The Cauchy problem for a combustion model in porous media

J. C. da Mota\textsuperscript{1,\$} M. M. Santos\textsuperscript{†} R. A. Santos\textsuperscript{‡,¶}

Abstract

We prove the existence of a global solution to the Cauchy problem for a nonlinear reaction-diffusion system coupled with a system of ordinary differential equations. The system models the propagation of a combustion front in a porous medium with two layers, as derived by J. C. da Mota and S. Schecter in \textit{Combustion fronts in a porous medium with two layers}, Journal of Dynamics and Differential Equations, 18(3) (2006). For the particular case, when the fuel concentrations in both layers are known functions, the Cauchy problem was solved by J. C. da Mota and M. M. Santos in \textit{An application of the monotone iterative method to a combustion problem in porous media}, Nonlinear Analysis: Real World Application, 12 (2010). For the full system, in which the fuel concentrations are also unknown functions, we construct an iterative scheme that contains a sequence which converges to a solution of the system, locally in time, under the conditions that the initial data are Hölder continuous, bounded and nonnegative functions. We also show the existence of a global solution, if the initial date are additionally in the Lebesgue space \( L^p \), for some \( p \in (1, \infty) \). Our proof of the local existence relies on a careful analysis on the construction of the fundamental solution for parabolic equations obtained by the parametrix method. In particular, we show the continuous dependence of the fundamental solution for parabolic equations with respect to the coefficients of the equations. To obtain the global existence, we employ the “method of auxiliary functions” as used by O. A. Oleinik and S. N. Kruzhkov in \textit{Quasi-linear second-order parabolic equations with many independent variables}, Russian Mathematical Surveys, 16(5) (1961). Furthermore, for a broad class of reaction-diffusion systems we show that the non negative quadrant is a positively invariant region, and, as a consequence, that classical solutions of similar systems, with the reactions functions being non decreasing in one unknown and semi-lipschitz continuous in the other, are bounded by lower and upper solutions for any positive time if so they are at time zero.

1 Introduction

We are mainly concerned with a specific system of the type

\[ (u_i)_t - \alpha_i(y_i)(u_i)_{xx} + \beta_i(y_i)(u_i)_x = f_i(y_i, u_1, u_2), \quad x \in \mathbb{R}, \quad t > 0 \tag{1.1} \]

for the unknowns \( u_i, y_i \), with \( i = 1, 2 \), where \( y_i \) satisfies an ordinary differential equation which can be solved depending on \( u_i \), and \( \alpha_i(y_i), \beta_i(y_i) \) are given functions of \( y_i \), and \( f_i(y_i, u_1, u_2) \) is a function (also given) of \( y_i, u_1 \) and \( u_2 \). For fixed \( y_i \), the equations (1.1) are a system of parabolic equations for \( u_1, u_2 \) coupled by the function \( f_i \). For the full system, in the unknowns \( u_1, u_2, y_1, y_2 \), since \( y_i \) can be expressed depending on \( u_i \), our system can be written in the unknowns \( u_1, u_2 \) only, but with coefficients depending in a peculiar way on \( u_1, u_2 \). In fact, the system we shall consider can be written in the form

\[ (u_i)_t - a(x, \int_0^t f(u_i) \, d\tau) (u_i)_{xx} + b(x, \int_0^t f(u_i) \, d\tau) (u_i)_x = F_i(x, u_1, u_2, \int_0^t f(u_i) \, d\tau) \tag{1.2} \]

for given functions \( a, b, f, \) and \( F_i \).

\textsuperscript{1}Departamento de Matemática, IME-UFG (Instituto de Matemática e Estatística–Universidade Federal de Goiás). Cx. Postal 131, Campus II, 74001-970 Goiânia, GO, Brazil. jesus@ufg.br, rasantos@ufg.br

\textsuperscript{2}Departamento de Matemática, IMECC-UNICAMP (Instituto de Matemática, Estatística e Computação Científica–Universidade Estadual de Campinas). Rua Sério Buarque de Holanda, 651, Cidade Universitária Zeferino Vaz, 13083-859 Campinas, SP, Brazil. msantos@ime.unicamp.br

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Specifically, the functions $\alpha_i$, $\beta_i$ and $f_i$ in (1.1) are given by

$$
\alpha_i(y_i) = \frac{\lambda_i}{a_i + b_i y_i}, \quad \beta_i(y_i) = \frac{c_i}{a_i + b_i y_i}
$$

and

$$
f_i(y_i, u_1, u_2) = \frac{b_i A_i u_i d_i}{a_i + b_i y_i} y_i f(u_i) + (-1)^i q \frac{u_1 - u_2}{a_i + b_i y_i},
$$

where $f(u_i)$ is the “Arrhenius function”

$$
f(u_i) = e^{-\frac{E_i}{T}},
$$

being $E$ is a positive constant\(^1\) and $\lambda_i, a_i, b_i, c_i, d_i, A_i$, $i = 1, 2$, and $q$ are positive constants.

The unknown $y_i$ satisfies the ordinary differential equation

$$
(y_i)_t = -A_i y_i f(u_i).
$$

Joint with equations (1.1) we add the initial data

$$
u_i|_{t=0} = u_{i,0}
$$

and

$$y_i|_{t=0} = y_{i,0},
$$

for given functions $u_{i,0}, y_{i,0}$. Solving (1.3) for $y_i$, we find

$$y_i = y_{i,0}(x)e^{-A_i \int_0^t f(u_i)\,d\tau}.
$$

Substituting (1.8) in (1.1) we obtain (1.2), with

$$a(x, \int_0^t f(u_i)\,d\tau) = \alpha_i(y_{i,0}(x)e^{-A_i \int_0^t f(u_i)\,d\tau}),
$$

$$b(x, \int_0^t f(u_i)\,d\tau) = \beta_i(y_{i,0}(x)e^{-A_i \int_0^t f(u_i)\,d\tau})
$$

and

$$F_i(x, u_1, u_2, \int_0^t f(u_i)\,d\tau) = f_i(y_{i,0}(x)e^{-A_i \int_0^t f(u_i)\,d\tau}, u_1, u_2).
$$

The system formed by the equations (1.1) and (1.5), with the constitutive functions (1.3) and (1.4), models the propagation of a combustion front in a porous medium with two layers [4]. The unknowns $u_1$ and $y_1$ stands for the temperature and the fuel concentration, respectively, in one layer, and $u_2$ and $y_2$ stands for the same in the other layer, and the constants $\lambda_i, a_i, \ldots$ are parameters related to the medium. We refer to [4] for a detailed derivation of this model.

In this paper we solve the Cauchy problem (1.1), (1.3)–(1.7) (or, equivalently, (1.2) joint with the initial data (1.6)), for given functions $u_{i,0}$ and $a, b, F_i$ in (1.9)–(1.11), being $\alpha_i, \beta_i, f_i$ and $f$ given in (1.3) and (1.4). Furthermore, for a broad class of reaction-diffusion systems (see (1.13) and (1.14)) we show that the non negative quadrant is a positively invariant region, and, as a consequence, that classical solutions of similar systems, with the reactions functions being non decreasing in one unknown and semi-lipschitz continuous in the other (see (1.18)), are bounded by lower and upper solutions for any positive time if so they are at time zero.

Setting some notations, we say that a function is of class $C^{2,1}$ in a set $S \subset \mathbb{R}^d \times [0, \infty)$ if it has continuous derivatives up to second order with respect to $x$ and up to first order with respect to $t$ for all $(x, t) \in S$, and denote this class by $C^{2,1}(S)$ (or simply by $C^{2,1}$), and of class $C^{\alpha, \beta}$ in $S$, for some $\alpha \in (0, 1]$, if it is bounded and Hölder continuous in $S$ with exponent $\alpha$ with respect to $x$ (Lipschitz continuous if $\alpha = 1$) and with exponent $\beta$ with respect to $t$, and denote this class by $C^{\alpha, \beta}(S)$ (or simply by $C^{\alpha, \beta}$), i.e., a function $u(x, t)$ is said to be in $C^{\alpha, \beta}(S)$, for some $\alpha \in (0, 1]$, if there is a constant $C > 0$ such that $|u(x, t)| \leq C$ for all $(x, t) \in S$ and $|u(x, t_1) - u(x, t_2)| \leq C(|x_1 - x_2|^\alpha + |t_1 - t_2|^\beta)$ for all $(x_1, t_1), (x_2, t_2) \in S$. The space $C^{\alpha, \beta}(S)$ is endowed with the norm $\|u\|_{C^{\alpha, \beta}(S)} := \sup_{(x, t) \in S} |u(x, t)| + \sup_{(x, t) \neq (y, s), (x, t), (y, s) \in S} \frac{|u(x, t) - u(y, s)|}{|x - y|^\alpha + |t - s|^\beta}$. For the space of lipschitzian bounded

\(^1\)We notice that the function $f(s) = e^{-\frac{s}{T}}$, $s \neq 0$, can be extended by zero continuously from $s > 0$ to $s = 0$. In fact, $\lim_{s \to 0} \frac{f(s) - f(0)}{s} = 0$ for any $k = 0, 1, 2, \ldots$. Despite the discontinuity when $s \to 0^-$ (lim$_{s \to 0^-} f(s) = \infty$), this will not cause problems in our analysis because essentially we will deal only with non negative functions $u_i$, $i = 1, 2$, cf. theorems 1\(^2\) and 2\(^3\).
functions $u$, defined in a set $S$ in $\mathbb{R}^d$ or $\mathbb{R}^d \times [0, \infty)$, we use the norm $\|u\|_1 := \sup_{x \in S} |u(x)| + \sup_{x \neq y, x, y \in S} \frac{|u(x) - u(y)|}{|x - y|}$.

Throughout the paper, $i, j = 1, 2$ with $j \neq i$.

We denote by $\varphi$ the “upper solution” $\varphi(t) = (M + \beta)e^{\alpha t} - \beta$ for the Cauchy problem (1.1), (1.0) with given $y_i$, satisfying $0 \leq y_i \leq \|y_i\|_{\infty}$, where

$$M = \max_{i=1,2} \|u_{i,0}\|_{\infty}, \quad \alpha = \max_{i=1,2} \frac{\lambda a_i |\partial \Omega|}{\lambda a_i}, \quad \text{and} \quad \beta = \max_{i=1,2} \frac{A_i}{\lambda a_i},$$

and, for $0 < T \leq \infty$, we denote by $(0, \varphi)_T$ the sector (set) of vector functions $u = (u_{1,2}) : \mathbb{R} \times [0, T) \rightarrow \mathbb{R}^2$ such that $0 \leq u_1(x,t) \leq \varphi(t)$ for $i = 1, 2$ and for all $(x, t) \in \mathbb{R} \times [0, T)$. It is easy to check that the pair of $(\varphi, \varphi')$ is an ordered pair (ordered in the sense that $\bar{u}_i \leq \bar{u}_j$) of lower and upper solutions to the system (1.1) and (1.3)–(1.11) (or, equivalently, (1.2)–(1.0), (1.9)–(1.11)) are the following theorems:

**Theorem 1.** (Local solution). Let $u_{i,0}$ and $y_{i,0}$ be nonnegative, lipschitz continuous and bounded functions in $\mathbb{R}$. Then there is a positive number $T$ such that the Cauchy problem (1.2)–(1.6), (1.9)–(1.11) has a solution $u = (u_{1,2})$ in the class $C^{2,1}((0, T) \times \mathbb{R})$ satisfying $0 \leq u_1(x,t) \leq \varphi(t)$ for all $(x, t) \in \mathbb{R} \times [0, T]$. Besides, if additionally $u_{i,0} \in L^p(\mathbb{R})$ for some $p \in (1, \infty)$ then $u \in L^\infty((0, T); L^p(\mathbb{R}))$, with a possible smaller $T$.

**Theorem 2.** (Global solution). Assume that the hypotheses of Theorem 1 are in force, including $u_{i,0} \in L^p(\mathbb{R})$ for some $p \in (1, \infty)$, and, in addition, that $y_{i,0} \in C^{2,1}(\mathbb{R})$ and $(y_{1,0})'$ is bounded. Then the Cauchy problem (1.2)–(1.6), (1.9)–(1.11) has a solution $u = (u_{1,2})$ in $C^{2,1}((0, \infty)) \cap C^{1,\frac{1}{2}}(\mathbb{R} \times [0, \infty)) \cap L^\infty_{loc}((0, \infty); L^p(\mathbb{R}))$ satisfying $0 \leq u_1(x,t) \leq \varphi(t)$, for all $(x, t) \in \mathbb{R} \times [0, \infty)$.

Furthermore, considering general parabolic operators

$$L_i = \partial_t - \sum_{k,j=1}^d a_{i,kj}(x,t) \partial_{x_k x_j} + \sum_{k=1}^d b_{i,k}(x,t) \partial_{x_k},$$

where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $0 < t < T \leq \infty$, $d \geq 1$, and the operator $L_i$ is uniformly parabolic, i.e. for some constant $\lambda > 0$, $\sum_{k,j=1}^d a_{i,kj}(x,t)\xi_k \xi_j \geq \lambda |\xi|^2$ for all $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ and all $(x, t) \in \Omega_T := \mathbb{R}^d \times (0, T)$, using the arguments on invariant regions given in [2, 15], which is the proof of the maximum principle for the heat equation, we state and prove Theorem 3 below, and as a consequence, Theorem 4. In these theorems we take (vector) functions $u = (u_{1,2})$ in the class $C^{1,\frac{1}{2}}(\Omega_T) \cap C^{1,\frac{1}{2}}(\mathbb{R} \times [0, \infty))$ satisfying the condition

$$\liminf_{|x| \to \infty, t \to 0^+} u_i(x,t) \geq 0$$

(cf. condition $K$ in [15]).

**Theorem 3.** Let $\delta$ be a positive number and $c_i(x,t)$ a bounded function in $\Omega_T$. If $f_i(x,t,u_{1,2})$ is a function such that, for some positive number $\varepsilon_0$, it satisfies $f_i \geq 0$ when $-\varepsilon_0 < u_0 < 0$ and $u_j > -\varepsilon_0$, for each $(x, t) \in \Omega_T$ (where $j \neq i, i, j = 1, 2$) then the quadrant $Q = \{(u_{1,2}) : u_1 \geq 0, u_2 \geq 0\}$ is a positively invariant region to the system

$$L_i(u_i) + c_i u_i = f_i(x,t,u_{1,2}) + \delta$$

for any classical solution $u = (u_{1,2})$ satisfying the condition (1.13). More precisely, under the above hypotheses, if $u = (u_{1,2}) \in C^{1,\frac{1}{2}}(\Omega_T) \cap C(\mathbb{R}^d \times [0, T])$ satisfies (1.13) and the inequality $(L_i + c_i u_i)(x,t) \geq f_i(x,t,u_{1,2},(x,t)) + \delta$, for all $(x, t) \in \Omega_T$, and $u(x,0) \in Q$ for all $x \in \mathbb{R}^d$, then $u(x,t) \in Q$ for all $(x, t) \in \mathbb{R}^d \times [0, T]$.

And as a corollary we obtain

\footnote{If $g$ is a bounded function defined in $\mathbb{R}$, $\|g\|_\infty := \sup_{x \in \mathbb{R}} |g(x)|$.}

\footnote{If $T < \infty$ and the function $u_i$ is defined and continuous in $\mathbb{R} \times [0, T]$, obviously we can extend the inequality $0 \leq u_i(x,t) \leq \varphi(t)$ for $t = T$.}

\footnote{Here the term “loc” stands for “locally” in time, i.e. a function $u \in C^{1,\frac{1}{2}}_{loc}(\mathbb{R} \times [0, \infty)) \cap L^\infty_{loc}((0, \infty); L^p(\mathbb{R}))$ if $u|\mathbb{R} \times [0, T] \in C^1(\mathbb{R} \times [0, T]) \cap L^\infty((0, T); L^p(\mathbb{R}))$, for any $T > 0$.}

\footnote{$C(\mathbb{R}^d \times [0, T])$ denotes the space of continuous vector functions in $\mathbb{R}^d \times [0, T]$.}
Theorem 4. Let $\delta$ be a positive number. Suppose that for each fixed $(x,t) \in \Omega_T$, $f_i(x,t,u_1,u_2)$ is a non decreasing function with respect to $u_j$ (where $j \neq i$, $i,j = 1,2$) and, for some positive number $\varepsilon_0$, it satisfies the “semi-lipschitz” condition

\begin{align}
&f_1(x,t,s + u_1,u_2) - f_1(x,t,u_1,u_2) \geq c_1(x,t)s, \\
&f_2(x,t,u_1,s + u_2) - f_2(x,t,u_1,u_2) \geq c_2(x,t)s
\end{align}

(1.15)

for all $s \in (-\varepsilon_0,0)$ and all $((x,t),(u_1,u_2)) \in \Omega_T \times \mathbb{R}^2$, where $c_i(x,t)$ is some bounded function in $\Omega_T$; and else

\begin{align}
&f_1(x,t,u_1,s + u_2) - f_1(x,t,u_1,u_2) \geq -\delta', \\
&f_2(x,t,s + u_1,u_2) - f_2(x,t,u_1,u_2) \geq -\delta'
\end{align}

(1.16)

for all $s \in (-\varepsilon_0,0)$ and all $((x,t),(u_1,u_2)) \in \Omega_T \times \mathbb{R}^2$, where $\delta'$ is some positive number less than $\delta$.

1. If $\bar{u} = (\bar{u}_1, \bar{u}_2) \in C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0,T])$ is a lower solution to the system

\begin{align}
\mathcal{L}_i(u_i) = f_i(x,t,u_1,u_2)
\end{align}

(1.17)

i.e. $(\mathcal{L}_i\bar{u}_i)(x,t) \leq f_i(x,t,\bar{u}_1(x,t),\bar{u}_2(x,t))$, for all $(x,t) \in \Omega_T$, and $u = (u_1,u_2) \in C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0,T])$ is an upper solution to the system

\begin{align}
\mathcal{L}_i(u_i) = f_i(x,t,u_1,u_2) + \delta
\end{align}

(1.18)

i.e. $\mathcal{L}_i(u_i)(x,t) \geq f_i(x,t,u_1(x,t),u_2(x,t)) + \delta$ for all $(x,t) \in \Omega_T$, and such that $u - \bar{u}$ satisfies the condition \((1.13)\), and $u_i(x,0) \geq \bar{u}_i(x,0)$ for all $x \in \mathbb{R}^d$, then $u_i(x,t) \geq \bar{u}_i(x,t)$ for all $(x,t) \in \Omega_T$.

2. Analogously, if $\bar{u} = (\bar{u}_1, \bar{u}_2) \in C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0,T])$ is an upper solution to the system \((1.17)\), i.e. $(\mathcal{L}_i\bar{u}_i)(x,t) \geq f_i(x,t,\bar{u}_1(x,t),\bar{u}_2(x,t))$, for all $(x,t) \in \Omega_T$ and $u = (u_1,u_2) \in C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0,T])$ is a lower solution to the system

\begin{align}
\mathcal{L}_i(u_i) = f_i(x,t,u_1,u_2) - \delta
\end{align}

(1.19)

i.e. $\mathcal{L}_i(u_i)(x,t) \leq f_i(x,t,u_1(x,t),u_2(x,t)) - \delta$ for all $(x,t) \in \Omega_T$, and such that $\bar{u} - u$ satisfies the condition \((1.13)\), and $\bar{u}_i(x,0) \geq u_i(x,0)$ for all $x \in \mathbb{R}^d$, then $\bar{u}_i(x,t) \geq u_i(x,t)$ for all $(x,t) \in \Omega_T$.

Next we give the main ideas to prove theorems \([1]\) and \([2]\).

From now on, we refer to problem \((1.1)\), \((1.3)\)-(\(1.7)\), or, equivalently, \((1.2)\) - \((1.6)\), \((1.9)-(1.11)\), simply as problem \((1.1)\) - \((1.7)\), or, \((1.2)\) - \((1.6)\).

We prove Theorem \([1]\) by taking the limit of a subsequence given by the iterative scheme

\begin{align}
\begin{cases}
(u^n_i)_t - \alpha_i(y^n_i)(u^n_i)_{xx} + \beta_i(y^n_i)(u^n_i)_x = \tilde{f}_i(y^n_i - 1, u^n_i - 1, u^n_i - 1) \\
(y^n_i)_t = -A_iy^n_i(f(u^n_i)) \\
(y^n_i)_t = (u_i,0, y_i,0)
\end{cases}
\end{align}

(1.20)

$n = 1,2,\ldots$, starting from an initial function $(u^n_1, u^n_2) \in C^{1,\frac{1}{2}}(\mathbb{R} \times [0,T])$ for some sufficiently small time $T > 0$, where $\tilde{f}$ is the function that coincides with the Arrhenius function $f(s) = e^{-s}$ for $s > 0$ and it is equal to zero for $s \leq 0$, and, $f_i$ is the function $f_i$ in \((1.3)\) except for the Arrhenius function $f$ which is replaced by $\tilde{f}$. More precisely, we show that there is a positive time $T$, depending on the initial data $u_i,0, y_i,0$ and on the parameters in the equations (i.e. on $\lambda, a_i, \text{etc.}$), such that the operator $A(u_1, u_2) = (w_1, w_2)$, where $(w_1, w_2)$ solves

\begin{align}
\begin{cases}
(w_1)_t = -\alpha_i(y_i)(w_i)_{xx} + \beta_i(y_i)(w_i)_x = \tilde{f}_i(y_i - 1, u_1, u_2) \\
(y_i)_t = -A_iy_i(f(u_i)) \\
(w_i, y_i)|_{t=0} = (u_i,0, y_i,0)
\end{cases}
\end{align}

(1.21)

is well defined in some ball $\Sigma := \{u = (u_1, u_2) \in C^{1,\frac{1}{2}}(\mathbb{R} \times [0,T]); \|u_i\|_{C^{1,\frac{1}{2}}(\mathbb{R} \times [0,T])} \leq R, \ i = 1,2 \}$, $R > 0$, i.e. there exist positive number $R,T$ such that $A(u) \in \Sigma$ for all $u \in \Sigma$. See Lemma \([6]\). In particular, the sequence $\{u^n\} = \{u^n_1, u^n_2\}$ given by $A(u^n) = (u^n - 1)$, starting from any $u^0 \in \Sigma$, is bounded in the norm $|| \cdot ||_{1,1/2}$. Therefore, by Arzela-Ascoli’s theorem, there exists a function $u = (u_1, u_2) \in \Sigma$ and a subsequence of $\{u^n\}$, which we still denote by $\{u^n\}$, that converges to $u$, uniformly on bounded sets in $\mathbb{R} \times [0,T]$. To show that the limit $u$ is a solution of \((1.2)\) and \((1.6)\), we use the integral representation

\begin{align}
\begin{align}
\begin{align}
&u^n_i(x,t) = \int_R \int_R \Gamma_{i,n}(x,t,\xi,0)u^n_i(\xi) \tr {d\xi} + \int_0^t \int_R \Gamma_{i,n}(x,t,\xi,\tau) \int_0^\tau \int_R \Gamma_{i,n}(x,t,\xi,\tau) f_i(y^n_i - 1, u^n_i - 1, u^n_i - 1)(\xi,\tau) d\xi d\tau,
\end{align}
\end{align}
\end{align}

(1.22)
for the solution $u_i^n$ of the parabolic equation

$$
(u_i^n)_t - \alpha_i(y_i^n)(u_i^n)_{xx} + \beta_i(y_i^n)(u_i^n)_x = f_i(y_i^n, u_i^n, w_i^n) \quad (1.23)
$$

occurring in (1.20), where $\Gamma_{i,n}$ denotes the fundamental solution of the associated homogeneous equation $\mathcal{L}_{i,n} w_i = 0$, for $\mathcal{L}_{i,n} := \partial_t - \alpha_i(y_i^n)\partial_{xx} + \beta_i(y_i^n)\partial_x$.

Now suppose that the sequence of fundamental solutions $\{\Gamma_{i,n}\}$ converges, in some appropriate sense, to the fundamental solution $\Gamma_i$ of the also parabolic equation $\mathcal{L}_i w_i = 0$, for $\mathcal{L}_i := \partial_t - \alpha_i(y_i^n)\partial_{xx} + \beta_i(y_i^n)\partial_x$, when $n$ tends to infinity, where $y_i = y_{i,0}(x)e^{-A_i} I^n f_i(u_i^n) ds$. Then, having that the sequence $\{u_i^n\}$ is bounded in $\mathbb{R} \times [0, T]$, for some positive $T$, and that it converges uniformly to $u_i \in C^{1,1/2}(\mathbb{R} \times [0, T])$ in bounded sets in $\mathbb{R} \times [0, T]$, it follows from (1.22) that $u_i$ satisfies the integral equation

$$
u_i(x,t) = \int_0^t \int_0^x \Gamma_i(x,t,\xi,0)u_i(\xi)d\xi + \int_0^t \int_0^x \Gamma_i(x,t,\xi,\tau)f_i(\xi, u_i, w_i)\xi d\xi d\tau, \quad (1.24)
$$

for $(x,t) \in \mathbb{R} \times [0, T]$.

Thus, by standard arguments, it follows that $u_i \in C^{2,1}(\mathbb{R} \times (0, T)) \cap C^1(\mathbb{R} \times [0, T])$ and it is a solution of (1.20). In Section 2 we show the continuous dependence of fundamental solutions of parabolic equations with respect to the coefficients of the equations and, as a consequence, the convergence of $\{\Gamma_{i,n}\}$ to $\{\Gamma_i\}$, when $n \to \infty$. To conclude the last assertion in Theorem 1.1 we shall show in Section 3.1 with the help of the “generalized Young’s inequality” [6, p. 9] and the fact that the fundamental solution $\Gamma_i$ occurring in (1.20), where $\Gamma_i$ is the supremum of the set of pairs $(i,j)$ (1.24) is increasing with respect to $i$ and $j$, with initial data $u_{i,0} \in L^p$ and $T$ sufficiently small. Then the assertion follows by Banach-Altoglu’s theorem. To show that the obtained solution $u = (u_1, u_2)$ is in the sector $\langle 0, \varphi \rangle_T$, we observe that $u = (u_1, u_2)$ is a solution of the Cauchy problem

$$
\begin{cases}
\mathcal{L}_i(w_i) = (u_i)_t - \alpha_i(y_i)(u_i)_{xx} + \beta_i(y_i)(u_i)_x = f_i(y_i, u_i, w_i), & x \in \mathbb{R}, t > 0 \\
(w_i(x,0)) = u_{i,0}(x), & x \in \mathbb{R}
\end{cases} \quad (1.25)
$$

in the unknown $w_i$, for $y_i$ given by (1.8), and in Section 3 we shall show that the function $f_i(x,t,u,w)$ satisfies all the hypotheses of Theorem 1.1 or, more precisely, Corollary 4.3 in Section 3.

Let us just mention here that the reaction function $f_i$ in (1.25) is increasing with respect to $u_i$ and $w_i$. Indeed, from (1.3) we have $\partial f_i / \partial w_i = \eta_i(A_i + B_i y_i)$ for all $w_i \in \mathbb{R}$. In Subsection 3.1 we shall show that the system (1.25) fulfills all the hypotheses of Corollary 4.

To prove Theorem 2 we let $\{0, T^*\}, 0 < T^* \leq \infty$, be a maximal interval in which there exists a solution $u^*$ to the problem (1.2)-1.0 in the space

$$
X_{T^*} := C^{2,1}(\mathbb{R} \times (0, T^*)) \cap C^1(\mathbb{R} \times [0, T^*]); L^p(\mathbb{R}) \quad (1.26)
$$

intercepted with the sector $\langle 0, \varphi \rangle_T$ in $X_T \cap \langle 0, \varphi \rangle_T$ that coincides with $u^*$ in $[0, T^*)$ then $T = T^*$. (The existence of $T^*$ can be assured in the standard way by Zorn’s lemma: we consider the set of pairs $(u, X_T \cap \langle 0, \varphi \rangle_T)$, such that $u$ is a solution of (1.2)-1.0 in $X_T \cap \langle 0, \varphi \rangle_T$, 0 < $T \leq \infty$, ordered with the relation $(u, X_T \cap \langle 0, \varphi \rangle_T) \leq (u^*, X_T \cap \langle 0, \varphi \rangle_T)$ if $T \leq T^*$ and $u^* \mid [0, T] = u$. Any subset $C$ of this set of pairs that is totally ordered has the upper bound $(\pi, X_T \cap \langle 0, \varphi \rangle_T)$, where $\pi$ is the supremum of the set of $T$ such that $(u, X_T \cap \langle 0, \varphi \rangle_T) \in C$. (Then $\pi$ is in $\mathbb{R}$ if this set of $T$ is unbounded) and $\pi$ is defined by $\pi [0, T] = u$ whatever it is $(u, X_T \cap \langle 0, \varphi \rangle_T) \in C$. Then, by Zorn’s lemma the above set of pairs has a maximum element, i.e. there exists a pair $(u^*, X_T \cap \langle 0, \varphi \rangle_T)$ such that if $(u, X_T \cap \langle 0, \varphi \rangle_T)$ is any other pair such that $(u, X_T \cap \langle 0, \varphi \rangle_T) \geq (u^*, X_T \cap \langle 0, \varphi \rangle_T)$ then $(u, X_T \cap \langle 0, \varphi \rangle_T) \leq (u^*, X_T \cap \langle 0, \varphi \rangle_T)$ i.e. if $u$ is a solution of (1.2)-1.0 in $X_T \cap \langle 0, \varphi \rangle_T$ such that $T \geq T^*$ and $u \mid [0, T^*) = u^*$ then $T = T^*$ and $u = u^*$. Then we shall show in Section 5 that if $T^* < \infty$ then we have a contradiction, by proving that, in this case, the maximal solution $u^*$ has a continuous extension up to the time $T^*$, and that this extension is lipschitz continuous and it is in $L^p$, as a function of $x \in \mathbb{R}$, for $t = T^*$, thus $u^*$ can be extended to a larger time, accordingly with Theorem 1.1. The idea to extend $u^*$ up to the time $T^*$ is, again, to use the integral representation (1.24), for $u_i = u_i^*$, $(x, t) \in \mathbb{R} \times [0, T^*)$, with $\Gamma_i$ being the fundamental solution of the equation $\mathcal{L}_i^* w_i = 0$, for $\mathcal{L}_i^* := \partial_t - \alpha_i(y_i)\partial_{xx} + \beta_i(y_i)\partial_x$, where $y_i^* = y_{i,0}(x)e^{-A_i} I^n f_i^*(u_i^n) ds$, and with $f_i^*(y_i, u_i, w_i) = f_i(y_i^*, u_i^*, w_i^*)$. To accomplish this, we need to prove that the derivatives $\partial^2 u_i^*$ are in $L^p$ on $\mathbb{R} \times (0, T^*)$ (see Corollary 5) and we do that by the “method of auxiliary functions” i.e. following [12] or [13, 14]; see [12, p. 107], we make a substitution $u_i^* = h_i(u_i)$.
for an appropriate function $h_1$ (in particular, such that $h'_1$ is positive and bounded) and estimate $|\partial_x v_i|$ (instead of trying to estimate $|\partial_x u^*_i|$) at a maximum point, by looking for the equation satisfied by $v_i$. This leads to some technical estimates where we use the explicit forms for the functions $\alpha(y_i), \beta_i(y_i)$ and $f_i(y_i, u_1, u_2)$ in (2.1) (see Section 5). Certainly, it would be a very interesting investigation to extend our main results regarding the system (1.1) (theorems 1 and 2) to more general functions $\alpha(y_i), \beta_i(y_i)$ and $f_i(y_i, u_1, u_2)$ (or functions $a, b$ and $F_i$ in (1.2)).

The preceding paragraphs give the fundamental and intuitive ideas to prove theorems 1 and 2. In the next sections we give the rigorous and complete proofs of all theorems stated above. In Section 4 we present a brief summary of the construction of fundamental solutions for parabolic equations by the parametrix method and main properties of the coefficients of the equations.

In this section, we present a summary on the construction by the parametrix method and main properties of the fundamental solution for parabolic equations, and show its continuous dependence with respect to the coefficients of the equations.

## 2 The fundamental solution

In this section, we present a summary on the construction by the parametrix method and main properties of the fundamental solution for parabolic equations, and show its continuous dependence with respect to the coefficients of the equations.

### 2.1 Definition and some properties

Consider the equation and the operator $\mathcal{L}$ given by

$$\mathcal{L}u \equiv \partial u / \partial t - a(x, t) \partial^2 u / \partial x^2 + b(x, t) \partial u / \partial x + c(x, t)u = 0, \quad (2.1)$$

in the set $\Omega_T := \{(x, t); x \in \mathbb{R}, 0 \leq t \leq T\}$, for some positive number $T$, with the coefficients $a, b, c$ in the class $C^{\infty}(\mathbb{R})$, for some $a \in (0, 1]$, with $\mathcal{L}$ being a uniform parabolic operator in $\Omega_T$, i.e., there are strictly positive constants $\lambda_0, \lambda_1$ such that

$$\lambda_0 \leq a(x, t) \leq \lambda_1 \quad (2.2)$$

for all $(x, t) \in \Omega_T$.

**Definition 1.** A fundamental solution of the parabolic equation $(2.1)$ is a function $\Gamma(x, t, \xi, \tau)$, defined for all $(x, t)$ and $(\xi, \tau)$ in $\Omega_T$ with $t > \tau$, such that $\mathcal{L} \Gamma = 0$ in $\Omega_T$, as a function of $(x, t)$, for each fixed $(\xi, \tau) \in \Omega_T$, and $\lim_{t \to \tau^+} \int_{\mathbb{R}} \Gamma(x, t, \xi, \tau) \psi(\xi)d\xi = \psi(x)$, for all $x \in \mathbb{R}$ and $\tau \in [0, T)$, for any continuous function $\psi(x)$ such that $|\psi(x)| \leq ce^{b|x|^2}$, for all $x \in \mathbb{R}$, for some positive constants $c$ and $h$ with $h < 1/(4\lambda_1 T)$.

Fundamental solutions for parabolic equations was found by E. E. Levi [11], using the parametrix method. Our presentation in this section follows mostly [8] and [10]. Accordingly, the fundamental solution to the equation (2.1) is given by

$$\Gamma(x, t, \xi, \tau) = Z(x, t, \xi, \tau) + \int_{\tau}^{t} \int_{\mathbb{R}} Z(x, t, y, \sigma) \phi(y, \sigma, \xi, \tau) dyd\sigma, \quad (2.3)$$

where $(x, t), (\xi, \tau) \in \Omega_T$, $t > \tau$, the function $Z(x, t, \xi, \tau)$, as a function $(x, t)$, is the fundamental solution of the heat equation $\partial u / \partial t - a(\xi, \tau) \partial^2 u / \partial x^2 = 0$, i.e.

$$Z(x, t, \xi, \tau) = \frac{1}{(4\pi a(\xi, \tau)(t - \tau))^{\frac{3}{2}}} e^{-\frac{(x - \xi)^2}{4a(\xi, \tau)(t - \tau)}}, \quad (2.4)$$

for each fixed $(\xi, \tau) \in \Omega_T$, and

$$\phi(x, t, \xi, \tau) = \sum_{m=1}^{\infty} (-1)^m (\mathcal{L}Z)_m(x, t, \xi, \tau), \quad (2.5)$$
where \( (LZ)_1 = LZ = (a(\xi, \tau) - a(x, t)) \frac{\partial Z}{\partial x} + b(\xi, \tau) + cZ \) and, for \( m \geq 1 \),
\[
(\mathcal{L}Z)_{m+1}(x, t, \xi, \tau) = \int_0^t \int_\mathbb{R} (\mathcal{L}Z(x, t, y, \sigma))(\mathcal{L}Z)(y, \sigma, \xi, \tau)dyd\sigma.
\] (2.6)

Next we give some important estimates, which, in particular, show that the function \( \Gamma \) given by (2.3) is well defined, i.e. the series in (2.4) converges and (2.5) yields a smooth function \( \Gamma \), for \( t > \tau \). In the sequel, \( (x, t), (\xi, \tau) \in \Omega_T, t > \tau \), and, \( K \) and \( C \) denote any positive constants.

For the function \( Z(x, t, \xi, \tau) \), we have the estimate
\[
|D^r_z D^s_t Z(x, t, \xi, \tau)| \leq K(t - \tau)^{-\frac{r+2s}{2}} e^{-C(\xi - \xi')^2},
\] (2.7)
for all nonnegative integers \( r, s \), where throughout \( D^r_z \) or \( \partial^r_z \) and \( D^s_t \) or \( \partial^s_t \) stand for the derivatives with respect to \( t \) and \( x \) of order \( r \) and \( s \), respectively. Besides, since \( \int_\mathbb{R} \dot{Z}(z, t, \xi, \tau)dz = 1 \), we have that
\[
\int_\mathbb{R} D^r_z D^s_t Z(z, t, \xi, \tau)dz = 0,
\] (2.8)
for all \( r, s \in \mathbb{Z}_+ \) such that \( 2r + s > 0 \). Finally, \( Z \) and its derivatives are Hölder continuous in \( \xi \), i.e.
\[
|D^r_z D^s_t Z(z, t, \xi, \tau) - D^r_z D^s_t Z(z, t, \xi', t, \tau)| \leq \frac{K|\xi - \xi'|^\alpha}{(t - \tau)^{\frac{r+2s}{2}}} e^{-C(\xi - \xi')^2},
\] (2.9)
where \( C = C(\lambda_1) \) and \( K = K(\lambda_0, \lambda_1, ||a||_0, \alpha) \). For the function \( \phi(x, t, \xi, \tau) \), we have the estimates
\[
|\phi(x, t, \xi, \tau)| \leq \frac{K}{(t - \tau)^{\frac{r+2s}{2}}} e^{-C(\xi - \xi')^2},
\] (2.10)
where \( C = C(\lambda_1) \) and \( K = K(\lambda_0, \lambda_1, ||a||_0, \alpha, ||b||_\infty, ||c||_\infty, T) \), and, for any \( \gamma \in (0, \alpha) \),
\[
|\phi(x, t, \xi, \tau) - \phi(y, t, \xi, \tau)| \leq \frac{K|x - y|^\gamma}{(t - \tau)^{\frac{r+2s}{2}}} \left(e^{-C(\xi - \xi')^2} + e^{-C(\xi - \xi')^2} \right),
\] (2.11)
where \( C \) and \( K \) are as in (2.10).

Finally, for the function \( \Gamma(x, t, \xi, \tau) \) we have the estimate (see also Corollary [1] in this paper)
\[
|D^r_z D^s_t \Gamma(x, t, \xi, \tau)| \leq \frac{K}{(t - \tau)^{\frac{r+2s}{2}}} e^{-C(\xi - \xi')^2},
\] (2.12)
for all \( r, s \in \mathbb{Z}_+ \) such that \( 2r + s \leq 2 \), and, again, \( C \) and \( K \) are as in (2.10). Besides, the fundamental solution \( \Gamma \) is nonnegative (see [1] and [9]).

Now consider the Cauchy problem
\[
\begin{cases}
\mathcal{L}u(x, t) = f(x, t), & \text{in } \mathbb{R} \times (0, T], \ T > 0 \\
u(x, 0) = u_0(x), & \text{in } \mathbb{R},
\end{cases}
\] (2.13)
where \( \mathcal{L} \) is defined in (2.1) and \( f \) and \( u_0 \) are given continuous functions, in \( \mathbb{R} \times (0, T] \) and \( \mathbb{R} \), respectively, bounded by the exponential growth
\[
|f(x, t)|, \ |u_0(x)| \leq ce^{hx^2}
\] (2.14)
for positive constants \( c \) and \( h \) such that \( h < 4/(\lambda_1 T) \), and for all \( x \in \mathbb{R} \) and \( t \in [0, T] \). The following theorem gives a representation formula for its solution using the fundamental solution.

**Theorem 5.** Let \( \Gamma \) be the fundamental solution of the equation \( \mathcal{L}u = 0 \), where \( \mathcal{L} \) is the parabolic operator in (2.13). If, besides (2.11), the function \( f \) is locally Hölder continuous in \( x \), uniformly with respect to \( t \), then the function
\[
u(x, t) = \int_\mathbb{R} \Gamma(x, t, \xi, \tau)u_0(\xi)d\xi = \int_0^t \int_\mathbb{R} \Gamma(x, t, \xi, \tau)f(\xi, \tau)d\xi d\tau
\] (2.15)
is the unique solution of the Cauchy problem (2.13) in \( C^{2,1}(\mathbb{R} \times (0, T]) \cap C(\mathbb{R} \times [0, T]) \) bounded by an exponential growth with respect to \( x \), as in (2.14).
2.2 Continuous dependence on the coefficients

Given positive numbers \( T, R, \lambda \) and \( \alpha \), with \( 0 < \alpha \leq 1 \) and \( \lambda < R \), let \( B(R, \lambda, \alpha) \) be the set of vector valued functions \( v = (a(x, t), b(x, t), c(x, t)) \) in \( C^{\alpha, 2}(\Omega_T) \) such that \( a \geq \lambda \) and \( \|a\|_{\alpha, 2}, \|b\|_{\alpha, 2}, \|c\|_{\alpha, 2} < R \). For \( v \in B(R, \lambda, \alpha) \), we define the norm \( \|v\|_{\alpha, 2} = \max\{\|a\|_{\alpha, 2}, \|b\|_{\alpha, 2}, \|c\|_{\alpha, 2}\} \). Any \( v = (a, b, c) \in B(R, \lambda, \alpha) \) defines a parabolic equation of the form (2.1) (with (2.2) satisfied with \( \lambda_0 = \lambda \) and \( \lambda_1 = R \)) and, reciprocally, any (uniformly) parabolic equation of the form (2.1) (satisfying (2.2)) yields a \( v = (a, b, c) \in B(R, \lambda, \alpha) \), for any \( \lambda \in (0, \lambda_0) \) and \( R > \|\alpha, 2\) . To highlight the dependence of the operator \( \mathcal{L} \) given in (2.1) on the coefficients \( a, b, c \equiv v \), we shall write \( \mathcal{L} = \mathcal{L}_{[\alpha]} \), i.e.

\[
\mathcal{L}_{[\alpha]}u \equiv \mathcal{L}u - \frac{\partial u}{\partial t} - a(x,t)\frac{\partial^2 u}{\partial x^2} + b(x,t)\frac{\partial u}{\partial x} + c(x,t)u
\]

and for the fundamental solution of \( \mathcal{L}_{[\alpha]}u = 0 \) we shall write \( \Gamma = \Gamma_{[\alpha]} \), i.e.

\[
\Gamma_{[\alpha]}(x,t,\xi,\tau) = Z_{[\alpha]}(x,t,\xi,\tau) = \int_\mathbb{R} Z_{[\alpha]}(x,y,t,\sigma)\phi_{[\alpha]}(y,\sigma,\xi,\tau)dyd\sigma,
\]

where \( Z_{[\alpha]} = Z_{[(\alpha,0,0)]} \equiv Z \) and \( \phi_{[\alpha]} \equiv \phi \) are given in (2.4) and (2.5).

In the next lemmas we establish some estimates for the fundamental solution (2.17) and its derivatives which take into account the dependence on its coefficients \( a, b, c \equiv v \). We first establish these estimates for the functions \( Z_{[\alpha]} \) and \( \phi_{[\alpha]} \) and then, using (2.17) and the series (2.2) for \( \phi_{[\alpha]} \), we extend them for \( \Gamma_{[\alpha]} \). These estimates are the key point to obtain the local solution stated in Theorem 1.

We shall write \( C \) and \( K \) to denote positive constants that might depend on the parameters \( R, \lambda, \alpha, T \), but not on the coefficients \( v \) neither on the solutions \( u \) or the data \( f, u_0 \), unless otherwise stated. Besides, \( K \) depends continuously on \( T \).

Lemma 1. Given \( \nu, \tau \in B(R, \lambda, \alpha) \), we have that

\[
|D_x^s Z_{[\alpha]}(x,t,\xi,\tau)| \leq K \|v\|_{\alpha, 2} \frac{1}{(t - \tau)^{\frac{s}{2}}} e^{-\frac{\lambda (t - \tau)\xi^2}{(t - \tau)}} ,
\]

for \( s = 0, 1, 2 \), where \( C < 1/(4R) \) and \( K = K(R, \lambda) \).

Proof. Since \( Z_{[\alpha]}(x,t,\xi,\tau) = \frac{1}{(4\pi (t - \tau))^{\frac{n}{2}}} e^{-\frac{\lambda (t - \tau)\xi^2}{(t - \tau)}} \) and its derivatives on \( x \) depends on the coefficient \( a \) of \( \mathcal{L}_{[\alpha]} \), we do not depend on the other coefficients \( b \) and \( c \), computing the derivative of \( D_x^s Z_{[\alpha]} \) with respect to \( a \), we find \( |D_x^s D_x^a Z_{[\alpha]}(x,t,\xi,\tau)| \leq \frac{K}{(t - \tau)^{\frac{s}{2}}} e^{-\frac{\lambda (t - \tau)\xi^2}{(t - \tau)}} \), for \( s = 0, 1, 2 \), and constants \( K \) and \( C \) as in statement of the lemma. Then the desired inequality follows by the Mean Value Theorem.

Lemma 2. Let \( L \) and \( \alpha \) be strictly positive numbers being \( \alpha \leq 1 \), and let \( g \) denote the gamma function \( g(x) := \int_\mathbb{R} \frac{1}{t^{\alpha} - 1} e^{-t} dt \). Then \( \sum_{m=1}^{\infty} m\lambda^{\alpha}/g(\frac{\lambda}{m}) \) is a convergent series.
Proof. We begin by recalling the relation \( \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = B(x, y) \) between the gamma function \( \Gamma \) and the beta function, \( B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt \) (see [15], p.41). Denoting the general term of the given series by \( b_m \), and using the above relation, we obtain

\[
\lim_{m \to \infty} \frac{b_{m+1}}{b_m} = A \lim_{m \to \infty} \frac{B(m, \pi)}{\Gamma(\frac{m}{2})} = A \lim_{m \to \infty} \frac{1}{\Gamma(\frac{m}{2})} \int_0^1 t^{\frac{m}{2}-1}(1-t)^{\frac{m}{2}-1} dt = 0.
\]

Therefore, the result follows. \( \square \)

**Lemma 3.** Let \( \beta \in [0, 1] \) and \( \gamma \in (0, \alpha) \). If \( v, \varphi \in B(R, \lambda, \alpha) \) then

\[
|\phi_v(x, t, \xi, \tau) - \phi_{\varphi}(x, t, \xi, \tau)| \leq K \|v - \varphi\|_{\alpha, \beta} e^{-C(x-\xi)^2/(t-\tau)} \tag{2.18}
\]

and

\[
|\phi_v(x, t, \xi, \tau) - \phi_{\varphi}(y, t, \xi, \tau) - \phi_{\varphi}(y, t, \xi, \tau)| \leq K \|v - \varphi\|_{\alpha, \beta}^2 |x - y|^{(1-\beta)} \left( e^{-C(x-\xi)^2/(t-\tau)} + e^{-C(y-\xi)^2/(t-\tau)} \right), \tag{2.19}
\]

where \( C < \frac{1}{t^2} \) and \( K = K(R, \lambda, \alpha, T) \).

**Proof.** The proof of (2.18) follows from the following inequality:

\[
|((LZ_v)_m - (LZ_{\varphi})_m)(x, t, \xi, \tau)| \leq mK^m \left( \frac{\pi}{2} \right)^{m-1} \|v - \varphi\|_{\alpha, \beta} \frac{g(\frac{\pi}{2})^m}{\Gamma(\frac{m}{2})} \left( \frac{1}{(t-\tau)^{m/2}} \right) e^{-C(x-\xi)^2/(t-\tau)}, \tag{2.20}
\]

where, for simplicity, we set \( L \equiv L_v \), and \( g \) denotes the gamma function; see Lemma [2]. We show this inequality by induction on \( m \). For \( m = 1 \), we have

\[
|((L_v)Z_v)_1 - (L_{\varphi}Z_{\varphi})_1|(x, t, \xi, \tau)| \leq |((v(x, t) - a(x, t))\partial_{xx}Z_v + b(x, t)\partial_x Z_v|
\]

\[
+ c(x, t)Z_v| - ((\varphi(x, t) - a(x, t))\partial_{xx}Z_{\varphi} + b(x, t)\partial_x Z_{\varphi} + \varphi(t, x)Z_{\varphi})| \leq |((v(x, t) - a(x, t)) - ((\varphi(x, t) - \varphi(t, x)))|\partial_{xx}Z_v| |
\]

\[
+ |(v(x, t) - \varphi(t, x))|\partial_{xx}Z_{\varphi} + \partial_x Z_v| + |b(x, t) - \varphi(t, x)|\partial_x Z_v|
\]

\[
+ |c(x, t) - \varphi(t, x)|Z_v| + |(v(x, t) - \varphi(t, x))|Z_v| + |\varphi(t, x)|Z_v| - Z_{\varphi} | \equiv I.
\]

Then from Lemma [1] (2.14), and the estimate

\[
|x - \xi|^\alpha e^{-C(x-\xi)^2/(t-\tau)} = \left( \frac{x-\xi}{t-\tau} \right)^{\alpha/2} \left( \frac{x-\xi}{t-\tau} \right)^{\alpha/2} e^{-C(x-\xi)^2/(t-\tau)},
\]

where \( C \) is a new constant which we shall continue denoting by \( C \), we obtain

\[
I \leq K \|v - \varphi\|_{\alpha, \beta} \left( \frac{2}{(t-\tau)^{3/2}} \right) e^{-C(x-\xi)^2/(t-\tau)} + \frac{2}{(t-\tau)^{3/2}} e^{-C(y-\xi)^2/(t-\tau)} + \frac{2}{(t-\tau)^{3/2}} e^{-C(x-\xi)^2/(t-\tau)}
\]

\[
\leq K \|v - \varphi\|_{\alpha, \beta} \frac{2}{(t-\tau)^{3/2}} e^{-C(x-\xi)^2/(t-\tau)},
\]
where $C < 1/(4R)$ and $K = K(R, \lambda, \alpha, T)$. Now, assuming that (2.20) is true for an $m \geq 1$, we obtain:

$$\left| ((L_{[\nu]}Z_{[\nu]})_{m+1} - (L_{[\nu]}Z_{[\nu]})_{m+1})(x, t, \xi, \tau) \right|$$

$$\leq \left| \int_{\tau}^{t} \int_{R} \left| L_{[\nu]}Z_{[\nu]}(x, y, t, \sigma) \right| \left| (L_{[\nu]}Z_{[\nu]})_{m}(y, \xi, \sigma, \tau) \right| d\sigma \right|$$

$$- \left| \int_{\tau}^{t} \int_{R} \left( L_{[\nu]}Z_{[\nu]}(x, y, t, \sigma) \right) \left( (L_{[\nu]}Z_{[\nu]})_{m}(y, \xi, \sigma, \tau) \right) d\sigma \right|$$

$$\leq \int_{\tau}^{t} \int_{R} \left| (L_{[\nu]}Z_{[\nu]} - L_{[\nu]}Z_{[\nu]})(x, y, t, \sigma) \right| (L_{[\nu]}Z_{[\nu]})_{m}(y, \xi, \sigma, \tau) d\sigma$$

$$+ \int_{\tau}^{t} \int_{R} \left| (L_{[\nu]}Z_{[\nu]}(x, t, \xi, \tau)) \right| (L_{[\nu]}Z_{[\nu]})_{m}(y, \xi, \sigma, \tau) d\sigma$$

$$\leq \int_{\tau}^{t} \int_{R} K \left| \frac{\|v - \tau\|_{C \alpha \phi \frac{g(\tau)}{g(\tau)}}}{(t - \sigma)^{\frac{\lambda}{\mu}} - \frac{\lambda}{\mu} \frac{m\alpha}{2}} e^{-C_1 \frac{(x - y)^2}{t - \tau}} \right| d\sigma$$

$$\times 1 \frac{1}{(t - \sigma)^{\frac{\lambda}{\mu}} - \frac{\lambda}{\mu} \frac{m\alpha}{2}} e^{-C_1 \frac{(y - \xi)^2}{t - \sigma}} d\sigma,$$

where we used that

$$\int_{\mathbb{R}} e^{-C_1 \frac{(x - y)^2}{t - \tau}} e^{-C_1 \frac{(y - \xi)^2}{t - \sigma}} dy = \left( \frac{\pi}{C} \right)^{\frac{1}{2}} \left( t - \sigma \right)^{\frac{1}{2}} e^{-C_1 \frac{(x - y)^2}{t - \tau}}$$

and

$$\int_{\tau}^{t} \left| \frac{1}{(t - \sigma)^{\frac{\lambda}{\mu}} - \frac{\lambda}{\mu} \frac{m\alpha}{2}} \right| d\sigma = \frac{1}{(t - \tau)^{\frac{\lambda}{\mu}} - \frac{\lambda}{\mu} \frac{m\alpha}{2}} \frac{g(\tau)g(\tau^\alpha)}{g(\tau^{\alpha\mu})} \left( t - \tau \right)^{-\frac{\lambda}{\mu}} \frac{g(\tau)g(\tau^\alpha)}{g(\tau^{\alpha\mu})}$$

(see [10] p. 362). So,

$$\left| ((L_{[\nu]}Z_{[\nu]})_{m+1} - (L_{[\nu]}Z_{[\nu]})_{m+1})(x, t, \xi, \tau) \right|$$

$$\leq (m + 1)K^{m+1} \left( \frac{\pi}{C} \right)^{\frac{1}{2}} \frac{g(\tau)g(\tau^\alpha)}{g(\tau^{\alpha\mu})} \left( t - \tau \right)^{-\frac{\lambda}{\mu}} \frac{g(\tau)g(\tau^\alpha)}{g(\tau^{\alpha\mu})} e^{-C_1 \frac{(x - y)^2}{t - \tau}}.$$
**Lemma 4.** Let \(v, \varpi \in B(R, \lambda, \alpha)\), \(\beta \in (0, 1)\), \(\gamma \in (0, \alpha)\), and \(\Gamma_{[\varpi]}, \Gamma_{[\varpi]}\), the fundamental solutions of the equations \(L_v u = 0\), \(L_{\varpi} w = 0\), as defined in (2.17) and (2.4)–(2.6). Then we have the following estimates:

\[
| (D^2_v \Gamma_{[\varpi]} - D^2_v \Gamma_{[\varpi]}) (x, t, \xi, \tau) | \leq \frac{K \|v - \varpi\|_{\alpha, \frac{3}{2}}}{(t - \tau)^{1 + \beta}} e^{-C \frac{(x - \xi)^2}{t - \tau}}, \quad s = 0, 1; \quad (2.21)
\]

\[
| (\partial_s x \Gamma_{[\varpi]} - \partial_s x \Gamma_{[\varpi]}) (x, t, \xi, \tau) | \leq \frac{1}{(t - \tau)^{2 + \beta}} e^{-C \frac{(x - \xi)^2}{t - \tau}};
\]

\[
(2.22)
\]

and

\[
| (\partial_t \Gamma_{[\varpi]} - \partial_t \Gamma_{[\varpi]}) (x, t, \xi, \tau) | \leq \frac{1}{(t - \tau)^{2 + \beta}} e^{-C \frac{(x - \xi)^2}{t - \tau}};
\]

\[
(2.23)
\]

where \(C < \frac{1}{4\pi^2} \) and \(K = K(R, \lambda, \alpha, T)\).

**Proof.** For \(s = 0\) we have that

\[
| (\Gamma_{[\varpi]} - \Gamma_{[\varpi]}) (x, t, \xi, \tau) | \leq | (Z_{[\varpi]} - Z_{[\varpi]}) (x, t, \xi, \tau) | +
\]

\[
+ \int_0^t \int_R | Z_{[\varpi]} (x, y, t, \sigma) \phi_{[\varpi]} (y, \sigma, \xi, \tau) - Z_{[\varpi]} (x, y, t, \sigma) \phi_{[\varpi]} (y, \sigma, \xi, \tau) | dy d\sigma \leq
\]

\[
\leq | Z_{[\varpi]} - Z_{[\varpi]} | + \int_0^t \int_R | Z_{[\varpi]} - Z_{[\varpi]} | \phi_{[\varpi]} | + | Z_{[\varpi]} | \phi_{[\varpi]} - \phi_{[\varpi]} | dy d\sigma.
\]

From estimates (2.7) and (2.10) and Lemmas 1 and 3 it follows that

\[
| (\Gamma_{[\varpi]} - \Gamma_{[\varpi]}) (x, t, \xi, \tau) | \leq \frac{K \|v - \varpi\|_{\alpha, \frac{3}{2}}}{(t - \tau)^{2 + \beta}} e^{-C \frac{(x - \xi)^2}{t - \tau}}
\]

\[
+ \int_0^t \int_R | K \|v - \varpi\|_{\alpha, \frac{3}{2}} e^{-C \frac{(x - y)^2}{(t - \sigma)^2}} K \frac{K}{(t - \tau)^{2 + \beta}} e^{-C \frac{(y - \xi)^2}{(t - \tau)^2}} dy d\sigma
\]

\[
+ \int_0^t \int_R \frac{K}{(t - \tau)^{2 + \beta}} e^{-C \frac{(y - \xi)^2}{(t - \tau)^2}} dy d\sigma.
\]

Since,

\[
\int_R e^{-C \frac{(y - \xi)^2}{(t - \tau)^2}} dy = \frac{\pi}{C^2} \frac{(t - \tau)}{2} e^{-C \frac{(x - \xi)^2}{(t - \tau)^2}}
\]

we obtain

\[
| (\Gamma_{[\varpi]} - \Gamma_{[\varpi]}) (x, t, \xi, \tau) | \leq \frac{1}{(t - \tau)^{2 + \beta}} \left( \int_0^t (t - \sigma)^{1 + \beta} d\sigma + \left( \frac{\pi}{C^2} \right) \frac{(t - \tau)^{1 + \beta}}{2} \right) e^{-C \frac{(x - \xi)^2}{(t - \tau)^2}}
\]

\[
(2.25)
\]

where \(K = K(R, \lambda, \alpha, T)\).

For the case \(s = 1\), we have

\[
(\partial_s x \Gamma_{[\varpi]} - \partial_s x \Gamma_{[\varpi]}) (x, t, \xi, \tau)
\]

\[
= (\partial_s x Z_{[\varpi]} - \partial_s x Z_{[\varpi]}) (x, t, \xi, \tau)
\]

\[
+ \int_0^t \int_R \partial_s Z_{[\varpi]} (x, y, t, \sigma) \phi_{[\varpi]} (y, \sigma, \xi, \tau) - \partial_s Z_{[\varpi]} (x, y, t, \sigma) \phi_{[\varpi]} (y, \sigma, \xi, \tau) dy d\sigma
\]

\[
= (\partial_s x Z_{[\varpi]} - \partial_s x Z_{[\varpi]}) + \int_0^t \int_R (\partial_s Z_{[\varpi]} - \partial_s Z_{[\varpi]}) \phi_{[\varpi]} dy d\sigma + \int_0^t \int_R \partial_s Z_{[\varpi]} (\phi_{[\varpi]} - \phi_{[\varpi]}) dy d\sigma
\]

\[
\equiv J_1 + J_2 + J_3.
\]
From Lemma 1, we have

\[ |J_1| = |(\partial_x Z_{[\mathcal{v}]} - \partial_x Z_{[\mathcal{w}]})(x, t, \xi, \tau)| \leq \frac{K \left\| a - \mathcal{w} \right\|_\infty}{t - \tau} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)}. \tag{2.26} \]

Using Lemma 1 the estimate (2.10) and the identity (2.24), we have

\[ |J_2| \leq \int_t^\infty \int \left| (\partial_x Z_{[\mathcal{v}]} - \partial_x Z_{[\mathcal{w}]})(x, y, t, \sigma) \right| |\phi_{[\mathcal{v}]}(y, \sigma, \xi, \tau)| dy d\sigma \tag{2.27} \]

\[ \leq \int_t^\infty \int \frac{K \left\| a - \mathcal{w} \right\|_\infty}{t - \sigma} e^{-C\left(\frac{(x - \xi)^2}{t - \sigma}\right)} \frac{K}{(\sigma - \tau)^{\frac{3}{2}}} e^{-C\left(\frac{(x - \xi)^2}{\sigma - \tau}\right)} dy d\sigma \]

\[ \leq \frac{K \left\| a - \mathcal{w} \right\|_\infty}{(t - \tau)^{\frac{1}{2}}} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)} \int_t^\infty (t - \sigma)^{-\frac{1}{2}} (\sigma - \tau)^{-\frac{3}{2}} d\sigma \]

\[ \leq \frac{K \left\| a - \mathcal{w} \right\|_\infty}{(t - \tau)^{\frac{1}{2}}} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)}. \]

Finally, using Lemma 3 (2.17) and (2.24), we obtain

\[ |J_3| \leq \int_t^\infty \int |\partial_x Z_{[\mathcal{v}]}(x, y, t, \sigma)| (|\phi_{[\mathcal{v}]} - \phi_{[\mathcal{w}]}|)(y, \sigma, \xi, \tau)| dy d\sigma \tag{2.28} \]

\[ \leq \int_t^\infty \int \frac{K}{t - \sigma} e^{-C\left(\frac{(x - \xi)^2}{t - \sigma}\right)} \frac{K}{(\sigma - \tau)^{\frac{1}{2}}} e^{-C\left(\frac{(x - \xi)^2}{\sigma - \tau}\right)} dy d\sigma \]

\[ \leq \frac{K}{(t - \tau)^{\frac{1}{2}}} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)} \int_t^\infty (t - \sigma)^{-\frac{1}{2}} (\sigma - \tau)^{-\frac{3}{2}} d\sigma \]

\[ \leq \frac{K}{(t - \tau)^{\frac{1}{2}}} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)}, \]

where \( K = K(R, \lambda, \alpha, T). \) From (2.26), (2.27) and (2.28), we get

\[ |(\partial_x \Gamma_{[\mathcal{v}]} - \partial_x \Gamma_{[\mathcal{w}]})(x, t, \xi, \tau)| \leq \frac{K \left\| v - \mathcal{w} \right\|_\alpha \frac{\alpha}{\tau} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)} \]

for a \( K \) as above.

Regarding the second derivative with respect to \( x \), we have

\[ (\partial_{xx} \Gamma_{[\mathcal{v}]} - \partial_{xx} \Gamma_{[\mathcal{w}]})(x, t, \xi, \tau) = (\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]})(x, t, \xi, \tau) \]

\[ + \int_t^\infty \int \partial_{xx} Z_{[\mathcal{v}]}(x, y, t, \sigma) \phi_{[\mathcal{v}]}(y, \sigma, \xi, \tau) - \partial_{xx} Z_{[\mathcal{w}]}(x, y, t, \sigma) \phi_{[\mathcal{w}]}(y, \sigma, \xi, \tau) dy d\sigma \]

\[ = (\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]}) + \int_t^\infty \int (\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]}) \phi_{[\mathcal{v}]} dy d\sigma \]

\[ + \int_t^\infty \int \partial_{xx} Z_{[\mathcal{w}]}(\phi_{[\mathcal{v}]} - \phi_{[\mathcal{w}]}) dy d\sigma \]

\[ \equiv I_1 + I_2 + I_3. \]

From Lemma 1,

\[ |I_1| = |(\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]})(x, t, \xi, \tau)| \leq \frac{K \left\| a - \mathcal{w} \right\|_\infty}{(t - \tau)^{\frac{3}{2}}} e^{-C\left(\frac{(x - \xi)^2}{t - \tau}\right)}. \tag{2.29} \]

To estimate \( I_2 \), we write

\[ I_2 = \int_t^\infty \int (\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]})(x, y, t, \sigma) \phi_{[\mathcal{v}]}(y, \sigma, \xi, \tau) dy d\sigma \]

\[ = \int_t^\infty \int (\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]}) \phi_{[\mathcal{v}]} dy d\sigma + \int_t^{t/2} \int (\partial_{xx} Z_{[\mathcal{v}]} - \partial_{xx} Z_{[\mathcal{w}]}) \phi_{[\mathcal{v}]} dy d\sigma \]

\[ \equiv I'_2 + I''_2. \]
Applying Lemma 1, (2.10) and (2.24) we get

\[
|I'_2| \leq \int_\tau^{t+\frac{\tau}{2}} \int \frac{K\|a - \overline{a}\|_\infty e^{-C\frac{(x-y)^2}{(t-\sigma)^2}}}{(t-\sigma)^2} \frac{K}{(\sigma - \tau)^{\frac{1}{2}}} e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}} dyd\sigma
\]

\[
\leq K\|a - \overline{a}\|_\infty e^{-C\frac{(x-y)^2}{(t-\sigma)^2}} \int_\tau^{t+\frac{\tau}{2}} (t-\sigma)^{-1}(\sigma - \tau)^{-1+\frac{1}{2}} d\sigma 
\]

\[
\leq \frac{K\|a - \overline{a}\|_\infty}{(t-\tau)^{\frac{3}{2}}} e^{-C\frac{(x-y)^2}{t-\tau}}
\]

Now,

\[
I''_2 = \int_\tau^{t+\frac{\tau}{2}} \int (\partial_{xx} Z_{[\nu]} - \partial_{xx} \overline{Z}_{[\nu]})(x, y, t, \sigma) \phi_{[\nu]}(y, \sigma, \xi, \tau) dyd\sigma
\]

\[
= \int_\tau^{t+\frac{\tau}{2}} \int (\partial_{xx} Z_{[\nu]} - \partial_{xx} \overline{Z}_{[\nu]})(x, y, t, \sigma)(\phi_{[\nu]}(y, \sigma, \xi, \tau) - \phi_{[\nu]}(x, \sigma, \xi, \tau)) dyd\sigma
\]

\[
= \int_\tau^{t+\frac{\tau}{2}} \int (\partial_{xx} Z_{[\nu]} - \partial_{xx} \overline{Z}_{[\nu]})(x, y, t, \sigma)(\phi_{[\nu]}(y, \sigma, \xi, \tau) - \phi_{[\nu]}(x, \sigma, \xi, \tau)) dyd\sigma
\]

\[
+ \int_\tau^{t+\frac{\tau}{2}} \int (\partial_{xx} Z_{[\nu]} - \partial_{xx} \overline{Z}_{[\nu]})(x, y, t, \sigma) \phi_{[\nu]}(x, \sigma, \xi, \tau) dyd\sigma,
\]

since \( \int (\partial_{xx} Z_{[\nu]})(x, x, t, \sigma) - \partial_{xx} \overline{Z}_{[\nu]}(x, x, t, \sigma)) dy = 0 \) (see (2.8)). Then, applying Lemma 1, (2.9), (2.10) and (2.11), we obtain

\[
|I''_2| \leq \int_\tau^{t+\frac{\tau}{2}} \int \frac{K\|a - \overline{a}\|_\infty}{(t-\sigma)^2} e^{-C\frac{(x-y)^2}{(t-\sigma)^2}} \frac{K|e - y|}{(\sigma - \tau)^{\frac{1}{2}}(t-\sigma)^{\frac{1}{2}}} \frac{e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}}}{(\sigma - \tau)^{\frac{1}{2}}} dyd\sigma
\]

\[
+ \int_\tau^{t+\frac{\tau}{2}} \int \frac{K\|a - \overline{a}\|_\infty}{(t-\sigma)^{\frac{3}{2}}} e^{-C\frac{(x-y)^2}{(t-\sigma)^2}} e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}} \frac{e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}}}{(\sigma - \tau)^{\frac{1}{2}}} dyd\sigma
\]

\[
\leq \int_\tau^{t+\frac{\tau}{2}} \int \frac{K\|a - \overline{a}\|_\infty}{(t-\sigma)^{\frac{3}{2}}} e^{-C\frac{(x-y)^2}{(t-\sigma)^2}} \frac{1}{(\sigma - \tau)^{\frac{1}{2}}} \frac{e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}}}{(\sigma - \tau)^{\frac{1}{2}}} dyd\sigma
\]

\[
+ \int_\tau^{t+\frac{\tau}{2}} \int \frac{K\|a - \overline{a}\|_\infty}{(t-\sigma)^{\frac{3}{2}}} e^{-C\frac{(x-y)^2}{(t-\sigma)^2}} \frac{e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}}}{(\sigma - \tau)^{\frac{1}{2}}} dyd\sigma
\]

\[
\leq K(\|a - \overline{a}\|_\infty + \|a - \overline{a}\|_\infty^\beta) e^{-C\frac{(x-y)^2}{t-\tau}} \frac{e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}}}{(t-\tau)^{\frac{1}{2}}},
\]

Thus,

\[
|I_2| \leq |I'_2| + |I''_2| \leq K(\|a - \overline{a}\|_\infty + \|a - \overline{a}\|_\infty^\beta) e^{-C\frac{(x-y)^2}{t-\tau}} \frac{e^{-C\frac{(y-\xi)^2}{(\sigma - \tau)^2}}}{(t-\tau)^{\frac{1}{2}}}. (2.30)
\]

To estimate \( I_3 \), we write

\[
I_3 = \int_\tau^{t} \int \partial_{xx} Z_{[\nu]}(x, y, t, \sigma)(\phi_{[\nu]} - \phi_{[\nu]}) (y, \sigma, \xi, \tau) dyd\sigma
\]

\[
= \int_\tau^{t} \int \partial_{xx} Z_{[\nu]}(x, y, t, \sigma)[(\phi_{[\nu]} - \phi_{[\nu]})(y, \sigma, \xi, \tau) - (\phi_{[\nu]} - \phi_{[\nu]})(x, \xi, \sigma, \tau)] dyd\sigma
\]

\[
+ \int_\tau^{t} \int (\partial_{xx} Z_{[\nu]}(x, y, t, \sigma) - \partial_{xx} Z_{[\nu]}(x, x, t, \sigma))(\phi_{[\nu]} - \phi_{[\nu]})(x, \xi, \sigma, \tau) dyd\sigma
\]

\[
\equiv I'_3 + I''_3,
\]
where we have used (2.28). Applying Lemma 3 and (2.27), we get

\[
|I_3'| \leq \int_\tau^t \int \frac{K}{(t - \sigma)\frac{3}{2}} e^{-C(t - \sigma)^2} K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} |x - y|^{1-\beta} (e^{-C(t - \sigma)^2} + e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq \int_\tau^t \int \frac{K}{(t - \sigma)\frac{3}{2}} e^{-C(t - \sigma)^2} K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} |x - y|^{1-\beta} (e^{-C(t - \sigma)^2} + e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
+ K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

In order to estimate \(I_3''\), we use Lemma 3 and (2.29) as follows:

\[
|I_3''| \leq \int_\tau^t \int \frac{K}{(t - \sigma)\frac{3}{2}} e^{-C(t - \sigma)^2} \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} e^{-C(t - \sigma)^2} dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

Then,

\[
|I_3| \leq |I_3'| + |I_3''| \leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

From the above estimates, (2.29), (2.30) and (2.31), we obtain

\[
||\partial_x \Gamma_{\xi}|| \leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

\[
\leq K \frac{v - \nabla}{(t - \sigma)\frac{3}{2}} \int_\tau^t \frac{1}{(t - \sigma)\frac{3}{2}} (e^{-C(t - \sigma)^2}) dyd\sigma
\]

Finally, the proof of (2.23) follows from (2.21), (2.22) and the equations \(L_{\xi} \Gamma_{\xi} = L_{\xi} \Gamma_{\xi} = 0\).
Corollary 1. For \( v \in B(R, \lambda, \alpha) \) we have the following uniform estimate:
\[
|D^2 \Gamma_{[v]}(x, t, \xi, \tau)| \leq \frac{K}{(t-\tau)^{\frac{n-1}{2}}} e^{-C|\xi-\tau|^2/(t-\tau)}, \quad s = 0, 1
\]
where \( K = K(R, \lambda, \alpha, T) \).

Proof. Take \( \varphi = (1, 0, 0) \) in (2.21).

We also have the following lemma.

Lemma 5. Let \( v, v \in B(R, \lambda, \alpha) \), \( n = 1, 2, \ldots \). If \( v_n(x,t) \) converges to \( v(x,t) \) pointwise in \( \mathbb{R} \times [0, T] \), as \( n \) goes to infinity, then \( \Gamma_{[v_n]}(x, t, \xi, \tau) \) converges to \( \Gamma_{[v]}(x, t, \xi, \tau) \), for any \( (x, t), (\xi, \tau) \in \mathbb{R} \times [0, T] \), with \( t > \tau \).

Proof. First we show the pointwise convergence of \( Z_{[v_n]} \) and \( \phi_{[v_n]} \). From (2.2) it is easy to see that
\[
D^r D^s Z_{[v_n]} \to D^r D^s Z_{[v]}
\]
pointwise, where \( r \) and \( s \) are nonnegative integers. To prove that \( \phi_{[v_n]} \) converges pointwise to \( \phi_{[v]} \), we notice that
\[
\mathcal{L}_{[v_n]}(Z_{[v_n]}(x, t, \xi, \tau)) = (a_n(\xi, \tau) - a_n(x, t))\partial_{x\xi} Z_{[v_n]} + b_n(x, t)\partial_z Z_{[v_n]} + c_n(x, t)Z_{[v_n]},
\]
so it follows from (2.33) that
\[
\mathcal{L}_{[v_n]}(Z_{[v_n]}) \to \mathcal{L}_{[v]}(Z_{[v]}),
\]
pointwise. Besides, we have
\[
\mathcal{L}_{[v_n]}(Z_{[v_n]}(x, t, \xi, \tau)) \leq K^m \frac{n!}{(C^m)} \frac{g(\varphi, n)}{g(\varphi, n)} \frac{1}{(t-\tau) \frac{n}{2}} e^{-C|\xi-\tau|^2},
\]
where \( K \) and \( C \) are positive constants which do not depend on \( n \). Now, recalling (2.6), one can show by induction on \( m \) that \( (\mathcal{L}_{[v_n]})(Z_{[v_n]})_m \) converges to \( (\mathcal{L}_{[v]})(Z_{[v]})_m \), pointwise, as \( m \) goes to infinity. Indeed, following the construction of the fundamental solution in [10, p. 362], we have
\[
|\mathcal{L}_{[v_n]}(Z_{[v_n]}(x, t, \xi, \tau))| \leq K^m \frac{n!}{(C^m)} \frac{g(\varphi, n)}{g(\varphi, n)} \frac{1}{(t-\tau) \frac{n}{2}} e^{-C|\xi-\tau|^2},
\]
where \( g \) is the gamma function. So,
\[
|\mathcal{L}_{[v_n]}(Z_{[v_n]}(x, y, t, \sigma))| \leq K^m \frac{n!}{(C^m)} \frac{g(\varphi, n)}{g(\varphi, n)} \frac{1}{(t-\tau) \frac{n}{2}} e^{-C|\xi-\tau|^2},
\]
and thus, by the induction hypothesis, we obtain \( (\mathcal{L}_{[v_n]})(Z_{[v_n]})(L_{[v_n]})_m(Z_{[v_n]}) \to (\mathcal{L}_{[v]})(Z_{[v]})(L_{[v]})_m(Z_{[v]}), \) pointwise. Then, by the Lebesgue’s Dominated Convergence Theorem, \( (\mathcal{L}_{[v_n]})(Z_{[v_n]}(x, t, \xi, \tau)) \) converges to \( (\mathcal{L}_{[v]})(Z_{[v]})(x, t, \xi, \tau) \).

Theorem 6. Let \( T > 0, \beta \in (0, 1) \), \( v = (a, b, 0, \varphi) = (\overline{v}, 0) \in B(R, \lambda, 1), f, \mathcal{f} \in C^1(\bar{T} \{0\} \subset \overline{\Omega_T}) \) and \( u_0, \overline{u}_0 \) be Lipschitz continuous and bounded functions in \( \Omega \). If \( u \) and \( \overline{u} \) are the solutions of the problems
\[
\mathcal{L}_{[u]} u = f, \quad in \ \mathbb{R} \times (0, T], \quad u(x, 0) = u_0, \quad x \in \mathbb{R}_x, \quad (2.37)
\]
\[
\mathcal{L}_{[\overline{u}]} \overline{u} = \mathcal{f}, \quad in \ \mathbb{R} \times (0, T], \quad u(x, 0) = \overline{u}_0, \quad x \in \mathbb{R}, \quad (2.38)
\]
then
\[
\|u - \overline{u}\|_{1, \frac{1}{2}} \leq K[\|v - \overline{v}\|_{1, \frac{1}{2}} + \|v - \overline{v}\|_{1, \frac{1}{2}}^\beta + \|u_0 - \overline{u}_0\|_1] + T^\frac{1}{2}(\|f\|_{1, \frac{1}{2}}^1 + \|v - \overline{v}\|_{1, \frac{1}{2}}^\beta + \|v - \overline{v}\|_{1, \frac{1}{2}}^\beta), \quad (2.39)
\]
where \( K = K(R, \lambda, T, \|u_0\|_1) \).
Proof. From Theorem 5 we have
\[
(u - \overline{u})(x, t) = \int_R \Gamma_{[\cdot]}(x, t, \xi, 0) u_0(\xi) - \Gamma_{[\cdot]}(x, t, \xi, 0) \overline{u}_0(\xi) d\xi \\
+ \int_0^t \int_R \Gamma_{[\cdot]}(x, t, \xi, \tau)f(\xi, \tau) - \Gamma_{[\cdot]}(x, t, \xi, \tau)\overline{f}(\xi, \tau) d\xi d\tau \\
\equiv V(x, t) + W(x, t).
\]
By Lemma 4 and (2.12), we get
\[
|V(x, t)| \leq \int_R |\Gamma_{[\cdot]}(x, t, \xi, 0) u_0(\xi) - \Gamma_{[\cdot]}(x, t, \xi, 0) \overline{u}_0(\xi)| d\xi
\]
\[
\leq \int_R \left[ |\Gamma_{[\cdot]} - \Gamma_{[\cdot]}(x, t, \xi, 0) u_0(\xi)| + |\Gamma_{[\cdot]}(x, t, \xi, 0)(u_0(\xi) - \overline{u}_0(\xi))| \right] d\xi
\]
\[
\leq \int_R \left[ K\|v - \overline{u}\|_{1, 1/2} e^{-C|x - \xi|^2} \right] d\xi + \frac{K}{t^{1/2}} e^{-C|x - \xi|^2} \left\| u_0 - \overline{u}_0 \right\|_\infty d\xi
\]
\[
\leq K \left( \|v - \overline{u}\|_{1, 1/2} + \|u_0 - \overline{u}_0\|_\infty \right),
\]
where \( K = K(R, \lambda, T, \|u_0\|_\infty) \). In view of Remark 1, we can write
\[
\partial_x V(x, t) = \int_R \partial_x \Gamma_{[\cdot]}(x, t, \xi, 0) u_0(\xi) - \partial_x \Gamma_{[\cdot]}(x, t, \xi, 0) \overline{u}_0(\xi) d\xi
\]
\[
= \int_R \left( \partial_x \Gamma_{[\cdot]} - \partial_x \Gamma_{[\cdot]}(x, t, \xi, 0)(u_0(\xi) - u_0(x)) \right) d\xi
\]
\[
+ \int_R \partial_x \Gamma_{[\cdot]}(x, t, \xi, 0) \left( u_0(\xi) - \overline{u}_0(\xi) \right) - \left( u_0(x) - \overline{u}_0(x) \right) d\xi,
\]
so, by Lemma 4 and estimate (2.12) and using that
\[
\frac{|x - \xi|}{t} e^{-C|x - \xi|^2} \leq \frac{1}{t^{1/2}} e^{-C|x - \xi|^2} \leq \text{const.} \frac{1}{t^{1/2}} e^{-C|x - \xi|^2},
\]
we get
\[
|\partial_x V(x, t)| \leq \int_R \left( \frac{K\|v - \overline{u}\|_{1, 1/2}\|u_0\|_1 |x - \xi|}{t} + K \left\| u_0 - \overline{u}_0 \right\|_1 |x - \xi| \right) e^{-C|x - \xi|^2} d\xi
\]
\[
\leq \int_R \left[ \frac{K\|u_0\|_1 \|v - \overline{u}\|_{1, 1/2}}{t^{1/2}} + K \left\| u_0 - \overline{u}_0 \right\|_1 \right] e^{-C|x - \xi|^2} d\xi
\]
\[
\leq K \left( \|v - \overline{u}\|_{1, 1/2} + \|u_0 - \overline{u}_0\|_1 \right),
\]
with \( K = K(R, \lambda, T, \|u_0\|_1) \).
In order to get the Hölder continuity with respect to \( t \), using again Remark 1, we write
\[
V(x, t) - V(x, t')
\]
\[
= \int_R \left( \Gamma_{[\cdot]}(x, t, \xi, 0) - \Gamma_{[\cdot]}(x, t', \xi, 0) \right) u_0(\xi) - \Gamma_{[\cdot]}(x, t', \xi, 0) \overline{u}_0(\xi) d\xi
\]
\[
= \int_R \int_{t'}^t \partial_t \Gamma_{[\cdot]}(x, s, \xi, 0) u_0(\xi) - \partial_t \Gamma_{[\cdot]}(x, s, \xi, 0) \overline{u}_0(\xi) ds d\xi
\]
\[
= \int_{t'}^t \int_R \left( \partial_t \Gamma_{[\cdot]} - \partial_t \Gamma_{[\cdot]}(x, s, \xi, 0) u_0(\xi) - \partial_t \Gamma_{[\cdot]}(x, s, \xi, 0) \overline{u}_0(\xi) \right) d\xi ds
\]
\[
= \int_{t'}^t \int_R \left( \partial_t \Gamma_{[\cdot]} - \partial_t \Gamma_{[\cdot]}(x, s, \xi, 0) \right) u_0(\xi) - \left( u_0(x) - \overline{u}_0(x) \right) d\xi ds
\]
\[
+ \int_{t'}^t \int_R \partial_t \Gamma_{[\cdot]}(x, s, \xi, 0) \left( u_0 - \overline{u}_0 \right)(\xi) - \left( u_0 - \overline{u}_0 \right)(x) d\xi ds.
\]
Hence, using Lemma $4$ and estimate (2.12), we obtain

\[
|V(x, t) - V(x, t')| \leq \int_0^t \int_R |\partial_t \Gamma_{\parallel} - \partial_t \Gamma_{\parallel} (x, s, \xi, 0)| |u_0(\xi) - u_0(x)| \, d\xi \, ds \\
+ \int_0^t \int_R |\partial_t \Gamma_{\parallel} (x, s, \xi, 0)| |(u_0 - \overline{u}_0(\xi)) - (u_0 - \overline{u}_0)(x)| \, d\xi \, ds \\
\leq \int_0^t \int_R K(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|v - \overline{v} \|_{1, \frac{1}{2}}^\beta) |u_0(\xi) - u_0(x)| + \xi \left( \frac{1}{|x - \xi|^\gamma(1- \beta) + \frac{1}{s^\beta}} \right) e^{-C^\beta \xi x^2} \, d\xi \, ds \\
+ \int_0^t \int_R K(u_0 - \overline{u}_0) |x - \xi| e^{-C^\beta \xi x^2} \, d\xi \, ds \\
\leq \int_0^t \int_R K(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|v - \overline{v} \|_{1, \frac{1}{2}}^\beta) |u_0 - \overline{u}_0(\xi)| + \xi \left( \frac{1}{s^\beta} \right) e^{-C^\beta \xi x^2} \, d\xi \, ds \\
+ \int_0^t \int_R K(u_0 - \overline{u}_0) e^{-C^\beta \xi x^2} \, d\xi \, ds \\
\leq K(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|v - \overline{v} \|_{1, \frac{1}{2}}^\beta + |u_0 - \overline{u}_0|) \int_0^t \int_R \frac{1}{s^\beta} \, d\xi \, ds \\
\leq K(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|v - \overline{v} \|_{1, \frac{1}{2}}^\beta + |u_0 - \overline{u}_0|) (t - t')^{\frac{1}{2}},
\]

where $K = K(R, \lambda, T, \|u_0\|_1)$.

From estimates (2.40), (2.41) and (2.42), we have

\[
\|V\|_{1, \frac{1}{2}} \leq K(R, \lambda, T, \|u_0\|_1)(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|v - \overline{v} \|_{1, \frac{1}{2}}^\beta + |u_0 - \overline{u}_0|),
\]

with a new $K$.

Similarly, we can estimate $W$:

\[
W(x, t) = \int_0^t \int_R \Gamma_{\parallel} (x, t, \xi, \tau) f(\xi, \tau) - \Gamma_{\parallel} (x, t, \xi, \tau) f(\xi, \tau) \, d\xi \, d\tau \\
= \int_0^t \int_R (\Gamma_{\parallel} - \Gamma_{\parallel}) (x, t, \xi, \tau) f(\xi, \tau) + \Gamma_{\parallel} f(\xi, \tau) - \Gamma_{\parallel} f(\xi, \tau) \, d\xi \, d\tau.
\]

Hence, using Lemma $4$ and (2.12), we have

\[
|W(x, t)| \leq \int_0^t \int_R |(\Gamma_{\parallel} - \Gamma_{\parallel}) (x, t, \xi, \tau) f(\xi, \tau)| + |\Gamma_{\parallel} f(\xi, \tau) - \Gamma_{\parallel} f(\xi, \tau)| \, d\xi \, d\tau \\
\leq \int_0^t \int_R K(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|f - \overline{f} \|_{\infty}) \frac{1}{(t - \tau)^{\beta}} e^{-C^\beta \xi x^2} \, d\xi \, d\tau \\
\leq K(R, \lambda, T)(\|f\|_{\infty} + 1) T^{\beta} (\|v - \overline{v} \|_{1, \frac{1}{2}} + \|f - \overline{f} \|_{\infty}).
\]

Besides,

\[
|\partial_x W(x, t)| \leq \int_0^t \int_R |(\partial_x \Gamma_{\parallel} - \partial_x \Gamma_{\parallel}) (x, t, \xi, \tau) f(\xi, \tau)| + |\partial_x \Gamma_{\parallel} f(\xi, \tau) - \partial_x \Gamma_{\parallel} f(\xi, \tau)| \, d\xi \, d\tau \\
\leq \int_0^t \int_R K(\|v - \overline{v} \|_{1, \frac{1}{2}} + \|f - \overline{f} \|_{\infty}) \frac{1}{(t - \tau)^{\beta}} e^{-C^\beta \xi x^2} \, d\xi \, d\tau \\
\leq K(R, \lambda, T)(\|f\|_{\infty} + 1) T^{\beta} (\|v - \overline{v} \|_{1, \frac{1}{2}} + \|f - \overline{f} \|_{\infty}).
\]
To prove the Hölder continuity with respect to $t$, we write

\[
W(x, t) - W(x, t') = \int_0^t \int_{\mathbb{R}} [(\Gamma_{[t]}(x, t, \xi, \tau)) f(\xi, \tau) - \Gamma_{[\tau]}(x, t, \xi, \tau) \mathbf{f}(\xi, \tau)] d\xi d\tau \\
- \int_0^t \int_{\mathbb{R}} [(\Gamma_{[t]}(x, t', \xi, \tau)) f(\xi, \tau) - \Gamma_{[\tau]}(x, t', \xi, \tau) \mathbf{f}(\xi, \tau)] d\xi d\tau \\
= \int_0^t \int_{\mathbb{R}} [(\Gamma_{[t]}) f(\xi, \tau) - \Gamma_{[\tau]}(x, t, \xi, \tau) \mathbf{f}(\xi, \tau)] d\xi d\tau \\
+ \int_0^t \int_{\mathbb{R}} [(\Gamma_{[t]}(x, t, \xi, \tau) - \Gamma_{[t]}(x, t', \xi, \tau)) f(\xi, \tau) - (\Gamma_{[\tau]}(x, t, \xi, \tau) - \Gamma_{[\tau]}(x, t', \xi, \tau)) \mathbf{f}(\xi, \tau)] d\xi d\tau \\
= \int_0^t \int_{\mathbb{R}} [(\Gamma_{[t]}) f(\xi, \tau) + \Gamma_{[\tau]}(x, t, \xi, \tau)(f - \mathbf{f})(\xi, \tau)] d\xi d\tau \\
+ \int_0^t \int_{\mathbb{R}} [(\Gamma_{[t]}(x, t, \xi, \tau) - \Gamma_{[t]}(x, t', \xi, \tau)) f(\xi, \tau) - (\Gamma_{[\tau]}(x, t, \xi, \tau) - \Gamma_{[\tau]}(x, t', \xi, \tau)) \mathbf{f}(\xi, \tau)] d\xi d\tau \\
\equiv W_1 + W_2 + W_3
\]

where $0 < \epsilon < t'$ is arbitrary. Using Lemma [4] and (2.28), we estimate

\[
|W_1| \leq \int_0^{t'} \int_{\mathbb{R}} (K\|v - \mathbf{f}\|_{1, t/2} + K\|f - \mathbf{f}\|_{\infty}) \frac{e^{-C\frac{(t' - t)^2}{(t - \tau)^{3/2}}}}{(t - \tau)^{1/2}} d\xi d\tau \leq K(R, \lambda, \mathbf{T})(\|f\|_{\infty} + 1)T^{1/2}(\|v - \mathbf{f}\|_{1, t/2} + \|f - \mathbf{f}\|_{\infty})(t - t')^{1/2}
\]  \hspace{1cm} (2.46)

Regarding $W_2$, we apply (2.28) to get

\[
|W_2| \leq \int_0^{t'} \int_{\mathbb{R}} \left( \frac{K}{(t - \tau)^{1/2}} e^{-C\frac{(t' - t)^2}{(t - \tau)^{3/2}}} + \frac{K}{(t' - \tau)^{1/2}} e^{-C\frac{(t' - t)^2}{(t - \tau)^{3/2}}} \right) (\|f\|_{\infty} + \|\mathbf{f}\|_{\infty}) d\xi d\tau \leq K(\|f\|_{\infty} + \|\mathbf{f}\|_{\infty})\epsilon.
\]  \hspace{1cm} (2.47)

The term $W_3$ can be estimated using Remark [1] as follows:

\[
W_3 = \int_0^{t'} \int_{\mathbb{R}} \int_0^t \left[ \partial_r \Gamma_{[v]}(x, \xi, s, \tau) f(\xi, \tau) - \partial_r \Gamma_{[\tau]}(x, \xi, s, \tau) \mathbf{f}(\xi, \tau) \right] ds d\xi d\tau \\
= \int_0^{t'} \int_{\mathbb{R}} \int_0^t \left[ \partial_r \Gamma_{[v]}(x, \xi, s, \tau) f(\xi, \tau) - \partial_r \Gamma_{[\tau]}(x, \xi, s, \tau) \mathbf{f}(\xi, \tau) - (f(x, \tau) - \mathbf{f}(x, \tau)) \right] ds d\xi d\tau
\]
Now, applying Lemma 4 and (2.28), and writing $K_1 = K(\|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta)\|f\|_{1,\frac{4}{3}}$, it follows that

$$
\|W_3\| \leq \int_0^{t'-\epsilon} \int_{\mathbb{R}} K_1 \left( \frac{1}{|x - \xi|^{(1-\beta)(s - \tau)}(s - \tau)^{2s - 2\beta - 1}} + \frac{1}{(s - \tau)^2} \right) e^{-C(s - \tau)^2|x - \xi|} \, d\xi \, ds \, d\tau
$$

\[ \leq K(1 + \|f\|_{1,\frac{4}{3}}) \|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta \]

\[+ \|f - \overline{f}\|_{1,\frac{4}{3}} \left( \int_0^{t'-\epsilon} \int_{\mathbb{R}} \left( \frac{1}{(s - \tau)^2} + \frac{1}{s - \tau} \right) e^{-C(s - \tau)^2} \, d\xi \, ds \right) \]

\[\leq K(1 + \|f\|_{1,\frac{4}{3}}) \|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta \]

\[+ \|f - \overline{f}\|_{1,\frac{4}{3}} \left( \int_0^{t'-\epsilon} \int_{\mathbb{R}} \left( \frac{1}{(s - \tau)^2} + \frac{1}{s - \tau} \right) e^{-C(s - \tau)^2} \, d\xi \, ds \right) \]

\[\leq K(1 + \|f\|_{1,\frac{4}{3}}) \|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta + \|f - \overline{f}\|_{1,\frac{4}{3}} \int_0^{t'-\epsilon} \int_{\mathbb{R}} \left( \frac{1}{(s - \tau)^2} \right) \, d\xi \, ds \]

\[\leq K(1 + \|f\|_{1,\frac{4}{3}}) \|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta + \|f - \overline{f}\|_{1,\frac{4}{3}} T(t - t')^{\frac{1}{2}},
\]

where for the last inequality we used that (2.47) is true for all $\epsilon \in (0, t')$. From (2.46), (2.47) and (2.48), we conclude that

$$
\|W(x, t) - W(x, t')\| \leq K(\|f\|_{1,\frac{4}{3}} + 1) \|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta + \|f - \overline{f}\|_{1,\frac{4}{3}} T(t - t')^{\frac{1}{2}},
$$

where $K = K(R, \lambda, T)$. It follows from (2.41), (2.45) and (2.49) that

$$
\|W\|_{1,\frac{4}{3}} \leq K(R, \lambda, T) T^{\frac{1}{2}}(\|f\|_{1,\frac{4}{3}} + 1)(\|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta + \|f - \overline{f}\|_{1,\frac{4}{3}}).
$$

Finally, from (2.43) and (2.50), we have

$$
\|u - \overline{u}\|_{1,\frac{4}{3}} \leq \|v - \overline{v}\|_{1,\frac{4}{3}} + \|W\|_{1,\frac{4}{3}} \leq K(\|v - \overline{v}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta + \|u_0 - \overline{u}_0\|_{1,\frac{4}{3}} + T^{\frac{1}{2}}(\|f\|_{1,\frac{4}{3}} + 1)(\|f - \overline{f}\|_{1,\frac{4}{3}} + \|v - \overline{v}\|_{1,\frac{4}{3}}^\beta))
$$

where $K = K(R, \lambda, T, \|u_0\|_{1,\frac{4}{3}})$.

In particular we have the following estimate for a solution of (2.37)

**Corollary 2.** In the same conditions of Theorem 3, if $u$ is a solution of (2.37), then

$$
\|u\|_{1,\frac{4}{3}} \leq K(R, \lambda, T, \|u_0\|_{1,\frac{4}{3}} + T^{\frac{1}{2}}(\|f\|_{1,\frac{4}{3}} + 1)(\|f - \overline{f}\|_{1,\frac{4}{3}}),
$$

where $K = K(R, \lambda, T, \|u_0\|_{1,\frac{4}{3}})$.

**Proof.** The proof follows from Corollary 3 by taking $\overline{v} = v, \overline{f} = 2f$ and $\overline{u}_0 = 2u_0$.

## 3 Local solution

In this section we prove Theorem 4. For simplicity we shall write $f$ and $f_1$ instead of $\overline{f}$ and $\overline{f}_1$, respectively.

Consider the operator $A$ given in (1.21). In the lemma below we construct an invariant set for $A$.

**Lemma 6.** Let $R_1 = \frac{\lambda}{\alpha(2\beta + 1)} \left( 1 + \frac{2\lambda}{\alpha} \right) K(R, a_0, b_0, \|y_0\|_{0, \frac{4}{3}}) T, \|u_0\|_1$ be the constant given in the Corollary 2. $K_1 = \sup_{0 \leq \tau \leq 1} K(R, a_0, b_0, \|y_0\|_{0, \frac{4}{3}}) T, \|u_0\|_1, M_1 > K_1 \|u_0\|_1$ and $\Sigma = \{(u_1, u_2) \in \left(C^{1,\frac{4}{3}}(\mathbb{R} \times [0, T])\right)^2 : \|u_1\|_{1,\frac{4}{3}} \leq M_1 \}$.

Then $A(\Sigma) \subset \Sigma$, if $T > 0$ is sufficiently small.
Proof. Given \((u_1, u_2) \in \Sigma\) we get \(y_t \equiv y_t(u_i)\) explicitly solving \((y_t)_t = -A_i y_t f(u_i)\), i.e.

\[
y_t(x, t) = y_{i,0}(x)e^{-A_i \int_0^t f(u_i(s)) \, ds}
\]

and, so, \(0 \leq y_t \leq \|y_{i,0}\|_{\infty}\), and else,

\[
\|y_t\|_{1, \frac{1}{2}} \leq \|y_{i,0}\|_1 \left\| e^{-A_i \int_0^t f(u_i(s)) \, ds} \right\|_{1, \frac{1}{2}} + \left\| \sup_{(x,t) \in \Omega_T} \left( e^{-A_i \int_0^t f(u_i(s)) \, ds} - e^{-A_i \int_0^t f(u_i(s)) \, ds} \right) \right\|_{1, \frac{1}{2}} \\
\leq \|y_{i,0}\|_1 \left( 1 + \sup_{(x,t), (\pi, \tau) \in \Omega_T} \left( A_i \int_0^t \left| f(u_i(x,s)) - f(u_i(\pi, s)) \right| \, ds \right) \right) + \sup_{(x,t), (\pi, \tau) \in \Omega_T} \left( A_i \left| A_i \int_0^t f(u_i(s)) \, ds \right| \right) \\
\leq \|y_{i,0}\|_1 \left( 1 + TA_i \|u_i\|_{1, \frac{1}{2}} \cdot f'_{\infty} + T \frac{1}{2} A_i \right) \\
\leq \|y_{i,0}\|_1 \left( 1 + TA_i M_i \|f'\|_{\infty} + T \frac{1}{2} A_i \right) \leq 2 \|y_{i,0}\|_1,
\]

where, for the last inequality, we took \(T > 0\) sufficiently small such that \(T \frac{1}{2} + TA_i M_i \|f'\|_{\infty} \leq 1\). Furthermore, \(v_i \equiv v_i(u_i) = \left( \frac{\lambda_i}{a_i + b_i y_i(u_i)} \right) \in B(R_i, \frac{\lambda_i}{a_i + b_i y_i(u_i)})\), see the definition of the set \(B(R, \lambda, \alpha)\) at the beginning of section 2.2. Indeed,

\[
\| \frac{\lambda_i}{a_i + b_i y_i(u_i)} \|_{1, \frac{1}{2}} \\
= \sup_{(x,t) \in \Omega_T} \frac{\lambda_i}{a_i + b_i y_i(u_i)} + \sup_{(x,t), (\pi, \tau) \in \Omega_T} \left| \frac{\lambda_i}{a_i + b_i y_i(u_i)} - \frac{\lambda_i}{a_i + b_i y_i(u_i)} \right| \\
\leq \frac{\lambda_i}{a_i} + \frac{b_i \lambda_i}{a_i^2} \sup_{(x,t), (\pi, \tau) \in \Omega_T} \left| y_i(u_i(x,t)) - y_i(u_i(\pi, \tau)) \right| \\
\leq \frac{\lambda_i}{a_i} + \frac{b_i \lambda_i}{a_i^2} \|y_i\|_{1, \frac{1}{2}} \leq \frac{\lambda_i}{a_i} + \frac{2b_i \lambda_i}{a_i} \|y_{i,0}\|_{1, \frac{1}{2}}.
\]

where we used (3.2). Analogously, we can verify that \(\| \frac{\lambda_i}{a_i + b_i y_i(u_i)} \|_{1, \frac{1}{2}} \leq \frac{\lambda_i}{a_i} + \frac{2b_i \lambda_i}{a_i^2} \|y_i\|_{1, \frac{1}{2}}\) and, so,

\[
\| \frac{\lambda_i}{a_i + b_i y_i(u_i)} \|_{1, \frac{1}{2}} + \| \frac{c_i}{a_i + b_i y_i(u_i)} \|_{1, \frac{1}{2}} \leq \frac{\lambda_i}{a_i} + \frac{c_i}{a_i} \|y_{i,0}\|_{1, \frac{1}{2}} \leq \frac{\lambda_i}{a_i} + \frac{\lambda_i}{a_i} \left( 1 + \frac{2b_i \lambda_i}{a_i} \|y_{i,0}\|_{1, \frac{1}{2}} \right) = R_i
\]

Adding to the fact that \(0 < \frac{\lambda_i}{a_i + b_i y_i(u_i)} \leq \frac{\lambda_i}{a_i + b_i y_i(u_i)}\), we conclude that \(v_i(u_i) \in B \left( R_i, \frac{\lambda_i}{a_i + b_i y_i(u_i)} \right)\).

From the above, the hypotheses of Theorem 5 are satisfied. Therefore, the problem

\[
\begin{align*}
\mathcal{L}_{v_i(u_i)}(u_i) & = f_i(y_i, u_1, u_2), \quad \text{in } \mathbb{R} \times (0, T], \\
\|w_i(x, 0) = u_{i,0}(x), \quad x \in \mathbb{R},
\end{align*}
\]

has a unique solution with an exponential growth in the space \(C^{2,1}(\mathbb{R} \times (0, T]) \cap C(\mathbb{R} \times [0, T])\).
From Corollary 2 we have
\[
\|u_i\|_{1,\frac{1}{2}} \leq K(R_i, \frac{\lambda_i}{a_i + b_i \|y_i,0\|_{\infty}}, T, \|u_{i,0}\|_1)\|\|u_{i,0}\|_1
+ T\|\|f_i(y_i, u_{i1}, u_{i2})\|_{1,\frac{1}{2}} + 1)\|\|F_i(y_i, u_{i1}, u_{i2})\|_{1,\frac{1}{2}}
\leq K_i \|\|u_{i,0}\|_1 + T\\|\|f_i(y_i, u_{i1}, u_{i2})\|_{1,\frac{1}{2}} + 1)\|\|F_i(y_i, u_{i1}, u_{i2})\|_{1,\frac{1}{2}}
\leq K_i [\|\|u_{i,0}\|_1 + T\Phi(M_1, M_2, \|y_i,0\|_1)] \leq M_i,
\]
provided that $T$ is sufficiently small.

\section{3. Proof of Theorem 1}

Let $T$, $\mathcal{A}$ and $\Sigma$ be as in Lemma 4 and for any fixed $(u_0^1, u_0^2) \in \Sigma$, let $(u_n^1, u_n^2)$, $n = 1, 2, \cdots$, be the sequence defined by $(u_n^1, u_n^2) = \mathcal{A}(u_{n-1}^1, u_{n-1}^2)$. From Lemma 5 we have that this sequence is bounded in $C^{1,\frac{1}{2}}((\Omega_T), (\Omega_T = \mathbb{R} \times (0, T)))$. Then, by Arzelà-Ascoli’s theorem (see [5, p. 635]), there exists a $(u_1, u_2) \in \Sigma$ and a subsequence of $(u_n^1, u_n^2)$, which we shall still denote by $(u_n^1, u_n^2)$, such that it converges to $(u_1, u_2)$, uniformly in compacts sets in $\mathbb{R} \times [0, T]$. By Theorem 5 we can write
\[
u_i(x, t) = \int_{\Omega_T} \Gamma_{\nu_i}(x, t, \xi, 0) u_i(\xi) d\xi + \int_0^t \int_{\Omega_T} \Gamma_{\nu_i}(x, t, \xi, \tau) f_i(y_i(u_i^n), u_i^n, u_i^n)(\xi, \tau) d\xi d\tau,
\]
and
\[
u_i(x, t) = y_i(\nu_i) e^{-A_i \int_0^t f_i(u_i^s(x, s)) ds}.
\]

As $u_n^i$ converges to $u_i$, we have that $y_i(u_n^i)$, $v_i(u_n^i)$ and $f_i(y_i(u_n^i), u_n^i, u_n^i)$ converge to $y_i(u_i)$, $v_i(u_i)$ and $f_i(y_i(u_i), u_i, u_i)$, respectively. Such convergences are uniform on compacts sets in $\mathbb{R} \times [0, T]$, because $u_n^i$ so converges, $f^i$ is bounded on $\mathbb{R}$, and $\nabla f_i$ is bounded on $[0, M_1] \times [0, M_2] \times [0, \|y_i,0\|_{\infty}]$. Moreover, as $\|u_i\|_{1,\frac{1}{2}} \leq M_i$, for all $n \in \mathbb{N}$, we have that $\|u_i\|_{1,\frac{1}{2}} \leq M_i$ and so, $v_i(u_n^i), v_i(u_i) \in B(R_i, \frac{\lambda_i}{a_i + b_i \|y_i,0\|_{\infty}}, 1)$. From Lemma 5 we have that $\Gamma_{\nu_i}(u_n^i)$ converges to $\Gamma_{\nu_i}(u_i)$ pointwise.

As $\Gamma_{\nu_i}(u_n^i)(x, t, \xi, 0) u_i(\xi) \leq K - \frac{t}{2} e^{-C \frac{\xi^2}{2}}$ and
\[
\Gamma_{\nu_i}(u_n^i)(x, t, \xi, \tau) f_i(y_i(u_n^i), u_n^i, u_n^i)(\xi, \tau) \leq K(1 - \frac{t}{2} e^{-C \frac{\xi^2}{2}}),
\]
where $K$ and $C$ are constants that do depend on $n$ (see Corollary 1), it follows, by the Lebesgue’s dominated convergence theorem, that
\[
u_i(x, t) = \int \Gamma_i(x, t, \xi, 0) u_{i,0}(\xi) d\xi + \int_0^t \int \Gamma_i(x, t, \xi, \tau) f_i(y_i(u_{i1}, u_{i2}), \xi, \tau) d\xi d\tau.
\]
where $\Gamma_i$ is the fundamental solution to the equation $(w_i)_t - \alpha_i(y_i)(w_i)_{xx} + \beta_i(y_i)(w_i) = 0$, with $y_i \equiv y_i(x, t) = y_i,0(0) e^{-A_i \int_0^t f_i(u_i^s(x, s)) ds}$, and, by Theorem 5 $u_i = (u_i^1, u_i^2)$ is a solution of the system $\{1.2\} - \{1.7\}$, with $u_i \in C^{2,1}(\mathbb{R} \times (0, T)) \cap C^{1,\frac{1}{2}}(\mathbb{R} \times [0, T])$.

To obtain that $u$ is in the sector $(0, \varphi)_T$, by what we discussed in the Introduction (see p. 5) we need to show the continuous dependence of the solution of the Cauchy problem $\{1.2\}$ with respect to reaction functions $f_i$ (here, denoted simply by $f_i$) i.e. (more precisely) that the solution $u^\delta = (u_1^\delta, u_2^\delta), \delta > 0$, of
\[
\begin{cases}
(w_1)_t - \alpha(y)(w_1)_{xx} + \beta(y)(w_1) = f_i(y_i, w_1, w_2), & x \in \mathbb{R}, \ t > 0 \\
w_i(x, 0) = u_{i,O}(x), & x \in \mathbb{R},
\end{cases}
\]
where $f_i(y_i, w_1, w_2) := f_i(y_i, w_1, w_2) + \delta y_i \equiv y_i(x, t) = y_i,0(x) e^{-A_i \int_0^t f_i(u_i^s(x, s)) ds}$, converges pointwise to $u$ when $\delta \to 0+$, and, that all hypotheses of Corollary 4 are fulfilled.

Let us first observe that $\tilde{u} = (0, 0)$ and $\tilde{u} = (\varphi, \varphi)$, where (see p. 3) $\varphi(t) = (M + \beta) e^{\alpha t} - \beta$ (being $M = \max_{i=1,2} \|u_{i,0}\|_{\infty}$, $\alpha = \max_{i=1,2} \{\frac{\beta_i \|y_i,0\|_{\infty}}{a_i}\}$ and $\beta = \max_{i=1,2} \{\frac{\beta_i}{a_i}\}$) are a pair of lower and upper solutions to the system
\[
\begin{cases}
w_i(x) = \alpha_i(y)(w_i)_{xx} + \beta_i(y)(w_i) = f_i(y_i, w_1, w_2) = f_i(x, t, w_1, w_2)
\end{cases}
\]
occurring in (1.28) (i.e. the system in (3.3) without delta). (See Lemma 2 in [3].) Indeed, it is obvious that 
$\tilde{u} = (0,0)$ is a lower solution (in fact, a solution) to this system, since $f_i(0,0) = 0$. Regarding $\tilde{u} = (\varphi, \varphi)$, notice that $f_i = \frac{b_iA_iw_i + d_i}{a_i - b_iy_i}g_i\tilde{f}(w_i) \leq \frac{b_iA_iw_i + d_i}{a_i}\|y_{i,0}\|\infty$ when $w_1 = w_2$ and $w_i \geq 0$ (recall that $\tilde{f}$ is the function that coincides with the “Arrhenius function” $e^{-\frac{s}{T}}$ for $s > 0$ and vanishes for $s \leq 0$) and $\mathcal{L}_i(\varphi) = \varphi'(t) = \alpha(M + \beta)e^{\alpha t}$, so,

$$\mathcal{L}_i(\varphi)(x,t) - f_i(x,t,\varphi,\varphi) \geq \alpha(M + \beta)e^{\alpha t} - \frac{A_ib_i\varphi(t) + d_i}{a_i}\|y_{i,0}\|\infty$$

$$= (M + \beta)(\alpha - \frac{A_ib_i\|y_{i,0}\|\infty}{a_i})e^{\alpha t} + \frac{A_ib_i}{a_i}(\beta - \frac{d_i}{A_ib_i})\|y_{i,0}\|\infty$$

$$\geq 0.$$

Next, as we noticed in the Introduction, we observe that $f_i$ is increasing with respect to $w_j$ ($i, j = 1,2 ; j \neq i$) for $(a_i + b_iy_i)\partial f_i/\partial w_j = q > 0$. On the other hand, $(a_i + b_iy_i)\|\partial f_i/\partial w_i\| = |b_iA_iy_i\tilde{f}(w_i) + (b_iA_iw_i + d_i)y_i\tilde{f}'(w_i) - q| \leq b_iA_iy_i + k_i(b_iA_i + d_i)y_i + q \leq b_iA_i\|y_{i,0}\|\infty + k_i(b_iA_i + d_i)\|y_{i,0}\|\infty + q$, where $k_i$ is some positive constant, so $|\partial f_i/\partial w_i|$ is bounded by a constant, i.e. $f_i$ is uniformly lipschitz continuous in the variable $w_i$, and thus the “semi-lipschitz” condition (1.15) is satisfied with $c_i$ being a constant, for an arbitrary $\varepsilon_0$ (in the notation of Theorem [4]). Concerning the condition (1.16), we have $f_i|_{w_j=s+u_j_i} = sq$, so, it is satisfied with any $\varepsilon_0 < \delta/q$ and $\delta' = \varepsilon_0q$.

Now, we notice that both the lower solution $\hat{u} = (0,0)$ and the upper solution $\bar{u} = (\varphi, \varphi)$ satisfy trivially the condition (1.13), since their components are non negative functions. As for $u$, using the integral representation (3.8), we also see easily that it satisfies (1.13), since the first part $\int\Gamma_i(x,t,\xi,0)u_{i,0}(\xi)d\xi$ is non negative ($\Gamma_i, u_{i,0} \geq 0$) and the modulus of the second part $\int_0^t \int\Gamma_i(x,t,\xi,\tau)f_i(y_i,u_1,u_2)(\xi,\tau)d\xi d\tau$ can be estimated by a constant times $t$, because $u$ is bounded and $\int\Gamma_i d\xi = 1$ (see Remark [4]). It remains to show the continuous dependence, i.e. that $u^\delta$ converges pointwise to $u$, but up to here, we can conclude, by Theorem [4] that $u^\delta \in (0,\varphi)_T$. In particular, $u^\delta$ is bounded, uniformly with respect to $\delta$.

To show the continuous dependence, using the integral representation (2.15), with $\Gamma_i$ being the fundamental solution to the equation $(w_i)_t - \alpha_i(y_i)(w_i)_{xx} + \beta_i(y_i)(w_i)_x = 0$, and again that $\int_0^t \int\Gamma_i(x,t,\xi,\tau)\xi d\xi d\tau = 1$ (see Remark [4]), we have

$$(u_i - u_i^\delta)(x,t) = \int_0^t \int\Gamma_i(x,t,\xi,\tau)[f(y_i,u_1,u_2) - f(y_i,u_i^\delta,u_2^\delta)](\xi,\tau)d\xi d\tau \pm \delta t$$

thus, using the lipschitz continuity of $f_i$ in bounded sets (recall that $u$ is bounded and $u^\delta$ is in the sector $(0,\varphi)_T$; the latter being a consequence of Theorem [4] we obtain $\sup_{\Gamma_i}(|u - u^\delta|(x,t)) \leq K\int_0^t \sup_{\Gamma_i}(|u - u^\delta|(x,\tau))d\tau + \delta T$, so, by Gronwall's lemma, $\sup_{\Gamma_i}(|u - u^\delta|(x,t)) \leq \delta T e^{KT}$, for some constant $K$. This shows that $\lim_{\delta \to 0^+} u^\delta = u$ pointwise (in fact, uniformly) in $\Omega_T = \mathbb{R} \times (0,T)$.

Now it remains to show the $L^p$ assertion (the last assertion) in Theorem [4]. This is essentially a consequence of the “generalized Young’s inequality” [4] p. 9] and the fact that the fundamental solution $\Gamma_{[v; u^\delta_{\xi}]}$ is a “regular kernel”, uniformly with respect to $n$. More precisely, we shall show in the next paragraph that there exist positive numbers $T \leq T$ and $S$ such that, if $\|u_i^\delta(t)\|_{L^p} \leq S$ for all $t \in [0,T]$ then $\||u_i^\delta(t)|_{L^p} \leq S$ for all $t \in [0,T]$ as well. Then the assertion follows by Banach-Alaoglu's theorem.

From (3.3), the “generalized Young’s inequality” [4] p. 9] and the Minkowski's inequality for integrals
In this section we are concerned with general parabolic operators $L_i$ given by \(1.12\). We prove theorems 3 and 4 and state and prove two corollaries which are version of these theorems in the case one has continuous dependence of the solution of the system with respect to the reaction functions, and also make three remarks giving alternative conditions for the hypotheses of theorems 3 and 4.

We begin by giving the main idea to prove Theorem 3 cf. [2 Theorem 4.1] (18 Theorem 14.7). Under the hypotheses of Theorem 3 except for the condition 1.13 for now, suppose for an arbitrary small positive number $\varepsilon (0 < \varepsilon < \varepsilon_0)$ there is a point $(x_0, t_0) \in \mathbb{R}^d \times (0, T)$ on which $u = (u_1, u_2)$ belongs to the boundary of the slightly enlarged quadrant $Q_\varepsilon := \{u_1 \geq -\varepsilon$ and $u_2 \geq -\varepsilon\}$ and such that $u(x, t)$ belongs to its interior for all $(x, t) \in \mathbb{R}^d \times (0, t_0)$. If $u(x_0, t_0)$ belongs to the vertical part $V_\varepsilon := \{u_1 = -\varepsilon$ and $u_2 \geq -\varepsilon\}$ of the boundary $\partial Q_\varepsilon$, taking the equation \(1.14\) at the the point $(x, t) = (x_0, t_0)$ we obtain $u_1 = -\varepsilon$ and $L_1(u) \leq 0$, so $f_1(x_0, t_0, -\varepsilon, u_2(x_0, t_0)) + \delta = (L_1(u_1) + c_1 u_1)(x_0, t_0) \leq -c_1 \varepsilon$, which contradicts the hypothesis $f_1(x, t, u_1, u_2) \geq 0$, when $-\varepsilon_0 < u_1 < 0$ and $u_2 > -\varepsilon_0$, since we can take $\varepsilon \in (0, \varepsilon_0)$ sufficiently small such that $-c_1 \varepsilon < \delta$. Analogously, we obtain a contradiction if $u(x_0, t_0)$ belongs to the horizontal part $H_\varepsilon := \{u_1 \geq -\varepsilon$ and $u_2 = -\varepsilon\}$. Thus, the crux point of this argument is to show the existence of the point $(x_0, t_0)$ having the above properties. The idea is that if we assume that $u(x, t)$ does not belong to $Q_\varepsilon$ for all $(x, t) \in \Omega_T = \mathbb{R}^d \times (0, T)$ then, since at $t = 0, u \in \text{int.}(Q_\varepsilon)$, there would exist this “first point” $(x_0, t_0) \in \Omega_T$ with $t_0 > 0$ on which $u$ belongs to the boundary of $Q_\varepsilon$, from thence we obtain the contradiction with the assumption $f_j \geq 0$ when $-\varepsilon_0 < u_i < 0$ and $u_j > -\varepsilon_0$ (being $j \neq i, i, j = 1, 2$). However, \(a \text{ priori}\) it might occur that $u(x_n, t_n) \notin Q_\varepsilon$ for a sequence of points $(x_n, t_n) \in \Omega_T$ with $t_n \nearrow \infty$ and $|x_n| \to \infty$, even though $u(x, 0) \in \text{int.}(Q_\varepsilon)$ for all $x \in \mathbb{R}^d$, and in this case, this point $(x_0, t_0)$ would not exist. This situation is avoided with the condition 1.13.

**Proof of Theorem 3.** Let us assume there is a point $(x, t) \in \Omega_T$ such that $u(x, t) \notin Q_\varepsilon$ and we shall obtain a contradiction. If this is the case then, by the continuity of $u$, there exists another point on which $u$ belongs to $V_\varepsilon$ or $H_\varepsilon$ (defined above). Consider the case that $u \in V_\varepsilon$ (the case $u \in H_\varepsilon$ is
Proof of Theorem 3. Theorem 3 is obtained by comparison, via Theorem 3. Indeed, in the case that \( u \) is an upper solution to (1.15), defining \( w_i = u - u_i \), we have (1.15) is a positively invariant region to the system (1.14). Moreover, let \( u = (u_1, u_2) \in C^1(\Omega_T) \cap C(\mathbb{R}^d \times [0, T]) \) be a solution to the system (1.14) such that \( u(x, 0) \in Q \) for all \( x \in \mathbb{R}^d \). If \( u \) is the pointwise limit, when \( \delta \to 0^+ \), of \( u^\delta \), where \( u^\delta \in C^1(\Omega_T) \cap C(\mathbb{R}^d \times [0, T]) \) satisfying (1.14) is a solution of the Cauchy problem (assuming it has such a solution).

Corollary 3. Corollary of Theorem 3 Under the hypotheses and notations of Theorem 3 but with \( \delta = 0 \), suppose we have a continuous dependence of the solutions of the system

\[
\mathcal{L}_i(u_i) + c_i u_i = f_i(x, t, u_1, u_2)
\]

for all \( (x, t) \in \Omega_T \) and \( 0 < T \leq \infty \), \( Q = \{(u_1, u_2) \mid u_1, u_2 \geq 0\} \) is a positively invariant region to the system (1.14). More precisely, let \( u = (u_1, u_2) \in C^1(\Omega_T) \cap C(\mathbb{R}^d \times [0, T]) \) be a solution to the system (1.14) such that \( u(x, 0) \in Q \) for all \( x \in \mathbb{R}^d \). If \( u \) is the pointwise limit, when \( \delta \to 0^+ \), of \( u^\delta \), where \( u^\delta \in C^1(\Omega_T) \cap C(\mathbb{R}^d \times [0, T]) \) satisfying (1.14) is a solution of the Cauchy problem (assuming it has such a solution).

Remark 2. As we can see from the proofs of Theorem 3 and Corollary 3, we can replace in these results the condition on the reaction functions \( f_i \geq 0 \) when \( -\varepsilon_0 < u_i < 0 \) and \( u_j > \varepsilon_0 \) \((i, j = 1, 2, j \neq i)\) by \( f_i > 0 \) when \( u_i = 0 \) and \( u_j \geq 0 \), if we assume that \( u(x, 0) \in \text{int}(Q) \) for all \( x \in \mathbb{R}^d \), or, if we assume a continuous dependence also on the initial data, i.e. \( u(x, 0) \in Q \) for all \( x \in \mathbb{R}^d \) and \( u \) is the pointwise limit, when \( \delta \to 0^+ \), of the solution \( u^\delta = (u_1^\delta, u_2^\delta) \) in the space \( C^1(\Omega_T) \cap C(\mathbb{R}^d \times [0, T]) \) and satisfying (1.14) of the Cauchy problem (assuming it has such a solution)
Corollary 4. (Corollary of Theorem 3) Let the hypotheses of Theorem 3 on the reactions functions \( f_i \) be in force and suppose we have a continuous dependence of the solution of (1.17) with respect to the reaction functions \( f_i \); more precisely, suppose \( u = (u_1, u_2) \in C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0, T)) \) \((0 < T \leq \infty, \Omega_T = \mathbb{R}^d \times (0, T))\) is a solution of (1.17), which is the pointwise limit, in \( \Omega_T \), of \( \{u^{\delta}\}_{\delta > 0} \) and also of \( \{u^{-\delta}\}_{\delta > 0} \), where \( u^{\delta} = (u_1^{\delta}, u_2^{\delta}) \) (respect. \( u^{-\delta} = (u_1^{-\delta}, u_2^{-\delta}) \)), in the space \( C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0, T)) \) and satisfying (1.13), is a solution of the Cauchy problem (assuming such a solution exists)

\[
\begin{align*}
\mathcal{L}_i(u_i^{\delta}) & = f_i(x,t,u_i^{\delta}, u_2^{\delta}) + \delta, \quad (x,t) \in \Omega_T, \\
\mathcal{L}_i(u_i^{-\delta}) & = f_i(x,t,u_i^{-\delta}, u_2^{-\delta}), \quad x \in \mathbb{R}^d,
\end{align*}
\]

(4.4)

\( \delta > 0 \). Then if \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2) \) (respect. \( \hat{u} = (\hat{u}_1, \hat{u}_2) \)), in the space \( C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0, T)) \) and satisfying (1.13), is a lower (respect. upper) solution to the system (1.17) (i.e. \( \mathcal{L}_i(u_i) \leq f_i(x,t,u_1(x,t),u_2(x,t)) \) for all \((x,t) \in \Omega_T\); respect. \( \mathcal{L}_i(u_i) \geq f_i(x,t,u_1(x,t),u_2(x,t)) \) for all \((x,t) \in \Omega_T\)) such that \( u_i(x,0) \leq \tilde{u}_i(x,0) \) (respect. \( u_i(x,0) \leq \hat{u}_i(x,0) \)) for all \( x \in \mathbb{R}^d \). Then, by Theorem 3 we have that \( u_i(x,t) \geq \tilde{u}_i(x,t) \) for all \( x \in \mathbb{R}^d \times [0, T) \). Since this is true for any \( \delta > 0 \) and \( u(x,t) = \lim_{\delta \to 0^+} u_i^{\delta}(x,t) \) for all \((x,t) \in \Omega_T\), we obtain the result.

Remark 4. In the statement 1 (respect. statement 2) of Theorem 4 we can replace the conditions (1.15) and (1.16) by

\[
\begin{align*}
&f_1(x,t,s + u_1(x,t),u_2(x,t)) - f_1(x,t,u_1(x,t),u_2(x,t) \geq c_1(x,t)s, \\
f_2(x,t,u_1(x,t),s + u_2(x,t)) - f_2(x,t,u_1(x,t),u_2(x,t) \geq c_2(x,t)s
\end{align*}
\]

(4.5)

and

\[
\begin{align*}
f_1(x,t,u_1(x,t),s + u_2(x,t)) - f_1(x,t,u_1(x,t),u_2(x,t) \geq -\delta, \\
f_2(x,t,s + u_1(x,t),u_2(x,t)) - f_2(x,t,u_1(x,t),u_2(x,t)) \geq -\delta
\end{align*}
\]

(4.6)

for all \((x,t) \in \Omega_T = \mathbb{R}^d \times (0, T)\), \( s \in (-\varepsilon, 0) \), and all \( u = (u_1, u_2) \in C^{1,2}(\Omega_T) \cap C(\mathbb{R}^d \times [0, T)) \) satisfying (1.13) and such that \( u_1 \geq \tilde{u}_i \) (respect. \( u_1 \leq \hat{u}_i \)). Cf. [16, §8.2]. Indeed, following the proof of Theorem 3 p. 25 if there was a point \((x,t) \in \Omega_T \) such that \( w(x,t) := (u - \tilde{u})(x,t) \notin \mathcal{Q}_\varepsilon \) (respect. \( w(x,t) := (u - \hat{u})(x,t) \notin \mathcal{Q}_\varepsilon \)) for some arbitrarily small \( \varepsilon \), then we would get the contradiction \( (\mathcal{L}_i - c_i)u \leq 0 \) and (see the proof of Theorem 3) \( (\mathcal{L}_i - c_i)w_i \geq f_i(x,t,u_1, u_2) - f_i(x,t, \tilde{u}_i, \hat{u}_2) - c_i w_i + \delta > 0 \) at some point \((x_0,t_0) \in \Omega_T \). Notice that \( u_1 \leq \tilde{u}_i \) (respect. \( u_1 \leq \hat{u}_i \)) for all \((x,t) \in \Omega_T\).

5 Global solution

In this section we prove Theorem 2. Let us denote in this section by \( u = (u_1,u_2) \) a maximal solution of (1.2)−(1.6), defined in a maximal interval \([0,T^*]\), (see the Introduction, p. 5), in the space \( X_{T^*} = C^{2,1}(\mathbb{R} \times (0,T^*)) \cap C^{1,2}_{loc}(\mathbb{R} \times [0,T^*]) \cap L^\infty((0,\infty);L^p(\mathbb{R})) \), which was also presented in the Introduction, intercepcted with the sector \((0,\varphi)_{T^*}\). Then we shall show that \( T^* = \infty \). Throughout this section we assume all the hypotheses in Theorem 2 specially \( u_{1,0} \in L^p(\mathbb{R}) \), for some \( p \in (1,\infty) \). We suppose that \( T^* < \infty \) and we shall obtain a contradiction.

Let us recall that the convolution product of functions in conjugate Lebesgue spaces on \( \mathbb{R}^n \) decay to zero at infinity, more precisely, if \( f \in L^p(\mathbb{R}^n) \) and \( g \in L^q(\mathbb{R}^n) \), with \( 1/p + 1/q = 1 \), then \( f * g \in C_0(\mathbb{R}^n) \), where \( C_0(\mathbb{R}^n) \) denote the space of continuous functions \( h \) on \( \mathbb{R}^n \) such that \( \lim_{|x| \to \infty} |h(x)| = 0 \). Besides, \( \sup_{x \in \mathbb{R}^n} |f * g(x)| \leq \|f\|_{L^p} \|g\|_{L^q} \). See [21, p. 214] Using this fact we can prove the following lemma.

Lemma 7. For any \( t \in (0, T^* ) \) and \( s = 0, 1 \), we have \( \partial_t^s u(x,t) \in C_0(\mathbb{R}) \). Furthermore, there exist the partial derivatives \( \partial_t u(x,t) \) and \( \partial_t \partial_t u(x,t) \), for any \( x \in \mathbb{R} \times (0, T^* ) \).

\(^9\)We would like to thank Prof. Lucas C. F. Ferreira for bringing our attention to this fact and suggesting us to take the initial data \( u_{1,0} \in L^p \).
Proof. To prove the first part, for fixed $t \in (0, T^*)$ we use (3.3) to write

$$\partial_x u_i(x, t) = \int \partial_x \Gamma(x, t, \xi, 0) u_{i,0}(\xi) d\xi + \int_0^t \int \partial_x \Gamma(x, t, \xi, \tau) F_i(\xi, \tau) d\xi d\tau \equiv V(x, t) + W(x, t)$$

where $F_i(\xi, \tau) = f_i(y_i, u_1, u_2)(\xi, \tau)$. Noting that $|V(x, t)| \leq \int \frac{K}{(t-\tau)^{n/2}} |u_{i,0}(\xi)| d\xi$ and using that $u_{i,0} \in L^p$ and $e^{-C\frac{r^2}{\varepsilon^2}} \in L^q$, for $1 < p < \infty$ and $q$ being the conjugate exponent of $p$, it follows that $V(., t)$ belongs to $C_0(\mathbb{R})$. Now, for fixed $\tau$, $\tau < t$, the same argument proves also that $G(., t, \tau):= \int \partial_x \Gamma(., t, \xi, \tau) F_i(\xi, \tau) d\xi \in C_0(\mathbb{R})$. Since $|G(x, t, \tau)| \leq \frac{K}{(t-\tau)^{n/2}}$ and this latter function is integrable on $[0, t]$, it follows from the Lebesgue’s dominated convergence theorem that $W(., t) \in C_0(\mathbb{R})$ as well.

To prove the second part, for fixed $T \in (0, T^*)$ we observe that $u_i \in C^{1, 1/2}(\mathbb{R} \times [0, T])$ and $w_i = u_i$ satisfies the parabolic equation $L_{(\varepsilon(u_i))} w_i = F_i(y_i, u_1, u_2)$ in $\mathbb{R} \times (0, T)$. Given that the coefficients of this equation and $F_i$ are Hölder continuous functions, it follows from Theorem [8] p. 72 that $\partial_x u_i$ is locally Hölder continuous, which implies that the coefficients of this equation has derivative with respect to $x$ locally Hölder continuous and $(F_i)_x$ is also locally Hölder continuous. Then, again from Theorem [8] p. 72 we obtain the existence of the derivatives $\partial_x^2 u_i$ and $\partial_x \partial_x u_i$.

Next we show that $\partial_x u_i$ is bounded, in $\mathbb{R} \times (0, T^*)$. It is enough to show this bound in $\mathbb{R} \times (T, T^*)$ for a $T \in (0, T^*)$, since $u_i \in C^{1, 1/2}(\mathbb{R} \times [0, T])$, for any $T \in (0, T^*)$. We start with the following lemma.

**Lemma 8.** Let $\varepsilon \in (0, T^*)$. There is a constant $K$ with the following property: Let $a < b$ in $\mathbb{R}$ and $T \in (\varepsilon, T^*)$. If the maximum value $q_i$ of $|\partial_x u_i|$ in $[a, b] \times [\varepsilon, T]$ is attained in a point in $(a, b) \times (\varepsilon, T)$, or, $q_i$ is attained in a point in $(a, b) \times (\varepsilon, T)$ and $\partial_x u_j$ is bounded in $\mathbb{R} \times (0, T^*)$, for $i, j = 1, 2$ and $i \neq j$, then $q_i \leq K$.

**Proof.** Following [12] (or [13; 14]; see [12, p. 107]), we define a new function $v_i$ by the equation

$$v_i = h_i(v_i) := K^*(-2 + 3 \varepsilon \int_0^\Psi e^{-s^{-m_i}} ds), \quad (5.1)$$

where $m_i$ is a sufficiently large constant and $K^* := \varphi(T^*)$. ($K^*$ is a constant that bounds $u_i$. Recall that the maximal solution is in the sector $(0, \varphi)_{T^*}$ and we are assuming that $T^* < \infty$, in order to obtain a contradiction.) We notice that, for any positive numbers $m_i$, we have $v_i \geq 2/3e$ and $v_i \leq \frac{\varphi(T^*)}{K^*} + 3/3e$ for $t \leq T \leq T^*$. Indeed, $-2 + 3 \varepsilon \int_0^\Psi e^{-s^{-m_i}} ds \leq -2 + 3 \varepsilon \int_0^\Psi ds = 0$, so it must be $v_i \geq 2/3e$ in order that $\partial_x v_i \geq 0$. On the other hand, if $t \leq T \leq T^*$ then $u_i \leq \varphi(T)$, since $u = (u_1, u_2)$ belongs to the sector $(0, \varphi)_{T^*}$, thus, from the equation (5.1) we have $\int_0^\Psi e^{-s^{-m_i}} ds \leq \varphi$. But $\int_0^\Psi e^{-s^{-m_i}} ds \geq \int_0^\Psi ds = \varphi$, then $v_i \leq \varphi$.

Now making the substitution $u_i = h_i(v_i)$ in the equation (1.1) (with the constitutive functions (1.3),) we have that $v_i$ satisfies the following equation:

$$\begin{align*}
(v_i)_{xx} + \left(\frac{\lambda_i b_i(y_i)}{a_i + b_i(y_i)}\right) (v_i)_{x} + \left(\frac{c_i}{a_i + b_i(y_i)}\right) v_i = \frac{f_i(y_i, u_1(x), u_2(x))}{h_i(v_i)}.
\end{align*}$$

Differentiating with respect to $x$ we have

$$\begin{align*}
(v_i)_{x} &= \frac{\lambda_i b_i(y_i)}{a_i + b_i(y_i)} (v_i)_{xx} - \frac{c_i}{a_i + b_i(y_i)} v_i + \frac{a_i b_i(y_i)}{(a_i + b_i(y_i))^2} v_i^2,
\end{align*}$$

$$\begin{align*}
(v_i)_{x} &= \frac{\lambda_i b_i(y_i)}{a_i + b_i(y_i)} (v_i)_{xx} - \frac{c_i}{a_i + b_i(y_i)} v_i + \frac{a_i b_i(y_i)}{(a_i + b_i(y_i))^2} v_i^2.
\end{align*}$$

Let $a, b, c, T$ as in the statement of the lemma. If the maximum value of $|u_i(x)|$ in $[a, b] \times (\varepsilon, T)$ is attained in $(a, b) \times (\varepsilon, T)$, then, the same holds for $\frac{1}{3eK^*} |u_i(x)|$ and, as a consequence, this is also true for $|(v_i)_{x}|$, if $m_i$ is chosen sufficiently large. Indeed, $|(u_i)_{x}|_{x,t} = 3eK^* e^{-v_i(x,t)\frac{m_i}{3}} |u_i(x,t)|$, so for any $(x, t) \in [a, b] \times (\varepsilon, T)$, we have

$$\begin{align*}
\frac{1}{3eK^*} |(u_i)_{x}(x, t)| - |(u_i)_{x}(x, t)| &\leq \frac{1}{3eK^*} |(u_i)_{x}(x, t)| - \frac{e^{v_i(x,t)\frac{m_i}{3}}}{3eK^*} |(u_i)_{x}(x, t)|
\leq \frac{1}{3eK^*} |(u_i)_{x}(x, t)| - \frac{1}{3eK^*} |(u_i)_{x}(x, t)|
\leq |(v_i)_{x}| \frac{1}{3eK^*} q_i.
\end{align*}$$
where, as in the statement of the lemma, \( q_i \) is the maximum of \(|(u_i)_x|\) in \([a, b] \times [\epsilon, T]\). Then, \(|(v_i)_x|\) converges to \( \frac{1}{K} \max_{[\epsilon, T]} |(u_i)_x(x, t)| \), uniformly in \((x, t)\), when \( m_i \) tends to infinity. The uniform convergence together with the fact that the maximum of \(|(v_i)_x|\) occurs only in \([a, b] \times (\epsilon, T]\) ensures that the maximum of \(|(v_i)_x|\) also occurs only in \((a, b) \times (\epsilon, T]\), if we take \( m_i \) sufficiently large.

Let \( p_i \) be the maximum value of \(|(v_i)_x|\) in \([a, b] \times [\epsilon, T]\). By the hypothesis, in the first alternative in the statement of the lemma, there is a point \((x, t)\) in \((a, b) \times (\epsilon, T]\) such that \((v_i)_x(x_1, t_1) = \pm p_i\). Since we want to estimate \(|\partial x|\), we can assume \( p_i > 0 \), without loss of generality.

Initially we consider the case where \((v_i)_x(x_1, t_1) = p_i\). Then, \((v_i)_x(x_1, t_1) \geq 0, (v_i)_x(x_1, t_1) = 0\) and \((v_i)_x(x_1, t_1) \leq 0\), so, computing (5.2) in \((x_1, t_1)\), we obtain

\[
-c_i b_i (y_i)_x + \frac{\lambda_i b_i (y_i)_x}{(a_i + b_i y_i)^2} \frac{h_i''}{h_i'(a_i + b_i y_i)^2} - \frac{\lambda_i a_i + b_i y_i (h_i''(a_i + b_i y_i)^2)}{a_i + b_i y_i} \leq \frac{(f_i(y_i, h_1(v_i), h_2(v_i)))}{h_i'(v_i)}. \tag{5.3}
\]

From the definition of \( h_i \), equation (5.1), we have

\[
h_i' = 3 e K^* |(v_i)|^{-1} e^{-v_i}, \quad h_i'' = -3 e K^* m_i v_i^{m_i - 1} e^{-v_i},
\]

and from (4.1),

\[
(y_i)_x = \int_{y_0(x)}^{y_i(x)} e^{-A_i f_i'(h_i(x))} dx + \int_{y_0(x)}^{y_i(x)} e^{-f_i'(h_i(x))} dx - (A_i \int_{h_i'(v_i)}^{h_i'(v_i)} f_i'(h_i(x)) h_i'(v_i) |(v_i)_x| (x, s) ds,
\]

So, at the point \((x_1, t_1)\), we obtain the estimate

\[
|(y_i)_x| \leq (1 + 3 T^* e K^* p_i) \leq K_1 (1 + p_i), \tag{5.6}
\]

where \( K_1 \) is a constant independent of the interval \([a, b]\) and of \( T \). On the other hand,

\[
(f_i(y_i, h_1(v_i), h_2(v_i))) = -[(b_i A_i h_i'(v_i) + a_i)(y_i f_i(h_i(v_i)) + (1)^i q(h_1(v_i) - h_2(v_i)))](a_i + b_i y_i) + \{b_i A_i h_i'(v_i) f_i(h_i(v_i)) + (b_i A_i h_i'(v_i) + d_j) [(y_i f_i(h_i(v_i)) h_i'(v_i) x] + (1)^i q(h_i'(v_i)(v_i)_x - h_i'(v_i)(v_i)_x'](a_i + b_i y_i) -1,
\]

therefore, at \((x, t) = (x_1, t_1)\), \(|f_i(y_i, h_1(v_i), h_2(v_i))| \leq (1 + p_i + p_j), i \neq j\), with \( K \) being a constant independent of \( a, b \) and \( T \). As \((f_i/h_i'(v_i)) = [(f_i)_x h_i'(v_i) - f_i h_i'(v_i) (v_i)_x (h_i'(v_i))^{-2}, it follows that\)

\[
|(f_i)_x h_i'(v_i)| \leq (1 + p_i + p_j), \tag{5.7}
\]

i.e.

\[
-c_i b_i K_1 (1 + p_i) \frac{h_i''}{(a_i + b_i y_i)^2} p_i - \frac{\lambda_i b_i K_1 (1 + p_i) m_i v_i^{m_i - 1}}{(a_i + b_i y_i)^2} p_i^2 + \frac{\lambda_i m_i (m_i - 1) v_i^{m_i - 2}}{(a_i + b_i y_i)^2} p_i^3 \leq K_1 (1 + p_i + p_j).
\]

As \( 0 \leq y_i \leq \|y_0, \|_{\infty} \) and \( 2/(3 e) \leq v_i \leq 1 \), we obtain

\[
-c_i b_i K_1 (1 + p_i) \frac{a_i}{a_i^2} p_i - \frac{\lambda_i b_i K_1 m_i v_i^{m_i - 1}}{a_i} p_i^2 + \frac{((m_i) - 1) - b_i K_1 \lambda_i m_i v_i^{m_i - 2}}{(a_i + b_i y_i)^2} p_i^3 \leq K_1 (1 + p_i + p_j).
\]

Recall that \( K_1 \) in the inequality (5.8) does not depend on \( m_i \). Thus, we can take \( m_i \) large enough such that \( p_i \) and \( p_2 \) satisfy

\[
\begin{align*}
&c_1 p_1^3 - d_1 y_i^2 - c_2 p_i - 1 \leq p_2 \\
&c_2 p_2^3 - d_2 y_i^2 - c_2 p_2 - 1 \leq p_i,
\end{align*}
\tag{5.8}
\]
where \( c_i, d_i, e_i \) are positive constants and independent of \( a, b \) and \( T \). In the first quadrant, i.e. \( p_1 \geq 0 \) and \( p_2 \geq 0 \), the region defined by \( \{5.8\} \) is bounded. Therefore, there is a constant \( \mathcal{K}(T^*) \), such that \( \eta < p_1 \leq \mathcal{K}(T^*) \).

Similarly, we can see that the same occurs when \( \frac{\partial x_v}{\partial t}(x_1, t_1) = -p_2 \) and \( \frac{\partial x_u}{\partial t}(x_1, t_1) = -p_1 \) and \( \frac{\partial x_v}{\partial t}(x_2, t_2) = -p_2 \). Indeed, suppose now that \( \frac{\partial x_v}{\partial t}(x_1, t_1) = p_1 \) and \( \frac{\partial x_v}{\partial t}(x_2, t_2) = -p_2 \). In this case we have \( (\nu_1)_{\nu t}(x_1, t_1) \geq 0, (\nu_1)_{\nu t}(x_1, t_1) = 0 \) and \( (\nu_1)_{\nu t}(x_1, t_1) \geq 0 \). While at the point \( (x_2, t_2) \), we have \( (\nu_2)_{\nu t}(x_2, t_2) \leq 0, (\nu_2)_{\nu t}(x_2, t_2) = 0 \) and \( (\nu_2)_{\nu t}(x_2, t_2) \geq 0 \). If we compute \( (5.2) \) at \( (x_1, t_1) \), we obtain

\[
\left\{ \begin{array}{l}
\frac{c_1 b_1(y_1)}{a_1 + b_1 y_1} \frac{h''_{y_1}}{h_{y_1}'} p_1 + \lambda_i b_i(y_1) \frac{h''_{y_i}}{h_{y_i}'} p_i^2 - \lambda_i \frac{h''_{y_i}}{h_{y_i}'} p_i^2 \leq \frac{f(y_1, y_1, y_1)}{h_{y_1}(y_1)} x_i,
\frac{c_2 b_2(y_2)}{a_2 + b_2 y_2} \frac{h''_{y_2}}{h_{y_2}'} p_2 + \lambda_i b_i(y_2) \frac{h''_{y_i}}{h_{y_i}'} p_i^2 \geq \frac{f(y_2, y_2, y_2)}{h_{y_2}(y_2)} x_i.
\end{array} \right.
\]

Again, for \( m_1 \) large enough, we have that \( p_1 \) and \( p_2 \) satisfy a system of the type \( \{5.8\} \), and therefore \( (p_1, p_2) \) is in a bounded region of the plane.

Finally, suppose that \( (\nu_1)_{\nu t}(x_1, t_1) = -p_1 \) and \( (\nu_2)_{\nu t}(x_2, t_2) = -p_2 \). In this case, at the point \( (x_1, t_1) \) we have that \( (\nu_1)_{\nu t}(x_1, t_1) \leq 0, (\nu_1)_{\nu t}(x_1, t_1) = 0 \) and \( (\nu_1)_{\nu t}(x_1, t_1) \geq 0 \). If we compute \( (5.2) \) at the point \( (x_1, t_1) \), we obtain

\[
\left\{ \begin{array}{l}
\frac{c_1 b_1(y_1)}{a_1 + b_1 y_1} \frac{h''_{y_1}}{h_{y_1}'} p_1 + \lambda_i b_i(y_1) \frac{h''_{y_i}}{h_{y_i}'} p_i^2 + \lambda_i \frac{h''_{y_i}}{h_{y_i}'} p_i^2 \geq \frac{f(y_1, y_1, y_1)}{h_{y_1}(y_1)} x_i,
\frac{c_2 b_2(y_2)}{a_2 + b_2 y_2} \frac{h''_{y_2}}{h_{y_2}'} p_2 + \lambda_i b_i(y_2) \frac{h''_{y_i}}{h_{y_i}'} p_i^2 \geq -K(1 + p_1 + p_2)
\end{array} \right.
\]

and, again, for \( m_1 \) large enough, analogously as in the previous cases, we infer that \( (p_1, p_2) \) is bounded, independently of \( a, b \) and \( T \).

To prove the lemma under the second hypothesis alternative, it is enough to notice that if \( \frac{\partial x_v}{\partial t} \) is bounded in \( \mathbb{R} \times (0, T^*) \), the first inequality in \( \{5.8\} \) is sufficient to assure that \( \frac{\partial x_v}{\partial t} \) is bounded, and similarly, for the case that \( \frac{\partial x_v}{\partial t} \) is bounded.

In the next lemma, using that \( u \in L^\infty_\nu([0, \infty); L^p(\mathbb{R})) \), we show that \( u(x, t) \) decay to zero when \( |x| \to \infty \), uniformly with respect to \( t \) in any compact interval in \( (0, T^*) \).

**Lemma 9.** Let \( [\tilde{t}, T] \subset (0, T^*) \) \( \tilde{t} < T < T^* \). Then \( \lim_{|x| \to \infty} |u(x, t)| = 0 \), uniformly with respect to \( t \in [\tilde{t}, T] \).

**Proof.** Let \( t \in [\tilde{t}, T] \). For \( 0 < \epsilon < t \), from \( \{3.8\} \) we have that

\[
\frac{\partial u}{\partial t}(x, t) = \int \frac{\partial_x \Gamma_{(v_1, u_1)}(x, t, \xi, \tau) f(y_1(u_1), u_1, u_2)(\xi, \tau) d\xi d\tau}{\int_0^t \partial_x \Gamma_{(v_1, u_1)}(x, t, \xi, \tau) f(y_1(u_1), u_1, u_2)(\xi, \tau) d\xi d\tau}
\]

therefore, using the estimate \( (2.12) \), we have

\[
\left| \frac{\partial u}{\partial t}(x, t) \right| \leq \int_0^t \frac{K}{t^\epsilon} e^{-c(x, t)^2} |v_1, 0(\xi)| d\xi + \int_0^{t-\epsilon} \frac{K}{t^\epsilon} e^{-c(x, t)^2} |f(y_1, u_1, u_2)(\xi, \tau)| d\xi d\tau
\]

\[
\leq \int_0^t \frac{K}{t^\epsilon} e^{-c(x, t)^2} |v_1, 0(\xi)| d\xi + \int_{t-\epsilon}^t \frac{K}{t^\epsilon} e^{-c(x, t)^2} |f(y_1, u_1, u_2)(\xi, \tau)| d\xi d\tau
\]

\[
\leq \int_{t-\epsilon}^t \frac{K}{t^\epsilon} e^{-c(x, t)^2} |v_1, 0(\xi)| d\xi + \int_{t-\epsilon}^t \frac{K}{t^\epsilon} e^{-c(x, t)^2} |f(y_1, u_1, u_2)(\xi, \tau)| d\xi d\tau
\]
Since $u = (u_1, u_2) \in \langle 0, \varphi \rangle_{T^*}$, $(y_i, u_1, u_2)$ belongs to a bounded region in $\mathbb{R}^3$. If $\|f_i\|_\infty$ is the sup of $|f_i|$ in this region, we have that

$$|\partial_x u_i(x,t)| \leq \int \frac{K e^{-C_2 x^2}}{t^3} |u_{i,0}(\xi)| d\xi + \int_0^t \int \frac{K e^{-C_2 (t-\tau)^2}}{t} |f_i(y_i(u_i), u_1, u_2)(\xi, \tau)| d\xi d\tau + K\|f_i\|_\infty \int_{t-\tau}^t \frac{1}{(t-\tau)} d\tau$$

$$\leq \int \frac{K e^{-C_2 x^2}}{t^3} |u_{i,0}(\xi)| d\xi + \int_0^t \int \frac{K e^{-C_2 (t-\tau)^2}}{t} |f_i(y_i(u_i), u_1, u_2)(\xi, \tau)| d\xi d\tau + K e^{\frac{1}{2}}$$

From [7, p. 241], the last two integrals of the above inequalities belong to $C_0$ and do not depend on $t \in [\bar{T}, \bar{T}]$. Thus, taking an interval $[a, b]$ for which these integrals are less than $e^{\frac{1}{2}}$ in $[a, b]^c$, we have $|\partial_x u_i(x,t)| \leq Ke^{\frac{1}{2}}$ for all $(x, t) \in [a, b]^c \times [\bar{T}, \bar{T}]$. The constant $K$ is essentially the same that appears in the estimate (2.12).

**Corollary 5.** $\partial_x u_1$ is bounded in $\mathbb{R} \times (0, T^*)$.

**Proof.** Let $T_1 \in (0, T^*)$ fixed. We take a constant $\bar{T} > 0$ strictly greater than the constant obtained in Lemma [5] and $sup_{x \in [0, T]}|\partial_x u_i|$. Initially, we assume that both $u_{1,0}$ and $u_{2,0}$ are unbounded in $\mathbb{R} \times (0, T^*)$. Let $T \in (0, T^*)$ such that $|u_{1,0}| > \bar{T}$ at some point of $\mathbb{R} \times (0, T)$. Consider $\bar{T} = \inf\{t; |\partial_x u_1(x,t)| > \bar{T} \text{ for some } x \in \mathbb{R}\}$ and let $(x_n, t_n) \in \mathbb{N}$ be a sequence such that $|\partial_x u_1(x_n, t_n)| > \bar{T}$, for all $n \in \mathbb{N}$, and $t_n \not\to \bar{T}$. It is clear that $\bar{T} > T_1$. Let $\varepsilon \in (0, T^* - T_1)$. From Lemma [5] there is an interval $[a, b]$ and a subsequence of $(x_n)$ such that $x_n \to \bar{x}$ and $\{\partial_x u_1\}$ is bounded in $\mathbb{R} \times (0, T)$. Similarly, assuming that $\partial_x u_1$ is unbounded, we obtain a point $(\bar{x}, \bar{T})$ such that $|\partial_x u_2(\bar{x}, \bar{T})| = \bar{T} = \max\{|\partial_x u_1| \in \mathbb{R} \times (0, \bar{T})\}$. From Lemma [5] there is an interval $(A, B)$ containing the points $\bar{x}$ and $\bar{T}$, we obtain a contradiction to Lemma [5] because both maximum points of $|\partial_x u_1|$ and $|\partial_x u_2|$ occur in $(A, B) \times (\frac{\bar{T}}{2}, \bar{T})$ and both are bounded by the constant given by Lemma [5]. If $\bar{T} < T_1$, Lemma [5] assures the existence of an interval $(A, B)$ for which $|\partial_x u_1| < \bar{T}$ in $[A, B]^c \times [T, \bar{T}]$. Thus, the maximum points of $|\partial_x u_1|$ and $|\partial_x u_2|$ in $(A, B) \times (\frac{\bar{T}}{2}, \bar{T})$ both occur in $(A, B) \times (\frac{\bar{T}}{2}, \bar{T})$ and they are not bounded by the constant given by Lemma [5].

Let us now assume that $\partial_x u_1$ is unbounded and $\partial_x u_2$ is bounded, in $\mathbb{R} \times (0, T^*)$. Repeating the initial argument for $\partial_x u_1$, we get that the maximum point $(\bar{x}, \bar{T})$ of $|\partial_x u_1|$ in $\mathbb{R} \times (0, \bar{T})$ occurs in $(A, B) \times (\frac{\bar{T}}{2}, \bar{T})$ and the maximum $|\partial_x u_1(\bar{x}, \bar{T})| = \bar{T}$ is greater than the constant given by Lemma [5]. Adding to this the fact that $\partial_x u_2$ is bounded in $\mathbb{R} \times (0, T^*)$, we again arrive at a contradiction with Lemma [5].

**Lemma 10.** The function $y_i = y_{i0}(x) e^{-A_i \int_0^t f(u_i(s)) ds}$ and the coefficients $\alpha_i(y_i), \beta_i(y_i) \in C^{1, \frac{1}{2}}(\mathbb{R} \times [0, T^*))$.

**Proof.** It is clear that $|y_i(x, t)| \leq \|y_i0\|_\infty$, for all $\mathbb{R} \times (0, T^*)$. Besides,

$$\partial_x y_i(x, t) = (y_{i0}(x) - A_i y_{i0}(x) \int_0^t f(u_i(x, s)) \partial_x u_i(x, s) ds) e^{-A_i \int_0^t f(u_i(x, s)) ds}$$

so, since $y_{i0}$ is bounded, by hypothesis, and we have Corollary [5] it follows that $(y_i)_x$ is bounded in $\mathbb{R} \times (0, T^*)$. Moreover,

$$|y_i(x, t) - y_i(x, t')| \leq y_{i0}(x)|e^{-A_i \int_0^t f(u_i(x, s)) ds} - e^{-A_i \int_0^t f(u_i(x, s)) ds}|$$

$$\leq K |\int_0^t f(u_i(x, s)) ds - \int_0^t' f(u_i(x, s)) ds| \leq K |\int_{t'}^t f(u_i(x, s)) ds|$$

$$\leq K (t - t') \leq K (t - t') \frac{1}{2}$$

for all $(x, t, x, t') \in \mathbb{R} \times (0, T^*)$, with $|t - t'| \leq 1$, for some constant $K$. Finally, as the composition of a Hölder continuous function with a function having a bounded derivative is also a Hölder continuous function, the result follows by using Lemma [5].
5.1 Proof of Theorem 2

Due to Lemma (10), we can consider the parabolic equation $\partial_t - \alpha_i(y_i)\partial_{xx} + \beta_i(y_i)\partial_x = 0$ in the domain $\mathbb{R} \times [0, T^*)$. Let us denote its fundamental solution by $\Gamma$. By Theorem 5 we can write

$$u_i(x, t) = \int_{\mathbb{R}} \Gamma(x, t, \xi, 0) u_{i,0}(\xi) d\xi + \int_0^t \int_{\mathbb{R}} \Gamma(x, t, \xi, \tau) f_i(y_i, u_1, u_2)(\xi, \tau) d\xi d\tau$$

(5.12)

for all $(x, t) \in \mathbb{R} \times [0, T^*)$. Now, since $u_{i,0}$ is bounded, by hypothesis, $u_i$ is bounded (recall that $u = (u_1, u_2)$ is in the sector $(0, \varphi)T^*$). and we have the estimate $\Gamma \leq K(t - \tau)^{-1/2} e^{-C(t-\tau)/\tau^2}$ (see (2.12)), for $x, \xi \in \mathbb{R}$ and $t, \tau \in [0, T^*)$, $\tau < t$, it follows that the functions $\Gamma(x, t, \xi, 0) u_{i,0}(\xi), \Gamma(x, t, \xi, \tau) f_i(y_i, u_1, u_2)(\xi, \tau)$ are integrable with respect to $\xi$ in $\mathbb{R}$ when $t = T^*$, and that $\int_0^t \int_{\mathbb{R}} \Gamma(x, T^*, \xi, \tau) f_i(y_i, u_1, u_2)(\xi, \tau) d\xi d\tau$ is integrable with respect to $\tau$ in $[0, T^*)$. Thus the right hand side of (5.12) is well defined for $t = T^*$ and we set $u_i(x, T^*)$ as being this value.

Next, with this definition, we show that $u_i(x, t)$ converges to $u_i(x, T^*)$ when $t \to T^*$ uniformly with respect to $x \in \mathbb{R}$. In fact, we have $\|u_i(\cdot, T^*) - u_i(\cdot, t)\|_{L^\infty(x)} \leq K(T^* - t)^{1/2}$, for some constant $K$. Indeed, $u_i(x, T^*) - u_i(x, t)$

$$= \int_{\mathbb{R}} \Gamma(x, T^*, 0) - \Gamma(x, t, 0) u_{i,0}(\xi) d\xi$$

$$+ \int_t^{T^*} \int_{\mathbb{R}} \Gamma(x, t, \xi, \tau) f_i(y_i, u_1, u_2)(\xi, \tau) d\xi d\tau$$

$$+ \int_0^t \int_{\mathbb{R}} \Gamma(x, \tau, \xi, \tau) - \Gamma(x, t, \xi, \tau) f_i(y_i, u_1, u_2)(\xi, \tau) d\xi d\tau$$

$$= I_1 + I_2 + I_3.$$

If $\frac{T^*}{s} \leq t \leq T^*$, we have

$$|I_1| = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma(x, t, \xi, 0) u_{i,0}(\xi) d\xi = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma(x, t, \xi, \tau) f_i(y_i, u_1, u_2)(\xi, \tau) d\xi d\tau$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K \|f_i\|_{L^\infty} |\xi - \tau|^{-1/2}$$

$$\leq K \|f_i\|_{L^\infty} |t|^2 |\xi - \tau|^{-1/2}$$

$$= K \|f_i\|_{L^\infty} (T^* - t)^{1/2}.$$
\[ K \int_0^t [(T^* - \tau)^{\frac{1}{2}} - (t - \tau)^{\frac{1}{2}}] d\tau \leq K \int_0^t (T^* - t)^{\frac{1}{2}} d\tau \leq KT^* (T^* - t)^{\frac{1}{2}}. \]

From the above convergence, we conclude that \( u_i(\cdot, T^*) \) is bounded and nonnegative (since \( u = (u_1, u_2) \in (0, \varphi, T^*) \) and it is Lipschitz continuous as well, by Corollary [3]. To end the proof of Theorem 2, it remains to prove that \( u_i(\cdot, T^*) \in L^P \). Using again the “generalized Young’s inequality” [6, p. 9] and the Minkowski’s inequality for integrals [7, p. 194] (see the proof of the \( L^P \) assertion (the last assertion) in Theorem 1), we obtain

\[
\| u_i(\cdot, \tau) \|_{L^P} \leq C_1 + C_2 \int_0^T (\| u_1(\cdot, \tau) \|_{L^P} + \| u_2(\cdot, \tau) \|_{L^P}) d\tau.
\]

Thus, \( \| u_1(\cdot, t) \|_{L^P} + \| u_2(\cdot, t) \|_{L^P} \leq C_1 + C_2 \int_0^T (\| u_1(\cdot, \tau) \|_{L^P} + \| u_2(\cdot, \tau) \|_{L^P}) d\tau \). Then, defining \( \phi(t) = \| u_1(\cdot, t) \|_{L^P} + \| u_2(\cdot, t) \|_{L^P} \), we have \( \phi(t) \leq C_1 + C_2 \int_0^t \phi(\tau) d\tau \). By the Gronwall’s inequality for integrals (see [5, p. 625]), it follows that \( \phi(t) \leq C_1 (1 + C_2 T^* e^{C_2 T^*}) \), for all \( t \in [0, T^*] \). Therefore, \( \| u_i(\cdot, t) \|_{L^P} \leq C_1 (1 + C_2 T^* e^{C_2 T^*}) \), for all \( t \in [0, T^*] \), and from the Fatou’s lemma, we have \( u_i(x, T^*) \in L^P \).

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