Abstract

We present a geometrical description of $N = 8$ supergravity, using central charge superspace. The essential properties of the multiplet, like self-duality properties of the vectors or the non-linear sigma model structure of the scalars, are deduced from constraints at 0 and 1/2 canonical dimension. We also present in detail how to derive from this geometrical formulation the supergravity transformations as well as the whole equations of motion for the component fields in order to compare them with the results already known and obtained in formulations on the component level.

Key-Words: extended supersymmetry, supergravity, central charge superspace, equations of motion.
1 Introduction

$N = 8$ extended supergravity is called maximal in the sense that it does not require helicities larger than two (gravitons). One of its intriguing properties is that it admits $SU(8)$ as intrinsic gauge invariance group. The spectrum of the multiplet looks rather complicated at the first sight: it contains a graviton, 8 Rarita-Schwinger fields, 28 graviphotons, 56 helicity 1/2 fields and 70 helicity 0 states, which are considered to be all scalars. However, these fields are organized in a quite simple structure which is recovered and described in a concise manner using the geometric description of the theory in central charge superspace. One of the main objectives of the present paper is to emphasize this feature.

The analysis of these theories began with the development of the basic structure of $SO(8)$ supergravity, realized by de Wit and Freedman [1]. They remarked the presence of complicated, non-polynomial structures in the scalars and pseudo-scalars of the theory. After that, Cremmer and Julia described in detail the component structure of this theory [2], [3], obtaining it by dimensional reduction of the 11 dimensional supergravity theory to 4 dimensions. In particular, they established that the scalars take part of a non-linear $\sigma$ model and they live on the $E_7/SU(8)$ coset space. De Wit and Nicolai also resumed the properties of this theory [4] with particular attention on separable properties and conjectures which yield the known and accepted structure.

The well-known methods of superspace geometry, as developed by Wess and Zumino [5] and resumed in [6] were generalized to extended supergravity and were applied to the $N = 8$ case [7], [8]. In these papers, the authors used ordinary extended superspace, without any additional bosonic coordinates. In this approach the identification of both the vectors and the scalars can be done through their field strengths in torsion components, nevertheless, for more clearness, the necessary number of gauge vectors is introduced explicitly in the covariant derivatives. In order to render the description more transparent, Howe and Lindström in the appendix of [8] and Siegel [9] introduce additional bosonic coordinates in superspace. This second approach is completely equivalent with the former one, though, it gives a more natural interpretation of graviphotons: they are identified as the gauge vectors corresponding to local translations in the direction of the additional bosonic coordinates. Concerning the scalars, they are identified in a non-trivial way in some components of the super-vielbein in the extra bosonic sector. In reference [10] a detailed presentation is given of this second approach.

The aim of this paper is to present an approach to the identification of the $N = 8$ supergravity in central charge superspace, generalizing the constructions applied to $N = 2$ [12] and $N = 4$ [13]. Here, the vectors are identified in the frame components with central charge flat index and the corresponding gauge transformations are realized as superspace diffeomorphisms in the central charge directions. Moreover, we show that it is possible to identify the scalars of the theory directly, in torsion components of 0 canonical dimension. Also, the three silver rules for maximally dualised extended supergravities [14] are recovered as consequences of natural and simple constraints on the geometry. Namely, we find that the moduli space of the scalar fields has a $E_{7(+7)}/SU(8)$ coset space structure, that these scalar fields serve as converters between $SU(8)$ indices and central charge indices (corresponding to $E_{7(+7)}$ representation indices), and also, that the vectors taking part of the multiplet are self–dual or anti–self–dual in the $SU(8)$ basis while they satisfy a twisted self–duality relation in the central charge basis.

The paper is organized as follows. In section 2, we review the basic structure of central charge superspace. Section 3 is devoted to the identification of the multiplet: physical fields are identified in the vielbein and in the torsion, and their main properties are deduced from the geometric structure. Then, we compute the supergravity transformations of the fields
in section 4. We finish with the equations of motion which are presented in section 5.

2 General geometric structure

The geometrical description of extended supergravity theories is based on a generalization to the extended case of standard $N = 1$ superspace methods [6], [15]. The Abelian gauge vectors which appear for higher $N$ have a natural interpretation in central charge superspace: they appear as components of the superspace frame. As such, their gauge transformations appear on the same footing as space-time diffeomorphisms and local supersymmetry transformations. General ideas about supergravity in central charge superspace are presented in detail in [12] and [13]. Nevertheless, we recall here briefly the basic notions we use throughout the article.

Consider the central charge superspace with frame $E^A = (E^a, E^a_{\dot{\alpha}}, E_\alpha^A, E^u)$, where $a, \alpha, \dot{\alpha}$ denote the usual vector and Weyl spinor indices, capital indices $A$ count the number of supercharges and boldface indices $u$ the number of central charges. The structure group, which is chosen to be $SL(2, C) \otimes SU(8)$, acts on the frame, $\delta X^A = E^B X_B^A$, and the corresponding covariant derivatives are defined in terms of the connection 1–form $\Phi_B^A$.

The representation of the structure group on flat indices is block-diagonal with respect to the space-time, spinorial and central charge sector. The a priori non-zero connection components $\Phi_B^A$ are then

$$\Phi_b^a, \quad \Phi_{\beta A}^{a} = \delta_{\beta}^{\alpha} \Phi_{\alpha A}^{a} + \delta_{\alpha}^{\beta} \Phi_{\dot{\alpha} A}^{a}, \quad \Phi_{\dot{\alpha} A}^{\dot{\beta}} = \delta_{\dot{\beta}}^{\dot{\alpha}} \Phi_{\dot{\alpha} A}^{\dot{\beta}} + \delta_{\dot{\alpha}}^{\dot{\beta}} \Phi_{\dot{\beta} A}^{\dot{\alpha}}, \quad \Phi_{u}^{z},$$

with $\Phi_{\alpha A}^{A} = 0$ and $\Phi_{\dot{\alpha} A}^{A} + \bar{\Phi}_{\dot{\beta} A}^{A} = 0$, as properties of the vector representation of the Lie algebra $su(8)$.

The torsion and the curvature are given as differential forms in central charge superspace

$$T^A = DE^A = dE^A + E^B \Phi_B^A,$$

$$R_B^A = d\Phi_B^A + \Phi_{BC}^A \Phi_B^C,$$

and satisfy the Bianchi identities

$$DT^A = E^B R_B^A,$$

$$DR_B^A = 0,$$

where the Bianchi identity of the torsion (4) is a consequence of the action of two covariant derivatives on a covariant vector $u^A$,

$$DDu^A = u^B R_B^A,$$

and the definition of the torsion (2).

The supergravity transformations (or Wess-Zumino transformations) are defined as usual [6], [16] to be a special combination of diffeomorphisms and gauge transformations such that they transform covariant vectors into covariant vectors:

$$\delta^{WZ}_{\xi} \equiv (L_{\xi} + \delta_{X})|_{X = \xi \Phi},$$

1We note $SU(8)$ the Lie group and $su(8)$ its Lie algebra.
acting on covariant vectors by the “covariant Lie derivative” \( \mathcal{L}_\xi = \iota_\xi D + D \iota_\xi \). In particular, the supergravity transformations of the vielbein and the connection are the following:

\[
\begin{align*}
\delta_W^\xi E^A &= D\xi^A + \iota_\xi T^A, \\
\delta_W^\xi \Phi_B^A &= \iota_\xi R_B^A.
\end{align*}
\]

(8)

(9)

The geometrical description of the \( N = 8 \) supergravity theory in this context, is to identify first the components of the multiplet in the geometrical objects reviewed above (frame, torsion, connection, curvature) in such a way that they transform under supergravity transformations between themselves. Once the identification of the component fields is done and in order to convince oneself that the identified theory is nothing else but the \( N = 8 \) supergravity described in the original works on the component level, one has to compare the supersymmetry transformations as well as the equations of motion we can deduce from the geometrical description with those given in the component formalism [2], [4].

This is the aim and the strategy of the next sections.

3 Identification and properties of the multiplet

We will identify in this section the component fields (one graviton, 8 Rarita-Schwinger fields, 28 graviphotons, 56 helicity 1/2 fields and 70 scalars) in the context of the basic geometrical objects of central charge superspace.

3.1 Identification of the gauge component fields in the super-vielbein

In analogy with general relativity, where the graviton is identified in the vierbein, central charge superspace provides a unified geometric interpretation of the graviton, gravitini and graviphotons in the frame \( E^A \),

\[
E^a \| = dx^m e_m^a, \quad E^\alpha_\Lambda \| = \frac{1}{2} dx^m \psi_m^\alpha_\Lambda, \quad E^z_\Lambda \| = \frac{1}{2} dx^m \bar{\psi}_{m\dot{\alpha}}^\Lambda, \quad E^u \| = dx^m \nu_m^u,
\]

(10)

where the double bar projects at the same time on the vector coefficient of the differential form and on the lowest superfield component [17].

Also, there are a priori other independent gauge fields in the theory, namely, the connection fields. Concerning the connection in the central charge sector, let us adopt in the following the requirement that the representation of the structure group on central charge indices be trivial,

\[
\Phi_u^z = 0.
\]

(11)

Concerning the other components, the double bar projections of the connection one–forms

\[
\Phi^a = dx^m \Phi^a_m \quad \text{and} \quad \Phi^\Lambda = dx^m \Phi^\Lambda_m,
\]

(12)

give the ordinary Lorentz and \( SU(8) \) connections. However, as it will be shown in the next sections (see equations (81) and (74)), once the constraint (11) adopted, in the case of the on–shell \( N = 8 \) supergravity both the Lorentz and the \( SU(8) \) connections are given as functions of the other component fields and their space-time derivatives.

As the remaining component fields of the multiplet, the scalars and the helicity 1/2 fields, have to complete the above gauge fields into a supersymmetry multiplet. We are looking for
them in the supergravity transform of the frame (8), that is in torsion components, which satisfy the Bianchi identity (4), or displayed on 3–form coefficients,

\[
\left(D_C B A\right)_T : \quad E^B E^C E^D \left(D_D T_{CB} A + T_{DC} F T_{FB} A - R_{DCB} A\right) = 0.
\] (13)

Torsion components also appear in the algebra of covariant derivatives (6)

\[
(D_C, D_B) u^A = -T_{CB} F D_F u^A + R_{CB} F A u^F,
\] (14)

when it is acting on a covariant vector \( u^A \).

Recall, that in general relativity the Bianchi identities (4) and (5) are independent, while for supergravity defined in ordinary extended superspace without central charge coordinates, Dragon’s theorem [18] tells us that the Bianchi identities for the curvature (5) are a consequence of the Bianchi identities for the torsion (4). In this latter case, the torsion is considered as the fundamental object of the geometry and all curvature components are expressed as functions of the torsion components and their covariant derivatives. However, as the theorem of Dragon is based on the relation between the representations of the structure group on bosonic and fermionic indices, in the presence of central charge coordinates in general it ceases to be valid as it stands [19]. Nevertheless, assuming trivial gauge structure (11) in the central charge sector, Dragon’s therem remains valid and it is sufficiant to investigate the first set (13) of Bianchi identities.

The constraints

\[
T^C_{\gamma b} = 0, \quad T^\gamma_{\dot{c} b} = 0, \quad T_{\gamma b} = 0,
\] (15)

are the usual conventional ones [20], [19], which are nothing else but some redefinitions of the supervielbein and of the Lorentz connection. Other less conventional constraints, as for example

\[
T_{zB} A = 0,
\] (16)

serve to reduce the number of independant fields and may also imply equations of motion.

The remaining constraints, at canonical dimension 0 and 1/2, allow the identification of the scalars and the 1/2 helicity fields. They will be presented in the next two paragraphs in some more detail. Once the constraints on torsion components are imposed, their consistency with Bianchi identities (13) must be checked, which has been carefully done.

### 3.2 Constraints at dimension 0, identification of the scalars

The geometrical description of the \( N = 8 \) supergravity multiplet in central charge superspace is based on a set of natural constraints at canonical dimension 0. This type of constraints was already used to identify the \( N = 2 \) minimal supergravity multiplet [12] as well as to identify the \( N = 4 \) supergravity with antisymmetric tensor [13], [21], the Nicolai–Townsend multiplet. They remain the same in the present case of \( N = 8 \) supergravity :

\[
T^{C}_{\gamma \beta} A = 0, \quad T^{C}_{\dot{\gamma} \beta} A = -2i \delta^{C}_{\dot{\gamma}}(\sigma^{a})_{\gamma} \dot{\beta}, \quad T^{\dot{\gamma} \beta}_{CB} A = 0,
\] (17)

\[
T^{CB}_{\gamma \beta} u = \epsilon_{\gamma \beta} T^{[CB]} u, \quad T^{C}_{\dot{\gamma} \beta} u = 0, \quad T^{\dot{\gamma} \beta}_{CB} u = \epsilon^{\dot{\gamma} \beta} T_{[CB]} u.
\] (18)

As in the \( N = 4 \) case, we expect that the objects \( T^{[CB]} u \) and \( T_{[CB]} u \), which can be organized in matrix form as

\[
T = \begin{pmatrix}
T_{[DC]} u \\
T_{[DC]} u
\end{pmatrix},
\] (19)
play an important role in the identification of the scalars. One of the most important
assumptions we make in order to identify the $N = 8$ supergravity multiplet is to suppose
that there exists a matrix $S$ with components
\[ S \equiv \begin{pmatrix} S_{u[dc]} & S_u^{[dc]} \end{pmatrix}, \tag{20} \]
such that the components of $T$ and $S$ satisfy
\[
\begin{pmatrix}
T_{[dc]}^{u} S_{u[ba]} & T_{[dc]}^{u} S_u^{[dc]}
\end{pmatrix} =
\begin{pmatrix}
\frac{1}{2} S_{[ba]}^{DC} & 0 \\
0 & \frac{1}{2} S^{BA}_{DC}
\end{pmatrix}, \tag{21}
\]
and
\[
S_{u[ba]} T^{[ba]} z + S_u^{[ba]} T_{[ba]} z = \delta^z_u. \tag{22}
\]
Recall, that we did not fix a priori the number of central charge indices $u$. However,
the assumptions above imply that the matrices $T$ and $S$ are square matrices of dimension
$N(N-1)$ and of maximal rank $2$. So, these assumptions fix the number of central charge
indices $u$ to be $N(N-1) = 56$, and we can write the relations (21) and (22) as
\[ TS = 1_{56}, \quad ST = 1_{56}. \tag{23} \]
In this sense $S$ is the inverse of the matrix $T$, constructed from the Lorentz scalars contained
in the torsion components of zero canonical dimension (18).

As a matter of fact, these matrices serve as converters between the central charge basis
(indices $u$) and the $SU(8)$ basis in the antisymmetric representation (indices $[dc]$ and $[^{dc}]$).
For instance all object $X^u$ can be converted in the $SU(8)$ basis using the matrix $S$:
\[
\begin{pmatrix} X_{[dc]} & X^{[dc]} \end{pmatrix} = X^u \begin{pmatrix} S_{u[dc]} & S_u^{[dc]} \end{pmatrix}. \tag{24} \]
Inversely, one can come back to the central charge basis using the matrix $T$:
\[
X^u = \begin{pmatrix} X_{[dc]} & X^{[dc]} \end{pmatrix} \begin{pmatrix} T_{[dc]}^{u} & T_{[dc]}^{u} \end{pmatrix} = X_{[dc]} T_{[dc]}^{u} + X^{[dc]} T_{[dc]}^{u}. \tag{25} \]
We identify the scalars of the multiplet as the lowest superfield components of the scalar
superfields $T$:
\[ T^| = T, \quad S^| = S. \tag{26} \]
However, at this stage we have a problem with the degrees of freedom contained in the
objects $S$. Namely, in the $N = 8$ supergravity multiplet we expect to have 70 scalars,
while the matrix $T$ identified in torsion components, and therefore its lowest superfield
components $T$, have a priori $56 \times 56$ independent components. This problem will be solved
in paragraph 3.5, at dimension 1, where we will rather count the degrees of freedom in the
field strength $(D_m T) S$.

### 3.3 Constraints at dimension 1/2, identification of the 1/2 helicity fields

In turn, the helicity 1/2 fields, or gravigini fields, are identified as usual [10], [22] in the
dimension 1/2 torsion component
\[
T_{\gamma}^{\beta \alpha} = \epsilon_{\beta \gamma} T_{[cba]}^{\alpha}, \quad T_{C}^{\gamma \delta \alpha} = \epsilon_{\gamma \delta} T_{[cba]}^{\alpha}, \tag{27}
\]

\[^2\text{The demonstration is immediate using the property } \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) \text{ for all matrices } A \text{ and } B \text{ with dimensions such that the product of them is well-defined.}\]
as the lowest superfield component of the gravitini superfields \(T_{[CBA]}^\alpha\) and \(T^{[CBA]}_{\dot{\alpha}}\),

\[
T_{[CBA]}^\alpha | = \lambda_{[CBA]}^\alpha, \quad T^{[CBA]}_{\dot{\alpha}} | = \lambda^{[CBA]}_{\dot{\alpha}}.
\]  

(28)

Recall that our aim is to describe the \(N = 8\) on–shell supergravity multiplet containing no other independent helicity 1/2 fields than these \(C_8^5 = 56\) ones in the totally antisymmetric representation. Therefore, the constraints at canonical dimension 1/2 are chosen in such a way that on one hand they eliminate all dimension 1/2 fields which are not in the good representation of \(SU(8)\), and, on the other hand all the remaining 1/2 helicity fields are linear combinations of \(T_{[CBA]}^\alpha\) and \(T^{[CBA]}_{\dot{\alpha}}\).

More precisely, we require first

\[
T_{CBA}^{\gamma \beta A} = 0, \quad T_{CBA}^{\gamma \dot{\beta} A} = 0, \quad T_{CBA}^{\gamma \dot{\beta} A} = 0, \quad T_{CBA}^{\gamma \dot{\beta} A} = 0.
\]  

(29)

The case of torsion components of the form \(T_{C[D}u^{\beta]}\) deserves more attention. Recall that we have already eliminated those with at least one central charge differential form index. As to those with a structure group central charge index, the possibility to pass from the central charge basis (indices \(u\) to the \(SU(8)\) basis (antisymmetric combination \([BA]\)) will be crucial for the following discussion. Candidates of torsion components which allow to accomodate \(T_{[CBA]}^\alpha\) or \(T^{[CBA]}_{\dot{\alpha}}\) are

\[
X_{[DCB]}^\gamma u = D_{\delta}^{[DCB]} u, \quad X_{[DCB]}^{\dot{\gamma}} u = D_{\delta}^{\dot{\gamma}} T_{[DCB]} u,
\]  

(30)

and \(T_{\alpha A}^u\), \(T_{\dot{\alpha} A}^u\), which appear in the expression of the torsion components

\[
T_{\gamma b}^C u = -2i(\sigma_b)^{\gamma \gamma} T_{CAB}^{\gamma C}, \quad T_{CAB}^{\gamma B} u = -2i(\tilde{\sigma}_b)^{\gamma \gamma} T_{[CBA]} u.
\]  

(31)

As a consequence, we require

\[
T_{\alpha A}^u \left( S_{u[B]}^A, S_{u[A]}^A \right) = \left( \alpha T_{[CBA]}^\alpha 0 \right)
\]  

(32)

\[
T_{\dot{\alpha} A}^u \left( S_{u[B]}^A, S_{u[A]}^A \right) = \left( 0 \quad \tilde{\alpha} T^{[CBA]}_{\dot{\alpha}} \right)
\]  

(33)

\[
X_{[FDC]}^\gamma u \left( S_{u[B]}^A, S_{u[A]}^A \right) = \left( 0 \quad \beta \epsilon_{FDCBAG2G3} T_{[G1G2G3]}^{[CBA]} \right)
\]  

(34)

\[
X_{[FDC]}^{\dot{\gamma}} u \left( S_{u[B]}^A, S_{u[A]}^A \right) = \left( \bar{\beta} \epsilon_{FDCBAG2G3} T_{[G1G2G3]}^{[CBA]} 0 \right).
\]  

(35)

These assumptions assure that the gravitini fields \(T_{[CBA]}^\alpha\) and \(T^{[CBA]}_{\dot{\alpha}}\) are the only 1/2 helicity fields in the geometry, and in particular, all the torsion components of canonical dimension 1/2 can be expressed using these fields (see appendix B). Also, with these additional assumptions the Bianchi identities \(\left(D_{\delta}^{DCB} u\right)_{\gamma}[\delta \gamma B] \left(D_{\delta}^{DCB} u\right)_{\gamma}[\delta \gamma A] T^{[CBA]}_{[EFG]}\delta\) with their complex conjugates imply that the Lorentz scalar \(T\) transforms under supersymmetry transformations into the helicity 1/2 fields:

\[
(D_{\delta}^{DCB} u) T = \left( \begin{array}{cc}
0 & 0 \\
-\frac{1}{2} \epsilon_{FDCBAG2G3} T_{[EFG]}^{[CBA]} \delta & 0
\end{array} \right)
\]  

(36)

\[
(D_{\delta}^{DCB} u) T = \left( \begin{array}{cc}
0 & -\frac{1}{2} \epsilon_{FDCBAG2G3} T_{[EFG]}^{[CBA]} \delta \\
-\frac{1}{2} \delta_{FDCBAG2G3} T_{[EFG]}^{[CBA]} \delta & 0
\end{array} \right)
\]  

(37)

with \(\alpha, \tilde{\alpha}, \beta\) and \(\bar{\beta}\) some complex parameters, which will be determined by consistency requirements at higher canonical dimensions, in sections 3.4 and 3.5.
3.4 Self-duality and anti-self-duality of the graviphotons

The use of central charge superspace allowed to identify the gauge vectors of the multiplet in the super-vielbein (10). Notice, on the one hand that at this stage we identified in the geometry as many gauge vectors \( v_m \) as the number of central charge indices, that is 56. On the other hand, the number of graviphotons taking part of the \( N = 8 \) supergravity multiplet is only the half of that, that is 28. The aim of this paragraph is to clarify this problem by analyzing how the Bianchi identities imply specific properties satisfied by the field strength of the gauge vectors \( v_m \), reducing their degrees of freedom to the half.

Since the vectors are identified in the vielbein, their super-covariant field strength is the torsion component \( T_{ba} \), which we denote in the following by \( F_{ba} \), and which in turn can be converted in the \( SU(8) \) basis by

\[
\begin{pmatrix}
F_{ba[DC]} & F_{ba[DC]}
\end{pmatrix} = F_{ba} \begin{pmatrix}
S_{u[DC]} & S_{u[DC]}
\end{pmatrix}.
\]

It is worthwhile to present here briefly the intriguing interplay of the Bianchi identities in order to determine the properties of this object.

First of all, recall the very general result of the Bianchi identities

\[
D^\alpha_\beta T^{\lambda\mu\nu}_\alpha = -i \delta^\alpha_\beta G_{\gamma\delta\epsilon\mu}\epsilon_{\epsilon\lambda\mu\nu},
\]

\[
D^\dot{\alpha}_\dot{\beta} T^{\lambda\mu\nu}_{\dot{\alpha}} = -i \delta^\dot{\alpha}\dot{\beta} G_{\dot{\lambda}\dot{\gamma}\dot{\delta}\epsilon\mu}\epsilon_{\epsilon\lambda\mu\nu}.
\]

Here the superfields \( G_{(\beta\dot{\alpha})[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) and \( G^{(\beta\dot{\alpha})[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) are the self–dual and respectively, the anti–self–dual part of the a priori arbitrary antisymmetric tensor superfields \( G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) and \( G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \), appearing in the torsion components

\[
T_{\dot{\gamma}\gamma}^\alpha_{\dot{\beta}} = \frac{1}{2} (\sigma^f)_{\dot{\gamma}\gamma}^\alpha (\eta_{fb} T^{(\gamma\alpha)}_{(\gamma\alpha)} - G_{f\dot{b}b(\gamma\alpha)}) ,
\]

\[
T_{\gamma\gamma}^\alpha_{\dot{\gamma}} = \frac{1}{2} (\sigma^f)_{\gamma\gamma}^\alpha (\eta_{fb} T^{(\gamma\alpha)}_{(\gamma\alpha)} - G_{f\dot{b}b(\gamma\alpha)}) ,
\]

as they are given by the Bianchi identities \( (\delta^\alpha_{\dot{\alpha}})_{\epsilon\lambda\mu\nu} \) and \( (\delta^\dot{\alpha}_{\dot{\beta}})_{\epsilon\lambda\mu\nu} \). Now the question is how to relate the antisymmetric tensors \( G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) and \( G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) to the field strength \( F_{ba} \) of the graviphotons, since a relation between these two objects would insure that, as expected, the supersymmetry transform of the 1/2 helicity fields contains the field strength of the gauge vectors.

The response is given by the Bianchi identity \( (\delta^\alpha_{\dot{\alpha}})_{\epsilon\lambda\mu\nu} \), which turns out to be a real mine of information. Namely, it implies that the antisymmetric tensors \( G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) and \( G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} \) are related to the \( SU(8) \) components of the field strength of the graviphotons in a very simple way:

\[
G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} = -8i F_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} ,
\]

\[
G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} = -8i F_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} .
\]

Moreover, it implies that the parts of the field strength which are not present in the spinorial derivative of the helicity 1/2 fields vanish in the linear approach. They are given as quadratic terms in the gravigini:

\[
F^{(\delta\dot{\beta})}_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} = \frac{G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu}^{(\delta\dot{\beta})}}{3!} \epsilon_{\epsilon\lambda\mu\nu} T^{[\epsilon\lambda\mu\nu]_{\epsilon\lambda\mu\nu}^{(\delta\dot{\beta})}} ,
\]

\[
F^{(\delta\dot{\beta})}_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu} = \frac{G_{ba[\beta\dot{\alpha}]\epsilon\lambda\mu\nu}^{(\delta\dot{\beta})}}{3!} \epsilon_{\epsilon\lambda\mu\nu} T^{[\epsilon\lambda\mu\nu]_{\epsilon\lambda\mu\nu}^{(\delta\dot{\beta})}} .
\]
These are the relations which permit us to write down the self–duality properties of the graviphotons (51) and explain the reduction of the graviphotons’ degrees of freedom.

Finally, the same Bianchi identity implies the vanishing of the superfields $T^{[\alpha]}$ and $T_{[\alpha]}$, giving the antisymmetric part of the spinorial derivative of the gravigini as a quadratic term in themselves

$$D^\beta \bar{D} T^{[\alpha]}_{[\alpha]} = -\frac{\bar{\beta}}{2} \varepsilon^{[\alpha \beta \gamma \delta \epsilon] T^{[\alpha]}_{[\gamma] \epsilon}}$$

and fixes the parameters $\alpha$ and $\bar{\alpha}$ by implying the relations

$$\left( \alpha - \frac{1}{16} \right) F^{(\delta \beta)}_{[\alpha]} = 0, \quad \left( \bar{\alpha} - \frac{1}{16} \right) F^{(\delta \beta)}_{[\alpha]} = 0.$$

Recall, that the parts of the field strength $F$ which appear in relations (48) are those, in which the gravigini fields transform under supersymmetry and we require that they do not vanish. So, the parameters $\alpha$ and $\bar{\alpha}$ are determined to be $\alpha = \bar{\alpha} = 1/16$.

As a conclusion of this paragraph let us denote the objects

$$F_{[\alpha \beta]}^{+} = F_{[\alpha \beta]} - \frac{1}{16} \frac{\bar{\beta}}{3!} \varepsilon^{[\alpha \beta \gamma \delta \epsilon] F_{[\gamma \epsilon]}}$$

$$F_{[\alpha \beta]}^{-} = F_{[\alpha \beta]} - \frac{1}{16} \frac{\beta}{3!} \varepsilon^{[\alpha \beta \gamma \delta \epsilon] F_{[\gamma \epsilon]}}$$

which in virtue of the relations (44) and (45) satisfy the self–duality and anti–self–duality relations

$$\frac{i}{2} \varepsilon^{dc \beta a} F_{[\alpha \beta]}^{+} = F^{+dc}_{[\alpha \beta]}, \quad \frac{i}{2} \varepsilon^{dc \beta a} F_{[\alpha \beta]}^{-} = -F^{-dc}_{[\alpha \beta]},$$

and are the parts of the graviphoton field strength which effectively take part of the multiplet. These self–duality relations can also be given in the central charge basis as

$$\frac{i}{2} \varepsilon^{dc \beta a} F_{[\alpha \beta]} u = F^{dc}_{[\alpha \beta]} \left( S \omega T \right)_{[u]}$$

$$= \frac{1}{8} \frac{\bar{\beta}}{3!} \varepsilon^{B F_{1} F_{2} F_{3}} (T^{[F_{1} F_{2} F_{3}]} \bar{\sigma}_{[F_{4} F_{5} F_{6}]} T^{[F_{4} F_{5} F_{6}]}) T_{[B A]} u + \frac{1}{8} \frac{\beta}{3!} \varepsilon^{B F_{1} F_{2} F_{3}} (T^{[F_{1} F_{2} F_{3}]} \sigma_{[F_{4} F_{5} F_{6}]} T^{[F_{4} F_{5} F_{6}]}) T_{[B A]} u,$$

with

$$\omega = \begin{pmatrix} \frac{\beta}{2} & 0 \\ 0 & -\frac{\beta}{2} \end{pmatrix}$$

an $56 \times 56$ matrix. In the linear approach one can recognize in (52) the twisted self–duality relation for graviphotons [14], which is also called the third silver rule of supergravity\textsuperscript{3}. Let

\textsuperscript{3}It is possible to construct an object $\tilde{F}_{[\alpha \beta]}$ satisfying the pure twisted self–duality relation

$$\frac{i}{2} \varepsilon^{dc \beta a} \tilde{F}_{[\alpha \beta]} u = \tilde{F}^{dc}_{[\alpha \beta]} \left( S \omega T \right)_{[u]},$$

even in the full non–linear case, using the self–dual and anti–self–dual field strengths $F_{[\alpha \beta]}^{+}$ and $F_{[\alpha \beta]}^{-}$:

$$\tilde{F}_{[\alpha \beta]} u = \begin{pmatrix} F_{[\alpha \beta]}^{+} [DC] & F_{[\alpha \beta]}^{-} [DC] \\ T^{[DC]}_{[u]} & T_{[DC]}^{[u]} \end{pmatrix}. $$
us anticipate here, that the matrix $\omega$ plays the rôle of an invariant operator acting on the 56 dimensional representation of the Lie group $K$ for a supergravity theory, where the scalars are organized in a $G/K$ non–linear sigma model. In our case one finds a $E_{7(+7)}/SU(8)$ non–linear sigma model structure for the scalars – but this will be the subject of the next paragraph.

3.5 The $E_{7(+7)}/SU(8)$ non–linear sigma model

As already observed on the component level [3], [4] and put in evidence in the former super–space approaches [7], [8], [10], the hypothesis that the scalars take part of an $E_{7(+7)}/SU(8)$ non–linear sigma model is compatible with the structure of the multiplet. General features of $G/K$ non–linear sigma models with $G$ a non–compact Lie group and $K$ its maximal compact sub–group can be found in [23], [24], [25].

Recall however, that in component approaches this structure is related to the existence of a duality invariance [3], [25], while the identification of $G/K$ as the duality group is based only on considerations on the dimensions of the interplaying Lie groups [3], [4]. Then, the existing geometrical descriptions, in order to recover this structure, use constraints inspired by the 56 dimensional, fundamental representation of the Lie algebra of $E_{7(+7)}$ on the supercovariant field strength of the scalars. The aim of this paragraph is to analyze the properties concerning these aspects implied naturally by the constraints presented so far.

As in the case of the gauge vectors there is an ensemble of Bianchi identities which interplay in order to give the properties of the field strength of the scalars identified in the $56 \times 56$ matrix $(D_0 T) S$.

First of all, as we expect to identify the field strength of the scalars in the supersymmetry transform of the gravition fields, let us recall the general result of the analysis of Bianchi identities $(\delta^C \gamma^D T)$ and $(\hat{\delta}^C \gamma^D T)$. Namely, that they are satisfied if and only if the covariant spinorial derivatives of the spinor fields $T_{[CBA]}\hat{\alpha}$ and $T^{[CBA]}\hat{\alpha}$ giving a Lorentz vector are totally antisymmetric in their $SU(8)$ indices:

$$D^i_\delta T^{[CBA]}\hat{\alpha} = P_{\hat{\alpha}}^{\hat{\alpha} [DCBA]}, \quad D^i_D T_{[CBA]}\alpha = P_{\alpha}^{\alpha [DCBA]}, \quad (54)$$

with $P_{\beta}^{\alpha [DCBA]}$ and $P_{\beta}^{\alpha [DCBA]}$ a priori some arbitrary superfields.

Now again, the essential question is how can we relate the field strength of the scalars, $(D_0 T) S$, to the superfields $P_{\beta}^{[DCBA]}$ and $P_{\alpha}^{[DCBA]}$. In order to answer this question, notice that the covariant derivatives $D_0 T^{[DC]} u$ and $D_0 T_{[DC]} u$ appear explicitly in the Bianchi identities $(\delta^C \gamma^D u) T^i$ and $(\hat{\delta}^C \gamma^D u) T^i$. These are the identities which imply that the field strength $(D_0 T) S$ of the scalars should take the form

$$(D_0 T) S = \left( \begin{array}{c}
-2\delta^D_{[b} T^c_{b]} \alpha & -\frac{i}{2} P_{\beta}^{[DCBA]} \\
-\frac{i}{2} P_{\alpha}^{[DCBA]} & 2\delta^{[b} P_{\alpha]}_{[b} c \end{array} \right), \quad (55)$$

that is, it contains the superfields $P$ as off–diagonal blocks. Moreover, using the algebra of covariant derivatives, one can easily show at this stage, that the superfields $P$ with upper $SU(8)$ indices are related to the superfields $\hat{P}$ with lower $SU(8)$ indices by the totally antisymmetric tensor,

$$P_{\beta}^{A_1...A_8} = -\frac{\beta}{2} \epsilon^{A_1...A_8} P_{\beta [A_1...A_8]}, \quad P_{\beta}^{A_1...A_8} = -\frac{\beta}{2} \epsilon_{A_1...A_8} P_{\beta [A_1...A_8]}, \quad (56)$$
and the consistency of these two relations fixes the parameters $\beta$ and $\bar{\beta}$ to be $\beta = -\eta/12$ and $\bar{\beta} = -\bar{\eta}/12$, with $\eta\bar{\eta} = 1$. As a consequence, one has the following $\eta$–duality relations:

$$P_b^{[A_1...A_4]} = \frac{\eta}{4!}\varepsilon^{A_1...A_8} P_b^{[A_5...A_8]}, \quad P_{\bar{b}}^{[A_1...A_4]} = \frac{\bar{\eta}}{4!}\varepsilon^{A_1...A_8} P_{\bar{b}}^{[A_5...A_8]}.$$ \hspace{1cm} (57)

Concerning the object $\hat{T}_b^{C_A}$, appearing on the diagonal of (55), is defined as a function of an a priori arbitrary superfield $T_b^{C_A}$,

$$\hat{T}_b^{C_A} = T_b^{C_A} + \frac{i}{16} T_{[\alpha\beta]} \sigma_{\beta} T^{[\alpha\beta]},$$ \hspace{1cm} (58)

which in turn appears in the decomposition of the torsion components $T_{a b}^{\alpha (a}$ and $T_{c b}^{\alpha (a}$ as it is given by the Bianchi identities ($\frac{\tilde{b}^{\alpha}}{\tilde{b}^{\alpha}}$)$_{\alpha}$ and ($\frac{\tilde{b}^{\alpha}}{\tilde{b}^{\alpha}}$)$_{\alpha}$:

$$T_{a b}^{\alpha (a} = -2T_{b a}^{C_A}, \quad T_{c b}^{\alpha (a} = 2T_{b c}^{C_A}. \hspace{1cm} (59)$$

Since this superfield $T_b^{C_A}$ is left undetermined by the constraints we put so far and since we do not need independent superfields any more in the geometry, we fix it in such a way that we have 0 on the diagonal of the $(D_b T)S$ matrix\(^4\). This choice is suggested by the analogy with the matrices $(D_b^{\alpha})S$ and $(D^b_T)S$ of (36) and (37). Therefore, we have

$$T_b^{C_A} = -\frac{i}{16} T_{[\alpha\beta]} \sigma_{\beta} T^{[\alpha\beta]},$$ \hspace{1cm} (60)

and since we also determined the values of the parameters $\alpha, \bar{\alpha}, \beta, \bar{\beta}$ in the expressions (36) and (37) of the components with spinorial indices, we can sum up the results concerning the 1–form $\Omega = (DT)S$ as follows:

$$\Omega_{\delta}^D = -\frac{1}{2} \frac{1}{3!} \left( \begin{array}{cc} 0 & \eta^{cBbA^DEFG} T_{[EFG]\delta} \\ \eta^{cBbA^DEFG} T_{[EFG]\delta} & 0 \end{array} \right),$$ \hspace{1cm} (61)

$$\Omega_{\delta}^D = -\frac{1}{2} \frac{1}{3!} \left( \begin{array}{cc} 0 & \delta^{BbA^DEFG} T_{[EFG]\delta} \\ \delta^{BbA^DEFG} T_{[EFG]\delta} & 0 \end{array} \right),$$ \hspace{1cm} (62)

$$\Omega_a = -\frac{i}{4} \left( \begin{array}{cc} 0 & P_{a[BAB']} \\ P_{a[BAB']} & 0 \end{array} \right),$$ \hspace{1cm} (63)

$$\Omega_{\mu} = 0.$$ \hspace{1cm} (64)

As a consequence, we have to just read out the general matricial structure of the entire form $\Omega$:

$$\Omega = (DT)S = \left( \begin{array}{cc} 0 & \Omega^{[DCBA]} \\ \Omega^{[DCBA]} & 0 \end{array} \right).$$ \hspace{1cm} (65)

where the forms $\Omega^{[DCBA]}$ and $\Omega^{[DCBA]}$ are related by the $\eta$–duality relations

$$\Omega_{[HGFE]} = \frac{\eta}{4!}\varepsilon_{HGFE} DCBA \Omega^{[DCBA]}, \quad \Omega_{[HGFE]} = \frac{\bar{\eta}}{4!}\varepsilon_{HGFE} DCBA \Omega^{[DCBA]}, \quad \eta\bar{\eta} = 1. \hspace{1cm} (66)$$

\(^4\)Recall that an analysis of conventional constraints on torsion components in extended supergravity [20], [19] shows that the fixing of the traceless part of $T_b^{C_A}$ is nothing but a redefinition of the $SU(8)$ connection component $\Phi_b^{C_A}$. However, as the structure group is only $SU(8)$ and not $U(8)$, a fixing of the trace part $T_b^{A\lambda}$ cannot be interpreted as a conventional constraint.
Let us emphasize here, that although an analogous one–form $\Omega$ is present in previous geometrical descriptions \cite{8}, \cite{10}, its matricial properties like the total antisymmetry of $\Omega_a^{[DCBA]}$ and $\Omega_a^{[DCBA]}$ as well as the $\eta$–type duality relations were imposed \cite{10} and not deduced from lower dimensional constraints. Concerning our approach we can say that the total antisymmetry as well as $\eta$–duality of the components of $\Omega$, defined as $\Omega = (DT)S$, have their origin in the requirement that there should be only one type of 1/2 helicity fields in the multiplet, that is in constraints at 1/2 canonical dimension.

Moreover, as a consequence of its definition, the one–form $\Omega$ has further remarkable properties. First of all, it has the decomposition

$$\Omega = (DT)S = (dT)S - \Phi_{SU(8)},$$

(67)

with $\Phi_{SU(8)}$ the $SU(8)$ connection in the $56 \times 56$ representation

$$\Phi_{SU(8)} = \begin{pmatrix} 2\delta_{[B}^{[D} \Phi_{C]}^A & 0 \\ 0 & -2\delta_{[D}^{[B} \Phi_{A]}^C \end{pmatrix}. \quad (68)$$

Since it is defined using the covariant derivative, the fact that $\Omega$ transforms in a covariant manner under $SU(8)$ gauge transformations,

$$\delta_X \Omega = -X_{SU(8)} \Omega + \Omega X_{SU(8)}, \quad (69)$$

with $X_{SU(8)}$ the gauge parameter in the $56 \times 56$ representation (68), is obvious. Finally, $\Omega$ satisfies the identity

$$D\Omega + \Omega \Omega + R_{SU(8)} = 0, \quad (70)$$

with again, $R_{SU(8)}$ the $SU(8)$ curvature in the $56 \times 56$ representation (68).

All these properties of $\Omega$ show that it is a super–analogue of the one–form (usually denoted by $P$) used to write down the Lagrangian of a non–linear sigma model on $G/K$ coset space with $G$ a non–compact Lie group and $K$ its maximal compact subgroup. Clearly, the gauge group $SU(8)$ plays the rôle of the compact group $K$, so what remains, is to identify the group $G$. As a matter of fact we only have access to the properties of a representation of the Lie algebra of $G$, where the object $(dT)S$ takes its values. Observe, that equation (67) gives the form of the matrix $(dT)S$:

$$(dT)S = \begin{pmatrix} 2\delta_{[B}^{[D} \Phi_{C]}^A & \Omega_{[DCBA]}^{[D}\Phi_{C]}^A \\ \Omega_{[DCBA]}^{DCBA} & -2\delta_{[D}^{[B} \Phi_{A]}^C \end{pmatrix}, \quad \text{with } \Omega^{[HGF]} = \frac{\eta}{4!} \varepsilon^{HGFEDCBA} \Omega_{[DCBA]}, \quad (71)$$

where one can recognize the $56 \times 56$ representation of the Lie algebra of $E_{7(+7)}$ (see appendix D of the article \cite{3}). Here the matrix decomposition on the diagonal and off–diagonal parts in the $28 \times 28$ blocks correspond to the decomposition of the Lie algebra of $E_{7(+7)}$ on its Lie subalgebra $su(8)$ and the orthogonal complement of $su(8)$ with respect to the Killing form. It is straightforward to verify that the $su(8)$ part corresponds to the compact, while its complement, the off–diagonal part, corresponds to the non–compact part of $E_{7(+7)}$. Therefore, we accept that indeed, the scalars live on the $E_{7(+7)}/SU(8)$ coset space, so we recovered the first silver rule for the maximally dualised form of the $N = 8$ supergravity \cite{14}.

As a final check, one may recall as usual that the dimension of $E_{7(+7)}$ and $SU(8)$ is equal to 133 and respectively 63, therefore, the dimension of the coset space parametrized by the scalar fields is 70: exactly the number of degrees of freedom in the field strength $\Omega_a$ of the scalars, that is $C^4_8$, and exactly the number of degrees of freedom associated to 0 helicity fields in an $N = 8$ supergravity multiplet.
3.6 The $SU(8)$ connection

Recall, that the $SU(8)$ connection 1–form $\Phi^B_A$ is introduced in the geometry as an a priori independent object. However, the constraints we imposed imply that it can be expressed as a function of the scalars and their derivatives. Indeed, the diagonal part of equation (67) gives immediately the expression

$$\Phi^B_A = \frac{1}{3} \left( dT^{[FB]} u \right) S_{u[F]} = -\frac{1}{3} \left( dT^{[FA]} u \right) S_{u^[FB]} ,$$

(72)

with the property

$$\left( dT^{[1]} u \right) S_{u[1]} = \left( dT^{[1]} u \right) S_{u^[1]} = 0 ,$$

(73)

which ensures that the connection is traceless. Therefore, the $SU(8)$ connection $\Phi^B_A$ is neither an independent field, it can be expressed in terms of the scalar fields of the multiplet and of their space–time derivatives:

$$\Phi^B_A = \frac{1}{3} dx^m \Phi^B_A = \frac{1}{3} dx^m \left( \partial_m T^{[FB]} u \right) S_{u[F]} = -\frac{1}{3} dx^m \left( \partial_m T^{[FA]} u \right) S_{u^[FB]} ,$$

(74)

relation, which is in agreement with the expression given by de Wit and Nicolai [4].

Also, it should be noted that the diagonal part of equation (70) expresses the $SU(8)$ curvature in terms of $\Omega$ in a simple way,

$$R^B_A = -\frac{1}{3} \Omega^{[BJK]} \Omega_{[AJK]} .$$

(75)

The identification of the $SU(8)$ connection was possible because of the vanishing of the central charge connection (11). With this condition, the Bianchi identities lead us to the following results: all the curvature components with at least one lower central charge index as well as the central charge derivatives of all torsion components vanish,

$$R_{\alpha CB}^A = 0 , \quad D_{\alpha} T_{CB}^A = 0 .$$

(76)

Due to the these equations and equation (11), the central charge sector could appear to be trivial. Nevertheless, it is essential on the one hand in identifying vectors in the vielbein and the scalars in the torsion, and on the other hand in deducing the essential properties as the self–duality properties of the vectors and the non–linear sigma model structure of the scalars.

4 Supergravity transformations of the component fields

Once the component fields of the supergravity multiplet are identified, the aim of the present section is to deduce their supergravity transformations in terms of component fields and compare these transformations with those found on the component level [3], [4].

Recall that one of the fundamental advantages of the geometrical description in central charge superspace is that space-time diffeomorphisms, supersymmetry and gauge transformations identified as central charge transformations are treated on the same footing as superspace diffeomorphisms. Given that component fields were identified in the super vielbein and in torsion components and also that we know how such geometrical objects transform under supergravity transformations (8), it is a straightforward exercise to write down how these transformations act on components.
Let us begin with the component fields identified in the frame (10): the graviton, the gravitini and graviphotons. Their supergravity transformations can be read out of the explicit component expansion

$$
\delta \xi^A E^m A = D^m \xi^A + E^m B^C T_{CB} A
$$

of the transformation (8) taking the lowest superfield component (noted here by the bar ) and choosing for $A$ either vector, spinorial or respectively central charge indices. As for the scalars and spinor fields, identified in torsion components, that is as the lowest superfield component of super–covariant fields $V$, their supergravity transformation is simply

$$
\delta \xi^A V = L_\xi V = \xi^A \mathcal{D}_A V.
$$

However, in order to make explicit these transformation laws, we need the expression in component fields of the lowest superfield components of the basic superfields appearing in torsion components of (77) and covariant derivatives of (78). The aim of the next subsection is precisely to give the list of the necessary expansions.

### 4.1 Supercovariant → component toolkit

The basic superfields appearing in torsion components (see appendix B) are all super–covariant quantities and their lowest superfield components inherit this property. Let us sum up in this paragraph the component expressions of the supercovariant field strengths, needed to write down supergravity transformations of the component fields. General formulas used to determine these expressions are easily written using the notation $E^A \| = e^A = \text{d}x^m e^m A$ [16].

Recall that the graviton, gravitini and graviphotons are identified in the super-vielbein. Thus, their field strength can be found in their covariant counterparts using

$$
T_{mn}^A = \frac{1}{2} \text{d}x^m \text{d}x^n (D_m e^m A - D_n e^n A) = \frac{1}{2} e^B e^C T_{CB} A [\text{79}].
$$

For $A = a$, one finds the relation

$$
D_m e^m a - D_n e^n a = i \bar{\psi}_{nA} \sigma^a \psi_m A,
$$

which determinates the Lorentz connection $\Phi_{mkl} = e^b e^l a \Phi_{mba}$ in terms of the vierbein, its derivatives and gravitini fields

$$
\Phi_{mkl} = \frac{1}{2} (e_m a \partial_k e_l a - e_l a \partial_m e_k a - e_k a \partial_l e_m a) - (k \leftrightarrow l)
$$

$$
- \frac{i}{4} (\psi_{m\Lambda} \sigma_k \bar{\psi}_l A - \psi_{l\Lambda} \sigma_m \bar{\psi}_k A + \psi_{k\Lambda} \sigma_l \bar{\psi}_m A) - (k \leftrightarrow l).
$$

For $A = u$, the central charge indices, we obtain in general the covariant field strength of the graviphotons

$$
F_{ba}^u = e_b^m e^m a T_{mn}^u + e_b^m e^m a \left[ \frac{1}{4} (\bar{\psi}_n C \psi_m B) + \frac{i}{8} (\bar{\psi}_{nA} \sigma_m A) \lambda^{[CBA]} \right] T_{[CB]} u
$$

$$
+ e_b^m e^m a \left[ \frac{1}{4} (\bar{\psi}_{nC} \psi_m B) + \frac{i}{8} (\bar{\psi}_{nA} \sigma_m A) \lambda^{[CBA]} \right] T^{[CB]} u,
$$

\footnote{One may observe that in [4], page 334, the Lorentz connection depends also on the spinor fields. However, this difference is just a matter of redefinition, it corresponds in our geometrical description to the replacement of the conventional constraint $T_{cb} a = 0$ by $T_{cb} a = \lambda \epsilon_{dcb} a (T_{[CBA]} \sigma^d T^{[CBA]}).
The field strength of the graviphotons \( F_{nm}^u \) is the field strength of the graviphotons, and it can be written in the \( SU(8) \) basis as:

\[
F_{ba[cb]} = e_b^ne^m_c F_{nm}^u S_{u[cb]} + e_b^ne^m_c \left[ \frac{1}{4} (\psi_{nC} \psi_{mB}) + \frac{i}{8} (\bar{\psi}_n^A \bar{\sigma}_m \lambda_{[CB]}^A) \right],
\]

(83)

and

\[
F_{ba[cb]} = e_b^ne^m_c F_{nm}^u S_{u[cb]} - \frac{1}{8} \text{tr}(\sigma_{nm} \sigma^kl) \left[ (\psi_{kC} \psi_{lB}) + \frac{i}{2} (\bar{\psi}_k^A \bar{\sigma}_l \lambda_{[CB]}^A) \right],
\]

(85)

However, as we already have seen in section 3.4, the field strength of the graviphotons satisfies self-duality relations, which reduce their degrees of freedom to the half. We have seen in particular, that the dynamic part of this field strength corresponds to the self-dual part \( F_{ba[BA]} \) of the component \( F_{ba[BA]} \) of the component \( F_{ba[BA]} \), while the remaining parts (the anti-self-dual part of the component \( F_{ba[BA]} \) and the self-dual part of the component \( F_{ba[BA]} \)) are given as quadratic terms in the spinor superfields (44), (45). Therefore, we are rather interested in the expression of lowest superfield components of the objects \( F_{ba[BA]} \) and \( F_{ba[BA]} \):

\[
F_{ba[CB]} = e_b^ne^m_c F_{nm}^u S_{u[CB]} - \frac{1}{8} \text{tr}(\sigma_{nm} \sigma^kl) \left[ (\psi_{kC} \psi_{lB}) + \frac{i}{2} (\bar{\psi}_k^A \bar{\sigma}_l \lambda_{[CB]}^A) \right],
\]

(87)

These are the objects, which correspond to the super-covariant field strength of the graviphotons \( \hat{F}_{nm[cb]} \) and \( \hat{F}_{nm[cb]} \) used on the component level in the article [4], which can be defined as

\[
\hat{F}_{nm[cb]} = e_b^ne^m_c F_{nm}^a F_{ba[cb]}, \quad \hat{F}_{nm[cb]} = e_b^ne^m_c F_{nm}^a F_{ba[cb]},
\]

(88)

and have the expressions

\[
\hat{F}_{nm[cb]} = F_{nm}^u S_{u[cb]} - \frac{1}{8} \text{tr}(\sigma_{nm} \sigma^kl) \left[ (\psi_{kC} \psi_{lB}) + \frac{i}{2} (\bar{\psi}_k^A \bar{\sigma}_l \lambda_{[CB]}^A) \right],
\]

(89)

In order to be able to compare our results to those of the component approach in [4], we will systematically use the objects \( \hat{F}_{nm[cb]} \) and \( \hat{F}_{nm[cb]} \) in our component formulas.

As for \( A = \lambda_\alpha \) and \( A = \lambda_\alpha \), we have the expression of the covariant field strength of the gravitini

\[
T_{cbk}^\alpha = e_b^m e_c^m D_{[n} \psi_{m\alpha]} - 4ie_b^m F_{c[a]} \left[ (\bar{\psi}_n^B \bar{\sigma}_n^A) + \frac{1}{4} e_b^m e_c^m (\bar{\psi}_n^B \bar{\psi}_n^B) \lambda_{[CB]}^A \right]
\]

\[
- \frac{i}{16} e_b^m e_c^m (\bar{\psi}_{nB} \bar{\sigma}_{nB}^A) \lambda_{[BEC]} \lambda_{[BEC]} + \frac{i}{48} e_b^m e_c^m (\bar{\psi}_{nB} \bar{\sigma}_{nB}^A) \lambda_{[BEC]} \lambda_{[BEC]},
\]

(90)

and

\[
T_{cbk}^\alpha = e_b^m e_c^m D_{[n} \psi_{m\alpha]} - 4ie_b^m F_{c[a]} \left[ (\bar{\psi}_n^B \bar{\sigma}_n^A) - \frac{1}{4} e_b^m e_c^m (\bar{\psi}_n^B \bar{\psi}_n^B) \lambda_{[CBA]} \right]
\]

\[
+ \frac{i}{16} e_b^m e_c^m (\bar{\psi}_{nB} \bar{\sigma}_{nB}^A) \lambda_{[CBA]} - \frac{i}{48} e_b^m e_c^m (\bar{\psi}_n^A \bar{\sigma}_{nB}^A) \lambda_{[BEC]} \lambda_{[BEC]},
\]

(91)

Defining \( \hat{\psi}_{nm}^\alpha = e_n^c e_m^b T_{cbk}^\alpha \) and respectively \( \hat{\psi}_{nm}^\alpha = e_n^c e_m^b T_{cbk}^\alpha \), one obtains the supercovariant field strength of the gravitini used in the component formalism in the article.
[4]:

\[ \dot{\Psi}_{nm}^\alpha = D_n [\dot{\psi}_m^\alpha]_A + \frac{1}{4} (\dot{\psi}_n^C \dot{\psi}_m^B) \lambda_{[CBA]}^\alpha \]

\[ + 4i (\dot{\psi}_n^B \dot{\sigma}^I) \left[ \dot{F}^+_m[t,BA] - \frac{n}{2(4!)^2} \varepsilon_{BAP} \varepsilon_{123456} (\lambda_{[P_1P_2P_3]} \sigma_m[I] \lambda_{[P_4P_5P_6]}^I) \right] \]

\[ + \frac{i}{16} \left( \dot{\psi}_n^{\bar{B}} \dot{\sigma}^I \alpha (\lambda_{[AEF]} \sigma^I \lambda_{[BFP]}^I) - \frac{i}{48} (\dot{\psi}_{[nA} \sigma_m[I] \lambda_{[EFG]}^I \lambda_{[F]}^I \lambda_{[G]}^I) \right), \]

\[ \dot{\Psi}_{nm}^{\dot{\alpha}} = D_n [\dot{\psi}_m^{\dot{\alpha}}]_A + \frac{1}{4} (\dot{\psi}_n^C \dot{\psi}_m^B) \lambda_{[CBA]}^{\dot{\alpha}} \]

\[ + 4i (\dot{\psi}_n^B \dot{\sigma}^I \dot{\alpha} \left[ \dot{F}^+_m[t,BA] - \frac{n}{2(4!)^2} \varepsilon_{BAP} \varepsilon_{123456} (\lambda_{[P_1P_2P_3]} \sigma_m[I] \lambda_{[P_4P_5P_6]}^I) \right] \]

\[ - \frac{i}{16} \left( \dot{\psi}_n^{\bar{B}} \dot{\sigma}^I \dot{\alpha} (\lambda_{[AEF]} \sigma^I \lambda_{[BFP]}^I) + \frac{i}{48} (\dot{\psi}_{[nA} \sigma_m[I] \lambda_{[EFG]}^I \lambda_{[F]}^I \lambda_{[G]}^I) \right). \]

Finally, for the field strength of the scalars \( \Omega_a \), we can use the definition of the double projection on the one–form \( \Omega \),

\[ \Omega_a = dx^m \Omega_m = e^A \Omega_A, \quad (90) \]

and obtain

\[ \Omega_a = e_a^m \left( \Omega_m - \frac{1}{2} \psi_m^{\alpha} \Omega_a^{\alpha} - \frac{1}{2} \dot{\psi}_m^{\dot{\alpha}} \Omega_a^{\dot{\alpha}} \right), \quad (91) \]

with \( \Omega_m = (D_m T) S \) the ordinary field strength of the scalars. Then, using the expressions of the matrix components in (61), (62), we obtain for the two o ff–diagonal blocks

\[ \Omega_a^{[DCBA]} = e_a^m \left( \Omega_m^{[DCBA]} + \frac{\eta}{4!} \varepsilon^{DCBAEFGH} \left( \psi_{mE} \lambda_{[FGH]}^I \right) + \left( \dot{\psi}_m^{[DCBA]} \right) \right), \quad (92) \]

\[ \Omega_a^{[DCBA]} = e_a^m \left( \Omega_m^{[DCBA]} + \left( \psi_m^{[DCBA]} \right) + \frac{\eta}{4!} \varepsilon^{DCBAEFGH} \left( \dot{\psi}_m^{[DCBA]} \right) \right). \quad (93) \]

Analogously to the previous case, the super–covariant field strength of the scalars is defined as \( \dot{\Omega}_m = e_m^{\alpha} \Omega_a \) and has the component expansions

\[ \dot{\Omega}_m^{[DCBA]} = \Omega_m^{[DCBA]} + \frac{\eta}{4!} \varepsilon^{DCBAEFGH} \left( \psi_{mE} \lambda_{[FGH]}^I \right) + \left( \dot{\psi}_m^{[DCBA]} \right), \quad (94) \]

\[ \dot{\Omega}_m^{[DCBA]} = \Omega_m^{[DCBA]} + \left( \psi_m^{[DCBA]} \right) + \frac{\eta}{4!} \varepsilon^{DCBAEFGH} \left( \dot{\psi}_m^{[DCBA]} \right). \quad (95) \]

We are now ready to explicite supergravity transformations of the component fields.

### 4.2 Supersymmetry transformations

Recall that in the superspace description supersymmetry transformations are supergravity transformations with only spinorial non–zero parameters

\[ \xi^A = (0, \xi^\alpha, \xi^{\dot{\alpha}}, 0). \quad (96) \]

Therefore, using the general expressions (77), (78) as well as the expressions of torsion components and spinorial derivatives of the basic fields summed up in appendix B, we have the following component transformation laws.
For the graviton, gravitini and graviphotons we have

\[ \delta^w_{\xi} e_m^a = i \left( \xi_a \sigma^a \psi_m \Lambda + \bar{\xi} \bar{\sigma}^a \bar{\psi}_{m\Lambda} \right), \]

\[ \frac{1}{2} \delta^w_{\xi} \tilde{\psi}_m^\alpha = D_m \tilde{\xi}^\alpha - \frac{1}{2} (\tilde{\xi} C \tilde{\psi}_m^B) \lambda_{[CB]}^\alpha \]

\[ + 4i (\tilde{\xi} C \sigma^n)^\alpha \hat{F}^+_{mn}[CA] + \frac{2i \eta}{(4i)^2} \varepsilon_{ACP1...F6} (\tilde{\xi} C \lambda [F_1 F_2 F_3]) (\lambda [F_4 F_5 F_6] \tilde{\sigma}_m)^\alpha \]

\[ - \frac{i}{16} (\xi_\nu \sigma_m \sigma_n)^\alpha (\lambda_{[CB]}^{\sigma^n})^{[PCH]} + \frac{i}{48} (\xi_\alpha \sigma_m)^\alpha (\lambda_{[PCH]}^{\sigma^n})^{[PCH]}, \]

\[ \frac{1}{2} \delta^w_{\xi} \tilde{\psi}_m^\dot{\alpha} = D_m \tilde{\xi}_{\dot{\alpha}} - \frac{1}{2} (\tilde{\xi} C \psi_mB) \lambda_{[CBA]}^{\dot{\alpha}} \]

\[ + 4i (\tilde{\xi} C \sigma^n)_{\dot{\alpha}} \hat{F}^-_{mn}[CA] + \frac{2i \eta}{(4i)^2} \varepsilon_{ACP1...F6} (\tilde{\xi} C \lambda [F_1 F_2 F_3]) (\lambda [F_4 F_5 F_6] \sigma_m)^{\dot{\alpha}} \]

\[ + \frac{i}{16} (\xi_\nu \sigma_m \sigma_n)_{\dot{\alpha}} (\lambda_{[PCH]}^{\sigma^n})^{[ACB]} - \frac{i}{48} (\xi_\alpha \tilde{\sigma}_m)_{\dot{\alpha}} (\lambda_{[PCH]}^{\sigma^n})^{[PCH]}, \]

\[ \delta^w_{\xi} \psi_m^u = - \frac{1}{2} \left[ (\xi_B \psi_m) + \frac{i}{4} (\tilde{\xi} C \sigma_m \lambda_{[CB]}^A) \right] T^{[BA]}^u \]

\[ - \frac{1}{2} \left[ (\xi_B \tilde{\psi}_m^A) + \frac{i}{4} (\xi C \sigma_m \lambda_{[CBA]}^A) \right] T_{[BA]}^u. \]

Using the spinorial derivatives (162)–(165) of the gravigini superfields we obtain for the helicity 1/2 fields the transformation law

\[ \delta^w_{\xi} \lambda_{[CBA]}^{\alpha} = 4 \delta^w_{\xi} (\xi D \sigma^{mn})^\alpha \hat{F}^+_{mn}[EF] + 4 i (\tilde{\xi} D \sigma^m)^\alpha \hat{\Omega}_m[DCBA] \]

\[ - \frac{i}{4} \varepsilon_{CBAE1...B5} \xi_D^\alpha (\lambda^{[DE1E2]} [E3E4E5]), \]

\[ \delta^w_{\xi} \lambda_{[CBA]}^{\dot{\alpha}} = 4 \delta^w_{\xi} (\xi D \sigma^{mn})_{\dot{\alpha}} \hat{F}^-_{mn}[EF] + 4 i (\xi D \sigma^m)_{\dot{\alpha}} \hat{\Omega}_m[DCBA] \]

\[ - \frac{i}{4} \varepsilon_{CBAE1...B5} \xi_D^{\dot{\alpha}} (\lambda^{[DE1E2]} [E3E4E5]). \]

As for the supersymmetry transformations of the scalar fields, one obtains

\[ \delta^w_{\xi} T = L \xi T = (i \xi \Omega) T, \]

that is, just a rotation by a matrix $\Sigma$, which is an element of the orthogonal complement of the Lie algebra $su(8)$ in the Lie algebra of $E_{7(7)}$,

\[ \delta^w_{\xi} T = \Sigma T, \quad \text{with} \quad \Sigma = i \xi \Omega \begin{pmatrix} 0 & \Sigma_{[DCBA]} \\ \Sigma_{[DCBA]} & 0 \end{pmatrix}, \]

or in matrix components,

\[ \delta^w_{\xi} T_{[DC]}^u = \Sigma_{[DCBA]} T_{[BA]}^u, \quad \delta^w_{\xi} T_{[DC]}^u = \Sigma_{[DCBA]} T_{[BA]}^u. \]
Here, of course, the objects \( \Sigma^{[DCBA]} \) and \( \Sigma_{[DCBA]} \) are related by the duality relation (66) and they are computed using the component expressions (61) and (62) of \( \Omega^A_\alpha \) and \( \Omega^\alpha_A \):

\[
\Sigma^{[DCBA]} = -2 \left[ \frac{\eta}{4!} \varepsilon^{DCBAEFGH} (\xi_{[B}[\lambda_{|FGH]}) + (\bar{\xi}_{[D}[\lambda_{|CBA]}]) \right], \tag{106}
\]

\[
\Sigma_{[DCBA]} = -2 \left[ (\xi_{[D}[\lambda_{|CBA]}]) + \frac{\bar{\eta}}{4!} \varepsilon_{DCBAEFGH} (\bar{\xi}_{E}[\lambda_{|FGH]}) \right]. \tag{107}
\]

Finally, we can also notice, that as pointed out on the component level in the article [4], the supersymmetry transformations of the SU(8) connection, which is a function of the scalar fields and their derivatives (72), can also be given in a simple way using the above defined field–dependent parameters \( \Sigma \). Indeed, using the expression (75) of the SU(8) curvature, we have

\[
\delta^WZ \xi^B A = \iota \xi R^B A = \frac{1}{3} \left[ \Sigma^{[BEFG]} \Omega_{[AEFG]} - \Sigma_{[AEFG]} \Omega^{[BEFG]} \right], \tag{108}
\]

while on the component level,

\[
\delta^WZ \xi^m_B = \frac{1}{3} \left[ \Sigma^{[BEFG]} \Omega_m^{[AEFG]} - \Sigma_{[AEFG]} \Omega_m^{[BEFG]} \right]. \tag{109}
\]

These transformation laws are in perfect concordance with those found at the component level in [4] (we prefer to make reference to this work, because the transformation laws given there contain all the non–linear terms even in the spinor fields, which are not given explicitly in the original works like [3]).

### 4.3 Central charge transformations

In the geometrical description central charge transformations are just supergravity transformations in the direction of central charge coordinates, that is, with parameters

\[
\zeta^A = (0, 0, 0, \zeta^u). \tag{110}
\]

Therefore, we can use the general formulas (77) and (78) with these parameters in order to give these transformations. However, as a consequence of the constraint \( T_{zB}^A = 0 \) as well as of the fact that the central charge derivative of all torsion components vanish, \( D_z T_{CB}^A = 0, \) all component fields but the graviphotons transform trivially under central charge transformations.

The transformation of the graviphotons is simply

\[
\delta^WZ v^u_m = D_m \zeta^u, \tag{111}
\]

since they are the gauge fields corresponding the central charge transformations.

Also, it was noticed in [4], that a gauge transformation with a scalar–dependent parameter appears in the commutator of two supersymmetry transformations. This feature appears naturally in our approach, since one handles with central charge transformations which are present in the algebra of supergravity transformations with a parameter which depends on the 0 dimensional torsion components, where the scalars were identified. In order to see this in detail, recall that the commutator of two supergravity transformations acting on the frame is

\[
\left[ \delta^WZ_\xi, \delta^WZ_\eta \right] E^A = \delta^WZ_\xi E^A - E^B \iota \xi E^B R^A, \tag{112}
\]
with
\[ [\xi, \eta]^A = \xi^B\eta^C \mathcal{D}_{CB} + \xi^B(\mathcal{D}_{B}\eta^A) - \eta^B(\mathcal{D}_B\xi^A). \] (113)

Then choosing for instance the only non–zero parameters \( \xi^A \) and \( \eta^A \), the commutator on a graviphoton becomes
\[
\left[ \delta_{\xi}^{WZ}, \delta_{\eta}^{WZ} \right] v_m^u = \mathcal{D}_m \left( (\xi^B\eta_A) T^{[BA]u} \right) - \frac{i}{8}(\xi^B\eta_A) (\lambda^{[BAF]}\sigma_m\lambda_{[FDC]}) T^{[DC]u}
- \frac{1}{2}(\xi^B\eta_A) (\lambda^{[BAD]}v_m^C) T_{[DC]}^u, \] (114)

and indeed, it contains a central charge transformation (111) with parameter \( (\xi^B\eta_A) T^{[BA]u} \) depending on scalars as well as a supersymmetry transformation (100) with parameter \( (\xi^B\eta_A) \lambda^{[BAF]} \) depending on the helicity 1/2 fields.

5 The equations of motion

The problem of the derivation of field equations of motion without the knowledge of a Lagrangian, using considerations on representations of the symmetry group, was considered for a long time [26], [27]. The question is particularly interesting for supersymmetric theories and for this case various approaches have been developed. Here we use the techniques of superspace geometry introduced by Wess and Zumino, which consist in looking to consequences of covariant constraints corresponding to on–shell field content of a representation of the supersymmetry algebra.

Indeed, the next and last step in the geometric description of the \( N = 8 \) on–shell supergravity theory is to deduce the equations of motion implied by the constraints we used to identify the multiplet in the geometry, and compare them with those found from the Lagrangian given in the original works in the component formalism [3], [4].

The method of deducing the equations of motion for \( N \geq 3 \) extended supergravity is similar to the case of the \( N = 1 \) Yang–Mills theory in the sense that the gravitino superfields \( T_{[CBA]} \), \( T^{[CBA]} \) play an analogous rôles to the gaugino superfields and all equations of motion but those for the graviton and gravitini are contained in their higher superfield components [21]. Therefore, these equations of motion are found by successively acting with spinorial derivatives on the spinorial derivatives of the gravitino superfields. This is the approach which was adopted in previous superspace descriptions of the \( N = 8 \) supergravity in order to derive the free equations of motion from the geometry [9], [8] (see also references [22] and [28]). Alternatively, one could also just pick out certain Bianchi identities, which give the equations of motion for the component fields. This strategy to obtain equations of motion is outlined in the article [7]. For the purpose of putting in evidence free equations of motion of component fields it is sufficient to consider only the linearized version and the calculations are simple. However, one has to consider the full theory if one wants to obtain all the non–linear terms which arise in equations of motion derived from a Lagrangian in component formalism.

To begin with let us explain in detail how one can derive the equations of motion for the gravitini. First, recall that the spinorial derivatives of the gravitino superfields \( T_{[CBA]} \) (see appendix B) have for instance the properties
\[
\mathcal{D}_{[CBA]} T_{[CBA]} = \frac{\eta}{12} \varepsilon_{CBAJKLM} \left( T^{[DL]} T^{[KLM]} \right), \] (115)
\[
\sum_{DC} \mathcal{D}_{D} T_{[CBA]} = 0. \] (116)
In order to get the Dirac equation of the 1/2 helicity field, one can just act on the last relation by the spinorial derivative $D_{\dot{a}}$ obtaining

$$\sum_{DC} \left( \{ D_{\varepsilon}^{\dot{b}} , D_{D}^{\dot{\alpha}} \} T_{[CBA]\alpha} - D_{\dot{D}}^{\dot{b}} \left( D_{\varepsilon}^{\dot{b}} T_{[CBA]a} \right) \right) = 0,$$

and take the antisymmetric part of this relation in the indices $\varepsilon$ and $\alpha$. Then, using the algebra of covariant derivatives as well as equation (115) and again the expressions of the spinorial derivatives of the gravitini superfields one obtains

$$D_{[\alpha}\dot{D}^{\dot{\alpha}]} T_{[CBA]a} = -\frac{i\eta}{3} \varepsilon_{CBA}^{\alpha\beta\gamma} F_{ba}^{\dot{[EF]}(\sigma^{ba} T^{[\dot{\gamma}\dot{H}]} )\dot{\dot{\alpha}}}$$

$$+ \frac{i}{16} \left( T_{[CBA]} T_{[EFG]} \right) T_{[EFG]a}^{\dot{\dot{\alpha}}} - \frac{i}{8} \oint_{CBA} \left( T_{[CEF]} T_{[BAG]} \right) T_{[EFG]a}^{\dot{[EF]}\dot{[\dot{\alpha}]}} .$$

Alternatively, it is worthwhile to observe that this equation of motion can also be deduced from the Bianchi identity $\left( \delta^{\dot{D}}_{\alpha} \right)$ of dimension 3/2 using the expressions of torsion components and of spinorial derivatives (see appendix B) of lower canonical dimension.

The conjugate Dirac equation can be obtained in an analogous way:

$$D_{\alpha\dot{D}} T_{[CBA]\dot{a}} = -\frac{i\eta}{3} \varepsilon_{CBA}^{\alpha\beta\gamma} F_{ba}^{\dot{[EF]}(\sigma^{ba} T^{[\dot{\gamma}\dot{H}]} )\dot{\dot{\alpha}}}$$

$$+ \frac{i}{16} \left( T_{[CBA]} T_{[EFG]} \right) T_{[EFG]a}^{\dot{\dot{\alpha}}} - \frac{i}{8} \oint_{CBA} \left( T_{[CEF]} T_{[BAG]} \right) T_{[EFG]a}^{\dot{[EF]}\dot{[\dot{\alpha}]}} .$$

Moreover, the equations of motion for the graviphotons and those of the scalars can be deduced by further acting with covariant spinorial derivatives on the equations of motion (118) and (119) of the gravitini fields as follows.

On the one hand the trace part in the $SU(8)$ indices of the spinorial derivative $D_{\dot{b}}^{\dot{a}}$ of the Dirac equation (118) gives the equations of motion for the self–dual field strength $F_{ba}^{\dot{[EF]}[BA]}$ of the graviphotons,

$$D_{f} F_{_[[BA]}^{\dot{[EF]}} = -D_{f} \left[ \frac{\eta}{2(4!)^2} \varepsilon_{BAF_{1}...F_{6}} \left( T_{[F_{1}F_{2}F_{3}] \sigma^{f} a T_{[F_{4}F_{5}F_{6}]} \right) \right]$$

$$+ \Omega_{f[BAL]} \left[ F_{-[1]}^{\dot{[EF]}\dot{[\dot{a}]}} + \frac{\eta}{2(4!)} \varepsilon_{IJV_{1}...V_{6}} \left( T_{[I_{1}V_{2}V_{3}] \sigma^{f} a T_{[V_{4}V_{5}V_{6}]} \right) \right]$$

$$- \frac{1}{2(4!)} \left( T_{[BA]} \sigma^{f} a T_{[LJK]} \right) \Omega_{f}_{[PJK]} + \frac{i}{16} \left( T_{[BA]} \sigma^{f} a T_{[F_{1}F_{2}F_{3}]} \right) F_{f g}^{[1]}$$

$$+ \frac{i\eta}{2(4!)^{3}} \varepsilon_{LJKLMEFG} \left( T_{[DLM]} T_{[EFG]} \right) \left( T_{[DBA]} \sigma^{a} T_{[LJK]} \right) ,$$

while the conjugate field equations for $F_{ba}^{\dot{[EF]}[BA]}$ can be obtained in an analogous way,

$$D_{f} F_{_[[BA]}^{\dot{[EF]}} = -D_{f} \left[ \frac{\eta}{2(4!)^2} \varepsilon_{BAF_{1}...F_{6}} \left( T_{[F_{1}F_{2}F_{3}] \sigma^{f} a T_{[F_{4}F_{5}F_{6}]} \right) \right]$$

$$+ \Omega_{f[BAL]} \left[ F_{-[1]}^{\dot{[EF]}\dot{[\dot{a}]}} + \frac{\eta}{2(4!)} \varepsilon_{IJV_{1}...V_{6}} \left( T_{[I_{1}V_{2}V_{3}] \sigma^{f} a T_{[V_{4}V_{5}V_{6}]} \right) \right]$$

$$- \frac{1}{2(4!)} \left( T_{[BA]} \sigma^{f} a T_{[LJK]} \right) \Omega_{f}_{[PJK]} + \frac{i}{16} \left( T_{[BA]} \sigma^{f} a T_{[F_{1}F_{2}F_{3}]} \right) F_{f g}^{[1]}$$

$$+ \frac{i\eta}{2(4!)^{3}} \varepsilon_{LJKLMEFG} \left( T_{[DLM]} T_{[EFG]} \right) \left( T_{[DBA]} \sigma^{a} T_{[LJK]} \right) .$$
Due to the self-duality properties of these field strengths, these equations of motion are related to their Bianchi identities as follows

$$\frac{i}{2} \epsilon^{fabc} D_f F^{[BA]}_{cb} = \mathcal{D}_f F^{[BA]} + \frac{i}{2} \epsilon^{fabc} D_f F^-_{cb}^{[BA]} = -D_f F^{-fa}_{[BA]} \,. \quad (122)$$

Again, it is worthwhile to notice that the Bianchi identities of the graviphotons can be found directly among the super Bianchi identities, namely they are the identities $\langle dbb \rangle$. As a matter of fact the form given here can be recovered from the Bianchi identity $\langle dbb \rangle$ after converting the central charge index into $SU(8)$ ones by multiplying by the scalar matrices $S^u_{[BA]}$ and $S_{u[BA]}$, as well as using the equations of motion for the gravitini fields presented below in equations (126) and (127).

On the other hand, take the spinorial derivative $\mathcal{D}_f^\delta$ of the same Dirac equation (118) and after commuting on the left-hand-side this derivative with that of the space-time one, take the totally antisymmetric part in the $SU(8)$ indices. Now, the antisymmetric part in the spinorial indices $\delta$ and $\alpha$ gives the equations of motion for the scalars written for the field strength component $\Omega_f^{[DCBA]}$

$$\mathcal{D}_f^\delta \Omega_f^{[DCBA]} = -2\delta^{E_1 \ldots E_4^{[BA}} F_{E_{5}}^+[E_{1}E_{2}] - 2\bar{\eta}\epsilon^{E_1 \ldots E_4^{[DCBA}} F_{E_{5}}^-[E_{1}E_{2}] F_{ba}^{[E_{3}E_{4}]}$$

$$+ \frac{i}{2(3!)(4!)} \delta^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{[E_{1}E_{2}E_{3}E_{4}]} \left( T_{[F_1F_2F_3]} \sigma^f T_{[F_1F_2F_3]} \right)$$

$$- \frac{i}{2(4!)} \delta^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{[E_{1}E_{2}E_{3}]} \left( T_{[E_4F_2F_3]} \sigma^f T_{[E_4F_2F_3]} \right)$$

$$+ \frac{i}{32} \delta^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{[E_{1}E_{2}E_{3}F_1]} \left( T_{[E_4F_2F_3]} \sigma^f T_{[E_4F_2F_3]} \right)$$

$$+ \frac{i}{2(3!)(4!)} \epsilon^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{E_{1}E_{2}E_{3}F_1} \left( T_{[E_4F_2F_3]} \sigma^f T_{[E_4F_2F_3]} \right)$$

$$+ \frac{\eta}{256(4!)} \delta^{E_1 \ldots E_4^{[DCBA}} \epsilon^{F_1 \ldots F_8} \left( T_{[E_1E_2E_3]} T_{[F_1F_2F_3]} \right) \left( T_{[E_4F_2F_3]} T_{[F_4F_7F_8]} \right)$$

The equations of motion for the field strength component $\Omega_f^{[DCBA]}$

$$\mathcal{D}_f^\alpha \Omega_f^{[DCBA]} = -2\delta^{E_1 \ldots E_4^{[DCBA}} F_{E_{5}}^-[E_{1}E_{2}] - 2\bar{\eta}\epsilon^{E_1 \ldots E_4^{[DCBA}} F_{E_{5}}^+[E_{1}E_{2}] F_{ba}^{[E_{3}E_{4}]}$$

$$+ \frac{i}{2(3!)(4!)} \delta^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{[E_{1}E_{2}E_{3}E_{4}]} \left( T_{[F_1F_2F_3]} \sigma^f T_{[F_1F_2F_3]} \right)$$

$$+ \frac{i}{2(4!)} \delta^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{[E_{1}E_{2}E_{3}]} \left( T_{[E_4F_2F_3]} \sigma^f T_{[E_4F_2F_3]} \right)$$

$$- \frac{i}{32} \delta^{E_1 \ldots E_4^{[DCBA}} \Omega_f^{[E_{1}E_{2}E_{3}F_1]} \left( T_{[E_4F_2F_3]} \sigma^f T_{[E_4F_2F_3]} \right)$$

$$+ \frac{\eta}{256(4!)} \delta^{E_1 \ldots E_4^{[DCBA}} \epsilon^{F_1 \ldots F_8} \left( T_{[E_1E_2E_3]} T_{[F_1F_2F_3]} \right) \left( T_{[E_4F_2F_3]} T_{[F_4F_7F_8]} \right)$$

\[ (123) \]
down the equations of motion of the gravitini fields. It turns out that the knowledge of these parts is sufficient to write the components \( T_{\alpha}^{\beta\gamma\delta} \) of the gravitini field strength \( T_{\alpha}^{\beta\gamma\delta} \), since the Bianchi identity for the graviton is just the Kähler condition (5) written for the Lorentz curvature. The equations of motion for the gravitini and graviton cannot be obtained by acting again by spinorial derivatives on the equations of motion obtained so far. One could obtain this way only the components \( T_{\alpha}^{\beta\gamma\delta} \) of the one–form \( \Omega \).

The symmetric part in the indices \( \delta \) and \( \alpha \) is the anti–self–dual part of the Bianchi identity for the scalars,

\[
\mathcal{D}_d\Omega^\mathrm{DCBA}_c - \mathcal{D}_c\Omega^\mathrm{DCBA}_d = -32 \left( T_{dc}^{[D}[\Omega] + \frac{\eta}{4!} \varepsilon^\mathrm{DCBAFLJK} (T_{dCF}[\Omega]) \right),
\]

which can be deduced also from the off–diagonal part of the superspace Bianchi identities (70) of the one–form \( \Omega \).

Unlike the equations of motion presented above, in Poincaré supergravity the equations of motion for the gravitini and graviton cannot be obtained by acting again by spinorial derivatives on the equations of motion obtained so far. One could obtain this way only the Bianchi identities for these fields (which can be obtained also in a much direct way: the Bianchi identities of the gravitini are exactly the superspace Bianchi identities \( (d_{\alpha})_\beta \) and \( (d_{\beta})_\alpha \), since the Bianchi identity for the graviton is just the \( d_{\alpha} \) component of the second Bianchi identity (5) written for the Lorentz curvature). The equations of motion for the gravitini and the graviton are directly given by the superspace Bianchi identities.

For example, the Bianchi identities \( \left( \frac{d}{\partial c}\beta\alpha \right)_T \) and \( \left( \frac{d}{\partial c}\beta\alpha \right)_T \) give the irreducible components \( T(\beta\alpha)^\Lambda \) and \( T(\hat{\beta}\hat{\alpha})^\Lambda \) of the gravitini field strength \( T_{\beta\alpha}^\Lambda \) and respectively the irreducible components \( T(\beta\alpha)^\Lambda \) and \( T(\hat{\beta}\hat{\alpha})^\Lambda \) of the gravitini field strength \( T_{\beta\alpha}^\Lambda \) as a non–linear function of basic superfields. It turns out that the knowledge of these parts is sufficient to write down the equations of motion of the gravitini fields:

\[
\varepsilon^{dcba} (\hat{\sigma} c T_{\beta\alpha}^\Lambda)_{\hat{\alpha}} = \frac{1}{3!} (\sigma^a \sigma^b d T_{[DCBA]}^{[\alpha]} \hat{\alpha}) \Omega_{a[B}\Omega]_{cA]} + i (\hat{\sigma} d \hat{\sigma} d T_{[CBA]}^{[\alpha]} \hat{\alpha}) F_{ba}^{+ [CB]} - \frac{i \eta}{3(4!)^2} \varepsilon^{F_1...F_8} (T_{[AF_1F_2]}^{[\Lambda]} T_{[F_3F_4]}^{[\beta\alpha]} (\sigma^d T_{[\beta\alpha]})^{\Lambda}) F_{ba}^{+ [CB]},
\]

\[
\varepsilon^{dcba} (\sigma c T_{\beta\alpha}^\Lambda)_{\alpha} = -\frac{1}{3!} (\sigma^a \sigma^d T_{[DCB]}^{[\alpha]} \alpha) \Omega_{a[B\Omega]_{cA]} - i (\sigma^a \sigma^b d T_{[CBA]}^{[\alpha]} \alpha) F_{ba}^{+ [CB]} + \frac{i \eta}{3(4!)^2} \varepsilon^{F_1...F_8} (T_{[AF_1F_2]}^{[\Lambda]} T_{[F_3F_4]}^{[\beta\alpha]} (\sigma^d T_{[\beta\alpha]})^{\Lambda})_{\alpha}.
\]

Finally, in order to give the equations of motion for the gravitron we need the expression of the supercovariant Ricci tensor, \( R_{df} = R_{dcba} \eta^{\alpha a} \), which is given (149) by the superspace Bianchi identities at canonical dimension 2. The corresponding Ricci scalar, \( R = R_{db} \eta^{db} \), is then also determined (150) and using the equations of motion for the spinor fields we find that the Einstein equation takes the form:

\[
R_{db} - \frac{1}{2} R_{db} \eta^{ab} = -\frac{1}{3!} \left( \Omega_{a[B\Omega]_{cA]} - \frac{1}{2} \eta_{db} \Omega_{f[I\Omega]L}^{[\alpha]} \Omega_{f[I\Omega]L}^{[\alpha]} \right) - 32 F_{(d}^{[f} F_{]b}^{f]} [\Omega] - \frac{1}{4!} \left( (T_{[LJK]}^{[\alpha]} (d D b) T_{[LJK]}^{[\alpha]} + (T_{[LJK]}^{[\alpha]} (d D b) T_{[LJK]}^{[\alpha]} \right) + \frac{\eta}{3(4!)^2} \varepsilon^{G_1...G_8} \left( F_{(d}^{+ f} [G_1G_2] (T_{[G_3G_4G_5]}^{[\alpha]} (d D b) T_{[G_6G_7G_8]}^{[\alpha]} \right)
\]
\[
\begin{align*}
&\quad -\frac{\eta_{db}}{4} \mathcal{F}^{+ef[G_1G_2] \left( T_{[G_3G_4G_5]} \sigma_{ef} T_{[G_6G_7G_8]} \right)} \\
&+ \frac{\eta_{db}}{3!} \epsilon_{G_1...G_8} \left[ \mathcal{F}^{-ef[G_1G_2] \left( T_{[G_3G_4G_5]} \bar{\sigma}_{bf} T_{[G_6G_7G_8]} \right)} \\
&- \frac{\eta_{db}}{4} \mathcal{F}^{-ef[G_1G_2] \left( T_{[G_3G_4G_5]} \bar{\sigma}_{ef} T_{[G_6G_7G_8]} \right)} \right] \\
&+ \frac{\eta_{db}}{64(4!)} \delta^{G_1...G_6}_{H_1...H_6} \left( T_{[G_1G_2G_3]} \sigma_d T_{[H_1H_2H_3]} \right) \left( T_{[G_4G_5G_6]} \sigma_b T_{[H_4H_5H_6]} \right) \\
&- \frac{\eta_{db}}{32(4!)} \delta^{G_1...G_5}_{H_1...H_5} \left( T_{[FG_1G_2]} T_{[G_3G_4G_5]} \right) \left( T_{[FH_1H_2]} T_{[H_3H_4H_5]} \right) \\
&+ \frac{1}{4(4!)} \left[ \left( T_{[IJL]} \sigma_d T_{[KL]} \right) \left( T_{[LMN]} \sigma_b T_{[MN]} \right) \\
&- 9 \left( T_{[GIL]} \sigma_d T_{[FL]} \right) \left( T_{[FKL]} \sigma_b T_{[GL]} \right) \right] \\
&- \frac{1}{(4!)^2} \eta_{db} \left( T_{[LK]} T_{[LMN]} \right) \left[ \left( T_{[ILJ]} T_{[LMN]} \right) + \frac{27}{4} \left( T_{[HL]} T_{[KMN]} \right) \right],
\end{align*}
\]

where one may recognize in the first two lines of the right–hand–side the usual terms of the energy–momentum tensor corresponding to matter fields: scalar fields, photon fields and spinor fields respectively. The contribution of the gravitini is hidden in the left–hand–side, in the component development \[21\] of \( R_{db} \).

The Einstein equation completes the ensemble of the equations of motion for the component fields. Fortunately, in the article \[4\], which contains a detailed list of the results obtained in the component formulation, most of the equations of motion are given in terms of super–covariant quantities, given in paragraph 4.1 as functions of the component fields. As a consequence, it is easy to compare the equations of motion deduced in this paragraph from the geometry with the component results of \[4\], and see that there is a perfect concordance between them.

## 6 Conclusion

We have presented here a new approach to the superspace formulation of \( N = 8 \) supergravity, using central charge superspace. The presence of the central charge coordinates is essential in the formulation. It permits to identify the gauge vectors of the theory in the super–vielbein on the same footing with the graviton and gravitini, and also, it allows to identify the scalars directly, as lowest canonical dimension torsion components. In addition, we recover the well–known essential properties of the multiplet as consequences of the geometric structure: we deduce the self–duality properties of the vectors as well as the \( E_7(+7)/SU(8) \) non–linear \( \sigma \) model structure for the scalars using straightforward superspace techniques in central charge superspace.

It is worthwhile to note here that there exists a formal correspondence between the formulation in central charge superspace presented here and the geometric formulation in superspace extended by 56 bosonic coordinates \[8\] presented in detail by Howe \[10\]. The correspondence, described in more detail in \[19\], is based on a simple redefinition of the frame of the type

\[
\tilde{E} = E \mathcal{X}, \quad \text{with} \quad \mathcal{X} = \begin{pmatrix} \delta_\mathcal{A} \\ 0 \\ S \end{pmatrix},
\]

that is, on a rotation by the scalars \( S \) in the central charge sector\footnote{Underlined indices denote the ordinary superspace sector \( \mathcal{A} = (a, \lambda, \dot{\lambda}) \).}. However, the difference
between the two approaches is conceptual in the sense that in Howe’s approach the translation generators in the extra bosonic coordinates are not "genuine" central charges, they transform under $SU(8)$. Indeed, the frame (129) in the extra bosonic sector ($\tilde{E}_{[bA]}$, $\tilde{E}^{[bA]}$) and thus also the translation generators in the direction of the extra bosonic coordinates carry $SU(8)$ representation indices.

The identification of the $N = 8$ supergravity in the geometry of central charge superspace described here is complete, since we also presented in detail the deduction from the geometry of both the supergravity transformation laws and the equations of motion for the component fields. Our results obtained from the geometric formulation are in perfect accord with the results in terms of super–covariant quantities obtained in component formulation given in the article [4].

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A Totally antisymmetric $U(N)$ tensors and deltas

In this appendix we collect some useful relations concerning totally antisymmetric tensors and deltas in $U(N)$ indices.

$\delta^A_{\delta}$ being the Kronecker symbol, we can define the objects with antisymmetric indices

\[
\delta_{B_1 B_2}^{A_1 A_2} = \delta_{B_1}^{A_1} \delta_{B_2}^{A_2} - \delta_{B_1}^{A_2} \delta_{B_2}^{A_1},
\]

\[
\delta_{B_1 B_2 B_3}^{A_1 A_2 A_3} = \delta_{B_1}^{A_1} \delta_{B_2}^{A_2} \delta_{B_3}^{A_3} + \delta_{B_1}^{A_2} \delta_{B_2}^{A_3} \delta_{B_3}^{A_1} + \delta_{B_1}^{A_3} \delta_{B_2}^{A_1} \delta_{B_3}^{A_2},
\text{etc.}
\]

In general, for $k$ antisymmetric indices we define the corresponding objects by recursion

\[
\delta_{B_1 \ldots B_k}^{A_1 \ldots A_k} = \sum_{i=1}^{k} (-1)^{i+k+1} \delta_{B_1 B_2}^{A_1 A_2} \delta_{B_1 B_2 B_3}^{A_1 A_2 A_3} \ldots \delta_{B_1 B_2 \ldots B_k}^{A_1 A_2 \ldots A_k} \ldots
\]

(130)

and call them "generalized deltas". These generalized deltas have the property

\[
\delta_{B_1 \ldots B_k}^{A_1 \ldots A_k} X^{B_1 \ldots B_k} = k! X^{[A_1 \ldots A_k]}, \quad \forall X^{[A_1 \ldots A_k]},
\]

(131)

\[
\delta_{B_1 \ldots B_k}^{A_1 \ldots A_k} X^{[B_1 \ldots B_k]} = k! X_{[A_1 \ldots A_k]}, \quad \forall X_{[A_1 \ldots A_k]},
\]

(132)

thus, the operator $\frac{1}{k!} \delta_{B_1 \ldots B_k}^{A_1 \ldots A_k}$ acts as the unit operator in the space of tensors which are antisymmetric in $k$ indices. If we contract an upper index with a lower one in a generalized delta we have

\[
\delta_{A_1 B_2 \ldots B_k}^{A_1 A_2 \ldots A_k} = (N - k + 1) \delta_{B_2 \ldots B_k}^{A_2 \ldots A_k},
\]

(133)

as for the contraction of $l$ indices, this implies

\[
\delta_{A_1 A_2 B_{l+1} \ldots B_k}^{A_1 A_2 \ldots A_k} = (N - k + 1)(N - k + 2)\ldots(N - k + l) \delta_{B_{l+1} \ldots B_k}^{A_{l+1} \ldots A_k}.
\]

(134)

Let $\epsilon^{A_1 A_2 \ldots A_N}$ be the totally antisymmetric tensor. Choosing $\epsilon^{1 \ldots N} = 1$ and $\epsilon_{1 \ldots N} = 1$ we have the relation

\[
\epsilon^{A_1 \ldots A_N} \epsilon_{B_1 \ldots B_N} = \delta_{B_1 \ldots B_N}^{A_1 \ldots A_N}.
\]

(135)
Then, if we contract $k$ indices on a product of two totally antisymmetric tensors $\varepsilon$, we obtain as consequence of (135) and of the property (134)

$$
\varepsilon^{\gamma_1\ldots\gamma_{k-1}\alpha_k} \varepsilon_{\alpha_1\ldots\alpha_{k-1}\gamma_{k+1}} = k! \delta^{\gamma_{k+1}\ldots\gamma_N}_{\alpha_{k+1}\ldots\alpha_N} \tag{136}
$$

Finally, since the property (132) is valid also for $\varepsilon$, we have

$$
\delta^{\gamma_{k+1}\ldots\gamma_N}_{\alpha_{k+1}\ldots\alpha_N} \varepsilon^{\gamma_1\ldots\gamma_{k-1}\alpha_k} = k! \varepsilon^{\gamma_1\ldots\gamma_{k-1}\alpha_k} \tag{137}
$$

**B Solution of the Bianchi identities**

In this appendix we give the solution of the Bianchi identities, which is compatible with the constraints presented in section 3 and corresponds to the $N = 8$ supergravity in central charge superspace. This means that all torsion and curvature components are expressed as polynomial functions of the scalar superfields $T^{[\mathcal{CB}]}u$ and $T^{[\mathcal{CB}]}u$ (or their super–covariant field strength $\Omega_\alpha$), the gravigini superfields $T^{[\mathcal{CB}]}u$ and $T^{[\mathcal{CB}]}u$, the gravitini Weyl tensors $\Sigma_{(\gamma\beta\alpha)}$ and $\Sigma^{(\gamma\beta\alpha)}$, the usual Weyl tensors $V_{(\delta\beta\alpha)}$ and $V_{(\delta\beta\alpha)}$, as well as the super–covariant field strength of the graviphotons, identified in the torsion component $T_{cb}u$, denoted also $F_{cb}u$. Actually, as we already have seen in section 3.4, the dynamical parts of this field strengths of graviphotons are only the fields $F_{cb}^{[BA]}$ and $F_{cb}^{[BA]}$, which are subject to the self–duality relations (51), and only these parts appear in torsion and curvature components. Moreover, the spinorial derivatives of these basic superfields are also expressed as polynomial functions of themselves.

In this appendix we will give all non–vanishing torsion and curvature components as well as the spinorial covariant derivatives of the basic superfields listed above.

**Non–vanishing torsion components**

The list of the non–vanishing torsion components is the following:

$$
T^{CB\alpha}_{\gamma \beta} = -2i \delta^{C}_{B} (\alpha) \varepsilon^{\beta}_{\gamma}, \quad T^{CBu}_{\gamma \beta} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}u, \quad T^{\dot{\gamma} \beta u}_{\gamma B} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}u, \quad T^{\dot{\gamma} \beta u}_{CB} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}u, \tag{138}
$$

$$
T^{CB\alpha}_{\gamma \beta} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}\alpha, \quad T^{CB\alpha}_{\gamma \beta} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}\alpha, \quad T^{\dot{\gamma} \beta u}_{\gamma B} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}u, \quad T^{\dot{\gamma} \beta u}_{CB} = \varepsilon_{\gamma \beta} T^{[\mathcal{CB}]}u, \tag{139}
$$

$$
T_{cb\alpha} = - (\varepsilon \sigma \sigma) \gamma \beta \gamma \beta \alpha + \frac{i}{18} (\delta^{fg}_{cd} - \frac{i}{2} \varepsilon^{fg}_{cd} \eta_{fg}) \Omega_{[ABCD]} (\sigma T^{[ABCD]}_{[ABCD]}), \tag{140}
$$

$$
T_{cb\dot{\alpha}} = - (\varepsilon \sigma \sigma) \gamma \beta \gamma \beta \alpha + \frac{i}{18} (\delta^{fg}_{cd} + \frac{i}{2} \varepsilon^{fg}_{cd} \eta_{fg}) \Omega_{[ABCD]} (\sigma T^{[ABCD]}_{[ABCD]}), \tag{141}
$$

25
where the superfields $T^B_A$ and $U^B_A$ are quadratic terms in the spinor superfields

$$
T^B_A = -\frac{i}{16} T_{[A][j]} a^i b^j T^{B[A]}_j, \quad U^B_A = \frac{i}{16} \left( T_{[A][j]} a^i b^j T^{B[A]}_j - \frac{1}{6} \delta^B_A T_{[j][k]} a^i b^j T^{B[A]}_k \right),
$$

(142)

while for the field strength components $F_{ba[BA]}$ and $F_{ba[BA]}$ we have

$$
F_{ba[BA]} = F_{ba[BA]}^+ - \frac{\eta}{2(4)!} \epsilon_{BAP} T^{[P_1P_2P_3]}_{ba} T^{[P_4P_5P_6]}_{BA},
$$

(143)

$$
F_{ba[BA]} = F_{ba[BA]}^- - \frac{\eta}{2(4)!} \epsilon_{BAP} T^{[P_1P_2P_3]}_{ba} T^{[P_4P_5P_6]}_{BA}.
$$

(144)

### Curvature components

Let us begin with giving the list of Lorentz curvature components:

$$
R_{\delta \gamma \delta \gamma} = -16 \epsilon_{\delta \gamma} F_{ba[DC]}^{[DC]}, \quad R^{\delta \gamma}_{\delta \gamma} = -16 \epsilon_{\delta \gamma} F_{ba[DC]}^{[DC]}, \quad R_{\delta \gamma \delta \gamma}^{\delta \gamma} = 4 \epsilon_{\delta \gamma \delta \gamma} U^{CD} C.
$$

(145)

$$
R_{\delta \gamma \delta \gamma} = \left( \epsilon_{\beta \delta} \right) (\epsilon_{\sigma \beta} \sigma \sigma) \left( \epsilon_{\sigma \beta} \sigma \sigma \right) V^{(\delta \gamma \beta \gamma)}
$$

(146)

$$
R_{\delta \gamma \delta \gamma} = \left( \epsilon_{\sigma \beta} \sigma \sigma \right) \left( \epsilon_{\sigma \beta} \sigma \sigma \right) V^{(\delta \gamma \beta \gamma)}
$$

(147)

$$
R_{\delta \gamma \delta \gamma} = \left( \epsilon_{\sigma \beta} \sigma \sigma \right) \left( \epsilon_{\sigma \beta} \sigma \sigma \right) V^{(\delta \gamma \beta \gamma)}
$$

(148)

with the supercovariant Ricci tensor, $R_{\delta \gamma} = R_{\delta \gamma \delta \gamma} \eta^\delta$, given by

$$
R_{\delta \gamma} = -\frac{1}{3!} \left( \Omega_{[d}^{[ijkl]} \Omega_{b][ijkl]} - 32 F_{(d}^{[BA]} F_{b)\gamma} [T_{[G3G4G5]} \sigma_b f T^{[G6G7G8]}_b] \right)
$$

$$
+ \frac{\eta}{3!} \epsilon_{G1...G8} \left[ F_{(d}^{[BA]} \sigma_b f T^{[G3G4G5]}_b T^{[G6G7G8]}_b \right]
$$

$$
+ \frac{1}{4} \eta_{b} \epsilon_{G1...G8} \left[ F_{(d}^{[BA]} \sigma_b f T^{[G3G4G5]}_b T^{[G6G7G8]}_b \right]
$$

$$
- \frac{1}{4!} \left( T_{[ijkl]} \sigma_{(d} D_{b)} T^{[ijkl]} \right) - \frac{1}{4} \eta_{b} \left( T_{[ijkl]} \sigma_{(d} D_{b)} T^{[ijkl]} \right)
$$

$$
+ \left( T_{[ijkl]} \sigma_{(d} D_{b)} T^{[ijkl]} \right) - \frac{1}{4} \eta_{b} \left( T_{[ijkl]} \sigma_{(d} D_{b)} T^{[ijkl]} \right)
$$

$$
+ \frac{1}{64(4)!} \delta_{G1...G6} \left( T_{[G1G2G3]}^{[BA]} T^{[h]} T^{[b]} T^{[h]} T^{[b]} \right) \left( T_{[G4G5G6]}^{[BA]} T^{[b]} T^{[h]} T^{[b]} \right)
$$

$$
+ \frac{1}{32(4)!} \eta_{b} \left( T_{[FG1G2]}^{[BA]} T^{[h]} T^{[b]} T^{[h]} T^{[b]} \right) \left( T_{[FG1G2]}^{[BA]} T^{[h]} T^{[b]} T^{[h]} T^{[b]} \right)
$$

$$
+ \frac{1}{4(4)!} \left[ T_{[ijkl]} \sigma_{(d} T^{[ijkl]} \right] T^{[ijkl]} \sigma_{(d} T^{[ijkl]} \right] - 9 \left( T_{[ijkl]} \sigma_{(d} T^{[ijkl]} \right) T^{[ijkl]} \sigma_{(d} T^{[ijkl]} \right]
$$

(149)
and the corresponding Ricci scalar, \( R = R_{db} \eta^{db} \), which is then

\[
R = -\frac{1}{3!} \Omega^{f[ILK]} \Omega_{f[ILK]} \\
+ \frac{1}{3(4!)} \eta^{G_1...G_8} F_{\alpha [G_1G_2} \left( T_{[G_3G_4G_5]} \sigma^a T_{[G_6G_7G_8]} \right) \\
+ \frac{1}{3(4!)} \bar{\eta}^{G_1...G_8} F_{\alpha \gamma} \left( T_{[G_3G_4G_5]} \bar{\sigma}^\alpha T_{[G_6G_7G_8]} \right) \\
+ \frac{1}{8(4!)} \delta^{G_1...G_8}_{H_1...H_5} \left( T_{[FG_1G_2]} T_{[G_3G_4G_5]} \right) \left( T^{[PH_1H_2]} T^{[H_3H_4H_5]} \right) \\
+ \frac{1}{16(4!)} \left( T_{[JLK]} T_{[LMN]} \right) \left[ \left( T^{[JLK]} T^{[LMN]} \right) - 9 \left( T^{[JL]} T^{[KMN]} \right) \right].
\]

The SU(8) curvature components are the following:

\[
R^{\delta CB}_{\Delta \gamma A} = \frac{\eta}{4!} \delta^\epsilon_{\gamma} \varepsilon_{DCBEFJKL} (T^{[AEF]} T_{[JLK]}) - \frac{\bar{\eta}}{(3!)} \delta_{A}^{(D} \varepsilon_{C)BF1...F6} T_{[F1F2F3]} \delta T_{[F4F5F6]} \gamma,
\]

\[
R^{\delta \lambda \beta}_{DC A} = -\frac{\bar{\eta}}{4!} \delta^{\epsilon}_{\lambda} \varepsilon_{DCBEFJKL} (T^{[AEF]} T_{[JLK]}) + \frac{\eta}{(3!)} \delta_{\lambda}^{(D} \varepsilon_{C)AF1...F6} T_{[F1F2F3]} \delta T_{[F4F5F6]} \gamma,
\]

\[
R^{\delta \beta}_{\Delta C A} = 2i(\delta_{A}^{D} T^{\gamma} T^{\beta} \gamma - \delta_{A}^{\gamma} U^{\gamma} \delta \delta_{\alpha}^{\beta} C - \delta_{A}^{\gamma} U^{\beta} \delta \eta \gamma) + 4i(\delta_{A}^{D} U^{\gamma} \delta \delta_{\alpha}^{\gamma} C + \delta_{C}^{\gamma} U^{\beta} \delta \gamma) - T^{[DBF]} \eta T_{[CAF]} \delta.
\]

\[
R^{\delta \beta}_{\Delta C A} = \frac{1}{18} \left( \delta_{A}^{D} T^{[BEF]} \Omega_{[C]BEFG} - 3 \delta_{A}^{D} \Omega_{C}^{[BEFG]} \right) T_{[JLK] \delta},
\]

\[
R^{\delta \beta}_{DC A} = -\frac{1}{18} \left( \delta_{D}^{[BEF]} \Omega_{C}^{[AEFG]} - 3 \delta_{D}^{[B} \Omega_{C}^{[AEFG]} \right) T_{[JLK] \delta}.
\]

\[
R^{\delta \beta}_{\Delta C A} = \frac{1}{3} \left( \Omega_{d}^{[BLJK]} \Omega_{e[A]JLJK} - \Omega_{e}^{[BLJK]} \Omega_{d[A]JLJK} \right).
\]

**Spinorial derivatives**

The expression of the spinorial derivatives of the basic superfields are obtained in general using the results of the Bianchi identities as well as the algebra of covariant derivatives and they are called also constituency equations. Let us sum up there here.

Spinorial derivatives of the scalars:

\[
D^\alpha T^{[dc]} u = -\frac{\eta}{2(3!)} \varepsilon^{FDCJKLM} T_{[KLM]} \sigma T_{[JL]} u
\]

\[
D^\alpha T^{[dc]} u = -\frac{1}{2(3!)} \delta_{F}^{DCJ L} T_{[KLM]} \sigma T_{[JL]} u
\]

\[
D^\phi T^{[dc]} u = -\frac{1}{2(3!)} \delta_{DCJ L}^{FKLM} T_{[KLM]} \sigma T_{[JL]} u
\]

\[
D^\phi T^{[dc]} u = -\frac{\bar{\eta}}{2(3!)} \varepsilon^{FDCJKLM} T_{[KLM]} \sigma T_{[JL]} u
\]

Notice, that the spinorial derivatives of the scalars \( T \) are contained by definition in the spinorial components of the one–form \( \Omega = (DT)S \). Therefore, the above expressions are equivalent to the relations

\[
D^\alpha T = \Omega^\alpha T, \quad D^\phi T = \Omega^\phi T,
\]

with \( \Omega^\alpha \) and \( \Omega^\phi \) given by the equations (61) and (62).
Spinorial derivatives of the gravigini superfields:

\[
D^\psi F^-[\alpha\beta] = -4\delta^F_{\alpha\beta} \epsilon_{[\alpha} \phi_{\beta]} + \frac{\eta}{4!} \epsilon_{\alpha\beta} \epsilon_{[\alpha} \phi_{\beta]} (T^[[\alpha\beta]]T^[\alpha\beta])
\]  
(162)

Spinorial derivatives of the self–dual and anti–self–dual super–covariant field strength:

\[
D^\dot{\psi} F^+[\alpha\beta] \equiv -4i(\dot{\phi}^\alpha \epsilon_{\beta} \Omega_f_{[\alpha\beta]} + \frac{\eta}{4!} \epsilon_{\beta} \epsilon_{\alpha\beta} \phi_{[\alpha\beta]} (T^[[\alpha\beta]]T^[\alpha\beta])
\]  
(163)

Spinorial derivatives of the gravigini superfields:

\[
D^\phi \Omega^a_{[\alpha\beta]} = -\frac{\eta}{12} \epsilon_{[\alpha\beta]} \epsilon_{[\alpha\beta]} \phi \alpha + \frac{i}{2} \delta^F_{[\alpha\beta]} \phi_f \epsilon_{[\alpha\beta]} (\sigma^F T^[\alpha\beta])
\]  
(164)

Spinorial derivatives of \( \Omega_a \), the field strength of the scalars:

\[
D^\phi \Omega^a_{[\alpha\beta]} = -\frac{\eta}{12} \epsilon_{[\alpha\beta]} \epsilon_{[\alpha\beta]} \phi \alpha + \frac{i}{2} \delta^F_{[\alpha\beta]} \phi_f \epsilon_{[\alpha\beta]} (\sigma^F T^[\alpha\beta])
\]  
(165)

Spinorial derivatives of the self–dual and anti–self–dual super–covariant field strength:

\[
D^\dot{\psi} F^-[\alpha\beta] \equiv -4i(\dot{\phi}^\alpha \epsilon_{\beta} \Omega_f_{[\alpha\beta]} + \frac{\eta}{4!} \epsilon_{\beta} \epsilon_{\alpha\beta} \phi_{[\alpha\beta]} (T^[[\alpha\beta]]T^[\alpha\beta])
\]  
(166)

Spinorial derivatives of \( \Omega_a \), the field strength of the scalars:

\[
D^\phi \Omega^a_{[\alpha\beta]} = -\frac{\eta}{12} \epsilon_{[\alpha\beta]} \epsilon_{[\alpha\beta]} \phi \alpha + \frac{i}{2} \delta^F_{[\alpha\beta]} \phi_f \epsilon_{[\alpha\beta]} (\sigma^F T^[\alpha\beta])
\]  
(167)

Spinorial derivatives of the self–dual and anti–self–dual super–covariant field strength:

\[
D^\dot{\psi} F^-[\alpha\beta] = \frac{1}{8} \phi^\alpha \delta^\beta \epsilon_{[\alpha\beta]} \phi \alpha \alpha + 2T^[\alpha\beta] \phi_f \epsilon_{[\alpha\beta]} (T - U) \phi_f
\]  
(170)

Spinorial derivatives of \( \Omega_a \), the field strength of the scalars:

\[
D^\phi \Omega^a_{[\alpha\beta]} = \frac{1}{8} \phi^\alpha \delta^\beta \epsilon_{[\alpha\beta]} \phi \alpha \alpha + 2T^[\alpha\beta] \phi_f \epsilon_{[\alpha\beta]} (T - U) \phi_f
\]  
(171)

Spinorial derivatives of the self–dual and anti–self–dual super–covariant field strength:

\[
D^\dot{\psi} F^+[\alpha\beta] = \frac{1}{8} \phi^\alpha \delta^\beta \epsilon_{[\alpha\beta]} \phi \alpha \alpha + 2T^[\alpha\beta] \phi_f \epsilon_{[\alpha\beta]} (T - U) \phi_f
\]  
(172)

Spinorial derivatives of \( \Omega_a \), the field strength of the scalars:

\[
D^\phi \Omega^a_{[\alpha\beta]} = \frac{1}{8} \phi^\alpha \delta^\beta \epsilon_{[\alpha\beta]} \phi \alpha \alpha + 2T^[\alpha\beta] \phi_f \epsilon_{[\alpha\beta]} (T - U) \phi_f
\]  
(173)
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