Pricing and hedging barrier options in a hyper-exponential additive model

Marc Jeannin†,* and Martijn Pistorius*

†Models and Methodology Group, Risk Management Department
Nomura International plc
Nomura House 1 St Martin’s-le-Grand, London EC1A 4NP, UK
E-mail: marc.jeannin@nomura.com

*Department of Mathematics, Imperial College London
South Kensington Campus, London SW7 2AZ, UK
E-mail: martijn.pistorius@imperial.ac.uk

Abstract

In this paper we develop an algorithm to calculate the prices and Greeks of barrier options in a hyper-exponential additive model with piecewise constant parameters. We obtain an explicit semi-analytical expression for the first-passage probability. The solution rests on a randomization and an explicit matrix Wiener-Hopf factorization. Employing this result we derive explicit expressions for the Laplace-Fourier transforms of the prices and Greeks of barrier options. As a numerical illustration, the prices and Greeks of down-and-in digital and down-and-in call options are calculated for a set of parameters obtained by a simultaneous calibration to Stoxx50E call options across strikes and four different maturities. By comparing the results with Monte-Carlo simulations, we show that the method is fast, accurate, and stable.

Keywords: Hyper-exponential additive processes, matrix Wiener-Hopf factorization, first passage times, barrier options, multi-dimensional Laplace transform, Fourier transform, sensitivities.

Acknowledgements: We would like to thank P. Howard and S. Obraztsov for their support, and also D. Madan for useful conversations. This research was supported by EPSRC grant EP/D039053, and was partly carried out while the authors were based at King's College London.
1 Introduction

Barrier options are contracts whose pay-offs are activated or de-activated when the underlying process crosses a pre-specified level. These contracts are among the most popular path-dependent options. To value barrier options, a model needs to be sufficiently flexible to calibrate call option prices at different strikes and maturities. However, it is desirable to maintain a degree of analytical tractability to facilitate the calculations, especially for the Greeks or the sensitivities. These sensitivities describe the change in the model price with respect to a change in the underlying parameter, and are important for an appreciation of the robustness of the model’s results. It is well known that the accurate evaluation of the Greeks is a challenging numerical problem, since standard PDE or Monte-Carlo methods are generally slow and unstable.

It is well established that the geometric Brownian motion model lacks the flexibility to capture features in financial asset return data such as the skewness and the excess kurtosis. It cannot calibrate simultaneously to a set of call option prices. To address these limitations, one of the approaches consists of introducing jumps in the price process by replacing the Brownian motion by a Lévy process. Lévy models, such as the VG, CGMY, NIG, KoBoL, generalised hyperbolic, and Kou’s double exponential model, have been successfully applied to the valuation of European-type options. We refer to Cont and Tankov [14], Boyarchenko and Levendorskii [7], and Schoutens [30] for background and references on the application of Lévy models in option pricing.

As observed by many authors, such as Eberlein and Kluge [16], or Carr and Wu [11], Lévy models are generally not capable of calibrating option prices simultaneously across strikes and maturities. Empirical studies of S&P500 index data by Carr and Wu [11], and Pan [26], show that the implied jump intensities and the implied jump size distributions vary greatly over time. The prices of short-dated options exhibit a significantly larger risk-premium than that of long-dated options. This is reflected in the thicker tails of the implied marginal risk-neutral distributions, especially at short maturities. For example, in the equity markets, short-dated out-of-the-money put options are relatively expensive since the risk of a large negative jump in the share is priced. Because of the stationarity and independence of the increments of a Lévy process, the moments exhibit a rigid term structure that is different from what is observed in market data. This lack of flexibility can be overcome by considering models driven by additive processes, which have independent and time-inhomogeneous increments.

Additive models have been used for equity option pricing by Carr et al. [10], Galloway and Nolder [17], and by Eberlein and Kluge [16] for interest rate option pricing. Motivated by modelling considerations, Carr [10] proposed a self-similar additive model for the log-price, and reported good calibration results across time. Galloway and Nolder [17] carried out a calibration study for various related models. Eberlein and Kluge [16] constructed an HJM model driven by an additive process with continuous characteristics, and they obtained a good fit for swaptions by using piecewise constant parameters.
In this paper we follow a similar approach: we model the share price by an additive process with hyper-exponential jumps. Hyper-exponential distributions are finite mean-mixtures of exponential distributions which can approximate monotone distribution arbitrarily closely. As first observed by Asmussen et al. [4], most of the popular Lévy models used in mathematical finance possess completely monotone Lévy densities and can therefore be approximated well by hyper-exponential Lévy models. A hyper-exponential additive model is sufficiently flexible to allow for an accurate calibration to European option prices across strikes and multiple maturities. In addition, if the parameters are piecewise constant, the model admits semi-analytical expressions for prices and Greeks of barrier options.

There currently is a body of literature devoted to various aspects of pricing barrier options. In the setting of Lévy models, a transform-based approach to price barrier options has been developed in a number of papers, including Geman and Yor [18], Kou and Wang [21], Davydov and Linetsky [15], Boyarchenko and Levendorskii [8]. In particular, Kou and Wang [21], Kou et al. [22], Sepp [29], Lipton [24], and Jeannin and Pistorius [19] considered the cases of Lévy processes with double-exponential and hyper-exponential jumps.

In this paper, the transform algorithm that we develop is based on a so-called matrix Wiener-Hopf factorization. Such matrix factorizations were first studied by London et al. [25] and Rogers [27] for (noisy) fluid models. Jiang and Pistorius [20] developed matrix-Wiener factorization results for regime-switching models with jumps. We show that by suitably randomizing the parameters the distributions of the infimum and supremum of the randomized hyper-exponential additive process can be explicitly expressed in terms of a matrix Wiener-Hopf factorization. We use these results to derive semi-analytical expressions for the first-passage time probabilities, for the prices, and for the Greeks of barrier options, up to a multi-dimensional transform. The actual prices are subsequently obtained by inverting this transform.

As a numerical illustration, we calibrate the hyper-exponential additive model to Eurostoxx prices quoted on 27 February 2007 at four different maturities. We calculate in this setting down-and-in digital and down-and-in call option prices and Greeks (delta and gamma). To invert the transform, we use a contour deformation algorithm and a fractional Fast Fourier Transform algorithm, developed by Talbot [31], Bailey and Swarztrauber [6], and Chourdakis [12], [13]. We also compare it to Monte-Carlo Euler scheme simulations. We find that the algorithm is accurate and stable, and much faster than Monte-Carlo simulations (especially for the Greeks). This method is suitable for applications in which the number of periods is not too large (up to four). When a larger number of periods is required, the direct inversion method used here is no longer feasible. The subject still needs to be further investigated and is left for future research.

The remainder of the paper is organized as follows. In Section 2 we define the hyper-exponential additive model and present its application to European call option pricing. In Sections 3 and 4 we derive semi-analytical expressions for the first-passage probabilities of a hyper-exponential additive process in terms of a matrix Wiener-Hopf factorisation, and for the prices and Greeks of barrier options. In Section 5 we present numerical results.
2 The model

2.1 Additive processes

We consider an asset price process $S$ modelled as the exponential

$$S_t = S_0 e^{X_t}$$

of an additive process $X$. Informally, an additive process can be described as a Lévy process with time-dependent characteristics or, equivalently, as a process with independent but non-stationary increments. We briefly review below some key properties of additive processes. For further background on additive processes and their applications in finance, we refer to Sato [28], and to Cont and Tankov [14]. An additive process can be defined more formally as follows.

**Definition 1** For a given $T > 0$, $X = \{X_t, t \in [0, T]\}$ is an additive process if

(i) $X_0 = 0$,

(ii) For any finite partition $0 \leq t_0 < t_1 \cdots < t_k \leq T$, the random variables $X_{t_k} - X_{t_{k-1}}, \cdots, X_{t_0}$ are independent,

(iii) The sample paths $t \mapsto X_t$ have càdlàg modifications almost surely.

If $X$ is an additive process, then, for every $t \in [0, T]$, $X_t$ has an infinitely divisible Lévy distribution with Lévy triplet $(M_t, \Sigma_t^2, \Lambda_t)$; that is, the characteristic function of $X_t$ is given by $\Phi_t(u) = \exp[\Psi_t(u)]$. According to the Lévy-Khintchine formula, $\Psi_t$ is the characteristic exponent given by

$$\Psi_t(u) = iuM_t - \frac{\Sigma_t^2}{2}u^2 + \int_{-\infty}^{\infty} \left\{ e^{iux} - 1 - iux1_{\{|x|<1\}} \right\} \Lambda_t(dx),$$

with $M_t, \Sigma_t \in \mathbb{R}$, and where $\Lambda_t$ the Lévy measure satisfies the integrability constraint

$$\int (1 \wedge x^2)\Lambda_t(dx) < \infty.$$

The law of the additive process $\{X_t, t \in [0, T]\}$ is determined by the collection of Lévy triplets $\{(M_t, \Sigma_t^2, \Lambda_t)\}$ for $t \in [0, T]$. If the Lévy triplets are time-independent, $X$ is a Lévy process. If the additive process has absolutely continuous characteristics, the Lévy triplets take the explicit form

$$M(t) = \int_0^t \mu(s)ds, \quad \Sigma(t) = \int_0^t \sigma^2(s)ds,$$

$$\Lambda_t(B) = \int_0^t \int_B g(s, x)dxds \quad \text{for Borel sets } B,$$

where $\mu, \sigma^2 : [0, T] \to \mathbb{R}$ and $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ are integrable functions, with $g$ and $\sigma^2$ non-negative. We call the functions $(\mu, \sigma^2, g)$ the local triplet of $X$. 

3
We assume that we have been given deterministic integrable functions \( r(t) \) and \( d(t) \) representing the short rate and the dividend yield, and that the characteristic exponent of \( X_t \) satisfies

\[
\Psi_t(-i) = \int_0^t [r(s) - d(s)] ds,
\]

or equivalently

\[
\mu(t) + \frac{\sigma^2(t)}{2} + \int_{-\infty}^{\infty} \left[ e^x - 1 - 1_{\{x < 1\}} x \right] g(t, x) dx = r(t) - d(t).
\]

It follows that the discounted process

\[
S_0 \exp \left( \int_0^t [r(s) - d(s)] ds \right)
\]

is a martingale if and only if (1) (or, equivalently, (2)) holds.

### 2.2 Hyper-exponential additive processes

In what follows we restrict the discussion to a hyper-exponential additive process \( X \) which is specified by its local triplet \((\mu, \sigma^2, g)\) where \( g \) is given by

\[
g(t, x) = \sum_{k=1}^{n^+} \pi^+_k(t) \alpha^+_k(t) e^{-\alpha^+_k(t) x} 1_{\{x > 0\}} + \sum_{j=1}^{n^-} \pi^-_j(t) \alpha^-_j(t) e^{-\alpha^-_j(t) |x|} 1_{\{x < 0\}},
\]

where \( \pi^\pm_k(t) \) and \( \alpha^\pm_k(t) \) are non-negative. The continuous part of \( X \) consists of a diffusion with time-dependent drift \( \mu(t) \) and volatility \( \sigma(t) \). The jump part of \( X \) is of finite activity and forms an inhomogeneous compound Poisson process where positive and negative jumps occur at the rates

\[
\lambda^+(t) := \sum_{k=1}^{n^+} \pi^+_k(t) \quad \text{and} \quad \lambda^-(t) := \sum_{j=1}^{n^-} \pi^-_j(t),
\]

and jump sizes are distributed according to a hyper-exponential distribution. Small random price movements are intuitively modelled by the diffusion part, whereas sudden changes of the price are captured by the jump-part of \( X \). If we take \( n^\pm = 1 \), the jump-sizes are exponentially distributed, and this model reduces to an extension of the Kou model with time-dependent parameters.

### 2.3 Piecewise constant parameters

To reduce the dimension of the available parameter set, we take the functions \( \mu(t), \sigma(t) \) and \( g(t, \cdot) \) to be piecewise constant. Given that we have a finite set of European call options with different maturities \( T_1, \ldots, T_N \), we take the local parameters to be constant between the different maturities \( T_i \). Then for all \( t \in (T_{i-1}, T_i) \), (with \( T_0 = 0 \)) we set

\[
\mu(t) = \mu^{(i)}, \quad \sigma^2(t) = \sigma^{2(i)}, \quad g(t, x) = g^{(i)}(x), \quad i = 1, \ldots, N.
\]
For $t \in (T_{i-1}, T_i]$ the characteristic exponent of $X_t - X_{T_{i-1}}$ is given by

$$\Psi_{T_{i-1},t}(u) =: \Psi^{(i)}(u),$$

where

$$\Psi^{(i)}(u) = \mu^{(i)} u - \frac{\sigma^{2(i)} u^2}{2} + \sum_{k=1}^{n^+} \pi^{(i)}_k \left( \frac{u}{\alpha^{(i)}_k - u} \right) - \sum_{j=1}^{n^-} \pi^{(i)}_j \left( \frac{u}{\alpha^{(i)}_j + u} \right). \quad (4)$$

### 3 First passage probabilities

The value of a digital barrier option can be expressed in terms of the distribution

$$F^+(x; T) = P(X_T \leq x)$$

of the running supremum

$$\overline{X}_T = \sup_{s \leq T} X_s$$

of $X$, or equivalently, the distribution of the first-passage time

$$T^+(x) = \inf \{ t \geq 0 : X_t > x \}$$

which is related to $F^+(x; T)$ by

$$P(T^+(x) \leq T) = 1 - F^+(x; T).$$

Whereas for a Lévy process the distributions of the infimum and supremum are linked to the characteristic exponent by the so-called Wiener-Hopf factorization, such a result does not exist for general additive processes, because of the time-dependence of the parameters. However, in the case of piecewise constant parameters, the triplet changes only at deterministic times, so that as a consequence the distribution function $F^+(x) = F^+(x; T^{(1)}, \ldots, T^{(N)})$ only depends on the inter-jump times $T^{(i)} = T_i - T_{i-1}$ (with $T_0 = 0$). In this case, as we show below, the $N$-dimensional Laplace transform $G^+(x, \mathbf{q})$ of $F^+(x)$, given by

$$G^+(x, \mathbf{q}) = \int e^{-(q_1 u_1 + \cdots + q_N u_N)} F^+(x; u_1, \ldots, u_N) du_1 \cdots du_N,$$

where $\mathbf{q} = (q_1, \ldots, q_N)$, is expressed explicitly in terms of a matrix Wiener-Hopf factorization. To state this result we need to introduce some further notation.

For any vector $\mathbf{v} = (v_1, \ldots, v_n)$, we denote by $\Delta_\mathbf{v}$ the diagonal matrix $\Delta_\mathbf{v} = (v_i, i = 1, \ldots, n)_{\text{diag}}$. Let $Q$ be the $N(1+n^++n^-) \times N(1+n^++n^-)$ matrix given in block notation by

$$Q = \begin{pmatrix} H^+ & D^- \\ C^- & T^- \end{pmatrix}, \quad (5)$$

where

$$H^+ = \begin{pmatrix} G - \Delta \lambda & b^+ \\ t^+ & T^+ \end{pmatrix}, \quad (6)$$
Here $\lambda = (\lambda_i^+ + \lambda_i^-, i = 1, \ldots, N)$, and $G$ and $b^+$ are the $N \times N$ and $N \times N n^+$ matrices in block notation given by

$$G = \begin{pmatrix} -q_1 & q_1 & q_2 & \cdots & -q_N \\ -q_2 & q_2 & \cdots & \cdots & -q_N \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -q_{N-1} & q_{N-1} & \cdots & \cdots & -q_N \\ -q_N & -q_N & \cdots & \cdots & -q_N \end{pmatrix}, \quad b^+ = \begin{pmatrix} \pi^{+(1)} \\ \pi^{+(2)} \\ \vdots \\ \pi^{+(N)} \end{pmatrix}. \quad (7)$$

Here $\pi^{+(i)}$ is the row-vector $\pi^{+(i)} = (\pi_i^{+(i)}, i = 1, \ldots, n)$, and where $t^+, T^+$ are given by

$$T^+ = \begin{pmatrix} -\Delta \alpha^+ & -\Delta \alpha^+ & \cdots & -\Delta \alpha^+ \\ -\Delta \alpha^+ & -\Delta \alpha^+ & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ -\Delta \alpha^+ & \cdots & \cdots & -\Delta \alpha^+ \end{pmatrix}, \quad t^+ = \begin{pmatrix} \alpha^+ \\ \alpha^+ \\ \vdots \\ \alpha^+ \end{pmatrix}, \quad (8)$$

where $\alpha^+$ is the column-vector $\alpha^+ = (\alpha_i^+, i = 1, \ldots, n^+)'$, and

$$C^- = (t^- \ O^-), \quad D^- = \begin{pmatrix} b^- \\ (O^+) \end{pmatrix}, \quad (9)$$

where $O^\pm$ are $n^+ N \times n^+ N$ zero matrices, and $b^-, T^-, \text{ and } t^-$ are given by (7) and (8) with $\pi^{+(i)}$ and $\alpha^+$ replaced by $\pi^{-(i)}$ and $\alpha^-$. The matrix $Q$ is a generator matrix, that is, a square matrix with non-negative off-diagonal elements and non-positive row sums, and defines a Markov chain. This Markov chain is associated to a randomization and embedding of the additive process $X$ (which will be illustrated with a concrete example below). We recall that a sub-probability matrix is a matrix with non-negative elements and row sums not larger than one. By applying the matrix Wiener-Hopf factorization results of Jiang and Pistorius [20] to the current setting we arrive at the following conclusion.

**Theorem 1** It holds that

$$G^{(+)}(x, q) = \frac{1}{q_1 \ldots q_N} \times \left[ 1 - e'_1 e^{Q^+ x} 1 \right],$$

where

$$e'_1 = (1, 0, \ldots, 0) \quad \text{and} \quad 1 = (1, \ldots, 1)'.$$
Here the O’s are zero matrices of appropriate sizes, and in block notation we have,

\[ S^2 = \begin{pmatrix} \Delta \sigma^2 \\ O^+ \end{pmatrix}, \quad V^+ = \begin{pmatrix} +\Delta \mu \\ I^+ \end{pmatrix}, \]

(12)

with

\[ \sigma^2 = (\sigma_i^2, i = 1, \ldots, N), \quad \mu = (\mu_i, i = 1, \ldots, N). \]

\( O^+ \) and \( I^+ \) represent \( n^+ N \times n^+ N \) zero and identity matrices, respectively.

By applying Theorem 1 to \(-X\), we find the corresponding pair of matrices \((Q_-, \eta_-)\). The quadruple \((Q_+, \eta_+, Q_-, \eta_-)\) is called a matrix Wiener-Hopf factorization of \(Q\).

**Example.** To illustrate this approach, we consider a hyper-exponential additive process \(X\) on \([0, T_2]\) whose parameters are constant during the periods \([0, T_1]\) and \([T_1, T_2]\). In the first period \(X\) evolves as a jump-diffusion with positive and negative exponential jumps with means and jump rates \(1/\alpha^+, \lambda^+\) and \(1/\alpha^-, \lambda^-\). In the second period \(X\) is a Brownian motion with drift. The idea is to randomize the times between maturities by replacing \(T(1) = T_1\) and \(T(2) = T_2 - T_1\) with independent exponential random variables having means \(q_1^{-1}\) and \(q_2^{-1}\). This results in a regime-switching jump-diffusion with the regime only jumping from state 1 to state 2, according to the generator matrix

\[ G = \begin{pmatrix} -q_1 & q_1 \\ 0 & -q_2 \end{pmatrix}. \]

We associate to the regime-switching process a continuous Markov additive process, which can be informally obtained by replacing positive and negative jumps with stretched slopes of +1 and −1 (see Asmussen [3] for background on this embedding). As described in [20], in this case the generator of the modulating Markov process is given by

\[ Q = \begin{pmatrix} H^+ \\ C^- \end{pmatrix} \begin{pmatrix} D^- \\ T^- \end{pmatrix} = \begin{pmatrix} -q_1 - \lambda^+ - \lambda^- & q_1 & \lambda^+ \\ 0 & -q_2 & 0 \\ \alpha^+ & 0 & -\alpha^+ \end{pmatrix}, \]

with the matrices \(S^2\) and \(V^+\) in Theorem 1 given by

\[ S^2 = \begin{pmatrix} \sigma_1^2 \\ \sigma_2^2 \\ 0 \end{pmatrix}, \quad V^+ = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}. \]

### 3.1 Solution of the matrix equation

To solve the system (11), which is a Ricatti-type matrix equation, we follow a spectral approach and determine the spectral decomposition of \(Q^+\). Denoting by \(h(\rho)\) a
Figure 1: Paths of the various processes related to log-price process $X$ are illustrated. Here $X$ is a hyper-exponential additive process on the period $[0, T_2]$ whose parameters are constant during the periods $[0, T_1]$ and $[T_1, T_2]$. During the first period $[0, T_1]$, the process $X$ evolves as a jump-diffusion with volatility $\sigma_1$, drift $\mu_1 = 0$ and exponentially-distributed jumps. The positive and negative jumps are exponentially-distributed with means and jump rates $1/\alpha^+, \lambda^+$ and $1/\alpha^-, \lambda^-$, respectively. During the second period $[T_1, T_2]$, $X$ evolves as a linear Brownian motion with volatility $\sigma_2$ and drift $\mu_2 = 0$. Associated to $X$ is a continuous Markov additive process $A$, which can be obtained from $X$ by replacing the positive and negative jumps with linear stretches of path of slopes $+1$ and $-1$, and replacing the fixed times $T_1, T_2 - T_1$ by independent exponential random times $e_1$ and $e_2$ with parameters $q_1, q_2$. The process $Y$ that records the current state or regime of $A$ is a Markov process with generating matrix $Q$. When $Y$ takes values 1 and 2, $A$ evolves as a linear Brownian motion with zero drift and volatility $\sigma_1$ and $\sigma_2$ respectively; and when $Y$ is 3 and 4, $A$ is a positive or negative unit drift. These linear stretches of paths of $A$ originate from the jumps of $X$. A jump of $Y$ from one state to another is induced either by a jump of $X$ or by a switch of the set of parameters that determine the dynamics of $X$. By time-changing $A$ by the time $T_0(t)$ up to time $t$ that $Y$ was equal to 1 and 2, we recover a regime-switching jump-diffusion; that is, the process $\{A(T_0(t)), t \geq 0\}$ is in law equal to a regime-switching jump-diffusion. Finally, replacement of the times at which a regime switch occurs by $T_1$ and $T_2$ results in a process that has the same distribution as $X$. 
(column) eigenvector of $Q^+$ corresponding to the eigenvalue $\rho$, one finds that it is a matter of algebra to verify that the system (11) can be equivalently rewritten as
\[
\frac{1}{2} \tilde{S}^2 \left( \begin{array}{c} I \\ \eta^+ \end{array} \right) Q^2_+ - \tilde{V} \left( \begin{array}{c} I \\ \eta^+ \end{array} \right) Q_+ + Q \left( \begin{array}{c} I \\ \eta^+ \end{array} \right) = O.
\]
Here $O$ is a $N(1 + n^+ + n^-)$ square zero matrix, $I$ is an $N(1 + n^+)$ identity matrix, and
\[
\tilde{S}^2 = \begin{pmatrix} \Delta \sigma^2 \\ O^+ \\ O^- \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} \Delta \mu \\ I^+ \\ -I^- \end{pmatrix}.
\]
Defining the matrix $K(s)$ by
\[
K(s) = \frac{s^2}{2} \tilde{S}^2 + s \tilde{V} + Q,
\]
we find that $h(\rho)$ solves the linear system
\[
K[\rho] \left( \begin{array}{c} I \\ \eta^+ \end{array} \right) h(\rho) = 0,
\]
which implies that $\rho$ is a root of the equation $\det K(s) = 0$. The following result characterizes the eigenvalues of $Q^+$ (see Appendix A):

**Lemma 1**

(i) It holds that
\[
| \det(K(s)) | = \prod_{i=1}^{N} \left| \Psi^{(i)}(-is) - q_i \prod_{k=1}^{n^+} (s - \alpha_k^+) \prod_{l=1}^{n^-} (s + \alpha_l^-) \right|,
\]
where $\Psi^{(i)}$ is given in (4).

(ii) The equation
\[
\det K(s) = 0
\]
has $N(1 + n^+)$ positive roots and $N(1 + n^-)$ negative roots.

Since $-Q^+$ is the negative of a generator matrix, it is non-negative definite, so its eigenvalues are non-negative and are given by the positive roots of (15). In particular, if the positive roots
\[
\rho_+ = (\rho_i^+: i = 1, \ldots, N(n^+ + 1))
\]
of equation (15) are distinct, it follows from Lemma 1 and Theorem 1 that
\[
G^{(+)}(x, q) = (q_1 \cdots q_N)^{-1} \times [1 - e_i^t U_+ e^{-\Delta \rho_i^+ x} U_+^{-1} 1],
\]
where $U_+ = (h(\rho_i^+), i = 1, \ldots, N(n^+ + 1))$. 

9
3.2 The final position and the first exit time

The valuation of barrier options involves the joint distribution of the final position at maturity $T$ and the first exit time. We will extend the results in the previous section by considering the following:

$$F^+(x, s) := E[e^{sX_T}1_{\{X_T > x\}}] = E[e^{sX_T}1_{\{T+(x) < T\}}].$$

$F^+(x, s)$ depends on time only through the inter-maturity times $(T^{(1)}, \ldots, T^{(N)})$.

The Laplace transform $G^+(x, s, q)$ of $F^+(x, s)$ in $(T^{(1)}, \ldots, T^{(N)})$, can be expressed in terms of $Q^+$ and $K(s)$ as follows:

**Proposition 1** It holds that

$$G^+(x, s, q) = e^{sx}e^q \times e^{x} K(s)^{-1} K(0) 1$$

for all $s \in \mathbb{C}$ with Re($s$) $\in (- \min_{j=1,\ldots,n^-} \alpha_j^-, \min_{k=1,\ldots,n^+} \alpha_k^+)$.

A proof is given in Appendix A.

3.3 First passage to a lower level

The form of the analogous distributions concerning the infimum

$$F^-(x) = P(-X_T \leq x), \quad F^-(x, s) = E[e^{sX_T}1_{\{-X_T > x\}}], \quad x > 0,$$

can be found by applying the results in the previous section to the process $-X$. More specifically, it is straightforward to check that the $N$-dimensional Laplace transforms $G^-(x, q)$ and $\overline{G}^-(x, s, q)$ are given by (10) and (16) replacing $Q^+$ by $Q^-$. $(Q^-, \eta^-)$ satisfies the system of matrix equations (11) with $V^+$, $H^+$, $D^-$, $C^-$, $T^-$ replaced by $V^-$, $H^-$, $D^+$, $C^+$, $T^+$, where the latter set is defined by interchanging $+$ and $-$ in equations (12), (9), (8) and (6). It is straightforward to verify that (i) an eigenvector $h(\rho)$ of $Q^-$ corresponding to eigenvalue $\rho$ satisfies

$$K[\rho] \left( \begin{array}{c} \eta^- \\ I \end{array} \right) h(\rho) = 0,$$

where $I$ is an $N(1 + n^-)$ identity matrix, and (ii) that, in view of Lemma (1) the eigenvalues of $Q^-$ are given by the negative roots

$$\rho^- = \left( \rho_j^-, j = 1, \ldots, N(n^- + 1) \right)$$

of det $K(\rho) = 0$.  

4 Prices and Greeks of digital and barrier options

Using the first-passage results from the previous section we derive semi-analytical expressions for the prices and sensitivities of down-and-in digital and knock-in call options. A down-and-in digital option at level \( H < S_0 \) is a contract that pays out one unit at maturity \( T \) if the price \( S \) has down-crossed the level \( H \) before \( T \). Similarly, a down-and-in call option at level \( H < S_0 \) and with strike \( K \) is a call option whose payoff is activated once \( S \) down-crosses \( H \). Taking the risk-free rate \( r \) and the dividend rate \( d \) to be constant, the arbitrage free prices of a down-and-in digital (DIC) and a call option (DIC) are given respectively by

\[
DIC(T, H, S_0) = e^{-(r-d)T}E\left[ 1_{\{\inf_{t \leq T} S_t < H\}} \right] = e^{-(r-d)T}P(X_T < h),
\]

where \( h = \log(H/S_0) \) is the log-barrier, and

\[
DIC(T, H, K, S_0) = e^{-(r-d)T}S_0E\left[ (e^{X_T} - e^K) + 1_{\{X_T < h\}} \right],
\]

where \( k = \log(K/S_0) \) denotes the log-strike. Let \( \hat{DID}(q) \) denote the joint Laplace transform of \( DID \) in the inter-maturity times \( (T^{(1)}, \ldots, T^{(N)}) \) (with \( T^{(N)} = T \)), and denote by \( \hat{DIC}^*(q, s) \) the Laplace-Fourier transform in \( (T^{(1)}, \ldots, T^{(N)}) \) and in the log-strike \( k \). Then we have the following result:

**Proposition 2** For \( h = \log(H/S_0) < 0 \) it holds that

\[
\hat{DID}(q) = \frac{1}{c(q)} \times e^q e^{qh} 1,
\]

\[
\hat{DIC}^*(q, s) = \frac{S_0 e^{b k}}{c(q) b(b-1)} \times e^q e^{qh} K(s)^{-1} K(0) 1,
\]

where \( k = \log(K/S_0) \), \( q = (q_1, \ldots, q_N) \), and \( b = \alpha + is + 1 \), \( c(q) = (q_1 + r) \cdots (q_N + r) \).

Before we give the proof we observe that from the explicit expressions [17] and [18] semi-analytical formulas can be obtained for the delta and gamma of the down-and-in digital and call options (i.e. the first and second derivatives of the option value with respect to the spot \( S_0 \)). Indeed, the derivatives of the expressions [17] and [18] with respect to \( S_0 \) are equal to the Laplace-Fourier transforms of the derivatives of the option, as integration and differentiation are interchangeable in this case. In the case of a down-and-in digital option we find that the Laplace transforms \( \hat{\Delta DID} \) and \( \hat{\Gamma DID} \) and gamma \( \Delta DID \) and gamma \( \Gamma DID \) are given by

\[
\hat{\Delta DID}(q) = -\frac{1}{c(q) S_0} \times e^q e^{qh} 1, \quad \hat{\Gamma DID}(q) = \frac{1}{c(q) S_0^2} \times e^q [(Q^{-2} + Q^{-1}) e^{qh}] 1.
\]

**Proof of Proposition 2**: The expression [17] is a direct consequence of Theorem [1] (see also Section 3.3). To verify [18] we start by taking the Fourier transform in \( k \) and find as in [21] that the Fourier transform \( DIC^* \) is given by

\[
DIC^*(\alpha + is) = \frac{F(-h, b)}{b(b-1)}.
\]
where \( b = \alpha + is + 1 \) and

\[
\overline{F}^{(-)}(x, b) = \mathbb{E}[e^{bx} 1_{\{X_T \geq x\}}].
\]

From Proposition 1 we deduce that the form the joint Laplace transform \( \overline{G}(x, b, q) \) of \( \overline{F}^{(-)}(x, b) \) in \((T^{(1)}, \ldots, T^{(N)})\) is given by

\[
\overline{G}(x, b, q) = e^{bx} c(q) \times e^{\frac{\partial}{\partial q} (Q^{-h} K(s))} K(0). \tag{20}
\]

Combining (19) and (20) completes the proof. \( \Box \)

5 Numerical results

5.1 Calibration

To determine a parameter set to test the method, we calibrate the hyper-exponential additive model to Eurostoxx call options at four different maturities, observed in the market on 20 February 2007. The spot price is EUR 4150, the risk-free rate is assumed to be fixed at \( r = 0.03 \), and the dividend rate is taken to be zero. As we find that inclusion of positive jumps does not substantially improve the calibration results, we only consider negative jumps, and we specify the jump size parameters to be \((\alpha_1 - 1, \alpha_2 - 2) = (3, 10)\). The jump arrival rates \( \pi^- \) and the volatility are piecewise constant in time, and are estimated by minimizing the root-mean-square error between model and observed market call prices. Using the well known Fourier transform method (briefly recalled in Appendix B) the calibration is carried out maturity by maturity under constraints through a bootstrapping method with well-defined local triplets:

i Calibrate call prices at \( T^{(1)} \) to obtain the parameters \((\sigma(T_1), \pi^+_1(T_1), \pi^{-1}_1(T_1))\).

ii For \( j = 2, \ldots, N \) calibrate call prices at \( T^{(j)} \) to obtain \((\sigma(T_j), \pi^+_j(T_j), \pi^{-j}_j(T_j))\).

In Figure 2 the calibration results are presented with plots of the market and model implied volatility surfaces corresponding to the four maturities 6m, 1Y, 3Y and 5Y. The root-mean-square error (RMSE) and the average relative percentage error (ARPE) are equal to 5.30 and 1.1%. We compare it to a price process that follows a Lévy process with hyper-exponential jumps (i.e. with constant parameters over time), and find that the calibration of the four maturities in that case give a RMSE of 9.82 and an ARPE of 2.9%. In Table 1 the resulting parameter sets are displayed under the hyper-exponential additive and Lévy models. In the case of the hyper-exponential model we observe a high jump intensity of small jumps for short maturities that decrease substantially over time. This is consistent with the finding of Carr and Wu [11], and Pan [26].
Figure 2: Calibration of Eurostoxx call option prices on February 20th 2007 for an additive hyper-exponential model with piecewise constant parameters. Crosses represent market implied volatilities and circles model implied volatilities for four maturities: 6 months (red), 1 year (orange), 3 years (green), 5 years (blue).

| Additive hyperexponential model | $T_{Start}$ | $T_{End}$ | $\sigma$ | $\pi_1$ | $\pi_2$ |
|--------------------------------|-------------|----------|----------|--------|--------|
| 0                              | 0.5         | 0.0995   | 0.0371   | 11.1819 |
| 0.5                            | 1           | 0.0759   | 0.2091   | 9.9540  |
| 1                              | 3           | 0.0786   | 0.4738   | 7.0322  |
| 3                              | 5           | 0.0858   | 0.8084   | 0.2361  |

| Lévy hyperexponential model     | $T_{Start}$ | $T_{End}$ | $\sigma$ | $\pi_1^-$ | $\pi_2^-$ |
|--------------------------------|-------------|----------|----------|----------|----------|
| 0                              | 5           | 0.1171   | 0.5693   | 0.0165   |

Table 1: Calibration of Eurostoxx call option prices quoted on February 20 2007 for an additive hyper-exponential model with piecewise constant parameters and for a Lévy hyper-exponential model with jump parameters $\alpha_1 = 3$ and $\alpha_2 = 10$. The interest and dividend rates are $r = 0.03$ and $d = 0$. The maturities are given in years.
5.2 Results for the barrier option prices and Greeks

Using the parameter set found in the calibration of the Eurostoxx call options, we value barrier and digital options on the Eurostoxx index, modelling its price process as the exponential of a hyper-exponential additive process. We use the semi-analytical results in Proposition 2. To invert the multi-dimensional Laplace transforms we choose Talbot’s method [31] (see also [12]) for down-and-in digital options. We combine it with the fractional FFT algorithm of Bailey and Swarztrauber [6], and Chourdakis [13] for down-and-in call options. See Appendix C for a detailed description of the implementation of these transform algorithms. We compare it to the same quantities calculated by Monte-Carlo simulations, using a standard Euler scheme.

5.2.1 Down-and-in digital options

We price down-and-in digital options with a maturity of five years for different spot levels. We evaluate the required 4-dimensional Laplace transform over two time increments of six months, and two of two years, that is, \(T^{(1)} = 0.5, T^{(2)} = 0.5, T^{(3)} = 2, T^{(4)} = 2\). We use Talbot’s algorithm with \(M = 6\) (see Appendix C for an explanation of this parameter). Using Mathematica to run the algorithm, the computation time was five minutes on a 3189 Mhz computer to calculate prices and Greeks for fourteen different spot levels. The calculation of first passage probabilities using Monte-Carlo simulations requires a large number of time steps and paths. We use one million paths with \(\delta t = 5 \times 10^{-5}\) and it takes several hours to obtain stable Greeks in C++. Error bounds cannot be obtained analytically, but we observe in Table 2 that the results of the transform method agree with Monte-Carlo simulation results. Figures 3 and 4 report prices and Greeks for down-and-in digital options. The options are expressed as a percentage of the spot price. The values of the sensitivities are expressed as fractions of the spot price \(S_0\).

This transform algorithm is particularly efficient at a book level, since once the generating matrices \(Q^-\) of the infimum have been calculated for different values of the vector \(q\) the calculation of prices and Greeks of any digital barrier product is just a matter of summation.

5.2.2 Down-and-in call options

We value down-and-in call options with a maturity of one year for different strike levels. In this case, a two-dimensional Laplace inversion is required over time increments \(T^{(1)} = 0.5\) and \(T^{(2)} = 0.5\). For the inversions of the Laplace transform and the Fourier transform, we set \(M = 7\) and \(N = 1024\) (refer to Appendix C for an explanation of these parameters). For Monte-Carlo simulations, we use one million paths with time step \(\delta t = 2.5 \times 10^{-5}\). The option prices and Greeks obtained by the two methods are reported in Table 3 and figures 4 and 5. We observe that the results of the transform method agree with the Monte-Carlo simulation results. Using Mathematica again to run the algorithm, the computation time is ten minutes to calculate the option prices, delta and gamma for eleven different levels of the strike.
Since the option prices and Greeks of a down-and-in call option are obtained via a Fourier-Laplace transform, it takes more time than in the case of a digital option (approximately twice as long), which is still much faster than a Monte-Carlo Euler scheme. We note that the transform algorithm is particularly efficient for the pricing of options with different strikes, as we obtain by FrFFT inversion the prices and Greeks of any down-and-in call options on a log-strike grid.
Figure 3: Prices of down-and-in digital options with a barrier $H$ set at 90% of EUR 4150 and maturity $T = 5$ years. Semi-analytical results are indicated with the symbol $\times$ and Monte-Carlo results with the symbol $\circ$.

Figure 4: Greeks of down-and-in digital options with a barrier $H$ set at 90% of EUR 4150 and maturity $T = 5$ years. Semi-analytical results are indicated with the symbol $\times$ and Monte-Carlo results with the symbol $\circ$. 
Figure 5: Prices of down-and-in call options with a barrier $H$ set at 90%, a spot at EUR 4150 and maturity $T = 1$ year. Semi-analytical results are indicated with the symbol $\times$ and Monte-Carlo results with the symbol $\circ$.

Figure 6: Greeks of down-and-in call options with a barrier $H$ set at 90%, a spot at EUR 4150 and maturity $T = 1$ year. Semi-analytical results are indicated with the symbol $\times$ and Monte-Carlo results with the symbol $\circ$. 
### Down-and-in digital options

| %    | Price (MC−, MC+) | Delta (%10⁻¹) | Gamma (%10⁻⁷) |
|------|-----------------|----------------|---------------|
| 92   | 0.5261 (0.5000, 0.7499) | -1.226 -1.232 | 7.34 7.54   |
| 94   | 0.6446 (0.6137, 0.6735) | -0.812 -0.819 | 3.64 3.70   |
| 96   | 0.5093 (0.5566, 0.6207) | -0.577 -0.579 | 1.92 1.94   |
| 98   | 0.5478 (0.5143, 0.5805) | -0.450 -0.451 | 1.09 1.11   |
| 100  | 0.5140 (0.4800, 0.5475) | -0.374 -0.377 | 0.67 0.67   |
| 102  | 0.4850 (0.4506, 0.5189) | -0.328 -0.330 | 0.46 0.45   |
| 104  | 0.4592 (0.4245, 0.4932) | -0.294 -0.297 | 0.34 0.33   |
| 106  | 0.4358 (0.4008, 0.4697) | -0.260 -0.271 | 0.28 0.30   |
| 108  | 0.4144 (0.3795, 0.4483) | -0.247 -0.247 | 0.24 0.24   |
| 110  | 0.3946 (0.3599, 0.4285) | -0.229 -0.238 | 0.21 0.21   |
| 112  | 0.3762 (0.3518, 0.4010) | -0.212 -0.212 | 0.19 0.19   |
| 114  | 0.3592 (0.3250, 0.3917) | -0.198 -0.197 | 0.17 0.17   |
| 116  | 0.3433 (0.3095, 0.3771) | -0.184 -0.182 | 0.15 0.16   |
| 118  | 0.3285 (0.3032, 0.3632) | -0.172 -0.170 | 0.14 0.14   |

Table 2: Down-and-in digital options with barrier $H$ set at 90% of EUR 4150. The first column contains the spot price as a percentage figure of 4150. The columns with TA contain the results obtained using the transform algorithm, whereas MC refers to Monte-Carlo results. In the case of the price the Monte-Carlo results are reported in the form of a 95% confidence interval, and in the other cases as a point estimate.

### Down-and-in call options

| %    | Price (MC−, MC+) | Delta (%10⁻¹) | Gamma (%10⁻⁷) |
|------|-----------------|----------------|---------------|
| 80   | 111.13 (108.85, 111.20) | -2.458 -2.440 | 9.49 9.39   |
| 82   | 93.48 (91.46, 93.55) | -2.124 -2.107 | 8.43 8.34   |
| 84   | 77.14 (75.35, 77.19) | -1.808 -1.792 | 7.45 7.37   |
| 86   | 62.28 (60.71, 62.33) | -1.514 -1.499 | 6.47 6.40   |
| 88   | 49.08 (47.71, 49.13) | -1.242 -1.231 | 5.57 5.51   |
| 90   | 37.63 (36.7, 38.76) | -0.991 -0.988 | 4.65 4.59   |
| 92   | 28.11 (27.43, 28.46) | -0.783 -0.776 | 3.88 3.84   |
| 94   | 20.94 (20.13, 21.01) | -0.599 -0.593 | 3.07 3.03   |
| 96   | 15.11 (14.45, 15.16) | -0.448 -0.443 | 2.45 2.43   |
| 98   | 10.66 (10.12, 10.72) | -0.326 -0.320 | 1.85 1.84   |
| 100  | 7.36 (6.92, 7.41) | -0.231 -0.226 | 1.37 1.36   |
| 102  | 4.97 (4.63, 5.02) | -0.159 -0.156 | 0.94 0.93   |
| 104  | 3.28 (3.03, 3.34) | -0.107 -0.104 | 0.64 0.63   |
| 106  | 2.12 (1.94, 2.18) | -0.070 -0.069 | 0.43 0.42   |
| 108  | 1.35 (1.22, 1.41) | -0.045 -0.044 | 0.28 0.28   |
| 110  | 0.84 (0.74, 0.89) | -0.028 -0.027 | 0.23 0.22   |
| 112  | 0.51 (0.47, 0.54) | -0.017 -0.017 | 0.18 0.18   |
| 114  | 0.30 (0.25, 0.34) | -0.010 -0.009 | 0.13 0.13   |
| 116  | 0.18 (0.16, 0.19) | -0.006 -0.006 | 0.10 0.10   |
| 118  | 0.10 (0.08, 0.11) | -0.003 -0.003 | 0.07 0.07   |
| 120  | 0.06 (0.03, 0.06) | -0.002 -0.002 | 0.03 0.03   |

Table 3: Down-and-in call options with a barrier $H$ set at 90%, a spot at EUR 4150 and maturity $T = 1$ year for a range of strikes. The first column contains the strike level as a percentage figure of the spot EUR 4150. The columns with TA contain the results obtained using the transform algorithm, whereas MC refers to Monte-Carlo results. In the case of the price the Monte-Carlo results are reported in the form of a 95% confidence interval, and in the other cases as a point estimate.
APPENDIX

A Proofs

Proof of Lemma 1:

(i) It is straightforward to verify that $K^#(s)$ can be obtained from $K(s)$ by interchanging some columns and rows, where $K^#(s)$ is given by

$$K^#(s) = \begin{pmatrix} K_1(s) & D_1 \\ K_2(s) & D_2 \\ & \ddots \\ K_{N-1}(s) & D_{N-1} \\ K_N(s) \end{pmatrix},$$

where $K_w(s)$ and $D_w$ are square matrices of dimension $n^+ + n^- + 1$. There are defined respectively by

$$K_w(s) = \begin{pmatrix} \frac{\sigma^2 s^2}{2} + \mu_w s - q_w - \lambda^+_w - \lambda^-_w \\ \alpha^-_1 \\ \vdots \\ \alpha^-_m \\ \alpha^+_1 \\ \vdots \\ \alpha^+_n \end{pmatrix} \begin{pmatrix} \pi^-_1(w) \\ -s - \alpha^-_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $\lambda^+_w = \sum_i \pi^+_i(w)$, and

$$(D_w)_{ij} = \begin{cases} q_w & \text{if } i = j = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore $|\det(K(s))|$ is equal to $|\det(K^#(s))|$. To proceed we recall an identity from matrix algebra. Let $M$ be a matrix of the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix},$$

in block notation, where $M_{22}$ is invertible. Then

$$\det(M) = \det(M_{22}) \det(M_{11} + M_{12}M_{22}^{-1}M_{21}).$$

Using this identity, it is a matter of algebra to verify by induction that

$$\det(K^#(s)) = \det(K_1(s)) \cdots \det(K_N(s)).$$
As a consequence we find, by applying this matrix identity, that
\[
\det(K_w(s)) = \left( \frac{\sigma_w^2}{2} s^2 + \mu_w s + \sum_{i=1}^{n^+} \pi_i^+ (w) \frac{\alpha_i^+}{s - \alpha_i^+} + \sum_{j=1}^{n^-} \pi_j^- (w) \frac{\alpha_j^-}{s - \alpha_j^-} - \lambda_w^+ - \lambda_w^- - q_w \right) \times \\
\times \prod_{i=1}^{n^+} (s - \alpha_i^+) \prod_{j=1}^{n^-} (-s - \alpha_j^-),
\]
and the assertion follows in view of (4).

(ii) Using the intermediate value theorem and the specific form of Ψ, it is straightforward to check that the equation Ψ(−ui) = q, q > 0 has n + 1 positive roots ρi+ and m + 1 negative roots ρj−, satisfying
\[
ρ_{m+1}^- < -ρ_m^- < . . . < -ρ_1^- < ρ_1^+ < 0 < ρ_i^+ < ρ_1^- < . . . < ρ_n^+ < α_n^- < ρ_{n+1}^+.
\]
Since det(K_w(s)) is a polynomial of degree n + m + 2, it follows that all the roots of det(K_w(s)) = 0 are given by (ρi+, i = 1, . . . , n + 1) and (ρj−, j = 1, . . . , m + 1).

In view of the form of det(K(s)) derived in (i) the assertion follows.

Proof of Proposition 1: Consider the following randomization of X obtained by randomizing the inter-maturity times T(i) by replacing them by independent exponential random times with means q−1, and call this process ˜X. The process ˜X is a regime-switching jump-diffusion, where the only regime switches that can occur are from i to i + 1 at rate qi (i = 1, . . . , N − 1), and from the final state N to an absorbing 'graveyard state' ζ. As shown in [20], the process ˜X is equal to a time-changed continuous process A, say. Denoting by ξ the epoch at which A is sent to η, by Y the modulating Markov chain, and by τ = inf{t ≥ 0 : Aτ = x}, we have
\[
q_1 \cdots q_N G(x, s, q) = E[e^{sA_\xi} 1_{\{X_\tau > x\}}] = E[e^{sA_\xi} 1_{\{\tau < \xi\}}] = E[e^{sA_\xi} 1_{\{\tau < \xi\}} f(Y_\tau)] = e^{st} E[1_{\{\tau < \xi\}} f(Y_\tau)],
\]
with
\[
f(y) = E[e^{sA_0} \mid A_0 = 0, Y_0 = y],
\]
where the last two lines follow by the Markov property of (A, Y) and the fact that A is continuous. To guarantee that all the expressions are well defined in this calculation s has to be such that E[e^{sN_1}] < ∞, which corresponds to the restriction that
\[
Re(s) \in (-\min_j \alpha_j^-, \min_i \alpha_i^+).
\]

In [20] it was shown that the vector f = (f(y), y ∈ N), where N denotes the state space of Y, is given by
\[
f = K(s)^{-1} Q1,
\]
where the matrix $K(s)$ is given in (13). Combining these results with Theorem 1 we find that

$$q_1 \cdots q_N \mathcal{G}(x, s, q) = e^{x^T q} K(s)^{-1} Q_1,$$

and the proof is complete. $\square$

B European call options

Under the hyper-exponential additive model with piecewise constant parameters (3), the characteristic function at time $T$ is explicitly given by

$$\Phi(i)(u) = \exp \left( \sum_{j=1}^{i} \Psi(j)(u) \right),$$

with $\Psi(j)$ as given in (4). The price of a European call with maturity $T_i$ can thus be efficiently calculated using a well-established Fourier transform method, which we briefly recall. The Fourier transform $C_{T_i}^*$ of $C_{T_i}(k)$, the price of a call option with log-strike $k = \log(K/S_0)$ and maturity $T_i$, can be explicitly expressed in terms of the characteristic function $\Phi(i)(u)$ as follows:

$$C_{T_i}^*(v - i\alpha) = S_0 e^{-\alpha k} \int_{-\infty}^{\infty} e^{ivk} E[e^{\alpha k}(e^{X_{T_i}} - e^k)^+] dk$$

$$= S_0 e^{-\alpha k} \frac{\Phi(i)(v - (\alpha + 1)i)}{(\alpha + iv)(\alpha + 1 + iv)}. \quad (21)$$

Since the call pay-off function itself is not square-integrable in the log-strike, the axis of integration is here shifted over $i\alpha$ which corresponds to exponentially dampening the pay-off function at a rate $\alpha$, which is usually taken to be $\alpha = 0.75$ (see Carr and Madan [9]). The call option prices are then determined by inverting the Fourier transform:

$$C_{T_i}(k) = S_0 e^{-\alpha k} e^{-\alpha T_i} \frac{1}{\pi} \int_{0}^{\infty} e^{-ivk} \frac{\Phi(i)(v - (\alpha + 1)i)}{(\alpha + iv)(\alpha + 1 + iv)} dv. \quad (22)$$

C Transform inversion algorithms

C.1 Multi-dimensional Laplace inversion

To evaluate down-and-in digital option prices (DID), we invert the multi-dimensional Laplace transform (17) to obtain

$$DID(S_0, h, \mathbf{T}) = \frac{1}{(2\pi i)^N} \int_{C_N} \cdots \int_{C_1} e^{\eta T_i + \cdots + q_N T_N} \overline{DID}(S_0, h, \mathbf{q}) d\mathbf{q}. \quad (23)$$

where $\mathbf{T} = (T_1, \ldots, T_N)$ and $C_n$ are vertical lines in the complex plane defined by $q_n = r_n + iy_n$ for $n = 1, \ldots, N$ with $-\infty < y_n < \infty$ and fixed values of $r_n$, chosen such that all the singularities of the transform $\overline{DID}(S_0, h, \mathbf{q})$ are coordinate-wise on the left
of the lines \( C_n \). Many algorithms approximate the integrals in (23) by a finite linear combination of the transform at some specific nodes with certain weights. Three approaches have been studied by Abate et al. [2], based on Fourier series expansion, combinations of Gaver functionals, and deformation of the integral contour. Here we concentrate on the last method developed by Talbot [31], since reports in the literature (e.g. [2]) suggest that this approach offers high performance for a short time of execution, which our numerical results confirm. We write

\[
DID(S_0, h, T) = \frac{1}{(2\pi)^N} \left( \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \beta_1(\theta) \cdots \beta_N(\theta) \widehat{\text{DID}}(S_0, h, q(\theta)) d\theta \right), \tag{24}
\]

with \( n = 1, \ldots N, \beta_n(\theta) = w_n e^{i \omega_n T_n}, q_n(\theta) = i \omega_n w_n, \) and

\[
w_n = -1 + i \theta + i(\theta \cot \theta - 1) \cot \theta.
\]

Since \( \text{DID} \) is a real valued function, \( \text{DID} \) is also equal to the real part of the integral on the right-hand side of (24), which can be used to reduce the calculation by a factor of two. To illustrate the evaluation of the integrals (24), we present concrete expressions for the approximating sums when \( N = 4 \) (which is the setting that will be implemented later on). Defining

\[
\theta_n^k = k\pi/M \quad \text{and} \quad r_n = \frac{2M}{5T_n},
\]

we obtain

\[
DID(S_0, h, T) \approx \frac{2}{5^4 T_1 T_2 T_3 T_4} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{M-1} \sum_{k_3=0}^{M-1} \sum_{k_4=0}^{M-1} \beta_{k_1, k_2, k_3, k_4} f(q_{k_1}/T_1, q_{k_2}/T_2, q_{k_3}/T_3, q_{k_4}/T_4) + \beta_{k_1, k_2, k_3, k_4} \widehat{\beta}(q_{k_1}/T_1, q_{k_2}/T_2, q_{k_3}/T_3, q_{k_4}/T_4) + \beta_{k_1, k_2, k_3, k_4} \widehat{\beta}(q_{k_1}/T_1, q_{k_2}/T_2, q_{k_3}/T_3, q_{k_4}/T_4) + \beta_{k_1, k_2, k_3, k_4} \widehat{\beta}(q_{k_1}/T_1, q_{k_2}/T_2, q_{k_3}/T_3, q_{k_4}/T_4),
\]

where \( f \) is equal to \( \widehat{\text{DID}} \). The weights and the nodes are given by

\[
q_0 = \frac{2M}{5}, \quad q_k = \frac{2k\pi}{5}(\cot(k\pi/M) + i), \quad 0 < k < M, \quad \beta_0 = 0.5 e^{i\omega_0}, \quad \beta_k = (1 + i(k\pi/M)(1 + [\cot(k\pi/M)]^2) - i \cot(k\pi/M)) e^{i\omega_k}.
\]

Since the weights and nodes are independent of the transform, the calculation time of the algorithm can be reduced by pre computing and storing weights and nodes. The speed of convergence and the accuracy of the Talbot algorithm will depend on the regularity of the Laplace transform \( f \). Although universal error bounds are not known, Abate et al. [1] showed numerically that the single parameter \( M \) can be used to control the error and can be seen as a measure for the precision. They found after extensive numerical experiments that for a large class of Laplace transforms the relative error is approximately \( 10^{-0.6M} \). For high dimensional inversion, extra accuracy in the inner sums may be needed to obtain a sufficient degree of precision for the outer sums, which can be achieved by increasing \( M \).
C.2 Fractional Fourier Transform

To evaluate down-and-in call option prices (DIC), we invert the Fourier-Laplace transform (18) over log-strike and time periods. For the inversion of the Laplace transform we again apply the Talbot algorithm. In the case of two time periods, with

$$ f_v(q_1, q_2) = \hat{DIC}^*(S_0, h, v, q), $$

we find that the Fourier transform $\hat{DIC}^*$ can be approximated by the following sums:

$$ DIC^*(S_0, h, v, T) \approx \frac{1}{5^2 T_1 T_2} \sum_{k_1=0}^{M-1} \sum_{k_2=0}^{M-1} \left\{ \beta_{k_1} \beta_{k_2} f_v(q_{k_1}/T_1, q_{k_2}/T_2) + \overline{\beta_{k_1}} \overline{\beta_{k_2}} f_v(\overline{q_{k_1}}/T_1, \overline{q_{k_2}}/T_2) ight\}. $$

Unlike the case of the inversion of $\hat{DID}$, we cannot reduce the calculation time by two by using complex conjugates, since the function $DIC^*$ is not real valued. Down-and-in call prices are then obtained by inverting the Fourier transform over strike:

$$ DIC(S_0, h, k, T) = e^{-\alpha k} \frac{\pi}{rT} \int_0^{\infty} e^{-i\nu v} DIC^*(v) dv, $$

where $\alpha$ is the rate of exponential dampening. This integral is approximated for a set of log-strikes between $(-x_0, x_0)$ as a summation:

$$ DIC(S_0, h, k, T) \approx \sum_{j=0}^{N-1} w_j e^{-i\nu \delta j(-x_0 + k \lambda)} DIC^*(\delta j) \delta, \quad k = 1 \cdots N - 1, $$

where $(w_j)_{j=0}^{N-1}$ are the integration weights defined by the trapezoidal rule with $w_0 = w_{N-1} = 0.5$ and 1 otherwise, $\lambda = 2x_0/N$ is the log-strike grid step-size and $\delta$ is the $v$-grid step-size. Carr and Madan [9] and Chourdakis [13] set $\delta = 0.25$.

To have accurate prices for any strike, the log-strike grid spacing $\lambda$ needs to be sufficiently small. A common approach is to apply directly the Fast Fourier Transform (FFT) and to compute the summation (25) on a fixed log-strike range $(-x_0, x_0)$ with $x_0 = \pi/\delta$ using many points $N$. Bailey and Swarztrauber [5], [6] propose an alternative approach, and define the Fractional Fast Fourier transform (FrFFT), which uses an arbitrary range. Chourdakis [13] showed that the FrFFT can be used to calculate option prices with less points without losing accuracy. He reported that the FrFFT is 45 times faster than the FFT for the calculation of European option prices. Since in our case the Fourier transform $DIC^*$ is obtained numerically, we chose to employ the FrFFT. We now briefly specify the form of this algorithm in our setting, and refer for further details to [5], [6], [13]. The resulting sum is then given by

$$ DIC(S_0, h, k, T) \approx \frac{S_0 e^{-\alpha k} e^{-rT}}{\pi} \sum_{j=0}^{N-1} \tilde{w}_j e^{-\pi i j^2 \nu} e^{-\pi i (k-j)^2 \nu} DIC^*(\delta j) \delta, $$

23
where \( k = 1 \cdots N - 1 \), \( \tilde{w}_j = w_j e^{ix_0 \delta j} \) and \( \nu = \delta x_0 / N \pi \). Extending this summation into a circular convolution over \( 2N \) yields

\[
\text{DIC}(S_0, h, k, T) \approx S_0 e^{-(\alpha k + i\nu k^2 \nu)} e^{-rT} \frac{2N-1}{\pi} \sum_{j=0}^{2N-1} y_j z_{k-j}, \quad k = 1 \cdots N - 1,
\]

where

\[
y_j = \tilde{w}_j e^{-\pi j^2 \nu} \text{DIC}^*(\delta j) \delta, \quad z_j = e^{-\pi j^2 \nu}, \quad j < N,
\]

and

\[
y_j = 0, \quad z_j = e^{-\pi (j-2N)^2 \nu}, \quad j \geq N.
\]

This equation can be rewritten in terms of three discrete Fourier transforms:

\[
\text{DIC}(k) \approx S_0 e^{-(\alpha k + i\nu k^2 \nu)} e^{-rT} \frac{1}{\pi} F_k^{-1}(F_k(y)F_k(z)), \quad k = 1 \cdots N - 1,
\]

with

\[
F(x) = \sum_{j=0}^{N-1} x_j e^{-2\pi ij k/N}, \quad F^{-1}(x) = \sum_{j=0}^{N-1} x_j e^{2\pi ij k/N}.
\]

Although the latter sum is computed by invoking two Fourier transforms and one inverse Fourier transform, this approach has the advantage of computing the option prices on a specific log-strike window \((-x_0, x_0)\) with independent grids \( \delta \) and \( \lambda \) and requires less points.
References

[1] J. Abate and P. P. Valko. Multi-precision Laplace transform. *International Journal for Numerical Methods in Engineering*, 60:979-993, 2004.

[2] J. Abate and W. Whitt. A unified framework for numerically inverting Laplace transforms. *INFORMS Journal on Computing*, 18:408-421, 2006.

[3] S. Asmussen. Ruin probabilities. World Scientific, Singapore, 2000.

[4] S. Asmussen, D. B. Madan, and M. R. Pistorius. Pricing equity default swaps under an approximation to the CGMY Lévy model. *Journal of Computational Finance* 11:79-93, 2008.

[5] D. H. Bailey and P. N. Swarztrauber. The fractional Fourier transform and applications. *SIAM Review*, 33:389-404, 1991.

[6] D. H. Bailey and P. N. Swarztrauber. A fast method for the numerical evaluation of continuous fourier and laplace transforms. *SIAM Journal on Scientific Computing*, 55:205-238, 1994.

[7] S. I. Boyarchenko and S. Levendorskii. Non-Gaussian Merton Black Scholes theory. World Scientific Publishing, 2002.

[8] M. Boyarchenko and S. Levendorskii. Prices and sensitivities of barrier and first-touch digital options in Lévy driven models. *Preprint downloadable at SSRN.com/abstract=1155149*

[9] P. Carr and D. Madan. Option valuation using the fast Fourier transform. *Journal of Computational Finance*, 2:61-73, 1998.

[10] P. Carr, D. Madan, H. German and M. Yor. Self decomposability and option pricing. *Mathematical Finance*, 17:31-57, 2007.

[11] P. Carr and L. Wu. What type of processes underlies options? A simple robust test. *Journal of Finance*, 58:2581-2610, 2003.

[12] G. L. Choudhury, W. Whitt, and G. L. Lucantoni. Multidimensional transform inversion with applications to the transient M/G/1 queue. *Annals of Applied Probability*, 4:719-740, 1994.

[13] K. Chourdakis. Option pricing using the fractional FFT. *Journal of Computational Finance*, 8:1-18, 2004.

[14] R. Cont and P. Tankov. Financial modelling with jump processes. Chapman and Hall, Boca Raton, 2004.

[15] D. Davydov and V. Linetsky. The Valuation and hedging of barrier and lookback options under the CEV process. *Management Science*, 47:949-965, 2001.
[16] E. Eberlein and W. Kluge. Exact pricing formulae for caps and swaptions in a Lévy term structure model. *Journal of Computational Finance*, 9:99-125, 2006.

[17] M. L. Galloway and C. A. Nolder. Subordination, self similarity, and option pricing. *Journal of Applied Mathematics and Decision Sciences*, 30-60, 2008.

[18] H. Geman and M. Yor. Pricing and hedging double barrier options: a probabilistic approach. *Mathematical Finance*, 6:365-378, 1996.

[19] M. Jeannin and M. R. Pistorius. A transform approach to calculating prices and Greeks of barrier options driven by a class of Lévy processes. *Quantitative Finance*, to appear.

[20] Z. Jiang and M. R. Pistorius. On perpetual american put valuation and first-passage in a regime-switching model with jumps. *Finance and Stochastics*, 12:331-355, 2008.

[21] S. G. Kou and H. Wang. Option pricing under a double exponential jump diffusion model. *Management Science*, 50:1179-1192, 2004.

[22] S. G. Kou, G. Petrella and H. Wang. Pricing path-dependent options with jump risk via Laplace transforms. *Kyoto Economic Review*, 74:1-23, 2005.

[23] R. Lee. Option pricing by transform method: Extension, unification and error control. *Journal of Computational Finance*, 7:51-86, 2004.

[24] A. Lipton Assets with jumps. *Risk*, 149-153, 2002.

[25] R. R. London, Mc Kean, L. C. G. Rogers and D. Williams. A martingale approach to some Wiener-Hopf problems I, II. *Seminar on Probability*, 41-67, 68-90, 1982.

[26] J. Pan. The jump-risk premia implicit in options: evidence from integrated time-series study. *Journal of Financial Economcis*, 63:3-50, 2002.

[27] L. C. G. Rogers. Fluid models in queueing theory and Wiener-Hopf factorization of Markov chains. *Annals of Applied Probability*, 4:390-413, 1994.

[28] K. Sato. Lévy processes and infinitely divisible distributions. Cambridge university press, Cambridge, 1999.

[29] A. Sepp. Analytical pricing of double barrier options under a double-exponential jump-diffusion process: Applications of Laplace transform. *International Journal of Theoretical and Applied Finance*, 7:151-175, 2004.

[30] W. Schoutens. Lévy processes in finance, Wiley, 2003.

[31] A. Talbot. The accurate numerical inversion of Laplace transforms. *Journal of the Institute of Mathematics and its Applications*, 23:97-120, 1979.