Abstract

Both the canonical and microcanonical ensembles are utilized to study the thermodynamic and evaporation properties of a closed black cosmic string whose spacetime is asymptotically anti deSitter. There are similarities and differences to the Schwarzschild-anti deSitter and 2+1 BTZ black hole solutions. It is found that there exist regimes of black string/thermal radiation equilibrium as well as stable remnant regimes. The relevance to black hole evaporation is discussed.

1 Introduction

The interesting field of black hole thermodynamics began with the discovery by Hawking [1] that black holes actually radiate energy as if they were hot bodies. The temperature being governed by a multiple of the surface gravity, κ, of the black hole. It is now also well know that the area of the event horizon is a direct measure of the entropy of the black hole [2] by the relation $S = G^{-1}A/4$, $A$ being the area of the event horizon.

The above properties were originally derived in spacetimes which are asymptotically space-like flat. It is useful, however, to study such phenomena in spacetimes which are not necessarily asymptotically flat since it is
unknown how good the assumption of asymptotic flatness may be. Black hole solutions also exist which tend either to deSitter spacetime (if the cosmological constant is positive) or anti-deSitter (adS) (if the cosmological constant is negative).

These studies have been extended to the Schwarzschild-deSitter as well as the Schwarzschild-adS case (the latter being of particular relevance to the study here). It was found that an identical area-entropy law holds in the asymptotically adS case as does in the case of asymptotic flatness.

Another solution which tends to adS is the 2+1 dimensional black hole formulated by Banados, Teitelboim and Zanelli (hereafter referred to as the BTZ black hole). This must be the case in lower dimensional gravity since for $D < 4$, $D$ being the number of spacetime dimensions, the Riemann curvature tensor must vanish if the Ricci tensor is zero. Thermodynamic studies of this system have also been done. The entropy in this case turns out to be measured by the circumference of the event horizon.

The metric of the spacetime will be flat torus model developed by Lemos and Zanchin with $S^1 \times S^1$ topology. Black cosmic strings have also been studied by Kaloper. The metric is given by

$$ds^2 = -\left(\alpha^2 \rho^2 - \frac{2GM}{\pi \rho}\right)dt^2 + \frac{d\rho^2}{\left(\alpha^2 \rho^2 - \frac{2GM}{\pi \rho}\right)} + \rho^2 \left(d\varphi^2 + d\vartheta^2\right),$$

(1)

where $M$ is the total mass of the string and $\alpha = -\frac{1}{3} \Lambda$ ($\Lambda$ being the cosmological constant). The coordinate ranges are as follows: $-\infty < t < +\infty$, $0 \leq \rho < +\infty$, $0 \leq \varphi < 2\pi$ and $0 \leq \vartheta < 2\pi$. By the redefinition $\vartheta \to \alpha z$ with $-\infty \leq z < +\infty$ the above metric describes an infinitely long black cosmic string with a mass per unit length of $\alpha M/2\pi$. The Kretschmann scalar is given by

$$R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = 24\alpha^2 \left(1 + \frac{(2GM)^2}{2\pi^2 \alpha^4 \rho^6}\right),$$

(2)

from which it can be seen that a true polynomial singularity exists at $\rho = 0$.

An event horizon is located at $\rho_H = \left(\frac{2GM}{\pi \alpha^2}\right)^\frac{1}{3}$.

\(^{1}\)Units are used here such that $c = k_B = \hbar = 1$. This gives the Planck mass a value of $m_p = G^{-\frac{1}{2}}$. 

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Since the background reference spacetime is adS, special boundary conditions must be imposed at the time-like surface $\rho = \infty$ so that a well defined Cauchy problem will exist [11] (see Fig.1). The standard “reflective” boundary condition will be used here for massless particles noting that massive particles do not reach spatial infinity. As pointed out in [6], this has some interesting effects on black hole evaporation as will be discussed below.

Both the canonical and $\mu$icrocanonical ensembles will be used in this study. Since reflection is imposed at infinity and the metric is static, energy is conserved. The total amount of energy is also bounded due to the presence of the cosmological constant which causes large (infinite) redshift effects away from the origin [12].

![Penrose diagram for adS spacetime](image)

Figure 1: Penrose diagram for adS spacetime. The time-like surface allows null information to “leak out” at spatial infinity (arrow) therefore a reflective condition must be enforced at this surface. The dotted line displays a representative time-like geodesic which is automatically reflected back in the spacetime.
2 Canonical Ensemble

Since there is no time periodicity in its Euclidean extension, adS spacetime has no natural temperature. Thermal states can be defined, however, by imposing a periodicity of $\beta = T^{-1}$ in imaginary time where $T$ is the desired temperature. The resulting states will have a local temperature $T_{\text{local}}$:

$$T_{\text{local}} = \frac{\beta^{-1}}{g_{00}}. \quad (3)$$

Unlike the asymptotically flat case, the canonical ensemble can be perfectly defined in a spacetime which is adS. The stress-energy tensor for a conformally coupled scalar field is given by

$$T^\mu_\nu = \frac{\pi^2}{90} \frac{gT^4}{(1 + \alpha^2 r^2)^2} \left( \delta^\mu_\nu - 4 \delta_0^\nu \delta_0^\mu \right) + O(\alpha^{-2} T^2), \quad (4)$$

where $g$ is the number of spin states. A mass integral can be formed by contraction with a properly normalised time-like killing vector. Integrating over all space gives the total energy of the thermal state,

$$E \approx \frac{\pi^4}{30} g T^4 \alpha^{-3}. \quad (5)$$

The integral of this quantity with respect to inverse temperature, $\beta$, yields the partition function $Z$.

$$\ln(Z) = \frac{\pi^4}{90} \frac{g}{(\alpha \beta)^3} + O(\alpha^{-1} \beta^{-1}), \quad (6)$$

from which the free energy may be calculated as

$$F = -T \ln(Z) = -\frac{\pi^4}{90} \frac{g}{\alpha^3 \beta^4} + O(\alpha^{-1} \beta^{-2}). \quad (7)$$

It should be noted that gravitational back reaction effects of the thermal radiation have been ignored. There is also a temperature above which the

\footnote{for a review of the problems associated with defining a canonical ensemble in asymptotically flat spacetimes see [2]}
radiation will be unstable to collapse forming a black hole. This temperature is given by

\[ T_{\text{collapse}} \approx (gG)^{-\frac{1}{4}} \alpha^\frac{1}{2}, \]  

and can be derived from the following argument. If the radiation is spherically symmetric, the Einstein field equations give

\[ g^{11} = A + \alpha^2 r^2 - \frac{8\pi G r}{r} \int r^2 dr \]  

where \( A \) is a constant and \( \varepsilon \) is given by

\[ \varepsilon = \frac{\pi^2}{30} gT^4. \]  

A horizon will form unless condition (8) holds.

Attention is now turned to the black string spacetime. The Euclidean extension is obtained by making the transformation \( \tau = it \) and the Hawking temperature can be calculated by demanding that the Euclidean metric be regular on the horizon. This gives a periodicity in imaginary time

\[ \beta = T^{-1} = \frac{2}{3(GM)^{\frac{1}{2}}} \left( \frac{2\pi^2}{\alpha^2} \right)^{\frac{2}{3}}. \]  

The expectation value of the energy is given by the total mass, \( M \), of the black string so that the heat capacity \( \partial M / \partial T \) is always positive and black hole states may be in thermal equilibrium with radiation. Note that unlike the Schwarzschild case (whose temperature is inversely proportional to the black hole mass) the temperature here is proportional to \( M^{1/3} \). For the BTZ black hole temperature is proportional to \( M^{1/2} \).

From the above properties one easily obtains the partition function

\[ \ln (Z) = \frac{16}{27} \frac{\pi^4}{G\alpha^4} \beta^{-2} \]  

and the free energy

\[ F = -\frac{16}{27} \frac{\pi^4}{G\alpha^4} \beta^{-3} = -\frac{M}{2}. \]  

\(^3\)Since the straight cosmic string is infinite, some quantities such as total energy are unbounded. This leads to infinite quantities when calculating certain thermodynamic functions. Intensive quantities, however, are still well defined.
By comparing free energies (7) and (13) it is noted that for $T > 160/(3\alpha gG)$ the radiation free energy is less than the black string free energy. A black string configuration with a temperature higher than this will therefore totally evaporate to a radiation state. This crossover occurs at a mass of

$$M \approx \frac{1.31 \times 10^8 \pi^4}{729 \alpha^4 gG^4}.$$  

(14)

However, since above temperature (8) the radiation self gravitation will cause it to collapse, only black hole solutions are stable if

$$\alpha < \left(\frac{160}{3}\right)^{2/3} (gG)^{-1/2}.$$  

(15)

The entropy of the black string is given by

$$S = \frac{M}{T} - \frac{F}{T} = \frac{16 \pi^4}{9 \alpha^4 T^2} = \frac{\pi G A}{G^4 A}$$  

(16)

where $A$ is the cross sectional area of a disk of radius $\rho = \rho_H$. (16) is therefore the analog of the entropy/area relation for spherically symmetric systems. This result shows that the canonical ensemble is well defined in the black string spacetime. The defining integral for the partition function is

$$Z = \int N(M) e^{-M\beta} dM,$$  

(17)

where $N(M)$ is the density of states given by

$$N(M) \propto e^{S} = e^{\alpha M^{2/3}}$$  

(18)

so that (17) is well defined. In asymptotically flat spacetimes the density of states goes as $\exp(\alpha M^2)$ so that (17) does not converge.

3  Microcanonical Ensemble

adS, unlike asymptotically flat, spacetime does not require one to introduce an artificial “box” to bound the system. This is due to the fact
that geodesics in adS automatically reflect massive particles. Particles of zero mass, which do propagate to spatial infinity, will be reflected by the appropriate boundary conditions (see Fig.1).

The density of states is given by the inverse LaPlace transform

\[ N(E) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Z(\beta) e^{\beta E} d\beta. \quad (19) \]

For adS the quantities of interest have been computed in [4] and are quoted here. For stable thermal radiation

\[ Z \approx \exp \left( \frac{\pi^4}{30} g(\alpha\beta)^{-3} \right). \quad (20) \]

A saddle point to (19) exists at

\[ \beta \approx \left( \frac{\pi^4}{30} g\alpha^{-3} E^{-1} \right)^{1/4} \quad (21) \]

so that in the stationary phase approximation the number of states is given by

\[ N(E) \approx \exp \left[ \frac{4\pi}{3} \left( \frac{g}{30\alpha^3} \right)^{1/4} E^{3/4} \right]. \quad (22) \]

For the black string spacetime recall that \( Z(\beta) \) is given by (12) so that a saddle point exists at

\[ \beta = \left( \frac{32 \pi^4}{27 G\alpha^4} \right)^{1/3} E^{-1/3} \quad (23) \]

yielding

\[ N_{\text{blackstring}}(E) = \exp \left[ \frac{3}{2} \left( \frac{32\pi^4}{27 G\alpha^4} \right)^{1/3} E^{2/3} \right]. \quad (24) \]

The energy dependence \( e^{\pi E^{2/3}} \) is similar to the Schwarzschild-adS black hole [4] for large mass. The BTZ black hole has an \( e^{\pi E^{1/2}} \) dependence [6]. From this relation it is determined that \( N_{\text{radiation}} > N_{\text{blackstring}} \) when

\[ E > \sim \frac{1.8 \times 10^6 \pi^4}{8.2 \frac{G^4\alpha^7 g^3}{3}}. \quad (25) \]
At the stationary phase point the temperature/energy relation of the system is given by

\[ E = M_{\text{blackstring}} + E_{\text{radiation}} \approx \frac{32\pi^4}{27G\alpha^4} \beta^{-3} + \frac{\pi^4}{30g\alpha^{-3}}\beta^{-4}. \] (26)

Therefore, when the radiation contribution is significant and there exists a state in which the black string is in thermal equilibrium with thermal radiation, the number of states is given by

\[ N(E) \approx \exp \left[ \left( \frac{\pi^2}{\alpha^2} \beta \right)^{2/3} G^{-1/3} + \frac{4\pi}{3} \left( \frac{g}{\alpha^3} \right) \right] E_{\text{radiation}}^{3/4}. \] (27)

4 Evaporation of the Black String

It is interesting to speculate about the effects of evaporation in a space-time which is asymptotically adS. An excellent discussion of the effects for the BTZ black hole can be found in [6]. The central question is whether or not mass loss from the black string can occur since, unlike the asymptotically flat case, massive particles automatically return to their original position on a time scale governed by the cosmological constant. Massless particles are reflected back at spatial infinity by the boundary condition on a similar timescale. Complete evaporation is therefore not possible in all situations.

For situations where total evaporation is possible there are several possibilities. There is the situation where evaporation is complete and the remnant will be pure radiation with no black string. The other possibility is the case where \( \alpha < \left( \frac{160}{3} \right)^{2/3} (gG)^{-1/2} \). In this case the resulting radiation is unstable to collapse and will therefore form a black hole. Since it is unlikely that the recollapsing matter will form a toroidal black hole the initial configuration again will have completely evaporated, the remnant most likely being a spherical black hole. It is interesting to speculate on the significance of such evaporation in the context of cosmic string theory.

It is well known from the theory of cosmic strings in flat spacetime [13] that oscillating string loops will lose energy\(^4\) via gravitational wave emission.

\(^4\)It is assumed that the cosmic string is not a superconducting or global string in which case there may be other significant methods of energy loss.
This occurs at a rate approximately given by

\[ \dot{E} = G K \mu^2 \]  (28)

where \( K \) is a geometric factor and \( \mu \) is the string tension. The lifetime of a loop losing energy via this mechanism is approximately

\[ t \approx \frac{L}{KG\mu} \]  (29)

\( L \) being the length of the loop. The static toroidal black string studied here admits another mechanism for decay.

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