Anomalous Higgs Couplings in the $SO(5) \times U(1)_{B-L}$ Gauge-Higgs Unification in Warped Spacetime

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The gauge couplings $WWZ$, $WWWW$, and $WWZZ$ in the gauge-Higgs unification scenario in the Randall-Sundrum warped spacetime remain almost universal as in the standard model, but substantial deviation results for the Higgs couplings. In the $SO(5) \times U(1)_{B-L}$ model, the couplings $WWH$ and $ZZH$ are suppressed by a factor $\cos \theta_H$ from the values in the standard model, while the bare couplings $WWH$ and $ZZHH$ are suppressed by a factor $1 - \frac{2}{3} \sin^2 \theta_H$. Here $\theta_H$ is the Yang-Mills AB phase (Wilson line phase) along the fifth dimension, which characterizes the electroweak symmetry breaking. The suppression can be used to test the gauge-Higgs unification scenario at LHC and ILC. It is also shown that the $WWZ$ coupling in flat spacetime deviates from the standard model value at moderate values of $\theta_H$, contradicting with the LEP2 data.

§1. Introduction

In the previous paper we showed that substantial deviation in the Higgs couplings to $W$ and $Z$ bosons from those in the standard model is expected as a general feature in the gauge-Higgs unification model in warped spacetime,\(^1\) which can be tested at LHC and ILC in the coming future. Further the deviation in the $WWZ$ coupling was shown to be very small in warped spacetime. In the present paper we give more thorough analysis of these couplings to strengthen the statements, in addition to give detailed account of the mass spectrum and wave functions in the gauge-Higgs sector in the $SO(5) \times U(1)_{B-L}$ model.

In the gauge-Higgs unification scenario the Higgs field in four dimensions is identified with the zero mode of the extra-dimensional component of gauge potentials in higher dimensional gauge theory. As such, the mass and couplings of the Higgs field are not arbitrary parameters in theory. They follow from the gauge principle. The original proposal by Fairlie and by Manton to unify the Higgs field in the six-dimensional gauge theory with $S^2$ as extra dimensions was unsatisfactory as it gives too low Kaluza-Klein energy scale and unrealistic couplings and spectrum.\(^2,3\) Shortly after their proposal it has been recognized that Wilson line phases, or Yang-Mills Aharonov-Bohm phases associated with non-simply connected extra dimensions can serve as the Higgs field in flat spacetime. These phases, denoted as $\theta_H$ in the present paper, label classically degenerate vacua. The value of $\theta_H$ is determined at the quantum level. When the value is nontrivial, the gauge symmetry is dynamically broken.\(^4,5\)

The scenario of identifying $\theta_H$ with a 4D Higgs field, which was applied first to GUT and then to the electroweak interactions, has many attractive features.\(^6-44\)

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Besides inducing dynamical gauge symmetry breaking, it predicts a finite mass for the Higgs field, independent of the cutoff scale.\(^45\)–\(^48\) In the electroweak theory, it can solve the gauge hierarchy problem.\(^7\) The dynamically determined value of \(\theta_H\) depends on the details of the theory, particularly in the fermion sector. Astonishingly many of the features in the gauge-Higgs sector such as the mass spectrum and couplings are determined once the value of \(\theta_H\) is specified. In this respect our analysis is robust. It is shown below that the \(WWH\), \(ZZH\), \(WWHH\) and \(ZZHH\) couplings are suppressed compared with those in the standard model. The predictions obtained for the gauge-Higgs couplings can be tested at LHC and ILC. If the deviation from the standard model is observed as indicated in the gauge-Higgs unification scenario, then it gives strong hint for the existence of extra dimensions. It is also confirmed that the \(WWZ\) coupling remains universal in warped spacetime, but it becomes smaller in flat spacetime compared with that in the standard model, thus already contradicting with the LEP2 data on \(W\) pair production. This strongly suggests that the extra-dimensional space is curved and warped, if it exists. There seems intimate connection between the gauge-Higgs unification scenario and the holography in the warped space.\(^31\), \(^33\), \(^39\), \(^42\), \(^43\)

The paper is organized as follows. The \(SO(5) \times U(1)_{B-L}\) model is set up in the next section. The spectrum and mode functions of gauge bosons are given in Sec. 3, whereas those of fermions are given in Sec. 4. Approximate masses and wave functions of gauge fields, Higgs field, and light fermions in four dimensions are given in Sec. 5. Gauge couplings and Higgs couplings are evaluated in Sec. 6 to make predictions described above. Gauge couplings of fermions are briefly discussed in Sec. 7 and a summary is given in Sec. 8. Useful formulas are collected in appendices.

\section{SO(5) \times U(1)_{B-L} model}

We consider an \(SO(5) \times U(1)_{B-L}\) gauge theory in the warped five-dimensional spacetime.\(^33\) The fifth dimension is compactified on an orbifold \(S^1/Z_2\) with a radius \(R\). We use, throughout the paper, \(M, N, \cdots = 0, 1, 2, 3, 4\) for the 5D curved indices, \(A, B, \cdots, = 0, 1, 2, 3, 4\) for the 5D flat indices in tetrads, and \(\mu, \nu, \cdots = 0, 1, 2, 3\) for 4D flat indices. The background metric is given by\(^49\)

\[
ds^2 = G_{MN}dx^M dx^N = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \tag{2.1}
\]

where \(\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1), \sigma(y) = \sigma(y + 2\pi R), \) and \(\sigma(y) \equiv k |y| \) for \(|y| \leq \pi R\). The cosmological constant in the bulk 5D spacetime is given by \(\Lambda = -k^2\). \((x'^\mu, y)\) and \((x'^\mu, y + 2\pi R)\) are identified with \((x'^\mu, y)\). The spacetime is equivalent to the interval in the fifth dimension \(y\) with two boundaries at \(y = 0\) and \(y = \pi R\), which we refer to as the Planck brane and the TeV brane, respectively.

There are \(SO(5)\) gauge fields \(A_M\) and \(U(1)_{B-L}\) gauge field \(B_M\). The former are decomposed as

\[
A_M = \sum_{I=1}^{10} A_M^I T^I + \sum_{a_L=1}^{3} A_M^{a_L} T^{a_L} + \sum_{a_R=1}^{3} A_M^{a_R} T^{a_R} + \sum_{\hat{a}=1}^{4} A_M^{\hat{a}} T^{\hat{a}}, \tag{2.2}
\]
where $T^{a_l, \bar{a}_l}$ ($a_l, a_R = 1, 2, 3$) and $T^a$ ($\bar{a} = 1, 2, 3, 4$) are the generators of $SO(4) \sim SU(2)_L \times SU(2)_R$ and $SO(5)/SO(4)$, respectively. The spinorial representation of $T^I$ is tabulated in (A.1) in appendix A. As a matter field we introduce a spinor field $\Psi$ in the spinorial representation of $SO(5)$ (i.e., 4 of $SO(5)$) in the bulk as an example.

The relevant part of the action in the bulk is

$$S = \int d^5 x \sqrt{-G} \left[ -\text{tr} \left( \frac{1}{2} F^{(A)MN} F_{MN}^{(A)} + \frac{1}{\xi} (f_{gf}^{(A)})^2 + \mathcal{L}_{gh}^{(A)} \right) \right]$$  \hspace{1cm} (2.3)

$$- \left( \frac{1}{4} F^{(B)MN} F_{MN}^{(B)} + \frac{1}{2\xi} (f_{gf}^{(B)})^2 + \mathcal{L}_{gh}^{(B)} \right) + i\bar{\Psi} \Gamma^N \mathcal{D}_N \Psi - iM_\phi \bar{\Psi} \Psi,$$

where $G \equiv \det(G_{MN})$ and $\Gamma^N \equiv e_A^N \Gamma^A$. The 5D $\gamma$-matrices $\Gamma^A$ are related to the 4D ones $\gamma^\mu$ by $\Gamma^\mu = \gamma^\mu$ and $\Gamma^4 = \gamma_5$ which is the 4D chiral operator. The gauge-fixing functions $f_{gf}^{(A,B)}$ are specified in the next section. $\mathcal{L}_{gh}^{(A,B)}$ are the associated ghost Lagrangians, and $M_\phi$ is a bulk (kink) mass parameter. Since the operator $\bar{\Psi} \Psi$ is $Z_2$-odd, we need the periodic sign function $\varepsilon(y) = \sigma'(y)/k$ satisfying $\varepsilon(y) = \pm 1$.

The field strengths and the covariant derivatives are defined by

$$F_{MN}^{(A)} \equiv \partial_M A_N - \partial_N A_M - ig_A [A_M, A_N],$$

$$F_{MN}^{(B)} \equiv \partial_M B_N - \partial_N B_M,$$

$$\mathcal{D}_M \Psi \equiv \left( \partial_M - \frac{1}{4} \omega_{AB} \Gamma_{AB} - ig_A A_M - i\frac{g_B}{2} \mathcal{Q}_{B-L} B_M \right) \Psi,$$  \hspace{1cm} (2.4)

where $g_A$ ($g_B$) is the 5D gauge coupling for $A_M$ ($B_M$), $\mathcal{Q}_{B-L}$ is a charge of $U(1)_{B-L}$, and $\Gamma^{AB} \equiv \frac{1}{4} [\Gamma^A, \Gamma^B]$. The spin connection 1-form $\omega^{AB} = \omega_{M}^{AB} dx^M$ determined from the metric (2.1) is

$$\omega^{\mu A} = -\sigma' e^{-\sigma} dx^\mu, \text{ other components } = 0.$$  \hspace{1cm} (2.5)

The boundary conditions at the fixed points $y_0 = 0$ and $y_\pi = \pi R$, which preserve the orbifold structure, are

$$\begin{pmatrix} A_\mu \\ A_y \end{pmatrix} (x, y_j - y) = P_j \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix} (x, y_j + y) P_j^{-1},$$

$$\begin{pmatrix} B_\mu \\ B_y \end{pmatrix} (x, y_j - y) = \begin{pmatrix} B_\mu \\ -B_y \end{pmatrix} (x, y_j + y),$$

$$\Psi(x, y_j - y) = \eta_j P_j \gamma_5 \Psi(x, y_j + y),$$  \hspace{1cm} (2.6)

where $\eta_j = \pm 1$ ($j = 0, \pi$). $P_j \in SO(5)$ are constant matrices satisfying $P_j^2 = 1$. In the present paper we take

$$P_0 = P_\pi = \begin{pmatrix} 1_2 \\ -1_2 \end{pmatrix}$$  \hspace{1cm} (2.7)
in the spinorial representation, or equivalently \( P_0 = P_\pi = \text{diag}(-1,-1,-1,-1,1) \) in the vectorial representation. Then the gauge symmetry is broken to \( SO(4) \times U(1)_{B-L} \sim SU(2)_L \times SU(2)_R \times U(1)_{B-L} \) at both boundaries. (The broken generators are \( T^\alpha; \hat{a} = 1,2,3,4. \) It is convenient to decompose \( \Psi \) as

\[
\Psi = \begin{pmatrix} q \\ Q \end{pmatrix},
\]

(2.8)

where \( q \) and \( Q \) belong to \((\frac{1}{2},0)\) and \((0,\frac{1}{2})\) of \( SU(2)_L \times SU(2)_R \), respectively.

Fields with Neumann boundary conditions at both boundaries have zero modes when perturbation theory is developed around the trivial configuration \( A_M = 0 \). With (2.6) and (2.7) there arise zero modes for \( A^\hat{a}_4 \) (\( \hat{a} = 1,2,3,4 \)). They are identified with the \( SU(2)_L \) doublet-Higgs field in the standard model; \( \Phi \propto (A^1_y + iA^2_y, A^3_y - iA^4_y)^t \).

A nonvanishing expectation value of \( A^4_y \) gives rise to a Wilson line phase or Yang-Mills Aharonov-Bohm phase, \( \theta_H \equiv gA \int^\pi_0 dy \langle A^4_y \rangle / 2\sqrt{2} = (gA/\sqrt{2}) \int^\pi_0 dy \langle A^4_y \rangle \).

More explicitly,

\[
\langle A^4_y \rangle = \frac{2\sqrt{2}ke^{2ky}}{gA(e^{2k\pi R} - 1)} \theta_H. \tag{2.9}
\]

Although \( \theta_H \neq 0 \) gives vanishing field strengths, it affects physics at the quantum level. The global minimum of the effective potential for \( \theta_H \) determines the quantum vacuum.\(^4\) The nonvanishing \( \theta_H \) induces dynamical electroweak gauge symmetry breaking.

There are residual gauge transformations which maintain the boundary condition \( (2.6) \).\(^5,36\) A large gauge transformation given by

\[
\Omega^{\text{large}}(y) = \exp \left\{ \pi \frac{e^{2ky} - 1}{e^{2k\pi R} - 1} \left( 2\sqrt{2} T^4 \right) \right\}, \tag{2.10}
\]

(0 \( \leq y \leq \pi R, n: \) an integer) shifts \( \theta_H \) by \( \theta_H + 2\pi n \), which implies that all physical quantities are periodic functions of \( \theta_H \). The large gauge invariance is vital to guarantee the finiteness of the Higgs boson mass.\(^{45,48}\) The \( \theta_H \)-dependent part of the effective potential diverges without the large gauge invariance.

The even-odd property in \( (2.6) \) does not completely fix boundary conditions of the fields. If there are no additional dynamics on the two branes, fields which are odd under parity at \( y = 0 \) or \( \pi R \) obey the Dirichlet boundary condition (D) so that they vanish there. On the other hand, fields which are even under parity obey the Neumann boundary conditions (N). For gauge fields the Neumann boundary condition is given by \( dA_\mu / dy = 0 \) or \( d(e^{-2ky}A_y) / dy = 0 \). As a result of additional dynamics on the branes, however, a field with even parity, for instance, can effectively obey the Dirichlet boundary condition. The field develops a cusp-type singularity there due to brane dynamics. As discussed below, the \( SO(4) \) symmetry on the Planck brane is broken to \( SU(2)_L \times U(1)_Y \) in this manner.

Let us define new fields \( A^3_R_M \) and \( A^Y_M \) by

\[
\begin{pmatrix} A^3_R_M \\ A^Y_M \end{pmatrix} = \begin{pmatrix} c_\phi & -s_\phi \\ s_\phi & c_\phi \end{pmatrix} \begin{pmatrix} A^3_R \\ B_M \end{pmatrix},
\]
As a result of additional dynamics on the Planck brane the even fields $A_\mu$ obey the same boundary conditions as the original fields. We remark that in Table I preserve the large gauge invariance, that is, new gauge potentials obtained by \( (2.10) \) obey the same boundary conditions as the original fields. We remark that the Neumann (N) boundary condition on the Planck brane cannot be imposed on $A^a$ so that the Neumann boundary condition for $A^a$ is given by \( \pi k_R \) at the boundary. Since \( A^a \) is odd, we suppose that as a result of additional dynamics on the Planck brane the even fields $A_{1}^{aR}$, $A_{2}^{aR}$, and $A_{3}^{aR}$ obey the Dirichlet (D) boundary condition there. The resultant symmetry of the theory is $SU(2)_L \times U(1)_Y$ at the Planck brane. The resultant symmetry of the theory is $SU(2)_L \times U(1)_Y$, which is subsequently broken to $U(1)_{EM}$ by nonvanishing $A^a_\mu$ \( (\hat{a} = 1, 2, 3, 4) \) or $\theta_H$. The weak hypercharge $Y$ is given by $Y = T^3 + g_{B-L}/2$.

With the boundary condition in Table I the gauge symmetry $SO(5) \times U(1)_{B-L}$ in the bulk is reduced to $SO(4) \times U(1)_{B-L}$ at the TeV brane and to $SU(2)_L \times U(1)_Y$ at the Planck brane. The resultant symmetry of the theory is $SU(2)_L \times U(1)_Y$, which is subsequently broken to $U(1)_{EM}$ by nonvanishing $A^a_\mu$ \( (\hat{a} = 1, 2, 3, 4) \) or $\theta_H$. The weak hypercharge $Y$ is given by $Y = T^3 + g_{B-L}/2$.

The boundary conditions of $A_{1}^{aL}$, $A_{2}^{aL}$, and $A_{3}^{aL}$ are changed from N to D on the Planck brane, if additional dynamics on the Planck brane spontaneously breaks $SU(2)_R \times U(1)_{B-L}$ to $U(1)_Y$ at relatively high energy scale $M$, say, near the Planck scale $M_P$ so that $A_{1}^{aL}$, $A_{2}^{aL}$, and $A_{3}^{aL}$ have masses of $O(M)$ on the Planck brane. Below the TeV scale, the mass terms on the Planck brane strongly suppress the boundary values of these fields, effectively changing the boundary conditions from N to D at the Planck brane. With the underlying gauge invariance it is expected that the tree unitarity for the gauge boson scatterings\(^{53-55}\) is preserved with these boundary conditions.

\( \S 3. \) Spectrum and mode functions of gauge bosons

In this section we derive the spectrum and mode functions of gauge fields with the boundary conditions listed in Table I. Although such quantities have been well
discussed in many papers in the case of vanishing \( \theta_H \) (see Refs. 50–52) for example), the case of nonzero \( \theta_H \) becomes highly nontrivial as the boundary conditions of 5D fields are twisted by the angle \( \theta_H \), i.e., they are no longer either the ordinary Neumann or Dirichlet boundary conditions. In fact the \( SU(3) \) model has been analysed in Ref. 37) where it is found that the wave functions have nontrivial \( \theta_H \) dependence. Here and in the next section we provide systematic KK analysis and obtained the full spectrum and the wave functions in the \( SO(5) \times U(1)_{B-L} \) model for a general value of \( \theta_H \). The results obtained here are used to estimate various coupling constants in Sec. 6.

3.1. General solutions in the bulk

The basic procedure is the same as in our previous papers. We employ the background field method, separating \( A_M, B_M \) into the classical part \( A_M^c, B_M^c \) and the quantum part \( A_M^q, B_M^q \); \( A_M = A_M^c + A_M^q \) and \( B_M = B_M^c + B_M^q \). It is convenient go over to the conformal coordinate \( z \equiv e^{\sigma(y)} \) for the fifth dimension;

\[
d s^2 = \frac{1}{z^2} \left( \eta_{\mu \nu} dx^\mu dx^\nu + \frac{dz^2}{k^2} \right),
\]

\[
\partial_y = k z \partial_z, \quad A_y = k z A_z, \quad B_y = k z B_z.
\] (3.1)

The boundaries are located at \( z = 1 \) and \( z = \pi \equiv e^{k \pi R} \). The gauge-fixing functions are chosen as

\[
f_{gf}^{(A)} = z^2 \left\{ \eta^{\mu \nu} \mathcal{D}_\mu A_3^\nu + \xi k^2 z \mathcal{D}_z \left( \frac{1}{z} A_3^z \right) \right\},
\]

\[
f_{gf}^{(B)} = z^2 \left\{ \eta^{\mu \nu} \partial_\mu B_3^\nu + \xi k^2 z \partial_z \left( \frac{1}{z} B_3^z \right) \right\}
\] (3.2)

where \( \mathcal{D}_\mu A_3^\mu \equiv \partial_\mu A_3^\mu - ig_A [A_3^\mu, A_3^\nu] \).

The quadratic terms for the gauge and ghost fields are simplified for \( \xi = 1 \),

\[
S = \int d^4x \frac{dz}{k z} \left[ \text{tr} \left\{ \eta^{\mu \nu} A_3^\mu (\Box + k^2 \mathcal{P}_4) A_3^\nu + k^2 A_3^\mu (\Box + k^2 \mathcal{P}_z) A_3^\mu + \mathcal{L}_{gh}^{(A)} \right\} + \eta^{\mu \nu} B_3^\mu (\Box + k^2 \mathcal{P}_4) B_3^\nu + k^2 B_3^\mu (\Box + k^2 \mathcal{P}_z) B_3^\mu + \mathcal{L}_{gh}^{(B)} \right],
\] (3.3)

where

\[
\Box \equiv \eta^{\mu \nu} \partial_\mu \partial_\nu, \quad \mathcal{P}_4 \equiv z \mathcal{D}_z^c \mathcal{D}_z^c, \quad \mathcal{P}_z \equiv z \mathcal{D}_z^c \mathcal{D}_z^z.
\] (3.4)

Here we have taken \( A_3^\mu = 0 \), respecting the 4D Poincaré symmetry. The surface terms at the boundaries at \( z = 1, z = \pi \) vanish thanks to the boundary conditions for each field.

The linearized equations of motion for \( A_M \) become

\[
\Box A_3^\mu + k^2 z \mathcal{D}_z^c \mathcal{D}_z^z A_3^\mu = 0,
\]

\[
\Box A_3^z + k^2 \mathcal{D}_z^c \mathcal{D}_z^z A_3^z = 0.
\] (3.5)
and those for $B_M$ have similar forms. The classical background is taken to be
\[ A_c^z = v z T^4 \] (v: real constant) below.

We move to a twisted basis by a gauge transformation\[ \tilde{A}_M \equiv \Omega A_M^q \Omega^{-1}, \quad \tilde{B}_M \equiv B_M^q, \]
\[ \Omega(z) \equiv \exp \left\{ i g A \int_z^{z_f} dz' A_c^z(z') \right\}. \tag{3.6} \]

As shown in Refs. 5, 56, 57, sets of boundary conditions related by gauge transformations form equivalence classes and all sets in each equivalence class are physically equivalent. Dynamics of the Yang-Mills AB phase $\theta_H$ guarantees the equivalence.

In the twisted basis the classical background of the gauge fields vanishes so that the linearized equations of motion reduce to free field equations, while the boundary conditions become more involved.

\[ \Box \tilde{A}_\mu + k^2 \left( \partial_z^2 - \frac{1}{z} \partial_z \right) \tilde{A}_\mu = 0, \quad \Box \tilde{A}_z + k^2 \left( \partial_z^2 - \frac{1}{z} \partial_z + \frac{1}{z^2} \right) \tilde{A}_z = 0, \tag{3.7} \]

Thus the equations for eigenmodes with a mass eigenvalue $m_n = k \lambda_n$ are

\[ \left\{ \frac{d^2}{dz^2} - \frac{1}{z} \frac{d}{dz} + \lambda_n^2 \right\} \tilde{h}^I_{A,n} = 0, \quad \left\{ \frac{d^2}{dz^2} - \frac{1}{z} \frac{d}{dz} + \lambda_n^2 \right\} \tilde{h}^I_{\phi,n} = 0. \tag{3.8} \]

With these eigenfunctions the gauge potentials are expanded as

\[ \tilde{A}_\mu(x,z) = \sum_n \tilde{h}^I_{A,n}(z) A_{\mu}^{(n)}(x), \quad \tilde{A}_z(x,z) = \sum_n \tilde{h}^I_{\phi,n}(z) \phi^{(n)}(x), \]
\[ \tilde{B}_\mu(x,z) = \sum_n \tilde{h}^B_{A,n}(z) A_{\mu}^{(n)}(x), \quad \tilde{B}_z(x,z) = \sum_n \tilde{h}^B_{\phi,n}(z) \phi^{(n)}(x). \tag{3.9} \]

The general solutions to Eq.(3.8) are expressed in terms of the Bessel functions as

\[ \tilde{h}^I_{A,n}(z) = z \left\{ \alpha_{A,n}^I J_1(\lambda_n z) + \beta_{A,n}^I Y_1(\lambda_n z) \right\}, \]
\[ \tilde{h}^I_{\phi,n}(z) = z \left\{ \alpha_{\phi,n}^I J_0(\lambda_n z) + \beta_{\phi,n}^I Y_0(\lambda_n z) \right\}, \tag{3.10} \]

where $I = (a_L, a_R, \hat{a}, B)$ and $\alpha_n$’s and $\beta_n$’s are constants to be determined by the boundary conditions.

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We define $\Omega(z)$ so that $\Omega(z_\pi) = 1_4$ in contrast to our previous work\[ 1, 37 \] where it is defined as $\Omega(1) = 1_4$.\[ ^*) \]
3.2. Mass eigenvalues and mode functions

To determine the eigenvalues $\lambda_n$’s and the corresponding mode functions (3.10), we need to take into account the boundary conditions listed in Table I. In this subsection we mainly examine the 4D components of the gauge fields ($A_\mu$, $B_\mu$). The mass spectrum and the mode functions for the extra-dimensional components ($A_z$, $B_z$) are examined in the next subsection.

At the boundaries

$$\partial_z A^a_L = \partial_z A^a_R = 0 \ , \ A^{1,2}_{\mu} = A^{3}_{\mu} = 0 \quad \text{at} \ z = 1 \ , \quad (3.11)$$

$$\partial_z A^a_L = \partial_z A^a_R = \partial_z B_\mu = 0 \ , \ A^a_\mu = 0 \quad \text{at} \ z = z_\pi \ . \quad (3.12)$$

We translate these conditions into those in the twisted basis ($\tilde{A}_M$, $\tilde{B}_M$). Among the extra-dimensional components of the gauge fields, only $A^a_z$ can have non-vanishing vacuum expectation values (VEV). With the residual $SU(2)_L \times U(1)_Y$ symmetry, we can restrict ourselves to

$$A^c_z = v z T^4 \ , \quad (3.13)$$

where a constant $v$ is related to $\theta_H$ by

$$\theta_H = \frac{g_A v}{2 \sqrt{2}} (z_\pi^2 - 1) \ . \quad (3.14)$$

The potential has a classical flat direction along $\theta_H$. The value for $\theta_H$ is determined at the quantum level. Using (3.13), the gauge transformation matrix $\Omega$ defined in (3.6) is calculated as

$$\Omega(z) = \exp \left\{ \frac{i}{2} \theta(z) (2 \sqrt{2} T^4) \right\} = \begin{pmatrix} c_\theta 1_2 & i s_\theta 1_2 \\ i s_\theta 1_2 & c_\theta 1_2 \end{pmatrix}, \quad (3.15)$$

where

$$\theta(z) = \frac{g_A}{\sqrt{2}} \int_z^{z_\pi} dz' A^4_z(z') = \frac{g_A v}{2 \sqrt{2}} (z_\pi^2 - z^2) = \theta_H \frac{z_\pi^2 - z^2}{z_\pi^2 - 1} \ , \quad (3.16)$$

and $c_\theta \equiv \cos \frac{1}{2} \theta(z)$, $s_\theta \equiv \sin \frac{1}{2} \theta(z)$. Thus the relation between $A_M$ and $\tilde{A}_M$ in Eq.(3.6) can be written as

$$\begin{pmatrix} \tilde{A}^{aL}_M \\ \tilde{A}^{aR}_M \\ \tilde{A}^a_M \end{pmatrix} = \begin{pmatrix} c_\theta^2 & s_\theta^2 & \sqrt{2} s_\theta c_\theta \\ s_\theta^2 & c_\theta^2 & -\sqrt{2} s_\theta c_\theta \\ \sqrt{2} s_\theta c_\theta & \sqrt{2} s_\theta c_\theta & c_\theta^2 - s_\theta^2 \end{pmatrix} \begin{pmatrix} A^{aL}_M \\ A^{aR}_M \\ A^a_M \end{pmatrix} \ , \quad (3.17)$$

where $a = 1, 2, 3$.

The boundary conditions (3.11) and (3.12) can be rewritten in terms of $\tilde{A}_\mu$ by using this relation. The condition (3.12) determines the ratios between $\alpha_{A,n}$’s and $\beta_{A,n}$’s in Eq.(3.10) so that the mode functions have the following forms.

$$\tilde{h}^{aL}_{A,n}(z) = C^{aL}_{A,n} z F_{1,0}(\lambda_n z, \lambda_n z_\pi) \ ,$$
3.2.1. Charged sector ($A_{\mu}^L, A_{\mu}^R, A_{\mu}^{\pm}$)

Here the functions $F_{\alpha,\beta}(u, v)$ are defined in (3.11). The mass eigenvalue $\lambda_n$ and the coefficients $C_n$'s are determined by the remaining boundary condition (3.11), which amount to

$$
\begin{pmatrix}
2 H \lambda_n F_{0,0} & s_H^2 \lambda_n F_{0,0} & -\sqrt{2} s_H c_H F_{0,1} \\
n_H^2 \lambda_n F_{1,0} & s_H^2 \lambda_n F_{1,0} & \sqrt{2} s_H c_H F_{1,1} \\
\sqrt{2} s_H c_H \lambda_n F_{1,0} & -\sqrt{2} s_H c_H \lambda_n F_{1,0} & (c_H^2 - s_H^2) F_{1,1}
\end{pmatrix}
\begin{pmatrix}
C_{A,n}^a \\
C_{A,n}^b \\
C_{A,n}^c
\end{pmatrix} = 0 ,
$$

for $a = 1, 2, 3$.

$$
\begin{pmatrix}
c_H^2 \lambda_n F_{0,0} & s_H^2 \lambda_n F_{0,0} & -\sqrt{2} s_H c_H F_{0,1} \\
\sqrt{2} s_H c_H \lambda_n F_{1,0} & s_H^2 \lambda_n F_{1,0} & \sqrt{2} s_H c_H F_{1,1} \\
s_H^2 \lambda_n F_{0,0} & s_H^2 \lambda_n F_{0,0} & (c_H^2 - s_H^2) F_{1,1}
\end{pmatrix}
\begin{pmatrix}
C_{A,n}^a \\
C_{A,n}^b \\
C_{A,n}^c
\end{pmatrix} = 0 ,
$$

where $s_H \equiv \sin \frac{\theta_H}{2}$, $c_H \equiv \cos \frac{\theta_H}{2}$. Here and henceforth, $F_{\alpha,\beta}$ without the argument denotes $F_{\alpha,\beta}(\lambda_n, \lambda_n z_{\pi})$.

$U(1)$ subgroup remains unbroken for any value of non-zero $\theta_H$, which is identified with the electromagnetic gauge group $U(1)_{EM}$. The gauge fields are classified in three sectors, the charged sector

$$(A_{\mu}^{\pmL}, A_{\mu}^{\pmR}, A_{\mu}^{\pm}) \equiv \frac{1}{\sqrt{2}} (A_{\mu}^{1L} \pm i A_{\mu}^{2L}, A_{\mu}^{1R} \pm i A_{\mu}^{2R}, A_{\mu}^{1L} \pm i A_{\mu}^{2R}) ,
$$

the neutral sector

$$(A_{\mu}^{3L}, A_{\mu}^{3R}, A_{\mu}^{3}) ,
$$

and the ‘singlet’ sector $A_{\mu}^{3}$. The latter two sectors are neutral under $U(1)_{EM}$. The orthonormal relations among the mode functions are

$$
\int \frac{dz}{kz} \left\{ \tilde{h}_{A,n}^{\pm L} \tilde{h}_{A,n}^{\pm L} + \tilde{h}_{A,n}^{\pm R} \tilde{h}_{A,n}^{\pm R} + \tilde{h}_{A,n}^{\pm} \tilde{h}_{A,n}^{\pm} \right\} = \delta_{n,l} ,
$$

$$
\int \frac{dz}{kz} \left\{ \tilde{h}_{A,n}^{3 L} \tilde{h}_{A,n}^{3 L} + \tilde{h}_{A,n}^{3 R} \tilde{h}_{A,n}^{3 R} + \tilde{h}_{A,n}^{3} \tilde{h}_{A,n}^{3} \right\} = \delta_{n,l} ,
$$

$$
\int \frac{dz}{kz} \tilde{h}_{A,n}^{\pm} \tilde{h}_{A,n}^{\pm} = \delta_{n,l} .
$$

3.2.1. Charged sector ($A_{\mu}^{\pmL}, A_{\mu}^{\pmR}, A_{\mu}^{\pm}$)

In order for nontrivial solutions to Eq. (3.19) to exist, the determinant of the $3 \times 3$ matrix must vanish, which leads to

$$
F_{1,0} \left\{ \pi^2 \lambda_n^2 z_{\pi} F_{0,0} F_{1,1} - 2 \sin^2 \theta_H \right\} = 0 .
$$
Once the spectrum $\lambda_n$ is determined by the above equation, the corresponding $C_{A,n}$’s are fixed by (3.19) with the normalization condition (3.24).

There are two cases for the mass spectrum.

**Case 1:** $F_{1,0} = 0$

There is no massless mode and the lightest mode has a mass of $O(m_{KK})$, where the Kaluza-Klein (KK) mass scale $m_{KK}$ is given by

$$m_{KK} \equiv \frac{k\pi}{z_\pi - 1}.$$  

(3.26)

The coefficients $C_{A,n}^I$ ($I = \pm_L, \pm_R, \pm_\perp$) in the mode functions are given by

$$C_{A,n}^{\pm_L} = (1 - \cos \theta_H) \hat{C}_1,$$

$$C_{A,n}^{\pm_R} = -(1 + \cos \theta_H) \hat{C}_1,$$

$$C_{A,n}^{\pm_\perp} = 0,$$

$$\hat{C}_1 = \frac{\sqrt{k}}{\sqrt{1 + \cos^2 \theta_H}} \left\{ \frac{4}{\pi^2 \lambda_n^2} - F_{0,0}^2 \right\}^{-1/2}.$$  

(3.27)

The mass spectrum is independent of $\theta_H$ and is the same as for the modes with the boundary condition (D,N) at $\theta_H = 0$. For nonzero $\theta_H$, however, the above modes do not have definite $Z_2$-parities since components with different boundary conditions mix with each other. This can be seen explicitly from the fact that the mode functions have nontrivial $\theta_H$-dependences.

**Case 2:** $\pi^2 \lambda_n^2 z_\pi F_{0,0} F_{1,1} = 2 \sin^2 \theta_H$

In this case the mass spectrum depends on $\theta_H$. The lightest mode is massless at $\theta_H = 0$, while it acquires a nonvanishing mass when $\theta_H \neq 0$. The lightest mode is identified with the $W$ boson. The coefficients in the mode functions are given by

$$C_{A,n}^{\pm_L} = (1 + \cos \theta_H) \hat{C}_2,$$

$$C_{A,n}^{\pm_R} = (1 - \cos \theta_H) \hat{C}_2,$$

$$C_{A,n}^{\pm_\perp} = -\sqrt{2} \sin \theta_H \frac{F_{1,0}}{F_{1,1}} \hat{C}_2,$$

$$\hat{C}_2 = \frac{\sqrt{k}}{\sqrt{1 + \cos^2 \theta_H}} \left\{ \frac{4}{\pi^2 \lambda_n^2} + \frac{\pi^2 \lambda_n^2 \sin^2 \theta_H (1 + \cos^2 \theta_H)}{\sin^2 \theta_H (1 + \cos^2 \theta_H)} \right\}^{-1/2}.$$  

(3.28)

One comment is in order about the behavior of $\hat{C}_2$ in the $\theta_H \to 0$ limit. For the KK excited states, or modes with $\lim_{\theta_H \to 0} \lambda_n \neq 0$, either $F_{0,0}$ or $F_{1,1}$ is $O(\theta_H^2)$. $C_{A,n}^{\pm_L}$ becomes dominant for the modes with $F_{0,0} = O(\theta_H^2)$, while $C_{A,n}^{\pm_R}$ becomes dominant for the modes with $F_{1,1} = O(\theta_H^2)$. For the zero mode $\lambda_0 = O(\theta_H)$ and $\hat{C}_2 \simeq \sqrt{\pi} \lambda_0 / 4\sqrt{R}$. 
3.2.2. Neutral sector \((A_{\mu}^{3L}, A_{\mu}^{3R}, A_{\mu}^{\hat{3}}, B_{\mu})\)

The determinant of the \(4 \times 4\) matrix in (3.20) must vanish, which leads to

\[
\lambda_n F_{0,0} F_{1,0} \left\{ \pi^2 \lambda_n^2 z_n F_{0,0} F_{1,1} - 2(1 + s_\phi^2) \sin^2 \theta_H \right\} = 0 \quad .
\]

(3.29)

\(s_\phi\) is defined in (2.11). This determines the mass spectrum. Once \(\lambda_n\) is determined, the coefficients \(C_{A,n}(\bar{I})\) in mode functions are fixed by (3.20) with the normalization condition (3.24).

The neutral sector is classified into three cases.

**Case 1:** \(\lambda_n F_{0,0} = 0\)

The massless mode \((\lambda_0 = 0)\) identified with the photon for the unbroken \(U(1)_{EM}\) has a constant mode function

\[
\tilde{h}_{A,0}^{3L} = \tilde{h}_{A,0}^{3R} = \frac{s_\phi}{\sqrt{(1 + s_\phi^2)\pi R}} \quad ,
\]

\[
\tilde{h}_{A,0}^{\hat{3}} = 0 \quad , \quad \tilde{h}_{A,0}^{B} = \frac{c_\phi}{\sqrt{(1 + s_\phi^2)\pi R}} \quad .
\]

(3.30)

The massive modes have

\[
\tilde{h}_{A,n}(z) = \tilde{h}_{A,0}(z) = \sqrt{2k s_\phi} \left\{ \frac{4}{\pi^2 \lambda_n^2} - F_{1,0}^2 \right\}^{-1/2} F_{1,0}(\lambda_n z, \lambda_n z_n) \quad ,
\]

\[
\tilde{h}_{A,n}(z) = 0 \quad , \quad \tilde{h}_{A,n}(z) = \frac{c_\phi}{s_\phi} \tilde{h}_{A,0}(z) \quad .
\]

(3.31)

(3.30) can be obtained also from (3.31) by taking the limit of \(\lambda_n \to 0\). Note that the mass spectrum and the mode functions in the photon sector are independent of \(\theta_H\). In fact we can extract the photon sector from the neutral sector by the following field redefinition.

\[
\begin{pmatrix}
A_{\mu}^{3V} \\
A_{\mu}^{3A} \\
A_{\mu}^{\hat{3}} \\
A_{\mu}^{B}
\end{pmatrix}
= \frac{1}{\sqrt{2(1 + s_\phi^2)}} \begin{pmatrix}
c_\phi \\
\sqrt{1 + s_\phi^2} \\
\sqrt{2s_\phi} \\
\sqrt{2c_\phi}
\end{pmatrix} \begin{pmatrix}
\frac{c_\phi}{\sqrt{1 + s_\phi^2}} - \frac{c_\phi}{\sqrt{2s_\phi}} \\
-\frac{2s_\phi}{\sqrt{2s_\phi}} \\
0 \\
\sqrt{2c_\phi}
\end{pmatrix} \begin{pmatrix}
A_{\mu}^{3L} \\
A_{\mu}^{3R} \\
\hat{C}_3 \\
B_{\mu}
\end{pmatrix} \quad .
\]

(3.32)

The photon field \(A_{\mu}^{3V}\) does not mix with the other components \((A_{\mu}^{3L}, A_{\mu}^{3R})\) under the \(\Omega\)-rotation in (3.15).

**Case 2:** \(F_{1,0} = 0\)

The equation that determines the mass spectrum is the same as in the case 1 in the charged sector. The coefficients in the mode functions are given by

\[
C_{A,n}^{3L} = (1 - \cos \theta_H) \tilde{C}_3 \quad ,
\]

\[
C_{A,n}^{3R} = -(1 + \cos \theta_H) \tilde{C}_3 \quad ,
\]

\[
C_{A,n}^{\hat{3}} = 0 \quad ,
\]

\[
C_{A,n}^{B} = 2t_\phi \cos \theta_H \tilde{C}_3 \quad ,
\]
\[ \hat{C}_3 = \frac{\sqrt{2\hat{k}}}{\sqrt{1 + (1 + 4t_\phi^2)\cos^2 \theta_H}} \left\{ \frac{4}{\pi^2 \lambda_n^2} - F_{0,0}^2 \right\}^{-1/2}, \]  

(3.33)

where \( t_\phi \equiv s_\phi/c_\phi \).

**Case 3:** \( \pi^2 \lambda_n^2 \pi^2 F_{0,0} F_{1,1} = 2(1 + s_\phi^2) \sin^2 \theta_H \)

In this case the mass spectrum depends on \( \theta_H \). The lightest mode becomes massless at \( \theta_H = 0 \) while it acquires a nonzero mass when \( \theta_H \neq 0 \). The mode is identified with the \( Z \) boson. The coefficients in the mode functions are given by

\[ C_{3A,n}^L = \left\{ c_\phi^2 + \cos \theta_H(1 + s_\phi^2) \right\} \hat{C}_4, \]
\[ C_{3A,n}^R = \left\{ c_\phi^2 - \cos \theta_H(1 + s_\phi^2) \right\} \hat{C}_4, \]
\[ C_{3A,n}^\hat{3} = -\sqrt{2}(1 + s_\phi^2) \sin \theta_H \frac{F_{1,0}}{F_{1,1}} \hat{C}_4, \]
\[ C_{3A,n}^B = -\frac{2s_\phi c_\phi}{c_\phi^2 + \cos \theta_H(1 + s_\phi^2)} C_{3A,n}^{3L}, \]

\[ \hat{C}_4 = \frac{\sqrt{\hat{k}}}{\sqrt{2(1 + s_\phi^2)\cos \chi}} \left\{ \frac{4}{\pi^2 \lambda_n^2} + \frac{\pi^2 \lambda_n^2 \pi^2 F_{1,0}^2}{\sin^2 2\chi} - \frac{F_{1,0}^2}{\cos^2 \chi} - \frac{F_{2,0}^2}{\sin^2 \chi} \right\}^{-1/2}, \]

\[ \sin^2 \chi \equiv \frac{1 + s_\phi^2}{2} \sin^2 \theta_H. \]

(3.34)

Note that the \( 4 \times 4 \) matrix in (3.20) reduces to a direct sum of \( 3 \times 3 \) and \( 1 \times 1 \) matrices and the former is identical to the \( 3 \times 3 \) matrix in (3.19) if we set \( (s_\phi, c_\phi) = (0, 1) \). Thus the spectrum and the mode functions of the charged sector \( (A_{\mu}^{3L}, A_{\mu}^{3R}, A_{\hat{3}}^\mu) \) are reproduced from those of \( (A_{\mu}^L, A_{\mu}^R, A_{\hat{3}}^\mu) \) by setting \( (s_\phi, c_\phi) = (0, 1) \).

### 3.2.3. Singlet sector \( A_{\mu}^4 \)

Finally there is the singlet sector \( A_{\mu}^4 \). There is no zero mode in this sector. From the normalization condition (3.24), the coefficient is determined as

\[ C_{4A,n}^4 = \sqrt{2\hat{k}} \left\{ \frac{4}{\pi^2 \lambda_n^2} - F_{0,1}^2 \right\}^{-1/2}. \]

(3.35)

In the gauge sector, there are some classes of K.K. modes whose spectra are independent of \( \theta_H \), i.e., the case 1 in the charged sector, the cases 1 and 2 in the neutral sector, and the singlet sector. In all these cases, the mode functions do not have nonzero components for \( T^a (a = 1, 2, 3, \text{or } \pm 3) \). This means that the modes in these classes do not have nonvanishing couplings to the Higgs field \( H = \varphi^0 \), which reflects the \( \theta_H \)-independence of the mass spectra. (The corresponding coupling constants are expressed like (6.8) in Sect. 6.2.)

### 3.3. Spectrum and mode functions of gauge scalars

In this subsection the spectrum and mode functions for the extra-dimensional components of gauge potentials, or gauge scalars, are examined. The boundary
conditions for $A_z$ and $B_z$ are given by

$$A_z^{aL} = A_z^{aR} = B_z = 0 \quad , \quad \partial_z \left( \frac{A^z}{z} \right) = 0 \quad (3.36)$$

at both boundaries. The conditions at $z = z_\pi$ determine the ratios between $\alpha_{\varphi,n}$’s and $\beta_{\varphi,n}$’s in Eq. (3.10) so that the mode functions have the following forms.

$$\tilde{h}_{\varphi,n}^a (z) = \tilde{C}_{\varphi,n}^a F_0,0 \left( \lambda_n z, \lambda_n z_\pi \right) ,$$

$$\tilde{h}_{\varphi,n}^a (z) = \tilde{C}_{\varphi,n}^a F_0,1 \left( \lambda_n z, \lambda_n z_\pi \right) ,$$

$$\tilde{h}_{\varphi,n}^a (z) = \tilde{C}_{\varphi,n}^a F_0,0 \left( \lambda_n z, \lambda_n z_\pi \right) ,$$

$$\tilde{h}_{\varphi,n}^a (z) = \tilde{C}_{\varphi,n}^a F_0,0 \left( \lambda_n z, \lambda_n z_\pi \right) . \quad (3.37)$$

To treat gauge scalars, it is convenient to define

$$A_z^{aL+R} = \frac{1}{\sqrt{2}} \left( A_z^{aL} \pm A_z^{aR} \right) , \quad (3.38)$$

in terms of which Eq (3.17) can be rewritten as

$$\tilde{A}_z^{aL+R}(z) = \frac{1}{\sqrt{2}} \left( A_z^{aL} \pm A_z^{aR} \right) , \quad (3.39)$$

where $a = 1, 2, 3$. In contrast to the 4D components $A_\mu^I$ and $B_\mu$, the boundary conditions in (3.36) do not mix $\tilde{A}_z^{aL+R}$, $\tilde{A}_z^{aL+R}$, $\tilde{A}_z^a$ and $B_z$. By making use of (3.37), the boundary conditions at the Planck brane is rewritten as

$$F_{0,0} C_{\varphi,n}^{aL+R} = 0 \quad , \quad (3.40)$$

$$\begin{pmatrix} \cos \theta_H F_{0,0} & -\sin \theta_H F_{0,1} \\ \sin \theta_H F_{1,0} & \cos \theta_H F_{1,1} \end{pmatrix} \begin{pmatrix} C_{\varphi,n}^{aL+R} \\ C_{\varphi,n}^a \end{pmatrix} = 0 \quad , \quad (3.41)$$

for $a = 1, 2, 3,$ and

$$\lambda_n F_{1,1} C_{\varphi,n}^a = 0 \quad , \quad (3.42)$$

$$F_{0,0} C_{\varphi,n}^{aL+R} = 0 \quad . \quad (3.43)$$

Here $F_{\alpha,\beta} = F_{\alpha,\beta}(\lambda_n, \lambda_n z_\pi)$. The orthonormal relations are given by

$$\int_1^{z_\pi} \frac{k dz}{z} \frac{\tilde{h}_{\varphi,n}^{aL+R}}{\tilde{h}_{\varphi,l}^{aL+R}} = \delta_{n,l} \quad ,$$

$$\int_1^{z_\pi} \frac{k dz}{z} \left\{ \frac{\tilde{h}_{\varphi,n}^{aL+R}}{\tilde{h}_{\varphi,l}^{aL+R}} + \frac{\tilde{h}_{\varphi,n}^a}{\tilde{h}_{\varphi,l}^a} \right\} = \delta_{n,l} \quad .$$
\[
\int_1^{z_\pi} \frac{kdz}{z} \tilde{h}_{\varphi,n}^z \tilde{h}_{\varphi,l}^z = \delta_{n,l}
\]
\[
\int_1^{z_\pi} \frac{kdz}{z} \tilde{h}_{\varphi,n}^{\hat{B}} \tilde{h}_{\varphi,l}^{\hat{B}} = \delta_{n,l}.
\] (3.44)

It follows from the conditions (3.40) and (3.41) that the spectra for the charged sector (3.22) and the neutral sector (3.23) are degenerate. The gauge scalar sector is classified into four cases specified by Eqs. (3.40)-(3.43).

**Case 1: Singlet sector I** \(A_{z}^{a_{L+R}}\)

For the \(A_{z}^{a_{L+R}}\) components the mass spectrum is determined by

\[
F_{0,0} = 0.
\] (3.45)

There are no zero-modes and the coefficients \(C_{\varphi,n}^{I}\) in the mode functions are determined by the normalization condition (3.44).

\[
C_{\varphi,n}^{a_{L+R}} = \sqrt{\frac{2}{k}} \left\{ \frac{4}{\pi^2 \lambda_n^2} - F_{1,0}^2 \right\}^{-1/2}
\]
\[
C_{\varphi,n}^{a_{L-R}} = C_{\varphi,n}^{\hat{a}} = C_{\varphi,n}^{B} = 0,
\] (3.46)

where \(a_{L\pm R} = 1, 2, 3\) and \(\hat{a} = 1, 2, 3, 4\).

**Case 2: Doublet sector** \((A_{z}^{a_{L-R}}, A_{\hat{a}}^{a})\)

For \((A_{z}^{a_{L-R}}, A_{\hat{a}}^{a})\) the mass spectrum is determined by

\[
F_{0,0} F_{1,1} = \frac{4 \sin^2 \theta_H}{\pi^2 \lambda_n^2 z_\pi}.
\] (3.47)

From Eq. (3.41) and the normalization condition (3.44), the mode functions are obtained as

\[
C_{\varphi,n}^{a_{L-R}} = \sqrt{\frac{2}{k}} \left\{ \frac{4}{\pi^2 \lambda_n^2} + \frac{\pi^2 \lambda_n^2 z_\pi^2 F_{1,0}^2 F_{1,0}^2}{\sin^2 2\theta_H} - \frac{F_{1,0}^2}{\cos^2 \theta_H} - \frac{F_{0,0}^2}{\sin^2 \theta_H} \right\}^{-1/2},
\]
\[
C_{\varphi,n}^{\hat{a}} = - \tan \theta_H \frac{F_{1,0}}{F_{1,1}} C_{\varphi,n}^{a_{L-R}},
\]
\[
C_{\varphi,n}^{a_{L+R}} = C_{\varphi,n}^{B} = 0.
\] (3.48)

**Case 3: Higgs sector** \(A_{z}^{\hat{a}}\)

The sector \(A_{z}^{\hat{a}}\) corresponds to the Higgs sector. The spectrum is determined by

\[
\lambda_n F_{1,1} = 0.
\] (3.49)

There is a zero-mode, which is identified as the 4D Higgs boson. It acquires a nonvanishing finite mass \(m_H\) by quantum effects at the one loop level. It has been
estimated in Refs. 36, 37) that $m_H \sim 0.1 \sqrt{\alpha_W} k \pi R m_W / |\sin \theta_H|$, which gives $m_H$ in a range $140 \sim 280$ GeV. The mode functions in this case are given by

$$\tilde{h}_{\varphi,0}^4(z) = \sqrt{\frac{2}{k(z^2_n - 1)}} z,$$

$$\tilde{h}_{\varphi,0}^{aL \pm R}(z) = \tilde{h}_{\varphi,0}^B(z) = 0,$$ (3.50)

for the zero-mode (the 4D Higgs field), and

$$C_{\varphi,n}^4 = \sqrt{\frac{2}{k}} \left\{ \frac{4}{\pi^2 a^2_n} - F^2_{0,1} \right\}^{-1/2},$$

$$C_{\varphi,n}^{aL \pm R} = C_{\varphi,n}^B = 0,$$ (3.51)

for other KK modes ($n \neq 0$).

**Case 4: Singlet sector II $B_z$**

The spectrum is the same as in Case 1, but the mode functions are non-vanishing only in the $B_z$ part. They are given by

$$C_{\varphi,n}^B = \sqrt{\frac{2}{k}} \left\{ \frac{4}{\pi^2 a^2_n} - F^2_{1,0} \right\}^{-1/2},$$

$$C_{\varphi,n}^{aL \pm R} = C_{\varphi,n}^B = 0,$$ (3.52)

where $a_{L \pm R} = 1, 2, 3$ and $\hat{a} = 1, 2, 3, 4$.

Notice that the spectrum depends on $\theta_H$ only in the doublet sector. In other words, the Higgs field couples only to the doublet sector.

### §4. Spectrum and mode functions of fermions

Masses of quarks and leptons can originate not only from gauge interactions and bulk kink masses in the fifth dimension, but also from brane interactions with additional fermion fields on the branes. Indeed such additional interactions seem necessary to realize the observed mass spectrum and gauge couplings in the quark and lepton sectors. The main focus in the present paper is gauge-Higgs interactions, and we defer, to a separate paper, detailed discussions about how to construct realistic models. At the moment we merely mention that one can introduce chiral spinor fields $\chi_R$ on the Planck brane and $\chi_L$ on the TeV brane, which have mixing terms with the bulk fermion $\Psi$. Let us take $\eta_{R,1} = +1$ in (2.26) so that the $Z_2$-parities are assigned as Table II for a fermion multiplet in the spinor representation of $SO(5)$.

We further suppose that $\chi_{Ri}$ ($i = 1, 2$) and $\chi_L$ have the same quantum number as $Q_{Ri}$ and $q_L$, where $i$ denotes the $SU(2)_R$-doublet index. The Lagrangian in the fermionic sector, then, would be

$$\mathcal{L}_{\text{ferm}} = \sqrt{-G} \left[ i\Phi^N \mathcal{D}_N \Phi - iM_{\Phi^N}\tilde{\Phi}^N \Phi \right]$$

where $\Phi$ is a fermionic field.
Table II. The $Z_2$-parities of the fermions. We take the same parity assignment at both boundaries.

| $q_R$ | $Q_R$ | $q_L$ | $Q_L$ |
|-------|-------|-------|-------|
| even  | odd   | odd   | even  |

\[
\begin{align*}
+ \sum_{i=1}^{2} \left\{ i \bar{\chi}_R \gamma^\mu D_\mu \chi_R - (i \mu_Q \bar{\chi}_R Q_L + \text{h.c.}) \right\} \delta(y) \\
+ \left\{ i \bar{\chi}_L \gamma^\mu D_\mu \chi_L - (i \mu_q \bar{\chi}_L q_R + \text{h.c.}) \right\} \delta(y - \pi R) \right], \tag{4.1}
\end{align*}
\]

where $\mu_Q$ and $\mu_q$ are brane-mass parameters of mass-dimension $1/2$. With these additional parameters a realistic spectrum can be reproduced.

In the subsequent discussions, however, we restrict ourselves to fermions without brane interactions, setting $\mu_Q = \mu_q = 0$ and dropping $\chi_R$ and $\chi_L$. Accordingly the index $i$ of $Q_{L,i}$ is suppressed. We describe below how the mass spectrum and gauge couplings are determined, and what kind of potential problems arise in the simplified model.

4.1. General solutions in the bulk

The linearized equations of motion are

\[
\begin{align*}
e^\sigma \gamma^\mu \partial_\mu q_R - (\partial_y - 2\sigma' + M_\phi \varepsilon)q_L + \frac{ig_A}{2\sqrt{2}} A_y^4 Q_L &= 0, \\
e^\sigma \gamma^\mu \partial_\mu Q_R - (\partial_y - 2\sigma' + M_\phi \varepsilon)Q_L + \frac{ig_A}{2\sqrt{2}} A_y^4 q_L &= 0, \\
e^\sigma \gamma^\mu \partial_\mu q_L + (\partial_y - 2\sigma' - M_\phi \varepsilon)q_R - \frac{ig_A}{2\sqrt{2}} A_y^4 Q_R &= 0, \\
e^\sigma \gamma^\mu \partial_\mu Q_L + (\partial_y - 2\sigma' - M_\phi \varepsilon)Q_R - \frac{ig_A}{2\sqrt{2}} A_y^4 q_R &= 0. \tag{4.2}
\end{align*}
\]

From the parity assignment and linearized equations of motion, the boundary conditions for the bulk fermions $(q, Q)$ are determined. In the conformal coordinate $z = e^{\sigma(y)}$, they are written as

\[
\begin{align*}
D_-(c) \hat{q}_R &= 0, \quad \hat{q}_L = 0, \\
\hat{Q}_R &= 0, \quad D_+(c) \hat{Q}_L = 0, \tag{4.3}
\end{align*}
\]

at both $z = 1$ and $z = z_\pi$. Here $\hat{q} \equiv z^{-2} q$, $\hat{Q} \equiv z^{-2} Q$, and

\[
D_\pm(c) \equiv \pm \frac{d}{dz} + \frac{c}{z}, \tag{4.4}
\]

where $c \equiv M_\phi / k$. We remark that when there are brane interactions with localized fermions, i.e. when $\mu_q, \mu_Q \neq 0$, the above boundary conditions are modified, becoming no longer Dirichlet- nor Neumann-type.
We expand the 5D fermion fields into 4D K.K. modes.

\[
\hat{q}_R(x, z) = \sum_n \hat{f}^q_{R,n}(z) \psi^{(n)}_R(x), \quad \hat{q}_L(x, z) = \sum_n \hat{f}^q_{L,n}(z) \psi^{(n)}_L(x),
\]

\[
\hat{Q}_R(x, z) = \sum_n \hat{f}^Q_{R,n}(z) \psi^{(n)}_R(x), \quad \hat{Q}_L(x, z) = \sum_n \hat{f}^Q_{L,n}(z) \psi^{(n)}_L(x).
\] (4.5)

It follows from (4.2) that equations for an eigenmode with a mass eigenvalue \( m_n = k\lambda_n \) are given by

\[
D_-(c) \hat{f}^q_{R,n}(z) - i\frac{\dot{\theta}}{2} \hat{f}^Q_{R,n}(z) = \lambda_n \hat{f}^q_{L,n}(z),
\]

\[
D_-(c) \hat{f}^q_{R,n}(z) + i\frac{\dot{\theta}}{2} \hat{f}^Q_{R,n}(z) = \lambda_n \hat{f}^q_{L,n}(z),
\]

\[
D_+(c) \hat{f}^q_{L,n}(z) + i\frac{\dot{\theta}}{2} \hat{f}^Q_{L,n}(z) = \lambda_n \hat{f}^q_{R,n}(z).
\] (4.6)

Here \( \dot{\theta} \equiv d\theta/dz \) where \( \theta(z) \) is defined in (3.10). The orthonormal relations are

\[
\int_1^{z_s} \frac{dz}{k} \left\{ (\hat{f}^q_{R,l})^* \hat{f}^q_{R,n} + (\hat{f}^Q_{R,l})^* \hat{f}^Q_{R,n} \right\} = \delta_{n,l},
\]

\[
\int_1^{z_s} \frac{dz}{k} \left\{ (\hat{f}^q_{L,l})^* \hat{f}^q_{L,n} + (\hat{f}^Q_{L,l})^* \hat{f}^Q_{L,n} \right\} = \delta_{n,l}.
\] (4.7)

In order to solve the mode equations, it is convenient to move to the twisted basis defined in (3.10) and (3.15), in which

\[
\left( \hat{q} \right)' \equiv \Omega(z) \left( \hat{q} \right).
\] (4.8)

The mode equations are simplified to

\[
D_-(c) \hat{f}^q_{R,n} = \lambda_n \hat{f}^q_{L,n}, \quad D_+(c) \hat{f}^q_{L,n} = \lambda_n \hat{f}^q_{R,n},
\]

\[
D_-(c) \hat{f}^q_{L,n} = \lambda_n \hat{f}^q_{R,n}, \quad D_+(c) \hat{f}^q_{R,n} = \lambda_n \hat{f}^q_{L,n},
\] (4.9)

where

\[
\begin{pmatrix}
\hat{f}^q_{R,n} \\
\hat{f}^q_{L,n}
\end{pmatrix}
= \begin{pmatrix}
c_{\theta} & i\theta \\
i\theta & c_{\theta}
\end{pmatrix}
\begin{pmatrix}
\hat{f}^q_{R,n} \\
\hat{f}^q_{L,n}
\end{pmatrix}, \quad
\begin{pmatrix}
\hat{f}^q_{L,n} \\
\hat{f}^q_{R,n}
\end{pmatrix}
= \begin{pmatrix}
c_{\theta} & i\theta \\
i\theta & c_{\theta}
\end{pmatrix}
\begin{pmatrix}
\hat{f}^q_{L,n} \\
\hat{f}^q_{R,n}
\end{pmatrix}.
\] (4.10)

Then (4.5) becomes

\[
\tilde{q}_R(x, z) = \sum_n \hat{f}^q_{R,n}(z) \psi^{(n)}_R(x), \quad \tilde{q}_L(x, z) = \sum_n \hat{f}^q_{L,n}(z) \psi^{(n)}_L(x),
\]

\[
\tilde{Q}_R(x, z) = \sum_n \hat{f}^Q_{R,n}(z) \psi^{(n)}_R(x), \quad \tilde{Q}_L(x, z) = \sum_n \hat{f}^Q_{L,n}(z) \psi^{(n)}_L(x).
\] (4.11)
The general solutions of Eqs. (4.9) are

\[
\begin{align*}
\tilde{j}_{R,n}(z) &= z^{\frac{1}{2}} \left\{ a_n^{\alpha} J_{\alpha-1}(\lambda_n z) + b_n^{\alpha} Y_{\alpha-1}(\lambda_n z) \right\}, \\
\tilde{j}_{Q,n}(z) &= z^{\frac{1}{2}} \left\{ a_n^{\alpha} J_{\alpha-1}(\lambda_n z) + b_n^{\alpha} Y_{\alpha-1}(\lambda_n z) \right\}, \\
\tilde{j}_{L,n}(z) &= z^{\frac{1}{2}} \left\{ a_n^{\alpha} J_\alpha(\lambda_n z) + b_n^{\alpha} Y_\alpha(\lambda_n z) \right\}, \\
\tilde{j}_{Q,n}(z) &= z^{\frac{1}{2}} \left\{ a_n^{\alpha} J_\alpha(\lambda_n z) + b_n^{\alpha} Y_\alpha(\lambda_n z) \right\},
\end{align*}
\]

(4.12)

where \( \alpha \equiv (M_\Psi/k) + \frac{1}{2} \). The eigenvalues \( \lambda_n \) and the coefficients \( a_n \)'s and \( b_n \)'s are determined by the boundary conditions (4.13).

4.2. Mass eigenvalues and mode functions

From the conditions (4.13) at \( z = z_\pi \), the ratios between \( a_n \)'s and \( b_n \)'s in (4.12) are determined so that the mode functions are written by using the function \( F_{\alpha,\beta}(u,v) \) defined in (C.1) as

\[
\begin{align*}
\tilde{j}_{R,n}(z) &= C_n^\alpha z^{\frac{1}{2}} F_{\alpha-1,\alpha}(\lambda_n z, \lambda_n z_\pi), \\
\tilde{j}_{Q,n}(z) &= C_n^\alpha z^{\frac{1}{2}} F_{\alpha-1,\alpha-1}(\lambda_n z, \lambda_n z_\pi), \\
\tilde{j}_{L,n}(z) &= C_n^\alpha z^{\frac{1}{2}} F_{\alpha,\alpha}(\lambda_n z, \lambda_n z_\pi), \\
\tilde{j}_{Q,n}(z) &= C_n^\alpha z^{\frac{1}{2}} F_{\alpha-1,\alpha}(\lambda_n z, \lambda_n z_\pi).
\end{align*}
\]

(4.13)

The eigenvalue \( \lambda_n \) and constants \( C_n^\alpha \) and \( C_n^\beta \) are determined by the remaining boundary conditions in (4.13) at \( z = z_\pi \). By making use of (4.13), the two conditions for right-handed components in (4.13) at \( z = z_\pi \) are rewritten as

\[
\begin{pmatrix}
c_H F_{\alpha,\alpha} & -is_H F_{\alpha,\alpha-1} \\
-is_H F_{\alpha-1,\alpha} & c_H F_{\alpha-1,\alpha-1}
\end{pmatrix}
\begin{pmatrix}
C_n^\alpha \\
C_n^\beta
\end{pmatrix} = 0.
\]

(4.14)

Here \( F_{\alpha,\beta} = F_{\alpha,\beta}(\lambda_n, \lambda_n z_\pi) \). For a nontrivial solution to exist, the determinant of the \( 2 \times 2 \) matrix in Eq. (4.14) must vanish, which leads to

\[
\pi^2 \lambda_n^2 z_\pi^2 F_{\alpha-1,\alpha-1} F_{\alpha,\alpha} = 4s_H^2.
\]

(4.15)

We have used the last formula in (C.2). This is the equation that determines the mass spectrum \( \{ \lambda_n \} \). Once \( \lambda_n \) is determined, the corresponding \( C_n^\alpha \) and \( C_n^\beta \) are fixed by (4.14) with the normalization condition (4.17). The result is

\[
C_n^\alpha = is_H F_{\alpha-1,\alpha} C_n^\beta,
\]

\[
C_n^\beta = \sqrt{2k} \left[ \frac{4}{\pi^2 \lambda_n^2} + \frac{\pi^2 \lambda_n^2 z_\pi^2 F_{\alpha-1,\alpha-1}^2 F_{\alpha,\alpha}^2}{4s_H^2} - \frac{F_{\alpha,\alpha}^2}{c_H^2} - \frac{F_{\alpha-1,\alpha-1}^2}{s_H^2} \right]^{-1/2}.
\]

(4.16)

* The remaining conditions in (4.13) at \( z = z_\pi \) provide the same conditions on \( (C_n^\alpha, C_n^\beta) \) as (4.14).
§5. Masses and wave functions of light particles

Approximate expressions of the masses and wave functions of light particles such as $W$, $Z$, quarks and leptons can be obtained. The mass of the 4D Higgs particle is generated at the quantum level. Its mass is estimated from the effective potential for the Yang-Mills AB phase $\theta_H$.

5.1. Gauge sector

The masses and wave functions of the $W$ and $Z$ bosons have nontrivial $\theta_H$ dependence. They belong to the case 2 of the charged sector in Sec. 3.2.1 and the case 3 of the neutral sector in Sec. 3.2.2 whose mass spectra are determined by

\[ \pi^2 \lambda_n^2 z_\pi F_{0,0} F_{1,1} = 2 \sin^2 \theta_H \]
\[ \pi^2 \lambda_n^2 z_\pi F_{0,0} F_{1,1} = 2(1 + \phi^2) \sin^2 \theta_H \]

respectively. Here $F_{\alpha,\beta} = F_{\alpha,\beta}(\lambda_n, \lambda_n z_\pi)$. These equations are similar to those in the $SU(3)$ model discussed in Ref. 37, but there is an important difference in the numerical factors on the right sides of the two equations above. In the $SU(3)$ model one has $4 \sin^2 \frac{\theta_H}{2}$ in place of $2 \sin^2 \theta_H$ in the equation determining the $W$ mass.\(^*)\) Fig. 1 depicts the masses of the KK tower of the $W$ boson as functions of $\theta_H$ for $k \pi R = 35, 3, 0.35$. Due to the numerical factor mentioned above, the mass spectrum does not approach a linear spectrum in $\theta_H$ in the flat limit ($k \pi R \to 0$) in contrast to the $SU(3)$ model. (See Fig. 1 in Ref. 37.) For $0 < k \pi R \ll 1$,

\[ \lambda_n z_\pi \gg \lambda_n \gg \frac{m_{KK}}{k} = \frac{\pi}{z_\pi - 1} \approx \frac{\pi}{k \pi R} \gg 1 \quad (n \geq 1) \]  

With (5.3), the left-hand side of (5.1) becomes

\[ \pi^2 \lambda_n^2 z_\pi F_{0,0} F_{1,1} \approx 4 \sin^2 (\pi R m_n) \quad (n \geq 1) \]  

The mass eigenvalues $m_n = k \lambda_n$ become linear functions of $\theta_H$ in the flat limit only if the numerical factor in the right-hand side of Eq. (5.1) is 4.\(^**)\) Therefore the K.K. level-crossing does not occur as $\theta_H$ increases from 0 to $\pi$ in our model even in the flat geometry. This is one of the distinctive properties of the $SO(5) \times U(1)_{B-L}$ model.

As can be seen from Fig. 1 the mass of the lightest mode is much lighter than the K.K. mass scale $m_{KK}$ when the warp factor $z_\pi = e^{k \pi R}$ is large, that is, when $\lambda_0, \lambda_0 z_\pi \ll 1$. Thus one may use (5.4) to obtain an approximate expression for the mass of the lightest mode in each category determined by (5.1) and the corresponding mode function from (3.28) or (3.34). For the $W$ boson the mass is given by

\[ m_W = \frac{m_{KK}}{\pi} \sqrt{\frac{1}{k \pi R}} \left| \sin \theta_H \right| \left\{ 1 + O\left(\frac{\pi^2 m_W^2}{m_{KK}^2}\right) \right\} \]  

\(^*)\) The masses of the $Z$ boson and its K.K. modes were not discussed in Ref. 37) as the correct value of the Weinberg angle $\theta_W$ is not obtained in the $SU(3)$ model.

\(^**)\) When the numerical factor is 4, it can be easily shown that the mass of the lightest ($n = 0$) mode also becomes a linear function of $\theta_H$ in the flat limit.
with its mode function
\[
\hat{h}_{A,0}^{±L}(z) \simeq \frac{1 + \cos \theta_H}{2\sqrt{\pi R}}, \quad \hat{h}_{A,0}^{±R}(z) \simeq \frac{1 - \cos \theta_H}{2\sqrt{\pi R}}, \quad \hat{h}_{A,0}^±(z) \simeq \frac{\sin \theta_H}{\sqrt{2\pi R}} \left( 1 - \frac{z^2}{z^2_π} \right).
\]

For the Z boson the mass is given by
\[
m_Z = \frac{m_{KK}}{\pi} \sqrt{\frac{1 + s_φ^2}{kπR}} \left| \sin \theta_H \right| \left\{ 1 + \mathcal{O}\left( \frac{π^2 m_Z}{m_{KK}^2} \right) \right\}
\]
with its mode function
\[
\hat{h}_{A,0}^{3L}(z) \simeq \frac{c_φ^2 + \cos \theta_H (1 + s_φ^2)}{2\sqrt{(1 + s_φ^2)πR}}, \quad \hat{h}_{A,0}^{3R}(z) \simeq \frac{c_φ^2 - \cos \theta_H (1 + s_φ^2)}{2\sqrt{(1 + s_φ^2)πR}},
\]
\[
\hat{h}_{A,0}^3(z) \simeq -\sin \theta_H \sqrt{\frac{1 + s_φ^2}{2πR}} \left( 1 - \frac{z^2}{z^2_π} \right), \quad \hat{h}_{A,0}^B(\bar{z}) \simeq -\frac{s_φ c_φ}{\sqrt{(1 + s_φ^2)πR}}.
\]

We stress that \(m_W\) and \(m_Z\) are not proportional to the VEV of the 4D Higgs field, or \(\theta_H\), in contrast to the ordinary Higgs mechanism in four dimensions. The mechanism of mass generation for the 4D gauge bosons in the gauge-Higgs unification scenario involves not only 4D gauge fields and scalar fields in each KK level, but also fields in other KK levels. The lowest mode in each KK tower necessarily mixes with heavy K.K. modes when \(\theta_H\) acquires a nonzero value. Furthermore, there is mixing with other components of the gauge group at non-vanishing \(\theta_H\) as well. The spectrum and mixing is such that they become periodic in \(\theta_H\) with a period \(2π\).

From \((5.4)\) and \((5.6)\), the Weinberg angle \(\theta_W\) determined from \(m_W\) and \(m_Z\) becomes
\[
\sin^2 \theta_W \equiv 1 - \frac{m_W^2}{m_Z^2}
\]
\[
\simeq \frac{s_φ^2}{1 + s_φ^2} = \frac{g_B^2}{g_A^2 + 2g_B^2} = \frac{g_Y^2}{g_A^2 + g_B^2}.
\]

The approximate equality in the second line is valid to the \(\mathcal{O}(0.1\%)\) accuracy for \(m_{KK} = \mathcal{O}(\text{TeV})\). In the last equality the relation \(g_Y = g_A g_B / \sqrt{g_A^2 + g_B^2}\) has been
made use of. Note that \( s_\phi \) defined in (2.11) satisfies \( s_\phi \simeq \tan \theta_W \). The Weinberg angle \( \theta_W \) may be determined from the vertices in the neutral current interactions. As we will see in (7.5) below, \( \theta_W \) determined this way coincides with that in (5.8) to good accuracy. Thus the \( \rho \) parameter is nearly one in our model. We remark that the \( \rho \) parameter substantially deviates from 1 in the flat limit \( (\pi kR \to 0) \) when \( \theta_H \) is nonvanishing.

According to the classification in the gauge sector, it is convenient to divide the K.K. modes of the charged sector into two classes: \( (\tilde{W}^{(n)}_{\mu}, W^{(n)}_{\mu}) \), corresponding to the cases 1 and 2 in the charged sector. Similarly the K.K. modes of the neutral sector are divided into three classes: \( (A^{(n)}_{\mu}, \tilde{Z}^{(n)}_{\mu}, Z^{(n)}_{\mu}) \), corresponding to the cases 1, 2, and 3 in the neutral sector. The \( W \) and \( Z \) bosons are \( W^{(0)}_{\mu} \) and \( Z^{(0)}_{\mu} \), respectively.

### 5.2. Fermion sector

The mass spectrum in the fermion sector is determined by Eq. (4.15). For \( \lambda_0 \lesssim 1 \) we obtain from (4.15) and (C.4)\(^{37} \),\(^{40} \)

\[
m_0 = k\lambda_0 \simeq k \left( \frac{\alpha(\alpha - 1)}{z \pi \sinh(\alpha k\pi R) \sinh((\alpha - 1)k\pi R)} \right)^{1/2} |s_H|. \quad (5.9)
\]

The mode functions for the lightest mode \( \psi^{(0)} \) are obtained from Eq. (4.14) with the normalization condition (4.7). For \( \alpha > 1 \), for example, they are approximately expressed as

\[
\tilde{f}^q_{R,0}(z) \simeq -i\sqrt{2k\alpha z^{-\alpha} z^{\frac{1}{2} - \alpha}}, \quad \tilde{f}^Q_{R,0}(z) \simeq s_H c_H \sqrt{2k\alpha z^{-\alpha} z^{\frac{3}{2} - \alpha}}, \\
\tilde{f}^q_{L,0}(z) \simeq i s_H \sqrt{2k(\alpha - 1)z^{\frac{1}{2} - \alpha}}, \quad \tilde{f}^Q_{L,0}(z) \simeq c_H \sqrt{2k(\alpha - 1)z^{\frac{3}{2} - \alpha}}. \quad (5.10)
\]

It follows that the right-handed component \( \psi^{(0)}_{R} \) is localized around the TeV brane while the left-handed component \( \psi^{(0)}_{L} \) around the Planck brane.\(^{\star \star} \) For \( \alpha < 0 \), on the other hand, \( \psi^{(0)}_{R} \) is localized around the Planck brane while \( \psi^{(0)}_{L} \) around the TeV brane.

With (5.9) the hierarchical fermion mass spectrum can be reproduced by choosing \( \alpha \) in an \( O(1) \) range.\(^{37} \) However there is a serious problem in this scenario as pointed out in Ref. 1). The gauge couplings to the \( W \) and \( Z \) bosons deviate from the observed values for at least one chiral component of the fermion when \( \theta_H \) is nonzero. This is due to the fact that the wave function of one of the chiral components is inevitably localized around the TeV brane where the mode functions for the \( W \) and \( Z \) bosons deviate from the constant values (see (5.5) and (5.7)).

This problem can be avoided by introducing boundary fields and turning on boundary mass terms with bulk fermions as described with the action (4.1). There

\[^{\star} \] The subleading terms in \( \tilde{f}^q_{L,0}(z) \) and \( \tilde{f}^Q_{R,0}(z) \) dropped in (5.10) become comparable to the leading terms at \( z = z_\pi \), but they remain suppressed in the bulk.

\[^{\star \star} \] \( \tilde{f}^q_{R,0}(z) \) can be localized at the Planck brane if \( \alpha > 3/2 \), but it is exponentially suppressed compared to \( \tilde{f}^q_{R,0}(z) \) for \( z > 1 \) in such a case.
are two origins for fermion masses; the Yang-Mills AB phase $\theta_H$ and boundary masses $\mu_q, \mu_Q$. It can be shown that for $\alpha > 1$ and $z_2^{(1-\alpha)} \ll (\mu_Q^2/k)^{2} \ll \mu_q^2/k \simeq O(1)$ the lightest mass becomes $m_0 \simeq \sqrt{k(\alpha - 1)}\mu_Q$ and its wave functions are suppressed at the TeV brane for both $L$ and $R$ chiral components. Further $SU(2)_R$ is broken at the Planck brane so that $\mu_Q$ can be chosen to be different for the upper and the lower components of $Q_L$. Thus one can realize the observed fermion masses with appropriate $\mu_Q$. The detailed discussions will be given in a separate paper.

§6. Gauge-Higgs self-couplings

Once wave functions of the $W$ and $Z$ bosons and the Higgs boson are determined, the effective four-dimensional couplings among them can be calculated as overlap integrals in the fifth dimension. As $\theta_H$ becomes nonvanishing, the form of wave functions substantially changes with large mixing so that the effective four-dimensional couplings are expected to have nontrivial $\theta_H$-dependence in general. This behavior of the couplings provides crucial tests for the gauge-Higgs unification scenario.

The $WWZ$ coupling has been indirectly measured in the LEP2 experiment of $W$ pair production. The standard model fits the data well within a few percents so that deviation from the value in the standard mode could rule out the model under consideration. As we shall see below, the $WWZ$ coupling in the gauge-Higgs unification in the Randall-Sundrum warped spacetime remains almost universal as in the standard model. In the gauge-Higgs unification in flat spacetime, however, substantial deviation results.

Important predictions from the gauge-Higgs unification scenario are obtained for the $WWH$, $ZZH$, $WWHH$ and $ZZHH$ couplings where $H$ stands for the Higgs boson. These couplings are suppressed compared with those in the standard model. This gives a crucial test to be performed at LHC in the coming years.

6.1. $WWZ$ coupling

The self-couplings among $W$, $Z$ and Higgs bosons are determined from the interaction terms in the twisted basis

$$\int_{z_0}^{z_f} \frac{dz}{kz} \left\{ i g_A \eta^{\mu \rho} \eta^{\sigma \nu} \text{tr}(\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu)[\tilde{A}_\rho, \tilde{A}_\sigma] + 2 ig_A k^2 \eta^{\mu \nu} \text{tr}(\partial_\mu \tilde{A}_z - \partial_z \tilde{A}_\mu)[\tilde{A}_\nu, \tilde{A}_z] + \frac{1}{2} g_A^2 \eta^{\mu \rho} \eta^{\sigma \nu} \text{tr}[\tilde{A}_\mu, \tilde{A}_\nu][\tilde{A}_\rho, \tilde{A}_\sigma] + g_A^2 k^2 \eta^{\mu \nu} \text{tr}[\tilde{A}_\mu, \tilde{A}_z][\tilde{A}_\nu, \tilde{A}_z] \right\}$$

(6.1)

by inserting the wave functions of $W$, $Z$ and $H$. The relevant part of the expansion of the gauge fields is

$$\tilde{A}_\mu = W_\mu^{(0)}(x) \left\{ \tilde{h}_W^{+R}(z)T^{-L} + \tilde{h}_W^{+R}(z)T^{-R} + \tilde{h}_W^+(z)T^0 \right\}$$

$$+ W_\mu^{(0)t}(x) \left\{ \tilde{h}_W^{R+}(z)T^{+L} + \tilde{h}_W^{R+}(z)T^{+R} + \tilde{h}_W^R(z)T^0 \right\}$$

$$+ Z_\mu^{(0)}(x) \left\{ \tilde{h}_Z^{3L}(z)T^{3L} + \tilde{h}_Z^{3L}(z)T^{3R} + \tilde{h}_Z^3(z)T^0 \right\} ,$$
\[ \tilde{A}_z = H^{(0)}(x) \xi^4 H(z)T^4. \] (6.2)

Here \( T^{\pm L} = (T^{1L} \pm i T^{2L})/\sqrt{2} \) etc.. The W wave functions \( \tilde{h}^{\pm R,L}_W(z) = \tilde{h}^{\pm R,L}_{A,0}(z) \) and \( \tilde{\tilde{h}}^{\pm}(z) = \tilde{\tilde{h}}^{\pm}_{A,0}(z) \) are given by (3.18) with (3.28). The Z wave functions \( \tilde{h}^{3R,L}_Z(z) = \tilde{h}^{3R,L}_{A,0}(z) \) and \( \tilde{\tilde{h}}^{3}(z) = \tilde{\tilde{h}}^{3}_{A,0}(z) \) are given by (3.18) with (3.34). The Higgs wave function \( \tilde{\tilde{h}}^{4}(z) = \tilde{\tilde{h}}^{4}_{A,0}(z) \) is given by (3.30).

The WWZ coupling is evaluated by inserting (6.2) into the first term in (6.1) and integrating over \( z \). The result is

\[ \mathcal{L}_{WWZ}^{(4)} = i g_{WWZ} \left\{ (\partial_\mu W^{(0)\dagger}_\nu - \partial_\nu W^{(0)\dagger}_\mu) W^{(0)\mu} Z^{(0)\nu} - (\partial_\mu W^{(0)}_\nu - \partial_\nu W^{(0)}_\mu) W^{(0)\dagger\mu} Z^{(0)\nu} \right\} \]

where the coupling \( g_{WWZ} \) is expressed as overlap integrals

\[ g_{WWZ} = g_A \int_1^{2\pi} \frac{dz}{k z} \left[ \tilde{h}^{3L}_W \left\{ (\tilde{h}^{\pm L}_W)^2 + \frac{1}{2} (\tilde{\tilde{h}}^{\pm L}_W)^2 \right\} + \tilde{h}^{3R}_W \left\{ (\tilde{h}^{\pm R}_W)^2 + \frac{1}{2} (\tilde{\tilde{h}}^{\pm R}_W)^2 \right\} \right. \]

\[ \left. + \tilde{\tilde{h}}^{4}_W \tilde{h}^{\pm}_W (\tilde{\tilde{h}}^{\pm L}_W + \tilde{\tilde{h}}^{\pm R}_W) \right]. \] (6.4)

Note that given \( \theta_H \) and \( k \pi R \), \( m_W \) determines \( k = O(M_{pl}) \) corresponds to \( k \pi R \approx 35 \). With these parameters we have exact wave functions as summarized in (3.18), (3.28) and (3.34). When \( k \pi R \approx 35 \) and the warp factor is large \( e^{k \pi R} \gg 1 \), the approximate expressions for the wave functions of \( W \) and \( Z \) given by (5.5) and (5.7) can be employed to find

\[ g_{WWZ} \simeq \frac{g_A}{\sqrt{(1 + s^2_\phi) \pi R}} \simeq g \cos \theta_W. \] (6.5)

Here a dimensionless coupling \( g \) is defined as

\[ g \equiv \frac{g_A}{\pi R}, \] (6.6)

which is the 4D \( SU(2)_L \) gauge coupling at \( \theta_H = 0 \). In the last equality in (6.5), (6.8) has been made use of. We have neglected corrections suppressed by a factor of \( (k \pi R)^{-1} \approx 1/35 \) in conformity with the approximation employed in deriving Eqs. (5.4)-(5.7).

These couplings (6.5) have the same values as those in the standard model. Although the wave functions vary substantially as \( \theta_H \), the triple gauge coupling \( WWZ \) remains almost universal. The statement can be strengthened by numerically integrating (6.4) with exact wave functions. In Table III the values of the ratio of the trilinear couplings (6.4) to \( g_A/\sqrt{(1 + s^2_\phi) \pi R} \) are shown for various values of \( \theta_H \) and \( k \pi R \). It is clearly seen that the relation (6.5) hold to extreme accuracy when the warp factor \( e^{k \pi R} \) is large. This is very important in the light of the experimental
Table III. The values of a ratio of $g^{(1)}_{WWZ}(= g^{(2)}_{WWZ})$ to $g_A/\sqrt{(1 + s^2_\varphi)}\pi R$ for $\theta_H = \pi/10, \pi/4, \pi/2$ and $k\pi R = 35, 3.5, 0.35$.

| $k\pi R$ | $\theta_H$ | $\pi/10$ | $\pi/4$ | $\pi/2$ |
|-----------|-------------|----------|----------|----------|
| 35        | 0.9999987  | 0.999964 | 0.99985  |
| 3.5       | 0.9999078  | 0.996993 | 0.98460  |
| 0.35      | 0.9994990  | 0.979458 | 0.83378  |

fact that $W$ pair production rate measured at LEP2 is consistent with the $WWZ$ coupling in the standard model. In the flat spacetime limit ($k\pi R \rightarrow 0$), however, substantial deviation from the standard model results with moderate $\theta_H$. When $\theta_H = O(1)$ gauge-Higgs unification in flat spacetime contradicts with the data for the $WWZ$ coupling.

As can be seen in the overlap integrals in (6.4), weight of wave functions is best measured in the $y = k^{-1} \ln z$ coordinate. The wave functions of the gauge bosons in the warped spacetime in the $y$ coordinate remain almost constants except in a tiny region near $y = \pi R$ even at $\theta_H \neq 0$. (See (5.5) and (5.7).) In the flat spacetime, however, wave functions are deformed substantially at $\theta_H \neq 0$ to have nontrivial $y$-dependence in the entire region, which contributes to substantial deviation in the $WWZ$ coupling from the standard model.

6.2. $WWH$ and $ZZH$ couplings

A striking prediction of the gauge-Higgs unification scenario is obtained for the Higgs couplings to $W$ and $Z$. Let us consider the $WWH$ and $ZZH$ couplings. The 4D Higgs field $H = \varphi^{(0)}$ is a part of the 5D gauge fields so that the Higgs couplings to $W$ and $Z$ are completely determined by the gauge principle once $\theta_H$ is given. Unlike the $WWZ$ coupling, substantial deviation from the standard model is predicted.

Indeed, by inserting (6.2) into the second term in (6.1), one finds that

$$\mathcal{L}^{(4)} = \lambda_{WWH} H^{(0)} W^{(0)\mu} W^{(0)}_\mu + \frac{1}{2} \lambda_{ZZH} H^{(0)} Z^{(0)\mu} Z^{(0)}_\mu + \cdots ,$$

(6.7)

where

$$\lambda_{WWH} = g_A k \int_1^{2\pi} \frac{dz}{z} \tilde{h}_H^{-1} \left\{ \tilde{h}_W^{\pm} \partial_z \left( \tilde{h}_W^{\pm} - \tilde{h}_W^{\pm} \right) - \partial_z \tilde{h}_W^{\pm} \left( \tilde{h}_W^{\pm} - \tilde{h}_W^{\pm} \right) \right\} ,$$

$$\lambda_{ZZH} = g_A k \int_1^{2\pi} \frac{dz}{z} \tilde{h}_H^{-1} \left\{ \tilde{h}_Z^{3} \partial_z \left( \tilde{h}_Z^{3} - \tilde{h}_Z^{3} \right) - \partial_z \tilde{h}_Z^{3} \left( \tilde{h}_Z^{3} - \tilde{h}_Z^{3} \right) \right\} .$$

(6.8)

Recall that the wave function of the Higgs field, $\tilde{h}_H(z) \propto z = e^{k y}$, is localized near the TeV brane at $z = z_x$ when evaluated in the $y$-coordinate relevant in the integrals (6.8). The behavior of the wave functions of $W$ and $Z$ bosons near the TeV brane sensitively depends on $\theta_H$ so that nontrivial $\theta_H$ dependence is expected for the $WWH$ and $ZZH$ couplings.

The integrals in (6.8) can be evaluated in a closed form. Consider $\lambda_{WWH}$. Inserting the wave functions into (6.8) and making use of the identity $\partial_z \left\{ z F_1(\lambda z, \lambda z_\pi) \right\} =
\( \lambda z F_{0, \theta}(\lambda z, \lambda z_\pi) \) and the last relation in \( (C.2) \), one finds

\[
\lambda_{WWH} = \frac{4g_A k^2 C^\pm_W (C^\pm_W - C^\pm_R)}{\pi^2 m_W z_\pi} \int_1^{z_\pi} dz \tilde{h}_H^\pm(z)
\]

\[
= \frac{2g_A k \sqrt{2} (z_\pi^2 - 4)}{\pi^2 m_W z_\pi} C^\pm_W (C^\pm_W - C^\pm_R)
\quad (6.9)
\]

The coefficients \( C^\pm_{W, R} \) are defined in \( (6.6) \) and \( C^\pm_{W, A} \) are given by \( (6.2) \). Similarly for \( \lambda_{ZZH} \) one finds

\[
\lambda_{ZZH} = \frac{2g_A k \sqrt{2} (z_\pi^2 - 4)}{\pi^2 m_Z z_\pi} C^\pm_Z (C^\pm_Z - C^\pm_R)
\quad (6.10)
\]

where the coefficients \( C^\pm_Z = C^\pm_{Z, A} \) and \( C^\pm = C^\pm_{Z, 0} \) are given by \( (6.3) \). The formulas \( (6.9) \) and \( (6.10) \) are exact. They are fairly well evaluated with the approximate formulas \( (5.4)-(5.7) \), leading to \( \tilde{C}_2 \sim \sqrt{\pi} m_W / 4 k \sqrt{R} \) for \( W \) and \( \tilde{C}_4 \sim \sqrt{\pi} m_Z / 4 k \sqrt{R} \) for \( \tilde{Z} \). Insertion of these gives

\[
\lambda_{WWH} \approx \frac{g_A k}{\pi R z_\pi} \sin \theta_H \cos \theta_H \approx g m_W \cdot p_H \cos \theta_H|
\]

\[
\lambda_{ZZH} \approx \frac{g_A k (1 + s_\phi^2)}{\pi R z_\pi} \sin \theta_H \cos \theta_H \approx \frac{g m_Z}{\cos \theta_W} \cdot p_H \cos \theta_H|
\quad (6.11)
\]

where \( g \) is defined in \( (6.6) \) and \( p_H \equiv \text{sgn}(\tan \theta_H) \). Both \( \lambda_{WWH} \) and \( \lambda_{ZZH} \) are suppressed by a factor \( \cos \theta_H \) compared with the corresponding couplings in the standard model. Unless \( \theta_H \) is very small, this gives substantial suppression which can be checked in the coming experiments at LHC. This is a generic prediction in the gauge-Higgs unification in the warped spacetime. It does not depend on the details of the model such as fermion content and couplings.

6.3. Vanishing \( WWH^{(n)} \) and \( ZZH^{(n)} \) couplings

There is a Kaluza-Klein (KK) tower of the 4D Higgs field. The 4D Higgs field is a part of the fifth dimensional component of gauge potentials. It is associated with the Yang-Mills AB phase along the fifth dimension, and is a physical degree of freedom. Its KK excited states, however, are unphysical. In the unitary gauge they are eliminated to be absorbed by the KK excited states of the four-dimensional gauge fields. One may wonder if these KK excited states of the 4D Higgs field, \( H^{(n)} \), have nontrivial couplings to \( W \) and \( Z \).

The \( WWH^{(n)} \) and \( ZZH^{(n)} \) couplings are evaluated in the same manner as the \( WWH \) and \( ZZH \) couplings in Sec. 6.2. They are given by

\[
\mathcal{L}^{(4)} = \lambda_{WWH^{(n)}} H^{(n)} W^{(n)\mu} W^{(n)\mu} + \frac{1}{2} \lambda_{ZZH^{(n)}} H^{(n)} Z^{(n)\mu} Z^{(n)\mu} + \cdots
\quad (6.12)
\]

where

\[
\lambda_{WWH^{(n)}} = g_A k \int_1^{z_\pi} \frac{dz}{z} \tilde{h}_H^{\pm(n)} \left\{ \tilde{h}_W^+ \partial_z \left( \tilde{h}_W^- - \tilde{h}_W^0 \right) - \partial_z \tilde{h}_W^+ \left( \tilde{h}_W^- - \tilde{h}_W^0 \right) \right\}
\]

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\[ \lambda_{ZZH^{(n)}} = g_A k \int_1^{z_{\pi}} \frac{dz}{z} \{ \tilde{h}_Z^4 \partial_z \left( \tilde{h}_Z^3 - \tilde{h}_Z^3 \right) - \partial_z \tilde{h}_Z^4 \left( \tilde{h}_Z^3 - \tilde{h}_Z^3 \right) \} . \] (6.13)

We first recall that the spectrum \( m_n^{W} = k \lambda_n^{W} \) for \( H^{(n)} \) \( (n \geq 1) \) is determined by the equation \( F_{1,1}(\lambda_n^{W}, \lambda_n^{Z}, \lambda_n^{Z}) = 0 \). The wave function is given by \( \tilde{h}_n^4(z) = C_{\varphi,n} z F_{0,1}(\lambda_n^{W}, \lambda_n^{Z}, \lambda_n^{Z}) \) where \( C_{\varphi,n} \) is given by \( (3.51) \). \( \lambda_{WWW} \) is evaluated as \( \lambda_{WWH} \) in the previous subsection. A similar expression to the first line in Eq. \( (6.9) \) is obtained where \( \tilde{h}_H^4 \) is replaced by \( \tilde{h}_H^4 \). Inserting the wave function, one finds

\[
\lambda_{WWH^{(n)}} = \frac{4g_A k^2 C_{\varphi,n} C_W (C_{W}^L - C_{W}^R)}{2 m_W z_{\pi}} \int_1^{z_{\pi}} dz z F_{0,1}(\lambda_n^{W}, \lambda_n^{Z}, \lambda_n^{Z})
= \frac{4g_A k^2 C_{\varphi,n} C_W (C_{W}^L - C_{W}^R)}{2 m_W z_{\pi}} \frac{-1}{\lambda_n^{W}} F_{1,1}(\lambda_n^{W}, \lambda_n^{Z}, \lambda_n^{Z})
= 0 .
\] (6.14)

Similarly one finds that \( \lambda_{ZZH^{(n)}} = 0 \). This proves that the \( WWH^{(n)} \) and \( ZZH^{(n)} \) couplings identically vanish.

6.4. \( W^4 \) and \( W^2Z^2 \) couplings

The four-dimensional gauge couplings \( WWWW \) and \( W^4 \) and \( W^2Z^2 \) are evaluated from the third term in \( (6.1) \). One finds that

\[
\mathcal{L}^{(4)} = \frac{1}{2} g_{WWW} \left\{ W_{\mu}^{(0)} W_{\nu}^{(0)} W_{\lambda}^{(0)} \right\} - (W_{\mu}^{(0)} W_{\nu}^{(0)} W_{\lambda}^{(0)})^2 \right\}
+ g_{WWZZ} \left\{ W_{\mu}^{(0)} W_{\nu}^{(0)} Z_{\lambda}^{(0)} \right\} - W_{\mu}^{(0)} W_{\nu}^{(0)} Z_{\lambda}^{(0)} Z_{\lambda}^{(0)} \right\}
\] (6.15)

where the couplings \( g_{WWW} \) and \( g_{WWZZ} \) are expressed as overlap integrals

\[
g_{WWW} = g_A^2 \int_1^{z_{\pi}} \frac{dz}{k z} \left\{ \left( \tilde{h}_W^{\pm} \right)^2 + \frac{1}{2} \left( \tilde{h}_W^3 \right)^2 \right\}^2 + \left\{ \left( \tilde{h}_W^{\pm} \right)^2 + \frac{1}{2} \left( \tilde{h}_W^3 \right)^2 \right\}^2 \right) .
\]

\[
g_{WWZZ} = g_A^2 \int_1^{z_{\pi}} \frac{dz}{k z} \left\{ \left( \tilde{h}_W^{\pm} \tilde{h}_W^3 \right) + \frac{1}{2} \left( \tilde{h}_W^3 \right)^2 \right\}^2 + \left\{ \left( \tilde{h}_W^{\pm} \tilde{h}_W^3 \right) + \frac{1}{2} \left( \tilde{h}_W^3 \right)^2 \right\}^2 \right) \right] .
\] (6.16)

Inserting \( (5.5) \) and \( (5.7) \) into \( (6.16) \), one finds

\[
g_{WWW} \simeq \frac{g_A^2}{\pi R} = g^2 .
\]
\[ g^4_{W W Z Z} \simeq \frac{g_A^2}{\pi R (1 + s^2_\phi)} = g^2 \cos^2 \theta_W \ , \quad (6.17) \]

which coincide with couplings in the standard model.

The quartic gauge couplings are approximately independent of \( \theta_H \) in the gauge-Higgs unification in the warped spacetime, while in flat spacetime they deviate from the values in the standard model. The approximate universality of the gauge couplings in the warped spacetime is guaranteed by the approximately uniform distribution of the wave functions of the gauge bosons in the entire fifth dimension except for the tiny vicinity of the TeV brane.

6.5. \( W W H H \) and \( Z Z H H \) couplings

The four-dimensional gauge-Higgs couplings \( W W \) and \( Z Z H H \) are evaluated from the fourth term in (6.1). One finds that

\[ \mathcal{L}^{(4)} = -\frac{1}{4} \lambda_{WWHH}^2 W_{\mu}^{(0)} W_{\mu}^{(0)} H(0) H(0) - \frac{1}{8} \lambda_{ZZHH}^2 Z_{\mu}^{(0)} Z_{\mu}^{(0)} H(0) H(0) \quad (6.18) \]

where the couplings \( \lambda_{WWHH}^2 \) and \( \lambda_{ZZHH}^2 \) are expressed as overlap integrals

\[
\lambda_{WWHH}^2 = k g_A^2 \int_1^{z_n} \frac{dz}{z} \left( \tilde{h}_H^4 \right)^2 \left\{ (\tilde{h}_W^4 - \tilde{h}_W^4)^2 + 2(\tilde{h}_W^4)^2 \right\} ,
\]

\[
\lambda_{ZZHH}^2 = k g_A^2 \int_1^{z_n} \frac{dz}{z} \left( \tilde{h}_Z^4 \right)^2 \left\{ (\tilde{h}_Z^4 - \tilde{h}_Z^4)^2 + 2(\tilde{h}_Z^4)^2 \right\} . \quad (6.19)
\]

As in the case of the cubic couplings \( \lambda_{WWW} \) and \( \lambda_{ZZZ} \), the Higgs wave function \( \tilde{h}_{H,0}^4 \) is localized near the TeV brane so that the overlap integrals suffer from nontrivial \( \theta_H \) dependence. With (5.5) and (5.7) inserted, the integrals in (6.19) are evaluated to be

\[
\lambda_{WWHH}^2 \simeq \frac{g_A^2}{\pi R} \left( 1 - \frac{2}{3} \sin^2 \theta_H \right) = g^2 \left( 1 - \frac{2}{3} \sin^2 \theta_H \right) ,
\]

\[
\lambda_{ZZHH}^2 \simeq \frac{g_A^2(1+ s^2_\phi)}{\pi R} \left( 1 - \frac{2}{3} \sin^2 \theta_H \right) \simeq \frac{g^2}{\cos^2 \theta_W} \left( 1 - \frac{2}{3} \sin^2 \theta_H \right) . \quad (6.20)
\]

Compared with the values in the standard model, these couplings are suppressed by a factor \( 1 - \frac{2}{3} \sin^2 \theta_H \).

One comment is in order. The couplings defined in (6.19) and (6.20) are to be called as the bare \( W W \) and \( Z Z H H \) couplings. In the effective theory at low energies where all heavy modes are integrated out, the effective \( W W \) and \( Z Z H H \) couplings contain contributions coming from tree diagrams involving \( W^{(n)} \) and \( Z^{(n)} \) as intermediate states. Their contributions may not be negligible, and need careful examination.

The suppression of the bare Higgs couplings is a generic feature of the gauge-Higgs unification, and should be used for testing the scenario by experiments.
§7. Gauge couplings of fermions

Inserting (3.30) and (4.11) into

$$ S_{\text{gc}} = \int \! dx \int \! dz \frac{k}{z} \left\{ g_A \bar{\psi} \gamma^\mu A_\mu \bar{\psi} + \frac{g_B}{2} \bar{\psi} \gamma^\mu B_\mu Q_{B-L} \bar{\psi} \right\} , $$

we obtain

$$ \mathcal{L}^{(4)}_{\text{gc}} = \sum_n W^{(n)}_\mu \left\{ \frac{g_L}{\sqrt{2}} \bar{\psi}_L^{(0)} \gamma^\mu \psi_L^{(0)} + \frac{g_R}{\sqrt{2}} \bar{\psi}_R^{(0)} \gamma^\mu \psi_R^{(0)} + \text{h.c.} \right\} $$

$$ + \sum_n Z^{(n)}_\mu \sum_{i=1}^2 \left\{ g_L Z^{(n)}_\mu \bar{\psi}_L^{(0)} \gamma^\mu \psi_L^{(0)} + g_R Z^{(n)}_\mu \bar{\psi}_R^{(0)} \gamma^\mu \psi_R^{(0)} \right\} $$

$$ + \sum_n A^{(n)}_\mu \sum_{i=1}^2 \left\{ g_L A^{(n)}_\mu \bar{\psi}_L^{(0)} \gamma^\mu \psi_L^{(0)} + g_R A^{(n)}_\mu \bar{\psi}_R^{(0)} \gamma^\mu \psi_R^{(0)} \right\} + \cdots , $$

where the ellipsis denotes terms involving the massive K.K. modes of the fermions. The 4D gauge couplings are given by

$$ g_L^{W(n)} \equiv g_A \int \! \frac{dz}{k} \left\{ |f_{L,0}|^2 h_{W(n)}^e + |f_{L,0}|^2 h_{W(n)}^\mu + \sqrt{2} \text{Im} \left\{ \left( f_{L,0}^* \right)^s h_{W(n)}^s \right\} \right\} , $$

$$ g_L^{\gamma,Z(n)} \equiv (-1)^i - \frac{1}{2} g_A \int \! \frac{dz}{k} \left[ |f_{L,0}|^2 h_{\gamma,Z(n)}^e + |f_{L,0}|^2 h_{\gamma,Z(n)}^\mu + \sqrt{2} \text{Im} \left\{ \left( f_{L,0}^* \right)^s h_{\gamma,Z(n)}^s \right\} \right] $$

$$ + \frac{g_B q_{B-L}}{2} \int \! \frac{dz}{k} h_{\gamma,Z(n)} \left\{ |f_{L,0}|^2 + |f_{L,0}|^2 \right\} , $$

where $q_{B-L}$ is an eigenvalue of $Q_{B-L}$. The index $i = 1, 2$ denotes the $SU(2)_R$-doublet index. The same expressions hold for the right-handed (R) components where $L$ is replaced by $R$. The wave functions $h_{L,\gamma,Z(n)}^e$, $h_{L,\gamma,Z(n)}^\mu$ etc. are given by (3.30), (3.31) and by (3.34), respectively. In a simplified model without boundary mass terms there results nontrivial couplings of right-handed fermions to W bosons.

From the approximate expressions of the mode functions (5.5), (5.7), and (5.10), the 4D gauge couplings are found to be

$$ g_L^{W(0)} \simeq \frac{g_A}{\sqrt{\pi R}} = g , $$

$$ g_L^{Z(0)} \simeq \frac{(-1)^i g_A - g_B q_{B-L} s_\phi c_\phi}{2 \sqrt{(1 + s_\phi^2) \pi R}} \simeq \frac{g}{\cos \theta_W} \left\{ \frac{(-1)^i}{2} - q_{\text{EM}} \sin^2 \theta_W \right\} , $$

$$ g_i^{(0)} = e q_{\text{EM}} , $$
where \( e \equiv g_A \sin \theta_W / \sqrt{\pi R} = g \sin \theta_W \) is the \( U(1)_\text{EM} \) gauge coupling constant and \( q_{\text{EM}} \equiv \{(-1)^{i-1} + q_{B-L}\}/2 \) is the electromagnetic charge. The relation (5.8) has been made use of in the second equality in (7.5). Note that Eqs. (7.4), (7.5) and (7.6) agree with the counterparts in the standard model. Rigorously speaking, the couplings \( g_W(0) \) and \( g_Z(0) \) have small dependence on the parameter \( \alpha \), which leads to tiny violation of the universality in weak interactions as discussed in Ref. 37). It was found that there results violation of the \( \mu - e \) universality of \( O(10^{-8}) \), which is well in the experimental bound.

\( g_W(0) \) and \( g_Z(0) \) for a multiplet \( \Psi \), however, substantially deviate from the standard model values. For instance, one finds \( g_W(0) \approx g (1 - \cos \theta_H)/2 \). This is because the mode functions of the right-handed fermions are localized near the TeV brane for \( \alpha > 1 \). Since K.K. excited states are also localized near the TeV brane, the mixing with K.K. excited states becomes strong, leading to the deviation. The problem can be avoided by introducing boundary fields \( \chi_{R,L} \) with boundary mass mixing with bulk fermions, i.e., \( \mu_Q, \mu_Q \neq 0 \) as in the action (4.1). It can be arranged such that right-handed components mainly consist of boundary fields on the Planck brane so that the deviation of the gauge couplings from the standard model become small enough.

§8. Summary

Gauge-Higgs unification in warped spacetime has many attractive features. It identifies the 4D Higgs field with a zero mode of the extra-dimensional components of the gauge fields, or the Yang-Mills AB phase in the extra dimensions. The Higgs couplings are determined by the gauge principle and the structure of background spacetime.

Although the wave functions of the \( W \) and \( Z \) bosons substantially vary as \( \theta_H \), the \( WWZ \), \( WWWW \), and \( WWZZ \) couplings in the warped spacetime remain nearly the same as in the standard model. However, these couplings considerably deviate from the standard model in the flat spacetime, thus contradicting the LEP2 data on the \( W \) pair production. These stem from the fact that the wave functions of the gauge bosons in the warped spacetime remain almost constants except in a tiny region near \( y = \pi R \) even at \( \theta_H \neq 0 \) while they are deformed substantially to have nontrivial \( y \)-dependence in the entire region in the flat spacetime. The warped spacetime saves the universality of the gauge couplings.

The important deviation from the standard model shows up in the Higgs couplings. We have shown that the \( WWH \) and \( ZZH \) couplings are suppressed by a factor \( \cos \theta_H \) compared with those in the standard model whereas the bare \( WWHH \) and \( ZZHH \) couplings are suppressed by a factor \( 1 - 2 \sin^2 \theta_H \). The precise content of matter fields affects the location of the global minimum of the effective potential \( V_{\text{eff}}(\theta_H) \). Once the value of \( \theta_H \) is determined, the wave functions of \( W \), \( Z \), and \( H \) are fixed as functions of \( \theta_H \). Hence the suppression of the Higgs couplings to \( W \) and \( Z \) is a generic feature of the gauge-Higgs unification, and is independent of the details of the model. It can be used to test the scenario. A similar suppression of these Higgs
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couplings are recently discussed in Ref. 43) with detailed phenomenological studies in
the context of models where the Higgs emerges from a strongly-interacting sector as
a pseudo-Goldstone boson, which are closely related to the gauge-Higgs unification
models. The phenomenological study of our results such as detailed detectability at
LHC/ILC is an important issue and need to be investigated.

As briefly discussed in the present paper, additional brane interactions are nec-
essary to have a realistic spectrum and gauge couplings of fermions. Wave functions
of fermions sensitively depend on such interactions, and so do Yukawa couplings. In
Ref. 37) it is shown that Yukawa couplings in a model without brane interactions
are also suppressed compared with those in the standard model. It is expected that
Yukawa couplings are suppressed in a more realistic model with brane interactions
as well, but definitive statements must be awaited until precise form of the action is
specified in the fermion sector.

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Appendix A

SO(5) generators

The spinorial representation of the SO(5) generators $T^I$ is given by

$$T^{a_L} = \frac{1}{2} \begin{pmatrix} \sigma_a & 0_2 \\ 0_2 & 0 \end{pmatrix}, \quad T^{a_R} = \frac{1}{2} \begin{pmatrix} 0_2 & \sigma_a \\ \sigma_a & 0_2 \end{pmatrix}, \quad T^{\hat{a}} = \frac{i}{2\sqrt{2}} \begin{pmatrix} -\sigma_{\hat{a}}^\dagger & \sigma_{\hat{a}} \\ \sigma_{\hat{a}} & -\sigma_{\hat{a}}^\dagger \end{pmatrix},$$

where $T^{\hat{a}}$ ($\hat{a} = 1, 2, 3, 4$) and $T^{a_L,a_R}$ ($a_L, a_R = 1, 2, 3$) are respectively the generators
of $SO(5)/SO(4)$ and $SO(4) \sim SU(2)_L \times SU(2)_R$, and $\sigma_{\hat{a}} \equiv (\hat{\sigma}, -i\hat{l}_2)$. They are
normalized as

$$\text{tr}(T^IT^J) = \frac{1}{2} \delta^{IJ},$$

where $I, J = (a_L, a_R, \hat{a})$.

Appendix B

Useful formulae for Bessel functions

Here we collect useful formulae for the Bessel functions. $J_\alpha(z)$ and $Y_\alpha(z)$ denote
the Bessel functions of the first and second kinds, respectively.

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-z^2/4)^n}{n! \Gamma(\alpha + n + 1)},$$

(B.1)
\[ Y_\alpha(z) \equiv \begin{cases} 
\frac{1}{\sin \pi \alpha} \{ \cos \pi \alpha \cdot J_\alpha(z) - J_{-\alpha}(z) \} & \text{for } \alpha \neq \text{ an integer}, \\
\frac{1}{\pi} \left[ \frac{\partial J_\alpha(z)}{\partial \alpha} - (-1)^n \frac{\partial J_{-\alpha}(z)}{\partial \alpha} \right] & \text{for } \alpha = n = \text{ an integer}.
\end{cases} \] (B.2)

Their behavior for \(|z| \gg 1\) is given by
\[ J_\alpha(z) \sim \sqrt{\frac{2}{\pi z}} \cos \left( z - \frac{(2\alpha + 1)\pi}{4} \right), \]
\[ Y_\alpha(z) \sim \sqrt{\frac{2}{\pi z}} \sin \left( z - \frac{(2\alpha + 1)\pi}{4} \right). \] (B.3)

These Bessel functions satisfy the following relations.
\[ Z_{\alpha-1}(z) + Z_{\alpha+1}(z) = 2\frac{\alpha}{z} Z_\alpha(z), \]
\[ \frac{dZ_\alpha(z)}{dz} = \frac{\alpha}{z} Z_\alpha(z) - Z_{\alpha+1}(z) = Z_{\alpha-1}(z) - \frac{\alpha}{z} Z_\alpha(z), \]
\[ J_\alpha(z)Y_{\alpha-1}(z) - Y_\alpha(z)J_{\alpha-1}(z) = \frac{2}{\pi z}, \] (B.4)
\[ \int^z dz z Z_\alpha(\lambda z) \tilde{Z}_\alpha(\lambda z) = \frac{2^2}{4} \left\{ 2Z_\alpha(\lambda z) \tilde{Z}_\alpha(\lambda z) - Z_{\alpha-1}(\lambda z) \tilde{Z}_{\alpha+1}(\lambda z) \right. \]
\[ - \tilde{Z}_{\alpha+1}(\lambda z) \tilde{Z}_{\alpha-1}(\lambda z) \right\}, \] (B.5)

where \(Z_\alpha(z), \tilde{Z}_\alpha(z)\) are linear combinations of \(J_\alpha(z)\) and \(Y_\alpha(z)\).

**Appendix C**

**Properties of \(F_{\alpha,\beta}(u, v)\)**

Using the Bessel functions, we define a function
\[ F_{\alpha,\beta}(u, v) \equiv J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v). \] (C.1)

Using \([B.4]\), this satisfies the following relations,
\[ F_{\alpha,\beta}(u, v) = -F_{\beta,\alpha}(v, u), \]
\[ F_{\alpha,\alpha-1}(u, u) = -F_{\alpha,\alpha+1}(u, u) = \frac{2}{\pi u}, \]
\[ F_{\alpha+1,\beta}(u, v) + F_{\alpha-1,\beta}(u, v) = \frac{2\alpha}{u} F_{\alpha,\beta}(u, v), \]
\[ F_{\alpha-1,\alpha}(u, v)F_{\alpha,\alpha-1}(u, u) = F_{\alpha-1,\alpha-1}(u, v)F_{\alpha,\alpha}(u, v) - \frac{4}{\pi^2uv}. \] (C.2)

For non-integer \(\alpha\), we can express \(F_{\alpha,\alpha}(u, v), F_{\alpha,\alpha-1}(u, v), \text{ and } F_{\alpha-1,\alpha}(u, v)\) solely in terms of the Bessel function of the first kind as
\[ F_{\alpha,\alpha}(u, v) = \frac{1}{\sin \pi \alpha} \{ J_{-\alpha}(u)J_\alpha(v) - J_\alpha(u)J_{-\alpha}(v) \}. \]
Using these expressions and \( (B.1) \), we obtain for \( F \):

\[
F_{\alpha,\alpha - 1}(u, v) = \frac{1}{\sin \pi \alpha} \left\{ J_\alpha(u) J_{1 - \alpha}(v) + J_{-\alpha}(u) J_{\alpha - 1}(v) \right\},
\]

\[
F_{\alpha - 1,\alpha}(u, v) = \frac{1}{\sin \pi \alpha} \left\{ J_{\alpha - 1}(u) J_{-\alpha}(v) + J_{1 - \alpha}(u) J_{\alpha}(v) \right\}.
\]

(C-3)

Using these expressions and \( (B.1) \), we obtain for \( z \geq 1 \) and \( \lambda z \ll 1 \),

\[
F_{\alpha,\alpha}(\lambda, \lambda z) = \frac{z^\alpha - z^{-\alpha}}{\alpha} \left\{ 1 + \mathcal{O}(\lambda^2 z^2) \right\},
\]

\[
F_{\alpha - 1,\alpha}(\lambda, \lambda z) = -\frac{2}{\pi \lambda z^\alpha} \left\{ \frac{\lambda}{2\pi\alpha(1 - \alpha)} \left\{ z^\alpha - (1 - \alpha)z^{-\alpha} - \alpha z^{2 - \alpha} \right\} \left\{ 1 + \mathcal{O}(\lambda^2 z^2) \right\} \right\},
\]

\[
F_{\alpha,\alpha - 1}(\lambda, \lambda z) = \frac{2}{\pi \lambda z^{-1 - \alpha}} \left\{ \frac{\lambda}{2\pi\alpha(1 - \alpha)} \left\{ z^{1 - \alpha} - \alpha z^{\alpha - 1} - (1 - \alpha)z^{\alpha + 1} \right\} \left\{ 1 + \mathcal{O}(\lambda^2 z^2) \right\} \right\},
\]

(C-4)

From \( (B.5) \), we obtain

\[
\int_1^{z\pi} dz \ z F_{\alpha,\beta}^2(\lambda_1 z, \lambda_2) = \left[ \frac{z^2}{2} \left\{ F_{\alpha,\beta}^2(\lambda_1 z, \lambda_2) - F_{\alpha + 1,\beta}(\lambda_1 z, \lambda_2) F_{\alpha - 1,\beta}(\lambda_1 z, \lambda_2) \right\} \right]_1^{z\pi},
\]

\[
= \left[ \frac{z^2}{2} \left\{ F_{\alpha,\beta}^2(\lambda_1 z, \lambda_2) + F_{\alpha + 1,\beta}(\lambda_1 z, \lambda_2) \right. \right.
\]

\[
\left. - \frac{2\alpha}{\lambda z} F_{\alpha,\beta}(\lambda_1 z, \lambda_2) F_{\alpha + 1,\beta}(\lambda_1 z, \lambda_2) \right\]_1^{z\pi}.
\]

(C-5)

In the second equality, we used the third equation of \( (C.2) \). Using this, we obtain the following integral formulae, which are useful for determining normalization factors of the mode functions.

\[
\int_1^{z\pi} dz \ z F_{\alpha,\beta}^2(\lambda z, \lambda z_\pi) = \frac{1}{2} \left\{ \frac{4}{\pi^2 \lambda^2} + \frac{2\alpha}{\lambda} F_{\alpha,\beta} F_{\alpha - 1,\beta} - F_{\alpha,\beta}^2 - F_{\alpha - 1,\beta}^2 \right\},
\]

\[
\int_1^{z\pi} dz \ z F_{\alpha - 1,\beta}^2(\lambda z, \lambda z_\pi) = \frac{1}{2} \left\{ \frac{4}{\pi^2 \lambda^2} + \frac{2(\alpha - 1)}{\lambda} F_{\alpha,\beta} F_{\alpha - 1,\beta} - F_{\alpha,\beta}^2 - F_{\alpha - 1,\beta}^2 \right\}.
\]

(C-6)

\( F_{\alpha,\beta} \) in the right-hand sides are understood as \( F_{\alpha,\beta}(\lambda, \lambda z_\pi) \).

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