THE COVERING NUMBER OF THE DIFFERENCE SETS IN PARTITIONS OF G-SPACES AND GROUPS

TARAS BANAKH AND MIKOLAJ FRĄCZYK

ABSTRACT. We prove that for every finite partition \( G = A_1 \cup \cdots \cup A_n \) of a group \( G \) either \( \text{cov}(A_i A_i^{-1}) \leq n \) for all cells \( A_i \) or else \( \text{cov}(A_i A_i^{-1}, A_i) < n \) for some cell \( A_i \) of the partition. Here \( \text{cov}(A) = \min\{|F| : F \subset G, \ G = FA\} \) is the covering number of \( A \) in \( G \). A similar result is proved also of partitions of \( G \)-spaces. This gives two partial answers to a problem of Protasov posed in 1995.

This paper was motivated by the following problem posed by I.V. Protasov in Kourovka Notebook [3].

**Problem 1.** (Protasov, 1995). Is it true that for any partition \( G = A_1 \cup \cdots \cup A_n \) of a group \( G \) some cell \( A_i \) of the partition has \( \text{cov}(A_i A_i^{-1}) \leq n \)?

Here for a non-empty subset \( A \subset G \) by

\[
\text{cov}(A) = \min\{|F| : F \subset G, \ G = FA\}
\]

we denote the covering number of \( A \).

In fact, Protasov’s Problem can be posed in a more general context of ideal \( G \)-spaces. Let us recall that a \( G \)-space is a set \( X \) endowed with an action \( G \times X \to X, (g, x) \mapsto gx \), of a group \( G \). An ideal \( G \)-space is a pair \((X, \mathcal{I})\) consisting of a \( G \)-space \( X \) and a \( G \)-invariant Boolean ideal \( \mathcal{I} \subset \mathcal{B}(X) \) in the Boolean algebra \( \mathcal{B}(X) \) of all subsets of \( X \). A Boolean ideal on \( X \) is a proper subfamily \( \mathcal{I} \subset \mathcal{B}(X) \) such that for any \( A, B \in \mathcal{I} \) any subset \( C \subset A \cup B \) belongs to \( \mathcal{I} \). A Boolean ideal \( \mathcal{I} \) is \( G \)-invariant if \( \{gA : g \in G, \ A \in \mathcal{I}\} \subset \mathcal{I} \). A Boolean ideal \( \mathcal{I} \subset \mathcal{B}(G) \) on a group \( G \) will be called \( \mathcal{I} \)-invariant if \( \{xAy : x, y \in G, \ A \in \mathcal{I}\} \subset \mathcal{I} \). By \( [X]^{<\omega} \) and \( [X]^{\leq \omega} \) we denote the families of all finite and countable subsets of a set \( X \), respectively. The family \( [X]^{<\omega} \) (resp. \( [X]^{\leq \omega} \)) is a Boolean ideal on \( X \) if \( X \) is infinite (resp. uncountable).

For a subset \( A \subset X \) of an ideal \( G \)-space \( (X, \mathcal{I}) \) by

\[
\Delta(A) = \{g \in G : gA \cap A \neq \emptyset\} \quad \text{and} \quad \Delta_\mathcal{I}(A) = \{g \in G : gA \cap A \notin \mathcal{I}\}
\]

we denote the difference set and \( \mathcal{I} \)-difference set of \( A \), respectively.

Given a Boolean ideal \( \mathcal{J} \) on a group \( G \) and two subsets \( A, B \subset G \) we shall write \( A = \mathcal{J} B \) if the symmetric difference \( A \Delta B = (A \setminus B) \cup (B \setminus A) \) belongs to the ideal \( \mathcal{J} \). For a non-empty subset \( A \subset G \) put

\[
\text{cov}_\mathcal{J}(A) = \min\{|F| : F \subset G, \ FA = \mathcal{J} G\}
\]

be the \( \mathcal{J} \)-covering number of \( A \). For the empty subset we put \( \text{cov}_\mathcal{J}(\emptyset) = \infty \) and assume that \( \infty \) is larger than any cardinal number.

Observe that for the left action of the group \( G \) on itself we get \( \Delta(A) = AA^{-1} \) for every subset \( A \subset G \). That is why Problem 1 is a partial case of the following general problem.

**Problem 2.** Is it true that for any partition \( X = A_1 \cup \cdots \cup A_n \) of an ideal \( G \)-space \( (X, \mathcal{I}) \) some cell \( A_i \) of the partition has \( \text{cov}(\Delta_\mathcal{I}(A_i)) \leq n \)?

This problem has an affirmative answer for \( G \)-spaces with amenable acting group \( G \), see [3, 4.3]. The paper [3] gives a survey of available partial solutions of Protasov’s Problems 1 and 2. Here we mention the following result of Banakh, Ravsky and Slobodianuk [1].

**Theorem 1.** For any partition \( X = A_1 \cup \cdots \cup A_n \) of an ideal \( G \)-space \( (X, \mathcal{I}) \) some cell \( A_i \) of the partition has

\[
\text{cov}(\Delta_\mathcal{I}(A_i)) \leq \max_{0 \leq k \leq n} \sum_{p=0}^{n-k} k^p \leq n!
\]

In this paper we shall give another two partial solutions to Protasov’s Problems 1 and 2.

**Theorem 2.** For any partition \( X = A_1 \cup \cdots \cup A_n \) of an ideal \( G \)-space \( (X, \mathcal{I}) \) either

- \( \text{cov}(\Delta_\mathcal{I}(A_i)) \leq n \) for all cells \( A_i \), or else

1991 Mathematics Subject Classification. 05E15, 05E18, 28C10.

Key words and phrases. \( G \)-space, difference set, covering number, compact right topological semigroup, minimal measure, idempotent measure, quasi-invariant measure.
For any subset $A$ for every $\mu$ cells

Proof. By Theorem 2 either $\text{cov}(\Delta_2(A_i)) \leq n$ for all cells $A_i$ or else $\text{cov}(\Delta_2(A_i) \cdot \Delta_2(A_i)) < n$ for some cell $A_i$. The Zorn's Lemma combined with the compactness of the orbits implies that the orbit $I(A_i)$ contains a minimal measure. For any partition $G$ considered as $G$-spaces endowed with the left action of $G$ on itself, we can prove a bit more:

Theorem 3. Let $G$ be a group and $I$ be an invariant Boolean ideal on $G$ with $|G|^{|\omega|} \not\subset I$. For any partition $G = A_1 \cup \cdots \cup A_n$ of $G$ either

- $\text{cov}(\Delta_2(A_i)) \leq n$ for all cells $A_i$ or else
- $\text{cov}(\Delta_2(A_i)) < n$ for some cell $A_i$ and for some $G$-invariant Boolean ideal $J \not\subset A_i^{-1}$ on $G$.

Corollary 2. For any partition $G = A_1 \cup \cdots \cup A_n$ of a group $G$ either $\text{cov}(A_iA_i^{-1}) \leq n$ for all cells $A_i$ or else $\text{cov}(A_iA_i^{-1}A_i) < n$ for some cell $A_i$ of the partition.

Proof. On the group $G$ consider the trivial ideal $I = \{\emptyset\}$. By Theorem 3 either $\text{cov}(A_iA_i^{-1}) \leq n$ for all cells $A_i$ or else $\text{cov}(A_iA_i^{-1}) < n$ for some cell $A_i$ and some $G$-invariant ideal $J \not\subset A_i^{-1}$ on $G$. In the first case we are done. In the second case choose a finite subset $F \subset G$ of cardinality $|F| < n$ such that the set $FA_iA_i^{-1} \in J$. Since $A_i^{-1} \in J$, for every $x \in G$ the set $xA_i^{-1}$ intersects $FA_iA_i^{-1}$ and hence $x = FA_iA_i^{-1}$. Therefore $\text{cov}(A_iA_i^{-1}A_i) < |F| < n$.

Taking into account that the ideal $J$ appearing in Theorem 3 is $G$-invariant but not necessarily invariant, we can ask the following question.

Problem 3. Is it true that for any partition $G = A_1 \cup \cdots \cup A_n$ of a group $G$ some cell $A_i$ of the partition has $\text{cov}(A_iA_i^{-1}) \leq n$ for some invariant Boolean ideal $J$ on $G$?

1. Minimal measures on $G$-spaces

Theorems 2 and 3 will be proved with help of minimal probability measures on $X$ and right quasi-invariant idempotent measures on $X$.

For a $G$-space $X$ by $P(X)$ we denote the (compact Hausdorff) space of all finitely additive probability measures on $X$. The action of the group $G$ on $X$ extends to an action of the convolution semigroup $P(G)$ on $P(X)$: for two measures $\mu \in P(G)$ and $\nu \in P(X)$ their convolution is defined as the measure $\mu * \nu \in P(X)$ assigning to each bounded function $\varphi : X \to \mathbb{R}$ the real number

$$\mu * \nu(\varphi) = \int_X \varphi(x) \, d\nu(x) \, d\mu(g).$$

The convolution map $* : P(G) \times P(X) \to P(X)$ is right-continuous in the sense that for any fixed measure $\nu \in P(X)$ the right shift $P(G) \to P(X)$, $\mu \mapsto \mu * \nu$, is continuous. This implies that the $P(G)$-orbit $P(G) * \nu = \{\mu * \nu : \mu \in P(G)\}$ of $\nu$ coincides with the closure $\text{cov}(G * \nu)$ of the convex hull of the $G$-orbit $G * \nu$ of $\nu$ in $P(X)$.

A measure $\mu \in P(X)$ will be called minimal if for any measure $\nu \in P(G) * \mu$ we get $P(G) * \nu = P(G) * \mu$. The Zorn's Lemma combined with the compactness of the orbits implies that the orbit $P(G) * \mu$ of each measure $\mu \in P(X)$ contains a minimal measure.

It follows from Day's Fixed Point Theorem [8, 114] that for a $G$-space $X$ with amenable acting group $G$ each minimal measure $\mu$ on $X$ is $G$-invariant, which implies that the set $\text{cov}(G * \mu)$ coincides with the singleton $\{\mu\}$.

For an ideal $G$-space $(X, I)$ let $P_1(X) = \{\mu \in P(X) : \forall A \in I \mu(A) = 0\}$.

Lemma 1. For any ideal $G$-space $(X, I)$ the set $P_1(X)$ contains some minimal probability measure.

Proof. Let $U$ be any ultrafilter on $X$, which contains the filter $\mathcal{F} = \{F \subset X : X \setminus F \in I\}$. This ultrafilter $U$ can be identified with the 2-valued measure $\mu_U : B(X) \to \{0, 1\}$ such that $\mu_U(\emptyset) = 1$. It follows that $\mu_U(A) = 0$ for any subset $A \in I$. In the $P(G)$-orbit $P(G) * \mu_U$ choose any minimal measure $\mu = \nu * \mu_U$ and observe that for every $A \in I$ the $G$-invariance of the ideal $I$ implies $\mu(A) = \int_G \mu_U(x^{-1}A) \, d\nu(x) = 0$. So, $\mu \in P_1(X)$. For a subset $A$ of a group $G$ put

$$I_{12}(A) = \inf_{\mu \in P(G)} \sup_{y \in G} \mu(Asy).$$
Lemma 2. If a subset $A$ of a group $G$ has $ls_1(A) = 1$, then $\text{cov}(G \setminus A) \geq \omega$.

Proof. It suffices to show that $G \neq F(G \setminus A)$ for any finite set $F \subset G$. Consider the uniformly distributed measure $\mu = \frac{1}{|F|} \sum_{x \in F} \delta_{x^{-1}}$ on the set $F^{-1}$. Since $ls_1(A) = 1$, for the measure $\mu$ there is a point $y \in G$ such that $1 - \frac{1}{|F|} < \mu(Ay) = \frac{1}{|F|} \sum_{x \in F} \delta_{x^{-1}}(Ay)$, which implies that $\mu(Ay) = 1$ and $\text{supp}(\mu) = F^{-1} \subset Ay$. Then

$F^{-1}y^{-1} \cap (G \setminus A) = \emptyset$ and $y^{-1} \notin F(G \setminus A)$. \hfill $\Box$

Remark 1. By Theorem 3.8 of [2], for every subset $A$ of a group $G$ we get $ls_1(A) = 1 - is_2(G \setminus A)$ where

$$is_2(B) = \inf_{\mu \in P_\nu(G)} \sup_{x \in G} \mu(xB)$$

for $B \subset G$ and $P_\nu(G)$ denotes the set of finitely supported probability measures on $G$.

For a probability measure $\mu \in P(X)$ on a $G$-space $X$ and a subset $A \subset X$ put

$$\tilde{\mu}(A) = \sup_{x \in G} \mu(xA).$$

2. A density version of Theorem 2

In this section we shall prove the following density theorem, which will be used in the proof of Theorem 2 presented in the next section.

Theorem 4. Let $(X, \mathcal{I})$ be an ideal $G$-space and $\mu \in P_\nu(X)$ be a minimal measure on $X$. If some subset $A \subset X$ has $\tilde{\mu}(A) > 0$, then the $\mathcal{I}$-difference set $\Delta_\mathcal{I}(A)$ has $\mathcal{J}$-covering number $\text{cov}_\mathcal{J}(\Delta_\mathcal{I}(A)) \leq 1/\tilde{\mu}(A)$ for some $G$-invariant ideal $\mathcal{J} \not\supset \Delta_\mathcal{I}(A)$ on $G$.

Proof. By the compactness of $P(G) * \mu = \text{conv}(G \cdot \mu)$, there is a measure $\mu' \in P(G) * \mu \subset P_\nu(X)$ such that

$$\mu'(A) = \sup_{\nu \in P(G) * \mu : \nu(A)} = \tilde{\mu}(A).$$

We can replace the measure $\mu$ by $\mu'$ and assume that $\mu(A) = \tilde{\mu}(A)$. Choose a positive $\epsilon$ such that $|\frac{1}{\mu(A)} - 1| = |\frac{1}{\mu(A)} - 1|$, where $|r| = \max\{n \in \mathbb{Z} : n \leq r\}$ denotes the integer part of a real number $r$.

Consider the set $L = \{x \in G : \mu(xA) > \tilde{\mu}(A) - \epsilon\}$ and choose a maximal subset $F \subset L$ such that $\mu(xA \cap yA) = 0$ for any distinct points $x, y \in L$. The additivity of the measure $\mu$ implies that $1 \geq \sum_{x \in F} \mu(xA) > |F|/(\tilde{\mu}(A) - \epsilon)$ and hence $|F| \leq |\frac{1}{\mu(A)}| = |\frac{1}{\mu(A)}|$. By the maximality of $F$, for every $x \in L$ there is a $y \in L$ such that $\mu(xA \cap yA) > \tilde{\mu}(A) - \epsilon$. Then $xA \cap yA \notin \mathcal{I}$ and $y^{-1}x \in \Delta_\mathcal{I}(A)$. It follows that $x \in y \cdot \Delta_\mathcal{I}(A) \subset F \cdot \Delta_\mathcal{I}(A)$ and $L \subset F \cdot \Delta_\mathcal{I}(A)$.

We claim that $ls_1(L) = 1$. Given any measure $\nu \in P(G)$, consider the measure $\nu^{-1} \in P(G)$ defined by $\nu^{-1}(B) = \nu(B^{-1})$ for every subset $B \subset G$. By the minimality of $\mu$, we can find a measure $\eta \in P(G)$ such that $\eta \ast \nu^{-1} \ast \mu = \mu$. Then

$$\tilde{\mu}(A) = \mu(A) = \eta \ast \nu^{-1} \ast \mu(A) = \int_G \mu(x^{-1}A) d\eta \ast \nu^{-1}(x) \leq$$

$$\leq (\tilde{\mu}(A) - \epsilon) \cdot \eta \ast \nu^{-1}(\{x \in G : \mu(x^{-1}A) \leq \tilde{\mu}(A) - \epsilon\}) + \tilde{\mu}(A) \cdot \eta \ast \nu^{-1}(\{x \in G : \mu(x^{-1}A) > \tilde{\mu}(A) - \epsilon\}) \leq$$

$$\leq \eta \ast \nu^{-1}(1 - \eta \ast \nu^{-1}(L^{-1})) + \tilde{\mu}(A) \cdot \eta \ast \nu^{-1}(L^{-1}) \leq \tilde{\mu}(A)$$

implies that $\eta \ast \nu^{-1}(L^{-1}) = 1$. It follows from

$$1 = \eta \ast \nu^{-1}(L^{-1}) = \int_G \nu^{-1}(y^{-1}L^{-1}) d\eta(y)$$

that for every $\delta > 0$ there is a point $y \in G$ such that $\nu(Ly) = \nu(y^{-1}L^{-1}) > 1 - \delta$. So, $ls_1(L) = 1$.

By Lemma 2 the family $\mathcal{J} = \{B \subset G : \exists E \in [G]^{<\omega} B \subset E(G \setminus L)\}$ is a $G$-invariant ideal on $G$, which does not contain the set $L \subset F \cdot \Delta_\mathcal{I}(A)$ and hence does not contain the set $\Delta_\mathcal{I}(A_i)$. It follows that $\text{cov}_\mathcal{J}(\Delta_\mathcal{I}(A_i)) \leq |F| \leq 1/\tilde{\mu}(\mathcal{A}_i)$. \hfill $\Box$

3. Proof of Theorem 2

Let $X = A_1 \cup \cdots \cup A_n$ be a partition of an ideal $G$-space $(X, \mathcal{I})$. By Lemma 1 there exists a minimal probability measure $\mu \in P(X)$ such that $\mathcal{I} \subset \{A \in B(G) : \mu(A) = 0\}$.

For every $i \in \{1, \ldots, n\}$ consider the number $\tilde{\mu}(A_i) = \sup_{x \in G} \mu(xA_i)$ and observe that $\sum_{i=1}^n \tilde{\mu}(A_i) \geq 1$. There are two cases.

1) For every $i \in \{1, \ldots, n\}$ $\tilde{\mu}(A_i) \leq \frac{1}{n}$. In this case for every $x \in G$ we get

$$1 = \sum_{i=1}^n \mu(xA_i) \leq \sum_{i=1}^n \tilde{\mu}(A_i) \leq n \cdot \frac{1}{n} = 1$$

and hence $\mu(xA_i) = \frac{1}{n}$ for every $i \in \{1, \ldots, n\}$. For every $i \in \{1, \ldots, n\}$ fix a maximal subset $F_i \subset G$ such that $\mu(xA_i \cap yA_i) = 0$ for any distinct points $x, y \in F_i$. The additivity of the measure $\mu$ implies that
1 \geq \sum_{x \in F_i} \mu(xA_i) \geq |F_i|/n$ and hence $|F_i| \leq n$. By the maximality of $F_i$, for every $x \in G$ there is a point $y \in F_i$ such that $\mu(xA_i \cap yA_i) > 0$ and hence $xA_i \cap yA_i \notin \mathcal{I}$. The $G$-invariance of the ideal $\mathcal{I}$ implies that $y^{-1}x \in \Delta_{\mathcal{I}}(A_i)$ and so $x \in y^{-1}\Delta_{\mathcal{I}}(A_i) \subset F_i \cdot \Delta_{\mathcal{I}}(A_i)$. Finally, we get $G = F_i \cdot \Delta_{\mathcal{I}}(A_i)$ and $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq |F_i| \leq n$.

2) For some $i$ we get $\bar{\mu}(A_i) > 1/n$. In this case Theorem guarantees that $\text{cov}(\Delta_{\mathcal{I}}(A_i)) \leq 1/\bar{\mu}(A_i) < n$ for some $G$-invariant ideal $\mathcal{J} \neq \Delta_{\mathcal{I}}(A_i)$ on $G$.

4. Applying idempotent quasi-invariant measures

In this section we develop a technique involving idempotent right quasi-invariant measures, which will be used in the proof of Theorem presented in the next section.

A measure $\mu \in P(G)$ on a group $G$ will be called right quasi-invariant if for any $y \in G$ there is a positive constant $c > 0$ such that $c \cdot \mu(Ay) \leq \mu(A)$ for any subset $A \subset G$.

For an ideal $G$-space $(X, \mathcal{I})$ and a measure $\mu \in P(X)$ consider the set

$$P_\mu(G; \mu) = \{\lambda \in P(G) : \forall y \in G \; \lambda \ast \delta_y \ast \mu \in P_\mu(X)\}$$

and observe that it is closed and convex in the compact Hausdorff space $P(G)$.

**Lemma 3.** Let $(X, \mathcal{I})$ be an ideal $G$-space with countable acting group $G$. If for some measure $\mu \in P(X)$ the set $P_\mu(G; \mu)$ is not empty, then it contains a right quasi-invariant idempotent measure $\nu \in P_\mu(G; \mu)$.

**Proof.** Choose any strictly positive function $c : G \to [0, 1]$ such that $\sum_{g \in G} c(g) = 1$ and consider the $\sigma$-additive probability measure $\lambda = \sum_{g \in G} c(g) \delta_{g^{-1}} \in P(G)$. On the compact Hausdorff space $P(G)$ consider the right shift $\Phi : P(G) \to P(G)$, $\Phi(\nu) = \nu \ast \lambda$.

We claim that $\Phi(P_\mu(G; \mu)) \subset P_\mu(G; \mu)$. Given any measure $\nu \in P_\mu(G; \mu)$ we need to check that $\Phi(\nu) = \nu \ast \lambda \in P_\mu(G; \mu)$, which means that $\nu \ast \lambda \ast \delta_x \ast \mu \in P_\mu(X)$ for all $x \in G$. It follows from $\nu \in P_\mu(G; \mu)$ that $\nu \ast \delta_{g^{-1}} \ast \mu \in P_\mu(X)$. Since the set $P_\mu(X)$ is closed and convex in $P(X)$, we get

$$\nu \ast \lambda \ast \delta_x \ast \mu = \sum_{g \in G} c(g) \cdot \nu \ast \delta_{g^{-1}} \ast \delta_x \ast \mu = \sum_{g \in G} \nu \ast g \delta_{g^{-1}} \ast \mu \in P_\mu(X).$$

So, $\Phi(P_\mu(G; \mu)) \subset P_\mu(G; \mu)$ and by Schauder Fixed Point Theorem, the continuous map $\Phi$ on the non-empty compact convex set $P_\mu(G; \mu) \subset P(G)$ has a fixed point, which implies that the closed set $S = \{\nu \in P_\mu(G; \mu) : \nu \ast \lambda = \nu\}$ is not empty. It is easy to check that $S$ is a semigroup of the convolution semigroup $(P(G), \ast)$. Being a compact right-topological semigroup, $S$ contains an idempotent $\nu \in S \subset P_\mu(G; \mu)$ according to Ellis Theorem. Since $\nu \ast \lambda = \nu$, for every $A \subset G$ and $x \in G$ we get

$$\nu(A) = \nu \ast \lambda(A) = \sum_{g \in G} c(g) \cdot \nu \ast \delta_{g^{-1}}(A) = \sum_{g \in G} c(g) \cdot \nu(Ag) \geq c(x) \cdot \nu(Ax),$$

which means that $\nu$ is right quasi-invariant.

**Remark 2.** Lemma does not hold for uncountable groups, in particular for the free group $F_\alpha$ with uncountable set $\alpha$ of generators. This group admits no right quasi-invariant measure. Assuming conversely that some measure $\mu \in P(F_\alpha)$ is right quasi-invariant, fix a generator $a \in \alpha$ and consider the set $A$ of all reduced words $w \in F_\alpha$ that end with $a^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. Observe that $F_\alpha = Aa \cup A$ and hence $\mu(A) > 0$ or $\mu(Aa) > 0$. Since $\mu$ is right quasi-invariant both cases imply that $\mu(A) > 0$ and then $\mu(\mathbb{F}b) > 0$ for any generator $b \in \alpha \setminus \{a\}$. But this is impossible since the family $(A\mathbb{F})_{b \in \alpha \setminus \{a\}}$ is disjoint and uncountable.

In the following lemma for a measure $\mu \in P(X)$ we put $\bar{\mu}(A) = \sup_{x \in G} \mu(xA)$.

**Lemma 4.** Let $(X, \mathcal{I})$ be an ideal $G$-space and $\mu \in P(X)$ be a measure on $X$ such that the set $P_\mu(G; \mu)$ contains an idempotent right quasi-invariant measure $\lambda$. For a subset $A \subset X$ and numbers $\delta \leq \varepsilon < \sup_{x \in G} \lambda \ast \mu(xA)$ consider the sets $M_\delta = \{x \in G : \mu(xA) > \delta\}$ and $L_\varepsilon = \{x \in G : \lambda \ast \mu(xA) > \varepsilon\}$. Then:

1. $\lambda(gM_\delta^{-1}) > (\varepsilon - \delta)/(\bar{\mu}(A) - \delta)$ for any point $g \in L_\varepsilon$;
2. the set $M_\delta$ does not belong to the $G$-invariant Boolean ideal $\mathcal{J}_\delta \subset P(G)$ generated by $G \setminus L_\delta$;
3. $\text{cov}(\delta)(\Delta_{\mathcal{I}}(A)) < 1/\delta$.

**Proof.** Consider the measure $\nu = \lambda \ast \mu$ and put $\bar{\nu}(A) = \sup_{x \in G} \nu(xA)$ for a subset $A \subset X$.

1. Fix a point $g \in L_\varepsilon$ and observe that

$$\varepsilon < \lambda \ast \mu(gA) = \int_{G} \mu(x^{-1}gA)d\lambda(x) \leq \lambda(\{x \in G : \mu(x^{-1}gA) \leq \delta\}) + \bar{\mu}(A) \cdot \lambda(\{x \in G : \mu(x^{-1}gA) > \delta\}) = \delta \cdot (1 - \lambda(gM_\delta^{-1})) + \bar{\mu}(A)\lambda(gM_\delta^{-1}) = \delta + (\bar{\mu}(A) - \delta)\lambda(gM_\delta^{-1})$$
which implies \( \lambda(gM_\delta^{-1}) > \gamma := \frac{-\varepsilon - \delta}{\mu(A) - \delta} \).

2. To derive a contradiction, assume that the set \( M_\delta \) belongs to the \( G \)-invariant ideal generated by \( G \setminus L_\delta \) and hence \( M_\delta \in E(G \setminus L_\delta) \) for some finite subset \( E \subset G \). Then

\[
M_\delta \subset E(G \setminus L_\delta) = G \setminus \bigcap_{\varepsilon \in E} eL_\delta.
\]

Choose an increasing sequence number sequence \( (\varepsilon_k)_{k=0}^\infty \) such that \( \delta < \varepsilon \leq \varepsilon_0 \) and \( \lim_{k \to \infty} \varepsilon_k = \nu(A) \). For every \( k \in \omega \) fix a point \( g_k \in L_{\varepsilon_k} \). The preceding item applied to the measure \( \nu \) and set \( L_\delta \) (instead of \( \mu \) and \( M_\delta \)) yields the lower bound

\[
\lambda(g_kL_\delta^{-1}) > \frac{\varepsilon_k - \delta}{\nu(A) - \delta}
\]
for every \( k \in \omega \). Then \( \lim_{k \to \infty} \lambda(g_kL_\delta^{-1}) = 1 \) and hence \( \lim_{k \to \infty} \lambda(z_kL_\delta^{-1}g) = 1 \) for every \( g \in G \) by the right quasi-invariance of the measure \( \lambda \). Choose \( k \) so large that \( \lambda(z_kL_\delta^{-1}g)^{-1} > 1 - \frac{1}{|G|} \gamma \) for all \( g \in E \). Then the set \( \bigcap_{g \in E} z_kL_\delta^{-1}g^{-1} \) has measure \( > 1 - \gamma \) and hence it intersects the set \( z_kM_a^{-1} \) which has measure \( \lambda(z_kM_a) \geq \gamma \). Consequently, the set \( M_a^{-1} \) intersects \( \bigcap_{g \in E} gL_\delta^{-1} \), and the set \( M_a \) intersects \( \bigcap_{g \in E} gL = G \setminus (E(G \setminus L_\delta)) \), which contradicts the choice of the set \( E \).

**Corollary 3.** Let \((X, \mathcal{I})\) be an ideal \( G \)-space with countable acting group \( G \) and \( \mu \in P(X) \) be a measure on \( X \) such that the set \( P_2(G; \mu) \) is not empty. For any partition \( X = A_1 \cup \cdots \cup A_n \) of \( X \) either:

1. \( \text{cov}(\Delta(A_i)) \leq n \) for all cells \( A_i \) or else
2. \( \text{cov}(\Delta(A_i)) < n \) for some cell \( A_i \) and some \( G \)-invariant Boolean ideal \( \mathcal{J} \subset \mathcal{P}(G) \) such that \( \{ x \in G : \mu(xA_i) > \frac{1}{n} \} \notin \mathcal{J} \).

**Proof.** By Lemma 8 the set \( P_2(G; \mu) \) contains an idempotent right quasi-invariant measure \( \lambda \). Then for the measure \( \nu = \lambda \ast \mu \in P_2(X) \) two cases are possible:

- Every cell \( A_i \) of the partition has \( \nu(A_i) = \sup_{x \in G} \nu(xA_i) \leq \frac{1}{n} \). In this case we can proceed as in the proof of Theorem 2 and prove that \( \text{cov}(\Delta(A_i)) \leq n \) for all cells \( A_i \) of the partition.
- Some cell \( A_i \) of the partition has \( \nu(A_i) > \frac{1}{n} \). In this case Lemma 2 guarantees that \( \text{cov}(\Delta(A_i)) < n \) for the \( G \)-invariant Boolean ideal \( \mathcal{J} \subset \mathcal{P}(G) \) generated by the set \( \{ x \in G : \nu(xA_i) \leq \frac{1}{n} \} \), and the set \( M = \{ x \in G : \mu(xA_i) > \frac{1}{n} \} \) does not belong to the ideal \( \mathcal{J} \).

Next, we extend Corollary 3 to \( G \)-spaces with arbitrary (not necessarily countable) acting group \( G \). Given a \( G \)-space \( X \) denote by \( \mathcal{H} \) the family of all countable subgroups of the acting group \( G \). A subfamily \( \mathcal{F} \subset \mathcal{H} \) will be called

- **closed** if for each increasing sequence of countable subgroups \( \{ H_n \}_{n \in \omega} \subset \mathcal{F} \) the union \( \bigcup_{n \in \omega} H_n \) belongs to \( \mathcal{F} \);
- **dominating** if every countable subgroup \( H \in \mathcal{H} \) is contained in some subgroup \( H' \in \mathcal{F} \);
- **stationary** if \( \mathcal{F} \cap \mathcal{C} \neq \emptyset \) for every closed dominating subset \( \mathcal{C} \subset \mathcal{H} \).

It is known (see [4], 4.3) that the intersection \( \bigcap_{n \in \omega} C_n \) of any countable family of closed dominating sets \( C_n \subset \mathcal{H}, n \in \omega \), is closed and dominating in \( \mathcal{H} \).

For a measure \( \mu \in P(X) \) and a subgroup \( H \in \mathcal{H} \) let

\[
P_2(H; \mu) = \{ \lambda \in P(H) : \forall x \in H \quad \lambda \ast \delta_x \ast \mu \notin P_2(X) \}.
\]

**Theorem 5.** Let \((X, \mathcal{I})\) be an ideal \( G \)-space and \( \mu \in P(X) \) be a measure on \( X \) such that the set \( \mathcal{H}_X = \{ H \in \mathcal{H} : P_2(H; \mu) \neq \emptyset \} \) is stationary in \( \mathcal{H} \). For any partition \( X = A_1 \cup \cdots \cup A_n \) of \( X \) either:

1. \( \text{cov}(\Delta(A_i)) \leq n \) for all cells \( A_i \) or else
2. \( \text{cov}(\Delta(A_i)) < n \) for some cell \( A_i \) and some \( G \)-invariant Boolean ideal \( \mathcal{J} \subset \mathcal{P}(G) \) such that \( \{ x \in G : \mu(xA_i) > \frac{1}{n} \} \notin \mathcal{J} \).

**Proof.** Let \( \mathcal{H}_\nu = \{ H \in \mathcal{H}_X : \forall i \leq n \quad \text{cov}(H \cap \Delta(A_i)) \leq n \} \) and \( \mathcal{H}_\delta = \mathcal{H}_X \setminus \mathcal{H}_\nu \). It follows that for every \( H \in \mathcal{H}_\nu \) and \( i \in \{1, \ldots, n\} \) we can find a subset \( f_i(H) \subset H \) of cardinality \( |f_i(H)| \leq n \) such that \( H \subset f_i(H) \cdot \Delta(A_i) \). The assignment \( f_i : H \mapsto f_i(H) \) determines a function \( f_i : \mathcal{H}_\nu \to [G]^{<\omega} \) to the family of all finite subsets of \( G \).

The function \( f_i \) is regressive in the sense that \( f_i(H) \subset H \) for every subgroup \( H \in \mathcal{H}_\nu \).
By Corollary 3 for every subgroup $H \in \mathcal{H}_3$, there are an index $i_H \in \{1, \ldots, n\}$ and a finite subset $f(H) \subset H$ of cardinality $|f(H)| < n$ such that the set $J_H = H \setminus \left( f(H) \cdot (H \cap \Delta_\mathcal{I}(A_{i_H})) \right)$ generates the $H$-invariant ideal $\mathcal{J}_H \subset \mathcal{P}(H)$ which does not contain the set $M_H = \{ x \in H : \mu(x_{A_{i_H}}) > \frac{1}{n} \}$.

Since $\mathcal{H}_3 = \mathcal{H}_1 \cup \mathcal{H}_3$ is stationary in $\mathcal{H}$, one of the sets $\mathcal{H}_1$ or $\mathcal{H}_3$ is stationary in $\mathcal{H}$.

If the set $\mathcal{H}_1$ is stationary in $\mathcal{H}$, then by Jech’s generalization [3, 8, 4.4] of Fodor’s Lemma, the stationary set $\mathcal{H}_1$ contains another stationary subset $S \subset \mathcal{H}_1$ such that for every $i \in \{1, \ldots, n\}$ the restriction $f_i|S$ is a constant function and hence $f_i(S) = \{ F_i \}$ for some finite set $F_i \subset G$ of cardinality $|F_i| \leq n$. We claim that $G = F_i \cdot \Delta_\mathcal{I}(A_i)$. Indeed, given any element $g \in G$, by the stationarity of $S$ there is a subgroup $H \subset S$ such that $g \in H$. Then $g \in H \subset f_i(H) \cdot \Delta_\mathcal{I}(A_i) = F_i \cdot \Delta_\mathcal{I}(A_i)$ and hence $\text{cov}(\Delta_\mathcal{I}(A_i)) \leq |F_i| \leq n$ for all $i$.

Now assume that the family $\mathcal{H}_3$ is stationary in $\mathcal{H}$. In this case for some $i \in \{1, \ldots, n\}$ the set $\mathcal{H}_i = \{ H \in \mathcal{H}_3 : i_H = i \}$ is stationary in $\mathcal{H}_3$. Since the function $f : \mathcal{H}_3 \to [G]^{<\omega}$ is regressive, by Jech’s generalization [3, 8, 4.4] of Fodor’s Lemma, the stationary set $\mathcal{H}_i$ contains another stationary subset $S \subset \mathcal{H}_i$ such that the restriction $f_i|S$ is a constant function and hence $f_i(S) = \{ F_i \}$ for some finite set $F \subset G$ of cardinality $|F| < n$. We claim that the set $J = G \setminus (F \cdot \Delta_\mathcal{I}(A_i))$ generates a $G$-invariant ideal $\mathcal{J}$, which does not contain the set $M = \{ x \in G : \mu(x_{A_i}) > \frac{1}{n} \}$. Assume conversely that $M \in \mathcal{J}$ and hence $M \subset EJ$ for some finite subset $E \subset G$. By the stationarity of the set $S$, there is a subgroup $H \subset S$ such that $E \cap H \subset H$. It follows $H \cap J = H \setminus (F \cdot (H \cap \Delta_\mathcal{I}(A_i))) = H \setminus (f(H) \cdot (H \cap \Delta_\mathcal{I}(A_i))) = J_H$ and $M_H = \{ x \in H : \mu(x_{A_i}) > \frac{1}{n} \} = H \cap M \subset H \cap EJ = EJ_H \in \mathcal{J}_H$, which contradicts the choice of the ideal $\mathcal{J}_H$. □

5. PROOF OF THEOREM 3

Theorem 3 is a simple corollary of Theorem 5. Indeed, assume that $G = A_1 \cup \cdots \cup A_n$ is a partition of a group and $\mathcal{I} \subset \mathcal{P}(G)$ is an invariant ideal on $G$ which does not contain any countable subset and hence does not contain any countable subgroup $H_0 \subset G$. Let $\mathcal{H}$ be the family of all countable subgroups of $G$ and $\mu = \delta_1$ be the Dirac measure supported by the unit 1 of the group $G$. We claim that that for every subgroup $H \subset \mathcal{H}$ that contains $H_0$ the set $P_\mu(H; \mu)$ is not empty. It follows from $H_0 \not\in \mathcal{I}$ that the family $\mathcal{H}_I = \{ H \cap A : A \in \mathcal{I} \}$ is an invariant Boolean ideal on the group $H$. Then the family $\{ H \cap A : A \in \mathcal{I} \}$ is a filter on $H$, which can be enlarged to an ultrafilter $\mathcal{U}_H$. The ultrafilter $\mathcal{U}_H$ determines a 2-valued measure $\mu_H : \mathcal{P}(H) \to \{0, 1\}$ such that $\mu_H(1) = \mathcal{U}_H$. By the right invariance of the ideal $\mathcal{I}$, for every $A \in \mathcal{I}$ and $x \in H$ we get $\mu_H \ast \delta_1 \ast \mu(A) = \mu_H(Ax) = 0$, which means that $\mu_H \in P_\mu(H; \mu)$. So, the set $\mathcal{H}_I = \{ H \in \mathcal{H} : P_\mu(H; \mu) \not= \emptyset \} \supset \{ H \in \mathcal{H} : H \supset H_0 \}$ is stationary in $\mathcal{H}$. Then by Theorem 5 either

(1) $\text{cov}(\Delta_\mathcal{I}(A_i)) \leq n$ for all cells $A_i$ or else
(2) $\text{cov}_{\mathcal{I}}(\Delta_\mathcal{I}(A_i)) < n$ for some cell $A_i$ and some $G$-invariant Boolean ideal $\mathcal{J} \subset \mathcal{P}(G)$ such that $A_i^{-1} = \{ x \in G : \delta_1(x_{A_i}) > \frac{1}{n} \} \not\in \mathcal{J}$.

REFERENCES

[1] T. Banakh, O. Ravsky, S. Slobodianiuk, On partitions of G-spaces and G-lattices, preprint (http://arxiv.org/abs/1303.1427).
[2] T. Banakh, Extremal densities and submeasures on groups and G-spaces and their combinatorial applications, preprint (http://arxiv.org/abs/1312.5078).
[3] T. Banakh, I. Protasov, S. Slobodianiuk, Densities, submeasures and partitions of groups, preprint (http://arxiv.org/abs/1303.3612).
[4] T. Jech, Some combinatorial problems concerning uncountable cardinals, Ann. Math. Logic 5 (1972/73), 165–198.
[5] T. Jech, Stationary sets. in: Handbook of set theory. Vol.1, 93–128, Springer, Dordrecht, 2010.
[6] N. Hindman, D. Strauss, Algebra in the Stone-Cech compactification, Walter de Gruyter & Co., Berlin, 1998.
[7] V.D. Mazurov, E.I. Khukhro, (eds.) Unsolved problems in group theory: the Kourovka notebook, Thirteenth augmented edition. Russian Academy of Sciences Siberian Division, Institute of Mathematics, Novosibirsk, 1995. 120 pp.
[8] A. Paterson, Amenability, Amer. Math. Soc. Providence, RI, 1988.

T. Banakh: Ivan Franko National University of Lviv (UKRAINE) AND Jan Kochanowski University in Kielce (POLAND)
E-mail address: t.o.banakh@gmail.com

M. Frączyk: Institute of Mathematics, Jagiellonian University, Kraków (POLAND) AND Institut Galilée, Université Paris 13, Paris (FRANCE)
E-mail address: mikolaj.fraczky@gmail.com