Strong coupling theory of heavy fermion criticality

Elihu Abrahams,1 Jörg Schmalian,2 and Peter Wölfle2,3

1Department of Physics and Astronomy, University of California Los Angeles, Los Angeles, CA 90095
2Institute for Theory of Condensed Matter, Karlsruhe Institute of Technology, 76049 Karlsruhe, Germany
3Institute for Nanotechnology, Karlsruhe Institute of Technology, 76031 Karlsruhe, Germany

(Dated: February 6, 2014)

We present a theory of the scaling behavior of the thermodynamic, transport and dynamical properties of a three-dimensional metal at an antiferromagnetic (AFM) critical point. We show how the critical spin fluctuations at the AFM wavevector \( q = Q \) induce energy fluctuations at small \( q \), giving rise to a diverging quasiparticle effective mass over the whole Fermi surface. The coupling of the fermionic and bosonic degrees of freedom leads to a self-consistent relation for the effective mass, which has a strong coupling solution in addition to the well-known weak-coupling spin-density-wave solution. We thereby use the recently-introduced concept of critical quasiparticles, employing a scale dependent effective mass ratio \( m^*/m \) and quasiparticle weight factor \( Z \). As a consequence of the diverging effective mass the Landau Fermi liquid interaction is found to diverge in all channels except the critical one, causing important vertex corrections. The ensuing spin fluctuation spectrum obeys \( \omega/T \)-scaling. Our results are in good agreement with experimental data on the heavy fermion compounds YbRh\(_2\)Si\(_2\) and CeCu\(_{6-x}\)Au\(_x\) assuming 3D and 2D spin fluctuations, respectively.

Introduction. Quantum phase transitions in metallic compounds have attracted considerable interest over the last two decades. These systems exhibit deviations from the standard Fermi liquid model. This “non-Fermi liquid” behavior is a consequence of the interaction of the fermionic (Landau) quasiparticles with bosonic critical fluctuations. Early theories of quantum critical behavior, formulated in the framework of a Ginzburg-Landau-Wilson action of the order parameter field \( \phi \) [1-2], found that the effective dimension of the corresponding \( \phi^4 \)-field theory is increased to \( d_{\text{eff}} = d + z \) where \( d, z \) are the spatial dimension of the fluctuations and the dynamical critical exponent, respectively. In many cases of interest \( d_{\text{eff}} \) is above the upper critical dimension, so that the fluctuations are effectively non-interacting and the theory is of the Gaussian type (for a review see [3]). While this theory is well founded in the case of non-metallic systems, for metallic systems the question arises whether the fermionic degrees of freedom may be easily integrated out. In this paper, we show that it is often not possible to reduce the description to a Hertz-Millis model. Rather, the interplay of fermionic and bosonic degrees of freedom generates critical behavior of the fermionic quasiparticles, acting back on the spectrum of bosonic fluctuations. Then, a strong coupling regime with respect to the fermion-boson coupling is reached, while the interaction of critical bosonic fluctuations with each other is still weak, such that the theory remains in a “renormalized Gaussian regime”.

Experimentally well-studied candidate systems we shall focus on are the heavy-fermion compounds CeCu\(_{6-x}\)Au\(_x\) (CCA) for which we postulate quasi-two-dimensional antiferromagnetic (AFM) spin fluctuations at the critical point \( x = 0.1 \) [4] and YbRh\(_2\)Si\(_2\) (YRS), for which we assume AFM fluctuations at the critical point at magnetic field \( H = H_c \) of three-dimensional character at temperatures \( T \) less than 0.3 K, crossing over into three-dimensional ferromagnetic fluctuations at higher \( T \). An issue that has hampered progress in developing a strong-coupling theory of criticality that involves AFM fluctuations has been that they transfer a large momentum of the order of the ordering wave vector \( q = Q \). As a consequence, the self-energy in one-loop order becomes highly anisotropic, being critical only at so-called “hot spots” on the Fermi surface that are connected by \( \bar{Q} \). However, the combined exchange of two AFM fluctuations, which may be viewed as a spin-exchange energy fluctuation [7], may transfer only a small momentum \( q \) and we argue here that the one-loop order process of such energy fluctuations is dominant in providing a renormalization of the fermionic quasiparticle effective mass that is uniform over the Fermi surface. Thus, we consider here the simplest case, in which the fermion self energy is only weakly momentum dependent. A different way by which the effect of the critical AFM fluctuations may be distributed all over the Fermi surface is by means of impurity scattering [8-9].

However, a problem with multiple fluctuation exchange is that each additional fluctuation propagator gives rise to an additional energy integration and will thus contribute a small phase space factor, anywhere from \( \omega^2 \) to \( \omega \), depending on the critical momentum dependence. As we will show below, such factors may be offset by Fermi liquid renormalization factors \( Z^{-n} \gg 1 \), provided the quasi-particle weight factor \( Z \) tends to zero, as the excitation energy \( \omega \) or the temperature \( T \) tends to zero. As discussed in our previous work [8-9], even if \( Z \to 0 \) in the limit \( \omega, T \to 0 \) the quasiparticle picture may still be applicable at non-zero \( \omega, T \), since the quasiparticle width \( \Gamma \) gets renormalized by a factor of \( Z \), which helps to keep it smaller than the excitation energy \( \omega \), a necessary condition for the existence of a quasiparticle peak in the
single-electron spectral function. A careful identification of the renormalization of the bosonic (AFM) propagator by $Z$-factors is at the heart of our theory. This includes the determination of critical vertex corrections of various types. For example, while it is well-known that the spin susceptibility at small $q$ receives a correction of the external vertex of order $1/Z$, which follows from a Ward identity, we argue that a similar vertex correction applies even at the AFM wave vector $Q$. The reason is that a diverging effective mass ($\propto 1/Z$) causes the Landau parameters (dimensionless Fermi-liquid interaction components) of all channels except the critical one to diverge as well, thus rendering the corresponding partial susceptibilities finite. Acting in the crossed particle-hole channel, these divergent interactions cause divergent vertex corrections $\propto 1/Z$. This means that a fermion-boson model in which only critical bosons are included, which has been used frequently at weak coupling, is no longer sufficient at strong coupling. The interplay of spin fluctuations and fermionic excitations has also been considered in $1/N$ expansion by Abanov and Chubukov [10] and a renormalization group formulation has been given by Metlitski and Sachdev [11]. However, these authors did not consider the quasiparticle renormalization in the strong coupling regime.

In the context of heavy fermion compounds, the quantum critical point has been often associated with a breakdown of the Kondo effect, and therefore a breakdown of the picture of heavy quasiparticles [12–14]. In this scenario, it is assumed that at the critical point the energy scales of the Kondo effect and the exchange interaction between the localized $f$-spins (in the absence of the Kondo effect) are approximately equal. Some of these scenarios have been developed enough to allow comparison with experimentally observed critical exponents, in particular for CeCu$_{6.9}$Au$_{0.1}$ and for YbRh$_2$Si$_2$ [15–16]. However, experimentally the quasiparticle mass does not appear to be drastically reduced (by orders of magnitude) when the QCP is approached, as would be expected if the Kondo effect were to be suppressed. We argue here that in the cases we consider the Kondo effect, or more precisely the heavy quasiparticle picture, remains intact. However, the quasiparticles experience an AFM spin-exchange interaction responsible for the ordering of their spins in the AFM state. In other words, we propose that the ordered state is an itinerant heavy-quasiparticle SDW state, at least in the neighborhood of the critical point. The resulting small ordered magnetic moment is in agreement with observation. Roughly speaking, the ordered moments provide a magnetic field acting on the Kondo ions. As long as the $f$-electron Zeeman splitting caused by this field is small compared to the Kondo temperature, the Kondo effect is only weakly suppressed.

In this report, we present a semi-phenomenological theory of the scaling behavior near an AFM QCP within an Anderson lattice model. As discussed above, we show how energy fluctuations may lead to a momentum-independent critical quasiparticle self energy. The feedback of the critical quasiparticle properties (the $Z$-factor) into the spin and energy fluctuation spectrum leads to a self-consistent equation for the quasiparticle self energy and effective mass $m^*/m \propto Z^{-1}$. This allows a strong-coupling solution in the form of a fractional power law $Z(\omega) \propto \omega^\alpha$ with $\alpha = 1/4$ for 3D and $\alpha = 1/8$ for 2-D AFM spin fluctuations. The free energy obeys scaling characterized by fractional power laws. The dynamical structure factor is found to satisfy $\omega/T$-scaling within the critical cone. Comparison of our theory with experimental data shows detailed agreement.

The above scenario depends sensitively on the detailed nature of spin fluctuations in a given system. For example, 3D AFM fluctuations do not lead to true critical behavior, i.e. a Gaussian fluctuation theory is applicable [11–12], provided the effective mass enhancement by critical fluctuations is not too large. We argue below that in CCA there is a wide region of quasi-2D antiferromagnetic fluctuations which gives rise to a substantial enhancement of the effective mass and may drive the system into a strong-coupling regime of 2D or 3D antiferromagnetic fluctuations.

Critical quasiparticle picture. Our starting point is a heavy Fermi liquid as found for an Anderson lattice model of correlated $f$-electrons (on-site interaction $U$) hybridizing with conduction electrons. The energy scale of the heavy-fermion band is given by the “coherence temperature” $T_{coh}$, which is well above the temperature regime for which “non-Fermi liquid” behavior is observed near a QCP. On top of the heavy-fermion quasiparticle renormalization, the critical fluctuations cause a further renormalization on which we focus below. The single-particle Green’s functions may be decomposed into a quasiparticle term and an incoherent contribution, $G(\mathbf{k}, \omega) = ZG^{qp} + G^{inc}$, where the quasiparticle weight factor $Z$ is defined by $Z^{-1} = 1 - \partial \text{Re} \Sigma(\omega)/\partial \omega$. The quasiparticle Greens function is given by $G^{qp}(\mathbf{k}, \omega) = [\omega - E_k - i\Gamma]^{-1}$, with $E_k = (m/m^*)v_F(k - k_F)$, where $v_F$ is the Fermi velocity of the heavy-fermion band and the quasiparticle width is $\Gamma = Z \text{Im} \Sigma(E_k)$.

The condition for the quasiparticle picture to be valid is $\Gamma < |E_k|$. In the Fermi-liquid regime, $\Gamma = c(E_k)^2 < |E_k|$ in the limit $E_k \rightarrow 0$. Here, we argue that the quasiparticle stability condition may be even satisfied in non-Fermi liquid situations. We extend the usual quasiparticle picture by recognizing that the parameter $Z = m/m^*$ depends on the energy scale, $Z = Z(\omega) = 1/[1 - \partial \Sigma(\omega)/\partial \omega]$. It is important to observe that the (retarded) self energy is an analytic function in the upper half plane, so that the real and imaginary parts of any nonanalytic term (in the lower half plane) are locally connected. Even in a true non-Fermi-liquid phase with $\Sigma(\omega) \propto -i|\omega|^{1-\alpha}$, $\alpha < 1$ so that $\text{Im} \Sigma(\omega) \propto \text{Re} \Sigma(\omega) \propto |\omega|^{1-\alpha}$ and $Z \propto (|E_k|)^\alpha$, one finds $\Gamma/|E_k| = \tan(\frac{\pi}{2}\alpha) < 1$.
for $0 < \alpha < 1/2$. In this case, although $Z = 0$ at the Fermi surface, the spectral function for non-zero excitation energy may be peaked sharply enough to separate a quasiparticle contribution from the incoherent part.

**Spectrum of critical fluctuations.** We assume that in the paramagnetic phase of a metal close to an antiferromagnetic quantum critical point, the self energy for the single-particle Green’s function is determined by the interaction with magnetic fluctuations. Here, we take the imaginary part of the renormalized retarded dynamical spin susceptibility for wave vectors near the AFM ordering wave vector $Q$ to be of the form

$$\text{Im} \chi(q, \omega) = \frac{(N_0/Z)(\nu/v_F^* Q)}{\nu + Zq^2 \xi_0^2 + (\nu/v_F^* Q)^2},$$

(1)

where $q$ is measured from the ordering wave vector $Q$. Here, $N_0$ is the bare and $N_0/Z$ the renormalized density of states at the Fermi surface, $v_F^*$ is the renormalized quasiparticle Fermi velocity, $\xi_0 \approx k_F^{-1}$ is the microscopic AFM correlation length and the control parameter $r = 0 + T/Z$, where $r_0(H) = 1 + F(Q, H) \propto H/H_c - 1$ and $T/Z$ is the temperature-dependent shift of the parameter $r$ \[2\] [3]. Here, $F(Q, H)$ is a dimensionless generalized Landau parameter, which tends to $-1$ at the critical point. $H$ is representative of the control parameter that tunes to the phase transition, and thus may be magnetic field, pressure, doping concentration, etc. The factor $Z$ multiplying $q^2$ in Eq. (1) is a crucial feature of our theory, see \[8\] [9].

We now define an energy-fluctuation propagator $\chi_E(k, \nu)$ by combining two spin fluctuation propagators, in the form

$$\text{Im} \chi_E(k, \nu) = \sum_{q_1, \nu_1} G_{k+q_1} G_{k+q_1-\kappa} \text{Im} \chi(q_1, \nu_1)$$

$$\times \text{Im} \chi(q_1 - k, \nu_1 - \nu) [b(\nu_1 - \nu) - b(\nu_1)],$$

(2)

where $b(\nu)$ is the Bose function. The Green’s functions $G_{k+q_1}, G_{k+q_1-\kappa}$ are off-shell and may be replaced by $1/\epsilon_p$. Performing the momentum integration by Fourier transform, we get

$$\text{Im} \Sigma(k, \omega) = -\lambda_F^2 \int \frac{d\nu}{\pi} \sum_q \text{Im} G(k + q, \omega + \nu)$$

$$\times \text{Im} \chi_E(q, \nu) [b(\nu) + f(\omega - \nu)]$$

$$\approx \omega_0 Z^{-2} (\omega/\omega_0 Z^2)^{d-1/2} \propto |\omega|^{d-1/2-\alpha(2d+1)},$$

(4)

where we used $\int d\cos \theta_{q,K} \text{Im} G_{k+q} \approx (1/Z_v f q)$, the dimensionful (Landau) interaction vertex is given by $\lambda_F \propto (ZN_0^2)^{-1}$, and $f(\omega), b(\omega)$ are Fermi and Bose functions, which at low $T$ confine the $\nu$-integration to the interval $[0, \omega]$. In the last equation we assumed the power law $Z(\omega) \propto |\omega|^{\alpha}$. The scale dependent contribution to $\text{Re} \Sigma(\omega)$ follows from analyticity as $\text{Re} \Sigma(\omega) \propto -|\omega|^{d-1/2-\alpha(2d+1)} Z^{-2} (\omega/\omega_0 Z^2)^{d-1/2}$. This leads to the self-consistency relation for $Z(\omega)$

$$Z^{-1}(\omega) = 1 - \frac{1}{\partial \text{Re} \Sigma(\omega)} \partial \omega \approx 1 + Z^{-2d-1} (\omega/\omega_0)^{d-3/2}$$

(5)

We now explore the consequences of the scale dependent $Z$. In general, the $\omega$ and $T$ dependence of $Z$ is obtained by substituting $\sqrt{\omega^2 + cT^2}$ for $\omega$, where $c$ is a constant of order unity. For frequencies less than the temperature, we may replace $\omega$ by $T$. As long as $Z^{-2d-1}(T)/\omega_0^{d-3/2} \ll 1$ for any $T$, the system will be in the Gaussian fluctuation regime all the way down to the critical point. If, however, the initial value of $Z^{-1}(T)$, when one enters the AFM fluctuation regime, is sufficiently large, such that $Z^{-2d-1}(T)/\omega_0^{d-3/2} \gg 1$, a new regime is accessed, which is of a strong-coupling nature. We find the characteristics of this new regime within the present approximation by solving the self-consistent Eq. (5), to get

$$Z(T) \propto (T/\omega_0)^{(1-3/2d)/2}.$$ 

(6)

The exponent $\alpha$ is therefore found to be $\alpha = 1/4$ for 3D and $\alpha = 1/8$ for 2D fluctuations.

In the case of only AFM fluctuations in a clean system, it is difficult to satisfy the strong-coupling condition of sufficiently large $Z^{-1}(T)$ . However, if on the initial approach to the critical point, fluctuations dominate that lead to a growing $Z^{-1}(T)$ with decreasing $T$, the condition may be met at some point. The precise crossover point is determined by the crossover of these precursor fluctuations to the critical AFM fluctuations and by the condition above that leads to Eq. (4). As discussed in \[8\] [9] impurity scattering helps to enhance the effect of AFM fluctuations on $Z(T)$ . In addition, there are clear indications in the data on YbRh$_2$Si$_2$ of 3D FM fluctuations. \[18\] In that case, one finds $Z^{-1}(T) \propto \ln(T_0/T)$, so that $Z^{-1}$ grows as $T \to 0$.

**Role of non-critical fluctuations and vertex corrections.** An effective mass that diverges uniformly over the Fermi
surface i.e. $Z \to 0$ seems to cause diverging susceptibilities in all channels, since in Fermi-liquid theory the partial susceptibilities are given by $\chi_l = N_l^0 / (1 + F_l)$, with $N_l^0 = N_0 / Z$ the renormalized DOS and $F_l$ the Landau interaction parameter in channel $l$. Anticipating that all susceptibilities except the critical one stay finite as the critical point is approached, we are forced to conclude that the Landau parameters diverge $F_l \propto 1 / Z$, compensating for the divergent DOS. This has been shown within microscopic theory in at least one case, a $d$-wave Pomeranchuk instability [19]. The diverging Landau interaction gives rise to a singular correction of the spin vertex at $q = Q, \Gamma_0^l \propto 1 / Z$, when substituted in the crossed p-h channel. The two vertex corrections at both ends of the particle-hole bubble $\chi_0(q, \omega) = \sum GG$ cancel with the qp weight factors of the two Green’s functions, in analogy to the known behavior at $q = 0$. The existence of these vertex corrections is an essential ingredient of our theory.

**Dynamical scaling.** The critical behavior of the spin-excitation spectrum as discussed above, see Eq. (1), is given by (employing units $(\omega_0, \xi_0^{-1})$ for $(\omega, q)$

$$\text{Im} \chi(q, \omega) \propto \frac{\omega / Z^2}{T + Zq^2 / (\omega / Z)^2}, \quad (7)$$

By equating the terms $Zq^2$ and $\omega / Z$ in the denominator, with $Z \propto \omega^\nu$, the dynamical critical exponent is found as $z = 4d / 3$. The correlation length $\xi(H, T)$ in the Fermi liquid (quantum disordered) regime follows from $r_0 \sim Zq^2 \sim q^{2+\alpha}$ as $\xi(H, T = 0) \sim (H - H_c)^{-\nu}$, where $\nu = 3 / (3 + 2d)$. In the critical regime, the leading temperature dependence of the control parameter $r(T) = r_0 + \delta r(T)$ is generated by the fluctuation interaction $(\delta^2)$ - term in lowest approximation: $\delta r(T) \propto T^{1-\alpha}$.

It follows that $\xi(H, T) \sim T^{-1/z}$. The boundary of the critical region - the critical cone - in the $(H, T)$ phase diagram is found from $\xi(H, T = 0) = \xi(H_c, T)$ as $T \sim |H - H_c|^{\nu / \alpha}$. Then $\xi(r_0, T)$ has the form $\xi(r_0, T) = T^{-1/z} g(r_0 T^{-1/\nu})$. The $(\omega, T)$ dependence of $Z$ may be accounted for with the form $Z(\omega, T) \propto |\omega^2 + (\alpha T)^2|^{\nu / 2} = T^{(z-2) / 2\nu} \xi(\omega / T)$. Here, $\alpha$ is a constant (perhaps $\pi$).

Then

$$\text{Im} \chi(q, \omega) \propto T^{-1} \frac{\omega / T^{\xi^2}}{[g^{1/\nu} + \xi^2 q^2 |q|^2]^{1/2} + (\omega / T^{\xi^2})^2}, \quad (8)$$

which shows the following general scaling relation

$$\text{Im} \chi(q, \omega) \propto T^{-1} \Phi \left( \frac{\omega}{T}, q \xi; r_0 T^{1/\nu} \right). \quad (9)$$

Inside the critical cone, where we may set $r_0 = 0$ and $\xi^{-1} \propto T^{1/z}$, with $x = \omega / T$, we find the scaling form

$$\text{Im} \chi(q, \omega) \propto \frac{\omega / T^{\xi^2}}{[g^{1/\nu} + \xi^2 q^2 |q|^2]^{1/2} + (\omega / T^{\xi^2})^2}. \quad (10)$$

$$\Phi(x, y) = \frac{x \xi^{-2}(x)}{(1 + c_q y^2)^2 + (x / \xi)^2}, \quad (11)$$

where $\xi(x) = c_Z [x^2 + a^2]^{\nu / 2}$ and $c_q, c_Z$ are constants.

Inside the critical cone and at the ordering wave vector the spin excitation spectrum obeys $\omega / T$-scaling

$$\text{Im} \chi(q, \omega) \sim \frac{1}{T} \frac{\omega / T^{\xi^2}}{1 + (\omega / T \xi)^2}, \quad (12)$$

A comparison of this scaling form with neutron scattering data on CeCuAu is shown in Fig. 1.

**Free energy.** The critical part of the free energy density may be derived from the expression for the entropy density in terms of the self-energy [21]:

$$S = \frac{1}{2\pi} N(0) \int \frac{d\omega}{T^2 \cosh^2 \frac{\omega}{T}} \frac{\omega}{T} \frac{\omega - \text{Re} \Sigma(\omega)}{T^{\xi^2}} \quad (13)$$

Substituting the critical self energy found above and integrating over temperature we find the scaling form of the free energy density in the case of $d$-dimensional spin fluctuations ($d = 2, 3$) in a 3D metal

$$f(H, T) = \xi^{-(2d+1)} \Phi \left( r_0 \xi^{1/\nu}, T \xi^2 \right) \quad (14)$$

The correlation volume $V_c \sim \xi^{(2d+1)}$ is understood as follows: The underlying critical degrees of freedom in the very low temperature Anderson lattice picture are the heavy fermionic quasiparticles described by the propagator $G(k, \omega)^{-1} = \omega - \epsilon k - \Sigma(\omega)$ with $\Sigma(\omega) \propto \omega^{1-\alpha}$. 

![FIG. 1. Inelastic neutron scattering: Comparison of theory Eq. (12) and experimental data [20] for CeCu$\text{_6}$Au$_x$ at the critical concentration $x = 0.1$](image-url)
Therefore, their dynamical exponent $z_f = 1/(1 - \alpha)$ and their dimensionality is $d_f = 1$. The entropy of the system is determined by hyperscaling for the fermions: $S \propto T^{d_f/z_f}$. Therefore, since $\xi \propto T^{-1/z}$, where $z = 4d/3$, the free energy density $f \propto T^{1+d_f/z_f} \sim \xi^{2d+1}$. Here, $z$ and $d$ are the bosonic ones discussed above.

The specific heat coefficient follows as

$$C/T \propto \begin{cases} T^{(2d+1-2z)/z}, & \text{critical regime} \\ \nu/(2d+1-2z), & \text{Fermi liquid regime}. \end{cases}$$

(15)

For the critical part of the magnetization we find

$$M - M(H_c,0) \propto \begin{cases} -T, & \text{critical regime} \\ -\nu/(2d+1-2), & \text{Fermi liquid regime}. \end{cases}$$

(16)

The susceptibility has a critical part

$$\chi - \chi(H_c,0) \propto \begin{cases} -T^{1-1/\nu}, & \text{critical regime} \\ -\nu/(2d+1-2), & \text{Fermi liquid regime}. \end{cases}$$

(17)

Using $\partial M/\partial T = -\text{const}$, we find the Grüneisen ratio in the critical regime:

$$\Gamma_G = -\frac{\partial M/\partial T}{C} \propto \frac{1}{C} \propto T^{(z-2d-1)/z}$$

(18)

while in the Fermi liquid regime, we have the universal result

$$\Gamma_G = -\frac{\rho_c}{H - H_c}, \quad \rho_c = -(z - 2d + 3/3) \nu$$

(19)

The critical scaling of the transport properties is obtained by observing that the quasiparticle relaxation rate scales as $\Gamma \propto (r_0 \xi^{1/\nu} T \xi^2)$. Then the resistivity behaves as

$$\rho - \rho(H_c,0) \propto \frac{m^*}{m} \Gamma \propto \begin{cases} T^{(z+2)/2}, & \text{critical regime} \\ -r_0/(z-2d-2\nu/2T^2), & \text{Fermi liquid regime}. \end{cases}$$

(20)

For the thermopower, $S$ we get, using $S \propto m^* T$, $S \propto C$ in both the critical regime and the Fermi liquid regime.

**Comparison with experiment.** The above results, in the case of 3D fluctuations, are identical to the ones obtained by two of us previously [8, 9], where impurity scattering was invoked to give rise to a critical, weakly momentum dependent self energy. In particular, the $Z$-exponent $\alpha = 1/4$ found there, and the ensuing critical indices $z = 4, \nu = 1/3$ are the same as the ones found in the present work for the clean system. The excellent agreement of the theory [8, 9] with the experimental data on YbRh$_2$Si$_2$ in the regime close to the QCP therefore applies to the present theory as well. In contrast to that earlier work, the present results do not depend on the impurity concentration. Indeed, in experiment, the critical parts of e.g. the specific heat do not show a dependence on the impurity content of the sample.

We turn to a different case, CeCu$_{6-x}$Au$_x$, for which a QCP has been found at the concentration $x = 0.1$ at ambient pressure in the absence of a magnetic field and at slightly different concentrations at a critical pressure and/or an applied critical field. As suggested by the neutron scattering data, the magnetic fluctuations there appear to be quasi-two-dimensional. We therefore compare our results for $d = 2$ with the available data. In Fig. 1, we have already compared theory and experiment for the dynamical spin susceptibility. In Fig. 2, we show the specific heat data [23] in comparison with $C/T \propto T^{-1/8}$ as obtained above in Eq. (15).

![Fig. 2. Specific heat: Comparison of theory Eq. (15) and experimental data [23] for CeCu$_{6-x}$Au$_x$ at the critical concentration $x = 0.1$.](image)

The resistivity result $\rho(T) - \rho(0) \propto T^{7/8}$ is fitted to the data in Fig. 3.

Our prediction for the uniform magnetization is $M(T) = M(0) - aT + bT^2$, from Eq. (16) augmented by a Fermi liquid correction $\propto T^2$. It fits the data [23] well, as shown in Fig. 4.

**Conclusion.** We presented a semi-phenomenological theory of quantum criticality near a critical point that separates antiferromagnetic and paramagnetic phases of a metal. Starting from the assumption that the Landau quasiparticle effective mass diverges on approaching the critical point, giving rise to critical quasiparticles, we identified the corresponding renormalization of the dynamical spin susceptibility near the antiferromagnetic wave vector. Critical
contributions to the electron self energy induced by antiferromagnetic fluctuations in a clean system are known to be strongly anisotropic (confined to the “hot spots”). However, impurity scattering may be shown [8,9] to distribute the effects of critical scattering all over the Fermi surface. Here we have shown that even in a clean system, the critical antiferromagnetic fluctuations give rise to a critical self energy uniformly over the Fermi surface. This is because the magnetic fluctuations generate energy fluctuations, which diverge in the long-wavelength limit. The scattering of quasiparticles off these energy fluctuations leads to a contribution to the effective mass which is nominally proportional to a positive power of energy, but is strongly enhanced by factors of the effective mass itself. This leads to a self-consistency relation for the effective mass, the quasiparticle Z-factor, which may have a strong-coupling solution provided the initial value of Z at the appropriate high-energy scale is sufficiently small (as may be caused by fluctuations leading to a weakly-diverging effective mass). While the critical quasiparticles living on the “cold” parts of the Fermi surface dominate most of the observable quantities, the “hot” quasiparticles may be shown to be even more singular, e.g. in d = 3, we find the equivalent of the exponent α to be =1/2. In this context, it is interesting to observe that the observed critical behavior of CCA depends on whether the QCP is tuned by varying the Au concentration, the pressure, or the magnetic field. It is conceivable that in the case of magnetic field tuning the precursor fluctuations necessary to access the strong coupling regime are too weak so that the system remains in the weak-coupling regime, as is apparently observed. A further condition for the applicability of the self-consistent solution is that the effective dimension of the bosonic fluctuations, z + d, is sufficiently above the upper critical dimension of the appropriate field theory (e.g. φ⁴ theory), such that boson-boson-interaction effects may be neglected.

Application of our theory to the cases of 3D and 2D fluctuations in a three-dimensional metal leads to a critically diverging effective mass m* ∝ T−α with α = 1/4 (3D) and α = 1/8 (2D), in good agreement with experimental data on the two heavy-fermion compounds YRS and CCA, where neutron scattering showed the presence of 3D and 2D AFM fluctuations. Our theory obeys hyperscaling, taking into account the scaling of the critical fermions. Further comparisons of our theory with experimental data on YRS and CCA show good agreement. In particular the universal behavior of the Grüneisen ratio in the quantum-disordered regime measured in YRS is in excellent agreement with our result [9].

We also note that we have found that the correlation length inside the critical cone grows unusually slowly with decreasing temperature, which appears to have been observed in neutron scattering and has sometimes been interpreted as supporting the notion of a “local quantum critical point.”

The results of this paper can, alternatively, be derived in terms of a renormalization group (RG) approach [25]. The RG analysis demonstrates that the theory is formally an expansion around the lower critical dimension where a marginal Fermi liquid behavior occurs and the notion of singular quasiparticles is well-defined. In our case, this corresponds to d_{c,uc} = 3/2, where the exponent α vanishes. The physics discussed in this paper then appears as an unstable fixed point. It implies that in the
limit where $Z \to 0$, the perturbation theory used here is ultimately expected to break down. Nevertheless, we expect the power laws obtained here to govern the relevant physics for a wide range of energies and temperatures.

Finally, we emphasize that we assumed the heavy quasiparticles to be robust, though modified by scattering from critical spin fluctuations. There is no breakdown of the Kondo effect nor an associated collapse of part of the Fermi surface in our scenario. Experimental features, such as the crossover behavior observed in transport properties across the “$T^*$-line” in the $T - H$ phase diagram of YRS, may be accounted for by a change in quasiparticle scattering strength associated with thermal activation of the (ESR) spin resonance [26].

While the good agreement of our theory with experiment across the board is encouraging, there is a need to put the phenomenological assumptions it involves on a firm microscopic basis. Work in this direction is in progress.

We acknowledge useful discussions with H. v. Loehneysen, F. Steglich, J, Thompson, Q. Si, C.M. Varma, and especially A.V. Chubukov. PW thanks the Department of Physics at the University of Wisconsin-Madison for hospitality during a stay as a visiting professor. Part of this work was performed during the summer of 2012 at the Aspen Center for Physics, which is supported by NSF grant No. PHY-1066293.

[1] J. A. Hertz, Phys. Rev. B 14, 1165 (1976).
[2] A. J. Millis, Phys. Rev. B 48, 7183 (1993).
[3] H. v. Lüschny, A. Rosch, M. Vojta, and P. Wölfle, Rev. Mod. Phys. 79, 1015 (2007).
[4] H. v. Lüschny, et al, J. Phys.: Condens. Matter, 8, 9689 (1996).
[5] O. Trovarelli et al, Phys. Rev. Lett. 85, 626 (2000).
[6] C. Stock et al, Phys. Rev. Lett. 109, 127201 (2012).
[7] This is similar to the composite operator discussion in S.A. Hartnoll, et al, Phys. Rev. B 84, 125115 (2011).
[8] Peter Wölfle, and Elihu Abrahams, Phys. Rev. B 84, 041101 (2011).
[9] Elihu Abrahams, and Peter Wölfle, Proc. Natl. Acad. Sci. USA 109, 3238 (2012).
[10] Ar. Abanov and A.V. Chubukov, Phys. Rev. Lett. 84, 5608 (2000); Phys. Rev. Lett. 93, 255702 (2004).
[11] Max A. Metlitski and Subir Sachdev, Phys. Rev. B 82, 075128 (2010).
[12] Q. Si, S. Rabello, K. Ingersent, and J. L. Smith, Nature (London) 413, 804 (2001).
[13] P. Coleman, C. Pepin, Q. Si, and R. Ramazashvili, J. Phys.: Condens. Matter 13, R723 (2001).
[14] T. Senthil, M. Vojta, and S. Sachdev, Phys. Rev. B 69, 035111 (2004); M. Vojta, J. Low Temp. Phys. 161, 203 (2010).
[15] Q. Si, J-X. Zhu, and D. R. Grempel, J. Phys.: Condens. Matter, 17, R1025 (2005).
[16] I. Paul, C. Pépin and M.R. Norman, Phys. Rev. Lett. 98, 026402 (2007); Phys. Rev. B 78, 035109 (2008); K-S. Kim and C. Pépin Phys. Rev. B 81, 205108 (2010) and references therein.
[17] Peter Wölfle and Elihu Abrahams, Ann. Phys. (Berlin) 523, 591 (2011).
[18] Ref. 6 and P. Gegenwartet al, Phys. Rev. Lett. 89, 056402 (2002); K. Ishida et al, Phys. Rev. Lett. 89, 107202 (2002); C. Krellner et al, Phys. Rev. Lett. 100, 066401 (2008).
[19] D. L. Maslov, and A. V. Chubukov, Phys. Rev. B 81, 045110 (2010).
[20] A. Schröder et al, Phys. Rev. Lett. 80, 5623 (1998).
[21] Here, the $T$-dependence of the self energy $\Sigma$ should be retained, see A. Chubukov et al, Phys. Rev. B 71, 205112 (2005).
[22] The scaling properties of what we call “critical quasiparticles” have been analyzed by T. Senthil, Phys. Rev. B 78, 035103 (2008)
[23] H. v. Lohneysen, et al, Physica B 223 & 224, 471 (1996).
[24] H. v. Lohneysen, private communication.
[25] J. Schmalian, P. Wölfle and E. Abrahams, in preparation.
[26] Peter Wölfle and Elihu Abrahams, Phys. Rev. B 80, 235112 (2009) and to be published.