1-MULTISOLITON AND OTHER INVARIANT SOLUTIONS OF COMBINED KDV-NKDV EQUATION BY USING LIE SYMMETRY APPROACH

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Abstract. Lie symmetry method is applied to investigate symmetries of the combined KdV-nKdV equation, that is a new integrable equation by combining the KdV equation and negative order KdV equation. Symmetries which are obtained in this article, are further helpful for reducing the combined KdV-nKdV equation into ordinary differential equation. Moreover, a set of eight invariant solutions for combined KdV-nKdV equation is obtained by using proposed method. Out of the eight solutions so obtained in which two solutions generate progressive wave solutions, five are singular solutions and one multisoliton solutions which is in terms of Weierstrass Zeta function.

1. Introduction

Korteweg and Vries derived KdV equation [6] to model Russell’s phenomenon of solitons like shallow water waves with small but finite amplitudes [26]. Solitons are localized waves that propagate without change of it’s shape and velocity properties and stable against mutual collision [7, 22]. It has also been used to describe a number of important physical phenomena such as magneto hydrodynamics waves in warm plasma, acoustic waves in an in-harmonic crystal and ion-acoustic waves [2, 8, 17]. A special class of analytical solutions of KdV equation, the so-called traveling waves, for nonlinear evolution equations (NEEs) is of fundamental importance because a lot of mathematical-physical models are often described by such a wave phenomena. Thus, the investigation of traveling wave solutions is becoming more and more attractive in nonlinear science nowadays. However, not all equations posed in these fields are solvable. As a result, many new techniques have been successfully developed by a diverse group of mathematicians and physicists, such as Rational function method [31, 32], Bäcklund transformation method [14], Hirota bilinear method [5, 13], Lie symmetry method [1, 9–11, 18, 30], Jacobi elliptic function method [12], Sine-cosine function method [27], Tanh-coth function method [15], Weierstrass function method [20], Homogeneous balance method [28], Exp-function method [4], (G'/G)-expansion method [21], etc. But, it is extremely difficult and time consuming to solve nonlinear problems with the well-known traditional methods. This work investigates the combined KdV-nKdV equation

\[ u_{xt} + 6u_x u_{xx} + u_{xxxx} + u_{xxxt} + 4u_x u_{xt} + 2u_{xx} u_t = 0, \]  

where \( u = u(x,t) \). We apply Lie symmetry analysis on combined KdV-nKdV equation, first constructed by Wazwaz [25] using recursion operator [19]. In addition, the combined KdV-nKdV equation [11] possesses the Painlevé property for complete
integrability [3]. In this paper, Lie point symmetry generators of the combined KdV-nKdV equation were derived. Similarity reductions and number of explicit invariant solutions for the equation using Lie symmetry method were obtained. All the new invariant solutions of combined KdV-nKdV were analyzed graphically. Also, 1-multisoliton solution obtained in terms of Weierstrass Zeta function which appear in classical mechanics such as, motion in cubic and quartic potentials, description of the movement of a spherical pendulum, and in construction of minimal surfaces [29]. Some of the outcomes are interesting in physical sciences and are beautiful in mathematical sciences.

The organization of the paper is as follows. In Sec. 2, we discuss the methodology of Lie symmetry analysis of the general case. In Sec. 3, we obtain infinitesimal generators and the Lie point symmetries of the Eq. (1). In Sec. 4, symmetry reductions and exact group invariant solutions for the combined KdV-nKdV equation were obtained. In Sec. 5, we discussed all the invariant solutions graphically by Figures 1, 2, 3 and 4. Finally, concluding remarks are summarized in Section 6.

2. Method of Lie Symmetries

Let us consider a system of partial differential equations as follows:

\[ \Lambda_\nu(x,u^{(n)}) = 0, \ \nu = 1, 2, ..., l, \]  

where \( u = (u^1, u^2, ..., u^q), x = (x^1, x^2, ..., x^p), u^{(n)} \) denotes all the derivatives of \( u \) of all orders from 0 to \( n \). The one-parameter Lie group of infinitesimal transformations for Eq. (2) is given by

\[ \tilde{x}^i = x^i + \epsilon \xi^i(x,u) + O(\epsilon^2); \ i = 1, 2, ..., p, \]
\[ \tilde{u}^j = u^j + \epsilon \phi^j(x,u) + O(\epsilon^2); \ j = 1, 2, ..., q, \]

where \( \epsilon \) is the group parameter, and the Lie algebra of Eq. (1) is spanned by vector field of the form

\[ V = \sum_{i=1}^{p} \xi^i(x,u) \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} \eta^j(x,u) \frac{\partial}{\partial u^j} \]  

A symmetry of a partial differential equation is a transformation which keeps the solution invariant in the transformed space. The system of nonlinear PDEs leads to the following invariance condition under the infinitesimal transformations

\[ P_r^{(n)} V [\Lambda_\nu(x,u^{(n)})] = 0, \ \nu = 1, 2, ..., l \]

along with \( \Lambda(x,u^{(n)}) = 0 \)

In the above condition, \( P_r^{(n)} \) is termed as \( n^{th} \)-order prolongation [16] of the infinitesimal generator \( V \) which is given by

\[ P_r^{(n)} V = V + \sum_{j=1}^{q} \sum_{J} \eta^j_J(x,u^{(n)}) \frac{\partial}{\partial u^j} \]

the second summation being over all (unordered) multi-indices \( J = (i_1, ..., i_k), \) \( 1 \leq i_k \leq p, 1 \leq k \leq n. \) The coefficient functions \( \eta^j_J \) of \( P_r^{(n)} V \) are given by the following expression

\[ \eta^j_J(x,u^{(n)}) = D_J \left( \eta_j - \sum_{i=1}^{p} \xi^i u^j_i \right) + \sum_{i=1}^{p} \xi^i u^j_{i,j,i} \]
where \( u^j_i = \frac{\partial u^j}{\partial x^i} \), \( u^J_i = \frac{\partial u^J}{\partial x^i} \) and \( D_J \) denotes total derivative.

### 3. Lie symmetry analysis for the combined KdV-nKdV equation

In this section, authors explained briefly all the steps of the STM method to keep this work self-confined. The Lie symmetries for the Eq. (1) have generated and then its similarity solutions are found. Therefore, one can consider the following one-parameter (\( \epsilon \)) Lie group of infinitesimal transformations

\[
\tilde{x} = x + \epsilon \xi(x, t, u) + O(\epsilon^2), \quad \tilde{t} = t + \epsilon \tau(x, t, u) + O(\epsilon^2), \quad \tilde{u} = u + \epsilon \eta(x, t, u) + O(\epsilon^2),
\]

(6)

where \( \xi, \tau \) and \( \eta \) are infinitesimals for the variables \( x, t \) and \( u \) respectively, and \( u(x, t) \) is the solution of Eq. (1). Therefore, the associated vector field is

\[
V = \xi(x, t, u) \frac{\partial}{\partial x} + \tau(x, t, u) \frac{\partial}{\partial t} + \eta(x, t, u) \frac{\partial}{\partial u},
\]

(7)

Lie symmetry of Eq. (1) will be generated by Eq. (7). Use fourth prolongation \( Pr^{(4)}V \) gives rise to the symmetry condition for Eq. (1) as follows:

\[
\eta_{tt} + 6\eta u_{xx} + 6\eta^x u_x + \eta^{xx} + 4\eta^t u_{xt} + 4\eta^x u_x + 2\eta^t u_{xx} + 2\eta^x u_t = 0,
\]

(8)

where \( \eta^x, \eta^t, \eta^{xx}, \eta^{xxx}, \) and \( \eta^{xxxt} \) are the coefficient of \( Pr^{(4)}V \), values are given in many references [1,16]. Incorporating all the expressions into Eq. (8), and then equating the various differential coefficients of \( u \) to zero, we derive following system of Eight determining equation

\[
\xi_u = \xi_{xx} = 0, \quad \xi_x + \xi_t = \tau_t, \quad \tau_x = \tau_u = 0, \quad \xi_x + 2\eta_x = \xi_x + \eta_u = 0, \quad \xi_x + \frac{1}{2} \tau_t = \eta_u.
\]

(9)

Solving the above system of equations, we obtain following infinitesimals for (1) using software Maple,

\[
\xi = (x - t)a_1 + a_2 + f(t), \quad \tau = f(t), \quad \eta = (t - \frac{x}{2} - u)a_1 + a_3 + \frac{1}{2} f(t),
\]

(10)

where \( a_1, a_2 \) and \( a_3 \) are arbitrary constants whereas \( f(t) \) is an arbitrary function.

The symmetries under which Eq. (1) is invariant can be spanned by the following four infinitesimal generators if we assume \( f(t) = c \), a constant Then all of the infinitesimal generators of Eq. (1) can be expressed as

\[
V = a_1 V_1 + a_2 V_2 + a_3 V_3 + c V_4.
\]
where
\begin{align*}
V_1 &= 2(x - t) \frac{\partial}{\partial x} + (t - \frac{x}{2} - u) \frac{\partial}{\partial u}, \\
V_2 &= \frac{\partial}{\partial x}, \\
V_3 &= \frac{\partial}{\partial u}, \\
V_4 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial t} + \frac{\partial}{\partial u}.
\end{align*}

(11)

Table 1. The commutator table of the vector fields (11)

\begin{tabular}{cccc}
* & $V_1$ & $V_2$ & $V_3$ & $V_4$ \\
\hline
$V_1$ & 0 & $-V_2 + \frac{1}{2}V_3$ & $V_3$ & $\frac{1}{2}V_3$ \\
$V_2$ & $V_2 - \frac{1}{2}V_3$ & 0 & 0 & 0 \\
$V_3$ & $-V_3$ & 0 & 0 & 0 \\
$V_4$ & $-\frac{1}{2}V_3$ & 0 & 0 & 0 \\
\end{tabular}

The vector field yield commutation relations through the Table 1. The $(i, j)$th entry in Table 1 is the Lie bracket $[V_i, V_j] = V_i \cdot V_j - V_j \cdot V_i$. Table 1 is skew-symmetric with zero diagonal elements. Table 1 shows that the generators $V_1, V_2, V_3$ and $V_4$ are linearly independent. Thus, to obtain the similarity solutions of Eq. (1), the corresponding associated Lagrange system is

\begin{equation}
\frac{dx}{\xi(x, t, u)} = \frac{dt}{\tau(x, t, u)} = \frac{du}{\eta(x, t, u)}.
\end{equation}

(12)

4. Invariant solutions of the combined KdV-nKdV equation

To proceed further, selection of $f(t)$ and by assigning the particular values to $a_i$’s $(1 \leq i \leq 3)$, provide new physically meaningful solutions of Eq. (1). In order to obtain symmetry reductions and invariant solutions, one has to solve the associated Lagrange equations given by (12). Now, let us discuss the following particular cases for various forms of $f(t)$:

Case 1: For $f(t) = at^2 + bt + c, a \neq 0, b \neq 0, c \neq 0$, then Eqs. (10) and (12) gives

\begin{equation}
\frac{dx}{(x-t)a_1 + a_2 + f(t)} = \frac{dt}{f(t)} = \frac{du}{\eta(x, t, u)}.
\end{equation}

(13)

The similarity form suggested by Eq. (13) is given by

\begin{equation}
u = \alpha + \frac{1}{4}(3t - x) + e^{-2a_1\beta \tan^{-1}(\beta T)}U(X),
\end{equation}

(14)

with similarity variable

\begin{equation}
X = (x - t + A)e^{-2a_1\beta \tan^{-1}(\beta T)},
\end{equation}

(15)

where

\begin{equation}
\alpha = \frac{a_2 + 4a_3}{4a_1}, \beta = \frac{1}{\sqrt{4ac - b^2}}, A = \frac{a_2}{a_1}, T = 2at + b.
\end{equation}

(16)
Progressive wave shown by Eq. (19)  

Singularity on \(1 + x - t = 0\) in Eq. (20)  

**Figure 1.** Invariant solution profiles for Eq. (19) and Eq. (20)

Inserting the value of \(u\) from Eq. (14) into Eq. (1), we get the following fourth order ordinary differential equation

\[
X U_{XXXX} + 4 U_{XXX} + 6 X U_X U_{XX} + 2 U_{XXX} + 8 U X^2 = 0,
\]

where \(X\) is given by Eq. (15) and \(U_X = \frac{dU}{dX}, U_{XX} = \frac{d^2U}{dX^2}\), etc.

Any how, we could not find the general solution of Eq. (17) still two particular solutions are found as below

\[
U(X) = c_1 \quad \text{and} \quad U(X) = c_2 \frac{X}{X},
\]

where \(c_1\) and \(c_2\) are arbitrary constants. Thus, from Eqs. (14) and (18), we get two invariant solutions of Eq. (1) given below

\[
u(x,t) = \alpha + \frac{1}{4}(3t - x) + c_1 e^{-2a_1\beta \tan^{-1}(\beta T)},
\]

\[
u(x,t) = \alpha + \frac{1}{4}(3t - x) + \frac{c_2}{A + x - t},
\]

where \(\alpha, \beta, T\) and \(A\) are given by Eq. (16).

**Case 2:** For \(f(t) = c, c \neq 0; a_1 = 0, a_2 \neq 0, a_3 \neq 0\), Eq. (12) are of the form

\[
\frac{dx}{c + a_2} = \frac{dt}{c} = \frac{du}{\frac{c}{2} + a_3}.
\]

The similarity solution to Eq. (21) can be written as

\[
u = \left(\frac{1}{2} + \frac{a_3}{c}\right) t + U(X),
\]

where \(X = x - (1 + \frac{a_2}{a_3})t\) is a similarity variable.

Inserting the value of \(u\) from Eq. (22) in Eq. (1), we get the fourth order ordinary differential equation in \(U\)

\[
a_2 U_{XXXX} + (a_2 - 2a_3 + 6a_2 U_X) U_{XX} = 0.
\]

Assume \(a_2 = 2a_3\). Without loss of generality, we can assume \(a_2 \neq 0\), then Eq. (34) reduces to

\[
U_{XXXX} + 6U_X U_{XX} = 0.
\]
The general solution of Eq. (24) in terms of WeierstrassZeta function as

\[ U(X) = c_3 + \left(-1\right)^{\frac{1}{2}} 2^\frac{3}{4} \text{WeierstrassZeta}[\alpha_1(x, t), \alpha_2], \tag{25} \]

where \( \alpha_1(x, t) = \left(-\frac{1}{2}\right)^{\frac{1}{2}} \left(x - \left(1 + \frac{n x}{c}\right)t + c_4\right) \), \( \alpha_2 = \{2 + \left(-1\right)^{\frac{1}{2}} 2^\frac{3}{4} c_4, c_5\} \) and \( c_3, c_4 \) and \( c_5 \) are arbitrary constants. From Eq. (25) with Eq. (22), we have another invariant solution of Eq. (1) as

\[ u(x, t) = c_3 + \left(\frac{1}{2} + \frac{a_3}{c}\right)t + \left(-1\right)^{\frac{1}{2}} 2^\frac{3}{4} \text{WeierstrassZeta}[\alpha_1(x, t), \alpha_2]. \tag{26} \]

### Case 2A: \( f(t) = c; a_2 = 0, a_1 \neq 0, a_3 \neq 0 \), Eq. (13) becomes

\[
\frac{dx}{(x-t)a_1 + c} = \frac{du}{\left(t - \frac{x}{2} - u\right)a_1 + a_3 + \frac{c}{2}}, \tag{27}
\]

In this case, we get

\[ u = \frac{a_3}{a_1} + \frac{1}{4}(3t - x) + e^{-\frac{a_3}{a_1}t}U(X), \tag{28} \]

where \( X = (x-t)e^{-\frac{a_3}{a_1}t} \). Substituting the value of \( u \) in Eq. (1), again we obtain the same fourth order ordinary differential equation (17). Some particular solutions are given below

\[ U(X) = c_6, \quad U(X) = \frac{c_7}{X}, \tag{29} \]

where \( c_6 \) and \( c_7 \) are arbitrary constants. Therefore, using Eq. (29) in Eq. (28), we obtain the following two exact solutions for Eq. (1)

\[ u(x, t) = \frac{a_3}{a_1} + \frac{1}{4}(3t - x) + c_6 e^{-\frac{a_3}{a_1}t}, \tag{30} \]

\[ u(x, t) = \frac{a_3}{a_1} + \frac{1}{4}(3t - x) + \frac{c_7}{x - t}. \tag{31} \]

### Case 3: For \( f(t) = 0; a_1 \neq 0, a_2 \neq 0, a_3 \neq 0 \), Eq. (13) modified as

\[
\frac{dx}{(x-t)a_1 + a_2} = \frac{dt}{0} = \frac{du}{(t - \frac{x}{2} - u)a_1 + a_3}. \tag{32}
\]

The group invariant solution is given as

\[ u = \frac{a_1 x(x - 4t) - 4a_3 x + 4U(T)}{4a_1(t - x) - 4a_2}, \tag{33} \]

with \( T = t \). Substituting the value of \( u \) from Eq. (33) into Eq. (1), we get the following reduced ordinary differential equation

\[
[3a_1^2 (t^2 - 1) - 2a_1 (a_2 - 2a_3 t + 2U(t)) - a_2 (a_2 + 4a_3)] [3a_1 t - a_2 + 2a_3 - 2U'(t)] = 0. \tag{34}
\]

Hence, from Eq. (34) we found two values of \( U \) given as

\[ U(T) = \frac{3a_1^2 (t^2 - 1) - 2(a_2 - 2a_3) a_1 t - a_2 (a_2 + 4a_3)}{4a_1}, \tag{35} \]

\[ U(T) = \frac{1}{4}(3a_1 t^2 - 2a_2 t + 4a_3 t + 4c_8), \tag{36} \]

\[ T = t \].
where $c_8$ is an arbitrary constant of integration. Here, Eq. (35) gives same solution as Eq. (20) with $c_2 = \frac{3}{4}$. Also, using Eq. (36) in Eq. (33) gives new invariant solution of Eq. (1) as

$$u(x,t) = 4 \left( a_3(t - x) + c_8 \right) + a_1 \left( 3t^2 - 4tx + x^2 \right) - 2a_2t,$$

where $\alpha$ and $A$ are given by Eq. (16).

**Case 3A:** For $f(t) = 0; a_2 = 0, a_1 \neq 0, a_3 \neq 0$, from Eq. (32), we have

$$\frac{dx}{(x - t)a_1} = \frac{dt}{0} = \frac{du}{(t - \frac{3}{4} - u)a_1 + a_4}.$$

Therefore, the similarity transformation method gives

$$u = \frac{4a_3(t - x) + c_8 + a_1 \left( 3t^2 - 4tx + x^2 \right) - 2a_2t}{4a_1(t - x) - 4a_2},$$

where $T = t$ and $U(T)$ is the similarity function. Substituting the value of $u$ in Eq. (33) we get reduced ordinary differential equation given as

$$\left[ a_3 \left( 3a_3 \left( t^2 - 1 \right) - 4U(t) \right) + 2a_3a_1t - 5a_1^2 \right] \left[ 3a_3t + a_1 - 2U'(t) \right] = 0.$$

Again it gives two values of $U$ as

$$U(T) = \frac{3a_3^2(t^2 - 1) + 2a_3a_1t - 5a_1^2}{4a_3},$$

$$U(T) = \frac{1}{4} \left( 2a_1t + 3a_3t^2 + 4c_9 \right),$$
(a) Progressive wave shown by solution Eq. (30) 
(b) Singularity shown by solution Eq. (31)
(c) Singularity on $1 + x - t = 0$ in Eq. (37) 
(d) Singularity in the solution Eq. (43)

Figure 3. Evolution profiles of various invariant solutions.

where $c_0$ is constant of integration. Here, Eq. (41) gives same solution as Eq. (20). Also, using (42) with (39) gives another new invariant solution of (1) as

$$u(x, t) = \frac{2a_1(2x - t) - a_3(3t^2 - 4tx + x^2) - 4c_9}{4(a_3(x - t) + a_1)}. \quad (43)$$

**Case 4:** For $f(t) = t^3; a_1 \neq 0, a_2 \neq 0, a_3 \neq 0$, from Eq. (32) we get similarity form suggested by Eq. (13) is given by

$$u = \alpha + \frac{1}{4}(3t - x) + e^{\frac{a_1}{2}t}U(X), \quad (44)$$

with similarity variable $X = (x - t + A)e^{\frac{a_1}{2}t}$ and $A$ is given by (16). In this case, we get same ordinary differential equation as Eq. (17), one particular solution is as follows

$$U(X) = c_{10}, \quad (45)$$

where $c_{10}$ is arbitrary constant. Thus, from Eqs. (44) and (45) we get another new invariant solution of Eq. (1) given below

$$u(x, t) = \alpha + \frac{1}{4}(3t - x) + c_{10}e^{\frac{a_1}{2}t}. \quad (46)$$
5. Discussion

The results of the combined KdV-nKdV equation presented in this paper have richer physical structure than earlier outcomes in the literature [25]. The recorded results are significant in the context of nonlinear dynamics, physical science, mathematical physics etc. The invariant solutions obtained can illustrate various dynamic behaviour due to existence of arbitrary constants. The nonlinear behaviour of the results are analyzed in the following manner:

**Figure 1:** Fig 1(a) shows progressive wave solution in Eq. (19) for $a_1 = 1.1, a_2 = 0.9, a_3 = 1.2, a = 0.88, b = 0.9$ and $c = 0.9$. Fig 1(b) shows presence of singularity in the plane $1 + x - t = 0$ in Eq. (20) with values $a_1 = 2.1, a_2 = 2.1, a_3 = 3.1$ and $c_2 = 0.9$.

**Figure 2:** The evolution profiles of 1-Multisoliton solution is given by Eq. (26) as shown in this Figure. We have recorded the physical nature with variation in parameters. Fig 2(a) For $a_2 = 1.9866, a_3 = 1.3241, c = 9.8654, c_4 = 1.2341$ and $c_5 = 0.8954$ a plot of WeiestrassZeta function is shown; Fig 2(b) For $a_2 = 5, c = 1, a_3 = 1, c_4 = 1$ and $c_5 = 1$ only few solitons are shown; Fig 2(c) Front orthographic projection is shown for $a_2 = 1.9866, a_3 = 1.3241, c = 9.8654, c_4 = 5.2341$ and $c_5 = 4.8954$; Fig 2(d) Front orthographic projection is shown for $a_2 = 1.767, a_3 = 1.3241, c = 9.8654, c_4 = 1.2341$ and $c_5 = 0.8954$.

**Figure 3:** Fig 3(a) For $a_1 = 0.8, a_3 = 0.8$ and $c = 1$, Eq. (30) exhibits progressive wave; Fig 3(b), shows singularity in planes for Eqs. (31) for $a_1 = 0.8, a_2 = 2, a_3 = 0.8$ and $c_7 = c_8 = c_9 = 1$; Fig 3(c, d) shows singularity wave profile for Eq. (37) and Eq. (43) which explain the transition of nonlinear behaviour in the form of opposite rotatory folded sheets.

**Figure 4:** $u(x, t)$ exhibits singularity near $t = 0$ but asymptotic structures is observed near $t = 0$ for parameters $a_1 = 1, a_2 = 1, a_3 = 1$ and $c_{10} = 1$.

6. Conclusion

In this paper, the similarity reductions and invariant solutions for the combined KdV-nKdV are presented. This paper obtained the 1-multisoliton and other invariant solution of the equation. The method that was used to obtain the exact group invariant solutions is the Lie symmetry analysis approach. All the solutions are different from earlier work which have been obtained by Wazwaz [25]. Eventually, the structure of combined KdV-nKdV equation is an non-trivial one, which can be

![Figure 4. Asymptotic structures at t = 0 for Eq. (46).](image-url)
clearly seen from the graphically results of invariant solutions. The Lie symmetry analysis method extracts the new forms of analytic solutions which are of physical importance such as condensed matter physics and plasma physics.

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