A DUAL FORMULA FOR THE SPECTRAL DISTANCE
IN NONCOMMUTATIVE GEOMETRY

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ABSTRACT. We present a dual formula for Connes’ spectral distance generalizing Beckmann’s formula for the Monge-Kantorovich-Wasserstein distance of order 1. We then discuss some examples with matrix algebras.

1. INTRODUCTION

In Connes’ framework [2], a noncommutative geometry is described by a spectral triple \((A, H, D)\), consisting in an associative complex \(\ast\)-algebra \(A\) that acts faithfully on a Hilbert space \(H\) through a representation \(\pi\), together with a (non-necessarily bounded) selfadjoint operator \(D\) on \(H\), such that \([D, \pi(a)]\) is bounded and \(\pi(a)(1 + D^2)^{-1}\) is compact for all \(a \in A\). The spectral distance between any two states \(\varphi_1, \varphi_2\) of \(\mathcal{A}\) (the \(C^*\)-closure of \(A\) with respect to the operator norm on \(H\)) is [3]:

\[
d_D(\varphi_1, \varphi_2) := \sup_{a \in A} \{ |\varphi_1(a) - \varphi_2(a)| : \|[D, \pi(a)]\| \leq 1 \}.
\] (1)

There is a natural spectral triple associated with a Riemannian spin manifold \(M\) of dimension \(m\), namely

\[
A = C_c^\infty(M), \quad H = L^2(M, S), \quad D = \mathbf{D} := -i \sum_{\alpha=1}^m \gamma^\alpha \nabla_\alpha
\] (2)

where the algebra \(C_c^\infty(M)\) of smooth compactly supported functions on \(M\) acts by pointwise multiplication on the Hilbert space \(L^2(M, S)\) of square integrable spinors,

\[
(\pi(f)\psi)(x) := f(x)\psi(x) \quad \forall f \in C_c^\infty(M), \psi \in L^2(M, S), x \in M,
\] (3)

while \(\mathbf{D}\) is the Dirac operator, expressed in local coordinates in terms of selfadjoint Dirac matrices \(\gamma^\alpha\) and spin covariant derivatives \(\nabla_\alpha\). In that case, \(\mathcal{A} = C_0(M)\) is the algebra of continuous functions on \(M\) vanishing at infinity.

States of \(C_0(M)\) – that is positive, normalized, linear functionals \(\varphi\) on \(C_0(M)\) – are in 1-to-1 correspondence with probability measures \(\mu\) on \(M\),

\[
\varphi(f) = \int_M f \, d\mu \quad \forall f \in C_0(M).
\] (4)
It was first noticed by Rieffel in [10] that the spectral distance (1) on a compact manifold is nothing but Kantorovich dual formulation of the Wasserstein distance $W_1$ of order 1 studied in optimal transport, with cost function given by the geodesic distance $d_{\text{geo}}$ on $M$. Namely one has

$$d_{\Theta}(\varphi_1, \varphi_2) = W_1(\mu_1, \mu_2)$$

with

$$W_1(\mu_1, \mu_2) := \inf_{\gamma} \int_{M \times M} d_{\text{geo}}(x, y) \, d\gamma(x, y),$$

where the infimum is on all the measures $\gamma$ on $M \times M$ with marginals the measures $\mu_1$ and $\mu_2$ defining the states $\varphi_1, \varphi_2$ through (4). The same is true on a locally compact manifold, as soon as it is complete [4].

For an arbitrary spectral triple $(A, H, D)$, Connes formula (1) provides a generalization of Kantorovich dual formula that makes sense also in the noncommutative framework. A natural question is whether there exists a “pre-dual” formula, that is a noncommutative version of (6) or, in other terms, a way to express the spectral distance (1) as an infimum rather than a supremum.

A first attempt starts with the following observation: on a manifold, the cost function is retrieved as the Wasserstein distance between pure states. On an arbitrary spectral triple, replacing states by probability measures on the space $\mathcal{P}(A)$ of pure states, a natural candidate for a pre-dual formula would be the Wasserstein distance $W_D$ on the space $\mathcal{S}(A)$ of states, with cost the spectral distance on pure states (but note that the map from probability measures on $\mathcal{P}(A)$ to $\mathcal{S}(A)$ is surjective but, in general, not injective). One shows [8] that $d_D(\varphi, \psi) \leq W_D(\varphi, \psi)$ for any probability measures $\varphi, \psi$ on $\mathcal{P}(A)$, with equality when $\varphi$ and $\psi$ are convex linear combinations of the same two pure states. However the equality does not hold in general (see the counterexample in [9]).

A similar approach is in [5, Sec. 4.6]: on the Berezin quantization $A$ of a compact homogeneous $G$-space $M$, with $G$ a connected compact semisimple Lie group, the symbol map associates to every quantum state a unique probability measure on $M$; one can then consider the Wasserstein distance between probability measures and show that it gives a distance on quantum states that is dual to a suitable seminorm on $A$, in the spirit of (1).

In this note, we show that there exists a dual formulation of the spectral distance (1) by adapting to noncommutative geometry the “dual of the dual” formula of the Wasserstein distance, also known as Beckmann’s problem (cf. [14, Sec. 4.2]). This is inspired by the recent work [1] on matrix algebras. Our main result is Theorem 1, which shows that the “dual of the dual” formula holds in full generality.
In the following, we will omit the representation symbol and identify an element of the algebra $A$ of a spectral triple with its representation as a bounded operator on $H$. By $A^+$ we shall mean the algebra generated by $A$ and $1 \in \mathcal{B}(\mathcal{H})$.

## 2. A DUAL FORMULATION FOR THE SPECTRAL DISTANCE

Let $(A, H, D)$ be a spectral triple, and

$$\Omega_D^1(A) := \text{Span}\{ a[D, b] : a, b \in A^+ \}$$

the $A$-bimodule of generalized 1-forms. Denote by $\nabla$ the derivation:

$$\nabla : A \to \Omega_D^1(A), \quad \nabla a := [D, a].$$

By definition of spectral triple, $[D, a]$ is bounded for any $a \in A$, so that $\Omega_D^1(A)$ contains $\text{Im}(\nabla)$, and therefore $\text{Im}(\nabla)$ is a subset of the algebra $\mathcal{B}(H)$ of bounded operators on $H$. We will assume that if $a \in A^+$ and $[D, a] = 0$, then $a = \lambda 1$ is proportional to the identity. This is sometimes referred to as “connectedness” condition.

Let $B$ be any (complex) Banach subspace of $\mathcal{B}(H)$ containing $\text{Im}(\nabla)$, $B^*$ its Banach dual, i.e. the set of linear functionals $\Phi : B \to \mathbb{C}$ that are bounded for the operator norm

$$\|\Phi\| := \sup_{b \in B : b \neq 0} \frac{|\Phi(b)|}{\|b\|},$$

and $\mathcal{L}(A, C)$ the set of linear maps $A \to C$. We denote by

$$\nabla^* : B^* \to \mathcal{L}(A, C)$$

the pullback of $\nabla$:

$$\nabla^* \Phi(a) := \Phi(\nabla a), \quad \forall \Phi \in B^*, a \in A.$$ 

Given a state $\varphi$ of $\overline{A}$ we will denote by $\varphi_0$ its restriction to $A$.

**Theorem 1.** Let $(A, H, D)$ be a connected spectral triple. For any $\varphi, \psi \in S(\overline{A})$ and any Banach space $B$ containing $\text{Im}(\nabla)$, define:

$$W(\varphi, \psi) := \inf_{\Phi \in B^*} \{ \|\Phi\| : \nabla^* \Phi = \varphi_0 - \psi_0 \}.$$ (12)

Then, (i) the above inf is well-defined (the set of $\Phi$ satisfying the side condition is nonempty), (ii) it is in fact a min and (iii) one has

$$W(\varphi, \psi) = d_D(\varphi, \psi) \quad \forall \varphi, \psi \in S(\overline{A}).$$ (13)

In particular, (12) gives a distance which is independent of the choice of $B$.

**Proof.** Let $B_0 = \text{Im}(\nabla)$ and $\Phi_0 : B_0 \to \mathbb{C}$ be the map given by:

$$\Phi_0(\nabla a) := \varphi_0(a) - \psi_0(a), \quad \forall a \in A.$$ (14)
Thanks to the connectedness condition for the spectral triple, such a map is well defined. Note that
\[
\|\Phi_0\| := \sup_{b \in B, b \neq 0} \frac{|\Phi_0(b)|}{\|b\|} = \sup_{a \in A : |[D,a]| \neq 0} \frac{|\varphi(a) - \psi(a)|}{\|[D,a]\|} = d_D(\varphi, \psi). \tag{15}
\]

By the Hahn-Banach theorem [13, Pag. 77, Cor. 1], the map \( \Phi_0 \) can be extended (non-uniquely) to a linear map in \( B^* \) with the same norm, which proves the first statement.

Let \( \Phi \in B^* \) be any of these extensions. Since by construction
\[
\nabla^* \Phi(a) = \Phi(\nabla a) = \Phi_0(\nabla a) = \varphi(a) - \psi(a) \quad \forall a \in A,
\]
we have from (15):
\[
W(\varphi, \psi) \leq \|\Phi\| = \|\Phi_0\| = d_D(\varphi, \psi). \tag{17}
\]

On the other hand, for all \( \Phi \in B^* \),
\[
\|\Phi\| = \sup_{b \in B, b \neq 0} \frac{|\Phi(b)|}{\|b\|} \geq \sup_{b \in B, b \neq 0} \frac{|\Phi(b)|}{\|b\|} = \sup_{a \in A : |a| \neq 0} \frac{|\nabla^* \Phi(a)|}{\|\nabla a\|}. \tag{18}
\]

In particular if \( \nabla^* \Phi(a) = \varphi(a) - \psi(a) \), the right hand side of previous equation is \( d_D(\varphi, \psi) \), which proves that \( W(\varphi, \psi) \geq d_D(\varphi, \psi) \). Hence the result.

Notice that the inf is attained on the map (14), thus is a min. \( \blacksquare \)

One can avoid the connectedness condition as follows. In order for the map (14) to exist it is enough that \( \nabla a = 0 \implies \varphi_0(a) = \psi_0(a) \), which is automatically satisfied if the states are at finite distance.

**Lemma 2.** Suppose \( d_D(\varphi, \psi) < \infty \). Then \( \varphi - \psi \) vanishes on \( \ker \nabla \).

**Proof.** Let \( b \in \ker \nabla \), with \( b \neq 0 \). Set \( a := \lambda b \), with \( \lambda \in \mathbb{C} \). Since \( [D,b] = 0 \), \( d_D(\varphi, \psi) \geq |(\varphi - \psi)(a)| = |\lambda| \cdot |(\varphi - \psi)(b)| \quad \forall \lambda \in \mathbb{C} \). If \( (\varphi - \psi)(b) \neq 0 \), the sup over all \( \lambda \) is infinite and we get a contradiction. \( \blacksquare \)

As a corollary, (13) holds in full generality for states that are at finite distance:

**Theorem 3.** Let \((A, H, D)\) be a (not necessarily connected) spectral triple and \( \varphi, \psi \) two states such that \( d_D(\varphi, \psi) < \infty \). Then \( d_D(\varphi, \psi) = W(\varphi, \psi) \) with \( W \) given by (12).

3. **Examples**

3.1. **Euclidean spaces.** In this section we recover Beckmann’s formula for the Wasserstein distance in \( \mathbb{R}^m \) from formula (12). More precisely we show that the r.h.s. of the latter coincides with Beckmann’s formula (given in (30) below).

Beckmann’s formula deals with real value functions, so first of all we need to check that the infimum in (12) can be equivalently searched on real valued
with Hilbert-Schmidt inner product:

\[ \langle m, \varphi_1, \varphi_2 \rangle = \sup_{a \in A^{s.a.}} \{ \varphi_1(a) - \varphi_2(a) : \|D, \pi(a)\| \leq 1 \}. \] (19)

Indeed, let \( \nabla_R \) be the restriction of (8) to the real vector space \( A^{s.a.} \). Take \( B_R \) to be any real Banach subspace of \( B(\mathbb{H}) \) containing \( \text{Im}(\nabla_R) \), denote by \( B^*_R \) its dual and \( \nabla^*_R : B^*_R \to \mathcal{L}(A^{s.a}, \mathbb{R}) \) the pullback of \( \nabla_R \). Then, starting from (19) and repeating almost verbatim the proof of the above theorems, using the Hahn-Banach theorem for real Banach algebras, one arrives at the following real version of Theorem 3.

**Theorem 4.** Let \((A, H, D)\) be a (not necessarily connected) spectral triple and \( \varphi, \psi \) two states such that \( d_D(\varphi, \psi) < \infty \). Then

\[ d_D(\varphi, \psi) = \inf_{\Phi \in B^*_R} \{ \|\Phi\| : \nabla^*_R \Phi = \varphi - \psi \} \] (20)

We now specialize to the spectral triple (2), with \( M = \mathbb{R}^m \). In such a case, \( \nabla f = -i \sum_{\alpha=1}^m \gamma^\alpha \partial_\alpha f \) for all \( f \in C^\infty_c(\mathbb{R}^m) \) and \( H = L^2(\mathbb{R}^m) \otimes \mathbb{R}^n \), where \( n = 2^{m/2} \) is the dimension of the spin representation. Let \( V \subset M_m(\mathbb{C}) \) be the real vector subspace of complex \( m \times m \) matrices spanned by \( i\gamma^1, \ldots, i\gamma^m \), equipped with Hilbert-Schmidt inner product:

\[ \langle a, b \rangle_{HS} := \frac{1}{m} \text{Tr}(a^*b), \quad \forall a, b \in M_m(\mathbb{C}). \] (21)

With such a normalization, gamma matrices \( i\gamma^\alpha \) form an orthonormal basis of \( V \):

\[ \langle i\gamma^\alpha, i\gamma^\beta \rangle_{HS} = \frac{1}{m} \text{Tr}(\gamma^\alpha \gamma^\beta) = \frac{1}{2m} \text{Tr}(\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha) = \frac{1}{m} \text{Tr}(\delta^\alpha\beta \mathbb{1}_m) = \delta^\alpha\beta. \] (22)

The image of \( \nabla_R \) is contained in the Banach space \( B_R := C_0(\mathbb{R}^m, V) \) of continuous functions \( \mathbb{R}^m \to V \) vanishing at infinity, with norm inherited from the operator norm on \( H \). Any \( b \in B_R \) has the form \( b = i \sum_\alpha f_\alpha \gamma^\alpha \), where \( f := (f_1, \ldots, f_m) \) is an \( m \)-tuple of real-valued \( C_0 \)-functions on \( \mathbb{R}^m \). Since the operator norm is a \( C^* \)-norm and

\[ b^*b = \sum_{\alpha, \beta} f_\alpha f_\beta \gamma^\alpha \gamma^\beta = \frac{1}{2} \sum_{\alpha, \beta} f_\alpha f_\beta (\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha) = \sum_\alpha |f_\alpha|^2 \mathbb{1} = |f|^2 \mathbb{1} \] (23)

with \( \mathbb{1} \) the identity matrix of dimension \( n \), the norm on \( B_R \) is just the Euclidean norm in \( \mathbb{R}^m \) composed with the sup norm:

\[ \|b\| = \sup_{x \in \mathbb{R}^m} |f(x)|. \] (24)

By Riesz Representation Theorem (see e.g. [12, Theorem 6.19]) the dual is \( B^*_R = \mathcal{M}(\mathbb{R}^m) \otimes V^*, \) with \( \mathcal{M}(\mathbb{R}^m) \) the set of (real-valued) Radon measures on \( \mathbb{R}^m \). Every
Φ ∈ B∗ can then be expressed in the form

\[ \Phi(\cdot) = i \sum_{\alpha} \int_{\mathbb{R}^m} \langle \gamma^\alpha, \cdot \rangle_{HS} \, dw_\alpha \]  

(25)

for some Radon measures \( w_1, \ldots, w_m \). Given two states \( \varphi, \psi \) corresponding to two measures \( \mu, \nu \), the side condition \( \nabla \Phi = \varphi_0 - \psi_0 \) in (12) becomes:

\[ \int_{\mathbb{R}^m} \nabla f \cdot dw = \int_{\mathbb{R}^m} (f \, d\mu - f \, d\nu), \quad \forall f \in C^\infty_c(M), \]  

(26)

where \( dw = (dw_1, \ldots, dw_m) \), \( \nabla = (\partial_1, \ldots, \partial_m) \) and “·” is the standard Euclidean scalar product. In particular, if \( dw = w(x) \, d^m x \), \( d\mu = \mu(x) \, d^m x \) and \( d\nu = \nu(x) \, d^m x \) for some \( C^1 \) functions, after integration by parts one gets:

\[ \int_{\mathbb{R}^m} f \{ \nabla \cdot w + \mu - \nu \} \, d^m x = 0 \quad \forall f \in C^\infty_c(M). \]  

(27)

Since \( C^\infty_c(\mathbb{R}^m) \) is dense in \( C_0(\mathbb{R}^m) \), the latter condition is equivalent to

\[ -\nabla \cdot w = \mu - \nu. \]  

(28)

For more general Radon measures, the divergence condition (28) is to be read in the weak sense, i.e. as in (26).

The norm on \( B^*_R \), dual to (24), is

\[ \| \Phi \| = \int_{\mathbb{R}^m} |w(x)| \, d^m x. \]  

(29)

Combining (29) and (28), after a replacement \( w \to -w \), Theorem 3 yields

\[ d_D(\varphi, \psi) = \min \left\{ \int_{\mathbb{R}^m} |w(x)| \, d^m x : \nabla \cdot w = \mu - \nu \right\}. \]  

(30)

The r.h.s. of the above equation is precisely Beckmann’s formula (4.4) in [14]. So we have recovered the later from the dual formulation of the spectral distance, as expected from Theorem 1.

3.1.1. Distributions on the real line. In simple cases, combining the formulas (30) and (1) one is able to explicitly compute the distance. For \( m = 1 \), for example, the difference between the cumulative distributions

\[ w(x) := \int_{-\infty}^{x} d(\mu - \nu) \]  

(31)

satisfies the side condition in (30), hence we immediately get

\[ d_D(\varphi, \psi) \leq \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{x} d(\mu - \nu) \right| \, dx. \]  

(32)

We know on the other hand that the sup in (1) can be searched, in the commutative case, among the set of all 1-Lipschitz functions (see e.g. [4]). If \( \mu - \nu \) has no singular continuous part, then (31) is piecewise continuous, its sign is piecewise
continuous and the primitive of the sign is a 1-Lipschitz functions $\phi$. From (1) it follows that:

$$d_D(\phi, \psi) \geq \left| \int_{-\infty}^{+\infty} \phi \, d(\mu - \nu) \right| = \int_{-\infty}^{+\infty} \left| \int_{-\infty}^{x} d(\mu - \nu) \right| \, dx \quad (33)$$

so that (32) is an equality.

### 3.2. Finite noncommutative spaces.

While (12) is always a minimum, the supremum in (1) is not always a maximum. Here we observe that it is a max in the finite-dimensional case.

Let us start with a slight generalization of the notion of spectral distance. Let $A^{s.a.}$ be an order unit space, and $L$ a Lipschitz seminorm on $A^{s.a.}$, that means $L(a) = 0 \Rightarrow a$ is proportional to the order unit $1$. A distance dual to $L$ is defined by [11]:

$$d(\phi, \psi) := \sup_{a \in A^{s.a.}} \{ \phi(a) - \psi(a) : L(a) \leq 1 \}, \quad (34)$$

for all states $\phi, \psi$ of $A^{s.a.}$. We recover the spectral distance of a connected spectral triple $(A, H, D)$ by taking as $A^{s.a.}$ the set of selfadjoint elements of $A^+$ and $L(a) := \|[D, a]\|$, and by recalling that the sup in (1) can be equivalently searched on the set of selfadjoint elements of $A^+$ [6].

If $A^{s.a.}$ is finite-dimensional, the usual interpretation is as a noncommutative metric space with a finite number of points. The next lemma expresses the fact that, roughly speaking, for finite noncommutative spaces there is always an “optimal transportation plan”.

**Proposition 5.** Let $A^{s.a.}$ be finite-dimensional. Then the sup in (34) is a max, i.e. for all $\phi, \psi$ there exists an element $a_{\text{max}} \in A^{s.a.}$ such that $d_D(\phi, \psi) = \phi(a_{\text{max}}) - \psi(a_{\text{max}})$.

**Proof.** Recall that $A^{s.a.}$ can always be realized as a real subspace of $B(H)^{s.a.}$, the set of selfadjoint operators on a finite-dimensional Hilbert space $H$, containing the identity operator $1$. We distinguish two cases: $L(1) \neq 0$ and $L(1) = 0$.

In the first case, $L$ is a norm on $A^{s.a.}$. Since $A^{s.a.}$ is a finite-dimensional vector space, the unit ball $B := \{ a \in A^{s.a.} : L(a) \leq 1 \}$ is a compact set. The map $f := \phi - \psi : B \to \mathbb{R}$ is continuous and its domain is a compact set, hence by Weierstrass theorem (see e.g. Appendix E of [7]) it attains its maximum on $B$, i.e. there exists $a_{\text{max}} \in B$ such that $d_D(\phi, \psi) = \phi(a_{\text{max}}) - \psi(a_{\text{max}})$.

Assume now $L(1) = 0$. For all $a \in A^{s.a.}$ and $\lambda \in \mathbb{R}:

$$L(a + \lambda 1) \leq L(a) + |\lambda|L(1) = L(a) \quad (35)$$

and

$$L(a) = L(a + \lambda 1 - \lambda 1) \leq L(a + \lambda 1) + |\lambda|L(1) = L(a + \lambda 1) \quad (36)$$
Hence $L(a) = L(a + \lambda 1)$. Since $\varphi(a - \lambda 1) - \psi(a - \lambda 1) = \varphi(a) - \psi(a)$, in (34) we can always replace $a$ by the traceless element $a - \frac{\text{Tr}(a)}{\text{Tr}(1)} 1$ proving that

$$d(\varphi, \psi) = \sup_{a \in S} \{ \varphi(a) - \psi(a) : L(a) \leq 1 \},$$

(37)

where $S = \{ a \in A^{\text{a.a.}} : \text{Tr}(a) = 0 \}$. Now $L$ is a norm on $S$, since by the Lipschitz condition $L(a) = 0$ implies $a = \lambda 1$ and this is traceless iff it is zero. We can then repeat the proof above with the unit ball $B := \{ a \in S : L(a) \leq 1 \}$. ■

3.2.1. Matrix algebras. Consider the spectral triple

$$A = M_n(\mathbb{C}), \quad H = \mathbb{C}^n \otimes \mathbb{C}^N, \quad D = \sum_{i=1}^{N} L_i \otimes E_{ii}$$

(38)

where $n, N \geq 1$, the algebra $A$ acts on the first factor of $H$, $L_1, \ldots, L_N \in M_n(\mathbb{C})$ are selfadjoint matrices and $E_{ij} \in M_N(\mathbb{C})$ is the matrix with 1 in position $(i, j)$ and zero everywhere else.

For such a spectral triple, we recover Theorem 1 of [1], cf. Prop. 6 below.

Recall that states $\varphi$ are in bijection with density matrices $\rho \in M_n(\mathbb{C})$, via the formula:

$$\varphi(a) = \text{Tr}(\rho a), \quad \forall a \in A.$$  

(39)

Choose the Banach space $B$ in (12) as follows:

$$B = M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C})$$

(40)

where we think of elements of $B$ as block diagonal matrices with $N$ blocks of type $n \times n$. The formula

$$\Phi = \sum_{i=1}^{N} \text{Tr}_{\mathbb{C}^n}(u_i \cdot )$$

(41)

gives a bijection between $u = (u_1, \ldots, u_N) \in B$ and $\Phi \in B^\ast$. From the cyclic property of the trace we deduce that

$$\nabla^\ast \Phi(a) = \Phi([D, a]) = \sum_{i=1}^{N} \text{Tr}_{\mathbb{C}^n}(u_i[L_i, a]) = -\sum_{i=1}^{N} \text{Tr}_{\mathbb{C}^n}([L_i, u_i]a)$$

(42)

for all $a \in A$, so that the side condition in (12) becomes

$$-\sum_{i=1}^{N} [L_i, u_i] = \rho_1 - \rho_2$$

(43)

for two states with density matrices $\rho_1$ and $\rho_2$.

The dual of the operator norm in the finite-dimensional case is the nuclear norm $\| \cdot \|_\ast:

$$\| \Phi \| = \| u \|_\ast := \text{Tr}(\sqrt{u^* u}).$$

(44)

Collecting all these informations, and since $u$ and $-u$ have the same norm, we arrive at the following theorem, specialization of Theorem 1 to the present case.
Proposition 6. The spectral distance of the triple (38) is given by
\[ d_D(\varphi_1, \varphi_2) = \inf \left\{ \|u\|_*: \sum_{i=1}^N [L_i, u_i] = \rho_1 - \rho_2 \right\}, \tag{45} \]
where \( \varphi_i = \text{Tr}(\rho_i \cdot) \), \( i = 1, 2 \), are any two states at finite distance and the infimum is over all \( u = (u_1, \ldots, u_N) \in M_n(\mathbb{C}) \oplus \ldots \oplus M_n(\mathbb{C}) \) satisfying the side condition.

Using (20) instead of (12) one can show that the inf in (45) can be searched in the set of antisymmetric matrices, thus getting the same formula that is in [1].

REFERENCES

[1] Y. Chen, T.T. Georgiou, L. Ning and A. Tannenbaum, *Matricial Wasserstein-1 Distance*, IEEE Control Systems Letters 1 (2017).

[2] Alain Connes, *Noncommutative geometry*, Academic Press, 1994.

[3] A. Connes and J. Lott, *The metric aspect of noncommutative geometry*, Nato ASI series B Physics 295 (1992), 53–93.

[4] F. D’Andrea and P. Martinetti, *A view on optimal transport from noncommutative geometry*, SIGMA 6 (2010), no. 057, 24 pages.

[5] F. D’Andrea, *Pythagoras Theorem in Noncommutative Geometry*, in “Noncommutative Geometry and Optimal Transport”, Contemporary Mathematics 676 (AMS, 2016), pp. 175–210.

[6] B. Iochum, T. Kräjewski, and P. Martinetti. *Distances in finite spaces from noncommutative geometry*, J. Geom. Phy. 31 (2001), 100–125.

[7] R.A. Horn and C.R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[8] P. Martinetti, *Towards a Monge-Kantorovich distance in noncommutative geometry*, Zap. Nauch. Semin. POMI 411 (2013).

[9] P. Martinetti, *Connes distance and optimal transport*, J. Phys. Conf. Series 968 (2018), 012007.

[10] M.A. Rieffel, *Metrics on states from actions of compact groups*, Documenta Math.3 (1998), 215–229.

[11] M.A. Rieffel, *Metric on state spaces*, Documenta Math. 4 (1999), 559–600.

[12] W. Rudin, *Real and complex analysis*, McGraw-Hill, 1987.

[13] M. Reed and B. Simon, *Methods of Modern Mathematical Physics I*, Academic Press, 1980.

[14] F. Santambrogio, *Optimal Transport for Applied Mathematicians*, PNLDE 87, Birkhäuser (2015).

[15] M. Takesaki, *Theory of operator algebras I*, Springer, 2001.

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