Liouville Quantum Mechanics on a Lattice
from Geometry of Quantum Lorentz Group

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Abstract

We consider the quantum Lobachevsky space $L^3_q$, which is defined
as subalgebra of the Hopf algebra $A_q(SL_2(C))$. The Iwasawa decom-
position of $A_q(SL_2(C))$ introduced by Podles and Woronowicz allows
to consider the quantum analog of the horospheric coordinates on $L^3_q$.
The action of the Casimir element, which belongs to the dual to $A_q$
ququantum group $U_q(SL_2(C))$, on some subspace in $L^3_q$ in these coordi-
nates leads to a second order difference operator on the infinite one-
dimensional lattice. In the continuos limit $q \to 1$ it is transformed
into the Schrödinger Hamiltonian, which describes zero modes into
the Liouville field theory (the Liouville quantum mechanics). We cal-
culate the spectrum (Brillouin zones) and the eigenfunctions of this
operator. They are $q$-continuos Hermit polynomials, which are partic-
ular case of the Macdonald or Rogers-Askey-Ismail polynomials. The
scattering in this problem corresponds to the scattering of first two
level dressed excitations in the $Z_N$ Baxter model in the very peculiar
limit when the anisotropy parameter $\gamma$ and $N \to \infty$, or, equivalently,
$(\gamma, N) \to 0$.

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1 Introduction

There are a lot of interrelations between integrable one-dimensional systems of particles and integrable 2d field theories. One of them is the similarity in dynamics between solitons and the classical Calogero-Moser-Sutherland-Toda particles and their r generalizations [1]. On a quantum level the similar phenomena has been observed recently in [2, 3]. In particular, it was discovered there that the scattering of some special excitations in the $Z_2$-Baxter model coincides with the scattering, which is defined by asymptotics of the Macdonald polynomials for the root system $A_1$ [4]. They depend on two parameters and in the simplest case define the usual zonal spherical functions on reducible symmetric spaces of rank one.

Recall that the zonal spherical functions are defined as a normalized eigenfunctions of the Laplace-Beltrami operator on a symmetric space, which are invariant with respect to a stationer subgroup and have a free asymptotic. In other words, the Laplace-Beltrami operator in the spherical coordinates coincides up to a simple gauge transformation with the generalized Calogero-Moser-Sutherland potential [5]. The corresponding Jost function is the famous Harish-Chandra $c$-function [6], which is factorizable [7]. This fact is an agreement with the complete integrability of the Calogero-Moser-Sutherland systems. In particular, the Jost functions for a symmetric noncompact space of rank one give rise to the scattering of a quantum particle on a semi-line on the Calogero-Moser potential $\frac{g^2}{\sinh^2 x}$. The same $S$-matrix arises in the scattering of two kinks in the XXX-model. In the similar fashion XXZ-model corresponds to the quantum symmetric space of rank one. But in this case the continuous semi-line is replaced on a semi-indefinite lattice.

The open quantum Toda Hamiltonian can be derived in the similar way [3]. To this end the same Laplace-Beltrami operator is considered in the horospherical coordinates. If its eigenfunctions are independent on the horospherical "angular" coordinates then the Laplace-Beltrami operator is reduced to a second order differential operator with constant coefficients. But if the functions have nontrivial multiplier when their arguments are shifted along horospheres (more exactly, they belong to a representation, induced by a character of a nilpotent subgroup), then the Laplace-Beltrami operator acquires a nontrivial potential term, which is nothing else as the open quantum dynamics was discussed partly in [1].

\footnote{Quantum dynamics was discussed partly in [1]}
Toda potential. The corresponding eigenfunctions are called the Whittaker functions \([9, 10]\). It was proved in \([8]\) that the corresponding Jost functions are also factorizable. The simplest \(SL_2\) case describes the scattering on the Liouville potential \(e^{-2x}\).

We can consider this model as a simplified version of the ubiquitous Liouville field theory, for \(\phi(\sigma, t)\) (the zero mode \(x(t) = \phi(\sigma, t)|_{\sigma=0}\)) , which describes 2D induced gravity in the conformal gauge. Then, \(e^{-2x}\) defines the circumference of 1d "universe". The quantum mechanics of the zero modes sheds light on the spectrum of the full theory in the quasiclassical approach \([11]\).

Our aim is to put this theory on a lattice. In other words, \(x\) will take discrete values. There are a few reasons to do it. First of all, introducing the new parameter - the step of a lattice, is equivalent to the generalizations of corresponding spin chain Lagrangians. Next, this model in the 2d version modifies the 2d gravity in the similar fashion as a finite group approximation of gauge groups in the Yang-Mills theories. This modification of 2d gravity can be in principle useful in the quantization procedure. Finally, classical solutions of models on lattices may lead presumably to deformed tau functions connected with new integrable hierarchies, as well as to partition functions of some topological theories.

Here we consider the very simple quantum theory, which turns out to be exactly solvable. To derive the model we proceed to "the second quantization" of the Liouville quantum mechanics using the formalism of quantum groups. Namely, we consider the quantum Lobachevsky space \(L^3_q\). It is defined as a subalgebra of the Hopf algebra \(A_q \supset L^3_q\), which is dual to the quantum Lorentz group \(U_q(SL_2(C))\). The algebra \(L^3_q\) is equipped with a right \(U_q(SL_2(C))\)-module structure. In \(L^3_q\) exists an analog of the horospherical coordinates, which are connected with the Iwasawa decomposition of \(A_q\) \([12]\). We calculate the Casimir element \(\Omega_q \in U_q(SL_2(C))\) in these coordinates. It is possible to reduce the operator to subspace, which is similar to the space of the induced representation of the nilpotent subgroup in the classical situation. The quantum horospherical variables are separated in this subspace. The operator \(\Omega_q\) becomes classical second order difference operator on a one-dimensional lattice with the Liouville "wall". It differs slightly from relativistic Liouville, introduced in \([13]\). In the limit, when a step of the lattice vanishes , \(\Omega_q|_{q\to 1}\) coincides with the Liouville Hamiltonian. The quantum Whittaker functions are the \(q\)-continuous Hermit polynomials, which are par-
ticular case of the Rogers-Askey-Ismail polynomials [14], or the Macdonald polynomials for the $A_1$ root system [4]. Thus, the Macdonald polynomials serve also to define the Whittaker functions, as well as the zonal spherical functions.

The Harish-Chandra c-function gives rise to the $S$-matrix, which coincides with the $S$-matrix for scattering of first two levels in the $Z_N$-Baxter model [15, 16] in the limit $N, \gamma \to \infty$, where $\gamma$ is an anisotropy parameter, or, equivalently, $\gamma, N \to 0$.

2 Classical Case

Let $\mathbf{L}^3 = SU_2 \backslash SL_2(\mathbb{C})$ be a homogeneous space of the second order unimodular Hermitian positive definite matrices, which is a model of the classical Lobachevsky space. Let

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \alpha \delta - \beta \gamma = 1.$$  

Then any $x \in \mathbf{L}^3$ can be represented as

$$x = g^* g = \begin{pmatrix} \bar{\alpha} \alpha + \bar{\gamma} \gamma & \bar{\alpha} \beta + \bar{\gamma} \delta \\ \bar{\beta} \alpha + \bar{\delta} \gamma & \bar{\beta} \beta + \bar{\delta} \delta \end{pmatrix}. \quad (1)$$

The Iwasawa decomposition

$$g = k b, \quad g \in SL_2(\mathbb{C}), \quad k \in SU_2, \quad b \in AN - Borel subgroup \quad (2)$$

allows to define the horospherical coordinates on $\mathbf{L}^3$. If

$$b = \begin{pmatrix} h & h z \\ 0 & h^{-1} \end{pmatrix},$$

then from (1)

$$x = b^* b = \begin{pmatrix} \bar{h} h & \bar{h} h z \\ \bar{z} \bar{h} h & \bar{z} \bar{h} h z + (\bar{h} h)^{-1} \end{pmatrix}. \quad (3)$$

The triple $(H = \bar{h} h, z, \bar{z})$ is uniquely determined by $x$. It is called the horospherical coordinates of $x$. It follows from (1) and (3) that

$$H = \bar{\alpha} \alpha + \bar{\gamma} \gamma,$$
\[ Hz = \bar{\alpha} \beta + \bar{\gamma} \delta, \]
\[ \bar{z} H = \bar{\beta} \alpha + \bar{\delta} \gamma. \]

Let for
\[ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_2(\mathbb{C}) \]
d\(_A, d\(_B, d\(_C\) and \(d\(_D\) be the Lie operators of the right shift on \(L^3\). In the horospherical coordinates they take the form
\[ d\(_A = \frac{1}{2} H \partial_H - z \partial_z, \quad d\(_D = -d\(_A, \]
\[ d\(_B = \partial_z, \]
\[ d\(_C = H z \partial_H - z^2 \partial_z + H^2 \partial_{\bar{z}}. \]

The second Casimir
\[ \Omega = d^2\(_A + d^2\(_D + d\(_B d\(_C + d\(_C d\(_B \]
(5)
in the horospherical coordinates takes the form
\[ \Omega = \frac{1}{2} H^2 \partial^2_H + \frac{3}{2} H \partial_H + 2H^{-2} \partial_{z\bar{z}}. \]
(6)
Consider the eigenvalue problem
\[ \frac{1}{2} \Omega \Phi_\lambda(H, z, \bar{z}) = -(\lambda^2 + \frac{1}{4}) \Phi_\lambda(H, z, \bar{z}), \]
(7)
and put
\[ \Phi_\lambda(H, z, \bar{z}) = H^{-1} \exp i\mu(z + \bar{z}) \phi_\lambda(H), \]
where \(\exp i\mu(z + \bar{z})\) can be consider as a unitary character of the nilpotent subgroup \(N\). Then for \(x = \frac{1}{2} \log H\) and \(\Psi_\lambda(x) = \phi_\lambda(H)\) \(\square\) is transformed to
\[ (-\frac{1}{4} d^2_{xx} + 4\mu^2 e^{-4x}) \Psi_\lambda(x) = 4\lambda^2 \Psi_\lambda(x). \]
(8)
The solution to this equation
\[ \Psi_\lambda(x) = K_{2\lambda}(2\mu e^{-2x}) \]
(9)
is the Bessel-Macdonald function with the asymptotic behavior

\[ K_{2i\lambda}(2\mu e^{-2x}) \sim \frac{\pi}{\sin 2\pi \lambda} \left( \frac{(\mu e^{2x})^{-2i\lambda}}{\Gamma(1-2i\lambda)} + \frac{(\mu e^{2x})^{2i\lambda}}{\Gamma(1+2i\lambda)} \right). \]  

(10)

This solution is the so called Whitteker function for \( SL_2(C) \) [9, 10]. The two-body \( S \)-matrix

\[ S(\lambda) = \frac{\Gamma(1+2i\lambda)}{\Gamma(1-2i\lambda)}, \]  

(11)

which is obtained from (10), describes the scattering of a quantum particle on the Liouville "wall" \( e^{-4x} \).

3 Quantum Lobachevsky Space

Let \( A_q(SL_2(C)) \) (0 < \( q \) ≤ 1) be the algebra of functions on \( SL_2(C) \) [12], which is defined as the factor algebra of the free associative \( C \)-algebra with generators \( \alpha, \beta, \gamma, \delta \), with an antiinvolution * : \( A_q \rightarrow A_q \), \((ab)^* = b^*a^*\) and the following relations

\[ \alpha\beta = q\beta\alpha, \quad \alpha\gamma = \gamma\alpha, \quad \beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \quad \beta\gamma = \gamma\beta, \]

\[ \alpha\delta - q\beta\gamma = 1, \quad \delta\alpha - q^{-1}\beta\gamma = 1, \]

\[ \beta\alpha^* = q^{-1}\alpha^*\beta + q^{-1}(1-q^2)\gamma^*\delta, \]

\[ \gamma^* = q^*q^* \gamma^*, \quad \beta^* = \beta^* \gamma^*, \quad \gamma^* = \gamma^* \gamma, \quad \gamma^* = \gamma^* \gamma, \quad \gamma^* = \gamma^* \gamma, \]

\[ \delta^* = \delta^* \gamma, \quad \delta\beta^* = \delta\beta^* \gamma, \quad \alpha^* = \alpha^* \beta + (1-q^2)(\delta^* \delta - \alpha^* \alpha) - (1-q^2)\gamma^* \gamma, \]

\[ \gamma^* \gamma = \gamma^* \gamma, \quad \delta^* \delta = \delta^* \delta - (1-q^2)\gamma^* \gamma. \]

The rest commutation relations can be read off from the rule \((ab)^* = b^*a^*\).

We cast the generators into the matrix form

\[ w = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad w^* = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix}. \]
With the comultiplication $\Delta : A_q \to A_q \otimes A_q$

$$\Delta \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \otimes \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right),$$

the antipode $S : A_q \to A_q$

$$S \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \left( \begin{array}{cc} \delta & -q^{-1} \beta \\ -q\gamma & \alpha \end{array} \right),$$

and the counit $\varepsilon : A_q \to \mathbb{C}$

$$\varepsilon \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).$$

$A_q$ becomes a Hopf algebra. In fact, it is a $*$-Hopf algebra, since

$$(\Delta(a))^* = \Delta(a^*),$$

and

$$S \circ * \circ S \circ * = \text{id.} \quad (13)$$

We define the $*$-Hopf subalgebra $A_q(SU_2)$ by the generators

$$A_q(SU_2) = \{ w_c = \left( \begin{array}{c} \alpha_c & -q \gamma_c^* \\ \gamma_c & \alpha_c^* \end{array} \right) \} \quad (14)$$

and the relations

$$\alpha_c^* \alpha_c + \gamma_c^* \gamma_c = 1, \quad \alpha_c \alpha_c^* + q^2 \gamma_c \gamma_c = 1,$$

$$\gamma_c^* \gamma_c = \gamma_c \gamma_c^*, \quad \alpha_c \gamma_c^* = q \gamma_c \alpha_c;$$

$$\alpha_c \gamma_c = q \gamma_c \alpha_c.$$

Then

$$w_c^* w_c = \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right). \quad (15)$$

In the similar way

$$A_q(AN_q) = \{ w_d = \left( \begin{array}{cc} h & z \\ 0 & h^{-1} \end{array} \right) \} \quad (16).$$
\[ hh^* = h^* h, \quad h z = q z h, \quad z h^* = q^{-1} h^* z, \]
\[ zz^* = z^* z + (1 - q^2)((h^* h)^{-1} - h^* h). \]

The Iwasawa decomposition in the quantum context takes the form \[ w = wcwd, \quad w \in A_q(SL_2(C)), \quad wc \in A_q(SU_2), \quad wd \in A_q(AN_q). \] (17)

Natural description of commutation relations (12) can be obtained from the construction of the quantum double. It was implemented in [17], where \( A_q(SL_2(C)) \) is described as a special quantum double of \( A_q(SU_2) \), and (13) are derived by means of corresponding \( R \)-matrix.

The quantum Lobachevsky space \( L^3_q \) is *-subalgebra of \( A_q(SL_2(C)) \) generated by the bilinear constituents
\[ w^* w = \begin{pmatrix} \alpha^* \alpha + \gamma^* \gamma & \alpha^* \beta + \gamma^* \delta \\ \beta^* \alpha + \delta^* \gamma & \beta^* \beta + \delta^* \delta \end{pmatrix} = \begin{pmatrix} p & s \\ s^* & r \end{pmatrix}. \]
(18)

Evidently, * acts as
\[ p^* = p, \quad (s)^* = s^*, \quad r^* = r. \]

We don’t need the explicit form of the commutation relations between \( p, s, s^* \) and \( r \) - they can be derived from (12).

Due to (13), (16) and (18)
\[ p = H = h^* h = hh^*, \quad s = H z, \quad s^* = z^* H, \quad r = z^* Hz + H^{-1}. \]
(19)

The triple \( (H, z, z^*) \) generates the horospherical coordinates in the algebra \( L^3_q \). It follows from (12) that
\[ Hz = q^2 z H, \quad H^{-1} z = q^{-2} z H^{-1}, \quad z^* H = q^{-2} H z^*, \]
\[ z^* H^{-1} = q^{-2} H^{-1} z^*, \]
\[ zz^* = q^2 z^* z + (q^2 - 1)(1 - H^{-2}). \]
(20)

Consider now the complex associative algebra \( U_q(SL_2(C)) \) with unit 1, generators \( A, B, C, D \) and the relations
\[ AD = DA = 1, \quad AB = q BA, \quad BD = q DB, \]
\[ AC = q^{-1} CA, \quad CD = q^{-1} DC, \]
(21)
\[ [B, C] = \frac{1}{q - q^{-1}}(A^2 - D^2). \]

In fact, it is the Hopf algebra, where

\[ \Delta(A) = A \otimes A, \quad \Delta(D) = D \otimes D, \]
\[ \Delta(B) = A \otimes B + B \otimes D, \]  
\[ \Delta(C) = A \otimes C + C \otimes D, \]  
\[ \epsilon \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]
\[ S \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} D & -q^{-1}B \\ -qC & A \end{pmatrix}. \]  

There exists a nondegenerate bilinear form \(< u, a >: U_q \times A_q \rightarrow \mathbb{C} \) such that

\[ < \Delta(u), a \otimes b > = < u, ab >, \quad < u \otimes v, \Delta(a) > = < uv, a >, \]
\[ < 1_U, a > = \varepsilon_A(a), \quad < u, 1_A > = \varepsilon_U(u), \quad < S(u), a > = < u, S(a) >. \]

It takes the form on the generators

\[ < A, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} > = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}, \quad < D, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} > = \begin{pmatrix} q^{-\frac{1}{2}} & 0 \\ 0 & q^{\frac{1}{2}} \end{pmatrix}, \]
\[ < B, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} > = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad < C, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} > = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]  

Moreover \( U_q(SL_2(\mathbb{C})) \) is *-Hopf algebra in duality, where the involution is defined by the pairing

\[ < u^*, a > = < u, (S(a))^* >. \]  

The element

\[ \Omega_q := \frac{q^{-1}A^2 + q^2D^2 - 2}{(q^{-1} - q)^2} + BC \]  

is a Casimir element, since it commutes with any \( u \in U_q(SL_2(\mathbb{C})) \).

The right action of \( u \in U_q(SL_2(\mathbb{C})) \) on \( A_q \) is defined as

\[ a.u := (u \otimes \text{id})(\Delta(a)). \]
It is the algebra action:
\[ a.(uv) = (a.u).v, \]  
which satisfies the "Leibnitz rule"
\[ (ab).u = \sum_j (a.u_1^j)(b.u_2^j), \text{ where } \Delta(u) = \sum_j u_1^j \otimes u_2^j. \]  
The left action is defined in the similar way.

The right action on the generators takes the form
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . A = \begin{pmatrix} q^{\frac{1}{2}} \alpha & q^{-\frac{1}{2}} \beta \\ q^2 \gamma & q^{-2} \delta \end{pmatrix},
\]
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . B = \begin{pmatrix} 0 & \alpha \\ 0 & \gamma \end{pmatrix},
\]
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . B = \begin{pmatrix} 0 & \alpha \\ 0 & \gamma \end{pmatrix},
\]
\[
\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} . D = \begin{pmatrix} q^{-\frac{1}{2}} \alpha & q^{\frac{1}{2}} \beta \\ q^{-2} \gamma & q^{2} \delta \end{pmatrix}.
\]

We will define now the right action of \( U_q(SL_2(C)) \) on \( L_q^3 \), which endows the later with the structure of the right *-module. For any \( a \in L_q^3 \) define the normal ordering using (12):
\[
: a : = \sum_k c_k a^*_{1,k} a_{2,k},
\]
where \( a^*_{1,k}, (a_{2,k}) \) are monoms depending on \( \alpha^*, \beta^*, \gamma^*, \delta^* \) \( (\alpha, \beta, \gamma, \delta) \). Then the right action on \( L_q^3 \), which will be denoted as \( (a).u \), is defined as follows
\[
(a).u = \sum_k c_k a^*_{1,k}(a_{2,k}.u).
\]
In particular, from (30)
\[
(p, s, s^*, r).A = (p, 0, s^*, 0), \quad (p, s, s^*, r).B = (o, p, 0, s^*),
\]
\[
(p, s, s^*, r).C = (s, 0, r, 0), \quad (p, s, s^*, r).D = (0, s, 0, r).
\]
To obtain the action of generators of $U_q(SL_2(\mathbb{C}))$ on the horospheric coordinates it is necessary to express them through $\alpha^*, \beta^*, \gamma^*, \delta^*$ and $\alpha, \beta, \gamma, \delta$, ordering them in correspondence with (31), and then to apply (32). Two expressions have the normal order from the very beginning (see (18) and (19))

$$
H = \alpha^* \alpha + \gamma^* \gamma,
$$

$$
Hz = \alpha^* \beta + \gamma^* \delta.
$$

For $H^{-1}$ we can write a representation as a formal series

$$
H^{-1} = (\alpha^*)^{-1} \sum_{k=0}^{\infty} (-1)^k q^{-2k} ((\alpha^*)^{-1} \gamma^*)^k (\gamma \alpha^{-1})^k \alpha^{-1}.
$$

It can be proved by the induction that

$$
H^l = (\alpha^*)^l \sum_{k=0}^{l} q^{(l-1)k} \frac{[l]_{q^2}!}{[k]_{q^2}![l-k]_{q^2}!} y^k y^l \alpha^l, \ l \geq 0, \ y = \gamma \alpha^{-1},
$$

$$
H^{-l} = (\alpha^*)^{-l} \sum_{k=0}^{\infty} (-1)^k q^{-(l+1)k} \frac{[l+k-1]_{q^2}!}{[k]_{q^2}![l-k]_{q^2}!} y^k y^l \alpha^l, \ l > 0.
$$

Here we use the standard notations

$$
[k]_{q^2} = \frac{q^k - q^{-k}}{q - q^{-1}} = \frac{\sinh k \hbar_2}{\sinh \hbar_2},
$$

where $q = \exp(-\hbar_2)$ and $[k]_{q^2}! = [k]_{q^2} [k-1]_{q^2} \ldots 1$.

The normal order for $z$ and $z^*$ can be derived from (33) and (34)

$$
z = \xi + \kappa, \ \xi = \alpha^{-1} \beta,
$$

and

$$
\kappa = \gamma^* H^{-1} \alpha^{-1} = \sum_{k=0}^{\infty} (-1)^k q^{-2k} (y^*)^{k+1} y^k \alpha^{-2}, \ y = \gamma \alpha^{-1}.
$$

Also

$$
\xi \kappa^k = q^{2k} \kappa^k \xi + (q^{2k} - 1) \kappa^{k+1},
$$

The subscribe 2 is used here to distinguish this constant from the usual Planck constant $\hbar_1$, which presents from the very beginning in front of derivatives in the classical group approach.
and therefore
\[
z^n = (\xi + \kappa)^n = \sum_{l=0}^{n} q^{l(n-1)} \frac{[n]_{q^2}!}{[l]_{q^2}! [l-n]_{q^2}!} \kappa^l \xi^{n-l}.
\] (39)

From (37)
\[
(z).A = q^{-1}z, \quad (z^*).A = z^*,
\]
\[
(z).B = q^{-1/2}, \quad (z^*).B = 0,
\]
\[
(z).C = -q^{1/2}z^2, \quad (z^*).C = q^{1/2}H^{-2},
\]
\[
(z).D = qz, \quad (z^*).D = qz^*.
\] (40)

In the same way from (35) for any \( r \in \mathbb{Z} \)
\[
(H^r).A = q^{r/2}H^r, \quad (H^r).B = 0,
\]
\[
(H^r).C = q^{r/2}H^r [r]_{q^2} H^r z, \tag{41}
\]
\[
(H^r).D = q^{-r/2}H^r.
\]

Formally, we can consider this relation for any \( r \in \mathbb{C} \), since \( H \) (35) can be defined as a positive definite Hermitian operator.

The relations (40) and (41) demonstrate the action of \( U_q(SL_2(\mathbb{C})) \) on the horospherical coordinates of \( L^3_q \). Moreover, using the same notations for this representation as for the generators of \( U_q(SL_2(\mathbb{C})) \), we can prove that in the classical limit (40) and (41) are transformed in the Lie derivatives (4)
\[
d_A = \lim_{q \to 1} \partial_q A, \quad d_D = \lim_{q \to 1} \partial_q D,
\]
\[
d_B = \lim_{q \to 1} B, \quad d_C = \lim_{q \to 1} C.
\]

As for the quantum Casimir (26), it is easy to check that
\[
\lim_{q \to 1} \Omega_q = \frac{1}{2} \Omega + \frac{1}{4}, \tag{42}
\]
(\( \Omega \) is the classical Casimir (3)), as it has to be.
4 Eigenfunctions of the quantum Casimir and scattering in spin chains

The eigenvalue problem

\[
(f(H, z, z^*)) \Omega_q = -\lambda^2 f(H, z, z^*),
\]

(43)

where \( f(H, z, z^*) \) is a Laurent series in \( H, z, z^* \), can be solved by means of the relations from the previous section (28) and (29).

In the similar fashion as in the classical case we choose a special form of \( f(H, z, z^*) \). Let

\[
e^v_q = \sum_{n=0}^{\infty} v^n (q; q)_n,
\]

where

\[
(a; q)_n = \begin{cases} 
1 & n = 0, \\
(1 - a)(1 - aq) \ldots (1 - aq^{n-1}) & n > 0,
\end{cases}
\]

be the q-exponent \([14]\). Then we assume

\[
f(H, z, z^*) = e^{i\mu q^{-\frac{2}{2}}(1-q^2)z} F_\lambda(H) e^{i\mu q^{-\frac{2}{2}}(1-q^2)z^*},
\]

and \( F_\lambda(H) = \sum_r a_{r+1}(\lambda) H^r \). Thus

\[
f(H, z, z^*) = \sum_{m=0}^{\infty} \sum_r \sum_{n=0}^{\infty} \frac{(i\mu)^{m+n} q^{-m-n}(1 - q^2)^{m+n}}{(q^2, q^2)_n (q^2, q^2)_n} a_{r+1}(\lambda) X^m H^r X^n,
\]

(44)

where \( X = q^{-\frac{2}{2}} z \).

From \([39]\)

\[
v^{(m,r,n)} = X^m H^r X^n = \sum_{k=0}^{m+n} \sum_{l=0}^{n} q^{k(m-1)+l(n-1)} [m]_q [n]_q [l]_q \xi^m \kappa^k \kappa^l \xi^n.
\]

As it follows from \([12]\)

\[
\kappa^k = (\alpha \gamma)^{k-k} \kappa^k (\alpha \gamma)^k.
\]

It can be find from \([28]\) and \([30]\) for \( n \geq 0 \)

\[
(\xi^n). A = q^{-n} \xi^n,
\]

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\[(\xi^n).B = q^{-1/2}[n]q^2 \xi^{n-1},\]
\[(\xi^n).C = -q^{1/2}[n]q^2 \xi^{n+1},\]
\[(\xi^n).D = q^n \xi^n,\]

For \(n \geq 0\) after some algebra we obtaine
\[(\kappa^n).A = q^{-n} \kappa^n,\]
\[(\kappa^n).B = 0,\]
\[(\kappa^n).C = -q^{n+1/2}[2n]q^2 \kappa^n \xi - q^{2n+1/2}[n]q^2 \kappa^{n+1},\]
\[(\kappa^n).D = q^n \kappa^n.\]

Then the following relation
\[(\alpha \gamma)^{-1} = \kappa^{-1} H^{-2} + \kappa\]
allows to derive the actions of the generators on the monom \(w^{(k,r,l)} = \kappa^k H^r \kappa^l\)
\[(w^{(k,r,l)}).A = q^{r/2-l} w^{(k,r,l)},\]
\[(w^{(k,r,l)}).B = 0,\]
\[(w^{(k,r,l)}).C = q^{-k+l+r+1/2} [k]q^2 w^{(k-1,r-2,l)} - q^{l-r+1/2} [2l-r]q^2 w^{(k,r,l)+1},\]
\[(w^{(k,r,l)}).D = q^{-r/2+l} w^{(k,r,l)}.\]

In agreement with (45)
\[(v^{(m,r,n)}).A = q^{r-n} v^{(m,r,n)},\]
\[(v^{(m,r,n)}).B = q^{r/2}[n]q^2 v^{(m,r,n-1)},\]
\[(v^{(m,r,n)}).C = q^{m+n+r-2}[m]q^2 v^{(m-1,r-2,n)} - q^{-r/2} [n-r]q^2 v^{(m,r,n+1)},\]
\[(v^{(m,r,n)}).D = q^{-r/2+n} v^{(m,r,n)}.

Using (28) and (29), we obtaine
\[(v^{(m,r,n)}).\Omega_q = [r + 1/2]q^2 v^{(m,r,n)} + q^{m+n+r-2}[m]q^2 [n]q^2 v^{(m-1,r-2,n-1)}.\]
Substituting it in (44) we come to
\[
(f(H, z, z^*), \Omega_q = e^{i\mu q \frac{3}{2}(1-q^2)x} \sum_r a_{r+1} \frac{r + 1}{2} q^r H^r e^{i\mu q \frac{3}{2}(1-q^2)x} = \]
\[
-\mu^2 e^{i\mu q \frac{3}{2}(1-q^2)x} \sum_r a_{r+1} q^{r-2} H^{r-2} e^{i\mu q \frac{3}{2}(1-q^2)x}.
\]
Therefore, (43) is reduced to
\[
\sum_r a_{r+1} \frac{r + 1}{2} q^r H^r - \mu^2 \sum_r a_{r+1} q^{r-2} H^{r-2} = -\lambda^2 \sum_r a_r H^r.
\]
This relation is equivalent to our main equation
\[
\frac{F_\lambda(qH) - 2F_\lambda(H) + F_\lambda(q^{-1})}{(q - q^{-1})^2} - \mu^2 q^{-3} H^{-2} F_\lambda(qH) = -\lambda^2 F_\lambda(H). \tag{46}
\]
It is the lattice counterpart of the Liouville equation (8) - in the limit \(q \to 1\), \((\bar{\hbar} \to 0)\) it is transformed in (44). Here the second derivative is replaced on the second difference operator. This naive modification of (8) can be easily predicted from the very beginning. The less obvious modification is the shift of the argument in the potential operator. It is of course will be crucial for the solving the eigenvalue problem exactly.

Using the shift operator in the form
\[
\exp(\bar{\hbar}_2 \partial) F(H) = F(qH),
\]
we can rewrite the last relation as
\[
\left[ \frac{(\sinh)^2(\bar{\hbar}_2 \partial)}{(\sinh)^2 \bar{\hbar}_2} - \mu^2 q^{-3} H^{-2} e^{\bar{\hbar}_2 \partial} \right] F_\lambda(H) = -\lambda^2 F_\lambda(H). \tag{47}
\]
To solve (46) assume that
\[
\mu^2 q^{-5} (q - q^{-1}) = 1, \tag{48}
\]
and replace the eigenvalue parameter
\[
\lambda = [i\theta]_{q^2} = \frac{\sin(\bar{\hbar}_2 \theta)}{\sinh \bar{\hbar}_2}, \quad x = \cos \bar{\hbar}_2 \theta, \tag{49}
\]
Since (46) is the difference operator, put

\[ H = q^{-n}, \]  \hspace{1cm} (50)

\[ c_n(x) = \frac{1}{(q^2; q^2)_n} F_\lambda(H)|_{H=q^{-n}}. \]  \hspace{1cm} (51)

In the new variables (46) takes the form

\[ (1 - q^{2n+2})c_{n+1}(x) + c_{n-1}(x) = 2xc_n(x). \]  \hspace{1cm} (52)

This equation has the form of recurrence relation for orthogonal polynomials. If we put \( c_{-1} = 0 \) and \( c_0 = 1 \), then (51) defines the \( q \)-contiguous Hermit polynomials, which are particular case of the Rogers-Askey-Ismail polynomials \([14]\), or, as the same, the Macdonald polynomials for the \( A_1 \) root system \([4]\). Namely,

\[ c_n(\cos \hbar_2 \theta) = C_n(\cos \hbar_2 \theta; t|q^2)|_{t=0}, \]  \hspace{1cm} (53)

where \( C_n(\cos \hbar_2 \theta; t|q^2) \) is Rogers-Askey-Ismail polynomial

\[ C_n(\cos \hbar_2 \theta; t|q^2) = \sum_{a+b=n, a, b \in \mathbb{Z}} \frac{(t; q^2)_a(t; q^2)_b}{(q^2; q^2)_a(q^2; q^2)_b} e^{i \hbar_2 \theta (a-b)}, \]

and for the Macdonald polynomials \( P_n \)

\[ F_\lambda(H)|_{H=q^{-n}} = (q^2; q^2)_n c_n(\cos \hbar_2 \theta) = \]

\[ (t; q^2)_n P_n(e^{i \hbar_2 \theta}; t|q^2)|_{t=0}. \]  \hspace{1cm} (54)

It is worthwhile to note, that the degrees of polynomials \( n \) play now the role of the "space" variable (53), while their arguments are related to the eigenvalue parameter \( \lambda \) (19), (31). Note that the argument \( \theta \), which defined the energy \( E = \lambda^2/2 \) by (19) lies in the interval \( \theta \in [0, \frac{2\pi}{\hbar_2}] \). The similar phenomena arises also for the zonal spherical functions on quantum "noncompact" symmetric spaces (see \([2, 3]\), where the case of \( U_q(SU(1, 1)) \) was considered).

The asymptotic behavior of \( c_n \) for \( n \to \infty \) is following (14)

\[ (q^2; q^2)_n c_n(\cos \hbar_2 \theta) \sim \left\{ \frac{q^{-in\theta}}{(q^{2i\theta}; q^2)_\infty} + \frac{q^{in\theta}}{(q^{-2i\theta}; q^2)_\infty} \right\}. \]  \hspace{1cm} (55)

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Thus, the Harish-Chandra $c$-function is equal to

$$
c(\lambda) = \frac{1}{(q^{-2i\theta}; q^2)_{\infty}}.
$$

To find a spin chain, which exhibits the same scattering, it is convenient to consider the extended theory and switch on the additional parameter $t$ \((53), (54)\). This theory was considered in \([3]\). It was demonstrated there that the Harish-Chandra $c$-functions for the generic $A_1$ Macdonald polynomials gives rise for the scalar $S$-matrix to the scattering of two special excitations in $Z_N$-Baxter model \([15, 16]\) in the $N \to \infty$ regime. It has two parameters - the modular parameter $\tau$ and the anisotropy parameter $\gamma$. In fact, $\tau$ is a modular parameter of an elliptic curve and $\gamma$ is a point on it. They are parameters of the Sklyanin algebra \([19, 20, 21]\), which allows a solution of corresponding Yang-Baxter equation. The parameters of the Macdonald polynomials are related to later as follows

$$
q = e^{2i\pi \tau}, \quad t = q^{\frac{i\gamma}{\pi}}.
$$

Some interesting solutions corresponding to points in the space $(\tau, \gamma)$ were considered in \([3]\). In our case $t = 0$. Since $\Im m\tau > 0$ and $\Re e\tau = 0$, then $\gamma \to \infty$. Simultaneously $N \to \infty$.

As it follows from the relation (5.2) \([3]\), there is the duality in the scattering picture in this model

$$
(N, \frac{-i\pi \tau}{\gamma}) \sim (\frac{-i\pi \tau}{\gamma}, N).
$$

Thus the same $S$-matrix can be realized in the $Z_N$-Baxter model in the regime

$$
\gamma \to 0, \quad N \to 0.
$$

5 Conclusion

We have found that the wave functions of the q-Liouville Hamiltonian lie in the same family of Macdonald polynomials as the wave functions of the q-Sutherland Hamiltonian \([2, 3, 18]\). In the former case $t = 0$ and $t = q$ in the later (see (54)). The Macdonald polynomials have an interpretation as
the "zonal spherical functions" on the Sklyanin quantum algebra \[3\]. Thus there exists the transition from the usual \( q = 1 \) Sutherland Hamiltonian to the Liouville quantum mechanics through the Sklyanin quantum algebra. It is natural to conjecture that this transition can be generalized on the arbitrary number of particles. Recall that the both Hamiltonians can be also considered as two special reductions of Casimirs on \( U_q(SL_2), \ q \leq 1 \). But this unification is performed in the different way by the increasing of degrees of freedom.

One of the way to study quantum integrable systems is to put the space variable on a lattice. In particular, for the Liouville field theory it was done firstly in \[22\], at least quasiclassically. Our approach is differ from it - instead of the discretization of the space variable we discretized the zero modes of the Liouville field. It is similar to the substitution a continuos gauge group on a discrete subgroup in the Monte-Carlo simulation of gauge theories. Introducing the lattice is equivalent to introducing an infinite hard wall instead of the Liouville exponential potential - the wave functions are vanish for \( n \leq 0 \). The theory, as a modified version of the zero modes dynamics of 2d gravity, is still ultraviolet free since for small distances the potential vanishes \( H^{-2} = q_{n \rightarrow +\infty}^{2n} = 0 \) (see \( \text{(46),(50)} \).

On the other hand the system under consideration resembles the Ruijsenaars relativistic Toda model \[13\] for two particles, though it does not coincide with it. In spite of later it allows to solve it exactly. It is possible also to find the explicit solution in the classical case after the substitution \( i\partial \rightarrow p \). Unfortunately, this solution has not clear group-theoretical interpretation, as the solutions of the open nonrelativistic Toda model \[3\]. At the same time the Ruijsenaars solutions are natural \( q \)-deformation of the non relativistic Toda solutions. It will be interesting to find a similar interpretation for the solution coming from the quantum Lorentz group.

Starting from the Hamiltonian \( \text{(46)} \) it is easy to guess the form of the lattice N-body Toda quantum mechanics. But it will not be easy to justify this Hamiltonian by means of the Iwasawa decomposition of \( A_q(SL(N, \mathbb{C})) \) \[17\] using the same type calculations as above. Nevertheless, investigations of \( q \)-Whitteker functions in a general case and their relations with representations of quantum groups is plausible.

The next desirable generalization of this model is its two-dimensional version in the quantum and classical form. Note that one of the possible versions of the q-deformed classical Liouville field theory can be reconstructed
in principle as in the classical theory from the q-deformed WZW model, proposed, for example, in [23].

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