A NOTE ON THE POINTS WITH DENSE ORBIT UNDER $\times 2$ AND $\times 3$ MAPS

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Abstract. It was conjectured by Furstenberg that for any $x \in [0, 1] \setminus \mathbb{Q}$,
$$\dim_H \{2^n x \pmod{1} : n \geq 1\} + \dim_H \{3^n x \pmod{1} : n \geq 1\} \geq 1.$$ When $x$ is a normal number, the above result holds trivially. In this note, we give explicit non-normal numbers for which the above dimensional formula holds.

1. Introduction

Let $b \geq 2$ be an integer and $T_b$ be the $b$-ary expansion given by
$$T_b x = bx \pmod{1}.$$ $x$ is a real number given by its $b$--ary expansion, that is,
$$x = \lfloor x \rfloor + \sum_{k \geq 1} \frac{a_k}{b^k} = \lfloor x \rfloor + 0.a_1a_2 \cdots$$
where the digit $a_1, a_2, \cdots$ are integers from $\{0, 1, \ldots, b-1\}$ and an infinity of the $a_k$ are not equal to $b-1$. Set
$$A_b(d, N, x) := \text{Card}\{j : 1 \leq j \leq N, a_j = d\}.$$ More generally, for a block $D_k = d_1 \cdots d_k$ of $k$ digits from $\{0, 1, \cdots, b-1\}$, set
$$A_b(D_k, N, x) := \text{Card}\{j : 0 \leq j \leq N - k, a_j + 1 = d_1, \cdots, a_{j+k} = d_k\}.$$ A real number $x$ is normal to base $b$, if and only if, for every $k \geq 1$, every block of $k$ digits from $\{0, 1, \ldots, b-1\}$ occurs in the $b$--ary expansion of $x$ with the same frequency $\frac{1}{b^k}$, that is, if and only if,
$$\lim_{n \to \infty} \frac{A_b(D_k, N, x)}{N} = \frac{1}{b^k}$$
for every $k \geq 1$ and every block $D_k$ of $k$ digits from $\{0, 1, \ldots, b-1\}$.

Given one integer $b \geq 2$, the dynamical properties of the orbit $T_b^n(x)$ were well studied in the literature and are clearly known at least for almost all $x$. However, as far as two integers $b_1$ and $b_2$ are concerned (assume that they are multiplicatively independent), the properties of the orbits $T_{b_1}^n(x)$ and $T_{b_2}^n(x)$ with a common point $x$ is far from being well understood. Very little is known on this topic. Intuitively, if
the expansion of $x$ under $T_{k_1}$ is simple, it cannot be too simple under $T_{k_2}$ (for some evidence see [3]). And the normality of a number is closely related to his expansion to different bases([2],[?]), nowadays, the investigate of normal number has been used to resolve the problems in dynamical system and ergodic.

Furstenberg conjectured that [1]: for any $x \in [0,1]\setminus \mathbb{Q}$,
\[
\dim_H \{2^n x (\mod 1) : n \geq 1 \} + \dim_H \{3^n x (\mod 1) : n \geq 1 \} \geq 1.
\]
It is clear that this is true for almost all $x$, since almost all $x$ are normal numbers. So, a natural question is to ask, besides normal numbers, can one give explicit examples fulfilling the above conjecture. In this short note, we proof there exist a set $E$ of full dimension which make Furstenberg conjectured hold.

2. Some notation

We use $C_1$ and $C_2$ to denote the collection of finite 2-adic and 3-adic words respectively, i.e.
\[
C_1 = \bigcup_{n \geq 1} \{0,1\}^n, \quad C_2 = \bigcup_{n \geq 1} \{0,1,2\}^n.
\]
Let $\mathcal{A} = \langle w_1, v_1, w_2, v_2, \cdots \rangle$, where $w_i \in C_1$ and $v_i \in C_2$, be an enumeration of the words in $C_1$ and $C_2$ such that each word in $C_1$ and $C_2$ appears infinitely many times in $\mathcal{A}$.

Let $T_2x = 2x (\mod 1)$ and $T_3x = 3x (\mod 1), x \in [0,1]$ be the $\times 2$ and $\times 3$ maps.

- $|w|$: the length of the word $w$;
- $\lfloor \xi \rfloor$: the integer part of a real number $\xi$;
- $0^m$: a word of length $m$ composed by 0.
- $r_i$: the integer $\lfloor |w_i| + (2 + |v_i|) \log_2 3 \rfloor + 2$;

For $u \in C_1$ and $v \in C_2$, the 2-adic cylinder and 3-adic cylinder are defined as
\[
[u]_2 := \{x \in [0,1] : \text{the 2-adic expansion of } x \text{ starts with } u\},
\]
\[
[v]_3 := \{x \in [0,1] : \text{the 3-adic expansion of } x \text{ starts with } v\}.
\]

3. Main results

**Theorem 1.** There exist a Cantor set $E$ composed of nonnormal numbers such that for all $x \in E$,

1. $\{2^n x (\mod 1) : n \geq 1 \} = [0,1]$ and $\{3^n x (\mod 1) : n \geq 1 \} = [0,1]$.
2. $\dim_H E = 1$.

The construction of the Cantor set $E$ is divided into three steps. Recall that $\mathcal{A} = \langle w_1, v_1, w_2, v_2, \cdots \rangle$ be an enumeration of the elements in $w \in C_1$ and $v \in C_2$ such that each element in $C_1 \cup C_2$ appears infinitely many times.

Fix an integer $m \geq 1$. Choose a sequence of integers $\{\ell_k\}_{k \geq 1}, \{n_k\}_{k \geq 1}$ such that
\[
(3.1) \quad n_k = \ell_k \cdot m, \quad \lim_{k \to \infty} \frac{r_1 + \cdots + r_k}{n_k} = 0.
\]

For each $k \geq 1$, write
\[
(3.2) \quad \widetilde{C}_1^{n_k} := \left\{(i_1, \cdots, i_{n_k}) \in C_1 : i_{\ell m + 1} = i_{\ell m + m} = 1, 0 \leq \ell < \ell_k \right\}.
\]

**Step 1:** the first level of the Cantor set.
For each \((i_1^{(1)} \cdots i_{n_1}^{(1)}) \in \widetilde{C}_1^{n_1}\), consider the 2-adic cylinder 

\[
[i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1]_2.
\]

Let \(r_1\) be the integer such that

\[
3 \left( \frac{1}{3} \right)^{r_1} \leq \left( \frac{1}{2} \right)^{n_1 + |\nu_1|} < 3 \left( \frac{1}{3} \right)^{r_1 - 1}.
\]

Then the 2-adic cylinder \([i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1]_2\) will contain at least one 3-adic cylinder of order \(r_1\). Denote by \([\epsilon_1]_3\) such a 3-adic cylinder (if there are many, just choose one).

It should be mentioned that \(r_1\) does not depend on \((i_1^{(1)} \cdots i_{n_1}^{(1)})\) but \(\epsilon_1\) does. This dependence will not play a role in the following argument, thus will not be explicitly addressed.

Now consider the 3-adic subcylinder \([\epsilon_1, \nu_1]_3\) contained in \([\epsilon_1]_3\). Similarly, let \(t_1\) be the integer such that

\[
2 \left( \frac{1}{2} \right)^{t_1} \leq \left( \frac{1}{3} \right)^{r_1 + |\nu_1|} < 2 \left( \frac{1}{2} \right)^{t_1 - 1}.
\]

Then choose a 2-adic cylinder \([\eta_1]_2\) of order \(t_1\) contained in \([\epsilon_1, \nu_1]_3\), so contained in the 2-adic cylinder \([i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1]_2\). Thus we can write this 2-adic cylinder \([\eta_1]_2\) as

\[
[\eta_1]_2 = [i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1, \tilde{\nu}_1]_2,
\]

to emphasize its dependence on \(\nu_1\). Now we have the following inclusions:

\[
(i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1, \tilde{\nu}_1]_2 \subset [\epsilon_1, \nu_1]_3 \subset [i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1]_2.
\]

A simple calculation can give us an estimation on the integer \(t_1\):

\[
n_1 + |\nu_1| + (|\nu_1| + 1) \frac{\log 3}{\log 2} + 1 \leq t_1 < n_1 + |\nu_1| + (|\nu_1| + 2) \frac{\log 3}{\log 2} + 2.
\]

The first level of \(E\) is defined as

\[
F_1 = \bigcup_{(i_1^{(1)} \cdots i_{n_1}^{(1)}) \in \widetilde{C}_1^{n_1}} [i_1^{(1)} \cdots i_{n_1}^{(1)}, \nu_1, \tilde{\nu}_1]_2,
\]

which is a collection of 2-adic cylinders of order \(t_1\).

**Step 2:** the \(k\)-th level of the Cantor set.

This is an inductive step. Assume that the \((k - 1)\)th level \(F_{k-1}\) has been constructed, which is a collection of 2-adic cylinders \([\eta_{k-1}]_1\) of order \(t_{k-1}\). Now we construct the \(k\)th level \(F_k\).

Fix an element \([\eta_{k-1}]_2 \in F_{k-1}\). For each \((i_1^{(k)} \cdots i_{n_k}^{(k)}) \in \widetilde{C}_1^{n_k}\), consider the 2-adic cylinder

\[
[\eta_{k-1}, i_1^{(k)} \cdots i_{n_k}^{(k)}, \nu_k]_2.
\]

Similar as in Step 1, let \(r_k\) be the integer such that

\[
3 \left( \frac{1}{3} \right)^{r_k} \leq \left( \frac{1}{2} \right)^{t_{k-1} + |\nu_k|} < 3 \left( \frac{1}{3} \right)^{r_k - 1}.
\]

Then there is a 3-adic cylinder, denoted by \([\epsilon_k]_3\), of order \(r_k\) contained in the 2-adic cylinder \([\eta_{k-1}, i_1^{(k)} \cdots i_{n_k}^{(k)}, \nu_k]_2\) (if there are many, just choose one).
It should also be mentioned that $r_k$ does not depend on $n_k$ for $1 \leq j \leq k$ but $\epsilon_k$ does. This dependence will not play a role in the following argument, thus will not be explicitly addressed.

Now consider the 3-adic subcylinder $[\epsilon_k, v_k]_3$ contained in $[\epsilon_k]_3$. Let $t_k$ be the integer such that

$$2 \left( \frac{1}{2} \right)^{t_k} \leq \left( \frac{1}{3} \right)^{r_k + |v_k|} < 2 \left( \frac{1}{2} \right)^{t_k - 1}.$$

Then choose a 2-adic cylinder $[\eta_2]_2$ of order $t_k$ contained in $[\epsilon_k, v_k]_3$, so contained in the 2-adic cylinder $[\eta_{k-1}, i_1^{(k)} \cdots i_{n_k}^{(k)}, w_k]_2$. Thus we can write this 2-adic cylinder $[\eta_k]_2$ as

$$[\eta_k]_2 = [\eta_{k-1}, i_1^{(k)} \cdots i_{n_k}^{(k)}, w_k, \tilde{v}_k]_2,$$

to emphasize its dependence on $v_k$. Now we have the following inclusions:

$$(3.4) \quad [\eta_{k-1}, i_1^{(k)} \cdots i_{n_{k-1}}^{(k)}, w_k, \tilde{v}_k]_2 \subset [\epsilon_k, v_k]_3 \subset [\eta_{k-1}, i_1^{(k)} \cdots i_{n_k}^{(k)}, w_k]_2.$$ 

A simple calculation can give us an estimation on the integer $t_k$ (replace the role of $n_1$ by $t_{k-1}$, $|w_1|$ by $|w_k|$, $|v_1|$ by $|v_k|$):

$$(3.5) \quad t_k - 1 + |w_k| + n_k + (|v_k| + 1) \frac{\log 3}{\log 2} + 1 \leq t_k < t_k - 1 + |w_k| + n_k + (|v_k| + 2) \frac{\log 3}{\log 2} + 2.$$

Let $p_i = |w_k, \tilde{v}_k| = t_k - t_{k-1} - n_k \leq r_i$. Thus it follows from $(3.3)$ and $(3.1)$ that

$$(3.6) \quad \lim_{k \to \infty} \frac{|w_1, \tilde{v}_1| + \cdots + |w_k, \tilde{v}_k|}{n_1 + \cdots + n_k} \leq \lim_{k \to \infty} \frac{r_1 + \cdots + r_k}{n_1 + \cdots + n_k} = 0.$$

The $k$th level of $E$ is defined as

$$F_k = \bigcup_{[\eta_{k-1}]_2 \in F_{k-1}} \bigcup_{(i_1^{(k)} \cdots i_{n_k}^{(k)}) \in \tilde{C}_1^{n_k}} [\eta_{k-1}, i_1^{(k)} \cdots i_{n_k}^{(k)}, w_k, \tilde{v}_k]_2,$$

which is a collection of 2-adic cylinders of order $t_k$.

Finally, the desired Cantor set $E$ is defined as

$$E = \bigcup_{m \geq 1} E_m, \quad E_m = \bigcap_{k \geq 1} \bigcup_{[\eta_k]_2 \in F_k} [\eta_k]_2.$$

The following three propositions correspond to the three items in Theorem 1.

**Proposition 1.** Every point in $E_m$ is nonnormal.

**Proof.** By the construction of $E_m$, for each $x \in E_m$, we express $x$ by its 2-adic expansion as follows:

$$x = [i_1^{(1)} \cdots i_{n_1}^{(1)}, w_1, \tilde{v}_1, \cdots, i_1^{(k)} \cdots i_{n_k}^{(k)}, w_k, \tilde{v}_k, \cdots]_2,$$

where $(i_1^{(k)} \cdots i_{n_k}^{(k)}) \in \tilde{C}_1^{n_k}$.

We count the frequency of the word $0^{2m}$ occurring in the 2-adic expansion of $x$. Recall the definition of $\tilde{C}_1^{n_k}$. Among the first $t_{k-1} + n_k$ terms in the 2-adic expansion of $x$, the number of the occurrence of $0^m$ will be less than $p_1 + \cdots + p_{k-1}$, Thus the frequency of the word $0^m$ occurring in the 2-adic expansion of $x$ is less than

$$\liminf_{k \to \infty} \frac{p_1 + \cdots + p_{k-1}}{n_k} = 0.$$

This shows that $x$ is nonnormal.

This shows that $x$ is nonnormal. □
Proposition 2. For any \( x \in E \),
\[
\{2^n x (\text{mod 1}) : n \geq 1\} = [0, 1], \quad \{3^n x (\text{mod 1}) : n \geq 1\} = [0, 1].
\]

Proof. Fix an \( x \in E \). It is sufficient to show that for any 2-adic finite word \( w \in C_1 \) and \( v \in C_2 \), they appear infinitely often in the 2-adic expansion and the 3-adic expansion of \( x \) respectively.

Recall the definition of \( A \). It suffices to show that for each \( k \geq 1 \), \( w_k \) and \( v_k \) appear in the 2-adic expansion and the 3-adic expansion of \( x \) respectively. This follows from (3.4) by noticing that
\[
T_{3}^{n_k} x \in [v_k]_3, \quad T_{2}^{n_k-1+n_k} (x) \in [w_k]_2.
\]

\( \square \)

Proposition 3. For each \( m \geq 2 \), \( \dim_H E_m \geq 1 - 2/m \).

Proof. Recall the definition of \( \tilde{C}_n \) (3.2). Write the points in \( E_m \) by its 2-adic expansion:
\[
\left\{ i_1^{(1)} \cdots i_{n_1}^{(1)}, i_1^{(2)} \cdots i_{n_2}^{(2)}, \cdots, i_1^{(k)} \cdots i_{n_k}^{(k)} \right\} \subset \tilde{C}_m, k \geq 1
\]

Now we will distribute a measure uniformly distributed on the cylinders with nonempty intersection with \( E_m \). Thus we count the number of \( t \)-th cylinders which have nonempty intersection with \( E_m \) for each \( t \geq 1 \). We denote this number by \( b_t \).

Recall the construction of the \( t \)-th level \( F_k \) of the Cantor set \( E_m \). We know that among the digit sequence of a point in \( E_m \), at the positions
\[
\{ t_k + \ell m + 1, t_k + (\ell + 1)m, t_k + n_k + 1, \cdots, t_k + n_k + p_k + 1, 0 \leq \ell < \ell_{k+1}, k \geq 0 \}
\]
the digits are fixed, while at other positions, the digits can be chosen arbitrarily in \( \{0, 1\} \). Thus the numbers \( b_t \) can be given as follows.

(i). When \( t \leq t_1 = n_1 + p_1 = \ell_1 m + p_1 \),
\[
b_t = \begin{cases} 2^{t-2}, & \text{when } t = \ell m \text{ for } 0 \leq \ell < \ell_1; \\ 2^{t-1}, & \text{when } \ell m < t < (\ell + 1)m \text{ for } 0 \leq \ell < \ell_1; \\ 2^{\ell m - 2}, & \text{when } \ell_1 m \leq t \leq n_1 + p_1. 
\end{cases}
\]

(ii). When \( t_k < t \leq t_{k+1} = t_k + n_k + p_k + 1 = t_k + \ell_{k+1} m + p_{k+1} \),
\[
b_t = \begin{cases} \prod_{j=1}^{k} 2^{t_j - 2}, & \text{when } t = t_k + \ell m \text{ for } 0 \leq \ell < \ell_{k+1}; \\ \prod_{j=1}^{k} 2^{t_j - 2} \cdot 2^{t_k - t_j - 2}, & \text{when } \ell m < t < (\ell + 1)m \text{ for } 0 \leq \ell < \ell_{k+1}; \\ \prod_{j=1}^{k+1} 2^{t_j - 2}, & \text{when } t_k + n_{k+1} \leq t \leq t_k + n_{k+1} + p_{k+1}. 
\end{cases}
\]

Thus for each 2-adic cylinder \( I_t \) with nonempty intersection with \( E_m \), its measure can be given explicitly as
\[
\mu(I_t) = b_t^{-1}.
\]

As a consequence, together with (3.3), we have
\[
(3.7) \quad \lim_{t \to \infty} \frac{\log \mu(I_t)}{\log |I_t|} = \lim_{t \to \infty} \frac{\log b_t}{\log 2} \geq 1 - \frac{2}{m}.
\]

So, using Billingsley’s lemma [7]: Let \( A \subset [0, 1] \) be Borel and let \( \mu \) be a finite Borel measure on \( [0, 1] \). Suppose \( \mu(A) \geq 0 \). If
\[
\alpha \leq \lim_{n \to \infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} \leq \beta
\]
for all $x \in A$, then $\alpha \leq \dim(A) \leq \beta$. we conclude that

$$\dim_H E_m \geq 1 - \frac{2}{m}$$

and then $\dim_H E = 1$. □

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