Which Nilpotent Groups are Self-Similar? *

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Abstract

Let $\Gamma$ be a finitely generated torsion free nilpotent group, and let $A^\omega$ be the space of infinite words over a finite alphabet $A$. We investigate two types of self-similar actions of $\Gamma$ on $A^\omega$, namely the faithfull actions with dense orbits and the free actions. A criterion for the existence of a self-similar action of each type is established.

Two corollaries about the nilmanifolds are deduced. The first involves the nilmanifolds endowed with an Anosov diffeomorphism, and the second about the existence of an affine structure.

Then we investigate the virtual actions of $\Gamma$, i.e. actions of a subgroup $\Gamma'$ of finite index. A formula, with some number theoretical content, is found for the minimal cardinal of an alphabet $A$ endowed with a virtual self-similar action on $A^\omega$ of each type.

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Introduction

1. General introduction

Let $A$ be a finite alphabet and let $A^\omega$ be the topological space of infinite words $a_1a_2\ldots$ over $A$, where the topology of $A^\omega = \varprojlim A^n$ is the pro-finite topology.

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An action of a group $\Gamma$ on $A^\omega$ is called self-similar iff for any $\gamma \in \Gamma$ and $a \in A$ there exists $\gamma_a \in \Gamma$ and $b \in A$ such that
\[ \gamma(aw) = b\gamma_a(w) \text{ for any } w \in A^\omega. \]

The group $\Gamma$ is called self-similar (respectively densely self-similar, respectively freely self-similar, respectively freely densely self-similar) if $\Gamma$ admits a faithful self-similar action (respectively a faithful self-similar action with dense orbits, respectively a free self-similar action, respectively a free self-similar action with dense orbits) on $A^\omega$ for some finite alphabet $A$.

Self-similar groups appeared in the early eighties, in the works of R. Grigorchuk [10] [11] and in the joint works of N. Gupta and S. Sidki [13] [14]. See also the monography [24] for an extensive account before 2005 and [25] [2] [9] [16] [12] for more recent works. A general question is

which groups $\Gamma$ are (merely, or densely ...) self-similar?

This paper brings an answer for finitely generated torsion-free nilpotent groups $\Gamma$, called FGTF nilpotent groups in the sequel. Then we will connect the main result with topics involving differential geometry and arithmetic groups.

The systematic study of self-similar actions of nilpotent groups started with [11], and the previous question has been raised in some talks of S. Sidki.

2. The main results
A few definitions are now required. A grading of a Lie algebra $\mathfrak{m}$ is a decomposition $\mathfrak{m} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{m}_n$ such that $[\mathfrak{m}_n, \mathfrak{m}_m] \subset \mathfrak{m}_{n+m}$ for all $n, m \in \mathbb{Z}$. It is called special if $\mathfrak{m}_0 \cap \mathfrak{z} = 0$, where $\mathfrak{z}$ is the center of $\mathfrak{m}$. It is called very special if $\mathfrak{m}_0 = 0$.

Let’s assume now that $\Gamma$ is a FGTF nilpotent group. By Malcev Theory [13] [26], $\Gamma$ is a cocompact lattice in a unique connected, simply connected (or CSC in what follows) nilpotent Lie group $N$. Let $\mathfrak{n}^\mathbb{R}$ be the Lie algebra of $N$ and set $\mathfrak{n}^\mathbb{C} = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{n}^\mathbb{R}$.

The main results, proved in Section 7, are the following

Theorem 2. The group $\Gamma$ is densely self-similar iff the Lie algebra $\mathfrak{n}^\mathbb{C}$ admits a special grading.

Theorem 3. The following assertions are equivalent
(i) The group $\Gamma$ is freely self-similar,
(ii) the group $\Gamma$ is freely densely self-similar, and
(iii) the Lie algebra $\mathfrak{n}^\mathbb{C}$ admits a very special grading.
As a consequence, let’s mention

**Corollary 4.** Let $M$ be a nilmanifold endowed with an Anosov diffeomorphism. Then there a free self-similar action of $\pi_1(M)$ with dense orbits on $A^\omega$, for some finite $A$.

**Corollary 8.** Let $M$ be a nilmanifold. If $\pi_1(M)$ is freely self-similar, then $M$ is affine complete.

Among FGTF nilpotent groups, some of them are self-similar but not densely self-similar. Some of them are not even self-similar, since Theorem 2 implies the next

**Corollary 7.** Let $M$ be one of the non-affine nilmanifolds appearing in [3]. Then $\pi_1(M)$ is not self-similar.

3. A concrete version of Theorems 2 and 3

Let $N$ be a CSC nilpotent Lie group, with Lie algebra $n^\mathbb{R}$. Let’s assume that $N$ contains some cocompact lattices $\Gamma$. By definition, the *degree* of a self-similar action of $\Gamma$ on $A^\omega$ is Card $A$. We ask the following question

For a given cocompact lattice $\Gamma \subset N$, what is the minimal degree

degree of a faithful (or a free) self-similar action with dense orbits?

More notions are now defined. Recall that the *commensurable class* $\xi$ of a cocompact lattice $\Gamma_0 \subset N$ is the set of all cocompact lattices of $N$ which share with $\Gamma_0$ a subgroup of finite index. The *complexity* $\text{cp} \xi$ (respectively the free complexity $\text{fcp} \xi$) of the class $\xi$ is the minimal degree of a self-similar action of $\Gamma$ with dense orbits (respectively a free self-similar action of $\Gamma$), for some $\Gamma \in \xi$.

For any algebraic number $\lambda \neq 0$, set $d(\lambda) = \text{Card } \mathcal{O}(\lambda)/\pi_\lambda$, where $\mathcal{O}(\lambda)$ is the ring of integers of $\mathbb{Q}(\lambda)$ and $\pi_\lambda = \{x \in \mathcal{O}(\lambda) | x\lambda \in \mathcal{O}(\lambda)\}$. For any isomorphism $h$ of a finite dimensional vector space over $\mathbb{Q}$, set

$$\text{ht } h = \prod_{\lambda \in \text{Spec } h/\text{Gal}(\mathbb{Q})} d(\lambda)^{m_\lambda},$$

where Spec $h/\text{Gal}(\mathbb{Q})$ is the list of eigenvalues of $h$ up to conjugacy by $\text{Gal}(\mathbb{Q})$ and where $m_\lambda$ is the multiplicity of the eigenvalue $\lambda$.

By Malcev’s Theory, the commensurable class $\xi$ determines a canonical $\mathbb{Q}$-form $n(\xi)$ of the Lie algebra $n^\mathbb{R}$. Let $\mathcal{S}(n(\xi))$ (respectively $\mathcal{V}(n(\xi))$) be the set of all $f \in \text{Aut } n(\xi)$ such that Spec $f|_{\mathbb{C}}$ (respectively Spec $f$) contains no algebraic integer.
Theorem 9. We have
\[
\text{cp} \xi = \min_{h \in S(n(\xi))} \text{ht} h, \quad \text{and}
\]
\[
\text{fcp} \xi = \min_{h \in V(n(\xi))} \text{ht} h.
\]

If, in the previous statement, \( S(n(\xi)) \) is empty, then the equality \( \text{cp} \xi = \infty \) means that no \( \Gamma \in \xi \) admits a faithfull self-similar action with dense \( \Gamma \)-orbits.

Theorem 9 answers the previous question only for the commensurable classes \( \xi \). For an individual \( \Gamma \in \xi \), it provides some ugly estimates for the minimal degree of \( \Gamma \)-actions, and nothing better can be expected.

The framework of nonabelian Galois cohomology shows the concreteness of Theorem 9. Up to conjugacy, the commensurable classes in \( N \) are classified by the \( \mathbb{Q} \)-forms of some classical objects with a prescribed \( \mathbb{R} \)-form, see Corollary 4 of ch. 9, and their complexity is an invariant of the arithmetic group \( \text{Aut} n(\xi) \).

As an illustration of the previous vague sentence, we investigate a class \( N \) of CSC nilpotent Lie groups \( N \), with Lie algebra \( n^\mathbb{R} \). The commensurable classes \( \xi(q) \) in \( N \) are classified, up to conjugacy, by the positive definite quadratic forms \( q \) on \( \mathbb{Q}^2 \). Then, we have
\[
\text{cp} \xi(q) = F(d)^{e(N)}
\]
where \( e(N) \) is an invariant of the special grading of \( \mathbb{C} \otimes n^\mathbb{R} \), where \(-d\) is the discriminant of \( q \), and where \( F(d) \) is the norm of a specific ideal in \( \mathbb{Q}(\sqrt{-d}) \), see Theorem 11 and Lemma 28.

In particular, \( N \) contains some commensurable classes of arbitrarily high complexity. In a forthcoming paper [22], more complicated examples are investigated, but the formulas are less explicit.

4. About the proofs. The proofs of the paper are based on different ideas. Theorem 1, which is a statement about rational points of algebraic tori, is the key step in the proof of Theorems 2, 3 and 11. It is based on standard results of number theory, including the Cebotarev’s Theorem. It is connected with the density of rational points for connected groups proved by A. Borel [6], see also [27].

Also, the proof of Corollary 4 is based on a paper of A. Manning [20] about Anosov diffeomorphisms. The proof of Corollary 7 is based on very difficult computations, which, fortunately, were entirely done in [3].
1 Self-similar actions and self-similar data

Let \( \Gamma \) be a group. This section explains the correspondence between the faithfull transitive self-similar \( \Gamma \)-actions and some virtual endomorphisms of \( \Gamma \), called self-similar data. Usually self-similar actions are actions on a rooted tree \( A^* \), see [24]. Here the groups are acting on the boundary \( A^\omega \) of \( A^* \). This equivalent viewpoint is better adapted to our setting.

1.1 Transitive self-similar actions

In addition of the definitions of the introduction, the following technical notion of transitivity will be used.

A self-similar action of \( \Gamma \) on \( A^\omega \) induces an action of \( \Gamma \) on \( A^\omega \). Indeed, for \( a, b \in A \) and \( \gamma \in \Gamma \), we have \( \gamma(a) = b \) if

\[
\gamma(aw) = b\gamma_a(w),
\]

for all \( w \in A^\omega \). A self-similar action is called transitive if the induced action on \( A \) is transitive. The group \( \Gamma \) is called transitive self-similar if it admits a faithfull transitive self-similar action.

Similarly the self-similar action of \( \Gamma \) on \( A^\omega \) induces an action of \( \Gamma \) on each level set \( A^n \). Such an action is often called level transitive if \( \Gamma \) acts transitively on each level \( A^n \). Obviously, the level transitive actions on \( A^* \) of [24] corresponds with the actions on \( A^\omega \) with dense orbits.

1.2 Core and \( f \)-core

Let \( \Gamma \) be a group and \( \Gamma' \) be a subgroup. The core of \( \Gamma' \) is the biggest normal subgroup \( K \trianglelefteq G \) with \( K \subseteq \Gamma' \). Equivalently the core is the kernel of the left action of \( \Gamma \) on \( \Gamma'/\Gamma' \).

Now let \( f : \Gamma' \to \Gamma \) be a group morphism. By definition the \( f \)-core is the biggest normal subgroup \( K \trianglelefteq G \) with \( K \subseteq \Gamma' \) and \( f(K) \subseteq K \).

1.3 Self-similar data

Let \( \Gamma \) be a group. A virtual endomorphism of \( \Gamma \) is a pair \( (\Gamma', f) \), where \( \Gamma' \) is a subgroup of finite index and \( f : \Gamma' \to \Gamma \) is a group morphism. A self-similar datum is a virtual endomorphism \( (\Gamma', f) \) with a trivial \( f \)-core.

Assume given a faithfull transitive self-similar action of \( \Gamma \) on \( A^\omega \). Let \( a \in A \), and let \( \Gamma' \) be the stabilizer of \( a \). By definition, for each \( \gamma \in \Gamma' \) there is a unique \( \gamma_a \in \Gamma \) such that

\[
\gamma(aw) = a\gamma_a(w),
\]

for any \( w \in A^\omega \). Let \( f : \Gamma' \to \Gamma \) be the map \( \gamma \mapsto \gamma_a \). Since the action is faithfull, \( \gamma_a \) is uniquely determined and \( f \) is a group morphism. Also it
follows from Proposition 2.7.4 and 2.7.5 of [24] that the $f$-core of $\Gamma'$ is the kernel of the action, therefore it is trivial. Hence $(\Gamma', f)$ is a self-similar datum.

Conversely, a virtual endomorphism $(\Gamma', f)$ determines a transitive self-similar action of $\Gamma$ on $A^\omega$, where $A \simeq \Gamma/\Gamma'$. Moreover the $f$-core is the kernel of the corresponding action, see ch 2 of [24] for details, especially subsection 2.5.5 of [24]). In conclusion, we have

**Lemma 1.** Let $\Gamma$ be a group. There is a correspondence between the self-similar data $(\Gamma', f)$ and the faithfull transitive self-similar actions of $\Gamma$ on $A^\omega$, where $A \simeq \Gamma/\Gamma'$.

This correspondence is indeed a bijection up to conjugacy, see [24] for a precise statement.

### 1.4 Good self-similar data

Let $\Gamma$ be a group, and let $(\Gamma', f)$ be a virtual endomorphism. Let $\Gamma_n$ be the subgroups of $\Gamma$ inductively defined by $\Gamma_0 = \Gamma$, $\Gamma_1 = \Gamma'$ and for $n \geq 2$

$$
\Gamma_n = \{ \gamma \in \Gamma_{n-1} | f(\gamma) \in \Gamma_{n-1} \}
$$

**Lemma 2.** The sequence $n \mapsto [\Gamma_n : \Gamma_{n+1}]$ is not increasing.

**Proof.** For $n > 0$, the map $f$ induces an injection of the set $\Gamma_n/\Gamma_{n+1}$ into $\Gamma_{n-1}/\Gamma_n$, thus we have $[\Gamma_n : \Gamma_{n+1}] \leq [\Gamma_{n-1} : \Gamma_n]$. \hfill $\Box$

The virtual endomorphism $(\Gamma', f)$ is called *good* if $[\Gamma_n : \Gamma_{n+1}] = [\Gamma / \Gamma']$ for all $n$.

Let $(\Gamma', f)$ be a self-similar datum, and let $A^*$ be the corresponding tree on which $\Gamma$ acts. If $a$ is the distinguished point in $A \simeq \Gamma/\Gamma'$, then $\Gamma_n$ is the stabilizer of $a^n$. If the self-similar datum $(\Gamma', f)$ is good, then $[\Gamma : \Gamma_n] = \text{Card } A^n$ and therefore $\Gamma$ acts transitively on $A^n$. Exactly as before, we have

**Lemma 3.** Let $\Gamma$ be a group. There is a correspondence between the good self-similar data $(\Gamma', f)$ and the faithfull self-similar actions of $\Gamma$ on $A^\omega$ with dense orbits, where $A \simeq \Gamma/\Gamma'$.

### 1.5 Fractal self-similar data

Let $\Gamma$ be a group. A *self-similar datum* $(\Gamma', f)$ is called *fractal* (or recurrent) if $f(\Gamma') = \Gamma$. A self-similar action of $\Gamma$ on some $A^\omega$ is called *fractal* if it is
transitive and the corresponding self-similar datum is fractal, see [24] section 2.8. Obviously a fractal action has dense orbits.

The group \( \Gamma \) is called \textit{fractal} (respectively \textit{freely fractal}) if \( \Gamma \) admits a faithfull (respectively free) fractal action on some \( A^\omega \).

2 Rational points of a torus

We are going to prove Theorem 1, about the rational points of algebraic tori.

For the whole chapter, let \( H \) be an algebraic torus defined over \( \mathbb{Q} \) and let \( X(H) \) be the group of characters of \( H \). For a number field \( K \), let’s denote by \( \text{Gal}(K) := \text{Gal}(\mathbb{Q}/K) \) its absolute Galois group. The group \( X(H) \) is a \( \text{Gal}(\mathbb{Q}) \)-module which is isomorphic to \( \mathbb{Z}^r \) as an abelian group, where \( r = \dim H \). The \textit{splitting field} of \( H \) is the smallest Galois extension \( L \) of \( \mathbb{Q} \) such that \( \text{Gal}(L) \) acts trivially on \( X(H) \), or, equivalently such that \( H \) is \( L \)-isomorphic to \( \mathbb{G}_m^r \), where \( \mathbb{G}_m \) denotes the multiplicative group. Moreover, we have

\[
\chi(h) \in L^*
\]

for any \( \chi \in X(H) \) and any \( h \in H(\mathbb{Q}) \).

Let \( \mathcal{O} \) be the ring of integers of \( L \). Recall that a \textit{fractional ideal} is a nonzero finitely generated \( \mathcal{O} \)-submodule of \( K \). A fractional ideal \( I \) is called \textit{integral} if \( I \subset \mathcal{O} \). If the fractional ideal \( I \) is integral and \( I \neq \mathcal{O} \), then \( I \) is merely an ideal of \( \mathcal{O} \).

Let \( \mathcal{I} \) be the set of all fractional ideals and \( \mathcal{I}^+ \) be the subset of all integral ideals. Given \( I \) and \( J \) in \( \mathcal{I} \), their \textit{product} is the \( \mathcal{O} \)-module generated by all products \( ab \) where \( a \in I \) and \( b \in J \). Since \( \mathcal{O} \) is a Dedekind ring, we have

\[
\mathcal{I} \simeq \bigoplus_{\pi \in \mathcal{P}} \mathbb{Z}[[\pi]]
\]

\[
\mathcal{I}^+ \simeq \bigoplus_{\pi \in \mathcal{P}} \mathbb{Z}_{\geq 0}[[\pi]],
\]

where \( \mathcal{P} \) is the set of prime ideals of \( \mathcal{O} \). Indeed the additive notation is used for for the group \( \mathcal{I} \) and the monoid \( \mathcal{I}^+ \): view as an element of \( \mathcal{I} \) the fractional ideal \( \pi_1^{m_1} \cdots \pi_n^{m_n} \) will be denoted as \( m_1[\pi_1] + \cdots + m_n[\pi_n] \).

Since \( \text{Gal}(L/\mathbb{Q}) \) acts by permutation on \( \mathcal{P} \), \( \mathcal{I} \) is a \( \mathbb{Z}\text{Gal}(L/\mathbb{Q}) \)-module. For \( S \subset \mathcal{P} \), set

\[
\mathcal{I}_S = \bigoplus_{\pi \in \mathcal{P}\setminus S} \mathbb{Z}[[\pi]].
\]

\textbf{Lemma 4.} Let \( S \subset \mathcal{P} \) be a finite subset and let \( r > 0 \) be an integer.

The \( \text{Gal}(L/\mathbb{Q}) \)-module \( \mathcal{I} \) contains a free \( \mathbb{Z}\text{Gal}(L/\mathbb{Q}) \)-module \( M(r) \) of rank \( r \) such that
(i) \( M(r) \cap \mathcal{I}^+ = \{0\} \), and
(ii) \( M(r) \subset \mathcal{I}_S \).

**Proof.** Let \( S' \) be the set of all prime numbers which are divisible by some \( \pi \in S \). Let \( \Sigma \) be the set of prime numbers \( p \) that are completely split in \( K \), i.e. such that \( \mathcal{O}/p\mathcal{O} \cong \mathbb{F}_p^{|L:Q|} \). For \( p \in \Sigma \), let \( \pi \in P \) be a prime ideal over \( p \).

When \( \sigma \) runs over \( \text{Gal}(L/Q) \) the ideals \( \pi^\sigma \) are all distinct, and therefore \( [\pi] \) generates a free \( \mathbb{Z}\text{Gal}(L/Q) \)-submodule of rank one in \( \mathcal{I} \).

By Cebotarev theorem, the set \( \Sigma \) is infinite. Choose \( r + 1 \) distinct prime numbers \( p_0, \ldots, p_r \) in \( \Sigma \setminus S' \), and let \( \pi_0, \ldots, \pi_r \in P \) such that \( \mathcal{O}/\pi_i = \mathbb{F}_{p_i} \). For \( 1 \leq i \leq r \), set \( \tau_i = [\pi_i] - [\pi_0] \) and let \( M(r) \) be the \( \mathbb{Z}\text{Gal}(L/Q) \)-module generated by \( \tau_1, \ldots, \tau_r \).

It remains to prove that \( M(r) \cap \mathcal{I}^+ = \{0\} \). Let \( \mathcal{A} = \sum_{1 \leq i \leq r, \sigma \in \text{Gal}(L/Q)} m_{i,\sigma} \tau_i^\sigma \) be an element of \( M(r) \cap \mathcal{I}^+ \). We have \( A = B - C \), where

\[
B = \sum_{1 \leq i \leq r, \sigma \in \text{Gal}(L/Q)} m_{i,\sigma} [\pi_i^\sigma],
\]

and

\[
C = \sum_{\sigma \in \text{Gal}(L/Q)} \left( \sum_{1 \leq i \leq r} m_{i,\sigma}^\sigma \right) [\pi_0^\sigma].
\]

Thus the condition \( A \in \mathcal{I}^+ \) implies that

\[
m_i^\sigma \geq 0, \text{ for any } 1 \leq i \leq k \text{ and } \sigma \in \text{Gal}(L/Q), \quad \sum_{1 \leq i \leq r} m_i^\sigma \leq 0, \text{ for any } \sigma \in \text{Gal}(L/Q).
\]

Thus all the integers \( m_i^\sigma \) vanish. Therefore \( M(r) \) intersects \( \mathcal{I}^+ \) trivially.

\( \square \)

For \( \pi \in P \), let \( v_\pi : L^* \rightarrow \mathbb{Z} \) be the corresponding valuation.

**Lemma 5.** Let \( S \subset P \) be a finite \( \text{Gal}(L/Q) \)-invariant subset and let \( r > 0 \) be an integer.

The \( \text{Gal}(L/Q) \)-module \( L^* \) contains a free \( \mathbb{Z}\text{Gal}(L/Q) \)-module \( N(r) \) of rank \( r \) such that

(i) \( N(r) \cap \mathcal{O} = \{1\} \), and
(ii) \( v_\pi(x) = 0 \) for any \( x \in N(r) \) and any \( \pi \in S \).

**Proof.** Set \( L^*_S = \{ x \in L^* | v_\pi(x) = 0, \forall \pi \in S \} \) and let \( \theta : L^*_S \rightarrow \mathcal{I}_S \) be the map \( x \mapsto \sum_{\pi \in P, \pi \neq \mathcal{P}} v_\pi(x) [\pi] \).

By Lemma 4, \( \mathcal{I}_S \) contains a free \( \mathbb{Z}\text{Gal}(L/Q) \)-module \( M(r) \) of rank \( r \) such that \( M(r) \cap \mathcal{I}^+ = \{0\} \). Let’s remark that \( \text{Coker} \theta \) is a subgroup of the class
group $\text{Cl}(L)$ of $L$. Since, by Dirichelet Theorem, $\text{Cl}(L)$ is finite, there is a positive integer $d$ such that $d.M(r)$ lies in the image of $\theta$. Since $M(r)$ is free, there is a free $\mathbb{Z}\text{Gal}(K/Q)$-module $N(r) \subset L^*_S$ of rank $r$ which is a lift of $d.M(r)$, i.e. such that $\theta$ induces an isomorphism $N(r) \simeq d.M(r)$. Since $\theta(O\setminus 0)$ lies in $I^+$, we have $\theta(N(r)\cap O) = \{0\}$. It follows that $N(r)\cap O = \{1\}$.

The second assertion follows from the fact that $N(r)$ lies in $L^*_S$. \hfill \Box

For $\pi \in \mathcal{P}$, let $O_\pi$ and $L_\pi$ be the $\pi$-adic completions of $O$ and $L$. Let $x, y \in L$ and let $n > 0$ be an integer. In what follows, the congruence $x \equiv y \pmod{nO_\pi}$ means $x_\pi \equiv y_\pi \pmod{nO_\pi}$, where $x_\pi$ and $y_\pi$ are the images of $x$ and $y$ in $L_\pi$.

The case $n = 1$ of the next statement will be used in further sections. In such a case, Assertion (ii) is tautological.

**Theorem 1.** Let $H$ be an algebraic torus defined over $\mathbb{Q}$, and let $L$ be its splitting field. Let $n > 0$ be an integer and let $S \subset \mathcal{P}$ be the set of prime divisors of $n$.

There exists $h \in H(\mathbb{Q})$ such that

(i) $\chi(h)$ is not an algebraic integer, for any non-trivial $\chi \in X(H)$, and

(ii) $\chi(h) \equiv 1 \pmod{nO_\pi}$ for any $\chi \in X(H)$ and any $\pi \in S$.

**Proof.**

**Step 1.** First an element $h' \in H(\mathbb{Q})$ satisfying Assertion (i) and

(iii) $v_\pi(\chi(h')) = 0$, for any $\pi \in S$ and any $\chi \in X(H)$

is found.

The abelian group $X(H)$ is free of rank $r$ where $r = \dim H$. Therefore, the comultiplication $\Delta : X(H) \rightarrow X(H) \otimes \mathbb{Z}\text{Gal}(L/Q)$ provides an embedding of $X(H)$ into a free $\mathbb{Z}\text{Gal}(L/Q)$-module of rank $r$. By lemma [5] there a free $\mathbb{Z}\text{Gal}(L/Q)$-module $N(r) \subset L^*_S$ of rank $r$ with $N(r) \cap O = \{1\}$. Let

$$\mu : X(H) \otimes \mathbb{Z}\text{Gal}(L/Q) \rightarrow N(r),$$

be an isomorphism of $\mathbb{Z}\text{Gal}(L/Q)$-modules, and set $h' = \mu \circ \Delta$.

Since $H(\mathbb{Q}) = \text{Hom}_{\text{Gal}(L/Q)}(X(H), L^*)$, the embedding $h'$ is indeed an element of $H(\mathbb{Q})$. Viewed as a map from $X(H)$ to $L^*$, $h'$ is the morphism $\chi \in X(H) \mapsto \chi(h')$.

Since $\text{Im} h' \cap O = 1$ and $h'$ is injective, $\chi(h')$ is not an algebraic integer if $\chi$ is a non-trivial character. Since $\text{Im} h' \subset L^*_S$, we have $v_\pi(\chi(h')) = 0$ for any $\chi \in X(H)$. Therefore $h'$ satisfies Assertions (i) and (iii).

**Step 2.** Let $\chi_1, \ldots, \chi_r$ be a basis of $X(H)$. Since $v_\pi(\chi_i(h')) = 0$ for any $\pi \in S$, the element $\chi_i(h') \mod nO_\pi$ is an invertible element in the finite ring $O_\pi/nO_\pi$. Therefore there are positive integers $m_{i,\pi}$ such that
\[ \chi_i(h')^{m_i, \pi} \equiv 1 \mod nO_{\pi}, \]
for all \(1 \leq i \leq r\) and all \(\pi \in S\). Set \(m = \text{lcm}(m_i, \pi)\) and set \(h = h^m\).
Obviously \(h\) satisfies Assertion (i). Moreover we have \(\chi_i(h) \equiv 1 \mod nO_{\pi}\),
for all \(\pi \in S\) and all \(1 \leq i \leq r\), and therefore \(h\) satisfies Assertion (ii) as well.

\[\square\]

3 Special Gradings

Let \(n\) be a finite dimensional Lie algebra defined over \(\mathbb{Q}\) and let \(z\) be its center. The relations between the gradings of \(\mathbb{C} \otimes n\) and the automorphisms of \(n\) are investigated now.

*The following important definitions will be used in the whole paper.* Let \(S(n)\) (respectively \(\mathcal{V}(n)\)) be the set of all \(f \in \text{Aut} n\) such that \(\text{Spec } f|_z\) (respectively \(\text{Spec } f\)) contains no algebraic integers. Moreover let \(\mathcal{F}(n)\) be the set of all \(f \in \mathcal{S}(n)\) such that all eigenvalues of \(f^{-1}\) are algebraic integers. Also set \(\mathcal{F}^+(n) = \mathcal{F}(n) \cap \mathcal{V}(n)\). Here, by eigenvalues of a \(\mathbb{Q}\)-linear endomorphism \(F\), we always mean the eigenvalues of \(F\) in \(\overline{\mathbb{Q}}\).

For any field \(K\) of characteristic zero, set \(n^K = K \otimes n\) and \(z^K = K \otimes z\). Let \(G = \text{Aut} n\) be the algebraic group of automorphisms of \(n\). By definition, \(G\) is defined over \(\mathbb{Q}\), and we have \(G(K) = \text{Aut} n^K\) for any field \(K\) of characteristic zero. The notation \(n\) underlines that \(n\) can be viewed as the functor in Lie algebras \(K \mapsto n^K_i\). Let \(H \subset G\) be a maximal torus defined over \(\mathbb{Q}\), whose existence is proved in [7], see also [6], Theorem 18.2.

By definition, a \(K\)-grading of \(n\) is is a decomposition of \(n^K\)
\[n^K = \bigoplus_{n \in \mathbb{Z}} n^n_{n}^{K}\]
such that \([n^n_{n}, n^n_{m}] \subset n^n_{n+m}\) for all \(n, m \in \mathbb{Z}\). A grading is called special (respectively very special) if \(z^K \cap n^n_{0} = 0\) (respectively if \(n^n_{0} = 0\)). A grading is called non-negative (respectively positive) if \(n^n_{n} = 0\) for \(n < 0\) (respectively \(n^n_{n} = 0\) for \(n \leq 0\)).

For any field \(K\) of characteristic zero, a \(K\)-grading of \(n\) can be identified with an algebraic group morphism
\[\rho : G_m \to G\]
defined over \(K\), where \(G_m\) denotes the multiplicative group.

Consider the following two hypotheses
(\(H_K\)) The Lie algebra \(n\) admits a special \(K\)-grading,
(\(H'_K\)) The Lie algebra \(n\) admits a very special \(K\)-grading.
Lemma 6. Let $K$ be the splitting field of $H$. Up to conjugacy, any grading of $n^C$ is defined over $K$. In particular

(i) The hypotheses $\mathcal{H}_C$ and $\mathcal{H}_Q$ are equivalent.
(ii) The hypotheses $\mathcal{H}_C^0$ and $\mathcal{H}_Q^0$ are equivalent.

Proof. Let

$$n^C = \oplus_{n \in \mathbb{Z}} n^C_n$$

be a grading of $n^C$ and let $\rho : \mathbb{G}_m \to G$ be the corresponding algebraic group morphism. Since any maximal torus of $G$ is $G(\mathbb{C})$-conjugate to $H$, it can be assumed that $\rho(\mathbb{G}_m) \subset H$.

Let $X(H)$ be the character group of $H$. The group morphism $\rho$ is determined by the dual morphism $L : X(H) \to \mathbb{Z} = X(\mathbb{G}_m)$. However, $\text{Gal}(K)$ acts trivially on $X(H)$. Thus $\rho$ is automatically defined over $K$.

Lemma 7. Let $\Lambda$ be a finitely generated abelian group and let $S \subset \Lambda$ be a finite subset containing no element of finite order. Then there exists a morphism $L : \Lambda \to \mathbb{Z}$ such that

$$L(\lambda) \neq 0$$

for any $\lambda \in S$.

Proof. Let $F$ be the subgroup of finite order elements in $\Lambda$. Using $\Lambda/F$ instead of $\Lambda$, it can be assumed that $\Lambda = \mathbb{Z}^d$ for some $d$ an $0 \notin S$. Let’s choose a positive integer $N$ such that $S \subset ]-N,N[^d$ and let $L : \Lambda \to \mathbb{Z}$ be the function defined by

$$L(a_1, \ldots, a_d) = \sum_{1 \leq i \leq d} a_i N^{i-1}.$$ 

For any $\lambda = (a_1, \ldots, a_d) \in S$, there is a smallest index $i$ with $a_i \neq 0$. We have $L(\lambda) = a_i N^{i-1}$ modulo $N^i$. Since $|a_i| < N$, it follows that $L(\lambda) \neq 0 \mod N^i$ and therefore $L(\lambda) \neq 0$.

Lemma 8. Let $f \in G(\mathbb{Q})$. There is a $f$-invariant $\mathbb{Z}$-grading of $n^\mathbb{Q}$ such that all eigenvalues of $f$ on $n^\mathbb{Q}_0$ are roots of unity.

In particular, if Spec $f$ contains no root of unity, then $n^\mathbb{Q}$ admits a very special grading.

Proof. Let $\Lambda \subset \mathbb{Q}^*$ be the subgroup generated by the Spec $f$. For any $\lambda \in \Lambda$ denote by $E(\lambda) \subset n^\mathbb{Q}$ the corresponding generalized eigenspace of $f$. Let $R$ be the set of all roots of unity in Spec $f$ and set $S = \text{Spec } f \setminus R$.

By Lemma 7, there is a morphism $L : \Lambda \to \mathbb{Z}$ such that $L(\lambda) \neq 0$ for any $\lambda \in S$. Let $G$ be the decomposition...
Let \( n^\mathbb{Q} \) be the Lie algebra of \( \mathfrak{sl}_n \), and let \( \mathfrak{g} \) be a finite-dimensional Lie algebra. We define \( \mathfrak{g}^\mathbb{Q} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^k \) of \( \mathfrak{g}^\mathbb{Q} \) defined by \( \mathfrak{g}^k = \bigoplus_{L(\lambda) = k} E(\lambda) \). Since \( [E(\lambda), E(\mu)] \subset E(\lambda \mu) \) and \( L(\lambda + \mu) = L(\lambda) + L(\mu) \) for any \( \lambda, \mu \in \Lambda \), it follows that \( \mathfrak{g} \) is a grading of the Lie algebra \( \mathfrak{g}^\mathbb{Q} \). Moreover we have \( \mathfrak{g}^0 = \bigoplus_{\lambda \in \mathbb{R}} E(\lambda) \), from which the lemma follows.

**Lemma 9.** With the previous notations

(i) the Lie algebra \( n^\mathbb{C} \) admits a special grading iff \( S(n) \neq \emptyset \).

(ii) the Lie algebra \( n^\mathbb{C} \) admits a very special grading iff \( \mathcal{V}(n) \neq \emptyset \).

**Proof.** In order to prove Assertion (i), let's consider the following assertion

\[(A) \quad H^0(\mathfrak{H}(\mathbb{Q}), \mathfrak{g}^\mathbb{Q}) = 0.\]

The proof is based on the following "cycle" of implications

\( n^\mathbb{C} \) has a special grading \( \Rightarrow (A) \Rightarrow S(n) \neq \emptyset \Rightarrow n^\mathbb{C} \) has a special grading.

**Step 1:** the existence of a special grading of \( n^\mathbb{C} \) implies \( (A) \). By hypothesis and Lemma 6, \( n^\mathbb{Q} \) admits a special grading. Let \( \rho : G_m \to G \) be the corresponding group morphism. Since all maximal tori of \( G \) are conjugate to \( H \), we can assume that \( \rho(G_m) \subset H \). Therefore we have

\[H^0(\mathfrak{H}(\mathbb{Q}), \mathfrak{g}^\mathbb{Q}) \subset H^0(\rho(\mathbb{Q}^*), \mathfrak{g}^\mathbb{Q}) = 0.\]

Thus Assertion \( A \) is proved.

**Step 2:** proof that \( (A) \) implies \( S(n) \neq \emptyset \). By Theorem 1, there exists \( f \in H(\mathbb{Q}) \) such that \( \chi(f) \) is not an algebraic integer for any non-trivial character \( \chi \in X(H) \). If we assume \( (A) \), then Spec \( f \mid \mathfrak{g} \) contains no algebraic integers and therefore \( S(n) \neq \emptyset \).

**Step 3:** proof that \( S(n) \neq \emptyset \) implies the existence of a special grading. For any \( f \in S(n) \), Since Spec \( f \mid \mathfrak{g} \) contains no roots of unity. It follows from Lemma 8 that the Lie algebra \( n^\mathbb{Q} \) (and therefore \( n^\mathbb{C} \)) admits a special grading. Therefore \( S(n) \neq \emptyset \) implies the existence of a special grading.

The proof of Assertion (ii) is almost identical. Instead of \( (A) \), the "cycle" of implications uses the following assertion

\[(A^0) \quad H^0(\mathfrak{H}(\mathbb{Q}), n^\mathbb{Q}) = 0.\]

**Lemma 10.** The following are equivalent:

(i) the Lie algebra \( n^\mathbb{Q} \) admits a non-negative special grading,

(ii) the Lie algebra \( n^\mathbb{C} \) admits a non-negative special grading, and

(iii) The set \( F(n) \) is not empty.
Proof. Proof that \((ii) \Rightarrow (iii)\). Let \(n^C = \bigoplus_{k \geq 0} n^C_k\) be a non-negative special grading of \(n^C\) and let \(\rho : \mathbb{G}_m \to G\) be the corresponding group morphism. Up to conjugacy, we can assume that \(\rho (\mathbb{G}_m) \subset H\). It follows that the grading is defined over the splitting field \(K\) of \(H\).

Let \(g_1 \in H(K)\) be the isomorphism defined by \(g_1 x = 2^k x\) if \(x \in n^C_k\). Set \(n = [K : \mathbb{Q}]\) and let \(g_1, g_2 \ldots g_n\) be the Gal\((L/\mathbb{Q})\)-conjugates of \(g_1\). Since all \(g_i\) belongs to \(H(K)\), the automorphisms \(g_i\) commute. Hence the product \(g := g_1 \cdots g_n\) is well defined and \(g\) belongs to \(H(\mathbb{Q})\). By hypotheses, all eigenvalues of \(g_i\) are power of 2, and all eigenvalues of \(g_i|_{z^C}\) are distinct from 1. Therefore all eigenvalues of \(g\) are integers, and all eigenvalues of \(g_i|_{z^C}\) are \(\neq \pm 1\). It follows that \(g^{-1}\) belongs to \(F(n)\). Therefore \(F(n) \neq \emptyset\)

Proof that \((iii) \Rightarrow (i)\). Let \(f \in F(n)\) and set \(g = f^{-1}\). Set \(K = \mathbb{Q}(\text{Spec} g)\) and let \(L : K^* \to \mathbb{Z}\) be the map defined by
\[
L(x) = \sum_p v_p (N_{K/\mathbb{Q}}(x))
\]
where the sum runs over all prime numbers \(p\) and where \(N_{K/\mathbb{Q}} : K^* \to \mathbb{Q}^*\) denotes the norm map.

For any integer \(k\), set
\[
n^C_k = \bigoplus_{L(x)=k} E(x)
\]
where \(E(x) \subset n^C\) denotes the generalized eigenspace associated to \(x \in \text{Spec } g\). We have \([E(x), E(y)] \subset E(xy)\) and \(L(xy) = L(x) + L(y)\), for any \(x, y \in K\).

Therefore the decomposition
\[
n^K = \bigoplus_{k \in \mathbb{Z}} n^C_k
\]
is a grading \(G\) of the Lie algebra \(n^C\). Since the function \(L\) is Gal\((\mathbb{Q})\)-invariant, the grading \(G\) is indeed defined over \(\mathbb{Q}\). It remains to prove that \(G\) is non-negative and special.

Since any \(x \in \text{Spec } g\) is an algebraic integer, we have \(L(x) \geq 0\) and the grading is non-negative. Since no \(x \in \text{Spec } g|_\mathbb{C}\) is an algebraic unit, we have \(N_{K/\mathbb{Q}}(x) \neq \pm 1\) and \(L(x) > 0\). Thus the grading is special, what proves that \((iii) \Rightarrow (i)\).

\[\square\]

Lemma 11. The following are equivalent:

(i) the Lie algebra \(n^\mathbb{Q}\) admits a positive grading,
(ii) the Lie algebra \(n^C\) admits a positive grading, and
(iii) The set \(F^+(n)\) is not empty.

Since the proof is almost identical to the previous proof, it will be skipped. The equivalence \((i) \Leftrightarrow (ii)\) also appears in [5].
4 Height and relative complexity

For the whole chapter, $V$ denotes a finite dimensional vector space over $\mathbb{Q}$. In this section, we define the notion of the \textit{height} of the isomorphisms $h \in GL(V)$ and the notion of a \textit{minimal lattice}.

4.1 Height, complexity and minimality

Let $h \in GL(V)$. Recall that a \textit{lattice} of $V$ is a finitely generated subgroup $\Lambda$ which contains a basis of $V$. Let $\mathcal{D}(h)$ be the set of all couple of lattices $(\Lambda, E)$ of $V$ such that $E \subset \Lambda$ and $h(E) \subset \Lambda$. By definition, the \textit{height} of $h$, is the integer

$$ht(h) := \min_{(\Lambda, E) \in \mathcal{D}(h)} [\Lambda : E].$$

Let $\mathcal{D}_{\text{min}}(h)$ be the set of all couples $(\Lambda, E) \in \mathcal{D}(h)$ such that $[\Lambda : E] = ht(h)$.

Similarly, for a lattice $\Lambda$ of $V$, the \textit{h-complexity} of $\Lambda$ is the integer

$$cp_h(\Lambda) := \min_{(\Lambda, E) \in \mathcal{D}(h)} [\Lambda : E].$$

It is clear that $cp_h(\Lambda) = [\Lambda : E]$, where $E = \Lambda \cap h^{-1}(\Lambda)$. The lattice $\Lambda$ is called \textit{minimal relative to} $h$ if $cp_h(\Lambda) = ht(h)$.

For the proofs, a technical notion of relative height is needed. Let $\text{End}_h(V)$ be the commutant of $h$ and let $A \subset C(h) \subset \text{End}_h(V)$ be a subring. By definition, an $A$-\textit{lattice} $\Lambda$ means a lattice $\Lambda$ which is an $A$-module. Let $\mathcal{D}^A(h)$ be the set of all couple of $A$-lattices $(\Lambda, E)$ in $\mathcal{D}(h)$. The $A$-\textit{height} of $h$ is the integer

$$ht_A(h) := \min_{(\Lambda, E) \in \mathcal{D}^A(h)} [\Lambda : E].$$

Obviously, we have $ht_A(h) \geq ht(h) = ht_Z(h)$. Let $\mathcal{D}^A_{\text{min}}(h)$ be the set of all couples $(\Lambda, E) \in \mathcal{D}^A(h)$ such that $[\Lambda : E] = ht_A(h)$.

4.2 Height and filtrations

Let $V$ be a finite dimensional vector space over $\mathbb{Q}$ and let $h \in GL(V)$. Let $A$ be a subring of $\text{End}_h(V)$ and let $A[h]$ be the subring of $\text{End}_h(V)$ generated by $A$ and $h$.

**Lemma 12.** Let $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ be a filtration of $V$, where each vector space $V_i$ is a $A[h]$-submodule. For $i = 1$ to $n$, set $h_i = h_{V_i/V_{i-1}}$. Then we have

$$ht_A(h) \geq \prod_{1 \leq i \leq n} ht_A(h_i).$$

Moreover if $V \simeq \bigoplus V_i/V_{i-1}$ as a $A[h]$-module, we have

$$ht_A(h) = \prod_{1 \leq i \leq n} ht_A(h_i).$$
Proof. Clearly it is enough to prove the lemma for \( n = 2 \). Let \((\Lambda, E) \in \mathcal{D}_{\min}^A(h)\). Set \( \Lambda_1 = \Lambda \cap V_1, E_1 = E \cap V_1, \Lambda_2 = \Lambda / \Lambda_1 \) and \( E_2 = E / E_1 \). We have
\[
[\Lambda : E] = [\Lambda_1 : E_1][\Lambda_2 : E_2].
\]
Since \((\Lambda_1, E_1) \in \mathcal{D}^A(h_1)\) and \((\Lambda_2, E_2) \in \mathcal{D}^A(h_2)\), we have
\[
ht_A(h) \geq ht_A(h_1) ht_A(h_2),
\]
what proves the first assertion.

Next, we assume that \( V \simeq V_1 \oplus V_2 \) as a \( A[h]\)-module. Let \((\Lambda_1, E_1) \in \mathcal{D}_{\min}^A(h_1), (\Lambda_2, E_2) \in \mathcal{D}_{\min}^A(h_2)\) and set \( \Lambda = \Lambda_1 \oplus \Lambda_2 \) and \( E = E_1 \oplus E_2 \). We have
\[
[\Lambda : E] = [\Lambda_1 : E_1][\Lambda_2 : E_2] = ht_A(h_1) ht_A(h_2).
\]
Therefore \( ht_A(h) \leq ht_A(h_1) ht_A(h_2) \). Hence \( ht_A(h) = ht_A(h_1) ht_A(h_2) \).

\[\square\]

Let \( h \in GL(V) \) as before. Its \textit{Chevalley decomposition} \( h = h_s h_u \) is uniquely defined by the following three conditions: \( h_s \) and \( h_u \) commutes, \( h_s \) is semi-simple and \( h_u \) is unipotent.

**Lemma 13.** We have \( ht(h) = ht(h_s) \).

Proof. By Lemma \([12]\) it can be assumed that the \( \mathbb{Q}[h] \)-module \( V \) is indecomposable. Therefore there is a vector space \( V_0 \), a semi-simple endomorphism \( h_0 \in \text{End}(V_0) \) and an isomorphism
\[
V \simeq V_0 \otimes \mathbb{Q}[t]/(t^n),
\]
relative to which \( h_s \) acts as \( h_0 \otimes 1 \) and \( h_u \) acts as \( 1 \otimes t \). Let \((\Lambda_0, E_0) \in \mathcal{D}_{\min}(h_0)\) and set \( \Lambda = \Lambda_0 \otimes \mathbb{Z}[t]/(t^n) \) and \( E = E_0 \otimes \mathbb{Z}[t]/(t^n) \). By Lemma \([12]\) we have
\[
ht(h) \geq ht(h_s) = ht(h_0)^n.
\]
Since \((\Lambda, E) \in \mathcal{D}(h)\) and
\[
[\Lambda : E] = [\Lambda_0 : E_0]^n = ht(h_0)^n,
\]
it follows that \( ht(h) = ht(h_s) \)

\[\square\]

### 4.3 Complexity of \( \mathcal{O}(h) \)-lattices

For any algebraic number \( \lambda \), let \( \mathcal{O}(\lambda) \) be the ring of integers of the number field \( \mathbb{Q}(\lambda) \). Set \( \pi_\lambda = \{ x \in \mathcal{O}(\lambda) | x \lambda \in \mathcal{O}(\lambda) \} \). Then \( \pi_\lambda \) is an integral ideal and its norm is the integer
\[
d(\lambda) := N_{\mathbb{Q}(\lambda)/\mathbb{Q}}(\pi_\lambda) = \text{Card} \mathcal{O}(\lambda) / \pi_\lambda.
\]

Let \( h \in GL(V) \) be semi-simple. Let \( P(t) \) be its minimal polynomial, let \( P = P_1 \ldots P_k \) be its factorization into irreducible factors. For \( 1 \leq i \leq k \), set
$K_i = \mathbb{Q}[t]/(P_i(t))$ and let $\mathcal{O}_i$ be the ring of integers of the number field $K_i$. Set $\mathcal{O}(h) = \oplus_{1 \leq i \leq k} \mathcal{O}_i$.

For each $\lambda \in \text{Spec } h$, let $m_\lambda$ be its multiplicity. Note that the functions $\lambda \mapsto m_\lambda$ and $\lambda \mapsto d(\lambda)$ are $\text{Gal}(\mathbb{Q})$-invariant, so they can be viewed as functions defined over $\text{Spec } h/\text{Gal}(\mathbb{Q})$.

**Lemma 14.** Let $\Lambda$ be an $\mathcal{O}(h)$-lattice of $V$. Then
\[
\text{cp}_h(\Lambda) = \prod d(\lambda)^{m_\lambda},
\]
where the product runs over $\text{Spec } h/\text{Gal}(\mathbb{Q})$.

**Proof.** With the previous notations, let $e_i$ be the unit of $\mathcal{O}_i$ and set $\Lambda_i = e_i \Lambda$. Since $\Lambda = \oplus_{1 \leq i \leq k} \Lambda_i$, it is enough to prove the lemma for $k = 1$, i.e. when the minimal polynomial of $h$ is irreducible.

Let $\lambda$ be one eigenvalue of $h$. With these new hypotheses, we have $\mathbb{Q}[h]/(P(t)) \simeq \mathbb{Q}(\lambda)$, $\mathcal{O}(h) \simeq \mathcal{O}(\lambda)$ and $V$ is a vector space of dimension $m_\lambda$ over $\mathbb{Q}(\lambda)$, relative to which $h$ is identified with the multiplication by $\lambda$. We have
\[
r_\lambda \Lambda = \Lambda \cap h^{-1} \Lambda.
\]
Since $\Lambda/r_\lambda \mathcal{I} \simeq (\mathcal{O}(\lambda)/r_\lambda)^{m_\lambda}$, it follows that $\text{cp}_h(\Lambda) = d(\lambda)^{m_\lambda}$. \hfill \Box

4.4 Computation of the height

Let $h \in GL(V)$ be semi-simple.

**Lemma 15.** We have
\[
\text{ht}(h) = \prod d(\lambda)^{m_\lambda},
\]
where the product runs over $\text{Spec } h/\text{Gal}(\mathbb{Q})$.

**Proof.** Using Lemmas 13 and Lemma 12, we can be assumed $V$ is a simple $\mathbb{Q}[h]$-module, and let $n$ be its dimension. The eigenvalues $\lambda_1, \ldots, \lambda_n$ of $h$ are conjugate by $\text{Gal}(\mathbb{Q})$. Under these simplifying hypotheses, the formula to be proved is
\[
\text{ht}(h) = d(\lambda_1).
\]

**Step 1: scalar extension.** Set $K = \mathbb{Q}(\lambda_1, \ldots, \lambda_n)$, let $U = K \otimes V$, let $\tilde{h} = 1 \otimes h$ be the extension of $h$ to $U$ and let $\{v_1, \ldots, v_n\}$ be a $K$ basis of $U$ such that $\tilde{h}.v_i = \lambda_i v_i$. We have $U = \oplus_{1 \leq i \leq n} U_i$, where $U_i = K v_i$.

Let $\mathcal{O}$ be the ring of integers of $K$. For each $1 \leq i \leq n$, set $\tilde{h}_i = h|_{U_i}$. Since each $U_i$ is a $\mathcal{O}[\tilde{h}]$-module, Lemma 12 shows that
\[
\text{ht}_{\mathcal{O}}(\tilde{h}) = \prod_{1 \leq i \leq n} \text{ht}_{\mathcal{O}}(\tilde{h}_i).
\]
Next, the integers \( \text{ht}_O(\tilde{h}_i) \) are computed. Let \( \Lambda_i \subset U_i \) be any \( O \)-lattice. Since \( O \) contains \( O(\lambda_i) = O(\tilde{h}_i) \), it follows from Lemma 14 that 
\[
\text{cp}_{\tilde{h}_i}(\Lambda_i) = d(\lambda_i)^r
\]
where \( r = \text{rk}_O(\Lambda_i) = [K : Q(\lambda_i)] \). Hence we have \( \text{ht}_O(\tilde{h}_i) = d(\lambda_i)^{[K : Q(\lambda_i)]} \). It follows that 
\[
\text{ht}_O(\tilde{h}) = \prod_{1 \leq i \leq n} d(\lambda_i)^{[K : Q(\lambda_i)]} = d(\lambda_1)^{[K : Q]} \]

**Step 2: end of the proof.** Now let \((\Lambda, E) \in D_{\text{min}}(h)\). Set \( \tilde{\Lambda} = O \otimes \Lambda \) and \( \tilde{E} = O \otimes E \). Since \( \tilde{E} \) is an \( O \)-module, we have \([\tilde{\Lambda} : \tilde{E}] \geq \text{ht}_O(\tilde{h})\). It follows that 
\[
\text{ht}(h)^{[K : Q]} = [\Lambda : E]^{[K : Q]} = [\tilde{\Lambda} : \tilde{E}] \geq \text{ht}_O(\tilde{h}) = d(\lambda)^{[K : Q]}.
\]
Thus we have \( d(\lambda_1) \leq \text{ht}(h) \). By Lemma 14 we have \( \text{ht}_O(h) = d(\lambda_1) \). It follows that 
\[
d(\lambda_1) \leq \text{ht}(h) \leq \text{ht}_O(\lambda)(h) = d(\lambda_1),
\]
what proves the formula.

\[\square\]

**Remark:** In number theory, the Weil height of an algebraic number \( \lambda \) is 
\( H(\lambda) = \theta d(\lambda)^{1/n} \), where \( \theta \) involves the norms at infinite places. Therefore \( \text{ht}(h) \) is essentially the Weil’s height of \( h \), up to the factor at infinite places.

### 4.5 A simple criterion of minimality

An obvious consequence of Lemmas 14 and 15 is

**Lemma 16.** Let \( h \in GL(V) \) be semi-simple and let \( \Lambda \) be an \( O(h) \)-lattice of \( V \). Then \( \Lambda \) is minimal relative to \( h \).

### 5 Malcev’s Theorem and self-similar data

In this chapter, we recall Malcev’s Theorem. Then we collect some related results, which are due to Malcev or viewed as folklore results. Then it is easy to characterize the self-similar data for FGTG nilpotent groups.

### 5.1 Three types of lattices

Let \( n \) be a finite dimensional be a nilpotent Lie algebra over \( Q \). The Lie algebra \( n \) is endowed with two group structures, the addition and the the Campbell-Hausdorff product. To avoid confusion, the Campbell-Hausdorff product is called the *multiplication* and it is denoted accordingly.
A multiplicative subgroup $\Gamma$ of $\mathfrak{n}$ means a subgroup relative to the Campbell-Hausdorff product. In general, a multiplicative subgroup $\Gamma$ is not an additive subgroup of $\mathfrak{n}$. However, notice that $\mathbb{Z}.x \subseteq \Gamma$ for any $x \in \Gamma$, because $x^n = nx$ for any $n \in \mathbb{Z}$.

A finitely generated multiplicative subgroup $\Gamma$ is called a multiplicative lattice if $\Gamma \mod [\mathfrak{n}, \mathfrak{n}]$ generates the $\mathbb{Q}$-vector space $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, or, equivalently, if $\Gamma$ generates the Lie algebra $\mathfrak{n}$. Let $N$ be the CSC nilpotent Lie group with Lie algebra $\mathfrak{n}^R = \mathbb{R} \otimes \mathfrak{n}$. A discrete subgroup $\Gamma$ of $N$ is called a cocompact lattice if $N/\Gamma$ is compact.

It should be noted that three distinct notions of lattices will be used in the sequel: the additive lattices, the multiplicative lattices and the cocompact lattices. When it is used alone, a lattice is always an additive lattice. This very common terminology could be confusing: the reader should read "multiplicative lattice" or "cocompact lattice" as single words.

5.2 Malcev’s Theorem
Any multiplicative lattice $\Gamma$ of a finite dimensional nilpotent Lie algebra over $\mathbb{Q}$ is a FGTF nilpotent group. Conversely, Malcev proved in [18] Malcev’s Theorem.

Malcev’s Theorem. Let $\Gamma$ be a FGTF nilpotent group.

1. There exists a unique nilpotent Lie algebra $\mathfrak{n}$ over $\mathbb{Q}$ which contains $\Gamma$ as a multiplicative lattice.
2. There exists a unique CSC nilpotent Lie group $N$ which contains $\Gamma$ as a cocompact lattice.
3. The Lie algebra of $N$ is $\mathbb{R} \otimes \mathfrak{n}$.

The Lie algebra $\mathfrak{n}$ of the previous theorem will be called the Malcev Lie algebra of $\Gamma$.

5.3 The coset index
From now on, let $\mathfrak{n}$ will be a finite dimensional nilpotent Lie algebra. The coset index, which is defined now, generalizes the notions of indices for additive lattices and for multiplicative lattices.

A subset $X$ of $\mathfrak{n}$ is called a coset union if $X$ is a finite union of $\Lambda$-coset for some additive lattice $\Lambda$.

Recall that the nilpotency index of $\mathfrak{n}$ is the smallest integer $n$ such that $C^{n+1}\mathfrak{n} = 0$, where $(C^n\mathfrak{n})_{n \geq 0}$ is its descending central series. The following lemma is easily proved by induction on the nilpotency index of $\mathfrak{n}$.

Lemma 17. Any multiplicative lattice $\Gamma$ of $\mathfrak{n}$ is a coset union.
Let $X \supset Y$ be two coset unions in $\mathfrak{n}$. Obviously, there is a lattice $\Lambda$ such that $X$ and $Y$ are both a finite union of $\Lambda$-coset. The *coset index* of $Y$ in $X$ is the number

$$[X : Y]_{\text{coset}} = \frac{\text{Card } X/\Lambda}{\text{Card } Y/\Lambda}.$$

The numerator and denominator of the previous expression depends on the choice of $\Lambda$, but $[X : Y]_{\text{coset}}$ is well defined. In general, the coset index is *not* an integer. Obviously if $\Lambda \supset \Lambda'$ are additive lattices in $\mathfrak{n}$, we have $[\Lambda : \Lambda']_{\text{coset}} = [\Lambda : \Lambda']$.

Similarly, for multiplicative lattices there is

**Lemma 18.** Let $\Gamma \supset \Gamma'$ be multiplicative lattices in $\mathfrak{n}$, we have $[\Gamma : \Gamma']_{\text{coset}} = [\Gamma : \Gamma']$.

The proof, done by induction on the nipotency index of $\mathfrak{n}$, is skipped.

### 5.4 Morphisms of FGTF nilpotent groups

**Lemma 19.** Let $\Gamma, \Gamma' \subset \mathfrak{n}$ be multiplicative lattices in $\mathfrak{n}$ and let $f : \Gamma' \to \Gamma$ be a group morphism. Then $f$ extends uniquely to a Lie algebra morphism $\tilde{f} : \mathfrak{n} \to \mathfrak{n}$.

Moreover $\tilde{f}$ is an isomorphism if $f$ is injective.

When $f$ is an isomorphism, the result is due to Malcev, see [18], Theorem 5. In general, the lemma is a folklore result and it is implicitly used in Homotopy Theory, see e.g. [1]. Since we did not found a precise reference, a proof, essentially based on Hall’s collecting formula (see Theorem 12.3.1 in [15]), is now provided.

**Proof.** Let $x \in \mathfrak{n}$. Since $\Gamma$ contains an additive lattice by Lemma 17, we have $m\mathbb{Z}x \subset \Gamma$ for some $m > 0$. Thus there is a unique map $\tilde{f} : \mathfrak{n} \to \mathfrak{n}$ extending $f$ such that $\tilde{f}(nx) = n\tilde{f}(x)$ for any $x \in \mathfrak{n}$ and $n \in \mathbb{Z}$. It remains to prove that

$$\tilde{f}(x + y) = \tilde{f}(x) + \tilde{f}(y), \text{ and } \tilde{f}([x, y]) = [\tilde{f}(x), \tilde{f}(y)],$$

for any $x, y \in \mathfrak{n}$.

Let $n$ be the nipotency index of $\mathfrak{n}$. Set $\mathcal{L}(2, n) = \mathcal{L}(2)/\mathcal{L}^{n+1}(2)$, where $\mathcal{L}(2)$ denotes the free Lie algebra over $\mathbb{Q}$ freely generated by $X$ and $Y$. Let $\Gamma(2, n) \subset \mathcal{L}(2, n)$ be the multiplicative subgroup generated by $X$ and $Y$.

As before, $m(X + Y)$ and $m[X, Y]$ belongs to $\Gamma(2, n)$ for some $m > 0$. Thus there are $w_1, w_2$ in the free group over two generators, such that $w_1(X, Y) = m(X + Y)$ and $w_2(X, Y) = m[X, Y]$.  


Since \( L(2, n) \) is a free in the category of nilpotent Lie algebras of nilpotency index \( \leq n \), we have
\[
w_1(x, y) = m(x + y) \text{ and } w_2(x, y) = m[x, y]
\]
for any \( x, y \in \mathfrak{n} \). From this it follows easily that \( \tilde{f} \) is a Lie algebra morphism.

5.6 Self-similar data for FGTF nilpotent groups
Let \( z \) be the center of \( \mathfrak{n} \). Recall that \( S(\mathfrak{n}) \) (respectively \( V(\mathfrak{n}) \)) is the set of all \( f \in \text{Aut} \mathfrak{n} \) such that Spec \( f|_z \) (respectively Spec \( f \)) contains no algebraic integers. Let \( \Gamma \supset \Gamma' \) be multiplicative lattices of \( \mathfrak{n} \), let \( f : \Gamma' \to \Gamma \) be a morphism and let \( \tilde{f} : \mathfrak{n} \to \mathfrak{n} \) be its extension.

Lemma 20. Let’s assume that \( f \) is injective. Then
(i) \((\Gamma', f)\) is a self-similar datum iff \( \tilde{f} \) belongs to \( S(\mathfrak{n}) \),
(ii) \((\Gamma', f)\) is a free self-similar datum iff \( \tilde{f} \) belongs to \( V(\mathfrak{n}) \)
(iii) if \((\Gamma', f)\) is a fractal datum, then \( f \) belongs to \( F(\mathfrak{n}) \).

Proof. Let \( V \) be a finite dimensional vector space over \( \mathbb{Q} \) and let \( f \in GL(V) \).
We will repeatedly use the fact that Spec \( f \) contains an algebraic integer iff \( V \) contains a finitely generated subgroup \( E \neq 0 \) such that \( f(E) \subset E \).

Proof of Assertion (i). Since \( \Gamma' \) contains a set of generators of \( \mathfrak{n} \), the subgroup \( Z(\Gamma') := \Gamma' \cap z \) is the center of \( \Gamma' \). Let \( K \) be the \( f \)-core of the virtual endomorphism \((\Gamma', f)\).

Let’s assume that \( (\Gamma', f) \) is not a self-similar datum. Since \( K \neq 1 \), the additive group \( K \cap Z(\Gamma') \) is non-trivial, finitely generated and \( \tilde{f} \)-invariant. Therefore \( \tilde{f} \notin S(\mathfrak{n}) \).

Conversely let’s assume that \( \tilde{f} \notin S(\mathfrak{n}) \). Then there is a nonzero finitely generated subgroup \( E \subset z \) such that \( \tilde{f}(E) \subset E \). By Lemma 17, \( Z(\Gamma') \) is an additive lattice of \( z \). Therefore we have \( mE \subset Z(\Gamma') \) for some \( m > 0 \). Since \( K \) contains \( mE \), it follows that \( (\Gamma', f) \) is not a self-similar datum.

Proof of Assertion (ii). Let \( A \subset \Gamma \) be a set of representatives of \( \Gamma/\Gamma' \). Let’s consider the action of \( \Gamma \) on \( A^\omega \) associated with the virtual endomorphism \((\Gamma', f)\).

Let’s assume that \( \tilde{f} \notin V(\mathfrak{n}) \). Then there is a nonzero finitely generated abelian subgroup \( F \subset \mathfrak{n} \) such that \( \tilde{f}(F) \subset F \). As before, it can be assumed \( F \) lies in \( \Gamma' \). Let \( e \in A \) be the representative of the trivial coset and let \( e^{\omega} = ee \ldots \) be the infinite word over the single letter \( e \). Since \( f(F) \subset F \), it follows that \( \gamma(e^{\omega}) = e^{\omega} \) for any \( \gamma \in F \). Hence \( \Gamma \) does not act freely on \( A^\omega \).
Conversely, let assume that $\Gamma$ does not act freely on $A$. Let’s define inductively the subsets $H(n) \subset \Gamma$ by
\[ H(1) = \bigcup_{a \in A} a \Gamma a^{-1} \quad \text{and} \quad H(n+1) = \{ \gamma \in \Gamma | \exists a \in A : a \gamma a^{-1} \in \Gamma' \land f(a \gamma a^{-1}) \in H(n) \}, \]
for $n \geq 1$. Indeed $H(n)$ is the set of all $\gamma \in \Gamma$ which have at least one fixed point on $A^\omega$. It follows easily that $H := \bigcap_{n \geq 1} H(n)$ is the set of all $\gamma \in \Gamma$ which have at least one fixed point on $A^\omega$. There is an integer $k$ such that $H \subset C^k$ but $H \not\subset C^{k+1}$.

Let $H$ be the image of $H$ in $C^k$ and let $F$ be the additive subgroup of $C^k$ generated by $H$. Since $\Gamma$ lies in a lattice, $F$ is finitely generated. Moreover we have $axa^{-1} \equiv x \mod C^{k+1}$, for any $x \in C^k$ and $a \in A$. It follows that $\tilde{f}_k(H) \subset \mathcal{H}$, where $\tilde{f}_k$ is the linear map induced by $\tilde{f}$ on $C^k/C^{k+1}$. Hence $\tilde{f}_k(F) \subset F$ and $\text{Spec } \tilde{f}_k$ contains an algebraic integer. Therefore $\tilde{f} \not\in \mathcal{V}(n)$.

Proof of Assertion (iii). Let $(\Gamma', f)$ be a fractal datum. Let $\Lambda$ be the additive lattice generated by $\Gamma$. Since
\[ \tilde{f}^{-1}(\Lambda) \subset \Lambda, \]
all $x \in \text{Spec } \tilde{f}^{-1}$ are algebraic integers. Therefore $\tilde{f}$ belongs to $\mathcal{F}(n)$. \qed

6 Relative complexity of multiplicative lattices

This chapter is the multiplicative analogue of ch. 4. The main result is the refined criterion of minimality. Together with Theorem 1, it is the main ingredient of the proof of Theorem 2 and 3.

Throughout the whole chapter, $\mathfrak{n}$ is finite dimensional nilpotent Lie algebra over $\mathbb{Q}$, and $\mathfrak{z}$ is its center.

6.1 Complexity of multiplicative lattices

Let $f \in \text{Aut } \mathfrak{n}$ and let $\Gamma$ be a multiplicative lattice of $\mathfrak{n}$. The complexity of $\Gamma$ relative to $f$ is the integer
\[ \text{cp}_f(\Gamma) = [\Gamma : \Gamma'], \]
where $\Gamma' = \Gamma \cap f^{-1}(\Gamma)$. The multiplicative lattice $\Gamma$ is called minimal relative to $f$ if $\text{cp}_f(\Gamma) = \text{ht}(f)$. Thanks to Lemma 18 the notation $\text{cp}_f(\Gamma)$ is unambiguous.

Lemma 21. Let $\Gamma$ be multiplicative lattices of $\mathfrak{n}$. Then we have
\[ \text{cp}_f(\Gamma) \geq \text{ht}(f). \]
Proof. The proof goes by induction on the nilpotency index of \( n \).

Let \( Z \) be the center of \( \Gamma \). Set \( \Gamma' = \Gamma \cap f^{-1}(\Gamma), Z' = Z \cap f^{-1}(Z), \Gamma = \Gamma/Z, \Gamma' = \Gamma'/Z' \). Also set \( \overline{\pi} = n/3 \) and let \( f : \overline{\pi} \to \overline{\pi} \) and \( f_0 : 3 \to 3 \) be the isomorphisms induced by \( f \).

By induction hypothesis, we have \( cp_f(\Gamma) \geq \text{ht}(f) \) and therefore \( [\Gamma : \Gamma'] \geq \text{ht}(f) \).

By definition, we have \( [Z : Z'] = cp_{f_0} Z \geq \text{ht}(f_0) \). Moreover by Lemma 15 we have \( \text{ht}(f) = \text{ht}(f_0) \text{ht}(f) \). It follows that
\[
[\Gamma : \Gamma'] = [Z : Z'] [\overline{\Gamma} : \overline{\Gamma'}] \geq \text{ht}(f_0) \text{ht}(f) = \text{ht}(f),
\]
and the statement is proved. \( \square \)

6.2 A property of the minimal multiplicative lattices

Let \( \Gamma \) be a multiplicative lattice of \( n \) and let \( h \in \text{Aut } n \). For simplicity, let’s assume that \( h \) is semi-simple.

Lemma 22. The following assertions are equivalent

(i) \( \Gamma \) is minimal relative to \( h \), and

(ii) the virtual morphism \((\Gamma', h)\) is good, where \( \Gamma' = \Gamma \cap h^{-1}(\Gamma) \).

In particular, there is a multiplicative lattice \( \tilde{\Gamma} \subset \Gamma \) which is minimal relative to \( h \).

Proof. By Lemma 17, \( \Gamma \) is a coset union. Any additive lattice contains a \( O(h) \)-module of finite index. Therefore there is an \( O(h) \)-lattice \( \Lambda \) such that \( \Gamma \) is an union of \( \Lambda \)-cosets.

Let \( \Gamma_0, \Gamma_1, \ldots \) be the multiplicative lattices inductively defined by \( \Gamma_0 = \Gamma, \Gamma_1 = \Gamma' \) and \( \Gamma_{n+1} = \Gamma_n \cap h^{-1}(\Gamma_n) \) for \( n \geq 1 \). Similarly let \( \Lambda_0, \Lambda_1, \ldots \) be the additive lattices defined by \( \Lambda_0 = \Lambda \) and \( \Lambda_{n+1} = \Lambda_n \cap h^{-1}(\Lambda_n) \) for \( n \geq 0 \).

By Lemma 2 the sequence \( [\Gamma_n : \Gamma_{n+1}] \) is not increasing. By Lemma 21 we have \( [\Gamma_n : \Gamma_{n+1}] \geq \text{ht}(f) \). Moreover, it follows from Lemma 16 that \( [\Lambda_n : \Lambda_{n+1}] = \text{ht}(h) \) for all \( n \).

Let’s assume now that \( \Gamma \) is minimal relative to \( h \). We have \( [\Gamma_n : \Gamma_{n+1}] = \text{ht}(f) \) for all \( n \), and therefore the virtual morphism \((\Gamma', h)\) is good.

Conversely, let’s assume that the virtual morphism \((\Gamma', h)\) is good. By hypotheses we have \( [\Gamma_n : \Gamma_{n+1}] = [\Gamma_0 : \Gamma_1]^n \) and \( [\Lambda_0 : \Lambda_n] = \text{ht}(h)^n \) for all \( n \geq 1 \).

It follows that
\[
[\Gamma_0 : \Lambda_{n+1}]_{\text{coset}} = [\Gamma_0 : \Lambda_0]_{\text{coset}} \text{ht}(h)^n.
\]
Since \( \Gamma_n \supset \Lambda_n \), we have \( [\Gamma_0 : \Gamma_n] \leq [\Gamma_0 : \Lambda_n]_{\text{coset}} \).

and therefore
\[ \Gamma_0 : \Gamma_1 \leq \left[ \Gamma_0 : \Lambda_0 \right]_{\text{coset}} \text{ht}(h)^n, \] for all \( n \geq 0. \)

Hence \( \Gamma_0 : \Gamma_1 \leq \text{ht}(f). \) It follows from Lemma 21 that \( \Gamma_0 : \Gamma_1 = \text{ht}(f), \) thus \( \Gamma \) is minimal relative to \( h. \)

In order to prove the last assertion, notice that the sequence \( \left[ \Gamma_n : \Gamma_{n+1} \right] \) is stationary for \( n \geq N, \) for some \( N > 0. \) Therefore \( (\Gamma_{N+1}, h) \) is a good virtual morphism of \( \Gamma_N. \) Thus the subgroup \( \tilde{\Gamma} = \Gamma_N \) is minimal relative to \( h. \)

6.3 A refined criterion of minimality

A refined version of Lemma 16 is now provided. Let \( \Gamma \) be a multiplicative lattice in \( n \) and let \( h \in \text{Aut} n \) be semi-simple. Let \( L \) be the field generated by \( \text{Spec} h, \) let \( \mathcal{O} \) be its ring of integers and let \( \mathcal{P} \) be the set of prime ideals of \( \mathcal{O}. \)

Let \( \Lambda \) be an \( \mathcal{O}(h) \)-lattice and let \( n > 0 \) be an integer. Let’s assume that \( \Lambda \supset \Gamma \) and \( \Gamma \) is an union of \( n \Lambda \)-cosets.

**Lemma 23.** Let \( S \) be the set of divisors of \( n \) in \( \mathcal{P}. \) Assume that

\[ \lambda \equiv 1 \mod n\mathcal{O}_\pi, \]

for any \( \lambda \in \text{Spec} h \) and any \( \pi \in S. \) Then \( \Gamma \) is minimal relative to \( h. \)

**Proof.**

**Step 1.** Since \( \text{Spec} h \) lies in \( \mathcal{O}_\pi \) for all \( \pi \in S, \) there exists a positive integer \( d, \) which is prime to \( n, \) such that \( d\lambda \in \mathcal{O} \) for all \( \lambda \in \mathcal{O}. \) Moreover we can assume that \( d \equiv 1 \mod n. \)

Let \( \lambda \in \text{Spec} h. \) We have \( d\lambda \equiv 1 \mod n\mathcal{O}_\pi \) for all \( \pi \in S. \) Therefore we have

\[ d\lambda \in 1 + n\mathcal{O}, \]

for all \( \lambda \in \text{Spec} h. \) Set \( H = dh. \) Since \( \text{Spec} dH \) and \( \text{Spec} (H - 1)/n \) lie in \( \mathcal{O}, \) it follows that

\[ H \in \mathcal{O}(h) \text{ and } H \in 1 + n\mathcal{O}(h). \]

**Step 2.** Set \( \Lambda' = \Lambda \cap h^{-1}\Lambda. \) Since all eigenvalues of \( h \) are units in \( \mathcal{O}_\pi \) whenever \( \pi \) divides \( n, \) the height of \( h \) is prime to \( n. \) By Lemma 16 we have \( [\Lambda : \Lambda'] = \text{ht}(h). \) Therefore we get

\[ \Lambda = \Lambda' + n\Lambda. \]

It follows that

\[ \Gamma = \coprod_{1 \leq i \leq k} g_i + n\Lambda \]

for some \( g_1, ..., g_k \in \Lambda', \) where \( k = [\Gamma : n\Lambda] \) and where \( \coprod \) is the symbol of the disjoint union. Since \( H(g_i) \equiv g_i \mod n\Lambda, \) we get that \( h(g_i) \in g_i + n\Lambda \subset \Gamma. \) Therefore we have

\[ 23 \]
Therefore we have \([\Gamma : n\Lambda] \geq k = [\Gamma : n\Lambda']\). It follows that 
\([\Gamma : \Gamma'] \leq [n\Lambda : n\Lambda'] = \text{ht}(h)\).
By Lemma 21 we have \([\Gamma : \Gamma'] = \text{ht}(h)\). Thus \(\Gamma\) is minimal relative to \(h\).

\[\Gamma' \supset \bigcup_{1 \leq i \leq k} g_i + n\Lambda',\]

7 Proof of Theorems 2 and 3

7.1 Proof of Theorem 2 and 3.
Let \(n\) be a finite dimensional nilpotent Lie algebra over \(\mathbb{Q}\) and let \(\mathfrak{z}\) be its center and let \(\Gamma\) be a multiplicative lattice of \(n\).

**Theorem 2.** The following assertions are equivalent

(i) The group \(\Gamma\) is transitive self-similar,
(ii) the group \(\Gamma\) is densely self-similar, and
(iii) the Lie algebra \(n\mathbb{C}\) admits a special grading.

**Proof.** Let’s consider the following assertion

\((A)\quad S(n) \neq \emptyset.\)

The implication \((ii) \Rightarrow (i)\) is tautological. Together with the Lemmas, the following implications are already proved

\((ii) \Rightarrow (i) \Rightarrow (A) \Leftrightarrow (iii).\)

Therefore, it is enough to prove that \((A) \Rightarrow (ii).\)

**Step 1. Definition of some \(h \in G(\mathbb{Q})\).** Let’s assume that \(S(n) \neq \emptyset\), and let \(f \in S(n)\). Since the semi-simple part of \(f\) is also in \(G(\mathbb{Q})\), it can be assumed that \(f\) is semi-simple. Let \(K \subset G\) be the Zariski closure of the subgroup generated by \(f\) and set \(H = K^0\).

Let \(\Lambda\) be the \(O(f)\)-module generated by \(\Gamma\). By Lemma 17, \(\Gamma\) is a coset union. Therefore \(\Lambda\) is a lattice and \(\Gamma\) is an union of \(n\Lambda\)-coset for some positive integer \(n\).

Let \(X(H)\) be the group of characters of \(H\), let \(K\) be the splitting field of \(H\), let \(O\) be the ring of integers of \(K\), let \(P\) be the set of prime ideals of \(O\) and let \(S\) be set set of all \(\pi \in P\) dividing \(n\).

By Theorem 1, there exists \(h \in H(\mathbb{Q})\) such that, for any non-trivial \(\chi \in X\) we have

(i) \(\chi(h)\) is not an algebraic integer, and
(ii) \(\chi(h) \equiv 1 \mod nO_\pi\) for any \(\pi \in S\).
Step 2. Let $\Gamma' = \Gamma \cap h^{-1}(\Gamma)$. We claim that the virtual morphism $(\Gamma', h)$ is a good self-similar datum.

Since $K \subset G$ is the Zariski closure of the subgroup generated by $f$, we have $\mathbb{Q}[h] \subset \mathbb{Q}[f]$ and therefore $\Lambda$ is a $\mathcal{O}(h)$-lattice. It follows from Lemma 24 that the virtual endomorphism $(\Gamma', h)$ is good.

Moreover, let $\Omega_0$ be the set of weights of $H$ over $\mathfrak{z} \mathbb{Q}$. There is an integer $l$ such that $f^l \in K^0 = H$. The spectrum of $f^l$ on $\mathfrak{z} \mathbb{Q}$ are the numbers $\chi(f^l)$ when $\chi$ runs over $\Omega_0$. Thus it follows that $\Omega_0$ does not contain the trivial character, hence $h$ belongs to $S(n)$.

Therefore by Lemma 20, the virtual endomorphism $(\Gamma', h)$ is a good self-similar datum. Thus by Lemma 3, $\Gamma$ is a densely self-similar group.

Theorem 3. The following assertions are equivalent

(i) The group $\Gamma$ is freely self-similar,

(ii) the group $\Gamma$ is freely densely self-similar, and

(iii) the Lie algebra $n \mathbb{C}$ admits a very special grading.

Proof. Let’s assume Assertion (i). Let’s consider a free self-similar action of $\Gamma$ on some $A^\omega$ and let $A'$ be any $\Gamma$-orbit in $A$. Then the action of $\Gamma$ on $A^\omega$ is free transitive self-similar, thus $\Gamma$ is freely transitive self-similar.

The rest of the proof is identical to the previous proof, except that

1) the assertion $(A)$ is replaced by $(A')$: $\mathcal{V}(n) \neq \emptyset$,

2) the Lemmas 3(ii) and 9(ii) are used instead of Lemmas 6(i) and 9(i) in order to prove that $(ii) \Rightarrow (i) \Rightarrow (A') \Leftrightarrow (iii)$,

3) the proof that $A' \Rightarrow (ii)$ uses the weights of $H$ and the eigenvalues of $f$ on $n$ instead of $\mathfrak{z}$.

7.2 Manning’s Theorem

Let $N$ be a CSC nilpotent Lie group $N$ and let $\Gamma$ be a cocompact lattice. The manifold $M = N/\Gamma$ is called a nilmanifold.

A diffeomorphism $f : M \to M$ is called an Anosov diffeomorphism if

(i) there is a continuous splitting of the tangent bundle $TM$ as $TM = E_u \oplus E_s$ which is invariant by $df$, and

(ii) there is a Riemannian metric relative to which $df|_{E_s}$ and $df^{-1}|_{E_u}$ are contracting.

For any $x \in M$, $f$ induces a group automorphism $f_*$ of $\Gamma \simeq \pi_1(M)$. By Lemma 19 $f_*$ extends to an isomorphism $\tilde{f}_* : n\mathbb{R} \to n\mathbb{R}$, where $n\mathbb{R}$ is
the Lie algebra of $N$. Strictly speaking, $\tilde{f}_*$ is only defined up to an inner automorphism. Since $f_*$ is well defined modulo the unipotent radical of $\text{Aut } n^\mathbb{R}$, the set $\text{Spec } f_*$ is unambiguously defined.

**Manning's Theorem.** The set $\text{Spec } f_*$ contains no root of unity.

See [18]. Later on, A. Manning proved a much stronger result. Namely $\text{Spec } f_*$ contains no eigenvalues of absolute value 1, and $f$ is topologically conjugated to an Anosov automorphism, see [19].

### 7.3 A Corollary for nilmanifolds with an Anosov diffeomorphism

**Corollary 4.** Let $M$ be a nilmanifold endowed with an Anosov diffeomorphism. Then $\pi_1(M)$ is freely densely self-similar.

**Proof.** By definition, we have $M = N/\Gamma$, where $N$ is a CSC nilpotent Lie group and $\Gamma \simeq \pi_1(M)$ is a cocompact lattice. Set $n^\mathbb{R} = \text{Lie } N$ and $n^\mathbb{C} = \mathbb{C} \otimes n^\mathbb{R}$. By Manning's Theorem and Lemma 8, $n^\mathbb{C}$ has a very special grading. Therefore $\Gamma$ is freely densely self-similar by Theorem 3.

### 7.4 Characterisation of fractal FGTF nilpotent groups

For completeness purpose, we will now investigate the non-negative gradings of $n^\mathbb{C}$. Unlike Theorems 2 and 3, the proof of Propositions 5 and 6 are quite obvious.

Let $n^\mathbb{Q}$ be a finite dimensional nilpotent Lie algebra and let $\Gamma$ be a multiplicative lattice in $n$. Set $n^\mathbb{C} = \mathbb{C} \otimes n^\mathbb{Q}$.

**Proposition 5.** The following assertions are equivalent

(i) The group $\Gamma$ is fractal

(ii) $n^\mathbb{C}$ admits a non-negative special grading.

(iii) $n^\mathbb{Q}$ admits a non-negative special grading.

**Proof.** It follows from Lemma 10 that Assertions (ii) and (iii) are equivalent.

**Proof that (i) $\Rightarrow$ (ii).** By assumption, there is a fractal datum $(\Gamma', f)$. Let $g : \Gamma \to \Gamma'$ be the inverse of $f$ and let $\tilde{g} \in \text{Aut } n$ be its unique extension.

Let $\Lambda \subset n$ be the additive subgroup generated by $\Gamma$. By Lemmas 17, $\Lambda$ is an additive lattice. Since we have $\tilde{g}(\Lambda) \subset \Lambda$, it follows that all eigenvalues of $\tilde{g}$ are algebraic integers.

Moreover $(\Gamma', g^{-1})$ is a self-similar datum, thus $\text{Spec } \tilde{g}^{-1}|_3$ contains no root of unity. Therefore, by Lemma 10, Assertion (ii) holds.
Proof that (iii) ⇒ (i). Let’s assume Assertion (iii) and let 
\[ n^Q = \oplus_{k \geq 0} n_k^Q \]
be a non-negative special grading of \( n^Q \).

By Lemma 17, \( \Gamma \) lies in a lattice \( \Lambda \). Since it is possible to enlarge \( \Lambda \), we can assume that
\[ \Lambda = \oplus_{k \geq 0} \Lambda_k, \]
where \( \Lambda_k = \Lambda \cap n_k^Q \). Since \( \Gamma \) is a coset union, there is an integer \( d \geq 1 \) such that \( \Gamma \) is an union of \( d\Lambda \)-cosets.

Let \( g \) be the automorphism of \( n^Q \) defined by \( g(x) = (d + 1)^k x \) if \( x \in n_k^Q \). We claim that \( g(\Gamma) \subset \Gamma \). Let \( x \in \Gamma \) and let \( x = \sum_{k \geq 0} x_k \) be its decomposition into homogenous components. We have
\[ g(x) = x + \sum_{k \geq 1} ((d + 1)^k - 1)x_k. \]
By hypothesis each homogenous component \( x_k \) belongs to \( \Lambda \). Since \( (d+1)^k - 1 \) is divisible by \( d \), we have \( g(x) \in x + d\Lambda \subset \Gamma \) and the claim is proved.

Set \( \Gamma' = g(\Gamma) \) and let \( f : \Gamma' \to \Gamma \) be the inverse of \( g \). It is clear that \( (\Gamma', f) \) is a fractal datum for \( \Gamma \), what proves Assertion (i). \( \square \)

**Proposition 6.** The following assertions are equivalent

(i) The group \( \Gamma \) is freely fractal
(ii) \( n^C \) admits a positive grading.
(iii) \( n^Q \) admits a positive grading.

Since the proof is strictly identical, it will be skipped.

## 8 Not self-similar FGTF nilpotent groups and affine nilmanifolds

This section provides an example of a FGTF nilpotent group which is not even self-similar, see subsection 8.6. The end of the section is about the Milnor-Scheuneman conjecture.

### 8.1 FGTF nilpotent groups with rank one center

Let \( \Gamma \) be a FGTF nilpotent group and let \( Z(\Gamma) \) be its center.

**Lemma 24.** Let’s assume that \( \Gamma \) is self-similar and \( Z(\Gamma) \simeq \mathbb{Z} \). Then \( \Gamma \) is transitive self-similar.
Proof. Assume that $\Gamma$ admits a faithful self-similar action on some $A^\omega$, where $A$ is a finite alphabet. Let $a_1, \ldots, a_k$ be a set of representatives of $A/\Gamma$, where $k$ is the number of $\Gamma$-orbits on $A$. For each $1 \leq i \leq k$, let $\Gamma_i$ be the stabilizer of $a_i$. For any $h \in \Gamma_i$, there is $h_i \in \Gamma$ such that $h(a_iw) = a_i h_i(w)$, for all $w \in A^\omega$. Since the action is faithfull $h_i$ is uniquely determined and the map $f_i : \Gamma_i \rightarrow \Gamma, h \mapsto h_i$ is a group morphism.

Let $n^Q$ be the Malcev Lie algebra of $\Gamma$, and let $z$ be its center, and let $z \neq 0$ be a generator of $\cap_i Z(\Gamma_i)$. By Lemma [9] the group morphism $f_i$ extends to a Lie algebra morphism $\tilde{f_i} : n^Q \rightarrow n^Q$. Since $z = Q \otimes Z(\Gamma)$ is one dimensional, it follows that either $\tilde{f_i}$ is an isomorphism or $\tilde{f_i}(z) = 0$. In any case, we have $\tilde{f_i}(z) = x_i z$, for some $x_i \in Q$. However $Qz$ is not invariant by all $\tilde{f_i}$, otherwise it would be in the kernel of the action. It follows that at least one $x_i$ is not an integer.

For such an index $i$, the $f_i$-core of $\Gamma_i$ is trivial, and the virtual morphism $(\Gamma_i, f_i)$ is a self-similar datum for $\Gamma$. Thus $\Gamma$ is transitive self-similar.

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8.2 Small representations

Let $N$ be a CSC nilpotent Lie group with Lie algebra $n^R$ and let $\Gamma$ be a cocompact lattice.

Lemma 25. If $\Gamma$ is transitive self-similar, then there exists a faithfull $n^R$-module of dimension $1 + \dim n^R$.

Proof. By hypothesis, $\Gamma$ is transitive self-similar. By Theorem 2, $\mathfrak{z}^C$ admits a special grading

$$n^C = \bigoplus_{n \in \mathbb{Z}} n_n^C.$$ 

Let $\delta : n^C \rightarrow n^C$ be the derivation defined by $\delta(x) = nx$ if $x \in n_n$. Since $\delta|\mathfrak{z}^C$ is injective, it follows that there is some $\partial \in \text{Der } n^R$ such that $\partial|\mathfrak{z}^R$ is injective.

Set $\mathfrak{m}^R = \mathbb{R}\partial \ltimes n^R$. Relative to the adjoint action, $\mathfrak{m}^R$ is a faithfull $\mathfrak{z}^R$-module. Therefore $\mathfrak{m}^R$ is a faithfull $n^R$-module with the prescribed dimension.

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\Box
\end{flushright}

8.3 Filiform nilpotent Lie algebras

Let $n$ be a nilpotent Lie algebra over $\mathbb{Q}$. Let $C^n n$ be the decreasing central series, which is inductively defined by $C^1 n = n$ and $C^{n+1} n = [n, C^n n]$. The
nilpotent Lie algebra $\mathfrak{n}$ is called \textit{filiform} if $\dim C^1\mathfrak{n}/C^2\mathfrak{n} = 2$ and $\dim C^k\mathfrak{n}/C^{k+1}\mathfrak{n} \leq 1$ for any $k > 1$. Set $n = \dim \mathfrak{n}$. It follows from the definition that $\dim C^k\mathfrak{n}/C^{k+1}\mathfrak{n} = 1$ for any $0 < k \leq n - 1$ and $C^k\mathfrak{n} = 0$ for any $k \geq n$.

**Lemma 26.** Let $\mathfrak{n}$ be a filiform nilpotent Lie algebra over $\mathbb{Q}$, with $\dim \mathfrak{n} \geq 3$. Then its center $\mathfrak{z}$ has dimension one.

\textit{Proof.} Let $z \in \mathfrak{n}$ be nonzero. Let $k$ be the integer such that $z \in C^k\mathfrak{n}\setminus C^{k+1}\mathfrak{n}$. Since $C^k\mathfrak{n} = C^{k+1}\mathfrak{n} \oplus \mathbb{Q}z$

$$C^{k+1}\mathfrak{n} = [n, C^k\mathfrak{n}] = [n, C^{k+1}\mathfrak{n}] + [n, z] = C^{k+2}\mathfrak{n}.$$ It follows that $C^{k+1}\mathfrak{n} = 0$. Therefore $\mathfrak{z}$ lies in $C^k\mathfrak{n}$, which is a one dimensional ideal. \qed

8.4 Benoist Theorem

\textbf{Benoist’s Theorem.} There is a nilpotent Lie algebra $\mathfrak{n}_B^\mathbb{R}$ of dimension 11 over $\mathbb{R}$, with the following properties

(i) The Lie algebra $\mathfrak{n}_B^\mathbb{R}$ has no faithfull representations of dimension 12,

(ii) the Lie algebra $\mathfrak{n}_B^\mathbb{R}$ is defined over $\mathbb{Q}$, and

(iii) the Lie algebra $\mathfrak{n}_B^\mathbb{R}$ is filiform.

The three assertions appear in different places of [3]. Indeed Assertion (i), which is explicitly stated in Theorem 2 of [3], hold for a one-parameter family of eleven dimensional Lie algebras, which are denoted $\mathfrak{a}_{-2,1,t}$ in section 2.1 of [3]. These Lie algebras are filiform by Lemma 4.2.2 of [3]. Moreover, when $t$ is rational, $\mathfrak{a}_{-2,1,t}$ is defined over $\mathbb{Q}$. Therefore the Benoist Theorem holds for the Lie algebras $\mathfrak{n}_B = \mathfrak{a}_{-2,1,t}$ where $t$ is any rational number.

8.5 A FGTF group which is not self-similar

Let $N_B$ the CSC nilpotent Lie group with Lie algebra $\mathfrak{n}_B^\mathbb{R}$. Since $\mathfrak{n}_B^\mathbb{R}$ is defined over $\mathbb{Q}$, $N_B$ contains some cocompact lattice.

**Corollary 7.** Let $\Gamma$ be any cocompact lattice in $N_B$. Then $\Gamma$ is not self-similar.

\textit{Proof.} Let’s assume otherwise. By Benoist Theorem and Lemma 26, the center of $\mathfrak{n}_B^\mathbb{R}$ is one dimensional. Thus the center of $\Gamma$ has rank one, and by Lemma 24, $\Gamma$ is transitive self-similar. By Lemma 25, $\mathfrak{n}_B^\mathbb{R}$ admits a faithfull representation of dimension 12, which contradicts Benoist Theorem. Therefore $\Gamma$ is not self-similar. \qed

29
8.6 On the Scheuneman-Milnor conjecture
A smooth manifold $M$ is called affine if it admits a torsion-free and flat connection. Scheuneman [28] and Milnor [23] asked the following question

is any nilmanifold $M$ affine?

The story of the Scheuneman-Milnor conjecture is quite interesting. For many years, there have been a succession of proofs followed by refutations, but there was no doubt that the conjecture should be ultimately proved... until a counterexample has been found by Benoist [3]. Indeed it is an easy corollary of his previously mentioned Theorem.

The following question is a refinement of the previous conjecture

if $\pi_1(M)$ is densely self-similar, is the nilmanifold $M$ affine?

A positive result in that direction is

**Corollary 8.** Let $M$ be a nilmanifold. If $\pi_1(M)$ is freely self-similar, then $M$ is affine complete.

**Proof.** Set $M = N/\Gamma$, where $N$ is a CSC nilpotent Lie group and $\Gamma$ is a cocompact lattice. Let $n^\mathbb{R}$ be the Lie algebra of $N$. By Theorem 3, $C \otimes n^\mathbb{R}$ admits a very special grading, what implies that a generic derivation is injective. Therefore there is a derivation $\delta$ of $n^\mathbb{R}$ which is injective. Set $m^\mathbb{R} = \mathbb{R} \delta \ltimes n^\mathbb{R}$. Then $N$ is equivariantly diffeomorphic to the affine space $\delta + n^\mathbb{R} \subset m^\mathbb{R}$. Therefore $M$ is affine complete. 

\[ \blacksquare \]

9 Absolute Complexities

For the whole chapter, $N$ will be a CSC nilpotent Lie groups, with Lie algebra $n^\mathbb{R}$. Let’s assume that $N$ contains some cocompact lattices.

Under the condition of Theorem 2 or 3 any cocompact lattice $\Gamma$ in $N$ admits a transitive or free self-similar action on some $A^\omega$. In this section, we try to determine the minimal degree of these actions.

9.1 Three type of absolute complexities

The complexity of a cocompact lattice $\Gamma \subset N$, denoted by $cp \Gamma$, is the smallest degree of a faithfull transitive self-similar action of $\Gamma$ on some $A^\omega$, with the convention that $cp \Gamma = \infty$ if $\Gamma$ is not transitive self-similar. Similarly, the free complexity of $\Gamma$, denoted by $fcp \Gamma$, is the smallest degree of a free self-similar action of $\Gamma$. Two cocompact lattices are called commensurable if they share a
commoun subgroup of finite index. The complexity and the free complexity of a commensurable class ξ are the integers
\[ cp_\xi = \min_{\Gamma \in \xi} cp_\Gamma, \]
\[ fcp_\xi = \min_{\Gamma \in \xi} fcp_\Gamma. \]

Then, the complexity of the nilpotent group \( N \) is
\[ cp_N = \max_\xi cp_\xi, \]
where \( \xi \) runs over all commensurable classes in \( N \). In what follows, we will provide a formula for the complexity of commensurable classes. The question
under which condition \( cp_N < \infty \)?
is not solved, but it is a deep question. In chapter 10, a class of CSC nilpotent Lie groups of infinite complexity is investigated.

9.2 Theorem
Let \( \xi \) be a commensurable class of cocompact lattices in \( N \), and let \( \Gamma \in \xi \). The Malcev Lie algebra \( \Gamma \) is a \( \mathbb{Q} \)-form of the Lie algebra \( n^R \). Since it depends only on \( \xi \), it will be denoted by \( n^R(\xi) \).

**Theorem 9.** We have
\[ cp_\xi = \min_{h \in S(n(\xi))} ht(h), \]
\[ fcp_\xi = \min_{h \in V(n(\xi))} ht(h). \]

**Proof.** Let \( h \in S(n(\xi)) \) be an isomorphism of minimal height. In order to show that \( cp_\xi = ht(h) \), we can assume that \( h \) is semi-simple, by Lemma 13.

Further, let \( \Gamma \) be any cocompact lattice in \( \xi \). By Lemma 20, we have \( cp_\Gamma = \min_{f \in S(n(\xi))} cp_f \Gamma \). By lemma 21, we have \( cp_f \Gamma \geq ht(f) \), therefore we have \( cp_\Gamma \geq ht(h) \). In particular \( cp_\xi \geq ht(h) \).

By Lemma 22, \( \Gamma \) contains a finite index subgroup \( \tilde{\Gamma} \) which is minimal relative to \( h \). Since \( cp_{\tilde{h}} \tilde{\Gamma} = ht(h) \), it follows that \( cp_\xi \leq ht(h) \).

Therefore \( cp_\xi = ht(h) \) and the first assertion is proved.

For the second assertion, let’s notice that an free action of minimal degree is automatically transitive, see the proof of Theorem 3. Then the rest of the proof is strictly identical to the previous proof. \( \square \)

9.3 Classification of lattices in a CSC nilpotent Lie groups
Obviously Malcev’s Theorem implies the following

**Malcev’s Corollary.** The map \( \xi \mapsto n(\xi) \) establishes a bijection between the commensurable classes of lattices and the \( \mathbb{Q} \)-forms of the Lie algebra \( n^R \).
For the next chapter, it is interesting to translate this into the framework of non-abelian Galois cohomology. Somehow, it is more concrete, since the non-abelian Galois cohomology classifies Q-forms of classical objects.

Set $G = \text{Aut}^n C$, let $U$ be its unipotent radical and set $\overline{G} = G/U$. From now on, fix once for all a commensurable class $\xi_0$ of cocompact lattices. Then $n(\xi_0)$ is a $Q$-form of $n^C$, what provides a $Q$-form of the algebraic groups $G$ and $\overline{G}$. It induces an action of $\text{Gal}(Q)$ over $\overline{G}(\overline{Q})$.

Set $Q_\text{re} = Q \cap \mathbb{R}$ and let

$$\pi : H^1(\text{Gal}(Q), \overline{G}(\overline{Q})) \to H^1(\text{Gal}(Q_\text{re}), \overline{G}(\overline{Q}))$$

be the natural map. Recall that these two non-abelian cohomologies are pointed sets, where the distinguished point $\ast$ comes from the given $Q$-form and the induced $Q_\text{re}$-form. Denote by Ker $\pi$ the kernel of $\pi$, i.e. the fiber $\pi^{-1}(\ast)$ of the distinguished point.

Let $L(N)$ be the set of all commensurable classes of lattices of $N$, up to conjugacy.

**Corollary 10.** There is a natural identification

$$L(N) \simeq \text{Ker } \pi.$$

**Proof.** For any field $K \subset \mathbb{C}$, set $n^K = K \otimes n(\xi_0)$. For any two fields $K \subset L \subset \mathbb{C}$, let $F(L/K)$ be the set of $K$-forms of $n^L$, up to conjugacy. Then $F(L/K)$ is a pointed set, whose distinguished point is the $K$-form $n^K$.

By the Lefschetz principle, the $Q$-forms of $n^C$ (up to conjugacy) are in bijection with the $Q$-forms of $n^\overline{Q}$. Similarly by the Tarski-Seidenberg principle the real forms (up to conjugacy) of $n^C$ are in bijection with the $Q_\text{re}$-forms of $n^\overline{Q}$. So we have

$$F(\mathbb{C}/Q) \simeq F(\overline{Q}/Q) \text{ and } F(\mathbb{C}/\mathbb{R}) \simeq F(\overline{Q}/Q_\text{re})$$

Since a Lie algebra is a vector space endowed with a tensor (its Lie bracket), it follows from [29], III-2, Proposition 1 that

$$F(\overline{Q}/Q) = H^1(\text{Gal}(Q), \overline{G}(\overline{Q})), \text{ and }$$

$$F(\overline{Q}/Q_\text{re}) = H^1(\text{Gal}(Q_\text{re}), \overline{G}(\overline{Q})).$$

Moreover since $U$ is unipotent, we have

$$H^1(\text{Gal}(Q), \overline{G}(\overline{Q})) \simeq H^1(\text{Gal}(Q), \overline{G}(\overline{Q})), \text{ and }$$

$$H^1(\text{Gal}(Q_\text{re}), \overline{G}(\overline{Q})) \simeq H^1(\text{Gal}(Q_\text{re}), \overline{G}(\overline{Q})).$$

There is a commutative diagram of pointed sets
where $\theta$ is the map $R \otimes \mathbb{Q} \rightarrow \mathbb{Q}$, $\theta'$ is the map $\mathbb{Q}_{\text{re}} \otimes \mathbb{Q} \rightarrow \mathbb{Q}$, $\pi$ and $\overline{\pi}$ are restrictions maps. It is tautological that $\mathcal{L}(N) = \text{Ker} \theta$. Since all vertical maps are bijective, it follows that $\mathcal{L}(N)$ is isomorphic to $\text{Ker} \overline{\pi}$.

\[ \square \]

10 Some Nilpotent Lie groups of infinite complexity.

This chapter is devoted to the analysis to a class of CSC nilpotent Lie groups $\mathcal{N}$, for which the classification of commensurable classes and the computation of their complexity are very explicitly connected with the arithmetic of complex quadratic fields.

For $K = \mathbb{R}$ or $\mathbb{C}$, let $O(2, K)$ be the group of linear automorphisms of $\mathbb{R}^2$ preserving the quadratic form $x^2 + y^2$. Let $\mathcal{L}$ be the class of nilpotent Lie algebras $n^R$ over $\mathbb{R}$ satisfying the following properties

(i) $n^R$ has a $\mathbb{Q}$-form

(ii) $n^R/[n^R, n^R] \simeq \mathbb{R}^2$ has dimension two

(iii) the Lie algebra $n^C := \mathbb{C} \otimes n^R$ has a special grading

(iv) for $K = \mathbb{R}$ or $\mathbb{C}$, the image of Aut $n^K$ in $GL(n^K/[n^K, n^K])$ is $O(2, K)$.

Let be the class of CSC nilpotent Lie groups $N$ whose Lie algebra $n^K$ is in $\mathcal{L}$.

It should be noted that the class $\mathcal{N}$ is not empty. There is one Lie group $N_{112} \in \mathcal{N}$ of dimension 112, see [22]. Indeed [22] contains a general method to find nilpotent Lie algebras with a prescribed group of automorphisms, modulo its unipotent radical. For the group $O(2, \mathbb{R})$, $N_{112}$ is the Lie group of minimal dimension obtained with this method. However it is difficult to provide more details, without going to very long explanations.

From now on, $N$ will be any Lie group in class $\mathcal{N}$, $\xi_0$ will be one commensurable class of lattices in $N$ and $n := n(\xi_0)$ will be the corresponding corresponding $\mathbb{Q}$-form of $n^R$. As before, set $n^K = K \otimes n$ for any field $K \subset \mathbb{C}$.
Let $G = \text{Aut} \ n$ the algebraic automorphism group of $n$, let $U$ be its unipotent radical and set $G = G/U$. By hypothesis, $\overline{G}$ is the algebraic group $O(2)$.

10.1 The $\mathbb{Z}$-grading of $n^C$

Since $\overline{G}(\mathbb{C}) = O(2, \mathbb{C})$, a maximal torus $H$ of $G(\mathbb{C})$ has dimension 1. Therefore $n^C$ has a $\mathbb{Z}$-grading

$$n^C = \bigoplus_{k \in \mathbb{Z}} n^C_k,$$

satisfying the following properties

(i) the grading is essentially unique, namely any other grading is a multiple of the given grading,

(ii) $\dim n^C_k = \dim n^C_{-k}$ for any $k$. In particular $n^C$ does not admit a (non-trivial) non-negative grading, and

(iii) the grading is not defined over $\mathbb{R}$.

Indeed since $\overline{G}(\mathbb{C}) = O(2, \mathbb{C})$, the normalizer $K(\mathbb{C})$ of $H(\mathbb{C})$ has two connected components, and any $\sigma \in K(\mathbb{C}) \setminus K(\mathbb{C})^0$ exchanges $n^C_k$ and $n^C_{-k}$, what shows Assertion (ii). Since $\overline{G}(\mathbb{R}) = O(2, \mathbb{R})$, no torus of $G(\mathbb{R})$ is split, what implies Assertion (iii).

Moreover, the grading is not very special, so $\text{fcp}(\xi) = \infty$ for any commensurable class $\xi$. For the forthcoming computation of $\text{cp}(\xi)$, the following quantity will be involved

$$e(N) = \sum_{k > 0} k \dim n^C_k.$$

For example, for the Lie group $N_{112}$ of [22], we have $e(N_{112}) = 126$.

10.2 Classification of commensurable lattices in $N$

**Lemma 27.** Let $N \in \mathcal{N}$. Up to conjugacy, there is a bijection between

(i) the commensurable class of cocompact lattices in $N$, and

(ii) the positive definite quadratic form on $\mathbb{Q}^2$.

**Proof.** Let $q_0$ be a given definite quadratic form on $\mathbb{Q}^2$. It determines a $\mathbb{Q}$-form of the algebraic group $O(2)$, and $H^1(\text{Gal}(\mathbb{Q}), O(2, \mathbb{Q}))$ classifies the quadratic forms on $\mathbb{Q}^2$, while the kernel of

$$H^1(\text{Gal}(\mathbb{Q}), O(2, \mathbb{Q})) \to H^1(\text{Gal}(\mathbb{Q}_{re}), O(2, \mathbb{Q}))$$

classifies the positive definite quadratic forms on $\mathbb{Q}^2$. Thus the lemma follows from Corollary 10.

$\square$

The classification of positive definite quadratic forms $q$ on $\mathbb{Q}^2$ is well known. Up to conjugacy, $q$ can be written as

34
where $a$, $d$ are positive and $d$ is a square-free integer. Then $q$ is determined by the following two invariants

(i) its discriminant $-d$, viewed as an element of $\mathbb{Q}^* / \mathbb{Q}^{*2}$,

(ii) the value $a$, viewed as an element in $\mathbb{Q}^*/\mathbb{N}_K/\mathbb{Q}(K^*)$, where $K = \mathbb{Q}(\sqrt{-d})$. Equivalently, this means that $q(\mathbb{Q}^2 \setminus 0) = a\mathbb{N}_K/\mathbb{Q}(K^*)$.

For any positive definite quadratic forms $q$ on $\mathbb{Q}^2$, let $\xi(q)$ be the corresponding commensurable class (or more precisely, the conjugacy class of the commensurable class). By Theorem 9, $cp\xi(q)$ only depends on $O(q)$, therefore it only depends on the discriminant $-d$.

10.3 The function $F(d)$

Let $d$ be a positive square-free integer. Set $K = \mathbb{Q}(\sqrt{-d})$, let $\mathcal{O}$ be its ring of integers, let $R$ be the set of roots of unity in $K$ and set $K_1 = \{ z \in K | z\overline{z} = 1 \}$. For $z \in K^*$, recall that the integer $d(z)$ is defined by $d(z) = \mathbb{N}_K/\mathbb{Q}(\pi_z) = \text{Card} \mathcal{O}/\pi_z$, where $\pi_z$ is the ideal $\pi_z = \{ a \in \mathcal{O} | az \in \mathcal{O} \}$. Set

$$F(d) = \text{Min}_{z \in K_1 \setminus R} d(z).$$

We will now show two formulas for $F(d)$. Indeed $F(d)$ is the norm of some specific ideal in $K = \mathbb{Q}(\sqrt{-d})$, and it is also the minimal solution of some diophantine equation.

Let $J$ be the set of all ideals $\pi$ of $\mathcal{O}$ such that $\pi$ and $\overline{\pi}$ are coprime and $\pi^2$ is principal.

Lemma 28. We have

$$F(d) = \text{Min}_{\pi \in J} N_{K/\mathbb{Q}}(\pi).$$

In particular, we have $F(1) = 5$ and $F(3) = 7$.

Proof. The map $z \mapsto \pi_z$ induces a bijection $(K_1 \setminus R)/R \simeq J$, from which the first assertion follows. Moreover, if $\text{Cl}(K') = \{0\}$, then $F(d)$ is the smallest split prime number. Therefore $F(1) = 5$ and $F(3) = 7$. $\square$

Let’s consider the following diophantine equation

$$4n^2 = a^2 + db^2,$$

with $n > 0$, $a > 0$ and $b \neq 0$. A solution $(n, a, b)$ of $(E)$ is called primitive if $\gcd(n, a) = 1$. Let $\text{Sol}(E)$ (respectively $\text{Sol}_{prim}(E)$) be the set of solutions (respectively of primitive solutions) of $(E)$.

Let $\pi \in J$. Since $\pi^2$ is principal, there are integers $a(\pi) > 0$ and $b(\pi)$ such that $a(\pi) + b(\pi)\sqrt{-d}$ is a generator of $4\pi^2$. Moreover, let’s assume that
Lemma 29. Under the hypothesis that \( d \neq 1 \) or \( 3 \), the map \( \theta \) induces a bijection from \( J \) to \( \text{Sol}_{\text{prim}}(E) \). In particular

\[
F(d) = \text{Min}_{(n,a,b) \in \text{Sol}(E)} n.
\]

**Proof.** Step 1: proof that \( \theta(J) \subset \text{Sol}_{\text{prim}}(E) \). An algebraic integer \( z \in \mathcal{O} \) is called primitive if there are no integer \( d > 1 \) such that \( \frac{z}{d} \) is an algebraic integer. Equivalently, there are no integer \( d > 1 \) such that \( d \mid z + \bar{z} \) and \( d^2 \mid z\bar{z} \).

Let \( \pi \in J \) and set \( z = 1/2(a(\pi) + b(\pi)\sqrt{-d}) \). Since \( z + \bar{z} = a(\pi) \) and \( z\bar{z} = N_{K/Q}(\pi^2) \), \( z \) is an algebraic integer which is a generator of \( \pi^2 \). Since \( \pi^2 \) and \( \pi^4 \) are coprime, \( z \) is primitive. Since \( z\bar{z} = N_{K/Q}(\pi) \), it follows that \( N_{K/Q}(\pi) \) and \( a(\pi) \) are coprime. Hence \( \theta(\pi) \in \text{Sol}_{\text{prim}}(E) \) and the claim is proved.

Step 2: proof that \( \theta(J) = \text{Sol}_{\text{prim}}(E) \). Let \( (n,a,b) \in \text{Sol}_{\text{prim}}(E) \) and \( z = 1/2(a(\pi) + b(\pi)\sqrt{-d}) \). Since \( z \neq \bar{z}, z + \bar{z} = a \) and \( z\bar{z} = n \), the number \( z \) is an algebraic integer. Set \( \tau = z\mathcal{O} \) and let

\[
\tau = \pi_1^{m_1} \cdots \pi_k^{m_k}
\]

be the factorization of \( \tau \) into a product of prime ideals of \( \mathcal{O} \), where, as usual we assume that \( \pi_i \neq \pi_j \) for \( i \neq j \) and all \( m_i \) are positive.

For \( 1 \leq i \leq k \), let \( p_i \) be the characteristic of the field \( \mathcal{O}/\pi_i \). Since \( n \) and \( a \) are coprime, \( \tau \) and \( \bar{\tau} \) are coprime. It follows that \( \bar{\pi}_i \) does not divide \( \tau \). In particular \( \pi_i \neq \bar{\pi}_i \) and \( N_{K/Q}(\pi_i) = p_i \). Since \( \pi_i \) and \( \bar{\pi}_i \) are the only two ideals over \( p_i \), we have \( m_i = v_{p_i}(N_{K/Q}(\pi)) = v_{p_i}(n^2) \). Since each \( m_i \) is even, we have \( \tau = \pi^2 \) for some ideal \( \pi \in J \). Therefore \( \theta(\pi) = (n,a,b) \), and the claim is proved.

Step 3. It follows easily that \( \theta \) is a bijection from \( J \) to \( \text{Sol}_{\text{prim}}(E) \). In particular \( F(d) = \text{Min}_{(n,a,b) \in \text{Sol}_{\text{prim}}(E)} n \), from which the lemma follows.

\[\square\]

10.4 Complexity computation

**Theorem 11.** Let \( q \) be a positive definite quadratic form on \( \mathbb{Q}^2 \) of discriminant \(-d\). Then we have

\[
\text{cp} \xi(q) = F(d)^{e(a^2)}
\]
Proof. Step 1. Let $G \subset \text{End}_Q(K)$ be the group generated by the multiplication by elements in $K_1$ and by the complex conjugation. We have $G \simeq O(2)$ and $SO(2) \simeq K_1$. As a $O(2)$-module, there is an isomorphism 

$$V \simeq \mathbb{Q}(\sqrt{-d}),$$

where $V = n(\xi(q))/[n(\xi(q)), n(\xi(q))]$.

Step 2. Let $\mathcal{S}(n(\xi(q)))$ be the image of $S(n(\xi(q)))$ in $O(q)$. We claim that $\mathcal{S}(n(\xi(q))) = K_1 \setminus R$.

Indeed $O(q)$ can be identified with a Levi factor of $G(Q)$ and let $\rho: O(q) \rightarrow G(Q)$ a corresponding lift. Any element in $R \cup O(q) \setminus SO(q)$ has finite order, hence we have

$$\mathcal{S}(n(\xi(q))) \subset K_1 \setminus R.$$

Let $z \in K_1 \setminus R$. It is clear that $z$ is not an algebraic integer. Since the grading is special, we have

$$\delta^C = \bigoplus_{k \neq 0} \delta_k^C.$$

Since the eigenvalues of $\rho(z)$ on $\delta_k$ is $z^k$, it follows that $z$ belongs to $\mathcal{S}(n(\xi(q)))$, what proves the point.

Step 3. Let $z \in K_1 \setminus R$. We have $\overline{z} = z^{-1}$. Therefore by Lemma [15] we have

$$\text{ht } \rho(z) = \prod_{k \geq 1} d(z^k)^{\dim n_k^c} = d(z)^{e(N)}.$$

Therefore Theorem 4 implies Theorem 5.

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