AN INEQUALITY IN NONCOMMUTATIVE $L_p$-SPACES

ÉRIC RICARD

Abstract. We prove that for any (trace-preserving) conditional expectation $\mathcal{E}$ on a noncommutative $L_p$ with $p > 2$, $Id - \mathcal{E}$ is a contraction on the positive cone $L^+_p$.

1. Introduction

It is plain that for positive real numbers $a, b \geq 0$ and $p \geq 2$, one has

$$(a - b)(a^{p-1} - b^{p-1}) \geq |a - b|^p.$$ Integrating the above inequality on some measure space $(\Omega, \mu)$ implies that for $f, g \in L_p(\Omega, \mu)^+$,

$$\int_{\Omega} (f(x) - g(x)) \left( f^{p-1}(x) - g^{p-1}(x) \right) \, d\mu(x) \geq \|f - g\|_p^p.$$ In [3], Mustapha Mokhtar-Kharroubi notices that this inequality may be used to get contractivity results on the positive cone of $L_p(\Omega, \mu)$. This note originates from the question whether its noncommutative analogue remains true. We provide a proof in the next section. We hope that the techniques involved there may be useful for further studies. We end up by making explicit some results from [3] for noncommutative $L_p$-spaces.

We refer the reader to [4] for the definitions of $L_p$-spaces associated to semifinite von Neumann algebras or more general ones. We also freely use basic results from [1].

2. Results

Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra. We denote by $f_p : \mathcal{M}^+ \to \mathcal{M}^+$, the $p^{|b|}$-power map and by $\mathcal{M}^{++}$ the set of positive invertible elements. We will often refer to positivity of the trace for the fact that if $a, b \in \mathcal{M}^+ \cap L_1(\mathcal{M})$, then $\tau(ab) \geq 0$.

Theorem 2.1. Let $p \geq 2$ and $a, b \in L_p(\mathcal{M})^+$, then

$$\tau(|a - b|^p) \leq \tau((a - b)(a^{p-1} - b^{p-1})).$$

Proof. First, as for any sequence of (finite) projections $p_i$ going to 1 strongly in $\mathcal{M}$ and any $x \in L_q(\mathcal{M})$ $(1 \leq q < \infty \|p_i x p_i - x\| \to 0$, we may assume that $\mathcal{M}$ is finite. Next by replacing $a$ and $b$ by $a + \varepsilon 1$ and $a + \varepsilon 1$ for some $\varepsilon > 0$, we may also assume that $[a, b] \subset \mathcal{M}^{++}$ to avoid any unnecessary technical complication.

We write $a = b + \delta$. To prove the result, we distinguish according to the values of $p$.

Case 1: $p \in [2, 3]$

Case 1.a: $a \geq b$, i.e. $\delta \geq 0$.

As $p - 1 = 1 + \theta$ with $\theta \in [0, 1]$, we use the well known integral formula

$$s^{1+\theta} = c_\theta \int_{R_+} \frac{t^\theta s^2}{s + t} \, dt, \quad \frac{s^2}{s + t} = s - t + \frac{t^2}{s + t}.$$ Hence

$$\tau(\delta(a^{1+\theta} - b^{1+\theta})) = c_\theta \int_{R_+} \frac{t^\theta \tau\left(\delta t^2(b + \delta + t)^{-1} - t^2(b + t)^{-1}\right)}{s + t} \, dt.$$ Recall the identity $(b + \delta + t)^{-1} - (b + t)^{-1} = -(b + \delta + t)^{-1}\delta(b + t)^{-1}$. Using positivity of the trace with $\delta(b + \delta + t)^{-1}\delta \leq \delta(b + t)^{-1}\delta$ and $(b + t)^{-1} \leq t^{-1}$:

$$\tau\left(\delta t^2(b + \delta + t)^{-1} - t^2(b + t)^{-1}\right) \geq \tau(\delta^2 - t\delta(b + t)^{-1}\delta) = \tau(\delta^3(b + t)^{-1}).$$

2010 Mathematics Subject Classification: 46L51; 47A30.

Key words: Noncommutative $L_p$-spaces.
Integrating, we get the desired inequality \( \tau(\delta(a^{1+\theta} - b^{1+\theta})) \geq \tau(\delta^2 b^{\theta - 2}) \).

Case 1.b: \( \delta \) arbitrary with decomposition \( \delta = \delta_+ - \delta_- \) into positive and negative parts. We reduce it to the previous case by introducing \( \alpha = a + \delta_- = b + \delta_+ \), so that \( \alpha \geq a, b \). We have

\[
\tau((a - b)(a^{p-1} - b^{p-1})) = \tau((a - \alpha)(a^{p-1} - \alpha^{p-1})) + \tau((a - \alpha)(\alpha^{p-1} - b^{p-1})) + \tau((a - b)(\alpha^{p-1} - b^{p-1})�(\alpha^{p-1} - b^{p-1})).
\]

The first and the third terms are bigger than \( \tau(\delta^p) \) and \( \tau(\delta^p) \) by Case 1.a. Hence it suffices to check that the two remaining terms are positive. We use again the integral formula (1) and \( \delta_+ \delta_- = 0 \) and positivity of the trace

\[
\tau(-\delta_-(a^{p-1} - b^{p-1})) = -c \int_{\mathbb{R_+}} t^\theta \tau(\delta_+ (b + \delta_+ + t)^{-1} - t^2(b + t)^{-1}) \frac{dt}{t} \\
= c \int_{\mathbb{R_+}} t^\theta t^2 \tau(\delta_+ (b + t)^{-1} - (b + \delta_+ + t)^{-1}) \frac{dt}{t} \geq 0.
\]

The last term is handled similarly.

Case 2: \( p \geq 3 \). First, for any \( n \in \mathbb{N}, n \geq 1 \), one easily checks by induction that we have the following identity

\[
\tau((a - b)(a^{p-1} - b^{p-1})) = \tau(\delta((b + \delta)^{p-1-n} - b^{p-1-n})b^n) + \sum_{k=1}^{n-1} \tau(\delta(b + \delta)^{p-1-k}\delta b^{k-1}).
\]

Let \( n \geq 1 \) be so that \( p - 1 - n = 1 + \theta \) with \( \theta \in [0, 1] \). By positivity of the trace, we get

\[
\tau((a - b)(a^{p-1} - b^{p-1})) \geq \tau(\delta((b + \delta)^{1+\theta} - b^{1+\theta})b^n) + \tau(\delta^2(b + \delta)^{p-2}).
\]

By the same computations as above thanks to (1)

\[
\tau(\delta((b + \delta)^{1+\theta} - b^{1+\theta})b^n) = c \int_{\mathbb{R_+}} t^\theta \tau(\delta + t^2(b + \delta + t)^{-1} - t^2(b + t)^{-1}) b^n \frac{dt}{t} \geq c \int_{\mathbb{R_+}} t^\theta \tau(\delta - t^2(b + \delta + t)^{-1}\delta(b + t)^{-1}) b^n \frac{dt}{t} \\
= c \int_{\mathbb{R_+}} t^\theta \tau(\delta^2 b(b + t)^{-1} b^n) \frac{dt}{t} = \tau(\delta^2 b^{n+\theta}) = \tau(\delta^2 b^{\theta-2}),
\]

where we used again positivity of the trace with \( (b + \delta + t)^{-1} \leq t^{-1} \) and \( 0 \leq (b + t)^{-1} b^n \).

To conclude let \( E \) to be the conditional expectation onto the subalgebra \( \mathcal{N} = \langle \delta \rangle \). As \( \mathcal{N} \) is commutative, the Jensen inequality is valid; for any \( \alpha \geq 1 \) and \( x \in \mathcal{M}^+ \): \( E(x^n) \geq (Ex)^n \). With \( \alpha = p - 2 \geq 1 \),

\[
\tau(\delta^2 b^{\theta-2}) = \tau(\delta^2 E(\delta b^{\theta-2})) \geq \tau(\delta^2 E(b^{\theta-2})), \quad \tau(\delta^2(b + \delta)^{p-2}) \geq \tau(\delta^2 E(b + \delta)^{p-2}).
\]

But with the usual decomposition \( \delta = \delta_+ - \delta_- \), as \( a, b \geq 0 \), \( \mathcal{E}b \geq \delta_- \) and \( \mathcal{E}(b + \delta) \geq \delta_+ \). By commutativity of \( \mathcal{N} \), we can conclude

\[
\tau((a - b)(a^{p-1} - b^{p-1})) \geq \tau(\delta^2(\delta_-^{p-2} + \delta_+^{p-2})) = \tau(|\delta|^p).
\]

□

We provide an alternative proof when \( p \in [3, 4] \).

Denote by \( R_x \) and \( L_x \) the right and left multiplication operators by \( x \in \mathcal{M} \) defined on all \( L_p(\mathcal{M}) \) (1 \( \leq p \leq \infty \)). When \( p = 2 \), for any \( x \in \mathcal{M}^\alpha \), the C*-algebra generated in \( \mathcal{B}(L_2(\mathcal{M})) \) by \( L_x \) and \( R_x \) is commutative and isomorphic to \( C(\sigma(x) \times \sigma(x)) \) where \( \sigma(x) \) is the spectrum of \( x \).

Lemma 2.2. For \( p \geq 1 \), the map \( f_p \) is Fréchet differentiable on \( \mathcal{M}^+ \). For \( \mathcal{M} \) finite, the derivative is given by the formula in \( \mathcal{L}_2(\mathcal{M})\):

\[
\forall x \in \mathcal{M}^+, \forall h \in \mathcal{M}^\alpha, \quad D_x f_p(h) = p \int_0^1 (tL_x + (1 - t)R_x)^{p-1}(h) \, dt.
\]
Proof. Assume $x \geq 3\delta$ for some $\delta > 0$. Taking $h \in \mathcal{M}^{sa}$ with $\|h\| < \delta$, we may compute $f_p(x+h)$ using the holomorphic functional calculus by choosing a curve $\gamma$ with index 1 surrounding the spectrum of $x$ with $\gamma \subset \{z \mid \text{Re } z > 0\}$ and $\text{dist}(\gamma, \sigma(x)) \geq 2\delta$:

$$(x+h)^p = \frac{1}{2i\pi} \int_{\gamma} z^p (z-(x+h))^{-1} \, dz$$

Hence

$$(x+h)^p - x^p = \frac{1}{2i\pi} \int_{\gamma} z^p (z-(x+h))^{-1} h(z-x)^{-1} \, dz$$

It follows directly that $f_p$ is Fréchet differentiable with derivative

$$D_x f_p(h) = \frac{1}{2i\pi} \int_{\gamma} z^p (z-x)^{-1} h(z-x)^{-1} \, dz = \frac{1}{2i\pi} \int_{\gamma} z^p L_{(z-x)^{-1}} R_{(z-x)^{-1}}(h) \, dz.$$ 

It then suffices to check that the two formulas coincide when $\mathcal{M}$ is finite; as $\mathcal{M} \subset L_2(\mathcal{M})$, we do it for $h \in L_2(\mathcal{M})$. But in $\mathcal{B}(L_2(\mathcal{M}))$, this boils down to an equality in $\mathcal{C}(\sigma(x) \times \sigma(x))$ so that we need only to justify that

$$\forall a, b \in \mathbb{R}^+, \quad \frac{1}{2i\pi} \int_{\gamma} \frac{z^p}{(z-a)(z-b)} \, dz = p \int_0^1 (ta + (1-t)b)^{p-1} \, dt.$$

The above computations yield that the left-hand side is $\frac{a^p - b^p}{a-b}$ if $a \neq b$ and $pa^{p-1}$ if $a = b$ which clearly coincide with the right-hand side. $\square$

Assuming $\mathcal{M}$ finite and $a = b + \delta, b \in \mathcal{M}^{+}$ as above, the alternative proof when $p \in [3, 4]$ relies on Lemma 2.2

$$\tau((a-b)(a^{p-1} - b^{p-1})) = p \int_0^1 \int_0^1 \tau(\delta (tL_{b+u\delta} + (1-t)R_{b+u\delta})^{p-2}(\delta)) \, dt \, du = p \int_0^1 \int_0^1 \tau(\delta (tL_{b+u\delta} + (1-t)R_{b+u\delta})^{p-2}(\delta))_{L_2(\mathcal{M})} \, dt \, du$$

As $p - 2 \in [1, 2]$, $f_{p-2}$ is operator convex, so that for any $m \in \mathcal{B}(L_2(\mathcal{M})^+)$ and any projection $\mathcal{E} \in \mathcal{B}(L_2(\mathcal{M}))$, we have $\mathcal{E} m^{p-2} \mathcal{E} \geq (\mathcal{E} m \mathcal{E})^{p-2}$. We choose $\mathcal{E}$ to be the $L_2$-conditional expectation onto the subalgebra generated by $\delta$.

$$\langle \delta, (tL_{b+u\delta} + (1-t)R_{b+u\delta})^{p-2}(\delta) \rangle_{L_2(\mathcal{M})} \geq \langle \delta, (tL_{E_{b+u\delta} + (1-t)R_{b+u\delta}}E_{b+u\delta})^{p-2}(\delta) \rangle_{L_2(\mathcal{M})}$$

where in the last inequality we have used that $R_x$ and $E$ commute if $x \in \delta''$. Tracking back the equalities, we obtain

$$\tau(\delta((b+\delta)^{p-1} - b^{p-1})) \geq \tau(\delta((E(b) + \delta)^{p-1} - E(b)^{p-1})) \geq \tau(\delta|\delta|),$$

where the last inequality comes from the result in the commutative case.

Remark 2.3. We point out that, for $p \in [2, 3]$, the result cannot be reduced to the commutative case as in the alternative proof. Indeed, $t \mapsto b^{p-2}$ is operator concave and the first inequality right above reverses.

Remark 2.4. Let $\varphi : L_p(\mathcal{M}) \to L_{p'}(\mathcal{M})$ be the duality map so that $(x, \varphi(x))_{L_p(\mathcal{M}), L_{p'}(\mathcal{M})} = \|x\|_p^{p'}$ and $\|\varphi(x)\|_{p'} = \|x\|_p^{p-1}$. When restricted to $L_p(\mathcal{M})^+$, it is exactly $f_{p-1}$, so the result can be written as: for $a, b \in \mathcal{L}_p(\mathcal{M})^{+}$

$$\langle a - b, \varphi(a) - \varphi(b) \rangle_{L_p(\mathcal{M}), L_{p'}(\mathcal{M})} \geq \|a - b\|_p^{p'}.$$ 

In this form, the inequality extends to general $L_p$-spaces in the sense of Haagerup, see [3, 2] for the arguments.
Corollary 2.5. Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra and $\mathcal{E} : \mathcal{M} \to \mathcal{M}$ be a $\tau$-preserving conditional expectation, then for all $p \geq 2$ and $x \in L_p(\mathcal{M})^+$,
\begin{equation}
\|x - \mathcal{E}x\|_p \leq \|x\|_p.
\end{equation}

Proof. Apply the above theorem with $a = x$ and $b = \mathcal{E}x$, as $\tau((x - \mathcal{E}x)(\mathcal{E}x)^{p-1}) = 0$, the Hölder inequality gives:
\[\|x - \mathcal{E}x\|_p^p \leq \tau((x - \mathcal{E}x)(\mathcal{E}x)^{p-1}) \leq \|x - \mathcal{E}x\|_p \|x\|_p^{p-1}.\]

\[\square\]

Remark 2.6. The inequality (2) does not hold for $p < 2$: a counterexample with $\mathcal{M} = \ell^2_\infty$ can be found in [3]. There, a slight extension of (2) is given; one can replace $\mathcal{E}$ by any positive contractive projection $\mathcal{C}$ on $L_p(\mathcal{M})$.

As explained in [3], the main inequality applies more generally to semigroups.

Corollary 2.7. Let $(\mathcal{M}, \tau)$ be a semifinite von Neumann algebra and $(T_t)_{t \geq 0}$ be a trace preserving unital positive strongly continuous semigroup on $\mathcal{M}$. For $\lambda > 0$, let $R_\lambda = \int_0^\infty e^{-\lambda T_t} dt$ be its resolvent. Then for all $p \geq 2$, $\lambda > 0$ and $x \in L_p(\mathcal{M})^+$,
\[\|x - \lambda R_\lambda x\|_p \leq \|x\|_p.
\]

Proof. We proceed as in Corollary 2.5. We apply Theorem 2.1 with $a = x$ and $b = \lambda R_\lambda x$ to get
\begin{align*}
\|x - \lambda R_\lambda x\|_p^p &\leq \tau((x - \lambda R_\lambda x)x^{p-1}) - \tau((x - \lambda R_\lambda x)(\lambda R_\lambda x)^{p-1}) \\
&\leq \|x - \lambda R_\lambda x\|_p \|x\|_p^{p-1} - \tau((x - \lambda R_\lambda x)(\lambda R_\lambda x)^{p-1}).
\end{align*}

To conclude, it suffices to note that $\tau((x - \lambda R_\lambda x)(\lambda R_\lambda x)^{p-1}) \geq 0$.

It can be checked by approximations thanks to the resolvent formula: $x - \lambda R_\lambda x = \lim_{t \to \infty} (1 - tR_t) \lambda R_\lambda x$. Indeed, recall that $tR_t$ is positive unital and trace preserving and hence a contraction on $L_p$ so that
\[\tau(tR_t(R_\lambda x)(R_\lambda x)^{p-1}) \leq \|R_\lambda x\|_p^p \quad \text{and} \quad \tau((t(1 - tR_t)R_\lambda x)(R_\lambda x)^{p-1}) \geq 0.\]

\[\square\]

References
[1] Rajendra Bhatia. Matrix analysis, volume 169 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1997.
[2] Uffe Haagerup, Marius Junge, and Quanhua Xu. A reduction method for noncommutative $L_p$-spaces and applications. Trans. Amer. Math. Soc., 362(4):2125–2165, 2010.
[3] Mustapha Mokhtar-Kharroubi. Contractivity theorems in real ordered Banach spaces with applications to relative operator bounds, ergodic projections and conditional expectations. Preprint, https://hal.archives-ouvertes.fr/hal-01148968.
[4] Gilles Pisier and Quanhua Xu. Non-commutative $L^p$-spaces. In Handbook of the Geometry of Banach Spaces, Vol. 2, pages 1459–1517. North-Holland, Amsterdam, 2003.
[5] Éric Ricard. Hölder estimates for the noncommutative Mazur maps. Arch. Math. (Basel) 104 (2015), no. 1, 37–45.

Laboratoire de Mathématiques Nicolas Oresme, Université de Caen Normandie, 14032 Caen Cedex, France.

E-mail address: eric.ricard@unicaen.fr