Trkalian fields: ray transforms and mini-twistors

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We study X-ray and Divergent beam transforms of Trkalian fields and their relation with Radon transform. We make use of four basic mathematical methods of tomography due to Grangeat, Smith, Tuy and Gelfand-Goncharov for an integral geometric view on them. We also make use of direct approaches which provide a faster but restricted view of the geometry of these transforms. These reduce to well known geometric integral transforms on a sphere of the Radon or the spherical Curl transform in Moses eigenbasis, which are members of an analytic family of integral operators. We also discuss their inversion. The X-ray (also Divergent beam) transform of a Trkalian field is Trkalian. Also the Trkalian subclass of X-ray transforms yields Trkalian fields in the physical space. The Riesz potential of a Trkalian field is proportional to the field. Hence, the spherical mean of the X-ray (also Divergent beam) transform of a Trkalian field over all lines passing through a point yields the field at this point. The pivotal point is the simplification of an intricate quantity: Hilbert transform of the derivative of Radon transform for a Trkalian field in the Moses basis. We also define the X-ray transform of the Riesz potential (of order 2) and Biot-Savart integrals. Then, we discuss a mini-twistor representation, presenting a mini-twistor solution for the Trkalian fields equation. This is based on a time-harmonic reduction of wave equation to Helmholtz equation. A Trkalian field is given in terms of a null vector in \( \mathbb{C}^3 \) with an arbitrary function and an exponential factor resulting from this reduction.

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I. INTRODUCTION

This is the second manuscript of a series1 aimed at studying the mathematical structure of Trkalian2,3 class of Beltrami fields (eigenvectors of Curl operator with constant eigenvalue) in integral geometric and twistor terms. Integral geometry and Twistor theory provide new mathematical methods for studying the geometry of Trkalian fields. These lead to a deeper understanding of their physical aspects.1,4–6

The Trkalian fields arise in different areas ranging from fluid dynamics and plasma physics to field theories. The field theoretic examples of Trkalian vectors are the force-free magnetic field and the Euclidean topologically massive Abelian gauge field.

In plasma physics, a Trkalian field simply corresponds to a force-free equilibrium state of a plasma. The mathematical tomography is based on applications of integral geometric methods in tomography. It makes use of both pure and applied techniques. It provides the mathematical basis for modern tomography in the realistic sense. The tomographical study of an equilibrium state of a plasma is an active field of research in plasma tomography.

The topologically massive gauge theories7–9 are qualitatively different from Yang-Mills type gauge theories besides their mathematical elegance and consistency.10 These are introduced as an alternative to the mechanism of spontaneous symmetry breaking for generation of mass. The study of their physical and mathematical aspects is an active and exciting field of research today providing new insight into the relation of gauge theories and gravity in low dimensions. In this context, Trkalian type solutions on 3-sphere $\mathbb{S}^3$, anti-de Sitter space $\mathbb{H}^3$ and other spaces in connection with contact geometry are discussed in Refs. 6, 10–12.

The Trkalian (Beltrami) fields equation also arises in connection with harmonic morphisms.13 See Refs. 14 and 15 and the reference therein for the solution on $\mathbb{S}^3$ (consisting of right/left-invariant 1-forms) and its uniqueness (upto permutation).

A purpose of this manuscript is to develop physical insight for Trkalian fields in integral geometric and twistor terms. Intuitively speaking, from a higher point of view, we expect these to be related to the representations of the group underlying Trkalian fields. Basically, we are trying to investigate the aspects of functions invariant under the Curl operator generating rotations. These integral transforms naturally arise in geometric analysis (in Fourier sense) of physical systems.

As a physical example of integral geometry in tomography, we shall frequently benefit the example of Lundquist16 field which is used to model solar magnetic clouds, (see Refs. 4, 5 and the references therein). We shall present a mathematical Röntgen of these clouds.

Mathematically, we aim to expose the interrelations of the most basic transforms in integral geometry, for Trkalian fields. We are motivated by the geometric picture4,5 that is provided by the Radon17 transform which is at a central place in integral geometry. This endows us with an intuition for Trkalian fields. These transforms which arise naturally in the study of integral geometric and tomographical aspects of Trkalian fields are intimately connected. The mathematical methods of tomography provide delicate ways for exposing their interrelations. This leads to an intuitive, integral understanding, besides its potential physical applications.

We also aim to discuss a mini-twistor representation for Trkalian fields presenting a mini-twistor solution for them. This leads to new mathematical challenges beside providing new physical insight into the Trkalian fields.

In the first part of this manuscript we shall study X-ray (John18) and Divergent beam transforms of Trkalian fields. Then we shall discuss the mini-twistor representation for Trkalian fields in the second part.

The Radon transformation provides a geometric formulation of Trkalian fields.14,4–6 Especially, Moses eigenfunctions19 of the Curl operator, which form a complete orthonormal basis, leads to a helicity decomposition of the Radon transform of Trkalian fields. The spherical Curl transform is a Radon probe transformation1–4,5 in this basis. The Radon transform of a Trkalian field is tangent to a sphere in the transform space. It satisfies a corresponding eigenvalue equation on this sphere.

Furthermore, we can associate an eigenvalue equation for Biot-Savart20,21 integral operator with Trkalian fields if they vanish at infinity. Meanwhile, the Riesz potential and Biot-Savart integrals naturally arise in Radon transform.1 The Radon-Biot-Savart (RBS) integral is defined as the Radon transform of the Biot-Savart operator.1 We can study Trkalian fields using the RBS operator in transform space.

First we shall study X-ray and Divergent beam transforms of Trkalian fields. The field is to be taken from Schwartz space $S[\mathbb{R}^3]$ of rapidly decreasing functions on $\mathbb{R}^3$.22

The X-ray and Divergent beam transforms are closely connected with the Radon transform. The mathematical methods of tomography (respectively Smith’ s and Tuy’ s methods discussed below) show that these transforms are basically in the form of a Minkowski-Funk23,24 and a closely related (well known, but no specific name in the literature known to the author) integral transform of certain intricate quantities (Hilbert transform of the derivative of Radon transform). The Moses eigenbasis is especially efficient in exhibiting this connection for Trkalian fields. Then, these naturally reduce to well known geometric integral transforms on a sphere (mentioned above) of the Radon25 or the spherical Curl transforms. More precisely, the X-ray transform reduces to Minkowski-Funk transform of the Radon...
transform on this sphere. Meanwhile, the Divergent beam transform reduces to another closely related (an extension of Minkowski-Funk) integral transform of the spherical Curl transform.

Moreover, these geometric integral transforms are members of an analytic family of integral operators. See Refs. 26 and 27 and the references therein. This leads us to inverse transforms which also belong to this family for the X-ray and Divergent beam transforms of Trkalian fields.

The connections of X-ray and Divergent beam transforms of Trkalian fields to their Radon transforms can be directly obtained without further ado, (a bit difficult to see but) simply substituting the relevant expressions without any sense. We shall first present direct proofs without logical motivation. Because the direct approach is more appropriate for exposing the relations of X-ray and Divergent beam transforms of Trkalian fields to the Radon and other transforms. However the underlying geometric structure can be investigated by using the mathematical methods of tomography. Because these connections and their inversion deserve a separate discussion in its own right, we have interchanged the logical order for a clean presentation below. Yet, the mathematical methods of tomography provides a logically unified view for our motivations and also for the development of geometric intuition along the manuscript.

The X-ray (also the Divergent beam) transform and its inversion intertwine the Curl operator ($\nabla \times$) and also the Divergence ($\nabla \cdot$), Gradient ($\nabla$) and Laplacian ($\nabla^2$) operators. Thus the X-ray (Divergent beam) transform of a Trkalian field is Trkalian. Also, the Trkalian subclass of X-ray transforms $XF$ (satisfying John’ s differential equation) yields Trkalian fields in the physical space. We shall also write John’ s differential equation for Trkalian fields in an equivalent form. Thus, we can study Trkalian fields either in physical space or in the transform space.

Another crucial quantity in integral geometry is the Riesz potential. The Riesz potential, of order $\alpha$ where $0 < \alpha < 3$, of a Trkalian field is proportional to the field. Hence, the spherical mean of X-ray (or Divergent beam) transform of a Trkalian field over all lines passing through a point yields the field itself at this point. This endows us with a new simple inversion formula for the X-ray (or Divergent beam) transform of Trkalian fields. This result is also logically implied by Gelfand-Goncharov’ s mathematical approach to tomography below.

Then we shall return back to the mathematical methods of tomography. The first purpose of this section is to provide a unified geometric view and motivation for the interrelations of integral transforms arising in our discussion, as mentioned above. The second purpose is to present a discussion of these mathematical methods with a view towards tomographical studies of Trkalian field models in nature. For this purpose, we shall study these mathematical methods using Trkalian fields. Especially for the sake of the second purpose and also for a clean presentation, this discussion will be postponed until the direct (but unmotivated) discussion of the geometry of X-ray and Divergent beam transforms finish.

We shall make use of four basic mathematical approaches of tomography due to Grangeat, Smith, Tuy and Gelfand-Goncharov for studying the X-ray and Divergent beam transforms of Trkalian fields and expressing their relations with the other transforms. These relations are outflow of a formula essentially obtained in Ref. 32. These methods basically make use of the Radon inversion (for tomographical reconstruction). They lead us to new inversion formulas for the X-ray and Divergent beam transforms of Trkalian fields with a view towards tomographical applications. They also provide the geometric motivation underlying the interrelations of the transforms mentioned. We shall adopt a mathematical approach rather than a tomographical implementation.

The Grangeat approach leads to another simple, direct inversion formula for the Divergent beam transform of Trkalian fields.

The Smith method reveals that the X-ray transform is in the form of a Minkowski-Funk transform of an intricate quantity related to the Radon transform. This simply reduces to Minkowski-Funk transform of the Radon transform on a sphere, yielding the result mentioned above. In this approach, the inversion formula can be expressed in terms of the Radon transform of the field.

The Tuy method enables us to investigate the Divergent beam transform in detail. It reveals that the Divergent beam transform is in the form of another closely related integral transform of a quantity related to the Radon transform. This reduces to the above mentioned integral transform of the spherical Curl transform. In this case, the inversion formula can be expressed in terms of the spherical Curl transform of the field.

We calculate the Divergent beam transform of the Lundquist field using the Tuy method. This yields a mathematical Röntgen of solar magnetic clouds.

Meanwhile, Gelfand-Goncharov approach leads to a direct inversion through the spherical mean that is mentioned above. This naturally makes use of the inverse transform that belongs to the above family of integrals operators.

The basic simplification in these approaches are due to the same intricate quantity: Hilbert transform of the derivative of Radon transform in the Moses basis.

The direct inversion formulas arising in Grangeat’ s and Gelfand-Goncharov’ s approaches mathematically seem more feasible than the inversions in Smith’ s and Tuy’ s methods.

These approaches provide different inversion formulas which may serve useful for designing reconstruction methods in tomographical studies of Trkalian field models in nature, depending on real physical situation. We shall not discuss tomographical implementations of these inversion formulas.
Furthermore, the Smith and Tuy methods mathematically enable us to define the X-ray and Divergent beam transforms of the Riesz potential (of order 2) and Biot-Savart integrals. The X-ray transform of the Biot-Savart integral of a Trkalian field reduces to the X-ray transform of the field.

In the second part of this manuscript we shall discuss Trkalian fields using (mini-)twistors. Twistor theory has been originally founded and developed by Penrose.\textsuperscript{34,35} It has led to a deeper understanding of nature. In simplest terms, this is based on writing contour integral solution for wave equation in (3 + 1) dimensional Minkowski space, using a holomorphic function. Similar formulas date back to Whittaker and Bateman.\textsuperscript{36,37}

The X-ray transform is a real analogue of the Penrose transform\textsuperscript{38,39} and a predecessor\textsuperscript{40} of Twistor theory. See for example Refs. 38–44 for relation of the X-ray transform and Twistor theory.

Mini-twistor space as an intrinsic structure has been introduced by Hitchin.\textsuperscript{45} The (mini-)twistor space of $\mathbb{R}^3$ is the space $TS^2$ of oriented lines in $\mathbb{R}^3$. This can be identified with $TCP^1$, the holomorphic tangent bundle of Riemann sphere $CP^1$. This is also given by the quotient of twistor space ($CP^3 \setminus CP^1$) of the Minkowski space by the action of time translation.\textsuperscript{45} The X-ray transform and the mini-twistors are both defined on the space $TS^2 \sim TCP^1$ of oriented lines in $\mathbb{R}^3$.

We shall discuss a mini-twistor representation, presenting a mini-twistor solution for the Trkalian fields equation. We shall make use of the solution\textsuperscript{46} of (vector) Helmholtz equation which is based on a time-harmonic reduction of the wave equation. A Trkalian field is given in terms of a null vector in $C^3$ with an arbitrary function and an exponential factor that results from the reduction.

The exponential factor contains the spatial part of an integrating factor for the time-harmonicity condition. The solution is of the same form containing the spatial part of any chosen integrating factor. We shall also use the general solution\textsuperscript{46} of this condition for writing the solution.

This solution can also be derived as a time-harmonic reduction of the twistor solution\textsuperscript{35} for electromagnetic fields in (3 + 1) dimensions.

We are led to a time-harmonic extension of Trkalian fields implicitly keeping this condition. This can be interpreted as a time-harmonic electromagnetic field.

We shall also present examples of Debye potentials for Chandrasekhar-Kendall\textsuperscript{47} (CK) type solutions using the twistor solution of (scalar) Helmholtz equation.

The relation of (mini-)twistors and ray transforms for Trkalian fields is beyond the limitations of this manuscript.

II. X-RAY AND DIVERGENT BEAM TRANSFORMS

A. Trkalian fields: Radon transform

Trkalian fields are eigenvectors of the curl operator

$$\nabla \times F(x) - \nu F(x) = 0,$$

with constant eigenvalue $\nu$. The Radon transform

$$F^R(p, \kappa) = R[F(x)](p, \kappa) = \int F(x)\delta(p - \kappa \cdot x)d^3x,$$

of a field $F(x)$ (that belongs to Schwartz class) on $\mathbb{R}^3$ is defined as the integral of the field over a hyperplane at (orthogonal) distance $p$ to the origin, with unit normal vector $\kappa$. The Radon transform $F^R(p, \kappa)$ of a Trkalian field satisfies

$$\Gamma \times F^R(p, \kappa) - \nu F^R(p, \kappa) = 0,$$

where $\Gamma = \kappa \partial/\partial p$.\textsuperscript{1} We also have: $\kappa \cdot F^R(p, \kappa) = 0$ which leads to $\Gamma \cdot F^R(p, \kappa) = 0$. Because the Radon transform intertwines the operator $\nabla$ with $\Gamma$. We can write this equation as

$$\frac{\partial}{\partial p} F^R(p, \kappa) + \nu \kappa \times F^R(p, \kappa) = 0.$$
The Curl transform is based on decomposing a vector field into helical eigenfunctions $\chi_{\lambda}(x|k) = (2\pi)^{-3/2} e^{i k \cdot x} Q_{\lambda}(k)$ of the curl operator, which form an orthogonal and complete set, in the fashion of a Fourier transform refining the Helmholtz decomposition. This is a helicity ($\lambda = -1, 0, 1$) decomposition in the basis $\{Q_{\lambda}(k)\}$.

A Trkalian field can be expressed as $F(x) = \nabla' F_{\lambda}(x)$ excluding the divergenceful component, where $F_{\lambda}(x) = (1/g) \int \chi_{\lambda}(x|k) f_{\lambda}(k) d^3 k$. Then we find $f_{\lambda}(k) = [\delta(k - \lambda \nu)/k^2] s_{\lambda}(k)$ relating the curl transform $f_{\lambda}(k)$ to the spherical curl transform $s_{\lambda}(k)$ of the field $F(x)$. Thus an arbitrary solution is given entirely in terms of its transform on a sphere of radius $k = \lambda \nu = |\nu|$ in transform space. Further, only the eigenfunctions for which $\lambda = sgn(\nu)$ contribute to the field. The Radon transform of a Trkalian field is tangent to this sphere

$$F_R^\lambda(p, \kappa) = (2\pi)^{1/2} \frac{1}{g} \frac{1}{\nu^2} [e^{i \lambda \nu p} Q_{\lambda}(\kappa) s_{\lambda}(\lambda \nu) + e^{-i \lambda \nu p} Q_{\lambda}(-\kappa) s_{\lambda}(-\lambda \nu)].$$

The factor $1/g$ is introduced for the sake of a proper strength for the gauge potential in topologically massive gauge theory. This can be taken as 1 for general Trkalian fields. The inverse transform is given as

$$F_{\lambda}(x) = \frac{1}{8\pi^2 \nu^2} \int_{S^2_{\kappa}} F_R^\lambda(\kappa \cdot x, \kappa) d\Omega_{\kappa},$$

using the adjoint Radon transform $R^1$, where $S^2_{\kappa}$ is the unit sphere in the transform space.

The simplest example of Trkalian fields is $F(x) = e^{ik_0 \cdot x} F_0$ where $k_0 = k_0 \kappa_0$, $k_0 = \lambda \nu > 0$ and $F_0 = Q_{\lambda}(\kappa_0)$: $\kappa_0 \times F_0 = -i \lambda F_0$, $\kappa_0 \cdot F_0 = 0$. Its Radon transform is

$$F_R^\lambda(p, \kappa) = (2\pi)^{1/2} \frac{1}{k_0} [e^{ik_0 \nu p} \delta(\kappa - \kappa_0) + e^{-ik_0 \nu p} \delta(\kappa + \kappa_0)] F_0.$$
B. X-ray and Divergent beam transforms

The X-ray transform of a vector-valued function \( F(x) \) (that belongs to Schwartz class) on \( \mathbb{R}^3 \) is defined as

\[
\mathcal{X}F(\theta, x) = \int_{-\infty}^{\infty} F(x + s\theta)ds,
\]

(12)

the integral of the field over line \( L \) passing through point \( x \) in the direction determined by the unit vector \( \theta \).\(^{22,33,48,49}\) This is a componentwise generalization of the X-ray transform of scalar fields. The X-ray transform is defined on the space of oriented lines in \( \mathbb{R}^3 \). Note, \( \mathcal{X}F(\theta, x) \) is unchanged if \( x \) is translated in the direction of \( \theta \). Therefore we restrict \( x \) to \( \theta \perp : x \cdot \theta = 0 \). Hence \( \mathcal{X}F(\theta, x) \) is a function defined on the tangent bundle \( \mathcal{T}\mathbb{S}^2 = \{ (\theta, x), \theta \in \mathbb{S}^2, x \in \theta \perp \} \) of the sphere \( \mathbb{S}^2. \) \(^{33,49} \) Also \( \mathcal{X}F(-\theta, x) = \mathcal{X}F(\theta, x) \).

The Divergent beam or Cone beam transform is defined as

\[
\mathcal{D}F(\theta, x) = \int_{0}^{\infty} F(x + s\theta)ds,
\]

(13)

the integral of the field over the half-line. We have: \( \mathcal{X}F(\theta, x) = \mathcal{D}F(\theta, x) + \mathcal{D}F(-\theta, x) \).

The X-ray transform (12) \([\text{also the Divergent beam transform (13)}]\) satisfies John’s equation below. The inversion problem of the X-ray transform is overdetermined,\(^{22} \) that is the data of all line integrals are redundant.\(^{50} \) There are various inversion methods for the X-ray (also Divergent beam) transform. For example, one can reconstruct \( F(x) \) knowing \( \mathcal{X}F(\theta, x) \) where \( \theta \in \mathbb{S}^2, x \in L \) and \( L \) is a suitable curve, (see Ref. 48, p. 52, p. 276 and Ref. 51).

Proposition 1: The X-ray transform of a Trkalian field is given as

\[
\mathcal{X}F(\lambda, x) = \frac{1}{(2\pi)^{1/2}} \frac{1}{g} \frac{1}{\lambda \nu} \int_{\mathbb{S}^2} e^{i\lambda \nu \cdot x} Q_{\lambda}(\kappa) s_{\lambda}(\lambda \nu \cdot \kappa) \delta(\kappa \cdot \theta) d\Omega_{\kappa}
\]

\[
= \frac{1}{4\pi} \lambda \nu \int_{\mathbb{S}^2} F_{\lambda}(\kappa \cdot x, \kappa) \delta(\kappa \cdot \theta) d\Omega_{\kappa},
\]

(17)

the Minkowski-Funk transform (in the transform space) of its Radon transform.
Proof: We substitute $F_\lambda(x + s\theta)$ using the second line of (6) in (12). See Appendix A 2a. Note that a result valid in a basis should be valid in any basis.

This proposition is based on the motivation in equation (59) below which makes use of Smith’s formula (55) in Section IV B. See equation (61) for the underlying intricate geometric quantity and its simplification.

The Minkowski-Funk 23,24 transform $\mathcal{M}$ of a continuous (even) function on a sphere is given by the integral of this function over great circles: geodesics in the sphere. This yields a function on the space of geodesics. This clearly annihilates odd functions.

We symbolically write $X = [\lambda \nu/(4\pi)]\mathcal{M}R$ respectively relating the X-ray (John18), Minkowski-Funk 23,24 and Radon 17 transforms for Trkalian fields. In $\mathbb{R}^3$, this corresponds to integral of the Radon transform of the field over a pencil of planes intersecting at a line (passing through the point $x$ in the direction of $\theta$) which are parametrized by a circle. Previously, Gonzales called this plane-to-line transform.25 See Ref. 43 for a twistor view of this transform.

There are various inversion methods for the Minkowski-Funk transform. We shall briefly discuss an inversion formula which is appropriate for our purpose below.

If we use the example (7) in the second line of (17) we find the result (14). Also, we find (16), $(\lambda = 1)$ substituting (9) in the second line of (17), where $\kappa_\tau = \sin \alpha = \sqrt{1 - \kappa_z^2}$, $\kappa_z = \cos \alpha$ and $\alpha$ is the polar angle on $S^2_\kappa$ with spherical angles $\psi, \alpha$. We shall avoid the details of these straightforward calculations.

We can arrive at the same result, as expressed in equation (A11), using the Curl expansion for Trkalian fields in Fourier slice-projection theorem 33

$$\mathcal{F}[\mathcal{X}F(\theta, x)](\theta, \xi) = (2\pi)^{1/2} \mathcal{F}[F(x)](\xi), \quad \xi \in \theta^\perp$$

for the X-ray transform. See Appendix A 2b. On the left-hand side $\mathcal{F}$ stands for Fourier transform in $\theta^\perp$ plane whereas on the right-hand side it is a Fourier transform in three dimensions.

**Proposition 2:** The Divergent beam transform of a Trkalian field is given as

$$\mathcal{D}F_\lambda(\theta, x) = \frac{1}{(2\pi)^{1/2}} \frac{1}{g \lambda \nu} \int_{S^2_\kappa} e^{i\lambda \nu \cdot x} Q_\lambda(\kappa) s_\lambda(\lambda \nu \kappa) \delta^+ (\kappa \cdot \theta) d\Omega_\kappa,$$

in the Moses basis.

Proof: We substitute $F_\lambda(x + s\theta)$ using the second line of (6) in (13) and use $\mathcal{F}[H(\pm x)](k) = \sqrt{-2\pi} \delta^\pm(k)$.

This proposition is based on the motivation in equation (69) below which makes use of (67) in Section IV C: Tuy’s method. See equation (74) for the underlying intricate geometric quantity and its simplification.

This is another basic transform (with no specific name in the literature known by the author) in integral geometry containing the Heisenberg delta function $\delta^\pm$. The first term in $\delta^\pm$ is associated with the Minkowski-Funk transform mentioned above. Hence, this is an extension of the Minkowski-Funk transform. The second term is to be understood in the sense of Cauchy principal value (see above). Roughly speaking, it provides a description of the behaviour of a function on the sphere except the geodesics, [see (21) below for the physical meaning].

This leads to

$$\mathcal{D}F_\lambda(-\theta, x) = \frac{1}{(2\pi)^{1/2}} \frac{1}{g \lambda \nu} \int_{S^2_\kappa} e^{i\lambda \nu \cdot x} Q_\lambda(\kappa) s_\lambda(\lambda \nu \kappa) \delta^-(\kappa \cdot \theta) d\Omega_\kappa,$$

since $\delta^+(\kappa \cdot \theta) = \delta^-(\kappa \cdot \theta)$.

For $F(x) = e^{ik_0 \cdot x} F_0$, $[k_0 = k_0 \kappa_0, k_0 = \lambda \nu > 0, F_0 = Q_\lambda(\kappa_0)]$, we find the result (15) substituting $s_\lambda(\lambda \nu \kappa) = (2\pi)^{3/2} \bar{g} \delta(\kappa - \kappa_0)^{1,4,5}$ in (19, 20).

In analogy with the X-ray transform (12), the difference

$$\mathcal{Y}F(\theta, x) = \mathcal{D}F(\theta, x) - \mathcal{D}F(-\theta, x) = \int_{-\infty}^{\infty} F(x + s\theta) sgn(s) ds,$$

where $sgn(s) = H(s) - H(-s)$ is the signum function, reduces to

$$\mathcal{Y}F_\lambda(\theta, x) = \frac{2}{(2\pi)^{3/2}} \frac{1}{g \lambda \nu} \int_{S^2_\kappa} e^{i\lambda \nu \cdot x} Q_\lambda(\kappa) s_\lambda(\lambda \nu \kappa) \frac{1}{\kappa \cdot \theta} d\Omega_\kappa,$$
For the Trkalian field $F(x) = e^{ik_0 x} F_0$, $|k_0| = k_0 \kappa_0$, $k_0 = \lambda \nu > 0$, $F_0 = Q_\lambda(\kappa_0)$, we find

$$\mathcal{Y} F_\lambda(\theta, x) = 2 \frac{1}{k_0} e^{ik_0 x} \frac{1}{\kappa_0 \cdot \theta} F_0. \quad (23)$$

We can write the equation (19) as

$$DF_\lambda(\theta, x) = \frac{1}{21/2} \int_{\frac{\lambda \nu}{g}} \mathcal{A}^0 G = \frac{1}{21/2} \int_{\frac{\lambda \nu}{g}} (\mathcal{U}^0 G + i \mathcal{V}^0 G), \quad (24)$$

where $G(\kappa, x) = e^{i \lambda \nu \kappa \cdot x} Q_\lambda(\kappa) s_\lambda(\lambda \nu \kappa)$, $\mathcal{A}^0 = \mathcal{U}^0 + i \mathcal{V}^0$ and

$$
\mathcal{U}^0 G = \frac{1}{2 \pi^{1/2}} \int_{S^2_\kappa} G(\kappa, x) \delta(\kappa \cdot \theta) d\Omega_\kappa, \quad \mathcal{V}^0 G = \frac{1}{2 \pi^{1/2}} \int_{S^2_\kappa} G(\kappa, x) \frac{1}{\kappa \cdot \theta} d\Omega_\kappa, \quad (25)
$$

with [a slight change of notation: $e_\nu(\phi) \rightarrow \kappa$ in (A11)] the integration measure $d\theta \kappa$ on the great circle $C$ determined by $\theta$. The integrals in $\mathcal{U}^0$ and $\mathcal{V}^0$ respectively correspond to the Minkowski-Funk transform in (17, A11) and the transform in (22). The Minkowski-Funk transform $\mathcal{U}^0$ describes behaviour of the function on great circles of the 2-sphere. Roughly, the transform $\mathcal{V}^0$ describes behaviour of the function on the 2-sphere except the great circles.

The transform $\mathcal{A}^0$ is a member (via analytic continuation) of an analytic family of integral operators $\mathcal{A}^\alpha = \mathcal{U}^\alpha + i \mathcal{V}^\alpha$ which arise in the study of Fourier transforms of homogeneous functions. Recently, these have been studied by Rubin.54,55 Because a detailed study of these would be distracting.

We have $XXF = [\lambda \nu/(2 \pi)] M[F^R]$, (17) and $\mathcal{U}^0 = [1/(2 \pi^{1/2})] M$, (25), hence $XXF = [\lambda \nu/(2 \pi^{1/2})] \mathcal{U}^0[F^R]$. The inverse of $\mathcal{U}^0$ is $(\mathcal{U}^0)^{-1} = \mathcal{U}^{-1}$ (Appendix A 3), hence $M^{-1} = [1/(2 \pi^{1/2})] \mathcal{U}^{-1}$. Thus

$$
F^R_\lambda(\kappa, x, \kappa) = -\frac{1}{2 \pi} \lambda \nu \int_{S^2_\theta} XFX_\lambda(\theta, x) \frac{1}{|\kappa \cdot \theta|^2} d\Omega_\theta, \quad (26)
$$

which is to be understood in a regularized sense.54,55 See equation (81) and the following discussion in Section IV D for a derivation of this inversion formula from the mathematical methods of tomography.

If we substitute (17) in (26) and interchange the order of integrations, we find

$$
F^R_\lambda(\kappa, x, \kappa) = -\frac{1}{(2 \pi)^{3/2}} \frac{1}{\nu^2} \int_{S^2_\kappa} e^{i \lambda \nu \kappa \cdot x} Q_\lambda(\kappa') s_\lambda(\lambda \nu \kappa') I(\kappa, \kappa') d\Omega_{\kappa'}. \quad (27)
$$

The integral

$$
I(\kappa, \kappa') = \int_{S^2_\theta} \frac{\delta(\kappa' \cdot \theta)}{|\kappa \cdot \theta|^2} d\Omega_\theta = -4 \pi^2 [\delta(\kappa - \kappa') + \delta(\kappa + \kappa')], \quad (28)
$$

can be evaluated using the plane wave decomposition of Dirac delta function. See Appendix A 3 a. Then the equation (27) yields $F^R_\lambda(\kappa, x, \kappa)$, (5). We can find $F_\lambda(x)$ using the inverse Radon transform (6). See Section IV D for this.

For $F(x) = e^{ik_0 x} F_0$, $|k_0| = k_0 \kappa_0$, $k_0 = \lambda \nu > 0$, $F_0 = Q_\lambda(\kappa_0)$, if we substitute (14) in (26), we find $F^R_\lambda(\kappa, x, \kappa)$, (7) with a similar reasoning as in (28), $(\kappa' \rightarrow \kappa_0)$. For the Lundquist solution (8), if we substitute (16), [that is (A9)] in (26) and use the plane wave decomposition of Dirac delta function, then we find $F^R_{L(\lambda=1)}(\kappa, x, \kappa)$, (9) for $\lambda = 1$. See Appendix A 3 b.
If we substitute (26) in (6), we find $F_{\lambda}(x)$ is given by the spherical mean of $\mathcal{X}F_{\lambda}(\theta, x)$. We shall derive this in a more direct way below, [see (43), (B35)].

As a different alternative, one can try expressing $F^{R}$ in terms of $\mathcal{X}F$: $F^{R}(\rho, \kappa) = [\mathcal{R}_{2}\mathcal{X}F(\theta, x)](\rho, \kappa)$ using a Radon transform: $\mathcal{R}_{2}$ in the plane $\theta$.\textsuperscript{19,56,57}

We can also write the Divergent beam transform (19, 24) as

$$\mathcal{D}F_{\lambda}(\theta, x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\gamma_{\nu}} \int_{R_{k}} \frac{1}{k^{2}} G(\kappa, x) e^{ik\eta} d^{3}k,$$

(29)

the equation (A17) with $Re(\alpha) = 0$, where $k = k\kappa$, $k(|k| = \kappa)$ and $\eta = \eta\theta$, $\eta = |\eta|$. It is straightforward to verify this for the field $F(x) = e^{ik_{0}x}F_{0}$, $[k_{0} = \kappa_{\theta}, k \kappa > 0, F_{0} = Q_{\lambda}(\kappa_{0})$ and $s_{\lambda}(\kappa_{\theta}) = (2\pi)^{3/2} g(\kappa_{\theta})$, using (15).

III. JOHN’ S EQUATION

The X-ray\textsuperscript{22} transform (12) [also the Divergent beam transform (13)] satisfies John’s equation\textsuperscript{18}

$$\frac{\partial^{2}}{\partial x^{j} \partial \theta^{j}} \mathcal{X}F(\theta, x) = \frac{\partial^{2}}{\partial x^{j} \partial \theta^{j}} \mathcal{X}F(\theta, x).$$

(30)

The John’s equation is the necessary and sufficient condition for a function to be expressible as an X-ray transform.\textsuperscript{22} This can be written as an ultrahyperbolic wave equation.

We can easily check (17) [also (19)] satisfies the equation (30). It is also easy to check (14) [also (15)].

**Proposition 3:** The X-ray (also Divergent beam) transform intertwines the Curl operator ($\nabla \times$):

$$\mathcal{X}[\nabla \times F](\theta, x) = (\nabla \times \mathcal{X}F)(\theta, x),$$

(31)

and also the Divergence ($\nabla \cdot$), Gradient ($\nabla$) and Laplacian ($\nabla^{2}$) operators.

**Proof:** Let $x' = x + s\theta$, then $\nabla_{x'} = \nabla_{x}$ and we have $\nabla_{x'} \times F(x') = \nabla_{x} \times F(x + s\theta)$. Then we find, the X-ray (Divergent beam) transform intertwines the Curl operator: $\mathcal{X}[\nabla \times F](\theta, x) = \nabla_{x} \times \mathcal{X}F(\theta, x)$, using the definition (12), [(13)]. The proofs for the Divergence: $\mathcal{X}[\nabla \cdot F](\theta, x) = (\nabla \cdot \mathcal{X}F)(\theta, x)$, Gradient: $\mathcal{X}[\nabla f](\theta, x) = (\nabla \mathcal{X}f)(\theta, x)$ and hence the Laplacian: $\mathcal{X}[\nabla^{2} f](\theta, x) = (\nabla^{2} \mathcal{X}f)(\theta, x)$ operators work with a similar reasoning. \(\square\)

**Proposition 4:** The X-ray (also Divergent beam) transform intertwines the operator ($\nabla \times - \nu = 0$ for constant $\nu$).

**Proof:** The X-ray (Divergent beam) transform is linear.\(\square\)

Thus the X-ray (Divergent beam) transform $\mathcal{X}F(\theta, x)$ of a Trkalian field $F(x')$, (1) is also Trkalian

$$\nabla_{x} \times \mathcal{X}F(\theta, x) - \nu \mathcal{X}F(\theta, x) = 0,$$

(32)

[ $\nabla_{x} \times \mathcal{D}F(\theta, x) - \nu \mathcal{D}F(\theta, x) = 0$ ] and $\nabla_{x} \cdot \mathcal{X}F(\theta, x) = 0$.

We also find $\nabla_{\theta} \times \mathcal{X}F(\theta, x) = (\nabla_{\theta} = s\nabla_{x} = s\nabla_{x})$ for a Trkalian field.

The X-ray transform (17) [also the Divergent beam transform (19)] satisfies (32) and $\nabla_{\theta} \cdot \mathcal{X}F(\theta, x) = 0$. We can easily check the X-ray transform $\mathcal{X}F(\theta, x)$, (14) [also the Divergent beam transform $\mathcal{D}F(\theta, x)$, (15)] of the field $F(x) = e^{ik_{0}x}F_{0}, [k_{0} = \kappa_{\theta}, k_{0} = \lambda \nu > 0, F_{0} = Q_{\lambda}(\kappa_{0})]$ is Trkalian and $\nabla_{\theta} \cdot \mathcal{X}F(\theta, x) = 0$. It is also straightforward to show that the X-ray transform $\mathcal{X}F_{L}(\theta, x)$, (16) of the Lundquist field is Trkalian and $\nabla_{\theta} \cdot \mathcal{X}F_{L}(\theta, x) = 0$.

We immediately find

$$\nabla_{\theta} \cdot (\nabla_{x} \times \mathcal{X}F) = (\nabla_{\theta} \times \nabla_{x}) \cdot \mathcal{X}F = -\nabla_{x} \cdot (\nabla_{\theta} \times \mathcal{X}F) = \nu \nabla_{\theta} \cdot \mathcal{X}F = 0,$$

(33)

using $\nabla_{\theta} \cdot \mathcal{X}F(x) = 0$, for a Trkalian field.

We find
\[ \nabla_x \times (\nabla_\theta \times \mathcal{X}F) - \nu \nabla_\theta \times \mathcal{X}F = 0, \] (34)

that is \( \nabla_\theta \times \mathcal{X}F \) is also Trkalian, using a straightforward reasoning similar to that above. This yields \( \nabla_x \cdot (\nabla_\theta \times \mathcal{X}F) = 0 \) as we expect, (33). We can also prove the equation (34) using John’s equation (30).

It is straightforward to verify (34) respectively for the X-ray transforms (14) of the field \( F(x) = e^{i k_0 \cdot x} F_0, [k_0 = k_0 \kappa_0, k_0 = \lambda \nu > 0, F_0 = \mathcal{Q}_\lambda (\kappa_0)] \) and (16) of the Lundquist field.

**Proposition 5:** For a Trkalian field the John’s equation (30) is equivalent (both necessary and sufficient) to \( \partial_{x_m} \epsilon_{jki} \partial_{\theta_n} (\mathcal{X}F)_j = \nu \partial_{\theta_m} (\mathcal{X}F)_k \) that is \( \partial_{x_m} (\nabla_\theta \times \mathcal{X}F) = \nu \partial_{\theta_m} \mathcal{X}F \).

**Proof:** We can write (30) as \( \nabla_\theta \times \nabla_x \mathcal{X}A = 0, \nabla_\theta \times \nabla_x \mathcal{X}B = 0, \nabla_\theta \times \nabla_x \mathcal{X}C = 0 \) where \( \mathcal{X}F = (\mathcal{X}A, \mathcal{X}B, \mathcal{X}C) \).

The proof is straightforward writing these and (32) and \( \nabla_\theta \cdot \mathcal{X}F(\theta, x) = 0 \) in components.

The difference of symmetric \((m \leftrightarrow k)\) equations in this yields (34). The sum of diagonal equations \((m = k)\) yields \( \nabla_x \cdot (\nabla_\theta \times \mathcal{X}F) = \nu \nabla_\theta \cdot \mathcal{X}F \) which identically vanishes, (33).

We also have \( \nabla_\theta \times (\nabla_x \times \mathcal{X}F) - \nu \nabla_\theta \times \mathcal{X}F = 0, \) (34). Then \( \nabla_x \cdot (\nabla_\theta \times \mathcal{X}F) = 0 \) trivially.

The X-ray transform and its formal adjoint

\[ \mathcal{X}^\dagger [G](x) = \int_{S^2_{\theta}} G(\theta, E_\theta x) d\Omega_\theta, \quad E_\theta x = x - (x \cdot \theta) \theta \] (35)

where \( [G(\theta, x) \text{ is to be taken in the range of X-ray transform;}] \ G(\theta, x) = \mathcal{X}[F](\theta, x), \) are related to the Riesz potential of order \( \alpha \) through \( \mathcal{T}^\dagger [F](x) = [1/(2\pi)^2] \mathcal{X}^\dagger \mathcal{T}^{\alpha-1} \mathcal{X}[F](x). \) Here \( \mathcal{T}^{\alpha-1} \) is the Riesz potential on \( S^2 \) acting on the second variable \( x \) of \( \mathcal{X}[F](\theta, x) \) and \( 0 < \alpha < n = 3. \) For \( \alpha = 1 \) this yields

\[ \mathcal{T}^\dagger [F](x) = \frac{1}{(2\pi)^2} \mathcal{X}^\dagger \mathcal{X}[F](x), \] (36)

where

\[ \mathcal{T}^\dagger [F](x) = \frac{1}{2\pi^2} \int \frac{F(y)}{|x - y|^2} d^3y. \] (37)

It is straightforward to prove

\[ \int_{S^2_{\theta}} \mathcal{D}F(x, \theta) d\Omega_\theta = 2\pi^2 \mathcal{T}^{\dagger -1} [F](x), \] (38)

see Ref. 58, p. 283. This leads to

\[ \mathcal{T}^{\dagger -1} [F](x) = \frac{1}{4\pi^2} \int_{S^2_{\theta}} \mathcal{X}F(\theta, x) d\Omega_\theta, \] (39)

using\(^{58} \theta \to - \theta \) in (38). We could infer this from (36) using invariance of \( \mathcal{X}F(\theta, x) \) under translation of \( x \) in the direction \( \theta: \mathcal{X}F(\theta, E_\theta x) = \mathcal{X}F(\theta, x). \) We can use equation (39) for inverting the X-ray transform.

**Proposition 6:** The inversion of X-ray transform intertwines the Curl \((\nabla \times)\) and also the Divergence \((\nabla \cdot)\), Gradient \((\nabla)\) and Laplacian \((\nabla^2)\) operators.

**Proof:** We can easily show: \( \nabla_x \times \mathcal{T}^{\dagger -1} [F](x) = \mathcal{T}^{\dagger -1} [\nabla \times \mathcal{X}F](x) \) for \( F(x) \) in the Schwartz class, using the Fourier transform of Riesz potential \( \mathcal{F} \mathcal{T}^{\dagger -1} [F](\xi) = |\xi|^{-\alpha} \mathcal{F} \mathcal{X}[F](\xi), \) \( \alpha < n = 3. \)\(^{33,49,58}\) We also have \( \mathcal{T}^{\dagger -1} \mathcal{T}^{\dagger -\alpha} = 1. \)\(^{49}\) Thus

\[ \mathcal{T}^{\dagger -1} \left[ \frac{1}{4\pi^2} \int_{S^2_{\theta}} \nabla_x \times \mathcal{X}F(\theta, x) d\Omega_\theta \right] = \nabla \times F, \] (40)

(39), \( \nabla_x \times \mathcal{X}F(\theta, x) \) satisfies (30), if \( \mathcal{X}F(\theta, x) \) does. The inversion of X-ray transform also intertwines the Divergence \((\nabla \cdot)\), Gradient \((\nabla)\) and hence the Laplacian \((\nabla^2)\) operators with a similar reasoning. \( \square \)
Proposition 7: The inversion of X-ray transform intertwines the operator \((\nabla \times) - \nu = 0\) for constant \(\nu\).

Proof: The transforms \(\mathcal{I}^{-1}\) and \(\mathcal{X}^1\) are linear. Thus the Trkalian subclass \((32)\) of functions \(\mathcal{X}\mathcal{F}(\theta, x)\) satisfying John’ s equation \((30)\) yields Trkalian fields in the physical space.

The propositions 3, 4, 5, 6 and 7 enable us to study Trkalian fields either in physical space or in the transform space.

Proposition 8: The Riesz potential for a Trkalian field is given by

\[
\mathcal{T}^\alpha[F_\lambda](x) = (\lambda^\nu)^{-\alpha}F_\lambda(x),
\]

where \(\alpha < 3\).

Proof: We are led to

\[
\mathcal{F}\{\mathcal{T}^\alpha[F]\}(\xi) = \frac{\delta(\xi - \lambda\nu)}{\xi^{\alpha+2}}Q_\lambda(u)s_\lambda(\lambda^\nu u),
\]

substituting the inverse spherical Curl transform \((6)\) into: \(\mathcal{F}\{\mathcal{T}^\alpha[F]\}(\xi) = |\xi|^{-\alpha}\mathcal{F}[F](\xi), \alpha < n = 3\) and using \(5\).

The result follows by inverting this. \(\square\)

The Riesz potential \((37)\) for \(F(x) = e^{ik_0 x}F_0, |k_0| = k_0|\mathbf{k}_0|, k_0 = \lambda^\nu > 0\) and the Lundquist field \((8)\) respectively lead to these fields themselves. We shall not present the details of these calculations here.

Proposition 9: The spherical mean of the X-ray (or Divergent beam) transform of a Trkalian field over all lines passing through a point yields the field at this point

\[
\int_{S^2_0} X_F(\theta, x) d\Omega_\theta = 4\pi^2 \frac{1}{\lambda^\nu}F_\lambda(x).
\]

Proof: The result follows from \((39)\) and Proposition 8. [Divergent beam transform: \(\mathcal{X}\mathcal{F}(\theta, x) = DF(\theta, x) + D\mathcal{F}(-\theta, x)\), the integrals of Divergent beam transforms in opposite directions are equal. See equation \((82)\) below.]

This provides us a simple inversion formula for Trkalian fields. This proposition is geometrically motivated by equation \((79)\) below in Section IV D: Gelfand-Goncharov’ s method. The underlying geometry is again based on the simplification in the same intricate geometric quantity: Hilbert transform of the derivative of Radon transform in the Moses basis.

The spherical mean of the X-ray transform \((14)\) yields the field \(F(x) = e^{ik_0 x}F_0,\) decomposing \(\theta\) into components which are respectively parallel and orthogonal to \(\mathbf{k}_0\). If we substitute \((17)\) in \((43)\), we find \((6)\) with a similar reasoning.

An analogous result for the Radon transform of Trkalian fields: \(\mathcal{T}^2[F_\lambda] = [1/(8\pi^2)]\mathcal{R}^\dagger\mathcal{R}[F_\lambda] = (1/\nu^2)F_\lambda\) also follows from equations \((34, 40)\) in Ref. 1.

IV. MATHEMATICAL METHODS OF TOMOGRAPHY

We shall use four basic mathematical approaches of tomography due to Grangeat, Tuy, Smith and Gelfand-Goncharov. These are based on relations of the X-ray and Divergent beam transforms to Hilbert transform of the derivative of Radon transform. Mathematically, these relations are outflow of the formula

\[
\int_{S^2_0} DF(\theta, x)h(\theta \cdot b)d\Omega_\theta = \int F^R(p, b)h(p - b \cdot x)dp,
\]

equations \((44)\) essentially obtained in Ref. 32, 33. Here we have introduced the normalization: \(\beta = \beta b, \beta = |\beta|, s = \beta p, F^R(s, \beta) = (1/\beta)F^R(p, b)\) for the sake of our conventions. \(33\) The distribution \(h\) satisfies \(h(ap) = (1/a^2)h(p), a > 0\).

We provide the mathematical derivations of these formulas for the sake of a self-contained presentation with a unique, consistent convention. See Appendix B 1 for a derivation of \((44)\) following Ref. 48, p. 276 and also Ref. 59. For a unified discussion of these approaches see Refs. 59, 60.
For the attentive reader, we remark that history has followed a path different from the mathematically logical one. These mathematical methods were developed independent of equation (44), only using practical tomographical ideas. We shall adopt a mathematical approach rather than a tomographical implementation.

The first purpose of this section is to provide a unified geometric motivation and intuition for the previous results in Sections II and III. The second purpose is to present a discussion of these mathematical methods with a view towards tomographical studies of Trkalian field models in nature. For this purpose, we shall study these mathematical methods using Trkalian fields. Especially for the sake of the second purpose and also for a clean presentation, this discussion was postponed until the direct (but unmotivated) discussion of the geometry of X-ray and Divergent beam transforms in Sections II and III finished.

These methods basically make use of the Radon inversion (for tomographical reconstruction). They lead us to new inversion formulas for the X-ray and Divergent beam transforms of Trkalian fields with a view towards tomographical applications. They also provide the geometric motivation underlying the interrelations of the transforms mentioned.

A. Grangeat’s method

We obtain Grangeat’s formula\(^28\)

\[
\frac{\partial}{\partial p} F^\mathcal{R}(p, \kappa) \bigg|_{p=\kappa \cdot x} = - \int_{S^2_\theta} \mathbf{D} F(\theta, x) \delta'(\kappa \cdot \theta) d\Omega_\theta, \tag{45}
\]

using \(h(p) = \delta'(p)\) in (44). This is a componentwise\(^33\) generalization of the formula for scalar fields. For a derivation of this formula see Appendix B 2 a.\(^59,61\) Thus we find

\[
\frac{\partial}{\partial p} F^\mathcal{R}(p, \kappa) \bigg|_{p=\kappa \cdot x} = \int_{S^2_{\theta|x=\kappa}} \frac{\partial}{\partial \kappa} \mathbf{D} F(\theta, x) d\theta, \tag{46}
\]

using the identity (B3).\(^33\) Here \(\partial/\partial \kappa\) denotes directional derivative along \(\kappa\).

We find

\[
F^\mathcal{R}(\kappa \cdot x, \kappa) = - \frac{1}{\nu} \kappa \times \int_{S^2_\theta} \mathbf{D} F(\theta, x) \delta'(\kappa \cdot \theta) d\Omega_\theta, \tag{47}
\]

for a Trkalian field using (3, 45). We also have \(\kappa \cdot \int_{S^2_\theta} \mathbf{D} F(\theta, x) \delta'(\kappa \cdot \theta) d\Omega_\theta = 0, (4, 45)\). The equation (47) leads to

\[
F^\mathcal{R}(\kappa \cdot x, \kappa) = - \frac{1}{2} \nu \kappa \times \int_{S^2_\theta} \mathbf{Y} F(\theta, x) \delta'(\kappa \cdot \theta) d\Omega_\theta, \tag{48}
\]

using \(\theta \rightarrow -\theta\) and rearranging.

We can check (45) for \(F(x) = e^{ik_0 \cdot x} F_0\), \([k_0 = k_0 \kappa_0, k_0 > 0]\). See Appendix B 2 b. If we substitute \(\mathbf{Y} F_\lambda(\theta, x), (23)\) in (48), we are similarly led to \(F^\mathcal{R}(\kappa \cdot x, \kappa), (7)\).

If we substitute (47) [or (48)] into the first line of (6), we find

\[
F(x) = - \frac{1}{8\pi^2} \int_{S^2_\theta} \kappa \times \int_{S^2_\theta} \mathbf{D} F(\theta, x) \delta'(\kappa \cdot \theta) d\Omega_\theta d\Omega_\kappa. \tag{49}
\]

This leads to

\[
F(x) = \pm \frac{1}{4\pi} \nu \int_{S^2_\theta} \theta \times \mathbf{D} F(\pm \theta, x) d\Omega_\theta, \tag{50}
\]

interchanging the order of integrations and using the identity (B3), (also \(\theta \rightarrow -\theta\)).

This is another simple, direct inversion (reconstruction) formula for the Divergent beam transform of Trkalian fields. Note that the simplification is basically due to the eigenvalue equation for the Radon transform of Trkalian fields. We shall not discuss the tomographical implementation of this inversion formula.

We can verify the formula (50) using (19,20). See Appendix B 2 c. If we substitute \(\mathbf{D} F(\pm \theta, x), (15)\) in (50), we similarly find \(F(x) = e^{ik_0 \cdot x} F_0, (\kappa \rightarrow k_0)\).
B. Smith’s method

Smith’s formula\(^{29}\) can be written in various ways.\(^{33,59,62}\) We shall follow Ref. 33, p. 24 (correcting misprints there) and Ref. 62. We extend the X-ray transform (12) as a function

\[ gF(\alpha, x) = \int_{-\infty}^{\infty} F(x + t\alpha)dt = \frac{1}{\alpha} \chi F(\theta, x), \]  

homogeneous of degree \(-1\), using a non-unit vector \(\alpha = \alpha \theta, \alpha = |\alpha|, s = \alpha t\). Then its Fourier transform is

\[ \mathcal{G}F(\beta, x) = \mathcal{F}[gF(\alpha, x)](\beta, x) = \frac{1}{(2\pi)^{3/2} \beta^2} \int_{S^2} \mathcal{D}F(\theta, x) h(\theta \cdot b)d\Omega_\theta. \]  

See Appendix B 3. Here

\[ h(p) = \int_{\alpha=-\infty}^{\infty} |\alpha|e^{-i\alpha p}d\alpha, \quad (p = \theta \cdot b) \]  

where \(h(ap) = (1/a^2)h(p), a > 0\) and \(h(-p) = h(p)\).

Thus, we find

\[ \mathcal{G}F(\beta, x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\beta^2} \int F^R(p, b)h(p - b \cdot x)dp, \]  

using (44). This leads to Smith’s formula

\[ \mathcal{G}F(\beta, x) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\beta^2} [\mathcal{H}\partial_p F^R(p, b)](b \cdot x, b), \]  

for vector fields, using the identity

\[ \int F^R(p, b)h(p - p_0)dp = 2\pi [\mathcal{H}\partial_p F^R(p, b)](p_0, b), \quad (p_0 = b \cdot x). \]  

Here \(\mathcal{H}\) is the Hilbert transform\(^{33}\) defined (on \(\mathbb{R}\)) by the principal-value integral

\[ \mathcal{H}[r(y)](x) = \frac{1}{\pi} \int \frac{r(y)}{x-y}dy. \]  

We can prove this identity noting \(h(p - p_0) = 2\partial_p[1/(p - p_0)]\). We also have

\[ [\mathcal{H}\partial_p r(y)](x) = \frac{1}{\pi} \int \frac{\partial_y r(y)}{x-y}dy = -\frac{1}{\pi} \int \frac{r(y)}{(x-y)^2}dy. \]  

If we invert (52) using (55) and (51), we find

\[ \chi F(\theta, x) = \alpha \mathcal{F}^{-1}[\mathcal{G}F(\beta, x)](\alpha, x) = \frac{1}{4\pi} \int_{S^2} [\mathcal{H}\partial_p F^R(p, b)](b \cdot x, b)\delta(b \cdot \theta)d\Omega_b, \]  

using \(\beta = \beta b \Rightarrow d^3\beta = \beta^2 d\beta d\Omega_b, \alpha = \alpha \theta\) and \([\mathcal{H}\partial_p F^R(p, b)](b \cdot x, b)\) is even under \(b \rightarrow -b\). Hence, the X-ray transform is in the form of a Minkowski-Funk transform: \(X F(\theta, x) = [1/(2\pi^{1/2})]\mathcal{U}^\alpha[(\mathcal{H}\partial_p F^R(p, b)](b \cdot x, b)\} \{\theta, x\}, (25)\) of \(\mathcal{H}\partial_p F^R(p, b)\).
For a Trkalian field \( (3), \mathcal{G}F(\beta, x), \) (54) satisfies: \( \nabla_x \times \mathcal{G}F(\beta, x) - \nu \mathcal{G}F(\beta, x) = 0, \) [equivalently from (52, 51) and (32)]. The equation (59) reduces to

\[
\mathcal{X}F(\theta, x) = -\frac{1}{4\pi} \nu \int_{S^2_b} b \times [\mathcal{H}F^R(p, b)][(b \cdot x, b)]\delta(b \cdot \theta) d\Omega_b,
\]

(60)

using (4). We find

\[
[\mathcal{H}\partial_p F^R_X(p, b)](p_0, b) = -\nu b \times [\mathcal{H}F^R_X(p, b)](p_0, b) = \lambda \nu F^R_X(p_0, b), \quad (p_0 = b \cdot x)
\]

(61)

using the Moses basis, (5). Thus (59), (60) reduce to (17), \( (b \rightarrow \kappa) \) which can be directly proven as in Proposition 1 in Section II. This can be inverted, for example using (26). The simplification here is basically due to the eigenvalue equation and the Hilbert transform of the derivative for the Radon transform of Trkalian fields in the Moses basis.

Smith’s inversion method\(^{62}\) makes use of the intermediate function

\[
K(\omega, \beta) = \int \mathcal{F}[F(x)](\tau\beta)\tau|e^{i\omega\tau}|d\tau = \frac{1}{(2\pi)^{3/2}} \int F^R(s, \beta)h(s - \omega) ds,
\]

(62)

\( \mathcal{G}F(\beta, x) = K(\beta \cdot x, \beta), \) (see also Ref. 59). Here we have used the Fourier slice theorem: \( \mathcal{F}[\mathcal{F}^R(s, \beta)](\tau, \beta) = 2\pi \mathcal{F}[\mathcal{F}^R(x)](\tau\beta). \)\(^1\) This leads to \( K(\omega, \beta) = [1/(2\pi)^{1/2}] [1/\beta^2] (\mathcal{H}\partial_p F^R(p, b)](\omega/\beta, b) \) using (66) and \( K_\lambda(\omega, \beta) = [1/(2\pi)^{1/2}] [1/\beta^2] \lambda \nu F^R_X(\omega/\beta, b) \) using (61), [or appropriately using (6) in (62)] for a Trkalian field. Hence \( K_\lambda(\beta \cdot x, \beta) = [1/(2\pi)^{1/2}] [1/\beta^2] \lambda \nu F^R_X(\beta \cdot x, b, b). \) Then the inversion formula\(^{62}\)

\[
F_\lambda(x) = \frac{1}{(2\pi)^{3/2}} \int K_\lambda(\omega, \beta)e^{-i(\omega - \beta \cdot x)} d\omega d^3\beta,
\]

(63)

can be expressed in terms of the Radon transform of the field. This reduces to (6). We shall not discuss its tomographical implementation.

C. Tuy’s method

We shall follow Ref. 59 for Tuy’s approach.\(^{30}\) We extend the Divergent beam transform (13) as a function

\[
gF(\alpha, x) = \int_0^\infty F(x + t\alpha) dt \frac{1}{\alpha} \mathcal{D}F(\theta, x),
\]

(64)

homogeneous of degree \(-1\), using a non-unit vector \( \alpha = \alpha \theta, \alpha = |\alpha|, s = \alpha t. \) Then its Fourier transform is

\[
\mathcal{G}F(\beta, x) = \mathcal{F}[gF(\alpha, x)](\beta, x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\beta^2} \int_{S^2_b} \mathcal{D}F(\theta, x)f(\theta \cdot b) d\Omega_b,
\]

(65)

See Appendix B 4. Here \( f(p) = 2\pi i \partial_p \delta^-(p), (p = \theta \cdot b) \) where \( f(ap) = (1/\alpha^2) f(p), a > 0 \) and \( f(-p) = -2\pi i \partial_p \delta^+(p). \) Thus we find

\[
\mathcal{G}F(\beta, x) = \frac{1}{(2\pi)^{3/2}} \frac{1}{\beta^2} \int \mathcal{F}^R(p, b)f(p - b \cdot x) dp,
\]

(66)

using (44). This leads to

\[
\mathcal{G}F(\beta, x) = \frac{\pi}{(2\pi)^{3/2}} \frac{1}{\beta^2} \left\{ [\mathcal{H}\partial_p F^R_X(p, b)](b \cdot x, b) - i \mathcal{H}\partial_p F^R_X(p, b)](b \cdot x, b) \right\},
\]

(67)

using the identity
We can prove this identity noting $f(p - p_0) = 2\pi i \partial_p \delta^-(p - p_0)$, upon simple manipulations.

If we invert (65) using (67) and (64), we find

$$\mathcal{D}F(\theta, x) = \alpha \mathcal{F}^{-1}[\mathcal{G}F(\beta, x)](\alpha, x) = \frac{\pi}{(2\pi)^2} \int_{S^b_\theta} \left\{ \left[ (\mathcal{H} - i) \partial_\beta \right] F^R(p, b) \right\}(p, b) \delta^+(b \cdot \theta) d\Omega_b,$$

(69)

using $\beta = \beta b \Rightarrow \delta^\beta_\beta = \beta^2 d\beta d\Omega_b$, $\alpha = \alpha \theta$ and $\mathcal{F}[H(-\beta')](p) = \sqrt{2\pi} \delta^+(p)$, $\beta' = \alpha \beta$. Hence, the Divergent beam transform is in the form of the integral transform: $\mathcal{D}F(\theta, x) = [1/(4\pi^{1/2})] A^0 \left\{ \left[ (\mathcal{H} - i) \partial_\beta \right] F^R(p, b) \right\}(b \cdot x, b)](\theta, x)\}, (25)$ of $[(\mathcal{H} - i) \partial_\beta] F^R(p, b)$. An easy check reveals $[\partial_\nu F^R(p, b)](b \cdot x, b)$ is odd under $b \rightarrow -b$ while $[\mathcal{H}\partial_\nu F^R(p, b)](b \cdot x, b)$ is even. Therefore only the even terms survive the integration. Thus we find

$$\mathcal{D}F(\theta, x) = \frac{1}{2} \chi F(\theta, x) + \frac{1}{2(2\pi)^2} \int_{S^b_\theta} \left[ \partial_\nu F^R(p, b) \right](b \cdot x, b) \frac{1}{b \cdot \theta} d\Omega_b,$$

(70)

using (59). The second term is associated with the difference

$$\mathcal{Y}F(\theta, x) = \frac{1}{(2\pi)^2} \int_{S^b_\theta} \left[ \partial_\nu F^R(p, b) \right](b \cdot x, b) \frac{1}{b \cdot \theta} d\Omega_b,$$

(71)

(21).

For a Trkalian field (3), $\mathcal{G}F(\beta, x)$, (66) satisfies: $\nabla_x \times \mathcal{G}F(\beta, x) - \nu \mathcal{G}F(\beta, x) = 0$, [equivalently from (65, 64) and (32)]. The equation (70) reduces to

$$\mathcal{D}F(\theta, x) = \frac{1}{2} \chi F(\theta, x) - \frac{1}{2(2\pi)^2} \nu \int_{S^b_\theta} b \times F^R(b \cdot x, b) \frac{1}{b \cdot \theta} d\Omega_b,$$

(72)

using (4). This leads to

$$\mathcal{Y}F(\theta, x) = -\frac{1}{(2\pi)^2} \nu \int_{S^b_\theta} b \times F^R(b \cdot x, b) \frac{1}{b \cdot \theta} d\Omega_b.$$

(73)

The equations (70, 72) reduce to (19), $(b \rightarrow \kappa)$ using the Moses basis (5), [also (17)] which can be directly proven as in Proposition 2 in Section II. Similarly (71, 73) reduce to (22). Note

$$\left\{ [(\mathcal{H} - i) \partial_\beta] F^R(\beta, b) \right\}(p_0, b) = -\nu b \times [(\mathcal{H} - i) F^R(\beta, b)](p_0, b)$$

$$= 2(2\pi)^{1/2}(1/g)(1/\lambda \nu) e^{i\lambda \nu p_0} Q_\lambda(b)s_\lambda(\lambda \nu b), \quad (p_0 = b \cdot x)$$

(74)

in the Moses basis. The simplification here is also due to the eigenvalue equation and the Hilbert transform of the derivative for the Radon transform of Trkalian fields in the Moses basis.

It is straightforward to find the difference for Lundquist field ($\lambda = 1$)

$$\mathcal{Y}F_{L(\lambda = 1)}(\theta, x) = -2F_0 \frac{1}{\nu \nu_r} \left\{ 2 \sum_{k=1}^{\infty} \sin[2k(\theta - \phi)] J_{2k}(\nu r) e_r(\theta) - J_0(\nu r) e_\theta \right\}$$

$$+ 2 \sum_{k=0}^{\infty} \cos[(2k + 1)(\theta - \phi)] J_{2k+1}(\nu r) e_2,$$

(75)

substituting the Radon transform (9) into (71). See Appendix B5. Then, we find the Divergent beam transform
\[ DF_{\lambda=1}(\theta, x) = F_0 \frac{1}{\nu r} \left\{ -2 \sum_{n=1}^{\infty} (-1)^n \sin[n(\theta - \phi)]J_n(\nu r)e_n(\theta) + J_0(\nu r)e_\theta \right\} \]

\[ + \left[ J_0(\nu r) + 2 \sum_{n=1}^{\infty} (-1)^n \cos[n(\theta - \phi)]J_n(\nu r) \right] e_z \],

of the Lundquist field using (B33) in (16) and then in (70). See p. 23 and p. 538 in Ref. 63 for these series.

Tuy’s inversion method makes use of the intermediate function

\[ K(\omega, \beta) = \frac{1}{2} \int \mathcal{F}\{F(x)\}(\tau \beta)(|\tau| + \tau)e^{i\omega\tau} d\tau = \frac{1}{(2\pi)^{3/2}} \int \mathcal{F}^R(s, \beta)f(s - \omega)ds, \]

\[ \mathcal{G}F(\beta, x) = K(\beta \cdot x, \beta). \] This yields \( K(\omega, \beta) = \frac{1}{(2\pi)^{3/2}}[1/\beta^2]I(\omega/\beta, b) \) where \( I(p, b) = \pi [(\mathcal{H} - i)\partial_p \mathcal{F}^R(q, b)] \) \( (p, b), \) [see (67)]. For a Trkalian field in the Moses basis \( \mathcal{G}F(\beta, x) = K(\beta \cdot x, \beta) = \frac{1}{(2\pi)^{3/2}}[1/\beta^2]I(\beta \cdot x, b) = \frac{1}{(2\pi)^{3/2}}[1/\beta^2] \{(\mathcal{H} - i)\partial_p \mathcal{F}^R(p, b)\}(\beta \cdot x, b) = (1/\nu)(1/\nu^2)e^{i\nu^2}Q_{\lambda}(\lambda \nu) \) \( \mathcal{F}_\lambda(\beta \cdot x, b) = (1/\nu)(1/\nu^2)e^{i\nu^2}Q_{\lambda}(\lambda \nu) \).

The inversion formula makes use of the intermediate function

\[ F(x) = \frac{1}{(2\pi)^3} \frac{1}{i} \int_{S^2_0} \int I(p, b)\delta'(p - b \cdot x)dpd\Omega_b, \]

can be expressed in terms of the spherical Curl transform of the field. This reduces to (6). We shall not discuss its tomographical implementation.

D. Gelfand-Goncharov’s method

If we use \( h(p) = 1/p^2, p = \theta \cdot b, \) then (44) leads to

\[ [\mathcal{H}\partial_p \mathcal{F}^R(p, b)](\beta \cdot x, b) = -\frac{1}{\pi} \int_{S^2_0} \frac{DF(\theta, x)}{(\theta \cdot b)^2} d\Omega_\theta. \]

see (58). This can also be inferred from (52, 55) and (B14).

For a Trkalian field (4), this yields

\[ b \times [\mathcal{H}\mathcal{F}^R(p, b)](\beta \cdot x, b) = \frac{1}{\nu} \int_{S^2_0} \frac{DF(\theta, x)}{(\theta \cdot b)^2} d\Omega_\theta. \]

This reduces to

\[ \mathcal{F}_\lambda^R(\beta \cdot x, b) = \frac{1}{\pi} \frac{1}{\lambda \nu} \int_{S^2_0} \frac{DF(\theta, x)}{(\theta \cdot b)^2} d\Omega_\theta, \]

using the Moses basis, (61). The simplification here is again due to the eigenvalue equation and the Hilbert transform of the derivative for the Radon transform of Trkalian fields in the Moses basis.

We can write similar formulas with \( X\mathcal{F}(\theta, x) \) using the substitution \( \theta \rightarrow -\theta \) through the equations (79, 80, 81). This leads to Semyanistyi’s inversion formula (26) in Section II.

If we substitute the equation (81), \( (b \rightarrow \kappa) \) in (6), we find

\[ \int_{S^2_0} DF(\theta, x)d\Omega_\theta = 2\pi^2 \frac{1}{\lambda \nu} F_\lambda(x), \]

which can be directly proved using the Riesz potential as in Proposition 9 in Section III. See Appendix B.6. We shall not discuss its tomographical implementation.
We also see that the substitution of Semyanistyi’s inversion formula (26) in the Radon inversion (6) leads to the inversion through spherical mean in Proposition 9.

If we use (15) in (81), the second term vanishes and we find

\[ F^R_\lambda (b \cdot x, b) = -\frac{1}{\nu^2} e^{i \kappa_0 \kappa_0' x} I(b, \kappa_0) F_0, \]  

(83)

where the integral \( I(b, \kappa_0) \) is given in (28), \((\kappa \rightarrow b, \kappa' \rightarrow \kappa_0)\). Then we find \( F^R_\lambda (b \cdot x, b) \), (7). Similarly, we are led to (5) substituting (19) in (81).

If we substitute (76) in (82) we are led to (8), \((\lambda = 1)\).

V. RIESZ POTENTIAL AND BIOT-SAVART INTEGRALS

We can write the X-ray transform of Riesz potential (of order 2) and Biot-Savart (BS) integrals using (59) in terms of their\(^1\) Radon transform.

If we replace \( F \rightarrow \mathcal{I}^2 [F] = 1/(8\pi^2) \mathcal{R}^\dagger \mathcal{R} [F] \Rightarrow F^R \rightarrow \mathcal{R} [\mathcal{I}^2 [F]] \) in (59) using (10), we find

\[ \mathcal{X} \mathcal{I}^2 [F](\theta, x) = \frac{1}{8\pi^2} \mathcal{X} \mathcal{R}^\dagger \mathcal{R} [F](\theta, x) = \frac{1}{\nu^2} \mathcal{X} F(\theta, x). \]  

(85)

We can easily verify this substituting (5) in (84) and comparing with (17).

The equation (11) yields \( \partial_\nu \mathcal{B} \mathcal{S} [F^R(p, \kappa)](p, \kappa) = -\kappa \times F^R(p, \kappa) \). If we replace \( F \rightarrow \mathcal{B} \mathcal{S} [F] = \nabla \times \mathcal{I}^2 [F] \Rightarrow F^R \rightarrow \mathcal{B} \mathcal{S} [F^R] \) in (59) using this, we find

\[ \mathcal{X} \mathcal{B} \mathcal{S} [F](\theta, x) = -\frac{1}{4\pi} \int_{S^2} \kappa \times \mathcal{H} F^R(p, \kappa) (\kappa \cdot x, \kappa) \delta(\kappa \cdot \theta) d\Omega_\kappa. \]  

(86)

Thus \( \mathcal{X} \mathcal{B} \mathcal{S} [F] = (1/4\pi) \mathcal{M} \{ \mathcal{H} \partial_\nu \mathcal{B} \mathcal{S} [F^R(p, \kappa)](p, \kappa) \} = -1/(4\pi) \mathcal{M} \{ \kappa \times [\mathcal{H} F^R(p, \kappa)] \} \). This is again in the form of a Minkowski-Funk transform. We call this John-Biot-Savart integral. Further, one can express \( F^R \) in terms of \( \mathcal{X} F \).\(^{49,56,57}\) The equation (86) also follows from (84): \( \mathcal{X} \mathcal{B} \mathcal{S} [F] = \mathcal{X} (\nabla \times \mathcal{I}^2 [F]) = \nabla \times \mathcal{X} \mathcal{I}^2 [F] \), (Proposition 3).

For Trkalian fields (4), we find

\[ \mathcal{X} \mathcal{B} \mathcal{S} [F](\theta, x) = \frac{1}{4\pi \nu} \int_{S^2} [\mathcal{H} \partial_\nu F^R(p, \kappa)](\kappa \cdot x, \kappa) \delta(\kappa \cdot \theta) d\Omega_\kappa = \frac{1}{\nu} \mathcal{X} F(\theta, x), \]  

(87)

(59) as we expect, since: \( \mathcal{B} \mathcal{S} [F] = (1/\nu) F \).

We can write \( \mathcal{D} \mathcal{B} \mathcal{S} [F](\theta, x) \) and \( \mathcal{Y} \mathcal{B} \mathcal{S} [F](\theta, x) \) integrals respectively using (69, 70) and (71), with a similar reasoning. For example, we find

\[ \mathcal{Y} \mathcal{B} \mathcal{S} [F](\theta, x) = -\frac{1}{(2\pi)^2} \int_{S^2} \kappa \times F^R(\kappa \cdot x, \kappa) \frac{1}{\kappa \cdot \theta} d\Omega_\kappa. \]  

(88)

We have \( \mathcal{Y} \mathcal{B} \mathcal{S} [F](\theta, x) = (1/\nu) \mathcal{Y} F(\theta, x) \), (73) for Trkalian fields.

These, together with the Radon transform,\(^4\) lead to an integral geometric understanding of these integrals. However, a physical or tomographical discussion of these integrals is beyond the scope of this manuscript.
VI. MINI-TWISTOR REPRESENTATION

The mini-twistor space as an intrinsic structure have been introduced by Hitchin. The Twistor theory, in simplest terms, is based on writing contour integral solutions for the wave equation in (3+1) dimensional Minkowski space, using holomorphic functions. We can write the solution of Helmholtz equation as a time-harmonic reduction of this, using mini-twistor space variables. See Appendix C.1. The quotient of twistor space (CP(1) \ CP(1)) of the Minkowski space by the action of time translation yields the mini-twistor space TCP(1).

The (mini-)twistor space of R^3 is the space TS^2 = \{(u, v) \in S^2 \times R^3, u, v \in R^3, |u| = 1, v \cdot u = 0\} \subset S^2 \times R^3 of oriented lines \( p = v + tu \) where \( u \) is the direction vector, \( v \) is the position (shortest) vector of \( l \) and \( p \) denotes a point of this line. This has a natural complex structure which can be identified with the holomorphic tangent bundle TCP^1 of projective line: the Riemann sphere CP^1, with local coordinates \((\eta, \omega)\), \([on S^2 - \{N\}, u \neq (0,0,1)]\). Here \( u \in S^2 \sim CP^1, \omega \) is the coordinate on the base CP^1 and \( \eta \) denotes the fiber coordinate which describes a holomorphic section.

We can regard a point \( p \) in R^3 as the intersection of all oriented straight lines through it which are parametrised by a 2-sphere S^2 in TS^2 \sim TCP^1. More precisely, each point \( p \) corresponds to a holomorphic section of TCP^1. These are fixed by an involutive map \( \tau \) on TS^2, \( \tau^2 = 1 \) reversing the orientation of lines which is called the real structure. The incidence relation between a point \( p(x, y, z) \) and a twistor \((\eta, \omega)\) that defines this section is given by \( \eta = (1/2) [(x + iy) + 2\omega - (x - iy)\omega^2] \). The set of twistors incident with a given point (the set of lines passing through this point) form a copy of CP^1 which lies as a real section of TCP^1. If we hold \((\eta, \omega)\) fixed, then \((x, y, z)\) satisfying the incidence relation defines a line in R^3. If we hold \((x, y, z)\) fixed, then \((\eta, \omega)\) satisfying the incidence relation parametrises the set of all lines through the point \( p(x, y, z) \). The local coordinates on S^2 - \{S\}, \([u \neq (0,0,-1)]\) are given by \( \omega' = 1/\omega, \eta' = -\eta/\omega^2 = (1/2) [(x - iy) - 2z\omega' - (x + iy)\omega^2] \). We shall ignore the factor 1/2 in \( \eta \).

We shall restrict a function defined on a domain of the mini-twistor space to a (projective) line and then integrate along a closed contour contained in this CP^1. We shall use (mini-)twistor solution of the Helmholtz equation for finding Trkalian fields. A Trkalian field (1) also satisfies the vector Helmholtz equation

\[
\nabla^2 F(x) = -k^2 F(x), \quad k = \nu
\]

but the converse is not necessarily true. We can write a solution of this equation as

\[
F(x) = [A(x), B(x), C(x)] = \int_C e^{-ikf(L, M, N)} d\omega,
\]

where \(L(\eta_\omega(\omega), \omega), M(\eta_\omega(\omega), \omega), N(\eta_\omega(\omega), \omega)\) are holomorphic functions of \(\eta_\omega(\omega) = z + iy + 2\omega - (x - iy)\omega^2\) and \(\omega\). Here \(f = \omega(x - iy) - z\) is, for example chosen as the spatial part of integrating factor \((C5, C8)\) for the time-harmonicity condition \((C4)\).

If we substitute (90) in (1), we find

\[
i \left(1 + \omega^2\right) N_\eta - 2\omega M_\eta = k(L + iM + \omega N),
2\omega L_\eta - \left(1 - \omega^2\right) N_\eta = -ik(L + iM + \omega N),
\left(1 - \omega^2\right) M_\eta - i \left(1 + \omega^2\right) L_\eta = -k[\omega(L - iM) - N].
\]

We shall avoid further considerations such as modifying this equation by introducing arbitrary holomorphic functions or modifying the contour \(C\) which calls for sheaf cohomology here.

The first and second equations in (91) yields \(-2\omega \left[\left(1 - \omega^2\right) M_\eta - i \left(1 + \omega^2\right) L_\eta\right] = 2k(L + iM + \omega N)\). This leads to

\[
\left(1 - \omega^2\right) L + i \left(1 + \omega^2\right) M + 2\omega N = 0,
\]

that is \(L + iM + 2N\omega - (L - iM)\omega^2 = 0\) using the third equation. If we substitute \(N\), (92) in (91), we find

\[
-i \left(1 + \omega^2\right) L + \left(1 - \omega^2\right) M = 0.
\]
The equations (92, 93) lead to: \( L(\eta, \omega) = (1 - w^2) u(\eta, \omega) \), \( M(\eta, \omega) = i (1 + w^2) u(\eta, \omega) \), \( N(\eta, \omega) = 2wu(\eta, \omega) \) where \( u(\eta, \omega) \) is an arbitrary holomorphic function (except some poles) of \( \eta \) and \( \omega \).

Thus a Trkalian field (1) is given by

\[
F(x) = [A(x), B(x), C(x)] = \int_C [(1 - \omega^2), i (1 + \omega^2), 2\omega] e^{-ikf}u(\eta, \omega)d\omega. \tag{94}
\]

Note \([(1 - \omega^2), i (1 + \omega^2), 2\omega]\) is a null vector in \( \mathbb{C}^3 \).

This solution is in the form of twistor solution to Maxwell equations in \((3 + 1)\) dimensions, as we expect (since Trkalian fields correspond to the spatial part of time-harmonic electromagnetic fields with no source). We can also derive this solution from the twistor solution of Maxwell equations (see Ref. 35, p. 33, pp. 206-207), using a similar time-harmonic reduction (with minor changes of conventions).

For example, we choose \( u(\eta, \omega) = g(\eta f(\omega))/h(\omega) \) with \( h(\omega) = (\omega - \omega_0)^m \). If \( g(\eta f(\omega)) = \eta f^m(\omega) \) where \( m \) is positive, \( \omega = 1 \) and \( \omega_0 = 0 \), that is \( u(\eta, \omega) = \eta f^m(\omega)/\omega \), we find

\[
F(x) = 2\pi i e^{i\nu z} \zeta/(1, i, 0), \quad \zeta = x + iy. \tag{95}
\]

If \( \omega_0 \neq 0 \), that is \( u(\eta, \omega) = \eta f^m(\omega)/(\omega - \omega_0) \), we find

\[
F(x) = 2\pi i e^{-i\nu[x-yi]} \eta f^m(\omega_0) \left( (1 - \omega_0^2), i(1 + \omega_0^2), 2\omega_0 \right). \tag{96}
\]

Hence for a holomorphic function \( g(\eta) = \sum_{n=0}^\infty a_n \eta^n \) and \( \omega_0 = 0 \) that is \( u(\eta, \omega) = g(\eta f(\omega))/\omega \) we find

\[
F(x) = 2\pi i e^{i\nu z} g(\zeta)/(1, i, 0). \tag{97}
\]

The orthogonality of real contact structures arising in case \( g(\zeta) \) is given by a derivative: \( g \rightarrow g' \) is discussed in Ref. 6. If we choose \( u(\eta, \omega) = g(\eta f(\omega))/\omega^2 \), we find

\[
F(x) = 2\pi i e^{i\nu z} \left\{ [-iv\zeta g(\zeta) + 2z g'(\zeta)] \right\} (1, i, 0) + 2g(\zeta)(0, 0, 1). \tag{98}
\]

If we choose \( u(\eta, \omega) = (1/\omega^2)e^{-i(\nu/2)\omega^{-1}}v \), \( -[(1/2)\omega^{-1}] \eta = f - g, \quad g = x(\omega + w^{-1})/2 - iy(\omega - w^{-1})/2 \) where \( x = (x, y, z) = (r \cos \phi, r \sin \phi, z) \) in cylindrical coordinates and \( \omega = e^{i\theta} \), \( C \) is a circle of unit radius about the origin, then we find the Lundquist solution (8) with \( F_0 = 4\pi i \) and \( \lambda = 1 \), using the integrals (C14) in Appendix C.2.

### A. Arbitrary integrating factor

We can choose different integrating factors for the time-harmonicity condition (C4). In fact, we do not have to choose an integrating factor initially. We can see this in a time-harmonic extension of the solution above. The integrating factor in (C5) yields a time-harmonic extension of Trkalian fields: \( F \rightarrow e^{-ikt}F \). This satisfies (vector) wave equation which reduces to (89).

If we use an arbitrary integrating factor

\[
g(p, q, \omega) = e^{-ikf(p,q,\omega)}h(p, q, \omega) = e^{-ikf(p,q,\omega)}H(\eta f(\omega), \omega), \tag{99}
\]

in (C4), we find

\[
\omega \frac{\partial \tilde{f}}{\partial p} + \frac{\partial \tilde{f}}{\partial q} = 1, \tag{100}
\]

that is \( \partial \tilde{f}/\partial t = 1 \) as a condition on the integrating factor. Then the field (90), (with \( f \rightarrow \tilde{f} \) containing both temporal and spatial pieces) satisfies the wave equation which reduces to (89) upon imposing the condition (100). If we substitute this field in (1), we find
This reduces to (91) for $\tilde{f} = q$. The equations (101) lead to the same equations (92, 93) using a similar reasoning. These yield the solution (94), \([f \rightarrow \tilde{f} \text{ satisfying (100)}]\) with a harmonic time dependence now.

Thus the solution is of the same form containing the spatial part of the chosen integrating factor.

Any solution of the time-harmonicity condition (C4) can be written using \(\tilde{f} = (1/2)(p/\omega + q)\), (C9), see (C11). This (excluding the temporal piece) leads to the solution (94) with \(f = (1/2)[\omega(x-iy) + (x+iy)/\omega]\), see (C10).

In this case, the Lundquist solution is simply given by \(u(\eta, \omega) = 1/\omega^2\), \((\omega = e^{i\theta}, C: \text{ unit circle about the origin})\).

If we use \(u = h(\omega')\) which has a Laurent series: \(h(\omega') = 1/\omega'^{n+1}\) with \(\omega = i\omega'\), \((k = \nu)\), we find

\[
F(x) = 4\pi e^{-im\varphi} \left[ \frac{im}{\nu} J_m(\nu r) e_r + J'_m(\nu r) e_\varphi - J_m(\nu r) e_z \right], \quad m = n - 1
\]

in cylindrical coordinates. See Appendix C.3.46 This is a circular cylindrical CK\(^70,71\) solution with no \(z\) dependence, upto conventions.

**B. Chandrasekhar-Kendall type solutions: Debye potentials**

We can use the solution

\[
\phi(x, y, z) = \int_C e^{-i\sigma f} H(\eta_x(\omega), \omega) d\omega, \quad f = \frac{1}{2} [i\omega(x-iy) + \frac{1}{\omega}(x+iy)]
\]

of the scalar Helmholtz equation \(\nabla^2 \phi = -k^2 \phi\), \((k = \sigma)\) as Debye potential for CK\(^47\) type Trkalian fields

\[
F(x) = -[\sigma \nabla \times (\phi w) + \nabla \times \nabla \times (\phi w)],
\]

where \(\omega\) is a fixed vector and \(\nabla \times F(x) - \sigma F(x) = 0\). It is straightforward to write a time-harmonic extension of the CK solution.

The potential for the circular cylindrical CK solution\(^70,71\) is effectively (apart from \(z\) coordinate) a 2 dimensional solution\(^46\) (satisfying the Helmholtz equation: \(\nabla^2 \phi + \nu^2 \phi = 0\) in 2 dimensions). We can express this as

\[
\phi(r, \varphi, z) = e^{-ikz} \int_C e^{-i\nu f} H d\omega = 2\pi i \left[ \frac{1}{lm} J_m(\nu r) e^{im\varphi - ikz} \right], \quad \sigma^2 = \nu^2 + k^2
\]

where \(H = \omega^{m-1}\), \(\omega = e^{i\theta}\), \(C: \text{ unit circle about the origin}\) and \(w = e_z\). Here we use (C12, C13).

This reduces to

\[
\phi(r, \varphi, z) = \int_C e^{-i\nu f} H d\omega = 2\pi i J_0(\nu r),
\]

the potential for the Lundquist solution, for \(m = 0, k = 0, (\sigma = \nu)\), \(H = \omega^{-1}\).

An interesting case is the class of axially symmetric potentials. If we assume axial symmetry about \(z\)-axis, then a rotation in \(xy\)-plane is given by \(\omega \rightarrow e^{i\theta} \omega\) (treated as a spinor coordinate) which induces the rotation \(x + iy \rightarrow e^{i\theta}(x + iy)\) (see Refs. 46 and also 45, 65) and \(\eta_x(\omega) \rightarrow e^{i\theta} \eta_x(\omega)\), \(d\omega \rightarrow e^{i\theta} d\omega\) while \(f = (1/2)[\omega(x-iy) + (x+iy)/\omega] \rightarrow f\). Hence we consider fields of the form

\[
\phi(x, y, z) = \int_C e^{-i\sigma f} G \left( \frac{\eta_x(\omega)}{\omega} \right) \frac{1}{\omega} d\omega,
\]
for some holomorphic $G$. We assume that $G$ has a Laurent series about the origin: $G = [\eta_\pi(\omega)/\omega]^n$, then

$$\phi(x, y, z) = \int_C e^{-i\sigma f} \left[ \frac{\eta_\pi(\omega)}{\omega} \right]^n \frac{1}{\omega} d\omega.$$  \hspace{1cm} (108)

For $n = 0$ ($\sigma = \nu$), this reduces to the potential (106).

We can use $\omega = -ie^{iu}$ for parametrizing (108). This yields

$$\phi(x, y, z) = 2^n i \int_C e^{-i\sigma f} (z + ix \cos u + iy \sin u)^n du,$$  \hspace{1cm} (109)

where $f = x \sin u - y \cos u$, with $\eta/\omega = 2(z + ix \cos u + iy \sin u)$. For $n = 1$, we find

$$\phi = 2iR \int_{\theta=0}^{\theta=2\pi} e^{-i\sigma R \sin \theta \sin(u-\varphi)} [\cos \theta + i \sin \theta \cos(u-\varphi)] du$$

$$= 2iR \int_{\theta=0}^{\theta=2\pi} e^{-i\sigma R \sin \theta \sin u} (\cos \theta + i \sin \theta \cos u) du,$$  \hspace{1cm} (110)

using spherical coordinates: $x = R \sin \theta \cos \varphi$, $y = R \sin \theta \sin \varphi$, $z = R \cos \theta$. The integral of the second term vanishes and the first term yields

$$\phi = 2\pi i z J_0(\sigma r), \quad r = R \sin \theta,$$  \hspace{1cm} (111)

in cylindrical coordinates, using (C15): ($\beta = \sigma r$). The potential (111), with $\omega = e_z$, leads to

$$F(x) = -4\pi i \sigma^2 \left\{ -\frac{1}{\sigma} J_1(\sigma r)e_r + z [J_1(\sigma r)e_z + J_0(\sigma r)e_y] \right\},$$  \hspace{1cm} (112)

a generalization of the Lundquist field (8), ($\sigma = \nu$).

In case $n = -1$, we use $f = \omega(x - iy) - z$ for the sake of simplicity.\textsuperscript{65,66} The denominator in (108) can be factored as $\eta_\pi(\omega) = -(x - iy)(\omega - \omega_1)(\omega - \omega_2)$ where $\omega_1 = (z - |x|)/(x - iy) = -e^{i\varphi} \tan(\theta/2)$, $\omega_2 = (z + |x|)/(x - iy) = e^{i\varphi} \cot(\theta/2)$. The integral branches depending on the sign of $z$.\textsuperscript{65} If $z > 0$, $|\omega_1| < 1$, then the single residue inside the unit circle $C$ yields

$$\phi = \frac{1}{2} e^{i\sigma |x|}.\hspace{1cm} (113)$$

This is the fundamental solution of the scalar Helmholtz equation: $\nabla^2 \phi + \sigma^2 \phi = -2\pi \delta(x)$. It is beyond the scope of this manuscript to provide a complete treatment of this case.\textsuperscript{65,66}

The classical spheromak equilibrium solution\textsuperscript{47,72} (see also Ref. 5 and the references therein) is given by

$$F = F_0 \left\{ \frac{2j_1(kR)}{kR} \cos \theta e_R + \frac{1}{kR} [j_1(kR) - \sin(kR)] \sin \theta e_\theta + j_1(kR) \sin \theta e_\phi \right\},$$  \hspace{1cm} (114)

in spherical coordinates, where $j_1(kR)$ is the spherical Bessel function, $\omega = R\omega$ (with the conventions of Ref. 72) and $\sigma = k$, (104). We need to consider an expression of the form (103) which reduces to\textsuperscript{73}

$$\phi = -i \frac{F_0}{k} \int_0^\pi e^{-ikR \cos \theta \cos \alpha} J_0(kR \sin \theta \sin \alpha) P_1^0(\cos \alpha) \sin \alpha d\alpha = -\frac{F_0}{k} J_1(kR) P_1^0(\cos \theta).$$  \hspace{1cm} (115)

An integral of this type was recently reconsidered by various authors,\textsuperscript{74–77} referring to Refs. 63, 78, and 79 and including alternative proofs. To the knowledge of the author, the simplest derivation of this integral expression is given in Ref. 73, p. 411 (with a misprint of coefficient).
VII. CONCLUSION

We have studied the X-ray and Divergent beam transforms of Trkalian fields in connection with their Radon transform. We remind that the Radon transform of a Trkalian field is defined on a sphere in the transform space and it satisfies a corresponding eigenvalue equation there. The mathematical methods of tomography (respectively Smith’s and Tuy’s methods) show that these transforms are basically in the form of a Minkowski-Funk and a closely related integral transform of certain intricate quantities (Hilbert transform of the derivative of Radon transform). The Moses eigenbasis is especially efficient in exhibiting this connection for Trkalian fields. These naturally reduce to well known geometric integral transforms on a sphere of the Radon or the spherical Curl transform.

More precisely, the X-ray transform of a Trkalian field is given by the Minkowski-Funk transform of its Radon transform on this sphere. In $\mathbb{R}^3$, this corresponds to the integral of the Radon transform of the field over a pencil of planes intersecting at a line. Previously, this transform was introduced by Gonzalez, called the plane-to-line transform, as an elementary geometric transform in integral geometry. This transform naturally arises for Trkalian fields. We refer the interested reader to Ref. 43 for a twistor approach to this transform.

Meanwhile the Divergent beam transform is given by another closely related (an extension of Minkowski-Funk) geometric integral transform of the spherical Curl transform of the field on the sphere. This also provides an extension over the plane-to-line transform. This seems a natural extension from the point of view of generalized functions.

We remark that these transforms are naturally defined on the sphere in the transform space. Geometrically, we are endowed with a simple picture showing the interrelations of these transforms for Trkalian fields on this sphere. This is made possible with the Moses basis.

Intuitively speaking, the X-ray or Divergent beam transform of a Trkalian field respectively integrates the field on a whole line or a half-line which is determined by a direction vector. This direction vector determines a great circle on the sphere in the transform space. Then these transforms are given by integrals of the Radon transform on this great circle. The X-ray: whole-line transform is given by a whole (in the distributional sense: Dirac delta) integral. This leads to the plane-to-line picture in $\mathbb{R}^3$. Meanwhile, the Divergent beam: half-line transform is given by a half (in the distributional sense: Heisenberg delta) integral, depending on the orientation. The picture in $\mathbb{R}^3$ for this is left to the imagination of the reader.

We can logically derive these results starting from the fundamental relation (44) of mathematical tomography in a unified manner. However, we have postponed the mathematical discussion originating from tomography until the basic investigation of X-ray and Divergent beam transforms of Trkalian fields had finished. This gave us the opportunity to consider the mathematical basis for tomographical studies of Trkalian field models in nature, for its own sake.

These transforms are members (via analytic continuation) of a well known analytic family of integral operators which arise in the study of Fourier transforms of homogeneous functions. Recently, these integral operators have been studied by Rubin. We have inverted the X-ray transform of a Trkalian field using Semayanti’s formula which also belongs to this family, so as to yield its Radon transform. This leads to an inversion through the spherical mean of the X-ray transform of Trkalian fields. The X-ray (also the Divergent beam) transform and its inversion intertwine the Curl operator and also the Divergence, Gradient and Laplacian operators. Thus the X-ray (Divergent beam) transform of a Trkalian field is Trkalian. Also, the Trkalian subclass of X-ray transforms $XF$ yields Trkalian fields in the physical space. We have also written the John’s equation for Trkalian fields in an equivalent form. Thus, we can study Trkalian fields either in physical space or in the transform space.

Another crucial quantity in integral geometry is the Riesz potential. The Riesz potential, of order $\alpha$ where $0 < \alpha < 3$, of a Trkalian field is proportional to the field. Hence, the spherical mean of the X-ray (Divergent beam) transform of a Trkalian field over all lines passing through a point yields the field at this point. This provides a new simple inversion formula for the X-ray (Divergent beam) transform of Trkalian fields. This result is also logically implied by Gelfand-Gonchorav’s mathematical approach (making use of the same intricate quantity) to tomography.

Then we have returned back to the mathematical methods of tomography. These methods provide us elegant mathematical tools for investigating the interrelations of these integral transforms in a unified view. First, these endowed us with an integral geometric view and motivation for the discussions above. Second, these enabled us to discuss these mathematical methods with a view towards tomographical studies of Trkalian field models in nature. For this purpose, we have studied these mathematical methods using Trkalian fields. We have adopted a mathematical approach rather than a tomographical implementation.

We have made use of four basic mathematical approaches of tomography due to Grangeat, Smith, Tuy and Gelfand-Goncharov. These methods are outflow of the fundamental relation of mathematical tomography. They are based on relations of the X-ray and Divergent beam transforms to the intricate quantities mentioned above: the Hilbert transform of the derivative of Radon transform.
These methods basically make use of the Radon inversion (for reconstruction). They lead us to new inversion formulas for the X-ray and Divergent beam transforms of Trkalian fields with a view towards tomographical applications.

In general, these simplify for Trkalian fields. The simplification arises in the crucial intricate quantities mentioned above. This is basically due to the eigenvalue equation (4) and the Hilbert transform of the derivative for Radon transform of Trkalian fields in the Moses basis. The Grangeat approach leads to another simple, direct inversion formula for the Divergent beam transform of Trkalian fields.

The Smith method reveals that the X-ray transform is in the form of a Minkowski-Funk transform of the intricate quantity mentioned above. This quantity reduces to the Radon transform for a Trkalian field, using the Moses basis. This provides the integral geometric view and motivation for the previous discussion of the X-ray transform of Trkalian fields. In this approach, the inversion formula can be expressed in terms of the Radon transform of Trkalian field.

The Tuy method enables us to study the Divergent beam transform in detail. It reveals that the Divergent beam transform is in the form of another closely related integral transform of a quantity related to the Radon transform. In this case, this quantity reduces to the spherical Curl transform, using the Moses basis. This leads us to the above mentioned integral geometric view of the Divergent beam transform of Trkalian fields. In this case, the inversion formula can be expressed in terms of the spherical Curl transform of the field.

We have calculated the Divergent beam transform of the Lundquist field which is used to model solar magnetic clouds, benefitting the Tuy method. This provides a mathematical Röntgen of these clouds.

Meanwhile, the Gelfand-Goncharov approach leads to a direct inversion through the spherical mean that is mentioned above. This naturally makes use of the inverse transform that belongs to the above family of integral operators. The simplification is again based on the same intricate quantity: Hilbert transform of the derivative of Radon transform in the Moses basis.

Briefly, the intricate quantity: Hilbert transform of the derivative of Radon transform which intrigued tomography simplifies for a Trkalian field in the Moses basis. The direct inversion formulas arising in Grangeat’s and Gelfand-Gonchorav’s approaches mathematically seem more feasible than the inversions in Smith’s and Tuy’s methods.

These approaches provide different inversion formulas which may serve useful for designing reconstruction methods in tomographical studies of Trkalian field models in nature, depending on real physical situation. The author expects that the Moses basis which has led to a drastical simplification in the crucial quantity may also be of practical use in tomographical studies. We shall not discuss tomographical implementations of these inversion formulas.

Furthermore, the Smith and Tuy methods mathematically enable us to define the X-ray and Divergent beam transforms of the Riesz potential (of order 2) and Biot-Savart integrals. The X-ray transform of the Biot-Savart integral of a Trkalian field reduces to the X-ray transform of the field. The Radon, X-ray and Divergent beam transforms of the Riesz potential (of order 2) and Biot-Savart integrals lead to an integral geometric understanding of these integrals. However, a physical or tomographical discussion of these integrals is beyond the scope of this manuscript.

In the second part of this manuscript we have discussed Trkalian fields using (mini-)twistors. The X-ray transform is a real analogue and a predecessor of Twistor theory. The X-ray transform and the mini-twistors are both defined on the space $\mathbb{T}\mathbb{S}^2 \sim \mathbb{T}\mathbb{C}^1$ of oriented lines in $\mathbb{R}^3$.

We have discussed a mini-twistor representation, presenting a mini-twistor solution for the Trkalian fields equation. This is based on twistor solution of the (vector) Helmholtz equation which makes use of a time-harmonic reduction of the wave equation. A Trkalian field is given in terms of a null vector in $\mathbb{C}^3$ with an arbitrary holomorphic function of two variables and an exponential factor that results from the reduction.

The exponential factor contains the spatial part of an integrating factor for the time-harmonicity condition. The solution is of the same form containing the spatial part of any chosen integrating factor. We have also used the general solution of this condition for writing our solution.

This solution can also be derived using a time-harmonic reduction of the twistor solution for electromagnetic fields in $(3 + 1)$ dimensions. We are led to a time-harmonic extension of Trkalian fields, implicitly keeping this condition. This can be interpreted as a time-harmonic electromagnetic field.

We have also presented examples of Debye potentials for CK type solutions using mini-twistors.

This manuscript is aimed at studying the most basic integral geometric aspects and the mini-twistor representation of Trkalian fields. We have made use of the mathematical methods of tomography (but not the tomography). These may serve useful for studying their physical properties in a realistic environment.

The Trkalian class of fields may also provide a simple and interesting example for studying the relation of ray transforms with Twistor theory. However, twistor tomography is beyond the limitations of this manuscript.
APPENDIX A: RAY TRANSFORMS

1. X-ray transform of Lundquist Field

We write \( x = re_r(\phi) + ze_z \), \( r > 0 \) and \( \theta = v_r e_r(\theta) + v_z e_z \), \( v_r > 0 \) in cylindrical coordinates. Then \( x' = x + s\theta = r'e_r(\phi') + z'e_z \) where \( r' \) is defined by \( r' \cos \phi' = r \cos \phi + sv_r \cos \theta \), \( r' \sin \phi' = r \sin \phi + sv_r \sin \theta \), \( r'^2 = r^2 + s^2v_r^2 + 2rsv_r \cos(\theta - \phi) \), \( z' = z + sv_z \) and

\[
\begin{align*}
e_r(\phi') &= \cos \phi' e_x + \sin \phi' e_y, \\
e_{\phi'} &= -\sin \phi' e_x + \cos \phi' e_y.
\end{align*}
\] (A1)

Hence

\[
F_L(x + s\theta) = F_0 \left[ \lambda J_1(\lambda \nu r') e_{\phi'} + J_0(\lambda \nu r') e_z \right] = F_0 \left[ \lambda \frac{1}{r'} J_1(\lambda \nu r') e_{\phi'} + \lambda v_r \frac{s}{r'} J_1(\lambda \nu r') e_\theta + J_0(\lambda \nu r') e_z \right],
\] (A2)

and we have

\[
\mathcal{X}F_L(\theta, x) = F_0 \left[ \lambda r \int_0^\infty \frac{1}{r'} J_1(\lambda \nu r') ds e_\phi + \lambda v_r \int_{-\infty}^\infty \frac{s}{r'} J_1(\lambda \nu r') ds e_\theta + \int_{-\infty}^\infty J_0(\lambda \nu r') ds e_z \right],
\] (A3)

where \( r'(s) \). We define a new variable: \( t = v_r s + r \cos(\theta - \phi) \) \( \Rightarrow ds = (1/v_r) dt \), \( r' = \sqrt{t^2 + u^2} \), \( u = r \sin(\theta - \phi) \). Then

\[
\mathcal{X}F_L(\theta, x) = F_0 \frac{1}{v_r} \left\{ \lambda r \int_{-\infty}^\infty \frac{1}{\sqrt{t^2 + u^2}} J_1(\lambda \nu \sqrt{t^2 + u^2}) dt \left[ e_\phi - \cos(\theta - \phi) e_\theta \right] + \lambda \int_{-\infty}^\infty \frac{t}{\sqrt{t^2 + u^2}} J_1(\lambda \nu \sqrt{t^2 + u^2}) dt e_\theta + \int_{-\infty}^\infty J_0(\lambda \nu \sqrt{t^2 + u^2}) dt e_z \right\}
\] (A4)

\[
= 2F_0 \frac{1}{v_r} \left\{ \lambda r \int_0^\infty \frac{1}{\sqrt{t^2 + u^2}} J_1(\lambda \nu \sqrt{t^2 + u^2}) dt \left[ e_\phi - \cos(\theta - \phi) e_\theta \right] + \int_0^\infty J_0(\lambda \nu \sqrt{t^2 + u^2}) dt e_z \right\}.
\]

We introduce another variable: \( \sinh y = t/|u| \) \( \Rightarrow dt = |u| \cosh dy \). This reduces to

\[
\mathcal{X}F_L(\theta, x) = 2F_0 \frac{1}{v_r} \left\{ \lambda u \int_0^\infty J_1(2\nu \cosh y) dy e_r(\theta) + |u| \int_0^\infty J_0(2\nu \cosh y) \cosh y dy e_z \right\},
\] (A5)

using \( e_\phi - \cos(\theta - \phi) e_\theta = \sin(\theta - \phi) e_r(\theta) \), where \( \omega = (1/2)\lambda \nu |u| > 0 \). We can evaluate these integrals using

\[
\int_0^\infty J_{m+n}(2z \cosh x) \cosh((m-n)x) dx = -\frac{\pi}{4} \left[ J_m(z)N_n(z) + J_n(z)N_m(z) \right], \quad z > 0
\] (A6)

the formula 6.663(3) in Ref. 79. Here \( J_n \) and \( N_m \) are respectively Bessel functions of the first and second (Neumann) kind. We also need

\[
\begin{align*}
N_{-1/2}(z) &= J_{1/2}(z), \\
J_{1/2}(z) &= \left( \frac{2}{\pi z} \right)^{1/2} \sin z, \\
J_{-1/2}(z) &= \left( \frac{2}{\pi z} \right)^{1/2} \cos z.
\end{align*}
\] (A7)

We respectively find
\[ \int_0^\infty J_1(2\omega \cosh y) dy = \frac{1}{2} \sin 2\omega, \quad \int_0^\infty J_0(2\omega \cosh y) \cosh y dy = \frac{1}{2} \cos 2\omega, \]  
(A8)

for \( m = 1/2, n = 1/2 \) and \( m = 1/2, n = -1/2 \). This leads to

\[ X F_L(\theta, x) = F_0 \frac{1}{\nu e_{\theta}} [\lambda u \sin 2\omega e_r(\theta) + |u| \cos 2\omega e_z], \]  
(A9)

that is equation (16) upon rearranging.

2. X-ray transform of a Trkalian Field

a. Reduction of the integral

We can simplify (17) decomposing \( \kappa \) into two components which are respectively parallel and orthogonal to \( \theta \):

\( \kappa = \kappa_0 + \kappa_\perp = u\theta + v e_r(\phi) \). That is \( \kappa_0 = u\theta, u = \cos \theta \) and \( \kappa_\perp = v e_r(\phi), v = \sqrt{1 - u^2} = \sin \theta \) where \( e_r(\phi) \) is the unit radial vector parametrized by angle \( \phi \) in the plane \( \theta \perp \) orthogonal to \( \theta \): \( e_r(\phi) \cdot \theta = 0 \). Then \( \kappa \cdot x = [u\theta + v e_r(\phi)] \cdot x \), \( \delta(\kappa \cdot \theta) = \delta(u) \) and \( d\Omega_\kappa = \sin \theta d\theta d\phi = -dud\phi \). Hence we can write (17) as

\[ X F_\Lambda(\theta, x) = \frac{1}{4\pi} \lambda \nu \int_C \int_{u=-1}^{u=1} \mathcal{F}^R_{\Lambda}([u\theta + \sqrt{1 - u^2} e_r(\phi)] \cdot x, u\theta + \sqrt{1 - u^2} e_r(\phi)) \delta(u) du d\phi, \]  
(A10)

where \( C \) is the unit circle in the plane \( \theta \perp \) which corresponds to a great circle on \( S^2_\kappa \) (the intersection of the plane with the sphere). This reduces to

\[ X F_\Lambda(\theta, x) = \frac{1}{4\pi} \lambda \nu \int_C \mathcal{F}^R_{\Lambda}(e_r(\phi) \cdot x, e_r(\phi)) d\phi = \frac{1}{(2\pi)^{1/2}} \frac{1}{\lambda \nu} \int_C e^{i\lambda \nu e_r(\phi) \cdot x} Q_\Lambda(e_r(\phi)) s_\Lambda(\lambda \nu e_r(\phi)) d\phi, \]  
(A11)

the integral along the great circle \( C \) in \( S^2_\kappa \) determined by \( \theta \) in the transform space.

One can further try writing (A11) in terms of rotation about the axis determined by \( \theta \) through angle \( \phi \) in the plane \( \theta \perp \) since \( Q_\Lambda(e_r(\phi)) \) can be regarded as an eigenfunction of this rotation.\(^{19}\)

b. Fourier slice-projection theorem

If we use the Curl expansion\(^4-6\) for a Trkalian field, we are led to

\[ \mathcal{F}[F_\Lambda(x)](\xi) = \frac{1}{(2\pi)^{3/2}} \int e^{-i\xi \cdot x} F_\Lambda(x) d^3x = \frac{1}{(2\pi)^{3/2}} \frac{1}{g} \int e^{-i\xi \cdot x} \int e^{i\kappa \cdot x} Q_\Lambda(k) f_\Lambda(k) d^3k d^3x \]  
(A12)

If we substitute this in Fourier slice-projection theorem (18), we immediately find

\[ X F_\Lambda(\theta, x) = (2\pi)^{1/2} \mathcal{F}^{-1} \{ \mathcal{F}[F_\Lambda(x)](\xi) \} = \frac{1}{(2\pi)^{1/2}} \frac{1}{g} \int e^{i\xi \cdot x} \mathcal{F}[F_\Lambda(x)](\xi) d^2\xi \]  
(A13)

\[ = \frac{1}{(2\pi)^{1/2}} \frac{1}{g \lambda \nu} \int_C e^{i\lambda \nu e_r(\phi) \cdot x} Q_\Lambda(e_r(\phi)) s_\Lambda(\lambda \nu e_r(\phi)) d\phi, \]

(A11), using \( \xi = \xi e_r(\phi) \in \theta \perp, d^2\xi = \xi d\xi d\phi \) and \( Q_\Lambda(\lambda \nu e_r(\phi)) = Q_\Lambda(e_r(\phi)) \) since \( \lambda \nu = |\nu| > 0 \).
3. The integral operators: $A^\alpha = U^\alpha + iV^\alpha$

The family of integrals $A^\alpha = U^\alpha + iV^\alpha$, $\alpha \in \mathbb{C}$, for $Re(\alpha) > 0$, $(n = 2)$ are given by

$$U^\alpha[G(\kappa)][\theta] = \frac{\Gamma((1 - \alpha)/2)}{2\pi \Gamma(\alpha/2)} \int_{S^2} G(\kappa) \frac{1}{|\theta \cdot \kappa|^{1 - \alpha}} d\Omega_\kappa, \quad \alpha \neq 1, 3, 5, \ldots$$  \hspace{1cm} (A14)

$$V^\alpha[G(\kappa)][\theta] = \frac{\Gamma((1 - \alpha)/2)}{2\pi \Gamma((1 + \alpha)/2)} \int_{S^2} G(\kappa) \frac{1}{|\theta \cdot \kappa|^{1 - \alpha}} \text{sgn}(\theta \cdot \kappa) d\Omega_\kappa, \quad \alpha \neq 2, 4, 6, \ldots$$

$G(\kappa) \in C^\infty(\mathbb{S}^2)$. See the Refs. 26 and 27 and the references therein. The transform $U^\alpha$ ($V^\alpha$) represents the even (odd) part of $A^\alpha$ and annihilates odd (even) functions. For $Re(\alpha) \leq 0$, these are to be understood in the sense of analytic continuation. In the case $\alpha \to 0$

$$U^0[G(\kappa)] = \frac{1}{2\pi^{3/2}} M[G(\kappa)],$$

and $V^1$ is related to hemispherical transform.\textsuperscript{26,27}

The inverse transforms\textsuperscript{26,27} are given as

$$(U^\alpha)^{-1} = U^{-1 - \alpha}, \quad (V^\alpha)^{-1} = V^{-1 - \alpha},$$

in the sense of analytic continuation for certain values of $\alpha$. This inversion of $U^\alpha$ was established by Semyanistyi who studied the connection of $U^\alpha$ with Fourier transform.\textsuperscript{54,55}

We have\textsuperscript{26,27} the relation

$$\int_{R^3} \frac{G(\kappa)}{k^{2 + \alpha}} e^{i k \cdot \eta} d^3 k = c_\alpha \eta^{\alpha - 1} A^\alpha[G(\kappa)][\theta, x], \quad c_{\alpha, n} = 2^{1 - \alpha} \pi^{3/2}$$

(A17)

where $k = k\kappa$, $\kappa = |k|$. $\eta = \eta\theta$, $\eta = |\eta|$. This can also be extended to all $\alpha \in \mathbb{C}$ by analytic continuation.

Note $\Gamma(1) = 1$, $\Gamma(-1/2) = -2\pi^{1/2}$ for our purpose.

a. Evaluation of an integral

We decompose $\theta$ and $\kappa$ into two components which are respectively parallel and orthogonal to $\kappa': \theta = \theta_\parallel + \theta_\perp = u\kappa' + v e_r(\phi)$. That is $\theta_\parallel = u\kappa'$, $u = \cos \theta$ and $\theta_\perp = v e_r(\phi)$, $v = \sqrt{1 - u^2} = \sin \theta$ where $e_r(\phi)$ is the unit radial vector parametrized by angle $\phi$, in the plane $\kappa'_\perp$ orthogonal to $\kappa'$: $e_r(\phi) \cdot \kappa' = 0$. Also $\kappa = \kappa_\parallel + \kappa_\perp$. Then $\kappa \cdot \theta = u\kappa \parallel \cdot \kappa' + \sqrt{1 - u^2} \kappa_\perp \cdot e_r(\phi)$, $\delta(\kappa' \cdot \theta) = \delta(u)$ and $d\Omega_\theta = \sin \theta d\theta d\phi = -dud\phi$. The integral (28) becomes

$$I(\kappa, \kappa') = \int_{\phi=0}^{2\pi} \int_{u=-1}^{u=1} \frac{\delta(u)}{|u\kappa \parallel \cdot \kappa' + \sqrt{1 - u^2} \kappa_\perp \cdot e_r(\phi)|^2} dud\phi.$$  \hspace{1cm} (A18)

This reduces to

$$I(\kappa, \kappa') = \int_{\phi=0}^{2\pi} \frac{1}{|\kappa_\perp \cdot e_r(\phi)|^2} d\phi.$$  \hspace{1cm} (A19)

If we use the planewave decomposition\textsuperscript{52,53} of Dirac delta function

$$\delta(x) = -\frac{1}{4\pi^2} \int_{\mathbb{S}^1} \frac{1}{|x \cdot \psi|^2} d\psi, \quad |\psi| = 1$$

(A20)

in the plane $\kappa_\perp$, we find
\[ I(\kappa, \kappa') = -4\pi^2\delta(\kappa_\perp). \]  

(A21)  

Thus we are led to
\[ I(\kappa, \kappa') = -4\pi^2[\delta(\kappa - \kappa') + \delta(\kappa + \kappa')], \]  

(A22)

using the fact that as \( \kappa_\perp = \kappa - \kappa_\parallel = 0 \), we have \( \kappa_\parallel = \pm \kappa' \) and hence the identity
\[ \delta(\kappa_\perp) = \delta(\kappa - \kappa_\parallel) = \delta(\kappa - \kappa') + \delta(\kappa + \kappa'). \]  

(A23)

[Simply: \( \kappa = \cos \alpha \kappa' + \sin \alpha e_r \) in the plane spanned by \( \kappa' \) and \( e_r \), thus \( \delta(\kappa_\perp) = \delta(\sin \alpha) = \delta(\alpha) + \delta(\alpha - \pi) = \delta(\kappa - \kappa') + \delta(\kappa + \kappa'). \)]

b. Example: Lundquist field

If we substitute the X-ray transform (16) [that is (A9)] of the Lundquist field in (26), we find

\[
F_L^{(\kappa \cdot x, \kappa)} = -\frac{1}{4\pi F_0} \frac{1}{\mu^2} \int_{\theta=0}^{2\pi} \left\{ \lambda \cos \theta \sin [\lambda \nu r \sin(\theta - \phi)] e_z + \lambda \sin \theta \sin [\lambda \nu r \sin(\theta - \phi)] e_y + \cos [\lambda \nu r \sin(\theta - \phi)] e_x \right\} I_1(\beta, \theta - \psi) d\theta,
\]  

(A24)

where

\[
I_1(\beta, \theta - \psi) = \int_{\alpha=0}^{\pi} \frac{1}{|\sin \alpha \sin \beta \cos(\theta - \psi) + \cos \alpha \cos \beta|^2} d\alpha.
\]  

(A25)

Here \( \theta = v_r e_r(\theta) + v_z e_z, v_r = \sin \alpha, v_z = \cos \alpha \Rightarrow d\Omega = \sin \alpha d\alpha d\theta \) and \( \kappa = \kappa_r e_r(\psi) + \kappa_z e_z, \kappa_r = \sin \beta, \kappa_z = \cos \beta \). Consider the integral

\[
I = \int_{\alpha=0}^{2\pi} \frac{1}{|y \cdot v(\alpha)|^2} d\alpha = I_1 + I_2,
\]  

(A26)

where \( y = \cos \beta e_x + \sin \beta \cos(\theta - \psi) e_y \) and \( v(\alpha) = \cos \alpha e_x + \sin \alpha e_y, |v| = 1 \). Here \( I_1 \) is the integral (A25) and

\[
I_2 = \int_{\alpha=\pi}^{2\pi} \frac{1}{|y \cdot v(\alpha)|^2} d\alpha.
\]  

(A27)

We immediately find \( I_2 = I_1 \) using: \( \alpha' = \alpha - \pi \). This leads to

\[
I_1 = \frac{1}{2}I = -2\pi^2 \delta(y) = -2\pi^2 \delta(\cos \beta) \delta(\sin \beta \cos(\theta - \psi)),
\]  

(A28)

using (A20) and \( \delta(y) = \delta(y_1 e_x + y_2 e_y) = \delta(y_1)\delta(y_2) \). This reduces to

\[
I_1 = -2\pi^2 \delta(\cos \beta) \delta(\cos(\theta - \psi)) = -2\pi^2 (\cos \beta) \left[ \delta(\theta - \psi - \pi/2) + \delta(\theta - \psi - 3\pi/2) \right].
\]  

(A29)

If we substitute this in (A24), then we find \( F_L^{(\kappa \cdot x, \kappa)}(\lambda = 1) \), (9) for \( \lambda = 1 \) after a straightforward manipulation.
APPENDIX B: MATHEMATICAL METHODS OF TOMOGRAPHY

1. The identity

The identity (44) follows as

\[
\int_{S^2_\theta} \mathcal{D} \mathbf{F}(\theta, x) h(\theta \cdot \mathbf{b}) d\Omega_\theta = \int_{S^2_\theta} \int_{t=0}^\infty \mathbf{F}(x + t\theta) h(t\theta \cdot \mathbf{b}) t^2 dt d\Omega_\theta, \quad x' = t\theta, x'' = x + x' \tag{B1}
\]

\[
= \int_{-\infty}^\infty \mathbf{F}(x'') \delta(p - b \cdot x'') h(p - b \cdot x) dp d^3 x''
\]

\[
= \int_{-\infty}^\infty \mathcal{F}^R(p, b) h(p - b \cdot x) dp,
\]

similar to the derivation for scalar fields in Ref. 48, p. 277, Ref. 59. We have only assumed \( h(ap) = \frac{1}{a^2} h(p), \) \( a > 0. \)

2. Grangeat’s method

a. Derivation

The derivation

\[
\frac{\partial}{\partial p} \mathcal{F}^R(p, \kappa) \bigg|_{p=\kappa \cdot x} = \int \mathbf{F}(r) \delta'(p - \kappa \cdot r) d^3 r \bigg|_{p=\kappa \cdot x} \tag{B2}
\]

\[
= -\int \mathbf{F}(x + r') \delta'(\kappa \cdot r') d^3 r', \quad r' = r - x, \delta'(-t) = -\delta'(t)
\]

\[
= -\int \mathcal{D} \mathbf{F}(\theta, x) \delta'(\kappa \cdot \theta) d\Omega_\theta, \quad r' = s\theta, s = |r'| > 0 \Rightarrow d^3 r' = s^2 ds d\Omega_\theta, \delta'(st) = \frac{1}{s^2} \delta'(t)
\]

follows the same reasoning for scalar fields.

The identities\(^{61}\)

\[
\int_{S^2_\theta} \mathbf{F}(\theta) \delta'(\kappa \cdot \theta) d\Omega_\theta = -\int_{S^2_\theta \cap \kappa^\perp} \frac{\partial}{\partial \kappa} \mathbf{F}(\theta) d\theta = -\int_{\psi=0}^{2\pi} \frac{\partial}{\partial q} \mathbf{F}(\theta = q\kappa + r\mathbf{v}(\psi)) \bigg|_{q=0} d\psi, \tag{B3}
\]

and

\[
\int_{S^2_\theta} \mathbf{F}(x \cdot \theta) \delta'(\kappa \cdot \theta) d\Omega_\theta = -x \cdot \kappa \int_{S^2_\theta \cap \kappa^\perp} \mathbf{F}'(x \cdot \theta) d\theta, \tag{B4}
\]

are useful in handling these type of integrals. Here \( \partial / \partial \kappa \) denotes directional derivative along \( \kappa, [|\theta| = 1, |\kappa| = 1 \) and \( \mathbf{v}(\psi) \) takes values on the unit circle: \(|\mathbf{v}| = 1 \Rightarrow q^2 + r^2 = 1 \), parametrized by angle \( \psi \) in the plane \( \kappa^\perp \) through the origin, Ref. 50, p. 71.\] One can derive (B4) with a reasoning similar to the derivation of (B3). See Ref. 48, p. 277.

b. Example

If we substitute \( \mathcal{D} \mathbf{F}(\theta, x), (15) \) in (45), the term containing \( \delta \delta' \) vanishes and we find

\[
\frac{\partial}{\partial p} \mathcal{F}^R(p, \kappa) \bigg|_{p=\kappa \cdot x} = -i \frac{1}{k_0} e^{ik_0 \kappa \cdot x} I' \mathbf{F}_0, \tag{B5}
\]
Trkalian fields: ray transforms and mini-twistors

where

\[
I' = \int_{S^2_0} \frac{1}{\kappa_0 \cdot \theta} \delta'(\kappa \cdot \theta) d\Omega_\theta = \kappa_0 \cdot \kappa \int_{S^2 \cap \kappa^\perp} \frac{1}{(\kappa_0 \cdot v)^2} d\psi = \kappa_0 \cdot \kappa I(\kappa_0, \kappa), \tag{B6}
\]

(A19). Here we have used the identity (B3) or (B4) and \(1/(\kappa_0 \cdot \theta)^2\) (in distributional sense). We decompose the vector \(\kappa_0 = \kappa_0 || + \kappa_0 \perp\) into components which are respectively parallel and orthogonal to \(\kappa\) and \(v(\psi) \in \kappa^\perp, |v| = 1\). This yields

\[
I' = -4\pi^2 \kappa_0 \cdot \kappa [\delta(\kappa - \kappa_0) + \delta(\kappa + \kappa_0)] = -4\pi^2 [\delta(\kappa - \kappa_0) - \delta(\kappa + \kappa_0)], \tag{B7}
\]

using the planewave decomposition of Dirac delta function (A20, A22). If we substitute this in (B5), we find

\[
\frac{\partial}{\partial p} F^R (p, \kappa) \bigg|_{p = \kappa \cdot x} = 4\pi^2 i \frac{1}{k_0} e^{ik_0 \kappa_0 \cdot x} [\delta(\kappa - \kappa_0) - \delta(\kappa + \kappa_0)] F_0, \tag{B8}
\]

(7).

c. Inversion for Trkalian fields

If we substitute (19,20) into the righthand side of (50) and interchange the order of integrations, we find

\[
\int_{S^2_0} \theta \times \mathcal{D} \mathcal{F}_\lambda(\pm \theta, x) d\Omega_\theta = \frac{1}{(2\pi)^{1/2}} g_{\lambda\nu} \int_{S^2_0} e^{i\nu \kappa \cdot x} I(\kappa) \times Q_\lambda(\kappa)s_{\lambda\nu}(\lambda\nu\kappa) d\Omega_\kappa, \tag{B9}
\]

where

\[
I(\kappa) = \int_{S^2_0} \theta \delta(\kappa \cdot \theta) d\Omega_\theta = \pm \frac{1}{2\pi} iI'. \tag{B10}
\]

Here the first integral vanishes. We can evaluate the second integral \(I'\) decomposing \(\theta\) into components which are respectively parallel and orthogonal to \(\kappa\). That is \(\theta = u\kappa + v e_r(\phi), u = \cos \theta, v = \sin \theta\) where \(e_r(\phi)\) is the unit radial vector parametrized by angle \(\phi\) in the plane \(\kappa^\perp\) orthogonal to \(\kappa\). Then \(\kappa \cdot \theta = u\kappa\) and \(d\Omega_\theta = -du d\phi\). Hence

\[
I' = \int_{S^2_0} \frac{\theta}{\kappa \cdot \theta} d\Omega_\theta = \int_{\phi=0}^{\phi=2\pi} \int_{u=-1}^{u=1} [u\kappa + v e_r(\phi)] \frac{1}{u} du d\phi = 4\pi \kappa, \tag{B11}
\]

since the second integral vanishes, \((e_r = \cos \phi e_1 + \sin \phi e_2\) in the plane \(\kappa^\perp\)). Thus we find

\[
\int_{S^2_0} \theta \times \mathcal{D} \mathcal{F}_\lambda(\pm \theta, x) d\Omega_\theta = \pm 4\pi \frac{1}{\nu} F_{\lambda}(x), \tag{B12}
\]

using \(\kappa \times Q_\lambda(\kappa) = -i\lambda Q_\lambda(\kappa)\) and (6).
3. Smith’s method

The equation (52) follows as

\[ GF(\beta, x) = F[gF(\alpha, x)](\beta, x), \quad \alpha = \alpha \theta \Rightarrow d^3 \alpha = \alpha^2 d\alpha d\Omega \theta \]

\[ = \frac{1}{2} \left( \frac{1}{(2\pi)^{3/2}} \int_{S^2} \int_{\alpha=-\infty}^{\infty} \chi F(\theta, x)|\alpha|e^{-i\alpha \theta \beta d\alpha d\Omega \theta} \right. \]

\[ + \left. \frac{1}{(2\pi)^{3/2}} \int_{S^2} \int_{\alpha=-\infty}^{\infty} \Phi(\theta, x)h(\theta \cdot \beta) d\Omega \theta, \quad \right. \]

where

\[ h(s) = \int_{\alpha=-\infty}^{\infty} | \alpha | e^{-i\alpha s} d\alpha = (2\pi)^{1/2} i \partial_s F[\text{sgn}(\alpha)](s) \]

\[ = 2 \partial_s \frac{1}{s} = -2 \frac{1}{s^2}, \quad (s = \theta \cdot \beta) \]

satisfies \( h(as) = (1/a^2) h(s), \ a > 0 \) and \( h(-s) = h(s) \). Hence we obtain (52), since \( \beta = \beta b, \ s = \beta p \).

4. Tuy’s method

The equation (65) follows as

\[ GF(\beta, x) = F[gF(\alpha, x)](\beta, x), \quad \alpha = \alpha \theta \Rightarrow d^3 \alpha = \alpha^2 d\alpha d\Omega \theta \]

\[ = \frac{1}{2} \left( \frac{1}{(2\pi)^{3/2}} \int_{S^2} \int_{\alpha=-\infty}^{\infty} \Phi(\theta, x)f(\theta \cdot \beta) d\Omega \theta, \right. \]

where

\[ f(s) = \frac{1}{2} \int_{\alpha=-\infty}^{\infty} (|\alpha| + \alpha)e^{-i\alpha s} d\alpha = (2\pi)^{1/2} F[R(\alpha)](s) \]

\[ = \frac{1}{2} \left( \frac{1}{2} \left[ h(s) - \frac{1}{i} k(s) \right] \right) \]

Here \( R(\alpha) = \alpha H(\alpha) \) is the Ramp function, \( h(s) = 2 \partial_s 1/s, \) (B14) and

\[ k(s) = \partial_s \int_{\alpha=-\infty}^{\infty} e^{i\alpha s} d\alpha = 2\pi \delta'(s). \]

Hence \( f(s) = 2\pi i \partial_s \delta^-(s) \). This also satisfies \( f(as) = (1/a^2) f(s), \ a > 0 \) and \( f(-s) = -2\pi i \partial_s \delta^+(s) \). Thus we obtain (65), since \( \beta = \beta b, \ s = \beta p \).

5. Divergent beam transform of Lundquist field

If we substitute the Radon transform (9) of the Lundquist field \( \lambda = 1 \) into the integral in (71) using \( x = x'e_{1}(\phi) + ze_{2}, \theta = v'e_{1}(\theta) + vze_{z}, \) and \( b = b'e_{1}(\psi) + b'ze_{z} \) where \( b_r = \sqrt{1 - b_z^2}, \ d\Omega_b = -db_z d\psi, \) the integration over \( b_z \) leads to

\[ I(\theta, x) = (2\pi)^2 \chi F(\theta, x) \]

\[ = -2\pi F_0 \int_{\psi'=-\infty}^{\infty} \left[ e^{i\nu r \cos(\psi' + \theta - \phi)} L(\psi' + \theta) e^{-i\nu r \cos(\psi' + \theta - \phi)} L'(\psi' + \theta) \right] \frac{1}{\cos \psi'} d\psi'. \]
Here $\psi' = \psi - \theta$ (and the integral over $\psi'$ is insensitive to shift of limits of integration). This yields

$$I(\theta, x) = -2\pi F_0 \frac{1}{\nu \nu r} \left[ (I_1^+ + I_1^-) e_r(\theta) - (I_1^+ - I_1^-) e_\theta - i(I_2^+ - I_2^-) e_z \right], \quad (B19)$$

where

$$I_1^\pm = \int_{\psi' = 0}^{2\pi} e^{\pm i \nu r \cos(\psi' + \theta - \phi)} d\psi', \quad I_2^\pm = \int_{\psi' = 0}^{2\pi} e^{\pm i \nu r \cos(\psi' + \theta - \phi)} \frac{1}{\cos \psi'} d\psi', \quad (B20)$$

$$I_3^\pm = \int_{\psi' = 0}^{2\pi} e^{\pm i \nu r \cos(\psi' + \theta - \phi)} \tan \psi' d\psi'.$$

We shall use the variable $\zeta = i\omega z$ where $z = e^{i\psi'}$ and $\omega = e^{i(\theta - \phi)}$. Hence $i \cos(\psi' + \theta - \phi) = (1/2)(\zeta - 1/\zeta)$, $\cos \psi' = (1/2)(1/i\omega)(1/\zeta)(\zeta^2 - \omega^2)$, $\tan \psi' = -i(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2)$, and $d\psi' = -i(1/\zeta) d\zeta$. These integrals become

$$I_1^\pm = -i \int_{C^\prime} e^{\frac{1}{2} u^+ (\zeta - 1)/\zeta} \frac{1}{\zeta} d\zeta, \quad I_2^\pm = 2\omega \int_{C^\prime} e^{\frac{1}{2} u^+ (\zeta - 1)/\zeta} \frac{1}{\zeta^2 - \omega^2} d\zeta, \quad (B21)$$

$$I_3^\pm = - \int_{C^\prime} e^{\frac{1}{2} u^+ (\zeta - 1)/\zeta} \frac{1}{\zeta^2 + \omega^2} d\zeta,$$

where $u^\pm = \pm \nu r$ and $C^\prime$ is a unit circle around the origin (the integrals are insensitive to shift of limits of integration). Note the notation: the Cauchy principal value is meant in these contour integrals.

We shall make use of the generating function$^{63}$ $e^{\frac{1}{2} u^+ (\zeta - 1)/\zeta} = \sum_{n=-\infty}^{\infty} \zeta^n J_n(u^\pm) = J_0(u^\pm) + \sum_{n=1}^{\infty} \left| \zeta^n + (-1)^n \zeta^{-n} \right| J_n(u^\pm)$ for Bessel functions.

In $I_1^\pm$ there are no poles on $C^\prime$ and only the term of order $n = 0$ in the generating function contributes to the integral

$$I_1^\pm = 2\pi J_0(\nu r). \quad (B22)$$

If we substitute the generating function in $I_2^\pm$, we get

$$I_2^\pm = 2\omega \left\{ I_{2a}^\pm J_0(u^\pm) + \sum_{n=1}^{\infty} \left[ I_{2b}^\pm + (-1)^n I_{2c}^\pm \right] J_n(u^\pm) \right\}. \quad (B23)$$

The contributions to the principal value integral $I_{2a}^\pm$ of the residues of simple poles at $\zeta = \pm \omega$ on $C^\prime$ cancel out

$$I_{2a}^\pm = \int_{C^\prime} \frac{1}{\zeta^2 - \omega^2} d\zeta = \pi i \text{Res} [f = 1/(\zeta^2 - \omega^2), \zeta = \omega] + \pi i \text{Res} [f = 1/(\zeta^2 - \omega^2), \zeta = -\omega] = 0. \quad (B24)$$

Meanwhile, for $n \geq 1$

$$I_{2b}^\pm = \int_{C^\prime} \frac{\zeta^n}{\zeta^2 - \omega^2} d\zeta = \pi i \text{Res} [g = \zeta^n/(\zeta^2 - \omega^2), \zeta = \omega] + \pi i \text{Res} [g = \zeta^n/(\zeta^2 - \omega^2), \zeta = -\omega] \quad (B25)$$

$$= \pi i \frac{1}{2} \left[ 1 + (-1)^{n+1} \right] \omega^{n-1},$$

and

$$I_{2c}^\pm = \int_{C^\prime} \frac{\zeta^{-n}}{\zeta^2 - \omega^2} d\zeta = 2\pi i \text{Res} [h = \zeta^{-n}/(\zeta^2 - \omega^2), \zeta = 0] + \pi i \text{Res} [h = \zeta^{-n}/(\zeta^2 - \omega^2), \zeta = \omega] \quad (B26)$$

$$+ \pi i \text{Res} [h = \zeta^{-n}/(\zeta^2 - \omega^2), \zeta = -\omega] \quad (B26)$$

$$= -\pi i \frac{1}{2} \left[ 1 + (-1)^{n+1} \right] \omega^{-(n+1)}.$$
Thus

\[ I_2^\pm = 2\pi i \omega \sum_{n=1}^{\infty} \frac{1}{2} [1 + (-1)^{n+1}] \left[ \omega^{n-1} - (-1)^n \omega^{-(n+1)} \right] J_n(u^\pm) = \pm 4\pi i \sum_{k=0}^{\infty} \cos[(2k+1)(\theta - \phi)]J_{2k+1}(\nu r). \] (B27)

The integral \( I_3^\pm \) reduces to

\[ I_3^\pm = -\left\{ I_{3a}^\pm J_0(u^\pm) + \sum_{n=1}^{\infty} [I_{3b}^\pm + (-1)^n I_{3c}^\pm] J_n(u^\pm) \right\}. \] (B28)

Here

\[ I_{3a}^\pm = \int_{C^\prime} \frac{1}{\zeta} \frac{\zeta^2 + \omega^2}{\zeta^2 - \omega^2} d\zeta = 2\pi \text{Res} [f = \zeta^{-1}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = 0] + \pi \text{Res} [f = \zeta^{-1}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = \omega] = 0, \]

and since \( n \geq 1 \)

\[ I_{3b}^\pm = \int_{C^\prime} \zeta^{n-1} \frac{\zeta^2 + \omega^2}{\zeta^2 - \omega^2} d\zeta = +\pi \text{Res} [g = \zeta^{n-1}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = \omega] + \pi \text{Res} [g = \zeta^{n-1}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = -\omega] = \pi i [1 + (-1)^n] \omega^n, \] (B30)

and

\[ I_{3c}^\pm = \int_{C^\prime} \frac{1}{\zeta^{n+1}} \frac{\zeta^2 + \omega^2}{\zeta^2 - \omega^2} d\zeta = 2\pi \text{Res} [h = \zeta^{-(n+1)}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = 0] + \pi \text{Res} [h = \zeta^{-(n+1)}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = \omega] + \pi \text{Res} [h = \zeta^{-(n+1)}(\zeta^2 + \omega^2)/(\zeta^2 - \omega^2), \zeta = -\omega] = -\pi i [1 + (-1)^n] \omega^{-n}. \] (B31)

Thus

\[ I_3^\pm = -\pi i \sum_{n=1}^{\infty} [1 + (-1)^n] \left[ \omega^n - (-1)^n \omega^{-n} \right] J_n(u^\pm) = 4\pi \sum_{k=1}^{\infty} \sin[2k(\theta - \phi)]J_{2k}(\nu r). \] (B32)

Then we obtain (75) using (B22, B27, B32) in (B18, B19).

We also need\(^3\)

\[ \sin(z \sin \alpha) = 2 \sum_{k=0}^{\infty} \sin[(2k+1)\alpha]J_{2k+1}(z), \quad \cos(z \sin \alpha) = J_0(z) + 2 \sum_{k=1}^{\infty} \cos(2k\alpha)J_{2k}(z), \] (B33)

for the Divergent beam transform (76) of the Lundquist field.
6. Gelfand-Goncharov’s method

If we substitute equation (81) in (6) and interchange the order of integrations, we find

\[ F_\lambda(x) = -\frac{1}{8\pi^3} \lambda \nu \int_{S_0^2} \mathcal{D} F_\lambda(\theta, x) I(\theta) d\Omega_\theta, \]  

where

\[ I(\theta) = \int_{S_0^2} \frac{1}{(\theta \cdot \kappa)^2} d\Omega_\kappa. \]

If we decompose \( \kappa \) into components which are respectively parallel and orthogonal to \( \theta \): \( \kappa = \kappa_\parallel + \kappa_\perp \) (\( \kappa_\parallel = u\theta \), \( u = \cos \theta \), and \( d\Omega_\kappa = \sin \theta d\theta d\phi = -dud\phi \)), then we find

\[ I(\theta) = 2\pi I, \]

where

\[ I = \int_{u=-1}^{1} \frac{1}{u^2} du. \]

The principal value of this integral can be easily calculated using a contour integral: \( I = -2 \). This leads to (82).

**APPENDIX C: MINI-TWISTORS**

1. Helmholtz equation: (mini-)twistor solution

We reproduce the following solution of Helmholtz equation from Ref. 46 which is partially based on Hitchin’s notes.

The solution of wave equation

\[ \Box^2 \psi = 0, \]  

\[ \Box^2 = \partial^2/\partial t^2 - \nabla^2, \quad \nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \]  

is given by

\[ \psi(x, y, z, t) = \int_C g[w(t + z) + (x + iy), \omega(x - iy) + (t - z), \omega] d\omega = \int_C g(p, q, \omega) d\omega, \]

where \( g \) is holomorphic in each of its arguments: \( p = x + iy + \omega(t + z), \quad q = t - z + \omega(x - iy) \) and \( \omega \). We can easily verify (C1) using \( \partial/\partial t = \omega \partial/\partial p + \partial/\partial q, \quad \partial/\partial x = \partial/\partial p + \omega \partial/\partial q, \quad \partial/\partial y = i(\partial/\partial p - \omega \partial/\partial q), \quad \partial/\partial z = \omega \partial/\partial p - \partial/\partial q. \)

We can write the wave equation (C1) as \( \partial^2\psi/\partial u \partial v - \partial^2\psi/\partial \zeta \partial \overline{\zeta} = 0 \) introducing new variables \( u = t - z, \quad v = t + z, \quad \zeta = x + iy, \quad \overline{\zeta} = x - iy \) and \( \psi(x, y, z, t) \to \psi(u, v, \zeta, \overline{\zeta}). \) This is immediately satisfied since: \( \partial/\partial u = \partial/\partial q, \quad \partial/\partial v = \omega \partial/\partial p, \quad \partial/\partial \zeta = \partial/\partial p, \quad \partial/\partial \overline{\zeta} = \omega \partial/\partial q. \) Then \( p = \zeta + \omega v, \quad q = \omega \overline{\zeta} + u \) in (C2).

For the Helmholtz equation, \(46\) we suppose

\[ \frac{\partial\psi}{\partial t} = -ik\psi. \]  

This leads to \( (\partial/\partial t)g[w(t + z) + (x + iy), \omega(x - iy) + (t - z), \omega] = -ikg \) using (C2), that is

\[ \left( \omega \frac{\partial}{\partial p} + \frac{\partial}{\partial q} \right) g(p, q, \omega) = -ikg. \]
A simple method for solving this equation is to use an integrating factor. For example we can use an integrating factor that is a function of $q$ so as to remove the right-hand side

$$g(p, q, \omega) = e^{-ikq}h(p, q, \omega).$$

(C5)

Then (C4) becomes

$$\omega \frac{\partial h}{\partial p} + \frac{\partial h}{\partial q} = 0.$$  

(C6)

The solution of this equation is given by

$$h(p, q, \omega) = H(\eta, \omega),$$

where $H$ is a holomorphic function of two arguments:

$$\eta_x(\omega) = p - \omega q = x + iy + 2\omega - (x - iy)^2$$

and $\omega$. Then $g(p, q, \omega) = e^{-ikq}H(\eta_x(\omega), \omega)$ and (C2) yields

$$\psi = e^{-ikt}\phi(x, y, z).$$

(C7)

where

$$\phi(x, y, z) = \int_C e^{-i\frac{k}{2}[\omega(x - iy) - z]}H(\eta_x(\omega), \omega)d\omega,$$

satisfies the Helmholtz equation $\nabla^2\phi = -k^2\phi$. This reduces to solution of Laplace equation for $k = 0$.

We could also use another integrating factor

$$g(p, q, \omega) = e^{-i\frac{k}{2}(p/\omega + q)}H(p - \omega q, \omega).$$

(C9)

This would yield

$$\phi = \int_C e^{-i\frac{k}{2}[\omega(x - iy) + (x + iy)/\omega]}H(\eta_x(\omega), \omega)d\omega.$$  

(C10)

Any solution of (C4) can be written in this form. Because introducing new variables: $\alpha = p - \omega q$, $\beta = p + \omega q$, the equation (C4) becomes

$$\frac{\partial g}{\partial \beta} = -i\frac{k}{2}\omega g,$$

and the general solution of this equation is $g(\alpha, \beta, \omega) = e^{-i\frac{k}{2}\omega}h(\alpha, \omega)$, $\alpha = \eta(\omega)$.

2. Contour integrals for Bessel functions

We need the following integrals

$$J_m(\nu r) \cos m\varphi = \frac{im}{2\pi} \int_{\theta=0}^{2\pi} e^{-im(\cos \theta + y \sin \theta)} \cos m\theta d\theta, \quad J_m(\nu r) \sin m\varphi = \frac{im}{2\pi} \int_{\theta=0}^{2\pi} e^{-im(\cos \theta + y \sin \theta)} \sin m\theta d\theta.$$  

(C12)

where $x = r \cos \varphi$, $y = r \sin \varphi$. These lead to

$$J_m(\nu r) e^{im\varphi} = \frac{im}{2\pi} \int_{\theta=0}^{2\pi} e^{-im(\cos \theta + y \sin \theta)} e^{im\theta} d\theta.$$  

Hence
3. Solution with Laurent series

If we use \( f = (1/2)[\omega(x - iy) + (x + iy)/\omega] \) with \( \omega = i\omega', \) \( (k = \nu) \) and choose \( u = h(\omega') \) which has a Laurent series: \( h(\omega') = 1/\omega'^{(n+1)} \), the equation (94) leads to

\[
F(x) = \int_C \left[ i \left( 1 + \omega'^2 \right), - \left( 1 - \omega'^2 \right), -2\omega' \right] e^{-i\varphi(x+iy)/\omega'-(x-iy)\omega'} \frac{1}{\omega'^{(n+1)}} d\omega'.
\]  

(16)

The generating function \( e^{\frac{1}{2}p(t-1/t)} = \sum_{m=-\infty}^{\infty} t^m J_m(p) \) for the Bessel functions yields

\[
e^{-\frac{1}{2}p[(x+iy)/\omega'-(x-iy)\omega']} = \sum_{m=-\infty}^{\infty} e^{-im\varphi} \omega'^m J_m(\nu r),
\]  

(17)

with \( p = -\nu r, t = -(x - iy)\omega'/r = -e^{-i\varphi}\omega', \) \( (x + iy = re^{i\varphi}) \) and \( J_m(\nu r) = (-1)^m J_m(x) \). The integrals in (16) can be evaluated using

\[
\int_C \sum_{m=-\infty}^{\infty} c_m \omega'^m \omega'^{(n+1)} d\omega' = 2\pi i c_n, \quad c_m = e^{-im\varphi} J_m(\nu r),
\]  

(18)

which is based on residues. \( e^{\varphi} \) We find

\[
F(x) = -2\pi e^{-i\varphi} \left( [J_\nu(\nu r) + e^{i2\varphi} J_{\nu-2}(\nu r)] , i \left[ J_\nu(\nu r) - e^{i2\varphi} J_{\nu-2}(\nu r) \right] , 2i e^{i\varphi} J_{\nu-1}(\nu r) \right).
\]  

(19)

This reduces to

\[
F(x) = 4\pi i e^{-i\varphi} \left[ im \frac{1}{\nu r} J_m(\nu r)e_r + J'_m(\nu r)e_\varphi - J_m(\nu r)e_z \right], \quad m = n - 1
\]  

(20)

in cylindrical coordinates: \( e_r = \cos \varphi e_x + \sin \varphi e_y, \) \( e_\varphi = -\sin \varphi e_x + \cos \varphi e_y, \) using the identities: \( J_n(x) - J_{n-2}(x) = -2J'_{n-1}(x), \) \( x[J_n(x) + J_{n-2}(x)] = 2(n-1)J_{n-1}(x), \) \( (x = \nu r). \) This is a circular cylindrical CK solution with no \( z \) dependence, up to conventions.
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