We investigate the influence of some natural types of subgroups on the structure of groups. A subgroup $H$ of the group $G$ is called core-free if $	ext{Core}_G(H) = \langle 1 \rangle$. We study the groups, in which every subgroup is either normal or core-free. More precisely, we obtain the structures of monolithic and non-monolithic groups with this property.

Keywords: normal subgroup, core-free subgroup, Dedekind group.

Let $G$ be a group. The following two normal subgroups are associated with any subgroup $H$ of the group $G$: $H^G$, the normal closure of $H$ in $G$, the least normal subgroup of $G$ including $H$, and $\text{Core}_G(H)$, the (normal) core of $H$ in $G$, the greatest normal subgroup of $G$, which is contained in $H$. We have

$$H^G = \langle H^x \mid x \in G \rangle$$

and

$$\text{Core}_G(H) = \bigcap_{x \in G} H^x.$$

A subgroup $H$ is normal if and only if $H = H^G = \text{Core}_G(H)$. In this sense, the subgroups $H$, for which $\text{Core}_G(H) = \langle 1 \rangle$, are the complete opposite of the normal subgroups. A subgroup $H$ of the group $G$ is called core-free in $G$ if $\text{Core}_G(H) = \langle 1 \rangle$.

There is a whole series of papers devoted to the study of groups with only two types of subgroups: subgroups with some property $\rho$ and subgroups with a property that is antagonistic to $\rho$ (see, for example, [1—6]). In particular, from the results of paper [3], it is possible to obtain a description of groups that have only two possibilities for each subgroup $H$: $H^G = H$ or $H^G = G$. In this connection, a dual question naturally arises on the structure of groups, in which, for each subgroup $H$, there are only two other possibilities: $\text{Core}_G(H) = H$ or $\text{Core}_G(H) = \langle 1 \rangle$. The finite groups having this property had been studied in [7]. Note at once that the groups, whose all subgroups are normal, possess this property.

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Recall that a group $G$ is called Dedekind, if every its subgroup is normal. The Dedekind group $G$ has the following structure: it is either Abelian or $G = Q_8 \times D \times P$, where $Q_8$ is a quaternion group of order 8, $D$ is an elementary Abelian 2-group, and $P$ is an Abelian $2'$-group [8].

Another extreme case that occurs here is the simple groups. In them, every proper subgroup is core-free. This fact immediately shows that the study of groups, in which $\text{Core}_G(H) = H$ or $\text{Core}_G(H) = \langle 1 \rangle$ for each subgroup $H$, makes sense for generalized soluble groups. The two key cases here are as follows: $G$ is a non-monolithic group or $G$ is a monolithic group. Let $G$ be a group. The intersection of all non-trivial normal subgroups $\text{Mon}(G)$ of $G$ is called the monolith of the group $G$. If $\text{Mon}(G) \neq \langle 1 \rangle$, then the group $G$ is called monolithic, and, in this case, $\text{Mon}(G)$ is the least non-trivial normal subgroup of $G$.

Our first main result is related to the non-monolithic case.

**Theorem A.** Let $G$ be an infinite group, whose non-normal subgroups are core-free. If $G$ is non-monolithic, then $G$ is a Dedekind group.

The following our main theorem considers the monolithic case. Here, we get a much more diverse situation. Separate considerations are required for non-periodic and periodic groups.

**Theorem B.** Let $G$ be a locally soluble non-periodic group, whose non-normal subgroups are core-free. Suppose that $G$ is not a Dedekind group. Then $G$ is monolithic, the factor-group $G/\text{Mon}(G)$ is non-periodic, $G = \text{Mon}(G) \times A$, and the following conditions hold:

(i) $\text{Mon}(G)$ is either torsion-free Abelian subgroup or elementary Abelian $p$-subgroup for some prime $p$;

(ii) $[G, G] = \text{Mon}(G) = C_G(\text{Mon}(G))$;

(iii) a subgroup $A$ is Abelian, and $\text{Tor}(A)$ is locally cyclic;

(iv) if $\text{Mon}(G)$ is an elementary Abelian $p$-subgroup, then $\text{Tor}(A)$ is a $p'$-subgroup;

(v) if $A$ has finite $0$-rank, then $\text{Mon}(G)$ is an elementary Abelian $p$-subgroup;

(vi) if $B$ is another complement to $\text{Mon}(G)$ in $G$, then the subgroups $A$ and $B$ are conjugate.

In turn, the case where $G$ is periodic also splits into two cases depending on whether the center includes a monolith or not. Recall that a $p$-group $G$ is called extraspecial, if $[G, G] = \zeta(G)$ is a subgroup of order $p$ and $G/\zeta(G)$ is an elementary Abelian $p$-group.

From this definition, we can see that the center of an extraspecial $p$-group $G$ is the least normal subgroup, so that if $H$ is a subgroup of $G$, and $H$ includes a non-trivial $G$-invariant subgroup, then $H$ includes $\zeta(G)$. The equality $[G, G] = \zeta(G)$ implies that $H$ is normal in $G$. In other words, every subgroup of $G$ is either normal or core-free.

**Theorem C.** Let $G$ be a periodic monolithic group, whose non-normal subgroups are core-free. Suppose that $G$ is not a Dedekind group. If the center of $G$ includes a monolith, then $G = KE$, where $K$ is a cyclic or quasicyclic $p$-subgroup, $E$ is an extraspecial $p$-subgroup, $K = \zeta(G)$, and $K \cap E = [G, G]$ is a subgroup of order $p$, $p$ is a prime.

**Theorem D.** Let $G$ be an infinite periodic locally soluble monolithic group, whose non-normal subgroups are core-free. Suppose that $G$ is not a Dedekind group and the monolith of $G$ is not central. Then $G = \text{Mon}(G) \times A$, and the following conditions hold:

(i) $\text{Mon}(G)$ is an infinite elementary Abelian $p$-subgroup for some prime $p$, and $A$ is an infinite periodic $p'$-group;

(ii) $[G, G] = \text{Mon}(G) = C_G(\text{Mon}(G))$;
(iii) whether the subgroup $A$ is locally cyclic, or $A = Q \times B$, where $Q$ is a quaternion group of order 8, and $B$ is a locally cyclic $2'$-subgroup;

(iv) if $C$ is another complement to $\text{Mon}(G)$ in $G$, then the subgroups $A$ and $C$ are conjugate.

Note that if $G/\text{Mon}(G)$ is finite or $\text{Mon}(G)$ is finite and non-central, then $G$ is finite (this follows from Theorem D). The last our result gives a description of the finite soluble group, whose non-normal subgroups are core-free. As was noted above, a finite group, whose non-normal subgroups are core-free, was studied in [7]. Our description is more detailed than the description given in Theorem 1 of that paper. We also note that the proof of Lemma 5 in [7] contains a gap (only the case where the both factor-groups $G/N_1$ and $G/N_2$ are non-Abelian was considered). In addition, there is a mistake there: the fact that $H$ is a subgroup of $T \times A$ does not implies that $H = H_1 \times H_2$, where $H_1 \leq T$ and $H_2 \leq A$. Therefore, we do not use the results of work [7]. We proved of the following result.

**Theorem E.** Let $G$ be a finite soluble group, whose non-normal subgroups are core-free. Suppose that $G$ is not a Dedekind group. Then $G$ is monolithic.

If the center of $G$ includes a monolith, then $G = KE$ where $K$ is a cyclic $p$-subgroup, $E$ is an extraspecial $p$-subgroup, $K = \zeta(G)$, and $K \cap E = [G, G]$ is a subgroup of order $p$, $p$ is a prime.

If the monolith of $G$ is not central, then $G = \text{Mon}(G) \times A$, and the following conditions hold:

(i) $\text{Mon}(G)$ is elementary Abelian $p$-subgroup for some prime $p$, and $A$ is a $p'$-group;

(ii) $[G, G] = \text{Mon}(G) = C_2(\text{Mon}(G))$;

(iii) whether a subgroup $A$ is cyclic or $A = Q \times B$, where $Q$ is a quaternion group of order 8, and $B$ is a cyclic $2'$-subgroup;

(iv) if $C$ is another complement to $\text{Mon}(G)$ in $G$, then the subgroups $A$ and $C$ are conjugate.

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ПРО СТРУКТУРУ ГРУП, ПІДГРУПІ ЯКИХ
АБО НОРМАЛЬНІ, АБО ВІЛЬНІ ВІД ЯДРА

Досліджується вплив деяких природних типів підгруп на структуру груп. Підгрупу $H$ групи $G$ називаємо вільною від ядра, якщо $\text{Core}_G(H) = \{1\}$. Вивчено групи, в яких кожна підгрупа або нормальна, або вільна від ядра. Точніше, одержано будову монолітичних та немонолітичних груп з цією властивістю.

Ключові слова: нормальна підгрупа, вільна від ядра підгрупа, дедекіндова група.

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О СТРУКТУРЕ ГРУПП, ПОДГРУППЫ КОТОРЫХ
ЛИБО НОРМАЛЬНЫ, ЛИБО СВОБОДНЫ ОТ ЯДРА

Исследуется влияние некоторых естественных типов подгрупп на структуру групп. Подгруппу $H$ группы $G$ называем свободной от ядра, если $\text{Core}_G(H) = \{1\}$. Изучены группы, в которых каждая подгруппа либо нормальна, либо свободна от ядра. Точнее, получена структура монолитических и немонолитических групп с этим свойством.

Ключевые слова: нормальная подгруппа, свободная от ядра подгруппа, дедекиндова группа.