System of Complex Brownian Motions Associated with the O’Connell Process

Makoto Katori *

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Abstract

The O’Connell process is a softened version (a geometric lifting with a parameter $a > 0$) of the noncolliding Brownian motion such that neighboring particles can change the order of positions in one dimension within the characteristic length $a$. This process is not determinantal. Under a special entrance law, however, Borodin and Corwin gave a Fredholm determinant expression for the expectation of an observable, which is a softening of an indicator of a particle position. We rewrite their integral kernel to a form similar to the correlation kernels of determinantal processes and show, if the number of particles is $N$, the rank of the matrix of the Fredholm determinant is $N$. Then we give a representation for the quantity by using an $N$-particle system of complex Brownian motions (CBMs). The complex function, which gives the determinantal expression to the weight of CBM paths, is not entire, but in the combinatorial limit $a \to 0$ it becomes an entire function providing conformal martingales and the CBM representation for the noncolliding Brownian motion is recovered.

Keywords The O’Connell process · Noncolliding Brownian motion · Geometric lifting · Combinatorial limit · Fredholm determinants · Quantum Toda lattice · Whittaker functions · Macdonald processes · Complex Brownian motions

1 Introduction

1.1 Background

A determinantal point process is a random ensemble of points in a space such that all correlation functions are given by determinants, whose matrix entries are values of a single continuous function called the correlation kernel [28, 26]. It can be generalized to space-time systems and if all spatio-temporal correlation functions are given by determinants, the process is also said to be determinantal [9, 15]. The noncolliding Brownian motion with

*Department of Physics, Faculty of Science and Engineering, Chuo University, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan; e-mail: katori@phys.chuo-u.ac.jp
a finite number of particles $N$ is determinantal for all deterministic initial configurations $\xi(\cdot) = \sum_{j=1}^{N} \delta_{r_j}(\cdot)$. (For each $A \subset \mathbb{R}$, $\xi(A) = \int_{A} \xi(dx)$ gives the number of particles in $A$ in the configuration $\xi$. Especially, for $r \in \mathbb{R}$, $\xi(\{r\})$ denotes the number of particles located at the point $r$.) In particular, if the initial positions of particles $\{r_j\}_{j=1}^{N}$ are all distinct (i.e., for $r \in \mathbb{R}$, $\xi(\{r\}) = 1$ if $r = r_j$, $1 \leq j \leq N$, and $\xi(\{r\}) = 0$ otherwise), the spatio-temporal correlation kernels is explicitly given as, for $(x, x') \in \mathbb{R}^2$, $(t, t') \in [0, \infty)^2$ [16]

$$K_N^\xi(t, x; t', x') = \sum_{j=1}^{N} \int_{\mathbb{R}} dy p(t, x|r_j)p(t', y|0)\Phi_\xi^r(x' + iy) - \mathbf{1}_{(t \geq t')}p(t - t', x|x') \quad (1.1)$$

with

$$\Phi_\xi^r(z) = \prod_{r \in \xi(\{r\})=1, r \neq r'} \frac{r - z}{r - r'}, \quad r', z \in \mathbb{C}, \quad (1.2)$$

where $i = \sqrt{-1}$, $p(t, y|x)$ denotes the transition probability density of the one-dimensional standard Brownian motion (BM)

$$p(t, y|x) = \frac{e^{-(x-y)^2/2t}}{\sqrt{2\pi t}} \mathbf{1}_{(t>0)} + \delta(x-y)\mathbf{1}_{(t=0)}, \quad (x, y) \in \mathbb{R}^2, \quad t \geq 0, \quad (1.3)$$

and $\mathbf{1}_{(\omega)}$ is the indicator function of a condition $\omega$; $\mathbf{1}_{(\omega)} = 1$ if $\omega$ is satisfied and $\mathbf{1}_{(\omega)} = 0$ otherwise. The results are extended to the infinite-particle systems, in which the function (1.2) is regarded as the Weierstrass canonical product representation of an entire function [16].

O’Connell introduced an $N$-component diffusion process, $N \geq 2$, which can be regarded as a stochastic version of a quantum open Toda-lattice [22]. The Hamiltonian of the GL($N, \mathbb{R}$)-quantum Toda lattice is given by

$$H_N^\omega = -\frac{1}{2} \Delta + \frac{1}{a^2} V_N(x/a), \quad x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \quad (1.4)$$

with the Laplacian $\Delta = \sum_{j=1}^{N} \partial^2/\partial x_j^2$ and the potential

$$V_N(x) = \sum_{j=1}^{N-1} e^{-(x_{j+1} - x_j)}. \quad (1.5)$$

The Weyl chamber of type $A_{N-1}$ is given by $\mathbb{W}_N = \{x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N\}$. For $\nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbb{W}_N$, the eigenfunction problem $H_N^\nu \psi^{(N)}(x/a) = \lambda(\nu) \psi^{(N)}(x/a)$ for the eigenvalue $\lambda(\nu) = -|\nu|^2/2$ is uniquely solved under the condition that $e^{-\nu \cdot x/a} \psi^{(N)}(x/a)$ is bounded and

$$\lim_{x \to \infty, x \in \mathbb{W}_N} e^{-\nu \cdot x/a} \psi^{(N)}(x/a) = \prod_{1 \leq j < k \leq N} \Gamma(\nu_k - \nu_j),$$

where $x \to \infty, x \in \mathbb{W}_N$ means $x_{j+1} - x_j \to \infty, 1 \leq j \leq N - 1$, and $\Gamma$ denotes the Gamma
function. The eigenfunction \( \psi^{(N)}_{\nu}(\cdot) \) is called the class-one Whittaker function [2, 23]. The infinitesimal generator of the O’Connell process is given by [22]

\[
\mathcal{L}_N^a = - (\psi^{(N)}_{\nu}(x/a))^{-1} \left( \mathcal{H}_N^a + \frac{1}{2} |\nu|^2 \right) \psi^{(N)}_{\nu}(x/a) = \frac{1}{2} \Delta + \nabla \log \psi^{(N)}_{\nu}(x/a) \cdot \nabla,
\]

where \( \nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N) \). This multivariate diffusion process is an extension of a one-dimensional diffusion studied by Matsumoto and Yor [20, 21]. (The Matsumoto-Yor process describes time-evolution of the relative coordinate of the \( N \)-particle system of BMs with drift vector \( \nu \), \( N \)-th component of the O’Connell process, \( 1 \leq j \leq N \) [12, 13, 14]. Let \( B_j(t), 1 \leq j \leq N \) be independent one-dimensional standard BMs started at \( B_j(0) = x_j \in \mathbb{R} \), and for \( \nu = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbb{R}^N \), \( B_j^{\nu}(t) = B_j(t) + \nu_j t, 1 \leq j \leq N \) be drifted BMs. We consider an \( N \)-particle system of BMs with drift vector \( \nu \), \( B^{\nu}(t) = (B_1^{\nu}(t), \ldots, B_N^{\nu}(t)), t \geq 0 \), such that the probability \( P_N^{a}(t|\{B^{\nu}(s)\}_{0 \leq s \leq t}) \) that all \( N \) particles survive up to time \( t \) conditioned on a path \( \{B^{\nu}(s)\}_{0 \leq s \leq t} \) decays following the equation

\[
\frac{d}{dt} P_N^a(t|\{B^{\nu}(s)\}_{0 \leq s \leq t}) = - \frac{1}{a^2} V_N(B^{\nu}(t)/a) P_N^a(t|\{B^{\nu}(s)\}_{0 \leq s \leq t}), \quad t \geq 0.
\]

It is a system of mutually killing BMs, in which the Toda-lattice potential (1.5) determines the decay rate of the survival probability depending on a configuration \( B^{\nu}(t) \) [13]. With the initial condition \( B^{\nu}(0) = x \in \mathbb{W}_N \), the survival probability \( P_N^a(t) = \mathbb{E}^x[P_N^a(t|\{B^{\nu}(s)\}_{0 \leq s \leq t})] \) is obtained by averaging over all paths of BMs started at \( x \), and we can show that [14, 23, 13]

\[
\lim_{t \to \infty} P_N^a(t) = c_1^a(N, \nu) e^{-\nu \cdot x/a} \psi^{(N)}_{\nu}(x/a), \quad \text{if } \nu \in \mathbb{W}_N, \nu \neq 0,
\]

\[
P_N^a(t) \sim c_2^a(N) t^{-N(N-1)/4} \psi^{(N)}_{0}(x/a) \quad \text{as } t \to \infty, \text{ if } \nu = 0,
\]

where \( c_1^a(N, \nu) \) and \( c_2^a(N) \) are independent of \( x \) and \( t \). Then, conditionally on surviving of all \( N \) particles, the equivalence of this vicious BM, which has a killing term given by the Toda-lattice potential, with the O’Connell process is proved. We note that the parameter \( a > 0 \) in the killing rate (1.7) with (1.5) indicates the characteristic range of interaction to kill neighboring particles as well as the characteristic length in which neighboring particles can exchange their order in \( \mathbb{R} \). It implies that if we take the limit \( a \to 0 \), the O’Connell process is reduced to the noncolliding BM. (The original vicious Brownian motion is a system of BMs such that if pair of particles collide they are annihilated immediately. The noncolliding BM is the vicious BM conditioned never to collide with each other, and thus all particles survive forever.)

In the present paper, the limit \( a \to 0 \) is called the combinatorial limit and an inverse of this procedure is said to be a geometric lifting in the sense of [4]. (See also [3].) Since determinantal functions associated with noncolliding diffusion processes (e.g., the Karlin-McGregor determinants, the Vandermonde determinants, the Schur functions) are replaced
by functionals of the class-one Whittaker functions in the geometric lifting [2, 22, 23], the O’Connell process is not a determinantal process.

Recently Borodin and Corwin [6] introduced the family of probability measures on sequences of partitions, which are written in terms of the Macdonald symmetric functions and specified by the Macdonald parameters $q, t \in [0,1] [19]$ as well as two Macdonald non-negative specializations [6]. This family of discrete measures is not determinantal. They showed, however, that if we consider a sub-family of processes with $t = 0$ called the $q$-Whittaker measures, and if we observe a special class of quantities, which are eigenvalues of Macdonald’s difference operators and called the Macdonald process observables [6], then some determinants appear, though still the processes are not determinantal. Taking a $q \to 1$ limit of the $q$-Whittaker measures leads to a family of Whittaker measures which are now continuous and supported in $\mathbb{R}^N$. The determinants survive this limit transition. The interesting and important fact is that the Whittaker measures are also realized as probability distributions of the O’Connell process started according to a special entrance law from $\rho$. Borodin and Corwin [6] proved that $E^{\nu,a}[\Theta^a(X^a_1(t) - h)]$, $h \in \mathbb{R}$ is given by a Fredholm determinant of a kernel $K_{e,h/a}$ for the contour integrals on $C(-\nu)$,

$$\Theta^a(x) = \exp(-e^{-x/a}).$$

Note that $\lim_{a \to 0} \Theta^a(x) = 1_{(x>0)}$, that is, (1.9) is a softening of an indicator function $1_{(x>0)}$.

For a configuration $\xi(\cdot) = \sum_{j=1}^N \delta_{r_j}(\cdot)$, $C(\xi)$ denotes a simple positively oriented contour on $\mathbb{C}$ containing the points $\{r_j\}_{j=1}^N$ located on $\mathbb{R}$. Let $\delta = \sup\{|\nu_j| : 1 \leq j \leq N\}$ and choose $0 < \delta < 1$ so that $\tilde{\delta} = \delta/2$. Borodin and Corwin [6] proved that $E^{\nu,a}[\Theta^a(X^a_1(t) - h)]$, $h \in \mathbb{R}$ is given by a Fredholm determinant of a kernel $K_{e,h/a}$ for the contour integrals on $C(-\nu)$,

$$\int_{\mathbb{R}^N} \det_{(v,v')\in C(-\nu)^2} \left[ \delta(v-v') + K_{e,h/a}(v, v') \right],$$

where

$$K_u(v, v') = \int_{-i\infty}^{i\infty} ds \frac{d\nu_s}{2\pi i} \Gamma(-s) \Gamma(1 + s) \prod_{\ell=1}^N \frac{\Gamma(v + \nu_s)}{\Gamma(s + v + \nu_\ell)} \frac{u^s e^{(u+a^2) \nu_s/2a^2}}{v + s - v'}, \quad u > 0.$$ (1.11)

Here the Fredholm determinant is defined by the sum of infinite series of multiple contour-integrals

$$\det_{(v,v')\in C(-\nu)^2} \left[ \delta(v-v') + K_u(v, v') \right] = \sum_{L=0}^{\infty} \frac{1}{L!} \prod_{j=1}^L \int_{C(-\nu)} d\nu_j \det_{1 \leq j,k \leq L} [K_u(\nu_j, \nu_k)],$$ (1.12)
where the term for $L = 0$ is assumed to be 1. Note that (1.11) depends on $\nu, a$ and $t$; $K_u(\cdot, \cdot) = K_u(\cdot, \cdot; \nu, a, t)$.

The Fredholm determinant formula (1.10) discovered by Borodin and Corwin [6] is surprising, since the O’Connell process is not determinantal as mentioned above. We would like to understand the origin of such a determinantal structure surviving in the geometric lifting from the noncolliding BM to the O’Connell process.

It is well-known in quantum mechanics that the wave function of free fermions is expressed by an $N \times N$ determinant called the Slater determinant. Then, a determinantal process is also called a fermion point process [26]. One should be careful, however, that the notion of fermion is not enough to formulate determinantal processes, since in the context of stochastic processes repulsive interactions between paths in a spatio-temporal plane should be described. In a previous paper [17], as an extension of notion of free fermions, we gave the complex Brownian motion (CBM) representation for the noncolliding BM. Let $Z_j(t), t \geq 0, 1 \leq j \leq N$ be a set of independent CBMs such that the real and imaginary parts, denoted by $V_j(t) = \mathbb{R}Z_j(t), W_j(t) = \mathbb{I}Z_j$, are independent one-dimensional standard BMs. Since $\Phi'(\cdot)$ given by (1.2) is entire, $\Phi'(Z_j(t))$ is a conformal map of a CBM, and hence it is a time change of a CBM. In other words, $\Phi'(Z_j(t)), 1 \leq j \leq N$ provide a set of independent complex local martingales, which are called conformal local martingales in Section V.2 of [24]. Therefore a determinant of $N \times N$ matrix, \( \det_{1 \leq j,k \leq N} \Phi'(Z_k(\cdot)) \), is a martingale for the system of independent CBMs. We proved that the noncolliding BM can be represented by the system of independent CBMs weighted by this determinantal martingale [17]. In comparison of the CBM representation for the noncolliding BM with a free fermion system, free-ness of particles is ensured by independence of CBMs, $Z_j(\cdot), 1 \leq j \leq N$ and fermionic property is dynamically expressed by the determinantal weight, \( \det_{1 \leq j,k \leq N} \Phi'(Z_k(T)) \), on paths \( \{Z_k(t), t \in [0, T]\}_{k=1}^N \) for any $0 < T < \infty$.

In the present paper, we would like to discuss the formula of Borodin and Corwin (1.10) for the O’Connell process from the viewpoint of our theory of determinantal processes [16, 17, 14]. In order to do that, we first rewrite their expression.

### 1.2 Main Results

In the present paper, we set $\nu = a\nu = (a\nu_1, a\nu_2, \ldots, a\nu_N) \in \mathbb{R}^N$ and $a\nu(\cdot) = \sum_{j=1}^N \delta_{a\nu_j}(\cdot)$. We first report the reexpression of the Fredholm determinant of Borodin and Corwin.

**Proposition 1.1** Assume that $\sup\{|\nu_j| : 1 \leq j \leq N\} < 1/(2a)$ and $\{\nu_j\}_{j=1}^N$ are all distinct. For $t \geq 0, (x, x') \in \mathbb{R}^2$, let

$$K^a_N(t; x, x') = \sum_{j=1}^N \int_{\mathbb{R}} dy p(t, x|\nu_j)p(t, y|0)\Phi^{\nu_j, a}_{\nu}(x' + iy),$$

where

$$\Phi^{\nu_j, a}_{\nu}(z) = \Gamma(1 - a(r' - z)) \prod_{r \in \nu\{r\} = \nu_j \cap \nu \neq \nu_j} \frac{\Gamma(a(r - r'))}{\Gamma(a(r - z))}, \quad r', z \in \mathbb{C},$$

(1.13)
and put
\[ K_N(x, x') = K_N(x, x'; t, \nu, a) = \frac{1}{t} K_N^\nu(t; x/t, x'/t). \] (1.15)

Then, for \( h \in \mathbb{R} \),
\[ \mathbb{E}^{\nu, a}[\Theta^a(X_1^a(t) - h)] = \text{Det} \left[ \delta(x - x') - K_N(x, x')1_{(x' < h)} \right] \]
\[ = \sum_{N' = 0}^{N} \frac{(-1)^{N'}}{N'} \prod_{j=1}^{N'} \int_{-\infty}^{h} dx_j \det_{1 \leq j, k \leq N'} \left[ K_N(x_j, x_k) \right]. \] (1.16)

The points are following.

(i) The expression of Borodin and Corwin can be rewritten as the Fredholm determinant of the rank \( N \) operator with a kernel \( K_N(x, x') \), \( (x, x') \in \mathbb{R}^2 \) multiplied by an indicator \( 1_{(x' < h)} \). Then the Fredholm series (1.16) has only \( N + 1 \) terms.

(ii) The kernel \( K_N^{\nu, a}(t; \cdot, \cdot, \cdot) \) is obtained from \( K_N^{\xi}(t, \cdot, t', \cdot) \) of (1.1) by setting \( \xi(\cdot) = \nu(\cdot) \) and \( t' = t \) and replacing the function \( \Phi^a(\cdot) \) by \( \Phi^a(\cdot) \). Equation (1.15) means that the kernel \( K_N \) is the reciprocal-time transform of \( K_N^{\nu, a} \) in the sense of [14].

Remark that an expression for the kernel \( K_N \), which is valid even when some of \( \tilde{\nu}_j \)'s coincide, is given by (3.5) in the proof of Proposition 1.1 in Section 3. It contains a contour integral on \( C(\tilde{\nu}) \) and, if \( \{\tilde{\nu}_j\}_{j=1}^{N} \) are all distinct, the Cauchy integral is readily performed and (1.13) is obtained. For simplicity of expressions and arguments, here we assume that \( \{\tilde{\nu}_j\}_{j=1}^{N} \) are all distinct.

By the fact \( \lim_{z \to 0} z \Gamma(z) = 1 \), in the combinatorial limit \( a \to 0 \), \( \Phi^a(\cdot) \to \Phi^a(\cdot) \), and thus
\[ \lim_{a \to 0} K_N^{\nu, a}(t; x, x') = \mathbb{E}^{\nu, a}[\Theta^a(X_1^a(t))], \quad (x, x') \in \mathbb{R}^2, \quad t \geq 0, \] (1.17)
where the rhs is the (equal time \( t' = t \)) correlation kernel (1.1) for the noncolliding BM without drift starting from a particle configuration \( \nu(\cdot) \). Then, the \( a \to 0 \) limit of the rhs of (1.16) gives the Fredholm determinantal expression for the probability that all particle-positions are greater than the value \( ht \), in the noncolliding BM without drift starting from \( \nu \), when we observe the configuration at the reciprocal time \( 1/t \);
\[ \lim_{a \to 0} \mathbb{E}^{\nu, a}[\Theta^a(X_1^a(t) - h)] = \mathbb{P}[X_1(1/t) > ht], \quad t \geq 0, h \in \mathbb{R}. \] (1.18)

Note that our noncolliding Brownian motion, \( X(t) = (X_1(t), X_2(t), \ldots, X_N(t)) \), is ordered as \( X_1(t) < X_2(t) < \cdots < X_N(t), t > 0 \) in labeled configurations. By the reciprocal time relation proved in [14], the rhs of (1.18) is equal to the probability that all particle-positions are greater than \( h \) at time \( t \), in the noncolliding BM with drift vector \( \nu \), where all particles
are started from the origin. This initial state is given by the delta measure at the origin with multiplicity $N$, expressed by $N\delta_0$, and we write the probability for this drifted noncolliding BM as $P^{N\delta_0}_\nu[\cdot]$. Then we have

$$P^{\hat{\nu}}_\nu[X_1(1/t) > ht] = P^{N\delta_0}_\nu[X_1(t) > h], \quad t \geq 0, h \in \mathbb{R}.$$  \hfill (1.19)

Combining (1.18) and (1.19) gives the relation

$$\lim_{a \to 0} E^{a\hat{\nu},\nu}[\Theta^a(X_1^a(t) - h)] = P^{N\delta_0}_\nu[X_1(t) > h], \quad t \geq 0, h \in \mathbb{R}. \hfill (1.20)$$

As mentioned before, $\Theta^a(\cdot)$ given by (1.9) is a geometric lifting of an indicator function $1_{(>0)}$. We will show that, in the combinatorial limit $a \to 0$, the transition probability density of the O'Connell process with $a\hat{\nu}$ started according to the entrance law coming from $"-\infty\rho"$ converges to that of the noncolliding BM with drift $\hat{\nu}$ started from $N\delta_0$ (see Lemma 2.1 in Section 2). Then, Proposition 1.1 will state that the result (1.10) by Borodin and Corwin is a geometrical lifting of the Fredholm determinantal expression for the probability $P^{N\delta_0}_\nu[X_1(t) > h]$ of the drifted noncolliding BM.

The complex function $\Phi^{\hat{\nu},\nu}(z)$ appears in the kernel (1.13) for the O'Connell process is not entire; as shown by (1.14), it has simple poles at

$$z_n = -\frac{n}{a} + r', \quad n \in \mathbb{N} \equiv \{1, 2, 3, \ldots\}. \hfill (1.21)$$

(Note that all poles go to infinity in the limit $a \to 0$ and the function becomes entire in the combinatorial limit.) Therefore, we will not obtain useful martingales to represent time evolutions of the system as in [17], but the single-time observables can have the CBM representations. The main result of the present paper is the following.

For a configuration $\hat{\nu}(\cdot) = \sum_{j=1}^N \delta_{\nu_j}(\cdot)$ with $\tilde{\nu}_j \in \mathbb{R}, 1 \leq j \leq N$, we consider the CBMs, $Z_j(t)$ starting from $\tilde{\nu}_j, 1 \leq j \leq N$. That is, $V_j(0) = \tilde{\nu}_j$ and $W_j(t) = 0, 1 \leq j \leq N$. The expectation with respect to the CBMs under such an initial condition is denoted by $E^{\hat{\nu}}[\cdot]$.

**Theorem 1.2** Under the same condition of Proposition 1.1,

$$E^{a\hat{\nu},\nu}[\Theta^a(X_1^a(t) - h)] = E^{\hat{\nu}}\left[\det_{1 \leq j,k \leq N} \left[\delta_{jk} - \Phi^{\hat{\nu},\nu}_\nu(Z_k(1/t))1_{(V_k(1/t)<ht)}\right]\right]. \hfill (1.22)$$

The observable $\Theta^a(X_1^a(t) - h), h \in \mathbb{R}$ is a softening of the indicator $1_{(X_1(t)>h)}$. Theorem 1.2 shows that its expectation for the O'Connell process started according to the entrance law coming from "$-\infty\rho$" has the determinantal CBM representation, in which the 'sharp' indicators $1_{(V_k(1/t)<ht)}, 1 \leq k \leq N$ are observed, but the complex weight on paths is 'softened' as $\det_{1 \leq j,k \leq N} \left[\Phi^{\hat{\nu},\nu}_\nu(Z_k(\cdot))\right]$, and the martingale property is lost. Further study of the maps of CBMs, $\Phi^{\hat{\nu},\nu}_\nu(Z_j(\cdot)), 1 \leq j,k \leq N$, and the system of CBMs with this determinantal weight is an interesting future problem. We hope that the present study will give some hint
for understanding why determinants appear in the processes which are not determinantal [6, 7, 1, 8].

The paper is organized as follows. In Section 2 preliminaries of the O’Connell process and the noncolliding BM are given. The derivation of Proposition 1.1 from the result by Borodin and Corwin [6] is given in Section 3. Section 4 is devoted to the proof of Theorem 1.2. Appendix A is prepared to give a sketch of a non-rigorous approach to deriving the result by Borodin and Corwin. (See the proof of Theorem 4.1.40 in [6] for a rigorous version of it.)

2 O’Connell Process and Noncolliding Brownian Motion

2.1 Orthogonality and Recurrence Relations of Class-One Whittaker Functions

Let $N = 2, 3, \ldots$ and $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. The class-one Whittaker function $\psi^{(N)}_{\mathbf{\nu}}(x)$ has several integral representations, one of which was given by Givental [11],

$$\psi^{(N)}_{\mathbf{\nu}}(x) = \int_{T_N(x)} \exp \left( F^{(N)}_{\mathbf{\nu}}(T) \right) \ dT.$$  

Here the integral is performed over the space $T_N(x)$ of all real lower triangular arrays with size $N$, $T = (T_{j,k}, 1 \leq k \leq j \leq N)$, with $T_{N,k} = x_k, 1 \leq k \leq N$, and

$$F^{(N)}_{\mathbf{\nu}}(T) = \sum_{j=1}^{N} \nu_j \left( \sum_{k=1}^{j} T_{j,k} - \sum_{k=1}^{j-1} T_{j-1,k} \right) - \sum_{j=1}^{N-1} \sum_{k=1}^{j} \left\{ e^{-(T_{j,k} - T_{j+1,k})} + e^{-(T_{j+1,k+1} - T_{j,k})} \right\}.$$  

We can prove that [22, 10]

$$\lim_{a \to 0} a^{N(N-1)/2} \psi^{(N)}_{\mathbf{\nu}}(x/a) = \frac{\det \left[ e^{\nu_j \nu_{\ell}} \right]_{1 \leq j, \ell \leq N}}{h_N(\mathbf{\nu})}, \quad (2.1)$$

where $h_N(\mathbf{\nu})$ is the Vandermonde determinant

$$h_N(\mathbf{\nu}) = \det_{1 \leq j, \ell \leq N} [\nu_{\ell}^{\nu_j - 1}] = \prod_{1 \leq j < \ell \leq N} (\nu_{\ell} - \nu_j). \quad (2.2)$$

The following orthogonality relation is proved for the class-one Whittaker functions [25, 29],

$$\int_{\mathbb{R}^N} \psi^{(N)}_{\mathbf{i}k}(x) \psi^{(N)}_{\mathbf{i}k'}(x) dx = \frac{1}{s_N(k) N!} \sum_{\sigma \in \mathcal{S}_N} \delta(k - \sigma(k')), \quad (2.3)$$
for $k, k' \in \mathbb{R}^N$, where $s_N(\cdot)$ is the density function of the Sklyanin measure [27]

$$s_N(\mu) = \frac{1}{(2\pi)^N N!} \prod_{1 \leq j < \ell \leq N} |\Gamma(i(\mu_\ell - \mu_j))|^{-2} \prod_{1 \leq j < \ell \leq N} \left\{ (\mu_\ell - \mu_j) \frac{\sin \pi(\mu_\ell - \mu_j)}{\pi} \right\}, \quad \mu \in \mathbb{R}^N, \quad (2.4)$$

$\mathcal{S}_N$ is the set of permutations of $N$ indices, and $\sigma(k') = (k'_{\sigma(1)}, \ldots, k'_{\sigma(N)})$ for $\sigma \in \mathcal{S}_N$. Borodin and Corwin proved that for a class of test functions, the orthogonality relation (2.3) can be extended for any $k, k' \in \mathbb{C}^N [6]$. Moreover, the following recurrence relations with respect to $\nu$ are established [18, 6]; for $1 \leq r \leq N - 1, \nu \in \mathbb{C}^N$,

$$\sum I \subset \{1, \ldots, N\}, |I| = r \prod_{j \in I, k \in \{1, 2, \ldots, N\}/I} \psi_{i\nu_{i\ell}}^{(N)}(x) = \exp \left( -\sum_{j=1}^{r} x_j \right) \psi_{i\nu}^{(N)}(x), \quad (2.5)$$

where $e_I$ is the vector with ones in the slots of label $I$ and zeros otherwise;

$$(e_I)_j = \begin{cases} 1, & j \in I, \\ 0, & j \in \{1, \ldots, N\} \setminus I. \end{cases}$$

In particular, for $r = 1$,

$$\sum_{j=1}^{N} \prod_{1 \leq k \leq N: k \neq j} \frac{1}{i(\nu_k - \nu_j)} \psi_{i\nu_{i\ell}}^{(N)}(x) = e^{-x_1} \psi_{i\nu}^{(N)}(x), \quad (2.6)$$

where the $\ell$-th component of the vector $e_{(j)}$ is $(e_{(j)})_\ell = \delta_{j\ell}, 1 \leq j, \ell \leq N$. As fully discussed by Borodin and Corwin [6], the recurrence relations (2.5) are derived as the $q \to 1$ limit of the eigenfunction equations associated to the Macdonald difference operators in the theory of symmetric functions [19]. For more details on Whittaker functions, see [18, 2, 12, 22, 13, 6] and references therein.

### 2.2 O’Connell Process

In order to discuss the relationship between the O’Connell process and the noncolliding BM, we have introduced the parameter $a > 0$. The transition probability density for the O’Connell process with $\nu$ is given by [14]

$$P^\nu_{N,a}(t, \nu|\nu) = e^{-t|\nu|^2/2a^2} \psi_{i\nu}^{(N)}(y/a) \psi_{i\nu}^{(N)}(x/a) Q_N^a(t, y|x), \quad x, y \in \mathbb{R}^N, t \geq 0, \quad (2.7)$$

with

$$Q_N^a(t, y|x) = \int_{\mathbb{R}^N} e^{-t|k|^2/2a^2} \psi_{iak}^{(N)}(x/a) \psi_{iak}^{(N)}(y/a) s_N(ak)dk. \quad (2.8)$$
(See also Proof of Proposition 4.1.32 in [6].) As a matter of fact, we can confirm that \( u(t, x) \equiv P_{\nu,a}^N(t, y|x) \) satisfies the Kolmogorov backward equation associated with the infinitesimal generator \( L_{\nu,a}^N \) given by (1.6),

\[
\frac{\partial u(t, x)}{\partial t} = L_{\nu,a}^N u(t, x) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2 u(t, x)}{\partial x_j^2} + \sum_{j=1}^{N} \frac{\partial \log \psi^{(N)}(x/a)}{\partial x_j} \frac{\partial u(t, x)}{\partial x_j},
\]

(2.9)

\( x \in \mathbb{R}^N, t \geq 0 \), under the condition \( u(0, x) = \delta(x - y) \equiv \prod_{j=1}^{N} (x_j - y_j), y \in \mathbb{R}^N \). Assume that the initial configuration \( x \in \mathbb{R}^N \) is given. Let \( M \in \mathbb{N} \) and \( 0 \leq t_1 < t_2 < \cdots < t_M < \infty \). Then, for this Markov process, the probability density function of the multi-time joint distributions is given by

\[
P_{\nu,a}^N(t_{m+1} - t_m, x^{(m+1)}|x^{(m)})P_{\nu,a}^N(t_1, x^{(1)}|x)
= e^{-t_M|V|^2/2a^2} \psi^{(N)}(x^{(M)}/a) \prod_{m=1}^{M-1} Q_{\nu,a}^N(t_m - t_{m+1}, x^{(m+1)}|x^{(m)})Q_{\nu,a}^N(t_1; x^{(1)}|x),
\]

(2.10)

\( x^{(m)} \in \mathbb{R}^N, 1 \leq m \leq M \).

Recall that we denote the O’Connell process by

\[
X^a(t) = (X_1^a(t), X_2^a(t), \ldots, X_N^a(t)), \quad t \geq 0.
\]

(2.11)

It is defined as an \( N \)-particle diffusion process in \( \mathbb{R} \) such that its backward Kolmogorov equation is given by (2.9) and the finite-dimensional distributions are determined by (2.10). Therefore, (2.11) is a unique solution of the following stochastic differential equation for given initial configuration \( X^a(0) = x \in \mathbb{R}^N \),

\[
dX_j^a(t) = dB_j(t) + \left[ F_{\nu,a}^N(X^a(t)) \right]_j dt, \quad 1 \leq j \leq N, t \geq 0
\]

(2.12)

with

\[
F_{\nu,a}^N(x) = \nabla \log \psi^{(N)}(x/a),
\]

(2.13)

where \( \{B_j(t)\}_{j=1}^N \) are independent one-dimensional standard BMs and \( [V]_j \) denotes the \( j \)-th coordinate of a vector \( V \).

### 2.3 Special Entrance Law

Let \( N \in \mathbb{N} \), and define

\[
\rho = \left( -\frac{N-1}{2}, -\frac{N-1}{2} + 1, \ldots, -\frac{N-1}{2} + 1, \frac{N-1}{2} \right).
\]

(2.14)
O’Connell considered the process starting from \( x = -M \rho \) and let \( M \to \infty \) [22]. It was claimed in [22] (see also [2]) that

\[
\psi^{(N)}_\nu(-M \rho) \sim C e^{-N(N-1)M/8} \exp\left(e^{M/2} \mathcal{F}_0(T^0)\right)
\]

(2.15)
as \( M \to \infty \), where the coefficient \( C \) and the critical point \( T^0 \) are independent of \( \nu \). Then as a limit of (2.7) with (2.8), we have a probability density function

\[
\mathcal{P}_{N}^{\nu,a}(t, x) \equiv \lim_{M \to \infty} P_{N}^{\nu,a}(t, x - M \rho)
\]

\[
e^{-t|\nu|^2/2a^2} \psi^{(N)}(x/a) \partial_N^a(t, x)
\]

(2.16)
with

\[
\partial_N^a(t, x) = \int \mathbb{R}^N e^{-t|k|^2/2} \psi^{(N)}(x/a)k_N(a k) dk
\]

(2.17)
for any \( t > 0 \). Since we have taken the limit \( M \to \infty \) for the state \(-M \rho\), we cannot speak of initial configurations any longer, but for an arbitrary series of increasing times, \( 0 < t_1 < t_2 < \cdots < t_M < \infty \), the probability density function of the multi-time joint distributions is given by

\[
\mathcal{P}_{N}^{\nu,a}(t_1, x^{(1)}; t_2, x^{(2)}; \ldots; t_M, x^{(M)}) = \prod_{m=1}^{M-1} P_{N}^{\nu,a}(t_{m+1} - t_m, x^{(m+1)}|x^{(m)}) \mathcal{P}_{N}^{\nu,a}(t_1, x^{(1)})
\]

(2.18)
for \( x^{(m)} \in \mathbb{R}^N, 1 \leq m \leq M \). We can call the probability measure \( \mathcal{P}_{N}^{\nu,a}(t, x)dx \) with (2.16) and \( dx = \prod_{j=1}^{N} dx_j \) an entrance law coming from \( \{ -M \rho \} \) [22] using a terminology of probability theory (see, for instance, Section XII.4 of [24]). We note that, by (2.7), (2.18) is written as

\[
\mathcal{P}_{N}^{\nu,a}(t_1, x^{(1)}; t_2, x^{(2)}; \ldots; t_M, x^{(M)})
\]

\[
e^{-t_M|\nu|^2/2a^2} \psi^{(N)}(x^{(M)}/a) \prod_{m=1}^{M-1} Q_N^a(t_{m+1} - t_m, x^{(m+1)}|x^{(m)}) \partial_N^a(t_1, x^{(1)}).
\]

The expectation with respect to the distribution of the present process started according to the special entrance law (2.16) is denoted by \( \mathbb{E}^{\nu,a}[\cdot] \). For measurable functions \( f^{(m)}, 1 \leq m \leq M \),

\[
\mathbb{E}^{\nu,a}\left[ \prod_{m=1}^{M} f^{(m)}(X^{a}(t_m)) \right]
\]

\[
e^{-t_M|\nu|^2/2a^2} \left\{ \prod_{m=1}^{M} \int \mathbb{R}^N dx^{(m)} \right\} f^{(M)}(x^{(M)}) \psi^{(N)}(x^{(M)}/a)Q_N^a(t_M - t_{M-1}, x^{(M)}|x^{(M-1)})
\]

\[
\times \prod_{m=2}^{M-1} f^{(m)}(x^{(m)})Q_N^a(t_m - t_{m-1}, x^{(m)}|x^{(m-1)}) f^{(1)}(x^{(1)}) \partial_N^a(t_1, x^{(1)}),
\]

(2.19)
0 < t_1 < \cdots < t_M < \infty$, where \( dx^{(m)} = \prod_{j=1}^{N} dx_j^{(m)}, 1 \leq m \leq M \).

**Remark 1** The present special entrance law (2.16) is called a Whittaker measure by Borodin and Corwin [6] and denoted by \( \text{WM}_{(\nu,t)}(x) \). Note that in the notation of [6], a Whittaker process is a ‘triangular array extension’ of the Whittaker measure and is not the same as the O’Connell process.

When \( M = 1 \), for \( t > 0 \), (2.19) gives
\[
\mathbb{E}^{\nu,a}[f(X^a(t))] = e^{-t|\nu|^2/2} a^{N} \int_{\mathbb{R}^{N}} dx f(x) \psi^{(N)}(x/a) \psi^{(N)}(t) = e^{-t|\nu|^2/2} a^{N} \int_{\mathbb{R}^{N}} dx f(x) \psi^{(N)}(x/a) \int_{\mathbb{R}^{N}} dk e^{-t|k|^2/2} \psi^{(N)}(k) s_N(a k).
\]

**2.4 \( a \to 0 \) Limit**

The transition probability density of the absorbing BM in \( \mathbb{W}_N \) is given by the Karlin-McGregor determinant of (1.3),
\[
q_N(t, y|x) = \det_{1 \leq j, k \leq N} [p(t, y_j|x_k)], \quad x, y \in \mathbb{W}_N, t \geq 0. \tag{2.21}
\]

Consider the drift transform of (2.21),
\[
q^\nu_N(t, y|x) = \exp \left\{ -\frac{t}{2} |\nu|^2 + \nu \cdot (y - x) \right\} q_N(t, y|x).
\]

Then, if \( \nu \in \mathbb{W}_N = \{ x \in \mathbb{R}^N : x_1 \leq x_2 \leq \cdots \leq x_N \} \), the transition probability density of the noncolliding BM with drift \( \nu \) is given by [3]
\[
p^\nu_N(t, y|x) = e^{-t|\nu|^2/2} \det_{1 \leq j, k \leq N} \left[ \frac{e^{\nu_j x_k}}{\psi^{(N)}(x_k)} \right] q_N(t, y|x), \quad x, y \in \mathbb{W}_N, \quad t \geq 0. \tag{2.22}
\]

In the limit \( \nu_j \to 0, 1 \leq j \leq N \), of (2.22) the transition probability density of the noncolliding BM is given by
\[
p_N(t, y|x) = \frac{h_N(y)}{h_N(x)} q_N(t, y|x), \quad x, y \in \mathbb{W}_N, t \geq 0. \tag{2.23}
\]

We prove the following. (The superscript \( a\nu \) is used for the processes with drift vector \( a\nu \).)

**Lemma 2.1** For \( \nu \in \mathbb{W}_N \),
\[
\lim_{a \to 0} \mathcal{P}^{a \nu, a}_N(t, x) = p_N(t^{-1}, x/t|\nu) d(x/t) = p'_N(t, x|0) d(x), \quad t > 0. \tag{2.24}
\]
Proof  By the asymptotics (2.1) and the definition (2.21) of $q_N$, we have

$$
\lim_{a \to 0} a^{N(N-1)/2} e^{-t|\nu|^2/2} \frac{\psi^{(N)}(x/a)}{h_N(\nu)} = \left(\frac{2\pi}{t}\right)^{N/2} e^{2|x|^2/t} \frac{q_N(t^{-1}, x/t|\nu)}{h_N(\nu)}. \tag{2.25}
$$

For $\partial^a_N$ defined by the integral (2.17), we can show that the Whittaker function with purely imaginary index multiplied by the Sklyanin density, $\psi^{(N)}_{-ia}(a\kappa)$, is uniformly integrable in $a > 0$ with respect to the Gaussian measure $e^{-t|a\kappa|^2/2}d\kappa$, $t > 0$. Then the integral and the limit $a \to 0$ is interchangeable. Since

$$
\psi^{(N)}_{-ia}(a\kappa) \sim (-ia)^{-N(N-1)/2} e^{-ix_j\kappa_j} \frac{\det e^{-ix_j\kappa_j}}{h(\kappa)}, \quad a \to 0
$$

by (2.1), and (2.4) gives $s_N(a\kappa) \sim a^{N(N-1)/2} \{h_N(\kappa)\}^2/(2\pi)^{N/2}$, as $a \to 0$, we have

$$
\lim_{a \to 0} a^{-N(N-1)/2} \partial^a_N(t, x) = \frac{1}{(2\pi)^N N!} \int_{\mathbb{R}^N} d\kappa e^{-t|\kappa|^2/2} \det_{1 \leq j, \ell \leq N} e^{-ix_j\kappa_j} h_N(\kappa)
$$

$$
= \frac{t^{-N(N+1)/4}}{(2\pi)^{N/2}} e^{-|x|^2/2t} \frac{1}{N!} \int_{\mathbb{R}^N} d(\sqrt{t}\kappa) \det_{1 \leq j, \ell \leq N} \left[ e^{-(\sqrt{t}k_\ell + ix_j/\sqrt{t})^2/2} \frac{1}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1} (i\sqrt{t}k_\ell - i\sqrt{t}k_m) \right].
$$

By multi-linearity of determinant,

$$
\frac{1}{N!} \int_{\mathbb{R}^N} d(\sqrt{t}\kappa) \det_{1 \leq j, \ell \leq N} \left[ e^{-(\sqrt{t}k_\ell + ix_j/\sqrt{t})^2/2} \frac{1}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1} (i\sqrt{t}k_\ell - i\sqrt{t}k_m) \right]
$$

$$
= \det_{1 \leq j, \ell \leq N} \left[ \int_{\mathbb{R}} d(\sqrt{t}\kappa) e^{-(\sqrt{t}k_\ell + ix_j/\sqrt{t})^2/2} \frac{1}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1} (i\sqrt{t}k_\ell - i\sqrt{t}k_m) \right]
$$

$$
= \det_{1 \leq j, \ell \leq N} \left[ \int_{\mathbb{R}} du e^{-(u+ix_j/\sqrt{t})^2/2} \frac{1}{\sqrt{2\pi}} \prod_{m=1}^{\ell-1} (i\sqrt{t}u - i\sqrt{t}k_m) \right]. \tag{2.26}
$$

The integral in the determinant (2.26) can be identified with an integral representation given by Bleher and Kuijlaars [5, 16] for the multiple Hermite polynomial of type II,

$$
P_{\ell-1}(x_j/\sqrt{t}) \quad \text{with} \quad \xi_{\ell-1}(\cdot) = \sum_{m=1}^{\ell-1} \delta_{i\sqrt{t}k_m}(\cdot).
$$

(We set $\xi_0(\cdot) \equiv 0$ and $\prod_{m=1}^{0}(\cdots) \equiv 1$.) It is a monic polynomial of $x_j/\sqrt{t}$ with degree $\ell - 1$. Then (2.26) is equal to the Vandermonde determinant

$$
h_N(x/\sqrt{t}) = t^{N(N-1)/4} h_N(x/t).
$$
Therefore, we obtain
\[
\lim_{a \to 0} a^{-N(N-1)/2} \varphi_N^a(t, x) = \frac{1}{(2\pi t)^{N/2}} e^{-|x|^2/2t} h_N(x/t). \tag{2.27}
\]

Combining (2.25) and (2.27), we obtain the equality
\[
\lim_{a \to 0} \mathcal{P}^{N,a}_N(t, x) = \frac{h_N(x/t)}{h_N(\nu)} q_N(t^{-1}, x/t|\nu)t^{-N}, \tag{2.28}
\]
which gives the first equality of (2.24) by the formula (2.23). The second equality is concluded by the reciprocal relation proved as Theorem 2.1 in [14]. The proof is then completed.

**Remark 2** Moreover, if we take the limit $\nu \to 0$ in (2.24), we have the following
\[
\lim_{\nu \to 0} \lim_{a \to 0} \mathcal{P}^{N,a}_N(t, x) = p_N(t, x|0)
= \frac{t^{-N^2/2}}{(2\pi)^{N/2}} \prod_{j=1}^{N} \Gamma(j) \frac{e^{-|x|^2/2t} (h_N(x))^{2}}{\nu^{N}}. \tag{2.29}
\]
This is the probability density of the eigenvalue distribution of the Gaussian unitary ensemble (GUE) with variance $\sigma^2 = t$ of random matrix theory. It implies that a geometric lifting of the GUE-eigenvalue distribution is the $\nu \to 0$ limit of the entrance law coming from “$-\infty \rho$”.

\[
\mathcal{P}^a_N(t, x) \equiv \lim_{\nu \to 0} \mathcal{P}^{N,a}_N(t, x)
= \psi_0^{(N)}(x/a) \varphi_N^a(t, x)
= \psi_0^{(N)}(x/a) \int_{\mathbb{R}^N} e^{-\|k\|^2/2} \psi^{(N)}_{-ia k}(x/a)s_N(a k) d k. \tag{2.30}
\]

### 3 Proof of Proposition 1.1

We start from the following result found as Theorem 4.1.40 in Borodin and Corwin [6]. (See Appendix A for a discussion of how this result relates to the properties of Whittaker functions as in (2.3) and (2.6).) Let $\widehat{\delta} = \sup\{|\nu_j| : 1 \leq j \leq N\}$ and choose $0 < \delta < 1$ so that $\widehat{\delta} < \delta/2$. Then for $u \in \mathbb{R}$
\[
\mathbb{E}^{N,a}[\exp(-ue^{-X^a_i(t)})] = \sum_{L=0}^{\infty} \frac{1}{L!} \prod_{j=1}^{L} \int_{C(-\nu)} \frac{dv_j}{2\pi i} \det [K_u(v_j, v_k)], \tag{3.1}
\]
where $K_u(v, v')$ is given by (1.11). By assumption on $\delta$, we can take the contour $C(-\nu)$ such that any pair of $v, v' \in C(-\nu)$ satisfies $|v - v'| < 1$. Then by using the identity
\[
\frac{1}{v + s - v'} = \int_0^\infty e^{-(v+s-v')b} db,
\]
and interchanging the $ds$ and $db$ integrals, which is justified by giving appropriate decay bounds on the integrand,

$$K_u(v, v') = \int_0^\infty db e^{v'b} \int_{-\infty+\delta}^{\infty+\delta} \frac{ds}{2\pi i} \Gamma(-s) \Gamma(1+s) \prod_{\ell=1}^N \frac{\Gamma(v + \nu_{\ell})}{\Gamma(s + v + \nu_{\ell})} u^s e^{-(v+s)b + tvs/a^2 + ts^2/2a^2}.$$  

By multi-linearity of determinants, the rhs of (3.1) is equal to

$$\sum_{L=0}^\infty \frac{1}{L!} \prod_{j=1}^L \left[ \int_{C(-\nu)}^{i\infty+\delta} \frac{ds}{2\pi i} \Gamma(-s) \Gamma(1+s) \prod_{\ell=1}^N \frac{\Gamma(v_j + \nu_{\ell})}{\Gamma(s + v_j + \nu_{\ell})} u^s e^{-(v_j+s)b_k + tv_j s/a^2 + ts^2/2a^2} \right]$$

$$= \sum_{L=0}^\infty \frac{1}{L!} \prod_{j=1}^L \int_0^\infty db_j \det_{1 \leq j, k \leq L} \left[ \tilde{K}_u(b_j, b_k) \right]$$  

(3.2)

with

$$\tilde{K}_u(b, b') = \int_{C(-\nu)}^{i\infty+\delta} \frac{dv}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} \frac{ds}{2\pi i} \Gamma(-s) \Gamma(1+s) \prod_{\ell=1}^N \frac{\Gamma(v + \nu_{\ell})}{\Gamma(s + v + \nu_{\ell})} u^s e^{-a\delta^2 + tvs/a^2 + ts^2/2a^2 - v(b-b')}.$$  

(3.3)

Here we set

$$\nu_j = a\hat{\nu}_j, \quad 1 \leq j \leq N, \quad u = e^{h/a},$$

and change the integration variables in (3.2) and (3.3) to be

$$b_j = (h - x_j)/a, \quad 1 \leq j \leq L, \quad v = -aw, \quad s = a\hat{s}.$$  

Then (3.2) is rewritten as

$$\sum_{L=0}^\infty \frac{1}{L!} \prod_{j=1}^L \int_{-\infty}^{h} dx_j \det_{1 \leq j, k \leq L} \left[ \tilde{K}(x_j, x_k) \right]$$  

(3.4)

with

$$\tilde{K}(x, x') = -a \int_{C(\rho)}^{i\infty+\hat{\delta}} \frac{dw}{2\pi i} \int_{-i\infty+\hat{\delta}}^{i\infty+\hat{\delta}} \frac{d\hat{s}}{2\pi i} \Gamma(-a\hat{s}) \Gamma(1+a\hat{s})$$

$$\times \prod_{\ell=1}^N \frac{\Gamma(a(\hat{\nu}_\ell - w))}{\Gamma(a(\hat{s} + \hat{\nu}_\ell - w))} e^{(x'-tw)\hat{s} + t\hat{s}^2/2 + w(x-x')}.$$  

(3.5)

where $\hat{\delta} = \delta/a$. Note that (3.5) is independent of $h$.  

15
By assumption, \( \{\hat{\nu}_j\}_{j=1}^N \) are all distinct. Then the Cauchy integral with respect to \( w \) on \( C(\hat{\nu}) \) is readily performed as follows. For each \( \hat{\nu}_j, 1 \leq j \leq N \),

\[
\text{Res}_{w=\hat{\nu}_j} \left( \frac{\Gamma(a(\hat{\nu}_j - w))}{\Gamma(a(\hat{s} + \hat{\nu}_j - w))} \right) = -\frac{1}{a\Gamma(a \hat{s})}.
\]

Since

\[
-\frac{\Gamma(-a \hat{s})\Gamma(1 + a \hat{s})}{a\Gamma(a \hat{s})} = \frac{1}{a}(1 - a \hat{s})
\]

by residue calculation, (3.5) becomes

\[
\hat{K}(x, x') = -\sum_{j=1}^{N} \int_{-\infty+i\hat{s}}^{\infty+i\hat{s}} \frac{d\hat{s}}{2\pi i} \Gamma(1 - a \hat{s}) \times \prod_{1 \leq \ell \leq N, \ell \neq j} \frac{\Gamma(a(\hat{\nu}_\ell - \hat{\nu}_j))}{\Gamma(a(\hat{s} + \hat{\nu}_\ell - \hat{\nu}_j))} e^{(x'-\hat{\nu}_j) \hat{s} + t \hat{s}^2/2 + \hat{\nu}_j (x-x')}.
\]

Next, in each term of the summation over \( j, 1 \leq j \leq N \), in (3.6), we change the integration variable, \( \hat{s} \to y \), as

\[
\hat{s} = -(x'/t + iy) + \hat{\nu}_j.
\]

Then (3.6) is written as

\[
\hat{K}(x, x') = -\sum_{j=1}^{N} \int_{-\infty+i(\hat{s} + x'/t - \hat{\nu}_j)}^{\infty+i(\hat{s} + x'/t - \hat{\nu}_j)} dy \Gamma(1 - a\{\hat{\nu}_j - (x'/t + iy)\}) \times \prod_{1 \leq \ell \leq N, \ell \neq j} \frac{\Gamma(a(\hat{\nu}_\ell - \hat{\nu}_j))}{\Gamma(a\{\hat{\nu}_\ell - (x'/t + iy)\})} \frac{e^{x'^2/2t} e^{-t(\hat{\nu}_j-x/t)^2/2} e^{-ty^2/2}}{\sqrt{2\pi}}.
\]

By definition of (1.3)

\[
\frac{e^{-t(\hat{\nu}_j-x/t)^2/2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{t}} p(t^{-1}, x/t|\hat{\nu}_j), \quad \frac{e^{-ty^2/2}}{\sqrt{2\pi}} = \frac{1}{\sqrt{t}} p(t^{-1}, y|0), \quad t > 0,
\]

and thus

\[
\hat{K}(x, x') = -\frac{1}{t} \frac{e^{x'^2/2t}}{e^{x'^2/2t}} \sum_{j=1}^{N} \int_{-\infty+i(\hat{s} + x'/t - \hat{\nu}_j)}^{\infty+i(\hat{s} + x'/t - \hat{\nu}_j)} dy p(t^{-1}, x/t|\hat{\nu}_j) p(t^{-1}, y|0) \times \Gamma(1 - a\{\hat{\nu}_j - (x'/t + iy)\}) \prod_{1 \leq \ell \leq N, \ell \neq j} \frac{\Gamma(a(\hat{\nu}_\ell - \hat{\nu}_j))}{\Gamma(a\{\hat{\nu}_\ell - (x'/t + iy)\})}.
\]

Here we consider each integral with respect to \( y \) in the summation. Note that \( p(t^{-1}, y|0) \) and \( 1/\Gamma(a\{\hat{\nu}_\ell - (x'/t + iy)\}), 1 \leq \ell \leq N, \ell \neq j \) are all entire functions of \( y \). The function
\(\Gamma(1-a\{\tilde{\nu}_j-(x'/t+iy)\})\) has simple poles, which are located at \(y_n = i(n/a+x'/t-\tilde{\nu}_j), n \in \mathbb{N}\). Since the assumption \(\delta < 1\) gives \(\tilde{\delta} = \delta/a < 1/a\), \(\Im y_n > \tilde{\delta} + x'/t - \tilde{\nu}_j, n \in \mathbb{N}\), and thus the integrand has no singularity in the strip between the line \(C' = \{z = y+i(\tilde{\delta}+x'/t-\tilde{\nu}_j) : y \in \mathbb{R}\}\) and the real axis \(\mathbb{R}\) in \(\mathbb{C}\), \(1 \leq j \leq N\). Owing to the Gaussian factor \(p(t^{-1}, y|0)\), the integral on \(C'\) can be replaced by that over \(\mathbb{R}\). Then we can conclude that

\[
\hat{K}(x, x') = -\frac{1}{t} \frac{e^{x^2/2t}}{t e^{(x')^2/2t}} K^\nu_a(1/t; x/t, x'/t),
\]

where \(K^\nu_a\) is given by (1.13). By the multi-linearity and the cyclic property (the gauge invariance) of determinants,

\[
det_{1 \leq j, k \leq L} \hat{K}(x_j, x_k) = (-1)^L t^{-L} \det_{1 \leq j, k \leq L} [K^\nu_a(1/t; x_j/t, x_k/t)].
\]

For fixed \(t > 0, a > 0\), consider the integral operator in \(L^2(\mathbb{R})\) with the kernel (1.13). It can be regarded as the projection on the subspace \(\text{Span} \{p(t, \cdot |\tilde{\nu}_j) : 1 \leq j \leq N\}\), and has a domain given by \(\text{Span} \{\int_\mathbb{R} dy p(t, y|0) \Phi^\nu_a(\cdot + iy) : 1 \leq j \leq N\}\). As both subspaces have dimensions \(N, \det_{1 \leq j, k \leq L} [K^\nu_a(1/t; x_j/t, x_k/t)] = 0\) for \(L > N\). Then (1.16) is valid and the proof is completed. \(\blacksquare\)

4 Proof of Theorem 1.2

Let \(\chi(\cdot)\) be a real integrable function and consider the following integral; for \(N' \leq N, t \geq 0, a > 0\),

\[
I_{N'}[\chi] = \int_{\mathbb{R}^{N'}} dx \prod_{j=1}^{N'} \chi(x_j) \det_{1 \leq j, k \leq N'} [K^\nu_a(t; x_j, x_k)].
\]  

The determinant is defined using the notion of permutations and any permutation \(\sigma \in \mathfrak{S}_{N'}\) can be decomposed into a product of cycles. Let the number of cycles in the decomposition be \(\ell(\sigma)\) and express \(\sigma\) by \(\sigma = c_1 c_2 \ldots c_{\ell(\sigma)}\). Here \(c_\lambda\) denotes a cyclic permutation, \(1 \leq \lambda \leq \ell(\sigma)\), and if the size of a cycle is \(q_\lambda\), it is written as \(c_\lambda = (c_\lambda(1)c_\lambda(2)\ldots c_\lambda(q_\lambda))\), \(c_\lambda(j) \in \{1, 2, \ldots, N'\}\). By definition, we can assume the periodicity \(c_\lambda(j + q_\lambda) = c_\lambda(j), 1 \leq j \leq q_\lambda\). Then

\[
\det_{1 \leq j, k \leq N'} [K^\nu_a(t; x_j, x_k)] = \sum_{\sigma \in \mathfrak{S}_{N'}} (-1)^{N'-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} \prod_{j=1}^{q_\lambda} K^\nu_a(t; x_{c_\lambda(j)}, x_{c_\lambda(j+1)}),
\]

and (4.1) is written as

\[
I_{N'}[\chi] = \sum_{\sigma \in \mathfrak{S}_{N'}} (-1)^{N'-\ell(\sigma)} \prod_{\lambda=1}^{\ell(\sigma)} G[c_\lambda, \chi]
\]

with

\[
G[c_\lambda, \chi] = \int_{\mathbb{R}^{N'}} \prod_{j=1}^{q_\lambda} \left\{ dx_{c_\lambda(j)} \chi(x_{c_\lambda(j)}) K^\nu_a(t; x_{c_\lambda(j)}, x_{c_\lambda(j+1)}) \right\}.
\]
Now we write (1.13) as
\[
K^\nu_a_N(t, x, x') = \int_{\mathbb{R}} \nu(dv) \int_{\mathbb{R}} dy p(t, x|v) p(t, y|0) \Phi^\nu_a(x' + iy) \tag{4.3}
\]
with \( \nu(\cdot) = \sum_{j=1}^N \delta^\nu_j(\cdot) \), and rewrite (4.2) as
\[
G[c_{\lambda}, \chi] = \int_{\mathbb{R}^{q_{\lambda}}} \prod_{j=1}^{q_{\lambda}} dx_{c_{\lambda}(j)} \chi(x_{c_{\lambda}(j)}) \int_{\mathbb{R}} \nu(dv_{c_{\lambda}(j)}) \times \int_{\mathbb{R}} dy_{c_{\lambda}(j+1)} p(t, x_{c_{\lambda}(j)}|v_{c_{\lambda}(j)}) p(t, y_{c_{\lambda}(j+1)}|0) \Phi_{v_{c_{\lambda}(j)}, a}^\nu(x_{c_{\lambda}(j+1)} + iy_{c_{\lambda}(j+1)}) \tag{4.4}
\]
Here note that, when we applied (4.3) to each \( 1 \leq j \leq q_{\lambda} \), we labeled the integration variables as \( v \rightarrow v_{c_{\lambda}(j)} \) and \( y \rightarrow y_{c_{\lambda}(j+1)} \) corresponding to \( x = x_{c_{\lambda}(j)} \) and \( x' = x_{c_{\lambda}(j+1)} \), respectively. By Fubini’s theorem, (4.4) is equal to
\[
\int_{\mathbb{R}^{q_{\lambda}}} \prod_{j=1}^{q_{\lambda}} \nu(dv_{c_{\lambda}(j)}) \int_{\mathbb{R}^{q_{\lambda}}} \prod_{k=1}^{q_{\lambda}} \left\{ dx_{c_{\lambda}(k)} p(t, x_{c_{\lambda}(k)}|v_{c_{\lambda}(k)}) \chi(x_{c_{\lambda}(k)}) \right\}
\times \int_{\mathbb{R}^{q_{\lambda}}} \prod_{\ell=1}^{q_{\lambda}} \left\{ dy_{c_{\lambda}(\ell+1)} p(t, y_{c_{\lambda}(\ell+1)}|0) \Phi_{v_{c_{\lambda}(\ell+1)}, a}^\nu(x_{c_{\lambda}(\ell+1)} + iy_{c_{\lambda}(\ell+1)}) \right\}
\]
\[
= \mathbb{E}^p \left[ \prod_{k=1}^{q_{\lambda}} \chi(V_{c_{\lambda}(k)}(t)) \Phi_{v_{c_{\lambda}(k)}, a}^\nu(Z_{c_{\lambda}(k+1)}(t)) \right].
\]
Then (4.1) becomes
\[
I_{N'}[\chi] = \mathbb{E}^p \left[ \det_{1 \leq j, k \leq N'} \left[ \Phi_{v_{c_{\lambda}(k)}, a}(Z_{c_{\lambda}(k)}(t)) \chi(V_{c_{\lambda}(k)}(t)) \right] \right]. \tag{4.5}
\]
By the Fredholm expansion formula for determinant, we obtain the equality
\[
\sum_{N'=0}^N \frac{(-1)^{N'}}{N'!} I_{N'}[\chi] = \mathbb{E}^p \left[ \det_{1 \leq j, k \leq N} \left[ \delta_{jk} - \Phi_{v_{c_{\lambda}(k)}, a}(Z_{c_{\lambda}(k)}(t)) \chi(V_{c_{\lambda}(k)}(t)) \right] \right]. \tag{4.6}
\]
By setting \( \chi(\cdot) = 1_{(\cdot<h)}, h \in \mathbb{R} \) and performing the reciprocal time transform, the combination of (1.16) and (4.6) gives (1.22). Then the proof is completed.

\section*{Appendix}

\section*{A Sketch of a Non-Rigorous Derivation of (3.1)}

Here we provide a sketch of a non-rigorous approach to deriving Borodin and Corwin’s Theorem 4.1.40 [6]. We work with the Whittaker functions and use their orthogonality (2.3).
and recurrence relations (2.6) to compute moments $\mathbb{E}^{\nu,a}[(e^{-X^a(t)/a})^\kappa]$, $\kappa \in \mathbb{N}_0 \equiv \{0, 1, 2, \ldots\}$. This computation can be done rigorously as Section 4.1.4 of [6]. We then take a power series of these moments in an attempt to recover the Laplace transform of the distribution of $e^{-X^a(t)/a}$, $\mathbb{E}^{\nu,a}[\exp(-ue^{-X^a(t)/a})]$, $\Re u > 0$. This is the place where the derivation becomes non-rigorous because the power series is divergent for all values of $u$ and the moments do not identify the Laplace transform of the distribution. Nevertheless, proceeding formally and working with the divergent series we can recover the formula from Theorem 4.1.40 of [6].

Borodin and Corwin work at the higher level of $q$-Whittaker measures where the analogues of the moments are bounded by one and can be used to rigorously compute the $q$-deformed version of the Laplace transform of the distribution, which can be written as a Fredholm determinant. They then proved that the $q$-Whittaker measure converges weakly to the Whittaker measure, the $q$-deformed Laplace transform converges to the Laplace transform, and the Fredholm determinant has a limit which yields Theorem 4.1.40 of [6]. The fact that the formal calculations we describe actually recover the correct answer can be attributed to the fact that these are limits of the rigorous calculations done one level higher.

The expectation at a single time $t > 0$ given by (2.20) can be written as

$$\mathbb{E}^{\nu,a}[f(X^a(t))] = e^{-t|\nu|^2/2a^2} \int_{\mathbb{R}^N} dk e^{-t|k|^2/2s_N(a_k)} \int_{\mathbb{R}^N} dx f(x) \psi^{(N)}_{\nu}(x/a) \psi_{-ia_k}^{(N)}(x/a). \quad (A.1)$$

First we consider the case with $f(x) = e^{-x_1/a}$. By (2.6),

$$e^{-x_1/a} \psi_{\nu}^{(N)}(x/a) = \sum_{j=1}^N \prod_{1 \leq \ell \leq N; \ell \neq j} \frac{1}{\nu_\ell - \nu_j} \psi_i^{(N)}(\nu_\ell + i(e_{(j)}))(x/a), \quad (A.2)$$

and

$$\int_{\mathbb{R}^N} dx \ e^{-x_1/a} \psi_{\nu}^{(N)}(x/a) \psi_{-ia_k}^{(N)}(x/a)$$

$$= a^N \sum_{j=1}^N \prod_{1 \leq \ell \leq N; \ell \neq j} \frac{1}{\nu_\ell - \nu_j} \int_{\mathbb{R}^N} d\left(\frac{x}{a}\right) \psi_i^{(N)}(-ia_k)(x/a) \psi_i^{(N)}(\nu_\ell + i(e_{(j)}))(x/a)$$

$$= a^N \sum_{j=1}^N \prod_{1 \leq \ell \leq N; \ell \neq j} \frac{1}{\nu_\ell - \nu_j} s_N(a_k) N! \sum_{\sigma \in S_N} \delta(a_k - \sigma(-i\nu + ie_{(j)})),$$

where we used the orthogonality relation (2.3) extended to complex indices as in Section 4.1.4 of [6]. Then (A.1) gives

$$\mathbb{E}^{\nu,a}[e^{-X^a(t)/a}] = e^{-t|\nu|^2/2a^2} \sum_{j=1}^N \prod_{1 \leq \ell \leq N; \ell \neq j} \frac{1}{\nu_\ell - \nu_j} \times \frac{1}{N!} \sum_{\sigma \in S_N} \exp \left\{ -\frac{t}{2a^2} \sum_{p=1}^N (-i\nu_{\sigma(p)} + i(e_{(j)})_{\sigma(p)})^2 \right\}.$$
We can see
\[
\frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \exp \left\{ -\frac{t}{2a^2} \sum_{p=1}^{N} \left( -i \nu_{\sigma(p)} + i (e_{\{j\}})_{\sigma(p)} \right)^2 \right\} = e^{i \nu^2/2a^2 - \nu_j/a^2 + t/2a^2}
\]
for \( \nu \in \mathbb{R}^N \). Then, if we set
\[
f_N^{\nu,t,a}(v) = e^{tv/a^2} \prod_{\ell=1}^{N} \frac{1}{\nu + v + \nu_{\ell}},
\]
we have the expression
\[
\mathbb{P}^{\nu,a}[e^{-X_1^2(t)/a}] = e^{t/2a^2} \int_{\mathcal{N}(v)} \frac{dv}{2\pi i} f_N^{\nu,t,a}(v) = e^{t/2a^2} \int_{\mathcal{N}(v)} \frac{dv}{2\pi i} v + 1 - v f_N^{\nu,t,a}(v), \quad t \geq 0. \tag{A.4}
\]

Next we consider (A.1) in the case \( f(x) = e^{-x_1/a} \). By (A.2),
\[
e^{-x_1/a} \psi^{(N)}(\nu)(x/a) = (e^{-x_1/a})^2 \psi^{(N)}(\nu)(x/a) = \sum_{j_1=1}^{N} \prod_{1 \leq \ell_1 \leq N: \ell_1 \neq j_1} \frac{1}{\nu_{\ell_1} - \nu_{j_1}} e^{-x_1/a} \psi^{(N)}(\nu - e_{\{j_1\}} - e_{\{j_2\}}) (x/a).
\]

Applying the recurrence relation (2.6), it becomes
\[
\sum_{j_1=1}^{N} \prod_{1 \leq \ell_1 \leq N: \ell_1 \neq j_1} \frac{1}{\nu_{\ell_1} - \nu_{j_1}} \sum_{j_2=1}^{N} \prod_{1 \leq \ell_2 \leq N: \ell_2 \neq j_2} \frac{1}{\nu_{\ell_2} - \nu_{j_2}} \psi^{(N)}(\nu - e_{\{j_1\}} - e_{\{j_2\}}) (x/a) = \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \prod_{1 \leq \ell_1 \leq N: \ell_1 \neq j_1} \frac{1}{\nu_{\ell_1} - \nu_{j_1}} \prod_{1 \leq \ell_2 \leq N: \ell_2 \neq j_2} \frac{1}{\nu_{\ell_2} - \nu_{j_2}} \psi^{(N)}(\nu - e_{\{j_1\}} - e_{\{j_2\}}) (x/a)
\]
\[
+ \sum_{j_1=1}^{N} \prod_{1 \leq \ell_1 \leq N: \ell_1 \neq j_1} \frac{1}{\nu_{\ell_1} - \nu_{j_1}} \prod_{1 \leq \ell_2 \leq N: \ell_2 \neq j_1} \frac{1}{\nu_{\ell_2} - \nu_{j_1} + 1} \psi^{(N)}(\nu - 2e_{\{j_1\}}) (x/a).
\]

Moreover, it is rewritten as
\[
\sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \frac{\nu_{j_2} - \nu_{j_1}}{\nu_{j_2} - \nu_{j_1} + 1} \prod_{r=1}^{2} \prod_{1 \leq \ell_r \leq N: \ell_r \neq j_r} \frac{1}{\nu_{\ell_r} - \nu_{j_r}} \psi^{(N)}_{i(-i\nu + i e_{\{j_1\}} + i e_{\{j_2\}})} (x/a)
\]
\[
+ \sum_{j_1=1}^{N} \prod_{1 \leq \ell_1 \leq N: \ell_1 \neq j_1} \frac{1}{\nu_{\ell_1} - \nu_{j_1}} \prod_{1 \leq \ell_2 \leq N} \frac{1}{\nu_{\ell_2} + 1 - \nu_{j_1}} \psi^{(N)}_{i(-i\nu + 2i e_{\{j_1\}})} (x/a).
\]
Then, by using the orthogonality relation (2.3) and following the similar procedure to the first case, we have

\[
E_{\nu,a}^\nu [e^{-2X_\nu^a(t)/a}]
= e^{2t/2a^2} \left[ \sum_{j_1=1}^N \sum_{j_2=1}^N \frac{\nu_{j_2} - \nu_{j_1}}{2} \prod_{r=1}^{2} \frac{1}{1-\nu_{j_r} + \nu_{\ell_r}} \right. \\
+ \sum_{j_1=1}^N \left. \left\{ e^{t(-\nu_{j_1})/a^2} \prod_{1 \leq \ell \leq N: \ell \neq j_1} \frac{1}{1-\nu_{j_1} + \nu_{\ell}} \right\} \left( e^{t(-\nu_{j_1}+1)/a^2} \prod_{1 \leq \ell \leq N} \frac{1}{1-\nu_{j_1} + \nu_{2}} \right) \right].
\]

Here we consider a determinant of a matrix of size two

\[
\det_{1 \leq j, \ell \leq 2} \left[ \frac{1}{v_j + 1 - v_\ell} \right] = \begin{vmatrix} 1 / (v_2 + 1 - v_1) & 1/(v_1 + 1 - v_2) \\ 1 & 1 \end{vmatrix} = -\frac{(v_1 - v_2)^2}{1 - (v_1 - v_2)^2},
\]

which is equal to the symmetrization of \((v_2 - v_1)/(v_2 - v_1 + 1)\) with respect to indices \(j \in \{1, 2\}\) of \(v_j\)’s,

\[
\frac{1}{2} \left[ \frac{v_2 - v_1}{v_2 - v_1 + 1} + \frac{v_1 - v_2}{v_1 - v_2 + 1} \right].
\]

Then we obtain the expression

\[
\frac{1}{2} E_{\nu,a}^\nu [e^{-2X_\nu^a(t)/a}] = e^{2t/2a^2} \left[ \frac{1}{2} \prod_{r=1}^{2} \int_{C(-\nu)} \frac{dv_r}{2\pi i} \frac{1}{1-v_j + v_\ell} \int_{1 \leq j, \ell \leq 2} \left[ \prod_{r=1}^{2} f_N^{\nu,t,a}(v_r) \right. \\
+ \int_{C(-\nu)} \frac{dv}{2\pi i} \frac{1}{v+1} f_N^{\nu,t,a}(v) f_N^{\nu,t,a}(v+1) \right].
\]

By the similar calculation with the orthogonality relation (2.3) and the recurrence relation (2.6) of the Whittaker functions using the symmetrization identity

\[
\frac{1}{\kappa!} \sum_{\sigma \in \Sigma_{\kappa}} \prod_{1 \leq p < q \leq \kappa} \frac{v_{\sigma(q)} - v_{\sigma(p)}}{v_{\sigma(q)} - v_{\sigma(p)} + 1} = \frac{1}{1 \leq j, k \leq \kappa} \left[ \frac{1}{v_j + 1 - v_\ell} \right],
\]

one can prove the following. For any \(\kappa \in \mathbb{N}\)

\[
\frac{1}{\kappa!} E_{\nu,a}^\nu [e^{-\kappa X_\nu^a(t)/a}] = \frac{1}{\kappa!} \prod_{\lambda | | \lambda | = \kappa} \left[ \prod_{r=1}^{l(\lambda)} \int_{C(-\nu)} \frac{dv_r}{2\pi i} \frac{1}{1 \leq j, k \leq \lambda} \left[ \frac{1}{v_j + 1 - v_\ell} \right] \right. \\
\times \left. \prod_{j=1}^{l(\lambda)} \left\{ f_N^{\nu,t,a}(v_j) f_N^{\nu,t,a}(v_j+1) \cdots f_N^{\nu,t,a}(v_j + \lambda_j - 1) \right\} \right],
\]

(A.8)
where the summation is over all partitions
\[ \lambda = (\lambda_1, \lambda_2, \ldots) = 1^{m_1}2^{m_2} \ldots, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq 0, \quad m_j \in \mathbb{N}_0, \quad j \geq 1 \]
conditioned that \(|\lambda| \equiv \sum_{j \geq 1} \lambda_j = \kappa\). Here \(l(\lambda)\) denotes the length of \(\lambda\). (Precisely speaking, by using the orthogonality relation (2.3) and the recurrence relation (2.6) of the Whittaker functions, Borodin and Corwin gave a multiple contour-integral representation for general moment, \(E^{\nu,a}[(e^{-X^a_i(t)/a})^\kappa], \kappa \in \mathbb{N}\), in Lemma 4.1.29 in [6]. This integral formula involves nested contours. Then by deforming them to all be a small circle, denoted here by \(C(-\nu)\), Borodin and Corwin derived the formula (A.8) (Proposition 6.2.7 in [6]), in which the identity (A.7) was used.)

One can rewrite (A.8) in a suggestive form as
\[
\sum_{L=0}^{\infty} \frac{1}{L!} \sum_{n=(n_1,n_2,\ldots,n_L) \in \mathbb{N}^L, \sum_{j=1}^L n_j = \kappa} \prod_{r=1}^{L} \int_{C(-\nu)} \frac{dv_r}{2\pi i} \left\{ \frac{e^{t/2a^2}}{v_j + n_j - v_k} \right\}^{n_r} \times \det_{1 \leq j,k \leq L} \left[ \frac{1}{v_j + n_j - v_k} \right]^{L} \left\{ f_{N}^{\nu,t,a}(v_j)f_{N}^{\nu,t,a}(v_j + 1) \cdots f_{N}^{\nu,t,a}(v_j + n_j - 1) \right\}.
\]

For \(\Re u > 0\) one would like to recover the Laplace transform of the distribution of \(e^{-X^a_i(t)/a}\) from the moments via
\[
\sum_{\kappa=0}^{\infty} \frac{(-u)^\kappa}{\kappa!} E^{\nu,a}[(e^{-X^a_i(t)/a})^\kappa] = E^{\nu,a}[\exp(-ue^{-X^a_i(t)/a})].
\]

One checks from (A.8) that the moments grow super-exponentially, so this interchange of expectation and summation is unjustifiable and constitutes the physics ‘replica trick’. Nevertheless we proceed now complete formally. By reordering terms in an unbounded manner, we arrive at the formula
\[
E^{\nu,a}[\exp(-ue^{-X^a_i(t)/a})] = \sum_{L=0}^{\infty} \frac{1}{L!} \sum_{n=(n_1,n_2,\ldots,n_L) \in \mathbb{N}^L} \prod_{r=1}^{L} \int_{C(-\nu)} \frac{dv_r}{2\pi i} \left( \sum_{j=1}^{L} \frac{e^{nt/2a^2}}{v_j + n_j - v_k} (-u)^{n_j} f_{N}^{\nu,t,a}(v_j)f_{N}^{\nu,t,a}(v_j + 1) \cdots f_{N}^{\nu,t,a}(v_j + n_j - 1) \right) \times \det_{1 \leq j,k \leq L} \left[ \frac{1}{v_j + n_j - v_k} \right]^{L} \left\{ f_{N}^{\nu,t,a}(v_j)f_{N}^{\nu,t,a}(v_j + 1) \cdots f_{N}^{\nu,t,a}(v_j + n_j - 1) \right\},
\]
which was given as (3.1) in the text, where
\[
K_u(v, v') = \sum_{n=1}^{\infty} \frac{e^{nt/2a^2}}{v + n - v'} (-u)^n f_{N}^{\nu,t,a}(v)f_{N}^{\nu,t,a}(v + 1) \cdots f_{N}^{\nu,t,a}(v + n - 1).
\]
By (A.3),
\begin{equation}
    f_N^{\nu,a}(v) f_N^{\nu,a}(v+1) \cdots f_N^{\nu,a}(v+n-1) = e^{tn/a^2+tn^2/2a^2} \prod_{\ell=1}^{N} \frac{\Gamma(v+\nu_\ell)}{\Gamma(n+v+\nu_\ell)}.
\end{equation}

Then (A.10) is equal to
\begin{equation}
    K_u(v,v') = \sum_{n\in\mathbb{N}} (-1)^n \prod_{\ell=1}^{N} \frac{\Gamma(v+\nu_\ell)}{\Gamma(n+v+\nu_\ell)} \frac{u^n e^{tn/a^2+tn^2/2a^2}}{v+n-v'}.
\end{equation}

Since $\Gamma(-s)\Gamma(1+s) = -\pi/\sin(\pi s)$ by Euler’s reflection formula and since $-\pi/\sin(\pi s)$ has simple poles at $s = n \in \mathbb{Z}$ with residues $(-1)^n$, (A.11) can be reexpressed as (1.11).

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References

[1] Auffinger, A., Baik, J., Corwin, I.: Universality for directed polymers in thin rectangles. arXiv:math.PR/1204.4445

[2] Baudoin, F., O’Connell, N.: Exponential functionals of Brownian motion and class-one Whittaker functions. Ann. Inst. H. Poincaré, B 47, 1096-1120 (2011)

[3] Biane, P., Bougerol, P., O’Connell, N.: Littelmann paths and Brownian paths. Duke Math. J. 130, 127-167 (2005)

[4] Biane, P., Bougerol, P., O’Connell, N.: Continuous crystal and Duistermaat-Heckman measure for Coxeter groups. Adv. Math. 221, 1522-1583 (2009)

[5] Bleher, P. M., Kuijlaars, A. B. J.: Integral representations for multiple Hermite and multiple Laguerre polynomials. Ann. Inst. Fourier. 55, 2001-2014 (2005)

[6] Borodin, A., Corwin, I.: Macdonald processes. arXiv:math.PR/1111.4408

[7] Borodin, A., Corwin, I., Ferrari, P.: Free energy fluctuations for directed polymers in random media in 1+1 dimension. arXiv:math.PR/1204.1024

[8] Borodin, A., Corwin, I., Sasamoto, T.: From duality to determinants for $q$-TASEP and ASEP. arXiv:math.PR/1207.5035

[9] Borodin, A., Rains, E. M.: Eynard-Mehta theorem, Schur process, and their pfaffian analogs, J. Stat. Phys. 121, 291-317 (2005)
[10] Corwin, I., O'Connell, N., Seppäläinen, T., Zygouras, N.: Tropical combinatorics and Whittaker functions. arXiv:math.PR/1110.3489

[11] Givental, A.: Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture. In: Topics in Singular Theory, AMS Trans. Ser. 2, vol. 180, pp.103-115, AMS, Rhode Island (1997)

[12] Katori, M.: O'Connell’s process as a vicious Brownian motion. Phys. Rev. E 84, 061144 (2011)

[13] Katori, M.: Survival probability of mutually killing Brownian motion and the O'Connell process. J. Stat. Phys. 147, 206-223 (2012)

[14] Katori, M.: Reciprocal time relation of noncolliding Brownian motion with drift. J. Stat. Phys. 148, 38-52 (2012).

[15] Katori, M., Tanemura, H.: Noncolliding Brownian motion and determinantal processes. J. Stat. Phys. 129, 1233-1277 (2007)

[16] Katori, M., Tanemura, H.: Non-equilibrium dynamics of Dyson’s model with an infinite number of particles. Commun. Math. Phys. 293, 469-497 (2010)

[17] Katori, M., Tanemura, H.: Complex Brownian motion representation of the Dyson model. arXiv:math.PR/1008.2821

[18] Kharchev, S., Lebedev, D.: Integral representations for the eigenfunctions of quantum open and periodic Toda chains from the QISM formalism. J. Phys. A: Math. Gen. 34, 2247-2258 (2001)

[19] Macdonald, I. G.: Symmetric Functions and Hall Polynomials. 2nd ed. Oxford University Press, New York (1999)

[20] Matsumoto, H., Yor, M.: An analogue of Pitman’s $2M - X$ theorem for exponential Wiener functionals, Part I: A time-inversion approach. Nagoya Math. J. 159, 125-166 (2000)

[21] Matsumoto, H., Yor, M.: Exponential functionals of Brownian motion I: Probability laws at fixed time. Probab. Surveys 2, 312-347 (2005)

[22] O'Connell, N.: Directed polymers and the quantum Toda lattice. Ann. Probab. 40, 437-458 (2012)

[23] O'Connell, N.: Whittaker functions and related stochastic processes. arXiv:math.PR/1201.4849

[24] Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion. 3rd ed., Springer, New York, (2005)
[25] Semenov-Tian-Shansky, M. A.: Quantization of open Toda lattices. In: Dynamical Systems VII: Integrable Systems, Nonholonomic Dynamical Systems. Edited by V. I. Arnol’d and S. P. Novikov. Encyclopaedia of Mathematical Sciences, vol.16. pp.226-259, Springer, Berlin (1994)

[26] Shirai, T., Takahashi, Y.: Random point fields associated with certain Fredholm determinants I: fermion, Poisson and boson point process. J. Funct. Anal. 205, 414-463 (2003)

[27] Sklyanin, E. K.: The quantum Toda chain. In: Non-linear Equations in Classical and Quantum Field Theory, Lect. Notes in Physics, 226, pp. 195-233, Springer, Berlin (1985)

[28] Soshnikov, A.: Determinantal random point fields. Russian Math. Surveys 55, 923-975 (2000)

[29] Wallach, N. R.: Real Reductive Groups II. Academic Press, San Diego CA, (1992)