FOURIER-JACOBI MODELS OF DELIGNE-LUSZTIG CHARACTERS AND DEPTH ZERO LOCAL DESCENT FOR UNITARY GROUPS

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Abstract. In this paper, we deduce explicit multiplicity formulas of the Fourier-Jacobi model for Deligne-Lusztig characters of finite symplectic groups, unitary groups, and general linear groups. We then apply these results to deduce the explicit depth zero local descent (à la Soudry and Tanay) for $p$-adic unitary groups. The result is a concrete example in the context of non-tempered Gan-Gross-Prasad program.

1. Introduction

This paper was motivated by the celebrated Gan-Gross-Prasad conjectures ([GGP1, GGP2]) in the Fourier-Jacobi case, which study certain branching laws for representations of classical groups over finite fields, local fields, or adele rings of global fields. We have a two-fold purpose in this paper.

We first determine the multiplicity of the Fourier-Jacobi model for two Deligne-Lusztig characters of a finite symplectic, unitary, or general linear group. The Bessel case was settled by Reeder [R], who in fact made a qualitative study for the restrictions of Deligne-Lusztig characters in a general setting. Since the characters of Weil representations are not uniform functions, Reeder’s approach does not apply to our situation directly. The new input is the geometrization of the character of Weil representations [GH]. As a consequence, the multiplicity formula has to be viewed as a function of geometric type (see Definition 2.1).

Secondly, we study the local descent of an irreducible depth zero distinguished supercuspidal representation of $GL_m(E)$ to the quasi-split unitary group $U_{2n}(F)$ in the sense of [ST], where $E/F$ is a quadratic extension of $p$-adic local fields, and $n := \lfloor \frac{m-1}{2} \rfloor$. This local descent problem is a special case of the non-tempered local Gan-Gross-Prasad conjecture proposed in [GGP2]. The non-tempered local GGP conjecture is recently proved for general linear groups over $p$-adic local fields in [Ch] and remains open in general. Supercuspidal types encode arithmetic data, rendering them more amenable in number-theoretical applications. Their relationship with the local Langlands correspondence has been deciphered in many cases [BH, Ka], covering all representations pertinent to our setting.

1.1. Overview of Reeder’s work and motivation. Let $G$ be a connected simple algebraic group and $H$ be a closed reductive subgroup of $G$ defined over $\mathfrak{f} := F_q$. Let $F$ denote the Frobenius automorphism so that $G^F$ is the $\mathfrak{f}$-points of $G$. Reeder’s method [R] for computing the pairing

$$\langle R_{T,X}^G, R_{S,H}^H \rangle_{H^F}$$

of two Deligne-Lusztig characters (as class functions) can be outlined in the following steps:

1. Interpret $R_{T,X}^G$ and $R_{S,H}^H$ as sequences of the Deligne-Lusztig characters of $G^{F^\nu}$ and $H^{F^\nu}$ respectively, where $\nu$ varies in a certain arithmetic progression.

2. Demonstrate that the pairing $\langle R_{T,X}^G, R_{S,H}^H \rangle_{H^F}$ is a polynomial function $M$ in $q^\nu$, see Section 3.

3. Show that the degree of $M$ measures the relative complexity between the groups $G$ and $H$, and the leading coefficient of $M$ can be calculated in terms of combinatorial data.
Under specific conditions, the polynomial $M$ is reduced to a constant. This property facilitated Reeder’s application of his method to the Gross-Prasad problem, thereby enabling him to derive a multiplicity formula for the Bessel models of finite special orthogonal groups.

Inspired by Reeder’s work, we formulate a framework that incorporates the Weil representation in the character pairing and investigates the Fourier-Jacobi case of the Gan-Gross-Prasad conjecture. It is worth mentioning that this framework also applies to more generalized models of spherical varieties (as seen in [SV]) over finite fields, which plays a crucial role in [Shi].

1.2. Fourier-Jacobi model for finite classical groups. Let $\mathfrak{f}$ be a finite field of characteristic $p > 2$ and cardinality $q$, along with a fixed algebraic closure $\overline{\mathfrak{f}}$. For a subfield $\mathfrak{e}$ contained in $\overline{\mathfrak{f}}$, designate $\mathfrak{e}_d$ ($d \geq 1$) to be the degree $d$ extension of $\mathfrak{e}$ in $\overline{\mathfrak{f}}$. Let

$$\mathfrak{e}^0 := \{ a \in \mathfrak{e} \mid a \text{ is not contained in a proper subfield of } \mathfrak{e} \}$$

be the set of “regular elements” of $\mathfrak{e}$. When $\mathfrak{e}$ is an even degree extension of $\mathbb{F}_p$, let

$$\mathfrak{e}^1 := \{ a \in \mathfrak{e} \mid a^{\sqrt{|\mathfrak{e}|} + 1} = 1 \}$$

denote the set of “norm one” elements.

Let $G = \text{Sp}_{2n}$, $U_n$ or $\text{GL}_n$ be defined over $\mathfrak{f}$ and let $F$ be the Frobenius endomorphism on $G$. For simplicity, we focus on the basic case of the Fourier-Jacobi model of a $G$-module (see [GGP1, Sections 12 and 19]) in this introduction. The general setting, for which the main reference is [GGP1, Section 12], will be addressed in Section 5.1 and Section 6.2 for the finite field and $p$-adic field cases, respectively.

In the basic case, we are interested in the pairing

$$m(\pi, \sigma) := \langle \pi \otimes \omega_\psi^\vee, \sigma \rangle_{G^F}.$$  \hspace{1cm} (1.1)

Here $\pi, \sigma$ are (virtual) characters of $G^F$ and $\omega_\psi$ denotes the Weil representation of $G^F$ attached to a non-trivial additive character $\psi$ of $\mathfrak{f}$ in the sense of [Ho1]. Note that $m(\pi, \sigma)$ is the multiplicity of $\sigma$ occurring in $\pi \otimes \omega_\psi^\vee$ when $\pi$ and $\sigma$ are irreducible representations of $G^F$. Our first main result, Theorem 4.3, evaluates the pairing (1.1) for arbitrary Deligne-Lusztig characters ([DL]) $\pi = R^G_{T, \chi}$ and $\sigma = R^G_{S, \eta}$, where $T$ and $S$ are $F$-stable maximal tori in $G$, and $\chi \in \text{Irr}(T^F)$, $\eta \in \text{Irr}(S^F)$. When $\chi$ and $\eta$ are regular in the sense that their stabilizers in the corresponding Weyl groups $W_G(T)^F$ and $W_G(S)^F$ are trivial, a straightforward multiplicity formula (4.18) is provided in Section 4.4 (c.f. [R, (9.9)]).

Now let $G = \text{Sp}_{2n}$ or $U_n$, and we evaluate (4.18) in a very special but important scenario. Assume that $T, S$ are anisotropic and $\chi, \eta$ are regular. So $(-1)^{rk G} R^G_{T, \chi}$ and $(-1)^{rk G} R^G_{S, \eta}$ are irreducible cuspidal representations of $G^F$, where $rk G$ denotes the $\mathfrak{f}$-rank of $G$ (see [DL, Theorem 8.3]). There are unique partitions $\lambda = (j^{\lambda_j})$ and $\mu = (j^{\mu_j})$ of $n$ such that

$$T^F \cong \prod_j (j^{l_j})^{\lambda_j}, \quad S^F \cong \prod_j (j^{l_j})^{\mu_j}.$$  

The characters $\chi$ and $\eta$ decompose into

$$\chi = \bigotimes_j \chi_{jk} \otimes \cdots \otimes \chi_{jk} \quad \text{and} \quad \eta = \bigotimes_j \eta_{jk} \otimes \cdots \otimes \eta_{jk}$$

accordingly with $\lambda_{jk}, \mu_{jk} \in \text{Irr}(j^{l_j})$.

**Definition 1.1.** Let $\vartheta_j' \in \text{Irr}(j^{l_j})$ denote the unique nontrivial quadratic character. We say that $\chi$ and $\eta$ intertwine if $\chi_{jk}$ is a $\text{Gal}(\overline{\mathfrak{f}}/\mathfrak{f})$-conjugate of $\eta_{jk} \otimes \vartheta'_j$ for some $1 \leq j \leq n$, $1 \leq k \leq \lambda_j$ and $1 \leq l \leq \mu_j$.

Then we have the following result, which is in analogy with [R, Theorem 1.2].

**Theorem 1.2.** Suppose that $G = \text{Sp}_{2n}$ or $U_n$, and that $T$ and $S$ are anisotropic $F$-stable maximal tori in $G$. For regular characters $\chi \in \text{Irr}(T^F)$, $\eta \in \text{Irr}(S^F)$, we have

$$\langle R^G_{T, \chi} \otimes \omega_\psi^\vee, R^G_{S, \eta} \rangle_{G^F} = \begin{cases} 0, & \text{if } \chi, \eta \text{ intertwine}, \\ 1, & \text{otherwise}. \end{cases}$$
1.3. Remarks on Theorem 4.3. In contrast to the $p$-adic case where the multiplicity one theorem holds ([Sun]), the multiplicity $m(\pi, \sigma)$ for general $\pi, \sigma \in \text{Irr}(G^F)$ could be greater than one in the finite field case by Theorem 4.3 or (4.18). As an application of Theorem 4.3, for $G = U_n$ or $\text{GL}_n$ we give a quick proof of a conjecture in [HS] which asserts that $m(\pi, \sigma)$ is bounded by a function of $n$ that does not depend on $q = |\mathbb{F}|$. This is stated as Theorem 4.5, where we do not attempt to optimize the upper bound however. There has been a good deal of interest and progress in the topic of Weil representations of finite classical groups (see e.g. [GHo, HS, HZ]), and the result given by Theorem 4.3 should have applications towards the relevant problems.

Let us sketch the proof of Theorem 4.3. Inspired by [R], we extend the pairing (1.1) to the group $G^F$ of $F$-points of $G$ and regard its value as a function $M(\nu)$ of $\nu$. As $\nu$ ranges over an arithmetic progression $\mathcal{P}_m := \{1 + md : d \geq 0\}$ with the common difference $m$ being sufficiently divisible, the Frobenius endomorphisms $F^\nu$ act compatibly on certain geometric objects. This ensures that the function $M(\nu)$ has some nice form and converges as $\nu \to \infty$ along $\mathcal{P}_m$. This, in turn, forces that $M(\nu)$ is a constant on $\mathcal{P}_m$ and thereby yields the desired multiplicity formula.

The theta correspondence and see-saw dual pairs might be used to translate the result on Bessel models [R] to the Fourier-Jacobi case. Indeed, such an approach has been employed to study the Gan-Gross-Prasad problems and descent problems over finite fields in [LW1, LW3, LW4, W]. Nonetheless, the theta correspondence is an intricate subject (cf. [AMR, LW2, MQZ, P2, P3]). Thus, it becomes preferable to provide a more direct and independent proof of the Fourier-Jacobi case. This is one of the motivations for the current work.

1.4. The descent of depth zero supercuspidal representations. The local descent construction (cf. [JS, JNQ, ST]) serves as the inverse of the the local Langlands functorial lift from classical groups to general linear groups. This construction is naturally tied with the non-tempered Gan-Gross-Prasad conjecture ([GGP2]). The descent construction from irreducible supercuspidal self-dual representations of general linear groups to quasi-split unitary groups in terms of supercuspidal types was studied in [ST]. Now we describe our second application of Theorem 4.3, which determines the descents of irreducible depth zero supercuspidal representations in terms of supercuspidal types.

Let $F$ be a $p$-adic field, and $E$ be a quadratic extension of $F$. Let $\mathcal{O}_E$ denote the ring of integers in $E$, and $\mathfrak{p}_E$ be the maximal ideal of $\mathcal{O}_E$. Let $\text{SC}_0(\text{GL}_m(E)/\text{GL}_m(F))$ denote the set of irreducible depth zero supercuspidal representations of $\text{GL}_m(E)$ distinguished by $\text{GL}_m(F)$. As a result of [CG], $\text{SC}_0(\text{GL}_m(E)/\text{GL}_m(F))$ is non-empty only if $E/F$ is unramified when $m$ is odd and $E/F$ is ramified when $m$ is even. In these cases, $\text{SC}_0(\text{GL}_m(E)/\text{GL}_m(F))$ is parameterized by the orbits of

$$\mathfrak{c}_{m}^{\circ} := \mathfrak{c}_{m} \cap \mathfrak{c}_{m}^{1}$$

under the $\text{Gal}(\mathbb{F}_{m}/\mathbb{F})$-action. In particular, for each $s \in \mathfrak{c}_{m}^{1}$, we attach an irreducible depth zero supercuspidal representation distinguished representation $\tau_s \in \text{SC}_0(\text{GL}_m(E)/\text{GL}_m(F))$ (see Theorem 6.6).

Let $g^t$ denote the transpose of a matrix $g$, and let $\iota \in \text{Gal}(E/F)$ be the nontrivial element that acts entrywise on any matrix over $E$. Let $W = E^{2n}$ be a $2n$-dimensional vector space endowed with a skew-Hermitian form

$$\langle v_1, v_2 \rangle = v_1^t J_{2n} \iota(v_2) \quad \text{for } v_1, v_2 \in W,$$

where

$$J_{2n} = \begin{pmatrix} \begin{pmatrix} w_n \\ w_n \end{pmatrix} \\ \begin{pmatrix} 1 \\ \cdots \\ 1 \end{pmatrix} \end{pmatrix}_{n \times n}.$$

Let $H$ be the quasi-split unitary group $U_{2n}(F)$ defined by

$$H := U(W) := \{ g \in \text{GL}_{2n}(E) \mid g J_{2n} \iota(g^t) = J_{2n} \}.$$
From now on, we fix a character $\mu$ of $E^\times$ such that $\mu|_{E^F}$ is the quadratic character corresponding to the quadratic extension $E/F$ via the local class field theory.

A lattice $L$ of $W$ is called self-dual if

$$L = \{ v' \in W \mid \langle v', v \rangle \in \mathcal{O}_E \text{ for every } v \in L \}.$$  

The set of self-dual lattices forms a single $H$-orbit. We fix a self-dual lattice $L$ from now on. Let $$H_L := \{ g \in H : gL = L \}, \quad H_{L,+} := \{ g \in H : gL \subset p_E L \} \quad \text{and} \quad H_L := H_L/H_{L,+}.$$  

For simplicity, we employ the same symbol to represent an $H_L$-module and its inflation to $H_L$ through the quotient map $H_L \to H_L$. According to the ramification of $E/F$, we have the following two cases:

- The extension $E/F$ is unramified and $m + 1 = 2n$: The group $H_L$ is naturally isomorphic to the unitary group $U_{2n}$ defined over $f$. The group $H_L$ has a maximal tori $S_0$ such that the set of $f$-points of the dual tori $S_0^*$ is naturally isomorphic to $\mathfrak{e}_m^1 \times \mathfrak{e}_1^1$ up to the $\text{Gal}(\mathfrak{e}_m/f) \times \text{Gal}(f/f)$-action. For $(s, a) \in \mathfrak{e}_m^1 \times \mathfrak{e}_1^1$, let $\sigma_{s,a}$ denote the the cuspidal representation $(-1)^n R_{S_0,(-s,a),1}^H$. Then the compactly induced module

$$\sigma_{s,a} := \text{c-Ind}^H_{H_L} (\sigma_{s,a} \otimes \xi_{\mu}^{-1})$$

is a depth zero supercuspidal representation of $H$ (see [MP, Proposition 6.6]). Note that the set

$$\{ \sigma_{s,a} \mid a \in \mathfrak{e}_1^1 \}$$

only depends on the $\text{Gal}(\mathfrak{e}_m/f)$-orbit of $s$ by [DL, Theorem 6.8].

- The extension $E/F$ is ramified and $m = 2n$: The group $H_L$ is naturally isomorphic to the symplectic group $Sp_{2n}$ defined over $f$. The group $H_L$ has a maximal tori $S_0$ such that the $f$-points of the dual tori $S_0^*$ is naturally isomorphic to $\mathfrak{e}_m^1$ up to $\text{Gal}(\mathfrak{e}_m/f)$-action. For $s \in \mathfrak{e}_m^1$, let $\bar{\sigma}_s$ denote the cuspidal representation $(-1)^n R_{S_0,(-s),1}^H$. Then the compactly induced module

$$\sigma_s := \text{c-Ind}^H_{H_L} (\bar{\sigma}_s \otimes \xi_{\mu}^{-1}).$$

is a depth zero supercuspidal representation of $H$. Note that the $H$-module $\sigma_s$ only depends on the $\text{Gal}(\mathfrak{e}_m/f)$-orbit of $s$.

The local descent construction $\mathcal{D}$ is a map sending certain irreducible representations of $GL_m(E)$ to representations of $U_{[\frac{m}{2}]_f}(F)$. We refer to (6.3) for the definition of $\mathcal{D}$ with respect to the choice of $\mu$.

Now we can state the main result on local descent for unitary groups.

**Theorem 1.3.** Retain the notation above. Let $s \in \mathfrak{e}_m^{1,1}$, with $m \geq 2$.

1. If $E/F$ is unramified and $m = 2n - 1$, then $\mathcal{D}(\tau_s)$ contains $\sigma_{s,a}$ as a multiplicity free direct summand for each $a \in \mathfrak{f}_1^{1,2}$.
2. If $E/F$ is ramified and $m = 2n$, then

$$\mathcal{D}(\tau_s) = \sigma_s.$$  

**Theorem 1.3** can be viewed as an explicit realization of supercuspidal representations, which might find applications in the study of Rankin-Selberg integrals, local L-functions, and other local factors (see e.g. [AKM+, ST]).

The approach of this paper was first used in [LMNW], a work in progress, to study the depth zero local descent for $p$-adic special orthogonal groups in the sense of [JNQ], where the upshot was to extend the descent method beyond the supercuspidal case and thereby give more examples for the non-tempered Gan-Gross-Prasad conjecture. In future works, we also hope to extend the current paper and [LMNW] to representations of positive depth.

1.5. **Organization of the paper.** The contents of the paper are divided into two parts: The first, spanning Section 2 to Section 5, focuses on the Fourier-Jacobi models for finite classical groups. The second, spanning Section 6 to Section 9, addresses the depth zero local descent for $p$-adic unitary groups.
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2. Preliminaries

In this section we make some preliminaries from algebraic geometry, and collect some facts about Weil representations of finite classical groups.

2.1. Functions of geometric type. We introduce the following notion.

**Definition 2.1.** Let $\mathcal{P} \subset \mathbb{Z}_+$ be an arithmetic progression. A function $M : \mathcal{P} \to \mathbb{C}$ is said to be of geometric type if it is of the form

$$M(\nu) = \frac{\sum_{i=1}^{k} a_i \alpha_i^\nu}{\sum_{j=1}^{d} b_j \beta_j^\nu}, \quad \nu \in \mathcal{P},$$

where $a_i, \alpha_i, b_j, \beta_j \in \mathbb{C}$, and the denominator is non-zero for every $\nu \in \mathcal{P}$.

We have the following elementary but crucial lemma, for which we will give a proof for completeness.

**Lemma 2.2.** Let $M$ be a function of geometric type defined on an arithmetic progression $\mathcal{P}$. If $M$ is integer-valued and has a finite limit as $\nu \to \infty$ along $\mathcal{P}$, then $M$ is a constant function.

**Proof.** By assumption, there is an integer $c$ such that $M(\nu) = c$ for all $\nu$ sufficiently large. Since $M(\nu) - c$ is also a function of geometric type, it suffices to prove that $M$ is zero when $c = 0$. Write $\mathcal{P} = \{ \mu, \mu + m, \mu + 2m, \ldots \}$ with $\mu, m \in \mathbb{Z}_+$. Define a function $\tilde{M}$ on the set $\mathbb{N}$ of natural numbers by $\tilde{M}(d) = M(\mu + md)$ for $d \in \mathbb{N}$. Without loss of generality, we can assume

$$\tilde{M}(d) = \frac{\sum_{i=1}^{k} a_i' \alpha_i'^d}{\sum_{j=1}^{d} b_j \beta_j'^d}$$

for $d \in \mathbb{N}$

such that $a_i', \alpha_i', b_j, \beta_j' \in \mathbb{C}$, and $\alpha_i'$s are distinct and non-zero. The assumption $M(\nu) = 0$ for $\nu$ sufficiently large implies that $\sum_{i=1}^{k} a_i' \alpha_i'^d = 0$ for $d$ sufficiently large. Note that the determinant of the $k \times k$ Vandermonde matrix $(\alpha_i'^j)$ is non-zero. It implies that all $a_i'$s must be zero, thereby concluding the proof. \hfill $\Box$

We will focus on functions of geometric type defined on arithmetic progressions $\mathcal{P}$ starting from $\nu = 1$, that is,

$$\mathcal{P} = \mathcal{P}_m := \{ 1 + md : d \geq 0 \}, \quad \text{with } m \in \mathbb{Z}_+.$$

If $M$ and $M'$ are two such functions defined on $\mathcal{P}_m$ and $\mathcal{P}_{m'}$ respectively, then any linear combination of $M$ and $M'$, and the product $M \cdot M'$, are still functions of geometric type defined on $\mathcal{P}_m \cap \mathcal{P}_{m'} = \mathcal{P}_{\text{lcm}(m,m')}$ where $\text{lcm}(m,m')$ denotes the least common multiple of $m$ and $m'$.

The Definition 2.1 is motivated from algebraic geometry as follows. Let $\mathfrak{f}$ be a finite field of characteristic $p$ and cardinality $q$, with a fixed algebraic closure $\overline{\mathfrak{f}}$. Fix a prime number $\ell$ different from $p$, and fix once for all an identification $\overline{\mathfrak{f}} \cong \mathbb{C}$. For a quasi-projective scheme $X$ defined over $\mathfrak{f}$, denote by $D^b(X, \overline{\mathfrak{f}})$ the bounded derived category of constructible $\ell$-adic sheaves on $X$. 


For \( \mathcal{F} \in D^b(X, \mathcal{O}_X) \), the Grothendieck-Lefschetz trace formula ([SGA5, (2') p471]) relates the local and global traces of the geometric Frobenius \( F \):

\[
\sum_{x \in X^{F^\nu}} \sum_{i} (-1)^i \text{Tr}(F_x^{\nu}, H^i(\mathcal{F}_x)) = \sum_{i} (-1)^i \text{Tr}(F^{\nu}, H^i_c(X \times_f \mathcal{F}_x)) \quad \text{for } \nu \in \mathbb{Z}_+.
\]

where \( \mathcal{F}_x \) and \( \mathcal{F}_f \) are the pullbacks of \( \mathcal{F} \) equipped with canonical Frobenius actions \( F_x^{\nu} \) and \( F^{\nu} \) respectively.

Examining the right-hand side, it is evident that (2.1) is a function of geometric type with respect to \( \nu \in \mathbb{Z}_+ \). In particular, by setting \( \mathcal{F} \) as the constant sheaf, one concludes that the cardinality \( |X^{F^\nu}| \) is a function of geometric type (defined on suitable \( \mathcal{F} \subset \mathbb{Z}_+ \)).

The inner sum on the left hand side of (2.1) is a function on \( X^{F^\nu} \), which we denote by

\[
f^{\mathcal{F},(\nu)}(x) := \sum_{i} (-1)^i \text{Tr}(F_x^{\nu}, H^i(\mathcal{F}_x)), \quad x \in X^{F^\nu}.
\]

The procedure of associating a function to \( \mathcal{F} \) is called is called Grothendieck’s \textit{faisceaux-fonction correspondence}. We favor sheaves over functions because there are more functorial operations on sheaves. Notably, the base change operation is crucial in this paper, namely the \textit{faisceaux-fonction correspondence} relates \( f^{\mathcal{F},(\nu)}(x) \) for \( \nu \in \mathbb{Z}_+ \) and their summations (2.1) in a natural way.

2.2. The Weil representation of a symplectic group and its geometrization. Assume from now on that \( p \) is odd. In the following, we view a symplectic space of dimension \( 2n \) defined over \( \mathcal{O}_k \) as an additive affine group scheme \( V \) defined over \( \mathcal{O}_k \) together with a symplectic from

\[
\langle \cdot , \cdot \rangle_V : V \times V \rightarrow \mathbb{A}^1 \quad (\mathbb{A}^1 \text{ is equipped with the natural additive group structure})
\]

so that, by restriction to the set \( V^F \) of \( f \)-points, \( \langle \cdot , \cdot \rangle_V \) gives the symplectic structure.

The Heisenberg group associated to \( V \) is the scheme \( \mathbb{H}_V := V \times \mathbb{A}^1 \) equipped with the following group law

\[
(v, z)(v', z') = (v + v', z + z' + \frac{1}{2} \langle v, v' \rangle_V) \quad \text{for all} \, v, v' \in V, \, \text{and} \, z, z' \in \mathcal{O}_k.
\]

Now let \( \nu \in \mathbb{Z}_+ \). Fix a nontrivial additive character \( \psi \) of \( \mathcal{O}_k \). Then

\[
\psi^{(\nu)} := \psi \circ \text{Tr}_{\mathcal{O}_k/\mathcal{O}_f}
\]

is a nontrivial additive character of \( \mathcal{O}_f \). By [G, Theorem 2.4 (a)], up to isomorphism there is a unique irreducible representation \( \omega^{(\nu)}_\psi \) of \( \text{Sp}(V^{F^\nu}) \times \mathbb{H}^\nu_V \), called the Weil representation, such that

- \( \omega^{(\nu)}_\psi |_{\mathbb{H}^\nu_V} \) is irreducible and the center of \( \mathbb{H}^\nu_V \) acts by \( \psi^{(\nu)} \),
- if \( \text{Sp}(V^{F^\nu}) \cong \text{SL}_2(\mathcal{O}_f) \), then \( \omega^{(\nu)}_\psi |_{\text{Sp}(V^{F^\nu})} \) is non-isomorphic to its complex conjugate.

The Weil representation is geometrized by Gurevich-Hadani [GH]. We need the following reformulation of their main result.

**Theorem 2.3.** There exists an object \( \mathcal{F} \in D^b(G \times V, \mathbb{Q}_l) \) of pure weight \( 2n \) such that \( \mathcal{F}[n^2 + n] \) is perverse and

\[
\omega^{(\nu)}_\psi(g) = f^{\mathcal{F},(\nu)}(g, 0), \quad \text{for } g \in G^{F^\nu}.
\]

Here \( \omega^{(\nu)}_\psi \) denote the character of the Weil representation and \( f^{\mathcal{F},(\nu)} \) is the function on \( G^{F^\nu} \times V^{F^\nu} \) given by Grothendieck’s \textit{faisceaux-fonction correspondence} (2.2).

**Proof.** By [GH, Theorem 3.2.2.1] and [GH, (1.2.2)] there is an object \( \mathcal{X} \in D^b(G \times V, \mathbb{Q}_l) \) of pure weight zero such that \( \mathcal{X}[n^2 + n] \) is perverse and

\[
f^{\mathcal{X},(\nu)}(g, 0) = \frac{1}{\dim \omega^{(\nu)}_\psi} \omega^{(\nu)}_\psi(g).
\]

Since \( \dim \omega^{(\nu)}_\psi = q^{nm} \), we conclude that the \((-n)\)-th Tate twist \( \mathcal{X}(-n) \) meets the requirement of the theorem. See [D] for the theory of weights of \( \ell \)-adic sheaves.
A beneficial consequence of Theorem 2.3, which is useful to us later, is that the sums (4.4) of character values involving ωψν are functions of geometric type by the Grothendieck-Lefschetz trace formula, see Section 4.1.

The character formula for ωψ := ωψ(1) has been understood through many works (see [Ho2, G, GH, T] for example). For our purpose, we recall the restriction of ωψν to semisimple elements as given in [G].

Let us represent a partition λ in the form λ = (jλ), indicating that λ contains λj parts each of size j. Then the size of λ is given by |λ| = \sum jλj. Following [R, Section 9], the G'-conjugacy classes of F'-stable maximal tori in G are parameterized by pairs of partitions (λ, λ') such that |λ| + |λ'| = n. Under this parameterization, an F'-stable maximal torus T of G corresponds to a pair of partitions (λ, λ') such that

\[ T^F \cong \prod_j [(j^s)^{\lambda_j} \times (\hat{j}^s)^{\lambda'_j}] \]  

(2.3)

Let \( \vartheta_j \) and \( \vartheta'_j \) represent the unique nontrivial quadratic characters of \( j^s \) and \( \hat{j}^s \), respectively. Define the quadratic character \( \vartheta_T \) of \( T^F \) as the product of the \( \vartheta_j \)'s and \( \vartheta'_j \)'s according to the isomorphism (2.3). In accordance with (2.3), for an element \( s \in T^F \), the component of \( s \) in the \( j \)-th block is expressed as

\[ (s_{j1}, \ldots, s_{j\lambda_j}; s'_{j1}, \ldots, s'_{j\lambda'_j}) \].

Following [G, Corollary 4.8.1], we have

\[ \omega_\psi(s) = (-1)^{l(T^F, s)} \vartheta_T(s) q^{\frac{1}{2} \dim V^*}, \]

(2.4)

where \( V^* = \ker(s - 1_V) \) represents the eigenspace of \( s \) acting on \( V \) with eigenvalue 1, and

\[ l(T^F, s) := |\{ (j, k) \mid 1 \leq k \leq \lambda'_j, \ s'_{jk} \neq 1 \}|. \]

### 2.3. Weil representations of \( U_n \) and \( GL_n \)

In this section, let \( G \) be the unitary group \( U_n \) or general linear group \( GL_n \) defined over \( \mathbb{F} \). Let \( V \) be a symplectic space of dimension \( 2n \) defined over \( \mathbb{F} \). Then \( G \) is naturally embedded in \( \operatorname{Sp}(V) \) and we let \( \omega_\psi^{\hat{\lambda}} \) denote the restriction of the Weil representation of \( \operatorname{Sp}(V^{F^{\hat{\lambda}}}) \) to \( G^{F^{\hat{\lambda}}} \). As before, let \( \omega_\psi := \omega_\psi^{(1)} \).

It is worth noting that the Weil representation of \( G^{F^{\hat{\lambda}}} \) defined in [G], denoted by \( \omega_\psi^{\hat{\lambda}} \), differs from \( \omega_\psi \). Their relationship can be expressed as

\[ \omega_\psi \cong \omega_\psi \otimes \chi_G, \]

where \( \chi_G \) represents the quadratic character of \( G^{F^{\hat{\lambda}}} \) defined by

\[ \chi_G(g) := \det(g)^{\frac{n}{2}}, \quad g \in G^{F^{\hat{\lambda}}}, \]

(2.5)

with \( \varepsilon = -1 \) for \( G = U_n \) and \( \varepsilon = 1 \) for \( G = GL_n \).

Let \( T \) be an \( F \)-stable maximal torus of \( G \). Then \( T \) is an \( F \)-stable maximal torus of \( \operatorname{Sp}(V) \) via the embedding \( G \hookrightarrow \operatorname{Sp}(V) \). Recall the character \( \vartheta_T \) of \( T^F \) defined in the previous subsection. We see that

\[ \vartheta_T = \chi_G \big|_{T^F}, \]

where \( \chi_G \) is given in (2.5). We now recall the character values of \( \omega_\psi \) when restricted to the tori in \( G \). We associate the pair \( (\lambda, \lambda') \) of partitions such that \( |\lambda| + |\lambda'| \) to an \( F \)-stable maximal torus in \( G \) if (2.3) is satisfied. This gives an identification of the \( G^{F^{\hat{\lambda}}} \)-conjugacy classes of \( F^{\hat{\lambda}} \)-stable maximal tori in \( G \) with a subset of the set of pairs of partitions. More precisely, we have the following results by [G, Corollaries 1.4 and 4.8.2], which can be also deduced from (2.4).

**Lemma 2.4.** Retain the above notation.

(i) If \( G = U_n \), then \( \lambda_j = 0 \) for \( j \) odd, and \( \lambda'_j = 0 \) for \( j \) even, and one has

\[ \omega_\psi(s) = (-1)^n \vartheta_T(s) (-q)^{\frac{1}{2} \dim V^*}, \quad s \in T^F. \]
(ii) If $G = \text{GL}_n$, then $\chi'_j = 0$ for all $j$, and one has
\[ \omega_\psi(s) = \vartheta_T(s) q^{\frac{1}{2} \dim V_s}, \quad s \in T^F. \]

We conclude this section with the following remarks on the character formula.

**Remark 2.5.** Suppose that $G$ is a finite symplectic, unitary, or general linear group.

1. There exists $m \in \mathbb{Z}_+$ (see Section 3.4.1 for the precise condition) such that for $\nu \in \mathcal{P}_m$, the character formula for $\omega_\psi^{(\nu)}$ at $s \in T^{F_\nu}$ is similar to (2.4) with $q$ replaced by $q'$ and $\vartheta_T$ replaced by
\[ \vartheta_T^{(\nu)} := \vartheta_T \circ N_\nu^T, \]
where $N_\nu^T : T^{F_\nu} \to T^F$ is the norm map.

2. For a general element $g \in G^{F_\nu}$, the formula for $\omega_\psi^{(\nu)}(g)$ involves the Weil index of $\psi$, and by $[G$, Theorem 4.4$]$ it holds that
\[ |\omega_\psi^{(\nu)}(g)| = q^{\frac{1}{2} \dim V_g}, \]
where $V_g = \text{Ker}(g - 1_V)$.

### 3. Reeder’s multiplicity formula

This section is purely expository. We give a brief survey for the general framework provided by Reeder in $[R]$, which is fundamental for the study of various branching problems related to Deligne-Lusztig characters of finite groups of Lie type.

#### 3.1. Maximal tori

Let $G$ be a connected reductive $\overline{\mathfrak{f}}$-algebraic group defined over $\mathfrak{f}$, with Frobenius $F$. Let $T$ be an $F$-stable maximal torus in $G$, with Weyl group $W_G(T) = N_G(T)/T$. Then $W_G(T)^F = N_G(T)^F/T^F$ by the Lang-Steinberg theorem. Let $s \in T^F$ be a semisimple element, and denote by $G_s := C_G(s)^0$ the identity component of the centralizer $C_G(s)$ of $s$ in $G$. Put
\[ N_G(s,T)^F := \{ \gamma \in G^F \mid s^\gamma \in T \}, \]
where $s^\gamma := \gamma^{-1}s \gamma$. Then $G_s^F \times N_G(s,T)^F$ acts on $N_G(s,T)^F$, and we put
\[ N_G(s,T)^F := G_s^F \backslash N_G(s,T)^F. \]

A formula for $[\overline{N}_G(s,T)^F]$ is given in $[R]$ as follows. Put $W_G := W_G(T_0)$ for a fixed $F$-stable maximal torus $T_0$ in $G$ that is contained in an $F$-stable Borel subgroup of $G$. One can associate to $T$ is a cohomology class
\[ \text{cl}(T, G) \in H^1(F, W_G). \]
Define $W_{G_s}$ in a similar way. A map
\[ j_{G_s} : H^1(F, W_{G_s}) \to H^1(F, W_G) \]
is defined in $[R]$, which sends the class of an $F$-stable maximal torus in $G_s$ to the class of the same torus in $G$. Noting that $G_s$ and $G$ are connected groups of the same absolute rank. Denote by $\mathcal{T}(G)$ the set of all $F$-stable maximal tori in $G$, and for $\omega \in H^1(F, W_G)$ put
\[ \mathcal{T}_\omega(G) := \{ T \in \mathcal{T}(G) \mid \text{cl}(T,G) = \omega \}. \]

By $[R$, Corollary 2.3$]$, for $T \in \mathcal{T}_\omega(G)$ the set $N_G(s,T)^F \neq \emptyset$ if and only if $j_{G_s}^{-1}(\omega) \neq \emptyset$, in which case
\[ [\overline{N}_G(s,T)^F] = \sum_{\nu \in j_{G_s}^{-1}(\omega)} \frac{|W_G(T)^F|}{|W_{G_s}(T_\nu)^F|}, \]
where $T_\nu$ is an arbitrary member in $\mathcal{T}_\nu(G_s)$, for each $\nu \in j_{G_s}^{-1}(\omega)$. 

3.2. Deligne-Lusztig characters. An element \( g \in G^F \) has the Jordan decomposition \( g = su \), where \( s \in G^F \) is semisimple and \( u \in G_s^F \) is unipotent. Let \( T \in \mathcal{T}_\omega(G) \) and let \( \chi \in \text{Irr}(T^F) \). The virtual character \( R_{T,\chi}^{G} \) of \( G^F \) has the reduction formula ([DL])

\[
R_{T,\chi}^{G}(su) = \sum_{\gamma \in N_{G}(s,T)^F} \gamma \chi(s)Q_{T}^{G}(u),
\]

where \( \gamma \in N_{G}(s,T)^F \) is a representative of \( \bar{\gamma} \), \( \gamma T := \gamma T \gamma^{-1} \), \( \chi := \chi \circ \text{Ad}(\gamma^{-1}) \in \text{Irr}(\gamma T^F) \), and \( Q_{T}^{G} \) denotes the Green function. Breaking \( N_{G}(s,T)^F \) into \( W_{G}(T)^F \)-orbits,

\[
R_{T,\chi}^{G}(su) = \sum_{\imath \in J_{G_{s}}(\omega)} Q_{T_{\imath}}^{G_{s}}(u)\chi_{\imath}(s),
\]

where we define \( \mathcal{O}_{\imath} \) to be the \( W_{G}(T)^F \)-orbit in \( N_{G}(s,T)^F \) corresponding to \( \imath \) as in [R, Lemma 2.2], and

\[
(3.3) \quad \chi_{\imath} := \sum_{\gamma \in \mathcal{O}_{\imath}} \gamma \chi.
\]

Note that \( \chi_{\imath} \) is a well-defined function on \( Z(G_{s})^{F} \), where \( Z(G_{s}) \) denotes the center of \( G_{s} \). It turns out that

\[
(3.4) \quad R_{T,\chi}^{G}(zu) = \sum_{\imath \in J_{G_{s}}(\omega)} Q_{T_{\imath}}^{G_{s}}(u)\chi_{\imath}(z), \quad \text{if} \ G_{z} = G_{s}.
\]

For later use, the value \( \chi_{\imath}(s) \) can be also unfolded as

\[
(3.5) \quad \chi_{\imath}(s) = \frac{1}{|W_{G_{s}}(T_{\imath})^{F}|} \sum_{x \in W_{G}(T)^{F}} \gamma x \chi(s),
\]

where \( \gamma \) is an arbitrary element of \( N_{G}(s,T)^F \) such that \( \bar{\gamma} \in \mathcal{O}_{\imath} \).

3.3. Multiplicity formula. Let \( S \in \mathcal{T}(G) \). Assume that \( f : G^F \to \mathbb{C} \) is a virtual character, supported on the set of elements \( g \in G^F \) whose Jordan decomposition \( g = su \) satisfies that \( \text{Ad}(G^F) \cdot s \cap S \neq \emptyset \). Let \( G_{s}^{\text{un}} \) be the set of unipotent elements of \( G_{s} \), and let \( \mathcal{U}(G_{s}^{\text{un}}) \) be the finite set of \( \text{Ad}(G_{s}^{\text{un}}) \)-orbits in \( (G_{s}^{\text{un}})^{F} \). By [R, (5.2)],

\[
(3.6) \quad \frac{1}{|G^F|} \sum_{g \in G^F} f(g) = \sum_{s \in S^F} \frac{1}{|N_{G}(s,S)^F|} \sum_{[u] \in \mathcal{U}(G^F)} \frac{f(su)}{|C_{G_{s}}(u)|}.
\]

Let \( I(S) \) be an index set for the set of subgroups \( \{ G_{s} \mid s \in S \} \), which is finite. For \( \imath \in I(S) \), let \( G_{\imath} \) be the corresponding connected centralizer, and put

\[
S_{\imath} := \{ s \in S \mid G_{s} = G_{\imath} \},
\]

so that \( S \) has a finite partition

\[
S = \bigcup_{\imath \in I(S)} S_{\imath}.
\]

The Frobenius \( F \) acts on \( S \), hence on \( I(S) \) as well. Thus (3.6) has a refinement

\[
(3.7) \quad \frac{1}{|G^F|} \sum_{g \in G^F} f(g) = \sum_{\imath \in I(S)^F} \frac{1}{|N_{G}(\imath,S)^F|} \sum_{s \in S_{\imath}} \frac{f(su)}{|C_{G_{s}}(u)|},
\]

where \( N_{G}(\imath,S)^F = N_{G}(s,S)^F \) for any \( s \in S_{\imath} \).

3.4. Progression of Frobenius. Recall the arithmetic progressions \( \mathcal{P}_m, m \in \mathbb{Z}_+ \), which start from \( \nu = 1 \). We have the following general remarks from [R, §5.4, §5.5].
3.4.1. Let $\mathcal{B}_G$ be the flag variety of $G$. For a unipotent class $[u] \in \mathcal{U}(G^F)$, let $\mathcal{B}_G^u$ be the variety of $u$-fixed points. Then $\mathcal{B}_G^u$ is equi-dimensional, of dimension

$$d_G(u) := \dim \mathcal{B}_G^u = \frac{1}{2}(\dim C_G(u) - \rk G),$$

where $\rk G$ denotes the absolute rank of $G$.

Assume that $p$ is a good prime for $G$. For the classical groups considered in this paper, it amounts to the condition that $p$ is odd. For $T \in \mathcal{T}(G)$, denote by $Q^G_{T,\nu}$ the Green function for $T$ on $G^{F\nu}$. Then there exist $m \in \mathbb{Z}_+$ and Green polynomials $Q_{\omega,u}(t) \in \mathbb{Z}[t]$ of degree at most $d_G(u)$, where $\omega \in H^1(F,W_G)$ and $[u] \in \mathcal{U}(G^F)$, such that the following hold for all $\nu \in \mathcal{P}_m$:

- $F^\nu = F$ on $W_G$, and the class $\text{cl}(T,G)$ is the same with respect to $F$ or $F^\nu$;
- $F^\nu = F$ on $A_G(C)$, where $C = \text{Ad}(G) \cdot u$ and $A_G(C)$ is the component group of the centralizer of some $F$-fixed element in $C$, and the class of $u$ in $G^F$ or $G^{F\nu}$ corresponds to the same class in $H^1(F,A_G(C))$;
- $Q^G_{T,\nu}(u) = Q_{\omega,u}(q^\nu)$, where $\omega = \text{cl}(T,G)$.

Moreover, if $u = 1$ then the leading term of $Q_{\omega,1}(t)$ is $\epsilon_G(\omega)t^{d_G(1)}$, where $\epsilon_G(\omega) = (-1)^{\rk G + \rk T}$ for $T \in \mathcal{T}_\omega(G)$, $\rk$ denotes the $f$-rank, and $d_G(1) = \dim \mathcal{B}_G$ is the number of positive roots of $G$.

3.4.2. There exist $m \in \mathbb{Z}_+$, and polynomials $P_{\nu,u} \in \mathbb{Z}[t]$ of degree equal to $\dim C_{G_\nu}(u)$ and leading coefficient $\left| A_{\nu}(u) \right|$, where $\nu \in I(S)^F$, $[u] \in \mathcal{U}(G^F)$ and $A_{\nu}(u)$ is the component group of $C_{G_\nu}(u)$, such that the following hold for all $\nu \in \mathcal{P}_m$:

- $F^\nu = F$ on $I(S)$, and the conditions on $m$ in Section 3.4.1 hold for every $G_\nu$, $\nu \in I(S)$;
- $\left| C_{G_\nu}(u) F^\nu \right| = P_{\nu,u}(q^\nu)$;
- $\left| G_{\nu}(\iota,S) F^\nu \right| = \left| F_{\nu}(\iota,S) F^\nu \right|$.

4. Multiplicity Formula for Fourier-Jacobi Models

Assume from now on that $G$ is one of the classical groups $\text{Sp}_{2n}$, $\text{U}_n$ or $\text{GL}_n$ as before. In this section, following Reeder’s method [R] we derive a formula for the multiplicity

$$M(1) := \langle R^G_{T,\chi^{(\omega)}} \otimes \omega_\psi^{(\nu)}, R^G_{S,\eta^{(\nu)}} \rangle_{G^F}$$

associated to two Deligne-Lusztig characters, where $S, T \in \mathcal{T}(G)$, $\chi \in \text{Irr}(T^F)$ and $\eta \in \text{Irr}(S^F)$. We outline the strategy as follows.

The method is to show that the integer valued function

$$(4.1) \quad M(\nu) := \langle R^G_{T,\chi^{(\nu)}} \otimes \omega_\psi^{(\nu)}, R^G_{S,\eta^{(\nu)}} \rangle_{G^{F\nu}}$$

is of geometric type and has a finite limit as $\nu \to \infty$ along some arithmetic progression $\mathcal{P}_m$ starting from $\nu = 1$. Here $\chi^{(\nu)} := \chi \circ N^T_T$ is a character of $T^{F\nu}$, and likewise $\eta^{(\nu)}$ is a character of $S^{F\nu}$. This implies that $M$ is constant on $\mathcal{P}_m$ by Lemma 2.2, and therefore

$$M(1) = \lim_{\nu \to \infty, \nu \in \mathcal{P}_m} M(\nu).$$

The main result is stated as Theorem 4.3, and an explicit formula for regular Deligne-Lusztig characters is given by (4.18). As an application we prove a conjecture of Hiss and Schröer [HS] in type $A$, which is presented as Theorem 4.5.

For convenience, we write $\nu \rightarrow \mathcal{P}_m$ to indicate that $\nu \to \infty$ along $\mathcal{P}_m$. For two functions $A(\nu)$ and $B(\nu)$ defined on $\mathcal{P}_m$, we write

$$A(\nu) \approx_{\nu, \mathcal{P}_m} B(\nu)$$

if $A(\nu) = B(\nu)C(\nu)$ for some function $C(\nu)$ which converges to 1 as $\nu \rightarrow \mathcal{P}_m, \infty$. 
4.1. Multiplicity as a function of geometric type. We first prove the following result.

**Proposition 4.1.** Let \( m \in \mathbb{Z}_+ \) be as in Section 3.4.2. Then (4.1) defines a function \( M \) of geometric type on \( \mathcal{P}_m \).

**Proof.** Define the function

\[
f^{(\nu)} : G^{F^\nu} \to \mathbb{C}, \quad f(g) = \frac{R_{\gamma}^{G}}{T,\chi^{(\nu)}(g)} \cdot R_{S,\eta^{\nu}}(g) \cdot \omega^{(\nu)}(g).
\]

Then we have that

\[
M(\nu) = \frac{1}{|G^{F^\nu}|} \sum_{g \in G^{F^\nu}} f^{(\nu)}(g).
\]

By (3.4) and (3.7), we find that

\[
M(1) = \sum_{\iota \in I(S)^F, [u] \in \mathcal{U}(G_{\iota}^F)} \sum_{v \in j_{G_1}^{-1}(\text{cl}(T, G))} \left| \mathcal{N}_{G(t, S)^F} \right| \sum_{s \in S_{G,\iota}^{F^\nu}} \chi^F_{\nu}(s) \cdot \eta_{\iota}(s) \cdot \omega_{\psi}(su),
\]

where the middle sum is over \( v \in j_{G_1}^{-1}(\text{cl}(T, G)) \) and \( \varsigma \in j_{G_1}^{-1}(\text{cl}(S, G)) \).

Denote \( \alpha = (\iota, u, \nu, \varsigma) \) the summation indices of quadruples where

\[
\iota \in I(S)^F, \quad [u] \in \mathcal{U}(G_{\iota}^F), \quad v \in j_{G_1}^{-1}(\text{cl}(T, G)), \quad \varsigma \in j_{G_1}^{-1}(\text{cl}(S, G)).
\]

They are unchanged if \( F \) is replaced by \( F^{\nu} \) with \( \nu \in \mathcal{P}_m \), thus for such \( \nu \) we have that

\[
M(\nu) = \sum_{\alpha} \Psi_{\alpha}(q^{\nu}) \Theta_{\alpha}(\nu),
\]

where

\[
(4.3) \quad \Psi_{\alpha}(t) := \frac{Q_{G_{\iota,\iota}^{F,\nu}}(t)Q_{S,\iota}^{F,\nu}(t)}{|\mathcal{N}_{G(t, S)^F}| |P_{\iota,\iota}(t)|},
\]

is a rational function of \( t \), and

\[
(4.4) \quad \Theta_{\alpha}(\nu) := \sum_{s \in S_{G,\iota}^{F^\nu}} \chi^{F^\nu}_{\nu}(s) \cdot \eta^{F^\nu}_{\iota}(s) \cdot \omega^{F^\nu}_{\psi}(su).
\]

Here \( Q_{G_{\iota,\iota}^{F,\nu}}(t) \) and \( Q_{S,\iota}^{F,\nu}(t) \) are the Green polynomials from Section 3.4.1, and \( P_{\iota,\iota}(t) \) is the polynomial from Section 3.4.2. Note that

\[
S_\iota \subset Z_\iota := Z(G_\iota).
\]

Recall from (3.3) that \( \chi_{\nu} \) and \( \eta_{\iota} \) are sums of characters on \( Z_\iota \), thus the values \( \chi_{\nu}^{(\nu)}(s) \) and \( \eta_{\iota}^{(\nu)}(s) \), \( s \in S_{G,\iota}^{F^\nu} \), are the local traces of \( F^{\nu} \) on the corresponding sheaves. By the geometrization Theorem 2.3 and the Grothendieck-Lefschetz trace formula, each \( \Theta_{\alpha}(\nu) \) is a function of geometric type on \( \mathbb{Z}_+ \). More precisely, denote by \( \mathcal{L}_{\gamma} \) and \( \mathcal{L}_{\delta} \) the rank one sheaves on \( Z_\iota \) corresponding to the characters \( \gamma \) and \( \delta \) respectively, where \( \gamma \in \mathcal{O}_{\nu} \) and \( \delta \in \mathcal{O}_{\iota} \). Then

\[
\sum_{s \in S_{G,\iota}^{F^\nu}} \chi_{\nu}^{(\nu)}(s) \cdot \delta_{\nu}^{(\nu)}(s) \cdot \omega_{\psi}^{(\nu)}(su) = \text{Tr} \left( F^{\nu}, RT_{\gamma}(S_\iota, u^*\mathcal{F} \otimes \mathcal{L}_{\gamma}(\mathcal{L}_{\delta})) \right),
\]

where \( \mathcal{F} \) is the sheave given in Theorem 2.3, and \( u : G \to G, x \mapsto xu \) is the right translation by \( u \). It follows that \( M(\nu) \) is a function of geometric type on \( \mathcal{P}_m \). \( \square \)

Thus it remains to show that, after further increasing the divisibility of \( m \) if necessary, \( M(\nu) \) converges as \( \nu \to m, \infty \), and evaluate its limit.
4.2. A partition of $S$. For the classical groups $G$ of our concern, the centralizers $G_G(s)$, $s \in S$, are connected hence $G_s = C_G(s)$. We first give the description of $G_s$ uniformly, following e.g., [AMR].

Recall that $G$ acts on the $2n$-dimensional symplectic space $V$. Define a group

$$\Gamma := \text{Gal}(\bar{f}/f) \times \langle \iota \rangle,$$

where $\iota$ is the involution $\iota : \bar{f}^* \to \bar{f}^*$, $x \mapsto x^{-1}$. Then $\Gamma$ naturally acts on $\bar{f}^*$. Denote by $\Lambda_s \subset \bar{f}^*$ the set of distinguished eigenvalues of $s$ acting on $V$. Clearly $\Gamma$ acts on $\Lambda_s$, and for $a \in \Lambda_s$ denote by $[a]$ the $\Gamma$-orbit of $a$. Then we have that

$$G_s = \prod_{[a] \in \Lambda_s/\Gamma} G_{s,[a]};$$

where

- $G_{s,[1]}$ and $G_{s,[-1]}$ are classical groups of the same type as $G$;
- $G_{s,[a]}$, $[a] \neq \{ \pm 1 \}$, are general linear groups or unitary groups.

In the above, $G_{s,[\pm 1]}$ is interpreted as the trivial group if $\pm 1 \notin \Lambda_s$. Let us write

$$(4.5) \quad G'_s := \prod_{[a] \neq \{ 1 \}} G_{s,[a]},$$

so that $G_s = G'_s \times G_{s,[1]}$.

Let $J(S)$ be an index set for the following set of symplectic subspaces of $V$:

$$\{ V^s \mid s \in S \}. $$

Then $J(S)$ is finite of cardinality $|J(S)| = 2^n$. The Frobenius $F$ acts naturally on $J(S)$, and it can be shown that the set $J(S)^F$ of fixed points corresponds to

$$\{ V^s \mid s \in S^F \}. $$

Following (2.3) and Lemma 2.4, if we assume that $S^F$ is of the form

$$(4.6) \quad S^F \simeq \prod_j (f_j^*)^{\nu_j} \times (f_j^*)^{\mu_j}, \quad |\mu| + |\mu'| = n,$$

then $J(S)^F$ can be taken to be the set of all subsets of $\prod_j [1, \mu_j] \times [1, \mu'_j]$ hence $|J(S)^F| = 2^{2 \sum_j (\mu_j + \mu'_j)}$. For $j \in J(S)$, denote by $V^j$ the corresponding symplectic subspace of $V$, and put

$$S_j := \{ s \in S \mid V^s = V^j \}. $$

Then we have a partition

$$S = \bigsqcup_{j \in J(S)} S_j.$$

We now increase the divisibility of $m$ such that $F^m$ acts trivially on $J(S)$. Then for any $\nu \in \mathcal{P}_m$ it holds that

$$S^F_{\nu} = \bigsqcup_{j \in J(S)^F} S^F_{j,\nu}. $$

Denote by $G_j$ the centralizer of $S_j$ in $G$:

$$(4.7) \quad G_j := C_G(S_j) = Z_j \times G_{s,[1]},$$

where $Z_j := \overline{S_j} \subset S$ can be viewed as a maximal torus in $\text{Sp}(V/V^j)$, and $G_{s,[1]} = G_{s,[1]} \subset \text{Sp}(V^j)$ for any $s \in S_j$.

Recall the index set $I(S)$. For each $\iota \in I(S)^F$, let $J(S_\iota) \subset J(S)$ be the index set for the following set of symplectic subspaces of $V$:

$$\{ V^s \mid s \in S_\iota \}. $$

Note that by the description of $G_s$, we have that $|J(S_\iota)^F| \leq 2$ if $G = \text{Sp}_{2n}$, and $|J(S_\iota)^F| \leq |\Lambda_s/\Gamma|$, $s \in S_\iota$, if $G = U_n$ or $\text{GL}_n$. For $j \in J(S_\iota)$, put

$$S_{\iota,j} := S_\iota \cap S_j = \{ s \in S_\iota \mid V^s = V^j \}. $$
This gives a finite partition
\[ S_t = \bigsqcup_{j \in J(S_t)} S_{t,j}, \]
which induces a partition for each \( \nu \in P_m \):
\[ S^{F\nu}_t = \bigsqcup_{j \in J(S_t)^F} S^{F\nu}_{t,j}. \]

For each \( j \in J(S_t)^F \) we have a decomposition
\[ G_t = G'_{t,j} \times G_{j,[1]}, \]
where \( G'_{t,s} = G'_s \) for any \( s \in S_{t,j} \). Note that \( G'_{j} \) can be naturally viewed as a subgroup of \( \text{Sp}(V/V^j) \), and we have the inclusion
\[ (4.9) \quad S_{t,j} \subseteq Z_{t,j} := Z(G'_{t,j}) \subseteq Z_j \subseteq G'_{t,j}. \]

4.3. Multiplicity formula.

4.3.1. From Section 3.4, it follows that
\[ \deg \Psi_\alpha(t) \leq -\sqrt{r_k} G_t = -\sqrt{r_k} G = -n. \]

By (4.8), we can write the function (4.4) as
\[ \Theta_\alpha(\nu) = \sum_{j \in J(S_t)} \Theta_{\alpha,j}(\nu), \]
where
\[ \Theta_{\alpha,j}(\nu) := \sum_{s \in S^{F\nu}_{t,j}} \chi^{(\nu)}(s) \cdot \eta^{(\nu)}(s) \cdot \omega^{(\nu)}(su). \]

By Remark 2.5 (2), at \( s \in S^{F\nu}_{t,j} \) we have
\[ (4.10) \quad \left| \omega^{(\nu)}(su) \right| = q^{\frac{d}{2}} \dim V^\nu V^\nu \leq q^{n \nu} \dim V^\nu = q^{n \nu} \tilde{r_k} G_{j,[1]} \]
This gives the estimate
\[ (4.11) \quad |\Theta_{\alpha,j}(\nu)| \leq |Z^{F\nu}_{t,j}| \cdot q^{n \nu} \tilde{r_k} G_{j,[1]} \approx_{\nu,p_m} q^{n \nu} \tilde{r_k} Z_{t,j} + \tilde{r_k} G_{j,[1]} \leq q^{n \nu} \tilde{r_k} G_t = q^{vn}. \]

It follows that
\[ \Psi_\alpha(\nu) \Theta_{\alpha,j}(\nu) \rightarrow 0, \quad \text{as} \quad \nu \xrightarrow{p_m} \infty \]
unless
\[ \tilde{r_k} Z_{t,j} = \tilde{r_k} G'_{t,j}, \]
which is equivalent to that \( G'_{t,j} = Z_j \) in view of (4.9).

At this point, we make the following

**Remark 4.2.**

1. For \( i \in I(S)^F \) and \( j \in J(S_t)^F \), the following are equivalent:
   - \( G'_{t,j} = Z_j \),
   - \( G_t = G_{j,[1]} \),
   - \( S_{t,j} = Z^\text{reg}_j \),
where \( Z^\text{reg}_j \) is the subset of elements of \( Z_j \) that are regular in \( \text{Sp}(V/V^j) \).

2. For every \( j \in J(S)^F \), there is a unique \( i \in I(S)^F \) such that \( j \in J(S_i)^F \) and the equivalent conditions in (1) hold. This gives a map
\[ \phi : J(S)^F \rightarrow I(S)^F \]
such that \( G_j = G_{\phi(j)}, \quad j \in J(S)^F \).
4.3.2. Assume that $G_i = G_j$, i.e., $\nu = \phi(j)$ where $\phi$ is the map in Remark 4.2 (2). Then $[u] \in \mathcal{U}(G_i^F) = \mathcal{U}(G_j^F_{J[1]})$. In this case, if $u \neq 1$ then for $s \in S_{i,j} = Z_j^\text{reg}$ we have that
\[
\dim V^u = \dim(V^s \cap V^u) < \dim V^s.
\]
It follows from (4.10) and a similar estimate as (4.11) that
\[
\Psi_\alpha(\nu)\Theta_{\alpha,j}(\nu) \to 0, \quad \text{as } \nu \xrightarrow{p_m} \infty.
\]
Thus we further assume that $u = 1$. By the character formula (2.4), at $s \in Z_j^\text{reg,F}^\nu$ we have
\[
\omega_\nu(\nu) = (-1)^{l(Z^\nu_j, s)} \chi^\nu(\nu)_{\nu_j} \frac{\dim V^s}{q^{\nu_j}}\chi^\nu_j.
\]
The properties of $m$ listed in Section 3.4.2 ensure that $l(Z^\nu_j, s) = l(Z^\nu_j, s)$ for all $\nu \in \mathcal{U}_m$, which is the number of anisotropic factors of $Z_j$. Denote this number by $l(Z_j)$. By Remark 4.2 and the fact that regular elements are Zariski open dense, we find that
\[
\Theta_{\alpha,j}(\nu) \approx_{\nu, p_m} (-1)^{l(Z_j)} q^{\nu_j} \chi^\nu_j \langle \chi^\nu_j, \eta_j \rangle Z_j^\nu
\]
noting that $\alpha_i(1)$ is trivial. Thus
\[
(4.12) \quad \Psi_\alpha(q^\nu)\Theta_{\alpha,j}(\nu) \to \frac{(-1)^{rk T + rk S + l(Z_j)}}{|N_G(t, S)|} \langle \chi^\nu_j, \eta_j \rangle Z_j^\nu \quad \text{as } \nu \xrightarrow{p_m} \infty.
\]
Since in this case $G_i = G_j = Z_j \times G_j[1]$, we observe that
\[
\left| j_{G_j}^{-1}(\text{cl}(S, G)) \right| = 1 \quad \text{and} \quad \left| j_{G_j}^{-1}(\text{cl}(T, G)) \right| \leq 1,
\]
with equality hold in the latter if and only if $G_j$ contains a $G^F$-conjugate of $T$. In this case we denote the unique elements of $j_{G_j}^{-1}(\text{cl}(T, G))$ and $j_{G_j}^{-1}(\text{cl}(S, G))$ by $\nu(j)$ and $\zeta(j)$ respectively.

4.3.3. Now we can present the multiplicity formula.

**Theorem 4.3.** Let the definitions be as above. Then we have the multiplicity
\[
\langle R_{T,\chi}^G \otimes \omega_\nu^\vee, R_{S,\eta}^G \rangle^G_{J[S,T]} = \sum_{j \in J(S,T)} \frac{(-1)^{rk T + rk S + l(Z_j)}}{|N(\phi(j), S)|} \langle \chi^{\nu_j}(\nu_j, \eta_j) Z_j^\nu \rangle
\]
\[
= \sum_{j \in J(S,T)} \frac{(-1)^{rk T + rk S + l(Z_j)}}{|W_{G_j}(T_j)|} \sum_{\nu \in \mathcal{U}_m} \langle \chi^{\nu_j}(\nu_j, \eta_j) Z_j^\nu \rangle,
\]
where $J(S, T) := \{ j \in J(S) \mid j_{G_j}^{-1}(\text{cl}(T, G)) \neq \emptyset \}$, $j_{G_j}^{-1}(\text{cl}(T, G)) = \{ \nu(j) \}$, $j_{G_j}^{-1}(\text{cl}(S, G)) = \{ \zeta(j) \}$, and $(T_j, \chi_j)$ is any $G^F$-conjugate of $(T, \chi)$ such that $T_j \subset G_j$.

**Proof.** In view of the above discussions, we see that each summand of
\[
M(\nu) = \sum_{\alpha} \sum_{j \in J(S_i)} \Psi_\alpha(q^\nu)\Theta_{\alpha,j}(\nu)
\]
converges as $\nu \xrightarrow{p_m} \infty$. For each $j \in J(S)^F$, there is at most one quadruple
\[
\alpha = (\phi(j), [1], \nu(j), \zeta(j)),
\]
which exists only if $j_{G_j}^{-1}(\text{cl}(T, G)) \neq \emptyset$, such that $\Psi_\alpha(q^\nu)\Theta_{\alpha,j}(\nu)$ contributes to the limit of $M$ and is given by (4.12). Thus the double sum $M(\nu)$, after taking limit as $\nu \xrightarrow{p_m} \infty$, reduces to a sum over $j \in J(S, T)$. The formula in the theorem follows from direct substitutions of (3.2) and (3.5).

\hfill \square
4.4. Regular case. Let \( T, S \in \mathcal{J}(G) \) be parametrized by pairs of partitions \( (\lambda, \lambda') \) and \( (\mu, \mu') \) as (2.3) and (4.6) respectively, subject to the constraints in Lemma 2.4. Likewise, for \( j \in J(S, T) \), \( Z_j^F \) is precisely of the form

\[
Z_j^F \cong \prod_j (\ell_j^x)^{\nu_j} \times (\ell_j^y)^{\nu'_j},
\]

where \( \nu_j \leq \lambda_j, \mu_j \) and \( \nu'_j \leq \lambda'_j, \mu'_j \) for each \( j \). Then

\[
l(Z_j) = \sum_j \nu'_j.
\]

There are elements \( j \in J(S, T) \) that give rise to \( (\nu, \nu') \). Here the notation for the partition \( \nu = (j^{\nu_j}) \) should not be confused with the variable \( \nu \in \mathcal{P}_m \) used earlier. Put

\[
\kappa_G := \begin{cases} 1, & \text{if } G \text{ is of type A,} \\ 2, & \text{if } G \text{ is of type C.} \end{cases}
\]

We have that

\[
|N(\phi(j), S)^F| = \frac{|W_G(S)^F|}{|W_{G_j}(S)^F|} = \left( \frac{\mu}{\nu} \right) \left( \frac{\mu'}{\nu'} \right) \prod_j \nu_j! \nu'_j! (\kappa_G \cdot j)^{\nu_j + \nu'_j}.
\]

Assume now that \( \chi \) and \( \eta \) are regular, that is, they have trivial stabilizers in \( W_G(T)^F \) and \( W_G(S)^F \) respectively. In this case \((-1)^{rkT + rkG} R_{T, \chi}^G \) and \((-1)^{rkS + rkG} R_{S, \eta}^G \) are irreducible representations of \( G^F \). Below we give an explicit formula for the multiplicity

\[
\langle R_{T, \chi}^G \otimes \omega_\nu^\gamma, R_{S, \eta}^G \rangle_{GF}
\]

using Theorem 4.3. As expected, the computation is similar to the case of Bessel models for special orthogonal groups given in [R, §9], and we give a sketch.

Recall the group \( \Gamma = \text{Gal}(\mathbb{f}/\mathbb{f}) \times \langle \iota \rangle \) from Section 4.2. It acts on each \( \text{Irr}(\ell_j^x) \) and \( \text{Irr}(\ell_j^y) \) in the obvious way. Put

\[
\Gamma_G := \begin{cases} \text{Gal}(\mathbb{f}/\mathbb{f}), & \text{if } G \text{ is of type A,} \\ \Gamma, & \text{if } G \text{ is of type C.} \end{cases}
\]

On the \( j \)-th blocks of \( T^F \) and \( S^F \) respectively, write

\[
\chi = \chi_{j1} \otimes \cdots \otimes \chi_{ji} \otimes \chi'_{j1} \otimes \cdots \otimes \chi'_{ji},
\]

\[
\eta = \eta_{j1} \otimes \cdots \otimes \eta_{ji} \otimes \eta'_{j1} \otimes \cdots \otimes \eta'_{ji}.
\]

Recall the quadratic characters \( \vartheta_j \) and \( \vartheta'_j \) of \( \ell_j^x \) and \( \ell_j^y \) respectively. Define

\[
I_j := \{ k \in [1, \mu_j] \mid \eta_{jk} \in \Gamma_G : \chi_{jl} \vartheta_j \text{ for some } l \in [1, \lambda_j] \}
\]

\[
I'_j := \{ k \in [1, \mu'_j] \mid \eta'_{jk} \in \Gamma_G : \chi'_{jl} \vartheta'_j \text{ for some } l \in [1, \lambda'_j] \}.
\]

By the regularity assumption,

\[
\langle \chi_{\nu(j)} \vartheta_{Z_i}, \eta_{\nu(j)} \rangle_{Z_j^F} = \prod_j \left( \frac{|I_j|}{\nu_j} \right) \left( \frac{|I'_j|}{\nu'_j} \right) \nu_j! \nu'_j! (\kappa_G \cdot j)^{\nu_j + \nu'_j}.
\]

Put

\[
e_{T, S} := (-1)^{rkT + rkS}.
\]

By (4.13), (4.14), (4.15), (4.16) and Theorem 4.3,

\[
e_{T, S} \cdot \langle R_{T, \chi}^G \otimes \omega_\nu^\gamma, R_{S, \eta}^G \rangle_{GF} = \sum_{\nu, \nu'} (-1)^{\Sigma_j \nu_j} \prod_j \left( \frac{|I_j|}{\nu_j} \right) \left( \frac{|I'_j|}{\nu'_j} \right) = \begin{cases} 2^r, & \text{if } I_j = \emptyset \text{ for all } j, \\ 0, & \text{otherwise,} \end{cases}
\]
where $r = \sum_{j} |I_j|$. 

In particular, further assume that $T$ and $S$ are anisotropic mod $Z(G)$, so that $(-1)^{rk T + rk G} R^G_{T, \chi}$ and $(-1)^{rk S + rk G} R^G_{S, \eta}$ are cuspidal. Then $r = 0$ when $G = \text{Sp}_{2n}$ or $U_n$, and $r \leq 1$ when $G = GL_n$. Note that $I_j = \emptyset$ for all $j$ in the latter case.

### 4.5. A conjecture of Hiss and Schröer.

The study of the multiplicity (1.1) of the basic Fourier-Jacobi model amounts to understanding the irreducible decomposition

$$\pi \otimes \omega^\times = \sum_{\sigma \in \text{Irr}(G^F)} m(\pi, \sigma)\sigma$$

for any $\pi \in \text{Irr}(G^F)$. The case that $\pi = \text{St}_G$ is the Steinberg representation of $G^F$ is studied in [HZ]. Two general conjectures are made in [HS]: one is about the structure of the algebra \( \text{End}_{CG^F}(\pi \otimes \omega^\times) \), and the other one is the following conjecture on the multiplicities $m(\pi, \sigma)$ in the decomposition of $\pi \otimes \omega^\times$. We note that $\omega^\times \cong \omega_{q-1}^\times$.

**Conjecture 4.4 ([HS]).** Let $\pi \in \text{Irr}(G^F)$, where $G = \text{Sp}_{2n}$, $U_n$ or $GL_n$ is defined over $\overline{\mathbb{f}}$. Then the multiplicities of the irreducible constituents of $\pi \otimes \omega^\times$ are bounded by a function of $n$ which is independent of $q = |\overline{\mathbb{f}}|$.

Conjecture 4.4 is stated only for $G = \text{Sp}_{2n}$ or $U_n$ in [HS], but it is natural to include the $GL_n$ case as well. Below we apply Theorem 4.3 and the classification of $\text{Irr}(G^F)$ in [LS] to give a quick proof of this conjecture for $G = U_n$ or $GL_n$. We restrict to type A because in this case virtual characters on $G^F$ are all uniform, i.e. they can be spanned by Deligne-Lusztig characters. It is also possible to prove the conjecture for $G = \text{Sp}_{2n}$, using Theorem 4.3, the classification in [L1] and some extra work (e.g. [W]), which will not be pursued here.

We now prove Conjecture 4.4 in type A with the following explicit bound for the multiplicities $m(\pi, \sigma)$, although it is by no means optimal.

**Theorem 4.5.** Assume that $G = U_n$ or $GL_n$ is defined over $\overline{\mathbb{f}}$. Then for any $\pi, \sigma \in \text{Irr}(G^F)$,

$$m(\pi, \sigma) = \langle \pi \otimes \omega_{q-1}^\times, \sigma \rangle_{G^F} \leq 2^n(n!)^2.$$ 

**Proof.** By [LS, Theorem 3.2], any irreducible representation of $G^F$ is of the form

$$R^G_{L,\chi,\rho} := \frac{(-1)^{rk G + rk L}}{|W_L|} \sum_{w \in W_L} \text{Tr}(w\tilde{F}, V_{\rho}) \cdot R^G_{L_w,\chi_w},$$

where

- $L$ is an $F$-stable reductive connected subgroup of $G$ of absolute rank equal to that of $G$,
- $\chi : L^F \rightarrow \mathbb{C}^\times$ is a homomorphism, satisfying certain properties,
- $(\rho, V_{\rho}) \in \text{Irr}(W_L)^F$ and $F$ is a finite order automorphism of $V_{\rho}$ such that $FwF^{-1} = F(w)$ on $V_{\rho}$ for all $w \in W_L$,
- $T_w \in T(L)$ is of class $[w] \in H^1(F, W_L)$, and $\chi_w := \chi|_{T_w}$.

By Theorem 4.3, for any Deligne-Lusztig characters $R^G_{T,\chi}$ and $R^G_{S,\eta}$ of $G^F$, we have the crude estimate

$$|m(R^G_{T,\chi}, R^G_{S,\eta})| \leq \sum_{j \in J(S,T)} \frac{|W_G(T_j)^F|}{|W_G(T_j)|^2} \leq |J(S, T)| \cdot n! \leq 2^n n!.$$

Assume that $\pi = R^G_{L,\chi,\rho}$ and $\sigma = R^G_{L',\chi',\rho'}$ as above. Since $\tilde{F}$ is a finite order automorphism of $V_{\rho}$, where $w \in W_L$, we have that

$$|\text{Tr}(w\tilde{F}, V_{\rho})| \leq \dim \rho \leq |W_L|,$$

and similar inequalities hold for $\rho'$. Thus

$$m(\pi, \sigma) \leq 2^n n! \cdot \dim \rho \cdot \dim \rho' \leq 2^n n! \sqrt{|W_L| \cdot |W_{L'}|} \leq 2^n (n!)^2,$$

noting that $|W_L|, |W_{L'}| \leq n!$. This finishes the proof. \qed
5. Two non-regular examples

In this section, we apply Theorem 4.3 to study two examples of Fourier-Jacobi models for certain non-regular Deligne-Lusztig characters. They arise from the depth zero case of the local descent for $p$-adic unitary groups given in [ST], which will be studied in the next section.

For convenience, we switch to another standard notation for Deligne-Lusztig characters as follows. Let $G$ be a connected reductive group over $F$ with Frobenius $F$. The pairs $(T, \chi)$ where $T \in \mathcal{T}(G)$ and $\chi \in \text{Irr}(T^F)$, are in natural correspondence with the pairs $(T^s, s)$ where $T^s$ is the dual torus of $T$ and $s \in T^s, F$. Then we write $R_{T, s}^F := R_{T^s, s}^G$.

The algebraic subgroups in this section are all assumed to be $F$-stable.

5.1. Unitary groups. For $m \geq 1$, put $G_m := R_{1/4} GL_m$, where $R_{1/4}$ denotes the Weil restriction of scalars for the quadratic extension $\mathbb{F}_2/\mathbb{F}_1$, so that $G_m^F = GL_m(\mathbb{F}_2)$. Let $\iota \in \text{Gal}(\mathbb{F}_2/\mathbb{F}_1)$ be the nontrivial Galois automorphism. A representation $\tau$ of $GL_m(\mathbb{F}_2)$ is called conjugate self-dual if $\tau^\vee \cong \tau^\iota$.

Let $G = U_{4n-2}$ be defined over $\mathbb{F}_1$. Let $P$ be the Siegel parabolic subgroup of $G$, with Levi subgroup $M = G_{2n-1}$. Let $T \in \mathcal{T}(M)$ be an elliptic torus (i.e. $T^F \cong \mathbb{F}_2^{1_{4n-2}}$), and let $s \in T^s, F \cong \mathbb{F}_2^{1_{4n-2}}$ be regular in $M^s$. Then $R_{T, s}^M \in \text{Irr}(M^F)$ is cuspidal, and all irreducible cuspidal representations of $M^F \cong GL_{2n-1}(\mathbb{F}_2)$ arise in this way. It is known that (e.g. [Pr2, Proposition 5.1]) $R_{T, s}^M$ is conjugate self-dual if and only if

$$s \in \mathbb{F}_2^{1_{4n-2}}.$$  

On the other hand, $GL_m(\mathbb{F}_2)$ has no conjugate self-dual irreducible cuspidal representations if $m$ is odd. In this subsection, we assume that (5.1) holds.

By induction in stages,

$$\text{Ind}_{G}^{G_m}(R_{T, s}^G) = R_{T, s}^G,$$

where $\text{Ind}_{G}^{G_m}$ denotes the parabolic induction from $G^F$ to $G_m^F$, and we have viewed $T \in \mathcal{T}(G)$. The decomposition of the non-regular Deligne-Lusztig representation $R_{T, s}^G$ is as follows. Let $T_\alpha \in \mathcal{T}(G)$ such that $T_\alpha^F \cong \mathbb{F}_2^{1_{4n-2}} \times \mathbb{F}_2^{1_{4n-2}}$ and view $s \in \mathbb{F}_2^{1_{4n-2}}$ as an element of $T_\alpha^s, F$ via diagonal embedding. Put

$$\pi_{\text{reg}} := \frac{R_{T, s}^G - R_{T_\alpha, s}^G}{2}, \quad \pi_{\text{ss}} := \frac{R_{T, s}^G + R_{T_\alpha, s}^G}{2}.$$  

By (10.7.3) and (10.7.4) in [DL],

$$(5.3) \quad R_{T, s}^G = \pi_{\text{reg}} \oplus \pi_{\text{ss}}$$

is the decomposition of $R_{T, s}^G$ into irreducible representations of $G^F$. Moreover $\pi_{\text{reg}}$ is generic, and $\pi_{\text{ss}}$ is non-generic.

We briefly recall the Fourier-Jacobi model in this case, and refer to [GGP1] for details. For $0 \leq l \leq 2n-1$, let $P_l \subset P$ be the standard parabolic subgroup with Levi decomposition $P_l = M_l N_l$, such that $M_l \cong G_l^1 \times H_l$, where $H_l := U_{4n-2l-2}$. Recall that $G_l = R_{1/4} GL_l$. Let $\pi$ and $\sigma$ be virtual characters of $G^F$ and $H_l^F$ respectively. Denote by $\omega_{k, \psi}$, $k \geq 0$, the Weil representation of $U_k(\mathbb{F})$ associated to $\psi$. The Fourier-Jacobi model of $\pi$ and $\sigma$ is as follows.

The basic case that $l = 0$ is given by (1.1). We recall it as

$$m(\pi, \sigma) := \langle \pi \otimes \omega_{4n-2, \psi}^\vee, \sigma \rangle_{G^F}.$$  

For $l > 0$, one can define an $M_l$-equivariant homomorphism

$$(5.4) \quad N_l \to Z_l \times \mathbb{H}_l,$$

where $Z_l$ is a maximal unipotent subgroup of $G_l$, and $\mathbb{H}_l$ denotes the Heisenberg group associated to $H_l$. Let $\psi_l$ be a generic character of $Z_l^F$ defined using $\psi$, and put

$$\eta_{l, \psi} := \psi_l \otimes \omega_{4n-2l-2, \psi},$$

which via (5.4) can be naturally viewed as a representation of $R_l^F$, with

$$R_l := H_l \times N_l.$$
The Fourier-Jacobi model is concerned with the multiplicity 
\[ m(\pi, \sigma) := \langle \pi \otimes \nu^1_{l,\psi}, \sigma \rangle_{R^n_F}. \]
In view of [JZ], for a representation \( \pi \) of \( G^F \), let \( \mathcal{D}_{l,\psi}(\pi) \) be the Jacquet module of \( \pi \otimes \nu^1_{l,\psi} \) with respect to \( N_\psi^F \), which is a representation of \( H^F_\psi \), called the \( l \)-th Fourier-Jacobi descent of \( \pi \) along \( \psi \). Then in this case we equivalently have that 
\[ m(\pi, \sigma) = \langle \mathcal{D}_{l,\psi}(\pi), \sigma \rangle_{H^F_\psi}. \]

For the example in this subsection, since the basic case that \( l = 0 \) has been addressed by Theorem 4.3, we only need to consider the case that \( 0 < l \leq 2n - 1 \). Then we have the following result, which in view of the decomposition (5.3) can be interpreted as a finite field analog of the GGP conjecture for the Lusztig series of \( s \in G^{*F} \).

**Theorem 5.1.** Let \( R^{G,ss}_{F,s} = \pi_{\text{reg}} + \pi_{ss} \) be given by (5.3). Assume that \( n \geq 2 \). Then

1. For \( 0 < l \leq 2n - 1 \) and any Deligne-Lusztig character \( R_{H_1}^{H_i}_{S_0,S_0} \) of \( H^F_i \), it holds that 
\[ m(R^{G,ss}_{F,s}, (-1)^{l+1+\text{rk} S_0} R_{H_1}^{H_i}_{S_0,S_0}) = 1. \]

2. \( \mathcal{D}_{l,\psi}(\pi_{ss}) = 0 \) for \( n - 1 < l \leq 2n - 1 \), and
\[ \mathcal{D}_{n-1,\psi}(\pi_{ss}) = \bigoplus_{a \in \mathbb{F}_2^l} (-1)^n R_{S_0,\langle -s,a \rangle}^{U_{2n}}, \]
where \( S_0 \in \mathcal{T}(U_{2n}) \) such that \( S_0^F \cong \mathbb{F}_2^{1n-2} \times \mathbb{F}_2^1 \).

To utilize Theorem 4.3, we first reduce the Fourier-Jacobi model to the basic case as follows.

**Lemma 5.2.** Let \( \tau \in \text{Irr}(G^F) \) be a cuspidal representation, where \( 0 < l < 2n - 1 \). For \( \sigma \in \text{Irr}(H^F_i) \), put
\[ I(\tau, \sigma) = \text{Ind}^G_P(\tau \otimes \sigma), \]
where \( Q \) is a maximal parabolic subgroup of \( G \) with Levi subgroup \( G_1 \times H_1 \). Then 
\[ m(R^{G,ss}_{F,s}, \sigma) = m(R^{G,ss}_{F,s}, I(\tau, \sigma)). \]

**Proof.** It is clear that \( R^{G,ss}_{F,s} = \text{Ind}^G_P(R^M_{F,s}) \) does not belong to the Harish-Chandra series of \((G_1 \times L, \tau \otimes \mu)\), where \( L \) is any Levi subgroup of \( H_1 \) and \( \mu \) is any cuspidal representation of \( L^F \). Thus the proof of [GGP1, Theorem 16.1] works verbatim in this case. \( \square \)

**Proof.** (of Theorem 5.1) When \( l = 2n - 1 \), it follows directly from the uniqueness of Whittaker models for standard modules that 
\[ \mathcal{D}_{2n-1}(\pi_{\text{reg}}) = C, \quad \mathcal{D}_{2n-1}(\pi_{ss}) = 0. \]

Below we assume that \( 0 < l < 2n - 1 \).

We first prove (1). Let \( \tau \in \text{Irr}(G^F) \) be a cuspidal representation 
\[ \tau = (-1)^l R^G_{S_1,s_1}, \]
where \( S_1 \in \mathcal{T}(G_1), S_1^F \cong \mathbb{F}_2^l \), and \( s_1 \in S_1^* \) is regular in \( G_1 \). By induction in stages, 
\[ \text{Ind}^G_P(\tau \otimes (-1)^{l+1+\text{rk} S_0} R_{H_1}^{H_i}_{S_0,S_0}) = (-1)^{l \cdot \text{rk} S_0} R^{G}_{S,s'}, \]
where \( Q \) is as in Lemma 5.2, \( S = S_1 \times S_0 \in \mathcal{T}(G) \) and \( s' = (s_1, s_0) \in S^{*F} \). By Lemma 5.2, it suffices to show that 
\[ m(R^{G}_{F,s}, (-1)^{l \cdot \text{rk} S_0} R^{G}_{S,s'}) = 1. \]

For this we apply Theorem 4.3. Since \( T \) and \( S \) have no common rational factors, \( J(S,T) \) consists of a single element \( j \) for which \( Z_j \) is trivial and \( G_j = G \). Thus it is easy to see that 
\[ m(R^{G}_{F,s}, R^{G}_{S,s'}) = e_{T,S} = (-1)^{l \cdot \text{rk} S_0}, \]
The proves Theorem 5.1 (1).
Next we prove Theorem 5.1 (2). First consider the case that \( l = n - 1 \), and put 
\[
\tilde{\sigma}_{s,a} := (-1)^n R_{S_0,(-s,a)}^{U_{2n}} \in \text{Irr}(U_{2n}),
\]
where \( S_0 \in \mathcal{T}(U_{2n}) \) such that \( S_0^F \cong \mathbb{F}_{2n-2} \times \mathbb{F}_2 \), and \( a \in \mathbb{F}_2 \). Similar to the above, let \( \tau = (-1)^n R_{S_1,s_1}^G \in \text{Irr}(G_{2n-1}) \) be a cuspidal representation, so that by induction in stages
\[
I(\tau, \tilde{\sigma}_{s,a}) = R_{S,s_a}^G,
\]
where \( S := S_1 \times S_0 \in \mathcal{T}(G) \) and \( S'_{s,a} := (s_1,-s,a) \in S^{s,F} \cong \mathbb{F}_{2n-2} \times \mathbb{F}_{4n-2} \times \mathbb{F}_2 \). By Lemma 5.2 again,
\[
m(\pi_{sa}, \tilde{\sigma}_{s,a}) = m(\pi_{sa}, I(\tau, \tilde{\sigma}_{s,a})) = \frac{1}{2} \left( m(R_{T,s}^G, R_{S,s_a}^G) + m(R_{T,s}^G, R_{S,s_a}^G) \right).
\]
We have seen that \( m(R_{T,s}^G, R_{S,s_a}^G) = 1 \). Again we apply Theorem 4.3 to evaluate \( m(R_{T,s}^G, R_{S,s_a}^G) \).

We have \( J(S, T_a) = \{ j_0, j_1 \} \), for which \( Z_{j_0} = \{ 1 \} \), \( G_{j_0} = G \) and
\[
Z_{j_1} = \mathbb{F}_{4n-2}, \quad G_{j_1} = Z_{j_1} \times U_{2n-1}.
\]
Denote the \( j \)-summand in Theorem 4.3 by \( m_j \) for short. Similar to the above, we see that
\[
m_{j_0} = e_{T_a,S} = -1.
\]
To compute \( m_{j_1} \), we may conjugate \( T_a \) into \( G_{j_1} \), and assume that \( Z_{j_1}^F \) is the first factor of \( T_a^F \cong \mathbb{F}_{4n-2} \times \mathbb{F}_{4n-2} \). Recall that \( s \) embeds into \( T_a^F \) diagonally as \( (s,s) \). We find that
\[
\left\{ w \in W_G(T_a)^F \left| w(s,s)|_{Z_{j_1}^F} = s \right. \right\} = 2(2n - 1) = 2|W_{G_{j_1}}(T_a)^F|.
\]
From this it follows easily that
\[
m_{j_1} = e_{T_a,S}(-1)^j(2) = 2.
\]
Thus
\[
m(R_{T,s}^G, R_{S,s_a}^G) = m_{j_0} + m_{j_1} = 1.
\]
This proves that \( m(\pi_{sa}, \tilde{\sigma}_{s,a}) = 1 \).

Note that the summand in (5.5) corresponding to \( a \in \mathbb{F}_2 \), which we denote by \( \tilde{\sigma}_{s,a} \) in the above, is the unique member of its own Lusztig series (see [L2, L3] for this notion). Thus it remains to show that for any character of \( H_l^F \) of the form
\[
\sigma = (-1)^{l+1+\text{rk} S_0} R_{S_0,s_0}^{H_l}, \quad S_0 \in \mathcal{T}(H_l), \quad s_0 \in S_{s,F}^s,
\]
where \( n - 1 \leq l < 2n - 1 \) and \( \sigma \) is not isomorphic to any summand \( \sigma_{s,a} \) in (5.5) when \( l = n - 1 \), it holds that
\[
m(\pi_{sa}, \sigma) = 0.
\]
Applying Lemma 5.2 again, it is reduced to proving that
\[
m(\pi_{sa}, R_{S,s_a}^G) = 0,
\]
where \( S = S_1 \times S_0, \ S_1 \in \mathcal{T}(G_l), \ S_1^F \cong \mathbb{F}_{2l}, \ s_1 \in S_{s,F}^s \) and \( s' = (s_1, s_0) \in S^{s,F} \). Using Theorem 4.3, similar calculations show that in this case
\[
m(R_{T,s}^G, R_{S,s_a'}^G) = e_{T,S} = -e_{T_a,S} = -m(R_{T,s}^G, R_{S,s_a'}^G),
\]
which implies (5.6), hence proves Theorem 5.1 (2).
5.2. **Symplectic groups.** For the second example, we let $G = \text{Sp}_{2n}$ be defined over $\mathfrak{f}$. Let $P$ be the Siegel parabolic subgroup of $G$ with Levi subgroup $M = \text{GL}_{2n}$. Let $L \cong \text{GL}_n \times \text{GL}_n$ be a Levi subgroup of $M$. A representation $\tau$ of $M^F$ is said to have a linear model if

$$\text{Hom}_{L^F}(\tau, 1) \neq 0.$$ 

It is known that a cuspidal representation $\tau \in \text{Irr}(M^F)$ has a linear model if and only if $\tau$ is self-dual, i.e. $\tau \cong \tau^\vee$. Let $T \in \mathcal{T}(M)$ be elliptic (i.e. $T^F \cong \mathfrak{t}_{2n}^\vee$), and let $s \in T^*F$ be regular in $M^*$, so that $-R_{T,s}^M \in \text{Irr}(M^F)$ is cuspidal. By [L4] (see also [H1]), $-R_{T,s}^M$ has a linear model if and only if

$$s \in \mathfrak{t}_{2n}^\vee.$$ 

In this subsection, we assume that (5.7) holds. We mention that, in [LMNW] a criterion for the existence of linear models is given for general irreducible Deligne-Lusztig characters of $M^F$ which are not necessarily cuspidal.

Let $T_a \in \mathcal{T}(G)$ such that $T_a^F \cong \mathfrak{t}_{2n} \times \mathfrak{t}_{2n}$, and embed $s$ into $T_a$ diagonally. We have

$$\text{Ind}^G_P(-R_{T,s}^M) = -R_{T,s}^G = \pi_{\text{reg}} \otimes \pi_{\text{ss}},$$

where

$$\pi_{\text{reg}} := -\frac{R_{T,s}^G}{2}, \quad \text{and} \quad \pi_{\text{ss}} := -\frac{R_{T,a}^G - R_{T,s}^G}{2}$$

are irreducible representations of $G^F$. Again $\pi_{\text{reg}}$ is generic and $\pi_{\text{ss}}$ is not.

For $0 \leq l \leq 2n$, let $P_l = M_l N_l \subset P$ be the standard parabolic subgroup of $G$ with Levi subgroup $M_l \cong \text{GL}_l \times H_l$, where $H_l := \text{Sp}_{4n-2l}$. Similar to the case of unitary groups, one can define a representation $\nu_{l,\psi}$ of

$$R_l := H_l \times N_l,$$

which is the Weil representation $\omega_{l,\psi}$ of $G^F$ in the case that $l = 0$. For virtual characters $\pi$ and $\sigma$ of $G^F$ and $H_l^F$ respectively, one defines the Fourier-Jacobi model

$$m(\pi, \sigma) := \langle \pi \otimes \nu_{l,\psi}^{\vee}, \sigma \rangle_{R_l^F}.$$ 

Again we refer to [GGP1] for details. If $\pi$ is a representation of $G^F$, the $l$-th Fourier-Jacobi descent of $\pi$ along $\psi$ is Jacquet module of $\pi \otimes \nu_{l,\psi}^{\vee}$ with respect to $N_l^F$, which is a representation of $H_l^F$, denoted by $\mathcal{D}_{l,\psi}(\pi)$. In this case $m(\pi, \sigma) = \langle \mathcal{D}_{l,\psi}(\pi), \sigma \rangle_{H_l^F}$.

We have the following result for this example.

**Theorem 5.3.** Let $-R_{T,s}^G = \pi_{\text{reg}} + \pi_{\text{ss}}$ be given by (5.8). Then

1. For $0 < l \leq 2n$ and any Deligne-Lusztig character $R_{S_0,S_0}^{H_l}$ of $H_l^F$, it holds that

$$m(-R_{T,s}^G, (-1)^{l+r} S_0 R_{S_0,S_0}^{H_l}) = 1.$$ 

2. $\mathcal{D}_{l,\psi}(\pi_{\text{ss}}) = 0$ for $n < l \leq 2n$, and

$$\mathcal{D}_{l,\psi}(\pi_{\text{ss}}) = (-1)^n R_{S_0,-S_0}^{\text{Sp}_{2n}},$$

where $S_0 \in \mathcal{T}($Sp$)_{2n}$ such that $S_0^F \cong \mathfrak{t}_{2n}$.

The proof of Theorem 5.3 is similar to that of Theorem 5.1. One can first reduce to the basic case using the following lemma, and then apply Theorem 4.3. The details will be omitted.

**Lemma 5.4.** Let $\tau \in \text{Irr}(G^F)$ be a cuspidal representation, where $0 < l < 2n$. For $\sigma \in \text{Irr}(H_l^F)$, put

$$I(\tau, \sigma) = \text{Ind}_Q^G(\tau \otimes \sigma),$$

where $Q$ is a maximal parabolic subgroup of $G$ with Levi subgroup $\text{GL}_d \times H_l$. Then

$$m(-R_{T,s}^G, \sigma) = m(-R_{T,s}^G, I(\tau, \sigma)).$$
6. Local descent for \( p \)-adic unitary groups

In the rest of this paper, we consider unitary groups over \( p \)-adic local fields. From now on we systematically change the notations and write \( G, T, U_n, \GL_n \) etc. for algebraic groups over a finite field \( k \), and reserve \( G, T, U_n, \GL_n \) etc. for \( p \)-adic groups.

This section is basically expository. Let \( E/F \) be a quadratic extension of \( p \)-adic local fields, with odd residue characteristic \( p \). Let \( \iota \in \Gal(E/F) \) be the nontrivial Galois automorphism. In this section we first collect some results on irreducible representations of \( \GL_m(E) \) distinguished by the Galois involution, namely, by the subgroup \( \GL_m(E)^{\iota} = \GL_m(F) \). Then we recall the local descent of irreducible distinguished supercuspidal representations to unitary groups from [ST], and explain the connections with the Langlands functoriality and the non-tempered local GGP conjecture [GGP2]. Finally, we give the distinction criterion in the depth zero case following [CG, HM1].

6.1. Galois distinction. Let \( \omega_{E/F} \) be the quadratic character of \( F^{\times} \) given by the local class field theory, and fix a character \( \mu \) of \( E^{\times} \) such that \( \mu|_{F^{\times}} = \omega_{E/F} \). By composition with determinant maps, we regard \( \omega_{E/F} \) and \( \mu \) as characters of \( \GL_m(F) \) and \( \GL_m(E) \) respectively. Let \( \tau \) be an irreducible admissible representation \( \tau \) of \( \GL_m(E) \). We are concerned with the following notions:

- \( \tau \) is called distinguished (resp. \( \omega_{E/F} \)-distinguished) if
  \[
  \text{Hom}_{\GL_m(F)}(\tau, 1) \neq 0 \quad \text{(resp. Hom}_{\GL_m(F)}(\tau, \omega_{E/F}) \neq 0),
  \]
  in which case the Hom space has dimension one. Note that \( \tau \) is \( \omega_{E/F} \)-distinguished if and only if \( \tau \otimes \mu \) is distinguished.

- \( \tau \) is called conjugate self-dual if \( \tau^\vee \cong \tau^\iota \).

It is well-known that (e.g. [F, Pr1]) if \( \tau \) is distinguished, then \( \tau \) is conjugate self-dual and the central character of \( \tau \) is trivial on \( F^{\times} \).

Let \( \tau \) be an irreducible square-integrable conjugate self-dual representation of \( \GL_m(E) \), whose central character is trivial on \( F^{\times} \). By [K, AKT], if \( m \) is odd, then \( \tau \) is distinguished; if \( m \) is even, then exactly one of \( \tau \) and \( \tau \otimes \mu \) is distinguished. We have that

\[
L(s, \tau \times \tau^\iota) = L(s, \tau, As)L(s, \tau \otimes \mu, As),
\]

where the left hand side is the Rankin-Selberg L-function, and factors in the right hand side are the Asai L-functions, with \( As = As^+ \). See e.g. [GGP1, §7] for the definition of two Asai representations \( As^+ \) and \( As^- \), which correspond to the stable and unstable base change \( lU_m \rightarrow \GL_m \) respectively ([F, KK]). It follows from [K] that \( L(s, \tau, As) \) has a pole at \( s = 0 \) if and only if \( \tau \) is distinguished.

6.2. Local descent and non-tempered GGP conjecture. Assume from now on that \( \tau \) is irreducible distinguished and is moreover supercuspidal. Then \( \tau \) (resp. \( \tau \otimes \mu^{-1} \)) is a stable base change of an irreducible generic supercuspidal representation of a quasi-split \( U_m(F) \) for \( m \) odd (resp. even). Indeed, let \( M_r \) be the L-parameter of \( \tau \) under the local Langlands correspondance, which is \( m \)-dimensional irreducible representation of the Weil group \( W_E \), extended trivially to the Weil-Deligne group \( WD_E = W_E \times \SL_2(C) \). Then \( As^+(M_r)^{WD_E} \neq 0 \). By [GGP1, Prop. 7.5], \( M_r \) is conjugate orthogonal and \( M_r(\mu^{-1}) := M_r \otimes \mu^{-1} \) is conjugate symplectic, which give an L-parameter of \( U_m(F) \) for \( m \) odd and even respectively. Here \( \mu \) is viewed as a character of \( W_E \) via the local class field theory.

From the representation \( \tau \) of \( \GL_m(E) \), the Fourier-Jacobi local descent construction in [ST] produces generic supercuspidal representations of \( U_{2n}(F) \), where \( n := \lceil m/2 \rceil \). In particular, it realizes the inverse map to the above base change for the representation \( \tau^\vee \otimes \mu \) when \( m = 2n \) is even. We shall briefly recall the construction below, with slightly different notation and convention.

Let \( P \) be the Siegel parabolic subgroup of \( U_{2n}(F) \) with Levi subgroup \( M \cong \GL_m(E) \). For \( s \in \mathbb{C} \), define the normalized parabolic induction

\[
\rho_{\tau,s} := \Ind_P^{U_{2n}(F)} \tau_{\delta}, \quad \tau_{\delta} := \tau \otimes | \det |_E^{s-1/2},
\]

for \( s \in \mathbb{C} \).
where $| \cdot |_E$ is the normalized absolute value on $E$. Then $\rho_{\tau, 1}$ is of length two, and we have a short exact sequence

$$
0 \longrightarrow \rho_2 \longrightarrow \rho_{\tau, 1} \longrightarrow \pi_\tau \longrightarrow 0
$$

where the subrepresentation $\rho_2$ of $\rho_{\tau, 1}$ is generic, and the Langlands quotient $\pi_\tau$ of $\rho_{\tau, 1}$ is non-generic.

Fix a nontrivial additive character $\psi_F$ of $F$. For $0 \leq l \leq m$, let $P_l = M_l N_l \subset P$ be the standard parabolic subgroup of $U_{2m}(F)$ with Levi subgroup $M_l \cong (E^\times)^l \times U_{2m-2l}(F)$. Similar to the finite field case, for $l > 0$ we have a homomorphism $N_l \rightarrow \mathbb{Z}_l \times \mathbb{H}_l$, where $\mathbb{Z}_l$ is a maximal unipotent subgroup of $\text{GL}_l(E)$ and $\mathbb{H}_l$ is the Heisenberg group associated to $U_{2m-2l}$. As in [GGP1], associated to $\psi_F$ and the character $\mu$, one has an irreducible unitary representation

$$
\pi_{l, \psi_F, \mu} = \psi_l \otimes \omega_{2m-2l, \psi_F, \mu}
$$

of $R_l := U_{2m-2l}(F) \times N_l$. Here $\psi_l$ is the generic character of $\mathbb{Z}_l$ defined using $\psi_F$, and $\omega_{2m-2l, \psi_F, \mu}$ is the Weil representation of $U_{2m-2l}(F) \times \mathbb{H}_l$ defined using $\psi_F$ and $\mu$.

**Definition 6.1.** The $l$-th Fourier-Jacobi local descent of a smooth representation $\pi$ of $U_{2m}(F)$, denoted by $\mathcal{D}_l(\psi_F, \mu)(\pi)$, is a smooth representation of $U_{2m-2l}(F)$ defined as the Jacquet module of $\pi \otimes \nu_{l, \psi_F, \mu}$ with respect to $N_l$.

**Remark 6.2.** Our formulation follows [GGP1], and differs from [ST] by taking the contragredient of the Weil representation. This can be easily rectified as follows.

1. The Whittaker datum for $U_{2m-2l}(F)$, as well as the Weil representation $\omega_{2m-2l, \psi_F, \mu}$ (for fixed $\mu$), is determined by the choice of $\psi_F$ up to a twist by an element of $N_{E,F}(E^\times)$.
2. We can use that $\omega_{2m-2l, \psi_F, \mu} \cong \omega_{2m-2l, \psi_F^{-1}, \mu^{-1}}$ to translate the results from [ST] accordingly.

In view of Remark 6.2, we can state the following

**Theorem 6.3 ([ST]).** Let $\tau$ be an irreducible supercuspidal representation of $\text{GL}_{m}(E)$ such that $L(\delta, \tau, \text{As})$ has a pole at $s = 0$. Put $n = \lceil \frac{m+1}{2} \rceil$ and $l_0 = m - n = \lfloor \frac{m}{2} \rfloor$. Then the following hold.

1. $\mathcal{D}_{l_0}(\psi_F, \mu)(\tau) = 0$ for $l > l_0$, and $\mathcal{D}_{l_0}(\psi_F, \mu)(\pi_\tau) \neq 0$ is the multiplicity free direct sum of irreducible $\psi_F^{-1}$-generic supercuspidal representations $\sigma$ of $U_{2n}(F)$ such that the local gamma factor $\gamma(s, \tau, \sigma \times (\tau \otimes \mu^{-1}), \psi_F^{-1})$ has a pole at $s = 1$.
2. If $m = 2n$ is even, then $\mathcal{D}_{n}(\pi_\tau)$ is irreducible.

This result is compatible with the non-tempered local GGP conjecture [GGP2]. Recall that the L-parameter $M_\nu$ of $\tau$ is conjugate orthogonal. The A-parameter of $\pi_\tau$ is the representation $M_\nu \otimes [2]$ of $W_{E,F} \times \text{SL}_2(C)$, where $[k], k \geq 1$, denotes the irreducible algebraic $k$-dimensional representation of the Arthur $\text{SL}_2(C)$. Note that the L-parameter associated to $M_\nu \otimes [2]$ is $M_{\nu^{1/2}} \otimes M_{\nu^{-1/2}}$, where $\nu = |\det|_E$. The non-tempered GGP conjecture [GGP2, Conjecture 6.1] predicts that for any irreducible quotient $\sigma$ of $\mathcal{D}_{l_0}(\pi_\tau)$, the A-parameter $N_\sigma$ of $\sigma$ and $M_{\nu^{-1}} \otimes [2]$ form a relevant pair.

By Theorem 6.3, we have a well-defined map

$$
\mathcal{D}: \{ \tau \in \text{Irr}((\text{GL}_m(E)) \mid \tau \text{ is supercuspidal and } \tau \text{ has a pole at } s = 0 \} \longrightarrow \text{Rep}(U_{2l-\lceil \frac{m+1}{2} \rceil}(F)),
$$

and the image of $\mathcal{D}$ consists of certain semisimple representations of $U_{2l-\lceil \frac{m+1}{2} \rceil}(F)$. By the definition of the relevant pair of A-parameters [GGP2, §3], the following are clear:

- If $m = 2n - 1$ is odd, then $N_\sigma$ is of the form $M_{\nu^{-1}} \otimes \zeta$, where $\zeta$ is a conjugate symplectic character of $E^\times$ (i.e. $\zeta|_{E^\times} = \omega_{E/F}$);
- If $m = 2n$ is even, then $N_\sigma = M_{\nu^{-1}}$. 
6.3. Distinction criterion: depth zero case. The general distinction problem for supercuspidal representations has been studied extensively (see e.g. [HM2, H1, H2, Z]). For the Galois distinction of our concern, a criterion for tame supercuspidal representations is given in [HM1], and the general case is given in [S]. In the depth zero case, we shall follow the result in [CG], which gives a necessary and sufficient condition for the Galois distinction. It recovers the depth zero case of the criterion in [HM1].

An irreducible depth zero supercuspidal representation \( \tau \) of \( \text{GL}_m(E) \) is regular in the sense of [Ka], and is associated to a pair \((T, \chi)\), where \( T \cong E^\times_m \) is an elliptic torus in \( \text{GL}_m(E) \) and \( \chi \) is a regular depth zero character of \( T \). Here \( E_m \) denotes the degree \( m \) unramified extension of \( E \). Then we write \( \tau = \tau(T, \chi) \).

**Theorem 6.4 ([CG, HM1]).** An irreducible depth zero supercuspidal representation \( \tau \) of \( \text{GL}_m(E) \) is distinguished if and only if

(1) \( E/F \) is unramified, \( m = 2n - 1 \) is odd, and \( \chi|_{F_{2n-1}^2} = 1 \).

(2) \( E/F \) is ramified, \( m = 2n \) is even, and \( \chi|_{F_{2n}^2} = \omega E_{2n}/F_{2n} \).

**Remark 6.5.** *Theorem 6.4 indicates that the sufficient condition for the Galois distinction given in [HM1, Theorem 1.1] is also necessary in the depth zero case. Note that in both cases (1) and (2) of Theorem 6.4, \( E \cap F_m = F \) and the nontrivial Galois automorphism \( \iota_m \in \text{Gal}(E_m/F_m) \) restricts to \( \iota \in \text{Gal}(E/F) \):

\[
\begin{array}{ccc}
E & \xrightarrow{\iota_m} & E_m \\
\downarrow \iota & & \downarrow \iota_m \\
F & \xrightarrow{\iota} & F_m
\end{array}
\]

To compute the local descent, we reformulate Theorem 6.4 in terms of compact induction. Let \( \varepsilon \) and \( \mathfrak{f} \) be the residue fields of \( E \) and \( F \) respectively, so that \( \varepsilon = \mathfrak{f}_2 \) or \( \mathfrak{f} \) according to whether \( E/F \) is unramified or not. Let \( M := \text{GL}_m(E) \) and denote by \( \mathcal{B}(M) \) the building of \( M \). For a hyperspecial point \( x \in \mathcal{B}(M) \), denote by \([x]\) the image of \( x \) in the reduced building of \( M \). Let \( M_x \) and \( M_{[x]} \) be the stabilizer of \( x \) and \([x]\) respectively. Then

\[
M_{[x]} = ZM_x = \langle \omega_E \rangle \times M_x,
\]

where \( Z \cong E^\times \) is the center of \( M \) and \( \omega_E \) is a fixed uniformizer of \( E \). The reductive quotient \( M_x/M_{[x]} \) of \( M_x \cong \text{GL}_m(O_E) \) is naturally isomorphic to

\[
M^F := \text{GL}_m(\varepsilon),
\]

where \( M := R_{\mathfrak{f}/\mathfrak{f}} \text{GL}_m \) is defined over \( \mathfrak{f} \), and \( O_E \) is the ring of integers of \( E \). See [MP] for the notion of Moy-Prasad filtration. Let \( T \) be an maximally anisotropic maximal tori in \( M \) so that \( T^F \) and \( T^sF \) are naturally identified with \( E_\varepsilon \). Let \( \varepsilon_m^0 := \{ a \in \varepsilon_m \mid \varepsilon(a) = \varepsilon_m \} \) denote the set of regular elements in \( \varepsilon_m \). The Galois group \( \text{Gal}(\varepsilon_m/\varepsilon) \) acts on \( \varepsilon_m^0 \). For \( s \in \varepsilon_m^0 \) and \( c \in \mathbb{C}^\times \), let \( \bar{\tau}_{s,c} \) be the irreducible representation of \( M_{[x]} \) such that

- \( \bar{\tau}_{s,c}|_{M_x} \) is the inflation of Deligne-Lusztig character \((-1)^{m+1}R_T^M_{s,c} \),
- and \( \bar{\tau}_{s,c}(\omega_E) = c \).

The following map is a bijection that gives the well-known classification of irreducible depth zero supercuspidal representation of \( M \) (see e.g. [BR]):

\[
\begin{array}{cccc}
(\text{Gal}(\varepsilon_m/\varepsilon) \times \mathbb{C}^\times) & \rightarrow & \{ \pi \in \text{Irr}(\text{GL}_m(E)) \mid \pi \text{ is supercuspidal and depth zero} \} \\
(\text{Gal}(\varepsilon_m/\varepsilon) \cdot s, c) & \mapsto & \tau_{s,c} := \text{c-Ind}_{M_{[x]}} M \bar{\tau}_{s,c}.
\end{array}
\]

Let

\[
\varepsilon_m^{0,1} := \varepsilon_m^0 \cap \varepsilon_m^1.
\]

The following result is equivalent to Theorem 6.4 and can be easily deduced from the main theorem of [CG].
Theorem 6.6. Suppose there is an irreducible depth zero supercuspidal representation of $M = GL_m(E)$ distinguished by $GL_m(F)$. Then $E/F$ is unramified if $m$ is odd, and $E/F$ is ramified if $m$ is even. Moreover, there is a bijection

$$\text{Gal}(\mathcal{m}/\mathcal{e}) \backslash \mathcal{e}_{m,1} \to \left\{ \pi \in \text{Irr}(GL_m(E)) \mid \pi \text{ is supercuspidal, depth zero and distinguished by } GL_m(F) \right\},$$

where

$$c_s := \begin{cases} 1 & \text{if } E/F \text{ is unramified}, \\ -\varphi'_n(s) & \text{if } E/F \text{ is ramified}. \end{cases}$$

7. Models and splittings of Weil representations

Let $p_E$ and $p_F$ be the maximal ideals of $\mathcal{O}_E$ and $\mathcal{O}_F$ respectively. Assume that the additive character $\psi_F$ of $F$ has conductor $p_F$, so that $\psi_F | \mathcal{O}_F$ is inflated from a nontrivial additive character $\psi$ of $\mathfrak{f}$. Let $W = E^{2n}$ be endowed with a skew-Hermitian form $\langle \cdot, \cdot \rangle$ represented by the matrix

$$J_{2n} = \begin{pmatrix} -w_n & w_n \\ -w_n & w_n \end{pmatrix}, \quad \text{with } w_n = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}_{n \times n}.$$

Then we have the realization

$$U_{2n}(F) = U(W) = \left\{ g \in GL_{2n}(E) \mid g J_{2n}(g') = J_{2n} \right\},$$

where $g'$ denotes the transpose of $g$. Let $L = \mathcal{O}_E^{2n}$, which is a self-dual lattice of $V$ in the sense of (1.2). Put $L^2 = p_E L$. Then $L/L^2 = \mathcal{O}_F$ is a skew-Hermitian or symplectic space according to $\mathfrak{e} = \mathfrak{f}_2$ or $\mathfrak{f}$, whose isometry group

$$H = U_{2n} \text{ or } Sp_{2n}$$

is defined over $\mathfrak{f}$. The stabilizer of $L$ in $U(W)$,

$$U(W)_L := \{ g \in U(W) : g L = L \},$$

is a maximal compact subgroup of $U(W)$ with finite reductive quotient $H^F$.

Let $\mathbb{H}_W := W \times F$ be the Heisenberg group associated to $V$. By the Stone-von Neumann Theorem, up to isomorphism, there is a unique irreducible smooth representation $\rho_{\psi_F}$ of $\mathbb{H}_W$ with central character $\psi$. We consider two realizations of $\rho_{\psi_F}$.

The Schrödinger model of $\rho_{\psi_F}$ is realized on the space $S(Y)$ of locally constant, compactly supported functions on a Lagrangian subspace $Y$ of $V$. Following [RR], there is a projective representation

$$U(W) \to GL(S(Y)), \quad g \mapsto M^Y_g$$

of $U(W)$, with Ranga-Rao cocycle $c_Y(g, g')$. Given a character $\mu$ of $E^\times$ satisfying that $\mu | E^\times = \omega_{E/F}$, a splitting

$$\beta^Y_\mu : U(W) \to \mathbb{C}^\times$$

of $c_Y(g, g')$ is constructed in [Ku]. This gives the Weil representation $\omega_{\psi_F, \mu} := \omega_{2n, \psi_F; \mu}$ of $U(W)$ on $S(Y)$, so that $g \in U(W)$ acts by $\beta^Y_\mu(g) M^Y_g$.

Let $(\rho_{\psi_{\mathfrak{f}}}, S)$ be the unique irreducible representation of $\mathbb{H}_{L/L^2} := L/L^2 \times \mathfrak{f}$ with central character $\psi_{\mathfrak{f}}$, inflated to a representation $\tilde{\rho}_{\psi_F}$ of $\mathbb{H}_L := L \times \mathcal{O}_F$. The generalized lattice model of $\rho_{\psi_F}$ is realized on the space $S(L)$ of locally constant, compactly supported maps $f : W \to S$ satisfying that

$$f(v + x) = \psi_F \left( \frac{1}{2} \text{tr}_{E/F}(\langle x, v \rangle) \right) \tilde{\rho}_{\psi_F}(v). f(x), \quad v \in L, \quad x \in W.$$

By [P1, Theorem 3.4], there is a projective representation

$$U(W) \to GL(S(L)), \quad g \mapsto M^L_g,$$
with a unique splitting \( \beta^L : U(W) \to \mathbb{C}^\times \) such that \( \beta^L|_{U(W)_L} = 1 \). Then \( g \in U(W) \) acts by \( \beta^L(g)M^L_g \) on \( S(L) \) as a genuine representation. Recall that \( S \) supplies the Weil representation \( \omega_{\psi_f} \) of \( \mathbb{H}^\ell \), and we inflate \( \omega_{\psi_f} \) to a representation \( \bar{\omega}_{\psi_f} \) of \( U(W)_L \). Following [P1],

\[
(7.3) \quad (M^L_kf)(x) = \bar{\omega}_{\psi_f}(k)f(k^{-1}x)
\]

for \( k \in U(W)_L \), \( f \in S(L) \) and \( x \in W \).

Let \( \Psi : S(L) \to \mathcal{S}(Y) \) be an isomorphism as irreducible representations of \( \mathbb{H}_V \), which is unique up to scalar. Define the function

\[
\alpha : U(W) \to \mathbb{C}^\times
\]

such that \( \alpha(g) \cdot \Psi \circ M^L_g = M^Y_g \circ \Psi \), \( g \in U(W) \), and define \( \xi_\mu \) to be the ratio of the splitting \( \alpha_\beta^Y \) to Pan’s splitting \( \beta^L \):

\[
(7.4) \quad \xi_\mu := \alpha_\beta^Y (\beta^L)^{-1} : U(W) \to \mathbb{C}^\times.
\]

Thus we have a commutative diagram

\[
\begin{array}{ccc}
S(L) & \xrightarrow{\xi_\mu} & S(Y) \\
\downarrow{\beta^L(g)M^L_g} & & \downarrow{\beta^Y(g)M^Y_g} \\
S(L) & \xrightarrow{\Psi} & S(Y)
\end{array}
\]

for all \( g \in U(W) \). By [P1, O], \( \xi_\mu|_{U(W)_L} \) is a character that factors through the determinant. That is,

\[
\xi_\mu|_{U(W)_L} = \xi'_\mu \circ \det : U(W)_L \to \mathbb{C}^\times
\]

for a character \( \xi'_\mu \) of \( E_L := \det(U(W)_L) \). We summary above discussions as follows.

**Proposition 7.1.** The Weil representation \( \omega_{\psi_f,\mu} \) of \( U(W) \) can be realized on the generalized lattice model \( S(L) \) such that

\[
\omega_{\psi_f,\mu}(g) = \xi_\mu(g)\beta^L(g)M^L_g, \quad g \in U(W).
\]

Moreover,

\[
\omega_{\psi_f,\mu}(k) = \xi'_\mu(\det k) \cdot M^L_k, \quad k \in U(W)_L.
\]

To ease the notation, we also write \( (\omega_{\psi_f,\mu}, S(L)) \) and \( (\bar{\omega}_{\psi_f}, S) \) as representations of \( U(W) \ltimes \mathbb{H}_V \) and \( U(W)_L \ltimes \mathbb{H}_L \) respectively. Recall that \( \mathbb{H}_V \) and \( \mathbb{H}_L \) act by \( \rho_{\psi_f} \) and \( \bar{\rho}_{\psi_f} \) respectively.

**Corollary 7.2.** There is a nonzero map

\[
\phi \in \text{Hom}_{U(W)_L \ltimes \mathbb{H}_L}((\omega_{\psi_f,\mu}, \bar{\omega}_{\psi_f} \otimes \xi_\mu), \phi(f) = f(0), \quad f \in S(L).
\]

**Proof.** This can be verified directly using Proposition 7.1, (7.2) and the action (7.3) of \( M^L_k \), \( k \in U(W)_L \). \( \square \)

Put \( E_L := \det(U(W)_L) \). Then we have that

\[
E_L = E^1 := \{ t \in E^\times \mid N_{E/F}(t) = 1 \}
\]

if \( E/F \) is unramified, and

\[
E_L = E^+ := E^1 \cap (1 + p_E)
\]

is an index 2 subgroup of \( E^1 \) otherwise. The restriction \( \xi'_\mu|_{E^+} \) is determined in [O], based on the calculations in [P1]. In particular, \( \xi'_\mu : E_L \to \mathbb{C}^\times \) is determined if \( E/F \) is ramified.

Let us sketch the description of \( \xi'_\mu|_{E^+} \). By Hilbert’s 90, there is a commutative diagram

\[
\begin{array}{cccc}
(1 + p_E)/(1 + p_F) \cong & 0_E^\times/0_F^\times & \cong & E^\times/F^\times \\
\cong & \cong & \cong & \\
E^+ & \cong & E_L & \cong & E^1
\end{array}
\]
where the vertical isomorphisms are induced by the map $k \mapsto k/\sigma(k)$. Since $\mu$ is trivial on $N_{E/F}(E^\times)$, $\mu|_{1+P_E}$ factors through the quotient $(1 + p_E)/(1 + p_F)$ and can be regarded as a character of $E^\times$. Denote this character by $\mu^\vee$. By [O, Proposition 3.1],

\[(7.5) \quad \xi_{\mu}^\vee|_{E^+} = \mu^\vee. \]

8. Minimal $K$-types of induced representations

Let $V = E^{2m}$ be endowed with a skew-Hermitian form represented by $J_{2m}$. Denote by

\[
\{e_1, \ldots, e_m, f_m, \ldots, f_1\}
\]

the standard basis of $V$ so that

\[
\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0, \quad \langle e_i, f_j \rangle = \delta_{ij}.
\]

As before we assume that $m \geq 2$, and put $n = \lfloor \frac{m+1}{2} \rfloor$. Let

\[
V = X + W + X^\vee
\]

be a polarization, where $X = \text{Span}_E\{e_1, \ldots, e_n\}$ and $X^\vee = \text{Span}_E\{f_1, \ldots, f_n\}$ are isotropic subspaces of dimension $l_0 = m - n$ and are dual to each other. Set

\[
G = U(V) = U_{2m}(F), \quad H = U(W) = U_{2n}(F), \quad M = \text{GL}(Y) = \text{GL}_{m}(E),
\]

where $Y = \text{Span}_E\{e_1, \ldots, e_m\}$. Following [BS], we identify a point in the building of a classical group with a lattice function. Assume that $x \in \mathcal{B}(M) \subset \mathcal{B}(G)$ is the hyperspecial point which corresponds to the self-dual lattice

\[
V_{x,0} = \mathcal{O}_{E}^{2m},
\]

so that

\[
V_{x,0} \cap W = \mathcal{O}_{E}^{2n} = L
\]

is the lattice of $W$ as in Section 7. Let $y \in \mathcal{B}(H)$ be the hyperspecial point corresponding to $L$. Then

\[
W_{y,0} = L, \quad H_y = U(W)_L, \quad H_y/H_{y,0+} = H^F = U_{2n}(f) \text{ or } \text{Sp}_{2n}(f)
\]

as in (7.1). Recall that $P$ is the Siegel parabolic subgroup of $G$ with Levi subgroup $M$. We also have the quotient

\[
G^F = G_x/G_{x,0+} = U_{2n}(f) \text{ or } \text{Sp}_{2n}(f)
\]

according to whether $E/F$ is unramified or not, which has the Siegel parabolic subgroup $P^F = P_x/P_{x,0+}$ and Levi subgroup

\[
M^F = M_x/M_{x,0+} = \text{GL}_{m}(E).
\]

Let $\tau = \varepsilon\text{-Ind}_{M[x]}M_{[x]}^\vee$ be an irreducible distinguished depth zero supercuspidal representation of $M$, where $\bar{\tau} = \bar{\tau}_{s,c_s}$ as given by Theorem 6.6. Then $\bar{\tau}|_{M_s}$ is inflated from an irreducible (conjugate) self-dual cuspidal representation

\[
\bar{\tau'} := (-1)^{m+1}P_{T,s}^{M}
\]

of $M^F$. We have

\[
\text{Ind}_P^G(\bar{\tau'}) = \pi_{\text{reg}} \oplus \pi_{\text{ss}}
\]

as in (5.3) and (5.8). Recall the parabolic induction $\rho_{\tau, s} = \text{Ind}_P^G(\tau_s), s \in \mathbb{C}$, given by (6.1), and the short exact sequence (6.2) at $s = 1$. For convenience, write

\[
\delta' = \delta + m/2.
\]

By abuse of notation, in the rest of the paper we often identify a representation of a finite quotient with its inflation. The main result of this section is the following.

**Proposition 8.1.** With above definitions, we have that

1. $\rho_{\tau, s}^{G_{x,0+}} \cong \pi_{\text{reg}} \oplus \pi_{\text{ss}}, s \in \mathbb{C},$
2. $\rho_{\pi, \delta}^{G_{x,0+}} \cong \pi_{\text{reg}}, \pi_{\pi, \delta, \pi}^{G_{x,0+}} \cong \pi_{\text{ss}}.$
Proof. (1) Let $N$ be the unipotent radical of $P$, and let $N^F = N_x/N_{x,0^+}$. Take $z \in B(M)$ such that the following hold:
- $G_{x,0^+} \subset G_{z,0^+} \subset G_z \subset G_x$,
- $G_{z,0^+}/G_{x,0^+} \cong NF \subset GF$,
- $G_z/G_{z,0^+} \cong M_z/M_{z,0^+}$.

Define the following representation of $M[z] = ZM_x$,

$$\tau_\delta := \bar{\tau} \otimes |\det|_E^{\delta-1/2}, \quad \delta \in \mathbb{C},$$

so that

$$\tau_\delta = c\text{-Ind}_{M[z]}^M \bar{\tau}_\delta.$$  

Denote by $\mathcal{J}_N$ the Jacquet functor with respect to $N$. By [MP, Proposition 6.7],

$$\rho_{\tau,\delta}^{G_{x,0^+}} \cong \mathcal{J}_N(\rho_{\tau,\delta})^{M_{x,0^+}} \cong \bar{\tau}' \oplus \bar{\tau}'.$$

On the other hand,

$$\rho_{\tau,\delta}^{G_{x,0^+}} = (c\text{-Ind}_{ZM_zN}^N(\bar{\tau}_\delta))^G_{x,0^+}
\supset (c\text{-Ind}_{ZM_{x,0}N}^N(\bar{\tau}_\delta))^G_{x,0^+}
\cong \text{Ind}_G^G(\bar{\tau}')
= \pi_{\text{reg}} \oplus \pi_{\text{ss}}.$$  

It is clear that

$$\pi_{\text{reg}}^{N^F} \cong \pi_{\text{ss}}^{N^F} \cong \bar{\tau}'$$

as $M^F$-modules. By the uniqueness of minimal $K$-types, every $G^F$-modules in $\rho_{\tau,\delta}^{G_{x,0^+}}$ has cuspidal support $\bar{\tau}'$. Thus it follows from (8.2) and (8.3) that equality must hold in (8.3), that is,

$$\rho_{\tau,\delta}^{G_{x,0^+}} \cong \pi_{\text{reg}} \oplus \pi_{\text{ss}}.$$

(2) We first note that

$$\mathcal{J}_N(\rho_\tau) \cong \mathcal{J}_N(\pi_\tau) \cong \tau$$

and therefore

$$\mathcal{J}_N(\rho_\tau)^{M_{x,0^+}} \cong \mathcal{J}_N(\pi_\tau)^{M_{x,0^+}} \cong \bar{\tau}'$$

It follows that exactly one of $\rho_\tau^{G_{x,0^+}}$ and $\pi_\tau^{G_{x,0^+}}$ is isomorphic to $\pi_{\text{reg}}$, and the other is isomorphic to $\pi_{\text{ss}}$.

Let $U$ be the unipotent radical of the standard Borel subgroup of $G$, and let $U^F = U_x/U_{x,0^+}$. Let $\psi$ be a generic character of $U$ of depth zero. By Frobenius reciprocity, we have

$$0 \neq \text{Hom}_G(\rho_{\tau,\delta}, \text{Ind}_U^G(\psi))
= \text{Hom}_G(c\text{-Ind}_{ZM_{x,0}N}^N(\bar{\tau}_\delta), \text{Ind}_U^G(\psi))
\supset \text{Hom}_{M_xN}(\bar{\tau}_\delta, \text{Ind}_{ZM_{x,0}U}^G(\psi))
= \text{Hom}_{M_x}(\bar{\tau}', \text{Ind}_{M_xU}^G(\psi))
\supset \text{Hom}_{M_xN,G_{x,0^+}}(\bar{\tau}', \text{Ind}_{G_{x,0^+}U}^G(\psi))
= \text{Hom}_U(\text{Ind}_F^G(\bar{\tau}'), \psi|_{U^F}).$$

By the uniqueness of Whittaker models, all the above spaces are one-dimensional and the inclusions are in fact equalities. Recall that $\pi_{\text{reg}}$ is generic and $\pi_{\text{ss}}$ is not. It follows that a nonzero Whittaker functional of $\rho_{\tau,\delta}$ is nonvanishing on $\pi_{\text{reg}} \subset \rho_{\tau,\delta}^{G_{x,0^+}}$. Following [Ki], there is an intertwining operator $\rho_{\tau,1} \to \rho_{\tau,-1}$, with image isomorphic to $\pi_\tau$. Hence $\pi_{\text{reg}} \nsubseteq \rho_{\tau}^{G_{x,0^+}}$, which implies that

$$\pi_{\text{reg}} \cong \rho_{\tau}^{G_{x,0^+}}, \quad \pi_{\text{ss}} \cong \pi_{\tau}^{G_{x,0^+}}.$$
9. Explicit depth zero local descent

We keep the notations from the previous sections. We first recap the irreducible supercuspidal representations \( \sigma_{s,a} \) and \( \sigma \) of \( H = U(W) = U_{2n}(F) \) given by (1.3) and (1.4) respectively. Recall the representation \( \tau_s \) of \( GL_m(E) \) given by Theorem 6.6, and the character \( \xi_\mu \) of \( H_y = U(W)_L \) defined in Section 7.

- \( E/F \) is unramified, \( m = 2n - 1 \). In this case, as in the proof of Theorem 5.1, for each \( a \in \mathfrak{o}_2^1 \) define
  \[ \tilde{\sigma}_{s,a} = (-1)^n R^{\mathfrak{u}_{2n}}_{S_0,(-s,a)} \]
  where \( S_0 \in \mathcal{T}(U_{2n}), S_0^F \cong \mathbf{f}_{2n-2}^1 \times \mathbf{f}_2^1 \) is as in Theorem 5.1 (2). Define
  \[ \sigma_{s,a} = c\operatorname{Ind}_{H_y}^{H} (\tilde{\sigma}_{s,a} \otimes \xi^{-1}_\mu). \]

- \( E/F \) is ramified, \( m = 2n \). In this case, define
  \[ \sigma_s = (-1)^n R^{Sp_{2n}}_{S_0,-s}, \]
  where \( S_0 \in \mathcal{T}(Sp_{2n}), S_0^F \cong \mathbf{f}_{2n}^1 \) is as in Theorem 5.3 (2). Define
  \[ \sigma_s = c\operatorname{Ind}_{H_y}^{H} (\sigma_s \otimes \xi^{-1}_\mu). \]

Recall that in this case \( \xi_\mu = \mu^+ \circ \det \), where \( \mu^+ \) is the character of \( E^+ \) defined at the end of Section 7.

Now we state and prove Theorem 1.3 as follows, which is the main result on the depth zero local descent for unitary groups.

**Theorem 9.1.** Let the notations be as above. In particular \( \tau := \tau_s \) is an irreducible distinguished depth zero supercuspidal representations of \( GL_m(E) \) as in Theorem 6.6, with \( m \geq 2 \). Then the following hold.

1. If \( E/F \) is unramified, \( m = 2n - 1 \), then \( \sigma_{s,a} \) is a multiplicity free direct summand of \( D_{n-1,\psi_{F,\mu}}(\tau_s) \) for each \( a \in \mathfrak{o}_2^1 \).
2. If \( E/F \) is ramified, \( m = 2n \), then
   \[ D_{n,\psi_{F,\mu}}(\tau_s) = \sigma_s. \]

**Proof.** We shall only give the proof of (1), and the proof of (2) is similar. By the uniqueness of Fourier-Jacobi models or the more specific Theorem 6.3, it suffices to show that
\[
\operatorname{Hom}_H(D_{n-1,\psi_{F,\mu}}(\tau_s), \sigma_{s,a}) = \operatorname{Hom}_{R_{n-1}}(\pi_{\tau,s,a} \otimes \nu_{n-1,\psi_{F,\mu}}) \neq 0.
\]
We refer to Section 6.2 for the definition of the subgroup \( R_{n-1} = H \rtimes N_{n-1} \) of \( G \) and its representation
\[
\nu_{n-1,\psi_{F,\mu}} = \psi_{n-1} \otimes \omega_{2n,\psi_{F,\mu}}.
\]
In particular, \( \omega_{2n,\psi_{F,\mu}} = \omega_{2n,\psi_{F,\mu}} \) is the Weil representation of \( H \rtimes \mathbb{H}_W \), and the character \( \psi_{n-1} \) of \( Z_{n-1} \subset GL_{n-1}(E) \) is of depth zero defined using \( \psi_F \).

Following [ST], we have the open \((P,R_{n-1})\)-double coset in \( G \),
\[
\mathcal{O} := P^\gamma R_{n-1} = P^\gamma H N_{n-1},
\]
where
\[
\gamma := \begin{pmatrix}
0 & I_n & 0 & 0 \\
0 & 0 & 0 & -I_{n-1} \\
I_{n-1} & 0 & 0 & 0 \\
0 & 0 & I_n & 0
\end{pmatrix}.
\]
Define a subspace of \( \rho_{\tau,\beta} \),
\[
S(\mathcal{O}, \tau_s) := \{ f \in \rho_{\tau,\beta} \mid \operatorname{supp}(f) \subset \mathcal{O} \}.
\]
By [ST, Proposition 2.1], we have a natural isomorphism
\[
D_{n-1,\psi_{F,\mu}}(\rho_{\tau,\beta}) \cong D_{n-1,\psi_{F,\mu}}(S(\mathcal{O}, \tau_s))
\]
of \( H \)-modules.
Put $P^\gamma := \gamma^{-1} P \gamma \cap R_{n-1}$, so that there is a natural bijection

$$P \setminus P \gamma R_{n-1} \to P^\gamma \setminus R_{n-1}, \quad P \gamma g \mapsto P^\gamma g, \quad g \in R_{n-1}.$$ 

It follows that

$$(9.3) \quad S(\emptyset, \tau_\emptyset) \cong c \text{-} \text{Ind}^{R_{n-1}}_{\tau_\emptyset} \chi_\psi,$$

where $\tau_\emptyset(p) = \tau_\emptyset(\gamma p \gamma^{-1})$, $p \in P^\gamma$. Put $P_{[x]} = M_{[x]} N$, and define the open subgroup of $P^\gamma$,

$$P^\gamma_{[x]} := \gamma^{-1} P \gamma \cap H_y N_{n-1}.$$ 

Direct calculation shows that $P^\gamma_{[x]}$ is compact, with finite quotient $(P^\gamma)^F$, where

$$P^\gamma := \gamma^{-1} P \gamma \cap H N.$$ 

Recall that $\tau_\emptyset = c \text{-} \text{Ind}^M_{M_{[x]}} \tau_{[x]}$, where $\tau_{[x]}$ is given by (8.1). Applying (9.2), (9.3) and Frobenius reciprocity,

$$\text{Hom}_H(D_{n-1, \psi_\psi}, \sigma_{\gamma, \mu}) \ni \text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau}' \sigma_{\gamma, \mu}) \cong \text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau} \sigma_{\gamma, \mu}) \cong \text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau} \sigma_{\gamma, \mu}) \cong \text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau} \sigma_{\gamma, \mu}),$$

where the last inclusion is induced by the homomorphism $\phi$ in Corollary 7.2.

The verbatim adaptation of the calculation in [ST] gives the finite field analog of (9.4) that

$$(9.5) \quad \text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau}') \sigma_{\gamma, \mu} = \text{Hom}_H(P_{[x]} \psi, \sigma_{\gamma, \mu}),$$

where $\tilde{\psi}_{n-1}$ denotes the reduction of $\psi_{n-1}$ to $Z_{n-1}^F$ (cf. Section 5.1). By Theorem 5.1 (1),

$$\text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau}') \sigma_{\gamma, \mu} = \text{Hom}_H(D_{n-1, \psi_\psi}, \text{Ind}^{P_{[x]}}_\psi \tilde{\tau}') \sigma_{\gamma, \mu},$$

is one-dimensional, and we take a generator $\varphi$. Then $\varphi$ through (9.5) inflates to a nonzero map in the last space in (9.4), which thereby induces a nonzero map

$$\varphi \in \text{Hom}_H(D_{n-1, \psi_\psi}, \sigma_{\gamma, \mu}) = \text{Hom}_{R_{n-1}}(\rho_{\gamma, \mu}, \sigma_{\gamma, \mu}) \varphi \in \text{Hom}_{R_{n-1}}(\rho_{\gamma, \mu}, \sigma_{\gamma, \mu}).$$

Recall from Proposition 8.1 that

$$\rho_{\gamma, \psi} \cong \text{Ind}_{\gamma \psi}^{P_\psi}(\tilde{\tau}') \cong \pi_{\psi \gamma} \oplus \pi_{\gamma}. \quad \pi_{\psi \gamma} \cong \pi_{\psi} \cong \pi_{\gamma},$$

By unfolding the Frobenius reciprocity, we see that $\varphi \mid _{\pi_{\psi \gamma}}$ is the inflation of $\varphi$. Theorem 5.1 (2) implies that $\varphi \mid _{\pi_{\psi \gamma}} \neq 0$, and therefore

$$\varphi \mid _{\pi_{\psi \gamma}} \neq 0.$$ 

Taking $\delta = -1$, it follows that the composition

$$\pi_{\gamma} \mapsto \rho_{\gamma, \psi} \mapsto \sigma_{\gamma, \mu} \varphi \mid _{\psi \gamma}$$

gives a nonzero map in $\text{Hom}_{R_{n-1}}(\pi_{\gamma, \sigma_{\gamma, \mu} \varphi \mid _{\psi \gamma}})$, hence (9.1) holds. \hfill $\Box$
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