Chapter 1

QUANTUM CLONING
WITH CONTINUOUS VARIABLES*

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1. INTRODUCTION

Quantum information theory has developed dramatically over the past decade, driven by the prospects of quantum-enhanced communication and computation systems. Among the most striking successes, one finds for example the discovery of quantum factoring, quantum key distribution, or quantum teleportation. Most of these concepts were initially developed for discrete quantum variables, in particular quantum bits, which have now become the symbol of quantum information. Recently, however, a lot of attention has been devoted to investigating the use of continuous-variable systems in quantum informational or computational processes. Continuous-spectrum quantum variables, for example the quadrature components of a light mode, may be easier to manipulate than quantum bits. It is actually sufficient to process squeezed states of light into linear optics circuits in order to perform various quantum information processes over continuous variables [1]. As reported in the present book, variables with a continuous spectrum have been shown to be useful to carry out quantum teleportation, quantum entanglement purification, quantum error correction, or even quantum computation.

In this Chapter, the issue of cloning a continuous-variable quantum system will be analyzed, and a Gaussian cloning transformation will be introduced. Cloning machines, that is, transformations that achieve the best approximate copying of a quantum state compatible with the no-cloning theorem, have been a fundamental research topic over the last five years (see e.g. [2] for an overview). This question is of particular significance given the close connection between

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quantum cloning and quantum cryptography: using an optimal cloner generally makes it possible to obtain a tight bound on the best individual eavesdropping strategy in a quantum cryptosystem. This provides a strong incentive to investigating continuous-variable cloning in view of the recent proposals for quantum key distribution relying on continuous (Gaussian) key carriers [3, 4].

Here, we will focus on a Gaussian cloning transformation, which copies equally well any two canonically conjugate continuous variables such as the two quadrature components of a light mode [5]. More precisely, it achieves the optimal cloning of a continuous variable that satisfies the requirement of covariance with respect to displacements and rotations in phase space. Consequently, this cloner duplicates all coherent states with a same fidelity ($F = 2/3$). The optical implementation of this cloner and its extension to N-to-M cloners will also be discussed. Finally, the use of this cloner for the security assessment of continuous-variable quantum key distribution schemes will be sketched.

2. LIMITS ON OPTIMAL CLONING

Let us start by stating the problem of continuous-variable cloning in physical terms. Consider, as an example of canonically conjugate continuous variables, the quadrature components of a light mode, denoted as $x$ and $p$. This notation reflects the fact that $x$ and $p$ behave just like the position and momentum of a particle in a one-dimensional space, namely their commutator is $[x, p] = i$ (we put $\hbar = 1$ in this paper). If the wave function is a Dirac delta function—the particle is fully localized in position space, then $x$ can be measured exactly, and several perfect copies of the system can be prepared. However, such a cloning process fails to exactly copy non-localized states, e.g., momentum states. Conversely, if the wave function is a plane wave with momentum $p$—the particle is localized in momentum space, then $p$ can be measured exactly and one can again prepare several perfect copies of this plane wave. However, such a “plane-wave cloner” is then unable to copy position states exactly. In short, it is impossible to copy perfectly the eigenstates of two conjugate variables such as $x$ and $p$: this is essentially the content of the so-called no-cloning theorem [6, 7].

In the next Section, we will show that a cloning transformation can nevertheless be found that provides two copies of a continuous system, but at the price of a non-unity cloning fidelity. In other words, the cloning machine yields two imperfect copies of the system. Before describing this cloning machine in details, let us find a lower bound on the cloning-induced noise by exploiting a connection with measurement theory. More specifically, we make use of the fact that measuring $x$ on one clone and $p$ on the other clone cannot beat the optimal joint measurement of $x$ and $p$ on the original system [8]. It is known that such a joint measurement of a pair of conjugate observables on a single quantum system obeys an inequality akin to the Heisenberg uncertainty relation.
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but with an extra contribution to the minimum variance [9]. Denoting by \(x\) and \(p\) the two quadratures of the input mode, and by \(X\) and \(P\) the corresponding jointly measured output quadratures, we have

\[
X = x + n_x \\
P = p + n_p
\]

where \(n_x\) and \(n_p\) stand for the excess noise that we have on the measured quadratures. Since we consider a joint measurement, the variables \(X\) and \(P\) must commute: they can be viewed respectively as the \(x\) and \(p\) quadratures of two distinct modes. Thus, we have

\[
[X, P] = [x, p] + [x, n_p] + [n_x, p] + [n_x, n_p] = 0
\]

Assuming that the excess noises \(n_x\) and \(n_p\) are independent of the input quadratures, i.e., \([x, n_p] = [n_x, p] = 0\), we get \([n_x, n_p] = -i\), implying that \(n_x\) and \(n_p\) must obey an uncertainty relation. Specifically, any attempt to measure \(x\) and \(p\) simultaneously on a quantum system is constrained by the inequality

\[
\Delta n_x \Delta n_p \geq 1/2
\]

where \(\Delta n_x^2\) and \(\Delta n_p^2\) denote the variances of the excess noises originating from the joint measurement device. If the variances of the \(x\) and \(p\) quadratures of the input state are denoted by \(\delta x^2\) and \(\delta p^2\), respectively, we thus have for the variances of the measured values \(\Delta X^2 = \delta x^2 + \Delta n_x^2\) and \(\Delta P^2 = \delta p^2 + \Delta n_p^2\). As a consequence, the Heisenberg uncertainty relation \(\delta x \delta p \geq 1/2\) together with inequality (1.3) implies the relation [9]

\[
\Delta X \Delta P \geq 1
\]

where we have used the inequality \(a^2 + b^2 \geq 2\sqrt{a^2 b^2}\). Thus, the best possible joint measurement of \(x\) and \(p\) with a same precision on both quadratures of a coherent state \((\delta x^2 = \delta p^2 = 1/2)\) gives

\[
\Delta X^2 = \Delta P^2 = 1
\]

Compared with the vacuum noise, we note that the joint measurement of \(x\) and \(p\) effects an additional noise of minimum variance 1/2, so that the measured values suffer twice the vacuum noise.

Inequality (1.3) immediately translates into a lower bound on the cloning-induced noise variance [8]. If we assume that the device that is used in order to perform the joint measurement of \(x\) and \(p\) is actually a cloning machine followed by two measuring apparatuses \((x\) being measured on one clone and \(p\) on the other clone), we conclude that the variance of the noise added by this
cloning machine cannot be lower than 1/2 in order to comply with Eq. (1.3), that is
\[ \Delta n_x^2 = \Delta n_p^2 \geq 1/2 \quad (1.6) \]
(We require here the same noise level on \( x \) and \( p \).) This can also be shown explicitly by writing the canonical transformation of the cloner [10]. Denoting by \( X_{a(b)} \) and \( P_{a(b)} \) the two quadratures of the output mode \( a \) (resp. \( b \)), we have
\[
\begin{align*}
  X_a &= x + n_{x,a} \quad (1.7a) \\
  P_a &= p + n_{p,a} \quad (1.7b) \\
  X_b &= x + n_{x,b} \quad (1.7c) \\
  P_b &= p + n_{p,b} \quad (1.7d)
\end{align*}
\]
where \( x \) and \( p \) are the two quadratures of the input mode and \( n_{x,p,a/b} \) stand for the excess noises. Since the clones are carried by different modes (\( a \) and \( b \)), we have \([X_a, P_b] = [X_b, P_a] = 0\). Assuming, as before, that the excess noises are independent of the input mode, we get \([n_{x,a}, n_{p,b}] = [n_{x,b}, n_{p,a}] = -i\). This gives rise to two no-cloning uncertainty relations
\[
\begin{align*}
  \Delta n_{x,a} \Delta n_{p,b} &\geq 1/2 \quad (1.8a) \\
  \Delta n_{x,b} \Delta n_{p,a} &\geq 1/2 \quad (1.8b)
\end{align*}
\]
which constrain the excess noise variances \( \Delta n_{x,p,a/b}^2 \) of the two clones [5, 10]. Consequently, if the cloning process induces a small position (momentum) error on the first copy, then the second copy is necessarily affected by a large momentum (position) error. The Gaussian cloner we will discuss in the next Session saturates these inequalities and is symmetric in \( a \) and \( b \) (and in \( x \) and \( p \)): \[ \Delta n_{x,a}^2 = \Delta n_{p,a}^2 = \Delta n_{x,b}^2 = \Delta n_{p,b}^2 = 1/2 \quad (1.9) \]
To simplify the notation, we will denote this cloning-induced excess noise variance as \( \sigma^2 \) in the following.

3. GAUSSIAN CLONING TRANSFORMATION

We will define a class of cloning machines that yield two imperfect copies of a continuous-variable system, the underlying cloning transformation being covariant with respect to displacements in phase space \((x, p)\). By this, we mean that any two input states that are related by a displacement result in copies that are related in the same way; hence, the resulting cloning fidelity is invariant under displacements in phase space. Specifically, let us seek for a displacement-covariant transformation which duplicates with a same fidelity all coherent states \( |\psi\rangle \). Thus, if two input states are identical up to a displacement \( \hat{D}(x', p') = e^{-ix'p'e^{ip'\hat{x}}} \), then their respective copies should be identical up to
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![Quantum circuit for the continuous-variable cloning transformation. It consists of four C-NOT gates preceded by a preparation stage. Here, the ancillas are prepared in the state given by Eq. (1.19). See [5, 12].](image)

As shown in [5], this cloning map can be achieved via a unitary transformation \(\hat{U}\) acting on three modes: the input mode (variable 1) supplemented with two auxiliary modes, the blank copy (variable 2) and an ancilla (variable 3). The two auxiliary variables must be initially prepared in the joint state

\[
|\chi\rangle_{2,3} = \int \int_{-\infty}^{\infty} dx \, dp \, f(x, p) \, |\Psi(x, -p)\rangle_{2,3}
\] (1.11)

where \(f(x, p)\) is an (arbitrary) complex amplitude function, and

\[
|\Psi(x, p)\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx' \, e^{ipx'} \, |x'\rangle |x' + x\rangle
\] (1.12)

are the EPR states (the maximally-entangled states of two continuous variables).

The cloning transformation is defined as

\[
\hat{U}_{1,2,3} = e^{-i(x_3 - x_2)p_1} \, e^{-i(x_2 + p_3)}
\] (1.13)

where \(\hat{x}_k (\hat{p}_k)\) is the position (momentum) operator for variable \(k\). As shown in Fig. 1.1, this can be interpreted as a sequence of four continuous-variable controlled-NOT (C-NOT) gates, each being defined as the unitary transformation \(e^{-i\hat{x}_k \hat{p}_l}\) with \(k \, (l)\) referring to the control (target) variable [11].

Remarkably, Eq. (1.13) coincides with the discrete C-NOT gate sequence that achieves the qubit cloning transformation [13], up to a sign ambiguity.
originating from the fact that a continuous C-NOT gate is not equal to its inverse. After applying $\hat{U}$ to the state $|\psi\rangle_1|\chi\rangle_{2,3}$, we get the joint state

$$\int \int dx \, dp \, f(x, p) \, \hat{D}(x, p)|\psi\rangle_1 |\Psi(x, -p)\rangle_{2,3}$$

(1.14)

where variables 1 and 2 are taken as the two outputs of the cloner (clones a and b), while variable 3 (the ancilla) must simply be traced over. This is a peculiar state in that it can be reexpressed in a similar form by exchanging the two clones, namely

$$\int \int dx \, dp \, g(x, p) \, \hat{D}(x, p)|\psi\rangle_2 |\Psi(x, -p)\rangle_{1,3}$$

(1.15)

with

$$g(x, p) = \frac{1}{2\pi} \int \int dx' \, dp' \, e^{ip(x-x')} \, f(x', p')$$

(1.16)

being the two-dimensional Fourier transform of $f(x, p)$. The resulting state of the individual clones can then be written as

$$\rho_a = \int \int dx \, dp \, |f(x, p)|^2 \, \hat{D}(x, p)|\psi\rangle \langle \psi| \hat{D}^\dagger(x, p)$$

(1.17a)

$$\rho_b = \int \int dx \, dp \, |g(x, p)|^2 \, \hat{D}(x, p)|\psi\rangle \langle \psi| \hat{D}^\dagger(x, p)$$

(1.17b)

which is consistent with tracing Eq. (1.10) over any one of the clones. Thus, the clones are affected by position and momentum errors that are distributed according to $|f(x, p)|^2$ and $|g(x, p)|^2$. A central point here is that interchanging the two clones amounts to substitute the function $f$ with its two-dimensional Fourier transform $g$. This property is crucial as it ensures that the two copies suffer from complementary position and momentum errors. Indeed, one can check [5] that the four excess noise variances defined as

$$\Delta n_{x,a}^2 = \int \int dx \, dp \, x^2 |f(x, p)|^2$$

(1.18a)

$$\Delta n_{p,a}^2 = \int \int dx \, dp \, p^2 |f(x, p)|^2$$

(1.18b)

$$\Delta n_{x,b}^2 = \int \int dx \, dp \, x^2 |g(x, p)|^2$$

(1.18c)

$$\Delta n_{p,b}^2 = \int \int dx \, dp \, p^2 |g(x, p)|^2$$

(1.18d)

obey the no-cloning inequalities (1.8a) and (1.8b). (Here, we assume that the first-order moments of $|f(x, p)|^2$ and $|g(x, p)|^2$ vanish, that is, the clones are not biased.)
Within this class of cloning machines parametrized by \( f(x, p) \), a particularly simple rotation-covariant cloner can be found that provides two identical copies of a continuous system with the same error distribution in position and momentum. It corresponds to the choice \( f(x, p) = g(x, p) = e^{-(x^2 + p^2)/2}/\sqrt{\pi} \).

This cloner is named “Gaussian” as it effects Gaussian-distributed position- and momentum-errors on the input mode: the excess noise on both clones is distributed as \( e^{-(x^2 + p^2)/\pi} \), that is, as a bi-variate rotational-invariant Gaussian of variance \( \sigma^2 = 1/2 \). This cloner is optimal, as it satisfies Eq. (1.9). Here, the two auxiliary variables must be prepared in the state

\[
|\chi\rangle_{2,3} = \frac{1}{\sqrt{\pi}} \int \int_{-\infty}^{\infty} dy \, dz \, e^{-\frac{y^2 + z^2}{2}} |y\rangle_2 |y + z\rangle_3 \tag{1.19}
\]

which is simply the product vacuum state \( |0\rangle_2 |0\rangle_3 \) processed by a c-NOT gate \( e^{-i\hat{x}\hat{p}} \). The resulting transformation effected by \( \hat{U} \) on an input position state \( |x\rangle \) is thus given by

\[
|x\rangle_1 |\chi\rangle_{2,3} \rightarrow \frac{1}{\sqrt{\pi}} \int \int_{-\infty}^{\infty} dy \, dz \, e^{-\frac{x^2 + y^2}{2}} |x + y\rangle_1 |x + z\rangle_2 |x + y + z\rangle_3 \tag{1.20}
\]

where the three variables denote the two clones and the ancilla, respectively.

For an arbitrary input state \( |\psi\rangle \), it is readily checked that this transformation outputs two clones whose individual states are Gaussian distributed with a variance \( \sigma^2 = 1/2 \), namely

\[
\rho_a = \rho_b = \frac{1}{\pi} \int \int_{-\infty}^{\infty} dx \, dp \, e^{-(x^2 + p^2)} \hat{D}(x, p) |\psi\rangle \langle \psi| \hat{D}^\dagger(x, p), \tag{1.21}
\]

In particular, if the input is a coherent state \( |\alpha\rangle \) with \( \alpha = (x + ip)/\sqrt{2} \), it is easy to calculate the fidelity of this cloner by using \( |\langle \alpha |\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2) \):

\[
F = |\langle \alpha |\rho_a b |\alpha\rangle| = \frac{1}{1 + \Delta n^2} = \frac{2}{3} \tag{1.22}
\]

This cloning fidelity does not depend on \( \alpha \), so this Gaussian cloner copies all coherent states with the same fidelity \( 2/3 \). It can be viewed as the continuous counterpart of the universal qubit cloner [13], as its cloning fidelity is invariant under rotations in phase space. The physical origin of the cloning noise becomes, however, much more evident in the case of continuous variables: the Gaussian noise that affects the clones can simply be traced back to the Gaussian wave function of the two ancillary modes, see (1.19). This suggests that the noise that inevitably arises when cloning is intrinsically linked to the vacuum fluctuations of the auxiliary modes.

Note finally that this formalism can easily be extended to the cloning of squeezed states instead of coherent states [5]. One simply unsqueeze the state.
before cloning and then squeeze the clones again. For any value of the squeezing parameter \( r \), one can then define a Gaussian cloner that copies with fidelity 2/3 all squeezed states of which the same quadrature is squeezed by the same amount \( r \). In contrast, cloning these squeezed states using the rotation-covariant cloner defined above results in a fidelity that decreases as \( r \) increases.

4. OPTICAL IMPLEMENTATION

It is very instructive to write the cloning transformation in the Heisenberg picture, that is, following the evolution of the annihilation operators associated with the modes that are involved. Again, mode 1 denotes the input mode, and modes 2 and 3 the ancillary modes. Mode 1’ and 2’ stand for the two clones, while 3’ is the ancilla that is traced over after cloning. Here, \( a_j = (x_j + i p_j)/\sqrt{2} \) stands for the annihilation operator for mode \( j \). We require that the cloning transformation conserves the mean values, i.e., \( \langle a'_1 \rangle = \langle a'_2 \rangle = \langle a_1 \rangle \), so that the clones are centered on the original coherent state. We also require that the cloning transformation is covariant under rotations in phase space. It is shown in [14] that the optimal transformation satisfying these requirements is

\[
\begin{align*}
a'_1 &= a_1 + \frac{a_2}{\sqrt{2}} + \frac{a_3^\dagger}{\sqrt{2}} \quad (1.23a) \\
a'_2 &= a_1 - \frac{a_2}{\sqrt{2}} + \frac{a_3^\dagger}{\sqrt{2}} \quad (1.23b) \\
a'_3 &= a_1^\dagger + \sqrt{2} a_3 \quad (1.23c)
\end{align*}
\]

where mode 1 is initially prepared in an arbitrary coherent state \( |\alpha\rangle \), with \( \alpha = (x + ip)/\sqrt{2} \), while modes 2 and 3 are prepared in the vacuum state. This transformation clearly satisfies the commutation rules \([a'_i, a'_j] = \delta_{i,j}\) and yields the correct mean values \((x, p)\) for the two clones (modes 1’ and 2’). Also, one can easily check that the quadrature variances of the clones are equal to twice the vacuum noise, in accordance with the cloning excess noise variance \( \sigma^2 = 1/2 \). This transformation actually coincides with the Gaussian cloner introduced in the previous Section. Interestingly, we note here that the state in which the ancilla 3 is left after cloning is centered on \((x, -p)\), that is the phase-conjugated state \(|\alpha^*\rangle\). This means that, in analogy with the universal qubit cloner, the Gaussian cloner generates an “anticlone” (or time-reversed state) together with the two clones.

As suggested by the above transformation, a possible optical implementation of this Gaussian cloner consists in processing the input mode \( a_1 \) into a linear phase-insensitive amplifier [15] of gain \( G = 2 \):

\[
a_{\text{out}} = \sqrt{2} a_1 + a_3^\dagger, \quad a'_3 = a_1^\dagger + \sqrt{2} a_3, \quad (1.24)
\]
with mode 3 denoting the idler mode. This amplifier is limited by the quantum noise so it naturally leads to an optimal cloner. A gain $G = 2$ is needed since the cloner doubles the energy by creating two clones with the same energy as the input state. One then produces these two clones simply by processing the output signal of the amplifier through a 50:50 phase-free beam splitter,

$$a'_1 = \frac{1}{\sqrt{2}} (a_{\text{out}} + a_2), \quad a'_2 = \frac{1}{\sqrt{2}} (a_{\text{out}} - a_2), \quad (1.25)$$

as shown in Fig. 1.2. The rotation covariance of the resulting cloner is ensured by the fact that the amplifier and the beam splitter are phase-insensitive. Actually, combining Eqs. (1.24) and (1.25) results in the same canonical transformation as above, so this optical setup indeed implements the optimal Gaussian cloner. It is readily checked that this setup leads to an equal $x$- and $p$-error variance of $1/2$ for both clones.

5. GAUSSIAN CLONERS WITH MULTIPLE INPUTS AND OUTPUTS

Let us now consider the general problem of optimal $N \to M$ cloning, extending what was done in [16] for the case of quantum bits. Consider a Gaussian transformation which, from $N \geq 1$ identical replicas of an original input state, produces $M \geq 2$ output copies whose individual states are again given by an expression similar to Eq. (1.21) but with an error variance $\sigma^2_{N,M}$. (For the $1 \to 2$ Gaussian cloner above, we had $\sigma^2_{1,2} = 1/2$.) Using an argument based on the concatenation of cloners, it is possible to derive a lower bound on $\sigma^2_{N,M}$, that is [8]

$$\sigma^2_{N,M} \geq \frac{1}{N} - \frac{1}{M}, \quad (1.26)$$
so that the corresponding cloning fidelity for coherent states satisfies

\[ F_{N,M} \leq \frac{MN}{MN + M - N}. \]  \hspace{1cm} (1.27)

The proof is connected to quantum state estimation theory, the key idea being that cloning should not be a way of circumventing the noise limitation encountered in any measuring process. More specifically, concatenating a \( N \to M \) cloner with a \( M \to L \) cloner results in a \( N \to L \) cloner that cannot be better that the optimal \( N \to L \) cloner. We then make use of the fact that the excess noise variance of this \( N \to L \) cloner simply is the sum of the excess noise variances of the two component cloners [8]. Denoting by \( \sigma^2_{N,M} \) the excess noise variance of the optimal \( N \to M \) cloner, we get the inequality \( \sigma^2_{N,L} \leq \sigma^2_{N,M} + \sigma^2_{M,L} \).

In particular, if \( L \to \infty \), we have

\[ \sigma^2_{N,\infty} - \sigma^2_{M,\infty} \leq \sigma^2_{N,M} \]  \hspace{1cm} (1.28)

Since the limit of cloning with an infinite number of clones corresponds to a measurement, Eq. (1.28) simply implies that cloning the \( N \) replicas before measuring the \( M \) resulting clones does not provide a mean to enhance the accuracy of a direct measurement of the \( N \) replicas. This limit is useful because the joint measurement of \( x \) and \( p \) on \( N \) identical replicas of a coherent state is known to give a minimum noise variance \( \sigma^2_{N,\infty} = 1/N \). This, combined with Eq. (1.28), gives the minimum noise variance induced by cloning, Eq. (1.26), along with the corresponding cloning fidelity, Eq. (1.27). Note that these bounds can also be derived when \( N = 1 \) using techniques similar to the ones used for describing quantum nondemolition measurements. This was done in a paper establishing a link between cloning and teleportation for continuous variables [10]: for the \( 1 \to 2 \) cloner, the teleportation fidelity must exceed \( F_{1,2} = 2/3 \) in order to guarantee that the teleported state is of better quality than the state kept by the emitter.

Just like for the \( 1 \to 2 \) cloner, the bounds Eqs. (1.26) and (1.27) can be attained by a transformation whose implementation requires only a phase-insensitive linear amplifier and beam splitters [14, 17]. Loosely speaking, the procedure consists in concentrating the \( N \) input modes into a single mode by use of a network of beam splitters, then in amplifying the resulting mode and distributing the output mode of the amplifier into \( M \) modes through a second network of beam-splitters. A convenient way to achieve these concentration and distribution stages is provided by networks of beam splitters that realize a Discrete Fourier Transform (DFT). Cloning is then achieved by the following three-step procedure (see Fig. 1.3). First step: the \( N \) input modes are concentrated into a single mode through a DFT (acting on \( N \) modes):

\[ a'_k = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} \exp(\frac{ikl2\pi}{N}) a_l, \]  \hspace{1cm} (1.29)
Figure 1.3 Implementation of an $N \to M$ continuous-variable cloning machine based on a phase-insensitive linear amplifier. Here, C stands for the amplitude concentration stage while D refers to amplitude distribution. Both can be realized using a network of beam-splitters that achieve a DFT. See [14].

with $k = 0 \ldots N - 1$. This operation concentrates the energy of the $N$ input modes $a_i$ into one single mode $a'_0$ (hereafter renamed $a_0$) and leaves the remaining $N - 1$ modes ($a'_1 \ldots a'_{N-1}$) in the vacuum state. Second step: the mode $a_0$ is amplified with a linear amplifier of gain $G = M/N$. This results in

$$a'_0 = \sqrt{\frac{M}{N}} a_0 + \sqrt{\frac{M}{N} - 1} a'^z_0,$$

(1.30a)

$$a'^z_0 = \sqrt{\frac{M}{N} - 1} a'^z_0 + \sqrt{\frac{M}{N}} a_z. \quad (1.30b)$$

Third step: amplitude distribution by performing a DFT (acting on $M$ modes) between the mode $a'_0$ and $M - 1$ blank modes in the vacuum state:

$$a''_k = \frac{1}{\sqrt{M}} \sum_{l=0}^{M-1} \exp(ikl2\pi/M) \ a'_l,$$

(1.31)

with $k = 0 \ldots M - 1$, and $a'_i = a_i$ for $i = N \ldots M - 1$. The DFT now distributes the energy contained in the output of the amplifier among the $M$ output clones.

It is readily checked that this procedure meets the requirements we put on the $N \to M$ cloner, and is optimal. Indeed the quadrature variance of the $M$ output modes gives $1/2 + 1/N - 1/M$, implying that the cloning-induced excess noise variance is $1/N - 1/M$. Furthermore, the transformation is rotation covariant since the amplifier and the beam splitters are phase insensitive. In conclusion, we see that the optimal $N \to M$ cloning transformation can be implemented using only passive elements except for a single linear amplifier.

The above cloning transformation can be extended even further by considering a generalized cloner that produces $M$ clones from $N$ replicas of a coherent
state and \(N')\) replicas of its complex conjugate [18]. It is again universal over the set of coherent states in the sense that the cloning fidelities are invariant for all input coherent states. Interestingly, it can be shown that supplementing the \(N\) input states \(|\psi\rangle^{\otimes N}\) with \(N'\) phase-conjugated input states \(|\psi^*\rangle^{\otimes N'}\) can, under certain circumstances, provide clones with a higher fidelity than the above \(N + N' \rightarrow M\) cloner. Note that, together with the \(M\) clones, this phase-conjugate input cloner also yields \(M'\) anticlones (approximate copies of \(|\psi^*\rangle\)) at no cost, with \(N - N' = M - M'\). The advantage of having phase-conjugated inputs for a continuous-variable cloner actually also has a counterpart in the context of qubit cloners. Indeed, motivated by this finding on continuous-variable cloners, an optimal universal cloning transformation was recently derived that produces \(M\) copies of an unknown pair of orthogonal qubits [19]. For \(M > 6\), the cloning fidelity for a pair of orthogonal qubits can be shown to be higher than that of the optimal cloning of a pair of identical qubits. This is a first example of a quantum informational process that was initially described for continuous-variable systems and only later on extended back to quantum bits.

6. **EAVESDROPPING IN CONTINUOUS-VARIABLE QUANTUM CRYPTOGRAPHY**

As mentioned above, quantum cloning can be viewed as an individual eavesdropping strategy in continuous-variable quantum cryptography. Consider a quantum key distribution scheme in which the key is encoded into the displacement of a coherent or a squeezed state that is drawn from a Gaussian distribution [3, 4]. In the continuous-variable protocol defined in [3], which we will analyze here, squeezed states need to be used. The emitter (Alice) prepares a squeezed state for which the quadrature that is squeezed, \(x\) or \(p\), is chosen at random, and then displaces it by \(\hat{D}(r, 0)\) or \(\hat{D}(0, r)\) depending on \(x\) or \(p\) is squeezed. Here, \(r\) is drawn from a Gaussian distribution, and constitutes a continuous key element. The receiver (Bob) then measures either the \(x\)- or \(p\)-quadrature of the state he received, this choice being again random. After Bob’s measurement, Alice reveals the quadrature she squeezed (and displaced) and Bob rejects the cases where he measured the wrong quadrature, this discussion being made over an authenticated public channel (this procedure is known as sifting). The subset of states that are accepted by Bob then constitutes a Gaussian raw key (correlated Gaussian data at Alice’s and Bob’s side). Indeed, denoting as \(\sigma\) the variance of the quadrature that is squeezed by Alice, Bob gets for his measured quadrature an outcome \(\sigma'\) that is Gaussian distributed around \(\sigma\) with a variance \(\sigma\) (assuming for the moment that the quantum channel is perfect and that there is no eavesdropping). If the variance of the random displacements \(r\) imposed by Alice is noted \(V\), then this raw key shared by Alice and Bob can be
Continuous-variable cloning viewed as resulting from a Gaussian additive-noise channel characterized by a signal-to-noise ratio of $V/v$.

The maximum amount of shared key bits that can be extracted from this Gaussian raw key can be analyzed by applying some standard notions of Shannon theory for continuous channels [see e.g. [20]]. Consider a discrete-time continuous channel that adds a Gaussian noise of variance $v$ to the signal. If the input $r$ of the channel is a Gaussian signal of variance $V$, the uncertainty on $r$ can be measured by its Shannon entropy $h(r) = 2^{-1} \log_2(2\pi e V)$ bits. Conditionally on $r$, the output $r'$ is distributed as a Gaussian of variance $v$, so that the entropy of $r'$ conditionally on $r$ becomes $h(r'|r) = 2^{-1} \log_2(2\pi e v)$ bits.

Now, the overall distribution of $r'$ is of course the convolution of these two distributions, i.e., a Gaussian of variance $V + v$, so that the output entropy is $h(r') = 2^{-1} \log_2(2\pi e (V + v))$ bits. According to Shannon theory, the information processed through this noisy channel $r \rightarrow r'$ can be expressed as the amount by which the uncertainty on $r'$ is reduced by knowing $r$, that is

$$ I \text{ (bits)} = h(r') - h(r'|r) = 2^{-1} \log_2 \left( 1 + \frac{V}{v} \right) $$

where $V/v$ is the signal-to-noise ratio. This is Shannon’s famous formula for the capacity of a Gaussian additive-noise channel. It is worth noticing that this capacity is achieved in the case where the input is distributed as a Gaussian, which is precisely the case under consideration here.

In the protocol analyzed in [3], the variances $v$ and $V$ are related by the constraint that Alice’s choice of encoding the key into either $x$ or $p$ should be invisible to a potential eavesdropper. In the first case, Alice applies a Gaussian-distributed displacement $\hat{D}(r, 0)$ on a squeezed state whose $x$ quadrature has a variance $v$, so that the quadratures $x$ and $p$ of this Gaussian mixture have a variance $V + v$ and $1/(4v)$, respectively. In the second case, Alice applies a displacement $\hat{D}(0, r)$ on a squeezed state in $p$, resulting in a Gaussian mixture with variances $1/(4v)$ and $V + v$ for $x$ and $p$. These two Gaussian mixtures are required to be indistinguishable, which simply translates into the requirement that they have the same $x$ variances and the same $p$ variances:

$$ V + v = \frac{1}{4v} $$

This gives for the information

$$ I = \log_2 \left( \frac{1/2}{v} \right) $$

which measures the maximum number of key bits that can be extracted asymptotically (at the limit of long sequences) per use of the channel. (The factor
1/2 here is just the vacuum noise, so we see that this protocol requires squeezing, that is, \( v < 1/2 \). The actual methods that may be used to discretize the Gaussian raw key and correct the resulting errors so as to extract a common bit string are known as reconciliation protocols [21].

Let us now consider the information that is transmitted in the presence of an eavesdropper. We assume that the eavesdropper (Eve) processes each key element into a Gaussian cloning machine, keeps one clone, and sends the other one to Bob. Once the quadrature that contains the key \((x \text{ or } p)\) is revealed by Alice and Bob, Eve properly measures her clone. Clearly, Eve needs to use an asymmetric version of the Gaussian cloner described above as she must be able to tune the information she gains, and therefore the disturbance she effects in the transmission. (A possible implementation of this asymmetric Gaussian cloner is discussed in [17].) Thus, Eve adds some extra noise on the quadrature encoding the key, which results in a reduced signal-to-noise ratio on Alice-Bob channel. Remember here, that the quality of the two clones obey a no-cloning uncertainty relation akin to the Heisenberg relation, implying that the product of the \(x\)-error variance on the first clone times the \(p\)-error variance on the second one remains bounded by \((1/2)^2\); see Eqs. (1.8a) and (1.8b). In particular, if \( x \) and \( p \) are treated symmetrically, we have

\[
\Delta n_B^2 \Delta n_E^2 \geq (1/2)^2
\]

This translates into a balance between the signal-to-noise ratio in Alice-Bob channel \(V/(v + \Delta n_B^2)\) and that in Alice-Eve channel \(V/(v + \Delta n_E^2)\). This latter channel is also a Gaussian channel so it can be treated similarly. Using Eq. (1.33), we can write the information processed respectively in Alice-Bob and Alice-Eve channels as

\[
I_{AB} = \frac{1}{2} \log_2 \left( \frac{1 + 4v \Delta n_B^2}{4v(v + \Delta n_B^2)} \right) \quad (1.36a)
\]

\[
I_{AE} = \frac{1}{2} \log_2 \left( \frac{1 + 4v \Delta n_E^2}{4v(v + \Delta n_E^2)} \right) \quad (1.36b)
\]

which gives

\[
I_{AB} + I_{AE} - I = \frac{1}{2} \log_2 \left( \frac{(1 + 4v \Delta n_B^2)(1 + 4v \Delta n_E^2)}{4(v + \Delta n_B^2)(v + \Delta n_E^2)} \right) \quad (1.37)
\]

One can then show that \(I_{AB} + I_{AE} - I \leq 0\) by checking that the quantity inside the logarithm is less or equal to one. This simplifies to the condition

\[
1 - 4v^2 \leq 4 \Delta n_B^2 \Delta n_E^2 (1 - 4v^2)
\]

which is indeed true as a consequence of Eq. (1.35) and \( v < 1/2 \). Consequently, we have proven that, in this quantum cryptographic protocol, the no-cloning
uncertainty relation translates into an information exclusion principle [3]

\[ I_{AB} + I_{AE} \leq I \]  

(1.39)

In other words, the information \( I_{AE} \) gained by Eve is upper bounded by the defect of information at Bob’s side, \( I - I_{AB} \), which implies that the security is guaranteed if \( I_{AB} \geq I/2 \) (since Bob then has an advantage over Eve, \( I_{AB} \geq I_{AE} \)). Note that the bound in Eq. (1.39) is saturated by the asymmetric Gaussian cloner discussed above, which strongly suggests that this is the optimal individual attack (this actually can be proven rigorously). In practice, Alice and Bob can estimate the potentially eavesdropped information in the following way. Alice discloses the values \( r \) she sent for a random subset of the raw key. Then, Bob compares them to the values \( r' \) he received, in order to estimate the variance of the distribution of the differences \( r' - r \), i.e., the excess noise variance \( \Delta n^2_B \). This is sufficient to estimate \( I_{AB} \), and, via Eq. (1.39), an upper bound on \( I_{AE} \).

An extended continuous-variable quantum key distribution protocol relying on Gaussian key carriers has recently been proposed in [4], where coherent states may be used instead of squeezed states. The encoding then consists in imposing a displacement \( \hat{D}(x, p) \) onto the vacuum state with \( x \) and \( p \) being drawn from a bi-variate Gaussian distribution. Here, the choice of the quadrature is made by Bob, who decides to measure \( x \) or \( p \) at random, and then discloses his choice on the public channel. The corresponding value of Alice’s displacement (\( x \) or \( p \)) together with Bob’s measured outcome again can be viewed as resulting from a Gaussian channel, so the above information-theoretic treatment can be extended. In particular, one can calculate \( I_{AB} \) and \( I_{AE} \) in the case of an individual attack based on asymmetric Gaussian cloners. The security analysis of this coherent-state protocol is beyond the scope of the present paper.

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