Quantum Fields in an Expanding Universe

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Abstract

We extend our analysis for scalar fields in a Robertson-Walker metric to the electromagnetic field and Dirac fields by the method of invariants. The issue of the relation between conformal properties and particle production is re-examined and it is verified that the electromagnetic and massless spinor actions are conformal invariant, while the massless conformally coupled scalar field is not. For the scalar field case it is pointed out that the violation of conformal symmetry due to surface terms, although inessential for the equations of motion, does lead to effects in the quantized theory.

1 Introduction

The interest in quantum field theory (henceforth QFT) in classical curved space-times has never ceased especially after the development of particle production in cosmological spacetimes, initiated by L. Parker in 1968 [1], and the discovery of black hole evaporation by S. W. Hawking in 1974 [2]. Such a semiclassical treatment for gravity should hold in a region between the Planck length and the Compton wavelength (and certainly greater than this) of the matter field considered and could be a useful route towards understanding the principal ingredients for a theory which unifies gravity and quantum mechanics. All the theoretical motivations for studying QFT in curved spacetimes are corroborated by a possible application of the results to cosmology: a spectrum of quantum fluctuations during inflation could be related to the primordial spectrum of cosmological perturbations [3].

For the cosmological case particle production is related to the simple and well studied case of a harmonic oscillator with time dependent parameters. The time dependence of the metric leads to the appearance of different vacua for differing times during the evolution
of a free field. When exact solutions to the field equations of motion are not known, a systematic treatment based on the adiabatic approximation can be found in the literature [4]. For this reason it is a usual practice to rescale fields by a time dependent factor so as to obtain suitable expressions in order to implement an adiabatic expansion around a time independent solution. This rescaling factor is also related to the conformal weight of the field which leaves the equations of motion invariant under a conformal transformation in both the metric and the field. Since the Robertson-Walker (henceforth RW) metric is conformally related to a Minkowski metric one has a correspondence between fields in a RW metric and in flat spacetimes with time dependent masses. For the case of massless conformally coupled fields the time dependent parameters in the equations of motion disappear and the theory in such a curved spacetime corresponds to a theory in Minkowski spacetime.

The method of invariants [5] allows one to exactly quantize a harmonic oscillator with time dependent coefficients. The application of this method in QFT improves on the adiabatic approximation and allows one to introduce a vacuum and a Fock space associated with the quantum invariants. The use of the method of quantum invariants [5] has been previously applied, within the context of cosmology, to a quantized scalar field in a de Sitter space-time with a flat spatial section [6, 7]. Such an approach also arises naturally within a Born-Oppenheimer context, for the matter-gravity system [9], in a simple minisuperspace model when the semiclassical limit is taken for gravity and fluctuations are neglected [10].

In the previously studied scalar case a massive scalar field with a non-minimal coupling was examined [7]. It was found that the expectation value of the Hamiltonian of a Fourier mode of the field grows with time even in the massless conformally coupled case. However, for a non-minimally coupled scalar field the Hamiltonian density differs from the 00 component of the energy-momentum tensor, and the latter vanishes for zero mass and conformal coupling. The usual statement of no particle production in the massless conformally coupled case should then be applied directly to the Einstein equations: only the conformal anomaly is a source for the Einstein tensor in this case. Further in [7] it was also pointed out that the usual conformal rescaling does not leave the action invariant because of the presence of a boundary term.

It is then natural to re-examine, within the context of quantum invariants, the results previously obtained for the electromagnetic field and spin 1/2 particles. This shall be done in the next two sections respectively. In section 4 we discuss the conformal properties of the action considered, while in section 5 we analyze in detail the problems arising from the quantization of rescaled and non-rescaled fields. Finally in section 6 we present our
conclusions.

2 Electromagnetic Field

In this and the following sections we consider a RW line element

\[ ds^2 = -d\tau^2 + a^2(\tau)g_{ij}dx^idx^j \]  

(1)

where \( g_{ij} \) is the three metric for a flat three-space. In the absence of charges the Lagrangian density for the electromagnetic potential \( A_\mu \) is given by:

\[ \mathcal{L}_{EM} = \sqrt{-g} \left[ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right] \]
\[ = \frac{a^3}{4} \left[ \frac{2}{a^2} F_{0j}^2 - \frac{1}{a^4} F_{ij}^2 \right] \]  

(2)

where \( F_{\mu\nu} \equiv \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu \) (with \( \nabla_\mu \) the covariant derivative) because of the symmetry in the lower indices of the Cristoffel connections. On choosing the generalized Lorentz gauge \( \nabla_\mu A_\mu = 0 \) and \( A_0 = 0 \) (which is always possible in the source-free case) and considering a RW metric, we obtain the so called radiation gauge \( (\vec{\nabla} \cdot \vec{A} = 0) \). On then substituting \( A_\mu = (A_0, \vec{A}) \) in Eq. (2) and using the radiation gauge one has

\[ F_{0j} = \dot{A}_j \]
\[ F_{ij} = (\vec{\nabla} \times \vec{A})_{ij}, \]  

(3)

where by the dot we denote a derivative with respect to the proper time \( \tau \). Let us note that the vector potential \( \vec{A} \) has been chosen to be the covariant \( A_i \), and this is a natural choice if we wish to identify \( F_{\mu\nu} \) with a two form (metric independent). Further the choice of \( \vec{A} \) as a generalized coordinate with its subsequent quantization allows us to use the method of invariants for all the Fourier components of \( \vec{A} \) (this would not be possible for the case of the identification with the covariant form \( A^\mu \)). The final expression one then obtains for the electromagnetic Lagrangian density is

\[ \mathcal{L}_{EM} = \frac{1}{2} \left\{ a\ddot{\vec{A}} + \frac{1}{a} \vec{A} \cdot \nabla^2 \vec{A} - \frac{1}{a} \vec{\nabla} \cdot \left[ A_i \vec{\nabla} A_i - (\vec{A} \cdot \vec{\nabla}) \vec{A} \right] \right\} \]  

(4)

We expand \( \vec{A} \) as

\[ \vec{A} = \frac{1}{\sqrt{V}} \sum_{\vec{k}, \lambda} \left[ c^{(\lambda)}(\tau)\vec{\varepsilon}^{(\lambda)} e^{i\vec{k} \cdot \vec{x}} + c^{(\lambda)*}(t)\tilde{\varepsilon}^{(\lambda)} e^{-i\vec{k} \cdot \vec{x}} \right], \]  

(5)

where \( \lambda \) runs over the (two) polarization states and \( \vec{\varepsilon}^{(\lambda)} \) is a unit vector which satisfies \( \vec{k} \cdot \vec{\varepsilon}^{(\lambda)} = 0 \) and \( \vec{\varepsilon}^{(\lambda)} \cdot \vec{\varepsilon}^{(\lambda')} = \delta_{\lambda \lambda'} \). On further separating \( c^{(\lambda)}(\tau) \) into real and imaginary parts

\[ c^{(\lambda)}_k(\tau) = \frac{1}{\sqrt{2}} \left( c^{(\lambda)}_{k1} + ic^{(\lambda)}_{k2} \right) \]  

(6)
the action becomes:

\[ S = \sum_{k,i,\lambda} S_{ki}^{(\lambda)} = \frac{1}{2} \sum_{k,i,\lambda} \int \left[ a\dot{c}_{ki}^{(\lambda)} + \frac{k^2}{a} c_{ki}^{(\lambda)} \right] d\tau. \]  

(7)

Thus we see that the different modes \( k, i, (\lambda) \) decouple and one then obtains the following Hamiltonian for each mode:

\[ H_{i,k}^{(\lambda)} = \frac{1}{2} \left( \frac{\pi_{i,k}^{(\lambda)}}{a} + a\omega_k^2 c_{ki}^{(\lambda)} \right), \]

(8)

where \( \pi_{i,k}^{(\lambda)} = a\dot{c}_{ki}^{(\lambda)} \) and \( \omega_k^2 = k^2/a^2 \). The classical equation of motion is:

\[ \ddot{c}_{ki}^{(\lambda)} + \dot{a} a\dot{c}_{ki}^{(\lambda)} + a\omega_k^2 c_{ki}^{(\lambda)} = 0 \]  

(9)

and we proceed in analogy with the scalar field case \([7]\). On canonically quantizing, the Hamiltonian in Eq. (8) can be factorized as (henceforth we shall denote collectively \( k, i, (\lambda) \) by \( \sigma \) and retain the subscript \( k \) only when relevant):

\[ \hat{H}_\sigma = \hbar \omega_k \left( \hat{a}_\sigma^\dagger \hat{a}_\sigma + \frac{1}{2} \right) \]

(10)

with

\[ \hat{a}_\sigma = \left( \frac{a\omega_k}{2\hbar} \right)^\frac{1}{2} \left( \hat{c}_\sigma + i \frac{\hat{\pi}_\sigma}{\omega_k} \right), \]

(11)

\[ \hat{a}_\sigma^\dagger = \left( \frac{a\omega_k}{2\hbar} \right)^\frac{1}{2} \left( \hat{c}_\sigma - i \frac{\hat{\pi}_\sigma}{\omega_k} \right), \]

and \([\hat{a}_\sigma, \hat{a}_\sigma^\dagger] = \delta_{\sigma, \sigma'}\).

As we mentioned, a suitable method for the study of time dependent quantum systems is that of invariants \([\]\). In particular a hermitian operator \( \hat{I} \) which satisfies:

\[ \frac{\partial \hat{I}_\sigma(\tau)}{\partial \tau} = -\frac{i}{\hbar} [\hat{I}_\sigma(\tau), \hat{H}_\sigma(\tau)] = 0 \]

(12)

is an invariant. The invariant \( \hat{I}_\sigma \) has real, time independent, eigenvalues and in our case, can be decomposed in terms of basic linear invariants \([\]\):

\[ \hat{I}_{b,\sigma}(\tau) \equiv e^{i\Theta_k(\tau)} \hat{b}_\sigma(\tau) \equiv e^{i\Theta_k} \left( \frac{\hat{c}_\sigma}{\rho_k} + i \left( \frac{\pi_k}{\rho_k} \dot{\pi}_\sigma - a\rho_k \dot{c}_\sigma \right) \right), \]

(13)

\[ \hat{I}_{b,\sigma}^\dagger(\tau) \equiv e^{-i\Theta_k(\tau)} \hat{b}_\sigma(\tau)^\dagger \equiv \frac{e^{-i\Theta_k}}{\sqrt{2\hbar}} \left( \frac{\hat{c}_\sigma}{\rho_k} - i \left( \frac{\pi_k}{\rho_k} \dot{\pi}_\sigma - a\rho_k \dot{c}_\sigma \right) \right), \]

where \( \rho_k(\tau) \) is real and satisfies \([\]\):

\[ \ddot{\rho}_k + \frac{\dot{a}}{a} \dot{\rho}_k + \omega_k^2 \rho_k = \frac{1}{a^2 \rho_k^3} \]

(14)

with:

\[ \Theta_k(\tau) = \int_{-\infty}^{t} \frac{d\tau'}{a(\tau')} \rho_k^2(\tau'). \]

(15)
and:

\[ [\hat{b}_\sigma, \hat{b}_\sigma^\dagger] = \delta_{\sigma\sigma'} . \]  

(16)

Let us now note that the above equations (13) may be rewritten in terms of the classical solutions \( c_\sigma \) to Eq. (9) through \( c = \rho_k e^{-i\Theta_k} \):

\[ \hat{I}_{b\sigma}(\tau) = i (c_\sigma^* \hat{\pi}_\sigma - \pi_\sigma^* \hat{c}_\sigma) \]  

(17)

and the quadratic, hermitian, adiabatic invariant originally introduced in [5] is given by:

\[ \hat{I}_\sigma(\tau) = \hbar \left( \hat{b}_\sigma^\dagger \hat{b}_\sigma + \frac{1}{2} \right) = \frac{1}{2} \left[ \frac{\dot{b}_\sigma^2}{\rho_k^2} + (\rho_k \dot{\pi}_\sigma - a \rho_k \dot{c}_\sigma)^2 \right] , \]  

(18)

The linearly independent solutions to the equation of motion (9) are the Bessel functions:

\[ c = \eta^{1/2} \left\{ \begin{array}{c} J_{1/2}(k\eta) \\ N_{1/2}(k\eta) \end{array} \right\} \]  

(19)

where \( \eta \) is the conformal time and the above solutions are true both for de Sitter \( a = -\frac{1}{H\eta} \) (\( H \) is the Hubble constant, \( -\infty < \eta < 0 \)) and for power behaviour \( a = \eta^p \) (with \( p \) a positive real number). The general solution to Eq. (14) can be written as a non-linear combination of the solutions to the equations of motion as shown in [7]:

\[ \rho = \eta^{1/2} \left[ AJ_{1/2}(k\eta) + BN_{1/2}(k\eta) + 2(AB - \frac{\pi^2}{4})J_{1/2}(k\eta)N_{1/2}(k\eta) \right]^{\frac{1}{2}} \]  

(20)

where \( A, B \) are real constants (\( k \) independent because of the spatial symmetry of RW spacetime). On choosing \( A = B = \pi/2 \) one has

\[ \rho = \frac{1}{\sqrt{k}} \]  

(21)

which is independent of \( \eta \). The choice \( A = B \) is associated with the adiabatic vacuum at early times or the adiabatic vacuum for wavelengths \( 2a\pi/k \) which are well inside the Hubble radius \( a/\dot{a} = H^{-1} \) (see [7]), i.e. the Bunch-Davies vacuum.

From Eq. (21) we see that the annihilation operators \( \hat{a} \) and \( \hat{b} \) coincide for all times which implies

\[ \hat{H}_{k\lambda}^{(\lambda)} = \omega_k \hat{I}_{k\lambda}^{(\lambda)} . \]  

(22)

Hence for a (massless) photon the initial adiabatic vacuum remains such for all times, leading to a null photon production in RW spacetime, and its energy is redshifted as expected for radiation. We end by observing that if one relaxes the assumption of an adiabatic vacuum at early times, i.e. allows \( A \neq B \), one has a number of photons oscillating around the initial number, without a net growth.
3 Dirac Spinor Field

For the case of a massive spin $\frac{1}{2}$ field $\Psi$ the lagrangian density is:

$$\mathcal{L} = -\sqrt{-g} \left\{ \frac{i}{2} [\bar{\Psi} \gamma^\mu (\nabla_\mu \Psi) - (\nabla_\mu \bar{\Psi}) \gamma^\mu \Psi] + \mu \bar{\Psi} \Psi \right\}$$  \hspace{0.5cm} (23)

which in RW spacetimes can be rewritten as

$$\mathcal{L} = a^3 \left\{ \frac{i}{2} (\dot{\Psi}^\dagger - \dot{\Psi} \bar{\Psi}) - \Psi^\dagger \gamma_4 (\vec{\alpha} \cdot \vec{\nabla} + \mu) \Psi \right\}$$

$$\equiv a^3 \left\{ \frac{i}{2} (\dot{\Psi}^\dagger - \dot{\Psi} \bar{\Psi}) - \Psi^\dagger M \Psi \right\}$$  \hspace{0.5cm} (24)

where $\mu$ is the inverse Compton wavelength of the spinor field and we shall use the Pauli-Dirac representation for the $\gamma$ matrices. On expanding

$$\Psi = \frac{1}{\sqrt{V}} \sum_k \Psi_k e^{i \vec{k} \cdot \vec{x}}$$  \hspace{0.5cm} (25)

one obtains an action:

$$S = \sum_k S_k = \frac{1}{\sqrt{V}} \sum_k \int d\tau a^3 \left[ \frac{i}{2} (\dot{\Psi}_k^\dagger \Psi_k - \dot{\Psi}_k \bar{\Psi}_k) - \Psi_k^\dagger \gamma_4 (\vec{\alpha} \cdot \vec{\nabla} + \mu) \Psi_k \right]$$

$$\equiv \frac{1}{\sqrt{V}} \sum_k \int d\tau a^3 \left[ \frac{i}{2} (\dot{\Psi}_k^\dagger \Psi_k - \dot{\Psi}_k \bar{\Psi}_k) - \Psi_k^\dagger M_k \Psi_k \right]$$  \hspace{0.5cm} (26)

and again one can consider each Fourier mode separately and obtain a single mode Hamiltonian:

$$H_k = \psi_k^\dagger M_k \psi_k$$  \hspace{0.5cm} (27)

where $\psi_k = a^{3/2} \Psi_k$ and the matrix $M_k$ is:

$$M_k = \begin{pmatrix} \mu & \vec{\alpha} \cdot \vec{k} \\ \vec{\alpha} \cdot \vec{k} & -\mu \end{pmatrix}$$  \hspace{0.5cm} (28)

From the Lagrangian density and eq.(25) one obtains a classical equation of motion:

$$\left[ -\frac{\partial}{\partial t} + i \gamma_4 \vec{\alpha} \cdot \vec{k} + \gamma_4 \mu \right] w_k^{(r)} = 0$$  \hspace{0.5cm} (29)

which has been previously solved for a de Sitter space-time obtaining:

$$w_{1,k}^{(r)} = \frac{1}{a^{1/2}} \left( \frac{i Z_{\nu}^{(r)}(r)}{\vec{\alpha} \cdot \vec{k} Z_{\nu-1}^{(r)}} \right)_{r=1,2}$$  \hspace{0.5cm} (30)

and

$$w_{2,k}^{(r)} = \frac{1}{a^{1/2}} \left( \frac{-\vec{\alpha} \cdot \vec{k} Z_{\nu}^{(r)}(r)}{-i Z_{\nu-1}^{(r)}} \right)_{r=3,4}$$  \hspace{0.5cm} (31)
where $r = 1, 3(2, 4)$ correspond to spin up (down) for the Pauli spinors $\chi^{(r)}$ and $Z_\nu(k|\eta|)$ are Bessel functions ($J_\nu$, $N_\nu$ or a combination of them). Further $\nu = \frac{1}{2} - i \frac{\mu}{H}$ and we introduce a normalization factor $N_{a k}$ determined by

$$N_{a k}^2 u_k^{(r)\dagger} u_k^{(r')} = \delta_{r r'}.$$  

(32)

It is now of interest to examine the adiabatic limit for which the solutions (30) and (31) reduce to the usual static solutions:

$$u_k^{(r)} = \left( \frac{\chi^{(r)}}{a(\omega k + \mu)} \chi^{(r)} \right) e^{-i\omega t} \quad r = 1, 2$$  

(33)

and

$$u_k^{(r)} = \left( \frac{-\bar{\sigma} \cdot k}{a(\omega k + \mu)} \chi^{(r)} \right) e^{i\omega t} \quad r = 3, 4$$  

(34)

respectively, where $\omega = (\frac{k^2}{\omega^2} + \mu^2)^{1/2}$ and $r = 1, 3(2, 4)$ again refers to spin up (down) for the Pauli spinors. As in the previous case we introduce a normalization factor $N_k$ determined by:

$$N_k^2 u_k^{(r)\dagger} u_k^{(r')} = \delta_{r r'}.$$  

(35)

We expect that the reduction to the usual static solutions occurs for very early times ($\tau \to -\infty$) or for wavelengths which are very small compared to the de Sitter horizon $H^{-1}$. In such a limit ($-k\eta = k \frac{\eta}{H} e^{-Ht} \to \infty$) Eq. (33) leads to

$$u_k^{(r)} \to \frac{1}{\sqrt{2}} \left( \frac{-\chi^{(r)}}{\bar{\sigma} \cdot k} \right) e^{ik\eta} \quad r = 1, 2$$  

(36)

which requires that one take the Hankel functions:

$$Z_\nu = H^{(1)}_\nu$$  

(37)

and a similar result only involving $H^{(2)}_\nu$ is obtained from Eqs. (31) and (34). Again, as in previous cases, we have a two parameter set of solutions: it is only by requiring agreement for a particular (early time) limit with the adiabatic solutions that we constrain them.

We may quantize the system by postulating the usual canonical anticommutation relations

$$\{ \hat{\psi}^{\dagger}_{k}, \hat{\psi}_{k} \} = \hbar$$  

(38)

with all the other anticommutators being zero. If we employ the adiabatic (static) solutions Eqs. (33) and (34) one may expand

$$\hat{\psi} = \frac{1}{\sqrt{V}} \sum_k e^{ik\vec{x}} \sum_r N_k u_k^{(r)} a_k^{(r)}$$  

(39)
with \( \{ \hat{a}_k^\dagger, \hat{a}_k \} = \hbar \) and on substituting in the quantum Hamiltonian we obtain:

\[
\hat{H}_k = \hat{\psi}_k^\dagger M_k \hat{\psi}_k = \sum_{r=1,2} \omega_k \hat{a}_k^{(r)} \hat{a}_k^{(r)} + \sum_{r=3,4} \omega_k \hat{a}_k^{(r)} \hat{a}_k^{(r)}
\]

(40)

On then introducing a vacuum consisting of a "Dirac sea" of negative energy states one has the usual interpretation of the destruction of a negative energy as the creation of a positive energy antiparticle. Thus through the usual replacements

\[
\hat{\psi}_k = \frac{1}{\sqrt{V}} \sum_k N_k \left( e^{i\vec{k} \cdot \vec{x}} \hat{a}_k^{(s)} \hat{\psi}_k^{(s)} + e^{-i\vec{k} \cdot \vec{x}} \hat{a}_k^{(s)} \hat{\psi}_k^{(s)} \right)
\]

(42)

For this case, as in the previous section, one may also construct linear invariants \( \hat{I} \) by using Heisenberg fields and classical solutions through:

\[
\hat{I}_k^{(r)} = w_k^{(r)} \hat{\psi}_k = N_{ak} \hat{b}_k^{(r)}
\]

(43)

where \( \{ \hat{b}_k^{(r)} \} = \hbar \) which, on using the anticommutation relations for \( \hat{\psi} \) and the classical equations of motions for \( w \), satisfy:

\[
\frac{\partial \hat{I}_k^{(r)}(t)}{\partial t} - \frac{i}{\hbar} [\hat{\psi}_k^{(r)}(t), \hat{H}_k(t)] = 0.
\]

(44)

Correspondingly one may expand

\[
\hat{\psi}_k = \frac{1}{\sqrt{V}} \sum_k e^{i\vec{k} \cdot \vec{x}} \sum_r N_{ak} w_k^{(r)} \hat{b}_k^{(r)}
\]

(45)

and again one may introduce a vacuum consisting of a 'sea' of the equivalent of the negative energy states \( (r = 3, 4) \) with the usual interpretation of a destruction operator as the creation of a 'hole'. Thus the relations equivalent to eqs. (41) hold for the \( b \) operators, where the \( d \) operators and the functions \( y \) are introduced for \( r = 3, 4 \) (corresponding to \( c \) and \( w \)), obtaining

\[
\hat{\psi}_k = \frac{1}{\sqrt{V}} \sum_k N_{ak} \left( e^{i\vec{k} \cdot \vec{x}} \hat{b}_k^{(s)} w_k^{(s)} + e^{-i\vec{k} \cdot \vec{x}} \hat{b}_k^{(s)} \hat{y}_k^{(s)} \right)
\]

(46)

It is clear that the creation and destruction operators in the two bases are related by a Bogoliubov transformation whose coefficients may be easily determined; thus for example the creation of an invariant fermion quantum will correspond to a mixture of the creation of a Dirac particle and the destruction of a Dirac antiparticle. Further the quantum
invariants allow us to define an invariant vacuum (the fermion equivalent of the Bunch-Davies vacuum [12]) by

\[ b_k^{(s)}|0\rangle_b = a_k^{(s)}|0\rangle_b = 0 \]  

(47)

We may now examine particle production from the vacuum during a de Sitter expansion. In order to do this it will be sufficient to consider the expectation value of the Hamiltonian with respect to the above vacuum. On using the \( \psi \) expansion in the invariant basis it is straightforward to obtain

\[ \lim_{a \to \infty} b(0)|\hat{H}_k(t)|0\rangle_b = \lim_{a \to \infty} N_{ak}^2 \sum_{r=3,4} w_k^{(r)\dagger} M_k w_k^{(r)} = 2 \mu \left\{ \left| \frac{\pi [1 + i \cot (\nu - 1)\pi]}{\Gamma(\nu)^2} \right|^2 - 1 \right\} \left\{ \left| \frac{\pi [1 + i \cot (\nu - 1)\pi]}{\Gamma(\nu)^2} \right|^2 + 1 \right\}^{-1} \]  

(48)

which is constant, that is the number of fermions does not increase in time, and in particular we note that the RHS of Eq. (48) is zero for \( \mu = 0 \) (actually there is a non-leading term \( O(k/a) \) corresponding to radiation). Let us note that we have not concerned ourselves with renormalizing the vacuum energy since we are just interested in changes in it.

One may also consider the expectation value of the Hamiltonian with respect to other states, such as coherent states. Such states, for fermions, essentially consist of a superposition of zero and one fermion states, because of the Pauli exclusion principle. It is straightforward to verify that in this case also the expectation value of the Hamiltonian is asymptotically constant in time and is of order \( O(k/a) \) for \( \mu = 0 \). This of course means that in the de Sitter expansion the number of fermions in a given mode is asymptotically a constant.

In the next section we shall re-examine the results of this and the previous section by examining the properties of the action rather than the equations of motions under conformal transformations.

4 Conformal Invariance

Let us consider a general line element:

\[ ds^2 = g_{\mu \nu} dx^\mu dx^\nu \]  

(49)

and the action of a massless conformally coupled scalar field:

\[ S_\Phi = - \int d^4x \sqrt{-g} \left[ \frac{1}{2} g^{\mu \nu} \nabla_\mu \Phi \nabla_\nu \Phi + \frac{1}{12} R \Phi^2 \right] . \]  

(50)

We may now perform a conformal trasformation:

\[ g_{\mu \nu} \rightarrow \tilde{g}_{\mu \nu} = \Omega^2 g_{\mu \nu} \]

\[ \Phi \rightarrow \tilde{\Phi} = \Omega^{-1} \Phi \]  

(51)
with $\Omega$ real, non-zero, continuous and obtain for the change in the action

$$
\tilde{S}_\Phi - S_\Phi = \frac{1}{2} \int_V d^4x \sqrt{-g} \nabla^\mu \left( \Phi^2 \nabla_\mu \ln \Omega \right)
$$

$$
= \frac{1}{2} \int_{\Sigma(V)} d^3\sigma \sqrt{-g} n^\mu \Phi^2 \nabla_\mu \ln \Omega
$$

(52)

where $n_\mu$ is a unit vector perpendicular to the three dimensional surface $\Sigma$ containing $V$.

In particular for the RW metric (1) the surface term becomes

$$
- \frac{1}{2} \int_V d^3x d\tau \frac{\partial}{\partial \tau} \left( \Phi^2 \dot{a} \right) = - \frac{1}{2} \int_V d\eta d^3x \frac{\partial}{\partial \eta} \left( \Phi^2 \frac{1}{a} \frac{\partial a}{\partial \eta} \right)
$$

(53)

in agreement with our previous result [7]. On the right hand side we have introduced the conformal time $\eta$ with $ad\eta = d\tau$ with which the metric (1) becomes conformally flat:

$$
ds^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2).
$$

(54)

The electromagnetic field case is particularly simple: $A_\mu$ has zero conformal weight, that is it is unchanged under conformal trasformations. This is also true for $F_{\mu\nu}$ and for the action associated with it. For the case of fermions, one also has that the massless spin 1/2 Lagrangian density is invariant under conformal trasformations with

$$
\psi \rightarrow \tilde{\psi} = \Omega^{-3/2} \psi
$$

(55)

Thus for the case of the electromagnetic field and massless fermions conformal invariance holds both for the action and the equations of motion. Accordingly one then has that for a conformally flat metric the flat space-times results are reproduced and there is no particle production as the metric changes.

For a massless conformally coupled scalar field, on the other hand, conformal invariance holds only for the equation of motion, while the action and its transformed version differ through the presence of a boundary (for the RW metric a total derivative) term. Boundary terms, which are classically associated with canonical transformations, do not change the equations of motion and conserved charges. However for a RW metric the matter Hamiltonian is not a conserved charge and hence the time evolution is changed. Thus any attempt to quantize employing Lagrangians obtained on neglecting surface terms in RW metrics leads to questionable results, as also stated, but for a different reason, in [13].

5 Rescaled scalar field

Let us further discuss the consequences of the presence of a (time derivative) surface term in the scalar field action. As we have mentioned such a term is classically associated with a canonical transformation and quantum mechanically will correspond to a
unitary transformation (as is expected through the Poisson bracket - canonical commutator correspondence, naturally on neglecting eventual anomalies). On expanding a massive non-minimally coupled scalar field in Fourier modes as in [7]:

\[
\Phi = \frac{1}{\sqrt{V}} \sum_k \left[ e^{i k \cdot x} \Phi_k(\tau) + e^{-i k \cdot x} \Phi^*_k(\tau) \right]
\]

(56)

and on separating real and imaginary parts:

\[
\Phi_k(\tau) = \frac{1}{\sqrt{2}} \left( \phi^+_k + i \phi^-_k \right)
\]

(57)

one obtains an action for each mode \(k, i\)

\[
S_{\phi,k} = \frac{1}{2} \int a^3 d\tau \left( \dot{\phi}_k^2 + \dot{\Phi}_k^2 - \omega^2_k \phi^2_k \right)
\]

(58)

with \(\omega^2_k = \frac{k^2}{a^2} + \mu^2 + \xi R\) and henceforth for simplicity we shall consider one mode \(i, k\) and omit all such indices. On rescaling \(\phi = \zeta/a\) one immediately obtains the corresponding action:

\[
S_{\zeta} = \frac{1}{2} \int a d\tau \left[ \dot{\zeta}^2 - (\omega^2 - \frac{1}{6} R)\zeta^2 - \frac{1}{a} \frac{d}{d \tau} \left( \dot{a} \zeta^2 \right) \right]
\]

(59)

From the above actions, Eqs. (58) and (59) one obtains the following Hamiltonians (related to evolution in proper time):

\[
H_{\phi} = \frac{1}{2a^3} \frac{2}{\pi^2} + \frac{a^3}{2} \omega^2 \phi^2
\]

(60)

\[
H_{\zeta} = \frac{\pi^2}{2a} + \frac{a}{2} \omega^2 \zeta^2 + \frac{\dot{a}}{2a} (\pi_\zeta \zeta + \zeta \pi_\zeta)
\]

(61)

where \(\pi_\phi = a^3 \phi, \pi_\zeta = a \left( \zeta - \frac{\dot{a}}{a} \zeta \right) = \phi_\phi/a\), we have used \(R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)\) and one has

\[
H_{\zeta} = H_{\phi} + \frac{\dot{a}}{2a} (\phi \dot{\pi}_\phi + \pi_\phi \phi)
\]

(62)

It is easy to verify that the transformation from \(\phi\) to \(\zeta\) is canonical:

\[
[\phi, \pi_\phi]_{PB} = [\zeta, \pi_\zeta]_{PB} = 1
\]

(63)

and one relates the Poisson brackets (PB) to commutators in order to canonically quantize the two systems. For the system with Hamiltonian (60) one can introduce the following operator, which factorizes the Hamiltonian,

\[
\hat{a}_\phi = \left( \frac{a^3 \omega}{2\hbar} \right)^{\frac{1}{2}} \left( \hat{\phi} + i \frac{\pi_\phi}{a^3 \omega} \right)
\]

(64)

and the operator related to the linear invariant

\[
\hat{b}_\phi = \frac{1}{\sqrt{2\hbar}} \left[ \frac{\hat{\phi}}{\rho} + i \left( \rho \pi_\phi - a^3 \rho \hat{\phi} \right) \right]
\]

(65)
Similarly one may introduce the corresponding operators for the Hamiltonian (61)

\[
\hat{a}_\zeta = \left( \frac{a \omega_D}{2 \hbar} \right)^{1/2} \left( \hat{\zeta} + i \frac{\pi_\zeta}{a \omega_D} \right),
\]

\[
\hat{b}_\zeta = \frac{1}{\sqrt{2 \hbar}} \left\{ \hat{\zeta} + i \left[ a \rho (\hat{\pi}_\zeta + \hat{\zeta}) - \left( \hat{a} + \hat{a}^\dagger \right) \right] \right\},
\]

where \(\omega_D = (\omega^2 - \hat{a}^2/a^3)^{1/2}\) and in both cases \(\rho\) satisfies

\[
\ddot{\rho} + 3 \frac{\dot{a}}{a} \dot{\rho} + \omega^2 \rho = \frac{1}{a^6 \rho^3}
\]

Of course the above operators satisfy \([\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1\) and remarkably \(\hat{b}_\phi\) and \(\hat{b}_\zeta\) coincide implying that they lead to the same vacuum and Fock space. Actually this should not be surprising since the classical solutions to the equation of motion for \(\phi\) and \(\zeta/a\) are the same.

On using the above operators one obtains:

\[
\hat{I} = \hbar (\hat{b}_\phi^\dagger \hat{b}_\phi + \frac{1}{2}) = \hbar (\hat{b}_\zeta^\dagger \hat{b}_\zeta + \frac{1}{2})
\]

\[
\hat{H}_\phi = \hbar \omega (\hat{a}_\phi^\dagger \hat{a}_\phi + \frac{1}{2})
\]

\[
\hat{H}_\zeta = \hbar \omega_D (\hat{a}_\zeta^\dagger \hat{a}_\zeta + \frac{1}{2})
\]

where \(\hat{I}\) is an invariant and has time-independent eigenvalues, while \(\hat{H}_\phi\) and \(\hat{H}_\zeta\) obviously do not and correspond to the number of \(\phi\) and \(\zeta\) quanta times their respective energies. It is clear from Eq. (68) that it is only the \(\phi\) quanta that have a particle interpretation and the \(\phi\) and \(\zeta\) quanta are related through a Bogoliubov transformation

\[
\hat{a}_\phi = \frac{1}{2} \hat{\zeta} \left[ \frac{\omega_0^{1/2}}{\omega_D^{1/2}} + \frac{\omega^{1/2}}{\omega_0^{1/2}} \right] + \frac{1}{2} \hat{\zeta} \left[ \frac{\omega_0^{1/2}}{\omega_D^{1/2}} - \frac{\omega^{1/2}}{\omega_0^{1/2}} \right]
\]

corresponding to a squeezing [14]. Naturally the \(\hat{b}\) and \(\hat{a}\) are also related through a Bogoliubov transformation and it is the \(\hat{b}\) Fock space that describes the correct evolution.

Let us end by commenting that if we had omitted the surface term in Eq. (59) we would have obtained a third different Hamiltonian (which, apart from overall scale factors, coincides with the 00 component of the scalar field energy-momentum tensor appearing in the Einstein equations for \(\xi = 1/6\))

\[
\hat{H}_\zeta = \frac{1}{2a} \pi_\zeta^2 + \frac{a}{2} (\omega^2 - \frac{R}{6}) \zeta^2
\]

where \(\pi_\zeta = a \dot{\zeta} = \pi_\zeta + \hat{a} \zeta\). Again one may study the quantum system and introduce both linear invariant operators and operators factorizing the Hamiltonian. It is straightforward to verify that the former agree with Eq. (67), that is one obtains the same invariant
vacuum and the Fock space. This is not surprising since the equations of motion again are unchanged. For the quantum Hamiltonian, on the other hand, one obtains:

\[ \hat{H}_\zeta = \hbar \omega_D \left( \hat{a}^\dagger_\zeta \hat{a}_\zeta + \frac{1}{2} \right) \]  

(74)

where \( \omega_D = (\omega^2 - R/6)^{1/2} \) and

\[ \hat{a}_\zeta^\dagger = \left( \frac{a_\omega D}{2\hbar} \right)^{1/2} \left[ \hat{\zeta} + \frac{i}{a_\omega D} \hat{\pi}_\zeta \right] \]  

(75)

which are related to the corresponding operators in Eq.(66) through a Bogolubov transformation:

\[ \hat{a}_\zeta = \frac{1}{2} \hat{a}_\zeta \left[ \left( 1 + i \frac{\dot{a}}{a_\omega D} \right) \left( \frac{\omega_D}{\omega} \right)^{1/2} + \left( \frac{\omega_D}{\omega} \right)^{1/2} \right] + \frac{1}{2} \hat{a}_\zeta^\dagger \left[ \left( 1 + i \frac{\dot{a}}{a_\omega D} \right) \left( \frac{\omega_D}{\omega} \right)^{1/2} - \left( \frac{\omega_D}{\omega} \right)^{1/2} \right] \]  

(76)

It is important to note the different spectrum obtained with respect to Eqs. (71) and (72). Thus rescaling and omitting the surface term obtained will modify the quantum system and the spectrum of the Hamiltonian, leading to questionable results for some physical quantities, while maintaining the invariant vacuum and Fock space which corresponds to leaving the equations of motion unchanged. For example we may consider the invariant vacuum (\( |0\rangle_b \)) expectation values of the quantum Hamiltonians given by eqs.(70) and eqs.(74) for \( \xi = 0 \) (thus we do not have the presence of derivatives of \( a \) in the matter Hamiltonian - we return to this at the end of the next section) in a de Sitter space and consider the limit as \( a \to \infty \). One obtains [7]:

\[ \langle 0 | \hat{H}_\phi | 0 \rangle_b \simeq \frac{\hbar}{4} B \left( \frac{\Gamma(\vec{\nu})}{\pi} \right)^2 \left( \frac{k}{2H} \right)^{-2\vec{\nu}} a^{2\vec{\nu}} \left[ \mu^2 + H^2 \left( \vec{\nu} - \frac{3}{2} \right)^2 \right] \]

\[ \langle 0 | \hat{H}_\zeta | 0 \rangle_b \simeq \frac{\hbar}{4} B \left( \frac{\Gamma(\vec{\nu})}{\pi} \right)^2 \left( \frac{k}{2H} \right)^{-2\vec{\nu}} a^{2\vec{\nu}} \left[ \mu^2 - 2H^2 + H^2 \left( \vec{\nu} - \frac{1}{2} \right)^2 \right] \]

where \( B \) is a constant and \( \vec{\nu}^2 = 9/4 - \mu^2/H^2 \). We immediately note that the two expressions differ and in particular the first one vanishes for \( \mu = 0 \) (of course non-leading terms remain). Similar considerations hold on considering expectation values with respect to other states. A detailed discussion of the 'squeezing' effect of the surface term for the \( \xi = \mu = 0 \) case has been previously done [15].

6 Conclusions

The method of invariants is a particularly suitable tool in order to investigate quantum effects in time-dependent external fields. On using a conserved operator, one can find an invariant vacuum and an invariant Fock space, which are preserved during the time
evolution. In this paper we have used this tool in order to investigate the relation between particle production and the conformal properties of fields.

Motivated by the non conformal invariance of the action for the scalar field case - shown in section 4 for a general space-time -, we have analyzed the electromagnetic and Dirac fields. For the electromagnetic and the massless spinor fields the action is conformally invariant, and we have verified that there is no particle production in the Hamiltonian (which is equal to the energy density in these cases). For the massive spinor case there is a non-trivial particle production, but, because of the Pauli exclusion principle, the number of particles in a given mode can never exceed one.

We have gone beyond our previous paper and investigated, for cosmological space-times, some effects due to the presence of the surface term which violates the conformal invariance of the action in the scalar field case with the following results:

1. We verified that the use of rescaled fields is classically a canonical transformation and a unitary transformation quantum mechanically. On retaining all terms in the action we used the rescaled fields in order to canonically quantize the system, and obtained the same invariant vacuum and the same invariant Fock space as resulted on quantizing the original (non-rescaled) fields.

2. The fact that the energy density of a non-minimally coupled scalar field (which is the source of the Einstein tensor) differs from the canonically obtained Hamiltonians of the field and of the rescaled field (which generate the correct time evolution for the two systems) implies that the usual statement of no particle production - on neglecting quantum anomalies - in the massless conformally coupled scalar field should only be applied to the source of Einstein tensor. Indeed as we pointed out in the previous section apart from overall scale factors the rescaled scalar field Hamiltonian, obtained on omitting surface terms, coincides with the 00 component of the scalar field energy momentum tensor appearing in the Einstein equations for $\xi = 1/6$.

3. The neglect of the surface term in the action for the rescaled fields classically leads to a third and different Hamiltonian, and so a different time evolution. On canonically quantizing this system, one however gets the same invariant vacuum and Fock space corresponding to the fact that the equations of motion are unchanged.

Let us end by noting that in all the above we have employed the usual canonical formalism for the scalar field starting from the action and obtaining the Hamiltonian describing time evolution. Time evolution for matter can also be obtained from an initial action containing both gravitation and matter leading to the Wheeler-De Witt equation which, of course, does not contain time. On performing a Born-Oppenheimer decomposition and considering the semiclassical limit for gravity one is led to the Schrodinger
Schwinger-Tomonaga equation for the evolution of matter [16]. We feel it would be interesting to obtain a similar approach for the case of a non-minimally coupled scalar field. However there is an essential difficulty since a non-minimal coupling leads to terms containing derivatives of the metric appearing in the matter lagrangian. This will lead to momenta conjugate to the metric appearing in the matter Hamiltonian in contrast with what is usually assumed for a Born-Oppenheimer (or adiabatic) approach. Nonetheless we hope to return to this.

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