We consider dynamics of a quantum scalar field, minimally coupled to classical gravity, in the near-horizon region of a Schwarzschild black-hole. It is described by a static Klein-Gordon operator which in the near-horizon region reduces to a scale invariant Hamiltonian of the system. This Hamiltonian is not essentially self-adjoint, but it admits a one-parameter family of self-adjoint extension. The time-energy uncertainty relation, which can be related to the thermal black-hole mass fluctuations, requires explicit construction of a time operator near-horizon. We present its derivation in terms of generators of the affine group. Matrix elements involving the time operator should be evaluated in the affine coherent state representation.

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1 Introduction

Einstein’s general theory of relativity [1] describes gravity as a manifestation of the curvature of spacetime. A fundamental instability against collapse implies the existence of black holes as stable solutions of Einstein’s equations. A black hole is formed if a massive object (e.g. a star) collapses into an infinitely dense state known as a singularity. In this picture the curvature of spacetime becomes extreme and prevents any particle even light from escaping to infinity. A black hole may have several horizons that fully characterize its structure.

The simplest three-dimensional geometry for a black hole is a sphere (known as a Schwarzschild sphere), its surface defines the event horizon. In the case of a spherical black hole, with $R_{\mu\nu} = 0$, all horizons coincide at the Schwarzschild’s critical radius $r_s = 2GMc^{-2}$.

The Quantum Field Theory (QFT) in curved spacetime with classical event horizon is, however, troubled by the singularity at the horizon [2]. This problem may be solved by treating the black hole as a quantum state which implies that the energy of the black hole and its corresponding time do not commute at the horizon [3].

In this picture we study the dynamics of a scalar field in the near-horizon region described by a static Klein-Gordon (KG) operator which in this case becomes the Hamiltonian of the system. The dynamics of a scalar field in the near-horizon region, and its associated SO(2,1) conformal symmetry have been studied in many papers [4],[5],[6],[7],[8] in which a complete treatments of conformal quantum mechanics and of near-horizon symmetry were made. In this letter, we present the explicit construction of the time operator in the near-horizon region in terms of the generators of the affine group, and discuss its self-adjointness [9].

2 Scalar Field in the Near-Horizon Region

The Schwarzschild geometry of a static spherical black hole is described by the metric ($c = \hbar = G = 1$)

$$ds^2 = -f(r)dt^2 + [f(r)]^{-1}dr^2 + r^2d\Omega^2,$$

where $rf(r) = r - r_s$, and $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. Near horizon ($r \sim r_s$), $f(r)$ behaves as $f(r) \sim 2\kappa(r - r_s)$, where $\kappa = 1/2r_s$ denotes the surface gravity.
The equation of motion of a free scalar field $\Phi(x)$ in this background metric is

$$-rac{1}{f} \Phi_{tt} + f \Phi_{rr} + (f' + 2 f/r) \Phi_r + \Delta_q \Phi/r^2 - m^2 \Phi = 0. \quad (2)$$

By separating the time and angular variables in $\Phi(x) = e^{-it\omega} \phi(r) Y_{lm}(\Omega)$, the mode $\phi$ with angular momentum $l$ and frequency $\omega$ satisfies the equation [10],[11],[12]

$$\frac{1}{f} \omega^2 \phi + f \phi_{rr} + (f' + 2 f/r) \phi_r - \frac{l(l+1)}{r^2} \phi - m^2 \phi = 0. \quad (3)$$

In the near-horizon region, the Schroedinger-like equation in the variable $x = r - r_s$ can be studied. For very small values of $x$, the angular and mass terms can be neglected [13]. In terms of a new field variable $\psi = \sqrt{x} \phi$, the KG equation in the near-horizon region, for small $x$, reduces to a scale invariant Schroedinger equation

$$\left[\frac{d^2}{dx^2} + \left(\frac{1}{4} + \Theta^2\right) x^2\right] \psi(x) = 0 \quad (4)$$

where $\Theta = r_s\omega$. This reduction to a Schroedinger-like equation can be described by an effective attractive inverse square potential $2V(x) = -(1/4 + \Theta^2)x^{-2}$ which is conformally invariant with respect to the near-horizon variable $x$. The corresponding quantum Hamiltonian is

$$\hat{H} = \frac{1}{2}(p^2 + g x^2), \quad (5)$$

where $g = -(1/4 + \Theta^2)$ is supercritical for all nonzero frequencies $\omega$ [14]. The static black hole Hamiltonian $H_{bh}$ near horizon can then be defined as

$$\hat{H}_{bh} = M + \hat{H}. \quad (6)$$

3 Time Operator and its Relation to Spacetime Non-commutativity

The coordinate singularity associated with horizon may be overcome by assuming spacetime non-commutativity on the horizon, the so called Quantum Horizon [2], [3]. If a black hole is considered as a quantum state, its energy $\hat{H}_{bh}$ and its conjugate time $\hat{t}$ are expected to become conjugate operators obeying formally

$$[\hat{H}, \hat{t}] = i. \quad (7)$$
Since $\hat{H}$ is x-coordinate dependent, we expect spacetime noncommutativity, $[\hat{t}, x] \neq 0$. What is $\hat{t}$ as an operator? Due to Pauli theorem [15] no such self-adjoint operator should exist if the spectrum of the self-adjoint Hamiltonian is semibounded or discrete. In quantum theory $\hat{H}$ is essentially self-adjoint only for $g > 3/4$ in the domain

$$D_0 = \{ \psi \in L^2(\mathbb{R}^+, dx), \psi(0) = \psi'(0) = 0 \}$$

(8)

In this domain it has a continuous spectrum for $g \geq 3/4$ with $E > 0$ but no ground state at $E = 0$ [4].

For $g \leq 3/4$, the Hamiltonian is not essentially self-adjoint [6],[8], but it admits a one-parameter family of self-adjoint extension labeled by a $U(1)$ parameter $e^{iz}$, where $z$ is a real number, which labels the domains $D_z$ of the extended Hamiltonian. The set $D_z$ contains all the vectors in $D_0$, and vectors of the form $\psi_z = \psi_+ + e^{iz}\psi_-$. 

For $g = -1/4$, as pointed out by Moretti and Pinamonti [16], the analysis of the spectrum of $\hat{H}_z$ in some papers [5],[6],[7],[8] is not completely correct. By interpreting the logarithm, appearing in $\psi_z$ near $x \sim 0$, as a one-valued function Moretti and Pinamonti were able to show that for each $\hat{H}_z$ with $z \neq 0$ there is exactly one proper bound state in $D_z$: 

$$\psi_z(x) = N_z \sqrt{x} K_0 \left( \sqrt{E_z} x \right),$$

(9)

with the eigenvalue 

$$E_z = \exp \left[ \frac{\pi}{2} \cot \frac{z}{2} \right],$$

(10)

where $N_z$ is a normalization factor, and $K_0$ is the modified Bessel function. The horizon, in this picture, is located at $x = 0$ where the wave function $\psi_z$ vanishes. For $x \sim 0$ the logarithmic term in (9) vanishes if $x = x_z = 1/\sqrt{E_z}$ and $\psi_z$ exhibits a scale behavior of the type $\psi_z \sim \sqrt{x}$. In order to achieve that $x_z$ belongs to the near horizon region, $z$ should be close to $z = 0 + \epsilon, 2\pi + \epsilon, 4\pi + \epsilon, \ldots$, where $\epsilon \sim 0$ [8]. Then, in a band-like region $\Delta_z = [x_z(1 - \delta), x_z(1 + \delta)]$, where $\delta \sim 0$, all the eigenfunctions of $\hat{H}_z$ exhibit a scaling behavior. However, the zero mode solution to (5) is obtained only for discrete values of $z_n = 2(2n + 1)\pi$, where $n = 0, 1, 2, \ldots$.

The asymptotic scale invariance of the KG operator in the near horizon region implies that the Hamiltonian $\hat{H}$, and the scaling operator $\hat{D} = -(xp + px)/4$ obey the affine algebra type commutation relation

$$[\hat{H}, \hat{D}] = i\hat{H}.$$ 

(11)
This algebra can be easily extended to the full $SO(2, 1)$ conformal algebra by adding to the set $(\hat{H}, \hat{D})$ the conformal generator $\hat{K} = x^2/2$. In this case the $g$-dependent constant quadratic Casimir operator is obtained

$$C_2 = \frac{1}{2}(\hat{K}\hat{H} + \hat{H}\hat{K}) - \hat{D}^2 = \frac{g}{4} - \frac{3}{16}.$$  \hspace{1cm} (12)

If $\hat{H}^{-1}$ exists, we can formally construct a time operator $\hat{t}$ \cite{9}

$$\hat{t} = \frac{1}{2} (\hat{D}\hat{H}^{-1} + \hat{H}^{-1}\hat{D}),$$  \hspace{1cm} (13)

which obeys the required commutation relation, $[\hat{H}, \hat{t}] = i$. Although, both $\hat{H}$ and $\hat{D}$ separately can be made self-adjoint operators in the domain $L^2(\mathbb{R}^+, dE)$ it is not true for a $\hat{t}$-operator which contains $\hat{H}^{-1}$ \cite{17}. It is clear that $\hat{t}$ is not a self-adjoint operator in the domain $L^2(\mathbb{R}^+, dE)$ where

$$\hat{H} \rightarrow E \quad \hat{D} \rightarrow -i(E \frac{d}{dE} + \frac{1}{2}) \quad \hat{t} \rightarrow -i \frac{d}{dE}.$$  \hspace{1cm} (14)

Following Klauder \cite{17}, the commutation relation between $\hat{H}$ and $\hat{t}$ should be considered on the Hilbert space spanned by the affine coherent states. They are defined by

$$|\tau, \lambda\rangle = e^{i\tau\hat{H}} e^{-2i(ln \lambda)\hat{D}} |\eta\rangle$$  \hspace{1cm} (15)

where $|\eta\rangle$ is a fiducial vector chosen in such a way as to satisfy $\langle \eta | \hat{H}^{-1} | \eta\rangle < \infty$. For example, if we choose $\eta(E) \equiv \langle E | \eta\rangle = NE^\alpha \exp(-\beta E)$, then the conditions $\langle H \rangle = 1$ and $\langle H^{-1} \rangle < \infty$ lead to $\beta - \frac{1}{2} = \alpha > 0$. This allow us to calculate all the matrix elements involving time operator and even to study their classical limit as $\hbar \rightarrow 0$ at the horizon.

In the limit $g \rightarrow 0$, we have $\hat{H} \rightarrow H_0 = p^2/2$ and $\hat{t} \rightarrow t_0$ where

$$\hat{t}_0 = -\frac{1}{2} (xp^{-1} + p^{-1}x)$$  \hspace{1cm} (16)

is exactly the time-of-arrival operator of Aharonov and Bohm \cite{18}. The time operators $\hat{t}$ and $\hat{t}_0$ can also be related to each other by means of a unitary operator \cite{9} that transforms $\hat{H} \rightarrow \hat{H}_0$:

$$\hat{H} = U\hat{H}_0 U^\dagger, \quad \hat{t} = U\hat{t}_0 U^\dagger.$$  \hspace{1cm} (17)
The Aharonov-Bohm time operator $\hat{t}_0$ is not self-adjoint and its eigenfunctions are not orthonormal.

4 Conclusion

In this paper, we have studied the properties of a scalar field in the near-horizon region of a massive Schwarzschild black hole. The quantum Hamiltonian governing the near-horizon dynamics is found to be scale invariant and has the full conformal group as a dynamical symmetry group. Using only the generators of the affine group, we constructed the time operator near-horizon. The self-adjointness of $\hat{H}$ and $\hat{t}$ is also discussed.

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