On the intersection points of two plane algebraic curves

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Abstract

We prove that a set $\mathcal{X} \subset \mathbb{C}^2$, $\#\mathcal{X} = mn$, $m \leq n$, is the set of intersection points of some two plane algebraic curves of degrees $m$ and $n$, respectively, if and only if the following conditions are satisfied:

a) Any curve of degree $m + n - 3$ containing all but one point of $\mathcal{X}$, contains all of $\mathcal{X}$,

b) No curve of degree less than $m$ contains all of $\mathcal{X}$.

Let us mention that the conditions a) and b) in the “only if” direction of this result follow from the Ceyley-Bacharach and Noether theorems, respectively.

Keywords: Plane algebraic curve, intersection point, $n$-poised set, $n$-independent set.

1 Introduction, $n$-independence

Let $\Pi_n$ be the space of all polynomials in two variables and of total degree less than or equal to $n$. Its dimension is given by

$$N := \dim \Pi_n = \binom{n+2}{2}.$$

A plane algebraic curve is the zero set of some bivariate polynomial. To simplify notation, we shall use the same letter $p$, say, to denote the polynomial $p$ and the curve given by the equation $p(x, y) = 0$. More precisely, suppose $p$ is a polynomial without multiple factors. Then the plane curve defined by the equation $p(x, y) = 0$ shall also be denoted by $p$.

So lines, conics, and cubics are equivalent to polynomials of degree 1, 2, and 3, respectively; a reducible conic is a pair of lines, and a reducible cubic is a triple of lines, or consists of a line and an irreducible conic.

The following is a Linear Algebra fact:

Lemma 1.1. For any $N - 1 = (1/2)k(k + 3)$ points in the plane there is a curve of degree $k$ passing through them.
Suppose a set of \( k \) distinct points is given:

\[
\mathcal{X}_k = \{(x_i, y_i) : i = 1, 2, \ldots, k\} \subset \mathbb{C}^2.
\]

The problem of finding a polynomial \( p \in \Pi_n \) which satisfies the conditions

\[
p(x_i, y_i) = c_i, \quad i = 1, \ldots, k,
\]

is called interpolation problem.

**Definition 1.2.** The set of points \( \mathcal{X}_k \) is called \( n \)-poised, if for any data \((c_1, \ldots, c_m)\), there is a unique polynomial \( p \in \Pi_n \) satisfying the conditions \((1.1)\).

By a Linear Algebra argument a necessary condition for \( n \)-poisedness is

\[
k = \#\mathcal{X}_k = \dim \Pi_n = N.
\]

A polynomial \( p \in \Pi_n \) is called \( n \)-fundamental polynomial of a point \( A \in \mathcal{X} \), if

\[
p(A) = 1 \quad \text{and} \quad p|_{\mathcal{X}\setminus\{A\}} = 0,
\]

where \( p|_{\mathcal{X}} \) means the restriction of \( p \) to \( \mathcal{X} \). We shall denote such a polynomial by \( p^*_A \).

Sometimes we call \( n \)-fundamental also a polynomial from \( \Pi_n \) that just vanishes at all the points of \( \mathcal{X} \) but \( A \), since such a polynomial is a nonzero constant multiple of \( p^*_A \). A fundamental polynomial can be described as a plane curve containing all but one point of \( \mathcal{X} \).

Next we consider an important concept of \( n \)-independence and \( n \)-dependence of point sets (see [1], [3] - [7]).

**Definition 1.3.** A set of points \( \mathcal{X} \) is called \( n \)-independent, if each its point has an \( n \)-fundamental polynomial. Otherwise, it is called \( n \)-dependent.

**Definition 1.4.** A set of points \( \mathcal{X} \) is called essentially \( n \)-dependent, if none of its points has an \( n \)-fundamental polynomial.

If a point set \( \mathcal{X} \) is \( n \)-dependent, then for some \( A \in \mathcal{X} \), there is no \( n \)-fundamental polynomial, which means that for any polynomial \( p \in \Pi_n \) we have that

\[
p|_{\mathcal{X}\setminus\{A\}} = 0 \implies p(A) = 0.
\]

Thus a set \( \mathcal{X} \) is essentially \( k \)-dependent means that any plane curve of degree \( k \) containing all but one point of \( \mathcal{X} \), contains all of \( \mathcal{X} \).

Since fundamental polynomials are linearly independent we obtain that a necessary condition for \( n \)-independence is

\[
\#\mathcal{X} \leq \dim \Pi_n = N.
\]
It is easily seen that \( n \)-independence of \( X_k \) is equivalent to the solvability of the interpolation problem (1.1), meaning that for any data \( \{c_1, \ldots, c_k\} \) there exists a (not necessarily unique) polynomial \( p \in \Pi_n \) satisfying the interpolation conditions (1.1). In the case \( k = N \), i.e., for a point set \( X_N \), the \( n \)-independence means \( n \)-poisedness.

We have the following

**Proposition 1.5** ([6], Lemma 2.2). Suppose that the point set \( X \) is \( n \)-independent and each point of the set \( Y \) has \( n \)-fundamental polynomial with respect to the set \( X \cup Y \). Then the latter point set is \( n \)-independent, too.

**Corollary 1.6** ([6], Prop. 2.3). Suppose that a curve \( \sigma_k \), of degree \( k \) contains an \( n \)-independent point set \( X \). Suppose also that a set \( Y \) is outside of \( \sigma_k \) and is \((n - k)\)-independent. Then the set \( X \cup Y \) is \( n \)-independent.

Below we give a characterization of \( n \)-dependence of point sets consisting of at most 3\( n \) points.

**Theorem 1.7** ([6], Thm. 5.1). A set \( X \) consisting of at most 3\( n \) points is \( n \)-dependent if and only if one of the following holds.

a) \( n + 2 \) points are collinear,

b) \( 2n + 2 \) points belong to a (possibly reducible) conic,

c) \( \#X = 3n \), and there exist \( \sigma_3 \in \Pi_3 \) and \( \sigma_n \in \Pi_n \) such that \( X = \sigma_3 \cap \sigma_n \).

Next we bring three corollaries of this result.

**Corollary 1.8.** A set \( X \) consisting of at most \( 2n + 1 \) points is \( n \)-dependent if and only if \( n + 2 \) points are collinear.

A generalization of this result allowing for multiple points can be found in ([2], Theorem 9). From Corollary 1.8 we get immediately the following result of Severi [9]:

**Corollary 1.9** (Severi, [9]). Any set \( X \) consisting of at most \( n + 1 \) points is \( n \)-independent.

**Corollary 1.10.** A set \( X \) consisting of at most \( 3n - 1 \) points is \( n \)-dependent if and only if one of the following holds.

(i) \( n + 2 \) points are collinear,

(ii) \( 2n + 2 \) points belong to a (possibly reducible) conic.

A special case of above result, when \( \#X \leq 2n + 2 \), can be found in ([11], Prop. 1).

**Lemma 1.11.** Assume that \( X \) is essentially \( k \)-dependent and \( \sigma_n \) is a curve of degree \( n \). Assume also that the point set \( Y := X \setminus \sigma_n \) is not empty. Then \( Y \) is essentially \((k - n)\)-dependent.
Indeed, assume conversely that a point \( A \in \mathcal{Y} \) has a \((k-n)\)-fundamental polynomial: \( p^*_A \). Then it is easily seen that the polynomial \( p := p^*_A \sigma \) is a \( k \)-fundamental polynomial of \( A \) in the set \( \mathcal{X} \), which is a contradiction.

Assume that a curve \( \sigma \) of degree \( n \) is reducible, i.e.,

\[
\sigma = \sigma_1 \cdots \sigma_s, \tag{1.2}
\]

where the component \( \sigma_i \) has degree \( n_i \).

Denote by \( \mathcal{X}_i, \ i = 1, \ldots, s \), the set of points from \( \mathcal{X} \cap \sigma \) which do not lay in other components \( \sigma_j, \ j \neq i \), i.e.,

\[
\mathcal{X}_i = \mathcal{X} \setminus \bigcup_{j \neq i}^{s} \sigma_j. \tag{1.3}
\]

We call a component \( \sigma_i \) not empty with respect to the set \( \mathcal{X} \) if \( \mathcal{X}_i \neq \emptyset \).

**Lemma 1.12.** Assume that \( \mathcal{X} \subset \sigma \) is essentially \( k \)-dependent, where the curve \( \sigma \) of degree \( n \) is reducible, given by \( (1.2) \). Assume also that all the components are not empty with respect to \( \mathcal{X} \).

Then each set \( \mathcal{X}_i \) given in \( (1.3) \) is essentially \((k - n + n_i)\)-dependent.

Indeed assume that for some \( i, \ 1 \leq s \leq s \) the set \( \mathcal{X}_i \) is not essentially \((k - n + n_i)\)-dependent, i.e., a point \( A \in \mathcal{X}_i \) has a \((k - n + n_i)\)-fundamental polynomial: \( p_i \). Then it is easily seen that the polynomial \( p := p_i \prod_{j \neq i}^{s} \sigma_j \) is a \( k \)-fundamental polynomial of \( A \) in the set \( \mathcal{X} \), which is a contradiction.

### 1.1 The completeness of point sets in plane curves

**Definition 1.13.** Let \( \sigma \) be a plane curve of degree \( k \), without multiple components. Then the point set \( \mathcal{X} \subset \sigma \) is called \( n \)-complete in \( \sigma \), if the following assertion holds:

\[
p \in \Pi_n, \ p|_{\mathcal{X}} = 0 \Rightarrow p = q\sigma, \ q \in \Pi_{n-k}.
\]

The \( n \)-completeness in the case \( k > n \) means that \( p \in \Pi_n, \ p|_{\mathcal{X}} = 0 \Rightarrow p = 0 \). Therefore we have

**Lemma 1.14.** Let \( k > n \). Then a set of points \( \mathcal{X} \subset \sigma \) is \( n \)-complete in \( \sigma \), if and only if \( \mathcal{X} \) contains an \( n \)-poised subset \( \mathcal{X}_0 \).

Consider the following two linear spaces of polynomials:

\[
\mathcal{P}_{n,\mathcal{X}} := \{ p \in \Pi_n : p|_{\mathcal{X}} = 0 \}, \quad \mathcal{P}_{n,\sigma} := \{ p\sigma : p \in \Pi_{n-k} \},
\]

where \( \mathcal{X} \) is a point set and \( \sigma \subset \Pi_k \). Note that

\[
\mathcal{P}_{n,\mathcal{X}} \supset \mathcal{P}_{n,\sigma} \text{ if } \mathcal{X} \subset \sigma. \tag{1.4}
\]
Then we have also that
\[ \mathcal{P}_{n,X} = \mathcal{P}_{n,\sigma_k} \iff X \subset \sigma_k \text{ is } n\text{-complete in } \sigma_k. \] (1.5)

Now, we readily conclude from (1.4) and (1.5) that
\[ \dim \mathcal{P}_{n,X} = \dim \mathcal{P}_{n,\sigma_k} \iff X \subset \sigma_k \text{ is } n\text{-complete in } \sigma_k. \] (1.6)

Evidently we have that
\[ \dim \mathcal{P}_{n,\sigma_k} = \dim \Pi_{n-k}. \] (1.7)

For \( \dim \mathcal{P}_{n,X} \) we have the following well-known (see e.g. [3], [6])

Proposition 1.15. Let \( X_0 \) be a maximal \( n\)-independent subset of \( X \), i.e., \( X_0 \) is \( n\)-independent and \( X_0 \cup \{A\} \) is \( n\)-dependent for any \( A \in X \setminus X_0 \). Then we have that
\[ \dim \mathcal{P}_{n,X} = \dim \mathcal{P}_{n,X_0} = \dim \Pi_n - \#X_0. \] (1.8)

Set
\[ d(k, n) := \dim \Pi_n - \dim \Pi_{n-k}. \]

It is easily seen that \( d(k, n) = (n+1) + n + \cdots + (n-k+2) = \frac{1}{2} k(2n-k+3) \), if \( k \leq n + 2 \).

Finally, in view of (1.5)-(1.7), we get the following simple criterium for the completeness:

Proposition 1.16 (e.g., [8], Prop. 3.1). Let \( k \leq n + 2 \) and \( \sigma_k \) be a plane curve of degree \( k \). Then the point set \( X \subset \sigma_k \) is \( n\)-complete in \( \sigma_k \), if and only if \( X \) contains an \( n\)-independent subset \( X_0 \subset X \) with \( \#X_0 = d(k, n) \).

Note that in the cases \( k = n + 1, n + 2 \), we have that \( d(k, n) = \dim \Pi_k \), and Proposition follows from Lemma 1.4

Theorem 1.17 (Ceyley-Bacharach). Suppose that a set \( X \), \( \#X = mn \), is the set of intersection points of some two plane curves \( \sigma_m \) and \( \sigma_n \), of degrees \( m \) and \( n \), respectively: \( X = \sigma_m \cap \sigma_n \). Then we have that
\[ \begin{align*}
& a) \text{ the set } \mathcal{X} \text{ is essentially } \kappa\text{-dependent;} \\
& b) \text{ the set } \mathcal{X} \text{ is } (k+1)\text{-independent;} \\
& c) \text{ for any point } A \in \mathcal{X} \text{ the point set } \mathcal{X} \setminus \{A\} \text{ is } \kappa\text{-independent.}
\end{align*} \]

Theorem 1.18 (Noether). Suppose that a set \( X \), \( \#X = mn \), is the set of intersection points of some two plane curves \( \sigma_m \) and \( \sigma_n \), of degrees \( m \) and \( n \), respectively: \( X = \sigma_m \cap \sigma_n \). Then for any polynomial \( p_k \in \Pi_k \), vanishing on \( X \), we have that
\[ p_k = A_{k-m}\sigma_m + B_{k-n}\sigma_n, \] (1.9)

where \( A_{k-m} \in \Pi_{k-m} \) and \( B_{k-n} \in \Pi_{k-n} \).
Corollary 1.19. Suppose that a set $\mathcal{X}$, $\#\mathcal{X} = mn$, $m \leq n$, is the set of intersection points of some two plane curves $\sigma_m$ and $\sigma_n$, of degrees $m$ and $n$, respectively: $\mathcal{X} = \sigma_m \cap \sigma_n$. Then no curve of degree less than $m$ contains all of $\mathcal{X}$.

Indeed, suppose conversely that a curve $p$ of degree less than $m$ contains all of $\mathcal{X}$. Then we get from (1.9) that $p = 0$, which is a contradiction.

2 Main results

Throughout this section let us set

$$\kappa := \kappa(m, n) := m + n - 3.$$ 

Theorem 2.1. A set $\mathcal{X}$ with $\#\mathcal{X} = mn$, $m \leq n$, is the set of intersection points of some two plane curves of degrees $m$ and $n$, respectively, if and only if the following conditions are satisfied:

a) Any plane curve of degree $\kappa$ containing all but one point of $\mathcal{X}$, contains all of $\mathcal{X}$,

b) No curve of degree less than $m$ contains all of $\mathcal{X}$.

Let us mention that the necessity of the conditions a) and b) follow from Theorem 1.17 and Corollary 1.19, respectively. Note also that the condition a) above means that the point set $\mathcal{X}$ is essentially $\kappa$-dependent, while the condition b) means that the set $\mathcal{X}$ contains an $(m - 1)$-poised set.

Next we prove the part of sufficiency in the cases $m = 1, 2, 3$.

2.1 The proof of Theorem 2.1 in the cases $m = 1, 2, 3$

The case $m = 1$.

In this case an essentially $(n - 2)$-dependent set $\mathcal{X} = \{A_1, \ldots, A_n\}$ is given. By Corollary 1.8 we get that the $n$ points are collinear, i.e., belong to a line $\sigma_1$. Hence we get that $\mathcal{X} = \sigma_1 \cap \sigma_n$, where $\sigma_n$ has $n$ line components intersecting $\sigma_1$ at the $n$ points of $\mathcal{X}$, respectively.

The case $m = 2$.

In this case an essentially $(n - 1)$-dependent set $\mathcal{X} = \{A_1, \ldots, A_{2n}\}$ is given. By Corollary 1.10 we get that either $n + 1$ points of $\mathcal{X}$ are collinear, i.e., belong to a line $\sigma_1$, or all $2n$ points of $\mathcal{X}$ belong to a conic $\sigma_2$. Suppose first that $n + 1$ points of $\mathcal{X}$ belong to a line $\sigma_1$. Then by denoting $\mathcal{Y} = \mathcal{X} \setminus \sigma_1$ we have that $\#\mathcal{Y} \leq n - 1$. By the condition b) we have that $\mathcal{Y} \neq \emptyset$. Now we get from Lemma 1.11 that the set $\mathcal{Y}$ is essentially $(n - 2)$-dependent, which contradicts Corollary 1.9.

Next, suppose that all the $2n$ points of $\mathcal{X}$ belong to the conic $\sigma_2$. First consider the case when the conic $\sigma_2$ is irreducible. Then we get that $\mathcal{X} =$
\(\sigma_2 \cap \sigma_n\), where \(\sigma_n\) has \(n\) line components intersecting \(\sigma_2\) at \(n\) disjoint couples of points, respectively.

Finally suppose that the conic \(\sigma_2\) is reducible, i.e., is a pair of lines: \(\sigma_2 = \sigma_1 \sigma_1'\). First let us prove that each of the component lines contains exactly \(n\) points from \(\mathcal{X}\) hence the intersection point of \(\sigma_1\) and \(\sigma_1'\) does not belong to \(\mathcal{X}\). Assume conversely that a component, say \(\sigma_1\), contains \(n + 1\) points from \(\mathcal{X}\). Then by denoting \(\mathcal{Y} = \mathcal{X} \setminus \sigma_1\) we have that \(#\mathcal{Y} \leq n - 1\). By the condition b) we have that \(\mathcal{Y} \neq \emptyset\). Now we get from Lemma 1.11 that the set \(\mathcal{Y}\) is essentially \((n - 2)\)-dependent, which contradicts Corollary 1.9.

Hence we get that \(\mathcal{X} = \sigma_2 \cap \sigma_n\), where \(\sigma_n\) has \(n\) line components intersecting \(\sigma_1\) and \(\sigma_1'\) at \(n\) disjoint couples of points, one from \(\sigma_1\) and another from \(\sigma_1'\).

The case \(m = 3\).

In this case an essentially \(n\)-dependent set \(\mathcal{X} = \{A_1, \ldots, A_{3n}\}\) is given.

By Theorem 1.7 we get that either \(n + 2\) points belong to a line \(\sigma_1\), \(2n + 2\) points belong to a conic \(\sigma_2\), or \(\mathcal{X} = \sigma_3 \cap \sigma_n\), where \(\sigma_i \in \Pi_i\). It is enough to exclude here the first two possibilities.

Suppose first that \(n + 2\) points of \(\mathcal{X}\) belong to a line \(\sigma_1\). Then by denoting \(\mathcal{Y} = \mathcal{X} \setminus \sigma_1\) we have that \(#\mathcal{Y} \leq 2n - 2\). By the condition b) we have that \(\mathcal{Y} \neq \emptyset\). Now we get from Lemma 1.11 that the set \(\mathcal{Y}\) is essentially \((n - 1)\)-dependent. Hence by Corollary 1.8 we get that \(n + 1\) points of \(\mathcal{Y}\) belong to a line \(\sigma_1'\). Then by denoting \(\mathcal{Z} = \mathcal{X} \setminus (\sigma_1 \cup \sigma_1')\) we have that \(#\mathcal{Z} \leq n - 3\). By the condition b) we have that \(\mathcal{Z} \neq \emptyset\). Now we get from Lemma 1.11 that the set \(\mathcal{Z}\) is essentially \((n - 2)\)-dependent, which contradicts Corollary 1.9.

Next, suppose that \(2n\) points belong to a conic \(\sigma_2\). Then by denoting \(\mathcal{Y} = \mathcal{X} \setminus \sigma_2\) we have that \(#\mathcal{Y} \leq n\), \(\mathcal{Y} \neq \emptyset\). Now we get from Lemma 1.11 that the set \(\mathcal{Y}\) is essentially \((n - 2)\)-dependent. Hence by Corollary 1.8 we get that the set \(\mathcal{Y}\) has exactly \(n\) collinear points, belonging to a line \(\sigma_1\). Then by denoting \(\mathcal{Z} = \mathcal{X} \setminus \sigma_1\) we have that \(#\mathcal{Z} \leq 2n\), \(\mathcal{Z} \neq \emptyset\). Now we get readily from Lemma 1.11 that the set \(\mathcal{Z}\) is essentially \((n - 1)\)-dependent. Next we conclude, as above, from Corollary 1.4, that \(\mathcal{Z}\) contains exactly \(2n\) points and in the case when the conic \(\sigma_2\) has two line components, then each of the component lines contains exactly \(n\) points from \(\mathcal{X}\).

Now, from Proposition 1.10 we get that \(\mathcal{X}\) is not \(n\)-complete in \(\sigma_3 := \sigma_1 \sigma_2\). Hence there is a polynomial \(\sigma_n \in \Pi_n\) vanishing at \(\mathcal{X}\) but not on whole \(\sigma_3\), in particular

\[\mathcal{X} \subset \sigma_3 \cap \sigma_n.\]

It remains to verify that \(\sigma_3\) and \(\sigma_n\) have no common components.

Indeed, suppose that the common component of the highest degree is \(\sigma\), where \(\sigma \in \Pi_2\). Then we have that \(\sigma_n = \sigma \sigma_{n-2}\), where \(\sigma_{n-2} \in \Pi_{n-2}\). Now consider the line component \(\sigma_1\), of \(\sigma_3\) which is not a component of \(\sigma\). On that component we have exactly \(n\) points which are outside of \(\sigma\). Hence these \(n\) points belong to the curves \(\sigma_1\) and \(\sigma_{n-1}\), which contradicts to the Bezout
theorem, since the curves have no common component.

Finally, suppose that $\sigma \in \Pi_1$. Then we have that $\sigma_n = \sigma_{n-1}$, where $\sigma_{n-1} \in \Pi_{n-1}$. Now consider a (the) component $\sigma_k$, $k \leq 2$ of $\sigma_3$ different from $\sigma$. On that component we have exactly $kn$ points which are outside of $\sigma$. Hence these $kn$ points belong to the curves $\sigma_k$ and $\sigma_{n-1}$, which contradicts to the Bezout theorem, since the curves have no common component.

2.2 The proof of Theorem 2.1 in the case $m \geq 4$.

The proof of the sufficiency part of Thorem 2.1 is completed in the forthcoming Theorem 2.10 at the end of the section.

Let us start the discussion with the following

**Theorem 2.2.** Suppose that an irreducible curve $\sigma_m$ of degree $m$ contains a set $X$ of $mn$ points. Then the following statements hold:

(a) If the set $X$ is $\kappa$-independent then it is $n$-complete in $\sigma_m$.

(b) Suppose that $3 \leq m \leq n + 2$. If the set $X$ is $n$-complete in $\sigma_m$ then it is $\kappa$-independent.

**Proof.** Part a): Suppose that a set $X \subset \sigma_m$ is not $n$-complete in $\sigma_m$. Then there is a polynomial $\sigma_n \in \Pi_n$ that vanishes on $X$ but not on $\sigma_m$. Then, since the curve $\sigma_m$ is irreducible we conclude from the Bezout theorem that $X = \sigma_m \cap \sigma_n$.

Now we get from Theorem 1.17, a), that $X$ is $\kappa$-dependent. More precisely, we get from Theorem 1.17, a), that $X$ is essentially $\kappa$-dependent and, from the item c), that for any point $A \in X$ the point set $X \setminus \{A\}$ is $\kappa$-independent.

Part b): Suppose that the set of points $X$ is $n$-complete in $\sigma_m$. Then, according to Proposition 1.16 we have that $X$ contains an $n$-independent subset $Y$ of $d(m, n)$ points. Since $m \leq n + 2$ the number of points in $Z := X \setminus Y$ equals

$$mn - d(m, n) = mn - \frac{1}{2}m(2n - m + 3) = \frac{1}{2}m(m - 3).$$

Thus in the case $m = 3$ we have that $X = Y$ is $\kappa = n$-independent. Now assume that $m > 3$. In view of Lemma 1.1 we have that there is a curve $\sigma_{m-3}$ of degree $m - 3$ containing all the points of $Z$. Denote by $\tilde{Z}$ the set of all points of $X$ in $\sigma_{m-3}$. Since the curve $\sigma_m$ is irreducible it has no common component with $\sigma_{m-3}$. Next, we have that $\tilde{Z} \subset \sigma_m \cap \sigma_{m-3}$.

Therefore by Theorem 1.17, b), the set $\tilde{Z}$ is $m + (m - 3) - 2 = (2m - 5)$-independent. On the other hand we have that $\kappa = m + n - 3 \geq 2m - 5$. Therefore the set $\tilde{Z}$ is $\kappa$-independent. Then, we have that the set $X \setminus \tilde{Z} \subset X \setminus Z = Y$ is $n$-independent. By Corollary 1.6 the set $X$ is $\kappa$-independent. □
We get immediately from the proof of the part a) (the last sentence there):

**Corollary 2.3.** Suppose that an irreducible curve $\sigma_m$ of degree $m$ contains a set $X$ of $mn$ points, which is not $n$-complete. Then the set $X$ is essentially $\kappa$-dependent and for any point $A \in X$ the set $X \setminus \{A\}$ is $\kappa$-independent.

We get from the proof of the part b) of Theorem 2.2 the following

**Proposition 2.4.** Suppose that $3 \leq m \leq n+2$ and a (not necessarily irreducible) curve $\sigma_m$ of degree $m$ contains a set $X$ of $\leq mn$ points, which is $n$-complete. Then the set $X$ is not essentially $\kappa$-dependent.

**Proof.** By proof of part b) of Theorem 2.2 we have that there is a curve $\sigma_{m-3}$ of degree $m-3$ passing through all the points of the set $Z = X \setminus Y$. In the case $m = 3$ we have that $X = Y$ and thus is $\kappa$-independent. Now suppose that $m > 3$. Let us show that $\sigma_{m-3}$ does not contain all of $X$. Indeed, if $X \subset \sigma_{m-3}$ then the polynomial $\sigma_{m-3}$ vanishes on $X$ but not on $\sigma_m$. Hence $X$ is not $n$-complete in $\sigma_m$ which is a contradiction. Next, choose a point $A \in X \setminus \sigma_{m-3}$. We have that $A \in Y$. Consider the fundamental polynomial $p_{A,Y}$. Finally, notice that $p := \sigma_{m-3}p_{A,Y}$ is a fundamental polynomial of $A$ in the set $X$ of degree $\kappa = m + n - 3$. Hence, the set $X$ is not essentially $\kappa$-dependent.

**Theorem 2.5.** Assume that $m \leq n+2$. Then we have that any set of points $X$, with $\#X \leq m(\kappa+3-m)-1 = mn-1$, in an irreducible curve $\sigma_m$, is $\kappa$-independent.

**Proof.** The cases $m = 1$ and $m = 2$ are evident. Suppose that $m \geq 3$. Let us add a point $A \in \sigma_m \setminus X$ to $X$. If the resulted set $Y := X \cup \{A\}$ is $\kappa$-independent then $X \subset Y$ is also $\kappa$-independent and Theorem is proved. Now, suppose that $Y$ is $\kappa$-dependent. Then, according to Theorem 2.2 (b), it is not $n$-complete in $\sigma_m$. Then we get from Corollary 2.3 that $Y$ is essentially $\kappa$-dependent and $X = Y \setminus \{A\}$ is $\kappa$-independent.

**Theorem 2.6.** Assume that $\sigma_m$ is a curve of degree $m$, which is either not reducible or is reducible such that all its irreducible components are not empty with respect to a set $X \subset \sigma_m$. Assume also that $\#X \leq mn - 1 = m(\kappa+3-m)-1$, where $m \leq n+2$. Then $X$ is not essentially $\kappa$-dependent.

**Proof.** The cases $m = 1$ and $m = 2$ are evident. Suppose that $m \geq 3$. The case when $\sigma_m$ is irreducible follows immediately from Theorem 2.5. Now assume that $\sigma_m$ is reducible, i.e.,

$$\sigma_m = \sigma_{m_1} \cdots \sigma_{m_s},$$

where the component $\sigma_{m_i}$ is irreducible and has degree $m_i$. 
Assume, by way of contradiction, that $\mathcal{X}$ is essentially $\kappa$-dependent. Consider the set $\mathcal{X}_i$, $i = 1, \ldots, s$, given in (1.3). By the hypothesis $\mathcal{X}_i \neq \emptyset$, $i = 1, \ldots, s$. Since $\mathcal{X}$ is essentially $\kappa$-dependent we get from Lemma 1.12 that the set $\mathcal{X}_i$ is essentially $(\kappa - m + m_i)$-dependent. Next we are going to apply here Theorem 2.5. Note that the condition $m \leq n + 2$ here reduces to $m_i \leq (\kappa - m + m_i) - m_i + 5$, which is satisfied, since in its turn it reduces to $m_i \leq \kappa - m + 5 = n + 2$. Now, we conclude from Theorem 2.5 that $\# \mathcal{X}_i \geq m_i(\kappa - m + m_i - m_i + 3) = m_i(\kappa - m + 3)$. From here, by summing up, we get $\# \mathcal{X} \geq m(\kappa - m + 3) = mn$, which is a contradiction. □

Proposition 2.7. Suppose that $m \leq n$. If a point set $\mathcal{X}$, with $\# \mathcal{X} \leq mn$, is essentially $\kappa$-dependent then all the points of $\mathcal{X}$ lay in a curve of degree $m$ or $n - 3$.

Proof. The cases $n = 1, 2, 3$, are evident. Thus assume that $n \geq 4$. Suppose conversely that there is no curve of degree $m$ containing all of $\mathcal{X}$. Then there is an $m$-poised subset $\mathcal{Y} \subset \mathcal{X}$ of $(1/2)m(m + 3) + 1$ points.

Set $\mathcal{Z} = \mathcal{X} \setminus \mathcal{Y}$. Next we are going to show that

$$\# \mathcal{Z} \leq \dim \Pi_{n-3} - 1. \quad (2.1)$$

We have that $\nu := \# \mathcal{Z} - \dim \Pi_{n-3} + 1 \leq mn - (1/2)m(m + 3) - 1 - (1/2)n(n - 3) - 1 = (1/2)m(2n - m - 3) - 1(1/2)m(2n - m - 3) - (1/2)n(n - 3) - 1 = -(1/2)(n - m - 3)(n - m) - 1$.

Now, evidently $\nu < 0$ if $n = m$ or $n \geq m + 3$. While $\nu = 0$ if $n = m + 1$ or $n = m + 2$. Thus (2.1) is proved.

By Lemma 1.1 there is a curve $\sigma_{n-3}$ of degree $n - 3$ passing through all the points of $\mathcal{Z}$. We claim that $\mathcal{X} \subset \sigma_{n-3}$. Suppose by contradiction that there is a point $A \in \mathcal{X} \setminus \sigma_{n-3}$. Recall that the set $\mathcal{Y}$ is $m$-poised and consider the $m$-fundamental polynomial $p_{\mathcal{A},\mathcal{Y}}^*$. Now, notice that $p := \sigma_{n-3}p_{\mathcal{A},\mathcal{Y}}^*$ is a $\kappa$-fundamental polynomial of the point $A$ in the set $\mathcal{X}$, which is a contradiction. □

Proposition 2.8. Suppose that $m \leq n$. If a set $\mathcal{X}$ of $mn$ points is essentially $\kappa$-dependent then all the points of $\mathcal{X}$ lay in a curve of degree $m$.

Proof. Assume by the way of contradiction that $\mathcal{X}$ does not lay in a curve of degree $m$.

First let us prove that there is a number $m_0 > m$ such that

1) $m_0 \leq \frac{\kappa + 3}{2}$, i.e., $m_0 \leq n_0 := \kappa + 3 - m_0$,

2) all the points of $\mathcal{X}$ lay in a curve of degree $m_0$,

3) no curve of degree less than $m_0$ contains all of $\mathcal{X}$.

To this end let us apply Theorem 2.4 for $\mathcal{X}$ and $m = m' = \left[\frac{\kappa + 3}{2}\right]$. If $m' = \frac{\kappa + 3}{2}$ then we get that $\mathcal{X}$ lies in a curve $\sigma_{m'}$ of degree $m'$ or in a curve $\sigma_{n' - 3}$ of degree $n' - 3$, where $n' := \kappa + 3 - m' = m'$, and conclude that $\mathcal{X}$ lies in a curve $\sigma_{m'}$.  

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If \( m' = \frac{\kappa + 3}{2} - \frac{1}{2} \) then we get that \( \mathcal{X} \) lies in a curve \( \sigma_{m'} \) of degree \( m' \) or in a curve \( \sigma_{n' - 3} \) of degree \( n' - 3 = m' - 2 \), and again conclude that \( \mathcal{X} \) lies in a curve \( \sigma_{m'} \).

In both cases \( m' \leq \frac{\kappa + 3}{2} \), so we have that \( \mathcal{X} \) lies in a curve \( \sigma_{m'} \), where \( m' \leq n' \).

Now denote by \( m_0 \) the minimal possible \( m' \) with above described property and \( \sigma_{m_0} \) be the corresponding curve of degree \( m_0 \). Then it is easily seen that \( m_0 > m \) and the above conditions 1), 2) and 3) are satisfied.

Let us verify that \( mn \leq m_0n_0 - 1 \). For this end consider the parabola \( y = x(\kappa + 3 - x) \). Now it is easily seen that

\[
mn = m(\kappa - m + 3) < m_0(\kappa - m_0 + 3),
\]

(2.2)

since we have \( y(m) = y(n) \) and \( m < m_0 < n \).

Next, suppose first that the curve \( \sigma_{m_0} \) is irreducible. In view of (2.2) we conclude from Theorem 2.5 that the set \( \mathcal{X} \) is \( \kappa \)-independent, which is a contradiction. Note that here \( m_0 \leq n_0 \).

Finally, suppose that \( \sigma_{m_0} \) is a reducible curve: \( \sigma_{m_0} = \sigma_{m_1} \cdots \sigma_{m_s} \), where each component \( \sigma_{m_i} \) has degree \( m_i \), and is irreducible. In view of the above condition 3) no component is empty with respect to the point set \( \mathcal{X} \). Now by Theorem 2.6 we get that \( \mathcal{X} \) is not essentially \( \kappa \)-dependent, which is a contradiction.

Remark 2.9. Suppose that \( m \leq n \) and a set \( \mathcal{X} \) of \( mn \) points is essentially \( \kappa \)-dependent. Suppose also that no curve of degree less than \( m \) contains all of \( \mathcal{X} \). Let \( \sigma_m \) be the curve of degree \( m \) from Proposition 2.8 containing all of \( \mathcal{X} \). Then if the curve is reducible: \( \sigma_m = \sigma_{m_1} \cdots \sigma_{m_s} \), where each component \( \sigma_{m_i} \) has degree \( m_i \) and is irreducible, then no point of \( \mathcal{X} \) is an intersection point of the components and each component \( \sigma_{m_i} \) contains exactly \( m_i(\kappa - m + 3) \) points of \( \mathcal{X} \) which are essentially \( (\kappa - m + m_i) \)-dependent.

Indeed, the proof coincides with the last paragraph of the proof of Proposition 2.8 with \( m_0 \) replaced by \( m \).

Theorem 2.10. Given a set \( \mathcal{X} \), \( \# \mathcal{X} = mn \), \( m \leq n \), satisfying the following conditions:

a) Any plane curve of degree \( \kappa = m + n - 3 \) containing all but one point of \( \mathcal{X} \), contains all of \( \mathcal{X} \).

b) No curve of degree less than \( m \) contains all of \( \mathcal{X} \).

Then \( \mathcal{X} \) is the set of intersection points of some two plain curves \( \sigma_m \) and \( \sigma_n \) of degrees \( m \) and \( n \), respectively:

\[
\mathcal{X} = \sigma_m \cap \sigma_n.
\]

(2.3)

Proof. The cases \( m = 1, 2, 3 \) were considered earlier. Hence, suppose that \( m \geq 4 \). We have from Proposition 2.8 that all the points of \( \mathcal{X} \) lay in a curve
σ_m of degree m. Then we get from Proposition 2.4 that the set $X$ is not $n$-complete in $σ_m$.

Thus the set $X$ is not $n$-complete on $σ_m$. Therefore there exists a curve $σ_n$ of degree $n$ which vanishes on all the points of $X$ but does not vanish on whole curve $σ_m$. It only remains to show that the curves $σ_m$ and $σ_n$ do not have a common component. Suppose by way of contradiction that

$$σ_m = σ_lσ_{m−l} \text{ and } σ_n = σ_lσ_{n−l},$$

where $σ_i$ has degree $i$ and the curves $σ_{m−l}, σ_{n−l}$ have no common component.

Denote $Y := σ_{m−l} \cap σ_{n−l} \cap X$. In view of the condition b) we have that $Y \neq \emptyset$. Let $A \in Y$. By the Cayley-Bacharach theorem we have that $A$ has a fundamental polynomial $p_{A,Y}^⋆$ of degree $m + n − 2l − 2$ in the set $Y$. Now notice that the polynomial $p = σ_lσ_{m−l}^⋆$ of degree $m + n − l − 2 \leq m + n − 3$ is a fundamental polynomial of $A$ in the set $X$, which contradicts the condition a). Therefore (2.3) is proved.

Now we get from Theorems 2.1 and 1.17 the following

**Corollary 2.11.** Given a set $X$, $\#X = mn$, $m \leq n$, satisfying the following conditions:

a) The set $X$ is essentially $κ$-dependent,

b) The set $X$ contains an $(m−1)$-poised subset.

Then for any point $A ∈ X$ the point set $X \setminus \{A\}$ is $κ$-independent.

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