EQUIVARIANT VECTOR BUNDLES OVER CLASSIFYING SPACES FOR PROPER ACTIONS

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Abstract. Let $G$ be an infinite discrete group and let $EG$ be a classifying space for proper actions of $G$. Every $G$-equivariant vector bundle over $EG$ gives rise to a compatible collection of representations of the finite subgroups of $G$. We give the first examples of groups $G$ with a cocompact classifying space for proper actions $EG$ admitting a compatible collection of representations of the finite subgroups of $G$ that does not come from a $G$-equivariant (virtual) vector bundle over $EG$. This implies that the Atiyah-Hirzebruch spectral sequence computing the $G$-equivariant topological $K$-theory of $EG$ has non-zero differentials. On the other hand, we show that for right angled Coxeter groups this spectral sequence always collapses at the second page and compute the $K$-theory of the classifying space of a right angled Coxeter group.

1. Introduction

Let $G$ be an infinite discrete group and $\mathcal{F}$ be the family of finite subgroups of $G$. Recall that a classifying space for proper actions of $G$, denoted by $EG$, is a proper $G$-CW-complex such that the fixed point set $EG^H$ is contractible for every $H \in \mathcal{F}$. The space $EG$ is said to be cocompact if the orbit space $G \backslash EG = BG$ is compact. Many interesting classes of groups $G$ have cocompact models for $EG$, for example cocompact lattices in Lie groups, mapping class groups of surfaces, Out($F_n$), CAT(0)-groups and word-hyperbolic groups. We refer the reader to [7] for more examples and details.

Now assume $G$ is an infinite discrete group admitting a cocompact classifying space for proper actions $EG$. If

$$\xi : E \to EG$$

is a $G$-equivariant complex vector bundle over $EG$ (see Definition 2.2) and $x$ is a point of $EG$, then the fiber $\xi^{-1}(x)$ is a complex representation of the finite isotropy group $G_x$. The connectivity of the fixed point sets of $EG$ ensures that these representations are compatible (see Definition 2.1) with one another as $x$ and hence $G_x$ varies. Therefore, every $G$-equivariant complex vector bundle over $EG$ gives rise to a compatible collection of complex representations of the finite subgroups of $G$, and hence to an element of

$$\lim_{G/H \in O_FG} R(H).$$

Here, $\lim_{G/H \in O_FG} R(H)$ is the limit over the orbit category $O_FG$ of the representation ring functor

$$R(\cdot) : O_FG \to \text{Ab} : G/H \mapsto R(H).$$

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Denoting the Grothendieck completion of the monoid of isomorphism classes of complex $G$-vector bundles over $EG$ by $K^0_G(EG)$, one obtains a map
\[ \varepsilon_G : K^0_G(EG) \to \lim_{G/H \in \mathcal{O}_G} R(H) \]
that maps a formal difference of (isomorphism classes) vector bundles (i.e. a virtual vector bundle) to a formal difference of (isomorphism classes) of compatible collections of representations of the finite subgroups of $G$. We say a compatible collection of representations of the finite subgroups of $G$ can be realized as a (virtual) $G$-equivariant vector bundle over $EG$ if there exists a (virtual) $G$-equivariant vector bundle over $EG$ that maps to this collection under $\varepsilon_G$. One can also look at the corresponding situation for real (orthogonal) vector bundles and real (orthogonal) representations and obtain the map
\[ \varepsilon_G : KO^0_G(EG) \to \lim_{G/H \in \mathcal{O}_G} RO(H). \]

The maps $\varepsilon_G$ are equal to the edge homomorphisms of certain Atiyah-Hirzebruch spectral sequences converging to $K^*_G(EG)$ and $KO^*_G(EG)$ (see (1) and (2)). Lück and Oliver proved that (see Proposition 2.4) the map $\varepsilon_G$ (real or complex) is rationally surjective, meaning that a high enough multiple of every element in the target of $\varepsilon_G$ is contained in the image of $\varepsilon_G$. In the last paragraph of [10, p. 596] Lück and Oliver ask for an example of a group $G$ admitting a cocompact classifying space for proper actions $EG$ such that $\varepsilon_G$ is not surjective. In Section 3 of this paper we give the first example of such a group in the complex case. In Section 4 we give the first example of such a group in the real case. In both cases we also explain how to construct an example of a group $G$ admitting a cocompact classifying space for proper actions $EG$ that maps to this collection of representations under $\varepsilon_G$. On the other hand, these examples are more explicit and lower dimensional.

In the last section we show that for a right angled Coxeter group $W$, every compatible collection of representations of the finite subgroups of $W$ can be realized as a $W$-equivariant vector bundle over $EW$, so that the map
\[ \varepsilon_W : K^0_W(EW) \to \lim_{W/H \in \mathcal{O}_W} R(H). \]
is always surjective. Moreover, we show that this map is actually an isomorphism and that the higher equivariant $K$-theory (see Theorem 2.3) vanishes, i.e.
\[ K^1_W(EW) = 0. \]

Using a version of the Atiyah-Segal completion theorem for infinite discrete groups proven by Lück and Oliver, we use these results to compute the complex $K$-theory of $BW$, the classifying space of $W$ (see Corollary 5.5).

2. $G$-VECTOR BUNDLES AND ISOPTROPY REPRESENTATIONS

Throughout, let $G$ be a discrete group and let $X$ be a cocompact proper $G$-CW-complex, i.e. $X$ has finite isotropy and the orbit space $G \backslash X$ has a finite number of cells. The family of finite subgroups of $G$ will be denoted by $\mathcal{F}$. Although it is not necessary for most of the statements in this section, we will also assume that for every $H \in \mathcal{F}$, the fixed point set $X^H$ is non-empty and connected. For $H \in \mathcal{F}$, we also fix a zero cell $e_H \in X^H$. Recall that the
orbit category $\mathcal{O}_F G$ is the category whose objects are the cosets $G/H$, for all $H \in F$, and whose morphism are all $G$-equivariant maps between the objects.

**Definition 2.1.** Let $\Gamma$ be either the unitary group $U(n)$ or the real orthogonal group $O(n, \mathbb{R})$. For $H \in F$, let

$$\text{Rep}_\Gamma(H) = \text{Hom}(H, \Gamma)/\text{Inn}(\Gamma).$$

Note that if $\Gamma = U(n)$ ($\Gamma = O(n)$), then $\text{Rep}_\Gamma(H)$ is the set of isomorphism classes of $n$-dimensional complex (real) representations of $H$. One can consider $\text{Rep}_\Gamma(-)$ as a functor from $\mathcal{O}_F G$ to Sets. An element of the limit

$$\alpha = ([\alpha_H])_{H \in F} \in \lim_{G/H \in \mathcal{O}_F G} \text{Rep}_\Gamma(G/H)$$

is a called a compatible collection of complex or real $n$-dimensional representations of the finite subgroups of $G$. For $H \in F$, let $R(H)$ ($RO(H)$) be the complex (real) representation ring of $H$, i.e. the Grothendieck completion of the abelian cancellative monoid of isomorphism classes of finite dimensional complex (real) representations of $H$. Note that $\text{Rep}_{U(n)}(H)$ is naturally a subset of $R(H)$ and $\text{Rep}_{O(n)}(H)$ is naturally a subset of $RO(H)$. One can consider $R(-)$ as a functor from $\mathcal{O}_F G$ to Ab. An element of the inverse limit

$$\alpha = ([\alpha_H])_{H \in F} \in \lim_{G/H \in \mathcal{O}_F G} R(H)$$

is a called a compatible collection of complex virtual representations of the finite subgroups of $G$. One has

$$\lim_{G/H \in \mathcal{O}_F G} \text{Rep}_{U(n)}(G/H) \subset \lim_{G/H \in \mathcal{O}_F G} R(H)$$

and every element of $\lim_{G/H \in \mathcal{O}_F G} R(H)$ can be written as the difference of elements in $\lim_{G/H \in \mathcal{O}_F G} \text{Rep}_{U(n)}(G/H)$ and $\lim_{G/H \in \mathcal{O}_F G} \text{Rep}_{U(m)}(G/H)$, for $n$ and $m$ large enough. The analogous statements for $O(n, \mathbb{R})$ and $RO$ also hold.

**Definition 2.2.** A complex (real) $G$-vector bundle over $X$ is a complex (real) vector bundle $\pi : E \rightarrow X$ such that $\pi$ is $G$-equivariant and each $g \in G$ acts on $E$ and $X$ via a bundle isomorphism. An isomorphism of $G$-vector bundles over $X$ is just an isomorphism of vector bundle that is $G$-equivariant. The set of isomorphisms classes of complex (real) $G$-vector bundles over $X$ will be denoted by $\text{Bdl}_G(X)$ ($\text{Brd}_G(X)$). For every $x \in X$, the fiber $\pi^{-1}(x)$ is denoted by $E_x$. We refer the reader to [10] Section 1] and [14 Section I.9] for elementary properties of $G$-vector bundles over proper (cocompact) $G$-CW complexes.

**Theorem 2.3.** [10 Th. 3.2 and 3.15] There exists a 2-periodic (8-periodic) equivariant cohomology theory $K^p_G(X, A)$ ($KO^p_G(X, A)$) on the category of proper $G$-CW-pairs such that $K^0_G(X)$ ($KO^0_G(X)$) is the Grothendieck completion of the monoid of isomorphism classes of complex (real) $G$-vector bundles over $X$. In particular, for every $H \in F$, $K^0_G(G/H)$ ($KO^0_G(G/H)$ ) equals $R(H)$ ($RO(H)$).

As usual (see [11 Section 6] and [3 Th. 4.7]), the skeletal filtration of $X$ induces Atiyah-Hirzebruch spectral sequences

$$E_2^{p,q} = H^p_G(X, K^q_G(G/-)) \Longrightarrow K^{p+q}_G(X),$$

and

$$E_2^{p,q} = H^p_G(X, KO^q_G(G/-)) \Longrightarrow KO^{p+q}_G(X),$$

where $H^p_G(X, -)$ denotes Bredon cohomology of $X$ (see [11]).
Proposition 2.4. [11 Prop 5.8] The spectral sequences (1) and (2) above rationally collapse, meaning that the images of all differentials in these spectral sequences consist of torsion elements.

By our assumptions on $X$, the degree zero Bredon cohomology group $H^0_G(X, R(-))$ (resp. $H^0_G(X, RO(-))$), equals the limit of the functor $R(-)$ (resp. $RO(-)$), over the orbit category $\mathcal{O}_F G$. Consider the edge homomorphisms

$$\varepsilon_G : K^0_G(X) \to H^0_F(X, R(-))$$

and

$$\varepsilon_G : KO^0_G(X) \to H^0_F(X, RO(-))$$

of the spectral sequences (1) and (2). If $[\pi]$ is the isomorphism class of an $n$-dimensional complex $G$-vector bundle $\pi : E \to X$, then $\varepsilon_G([\pi])$ equals

$$(E_H)_H \in F \lim_{G/H \in \mathcal{O}_F G} \text{Rep}_{U(n)}(H) \subset H^0_G(X, R(-))$$

where $[E_H]$ denotes the isomorphism class in $R(H)$ of the $H$-representation $E_H$. The corresponding statement for real $G$-vector bundles also holds. Note that it follows from Proposition 2.4 that a suitable multiple of every compatible collection of (virtual) real or complex representations of the finite subgroups of $G$ is contained in the image of the edge homomorphism $\varepsilon_G$.

Recall that the classifying space for proper actions $EG$ is a terminal object in the homotopy category of proper $G$-CW complexes (e.g. [7 Th. 1.9]). Hence, if $X$ is any proper cocompact $G$-CW complex such that $X^H$ is non-empty and connected for each $H \in F$, then there exists a $G$-map $X \to EG$ that is unique up to $G$-homotopy and induces commutative diagrams

$$
\begin{array}{ccc}
K^0_G(X) & \to & \lim_{G/H \in \mathcal{O}_F G} R(H) \\
\downarrow K^0_G(EG) & & \downarrow \lim_{G/H \in \mathcal{O}_F G} RO(H) \\
KO^0_G(X) & \to & KO^0_G(EG). \\
\end{array}
$$

Hence, if a compatible collection $\alpha$ of representations can be realized as a $G$-vector bundle over $EG$, it can also be realized as a $G$-vector bundle over $X$.

3. Complex vector bundles

The purpose of this section is to construct a group $G$ with a cocompact classifying space for proper actions $EG$ admitting a compatible collection of complex representations of the finite subgroups of $G$ that cannot be realized as $G$-equivariant virtual complex vector bundle over $EG$, i.e. so that the edge homomorphism

$$\varepsilon_G : K^0_G(EG) \to \lim_{G/H \in \mathcal{O}_F G} R(H).$$

is not surjective.

Let $F = C_4 \rtimes C_2$ be the dihedral group of order 8 where $\sigma$ is generator for $C_4$ and $\varepsilon$ is a generator of $C_2$. Let $H = \langle \sigma^2 \rangle$ be the center of $F$, which has order two and denote the
$n$-skeleton of the universal $F/H$-space $E(F/H)$ by $X^n$. We let $F$ act on $X$ and $X^n$ via the projection onto $F/H$. Consider the complex 1-dimensional representation

$$\lambda : H = (\sigma^2) \to U(1) = S^1 : \sigma^2 \mapsto -1.$$  

Lemma 3.1. The isomorphism class $[\lambda]$ is contained in $R(H)^{F/H}$. For $k \in \mathbb{Z}$, the multiple $[\lambda^k]$ is contained in the image of the restriction map $\text{res} : R(F) \to R(H)$ if and only if $k$ is even.

Proof. Since $H$ is the center of $F$ it follows that the conjugation action of $F/H$ on $R(H)$ is trivial, hence $[\lambda] \in R(H)^{F/H} = R(H)$. One easily verifies that the representation

$$\mu : F \to U(2)$$

defined by

$$\mu(\sigma) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \mu(\varepsilon) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $\text{res}([\mu]) = [\lambda^2]$. Hence, $[\lambda^k]$ is contained in the image of $\text{res}$ for every even $k \in \mathbb{Z}$. Note that, as a free abelian group, $R(H)$ is generated by $[\lambda]$ and the isomorphism class of the 1-dimensional complex trivial representation $[\tau]$. Now suppose $k$ is odd and there exists an element $[\mu] - [\rho] \in R(F)$ such that $\text{res}([\mu] - [\rho]) = [\lambda^k]$. There are integers $l, m, n$ and such that $\text{res}([\mu]) = [\tau^l] + [\lambda^m]$, $\text{res}([\rho]) = [\tau^l] + [\lambda^n]$ and $m - n = k$. By changing the representative of $[\mu]$, we may also assume that

$$\mu : F \to U(l + m)$$

where $\mu(\sigma)$ is a diagonal matrix. Since $\mu(\sigma^2)$ has an $m$-dimensional eigenspace with eigenvalues $-1$ and an $l$-dimensional eigenspace with eigenvalue $1$, it follows that $\mu(\sigma)$ has an $s$-dimensional eigenspace with eigenvalue $i$ and a $t$-dimensional eigenspace with eigenvalue $-i$ such that $s + t = m$. Moreover, $\mu(\sigma^3)$ has an $s$-dimensional eigenspace with eigenvalue $-i$ and a $t$-dimensional eigenspace with eigenvalue $i$. Since $\sigma$ and $\sigma^3$ are conjugate in $F$, it follows that $s = t$ proving that $m$ is even. A similar argument shows that $n$ is also even. But this contradicts the fact that $k = m - n$ is odd. Hence, there does not exist an element $[\mu] - [\rho] \in R(F)$ such that $\text{res}([\mu] - [\rho]) = [\lambda^k]$, if $k$ is odd.

The following lemma uses the notation introduced above and will be cited in the next section.

Lemma 3.2. Every complex one-dimensional $F$-bundle over $X^3$ is isomorphic to the pullback of a complex one-dimensional $F$-bundle over $E(F/H)$ along the inclusion $i : X^3 \to E(F/H)$.

Proof. Let $S$ be the family of subgroups of $F$ containing only $H$ and the trivial subgroup. Note that isomorphism classes of complex one-dimensional $F$-vector bundles are the same as isomorphism classes of $F$-equivariant principal $S^1$-bundles, or $(F,S^1 = U(1))$-bundles, in the sense of [10, Section 2] and [12, Section 2,3]. Let $\pi : E \to X^3$ be a complex one-dimensional $F$-bundle and let $[\alpha_H : H \to U(1) = S^1]$ be the isomorphism class in $\text{Rep}_{S^1}(H)$ of the $H$-representation induced on the fibers of $\pi$. If we set $\alpha_{\{e\}} : \{e\} \to S^1$, then $A = ([\alpha_K])_{K \in S} \in \lim_{S \in S} \text{Rep}_{S^1}(K)$. It follows from [10, Lemma 2.4(b)] for $\Gamma = S^1$, that in order to show that $p$ is the pullback of a complex one-dimensional $F$-bundle over $E(F/H)$ along the inclusion $i : X^3 \to E(F/H)$, it suffices to show that every $F$-map from $X^3$ to $B_S(F,A)$
can be extended to an $F$-map from $E(F/H)$ to $BS(F,A)$. Here $BS(F,A)$ is a certain $F$-CW-complex such that $BS(F,A)^S$ is homotopy equivalent to $BS^1 = \mathbb{C}P^\infty$ for all $S \in S$, by [10, Lemma 2.4(a)] for $\Gamma = S^1$. It follows from Bredon’s equivariant obstruction theory (see [1]) that the potential obstruction for doing this lies in the relative Bredon cohomology groups $H^{n+1}_F(E(F/H),X;\pi_n(BS(F,A)^-))$ for $n \geq 3$. Since $\pi_n(\mathbb{C}P^\infty)$ is zero unless $n = 2$, the lemma is proven.

The idea for the following lemma is contained in [10, p 596].

**Lemma 3.3.** There exists an $n \geq 1$ such that $[\lambda]$ is not contained in the image of the edge homomorphism

$$K^n_F(X^n) \to R(H)^{F/H}.$$  

**Proof.** By [5, Theorem 5.1] for $X = \{e\}$, $\mathcal{F} = \{e, H\}$ and $E\mathcal{F} = E(F/H)$, there are maps

$$\alpha_n : \{R(F)/I^n\}_{n \geq 0} \xrightarrow{\cong} \{K^n_F(X^n)\}_{n \geq 0}$$

that induce an isomorphism of pro-rings. Here $I$ is the kernel of the restriction map $R(F) \to R(H)$. This implies that for sufficiently large $n \geq 1$ there exists a map $\beta_1 : K^n_F(X^n) \to R(F)/I$ making the following diagram commute

$$\begin{array}{ccc} R(F)/I^n & \xrightarrow{\alpha_n} & K^n_F(X^n) \\ & \searrow \downarrow \swarrow \downarrow \swarrow & \\ R(F)/I & \xrightarrow{\alpha_1} & K^n_F(X^1). \end{array}$$

This shows that the image of the restriction map

$$R(F) \to R(H)^{F/H}$$

coincides with the image of the edge homomorphism

$$K^n_F(X^n) \to R(H)^{F/H}.$$  

Since $[\lambda]$ does not lie in the image of $R(F) \to R(H)^{F/H}$ by Lemma 3.1, the lemma follows. \qed

Let $n \geq 3$. By [6, Th. A & Th. 8.3] there exists a compact $n$-dimensional locally CAT(0)-cubical complex $T_X^n$ equipped with a free cellular $F/H$-action and an $F/H$-equivariant map $t_X^n : T_X^n \to X^n$ that induces an isomorphism

$$H^*_F(X^n) \cong H^*_F(T_X^n)$$

for any equivariant cohomology theory $H^*_F(\cdot)$ (e.g. see [9, section 1]). (We remark that [6, Th. 8.3] is stated for equivariant homology theories, but the analogous statement holds for equivariant cohomology theories by essentially the same proof.) The action of $F$ on $T_X^n$ in the above is via the projection $F \to F/H$. Now let $Y^n$ be the univeral cover of $T_X^n$ and let $\Gamma_n$ be the group of self-homeomorphisms of $Y^n$ that lift the action of $F/H$ on $T_X^n$. Since $F/H$ acts freely on $T_X^n$, $\Gamma_n$ acts freely on $Y^n$. In particular, $\Gamma_n$ is torsion-free and $Y^n$ is an $n$-dimensional CAT(0)-cubical complex on which $\Gamma_n$ acts freely, compactly and cellulary. By construction there is a surjection $\Gamma_n \to F/H$ whose kernel $N_n$ is the torsion-free group...
of deck transformation of the covering \( Y^n \rightarrow T_X^n \). Now define the group \( G_n \) to be the pullback of \( \pi_n : \Gamma_n \rightarrow F/H \) along \( F \rightarrow F/H \). Then \( G_n \) acts on \( Y^n \) via the quotient map \( G_n \rightarrow G_n/H = \Gamma_n \) and fits into the short exact sequence

\[
1 \rightarrow N_n \rightarrow G_n \xrightarrow{p_n} F \rightarrow 1.
\]

Note that the only non-trivial finite subgroup of \( G_n \) is \( H \cong C_2 \) and that since \( N_n \) acts freely on \( Y^n \), the \( F \)-equivariant quotient map \( Y^n \rightarrow N_n \setminus Y^n = T_X^n \) induces an isomorphism (Lemma 3.5])

\[
(4) \quad K^*_F(T_X^n) \cong K^*_G(Y^n).
\]

Applying (3) and (4) to the composition \( Y^n \rightarrow T_X^n \rightarrow X^n \) and the equivariant cohomology theories \( K^*_F(\cdot) \) and \( H^*_F(\cdot, R(\cdot)) \) with \( * = 0 \), we obtain a commutative diagram

\[
\begin{array}{ccc}
K^0_F(X^n) & \xrightarrow{\cong} & K^0_G(Y^n) \\
\downarrow{\varepsilon_F} & & \downarrow{\varepsilon_G} \\
R(H)^{F/H} & \xrightarrow{\cong} & \lim_{G/H \in \mathcal{O}G} R(S).
\end{array}
\]

Since we proved in Lemma 3.3 that, for \( n \) large enough, the isomorphism class of \( \lambda \) does not lie in the image of the edge homomorphism

\[
K^0_F(X^n) \rightarrow R(H)^{F/H}
\]

it follows from the commutative diagram above that the compatible system of representations

\[
(\lambda \circ p_n[S])_{S \in \mathcal{F}} \in \lim_{G_n/S \in \mathcal{O}G_n} R(S) = H^0_F(G_n, R(\cdot)).
\]

does not lie in the image of the edge homomorphism

\[
\varepsilon_G_n : K^0_G(Y^n) \rightarrow \lim_{G_n/S \in \mathcal{O}G_n} R(S).
\]

Recall from [2] that non-empty CAT(0)-cube complexes are contractible and that the fixed point set for a finite group action on a CAT(0)-cube complex is contractible. Since \( G_n \) acts cellularly properly and cocompactly on the CAT(0)-cube complex \( Y_n \), we deduce that \( Y_n \) is a cocompact model for \( E\Gamma_n \). To summarize, we have constructed a group \( G = G_n \) with a cocompact classifying space for proper actions \( EG \) admitting a compatible collection of complex representations of the finite subgroups of \( G \) that cannot be realized as \( G \)-equivariant virtual complex vector bundle over \( EG \).

### 4. Real vector bundles

One could apply the same technique of the previous section in the real setting to obtain a group \( G \) with cocompact classifying space for proper actions \( EG \) so that the edge homomorphism

\[
\varepsilon_G : KO^0_G(EG) \rightarrow \lim_{G/H \in \mathcal{O}G} R(O(H))
\]

is not surjective. Instead we give an explicit description of a group \( G \) that admits \( \mathbb{R}^2 \) as a cocompact model for \( EG \) and admits a compatible collection of real representations of its finite subgroups that cannot be realized as a real \( G \)-vector bundle over \( \mathbb{R}^2 \).

We start by describing a related group \( \Gamma \) that is a 2-dimensional crystallographic group, or wallpaper group; this group is known as \( p2gg \), but we will describe it explicitly. Endow \( \mathbb{R}^2 \)
Figure 1. A wallpaper pattern for $\Gamma = p2gg$

with the CW-structure coming from the standard tessellation by unit squares with vertices at $\mathbb{Z}^2$, and let $\Gamma$ be the group of automorphisms of this CW-structure that preserves the pattern shown in Figure [1]. The stabilizer of a 2-cell is clearly trivial, and so the 2-cells form a single free $\Gamma$-orbit. There are two orbits of 1-cells, the vertical and horizontal edges, and again each orbit is free. There are two orbits of 0-cells, and the stabilizer of a 0-cell is cyclic of order two, generated by the rotation of order two fixing the point. Since the stabilizer of each cell acts trivially on that cell, the given CW-structure makes $\mathbb{R}^2$ into a $\Gamma$-CW-complex.

The translation subgroup $T$ of $\Gamma$ has index four, and consists of the elements $(x, y) \mapsto (x + 2m, y + 2n)$. The orientation-preserving subgroup $N$ of $\Gamma$ has index two, and consists of $T$ together with the rotations through $\pi$ about some point of $\mathbb{Z}^2$, which are of the form $(x, y) \mapsto (2m - x, 2n - y)$. Finally the elements of $\Gamma - N$ are the glide reflections whose axes bisect the sides of the 2-cells: $(x, y) \mapsto (2m + 1 - x, 2n + 1 + y)$ and $(x, y) \mapsto (2m + 1 + x, 2n + 1 - y)$. The quotients $T \backslash \mathbb{R}^2, N \backslash \mathbb{R}^2$ and $\Gamma \backslash \mathbb{R}^2$ are respectively a torus consisting of four squares, an $S^2$ obtained by identifying the boundaries of two squares, and a copy of $\mathbb{R}P^2$ obtained by identifying the edges of a square in pairs. The fact that $\Gamma - N$ contains no torsion elements is reflected in the fact that $\Gamma/N$ acts freely on the sphere $N \backslash \mathbb{R}^2$.

Now let $F$ be a copy of $C_4$ and let $H \cong C_2$ be the index two subgroup of $F$. The group $G$ is defined as the pullback of the two maps $\Gamma \to \Gamma/N \cong C_2$ and $F \to F/H \cong C_2$. By construction the group $G$ admits $\mathbb{R}^2$ as a cocompact model for $EG$, and fits into a short exact sequence

$$ 1 \to N \to G \to F \to 1 $$

such that $p(F) = H$.

Now let

$$ \lambda : H \to O(1, \mathbb{R}) = C_2 $$

be the 1-dimensional real sign representation of $H$, i.e. $\lambda$ is the identity map. The isomorphism class $[\lambda]$ is clearly contained in $RO(H)^{F/H}$, since $F$ is abelian.
Lemma 4.1. The isomorphism class \([\lambda^k]\) is contained in the image of the restriction map \(RO(F) \to RO(H)^{F/H}\).

if and only if \(k\) is even.

Proof. Recall that the irreducible real representations of \(C_4\) are up to isomorphism the one-dimensional trivial representation and the one-dimensional sign representation of \(F/H = C_2\) and one 2-dimensional faithful representation in which the elements of order four act as rotations by \(\pm \frac{\pi}{2}\). The restriction of the first two of the representations to \(H\) gives the trivial one-dimensional representation of \(H\), while the restriction of \(H\) of the third is \(\lambda \oplus \lambda\). We therefore conclude that the image of \(RO(F) \to RO(H)^{F/H}\) consists of element of the form \(2n[\lambda] + m[\text{tr}]\), where \(\text{tr}\) is the trivial one-dimensional representation of \(H\) and \(n, m \in \mathbb{Z}\). This shows that \([\lambda^k]\) is contained in the image of the restriction map \(RO(F) \to RO(H)^{F/H}\) if and only if \(k\) is even.

Lemma 4.2. Let \(F\) act on the infinite dimensional sphere \(S^\infty\) by first projecting onto \(F/H = C_2\) and then acting via the antipodal map. View \(S^2\) as the 2-skeleton of \(S^\infty\). Every \(F\)-equivariant orthogonal real bundle over \(S^2\) is isomorphic to the pullback of a \(F\)-equivariant orthogonal real bundle over \(S^\infty\) along the inclusion \(S^2 \to S^\infty\).

Proof. Let \(S\) be the family of subgroups of \(F\) containing \(H\) and the trivial subgroup. Note that \(F\)-equivariant orthogonal real bundles are the same as \(F\)-equivariant \(O(1, \mathbb{R}) = C_2\)-bundles, or \((F,C_2)\)-bundles, in the sense of [10] section 2. Now let \(\xi\) be an \((F,C_2)\)-bundle over \(S^\infty\) with fibers \(A = (\xi_S) \in \lim_{S \leq S} \text{Rep}_{C_2}(S)\). Note that \(B_S(F,A)^\infty \cong \mathbb{R}P^\infty_\infty\) for all \(S \in S\) (see [10] Def. 2.3 and Lemma 2.4). By [10] Lemma 2.4 it suffices to show that every \(F\)-map \(f : S^2 \to B_S(F,A)\) can be extended to an \(F\)-map \(\tilde{f} : S^\infty \to B_S(F,A)\). It follows from Bredon’s equivariant obstruction theory that the obstruction for doing this lies in the relative Bredon cohomology groups \(H^{n+1}_F(S^\infty, S^2; \pi_n(B_S(F,A)^\infty))\) for \(n \geq 2\). Since \(\pi_n(\mathbb{R}P^\infty_\infty)\) is zero unless \(n = 1\), the lemma is proven.

Lemma 4.3. Let \(F\) act on \(S^2\) by first projecting onto \(F/H = C_2\) and then acting via the antipodal map. There does not exist a real \(F\)-vector bundle \(\xi : E \to S^2\) such that the representation of \(H\) on the fibers of \(\xi\) is isomorphic to \(\lambda\).

Proof. Consider the infinite dimensional sphere \(S^\infty\) as a the universal \(C_2\)-space \(EC_2\), where \(C_2\) acts via the antipodal map and let \(F\) act on \(S^\infty\) via first projection onto \(F/H = C_2\) and then acting via \(C_2\). Now assume that there exists a real \(F\)-vector bundle \(\xi : E \to S^2\) such that the representation of \(H\) on the fibers of \(\xi\) is isomorphic to \(\lambda\). By Lemma [12] there also exists a real \(F\)-vector bundle \(\xi : E \to S^\infty\) such that the representation of \(H\) on the fibers of \(\xi\) is isomorphic to \(\lambda\). By [23] Corollary 5.2 (and the comments below) for \(X = \{\ast\}\) and \(E\xi = S^\infty\), there is an isomorphism

\[p : RO(F) \overset{\sim}{\to} \mathbb{KO}_F^0(S^\infty),\]

where the completion is the I-adic completion with respect to the kernel of the restriction map \(RO(F) \to RO(H)\). Moreover, as explained in [23] Example 5.5, the edge homomorphism of the equivariant Atiyah-Hirzebruch spectral sequence provides a map

\[\mathbb{KO}_F^0(S^\infty) \to RO(H)^{C_2}\]

such that composition with \(p\) is the restriction map \(RO(F) \overset{\sim}{\to} RO(H)^{F/H}\). This implies that the fibers of any \(F\)-vector bundle over \(S^\infty\), considered as \(H\)-representations, always lie
in the image of the restriction map $\text{RO}(F) \to \text{RO}(H)$. Since $\lambda$ does not lie in the image of $\text{RO}(F) \to \text{RO}(H)$ by Lemma 4.1, we arrive at a contradiction and conclude that there does not exist a real $F$-vector bundle $\xi : E \to S^2$ such that the representation of $H$ on the fibers of $\xi$ is isomorphic to $\lambda$.

Now consider the compatible system of real orthogonal representations

$$(\lambda \circ p|_{S\times F})_{S \in F} \in \varprojlim_{G/S \in \text{O}(G)} \text{RO}(S) = H^0_{\text{F}}(G, \text{RO}(-))$$

and assume that there exists a real $G$-vector bundle $\xi : E \to \mathbb{R}^2$ that realizes it. Since the kernel of $p$ is $N$ it follows from the lemma below and our observations above that $N \setminus \xi : N \setminus E \to N \setminus X$ is an $F$-vector bundle over $S^2$, where $F$ acts on $S^2$ via projection onto $F/H = C_2$, followed by the antipodal map. Moreover, the representation of $H$ on the fibers of $N \setminus \xi$ is by construction exactly $\lambda$. This however contradicts Lemma 4.3, so we conclude that there does not exist a real $G$-vector bundle $\xi : E \to \mathbb{R}^2$ that realizes the compatible system of real orthogonal representations $(\lambda \circ p|_{S\times F})_{S \in F}$.

**Lemma 4.4.** If $\xi : E \to X$ is a $G$-vector bundle over $X$ and $N$ is a normal subgroup of $G$ such that $N \cap G_{\xi}$ acts trivially on $\xi^{-1}(x)$ for every $x \in X$, then

$$N \setminus \xi : N \setminus E \to N \setminus X$$

is a $G/N$-vector bundle over $N \setminus X$.

**Proof.** Denote the projection $G \to G/N = Q$ by $\pi$. Let us first consider the case where $\xi$ is trivial (trivial in the sense of Section 6.1), i.e. assume $\xi$ is a pullback

$$
\begin{array}{ccc}
G \times_H V & \longrightarrow & G/H \\
\uparrow r & \downarrow p & \\
E & \longrightarrow & X \\
\end{array}
$$

of the $G$-vector bundle $G \times_H V \to G/H$ along the $G$-map $p : X \to G/H$ where $H$ is some point stabilizer of $X$. Note that $H \cap N$ acts trivially on $V$ and consider the pullback diagram

$$
\begin{array}{ccc}
Q \times_{\pi(H)} V & \longrightarrow & Q/\pi(H) \\
\uparrow w & \downarrow \uparrow N \setminus p & \\
P & \longrightarrow & N \setminus X \\
\end{array}
$$

of the $Q$-vector bundle $Q \times_{\pi(H)} V \to Q/\pi(H)$ along the $Q$-map $N \setminus p : N \setminus X \to Q/\pi(H)$. We define the map

$$
\psi : N \setminus E \to P : (g, v, x) \mapsto (\pi(g), v, x).
$$

It is easy to check that $\psi$ is a well-defined $Q$-equivariant open map. We claim that $\psi$ is a bijection. To prove surjectivity, let $(\pi(g), v, x) \in Q \times_{\pi(H)} V \times (N \setminus X)$ be an element of $P$. This means that there exists an $n \in N$ such that $ngH = p(x)$. Hence, $(ng, v, x) \in N \setminus E$ and $\psi((ng, v, x)) = (\pi(g), v, x)$. To prove injectivity, consider $(g, v, x)$ and $(g', v', x)$ in $N \setminus E$ and assume that $\psi((g, v, x)) = \psi((g', v', x'))$, i.e. $(\pi(g), v, x) = (\pi(g'), v', x')$. This implies that there exists an $h \in H$ such that $\pi(h)v = v'$ and $\pi(g) = \pi(g'h)$. Also, there exists an $n \in N$
such that $nx = x'$. Since $(g, v, x)$ and $(g', v', x)$ in $N \setminus E$ we have $gH = p(x)$ and $g'H = p(x')$, which implies that $g'h' = ng$ for some $h' \in H$. Now

$$n(g, v, x) = (ng, v, x') = (g'h', v, x') = (g', h'v, x').$$

We also compute,

$$ng = g'h' = g'hh'^{-1}h' = n'gh'^{-1}h'$$

for some $n' \in N$. Multiplying this last equation by $g^{-1}$ on the left and using normality of $N$, we conclude that $h'^{-1}h' \in H \cap N$ and hence $h'^{-1}v'v = v$. Since $hv = v'$, this implies that $n(g, v, x) = (g', v', x')$. Hence $(g, v, x) = (g', v', x')$, proving injectivity and the claim. Since $q \circ \psi = N \setminus \xi$ and $w \circ \psi = N \setminus r$, we conclude that the lemma holds in case $\xi$ is trivial.

Now consider the general case. Let $x \in N \setminus X$. Since $\xi : E \to X$ is locally trivial, $x \in X$ has an open $G$-neighbourhood $U$ such that there is a $G$-map $p : U \to G/H$ where $H = G_x$ and $\xi|_U$ is (homeomorphic to) the pullback

$$G \times_H V \to G/H$$

of the $G$-vector bundle $G \times_H V \to G/H$ along the $G$-map $p : X \to G/H$, where $V = \xi^{-1}(x)$. By the above, the quotient diagram

$$Q \times_{\pi(H)} V \to Q/\pi(H)$$

$$N \setminus \xi|_U \to N \setminus U$$

is a pullback diagram. Since $N \setminus U$ is an open $Q$-neighbourhood of $\xi$ and $Q_{\xi} = \pi(H)$, it follows that $N \setminus \xi : N \setminus E \to N \setminus X$ is a $Q$-vector bundle.

We finish this section by noting that a similar approach to the above together with Lemma 3.2 (which is the complex version of Lemma 4.2) and an application of [6, Th. A] can be used to produce a group $G$ admitting a three dimensional cocompact model for $EG$ that has a compatible system of one-dimensional complex representations that cannot be realized as a complex $G$-vector bundle over $EG$.

5. Right angled Coxeter groups

Let $\Gamma$ be a finite graph. We denote the vertex set of $\Gamma$ by $S = V(\Gamma)$ and the set edges of $\Gamma$ by $E(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. The right angled Coxeter group determined by $\Gamma$ is the Coxeter group $W$ with presentation

$$W = \langle S \mid s^2 \text{ for all } s \in V(\Gamma) \text{ and } (st)^2 \text{ if } (s, t) \in E(\Gamma) \rangle.$$
Note that $W$ fits into the short exact sequence

$$1 \rightarrow N \rightarrow W \xrightarrow{p} F = \bigoplus_{s \in S} C_2 \rightarrow 1$$

where $p$ takes $s \in S$ to the generator of the $C_2$-factor corresponding to $s$. A subset $J \subseteq S$ is called spherical if the subgroup $W_J = \langle J \rangle$ is finite (and hence isomorphic to $\bigoplus_{s \in J} C_2$). The empty subset of $J$ is by definition spherical. We denote the poset of spherical subsets of $S$ ordered by inclusion by $\mathcal{S}$. If $J \in \mathcal{S}$, then $W_J$ is called a spherical subgroup of $W$, while a coset $wW_J$ is called spherical coset. We denote the poset of spherical cosets, ordered by inclusion, by $W\mathcal{S}$. Note that $W$ acts on $W\mathcal{S}$ by left multiplication, preserving the ordering. The Davis complex $\Sigma$ of $W$ is the geometric realization of $W\mathcal{S}$. One easily sees that $\Sigma$ is a proper cocompact $W$-CW-complex. Since $\Sigma$ admits a complete CAT(0)-metric such that $W$ acts by isometries, it follows that $\Sigma$ is a cocompact model for $E_W$ (see [3, Th. 12.1.1 & Th. 12.3.4]). A consequence of this fact is that every finite subgroup of $W$ is subconjugate to some spherical subgroup of $W$. This implies that the group $N$ defined above is torsion-free.

Let $\mathcal{F}$ be the family of finite subgroups of $W$ and let $\Gamma$ be either the orthogonal group $O(n, \mathbb{R})$ or the unitary group $U(n)$. Given an abelian group $A$, we denote by

$$\mathbf{A} : \mathcal{O}_F W \rightarrow \text{Ab}$$

the trivial functor that takes all objects to $A$ and all morphism to the identity map. One can verify that

$$H^*_G(EF, \mathbf{A}) \cong H^*(BG, A).$$

**Lemma 5.1.** Every element of

$$\lim_{W/H \in \mathcal{O}_F W} \text{Rep}_\Gamma(H)$$

is of the form $([\lambda \circ p_H])_{H \in \mathcal{F}}$ for some group homomorphism $\lambda : F \rightarrow \Gamma$.

**Proof.** Every finite subgroup $H$ of $W$ is isomorphic to a finite direct sum of $C_2'$s. Since every element of order 2 in $\Gamma$ is conjugate in $\Gamma$ to a diagonal matrix with $\pm 1$ on the diagonal and commuting matrices can be simultaneously diagonalized, it follows that the image of every homomorphism $H \rightarrow \Gamma$ is conjugate to a finite subgroup of $\Gamma$ consisting of diagonal matrices with diagonal entries equal to $\pm 1$. It follows that every element of $\lim_{W/H \in \mathcal{O}_F W} \text{Rep}_\Gamma(H)$ is of the form $([\alpha_H])_{H \in \mathcal{F}}$ where $\alpha_H : H \rightarrow \Gamma$ is a homomorphism whose image lands in the finite abelian subgroup of $\Gamma$ consisting of diagonal matrices with diagonal entries equal to $\pm 1$. Since every finite subgroup of $W$ is subconjugate to a special subgroup $W_J$, the compatibility of the representations tells us that $([\alpha_H])_{H \in \mathcal{F}}$ is completely determined by the homomorphisms $\alpha_{(s)} : \langle s \rangle \rightarrow \Gamma$, for $s \in S$. Now define the homomorphism

$$\lambda : F = \bigoplus_{s \in S} C_2 \rightarrow \Gamma : (\sigma_s)_{s \in S} \mapsto \sum_{s \in S} \alpha_{(s)}(\sigma_s).$$

Then the compatibility of the representations implies that

$$([\lambda \circ p_H])_{H \in \mathcal{F}} = ([\alpha_H])_{H \in \mathcal{F}},$$
proving the following lemma. □

The following theorem applies to both complex and real representations and vector bundles.

**Theorem 5.2.** Let $W$ be a right angled Coxeter group. Every compatible collection of representations of the finite subgroups of $W$ can be realized as a $W$-equivariant vector bundle over the Davis complex $\Sigma = E\omega W$.

**Proof.** Let $A = ([\alpha_H])_{H \in F} \in \lim_{W/H \in O \omega W} \text{Rep}_F(H)$. It follows from [10, Lemma 2.4(b)] that the existence of a $(W,A)$-bundle over $\Sigma$ (see [10, p. 591]) follows from the existence a $W$-map $\Sigma \to B_F(G,A)$. Since $B_F(G,A)^{H}$ is homotopic to $BC_F(\alpha_H) = BF$ for all $H \in F$, it follows that the functor

$$\pi_k(B_F(W,A)^-): O_F(W) \to \text{Ab} : W/H \mapsto \pi_k(B_F(G,A)^H)$$

equals the trivial functor $\pi_k(BF)$ for all $k \geq 0$. From (5) and the contractibility of $BW$, it follows that the Bredon cohomology groups $H^{k+1}_W(\Sigma, \pi_k(B_F(W,A)^-)) : O_F(W) \to \text{Ab}$ are zero for all $k \geq 0$. Since there certainly exists a $W$-map from the 0-skeleton of $\Sigma$ to $B_F(G,A)$, it follows from Bredon’s equivariant obstruction theory that there exists a $W$-map $\Sigma \to B_F(G,A)$. In particular, we conclude that there exists a $(W,A)$-bundle over $\Sigma$, where $A = ([\alpha_H])_{H \in F} \in \lim_{W/H \in O \omega W} \text{Rep}_F(H)$.

Now consider a compatible collection of representations of the finite subgroups of $W$. By Lemma 5.1(i) above, this collection is of the form $([\lambda \circ p_H])_{H \in F} \in \lim_{W/H \in O \omega W} \text{Rep}_F(H)$ for some group homomorphism $\lambda : F \to \Gamma$. Since $A = ([p_H])_{H \in F} \in \lim_{W/H \in O \omega W} \text{Rep}_F(H)$, it follows from the above that there exists a $(W,A)$-bundle $\xi : E \to \Sigma$. If $\Gamma = O(n,\mathbb{R})$ then

$$\xi : E \times F \mathbb{R}^n \to \Sigma$$

is a real $W$-vector bundle over $\Sigma$ that realizes $([\lambda \circ p_H])_{H \in F}$, and if $\Gamma = U(n)$ then

$$\xi : E \times F \mathbb{C}^n \to \Sigma$$

is a complex $W$-vector bundle over $\Sigma$ that realizes $([\lambda \circ p_H])_{H \in F}$. Here $F$ acts on $\mathbb{R}^n$ or $\mathbb{C}^n$ via the map $\lambda$. □

**Lemma 5.3.** If $W$ is a right angled Coxeter group, then $H^n_W(\Sigma, R(-)) = 0$ for all $n > 0$, and $H^n_W(\Sigma, R(-))$ is free abelian of rank equal to the number of spherical subgroups of $W$.

**Proof.** This is proven in much the same way as the corresponding result for homology in [13]. In more detail, one uses the cubical structure on $\Sigma$, in which there is one orbit of $n$-cubes with stabilizer isomorphic to $(C_2)^n$ for each $n$-tuple of commuting elements of $S$. Since the stabilizer of a cube of strictly positive dimension acts non-trivially on the cube, this is not a $W$-CW-structure on $\Sigma$. However, its barycentric subdivision is isomorphic to the realization of the poset $WS$ as described in the introduction to this section.

Let $\Sigma^n$ denote the $n$-skeleton of $\Sigma$ with this cubical structure. Firstly, $\Sigma^0$ consists of a single free $W$-orbit of vertices, so $H^0_W(\Sigma^0; R(-))$ is isomorphic to the ordinary cohomology of
a point; since \( W \) acts freely the calculation reduces to an equivariant cohomology calculation for the trivial group action.

Let \( I = [-1, 1] \) be an interval, with \( C_2 \) acting by \( x \mapsto -x \) (i.e., swapping the ends of the interval). Note that \( I \) is isomorphic to the Davis complex for the Coxeter group \( C_2 \). Let \( \partial I \) denote the two end points \( \{-1, 1\} \). A direct computation shows that \( H^m_{C_2}(I, \partial I; R(-)) \) is isomorphic to \( \mathbb{Z} \) for \( m = 0 \) and is zero for \( m > 0 \).

Next consider \( I^n \) with \( C_2^n \) acting as the direct product of \( n \) copies of the above action of \( C_2 \) on \( I \). This is the Davis complex for the Coxeter group \( C_2^n \). Since the representation ring of a direct product of finite groups is naturally identified with the tensor product of the representation rings, the \( C_2^n \)-Bredon cochain complex for the pair \( (I^n, \partial I^n) \) with coefficients in \( R(-) \) is naturally isomorphic to the tensor product of \( n \) copies of the \( C_2 \)-Bredon cochain complex for \((I, \partial I)\) with coefficients in \( R(-) \). Since \( H^*_C(I, \partial I; R(-)) \) is free abelian one obtains a K"unneth formula

\[
H^*_C(I^n, \partial I^n, R(-)) \cong \bigotimes_{i=1}^{n} H^*_C(I, \partial I; R(-)).
\]

It follows that for each \( n \), \( H^m_{C_2}(I^n, \partial I^n; R(-)) \) is isomorphic to \( \mathbb{Z} \) for \( m = 0 \) and is zero for \( m > 0 \).

From these computations, it follows easily that \( H^m_{C_2}(\Sigma^n, \Sigma^{n-1}; R(-)) \) is zero for \( m > 0 \) and is isomorphic to a direct sum of copies of \( \mathbb{Z} \) indexed by the \( W \)-orbits of \( n \)-cubes in \( \Sigma \). By induction on \( n \) one sees that \( H^m_{C_2}(\Sigma^n; R(-)) \) is zero for \( m > 0 \) and isomorphic to a direct sum of copies of \( \mathbb{Z} \) indexed by the \( W \)-orbits of cubes of dimension at most \( n \) for \( m = 0 \). The claimed result follows, since the \( W \)-orbits of cubes in \( \Sigma \) are in bijective correspondence with the spherical subgroups of \( W \).

\( \square \)

**Theorem 5.4.** Let \( W \) be the right angled Coxeter group determined by a finite graph \( \Gamma \). Then \( K^0_W(\mathbb{Z}[V(\Gamma)]) = 0 \) and there is a ring isomorphism

\[
K^0_W(\mathbb{Z}[V(\Gamma)])(E^q)^{\mathbb{Z}[V(\Gamma)]} \cong \mathbb{Z}[V(\Gamma)]/\langle s^2 - 1, st - s - t + 1 \mid s \in V(\Gamma), (s, t) \notin E(\Gamma) \rangle.
\]

Here, \( \mathbb{Z}[V(\Gamma)] \) is the polynomial ring with variables in \( V(\Gamma) \) and integer coefficients. In particular,

\[
K^0_W(\mathbb{Z}[V(\Gamma)]) \cong \mathbb{Z}^d
\]
as an abelian group, where \( d \) is the number of spherical subgroups of \( W \).

**Proof.** Consider the Atiyah-Hirzebruch spectral sequence

\[
E_2^{p, q} = H^p_W(\mathbb{Z}[V(\Gamma)], K_0^q_W(W/-)) \Rightarrow K^{p+q}_W(\mathbb{Z}[V(\Gamma)])
\]

where \( K^0_W(W/-) = R(-) \) if \( q \) is odd and \( K^0_W(W/-) = 0 \) if \( q \) is even. In the lemma above, we proved that \( H^p_W(\Sigma, R(-)) = 0 \) for \( k > 0 \). It therefore follows that

\[
K^*_W(\mathbb{Z}[V(\Gamma)]) = \begin{cases} 
H^0_W(\mathbb{Z}[V(\Gamma)], R(-)) = \lim_{H \in O \cap W} R(H) & \text{if } n = 0 \\
0 & \text{if } n = 1.
\end{cases}
\]

Let \( I \) be the ideal

\[
\langle s^2 - 1, st - s - t + 1 \mid s \in V(\Gamma), (s, t) \notin E(\Gamma) \rangle
\]
in the polynomial ring \( \mathbb{Z}[V(\Gamma)] \). Note that as an abelian group \( \mathbb{Z}[V(\Gamma)]/I \) is free, with basis elements the commuting products \( s_1 \ldots s_k \) for all \( J = \{s_1, \ldots, s_k\} \in S \) \((J = \emptyset \) corresponds to the unit of \( \mathbb{Z}[V(\Gamma)]/I \)). This shows that

\[
\mathbb{Z}[V(\Gamma)]/I \cong \mathbb{Z}^d
\]
as an abelian group, where $d$ is the number of spherical subgroups of $W$.

We claim there is an isomorphism of rings

$$
\lim_{W/H \in O_{F}W} R(H) \cong \mathbb{Z}[V(\Gamma)]/I.
$$

Since $\lim_{W/H \in O_{F}W} R(H) = H^0_W(\Sigma, R(-)) = H^0_W(\Sigma^1, R(-))$, one can use the explicit description of $\Sigma$ to see that

$$
\lim_{W/H \in O_{F}W} R(H) \cong \lim_{J \in S} R(W_J).
$$

as rings. Since $W_J = \bigoplus_{s \in J} C_2$ and $R(C_2) = \mathbb{Z}[X]/(X^2 - 1)$, we have that

$$
R(W_J) = \bigotimes_{s \in J} \mathbb{Z}[s]/(s^2 - 1).
$$

as rings. For each $J \subseteq S$, we define the unital ring homomorphism

$$
\rho_J : \mathbb{Z}[V(\Gamma)]/I \to R(W_J)
$$

by setting

$$
\rho_J(s) = \begin{cases} 
1 \otimes \ldots \otimes 1 \otimes s \otimes 1 \otimes \ldots \otimes 1 \in R(W_J) & \text{if } s \in J \\
1 \otimes \ldots \otimes 1 \otimes 1 \in R(W_J) & \text{if } s \notin J.
\end{cases}
$$

One easily verifies that the maps $\rho_-$ are compatible with the restriction maps of $R(-)$, i.e. for $J \subset T \in S$, we have a commutative diagram

$$
\begin{array}{ccc}
Z[V(\Gamma)]/I & \xrightarrow{\rho_T} & R(W_T) \\
\downarrow{\rho_J} & & \downarrow{R_J} \\
R(W_J) & & \end{array}
$$

This implies that there is a ring homomorphism

$$
\rho : \mathbb{Z}[V(\Gamma)]/I \to \lim_{J \in S} R(W_J) \subseteq \bigoplus_{J \in S} R(W_J) : f \mapsto (\rho_J(f))_{J \in S}.
$$

Using the explicit basis for $\mathbb{Z}[V(\Gamma)]/I$ given above, one easily verifies that $\rho$ is an isomorphism. \qed

**Corollary 5.5.** Let $W$ be a right angled Coxeter group determined by a finite graph $\Gamma$. Then there is an isomorphism of rings

$$
K^n(BW) \cong \begin{cases} 
\mathbb{Z}[[V(\Gamma)]]/(s(s + 2), st \mid s \in V(\Gamma), (s, t) \notin E(\Gamma)) & \text{if } n = 0 \\
0 & \text{if } n = 1.
\end{cases}
$$

Here, $\mathbb{Z}[[V(\Gamma)]]$ is the formal power series ring with variables in $V(\Gamma)$ and integer coefficients.

**Proof.** The version of the Atiyah-Segal completion theorem that is proven for infinite discrete groups admitting a cocompact model for the classifying space for proper actions in [10, Theorem 4.4.(b)] implies that

$$
K^n(BW) = K^n_W(EW)_J,
$$

where $E = \bigoplus_{s \in \Sigma} C_2$. \qed
where the ideal $J$ is the kernel of the augmentation map $K^n_W(EW) \to \mathbb{Z}$ that maps vector bundles to their dimension. It therefore follows from the theorem above that

$$K^n(BW) = \begin{cases} \left( \mathbb{Z}[V(\Gamma)]/I\right)_J & \text{if } n = 0 \\ 0 & \text{if } n = 1. \end{cases}$$

where the ideal $J$ is the kernel of the ring homomorphism

$$\mathbb{Z}[V(\Gamma)]/I \to \mathbb{Z}$$

that takes $s \in V(\Gamma)$ to $1 \in \mathbb{Z}$ and

$I = (s^2 - 1, st - s - t + 1 \mid s \in V(\Gamma), (s, t) \notin E(\Gamma))$.

Using the map $s \mapsto s + 1$, we obtain an isomorphism

$$\left( \mathbb{Z}[V(\Gamma)]/I\right)_J \cong \left( \mathbb{Z}[V(\Gamma)]/(s(s + 2), st \mid s \in V(\Gamma), (s, t) \notin E(\Gamma)) \right)_{(s, s \in V(\Gamma))}.$$

Since there is an inclusion of ideals

$$(s(s + 2), st \mid s \in V(\Gamma), (s, t) \notin E(\Gamma)) \subseteq (s, s \in V(\Gamma)),$$

and it is well-known that $\mathbb{Z}[V(\Gamma)]_{(s, s \in V(\Gamma))} = \mathbb{Z}[\{V(\Gamma)\}]$, we obtain the isomorphism

$$\left( \mathbb{Z}[V(\Gamma)]/I\right)_J \cong \mathbb{Z}[\{V(\Gamma)\}]/(s(s + 2), st \mid s \in V(\Gamma), (s, t) \notin E(\Gamma)).$$

□

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