Maxwell, Yang-Mills, Weyl and eikonal fields defined by any null shear-free congruence

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Abstract

We show that (specifically scaled) equations of shear-free null geodesic congruences on the Minkowski space-time possess intrinsic self-dual, restricted gauge and algebraic structures. The complex eikonal, Weyl 2-spinor, \( SL(2,\mathbb{C}) \) Yang-Mills and complex Maxwell fields, the latter produced by integer-valued electric charges ("elementary" for the Kerr-like congruences), can all be explicitly associated with any shear-free null geodesic congruence. Using twistor variables, we derive the general solution of the equations of the shear-free null geodesic congruence (as a modification of the Kerr theorem) and analyze the corresponding “particle-like” field distributions, with bounded singularities of the associated physical fields. These can be obtained in a straightforward algebraic way and exhibit non-trivial collective dynamics simulating physical interactions.

Keywords—Twistors; Kerr theorem; Weyl equation; Kerr-Schild metrics; dynamics of singularities.

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1 Introduction and motivation

It is well known that the structure of null geodesic congruences is deeply related to the geometry and physics of space-time itself (see, e.g. [1, 2]). In particular, shear-free null congruences [1, 2, 4, 5] stand out for a number of remarkable properties, even on a flat Minkowski background \( M \). For example, numerous field structures, electromagnetic in particular, can be associated with a shear-free congruence [1, 2, 3]. Moreover, the geodesic and shear-free conditions are both invariant under deformations of the metric of the Kerr-Shild type. If the congruence is shear-free, then the entire system of vacuum Einstein or Einstein-Maxwell equations can often be satisfied with an appropriate choice of a single “potential” function. In this way, an important Kerr-like class of (algebraically special) solutions can be constructed [10, 12].

On the other hand, the equations of a shear-free null geodesic congruence possess a natural twistor structure. In fact, these equations have the form: \( \xi^A, D\xi_B - \nabla_{AA'}\xi_ {B'} = 0, \)

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*Throughout the paper the standard two-component spinor formalism is used. Upper case Latin indices range and sum over 0 and 1, and are raised and lowered by the symplectic spinors \( \epsilon^{AB} \) and \( \epsilon_{AB} \) respectively, as in [1]. The symbol \( \nabla_{AA'} \) denotes the usual spinor derivative operator in Minkowski space.
where the 2-spinor $\xi(x)$ corresponds to the tangent 4-vector field of the congruence $k_\mu = \xi_A \xi_{A'}$. According to the celebrated **Kerr theorem** \[1, 10\], the general (analytical) solution of (1) can be obtained in an implicit algebraic form

$$\Pi(T) = \Pi(\xi, iX\xi) = 0,$$

(2)

where $\Pi$ is a **homogeneous** holomorphic function of twistor components $T = \{\xi_A, \tau^A\}$ while the spinors $\xi, \tau$ are related to the hermitian matrix of space-time coordinates $X = \{X^{AA'}\}$ via the **incidence condition**

$$\tau = iX\xi \quad (\tau^A = iX^{AA'}\xi_A).$$

(3)

Since both (1) and (2) are invariant under generic local rescalings of the spinors $\xi \mapsto \alpha(x)\xi$, $\tau \mapsto \alpha(x)\tau$, in (2) we are actually dealing with an **arbitrary** holomorphic function $\Pi$ of three projective twistor coordinates geometrically defining a smooth 2-surface in $\mathbb{C}P^3$ \[13\].

However, there exists another, in essence equivalent to (2), form of the Kerr theorem \[14, 15\] which can be obtained via the transition to the **full twistor space** $T$ and explicitly exhibits a number of non-canonical symmetries, algebraic and geometric structures, themselves leading to a self-induced field and particle-like dynamics. Those, rather unfamiliar properties of the shear-free null geodesic congruence, will be explored in this paper.

Briefly, consider a smooth 2-surface in a 4-dimensional twistor space $T$ defined by two algebraic equations

$$\Pi^{(C)}(T) = \Pi^{(C)}(\xi, iX\xi) = 0, \quad (C = 0, 1),$$

(4)

with $\Pi^{(C)}$ being two **arbitrary** and independent holomorphic functions of the four twistor coordinates. For a couple of functions $\Pi^{(C)}$ the system (4) determines locally on $M$ a 2-spinor field $\xi(x)$. Differentiating (4) w.r.t. the coordinates $X^{AA'}$, we get for the spinor (except at the singular points, see below)

$$\xi^{A'} \nabla_{AA'} \xi_B = 0,$$

(5)

which, compared with (1), evidently defines a shear-free null geodesic congruence and, moreover, under special scaling of $\xi(x)$ is actually equivalent \[3\] to its equations (1) . In turn, (5) defines a new spintensor $\Phi_{A'A} \[16\]$ since it can be equivalently represented in the form

$$\nabla_{AA'} \xi_{B'} = \Phi_{B'A} \xi_{A'}.$$

(6)

Finally, we observe that the last system (6) admits an invariant Pfaffian (matrix) form \[17, 19\]

$$d\xi = \Phi dX\xi$$

(7)

for a 2-spinor $\xi(x) = \{\xi_{A'}\}$ and a $\text{Mat}(2, \mathbb{C})$-valued vector field $\Phi(x) = \{\Phi_{A'A}\}$. In the case of flat space-time, we have $dX = \{dX^{AA'}\}$ and the symbol $d$ in the left-hand side of (7) denotes the ordinary operator of external differentiation. System (7), its solutions and related geometric and physical structures are the main concern of the present paper.

Geometrically, system (7) defines a covariantly-constant spinor field with respect to an effective matrix-valued connection 1-form $\Omega = \Phi dX$. Spinor connections of similar type (on a generic Riemannian background) have been considered by K.P. Tod \[20\] in their relations to self-dual Einstein-Weyl manifolds and other geometric structures.

System (7) is overdetermined, so both the 2-spinor and the gauge fields are to be found from it in a self-consistent manner. The integrability conditions of (7) take the form

$$dd\xi \equiv 0 = (d\Phi - \Phi dX\Phi) \wedge dX\xi \equiv R\xi$$

(8)

and, as will be shown below, lead to self-duality of the curvature 2-form $R$ of the connection $\Omega$. Consequently, **the source-free Maxwell and Yang-Mills equations are satisfied identically** on the solutions to (7),
for the trace and trace-free parts of the curvature $R$ respectively. For this reason, in what follows system (7), equivalent to (6), (5) and, actually, to the defining equations of shear-free null geodesic congruence (1), is referred to as the \textit{generating system of equations}. Thus, the source-free Maxwell, the complex Yang-Mills and the Kerr-Schild metric fields can all be naturally associated with the solutions of the generating system of equations (7).

Moreover, the singularities of the strengths (curvatures) of all these fields coincide in space and time, being all determined by a single condition

$$\det \left[ \frac{d\Pi^{(C)}}{d\xi_{\alpha'}} \right] = 0,$$

which specifies the points where equations (4) admit multiple solutions. Geometrically, these points constitute a \textit{caustic} of the corresponding shear-free light bundle. As we shall see, this singular locus may be point-like, string-like or even a 2-dimensional surface. In the case when the set is bounded in 3-space we can identify it with a \textit{particle-like object} whose time evolution is fully governed by the field equations (7) and may be obtained from the algebraic system (4) along with the condition (9).

These singular objects exhibit, at a purely classical level, some properties of the real quantum particles. In particular, the value of electric charge $q$ is either zero or a whole multiple of the charge of the fundamental static solution; the latter corresponds just to the Kerr shear-free congruence with a ring-like singularity. The associated metric and electromagnetic field are identical to those of the Kerr-Newman solution in the general theory of relativity, except for the restrictions on the admissible value of the electric charge $q_0 = \pm 1/4$, which can thus be naturally identified with the \textit{elementary charge}. It follows, therefore, that the \textit{Kerr congruence can be equipped with an electromagnetic field defined directly via its principal spinor and carrying necessarily a “unit” electric charge}.

On the other hand, system (7), though equivalent to (5) or (6) and thus to the equations of the shear-free null geodesic congruence (1), was originally considered as a particular case of the differentiability conditions in the algebra of complex quaternions, isomorphic to the full algebra of $2 \times 2$ complex matrices, $\mathbb{B} \cong \text{Mat}(2, \mathbb{C})$ (see e.g. [22, 23]). In the present paper, however, we do not intend to discuss the problem of noncommutative analysis, referring the interested reader, e.g. to the profound paper of A.Sudbery [22], to the reviews [24], and to our early contributions on the subject [17–19, 21].

Let us outline the organization of the article. In the second section we obtain the 4-eikonal equation for the components of the basic 2-spinor, demonstrate the functional dependence of the components of the twistor associated with the generating system of equations and, thus, obtain the general solution of the latter in the form of the \textit{algebraic} system (4). The third section is devoted to the analysis of the “restricted” gauge symmetry of the generating system of equations (for which the gauge parameter depends only on the components of the field under transform) and its specific relation to the projective transformations in associated twistor space. In Sec. 4 we study the self-dual structure of the generating system of equations which follows from its integrability conditions and guarantees that the source-free Maxwell and Yang-Mills equations hold for the respective fields associated with the solutions of the generating system of equations. We also demonstrate that a 2-spinor composed of the components of the 4-potential matrix $\Phi$ obeys the Weyl equation for spin-1/2 massless fields. In Sec. 5 we rigorously study the relations of the generating system of equations to those of shear-free null geodesic congruences, write out the explicit expressions for the strengths of the associated Maxwell and Yang Mills fields and introduce Riemannian metrics of the Kerr-Schild type. A brief review of the previously obtained “particle-like” solutions of the generating system of equations and of the associated fields, together with the property of electric charge “self-quantization”, is presented in Sec. 6. In the last section we summarize our findings and conclude the paper.

\footnote{The Yang-Mills field possesses additional string-like singularities, see Section 5}

\footnote{In the dimensionless units we use; the numerical value itself is of no significance.}
2 The Eikonal Equation and General Solution to the Generating System of Equations

In the matrix representation of the biquaternion algebra $\mathbb{B}$ used in (7), we regard $\xi$ as a column $(2 \times 1$ matrix over $\mathbb{C}$), $\Phi$ as a $2 \times 2$ complex matrix of the general type, and $X$ as a hermitian matrix representing the coordinates of Minkowski space-time. System (7) is evidently invariant under the global Lorentz transformations of coordinates

$$X \rightarrow LXL^+, \quad \xi \rightarrow \bar{L}^+\xi, \quad \Phi \rightarrow \bar{L}^+\Phi \bar{L},$$

(10)

where $L \in SL(2, \mathbb{C})$ and $\bar{L}$ is the inverse of $L$. We see that the field $\xi(x)$ really transforms as a 2-spinor whereas the components of the field $\Phi(x)$ constitute a complex 4-vector (later identified with the electromagnetic 4-potential).

In view of the form of transformations (10), $\xi, \Phi$ are mapped respectively to matrices $\xi_{A'}, \Phi_{B',A}$, whereas $dX \rightarrow$ to a hermitian matrix $dX^{AA'}$, with (un)primed indices $A, A', \ldots = 0, 0'; 1, 1'$, in the usual 2-spinor notation. In this notation, system (7) takes the aforementioned form (6), equivalent to a system of eight partial differential equations.

Throughout the paper, we assume that both spinor components $\xi_{A'}$ are nonzero in the considered region of spacetime (otherwise, one can easily show, that the electromagnetic and other associated fields vanish trivially). We also assume all the functions $\{\xi_{A'}(x), \Phi_{B',A}(x)\}$ to be analytic everywhere except at some subset of zero measure where they may be singular.

Some properties of the solution $\xi(x)$ can be inferred directly from (6). Using the orthogonality identity $\xi^A \xi_{A'} = 0$, one readily finds

$$\nabla^{AA'} \xi_{C'} \nabla_{AA'} \xi_{B'} = 0,$$

(11)

which in particular implies the eikonal equation for any function $\lambda(\xi_{A'})$ of spinor components

$$\nabla^{AA'} \lambda \nabla_{AA'} \lambda = 0.$$

(12)

Returning to system (6) and multiplying it by $\xi^{A'}$ we reduce it back to system (5) where the 4-vector field $\Phi_{B',A}$ has been dispensed with. The latter may be recovered by multiplying (6) by $\xi^{A'}$,

$$\Phi_{B',A} = \xi^{A'} \nabla_{AA'} \xi_{B'},$$

(13)

with $\xi$ being a 2-spinor conjugate to $\xi$,

$$\xi_{A'} \xi^{A'} = 1,$$

(14)

defined up to the transformation

$$\xi \rightarrow \xi + \alpha \xi,$$

(15)

where $\alpha = \alpha(x) \in \mathbb{C}$ is an arbitrary scalar field.

Conversely, it follows from (10) that the spin-tensor $\Phi_{A,A'}$ can be defined by (13), implying that (13) is equivalent to the original spinor system (6), hence to the generating system of equations (7).

We now turn to the solutions of (3), or equivalently, of the generating system of equations (6). Since (4) and (5) are possibly related, one might expect the latter to possess a twistor structure. To show this, let us introduce a 2-spinor $\tau^A$ related to $\xi_{A'}$ via the Penrose correspondence (3). Then the pair of spinors $T = \{T^a\} = \{\xi_{A'}, \tau^A\}, \quad a = 0, 1, 0', 1'$, constitute a null twistor $T$ incident with a (real) Minkowski space-time point represented by $X^{AA'}$.

According to (3), the wedge product of the differentials $d\xi_{A'}$ and $d\tau^A$ is given by

$$d\tau^A \wedge d\xi_{B'} = X^{AA'} d\xi_{A'} \wedge d\xi_{B'} + \xi_{A'} dX^{AA'} \wedge d\xi_{B'}. \quad \text{(16)}$$

Hereafter, we shall ignore the imaginary unit $i$ in (3), the condition for a twistor to be null following from (3) takes then the form $\tau^A \xi - \xi^A \tau = 0$. 

4
From (16) a rather obvious property of the (nontrivial) twistors immediately follows: at least a pair of components of the twistor field \( T^a \) must be functionally independent (with respect to their dependence on coordinates \( X^{AA'} \)). Indeed, if we assume that the exterior products \( d\tau^A \wedge d\xi_{B'} \) and \( d\xi_{A'} \wedge d\xi_{B'} \) all vanish, then by (16) we find \( w^A \wedge d\xi_{B'} = 0 \), where the 1-forms \( w^A = dX^{AA'} \xi_{A'} \) have been introduced. Together with (7) this implies the condition \( \Phi_{B'C} w^A \wedge w^C = 0 \), whose solutions are \( w^A \wedge w^C = 0 \) or \( \Phi_{B'C} = 0 \). It is easy to see that the first solution leads to \( \xi_{A'} = 0 \), while combining the second solution with (14) gives \( d\xi_{B'} = 0 \). Excluding the first trivial case, we conclude that \( \xi_{A'} \) are constant, and consequently that the two components of the spinor \( \tau^A \) (3) are functionally independent.

If we now subject the spinor \( \xi_{A'} \) to the basic system (5), then the other pair of twistor components must depend on the first pair. Precisely, we have the following:

**Proposition.1** If \( \xi_{A'} \) is a solution of (2) then the corresponding twistor \( T^a \) has two and only two functionally independent components.

Going to differentials \( d\xi_{A'} \), \( d\tau^A \) in (16) and in (4) respectively, we arrive at the following system:

\[
\begin{align*}
d\xi_{B'} &= \Phi_{B'A} w^A, \\
d\tau^B &= X^{BB'} \Phi_{B'A} w^A + w^B,
\end{align*}
\] (17)

where as before \( w^A = dX^{AA'} \xi_{A'} \). Since the differentials of all twistor components are linear functions of the two 1-forms \( w^A \) only, it becomes evident that the exterior product of any three is zero, resulting in the desired functional dependence. This functional dependence can be expressed \[14\] in a more symmetric form \[13\] by a pair of holomorphic functions \( \{\Pi^{(C)}\} \), \( C = 0, 1 \) of four complex variables.

Conversely, the algebraic system (4) implicitly determines \( \xi_{A'} \) and it is easy to check, by differentiating (13) and multiplying the resulting system by \( \xi_{A'} \), that \( \xi_{B'} \) satisfies

\[
\frac{d\Pi^{(C)}}{d\xi_{B'}} \xi_{A'} \nabla_{AA'} \xi_{B'} = 0,
\] (19)

which, indeed, is equivalent to (5) except at some singular points (see below). Successively resolving system (13) at each space-time point \( X^{AA'} \) with respect to \( \xi_{A'} \) and substituting the resulting solution in (16) to find the corresponding “potentials” \( \Phi_{B'A} \) we obtain a solution to the generating system of equations starting from the algebraic constraints (4). This furnishes the proof of proposition.1.

Thus, system (13) implicitly determines the general (analytical) solution \( \{\xi_{A'}, \Phi_{B'A}\} \) of the generating system of equations. Points where equations (4) have multiple roots, i.e. cannot be uniquely resolved for \( \xi_{A'} \) according to (19) are defined by the condition (compare with (9))

\[
\det \left| \frac{d\Pi^{(C)}}{d\xi_{A'}} \right| = 0.
\] (20)

These points constitute a singular set of the electromagnetic field, which in the next Section will be associated with \( \Phi_{A'A} \). Together with (13) the last equation allows us to determine the shape and the time evolution of singularities (see Sec. 6 below).

Geometrically, system (13) defines a 2-dimensional complex surface in twistor space \( \mathbb{C}^4 \) (precisely, in the subspace of null twistors). Points of intersection of this surface with 2-dimensional planes formed by (null) twistors (4) represent the solution \( \xi_{A'} \) to the generating system of equations (multivalued in general) for each fixed space-time point \( X^{AA'} \). Singularities are then the pre-images (in \( \mathbb{M} \)) of the points of twistor space at which the planes (4) are tangent to the surfaces (13). Note that we ignore here the generally considered projective structure of the twistors which, in the present approach, has some peculiarities and is related to an unconventional gauge symmetry of the generating system of equations. These issues are discussed in the next section.
In passing, we shall establish an auxiliary result, which will come in handy in the next Section. Rewriting (18) in partial derivatives we get

$$\nabla_{AA'} \tau^B = X^{BB'} \Phi_{B'A} \xi_{A'} + \xi_{A'} \delta_{AA'}^B. \quad (21)$$

Using the orthogonality $\xi^A \xi_A = 0$, we immediately verify the validity of the following relations:

$$\nabla^{AA'} \xi_B \nabla_{AA'} \tau^B = 0, \quad (22)$$
$$\nabla^{AA'} \tau_B \nabla_{AA'} \tau_C = 0, \quad (23)$$

which along with (11) lead to the eikonal equation

$$\nabla^{AA'} \lambda \nabla_{AA'} \lambda = 0, \quad (24)$$

for any function $\lambda(T^a)$ of the four twistor components.

3 Projective transformations of twistors and the “weak” gauge symmetry of the generating system of equations

For an appropriate electrodynamical interpretation, we need to establish the gauge invariance of the generating system of equations. This is addressed in this section. More precisely, we shall study the symmetry of (6) under transformations

$$\xi_{A'} \rightarrow \xi'_{A'} = \alpha(x) \xi_{A'}, \quad (25)$$

where $\alpha(x)$ is a smooth complex function of coordinates. Using (5), it’s readily seen that $\alpha$ cannot be an arbitrary function of coordinates; it should rather satisfy the constraint

$$\xi^A \nabla_{AA'} \alpha = 0, \quad (26)$$

from which, in view of 2-spinors’ properties, it follows

$$\nabla_{AA'} \alpha = \rho_A \xi_A. \quad (27)$$

for some $\rho_A$ and, consequently, the eikonal equation for $\alpha(x)$

$$\nabla^{AA'} \alpha \nabla_{AA'} \alpha = 0. \quad (28)$$

Together with (24) this leads us to consider the following:

**Proposition.2** Transformations of the type (25) are symmetries of (6) iff $\alpha$ is a function of $T^a$ and $\Phi_{B'A}$ transform according to

$$\Phi_{B'A} \rightarrow \Phi'_{B'A} = \Phi_{B'A} + \nabla_{AB'} \ln \alpha. \quad (29)$$

Replacing in (6) $\xi_{A'}$ and $\Phi_{B'A}$ by their transformed values, after some simplification we obtain the following condition of the form-invariance of (6):

$$\xi_B' \nabla_{AA'} \alpha - \xi_A' \nabla_{AB'} \alpha = 0, \quad (30)$$

which is skew-symmetric in $A', B'$ and therefore equivalent to equation (26) for $\alpha(x)$. Assuming now that $\alpha = \alpha(T^a) = \alpha(\xi_{A'}, \tau^B)$, taking its derivatives and using (6) and (21) we prove that (26) is identically
satisfied, and thus every $\alpha(T^a)$ realizes a symmetry of the generating system of equations. This much takes care of the sufficient part of the proposition.

To prove the converse, suppose that transformation (25) is a symmetry of (6). This yields the following:

$$\xi_B' d\alpha = \alpha(\Phi'_{B'A} - \Phi_{B'A}) w^A,$$

where as before $w^A = dX^{AA'} \xi_{A'}$. Making use of (17) and (18), it’s easy to see that the exterior product of equation (31) with any two differentials of the twistor components vanishes, leading to the functional dependence of $\alpha$ on the corresponding twistor components. More symmetrically, this can be expressed as asserted above, i.e. $\alpha = \alpha(T^a)$. Passing then to partial derivatives in (31) we obtain the relation

$$\xi_{B'} \nabla_{AA'} \ln \alpha = (\Phi'_{B'A} - \Phi_{B'A}) \xi_{A'},$$

from which the desired transformation rule (29) for the “potentials” $\Phi_{B'A}$ follows (in a special case when the primed subscripts are equal). This completes the proof of proposition 2. ■

A few words are in order about the nature of transformations

$$\xi \to \xi' = \alpha(T^a)\xi,$$

which may be called restricted, or weak gauge transformations. Note that according to its definition the conjugate spinor $\tau^B$ transforms in a similar way

$$\tau \to \tau' = \alpha(T^a)\tau,$$

so that together (33) and (34) imply that the gauge symmetry of the generating system of equations may be expressed in twistor form

$$T \to T' = \alpha(T^a)T.$$

It’s clear that composition of transformations (35) is a transformation of the same type. Their associativity is also obvious. But the existence of inverse transformation is not so evident. However, according to proposition 1, both $T^a$ and its image $T'^a$ have only two functionally independent components, and $\alpha(T^a)$ depends, essentially, on these two components. So we can always express the two independent components of $T^a$ through those of $T'^a$. Substituting them in $\alpha^{-1}(T^a)$ results in the inverse transformation $T = \alpha^{-1}(T'^a)T'$ which is of the same type as (35). Hence these transformations constitute a group! In fact it is a proper subgroup of the full $\mathbb{C}(1)$-gauge group of transformations (25), the latter itself not being generally a symmetry of the generating system of equations. The statement that this subgroup is proper, becomes quite evident if we recall that $\alpha(T^a)$ necessarily obeys the eikonal equation (28).

Finally, we note that under transformations (35) the trace of the matrix 1-form $A := Tr(\Phi dX) = \Phi_{A'A} dX^{AA'}$ transforms gradient-wise

$$A \to A + d \ln \alpha,$$

just as the electromagnetic potential 1-form does under ordinary gauge transformations. Together with the 4-vector properties of $\Phi$ under the Lorentz transformations (10) this leads us to adopt the interpretation of the 1-form $A$ and the corresponding gauge-invariant 2-form $F = dA$ as the electromagnetic 4-potential and the electromagnetic field strengths respectively (of course, up to an arbitrary scale factor). In the following Section, we further elaborate on this electromagnetic interpretation by obtaining Maxwell equations for the 2-form $F$. Let us now look at the gauge transformations (35) from the standpoint of the geometry of twistor space. The Abelian nature of these transformations results in the ratio of any two twistor components $T^a$ remaining invariant, thus hinting at their projective origin. Not only the planes (3), but also the surfaces (4) are form-invariant under transformations (35) and, consequently, give rise to another solution of the generating system of equations (with the same electromagnetic 2-form $F$). So we may consider the
equivalence classes of the solutions (and of the surfaces respectively) which can be obtained one from another via the gauge transformations. That’s why, we may restrict ourselves to consider only the projective twistor space \( \mathbb{CP}^3 \). However, a projective structure of this type differs essentially from the conventional one, which originates from the transformations of the full gauge group. We shall return to this issue in Sec. 5. For the time being we shall deal with the structure of the full space of (null) twistors.

4 Self-duality, source-free gauge fields and Weyl equation

As mentioned in the introduction, from a geometric point of view, the generating system of equations (7) is the condition which must be met for a spinor \( \xi(x) \) to be covariantly constant with respect to the matrix-connection 1-form

\[ \Omega = \Phi dX. \]  

(37)

In view of (37), the original generating system of equations (7) may be written as follows:

\[ d\xi = \Omega \xi. \]  

(38)

Before we proceed further, it is worth noting that in a 4-vector representation the generating system of equations is equivalent to a system of equations defining a covariantly constant vector field w.r.t. a specific affine connection. Its particular form follows from a relativistic version of biquaternion algebra introduced originally in [27] and is considered in [30]. It is, however, worth noting that equations for covariantly constant fields on the background of Weyl geometry or a geometry with torsion determined by its trace can also be used for a transparent geometric treatment of electromagnetism and possess a number of properties closely related to those of the generating system of equations. In particular, any covariantly constant vector field in ordinary Weyl space is, remarkably, a shear-free null geodesic congruence on the underlying Minkowski background.

Reverting to the generating system of equations, we recall that it is an overdetermined system (comprising eight equations for six unknown functions), from which both the spinor and the “connection” gauge fields are to be determined. The dynamics of the connection field \( \Omega(x) \) can be obtained by external differentiation of (38), which yields

\[ R\xi \equiv (d\Omega - \Omega \wedge \Omega)\xi = 0, \]  

(39)

where in parentheses is the matrix curvature 2-form \( R \) of the connection. Since the spinor \( \xi \) is not arbitrary, but subject to (38), the integrability conditions (39) don’t imply the triviality of curvature, instead they lead to its self-duality [18, 19].

To demonstrate this, we note that for connection (37), the curvature \( R \) is of the following, rather specific form

\[ R = (d\Phi - \Phi dX\Phi) \wedge dX \equiv \pi \wedge dX, \]  

(40)

where a new matrix-valued 1-form \( \pi \) emerges, with the components

\[ \pi_{A'C} = \pi_{A'CBB'}dX^{BB'} = (\nabla_{BB'}\Phi_{A'C} - \Phi_{A'B}\Phi_{B'C})dX^{BB'}. \]  

(41)

In terms of \( \pi \), the integrability conditions (39) read \((\pi \wedge dX)\xi = 0\), or in spinor notation

\[ \pi_{A'CBB'}dX^{BB'} \wedge dX^{CC'}\xi_{C'} = 0. \]  

(42)

Making use of symmetry properties, we obtain from the last relation

\[ \pi_{A'C(B'\xi_{C'})} = 0, \]  

(43)
so that for any nontrivial solution \( \xi(x) \) it follows

\[
\pi^{A C}_{CB'} \equiv \nabla_{CB'} \Phi_{A'C} + \Phi_{B'C} \Phi_{A'C} = 0. \tag{44}
\]

Decomposing, in the usual way, the curvature \( (40) \) into the self- and antiself-dual parts it is easy to verify that equations \((44)\) are just the conditions for its self-dual part to vanish, whereas the other (antiself-dual) part \( \bar{R} \) takes the form

\[
\bar{R}^{C'}_{A'(BC)} = \nabla^{C'}_{(B \Phi_{A'|C})} - \Phi_{A'|C} \Phi_{A'C}
\]

and satisfies additional integrability conditions \( \bar{R} \xi = 0 \) (we’ll not make use of them below).

Thus, though the curvature 2-form \((40)\) of the connection 1-form \((37)\) is not identically (anti)self-dual (that is, (anti)self-dual in the “strong” sense), it necessarily becomes (anti)self-dual on the solutions of the generating system of equations. For this reason, the present property has been called weak self-duality \([14]\).

Physically, expression \((45)\) represents the strength of a matrix gauge field. In particular, its trace

\[
\mathcal{F}_{(BC)} = \bar{R}^{A'}_{A'(BC)} = \nabla^{A'}_{(B \Phi_{A'|C})}
\]

corresponds to the aforementioned electromagnetic field strength \( \mathcal{F} = d\alpha \), whereas the trace-free part of \((45)\) defines the strength of a Yang-Mills type’ field \([5]\).

Indeed, in view of the Bianchi identities

\[
dR \equiv \Omega \wedge R - R \wedge \Omega, \tag{46}
\]

self-duality of curvature \( R + iR^* = 0 \) leads to the source-free Maxwell equations for the electromagnetic 2-form \( F = Tr(R) = R^{A'}_{A'} \)

\[
dF^* = 0 = dF \equiv 0, \tag{47}
\]

and to equations of the Yang-Mills type \([19]\)

\[
dF^* - [\Omega, F^*] = 0, \tag{48}
\]

for the trace-free part of the curvature

\[
F^{B'}_{A'} = R^{B'}_{A'} - \frac{1}{2} F^{B'}_{A'}, \tag{49}
\]

corresponding to the trace-free part \( \Omega \) of the connection 1-form \([37]\).

Generally speaking, the electromagnetic 2-form \( F \) is a \( \mathbb{C} \)-valued field, yet in view of its self-duality it reduces to an \( \mathbb{R} \)-valued 2-form \( F \) related to \( F \) through

\[
F = F - iF^*, \tag{50}
\]

for which Maxwell equations in free space, in view of their linearity, hold too, so that the number of its degrees of freedom is just that of the ordinary electromagnetic field.

Explicitly, from the symmetric part of the integrability conditions \((44)\) we get

\[
\nabla_{C(B \Phi_{A'|C})} = 0 \quad \iff \quad F + iF^* = 0. \tag{51}
\]

The complex self-duality conditions \((51)\) are (locally) completely equivalent to the real vacuum Maxwell equations \([31]\) so that all the solutions of the latter can be obtained from them and vice versa. However, not every solution of \((51)\) can be augmented to a solution of the overdetermined generating system of equations itself. Such a state of affairs leads, in particular, to the fundamental property of charge quantization (see Sec. 6 below).

\*Owing to the restricted (weak) gauge symmetry this is not, precisely, what is usually regarded as, say, the \( SL(2, \mathbb{C}) \) Yang-Mills fields; nonetheless, these equations are identical in form, and the restrictions are imposed merely on the set of solutions admissible by the generating system of equations.
Note also that from the skew-symmetric part of (44) follows an additional “inhomogeneous Lorentz condition” \[18, 19\] for the $C$-valued electromagnetic potentials $\Phi^A_A = A_\mu$:

$$\nabla_{CC'} \Phi^{C'C} + \Phi^{C} C \Phi^{C'C} = 0 \iff \partial_\mu A^\mu + 2A_\mu A^\mu = 0,$$

which should also hold identically on every solution of the generating system of equations. Condition (52) is by no means gauge invariant in the ordinary sense, but it is invariant w.r.t. the weak gauge transformations \[83\], provided the potentials together with the principal spinor satisfy the generating system of equations.

Yet another remarkable constraint follows from the structure of the self-duality conditions (51) together with the additional condition (52). Indeed, making use of the weak gauge transformations (33), (29), one can get rid of a pair of components of the 4-potential matrix $\Phi^{C'C}$ (this will be demonstrated explicitly in Sec. 5), in such a way that the norm $\Phi^{C'C} \Phi^{C'C}$ in (52) vanishes. The joint system of equations (51) and (52) then yields

$$\nabla_{AB} \phi^A = 0,$$

where $\phi(x) = \{\phi^A\}$ stands for a 2-spinor field composed of two nonvanishing components of the spintensor $\Phi_{A'A}$. Equation (53) evidently reproduces the Weyl equation in its canonical form. Thus, for any solution of the generating system of equations, the components of the 4-potential matrix form a 2-spinor field which necessarily satisfies the Weyl equation for a spin-1/2 massless field.

As to the Yang-Mills type fields $F_{AB}A'$, they may always be expressed via the electromagnetic field strengths and the ratio of the spinor components (see Sec. 5 below). Note also that neither the real nor the imaginary part of the trace-free curvature $F_{AB}A'$ taken separately satisfy the source-free YM equations, in view of the nonlinearity of the latter. Therefore, the Yang-Mills-like fields here are necessarily complex. Further details on the properties of Yang-Mills fields in the present approach can be found in [19].

### 5 The generating system of equations and projective structure of null shear-free congruences

Recall that via the elimination of potentials $\Phi_{A'A}$, the generating system of equations (7) takes the form (5). In this form, the generating system of equations is, in fact, equivalent to the canonical equations defining shear-free null geodesic congruences so that every, suitably rescaled, shear-free null geodesic congruence gives a 2-spinor solution of the generating system of equations.

Indeed, once the solution $\xi_{A'}(x)$ to (5) is found, then a field of a null 4-vector $k_\mu(x), k_\mu k^\mu = 0$, can be defined as

$$k = k_\mu dx^\mu = \xi_{A'} \xi_A dX^{A'A'}.$$

Its field lines define a null congruence for which the shear-free criterion follows readily from (44). Hence, every solution $\xi_{A'}$ of the generating system of equations actually defines a shear-free null geodesic congruence. Contrary to (55), the equations of a shear-free null geodesic congruence (55) are invariant under the full complex Abelian group of rescaling (25) and reduce to the system of two equations in partial derivatives

$$\nabla_{\bar{w}} G = G \nabla_w G, \quad \nabla_{\bar{v}} G = G \nabla_v G,$$

where $G = \xi_{A'} / \xi_0'$ is the gauge invariant, and the following generally accepted notation has been used:

$$X^{A'A} = \begin{pmatrix} u & w \\ \bar{w} & v \end{pmatrix} \equiv \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix}.$$
The four real quantities \( \{ x^i; x^0 \} \), \( i = 1, 2, 3 \) correspond to the Cartesian space and time coordinates, respectively. Note that the individual spinor components \( \xi_{\alpha'} \) remain indeterminate by the equations of a shear-free null geodesic congruence which impose restrictions only on their quotient \( G(x) \).

Let us compare this with the generating system of equations (5). The latter is equivalent to a system of four equations for only two spinor components \( \xi_{\alpha'} \)

\[
\nabla_w \xi_{\alpha'} = G \nabla_u \xi_{\alpha'}, \quad \nabla_v \xi_{\alpha'} = G \nabla_\bar{w} \xi_{\alpha'}
\]

from which, of course, (58) follow for the quotient \( G(x) \). Multiple solutions of (58) with the same \( G \) correspond to different potentials but have the same strength of the associated electromagnetic field.

In view of this, from now on, we identify the generating system of equations with those of a shear-free null geodesic congruence by considering only the projectively (gauge) invariant part of the generating system of equations represented by system (56) (one may regard this as fixing the gauge to \( \xi_{\alpha'} = 1 \)). In particular, the gauge-invariant strength of the electromagnetic field should depend only on the quotient \( G(x) \) and, therefore, can be defined for a generic shear-free null geodesic congruence (see (65) below).

The general analytical solution of (56) for \( G(x) \) immediately follows from Proposition.1 in the form of an algebraic equation

\[
\Pi(G, \tau^0, \tau^1) = \Pi(G, u + wG, \bar{w} + vG) = 0
\]

which implicitly determines the function \( G(x) \). Here, \( \Pi \) is an arbitrary holomorphic function of three complex variables. Equation (65) expresses the functional dependence of the three components \( G, \tau^0, \tau^1 \) of the projective twistor \( \mathbf{T}^a \) associated with the solutions of the generating system of equations. For the shear-free null geodesic congruence the equivalent result is the well known Kerr theorem [1]. Note that the solutions of (56) in the form (59) are defined everywhere except at the points of the singular set (20) whose equation now reduces to

\[
P := \frac{d\Pi}{dG} = 0
\]

By multiplying the two equations of (56), we obtain once more the 4-eikonal equation for \( G(x) \) in the form

\[
\nabla_u \nabla_v G - \nabla_w G \nabla_{\bar{w}} G = 0
\]

while by differentiating them, we verify that \( G(x) \) satisfies also the linear d’Alembert equation [3, 14] (see also [32])

\[
\Box G(x) \equiv (\nabla_u \nabla_v - \nabla_w \nabla_{\bar{w}}) G(x) = 0
\]

Note that in view of (61) every \( C^2 \)-function \( \lambda(G) \) is also harmonic on the solutions of the generating system of equations,

\[
\Box \lambda(G) = 0
\]

Using now expression (13) for the potentials \( \Phi_{\alpha' \alpha} \) and taking into account (61), one can express the electromagnetic field strengths via the 2-nd order derivatives of \( \Lambda := \ln G \) as

\[
F_{00} = \nabla_u \nabla_{\bar{w}} \Lambda, \quad F_{11} = \nabla_v \nabla_w \Lambda, \quad F_{01} = \nabla_w \nabla_{\bar{w}} \Lambda
\]

so that the source-free Maxwell equations are guaranteed to hold for (64) in view of the wave equation (63) for \( \Lambda = \lambda(G) \). Twice differentiating the identity (59) w.r.t. the space-time coordinates, we finally obtain for the strengths (64) the following symmetric expression:

\[
F_{AB} = \frac{1}{P} \left( \Pi_{AB} - \frac{d}{dG} \left( \frac{\Pi_A \Pi_B}{P} \right) \right)
\]

where the function \( P \) is defined by (60) and \( \{ \Pi_A, \Pi_{AB} \} \), \( A, B = 0, 1 \) denote the (1-st and 2-nd order) partial derivatives of \( \Pi \) w.r.t. its twistor arguments \( \tau^0, \tau^1 \).
The strengths of the triplet of the Yang-Mills field defined by the 2-form (49) can now be represented in explicit form [33]
\[ F = \{-GF, iGF, -F\} \] (66)
its modulus (in isotopic 3-space) being equal to that of the Maxwell field, \( F^2 = F^2 \).

It is worth noting that in the above gauge \( \xi_0' = 1 \), as it follows from (13), the two components \( \Phi_{\nu'} \)
vanish, leaving \( \phi_A := \Phi_{A'1} \). Upon differentiating these expressions, in view of the d’Alembert equation (63), one obtains
\[ \nabla_w \phi_0 = \nabla_u \phi_1, \quad \nabla_v \phi_0 = \nabla_w \phi_1. \] (67)
These are precisely the Weyl equations (53) for the 2-spinor field \( \phi(x) \), which in the considered gauge are equivalent to the self-duality conditions (51) supplemented with the “inhomogeneous Lorentz condition” (52).

Finally, the intimate relation between the generating system of equations and the shear-free null geodesic congruence makes it possible to introduce one more geometrophysical structure – an effective Riemannian metric. In fact, it’s well-known [10, 35] that the deformation of the flat Minkowski metric \( \eta_{\mu\nu} \) into a metric \( g_{\mu\nu} \) of the Kerr-Schild type
\[ g_{\mu\nu} = \eta_{\mu\nu} + h k_{\mu} k_{\nu}, \] (68)
preserves the main characteristics of a shear-free null geodesic congruence (geodesity, twist and shear). Here \( h \) is a scalar field and the congruence \( k \) given by (64) takes a projectively invariant form
\[ k = du + \bar{G} dw + Gdw + G\bar{G} dv, \] (69)
\( \bar{G} \) being the complex conjugate of \( G \). We now make use of the results of the classical paper [10], where it was proved that the metric (68) satisfies Einstein-Maxwell electrovacuum system for any function \( G \) obeying the algebraic constraint (59), with a function \( \Pi \) linear in twistor arguments \( \tau_0, \tau_1 \):
\[ \Pi = \phi + (qG + s)\tau_1 - (pG + \bar{q})\tau_0. \] (70)
Here, \( \varphi = \varphi(G) \) is an arbitrary analytic function of the complex variable \( G \), \( s \) and \( p \) are real constants and \( q \) is a complex constant. Without going into the details, we note that according to the results of [10], the scalar field \( h \) in (65) is determined, up to an arbitrary real constant, by the function \( \Pi \) and another function \( \Psi(G) \) independent of \( \varphi(G) \) and related to the electromagnetic field arising therein. These electromagnetic fields satisfy Maxwell equations in the curved space-time with metric (65), and generally are distinct from those emerging in our approach and defined in the flat space-time [3]. However, for the most fundamental Reisner-Nordstrom and Kerr-Newman solutions these fields coincide [20] the only (but important!) difference being in that in our approach the electric charge is fixed in magnitude by the generating system of equations itself (see the next Sec. 6).

Moreover, it was shown in [11, 12, 22] that the singularities of the Riemann curvature of the Kerr-Schild metric (65) are given by (60), which, according to (65), also determines the singular locus of the electromagnetic field [2].

Hence, with every solution of the generating system of equations an electromagnetic, a \( \mathbb{C} \)-valued Yang-Mills and a Kerr-Schild metric (precisely, curvature) fields can be naturally associated, their principal singularities being determined by (60) and therefore coincide in space and time giving rise to a unique

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1. At the same time both these fields are generally different from the fields which may be defined for shear-free null geodesic congruence using the Penrose twistor transform [4] as well as from the null fields constructed via the Robinson procedure [5].
2. The associated Yang-Mills field (65) possesses additional string-like singularities defined by those of the function \( G \) itself and resembling the well-known Dirac string. The Weyl 2-spinor field (63) defined by the 4-potential matrix exhibits singularities of similar string-like type.
singular source for a number of fundamental physical fields with well-defined shape and dynamics. This makes it possible, in the framework of the classical field model based on the generating system of equations, to consider “particle-like” field distributions represented by common singularities of all the fields involved. This is illustrated on concrete examples in the next section.

6 Electric Charge and Particle-Like Solutions of the Generating System of Equations

In this Section we present a few solutions of the generating system of equations (and thus of the equations of a shear-free null geodesic congruence). They all may be obtained by an appropriate choice of the function $\Pi$ and the subsequent resolution of the algebraic constraint (59). To keep things as simple as possible we confine ourselves to functions $\Pi$ quadratic in $G$; more complicated examples are presented, e.g. in [31].

The fundamental static solution is generated by the following function:

$$\Pi = Gr^0 + \tau^1 + 2ia \equiv G(u + wG) - (\bar{w} + vG) + 2ia = 0,$$

(a = Const $\in \mathbb{R}$), from which we get

$$G = \frac{\bar{w}}{(z + ia) \pm r_s} \equiv \frac{x + iy}{(z + ia) \pm \sqrt{x^2 + y^2 + (z + ia)^2}}.$$  \hspace{1cm} (71)

The EM field (64) associated with this solution takes the form

$$\vec{E} - i\vec{H} = \pm \frac{\vec{r}_s}{4(r_s)^{3/2}}, \hspace{1cm} (\vec{E} + i\vec{H} = 0),$$  \hspace{1cm} (72)

where $\vec{r}_s = \{x, y, z + ia\}$, has a ring singularity of radius $a$, with an electric charge $q = \pm 1/4$ (in the dimensionless units we use), and a magnetic dipole and an electric quadrupole moments equal to $qa$ and $qa^2$, respectively [41]. Apart from the restriction on charge, the electromagnetic field (72) and the Riemannian metric defined by (68) via the shear-free null geodesic congruence (69), under the appropriate choice of the scalar function $h(x)$, reproduce exactly the EM field and metric of the Kerr-Newman solution (in the coordinates used in [10]). Particularly, for $a = 0$ the solution (71) represents the stereographic map $S^2 \to \mathbb{C}$, while the associated fields are the Coulomb field and the Reisner-Nordstrom metric respectively.

The self-duality condition (51) together with the gauge symmetry of the generating system of equations ensures the relation $q = N/4, N \in \mathbb{Z}$ for the values of electric charge associated with every solution of the generating system of equations. This property has both topological and dynamical origins, the latter being related to the overdetermined structure of the generating system of equations. The proof of the general theorem on integer-valued charge can be found in [31]. Unlike the purely topological approaches [37, 38] to the problem of quantization of electric charge, in the present framework, the charge of the fundamental static solution (71) can take only one fixed value and can thus be naturally identified with the elementary charge. Together with the well-known property of the Kerr-Newman solution to fix the gyromagnetic ratio $g = 2$ (equal to that of the Dirac particle [31]), the emergence of an elementary electric charge within the theory makes it more legitimate to interpret the fundamental solution (71) as a classical model of the electron (in comparison, say, with the models of Lopez [40], Israel [41], Burinskii [42, 45] and Newman [44] based on Einstein-Maxwell theory itself).

According to a general theorem proved in [32] the static solutions to the shear-free null geodesic congruence equations (and thus to the generating system of equations) with a bounded in 3-space singular
set are exhausted by the Kerr solution (71) (up to 3-translations and 3-rotations). If, however, we relax the static condition and look outside the class of functions (70) dealt with in [10], we discover a lot of time-dependent solutions with bounded singularities of different dimensions, 3-shapes and time evolutions. Such solutions with bounded structure of the common singular locus may be called particle-like [34]. They constitute a wide and physically interesting class of solutions of the equations of shear-free null geodesic congruence (the generating system of equations) and of the associated field equations, in particular Maxwell source-free equations. Some of these solutions seem to be quite unfamiliar in classical electrodynamics.

As an example, consider the axisymmetric solution of the particle-like type generated by the function [14, 33]

\[ \Pi = \tau^0 \tau^1 + b^2 G^2 = 0, \quad b = \text{Const.} \tag{73} \]

For real \( b \), it admits two singularities, with necessarily “elementary” charges \( +1/4 \) and \( -1/4 \), undergoing head-on hyperbolic motion. The corresponding electromagnetic field

\[
E_\rho = \pm \frac{8b^2 \rho z}{\Delta^{3/2}}, \quad E_z = \mp \frac{4b^2 M}{\Delta^{3/2}}, \quad H_\varphi = \pm \frac{8b^2 \rho t}{\Delta^{3/2}},
\]

(74)
is the well-known Born solution [36]. Here the following notation is used:

\[
\rho^2 = x^2 + y^2, \quad s^2 = t^2 - z^2, \quad M = s^2 + \rho^2 + b^2, \quad \Delta = M^2 - 4s^2 \rho^2,
\]

and the singularities are defined by the condition \( \Delta = 0 \). However, for a general complex \( b = b_0 + ib_1 \), with real \( b_0 \) and \( b_1 \), the singular set consists of two Kerr-like rings (see Fig. [1]), each of radius \( |b_1| \), carrying opposite “elementary” charges and executing hyperbolic motion along the \( z \)-axis: \( z = \pm \sqrt{b_0^2 + t^2} \). This, to the best of our knowledge, previously unknown distribution of electromagnetic field could be interesting in the general context of the Einstein-Maxwell theory. Moreover, for imaginary \( b = ia, a \in \mathbb{R} \) one obtains another, electrically neutral solution with a ring-like singularity at \( t = 0 \) which then expands into a torus. At times \( t > |a| \) the singularity turns into a self-intersecting torus, as depicted in Fig. [2].

Other particle-like solutions for which the singularity has a plane \( 8 \)-figure shape at \( t = 0 \), as well as a wave-type solution with a helix-like singularity were also presented in [34]. The latter is the counterpart of the electromagnetic wave in the present approach. A more complicated example simulating the process of pair annihilation can be found in [31].

It should be emphasized however that the singularities defined by the condition (60) are, in general, string-like, i.e. form a set of 1-dimensional curves moving in 3-space. Indeed, this C-condition reduces to 2 real equations for 4 coordinates on \( M \). Some properties of the arising string structures are presented and explored in [43].

We see that a number of exact solutions to the source-free Maxwell equations can be obtained in a purely algebraic way, some of which are previously unknown. These solutions are defined everywhere except at the points where the electromagnetic field blows up to infinity. These points constitute a locus which may be 0-, 1- or 2-dimensional and for a “particle-like” solution is bounded in 3-space. In general, it is not possible to cover such a set with a \( \delta \)-like source (due to the multivaluedness of the Kerr-type solutions). Nonetheless, the 3-shape and time evolution of the singularities are well defined and nontrivial due to the "hidden nonlinearity" [37] of Maxwell equations in this theory, “inherited” from the underlying generating system of equations (the equations of shear-free null geodesic congruence). While the latter ensures the existence of some “selection rules” for the solutions of Maxwell equation compatible with
the spinor structure of the generating system of equations, primarily for the electric charge, it also leads to the violation of the superposition principle (superposed solutions satisfy Maxwell equations but not necessarily the generating system of equations itself). A detailed discussion of the status of singular particle-like solutions may be found in [34].

7 Conclusion

Shear-free congruences of rectilinear light-like rays on the Minkowski space-time $\mathbf{M}$ (or its Kerr-Schild deformations) seem to be one of the simplest geometrical objects. Nonetheless, they turn out to be deeply related to a lot of other fundamental geometrical and physical structures. Some of them, in particular twistor geometry, have been known for a long time, as famously expressed by the Kerr theorem (71). However, the situation is even more remarkable. The shear-free null geodesic congruence equations turn out to be closely related to the form of the generating system of equations (7), which explicitly exhibits a number of fundamental symmetries and connections with relativistic fields. Specifically, the complex eikonal, wave, and Weyl equations as well as $SL(2, \mathbb{C})$ Yang-Mills and Maxwell source-free equations hold identically on the solutions of generating system of equations (the equations of shear-free null geodesic congruence).

On the other hand, the generating system of equations select from the solutions of relativistic field equations a peculiar and physically appropriate class. Together with the nontrivial time-dependent structure of common (bounded and carrying integer-valued electric charge) singularities of these fields, we arrive at a self-consistent Lorentz invariant dynamics of a set of such “particle-like” formations which is purely algebraic/geometric in origin. Making use of the asymptotic behavior of Kerr-Shild type solutions of Einstein equations explicitly defined by the generating system of equations, one could naturally endow each of the particle-like formations with spin and mass.

Moreover, such collective dynamics appear to be completely conservative. Indeed, consider a shear-
free null geodesic congruence generated by a point-like source moving arbitrary along a curve in \( M \) (or its complex extension \([46, 47]\)). Then, in the spirit of the Wheeler-Feynman conjecture, a set of identical point-like copies of a single particle will be detected by an observer on a single worldline via his/her light cone (i.e. as the set of solutions of the “retardation equation”). Remarkably \([50]\), for an inertial observer and arbitrary polynomially parameterized worldline a complete set of Lorentz invariant conservation laws holds within the emerging collective dynamics, along with some additional properties, such as “clusterization”.

However, as mentioned in Sec. 6, the particles-singularities defined by the generating system of equations (the equations of shear-free null geodesic congruence) generally exhibit a much more sophisticated collective string dynamics, the exploration and elaboration of which will hopefully be undertaken in the future.

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\(^1\)There exists another mechanism to obtain a collection of identical particles on a unique worldline that requires the implicit definition of the latter. This also leads to a collective conservative algebraic dynamics \([48, 49]\).
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