Darboux transformations, finite reduction groups and related Yang–Baxter maps

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Abstract
In this paper, we construct Yang–Baxter (YB) maps using Darboux matrices which are invariant under the action of finite reduction groups. We present six-dimensional YB maps corresponding to Darboux transformations for the nonlinear Schrödinger (NLS) equation and the derivative nonlinear Schrödinger (DNLS) equation. These YB maps can be restricted to four-dimensional YB maps on invariant leaves. The former are completely integrable and they also have applications to a recent theory of maps preserving functions with symmetries (Fordy A and Kassotakis A 2013 J. Phys. A: Math. Theor. 46 205201). We give a six-dimensional YB-map corresponding to the Darboux transformation for a deformation of the DNLS equation. We also consider vector generalizations of the YB maps corresponding to the NLS and DNLS equations.

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1. Introduction

The Yang–Baxter (YB) equation has a fundamental role in the theory of quantum and classical integrable systems. In particular, YB maps, namely the set-theoretical solutions [12] of the YB equation, have been of great interest for several researchers in the area of mathematical physics. They are related to several concepts of integrability such as, for instance, the multidimensionally consistent equations [3, 4, 7, 30–32]. In particular, for those YB maps which admit Lax representation [37], there are corresponding hierarchies of commuting transfer maps which preserve the spectrum of their monodromy matrix [39, 40]. Therefore, the construction of YB maps is important.

Here, we construct six-dimensional YB maps for all the cases corresponding to a recent classification of automorphic Lie algebras. Some of these maps can be reduced to completely integrable maps on invariant leaves.

The YB equation
\[ Y^{12} \circ Y^{13} \circ Y^{23} = Y^{23} \circ Y^{13} \circ Y^{12}, \] (1.1)
originates in the works of Yang [41] and Baxter [6]. Here, $Y_{ij}$ denotes the action of a linear operator $Y : U \otimes U \to U \otimes U$ on the $ij$ factor of the triple tensor product $U \otimes U \otimes U$, where $U$ is a vector space. In this form, equation (1.1) is known in the literature as the quantum YB equation.

In 1992, Drinfel’d [12] proposed to replace $U$ by an arbitrary set $A$ and, therefore, the tensor product $U \otimes U$ by the Cartesian product $A \times A$. In our paper, $A$ is an algebraic variety in $K^N$, where $K$ is any field of zero characteristic, such as $\mathbb{C}$ or $\mathbb{Q}$.

In [39], Veselov proposed the term Yang–Baxter map for the set-theoretical solutions of the quantum YB equation. Specifically, we consider the map $Y : A \times A \to A \times A$,

\[
Y : (x, y) \mapsto (u(x, y), v(x, y)).
\]  

(1.2)

Furthermore, we define the maps $Y_{ij} : A \times A \times A \to A \times A \times A$ for $i, j = 1, 2, 3, i \neq j$, which appear in equation (1.1), by the following relations

\[
Y_{12}^{ij}(x, y, z) = (u(x, y), v(x, y), z),
\]  

(1.3a)

\[
Y_{13}^{ij}(x, y, z) = (u(x, z), v(x, z)),
\]  

(1.3b)

\[
Y_{23}^{ij}(x, y, z) = (x, u(y, z), v(y, z)),
\]  

(1.3c)

where $x, y, z \in A$. The variety $A$, in general, can be of any dimension. We shall consider finite-dimensional varieties in $K^N$. The map $Y_{ij}$, $i < j$, is defined as $Y_{ij}$ where we swap $u(k, l) \leftrightarrow v(l, k), k, l = x, y, z$. For example, $Y_{31}^{ij}(x, y, z) = (v(y, x), u(y, x), z)$.

The map (1.2) is a YB map, if it satisfies the YB equation (1.1). Moreover, it is called reversible if the composition of $Y_{ij}$ and $Y_{ji}$ is the identity map,

\[
Y_{ij} \circ Y_{ji} = Id.
\]  

(1.4)

We use the term parametric YB map when $u$ and $v$ are attached with parameters $a, b \in K^n$, namely $u = u(x, y; a, b)$ and $v = v(x, y; a, b)$, meaning that the following map

\[
Y_{a,b} : (x, y; a, b) \mapsto (u(x, y; a, b), v(x, y; a, b)),
\]  

(1.5)

satisfies the parametric YB equation

\[
Y_{12}^{ab} \circ Y_{13}^{cd} \circ Y_{23}^{bc} = Y_{23}^{cb} \circ Y_{13}^{dc} \circ Y_{12}^{ab}.
\]  

(1.6)

Following Suris and Veselov in [37], we call a Lax matrix for a parametric YB map, a matrix $L = L(x; c, \lambda)$ depending on a variable $x$, a parameter $c$ and a spectral parameter $\lambda$, such that the Lax equation

\[
L(u(a, \lambda)L(v; b, \lambda) = L(y; b, \lambda)L(x; a, \lambda), \quad \text{for any } \lambda \in K,
\]  

(1.7)

is satisfied due to the YB map. The above is also called a refactorization problem.

It is obvious that the Lax equation (1.7) does not always have a unique solution, which motivated Kouloukas and Papageorgiou in [21] to propose the term strong Lax matrix for a YB map. This is when the Lax equation is equivalent to a YB map

\[
(u, v) = Y_{a,b}(x, y).
\]  

(1.8)

Actually, the uniqueness of refactorization (1.7) is a sufficient condition for the solutions of the Lax equation to define a reversible YB map [19, 21, 40] of the form (1.8). In the opposite case, one may need to check if the obtained map satisfies the YB equation.

One of the most famous parametric YB maps is Adler’s map [1]

\[
(x_1, x_2) \mapsto (u, v) = \left( x_2 - \frac{a - b}{x_1 + x_2}, x_1 + \frac{a - b}{x_1 + x_2} \right).
\]  

(1.9)
which is related to the 3D consistent discrete potential KDV equation [29, 33]. In terms of Lax matrices, Adler’s map (1.9) can be obtained from the following strong Lax matrix [37, 40]:

$$L(x; a, \lambda) = \begin{pmatrix} x & 1 \\ x^2 + a - \lambda & x \end{pmatrix}.$$  

(1.10)

In [34, 35], a variety of YB maps is constructed using the symmetries of multi-field equations on quad graphs.

Now, since the Lax equation, (1.7), has the obvious symmetry

$$(a, v, a, b) \leftrightarrow (y, x, b, a)$$

(1.11)

we have the following.

**Proposition 1.0.1.** If a matrix refactorization problem (1.7) yields a rational map (1.8), then this map is birational.

**Proof.** Let $Y : (x, y) \mapsto (u, v)$ be a rational map corresponding to a refactorization problem (1.7), i.e.

$$x \mapsto u = \frac{n_1(x, y; a, b)}{d_1(x, y; a, b)}, \quad y \mapsto v = \frac{n_2(x, y; a, b)}{d_2(x, y; a, b)},$$

(1.12)

where $n_i, d_i, i = 1, 2$, are polynomial functions of their variables.

Due to the symmetry (1.11) of the refactorization problem (1.7), the inverse map of $Y$, $Y^{-1} : (x, y) \mapsto (u, v)$, is also rational and, in fact,

$$u \mapsto x = \frac{n_1(v, u; b, a)}{d_1(v, u; b, a)}, \quad v \mapsto y = \frac{n_2(v, u; b, a)}{d_2(v, u; b, a)}.$$  

(1.13)

Therefore, $Y$ is a birational map. \qed

**Proposition 1.0.2.** If $L = L(x, a; \lambda)$ is a Lax matrix with corresponding YB map, $Y : (x, y) \mapsto (u, v)$, then the $\text{tr}(L(y, b; \lambda)L(x, a; \lambda))$ is a generating function of invariants of the YB map.

**Proof.** Since,

$$\text{tr}(L(u, a; \lambda)L(v, b; \lambda)) \equiv \text{tr}(L(y, b; \lambda)L(x, a; \lambda)) = \text{tr}(L(x, a; \lambda)L(y, b; \lambda)) \equiv \sum \lambda^I I_i(x, y; a, b),$$

(1.14)

and the function $\text{tr}(L(x, a; \lambda)L(y, b; \lambda))$ can be written as $\text{tr}(L(x, a; \lambda)L(y, b; \lambda)) = \sum \lambda^I I_i(x, y; a, b)$, from (1.14) follows that

$$I_i(u, v; a, b) = I_i(x, y; a, b),$$

(1.15)

which are invariants for $Y$. \qed

The invariants of a YB map, $I_i(x, y; a, b)$, may not be functionally independent. The invariants are useful if one is interested in the dynamics of such maps. In terms of dynamics, the most interesting maps are those which are not involutive, although involutive maps also have useful applications [16]. In all the cases presented in the next sections, our YB maps are not involutive.

In this paper, we are interested in the integrability of the YB maps as finite discrete maps. The transfer dynamics of YB maps is discussed in [39, 40].

Now, following [13, 38] we define integrability for YB maps.

**Definition 1.0.1.** A $2N$-dimensional YB map,

$$Y : (x_1, \ldots, x_{2N}) \mapsto (u_1, \ldots, u_{2N}), \quad u_i = u_i(x_1, \ldots, x_{2N}), \quad i = 1, \ldots, 2N,$$

is said to be completely integrable or Liouville integrable if
(i) there is a Poisson matrix $J_{ij} = \{x_i, x_j\}$, of rank $2r$, which is invariant under $Y$, namely $J \circ Y = \tilde{J}$, where $\tilde{J}_{ij} = \{u_i, u_j\}$;

(ii) map $Y$ has $r$ functionally independent invariants, $I_i$, namely $I_i \circ Y = I_i$, which are in involution with respect to the corresponding Poisson bracket, i.e. $\{I_i, I_j\} = 0$, $i, j = 1, \ldots, r$, $i \neq j$;

(iii) there are $k = 2N - 2r$ Casimir functions, namely functions $C_i$, $i = 1, \ldots, k$, such that $\{C_i, f\} = 0$, for any arbitrary function $f = f(x_1, \ldots, x_{2N})$. These are invariant under $Y$, namely $C_i \circ Y = C_i$.

Liouville integrability of a YB map is important for the construction of integrable lattices. In particular, for those YB maps which admit Lax representation, one could consider a family of integrable maps which preserve the spectrum of the corresponding monodromy matrix [39, 40]. The trace of the former provides us with invariants and one can claim integrability of the corresponding lattice, if the invariants are in involution with respect to a Poisson bracket.

2. Organization of this paper

In the next section, we briefly give some introduction to the notions of the reduction group [27], automorphic Lie algebras [8, 9, 22, 24] and Darboux transformations to make this text self-contained. We study the case associated with $sI_2(\mathbb{K})$ and, therefore, we can restrict ourselves to three distinct reduction groups, namely the trivial one, $Z_2$ and $Z_2 \times Z_2$ [8, 9, 24].

In section 4, we use Darboux transformations presented in [18] to derive YB maps. In particular, we consider Darboux matrices for the nonlinear Schrödinger (NLS) equation, the derivative nonlinear Schrödinger (DNLS) equation and a deformation of the DNLS equation. For these Darboux matrices, the refactorization is not unique. Therefore, for the corresponding six-dimensional YB maps which are derived from the refactorization problem, in principle, one needs to check the YB property separately. Yet, the entries of these Darboux matrices obey certain differential equations which possess first integrals. There is a natural restriction of the Darboux map on the affine variety corresponding to a level set of these first integrals. These restrictions make the refactorization unique and this guarantees that the induced four-dimensional YB maps satisfy the YB equation and they are reversible [40]. We show that these YB maps have Poisson structure. However, the first integrals are not always very useful for the reduction because, in general, they are polynomial equations. In particular, in the cases of NLS and DNLS equations we present six-dimensional YB maps and their four-dimensional restrictions on invariant leaves. These four-dimensional restrictions are birational YB maps and we prove that they are integrable in the Liouville sense. In the case of the deformation of the DNLS equation, we present a six-dimensional YB map and a linear approximation to the four-dimensional YB map.

In section 5, we consider the vector generalizations of the Adler–Yamilov YB map and the four-dimensional YB map corresponding to the DNLS equation.

3. Automorphic Lie algebras and Darboux transformations and reduction groups with degenerate orbits

The reduction group was first introduced in [26, 27]. It is a discrete group of automorphisms of a Lax operator, and its elements are simultaneous automorphisms of the corresponding Lie algebra and fractional-linear transformations of the spectral parameter.
Automorphic Lie algebras were introduced in [22, 23] and studied in [8, 9, 22–24]. These algebras constitute a subclass of infinite-dimensional Lie algebras and their name is due to their construction, which is very similar to the one for automorphic functions.

Darboux transformations and their relations to the theory of integrable systems have been extensively studied [25, 36]. Such transformations can be derived from Lax pairs as, for instance, in [36], or in a more systematic algebraic manner in [11, 18].

We are interested in Darboux transformations corresponding to Lax operators of the following form

$$L = L(p(x); \lambda) = D_x + U(p(x); \lambda),$$  \hspace{1cm} (3.1)

where $U$ belongs to an automorphic Lie algebra.

In the rest of the text, we use ‘$L$’ for Lax operators and “$L$” for Lax matrices of the refactorization problem (1.7).

By Darboux transformations, we understand maps

$$L(p(x); \lambda) \rightarrow \tilde{L} := L(\tilde{p}(x); \lambda) = MLM^{-1},$$  \hspace{1cm} (3.2)

$M$ is a matrix called the Darboux matrix. They map fundamental solutions, $\Psi$, of the equation $L\Psi = 0$ to other fundamental solutions, $\tilde{\Psi} = M\Psi$, of the equation $\tilde{L}\tilde{\Psi} = 0$.

The structure of Lax operators has a natural Lie algebraic interpretation in terms of Kac–Moody algebras and automorphic Lie algebras [8, 9, 22, 23]. While a Kac–Moody algebra is associated with an automorphism of finite order, an automorphic Lie algebra corresponds to finite groups of automorphisms, namely the reduction group [26, 27].

It has been shown that in the case of $2 \times 2$ matrices, which we study in this paper, the essentially different reduction groups are the trivial group (with no reduction), the cyclic group $\mathbb{Z}_2$ (leading to the Kac–Moody algebra $A_1^1$) and the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$ [8, 9, 24, 28].

We shall present four- and six-dimensional YB maps for all the following cases: the trivial group, associated with the NLS equation [42],

$$p_t = p_{xx} + 4p^2q, \quad q_t = -q_{xx} - 4pq^2;$$  \hspace{1cm} (3.3)

the $\mathbb{Z}_2$ group, associated with the DNLS equation [17],

$$p_t = p_{xx} + 4(p^2q)_x, \quad q_t = -q_{xx} + 4(pq^2)_x;$$  \hspace{1cm} (3.4)

and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group, which yields a deformation of the DNLS equation [28]

$$p_t = p_{xx} + 8(p^2q)_x - 4q, \quad q_t = -q_{xx} + 8(pq^2)_x - 4p.$$  \hspace{1cm} (3.5)

The above mentioned groups are representative of all the finite reduction groups with degenerate orbits, namely orbits corresponding to the fixed points of the fractional-linear transformations of the spectral parameter.

4. Derivation of YB maps

In [18], we used Darboux transformations to construct integrable systems of discrete equations, which have the multidimensional consistency property [3, 4, 7, 30–32]. The compatibility condition of Darboux transformations around the square is exactly the same with the Lax equation (1.7). Therefore, in this paper, we use Darboux transformations to construct YB maps.

We begin with the well-known example of the Darboux transformation for the NLS equation and construct its associated YB map.
4.1. The nonlinear Schrödinger equation

In the case of NLS equation, the Lax operator is given by

\[ \mathcal{L}(p, q; \lambda) = D_x + \lambda U_1 + U_0, \quad \text{where} \quad U_1 = \sigma_3 = \text{diag}(1, -1), \quad U_0 = \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix}. \quad (4.1) \]

The elementary Darboux transformation, \( M \), of \( \mathcal{L} \) is given by [18]

\[ M = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} f & p \\ \bar{q} & 1 \end{pmatrix}. \quad (4.2) \]

The entries of (4.2), according to definition (3.2), must satisfy the following system of equations

\[ \partial_x f = 2(pf - \bar{p}\bar{q}), \quad \partial_x p = 2(pf - \bar{p}), \quad \partial_x \bar{q} = 2(q - \bar{q}f), \quad (4.3) \]

which admits the following first integral

\[ \partial_x (f - \bar{p}\bar{q}) = 0. \quad (4.4) \]

This integral implies that \( \partial_x \det M = 0 \).

In correspondence with (4.2), we define the matrix

\[ M(x; \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} X & x_1 \\ x_2 & 1 \end{pmatrix}, \quad x = (x_1, x_2, X), \quad (4.5) \]

and substitute it into the Lax equation (1.7)

\[ M(u; \lambda)M(v; \lambda) = M(y; \lambda)M(x; \lambda), \quad (4.6) \]

to derive the following system of equations

\[ \begin{align*}
    u_1 &= x_1, \quad u_2 = y_2, \quad U + V = X + Y, \quad u_2v_1 = x_1y_2, \\
    u_1 + Uv_1 &= y_1 + x_1Y, \quad u_1v_2 + UV = x_2y_1 + XY, \quad v_2 + u_2V = x_2 + Xy_2.
\end{align*} \]

The corresponding algebraic variety is a union of two six-dimensional components. The first one is obvious from the refactorization problem (4.6), and it corresponds to the permutation map

\[ x \mapsto u = y, \quad y \mapsto v = x, \]

which is a (trivial) YB map. The second one can be represented as a rational six-dimensional non-involutive map of \( K^3 \times K^3 \rightarrow K^3 \times K^3 \)

\[ \begin{align*}
    x_1 &\mapsto u_1 = \frac{y_1 + x_1^2y_2 - x_1X + x_1Y}{1 + x_1y_2}, \quad y_1 \mapsto v_1 = x_1, \\
    x_2 &\mapsto u_2 = y_2, \quad y_2 \mapsto v_2 = \frac{x_2 + y_1^2y_2 + y_2X - y_2Y}{1 + x_1y_2}, \\
    X &\mapsto U = \frac{y_1y_2 - x_1x_2 + X + x_1y_2Y}{1 + x_1y_2}, \quad Y \mapsto V = \frac{x_1x_2 - y_1y_2 + x_1y_2X + Y}{1 + x_1y_2}, \quad (4.7)
\end{align*} \]

which, as one can easily check, satisfies the YB equation.

The trace of \( M(y; \lambda)M(x; \lambda) \) is a polynomial in \( \lambda \) whose coefficients are

\[ \text{tr}(M(y; \lambda)M(x; \lambda)) = \lambda^2 + \lambda I_1(x, y) + I_2(x, y), \]

where

\[ I_1(x, y) = X + Y \quad \text{and} \quad I_2(x, y) = x_2y_1 + x_1y_2 + XY, \quad (4.8) \]

and those, according to proposition 1.0.2, are invariants for the YB map (4.7).

In the following section, we show that the YB map (4.7) can be restricted to a four-dimensional YB map which has Poisson structure.
4.1.1. Restriction on symplectic leaves: the Adler–Yamilov map. In this section, we show that map (4.7) can be restricted to the Adler–Yamilov map on symplectic leaves, by taking into account the first integral, (4.4), of system (4.3).

In particular, we have the following.

Proposition 4.1.1

(i) \( \Phi = X - x_1x_2 \) and \( \Psi = Y - y_1y_2 \) are invariants (first integrals) of the map (4.7).

(ii) The six-dimensional map (4.7) can be restricted to a four-dimensional map \( Y_{a,b} : A_a \times A_b \rightarrow A_a \times A_b \), where \( A_a, A_b \) are level sets of the first integrals \( \Phi \) and \( \Psi \), namely

\[
A_a = \{ (x_1, x_2, X) \in K^3; X = a + x_1x_2 \}, \quad A_b = \{ (y_1, y_2, Y) \in K^3; Y = b + y_1y_2 \}.
\]  

(iii) Map \( Y_{a,b} \) is the Adler–Yamilov map.

Proof.

(i) It can be readily verified that (4.7) implies \( U - v_1v_2 = X - x_1x_2 \) and \( V - v_1v_2 = Y - y_1y_2 \). Thus, \( \Phi \) and \( \Psi \) are invariants, i.e. first integrals of the map.

(ii) The existence of the restriction is obvious. Using the conditions \( X = x_1x_2 + a \) and \( Y = y_1y_2 + b \), one can eliminate \( X \) and \( Y \) from (4.7). The resulting map, \( x \rightarrow u(x, y) \), \( y \rightarrow v(x, y) \), is given by

\[
(y, x) \xrightarrow{Y_{a,b}} \left( y - \frac{a - b}{1 + x_1y_2}x_1, y_2, x_1, x_2 + \frac{a - b}{1 + x_1y_2}y_2 \right).
\]  

(iii) Map (4.10) coincides with the Adler–Yamilov map.

Map (4.10) first appeared in the work of Adler and Yamilov [5]. It also appears in [20, 34].

Now, one can use the condition \( X = x_1x_2 + a \) to eliminate \( X \) from the Lax matrix (4.5), i.e.

\[
M(x; a, \lambda) = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a + x_1x_2 & x_1 \\ x_2 & 1 \end{pmatrix}, \quad x = (x_1, x_2).
\]  

The form of Lax matrix (4.11) coincides with the well-known Darboux transformation for the NLS equation (see [36] and references therein). Now, the Adler–Yamilov map follows from the strong Lax representation

\[
M(u; a, \lambda)M(v; b, \lambda) = M(y; b, \lambda)M(x; a, \lambda).
\]  

Therefore, the Adler–Yamilov map (4.10) is a reversible parametric YB map with strong Lax matrix (4.11). Moreover, it is easy to verify that it is not involutive.

For the integrability of this map we have the following.

Proposition 4.1.2. The Adler–Yamilov map (4.10) is completely integrable.

Proof. From the trace of \( M(y; b, \lambda)M(x; a, \lambda) \), we obtain the following invariants for the map (4.10)

\[
I_1(x, y) = x_1x_2 + y_1y_2 + a + b, \\
I_2(x, y) = (a + x_1x_2)(b + y_1y_2) + x_1y_2 + x_2y_1 + 1.
\]  

The constant terms in \( I_1, I_2 \) can be omitted. It is easy to check that \( I_1, I_2 \) are in involution with respect to invariant Poisson brackets defined as

\[
\{x_1, x_2\} = \{y_1, y_2\} = 1, \quad \text{and all the rest} \quad \{x_i, y_j\} = 0,
\]
and the corresponding Poisson matrix is invariant under the YB map (4.10). Therefore, the map (4.10) is completely integrable.

The above proposition implies the following.

**Corollary 4.1.3.** The invariant leaves $A_a$ and $B_b$, given in (4.9), are symplectic.

### 4.2. Derivative NLS equation: $\mathbb{Z}_2$ reduction

The Lax operator for the DNLS equation [10, 17] is given by

$$\mathcal{L}(p, q; \lambda) = D_4 + \lambda^2 U_2 + \lambda U_1,$$

where $U_2 = \sigma_3$, $U_1 = \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix}$.

and $\sigma_3$ is a Pauli matrix. Operator $\mathcal{L}$ is invariant with respect to the following involution

$$\mathcal{L}(\lambda) = \sigma_3 \mathcal{L}(-\lambda) \sigma_3,$$

where $\mathcal{L}(\lambda) = \mathcal{L}(p, q; \lambda)$. Involution (4.17) generates the so-called reduction group [23, 27] and it is isomorphic to $\mathbb{Z}_2$.

The Darboux matrix in this case is given by [18]

$$M := \lambda^2 \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & fp \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix},$$

(4.18)

whose entries $p, \tilde{q}$ and $f$ obey the following system of equations

$$\partial_\lambda p = 2p(\tilde{p}q - pq) - \frac{2}{f}(\tilde{p} - cp),$$

(4.19a)

$$\partial_\lambda \tilde{q} = 2\tilde{q}(\tilde{p}q - pq) - \frac{2}{f}(c\tilde{q} - q),$$

(4.19b)

$$\partial_\lambda f = 2f(pq - \tilde{p}q).$$

(4.19c)

System (4.19a)–(4.19c) has a first integral which obliges the determinant of matrix (4.18) to be $x$-independent, and it is given by

$$\partial_\lambda (f^2 \tilde{p}q - f) = 0.$$

(4.20)

Using the entries of (4.18) as variables, namely $(p, \tilde{q}, f; c) \rightarrow (x_1, x_2, X; 1)$, we define the matrix

$$M(x; \lambda) = \lambda^2 \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1X \\ x_2X & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x = (x_1, x_2, X).$$

(4.21)

The Lax equation implies the following equations:

$$u_1 V + v_1 V = x_1 X + y_1 Y, \quad u_2 U + v_2 V = x_2 X + y_2 Y,$$

$$UV = XY, \quad v_1 UV = x_1 XY, \quad u_2 UV = y_2 XY, \quad u_2 v_1 UV = x_1 y_2 XY,$$

(4.22)

$$U + V + u_1 v_2 UV = X + Y + x_2 y_1 XY.$$

As in the case of the NLS equation, the algebraic variety consists of two components. The first six-dimensional component corresponds to the permutation map

$$x \mapsto u = y, \quad y \mapsto v = x,$$

(4.23)

and the second corresponds to the following six-dimensional YB map

$$x_1 \mapsto u_1 = f_1(x, y), \quad y_1 \mapsto v_1 = f_2(y, \pi x),$$

$$x_2 \mapsto u_2 = f_2(x, y), \quad y_2 \mapsto v_2 = f_1(y, \pi x),$$

$$X \mapsto U = f_1(x, y), \quad Y \mapsto V = f_3(y, \pi x),$$

(4.24)
where \( \pi \) is the permutation function, \( \pi(x_1, x_2, X) = (x_2, x_1, X) \), \( \pi^2 = 1 \) and \( f_1, f_2 \) and \( f_3 \) are given by

\[
\begin{align*}
    f_1(x, y) &= -\frac{x_1X + (y_1 - x_1)Y - x_1x_2y_1XY - x_1^2x_2X^2}{x_1x_2X + x_1y_2Y - 1}, \\
    f_2(x, y) &= y_2, \\
    f_3(x, y) &= \frac{x_1x_2X + x_1y_2Y - 1}{x_1x_2X + x_1y_2Y - 1}.
\end{align*}
\] (4.25a, 4.25b, 4.25c)

One can verify that the above map is a non-involutive YB map. The invariants of this map are given by

\[
I_1(x \cdot \pi y) = XY \quad \text{and} \quad I_2(x, y) = (x \cdot \pi y)XY + X + Y.
\] (4.26)

The map \( Y : K^3 \times K^3 \rightarrow K^3 \times K^3 \), given by \( \{ (4.24), (4.25a) - (4.25c) \} \), can be restricted to a map of the Cartesian product of two two-dimensional affine varieties

\[
A_a = \{(x_1, x_2, X) \in K^3; X - X^2x_1x_2 = a \in K\},
\] (4.27)

\[
A_b = \{(y_1, y_2, Y) \in K^3; Y - Y^2y_1y_2 = b \in K\},
\] (4.28)

which are invariant varieties of the map \( Y \). Thus, the YB map, \( Y_{a,b} \), is a birational map \( Y_{a,b} : A_a \times A_b \rightarrow A_a \times A_b \).

It is easy to uniformize the rational variety \( A_a \) and express the YB map explicitly. The equations defining the varieties \( A_a \) and \( A_b \) are linear in \( x_1, x_2 \) and \( y_1, y_2 \), respectively. Hence, we can express

\[
x_2 = \frac{1}{x_1}X - \frac{a}{x_1X^2}, \quad y_2 = \frac{1}{y_1Y} - \frac{b}{y_1Y^2}.
\] (4.29)

The resulting map is given by

\[
x_1 \mapsto u_1 = \frac{h_1}{h_2}, \quad X \mapsto U = h_2Y, \quad y_1 \mapsto v_1 = x_1, \quad Y \mapsto V = \frac{1}{h_2}X
\] (4.30)

where the quantities \( h_i, i = 1, 2 \), are given by

\[
h_1 = \frac{ay_1Y + x_1X(a - Y)}{ay_1Y + x_1X(b - Y)}, \quad h_2 = \frac{ay_1Y + x_1X(b - Y)}{by_1Y + x_1X(b - X)}.
\] (4.31)

Nevertheless, in the next section, we present a more symmetric way to parametrize the varieties \( A_a, A_b \) and the Lax matrix.

4.2.1. \( \mathbb{Z}_2 \) reduction: a reducible six-dimensional YB map. Now, let us go back to the Darboux matrix (4.18) and replace \((f, p, f, f, f, c) \rightarrow (x_1, x_2, X; 1) \), namely

\[
M(x; \lambda) = \lambda^2 \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad x = (x_1, x_2, X).
\] (4.32)

From the Lax equation, we obtain the following equations

\[
w_2v_1 = x_1y_2, \quad u_2V = Xy_2, \quad Uv_1 = x_1Y, \quad UV = XY
\]

\[
u_1 + v_1 = x_1 + y_1, \quad U + uv_2 + V = x + x_2y_1 + Y, \quad u_2 + v_2 = x_2 + y_2.
\]

Now, the first six-dimensional component of the algebraic variety corresponds to the trivial map (4.23) and the second component corresponds to a map of the form (4.24), with \( f_1, f_2 \) and \( f_3 \) now given by

\[
f_1(x, y) = \frac{(x_1 + y_1)X - x_1Y - x_1x_2(x_1 + y_1)}{X - x_1(x_2 + y_2)},
\] (4.33a)
Now, map (4.37) follows from the strong Lax representation (4.12). Therefore, it is a reversible parametric YB map. It can also be verified that it is not involutive.

For the integrability of map (4.37), we have the following.

Proposition 4.2.2. Map (4.37) is completely integrable.

Proof.
(i) Map (4.24), (4.33a)–(4.33c) implies $U - u_1 u_2 = X - x_1 x_2$ and $V - v_1 v_2 = Y - y_1 y_2$. Therefore, $\Phi$ and $\Psi$ are first integrals of the map.

(ii) The conditions $X = x_1 x_2 + a$ and $Y = y_1 y_2 + b$ define the level sets, $A_a$ and $A_b$, of $\Phi$ and $\Psi$, respectively. Using these conditions, we can eliminate $X$ and $Y$ from map (4.24), (4.33a)–(4.33c). The resulting map, $Y_{a,b} : A_a \times A_b \rightarrow A_a \times A_b$, is given by (4.37).

Now, using condition $X = x_1 x_2 + a$, matrix (4.32) takes the following form:

$$M(x; k; \lambda) = \lambda^2 \begin{pmatrix} k + x_1 x_2 & 0 \\ 0 & \lambda \end{pmatrix} + \lambda \begin{pmatrix} 0 & x_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.38)$$

Now, map (4.37) follows from the strong Lax representation (4.12). Therefore, it is a reversible parametric YB map. It can also be verified that it is not involutive.

For the integrability of map (4.37), we have the following.

Proposition 4.2.1

(i) $\Phi = X - x_1 x_2$ and $\Psi = Y - y_1 y_2$ are invariants of the map \{(4.24), (4.33a)–(4.33c)\}.

(ii) The six-dimensional map \{(4.24), (4.33a)–(4.33c)\} can be restricted to a four-dimensional map $Y_{a,b} : A_a \times A_b \rightarrow A_a \times A_b$, given by

$$(x, y) \xrightarrow{Y_{a,b}} \begin{pmatrix} x_1 + \frac{a - b}{a - x_1 y_2} x_1, & \frac{a - x_2 y_2}{b - x_1 y_2} x_2, & \frac{b - x_1 y_2}{a - x_1 y_2} x_1, & x_2 + \frac{b - a}{b - x_1 y_2} y_2 \end{pmatrix}. \quad (4.37)$$

and $A_a$, $A_b$ are given by (4.9).
Proof. The invariants of map (4.37) which we retrieve from the trace of $M(y; b, \lambda)M(x; a, \lambda)$ are

$$I_1(x, y) = (a + x_1x_2)(b + y_1y_2), \quad I_2(x, y) = (x_1 + y_1)(x_2 + y_2) + a + b. \quad (4.39)$$

However, the quantities $x_1 + y_1$ and $x_2 + y_2$ in $I_2$ are invariants themselves. The Poisson bracket in this case is given by

$$\{x_1, y_2\} = \{y_1, x_2\} = \{y_2, x_1\} = 1, \quad \text{and all the rest } \{x_i, y_j\} = 0. \quad (4.40)$$

The rank of the Poisson matrix is 2, $I_1$ is one invariant and $I_2 = C_1C_2 + a + b$, where $C_1 = x_1 + y_1$ and $C_2 = x_2 + y_2$ are Casimir functions. The latter are preserved by (4.37), namely $C_i \circ Y_{a,b} = C_i$, $i = 1, 2$. Therefore, map (4.37) is completely integrable. □

Corollary 4.2.3. Map (4.37) can be expressed as a map of two variables on the symplectic leaf

$$x_1 + y_1 = c_1, \quad x_2 + y_2 = c_2. \quad (4.41)$$

4.3. A deformation of the DNLS equation: dihedral group

In the case of dihedral reduction group, the Lax operator is given by

$$\mathcal{L}(p, q; \lambda) = D_\lambda + \lambda^2 U_2 + \lambda U_1 + \lambda^{-1}U_{-1} - \lambda^{-2}U_{-2},$$

where $U_2 = U_{-2} = \sigma_3, \quad U_1 = \begin{pmatrix} 0 & 2p \\ 2q & 0 \end{pmatrix}, \quad U_{-1} = \sigma_1 U_1 \sigma_1, \quad (4.42)$$

and $\sigma_1, \sigma_3$ are Pauli matrices. Here, the reduction group consists of the following set of transformations acting on the Lax operator (4.42),

$$\mathcal{L}(\lambda) = \sigma_3 \mathcal{L}(-\lambda) \sigma_3 \quad \text{and} \quad \mathcal{L}(\lambda) = \sigma_1 \mathcal{L}(\lambda^{-1}) \sigma_1, \quad (4.43)$$

and it is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong D_2$. [23].

In this case, the Darboux matrix is given by [18]

$$M = f \left( \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} + \lambda \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} + g \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \right), \quad (4.44)$$

where its entries obey the following equations

$$\partial_q p = 2((\tilde{p}q - pq) + (p - \tilde{p})g + q - \tilde{q}), \quad (4.45a)$$

$$\partial_q \tilde{q} = 2((\tilde{p}q - pq)\tilde{q} + p - \tilde{p} + (q - \tilde{q})g), \quad (4.45b)$$

$$\partial_q g = 2((\tilde{p}q - pq)g + (p - \tilde{p})p + (q - \tilde{q})\tilde{q}), \quad (4.45c)$$

$$\partial_q f = -2(\tilde{p}q - pq)f. \quad (4.45d)$$

It can be shown that the above system of differential equations admits two first integrals, $\partial_q \Phi_1 = 0, \quad \Phi_1 \equiv f^2(g - p\tilde{q})$ and $\Phi_2 := f^2(g^2 + 1 - p^2 - \tilde{q}^2). \quad (4.46)$

In the next section, we construct a six-dimensional map from (4.44).
4.3.1. Dihedral group: a six-dimensional YB map. We consider the matrix $N := fM$, where $M$ is given by (4.44), and we change $(p, q, f^2) \rightarrow (x_1, x_2, X)$. Then,

$$N(x, X; \lambda) = \begin{pmatrix} \lambda^2 X + x_1 x_2 X + 1 & \lambda x_1 X + \lambda^{-1} x_2 X \\ \lambda x_2 X + \lambda^{-1} x_1 X & \lambda^{-2} X + x_1 x_2 X + 1 \end{pmatrix},$$

(4.47)

where we have substituted the product $f^2g$ by

$$f^2g = 1 + x_1 x_2 X,$$

(4.48)

using the first integral, $\Phi_1$, in (4.46), and having rescaled $c_1 \rightarrow 1$.

The Lax equation for the Darboux matrix (4.47) reads

$$N(u; \lambda)N(v; \lambda) = N(y; \lambda)N(x; \lambda),$$

(4.49)

from where we obtain an algebraic system of equations, omitted because of its length.

The first six-dimensional component of the corresponding algebraic variety corresponds to the trivial YB map

$$x \mapsto u = y, \quad y \mapsto v = x,$$

and the second component corresponds to the following map

$$x_1 \mapsto u_1 = \frac{f(x, y)}{g(x, y)}, \quad y_1 \mapsto v_1 = x_1,$$

$$x_2 \mapsto u_2 = y_2, \quad y_2 \mapsto v_2 = \frac{f(\pi y, \pi x)}{g(\pi y, \pi x)},$$

$$X \mapsto U = \frac{g(x, y)}{h(x, y)}, \quad Y \mapsto V = \frac{g(\pi y, \pi x)}{h(\pi y, \pi x)},$$

(4.50)

where $f, g$ and $h$ are given by

$$f(x, y) = x_1 X + \left[ x_2 - y_2 + 2x_1 x_2 y_1 + x_1^2 (y_2 - 3x_2) \right]XY$$

$$+ (x_2^2 - 1)[y_1(1 + x_1^2) - x_1(1 + x_1^2)]XY^2 - (x_1^2 - 1)(y_2 - x_2)X^2$$

$$- (x_2^2 - 1)(y_2^2 - 1)[y_2(y_2^2 - 1) + x_2(y_2^2 - 2x_1 y_2 + 1)]X^2 Y$$

$$+ y_1(x_1^2 - 1)^2 (y_2^2 - 1)(y_2^2 - 1)X^3 Y^2 + (x_1^2 - 1)^2 (x_2^2 - 1)(y_2 - x_2)X^3 Y$$

$$+ (y_1 - x_1)Y,$$

$$g(x, y) = X + 2y_2(y_1 - x_1)XY + (y_1^2 - 1)(x_1 - y_1)^2XY^2$$

$$+ 2(x_1^2 - 1)(x_1 - x_2 y_2)X^2 Y + 2x_2(x_1^2 - 1)(y_1^2 - 1)(x_1 - y_1)X^2 Y^2$$

$$+ (x_1^2 - 1)^2 (x_2^2 - 1)(y_2^2 - 1)X^3 Y^2,$$

$$h(x, y) = 1 - 2x_1(y_2 - x_2)X - 2(x_1 y_1 - 1)(y_2 - 1)XY$$

$$+ (x_2^2 - 1)(x_2 - y_2)^2 X^2 - 2y_1(x_2 - y_2)(x_1^2 - 1)(y_2 - 1)X^2 Y$$

$$+ (x_1^2 - 1)(y_2^2 - 1)(y_1^2 - 1)X^2 Y^2.$$

(4.51)

It can be verified that this is a parametric YB map. From $\text{tr}(N(x, X; \lambda)N(y, Y; \lambda))$, we extract the following invariants for the above map

$$I_1(x, y) = XY,$$

(4.52a)

$$I_2(x, y) = X + Y + (x_1 + y_1)(x_2 + y_2)XY,$$

(4.52b)

$$I_3(x, y) = 2x_1 x_2 X + 2y_1 y_2 Y + 2(x \cdot y + x_1 x_2 y_1 y_2)XY + 2.$$

(4.52c)
4.4. Dihedral group: a linearized YB map

We replace \( f \bar{q}, f p \) \( \rightarrow (\epsilon x_1, \epsilon x_2) \) in the Darboux matrix (4.44) and let us linearize the corresponding map around \( \epsilon = 0 \).

It follows from \( \Phi_1 = \frac{1-k}{4} \) and \( \Phi_2 = \frac{1+k}{4} \) that the quantities \( f \) and \( f g \) are given by

\[
 f = \frac{1+k}{2} + \mathcal{O}(\epsilon) \quad \text{and} \quad f g = \frac{1-k}{2} + \mathcal{O}(\epsilon),
\]

and, therefore, the Lax matrix by

\[
 M(x; k, \lambda) = \frac{1+k}{2} \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix} + \begin{pmatrix} 0 & \lambda x_1 + \lambda^{-1} x_2 \\ \lambda x_2 + \lambda^{-1} x_1 & 0 \end{pmatrix} + \frac{1-k}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon). \tag{4.53}
\]

The linear approximation to the YB map is given by

\[
 \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} \approx \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \frac{(a-1)(a-b)}{a+b} & \frac{a-b}{a+b} & \frac{2a}{a+b} & \frac{(a+1)(b-a)}{a+b} \\ \frac{a}{a+b} & \frac{2a}{a+b} & \frac{a+b}{a+b} & \frac{(a+1)(b-a)}{a+b} \\ \frac{a}{a+b} & \frac{2a}{a+b} & \frac{a+b}{a+b} & \frac{(a+1)(b-a)}{a+b} \\ \frac{a}{a+b} & \frac{2a}{a+b} & \frac{a+b}{a+b} & \frac{(a+1)(b-a)}{a+b} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}, \tag{4.54}
\]

which is a linear parametric YB map and it is not involutive.

5. \( 2N \times 2N \)-dimensional YB maps

In this section, we consider the vector generalizations of the YB maps (4.10) and (4.37). We replace the variables, \( x_1 \) and \( x_2 \), in the Lax matrices with \( N \)-vectors \( w_1 \) and \( w_2 \) to obtain \( 2N \times 2N \) YB maps. In what follows, we use the following notation for a \( n \)-vector \( w = (w_1, \ldots, w_n) \)

\[
w = (w_1, w_2), \quad \text{where} \quad w_1 = (w_1, \ldots, w_N), \quad w_2 = (w_{N+1}, \ldots, w_{2N}) \tag{5.1}
\]

and also

\[
\langle u_i \rangle := u_i, \quad |w_i| := w_i^T \quad \text{and their dot product with} \quad \langle u_i, w_i \rangle. \tag{5.2}
\]

5.1. NLS equation

Replacing the variables in (4.11) with \( N \)-vectors, namely

\[
 M(w; a, \lambda) = \begin{pmatrix} \lambda + a + \langle w_1, w_2 \rangle & \langle w_1 \rangle \\ \langle w_2 \rangle & 1 \end{pmatrix}, \tag{5.3}
\]

we obtain a unique solution of the Lax equation given by the following \( 2N \times 2N \) map

\[
 \begin{cases} 
 u_1 = \langle y_1 \rangle + f(z; a, b)|x_1|, \\
 u_2 = \langle y_2 \rangle, \tag{5.4a}
 \end{cases}
\]

and

\[
 \begin{cases} 
 v_1 = \langle x_1 \rangle, \\
 v_2 = \langle x_2 \rangle + f(z; b, a)|y_2|, \tag{5.4b}
 \end{cases}
\]

where \( f \) is given by

\[
f(z; b, a) = \frac{b-a}{1+z}, \quad z := \langle x_1, y_2 \rangle. \tag{5.4c}
\]

The above is a non-involutive parametric \( 2N \times 2N \) YB map with strong Lax matrix given by (5.3). As a YB map it appears in [34], but it is originally introduced by Adler [2]. Moreover,
one can construct the above $2N \times 2N$ map for the $N \times N$ Darboux matrix (5.3) by taking the limit of the solution of the refactorization problem in [21].

Two invariants of this map are given by

$$I_1(x, y; a, b) = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle,$$

(5.5a)

and

$$I_2(x, y; a, b) = b(x_1, x_2) + a(y_1, y_2) + \langle x_1, y_2 \rangle + \langle x_2, y_1 \rangle + \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle.$$

(5.5b)

These are the invariants which are obtained from the trace of $M(y; b, \lambda)M(x; a, \lambda)$ and they are not enough to claim Liouville integrability.

5.2. $\mathbb{Z}_2$ reduction

In the case of $\mathbb{Z}_2$ we consider, instead of (4.38), the following matrix:

$$M(w; a, \lambda) = \left( \begin{array}{cc} \lambda^2(a + \langle w_1, w_2 \rangle) & \lambda \langle w_1 \rangle \\ \lambda \langle w_2 \rangle & I \end{array} \right).$$

(5.6)

we obtain a unique solution for the Lax equation given by the following $2N \times 2N$ map

$$\begin{cases} (u_1) & = (y_1) + f(z; a, b)\langle x_1 \rangle, \\ (u_2) & = g(z; a, b)\langle y_2 \rangle, \end{cases}$$

(5.7a)

and

$$\begin{cases} (v_1) & = g(z; b, a)\langle x_1 \rangle, \\ (v_2) & = (x_2) + f(z; b, a)\langle y_2 \rangle, \end{cases}$$

(5.7b)

where $f$ and $g$ are given by

$$f(z; a, b) = \frac{a - b}{a - z}, \quad g(z; a, b) = \frac{a - z}{b - z}, \quad z := \langle x_1, y_2 \rangle.$$

(5.7c)

The above map is a non-involutive parametric $2N \times 2N$ YB map with a strong Lax matrix given by (5.6).

It follows from the trace of $M(y; b, \lambda)M(x; a, \lambda)$ that two invariants for (5.7a)–(5.7b) are given by

$$I_1(x, y; a, b) = b\langle x_1, x_2 \rangle + a\langle y_1, y_2 \rangle + \langle x_1, x_2 \rangle \langle y_1, y_2 \rangle,$$

(5.8a)

and

$$I_2(x, y; a, b) = (x_1 + y_1, x_2 + y_2).$$

(5.8b)

In fact, both vectors of the inner product in $I_2$ are invariants. However, as in the case of NLS vector generalization, the invariants are not enough to claim Liouville integrability.

6. Conclusions

In this paper, we used Darboux matrices, which are invariant under the action of reduction groups, as the main tool to connect integrable PDEs with discrete integrable maps. Specifically, solving a matrix refactorization problem for a Darboux matrix, we constructed six-dimensional Yang–Baxter (YB) maps which can be restricted to four-dimensional completely integrable YB maps on invariant leaves.

Our YB maps also have applications to a recent theory of maps preserving functions with symmetries [14]. The Liouville integrability of these maps is important if one is interested in the transfer dynamics of these maps.

The results obtained in this paper can be developed in several ways:

(i) study the transfer dynamics and the corresponding integrable lattices;

(ii) study the entwining YB maps;
(iii) examine the possibility of deriving Bäcklund transformations for PDEs from the corresponding YB map;
(iv) extend all the results for YB maps on Grassmann algebras [15].

Specifically, for (i), Liouville integrability is essential for the corresponding lattices to be integrable. In case (ii), one could consider YB maps which result as solutions of a refactorization problem between two Darboux matrices corresponding to two different PDEs. For case (iii), recall that the variables of the YB map are solutions of integrable PDEs. Then, demanding that the YB map preserves these solutions, we obtain relations between them. One could examine in which cases these relations are auto-Bäcklund transformations for the corresponding PDE. In the case of the entwining YB maps, this demand could lead to hetero-Bäcklund transformations, namely relations between the solutions of two different PDEs.

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