Estimates for first-order homogeneous linear characteristic problems

Simonetta Frittelli
Department of Physics, Duquesne University, Pittsburgh, PA 15282 and
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260
(Dated: July 18, 2018)

An algebraic criterion that is sufficient to establish the existence of certain \textit{a priori} estimates for the solution of first-order homogeneous linear characteristic problems is derived. Estimates of such kind ensure the stability of the solutions under small variations of the data. Characteristic problems that satisfy this criterion are, in a sense, \textit{manifestly well posed}.

I. INTRODUCTION

Any system of hyperbolic partial differential equations can be written in a peculiarly wicked form, namely: in characteristic form \cite{1}. In order to do this, a one-parameter family of characteristic surfaces is chosen as the level surfaces of a coordinate \( u \), referred to as the null coordinate or \textit{retarded time}. Instead of evolution from one time-level to the next, one obtains evolution from one characteristic slice to the next. However, characteristic surfaces are special: they are the only data surfaces for which the standard Cauchy problem cannot be solved, because the differential operator of the system is internal to these surfaces, failing to evolve some data out of the surface. Naturally, the equations “degenerate”, turning into a system where not all the equations can be integrated forward in retarded time, but there are “rules” that allow one to solve the problem in a hierarchical manner, zig-zagging back and forth between the equations \cite{2}. These hierarchical rules are the hallmark of characteristic evolution, making it significantly different from an initial value problem.

It has been known for quite some time \cite{2} that the characteristic “Cauchy problem” -- to obtain a unique solution from data given on a characteristic surface-- can be solved so long as the data are split into two separate sets: some data are given on the initial null slice, and the rest of the data are given on another slice that must be transverse to the initial null slice. An issue that has attracted less attention is: if the data are perturbed slightly, under what circumstances is the variation of the solution under control? Equivalently, will almost-zero data evolve into a solution that is also close to zero? We set up a control? Equivalently, will almost-zero data evolve into a solution that is also close to zero? We set up a

\textit{equivale}nct, turning into a system where not all the equations can be solved, because the differential operator of the system is internal to these surfaces, failing to evolve some data out of the surface. Naturally, the equations “degenerate”, turning into a system where not all the equations can be integrated forward in retarded time, but there are “rules” that allow one to solve the problem in a hierarchical manner, zig-zagging back and forth between the equations \cite{2}. These hierarchical rules are the hallmark of characteristic evolution, making it significantly different from an initial value problem.

It has been known for quite some time \cite{2} that the characteristic “Cauchy problem” -- to obtain a unique solution from data given on a characteristic surface-- can be solved so long as the data are split into two separate sets: some data are given on the initial null slice, and the rest of the data are given on another slice that must be transverse to the initial null slice. An issue that has attracted less attention is: if the data are perturbed slightly, under what circumstances is the variation of the solution under control? Equivalently, will almost-zero data evolve into a solution that is also close to zero? We set up a framework in which to address this question by defining certain types of estimates of the solution in terms of the free data, after \cite{3}. Subsequently we derive an algebraic criterion that is sufficient to determine whether the solutions satisfy such an \textit{a priori} estimate, thus establishing their stability with respect to small variations of the free data. This kind of stability is of relevance to numerical applications. A prominent instance of the use of the characteristic problem for numerical applications is that of the simulation of gravitational waves by numerically integrating the characteristic formulation of the Einstein equations \cite{1}.

As argued extensively in \cite{2}, characteristic problems for which the estimate can be established may be considered to be \textit{well posed} in the sense that for each set of data the solution exists, is unique and depends continuously on the free data. In addition, characteristic problems that satisfy the algebraic criterion developed here can be thought of as \textit{manifestly well posed}. Manifest well-posedness in the sense defined here is to characteristic problems as symmetric hyperbolicity \cite{2} is to initial-value problems.

Section \textbf{II} describes first-order linear characteristic problems after Duff \cite{2}. The estimates of interest are defined in Section \textbf{III} where the algebraic criterion is developed as well. Concluding remarks are offered in Section \textbf{IV}.

\section*{II. HOMOGENEOUS LINEAR CHARACTERISTIC PROBLEMS IN CANONICAL FORM}

Consider a generic homogeneous hyperbolic system of linear partial differential equations for \( m \) functions \( v = (v^a) \) of \( n \) variables \( y^a \), which can be written in matrix form as follows

\begin{equation}
A^a \frac{\partial v^a}{\partial y^a} + Dv = 0,
\end{equation}

where summation over repeated indices is understood. A characteristic surface \( \mathcal{N} \) is a surface given by \( \phi(y^a) = 0 \) such that

\begin{equation}
\det \left( A^a \frac{\partial \phi}{\partial y^a} \right) = 0.
\end{equation}

Denote by \( m \) the multiplicity of this characteristic surface (so that the rank of \( A^a \partial \phi / \partial y^a \) is \( n - m \)). Suppose \( \mathcal{T} \) given by \( \psi(y^a) = 0 \) is another surface intersecting \( \mathcal{N} \) at a submanifold of dimension \( n - 2 \), whose further properties are to be determined. We choose a suitable coordinate system \((u, x, x^i)\) \( i = 1 \ldots n - 2 \) for \( \mathbb{R}^n \) adapted to these two surfaces; i.e., such that

\begin{align}
u &\equiv \phi(y^a), \quad (3a) \\
x &\equiv \psi(y^a). \quad (3b)
\end{align}
In these coordinates \( v_a \) reads
\[
B^\alpha \partial_\alpha v + B^x \partial_x v + B^i \partial_i v + D v = 0,
\]
with
\[
B^\alpha = A^\alpha \frac{\partial \phi}{\partial y^\alpha},
\]
\[
B^x = A^x \frac{\partial \psi}{\partial y^x},
\]
\[
B^i = A^i \frac{\partial x^i}{\partial y^a}.
\]

By \( \beta \), there are \( m \) linearly independent left null vectors \( \tilde{z}_\nu \) and also \( m \) linearly independent right null vectors \( \tilde{z}_\nu \) (with \( \nu = 1 \ldots m \)) of the matrix \( B^u \), namely
\[
\tilde{z}_\nu B^u = 0,
\]
\[
B^u \tilde{z}_\nu = 0.
\]

We choose the right null vectors to be orthonormal in the sense that
\[
\tilde{z}_\alpha^\alpha \delta_\nu^\mu = \delta_{\mu \nu}.
\]

Multiplying \( \beta \) on the left with \( \tilde{z}_\nu \) we find that of the equations in the system do not involve derivatives with respect to \( u \):
\[
\tilde{z}_\nu B^x \partial_x v + \tilde{z}_\nu B^i \partial_i v + \tilde{z}_\nu D v = 0.
\]

Our aim is now to find a convenient transformation of variables that takes advantage of this split of the equations. We start by noticing that, using the variables that takes advantage of this split of the equation
\[
\tilde{z}_\nu B^u \partial_\nu v = 0,
\]
\[
B^u \tilde{z}_\nu = 0.
\]

We find that
\[
\tilde{z}_\nu B^x \partial_x v + \tilde{z}_\nu B^i \partial_i v + \tilde{z}_\nu D v = 0.
\]

Inverting \( \beta \) we have \( v_\alpha = S_\beta \nu^\nu \), which can be used into \( \gamma \) to obtain a set of equations in the transformed variables:
\[
\tilde{z}_\nu B^x S^t \partial_x v' + \tilde{z}_\nu B^i S^t \partial_i v' + \tilde{z}_\nu D S^t v' = 0.
\]

We’d like for these equations to be solvable for the \( x \)-derivatives of all the variables \( w_\mu \), that is: the ones that do not evolve out of the initial characteristic surface. The first \( m \) terms in each equation for fixed \( \nu \) are
\[
\tilde{z}_\nu B^x S^t v' = \tilde{z}_\nu B^x S^t w_\mu = \tilde{z}_\nu B^x S^t z_\nu^\nu \partial_\nu w_\mu
\]

Thus the set of \( m \) equations \( \beta \) can be solved for the \( m \) variables \( \partial_x w_\mu \) if and only if
\[
\det \{ \tilde{z}_\nu B^x S^t z_\nu^\nu \partial_\nu w_\mu \} \neq 0
\]

This is a restriction on the choice of \( \psi(y^\nu) \). For this restriction to hold it is sufficient, but not necessary, that the level surfaces of \( \psi(y^\nu) \) be non-characteristic. In many applications, the level surfaces of \( \psi \) are chosen to be time-like. For now on we assume that \( \beta \) holds. This allows us to interpret the \( m \) variables \( w_\nu \) as the null variables of the problem.

We have shown that under very weak conditions for the surface \( T \), the most general characteristic problem takes the following form
\[
N^u \partial_a q + N^x \partial_x q + N^i \partial_i v + N^0 v = 0
\]
\[
\partial_x w + L^u \partial_a q + L^i \partial_i v + L^0 v = 0
\]

Clearly the null variables \( w \) can be redefined by \( \hat{w} \equiv w + L^u q \) so that none of the Eqs. \( \beta \) contains \( x \)-derivatives of the normal variables. Additionally, since \( N^u \) is non-singular, we can choose normal variables \( \hat{q} \equiv N^u q \). In terms of these special choices of null and normal variables, Eqs. \( \gamma \) and \( \beta \) assume what is referred to as the canonical form:
\[
\partial_a \hat{q} + \hat{N}^x \partial_x \hat{q} + \hat{N}^i \partial_i \hat{v} + \hat{N}^0 \hat{v} = 0
\]
\[
\partial_x \hat{w} + \hat{L}^u \partial_a \hat{q} + \hat{L}^i \partial_i \hat{v} + \hat{L}^0 \hat{v} = 0
\]

where \( \hat{v} \equiv (\hat{w}, \hat{q}) \). We refer to \( \gamma \) as the evolution equations, and to \( \beta \) as the hypersurface equations. For a unique solution to exist, one must prescribe the values of \( \hat{w} \) on the surface \( x = 0 \) and the values of \( \hat{q} \) on the surface \( u = 0 \). The solution can then be constructed in a hierarchical manner. Since \( q \) is a known source for \( \delta \)
at \( u = 0 \), then \( \hat{w} \) can be found on the entire surface \( u = 0 \). Once \( \hat{w} \) is known at \( u = 0 \), it can be used as a given source for \( \hat{w} \) in order to find the values of the normal variables \( \hat{q} \) on the next surface at \( u = du \). These are then used into \( B \) to obtain \( \hat{w} \) on the surface \( u = du \). And so forth. In fact, Duff proves a theorem of existence and uniqueness of the solution given the canonical form of the characteristic problem.

As an example, consider the following equations for four unknowns \( v \) as functions of four variables \( x^a = (t, x, y, z) \):

\[
\begin{align*}
\partial_t v_1 &= \partial_x v_2 + \partial_y v_3 + \partial_z v_4, \\
\partial_t v_2 &= \partial_x v_1, \\
\partial_t v_3 &= \partial_y v_1, \\
\partial_t v_4 &= \partial_z v_1.
\end{align*}
\]

(19a, 19b, 19c, 19d)

These equations constitute a first-order version of the wave equation in three spatial dimensions (if we interpret the variables \( v_a \) as the derivatives of a single function \( f \)). However, this first-order version of the wave equation has characteristic speeds of \( 0 \) (rest) in addition to \( 1 \) (light). The level surfaces of \( \phi \equiv t - x \) are null planes, so they are characteristic and intersect the surfaces of fixed value of \( x \). We change coordinates \( (t, x, y, z) \rightarrow (u, x, y, z) \) with

\[
u = t - x
\]

(20)

which implies that \( \partial_t \rightarrow \partial_u \) and \( \partial_x \rightarrow \partial_x - \partial_u \). The system turns into

\[
\begin{align*}
\partial_u v^1 + \partial_u v^2 &= \partial_x v^2 + \partial_y v^3 + \partial_z v^4, \\
\partial_u v^2 + \partial_u v^1 &= \partial_x v^1, \\
\partial_u v^3 &= \partial_y v^3, \\
\partial_u v^4 &= \partial_z v^4.
\end{align*}
\]

(21a, 21b, 21c, 21d)

We can read off the matrix \( B^u \):

\[
B^u = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

(22)

The null variable of the problem is \( w = (v_1 - v_2)/\sqrt{2} \) and the normal variables are \( q_a = ((v_1 + v_2)/\sqrt{2}, v_3, v_4) \), in terms of which the system (21) takes the almost canonical form:

\[
\begin{align*}
2\partial_u q_2 - \partial_x q_2 - \frac{1}{\sqrt{2}} \partial_y q_3 - \frac{1}{\sqrt{2}} \partial_z q_4 &= 0, \\
\partial_u q_3 - \frac{1}{\sqrt{2}} \partial_y q_2 - \frac{1}{\sqrt{2}} \partial_z w &= 0, \\
\partial_u q_4 - \frac{1}{\sqrt{2}} \partial_z q_3 - \frac{1}{\sqrt{2}} \partial_z w &= 0, \\
\partial_x w - \frac{1}{\sqrt{2}} \partial_y q_3 - \frac{1}{\sqrt{2}} \partial_z q_4 &= 0.
\end{align*}
\]

(24a, 24b, 24c, 24d)

For a unique solution, we need to prescribe the value of \( w \) on the surface \( x = 0 \), and the values of \( q_2, q_3 \) and \( q_4 \) on the surface \( u = 0 \). Notice that, in this example, the surface \( x = 0 \) is not timelike with respect to the hyperbolic operator \( A^u \), but is also characteristic, as can be seen by inspection of the matrix \( B^x \):

\[
B^x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

(25)

However, we have \( \hat{z} B^x z = 2 \neq 0 \). Therefore the condition is satisfied in spite of the fact that the surfaces of fixed value of \( x \) are not timelike.

### III. WELL-POSEDNESS OF HOMOGENEOUS LINEAR CHARACTERISTIC PROBLEMS

The canonical system (18) can be written in the compact form

\[
C^u \partial_u v + Dv = 0
\]

(26)

where \( C^u \) and \( C^x \) have block-diagonal forms of a special type:

\[
C^u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C^x = \begin{pmatrix} N^x & 0 \\ 0 & 1 \end{pmatrix}
\]

(27)

where \( 1 \) is the identity of dimension \( n - m \) in the case of \( C^u \), and of dimension \( m \) in the case of \( C^x \). The matrix \( N^x \) is square, of dimension \( n - m \), and the various rectangular blocks \( 0 \) are vanishing matrices whose dimensions are clear from the context. Dropping the hats (\( \hat{\} \) ) for ease of notation, the variable \( v \) represents the set of normal variables \( q \) and null variables \( w \) of the characteristic problem in canonical form. Additionally, \( m \) functions \( w_0 \equiv (w_0^u(u, x^i)) \) are given as data on the surface \( T \) and \( n - m \) functions \( q_0 \equiv (q_0^u(x, x^i)) \) are given as data on the surface \( N \).

For the remainder of this section, we make the strong assumption that \( N^x \) and \( C^x \) are symmetric. Multiplication of (26) by \( v \) on the left then leads to a “conservation law” of the form

\[
\partial_a (\nu C^a v) + v R v = 0
\]

(28)
where \( \mathbf{R} \equiv 2\mathbf{D} - \partial_n \mathbf{C} \). We now integrate this conservation
law in an appropriate volume \( V \) of \( \mathbb{R}^n \). Our volume is a “hyperprism” limited by the surface \( u = 0 \) from “below”, the surface \( x = 0 \) on the “left”, and the surface \( u + x = T \), for an arbitrary constant \( T \), on the “top”. We assume there are no boundaries in the remaining co-
ordinate directions, in the sense that the solutions \( v \) will either be periodic functions of \( x' \) or will decay sufficiently fast at large values of \( x' \) in order for the integrals of their
squares to exist. The integration yields

\[
\int_{\Sigma_T} v(C^u + C^x) v \, d\Sigma_T - \int_{\mathcal{N}} (vC^u v) d\mathcal{N} - \int_{T} (vC^x v) \, dT + \int_{V} vRv \, dV = 0. \tag{29a}
\]

Clearly

\[
\int_{\mathcal{N}} vC^u v \, d\mathcal{N} = \int_{\mathcal{N}} \sum_{\nu=m+1}^{n} (q_\nu^0)^2 \, d\mathcal{N} \equiv \|q_0\|^2, \tag{30}
\]

and

\[
\int_{T} vC^x v \, dT = \int_{T} qN^x q \, dT + \int_{T} \sum_{\nu=1}^{m} (w_\nu^0)^2 \, dT = \int_{T} qN^x q \, dT + \|w_0\|^2 \tag{31a}
\]

Thus Eq. \( 29a \) is rearranged to read

\[
\int_{\Sigma_T} v(C^u + C^x) v \, d\Sigma_T = \|q_0\|^2 + \|w_0\|^2 + \int_{T} qN^x q \, dT - \int_{V} vRv \, dV. \tag{32a}
\]

If \( N^x \) is non-positive definite but also such that \( 1 + N^x \) is positive definite, we can define the norm of the solution \( v \) on the surface \( \Sigma_T \) by

\[
\|v\|^2_T \equiv \int_{\Sigma_T} v(C^u + C^x) v \, d\Sigma_T, \tag{33}
\]

and Eq. \( 32a \) implies

\[
\|v\|^2_T \leq \|q_0\|^2 + \|w_0\|^2 - \int_{V} vRv \, dV. \tag{34}
\]

In special case of constant coefficients with no undiffer-
entiated terms, namely \( \mathbf{R} = 0 \), Eq. \( 34 \) takes the form

\[
\|v\|^2_T \leq \|q_0\|^2 + \|w_0\|^2, \tag{35}
\]

which represents an \textit{a priori} estimate of the solution
in terms of the free data. It implies that the “size” of
the solution is controlled by the “size” of the data. We
may interpret it as a statement of well-posedness of the
characteristic problem. Clearly the estimate holds in the
presence of non-constant coefficients and undifferentiated
terms as long as \( \mathbf{R} \) is non-negative definite.

An estimate can still be drawn in the presence of a
negative definite bounded \( \mathbf{R} \), but it is weaker and holds
only for small values of \( T \), as we show next.

Since \( \mathbf{R} \) is negative definite, then

\[
-v\mathbf{R} v \leq r \sum_{\nu=1}^{n} (v_\nu^0)^2 \tag{36}
\]

where \( r = \max(\|R_{ij}\|) \) in the volume \( V \), assuming that
such a number \( r \) exists. On the other hand, since \( C^u + C^x \) is positive definite and symmetric then all its
eigenvalues are positive and we have

\[
v(C^u + C^x) v \geq c \sum_{\nu=1}^{n} (v_\nu^0)^2 \tag{37}
\]

with \( c \) being the smallest eigenvalue of \( C^u + C^x \). This implies

\[
-v\mathbf{R} v \leq \frac{r}{c} v(C^u + C^x) v \tag{38}
\]

Thus

\[
- \int_{V} vRv \, dV \leq \frac{r}{c} \int_{0}^{T} \|v\|^2_T \, dt \tag{39}
\]

where \( \|v\|^2_T \) is the norm of the solution on the surface \( \Sigma_T \)
given by \( u + x = t \) for fixed value of \( t < T \). Thus the
inequality \( 39 \) implies

\[
\|v\|^2_T \leq \|q_0\|^2 + \|w_0\|^2 + \frac{r}{c} \int_{0}^{T} \|v\|^2_T \, dt \tag{40}
\]

Here \( \|q_0\|^2 \) and \( \|w_0\|^2 \) are the norms of the normal and null variables with respect to the surfaces \( \mathcal{N} \) and \( T \) both bounded by the spatial surface at \( u + x = T \). For any value of \( t \leq T \) we can write down the same inequality

\[
\|v\|^2_t \leq \int_{\mathcal{N}_t} \sum_{\nu=1}^{m} (q_\nu^t)^2 \, d\mathcal{N}_t + \int_{T_t} \sum_{\nu=1}^{m} (w_\nu^t)^2 \, dT_t + \frac{r}{c} \int_{0}^{t} \|v\|^2_t \, dt' \tag{41}
\]

where \( \mathcal{N}_t \) and \( T_t \) are the subsets of \( \mathcal{N} \) and \( T \) bounded by \( \Sigma_t \), respectively. Since both integrals indicated are less than the norms \( \|q_0\|^2 \) and \( \|w_0\|^2 \) respectively, this implies

\[
\|v\|^2_t \leq \|q_0\|^2 + \|w_0\|^2 + \frac{r}{c} \int_{0}^{t} \|v\|^2_t \, dt' \tag{42}
\]

Using this inequality recursively into the right-hand side of \( 40 \) we have

\[
\|v\|^2_T \leq \left( 1 + \frac{rT}{c} + \frac{(rT)^2}{2c^2} + \ldots + \frac{(rT)^j}{j!c^j} \right) \times \left( \|q_0\|^2 + \|w_0\|^2 + \frac{r}{c} \int_{0}^{T} \|v\|^2_t \, dt \right) \tag{43}
\]
for any given non-negative integer $j$. In the limit for $j \to \infty$ the sequence in the right-hand side converges if $(rT/c) < 1$, in which case we have

$$\|v\|^2_T \leq e^{(r/c)T}(\|q_0\|^2 + \|w_0\|^2) \quad (44)$$

This is our final estimate for the solution in terms of the free data on the surfaces $\mathcal{N}$ and $\mathcal{T}$. The estimate involves an exponential factor essentially due to the presence of undifferentiated terms. The exponential factor depends on the properties of the system of equations (the principal matrices and the undifferentiated terms), but not on the choice of data. This is analogous to the a priori estimates for Cauchy problems with undifferentiated terms. As usual in such cases, the estimate is useless for large $T$, and, in particular, our proof only guarantees the estimate for $T < c/r$. Perhaps with greater care the estimate can be extended to longer values of $T$.

Because the a priori estimates (44) are independent of the choice of data, we can say that our characteristic problem is well-posed. The conditions under which we are able to derive a priori estimates thus become our criteria for well-posedness of linear homogeneous characteristic problems in canonical form:

i) The principal matrices $C^a$ are symmetric.

ii) The normal block of the principal $x-$matrix, denoted $N^x$, is non-positive definite but such that $1 + N^x$ is positive definite (namely, $-1 < N^x \leq 0$).

There is, clearly, no obstacle in generalizing the construction slightly to the case where the characteristic problem is cast into “almost” canonical form, namely, the case when the principal matrices are

$$C^u = \begin{pmatrix} N^u & 0 \\ 0 & 0 \end{pmatrix}, \quad C^x = \begin{pmatrix} N^x & 0 \\ 0 & 1 \end{pmatrix}, \quad (45)$$

which corresponds to a strictly canonical form up to a transformation of normal variables among themselves. In this case, the criterion is

i) The principal matrices $C^a$ are symmetric and

ii) The normal block of the principal $u-$matrix, denoted $N^u$, is positive definite. The normal block of the principal $x-$matrix, denoted $N^x$, is non-positive definite but such that $N^u + N^x$ is positive definite (namely, $-N^u < N^x \leq 0$).

If i) and ii) hold for a linear homogeneous characteristic problem in “almost” canonical form, then the problem is well posed in the sense that there exist a priori estimates of the kind $\|v\|^2_T \leq e^{KT}(\|q_0\|^2 + \|w_0\|^2)$, where $K$ is a constant independent of the data. This inequality is sufficient to establish the stability of the solutions under small variations of the data. Notice that $-N^u < N^x \leq 0$ is equivalent to the requirement that the surfaces given by $\phi(y^x) + \psi(y^u) = T$ with fixed value of $T$ are spatial with respect to the hyperbolic operator $A^a$, which in turn means that they can be interpreted as the level surfaces of a time function $t(y^a) \equiv \phi(y^x) + \psi(y^u)$.

As an illustration, we can see that the first-order form of the wave equation, Eqs. (24), is well posed. For Eqs. (24) we have

$$N^u = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N^x = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (46)$$

and also

$$C^u = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad C^x = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (47)$$

Therefore all the conditions are satisfied, and the estimates follow. In this case, the estimates are of the form (35) and hold for any chosen $T$ because all the principal matrices have constant coefficients and there are no undifferentiated terms ($R = 0$).

IV. CONCLUDING REMARKS

Characteristic problems for hyperbolic equations are rarely discussed in the literature. In fact, prior to Balean’s work [3, 4, 5], practically nothing was known about the characteristic problem of the simplest hyperbolic equation, that is, the wave equation. Balean discussed how to derive estimates for the solutions of the wave equation in its standard second-order form. Balean’s estimates differ markedly from ours. The estimates for general linear characteristic problems of the first-order that we present here constitute a direct generalization of the estimates that we recently derived for the particular case of solutions of the characteristic problem of the wave equation as a first-order system of PDE’s [3].

The value of the generalization that we present here resides in the formulation of algebraic criteria sufficient for the existence of the a priori estimates. We demonstrate elsewhere [6] that these criteria allow us to formulate the characteristic problem of the linearized Einstein equations in a form that is guaranteed to be well posed.

Several issues of interest remain wide open. First, given a general characteristic problem that is well posed in the sense that we introduce here, it is not at all clear as yet whether estimates of the derivatives of the solution in terms of the derivatives of the data would exist as well. We have succeeded in deriving estimates for the derivatives in the particular case of the characteristic problem of the wave equation [3]. However, the derivation depends strongly on the particular form of the hyperbolic operator of the wave equation, and its generalization to arbitrary characteristic problems is far from straightforward, quite unfortunately.

Secondly, a sufficient criterion to establish well posedness of a characteristic problem is useful, but a necessary criterion would, perhaps, be invaluable as a means to rule
out unstable problems with an eye towards numerical applications.

Thirdly, whether or not all well-posed hyperbolic problems admit well-posed characteristic problems in our sense might well be the most intriguing open question at this time.

Acknowledgments

I am indebted to Roberto Gómez for numerous conversations. This work was supported by the NSF under grants No. PHY-9803301, No. PHY-0070624 and No. PHY-0244752 to Duquesne University.

[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics*, vol. II (Interscience Publishers, New York-London, 1962).
[2] G. F. D. Duff, Can. J. Math. 10, 127 (1958).
[3] S. Frittelli, *Estimates for the characteristic problem of the first-order reduction of the wave equation*, to appear in J. Phys. A, math-ph/0408007.
[4] J. Winicour, Living Reviews in Relativity 4, 3 (2001), http://www.livingreviews.org/.
[5] B. Gustafsson, H.-O. Kreiss, and J. Oliger, *Time-dependent problems and difference methods* (Wiley, New York, 1995).
[6] R. M. Balean, Ph.D. thesis, University of New England, Armidale, NSW Australia (1996).
[7] R. Balean, Commun. PDE 22, 1325 (1997).
[8] R. Balean and R. Bartnik, P. Roy. Soc. Lond. A 454, 2041 (1998).
[9] S. Frittelli, *Well-posed first-order reduction of the characteristic problem of the linearized Einstein equations*, in preparation.