma^{(n)}\), \(\gamma_1^{(n)}\), \(\gamma_2^{(n)}\) given by lemma, \(\delta^{(n)} = \frac{1}{2}||\rho_1^{(n)} - \rho_2^{(n)}||\), and \(x_i^{(n)}\) being analogues of \(x_i\). The formula \(15\) applied to those states together with \(16\) implies that \(|x_1^{(n)} - x_2^{(n)}| \geq (\Delta - 2\delta^{(n)} M) \log d_n\). Thus we see that at least one of \(x_i\) must have arbitrary large modulus for large \(n\) (i.e. small \(\delta^{(n)}\)). Without loss of generality, we can assume it is \(x_1\). Then we get that one of two possibilities holds:

(i) \(x_1 \leq (-\Delta/2 + \delta^{(n)} M) \log d_n\)

(ii) \(x_1 \geq (\Delta/2 - \delta^{(n)} M) \log d_n\)

In case (i) the function is too concave, while in case (ii) it is too convex. In both cases, the function upon mixing two states can be arbitrarily different from the average of the function.

Let us discuss the first case. We have a situation where upon mixing two states, a function can go up an arbitrary amount. If \(f\) represents e.g. something which is not a valuable resource, then it seems not surprising that it can go highly up after forgetting, as we expect forgetting is not a useful operation. If the function is some useful resource, this means that forgetting may be very good. On the other hand, we have the impression that forgetting cannot be good for obtaining a resource. Let us explain, that the last statement need not be in contradiction with an arbitrarily large increase of the function \(f\). Namely, as noted in \(15\) a function that has such property, and is useful is entanglement of distillation of pure bipartite entanglement, from multipartite states. Does it mean that forgetting is useful for distillation? It is
easily to see that it is not the case (we will consider for simplicity two parties). Simply distillable entanglement of state $\rho$ is calculated by taking the product $\rho^{\otimes n}$. So $D(\rho)$ represents the amount of singlets drawn by Alice and Bob from state $\rho^{\otimes n}$ per copy, while $D(\sigma)$ the same for state $\sigma^{\otimes n}$. $D(\frac{1}{2}\rho + \frac{1}{2}\sigma)$ represents the amount of singlets drawn from state $(\frac{1}{2}\rho + \frac{1}{2}\sigma)^{\otimes n}$. We see that the latter state cannot be created out of two former states by forgetting one bit. The latter would give the much different state $\frac{1}{2}\rho^{\otimes n} + \frac{1}{2}\sigma^{\otimes n}$. Thus for reasonable quantities $f$, the effect (i) should be regarded as a type of activation.

Let us now discuss the case (ii). We have that upon mixing, the function goes arbitrarily down. If $f$ is convex, then of course only (ii) can occur, and together with convexity, it gives having the following:

**Proposition 3** A convex LOCC monotone $E$ that satisfies $E(\rho) \leq M \log d$ for some constant $M$, and that is not asymptotically continuous, admits locking.

**Proof.** From assumptions it follows that there must exist states $\rho_1$ and $\gamma_1$ and weights $1 - \epsilon$, $\epsilon$ such that the difference

$$x = [\epsilon E(\rho_1) + (1 - \epsilon) E(\gamma_1)] - E(\epsilon \rho_1 + (1 - \epsilon) \gamma_1) \quad (18)$$

can be arbitrarily large. Now let us note that a convex entanglement measure satisfies

$$E(p\rho_{AB} \otimes |0⟩⟨0|_{A'} + (1 - p)\rho_{A'B} \otimes |1⟩⟨1|_{A'}) = pE(\rho) + (1 - p)E(\tilde{\rho}) \quad (19)$$

One way follows from convexity and from nonincreasing of $E$ under tracing out a local qubit. Second - from the fact that state on the left-hand-side of inequality can be transformed into ensemble $\{(p, \rho), (1 - p, \tilde{\rho})\}$. Consider now the state

$$\rho_{ABA'} = (1 - \epsilon)\rho_1 \otimes |0⟩⟨0|_{A'} + \epsilon \gamma_1 \otimes |1⟩⟨1|_{A'} \quad (20)$$

where $A'$ is one qubit system. Its reduction is given by

$$\rho_{AB} = (1 - \epsilon)\rho_1 + \epsilon \gamma_1 \quad (21)$$

Hence following [13] we obtain that the difference

$$E(\rho_{ABA'}) - E(\rho_{AB}) \quad (22)$$

can be arbitrarily large, which is locking.

**Examples:** Consider so called convex roof measures [4], based on Renyi entropy with $0 \leq \alpha < 1$. Such measures are convex by definition, and on pure states they are equal to the Renyi entropy $S_\alpha = \frac{1}{1-\alpha} \log \text{Tr} \rho^\alpha$ of subsystem. For our choice of $\alpha$ Renyi entropy is greater than von Neumann entropy. It is easy to check that for a compressed version of state $\rho^{\otimes n}$ (denote it by $\rho_{\text{ppy}}$) where only typical eigenvalues are kept, the Renyi entropy for large $n$ tends to the von Neumann entropy $nS(\rho)$. On the other hand for the original state, it is equal to $nS_\alpha(\rho)$. As we know, the states $\rho_{\text{ppy}}$ and $\rho^{\otimes n}$ converge to each other. However for Renyi entropy we obtain that $\Delta = S_\alpha(\rho) - S(\rho)$. Thus Renyi entropy is not asymptotically continuous, and since we pointed out states on which it diverges, one can construct the states, on which we have locking effect.

Let us mention that the above theorem does not say anything about measures which are asymptotically continuous. Thus the case of $E_r$ [16] and $E_c$ which are asymptotically continuous had to be treated separately. Also the theorem does not say anything about measures that are not subextensive. Therefore the case of negativity was also treated separately. We believe that measures such as the distillable entanglement will not be lockable, but did not prove so here.

Finally we propose a definition of nonlockable version of entanglement measure:

**Definition 1** For any entanglement measure $E(\rho)$ the reduced entanglement measure $E \downarrow (\rho)$ is defined as

$$E \downarrow (\rho) = \inf_{\Lambda \in \text{CLOCC}} E(\Lambda(\rho) + S) \quad (23)$$

Here CLOCC is a class of LOCC operations in a closed system and $\Delta S = S(\Lambda(\rho)) - S(\rho)$ is the increase of entropy produced by measurement. In fact this is quantum analogue of reduced intrinsic information defined in [17]. One can also consider other versions of such reduction, choosing maps $\Lambda$ e.g. to be local bistochastic ones or local dephasings.

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