On the Identifiability of Finite Mixtures of Finite Product Measures

Behrooz Tahmasebi, Seyed Abolfazl Motahari, and Mohammad Ali Maddah-Ali

Abstract

The problem of identifiability of finite mixtures of finite product measures is studied. A mixture model with \( K \) mixture components and \( L \) observed variables is considered, where each variable takes its value in a finite set with cardinality \( M \). The variables are independent in each mixture component. The identifiability of a mixture model means the possibility of attaining the mixture components parameters by observing its mixture distribution. In this paper, we investigate fundamental relations between the identifiability of mixture models and the separability of their observed variables by introducing two types of separability: strongly and weakly separable variables. Roughly speaking, a variable is said to be separable, if and only if it has some differences among its probability distributions in different mixture components. We prove that mixture models are identifiable if the number of strongly separable variables is greater than or equal to \( 2K - 1 \), independent form \( M \). This fundamental threshold is shown to be tight, where a family of non-identifiable mixture models with less than \( 2K - 1 \) strongly separable variables is provided. We also show that mixture models are identifiable if they have at least \( 2K \) weakly separable variables. To prove these theorems, we introduce a particular polynomial, called characteristic polynomial, which translates the identifiability conditions to identity of polynomials and allows us to construct an inductive proof.

Index terms—Identifiability, mixture models, characteristic polynomial, separable variables.

1 Introduction

Statistical analysis of samples from mixtures of subpopulations is usually carried out by assuming that the underlying probability law is governed by a mixture model. There exists a large number of applications where mixture models are central part of the data analyses [2–5]. An important one is in population genetics, where the mixed population datasets are modeled by finite mixture models [4]. A finite mixture model can be represented by

\[
\mathbb{P}_\theta = \sum_{k=1}^{K} w_k \mathbb{P}_k,
\]

where \( \mathbb{P}_k \) and \( w_k \) are, respectively, the probability law governing and the relative population size of the \( k^{th} \) subpopulation. \( \theta \) encapsulates all the latent parameters used to specify the model including \( w_k \)s.

In a common setting, a mixture model with latent parameters is assumed to be the law governing a given dataset. Then, latent parameters are estimated by various methods such as maximum likelihood.

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This paper has been presented in part at IEEE International Symposium on Information Theory (ISIT) 2018 [1].
estimation. However, before starting the parameter estimation one needs to answer a very important and fundamental question: is the model identifiable? Identifiability means that there is a one to one map between the latent parameter θ and the probability law \( P_\theta \) (up to permutations of the subpopulations which will be discussed later.) [6–9].

In this paper, we consider finite mixture models with \( L \) observed variables which have the conditional independence property. This means that \( L \) variables are distributed in each mixture component according to a product probability measure, i.e., \( P_k = \bigotimes_{\ell \in \{L\}} P_{k\ell} \) for any \( k \in [K] \) (see Section 2 for more explanations). In addition, we assume that the observed variables are discrete random variables and take values in a finite set with cardinality \( M \). This finite set is also refereed to as the state space in the literature. This class of mixture models are called finite mixtures of finite product measures. In this paper, by mixture models we mean this class of distributions.

Studying identifiability of statistical models has a long history [6–9]. For finite mixtures of finite product measures the problem has been studied in various articles (see for instance [10, 11]). It is well known that there are some Bernoulli Mixture Models (BMMs) which are not identifiable [12, 13]. We note that, as mentioned in [11], only counting the dimensionality of the set of the latent parameters and then comparing it to the dimension of the mixture distribution is not sufficient to establish an identifiability argument. A novel method has been used in [10], based on the seminal result of Kruskal [14,15], where the authors show that the parameters of a finite mixture of finite product measures are generically identifiable if \( L \geq 2\lceil \log M(K) \rceil + 1 \). This means that if \( L \geq 2\lceil \log M(K) \rceil + 1 \), then the measure of the parameters that make the problem non-identifiable is zero.

In many applications, such as population genetics [4], after modeling the drawn dataset by mixture models, the next step is to cluster the given dataset into subpopulations (called population stratification in the population genetics literature). The key observation is that many observed variables are not useful for clustering, which means their probability distributions (pmfs) are (exactly) same in the different mixture components. This means the set of possible parameters for this type of mixture models has a zero measure. On the other side, a small fraction of the observed variables are useful for clustering, which means that they have enough separation among their pmfs to make the reliable clustering possible [16].

A fundamental (and also theoretical) question is that what is the number of required separable variables for a mixture model to be identifiable? We note that, for this case, we can not use the result of generic identifiability because the parameters are in a zero measure set. In this paper we establish connections between identifiability of the parameters of a mixture model and the separability of its variables. In this way, two notions of separability are introduced: strongly and weakly separable variables, which will be defined in the next two paragraphs. We note that our studies in this paper can be taught as a worst-case analysis of the identifiability of the finite mixtures of finite product measures.

In a strongly separable variable, the probability distributions (pmfs) of the variable in the \( K \) mixture components are strictly different from each other, for each element of the state space. This is a natural definition for good variables, as the strongly separable variables are essentially useful for clustering the datasets drawn by mixture models. In [16], a mathematical analysis of the problem of reliable clustering of datasets drawn by mixture models based on the separability of mixture components is presented. However, the identifiability of mixture models based on the separability conditions is not explored in [16]. We establish a condition on the number of strongly separable variables that results the identifiability of the parameters of the corresponding mixture model. In particular, we find a sharp threshold that the identifiability is guaranteed, if the number of strongly separable variables is greater than or equal to it. The threshold is \( 2K - 1 \), where \( K \) is the number of mixture components. We note that this threshold is independent of \( M \), the size of the state space of the variables, which is in contrast to the result of the generic identifiability. Also we show that this threshold is tight by introducing a family of non-identifiable
mixture models with less than $2K - 1$ strongly separable variables.

We note that for a strongly separable variable, the number of conditions which are needed to be satisfied scales by $M$. To study the effect of this scaling on identifiability we introduce weakly separable variables. In a weakly separable variable, the probability distributions (pmfs) of the variable in the $K$ mixture components are strictly different from each other, for at least one (not necessarily all) state space element. This means that for any weakly separable variable, the number of required conditions does not scale with $M$. Observe that by definitions any strongly separable variable is also weakly separable. Therefore, naturally we expect that the number of required weakly separable variables for the identifiability would be very larger than $2K - 1$, due to the order-wise fewer number of conditions. However, we prove that if a mixture model has at least $2K$ weakly separable variables, then it is identifiable. This means that the penalty of considering weakly separable variables instead of strongly separable variables is at most one.

We notice that the threshold of $2K - 1$ is also observed in the identifiability of the other problems, like the Latent Block Models (LBMs) [17], the binomial mixture models [18], the mixture models from grouped samples [19, 20] and also the topic modeling problem [21]. However, the model of this paper is essentially different from them and their proofs ideas are also very different from what we develop in this paper. For the case of binomial mixture models, the result of $2K - 1$ can be followed as a special case of our results, when the mixture components have i.i.d. variables.

To prove the sufficiencies in both $2K - 1$ strongly and $2K$ weakly separable variables, we introduce a multi-variable polynomial, called the characteristic polynomial representing a mixture model. Then, we prove that the identifiability of the mixture model is equivalent to the identifiability of its characteristic polynomial in the polynomials space. This allows us to exploit properties of polynomials to proof the identifiability argument. For the converse, we introduce a family of non-identifiable mixture models that have less than $2K - 1$ strongly separable variables for arbitrary $K, L$ and $M$.

The rest of the paper is organized as follows. In Section 2, we define the problem. In Section 3, we describe the main results of the paper. Finally, Sections 4 and 5 contain the proofs of the main theorems of the paper.

**Notation.** In this paper all vectors are columnar and they are denoted by bold letters, like $\mathbf{x}$. For any positive integer $N$, we define $[N] := \{1, 2, \ldots, N\}$. The transpose operator is denoted by $(\cdot)^T$. The polynomial identity is denoted by $\equiv$ which is used when two polynomials have the same coefficients. We use the notation $\mathbf{x} \otimes \mathbf{y}$ for the Kronecker product of two vectors $\mathbf{x}$ and $\mathbf{y}$. For any vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$, we define $\|\mathbf{x}\|_0 := \sum_{i=1}^n 1\{x_i \neq 0\}$. Let us define

$$\Delta_M = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_M)^T \in \mathbb{R}^M \bigg| \sum_{m=1}^M x_m = 1, \forall m \in [M] : x_m > 0 \right\}.$$  

Also $\overline{\Delta}_M$ denotes the closure of $\Delta_M$.

## 2 Problem Statement

We consider a finite mixture model with $K$ mixture components and $L$ observed variables where each observed variable takes its value in a finite set (finite state space) with cardinality $M$. For any mixture component, we have a generative model that the observed variables attain their realizations based on that. In this model, the $\ell^{th}$ variable in the $k^{th}$ mixture component is generated from a finite probability measure denoted by $\mathbf{f}_{kl} \in \overline{\Delta}_M$ (we call $\mathbf{f}_{kl}$ the frequency vector of the $\ell^{th}$ variable in the $k^{th}$ mixture component). We assume the conditional independence structure in the mixture model, which means
that the observed variables are independent in each mixture component. Let us denote the frequency of the $\ell$th variable, in the $k$th mixture component, for any $m \in [M]$, by $f_{k\ell m}$, as it is also denoted by $f_{k\ell}(m)$. We denote the collection of the frequencies specifically for the $k$th mixture component by a matrix $F_k = [f_{k1}, f_{k2}, \ldots | f_{kL}]^T = (f_{k\ell m})_{L \times M} \in [0, 1]^{L \times M}$. Let $F_{1:K} := (F_1, F_2, \ldots, F_K)$ denotes the collection of the frequencies in the mixture model in a specific order.

In our model, each data instance is generated from one of the mixture components according to a sampling distribution with pmf $w = (w_1, w_2, \ldots, w_K)^T \in \Delta_K$. Let us denote the latent parameters of the mixture model by $\theta = (F_{1:K}; w)$. We also denote the set of all possible latent parameters of the mixture models (as defined above) by $\Theta_{K,L,M}$. Also let $\overline{\Theta}_{K,L,M}$ denotes the closure of $\Theta_{K,L,M}$. Here, $\theta \in \Theta_{K,L,M} \subseteq \overline{\Theta}_{K,L,M}$.

In this paper, our objective is to attain the latent parameters $\theta = (F_{1:K}; w)$, using the mixture distribution. The formal definition of the mixture distribution is as follows.

**Definition 1.** The mixture distribution $f \in \overline{\Delta}_{M,L}$ of a mixture model with latent parameters $\theta = (F_{1:K}; w)$ is defined as

$$f := \sum_{k=1}^{K} \theta_k \times (f_{k1} \otimes f_{k2} \otimes \cdots \otimes f_{kL}).$$

In other words, for any $(m_1, m_2, \ldots, m_L) \in [M]^L$, we define

$$f(m_1, m_2, \ldots, m_L) := \sum_{k=1}^{K} \theta_k \prod_{\ell=1}^{L} f_{k\ell}(m_\ell).$$

Here $f$ is the vector containing all $f(m_1, m_2, \ldots, m_L)$ for any $(m_1, m_2, \ldots, m_L) \in [M]^L$ in a specific order.

In the following definitions, we define the notion of the identifiability of a mixture model.

**Definition 2.** For any $(F_{1:K}; w) \in \overline{\Theta}_{K,L,M}$ and any permutation $\pi$ on $[K]$, we define $(F_{1:K;}^\pi; w^\pi) \in \overline{\Theta}_{K,L,M}$ as follows.

$$F_{1:K;}^\pi := (F_{\pi(1)}, F_{\pi(2)}, \ldots, F_{\pi(K)}),$$

and

$$w^\pi := (w_{\pi(1)}, w_{\pi(2)}, \ldots, w_{\pi(K)})^T.$$

**Definition 3.** For any $(F_{1:K}; w), (G_{1:K}; z) \in \overline{\Theta}_{K,L,M}$, if there exists a permutation $\pi$ on $[K]$ such that $(F_{1:K}; w) = (G_{1:K;}^\pi; z^\pi)$, then we write $(F_{1:K}; w) \approx (G_{1:K}; z)$.

**Definition 4.** A mixture model with latent parameters $\theta = (F_{1:K}; w) \in \Theta_{K,L,M}$ and mixture distribution $f$ is said to be identifiable, iff for any $(G_{1:K}; z) \in \overline{\Theta}_{K,L,M}$ with mixture distribution $g$, such that $f = g$ (they have the same mixture distributions), we have $F_{1:K} \approx (G_{1:K}; z)$. In other words, a mixture model is identifiable, iff its latent parameters $\theta = (F_{1:K}; w)$ can be identified from its mixture distribution $f$, up to a label swapping.

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1 The main difference of $\Theta_{K,L,M}$ and $\overline{\Theta}_{K,L,M}$ is that in the first, the probability of sampling from each mixture component is positive, but in the second, it may be zero.
For the identifiability of a mixture model it is essential that the mixture components separate in some measures. In this paper, first we define the notion of strongly separable variables as the measure of difference among mixture components. In our definition, the \( \ell \)th variable is said to be strongly separable, if and only if the frequencies of it are different for any distinct mixture components and also for any state space element. In what follows, we mathematically define the notion of the strongly separable variables of a mixture model.

**Definition 5.** Consider a mixture model with latent parameters \( \theta = (F_{1:K}; w) \in \Theta_{K,L,M} \). In this mixture model, the \( \ell \)th variable is said to be strongly separable, iff for any \( m \in [M] \) and any distinct \( k_1, k_2 \in [K] \), we have \( f_{k_1m} \neq f_{k_2m} \). In other words, the \( \ell \)th variable is strongly separable, iff for any \( k_1, k_2 \in [K] \) we have \( \|f_{k_1\ell} - f_{k_2\ell}\|_0 = M \). Also the number of strongly separable variables of a mixture model with latent parameters \( \theta = (F_{1:K}; w) \) is denoted by \( \mathcal{L}_s(F_{1:K}; w) \).

**Remark 1.** We note that strongly separable variables are global. In other words, in a strongly separable variable, any pair of mixture components have different frequencies.

We note that the number of conditions that must be satisfied for the strong separability of a variable scales with the state space size \( M \). For studying the effect of this scaling on identifiability, we define weakly separable variables in this paper. The definition of the weakly separable variables is as follows.

**Definition 6.** Consider a mixture model with latent parameters \( \theta = (F_{1:K}; w) \in \Theta_{K,L,M} \). In this mixture model, the \( \ell \)th variable is said to be weakly separable, iff there is an \( m \in [M] \), such that for any distinct \( k_1, k_2 \in [K] \) we have \( f_{k_1m} \neq f_{k_2m} \). Also the number of weakly separable variables of a mixture model with latent parameters \( \theta = (F_{1:K}; w) \) is denoted by \( \mathcal{L}_w(F_{1:K}; w) \).

**Remark 2.** Observe that any strongly separable variable is also weakly separable. This means that we have the inequality \( \mathcal{L}_w(F_{1:K}; w) \geq \mathcal{L}_s(F_{1:K}; w) \) for any latent parameters \( (F_{1:K}; w) \in \Theta_{K,L,M} \). Specially for the case of binary state spaces (\( M = 2 \)) the definitions of the weakly separable variables and the strongly separable variables are same, resulting \( \mathcal{L}_w(F_{1:K}; w) = \mathcal{L}_s(F_{1:K}; w) \) for any \( (F_{1:K}; w) \in \Theta_{K,L,2} \).

In the next section, we analyze the problem of identifiability of the mixture models and its relation to the number of separable variables (strong or weak). In particular, we show that there is a sharp threshold on the number of separable variables (strongly or weak), that implies the identifiability of the corresponded mixture model.

### 3 Main Results

The main result of this paper for the strongly separable variables is summarized in the following theorem.

**Theorem 1.** (Necessary and sufficient condition for the identifiability based on the strongly separable variables) Any mixture model with latent variables \( \theta = (F_{1:K}; w) \in \Theta_{K,L,M} \), such that \( \mathcal{L}_s(F_{1:K}; w) \geq 2K - 1 \), is identifiable. Conversely, for any \( \mathcal{L}_s(F_{1:K}; w) \leq 2K - 2 \), there is a mixture model with latent variables \( \theta = (F_{1:K}; w) \in \Theta_{K,L,M} \), which has \( \mathcal{L}_s(F_{1:K}; w) \) strongly separable variables and is not identifiable.

**Remark 3.** The theorem states that if \( \mathcal{L}_s(F_{1:K}; w) \geq 2K - 1 \) then the identifiability of the parameters from the mixture distribution is guaranteed. On the other hand, if \( \mathcal{L}_s(F_{1:K}; w) \leq 2K - 2 \) there is no guarantee that the problem is identifiable. Hence, in the worst-case analysis, the identifiability is possible if and only if we have at least \( 2K - 1 \) strongly separable variables.

\[ \text{Note that in this case, for any distinct } k_1, k_2 \in [K], \text{ we have } \|f_{k_1\ell} - f_{k_2\ell}\|_0 \geq 1. \]
Remark 4. Our threshold depends only on the number of mixture components $K$, and it is independent of $M$. Naturally, we expect that our threshold varies with the size of the state space, similar to the generic identifiability result, which is $2\lceil \log^*_M(K) \rceil + 1$. Also, the threshold is $\Theta(K)$, while in the generic identifiability, the threshold is $\Theta(\log^*_M(K))$. This means that by relying on strongly separable variables order-wise more variables are required in comparison with generic identifiability.

Remark 5. For the proof of the theorem, first we introduce a multi-variable polynomial which is made up by the parameters of the problem. We show that the identifiability of a mixture model follows by the identifiability of its characteristic polynomial in the class of polynomials. Then, by exploiting the properties of the polynomials, we prove the sufficiency part of the theorem. For the necessary part, we introduce a family of non-identifiable mixture models that shows the necessity of the condition in the worst-case regime.

The result for the weakly separable variables is also provided in the next theorem.

**Theorem 2.** (Sufficient condition for the identifiability based on the weakly separable variables) Any mixture model with latent variables $\theta = (F_{1,K}; w) \in \Theta_{K,L,M}$ such that $\mathcal{L}_w(F_{1,K}; w) \geq 2K$ is identifiable.

Remark 6. The theorem states that although the weakly separable variables satisfy order-wise less conditions, but to use them, it suffices to have just one extra variable, in comparison with the strong separability measure.

Remark 7. It is easy to see that in a more general model, if the observed variables have state spaces with (possibly) different sizes denoted by $\{M_\ell\}_\ell \in [L]$, then the notion of weakly separable variables can be defined for them similarly and also the sufficiency of $2K$ weakly separable variables for the identifiability of them holds. For this matter, it just suffices to set $M = \max_{\ell \in [L]} M_\ell$ and then consider each observed variable as an instance with state space of cardinality $M$, by setting $M - M_\ell$ frequencies to be zero in the $\ell^{th}$ variable.

The proofs of the theorems are available in the next two sections.

**4 Proof of Theorem 1**

The proof consists of three steps. First we prove the sufficiency part of theorem for the binary state space, i.e., the case $M = 2$. Then we extend the proof for non-binary state spaces. The necessity part of theorem is also proved via providing a class of non-identifiable mixture models. This three steps of the proof are available in the next three subsections.

**4.1 Proof of the sufficiency part of Theorem 1 for $M = 2$**

In this part, we use the notation $f_{k \ell}$ instead of $f_{k \ell 1}$. Note that we have $f_{k \ell 1} + f_{k \ell 2} = 1$ in the binary case. Also, we use $\Theta_{K,L}$ and $\overline{\Theta}_{K,L}$ instead of $\Theta_{K,L,2}$ and $\overline{\Theta}_{K,L,2}$, respectively. First we need to define the characteristic polynomial of the latent parameters.

**Definition 7.** The characteristic polynomial of latent parameters $(F_{1,K}; w) \in \overline{\Theta}_{K,L}$ is an $L$-variable polynomial that is defined as follows.

$$C_{(F_{1,K}; w)}(x_1, x_2, \ldots, x_L) := \sum_{k=1}^{K} w_k \prod_{\ell=1}^{L} (x_\ell - f_{k \ell}).$$

We also denote the characteristic polynomials by $C_{(F_{1,K}; w)}(x_1:L)$ in a brief way.
The characteristic polynomial of latent parameters \((F_{1:K}; w) \in \Theta_{K,L}\) has an important role in our proofs. In particular, in the next lemma, we prove that the identifiability of the characteristic polynomial of a mixture model implies the the identifiability of the corresponding mixture model.

**Lemma 1.** For any \((F_{1:K}; w) \in \Theta_{K,L}\), the following propositions are equivalent.

(i) A mixture model with the latent parameters \((F_{1:K}; w) \in \Theta_{K,L}\) is identifiable.

(ii) For any \((G_{1:K}; z) \in \Theta_{K,L}\) satisfying the polynomial identity \(C_{(G_{1:K};z)}(x_{1:L}) \equiv C_{(F_{1:K};w)}(x_{1:L})\), we have \((F_{1:K}; w) \approx (G_{1:K}; z)\).

**Proof.** See appendix A. \(\square\)

**Remark 8.** Lemma 1 shows that the identifiability of mixture models of products of Bernoulli measures is equivalent to the identifiability of a class of multi-variable polynomials. This connection makes it possible to prove an identifiability result in the class of multi-variable polynomials and use it to prove the identifiability of mixture models.

Now, we state the following theorem which concludes the proof of the sufficiency part of Theorem 1 for binary case.

**Theorem 3.** Let \((F_{1:K}; w) \in \Theta_{K,L}\) with \(\mathcal{L}(F_{1:K}; w) \geq 2K - 1\). Then, for any \((G_{1:K}; z) \in \Theta_{K,L}\) the identity \(C_{(G_{1:K};z)}(x_{1:L}) \equiv C_{(F_{1:K};w)}(x_{1:L})\) implies \((F_{1:K}; w) \approx (G_{1:K}; z)\).

**Remark 9.** Observe that the sufficiency part of Theorem 1 for binary state spaces follows by Theorem 3 and Lemma 1. Also, note that Theorem 3 establishes an identifiability argument for a class of multi-variable polynomials, named by characteristic polynomials.

**Proof.** To prove Theorem 3 we establish an stronger argument. We relax the condition \(w \in \Delta_K\) in the definition of \(\Theta_{K,L}\) to \(\|w\|_0 = K\) and form a new set of latent parameters, denoted by \(\Theta^*_{K,L}\). Note that we have \(\Theta_{K,L} \subseteq \Theta^*_{K,L}\). Also \(\overline{\Theta}_{K,L}\) denotes the closure of \(\Theta^*_{K,L}\). Now we prove the statement of the theorem, for the cases that \((F_{1:K}; w) \in \Theta^*_{K,L}\) and \((G_{1:K}; z) \in \overline{\Theta}_{K,L}\), which is stronger than the theorem 3. The reason that we use this modification is that this allows us to establish an inductive proof.

The proof is based on an induction on \(K\). The case \(K = 1\) is trivial. Assume that the theorem is proved for any \(K < \tilde{K}\). We will prove the theorem for \(K = \tilde{K}\). Assume that for some \((F_{1:K}; w) \in \Theta^*_{K,L}\) and \((G_{1:K}; z) \in \overline{\Theta}_{K,L}\), we have the identity \(C_{(F_{1:K};w)}(x_{1:L}) \equiv C_{(G_{1:K};z)}(x_{1:L})\) and also \(\mathcal{L}(F_{1:K}; w) \geq 2\tilde{K} - 1\). We will show that \((F_{1:K}; w) \approx (G_{1:K}; z)\). Note that we have

\[
\sum_{k=1}^{\tilde{K}} w_k \prod_{\ell=1}^{L} (x_{\ell} - f_{k\ell}) \equiv \sum_{k=1}^{K} z_k \prod_{\ell=1}^{L} (x_{\ell} - g_{k\ell}). \tag{1}
\]

Without loss of generality, assume that the variable \(\ell = L\) is strongly separable in \((F_{1:K}; w)\). Letting \(x_L = \tilde{f}_{KL}\) in (1) results

\[
\sum_{k=1}^{\tilde{K}-1} w_k (f_{KL} - f_{kL}) \prod_{\ell=1}^{L-1} (x_{\ell} - f_{k\ell}) \equiv \sum_{k=1}^{K} z_k (f_{KL} - g_{kL}) \prod_{\ell=1}^{L-1} (x_{\ell} - g_{k\ell}). \tag{2}
\]

Note that all of the prior definitions for \((F_{1:K}; w) \in \Theta_{K,L}\) naturally extend to the elements of \(\Theta^*_{K,L}\).
Note that the term \( k = \tilde{K} \) in LHS of (1) becomes zero by choosing \( x_L = f_{KL} \). Also notice that for any \( k \in [\tilde{K} - 1], f_{KL} \neq f_{KL} \), because the variable \( \ell = L \) is strongly separable. Now two cases may happen.

**Case one.** First assume that the RHS of the summation in (2) has less than \( \tilde{K} \) non-zero terms\(^4\), i.e., there is a \( k \in [\tilde{K}] \), such that \( z_k(f_{KL} - g_{KL}) = 0 \). Without loss of generality, assume that \( z_\tilde{K}(f_{KL} - g_{KL}) = 0 \). In this case, we can use the induction hypothesis on the identity in (2). More precisely, since we have set \( z_\tilde{K}(f_{KL} - g_{KL}) = 0 \), the identity (2) can be written as

\[
\sum_{k=1}^{\tilde{K}-1} w_k(f_{KL} - f_{KL}) \prod_{\ell=1}^{L-1} (x_\ell - f_{K\ell}) \equiv \sum_{k=1}^{\tilde{K}-1} z_k(f_{KL} - g_{KL}) \prod_{\ell=1}^{L-1} (x_\ell - g_{K\ell}).
\]

We note that \( w'_k \neq 0 \), since the variable \( \ell = 1 \) has been assumed to be strongly separable and \( w_k \neq 0 \). Now we consider two sets of latent parameters corresponded to the two sides of (3). Observe that the number of strongly separable variables in the corresponding problem of the LHS of (3) is at least \( 2\tilde{K} - 2 \), which is greater than \( (\tilde{K} - 1) - 1 \). This shows that we can use the induction hypothesis in this case.

Hence, by using the induction hypothesis, we conclude that there is a permutation \( \pi_1 \) on \( [\tilde{K} - 1] \), such that for any \( k \in [\tilde{K} - 1] \) and \( \ell \in [L - 1] \), we have \( f_{K\ell} = g_{\pi_1(k)\ell} \). Now let \( 1 \leq \ell_1 < \ell_2 < \ldots < \ell_{2\tilde{K} - 2} \leq L - 1 \) be some strongly separable variables of \( (F_{1;\tilde{K}}; w) \). The existence of them is guaranteed by the assumption of the induction. Now if we let \( x_{\ell_k} = f_{k\ell_k} \) for any \( k \in [\tilde{K} - 1] \) in (1), we have

\[
\begin{align*}
&\tilde{w}_\tilde{K} \left( \prod_{k=1}^{\tilde{K}-1} (f_{k\ell_k} - f_{\tilde{K}\ell_k}) \right) \left( \prod_{\ell \in [L] \setminus \{\ell_1, \ell_2, \ldots, \ell_{\tilde{K}-1}\}} (x_\ell - f_{K\ell}) \right) \\
&\equiv \tilde{z}_\tilde{K} \left( \prod_{k=1}^{\tilde{K}-1} (f_{k\ell_k} - g_{\tilde{K}\ell_k}) \right) \left( \prod_{\ell \in [L] \setminus \{\ell_1, \ell_2, \ldots, \ell_{\tilde{K}-1}\}} (x_\ell - g_{K\ell}) \right).
\end{align*}
\]

We notice that the above polynomial identity holds due to the fact that each term in the summation in the LHS of (1), which corresponds to some \( k \in [\tilde{K} - 1] \), becomes zero, since we have set \( x_{\ell_k} = f_{k\ell_k} \). Also, for the RHS of (1), the \( k \)th term in the summation, for any \( k \in [\tilde{K} - 1] \) becomes zero, due to the fact that we have set \( x_{\ell_k} = f_{k\ell_k} = g_{k\ell_k} \) in (1), where \( k' = \pi_1^{-1}(k) \).

Since the variable \( \ell_k \) has been assumed to be strongly separable in \( (F_{1;\tilde{K}}; w) \), for any \( k \in [\tilde{K} - 1] \), we conclude that \( w' := w_\tilde{K} \prod_{k=1}^{\tilde{K}-1} (f_{k\ell_k} - f_{\tilde{K}\ell_k}) \neq 0 \). This shows that (4) is a non-zero polynomial. Hence by the identity of two (non-zero) polynomials in (4), we conclude that \( f_{K\ell} = g_{K\ell} \) for any \( \ell \in [L] \setminus \{\ell_1, \ell_2, \ldots, \ell_{\tilde{K}-1}\} \) and also \( w' = z' \).

A similar argument can be established to show that we have \( f_{K\ell} = g_{K\ell} \) for any \( \ell \in [L] \setminus \{\ell_{\tilde{K}}, \ell_{\tilde{K}+1}, \ldots, \ell_{2\tilde{K}-2}\} \). Combining the results shows that \( f_{K\ell} = g_{K\ell} \) for any \( \ell \in [L] \), since we have

\[
[L] = ([L] \setminus \{\ell_1, \ell_2, \ldots, \ell_{\tilde{K}-1}\}) \cup ([L] \setminus \{\ell_{\tilde{K}}, \ell_{\tilde{K}+1}, \ldots, \ell_{2\tilde{K}-2}\}).
\]

By applying this fact to (1) and using the identity \( w' = z' \), we conclude that that \( w_\tilde{K} = z_{\tilde{K}} \). Hence,

\(^4\) Note that the LHS of (2) contains \( \tilde{K} - 1 \) terms, which means that the number of distinct non-zero polynomials in the summation is \( \tilde{K} - 1 \). This is due to the strong separability of the variable \( \ell = 1 \).
the term \( k = \hat{K} \) in the summation in (I) can be canceled from two sides. This yields

\[
\sum_{k=1}^{\hat{K}-1} w_k \prod_{\ell=1}^{L} (x_\ell - f_{k\ell}) \equiv \sum_{k=1}^{\hat{K}-1} z_k \prod_{\ell=1}^{L} (x_\ell - g_{k\ell}). \tag{6}
\]

By applying the induction hypothesis to (6), we conclude that there is a permutation \( \psi \) on \([\hat{K} - 1]\) with the following property\(^5\). For any \( k \in [\hat{K} - 1] \) and \( \ell \in [L] \), we have \( w_k = z_{\psi(k)} \) and \( f_{k\ell} = g_{\psi(k)\ell} \). Combining the results shows that for the permutation \( \pi \) on \([\hat{K}]\), which is defined as

\[
\pi(k) = \begin{cases} 
\psi(k) & \text{if } k \in [\hat{K} - 1] \\
\hat{K} & \text{if } k = \hat{K} 
\end{cases}
\]

we have \((F_{1;\hat{K}}; w) = (G^\pi_{1;\hat{K}}; z^\pi)\). This shows that \((F_{1;\hat{K}}; w) \approx (G^\pi_{1;\hat{K}}; z)\) and completes the proof.

Case two. Now assume that the RHS of the summation in (2) has exactly \( \hat{K} \) non-zero terms. We notice that the proof of the case one does not depend on the location of the first chosen strongly separable variable. In other words, if for some \( \ell \in [L] \) and some \( k \in [\hat{K}] \), where \( \ell \) is the location of a strongly separable variable in \((F_{1;\hat{K}}; w)\), it is possible to set \( x_\ell = f_{k\ell} \) in (I), such that the assumption in the case one holds, then the proof is completed. Hence, we assume that for any \( \ell \), which is the location of a strongly separable variable of \((F_{1;\hat{K}}; w)\), and for any \( k \in [\hat{K}] \), if we set \( x_\ell = f_{k\ell} \) in (I), then the RHS of the result has exactly \( \hat{K} \) non-zero terms, i.e., \( f_{k\ell} \neq g_{k\ell} \) for any \( k', k \in [\hat{K}] \) and \( z_k \neq 0 \) for any \( k \in [\hat{K}] \). Denote the locations of some strongly separable variables in \((F_{1;\hat{K}}; w)\) by \( 1 \leq \ell_1 < \ell_2 < \ldots < \ell_{2\hat{K} - 2} \leq L - 1 \). Note that we still assumed that the variable \( \ell = L \) is strongly separable. Without loss of generality, assume that \( \ell_{2\hat{K} - 2} = L - 1 \). Following by these assumptions, we set \( x_{L-1} = g_{(K-1)(L-1)} \) in (2) and conclude that

\[
\sum_{k=1}^{\hat{K}-1} \frac{w_k a_k}{w_k'} \prod_{\ell=1}^{L-2} (x_\ell - f_{k\ell}) \equiv \sum_{k \in [\hat{K}] \setminus \{K-1\}} \frac{z_k b_k}{z_k'} \prod_{\ell=1}^{L-2} (x_\ell - g_{k\ell}), \tag{7}
\]

where \( a_k := (f_{KL} - f_{kL})(g_{(K-1)(L-1)} - f_{k(L-1)}) \) and \( b_k := (f_{KL} - g_{KL})(g_{(K-1)(L-1)} - g_{k(L-1)}) \) for any \( k \in [\hat{K}] \). Note that because of the discussed considerations, two sides of the summation in (7) have exactly \( \hat{K} - 1 \) terms in two sides, i.e., \( a_k \neq 0 \) for any \( k \in [\hat{K} - 1] \) and \( b_k \neq 0 \) for any \( k \in [\hat{K}] \setminus \{K-1\} \). Also, the number of the residual strongly separable variables of the corresponding problem of the LHS of (7) is at least \( 2\hat{K} - 3 \). Hence, using the induction hypothesis, we conclude that there is a bijection mapping \( \phi : [\hat{K} - 1] \to [\hat{K}] \setminus \{K-1\} \) such that for any \( \ell \in [L - 2] \) and \( k \in [\hat{K} - 1] \) we have \( f_{k\ell} = g_{\phi(k)\ell} \). Observe that because \( \hat{K} \geq 2 \), we have \( \ell_1 \in [L - 2] \). This shows that \( f_{k\ell_1} = g_{\phi(k)\ell_1} \) for any \( k \in [\hat{K} - 1] \). Note that this is contradiction with the assumption that we make for the case two, where now if we set \( x_{\ell_1} = f_{1\ell_1} \) in (I), then the RHS of the result has less than \( \hat{K} \) terms. This shows that it is impossible that the first case does not happen. This completes the proof.

\[\square\]

4.2 Proof of the sufficiency part of Theorem 1 for \( M \geq 3 \)

In this part, we focus on larger state spaces than binaries. The main idea for the proof is that by introducing some auxiliary binary mixture models, we can use the result of Theorem 3. The auxiliary

\[\footnote{Actually the permutation \( \psi \) is equal to the permutation \( \pi_1 \), which is defined previously. However, we do not use this identity and so it is not needed to prove it.}\]
binary mixture models are generated based on the projection of the variables into binary spaces. This allows us to use Theorem 3.

Consider latent variables \((F_{1:K}; w) \in \Theta_{K,L,M}\), where \(\mathcal{L}_s(F_{1:K}; w) \geq 2K - 1\) and \(M \geq 3\). Also assume that \((G_{1:K}; z) \in \Theta_{K,L,M}\) and \(f = g\). Note that \(f, g\) are the mixture distributions of the problems. We want to show that \((G_{1:K}; z) \approx (F_{1:K}; w)\). We introduce two auxiliary binary mixture models \((\tilde{F}_{1:K}; w) \in \Theta_{K,L,2}\) and \((G_{1:K}; z) \in \Theta_{K,L,2}\) as follows. For any \(k \in [K]\) and \(\ell \in [L]\), let \(\tilde{f}_{k\ell1} = f_{k\ell1}\) and \(\tilde{g}_{k\ell1} = g_{k\ell1}\). Also let \(f_{k\ell2} = 1 - f_{k\ell1}\) and \(g_{k\ell2} = 1 - g_{k\ell1}\). Note that by the definition of the strongly separable variables, they do not change by this transformation. In other words, if the \(\ell\text{th}\) variable is strongly separable in \((F_{1:K}; w)\), then it is strongly separable in \((\tilde{F}_{1:K}; w)\), too. Hence, we have \(\mathcal{L}_s(\tilde{F}_{1:K}; w) \geq \mathcal{L}_s(F_{1:K}; w) \geq 2K - 1\).

**Lemma 2.** \(C(\tilde{F}_{1:K}; w)(x_{1:L}) \equiv C(G_{1:K}; z)(x_{1:L})\).

**Proof.** See appendix B. \(\square\)

Using Lemma 2 and Theorem 3 we conclude that \((\tilde{F}_{1:K}; w) \approx (G_{1:K}; z)\). This implies that there is a permutation \(\pi\) on \([K]\) such that \((\tilde{F}_{1:K}; w) = (G^\pi_{1:K}; z^\pi)\). This shows that for any \(\ell \in [L]\) and any \(k \in [K]\) we have \(\tilde{f}_{k\ell1} = \tilde{g}_{k\ell1}\). Using the definitions of the auxiliary problems, we conclude that \(f_{k\ell1} = g_{\pi(k)\ell1}\) for any \(\ell \in [L]\) and \(k \in [K]\). It is also concluded that \(w_k = z_{\pi(k)}\) for any \(k \in [K]\).

Now we claim that \((F'_{1:K}; \omega) = (G^\pi_{1:K}; z^\pi)\) and this completes the proof. For this purpose, we need to show that for any \(\ell \in [L]\), \(k \in [K]\) and \(m \in [M] \setminus \{1\}\) we have \(f_{k\ell m} = g_{\pi(k)\ell m}\). Using the symmetry in the problem, it suffices to prove that \(f_{112} = g_{\pi(1)12}\).

Again, we introduce two auxiliary binary mixture models \((F'_{1:K}; w) \in \Theta_{K,L,2}\) and \((G'_{1:K}; z) \in \Theta_{K,L,2}\) as follows. For any \(k \in [K]\) and \(\ell \in [L] \setminus \{1\}\), let \(f'_{k\ell1} = f_{k\ell1}\) and \(g'_{k\ell1} = g_{k\ell1}\). For any \(k \in [K]\), let \(f'_{k11} = f_{k11}\) and \(g'_{k11} = g_{k11}\). Also, let \(f'_{k\ell2} = 1 - f'_{k\ell1}\) and \(g'_{k\ell2} = 1 - g_{k\ell1}\) for any \(k \in [K]\) and \(\ell \in [L]\).

Similarly, we have \(\mathcal{L}_s(F'_{1:K}; w) \geq \mathcal{L}_s(F_{1:K}; w) \geq 2K - 1\).

**Lemma 3.** \(C(F'_{1:K}; w)(x_{1:L}) \equiv C(G'_{1:K}; z)(x_{1:L})\).

**Proof.** See appendix C. \(\square\)

Using Lemma 3 and Theorem 3 we conclude that \((F'_{1:K}; w) \approx (G'_{1:K}; z)\). This yields that there is a permutation \(\psi\) on \([K]\), such that we have \((F'_{1:K}; w) = (G^\psi_{1:K}; z^\psi)\). Hence, for any \(\ell \in [L]\) and any \(k \in [K]\), we have \(f'_{k\ell1} = g_{\psi(k)\ell1}\). Thus, we conclude that \(f_{k\ell1} = g_{\psi(k)\ell1}\) for any \(\ell \in [L] \setminus \{1\}\) and \(k \in [K]\). Choose \(\ell \in [L] \setminus \{1\}\) such that the \(\ell\text{th}\) variable is strongly separable in \((F_{1:K}; w)\). This is possible due to the assumption of the existence of at least \(2K - 1\) strongly separable variables in \((F_{1:K}; w)\). We claim that \(\psi = \pi\). Note that for any \(k \in [K]\) we have \(f_{k\ell1} = g_{\pi(k)\ell1} = f_{\psi^{-1}(\pi(k))\ell1}\). Note that because of the strong separability of the \(\ell\text{th}\) variable, the set \(\{f_{k\ell1} \in [0,1] | k \in [K]\}\) has exactly \(K\) elements. This shows that \(\psi^{-1}(\pi(k)) = k\) or \(\psi(k) = \pi(k)\) for any \(k \in [K]\). Hence we have \(\psi = \pi\).

Now using \((F'_{1:K}; w) = (G^\psi_{1:K}; z^\psi) = (G_{1:K}^\pi; z^\pi)\), we conclude that \(f'_{111} = g'_{\pi(1)11}\). It is also assumed that \(f'_{111} = f_{112}\) and \(g_{\pi(1)11} = g_{\pi(1)12}\). Hence, we have \(f_{112} = g_{\pi(1)12}\). Similarly, we can conclude that \((F_{1:K}; w) = (G_{1:K}^\pi; z^\pi)\) and hence, the proof is completed.

### 4.3 Proof of the necessary part of Theorem 1

In this part, for any \(\overline{L} \in [\min(2K - 2, L)] \cup \{0\}\) we introduce a problem \((F_{1:K}; w) \in \Theta_{K,L,M}\) such that \(\mathcal{L}_s(F_{1:K}; w) = \overline{L}\) and \((F_{1:K}; w)\) is not identifiable. In particular, we introduce two problems

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For the case \(K = 1\), this assumption may be incorrect. However, the proof for the case \(K = 1\) is trivial and can be done directly.
Let \((F_1; w), (G_1; z) \in \Theta_{K,L,M}\), where \(L_s(F_1; w) = L_s(G_1; z) = L\), such that \(f = g\) (they have the same mixture distributions) and \((F_1; w) \neq (G_1; z)\).

For any positive constants \(\alpha\) and \(\beta\), we introduce two problems \((F_1; w), (G_1; z) \in \Theta_{K,L,M}\) as follows. First, we set \(w = (2K-1)/2^{2K-1}\) and \(z = (2K-1)/2^{2K-1}\) for any \(k \in [K]\). Note that \(\sum_{k=1}^{K} w_k = \sum_{k=1}^{K} z_k = 1\). Then, for any \(k \in [K]\), \(\ell \in [L]\) and \(m \in [M-1]\), we define \(f_{k\ell m} = \alpha(2k-2) + \beta m\) and \(g_{k\ell m} = \alpha(2k-1) + \beta m\). Also for any \(k \in [K]\), \(\ell \in [L] \setminus [L']\) and \(m \in [M-1]\), we set \(f_{k\ell m} = g_{k\ell m} = 1/M\).

The values of \(f_{k\ell m}\) and \(g_{k\ell m}\) are determined by the condition \(\sum_{m\in[M]} f_{k\ell m} = \sum_{m\in[M]} g_{k\ell m} = 1\). Note that in this definition, \((F_1; w), (G_1; z) \in \Theta_{K,L,M}\) if we choose \(\alpha\) and \(\beta\) positive and small enough. Now we claim that these two problems work for the proof of the necessary part of Theorem 1.

First we notice that \(L_s(F_1; w) = L_s(G_1; z) = L\). Second, it is obvious that \((F_1; w) \neq (G_1; z)\), because the frequencies of two problems in the first \(\ell \in [L]\) variables are essentially different from each other. Hence, for completing the proof, it remains to show that we have \(f = g\). In particular, we are interested to show that for any \(m = (m_1, m_2, \ldots, m_L) \in [M]^L\), we have \(f(m_1, m_2, \ldots, m_L) = g(m_1, m_2, \ldots, m_L)\), or equivalently, \(\sum_{k=1}^{K} w_k \Pi_{\ell \in I} f_{k\ell m} = \sum_{k=1}^{K} z_k \Pi_{\ell \in I} g_{k\ell m}\). First we state the following lemma about the defined problems \((F_1; w)\) and \((G_1; z)\).

**Lemma 4.** If for any \(I \subseteq [L]\) and \(m = (m_\ell)_{\ell \in I} \in [M-1]^{|I|}\) we have \(\sum_{k=1}^{K} w_k \Pi_{\ell \in I} f_{k\ell m} = \sum_{k=1}^{K} z_k \Pi_{\ell \in I} g_{k\ell m}\), then \(f = g\).

**Proof.** See appendix D.

Using Lemma 4, for the proof of \(f = g\), it suffices to prove that for any \(I \subseteq [L]\) and \(m = (m_\ell)_{\ell \in I} \in [M-1]^{|I|}\), the identity \(\sum_{k=1}^{K} w_k \Pi_{\ell \in I} f_{k\ell m} = \sum_{k=1}^{K} z_k \Pi_{\ell \in I} g_{k\ell m}\) holds. Observe that

\[
\sum_{k=1}^{K} w_k \prod_{\ell \in I} f_{k\ell m} - \sum_{k=1}^{K} z_k \prod_{\ell \in I} g_{k\ell m} = \sum_{k=1}^{K} \frac{(2K-1)}{2^{2K-1}} \prod_{\ell \in I} (\alpha(2k-2) + \beta m_\ell) - \sum_{k=1}^{K} \frac{(2K-1)}{2^{2K-1}} \prod_{\ell \in I} (\alpha(2k-1) + \beta m_\ell) = \sum_{i=0}^{2K-1} \frac{(2K-1)}{2^{2K-1}} (-1)^i \prod_{\ell \in I} (\alpha i + \beta m_\ell). \tag{8}
\]

We want to show that (8) is equal to zero. Let us define an \(M-1\) variable real function \(h(x_{1:M-1})\) as

\[
h(x_{1:M-1}) := \exp(\beta \times \sum_{m=1}^{M-1} m x_m)(1 - \exp(\alpha \times \sum_{m=1}^{M-1} x_m))^{2K-1}. \tag{10}
\]

**Lemma 5.** For any \(t = (t_1, t_2, \ldots, t_{M-1})^T \in \{0, 1, \ldots, 2K-2\}^{M-1}\), such that \(t = \sum_{m=1}^{M-1} t_m \leq 2K-2\), we have

\[
\frac{\partial^t h}{\partial x_1^{t_1} \partial x_2^{t_2} \cdots \partial x_{M-1}^{t_{M-1}}}(0, 0, \ldots, 0) = 0. \tag{11}
\]

7 The case \(L = 0\) is trivial. Note that if two mixture components have the same frequencies, then \(L = 0\) and \((F_1; w)\) is not identifiable, where the probability of sampling from each of that two mixture components can not be determined from the mixture distribution. Hence, we assume that \(L \geq 1\).

8 If \(I = \emptyset\), we define \(\prod_{\ell \in I} f_{k\ell m} = 1\).
Proof. See appendix E. □

Using Lemma 5, we aim to prove that (9) is equal to zero. Note that if we define \( t_m := |\{ \ell \in I : m_\ell = m \}| \in [2K - 2] \cup \{0\} \) for any \( m \in [M - 1] \) and \( t := \sum_{m=1}^{M-1} t_m = |I| \leq T \leq 2K - 2 \), then we have

\[
\sum_{i=0}^{2K-1} \frac{(2K-1)_i}{22K-1} (-1)^i \prod_{\ell \in I} (\alpha_i + \beta m_\ell) = \sum_{i=0}^{2K-1} \frac{(2K-1)_i}{22K-1} (-1)^i \prod_{m=1}^{M-1} (\alpha_i + \beta m)^{t_m}
\]

(12)

which is equal to zero based on Lemma 3. Note that (a) follows from the expansion of the function \( h(x_{1:M-1}) \) as follows

\[
h(x_{1:M-1}) = \exp(\beta \sum_{m=1}^{M-1} mx_m)(1 - \exp(\alpha \sum_{m=1}^{M-1} x_m))^{2K-1} = \sum_{i=0}^{2K-1} \frac{(2K-1)_i}{22K-1} (-1)^i \exp((\alpha + \beta m) \times (\sum_{m=1}^{M-1} x_m)).
\]

We are done.

5 Proof of Theorem 2

In this section, we prove the sufficiency of 2\( K \) weakly separable variables for the identifiability. First we note that for the binary state spaces two notions of weakly separable variables and strongly separable variables are equivalent. Therefore, according to the proof of Theorem 1, we conclude the desired result for \( M = 2 \). Hence we consider the cases that \( M \geq 3 \).

We establish a proof based on introducing the auxiliary binary mixture models, which is very similar to the proof of Theorem 1 for the case \( M \geq 3 \). Consider latent variables \( (F_{1:K}; w) \in \Theta_{K,L,M} \), where \( L \in [2K - 1] \) and \( M \geq 3 \). Also assume that \( (G_{1:K}; z) \in \overline{\Theta}_{K,L,M} \) and \( f = g \). This means that the two problems have the same mixture distributions. We aim to prove that \( (G_{1:K}; z) \approx (F_{1:K}; w) \).

Without loss of generality, assume that the variables \( 1 \leq \ell_1 < \ell_2 < \ldots < \ell_{2K} \leq L \) are weakly separable in \((F_{1:K}; w)\). This means that for any \( r \in [2K] \), there is an \( m_r \in [M] \) such that \( f_{k_\ell,m_r} \neq f_{k_\ell',m_r} \) for any distinct \( k, k' \in [K] \). Without loss of generality, assume that \( m_r = 1 \) for any \( r \in [2K] \). Let us define the set \( I = \{ \ell_r | r \in [2K] \} \).

Similar to the proof of Theorem 1 for \( M \geq 3 \), we introduce two auxiliary binary mixture models \((\tilde{F}_{1:K}; w) \in \Theta_{K,L,2}\) and \((\tilde{G}_{1:K}; z) \in \overline{\Theta}_{K,L,2}\) as follows. For any \( k \in [K] \) and \( \ell \in [L] \), let \( \tilde{f}_{k_\ell} = f_{k_\ell} \) and \( \tilde{g}_{k_\ell} = g_{k_\ell} \). Also let \( \tilde{f}_{k_\ell} = 1 - f_{k_\ell} \) and \( \tilde{g}_{k_\ell} = 1 - g_{k_\ell} \). Note that by the definition of the weakly separable variables, all of the variables \( \ell \in I \) are strongly separable in \((\tilde{F}_{1:K}; w)\). This means that \( L_s(\tilde{F}_{1:K}; w) \geq 2K \).

Using Lemma 2 and Theorem 1, we conclude that \((\tilde{F}_{1:K}; w) \approx (\tilde{G}_{1:K}; z)\). This means that there is a permutation \( \pi \) on \([K]\) such that \((\tilde{F}_{1:K}; w) = (\tilde{G}_{1:K}; z^\pi)\). This shows that for any \( \ell \in [L] \) and any \( k \in [K] \) we have \( \tilde{f}_{k_\ell} = \tilde{g}_{\pi(k)_\ell} \). Using the definitions of the auxiliary problems, we conclude that \( f_{k_\ell} = g_{\pi(k)_\ell} \) for any \( \ell \in [L] \) and \( k \in [K] \). It is also concluded that \( w_k = z_{\pi(k)} \) for any \( k \in [K] \).
Now we claim that \((F_{1:K}; \mathbf{w}) = (G^\pi_{1:K}; \mathbf{z}^\pi)\) and this completes the proof. For this purpose, we need to show that for any \(\ell \in [L]\), \(k \in [K]\) and \(m \in [M] \setminus \{1\}\) we have \(f_{k\ell m} = g_{\pi(k)\ell m}\). Using the symmetry in the problem, it suffices to prove that \(f_{112} = g_{\pi(1)12}\).

Again, we introduce two auxiliary binary mixture models \((F'_{1:K}; \mathbf{w}) \in \Theta_{K,L,2}\) and \((G'_{1:K}; \mathbf{z}) \in \overline{\Theta}_{K,L,2}\) as follows. For any \(k \in [K]\) and \(\ell \in [L] \setminus \{1\}\), let \(f'_{k\ell 1} = f_{k\ell 1}\) and \(g'_{k\ell 1} = g_{k\ell 1}\). For any \(k \in [K]\), let \(f'_{k11} = f_{k12}\) and \(g'_{k11} = g_{k12}\). Also, let \(f'_{k22} = 1 - f'_{k11}\) and \(g'_{k22} = 1 - g'_{k11}\) for any \(k \in [K]\) and \(\ell \in [L]\).

Now two cases may occur. First assume that \(1 \notin I\). In this case, we have the inequality \(L_s(F'_{1:K}; \mathbf{w}) \geq 2K\), due to the weak separability of the variables of the set \(I\).

For the second case, assume that \(1 \in I\). This shows that when we project the first variable into the specific binary space, which is defined, the first weakly separable variable may waste. However, the other weakly separable variables of \((F_{1:K}; \mathbf{w})\) hold in \((F'_{1:K}; \mathbf{w})\). Thus, we conclude that \(L_s(F'_{1:K}; \mathbf{w}) \geq 2K - 1\).

Hence, for the two cases we conclude that \(L_s(F'_{1:K}; \mathbf{w}) \geq 2K - 1\). The rest of the proof is exactly similar to the sufficiency proof of Theorem 1 for the case \(M \geq 3\) and so it is omitted. We are done.

**Remark 10.** In the worst-case regime, our counterexample for the identifiability of mixture models with less than \(2K - 1\) strongly separable variables is also valid for studying the weakly separable variables. In other words, there are mixture models with less than \(2K - 1\) weakly separable variables which are not identifiable. Hence, the optimal threshold for the weakly separable variables is achieved in this paper within at most one variable.

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A Proof of Lemma 1

In order to prove, it suffices to show that for any \((F_{1,K};w),(G_{1,K};z)\) \(\in \Theta_{K,L}\) with mixture distributions \(f\) and \(g\), the identity \(f = g\) implies \(C_{(F_{1,K};w)}(x_{1:L}) \equiv C_{(G_{1,K};z)}(x_{1:L})\) and vice versa. This is due to the definitions of the paper.

First assume that we have the identity \(C_{(F_{1,K};w)}(x_{1:L}) \equiv C_{(G_{1,K};z)}(x_{1:L})\). Note that

\[
C_{(F_{1,K};w)}(x_{1:L}) = \sum_{k=1}^{K} w_k \prod_{\ell=1}^{L} (x_{\ell} - f_{k\ell})
\]

\[
= \sum_{k=1}^{K} w_k \left\{ \sum_{I \subseteq [L]} (-1)^{|I|} \left( \prod_{\ell \in [L] \setminus I} x_{\ell} \right) \left( \prod_{\ell \in I} f_{k\ell} \right) \right\}
\]

\[
= \sum_{I \subseteq [L]} (-1)^{|I|} \left( \prod_{\ell \in [L] \setminus I} x_{\ell} \right) \left( \sum_{k=1}^{K} w_k \prod_{\ell \in I} f_{k\ell} \right). \tag{14}
\]
This implies that for any $I \subseteq [L]$ we have
\[ \sum_{k=1}^{K} w_k \prod_{\ell \in I} f_{k\ell} = \sum_{k=1}^{K} z_k \prod_{\ell \in I} g_{k\ell}. \] (15)

For any $m = (m_1, m_2, \ldots, m_L) \in \{1, 2\}^L$, define $I_m := \{ \ell \in [L] : m_\ell = 1 \}$. Observe that
\[ f(m_1, m_2, \ldots, m_L) = \sum_{k=1}^{K} w_k \left( \prod_{\ell=1}^{L} f_{k m_\ell} \right) \]
\[ = \sum_{k=1}^{K} w_k \left( \prod_{\ell \in I_m} f_{k\ell} \prod_{\ell \in [L] \setminus I_m} (1 - f_{k\ell}) \right) \]
\[ = \sum_{k=1}^{K} w_k \left( \sum_{I_m \subseteq I \subseteq [L]} (-1)^{|I|-|I_m|} \prod_{\ell \in I} f_{k\ell} \right) \]
\[ = \sum_{I_m \subseteq I \subseteq [L]} (-1)^{|I|-|I_m|} \left( \sum_{k=1}^{K} w_k \prod_{\ell \in I} f_{k\ell} \right). \] (16)

Using (15) and (16), we conclude that $f(m_1, m_2, \ldots, m_L) = g(m_1, m_2, \ldots, m_L)$ for any $m = (m_1, m_2, \ldots, m_L) \in \{1, 2\}^L$. This shows that $f = g$ and concludes the desired result.

Now for the other side, assume that $f = g$. We will show that $C_{(F_1:K;w)}(x_1:L) \equiv C_{(G_1:K;z)}(x_1:L)$. Using the identity in (14), it suffices to show that for any $I \subseteq [L]$ we have
\[ \sum_{k=1}^{K} w_k \prod_{\ell \in I} f_{k\ell} = \sum_{k=1}^{K} z_k \prod_{\ell \in I} g_{k\ell}. \] (17)

Note that for any $I \subseteq [L]$ we have
\[ \sum_{k=1}^{K} w_k \prod_{\ell \in I} f_{k\ell} = \sum_{k=1}^{K} w_k \left( \prod_{m \in \{1,2\}^L, I \subseteq I_m} \prod_{\ell=1}^{L} f_{k m_\ell} \right) \]
\[ = \sum_{m \in \{1,2\}^L, I \subseteq I_m} \left( \sum_{k=1}^{K} w_k \prod_{\ell=1}^{L} f_{k m_\ell} \right) \]
\[ = \sum_{m \in \{1,2\}^L, I \subseteq I_m} f(m_1, m_2, \ldots, m_L). \] (18)

This shows that if $f = g$, then for any $I \subseteq [L]$ we can conclude (17). Hence, we have $C_{(F_1:K;w)}(x_1:L) \equiv C_{(G_1:K;z)}(x_1:L)$ and this completes the proof.

### B Proof of Lemma 2

Using (14), it suffices to prove that for any $I \subseteq [L]$ we have
\[ \sum_{k=1}^{K} w_k \prod_{\ell \in I} \hat{f}_{k\ell} = \sum_{k=1}^{K} z_k \prod_{\ell \in I} \hat{g}_{k\ell}. \] (19)
For any \( m = (m_1, m_2, \ldots, m_L) \in [M]^L \), define \( I_m := \{ \ell \in [L] : m_\ell = 1 \} \). Note that for any \( I \subseteq [L] \) we have

\[
\sum_{k=1}^K w_k \prod_{\ell \in I} f_{k\ell 1} = \sum_{k=1}^K w_k \left( \sum_{m \in [M]^L : I \subseteq I_m} \prod_{\ell = 1}^L f_{k\ell m_\ell} \right)
\]
\[
= \sum_{m \in [M]^L \atop I \subseteq I_m} \left( \sum_{k=1}^K w_k \prod_{\ell = 1}^L f_{k\ell m_\ell} \right)
\]
\[
= \sum_{m \in [M]^L \atop I \subseteq I_m} f(m_1, m_2, \ldots, m_L).
\]

Hence, using the assumption \( f = g \), for any \( I \subseteq [L] \) we have \( \sum_{k=1}^K w_k \prod_{\ell \in I} f_{k\ell 1} = \sum_{k=1}^K z_k \prod_{\ell \in I} g_{k\ell 1} \), which implies \( (19) \). Thus, the proof is completed.

C Proof of Lemma 3

Similar to the proof of Lemma 2, it suffices to show that for any \( I \subseteq [L] \) we have

\[
\sum_{k=1}^K w_k \prod_{\ell \in I} f'_{k\ell 1} = \sum_{k=1}^K z_k \prod_{\ell \in I} g'_{k\ell 1}.
\]

If \( 1 \notin I \), similar to the proof of Lemma 2, we conclude \( (20) \). For any \( m = (m_1, m_2, \ldots, m_L) \in \{1, 2\}^L \), define \( I_m := \{ \ell \in [L] : m_\ell = 1 \} \). Assume that \( 1 \in I \). We have

\[
\sum_{k=1}^K w_k \prod_{\ell \in I} f'_{k\ell 1} = \sum_{k=1}^K w_k \left( \sum_{m \in [M]^L \atop I \setminus \{1\} \subseteq I_m, m_1 = 2} \prod_{\ell = 1}^L f'_{k\ell m_\ell} \right)
\]
\[
= \sum_{m \in [M]^L \atop I \setminus \{1\} \subseteq I_m, m_1 = 2} \left( \sum_{k=1}^K w_k \prod_{\ell = 1}^L f'_{k\ell m_\ell} \right)
\]
\[
= \sum_{m \in [M]^L \atop I \setminus \{1\} \subseteq I_m, m_1 = 2} f(m_1, m_2, \ldots, m_L).
\]

Hence, using the assumption \( f = g \), for any \( I \subseteq [L] \) we have \( \sum_{k=1}^K w_k \prod_{\ell \in I} f'_{k\ell 1} = \sum_{k=1}^K z_k \prod_{\ell \in I} g'_{k\ell 1} \). This completes the proof.
D  Proof of Lemma 4

For the proof of the lemma, it suffices to show that for any \( \mathbf{m} = (m_1, m_2, \ldots, m_L) \in [M]^L \), we have \( f(m_1, m_2, \ldots, m_L) = g(m_1, m_2, \ldots, m_L) \). Let us define \( B_m := \{ \ell \in [L] : m_\ell \neq M \} \). We write

\[
    f(m_1, m_2, \ldots, m_L) = \sum_{k=1}^K w_k \prod_{\ell=1}^L f_{k\ell m_\ell}
    = \sum_{k=1}^K w_k \prod_{\ell=1}^L f_{k\ell m_\ell} \times \left( \frac{1}{M^L - L} \right)
    = \sum_{k=1}^K w_k \left( \frac{1}{M^L - L} \right) \prod_{\ell \in B_m} f_{k\ell m_\ell} \times \prod_{\ell \in [L] \setminus B_m} \left( 1 - \sum_{m \in [M-1]} f_{k\ell m} \right)
    = \left( \frac{1}{M^L - L} \right) \sum_{k=1}^K w_k \sum_{B_m \subseteq I \subseteq [L]} (-1)^{|I| - |B_m|} \prod_{\ell \in I} f_{k\ell m_\ell}
    = \left( \frac{1}{M^L - L} \right) \sum_{k=1}^K w_k \sum_{B_m \subseteq I \subseteq [L]} (-1)^{|I| - |B_m|} \prod_{\ell \in I} f_{k\ell m_\ell}.
\]

Hence, if \( \sum_{k=1}^K w_k \prod_{\ell \in I} f_{k\ell m_\ell} = \sum_{k=1}^K z_k \prod_{\ell \in I} g_{k\ell m_\ell} \) for any \( I \subseteq [L] \) and \( (n_\ell)_{\ell \in I} \in [M-1]^{|I|} \), then \( f(m_1, m_2, \ldots, m_L) = g(m_1, m_2, \ldots, m_L) \) for any \( \mathbf{m} \in [M]^L \). This completes the proof.

E  Proof of Lemma 5

First define the two functions

\[
    f(x_{1:M-1}) := \exp(\beta \times \sum_{m=1}^{M-1} m x_m)
\]

and

\[
    g(x_{1:M-1}) := (1 - \exp(\alpha \times \sum_{m=1}^{M-1} x_m))^{2K-1}.
\]

Observe that \( h(x_{1:M-1}) = f(x_{1:M-1}) \times g(x_{1:M-1}) \). Now we write

\[
    \frac{\partial^h}{\partial x_1 \partial x_2 \cdots \partial x_{M-1}} (0, 0, \ldots, 0) = 
    \sum_{u, v \in \{0, 1, \ldots, 2K-2\}^{M-1}} C(u, v) \frac{\partial^u f}{\partial x_1^{u_1} \partial x_2^{u_2} \cdots \partial x_{M-1}^{u_{M-1}}} (0, 0, \ldots, 0) \times \frac{\partial^v g}{\partial x_1^{v_1} \partial x_2^{v_2} \cdots \partial x_{M-1}^{v_{M-1}}} (0, 0, \ldots, 0),
\]

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where \( u = \sum_{m=1}^{M-1} u_m \) and \( v = \sum_{m=1}^{M-1} v_m \). Also \( C(u, v) \in \mathbb{N} \) is a positive integer constant, which is independent of \( x_m \).

For completing the proof, it suffices to show that

\[
\frac{\partial^v g}{\partial x_1^{v_1} \partial x_2^{v_2} \cdots \partial x_{M-1}^{v_{M-1}}}(0, 0, \ldots, 0) = 0,
\]

for any \( v \in \{0, 1, \ldots, 2K - 2\}^{M-1} \) such that \( v = \sum_{m=1}^{M-1} v_m \leq 2K - 2 \). Note that we have

\[
\frac{\partial^v g}{\partial x_1^{v_1} \partial x_2^{v_2} \cdots \partial x_{M-1}^{v_{M-1}}} = C(v) \times (1 - \exp(\alpha \times \sum_{m=1}^{M-1} x_m))^{2K-1-v},
\]

for some constant \( C(v) \). We conclude the desired result as we have \( v \leq 2K - 2 < 2K - 1 \).