The following problem (proposed by Stanley Rabinowitz), appeared as problem 1325 in *Crux Mathematicorum*. We call this the *Crux Problem* since the accompanying diagram contains a shaded ‘cross’=crux.

**Crux problem.** Let $P$ be any point inside a unit circle with center $C$. Perpendicular chords are drawn through $P$. Rotation of these chords counterclockwise about $P$ through an angle $\theta$ sweep out the shaded area shown in Figure 1. Show that this shaded area only depends on $\theta$, but not on $P$ (and hence is easily seen to be $2\theta$ by taking $P = C$).

![Figure 1](image_url)

**Figure 1** Illustrating the Crux problem.

Two solutions were subsequently published [1, pp. 120-122]:

(I) Jörg Hätherich presented a solution using calculus and Archimedes’ theorem,
(II) Shiko Iwata presented a non-calculus solution based on trigonometry.

The accompanying editor’s note mentioned that Murray Klamkin generalised the problem to $n$ chords through $P$ with equal angles of $\pi/n$ between successive chords, with the area swept out, when these chords are rotated through an angle of $\theta$ about $P$, then being $n\theta$. The editor’s note ended with the following parenthetical remark: “This can also be proved using the solution II. Can it be proved as in solution I?”

In this note, we present a calculus-based solution, based on a special case of a generalization of “Archimedes’ theorem,” which is proved by employing vectors. We believe that this solution captures in some sense the crux of the matter.

We begin with a calculus-based proof along lines similar to the first solution given in [1].

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*First ‘Crux’ in the title.
†Second ‘crux’ in the title.
‡Third ‘crux’ in the title.
A calculus-based proof of the Crux problem

We will use the following result. We call it Archimedes’ theorem as it is Proposition 11 in Archimedes’ work *The Book of Lemmas* [2, p. 312].

**Theorem 1.** (Archimedes) If two mutually perpendicular chords $A_1B_1$ and $A_2B_2$ in a unit circle with center $C$ meet at $P$ (see Figure 2), then

$$PA_1^2 + PB_1^2 + PA_2^2 + PB_2^2 = 4.$$ 

![Figure 2](image_url) Archimedes’ theorem.

**Proof.** $\triangle B_2PA_1$ is similar to $\triangle B_2B_1D$ since we have

$$\angle B_2A_1B_1 = \angle B_2DB_1 \quad \text{and} \quad \angle B_2PA_1 = 90^\circ = \angle B_2B_1D.$$ 

So

$$\angle PB_2A_1 = \angle B_1B_2D.$$ 

This implies that

$$\angle A_1CA_2 = \angle B_1CD,$$ 

and so we obtain $A_1A_2 = B_1D$. By Pythagoras’ theorem, we have

$$PB_2^2 + PB_1^2 = B_2B_1^2 \quad \text{and} \quad PA_1^2 + PA_2^2 = A_1A_2^2.$$ 

Adding these, we obtain that

$$PA_1^2 + PB_1^2 + PA_2^2 + PB_2^2 = B_2B_1^2 + A_1A_2^2$$

$$= B_1B_2^2 + B_1D^2$$

$$= B_2D^2 = 2^2 = 4.$$
Now we give a calculus argument as follows: Rotating $A_1B_1$ and $A_2B_2$ about $P$ through an infinitesimal angle $d\theta$, we obtain four sectors, with areas given by

$$\frac{1}{2} PA_1^2 d\theta, \quad \frac{1}{2} PB_1^2 d\theta, \quad k = 1, 2.$$ 

By adding and using Archimedes’ theorem, we obtain the rate of change of area

$$\frac{dA}{d\theta} = \frac{1}{2} (PA_1^2 + PB_1^2 + PA_2^2 + PB_2^2) = \frac{1}{2} 4 = 2,$$

and so the total area, if the chords are rotated through an angle $\theta$, is given by

$$A = \int_0^\theta \frac{dA}{d\theta} d\theta = \int_0^\theta 2 d\theta = 2\theta.$$

### A vector calculus proof

We will first show the following:

**Proposition 1.** Let $P$ be any point inside a unit circle, and, through $P$, let there be $n$ chords $A_1B_1, \ldots, A_nB_n$ such that there are equal angles of $\pi/n$ between successive chords. Suppose moreover that $A_1B_1$ is a diameter. (See Figure 3.) If each chord is rotated counterclockwise through an angle $\theta$, then the total area formed by the resulting sectors is $n\theta$.

![Figure 3](image)

**Figure 3** Multiple chords sweeping out equal angles.

This will be shown to yield the generalization (given in Theorem 2) of the Crux problem, where as opposed to the situation above, one of the chords need not be a diameter.

In order to prove Proposition 1, we will first prove a special case of a generalization of Archimedes’ theorem (Theorem 3 in the next section, asserting that the sum of the squared distances from a point inside a unit circle to the vertices of $n$ equally angularly spaced chords passing through that point is $2n$), when one of the chords $A_1B_1$ is the diameter.
Lemma 1 (Generalised Archimedes’ theorem, special case). Let \( P \) be any point inside a unit circle, and let there be \( n \) chords \( A_1B_1, \ldots, A_nB_n \) through \( P \) such that there are equal angles of \( \pi/n \) between successive chords. Suppose, moreover, that \( A_1B_1 \) is a diameter. Then

\[
PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2 = 2n.
\]

Proof. Let \( C_1, \ldots, C_n \) be the centers of \( A_1B_1, \ldots, A_nB_n \). As \( A_1B_1 \) is the diameter, \( C_1 \) is the center of the circle. See Figure 4. We know that for all \( 1 \leq k \leq n \),

\[
\langle \overrightarrow{PA_k} - \overrightarrow{PC_1}, \overrightarrow{PA_k} - \overrightarrow{PC_1} \rangle = \| \overrightarrow{PA_k} - \overrightarrow{PC_1} \|_2^2 = 1,
\]

\[
\langle \overrightarrow{PB_k} - \overrightarrow{PC_1}, \overrightarrow{PB_k} - \overrightarrow{PC_1} \rangle = \| \overrightarrow{PB_k} - \overrightarrow{PC_1} \|_2^2 = 1.
\]

By expanding, adding, and rearranging, we obtain

\[
\sum_{k=1}^{n} (PA_k^2 + PB_k^2) = 2 \left( \sum_{k=1}^{n} \overrightarrow{PC_k} \right) - 2n \overrightarrow{PC_1}^2 + 2n. \tag{1}
\]

We need to determine the inner product on the right-hand side. We have

\[
\overrightarrow{PA_k} = \overrightarrow{PC_k} + \overrightarrow{C_kA_k}, \quad \text{and} \quad \overrightarrow{PB_k} = \overrightarrow{PC_k} + \overrightarrow{C_kB_k}.
\]

But since \( \overrightarrow{C_kA_k} + \overrightarrow{C_kB_k} = 0 \), we obtain \( \overrightarrow{PA_k} + \overrightarrow{PB_k} = 2\overrightarrow{PC_k} \). Hence

\[
\sum_{k=1}^{n} (\overrightarrow{PA_k} + \overrightarrow{PB_k}) = 2 \sum_{k=1}^{n} \overrightarrow{PC_k} = \sum_{k=1}^{n} \overrightarrow{PC_k} + \sum_{k=1}^{n} \overrightarrow{PC_{n-k}} = \sum_{k=1}^{n} (\overrightarrow{PC_k} + \overrightarrow{PC_{n-k}}).
\]

By referring to Figure 4, we see that for all \( 1 \leq k \leq n \),

\[
\overrightarrow{PC_k} + \overrightarrow{PC_{n-k}} = 2PC_k \left( \cos \frac{k\pi}{n} \right) \frac{\overrightarrow{PC_1}}{PC_1}
\]

\[
= 2 \left( \cos \frac{k\pi}{n} \right) PC_1 \left( \cos \frac{k\pi}{n} \right) \frac{\overrightarrow{PC_1}}{PC_1} = 2 \left( \cos \frac{k\pi}{n} \right)^2 \overrightarrow{PC_1}.
\]

So

\[
\sum_{k=1}^{n} (\overrightarrow{PA_k} + \overrightarrow{PB_k}) = \sum_{k=1}^{n} (\overrightarrow{PC_k} + \overrightarrow{PC_{n-k}}) = \sum_{k=1}^{n} 2 \left( \cos \frac{k\pi}{n} \right)^2 \overrightarrow{PC_1}. \tag{2}
\]

Now

\[
\sum_{k=1}^{n} 2 \left( \cos \frac{k\pi}{n} \right)^2 = \sum_{k=1}^{n} \left( 1 + \left( \cos \frac{2\pi}{n} \right) \right) = n + 0 = n,
\]

where we have used

\[
\sum_{k=1}^{n} \cos \left( \frac{2\pi}{n} \right) = 0. \tag{4}
\]
To see equation (4), we first note that this sum is the horizontal component of the sum $\mathbf{S}$ of $n$ vectors whose tails lie at the center of the unit circle and whose tips lie on the vertices of a regular $n$-gon inscribed in the circle. To see that $\mathbf{S}$ is zero, imagine rotating each vector counterclockwise through an angle of $\frac{2\pi}{n}$, and let the sum of the rotated vectors be $\mathbf{S'}$. On grounds of symmetry of the regular polygon, $\mathbf{S} = \mathbf{S'}$. On the other hand $\mathbf{S'}$ ought to be a rotated version of $\mathbf{S}$ through an angle of $\frac{2\pi}{n}$. This can only happen if $\mathbf{S} = 0$. (Alternative justifications of equation (4) can be given by first summing the geometric series

$$
\sum_{k=1}^{n} e^{i\frac{2\pi}{n}k} = e^{i\frac{2\pi}{n}} - \frac{1 - e^{i(2\pi)k}}{1 - e^{i\frac{2\pi}{n}}} = 0
$$

and taking real parts, or by noticing the sum of the $n$th roots of unity must add up to 0 since the coefficient of $e^1$ in $e^n - 1$ is 0, and again taking real parts.)

Consequently, using equations (1), (2), and (3), we obtain

$$
\sum_{k=1}^{n} (PA_k^2 + PB_k^2) = 2\left(\mathbf{PC_1}, \sum_{k=1}^{n} (PA_k + PB_k)\right) - 2nPC_1^2 + 2n
$$

$$
= 2\left(\mathbf{PC_1}, n\mathbf{PC_1}\right) - 2nPC_1^2 + 2n
$$

$$
= 2nPC_1^{2\hat{e}} - 2nPC_1^{2\hat{e}} + 2n = 2n.
$$

We are now ready to prove Proposition 1.

**Proof of Proposition 1.** Rotating $A_1B_1, \ldots, A_nB_n$ anticlockwise about $P$ through an infinitesimal angle $d\theta$, we obtain $2n$ sectors, with areas given by

$$
\frac{1}{2} PA_k^2 d\theta, \quad \frac{1}{2} PB_k^2 d\theta, \quad k = 1, \ldots, n.
$$

By adding and using Lemma 1, the rate of change of the total area is seen to be

$$
\frac{dA}{d\theta} = \frac{1}{2} \sum_{k=1}^{n} (PA_k^2 + PB_k^2) = \frac{1}{2} 2n = n.
$$
and so the total area, if the chords are rotated through an angle $\theta$, is given by

$$A = \int_0^\theta \frac{dA}{d\theta} d\theta = \int_0^\theta n d\theta = n\theta.$$  

\[\blacksquare\]

**Theorem 2.** Let $P$ be any point inside a unit circle, and let there be $n$ chords through $P$ such that there are equal angles of $\pi/n$ between successive chords. If each chord is rotated counterclockwise through an angle $\theta$, then the total area formed by the resulting sectors is $n\theta$.

**Proof.** To see how this follows from Proposition 1, we first construct the diameter $A_1B_1$ through $P$, and consider successive anticlockwise rotations of this diameter through angles of $\pi/n$, resulting in the chords $A_2B_2, \ldots, A_nB_n$. Let the given chords from the theorem statement be labeled as $A_1'B_1', \ldots, A_n'B_n'$, and let their rotated versions (though an angle $\theta$) be labeled as $A_1''B_1'', \ldots, A_n''B_n''$. See Figure 5.

![Figure 5](image-url)

**Figure 5** Illustration for the proof of Theorem 2.

Let the angle between $A_1B_1$ and $A_1'B_1'$ be $\theta'$, and that between $A_1B_1$ and $A_1''B_1''$ be $\theta''$. Then for all $1 \leq k \leq n$, we use the notation $\widehat{A_kB_k'}$ for the sector formed by the corresponding arc with $P$, and denote the area of the sector by $A(\widehat{A_kB_k'})$. Then:

$$\sum_{k=1}^n [A(\widehat{B_k'B_k''}) + A(\widehat{A_k'A_k''})]$$

$$= \sum_{k=1}^n [A(\widehat{B_kB_k'}) - A(\widehat{B_k'B_k'}) + A(\widehat{A_kA_k''}) - A(\widehat{A_k'A_k'})]$$
\[
= \sum_{k=1}^{n} [A(B_k B_k'') + A(A_k A_k'')] - \sum_{k=1}^{n} [(A(B_k B_k') + A(A_k A_k'))]
= n\theta'' - n\theta' = n(\theta'' - \theta') = n\theta.
\]

**Archimedes' theorem**

A consequence of Theorem 2 is the following generalization of Archimedes’ theorem from the \( n = 2 \) chord case considered earlier.

**Theorem 3** (Generalised Archimedes’ theorem). *Let \( P \) be any point inside a unit circle, and let there be \( n \) chords \( A_1 B_1, \ldots, A_n B_n \) through \( P \) such that there are equal angles of \( \pi/n \) between successive chords. Then\]
\[
PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2 = 2n.
\]

*Proof.* By Theorem 2, we know that if the chords are rotated through an infinitesimal angle \( d\theta \), then the sum of the areas of the resulting sectors is \( n \, d\theta \). But this area is also equal to
\[
\frac{1}{2} (PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2) d\theta.
\]
Consequently, we obtain that
\[
PA_1^2 + PB_1^2 + \cdots + PA_n^2 + PB_n^2 = 2n.
\]

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**REFERENCES**

[1] Solution to problem 1325. (1989). *Crux Mathematicorum*. 15(4): 120–121.

[2] The works of Archimedes. (2002). Reprint of the 1897 edition and the 1912 supplement, edited by T. L. Heath. New York: Dover.

**Summary.** A result of Archimedes states that for perpendicular chords passing through a point \( P \) in the interior of the unit circle, the sum of the squares of the lengths of the chord segments from \( P \) to the circle is equal to 4. A generalization of this result to \( n \geq 2 \) chords is given. This is done in the backdrop of revisiting Problem 1325 from *Crux Mathematicorum*, for which a new solution is presented.

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