SPHERICAL FUNCTORS

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Abstract. In this paper we describe some examples of so called spherical functors between triangulated categories, which generalize the notion of a spherical object ([5]). We also give sufficient conditions for a collection of spherical functors to yield a weak representation of the category of tangles, and prove a structure theorem for such representations under certain restrictions.

1. Introduction

Fix some 2-category $\mathcal{U}$ of triangulated categories, functors and natural transformations, which would be our universe. Suppose that there is another universe $\mathcal{T}$, where 1-morphisms (i.e. functors) form triangulated categories, and 2-morphisms are exact functors between these triangulated categories. Fix also a morphism of 2-categories $\mathcal{T} \to \mathcal{U}$.

An example of such a construction is the 2-category $\mathcal{U}_{\text{ag}}$ of bounded derived categories of coherent sheaves on algebraic varieties, all exact functors and all natural transformations between them, together with the 2-category $\mathcal{T}_{\text{ag}}$ of bounded derived categories of coherent sheaves on algebraic varieties, Fourier-Mukai transforms and natural transformations that come from morphisms of Fourier-Mukai kernels.

Another example is the 2-category $\mathcal{U}_{\text{DG}}$ of derived categories of DG categories, all exact functors and natural transformations between them, together with the 2-category $\mathcal{T}_{\text{DG}}$ of pre-triangulated DG categories, and DG functors and natural transformations between them.

In fact, when speaking about functors in the universe $\mathcal{U}$ and natural transformations between them, we will mean functors and natural transformations in the universe $\mathcal{T}$.

In the following text all functors are supposed to be derived. To simplify the notation, we will write $\mathcal{D}(X)$ for the bounded derived category $\mathcal{D}^b(\text{Coh}(X))$ of coherent sheaves on an algebraic variety $X$ and $\mathcal{D}_Y(X)$ for its subcategory which consists of complexes with cohomology supported set-theoretically on $Y \subset X$.

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2. Spherical functors.

Let $\mathcal{D}$, $\mathcal{D}_0$ be triangulated categories, $S : \mathcal{D}_0 \to \mathcal{D}$ a functor. Suppose that $S$ has left and right adjoints $L$, $R$. Then there are four natural morphisms of functors:

$$LS \to id \quad id \to SL \quad id \to RS \quad SR \to id.$$ 

Define the twist functor $T_S$ to be the cone of $SR \to id$, and the dual twist functor $T'_S$ to be the cone of $id \to SL$ shifted by $[-1]$, so that there are exact triangles of functors:

(1) $SR \to id \to T_S; \quad T'_S \to id \to SL.$

Call the functor $S$ spherical if it satisfies two following conditions:

(1) the cone of $id \to RS$ is an exact autoequivalence of $\mathcal{D}_0$. Let us call this functor $F_S$.

(2) the natural map $R \to F_SL$ induced by $R \to RSL$ is an isomorphism of functors.

**Proposition 1.** If $S$ is spherical, both $T_ST'_S$ and $T'_ST_S$ are naturally isomorphic to $id_D$.

**Proof.** This proof is in fact an adaptation of the results of Seidel and Thomas on spherical objects. The following diagram is commutative:

(2) $\begin{array}{ccc}
SR & \longrightarrow & SRL \\
\downarrow & & \downarrow \\
SL & \longrightarrow & T'[1] \\
\downarrow & & \downarrow \\
T S & \longrightarrow & T'SL \\
\end{array}$

Consider another diagram

(3) $\begin{array}{ccc}
0 & \longrightarrow & SL \\
\downarrow & & \downarrow \\
0 & \longrightarrow & SL \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0 \\
\end{array}$

There is a natural map $SL \to SRSL$ induced by $id \to RS$. It defines a map from the diagram (3) to (2). This map is commutative by the following lemma:

**Lemma 1.** (4) The composition map $S \to S(RS) \simeq (SR)S \to S$ is an isomorphism of functors. \qed
The map of first lines is included into another diagram

\[
\begin{array}{c}
0 \rightarrow SL \rightarrow SL \\
SR \rightarrow SRL \rightarrow SRT_S[1] \\
SR \rightarrow SF_SL
\end{array}
\]

The induced map \(SR \rightarrow SF_SL\) is an isomorphism, hence \(SL \rightarrow SRT_S[1]\) is also an isomorphism. Then from the diagram of last columns

\[
\begin{array}{c}
SL \rightarrow SL \rightarrow 0 \\
SRT_S'[S[1]] \rightarrow T_S'[S[1]] \rightarrow T_ST_S'[1] \\
0 \rightarrow id[1] \rightarrow T_ST_S'[1]
\end{array}
\]

we see that \(T_ST_S' \simeq id\). The second isomorphism \(T_ST_S \simeq id\) is proved the same way.

\[\square\]

**Corollary 1.** If \(S\) is a spherical functor, then the corresponding twist functor \(T_S\) is an exact autoequivalence of the category \(\mathcal{D}\).

**Proposition 2.** Let \(S_1 : \mathcal{D}_1 \rightarrow \mathcal{D}\) and \(S_2 : \mathcal{D}_2 \rightarrow \mathcal{D}\) be spherical functors.

1. If there exists an equivalence of categories \(X : \mathcal{D}_1 \rightarrow \mathcal{D}_2\), and \(S_1 \simeq S_2X\), then \(T_{S_1} \simeq T_{S_2}\).
2. If \(Y : \mathcal{D} \rightarrow \mathcal{D}\) is an autoequivalence, then \(YS_1\) is also a spherical functor, and, moreover, \(T_{YS_1} \simeq YT_{S_1}Y^R\).
3. \(T_{S_1}T_{S_2} \simeq T_{T_{S_1}S_2}T_{S_1} \simeq T_{S_2T_{S_2}S_1}\).

**Proof.**

1. Draw a diagram:

\[
\begin{array}{c}
S_2XX^R R_2 \rightarrow id \rightarrow T_{S_1} \\
S_2R_2 \rightarrow id \rightarrow T_{S_2}
\end{array}
\]

The map \(S_2XX^R R_2 \rightarrow S_2R_2\) is an isomorphism of functors, as is the identity map \(id \rightarrow id\), and this diagram is commutative; hence, there exists an isomorphism \(T_{S_1} \rightarrow T_{S_2}\).
(2) The triangle $YT_S Y^R[-1] \to YS_i R_i Y^R \to \text{id} \to YT_S Y^R$ is isomorphic to the triangle $YT_S Y^R[-1] \to YS_i R_i Y^R \to YY^R$, hence is distinguishable.

(3) This is implied by the previous part, with $Y = T_{S_1}$ or $Y = T_{S_2}$.

\[\square\]

**Corollary 2.** If there exists an autoequivalence $Y$ of the category $\mathcal{D}_2$ such that $T_{S_1} S_2 \simeq S_2 Y$, then the twist functors $T_{S_1}, T_{S_2}$ commute.

**Proposition 3.** If $S$ is a spherical functor, then the following commutation relations hold:

1. $T'_S S F S[1] \simeq S \simeq T'_S S F S[-1]$;
2. $F'_S L T'_S[1] \simeq L \simeq F'_S L T'_S[-1]$;
3. $F'_S R T'_S[1] \simeq R \simeq F'_S R T'_S[-1]$.

**Proof.** For a spherical functor $S$, both $T_S$ and $F_S$ are autoequivalences of the corresponding categories. Let us prove that $T_S S \simeq S F S[1]$:

$$T_S S = \{(SR)S \to S\} \simeq \{S \to S(RS)\}[1] = S F S[1].$$

Then the remaining two parts are done the same way:

$$F'_S L = \{(LS)L \to L\}[-1] \simeq \{L \to L(SL)\} = LT'_S[-1];$$

$$F'_S R = \{R \to (RS)R\} \simeq \{R(SR) \to R\}[-1] = RT'_S[-1].$$

### 3. Examples.

(1) Let $\mathcal{E}$ be an object of $\mathcal{D}$, and define a functor $S$ from the category $\text{Vect}$ of vector spaces to $\mathcal{D}$ by $S V = V \otimes \mathcal{E}$. Then for $F \in \text{Ob} \mathcal{D}$ we have $R F = \text{Hom}(\mathcal{E}, F)$ and $L F = \text{Hom}^*(F, \mathcal{E})$. The functor $S$ is spherical iff, first, the cone of $\text{id} \to \otimes \text{Hom}(\mathcal{E}, \mathcal{E})$ is an isomorphism, which means that $\text{Hom}(\mathcal{E}, \mathcal{E})$ is 2-dimensional, being a sum of $\text{Hom}^0(\mathcal{E}, \mathcal{E}) = \text{id} \cdot \mathbb{K}$ and $\text{Hom}^n(\mathcal{E}, \mathcal{E})$ for some $n > 0$, and second, for any $F$ the map $R F \to F S L F$, i.e. the map

$$\text{Hom}(\mathcal{E}, F) \to \text{Hom}(\mathcal{E}, \mathcal{E} \otimes \text{Hom}^*(F, \mathcal{E}))/\text{id} \simeq \text{Hom}^*(F, \mathcal{E}) \otimes \text{Hom}^n(\mathcal{E}, \mathcal{E})$$

is a quasiisomorphism, which is equivalent to the condition that for any $F$ the natural map

$$\text{Hom}^i(\mathcal{E}, F) \otimes \text{Hom}^{n-i}(F, \mathcal{E}) \to \text{Hom}^n(\mathcal{E}, \mathcal{E}) \simeq \mathbb{K}$$

is a non-degenerate pairing. We see that the functor $S$ is spherical if and only if the object $\mathcal{E}$ is spherical in the sense of Seidel and Thomas [5]. Two twists $T_{S_1}$ and $T_{S_2}$ satisfy braid relations when the functor $L(S_1) S_2 = \otimes \text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ is an autoequivalence of $\text{Vect}$, i.e. $\text{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ is one-dimensional, which also agrees with the results of Seidel and Thomas.
(2) Let $X$ be an algebraic variety, $D$ a divisor on $X$. Denote by $i : D \to X$ the embedding of $D$ into $X$. Let $S = i_* : \mathcal{D}(D) \to \mathcal{D}(X)$. Then $L = i^*$ and $R = i!$. Since $D$ is a divisor, we have

$$F_S = \{id \to RS\} = \{id \to i^*i_*\} = id \otimes \mathcal{O}_D(D)[-1]$$

and

$$R = i^* = i^* \otimes \mathcal{O}_D(D)[-1] = F_SL$$

which implies that $S$ is spherical. The twist functor is then a tensor multiplication by the line bundle $\mathcal{O}_X(D)$.

(3) Suppose $D$ is a divisor on $X$ and $D$ has a structure of a projective vector bundle $\mathbb{P}(E)$ of rank $r$ over some algebraic variety $M$. Denote by $\pi$ the projection $D \to M$, denote by $i$ the closed embedding $D \to X$. Then there is a functor $S = i_*\pi^* : \mathcal{D}(M) \to \mathcal{D}(X)$. Its left and right adjoint functors are $R = \pi^*i^!$ and $L = L(\pi^*)i^*$. As above,

$$i^* = i^* \otimes \mathcal{O}_D(D)[-1].$$

The composition $\pi_*\pi^*$ is isomorphic to $id$, and applying Serre duality one obtains

$$L(\pi^*) \cong \pi_*(id \otimes \omega_D[\dim D]) \otimes \omega_M^{-1}[\dim M] \cong \pi_*(id \otimes \mathcal{O}_\pi(-r))[r-1].$$

Then

$$F_S = \{id \to \pi_*i^*\pi^*\} = \pi_*(\pi^*id \otimes \mathcal{O}_D(D)[-1]) = id \otimes \pi_*\mathcal{O}_D(D)[-1].$$

In order for $S$ to be spherical, $\pi_*\mathcal{O}_D(D)$ must be a line bundle. This means that $\mathcal{O}_D(D) = \pi^*\mathcal{L} \otimes \mathcal{O}_\pi(-r)$ for some line bundle $\mathcal{L}$ on $M$. Then we have

$$R = \pi_*i^* = \pi_*i^* \otimes \mathcal{O}_D(D)[-1])$$

$$L = \pi_!i^* = \pi_!(i^* \otimes \mathcal{O}_\pi(-r))[r-1]$$

and the condition $R = F_SL$ holds. This proves the following

**Claim 1.** The functor $i_*\pi^*$ is spherical if and only if $\mathcal{O}_D(D) = \pi^*\mathcal{L} \otimes \mathcal{O}_\pi(-r)$ for some line bundle $\mathcal{L}$ on $M$.

If $M$ is a point and $D = \mathbb{P}^1$ is a projective line on a surface, then the functor $S$ maps $\mathcal{D}(pt) = Vect$ to a subcategory generated by $i_*\mathcal{O}_D$. It is well-known that $i_*\mathcal{O}_D$ is a spherical object if and only if $D \cdot D = -2$, which agrees with the above result.

4. Tangle representations.

It turns out that a special kind of spherical functors is especially useful in finding weak representations of the category of tangles.

**Definition 1.** A spherical functor $S$ is **strongly spherical**, if $F_S = [-2]$. 
Recall that the category $\text{Tan}$ of smooth tangles has natural numbers for objects and smooth $(n, m)$ tangles for morphisms. A weak representation of $\text{Tan}$ is an assignment of a triangulated category $\mathcal{D}_n$ to each $n$ and a functor $\Psi(\alpha) : \mathcal{D}_n \rightarrow \mathcal{D}_m$ to each $(n, m)$ tangle $\alpha$, so that relations between tangles hold for these functors up to a natural isomorphism of functors. Denote by $\text{FTan}$ an analogous category of framed tangles.

The standard set of generators for $\text{FTan}$ (illustrated in Figure 1) contains

- “cups” $g_n^i$, which generate strands $i$ and $i + 1$ in an $(n - 2, n)$ tangle (we adopt the convention of reading the tangle diagram from bottom to top)
- “caps” $f_n^i$ that connect strands $i$ and $i + 1$ in an $(n, n - 2)$ tangle
- “positive crossings” $t_n^i(+)$ that cross strands $i$ and $i + 1$ in an $(n, n)$ tangle with the $i$th strand passing over
- “negative crossings” $t_n^i(-)$ that cross strands $i$ and $i + 1$ in an $(n, n)$ tangle with the $i$th strand passing under
- “positive twists” $w_n^i(+) = (+)$ that twist the framing of the $i$th strand in an $(n, n)$ tangle by $+1$
- “negative twists” $w_n^i(-) = (-)$ that twist the framing of the $i$th strand in an $(n, n)$ tangle by $−1$

These generators obey a set of relations (cf. Appendix A). To construct a weak representation of $\text{Tan}$ it is sufficient to assign a functor to each generator so that tangle relations hold up to a natural isomorphism of functors.

**Claim 2.** Let $S_n^i : \mathcal{D}_{n-2} \rightarrow \mathcal{D}_n$, $1 \leq i < n$ be strongly spherical functors, $L_n^i$ (resp. $R_n^i$, resp. $T_n^i$, resp. $(T_n^i)'$) be their left adjoints (resp. right adjoints, resp. twists, resp. inverse twists). If the following conditions hold:

1. $S_n^i L_n^{i \pm 1}[-1] \simeq id$
2. $S_n^i S_n^j \simeq S_n^i S_n^j$ for $|i - j| > 1$
3. $L_n^i S_n^j \simeq S_n^i L_n^j$, $L_n^i S_n^j \simeq S_n^i L_n^j$ for $i - j > 1$

then the assignment
• \( \Psi(g_i^n) = S_i^n \)
• \( \Psi(f_i^n) = L_i^n[-1] \cong R_i^n[1] \)
• \( \Psi(t_i^n(+)) = T_i^n, \quad \Psi(t_i^n(-)) = (T_i^n)' \)
• \( \Psi(w_i^n(+)) = [-1], \quad \Psi(w_i^n(-)) = [1] \)

defines a weak representation of \( FTan \).

Proof. Let us check that the relations (19), (21)-(35) hold for the above choice of functors.

The Reidemeister move 0, cup-cup isotopy and cup-cap isotopy relations hold by the assumptions of the claim, and the cap-cap isotopy relation follows immediately from the cup-cup isotopy relation and the fact that caps are adjoint to cups up to a shift.

The cap-crossing isotopy, cup-crossing isotopy and crossing-crossing isotopy relations follow then from the above relations and the definition of a twist.

The Reidemeister move II relation \( T_i^n(T_i^n)' \cong id \cong (T_i^n)'T_i^n \) follows from the fact that \( S_i^n \) are spherical functors, hence \( T_i^n \) are equivalences of categories.

The commutation relations with twists (30)-(34) hold because all exact functors commute with shifts.

The remaining relations (35) (Reidemeister move I), (22) (Reidemeister move III) and (29) (the pitchfork move) are less trivial. For simplicity of notation assume that \( n = 3 \) and denote \( \Upsilon_j^3 \) by \( \Upsilon_i \), where \( \Upsilon \) stands for \( L, R, T \) or \( T' \).

4.1. Reidemeister move I: \( L_2T_1S_2[-1] \cong [1] \). By definition, the triangle

\[
L_2S_1R_1S_2 \rightarrow L_2S_2 \rightarrow L_2T_1S_2
\]

is exact. Observe that \( L_2S_1R_1S_2 = (L_2S_1)(L_2S_1)^R \), and since \( L_2S_1 \cong [1] \) is an equivalence, this functor is mapped to \( id \) with zero cone; moreover, this morphism of functors commutes with (factors through) the natural morphism \( L_2S_2 \rightarrow id \). From the triangle \( [2] \rightarrow L_2S_2 \rightarrow id \) we deduce then that \( L_2T_1S_2 \cong [2] \), qed.

4.2. Pitchfork move: \( T_1S_2 \cong T_2'S_1 \). Consider the exact triangle

\[
T_2'S_1 \rightarrow S_1 \rightarrow S_2L_2S_1.
\]

Applying the functor \( R_1S_2 \cong [-1] \) to it on the right, we get

\[
T_2'S_1[-1] \rightarrow S_1R_1S_2 \rightarrow S_2L_2S_1R_1S_2.
\]

On the other hand, there is another exact triangle

\[
S_1R_1S_2 \rightarrow S_2 \rightarrow T_1S_2,
\]

and the isomorphisms of functors \( id : S_1R_1S_2 \rightarrow S_1R_1S_2 \) and the natural map \( S_2L_2S_1R_1S_2 \cong S_2(L_2S_1)(L_2S_1)^R \rightarrow S_2 \) commute with the horizontal maps in the triangles, hence there is an isomorphism \( T_1S_2 \cong T_2'S_1 \), q.e.d.
4.3. Reidemeister move III: $T_1T_2T_1 \simeq T_2T_1T_2$.

**Proposition 4.** Let $S_1 : D_1 \to D$ and $S_2 : D_2 \to D$ be spherical functors. If there exists an exact equivalence of categories $Z : D_2 \to D_1$ such that $T_{S_1}S_2 \simeq T_{S_2}S_1Z$ then twist functors $T_{S_1}, T_{S_2}$ satisfy the braid relation $T_{S_1}T_{S_2}T_{S_1} \simeq T_{S_2}T_{S_1}T_{S_2}$.

**Proof.** By Proposition 2 the functors $T_{S_2}S_1$ and $T_{T_{S_2}S_1}Z = T_{T_{S_2}S_1}$ are isomorphic. Hence

$$T_{S_2}T_{S_1}T_{S_2} \simeq T_{S_2}T_{S_1}T_{S_2} \simeq T_{S_2}T_{S_1}T_{S_2} \simeq T_{S_2}T_{S_1}T_{S_2} \simeq T_{S_2}T_{S_1}T_{S_2}$$

q.e.d.

The conditions of the preceding proposition hold by the pitchfork move argument.

□

**Remark 1.** The condition on $S_i$ to be strongly spherical may be relaxed. Let $F_{S_n} \simeq [m]$ for all $n, i$. We still want $m$ to be the same for all $n, i$, otherwise isotopic links $f_n^i g_n^i$ would be sent to functors with different cohomology. Then we can set $\Psi(g_n^i) = S_n^i$, $\Psi(f_n^i) = (S_n^i)^R[-k_n^i] \simeq (S_n^i)^L[m - k_n^i]$, $\Psi(t_n^i(+)) = T_{S_n^i}[l_n^i]$, $\Psi(t_n^i(-)) = T_{S_n^i}[l_n^i]$, for some integers $k_n^i, l_n^i$, and $\Psi(w_n^i(\pm)) = [\pm(1 - m)]$. We need that $\Psi(f_n^i g_n^i) \simeq id \simeq \Psi(f_n^i g_n^i)$, which implies that $(S_n^{i+1})^R S_n^i [-k_n^{i+1}] \simeq id \simeq (S_n^{i+1})^L S_n^{i+1} [m - k_n^{i+1}]$: this gives us the condition $k_n^{odd} = m - k_n^{even}$. Then the pitchfork move requires $l_n^i + l_n^{i+1} = k_n^i - 1 = k_n^{i+1} - 1$, so we have $k_n^i = m/2$. Furthermore, Reidemeister move III implies that $l_n^i = l_n^{i+1}$. We arrive at the conditions $m = 4l + 2$, $\Psi(g_n^i) = S_n^i$, $\Psi(f_n^i) = (S_n^i)^R[-m/2] \simeq (S_n^i)^L[m/2]$, $\Psi(t_n^i(+)) = T_{S_n^i}[l]$, $\Psi(t_n^i(-)) = T_{S_n^i}[-l]$, and $\Psi(w_n^i(\pm)) = [\pm(1 - m)]$.

5. A computation of Ext groups.

Let us fix a weak triangulated representation $(D_n, \Psi)$ of $FTan$ such that $D_0$ is the bounded derived category $D^b(Vect)$ of vector spaces. To any $(0, n)$ tangle $\alpha$ we assign an object $E_\alpha = [\Psi(\alpha)](k)$ of $D_n$. Here $k$ denotes a one-dimensional vector space viewed as an object of the derived category. Assume that the conditions of Claim 2 hold. It turns out that several additional conditions suffice to determine uniquely the algebra $Ext^\bullet(\bigoplus_\alpha E_\alpha, \bigoplus_\alpha E_\alpha)$, where the index $\alpha$ runs through all flat $(0, n)$ tangles (otherwise called cup diagrams).

Note that the resulting graded algebra is isomorphic as a graded vector space, but strictly not isomorphic as an algebra, to Khovanov’s algebra described in 3; that is because Khovanov uses abelian categories, where it is natural to impose the skein relation in a form that virtually differs by sign from our condition 3 of Proposition 5.

Let us first define our algebra in combinatorial terms.
For a cup diagram $\alpha$, denote by $\alpha^\vee$ a cap diagram that is the mirror image of $\alpha$ with respect to the horizontal axis. For two cup diagrams $\alpha, \beta$ define a graded vector space $\mathcal{SW}^*(\alpha, \beta)$ as a tensor power $\bigotimes A_{\lambda_i}$, where labels $\lambda_i$ correspond to loops in $\alpha^\vee \circ \beta$, and each graded vector space $A_{\lambda_i}$ has a basis of two elements: 1 in degree 0 and $X_{\lambda_i}$ in degree 2.

Let labels $\mu_i$ enumerate caps in $\beta$, ordered so that if $i < j$, then the cap $\mu_i$ does not lie inside $\mu_j$. Let us consider a sequence of flat links $\alpha^\vee \circ \beta \circ \beta^\vee \circ \gamma = \xi_1, \xi_2, \ldots, \xi_{n+1} = \alpha^\vee \circ \gamma$ such that $\xi_j$ and $\xi_{j+1}$ differ by a saddle cobordism that replaces the cap $\mu_j$ and its mirror in $\beta \circ \beta^\vee$ by two vertical strands. Let $\Xi_j = \bigotimes A_{\lambda_i}$, where labels $\lambda_i$ correspond to loops in $\xi_j$, be a graded vector space associated to $\xi_j$. Then $\Xi_1 = \mathcal{SW}^*(\alpha, \beta) \otimes \mathcal{SW}^*(\beta, \gamma)$ and $\Xi_{n+1} = \mathcal{SW}^*(\alpha, \gamma)$. Define a map $\phi_j : \Xi_j \to \Xi_{j+1}$ as follows:

1. If the $j$-th saddle cobordism merges two loops labeled by $\lambda_1$ and $\lambda_2$ in $\xi_j$ into a loop labeled $\lambda_3$ in $\xi_{j+1}$, the map
   
   $$\phi_j : A_{\lambda_1} \otimes A_{\lambda_2} \to A_{\lambda_3}$$
   
   is defined by the following formulas:
   
   (a) If loops $\lambda_1$ and $\lambda_2$ do not lie inside each other:
   
   $$1 \otimes 1 \mapsto 1, \quad 1 \otimes X_{\lambda_2} \mapsto X_{\lambda_3}, \quad X_{\lambda_1} \otimes 1 \mapsto X_{\lambda_3}, \quad X_{\lambda_1} \otimes X_{\lambda_2} \mapsto 0.$$ 
   
   (b) If the loop $\lambda_1$ lies inside the loop $\lambda_2$:
   
   $$1 \otimes 1 \mapsto 1, \quad 1 \otimes X_{\lambda_2} \mapsto X_{\lambda_3}, \quad X_{\lambda_1} \otimes 1 \mapsto -X_{\lambda_3}, \quad X_{\lambda_1} \otimes X_{\lambda_2} \mapsto 0.$$ 

2. If the $j$-th saddle cobordism splits a loop labeled by $\lambda_3$ in $\xi_j$ into two loops labeled by $\lambda_1$ and $\lambda_2$ in $\xi_{j+1}$, the map
   
   $$\phi_j : A_{\lambda_3} \to A_{\lambda_1} \otimes A_{\lambda_2}$$
   
   is defined by the following formulas:
   
   (a) If loops $\lambda_1$ and $\lambda_2$ do not lie inside each other:
   
   $$1 \mapsto 1 \otimes X_{\lambda_2} + X_{\lambda_1} \otimes 1, \quad X_{\lambda_3} \mapsto X_{\lambda_1} \otimes X_{\lambda_2}.$$ 
   
   (b) If the loop $\lambda_1$ lies inside the loop $\lambda_2$:
   
   $$1 \mapsto 1 \otimes X_{\lambda_2} - X_{\lambda_1} \otimes 1, \quad X_{\lambda_3} \mapsto X_{\lambda_1} \otimes X_{\lambda_2}.$$ 

Define a map $\phi_{\alpha\beta\gamma} : \mathcal{SW}^*(\alpha, \beta) \otimes \mathcal{SW}^*(\beta, \gamma) \to \mathcal{SW}^*(\alpha, \gamma)$ as a composition $\phi_n \circ \ldots \circ \phi_1 : \Xi_1 \to \Xi_n$.

**Definition 2.** Define an algebra $\mathcal{SW}^*$ as a direct sum $\bigoplus_{\alpha, \beta} \mathcal{SW}^*(\alpha, \beta)$ with multiplication induced by maps $\phi_{\alpha\beta\gamma}$.

This algebra was first described in [6], so the abbreviation $\mathcal{SW}$ stands for Stroppel-Webster.

**Proposition 5.** If the representation $\Phi$ above sends generators $g_i^i$ (resp. $f_i^i$, resp. $t_i^i(n)$) to functors $G^i_n$ (resp. $F^i_n$, resp. $T^i_n(m)$) that satisfy the following conditions:

1. $F^i_n \simeq (G^i_n)^R[1] \simeq (G^i_n)^L[-1]$;
(2) for every \( n, i \) there exists an exact triangle of functors
\[
\text{id}[1] \to F^i_n G_n \to \text{id}[-1],
\]
where morphisms are adjunction morphisms;

(3) for every \( n, i \) there exist exact triangles of functors
\[
G^i_n F^i_n [-1] \to \text{id} \to T^i_n (+) \to T^i_n (-) \to G^i_n F^i_n [1],
\]
where morphisms between \( \text{id} \) and \( G^i_n F^i_n [\pm 1] \) are adjunction morphisms;

(4) for every \( n, i, j \) such that \( j > i + 1 \) there are isomorphisms of functors
\[
F^i_n G_n \to \text{id}, \quad G^i_n G^j_{n+2} G^j_{n+2} \to G^i_n G^j_{n+2} G^j_{n+2}, \quad F^i_n G_n \to G^i_n [-2] F^i_n \]
(when applicable) such that the following diagrams are commutative:

\[
\begin{array}{ccc}
F^i_{n+2} G^i_{n+2} G^i_n & \to & F^i_{n+2} G^i_{n+2} G^i_n \\
\downarrow & & \downarrow \\
G^i_n & \to & G^i_n
\end{array}
\]

\[
\begin{array}{ccc}
G^i_{n+2} G^j_{n-2} G^j_{n-2} & \to & G^j_{n+2} G^i_n G^k_{n-2} G^k_{n-2} \\
\downarrow & & \downarrow \\
G^k_{n+2} G^k_{n-2} G^j_{n-2} & \to & G^k_{n+2} G^j_{n-2} G^j_{n-2}
\end{array}
\]

then there is a graded algebra isomorphism
\[
\text{Ext}^* (\bigoplus_{\alpha} E_{\alpha} \bigoplus_{\alpha} E_{\alpha}) \simeq SW^*.
\]

**Corollary 3.** If a weak representation of the category of framed tangles is generated by a collection of spherical functors as in Claim 2 and there is a collection of isomorphisms of functors \( F^i_{n+2} G^i_n \to \text{id}, G^i_{n+2} G^j_{n-2} G^j_{n-2} \to G^i_{n+2} G^j_{n-2} G^j_{n-2} \) such that (12) holds, then there is a graded algebra isomorphism (13).

To prove Proposition 5, let us start with a lemma.

**Lemma 2.** Under the conditions of Proposition 5 any two flat isotopies (cf. Appendix B) of any two given tangles give the same morphism of functors.

**Proof.** We need to check that for elementary flat isotopy moves (cf. Appendix) two sides give the same morphism of tangles.

For move 0 the statement is part of the conditions of Proposition 5.

For other moves, the proofs go as follows: we present all functors as cones and prove that sequences of maps on both sides of the moves give the same isomorphisms for two components of the cones, hence for the cones themselves. Let us write this down for move 1. The rows in the diagram below are exact triangles of functors, and the last column represents isotopy move 1 (cf. Figure 5 in Appendix B).
We shall prove that the vertical composition maps $G_n^{i,i^2}F_n[-1] \rightarrow G_n^{i,i^2}F_n[-1]$ and $id \rightarrow id$ in this diagram are identities. The proof is illustrated in Figure 2.

The first column of the diagram maps isomorphically onto

\[
G_n^{i,i^2}F_n[-1] \rightarrow G_n^{i,i^2}F_n[-1] \rightarrow G_n^{i,i^2}F_n[-1] \rightarrow G_n^{i,i^2}F_n[-1] \rightarrow G_n^{i,i^2}F_n[-1]
\]

and the composition of the maps in this expression is the iterated move 11, hence is equal to the identity map. The map $id \rightarrow id$ is proved to be the identity in a similar way. It follows that the functorial map $T_n^i(+) \rightarrow T_n^i(+$) is also the identity, which proves the lemma for move 11.

In move 2 an argument similar to the proof of Proposition 1 shows that the functor $F_{n+2}^{i,i^2}G_{n+2}[-1] \rightarrow F_{n+2}^{i,i^2}G_{n+2}[-1]$ is naturally isomorphic to the functor
Figure 2. Isotopy move 1 gives the identity morphism of functors.

Then all we need to prove is the commutativity of the following square:
which follows from move $\Box$ and the fact that both functors in the left column are isomorphic to $F_{n+2}^i G_{n+2}^i F_{n+2}^{i+2} G_{n+2}^{i+1}$.

In move $\blacksquare$ we have a picture similar to that of the move $\Box$. We expand $T_n^i T_n^{i+1}(+) as the cone $G_n^i F_n^i T_n^{i+1}(+)[-1] \to T_n^{i+1}(+)$. Under both sides of the move this goes to the cone $G_n^i F_n^i T_n^{i+1}(+) T_n^i T_n^{i+1}(+)[-1] \to T_n^{i+1}(+) \to T_n^i T_n^{i+1}(+)$. We prove that the maps of functors $G_n^i F_n^i T_n^{i+1}(+)[-1] \to G_n^i F_n^i T_n^{i+1}(+) T_n^i T_n^{i+1}(+)[-1]$ and $T_n^{i+1}(+) \to T_n^{i+1}(+) T_n^i T_n^{i+1}(+)$ are the same on both sides of the move by proving that those maps coincide after being composed with the isomorphisms of functors $G_n^i F_n^i T_n^{i+1}(+) T_n^i T_n^{i+1}(+)[-1] \simeq G_n^i F_n^i T_n^{i+1}(+)[-1]$ and $T_n^{i+1}(+) \simeq T_n^{i+1}(+)$ respectively.

The move $\square$ reduces to the move $\blacksquare$ after expanding as a cone the first twist $T_{n-2}^i$ in $T_{n-2}^i F_{n}^{i+2} T_{n}^{i+1}(+)$. The move $\triangle$ reduces to the move $\Box$ after expanding as a cone the leftmost $T_n^i$ in the expression $T_n^i T_n^{i+2}(+) T_n^i T_n^{i+2}(+)$. 

$\Box$

**Proof of the Proposition.** Let us first establish a canonical isomorphism

$$E^{\bullet}(\bigoplus_{\alpha} \mathcal{E}_\alpha, \bigoplus_{\alpha} \mathcal{E}_\alpha) \to SW^\bullet$$

as graded vector spaces. For two cup diagrams $\alpha, \beta$ we have

$$E^{\bullet}(\mathcal{E}_\alpha, \mathcal{E}_\beta) = E^{\bullet}([\Psi(\alpha)](k), [\Psi(\beta)](k)) \cong$$

$$\cong H^\bullet([\Psi(\alpha)^F \Psi(\beta)](k)) \cong H^\bullet([\Psi(\alpha^\vee \circ \beta)](k)[-n]).$$

The link $\alpha^\vee \circ \beta$ is flat isotopic to the link $(f^1_2 g^1_2)^{\circ N}$, where $N$ is the number of loops in $\alpha^\vee \circ \beta$. By Lemma $\blacksquare$ there is a canonical isomorphism between $H^\bullet([\Psi(\alpha^\vee \circ \beta)](k))$ and $H^\bullet([\Psi((f^1_2 g^1_2)^{\circ N})](k))$ which in turn is canonically isomorphic to $H^\bullet([\Psi(f^1_2 g^1_2)^{\circ N}](k)) \cong A^{\otimes N}$.

The space $SW^\bullet$ has a distinguished basis. To establish the graded algebra isomorphism, let us pick three cup diagrams $\alpha, \beta$ and $\gamma$, and compute the composition map

$$(15) \quad SW^\bullet(\alpha, \beta) \otimes SW^\bullet(\beta, \gamma) \to E^{\bullet}(\mathcal{E}_\alpha, \mathcal{E}_\beta) \otimes E^{\bullet}(\mathcal{E}_\beta, \mathcal{E}_\gamma) \to$$

$$\to E^{\bullet}(\mathcal{E}_\alpha, \mathcal{E}_\gamma) \to SW^\bullet(\alpha, \gamma)$$

in that distinguished basis. To do that, consider a sequence of flat links $\alpha^\vee \circ \beta \circ \beta^\vee \circ \gamma = \xi_1, \xi_2, \ldots, \xi_{n+1} = \alpha^\vee \circ \gamma$ as in the definition of the algebra $SW^\bullet$. Let us construct the map

$$(16) \quad \Psi(\xi_0) \simeq E^{\bullet}(\mathcal{E}_\alpha, \mathcal{E}_\beta) \otimes E^{\bullet}(\mathcal{E}_\beta, \mathcal{E}_\gamma) \to E^{\bullet}(\mathcal{E}_\alpha, \mathcal{E}_\gamma) \simeq \Psi(\xi_n)$$

as a composition of maps $\Psi(\xi_0) \to \Psi(\xi_1) \to \ldots \to \Psi(\xi_n)$. Consider the map $\Psi(\xi_k) \to \Psi(\xi_{k+1})$. The links $\xi_k$ and $\xi_{k+1}$ differ only inside a simply connected domain $O$ with 4 marked points $p_1, p_2, p_3, p_4$ on its boundary, so that $\xi_k$ contains arcs $[p_1, p_2]$ and $[p_3, p_4]$, while $\xi_{k+1}$ contains arcs $[p_1, p_4]$ and $[p_2, p_3]$. Consider a flat isotopy that stabilizes $O$ pointwise, moves away
all loops that do not intersect $O$, and minimizes the number of points of inflection on all arcs outside $O$. We arrive at one of the four cases: first, either the points $p_i$ belong to the same component of the link $\xi_k$, and to different components of $\xi_{k+1}$, or vice versa; second, in the link where the points are in different components, the components may be separated or one may lie inside the other. These four are represented by four maps exactly as in the definition of the algebraic structure on $SW^\bullet$. The maps themselves are computed using the 2-point or 4-point examples.

Let us provide a 4-point example of merging two loops one inside another, assuming that merging (resp. splitting) of separated loops is already proved to be the multiplication (resp. comultiplication) in $A$. We need the map

$$F_2^1 F_4^2 G_3^1 G_2^1 \rightarrow F_2^1 F_4^1 F_4^1 G_2^2 G_1^1 [1]$$

that comes from the adjunction map $id \rightarrow G_1^1 F_4^1 [1]$. The diagram

(17) $$\begin{array}{c}
F_2^1 F_4^2 G_2^2 G_1^1 \rightarrow F_2^1 F_4^1 F_4^1 G_2^2 G_1^1 [1] \\
\downarrow \\
F_2^1 T_2^1 (-) F_3^2 G_4^3 T_2^1 (+) G_2^1 \rightarrow F_2^1 T_2^1 (-) F_4^2 G_4^3 G_4^2 T_2^1 (+) G_1^1 [1]
\end{array}$$

is commutative by Lemma 2 applied to the links $f_2^1 t_1^1 (-) g_2^2 g_1^3$ and $f_2^1 t_1^1 (-) f_3^2 t_1^2 (-) g_2^3 t_2^1 (+) g_1^1$. This diagram is illustrated in Figure 3.

Consider the graded vector space $M$ that corresponds to the link $f_2^1 t_1^1 (-) g_2^1$. By presenting $T_2^1 (-)$ as a cone we see that $M$ is a shift of the cokernel of the comultiplication map $A \rightarrow A \otimes A$. Therefore, it is a two-dimensional graded vector space generated by $1_M = 1 \otimes 1$ in degree $-1$ and $X_M = 2(1 \otimes X - X \otimes 1)$ in degree 1. The space $M$ has a structure of an
A-bimodule given by merging this link with a circle from the bottom or from the top. $1$ acts trivially on both sides, and we have $X \cdot X_M = X_M \cdot X = 0$, $X \cdot 1_M = X_M$, and $1_M \cdot X = -X_M$. There is an exact sequence of $A$-bimodules $A \to M \otimes A \to M$ that corresponds to Reidemeister move II. The map $A \to M \otimes A$ here sends $1$ to $X \otimes 1 + 1 \otimes X$, and $X$ to $X \otimes X$, and the map $M \otimes A \to M$ is the right action. Expanding $T^+_2(+) \text{ in the second line of diagram (17)}$ as a cone, we see that our required map $A \otimes A \to A$ comes from a morphism of exact triangles:

\[ \begin{array}{c}
A \otimes A \to M \otimes A \otimes A \to M \otimes A \\
A \to M \otimes A \to M
\end{array} \]

where the first line is the second line tensored by $A$, and vertical maps are right actions of the rightmost copy of $A$ on $M$. A simple computation shows that our map $A \otimes A \to A$ then sends $1 \otimes X$ to $-X$, and $X \otimes 1$ to $X$. □

6. Appendix A: Tangle relations.

Consider a category $\text{Tan}$ of tangles, whose objects are natural numbers, and morphisms from $\lbrack n \rbrack$ to $\lbrack m \rbrack$ are isotopy classes of $(n,m)$ tangles. The set of generators and relations for this category is long known ([2]). The standard generators are:

- "cups" $g^i_n$, which generate strands $i$ and $i+1$ in an $(n-2,n)$ tangle (we adopt the convention of reading the tangle diagram from bottom to top)
- "caps" $f^i_n$ that connect strands $i$ and $i+1$ in an $(n,n-2)$ tangle
- "positive crossings" $t^+_n(\pm)$ that cross strands $i$ and $i+1$ in an $(n,n)$ tangle with the $i$th strand passing over
- "negative crossings" $t^-_n(\pm)$ that cross strands $i$ and $i+1$ in an $(n,n)$ tangle with the $i$th strand passing under

The relations between them are described by the following lemma ([2], Lemma X.3; cf. [1]):

**Lemma 3.** Every isotopy of tangles is a composition of the following elementary isotopies up to isotopies of tangle diagrams:

\[ \begin{align*}
(19) \quad & \text{Reidemeister (0)} : \quad f^i_n \circ g^{i+1}_n = id = f^{i+1}_n \circ g^i_n; \\
(20) \quad & \text{Reidemeister (I)} : \quad f^i_n \circ t^\pm_n(\mp) \circ g^i_n = id = f^i_n \circ t^{i\pm}_n(+) \circ g^i_n; \\
(21) \quad & \text{Reidemeister (II)} : \quad t^+_n(\mp) \circ t^+_n(+) = id = t^+_n(+) \circ t^+_n(\mp);
\end{align*} \]
(22) Reidemeister (III):
\[ t_{i}^{n}(+) \circ t_{i}^{i+k}(+) \circ t_{i}^{i}(+) = t_{i}^{i+1}(+) \circ t_{i}^{i}(+) \circ t_{i}^{i+k}(+) ; \]

(23) cup–cup isotopy:
\[ g_{i}^{i+k} \circ g_{i}^{i} = g_{i+2}^{i+k} \circ g_{i+2}^{i+k-2} ; \]

(24) cap–cap isotopy:
\[ f_{i}^{i+k-2} \circ f_{i+2}^{i} = f_{i}^{i} \circ f_{i+2}^{i+k} ; \]

(25) cup–cap isotopy:
\[ g_{i}^{i+k-2} \circ f_{i}^{i} = f_{i+2}^{i} \circ g_{i+2}^{i+k} ; \]

(26) cup–crossing isotopy:
\[ g_{i}^{i} \circ t_{i-2}^{i+k-2} (\pm) = t_{i}^{i+k} (\pm) \circ g_{i}^{i} , \quad g_{i}^{i+k} \circ t_{i-2}^{i} (\pm) = t_{i}^{i} (\pm) \circ g_{i+2}^{i+k} ; \]

(27) cap–crossing isotopy:
\[ f_{i}^{i} \circ t_{i-2}^{i+k} (\pm) = t_{i-2}^{i+k-2} (\pm) \circ f_{i}^{i} , \quad f_{i}^{i+k} \circ t_{i-2}^{i} (\pm) = t_{i}^{i-2} (\pm) \circ f_{i}^{i+k} ; \]

(28) crossing–crossing isotopy:
\[ t_{i}^{i} (\pm) \circ t_{i}^{i+k} (\pm) = t_{i}^{i+k} (\pm) \circ t_{i}^{i} (\pm) ; \]

(29) pitchfork move:
\[ t_{n}^{i} (\pm) \circ g_{n}^{i+1} = t_{n}^{i+1} (\pm) \circ g_{n}^{i} , \quad t_{n}^{i} (\pm) \circ g_{n}^{i} = t_{n}^{i+1} (\pm) \circ g_{n}^{i} ; \]

where \( k \geq 2 \).

One could also consider the category \textbf{OTan} of oriented tangles, or the category \textbf{FTan} of framed tangles. In the latter case, the existing generators should be supplied with the blackboard framing, and two series of twist generators should be added:

- "positive twists" \( w_{n}^{i} (+) \) that twist the framing of the \( i \)th strand in an \((n, n)\) tangle by +1
- "negative twists" \( w_{n}^{i} (-) \) that twist the framing of the \( i \)th strand in an \((n, n)\) tangle by −1
with the obvious relations

\[ w^i_n(+) \circ w^i_n(-) = w^i_n(-) \circ w^i_n(+) = \text{id}; \]

\[ w^i_n(k) \circ g^i_n = w^{i+1}_n(k) \circ g^i_n; \quad w^i_n(k) \circ g^j_n = g^j_n \circ w^{i-1\pm}_n(k), \quad i \neq j, j + 1; \]

\[ f^i_n \circ w^i_n(k) = f^i_n \circ w^{i+1}_n(k); \quad w^i_n(k) \circ f^j_n = f^j_n \circ w^{i+1\pm}_n(k), \quad i \neq j, j + 1; \]

\[ w^i_n(k) \circ t^i_n(\pm) = t^i_n(\pm) \circ w^{i+1}_n(k); \quad w^{i+1}_n(k) \circ t^i_n(\pm) = t^i_n(\pm) \circ w^i_n(k); \]

\[ w^i_n(k) \circ t^i_n(\pm) = t^i_n(\pm) \circ w^i_n(k), \quad i \neq j, j + 1. \]

The relations (19), (21)-(29) remain unchanged, and (20) turns into

\[ f^i_n \circ t^{i\pm}_n(k) \circ g^i_n = w^i_n(k). \]

7. Appendix B. Flat tangle isotopies.

In this section we prove a technical result concerning elementary moves for a special class of tangle isotopies.

Definition 3. A flat tangle isotopy is an isotopy that does not involve Reidemeister type I moves.

If we supply our tangles with blackboard framing, flat isotopies would be those that preserve the framing.

Let us call two isotopies equivalent, if they are isomorphic as tangle cobordisms.

Proposition 6. Any two equivalent flat isotopies are equivalent via a sequence of the following elementary moves:

(0) \( (g^i_n \simeq f^{i+1}_n, g^i_n \simeq f^{i+1\pm}_n, g^i_n \simeq g^i_n) \leftarrow (g^i_n \simeq g^i_n) \)

(1) \( (t^{i+1}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^i_n) \leftarrow (t^i_n, t^i_n) \)

(2) \( (g^i_n, f^i_n, f^{i+1\pm}_n, f^{i+1\pm}_n, f^{i+1\pm}_n, f^{i+1\pm}_n, f^{i+1\pm}_n) \leftarrow (g^i_n, f^i_n, f^{i+1\pm}_n, f^{i+1\pm}_n, f^{i+1\pm}_n, f^{i+1\pm}_n, f^{i+1\pm}_n) \)

(3) \( (t^i_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n) \leftarrow (t^i_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n, t^{i+1\pm}_n) \)

(4) \( (t^{i+2}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n) \leftarrow (t^{i+2}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n, t^{i+2\pm}_n) \)
and the fact that any tangle isotopy is representable as a composition of finite. Then tangle isotopies project to paths in this compactified space, coordinate, but keeping the requirement on the number of critical points to order more than 2, and allowing several critical points with the same time can compactify it by adding singular diagrams: allowing critical points of fix critical points. It is a union of open balls of various dimensions. We

\[ t_{n-2}^i (1) f_n^{i+1} t_n^{i+2} (-) t_n^i (1) \simeq t_{n-2}^i (1) f_n^{i+1} t_n^i (1) t_n^{i+2} (-) \simeq \\
 t_{n-2}^i (1) f_n^{i+1} t_n^{i+2} (-) \simeq f_n^{i+2} (1) t_n^i (-) t_n^{i+2} (-) \simeq \\
 f_n^{i+1} (-) t_n^{i+2} (-) t_n^{i+1} (+) \]

\[ (t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) \simeq \\
 t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) \simeq \\
 t_n^{i+2} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) \simeq \\
 t_n^{i+2} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) \simeq \\
 (t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) \simeq \\
 t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) \simeq \\
 t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) \simeq \\
 t_n^i (+) t_n^{i+1} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) \simeq \\
 t_n^{i+2} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) \simeq \\
 t_n^{i+2} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) \simeq \\
 t_n^{i+2} (+) t_n^i (+) t_n^{i+2} (+) t_n^i (+) t_n^{i+1} (+) t_n^i (+) \]

**Proof.** Consider a moduli space of tangle diagrams up to diagram isotopies that fix critical points. It is a union of open balls of various dimensions. We can compactify it by adding singular diagrams: allowing critical points of order more than 2, and allowing several critical points with the same time coordinate, but keeping the requirement on the number of critical points to be finite. Then tangle isotopies project to paths in this compactified space, and the fact that any tangle isotopy is representable as a composition of

\[
\begin{align*}
\text{Figure 4. Isotopy move 0.} \\
\text{Figure 5. Isotopy move 1.}
\end{align*}
\]
elementary isotopies amounts to the fact that any path is homotopic to a path that only crosses domain walls at generic points, where two domains meet.

For a singular diagram we define the order of singularity to be the sum $\sum (\lambda_p - 2)$ over all critical points $p$, where $\lambda_p$ is the index of ramification for the projection onto the time axis. Then the set of singular diagrams is stratified by the order of singularity. Non-singular diagrams have order 0; generic points of domain walls have order 1, and the set $\text{Sing}_{\geq 2}$ of diagrams of order greater than 2 has local codimension at least 3; in a non-equidimensional space it means that there exists a subspace $U$ isomorphic to a manifold, such that the set $\text{Sing}_{\geq 2}$ lies in $U$ and has local codimension at least 3 in $U$. 
Equivalent isotopies correspond to homotopic paths in this compactified space. By the codimension argument, we can choose a homotopy that is separated from the set $\text{Sing}_{\geq 2}$, and intersects components of the set $\text{Sing}_2$ of diagrams with a singularity of order exactly 2 transversally at a generic point. Then the corresponding isotopy move is equivalent to a composition of elementary moves that correspond to small disks that cross components of $\text{Sing}_2$ transversally at a generic point, and those correspond to the list above.

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