TEICHMÜLLER’S PROBLEM
FOR GROMOV HYPERBOLIC DOMAINS

BY

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ABSTRACT

Let $T_K(D)$ be the class of $K$-quasiconformal automorphisms of a domain $D \subset \mathbb{R}^n$ with identity boundary values. Teichmüller’s problem is to determine how far a given point $x \in D$ can be mapped under a mapping $f \in T_K(D)$. We estimate this distance between $x$ and $f(x)$ from the above by using two different metrics, the distance ratio metric and the quasihyperbolic metric. We study Teichmüller’s problem for Gromov hyperbolic domains in $\mathbb{R}^n$ with identity values at the boundary of infinity. As applications, we obtain results on Teichmüller’s problem for $\psi$-uniform domains and inner uniform domains in $\mathbb{R}^n$.

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1. Introduction and main results

Teichmüller’s problem concerns finding a lower bound for the maximal dilation of the class of quasiconformal self-maps of a domain $D$, with identity boundary values, moving a point $x$ in the domain to a given point. Following [35], suppose that $D \subseteq \mathbb{R}^n \ (n \geq 2)$ is a domain. Note that the boundary of a domain in $\mathbb{R}^n$ is taken in the topology of the Riemann sphere $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, so $\partial D$ contains at least two points. Let

$$\mathcal{T}_K(D) = \{D \xrightarrow{f} \overline{D} \mid f \text{ is a homeomorphism s.t. } f|_D \text{ is } K\text{-QC and } f|_{\partial D} = \text{id}_{\partial D}\},$$

where $K$-QC means $K$-quasiconformal and $\text{id}_{\partial D}$ denotes the identity map on $\partial D$.

In [24], Teichmüller considered the above class of maps $\mathcal{T}_K(D)$ with

$$D = \mathbb{R}^2 \setminus \{(0,0),(1,0)\},$$

and obtained the following sharp inequality:

$$h_D(x, f(x)) \leq \log K$$

for all $x \in D$, where $h_D$ is the hyperbolic metric of $D$. This result may be regarded as a stability result for quasiconformal homeomorphisms, which hold the boundary pointwise fixed and map the domain onto itself.

For Teichmüller type results concerning the same problem in the case of the unit balls of $\mathbb{R}^n$, $n \geq 2$, we refer to [1, 16, 18, 20, 21, 22]. Vuorinen and Zhang have further studied Teichmüller’s problem for other domains in $\mathbb{R}^n$, such as convex domains and uniform domains with uniformly perfect boundaries; see [32, 35]. We also note that Teichmüller type results are applicable to questions related to the homogeneity of domains; see [9, 19] and references therein.

In this paper, we investigate Teichmüller’s problem for domains in $\mathbb{R}^n$ with uniformly perfect (see Definition 2.1) boundaries. We prove that the distance between $x$ and its quasiconformal image $f(x)$ is uniformly bounded with respect to the distance ratio metric $j$. For the definition of the metric $j$ see (2.1).

**Theorem 1.1:** Let $n \geq 2$, $C > 1$, and $K \geq 1$. There exists a constant $M = M(n,C,K)$ such that: If

1. $D \subsetneq \mathbb{R}^n$ is a domain,
2. $\partial D$ is a $C$-uniformly perfect set,
3. $f \in \mathcal{T}_K(D),$

then $j_D(x,f(x)) \leq M$ for all $x \in D$. 


Remark 1.1: Let $D \subset \mathbb{R}^n$ be a domain. We say that $\partial D$ is $C$-uniformly perfect, if $\partial D$ is $C$-uniformly perfect with respect to the spherical metric. It follows from [36, Lemma C] that uniform perfectness is preserved by quasimöbius transformations, and thus $\partial D$ is a $C'$-uniformly perfect set with respect to the Euclidean metric of $\mathbb{R}^n$, where $C'$ depends only on $C$.

Note that Theorem 1.1 does not hold for domains in $\mathbb{R}^n$ whose boundary is not uniformly perfect. Let $o = (0,0)$, $D = B(o,1) \setminus \{o\} \subset \mathbb{R}^2$, and $x_m = (1/m,0)$ for $m \geq 3$. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x) = |x|x$$

for all $x \in \mathbb{R}^2$. Therefore it follows from [29, 5.21] that $f \in \mathcal{T}_K(D)$ with $K = 2$. However

$$j_D(x_m, f(x_m)) \to \infty,$$

as $m \to \infty$.

Remark 1.2 ([32, Remark 3.5]): Next we give an example to show that the constant $M$ of Theorem 1.1 is strictly greater than $\log 3$ even if $f$ is a conformal map. Let $D = \mathbb{R}^3 \setminus Z$ and $Z = \{(0,0,z) : z \in \mathbb{R}\}$. Let $e_1 = (1,0,0)$, and let $f$ be the rotation around the line $Z$ with $f(e_1) = -e_1$ such that $f$ keeps the line $Z$ pointwise fixed. Then $f$ is conformal. However, we have $j_D(x, f(x)) = \log 3$ for all $x \in D$.

We may apply Theorem 1.1 to Teichmüller’s problem for $\psi$-uniform domains (see Definition 2.5) and deduce that the quasihyperbolic distance $k$ (see Definition 2.4) between $x$ and its image point $f(x)$ is uniformly bounded.

**Corollary 1.1:** Let $n \geq 2, C > 1, K \geq 1$, and let $\psi : [0, \infty) \to [0, \infty)$ be a homeomorphism. There exists a constant $M' = M'(n, C, K, \psi)$ such that: If

1. $D \subset \mathbb{R}^n$ is a $\psi$-uniform domain,
2. $\partial D$ is a $C$-uniformly perfect set,
3. $f \in \mathcal{T}_K(D)$,

then $k_D(x, f(x)) \leq M'$ for all $x \in D$.

Remark 1.3: The class of $\psi$-uniform domains in $\mathbb{R}^n$ was introduced by Vuorinen in [33]. In [10], Hästö, Klén, Sahoo and Vuorinen studied certain geometric properties of these domains. Because convex domains and uniform domains are both $\psi$-uniform, Corollary 1.1 holds for all convex domains and for all uniform domains in $\mathbb{R}^n$. 
Recently, Bonfert-Taylor, Canary, Martin and Taylor studied Teichmüller’s problem in the case of the classical hyperbolic space $\mathbb{H}^n$, and they proved that if the boundary extension of a quasiconformal map is the identity on $\partial \mathbb{H}^n$, then it is uniformly close to the identity map on $\mathbb{H}^n$; see [3, Lemma 4.1]. In [4], Bonk, Heinonen and Koskela proved that every uniform domain in $\mathbb{R}^n$ is Gromov hyperbolic with respect to the quasihyperbolic metric. This motivates us to study Teichmüller’s problem for Gromov hyperbolic domains.

As the second main aim of this paper, we consider Teichmüller’s problem for Gromov hyperbolic domains in $\mathbb{R}^n$. Let $D \subset \mathbb{R}^n$ ($n \geq 2$) be a domain and $k_D$ its quasihyperbolic metric. If $(D, k_D)$ is a $\delta$-hyperbolic metric space for some $\delta \geq 0$, then we call $D$ a **Gromov hyperbolic domain** or a **$\delta$-hyperbolic domain**. Denote by $\partial^*D$ the Gromov boundary of the hyperbolic space $(D, k_D)$ and by $D^*$ its Gromov closure. For more information about Gromov hyperbolic spaces see Subsection 2.4.

Let $D \subset \mathbb{R}^n$ be a $\delta$-hyperbolic domain and let $f : D \to D$ be a $K$-quasiconformal homeomorphism. Note that $(D, k_D)$ is a proper geodesic metric space by [4, Proposition 2.8], and it is not difficult to see from [8, Theorem 3] that $f : (D, k_D) \to (D, k_D)$ is a rough quasi-isometry. By combining these two facts with [5, Proposition 6.3], we find that the image of any Gromov sequence under $f$ is also Gromov, and so, $f$ induces a boundary map

$$\partial f : \partial^* D \to \partial^* D.$$ 

Now define

$$T^*_K(D) = \{D \xrightarrow{f} D \mid f \text{ is } K\text{-QC so that } \partial f = \text{id}_{\partial^* D}\},$$

where $f|_D$ is $K$-quasiconformal with respect to the Euclidean metric of $\mathbb{R}^n$. Our second main result reads as follows:

**Theorem 1.2:** Let $n \geq 2$, $\delta \geq 0$, $C > 1$, and $K \geq 1$. There exists a constant $L = L(n, \delta, C, K)$ such that: If

1. $D \subset \mathbb{R}^n$ is a $\delta$-hyperbolic domain,
2. $\partial^* D$ equipped with a visual metric $\rho$ is $C$-uniformly perfect,
3. $f \in T^*_K(D),$

then

$$k_D(x, f(x)) \leq L$$

for all $x \in D$. 

Remark 1.4: The definition of visual metrics on $\partial^* D$ is given by Definition 2.11. By [7, Corollary 5.2.9], we see that $\partial^* D$ endowed with any two visual metrics are quasimöbius equivalent. It follows from [36, Lemma C] that uniform perfectness is a quasimöbius invariant and thus $\partial^* D$ is $C'$-uniformly perfect with respect to any visual metric, where $C'$ depends only on $C$, $\delta$ and the parameters of the visual metrics.

It follows from [5, Theorem 6.5] that if two Gromov hyperbolic spaces are roughly isometrically equivalent, then their boundaries at infinity are bilipschitzly equivalent with respect to visual metrics based on the same parameters. Conversely, if the boundaries at infinity of two roughly starlike Gromov hyperbolic geodesic spaces are bilipschitz equivalent, then these spaces are roughly isometrically equivalent; see [7, Theorem 7.1.2]. Indeed, we know from Theorem 1.2 that a quasiconformal map $f \in T^*_K(D)$ is a rough isometry with respect to the quasihyperbolic metric.

**Corollary 1.2:** Let $n \geq 2$, $\delta \geq 0$, $C > 1$, and $K \geq 1$. There exists a constant $L' = L'(n, \delta, C, K)$ such that: If

1. $D \subsetneq \mathbb{R}^n$ is a $\delta$-hyperbolic domain,
2. $\partial^* D$ is a $C$-uniformly perfect set,
3. $f \in T^*_K(D),$

then for all $x, y \in D$, $$k_D(x, y) - L' \leq k_D(f(x), f(y)) \leq k_D(x, y) + L'.$$

As the second application of Theorem 1.2, we investigate Teichmüller’s problem for inner uniform domains $D$ in $\mathbb{R}^n$. For $x, y \in D$, the inner Euclidean metric $d_I$ of $D$ is given by

$$d_I(x, y) := \inf\{\ell(\gamma_{x,y})\},$$

where the infimum is taken over all rectifiable curves $\gamma_{x,y}$ in $D$ with endpoints $x$ and $y$.

Let $D_I = (D, d_I)$ and let $\overline{D}_I$ be the metric completion of $D$ with respect to $d_I$. Following [12, Section 6], let $\widehat{D}_I$ be the one point compactification $\overline{D}_I \cup \{\infty\}$ of $\overline{D}_I$ if $\overline{D}_I$ is unbounded, and $\widehat{D}_I = \overline{D}_I$ if $\overline{D}_I$ is bounded. Denote by $\partial_I D$ the topological boundary of $D_I$ in $\widehat{D}_I$. Thus $$\partial_I D = \overline{D}_I \setminus D$$
if $(D, d_I)$ is bounded, and
\[ \partial_I D = (\overline{D_I} \cup \{\infty\}) \setminus D \]
if $(D, d_I)$ is unbounded. Now define
\[ T_K(D_I) = \{ \tilde{f} : \tilde{D_I} \to \tilde{D_I} \mid f \text{ is a homeomorphism s.t. } f|_D \text{ is } K\text{-QC} \text{ and } f|_{\partial D} = \text{id}_{\partial D} \}, \]
where $f|_D$ is $K$-quasiconformal with respect to the Euclidean metric of $\mathbb{R}^n$. Our result concerning Teichmüller’s problem for inner uniform domains (see Definition 2.3) is the following:

**Corollary 1.3:** Let $n \geq 2$, $A \geq 1$, $C > 1$, and $K \geq 1$. There exists a constant $H = H(n, A, C, K)$ such that: If

1. $D \subsetneq \mathbb{R}^n$ is an inner $A$-uniform domain,
2. $\partial D$ is a $C$-uniformly perfect set,
3. $f \in T_K(D_I)$,

then for all $x \in D$,
\[ k_D(x, f(x)) \leq H. \]

**Remark 1.5:** We are grateful to Matti Vuorinen for pointing out that the main motivation in studying Teichmüller’s problem is to find sharp estimates, as was already done in Teichmüller’s original work [24]. More precisely, it is interesting to understand the convergence behavior of the bounds in the stability theory whenever the quasiconformality coefficient $K$ tends to 1; see, e.g., [16, 21, 35]. In light of the proofs of Theorems 1.1 and 1.2, the related bounds are unlikely to be sharp.

Indeed, in studying problems on Gromov hyperbolic spaces, properties of geometries at large scales are the main area of concern. In particular, in order to show that any quasiconformal self-map of a uniformly quasiconformally homogeneous manifold is uniformly close to an isometry, Bonfert-Taylor, Canary, Martin, and Taylor [3] only needed a suitable bound, but sharpness was not required in their study of Teichmüller’s problem for hyperbolic manifolds.

The rest of this paper is organized as follows. In Section 2, we recall necessary definitions and preliminary results. The proof of Theorem 1.1 is given in Section 3. Section 4 is devoted to the proof of Theorem 1.2, and the proofs of Corollaries 1.1, 1.2 and 1.3 are presented in Section 5.
2. Preliminaries and auxiliary results

2.1. Notation. Let the letters $A, B, C, \ldots$ denote positive numerical constants. Similarly, $C(a, b, c, \ldots)$ denotes universal positive functions of the parameters $a, b, c, \ldots$. Sometimes we write $C = C(a, b, c, \ldots)$ to emphasize the parameters on which $C$ depends and abbreviate $C(a, b, c, \ldots)$ to $C$.

2.2. Metric geometry. Let $(X, |\cdot|)$ be a metric space, and let

$$B(x, r) = \{z \in X \mid |z - x| < r\}.$$  

The metric space $X$ is called proper if its closed balls are compact. For a bounded set $S \subseteq X$, we denote the diameter of $S$ by $\text{diam}(S)$. We use $\overline{X}$ to denote the metric completion of $X$ and $\partial X = \overline{X} \setminus X$ to be its metric boundary. A metric space $X$ is called incomplete if it is not complete. Thus incompleteness of $X$ implies that $\partial X \neq \emptyset$. The identity map of $X$ is denoted by $\text{id}_X$.

A domain $D \subseteq X$ is an open and connected non-empty set. Let $X$ be a connected and complete metric space, and let $D \subsetneq X$ be a domain. Thus $\partial D \neq \emptyset$, and we write

$$d(x) = \text{dist}(x, \partial D)$$

for all $x \in D$. For $x, y \in D$, the distance ratio distance $j_D(x, y)$ is defined by

$$(2.1) \quad j_D(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d(x), d(y)\}}\right).$$

Definition 2.1: Let $C > 1$. A metric space $X$ is $C$-uniformly perfect, if for each $x \in X$ and every $r > 0$,

$$B(x, r) \setminus B(x, r/C) \neq \emptyset$$

provided $X \setminus B(x, r) \neq \emptyset$.

We also record the following invariance property of uniform perfectness of metric spaces under quasimöbius maps (see Definition 2.9) for later use.

Lemma 2.1 ([36, Lemma C]): Suppose that $f : X \to Y$ is a $\theta$-quasimöbius homeomorphism between two metric spaces. If $X$ is $C$-uniformly perfect, then $Y$ is $C'$-uniformly perfect with $C' = C'(C, \theta)$. 

A curve is a continuous function $\gamma : \mathbb{R} \supset [a, b] \to X$. The length of $\gamma$ is defined by

$$\ell(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i-1})| \right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < t_2 < \cdots < t_n = b$. The curve $\gamma$ is called rectifiable if $\ell(\gamma) < \infty$. The metric space $X$ is called rectifiably connected if each pair of points can be connected by a rectifiable curve.

The length function associated with a rectifiable curve $\gamma : [a, b] \to X$ is $s_\gamma : [0, \ell(\gamma)] \to [0, \ell(\gamma)]$, defined by

$$s_\gamma(t) = \ell(\gamma|[a,t]) \quad \text{for } t \in [a, b].$$

For any rectifiable curve $\gamma : [a, b] \to X$, there is a unique parametrization $\gamma_s : [0, \ell(\gamma)] \to X$ such that $\gamma = \gamma_s \circ s_\gamma$. Obviously, $\ell(\gamma_s|[0,t]) = t$ for $t \in [0, \ell(\gamma)]$. The parametrization $\gamma_s$ is called the arclength parametrization of $\gamma$. For a rectifiable curve $\gamma$ in $X$, following [4, Section 10], the line integral over $\gamma$ of each Borel function $\varrho : X \to [0, \infty)$ is

$$\int_\gamma \varrho \, ds = \int_0^{\ell(\gamma)} \varrho \circ \gamma_s(t) \, dt.$$

In 1978, uniform domains in $\mathbb{R}^n$ were introduced by Martio and Sarvas [23]. In order to establish their uniformization theory of Gromov hyperbolic spaces, Bonk, Heinonen and Koskela [4] generalized this concept to the setting of metric spaces.

**Definition 2.2:** Let $A \geq 1$, and let $X$ be a rectifiably connected and complete metric space. A domain $D \subseteq X$ is called $A$-uniform if each pair of points $x, y$ in $D$ can be joined by a rectifiable arc $\gamma$ in $D$ satisfying:

1. $\ell(\gamma) \leq A |x - y|$, and
2. $\min\{\ell(\gamma[x,z]), \ell(\gamma[z,y])\} \leq A d(z)$ for all $z \in \gamma$,

where $\gamma[x, z]$ is the part of $\gamma$ between $x$ and $z$.

We remark that $D$ is called $A$-quasiconvex, if any two points of $D$ can be connected by a curve satisfying the condition (1) above. We call a domain uniform if it is $A$-uniform for some constant $A \geq 1$ and quasiconvex if it is $A$-quasiconvex for some $A \geq 1$. 
Definition 2.3: We say that a domain $D \subseteq \mathbb{R}^n$ is **inner uniform**, if $(D, d_I)$ is $A$-uniform for some $A \geq 1$, where $d_I$ is the inner Euclidean metric $d_I$ of $D$.

Definition 2.4: Let $X$ be a connected and complete metric space, and let $D \subseteq X$ be a rectifiably connected domain. The **quasihyperbolic metric** $k_D$ of $D$ is defined as

$$k_D(x, y) = \inf \int_\gamma \frac{|dz|}{d(z)},$$

where the infimum is taken over all rectifiable curves $\gamma$ joining $x$ and $y$ in $D$.

There is an important property of uniform domains associated to $k_D$ and the distance ratio metric $j_D$. The statement is as follows.

Lemma 2.2 ([4, Lemma 2.13]): Let $X$ be a locally compact, rectifiably connected metric space and $D \subseteq X$ an $A$-uniform domain. Then for all $x, y \in D$,

$$k_D(x, y) \leq 4A^2j_D(x, y).$$

Definition 2.5: Let $\psi : [0, \infty) \to [0, \infty)$ be a homeomorphism. A domain $D \subseteq \mathbb{R}^n$ is called $\psi$-**uniform** if for all $x, y$ in $D$,

$$k_D(x, y) \leq \psi(r_D(x, y)) \text{ where } r_D(x, y) = \frac{|x - y|}{\min\{d(x), d(y)\}}.$$ 

Note that an $A$-uniform domain is $\psi$-uniform with $\psi(t) = 4A^2\log(1 + t)$, which follows from Lemma 2.2.

2.3. Maps on metric spaces. Assume that $X$ and $Y$ are metric spaces. For the basic theory of quasiconformal maps we refer to [11, 26]. There are several equivalent definitions for quasiconformality in $\mathbb{R}^n$. We adopt a version of the metric definition.

Definition 2.6: Let $n \geq 2$, let $D$ and $D'$ be domains in $\mathbb{R}^n$, and let $f : D \to D'$ be a homeomorphism. For $1 \leq K < \infty$, we say that $f$ is $K$-**quasiconformal** if

$$H(x) := \limsup_{r \to 0} \frac{\sup\{|f(x) - f(y)| | |x - y| = r\}}{\inf\{|f(x) - f(z)| | |x - z| = r\}} \leq K,$$

for all $x \in D$, and that $f$ is **quasiconformal** if it is $K$-quasiconformal for some $K$.

For $K$-quasiconformal maps we have the following property:
Theorem 2.1 ([8, Theorem 3]): For \( n \geq 2, K \geq 1 \), there are constants \( C \geq 1, \mu \in (0, 1] \) depending only on \( n \) and \( K \) such that if \( D, D' \subseteq \mathbb{R}^n \) and \( f : D \to D' \) is a \( K \)-quasiconformal map, then for all \( x, y \in D \),
\[
k_{D'}(f(x), f(y)) \leq C \max\{k_D(x, y), k_D(x, y)\mu\}.
\]

Following notations and terminology of [11, 13, 25, 27, 29, 34], we next recall the definitions of quasisymmetric and quasimöbius maps:

Definition 2.7: A homeomorphism \( f \) from \( X \) to \( Y \) is said to be

1. \( \eta \)-quasisymmetric if there is a homeomorphism \( \eta : [0, \infty) \to [0, \infty) \) such that
   \[
   |x - a| \leq t|x - b| \text{ implies } |f(x) - f(a)| \leq \eta(t)|f(x) - f(b)|
   \]
   for each \( t > 0 \) and for each triplet \( x, a, b \) of points in \( X \);

2. weakly \( H \)-quasisymmetric if there is a constant \( H < \infty \) such that
   \[
   |x - a| \leq |x - b| \text{ implies } |f(x) - f(a)| \leq H|f(x) - f(b)|
   \]
   for each triplet \( x, a, b \) of points in \( X \).

Definition 2.8: Let \( X \) and \( Y \) be incomplete and connected metric spaces. Let \( 0 < q < 1, H \geq 1, \) and \( \eta : [0, \infty) \to [0, \infty) \) a homeomorphism. Suppose \( f : X \to Y \) is a homeomorphism.

The map \( f \) is said to be \( q \)-locally \( \eta \)-quasisymmetric if the restrictions \( f|_{B(z, qd(z))} \) of \( f \) to \( B(z, qd(z)) \) are \( \eta \)-quasisymmetric for all \( z \in X \), where
\[
d(z) = \text{dist}(z, \partial X).
\]

Similarly, \( f \) is called \( q \)-locally weakly \( H \)-quasisymmetric if the restrictions \( f|_{B(z, qd(z))} \) of \( f \) to \( B(z, qd(z)) \) are weakly \( H \)-quasisymmetric for all \( z \in X \).

The following result is needed in the proof of Theorem 1.1.

Theorem 2.2 ([28, Theorem 3.12]): Let \( n \geq 2, K \geq 1 \) and \( \eta : [0, \infty) \to [0, \infty) \) a homeomorphism. Suppose that \( D \) and \( D' \) are domains in \( \mathbb{R}^n \) and \( f : \overline{D} \to \overline{D'} \) is a homeomorphism such that \( f|_D \) is \( K \)-quasiconformal and \( f|_{\partial D} \) is \( \eta \)-quasisymmetric. Then \( f \) is \( \eta_1 \)-quasisymmetric with \( \eta_1 = \eta_1(K, \eta, n) \).

A quadruple in \( X \) is an ordered sequence \( Q = (a, b, c, d) \) of four distinct points in \( X \). The cross ratio of \( Q \) is defined to be the number
\[
\tau(Q) = |a, b, c, d| = \frac{|a - c|}{|a - b|} \cdot \frac{|b - d|}{|c - d|}.
\]
Definition 2.9: Note that a homeomorphism $f$ from $X$ to $Y$ is said to be \( \theta\)-\textit{quasimöbius} if \( \theta : [0, \infty) \to [0, \infty) \) is a homeomorphism such that

\[
\tau(f(Q)) \leq \theta(\tau(Q))
\]

holds for each quadruple $Q \subseteq X$.

There is a criterion for quasimöbius maps between two bounded metric spaces to be quasisymmetric given by Väisälä. For later use we record this result as follows.

**Theorem 2.3 ([27, Theorem 3.12]):** Suppose that $X$ and $Y$ are two bounded metric spaces, that \( \lambda > 1 \), that $z_1, z_2, z_3$ are in $X$, and that $f : X \to Y$ is a \( \theta\)-quasimöbius homeomorphism satisfying the three-point condition:

\[
|z_i - z_j| \geq \frac{1}{\lambda} \text{diam}(X) \quad \text{and} \quad |f(z_i) - f(z_j)| \geq \frac{1}{\lambda} \text{diam}(Y)
\]

for all \( i \neq j \in \{1, 2, 3\} \). Then $f$ is \( \eta\)-quasisymmetric with \( \eta = \eta(\theta, \lambda) \).

Definition 2.10: Let $f : X \to Y$ be a map (not necessarily continuous) between metric spaces $X$ and $Y$, and let $L \geq 1$ and $M \geq 0$ be constants.

(1) If

(a) for each $x' \in Y$, there is $x \in X$ with $|x' - f(x)| \leq M$, and

(b) for all $x, y \in X$, 

\[
L^{-1}|x - y| - M \leq |f(x) - f(y)| \leq L|x - y| + M,
\]

then $f$ is called an \( (L, M)\)-\textit{roughly quasi-isometric map} (cf. [5]). If $L = 1$, then $f$ is called an \textit{M-roughly isometric map}.

(2) Moreover, if $f$ is a homeomorphism and $M = 0$, then it is called an \textit{L-bilipschitz map}.

2.4. GROMOV HYPERBOLIC SPACES. In this subsection, we recall some necessary terminology concerning Gromov hyperbolic spaces (cf. [4, 5, 6, 7]). Let $(X, d)$ be a metric space. Fix a base point $w$ in $X$.

(1) For $x, y \in X$, let 

\[
(x|y)_w = \frac{1}{2}(d(x, w) + d(y, w) - d(x, y)).
\]

This number is called the \textit{Gromov product} of $x, y$ with respect to $w$. 
(2) The space $X$ is called **geodesic**, if each pair of points $x, y \in X$ can be joined by a geodesic $[x, y]$; that is, a curve whose length is precisely the distance between $x$ and $y$. Moreover, a **geodesic triangle** $\Delta$ is a set

$$\Delta = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1] \subseteq X.$$ 

(3) Suppose $(X, d)$ is geodesic. The metric space $X$ is called **$\delta$-hyperbolic** ($\delta \geq 0$) if each point on the edge of any geodesic triangle in $X$ is within distance $\delta$ of some point on one of the other two edges. If $X$ is $\delta$-hyperbolic for some $\delta \geq 0$, we also say that it is Gromov hyperbolic.

(4) Suppose $(X, d)$ is $\delta$-hyperbolic.

(a) A sequence $\{x_i\}$ in $X$ is called a **Gromov sequence** if $(x_i | x_j)_w \to \infty$ as $i, j \to \infty$.

(b) Two such sequences $\{x_i\}$ and $\{y_j\}$ are said to be **equivalent** if $(x_i | y_j)_w \to \infty$ as $i \to \infty$.

(c) The **Gromov boundary** $\partial^* X$ of $X$ is defined to be the set of all equivalence classes of Gromov sequences, and $X^* = X \cup \partial^* X$ is called the **Gromov closure** of $X$. For the description of the topology of $X^*$ we refer to [6, Page 429].

(d) For $a \in X$ and $b \in \partial^* X$, the **Gromov product** $(a | b)_w$ of $a$ and $b$ is defined by

$$ (a | b)_w = \inf \{ \liminf_{i \to \infty} (a | b_i)_w \mid \{b_i\} \in b \}. $$

(e) For $a, b \in \partial^* X$, the **Gromov product** $(a | b)_w$ of $a$ and $b$ is defined by

$$ (a | b)_w = \inf \{ \liminf_{i \to \infty} (a_i | b_i)_w \mid \{a_i\} \in a \text{ and } \{b_i\} \in b \}. $$

Now, we define a metric on the boundary at infinity of a Gromov hyperbolic space via the extended Gromov products; see [6, 7].

**Definition 2.11:** Let $X$ be a $\delta$-hyperbolic space with $\delta > 0$ and $w \in X$ a fixed point. For $0 < \varepsilon < \min\{1, \frac{1}{5\delta}\}$, define

$$ \rho_{w, \varepsilon}(\xi, \zeta) = e^{-\varepsilon(\xi | \zeta)_w} $$

for all $\xi, \zeta \in \partial^* X$ with the convention $e^{-\infty} = 0$.

Let

$$ d_{w, \varepsilon}(\xi, \zeta) := \inf \left\{ \sum_{i=1}^{n} \rho_{w, \varepsilon}(\xi_{i-1}, \xi_i) \mid n \geq 1, \xi = \xi_0, \xi_1, \ldots, \xi_n = \zeta \in \partial^* X \right\}. $$
Then $(\partial^* X, d_{w, \varepsilon})$ is a metric space with
\[ \frac{\rho_{w, \varepsilon}}{2} \leq d_{w, \varepsilon} \leq \rho_{w, \varepsilon}, \]
and we call $d_{w, \varepsilon}$ the visual metric on $\partial^* X$ with respect to $w \in X$ and the parameter $\varepsilon$.

In [4], Bonk, Heinonen and Koskela introduced the concept of rough starlikeness for Gromov hyperbolic space with respect to a given base point in the space. They also proved that both bounded uniform spaces and Gromov hyperbolic domains in $\mathbb{R}^n$ are roughly starlike. It turns out that this property is very useful; see for instance [2].

**Definition 2.12**: Let $\kappa \geq 0$. Suppose that $(X, d)$ is a proper, geodesic $\delta$-hyperbolic metric space and that $w \in X$. We say that $X$ is $\kappa$-roughly starlike with respect to $w$ if for each $x \in X$, there exists a point $\xi \in \partial^* X$ and a geodesic ray $\alpha = [w, \xi]$ satisfying
\[ \text{dist}(x, \alpha) \leq \kappa. \]

Further, Väisälä extended their ideas and introduced the following definition in [30].

**Definition 2.13**: Let $\kappa \geq 0$. Suppose that $(X, d)$ is a proper, geodesic $\delta$-hyperbolic metric space and that $\xi \in \partial^* X$. We say that $X$ is $\kappa$-roughly starlike with respect to $\xi$ if for each $x \in X$, there is a point $\eta \in \partial^* X$ and a geodesic line $\gamma = [\xi, \eta]$ joining $\xi$ and $\eta$ such that
\[ \text{dist}(x, \gamma) \leq \kappa. \]

### 2.5. Bonk–Heinonen–Koskela Uniformization

We now recall the following conformal deformations of proper geodesic Gromov hyperbolic spaces that were introduced by Bonk, Heinonen and Koskela; see [4, Chapter 4]. We remark that this uniformization theory has many applications; see, e.g., [2, 17, 37].

Let $D \subseteq \mathbb{R}^n$ be a $\delta$-hyperbolic domain, and $k$ its quasihyperbolic metric. Fix a base point $w \in D$, and consider the family of conformal deformations of $(D, k)$ by the densities
\[ \rho_\varepsilon(x) = e^{-\varepsilon k(x, w)} \quad (\varepsilon > 0). \]

For $x, y \in D$, let
\[ d_\varepsilon(x, y) = \inf_{\gamma} \int_{\gamma} \rho_\varepsilon \, ds_k, \]

(2.3)
where \( ds_k \) is the arclength element with respect to the metric \( k \) and the infimum is taken over all rectifiable curves \( \gamma \) in \( D \) with endpoints \( x \) and \( y \).

Then \( d_\epsilon \) are metrics on \( D \), and we denote the resulting metric spaces by \( D_\epsilon = (D, d_\epsilon) \). Moreover, \( k_\epsilon \) is the quasihyperbolic metric of \( D_\epsilon \) and \( \partial_\epsilon D \) denote the metric completion and metric boundary of \( D \) with respect to \( d_\epsilon \), respectively.

Finally, we conclude with some auxiliary results for our needed

**Lemma 2.3:** Let \( D \subsetneq \mathbb{R}^n \) be a \( \delta \)-hyperbolic domain with \( n \geq 2 \), and \( k \) its quasihyperbolic metric. There are constants \( A, \kappa, M, C, \varepsilon_0 \) that depend only on \( \delta \) such that for a fixed \( \epsilon \in (0, \varepsilon_0] \) we have:

(a) \( (D, k) \) is a complete, proper and geodesic metric space.

(b) \( (D, k) \) is \( \kappa \)-roughly starlike with respect to each point of \( D^* \).

(c) \( D_\epsilon \) is \( A \)-uniform and bounded (with diameter at most \( 2/\epsilon \)).

(d) The identity map \( (D, k) \to (D, k_\epsilon) \) is \( M \)-bilipschitz.

(e) For all \( x, y \in D \), we have

\[
C^{-1}d_\epsilon(x, y) \leq e^{-\epsilon(x \mid y_\epsilon)} \min \{1, \epsilon k(x, y)\} \leq Cd_\epsilon(x, y).
\]

(f) There is a natural map \( D^* \xrightarrow{\varphi} \overline{D_\epsilon} \) that is a bijection.

(g) There is a natural \( \theta \)-quasimöbius identification \( (\partial^* D, \rho) \xrightarrow{\psi} \partial_\epsilon D \) with \( \psi = \varphi|_{\partial^* D} \), where \( \rho \) is a visual metric on \( \partial^* D \) and \( \theta \) is a self-homeomorphism of \([0, \infty)\) depending only on \( \delta \) and the parameter of \( \rho \).

**Proof.** (a) See [4, Proposition 2.8].

(b) See [30, Theorem 3.22] and also [4, Theorem 3.6].

(c) See [4, Proposition 4.5].

(d) See [4, Proposition 4.37].

(e) See [4, Lemma 4.10].

(f) See [30, Section 2.21] or [12, Section 6] for the notion of a natural map; this is a continuous extension of the identity map

\[
D^* \supset D \xrightarrow{id} D \subset \overline{D_\epsilon},
\]

where the topology is given as described in [6, 31]. In fact, by the fact (e) or the proof of [4, Proposition 4.13], we have the following:

There is a bijection \( D^* \xrightarrow{\varphi} \overline{D_\epsilon} \) that satisfies \( \varphi|_D = \text{id}_D \) and a sequence \( \overline{\pi} = \{x_i\} \) in \( D \) that satisfies \( d_\epsilon(x_i, \xi) \to 0 \) for some \( \xi \in \partial_\epsilon D \) if and only if \( \overline{\pi} \) is a Gromov sequence in \( (D, k) \) and \( \varphi(\hat{x}) = \xi \) where \( \hat{x} \in \partial^* D \) is the equivalence class of \( \overline{\pi} \).
(g) Note that $D_\epsilon$ is induced by the density $\rho_\epsilon(x) = e^{-\epsilon k(x,w)}$. Thus it follows from [4, Proposition 4.13] that there is a natural $\eta$-quasisymmetric identification $(\partial^* D, \rho_{w,\epsilon}) \to \partial_\epsilon D$, where $\rho_{w,\epsilon}$ is a visual metric on $\partial^* D$ based at $w$ with parameter $\epsilon$. By [7, Corollary 5.2.9], we see that $\partial^* D$ endowed with any two visual metrics are quasimöbius equivalent. Hence there is a natural $\theta$-quasimöbius identification $(\partial^* D, \rho) \to \partial_\epsilon D$ for all visual metric $\rho$ on $\partial^* D$, as desired.

3. Proof of Theorem 1.1

Here we assume that

1. $D \subset \mathbb{R}^n$ is a domain,
2. $\partial D$ is a $C$-uniformly perfect set,
3. $f \in T_K(D) = \{ \overline{D} \xrightarrow{\phi} \overline{D} \mid \phi|_D$ is $K$-QC and $\phi|_{\partial D} = \text{id}_{\partial D} \}$.

First, by Theorem 2.2, we know that there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $f$ is $\eta$-quasisymmetric on $\overline{D}$ with $\eta = \eta(K,n)$. We show that for all $x \in D$,

$$j_D(x, f(x)) \leq M := 2 \log(1 + 4\eta(C)).$$

For a given point $x \in D$, take a point $x_0 \in \partial D$ such that $|x - x_0| = d(x)$.

Next, we claim that there is a point $x_1 \in \partial D$ satisfying

$$\frac{d(x)}{C} \leq |x_0 - x_1| \leq 4d(x).$$

The proof of the claim is divided into two cases.

For the first case, suppose $D \subseteq B(x, 2d(x))$. Thus

$$\text{diam}(\partial D) = \text{diam}(D) \geq 2d(x).$$

Then we may pick a point $x_1 \in \partial D$ with

$$|x_0 - x_1| \geq \frac{1}{2} \text{diam}(D) \geq d(x).$$

Moreover, because $D \subseteq B(x, 2d(x))$, we have $|x_0 - x_1| \leq 4d(x)$, as desired.

For the remaining case, suppose $D \not\subseteq B(x, 2d(x))$. Note that the boundary of $D$ is taken in the topology of the Riemann sphere $\mathbb{R}^n \cup \{\infty\}$. Since $B(x_0, d(x)) \subseteq B(x, 2d(x))$, we have $\partial D \setminus B(x_0, d(x)) \neq \emptyset$. Because $\partial D$ is $C$-uniformly perfect with respect to the spherical metric, by Remark 1.1 we know
that $\partial D$ is uniformly perfect with respect to the Euclidean metric of $\mathbb{R}^n$. So there is no loss of generality in assuming that $\partial D$ is also $C$-uniformly perfect with the same constant in $\mathbb{R}^n$. Thus we see that there exists $x_1 \in \partial D$ such that
\[
\frac{d(x)}{C} \leq |x_0 - x_1| \leq d(x),
\]
and we obtain (3.1).

Furthermore, we are going to show that there exists a constant $C_1 \geq 1$ depending on $\eta$ and $C$ such that
\[
\frac{d(x)}{C_1} \leq d(f(x)) \leq |f(x) - f(x_0)| \leq C_1 d(x).
\]
Indeed, we only need to find a constant $C_1 \geq 1$ satisfying
\[
d(f(x)) \leq |f(x) - f(x_0)| \leq C_1 d(x),
\]
because the other direction follows from a symmetric argument along with the fact that the inverse map $f^{-1}$ is $\eta'$-quasisymmetric with identity boundary values, where $\eta'(t) = \eta^{-1}(t^{-1})^{-1}$ for all $t > 0$ [25].

Because $f$ is $\eta$-quasisymmetric on $\overline{D}$ and $f|_{\partial D} = \text{id}_{\partial D}$, we may compute from (3.1) that
\[
|f(x) - f(x_0)| \leq \eta\left(\frac{|x - x_0|}{|x_1 - x_0|}\right)|f(x_1) - f(x_0)|
\leq \eta(C)|x_1 - x_0|
\leq 4\eta(C)d(x),
\]
which yields (3.2) by taking $C_1 = 4\eta(C)$.

Since $f|_{\partial D} = \text{id}_{\partial D}$, by (3.2) we find that
\[
|f(x) - x| \leq |f(x) - f(x_0)| + |x_0 - x|
\leq C_1 d(x) + d(x)
\leq C_1(C_1 + 1)\min\{d(x), d(f(x))\},
\]
which implies
\[
j_D(x, f(x)) = \log \left(1 + \frac{|f(x) - x|}{\min\{d(x), d(f(x))\}}\right) \leq 2\log(1 + C_1) = M,
\]
as desired. □
4. Proof of Theorem 1.2

Throughout this section, we assume that

1. $D \subseteq \mathbb{R}^n$ is a $\delta$-hyperbolic domain,
2. $\partial^* D$ equipped with a visual metric $\rho$ is $C$-uniformly perfect,
3. $f \in \mathcal{T}_K^*(D) = \{D \xrightarrow{\phi} D \mid \phi$ is $K$-QC so that $\partial \phi = \text{id}_{\partial^* D}\}$.

Our goal is to find a constant $L$ such that, for all $x \in D$,

\[ k_D(x, f(x)) \leq L. \]

Fix a point $w \in D$. Because $D$ is $\delta$-hyperbolic, we know from Lemma 2.3(a) and (b) that $(D, k)$ is a proper geodesic metric space, and there is a constant $\kappa = \kappa(\delta) \geq 0$ such that $D$ is $\kappa$-roughly starlike with respect to $w$.

Let $D_\epsilon = (D, d_\epsilon)$ be the BHK-uniformization of $(D, k)$ obtained via the dampening conformal deformation as described in Subsection 2.5 with $d_\epsilon$ defined as in (2.3). We record some required auxiliary results. By Lemma 2.3, there are constants $A, M, C_1$ that depend only on $\delta$ such that:

a. $D_\epsilon$ is $A$-uniform and bounded (with diameter at most $2/\epsilon$).

b. The identity map $\vartheta : (D, k) \to (D, k_\epsilon)$ is $M$-bilipschitz, where $k_\epsilon$ is the quasihyperbolic metric of $D_\epsilon$.

c. For all $x, y \in D$, we have

\[ C_1^{-1} d_\epsilon(x, y) \leq \epsilon^{-1} e^{-\epsilon|x|y} \min\{1, \epsilon k(x, y)\} \leq C_1 d_\epsilon(x, y). \]

d. There is a natural map $D^* \xrightarrow{\varphi} \overline{D_\epsilon}$ that is a bijection.

e. There is a natural $\theta$-quasimöbius identification $(\partial^* D, \rho) \xrightarrow{\psi} \partial D_\epsilon$ with $\psi = \varphi|\partial^* D$, where $\theta$ is a self-homeomorphism of $[0, \infty)$ depending only on $\delta$ and the parameter of $\rho$.

Moreover, by the assumption that $(\partial^* D, \rho)$ is $C$-uniformly perfect, it follows from the fact (e) and Lemma 2.1 that

f. $\partial D_\epsilon$ is $C_0$-uniformly perfect with $C_0$ depending only on $\theta$ and $C$.

Since $f \in \mathcal{T}_K^*(D)$, we define a map $f^* : D^* \to D^*$

\[ f^* := \begin{cases} f & \text{in } D, \\ \partial f & \text{in } \partial^* D. \end{cases} \]

Further, we consider the map

\[ g := \varphi \circ f^* \circ \varphi^{-1} : \overline{D_\epsilon} \to \overline{D_\epsilon} \]

induced by $f^*$. For later use we need to prove the following lemma.
Lemma 4.1: The map $g : \overline{D_\epsilon} \to \overline{D_\epsilon}$ satisfies the following properties:

1. $g$ is a homeomorphism with $g|_{\partial_\epsilon D} = \text{id}_{\partial_\epsilon D}$;
2. $g|_{(D, d_\epsilon)}$ is $\eta_0$-quasisymmetric with $\eta_0 = \eta_0(\delta, K, n)$.

The proof of Lemma 4.1 is divided into two parts.

4.1. Proof of Lemma 4.1 (1). The assertion follows from the following statements:

(i) $g$ is a bijection,
(ii) $g|_D$ is a homeomorphism,
(iii) the continuous extension of $g|_{D_\epsilon}$ to the boundary $\partial_\epsilon D$ satisfies $g|_{\partial_\epsilon D} = \text{id}_{\partial_\epsilon D}$.

The statement (i) follows from the fact that both $f^*$ and $\varphi$ are bijections. For (ii), the statement (d) implies that $\varphi$ is a natural map with $\varphi|_D = \text{id}_D$. Thus by the definitions of $g$ and $f^*$, we know that $g|_{D_\epsilon} = f|_D$. Now it follows from [4, Proposition 2.8] that the identity maps

$$(D, | \cdot |) \to (D, k) \quad \text{and} \quad (D, d_\epsilon) \to (D, k_\epsilon)$$

are both homeomorphisms. Since $f|_{(D, | \cdot |)}$ is a homeomorphism, by the statement (b), $g|_{D_\epsilon}$ is a homeomorphism as well.

It remains to show (iii). For any sequence $\{x_i\}$ in $D_\epsilon$ with $d_\epsilon(x_i, p) \to 0$ for some $p \in \partial_\epsilon D$, the statement (d) implies that $\{\varphi^{-1}(x_i)\} = \{x_i\}$ is a Gromov sequence in $(D, k)$ with

$$\varphi^{-1}(p) = \xi \in \partial^* D.$$

Since $f \in T^*_K(D)$, we see from Theorem 2.1 that $f : (D, k) \to (D, k)$ is a rough quasi-isometry. Then by [5, Proposition 6.3], $\{f \circ \varphi^{-1}(x_i)\} = \{f(x_i)\}$ is also Gromov in $(D, k)$ with $f^*(\xi) = \xi \in \partial^* D$. Again by the statement (d), the sequence $\{g(x_i) = \{\varphi \circ f \circ \varphi^{-1}(x_i)\}\} \text{satisfies}$

$$d_\epsilon(g(x_i), p) = d_\epsilon(\varphi \circ f \circ \varphi^{-1}(x_i), \varphi(\xi)) = d_\epsilon(f(x_i), p) \to 0.$$ 

This ensures that the continuous extension of $g|_{D_\epsilon}$ to the boundary $\partial_\epsilon D$ satisfies $g|_{\partial_\epsilon D} = \text{id}_{\partial_\epsilon D}$, as required.
4.2. Proof of Lemma 4.1(2). We begin with some preparations and divide the proof into several steps.

Let $D'_\epsilon = (D, d'_\epsilon)$ be the BHK-uniformization of $(D, k)$ obtained via the dampening conformal deformation as described in Subsection 2.5 with $d'_\epsilon$ defined as in (2.3), but now we use the base point $w' = f(w)$. Again by Lemma 2.3(b), (c) and (d), we know that

1. $(D, k)$ is $\kappa$-roughly starlike with respect to $w'$,
2. $(D, d'_\epsilon)$ is $A$-uniform, and
3. the identity map $\vartheta' : (D, k) \to (D, k'_\epsilon)$ is $M$-bilipschitz, where $k'_\epsilon$ is the quasihyperbolic metric of the space $(D, d'_\epsilon)$.

Moreover, by Lemma 2.3(e), a direct computation shows that the identity map

$$\phi : (D, d_\epsilon) \to (D, d'_\epsilon)$$

is $\theta_0$-quasisymmetric with $\theta_0(t) = C't$ and $C'$ depending only on $\delta$. Then $g$ induces a map

$$h := \phi \circ g : (D, d_\epsilon) \to (D, d'_\epsilon).$$

Outline of the proof of Lemma 4.1(2). In the following, we first show that $h$ is quasisymmetric; see Lemma 4.2. Because the composition of a quasisymmetric map and a quasisymmetric map is also quasisymmetric (cf. [27, 36]), we observe that $g = \phi^{-1} \circ h$ is quasimöbius. After that, we want to use Theorem 2.3 to check the quasisymmetry of $g$. So we only need to verify the three-point condition (2.2) stated in Theorem 2.3; see Lemma 4.3.

Lemma 4.2: The map $h : (D, d_\epsilon) \to (D, d'_\epsilon)$ is $\theta$-quasisymmetric with $\theta$ depending only on $n, \delta$ and $K$.

Proof. First, we record some results from the previous arguments:

1. $(D, d_\epsilon)$ and $(D, d'_\epsilon)$ are both $A$-uniform and so they are $A$-quasiconvex,
2. the identity maps $\vartheta : (D, k) \to (D, k_\epsilon)$ and $\vartheta' : (D, k) \to (D, k'_\epsilon)$ are both $M$-bilipschitz.

Consider the identity maps

$$\tau : (D, |\cdot|) \to (D, d_\epsilon)$$

and

$$\tau' = \phi \circ \tau : (D, |\cdot|) \to (D, d'_\epsilon).$$
Since $\varphi|_D = \phi = \text{id}_D = \tau$, we have $h(x) = g(x) = f(x)$ for all $x \in D$. Therefore, for all $x \in D$,

$$h(x) = \phi \circ g(x) = \phi \circ \varphi \circ f \circ \varphi^{-1}(x) = \tau' \circ f \circ \tau^{-1}(x).$$

Then we show

**Claim 4.1:** The map $h:(D, d) \to (D, d')$ is $q$-locally weakly $H$-quasisymmetric with $q$ and $H$ depending only on $n, \delta$ and $K$.

By argument (2) and [14, Theorem 3.7], we see that the restrictions of $\tau^{-1}$ and $\tau'$ on each subdomain of $D$ are both $M'$-bilipschitz with respect to the quasihyperbolic metrics, where $M'$ depends only on $M$ and $A$. Moreover, it follows from [15, Theorem 1.8] that both $\tau^{-1}$ and $\tau'$ are $q_1$-locally weakly $H_1$-quasisymmetric on $D$, where $q_1$ and $H_1$ depend only on $M'$ and $A$.

Furthermore, because $f|_D$ is $K$-quasiconformal, we see from [11, Theorem 11.14] that $f|_D$ is $\frac{1}{2}$-locally $\eta_1$-quasisymmetric for some homeomorphism $\eta_1 : [0, \infty) \to [0, \infty)$ with $\eta_1$ depending only on $K$ and $n$.

By [15, Theorem 1.12], we know that the composition of two locally weakly quasisymmetric maps is also locally weakly quasisymmetric. This fact, together with the locally weak quasisymmetry of $\tau', f$ and $\tau^{-1}$, shows that $h:(D, d) \to (D, d')$ is $q$-locally weakly $H$-quasisymmetric because

$$h = \tau' \circ f \circ \tau^{-1},$$

which shows Claim 4.1.

Note that $(D, d_\epsilon)$ and $(D, d'_\epsilon)$ are both $A$-quasiconvex. To prove that $h$ is quasisymmetric, it follows from [29, Theorem 6.6] that we only need to find a constant $H_0 \geq 1$ depending only on $n, \delta$ and $K$ such that $h$ is weakly $H_0$-quasisymmetric.

For each triple of distinct points $x, y, z \in D$ with $d_\epsilon(x, y) \leq d_\epsilon(x, z)$, we show that

$$d'_\epsilon(h(x), h(y)) \leq H_0 d'_\epsilon(h(x), h(z))$$

for some constant $H_0$.

Let

$$t = \frac{\epsilon}{M} \log(1 + q) < 1.$$
Since \( f \mid_D \) is \( K \)-quasiconformal, we know that the inverse map \( f \mid_D^{-1} \) is \( K' \)-quasiconformal with \( K' = K'(K, n) \) because the inverse map of a quasiconformal homeomorphism is again quasiconformal. Thus it follows from Theorem 2.1 that there is a homeomorphism \( \psi = \psi(n, K) : [0, \infty) \to [0, \infty) \) such that for all \( u, v \in D \),

\[
(4.2) \quad \psi^{-1}(k(u, v)) \leq k(f(u), f(v)) \leq \psi(k(u, v)).
\]

We divide the proof of (4.1) into three cases.

**Case 4.1:** Suppose \( \epsilon k(x, z) < t \).

In this case, by the elementary inequality [4, (2.4)] and the choice of \( t \) we have

\[
\frac{d_\epsilon(x, y)}{d_\epsilon(x)} \leq \frac{d_\epsilon(x, z)}{d_\epsilon(x)} \leq \epsilon^{k(x, z)} - 1 < q,
\]

which implies that \( x, y, z \in B_\epsilon(x, qd_\epsilon(x)) \), where \( d_\epsilon(x) = d_\epsilon(x, \partial D) \) and \( B_\epsilon(x, qd_\epsilon(x)) = \{ y \in D \mid d_\epsilon(x, y) < qd_\epsilon(x) \} \).

Hence we obtain (4.1) from Claim 4.1 by choosing \( H_0 = H \).

**Case 4.2:** Suppose \( \epsilon k(x, z) \geq t \) and \( \epsilon k(x, y) < 1 \).

Because \( \epsilon k(x, z) \geq t \), we see from (4.2) that

\[
(4.3) \quad \epsilon k(h(x), h(z)) = \epsilon k(f(x), f(z)) \geq \epsilon \psi^{-1}(t/\epsilon) =: 1/C_2,
\]

and similarly

\[
(4.4) \quad \epsilon k(h(x), h(y)) = \epsilon k(f(x), f(y)) \leq \epsilon \psi(1/\epsilon) =: C_3.
\]

Then, by using (4.3), (4.4) and the statement (c), we have

\[
\frac{d'_\epsilon(h(x), h(y))}{d'_\epsilon(h(x), h(z))} \leq C_1^2 e^{\epsilon h(x)} = e^{\epsilon k(h(x), h(y))} \leq C_1^2 C_2 e^{C_3},
\]

where \( C_1 = C_1(\delta) \) is the constant of the statement (c). By setting

\[
H_0 = C_1^2 C_2 e^{C_3},
\]

we obtain (4.1).
CASE 4.3: Suppose $\epsilon k(x, z) \geq t$ and $\epsilon k(x, y) \geq 1$.

Because $d_\epsilon(x, y) \leq d_\epsilon(x, z)$, we note again from the statement (c) that

$$e^{(x|z)_w - (x|y)_w} \frac{\min\{1, \epsilon k(x, y)\}}{\min\{1, \epsilon k(x, z)\}} \leq C_1^2$$

and so

$$(x|z)_w - (x|y)_w \leq \frac{2\log C_1}{\epsilon}.$$  

It follows from (4.2) that $f : (D, k) \rightarrow (D, k)$ and its inverse map are both $\psi$-uniformly continuous (for the definition see [29, Section 2]). Because $\epsilon k(x, z) \geq t$, by (4.2) we obtain

$$\epsilon k(h(x), h(z)) \geq 1/C_2.$$  

Moreover, as $(D, k)$ is geodesic, by [29, Theorem 2.5] we know that there are positive constants $\lambda$ and $\mu$ depending only on $\psi$ such that $f : (D, k) \rightarrow (D, k)$ is a $(\lambda, \mu)$-rough quasi-isometry. Consequently, we see from (4.5) and [5, Proposition 5.5] that there is a constant $C_4 = C_4(\lambda, \mu, \delta, C_1, \epsilon)$ such that

$$\epsilon(h(x)|h(z))_{w'} - \epsilon(h(x)|h(y))_{w'} \leq C_4.$$  

So again by the statement (c) and (4.6), we obtain

$$\frac{d'_\epsilon(h(x), h(y))}{d'_\epsilon(h(x), h(z))} \leq C_1^2 e^{\epsilon(h(x)|h(z))_{w'} - \epsilon(h(x)|h(y))_{w'}} \frac{\min\{1, \epsilon k(h(x), h(y))\}}{\min\{1, \epsilon k(h(x), h(z))\}}$$

$$\leq C_1^2 C_2 e^{C_4},$$

as needed.  

As mentioned before, to prove the quasisymmetry of $g$, it suffices to show that $g$ satisfies the three-point condition (2.2) stated in Theorem 2.3.

LEMA 4.3: There are three distinct points $\xi_1, \xi_2, \xi_3 \in \partial_\epsilon D$, and a number $\lambda_0 > 0$ such that

$$d_\epsilon(g(\xi_i), g(\xi_j)) = d_\epsilon(\xi_i, \xi_j) \geq \lambda_0 \text{diam}_\epsilon(D_\epsilon),$$

for all $i \neq j \in \{1, 2, 3\}$.

Proof. To this end, we first show that

$$\text{diam}_\epsilon(\partial_\epsilon D) \leq \text{diam}_\epsilon(D_\epsilon) \leq M_1 \text{diam}_\epsilon(\partial_\epsilon D),$$

where $M_1 = Ae^{\epsilon k}$. 


Because $\text{diam}_\epsilon(\partial \epsilon D) \leq \text{diam}_\epsilon(D\epsilon)$, it suffices to check the second inequality.

We note from the statement (d) that there is a natural identification $\varphi : \partial^* D \to \partial \epsilon D$. Fix a point $\xi' \in \partial \epsilon D$ and take $\xi \in \partial^* D$ with $\varphi(\xi) = \xi'$. By Lemma 2.3(b), $(D, k)$ is $\kappa$-roughly starlike with respect to $\xi$. It follows that for $w \in D$, there is another point $\xi \in \partial^* D$ and a quasihyperbolic geodesic line $\gamma = [\xi, \zeta]_k$ joining $\xi$ and $\zeta$ such that $\{\gamma(n)\}_{n=1}^{\infty} \in \xi$, $\{\gamma(n)\}_{n=1}^{\infty} \in \zeta$ and the quasihyperbolic distance $\text{dist}_k(w, [\xi, \zeta]_k) \leq \kappa$.

This shows that there is a point $w_0 \in [\xi, \zeta]_k$ satisfying $k(w, w_0) \leq \kappa$.

Let $(\mathbb{R}, |\cdot|) \xrightarrow{\gamma} (D, k)$ be a $k$-arc-length parametrization of the quasihyperbolic geodesic line $(\xi, \zeta)_k$ with $\gamma(0) = w_0$, $\gamma(-\infty) = \xi$ and $\gamma(+\infty) = \zeta$. Then for each $x \in (\xi, \zeta)_k$,

$$k(x, w) \leq k(x, w_0) + k(w_0, w) \leq \kappa + k(x, w_0).$$

Therefore,

$$\ell_{\epsilon}(\gamma) = \int_{\gamma} e^{-\epsilon k(w, x)} \, ds_k \geq e^{-\epsilon k} \int_{\gamma} e^{-\epsilon k(w_0, x)} \, ds_k,$$

$$= e^{-\epsilon k} \int_{-\infty}^{+\infty} e^{-\epsilon |t|} \, dt = 2e^{-\epsilon k} \epsilon^{-1}.$$

Moreover, by the proof of [4, Proposition 4.5], $\gamma$ is an $A$-uniform arc in $D\epsilon$. This gives

$$\text{diam}_\epsilon(\partial \epsilon D) \geq d_\epsilon(\xi, \zeta) \geq A^{-1} \ell_{\epsilon}(\gamma) \geq \frac{e^{-\epsilon k}}{A} \frac{2}{\epsilon} \geq \frac{e^{-\epsilon k}}{A} \text{diam}_\epsilon(D\epsilon),$$

where the last inequality follows from the statement (a) that $\text{diam}_\epsilon(D\epsilon) \leq 2/\epsilon$. Hence we obtain (4.7).

Next, take two points $\xi_1$ and $\xi_2$ in $\partial \epsilon D$ with

$$d_\epsilon(\xi_1, \xi_2) = \text{diam}_\epsilon(\partial \epsilon D).$$

The statement (f) guarantees that $\partial \epsilon D$ is $C_0$-uniformly perfect. Because

$$\partial \epsilon D \setminus B_\epsilon(\xi_1, \frac{1}{2}d_\epsilon(\xi_1, \xi_2)) \neq \emptyset,$$

it follows that there is a point $\xi_3 \in \partial \epsilon D$ with

$$\frac{d_\epsilon(\xi_1, \xi_2)}{2C_0} \leq d_\epsilon(\xi_1, \xi_3) \leq \frac{1}{2}d_\epsilon(\xi_1, \xi_2)$$
and therefore
\[ d_\varepsilon(\xi_3, \xi_2) \geq d_\varepsilon(\xi_1, \xi_2) - d_\varepsilon(\xi_1, \xi_3) \geq \frac{1}{2} d_\varepsilon(\xi_1, \xi_2). \]
This, together with (4.7), shows that for all \( i \neq j \in \{1, 2, 3\}, \)
\[ d_\varepsilon(\xi_i, \xi_j) \geq \frac{\text{diam}_\varepsilon(\partial \varepsilon D)}{2C_0} \geq \frac{\text{diam}_\varepsilon(D_\varepsilon)}{2C_0 Ae^{\varepsilon\kappa}}. \]
Hence \( \lambda_0 = (2C_0 Ae^{\varepsilon\kappa})^{-1} \) is as needed.

In the following, we continue the proof of Theorem 1.2.

Note that we only need to check that there is a constant \( \Lambda \geq 0 \) depending only on \( n, K, \delta \) and \( C \) such that for all \( x \in D, \)
\[ (4.8) \quad k_\varepsilon(x, g(x)) \leq \Lambda. \]
Indeed, we see from the statement (b) and (4.8) that
\[ k(x, f(x)) \leq Mk_\varepsilon(x, g(x)) \leq \Lambda M, \]
which is the required estimate in Theorem 1.2 with the choice of \( L = MA. \)
Thus, it remains to prove (4.8).

4.3. PROOF OF (4.8). The following arguments for (4.8) are similar to the proof of Theorem 1.1. For completeness, we show the details. Before proceeding further, we need to prove some technical statements.

Fix \( x \in D \) and take \( x_0 \in \partial_\varepsilon D \) such that
\[ d_\varepsilon(x) = d_\varepsilon(x, \partial_\varepsilon D) = d_\varepsilon(x, x_0). \]
We first show that there is a constant \( M_2 = M_2(M_1, C_0) \geq 1 \) and a point \( x_1 \in \partial_\varepsilon D \) satisfying
\[ (4.9) \quad \frac{1}{M_2} d_\varepsilon(x) \leq d_\varepsilon(x_1, x_0) \leq 4d_\varepsilon(x). \]
We consider two possibilities. If \( \partial_\varepsilon D \subseteq B_\varepsilon(x, 2d_\varepsilon(x)) \), then there is a point \( x_1 \in \partial_\varepsilon D \) with
\[ d_\varepsilon(x_1, x_0) \geq \frac{1}{2} \text{diam}_\varepsilon(\partial_\varepsilon D) \geq \frac{1}{2M_1} \text{diam}_\varepsilon(D_\varepsilon) \geq \frac{d_\varepsilon(x)}{2M_1}, \]
where the penultimate inequality follows from (4.7). Moreover, we have
\[ d_\varepsilon(x_1, x_0) \leq d_\varepsilon(x_1, x) + d_\varepsilon(x, x_0) \leq 4d_\varepsilon(x), \]
which implies (4.9).
If \( \partial \epsilon D \not\subseteq B_\epsilon(x, 2d_\epsilon(x)) \), then we have
\[
\partial \epsilon D \setminus B_\epsilon(x_0, d_\epsilon(x)) \neq \emptyset,
\]
because \( B_\epsilon(x_0, d_\epsilon(x)) \subseteq B_\epsilon(x, 2d_\epsilon(x)) \). Moreover, as \( \partial \epsilon D \) is \( C_0 \)-uniformly perfect, it follows that there exists some point \( x_1 \in \partial \epsilon D \) such that
\[
\frac{d_\epsilon(x)}{C_0} \leq d_\epsilon(x_1, x_0) \leq d_\epsilon(x),
\]
as desired. Hence we obtain (4.9) with the choice of \( M_2 = 2M_1C_0 \).

Next, we show that there is a constant \( M_3 \geq 1 \) such that
\[
(4.10) \quad \frac{d_\epsilon(x)}{M_3} \leq d_\epsilon(g(x)) \leq d_\epsilon(g(x), g(x_0)) \leq M_3d_\epsilon(x).
\]
By symmetry, we only need to show that
\[
d_\epsilon(g(x), g(x_0)) \leq M_3d_\epsilon(x),
\]
because the inverse map of an \( \eta \)-quasisymmetric homeomorphism is \( \eta' \)-quasisymmetric with \( \eta'(t) = \eta^{-1}(t^{-1})^{-1} \) for all \( t > 0 \) (cf. [25]).

It follows from Lemma 4.1 and [25, Theorem 2.25] that \( g : \overline{D}_\epsilon \to \overline{D}_\epsilon \) is \( \eta_0 \)-quasisymmetric with \( g|_{\partial \epsilon D} = \text{id}_{\partial \epsilon D} \), where \( \eta_0 = \eta_0(\delta, K, n, C) \). Now by (4.9), we have
\[
d_\epsilon(g(x), g(x_0)) \leq \eta_0 \left( \frac{d_\epsilon(x, x_0)}{d_\epsilon(x_1, x_0)} \right) d_\epsilon(g(x_1), g(x_0))
\]
\[
\leq \eta_0(M_2)d_\epsilon(x_1, x_0)
\]
\[
\leq 4\eta_0(M_2)d_\epsilon(x),
\]
which shows (4.10) by taking \( M_3 = 4\eta_0(M_2) \).

Because \( g|_{\partial \epsilon D} = \text{id}_{\partial \epsilon D} \), we obtain from (4.10) that
\[
d_\epsilon(g(x), x) \leq d_\epsilon(x_0, x) + d_\epsilon(g(x), g(x_0))
\]
\[
\leq d_\epsilon(x) + M_3d_\epsilon(x)
\]
\[
\leq M_3(M_3 + 1) \min\{d_\epsilon(x), d_\epsilon(g(x))\}.
\]

Moreover, as \( (D, d_\epsilon) \) is \( A \)-uniform, we see from Lemma 2.2 and (4.11) that
\[
k_\epsilon(x, g(x)) \leq 4A^2 \log \left( 1 + \frac{d_\epsilon(g(x), x)}{\min\{d_\epsilon(x), d_\epsilon(g(x))\}} \right)
\]
\[
\leq 4A^2 \log[1 + M_3(M_3 + 1)] =: \Lambda,
\]
as desired. This proves (4.8).
5. Proofs of Corollaries 1.1, 1.2 and 1.3

5.1. Proof of Corollary 1.1. Assume that $D \subset \mathbb{R}^n$ is a $\psi$-uniform domain, that $\partial D$ is a $C$-uniformly perfect set, and that $f \in \mathcal{T}_K(D)$. We first see from Theorem 1.1 that there is a constant $M = M(n, C, K)$ such that for all $x \in D$,

$$j_D(x, f(x)) \leq M.$$

This yields

$$k_D(x, f(x)) \leq \psi(r_D(x, y))$$

$$= \psi(e^{j_D(x, f(x))} - 1)$$

$$\leq \psi(e^M - 1) =: M',$$

due to $D$ being $\psi$-uniform.

5.2. Proof of Corollary 1.2. Assume that $D \subset \mathbb{R}^n$ is a $\delta$-hyperbolic domain, that $\partial^* D$ is a $C$-uniformly perfect set, and that $f \in \mathcal{T}_K^*(D)$. By Theorem 1.2 we know that there is a constant $L = L(n, \delta, C, K)$ such that for all $x \in D$,

$$k_D(x, f(x)) \leq L.$$

Then for all $x, y \in D$, we obtain

$$|k_D(f(x), f(y)) - k_D(x, y)| \leq k_D(x, f(x)) + k_D(y, f(y)) \leq 2L =: L',$$

as desired.

5.3. Proof of Corollary 1.3. Assume that $D \subset \mathbb{R}^n$ is an inner $A$-uniform domain, that $\partial I D$ is a $C$-uniformly perfect set, and that $f \in \mathcal{T}_K(D_I)$.

Note first that $(D, d_I)$ is an $A$-uniform incomplete locally compact metric space. It follows from [4, Theorem 3.6] that $(D, k)$ is a proper and geodesic $\delta$-hyperbolic space with $\delta = \delta(A) \geq 0$, i.e., $D$ is a $\delta$-hyperbolic domain in $\mathbb{R}^n$.

Next, we see from [12, Theorem 6.2] that there is a natural map $\phi : D^* \to \overline{D}_I$ such that

$$\phi|_D = \text{id}_D$$

and $\phi : (\partial^* D, d_{w, \varepsilon}) \to (\partial I D, d_I)$ is $\eta$-quasimöbius with $\eta = \eta(A)$, where $\partial^* D$ is the Gromov boundary of $(D, k)$ and $d_{w, \varepsilon}$ is a visual metric on $\partial^* D$ with base point $w \in D$ and parameter $\varepsilon = \varepsilon(A) > 0$. Then by Lemma 2.1, we see that $\partial^* D$ is $C_0$-uniformly perfect with constant $C_0 = C_0(C, \eta) = C_0(C, A)$. 
Now we define
\[ g := \phi \circ f \circ \phi^{-1} : D^* \to D^*. \]
As \( f \in T_K(D_I) \) and \( \phi|_D = \text{id}_D \), we know that \( g \in T^*_K(D) \). Consequently, by Theorem 1.2, we see that there is a constant \( H = H(n, \delta, C_0, K) = H(n, A, C, K) \) such that for all \( x \in D \),
\[ k_D(x, f(x)) = k_D(x, g(x)) \leq H, \]
as desired.

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**References**

[1] G. D. Anderson and M. K. Vamanamurthy, *An extremal displacement mapping in n-space*, in *Complex Analysis Joensuu 1978 (Proc. Colloq., Univ. Joensuu, 1978)*, Lecture Notes in Mathematics, Vol. 747, Springer, Berlin, 1979, pp. 1–9.

[2] Z. M. Balogh and S. M. Buckley, *Geometric characterizations of Gromov hyperbolicity*, Inventiones Mathematicae 153 (2003), 261–301.

[3] P. Bonfert-Taylor, R. D. Canary, G. Martin and E. Taylor, *Quasiconformal homogeneity of hyperbolic manifolds*, Mathematische Annalen 331 (2005), 281–295.

[4] M. Bonk, J. Heinonen and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, Astériques 270 (2001).

[5] M. Bonk and O. Schramm, *Embeddings of Gromov hyperbolic spaces*, Geometric and Functional Analysis 10 (2000), 266–306.

[6] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften, Vol. 319, Springer, Berlin, 1999.

[7] S. Buyalo and V. Schroeder, *Elements of Asymptotic Geometry*, EMS Monographs in Mathematics, European Mathematical Society, Zürich, 2007.

[8] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasi-hyperbolic metric*, Journal d’Analyse Mathématique 36 (1979), 50–74.

[9] F. W. Gehring and B. P. Palka, *Quasiconformally homogeneous domains*, Journal d’Analyse Mathématique 30 (1976), 172–199.

[10] P. Hästö, R. Klén, S. K. Sahoo and M. Vuorinen, *Geometric properties of \( \varphi \)-uniform domains*, Journal of Analysis 24 (2016), 57–66.

[11] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Universitext, Springer, Berlin–Heidelberg–New York, 2001.

[12] D. Herron, N. Shanmugalingam and X. Xie, *Uniformity from Gromov hyperbolicity*, Illinois Journal of Mathematics 52 (2008), 1065–1109.
[13] X. Huang and J. Liu, Quasihyperbolic metric and quasisymmetric mappings in metric spaces, Transactions of the American Mathematical Society 367 (2015), 6225–6246.

[14] X. Huang, H. Liu and J. Liu, Local properties of quasihyperbolic mappings in metric spaces, Annales Academiae Scientiarum Fennicae. Mathematica 41 (2016), 23–40.

[15] M. Huang, A. Rasila, X. Wang and Q. Zhou, Semisolidity and locally weak quasisymmetry of homeomorphisms in metric spaces, Studia Mathematica 242 (2018), 267–301.

[16] R. Klén, V. Todorčević and M. Vuorinen, Teichmüller’s problem in space, Journal of Mathematical Analysis and Applications 455 (2017), 1297–1316.

[17] P. Koskela, P. Lammi and V. Manojlović, Gromov hyperbolicity and quasihyperbolic geodesics, Annales Scientifiques de l’École Normale Supérieure 47 (2014), 975–990.

[18] J. Krzyż, On an extremal problem of F. W. Gehring, Bulletin de l’Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques 16 (1968), 99–101.

[19] F. Kwakkel and V. Markovic, Quasiconformal homogeneity of genus zero surfaces, Journal d’Analyse Mathématique 113 (2011), 173–195.

[20] Y. Li, M. Vuorinen and X. Wang, Quasiconformal maps with bilipschitz or identity boundary values in Banach spaces, Annales Academiae Scientiarum Fennicae. Mathematica 39 (2014), 905–917.

[21] V. Manojlović and M. Vuorinen, On quasiconformal maps with identity boundary values, Transactions of the American Mathematical Society 363 (2011), 2467–2479.

[22] G. J. Martin, The Teichmüller problem for mean distortion, Annales Academiae Scientiarum Fennicae. Mathematica 34 (2009), 233–247.

[23] O. Martio and J. Sarvas, Injectivity theorems in plane and space, Annales Academiae Scientiarum Fennicae. Mathematica 4 (1978), 383–401.

[24] O. Teichmüller, Ein Verschiebungssatz der quasikonformen Abbildung, Deutsche Mathematik 7 (1944), 336–343.

[25] P. Tukia and J. Väisälä, Quasisymmetric embeddings of metric spaces, Annales Academiae Scientiarum Fennicae. Mathematica 5 (1980), 97–114.

[26] J. Väisälä, Lectures on n-Dimensional Quasiconformal Mappings, Lecture Notes in Mathematics, Vol. 229, Springer, Berlin–New York, 1971.

[27] J. Väisälä, Quasimöbius maps, Journal d’Analyse Mathématique 44 (1985), 218–234.

[28] J. Väisälä, Quasisymmetry and unions, Manuscripta Mathematica 68 (1990), 101–111.

[29] J. Väisälä, The free quasigroup, freely quasiconformal and related maps in Banach spaces, in Quasiconformal geometry and dynamics (Lublin, 1996), Banach Center Publications, Vol. 48, Institute of Mathematics, Polish Academy of Sciences, Warsaw, 1999, pp. 355–118.

[30] J. Väisälä, Hyperbolic and uniform domains in Banach spaces, Annales Academiae Scientiarum Fennicae. Mathematica 30 (2005), 261–302.

[31] J. Väisälä, Gromov hyperbolic spaces, Expositiones Mathematicae 23 (2005), 187–231.

[32] M. Vuorinen, A remark on the maximal dilatation of a quasiconformal mapping, Proceedings of the American Mathematical Society 92 (1984), 505–508.

[33] M. Vuorinen, Conformal invariants and quasiregular mappings, Journal d’Analyse Mathématique 45 (1985), 69–115.
[34] M. Vuorinen, *Conformal Geometry and Quasiregular Mappings*, Lecture Notes in Mathematics, Vol. 1319, Springer, Berlin, 1988.

[35] M. Vuorinen and X. Zhang, *Distortion of quasiconformal mappings with identity boundary values*, Journal of the London Mathematical Society **90** (2014), 637–653.

[36] X. Wang and Q. Zhou, *Quasimöbius maps, weakly quasimöbius maps and uniform perfectness in quasi-metric spaces*, Annales Academiae Scientiarum Fennicæ. Mathematica **42** (2017), 257–284.

[37] Q. Zhou, Y. Li and A. Rasila, *Gromov hyperbolicity, John spaces and quasihyperbolic geodesic*, Journal of Geometric Analysis **32** (2022), Article no. 228.