Defocusing gravitational microlensing

S. Capozziello\textsuperscript{1,2}, R. de Ritis\textsuperscript{1,2}, V.I. Mank’o\textsuperscript{3,4}, A.A. Marino\textsuperscript{1,2}, G. Marmo\textsuperscript{1,2}

\textsuperscript{1}Dipartimento di Scienze Fisiche, Università di Napoli,
\textsuperscript{2}Istituto Nazionale di Fisica Nucleare, Sezione di Napoli,
Mostra d’Oltremare pad. 19 I-80125 Napoli, Italy,
\textsuperscript{3}Osservatorio Astronomico di Capodimonte,
Via Moiariello 16 I-80131 Napoli, Italy
\textsuperscript{4}Lebedev Physical Institute, Leninsky Pr., 53, Moscow 117924, Russia.

Abstract

We introduce the notion of defocusing gravitational lens considering a MA-CHO located behind a light source with respect to an observer. The consequence of defocusing effect is a temporal variability of star luminosity which produces a gap instead of a peak as tell–tale signature in the light curve. General theory of (de)focusing rays (geodesics) in a gravitational field is presented. Furthermore, we give estimations of the mass of the lens and the optical depth connected to such a phenomenon.

PACS: 95.30 Sf

e–mail address:
capozziello@axpna1.na.infn.it
deritis@axpna1.na.infn.it
manko@astrna.na.astro.it
marino@axpna1.na.infn.it
gimarmo@axpna1.na.infn.it
1 Introduction

Recently, gravitational lensing has become one of the most powerful tools in astrophysics and cosmology to investigate the mass distributions and the presence of dark matter in the universe \cite{1,2,3}. In principle, it allows to estimate the gravitational mass of all large-scale structures, starting from galaxies to super clusters, and, in the specific application called \textit{microlensing}, it can be used to search for the so-called MACHOs (\textit{Massive Astrophysical Compact Halo Objects}) \cite{4}, considered the most probable candidates for baryonic dark matter of Galaxy halo \cite{5} (however other possibilities are also explored \cite{6,7,8}).

Such objects may be considered as the main constituents of the dark halo of spiral galaxies (in particular of our Galaxy) and, from theoretical constraints, could have a very large mass range \((10^{-8} \div 10^{2} M_{\odot})\), so that they could be little planets, big planets as Jupiter, brown dwarfs, or massive black holes \cite{9}.

The fundamental issue in this approach is how lensing by a point-like mass can be detected. Unless the lens is very massive \((M > 10^{6} M_{\odot})\), the angular separation of two images (usually produced by a point lens) is too small to be resolved (the angular separations of images are of the order \(\sim 10^{-6}\) arcsec, that is the reason for the term \textit{microlensing}). However, when it is not possible to detect multiple images, the magnification can still be seen if the lens and the source move relatively to each other: this motion gives rise to a lensing–induced time variability of the source luminosity \cite{10}. Such an effect was first observed for the quasars QSO 2237+0305 and QSO 0957+561 \cite{11,12}; so that we have to distinguish \textit{galactic} microlensing and \textit{extragalactic} or \textit{cosmological} microlensing.

In the first case, the light sources are stars and the angular separations involved are \(\sim 10^{-3}\) arcsec, in the second case, the sources are very distant quasars and the angular separations involved are \(\sim 10^{-6}\) arcsec. In both cases the term "microlensing" is used.

The principle on which microlensing lies is quite simple. If the closest approach between a point mass lens and a source is equal or less than \(\theta_{E}\), the Einstein angular radius, the peak magnification in lensing–induced light curve corresponds to a brightness enhancement (\(e.g. \sim 0.3\) magnitudes is a good number), which can be easily detected. The Einstein angular radius \(\theta_{E}\), as we shall discuss below, is a property of the system lens–source which furnishes the natural angular scale to describe the lensing geometry. In fact, for multiple imaging, it gives the typical angular separation among the single images; for axisymmetric lens–source–observer systems, it gives the aperture of the circular bright image, called \textit{Einstein ring} (the Einstein ring, as a geometric construction, can be defined in any case, that is also if a luminous circular image is not produced). However, sources which are closer than \(\theta_{E}\) to the optical axis experience strong lensing effect and are hardly magnified, sources which are located well outside of the Einstein ring are not very much magnified. In other words, for a lot of lens models, the Einstein ring represents the boundary between the zones where sources are strongly magnified or multiply–imaged and those where they are softly magnified or singly–magnified (actually the situation is very complicated depending on caustics and Fermat’s potential. For a detailed exposition see for example \cite{2}).
In order to detect microlensing, the first proposal was to monitor millions of stars in the Large Magellanic Cloud (LMC), or in the bulge of Galaxy in order to look for such magnifications. If enough events are detected, it should be possible to map the distribution of (dark) stellar–mass objects in the halo of Galaxy (due to the fact that LMC is near us and the halo of our galaxy is between) or between the Solar System and the bulge of Galaxy. The two approaches involve some care in the selection of distances between source and observer. In fact, the distance between the Sun and the center of LMC is \( \sim 55 \text{Kpc} \) while the distance between the Sun and the bulge of Galaxy is \( \sim 8.5 \text{Kpc} \): this difference of size gives Einstein radii for the selected sources which could differ of about one order of magnitude. Furthermore, the halo of Galaxy is supposed to extend of approximately \( \sim 50 \text{Kpc} \) so that the zone where MACHOs can pass is very large. However, both approaches can be used for "galactic microlensing" and, if we consider the Einstein radius \( r_E \sim 1 \div 10 \text{AU} \), the distances of the source–lens–observer system \( D \sim 1 \div 50 \text{Kpc} \), and the velocities of passing MACHOs \( v \sim 100 \div 500 \text{Km} \, \text{s}^{-1} \), we are going to give good numbers which can produce observable effects.

The biggest trouble of such a proposal was to distinguish the intrinsic variable stars (which are very numerous in a normal galaxy) and the lensing–induced variables. Fortunately, the light curves of lensed stars have certain features which allow to separate induced variability from intrinsic variability (e.g. the light curves are symmetric in time and there are no chromatic effects since light deflection does not depends on wavelength; on the contrary, intrinsic variables have asymmetric light curves; furthermore, magnification produces chromatic effects due to variability).

The expected time scale for microlensing–induced variations is given in terms of the typical angular scale \( \theta_E \), the relative velocity \( v \) between source and lens, and the distance of the observer to the lens \( D_{ol} \):

\[
\Delta t = \frac{D_{ol} \theta_E}{v} .
\]  

(1)

If light curves are sampled with time intervals between the hour and the year, the mass range of MACHOs is \( 10^{-6} \div 10^2 M_\odot \). Such a time is directly connected to the so called Shapiro delay defined as

\[
\Delta t = \int_{\text{source}}^{\text{observer}} \frac{2}{c^2} |\Phi| \, dl ,
\]  

(2)

which gives the total time delay obtained by integrating over a light path modified by the Newtonian potential \( \Phi \) from the source to the observer.

We have to note that we cannot get the mass of MACHO \( M \) directly from (1) since we have the combination \( M, D_{ol}, D_{ls}, D_{os} \) (in the definition of \( \theta_E \), see below) and \( v \) from which we have to extrapolate \( M \). This is a difficulty of the theory since we need also accurate distance indicators and accurate methods to calculate velocities of stars in the Galaxy.

Furthermore, we have to take into consideration the approximation we used: \( i.e. \) the system lens–source is considered as formed by point–like objects. In order to satisfy such
an approximation, we need that
\[ r_{\text{lens}} \ll r_E ; \]  
so if we are considering galactic microlensing with \( r_E \sim 1 \div 10 \text{AU} \), giants and supergiants stars are excluded since they have sizes of this order of magnitude. If we require, for example, \( r_{\text{lens}} \leq 10^{-3} r_E \), this implies mass densities of the order \( \rho \geq 1 \text{g}/\text{cm}^3 \) and then low mass stars (like those of Main Sequence), brown and white dwarfs pass the requirement. By using just Main Sequence stars as point–sources, as in most of the running experiments, we get a lower limit for the detectable lens mass coming from
\[ r_E \geq R_\odot ; \]  
which implies for the mass
\[ M_{\text{lens}} \geq 10^{-6} M_\odot , \]  
being \( M = \frac{4}{3} \pi r^3 \rho \) the mass contained in a sphere. This means that if MACHOs are extremely light objects (e.g. snowballs with \( M \ll 10^{-7} M_\odot \)) they cannot be detected.

Finally, we need also a statistical approach to microlensing since very little information can be obtained by a single event; then we have to consider some other questions essentially connected to: \( i \) the expected rate of events; \( ii \) the distributions at different \( \Delta t \) (the situation change if we look toward the bulge of the Galaxy or toward the halo); \( iii \) the seasonal modulations due to the Earth motions; \( iv \) the effects of binary (or multiple) stellar systems acting as lenses or as sources; \( v \) the absorption effects which can drop drastically the possibility of observations in certain zones of Galaxy.

Furthermore, the chance of seeing microlensing events depends on the optical depth, which is the probability that at any instant of time a given source is within the angle \( \theta_E \) of a lens. The optical depth is the integral over the number density \( n(D_{\text{ol}}) \) of lenses times the area enclosed by the Einstein ring of each lens, \( \text{i.e.} \)
\[ \tau = \frac{1}{\Omega} \int dV n(D_{\text{ol}}) \pi \theta_E^2 , \]  
where \( dV \) is the volume of an infinitesimal spherical shell with radius \( D_{\text{ol}} \) which covers a solid angle \( \Omega \). Eq.(6) may assume a very simple form if the sources are distant and compact objects, that is if sources and lenses have angular sizes smaller than \( \theta_E \).

Several groups are searching for MACHOs in the Galactic halo (by monitoring stars in LMC) or in the Galactic bulge; among them we have: MACHO [13], EROS [14], OGLE [15], and DUO [16]. So far, about 100 events have been detected and their number is increasing rapidly. Most of them have been seen toward the Galactic bulge. The majority of events have been caused by single lenses, but some of them are due to binary lenses (which are distinguishable from single lens events by characteristic double–peaked light curves). However, we have, until now, few experimental data which can be considered statistically relevant and allow to draw conclusions on the physical properties of MACHOs like their mass.
What we want to stress in our paper is the possibility of looking at the microlensing from a different point of view.

Microlensing is always discussed for lenses which focus light rays. On the other hand, in optics, we know that there exists the opposite effect if the refraction index of media is appropriately chosen and if the relative positions of the source and the lens is changed with respect to the observer. That is, it seems natural to us to ask the question whether there exist or not distributions of matter producing gravitational fields which deflect the light rays in a manner which mimics defocusing lenses of standard optics.

The wish to introduce and to study the notion of defocusing gravitational lens is motivated by the hypothesis that the microlensing events with luminosity peak may be accompanied by the existence of events with valley in the luminosity curve. This inverse phenomenon, in principle, may be understood if the relative positions of MACHOs (lenses), stars (standard sources) and the observer are taken into consideration. Usually, the studied situation is that a MACHO is between the source and the observer. The emitted rays by the source are slightly curved in the direction of the observer and such a fact produces the effect of luminosity magnification. Obviously, the opposite situation is statistically as probable as previous one when a MACHO is located behind the source with respect to the observer. Then, the source rays are slightly curved out of the observer direction which may detect a decreasing luminosity. In other words, when a MACHO moves behind the source, its gravitational field produces a defocusing action.

The aim of this paper is to deal with both defocusing and focusing effects using a unique model. Thus we will describe the situation in which a ”beam” of initial geodesics is squeezed by the gravitational field (or focused by the field) and produces an increasing of detected luminosity as well as the opposite case, when the initial ray beam radiated by source is enlarged by the gravitational field, that is when we have a decreasing (or defocusing) of detected luminosity. This general discussion may be precisely formulated using the equations for geodesics in a generic Schwarzschild gravitational field.

An important implication of our approach is that in this way it is possible to improve the number of detected events of MACHOs which produces microlensing effects since it would be taken into account valleys as well as peaks in light curves. We study both phenomena (focusing and defocusing) describing light trajectories starting from sources that, at the beginning, are straight lines and, passing near the gravitational field of a deflecting point mass (MACHO), differ from straight lines becoming, for example, a bundle of hyperbolic–like geodesics converging or diverging toward the observer.

The paper is organized as follows. In Sect.2, we deal with the generalities of point mass lenses obtaining the characteristic Einstein radius, the magnification, and introducing the concept of optical depth. Sect.3, is devoted to the discussion of geodesics in a point–like gravitational field giving the trajectories of (de)focused light rays. In Sect.4, we discuss how (de)focusing detection can be realized considering the deviation angles of ray paths. We apply the obtained results in Sect.5 by calculating the mass of the lens (MACHO) by (de)focusing and the optical depth, that is the probability to get significant lensing for randomly located compact sources. We draw conclusions in Sect.6.
2 Generalities on point mass lens model

Gravitational lensing essentially consists in the deflection of light in gravitational fields as predicted by the theory of General Relativity [18]. For small deflection angles and weak gravitational fields, which are the regimes of practical interest, the true position of a light source on the sky with respect to the position of its image(s) can be defined by the lens equation

$$\vec{\theta} - \vec{\theta}_s = -\left( \frac{D_{ls}}{D_{os}} \right) \vec{\alpha}(\vec{\theta}) = \vec{\alpha} ,$$

where $\vec{\theta}$ defines the position(s) of image(s) with respect to the optical axis, $\vec{\theta}_s$ the position of the source, and $\vec{\alpha}$ is the displacement angle. $D_{ls}$ and $D_{os}$ are respectively the distances between the lens and the source and the distance between the observer and the source.

We have to note that a given image position always corresponds to a specific source position whereas a given source position may correspond to several distinct image positions. Then we can have multiply imaged sources. In the case of a point mass lens, as we will precisely show in the next section, the deflection angle is given by

$$\alpha = \frac{4GM}{c^2r_0} ,$$

where $M$ is the mass of deflecting body; $r_0$ is the minimal distance between the passing light ray and the deflecting body [2],[18].

For point mass lenses, the geometry of the system is simplified and we do not need the full vector equation (7). By writing $r_0 = \theta D_{ol}$, the lens equation for a point–mass lens assumes the form

$$\alpha = \left( \frac{4GM}{c^2} \right) \left( \frac{D_{ls}}{D_{os}D_{ol}} \right) = \theta - \theta_s ,$$

which can be rewritten as:

$$\theta^2 - \theta_s \theta - \theta_E^2 = 0 ,$$

where

$$\theta_E = \sqrt{\frac{4GM(\leq r_E)D_{ls}}{c^2D_{ol}D_{os}}} ,$$

is the Einstein angle which corresponds to the Einstein radius

$$r_E = \theta_E D_{ol} ,$$

already introduced.

We see that it strictly depends on the distances involved and the mass of the deflector. The symbol $M(\leq r_E)$ means that the mass of the lens has to be contained inside a sphere whose radius is the Einstein one.

Before solving and discussing the algebraic Eq.(11), we have to spend some words on an important parameter connected to the lensing effect, the magnification. In fact, gravitational lensing preserves the surface brightness of a source and then the ratio (magnification) between the solid angle $d\Omega_i$ covered by the lensed image and that of the unlensed
source $d\Omega_s$ gives the flux amplification due to the lensing; this is given by the Jacobian of the transformation matrix between the source and the image(s), that is

$$\mu = \frac{d\Omega_i}{d\Omega_s} = \left| \det \left( \frac{\partial \vec{\theta}_s}{\partial \vec{\theta}_i} \right) \right|^{-1}. \quad (13)$$

If there are more than one images of a source, the total magnification is the sum of all image magnifications. Considering, as we are actually doing, a gravitational point mass lens system which is axially symmetric with respect to the line–of–sight, we can use for deflection the scalar angle $\vec{\theta}$ and apply Gauss’s law for the total flux. The light deflection reduces to a one–dimensional problem and Eq.(13) becomes

$$\mu = \frac{\theta_i d\theta_i}{\theta_s d\theta_s}, \quad (14)$$

which is easily appliable [2].

Let us now solve Eq.(10). We get

$$\theta_{\pm} = \frac{\theta_s}{2} \pm \sqrt{\frac{\theta_s^2}{4} + \theta_E^2}; \quad (15)$$

from which we see that

$$\theta_s = 0; \quad \rightarrow \quad \theta_{\pm} = \pm \theta_E. \quad (16)$$

Eqs.(15), (16) tell us that we have to expect at least two images from the same source which lie on the same plane of the source. In microlensing, as we discussed, it is difficult to separate them and the effect results in a luminosity enhancement of source. The magnification corresponding to Eq.(15) is

$$\mu_{\pm} = \left[ 1 - \left( \frac{\theta_E}{\theta_{\pm}} \right)^4 \right]^{-1}, \quad (17)$$

which tells us that if $\theta_s$ is zero, the magnification becomes singular; physically, this means that when the optical system source–lens–observer is aligned, we can get a huge magnification. The total amplification due to both images is

$$\mu = |\mu_-| + |\mu_| = \frac{\chi^2 + 2}{\chi \sqrt{\chi^2 + 4}}, \quad (18)$$

where

$$\chi = \frac{\theta_s}{\theta_E}. \quad (19)$$

Immediately, we see that

$$\theta_s \leq \theta_E \quad \rightarrow \quad \mu \geq 1.34, \quad (20)$$
which is the condition on the magnification inside the Einstein ring: we have that a magnification \( \mu \sim 1.34 \) corresponds to a magnitude enhancement of \( \Delta m \sim 0.32 \) as required in microlensing experiments. In other words, we can say that when the true position of a light source lies inside the Einstein ring, the total magnification of the two images that it yields amounts to \( \mu \geq 1.34 \). This means that the angular cross section for having significant lensing (i.e. \( \mu \sim 1.34 \) and \( \Delta m \sim 0.32 \)), is equal to \( \pi \theta_E^2 \), which from (11), is proportional to the mass \( M \) of the deflector and to the ratio of the distances involved. Such considerations allow to calculate one of the most useful quantities for lensing detection: the optical depth. Let us consider the case of randomly distributed point–mass lenses: it is possible to estimate the frequency of significant gravitational lensing events from the observations of distant compact sources, that is we are considering optical systems where the involved angular sizes are much smaller than \( \theta_E \). In this situation, the magnification of a compact source is equal or greater than 1.34 (since \( \theta_s < \theta_E \)) and the probability \( P \) to have significant lensing for a randomly located compact source at a distance \( D_{os} \) is given by

\[
P = \frac{\pi \theta_E^2}{4\pi} = \left( \frac{D_{ls}}{D_{os}D_{ol}} \right) \left( \frac{GM}{c^2} \right),
\]

where we have used the definition (11). Such a probability is linear in the mass \( M \) of deflector so that it holds also when several point–mass lenses are acting since the masses can be summed up. Assuming a constant density for the lens(es) and a static background (this last assumption surely holds for galactic distances), averaging on the distances \( D_{ls}, D_{ol}, D_{os} \), the probability (21) can be interpreted as the optical depth \( \tau \) for lensing [19],[20],[21],[22].

\[
P = \tau = -\left( \frac{D_{ls}}{D_{os}} \right) \frac{U}{c^2},
\]

where

\[
U = -\frac{GM}{D_{ol}},
\]

is the Newton potential due to the lens and measured by the observer. If inside the Einstein ring there are several deflecting bodies, \( U \) is their additive Newtonian potential. In other words, \( \tau \) corresponds to the fraction of sky covered by the Einstein ring. Due to the fact that the deflecting masses change the path of light rays, the observer will detect different luminosities for a given source when the deflector is present and when it is not present: then, the optical depth will depend on such a relative luminosity change.

In Sect. 5, we discuss some quantitative estimations of such quantities in connection to lenses which focus or defocus light rays.

3 Geodesics and light ray paths in a gravitational field

Before considering how to realize (de)focusing, it is useful to discuss the motion of light ray paths in a gravitational field since this fact allows to derive the luminosity variations due to the presence of light deflecting gravitational masses (in our case MACHOs).
In General Relativity, light rays move along geodesics [18], [23]. This fact means that, given the line element, \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \), they have to satisfy the equations

\[
\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0,
\]  

(24)

where

\[
\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\delta} (g_{\delta\mu,\nu} + g_{\delta\nu,\mu} - g_{\mu\nu,\delta}),
\]

(25)

are the Christoffel symbols and \( s \) is the parameter chosen along the trajectory.

Here, we are in the geometric optic approximation so that the light propagates as rays and we have not to take into consideration chromatic effects. For weak gravitational fields (usually considered in gravitational lensing effects), the metric tensor components can be expressed in terms of Newton gravitational potential \( \Phi \) as \( g_{00} \approx 1 + 2 \frac{\Phi(r)}{c^2} \) and \( g_{ik} \approx -\delta_{ik} \left( 1 - 2 \frac{\Phi(r)}{c^2} \right) \), the approximation \( \Phi/c^2 \ll 1 \) holds.

As it is well known [2], a gravitational field has the same effect of a medium (different from vacuum) in which light rays propagates and the Fermat principle holds. The refraction index \( n \) can be expressed in terms of the gravitational potential \( \Phi(r) \) produced by some matter distribution [4]), that is

\[
n = 1 - 2 \frac{\Phi(r)}{c^2}.
\]

(26)

If the rays pass near a body of mass \( M \), they will undergo the action of a Schwarzschild gravitational field described by the metric element

\[
ds^2 = \left( 1 - \frac{R_s}{r} \right) c^2 dt^2 - \frac{dr^2}{\left( 1 - \frac{R_s}{r} \right)} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right),
\]

(27)

where

\[
R_s = \frac{2MG}{c^2},
\]

(28)

is the Schwarzschild radius which tells us where the metric becomes singular. We get the trajectories of light rays by the Lagrangian

\[
\mathcal{L} = \left( 1 - \frac{R_s}{r} \right) (\dot{x}^0)^2 - \frac{\dot{r}^2}{\left( 1 - \frac{R_s}{r} \right)} - r^2 \left( \dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2 \right),
\]

(29)

obtained by line element (27). The derivative is with respect to \( s \). Its Euler–Lagrange equations are nothing else but the geodesics equations (24).

They give the condition for the planar motion, the conservation of angular momentum \( r^2 \dot{\phi} = k \), and the conservation of energy \( \left( 1 - \frac{R_s}{r} \right) \dot{x}^0 = E \). \( E \) and \( k \) are integration constants.
Substituting such results into the equation for \( r \), we get the equation for the 3-dimensional trajectories of rays as function of \( \phi \), that is

\[ u'' + u = \frac{3}{2} R_s u^2, \quad \text{where} \quad u = \frac{1}{r}. \tag{30} \]

The prime indicates the derivative with respect to \( \phi \). The rhs of \((30)\) gives rise to the relativistic effects of deflection of light. In fact, a particular solution of \((30)\) is \( u' = 0 \) which tells us that photons can stay in a circular orbit having the radius \( r = \frac{3}{2} R_s \).

The general solution of \((30)\) is an elliptical integral \([2]\) of the form

\[ \mathcal{K}(u, A_0) = \int \frac{du}{\sqrt{R_s u^3 - u^2 + A_0}} = \phi - \phi_0, \tag{31} \]

where \( A_0 \) and \( \phi_0 \) are integration constants. The integral \((31)\) cannot be inverted unless we approximate it or choose particular initial conditions.

However, we are considering weak gravitational fields which act as perturbations on the straight light ray trajectories, i.e. we are in the regime

\[ \frac{3R_s u^2}{2u} = \frac{3}{2} \frac{R_s}{r} \ll 1. \tag{32} \]

Condition \((32)\) means that the light rays pass far from the critical radius \( R_s \) where the gravitational field becomes singular (this fact is quite obvious since for the usual astrophysical bodies we have \( R_s \ll R_0 \) where \( R_0 \) is the surface radius).

If the gravitational source is absent (i.e. \( M = 0 \)), the general solution of \((30)\) is

\[ \tilde{u} = \frac{1}{r} = \frac{1}{r_0} \cos(\phi - \phi_0), \quad -\frac{\pi}{2} \leq \phi - \phi_0 \leq \frac{\pi}{2}, \tag{33} \]

which is a straight line in polar coordinates. The parameters \( r_0 \) and \( \phi_0 \) are the initial data of the problem; \( r_0 \) is the distance of the line from the origin of coordinates, \( \phi_0 \) is a given angle which tells us how much the line is tilted with respect to the polar axis.

If condition \((32)\) holds, the rhs of Eq.\((30)\) can be treated as a small perturbation and then we can search for solutions of the form

\[ u = \tilde{u} + \epsilon u_1, \tag{34} \]

where

\[ \epsilon = \frac{3R_s}{2r_0} \ll 1. \tag{35} \]

Eq.\((30)\) becomes

\[ u''_1 + u_1 = \frac{1}{r_0} \cos^2(\phi - \phi_0), \tag{36} \]
which admits the solution

\[
    u_1 = \frac{1}{3r_0} \left[ 2 - \cos^2(\phi - \phi_0) \right] + A_1 \cos(\phi - \phi_0) + A_2 \sin(\phi - \phi_0). \tag{37}
\]

The term in \( A_1 \) can be interpreted as a redefinition of \( r_0 \), the term in \( A_2 \) as the consideration of the straight line perpendicular to (33): they can be both absorbed by redefining the initial data. Finally, in the limit (32), the solution of (30) is

\[
    u(\phi) = \frac{1}{r} = \frac{1}{r_0} \left\{ \cos(\phi - \phi_0) + \frac{R_s}{2r_0} \left[ 2 - \cos^2(\phi - \phi_0) \right] \right\}, \tag{38}
\]

which is nothing else but a straight line corrected by a hyperbolic–like term in polar coordinates; the deflecting mass is set at the origin of reference frame. The amount of deviation from the rectilinear behaviour depends on the ratio \( \frac{R_s}{r_0} \), that is on the mass \( M \) of the gravitational source and on the parameter \( r_0 \).

Conversely, passing to Cartesian coordinates

\[
x = r \cos \phi, \quad y = r \sin \phi; \tag{39}
\]

Eq. (38) becomes

\[
r_0 = Ax + By + \left( \frac{R_s}{r_0} \right) \sqrt{x^2 + y^2} - \left( \frac{R_s}{2r_0} \right) \frac{(Ax + By)^2}{\sqrt{x^2 + y^2}}, \tag{40}
\]

where

\[
A = \cos \phi_0, \quad B = \sin \phi_0. \tag{41}
\]

The formula (40) will be useful for the discussion below.

Let us consider now the limit \( r \to \infty \). Eq. (38) is an algebraic equation for \( \cos(\phi - \phi_0) \) whose solutions are

\[
    \cos(\phi - \phi_0) = \left[ \frac{r_0}{R_s} \pm \sqrt{\frac{r_0^2}{R_s^2} + 2} \right]. \tag{42}
\]

Neglecting the positive sign solution which is without meaning, approximating the term under the square root, we get

\[
    \cos(\phi - \phi_0) \simeq -\frac{2MG}{c^2r_0}, \tag{43}
\]

which indicates how the presence of gravitational field \( M \neq 0 \) deviates the rays from the straight line direction. If \( M = 0 \) or \( r_0 \to \infty \) (that is in absence of gravitational field or when \( r_0 \) is very large), we have

\[
    \cos(\phi - \phi_0) = 0, \quad \phi - \phi_0 = \pm \frac{\pi}{2}, \tag{44}
\]

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while, if the gravitational field is weak in the limit \( r \to \infty \), we have

\[
\phi - \phi_0 = \pm \left( \frac{\pi}{2} + \delta \right); \quad (45)
\]

from which, by substituting into (38), we get \( \sin \delta \simeq \delta = \frac{R_s}{r_0} \) being \( \delta \) small. The total amount of ray deviation gives the standard result

\[
2\delta \simeq \frac{4MG}{c^2r_0}, \quad (46)
\]

which is the deflection angle due to a point mass acting as a gravitational lens (see Eq.(8)). If \( r_0 \simeq R_\odot \) and \( M \sim M_\odot \), we get the classical Eddington–Einstein result of \( \delta \sim 1.75" \).

4 (De)focusing and luminosity variation of the source

Now, taking in mind the results of previous section, we want to obtain the general formula describing the variation of luminosity of a radiation source in the sky induced by a gravitational microlensing effect. We will show that such a variation is due to the change of direction of light rays (geodesics) which move in a given nonstationary matter distribution and the effect is observable for a time \( \Delta t \) given by (1). In other words, we are supposing that a given background metric \( g^{(1)}_{\mu\nu} \) is modified by a passing heavy body (a MACHO) which locally perturbs it so that we have to consider a new metric \( g^{(2)}_{\mu\nu} \). The effect of such a background change is a deviation in the direction of geodesics which can result, as above shown, a bundle of hyperbolic–like curves instead of a bundle of straight lines.

We will consider the following two cases: the observable variation of source luminosity is due to a microlensing focusing effect by a gravitational lens and by a microlensing defocusing effect. In the first case, a MACHO is between the source and the observer producing focusing; in the second case, a MACHO is behind the source and light rays are defocused toward the observer. In the first case, the observer detects an increasing luminosity, in the second case, he detects a decreasing one. The problem can be easily formulated by a geometric model in which, given a reference frame, we assign the position of the light source and the position of the detector (a telescope) in a background metric \( g^{(1)}_{\mu\nu} \). Then we calculate the geodesics which give the light–ray paths. Then, considering a MACHO passing between the source and the observer or behind the source (with respect to the observer), the metric becomes \( g^{(2)}_{\mu\nu} \) and the geodesics will change giving focusing or defocusing of light rays.

Let us start by choosing a system of Cartesian coordinates. We put the source in

\[
(x_S, y_S) = (-a, 0), \quad (47)
\]

and the telescope in

\[
(x_T, y_T) = (R, h), \quad (48)
\]
as shown in Fig.1. There exists a unique light path (a unique geodesic) which intersects the source and the upper limit of telescope aperture (see again Fig.1).

Let us now suppose that, due to a redistribution of matter, the metric becomes $g^{(2)}_{\mu\nu}$, (the simplest case, as we said, is to consider a MACHO passing either between the source and the observer or behind the source). This fact modifies the structure of geodesic bundle from the source to the observer. Schematically, we have a new geodesic between the source and the upper limit of the aperture of telescope as shown in Figs.2 and 3.

The ray which reaches the upper limit of telescope in the metric $g^{(1)}_{\mu\nu}$ is emitted at the angle $\alpha_1$ while it is emitted at the angle $\alpha_2$ in the metric $g^{(2)}_{\mu\nu}$.

In the first case, geodesic is given by a function $y_1(x)$ in Cartesian coordinates; in the second one by a function $y_2(x)$. The angles $\alpha_1$ and $\alpha_2$ are given by the derivatives

$$\tan \alpha_1 = \frac{dy_1}{dx}\bigg|_{(-a,0)}, \quad \tan \alpha_2 = \frac{dy_2}{dx}\bigg|_{(-a,0)},$$

(49)

calculated in the coordinates of the source. Since the distances are very large the angles are small, so we have

$$R \gg h, \quad \text{and} \quad R + |a| \gg h,$$

(50)

and then

$$\alpha_1 \simeq \frac{dy_1}{dx}\bigg|_{(-a,0)}, \quad \alpha_2 \simeq \frac{dy_2}{dx}\bigg|_{(-a,0)}.$$  

(51)

In Fig.2, we show a focusing situation where metric is changing so that more light rays reach the telescope than in initial metric (Fig.1). It means that the luminosity of the source detected by the telescope becomes larger. In Fig.3, we show a defocusing situation: it is worthwhile to note that the formula defining the angle $\alpha_2$ is given by the same $x-$derivative of geodesic calculated in the source coordinates. Thus, the relative change of luminosity in the planar configuration is equal to

$$\frac{l_2 - l_1}{l_1} = \frac{\alpha_2 - \alpha_1}{\alpha_1}.\quad (52)$$

We have either an increasing luminosity for $\alpha_2 > \alpha_1$, (focusing case) or a decreasing one for $\alpha_2 < \alpha_1$ (defocusing case). Actually, to obtain an observable change of luminosity, we have to consider the square of (52) since we have to take into account solid angles in the space, that is

$$\frac{L_2 - L_1}{L_1} = \frac{\Delta L}{L} = \pm \left(\frac{\alpha_2 - \alpha_1}{\alpha_1}\right)^2.\quad (53)$$

Plus sign corresponds to the focusing situation, minus sign to the defocusing one. A general formula for relative change of luminosities is

$$\frac{\Delta L}{L} = \pm \left(\frac{dy_2(x)}{dx} - \frac{dy_1(x)}{dx}\right)\left(\frac{dy_1(x)}{dx}\right)^{-1}\bigg|_{(xS,yS)}^2, \quad (54)$$

(54)
where the two derivatives of geodesics are calculated in the coordinates of the source.

Let us now apply these general considerations to the case of a flat metric which is perturbed by the gravitational field of a moving MACHO. This means that the initial metric $g_{\mu\nu}^{(1)}$ is a Minkowski one while the metric $g_{\mu\nu}^{(2)}$ is a Schwarzschild one. Without MACHO, geodesics are straight lines emitted by the source (Fig.1), that is

$$r = \frac{r_0}{\cos(\phi - \phi_0)},$$  

(55)
in polar coordinates, or

$$r_0 = Ax + By,$$  

(56)
in Cartesian coordinates. The constants $A$ and $B$ are the same as (11).

When a MACHO perturbs the flat background, the geodesics are given by Eq.(38) (or (40)).

Let us now consider the simplest case of a MACHO of mass $M$ posed in the origin of coordinate. In the focusing situation (see Fig.2), the lens (a MACHO with Cartesian coordinates $\{0, 0\}$) is between the stellar source (posed in $\{-a, 0\}$) and the observer (with the aperture of telescope in $\{R, h\}$). We calculate, in the source coordinates, the derivative of light ray trajectory (55) or (56), when the MACHO is not present obtaining

$$\left.\frac{dy_1}{dx}\right|_{(-a,0)} = -\left(\frac{A}{B}\right),$$  

(57)

and when it is present (by using (38) or (40))

$$\left.\frac{dy_2}{dx}\right|_{(-a,0)} = -\left\{ A + \left(\frac{R_s}{2r_0}\right) \left(\frac{A^2 - 2}{B}\right) \right\}.$$  

(58)

From (38) (or (44)), we immediately obtain the relative change of luminosity induced on the telescope, that is

$$\frac{\Delta L}{L} = \left(\frac{R_s}{2r_0}\right)^2 \left\{ \frac{A^2 + 2}{A \left[1 + A \left(\frac{R_s}{2r_0}\right)\right]} \right\}^2.$$  

(59)

For defocusing, as shown in Fig.3, the source is in $\{a, 0\}$. Performing the same calculation as above, we get

$$\left.\frac{dy_1}{dx}\right|_{(a,0)} = -\left(\frac{A}{B}\right),$$  

(60)

when the MACHO is not present and

$$\left.\frac{dy_2}{dx}\right|_{(a,0)} = -\left\{ A - \left(\frac{R_s}{2r_0}\right) \left(\frac{A^2 - 2}{B}\right) \right\},$$  

(61)
when it is present. The relative change of luminosity induced on the telescope is
\[
\frac{\Delta L}{L} = -\left( \frac{R_s}{2r_0} \right)^2 \left\{ \frac{A^2 + 2}{A \left[ 1 - A \left( \frac{R_s}{r_0} \right) \right]} \right\}^2.
\]

(62)

It is worthwhile to note that if \( A \approx \frac{r_0}{R_s} \) in Eq. (62), the relative change can be huge. The couples of Eqs. (58), (59) and (61), (62) show that the variation of luminosity strictly depends on the relative positions of the MACHO (lens) and the ray \((r_0, \phi_0)\), on the mass of the MACHO (to be more precise on the product \(GM\) where the gravitational coupling \(G\) is supposed constant) and on the relative position of the lens and the light source (the signs inside Eqs.(59) and (62) depend on taking \(x_s = -a\) or \(x_s = a\) and show that the problem of focusing and defocusing is not completely symmetric). Such calculations can be performed in any position of source and lens, here, for simplicity, we have taken into account source, lens and observer which lie on the same line. These results can be used to estimate the quantities of microlensing theory as we shall do in next section.

5 The mass of MACHO and the optical depth

First of all, by the above formulas, we can calculate the mass of MACHO (lens) both for focusing and defocusing cases; in fact from Eq. (59) and (62), using (28), we obtain
\[
M = \left( \frac{c^2 r_0}{G} \right) \left[ \frac{A \sqrt{|\Delta L/L|}}{2 + A^2 \pm 2A^2 \sqrt{|\Delta L/L|}} \right].
\]

(63)

Now minus sign refers to focusing and plus to defocusing; furthermore, we are considering the absolute relative variation of light intensity, in the sense that, given a luminosity curve of a source, both the peak or the valley can give indications on the MACHO mass.

We have to stress that \( M = M(\Delta L/L, r_0, \phi_0) \), that is, in principle, we can get the mass of the MACHO only by knowing its position with respect to the source \( \{r_0, \phi_0\} \) and the relative variation of intensity. The term in \( A^2 = \cos^2 \phi_0 \) and \( \sqrt{|\Delta L/L|} \) at the denominator tells us whether the deflecting body is between the light source and the observer or behind the light source.

By using Eqs. (21) and (22), the optical depth is given by
\[
\tau_\pm = \left( \frac{r_0 D_{ls}}{D_{ol}^2} \right) \left[ \frac{A \sqrt{|\Delta L/L|}}{2 + A^2 \pm 2A^2 \sqrt{|\Delta L/L|}} \right],
\]

(64)

where \( \tau_+ \) is the optical depth (probability) connected to a focusing event while \( \tau_- \) is associated to a defocusing one. If \( r_E = \theta_E D_{ol} \), the direct dependence on \( \theta_E \) appears in (64). It is worthwhile to note that now the distances \( D_{ls}, D_{os}, D_{ol} \) explicitly give a
contribution telling us that optical depth (i.e. the probability to obtain lensing events) strictly depends on the geometry of the optical system source–lens–observer.

Another important issue is the duration of the relative luminosity variation of the source. If we consider $r_0 \simeq r_E$ and approximate Eqs. (59) and (62) in terms of $\left( \frac{R_s}{r_0} \right)$, we get

$$\frac{\Delta L}{L} \simeq \pm \left( \frac{\tilde{A} R_s}{v \Delta t} \right)^2,$$

where $\tilde{A} = \left( \frac{A^2 + 2}{A} \right)^2$. From (65), it is easy to see that the luminosity variation strictly depends on the velocity of a passing MACHO and on the time in which it remains in the Einstein ring. A fast passing MACHO will produce a sharp peak (or valley) of luminosity.

In order to give some estimation let us consider Eq. (63): we obtain a MACHO of mass $M \sim 0.5 \div 1 M_\odot$, if $\frac{\Delta L}{L} \sim 10^{-2}$, $r_0 \simeq r_E \sim 1$ AU and $\phi_0 \sim \pi \pm |\delta|$, with $|\delta| \sim 10^{-5}$. Such result holds for focusing and defocusing MACHOs. On the other hand, it is easy to obtain optical depth of the order $\tau \sim 10^{-6}$ as determined by the OGLE and MACHO collaborations toward the Galactic bulge [4], [13], or $\tau \sim 10^{-7} \div 10^{-8}$ as estimated toward the LMC [25]. It is interesting to see that in this second case the ratio $D_{ls} \over D_{os}$ is close to the unity since the distances in the Galactic halo and in LMC are similar, so being $r_0 \sim r_E$, $\tau$ depends only on the two angles $\theta_E$, $\phi_0$ and on the relative luminosity variation. The same results are also obtained for $\frac{\Delta L}{L} \sim 10^{-4}$ and $|\delta| \sim 10^{-3}$.

In principle, we can cover all the potentially detectable mass range $10^{-6} M_\odot$ to $10^2 M_\odot$ expected for MACHOs.

However, we can use also the relative source–lens velocity $v$ and the duration of luminosity variation $\Delta t$. The above mass $M \sim M_\odot$ is detected for $v \sim 200$ Km s$^{-1}$ and $\Delta t \sim 10^6$s.

This second method is good for measurements inside the Galaxy since the velocities are quite well known [26] and the distances $r_0 \sim r_E = r_E(D_l, D_{ls}, D_s)$ can be accurately estimated.

6 Conclusions

In this paper we have pointed out that we can get a microlensing effect not only if we detect an increasing luminosity for a given source, but also if we detect a decreasing one. Furthermore, by the knowledge of the geometry of the system source–lens–observer, we can estimate both mass and optical depth of a given lens. These facts could contribute to bypass one of the lack of microlensing detecting experiments: the low number of observed events (till now about 100, not all exactly tested, for millions of detected source stars). Roughly speaking, one could expect to double the number of successful detections including also defocusing events.
It is worthwhile to note that when several MACHOs are present, the previous discussion still holds due to the Fermat principle (see, for example \[\text{2}\]). In fact, any compact object perturbs the flat gravitational background and a light ray passing through the locally perturbed metrics \(g^{(2)}_{\mu\nu}, \ldots, g^{(j)}_{\mu\nu}\) undergoes \(j-1\) deviations. The effect is additive and it is similar to that of a light ray passing through different media with refraction indexes \(n_1, \ldots, n_j\). Then, in principle, it is possible to evaluate the total deviation of a light ray by summing up the effects of the various deflectors.

We have to stress that in a statistical approach to the microlensing, we have two contributions to the number density \(n(D_l)\) of lenses, one coming from focusing objects \(n_+(D_l)\) and another coming from defocusing objects \(n_-(D_l)\). As a final remark, we would like to stress that the approach we have developed in this paper could be useful to reconsider some faint sources which are expected to be brighter because of their mass.

In a forthcoming paper, we will develop such a statistical approach considering focusing and defocusing lenses and giving workable models which can be used in the measurements toward the bulge of Galaxy or toward the LMC.

ACKNOWLEDGMENTS

The authors want to thank Michal Jaroszyński for the useful discussions on the topic. V.I.M. thanks Prof. M. Capaccioli for the hospitality in Osservatorio Astronomico di Capodimonte.

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FIGURE CAPTIONS

Fig.1 Schematic representation of the system source–observer in a Cartesian reference frame. In this case the lens is not present and the metric $g_{\mu\nu}^{(1)}$ can be assumed to be Minkowski. The source is in $\{-a, 0\}$, the upper edge of the telescope (collecting the last light ray) is in $\{R, h\}$.

Fig.2 As above, but now a MACHO (lens) is present in the origin of coordinates. The metric is $g_{\mu\nu}^{(2)}$ which is the Schwarzschild local perturbation of $g_{\mu\nu}^{(1)}$. This is a focusing situation since the telescope collects more light than above (i.e. more geodesics, due to the action of the lens in the origin, converge in the telescope).

Fig.3 The defocusing situation. The positions of the source and of the lens are inverted with respect to the observer. The telescope collects less light than in the case shown in Fig.1 (and obviously in the case in Fig.2).