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Bounds for sets with few distances distinct modulo a prime ideal

Hiroshi Nozaki

Abstract Let $\mathcal{O}_K$ be the ring of integers of an algebraic number field $K$ embedded into $\mathbb{C}$. Let $X$ be a subset of the Euclidean space $\mathbb{R}^d$, and $D(X)$ be the set of the squared distances of two distinct points in $X$. In this paper, we prove that if $D(X) \subset \mathcal{O}_K$ and there exist $s$ values $a_1, \ldots, a_s \in \mathcal{O}_K$ distinct modulo a prime ideal $p$ of $\mathcal{O}_K$ such that each $a_i$ is not zero modulo $p$ and each element of $D(X)$ is congruent to some $a_i$, then $|X| \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}$.

1. Introduction

This paper is devoted to giving an upper bound on the cardinalities of certain finite sets $X$ in a metric space $M$, that have some special properties of the values of distances appearing in $X$. A finite set $X$ in $M$ is called an $s$-distance set if the number of distances of two distinct points in $X$ is equal to $s$. One of the major problems for $s$-distance sets is to determine the largest possible $s$-distance sets for given $s$, that is motivated from the extremal set theory. For this purpose, we need to give (or improve) upper bounds on the size and construct large sets. For small $s$, there are remarkable successful cases that can determine the largest sets, for example, 2-distance sets on a Euclidean sphere [9], sets of equiangular lines in Euclidean spaces [14], and several $s$-distance sets in the real, complex, or quaternionic projective spaces [17] or in polynomial association schemes [5, 6, 7].

In the literature of combinatorial geometry, upper bounds for $s$-distance sets for $L$-intersecting families have been obtained. An $L$-intersecting family $\mathfrak{F}$ is a family of subsets of a finite set $F$ that satisfies $|A \cap B| \in L$ for any distinct $A, B \in \mathfrak{F}$ for some $L \subset \{0, 1, \ldots, n-1\}$, where $|F| = n$. An $L$-intersecting family $\mathfrak{F}$ is said to be $k$-uniform if $|A| = k$ for each $A \in \mathfrak{F}$ for some constant $k$. For $k$-uniform $L$-intersecting families $\mathfrak{F}$, Ray-Chaudhuri and Wilson [21] proved an upper bound $|\mathfrak{F}| \leq \binom{n}{k}$, where $|L| = s$. This case corresponds to $s$-distance sets in Johnson association schemes. After this work, Frankl and Wilson [8] obtained $|\mathfrak{F}| \leq \sum_{i=0}^{s} \binom{n}{i}$ without the assumption of $k$-uniform.

Frankl and Wilson [8] also proved a modular version of the upper bound for $k$-uniform $L$-intersecting families. Namely they interpreted the sizes of the intersections as elements of $\mathbb{Z}/p\mathbb{Z}$ for some prime number $p$. Suppose the set $L$ has only $s$ elements distinct modulo $p$, and note that $|L|$ may be greater than $s$. For $k$-uniform $L$-intersecting families $\mathfrak{F}$, if $k \not\equiv a \pmod{p}$ for each $a \in L$, then Frankl–Wilson [8] showed that $|\mathfrak{F}| \leq \binom{n}{k}$. Note that this is the same upper bound as that obtained...
under the assumption $|L| = s$. For $L$-intersecting families $\mathfrak{F}$ with $r$ different sizes of elements of $\mathfrak{F}$ modulo $p$, Alon, Babai, and Suzuki [1] proved that $|\mathfrak{F}| \leq \sum_{i=0}^{r-1} \binom{s}{a_i}$ under a certain weak assumption which is simplified by [13]. The upper bound in [1] is proved by Koornwinder’s method [15], which gives upper bounds on the size $|X|$ by proving the linear independence of some polynomial functions that have a bijective correspondence to $X$.

We have upper bounds for Euclidean $s$-distance sets with several conditions, which are counterparts of that of $L$-intersecting families. Let $X$ be an $s$-distance set in the Euclidean space $\mathbb{R}^d$. For $X$ in the unit sphere $S^{d-1}$, which corresponds to the condition of $k$-uniform, Delsarte, Goethals, and Seidel [6] proved that $|X| \leq \binom{d+s-1}{s} + \binom{d+s-2}{s-1}$.

With no assumption, Bannai, Bannai, and Stanton [2] proved that $|X| \leq \binom{d+s}{s}$. For $X$ in $r$ concentric spheres, which corresponds to the condition of $r$ different sizes, Bannai, Kawasaki, Nitamizu, and Sato [3] obtained that $|X| \leq \sum_{i=0}^{2^r-1} \binom{d+s-1-i}{s-1}$. Recently simple alternative proofs of these upper bounds are given in [12, 20].

Blokhuis [4] gave a modular version of upper bounds for Euclidean sets, assuming that the squared distances are rational integers. Let $D(X)$ be the set of the squared Euclidean distances between two distinct points of $X$.

**Theorem 1.1 (mod-$p$ bound [4]).** Let $X$ be a subset of $\mathbb{R}^d$, and $p$ a prime number. Suppose $D(X)$ is a subset of rational integers $\mathbb{Z}$. If there exist $a_1, \ldots, a_s \in \mathbb{Z}$ distinct modulo $p$ such that

1. for each $i \in \{1, \ldots, s\}$, $a_i \not\equiv 0$ (mod $p$) and
2. for each $\alpha \in D(X)$, there exists $i \in \{1, \ldots, s\}$ such that $\alpha \equiv a_i$ (mod $p$),

then $|X| \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}$.

For sets in a sphere, several projective spaces, or $Q$-polynomial association schemes, Theorem 1.1 can be analogously obtained. However, the $r$-concentric spherical version is still open. The assumption $D(X) \subseteq \mathbb{Z}$ in Theorem 1.1 is surely strong, and the sets to which the theorem can be applied are restricted.

In this paper, we extend Theorem 1.1 to the ring of integers $\mathcal{O}_K$ of an algebraic number field $K$, and any prime ideal $p$ of it. Note that throughout this paper, we fix an embedding of $K$ into $\mathbb{C}$, and $K$ is interpreted as a subfield of $\mathbb{C}$. We use Koornwinder’s method to prove this mod-$p$ upper bound. However, the method to prove the linear independence of polynomial functions is new. In the proof, the localization $A_p$ of $A = \mathcal{O}_K$ by a prime ideal $p$ plays a key role and Nakayama’s lemma is applied for a certain finitely generated $A_p$-module. This method is purely algebraic and uniformly applicable to the polynomial spaces [10], [11, Sections 14–16], which includes the Euclidean sphere, the real, complex or quaternionic projective spaces (see [7, 17] for the theory of $s$-distance set in these projective spaces), or $Q$-polynomial association schemes (which include the theory of $L$-intersecting family as codes of Johnson or Hamming schemes) [5].

The paper is organized as follows. In Section 2, we introduce basic terminology and results about algebraic number fields. In Section 3, we prove a generalization of Theorem 1.1 (mod-$p$ bound) for the ring of integers $\mathcal{O}_K$ of an algebraic number field $K$ and a prime ideal $p \subseteq \mathcal{O}_K$. We also comment on the version of the theorem for an ideal that may not be prime. In Section 4, we extend the LRS type theorem proved in [16, 19]. Namely, if the cardinality of an $s$-distance set is relatively large, then a certain ratio of squared distances must be an algebraic integer (see Theorem 4.1). We explain the relationship between the LRS type theorem and mod-$p$ bound, which can refine an upper bound on the size of an $s$-distance set with given distances.
2. Preliminaries

An extension field $K$ of rationals $\mathbb{Q}$ is an algebraic number field if the degree $[K : \mathbb{Q}]$ is finite. An algebraic number field $K$ can be embedded into $\mathbb{C}$, and $K$ is always identified with a fixed specific subfield of $\mathbb{C}$. The ring of integers $\mathcal{O}_K$ is the ring consisting of all algebraic integers in $K$, where an algebraic integer is a complex number which is a root of a monic polynomial with integer coefficients. It is well known that $K$ is the quotient field of $\mathcal{O}_K$, a prime ideal of $\mathcal{O}_K$ is maximal, $\mathcal{O}_K$ is a finitely generated free $\mathbb{Z}$-module, and $\mathcal{O}_K$ may not be a principal ideal domain. For easy examples, if $K = \mathbb{Q}$, then $\mathcal{O}_K = \mathbb{Z}$. If $K = \mathbb{Q}(\sqrt{d})$ for a square-free integer $d$, then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \frac{1+\sqrt{d}}{2}\mathbb{Z} & \text{if } d \equiv 1 \pmod{4}, \\ \mathbb{Z} + \sqrt{d}\mathbb{Z} & \text{if } d \equiv 2, 3 \pmod{4}. \end{cases}$$

Suppose a ring is commutative and contains the identity. For a ring $A$, $(A, m)$ is a local ring if $A$ has a unique maximal ideal $m$. It is well known that for a ring $A$ and its maximal ideal $p \subset A$, we can construct a local ring $(A_p, pA_p)$. For $A = \mathcal{O}_K$, the local ring is $A_p = S^{-1}A = \{a/s \in K \mid a \in A, s \in S\}$, where $S = A \setminus p$. Its unique maximal ideal is $pA_p$, which is the ideal of $A_p$ generated by the elements of $p$. Note that $A_p$ is a principal ideal domain. The natural map $f : A/p \to A_p/pA_p$ is a field isomorphism.

The following theorem is called Nakayama’s lemma, which plays a key role in a proof of the main theorem. For a local ring $(A, m)$, the ideal $I$ in Theorem 2.1 is $m$.

**Theorem 2.1.** Let $A$ be a ring. Let $I$ be an ideal that is contained in all maximal ideals of $A$. Let $M$ be a finitely generated $A$-module. If $IM = M$, then $M = \{0\}$.

In order to prove the main theorem, we use the polynomial

$$f_x(\xi) = \sum_{i=1}^{s}((|x| - \xi_i|^2 - a_i),$$

for $x \in \mathbb{R}^d$, $a_i \in \mathbb{R}$, and variables $\xi = (\xi_1, \ldots, \xi_d)$, where $||x||$ is the Euclidean norm of $x$. By proving the linear independence of $\{f_x\}_{x \in X}$ as polynomial functions, the cardinality $|X|$ can be bounded above by the dimension of a certain linear space that contains $f_x$. We use the same polynomial space used in Bannai–Bannai–Stanton [2]. For $\xi_0 = \xi_1^0 + \cdots + \xi_d^0$, we define the polynomial space $P_s(\mathbb{R}^d)$ that consists of all real polynomial functions on $\mathbb{R}^d$ which are spanned by $\xi_0^{\lambda_1} \cdots \xi_d^{\lambda_d}$ with $\sum_{i=0}^{d} \lambda_i \leq s$. The dimension of $P_s(\mathbb{R}^d)$ is equal to $(d+s)^s + \binom{d+s-1}{s-1}$.

3. Bounds for $s$-distance sets modulo $p$

The following is the main theorem in this paper.

**Theorem 3.1 (mod-$p$ bound).** Let $X$ be a subset of $\mathbb{R}^d$, and $A = \mathcal{O}_K$ the ring of integers of an algebraic number field $K$. Let $p$ be a prime ideal of $A$. Suppose $D(X) \subset A_p$. If there exist $a_1, \ldots, a_s \in A_p$ distinct modulo $pA_p$ such that

1. for each $i \in \{1, \ldots, s\}$, $a_i \not\equiv 0 \pmod{pA_p}$
2. for each $\alpha \in D(X)$, there exists $i \in \{1, \ldots, s\}$ such that $\alpha \equiv a_i \pmod{pA_p}$,

then

$$|X| \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.$$
Proof. For each \( x \in X \), we define the polynomial \( f_x(\xi) \in P_s(\mathbb{R}^d) \) as
\[
f_x(\xi) = \prod_{i=1}^{s}(||x - \xi||^2 - a_i),
\]
where if needed, we replace \( a_i \) with a real value equivalent to \( a_i \) modulo \( pA_p \). These polynomials satisfy
\[
f_x(x) = (-1)^s \prod_{i=1}^{s} a_i \not\equiv 0 \pmod{pA_p},
\]
and
\[
f_x(y) \equiv 0 \pmod{pA_p}
\]
for \( x \neq y \in X \).

We prove \( \{f_x\}_{x \in X} \) is linearly independent as polynomial functions on \( \mathbb{R}^d \). Assume there exist \( m_x \in \mathbb{R} \) such that
\[
\sum_{x \in X} m_x f_x(\xi) = 0.
\]
Let \( M \) be an \( A_p \)-module generated by a finite set \( \{m_x\}_{x \in X} \), namely
\[
M = \sum_{x \in X} m_x A_p.
\]
From equalities (3.3) and (3.2), for each \( y \in X \),
\[
m_y f_y(y) = - \sum_{y \neq x \in X} m_x f_x(y) \in pA_p M.
\]
Since \( f_y(y) \in A_p \setminus pA_p \) from equality (3.1), it follows that \( f_y(y) \in A_p^s \) and
\[
m_y = - \sum_{y \neq x \in X} m_x f_x(y) f_y(y)^{-1} \in pA_p M.
\]
This implies that \( M \subset pA_p M \), and hence \( M = pA_p M \). By Nakayama’s lemma, \( M = \{0\} \) and \( m_x = 0 \) for each \( x \in X \). Therefore \( \{f_x\}_{x \in X} \) is linearly independent, and
\[
|X| = |\{f_x\}_{x \in X}| \leq \dim P_s(\mathbb{R}^d) = \binom{d + s}{s} + \binom{d + s - 1}{s - 1}
\]
as desired. \qed

**Corollary 3.2.** Let \( X \) be a subset of \( \mathbb{R}^d \), and \( O_K \) the ring of integers of an algebraic number field \( K \). Let \( p \) be a prime ideal of \( O_K \). Suppose \( D(X) \subset O_K \). If there exist \( a_1, \ldots, a_s \in O_K \) distinct modulo \( p \) such that
\begin{enumerate}
\item for each \( i \in \{1, \ldots, s\} \), \( a_i \not\equiv 0 \pmod{p} \) and
\item for each \( \alpha \in D(X) \), there exists \( i \in \{1, \ldots, s\} \) such that \( \alpha \equiv a_i \pmod{p} \),
\end{enumerate}
then
\[
|X| \leq \binom{d + s}{s} + \binom{d + s - 1}{s - 1}.
\]

**Proof.** Since \( A = O_K \subset A_p \) and \( A/p \cong A_p/pA_p \), this corollary is immediate from Theorem 3.1. \qed

**Example 3.3.** For \( X = \{(0,0), (1,0), (-\sqrt{3}/2, 1/2), (-\sqrt{3}/2, -1/2)\} \subset \mathbb{R}^2 \), the squared distances are \( D(X) = \{1, 2 + \sqrt{3}\} \). We take the algebraic number field \( K = \mathbb{Q}(\sqrt{3}) \). Then the ring of integers is \( O_K = \mathbb{Z} + \sqrt{3}\mathbb{Z} \), and \( p = (1 + \sqrt{3}) \) is a prime ideal of \( O_K \). Since \( 1 \equiv 2 + \sqrt{3} \pmod{p} \) holds, we have \( |X| \leq \binom{d + s}{s} + \binom{d + s - 1}{s - 1} = 4 \). The set \( X \) is an example attaining the upper bound in Corollary 3.2.
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We can prove a similar theorem to Theorem 3.1 for an ideal $I \subset \mathcal{O}_K$ which may not be prime as follows.

**Theorem 3.4.** Let $X$ be a subset of $\mathbb{R}^d$, and $A = \mathcal{O}_K$ the ring of integers of an algebraic number field $K$. Let $I$ be an ideal of $A$, and $I = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ the prime decomposition of $I$. Let $A_I = S^{-1}A = \{a/s \mid a \in A, s \in S\}$, where $S = \bigcup_{R \in (A/I)^\times} R$. Suppose $D(X) \subset A_I$. If there exist $a_1, \ldots, a_s \in A_I$ distinct modulo $IA_I$ such that

1. for each $i \in \{1, \ldots, s\}$, $a_i \in A_I$ and
2. for each $\alpha \in D(X)$, there exists $i \in \{1, \ldots, s\}$ such that $\alpha \equiv a_i$ (mod $IA_I$),

then

$$|X| \leq \binom{d+s}{s} + \binom{d+s-1}{s-1}.$$

**Proof.** The proof is similar to that of Theorem 3.1, but we use $p_1 \cdots p_r A_I$ instead of $pA_p$ as the ideal that is contained in all maximal ideals in Nakayama’s lemma. □

For Theorem 3.4, we must choose squared distances $a_i$ from $A_I$. Such distances $a_i$ can be expressed by $a_i = s_1/s_2$ for some $s_1, s_2 \in S = \bigcup_{R \in (A/I)^\times} R$. Since $S \subset \bigcup_{R \in (A/p_i)^\times} R$ for any $j$, the squared distances $a_i$ are also elements of $A_{p_j} = A \setminus p_j$. The natural homomorphisms

$$A_I/IA_I \to A_I/p_1^{\lambda_i} A_I \times \cdots \times A_I/p_r^{\lambda_i} A_I$$

$$\to A_I/p_1 A_I \times \cdots \times A_I/p_r A_I$$

$$\to A_{p_1} \times \cdots \times A_{p_r}$$

imply that the number of squared distances distinct modulo $IA_I$ is greater than or equal to that modulo $p_i A_{p_i}$ for any $i \in \{1, \ldots, r\}$. Therefore, Theorem 3.1 corresponding to the prime-ideal version gives the strongest upper bound for any ideal under our condition.

4. **LRS Type Theorem**

We now generalize the LRS type theorem proved in [19] as follows. The absolute bound $|X| \leq \binom{d+s}{s}$ is improved by this generalization.

**Theorem 4.1.** Suppose $s \geq 2$. Let $X$ be an $s$-distance set in $\mathbb{R}^d$ and $N = \dim P_{s-1}(\mathbb{R}^d) = \binom{s-1}{d-1} + \binom{d+s-2}{s-2}$. If $|X| \geq N + (N+1)/t$ for some $t \in \mathbb{N}$, then

$$K_j = \prod_{i=1, i \neq j}^{s} \frac{\alpha_i}{\alpha_i - \alpha_j}$$

is an algebraic integer of degree at most $t$ for each $j \in \{1, \ldots, s\}$.

**Proof.** Fix $j \in \{1, \ldots, s\}$. Define the polynomial

$$f(x, \xi) = \prod_{i=1, i \neq j}^{s} \left( \frac{\alpha_i - ||x - \xi||^2}{\alpha_i - \alpha_j} \right)$$

for each $x \in X$. Since $f(x, \xi) \in P_{s-1}(\mathbb{R}^d)$, the rank of the matrix $M = (f(x, y))_{x, y \in X}$ is at most $N$ [19]. The matrix can be expressed by $M = K_j I + A_j$, where $I$ is the identity matrix and $A_j$ is a $(0, 1)$-matrix with off diagonals. Since the size of $M$ is at least $N + (N+1)/t > N$, the matrix has $0$ eigenvalue whose multiplicity is at least $(N+1)/t$. This implies $-K_j$ is the eigenvalue of $A_j$, and hence $K_j$ is an algebraic integer.
Assume $K_j$ is an algebraic integer of degree larger than $t$. Then the number of the conjugates of $-K_j$ is at least $t$, and the conjugates are also eigenvalues of $A_j$. Since $A_j$ has the eigenvalue $-K_j$ with multiplicity at least $(N+1)/t$, the size of $A_j$ is at least $(t+1)(N+1)/t = N + 1 + (N+1)/t$, which contradicts our assumption. Therefore $K_j$ is an algebraic integer of degree at most $t$. \hfill\Box

For $t=1$, the values $K_j$ are integers under the condition in Theorem 4.1, which is the previous result proved in [19]. The following corollaries are immediate from Theorem 4.1.

**Corollary 4.2.** If $K_j$ is not an algebraic integer for some $j \in \{1, \ldots, s\}$, then $|X| \leq N$.

**Corollary 4.3.** Suppose $K_j$ is an algebraic integer for each $j \in \{1, \ldots, s\}$. Let $t$ be the maximum value of the degrees of $K_j$. If $t>1$ holds, then $|X| < N + (N+1)/(t-1)$.

Corollary 4.3 is an improvement of the absolute bound for $s$-distance sets with the LRS ratios.

If there exist $\alpha_i, \alpha_j \in D(X) \subset \mathcal{O}_K$ such that $\alpha_i$ is congruent to $\alpha_j$ modulo some prime ideal $\mathfrak{p}$ and $\alpha \not\equiv 0 \pmod{\mathfrak{p}}$ for each $\alpha \in D(X)$, then the LRS ratio $K_j$ is not an algebraic integer. Indeed, if $K_j \in \mathcal{O}_K$, then

\begin{equation}
0 \equiv K_j \prod_{i \neq j} (\alpha_i - \alpha_j) = \prod_{i \neq j} \alpha_i \not\equiv 0 \pmod{\mathfrak{p}},
\end{equation}

which is a contradiction. When $K_j$ is not an algebraic integer for some $j$, we obtain the bound $|X| \leq N$ by Theorem 4.1, and we may obtain a better bound depending on the number of the elements of $D(X)$ distinct modulo $\mathfrak{p}$.

The results proved in this paper – mod-$\mathfrak{p}$ bound and LRS type theorems – are analogously obtained for the sphere $S^{d-1}$ [6], several projective spaces [7, 17], or Q-polynomial association schemes [5, 7]. For spherical case, the LRS type theorem with $\mathcal{O}_K = \mathbb{Z}$ is useful to determine largest spherical $s$-distance sets for $s = 2, 3$. In [9, 18], several largest $s$-distance sets are determined by a computer assistance. The possibilities of choices of integers $K_i$ are finite, and we can take the finite choices of distances from $K_i$. Reducing the number of the possible distances is helpful to cut the computational cost by a computer. However, Equation (4.1) implies that it is impossible to reduce the choices of distances by our results.

**Remark 4.4.** Akihiro Munemasa, one of the editors of the journal, communicated to the author the following idea to prove Theorem 3.1 without the use of Nakayama’s lemma. Let $f_\mathfrak{p}(\xi)$ be the same as in the proof of Theorem 3.1. We consider the matrix $M = (f_\mathfrak{p}(y))_{x,y \in X}$, where $X$ satisfies the condition of the theorem. In order to prove the linear independence of $\{f_\mathfrak{p}\}_{x \in X}$, it suffices to show that the determinant of $M$ is non-zero. The entries of $M$ are elements of $A_\mathfrak{p}$, and $M$ is congruent to some diagonal matrix modulo $\mathfrak{p}A_\mathfrak{p}$ whose diagonal entries are units in $A_\mathfrak{p}$. The determinant $M$ is not congruent to 0 modulo $\mathfrak{p}A_\mathfrak{p}$, in particular, it is non-zero.

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