Abstract

I give a basic introduction to precision electroweak analysis, beginning with calculation at tree-level which most simply illustrates the procedure. I then work out the formalism for one-loop corrections to the vector boson self energies (oblique corrections). This is a tractable subset of the complete electroweak program. Not only does this exercise provide an analytically accessible demonstration of the theory involved in electroweak precision analyses, it also teaches students a useful technique to analyze a large class of theories beyond the Standard Model.
1 Symmetries, dynamics and observables

Voltaire said that natural philosophy is about calculating and measuring – almost everything else is chimera. I suppose our three lectures on precision electroweak analysis will be with Voltaire’s blessing, as it involves much calculating and measuring. We focus mostly on the calculating activity in these lectures, and rely on the measuring results produced by our experimental colleagues.

Our discussion starts with observables. We have fancy names for observables, such as “total hadronic cross-section”, and “leptonic partial width” and “effective weak mixing angle”. However, observables quantify very tangible events that happen in nature, albeit we humans forced the action: A lit up when B hit it, etc. That is a rather crude way of thinking about it, but in the end we must remember that measurement is about stuff slapping or pulling or yanking other stuff. Observables are just measurements with the blood wiped off.

Defining an observable in modern elementary particle physics involves sophistication and a fair amount of theoretical knowledge, but we will not get into a chicken and egg discussion. Let us suppose we have a nice collection of well-defined observables \( O_i \), presented within the context of a theory, and we wish to determine if it all makes sense. In other words, we wish to determine if our theory can explain the observables. Finding a theory that matches observables is not hard at all. Give me any set of \( n \) observables \( \{O_i\} \) and I can give you this theory: For every \( O_i \) we posit the reason \( R_i \), which simply states that \( O_i \) is true. This is a sort of “intelligent design” theory that cannot be ruled out as untrue but is clearly unsatisfactory and not useful to modern scientists.

Within particle physics today, we posit that symmetries (relations) and dynamics (strengths) produce nature, and observables are mere manifestations of these qualities. Observables have relations among themselves dictated by the symmetries and dynamics. Nature automatically gets all the relations right, and there is no intermediary needed by nature to calculate results. However, we humans need an intermediate step to understand the relations of observables. We (usually) need a lagrangian field theory to tie the symmetries and dynamics to the observables. The drawing on the next page visualizes this connection.
The symmetries and dynamics that we posit in these lectures are the ones of the Standard Model. At the core of the SM is its gauge groups $SU(3) \times SU(2)_L \times U(1)_Y$ and their corresponding gauge coupling strengths, $g_3$, $g_2$, and $g'$. At first we perform a tree-level analysis of the SM. This will be used to show how the lagrangian is a mere catalyst to relating observables in terms of observables. We will be able to show that the predicted relations at tree level are not satisfied by the data. Our next topic will be to discuss one loop corrections to the tree-level result. We will focus on the interesting subset of vector boson self-energies, which is illustrative of the general method and useful in beyond-the-SM analyses. We then will compute an example of the one-loop corrections due to fermion loops in the vector boson-self energies. We will show that when we write observables in terms of observables, there are no infinities. They cancel out automatically – we do not have to subtract them, they just are not there in physical calculations. Finally, we describe the utility of the techniques that we develop in these few short lectures. It is not all just for pedagogy. There is some direct use to what we say for analysing some theories beyond the Standard Model.
2 Tree-level analysis of SM precision electroweak observables

We will use the notation that observables are written with a hat on top of them. For example, we will denote the measured $Z$ boson mass observable as $\hat{m}_Z$. The observables that we are primarily interested in are $\hat{\alpha}$ (from Thomson limit of $\gamma^* \rightarrow e^+ e^-$ scattering), $\hat{G}_F$ (from muon decay), $\hat{m}_Z$ ($Z$ boson mass), $\hat{m}_W$ ($W$ boson mass), $\hat{\Gamma}_{l^+ l^-}$ (leptonic partial width of the $Z$ boson), and $s_{\text{eff}}^2$ (effective $\sin^2 \theta_W$). The value of $s_{\text{eff}}^2$ is defined to be the all-orders rewriting of $\hat{A}_{LR} (= A_e)$ as

$$\hat{A}_{LR} \equiv \frac{(1/2 - s_{\text{eff}}^2)^2 - s_{\text{eff}}^4}{(1/2 - s_{\text{eff}}^2)^2 + s_{\text{eff}}^4}.$$  

The measured values of all these observables\[1, 2\] are

\begin{align*}
\hat{\alpha} & = 1/137.0359895(61) \quad (2) \\
\hat{G}_F & = 1.16639(1) \times 10^{-5} \text{GeV}^{-2} \quad (3) \\
\hat{m}_Z & = 91.1875 \pm 0.0021 \text{GeV} \quad (4) \\
\hat{m}_W & = 80.426 \pm 0.034 \text{GeV} \quad (5) \\
\hat{s}_{\text{eff}}^2 & = 0.23150 \pm 0.00016 \quad (6) \\
\hat{\Gamma}_{l^+ l^-} & = 83.984 \pm 0.086 \text{MeV} \quad (7)
\end{align*}

At tree level we need only three lagrangian parameters to compute the six observables listed above. The three parameters are $g$ ($SU(2)$ gauge coupling), $g'$ ($U(1)_Y$ gauge coupling) and $v$ (Higgs vacuum expectation value). In anticipation of the convenience we will wish upon our one-loop discussion later, we cash in these three parameters for an equivalent set $e$, $s (= \sin \theta)$, and $v$, where $g = e/s$ and $g' = e/c$.

We can now compute all the observables in terms of these three lagrangian parameters:

\begin{align*}
\hat{\alpha} & = \frac{e^2}{4\pi} \quad (8) \\
\hat{G}_F & = \frac{1}{\sqrt{2}v^2} \quad (9) \\
\hat{m}_Z^2 & = \frac{e^2 v^2}{4s^2 c^2} \quad (10) \\
\hat{m}_W^2 & = \frac{e^2 v^2}{4s^2} \quad (11)
\end{align*}
\[ s_{\text{eff}}^2 = s^2 \quad (12) \]
\[ \Gamma_{l^+l^-} = \frac{v \ e^3}{96\pi \ s^3 c^3} \left( \left( -\frac{1}{2} + 2s^2 \right)^2 + \frac{1}{4} \right) \quad (13) \]

The equations above have measurements on the LHS and theory computations in terms of lagrangian parameters on the RHS. Staring at these equations enables us to scoff at questions like, *Does the SM predict the correct W mass?* Well, yes it does, if that’s the only observable we care about. We have an infinite number of ways (choices of \(e, s, \) and \(v\)) to reproduce the \(W\) mass. The real question that a theory must answer is, *Can we reproduce all experimental results with suitable choices of our input parameters?* This is a serious question requiring analysis.

The standard way to test the ability of a theory to reproduce data is via the \(\chi^2\) analysis. We have a set of observables \(\hat{O}_i^{\text{expt}}\) with uncertainties \(\Delta \hat{O}_i^{\text{expt}}\). The theory makes predictions \(O_i^{\text{th}}\) for the observables that depend on the lagrangian parameters. We find the best possible choices of the lagrangian parameters that fit the data by minimizing the \(\chi^2\) function

\[ \chi^2(e, s, v) = \sum_i \frac{(\hat{O}_i^{\text{expt}} - O_i^{\text{th}}(e, s, v))^2}{(\Delta \hat{O}_i^{\text{expt}})^2} \quad (14) \]

where \(i\) sums over the observables \(\hat{m}_W, s_{\text{eff}}^2, \) etc. A good discussion of how to interpret the statistics of the \(\chi^2\) distribution can be found in the PDG[1].

In our list of six observables, three of them are measured extraordinarily well: \(\hat{\alpha}, \hat{G}_F\) and \(\hat{m}_Z\). We can get a feel for how well the tree-level SM predictions match data by fixing the lagrangian parameters \(e, s\) and \(v\) in terms of these three observables, and then writing the remaining observables in terms of the \(\hat{\alpha}, \hat{G}_F\) and \(\hat{m}_Z\).

With simple algebra we find that

\[ e^2 = 4\pi \hat{\alpha} \quad (15) \]
\[ v^2 = \frac{\hat{G}_F^{-1}}{\sqrt{2}} \quad (16) \]
\[ s^2 c^2 = \frac{\pi \hat{\alpha}}{\sqrt{2}G_F \hat{m}_Z^2} \quad (17) \]

The last equation is equivalent to

\[ s^2 = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\hat{x}} \quad \text{where} \quad \hat{x} = \frac{\pi \hat{\alpha}}{\sqrt{2}G_F \hat{m}_Z^2} \quad (18) \]
We can now write the observables \( \hat{m}_W \), \( s^2_{\text{eff}} \) and \( \hat{\Gamma}_{l+l-} \) in terms of \( \hat{\alpha} \), \( \hat{m}_Z \) and \( \hat{G}_F \):

\[
\hat{m}_W^2 = \pi \sqrt{2} \hat{G}_F^{-1} \hat{\alpha} \left( 1 - \sqrt{1 - \frac{4\pi \hat{\alpha}}{\sqrt{2} \hat{G}_F \hat{m}_Z^2}} \right)^{-1} \quad (19)
\]

\[
s^2_{\text{eff}} = \frac{1}{2} - \frac{1}{2} \left[ 1 - \frac{4\pi \hat{\alpha}}{\sqrt{2} \hat{G}_F \hat{m}_Z^2} \right] \quad (20)
\]

\[
\hat{\Gamma}_{l+l-} = \frac{\sqrt{2} \hat{G}_F \hat{m}_Z^3}{12\pi} \left\{ \left( \frac{1}{2} - \sqrt{1 - \frac{4\pi \hat{\alpha}}{\sqrt{2} \hat{G}_F \hat{m}_Z^2}} \right)^2 + \frac{1}{4} \right\} \quad (21)
\]

If we plug in the very precisely known experimental values for \( \hat{\alpha} \), \( \hat{m}_Z \) and \( \hat{G}_F \), we find predictions for \( \hat{m}_W \), \( s^2_{\text{eff}} \) and \( \hat{\Gamma}_{l+l-} \):

\[
\text{Prediction of } \hat{m}_W = 80.939 \pm 0.003 \text{ GeV} \quad (22)
\]

\[
\text{Prediction of } s^2_{\text{eff}} = 0.21215 \pm 0.00003 \quad (23)
\]

\[
\text{Prediction of } \hat{\Gamma}_{l+l-} = 84.843 \pm 0.012 \text{ MeV} \quad (24)
\]

The predictions of \( \hat{m}_W \), \( s^2_{\text{eff}} \) and \( \hat{\Gamma}_{l+l-} \) in this particular tree-level procedure are approximately 15\( \sigma \), 120\( \sigma \) and 10\( \sigma \) off from their experimentally measured values. Statistically speaking, these are unacceptably large deviations of the theory from the experiment. We therefore conclude that the theory is not compatible with experiment.

However, we have only worked up to tree-level in the perturbative expansion of the theory. We must go to higher-order in the coupling constants to truly test the viability of the SM when confronting all the experimental data. This analysis has been applied to the Standard Model, and one finds that it is compatible with the precision electroweak data provided the Higgs boson mass is between about 114 GeV (direct bound) and 219 GeV (95\% C.L. upper bound)\[2\].

In the following sections we will consider how one-loop self-energies slightly alter the relationship between the lagrangian parameters and measured observables. In other words, the relationships among observables are slightly different than what we found above doing a tree-level analysis, and the theory predictions come closer to the experimental measurements.
3 One-loop self-energy corrections

In this lecture we focus on the class of corrections that arise solely from the self-energy corrections of the $\gamma$, $W^\pm$, and $Z$ vector bosons. Restricting our analysis to this class of corrections enables us to do something complete and meaningful in the short time we have together. A full-scale renormalization of the SM with all corrections explicitly calculated is a significantly more time-consuming project without significantly enhancing the conceptual learning. Furthermore, many of the most interesting ideas of physics beyond the SM require only analysing self-energy corrections to the vector bosons. For example, additional exotic states that do not couple directly to the SM fermions but have charges under the SM gauge symmetries qualify to be analysed in this manner.

Even in beyond-the-SM theories which have exotic states that do interact with the external fermions involved in precision electroweak analysis, it is most common that the non-oblique corrections have a small effect compared to the oblique corrections. This is generally true in supersymmetry, with the notable exception of the $Z \rightarrow b\bar{b}$ coupling that participates in $R_b$, $A_b$ and $A_{FB}^b$, which can (but generically does not) have large vertex corrections due to superpartners. One main reason for the dominance of oblique corrections over non-oblique corrections is that any charged object couples to the vector bosons, whereas usually only one or two particles in a theory couple to a specific fermion species. In other words, the summing over all contributors in self-energies wins out over the one or two diagrams that couple to an individual final state fermion.

To begin our analysis we stipulate that the lagrangian is the SM lagrangian and the couplings that affect the precision electroweak observables $s^2_{\text{eff}}$, $\hat{m}_W$, and $\hat{\Gamma}_{l+l-}$ are $\{e, s^2, v^2\}$. The relevant Feynman rules for our analysis are
By convention the one-loop corrections to the vector boson self-energies

\[ i [\Pi_{VV'}(q^2) g^{\mu\nu} - \Delta_{VV'}(q^2) q^\mu q^\nu]. \]  

(25)

Only the \( \Pi_{VV'} \) piece of the self-energies matters for our analysis since the \( q^\mu \) part of the second term is dotted into a light-fermion current and is zero by the Dirac equation, since the corresponding fermion masses is well-approximated to be zero:

\[ q^\mu J_\mu^{\text{light fermion}} \rightarrow \bar{f} \gamma^\mu q_\mu f \rightarrow \bar{f} m f \rightarrow 0. \]  

(26)

The way the self-energies are defined, they add to the vector boson masses by convention:

\[ m_V^2 \rightarrow m_V^2 + \Pi_{VV}(q^2 = m_V^2) \]  

(27)

Because the photon is massless we know that \( \Pi_{\gamma\gamma}(0) = 0 \) and \( \Pi_{\gamma Z}(0) = 0 \), and so we do not have to compute them. There is one subtlety to keep in mind. \( \Pi_{\gamma Z}(0) \) is not zero when the \( W^{\pm} \) bosons is included in the loop. The procedure that we outline
below gets slightly more complicated when we take that into account, and the details of that procedure can be found, for example, in several studies\[3, 4\]. However, this is special to the \( W^\pm \) bosons (gauge degree of freedom partners of the \( W^3 \)). In new physics scenarios (e.g., supersymmetry) there are no additional one-loop contributions to \( \Pi_{\gamma Z}(0) \), and it is usually appropriate in analyses of beyond-the-SM contributions to precision EW observables to ignore it.

### 3.1 Theoretical predictions for observables at one loop

The computation of the \( Z \) and \( W \) masses is straightforward. The resulting theoretical prediction of \( m_Z \) and \( m_W \) in terms of the lagrangian parameters and the one-loop self-energy corrections is

\[
(\hat{m}_Z)^{\text{th}} = \frac{e^2 v^2}{4 s^2 c^2} + \Pi_{ZZ}(m_Z^2) \\
(\hat{m}_W)^{\text{th}} = \frac{e^2 v^2}{4 s^2} + \Pi_{WW}(m_W^2)
\]

We next compute the theory prediction for \( \alpha \). It sounds odd to use the words “theory prediction of \( \alpha \)” since we often are sloppy in our wording (or thinking) and view \( \alpha \) as just a coupling. In reality, it is an observable defined in the Thomson limit of Compton scattering and probes the Coulomb potential at \( q^2 \to 0 \):

\[
\begin{align*}
- i \left. \frac{4 \pi \hat{\alpha}}{q^2} \right|_{q^2 \to 0} &= -i \frac{e^2}{q^2} \left. \left[ 1 + \frac{\Pi_{\gamma\gamma}(q^2)}{q^2} \right] \right|_{q^2 \to 0} \\
\text{If we define} & \quad \Pi_{\gamma\gamma}'(0) = \lim_{q^2 \to 0} \frac{\Pi_{\gamma\gamma}(q^2)}{q^2} \\
\text{then we can write the theory prediction for} \ \alpha \ \text{as} & \quad (\hat{\alpha})^{\text{th}} = \frac{e^2}{4 \pi} \left( 1 + \Pi_{\gamma\gamma}'(0) \right)
\end{align*}
\]

The muon decay observable \( \hat{G}_F \) is computed from the lifetime of the muon
which is proportional to $\hat{G}_F/\sqrt{2}$. This amplitude is then used to compute the muon lifetime

$$\tau_\mu^{-1} = \frac{\hat{G}_F^2 m_\mu^5}{192\pi^3} K(\alpha, m_e, m_\mu, m_W) \quad (33)$$

where the function $K$ is mainly a kinematics function and can be obtained from the electroweak chapter in the PDG [1]. The theory prediction for $\hat{G}_F$ is

$$\left(\frac{\hat{G}_F}{\sqrt{2}}\right)^{\text{th}} = \frac{g^2}{8 m_W^2} \left[ 1 + i \Pi_{WW}(q^2) \left( \frac{-i}{q^2 - m_W^2} \right) \right]_{q \to 0} = \frac{1}{2v^2} \left[ 1 - \frac{\Pi_{WW}(0)}{m_W^2} \right]. \quad (34)$$

The observable associated with $\hat{s}_{\text{eff}}^2$ is a little trickier than the other ones. For one, there are many different types of $\hat{s}_{\text{eff}}^2$ observables, depending on the final state fermion. We will define $\hat{s}_{\text{eff}}^2$ to be the observable associated with the left-right asymmetry of $Z$ decays to leptons. We assume universality of the leptons. The left-right asymmetry is defined to be the $Z$-pole production cross-section asymmetry of leptons produced from left polarized electron-positron collisions versus those produced from right polarized collisions,

$$A_{LR}^L = \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R} = \frac{c_L^2 - c_R^2}{c_L^2 + c_R^2} \quad (35)$$

where at tree-level the $c_L$ and $c_R$ couplings are defined by

$$Z_\mu \rightarrow f \bar{f} \quad i\gamma_\mu (c_L P_L + c_R P_R)$$

and

$$c_L = \frac{e}{s_c} (T^3 - Qs^2) \quad \text{and} \quad c_R = -\frac{eQs^2}{s_c} \quad (36)$$
The definition of $s_{\text{eff}}^2$ is chosen such that observable $\hat{A}_{LR}^1$ is written in terms of $\hat{s}_{\text{eff}}^2$ using the tree-level expression above with $s^2 \rightarrow \hat{s}_{\text{eff}}^2$. This is an unambiguous definition since the charges $Q$ and $T^3$ do not get renormalized. This definition of $s_{\text{eff}}^2$ will become clearer below as we compute it at one loop.

At this point, we need to compute the one-loop shifts in $c_L$ and $c_R$. We can neglect all $\Pi_{ZZ}$ contributions since they will only affect the overall factor of $c_L$ and $c_R$ which cancels. On the other hand, the $Z - A$ mixing self-energy does contribute to the $c_L$ and $c_R$ couplings:

$$c_L = \frac{e}{s c} (T^3 - Q s^2) + i \Pi_{\gamma Z}(m_Z^2) \left( \frac{-i}{m_Z^2} \right) (eQ)$$ (37)

$$= \frac{e}{s c} \left[ T^3 - Q \left( s^2 - s c \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} \right) \right]$$ (38)

$$c_R = \frac{-e Q s^2}{s c} + i \Pi_{\gamma Z}(m_Z^2) \left( \frac{-i}{m_Z^2} \right) (eQ)$$ (39)

$$= \frac{-e Q}{s c} \left[ s^2 - s c \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} \right].$$ (40)

The above $c_L$ and $c_R$ expressions are exactly the same as the tree-level expressions except $s^2 \rightarrow s^2 - s c \Pi_{\gamma Z}(m_Z^2)/m_Z^2$ in the numerator. Thus, at the $Z$-pole

$$(s_{\text{eff}}^2)^{\text{th}} = s^2 - s c \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}$$ (41)

where $\hat{A}_{LR} = \frac{(1/2 - \hat{s}_{\text{eff}}^2)^2 - (\hat{s}_{\text{eff}}^2)^2}{(1/2 - \hat{s}_{\text{eff}}^2)^2 + (\hat{s}_{\text{eff}}^2)^2}$. (42)

Now we compute $\hat{\Gamma}_{l^+l^-}$ from
The theoretical prediction for this observable in terms of independent lagrangian parameters and one-loop self-energies is

\[
(\hat{\Gamma}_{l+l}^{\text{th}})^{\mu} = \frac{Z_Z}{48\pi s^2 c^2} m_Z \left[ \left( \frac{1}{2} + 2(\hat{s}^2_{\text{eff}})^{\text{th}} \right)^2 + \frac{1}{4} \right]
\]  

(43)

Recall that \(\Pi_{\gamma Z}\) had the effect of just putting \(s^2 \rightarrow (\hat{s}^2_{\text{eff}})^{\text{th}}\) into the numerator of the \(c_L\) and \(c_R\) expressions. The \(\hat{m}_Z\) comes as a kinematical phase space mass of the \(Z\) decay.

Since we are computing a partial width and not a ratio of couplings, the \(\Pi_{ZZ}\) contribution must now be taken into account. The parameter \(Z_Z\) in the \((\hat{\Gamma}_{l+l}^{\text{th}})^{\mu}\) expression results from this contribution. It is a wavefunction residue piece. To compute this contribution we first must recognize that the \(\Pi_{ZZ}(q^2)\) self-energy when resummed affects the \(Z\) boson propagator in a simple way

\[
\text{Resummed Propagator} \quad \rightarrow \quad P_Z^{\mu\nu}(q^2) = \frac{-ig^{\mu\nu}}{q^2 - m_Z^2 - \Pi_{ZZ}(q^2)}.
\]  

(44)

But,

\[
\Pi_{ZZ}(q^2) = \Pi_{ZZ}(m_{\text{phys}}^2) + \Pi'_{ZZ}(m_{\text{phys}}^2)(q^2 - m_{\text{phys}}^2) + \cdots
\]  

(45)

where \(m_{\text{phys}}^2\) is really just the physical \(Z\) mass, \(\hat{m}_Z\) (I am writing \(m_{\text{phys}}\) here just for emphasis). The mass of the \(Z\) is defined to be the position of the real part of the pole of the propagator. From that definition and the expansion given above, we find that in the neighborhood of \(q^2 = m_{\text{phys}}^2\)

\[
q^2 - m_Z^2 - \Pi_{ZZ}(q^2) = q^2 - m_Z^2 - \Pi_{ZZ}(m_{\text{phys}}^2) - \Pi'_{ZZ}(m_{\text{phys}}^2)(q^2 - m_{\text{phys}}^2) + \cdots
\]  

\[
= (q^2 - m_{\text{phys}}^2)(1 - \Pi'_{ZZ}(m_{\text{phys}}^2)) + \cdots
\]  

(46)

Therefore, in the neighborhood of \(q^2 = m_{\text{phys}}^2\) the \(Z\) propagator can be written as

\[
\frac{-ig^{\mu\nu}}{(q^2 - m_{\text{phys}}^2)(1 - \Pi'_{ZZ}(m_{\text{phys}}^2))} = \frac{-iZ_Z g^{\mu\nu}}{(q^2 - m_{\text{phys}}^2)}
\]  

(47)
where
\[ Z_Z = 1 + \Pi'_{ZZ}(m_Z) + \text{higher order terms} \quad (48) \]

A standard approximation for \( \Pi'_{ZZ}(m_Z^2) \) is
\[ \Pi'_{ZZ}(m_Z^2) = \frac{\Pi_{ZZ}(m_Z^2) - \Pi_{ZZ}(0)}{m_Z^2} \quad (49) \]

This is a good approximation for many scenarios, and we will employ it hereafter just so we can match up with results published by others. However, I would like to emphasize that there is no reason why one needs to use this approximation, especially since there are now many good numerical and analytic tools to evaluate the one-loop self-energies. Sometimes we will also utilize the variable \( \delta_Z \) which is defined as
\[ Z_Z = 1 + \delta_Z, \]
where
\[ \delta_Z = \Pi'_{ZZ}(m_Z^2) \simeq \frac{\Pi_{ZZ}(m_Z^2) - \Pi_{ZZ}(0)}{m_Z^2} = \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} \quad (50) \]

### 3.2 Observables in terms of observables

At this point we have written all of our observables in terms of lagrangian parameters and \( \Pi \) functions (one-loop corrections). We now wish to do some analytical inversions of these expressions and compute observables in terms of other observables, similar to what we did in the tree-level analysis at the beginning of these lectures. There is no need for us to do this in principle. We are perfectly set now to compute the one-loop self-energies in our favorite theory and then try to fit the lagrangian parameters in a total \( \chi^2 \) analysis. Indeed, a full renormalization of the SM or any other theory of equivalent complexity is virtually impossible to analytically invert in order to write observables in terms of observables. However, we can do it here, by virtue of the relatively noncomplex nature of the one loop self-energies. Furthermore, as emphasized at the beginning, I wish to do this for pedagogical reasons, to show that one role of theories is to be able to express observables in terms of other observables. This knowledge may give one a different perspective about the infinities that supposedly afflict our theories.

Before we do those calculations, we need to say a few more things about the \( \hat{\alpha} \) observable. It is an unusual observable among our list, because it is obviously
incalculable. Recall from before that we found
\[ e^2 = \frac{4\pi\hat{\alpha}}{1 + \Pi'_{\gamma\gamma}(0)} \]  

(51)

The problem is with \( \Pi'_{\gamma\gamma}(0) \), which requires us to know the result of the photon self energy as \( q^2 \to 0 \):

\[ \begin{array}{c}
A_\mu \\
\text{had} \\
A_\mu \\
q^2 \to 0
\end{array} \]

Of course we know from the beginning of this section that
\[ \Pi_{\gamma\gamma}(q^2) \to q^2 B \text{ as } q^2 \to 0, \]

(52)

where \( B \) is some constant. There is no reason for \( B \) to be zero, and so there is no reason for the derivative of the self-energy \( \Pi'_{\gamma\gamma}(0) \to B \) to be zero. Unfortunately, however, it is not calculable.

The incalculability of \( \Pi'_{\gamma\gamma}(0) \) threatens to derail our precision electroweak analysis. However, it has been known for some time now that we can get at this value by using a combination of theory tricks and experimental data. The first thing we do is to rewrite \( \Pi'_{\gamma\gamma}(0) \) by adding and subtracting the self-energy at the higher scale \( q^2 = m_Z^2 \):

\[ \Pi'_{\gamma\gamma}(0) = \text{Re} \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \left[ \text{Re} \frac{\Pi_{\gamma\gamma}}{m_Z^2} - \Pi'_{\gamma\gamma}(0) \right] \]  

(53)

The first term is calculable as computations are done at the scale \( q^2 = m_Z^2 \) where all interactions are perturbative in the SM. The two terms in the bracket are not calculable, but we will give it a name \( \Delta\alpha(m_Z) \). There are three main contributions to \( \Delta\alpha(m_Z) \):

\[ \Delta\alpha(m_Z) = \Delta\alpha_l(m_Z) + \Delta\alpha_{\text{top}}(m_Z) + \Delta\alpha_{\text{had}}^{(5)}(m_Z) \]  

(54)

where

\[ \begin{align*}
\Delta\alpha_l(m_Z) &= 0.03150 \text{ with essentially no error} \\
\Delta\alpha_{\text{top}}(m_Z) &= -0.0007(1) \text{ } m_t \text{ dependent but negligible} \\
\Delta\alpha_{\text{had}}^{(5)} &= \text{incalculable light hadrons contributions}
\end{align*} \]
Fortunately, there is a way to measure $\Delta \alpha^{(5)}_{\text{had}}$. From the optical theorem and the methods of analytic continuation, one finds that

$$
\Delta \alpha^{(5)}_{\text{had}} = -\frac{m_Z^2}{3\pi} \int_{4m_Z^2}^{\infty} \frac{R_{\text{had}}(q^2) dq^2}{q^2(q^2 - m_Z^2)} \quad \text{where} \quad R_{\text{had}}(q^2) = \frac{\sigma_{\text{had}}(q^2)}{\sigma_{t+t}(q^2)}.
$$

(55)

Therefore, to get a numerical value for $\Delta \alpha^{(5)}_{\text{had}}$ one must integrate over the experimental hadronic cross-section over a wide energy range. As soon as $q^2$ is significantly above $\Lambda_{\text{QCD}}$ the theoretical cross-section can be used without concern. However, for lower $q^2$ (lower than about 5 GeV in practice), only the experimental data can be used. There are numerous experiments that contribute data for this integral in differing energy bins, and it is a challenge to understand all the systematics and statistical errors that go into the final number for $\Delta \alpha^{(5)}_{\text{had}}$. Many groups have gone through this difficult exercise and there are many different values obtained. The one the LEP Electroweak Working Group has been using is by Burkhardt and Pietrzyk[5], who conclude that

$$
\Delta \alpha^{(5)}_{\text{had}} = 0.02761 \pm 0.0036.
$$

(56)

We will now trade in the incalculable $\hat{\alpha}$ for the calculable/measured $\hat{\alpha}(m_Z)$, which is related to the lagrangian parameters and $\Pi$’s by

$$
\hat{\alpha}(m_Z) = \frac{\hat{\alpha}}{1 - \Delta \alpha(m_Z)} = \frac{e^2}{4\pi \left[ 1 + \frac{\Pi_{\gamma\gamma}(m_Z)}{m_Z^2} \right]}.
$$

(57)

Always remember, $\hat{\alpha}(m_Z)$ is an observable, which is a meaningful combination of many different experiments (Thomson scattering cross-section plus integration over $R_{\text{had}}(q^2)$), and its experimental value is

$$
\frac{1}{\hat{\alpha}(m_Z)} = 128.936 \pm 0.046.
$$

(58)

As for determining $v^2$ from observables, we can get it directly and simply from the $\hat{G}_F$ equation

$$
v^2 = \frac{1}{\sqrt{2} \hat{G}_F} \left[ 1 - \frac{\Pi_{WW}(0)}{m_W^2} \right].
$$

(59)

At this point we have $e^2$ and $v^2$ in terms of $\hat{\alpha}(m_Z)$, $\hat{m}_Z$ and $\hat{G}_F$, but we still do not have the lagrangian parameter $s^2$ in terms of those three key observables. To do this, we need to go to the theory prediction equation for $\hat{m}_Z$ and solve for $s^2$.

$$
\hat{m}_Z^2 = -\frac{e^2}{4s^2c^2}v^2 + \Pi_{ZZ}(m_Z^2) \quad \rightarrow \quad s^2c^2 = \frac{e^2v^2}{4 \left[ \frac{1}{\hat{m}_Z^2} - \frac{1}{\Pi_{ZZ}(m_Z^2)} \right]}.
$$

(60)
After plugging in our previously obtained expressions for $e^2$ and $v^2$ in terms of observables we get after some algebra

$$s^2c^2 = \frac{\pi \hat{\alpha}(\hat{m}_Z^2)}{\sqrt{2}G_F\hat{m}_Z^2}(1 + \delta_S) \tag{61}$$

where

$$\delta_S = \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2}. \tag{62}$$

A convenient definition that I will sometimes use is

$$s_0^2c_0^2 = \frac{\pi \hat{\alpha}(\hat{m}_Z^2)}{\sqrt{2}G_F\hat{m}_Z^2}. \tag{63}$$

With this definition

$$s^2 = s_0^2 + c_0^2 \frac{s_0^2c_0^2}{c_0^2 - s_0^2} \delta_S. \tag{64}$$

We now have expressions for each of the lagrangian parameters in terms of the three exceptionally well-measured observables $\{\hat{m}_Z, \hat{\alpha}(m_Z), \hat{G}_F\}$ and the self-energy correction $\Pi$’s and are ready to directly compute the theoretical prediction for each of the remaining observables. After some more algebra, which the student should do himself/herself, here are the answers:

$$\langle \hat{m}_W \rangle_{\text{th}} = \frac{\pi \hat{\alpha}(\hat{m}_Z^2)}{\sqrt{2}G_Fs_0^2} \left[ 1 - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \frac{c_0^2}{c_0^2 - s_0^2} \delta_S - \frac{\Pi_{WW}(0)}{m_W^2} + \frac{\Pi_{WW}(m_W^2)}{m_W^2} \right] \tag{65}$$

$$\langle s_{\text{eff}}^2 \rangle_{\text{th}} = s_0^2 + \frac{s_0^2c_0^2}{c_0^2 - s_0^2} \left[ \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{WW}(0)}{m_W^2} - \frac{(c_0^2 - s_0^2)}{s_0c_0} \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} \right] \tag{66}$$

$$\langle \hat{\Gamma}_{l\rightarrow}\rangle_{\text{th}} = \hat{\Gamma}_{l\rightarrow}^0 \left[ 1 - \frac{as_0^2c_0^2}{c_0^2 - s_0^2} \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} + \left( 1 + \frac{as_0^2c_0^2}{c_0^2 - s_0^2} \right) \frac{\Pi_{WW}(0)}{m_W^2} \right. \nonumber \\
+ \left. as_0c_0 \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} + a \frac{s_0^2c_0^2}{c_0^2 - s_0^2} \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} \right] \tag{67}$$

where

$$a = \frac{-8(-1 + 4s_0^2)}{(-1 + 4s_0^2)^2 + 1} \simeq 0.636. \tag{68}$$
3.3 Summary of results

In summary, the theoretical predictions for $\hat{s}_{\text{eff}}^2$, $\hat{m}_W$ and $\hat{\Gamma}_{l+l-}$ can be rewritten as

\[
(\hat{s}_{\text{eff}}^2)^{\text{th}} = s_0^2 - (0.328) \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - (0.421) \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}
\]
\[
-(0.328) \frac{\Pi_{WW}(0)}{m_W^2} + (0.328) \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2}
\]

\[
(\hat{m}_W)^{\text{th}} = \hat{m}_W^0 + (17.0 \text{ GeV}) \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} + (17.0 \text{ GeV}) \frac{\Pi_{WW}(0)}{m_W^2}
\]
\[
 + (40.0 \text{ GeV}) \frac{\Pi_{WW}(m_W^2)}{m_W^2} - (57.1 \text{ GeV}) \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2}
\]

\[
(\hat{\Gamma}_{l+l-})^{\text{th}} = \hat{\Gamma}_{l+l-}^0 + (17.5 \text{ MeV}) \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} + (22.5 \text{ MeV}) \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2}
\]
\[
 + (101 \text{ MeV}) \frac{\Pi_{WW}(0)}{m_W^2} - (83.9 \text{ MeV}) \frac{\Pi_{ZZ}(0)}{m_Z^2} - (17.5 \text{ MeV}) \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2}
\]

where

\[
\hat{c}_0^2 \hat{s}_0^2 = \frac{\pi \hat{\alpha}(m_Z^2)}{\sqrt{2} G_F \hat{m}_Z}
\]

\[
(\hat{m}_W^0)^2 = \frac{\pi \hat{\alpha}(m_Z^2)}{\sqrt{2} G_F \hat{s}_0^2}
\]

\[
\hat{\Gamma}_{l+l-}^0 = \frac{\hat{\alpha}(m_Z^2) \hat{m}_Z}{12 \hat{s}_0^2 \hat{c}_0^2} \left[ \left( -\frac{1}{2} + 2 \hat{s}_0^2 \right)^2 + \frac{1}{4} \right]
\]

with

\[
1/\hat{\alpha}(m_Z^2) = 128.936 \pm 0.046
\]

\[
\hat{G}_F = 1.16639(1) \times 10^{-5} \text{ GeV}^{-2}
\]

\[
\hat{m}_Z = 91.1875 \pm 0.0021 \text{ GeV}
\]

3.4 Connection of our results to the STU formalism

A convenient parametrization of one-loop oblique corrections is given by the STU formalism[4]. In the limit that the new physics scales are much larger than $m_Z$ one
finds that the oblique corrections to all Z-pole observables are expressable in terms of just three universal parameters, $S$, $T$ and $U$. The reason why we need $m_{\text{new}} \gg m_Z$ is because $S$, $T$ and $U$ are valid only in the approximation that all derivatives of self-energies can be Taylor expanded to leading order in $m_Z/m_{\text{new}} \ll 1$. (If the masses of the new particles are close to $m_Z$, the STU parameters can be augmented by additional parameters[7] to match the full one-loop results.)

In terms of the self-energy $\Pi$’s, the $S$, $T$ and $U$ parameters are

$$S = \frac{\alpha}{4s^2} \left[ c^2 \left( \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} - \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} \right) \right. \\
- \left. \frac{c}{s} (c^2 - s^2) \left( \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} - \frac{\Pi_{\gamma Z}(0)}{m_Z^2} \right) \right]$$

(78)

$$T = \frac{1}{\alpha} \left[ \frac{\Pi_{WW}(0)}{m_W^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} - \frac{2s}{c} \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} \right]$$

(79)

$$U = \frac{\alpha}{4s^2} \left[ \frac{\Pi_{WW}(m_W^2)}{m_W^2} - \frac{\Pi_{WW}(0)}{m_W^2} - c^2 \left( \frac{\Pi_{ZZ}(m_Z^2)}{m_Z^2} - \frac{\Pi_{ZZ}(0)}{m_Z^2} \right) \right. \\
- \left. s^2 \frac{\Pi_{\gamma\gamma}(m_Z^2)}{m_Z^2} - 2sc \left( \frac{\Pi_{\gamma Z}(m_Z^2)}{m_Z^2} - \frac{\Pi_{\gamma Z}(0)}{m_Z^2} \right) \right]$$

(80)

The shifts in the observables $\hat{s}_{\text{eff}}^2$, $\hat{m}_W$ and $\hat{\Gamma}_{l^+l^-}$ can all be written as expansions in $S$, $T$ and $U$:

$$\Delta(\hat{s}_{\text{eff}}^2)_{\text{th}} = (3.59 \times 10^{-3}) S - (2.54 \times 10^{-3}) T$$

(81)

$$\Delta \left( \frac{\hat{m}_W}{m_Z} \right)_{\text{th}} = -(3.15 \times 10^{-3}) S + (4.86 \times 10^{-3}) T + (3.70 \times 10^{-3}) U$$

(82)

$$\Delta(\hat{\Gamma}_{l^+l^-})_{\text{th}} = -(1.91 \times 10^{-4}) S + (7.83 \times 10^{-4}) T$$

(83)

If we plug in the expressions of $S$, $T$ and $U$ into these above equations we will find that the result is equivalent to the expressions given by eqs. 69-72 with $\Pi_{\gamma Z}(0) = 0$. 17
4 Cancellation of infinities

If we think in terms of observables only, there is no issue with infinities. Infinities come about from intermediate steps only when we must compute renormalized parameters in the lagrangian. The schemes we use to bookkeep the infinities are important, especially when one goes to higher loop order, but in the end we should remember that they are simply not there in physical processes. Any theoretical framework that we use must respect this obvious requirement.

We will show that the one-loop results we have computed do not introduce infinities into observables. The example we use for this purpose is a top-bottom fermion loop in the vector-boson self-energies. Before going straight to that calculation, I wish to take a short detour and describe Passarino-Veltman functions, which make one-loop analyses much more convenient.

4.1 Passarino-Veltman functions

In calculating vector boson self-energies we come across the same general two-point functions over and over again. It is useful to define these functions carefully, make a numerical program package to evaluate them, and never recalculate them again. The functions that come out from this analysis are usually called Passarino-Veltman functions, as they were the first to systematically define them.[8]

There are several conventions for Passarino-Veltman functions in use, and we use the one consistent with[9]:

\[
16\pi^2 \mu^{4-n} \int \frac{d^n q}{i(2\pi)^n} \frac{1}{q^2 - m^2 + i\varepsilon} = A_0(m^2) \tag{84}
\]

\[
16\pi^2 \mu^{4-n} \int \frac{d^n q}{i(2\pi)^n} \frac{1}{[q^2 - m_1^2 + i\varepsilon][((q-p)^2 - m_2^2 + i\varepsilon]} = B_0(p^2, m_1^2, m_2^2) \tag{85}
\]

\[
16\pi^2 \mu^{4-n} \int \frac{d^n q}{i(2\pi)^n} \frac{q_\mu}{[q^2 - m_1^2 + i\varepsilon][((q-p)^2 - m_2^2 + i\varepsilon]} = p_\mu B_1(p^2, m_1^2, m_2^2) \tag{86}
\]

\[
16\pi^2 \mu^{4-n} \int \frac{d^n q}{i(2\pi)^n} \frac{q_\mu q_\nu}{[q^2 - m_1^2 + i\varepsilon][((q-p)^2 - m_2^2 + i\varepsilon]} = p_\mu p_\nu B_{21}(p^2, m_1^2, m_2^2) + g_{\mu\nu} B_{22}(p^2, m_1^2, m_2^2) \tag{87}
\]

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Some of these functions have poles at $n = 4$, and thus have an “infinite” piece proportional to
\[ \Delta \equiv \frac{1}{4-n} - \gamma_E + \ln 4\pi \] (88)
where $\gamma_E \simeq 0.5772$ is the Euler-Mascheroni constant that always accompanies the $1/(4-n)$ pole term just as the $\ln 4\pi$ factor does.

The primitive one-point and two-point functions have analytic solutions
\[ A_0(m^2) = m^2 \left( \Delta + 1 - \ln \frac{m^2}{\mu^2} \right) \] (89)
\[ B_0(p^2, m_1^2, m_2^2) = \Delta - \int_0^1 \ln \frac{(1-x)m_1^2 + x m_2^2 - x(1-x)p^2 - i\varepsilon}{\mu^2} \, dx \]
\[ = \Delta - \ln \left( \frac{p^2}{\mu^2} \right) - I(x_+) - I(x_-) \] (90)
where
\[ x_{\pm} = \frac{(p^2 - m_2^2 + m_1^2) \pm \sqrt{(p^2 - m_2^2 + m_1^2)^2 - 4p^2(m_1^2 - i\varepsilon)}}{2p^2}, \quad \text{and} \]
\[ I(x) = \ln(1-x) - x \ln(1-x^{-1}) - 1. \] (91)

All remaining two-point functions can be written entirely in terms of $A_0$ and $B_0$ with various arguments[10]. Note, $A_0(m^2)$ can also be written in terms of $B_0$:
\[ A_0(m^2) = m^2[1 + B_0(0, m^2, m^2)] \quad \text{since} \]
\[ B_0(0, m^2, m^2) = \Delta - \ln \frac{m^2}{\mu^2}, \] (93)
and so all two-point functions can be written in terms of $B_0$ functions.

One is often interested in just the infinite pieces of these functions, as the infinite pieces dictate the renormalization group equations. They are also helpful to check calculations, since all infinities must cancel in observables. Shortly we will investigate the latter.

The relevant two-point functions in terms of their $\Delta$-dependent “infinite pieces” and their finite function pieces (written as lower-case) are
\[ A_0(m^2) = m^2 \Delta + a_0(m^2) \] (95)
\[ B_0(p^2, m_1^2, m_2^2) = \Delta + b_0(p^2, m_1^2, m_2^2) \] (96)
\[ B_1(p^2, m_1^2, m_2^2) = \frac{1}{2} \Delta + b_1(p^2, m_1^2, m_2^2) \]  
\[ B_{21}(p^2, m_1^2, m_2^2) = \frac{1}{3} \Delta + b_{21}(p^2, m_1^2, m_2^2) \]  
\[ B_{22}(p^2, m_1^2, m_2^2) = \left( \frac{m_1^2 + m_2^2}{4} - \frac{p^2}{12} \right) \Delta + b_{22}(p^2, m_1^2, m_2^2) \]

### 4.2 Fermion loop calculation

Now we come to our example. Let us compute the one-loop self energy due to a fermion loop with arbitrary vector boson external legs. The basic Feynman rule notation that we use for this calculation is

\[ V^\mu \begin{array}{c} \Downarrow \quad f \quad \bar{f} \end{array} iA\gamma^\mu(v - a\gamma_5) \]

where \( A, v \) and \( a \) are parametrizations of the coupling. The fermion couplings to a \( V' \) vector boson are \( A', v', \) and \( a' \). With these basic rules we are ready to compute the one-loop function \( \Pi_{VV'}(p^2) \):

\[ i\Pi_{VV'}^{\nu\nu} = - \int \frac{d^n q}{(2\pi)^n} \text{Tr} \left[ iA\gamma^\mu(v - a\gamma_5) \frac{i[(\not{q} - \not{p}) + m_2]}{(q - p)^2 - m_2^2} iA'\gamma^\nu(v' - a'\gamma_5) \frac{i(\not{q} + m_1)}{q^2 - m_1^2} \right] \]

After some gamma-trace algebra and manipulations one finds that

\[ \Pi_{VV'}^{\nu\nu} = \frac{AA'}{4\pi^2} \left\{ (vv' + aa') \left[ 2p^\nu p^\nu (B_{21} - B_1) + g^{\mu\nu}(-2B_{22} - p^2B_{21} + p^2B_1) \right] \ight. \\
+ \left. m_1m_2(vv' - aa')g^{\mu\nu}B_0 \right\} (p^2, m_1^2, m_2^2). \]

As we discussed at the beginning, in precision electroweak analysis the only piece of the self-energy that has a substantial influence on the observables is the part proportional to \( g^{\mu\nu} \): \( \Pi^{\mu\nu}(p^2) = \Pi(p^2)g^{\mu\nu} + \cdots \). As a check of our computations above, when we calculated the shifts in observables due to self-energy corrections, we should check that all \( \Delta \)-divergences cancel. After all, the partial width of the Z
boson into leptons should not divergence as the dimensions approach \( n \to 4 \) in the calculation.

For fermion self-energies, we can check for finiteness of the theory predictions given the expressions above. The \( \Delta \)-divergence part of \( \Pi_{VV'} \) is

\[
\Pi_{VV'}^\Delta(p^2) = \frac{AA'}{4\pi^2} \left\{ (vv' + aa') \left( -\frac{1}{2}(m_1^2 + m_2^2) + \frac{p^2}{3} \right) + (vv' - aa')m_1m_2 \right\} \Delta \tag{102}
\]

For the top-bottom quark doublet, we can compute these \( \Delta \)-divergence pieces. The nonzero contributions are

\[
\Pi_{ZZ}^\Delta(m_Z^2) = m_Z^2 \sum_{i=t,b} \frac{e^2}{4s_c^2c^2} \left[ (T_i^3 - 2Q_i s^2)^2 + (T_i^3)^2 \right] \Delta \tag{103}
\]

\[
\Pi_{\gamma\gamma}^\Delta(m_Z^2) = m_Z^2 \sum_{i=t,b} (eQ_i)^2 \Delta \tag{104}
\]

\[
\Pi_{\gamma Z}^\Delta(m_Z^2) = m_Z^2 \sum_{i=t,b} \frac{e^2Q_i}{2s_c}(T_i^3 - 2Q_i s^2) \Delta \tag{105}
\]

\[
\Pi_{WW}^\Delta(m_W^2) = m_W^2 \frac{e^2}{4s^2} \Delta \tag{106}
\]

Substituting these expressions into the eqs. (69-72), one finds that all \( \Delta \)-divergent terms cancel identically, as they should.

## 5 Note on the utility of these techniques

Finally, I would like to emphasize how useful the oblique correction calculations can be to research. Suppose you have a beyond-the-SM (BSM) theory that for one reason or another induces small corrections to the theoretical predictions of observables compared to the SM, and all those corrections can be expressed entirely in terms of vector boson self-energies. In this case, one can perform a rigorous precision electroweak analysis of the theory by following the techniques described above.

This case is applicable when the set of beyond-the-SM states under consideration cannot couple directly to the final state fermions. An example of this would be split supersymmetry, where the gauginos and higgsinos are light but the sfermions are decoupled[11, 12]. In that case, there is no way to couple the higgsinos and gauginos directly to the final states, because there are no superpartners of the fermions to
complete the vertex. Thus, oblique corrections as discussed here are the way to go.

Another example is in some strongly coupled theories that have pseudo-Nambu-Goldstone bosons with gauge quantum numbers but no flavor quantum number to complete a sizeable vertex correction. Also, if we have a collection of numerous beyond-the-SM states, almost all will couple to the vector bosons in some way (representations under their gauge groups), whereas only a few at most will couple to any given final state. Thus, we expect oblique corrections to often be the most important corrections even when the set of beyond-the-SM states do contain fields that couple to external fermions.

Once we decide that oblique corrections are the appropriate set of corrections to apply to our observables in a beyond-the-SM setting, we should follow these practical steps to perform a precision electroweak analysis.

First, one must get control of the SM observable predictions. The full SM precision electroweak analysis, with all vertex corrections, and higher-order QCD corrections, etc., is a very involved process. If your goal is not to redo this analysis, you can find an analysis that you trust. For example, the values of the observables given by the LEP Electroweak Working Group for reference values of the input parameters (\(\alpha_{\text{had}}, m_t, \alpha_s, m_h\), etc.) can often be used for the starting point. Likewise, the many precision electroweak programs available on the market, such as ZFITTER, can be used. Using \(a_i\) as a generic symbol for a SM input parameter, one finds

\[
O_{\text{SM}}(\{a_i\}) = O(\{a_i^{\text{ref}}\}) + \sum_i c_i(a_i - a_i^{\text{ref}}) + \cdots
\]

The coefficients \(a_i\) can be found in many publications. The key to this step is to compute or get the state-of-the-art computation for \(O_{\text{SM}}(\{a_i\})\) given various \(a_i\) inputs.

Second, compute the self-energy corrections due to the new BSM states. Using \(\eta_j\) as a generic symbol for a BSM input parameter, we can write our full expression for the observable \(O\) in terms of both the SM and BSM input parameters

\[
O_{\text{th}}(\{a_i\}, \{\eta_j\}) = O_{\text{SM}}(\{a_i\}) + \delta O_{\text{BSM}}(\{a_i\}, \{\eta_j\})
\]

Notice, our notation illustrates that we are viewing the BSM contributions as small shifts to the SM predictions. This is likely to be true given the apparently good
agreement the SM has with the precision electroweak data. If this is not true, a full renormalization procedure *de novo* must be carried out to have confidence in the result.

Third, set up a $\chi^2$ analysis of the full BSM theory:

$$\chi^2 = \sum_k \frac{(O_k^{\text{th}}(\{a_i\}, \{\eta_j\}) - O_k^{\text{expt}})^2}{(\Delta O_k)^2}$$ (109)

Minimizing the $\chi^2$ enables one to find the best-fit values of the $\{a_i\}$ and $\{\eta_j\}$ parameters. If the $\chi^2$ per degree of freedom is good, the theory survives. An interesting application of this type of analysis is to add additional new physics contributions (turning on some $\eta_j$ contributions) and see if the SM Higgs mass (one of the $a_i$ parameters) can be significantly heavier at the 95% confidence level than its value of about 200 GeV in the pure SM analysis (all $\eta_j$ decoupled)[16].

The student should look out for two common exceptions: nonuniversal corrections to the $Zb\bar{b}$ vertex and $Z'$ corrections. $Zb\bar{b}$ vertex corrections are often present in beyond-the-SM theories that treat the 3rd family special in any way, and one must be careful to take them into account. Luckily, these corrections are often finite, gauge invariant corrections all by themselves and can be inserted into the analysis rather easily. As for $Z'$, many of its effects are more akin to our beginning tree-level analysis than one-loop corrections. Using the techniques of these lectures, combined with an understanding of all ways a $Z'$ boson can interact with SM states, enables one to do an analysis of $Z'$ implications to precision electroweak observables[17]. I recommend the student make up a $Z'$ boson with his/her favorite couplings and compute all the corrections to observables as an exercise.

Finally, at the end of the lectures we spent some time with the full numerical results of the precision electroweak fits to the standard model. Many of the figures that I showed were from the annual LEP Electroweak Working Group report[2]. I encourage students who are interested in delving deeper into the technical aspects of this topic to read carefully this important document.

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References

[1] The Particle Data Group, Full Listings, http://pdg.lbl.gov.

[2] LEP Electroweak Working Group et al., “A combination of preliminary electroweak measurements and constraints on the standard model,” hep-ex/0312023, hep-ex/0412015.

[3] W. F. L. Hollik, “Radiative Corrections In The Standard Model And Their Role For Precision Tests Of The Electroweak Theory,” Fortsch. Phys. 38, 165 (1990).

[4] D. C. Kennedy, “Renormalization of electroweak gauge interactions,” FERMILAB-CONF-91-271-T, Invited lectures given at 1991 Theoretical Advanced Studies Inst., Boulder, CO, Jun 2-28, 1991.

[5] H. Burkhardt and B. Pietrzyk, “Update of the hadronic contribution to the QED vacuum polarization,” Phys. Lett. B 513, 46 (2001).

[6] M. E. Peskin and T. Takeuchi, “Estimation of oblique electroweak corrections,” Phys. Rev. D 46, 381 (1992).

[7] I. Maksymyk, C. P. Burgess and D. London, “Beyond \( S, T \) and \( U \),” Phys. Rev. D 50, 529 (1994) [hep-ph/9306267].

[8] G. Passarino and M. J. G. Veltman, “One Loop Corrections For \( e^+e^- \) Annihilation Into \( \mu^+\mu^- \) In The Weinberg Model,” Nucl. Phys. B 160, 151 (1979).

[9] D. M. Pierce, J. A. Bagger, K. T. Matchev and R. j. Zhang, “Precision corrections in the minimal supersymmetric standard model,” Nucl. Phys. B 491, 3 (1997) [hep-ph/9606211].

[10] R. G. Stuart, “Algebraic Reduction Of One Loop Feynman Diagrams To Scalar Integrals,” Comput. Phys. Commun. 48, 367 (1988); R. G. Stuart and A. Gongora, “Algebraic Reduction Of One Loop Feynman Diagrams To
Scalar Integrals. 2,” Comput. Phys. Commun. 56, 337 (1990); G. Devaraj and R. G. Stuart, “Reduction of one-loop tensor form-factors to scalar integrals: A general scheme,” Nucl. Phys. B 519, 483 (1998) [hep-ph/9704308].

[11] N. Arkani-Hamed and S. Dimopoulos, [hep-th/0405159] G. F. Giudice and A. Romanino, “Split supersymmetry,” [hep-ph/0406088]

[12] J. D. Wells, “Implications of supersymmetry breaking with a little hierarchy between gauginos and scalars,” [hep-ph/0306127] and “PeV-scale supersymmetry,” [hep-ph/0411041]

[13] S. P. Martin, K. Tobe and J. D. Wells, “Virtual effects of light gauginos and higgsinos: A precision electroweak analysis of split supersymmetry,” [hep-ph/0412424] See also, G. Marandella, C. Schappacher and A. Strumia, [hep-ph/0502095]

[14] D. Y. Bardin et al., “ZFITTER v.6.21: A semi-analytical program for fermion pair production in $e^+e^-$ annihilation,” Comput. Phys. Commun. 133, 229 (2001) [hep-ph/9908433].

[15] See, for example, A. Ferroglia, G. Ossola, M. Passera and A. Sirling, “Simple formulae for $\sin^2\theta_{\text{eff}}^\text{lep}$, $M_W$, $\Gamma_l$, and their physical applications,” Phys. Rev. D 65, 113002 (2002) [hep-ph/0203224].

[16] M. E. Peskin and J. D. Wells, “How can a heavy Higgs boson be consistent with the precision electroweak measurements?,” Phys. Rev. D 64, 093003 (2001) [hep-ph/0101342].

[17] K. S. Babu, C. F. Kolda and J. March-Russell, Phys. Rev. D 57, 6788 (1998) [hep-ph/9710441].

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