HIGHER LAWRENCE CONFIGURATIONS

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Abstract. Any configuration of lattice vectors gives rise to a hierarchy of higher-dimensional configurations which generalize the Lawrence construction in geometric combinatorics. We prove finiteness results for the Markov bases, Graver bases and face posets of these configurations, and we discuss applications to the statistical theory of log-linear models.

1. Introduction

Fix a configuration \( A = \{a_1, \ldots, a_n\} \) of lattice vectors spanning \( \mathbb{Z}^d \), and let \( \mathcal{L}(A) \subset \mathbb{Z}^n \) be the lattice of linear relations on \( A \). We introduce a hierarchy of configurations \( A^{(2)}, A^{(3)}, A^{(4)}, \ldots \), as follows. The configuration \( A^{(r)} \) consists of \( r \cdot n \) vectors in \( \mathbb{Z}^{dr+n} = (\mathbb{Z}^d \otimes \mathbb{Z}^r) \oplus \mathbb{Z}^n \), namely,

\[
A^{(r)} = \left\{ (a_i \otimes e_j) \oplus \delta_i : i = 1, \ldots, n, j = 1, \ldots, r \right\}
\]

where \( e_i \) and \( e_j \) denote unit vectors in \( \mathbb{Z}^n \) and \( \mathbb{Z}^r \) respectively. The first object in this hierarchy is \( A^{(2)} \), which is a configuration of \( 2n \) vectors isomorphic to the Lawrence lifting \( \Lambda(A) \) of the given configuration \( A \). See [10, §7] or [11, §6.6]. We call \( A^{(r)} \) the \( r \)-th Lawrence lifting of \( A \). In this paper we study the Lawrence hierarchy \( A^{(2)}, A^{(3)}, A^{(4)}, \ldots \) from the perspective of toric algebra, geometric combinatorics and applications to statistics [4].

The \( r \)-th Lawrence lifting \( A^{(r)} \) is characterized as the configuration whose linear relations are \( r \)-tuples of linear relations on \( A \) that sum to zero. Indeed, the lattice of linear relations on \( A^{(r)} \) has rank \( (r - 1)(n - d) \) and equals

\[
\mathcal{L}(A^{(r)}) = \left\{ (u^{(1)}, u^{(2)}, \ldots, u^{(r)}) \in (\mathbb{Z}^n)^r : u^{(i)} \in \mathcal{L}(A) \forall i, \sum u^{(i)} = 0 \right\}.
\]

We think of the elements of \( \mathcal{L}(A^{(r)}) \) as integer \( r \times n \)-tables whose column sums are zero and whose \( A \)-weighted row sums are zero. The type of such a table is the number of non-zero row vectors \( u^{(i)} \). Given any basis \( \{b^{(1)}, \ldots, b^{(n-d)}\} \subset \mathbb{Z}^n \) for the lattice \( \mathcal{L}(A) \) of linear relations on \( A \), it is easy to derive a lattice basis of \( \mathcal{L}(A^{(r)}) \) consisting of tables of type 2. For instance, take tables with first row some \( b^{(i)} \) and some other row \(-b^{(i)}\).

2000 Mathematics Subject Classification. Primary 52B20; Secondary 13P10, 62H17.

Key words and phrases. Markov basis, Graver basis, toric ideal, Lawrence polytope.

The first author was supported by grant BFM2001–1153 of the Spanish Dirección General de Investigación. The second author was partially supported by grants DMS-0200729 and DMS-0138323 of the U.S. National Science Foundation.
In toric algebra and its statistics applications we are interested in larger subsets of \( \mathcal{L}(\mathcal{A}(r)) \) which generate the lattice in a stronger sense. A *Markov basis* of \( \mathcal{A}(r) \) is a finite subset of \( \mathcal{L}(\mathcal{A}(r)) \) which corresponds to a minimal set of generators of the toric ideal \( I_{\mathcal{A}(r)} \) as in [10, §4], or, equivalently, to a minimal set of moves which connects any two nonnegative integer \( r \times n \)-tables that have the same column sums and the same \( \mathcal{A} \)-weighted row sums [1], [4], [6], [9]. We prove that the Markov bases stabilize for \( r \gg 0 \).

**Theorem 1.** For any configuration \( \mathcal{A} = \{a_1, \ldots, a_n\} \) in \( \mathbb{Z}^d \), there exists a constant \( m = m(\mathcal{A}) \) such that any higher Lawrence lifting \( \mathcal{A}(r) \), for any \( r \geq 2 \), has a Markov basis consisting of tables having type at most \( m \).

We call the minimum value \( m(\mathcal{A}) \) the *Markov complexity* of \( \mathcal{A} \). This paper was inspired by recent work of the statisticians Aoki and Takemura [1]. Their result states, in our notation, that the product of two triangles \( \Delta_2 \times \Delta_2 = \{ e_i \oplus e_j \in \mathbb{Z}^3 \oplus \mathbb{Z}^3 : 1 \leq i, j \leq 3 \} \) has Markov complexity 5. Indeed, \( \mathcal{L}((\Delta_2 \times \Delta_2)(r)) \) consists of integer \( 3 \times 3 \times r \)-tables with zero line sums in the three directions. These are all possible moves for the no-three-way interaction model [4]. Our Theorem 1 implies:

**Corollary 2.** For any positive integers \( p \) and \( q \) there exists an integer \( m \) such that the Markov basis for \( p \times q \times r \)-tables (in the no three-way interaction model, for arbitrary \( r \)) consists of tables of format \( p \times q \times m' \) with \( m' \leq m \).

We often use the phrase “the Markov basis” instead of “a Markov basis”. The definite article is justified because the minimal generating set of a homogeneous toric ideal is unique up to minor combinatorial modifications.

We prove Theorem 1 by providing an explicit upper bound for \( m(\mathcal{A}) \). Recall that the *Graver basis* of \( \mathcal{A} \) is the set of minimal elements in \( \mathcal{L}(\mathcal{A}) \setminus \{0\} \), where \( \mathbb{Z}^n \) is partially ordered by setting \( a \leq b \) if \( b \) is the conformal sum of \( a \) and \( b - a \). This condition means that, for every \( i \in \{1, \ldots, n\} \), either \( 0 \leq a_i \leq b_i \) or \( 0 \geq a_i \geq b_i \) holds. The Graver basis is unique, finite, and contains Markov bases for all subconfigurations of \( \mathcal{A} \). See [10] for bounds, algorithms and many details. We define the *Graver complexity* \( g(\mathcal{A}) \) to be the maximum type of any table in the Graver basis of some higher Lawrence lifting \( \mathcal{A}(r) \). Clearly, \( m(\mathcal{A}) \leq g(\mathcal{A}) \). We now state our main result. The phrase “the Graver basis of the Graver basis” is not a typo but it is the punchline. We regard the elements in the Graver basis of \( \mathcal{A} \) as the columns of some big matrix and then we compute the Graver basis of that big matrix.

**Theorem 3.** The Graver complexity \( g(\mathcal{A}) \) of a configuration \( \mathcal{A} \) is the maximum 1-norm of any element in the Graver basis of the Graver basis of \( \mathcal{A} \).

This paper is organized as follows. In Section 2 we present a few examples to illustrate the notions of Markov complexity and Graver complexity. The proof of Theorem 3 (and hence of Theorem 1) will be given in Section 3. Section 4 deals with applications to statistics. We show that if \( \mathcal{A} \) is any...
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log-linear hierarchical model (in the notation of [6, 9]) then $A(r)$ is the corresponding logit model (in the sense of [3, §VII], [5, §6]) where the response variable has $r$ levels. Thus Theorem [11] implies the existence of a finite Markov basis for logit models where the response variable has an unspecified number of levels. In Section 5 we prove an analogue of Theorem [8] for circuits, and we examine the convex polytopes arising from higher Lawrence liftings $A(r)$.

2. Examples

The first three examples below show that the Markov and Graver complexities of a configuration may coincide or differ a lot. After this we work out in detail the twisted cubic curve, a familiar example in toric algebra.

Example 4. Let $A$ be any configuration and $\Lambda(A) = A(2)$ its usual Lawrence lifting. Then $\Lambda(A)(r)$ is the Lawrence lifting of $A(r)$, since $(A(s))(r) = (A(s))(r)$ for all $r, s$. By [10, Theorem 7.1], the Markov and Graver complexity of $\Lambda(A)$ coincide. They are equal to the Graver complexity of $A$.

Example 5. Let $A = \{1, \ldots, 1\}$ consist of $n$ copies of the vector 1 in $\mathbb{Z}^1$. The $r$-th Lawrence lifting $A(r)$ is the product of two simplices $\Delta^{n-1} \times \Delta^{r-1}$. In statistics, this corresponds to two-dimensional tables of size $n \times r$. The Graver basis of $A(r)$ consists of the circuits in the complete bipartite graph $K_{n,r}$, and the Markov basis consists of circuits which fit in a subgraph $K_{n,2}$. The Graver complexity of $A$ is $n$, and the Markov complexity of $A$ is 2.

Example 6. Take $d = 1$, $n = 3$ and $A = \{k, l, m\}$, where $k$, $l$ and $m$ are pairwise relatively prime. Using Theorem [3] it can be shown that the Graver complexity $g(A)$ equals $k + l + m$. We invite the reader to write down the Graver basis element of type $k + l + m$ for $A^{(k+l+m)}$. It would be interesting to find a formula, in terms of $k, l$ and $m$, for the Markov complexity $m(A)$.

Example 7. (Twisted Cubic) Let $d = 2$ and $n = 4$ and fix the configuration

(2) $A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix}$

The corresponding statistical model is Poisson regression with four levels. The toric ideal $I_A$ of this configuration consists of the algebraic relations among the four cubic monomials in two unknowns $s$ and $t$:

$I_A = \langle x_1 x_3 - x_2^2, x_1 x_4 - x_2 x_3, x_2 x_3 - x_3^2 \rangle = \ker(k[x_1, x_2, x_3, x_4] \to k[s, t], x_1 \mapsto s^3, x_2 \mapsto s^2 t, x_3 \mapsto s t^2, x_4 \mapsto t^3).$

The Markov basis of $A$ is the set of three vectors $(1, -2, 1, 0)$, $(1, -1, -1, 1)$, and $(0, 1, -2, 1)$ corresponding to the minimal generators of $I_A$. The Graver basis of $A$ has two additional elements, namely $(1, 0, -3, 2)$ and $(2, -3, 0, 1)$. 


The “classical” Lawrence lifting is isomorphic to the eight column vectors of
\[ \mathcal{A}^{(2)} = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \\ 1 & 1 \end{pmatrix}, \]
where \( \mathbf{1} \) is the identity matrix of size 4 \( \times \) 4. This 8 \( \times \) 8-matrix has rank 6. Its kernel \( \mathcal{L}(\mathcal{A}^{(2)}) \) is a rank 2 lattice whose elements are identified with 2 \( \times \) 4-integer tables \( \mathbf{T} \) with \((\mathbf{1} \ 1) \cdot \mathbf{T} = \mathbf{0}\) and \(\mathbf{T} \cdot \mathbf{A}^t = \mathbf{0}\). It is spanned by
\[ \begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 2 & -1 & 0 \end{pmatrix} \] and \[ \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}. \]
By [10, Theorem 7.1], the Markov basis of \( \mathcal{A}^{(2)} \) equals the Graver basis of \( \mathcal{A}^{(2)} \). It consists of the five tables constructed from the Graver basis of \( \mathcal{A} \):
\[ \begin{pmatrix} 1 & -2 & 1 & 0 \\ -1 & 2 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -3 & 2 \\ -1 & 0 & 3 & -2 \end{pmatrix}, \begin{pmatrix} 2 & -3 & 0 & 1 \\ -2 & 3 & 0 & -1 \end{pmatrix}. \]

The third Lawrence lifting consists of the columns of the 10 \( \times \) 12-matrix
\[ \mathcal{A}^{(3)} = \begin{pmatrix} \mathcal{A} & 0 & 0 \\ 0 & \mathcal{A} & 0 \\ 0 & 0 & \mathcal{A} \\ 1 & 1 & 1 \end{pmatrix}. \]
Its kernel \( \mathcal{L}(\mathcal{A}^{(3)}) \) is the rank 4 lattice consisting of 3 \( \times \) 4-integer tables \( \mathbf{T} \) with \((1 \ 1 \ 1) \cdot \mathbf{T} = \mathbf{0}\) and \(\mathbf{T} \cdot \mathbf{A}^t = \mathbf{0}\). The Markov basis of \( \mathcal{A}^{(3)} \) has 21 tables. Fifteen of them are gotten from the five tables above by adding a row of zeros. The other six Markov basis elements are row permutations of
\[ \begin{pmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \\ -1 & 0 & 3 & -2 \end{pmatrix}. \]

It can be checked that no new Markov basis elements are needed for \( \mathcal{A}^{(r)} \), \( r \geq 4 \). Any two \( r \times 4 \)-tables of non-negative integers which have the same column sums and the same \( \mathcal{A} \)-weighted row sums can be connected by the known moves involving only two or three of the rows. Equivalently:

**Remark 8.** The twisted cubic curve \( \mathcal{A} \) has Markov complexity \( m(\mathcal{A}) = 3 \).

We next discuss the Graver complexity of \( \mathcal{A} \). The Graver basis of \( \mathcal{A}^{(3)} \) consists of 87 tables. Every other table in \( \mathcal{L}(\mathcal{A}^{(r)}) \) can be expressed as an \( \mathbb{N} \)-linear combination of these 87 without cancellation in any coordinate. In addition to the 21 Markov basis elements, the Graver basis of \( \mathcal{A}^{(3)} \) contains the following 66 tables which come in 6 symmetry classes (with respect to
permutations of the three rows and mirror reflection of the columns):

Class 1 (12 tables, degree 7):
\[
\begin{pmatrix}
-1 & 1 & 1 & -1 \\
0 & -1 & 2 & -1 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

Class 2 (12 tables, degree 9, circuit):
\[
\begin{pmatrix}
0 & -2 & 4 & -2 \\
-1 & 2 & -1 & 0 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

Class 3 (12 tables, degree 9):
\[
\begin{pmatrix}
-2 & 2 & 2 & -2 \\
1 & -2 & 1 & 0 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

Class 4 (6 tables, degree 10):
\[
\begin{pmatrix}
-2 & 3 & 0 & -1 \\
1 & -3 & 3 & -1 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

Class 5 (12 tables, degree 12):
\[
\begin{pmatrix}
-2 & 2 & 2 & -2 \\
1 & -2 & 1 & 0 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

Class 1 (12 tables, degree 15, circuit)
\[
\begin{pmatrix}
-2 & 2 & 2 & -2 \\
1 & -2 & 1 & 0 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

Here “degree” refers to the total degree of the associated binomial, and “circuit” means that the table has minimal support with respect to inclusion \([10, \S 4]\). For instance, the binomial of degree 15 for the table in Class 3 is
\[
x_{12}^6x_{21}^3x_{23}^3x_{31}x_{34}^2 - x_{11}^4x_{14}^2x_{22}^6x_{33}^3
\]

The Graver bases of \(A^{(4)}\) has 240 elements of type four, and hence it has
\[
240 + \binom{4}{3} \cdot 87 + \binom{4}{2} \cdot 5 = 558
\]

elements in total. We similarly compute the Graver bases for the higher Lawrence liftings \(A^{(5)}, A^{(6)}, \ldots\), for instance, using Hemmecke’s program 4ti2 [7]. The Graver basis of \(A^{(6)}\) contains the following table of type 6:
\[
\begin{pmatrix}
-2 & 3 & 0 & -1 \\
-2 & 3 & 0 & -1 \\
1 & -2 & 1 & 0 \\
1 & -2 & 1 & 0 \\
1 & -2 & 1 & 0 \\
1 & 0 & -3 & 2
\end{pmatrix}
\]

(5) (120 tables, degree 15, type 6)

The Graver basis element [15] shows that the Graver complexity of \(A\) is at least six. Using Theorem [3] we can check that this is the correct bound.

**Remark 9.** The twisted cubic curve \(A\) has Graver complexity \(g(A) = 6\).
3. Proofs

We first note that Theorem 3 implies Theorem 1 and hence also Corollary 2. The point is that the Graver basis of a toric ideal contains a subset of minimal generators (i.e. a Markov basis), and therefore \( m(\mathcal{A}) \leq g(\mathcal{A}) \). So, in order to show that \( m(\mathcal{A}) \) is finite, it suffices to show that \( g(\mathcal{A}) \) is finite.

To derive the exact formula for \( g(\mathcal{A}) \) given in Theorem 3, we begin with the observation that Graver basis elements of \( \mathcal{A} \) are those vectors \( a \) in \( \mathcal{L}(\mathcal{A}) \setminus \{0\} \) that cannot be decomposed as a conformal sum \( a = b + c \) with \( b, c \in \mathcal{L}(\mathcal{A}) \setminus \{0\} \). Conformal means \( |b_i + c_i| = |b_i| + |c_i| \) for all \( i \).

**Lemma 10.** Let \( u \) be a Graver basis element of \( \mathcal{A}^{(r)} \) and suppose that one of its rows, say \( a = u^{(i)} \), has a conformal decomposition \( a = a_1 + \cdots + a_k \), where the \( a_i \)'s are in \( \mathcal{L}(\mathcal{A}) \). Then the table \( u' \), gotten by removing the row \( a \) from \( u \) and inserting the rows \( a_1, \ldots, a_k \), is in the Graver basis of \( \mathcal{A}^{(r+k-1)} \).

**Proof.** If \( u' \) is not in the Graver basis, then it has a non-trivial conformal decomposition \( u' = v' + w' \) with \( v', w' \in \mathcal{L}(\mathcal{A}^{(r+k-1)}) \). Then, adding up the relevant \( k \) rows of \( v' \) to become a single row, and the same for \( w' \), we get two tables \( v, w \in \mathcal{L}(\mathcal{A}^{(r)}) \) and a non-trivial conformal decomposition \( u = v + w \) which proves that \( u \) is not in the Graver basis of \( \mathcal{A}^{(r)} \) either. \( \square \)

**Corollary 11.** Every Graver basis element \( u \) of some \( \mathcal{A}^{(r)} \) can be obtained by conformal addition of rows from a Graver basis element \( u' \) of some \( \mathcal{A}^{(r')} \) which has the property that each row of \( u' \) lies in the Graver basis of \( \mathcal{A} \).

Note that the implication of Lemma 10 works only in one direction. If \( u \) is Graver then \( u' \) is Graver, but the converse is generally not true.

**Example 12.** Let \( \mathcal{A} = \{1, 2, 1\} \). The first of the following two tables is in the Graver basis of \( \mathcal{A}^{(4)} \) but the second is not in the Graver basis of \( \mathcal{A}^{(3)} \).

\[
\begin{pmatrix}
0 & -1 & 2 \\
2 & -1 & 0 \\
-1 & 1 & -1 \\
-1 & 1 & -1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & -1 & 2 \\
2 & -1 & 0 \\
-2 & 2 & -2
\end{pmatrix}.
\]

**Proof of Theorem 3.** Let \( \mathcal{B} = \{b_1, \ldots, b_k\} \) be the Graver basis of \( \mathcal{A} \). Corollary 11 tells us that in computing the Graver complexity \( g(\mathcal{A}) \) we only need to consider tables all of whose rows lie in \( \mathcal{B} \). Let \( u = (u^{(1)}, \ldots, u^{(r)}) \in \mathcal{B}^r \) be such a table, for \( r \geq 3 \), and suppose that \( u^{(i)} \neq -u^{(j)} \) for all \( i, j \). We define \( \psi_u \) to be the integer vector of length \( k \) whose \( i \)-th entry counts (with sign) how many times \( b_i \) appears as a row in \( u \). Then the 1-norm of the vector \( \psi_u \) equals the number \( r \), which is the type of the table \( u \). Hence the following claim will imply Theorem 3. The table \( u \) is in the Graver basis of \( \mathcal{A}^{(r)} \) if and only if the vector \( \psi_u \) is in the Graver basis of \( \mathcal{B} \).

To prove this claim, first suppose that \( \psi_u \) is not in the Graver basis of \( \mathcal{B} \). Any conformal decomposition of \( \psi_u \) provides a conformal decomposition of \( u \) (into tables of smaller type), so that \( u \) is not in the Graver basis of \( \mathcal{A}^{(r)} \).
For the converse, we note that any conformal decomposition of \( u \) arises in this manner from some conformal decomposition of \( \psi_u \), because single rows of \( u \) admit no conformal decomposition. Hence any non-trivial conformal decomposition of \( u \) gives a non-trivial conformal decomposition of \( \psi_u \). \( \square \)

It is instructive to examine the proof of Theorem 3 for each of the examples discussed in Section 2. For instance, if \( A \) is the twisted cubic in (2) then

\[
\mathcal{B} = \left( \begin{array}{cccc}
1 & 2 & 1 & 0 \\
-2 & -3 & -1 & 1 \\
1 & 0 & -1 & -3 & -2 \\
0 & 1 & 1 & 2 & 1
\end{array} \right).
\]

The Graver basis of \( \mathcal{B} \) consists of 13 vectors, ten of which are the circuits. The vector of maximum 1-norm among these 13 vectors occurs for \( \psi_u = (3, -2, 0, 1, 0) \), the vector associated with the 6 \( \times \) 4-table \( u \) in (3).

We can now derive a bound for \( g(A) \) in terms of \( n, d \) and the maximum size of the entries in \( A \), which we denote \( s \). Theorem 4.7 in [10] says that the maximum 1-norm of the vectors in \( \mathcal{B} \) is at most \((d+1)(n-d)D(A)\), where \( D(A) \leq (ds)^{d/2} \) is the maximum absolute value among the full-dimensional minors of \( A \). This implies bounds for the cardinality \( N \) of \( \mathcal{B} \) and the maximum size \( D(\mathcal{B}) \) of a subdeterminant of \( \mathcal{B} \). For example:

\[
N \leq (2(d+1)(n-d)D(A))^n, \quad D(\mathcal{B}) \leq ((d+1)(n-d)D(A))^{n-d}
\]

Since \( \mathcal{B} \) has dimension \( n-d \), the same theorem cited above implies

\[
g(A) \leq (n-d+1)(N-(n-d))D(\mathcal{B}) \leq n(2(d+1)(n-d)D(A))^{2n-d}.
\]

**Remark 13.** The finiteness of \( g(A) \) can also be derived from a result about partially ordered sets (posets) proved in 1952 by Higman [5, Theorem 4.2]. We briefly present this approach which was suggested to us by Matthias Aschenbrenner. For any poset \( S \), we can define a new poset \( \tilde{S} \) as follows. The elements of \( \tilde{S} \) are the finite multisubsets of \( S \), and the order is

\[
(7) \quad V \leq V' \iff \exists f : V \to V' \text{ injective and with } \forall v \in V : v \leq f(v)
\]

A poset \( S \) is said to be **Noetherian** if every non-empty subset of \( S \) has at least one, but at most finitely many minimal elements. Higman proved that if \( S \) is a Noetherian poset then \( \tilde{S} \) is also Noetherian. In his paper [5], he attributes this result to an earlier unpublished manuscript of Erdős and Rado.

We apply this to the poset \( S = \mathbb{Z}^n \), defined as in the introduction:

\[
a \leq b \iff \text{for all } i \in \{1, \ldots, n\} : 0 \leq a_i \leq b_i \text{ or } 0 \geq a_i \geq b_i
\]

The poset \( \mathbb{Z}^n \) is known to be Noetherian. The poset \( \tilde{\mathbb{Z}}^n \) consists of all finite multisubsets of \( \mathbb{Z}^n \). Higman’s result implies that \( \tilde{\mathbb{Z}}^n \) is Noetherian.

There is a canonical map \( \phi_r \) from the lattice \((\mathbb{Z}^n)^\ast\) of \( r \times n \)-tables to \( \tilde{\mathbb{Z}}^n \). This map takes \( u = (u^{(1)}, u^{(2)}, \ldots, u^{(r)}) \in (\mathbb{Z}^n)^\ast \) to the multiset of its
non-zero row vectors $\phi_r(u) = \{u^{(1)}, u^{(2)}, \ldots, u^{(r)}\}\{0\}$. The union of the images of the maps $\phi_r$, as $r$ ranges over $\mathbb{N}$, is the following subset of $\mathbb{Z}^n$:

$$\mathcal{P}(A) = \{ V \in \mathbb{Z}^n : \text{the elements of } V \text{ lie in } L(A)\{0\} \text{ and sum to zero} \}.$$ 

**Corollary 14.** The infinite set $\mathcal{P}(A)$ has only finitely many minimal elements, with the partial order induced from $\mathbb{Z}^n$. The Graver complexity $g(A)$ is the maximum of their cardinalities.

**Proof.** The first assertion follows from Higman’s result, which implies that $\mathbb{Z}^n$ is Noetherian. For the last assertion, just observe that an $r \times n$-table $u \in L(A^{(r)})$ lies in the Graver basis of $A^{(r)}$ if and only if the multiset $\phi_r(u)$ is minimal in $\mathcal{P}(A)$, and the type of $u$ is the cardinality of $\phi_r(u)$. \hfill $\square$

4. Statistics

In this section we apply our results on higher Lawrence configurations to the statistical context of log-linear models. We consider hierarchical log-linear models for $m$-dimensional contingency tables. Such a model is specified by a collection $\Delta$ of subsets of $\{1, 2, \ldots, m\}$. The standard notation for $\Delta$, used in the books of Christensen [3], Fienberg [5] and other texts on cross-classified data, is a string of brackets each containing the elements of a subset in $\Delta$. For instance, the four-cycle model for $\Delta$ is $\Delta = [12][23][34][41]$. (This is the smallest non-decomposable graphical model.)

If the format of the table is specified, say $r_1 \times r_2 \times \cdots \times r_m$, then the model $\Delta$ is represented in toric algebra by a configuration $A$ as above, where $n = r_1 r_2 \cdots r_m$ and $d$ is the sum of the products $r_{i_1} r_{i_2} \cdots r_{i_p}$ where $\{i_1, i_2, \ldots, i_p\}$ runs over $\Delta$. Each coordinate in $A$ is either 0 or 1. For instance, the four-cycle model $\Delta = [12][23][34][41]$ for $2 \times 2 \times 2$-tables is represented by (the columns of) a $16 \times 16$-matrix with $0 - 1$-entries. Here the lattice $L(A)$ consists of all $2 \times 2 \times 2$-integer tables whose $[12]$-margins, $[23]$-margins, $[34]$-margins and $[41]$-margins are zero. See [3, 9].

The passage from the matrix $A$ to its Lawrence lifting $A^{(r)}$ has the following statistical interpretation. Think of the $m$ given random variables as explanatory variables, and consider an additional $(m+1)^{st}$ random variable, the response variable, which has $r$ levels. From the model $\Delta$ for $m$-dimensional tables, we construct the following model for $(m+1)$-dimensional tables:

$$\Delta_{\logit} = \{\{1, 2, \ldots, m\}\} \cup \{\sigma \cup \{m + 1\} : \sigma \in \Delta\}.$$ 

This is the logit model described in [3, VII.1]. For example, if $\Delta$ is the four-cycle model and the index “5” indicates the additional response variable then, in standard notation, $\Delta_{\logit} = [125][235][345][415][1234]$. We shall prove that the passage from a log-linear model $\Delta$ to the associated logit model $\Delta_{\logit}$ is described in toric algebra precisely by the Lawrence hierarchy.

**Theorem 15.** If $A$ represents a hierarchical log-linear model $\Delta$ for $r_1 \times \cdots \times r_m$-tables then $A^{(r)}$ represents the logit model $\Delta_{\logit}$ for $r_1 \times \cdots \times r_m \times r$-tables.
Proof. We think of an \( r_1 \times \cdots \times r_m \times r \)-table as a two-dimensional matrix with \( r_1r_2 \cdots r_m \) columns and \( r \) rows. Computing the \( A \)-weighted row sums of such a matrix means computing the \( (\sigma \cup \{m+1\}) \)-marginals for any \( \sigma \in \Delta \). Computing the column sums of such a matrix means computing the \( \{1,2,\ldots,m\} \)-marginals of the \( r_1 \times \cdots \times r_m \times r \)-table. Thus \( \mathcal{L}(A^{(r)}) \) is identified with the lattice of integer \( r_1 \times \cdots \times r_m \times r \)-tables whose margins in the model \( \Delta_{\logit} \) are zero. This is precisely the claim.

From Theorems 1 and 3, we obtain the following corollary.

**Corollary 16.** Consider a logit model \( \Delta_{\logit} \) where the numbers \( r_1,\ldots,r_m \) of levels of the explanatory variables are fixed, and the number \( r \) of levels of the response variable is allowed to increase. Then there exists a finite Markov basis which is independent of \( r \), and independent of possible structural zeros.

We need to explain the last subclause. Imposing structural zeros in the model \( A^{(r)} \) means to consider the toric model defined by a subconfiguration \( C \subset A^{(r)} \). The Graver basis of \( A^{(r)} \) is a universal Gröbner basis \([10, \S 7]\), and hence it contains generators for all elimination ideals. This implies:

**Remark 17.** The Graver basis of \( A^{(r)} \) contains a Markov basis for any subconfiguration \( C \subset A^{(r)} \). (It works even if structural zeros are imposed.)

Hence to get the last assertion in Corollary 16 one takes the Graver basis of \( A^{(r)} \) for \( r \gg 0 \). The prototype of such a finiteness result was obtained by Aoki and Takamura in \([1]\). They considered the no-three-way interaction model for three-dimensional contingency tables. This is the logit model

\[
\Delta_{\logit} = [12][13][23]
\]

derived from the most classical independence model

\[
\Delta = [1][2]
\]

for two-dimensional tables. If \( r_1 = 2 \leq r_2 \) then it was known from \([4]\) that the Markov basis stabilizes for \( r \geq r_2 \). Aoki and Takamura \([1]\) considered the case \( r_1 = r_2 = 3 \), and they constructed the Markov basis which stabilizes for \( r \geq 5 \). Using Theorem 3 and Hemmecke’s program 4ti2 \([7]\), we found that the Graver basis for \( 3 \times 3 \times r \)-tables stabilizes for \( r \geq 9 \). In symbols,

\[
g(\Delta_2 \times \Delta_2) = 9.
\]

An element of the Graver basis of the Graver basis of \( \Delta_2 \times \Delta_2 \) which attains this bound is gotten from the following representation of the zero matrix:

\[
3 \cdot \begin{pmatrix}
1 & -1 & 0 \\
0 & 0 & 0 \\
-1 & 1 & 0
\end{pmatrix} + 2 \cdot \begin{pmatrix}
-1 & 1 & 0 \\
0 & -1 & 1 \\
1 & 0 & -1
\end{pmatrix} + 1 \cdot \begin{pmatrix}
1 & 0 & -1 \\
-1 & 1 & 0 \\
0 & -1 & 1
\end{pmatrix} + 2 \cdot \begin{pmatrix}
-1 & 0 & 1 \\
0 & 1 & -1 \\
1 & -1 & 0
\end{pmatrix} + 1 \cdot \begin{pmatrix}
0 & 1 & -1 \\
1 & -1 & 0 \\
-1 & 0 & 1
\end{pmatrix}.
\]
5. Geometric Combinatorics

A non-zero table $u$ in $\mathcal{L}(\mathcal{A}(r))$ is a circuit if the entries of $u$ are relatively prime and the support of $u$ is minimal with respect to inclusion. We define the circuit complexity $c(\mathcal{A})$ as the maximum type of any table that is a circuit of some higher Lawrence lifting $\mathcal{A}(r)$. Since the set of circuits of $\mathcal{A}(r)$ is a subset of the Graver basis of $\mathcal{A}(r)$, by [10, Proposition 4.11], we have

$$c(\mathcal{A}) \leq g(\mathcal{A}).$$

Example 6 shows that there is no bound for $g(\mathcal{A})$ in terms of $n$ and $d$ alone. On the other hand, such a bound does exist for the circuit complexity $c(\mathcal{A})$:

**Theorem 18.** The circuit complexity of $\mathcal{A}$ is bounded above by $n - d + 1$.

We shall derive this theorem from the following lemma, which can be rephrased as “the circuits of any $\mathcal{A}(r)$ are circuits among the circuits of $\mathcal{C}$”.

**Lemma 19.** Let $C$ be the configuration consisting of all circuits of $\mathcal{A}$. The non-zero rows of any circuit of type at least 3 of $\mathcal{A}(r)$ are distinct (and not opposite) elements of $C$ multiplied by numbers which form a circuit of $C$.

**Proof.** Let $u \in \mathcal{L}(\mathcal{A}(r))$ represent a circuit of $\mathcal{A}(r)$. If two rows are opposite, then these two rows have the sign pattern of an element of $\mathcal{L}(\mathcal{A}(2))$, and all other rows must be zero. If some row $u(i)$ is not a multiple of a circuit of $\mathcal{A}$, then, by [10, Lemma 4.10], $u(i)$ can be written as a non-negative rational conformal combination of circuits. We can write $\alpha_0 u(i) = \alpha_1 c_1 + \cdots + \alpha_k c_k$, where the $\alpha_j$'s are positive integers and each $c_j \in \mathcal{L}(\mathcal{A})$ is a circuit conformal to $u(i)$. Then, $\alpha_0 u$ can be decomposed as a sum of tables with support strictly contained in that of $u$, a contradiction.

Let us now write $u(i) = \alpha_i c_i$, with $c_i \in C$. The vector of coefficients (the $\alpha_i$'s) lies in $\mathcal{L}(C)$. Again by [10, Lemma 4.10], if it is not a circuit of $C$ then it can be decomposed as a non-negative rational conformal combination of circuits of $C$. As before, this decomposition translates into a decomposition of some multiple of $u$ into tables with strictly smaller support.

**Proof of Theorem 18.** The configuration $C$ of Lemma 19 has rank $n - d$. □

Recall (e.g. from [2] or [11]) that the oriented matroid of $\mathcal{A}(r)$ is specified by the collection of all sign patterns of circuits of $\mathcal{A}(r)$. Theorem 18 implies:

**Corollary 20.** If $c = c(\mathcal{A}) < r$ then the oriented matroid of the higher Lawrence lifting $\mathcal{A}(r)$ is determined by the oriented matroid of $\mathcal{A}(c)$.

The convex hull $\text{conv}(\mathcal{A}(r))$ of the higher Lawrence configuration $\mathcal{A}(r)$ is a convex polytope of dimension $dr + n - 1$ in $\mathbb{R}^{dr+n}$. A subset $C$ of $\mathcal{A}(r)$ is a face of $\mathcal{A}(r)$ if there exists a linear functional $\ell$ on $\mathbb{R}^{dr+n}$ whose minimum over $\mathcal{A}(r)$ is attained precisely at the subset $C$. Equivalently, the convex polytope $\text{conv}(\mathcal{A}(r))$ has a (geometric) face $F$ such that $F \cap \mathcal{A}(r) = C$. 
Corollary 21. If \( c = c(A) < r \) then the set of faces of \( A^{(r)} \) is determined by the set of faces of \( A^{(c)} \). In particular, a subset of \( A^{(r)} \) is a face if and only if its restriction to any subtable with only \( c \) rows is a face of \( A^{(c)} \).

Proof. We use oriented matroid arguments as in \([2, \S 9]\). The faces of \( A^{(r)} \) are the complements of the positive covectors of \( A^{(r)} \). Now an \( r \times n \)-table of signs is a covector of \( A^{(r)} \) if and only if it is orthogonal (in the combinatorial sense of \([2, \S 3]\)) to all circuits of \( A^{(r)} \). Since circuits have type at most \( c \), the orthogonality relation can be tested by restricting to subtables with \( c \) rows only. Hence an \( r \times n \)-table of signs is a (positive) covector of \( A^{(r)} \) if and only if every \( c \times n \)-subtable is a (positive) covector of \( A^{(c)} \). \( \square \)

A basic result concerning the “classical” Lawrence construction is that the oriented matroid of \( A \) can be recovered from the set of faces of \( A^{(2)} \), and vice versa. This statement is no longer true for higher Lawrence liftings.

Example 22. We present a configuration \( A \) which has the property that the set of faces of \( A^{(3)} \) cannot be recovered from the oriented matroid of \( A^{(2)} \). Let \( d = 3, n = 6 \) and consider the configurations

\[
A = \begin{pmatrix}
4 & 0 & 0 & 2 & 1 & 1 \\
0 & 4 & 0 & 1 & 2 & 1 \\
0 & 0 & 4 & 1 & 1 & 2
\end{pmatrix} \quad \text{and} \quad A' = \begin{pmatrix}
6 & 0 & 0 & 2 & 1 & 1 \\
-1 & 4 & 0 & 1 & 2 & 1 \\
-1 & 0 & 4 & 1 & 1 & 2
\end{pmatrix}.
\]

These two matrices determine the same oriented matroid on \( \{1, 2, 3, 4, 5, 6\} \). The configuration \( A^{(3)} \) consists of the 18 column vectors of a \( 15 \times 18 \)-matrix formed as in \([3]\). These vectors are indexed by the entries of a \( 3 \times 6 \)-table. The following table is a positive covector of the oriented matroid of \( A^{(3)} \):

\[
\begin{pmatrix}
+ & 0 & 0 & + & 0 & 0 \\
0 & + & 0 & 0 & + & 0 \\
0 & 0 & + & 0 & 0 & +
\end{pmatrix}
\]

To see this, multiply the \( 15 \times 18 \)-matrix \( A^{(3)} \) on the left by the vector

\[
\ell = \begin{pmatrix}
3, -1, -1; -1, 3, -1; -1, -1, 3; 4, 4, 4, 0, 0, 0
\end{pmatrix}.
\]

This vector supports a face of \( \text{conv}(A^{(3)}) \), and the elements of \( A^{(3)} \) on that face are indexed by the twelve zeros in \([9]\). We claim that the sign table \([9]\) is not a covector of \((A')^{(3)}\). If it were, then there exists an analogous vector \( \ell' \) such that \( \ell' \cdot (A')^{(3)} \) has the same support as \( \ell \cdot A^{(3)} \). This requirement leads to an inconsistent system of linear equations for \( \ell' \). We conclude that while \( A \) and \( A' \) share the same rank 3 oriented matroid, the two polytopes \( \text{conv}(A^{(3)}) \) and \( \text{conv}((A')^{(3)}) \) are not combinatorially isomorphic.
References

[1] S. Aoki and A. Takemura, Minimal basis for connected Markov chain over \(3 \times 3 \times K\) contingency tables with fixed two-dimensional marginals, University of Tokyo, Technical Report METR 02-02, February 2002, http://www.e.u-tokyo.ac.jp/~takemura/.

[2] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, Oriented Matroids, Cambridge University Press, Cambridge 1992.

[3] R. Christensen, Log-Linear Models, Springer Texts in Statistics. New York, 1990.

[4] P. Diaconis and B. Sturmfels, Algebraic algorithms for sampling from conditional distributions, *Annals of Statistics* 26:1 (1998) 363–397.

[5] S.E. Fienberg, The Analysis of Cross-Classified Categorical Data, MIT Press, Cambridge, Massachusetts, 1977.

[6] D. Geiger, C. Meek and B. Sturmfels, On the toric algebra of graphical models, Microsoft Research Preprint, 2002.

[7] R. Hemmecke, 4ti2: Computation of Hilbert bases, Graver bases, toric Gröbner bases, and more. Software freely available at http://www.4ti2.de/.

[8] G. Higman, Ordering by divisibility in abstract algebras, *Proc. London Math. Soc.* 2:3, (1952). 326–336.

[9] S. Hoşten and S. Sullivant, Gröbner bases and polyhedral geometry of reducible and cyclic models, *Journal of Combinatorial Theory, Series A* 100 (2002) 277–301.

[10] B. Sturmfels, Gröbner Bases and Convex Polytopes, University Series Lectures 8, American Mathematical Society, Providence, 1995.

[11] G. M. Ziegler, Lectures on Polytopes, Springer Graduate Texts in Mathematics, Vol. 152, 1994.

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