EXISTENCE AND CONTINUATION OF PERIODIC SOLUTIONS OF AUTONOMOUS NEWTONIAN SYSTEMS

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Abstract. In this article we study the existence and the continuation of periodic solutions of autonomous Newtonian systems. To prove the results we apply the infinite-dimensional version of the degree for $SO(2)$-equivariant gradient operators defined by the third author in [23] and developed in [24]. Using the results due to Rabier [19] and Wang [26] we show that the Leray-Schauder degree is not applicable in the proofs of our theorems, because it vanishes.

1. Introduction

The first aim of this article is to study the existence of periodic solutions of the following system

$$\ddot{x} = -V'(x) \quad (1.1)$$

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$ and $V'$ denotes the gradient of $V$. We assume that $(V')^{-1}(0) = \{p_1, \ldots, p_q\}$ is a finite set and that $V'(x) = V''(\infty) \cdot x + o(\|x\|)$ as $\|x\| \to \infty$, where $V''(\infty)$ is a real symmetric $(n \times n)$-matrix.

Such a problem has been considered for $q = 1$ by Amann and Zehnder, see [2], and by Benci and Fortunato, see [5], for any $q \in \mathbb{N}$.

Benci and Fortunato estimated the number of non-stationary $T$-periodic solutions of (1.1) as $T \to \infty$. To avoid some technicalities and to make the proofs more transparent they assumed that all the non-stationary $T$-periodic solutions are not $T$-resonant and that potential $V$ is a Morse function. These assumptions seems to be restrictive and rather difficult to verify.

We relax these assumptions and therefore we obtain only the existence of at least one non-stationary $T$-periodic solution of (1.1). We formulate the sufficient conditions for the existence of non-stationary $T$-periodic solutions of (1.1) in terms of $V''(p)$ and $\text{ind}(-V', p)$, where $p \in \{p_1, \ldots, p_q, \infty\}$. It is worth to point out that we can treat problems with resonance at stationary solutions and at the infinity. As a basic tool we use the degree for $SO(2)$-equivariant gradient maps, see [23], [24].

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The second aim of this article is to study the continuation of non-stationary $T$-periodic solutions of the following system

$$\ddot{x} = -V'_\lambda(x)$$

(1.2)

where $V_\lambda \in C^2(\mathbb{R}^n, \mathbb{R})$ for $\lambda \in \mathbb{R}$ and potential $V_0$ possesses all the properties of potential $V$ in (1.1). We formulate sufficient conditions for the existence of connected sets of $T$-periodic solutions of (1.2) emanating from level $\lambda = 0$.

We consider solutions of (1.1) and (1.2) as critical points of $SO(2)$-invariant functional defined on a suitably chosen Hilbert space, which is an orthogonal representation of the group $SO(2)$, see also [18, 20]. Gradient of this functional is an $SO(2)$-equivariant map in the form of a compact perturbation of the identity.

It is known that the Conley index and the Morse theory are not suitable tools for the study of global bifurcations and the continuation of critical points of functionals, see [3, 6, 14, 17, 25] for discussion and examples. Since considered gradient is $SO(2)$-equivariant, the Leray-Schauder degree is not applicable in our approach because it vanishes, see [19, 26] and Remark 5.2.6. Therefore to prove our results we apply the degree for $SO(2)$-equivariant gradient maps. Degrees for $G$-equivariant gradient maps has been defined in [8] for $G = SO(2)$. Next it was improved in [23] and in [12] for symmetries of any compact Lie group $G$.

After introduction this article is organized in the following way.

In Section 2 we summarize without proofs the relevant material on the degree for $SO(2)$-equivariant gradient maps. We finish this section with a continuation theorem of critical orbits of $SO(2)$-invariant functionals and the Rabier and Wang result concerning computation of the Leray-Schauder degree for $SO(2)$-equivariant operators.

The main results of Section 3 are Lemmas 3.2, 3.3. We construct in these lemmas admissible $G$-equivariant gradient homotopies for the class of operators in the form of a compact perturbation of a $G$-equivariant self-adjoint Fredholm operator $L$.

We use these homotopies in Section 4 in the case $L = Id$ and $G = SO(2)$. Namely, we simplify computation of the degree for $SO(2)$-equivariant gradient maps at an isolated degenerate critical point i.e. at a critical point with the isotropy group $SO(2)$ or at the infinity.

In Section 5 we formulate and prove the main results of this article. In Subsection 5.1 we study properties of the functional $\Phi_A$ associated to the linear system (5.1.1). In Subsection 5.2 we formulate and prove the sufficient conditions for the existence of non-stationary $T$-periodic solutions of the nonlinear system (5.2.1), see Theorems 5.2.1, 5.2.2. We would like to emphasize that we cannot use in our approach the Leray-Schauder degree, see Remark 5.2.6. In Subsection 5.3 we study continuation of non-stationary $T$-periodic solutions of family of nonlinear equations (5.3.1), see Theorem 5.3.1.

In Section 6 we illustrate results proved in Section 5.

### 2. Preliminaria

In this section, for the convenience of the reader, we remind the main properties of the degree for $SO(2)$-equivariant gradient maps defined in [23]. This degree will be denoted...
briefly by $\nabla_{SO(2)}-\text{deg}$. We finish this section with a theorem due to Rabier [19] and Wang [20].

Put $U(SO(2)) = \mathbb{Z} \oplus \bigoplus_{k=1}^{\infty} \mathbb{Z}$ and define actions

$$
+, \star : U(SO(2)) \times U(SO(2)) \to U(SO(2))
$$

$$
\cdot : \mathbb{Z} \times U(SO(2)) \to U(SO(2))
$$

as follows

$$
\alpha + \beta = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \ldots, \alpha_k + \beta_k, \ldots) \quad (2.1)
$$

$$
\alpha \star \beta = (\alpha_0 \cdot \beta_0, \alpha_0 \cdot \beta_1 + \beta_0 \cdot \alpha_1, \ldots, \alpha_0 \cdot \beta_k + \beta_0 \cdot \alpha_k, \ldots) \quad (2.2)
$$

$$
\gamma \cdot \alpha = (\gamma \cdot \alpha_0, \gamma \cdot \alpha_1, \gamma \cdot \alpha_k, \ldots) \quad (2.3)
$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k, \ldots), \beta = (\beta_0, \beta_1, \ldots, \beta_k, \ldots) \in U(SO(2))$ and $\gamma \in \mathbb{Z}$. It is easy to check that $(U(SO(2)), +, \star)$ is a commutative ring with unit $I = (1, 0, \ldots) \in (SO(2))$. Ring $(U(SO(2)), +, \star)$ is known as the tom Dieck ring of the group $SO(2)$. For a definition of the tom Dieck ring $U(G)$, where $G$ is any compact Lie group, we refer the reader to [11].

If $\delta_1, \ldots, \delta_q \in U(SO(2))$, then we write $\prod_{j=1}^{q} \delta_j$ for $\delta_1 \star \ldots \star \delta_q$. Moreover, it is understood that $\prod_{j \in \emptyset} \delta_j = \emptyset \in U(SO(2))$.

Let $V$ be a real, finite-dimensional and orthogonal representation of the group $SO(2)$. If $v \in V$ then the subgroup $SO(2)_v = \{g \in SO(2) : g \cdot v = v\}$ is said to be the isotropy group of $v \in V$. Let $\Omega \subset V$ be an open, bounded and $SO(2)$-invariant subset and let $H \subset SO(2)$ be closed subgroup. Then we define

- $\Omega^H = \{v \in \Omega : H \subset SO(2)_v\} = \{v \in \Omega : g v = v \forall g \in H\}$,
- $\Omega_H = \{v \in \Omega : H = SO(2)_v\}$.

Fix $k \in \mathbb{N}$ and set $C^k_{SO(2)}(V, \mathbb{R}) = \{f \in C^k(V, \mathbb{R}) : f \text{ is } SO(2) - \text{invariant}\}$. Let $f \in C^1_{SO(2)}(V, \mathbb{R})$. Since $V$ is an orthogonal representation, gradient $\nabla f : V \to V$ is an $SO(2)$-equivariant $C^0$-map. If $H \subset SO(2)$ is a closed subgroup then $V^H$ is a finite-dimensional representation of the group $SO(2)$ and $(\nabla f)^H = \nabla (f|_{V^H}) : V^H \to V^H$ is well-defined $SO(2)$-equivariant gradient map. Choose an open, bounded and $SO(2)$-invariant subset $\Omega \subset V$ such that $(\nabla f)^{-1}(0) \cap \partial \Omega = \emptyset$. Under these assumptions we have defined in [23] the degree for $SO(2)$-equivariant gradient maps $\nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) \in U(SO(2))$ with coordinates

$$
\nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) =
$$

$$
= (\nabla_{SO(2)} - \text{deg}_{SO(2)}(\nabla f, \Omega), \nabla_{SO(2)} - \text{deg}_{Z_1}(\nabla f, \Omega), \ldots, \nabla_{SO(2)} - \text{deg}_{Z_k}(\nabla f, \Omega), \ldots).
$$
For $\gamma > 0$ and $v_0 \in V^{SO(2)}$ we put $D_\gamma(V, v_0) = \{v \in V : |v - v_0| < \gamma\}$. In the following theorem we formulate the main properties of the degree for $SO(2)$-equivariant gradient maps.

**Theorem 2.1 (23).** Under the above assumptions the degree for $SO(2)$-equivariant gradient maps has the following properties

1. If $\nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) \neq \Theta$, then $(\nabla f)^{-1}(0) \cap \Omega \neq \emptyset$.
2. If $\nabla_{SO(2)} - \text{deg}_H(\nabla f, \Omega) \neq 0$, then $(\nabla f)^{-1}(0) \cap \Omega^H \neq \emptyset$.
3. If $\Omega = \Omega_0 \cup \Omega_1$ and $\Omega_0 \cap \Omega_1 = \emptyset$, then
   \[
   \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) = \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega_0) + \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega_1),
   \]
4. If $\Omega_0 \subset \Omega$ is an open $SO(2)$-invariant subset and $(\nabla f)^{-1}(0) \cap \Omega \subset \Omega_0$, then
   \[
   \text{DEG}(\nabla f, \Omega) = \text{DEG}(\nabla f, \Omega_0),
   \]
5. If $f \in C^1_{SO(2)}(V \times [0, 1], \mathbb{R})$ is such that $(\nabla_v f)^{-1}(0) \cap (\partial \Omega \times [0, 1]) = \emptyset$, then
   \[
   \nabla_{SO(2)} - \text{deg}(\nabla f_0, \Omega) = \nabla_{SO(2)} - \text{deg}(\nabla f_1, \Omega),
   \]
6. If $W$ is an orthogonal representation of the group $SO(2)$, then
   \[
   \nabla_{SO(2)} - \text{deg}((\nabla f, Id), \Omega \times D_\gamma(W, 0)) = \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega),
   \]
7. If $f \in C^2_{SO(2)}(V, \mathbb{R})$ is such that $\nabla f(0) = 0$ and $\nabla^2 f(0)$ is an $SO(2)$-equivariant self-adjoint isomorphism then there is $\gamma > 0$ such that
   \[
   \nabla_{SO(2)} - \text{deg}(\nabla f, D_{\gamma}(V, 0)) = \nabla_{SO(2)} - \text{deg}(\nabla^2 f(0), D_{\gamma}(V, 0)).
   \]

**Remark 2.1.** Directly from the definition of the degree for $SO(2)$-equivariant gradient maps, see [23], it follows that

1. If $H \subset SO(2)$ is a closed subgroup and for any $v \in \Omega SO(2)_v \neq H$, then
   \[
   \nabla_{SO(2)} - \text{deg}_H(\nabla f, \Omega) = 0.
   \]
2. If $\Omega_{SO(2)} = \emptyset$, then
   \[
   \nabla_{SO(2)} - \text{deg}(\nabla f, \Omega) = \begin{cases} 0 & \text{if } \Omega_{SO(2)} = \emptyset, \\ 1 & \text{if } \Omega_{SO(2)} = \{0\} \subset V, \\ \deg_B(\nabla f_{SO(2)}, \Omega_{SO(2)}, 0) & \text{otherwise}, \end{cases}
   \]

where $\deg_B$ denotes the Brouwer degree.

Below we formulate product formula for the degree for $SO(2)$-equivariant gradient maps.

**Theorem 2.2 (24).** Let $\Omega_i \subset V_i$ be an open, bounded and $SO(2)$-invariant subset of a finite-dimensional, orthogonal representation $V_i$ of the group $SO(2)$, for $i = 1, 2$. Let $f_i \in C^1_{SO(2)}(V_i, \mathbb{R})$ be such that $(\nabla f_i)^{-1}(0) \cap \partial \Omega_i = \emptyset$, for $i = 1, 2$. Then
\[
\nabla_{SO(2)} - \text{deg}((\nabla f_1, \nabla f_2), \Omega_1 \times \Omega_2) = \nabla_{SO(2)} - \text{deg}(\nabla f_1, \Omega_1) \times \nabla_{SO(2)} - \text{deg}(\nabla f_2, \Omega_2).
\]
For \( k \in \mathbb{N} \) define a map \( \rho^k : SO(2) \to GL(2, \mathbb{R}) \) as follows

\[
\rho^k(e^{i\theta}) = \begin{bmatrix} \cos(k \cdot \theta) & -\sin(k \cdot \theta) \\ \sin(k \cdot \theta) & \cos(k \cdot \theta) \end{bmatrix} \quad 0 \leq \theta < 2 \cdot \pi.
\]

For \( j, k \in \mathbb{N} \) we denote by \( \mathbb{R}[j, k] \) the direct sum of \( j \) copies of \((\mathbb{R}^2, \rho^k)\), we also denote by \( \mathbb{R}[j, 0] \) the trivial \( j \)-dimensional representation of \( SO(2) \). We say that two representations \( V \) and \( W \) are equivalent if there exists an equivariant, linear isomorphism \( T : V \to W \).

The following classic result gives a complete classification (up to equivalence) of finite-dimensional representations of the group \( SO(2) \) (see [1]).

**Theorem 2.3 ([2]).** If \( V \) is a finite-dimensional representation of \( SO(2) \) then there exist finite sequences \( \{j_i\}, \{k_i\} \) satisfying:

\[
(*) \quad k_i \in \{0\} \cup \mathbb{N}, \quad j_i \in \mathbb{N}, \quad 1 \leq i \leq r, \quad k_1 < k_2 < \cdots < k_r
\]

such that \( V \) is equivalent to \( \bigoplus_{i=1}^r \mathbb{R}[j_i, k_i] \). Moreover, the equivalence class of \( V \), \( (V \approx \bigoplus_{i=1}^r \mathbb{R}[j_i, k_i]) \) is uniquely determined by \( \{k_i\}, \{j_i\} \) satisfying (*).

We will denote by \( m^-(L) \) the Morse index of a symmetric matrix \( L \).

To apply successfully any degree theory we need computational formulas for this invariant. Below we show how to compute degree for \( SO(2) \)-equivariant gradient maps of a linear, self-adjoint, \( SO(2) \)-equivariant isomorphism.

**Lemma 2.1 ([3]).** If \( V \approx \mathbb{R}[j_0, 0] \oplus \mathbb{R}[j_1, k_1] \oplus \cdots \oplus \mathbb{R}[j_r, k_r], \) \( L : V \to V \) is a self-adjoint, \( SO(2) \)-equivariant, linear isomorphism and \( \gamma > 0 \) then

\[ (1) \quad L = \text{diag} \,(L_0, L_1, \ldots, L_r), \]
\[ (2) \quad \nabla_{SO(2)}^{-\deg_H}(L, D_\gamma(V, 0)) = \begin{cases} (-1)^{m^-(L_0)}, & \text{for } H = SO(2), \\ (-1)^{m^-(L_0)} \cdot \frac{m^-(L_i)}{2}, & \text{for } H = \mathbb{Z}_{k_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_r}\}, \end{cases} \]
\[ (3) \quad \text{in particular, if } L = -I, \text{ then} \]
\[ \nabla_{SO(2)}^{-\deg_H}(-I, D_\gamma(V, 0)) = \begin{cases} (-1)^{j_0}, & \text{for } H = SO(2), \\ (-1)^{j_0} \cdot j_i, & \text{for } H = \mathbb{Z}_{k_i}, \\ 0, & \text{for } H \notin \{SO(2), \mathbb{Z}_{k_1}, \ldots, \mathbb{Z}_{k_r}\}. \end{cases} \]

Let \((H, \langle \cdot, \cdot \rangle)_H\) be an infinite-dimensional, separable Hilbert space which is an orthogonal representation of the group \( SO(2) \) and let \( C^1_{SO(2)}(H, \mathbb{R}) \) denote the set of \( SO(2) \)-invariant \( C^1 \)-functionals. Fix \( \Phi \in C^1_{SO(2)}(H, \mathbb{R}) \) such that

\[
\nabla \Phi(u) = u - \nabla \eta(u), \quad (2.4)
\]

where \( \nabla \eta : H \to H \) is an \( SO(2) \)-equivariant compact operator. Let \( U \subset H \) be an open, bounded and \( SO(2) \)-invariant set such that \( (\nabla \Phi)^{-1}(0) \cap \partial U = \emptyset \). In this situation
Theorem 2.4. Let $\mathcal{U}_i \subset \mathbb{H}_i$, be an open, bounded and $SO(2)$-invariant subset of infinite-dimensional, orthogonal representation $\mathbb{H}_i$ of the group $SO(2)$, for $i = 1, 2$. Let $f_i \in C^1_{SO(2)}(\mathbb{H}_i, \mathbb{R})$ be such that

1. $\nabla f_i = Id - \nabla \eta_i$ is an operator in the form of a compact perturbation of the identity, for $i = 1, 2$,
2. $(\nabla f_i)^{-1}(0) \cap \partial \mathcal{U}_i = \emptyset$, for $i = 1, 2$.

Then the following formula holds true

$$
\nabla_{SO(2)} - \text{deg}((Id - \nabla \eta_1, Id - \nabla \eta_2), \mathcal{U}_1 \times \mathcal{U}_2) = 
\nabla_{SO(2)} - \text{deg}(Id - \nabla \eta_1, \mathcal{U}_1) \times \nabla_{SO(2)} - \text{deg}(Id - \nabla \eta_2, \mathcal{U}_2).
$$

Let $L : \mathbb{H} \to \mathbb{H}$ be a linear, bounded, self-adjoint, $SO(2)$-equivariant operator with spectrum $\sigma(L) = \{\lambda_i\}$. By $V_L(\lambda_i)$ we will denote eigenspace of $L$ corresponding to the eigenvalue $\lambda_i$ and we put $\mu_L(\lambda_i) = \dim V_L(\lambda_i)$. In other words $\mu_L(\lambda_i)$ is the multiplicity of the eigenvalue $\lambda_i$. Since operator $L$ is linear, bounded, self-adjoint, and $SO(2)$-equivariant, $V_L(\lambda_i)$ is a finite-dimensional, orthogonal representation of the group $SO(2)$. For $\gamma > 0$ and $v_0 \in \mathbb{H}^{SO(2)}$ put $D_\gamma(\mathbb{H}, v_0) = \{v \in \mathbb{H} : |v - v_0| < \gamma\}$.

Combining Theorem 4.5 in [23] with Theorem 2.2 we obtain the following theorem.

Theorem 2.5. Under the above assumptions if $1 \notin \sigma(L)$, then

$$
\nabla_{SO(2)} - \text{deg}(Id - L, D_\gamma(\mathbb{H}, 0)) = \prod_{\lambda_i > 1} \nabla_{SO(2)} - \text{deg}(Id - L, D_\gamma(V_L(\lambda_i), 0)) \in U(SO(2)).
$$

It is understood that if $\sigma(L) \cap [1, +\infty) = \emptyset$, then

$$
\nabla_{SO(2)} - \text{deg}(Id - L, D_\gamma(\mathbb{H}, 0)) = \mathbb{1} \in U(SO(2)).
$$

Below we formulate the continuation theorem for $SO(2)$-equivariant gradient operators in the form of a compact perturbation of the identity. In other words we study continuation of critical orbits of $SO(2)$-invariant $C^1$-functionals. The proof this theorem is standard, but in this proof we have to replace the Leray-Schauder degree with the degree for $SO(2)$-equivariant gradient operators.

Theorem 2.6. Let $\Phi \in C^1_{SO(2)}(\mathbb{H} \times \mathbb{R}, \mathbb{R})$ be such that $\nabla_u \Phi(u, \lambda) = u - \nabla_u \eta(u, \lambda)$, where $\nabla \eta : \mathbb{H} \times \mathbb{R} \to \mathbb{H}$ is an $SO(2)$-equivariant compact operator. Fix an open, bounded and $SO(2)$-invariant subset $\mathcal{U} \subset \mathbb{H}$ and $\lambda_0 \in \mathbb{R}$ such that

1. $(\nabla_u \Phi(\cdot, \lambda_0))^{-1}(0) \cap \partial \mathcal{U} = \emptyset$,
2. $\nabla_{SO(2)} - \text{deg}(\nabla_u \Phi(\cdot, \lambda_0), \mathcal{U}) \neq \Theta \in U(SO(2))$.

Then there exists continua (closed connected sets) $C^\pm \subset \mathbb{H} \times \mathbb{R}$, with

$$
C^- \subset ((-\infty, \lambda_0] \times \mathbb{H}) \cap (\nabla_u \Phi(\cdot, \lambda_0))^{-1}(0),
C^+ \subset ([\lambda_0, +\infty) \times \mathbb{H}) \cap (\nabla_u \Phi(\cdot, \lambda_0))^{-1}(0),
$$
and for both $\mathcal{C} = \mathcal{C}^{\pm}$ the following statements are valid

1. $\mathcal{C} \cap (\{\lambda_0\} \times \mathcal{U}) \neq \emptyset$,
2. either $\mathcal{C}$ is unbounded or else $\mathcal{C} \cap (\mathbb{H} \setminus \text{cl}(\mathcal{U})) \neq \emptyset$.

We finish this section with a special case, $G = SO(2)$, of the theorem due to Rabier and Wang, see [19, 26].

**Theorem 2.7.** Let $\mathcal{U} \subset \mathbb{H}$ be an open, bounded and $SO(2)$-invariant set and let $f \in C^0_{SO(2)}(\text{cl}(\mathcal{U}), \mathbb{H})$ be an operator in the form of a compact perturbation of the identity such that $0 \notin f(\partial \mathcal{U})$. Then $\deg_{LS}(f, \mathcal{U}, 0) = \deg_{LS}(f^{SO(2)}, \mathcal{U}^{SO(2)}, 0)$, where $\deg_{LS}$ denotes the Leray-Schauder degree.

### 3. Admissible G-equivariant Gradient Homotopies

This section is of technical nature. We prove here the splitting lemma at the origin and at the infinity. In fact we construct admissible $G$-equivariant gradient homotopies. In the next sections, using these homotopies, we will compute the degree for $SO(2)$-equivariant gradient maps.

Let $G$ be any compact Lie group and let $\Phi \in C^2_G(\mathbb{H}, \mathbb{R})$ has the following form

$$\Phi(x) = \frac{1}{2} \langle Lx, x \rangle_{\mathbb{H}} - g(x),$$

where $\nabla g : \mathbb{H} \to \mathbb{H}$ is a $G$-equivariant compact operator and

(F.1) $L : \mathbb{H} \to \mathbb{H}$ is a $G$-equivariant self-adjoint Fredholm operator.

Assume that for $p \in \{0, \infty\}$

$$\Phi(x) = \frac{1}{2} \langle (L - L_p)x, x \rangle_{\mathbb{H}} + \eta_p(x),$$

i.e.

$$\Phi(x) = \frac{1}{2} \langle (L - L_0)x, x \rangle_{\mathbb{H}} + \eta_0(x) = \frac{1}{2} \langle (L - L_\infty)x, x \rangle_{\mathbb{H}} + \eta_\infty(x),$$

where

(F.2) $\nabla \eta_p : \mathbb{H} \to \mathbb{H}$ is a $G$-equivariant, compact operator,
(F.3) $L_p : \mathbb{H} \to \mathbb{H}$ is a linear, $G$-equivariant, self-adjoint and compact operator,
(F.4) $\|\nabla^2 \eta_p(x)\| \to 0$ as $\|x\| \to p$,
(F.5) $0 \in \sigma(L - L_p)$,
(F.6) $p$ is an isolated critical point of $\Phi$.

We treat $p = \infty$ as a critical point of $\Phi$ with Hessian $\nabla^2 \Phi(\infty) = L - L_\infty$. Moreover, we say that $\infty$ is an isolated critical point if $(\nabla \Phi)^{-1}(0)$ is bounded.

We will denote by $V_p$ and $W_p$ the kernel and the image of $\nabla^2 \Phi(p) = L - L_p$, respectively. Notice that $V_p$ and $W_p$ are finite and infinite-dimensional orthogonal representation of the group $G$, respectively. Since the operator $L - L_p$ is self-adjoint, $\mathbb{H} = V_p \oplus W_p$. Put $A_p = (L - L_p)|_{W_p} : W_p \to W_p$. Notice that the operator $A_p$ is an isomorphism. From now on $\pi_p : \mathbb{H} \to W_p$ and $Id - \pi_p : \mathbb{H} \to V_p$ stand for $G$-equivariant, orthogonal projections. Set $\Phi_1^p = (Id - \pi_p) \circ \nabla \Phi$ and $\Phi_2^p = \pi_p \circ \nabla \Phi$.
The following two versions of the implicit function theorem will allow us to construct admissible \( G \)-equivariant gradient homotopies.

**Theorem 3.1.** Let \( \Phi \in C^1_G(\mathbb{H}, \mathbb{R}) \) be a functional given by (3.1). Suppose that \( \Phi(x) = \frac{1}{2} \langle (L - L_0)x, x \rangle_{\mathbb{H}} + \eta_0(x) \) and that assumptions (F.1) – (F.5) are satisfied for \( p = 0 \). Then there exist \( \varepsilon > 0 \) and \( G \)-equivariant, \( C^1 \)-mapping \( w_0 : D_\varepsilon(V_0, 0) \rightarrow W_0 \) such that

\[
\begin{align*}
(i) & \quad w_0(0) = 0, Dw_0(0) = 0, \\
(ii) & \quad \Phi^0_\beta(v, w) = 0 \text{ for } v \in D_\varepsilon(V_0, 0) \text{ iff } w = w_0(v).
\end{align*}
\]

*Proof.* The existence of \( w_0 : D_\varepsilon(V_0, 0) \rightarrow W_0 \) we obtain from the nonequivariant version of the implicit function theorem, where \( \varepsilon > 0 \) is sufficiently small. What is left is to show that \( w_0 \) is \( G \)-equivariant. Since the operator \( \Phi^0_\beta \) is \( G \)-equivariant, \( \Phi^0_\beta(gv, gw_0(v)) = 0 \) for all \( v \in D_\varepsilon(V_0, 0), g \in G \). Moreover, we have and \( \Phi^0_\beta(gv, w_0(gv)) = 0 \) for all \( v \in D_\varepsilon(V_0, 0), g \in G \). From the uniqueness of \( w_0 \) we obtain \( gw_0(v) = w_0(gv) \) for all \( v \in D_\varepsilon(V_0, 0) \) and \( g \in G \), which completes the proof. \( \square \)

**Theorem 3.2.** Let \( \Phi \in C^1_G(\mathbb{H}, \mathbb{R}) \) be a functional given by (3.1). Suppose that \( \Phi(x) = \frac{1}{2} \langle (L - L_\infty)x, x \rangle_{\mathbb{H}} + \eta_\infty(x) \) and that assumptions (F.1) – (F.5) are satisfied for \( p = \infty \). Then there is \( \beta_0 > 0 \) and \( G \)-equivariant \( C^1 \)-mapping \( w_\infty : V_\infty \setminus cl(D_{\beta_0}(V_\infty, 0)) \rightarrow W_\infty \) such that \( \Phi^\infty_\beta(v, w) = 0 \) for \( v \in V_\infty \setminus cl(D_{\beta_0}(V_\infty, 0)) \) iff \( w = w_\infty(v) \).

The above theorem has been proved as a part of the proof of Lemma 4.3 in [4]. \( G \)-equivariance follows in the same way as in the proof of Theorem 3.1.

As a consequence of Theorems 3.1 3.2 we obtain the following corollary.

**Corollary 3.1.** Let \( \Phi \in C^1_G(\mathbb{H}, \mathbb{R}) \) be a functional given by (3.1). Fix \( p \in \{0, \infty\} \) and assume that functional \( \Phi(x) = \frac{1}{2} \langle (L - L_p)x, x \rangle_{\mathbb{H}} + \eta_p(x) \) satisfies assumptions (F.1) – (F.5). If \( \{x_n\} \subset (\nabla \Phi)^{-1}(0) \) converges to \( p \) then there is \( n_0 \in \mathbb{N} \) such that for any \( n > n_0 \) there is \( v_n \in V_p \) such that \( G_{x_n} = G_{v_n} \).

*Proof.* Fix \( p = 0 \). From Theorem 3.1 it follows that there is \( \varepsilon > 0 \) such that if \( x_n \in (\nabla \Phi)^{-1}(0) \cap D_\varepsilon(\mathbb{H}, p) \) then \( x_n = (v_n, w_p(v_n)) \in \mathbb{H} \), where \( w_p : D_\varepsilon(V_p, p) \rightarrow W_p \) is a \( G \)-equivariant map. Since \( w_p \) is \( G \)-equivariant, \( G_{v_n} \subset G_{w_p(v_n)} \) for any \( v_n \in D_\varepsilon(V_p, p) \). Consequently, \( G_{x_n} = G_{(v_n, w_p(v_n))} = G_{v_n} \cap G_{w_p(v_n)} = G_{v_n} \), which completes the proof. The same proof remains valid for \( p = \infty \), but we have to replace Theorem 3.1 with Theorem 3.2. \( \square \)

Let us consider continuous, \( G \)-equivariant extensions \( \tilde{w}_p : V_p \rightarrow W_p \) of \( w_p, p = 0, \infty \), defined in Theorems 3.1 3.2. To shorten notation we continue to write \( w_p \) for \( \tilde{w}_p, p = 0, \infty \). For \( p \in \{0, \infty\} \) consider a family of \( G \)-invariant \( C^1 \)-functionals \( H_p \in C^1_G(\mathbb{H} \times [0, 1], \mathbb{R}) \), defined in a following way

\[
H_p((v, w), t) = \frac{1}{2} \langle A_p(w), w \rangle_{\mathbb{H}} + \frac{1}{2} t(2 - t) \langle A_p(w_p(v)), w_p(v) \rangle_{\mathbb{H}} + t \eta_p(v, w_p(v)) + (1 - t) \eta_p(v, w + tw_p(v)),
\]

(3.3)
where \((v, w) \in \mathbb{H} = V_p \oplus W_p\).

This family has been introduced by Dancer in [9]. Recall that homotopy \(H : \Omega \times [0, 1] \to \mathbb{R}\), where \(\Omega\) is an open, bounded, \(G\)-invariant subset of an orthogonal \(G\)-representation \(\mathbb{H}\), is said to be \(\Omega\)-admissible iff

\[
\begin{align*}
&\text{(2)} \quad \nabla H(\cdot, t) : (\text{cl}(\Omega), \partial \Omega) \to (\mathbb{H}, \mathbb{H} \setminus \{0\}) \text{ is a gradient, } G\text{-equivariant mapping for all } t \in [0, 1], \\
&\text{(3)} \quad \nabla H(\cdot, t) = L - \nabla g_t, \text{ where } \nabla g(\cdot, t) = \nabla g_t \text{ and } \nabla g : \mathbb{H} \times [0, 1] \to \mathbb{H} \text{ is a compact operator.}
\end{align*}
\]

By \(\nabla H\) we denote the gradient of \(H\) with respect to the coordinate \(x \in \mathbb{H}\).

**Lemma 3.1.** For \(p \in \{0, \infty\}\) the family \(H_p \in C^1_G(\mathbb{H} \times [0, 1], \mathbb{R})\) given by the formula \((3.3)\) is of the form \(\nabla H_p(\cdot, t) = L - \nabla g_t\) for \(t \in [0, 1]\), where \(\nabla g : \mathbb{H} \times [0, 1] \to \mathbb{H}\) is a compact operator.

**Proof.** Observe that

\[
\nabla H_p((v, w), t) = A_p(w) + t(2 - t)[Dw_p(v)]^T A_p(w_p(v)) + t(Id - \pi_p)\nabla \eta_p(v, w_p(v)) + t[Dw_p(v)]^T \pi_p \nabla \eta_p(v, w_p(v)) + (1 - t)\nabla \eta_p(v, w + tw_p(v)) + (1 - t)t[Dw_p(v)]^T \pi_p \nabla \eta_p(v, w + tw_p(v)).
\]

From definition we have \(A_p(w) = (L - L_p)(v, w)\). Recall that \(L_p\) and \(\nabla \eta_p\) are compact operators and \(V_p\) is finite dimensional space. To complete the proof we use following facts

(i) superposition of compact and continuous mappings is compact,

(ii) continuous, finite-dimensional mapping is compact,

(iii) continuous mapping defined on a finite dimensional Banach space is compact.

We finish this section with splitting lemmas at the origin and at the infinity.

**Lemma 3.2.** (Splitting lemma at the origin) Suppose that functional \(\Phi \in C^2_G(\mathbb{H}, \mathbb{R})\) is given by formula \((3.1)\). Assume additionally that for \(p = 0\) there is representation \(\Phi(x) = \frac{1}{2}\langle (L - L_0)x, x \rangle_{\mathbb{H}} + \eta_0(x)\) and assumptions \((F.1)-(F.6)\) hold. Then there exist \(\alpha_0 > 0\) and \(G\)-equivariant gradient homotopy \(\nabla H_0 : (V_0 \oplus W_0) \times [0, 1] \to \mathbb{H}\), satisfying the following conditions

\[
\begin{align*}
&\text{(1)} \quad \nabla H_0^{-1}(0) \cap (\text{cl}(D_{\alpha_0}(V_0, 0)) \times \text{cl}(D_{\alpha_0}(W_0, 0)) \times [0, 1]) = \{0\} \times [0, 1], \\
&\text{(2)} \quad \nabla H_0((v, w), t) = (L - \nabla g_t)(v, w), \text{ for } t \in [0, 1], (v, w) \in V_0 \oplus W_0, \text{ where } \nabla g_t = \nabla g(\cdot, t) \text{ and } \nabla g : \mathbb{H} \times [0, 1] \to \mathbb{H} \text{ is a compact mapping}, \\
&\text{(3)} \quad \nabla H_0((v, w), 0) = \nabla \Phi(v, w), \\
&\text{(4)} \quad \text{there exists a } G\text{-equivariant, gradient mapping } \nabla \varphi_0 : (V_0, 0) \to (V_0, 0) \text{ such that } \\
&\nabla H_0((v, w), 1) = (\nabla \varphi_0(v), A_0(w)) = (\nabla \varphi_0(v), (L - L_0)|_{W_0}(w)).
\end{align*}
\]

**Proof.** Let \(H_0 : (V_0 \oplus W_0) \times [0, 1] \to \mathbb{R}\) will be defined by formula \((3.3)\) for \(p = 0\), i.e.

\[
\begin{align*}
H_0((v, w), t) &= \frac{1}{2}\langle A_0(w), w \rangle_{\mathbb{H}} + \frac{1}{2}t(2 - t)\langle A_0(w_0(v)), w_0(v) \rangle_{\mathbb{H}} + t\eta_0(v, w_0(v)) + (1 - t)\eta_0(v, w + tw_0(v)).
\end{align*}
\]
Lemma 3.1 yields (2). To complete the proof notice that

\[ \alpha(1) \text{ has been verified for example in [15], for } \alpha_0 > 0 \text{ taken as a sufficiently small number.} \]

Lemma 3.1 yields (2). To complete the proof notice that

\[ \nabla H_0((v, w), 0) = \nabla \Phi(v, w), \]
\[ \nabla H_0((v, w), 1) = (\nabla \varphi_0(v), A_0(w)), \text{ where } \nabla \varphi_0(v) = \Phi^0_1(v, w_0(v)). \]

\[ \square \]

**Lemma 3.3.** (Splitting lemma at the infinity) Suppose that functional \( \Phi \in C^2_{\mathcal{G}}(\mathbb{H}, \mathbb{R}) \) is given by formula (3.1). Assume additionally that for \( p = \infty \) there is representation \( \Phi(x) = \frac{1}{2}((L - L_\infty)x, x)_\mathbb{H} + \eta_\infty(x) \) and assumptions (F.1)-(F.6) hold. Then there exist number \( \alpha_\infty > 0 \) and \( G \)-equivariant gradient homotopy \( \nabla H_\infty : (V_\infty \oplus W_\infty) \times [0, 1] \to \mathbb{H} \), satisfying the following conditions

1. \( \nabla H_\infty^{-1}(0) \subset cl(D_{\alpha_\infty}(V_\infty, 0)) \times cl(D_{\alpha_\infty}(W_\infty, 0)) \times [0, 1], \)
2. \( \nabla H_\infty((v, w), t) = (L - \nabla g_t)(v, w), \text{ for } t \in [0, 1], (v, w) \in V_\infty \oplus W_\infty, \) where \( \nabla g_t = \nabla g(\cdot, t) \) and \( \nabla g : \mathbb{H} \times [0, 1] \to \mathbb{H} \) is a compact mapping,
3. \( \nabla H_\infty((v, w), 0) = \nabla \Phi(v, w) \)
4. there exists a \( G \)-equivariant, gradient mapping \( \nabla \varphi_\infty : V_\infty \to W_\infty \) such that

\[ \nabla H_\infty((v, w), 1) = (\nabla \varphi_\infty(v), A_\infty(w)) = (\nabla \varphi_\infty(v), (L - L_\infty)|_{W_\infty}(w)). \]

**Proof.** Recall, that \( \Phi^\infty = (Id - \pi_\infty) \circ \nabla \Phi : \mathbb{H} \to V_\infty \) and \( \Phi^\infty = \pi_\infty \circ \nabla \Phi : \mathbb{H} \to W_\infty. \) Applying Theorem 3.2 to \( \Phi, \) we obtain \( \beta_0 > 0 \) and a \( G \)-equivariant \( C^1 \)-mapping \( w_\infty : V_\infty \setminus cl(D_{\beta_0}(V_\infty, 0)) \to W_\infty \) such that \( \Phi^\infty_2(v, w) = 0 \) for \( v \in V_\infty \setminus cl(D_{\beta_0}(V_\infty, 0)) \) iff \( w = w_\infty(v). \)

Fix \( \beta_1 > \beta_0 \) such that

\[ (Id - \pi_\infty)(\nabla \Phi^{-1}(0)) \subset cl(D_{\beta_1}(V_\infty, 0)) \]
and

\[ \sup \{ \| \nabla^2 \eta_\infty(v, w) \| ; (v, w) \in \mathbb{H} \text{ and } \| (v, w) \|_\mathbb{H} > \beta_1 \} \leq \frac{1}{2}\| A_\infty^{-1} \|^{-1}. \]

Consider the family \( H_\infty : (V_\infty \oplus W_\infty) \times [0, 1] \to \mathbb{H} \) defined by formula (3.3) for \( p = \infty, \) i.e.

\[ H_\infty((v, w), t) = \frac{1}{2}(A_\infty(w, w))_\mathbb{H} + \frac{1}{2}t(2 - t)(A_\infty(w_\infty(v)), w_\infty(v))_\mathbb{H} + t\eta_\infty(v, w_\infty(v)) + (1 - t)\eta_\infty(v, w + tw_\infty(v)). \]

We claim that there exists \( \beta_2 > 0 \) such that if \( (v, w) \in \mathbb{H} \setminus (cl(D_{\beta_1}(V_\infty, 0)) \times cl(D_{\beta_2}(W_\infty, 0))) \) then \( \nabla H_\infty((v, w), t) \neq 0. \) Notice that \( (v, w) \in \mathbb{H} \setminus (cl(D_{\beta_1}(V_\infty, 0)) \times cl(D_{\beta_2}(W_\infty, 0))) \) iff

(i) either \( \| v \|_\mathbb{H} < \beta_1 \) and \( \| w \|_\mathbb{H} > \beta_2, \)
(ii) or \( \| v \|_\mathbb{H} > \beta_1. \)

Case (i). For \( \beta_1 \) there exists \( K \) such that if \( \| v \|_\mathbb{H} < \beta_1 \) then \( \| w_\infty(v) \|_\mathbb{H} \leq K. \) Moreover, from (F.4), for fixed \( \varepsilon > 0 \) there exists \( M > 0 \) such that if \( \| x \|_\mathbb{H} > M \) then \( \| \nabla \eta_\infty(x) \|_\mathbb{H} <
Case (ii). Fix \((v, t) \in (V_\infty \setminus cl(D_{\beta_1}(V_\infty, 0))) \times [0, 1]\). We claim that
\[
\pi_\infty(\nabla H_\infty((v, w), t)) = 0
\] (3.6)
iff \(w = 0\). Indeed, \(w = 0\) is a solution of (3.6). We proceed to show that it is unique.
\[
\pi_\infty(\nabla H_\infty((v, w), t)) = 0 \iff A_\infty(w) + (1 - t) \pi_\infty(\nabla \eta_\infty(v, w + tw_\infty(v))) = 0 \iff w = -(1 - t) A^{-1}_\infty \circ \pi_\infty(\nabla \eta_\infty(v, w + tw_\infty(v))).
\]
Denote the right side of the above equality by \(\phi(v, t)(w)\). From the mean value theorem and (3.5) we obtain
\[
\|\phi(v, t)(w_1) - \phi(v, t)(w_2)\|_H \leq \|A^{-1}_\infty\| \cdot \|\nabla \eta_\infty(v, tw_1 + w_\infty(v)) - \nabla \eta_\infty(v, tw_2 + w_\infty(v))\|_H \leq \\
\leq \|A^{-1}_\infty\| \cdot \|w_1 - w_2\|_H \cdot \sup\{\|\nabla^2 \eta_\infty(u)\|; \ u \in \mathbb{H} \text{ and } \|u\|_H > \beta_1\} \leq \frac{1}{2}\|w_1 - w_2\|_H.
\]
Hence \(\phi(v, t) : W_\infty \to W_\infty\) is a contraction. Using the Banach fixed point theorem we conclude that \(w = 0\) is the unique solution of (3.6). If \(w = 0\) then \((Id - \pi_\infty)(\nabla H_\infty((v, 0), t)) = \Phi_1^\infty(v, w(v)) \neq 0\) for \(\|v\| \geq \beta_1\). Now it suffices to take \(\alpha_\infty = \max\{\beta_1, \beta_2\}\).

Lemma 3.1 yields (2). To complete the proof notice that
\[
\begin{align*}
(1) \quad &\nabla H_\infty(\cdot, 0) = \nabla \Phi(v, w), \\
(2) \quad &\nabla H_\infty(\cdot, 1) = (\nabla \varphi_\infty(v), A_\infty(w)), \quad \text{where } \nabla \varphi_\infty(v) = \Phi_1^\infty(v, w_\infty(v)).
\end{align*}
\]

\[\square\]

Remark 3.1. Notice that from conditions (1) in Lemmas 3.2 and 3.3 for \(p = 0, \infty\) respectively, we have
\[
\nabla H_p(\cdot, t) : (cl(D_{\alpha_p}(V_p, 0)) \times cl(D_{\alpha_p}(W_p, 0)), \partial(cl(D_{\alpha_p}(V_p, 0)) \times cl(D_{\alpha_p}(W_p, 0)))) \to \\
\to (\mathbb{H}, \mathbb{H} \setminus \{0\}).
\]

Hence both homotopies \(H_p\) are \(\Omega\)-admissible for \(\Omega = D_{\alpha_p}(V_p, 0) \times D_{\alpha_p}(W_p, 0)\).
4. Applications of Splitting Lemmas

In this section we compute the indices of isolated critical points of $SO(2)$-invariant functionals in terms of the degree for $SO(2)$-equivariant gradient maps. Throughout this section we assume that $L = Id$, $G = SO(2)$ and $\dim H^{SO(2)} < \infty$. Let functional $\Phi \in C^2_{SO(2)}(H, R)$ be given by (3.1). Suppose that $\Phi(x) = \frac{1}{2}((L - L_p)x, x)_{\mathbb{H}} + \eta_p(x)$ for $p \in \{0, \infty\}$, and that assumptions $(F.1) - (F.6)$ are satisfied. Notice that if $p = 0$, functional $\Phi$ satisfies assumptions of Lemma $3.2$ and if $p = \infty$, assumptions of Lemma $3.3$. Hence, further consideration we can carry out parallel for $p \in \{0, \infty\}$. From Lemmas $3.2$ $3.3$ and Theorem $2.4$ we have

$$\nabla_{SO(2)} - \deg(\nabla \Phi, D_{\alpha_p}(H, p)) =$$

$$= \nabla_{SO(2)} - \deg((\nabla \phi_p, A_p), D_{\alpha_p}(V_p, p) \times D_{\alpha_p}(W_p, p)) =$$

$$= \nabla_{SO(2)} - \deg((\nabla \phi_p, D_{\alpha_p}(V_p, p)) \times \nabla_{SO(2)} - \deg(A_p, D_{\alpha_p}(W_p, p))$$

Lemma 4.1. Fix $p \in \{0, \infty\}$. Let $\Phi \in C^2_{SO(2)}(H, R)$, admits the representation $\Phi(x) = \frac{1}{2}((Id - L_p)x, x)_{\mathbb{H}} + \eta_p(x)$, and assumptions $(F.1) - (F.6)$ are fulfilled. Moreover, assume that $V_p = \ker(Id - L_p) \subset H^{SO(2)}$. Then

$$\nabla_{SO(2)} - \deg(\nabla \Phi, D_{\alpha_p}(H, p)) = \deg_B(\nabla \Phi_{SO(2)}, D_{\alpha_p}(H^{SO(2)}, p), 0) \cdot$$

$$\cdot \nabla_{SO(2)} - \deg((Id - L_n)((H^{SO(2)})^\perp, D_{\alpha_p}(H^{SO(2)}, 0))).$$

Proof. Since $A_p = ((A_p)|_{W_p^{SO(2)}}, (A_p)|_{W_p^{SO(2)} \cap W_p^{SO(2)}}), V_p \subset H^{SO(2)}$ and (4.1) we have

$$\nabla_{SO(2)} - \deg(\nabla \phi_p, (A_p)|_{W_p^{SO(2)}}, D_{\alpha_p}(W_p^{SO(2)}, 0))$$

$$= \nabla_{SO(2)} - \deg((\nabla \phi_p, D_{\alpha_p}(V_p, p)) \times \nabla_{SO(2)} - \deg((A_p)|_{W_p^{SO(2)}}, D_{\alpha_p}(W_p^{SO(2)}, 0)) \times$$

$$\times \nabla_{SO(2)} - \deg((A_p)|_{W_p^{SO(2)}}, D_{\alpha_p}(W_p \cap W_p^{SO(2)}, 0)))$$

$$= \nabla_{SO(2)} - \deg((\nabla \phi_p, D_{\alpha_p}(W_p \cap W_p^{SO(2)}))), D_{\alpha_p}(V_p, p) \times D_{\alpha_p}(W_p \cap W_p^{SO(2)}, 0) \times$$

$$\times \nabla_{SO(2)} - \deg((A_p)|_{W_p^{SO(2)}}, D_{\alpha_p}(W_p^{SO(2)}), 0)) = \nabla_{SO(2)} - \deg(\nabla \Phi_{SO(2)}, D_{\alpha_p}(H^{SO(2)}, p)) \times$$

$$\times \nabla_{SO(2)} - \deg((Id - L_p)|_{W_p^{SO(2)}}, D_{\alpha_p}(H^{SO(2)}, 0)))$$

(4.2)

Recall that $(\nabla \Phi)_{SO(2)} = (\nabla \Phi)|_{H^{SO(2)}} = \nabla(\Phi|_{H^{SO(2)}}) : H^{SO(2)} \rightarrow H^{SO(2)}$ is well-defined gradient map and notice that by Remark $2.1$ we have

$$\nabla_{SO(2)} - \deg(\nabla \Phi_{SO(2)}, D_{\alpha_p}(H^{SO(2)}, 0)) = (\deg_B(\nabla \Phi_{SO(2)}, D_{\alpha_p}(H^{SO(2)}, 0)), 0, \ldots)$$

(4.3)

Taking into account formulas (2.2), (4.2) and (4.3) we complete the proof. □
Recall, that by $V_L(\lambda_i)$ we denote the eigenspace of the operator $L$ corresponding to the eigenvalue $\lambda_i$. Using Theorem 2.3 we obtain

$$\nabla_{SO(2)} - \text{deg} (\nabla \Phi, D_{\alpha_p}(H, p)) =$$

$$= \nabla_{SO(2)} - \text{deg} (\nabla \varphi_p, D_{\alpha_p}(V_p, p)) \ast \nabla_{SO(2)} - \text{deg} (-Id, D_{\alpha_p}(\bigoplus_{\lambda_i > 1} V_L(\lambda_i), 0)).$$

Since $V_p = \ker(Id - L_p)$ is a finite-dimensional representation of the group $SO(2)$, from Theorem 2.3 we obtain numbers $j_0 \geq 0, j_1, \ldots, j_r > 0$ and $k_r > \ldots > k_1 > k_0 = 0$ such that $V_p \approx \mathbb{R}[j_0, k_0] \oplus \mathbb{R}[j_1, k_1] \oplus \ldots \oplus \mathbb{R}[j_r, k_r]$. Let $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$. Denote $k_{i_1 \ldots i_s} = \gcd\{k_{i_1}, \ldots, k_{i_s}\}$. Possible isotropy groups of points of $V_p$ are $SO(2)$ and groups $\mathbb{Z}_{k_{i_1 \ldots i_s}}$, for arbitrary $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$. From Remark 2.1 we have

$$\nabla_{SO(2)} - \text{deg}_H(\nabla \varphi_p, D_{\alpha_p}(V_p, p)) = 0$$

for $H \notin \{SO(2)\} \cup \bigcup_{\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}} \{\mathbb{Z}_{k_{i_1 \ldots i_s}}\}$. Hence we have proved the following lemma.

**Lemma 4.2.** Let $p \in \{0, \infty\}$. Let $\Phi \in C^2_{SO(2)}(\mathbb{H}, \mathbb{R})$, admits the representation $\Phi(x) = \frac{1}{2}((Id - L_p)x, x)_H + \eta_p(x)$, and assumptions (F.1) – (F.6) hold. Then

$$\nabla_{SO(2)} - \text{deg} (\nabla \Phi, D_{\alpha_p}(H, p)) =$$

$$= \nabla_{SO(2)} - \text{deg} (\nabla \varphi_p, D_{\alpha_p}(V_p, p)) \ast \nabla_{SO(2)} - \text{deg} (-Id, D_{\alpha_p}(\bigoplus_{\lambda_i > 1} V_L(\lambda_i), 0)).$$

Moreover, by formula (2.2), we have

$$\nabla_{SO(2)} - \text{deg}_H(\nabla \Phi, D_{\alpha_p}(H, p)) =$$

$$= \nabla_{SO(2)} - \text{deg}_{SO(2)}(\nabla \varphi_p, D_{\alpha_p}(V_p, p)) \ast \nabla_{SO(2)} - \text{deg}_H(-Id, D_{\alpha_p}(\bigoplus_{\lambda_i > 1} V_L(\lambda_i), 0)) \quad (4.4)$$

for $H \notin \{SO(2)\} \cup \bigcup_{\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}} \{\mathbb{Z}_{k_{i_1 \ldots i_s}}\}$.

**Corollary 4.1.** Let assumptions of Lemma 4.2 be satisfied. Assume additionally that $V_p^{SO(2)} = \ker(Id - L_p)^{SO(2)} = \{0\}$. Then

$$\nabla_{SO(2)} - \text{deg}_H(\nabla \Phi, D_{\alpha_p}(H, p)) = \nabla_{SO(2)} - \text{deg}_H(-Id, D_{\alpha_p}(\bigoplus_{\lambda_i > 1} V_L(\lambda_i), 0))$$

for $H \notin \{SO(2)\} \cup \bigcup_{\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}} \{\mathbb{Z}_{k_{i_1 \ldots i_s}}\}$.

**Proof.** Since $V_p^{SO(2)} = \ker(Id - L_p)^{SO(2)} = \{0\}$, by Remark 2.1 we have

$$\nabla_{SO(2)} - \text{deg}_{SO(2)}(\nabla \varphi_p, D_{\alpha_p}(V_p, p)) = 1.$$

Applying formula (4.4) we complete the proof. \qed
5. Nonstationary Periodic Solutions of Autonomous Newtonian Systems

Throughout this section we study periodic solutions of autonomous Newtonian systems. We define an $SO(2)$-invariant functional on a suitably chosen infinite-dimensional Hilbert space which is an infinite-dimensional, orthogonal representation of the group $SO(2)$. Critical orbits of this functional are in one-to-one correspondence with solutions of a considered system. Therefore for our purpose it is enough to study only the critical orbits of this functional.

We begin this section with a definition of an appropriate Hilbert space. Fix $T > 0$ and define

$$H^1_T = \{ u : [0, T] \to \mathbb{R}^n : u \text{ is abs. cont., } u(0) = u(T), \dot{u} \in L^2([0, T], \mathbb{R}^n) \}.$$

It is known that $H^1_T$ is a separable Hilbert space with a scalar product given by the formula

$$\langle u, v \rangle_{H^1_T} = \int_0^T (\dot{u}(t), \dot{v}(t)) + (u(t), v(t)) \, dt,$$

where $(\cdot, \cdot)$ and $\| \cdot \|$ are the usual scalar product and norm in $\mathbb{R}^n$, respectively. It is easy to show that $(H^1_T, \langle \cdot, \cdot \rangle_{H^1_T})$ is an orthogonal representation of the group $SO(2)$ with an $SO(2)$-action given by shift in time.

Let us consider the following Newtonian system

$$\begin{cases}
\ddot{u} = -V'(u) \\
u(0) = u(T) \\
\dot{u}(0) = \dot{u}(T)
\end{cases}$$

(5.1)

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$. Solutions of (5.1) are in one to one correspondence with critical points of an $SO(2)$-invariant $C^2$-functional $\Phi_V : H^1_T \to \mathbb{R}$ defined as follows

$$\Phi_V(u) = \int_0^T \frac{1}{2} \| \dot{u}(t) \|^2 - V(u(t)) \, dt$$

(5.2)

Notice that for any $u, v \in H^1_T$ we have $\Phi'_V(u)(v) = \langle \nabla \Phi_V(u), v \rangle_{H^1_T} = \langle u - \nabla \zeta(u), v \rangle_{H^1_T}$, where $\nabla \zeta : H^1_T \to H^1_T$ is an $SO(2)$-equivariant, compact, gradient operator given by the formula $\langle \nabla \zeta(u), v \rangle_{H^1_T} = \int_0^T (u(t) + V(u(t)), v(t)) \, dt$. In the other words the gradient $\nabla \Phi_V : H^1_T \to H^1_T$ is an $SO(2)$-equivariant $C^1$-operator in the form of a compact perturbation of the identity.

5.1. Linear Equation. In this section we carry on detailed analysis of a linear system of the form

$$\begin{cases}
\ddot{u} = -Au \\
u(0) = u(T) \\
\dot{u}(0) = \dot{u}(T)
\end{cases}$$

(5.1.1)

where $A$ is a real, symmetric $(n \times n)$-matrix. Moreover, we study properties of a functional associated with equation (5.1.1).
Define the corresponding functional \( \Phi_A : \mathbb{H}^1_T \to \mathbb{R} \) as follows

\[
\Phi_A(u) = \frac{1}{2} \int_0^T \| \dot{u}(t) \|^2 - (Au(t), u(t)) \, dt = \frac{1}{2} \langle u - L_A(u), u \rangle_{\mathbb{H}^1_T} \quad (5.1.2)
\]

and notice that \( \nabla \Phi_A = Id - L_A \), where \( L_A : \mathbb{H}^1_T \to \mathbb{H}^1_T \) is a linear, self-adjoint, \( SO(2) \)-equivariant and compact operator defined by the formula

\[
\langle L_A(u), v \rangle_{\mathbb{H}^1_T} = \int_0^T (u(t) + Au(t), v(t)) \, dt.
\]

To study the linear eigenvalue problem \( u - \lambda L_{Id_n}(u) = 0 \) it is enough to consider the following system

\[
\begin{cases}
\ddot{u} = -(2\lambda - 1)u \\
u(0) = u(T) \\
\dot{u}(0) = \dot{u}(T)
\end{cases}
\quad (5.1.3)
\]

It is easy to check that eigenvalues and eigenspaces of the operator \( L_{Id_n} : \mathbb{H}^1_T \to \mathbb{H}^1_T \) are of the following form

1. \( \sigma (L_{Id_n}) = \left\{ \lambda_k = \frac{2T^2}{T^2 + 4k^2\pi^2} \right\}_{k \in \mathbb{N}\cup\{0\}} \),
2. \( V_{L_{Id_n}}(\lambda_0) = \mathbb{R}^n \approx \mathbb{R}[n, 0] \),
3. \( V_{L_{Id_n}}(\lambda_k) = \left\{ a_k \cos \left( \frac{2k\pi}{T} \right) t + b_k \sin \left( \frac{2k\pi}{T} \right) t : a_k, b_k \in \mathbb{R}^n \right\} \approx \mathbb{R}[n, k], k \in \mathbb{N} \).

From the above we obtain that an orthonormal basis in \( \mathbb{H}^1_T \) can be chosen as follows

\[
\sqrt{\frac{1}{T}} \cdot e_i, \sqrt{\frac{2T}{T^2 + 4k^2\pi^2}} \cdot \cos \left( \frac{2k\pi}{T} t \right) \cdot e_i, \sqrt{\frac{2T}{T^2 + 4k^2\pi^2}} \cdot \sin \left( \frac{2k\pi}{T} t \right) \cdot e_i,
\]

or in equivalent way

\[
\sqrt{\frac{\lambda_0}{2T}} \cdot e_i, \sqrt{\frac{\lambda_k}{T}} \cdot \cos \left( \frac{2k\pi}{T} t \right) \cdot e_i, \sqrt{\frac{\lambda_k}{T}} \cdot \sin \left( \frac{2k\pi}{T} t \right) \cdot e_i,
\]

where \( i = 1, \ldots, n \) and \( k \in \mathbb{N} \).

It is clear that \( u \in \mathbb{H}^1_T \) possesses Fourier series of the form

\[
u(t) = \tilde{a}_0 \sqrt{\frac{\lambda_0}{2T}} + \sum_{k \in \mathbb{N}} \tilde{a}_k \cdot \left( \sqrt{\frac{\lambda_k}{T}} \cdot \cos \left( \frac{2k\pi}{T} t \right) \right) + \tilde{b}_k \cdot \left( \sqrt{\frac{\lambda_k}{T}} \cdot \sin \left( \frac{2k\pi}{T} t \right) \right) =
\]

\[
a_0 + \sum_{k \in \mathbb{N}} a_k \cdot \cos \left( \frac{2k\pi}{T} t \right) + b_k \cdot \sin \left( \frac{2k\pi}{T} t \right).
\]

In the following lemma we study properties of the operator \( \nabla \Phi_A = Id - L_A : \mathbb{H}^1_T \to \mathbb{H}^1_T \).

**Lemma 5.1.1.** If \( u \in \mathbb{H}^1_T \) with Fourier series

\[
u(t) = a_0 + \sum_{k \in \mathbb{N}} a_k \cdot \cos \left( \frac{2k\pi}{T} t \right) + b_k \cdot \sin \left( \frac{2k\pi}{T} t \right),
\]
then

\[ \nabla \Phi_A(u) = u - L_A(u) = -A \cdot a_0 + \sum_{k=1}^{\infty} (\Lambda(k) \cdot a_k) \cdot \cos\left(\frac{2k\pi}{T}t\right) + (\Lambda(k) \cdot b_k) \cdot \sin\left(\frac{2k\pi}{T}t\right), \]

where \( \Lambda(k) = \left(\frac{4k^2\pi^2}{4k^2\pi^2 + T^2}Id - \frac{T^2}{4k^2\pi^2 + T^2}A\right) \).

**Proof.** Fix \( u \in \mathbb{H}_T^1 \) with Fourier series \( u(t) = a_0 + \sum_{k \in \mathbb{N}} a_k \cdot \cos\left(\frac{2k\pi}{T}t\right) + b_k \cdot \sin\left(\frac{2k\pi}{T}t\right) \). Then

\[ \nabla \Phi_A(u) = u - L_A(u) = (Id - L_A)(a_0) + \sum_{k \in \mathbb{N}} (Id - L_A) \left( a_k \cdot \cos\left(\frac{2k\pi}{T}t\right) + b_k \cdot \sin\left(\frac{2k\pi}{T}t\right) \right). \]

What is left is to compute

1. \( (Id - L_A)(a_0) \in V_{L_{Id^i_n}}(\lambda_0) \),
2. \( (Id - L_A) \left( a_k \cdot \cos\left(\frac{2k\pi}{T}t\right) + b_k \cdot \sin\left(\frac{2k\pi}{T}t\right) \right) \in V_{L_{Id^i_n}}(\lambda_0) \).

Put \( u_0 = a_0 \), fix \( v \in \mathbb{H}_T^1 \) and notice that

\[ \langle \nabla \Phi_A(u_0), v \rangle_{\mathbb{H}_T^1} = \langle u_0 - L_A(u_0), v \rangle_{\mathbb{H}_T^1} = \int_0^T -(A(u_0), v(t)) \, dt = \]

\[ = \int_0^T (\dot{u}_0, \dot{v}(t)) + (u_0, v(t)) - ((Id + A)(u_0), v(t)) \, dt = \]

\[ = \langle u_0 - (Id + A)(u_0), v \rangle_{\mathbb{H}_T^1} = \langle -A(u_0), v \rangle_{\mathbb{H}_T^1}. \]

Summing up, we obtain \( \nabla \Phi_A(u_0) = u_0 - L_A(u_0) = -A(u_0) \).

For simplicity of notation, we let \( u_k(t) \) stand for \( a_k \cdot \cos\left(\frac{2k\pi}{T}t\right) + b_k \cdot \sin\left(\frac{2k\pi}{T}t\right) \). Fix \( v \in \mathbb{H}_T^1 \) and notice that

\[ \langle \nabla \Phi_A(u_k), v \rangle_{\mathbb{H}_T^1} = \langle u_k - L_A(u_k), v \rangle_{\mathbb{H}_T^1} = \int_0^T (\dot{u}_k(t), \dot{v}(t)) - (Au_k(t), v(t)) \, dt = \]

\[ = \int_0^T (\dot{u}_k(t), \dot{v}(t)) + (u_k(t), v(t)) - (u_k(t) + Au_k(t), v(t)) \, dt = \int_0^T (-\ddot{u}_k(t) + u_k(t), v(t)) - \]

\[ -(u_k(t) + Au_k(t), v(t)) \, dt = \int_0^T \left( \frac{4k^2 \pi^2 + T^2}{T^2} \right) (u_k(t), v(t)) \, dt - \]

\[ - \int_0^T \left( \frac{T^2}{4k^2 \pi^2 + T^2} \right) \left( \frac{4k^2 \pi^2 + T^2}{T^2} \right) (u_k(t) + Au_k(t), v(t)) \, dt = \]

\[ = \langle u_k, v \rangle_{\mathbb{H}_T^1} - \left\langle \left( \frac{T^2}{4k^2 \pi^2 + T^2} \right) (Id + A)u_k, v \right\rangle_{\mathbb{H}_T^1} = \]

\[ = \left\langle \left( \frac{4k^2 \pi^2}{4k^2 \pi^2 + T^2}Id - \frac{T^2}{4k^2 \pi^2 + T^2}A \right) u_k, v \right\rangle_{\mathbb{H}_T^1}. \]
Summing up, we obtain
\[ \nabla \Phi_A(u_k) = u_k - L_A(u_k) = \left( \frac{4k^2\pi^2}{4k^2\pi^2 + T^2} I - \frac{T^2}{4k^2\pi^2 + T^2} A \right) u_k = \]
\[ = (\Lambda(k) \cdot a_k) \cdot \cos \frac{2k\pi}{T} t + (\Lambda(k) \cdot b_k) \cdot \sin \frac{2k\pi}{T} t = \Lambda(k) u_k, \]
which completes the proof. \[\square\]

As a direct consequence of Lemma 5.1.1 we obtain the following two corollaries.

**Corollary 5.1.1.** The following conditions are equivalent

1. Operator $\nabla \Phi_A = I - L_A : \mathbb{H}^1_T \rightarrow \mathbb{H}^1_T$ is an isomorphism,
2. $\ker \Lambda(k) = \{0\}$ for any $k \in \mathbb{N} \cup \{0\}$,
3. $\sigma(A) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \cup \{0\} \right\} = \emptyset$.

For $\alpha \in \mathbb{R}$ we will denote by $\mu_A(\alpha)$ the multiplicity of $\alpha$ considered as an eigenvalue of matrix $A$. If $\alpha \notin \sigma(A)$ then it is understood that $\mu_A(\alpha) = 0$. Moreover, if $\alpha \in \sigma(A)$ then we will denote by $V_A(\alpha)$ the eigenspace of $A$ corresponding to the eigenvalue $\alpha$. For any $k \in \mathbb{N} \cup \{0\}$ define

1. $\sigma_k(A,T) = \sigma(A) \cap \left( \frac{4k^2\pi^2}{T^2}, +\infty \right)$,
2. $j_k(A,T) = \sum_{\alpha \in \sigma_k(A,T)} \mu_A(\alpha)$.

**Remark 5.1.1.** Notice that
\[ \nu(A,T) = \mu_A(0) + 2 \sum_{k=1}^{\infty} \mu_A \left( \frac{4k^2\pi^2}{T^2} \right) ; j(A,T) = j_0(A,T) + 2 \sum_{k=1}^{\infty} j_k(A,T), \]
where the numbers $\nu(A,T), j(A,T)$ are defined in [18] on page 207. Since we are going to apply the degree for $SO(2)$-equivariant gradient maps, we have to describe the finite-dimensional spaces $\ker I - L_A$ and $\bigoplus_{\lambda_i > 1} V_{L_A}(\lambda_i)$ as representations of the group $SO(2)$.

It is not enough for our purpose to know only the dimensions $\nu(A,T) = \dim \ker I - L_A$ and $j(A,T) = \dim \bigoplus_{\lambda_i > 1} V_{L_A}(\lambda_i)$.

The following corollary will prove extremely useful in the next sections.

**Corollary 5.1.2.** Operator $\nabla \Phi_A = I - L_A : \mathbb{H}^1_T \rightarrow \mathbb{H}^1_T$ has the following properties

1. $\mathbb{H}^1_{T,0} = \ker \nabla \Phi_A = V_{L_A}(1) = \ker A \bigoplus_{k=1}^{\infty} \left\{ a_k \cdot \cos \frac{2k\pi}{T} t + b_k \cdot \sin \frac{2k\pi}{T} t : a_k, b_k \in \mathbb{R} \left( \frac{4k^2\pi^2}{T^2} \right) \right\}$,
(2) $H^1_{T,0} \approx \bigoplus_{k=0}^{\infty} \mathbb{R} \left[ \mu_A \left( \frac{4k^2 \pi^2}{T^2} \right), k \right]$.

(3) $\dim H^1_{T,0} = \mu_A(0) + 2 \sum_{k=1}^{\infty} \mu_A \left( \frac{4k^2 \pi^2}{T^2} \right)$.

(4) $H^1_{T,-} = \bigoplus_{\lambda_i > 1} V_{LA}(\lambda_i) = \bigoplus_{\alpha \in \sigma_0(A)} V_A(\alpha) \oplus \bigoplus_{k=1}^{\infty} \bigoplus_{\alpha \in \sigma_k(A,T)} \left\{ a_k \cdot \cos \frac{2k\pi}{T} t + b_k \cdot \sin \frac{2k\pi}{T} t : a_k, b_k \in V_A(\alpha) \right\}$.

(5) $H^1_{T,-} \approx \bigoplus_{k=0}^{\infty} \bigoplus_{\alpha \in \sigma_k(A,T)} \mathbb{R} [\mu_A(\alpha), k] = \bigoplus_{k=0}^{\infty} \mathbb{R}[j_k(A,T), k]$, where $\sigma_k(A,T) = \{ k \pi / T : k \in \mathbb{N} \}$.

(6) $\dim H^1_{T,-} = \sum_{\alpha \in \sigma_0(A)} \mu_A(\alpha) + 2 \sum_{k=1}^{\infty} \sum_{\alpha \in \sigma_k(A,T)} \mu_A(\alpha) = j_0(A,T) + 2 \sum_{k=1}^{\infty} j_k(A,T)$.

(7) $H^1_{T,+} = \bigoplus_{\lambda_i < 1} V_{LA}(\lambda_i)$.

(8) $H^1_T = H^1_{T,-} \oplus H^1_{T,0} \oplus H^1_{T,+}$.

The following fact is a direct consequence of Theorem \ref{thm:main} Lemma \ref{lem:oper} and Corollaries \ref{cor:1} \ref{cor:2}.

**Fact 5.1.1.** If $\sigma(A) \cap \left\{ \frac{4k^2 \pi^2}{T^2} : k \in \mathbb{N} \cup \{0\} \right\} = \emptyset$, then the operator $\nabla \Phi_A = Id - L_A : H^1_T \to H^1_T$ is an isomorphism. Additionally, for $\gamma > 0$

\[ \nabla_{SO(2)} - \deg_H \left( \nabla \Phi_A, D_\gamma (H^1_{T,0}, 0) \right) = \nabla_{SO(2)} - \deg_H \left( -Id, D_\gamma (H^1_{T,-}, 0) \right) = \begin{cases} (-1)^{j_0(A,T)} & \text{for } H = SO(2), \\ (-1)^{j_k(A,T)} \cdot j_k(A,T) & \text{for } H = \mathbb{Z}_k. \end{cases} \]

It is understood that if $H^1_{T,-} = \{0\}$ then $\nabla_{SO(2)} - \deg \left( \nabla \Phi_A, D_\gamma (H^1_{T,0}) \right) = I \in U(SO(2))$.

5.2. Existence of Periodic Solutions of Nonlinear Equation. In this section we formulate sufficient conditions for the existence of non-stationary $T$-periodic solutions of autonomous Newtonian systems.

Let us consider the following Newtonian system

\[ \begin{cases} \ddot{u} = -V'(u) \\ u(0) = u(T) \\ \dot{u}(0) = \dot{u}(T) \end{cases} \quad (5.2.1) \]

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$. Suppose

(i) $(V')^{-1}(0) = \{p_1, \ldots, p_q\}$,

(ii) $V'(x) = V''(\infty) \cdot x + o(||x||)$ as $||x|| \to \infty$. 

\[ \begin{cases} (V')^{-1}(0) = \{p_1, \ldots, p_q\} \\ V'(x) = V''(\infty) \cdot x + o(||x||) \end{cases} \quad (5.2.1) \]

where $V \in C^2(\mathbb{R}^n, \mathbb{R})$. Suppose

\[ \begin{cases} (V')^{-1}(0) = \{p_1, \ldots, p_q\} \\ V'(x) = V''(\infty) \cdot x + o(||x||) \end{cases} \quad (5.2.1) \]
Recall that for \( p \in \mathbb{H}^{SO(2)} \) and \( \gamma > 0 \) we set \( D_\gamma(\mathbb{H}, p) = \{ v \in \mathbb{H} : \|v - p\|_\mathbb{H} < \gamma \} \). Moreover, if \( p = \infty \) then \( D_\gamma(\mathbb{H}, \infty) := D_\gamma(\mathbb{H}, 0) \). Define \( \text{ind}(-V', p_i) = \lim_{\alpha \to 0} \deg_B(-V', D_\alpha(\mathbb{R}^n, p_i), 0) \) for \( i = 1, \ldots, q \) and \( \text{ind}(-V', \infty) = \lim_{\alpha \to \infty} \deg_B(-V', D_\alpha(\mathbb{R}^n, \infty), 0) \).

**Remark 5.2.1.** Under the above assumptions \( \text{ind}(-V', \infty) = \sum_{i=1}^q \text{ind}(-V', p_i) \).

**Definition 5.2.1.** Let \( p \in \{p_1, \ldots, p_q, \infty\} \) and \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} \neq \emptyset \). Define \( I_V(p, T) = (I_V(p, T)_{SO(2)}, I_V(p, T)_{\mathbb{Z}_1}, \ldots, I_V(p, T)_{\mathbb{Z}_k}, \ldots) \in U(SO(2)) \) in the following way

\[
I_V(p, T)_H = \begin{cases} 
\text{ind}(-V', p) & \text{for } H = SO(2), \\
\text{ind}(-V', p) \cdot j_k(V''(p), T) & \text{for } H = \mathbb{Z}_k.
\end{cases}
\]  

(5.2.2)

**Remark 5.2.2.** Notice that, if moreover we assume \( \det V''(p) \neq 0 \), then formula (5.2.2) becomes

\[
I_V(p, T)_H = \begin{cases} 
(-1)^{\mu(V''(p), T)} & \text{for } H = SO(2), \\
(-1)^{\mu(V''(p), T)} \cdot j_k(V''(p), T) & \text{for } H = \mathbb{Z}_k.
\end{cases}
\]  

(5.2.3)

**Lemma 5.2.1.** If \( p \in \{p_1, \ldots, p_q, \infty\} \) and \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \emptyset \) then the stationary solution \( p \) is an isolated critical point of \( \Phi_V \).

**Proof.** Since \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \emptyset \), from Corollary 5.1.2 we conclude that \( \ker(Id - L_p) \subset (\mathbb{H}^1_T)^{SO(2)} \). Suppose, contrary to our claim that \( p \) is not isolated in \( (\nabla \Phi)^{-1}(0) \). Taking into account that \( \ker(Id - L_p) \subset (\mathbb{H}^1_T)^{SO(2)} \) and Corollary 5.1, we obtain that \( p \) is not isolated in \( (\nabla \Phi)^{-1}(0) \cap (\mathbb{H}^1_T)^{SO(2)} \). But there is only a finite number of stationary solutions of (5.2.1), a contradiction. \( \square \)

**Lemma 5.2.2.** If \( p \in \{p_1, \ldots, p_q, \infty\} \) and \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \emptyset \) then

\[
I_V(p, T) = \nabla_{SO(2)} \deg(\nabla \Phi_V, D_{\alpha_p}(\mathbb{H}^1_T, p)),
\]

where \( \alpha_p \) is sufficiently small if \( p = p_i \) for \( i = 1, \ldots, q \) and sufficiently large if \( p = \infty \).

**Proof.** From Lemma 5.2.1 it follows that the functional \( \Phi_V \) satisfies conditions (F.1)-(F.6). Hence we can apply Lemma 4.1 to compute \( \nabla_{SO(2)} \deg(\nabla \Phi_V, D_{\alpha_p}(\mathbb{H}^1_T, p)) \). We obtain

\[
\nabla_{SO(2)} \deg(\nabla \Phi_V, D_{\alpha_p}(\mathbb{H}^1_T, p)) = \deg_B(\nabla \Phi_V^{SO(2)}, D_{\alpha_p}(\mathbb{H}^1_T^{SO(2)}), 0),
\]

\[
\cdot \nabla_{SO(2)} \deg((Id - L_p)_{((\mathbb{H}^1_T)^{SO(2)})}, D_{\alpha_p}(((\mathbb{H}^1_T)^{SO(2)})^\perp, 0)).
\]
Since \((Id - L_p)|_{((\mathbb{H}^1_T)_{SO(2)})^\perp}\) is an isomorphism, from Fact 5.1.1 we have
\[
\nabla_{SO(2)} - \deg((Id - L_p)|_{((\mathbb{H}^1_T)_{SO(2)})^\perp}, D_{\alpha_p}((\mathbb{H}^1_T)_{SO(2)})^\perp, 0)) = \sum_{i=1}^{q} D_{\alpha_{p_i}}((\mathbb{H}^1_T, p_i)))
\]
Moreover, since \((\mathbb{H}^1_T)^{SO(2)}\) = \(\{u \in H^1_T : u = const\}\) it is easily seen that
\[
\deg_B(\nabla_{SO(2)} \Phi_V, D_{\alpha_p}(\mathbb{H}^1_T, p), 0) = \deg_B(-V', D_{\alpha_p}(\mathbb{R}^n, p), 0)\]
Applying (2.2) we obtain the assertion. □

The following theorem ensures the existence of at least one non-stationary \(T\)-periodic solution of system (5.2.1). See [21, 22] for related results.

**Theorem 5.2.1.** Suppose that

(i) \((V')^{-1}(0) = \{p_1, \ldots, p_q\}\),
(ii) \(V'(x) = V''(\infty) \cdot x + o(\|x\|)\) as \(\|x\| \to \infty\).
Assume additionally that \(\sigma(V''(p)) \cap \left\{\frac{4k^2\pi^2}{T^2} : k \in \mathbb{N}\right\} = \emptyset\) for any \(p \in \{p_1, \ldots, p_q, \infty\}\) and that
\(I_V(\infty, T) \neq \sum_{i=1}^{q} I_V(p_i, T)\). Then there exists at least one non-stationary \(T\)-periodic solution of (5.2.1).

**Proof.** Notice that stationary solutions \(p_1, \ldots, p_q\) and \(\infty\) are isolated critical points of functional \(\Phi_V\) given by (5.2). Suppose, contrary to our claim, that \(p_1, \ldots, p_q\) are the only \(T\)-periodic solutions of (5.2.1). Thus we can choose \(\alpha_\infty, \alpha_{p_i} > 0, i = 1, \ldots, q\) such that

(i) \((\nabla_{SO(2)} \Phi_V)^{-1}(0) \cap (\mathbb{H}^1_T \setminus D_{\alpha_\infty}(\mathbb{H}^1_T, \infty)) = \emptyset\),
(ii) \((\nabla_{SO(2)} \Phi_V)^{-1}(0) \cap D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i) = \{p_i\}, i = 1, \ldots, q\),
(iii) \(D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i) \cap D_{\alpha_{p_j}}(\mathbb{H}^1_T, p_j) = \emptyset, i \neq j\),
(iv) \(cl(D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i)) \subset D_{\alpha_\infty}(\mathbb{H}^1_T, \infty)\) for \(i = 1, \ldots, q\). From the Theorem 2.1 (3) and (4) we obtain
\[
\nabla_{SO(2)} - \deg(\nabla_{SO(2)} \Phi_V, D_{\alpha_\infty}(\mathbb{H}^1_T, \infty)) = \sum_{i=1}^{q} \nabla_{SO(2)} - \deg(\nabla_{SO(2)} \Phi_V, D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i)).
\]
But since \(\Phi_V\) satisfies assumptions of the Lemma 5.2.2 we have
\[
\nabla_{SO(2)} - \deg(\nabla_{SO(2)} \Phi_V, D_{\alpha_\infty}(\mathbb{H}^1_T, \infty)) = I_V(\infty, T) \neq \sum_{i=1}^{q} I_V(p_i, T) = \sum_{i=1}^{q} \nabla_{SO(2)} - \deg(\nabla_{SO(2)} \Phi_V, D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i)),
\]
a contradiction. □
Remark 5.2.3. Notice that the above theorem can be formulated in the following equivalent way. Assume that for \( p \in \{p_1, \ldots, p_q, \infty\} \), \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \emptyset \). If there exists \( k \in \mathbb{N} \) such that

\[
\text{ind}(-V', \infty) \cdot j_k(V''(\infty), T) \neq \sum_{i=1}^{q} \text{ind}(-V', p_i) \cdot j_k(V''(p_i), T)
\]

then there exists at least one non-stationary \( T \)-periodic solution of \((5.2.1)\).

Suppose now that \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} \neq \emptyset \). In that case we cannot define \( I_V(p, T) \) such as in Definition 5.2.1. However, we can define almost all the coordinates of it. Recall that for \( \{k_1, \ldots, k_r\} \) and \( \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\} \) we have \( k_{i_1 \ldots i_s} = \gcd\{k_{i_1}, \ldots, k_{i_s}\} \).

Definition 5.2.2. Let \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\} \), where \( p \in \{p_1, \ldots, p_q, \infty\} \). Put \( \mathbb{K} = \bigcup_{\{i_1, \ldots, i_s\} \in \{1, \ldots, r\}} \{k_{i_1 \ldots i_s}\} \) and define

\[
I_V(p, T)_H = \begin{cases} 
\text{ind}(-V', p) & \text{for } H = \text{SO}(2), \\
\text{ind}(-V', p) \cdot j_k(V''(p), T) & \text{for } H = \mathbb{Z}_k \text{ and } k \notin \mathbb{K}.
\end{cases}
\]

Remark 5.2.4. Observe that if \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\} \) and moreover \( \det V''(p) \neq 0 \), then \( \deg_{\mathbb{Z}_k}(\mathbb{R}^n, p, 0) = (-1)^{j_0(V''(p))} \) and consequently for \( \mathbb{K} = \bigcup_{\{i_1, \ldots, i_s\} \in \{1, \ldots, r\}} \{k_{i_1 \ldots i_s}\} \) we have

\[
I_V(p, T)_H = \begin{cases} 
(-1)^{j_0(V''(p), T)} & \text{for } H = \text{SO}(2), \\
(-1)^{j_0(V''(p), T)} \cdot j_k(V''(p), T) & \text{for } H = \mathbb{Z}_k \text{ and } k \notin \mathbb{K}.
\end{cases}
\]

Lemma 5.2.3. Fix \( p \in \{p_1, \ldots, p_q, \infty\} \) and assume that \( \sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\} \). If the stationary solution \( p \in \mathbb{H}_T^1 \) is an isolated critical point of \( \Phi_V \) then for \( k \notin \mathbb{K} = \bigcup_{\{i_1, \ldots, i_s\} \in \{1, \ldots, r\}} \{k_{i_1 \ldots i_s}\} \) we have

\[
I_V(p, T)_{z_k} = \nabla_{\text{SO}(2)} \deg_{\mathbb{Z}_k}(\nabla \Phi_V, D_{\alpha_s}(\mathbb{H}_T^1, p)),
\]

where \( \alpha \) is a conveniently chosen radius and \( \Phi_V \) is a functional given by \((5.2)\).

\[\text{Proof.}\] Notice that if \( p \) is an isolated critical point of \( \Phi_V \), then functional \( \Phi_V \) satisfies conditions \((F.1)-(F.6)\). Applying Lemma 5.2.2 we conclude
As in the proof of Theorem 5.2.1 we obtain
\[ \nabla_{SO(2)} - \text{deg}(\nabla \varphi, D_{ap}(\mathbb{H}_T^1, p)) = \]
\[ = \nabla_{SO(2)} - \text{deg}(\nabla \varphi, D_{ap}(V_p, p)) \cdot \nabla_{SO(2)} - \text{deg}(-\text{Id}, D_{ap}(\mathbb{H}_T^1, 0)). \]

Since \( k \notin \mathbb{K} \), \((V_p)_{\mathbb{Z}_k} = 0 \). Hence by Remark 2.1 we have \( \nabla_{SO(2)} - \text{deg}_{\mathbb{Z}_k}(\nabla \varphi, D_{ap}(V_p, p)) = 0 \). Therefore taking into account formula (2.2) we obtain the following
\[ \nabla_{SO(2)} - \text{deg}_{\mathbb{Z}_k}(\nabla \varphi, D_{ap}(\mathbb{H}_T^1, p)) = \]
\[ = \nabla_{SO(2)} - \text{deg}_{\mathbb{Z}_k}(\nabla \varphi, D_{ap}(V_p, p)) \cdot \nabla_{SO(2)} - \text{deg}_{\mathbb{Z}_k}(-\text{Id}, D_{ap}(\mathbb{H}_T^1, 0)) \]
for \( k \notin \mathbb{K} \). But from Fact 5.1.1 we obtain
\[ \nabla_{SO(2)} - \text{deg}(-\text{Id}, D_{ap}(\mathbb{H}_T^1, 0)) = \]
\[ = ((-1)^{j_0(V''(p), T)}, (-1)^{j_0(V''(p), T)} \cdot j_1(V''(p), T), \ldots, (-1)^{j_0(V''(p), T)} \cdot j_k(V''(p), T), \ldots). \]

Taking into consideration that
\[ \nabla_{SO(2)} - \text{deg}_{\mathbb{Z}_k}(\nabla \varphi, D_{ap}(V_p, p)) \cdot (-1)^{j_0(V''(p), T)} = \text{deg}_B(-V', D_{ap}(\mathbb{R}^n, p), 0) \]
we complete the proof. \( \square \)

Combining the above considerations with Theorem 5.2.1 we can formulate its more general version.

**Theorem 5.2.2.** Suppose that

(i) \((V')^{-1}(0) = \{p_1, \ldots, p_q\}\),
(ii) \(V'(x) = V''(\infty) \cdot x + o(\|x\|) \) as \( \|x\| \to \infty \).

If there exists \( k \in \mathbb{N} \) such that \( I_V(p, T)_k \) is defined for all \( p \in \{p_1, \ldots, p_q, \infty\} \) and \( I_V(\infty, T)_k \neq \sum_{i=1}^{q} I_V(p_i, T)_k \), then there exists at least one non-stationary \( T \)-periodic solution of (5.2.1). Moreover, if \( p_0 \in \{p_1, \ldots, p_q, \infty\} \) is not an isolated \( T \)-periodic solution of (5.2.1), then there are numbers \( k_1, \ldots, k_r \in \mathbb{N} \) such that \( \sigma(V''(p_0)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\} \) and the minimal period of any solution sufficiently close to \( p_0 \) equals
\[ \frac{T}{k_{i_1 \ldots i_s}} \]
for some \( \{i_1, \ldots, i_s\} \subset \{1, \ldots, r\} \).

**Proof.** Assume that stationary solutions \( p_1, \ldots, p_q \) and \( \infty \) are isolated critical points of \( \Phi_V \). If they were not isolated, we would obtain an infinite sequence of critical points of \( \Phi_V \), i.e. a sequence of \( T \)-periodic non-stationary solutions of (5.2.1), which is our assertion. Suppose, contrary to our claim, that \( p_1, \ldots, p_q \) are the only \( T \)-periodic solutions of (5.2.1). As in the proof of Theorem 5.2.1 we obtain
\[ \nabla_{SO(2)} - \text{deg}(\nabla \Phi, D_{ap}(\mathbb{H}_T^1, \infty)) = \sum_{i=1}^{q} \nabla_{SO(2)} - \text{deg}(\nabla \Phi, D_{ap_i}(\mathbb{H}_T^1, p_i)). \]
But since $\Phi_V$ satisfies assumptions of the Lemma 5.2.3 we have
\[
\nabla_{SO(2)} - \deg_{Z_k}(\nabla \Phi_V, D_{\alpha_{\infty}}(\mathbb{H}^1_T, \infty)) = I_V(\infty, T)_{Z_k} \neq \\
\sum_{i=1}^q I_V(p_i, T)_{Z_k} = \sum_{i=1}^q \nabla_{SO(2)} - \deg_{Z_k}(\nabla \Phi_V, D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i)).
\]
Hence, $\nabla_{SO(2)} - \deg(\nabla \Phi, D_{\alpha_{\infty}}(\mathbb{H}^1_T, \infty)) \neq \sum_{i=1}^q \nabla_{SO(2)} - \deg(\nabla \Phi_V, D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i))$, a contradiction.

Fix $p_0 \in \{p_1, \ldots, p_q, \infty\}$ and assume that $p_0$ is not isolated in $(\nabla \Phi_V)^{-1}(0)$. Then from Lemma 5.2.1 it follows that $\sigma(V''(p_0)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\}$.

Hence, \( \ker \nabla^2 \Phi_V(p_0) \approx \bigoplus_{i=1}^r \mathbb{R}[\mu_{V''(p_0)}(\frac{4k_i^2\pi^2}{T^2}), k] \). That is why the isotropy group $SO(2)_x$ of any element of $x \in \ker \nabla^2 \Phi_V(p_0) \backslash \{0\}$ is equal to $\mathbb{Z}_{k_{i_1}\ldots i_s}$ for some $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$. It is clear that if $x \in \mathbb{H}_T^1$ and $SO(2)_x = \mathbb{Z}_{k_{i_1}\ldots i_s}$ then the minimal period of $x$ is equal to $\frac{T}{k_{i_1}\ldots i_s}$.

The rest of the proof is a direct consequence of Corollary 3.1. \( \square \)

**Remark 5.2.5.** Notice that the above theorem one can formulate in the following equivalent way. Assume that for $p \in \{p_1, \ldots, p_q, \infty\}$
\[
\sigma(V''(p)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\}
\]
and define $\mathbb{K} = \bigcup_{p \in \{p_1, \ldots, p_q, \infty\}} \bigcup_{\{i_1, \ldots, i_s\} \in \{1, \ldots, r\}} \{ \gcd\{k_{i_1, p}, \ldots, k_{i_s, p}\} \}$. If there exists $k \notin \mathbb{K}$ such that
\[
\text{ind}(-V', \infty) \cdot j_k(V''(\infty), T) \neq \sum_{i=1}^q \text{ind}(-V', p_i) \cdot j_k(V''(p_i), T)
\]
then there exists at least one non-stationary $T$-periodic solution of (5.2.1).

**Remark 5.2.6.** Notice that we cannot prove Theorem 5.2.2 using the Leray-Schauder degree because it vanishes. In fact, the assumption of Theorem 5.2.2 can be rewritten in the following way
\[
\nabla_{SO(2)} - \deg \left( \nabla \Phi, D_{\alpha_{\infty}}(\mathbb{H}^1_T, \infty) \backslash \text{cl} \left( \bigcup_{i=1}^p D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i) \right) \right) \neq \Theta \in U(SO(2)).
\]
From Theorem 2.7 it follows that
\[
\deg_{LS} \left( \nabla \Phi, D_{\alpha_{\infty}}(\mathbb{H}^1_T, \infty) \backslash \text{cl} \left( \bigcup_{i=1}^p D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i) \right), 0 \right) =
\]
and the minimal period of any solution sufficiently close to $p$ because

$$
\left. \left( - V', D_{\alpha_{\infty}}(\mathbb{R}^n, \infty) \right) \cap \left( \bigcup_{i=1}^{p} D_{\alpha_{p_i}}(\mathbb{R}^n, p_i) \right), 0 \right) = 0 \in \mathbb{Z}
$$

5.3. Continuation of Periodic Solutions of Nonlinear Equation. In this section we study continuation of non-stationary $T$-periodic solutions of the family of Newtonian systems of the form

$$
(F)_\lambda \begin{cases}
\ddot{u} = -V'_\lambda(u) \\
u(0) = u(T) \\
\dot{u}(0) = \dot{u}(T)
\end{cases}
$$

where $V_\lambda \in C^2(\mathbb{R}^n, \mathbb{R}), \lambda \in \mathbb{R}$. By $I_{V_\lambda}(p)$ we denote the index defined in the previous section.

**Theorem 5.3.1.** Assume that

1. $(V_0'^{-1})^{-1}(0) = \{p_1, \ldots, p_q\}$,
2. $V_0'(x) = V_0''(\infty) \cdot x + o(||x||)$ as $||x|| \to \infty$,
3. there exists $k \in \mathbb{N}$ such that $I_{V_0}(\infty, T) z_k \neq \sum_{i=1}^{q} I_{V_0}(p_i, T) z_k$.

Then there exists an infinite sequence of non-stationary $T$-periodic solutions of $(F)_0$ converging to some $p \in \{p_1, \ldots, p_q, \infty\}$ or there exist closed, connected sets $\mathcal{C}^\pm$ such that

$$
\mathcal{C}^- \subset (\mathbb{H}^1_T \times (-\infty, 0]) \cap (\nabla \Phi_{V_\lambda})^{-1}(0),
\mathcal{C}^+ \subset (\mathbb{H}^1_T \times [0, +\infty)) \cap (\nabla \Phi_{V_\lambda})^{-1}(0).
$$

Moreover, for $\mathcal{C} = \mathcal{C}^\pm$

1. $\mathcal{C} \cap \left( \left( D_{\alpha_{\infty}}(\mathbb{H}^1_T, \infty) \setminus \bigcup_{i=1}^{q} D_{\alpha_{p_i}}(\mathbb{H}^1_T, p_i) \right) \times \{0\} \right) \neq \emptyset$,
2. either $\mathcal{C}$ is not bounded or $\mathcal{C} \cap \{p_1, \ldots, p_q\} \neq \emptyset$.

Additionally, if $p_0 \in \{p_1, \ldots, p_q, \infty\}$ is not an isolated $T$-periodic solution of $(F)_0$, then there are numbers $k_1, \ldots, k_r \in \mathbb{N}$ such that

$$
\sigma(V''(p_0)) \cap \left\{ \frac{4k^2\pi^2}{T^2} : k \in \mathbb{N} \right\} = \left\{ \frac{4k_1^2\pi^2}{T^2}, \ldots, \frac{4k_r^2\pi^2}{T^2} \right\}
$$

and the minimal period of any solution sufficiently close to $p_0$ equals $\frac{T}{k_{i_1...i_s}}$, for some $\{i_1, \ldots, i_s\} \subset \{1, \ldots, r\}$. 
Proof. Consider functional $\Phi_{V_0} \in C^2_{SO(2)}(\mathbb{H}^1, \mathbb{R})$ given by (5.2) and suppose that stationary solutions $p_1, \ldots, p_q$ and $\infty$ are isolated solutions of (5.3.1) on level $\lambda_0 = 0$, i.e. they are isolated critical points of $\Phi_{V_0}$. Thus we can choose $\alpha_\infty, \alpha_{p_i} > 0, i = 1, \ldots, q$ such that

(i) $\left(\nabla \Phi_{V_0}\right)^{-1}(0) \cap \left(\mathbb{H}_T^1 \setminus D_{\alpha_\infty}(\mathbb{H}_T^1, \infty)\right) = \emptyset,$

(ii) $\left(\nabla \Phi_{V_0}\right)^{-1}(0) \cap D_{\alpha_{p_i}}(\mathbb{H}_T^1, p_i) = \{p_i\}, i = 1, \ldots, q,$

(iii) $D_{\alpha_{p_i}}(\mathbb{H}_T^1, p_i) \cap D_{\alpha_{p_j}}(\mathbb{H}_T^1, p_j) = \emptyset, \text{ for } i \neq j.$

Set $\Omega = D_{\alpha_\infty}(\mathbb{H}_T^1, \infty) \setminus \text{cl} \left(\bigcup_{i=1}^q D_{\alpha_{p_i}}(\mathbb{H}_T^1, p_i)\right)$. Notice that from the assumption (3) and Facts 5.2.2 5.2.3 we obtain $\nabla_{SO(2)} \text{deg}(\nabla \Phi_{V_0}, \Omega) \neq \emptyset \in U(SO(2))$. The rest of the proof is a direct consequence of Theorem 5.2.2. The second part of the proof is in fact the same as the proof of Theorem 5.2.2. \qed

Corollary 5.3.1. Let assumptions of Theorem 5.3.1 be satisfied. If moreover $\sigma(V''(p)) \cap \left\{\frac{4k^2\pi^2}{T^2} : \ k \in \mathbb{N}\right\} = \emptyset$ for any $p \in \{p_1, \ldots, p_q, \infty\}$ then there exist closed connected sets $C^- \subset (\mathbb{H}_T^1 \times (-\infty, 0]) \cap \left(\nabla \Phi_{V_0}\right)^{-1}(0), C^+ \subset (\mathbb{H}_T^1 \times [0, +\infty)) \cap \left(\nabla \Phi_{V_0}\right)^{-1}(0)$ with properties (C1), (C2).

Proof. From Lemma 5.2.1 we conclude that any $p \in \{p_1, \ldots, p_q, \infty\}$ is an isolated critical point of $\Phi_{V_0}$ i.e. an isolated solution of $\ddot{u} = -V'_0(u)$. \qed

Remark 5.3.1. If $C = C^\pm$ in Theorem 5.3.1 is bounded, then symmetry breaking phenomenon occurs, i.e. $C$ contains solutions with different minimal periods. Indeed, in this case from (C2) we obtain that $C$ contains stationary solutions (whose isotropy group in $\mathbb{H}_T^1$ is equal to $SO(2)$) and from (C1) - non-stationary solution (with isotropy group $\mathbb{Z}_k$ for some $k \in \mathbb{N}$, which means that its minimal period is equal to $\frac{T}{k}$).

Remark 5.3.2. Similarly as in the previous section the Leray-Schauder degree is not applicable in the proof of Theorem 5.3.1.

6. Illustration

In this section we illustrate abstract results proved in the previous parts of this article. Namely, we study non-stationary $T$-periodic solutions of the following system

$$
\begin{align*}
\dot{u} &= -V'(u) \\
u(0) &= u(T) \\
\dot{u}(0) &= \dot{u}(T)
\end{align*}
$$

where potential $V : \mathbb{R}^n \to \mathbb{R}$ is defined as follows

$$V(x) = \frac{1}{2} < V''(\infty)x, x > + W(x) = \frac{1}{2} < V''(\infty)x, x > + \frac{1}{\sqrt{\|x\|^2 + a}}$$

where $a > 0$ and $V''(\infty)$ is a real symmetric $(n \times n)$-matrix.

In the following four lemmas we study properties of the functional $V$.
Lemma 6.1. If potential $V$ is given by (6.2), then

$$V'(x) = V''(\infty) \cdot x + W'(x) = V''(\infty) \cdot x + \frac{1}{(\|x\|^2 + a)^{3/2}} x = V''(\infty) \cdot x + o(\|x\|) \text{ as } \|x\| \to \infty.$$ 

Additionally, there is an orthogonal matrix $P \in O(n, \mathbb{R})$ such that

$$V'(x) = P \cdot \left( J(V''(\infty)) + \frac{1}{(\|x\|^2 + a)^{3/2}} \cdot \text{Id}_{\mathbb{R}^n} \right) \cdot P^{-1} \cdot x,$$

where $J(V''(\infty)) = \text{diag } \{\lambda_1, \ldots, \lambda_n\}$. Moreover, we have

1. $(W)^{\prime \prime}_{x_i x_i}(x) = -3x_i^2$, 
2. $(W)^{\prime \prime}_{x_i x_i}(x) = -3x_i^2 + \frac{1}{(\|x\|^2 + a)^{3/2}}$ for $i = 1, \ldots, n$, 
3. $(W)^{\prime \prime}_{x_i x_j}(x) = -3x_i x_j (\|x\|^2 + a)^{5/2}$ for $i \neq j, i, j = 1, \ldots, n$.

The easy proof of the above lemma is left to the reader.

Lemma 6.2. Assume that $\mu_{V''(\infty)}(\lambda_i) = 1$ for any $\lambda_i \in \sigma(V''(\infty)) \cap \left[ -\frac{1}{\sqrt{a^3}}, 0 \right)$. Then $\#(V')^{-1}(0) < \infty$. Moreover, if $x \in (V')^{-1}(0)$ then $x = Py$ where for $i = 1, \ldots, n$ we have

$$y_i = \begin{cases} 
0, & \text{if } \lambda_i \geq 0 \text{ or } \lambda_i < -\frac{1}{\sqrt{a^3}} \\
0 \text{ or } \pm \sqrt{\frac{1}{\sqrt{\lambda_i^2}} - a}, & \text{otherwise.} 
\end{cases}$$

Proof. Let $P \in O(n, \mathbb{R})$ be as in Lemma 6.1. Hence $V'(y) = 0$ iff $P^{-1}V'(Py) = 0$ and moreover

$$P^{-1}V'(Py) = \left( J(V''(\infty)) + \frac{1}{(\|y\|^2 + a)^{3/2}} \cdot \text{Id}_{\mathbb{R}^n} \right) \cdot y.$$ 

Fix $\lambda_i \in \sigma_+(V''(\infty)) \cup \{0\}$ and notice that $\lambda_i + \frac{1}{(\|y\|^2 + a)^{3/2}} > 0$ for any $y \in \mathbb{R}^n$. Assume now that $\lambda_i \in \sigma_-(V''(\infty))$. It is clear that

$$\lambda_i + \frac{1}{(\|y\|^2 + a)^{3/2}} = 0 \iff \|y\|^2 = \frac{1}{\sqrt{\lambda_i^2}} - a$$

and consequently if $a > \frac{1}{\sqrt{\lambda_i^2}}$ then $\lambda_i + \frac{1}{(\|y\|^2 + a)^{3/2}} < 0$.

Taking into account that $\mu_{V''(\infty)}(\lambda_i) = 1$ for any $\lambda_i \in \sigma_-(V''(\infty))$ and the above we obtain that

$$\left( J(V''(\infty)) + \frac{1}{(\|y\|^2 + a)^{3/2}} \cdot \text{Id}_{\mathbb{R}^n} \right) \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$
if and only if for \( i = 1, \ldots, n \) we have

\[
y_i = \begin{cases} 
0, & \text{if } \lambda_i \geq 0 \text{ or } \lambda_i < -\frac{1}{\sqrt{a^3}}, \\
0 \text{ or } \pm \sqrt{\frac{3\lambda_i^2}{a}}, & \text{otherwise},
\end{cases}
\]

which completes the proof. \( \square \)

**Lemma 6.3.** If \( \sigma(V''(\infty)) \cap \left[-\frac{1}{\sqrt{a^3}}, 0\right) = \emptyset \) then \( (V')^{-1}(0) = \{\Theta\} \).

**Proof.** By Lemma 6.4 we have

\[
V'(x) = P \cdot \left( J(V''(\infty)) + \frac{1}{(\|x\|^2 + a)^{3/2}} \cdot I_{\mathbb{R}^n} \right) \cdot P^{-1} \cdot x \quad (6.3)
\]

Since \( \sigma(V''(\infty)) \cap \left[-\frac{1}{\sqrt{a^3}}, 0\right) = \emptyset \) for \( i = 1, \ldots, n \) and any \( x \in \mathbb{R}^n \). Hence matrix \( P \cdot \left( J(V''(\infty)) + \frac{1}{(\|x\|^2 + a)^{3/2}} \cdot I_{\mathbb{R}^n} \right) \cdot P^{-1} \) is nondegenerate for any \( x \in \mathbb{R}^n \). Taking into account (6.3) we complete the proof. \( \square \)

**Lemma 6.4.** Under the above assumptions \( \text{ind}(-V', \infty) = (-1)^{n-m(V''(\infty))} \).

**Proof.** Notice that for sufficiently large \( \|x\| \) and \( \lambda_i \in \sigma(V''(\infty)) \setminus \{0\} \) we have

\[
\text{sign} \left( \lambda_i + \frac{1}{(\|x\|^2 + a)^{3/2}} \right) = \text{sign } \lambda_i.
\]

For \( i = 1, \ldots, n \) define \( \psi_i : [0, 1] \to \mathbb{R} \setminus \{0\} \) as follows

\[
\psi_i(t) = \begin{cases} 
t \cdot \text{sign } \lambda_i + (1-t) \cdot \left( \lambda_i + \frac{1}{(\|x\|^2 + a)^{3/2}} \right), & \text{if } \lambda_i \neq 0, \\
t + \frac{1-t}{(\|x\|^2 + a)^{3/2}}, & \text{if } \lambda_i = 0.
\end{cases}
\]

Since the above, for any \( t \in [0, 1] \) and sufficiently large \( \|x\| \) we have

\[
\det \left( P \cdot \begin{bmatrix} \psi_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_n(t) \end{bmatrix} \cdot P^{-1} \right) = \prod_{i=1}^n \psi_i(t) \neq 0.
\]

Define a map \( \Psi : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n \) as follows

\[
\Psi(x, t) = \left( P \cdot \begin{bmatrix} \psi_1(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \psi_n(t) \end{bmatrix} \right) \cdot P^{-1} \cdot x.
\]

It is easy to verify that

(1) \( \Psi(\cdot, 0) = V'(\cdot) \).
From Lemma 6.2 it follows that
\[ \det(\Psi(x,t)) \neq 0, \]
(3) \( \text{ind}(-V', \infty) = \text{ind}(-\Psi(\cdot, 0), \infty) = \text{ind}(-\Psi(\cdot, 1), \infty) = (-1)^{n-m-V''(\infty)}, \]
which completes the proof. \( \Box \)

**Example 6.1.** In this example we study system (6.1) with resonance at the infinity \( V''(\infty) \) is degenerate. Potential \( V \) is a Morse function i.e. all the critical points of \( V \) are nondegenerate. The origin \( \Theta \in \mathbb{R}^n \), treated as a constant function, is a resonant stationary solution of (6.1) i.e. \( \sigma(V''(\Theta)) \cap \{k^2 : k \in \mathbb{N}\} \neq \emptyset. \) Consider system (6.1) with \( n = 4, a = 1, T = 2\pi \) and

\[ V''(\infty) = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2\sqrt{2} \end{bmatrix}. \]

From Lemma 6.2 it follows that \( (V')^{-1}(0) = \{\Theta, \pm e_4\} \). Moreover, by Lemma 6.1 we have

\[ V''(\Theta) = V''(\infty) + W''(\Theta) = V''(\infty) + Id = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - 1/2\sqrt{2} \end{bmatrix}, \]

and

\[ V''(\pm e_4) = V''(\infty) + W''(\pm e_4) = \begin{bmatrix} 7 & 0 & 0 & 0 \\ 0 & -2 + 1/2\sqrt{2} & 0 & 0 \\ 0 & 0 & 1/2\sqrt{2} & 0 \\ 0 & 0 & 0 & -3/4\sqrt{2} \end{bmatrix}. \]

Since \( \det(-V''(\Theta)) < 0, \text{ind}(-V', \Theta) = -1. \) Additionally condition \( \det(-V''(\pm e_4)) > 0 \) implies \( \text{ind}(-V', \pm e_4) = 1. \) Moreover, from Lemma 6.4 it follows that \( \text{ind}(-V', \infty) = 1. \) Finally notice that

\[ j_k(V''(\Theta), 2\pi) = \begin{cases} 1 & \text{if } k = 1, 2 \\ 0 & \text{otherwise,} \end{cases} \quad j_k(V''(\Theta), 2\pi) = j_k(V''(\pm e_4), 2\pi) = \begin{cases} 1 & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases} \]

It is clear that \( \sigma(V''(\Theta)) \cap \{k^2 : k \in \mathbb{N}\} = \{1\} \) and that

\[ \sigma(V''(\pm e_4)) \cap \{k^2 : k \in \mathbb{N}\} = \sigma(V''(\infty)) \cap \{k^2 : k \in \mathbb{N}\} = \emptyset. \]

Moreover,

\[ \text{ind}(-V', \infty) \cdot j_2(V''(\infty), 2\pi) = 1 \cdot 0 = 0 \neq -1 = (-1) \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = \text{ind}(-V', \Theta) \cdot j_2(V''(\Theta), 2\pi) + \text{ind}(-V', e_4) \cdot j_2(V''(e_4), 2\pi) + \]
there is at least one non-stationary. We have just shown that all the assumption of Theorem 5.2.2 are satisfied. Therefore there is at least one non-stationary 2π-periodic solution of system \(6.1\).

**Example 6.2.** In this example we study system \(6.1\) with resonance at the infinity \((V'')(\infty)\) is degenerate). Potential \(V\) is not a Morse function because \(\Theta \in \mathbb{R}^4\) is a degenerate critical point of \(V\). The origin \(\Theta \in \mathbb{R}^n\), treated as a constant function, is a resonant stationary solution of \(6.1\) i.e. \(\sigma(V''(\Theta)) \cap \{k^2 : k \in \mathbb{N}\} \neq \emptyset\). Consider system \(6.1\) with \(n = 4, a = 1, T = 2\pi\) and

\[
V''(\infty) = \begin{bmatrix}
7/2 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1/2 \sqrt{2}
\end{bmatrix}.
\]

From Lemma 6.1 it follows that \((V')^{-1}(0) = \{\Theta, \pm e_4\}\). Moreover, by Lemma 6.1 we have

\[
V''(\Theta) = V''(\infty) + W''(\Theta) = V''(\infty) + \text{Id} = \begin{bmatrix}
9/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 - 1/2 \sqrt{2}
\end{bmatrix}
\]

and

\[
V''(\pm e_4) = V''(\infty) + W''(\pm e_4) = \begin{bmatrix}
7/2 + 1/2 \sqrt{2} & 0 & 0 & 0 \\
0 & -1 + 1/2 \sqrt{2} & 0 & 0 \\
0 & 0 & 1/2 \sqrt{2} & 0 \\
0 & 0 & 0 & -3/4 \sqrt{2}
\end{bmatrix}.
\]

Since \(\det(-V''(\pm e_4)) > 0\), \(\text{ind}(-V', \pm e_4) = 1\). By Lemma 6.1 we have \(\text{ind}(-V', \infty) = 1\). Consequently by Remark 6.2.1 we obtain that \(\text{ind}(-V', \Theta) = -1\). Finally notice that

\[
j_k(V''(\Theta), 2\pi) = \begin{cases}
1 & \text{if } k = 1, 2 \\
0 & \text{otherwise},
\end{cases} \quad j_k(V''(\infty), 2\pi) = j_k(V''(\pm e_4), 2\pi) = \begin{cases}
1 & \text{if } k = 1, 2 \\
0 & \text{otherwise}.
\end{cases}
\]

It is evident that \(\sigma(V''(\Theta)) \cap \{k^2 : k \in \mathbb{N}\} = \{1\}, \det(V''(\Theta)) = 0\) and that \(\sigma(V''(\pm e_4)) \cap \{k^2 : k \in \mathbb{N}\} = \sigma(V''(\infty)) \cap \{k^2 : k \in \mathbb{N}\} = \emptyset\). Moreover,

\[
\text{ind}(-V', \infty) \cdot j_2(V''(\infty), 2\pi) = 1 \cdot 0 = 0 \neq -1 = (-1) \cdot 1 + 1 \cdot 0 + 1 \cdot 0 = \text{ind}(-V', \Theta) \cdot j_2(V''(\Theta), 2\pi) + \text{ind}(-V', e_4) \cdot j_2(V''(e_4), 2\pi) + \text{ind}(-V', -e_4) \cdot j_2(V''(-e_4), 2\pi).
\]

We have just shown that all the assumption of Theorem 5.2.2 are satisfied. Therefore there is at least one non-stationary 2π-periodic solution of system \(6.1\).
Example 6.3. Let us put in \((6.1)\) \(n = 1, V''(\infty) = 0, a = \frac{1}{4}\). Under these assumptions \((6.1)\) is the Sitnikov circular problem studied for example in \[7, 16\]. It is easy to check that \((V')^{-1}(0) = \{0\}, V''(0) = 8, \text{ind}(-V', 0) = \text{ind}(-V', \infty) = -1\). Finally notice that

\[ j_k(V''(0), T) = j_k(8, T) = \begin{cases} 1 & \text{if } k < \frac{T \sqrt{2}}{\pi} \\ 0 & \text{otherwise}. \end{cases} \]

and \(j_k(V''(\infty), T) = j_k(0, T) = 0\) for any \(k \in \mathbb{N}\). Summing up, if \(T > \frac{\pi}{\sqrt{2}}\) then

\[ \text{ind}(-V', 0) \cdot j_1(V''(0), T) = (-1) \cdot 1 = -1 \neq 0 = (-1) \cdot 0 = \text{ind}(-V', \infty) \cdot j_1(V''(\infty), T). \]

We have just shown that all the assumption of Theorem 5.2.2 are satisfied. Therefore there is at least one non-stationary \(T\)-periodic solution of the circular Sitnikov problem for any \(T > \frac{\pi}{\sqrt{2}}\).

In the rest of this section we study continuation of non-stationary \(T\)-periodic solutions of the family of Newtonian systems of the form

\[
\begin{aligned}
\dot{u} &= -V'_\lambda(u) \\
u(0) &= u(T) \\
\dot{u}(0) &= \dot{u}(T)
\end{aligned}
\]

where \(V_\lambda \in C^2(\mathbb{R}^n, \mathbb{R}), \lambda \in \mathbb{R}\) and \(V_0\) is given by formula \((6.2)\).

Example 6.4. Let us consider system \((6.4)\) with \(V_0\) satisfying all the assumptions of potential \(V\) considered in Example 6.1 or in Example 6.2. It is easy to verify that system \((6.4)\) satisfies all the assumptions of Theorem 5.3.1. That is why the set of non-stationary \(2\pi\)-periodic solutions of system \((6.4)\) satisfies the alternative from of Theorem 5.3.1.

Example 6.5. Let us consider system \((6.4)\) with \(V_0\) satisfying all the assumptions of the Sitnikov potential \(V\) considered in Example 6.3. It is easy to verify that system \((6.4)\) satisfies all the assumptions of Theorem 5.3.1 and Corollary 5.3.1. Hence for any \(T > \frac{\pi}{\sqrt{2}}\) there are closed connected sets \(C^\pm\) of non-stationary \(T\)-periodic solutions of system \((6.4)\) with properties \((C1), (C2)\) from Theorem 5.3.1.

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