On the construction of fully interpreted formal languages which posses their truth predicates

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Abstract
We shall first construct by ordinary recursion method subsets to the set $D$ of Gödel numbers of the sentences of a language $L$. That language is formed by the sentences of a fully interpreted formal language $L$, called an MA language, and sentences containing a monadic predicate letter $T$. From the class of the constructed subsets of $D$ we extract one set $U$ by transfinite recursion method. Interpret those sentences whose Gödel numbers are in $U$ as true, and their negations as false. These sentences together form an MA language. It is a sublanguage of $L$ having $L$ as its sublanguage, and $T$ is its truth predicate.

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1 Introduction

In [4] a theory of truth is defined for certain sublanguages of a language $L$ which is the first order language $L = \{\in\}$ of set theory augmented by a monadic predicate $T$. The interpretation of $L$ is determined by the minimal model $M$ constructed in [2] for ZF set theory. This interpretation makes $L$ fully interpreted, i.e., its sentences are either true or false. The sublanguages for which a theory of truth is defined belong to a class of sublanguages of $L$. Languages of that class are denoted by $L_U$, where $U$ is a subset of the set $D$ of the Gödel numbers of sentences of $L$. The Gödel numbers of sentences of $L_U$ belong to the set $G(U) \cup F(U)$, where the subsets $G(U)$ and $F(U)$ of $D$ are told to satisfy the following rules ('iff' abbreviates 'if and only if'):

(r1) If $A$ is a sentence of $L$, then the Gödel number $\# A$ of $A$ is in $G(U)$ iff $A$ is true in the interpretation of $L$, and in $F(U)$ iff $A$ is false in the interpretation of $L$.

(r2) Let $n$ be a numeral. $\#T(n)$ is in $G(U)$ iff $n$ is the numeral $[A]$ of the Gödel number of a sentence $A$ of $L$ and $\#A$ is in $U$. $\#T(n)$ is in $F(U)$ iff $n = [A]$, where $A$ is a sentence of $L$ and $\#[-A]$ is in $U$.

In the next rules (r3)–(r7) $A$ and $B$ denote sentences of $L$.

(r3) Negation rule: $\#[-A]$ is in $G(U)$ iff $\#A$ is in $F(U)$, and in $F(U)$ iff $\#A$ is in $G(U)$.

(r4) Disjunction rule: $\#[A \lor B]$ is in $G(U)$ iff $\#A$ or $\#B$ is in $G(U)$, and in $F(U)$ iff $\#A$ and $\#B$ are in $F(U)$.

(r5) Conjunction rule: $\#[A \land B]$ is in $G(U)$ iff both $\#A$ and $\#B$ are in $G(U)$. $\#[A \land B]$ is in $F(U)$ iff $\#A$ or $\#B$ is in $F(U)$.

(r6) Implication rule: $\#[A \implies B]$ is in $G(U)$ iff $\#A$ is in $F(U)$ or $\#B$ is in $G(U)$. $\#[A \implies B]$ is in $F(U)$ iff $\#A$ is in $G(U)$ and $\#B$ is in $F(U)$.

(r7) Biconditionality rule: $\#[A \iff B]$ is in $G(U)$ iff $\#A$ and $\#B$ are both in $G(U)$ or both in $F(U)$, and in $F(U)$ iff $\#A$ is in $G(U)$ and $\#B$ is in $F(U)$ or $\#A$ is in $F(U)$ and $\#B$ is in $G(U)$.

Assuming that the set $X$ of numerals of Gödel numbers of sentences of $L$ is the intended domain of discourse for $T$, the following rules are presented for $\exists x T(x)$ and $\forall x T(x)$:

(r8) $\#[\exists x T(x)]$ is in $G(U)$ iff $\#T(n)$ is in $G(U)$ for some $n \in X$, and $\#[\exists x T(x)]$ is in $F(U)$ iff $\#T(n)$ is in $F(U)$ for every $n \in X$.

(r9) $\#[\forall x T(x)]$ is in $G(U)$ iff $\#T(n)$ is in $G(U)$ for every $n \in X$, and $\#[\forall x T(x)]$ is in $F(U)$ iff $\#T(n)$ is in $F(U)$ at least for one $n \in X$.

In [3] the above considerations are extended to the case when the language $L$ is assumed to be mathematically agreeable (shortly MA). By Chomsky’s definition (cf. [1]) a “language is a set (finite or infinite) of sentences of finite length, and constructed out of finite sets of symbols”. Allowing also countable sets of symbols, we say that $L$ is an MA language if it satisfies the following conditions.
(i) The syntax of $L$ contains a countable syntax of the first-order predicate logic with equality (cf., e.g., [6, II.5]), natural numbers in variables and their names, numerals in terms.

(ii) $L$ is fully interpreted.

(iii) Classical truth tables (cf. e.g., [6, p.3]) are valid for the logical connectives $\neg$, $\lor$, $\land$, $\to$ and $\leftrightarrow$ of sentences of $L$.

(iv) Classical rules of truth hold for $\forall x P(x)$ and $\exists x P(x)$ where $P$ is a predicate of $L$.

Any countable first-order formal language, equipped with a consistent theory interpreted by a countable model, and containing natural numbers and numerals, is an MA language. A classical example is the language of arithmetic with its standard model and interpretation.

Basic ingredients in the approach of [5] are:
1. An MA language $L$ (base language).
2. A monadic predicate letter $T$.
3. The language $\mathcal{L}$, which has sentences of $L$, $T(n)$, where $n$ is a numeral, $\forall x T(x)$ and $\exists x T(x)$ as its basic sentences, and which is closed under logical connectives $\neg$, $\lor$, $\land$, $\to$ and $\leftrightarrow$.
4. The set $D$ of Gödel numbers of sentences of $\mathcal{L}$ in its fixed Gödel numbering.

Neither in [4] nor in [5] the sets $G(U)$ and $F(U)$ satisfying rules (r1)–(r9) are shown to exist. Our main task is to construct sets $G(U)$ and $F(U)$, and prove that they satisfy rules (r1)–(r9), and the following rules when $n \in X$, and $\#\exists x \neg T(x)$ is in $F(U)$ if $\#T(n)$ is in $G(U)$ for every $n \in X$.

(r10) $\#\exists x \neg T(x)$ is in $G(U)$ iff $\#T(n)$ is in $F(U)$ for some $n \in X$, and $\#\exists x \neg T(x)$ is in $F(U)$ if $\#T(n)$ is in $G(U)$ for every $n \in X$.

Because of the recursive construction of sets $G(U)$ we revise proofs given in [4, 5] for properties of sets $G(U)$ and $F(U)$. Some of them are used in [4, Theorem 4.1] to prove by transfinite recursion method the existence of consistent fixed points $U$ of $G$, i.e., subsets $U$ of $D$ which satisfy $U = G(U)$, and for no sentence $A$ of $\mathcal{L}$ the Gödel numbers of both $A$ and $\neg A$ are in $U$.

Among them there is the smallest one which is contained in every consistent fixed point of $G$.

To the smallest consistent fixed point $U$ of $G$ there corresponds a sublanguage $\mathcal{L}_0$ of $\mathcal{L}$ which has $G(U) \cup F(U)$ as the set of Gödel numbers of its sentences. As in [4, 5], define an interpretation for sentences of $\mathcal{L}_0$ as follows. A sentence $A$ of $\mathcal{L}_0$ is interpreted as true if its Gödel number $\#A$ is in $G(U)$, and as false if $\#A$ is in $F(U)$. The so defined theory of truth for $\mathcal{L}_0$ is shown in [4, Theorem 3.1] to conform well with the eight norms presented for theories of truth in [7].

$T$ is called a truth predicate for $\mathcal{L}_0$, because $T$-biconditionality: $A \leftrightarrow T([A])$, is shown in [5, Lemma 4.1] to be true in that interpretation for all sentences $A$ of $\mathcal{L}_0$ ($T([A])$ stands for ‘$A$ is true’). $\mathcal{L}_0$ is fully interpreted because $G(U)$ and $F(U)$ are separated by Lemma 4.1.

$L$ is by rule (r1) a sublanguage of $\mathcal{L}_0$. Moreover, a sentence $A$ of $L$ is by [5, Lemma 4.2] true, respectively false, in the interpretation of $L$ if and only if $A$ is true, respectively false, in the interpretation of $\mathcal{L}_0$.

$\mathcal{L}_0$ is an MA language. By above results it satisfies properties (i) and (ii). Rules (r1)–(r11) and the above defined interpretation imply that properties (iii) and (iv) are valid by assuming that $T$ and $\neg T$ are the only predicates of $\mathcal{L}_0$ which are not predicates of $L$.

Main tools used in proofs are ZF set theory and classical logic.
2 Recursive construction of sets $G(U)$

Let $L$, $T$, $\mathcal{L}$ and $D$ be as in the Introduction, and let $W$ denote the set of Gödel numbers of all those sentences of $L$ which are true in the interpretation of $L$. Given a subset $U$ of $D$, denote

$$
D_1(U) = \{ \#T(n) : n = [A], \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#A \text{ is in } U \},
$$

$$
D_2(U) = \{ \#[-T(n)] : n = [A], \text{ where } A \text{ is a sentence of } \mathcal{L} \text{ and } \#[-A] \text{ is in } U \}.
$$

We shall construct subsets $G_n(U), n \in \mathbb{N}_0$, of $D$ recursively as follows. Define

$$
G_0(U) = \begin{cases}
W & \text{if } U = \emptyset, \\
W \cup D_1(U) \cup \{ \#[\exists x T(x)], \#[-\forall x \neg T(x)] \} & \text{if } \emptyset \subset U \subset D, \text{ and } D_2(U) = \emptyset, \\
W \cup D_1(U) \cup D_2(U) \cup \{ \#[\exists x T(x)], \#[-\forall x \neg T(x)] \} \\
\cup \{ \#[-\forall x T(x)], \#\exists x T(x) \} & \text{if } \emptyset \subset U \subset D, \text{ and } D_2(U) \neq \emptyset, \\
W \cup D_1(U) \cup D_2(U) \cup \{ \#[\exists x T(x)], \#[-\forall x \neg T(x)], \#[\forall x T(x)], \#[-\forall x T(x)] \} \\
\cup \{ \#[-\forall x T(x)], \#\exists x T(x) \} \cup \{ \#\exists x T(x) \} & \text{if } U = D.
\end{cases}
$$

(2.2)

Let $A$ and $B$ denote sentences of $\mathcal{L}$. When $n \in \mathbb{N}_0$, and $G_n(U)$ is defined, denote

$$
G^1_n(U) = \{ \#[A \lor B] : \#A \text{ or } \#B \text{ is in } G_n(U) \},
$$

$$
G^2_n(U) = \{ \#[A \land B] : \#A \text{ and } \#B \text{ are in } G_n(U) \},
$$

$$
G^3_n(U) = \{ \#[A \rightarrow B] : \#[-A] \text{ or } \#B \text{ is in } G_n(U) \},
$$

$$
G^4_n(U) = \{ \#[A \leftrightarrow B] : \text{ both } \#A \text{ and } \#B \text{ or both } \#[-A] \text{ and } \#[-B] \text{ are in } G_n(U) \},
$$

$$
G^5_n(U) = \{ \#[-(A \lor B)] : \#[-A] \text{ and } \#[-B] \text{ are in } G_n(U) \},
$$

$$
G^6_n(U) = \{ \#[-(A \land B)] : \#[-A] \text{ or } \#[-B] \text{ is in } G_n(U) \},
$$

$$
G^7_n(U) = \{ \#[-(A \rightarrow B)] : \#A \text{ and } \#[-B] \text{ are in } G_n(U) \},
$$

$$
G^8_n(U) = \{ \#[-(A \leftrightarrow B)] : \text{ both } \#A \text{ and } \#[-B] \text{ or both } \#[-A] \text{ and } \#B \text{ are in } G_n(U) \},
$$

$$
G^9_n(U) = \{ \#[-(-A)] : \#A \text{ is in } G_n(U) \},
$$

and define

$$
G_{n+1}(U) = G_n(U) \cup \bigcup_{k=1}^{9} G^k_n(U).
$$

(2.4)

Because $\mathcal{L}$ is closed with respect to connectives $\neg$, $\lor$, $\land$, $\rightarrow$ and $\leftrightarrow$, it follows from the above construction that $G_n(U)$ is defined for every $n \in \mathbb{N}_0$. Moreover, $G_n(U) \subseteq G_{n+1}(U)$ and $G^k_n(U) \subseteq G^k_{n+1}(U)$ for all $n \in \mathbb{N}_0$ and $k = 1, \ldots, 9$. In particular, we can define

$$
G(U) = \bigcup_{n=0}^{\infty} G_n(U).
$$

(2.5)

Because every set $G_n(U)$ is a subset of $D$, then also $G(U)$ is contained in $D$. 

4
3 Validity of rules (r1)–(r11)

Now we are ready to prove our main result.

**Theorem 3.1.** Let $U$ be a subset of $D$, and let the subsets $G(U)$ and $F(U)$ of $D$ be defined by (2.5), and by

$$F(U) = \{\#A : \#\neg A \in G(U)\}. \quad (3.1)$$

Then rules (r1)–(r11) are valid.

**Proof.** Let $A$ be a sentence of $L$. It follows from the construction of $G(U)$ that $\#A$ is in $G(U)$ iff $\#A$ is in $W$, i.e., iff $A$ is true in the interpretation of $L$. Definition (3.1) of $F(U)$ implies that $\#A$ is in $F(U)$ iff $\#\neg A$ is in $G(U)$ iff $\neg A$ is true in the interpretation of $L$. Because $L$ is an MA language, then $\neg A$ is true in the interpretation of $L$ iff $A$ is false in the interpretation of $L$. This proves (r1).

Let $n$ be a numeral. The construction of $G(U)$ implies that $\#T(n)$ is in $G(U)$ iff it is in $D_1(U)$, i.e., iff $n = \lfloor A \rfloor$, where $A$ is a sentence of $L$ and $\#A$ is in $U$. $\#T(n)$ is by (3.1) in $F(U)$ iff $\#\neg T(n)$ is in $G(U)$ iff (by the construction of $G(U)$) $\#\neg T(n)$ is in $D_2(U)$ iff (by the definition of $D_2(U)$) $n = \lfloor A \rfloor$, where $A$ is a sentence of $L$ and $\#\neg A$ is in $U$. This ends the proof of (r2).

In the proof of (r3) we need the following auxiliary result.

(r0) If $A$ is a sentence of $L$, then $\#\neg\neg A$ is in $G(U)$ iff $\#A$ is in $G(U)$.

Assume first that $\#\neg\neg A$ is in $G_0(U)$. Then $\#\neg\neg A$ is in $W$, so that sentence $\neg A$ is true in the interpretation of $L$. Since $L$ is an MA language, then $A$ is true in the interpretation of $L$. Thus $\#A$ is in $W$, and hence in $G(U)$.

Assume next that the least of those $n$ for which $\#\neg\neg A$ is in $G_n(U)$ is $> 0$. The definition of $G_n(U)$ implies that if $\#\neg\neg A$ is in $G_n(U)$, then $\#\neg\neg A$ is in $G^0_{n-1}(U)$, so that $\#A$ is in $G_{n-1}(U)$, and hence in $G(U)$.

Thus $\#A$ is in $G(U)$ if $\#\neg\neg A$ is in $G_n(U)$ for some $n \in N_0$, or equivalently, if $\#\neg\neg A$ is in $G(U)$.

Conversely, if $\#A$ is in $G(U)$, then it is in $G_n(U)$ for some $n$, so that $\#\neg\neg A$ is in $G^0_n(U)$, and hence in $G_{n+1}(U)$, and thus in $G(U)$. This concludes the proof of (r0).

To prove (r3), let $A$ be a sentence of $L$. It follows from (3.1) that $\#\neg\neg A$ is in $G(U)$ iff $\#A$ is in $F(U)$. Consider next the case when $\#\neg A$ is in $F(U)$. By (3.1) this holds iff $\#\neg A$ is in $G(U)$ iff (by (r0)) $\#A$ is in $G(U)$. This ends the proof of (r3).

Let $A$ and $B$ be sentences of $L$. If $\#A$ or $\#B$ is in $G(U)$, there is by (2.5) an $n \in N_0$ such that $\#A$ or $\#B$ is in $G_n(U)$. Thus $\#A \lor B$ is in $G^0_n(U)$, and hence in $G(U)$.

Conversely, assume that $\#A \lor B$ is in $G(U)$. Then there is by (2.5) an $n \in N_0$ such that $\#A \lor B$ is in $G_n(U)$. Assume first that $n = 0$. If $\#A \lor B$ is in $G_0(U)$, it is in $W$. Thus $A \lor B$ is true in the interpretation of $L$. Because $L$ is an MA language, then $A$ or $B$ is true in the interpretation of $L$, i.e., $\#A$ or $\#B$ is in $W$, and hence in $G(U)$.

Assume next that the least of those $n$ for which $\#A \lor B$ is in $G_n(U)$ is $> 0$. Then $\#A \lor B$ is in $G^0_{n-1}(U)$, so that $\#A$ or $\#B$ is in $G_{n-1}(U)$, and hence in $G(U)$.

Consequently, $\#A \lor B$ is in $G(U)$ iff $\#A$ or $\#B$ is in $G(U)$.
It follows from (3.1) that
(a) \( \#[A \lor B] \) is in \( F(U) \) iff \( \#[\neg(A \lor B)] \) is in \( G(U) \).
If \( \#[\neg(A \lor B)] \) is in \( G(U) \), there is by (2.3) an \( n \in \mathbb{N}_0 \) such that \( \#[\neg(A \lor B)] \) is in \( G_n(U) \).
Assume that \( n = 0 \). If \( \#[\neg(A \lor B)] \) is in \( G_0(U) \), it is in \( W \). Then \( \neg(A \lor B) \) is true in the interpretation of \( L \), so that \( (L \) is an MA language) \( \neg A \) and \( \neg B \) are true in the interpretation of \( L \), i.e., \( \#[\neg A] \) and \( \#[\neg B] \) are in \( W \), and hence in \( G(U) \).
Assume next that the least of those \( n \) for which \( \#[\neg(A \lor B)] \) is in \( G_n(U) \) is \( > 0 \). Then \( \#[\neg(A \lor B)] \) is in \( G_{n-1}(U) \), so that \( \#[\neg A] \) and \( \#[\neg B] \) are in \( G_{n-1}(U) \), and hence in \( G(U) \). Thus, \( \#[\neg A] \) and \( \#[\neg B] \) are in \( G(U) \) if \( \#[\neg(A \lor B)] \) is in \( G(U) \).
Conversely, if \( \#[\neg A] \) and \( \#[\neg B] \) are in \( G(U) \), there exist by (2.3) \( n_1, n_2 \in \mathbb{N}_0 \) such that \( \#[\neg A] \) is in \( G_{n_1}(U) \) and \( \#[\neg B] \) is in \( G_{n_2}(U) \). Denoting \( n = \max\{n_1, n_2\} \), then both \( \#[\neg A] \) and \( \#[\neg B] \) are in \( G_n(U) \). This result and the definition of \( G_n^5(U) \) imply that \( \#[\neg(A \lor B)] \) is in \( G_n^5(U) \), and hence in \( G(U) \). Consequently,
(b) \( \#[\neg(A \lor B)] \) is in \( G(U) \) iff \( \#[\neg A] \) and \( \#[\neg B] \) are in \( G(U) \) iff (by (3.1)) \( \#A \) and \( \#B \) are in \( F(U) \).
Thus, by (a) and (b), \( \#[A \lor B] \) is in \( F(U) \) iff \( \#A \) and \( \#B \) are in \( F(U) \). This concludes the proof of (r4).

The proofs for the validity of rules (r5)–(r7) are similar to that given above for rule (r4).
\( \#[\exists x T(x)] \) is in \( G(U) \) iff (by construction of \( G(U) \)) \( \#[\exists x T(x)] \) is in \( G_0(U) \) iff \( U \) is nonempty iff (by (r2)) \( \#T(n) \) is in \( G(U) \) for some numeral \( n \in X \).
\( \#[\exists x T(x)] \) is in \( F(U) \) iff (by (r3)) \( \#[\neg \exists x T(x)] \) is in \( G(U) \) iff (by the construction of \( G(U) \)) \( \#[\neg \exists x T(x)] \) is in \( G_0(U) \) iff \( U = D \) iff (by (r2)) \( \#T(n) \) is in \( F(U) \) for every numeral \( n \in X \). This concludes the proof of (r8).
\( \#[\forall x T(x)] \) is in \( G(U) \) iff (by construction of \( G(U) \)) \( \#[\forall x T(x)] \) is in \( G_0(U) \) iff \( U = D \) iff (by (r2)) \( \#T(n) \) is in \( G(U) \) for every numeral \( n \in X \).
\( \#[\forall x T(x)] \) is in \( F(U) \) iff (by (r3)) \( \#[\neg \forall x T(x)] \) is in \( G(U) \) iff (by the construction of \( G(U) \)) \( \#[\neg \forall x T(x)] \) is in \( G_0(U) \) iff \( D_2(U) \) is nonempty iff \( \#[\neg T(n)] \) is in \( G(U) \) at least for one numeral \( n \in X \) iff \( \#T(n) \) is in \( F(U) \) at least for one numeral \( n \in X \). This ends the proof of rule (r9).
Similar reasoning as in the above proofs of (r8) and (r9) can be used to verify that rules (r10) and (r11) are valid.

\[ \square \]

4 Properties of \( G(U) \) and \( F(U) \) when \( U \) is consistent

In this section we shall prove some properties of \( G(U) \) and \( F(U) \), where \( U \) is consistent. They are used in \( [1] \) to prove the existence of consistent fixed points of \( G \).

**Lemma 4.1.** Let \( U \) be a consistent subset of \( D \). Then \( G(U) \cap F(U) = \emptyset \).

**Proof.** If \( A \) is in \( L \), then by rule (r1) \( \#A \) is not in \( G(U) \cap F(U) \) because \( L \) is fully interpreted. Let \( n \) be a numeral. \( \#T(n) \) is in \( G_0(U) \) iff it is in \( D_1(U) \) iff \( n = [A] \), where \( \#A \) is in \( U \). \( \#T(n) \) is in \( F(U) \) iff \( \#[\neg T(n)] \) is in \( G(U) \), or equivalently, in \( D_2(U) \) iff \( n = [A] \), where \( \#[\neg A] \) is in \( U \). Thus \( \#T(n) \) cannot be both in \( G(U) \) and in \( F(U) \), and hence not in \( G_0(U) \cap F(U) \), because the consistency of \( U \) implies that \( \#A \) and \( \#[\neg A] \) cannot be both in \( U \).

6
If $U$ is empty, then none of the Gödel numbers $\#[\exists x T(x)], \#[-\exists x T(x)], \#[\forall x T(x)], \#[-\forall x T(x)], \#[\forall x - T(x)], \#[-\forall x - T(x)], \#[\exists x - T(x)], \#[-\exists x - T(x)]$ is in $G_0(U)$. Hence they are not in $G_0(U) \cap F(U)$.

Assume next that $U$ is not empty. Because $U$ is consistent, it is a proper subset of $D$. Then rules (r1), (r8)-(r11), result (r0) and definitions (2.2) and (3.1) imply that not any of the above listed Gödel numbers is both in $G_0(U)$ and in $F(U)$, and hence in $G_0(U) \cap F(U)$.

The above results and the definition of $G_0(U)$ imply that the induction hypothesis:

$$(h0) \quad G_n(U) \cap F(U) = \emptyset$$

holds for $n = 0$. If $\# [A \land B]$ is in $G_n(U) \cap F(U)$, then $\# A$ or $\# B$ is in $G_n(U)$, and both $\# A$ and $\# B$ are in $F(U)$ by (r4), so that $\# A$ or $\# B$ is in $G_n(U) \cap F(U)$. Hence, if (h0) holds, then $G_n(U) \cap F(U) = \emptyset$.

$\# [A \lor B]$ cannot be in $G_n(U) \cap F(U)$, for otherwise both $\# A$ and $\# B$ are in $G_n(U)$, and at least one of $\# A$ and $\# B$ is in $F(U)$, so that $\# A$ or $\# B$ is in $G_n(U) \cap F(U)$, contradicting with (h0). Thus $G_n(U) \cap F(U) = \emptyset$ if (h0) holds.

If $\# [A \rightarrow B]$ is in $G_n(U) \cap F(U)$, then $\# [\neg A]$ or $\# B$ is in $G_n(U)$ and both $\# [\neg A]$ and $\# B$ are in $F(U)$. Then $\# [\neg A]$ or $\# B$ is in $G_n(U) \cap F(U)$. Thus $G_n(U) \cap F(U) = \emptyset$ if (h0) holds.

If $\# [A \leftrightarrow B]$ is in $G_n(U) \cap F(U)$, then both $\# A$ and $\# B$ or both $\# [\neg A]$ and $\# [\neg B]$ are in $G_n(U)$, and both $\# A$ and $\# [\neg B]$ or both $\# [\neg A]$ and $\# [B]$ are in $F(U)$. Then one of Gödel numbers $\# A$, $\# B$, $\# [\neg A]$ and $\# [\neg B]$ is in $G_n(U) \cap F(U)$, contradicting with (h0). Consequently, $G_n(U) \cap F(U) = \emptyset$ if (h0) holds.

If $\# [\neg (A \lor B)]$ is in $G_n(U) \cap F(U)$, then $\# [\neg A]$ and $\# [\neg B]$ are in $G_n(U)$, and $\# [A \lor B]$ is in $G(U)$, i.e., $\# A$ or $\# B$ is in $G(U)$, or equivalently, $\# [\neg A]$ or $\# [\neg B]$ is in $F(U)$. Thus $\# [\neg A]$ or $\# [\neg B]$ is in $G_n(U) \cap F(U)$. Hence, if (h0) holds, then $G_n(U) \cap F(U) = \emptyset$.

If $\# [\neg (A \land B)]$ is in $G_n(U) \cap F(U)$, then $\# [\neg A]$ or $\# [\neg B]$ is in $G_n(U)$, and $\# [A \land B]$ is in $G(U)$, or equivalently, $\# A$ and $\# B$ are in $G(U)$, i.e., $\# [\neg A]$ and $\# [\neg B]$ are in $F(U)$. Consequently, $\# [\neg A]$ or $\# [\neg B]$ is in $G_n(U) \cap F(U)$, contradicting with (h0). Thus $G_n(U) \cap F(U) = \emptyset$ if (h0) holds.

If $\# [\neg (A \rightarrow B)]$ is in $G_n(U) \cap F(U)$, then $\# A$ and $\# [\neg B]$ are in $G_n(U)$, and $\# [A \rightarrow B]$ is in $G(U)$, i.e., $\# [\neg A]$ or $\# B$ is in $G(U)$, or equivalently, $\# A$ or $\# [\neg B]$ is in $F(U)$. Thus $\# A$ or $\# [\neg B]$ is in $G_n(U) \cap F(U)$. Hence, if (h0) holds, then $G_n(U) \cap F(U) = \emptyset$.

$\# [\neg (A \leftrightarrow B)]$ cannot be in $G_n(U) \cap F(U)$, for otherwise both $\# [A]$ and $\# [\neg B]$ or both $\# [\neg A]$ and $\# [B]$ are in $G_n(U)$, and $\# [A \leftrightarrow B]$ is in $G(U)$, i.e., both $\# [A]$ and $\# [\neg B]$ or both $\# A$ and $\# B$ are in $F(U)$. Thus one of Gödel numbers $\# A$, $\# B$, $\# [\neg A]$ and $\# [\neg B]$ is in $G_n(U) \cap F(U)$, contradicting with (h0). Thus (h0) implies that $G_n(U) \cap F(U) = \emptyset$.

If $\# [\neg (\neg A)]$ would be in $G_n(U) \cap F(U)$, then $\# A$ would be in $G_n(U)$ and $\# [\neg (\neg A)]$, or equivalently, by (r0), $\# A$ would be in $F(U)$, so that $\# A$ would be in $G_n(U) \cap F(U)$. Consequently, $G_n(U) \cap F(U) = \emptyset$ if (h0) holds.

Because $G_{n+1}(U) = G_n(U) \cup \bigcup_{k=1}^n G_k(U)$, the above results imply that $G_{n+1}(U) \cap F(U) = \emptyset$ if (h0) holds. Since it holds when $n = 0$, the above proof shows by induction that it holds for all $n \in \mathbb{N}_0$.

If $\# A$ is in $G(U)$, it is by (2.2) in $G_n(U)$ for some $n \in \mathbb{N}_0$. Because (h0) holds, then $\# A$ is not in $F(U)$. Consequently, $G(U) \cap F(U) = \emptyset$. \qed
**Lemma 4.2.** If $U$ is a consistent subset of $D$, then both $G(U)$ and $F(U)$ are consistent.

*Proof.* If $G(U)$ is not consistent, then there is such a sentence $A$ of $L$, that $\#A$ and $\#[-A]$ are in $G(U)$. Because $\#[-A]$ is in $G(U)$, then $\#A$ is also in $F(U)$ by (3.1), and hence in $G(U) \cap F(U)$. But then, by Lemma 4.1, $U$ is not consistent. Consequently, if $U$ is consistent, then $G(U)$ is consistent. The proof that $F(U)$ is consistent if $U$ is, is similar. $\square$

**Lemma 4.3.** Assume that $U$ and $V$ are consistent subsets of $D$, and that $U \subseteq V$. Then $G(U) \subseteq G(V)$ and $F(U) \subseteq F(V)$.

*Proof.* Assume that $U$ and $V$ are consistent subsets of $D$, and that $U \subseteq V$. Let $A$ be a sentence of $L$. By rule (r1) $\#A$ is in $G(U)$ and also in $G(V)$ iff $\#A$ is in $W$.

Let $n$ be a numeral. If $\#T(n)$ is in $G_0(U)$, then $n = [A]$, where $\#A$ is in $U$. Because $U \subseteq V$, then $\#A$ is also in $V$, whence $\#T(n)$ is in $G_0(V)$. If $\#[-T(n)]$ is in $G_0(U)$, then $n = [A]$, where $\#[-A]$ is in $U$. Because $U \subseteq V$, then $\#[-A]$ is also in $V$, whence $\#[-T(n)]$ is in $G_0(V)$.

If $\#[\exists x T(x)]$ is in $G_0(U)$, then $U$ is nonempty. Because $U \subseteq V$, then also $V$ is nonempty, whence $\#[\exists x T(x)]$ is in $G_0(V)$. Consequently, $\#[\exists x T(x)]$ is in $G_0(V)$ whenever it is in $G_0(U)$.

The similar reasoning shows that $\#[-(\forall x T(x))]$ is in $G_0(V)$ whenever it is in $G_0(U)$.

$\#[\exists x \neg T(x)]$ is in $G_0(U)$ iff $D_2(U)$ is nonempty, i.e., such a sentence $A$ exists in $L$ that $\#[-A]$ is in $U$. But then $\#[-A]$ is also in $V$, i.e., $D_2(V)$ is nonempty, whence $\#[-\exists x \neg T(x)]$ is in $G_0(V)$. Similarly it can be shown that $\#[-\forall x T(x)]$ is in $G_0(V)$ whenever it in $G_0(U)$.

As consistent sets both $U$ and $V$ are proper subsets of $D$. Thus the Gödel numbers $\#[\forall x T(x)]$, $\#[-(\exists x \neg T(x))], \#[-\exists x T(x)]$ and $\#[-\forall x T(x)]$ are neither in $G_0(U)$ nor in $G_0(V)$.

The above results imply that $G_0(U) \subseteq G_0(V)$. Make an induction hypothesis:

(h1) $G_n(U) \subseteq G_n(V)$.

The definitions of the sets $G^k_n(U)$, $k = 1, \ldots, 9$, given in (2.3), together with (h1), imply that $G^k_n(U) \subseteq G^k_n(V)$ for each $k = 1, \ldots, 9$. Thus

$$G_{n+1}(U) = G_n(U) \cup \bigcup_{k=1}^9 G^k_n(U) \subseteq G_n(V) \cup \bigcup_{k=1}^9 G^k_n(V) = G_{n+1}(V).$$

Because (h1) is shown to hold when $n = 0$, then it holds for every $n \in \mathbb{N}_0$.

If $\#A$ is in $G(U)$, it is by (2.5) in $G_n(U)$ for some $n$. Thus $\#A$ is in $G_n(V)$ by (h1), and hence in $G(V)$. Consequently, $G(U) \subseteq G(V)$.

If $\#A$ is in $F(U)$, it follows from (3.1) that $\#[-A]$ is in $G(U)$. Because $G(U) \subseteq G(V)$, then $\#[-A]$ is in $G(V)$ This implies by (3.1) that $\#A$ is in $F(V)$. Thus $F(U) \subseteq F(V)$. $\square$

Let $P$ denote the family of all consistent subsets of the set $D$ of Gödel numbers of sentences of $L$. According to Lemma 4.2 the mapping $G := U \mapsto G(U)$ maps $P$ into $P$. Assuming that $P$ is ordered by inclusion, it follows from Lemma 4.3 that $G$ is order preserving.

In the formulation and the proof of Theorem 4.1 below transfinite sequences of $P$ indexed by von Neumann ordinals are used. Such a sequence $(U_\lambda)_{\lambda \in \alpha}$ of $P$ is said to be increasing if $U_\mu \subseteq U_\nu$ whenever $\mu \in \nu \in \alpha$, and strictly increasing if $U_\mu \subset U_\nu$ whenever $\mu \in \nu \in \alpha$. 

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Lemma 4.4. Assume that \((U_\lambda)_{\lambda \in \alpha}\) a strictly increasing sequence of consistent subsets of \(D\). Then

(a) \((G(U_\lambda))_{\lambda \in \alpha}\) is an increasing sequence of consistent subsets of \(D\).

(b) The set \(U_\alpha = \bigcup_{\lambda \in \alpha} G(U_\lambda)\) is consistent.

Proof. (a) Consistency of the sets \(G(U_\lambda), \lambda \in \alpha\), follows from Lemma 4.2 because the sets \(U_\lambda, \lambda \in \alpha\), are consistent.

Because \(U_\mu \subseteq U_\nu\) whenever \(\mu \in \nu \in \alpha\), then \(G(U_\mu) \subseteq G(U_\nu)\) whenever \(\mu \in \nu \in \alpha\), by Lemma 4.3, whence the sequence \((G(U_\lambda))_{\lambda \in \alpha}\) is increasing. This proves (a).

(b) To prove that the set \(\bigcup_{\lambda \in \alpha} G(U_\lambda)\) is consistent, assume on the contrary that there exists such a sentence \(A\) in \(\mathcal{L}\) that both \(#A\) and \(#\lnot A\) are in \(\bigcup_{\lambda \in \alpha} G(U_\lambda)\). Thus there exist \(\mu, \nu \in \alpha\) such that \(#A\) is in \(G(U_\mu)\) and \(#\lnot A\) is in \(G(U_\nu)\). Because \(G(U_\mu) \subseteq G(U_\nu)\) or \(G(U_\nu) \subseteq G(U_\mu)\), then both \(#A\) and \(#\lnot A\) are in \(G(U_\mu)\) or in \(G(U_\nu)\). But this is impossible, since both \(G(U_\mu)\) and \(G(U_\nu)\) are consistent by (a). Thus, the set \(\bigcup_{\lambda \in \alpha} G(U_\lambda)\) is consistent, so that the conclusion of (b) holds. \(\square\)

A subset \(V\) of \(D\) is called sound if \(V \subseteq G(V)\). Let \(V\) be a subset of the set \(W\) of G"odel numbers of all those sentences of \(L\) which are true in the interpretation of \(L\). Since \(L\) is an MA language, then \(W\) is consistent. Thus also \(V\) is consistent. Because \(W = G_0(\emptyset) \subseteq G(\emptyset)\), then \(V \subseteq G(\emptyset) \subseteq G(V)\). Thus \(V\) is also sound.

The following fixed point theorem is an application of Lemmas 4.2, 4.3 and 4.4 and is proved in [4].

Theorem 4.1. If \(V \in \mathcal{P}\) is sound, there exists the smallest of those fixed points of \(G\) which contain \(V\). This fixed point is the last member of the union of those transfinite sequences \((U_\lambda)_{\lambda \in \alpha}\) of \(\mathcal{P}\) which satisfy

\[
(C) \quad (U_\lambda)_{\lambda \in \alpha}\text{ is strictly increasing, } U_0 = V, \text{ and if } 0 \in \mu \in \alpha, \text{ then } U_\mu = \bigcup_{\lambda \in \mu} G(U_\lambda).
\]

The union \((U_\lambda)_{\lambda \in \gamma}\) of the transfinite sequences satisfying (C) can be characterized as follows (cf. [3]).

(I) \(U_0 = V\). If \(\lambda\) is in \(\gamma\), then \(\lambda + 1\) is in \(\gamma\) iff \(U_\lambda \subseteq G(U_\lambda)\), in which case \(U_{\lambda + 1} = G(U_\lambda)\).

If \(\alpha\) is a limit ordinal, and \(\lambda\) is in \(\gamma\) for each \(\lambda \in \alpha\), then \(\alpha\) is in \(\gamma\), and \(U_\alpha = \bigcup_{\lambda \in \alpha} U_\lambda\).

It follows from (I) that the sequence \((U_\lambda)_{\lambda \in \gamma}\) begins with \(U_0 = V, U_{n+1} = G(U_n), n = 0, 1, \ldots, U_\omega = \bigcup_{n \in \omega} U_n, U_{\omega+n+1} = G(U_{\omega+n}), n = 0, 1, \ldots, \) e.t.c., as long as the so defined sets exist and contain strictly previous sets. Because \((U_\lambda)_{\lambda \in \gamma}\) is a strictly increasing sequence of subsets of a countable set \(D\), then \(\gamma\) is a countable ordinal. In this sense the smallest fixed point of \(G\) that contains \(V\) is determined by a countable recursion method. By [5, Corollary 3.1] this fixed point \(U\) is the smallest of all fixed points of \(G\) if \(V\) is a subset of \(W\). Those sentences of \(\mathcal{L}\) whose G"odel numbers are in \(U\), or equivalently, in \(G(U)\), and their negations, whose G"odel numbers belong by (r3) to \(F(U)\), form an MA language \(\mathcal{L}_0\) which contains its truth predicate.
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