Cylindrically symmetric solitons in Einstein–Yang–Mills theory

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Recently new Einstein-Yang-Mills (EYM) soliton solutions were presented which describe superconducting strings with Kasner asymptotics [hep-th/0610183]. Here we study the static cylindrically symmetric SU(2) EYM system in more detail. The ansatz for the gauge field corresponds to superposition of the azimuthal $B_\phi$ and the longitudinal $B_z$ components of the color magnetic field. We derive sum rules relating data on the symmetry axis to asymptotic data and show that generic asymptotic structure of regular solutions is Kasner. Solutions starting with vacuum data on the axis generically are divergent. Regular solutions correspond to some bifurcation manifold in the space of parameters which has the low-energy limiting point corresponding to string solutions in flat space (with the divergent total energy) and the high-curvature point where gravity is crucial. Some analytical results are presented for the low energy limit, and numerical bifurcation curves are constructed in the gravitating case. Depending on the parameters, the solution looks like a straight string or a pair of straight and circular strings. The existence of such non-linear superposition of two strings becomes possible due to self-interaction terms in the Yang-Mills action which suppress contribution of the circular string near the polar axis.

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I. INTRODUCTION

The discovery by Bartnik and McKinnon [1] of a particle-like solution in the Einstein-Yang-Mills (EYM) theory and construction of the corresponding black holes [2] stimulated wide research of Einstein-Yang-Mills (EYM) solitons and their generalization in supergravities and other related models (for a review see [3]). Between them, cylindrically symmetric four-dimensional configurations remained relatively less explored. Useful general analysis can be found in the study of Einstein–Maxwell geons [4,5], the Melvin magnetic universe [6] and the abelian cosmic strings [7]. Other configurations such as the Yang–Mills vortex in 4+1 dimensions [8], cylindrical black holes [9,10] and some numerical solution in [11] are also relevant. Recently a new class of the EYM cylindrically-symmetric solutions was reported [12] describing the Einstein-Yang-Mills strings. The purpose of this paper is to study the cylindrically symmetric four-dimensional EYM system in more detail.

Our ansatz for the gauge field contains two functions which describe the longitudinal and the azimuthal magnetic field components. The stress-energy tensor does not exhibit the boost symmetry as in the case of cosmic strings, so the metric is parametrized by three different functions of the radial variable. We analyze first the solutions in flat space-time imposing the regularity conditions on the polar axis. Generically, the integral curves starting with such boundary data on the axis diverge at finite distance exhibiting the square-root singularity of the Yang-Mills field, reminiscent of that in the spherically symmetric case [13]. There exist, however, the bifurcation curve in the parameter space which correspond to regular solutions. The energy density does not fall sufficiently fast at infinity, so the total energy diverges.

When gravity is switched on, field configurations become effectively compact and the metric is asymptotically Ricci-flat. It is, however, not Minkowskian, but Kasner. We derive the integral sum rules relating the boundary values of the metric functions on the polar axis with the Kasner parameters which show the absence of asymptotically flat configurations of non-zero mass. We also present a (non-rigorous) proof of existence of regular solutions with Kasner asymptotic. Numerical integration reveals the following typical behavior of the solutions. In the vicinity of the polar axis the space-time is flat. With growing distance the longitudinal magnetic field is dominant within a certain region. The YM field there is effectively U(1), while the metric is approximately Melvin. With the further growing distance the azimuthal component of the YM magnetic field comes into play and the YM non-linearity becomes manifest. In this region the solution may be interpreted as describing the circular magnetic string. Finally, in the asymptotic region the metric acquires the boost symmetry in the $t-\varphi$ plane exactly in the same way as the metric of the cosmic string is boost-symmetric in the $t-z$ plane. Moreover, the metric component $g_{zz}$ asymptotically tends to zero, so the space-time becomes effectively 1 + 2 dimensional. This effect is similar to the asymptotical contraction of the azimuthal dimension in the Melvin solution. Thus, our solution may be loosely interpreted as describing the system of interacting straight and circular magnetic strings in the four-dimensional EYM theory.

The paper is organized as follows. In Sec. II we introduce the gauge field ansatz and discuss the symmetries of the one-dimensional reduced action. We show the existence of two scaling symmetries and derive the corresponding Noether currents. In Sec. III we discuss the
equations of motions and formulate the boundary conditions of regularity on the polar axis both in the flat space and within the full self-gravitating treatment. Using them, we derive the sum rules relating the local values of some combinations of the metric functions and the YM variables to the radial integrals of quantities which admit simple physical interpretation in terms of the energy density and principal pressures. Sec. IV is devoted to study of the asymptotic conditions and investigation of the existence of the desired asymptotically vacuum solutions. Qualitative analysis of the field equations and the results of the numerical integration are presented in Sec. V. In conclusion we briefly formulate our results and sketch further prospects.

II. THE MODEL

A. Einstein-Yang-Mills action

We consider the pure SU(2) Einstein-Yang-Mills action

\[ S = \int dx^4 \sqrt{-g} \left( -\frac{1}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \] (1)

with the field tensor

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + \varepsilon_{abc} A_{\mu}^b A_{\nu}^c. \] (2)

Here \( e \) is the gauge coupling constant, and

\[ A_{\mu} = T_a A_{\mu}^a \] (3)

is the matrix-valued gauge potential, with the group generators \( T_a \) normalized as

\[ T_a T_b = \frac{1}{2} \varepsilon_{abc} T_c - \frac{1}{4} \delta_{ab}. \] (4)

Variation of the action (1) with respect to the metric \( g^{\mu\nu} \) leads to the Einstein equations

\[ R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^\lambda_\lambda \right) \] (5)

with the stress-energy tensor

\[ T_{\mu\nu} = -F_{\mu\rho}^a F_{\nu}^{\sigma a} + \frac{1}{4} g_{\mu\nu} F_{\lambda\sigma}^a F^{\lambda\sigma a}, \] (6)

which is traceless:

\[ T^\lambda_\lambda = 0. \] (7)

Variation with respect to the gauge field \( A_{\mu}^a \) gives the Yang-Mills equations

\[ \left( D_{\mu} \left( \sqrt{-g} F^{\mu\nu} \right) \right)^a = 0, \] (8)

where \( D_{\mu} \) is the total covariant derivative with respect to both the gauge group and metric connections.

B. Ansatz for the metric and the gauge field

We shall use the cylindrical coordinates \( x^\mu = (t, r, \varphi, z) \). The space-time generated by the static cylindrically symmetric source is diagonal and possesses three commuting Killing vectors \( \partial_t, \partial_\varphi, \partial_z \). Its explicit form depends on the chosen gauge, here we will use the Kasner gauge \( g_{rr} = -1 \):

\[ ds^2 = N^2(r) dt^2 - dr^2 - L^2(r) d\varphi^2 - K^2(r) dz^2. \] (9)

The corresponding non-zero components of the Ricci tensor are:

\[ R_0^0 = \frac{(N' L K')'}{N L K}, \quad R_\varphi^\varphi = \frac{N''}{N} + \frac{L''}{L} + \frac{K''}{K}, \]

\[ R_z^z = \frac{(N L K')'}{N L K}. \] (10)

To construct an appropriate ansatz for the gauge field we have to ensure that the effect of the field transformation upon the coordinate translation along the Killing vector \( K \),

\[ A_{\mu} \rightarrow A_{\mu} - \epsilon \mathcal{L}_K A_{\mu}, \] (11)

can be compensated by a suitable gauge transformation

\[ A_{\mu} \rightarrow A_{\mu} + \epsilon D_{\mu} W, \] (12)

with \( W \) being a Lie algebra valued function such that

\[ \mathcal{L}_K A_{\mu} = D_{\mu} W. \] (13)

With three Killing vectors we will thus have for the SU(2) YM field nine independent functions. Imposing as usual for the static axially-symmetric configurations a discrete symmetry \( M_{xz} \otimes C \), where the first factor stands for reflection in the \( xz \)-plane and the second is charge conjugation, we reduce the number of independent components to six. There are different gauge equivalent ways to parameterize the field potential, we choose the following gauge:

\[ e A_t = T_t w_0^1 + T_z w_3^3, \] (14)

\[ e A_r = T_r w_3^1, \] (15)

\[ e A_\varphi = T_r w_3^1 + T_z w_3^3, \] (16)

\[ e A_z = T_\varphi w_3^2, \] (17)

where the rotated generators \( T_t, T_\varphi \) and \( T_z \) are used:

\[ T_n = \frac{1}{2i} \tau_n \epsilon_n, \quad \epsilon_n = (\cos n \varphi, \sin n \varphi, 0), \]

\[ \epsilon_\varphi = (-\sin n \varphi, \cos n \varphi, 0), \quad \epsilon_z = (0, 0, 1), \] (18)

\( \tau_n \) being the Pauli matrices. The real number \( n \) must be integer to avoid multivaluedness of the potential. Two temporal components in (14-17) can be regarded as electric potentials, while the other four are the magnetic
ones. Thus purely magnetic static axially symmetric configurations can be described by four real functions of two variables $w_\alpha^\beta (r, z)$: [15, 16]:

$$eA_\mu dx^\mu = T_\varphi \left(w_1^2 dr + w_3^2 dz\right) + \left(T_r w_2^3 + T_z w_3^3\right) d\varphi.$$  \hfill (19)

The corresponding gauge fixing condition can be chosen in the form

$$r \partial_r w_3^2 + \partial_z w_3^3 = 0.$$  \hfill (20)

This ansatz has a residual form-invariance with respect to the transformation

$$U = \exp (T_\varphi \lambda(r, z)) = \cos \frac{\lambda}{2} + 2 T_\varphi \sin \frac{\lambda}{2}.$$  \hfill (21)

under which the functions $(w_1^2, w_3^3)$ transform as 2-dimensional gauge field

$$w_\alpha^\beta \to w_\alpha^\beta + \partial_\alpha \lambda, \quad \alpha = 1, 3,$$  \hfill (22)

while the functions $(w_1^1, w_3^3 - \nu)$ behave as scalar doublet:

$$\left(\begin{array}{c} w_1^1 \\ w_3^3 - \nu \end{array}\right) \to \left(\begin{array}{c} \cos \lambda & \sin \lambda \\ -\sin \lambda & \cos \lambda \end{array}\right) \left(\begin{array}{c} w_1^1 \\ w_3^3 - \nu \end{array}\right).$$  \hfill (23)

Now we impose the translational symmetry along the polar axis assuming that all gauge functions depend only on radial variable: $w_\alpha^\beta = w_\alpha^\beta (r)$. Performing the gauge transformation

$$U = \exp (T_\varphi \lambda(r)), \quad \tan \lambda = -\frac{w_1^1}{w_3^3 - \nu}$$  \hfill (24)

we can exclude the component $w_3^3$. Furthermore, the radial component of the gauge field vanishes by virtue of the YM equation $w_1^1 + \lambda' = 0$. Therefore in the static cylindrically symmetric magnetic case we are left with only two independent functions:

$$eA_\mu dx^\mu = T_2 w_3^3 dx^3 + T_3 w_3^3 dx^2.$$  \hfill (25)

It is worth noting that with this parametrization the equations of motion are invariant under discrete transformation interchanging both the space and color indexes $\varphi \leftrightarrow z$. This symmetry, however, is broken by the boundary conditions on the polar axis which is different for $w_3^3$ and $w_3^3$. Renaming the functions $w_3^3 (r)$ as in the cosmon string theory, we rewrite the final ansatz in the form

$$eA_\mu dx^\mu = T_\varphi R(r) dz + T_z (P(r) - \nu) d\varphi.$$  \hfill (26)

For another way to arrive at this representation see [17].

The corresponding matrix-valued field tensor has the following non-zero components:

$$eF_{\varphi \varphi} = T_2 P', \quad eF_{\varphi z} = T_\varphi R', \quad eF_{\varphi z} = -T_r P R.$$  \hfill (27)

Substituting them into the action [11] we obtain

$$S = \int d^4x \sqrt{-g} \left\{ -\frac{R}{16\pi G} - \frac{1}{2e^2} \left( \frac{P^2}{L^2} + \frac{R^2}{K^2} + \frac{P^2 R^2}{L^2 K^2} \right) \right\}.$$  \hfill (28)

The stress-energy tensor [6] for the gauge field is diagonal and can be expressed in terms of the energy density and the principal pressures as follows:

$$T_0^0 = \rho = \epsilon_\varphi + \epsilon_z + \epsilon_w,$$  \hfill (29)

$$T_r^r = -p_r = -\epsilon_\varphi - \epsilon_z + \epsilon_w,$$  \hfill (30)

$$T_\varphi^\varphi = -p_\varphi = \epsilon_\varphi - \epsilon_z - \epsilon_w,$$  \hfill (31)

$$T_z^z = -p_z = -\epsilon_\varphi + \epsilon_z - \epsilon_w.$$  \hfill (32)

where

$$\epsilon_\varphi = \frac{R^2}{2e^2 K^2}, \quad \epsilon_z = \frac{P^2}{2e^2 L^2}, \quad \epsilon_w = \frac{P^2 R^2}{2e^2 L^2 K^2}.$$  \hfill (33)

Its tracelessness implies

$$\rho = p_r + p_\varphi + p_z.$$  \hfill (34)

C. Symmetries of the reduced action

It is convenient to pass to the dimensionless quantities introducing the scale factor $l = e^{-1}\sqrt{4\pi G}$ and performing the rescalings

$$r \to l^{-1} r, \quad R \to l R, \quad P \to P, \quad N \to N, \quad L \to l^{-1} L, \quad K \to K.$$  \hfill (35)

Integrating the action (26) over $t, z, \varphi$ we obtain the one-dimensional action

$$S = \frac{\text{Vol}(t, z, \varphi)}{e^2 l^2} S_1, \quad S_1 = \int L dr,$$  \hfill (36)

where $\text{Vol}(t, z, \varphi)$ is the normalization volume, and the one-dimensional field lagrangian is

$$L = \frac{NLK}{2} \left( \frac{N' L'}{N L} + \frac{N' K'}{N K} + \frac{L' K'}{L K} - \frac{P^2}{L^2} - \frac{R^2}{K^2} - \frac{P^2 R^2}{L^2 K^2} \right).$$  \hfill (37)

As was argued above, this lagrangian is invariant under a discrete $\varphi \leftrightarrow z$ transformation which now reads as

$$\left(\begin{array}{c} P \\ L \end{array}\right) \leftrightarrow \left(\begin{array}{c} R \\ K \end{array}\right).$$  \hfill (38)

In terms of the rescaled quantities the stress-energy tensor will still be given by the Eq. (32) with

$$\epsilon_\varphi = \frac{R^2}{2K^2}, \quad \epsilon_z = \frac{P^2}{2L^2}, \quad \epsilon_w = \frac{P^2 R^2}{2L^2 K^2}.$$  \hfill (39)

The dimensional reduction over $t$, $\varphi$ and $z$ leads to two continuous symmetries. To make this more transparent, it is convenient to use an exponential parametrization for the metric and gauge functions:

$$N = e^\nu, \quad L = e^\lambda, \quad K = e^\xi,$$  \hfill (40)

$$R = e^\nu, \quad P = e^\lambda.$$  \hfill (41)
In terms of these new variables the lagrangian (67)
\[ L = \frac{1}{2} e^{\nu + \lambda + \xi} \left( \nu' \lambda' + \nu' \xi' + \lambda' \xi' - 2 e^{2(\beta - \lambda)} \beta^2 e^{2(\alpha - \xi)} - e^{(\alpha - \xi) + 2(\beta - \lambda)} \right), \] (41)
contains only three independent linear combinations of metric functions in the exponential. Choosing them as the new variables:
\[ \begin{align*}
\tau &= \nu + \lambda + \xi \\
\sigma &= -\xi + \alpha \\
\psi &= -\lambda + \beta \\
\tilde{\alpha} &= \alpha \\
\tilde{\beta} &= \beta
\end{align*} \] (42)
one finds that \( \tilde{\alpha} \) and \( \tilde{\beta} \) are cyclic. Therefore the lagrangian is invariant under the infinitesimal translations of \( \tilde{\alpha} \) and \( \tilde{\beta} \) which leave invariant \( \tau, \sigma \) and \( \psi \):
\[ \begin{align*}
\delta \alpha &= 0 \\
\delta \beta &= \varepsilon_1 \\
\delta \psi &= -\varepsilon_1 \\
\delta \xi &= 0 \\
\delta \lambda &= \varepsilon_1
\end{align*} \] (43)
The corresponding Noether charges are
\[ \begin{align*}
Q_1 &= \frac{1}{2} e^{\nu + \lambda + \xi} \left( \nu' - \lambda' - 2 e^{2(\beta - \lambda)} \beta^2 \right), \\
Q_2 &= \frac{1}{2} e^{\nu + \lambda + \xi} \left( \nu' - \xi' - 2 e^{2(\alpha - \xi)} \alpha^2 \right)
\end{align*} \] (44)
or, in the previous terms
\[ \begin{align*}
Q_1 &= \frac{1}{2} NLK \left( \frac{N'}{N} \frac{L'}{L} - 2 \frac{P' P}{L^2} \right), \\
Q_2 &= \frac{1}{2} NLK \left( \frac{N'}{N} \frac{K'}{K} - 2 \frac{R' R}{K^2} \right)
\end{align*} \] (46)
These are conserved on shell together with the total energy (hamiltonian), which for the static system differs from the one-dimensional lagrangian by the sign of the potential term:
\[ H = \frac{NLK}{2} \left( \frac{N'}{NL} + \frac{N' K'}{NK} + \frac{L' K'}{LK} - \frac{P'^2}{L^2} - \frac{R'^2}{K^2} + \frac{P^2 R^2}{L^2 K^2} \right) = 0. \] (48)
Let us discuss some cases studied previously. In some gauge models in curved space-time with cylindrical symmetry, such as cosmic strings, the stress-energy tensor satisfies the condition
\[ T_0^0 = T_z^z, \] (49)
implying the full Poincare symmetry of the \((t, z)-plane.\) In this case \( g_{tt} = g_{zz} \) and the metric is described by only two functions. From the Eq. (52) it is clear that this corresponds to \( \varepsilon_1 = 0, \) i.e. \( P = \nu \) (to ensure the regularity of the axis), in which case we find the Melvin solution \( \xi, \psi \) which we will call here “\( z\)-string”. In the general case, in view of the \( z - \varphi \) symmetry mentioned above, we deal with the non-linear superposition of the \( z\)-string and the \( \varphi\)-string. These are interacting via the non-linear terms in the Yang-Mills equations.

It is easy to see that a single \( \varphi\)-string is singular on the axis \( r = 0 \). Moreover, within the Einstein-Maxwell theory a configuration which contains both strings inherits the \( \varphi\)-string singularity. However the self-interaction in the non-Abelian theory makes possible the superposition of both strings regular on the polar axis. As we will see, in such configurations the contribution of the \( \varphi\)-string is suppressed near the polar axis, this ensures the regularity of the full solution.

III. EQUATIONS OF MOTION AND CONDITIONS ON THE POLAR AXIS

In view of the tracelessness of the stress-energy tensor of the gauge field we have three independent Einstein equations which read (5)
\[ \begin{align*}
(N' L K)'' &= \frac{P'^2}{L^2} + \frac{R'^2}{K^2} + \frac{P^2 R^2}{L^2 K^2}, \\
(N L' K)'' &= -\frac{P'^2}{L^2} + \frac{R'^2}{K^2} - \frac{P^2 R^2}{L^2 K^2}, \\
(N L K'') &= \frac{P'^2}{L^2} - \frac{R'^2}{K^2} - \frac{P^2 R^2}{L^2 K^2}.
\end{align*} \] (50)
In the Kasner gauge (9) the \((rr)\) Einstein equation is a first order equation which is the equivalent to the hamiltonian constraint (13).

The Yang-Mills equations have a symmetric form
\[ \begin{align*}
\left( R' \frac{NL}{K} \right)' &= \frac{N}{LK} P R^2, \\
\left( \frac{P' NL}{K} \right)' &= \frac{N}{LK} P R^2.
\end{align*} \] (53)
As a whole we have the system of five non-linear second-order differential equations \( (50, 51, 52, 53, 54) \), which must be completed by the set of boundary conditions. In this paper, we will be interested in the globally regular asymptotically Ricci-flat solutions. So we have to formulate the regularity conditions on the symmetry axis and to analyze the asymptotic behavior compatible with the ansatze for the metric and the YM field. The regularity conditions on the polar axis will be discussed separately for the flat-space problem and within the full self-gravitating treatment.
A. Expansion near the polar axis in flat space

In the flat space-time $N = K = 1$, $L = r$, and the equations of motion (53), (54) read

\[ r (r R')' = RP^2, \]
\[ r \frac{(P')'}{r} = R^2 P. \]

For the system of two second-order differential equations we have to impose four boundary conditions on the polar axis. Requiring the solutions to be regular on the polar axis, we find the conditions $PR = 0$ at $r = 0$: \[ P(0) = k, \quad R(0) = 0, \]

where $k$ is a real parameter. We will consider $k \neq 0$, since for $k = 0$ and $R(0) = \text{const}$ the solution is trivial $R(0) = P(0) = 0$.

Note that in the Abelian case the right hand side of the equation (53) is zero. Integrating this equation one finds a singularity $R'(r) = R'(0)/r$ which can be avoided only by excluding the $\varphi$-component of the gauge field from the ansatz. In the non-Abelian case this singularity is avoided by imposing the above conditions.

To proceed further, we look for the power series solutions for $R$ and $P$ near the polar axis. The first coefficient in the right hand side of the equation for $R$ is proportional to $k^2$. Thus in the left hand side of the equation the leading term must be proportional to $r^k$. One can find that the other terms in the expansions for $R$ and $P$ are proportional to $r^{2n+k+m}$, where $n$ and $m$ are integers. In the lowest order one obtains the following expansion:

\[ R = \frac{q}{|k|} r^k + \frac{q p k}{4(k+1)} r^{2+k} + O(r^{2+2k}), \]
\[ P = k + \frac{D^2}{2} + \frac{q^2}{4(k+1)} r^{2+k} + O(r^{3+2k}), \]

where $|k|$ denotes the integer part of $k$. The free parameters $q$ and $p$ represent the boundary conditions to the derivatives on the axis $r = 0$. Together with (57) one has four boundary conditions which are sufficient to formulate the boundary problem.

For an integer $k$ the parameters $q$ and $p$ are equal to the higher order derivatives of $R$ and $P$ as follows

\[ p = P''(0), \quad q = R^{(k)}(0), \]

and the expansions contain only integer powers of $r$, therefore the solutions for $R$ and $P$ are analytic functions. In this case we can obtain the higher order terms in series of expansions which are presented in Tab. 1.

These have the following structure:

\[ R = \sum_{j=0}^{\infty} R_{k+2j} \frac{r^{k+2j}}{(k+2j)!}, \]
\[ P = k + p r^2 + \sum_{j=0}^{\infty} P_{2(k+j+1)} \frac{r^{2(k+j+1)}}{2(k+j+1)!}, \]

where the coefficients of any order can be expressed through the free parameters $p, q$ as follows

\[ R_{k+2j} = \sum_{i=0}^{\infty} C_{ij}^{(k)} q^{2i+1} p^{-i(k+1)}, \]
\[ P_{2(k+j+1)} = \sum_{i=0}^{\infty} D_{ij}^{(k)} q^{2(i+1)} p^{-i(k+1)}. \]

Thus we have the three-parameter family of solutions to the system (55) specified by the set $\{q, p, k\}$. But as will be shown below one can fix e.g. $q = 1$ and consider only the two-parameter family $\{p, k\}$, while other solutions with $q \neq 1$ can be obtained by transformations of the family $q = 1$.

It is easy to see that if $\{R(r), P(r)\}$ is a solution to the system (55) then $\{IR(lr), P(lr)\}$ will be also a solution, where $l$ is some constant.

The question arises whether this rescaling affect parameters $\{q, p, k\}$. Consider the solution $\{R(q_0, p_0, k_0; r), P(q_0, p_0, k_0; r)\}$ and make the scale transformation $\{IR(q_0, p_0, k_0; lr), P(q_0, p_0, k_0; lr)\}$. The latter solution must coincide with some solution $\{R(q_0, p_0, k_0; r), P(q_0, p_0, k_0; r)\}$. In the leading orders of expansion one has

\[ q_0 r^k = l q_0 r^k l^k, \]
\[ k + p r^2 / 2 = k_0 + p_0 r^2 l^2 / 2. \]

This implies that $k_0 = k, p_0 = pl^{-2}$ and $l = (q_0 / q_0) l$. Extracting $l$ one obtains

\[ p = p_0 (q_0 / q_0) l. \]

Thus all solutions belonging to the curve (67) in the plane $k = \text{const}$ of the parameter space can be obtained from the solution $\{q_0 = 1, p_0, k\}$ only by the scale transformation

\[ R(q, p, k; r) = q^{\frac{e}{2}} R\left(1, q^{-\frac{e}{2}}, k; q^{\frac{e}{4}} r\right), \]
\[ P(q, p, k; r) = P\left(1, q^{-\frac{e}{2}}, k; q^{\frac{e}{4}} r\right). \]

In the following we will restrict to the case $q = 1$.

B. Curved space

Near the polar axis the metric component $g_{\varphi \varphi} = L^2(r)$ must vanish. To ensure the finiteness of the curvature the regularity conditions on the metric functions and their derivatives must be imposed. To find them we consider the Kretschmann scalar,

\[ R_{\mu \nu \lambda \tau} R^{\mu \nu \lambda \tau} = \left( \frac{N' L'}{N L} \right)^2 + \left( \frac{N' K'}{N K} \right)^2 + \left( \frac{L' K'}{L K} \right)^2 + \left( \frac{N''}{N} \right)^2 + \left( \frac{L'}{L} \right)^2 + \left( \frac{K''}{K} \right)^2, \]
which must be finite on the polar axis. Since \( L \) vanishes there, one has to require \( L'(0) = 0 \) as well. This implies that \( L'(0) \) must be non-zero, otherwise \( g_{\Phi \Phi} \) will be identically zero. Furthermore, vanishing of denominators in the upper string of (70) must be compensated by vanishing of numerators, which implies \( N'(0) = K'(0) = 0 \). Finally, \( N \) and \( K \) cannot vanish on the axis or they will be equal to zero everywhere. Therefore the locally flat metric starting with

\[
N(0) = K(0) = 1, \quad L(0) = 0, \\
N'(0) = K'(0) = 0, \quad L'(0) = 1. \tag{71}
\]

is the only one (up to rescaling) which provide finiteness of the expression (70) one the polar axis. Note that from the azimuthal Einstein equation one can show that the condition

\[
L''(r = 0) = 0 \tag{72}
\]

is satisfied if the azimuthal pressure is finite near the polar axis.

Now we use the same strategy to find the appropriate condition on the gauge field (24), constructing the power series expansions near \( r = 0 \) for all variables satisfying the metric conditions (71) directly from the system (50–54). One can expect similar results for the field functions (57) and the six conditions on the metric axis

\[
\text{conditions consists of the previously obtained two conditions} \quad \text{for the field functions} \quad \text{and the six conditions on the metric} \quad (74), \quad \text{with three parameters} \quad \{p, q, k\} \quad \text{remaining free (recall that to ensure analyticity of the solution near the polar axis one has to take integer} \quad k \text{).}
\]

Unlike the flat case, now the series expansions contain some additional terms. But all such terms include only higher powers of parameters \( p \) and \( q \) apart from the main terms from (58, 59). For this reason in the weak field case \( p \to 0, q \to 0 \) the curved space expansions are dominated by the flat space ones (58, 59), i.e. the solutions satisfying (77) will differ only by the scale factors. This scale invariance does not hold in general, however.

Now we would like to clarify the physical meaning of the parameter \( p \). Using the Tab. (11) one finds

\[
p = \lim_{r \to 0} \frac{P'(r)}{L(r)}. \tag{75}
\]

Comparing this expression with the component \( F_{r\phi} \) of the field tensor (27), one can see that this quantity is nothing else than the corresponding tetrad component with the color index \( \phi \). It is therefore the limiting value of the physical magnetic field component along the symmetry axis

\[
B_z(r) = \frac{P'(r)}{\epsilon L(r)}. \tag{76}
\]

After rescaling, we obtain the dimensionless parameter

\[
\tilde{B} = 4\pi G e^{-1} B(0), \tag{77}
\]

therefore the parameter \( p \) is equal up to sign to the (dimensionless) value of \( z \)-component of the magnetic field on the polar axis:

\[
p = -\tilde{B}. \tag{78}
\]

\[\textbf{C. Sum rules}\]

To relate the expansion on the symmetry axis to the asymptotic form of the solution on can derive the sum rules following the approach of the Ref. (18). Multiplying the Einstein equations (80, 82) by

\[
\sqrt{-g} = NLK, \tag{79}
\]

we integrate them from \( r = 0 \) to \( r \) using the expressions (42, 49) for the stress-energy tensor. This gives

\[
N'LK = M, \tag{80}
\]

\[
NLK' = W + 1, \tag{81}
\]

\[
NLK'' = X, \tag{82}
\]

where three new functions are introduced to the integrals

\[
M = \int_0^r dr \sqrt{-g}(\rho + p_r + p_\phi + p_z), \tag{83}
\]

\[
W = -\int_0^r dr \sqrt{-g}(\rho - p_r + p_\phi - p_z), \tag{84}
\]

\[
X = -\int_0^r dr \sqrt{-g}(\rho - p_r - p_\phi + p_z). \tag{85}
\]

The function \( M(r) \) is the Tolman mass of the gravitating YM field within the cylinder of the radius \( r \). Two other quantities are related to integral pressures in the longitudinal and azimuthal directions, their more precise meaning will be clear later on. The sum rules (80, 82) express the metric functions through the integral characteristics of the matter field.

Two other some rules can be obtained by integrating the YM equations. Multiplying (53, 54) on \( R \) and \( P \)
respectively, we can rewrite the YM equations as follows
\[
\left(\frac{NL}{K} RR\right)' = NKL \left(\frac{R^2}{K^2} + \frac{P^2 R^2}{L^2 K^2}\right), \quad (86)
\]
\[
\left(\frac{NK}{L} PP\right)' = NKL \left(\frac{P^2}{L^2} + \frac{P^2 R^2}{L^2 K^2}\right). \quad (87)
\]
Then we integrate in the same way as the Einstein equations obtaining the sum rules:
\[
\frac{NL}{K} \left(\frac{R^2}{K^2}\right) = \int_0^r dr \sqrt{-g} \left(\frac{R^2}{K^2} + \frac{P^2 R^2}{L^2 K^2}\right) = \frac{M - X}{2}, \quad (88)
\]
\[
\frac{NK}{L} \left(\frac{P^2}{L^2}\right) = \int_0^r dr \sqrt{-g} \left(\frac{P^2}{L^2} + \frac{P^2 R^2}{L^2 K^2}\right) + \lim_{r \to 0} \left(\frac{NK P P'}{L} \right) = \frac{M - W}{2} - k_0 B. \quad (89)
\]
From the three integral functions \( M, W, X \) only two are independent: the constraint equation (88) implying
\[
MX + M(W + 1) + X(W + 1) = 0. \quad (90)
\]

### IV. ASYMPTOTIC BEHAVIOR

Methods of investigations of the asymptotical geometry generated by cylindrically symmetric field configurations were developed within the model of Abelian cosmic strings \[7, 19, 20, 21\], for a more general model-independent approach see \[18\]. Here we shall use the same technique with some modifications appropriate to the YM field dynamics.

#### A. Kasner solution

It is well known that the general vacuum solution of Einstein equations for the ansatz \[9\] is given by the Kasner metric
\[
d_s^2 = r^{2a} dt^2 - dr^2 - \gamma^2 r^{2b} d\varphi^2 - r^{2c} dz^2, \quad (91)
\]
where constants \( a, b \) and \( c \) satisfy the conditions
\[
a + b + c = a^2 + b^2 + c^2 = 1. \quad (92)
\]
Consider the cylindrically symmetric configurations of the YM field for which the stress-energy tensor vanishes at \( r \to \infty \). Then the metric in the asymptotical region will approach \[91\], where parameters \( a, b \) and \( c \) must be specified in terms of parameters of the field configuration. For this purpose we define the asymptotic values of the integral functions \( M, W, X \) assuming them to be finite
\[
M_\infty = \lim_{r \to \infty} M(r), \quad W_\infty = \lim_{r \to \infty} W(r), \quad X_\infty = \lim_{r \to \infty} X(r), \quad (93)
\]
and denote \( \Sigma_\infty = M_\infty + W_\infty + X_\infty + 1 \). These three parameters satisfy a constraint \[80\]:
\[
M_\infty X_\infty + M_\infty(W_\infty + 1) + X_\infty(W_\infty + 1) = 0. \quad (94)
\]
Together with \( B \), we thus have four parameters to be related to the Kasner parameters. From the Eqs. (80—83) it follows
\[
N'LK \to M_\infty, \quad (95)
\]
\[
NL'K \to W_\infty + 1, \quad (96)
\]
\[
NL'K \to X_\infty. \quad (97)
\]
It is easy to find the solution to these equations assuming \( \Sigma_\infty \neq 0 \):
\[
N = \omega r^a, \quad a = M_\infty/\Sigma_\infty, \quad (98)
\]
\[
L = \gamma r^b, \quad b = (W_\infty + 1)/\Sigma_\infty, \quad (99)
\]
\[
K = (\omega \gamma)^{-1} r^c, \quad c = X_\infty/\Sigma_\infty, \quad (100)
\]
\[
\sqrt{-g} = NLK = \Sigma_\infty r. \quad (101)
\]
where \( \omega, \gamma \) are some constants. The first Kasner condition \[92\] is obviously satisfied, while the second is equivalent to \[91\].

So we are left with only two independent parameters describing the asymptotically vacuum metric. The condition \[84\] means the fulfillment of the radial Einstein equation. From \[104\] it is clear that the condition \( \Sigma_\infty \neq 0 \) is the requirement of the non-degeneracy of the metric at infinity. On the polar axis \( r = 0 \) the boundary condition for \( \sqrt{-g} \) read that its derivative is positive. Since \( \sqrt{-g} \) must be non-vanishing at any point except \( r = 0 \), one has to require \( \Sigma_\infty > 0 \).

#### B. Existence of asymptotically vacuum solutions

Now let us turn to the dynamic of the gauge field. The assumption of the finiteness of \( M_\infty, W_\infty \) and \( X_\infty \) imposes some conditions on the gauge field asymptotics. Actually, the energy density must decrease fast enough for \( M_\infty \) to be finite. From Eqs. (83—83) it is clear that the absolute values of \( W_\infty \) and \( X_\infty \) are smaller than \( M_\infty \). The total energy density is the sum of three components \[20\], which asymptotically can be written as
\[
\varepsilon_\varphi \sim R^2 r^{-2c}, \quad \varepsilon_\varphi \sim P^2 r^{-2b}, \quad \varepsilon_w \sim P^2 R^4 r^{2b-2}. \quad (102)
\]
All these quantities must fall faster than \( r^{-2} \) to ensure finiteness of \( M_\infty \). Notice that \( 0 < a = M_\infty/\Sigma_\infty \leq 1 \) since \( \Sigma_\infty > 0 \), and \( M_\infty > 0 \) because it is proportional to the
energy of the configuration per unit length. Therefore $2a - 2 > -2$ and $RP = o(1)$ as $r \to \infty$.

Now we can obtain further information invoking the sum rules \(88, 89\) in combination with the metric asymptotics \(98, 100\):

\[
(R^2)' = \frac{(M_\infty - X_\infty)K^2}{\Sigma_\infty r}, \quad (P^2)' = \frac{(M_\infty - W_\infty - B)L^2}{\Sigma_\infty r},
\]

(103) (104)

Integrating these equations in the leading order with account for \(88, 100\) we obtain

\[
R^2 = C_1^{(R)} + C_2^{(R)}r^{2c},
\]

(105)

\[
P^2 = C_1^{(P)} + C_2^{(P)}r^{2b},
\]

(106)

with four constant parameters, which can be determined using \(102\). The integrand in \(88\) is positively definite, so the integral is non-zero. Since $C_2^{(R)}$ is proportional to this integral, it must be non-zero too. To satisfy the finiteness condition for $M_\infty$ with asymptotic behavior of the first exponents in Eqs.\(102\) the quantity $2(2c - 1) - 2c = 2c - 2$ has to be smaller than $-2$, so $c$ must be negative. Since among the set of parameters $a, b$ and $c$ satisfying the Kasner conditions there can not be two negative quantities, $b$ has to be positive and the corresponding exponent in Eqs.\(112\) is equal to $2b - 2 > -2$. Thus to ensure the finiteness condition for $M_\infty$ we must take $C_2^{(P)} = 0$, or, in other terms,

\[
\int_0^\infty drNLK \left( \frac{P^2}{L^2} + \frac{P^2 R^2}{L^2 K^2} \right) - kB = 0,
\]

(107)

what is equivalent to one more constraint on the parameters:

\[
M_\infty - W_\infty = 2kB.
\]

(108)

Note that a similar relation (up to some factor) was obtained within the model of Abelian cosmic strings \(21\) as the condition of existence of the Melvin-like solutions.

To get further information about asymptotics of the gauge functions we apply the same integration procedure as described in section III to the non-transformed equation of motion \(55\). This gives

\[
R = C_1 + C_2r^{2c}
\]

(109)

with two more constants. Since $c < 0$, for the square roots of \(105\) one has two possibilities:

\[
R = \sqrt{\frac{C_1^{(R)}}{C_1^{(R)}}} r^c, \quad C_1^{(R)} = 0,
\]

(110)

\[
R = \sqrt{\frac{C_1^{(R)}}{2C_1^{(R)}}} r^{2c}, \quad C_1^{(R)} \neq 0.
\]

(111)

Comparing with \(109\) we see that the second case is realized. But this implies that \(C_1^{(P)} = 0\) and \(P \to 0\) at infinity faster than power-law, while $R$ approaches a constant as a power with the exponent $2c$. Finally one finds the following explicit expression for $C_2^{(R)}$:

\[
C_2^{(R)} = \frac{M_\infty - X_\infty}{4C_1 X_\infty \omega^2 \gamma^2}.
\]

(112)

Now consider the flat space case. We are looking for asymptotically vacuum configurations, which implies vanishing energy density at infinity:

\[
rR^2 + \frac{P^2}{r} + \frac{R^2 P^2}{r^2} \to 0.
\]

(113)

Integrating the equations of motion \(100, 100\) with zero right hand sides, one obtains the asymptotical expressions for field functions

\[
R = C_1 \ln r + C_1,
\]

(114)

\[
P = C_3 r^2 + C_4,
\]

(115)

which satisfy \(112\) if \(C_3 = 0\). Also there must be \(C_4 = 0\) otherwise the right hand side in the equation for $P$ will be non-zero. Thus the conditions for the field components at infinity read

\[
R \sim \ln r, \quad P \to 0.
\]

(116)

These are the necessary conditions of vanishing of the energy density. For more detailed expansion see sec.V.

Performing the same steps with the equation for $P$, one obtains the following relation:

\[
\lim_{r \to \infty} \frac{(P^2)'}{2P} = \frac{\int_0^\infty drrNLK \left( \frac{P^2}{r^2} + \frac{R^2 P^2}{r^2} \right)}{\Sigma_\infty} = \lim_{r \to \infty} \frac{P'}{P}.
\]

(117)

Therefore the condition \(116\) is satisfied if the integral exists and the whole expression is equal to zero. This is the necessary condition of the existence of asymptotically vacuum solutions:

\[
\int_0^\infty drrNLK \left( \frac{P^2}{r^2} + \frac{R^2 P^2}{r^2} \right) - kB = 0.
\]

(118)

In the flat case all values $M(r), W(r)$ and $X(r)$ diverges as $\ln r$ at infinity because of the term $R^2/r \sim r^{-1}$ in the integrand. But the difference $(M - W)$ remains finite and it is equal to the integral in the left hand side of Eq.\(118\). So the form of the bifurcation condition \(108\) remains the same though the separate values of $M_\infty$ and $W_\infty$ are not well defined.

### C. Kasner parameters

So far we have obtained the asymptotic behavior of the gauge and metric functions consistent with the equations
of motion. The solution for the metric is determined by three parameters \(a\), \(b\) and \(c\) satisfying two constraints. Now recall that there are also two conserved Noether charges (14), (17). This enables us to expose the asymptotic parameters of the solution through the data on the polar axis.

Equating the Noether charges \(Q_{\mu}|_{r=0} = Q_{\mu}|_{r \to \infty}\) one obtains further two constraints:

\[
2kB - 1 = \Sigma_{\infty}(a - b),
\]

\[
0 = \Sigma_{\infty}\left(a - c - \frac{M_{\infty} - X_{\infty}}{\Sigma_{\infty}}\right).
\]

These relations can be used to express \(b\) and \(c\) in terms of two new parameters:

\[
b = a + \beta, \quad \beta = \frac{1 - 2kB}{\Sigma_{\infty}},
\]

\[
c = a - \sigma, \quad \sigma = \frac{M_{\infty} - X_{\infty}}{\Sigma_{\infty}}.
\]

Now, from the first Kasner condition one has

\[
a = \frac{1}{3}(1 + \sigma - \beta).
\]

Finally, the quadratic Kasner condition allows us to express all parameters in terms of the only parameter \(\beta\):

\[
a = \frac{1}{3}(1 - 3(\beta/2) + \sqrt{1 - 3(\beta/2)^2}),
\]

\[
b = \frac{1}{3}(1 + 3(\beta/2) + \sqrt{1 - 3(\beta/2)^2}),
\]

\[
c = \frac{1}{3}(1 - 2\sqrt{1 - 3(\beta/2)^2}).
\]

In the previous subsection we have showed that \(c\) must be negative, this determines the sign of square root, an alternative choice of a sign would give \(c \geq 1/3\). We will show later that the expression under the square root is always positive. Thus the values \(B\) of the magnetic field value on the symmetry axis along with \(\Sigma_{\text{infty}}\) fully specify the solution in the asymptotic region.

D. Asymptotic YM vacua

As was shown above the real number \(k\) specifies the family of solutions for \(R\) and \(P\) i.e. the radial part of the ansatz. Since \(R \sim r^k\) in the vicinity of the axis, the azimuthal component of the magnetic field \(B_{\phi} \sim R' \sim r^{k-1}\) diverges for non-integer \(k\) on the interval \(0 < k < 1\) and so does the Kretchmann scalar (70). To ensure analyticity of the solution near the polar axis \(k\) should be chosen integer. Thus \(k\) can be regarded as a "regular" number of the configuration. If \(k = 0\) there are no regular solutions except the trivial one, since the condition (108) implies vanishing of the energy density.

The parameter \(\nu\) does not enter the equations of motion. This parameter must be integer to avoid the multivaluedness of the solution. Actually it (and only it) specifies the azimuthal part of the ansatz, while \(k\) (and only \(k\)) specifies the radial part. The full analytic solution comes as a product of both parts and thus is characterized by two integers. The natural question arises whether the integers \(\nu\) and \(k\) can be given any topological interpretation.

Using the asymptotic expansion for the gauge potential \(A_\mu\) (recall that for large \(r\) the function \(P\) vanishes while \(R\) tends to a constant, \(C\)), we find asymptotically the potential one-form as

\[
A_{\infty} = CT_\varphi dz - \nu T_z d\varphi,
\]

and the field two-form \(F = dA + A \wedge A\) vanishes. This implies that \(A\) in (127) describes a vacuum state and can be expressed through the matrix-valued gauge function:

\[
A_{\infty} = UdU^{-1}.
\]

This equation can be solved as follows:

\[
U = \frac{1}{\sqrt{2}}(\cos f(1 - 2T_r) - 2\sin f(T_\varphi + T_z)),
\]

where

\[
f = (Cz - \nu \varphi)/2.
\]

Now consider the potential on the polar axis. Here it is also a pure gauge:

\[
A_{\text{axis}} = (k - \nu)T_z d\varphi = \tilde{U}d\tilde{U}^{-1}
\]

with

\[
\tilde{U} = \cos \tilde{f} - 2\sin \tilde{f}T_z, \quad \tilde{f} = (k - \nu)\varphi/2.
\]

These two vacua are related by the gauge transformation (129) with \(f = (Cz - k\varphi)/2\). The winding number of this transformation is zero and the transformation is "small". This reflects the fact that the mapping of the asymptotic cylinder \((r \to \infty, z)\) into the gauge group \(\text{SU}(2) \sim S^3\) is always topologically trivial. Thus we can conclude that the integers \(\nu\) and \(k\) do not have any topological meaning. Note also that the divergence of the Chern–Simons current

\[
\nabla_\mu K^\mu = \frac{e^2}{10\pi^2} \text{tr} F_{\mu\nu} \tilde{F}^{\mu\nu}
\]

is zero because of the absence of electric part in the ansatz.

It is instructive to compare the situation with that in the case of the EYM sphalerons of Bartnik-McKinnon’s type. Sphalerons are the finite energy saddle points in the configuration space of the system, whose existence can be revealed by the constructing the non-contractible loops of configurations (22) which alternatively can be
seen as connecting topologically distinct vacua. For the spherically symmetric four-dimensional EYM configurations this was done explicitly in [23]. Configurations of finite energy along the loop form a manifold homeomorphic to $S^3$, and the existence of non-contractible loops is a consequence of non-triviality of the mapping of this $S^3$ to the gauge group $SU(2) \sim S^3$. In the cylindrical case one considers the loop of configurations of finite energy per unit length which form the manifold equivalent to $S^2$. Thus instead of the mapping $S^3 \to S^2$ one deals with the mapping $S^2 \to S^3$ which is topologically trivial. Therefore in the cylindrical case the solutions do not admit sphaleronic interpretation.

V. EXISTENCE OF GLOBAL SOLUTIONS AND NUMERICAL INTEGRATION

In the previous sections we have shown that global solutions asymptotically approaching vacuum configurations, if exist, should be fully specified by the boundary conditions on the polar axis. Up to rescaling, the only relevant quantity is the parameter $p = -B$ which is the second derivative of $P$ on the axis (see Sec. III, we set the scale taking $q = 1$). It turns out that the family of solutions specified by different $p$ exhibits the bifurcation structure similar to that observed in the Bartnik-McKinnon case, so the existence of regular solutions can be proved by similar methods. Here we sketch a proof and present the results of numerical integration which strongly supports this analysis.

A. Flat case

To better understand the dynamics of the gauge field we start with the flat-space problem. Our goal is to investigate the bifurcating behavior of the system \[ \frac{dp}{dr} = P, \quad \frac{dp}{dr} = P. \] Unfortunately the dynamical systems technique does not provide enough information about solutions. There are three special points of the system: two at the the origin ($r = 0, P = 0$) and ($r = 0, R = 0$) and one at infinity ($1/r = 0, P = 0$). The corresponding linearized systems are degenerate and phase portraits in the vicinity of the special points are trivial. It turns out that the values of parameters $p$ and $q$ are non-small on the bifurcation curve, so we cannot use the asymptotic methods for the analysis. Our reasoning will based on the fact that the equations \[ \frac{dp}{dr} = P, \quad \frac{dp}{dr} = P \] are linear with respect to the variable entering with its derivative, if the other variable is considered as a given function.

First, let us investigate properties of the solutions of the Eq. (55) with an arbitrary regular function $P(r)$. In what follows we assume $P(0) = k > 0$.

**Lemma 1.** The solution for $R(r)$ with the boundary conditions $R(0) = 0$, $R'(0) = q > 0$ is monotonously increasing function for $r > 0$.

**Proof.** Since $R'(0) > 0$, it remains positive in some region $0 < r < a$. Assuming that it vanishes at $r = a$, we integrate the Eq. (55) on the segment $[0, a]$:

\[
R'(a) = \frac{1}{a} \int_0^a \frac{R P^2}{r} dr. \quad (134)
\]

Since $R'$ is positive on $[0, a)$, the function $R$ itself is also positive, $R(r) > 0$ for $r \in (0, a)$. Due to the assumption $P(0) > 0$ the function $P^2(r)$ is not identically zero on this segment. Therefore the integral at the right hand side of the Eq. (134) is strictly positive and hence $R'(a) \neq 0$, what contradicts to the initial assumption. \qed

**Lemma 2.** Consider two positive functions satisfying $P_1(r) > P_2(r)$ for $r > 0$. Then the difference $\Delta R \equiv R[q, P_1] - R[q, P_2]$ is monotonously increasing function for $r > 0$.

**Proof.** Using the expansions near $r = 0$ one can see that the rate of growth of $\Delta R$ near the origin is proportional to the value of $P_1^2 - P_2^2$ at the origin. Therefore there exists some small value $r = a$ for which $\Delta R(a) > 0$, $\Delta R'(a) > 0$. Again assuming that at some point $r = b$ the derivative $\Delta R'$ vanishes and integrating the equation for this quantity (which is essentially the same as for $R'$ due to linearity of the Eq. (55)) we obtain

\[
\Delta R'(b) = \Delta R'(a) \frac{a}{b} + \frac{1}{b} \int_a^b \frac{\Delta R(P_1^2 - P_2^2)}{r} dr. \quad (135)
\]

Similarly to the previous lemma, we obtain $\Delta R'(b) > 0$ what contradicts to the assumption. Thus $\Delta R(r)$ is positive and $\Delta R(r)$ is monotonously increasing for $r > 0$. \qed

Now let us turn to the properties of the solution of the Eq. (55) for $P$ with a given arbitrary regular function $R(r)$.

**Lemma 3.** If at some point $r = a$ one has $P(a) \geq 0$, $P'(a) > 0$, then both $P(r)$ and $P'(r)$ are positive monotonously increasing functions for $r > a$.

**Proof.** Assume that there is some point $b > a$ such that $P'(a \leq r < b) > 0$, $P'(b) = 0$. Then $P$ will be a non-decreasing function on the segment $[a, b]$ implying $P \geq 0$. Direct integration of the Eq. (55) leads to

\[
P'(b) = P'(a) b/a + b \int_a^b R^2 P dr. \quad (136)
\]

The integrand at the right hand side is non-negative, therefore $P'(b) > 0$ contradicting to the assumption. Thus $P(r)$ is a monotonously increasing function and from the Eq. (136) it follows that $P'$ is monotonously increasing for $r > a$ too. \qed
Since the Eq. (26) is linear in $P$, it is invariant under inversion $P \rightarrow -P$, therefore if $P(a) \leq 0$, $P'(a) < 0$ then $-P(r)$ and $-P'(r)$ are positive monotonously increasing functions for $r > a$.

Now let us turn to the full system (53)–(59). One can easily see that there exist integral curves corresponding to both cases considered above. Indeed, for small values of $|p|$ and large values of $q$ the derivative $P'$ with the growing of $r$ soon becomes positive with $P$ also remaining positive. Vice versa, for large $|p|$ and small $q$ the function $P$ soon changes the sign while $P'$ also remains negative. In both cases we will observe an infinite growth of $|P|$ as $r \rightarrow \infty$. Between these two sets of $p, q$ there is a bifurcation set such that the corresponding integral curve satisfies the inequalities $P(r) > 0$, $P'(r) < 0$ as $r \rightarrow \infty$. Then $|P|$ will be a monotonously decreasing function at infinity. Thus we will obtain the heteroclinic trajectory connecting two special points of the system: the origin $(r = 0, R = 0)$ and infinity $(1/r = 0, P = 0)$. Using the above lemmas one can prove that this solution is unique:

**Theorem 1.** For each set of parameters $q, k$ there is a unique value of parameter $p$ corresponding to the heteroclinic trajectory connecting points $(r = 0, R = 0)$ and $(1/r = 0, P = 0)$.

**Proof.** Let us assume that there are such two trajectories. Since the solution of the system is fully specified by the set of parameters $(q, p, k)$, different trajectories must correspond to different values of $p$, say $p_1 > p_2$. The series expansion for $P$ near the origin implies $\Delta P \equiv P_1 - P_2 \approx (p_1 - p_2)r^2/2 > 0$. This means that there is some point $r = a$, where $\Delta P(a) > 0$, $\Delta P'(a) > 0$. Now consider the point $r = b$ where $\Delta P''$ vanishes: $\Delta P'(a < r < b) > 0$, $\Delta P'(b) = 0$. Since $\Delta P'$ is positive, the function $\Delta P$ is monotonously increasing which implies that $P_1 > P_2$ on the segment $[a, b]$ (with both $P_1$, $P_2$ remaining positive). Using the lemma 2, we find that $R_1 > R_2$ on this segment. Integration of the equation for $\Delta P$ gives

$$\Delta P'(b) = \Delta P'(a)b/a + b \int_a^b (R_1^2 P_1 - R_2^2 P_2)d\tilde{r}. \quad (137)$$

The integrand is positive and the quantity $\Delta P'(a)$ is positive too, therefore $\Delta P'(b) > 0$ what contradicts to the assumption. This means that $\Delta P(r)$ is monotonously growing function for $r > a$ with the rate of growth not slower than $r^2$. But as $r \rightarrow \infty$, both $P_1$, $P_2$ vanish and $\Delta P$ has to vanish too. Thus we arrive at the contradiction again, hence the assumption about the existence of several heteroclinic trajectories is wrong. Therefore there is a unique value $p = p_0$ for each set $(q, k)$ for which the corresponding solution satisfies the boundary conditions $R(0) = 0$, $P(0) = k \neq 0$ and monotonously reaches its asymptotic form (110).

Thus the full solution space consists of the following sets:

- Solutions with monotonously increasing $|P|$. Numerical calculations show that these grow infinitely at some finite $r$, i.e. they are singular.
- Solutions with monotonously decreasing $|P|$. There is a unique bifurcation value $p = p_0$ for each set of $(q, k)$ for which such a solution can arise. These solutions are globally regular.

Numerical integration of radial equations starting from the polar axis $r = 0$ shows that for any fixed value of $q$ both types of solutions are realized. Singular solutions diverge at finite distance $r = r_s$ as follows:

$$R(r) \rightarrow +\infty, \quad P(r) \rightarrow +\infty, \quad \text{as} \quad r \rightarrow r_s, \quad (138)$$

$$R(r) \rightarrow +\infty, \quad P(r) \rightarrow -\infty, \quad \text{as} \quad r \rightarrow r_s, \quad (139)$$

In the vicinity of the singularity, the system (55)–(59) can be integrated analytically. In the leading order one has

$$R'' = R\hat{P}^2, \quad (140)$$

$$\hat{P}'' = R\hat{P}, \quad (141)$$

where $\hat{P} = P/r_s$. This system is invariant under the transformation $R \leftrightarrow \hat{P}$ (actually this is the symmetry (33)). Multiplying these equations on $R$ and $\hat{P}$ respectively one obtains

$$RR'' = \hat{P}\hat{P}'', \quad (142)$$

and thus there exist solutions $\hat{P} = \pm R$. Then the equation

$$R'' = R^3 \quad (143)$$

can be easily solved, and find we the divergent terms as follows:

$$R = \frac{\sqrt{2}}{r_s - r}, \quad P = \pm \frac{\sqrt{2}r_s}{r_s - r}. \quad (144)$$

Now discuss globally regular solutions. It was confirmed numerically that for any fixed value of $q$ there is a unique value $p = p_0$ such that for any $p > p_0$ and $p < p_0$ the integral curves correspond to singular solutions of the first and second types respectively. As $p \rightarrow p_0$ the singularity point $r_s$ monotonously moves away from the origin presumably to infinity. In order to reconfirm the existence of regular solutions numerically we have performed integration from infinity gluing it with the solution started from the polar axis. Asymptotically the regular solution to the system (55)–(59) is given by the following three-parameter family:

$$R = (C_1 \ln r + C_2)\alpha(r), \quad (145)$$

$$P = D\sqrt{\frac{r}{\ln r}} \exp\left[-C_1 r \ln r + (C_1 - C_2) r\right]\beta(r), \quad (146)$$

where some functions $\alpha(r), \beta(r)$ satisfying the boundary conditions $\alpha, \beta \rightarrow 1$ as $r \rightarrow \infty$. The expansions for
\( \alpha, \beta \) are rather complicated because of the presence of logarithmic terms \( R \) and exponential terms for \( P \). In particular,

\[
\beta(r) = 1 - \frac{C_2}{2C_1 \ln r} + o(\ln^{-1} r),
\]

(147)

\[
\alpha(r) = 1 + \left( \frac{P}{2C_1 r \ln r} \right)^2 (1 + o(\ln^{-1} r)).
\]

(148)

Therefore any globally regular solution is fully specified by the set parameters \((C_1, C_2, D)\) as well as by set \((q, p, k)\). In the asymptotic set the parameter \( D \) defines a scale, reflecting the situation with the set \((q, p, k)\). Numerical integration provides matching of these two sets of parameters with each other.

Previously we have obtained the necessary condition \(118\) of existence of regular asymptotically vacuous solution. Numerical integration confirms that for the solutions at the bifurcation point \( p = p_b \) this condition is satisfied indeed. Both regular and singular numerical solutions for \( q = 1, k = 1 \) are shown in Fig. 1. For other values of \( k \) the behavior of solutions is similar. Also, as we have argued there are families of the solutions related by the scale transformation. Numerical computations confirm that all solutions with

\[
p = p_b q^{2/k}
\]

(149)

are regular. They correspond to the bifurcation curves of in the parameter space shown in Fig. 2.

### B. Self-gravitating case

As was argued before, the behavior of the full self-gravitating solutions corresponding to small values of parameters \( q \rightarrow 0, p \rightarrow 0 \) must be the same as in the flat space. This implies the existence of both regular and singular solutions in the gravitating case too. Numerical integration of the system \(153\) proceeds along the same lines with the boundary conditions obtained from the expansions on the polar axis with free parameters \((q, p, k)\) (Tab. II). The main difference is that now the asymptotic behavior of the metric functions and \( R \) at infinity is power-low with the linear combinations of the Kasner parameters \((a, b, c)\) in the exponent. For different sets of these parameters the linear combinations with the same coefficients can be positive or negative which mean either infinite growth or vanishing of the corresponding terms in the expansion. For this reason the asymptotic solutions will be different for different sets of \((a, b, c)\). We give here the asymptotic expansions in the most symmetric case of the values of parameters: \( a = 2/3, b = 2/3, c = -1/3 \). In this case the functions \((N, L, K, R)\) expand in powers of

\[
x = r^{-1/3};
\]

\[
R = C_1 + C_2 x^2 + \frac{24C_2^2 x^4}{\omega^2 \gamma^2} + O(x^5),
\]

(150)

\[
N = \frac{\omega}{x^2} - \frac{C_2^2}{\omega \gamma^2} + \frac{\omega C_1}{\gamma} x + \frac{2C_2^2 x^3}{\omega^2 \gamma^2} + O(x^5),
\]

(151)

\[
L = N^2 \omega,
\]

(152)

\[
K = \omega x - 12 \omega C_3 x^4 + O(x^5).
\]

(153)

The YM function \( P \) has an exponential form:

\[
P = \exp \left\{ -\frac{3}{4 \omega \gamma} \left( C_1 \frac{x^4}{x^4} + \frac{2C_2}{x^2} - \frac{4C_2^2}{\omega^2 \gamma^2} \ln x \right) \right\} (D + O(x)).
\]

(154)

Altogether we have six parameters: \( C_1, C_2, D \) for the YM functions as in the flat space, and four metric parameters: \( \omega, \gamma, C_3, \) two of which can be changed by rescalings of \( t \) and \( z \), and one independent Kasner parameter. Together with three parameters on the axis \( q, p, k \) one has ten parameters to match the solutions obtained by the integration from the axis and from the infinity. Since these solutions come from the system of five second order differential equations the number of parameters coincide with the number of matching conditions. The numerical integration scheme is the same as in the flat case, and the conservation of charges \( Q_1, Q_2, \Sigma \) along the integral curves is used to control the numerical error.

It turns out that in the gravitating case the bifurcation parameters do not satisfy the condition \(149\) anymore. The bifurcation curve has to be found numerically. Actual calculations show that it is bounded within the finite region

\[
0 \leq q \leq q_{\max}, \quad 0 \geq p \geq p_{\max},
\]

(155)

and for each \( q \) in this region there are two bifurcation parameters \( p \). The shapes of the bifurcations curves are similar for all \( k \), but with growing \( k \) the region of parameters becomes very narrow, this is illustrated on Fig. 3 for three lowest \( k \). The condition \(107\) is satisfied indeed for the regular solutions, and all other analytical results of the Sec. IV are also confirmed by numerical computations.

Using the Eqs. \(124,126\) we can relate the asymptotic Kasner parameters to the value of the magnetic field on the symmetry axis. The weak field limit \( B \rightarrow 0 \) corresponds to the region \( q \rightarrow 0, p \rightarrow 0 \) on the bifurcation curve. In this case the metric approaches the Minkowski metric. From the asymptotic relations \(98,101\) one can see that this implies \( M, W, X, Z \rightarrow 0, \Sigma \rightarrow 1, \) and the parameter \( \beta \) in Eq. \(124,126\) approaches the value \(1 - \delta, \delta \rightarrow 0\). In the first order in \( \delta \):

\[
N \sim r^\delta, \quad L \sim r, \quad K \sim r^{-\delta}.
\]

(156)

As we move along the bifurcation curve, the field parameter \( B \) grows up, the exponent of \( L \) decreases, while
the exponent of \( N \) increases. For \( B = 1/(2k) \) the metric acquires an additional boost symmetry: from Eq. (121) it follows that \( \beta = 0 \) so the Kasner parameters are \( a = b = 2/3, c = -1/3 \). For this particular solution the metric possesses the asymptotical boost symmetry in the plane \((t, \varphi)\) since \( g_{tt} \) becomes proportional to \( g_{\varphi\varphi} \) as \( r \to \infty \).

With \( B \) growing further, the Kasner parameters satisfy \( a > b \). But still \( g_{tt} \) and \( g_{\varphi\varphi} \) grow at infinity while \( g_{zz} \) decreases. For the Melvin universe one has growing \( g_{zz} = g_{rr} \) and decreasing \( g_{\varphi\varphi} \). When the bifurcation curve approaches the \( p \)-axis, which corresponds to the Abelian case with \( R = 0 \), the metric is of the Melvin type. It is interesting to observe how these different types of metric transform into each other. It turns out that for \( q \to 0 \) but finite \( p \) the functions \( R, R' \) persist while \( P, P' \) are not. This is why only the \( z \)-string (associated with the function \( P \)) persists in the vicinity of the symmetry axis. It generates the Melvin metric with growing \( r \), while the energy density decreases as a power-law. In this region the stress-energy tensor exhibits the Melvin symmetry:

\[
T^0_0 = -T^r_r = -T^\varphi_\varphi = T^z_z. \tag{157}
\]

But for larger \( r \), the functions \( R \) and \( R' \) become non-negligible, and the energy density of the \( \varphi \)-string increases and reaches its maximum Fig. 9. In this region the energy density of the \( z \)-string decreases exponentially. So at the small distances from the axis the \( z \)-string dominates and generates the Melvin metric, while at larger distances the \( \varphi \)-string dominates, transforming the metric asymptotically into the form (124–126). In this region the components of the stress-energy tensor satisfy the relations

\[
T^0_0 = -T^r_r = T^\varphi_\varphi = -T^z_z. \tag{158}
\]

As \( p \to p_{\text{max}} \), the boundary between these two regimes moves to infinity. Since strings cores are largely separated, one can introduce the effective radius \( r_0 \) such that the mass of the \( z \)-string \( M_z \) is concentrated in the region \((0, r_0)\), while the mass of the \( \varphi \)-string \( M_\varphi \) corresponds to the region \((r_0, \infty)\). We define then

\[
M_z = M(r_0), \quad M_\varphi = M_\infty - M(r_0), \tag{159}
\]

and similarly for the quantities \( W, X, \Sigma \). The solution within the region \((0, r_0)\) can be considered as Melvin, so it is known that \( M_z = 2 \), and due to the symmetry (157) one has \( M_z = -W_\varphi = X_\varphi = 2, \Sigma_\varphi = 3 \). The integration of the Eqs. (95–97) gives:

\[
N^N\text{LK} \rightarrow M_\varphi + M_z = M_\varphi + 2, \tag{160}
\]

\[
N^N\text{LK}' \rightarrow W_\varphi + W_z + 1 = W_\varphi - 1, \tag{161}
\]

\[
N^N\text{LK}' \rightarrow X_\varphi + X_z = X_\varphi + 2. \tag{162}
\]

Using the asymptotic symmetry (158), one finds \( M_\varphi = W_\varphi = -X_\varphi \) and the Kasner parameters can be expressed in terms of the \( \varphi \)-string mass \( M_\varphi \) only:

\[
a = \frac{M_\varphi + 2}{M_\varphi + 3}, \quad b = \frac{M_\varphi - 1}{M_\varphi + 3}, \quad c = \frac{2 - M_\varphi}{M_\varphi + 3}. \tag{163}
\]

For them the first Kasner condition is satisfied automatically, and the second gives the following equation for \( M_\varphi \):

\[
M_\varphi(M_\varphi - 4) = 0. \tag{164}
\]

So we have two possibilities: either \( M_\varphi = 0 \) implying the pure Melvin metrics, or \( M_\varphi = 4 \) (in our dimensionless units) which gives the limiting solution \( a = 6/7, b = 3/7, c = -2/7 \). For small values of the parameter \( q \to 0 \) when the above picture of well separated strings is valid, the mass of the \( \varphi \)-string approaches the value 4. Recall that the pure Melvin case, the mass of the \( z \)-string approaches the value 2 as \( p \to 0 \).

The numerical results for \( k = 1 \) are presented on Fig. 4 where the dependence of Kasner parameters on the field strength \( p \) is shown. A typical numerical solution is shown on Figs. 5,6,7. The numerical solution for flat space of one dimension (2+1) has shown that the corresponding asymptotic value is \( -k \). This means that

\[
B_{\text{max}} = -p_{\text{max}} = 2/k. \tag{165}
\]

Numerical computations confirm exactly this result, as illustrated on Fig. 9. Furthermore, in the Abelian case the bifurcation condition (165) gives again (166). In this limit the following inequality holds \( \Sigma_\infty \geq 3 \) (the strict equality holds for the Melvin solution) and thus \( \beta \geq -1 \). Therefore the parameter \( \beta \) in Eq. (124–126) takes values within the interval \([-1, 1]\), which ensures positivity of the expression under the square root. But the signs before the square root are different in the Abelian and non-Abelian cases.

The space generated by the \( \varphi \)-string can be visualized as the surface of a cup. Since \( 0 < b < 1 \), the circumference of the circle bounding the disk of the radius \( r \) centered on the axis and perpendicular to it is growing up slower than \( r \). If we embed the surface \((r, \varphi)\) carrying the corresponding metric into the flat space of one dimension more, it will look like a cup whose shape looks asymptotically as a figure of rotation of the curve \( z_{\text{flat}} = r_{\text{flat}}^{1/b} \).
Solutions corresponding to any set of parameters \((q, p, k)\) not belonging to the bifurcation curve are singular. Their behavior is similar to that in the flat case, but now there is a singularity of the metric, not of the gauge field. At the singularity the determinant of the metric vanishes, \(g(r_s) = 0\), while the Kretschmann scalar \((70)\) diverges as \((r_s - r)^{-4}\). The corresponding expansions of the metric and the field functions in terms of the deviation \(x = r_s - r\) can be found analytically. It turns out that the expansion of field functions contains only the integer exponents while that of the metric functions — only half-integer exponents. In the leading order in \(x\)

\[
P, R \rightarrow \text{const}, N \rightarrow x^{-1/2}, L, K \rightarrow x^{1/2}.
\]

An example of such an expansion for \(k = 1\) is given in Tab. III.

At the end of this section we would like to mention that the obtained results for the bifurcation conditions \([12, 13, 18, 19]\), as well as the asymptotic expansions \([14, 15, 19, 20, 21]\), do not depend on the assumed discrete nature of \(k\). The asymptotic behavior of the system will be the same for non-integer values of \(k\).

VI. CONCLUSION

In this paper, we have continued investigation of static cylindrically symmetric purely magnetic SU(2) EYM configurations started in \([12]\). We extended analysis of bifurcation manifolds, gave a (non-rigorous) proof of existence, considered in more detail structure of solution space near the high-curvature limit and described in detail singular solutions. Our set of the integral sum rules allows one to relate the asymptotic parameters of the solution to the boundary data on the polar axis and to simplify the numerical integration. The values of the Kasner exponents can be found analytically in some cases. Otherwise they are obtained numerically starting with the data on the symmetry axis parameterized by the value of the magnetic field.

The geometric structure and the matter distribution for soliton solutions obtained suggest an interpretation of the solution as describing a pair of straight and circular magnetic strings. Asymptotically the metric exhibits the boost symmetry in the plane \((t, \varphi)\) similar to the \((t, z)\) boost symmetry of the usual Melvin solution.

Some generalizations can be suggested, in particular, similar structure should exist in the bosonic sector of the Freedman–Shwartz SU(2)×SU(2) gauged supergravity. An intriguing question is whether the supersymmetric solutions may exist. The corresponding Abelian U(1)×U(1) configurations were considered in \([27]\) showing the absence of supersymmetry for the Melvin-type configuration. But as was shown in \([28, 29]\), spherically symmetric static purely magnetic configurations admits supersymmetry of gauged supergravity only in the case of truncation to SU(2)×U(1). So supersymmetry for cylindrically symmetric gauge field configurations is still not excluded in the non-Abelian sector.

As other directions we could mention an inclusion of the electric component of the YM field, as well as dressing the solutions with additional structure such as propagating gravitational waves \([30]\) and black holes \([31, 32]\), similar to that found for the Melvin background.

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TABLE I: Expansions at the origin in flat space

| k | $R$ | $P$ |
|---|---|---|
| 1 | $qr + \frac{3}{4} pq r^3 + \left( \frac{15}{8} p^2 q + \frac{5}{2} q^3 \right) \frac{r^5}{5!} + \left( \frac{315}{64} p^3 q + \frac{385}{16} pq \right) \frac{r^7}{7!} + O(r^9)$ | $1 + \frac{p^2}{2!} + \frac{3q^2 p^2 q - 3q^2}{4!} r^3 + O(r^5)$ |
| 2 | $q^2 r^2 + 2pq \frac{r^4}{4!} + \frac{105}{16} p^2 q^6 \frac{r^6}{6!} + \left( \frac{105}{4} p^3 q + 28q^4 \right) \frac{r^8}{8!} + \left( \frac{17325}{128} p^4 q + \frac{9435}{8} pq \right) \frac{r^{10}}{10!} + O(r^{12})$ | $1 + \frac{p^2}{2!} + \frac{15q^2 p^2 q - 15q^2}{8!} r^6 + \frac{273}{16} p^2 q^7 \frac{r^7}{7!} + \frac{1449}{16} p^3 q^2 \frac{r^9}{9!} + \left( \frac{72765}{128} p^4 q + \frac{2475}{4} pq \right) \frac{r^{11}}{11!} + O(r^{13})$ |
| 3 | $q^3 \frac{r^3}{3!} + \frac{15}{4} pq \frac{r^5}{5!} + \frac{273}{16} p^2 q^7 \frac{r^7}{7!} + \frac{1449}{16} p^3 q^2 \frac{r^9}{9!} + \left( \frac{72765}{128} p^4 q + \frac{2475}{4} pq \right) \frac{r^{11}}{11!} + O(r^{13})$ | $1 + \frac{p^2}{2!} + \frac{15q^2 p^2 q - 15q^2}{8!} r^6 + \frac{273}{16} p^2 q^7 \frac{r^7}{7!} + \frac{1449}{16} p^3 q^2 \frac{r^9}{9!} + \left( \frac{72765}{128} p^4 q + \frac{2475}{4} pq \right) \frac{r^{11}}{11!} + O(r^{13})$ |
| 4 | $q^4 \frac{r^4}{4!} + 6pq \frac{r^6}{6!} + \frac{147}{4} p^2 q^8 \frac{r^8}{8!} + \frac{495}{2} p^3 q^2 \frac{r^{10}}{10!} + \frac{479655}{256} p^4 q^3 \frac{r^{12}}{12!} + O(r^{14})$ | $1 + \frac{p^2}{2!} + \frac{3q^2 p^2 q - 3q^2}{4!} r^4 + O(r^6)$ |

TABLE II: Expansion at the origin in curved space ($k = 1$)

| R | $qr + \left( \frac{3}{4} pq + p^2 q - \frac{3}{2} q^2 \right) \frac{r^3}{3!} + O(r^5)$ |
|---|---|
| P | $1 + \frac{p^2}{2!} + \left( 3q^2 - 5p^3 \right) \frac{r^4}{4!} + O(r^6)$ |
| N | $1 + \frac{1}{2} p^2 + q^2 \frac{r^2}{2!} + \left( 4p^2 q^2 - p^4 + \frac{9}{2} pq^2 + 3q^4 \right) \frac{r^4}{4!} + O(r^6)$ |
| L | $r - 2p^2 \frac{r^3}{3!} + \left( \frac{35}{2} p^4 - 12pq^2 - 6q^4 \right) \frac{r^5}{5!} + O(r^7)$ |
| K | $1 + \frac{1}{2} p^2 + q^2 \frac{r^2}{2!} + \left( 3q^3 - 4p^2 q^2 - p^4 + \frac{9}{2} pq^2 \right) \frac{r^4}{4!} + O(r^6)$ |

TABLE III: Expansion at the singular point in curved space ($k = 1$)

| R | $R_0 - \frac{K^2}{2R_0} x + \frac{5L_1 K_1^2}{4L_1 R_0^2} x^2 + \frac{2K_1^2 P_2 R_0}{L_1 R_0^2} x^3 + O(x^3)$ |
| P | $L_1 K_1 \frac{R_0}{R_0} - \frac{L_1 R_0}{K_1} x + \frac{P_2 x^2}{2!} + O(x^3)$ |
| N | $N_0 x^{-1/2} + \frac{N_0 (3L_1 K_1^2 + 2R_0^2 L_1 + K_1^2 P_2 R_0)}{4L_1 R_0^2 K_1^2} x^{1/2} + O(x^{3/2})$ |
| L | $L_1 x^{-1/2} + \frac{L_1 K_1 + 2R_0^2 L_1 - K_1^2 P_2 R_0}{4R_0^2 K_1^2} x^{3/2} + O(x^{5/2})$ |
| K | $K_1 x^{1/2} + \frac{3L_1 K_1 - 14R_0^2 L_1 + K_1^2 P_2 R_0 x^{3/2}}{4R_0^2 L_1 K_1^2} + O(x^{5/2})$ |
FIG. 1: Bifurcation of solutions in flat space. Dashed lines — the two singular solutions corresponding to the parameter values $p = -0.9076$ and $p = -0.9078$ respectively. Solid lines correspond to the regular solution with the bifurcational value $p = -0.90777863505\ldots$

FIG. 2: Bifurcation curves for $k = 1, 2, 3, 4$ in the flat space.

FIG. 3: Bifurcation curves for $k = 1, 2, 3$ in the self-gravitating case (solid lines) and in the flat space (dashed lines).
FIG. 4: Kasner asymptotic parameters corresponding to the solutions with $k = 1$.

FIG. 5: The metric and field functions for the self-gravitating solutions. Dashed lines — the singular solution ($q = 0.39$, $p = -0.5$), solid lines — the regular solution ($q = 0.3862649505\ldots$, $p = -0.5$).
FIG. 6: The square root of the metric tensor $\sqrt{-g}$ and the energy density $\sigma = T_0^0 \sqrt{-g}$ for self-gravitating solutions. Dashed lines — the singular solution ($q = 0.39, p = -0.5$), solid lines — the regular solution at bifurcation ($q = 0.3862649505 \ldots, p = -0.5$).

FIG. 7: The components of the stress-energy tensor $\sigma = T_0^0 \sqrt{-g}$, $\tau_i = T_i^i \sqrt{-g}$ for the self-gravitating configuration.
FIG. 8: The metric of the regular solution with $p = -1.85$. Solid lines correspond to the effective exponents of metric functions $\tilde{a}_i = (\ln N_i)^r$ where $i$ labels $a, b, c$ and $N, L, K$ respectively. When $\tilde{a}_i \approx \text{const}$, the metric functions $N_i \sim r^{\tilde{a}_i}$. Dashed line — the energy density $\sigma = T_0 \sqrt{-g}$.

FIG. 9: Continuation of the previous plot Fig. 8 with an enhanced (factor of 100) scale for $\sigma$. 
FIG. 10: The geometry of the section $z = \text{const.}$ of the solution carrying the metric $dl^2 = dr^2 + L^2(r) d\varphi^2$ as embedded into the three-dimensional flat space (the desired surface is the figure of rotation of this curve around the vertical axis). The coordinate along the horizontal axis is $r_{\text{flat}} = \sqrt{g_{\varphi\varphi}}$, the vertical coordinate is defined by $dz_{\text{flat}}^2 = dr^2 - dr_{\text{flat}}^2$. Three regions correspond to the nearly flat metric, and that of the Melvin straight and circular strings respectively.

FIG. 11: Plot of the metric function $K(r)$ showing an effective longitudinal thickness of the configuration along the $z$–axis as a function of the radial. Three regions are shown which correspond to the nearly flat metric, the straight string core and the circular string.