Electromagnetic field of a charge moving in a waveguide and intersecting a boundary between vacuum and a resonance dispersion medium

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Abstract. Electromagnetic field of a charge that uniformly moves in a circular waveguide from a vacuum into a dielectric possessing frequency dispersion of a resonant type is under consideration. The main attention is paid to the analytical and numerical investigation of the waveguide modes excited by the charge. It is shown that there is a region in the medium where partial compensation of Cherenkov radiation (CR) takes place similar to the case of a nondispersive dielectric. However, the electromagnetic field pattern is complex and the role of resonant dispersion is essential. In the vacuum area of the waveguide, the main part of the field is a response from the resonant medium with frequency which is close to the resonance frequency. This effect might be used for diagnostics of medium properties. The typical plots for the field demonstrate the process of forming the so-called ‘wakefield’ in the dielectric part of waveguide.

1. Introduction

In this paper, the electromagnetic field of a charged particle that moves along the axis of a circular waveguide and flights from a vacuum area into an area filled up with a medium possessing frequency dispersion of a resonant type is under investigation. Note that the case when the charge flights from the medium into vacuum has been considered in paper [1], and such an interesting effect as a single mode Cherenkov-transition radiation (CTR) has been pointed out. This phenomenon is interesting in connection with development of new methods of generation of gigahertz and terahertz radiation [2-4].

The problems with one or two transversal boundaries between a nondispersive dielectric and a vacuum in a waveguide were analyzed in a series of papers [5-9]. In the case of a charge flying from the vacuum area into a dielectric one [5, 7], the CTR (which is Cherenkov radiation (CR) generated in a dielectric and reflected off the boundary) compensates CR (or the so-called wakefield) in some area near the border. This effect is important for the wakefield acceleration technique [10, 11]. However, the medium dispersion that was not taken into account in mentioned works can radically influence the radiation. We consider here the case of a semi-infinite dielectric that possesses a resonant dispersion. The field of a charge in the unbounded resonant medium was analyzed in [12-14]. The case of a regular waveguide filled with this medium was analyzed in [15-17]. These papers show that the role of resonant dispersion is essential in such problems.

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2. Analytical investigation

2.1. General aspects

Let us consider a small charge particle $q$ that uniformly moves with the velocity $\vec{V} = c\beta \hat{z}$ along the $z$-axis of a metal circular waveguide with the radius $a$ and intersects the border ($z = 0$) between two semi-infinite media (figure 1). The medium on the left ($z < 0$) is a vacuum with $\varepsilon_1 = 1$.

![Figure 1. Geometry of the problem.](image)

The nonmagnetic homogeneous isotropic medium on the right from the boundary ($z > 0$) is described by the following frequency-dependent electric permittivity:

$$\varepsilon_2 = \varepsilon_2(\omega) = 1 + \frac{\omega_p^2}{\omega_l^2 - \omega^2},$$

where $\omega_p$ and $\omega_l$ represent a plasma frequency and a resonant frequency, respectively. Note that the permittivity at zero frequency is $\varepsilon_2 = \varepsilon_0 = 1 + \omega_p^2 / \omega_l^2$, and $\varepsilon_2 \rightarrow 1$ at $\omega \rightarrow \infty$. The charge intersects the boundary at the moment $t = 0$.

The general analytical solution to the problem for two arbitrary isotropic homogeneous media is given in the form of a decomposition in infinite series of normal modes [18]. In this paper, we write only expressions for the magnetic field:

$$\vec{H}_{1,2} = \vec{H}_{1,2}^f + \vec{H}_{1,2}^b,$$

where subscript 1 and subscript 2 relate to the areas $z < 0$ and $z > 0$, respectively. The first summand in (2) is the ‘forced’ field, which is the field of a charge in a regular waveguide (we use the terms following V.A. Ginzburg [19]). The forced field is written in the form [5]:

$$H_{\phi,1,2}^f = \frac{2q\beta}{\pi a^2} \sum_{n=1}^\infty \frac{\omega_n J_1(\chi_n r/a)}{J_1^2(\chi_n)} \int_{-\infty}^{\infty} \exp\left[-i(\omega t - z/c\beta)\right] d\omega,$$

where $\omega_n = \chi_n c / a$, $\chi_n$ is the $n$th zero of the Bessel function ($J_0(\chi_n) = 0$).

The second summand in (2) is the ‘free’ field, which is connected with the influence of the boundary. It is written in the form [5]

$$H_{\phi,1,2}^b = \frac{2q\beta}{\pi c a^2} \sum_{n=1}^\infty \frac{\omega_n J_1(\chi_n r/a)}{J_1^2(\chi_n)} B_{n,1,2} \exp\left[i(k_{s,1,2} z - \omega t)\right] d\omega,$$

where

$$B_{n,1,2} = \frac{\omega (\varepsilon_{1,2} - \varepsilon_{2,1})}{(c\varepsilon_{1,2} k_{1,2} + c k_{1,2})} \frac{\omega (\pm \varepsilon_{1,2} \beta^2 + 1 - c \beta k_{1,2})}{\omega^2 (1 - \varepsilon_{1,2} \beta^2) + \beta^2 \varepsilon_{2,1} \omega^2 (\omega \pm c \beta k_{1,2})},$$

$$k_{s,1,2} = \sqrt{\omega^2 \varepsilon_{1,2} - \omega_0^2 / c}, \quad \text{Im} k_{s,1,2} > 0.$$

The forced field was earlier investigated in detail [1, 16-18]. Note that if the condition $\varepsilon_2(\omega) \beta^2 > 1$ is fulfilled, the forced field in the medium includes CR:
\[ H_{\text{CR}}^{\alpha} = \frac{q}{a^2} \sum_{n=1}^{\infty} \frac{\omega_n J_1(\chi_n r/a)}{J_1^2(\chi_n)} A_{\text{CR}}^{\alpha} \sin \left[ \omega_n x_{0n} \left( \frac{z}{c\beta} - t \right) \right] 0(\beta ct - z), \]  

where \[ A_{\text{CR}}^{\alpha} = \frac{2\sqrt{2} \beta^2}{\omega_n \sqrt{1 - \beta^2}} \frac{S + \xi_n - 2(1 - \beta^2)}{S \sqrt{S + \xi_n}}, \]  
\[ x_{0n} = (S + \xi_n)^{1/2}(2 - 2\beta^2)^{1/2}, \]  
\[ y_n = \omega_n / \omega_r = \chi_n / x_r, \]  
\[ x_r = a_0 \omega_r / c, \]  
\[ \xi_n = 1 - \beta^2(e_{0n}^2 + y_n^2), \]  
\[ S = \sqrt{(1 + \xi_n)^2 + 4\beta^2(1 - \beta^2)y_n^2}, \]  
\[ \theta(x) \] is the Heaviside step function. One can see that the frequency of the radiated waves lays within the interval \( \omega_c < \omega < \omega_r \), where \( \omega_c = \omega_r(1 - e_{0n}\beta^2)^{1/2}(1 - \beta^2)^{1/2} \) if \( e_{0n}\beta^2 < 1 \) and \( \omega_c = 0 \) if \( e_{0n}\beta^2 > 1 \). Thus, CR is generated in a medium with resonance dispersion for any velocity of the charge, in contrast to the nondispersive dielectric, when the CR is excited if the charge velocity exceeds the Cherenkov threshold, that is \( e_{0n}\beta^2 > 1 \). Note that the considered case can be reduced to the nondispersive dielectric case with the permittivity \( \varepsilon_0 \) in the limiting process when \( \omega_r \rightarrow \infty, \omega_c \rightarrow \infty \) and \( e_2 \rightarrow \varepsilon_0 \).

The forced field in the vacuum part of the waveguide and in the medium near the charge has a quasi-Coulomb character. It is written in the form

\[ H_{\text{vol},2}^{\alpha} = \frac{q}{a^2} \sum_{n=1}^{\infty} \frac{J_1(\chi_n r/a)}{J_1^2(\chi_n)} A_{\text{vol},2}^{\alpha} \exp \left\{ -\omega_n x_{0n}^{(1/2)}(z/c\beta) - t \right\}, \]  

where \[ x_{0n}^{(1)} = \beta y_n(1 - \beta^2)^{3/2}, \quad x_{0n}^{(2)} = (S - \xi_n)^{1/2}(2 - 2\beta^2)^{1/2}, \]  
\[ A_{\text{vol}}^{\alpha} = 2\beta(1 - \beta^2)^{1/2}, \quad A_{\text{vol}}^{\alpha} = A_{\text{vol}}^{\alpha} y_n \left[ S - \xi_n + 2(1 - \beta^2) \right] S^{-1} \left[ 2(S - \xi_n) \right]^{1/2}. \]  

In subsequent research, the main attention is focused on the free-field components (4). They are analyzed using two methods: an analytical method and a numerical one (they were developed in our papers [1, 5-9, 20] for different problems with sectionally homogeneous waveguides). In this paper, we only provide the most important results.

2.2. The field in the vacuum area

The analytical investigation is based on the complex variable function theory. Asymptotic expressions for the free field components of each mode can be obtained with the steepest decent technique [21], as it was made in our previous papers [5-9, 20]. The first step in such research is to study the integrand singularities in (4) in the complex plane of \( \omega \) which are listed for the field in the vacuum area as follows:

- the branch points of the radical \( k_{z1} = \pm \omega_n - i0; \)
- the branch points of the radical \( k_{z2} \)
- \( \pm \alpha_{0n}^{(1)} = \pm \omega_n - i0; \)
- the poles \( \pm \alpha_{0n}^{(1)} = \pm i\omega_n, \pm \alpha_{0n}^{(2)} = \pm i\omega_n, \) where \( \omega_{0n}^{(1/2)} \) described by equations (10). The singularities placed on the real axis are slightly shifted downward from the axis if small losses are taken into account (figure 2(a) shows the comparative disposition of an initial integration pass and the singularities). Note that, further, we tend these losses to zero.
The branch cuts are defined by the following equations: \( \Re \sqrt{\omega^2 - \omega_n^2} = 0 \), \( \Re \sqrt{\omega^2 c^2 - \omega_n^2} = 0 \), and the initial integration path goes along the upper edge of the cuts. Before using the steepest decent technique, it is convenient to make the following replacement of variables:

\[
\omega = \omega_n \cosh \chi, \quad \sqrt{\omega^2 - \omega_n^2} = \omega_n \sinh \chi.
\]

(12)

This replacement removes the pair of branch points \( \pm \omega_n \). Disposition of singularities and the integration path in a complex plane of \( \omega \) and \( \chi \) are shown in figure 2. There are two saddle points on contour \( \Gamma' \): \( \chi_0 = \chi_0 + i\pi \), where \( \cosh(\chi_0) = ct/R \), \( R = \sqrt{c^2 t^2 - z^2} \). The steepest descent paths (SDPs) consist of two branches (\( \Gamma_0' \) and \( \Gamma_1' \)). The contributions of the saddle points \( \chi_{0,1} \) represent the space transition radiation and the front of TR propagates with the light velocity \( c \). Singularities (in the case under consideration - the branch points \( \chi_n \), which are analogues of \( \pm \omega_n^{(2)} \), \( \pm \omega_i \)) can be crossed in the transformation of contour \( \Gamma' \) into the SDP \( \Gamma_{0,1}' \) passing through the saddle points \( \chi_{0,1} \), and the contributions from the corresponding singularities should be included in asymptotic expressions.

So, if \( \Re \chi_{0,1} > \Re \chi_n^{(2)} \), then there is no intersection of the branch points and the saddle points contributions only give asymptotic expression for the free field in the vacuum part of the waveguide. However, in the case \( \Re \chi_{0,1} < \Re \chi_n^{(2)} \), the saddle points can lie on the edges of the branch cuts, and the branch points \( \chi_n^{(2)} \) are crossed in transformation of \( \Gamma' \) into \( \Gamma_{0,1}' \). If \( \Re \chi_{0,1} < \Re \chi_n^{(2)} \), then the branch cut contributions are added to the free field (figure 2(b)). The branch cut contributions are included into the free field in the area \( |z| < \bar{z} \), \( \bar{z} = ct \bar{\omega}_i^{(1)} / \bar{\omega}_n^{(2)} \) and the frequencies of this part of the field lay within the interval \( [\bar{\omega}_i, \bar{\omega}_n^{(2)}] \), where \( \bar{\omega}_i^{(1)}, \bar{\omega}_n^{(2)} \) described by formulas (11). The contributions from the branch points can be evaluated analytically by the Laplace method [20,21] or numerically by integration along the branch cuts.

![Figure 2](https://example.com/figure2.png)

**Figure 2.** Disposition of singularities of integrands, branch cuts, and the integration paths in a complex plane of (a) \( \omega \) and (b) \( \chi \) for the field in a vacuum (the case of \( \Re \chi_{0,1} < \Re \chi_n^{(2)} \)). The dashed parts of the SDP \( \Gamma_{0,1}' \) are situated in the lower sheet of the Riemann surface.
2.3. The field in the resonant medium

For the free-field component in the resonant medium, analysis of the singularities of integrand in (4) in the complex plane of $\omega$ shows that, in addition to singularities described above, there is one more pair of poles $\pm \omega_{0n}^{(2)} = \omega_n x_{0n}$, where $x_{0n}$ is defined by formula (7). These poles coincide with ones in the forced field (they are connected with CR in the medium). Note that magnitude of $\omega_{0n}^{(2)}$ tends to $\omega$ with increasing in the mode number $n$ and, for some $n$, is decreasing in increase in $\beta$. If $\beta \ll 1$, then $x_{0n} \approx 1 - (\varepsilon_0 - 1)\beta^2 / 2$; if $\beta \equiv 1$, then $x_{0n} = y_n (y_n^2 + \varepsilon_0 - 1)^{1/2} [1 + O(y_n^{-2})]$, where $y = (1 - \beta^2)^{1/2}$.

Disposition of singularities and the integration path in a complex plane of $\omega$ is presented in figure 3. In this situation, it is not convenient to make any change of variables (like (1.2)), and we apply the steepest decent technique directly in the complex plane of $\omega$.

The saddle point analysis shows a complex wave picture because there are three pairs of saddle points with different frequencies $\pm \Omega_n^{(1)}$, $\pm \Omega_n^{(2)}$, and $\pm \Omega_n^{(3)}$ in some domain near the boundary at $z < z_0$. These frequencies can be approximately described by following expressions:

$$\Omega_n^{(1)} \approx \omega_n^{(1)} ct / R, \quad \Omega_n^{(2)} \approx \omega_n^{(2)} ct / R \quad \text{and} \quad \Omega_n^{(3)} \approx \omega_n (ct - z) / R,$$

where $R = \sqrt{c^2 t^2 - z}$. However, there is only one pair of saddle points $\Omega_n^{(2)} \approx \omega_n^{(2)} ct / R$ in the area $z_0 < z < ct$. In figure 4, dependences of the saddle point frequencies on distance are shown. Magnitude $z_0$ can be evaluated numerically as analytical approximation for $\Omega_n^{(3)}$ is not quite correct in this domain.

In addition to the saddle point contributions and the branch cut contributions, the poles $\pm \omega_{0n}^{(2)}$ can contribute into the free field under some condition. These contributions (calculated with the residue theorem) give the CTR, which is described by formula (6) but taken with an opposite sign. So, there is a compensation for the forced field with a part of free one in some domain from the boundary at $z < z_2$. However, it is difficult to estimate the dimension of this zone in general. We managed to find it analytically for rather greater velocities of the charge motion only ($\beta \approx 1$). In this situation, the disposition of the poles $\pm \omega_{0n}^{(2)}$ in close proximity to just one pair of saddle points $\pm \Omega_n^{(1)}$ is only

![Figure 3. Singularities, branch cuts and integration paths in the complex plane of $\omega$ for the field components in the resonant dielectric.](image)

![Figure 4. Frequencies of saddle points at different distances in the medium (solid lines are for numerical results, dot-dashed lines are for analytical approximations): $n = 1$; $x = 5$; $\omega_p = 0.8 \omega_1$.](image)
important (figure 4). Therefore, for $\beta \approx 1$, we have a generalization of the case of nondispersive dielectric [5]:

$$z_2 \approx c t \varepsilon_2^{-1}$$

where the dimension of this zone $z_2$ is determined with the group velocity $v_{g2} \approx c \varepsilon_2^{-1}$ in a regular waveguide.

3. Numerical investigation, results and discussion

In the numerical approach, the exact integral representations (4) are used to calculate the free-field components. The numerical algorithm is based on a certain transformation of the initial integration path in the complex plane of $\omega$ as it was earlier used for the computation of the field in different dispersive unbounded [14] or semi-bounded [22] media including different problems with waveguides [1, 4-9, 20]. First, we rewrite the exact formulas into the integrals between the limits 0 and $+\infty$. Second, we transform the integration path into a new contour in the upper-half plane for $z > ct$ (before the ‘wave front’) and into another contour in the lower-half plane for $z < ct$ (behind the ‘wave front’). These new contours are presented in figure 3 with green dashed line and red dotted line. They bypass all of the singularities and subsequently go parallel to the SDP. Note that calculations can be optimized with matching of parameters of new contours.

The results obtained are presented in figure 5. Ammonia [23] is considered to be a typical medium; it has a resonance of 23.87 GHz and nondimensional value $x_i = 5$ (refer to expression (8)) at $a \approx 1$ cm. Figure 5 shows the component $H_\phi$ of the first mode of the total field in the vacuum area and in the resonant medium for different values of velocity $\beta$ at different time moments. The forced field (6) and (9) in the medium and the branch cuts contributions (evaluated numerically) in a vacuum are also presented.

In the vacuum part of the waveguide ($z < 0$) the free field consist of TR and the branch cuts contributions. As can be seen, the branch cuts contributions are the main part of the field in a vacuum especially for relatively greater velocities (figure 5(d)-(i)). Therefore, the branch cuts contributions give a response from the resonant medium with frequencies being about resonance frequency. The maximum amplitude of this response is comparable with the amplitude of CR in the medium. The forced field, which consists of CR (wakefield) and a quasi-Coulomb field, form practically the total field in some area after the charge propagation $c \varepsilon_2^{-1} > z < c \beta t$. However, there is also domain $z < z_2$ where the influence of the boundary is principal. In this domain, the part of the free field (the CTR) compensates the wakefield (CR) as well as in the case of nondispersive dielectric. However, for relatively low velocity of the charge motion (figure 5(a)-(c)), the compensation is not quite visible because of the effect of TR. Note that the dimension of the compensation area is smaller.
if \( \varepsilon_2 \) takes on a larger value (14). It might be achieved, for example, for greater mode number when the CR frequency tends to the resonant frequency, and \( \varepsilon_2\mid_{(12)} \) is considerably increasing.

\[
\omega_0 (n - 1)^2 \approx \frac{2}{\varepsilon_2}.
\]

\[\frac{a z}{a} = 10, \quad \beta = 0.5, \quad \beta = 0.9, \quad \gamma = 100, \quad \omega_0 \approx 0.8\omega_c, \quad a = 1\text{cm}.\]

\[
\text{Sighting point is } r = 0.5a.
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Dependence of normalized component \( \tilde{H}_\phi = H_\phi a^2 q^{-1} \) of the first mode of the total field (red solid line 1) on distance \( z/a \) for different dimensionless times \( ct/a \) and velocities \( \beta \) (or \( \gamma = (1 - \beta^2)^{-1/2} \)). The branch point contributions (blue dashed line 2) and the forced field (green dot-dashed line 3) are presented as well; \( n = 1; \ x_z = 5; \ \omega_0 = 0.8\omega_c; \ a = 1\text{cm}. \) Sighting point is \( r = 0.5a. \)
\end{figure}

Note that, for ultrarelativistic particle (figure 5(g)-(i)), the TR increases and concentrates near the front which propagates with the light velocity \( c \) [20].

4. Conclusion

We have analyzed the electromagnetic field of a small charged bunch that moves in a circular waveguide and intersects the boundary between a vacuum and a resonant medium. The analysis of the field components has been performed both analytically and numerically. The analytical investigation has been based on the steepest descent method. The algorithm for the computation has been also presented.

It has been shown that, in any cross-section in the vacuum area, the radiation field consists of two parts. The first one is the single-frequency field (the contribution of the saddle points), and the second one is the field having certain limited continuous spectrum lying near the resonance frequency (the
contribution of the branch cuts). It is interesting that, as a rule, the second part is predominant. This result might be used for diagnostics of material properties.

In the resonant medium, there is the area after the bunch where the wave field practically coincides with the wakefield in the regular waveguide (the ‘wakefield area’) similar to the case of nondispersive dielectric. There is also the domain where the boundary influence is principal, the total field is relatively small, and wave picture is rather complex (the ‘compensation area’). Note that the compensation area is decreasing and, correspondingly, the wakefield area is increasing with increase in the mode number and the velocity of the bunch motion. The results obtained can be useful for wakefield acceleration technique.

Acknowledgments
This research was supported by the Russian Foundation for Basic Research (grant 15-02-03913a).

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