Abstract

In discrete convex analysis, various convexity concepts are considered for discrete functions such as separable convexity, L-convexity, M-convexity, integral convexity, and multimodularity. These concepts of discrete convex functions are not mutually independent. For example, M$^\natural$-convexity is a special case of integral convexity, and the combination of L$^\natural$-convexity and M$^\natural$-convexity coincides with separable convexity. This paper aims at a fairly comprehensive analysis of the inclusion and intersection relations for various classes of discrete convex functions. Emphasis is put on the analysis of multimodularity in relation to L$^\natural$-convexity and M$^\natural$-convexity.

Keywords: Discrete convex analysis, L-convex function, M-convex function, Multimodular function, Separable convex function

1 Introduction

A function defined on the integer lattice $\mathbb{Z}^n$ is called a discrete function. Various types of convexity have been defined for discrete functions in discrete convex analysis [6, 15, 16, 18, 19], such as L-convex functions, L$^\natural$-convex functions, M-convex functions, M$^\natural$-convex functions, integrally convex functions, and multimodular functions. L$^\natural$-convex functions have applications in several fields including operations research (inventory theory, scheduling, etc.) [3, 4, 33], economics and auction theory [20, 31], and computer vision [31]. M$^\natural$-convex functions have applications in operations research [4] and economics [16, 19, 20, 25, 32]. Multimodular functions are used for queueing, Markov decision processes, and discrete-event systems [1, 2, 8].

The classes of discrete convex functions are not mutually independent. In some cases, two classes of discrete convex functions have an inclusion relation as an (almost) immediate consequence of the definitions. This is the case, for example, with the inclusion of M-convex functions in the set of M$^\natural$-convex functions. In other cases, inclusion relations are established as theorems. For example, it is a theorem [16, Theorem 6.42] that M$^\natural$-convex functions are integrally convex functions. This paper aims at a fairly comprehensive analysis of the inclusion relations for various classes of discrete convex functions.

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M♮-convex and multimodular

Figure 1: M♮-convex sets (polymatroids) with and without multimodularity

In this paper we are also interested in what is implied by the combination of two convexity properties. For example, it is known ([23, Theorem 3.17], [16, Theorem 8.49]) that a function is both L♮-convex and M♮-convex if and only if it is separable convex, where a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is called separable convex if it is represented as

\[
f(x) = \sum_{i=1}^{n} \varphi_i(x_i) \quad (x \in \mathbb{Z}^n)
\]

with univariate functions \( \varphi_i \) \((i = 1, 2, \ldots, n)\) that satisfy \( \varphi_i(t-1) + \varphi_i(t+1) \geq 2\varphi_i(t) \) for all \( t \in \mathbb{Z} \) and are finite-valued on intervals of integers. It is easy to see that a separable convex function is both L♮-convex and M♮-convex, and hence the content of the above statement lies in the claim that the combination of L♮-convexity and M♮-convexity implies separable convexity. In this paper we are interested in such statements for other classes of discrete convex functions. In particular, we show in Section 5.3 that a function is both L♮-convex and multimodular if and only if it is separable convex.

It is noted, however, that such relationship may not be so simple for other pairs of function classes. As an example, consider M♮-convex functions and multimodular functions. It is known [9] that M♮-convex functions and multimodular functions are the same for functions in two variables. For functions in more variables, this is not the case, and there is no inclusion relation between the classes of M♮-convex functions (resp., sets) and that of multimodular functions (resp., sets). Figure 1 shows examples of integral polymatroids (M♮-convex sets) with and without multimodularity (see Examples 5.3 and 5.4 for details). The indicator function of the set in (a) is both M♮-convex and multimodular, but not separable convex. A systematic study in Section 5.4 will show that there are infinitely many functions (resp., sets) that are both M♮-convex and multimodular, but not separable convex.

This paper is organized as follows. Section 2 introduces notations and summarizes the known pairwise inclusion relations between classes of discrete convex functions and shows how the analysis for functions can be reduced to that for sets. The main results are given in Sections 3 and 4 with illustrative examples. Section 5 presents technical contributions, containing theorems and proofs about multimodularity. Definitions of various concepts of discrete convex functions are given in Appendix for the convenience of reference.
2 Preliminaries

2.1 Notation

Let \( n \) be a positive integer and \( N = \{1, 2, \ldots, n\} \). For a subset \( A \) of \( N \), we denote by \( e^A \) the characteristic vector of \( A \); the \( i \)-th component of \( e^A \) is equal to 1 or 0 according to whether \( i \in A \) or not. We use a short-hand notation \( e^i \) for \( e^{\{i\}} \), which is the \( i \)-th unit vector. The vector with all components equal to 1 is denoted by \( 1 \), that is, \( 1 = (1, 1, \ldots, 1) = e^N \). We sometimes use \( e^0 \) to mean the zero vector 0.

For a vector \( x = (x_1, x_2, \ldots, x_n) \) and a subset \( A \) of \( N \), we use notation \( x(A) = \sum_{i \in A} x_i \). For two vectors \( x, y \in \mathbb{R}^n \), the vectors of componentwise maximum and minimum of \( x \) and \( y \) are denoted, respectively, by \( x \vee y \) and \( x \wedge y \), i.e.,

\[
(x \vee y)_i = \max(x_i, y_i), \quad (x \wedge y)_i = \min(x_i, y_i) \quad (i = 1, 2, \ldots, n).
\]  

(2.1)

For a real number \( t \in \mathbb{R} \), \([t]\) denotes the smallest integer not smaller than \( t \) (rounding-up to the nearest integer) and \([t]\) the largest integer not larger than \( t \) (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise applications. That is,

\[
[x] = ([x_1], [x_2], \ldots, [x_n]), \quad \lfloor x \rfloor = ([\lfloor x_1 \rfloor], [\lfloor x_2 \rfloor], \ldots, [\lfloor x_n \rfloor]).
\]  

(2.2)

For integer vectors \( a \in (\mathbb{Z} \cup \{-\infty\})^n \) and \( b \in (\mathbb{Z} \cup \{+\infty\})^n \) with \( a \leq b \), the box of reals and the box of integers (discrete rectangle, integer interval) between \( a \) and \( b \) are denoted by \([a, b]_{\mathbb{R}}\) and \([a, b]_{\mathbb{Z}}\), respectively, i.e.,

\[
[a, b]_{\mathbb{R}} = \{ x \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \ (i = 1, 2, \ldots, n) \},
\]  

(2.3)

\[
[a, b]_{\mathbb{Z}} = \{ x \in \mathbb{Z}^n \mid a_i \leq x_i \leq b_i \ (i = 1, 2, \ldots, n) \}.
\]  

(2.4)

We consider functions defined on integer lattice points, \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \), where the function may possibly take \( +\infty \). The effective domain of \( f \) means the set of \( x \) with \( f(x) < +\infty \) and is denoted as

\[
\operatorname{dom} f = \{ x \in \mathbb{Z}^n \mid f(x) < +\infty \}.
\]  

(2.5)

We always assume that \( \operatorname{dom} f \) is nonempty. The set of the minimizers of \( f \) is denoted by \( \operatorname{arg\ min} f \). For a function \( f \) and a vector \( p \), \( f[-p] \) denotes the function defined by

\[
f[-p](x) = f(x) - \sum_{i=1}^{n} p_i x_i.
\]  

(2.6)

The convex hull of a set \( S \subseteq \mathbb{Z}^n \) is denoted by \( \overline{S} \). The indicator function of a set \( S \subseteq \mathbb{Z}^n \) is denoted by \( \delta_S \), which is the function \( \delta_S : \mathbb{Z}^n \to \{0, +\infty\} \) defined by

\[
\delta_S(x) = \begin{cases} 
0 & (x \in S), \\
+\infty & (x \not\in S).
\end{cases}
\]  

(2.7)

2.2 Classes of discrete convex functions

In this paper we investigate pairwise relations for the following 13 classes of discrete convex functions: separable convex functions, integrally convex functions, L-convex functions, \( L^2 \)-convex functions, \( L_2 \)-convex functions, \( L^3 \)-convex functions, M-convex functions, M^3-convex
functions, \( M_2 \)-convex functions, \( M_2^2 \)-convex functions, multimodular functions, globally discrete midpoint convex functions, and directed discrete midpoint convex functions. The separable convex functions are defined by (1.1) in Introduction. The definitions of other functions are given in Appendix A, except that the definition of multimodular functions is given in Section 5.1.

The inclusion relations between these function classes are summarized in the following proposition, with Remark 2.1 giving the references. The inclusion relations (2.8)–(2.11) are depicted in Fig. 2.

**Proposition 2.1.** The following inclusion relations hold for functions on \( \mathbb{Z}^n \):

\[
\begin{align*}
\{\text{separable conv.}\} & \subseteq \{L^2\text{-conv.}\} \subseteq \{\text{integrally conv.}\}, \\
\{\text{separable conv.}\} & \subseteq \{M^2\text{-conv.}\} \subseteq \{\text{integrally conv.}\}, \\
\{L\text{-conv.}\} & \subseteq \left\{ \begin{array}{l}
\{L^2\text{-conv.}\} \\
\{L_2\text{-conv.}\}
\end{array} \right\} \subseteq \{L_2^2\text{-conv.}\} \subseteq \{\text{integrally conv.}\}, \\
\{M\text{-conv.}\} & \subseteq \left\{ \begin{array}{l}
\{M^2\text{-conv.}\} \\
\{M_2\text{-conv.}\}
\end{array} \right\} \subseteq \{M_2^2\text{-conv.}\} \subseteq \{\text{integrally conv.}\}, \\
\{\text{separable conv.}\} & \subseteq \{\text{multimodular}\} \subseteq \{\text{integrally conv.}\}, \\
\{L^2\text{-conv.}\} & \subseteq \{\text{globally d.m.c.}\} \subseteq \{\text{integrally conv.}\}, \\
\{L^2\text{-conv.}\} & \subseteq \{\text{directed d.m.c.}\} \subseteq \{\text{integrally conv.}\}.
\end{align*}
\]

(2.8) (2.9) (2.10) (2.11) (2.12) (2.13) (2.14)

**Remark 2.1.** Here is a supplement to Proposition 2.1. Integral convexity is established for \( L^2 \)-convex functions in [16, Theorem 7.20], for \( M^2 \)-convex functions in [23, Theorem 3.9] (see also [16, Theorem 6.42]), and for globally discrete midpoint convex functions in [13, Theorem 6]. The inclusion relations (2.10) and (2.11) concerning \( L_2 \)-convex and \( M_2 \)-convex functions are given in [16, Section 8.3]. The integral convexity of multimodular functions in (2.12) was pointed out first in [18, Section 14.6], while this is implicit in the construction...
of the convex extension given earlier in [8, Theorem 4.3]. The first inclusion in (2.12) for multimodular functions is given in [9, Proposition 2]. The inclusion relations in (2.13) are given in [13, Theorem 6], and those in (2.14) are in [35].

2.3 Our approach to identify the intersection of two classes

When two function classes have no inclusion relation between themselves, we are interested in identifying the intersection of these classes. A typical (known) statement of this kind is that a function is both \( L^\natural \)-convex and \( M^\natural \)-convex if and only if it is separable convex. In this paper (Section 4) we are concerned with similar statements:

\[
\text{A function is both A-convex and B-convex if and only if it is C-convex,} \tag{2.15}
\]

where A-, B-, and C-convex denote certain specified discrete convexity from among those treated in this paper.

In discrete convex analysis, it is generally true that a set \( S \subseteq \mathbb{Z}^n \) has a certain discrete convexity if and only if its indicator function \( \delta_S \) has the same discrete convexity. Therefore, the statement (2.15) implies the corresponding statement for a set:

\[
\text{A set is both A-convex and B-convex if and only if it is C-convex.} \tag{2.16}
\]

While the implication “(2.15) \( \Rightarrow \) (2.16)” is obvious, the converse is also true under mild assumptions. In such cases, we can reduce the proof of (2.15) for functions to that of (2.16) for sets.

As a technical assumption for the reverse implication “(2.16) \( \Rightarrow \) (2.15)” we introduce the condition on a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) that

(i) \( \text{dom} f \) is bounded or (ii) \( \text{dom} f \) is a polyhedron and \( f \) is convex-extensible,

\[
\tag{2.17}
\]

where, in this paper, we call a function \( f \) convex-extensible if it is extensible to a locally polyhedral convex function in the sense of [6, Section 15]. For the set version, we consider the condition on a set \( S \subseteq \mathbb{Z}^n \) that

(i) \( S \) is bounded or (ii) \( \overline{S} \) is a polyhedron and \( S = \overline{S} \cap \mathbb{Z}^n \).

\[
\tag{2.18}
\]

For the sake of exposition, let us choose separable convexity as “C-convexity” in the above generic statement. Note that a box of integers is the set version of a separable convex function. We use the following fact, where \( \text{arg min} f[-p] \) means the set of minimizers of the function \( f[-p] \) defined in (2.6).

Proposition 2.2. Under the assumption (2.17), a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is separable convex if and only if, for any vector \( p \in \mathbb{R}^n \), \( \text{arg min} f[-p] \) is a box of integers or an empty set.

Proof. See Remark 2.2 at the end of Section 2.3.

For “A-convexity” and “B-convexity” in the generic statement, we assume the following natural properties:

If a function \( f \) is A-convex (resp., B-convex), then \( f \) satisfies (2.17) and, for any \( p \in \mathbb{R}^n \), \( \text{arg min} f[-p] \) is A-convex (resp., B-convex).

\[
\tag{2.19}
\]

By Proposition 2.2 (only-if part), separable convexity has this property. Moreover, it is known that all kinds of discrete convexity treated in this paper have this property.
Proposition 2.3. Under the assumptions \((2.17)–(2.19)\), the following two statements are equivalent:

1. A set which is both \(A\)-convex and \(B\)-convex is a box of integers.
2. A function which is both \(A\)-convex and \(B\)-convex is a separable convex function.

Proof. The implication \((2) \implies (1)\) is obvious, as \((1)\) for a set \(S\) follows from \((2)\) for its indicator function \(\delta_{S}\). The converse, \((1) \implies (2)\), can be shown as follows. Let \(f\) be both \(A\)-convex and \(B\)-convex. By the assumption \((2.19)\), the set \(S = \arg \min f[-p]\) is both \(A\)-convex and \(B\)-convex for each \(p \in \mathbb{R}^n\). Then, by \((1)\), \(\arg \min f[-p]\) is a box for each \(p \in \mathbb{R}^n\). It follows from this and Proposition 2.2 that \(f\) is separable convex.

Similarly, we can obtain the following propositions, where “\(C\)-convexity” in the generic statement is replaced by \(L\)-convexity and \(M\)-convexity, respectively.

Proposition 2.4. Under the assumptions \((2.17)–(2.19)\), the following two statements are equivalent:

1. A set which is both \(A\)-convex and \(B\)-convex is an \(L\)-convex set.
2. A function which is both \(A\)-convex and \(B\)-convex is an \(L\)-convex function.

Proof. The proof is the same as that of Proposition 2.3 except that Proposition 2.2 is replaced by Theorem A.7.

Proposition 2.5. Under the assumptions \((2.17)–(2.19)\), the following two statements are equivalent:

1. A set which is both \(A\)-convex and \(B\)-convex is an \(M\)-convex set.
2. A function which is both \(A\)-convex and \(B\)-convex is an \(M\)-convex function.

Proof. The proof is the same as that of Proposition 2.3 except that Proposition 2.2 is replaced by Theorem A.8.

Proposition 2.3 will be used in the proof of Theorem 5.7 and Propositions 2.4 and 2.5 will be used in the proofs of Theorems 4.1 and 4.2, respectively.

Remark 2.2. A proof of Proposition 2.2 is outlined here for completeness. Suppose that \(f\) is a separable convex function represented as \(f(x) = \sum_{i=1}^{n} \varphi_i(x_i)\) in \((1.1)\). Then \(f[-p](x) = \sum_{i=1}^{n} \varphi_i[-p_i](x_i)\), from which \(x\) is a minimizer of \(f[-p]\) if and only if, for each \(i\), \(x_i\) is a minimizer of \(\varphi_i[-p_i]\). Therefore \(\arg \min f[-p]\) is a box \([a, b]_{\mathbb{R}}\), where \(a\) and \(b\) are the vectors whose components are defined by \([a_i, b_i]_{\mathbb{Z}} = \arg \min \varphi_i[-p_i] (i = 1, 2, \ldots, n)\). To show the converse under \((2.17)\), suppose that \(\arg \min f[-p]\) is a box of integers for any \(p \in \mathbb{R}^n\).

First we consider the case (ii) where \(\text{dom } f\) is a polyhedron and \(f\) is convex-extensible. Let \(\overline{f}\) denote the convex extension of \(f\). We use notation \(S_p := \arg \min f[-p]\). By the assumption, \(S_p\) is a box of integers and \(\overline{S_p} = \arg \min \overline{f[-p]} = \arg \min \overline{f[-p]}\) is a box of reals. We have \(\text{dom } f = \bigcup_p S_p\), which implies that \(\text{dom } f\) is a box of reals and hence \(\text{dom } f\) is a box of integers. Fix an arbitrary \(z \in \text{dom } f\). For \(i = 1, 2, \ldots, n\), define \(\varphi_i : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}\) by \(\varphi_i(t) = f(z_i, \ldots, z_{i-1}, t, z_{i+1}, \ldots, z_n)\) for \(t \in \mathbb{Z}\). By the assumed convex-extensibility of \(f\), we have that \(\text{dom } \varphi_i\) is an interval of integers and \(\varphi_i(t - 1) + \varphi_i(t + 1) \geq 2\varphi_i(t)\) for all \(t \in \mathbb{Z}\). Using these univariate functions we obtain the desired expression \(f(x) = \sum_{i=1}^{n} \varphi_i(x_i)\).

Next, we turn to the other case (i) where \(\text{dom } f\) is assumed to be bounded. Let \(\tilde{f}\) denote the convex envelope of \(f\), by which we mean that \(\tilde{f}(x)\) is equal to the maximum of \(g(x)\) taken over all closed convex functions \(g\) satisfying \(g(y) \leq f(y)\) for all \(y \in \text{dom } f\). Since \(\text{dom } f\) is
bounded, \( \mathcal{T} \) is a polyhedral convex function and \( \text{dom} \ f = \text{dom} \ f \) is a polyhedron. To show \( \mathcal{T} = f \) (convex-extensibility of \( f \)) by contradiction, suppose that \( \mathcal{T}(y) < f(y) \) for some \( y \in \text{dom} \ f \). Let \( p \) be a subgradient of \( \mathcal{T} \) at \( y \), which is equivalent to saying that \( y \in \arg \min \mathcal{T}[-p] \). Since \( \arg \min \mathcal{T}[-p] = \arg \min f[-p] \), this implies \( y \in \arg \min f[-p] \), whereas \( y \notin \arg \min f[-p] \) by \( \mathcal{T}(y) < f(y) \). This is a contradiction to the assumption that \( \arg \min f[-p] \) is a box of integers. Hence \( f \) is convex-extensible, and the proof in the case of (i) is reduced to that in the case of (ii).

\[ \text{Example 5.4 in Section 5.4.1.} \]

### 3 Inclusion Relations Between Convexity Classes

In Section 2, we have presented basic inclusion relations for fundamental classes of discrete convex functions (Proposition 2.1 and Fig. 2). The objective of this section is to offer supplementary facts. In particular, we present examples to demonstrate proper inclusion (\( \subset \)), and cover (globally, directed) discrete midpoint convex functions. The relation of multimodularity to \( L^3 \)- and \( M^3 \)-convexity will be investigated later in Sections 5.3 and 5.4.

Table 1 is a summary of the inclusion and intersection relations for various classes of discrete convex functions. In each row, corresponding to a class of discrete convex functions, the first (upper) line shows its inclusion relation by \( \subset \) (resp., \( \supset \)) means that the class of the row is properly contained in (resp., properly contains) the class of the column. The symbols \( \triangle \) and \( \triangledown \) indicate that there is no inclusion relation between the classes of the row and the column; \( \triangle \) or \( \triangledown \) is used depending on whether their intersection includes all separable convex functions or not. The second (lower) line of each row shows what their intersection is, to be treated in Section 4.

A set \( S \) is always assumed to be a subset of \( \mathbb{Z}^n \) and a function \( f \) is defined on \( \mathbb{Z}^n \), that is, \( f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\} \). We can demonstrate that a certain class of discrete convex functions, say, A-convex functions is not contained in another class of discrete convex functions, say, B-convex functions by exhibiting a set \( S \) that is A-convex and not B-convex. Note that, in this case, the indicator function \( \delta_S \) is an A-convex function which is not B-convex.

#### 3.1 L-convexity, M-convexity, and multimodularity

The following examples are concerned with L-convexity, M-convexity, and multimodularity (m.m.). For function classes \( A, B, C \) with \( \{A, B, C\} = \{L, M, \text{m.m.}\} \), we have \( A \setminus (B \cup C) \neq \emptyset \).

**Example 3.1.** Let \( S = \{x \in \mathbb{Z}^3 \mid x = t(1, 1, 1), t \in \mathbb{Z}\} \). This set is L-convex (hence \( L^3 \)-convex), but it is neither \( M^3 \)-convex nor multimodular. Multimodularity of \( S \) is denied in Example 5.1 in Section 5.4.1.

**Example 3.2.** Let \( S = \{x \in \mathbb{Z}^3 \mid x = t(1, -1, 1), t \in \mathbb{Z}\} \). This set is multimodular but it is neither \( L^3 \)-convex nor \( M^3 \)-convex. Multimodularity of \( S \) will be demonstrated in Example 5.2 in Section 5.2.2.

**Example 3.3.** Let
\[
S = \{x \in \mathbb{Z}^3 \mid 0 \leq x_i \leq 3 (i = 1, 2, 3), x(X) \leq 5 (|X| = 2), x_1 + x_2 + x_3 \leq 6\}, \quad (3.1)
\]
which is depicted in Fig. 1(b). This set is \( M^3 \)-convex, but it is neither \( L^3 \)-convex nor multimodular. For \( x = (1, 2, 3) \) and \( y = (2, 2, 2) \), we have \( (x + y)/2 = (3/2, 2, 5/2) \) and \( [(x + y)/2] = (2, 2, 3) \notin S \). Hence \( S \) is not \( L^3 \)-convex. Multimodularity of \( S \) is denied in Example 5.4 in Section 5.4.1.
Table 1: Relations between classes of discrete convex functions

|        | sep | int-c | L   | L₂   | L₃   | M   | M₂   | M₃   | m.m. | g-dmc | d-dmc |
|--------|-----|-------|-----|------|------|-----|------|------|------|-------|-------|
| sep    | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| int-c  | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| L      | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| L₂     | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| L₃     | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| M      | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| M₂     | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| M₃     | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| m.m.   | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| g-dmc  | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |
| d-dmc  | ⊆   | ⊆     | ⊆   | ⊆    | ⊆    | ⊆   | ⊆    | ⊆    | ⊆    | ⊆     | ⊆     |

sep = separable convex, int-c = integrally convex, m.m. = multimodular.
g-dmc = globally discrete midpoint convex, d-dmc = directed discrete midpoint convex.
⊂: The class of the row is properly contained (⊂) by the class of the column.
⊃: The class of the row properly contains (⊃) the class of the column.
△: There is no inclusion relation between the classes of the row and the column, and their intersection includes all separable convex functions.
▽: There is no inclusion relation between the classes of the row and the column, and their intersection does not include the set of separable convex functions.
lin: linear function on \( \mathbb{Z}^n \), point: function defined on a single point.
*: results of this paper, ◦: results in previous studies, Unmarked: easy observations.
Example 3.4. From Example 3.3 we can make an example of an M-convex set that is neither \( L^2 \)-convex nor multimodular. Using \( S \) in (3.1), define
\[
\hat{S} = \{ x \in \mathbb{Z}^4 \mid (x_1, x_2, x_3) \in S, \ x_4 = 6 - (x_1 + x_2 + x_3) \},
\]
which is indeed M-convex by (A.37). This set is not \( L^2 \)-convex since discrete midpoint convexity fails for \( x = (1, 2, 3, 0) \) and \( y = (2, 2, 2, 0) \). Multimodularity of \( \hat{S} \) is denied in Example 5.5 in Section 5.4.1.

Remark 3.1. When \( n = 2 \), every \( M^2 \)-convex function is multimodular, and the converse is also true [9, Remark 2.2]. When \( n = 3 \), every M-convex function is multimodular (Proposition 5.10 in Section 5.4.1), but the converse is not true.

3.2 Global and directed discrete midpoint convexity

The following examples show that there is no inclusion relation between global d.m.c. and directed d.m.c., that is, \( A \setminus B \neq \emptyset \) when \( \{ A, B \} = \{ g\text{-dmc}, d\text{-dmc} \} \).

Example 3.5 ([35] Example 3). Let
\[
S = \{(0, 0, 0), (1, 1, 0), (1, 0, -1), (2, 1, -1)\}.
\]
This set is globally d.m.c. but not directed d.m.c. Indeed, \( x = (0, 0, 0) \) and \( y = (2, 1, -1) \) are the only pair with \( \|x - y\|_\infty \geq 2 \). We have \( (x + y)/2 = (1, 1/2, -1/2) \), \( [(x + y)/2] = (1, 1, 0) \in S \) and \( [(x + y)/2] = (1, 0, -1) \in S \). On the other hand, using notation \( \tilde{\mu} \) defined in (A.26) and (A.27), we have \( \tilde{\mu}(x, y) = (1, 0, 0) \notin S \) and \( \tilde{\mu}(y, x) = (1, 1, -1) \notin S \). Hence \( S \) is not directed d.m.c.

Example 3.6 ([35] Example 3). Let
\[
S = \{(0, 0, 0), (1, 0, 0), (1, 1, 1), (2, 1, 1), (1, 1, -1), (2, 1, -1), (1, 1, 0), (2, 1, 0)\}.
\]
This set is directed d.m.c. but not globally d.m.c. For \( x = (0, 0, 0) \) and \( y = (2, 1, -1) \) with \( \|x - y\|_\infty \geq 2 \), we have \( (x + y)/2 = (1, 1/2, -1/2) \) and \( [(x + y)/2] = (1, 0, -1) \notin S \). Hence \( S \) is not globally d.m.c.

Example 3.7. Let
\[
f(x_1, x_2, x_3) = (x_1 - x_2)^2 + (x_1 + x_3)^2 + (x_2 + x_3)^2.
\]
This function is not globally d.m.c. Indeed, for \( x = (0, 0, 0) \) and \( y = (2, 1, -1) \) with \( \|x - y\|_\infty \geq 2 \), we have \( (x + y)/2 = (1, 1/2, -1/2) \), \( u := [(x + y)/2] = (1, 1, 0) \), and \( v := [(x + y)/2] = (1, 0, -1) \). Then
\[
f(x) + f(y) = (0 + 0 + 0) + (1 + 1 + 0) < f(u) + f(v) = (0 + 1 + 1) + (1 + 0 + 1),
\]
which is a violation of discrete midpoint convexity. This function \( f \) is a quadratic function represented as \( f(x) = x^\top Qx \) with a diagonally dominant matrix \( Q = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \). It is known [35, Theorem 9] that a quadratic function defined by a diagonally dominant matrix with nonnegative diagonal elements is directed d.m.c. More generally, 2-separable convex functions are directed d.m.c. [35, Theorem 4].
Example 3.8. Let

\[ f(x_1, x_2, x_3) = (x_1 + x_2)^2 = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \]

This function is directed d.m.c. (by diagonal dominance) but not globally d.m.c. Indeed, for \( x = (1, 0, 0) \) and \( y = (0, 1, 2) \), we have \( u := [(x + y)/2] = (1, 1, 1) \), \( v := [(x + y)/2] = (0, 0, 1) \), and

\[ f(x) + f(y) = 1 + 1 < f(u) + f(v) = 4 + 0, \]

which is a violation of discrete midpoint convexity.

3.3 \textbf{L}_2\text{-convexity and discrete midpoint convexity}

The following examples show that there is no inclusion relation between \( \text{L}_2\)-convexity and (global, directed) discrete midpoint convexity, that is, \( A \setminus B \neq \emptyset \) when \( \{A, B\} = \{\text{L}_2, (g-, d-)dmc\}. \)

Example 3.9 ([13] Remark 2, [35] Example 2). Let

\[ S = \{(0, 0, 0), (0, 1, 1), (1, 1, 0), (1, 2, 1)\}. \]

This set is \( \text{L}_2^2 \)-convex, since \( S = S_1 + S_2 \) for two \( \text{L}_2 \)-convex sets \( S_1 = \{(0, 0, 0), (0, 1, 1)\} \) and \( S_2 = \{(0, 0, 0), (1, 1, 0)\} \). The set \( S \) is not (globally, directed) d.m.c., since, for \( x = (0, 0, 0) \) and \( y = (1, 2, 1) \) with \( x \leq y \) and \( \|x - y\|_\infty \geq 2 \), we have \( (x + y)/2 = (1/2, 1, 1/2) \), \( [(x + y)/2] = (1, 1, 1) \) \( \notin S \) and \( [(x + y)/2] = (0, 1, 0) \) \( \notin S \). From this \( \text{L}_2^2 \)-convex set \( S \), we define a set \( T \subseteq \mathbb{Z}^4 \) by

\[ T = \{(x, 0) + \mu(1, 1) \mid x \in S, \; \mu \in \mathbb{Z}\}, \]

which is \( \text{L}_2 \)-convex and not (globally, directed) d.m.c.

Example 3.10. Let

\[ S = \{x \in \mathbb{Z}^4 \mid x_1 + x_2 - x_3 - x_4 = 0\}, \]

which is \( \text{L}_2 \)-convex, since \( S = S_1 + S_2 \) for two \( \text{L} \)-convex sets

\[ S_1 = \{x \in \mathbb{Z}^4 \mid x_1 = x_3, \; x_2 = x_4\}, \]

\[ S_2 = \{x \in \mathbb{Z}^4 \mid x_1 = x_4, \; x_2 = x_3\}. \]

The set \( S \) is not (globally, directed) d.m.c., since, for \( x = (0, 0, 0, 0) \) and \( y = (2, 2, 1, 3) \) with \( x \leq y \) and \( \|x - y\|_\infty \geq 2 \), we have \( (x + y)/2 = (1, 1, 1/2, 3/2) \), \( [(x + y)/2] = (1, 1, 1, 2) \) \( \not\in S \) and \( [(x + y)/2] = (1, 1, 0, 1) \) \( \not\in S \).

Example 3.11. The set \( S = \{(1, 0), (0, 1)\} \) is (globally, directed) d.m.c. but not \( \text{L}_2^2 \)-convex (hence not \( \text{L}_2 \)-convex).
\section{M\textsubscript{2}-convexity and discrete midpoint convexity}

The following examples show that there is no inclusion relation between \(M\textsubscript{2}\)-convexity and (global, directed) discrete midpoint convexity, that is, \(A \setminus B \neq \emptyset\) when \(\{A, B\} = \{M\textsubscript{2}, (g-, d-)dmc\}\).

\textbf{Example 3.12.} Let \(S\) be the set of integer points on the maximal face in Fig. \(\Pi(a)\), that is,
\[
S = \{x \in \mathbb{Z}^3 | x_i \leq 3 \ (i = 1, 2, 3), \ x_1 + x_2 \leq 5, \ x_2 + x_3 \leq 5, \ x_1 + x_2 + x_3 = 6\} \\
= \{(3, 0, 3), (1, 2, 3), (3, 2, 1), (1, 3, 2), (2, 3, 1), (2, 1, 3), (2, 2, 2), (3, 1, 2)\}.
\]

This set \(S\) is \(M\)-convex and hence \(M\textsubscript{2}\)-convex. However, it is not (globally, directed) d.m.c. For \(x = (3, 0, 3)\) and \(y = (2, 2, 2)\) with \(|x - y|\)\(\text{infty} \geq 2\), we have \((x + y)/2 = (5/2, 1, 5/2)\), \([x + y)/2] = \tilde{\mu}(x, y) = (3, 1, 3) \not\in S\) and \([(x + y)/2] = \tilde{\mu}(y, x) = (2, 1, 2) \not\in S\) using notation \(\tilde{\mu}\) defined in (A.26) and (A.27). Hence \(S\) is neither globally d.m.c. nor directed d.m.c.

\textbf{Example 3.13.} The set \(S = \{(0, 0), (1, 1)\}\) is (globally, directed) d.m.c. but not \(M\textsubscript{2}\)-convex (hence not \(M\textsubscript{2}\)-convex).

\section{Intersection of Two Convexity Classes}

The following are the major findings of this paper concerning the intersection of two convexity classes of functions \(f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}\).

- A function which is both \(L\textsuperscript{2}\)-convex and multimodular is separable convex (Theorem 5.8).
- Furthermore, a function which is both \(L\textsuperscript{2}\)-convex and multimodular is separable convex (Theorem 5.7).
- A function which is both \(M\textsuperscript{2}\)-convex and multimodular is not necessarily separable convex (Section 5.4).

For the ease of reference we summarize remarkable relations below:
\[
\begin{align*}
\{L\textsubscript{2}\text{-convex}\} \cap \{L\textsuperscript{2}\text{-convex}\} &= \{L\text{-convex}\}, \quad (4.1) \\
\{M\textsubscript{2}\text{-convex}\} \cap \{M\textsuperscript{2}\text{-convex}\} &= \{M\text{-convex}\}, \quad (4.2) \\
\{L\text{-convex}\} \cap \{M\text{-convex}\} &= \emptyset, \quad (4.3) \\
\{L\textsubscript{2}\text{-convex}\} \cap \{M\textsubscript{2}\text{-convex}\} &= \emptyset, \quad (4.4) \\
\{L\textsuperscript{2}\text{-convex}\} \cap \{M\textsuperscript{2}\text{-convex}\} &= \{\text{separable convex}\}, \quad (4.5) \\
\{L\textsuperscript{2}\text{-convex}\} \cap \{M\textsuperscript{2}\text{-convex}\} &= \{\text{separable convex}\}, \quad (4.6) \\
\{L\text{-convex}\} \cap \{M\textsuperscript{2}\text{-convex}\} &= \{\text{linear on } \mathbb{Z}^n\}, \quad (4.7) \\
\{L\textsuperscript{2}\text{-convex}\} \cap \{M\text{-convex}\} &= \{\text{singleton effective domain}\}, \quad (4.8) \\
\{\text{multimodular}\} \cap \{L\textsuperscript{2}\text{-convex}\} &= \{\text{separable convex}\}, \quad (4.9) \\
\{\text{multimodular}\} \cap \{L\textsuperscript{2}\text{-convex}\} &= \{\text{separable convex}\}, \quad (4.10) \\
\{\text{multimodular}\} \cap \{M\textsuperscript{2}\text{-convex}\} \supset \{\text{separable convex}\}, \quad (4.11) \\
\{\text{globally d.m.c.}\} \cap \{\text{directed d.m.c.}\} \supset \{L\textsuperscript{2}\text{-convex}\}. \quad (4.12)
\end{align*}
\]
The relations (4.1) and (4.2) will be established in Theorems 4.1 and 4.2, respectively. The relations (4.3) and (4.4) are obvious. The relations (4.5) and (4.6) are well known, originating in [23, Theorem 3.17] and stated in [16, Theorem 8.49]. The relations (4.7) and (4.8) will be discussed in Section 4.3. The relations (4.9) and (4.10) will be established in Theorems 5.8 and 5.7, respectively, and (4.11) is discussed in Section 5.4. The relation (4.12) is treated in Section 4.5.

In the following we make observations, major and minor, for each case of intersection.

4.1 L-convexity

In this section we deal with intersection of classes related to L-convexity (and its relatives like $L^2$, $L^2$-, and $L^2$-convexity).

- **L-convexity & separable convexity**: An L-convex set $S$ is a box if and only if $S = \mathbb{Z}^n$. Hence, a function $f$ is both L-convex and separable convex if and only if $f$ is a linear function with $\text{dom } f = \mathbb{Z}^n$.

  ($\because$ An L-convex set $S$ has translation invariance ($x \in S, \mu \in \mathbb{Z} \Rightarrow x + \mu \mathbf{1} \in S$) in (A.23). If a box has this property, it must be $\mathbb{Z}^n$. The statement for a function follows from the statement for a set.)

- **$L_2$-convexity & separable convexity**: An $L_2$-convex set $S$ is a box if and only if $S = \mathbb{Z}^n$. Hence, a function $f$ is both $L_2$-convex and separable convex if and only if $f$ is a linear function with $\text{dom } f = \mathbb{Z}^n$.

  ($\because$ An $L_2$-convex set $S$ also has translation invariance.)

- **$L_2$-convexity & $L^2$-convexity**: A set $S$ is both $L_2$-convex and $L^2$-convex if and only if $S$ is L-convex. Hence, a function $f$ is both $L_2$-convex and $L^2$-convex if and only if $f$ is L-convex. See Theorem 4.1 below.

The following theorem shows that the combination of $L_2$-convexity and $L^2$-convexity is equivalent to L-convexity.

**Theorem 4.1.**

1. A set $S$ ($\subseteq \mathbb{Z}^n$) is L-convex if and only if it is both $L_2$-convex and $L^2$-convex.
2. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is L-convex if and only if it is both $L_2$-convex and $L^2$-convex.

*Proof.* (1) First note that the only-if-part of (1) is obvious. To prove the if-part, suppose that a set $S$ is both $L_2$-convex and $L^2$-convex. Recall that an $L^2$-convex set $S$ is L-convex if and only if it is translation invariant in the sense that $x + \mu \mathbf{1} \in S$ for every $x \in S$ and every integer $\mu$ (see Theorem A.6). Since $S$ is $L_2$-convex, it can be represented as $S = S_1 + S_2$ with two $L$-convex sets $S_1$ and $S_2$. Since $S_1$ and $S_2$ are translation invariant, $S$ is also translation invariant. Therefore, $S$ must be L-convex.

(2) The statement (2) for functions follows from (1) for sets by the general principle given in Proposition 2.4. Note that the assumption (2.19) holds for $L_2$-convexity and $L^2$-convexity (see [16, Proposition 8.40] for this statement about $L_2$-convexity).
4.2 M-convexity

In this section we deal with intersection of classes related to M-convexity (and its relatives like $M^2$, $M_2^-$, and $M_2^{\natural}$-convexity).

- **M-convexity & separable convexity**: An M-convex set $S$ is a box if and only if $S$ is a singleton (a set consisting of a single point). Hence, a function $f$ is both M-convex and separable convex if and only if $f$ is a function whose effective domain is a singleton. ($\because$ An M-convex set $S$ consists of vectors with a constant component sum, i.e., $x(N) = y(N)$ for all $x, y \in S$. If a box has this property, it must be a singleton. The statement for a function follows from the statement for a set.)

- **$M_2^-$-convexity & separable convexity**: An $M_2^-$-convex set $S$ is a box if and only if $S$ is a singleton. Hence, a function $f$ is both $M_2^-$-convex and separable convex if and only if $f$ is a function whose effective domain is a singleton. ($\because$ An $M_2^-$-convex set $S$ also consists of vectors with a constant component sum.)

- **$M_2^-$-convexity & $M^{\natural}$-convexity**: A set $S$ is both $M_2^-$-convex and $M^{\natural}$-convex if and only if $S$ is M-convex. Hence, a function $f$ is both $M_2^-$-convex and $M^{\natural}$-convex if and only if $f$ is M-convex. See Theorem 4.2 below.

The following theorem shows that the combination of $M_2^-$-convexity and $M^{\natural}$-convexity is equivalent to M-convexity.

**Theorem 4.2.**

(1) A set $S (\subseteq \mathbb{Z}^n)$ is M-convex if and only if it is both $M_2^-$-convex and $M^{\natural}$-convex.

(2) A function $f : \mathbb{Z}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is M-convex if and only if it is both $M_2^-$-convex and $M^{\natural}$-convex.

**Proof.** (1) First note that the only-if-part of (1) is obvious. To prove the if-part, suppose that a set $S$ is both $M_2^-$-convex and $M^{\natural}$-convex. Recall that an $M^{\natural}$-convex set $S$ is M-convex if and only if $x(N) = y(N)$ for all $x, y \in S$ (see Section A.4.2). On the other hand, an $M_2^-$-convex set has this property. Therefore, $S$ must be M-convex.

(2) The statement (2) for functions follows from (1) for sets by the general principle given in Proposition 2.3. Note that the assumption (2.19) holds for $M_2^-$-convexity and $M^{\natural}$-convexity (see [16, Theorems 8.17] for this statement about $M_2^-$-convexity). \hfill \Box

4.3 L-convexity and M-convexity

In this section we deal with intersection of classes of L-convexity and M-convexity (including their variants).

- **L-convexity & M-convexity**: There exists no set that is both L-convex and M-convex. Hence, there exists no function that is both L-convex and M-convex. ($\because$ An L-convex set $S$ has translation invariance ($x \in S, \mu \in \mathbb{Z} \Rightarrow x + \mu 1 \in S$) in [A.23]. An M-convex set $S$ consists of vectors with a constant component sum, i.e., $x(N) = y(N)$ for all $x, y \in S$. These two properties are not compatible.)
- **L\(^2\)**-convexity & M\(^2\)-convexity: A set \(S\) is both L\(^2\)-convex and M\(^2\)-convex if and only if \(S\) is a box. Hence, a function \(f\) is both L\(^2\)-convex and M\(^2\)-convex if and only if \(f\) is separable convex. See [23, Theorem 3.17] and [16, Theorem 8.49].

- **L-convexity & M\(^0\)-convexity**: A set \(S\) is both L-convex and M\(^0\)-convex if and only if \(S = \mathbb{Z}^n\). Hence, a function \(f\) is both L-convex and M\(^0\)-convex if and only if \(f\) is a linear function with \(\text{dom } f = \mathbb{Z}^n\).

  (\(\because\) An M\(^0\)-convex set \(S\) has the property that if \(x, y \in S\) and \(x \leq y\), then \([x, y]_Z \subseteq S\), whereas an L-convex set has translation invariance. Hence \(S = \mathbb{Z}^n\).)

- **L\(^2\)**-convexity & M-convexity: A set \(S\) is both L\(^2\)-convex and M-convex if and only if \(S\) is a singleton. Hence, a function \(f\) is both L\(^2\)-convex and M-convex if and only if \(f\) is a function whose effective domain is a singleton.

  (\(\because\) An L\(^2\)-convex set \(S\) has the property that \(x, y \in S\) implies \(x \lor y, x \land y \in S\), whereas an M-convex set consists of vectors with a constant component sum. Hence \(S\) must be a singleton.)

The four statements above can be extended to L\(_2\)-convexity and M\(_2\)-convexity without substantial changes.

- **L\(_2\)**-convexity & M\(_1\)-convexity: There exists no set that is both L\(_2\)-convex and M\(_1\)-convex. Hence, there exists no function that is both L\(_2\)-convex and M\(_1\)-convex. This implies that there exists no function that is both L\(_2\)-convex and M-convex (or L-convex and M\(_2\)-convex).

  (\(\because\) An L\(_2\)-convex set has translation invariance and an M\(_2\)-convex set consists of vectors with a constant component sum.)

- **L\(_2\)**-convexity & M\(_2\)-convexity: A set \(S\) is both L\(_2\)-convex and M\(_2\)-convex if and only if \(S\) is a box. Hence, a function \(f\) is both L\(_2\)-convex and M\(_2\)-convex if and only if \(f\) is separable convex. See [23, Theorem 3.17], [16, Theorem 8.49], and [11]. This implies that a function \(f\) is both L\(_2\)-convex and M\(_2\)-convex (or M\(_2\)-convex and L\(_2\)-convex) if and only if \(f\) is separable convex.

- **L\(_2\)**-convexity & M\(_3\)-convexity: A set \(S\) is both L\(_2\)-convex and M\(_3\)-convex if and only if \(S = \mathbb{Z}^n\) (proved below). Hence, a function \(f\) is both L\(_2\)-convex and M\(_3\)-convex if and only if \(f\) is a linear function with \(\text{dom } f = \mathbb{Z}^n\). This implies that a function \(f\) is both L\(_2\)-convex and M\(_3\)-convex (or L-convex and M\(_2\)-convex) if and only if \(f\) is a linear function with \(\text{dom } f = \mathbb{Z}^n\).

  (\(\because\) An M\(_3\)-convex set \(S\) has the property that if \(x, y \in S\) and \(x \leq y\), then \([x, y]_Z \subseteq S\), whereas an L\(_2\)-convex set has translation invariance. Hence \(S = \mathbb{Z}^n\).)

- **L\(_3\)**-convexity & M\(_2\)-convexity: A set \(S\) is both L\(_3\)-convex and M\(_2\)-convex if and only if \(S\) is a singleton (proved below). Hence, a function \(f\) is both L\(_3\)-convex and M\(_2\)-convex if and only if \(\text{dom } f\) is a singleton. This implies that a function \(f\) is both L\(_3\)-convex and M-convex (or L-convex and M\(_2\)-convex) if and only if \(\text{dom } f\) is a singleton.

  (\(\because\) Let \(S = S_1 + S_2\) with L\(_3\)-convex sets \(S_k\) for \(k = 1, 2\). Take \(x, y \in S\) and represent them as \(x = x_1 + x_2\) and \(y = y_1 + y_2\) with \(x_k, y_k \in S_k\) for \(k = 1, 2\). Define \(z_k = x_k \lor y_k\) for
\[ k = 1, 2, \text{ and } z = z_1 + z_2. \text{ Since } z_k \in S_k \text{ for } k = 1, 2, \text{ we have } z \in S. \text{ By } M_2\text{-convexity of } S, \text{ we have } z(N) = x(N), \text{ while}
\]
\[ z(N) = z_1(N) + z_2(N) = (x_1 \lor y_1)(N) + (x_2 \lor y_2)(N) \geq x_1(N) + x_2(N) = x(N). \]

It then follows that \( x_k \lor y_k = x_k \) for \( k = 1, 2 \), that is, \( y_k \leq x_k \) for \( k = 1, 2 \). By symmetry we also have \( y_k \geq x_k \) for \( k = 1, 2 \), and hence \( x = y \).

### 4.4 Multimodularity

In this section we deal with intersection of classes related to multimodularity.

- **Multimodularity & L-convexity**: A set \( S \) is both multimodular and L-convex if and only if \( S = \mathbb{Z}^n \). Hence, a function \( f \) is both multimodular and L-convex if and only if \( f \) is a linear function with \( \text{dom } f = \mathbb{Z}^n \).

  \( \therefore \) A multimodular set \( S \) has the property that if \( x, y \in S \) and \( x \leq y \), then \([x, y]_Z \subseteq S \) (Proposition 5.6), whereas an L-convex set has translation invariance. Hence \( S = \mathbb{Z}^n \).

- **Multimodularity & \( L_2 \)-convexity**: A set \( S \) is both multimodular and \( L_2 \)-convex if and only if \( S = \mathbb{Z}^n \). Hence, a function \( f \) is both multimodular and \( L_2 \)-convex if and only if \( f \) is a linear function with \( \text{dom } f = \mathbb{Z}^n \).

  \( \therefore \) A multimodular set \( S \) has the property that if \( x, y \in S \) and \( x \leq y \), then \([x, y]_Z \subseteq S \) (Proposition 5.6), whereas an \( L_2 \)-convex set has translation invariance. Hence \( S = \mathbb{Z}^n \).

- **Multimodularity & \( L^3 \)-convexity**: A set \( S \) is both multimodular and \( L^3 \)-convex if and only if \( S \) is a box. Hence, a function \( f \) is both multimodular and \( L^3 \)-convex if and only if \( f \) is separable convex. See Theorem 5.8 in Section 5.3.

- **Multimodularity & \( L^3 \)-convexity**: A set \( S \) is both multimodular and \( L^3 \)-convex if and only if \( S \) is a box. Hence, a function \( f \) is both multimodular and \( L^3 \)-convex if and only if \( f \) is separable convex. See Theorem 5.7 in Section 5.3.

- **Multimodularity & \( M^\bullet \)-convexity**: A function in two variables is multimodular if and only if it is \( M^\bullet \)-convex [9, Remark 2.2]. For functions in more than two variables, these classes are distinct, without inclusion relation. The intersection of these classes can be analyzed and understood fairly well (Section 5.4.1). Figure 1(a) demonstrates their nonempty intersection, giving a concrete example of a set that is both multimodular and \( M^\bullet \)-convex (see Example 5.3). This is also an instance of a set that is both multimodular and \( M^\bullet_2 \)-convex.

- **Multimodularity & \( M \)-convexity**: The intersection of these two classes can be analyzed and understood fairly well (Section 5.4.1). Their intersection is nonempty, which is demonstrated by the set \( S \) in (3.2). That is, the set of integer points on the maximal face \( (x_1 + x_2 + x_3 = 6) \) in Fig. 1(a) is both multimodular and \( M \)-convex (see Example 5.3). This is also an instance of a set that is both multimodular and \( M_2 \)-convex.
4.5 Discrete midpoint convexity

In this section we deal with intersection of classes related to discrete midpoint convexity. We first note that the indicator function of any nonempty subset of \([0, 1]^n\) is (globally, directed) discrete midpoint convex. We also note that the indicator function of \(S = \{(2, 0), (1, 1), (0, 2)\}\) is (globally, directed) discrete midpoint convex.

- **d.m.c. & \(L_2\)-convexity**: An \(L\)-convex function is both \(L_2\)-convex and (globally, directed) discrete midpoint convex, but it is not known whether the converse is true or not.

- **d.m.c. & \(L^\natural_2\)-convexity**: An \(L^\natural\)-convex function is both \(L^\natural_2\)-convex and (globally, directed) discrete midpoint convex, but it is not known whether the converse is true or not.

- **d.m.c. & \(M\)-convexity**: For every matroid, the indicator function of the set of its bases is both \(M\)-convex and (globally, directed) discrete midpoint convex. The indicator function of \(S = \{(2, 0), (1, 1), (0, 2)\}\) is both \(M\)-convex and (globally, directed) discrete midpoint convex. Note that general separable convex functions do not belong to this class because a separable convex function \(f\) is \(M\)-convex only if \(\text{dom } f\) is a singleton.

- **d.m.c. & \(M^2\)-convexity**: For every pair of matroids, the indicator function of the set of their common bases is both \(M^2\)-convex and (globally, directed) discrete midpoint convex. For example, the indicator function of \(S = \{(1, 1, 0, 0), (0, 0, 1, 1)\}\) is both \(M^2\)-convex and (globally, directed) discrete midpoint convex.

- **d.m.c. & \(M^\natural\)-convexity**: A separable convex function is both \(M^\natural\)-convex and (globally, directed) discrete midpoint convex. The converse is not true, as shown by the indicator function of \(S = \{(1, 0), (0, 1)\}\) or \(S = \{(2, 0), (1, 1), (0, 2)\}\).

- **d.m.c. & \(M^\natural_2\)-convexity**: A separable convex function is both \(M^\natural_2\)-convex and (globally, directed) discrete midpoint convex. The converse is not true, as shown by the indicator function of \(S = \{(1, 0), (0, 1)\}\) or \(S = \{(2, 0), (1, 1), (0, 2)\}\).

- **d.m.c. & multimodularity**: A separable convex function is both multimodular and (globally, directed) discrete midpoint convex. The converse is not true, as shown by the indicator function of \(S = \{(1, 0), (0, 1)\}\) or \(S = \{(2, 0), (1, 1), (0, 2)\}\).

- **globally d.m.c. & directed d.m.c.**: An \(L^\natural\)-convex function is both globally and directed discrete midpoint convex. The converse is not true, as shown by the indicator function of \(S = \{(1, 0), (0, 1)\}\) or \(S = \{(2, 0), (1, 1), (0, 2)\}\).

5 Results Concerning Multimodularity

In this section we present results concerning multimodularity, which constitute the major technical contributions of this paper.
5.1 Definition of multimodularity

Recall that $e^i$ denotes the $i$th unit vector for $i = 1, 2, \ldots, n$, and let $F \subseteq \mathbb{Z}^n$ be a set of vectors defined by

$$F = \{-e^1, e^1 - e^2, e^2 - e^3, \ldots, e^{n-1} - e^n, e^n\}. \tag{5.1}$$

A finite-valued function $f : \mathbb{Z}^n \to \mathbb{R}$ is said to be modular if it satisfies

$$f(z + d) + f(z + d') \geq f(z) + f(z + d + d') \tag{5.2}$$

for all $z \in \mathbb{Z}^n$ and all distinct $d, d' \in F$. It is known [8, Proposition 2.2] that $f : \mathbb{Z}^n \to \mathbb{R}$ is modular if and only if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R}$ defined by

$$\tilde{f}(x_0, x) = f(x_0 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}) \quad (x_0 \in \mathbb{Z}, x \in \mathbb{Z}^n) \tag{5.3}$$

is submodular in $n + 1$ variables. This characterization enables us to define multimodularity for a function that may take the infinite value $+\infty$. That is, we say [9, 17] that a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom} f \neq \emptyset$ is multimodular if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{+\infty\}$ associated with $f$ by (5.3) is submodular.

Multimodularity and $L^2$-convexity have the following close relationship.

**Theorem 5.1** ([17]). A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is multimodular if and only if the function $g : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$g(y) = f(y_1, y_2 - y_1, y_3 - y_2, \ldots, y_n - y_{n-1}) \quad (y \in \mathbb{Z}^n) \tag{5.4}$$

is $L^2$-convex.

The relation (5.4) between $f$ and $g$ can be rewritten as

$$f(x) = g(x_1, x_1 + x_2, x_1 + x_2 + x_3, \ldots, x_1 + \cdots + x_n) \quad (x \in \mathbb{Z}^n). \tag{5.5}$$

Using a bidiagonal matrix $D = (d_{ij} \mid 1 \leq i, j \leq n)$ defined by

$$d_{ii} = 1 \quad (i = 1, 2, \ldots, n), \quad d_{i+1,i} = -1 \quad (i = 1, 2, \ldots, n-1), \tag{5.6}$$

we can express (5.4) and (5.5) more compactly as $g(y) = f(Dy)$ and $f(x) = g(D^{-1}x)$, respectively. The matrix $D$ is unimodular, and its inverse $D^{-1}$ is an integer lower-triangular matrix with $(D^{-1})_{ij} = 1$ for $i \geq j$ and $(D^{-1})_{ij} = 0$ for $i < j$. For $n = 4$, for example, we have

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{5.7}$$

A nonempty set $S$ is called multimodular if its indicator function $\delta_S$ is multimodular. The effective domain of a multimodular function is a multimodular set. The following proposition, a special case of Theorem 5.1 connects multimodular sets with $L^2$-convex sets.

**Proposition 5.2** ([9, 17]). A set $S \subseteq \mathbb{Z}^n$ is multimodular if and only if it can be represented as $S = \{Dy \mid y \in T\}$ for some $L^2$-convex set $T$, where $T$ is uniquely determined from $S$ as $T = \{D^{-1}x \mid x \in S\}$. \hfill \blacksquare
Theorem 5.3. Under the assumption (2.17), a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is multimodular if and only if, for any vector \( p \in \mathbb{R}^n \), \( \arg \min f[-p] \) is a multimodular set or an empty set.

Proof. By Theorem 5.1 and Proposition 5.2 this follows from the corresponding statement (Theorem A.7) for \( L^3 \)-convex functions.

Example 5.1. Proposition 5.2 is applied to the set \( S = \{ x \in \mathbb{Z}^3 \mid x = t(1, 1, 1), t \in \mathbb{Z} \} \) treated in Example 5.1. We have

\[
T = \{ D^{-1}x \mid x \in S \} = \{ x \in \mathbb{Z}^3 \mid x = t(1, 2, 3), t \in \mathbb{Z} \},
\]

which is not \( L^3 \)-convex. Hence \( S \) is not multimodular. □

The reader is referred to [1, 2, 8, 9, 17, 18] for more about multimodularity.

5.2 Polyhedral description of multimodular sets

As a technical basis for our argument in Sections 5.3 and 5.4, we show a description of multimodular sets by inequalities. A subset \( I \) of the index set \( N = \{1, 2, \ldots, n\} \) is said to be consecutive if it consists of consecutive numbers, that is, it is a set of the form \( I = \{k, k + 1, \ldots, l - 1, l\} \) for some \( k \leq l \).

Theorem 5.4.
(1) If \( S \subseteq \mathbb{Z}^n \) is a multimodular set, then \( S = \overline{S} \cap \mathbb{Z}^n \) and its convex hull \( \overline{S} \) is represented as

\[
\overline{S} = \{ x \in \mathbb{R}^n \mid a_I \leq x(I) \leq b_I \ (I: \text{consecutive subset of } N) \},
\]

where \( a_I = \inf \{ x(I) \mid x \in S \} \) and \( b_I = \sup \{ x(I) \mid x \in S \} \) for consecutive subsets \( I \) of \( N \).

(2) For any \( a_I \in \mathbb{Z} \cup \{-\infty\} \) and \( b_I \in \mathbb{Z} \cup \{+\infty\} \) indexed by the family of consecutive subsets \( I \) of \( N \), the polyhedron \( P \) defined by

\[
P = \{ x \in \mathbb{R}^n \mid a_I \leq x(I) \leq b_I \ (I: \text{consecutive subset of } N) \}
\]

is an integer polyhedron, and \( S := P \cap \mathbb{Z}^n \) is a multimodular set, provided \( P \neq \emptyset \).

(3) A nonempty set \( S \subseteq \mathbb{Z}^n \) is multimodular if and only if it can be represented as

\[
S = \{ x \in \mathbb{Z}^n \mid a_I \leq x(I) \leq b_I \ (I: \text{consecutive subset of } N) \}
\]

for some \( a_I \in \mathbb{Z} \cup \{-\infty\} \) and \( b_I \in \mathbb{Z} \cup \{+\infty\} \) indexed by the family of consecutive subsets \( I \) of \( N \).

Proof. First we mention some basic facts about the transformation \( x = Dy \) connecting multimodularity and \( L^3 \)-convexity. Define \( T = \{ y \mid y = D^{-1}x, \ x \in S \} \) for any nonempty set \( S \subseteq \mathbb{Z}^n \). Then the convex hulls of \( T \) and \( S \) correspond to each other by the same transformation, that is, \( T = \overline{T} \cap \mathbb{Z}^n \). Since \( D \) is unimodular, \( T = \overline{T} \cap \mathbb{Z}^n \) holds if and only if \( S = \overline{S} \cap \mathbb{Z}^n \). By Theorem A.3 \( T \) is \( L^3 \)-convex if and only if \( T = \overline{T} \cap \mathbb{Z}^n \) and its convex hull \( \overline{T} \) can be described as

\[
\alpha_i \leq y_i \leq \beta_i \quad (i = 1, 2, \ldots, n),
\]

\[
\alpha_{ij} \leq y_j - y_i \leq \beta_{ij} \quad (1 \leq i < j \leq n).
\]
By substituting \( y_i = x_1 + x_2 + \cdots + x_i \) \( (i = 1, 2, \ldots, n) \), i.e., \( y = D^{-1}x \), into these inequalities we obtain the following inequalities
\[
\begin{align*}
\alpha_i & \leq x_1 + x_2 + \cdots + x_i \leq \beta_i & (i = 1, 2, \ldots, n), \tag{5.10} \\
\alpha_{ij} & \leq x_{i+1} + x_{i+2} + \cdots + x_j \leq \beta_{ij} & (1 \leq i < j \leq n) \tag{5.11}
\end{align*}
\]
to describe \( S \).

(1) If \( S \) is multimodular, then \( T \) is \( L^2 \)-convex by Proposition 5.2. By the above argument, \( S \) is described by (5.10) and (5.11). These inequalities are of the form of \( \alpha_i \leq x(I) \leq \beta_i \) for consecutive subsets \( I \); (5.10) corresponding to \( I = \{1, 2, \ldots, i\} \) and (5.11) to \( I = \{i + 1, i + 2, \ldots, j\} \). Thus we obtain (5.7). Then the validity of the expressions \( \alpha_i = \inf\{x(I) \mid x \in S\} \) and \( \beta_i = \sup\{x(I) \mid x \in S\} \) is obvious.

(2) Let \( I_1 \) (resp., \( I_2 \)) denote the family of consecutive subsets \( I \) for which \( b_I \) (resp., \( a_I \)) is finite. Let \( A_1 \) (resp., \( A_2 \)) denote the matrix whose rows are indexed by \( I_1 \) (resp., \( I_2 \)) and whose row vector corresponding to \( I \) is the characteristic vector \( e^I \). Let \( b \) (resp., \( a \)) be the vector \( (b_I \mid I \in I_1) \) (resp., \( (a_I \mid I \in I_2) \)). Then the system of inequalities in (5.8) can be expressed as
\[
\begin{bmatrix}
A_1 \\
-A_2
\end{bmatrix} x \leq \begin{bmatrix}
b \\ -a
\end{bmatrix},
\tag{5.12}
\]
Each of the matrices \( A_1 \) and \( A_2 \) is a so-called interval matrix (row-oriented), or a matrix with consecutive ones property. This implies that the combined matrix \( \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \) is also an interval matrix, which is known to be totally unimodular [29, p.279, Example 7]. Therefore, the coefficient matrix \( \begin{bmatrix} A_1 \\ -A_2 \end{bmatrix} \) is also totally unimodular, and hence (5.12) determines an integer polyhedron, which shows the integrality of \( P \) in (5.8). The integrality of \( P \) implies \( P = P \cap \mathbb{Z}^n \), which is equivalent to saying that the convex hull of \( S = P \cap \mathbb{Z}^n \) is described by (5.10) and (5.11). Then the basic facts presented at the beginning of the proof shows that \( T = \{y \mid y = D^{-1}x, x \in S\} \) is \( L^2 \)-convex, which implies, by Proposition 5.2, that \( S \) is multimodular.

(3) If \( S \) is multimodular, the statement (1) shows that \( S = S \cap \mathbb{Z}^n \) and \( S \) is described as (5.7). Hence follows (5.9). Conversely, suppose that (5.9) holds. Then \( P := \overline{S} \) is described as (5.8) by the integrality of a polyhedron of the form of (5.8) shown in the statement (2). The expression (5.9) implies \( S = S \cap \mathbb{Z}^n = P \cap \mathbb{Z}^n \), while \( P \cap \mathbb{Z}^n \) is multimodular by the statement in (2). Thus \( S \) is multimodular if it is represented as (5.9). \( \square \)

It is worth while mentioning box-total dual integrality of the inequality system given in Theorem 5.4. A linear inequality system \( Ax \leq b \) (in general) is said to be \textit{totally dual integral (TDI)} if the entries of \( A \) and \( b \) are rational numbers and the minimum in the linear programming duality equation
\[
\max\{w^\top x \mid Ax \leq b\} = \min\{y^\top b \mid y^\top A = w^\top, y \geq 0\}
\]
has an integral optimal solution \( y \) for every integral vector \( w \) such that the minimum is finite ([29, 30]). A linear inequality system \( Ax \leq b \) is said to be \textit{box-totally dual integral (box-TDI)} if the system \( [Ax \leq b, d \leq x \leq c] \) is TDI for each choice of rational (finite-valued) vectors \( c \) and \( d \). It is known [30, Theorem 5.35] that the system \( Ax \leq b \) is box-TDI if the matrix \( A \) is totally unimodular. A polyhedron is called a \textit{box-TDI polyhedron} if it can be described by a box-TDI system.
We show the multimodularity of Example 5.2. This set admits a representation of the form of (5.9), and therefore it is multimodular by Theorem 5.4. Multimodularity of $S$ is box-TDI, and the convex hull of a multimodular set is a box-TDI integer polyhedron.

\[ S = \{ x \in \mathbb{Z}^3 \mid x_1 + x_2 = 0, x_2 + x_3 = 0 \} \]

of the form of (5.9), and therefore it is multimodular by Theorem 5.4. Multimodularity of $S$ can also be verified by Proposition 5.2. Indeed we have

\[ T = \{ D^{-1} x \mid x \in S \} = \{ x \in \mathbb{Z}^3 \mid x = t(1, 0, 1), t \in \mathbb{Z} \}, \]

which is $L_2^b$-convex.

\section{Multimodularity and $L_2^b$-convexity}

The following theorem shows the combination of multimodularity and $L_2^b$-convexity is equivalent to separable convexity.

**Theorem 5.7.**

1. A set $S (\subseteq \mathbb{Z}^n)$ is a box of integers if and only if it is both multimodular and $L_2^b$-convex.
2. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is separable convex if and only if it is both multimodular and $L_2^b$-convex.

**Proof.** (1) First note that the only-if-part of (1) is obvious. The if-part can be shown as follows. Let $S_1, S_2 \subseteq \mathbb{Z}^n$ be $L_2^b$-convex sets such that $S = S_1 + S_2$. In the case of bounded $S$, the proof is quite easy as follows. Each $S_k$ has the unique minimum element $a^k \in S_k$ and the unique maximum element $b^k \in S_k$. Then $a = a^1 + a^2$ is the unique minimum of $S$ and $b = b^1 + b^2$ is the unique maximum of $S$, for which we have $S \subseteq [a, b]_\mathbb{Z}$. By Proposition 5.6 on the other hand, it follows from $a, b \in S$ and $a \leq b$ that $[a, b]_\mathbb{Z} \subseteq S$. Thus we have proved $S = [a, b]_\mathbb{Z}$.

The general case with (possibly) unbounded $S$ can be treated as follows. For each $i \in N$, put $a_i = \inf_{y \in S} y_i$ and $b_i = \sup_{y \in S} y_i$, where we have the possibility of $a_i = -\infty$ and/or $b_i = +\infty$. Obviously, $S \subseteq [a, b]_\mathbb{Z}$ holds. To prove $[a, b]_\mathbb{Z} \subseteq S$, take any $x \in [a, b]_\mathbb{Z}$. For each $i \in N$, there exist vectors $p^i, q^i \in S$ such that $p^i \leq x_i \leq q^i$, where $p^i, q^i$ denote the $i$th component...
of vectors $p'$, $x$, and $q'$, respectively. Since $p', q' \in S = S_1 + S_2$, we can express them as $p' = p'^1 + p'^2$, $q' = q'^1 + q'^2$ with some $p'^k, q'^k \in S_k (k = 1, 2)$. Consider

$$
\hat{p}^k = \bigvee_{i \in N} p'^k \in S_k \quad (k = 1, 2), \quad \hat{p} = \hat{p}^1 + \hat{p}^2 \in S,
$$
$$
\hat{q}^k = \bigvee_{i \in N} q'^k \in S_k \quad (k = 1, 2), \quad \hat{q} = \hat{q}^1 + \hat{q}^2 \in S.
$$

Then, for each component $i \in N$, we have

$$
\hat{p}_i = \hat{p}^1_i + \hat{p}^2_i \leq p'^1_i + p'^2_i = p_i \leq x_i,
$$
$$
\hat{q}_i = \hat{q}^1_i + \hat{q}^2_i \geq q'^1_i + q'^2_i = q_i \geq x_i,
$$

which shows $x \in [\hat{p}, \hat{q}]_\mathbb{Z}$. By Proposition 5.6 it follows from $\hat{p}, \hat{q} \in S$ and $\hat{p} \leq \hat{q}$ that $[\hat{p}, \hat{q}]_\mathbb{Z} \subseteq S$. Therefore, $x \in [\hat{p}, \hat{q}]_\mathbb{Z} \subseteq S$, where $x$ is an arbitrarily chosen element of $[a, b]_\mathbb{Z}$. Hence $[a, b]_\mathbb{Z} \subseteq S$. Thus we complete the proof of $S = [a, b]_\mathbb{Z}$.

(2) The statement (2) for functions follows from (1) for sets by the general principle given in Proposition 2.3. Note that the assumption (2.19) holds for multimodularity and $L^\mathbb{Z}$-convexity.

The following statement, with $L^\mathbb{Z}$-convexity in place of $L^\mathbb{Z}_+$-convexity, is an immediate corollary of Theorem 5.7 since separable convex functions are $L^\mathbb{Z}$-convex.

**Theorem 5.8.**

1. A set $S (\subseteq \mathbb{Z}^n)$ is a box of integers if and only if it is both multimodular and $L^\mathbb{Z}$-convex.
2. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ is separable convex if and only if it is both multimodular and $L^\mathbb{Z}$-convex.

### 5.4 Multimodularity and $M^\mathbb{Z}$-convexity

#### 5.4.1 Theorem

For $a, b \in N = \{1, 2, \ldots, n\}$, we denote the set of integers between $a$ and $b$ by $I(a, b)$, that is,

$$
I(a, b) = \{i \in \mathbb{Z} \mid a \leq i \leq b\}.
$$

In particular, $I(a, b) = \emptyset$ if $a > b$. We consider an integer-valued function $r(a, b)$, defined for all $(a, b)$ with $1 \leq a \leq b \leq n$, satisfying the following conditions:

$$
r(a, a) \geq 0 \quad (a \in N),
$$
$$
\max\{r(a, b - 1), r(a + 1, b)\} \leq r(a, b) \quad (a < b),
$$
$$
r(a, b) \leq r(a, b - 1) + r(a + 1, b) - r(a + 1, b - 1) \quad (a < b).
$$

Recall the relation of $M^\mathbb{Z}$-convexity and polymatroids that the set of integer points of an integral polymatroid is precisely a bounded $M^\mathbb{Z}$-convex set containing the origin $0$ and contained in the nonnegative orthant. The following theorem characterizes an integral polymatroid that is also multimodular. The proof is given in Section 5.4.2.

**Theorem 5.9.** A bounded set $S$ containing the origin $0$ and contained in the nonnegative orthant ($0 \in S \subseteq \mathbb{Z}^n_+$) is both $M^\mathbb{Z}$-convex and multimodular if and only if it is described as

$$
S = \{x \in \mathbb{Z}^n_+ \mid x(I(a, b)) \leq r(a, b) \quad (1 \leq a \leq b \leq n)\}
$$

with a function $r$ satisfying (5.13)–(5.15).
Remark 5.1. The function $r$ can be interpreted as a set function $\rho$ defined for consecutive subsets of $N$ by $\rho(I(a,b)) = r(a,b)$. The conditions (5.13), (5.14), and (5.15) correspond, respectively, to nonnegativity, monotonicity, and submodularity of $\rho$ (see Section 5.4.2 for the precise meaning). Every function $r$ satisfying (5.13)–(5.15) can be constructed as follows. First assign arbitrary nonnegative integers to $r(a,a)$ for $a \in N$. Next, for each $a \in N - \{n\}$, assign to $r(a,a+1)$ an arbitrary integer between $\max\{r(a,a), r(a+1,a+1)\}$ and $r(a,a) + r(a+1,a+1)$. Then, for $k = 2, 3, \ldots, n - 1$, define $r(a,b)$ with $b - a = k$ that satisfy (5.14) and (5.15), which is possible because the right-hand side of (5.15) is not smaller than the left-hand side of (5.14).

It follows from Theorem 5.9 with Theorem 5.4 that, if a bounded set $S$ is $M^2$-convex and multimodular, then the set of its maximal elements $S_{\text{max}}$ is $M$-convex and multimodular (Example 5.3). However, the converse is not true (Example 5.4).

Example 5.3. Let $S$ be the set of integer points of the polymatroid in Fig. 1(a). This set can be described in the form of (5.16) as

$$S = \{x \in \mathbb{Z}^3_+ \mid x_i \leq 3 \ (i = 1, 2, 3), \ x_1 + x_2 \leq 5, \ x_2 + x_3 \leq 5, \ x_1 + x_2 + x_3 \leq 6\},$$

which does not involve an inequality for $x_1 + x_3$ corresponding to a non-consecutive subset $\{1, 3\}$. By Theorem 5.9, $S$ is both $M^3$-convex and multimodular. The set of the maximal elements

$$S_{\text{max}} = \{(3,0,3), (1,2,3), (3,2,1), (1,3,2), (2,3,1), (2,1,3), (2,2,2), (3,1,2)\}$$

is both $M$-convex and multimodular.

Example 5.4. Let $S$ be the set of integer points of the polymatroid in Fig. 1(b). This set can be described as

$$S = \{x \in \mathbb{Z}^3 \mid x_i \leq 3 \ (i = 1, 2, 3), \ x_1 + x_2 \leq 5, \ x_1 + x_3 \leq 5, \ x_2 + x_3 \leq 5, \ x_1 + x_2 + x_3 \leq 6\}. \quad (5.17)$$

Since this expression involves a (non-redundant) inequality for $x_1 + x_3$ corresponding to a non-consecutive subset $\{1, 3\}$, Theorem 5.9 (or Theorem 5.4) shows that $S$ is not multimodular. While $S$ is not multimodular, the set of its maximal elements

$$S_{\text{max}} = \{x \in \mathbb{Z}^3 \mid x_i \leq 3 \ (i = 1, 2, 3), \ x_1 + x_2 \leq 5, \ x_1 + x_3 \leq 5, \ x_2 + x_3 \leq 5, \ x_1 + x_2 + x_3 = 6\}$$

$$= \{x \in \mathbb{Z}^3 \mid 1 \leq x_i \leq 3 \ (i = 1, 2, 3), \ x_1 + x_2 + x_3 = 6\}$$

is multimodular by Theorem 5.4. Thus the set $S_{\text{max}}$ is both $M$-convex and multimodular in spite of the fact that $S$ itself is $M^2$-convex and not multimodular.

Example 5.5. Here is an example of an $M$-convex set that is not multimodular. Using the set $S$ in (5.17), consider

$$\hat{S} = \{x \in \mathbb{Z}^4 \mid (x_1, x_2, x_3) \in S, \ x_4 = 6 - (x_1 + x_2 + x_3)\},$$

which is the $M$-convex set treated in Example 3.4. Here we show that $\hat{S}$ is not multimodular. To use Proposition 5.2, consider the set $\hat{T} = \{D^{-1}x \mid x \in \hat{S}\}$. Discrete midpoint convexity fails for $y = (2, 2, 5, 6) \in \hat{T}$ (corresponding to (2,0,3,1) $\in \hat{S}$) and $z = (3, 4, 6, 6) \in \hat{T}$ (corresponding to (3,1,2,0) $\in \hat{S}$) with $(y+z)/2 = (5/2, 3, 11/2, 6)$ and $[(y+z)/2] = (3, 3, 6, 6)$, where $w = (3, 3, 6, 6)$ is not contained in $\hat{T}$ since the corresponding vector $Dw = (3, 0, 3, 0)$ is not in $\hat{S}$.
Finally, we make an observation for the case of three variables.

**Proposition 5.10.**
(1) When \( n = 3 \), every M-convex set is multimodular.
(2) When \( n = 3 \), every M-convex function is multimodular.

**Proof.** (1) Let \( S \) be an M-convex set in \( \mathbb{Z}^3 \). Then \( S \) is described (cf., (A.36)) as
\[
S = \{ x \in \mathbb{Z}^3 \mid x(X) \leq \rho(X) \ (X \subseteq N), \ x_1 + x_2 + x_3 = \rho(N) \},
\]
where \( N = \{1, 2, 3\} \). Among the subsets of \( N \), \( X = \{1, 3\} \) is the only non-consecutive subset. With the use of the equality constraint \( x_1 + x_2 + x_3 = \rho(N) \), the corresponding inequality \( x_1 + x_3 \leq \rho(\{1, 3\}) \) can be rewritten as \( x_2 \geq \rho(N) - \rho(\{1, 3\}) \), which is an inequality constraint for a consecutive subset \( \{2\} \). Then \( S \) is multimodular by Theorem 5.4.

(2) By Theorem 5.3, the statement (2) for functions follows from (1) for sets. \( \square \)

### 5.4.2 Proof

This section is devoted to the proof of Theorem 5.9.

For the proof of the if-part, suppose that \( S \) is described as (5.16) with \( r \) satisfying (5.13)–(5.15), where \( S \) is nonempty since \( 0 \in S \). Then \( S \) is multimodular by Theorem 5.4. To discuss \( M^r \)-convexity, we define a set function \( \rho \) from the given \( r \). We represent a subset \( X \) of \( N \) as a union of disjoint consecutive subsets:
\[
X = I(a_1, b_1) \cup I(a_2, b_2) \cup \cdots \cup I(a_m, b_m)
\]
with \( a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_m \leq b_m \), and define \( \rho(X) \) by
\[
\rho(X) = r(a_1, b_1) + r(a_2, b_2) + \cdots + r(a_m, b_m), \tag{5.18}
\]
where \( \rho(0) = 0 \). Then we have
\[
\rho(X) = \rho(I(a_1, b_1)) + \rho(I(a_2, b_2)) + \cdots + \rho(I(a_m, b_m)). \tag{5.19}
\]
For any \( x \in S \) and \( X \subseteq N \), we have
\[
x(X) = \sum_{j=1}^{m} x(I(a_j, b_j)) \leq \sum_{j=1}^{m} r(a_j, b_j) = \rho(X),
\]
which shows that inequalities \( x(X) \leq \rho(X) \) for non-consecutive subsets \( X \) may be added to the expression in (5.16). Therefore we have
\[
S = \{ x \in \mathbb{Z}^n_+ \mid x(X) \leq \rho(X) \ (X \subseteq N) \}.
\]

By the construction, \( \rho \) is monotone (nondecreasing) with \( \rho(0) = 0 \). Moreover, \( \rho \) is submodular, which is shown in Lemma 5.11 at the end of this section. Therefore, by (A.32), \( S \) is an \( M^r \)-convex set. This completes the proof of the if-part of Theorem 5.9.

For the proof of the only-if part, let \( S \) be a bounded \( M^r \)-convex and multimodular set satisfying \( 0 \in S \subseteq \mathbb{Z}^n_+ \). By (A.32), for an \( M^r \)-convex set, we have
\[
S = \{ x \in \mathbb{Z}^n_+ \mid x(X) \leq \rho(X) \ (X \subseteq N) \}
\]

with a nondecreasing integer-valued submodular function \( \rho : 2^N \to \mathbb{Z} \) with \( \rho(\emptyset) = 0 \), where we may assume that \( \rho \) is tight in the sense of \( \rho(X) = \max \{ x(X) \mid x \in S \} \) \((X \subseteq N)\). Since \( S \) is multimodular, every inequality \( x(X) \leq \rho(X) \) for a non-consecutive subset \( X \) must be redundant by Theorem 5.4. Hence we have

\[
S = \{ x \in \mathbb{Z}_+^N \mid x(I) \leq \rho(I) \ (I: \text{consecutive subset of } N) \}.
\]

This coincides with the expression (5.16) for \( r(a, b) \) defined by \( r(a, b) = \rho(I(a, b)) \). Note that \( r(a, b) \) satisfies (5.13)–(5.15) by the properties (A.33)–(A.35) of \( \rho \). This completes the proof of the only-if-part of Theorem 5.9.

Finally we show the submodularity of \( \rho \) in (5.18).

**Lemma 5.11.** \( \rho \) in (5.18) is submodular.

**Proof.** For any subset \( Z \) of \( N \), we consider submodularity of \( \rho \) restricted to subsets of \( Z \):

\[
\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) \quad (X, Y \subseteq Z). \tag{5.20}
\]

We prove this by induction on \( \alpha = |Z| \).

Obviously, (5.20) is true when \( |Z| = 1 \). When \( |Z| = 2 \), we have two cases, depending on whether \( Z = \{ a, b \} \) is consecutive or not. If \( a + 1 = b \), we may assume \( X = \{ a \} \) and \( Y = \{ a + 1 \} \), for which (5.20) follows from (5.15) since \( \rho(X) = r(a, a) \), \( \rho(Y) = r(a + 1, a + 1) \), \( \rho(X \cup Y) = r(a, a + 1) \), and \( \rho(X \cap Y) = 0 \). If \( a + 1 < b \), we may assume \( X = \{ a \} \) and \( Y = \{ b \} \), for which (5.20) holds with equality by (5.19) and \( \rho(X \cap Y) = 0 \).

Now assume \( \alpha = |Z| \geq 3 \). To prove (5.20) under the induction hypothesis, it suffices to show

\[
\rho(Z) + \rho(Z - \{ p, q \}) - \rho(Z - p) - \rho(Z - q) \leq 0 \quad (p < q; p, q \in Z), \tag{5.21}
\]

where \( \rho(Z - p) \) means \( \rho(Z \setminus \{ p \}) \), etc. We divide the subsequent argument into three cases.

**Case (a):** \( Z \) is a consecutive set; **Case (b):** \( Z = I_1 \cup I_2 \cup \cdots \cup I_m \) with consecutive sets \( I_1, I_2, \ldots, I_m \) \((m \geq 2)\), and \( p, q \in I_k \) for some \( k \); **Case (c):** \( Z = I_1 \cup I_2 \cup \cdots \cup I_m \) as in (b), but \( p \in I_l \) and \( q \in I_l \) for some \( k \neq l \). It turns out that Case (a) is the essential case.

**Case (a):** Let \( Z = I(a, b) \). We have \( a \leq p < q \leq b \). Define

\[
I = \{ i \in Z \mid i < p \}, \quad J = \{ i \in Z \mid p < i < q \}, \quad K = \{ i \in Z \mid i > q \}.
\]

We have

\[
\rho(Z) \leq \rho(Z - a) + \rho(Z - b) - \rho(Z - \{ a, b \})
\]

by (5.15), whereas (5.19) gives

\[
\rho(Z - \{ p, q \}) = \rho(I) + \rho(J) + \rho(K),
\]

\[
\rho(Z - p) = \rho(I) + \rho(JqK),
\]

\[
\rho(Z - q) = \rho(IpJ) + \rho(K),
\]

and the result follows.
where \(\rho(JqK)\) means \(\rho(J \cup \{q\} \cup K)\), etc. Substituting these into the left-hand side of (5.21), we obtain

LHS of (5.21)

\[
\leq \rho(Z - a) + \rho(Z - b) - \rho(Z - (a, b)) + \rho(J) - \rho(JqK) - \rho(1pJ)
\]

\[
= [\rho(Z - a) + \rho(JqK - b) - \rho(Z - (a, b)) - \rho(1pJ)]
\]

\[
+ [\rho(Z - b) + \rho(J) - \rho(1pJ) - \rho(JqK - b)]
\]

\[
= [\rho(X' \cup Y') + \rho(X' \cap Y') - \rho(X') - \rho(Y')]
\]

\[
+ [\rho(X'' \cup Y'') + \rho(X'' \cap Y'') - \rho(X'') - \rho(Y'')],
\]

(5.22)

where \(X' := Z - \{a, b\}\) and \(Y' := JqK\), for which \(Z - a = X' \cup Y'\) and \(JqK - b = X' \cap Y'\), and \(X'' := 1pJ\) and \(Y'' := JqK - b (= X' \cap Y')\), for which \(Z - b = X'' \cup Y''\) and \(J = X'' \cap Y''\). Since \(|Z - a| = |Z - b| = \alpha - 1\), (5.22) is non-positive (\(\leq 0\)) by the induction hypothesis (5.20) for \(\alpha - 1\).

Case (b): By (5.19) we have

\[
\rho(Z) = \rho(I_k) + \sum_{j \neq k} \rho(I_j), \quad \rho(Z - (p, q)) = \rho(I_k - (p, q)) + \sum_{j \neq k} \rho(I_j),
\]

\[
\rho(Z - p) = \rho(I_k - p) + \sum_{j \neq k} \rho(I_j), \quad \rho(Z - q) = \rho(I_k - q) + \sum_{j \neq k} \rho(I_j).
\]

Therefore,

LHS of (5.21) = \(\rho(I_k) + \rho(I_k - (p, q)) - \rho(I_k - p) - \rho(I_k - q)\),

which is non-positive (\(\leq 0\)) by the induction hypothesis (5.20) since \(|I_k| \leq \alpha - 1\).

Case (c): By (5.19) we have

\[
\rho(Z) = \rho(I_k) + \rho(I_l) + \sum_{j \neq k, l} \rho(I_j), \quad \rho(Z - (p, q)) = \rho(I_k - p) + \rho(I_l - q) + \sum_{j \neq k, l} \rho(I_j),
\]

\[
\rho(Z - p) = \rho(I_k - p) + \rho(I_l) + \sum_{j \neq k, l} \rho(I_j), \quad \rho(Z - q) = \rho(I_k) + \rho(I_l - q) + \sum_{j \neq k, l} \rho(I_j).
\]

Therefore, LHS of (5.21) = 0.

Thus we have shown (5.21) in all cases (a), (b), and (c). Therefore, (5.20) holds when \(|Z| = \alpha\).

\[\square\]

### A Definitions of Discrete Convex Functions

This section offers definitions of various concepts of discrete convex functions such as integrally convex, L-convex, and M-convex functions. The definition of multimodular functions is given in Section 5.1. We consider functions defined on integer lattice points, \(f : \mathbb{Z}^n \to \mathbb{R} \cup [+\infty]\), where the function may possibly take \(+\infty\) but it is assumed that the effective domain, \(\text{dom } f = \{x \mid f(x) < +\infty\}\), is nonempty.

#### A.1 Separable convexity

For integer vectors \(a \in (\mathbb{Z} \cup [-\infty])^n\) and \(b \in (\mathbb{Z} \cup [+\infty])^n\) with \(a \leq b\), \([a, b]_\mathbb{Z}\) denotes the box of integers (discrete rectangle, integer interval) between \(a\) and \(b\), i.e.,

\[
[a, b]_\mathbb{Z} = \{x \in \mathbb{Z}^n \mid a_i \leq x_i \leq b_i \ (i = 1, 2, \ldots, n)\}.
\]
A function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) in \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{Z}^n \) is called separable convex if it can be represented as
\[
f(x) = \varphi_1(x_1) + \varphi_2(x_2) + \cdots + \varphi_n(x_n)
\] (A.2)
with univariate functions \( \varphi_i : \mathbb{Z} \to \mathbb{R} \cup \{+\infty\} \) satisfying
\[
\varphi_i(t - 1) + \varphi_i(t + 1) \geq 2\varphi_i(t) \quad (t \in \mathbb{Z}),
\] (A.3)
where \( \text{dom} \varphi_i \) is an interval of integers.

### A.2 Integral convexity

For \( x \in \mathbb{R}^n \) the integral neighborhood of \( x \) is defined as
\[
N(x) = \{z \in \mathbb{Z}^n \mid |x_i - z_i| < 1 \ (i = 1, 2, \ldots, n)\}.
\] (A.4)
It is noted that strict inequality “\(<\)” is used in this definition and hence \( N(x) \) admits an alternative expression
\[
N(x) = \{z \in \mathbb{Z}^n \mid x_i \leq z_i \leq \lfloor x_i \rfloor \ (i = 1, 2, \ldots, n)\}.
\] (A.5)

For a set \( S \subseteq \mathbb{Z}^n \) and \( x \in \mathbb{R}^n \) we call the convex hull of \( S \cap N(x) \) the local convex hull of \( S \) at \( x \). A nonempty set \( S \subseteq \mathbb{Z}^n \) is said to be integrally convex if the union of the local convex hulls \( S \cap N(x) \) over \( x \in \mathbb{R}^n \) is convex \([16]\). This is equivalent to saying that, for any \( x \in \mathbb{R}^n \), \( x \in S \) implies \( x \in S \cap N(x) \).

For a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) the local convex extension \( \tilde{f} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) of \( f \) is defined as the union of all convex envelopes of \( f \) on \( N(x) \). That is,
\[
\tilde{f}(x) = \min \left\{ \sum_{y \in N(x)} \lambda_y f(y) \mid \sum_{y \in N(x)} \lambda_y y = x, \ (\lambda_y) \in \Lambda(x) \right\} \quad (x \in \mathbb{R}^n),
\] (A.6)
where \( \Lambda(x) \) denotes the set of coefficients for convex combinations indexed by \( N(x) \):
\[
\Lambda(x) = \{ (\lambda_y \mid y \in N(x)) \mid \sum_{y \in N(x)} \lambda_y = 1, \lambda_y \geq 0 \text{ for all } y \in N(x) \}.
\]

If \( \tilde{f} \) is convex on \( \mathbb{R}^n \), then \( f \) is said to be integrally convex \([5]\). The effective domain of an integrally convex function is an integrally convex set. A set \( S \subseteq \mathbb{Z}^n \) is integrally convex if and only if its indicator function \( \delta_S : \mathbb{Z}^n \to \{0, +\infty\} \) is an integrally convex function.

Integral convexity of a function can be characterized by a local condition under the assumption that the effective domain is an integrally convex set.

**Theorem A.1** (\([5] [12]\)). Let \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) be a function with an integrally convex effective domain. Then the following properties are equivalent:

(a) \( f \) is integrally convex.

(b) For every \( x, y \in \mathbb{Z}^n \) with \( \|x - y\|_\infty = 2 \) we have
\[
\tilde{f} \left( \frac{x + y}{2} \right) \leq \frac{1}{2} (f(x) + f(y)).
\] (A.7)

The reader is referred to \([5] [10] [12] [26] [27] [28], [16] Section 3.4, and [20] Section 13\) for more about integral convexity.
A.3 L-convexity

L- and $L^\#$-convex functions form major classes of discrete convex functions [16, Chapter 7]. The concept of $L^\#$-convex functions was introduced in [7] as an equivalent variant of L-convex functions introduced earlier in [15].

A.3.1 $L^\#$-convex functions

A nonempty set $S \subseteq \mathbb{Z}^n$ is called $L^\#$-convex if

$$x, y \in S \implies \left\lfloor \frac{x + y}{2} \right\rfloor, \left\lceil \frac{x + y}{2} \right\rceil \in S,$$  \hspace{1cm} (A.8)

where, for $t \in \mathbb{R}$ in general, $\left\lfloor t \right\rfloor$ denotes the smallest integer not smaller than $t$ (rounding-up to the nearest integer) and $\left\lceil t \right\rceil$ the largest integer not larger than $t$ (rounding-down to the nearest integer), and this operation is extended to a vector by componentwise applications.

The property (A.8) is called discrete midpoint convexity (in the original sense of the word).

A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$ with $\text{dom } f \neq \emptyset$ is said to be $L^\#$-convex if it satisfies a quantitative version of discrete midpoint convexity, i.e., if

$$f(x) + f(y) \geq f\left(\left\lfloor \frac{x + y}{2} \right\rfloor\right) + f\left(\left\lceil \frac{x + y}{2} \right\rceil\right)$$  \hspace{1cm} (A.9)

holds for all $x, y \in \mathbb{Z}^n$. The effective domain of an $L^\#$-convex function is an $L^\#$-convex set. A set $S$ is $L^\#$-convex if and only if its indicator function $\delta_S$ is an $L^\#$-convex function.

It is known [16, Section 7.1] that $L^\#$-convex functions can be characterized by several different conditions, stated in Theorem A.2 below. The condition (b) in Theorem A.2 imposes discrete midpoint convexity (A.9) for all points $x, y$ at $\ell_\infty$-distance 1 or 2. The condition (c) refers to submodularity, which means that

$$f(x) + f(y) \geq f(x \lor y) + f(x \land y)$$  \hspace{1cm} (A.10)

holds for all $x, y \in \mathbb{Z}^n$, where $x \lor y$ and $x \land y$ denote, respectively, the vectors of componentwise maximum and minimum of $x$ and $y$; see (2.1). The condition (d) refers to a generalization of submodularity called translation-submodularity, which means that

$$f(x) + f(y) \geq f((x - \mu \mathbf{1}) \lor y) + f(x \land (y + \mu \mathbf{1}))$$  \hspace{1cm} (A.11)

holds for all $x, y \in \mathbb{Z}^n$ and nonnegative integers $\mu$, where $\mathbf{1} = (1, 1, \ldots, 1)$. The condition (e) refers to the condition that, for any $x, y \in \mathbb{Z}^n$ with $\text{supp}^+(x - y) \neq \emptyset$, the inequality

$$f(x) + f(y) \geq f(x - e^A) + f(y + e^A)$$  \hspace{1cm} (A.12)

holds with $A = \arg \max_i \{x_i - y_i\}$, where $\text{supp}^+(x - y) = \{i \mid x_i > y_i\}$ and $e^A$ denotes the characteristic vector of $A$. The condition (f) refers to submodularity of the function

$$\tilde{f}(x_0, x) = f(x - x_0 \mathbf{1}) \quad (x_0 \in \mathbb{Z}, x \in \mathbb{Z}^n)$$  \hspace{1cm} (A.13)

in $n + 1$ variables associated with the given function $f$.

---

1This condition (A.12) is labeled as (L$^\#$-APR[\mathbb{Z}]) in [16, Section 7.2].
Theorem A.2 ([21, Theorem 2.2]). For $f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\}$, the following conditions, (a) to (f), are equivalent:

(a) $f$ is an $L^3$-convex function, that is, it satisfies discrete midpoint inequality (A.9) for all $x, y \in \mathbb{Z}^n$.

(b) dom $f$ is an $L^2$-convex set, and $f$ satisfies discrete midpoint inequality (A.9) for all $x, y \in \mathbb{Z}^n$ with $\|x - y\|_\infty \leq 2$.

(c) $f$ is integrally convex and submodular (A.10).

(d) $f$ satisfies translation-submodularity (A.11) for all nonnegative $\mu \in \mathbb{Z}$.

(e) $f$ satisfies the condition (A.12).

(f) $\tilde{f}$ in (A.13) is submodular (A.10).

For a set $S \subseteq \mathbb{Z}^n$ we consider conditions

$$x, y \in S \implies x \lor y, x \land y \in S, \quad \text{ (A.14)}$$

$$x, y \in S \implies (x - \mu \mathbf{1}) \lor y, x \land (y + \mu \mathbf{1}) \in S, \quad \text{ (A.15)}$$

$$x, y \in S, \supp^+(x - y) \neq \emptyset \implies x - e^i, y + e^i \in S \text{ for } A = \arg \max_i \{x_i - y_i\}. \quad \text{ (A.16)}$$

The first condition (A.14), meaning that $S$ is a sublattice of $\mathbb{Z}^n$, corresponds to submodularity (A.10), whereas (A.15) and (A.16) correspond to (A.11) and (A.12), respectively.

Theorem A.3 ([21, Proposition 2.3]). For a nonempty set $S \subseteq \mathbb{Z}^n$, the following conditions, (a) to (d), are equivalent:

(a) $S$ is an $L^3$-convex set, that is, it satisfies (A.8).

(b) $S$ is an integrally convex set that satisfies (A.14).

(c) $S$ satisfies (A.15) for all nonnegative $\mu \in \mathbb{Z}$.

(d) $S$ satisfies (A.16).

The following polyhedral description of an $L^k$-convex set is known [16, Section 5.5].

Theorem A.4. A set $S \subseteq \mathbb{Z}^n$ is $L^3$-convex if and only if $S = \overline{S} \cap \mathbb{Z}^n$ and its convex hull $\overline{S}$ can be represented as

$$\overline{S} = \{ x \in \mathbb{R}^n \mid \alpha_i \leq x_i \leq \beta_i \ (i \in N), \ x_j - x_i \leq \gamma_{ij} \ (i, j \in N) \}$$

for some $\alpha_i \in \mathbb{Z} \cup \{-\infty\}$, $\beta_i \in \mathbb{Z} \cup \{+\infty\}$, and $\gamma_{ij} \in \mathbb{Z} \cup \{+\infty\}$ ($i, j \in N$) with $\gamma_{ii} = 0$ ($i \in N$) such that $\tilde{\gamma}_{ij}$ defined for $i, j \in N \cup \{0\}$ by

$$\tilde{\gamma}_{00} = 0, \quad \tilde{\gamma}_{ij} = \gamma_{ij}, \quad \tilde{\gamma}_{0j} = -\alpha_i, \quad \tilde{\gamma}_{ij} = \beta_j \quad (i, j \in N)$$

satisfies the triangle inequality:

$$\tilde{\gamma}_{ij} + \tilde{\gamma}_{jk} \geq \tilde{\gamma}_{ik} \quad (i, j, k \in N \cup \{0\}). \quad \text{ (A.19)}$$

Such $\alpha_i, \beta_i, \gamma_{ij}$ are determined from $S$ by

$$\alpha_i = \min \{ x_i \mid x \in S \}, \quad \beta_i = \max \{ x_i \mid x \in S \} \quad (i \in N),$$

$$\gamma_{ij} = \max \{ x_j - x_i \mid x \in S \} \quad (i, j \in N). \quad \text{ (A.20)}$$

Remark A.1. Here are additional remarks about the polyhedral descriptions in Theorem A.4.
• The correspondence between \( S \) and integer-valued \((\alpha, \beta, \gamma)\) with (A.19) is bijective (one-to-one and onto) through (A.17), (A.20), and (A.21).

• For any integer-valued \((\alpha, \beta, \gamma)\) (independent of the triangle inequality), \( S \) in (A.17) is an \( L^3 \)-convex set if \( S \neq \emptyset \). We have \( S \neq \emptyset \) if and only if there exists no negative cycle with respect to \( \tilde{\gamma} \), where a negative cycle means a set of indices \( i_1, i_2, \ldots, i_m \) such that

\[
\tilde{\gamma}_{i_1 i_2} + \tilde{\gamma}_{i_2 i_3} + \cdots + \tilde{\gamma}_{i_m i_1} + \tilde{\gamma}_{i_m i_1} < 0.
\]

### A.3.2 \( L \)-convex functions

A function \( f(x_1, x_2, \ldots, x_n) \) with \( \text{dom} \ f \neq \emptyset \) is called \( L \)-convex if it is submodular (A.10) and there exists \( r \in \mathbb{R} \) such that

\[
f(x + \mu \mathbf{1}) = f(x) + \mu r \quad \text{(A.22)}
\]

for all \( x \in \mathbb{Z}^n \) and \( \mu \in \mathbb{Z} \). If \( f \) is \( L \)-convex, the function \( g(x_2, \ldots, x_n) := f(0, x_2, \ldots, x_n) \) is an \( L^5 \)-convex function, and any \( L^5 \)-convex function arises in this way. The function \( \tilde{f} \) in (A.13) derived from an \( L^5 \)-convex function \( f \) is an \( L \)-convex function with \( \tilde{f}(x_0 + \mu, x + \mu \mathbf{1}) = \tilde{f}(x_0, x) \), and we have \( f(x) = \tilde{f}(0, x) \).

**Theorem A.5** ([21, Theorem 2.4]). For \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \), the following conditions, (a) to (c), are equivalent:

(a) \( f \) is an \( L \)-convex function, that is, it satisfies (A.10) and (A.22) for some \( r \in \mathbb{R} \).

(b) \( f \) is an \( L^5 \)-convex function that satisfies (A.22) for some \( r \in \mathbb{R} \).

(c) \( f \) satisfies translation-submodularity (A.11) for all \( \mu \in \mathbb{Z} \) (including \( \mu < 0 \)).

A nonempty set \( S \) is called \( L \)-convex if its indicator function \( \delta_S \) is an \( L \)-convex function. This means that \( S \) is \( L \)-convex if and only if it satisfies (A.14) and

\[
x \in S, \ \mu \in \mathbb{Z} \implies x + \mu \mathbf{1} \in S. \quad \text{(A.23)}
\]

This property is sometimes called the translation invarian ce in the direction of \( \mathbf{1} \). The effective domain of an \( L \)-convex function is an \( L \)-convex set.

The following theorem gives equivalent conditions for a set to be \( L \)-convex.

**Theorem A.6** ([21, Proposition 2.5]). For a nonempty set \( S \subseteq \mathbb{Z}^n \), the following conditions, (a) to (c), are equivalent:

(a) \( S \) is an \( L \)-convex set, that is, it satisfies (A.14) and (A.23).

(b) \( S \) is an \( L^5 \)-convex set that satisfies (A.23).

(c) \( S \) satisfies (A.15) for all \( \mu \in \mathbb{Z} \) (including \( \mu < 0 \)).

The following theorem reduces the concept of \( L \)-convex functions to that of \( L \)-convex sets.

**Theorem A.7** ([6, Section 16.2], [16, Theorem 7.17]). Under the assumption (2.17), a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is \( L \)-convex (resp., \( L^5 \)-convex) if and only if, for any vector \( p \in \mathbb{R}^n \), arg min \( f[-p] \) is an \( L \)-convex (resp., \( L^5 \)-convex) set or an empty set.
A.3.3 Discrete midpoint convex functions

A nonempty set \( S \subseteq \mathbb{Z}^n \) is said to be discrete midpoint convex if

\[
x, y \in S, \quad \|x - y\|_\infty \geq 2 \implies \left\lfloor \frac{x + y}{2} \right\rfloor, \left\lceil \frac{x + y}{2} \right\rceil \in S.
\]  

(A.24)

This condition is weaker than the defining condition (A.8) for an \( L^\& \)-convex set, and hence every \( L^\& \)-convex set is a discrete midpoint convex set.

A function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is called globally discrete midpoint convex \([13]\) if the discrete midpoint convexity

\[
f(x) + f(y) \geq f\left(\left\lfloor \frac{x + y}{2} \right\rfloor\right) + f\left(\left\lceil \frac{x + y}{2} \right\rceil\right)
\]  

(A.25)

is satisfied by every pair \((x, y)\) with \( \|x - y\|_\infty \geq 2 \). The effective domain of a globally discrete midpoint convex function is necessarily a discrete midpoint convex set. Obviously, every \( L^\& \)-convex function is globally discrete midpoint convex. We sometimes abbreviate “discrete midpoint convex(ity)” to “d.m.c.”

We define notation \( \tilde{\mu}(a, b) \) for a pair of integers \((a, b)\) by

\[
\tilde{\mu}(a, b) = \begin{cases} 
\left\lfloor \frac{(a + b)/2}{2} \right\rfloor & (a \geq b), \\
\left\lceil \frac{(a + b)/2}{2} \right\rceil & (a \leq b),
\end{cases}
\]  

(A.26)

and extend this notation to a pair of integer vectors \((x, y)\) as

\[
\tilde{\mu}(x, y) = (\tilde{\mu}(x_1, y_1), \tilde{\mu}(x_2, y_2), \ldots, \tilde{\mu}(x_n, y_n)).
\]  

(A.27)

A function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is called directed discrete midpoint convex \([35]\) if the inequality

\[
f(x) + f(y) \geq f(\tilde{\mu}(x, y)) + f(\tilde{\mu}(y, x))
\]  

(A.28)

is satisfied by every pair \((x, y)\) in \( \mathbb{Z}^n \times \mathbb{Z}^n \). A nonempty set \( S \) is called directed discrete midpoint convex if its indicator function \( \delta_S \) is a directed discrete midpoint convex function. \( L^2 \)-convex functions are directed discrete midpoint convex functions, and \( L^\& \)-convex sets are directed discrete midpoint convex sets (see (2.14) in Proposition 2.1).

A.3.4 \( L^2 \)-convex functions

A nonempty set \( S \subseteq \mathbb{Z}^n \) is called \( L^2 \)-convex (resp., \( L^\& \)-convex) if it can be represented as the Minkowski sum (vector addition) of two \( L \)-convex (resp., \( L^\& \)-convex) sets \([16]\, \text{Section 5.5}\). That is,

\[
S = \{x + y \mid x \in S_1, y \in S_2\},
\]

where \( S_1 \) and \( S_2 \) are \( L \)-convex (resp., \( L^\& \)-convex) sets. An \( L^\& \)-convex set is the intersection of an \( L^2 \)-convex set with a coordinate hyperplane. That is, for \( L \)-convex sets, the operations of the Minkowski addition and the restriction to a coordinate hyperplane commute with each other. This fact is stated in \([16\, \text{p.129}]\) without a proof, and a proof can be found in \([11\, \text{Section 2.3}]\). The polyhedral description of \( L^2 \)-convex and \( L^\& \)-convex sets is given in \([11]\). An \( L^2 \)-convex set has the property of translation invariance in (A.23).
A function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be \( L_2\)-convex if it can be represented as the (integral) infimal convolution \( f_1 \triangle f_2 \) of two \( L \)-convex functions \( f_1 \) and \( f_2 \), that is, if

\[
f(x) = (f_1 \triangle f_2)(x) = \inf \{f_1(y) + f_2(z) \mid x = y + z; \ y, z \in \mathbb{Z}^n\} \quad (x \in \mathbb{Z}^n).
\]

It is known [34] (see also [16, Note 8.37]) that the infimum is always attained as long as it is finite. The (integer) infimal convolution of two \( L^3 \)-convex functions is called an \( L_3^2 \)-convex function. A nonempty set \( S \) is \( L_2 \)-convex (resp., \( L^3_2 \)-convex) if and only if its indicator function \( \delta_S \) is \( L_2 \)-convex (resp., \( L^3_2 \)-convex).

### A.4 M-convexity

M- and \( M^k \)-convex functions form major classes of discrete convex functions [16, Chapter 6]. The concept of \( M^k \)-convex functions was introduced in [22] as an equivalent variant of M-convex functions introduced earlier in [14].

#### A.4.1 \( M^k \)-convex functions

For two vectors \( x, y \in \mathbb{Z}^n \) we use notations

\[
\text{supp}^+(x - y) = \{i \mid x_i > y_i\}, \quad \text{supp}^-(x - y) = \{i \mid x_i < y_i\}.
\]

We say that a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) with \( \text{dom} f \neq \emptyset \) is \( M^k \)-convex, if, for any \( x, y \in \mathbb{Z}^n \) and \( i \in \text{supp}^+(x - y) \), we have (i)

\[
f(x) + f(y) \geq f(x - e^i) + f(y + e^i)
\]

or (ii) there exists some \( j \in \text{supp}^-(x - y) \) such that

\[
f(x) + f(y) \geq f(x - e^j + e^i) + f(y + e^j - e^i).
\]

This property is referred to as the exchange property. A more compact expression of the exchange property is as follows:

\[(M^k\text{-EXC}) \text{ For any } x, y \in \mathbb{Z}^n \text{ and } i \in \text{supp}^+(x - y), \text{ we have}
\]

\[
f(x) + f(y) \geq \min_{j \in \text{supp}^-(x - y) \cup \{0\}} \{f(x - e^j + e^i) + f(y + e^j - e^i)\},
\]

where \( e^0 = 0 \) (zero vector). In the above statement we may change “For any \( x, y \in \mathbb{Z}^n \)” to “For any \( x, y \in \text{dom } f \)” since if \( x \notin \text{dom } f \) or \( y \notin \text{dom } f \), the inequality (A.31) trivially holds with \( f(x) + f(y) = +\infty \).

\( M^k \)-convex functions can be characterized by a number of different exchange properties including a local exchange property under the assumption that function \( f \) is (effectively) defined on an \( M^k \)-convex set. See [32, Theorem 6.8], [16, Chapters 4 and 6], [20, Section 4], and [24] for detailed discussion about the exchange properties.

It follows from (\( M^k\)-EXC) that the effective domain \( S = \text{dom } f \) of an \( M^k \)-convex function \( f \) has the following exchange property:

\[(B^k\text{-EXC}) \text{ For any } x, y \in S \text{ and } i \in \text{supp}^+(x - y), \text{ we have (i) } x - e^i \in S \text{ and } y + e^i \in S \text{ or}
\]

(ii) there exists some \( j \in \text{supp}^-(x - y) \) such that \( x - e^j + e^i \in S \) and \( y + e^j - e^i \in S \).
A nonempty set $S \subseteq \mathbb{Z}^n$ having this property is called an $M^\circ$-convex set.

An $M^\circ$-convex set is nothing but the set of integer points in an integral generalized polymatroid [16, Section 4.7]. In particular, an integral polymatroid can be identified with a bounded $M^\circ$-convex set containing $0$ and consisting of nonnegative vectors. As is well known [6, Section 2.2], the set of integer points of an integral polymatroid is described as

$$S = \{ x \in \mathbb{Z}^n | x \leq 0, \, x(X) \leq \rho(X) (X \subseteq N) \} \quad (A.32)$$

with a nondecreasing integer-valued submodular function $\rho$. To be more specific, the set function $\rho : 2^N \to \mathbb{Z}$ should satisfy

$$\rho(0) = 0, \quad (A.33)$$

$$X \subseteq Y \implies \rho(X) \leq \rho(Y), \quad (A.34)$$

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y) \quad (X, Y \subseteq N). \quad (A.35)$$

Moreover, the convex hull $\overline{S}$ of $S$ is described similarly as $\overline{S} = \{ x \in \mathbb{R}^n | x \geq 0, \, x(X) \leq \rho(X) (X \subseteq N) \}$ and we can determine $\rho$ from $S$ by $\rho(X) = \max \{ x(X) | x \in S \} (X \subseteq N)$.

### A.4.2 M-convex functions

If a set $S \subseteq \mathbb{Z}^n$ lies on a hyperplane with a constant component sum (i.e., $x(N) = y(N)$ for all $x, y \in S$), the exchange property (B$^\circ$-EXC) takes a simpler form (without the possibility of the first case (i)):

**B-EXC** For any $x, y \in S$ and $i \in \text{supp}^+ (x - y)$, there exists some $j \in \text{supp}^- (x - y)$ such that $x - e^i + e^j \in S$ and $y + e^i - e^j \in S$.

A nonempty set $S \subseteq \mathbb{Z}^n$ having this exchange property is called an $M$-convex set, which is an alias for the set of integer points in an integral base polyhedron. An $M$-convex set $S$ contained in the nonnegative orthant $\mathbb{Z}^n_+$ can be described as

$$S = \{ x \in \mathbb{Z}^n | x(X) \leq \rho(X) (X \subseteq N), \, x(N) = \rho(N) \} \quad (A.36)$$

with a set function $\rho : 2^N \to \mathbb{Z}$ satisfying (A.33), (A.34), and (A.35).

An $M^\circ$-convex function whose effective domain is an $M$-convex set is called an $M$-convex function [14] [15] [16]. In other words, a function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{ +\infty \}$ is $M$-convex if and only if it satisfies the exchange property:

**M-EXC** For any $x, y \in \text{dom} \, f$ and $i \in \text{supp}^+ \, (x - y)$, there exists $j \in \text{supp}^- \, (x - y)$ such that (A.30) holds.

M-convex functions can be characterized by a local exchange property under the assumption that function $f$ is (effectively) defined on an $M$-convex set. See [16, Section 6.2].

M-convex functions and $M^\circ$-convex functions are equivalent concepts, in that $M^\circ$-convex functions in $n$ variables can be obtained as projections of $M$-convex functions in $n + 1$ variables. More formally, let “0” denote a new element not in $N$ and $\tilde{N} = \{ 0 \} \cup N = \{ 0, 1, \ldots, n \}$. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{ +\infty \}$ is $M^\circ$-convex if and only if the function $\tilde{f} : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \{ +\infty \}$ defined by

$$\tilde{f}(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -x(N) \\ +\infty & \text{otherwise} \end{cases} \quad (x_0 \in \mathbb{Z}, \, x \in \mathbb{Z}^n) \quad (A.37)$$

is an $M$-convex function.

The following theorem reduces the concept of M-convex functions to that of $M$-convex sets.

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Theorem A.8 ([6, Section 17], [16, Theorem 6.30]). Under the assumption (2.17), a function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is \( M \)-convex (resp., \( M^{\natural} \)-convex) if and only if, for any vector \( p \in \mathbb{R}^n \), \( \arg \min f[-p] \) is an \( M \)-convex (resp., \( M^{\natural} \)-convex) set or an empty set.

A.4.3 \( M_2 \)-convex functions

A nonempty set \( S \subseteq \mathbb{Z}^n \) is called \( M_2 \)-convex if it can be represented as the intersection of two \( M \)-convex sets [16, Section 4.6]. Similarly, a nonempty set \( S \subseteq \mathbb{Z}^n \) is called \( M^2 \)-convex if it can be represented as the intersection of two \( M^2 \)-convex sets [16, Section 4.7]. An \( M_2 \)-convex set \( S \) lies on a hyperplane with a constant component sum (i.e., \( x(N) = y(N) \) for all \( x, y \in S \)).

A function \( f : \mathbb{Z}^n \to \mathbb{R} \cup \{+\infty\} \) is said to be \( M_2 \)-convex if it can be represented as the sum of two \( M \)-convex functions \( f_1 \) and \( f_2 \), that is, if

\[
f(x) = f_1(x) + f_2(x) \quad (x \in \mathbb{Z}^n).
\]

The sum of two \( M^2 \)-convex functions is called an \( M^2_2 \)-convex function. A nonempty set \( S \) is \( M_2 \)-convex (resp., \( M^2_2 \)-convex) if and only if its indicator function \( \delta_S \) is \( M_2 \)-convex (resp., \( M^2_2 \)-convex).

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