I. POVMs FOR PHOTO DETECTION

Photo detection is at its core an information theoretic process; a measurement outcome—a click—reveals information about the outside world quantifiable in bits [1]. In the case of a single-photon detector (SPD), a click is correlated (imperfectly) with the presence of a particular type of photon, thus revealing information about the presence of photons of that type along with whatever else in the world such a photon is correlated with. The most general quantum description of this process is in terms of a positive operator-valued measure (POVM), a set of positive operators \( \hat{\Pi}_k \) that sum to the identity, where each \( k \) corresponds to a different measurement outcome. Given an arbitrary input state \( \hat{\rho} \) the probability to obtain outcome \( k \) is given by the Born Rule

\[
P(k) = \text{Tr}(\hat{\rho} \hat{\Pi}_k).
\]

Generically, each POVM element \( \hat{\Pi}_k \) can be written as a weighted sum over orthonormal quantum states

\[
\hat{\Pi}_k = \sum_i w_i^{(k)} \left| \phi_i^{(k)} \right\rangle \left\langle \phi_i^{(k)} \right|
\]

reducing to an ideal Von Neumann measurement only when the sum contains a single term with its weight \( w_i^{(k)} \) equal to 1 [2]. The weight \( w_i^{(k)} \) equals the conditional probability to obtain measurement outcome \( k \) given input \( i \). The posterior conditional probability that, given an outcome \( k \), we project onto input \( i \) is given by Bayes’ theorem [3]

\[
P(i|k) = \frac{w_i^{(k)} P(i)}{P(k)}
\]

with \( P(k) = \sum_i w_i^{(k)} P(i) \) and \( P(i) \) the a priori probabilities to get outcome \( k \) and for input \( i \) to be present, respectively [4]. Through Bayes’ theorem, an experimentalist is able to retrodict—that is, update their probability distribution over possible inputs—but only if they know what measurement their detector actually performs.

Knowledge of the POVM is essential for both gaining information from a measurement device and characterizing detector performance, hence the experimental need for detector tomography [5–10]. Commercial photo detectors are characterized by industry-standard figures of merit [11], which can be calculated from a POVM (for an in-depth review, see Ref. [12]). Here we will concern ourselves mostly with two figures of merit, detection efficiency and time-frequency uncertainty:

**Efficiency.**—The maximum efficiency with which an SPD outcome \( k \) (for instance, a single click) can be triggered by input single photon states is the maximum relative weight in \[2\] \( \eta_{\text{max}} = \max_i \{ w_i^{(k)} \} \). The maximum efficiency is achieved only when the input quantum state is one the measurement projects onto. This follows directly from the Born rule; \( P(k|i) = \text{Tr}(\hat{\Pi}_k \hat{\rho}) \rightarrow w_i^{(k)} \) if and only if \( \hat{\rho} = \left| \phi_i^{(k)} \right\rangle \left\langle \phi_i^{(k)} \right| \).

**Time-Frequency Uncertainty.**—The spectral uncertainty and (input-independent) timing jitter are determined entirely by the spectral and temporal widths of the states projected onto by the measurement outcome \( k \) [13], which form a retrodictive probability distribution. For any continuous variable \( X \) (here either time \( t \) or frequency \( \omega \)), we find it less convenient to use the variance as measure of uncertainty and instead define the uncertainty entropically [12–14]

\[
\Delta X^{(k)} = 2 H_X^{(k)} \delta X.
\]

Here \( H_X^{(k)} \) is the Shannon entropy defined as

\[
H_X^{(k)} = -\sum_j p(j|k) \log_2 p(j|k)
\]

with the sum over discretized \( X \)-bins of size \( \delta X \). \( p(j|k) \) is the a posteriori probability for the detected photon to be in bin \( j \) given outcome \( k \), and is calculated as

\[
p(j|k) = \int_{(j-1)\delta X}^{j\delta X} dX \sum_i P(i|k) |\phi_i(X)|^2
\]

where we have defined a normalized distribution over \( X \) given by the norm squared of the quantum state \( |\phi_i(X)|^2 \).
(where \( \phi_i(X) \equiv \langle X | \phi_i \rangle \)). The conditional probability \( P(i | k) \) is precisely the one from Bayes theorem \(5\): \( P(i | k) \) reduces to \( w_i^{(k)} / \Omega(k) \) in the case of a uniform prior \(17\), where \( \Omega(k) = \sum_i w_i^{(k)} \) is the bandwidth \(12\). Critically, \( \Delta X(k) \) is independent of the bin size \( \delta X \) in the small-bin limit, even though the entropy \( H_X(k) \) is strongly dependent on the bin size. One can verify that this definition of uncertainty yields a Heisenberg uncertainty relation \(14\)

\[
\Delta \omega \Delta t \geq \hbar / 2.
\]

The construction of measurements projecting onto arbitrary single-photon states is critical in quantum optical and quantum communication experiments. Mis-math between the single-photon state generated and the state projected onto by the measurement induces an irreversible degradation in efficiency. Furthermore, the capacity to efficiently project onto orthogonal single-photon states enables a wide range of quantum information and quantum optical applications, as we will discuss in section IV. From a foundational perspective, a procedure to build measurements projecting onto minimum-uncertainty Gaussian single-photon wavepackets paves the way for future tests of fundamental quantum theory.

II. SIMPLIFIED MEASUREMENTS PROJECTING ONTO ARBITRARY SINGLE-PHOTON STATES

We will now discuss how to construct a simple POVM that efficiently projects onto an arbitrary single-photon wavepacket. To aid us, we will now make four simplifying assumptions. First, we will consider only the time-frequency degree of freedom of the electromagnetic field, as the other degrees of freedom (e.g., polarization) can be efficiently sorted prior to detection in a pre-filtering process \(18\)\(20\). Second, we consider only a single excitation incident to the photo detector. Multiple photons can always be efficiently multiplexed to achieve a photon number resolution using SPD pixels \(21\). Third, we will not model a continuous measurement (as briefly discussed in the appendix of \(22\)), but instead a discretized measurement where at a particular time \(T\) we ascertain whether or not a photon has interacted with the SPD, ending the measurement. Lastly, we will consider only a binary-outcome photo detector, “click” or “no click.” This simplifies the POVM so that it only contains the two elements \( \Pi_T \) and \( \Pi_0 \), both projecting onto the Hilbert space of single photon states and the vacuum state. Generalizations to non-binary-outcome SPDs are straightforward: one can concatenate binary-outcome POVMs to generate non-binary-outcome experiments.

We now begin construction of the POVM \( \{ \Pi_T, \Pi_0 \} \) in earnest. Consider a two-level system with time-dependent transition frequency \(\Delta(t)\), with time-dependent coupling to a Markovian external electromagnetic continuum of states \(23\). Experimentally, a time-dependent decay rate \(\kappa(t)\) is induced by a rapid variation of density of states \(24\)\(25\) and a time-dependent resonance \(\Delta(t)\) can be varied with a time-dependent external electric field (Stark effect, \(26\)) or through a two-channel Raman transition \(27\).

The general state of the two-level system can be written in the Schrödinger picture \(|\psi(t)\rangle = C_0(t) |0\rangle + C_1(t) |1\rangle\). In the quantum trajectory picture, there are two types of evolution of \(|\psi(t)\rangle\): Schrödinger-like smooth evolution with a non-Hermitian effective Hamiltonian and quantum jumps (at random times) \(28\)\(29\). A quantum jump will always correspond to the excitation leaking out of the system and so, in the absence of a dark counts, we only need consider the Schrödinger-like evolution. In this picture, the quantum state of the two-level system remains pure with the time-dependent excited state amplitude \(C_1(t)\) having the form of a Langevin equation

\[
\dot{C}_1(t) = -\frac{\kappa(t)}{2} C_1(t) - i \Delta(t) C_1(t) + \sqrt{\kappa(t)} f(t) \tag{8}
\]

where \(f(t)\) is a normalized input photon wavepacket \(30\). We can solve this equation with the result

\[
C_1(t) = \int_{T_0}^t dt'' f(t'') \sqrt{\kappa(t'')} \exp \left[ - \int_{t''}^t dt' D(t') \right], \tag{9}
\]

where

\[
D(t) = i \Delta(t) + \frac{\kappa(t)}{2}, \tag{10}
\]

and where \(T_0\) is a time in the distant past where our photodetector was still off, so that \(\kappa(T_0) = 0\) and \(C_1(T_0) = 0\). Our measurement consists in checking if the system is in the excited state at time \(T\). The probability to obtain a positive result (corresponding to detecting the incident photon wavepacket) is \(|C_1(T)|^2\). We can write

\[
C_1(T) = \int_{T_0}^T dt \Psi^*(t) f(t) \tag{11}
\]

with

\[
\Psi^*(t) = \sqrt{\kappa(t)} \exp \left[ - \int_t^T dt' D(t') \right]. \tag{12}
\]

Whereas \(f(t)\) is a normalized wave function, \(\Psi(t)\) is sub-normalized for finite \(T_0\), since

\[
\mathcal{W} = \int_{T_0}^T dt |\Psi(t)|^2 = 1 - \exp \left[ - \int_{T_0}^T dt \kappa(t) \right]. \tag{13}
\]

We can interpret \(\Psi(t)\) as a retrodictive probability amplitude (for simple examples, see Fig. 1 and Fig. 2), identifying at which times a photon likely entered the system given a detector “click” at \(t = T\).
We can define a normalized single-photon state

$$|\Psi_T\rangle = \mathcal{W}^{-1/2} \int_{T_0}^T dt \Psi(t)\hat{a}_\text{in}^\dagger(t)|\text{vac}\rangle$$

(14)

with the creation operator \(\hat{a}_\text{in}^\dagger(t)\) acting on the input continuum of states. The arbitrary input single-photon state (which may have been created long before our detector was turned on at \(T_0\) or long after the measurement ended at time \(T\)) is

$$|f\rangle = \int_{-\infty}^\infty dt f(t)\hat{a}_\text{in}^\dagger(t)|\text{vac}\rangle.$$  

(15)

The commutator relation for the input field operator is

$$[\hat{a}_\text{in}(t), \hat{a}_\text{in}^\dagger(t')] = \delta(t - t').$$

The probability for an arbitrary input photon wavepacket \(f(t)\) to result in the system being found in the excited state at a time \(T\) is \(|C_1(T)|^2 = \mathcal{W} \langle f|\Psi_T\rangle \langle\Psi_T|f\rangle\). The measurement does not project onto times after we have checked if the system is in the excited state, nor onto times before the detector was turned on.

We rewrite this probability in terms of a POVM element containing a single element

$$\hat{\Pi}_T = \mathcal{W}|\Psi_T\rangle \langle\Psi_T|.$$  

(16)

To the extent our detector has been open long enough, such that \(\mathcal{W} \to 1\), our detector could act as a perfectly efficient detector for a specific single-photon wavepacket with temporal mode function \(\Psi(t)\) [31]. This wavepacket is the time reverse of the wavepacket that would be emitted by our system if it started in the state \(|1\rangle\) [32].

For this simple system, the POVM element is both pure (containing just one term [33]) and (almost) maximally efficient (the weight \(\mathcal{W}\) may approach unity as close as we wish).

Here we observe an obvious trade-off between efficiency and photon counting rate: one cannot project onto a long single-photon wavepacket in a short time interval without cutting off the tails, lowering the overall detection efficiency [34].

The two-level system described in Eq. (8) is a special case but an important one; the two-level system is often a very good approximation of more complicated systems near-resonance [35]. In this paper, we will focus on the simple time-dependent system [8] as it is sufficiently general to perform a measurement described by any time-independent system, and more [36]. Indeed, [8] is general enough to project onto a completely arbitrary single-photon wavepacket, a result we will now prove.
\[ \Psi^*(t) = A(t)e^{i\phi(t)}, \text{ positive amplitude } A(t), \text{ and phase } \phi(t). \] Inserting this into (12), we arrive at two separate expressions

\[
A(t) = \sqrt{\kappa(t)}e^{-\int_{t}^{T} dt' \frac{\kappa(t')}{2}} \\
\phi(t) = -\int_{t}^{T} dt' \Delta(t').
\] (17)

The second line is always solvable by \[\Delta(t) = \hat{\phi}(t)\] up to a constant global phase shift provided \(\phi(t)\) is everywhere differentiable (smooth). We now focus on the first line. Taking the natural logarithm we arrive at an expression

\[
2\log[A(t)] - \log[\kappa(t)] = -\int_{t}^{T} dt' \kappa(t').
\] (18)

Taking the time derivative of both sides, we arrive at a Bernoulli differential equation [37]

\[
\kappa^{-2}(t) \frac{d\kappa(t)}{dt} - \frac{2}{A(t)\kappa(t)} \frac{dA(t)}{dt} = -1.
\] (19)

Provided \(\frac{1}{A(t)} \frac{dA(t)}{dt}\) is continuous, this is solved by

\[
\kappa(t) = \frac{A^2(t)}{1 - \int_{t}^{T} A^2(t')dt'}
\] (20)

Here, \(\kappa(t)\) is given by the square of the electromagnetic field, divided by a correction factor accounting for the finite response time imposed by \(\kappa(t)\) itself [38]. From [20], we observe that the only condition imposed on \(A(t)\) is that \(A^2(t)\) have an antiderivative. We simply require \(A^2(t)\) be continuous, which in turn requires \(A(t)\) to be continuous. Thus, any wavepacket with smooth phase profile \(\phi(t)\) and smooth amplitude \(A(t)\) is projected onto by some physically realizable single photon detection scheme.

**Special Case: Heisenberg-Limited Measurement**

A Heisenberg-limited simultaneous measurement of time and frequency is achieved with a Gaussian time-frequency distribution. We want a temporal wavepacket \(\Psi^*(t)\) that is the complex square root a Gaussian distribution

\[
\Psi^*(t) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(t-t_0)^2}{4\sigma^2}} e^{i\frac{\omega_0}{2}t}
\] (21)

where \(\sigma\) is the temporal half-width, and \(t_0\) and \(\omega_0\) are the central time frequency of the Gaussian distribution. We find that this wavepacket is projected onto by a time-dependent system with constant resonance \(\Delta(t) = \omega_0\) and a time-dependent coupling

\[
\kappa(t) = \frac{e^{-\frac{(t-t_0)^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}(1 + \frac{1}{2} \text{Erf}\left[\frac{t_0-T}{\sqrt{2}\sigma}\right] - \frac{1}{2} \text{Erf}\left[\frac{t_0-T}{\sqrt{2}\sigma}\right])}
\] (22)

as in Fig. 3. Note that the coupling \(\kappa(t)\) is \(T\)-dependent even though the projected state (21) is \(T\)-independent, in agreement with the general case (20).

**III. REALISTIC MEASUREMENTS PROJECTING ONTO ARBITRARY SINGLE-PHOTON STATES**

The model of a SPD as an isolated two-level system is highly idealized. In a more realistic system, photo detection is an extended process wherein a photon is transmitted into the detector, interacting with the system and triggering a macroscopic change of the photo detector state (amplification) which can then be measured classically. Many theories of single photon detection have been developed over the past century, 39-50 and indeed there are numerous implementations of SPD technology 51-55. Across all systems, we identify these
three stages of transmission, amplification, and measurement as universal. In this section, we derive a POVM that incorporates all three stages quantum mechanically, at the end of the section extending the model to include fluctuations of system parameters. The time-dependent two-level system from the previous section enabling arbitrary wavepacket projection is incorporated into the three-stage model as the trigger for the amplification mechanism. We will assume in this analysis that the three-stage model as the trigger for the amplification process. An inefficiency $1 - \eta$ affects photo detection by changing which post-amplification Fock states are projected onto: the larger $1 - \eta$ is, the more post-amplification states $|n, T\rangle \langle n, T|$ will be projected onto immediately by the same POVM element $\hat{\Pi}_T$, making it harder to distinguish between signal and noise.

We now move one step further back in the chain of inference (Fig. 4) so the POVM element $\hat{\Pi}_T$ projects onto the number of excitations $m$ input to the amplification trigger. Amplification is a generic feature of photo detection; without a macroscopic change in the internal state of a photo detector, there is no way to correlate detection by changing which post-amplification Fock states are projected onto: the presence of a single photon. Indeed, this is what we would expect; we do not need to know precisely the internal state of the photo detector in order to use it to efficiently detect the presence of a single photon.

The macroscopic measurement performed on the amplified signal will, in general, be inefficient. We model this in a standard way [57], using a beamsplitter with frequency independent transmission amplitude $\sqrt{\eta}$. We can then rewrite the POVM element [23] so that it projects onto Fock states in the amplification target mode prior to the measurement

$$\hat{\Pi}_T = \sum_{k \in K_{\text{click}}} \sum_{n=k}^{\infty} \Pr(n | k) |n, T\rangle \langle n, T|$$

where we have defined

$$\Pr(n | k) = \binom{n}{k} \eta^k (1 - \eta)^{n-k}$$

the probability to detect $k$ excitations given that there were $n$ excitations in the output mode of the amplification process. An inefficiency $1 - \eta$ affects photo detection by changing which post-amplification Fock states are projected onto: the larger $1 - \eta$ is, the more post-amplification states $|n, T\rangle \langle n, T|$ will be projected onto immediately by the same POVM element $\hat{\Pi}_T$, making it harder to distinguish between signal and noise.

Consider a macroscopic measurement performed at time $T$ with a binary response triggered by $k$ excitations measured in the amplified signal [56]. Such a POVM can be written as a projector onto Fock states in the Hilbert space internal to the system

$$\hat{\Pi}_T = \sum_{k \in K_{\text{click}}} |k, T\rangle \langle k, T|.$$ 

Here, we have defined a set $K_{\text{click}}$ that sets the threshold for how many amplification excitations must be measured to trigger a macroscopic detection event. At this stage, we can already see that the internal state the POVM projects onto is highly mixed, but this will not directly translate to an impure measurement on the Hilbert space of input photons. Indeed, this is what we would expect; we do not need to know precisely the internal state of the photo detector in order to use it to efficiently detect the presence of a single photon.

FIG. 4. A POVM description of the three-stage model of photo detection, where the chain of inference (left to right) moves opposite the arrow of time, connecting a macroscopic “click” outcome to the state of the input field. A “click” outcome represented by the POVM element $\hat{\Pi}_T$ indicates $k \in K_{\text{click}}$ excitations were measured post amplification with detection efficiency $\eta$. This suggests $n \approx k/\eta$ (and, strictly, $n \geq k$) excitations were present after in the target mode for amplification. In turn, this indicates that $m \approx (n - \bar{m}_{\text{th}})/G \leq n$ excitations were likely incident to the amplification process trigger, with $\bar{m}_{\text{th}}$ the expected number of thermal excitations already in the amplification target mode. If $m$ is larger than the expected number of thermal excitations in the amplification trigger mode $\bar{m}_{\text{th}}$, we conclude that one (or more) input photon of the form $|T \Psi_T\rangle \langle T \Psi_T|$ was likely present. In addition to the states written explicitly, there are other possible states where $k$, $n$, and $m$ deviate from their most likely values. These states (denoted by parallelograms) contribute to the POVM element, as the internal state of the photo detector is in general highly mixed. Nonetheless, in the end a SPD POVM element $\hat{\Pi}_T$ only projects onto the two input states $|T \Psi_T\rangle \langle T \Psi_T|$ and $|\text{vac}\rangle \langle \text{vac}|$, remaining relatively pure.
the trigger mode. In using this expression we do impose a restriction that there must be $M > G n$ excitations in the reservoir mode, but restrictions of this type are to be expected (the energy for amplification must come from somewhere) and we will be most interested in few photons ($n = 0, 1, 2$) in this analysis. In most physical platforms $G$ will fluctuate [66], as will other (classical) system parameters which we will return to at the end of this section. (Exceptions do exist; for Hamiltonians that implement deterministic amplification schemes [with small integer values for $G$] see Ref. [67].) However, even with a definite gain factor $G$ and number of input excitations $m$, we will still not end up with exactly $n = N + G m$ excitations if the target mode is initially in a thermal state 

$$P^\text{th}_{N,T} = \frac{1}{1 + N} \left( \frac{\tilde{N}}{1 + \tilde{N}} \right)^N; \tilde{N} = \frac{1}{e^{\frac{\hbar \omega}{k_b T}} - 1}, \quad (28)$$

where $\omega'$ and $k_B T$ are the frequency and the temperature of the target mode. Assuming the ideal amplification scheme in [26], we now write the POVM element $\hat{\Pi}_k$ in terms of the number $m$ excitations that trigger the application mechanism

$$\hat{\Pi}_T = \sum_{k \in K_{\text{click}}} \sum_{n=m}^{\infty} \frac{\text{Pr}(n|k)}{\sum_{n=1}^{\text{Int}_{-\frac{G_m}{2}}} P^\text{th}_{n-G_m m}|\Psi_T\rangle \langle T\Psi_T| \otimes |R \Psi_T\rangle \langle R \Psi_T|} + \sum_{m=1}^{\text{Int}_{-\frac{G_m}{2}}} m P^\text{th}_{n-G_m m} \rho^{2(m-1)} |T\Psi_T\rangle \langle T\Psi_T| \otimes |R \Psi_T\rangle \langle R \Psi_T| m^{-1} \quad (30)$$

The first line corresponds to dark counts generated from thermal excitations post-amplification and the second line corresponds to dark counts generated by thermal excitations that then trigger the amplification mechanism. Only the third line contains a projection onto a photon to be detected. (The multiplicative factor $m$ in the third line is combinatorial in origin: $m$ total excitations in the trigger mode with $m - 1$ generated from thermal fluctu-
where $\hat{a}^\dagger$ and $\hat{b}^\dagger$ are the creation operators for the external and internal continua and we have defined a Fourier-transformed wavepacket for the amplification trigger mode $\hat{\Psi}(\omega) = \text{FT}[\sqrt{\kappa(t)}\hat{\Psi}(t)]$. We can now see how pre-amplification dark counts (the second line of (30)) can be suppressed: by reducing the overlap of $|\hat{\Psi}(\omega)|^2$ and $|R(\omega)|^2$, that is, by only amplifying the frequencies we wish to detect so that $\rho^2 \ll 1$. In this case, the POVM element (30) will be dominated by the $m=1$ term of the third line (the signal to be detected with no thermal excitations), as well as potentially the first line. (To reiterate, these are dark counts post-amplification, but these can be reduced by amplifying at a high frequency such that $\hbar\omega' \gg k_B T$, where $\omega'$ and $T$ are the frequency and temperature of the target mode.)

Finally, we trace over the internal continua, which we assume is in a thermal state with temperature $k_B T$. The POVM projects onto the external continua only

$$\hat{\Pi}_T = \sum_{k \in K_{\text{click}}} \sum_{n=k}^{\infty} \text{Pr}(n|k) \left( \sum_{m=0}^{\infty} P_{n-Gm}^{\text{th}} \rho^{2m} |\text{vac}\rangle \langle \text{vac}| + \sum_{m=1}^{\infty} m P_{n-Gm}^{\text{th}} P_m^{\text{th}} \rho^2 |\text{vac}\rangle \langle \text{vac}| \right)$$

$$\equiv w_0|\text{vac}\rangle \langle \text{vac}| + w_T \langle T \Psi_T | T \Psi_T \rangle$$

where in the last line we have absorbed the sums in front of the two projectors into weights so that the POVM element has the form of (2) and with $P_j^{\text{th}}$ the probability to have $j$ excitations (now in the non-monochromatic reflected mode defined in (31)). For a finite detector on-time $T_0 > -\infty$, the weights $w_0$ and $w_T$ will be slightly less than in (32) due to wavepacket sub-normalization [13]. However, this deviation is negligible provided the detector is left on for a time comparable to the temporal mode's width. We now reconsider the question of projecting onto an arbitrary wavepacket, including the full quantum description. We find that this is possible to do in principle, 

**Proof.** Consider a photon with complex normalized spectral wavepacket $\tilde{f}(\omega)$. If detection is achieved with a time-dependent two-level system preceded by a quantum network with filtering transmission function $T(\omega)$, the system will project onto a state $|T \Psi_T \rangle$ as defined in (31). In the low-noise limit this will be the only state projected onto by the (pure) POVM element. From the Born rule, the probability of detection will be

$$P_T = w_T \frac{1}{\tau^2} \left| \int_{-\infty}^{\infty} d\omega \tilde{f}(\omega) \hat{\Psi}(\omega) T^*(\omega) \right|^2$$

with $w_T$ the overall weight given by (32) and maximum possible detection efficiency, which can be arbitrarily close to unity. It is possible to achieve $P_T = w_T$ (mode-matched detection) in (33) if and only if

$$\hat{\Psi}(\omega) = \tilde{f}^*(\omega) \frac{1}{T(\omega)} e^{i\omega T}.$$  

(34)

From [12], we know that it is possible to generate an arbitrary temporal wavepacket $\text{FT}^{-1}[\hat{\Psi}(\omega)] = \sqrt{\kappa(t)}\hat{\Psi}(t)$ from a time-dependent two-level system. The Fourier transform of a continuous smooth function is itself smooth and continuous. Thus, if the right hand side of (34) is a well-defined spectral wavepacket (smooth and continuous), one can find functions $\kappa(t)$ and $\Delta(t)$ such that $\hat{\Psi}(\omega)$ has the form of (34). □

**Remark.** Arbitrary wavepacket detection (and thus Heisenberg-limited simultaneous measurements of time and frequency) is in principle possible only when there are no photonic band gaps induced by the filter; if $T(\omega') = 0$ for some frequency $\omega'$, there is simply no way to compensate for the lost information about $\omega'$. Photonic band gaps are a generic feature of parallel (and hybrid) quantum networks [22] as well as certain non-Markovian systems [69]. Network/reservoir engineering must be employed to ensure any $\omega'$ where $T(\omega') = 0$ is not a frequency of interest.

The POVM $\{\hat{\Pi}_T, \hat{\Pi}_0\}$ with $\hat{\Pi}_T$ defined in (32) and $\hat{\Pi}_0 = \hat{1} - \hat{\Pi}_T$ provides a complete description of the single photon detection process that is fully quantum from beginning to end (Fig. 4). However, there is a final element that must be considered to make the description applicable to laboratory systems: classical parameter fluctuations. For continuous parameter fluctuations over any system parameter or set of system parameters $X$, these are naturally incorporated

$$\hat{\Pi}_T = \int dX \text{Pr}(X) (w_0|\text{vac}\rangle \langle \text{vac}| + w_T \langle T \Psi_T | T \Psi_T \rangle)$$

(35)

where we have assumed a (known) probability distribution $\text{Pr}(X)$. In (35), the system parameter(s) $X$ could be...
such that only the weights \( w_0 \) and \( w_T \) depend on \( X \), or \( X \) could be such that the state \( |T \Psi_T \rangle \) depends on \( X \) as well (for a summary, see Fig. 5). In the case of the latter, the POVM will become less pure and will need rediagonalization to determine which states are projected onto \([70]\). This final POVM not only includes ignorance about the internal state of the photo detector as was depicted in Fig. 4 but also classical ignorance about the state of the photo detector due to system-lab interactions.

IV. APPLICATIONS

Using the time-dependent two-level system, we are able to project onto orthogonal quantum states (Fig. 6). This enables efficient detection of photonic qubits, an essential component of any quantum internet \([71, 72]\). More generally, temporal modes provide a complete framework for quantum information science \([73]\), with efficient detection of orthogonal modes (and their superpositions to create mutually unbiased bases) a key ingredient. Fully manipulable temporal modes also play a key role in error-corrected quantum transduction \([74]\), where a time-reversed temporal mode can restore an unknown superposition in a qubit. Here, efficient detection of arbitrary temporal modes is essential so that quantum jumps out of the dark state are efficiently heralded.

High-purity measurements that project onto orthogonal single-photon wavepackets also enable super-resolved measurements \([77]\). Suppose we have two single-photon sources emitting almost identical pure states differing slightly in either emission time or central frequency

\[
\hat{\phi}_1 = \frac{|\phi_1\rangle + \sqrt{\epsilon}|\phi_1\rangle}{\sqrt{1+\epsilon}}, \\
\hat{\phi}_2 = \frac{|\phi_1\rangle - \sqrt{\epsilon}|\phi_1\rangle}{\sqrt{1+\epsilon}},
\]

with \( \langle \hat{\phi}_1 | \hat{\phi}_2 \rangle \) real, \( \epsilon \ll 1 \), and \( \langle \phi_1 | \phi_2 \rangle = 0 \). Alternatively, we may imagine a single source of light but the light we receive may have either been slightly Doppler-shifted or it may have been slightly delayed.

Suppose now that we receive one photon that could equally likely be from either source so that our input state is

\[
\hat{\rho} = \frac{1}{2} |\hat{\phi}_1\rangle \langle \hat{\phi}_1 | + \frac{1}{2} |\hat{\phi}_2\rangle \langle \hat{\phi}_2 |.
\]

If we can measure both \( \hat{\Pi}_1 = \eta |\phi_1\rangle \langle \phi_1 | \) and \( \hat{\Pi}_1 = \eta |\phi_2\rangle \langle \phi_2 | \) (that is, if we have separate photodetectors
with these (pure) POVM elements, or a single non-binary-outcome photo detector, then we find the probability of clicks

\[ P_1 = \text{Tr} \left[ \hat{\Pi}_1 \hat{\rho} \right] = \eta \frac{1}{1 + \epsilon} \]
\[ P_2 = \text{Tr} \left[ \hat{\Pi}_2 \hat{\rho} \right] = \eta \frac{\epsilon}{1 + \epsilon} \]  \hspace{1cm} (38)

so that the ratio of clicks gives a direct estimate of \( \epsilon \), even for low efficiency \( \eta \). Here all that is needed for time-frequency domain super-resolved measurement of \( \epsilon \) are SPDs with time-dependent couplings and resonance frequencies as opposed to nonlinear optics \[78].

In traditional quantum key distribution (QKD) schemes (that is, not measurement device independent (MDI-QKD), specification of the measurement POVM is essential to robust security proofs \[79][81]. Here, we have verified several assumptions about the eavesdropper’s capabilities common in security proofs: that high-purity measurements are possible, that high efficiency measurements are possible, and (for continuous-variable (CV)-QKD proofs) minimum time-frequency uncertainty measurements are possible. In particular for CV-QKD, an eavesdropper can perform measurements that project onto variable-width spectral modes, disrupting temporal correlations between Alice and Bob (who are assumed to use fixed time-frequency bins) \[82]. Here, the capacity to adjust the width of the spectral mode \( \Psi(\omega) \) provides Alice and Bob a new strategy to mitigate Eve’s attack and extract a secure key.

More generally, detector tomography is an important tool across implementations of single-photon and number-resolved photo detection \[7][8][53][54]. Real-time tomography could be useful in QKD protocols resistant to “trojan-horse attacks” \[55] or any SPD platform subject to time-dependent environmental parameter fluctuations: for instance, atmospheric turbulence in MDI-QKD \[56] or interplanetary medium in deep space classical communications \[57]. Recently tomography speed-ups have been achieved using machine learning assisted tomography protocols \[58]. The POVMs derived in this paper provide priors which can further speed up detector tomography \[59]. These include approximate effects of environmental fluctuations as outlined in Fig. 5 and a global optimum POVM for single photon detection \[62] which can be used to incorporate detector calibration and optimization into \textit{in situ} tomographic protocols.

V. CONCLUSIONS

Having gone through applications of our work, we return to the fundamental (as opposed to pratical) limits to single photon detection and their implications.

Here we have constructed single-photon measurements that are Heisenberg-limited in two ways: the first is that they can project onto Gaussian time-frequency states as illustrated in Fig. 3 and the second is that the amplification scheme reaches a Heisenberg-limited (linear in the gain \( G \)) signal-to-noise ratio, surpassing the standard quantum limit (a signal-to-noise ratio going like of \( \sqrt{G} \)) \[63]. Achieving these simultaneously is possible in principle with no drawback. Indeed, the only stringent tradeoff we encounter in this analysis is between efficiency and photon counting rate, which becomes substantial when an SPD is reset at a faster rate than \( \sim 1/\Delta t \). (The photon does not have sufficient time to excite the two-level system with high probability before the system is reset.) For other figures of merit, we find that they are either independent, or deteriorate together \[90]. While it does appear from \[32] that improving efficiency also increases dark counts, these are decoupled by ensuring the coefficient \( \rho \ll 1 \)—that is, by making \( T(\omega) \) broader than \( \Psi(\omega) \). While it is commonsense that one should only amplify the frequencies they wish to detect, our work clarifies how enormously important this is. The dark counts produced in this way are insuperable; they cannot be removed post-amplification without removing the single-photon signal as well.

Another conclusion from this work is rather optimistic: Here we have given a quantum description of an entire single photon detection process projecting onto arbitrary single photon states and the only fundamental limitations encountered are Heisenberg limits. Incorporating realistic descriptions of amplification and a final measurement reduce efficiency and increase dark counts, but even so a Heisenberg-limited measurement is still achievable in principle. Similarly, incorporating the filtering of a first irreversible step does not impede implementation of Heisenberg-limited measurements provided no frequencies are completely blocked from entering the trigger mechanism. Even considering parameter fluctuations \[35] in internal temperatures \( k_B T \) and \( k_B T' \), amplification frequency \( \omega' \), and amplification gain factor \( G \)—which are unavoidable in any realistic system—Heisenberg limited time-frequency measurements are achieved. To the authors’ knowledge, this is first proposed quantum procedure for reaching Heisenberg limited time-frequency measurements in a realistic quantum system. In addition to being a fundamental limit to SPD performance, probing Heisenberg limits paves the way for future experimental tests of foundational quantum theory.

This work is supported by funding from DARPA under Contract No. W911NF-17-1-0267.
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[31] For measurements projecting onto a Gaussian wavepacket as in [22] and Fig. 3, the weight (77) has the simple form $W = \frac{1}{\pi} \text{Erf} \left( \frac{\Delta t}{\sqrt{2} \sigma} \right) + \frac{1}{2} \text{Erf} \left( \frac{T - \Delta t}{\sqrt{2} \sigma} \right)$, going to unity for $T, \Delta t \gg \sigma$.
[32] An alternative to directly solving (5) is to find the Green’s function $G(t)$ of the time-reversed problem: at $t_0$ the two-level system is started in the excited state and at time $T$ we check whether the excitation has leaked out. Taking $t \rightarrow T - t$, one arrives at (14) with $\Psi(t) = G(T - t)$. This approach is less direct but it does clarify the role of the Green’s function: propagating back in time starting from $t = T$ (when the photon is detected) back to the infinite past, which indeed is what the POVM does as well (Fig. 3).
[33] We define purity of the POVM element $\text{Pur}[\hat{P}_{ik}] = \frac{1}{\sqrt{T(0)}^2} (\langle 0|\hat{a}_0^\dagger \hat{a}_0 |0 \rangle) \leq 1$ where the upper limit is reached only when the POVM element projects onto a single state.
[34] The limitation to photon counting rate imposed by efficient detection of long temporal wavepackets is avoided via signal multiplexing, see Ref. [21].
[35] For an arbitrary multi-level time-independent structure, we will end up with a system of equations governing discrete state evolution $\tilde{C}(t) = M\tilde{C}(t) + \tilde{S}(t)$ (39) with $M$ a time-independent matrix and $\tilde{S}(t)$ a time-dependent (inhomogeneous) source term describing the input photon. The solution is then always of the form $\tilde{C}(t) = e^{Mt}\tilde{C}(t_0) + \int_{t_0}^{t} dt' e^{M(t-t')}\tilde{S}(t')$. (40)

Writing $e^{M(t-t')}$ as a Green’s matrix, we can identify elements that correspond to transitions to the final monitored discrete state (detector outcomes) through standard numerical techniques [97].

In particular, time-independent systems cannot achieve Heisenberg-limited measurements of time and frequency. This is because networks of discrete states experience a natural spectral broadening that is Lorentzian. While Gaussian broadening can additionally occur (for instance, due to Doppler shifts in atomic distributions [89]) this only increases the product uncertainty further from the minimum of $\Delta \omega \Delta t = \epsilon \pi$ [14], attained only by pure measurements projecting onto Gaussian wavepackets.

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By varying certain key parameters, it is possible to induce an exceptional-point structure in [35], for instance, by introducing a discrete probability distribution over resonance frequencies corresponding to classical ignorance about a discrete set of detector settings. Here the exceptional point occurs when the frequencies are made degenerate, which (since the discrete states have identical quantum numbers) is forbidden by unitarity. Here the range over the resonances are distributed is the exceptional-point parameter [100].

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