Cosmological scenarios with bounce and Genesis in Horndeski theory and beyond

An essay in honor of I.M. Khalatnikov on the occasion of his 100th birthday

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Abstract

This essay is a brief review of the recent studies of non-singular cosmological scenarios with bounce and Genesis and their stability in a subclass of scalar-tensor theories with higher derivatives – beyond Horndeski theories. We discuss the general results of stability analysis of the non-singular cosmological solutions in beyond Horndeski theories, as well as other closely related topics: 1) the no-go theorem, which is valid in the general Horndeski theories but not in their extensions, 2) singularities in disformal transformations relating beyond Horndeski theories with general ones, 3) healthy behaviour of the scalar sector in the unitary gauge despite divergencies of coefficients in the quadratic action for perturbations ("γ-crossing"). We describe several specific examples of bouncing cosmologies and models with Genesis epoch which have neither ghosts nor gradient instabilities among the linearized perturbations about the homogeneous isotropic background during entire evolution.

1 Introduction

Cosmological scenarios with the bouncing or Genesis stage serve as possible extensions of the standard hot Big Bang theory. In both of these scenarios, space-time has vanishing 4D curvature at early times, i.e., the Hubble parameter and its time derivatives take on small values. The bouncing model implies that the Universe undergoes a contracting stage at early times, which terminates at some moment of time (the bounce) and the Universe transits to the expansion.
epoch (see Refs. [1, 2, 3] for reviews). The Genesis scenario describes the accelerated expansion of the Universe from the asymptotically empty Minkowski space: the energy density of exotic matter, which drives the evolution during the Genesis epoch, grows in time, and so does the expansion rate (the Hubble parameter); at some stage, when the energy density and Hubble parameter have grown sufficiently large, the energy of the exotic matter gets transformed into heat, so that the Universe transits to the standard hot stage [4, 5].

The specific feature of both scenarios is the absence of initial singularity, whose inevitable presence in the hot Big Bang theory has not been overcome even with the invention of inflation [6, 7]. In fact, non-singular cosmologies with bounce or Genesis may be equally well considered as complementary or alternative to inflationary scenario [3, 8].

One of the issues one should take care of when constructing bouncing models is Belinskii-Khalatnikov-Lifshitz (BKL) phenomenon [9]. It may lead to strong inhomogeneity and anisotropy of space at the end of contraction stage, which are unacceptable in a self-consistent cosmological model, see discussion in Refs. [10] and [1]. One of the possible solutions to the BKL problem within General Relativity (GR) is to introduce a matter component with a super stiff equation of state $p \geq \rho$ during the contraction stage, where $\rho$ and $p$ are energy density and effective pressure, respectively. One of the simple options involves a homogeneous massless scalar field with the equation of state $p = \rho$ dominating during the contraction epoch; this option follows from the results obtained by Khalatnikov and Kamenshchik in Ref. [11]. Another approach to solving the BKL problem is realized, for example, in the ekpyrosis scenario [12]. In any case, one of the viability criteria for a bouncing model is the absence of the BKL behaviour during contraction.

An important property of non-singular cosmologies with bounce or Genesis is the necessity to introduce a specific matter component, which, unless one abandons GR or relies upon the 3D spatial curvature, has to violate the Null Energy Condition (NEC), see, for instance, Ref. [13] for a review,

$$T_{\mu\nu}n^\mu n^\nu > 0,$$

where $T_{\mu\nu}$ is the energy-momentum tensor, and $n^\mu$ is any null vector ($g_{\mu\nu}n^\mu n^\nu = 0$). In a general case, when gravity is modified, NEC is replaced with the Null Convergence Condition (NCC) [14]

$$R_{\mu\nu}n^\mu n^\nu > 0,$$

where $R_{\mu\nu}$ is Ricci tensor. The need to introduce the NEC-violating matter becomes evident upon considering the combination of Einstein equations for a spatially-flat, homogeneous and isotropic Universe:

$$\dot{H} = -4\pi G(\rho + p),$$

where $H$ denotes the Hubble parameter. Indeed, by choosing $n^\mu = (1, a^{-1}q^i)$, where $q^2 = 1$, in the cosmological background described above the NEC (1) takes the form

$$p + \rho > 0.$$  

If the NEC (4) is satisfied, it follows from eq. (3) that $\dot{H} < 0$, hence, if the Universe was contracting
in the past, it would continue contraction until it would reach singularity \(^1\). Likewise, NEC-violating matter is necessary for the Universe starting off with Genesis: this scenario implies growing Hubble parameter during the Genesis epoch which is forbidden by eqs. (3), (4). Therefore, the cosmological solutions with bounce and Genesis make it necessary to consider NEC-violating matter (or NCC-violation in the case of modified gravity). The latter feature makes bouncing and Genesis models non-standard, since the majority of known types of matter comply with the NEC/NCC while the attempts to violate these conditions often result in pathologies like ghosts, gradient instabilities and tachyons among the linearized perturbations about the homogeneous isotropic background, see, for instance, Refs. [13, 16] for reviews.

One of the possible ways to obtain the NEC/NCC-violation is to invoke the generalized Galileon theory [17, 18, 19, 20] or, equivalently, Horndeski theory [21]. Horndeski theories are the most general scalar-tensor theories of modified gravity with second derivative terms in the Lagrangian, whose presence, however, does not affect the order of differential equations of motion – they are still second order in derivatives. Therefore, due to a specifically designed structure of the Lagrangian, Horndeski theories are free of Ostrogradsky ghosts and have \((2 + 1)\) dynamical degrees of freedom (DOF), two tensor modes and one scalar mode about the homogeneous isotropic background. Quite recently an even more general class of scalar-tensor theories with second derivatives in the Lagrangian but without Ostrogradsky instabilities has been discovered – the so-called "degenerate higher order scalar-tensor theories" (or DHOST theories) [22, 23, 24, 25, 26, 27, 28, 29, 30] and "U-degenerate theories" [31]. The important difference between these generalizations and Horndeski theories is the fact that the former have third order equations of motion while propagating the same three DOF as Horndeski theories do. Moreover, there is a non-trivial relation between some of the subclasses of DHOST theories and Horndeski theories via the invertible disformal transformation of metric \([32, 33, 34, 35]\)

\[
g_{\mu\nu} \rightarrow \Omega^2(\pi, X)g_{\mu\nu} + \Gamma(\pi, X)\partial_\mu \pi \partial_\nu \pi, \quad (5)
\]

where \(\pi\) denotes a scalar field (Galileon field), \(X = g^{\mu\nu}\partial_\mu \pi \partial_\nu \pi\), while \(\Omega^2(\pi, X)\) and \(\Gamma(\pi, X)\) are arbitrary functions. Interestingly, one of the first examples of DHOST theories from the subclass called "beyond Horndeski theories" was discovered by applying the disformal transformation (5) to a certain Lagrangian of Horndeski type [22].

Horndeski theories and their generalizations possess a specific feature of admitting healthy NEC/NCC-violating regimes. Here healthy means that violation of the NEC/NCC does not forbid the solution stability at the linearized level [16]; in what follows stability of a solution means that there are no ghost or gradient instabilities. The above feature makes this class of scalar-tensor theories interesting from the viewpoint of construction of non-standard cosmologies like bounce and Genesis. And, indeed, a significant number of bouncing scenarios were suggested where the stage with the NEC/NCC-violation was driven by the Galileon field of Horndeski type [36, 37, 38, 39, 40, 41, 42, 43]. These solutions were shown to be stable during some finite period of time, including the stage with the NEC/NCC-violation. The Universe with Genesis was

\(^1\)This reasoning does not apply to the case of the closed Universe, where the bounce is possible if the energy density and effective pressure grow slower than \(a^{-2}\) during contraction [15].
also studied in various subclasses of Horndeski theories [5, 44, 45, 46, 47, 48, 49, 50]. The issue of superluminal propagation of perturbation modes in the original Genesis model was specifically addressed in Refs. [51, 52].

Further studies have shown, however, that there is stability-related obstruction to constructing complete non-singular cosmological scenarios in Horndeski theories, i.e., the models whose evolution can be followed from \( t \to -\infty \) to \( t \to +\infty \). Initially, a no-go theorem has been established for a cubic subclass of Horndeski theories: it stated that there are no completely stable solutions with bounce or Genesis since gradient instabilities and/or ghosts inevitably arise sooner or later among the perturbations about the homogeneous isotropic background [53]. Similar property has been found in the case when along with the Galileon field, there is an additional scalar field obeying the NEC [54]. These no-go theorems were further generalized to Horndeski theories of the most general form [55] as well as multi-Galileon theories [56]. Hence, it has been shown on general grounds that Horndeski theories are not suitable for constructing non-singular cosmological solutions that are stable during entire evolution (see also Refs. [16, 57]).

The topic got new twist when the no-go argument and ways to circumvent it were analyzed within one of the subclasses of DHOST/U-degenerate theories dubbed beyond Horndeski theories or GLVP [23, 24]. Within the effective field theory (EFT) approach, it was shown in Refs. [58, 59] that the beyond Horndeski terms in the Lagrangian introduce significant changes to the stability condition, which is a crucial ingredient of the no-go theorem discussed above. This was a strong indication that the no-go theorem might be evaded by going beyond Horndeski. The first explicit examples of stable non-singular cosmological solutions were suggested in Refs. [60, 61], where a covariant approach was used instead of EFT. The covariant formalism has the advantage of dealing directly with the Lagrangian of the theory, and, hence, enabling one to check the solutions against the equations of motion, which is impossible in the EFT approach. As a result, Refs. [60, 61] give explicit Lagrangians of beyond Horndeski type, which admit completely stable bouncing solutions; in Ref. [60] a complete stable Genesis-like solution was constructed as well.

The solutions in Ref. [60] have a specific property that has to do with the asymptotic behaviour of the theory as \( t \to \pm \infty \): at late times the beyond Horndeski theory transforms into the theory of a conventional massless scalar field within GR, while at early times \( (t \to -\infty) \) the Lagrangian does not simplify and remains of beyond Horndeski type, which significantly differs from GR + conventional scalar field. The simple form of the theory in both asymptotics \( t \to \pm \infty \) is not an obligatory requirement, but it may be an advantage in the context of the further applications towards constructing realistic models of the early Universe. As discussed in Ref. [60] having the bounce or Genesis with simple form of asymptotics is non-trivial because of the so-called \( \gamma \)-crossing phenomenon, which was considered unacceptable at the time. We discuss in detail the issue of \( \gamma \)-crossing in Sec. 2.3. There we stress that, in fact, there is nothing wrong with \( \gamma \)-crossing, as shown in Refs. [62, 64]. Once the healthy nature of \( \gamma \)-crossing was understood, there were suggested completely stable bouncing and Genesis scenarios in beyond Horndeski theory, whose asymptotics at both \( t \to +\infty \) and \( t \to -\infty \) are described by GR with a massless scalar field [64, 65]. The additional advantage of the bouncing model of Ref. [64] is the absence of the BKL phenomenon during the contracting stage due to the domination of the massless scalar field.
prior to the bounce.

To summarize, today there are examples of completely stable cosmological solutions of bouncing and Genesis types within beyond Horndeski theory. However, one may be puzzled by an apparent contradiction: on the one hand, the no-go theorem valid in Horndeski subclass is evaded by going beyond Horndeski, but, on the other hand, Horndeski and beyond Horndeski subclasses are related by disformal transformation (5). Indeed, the disformal transformation (5) is a mere field redefinition, which at a glance cannot affect the stability of a solution. The resolution of this apparent paradox is that the disformal transformation from the beyond Horndeski theory with a completely stable solution to the Horndeski theory with no-go theorem turns out to be singular at a certain moment of time. The latter result has been obtained within the EFT approach in Ref. [59]. One of the purposes of this review is to confirm this singular character of the disformal transformation in the covariant formalism utilised in Refs. [60, 64, 65], see Sec. 3 for details.

This brief review has the following structure. In Sec. 2 we revisit the construction and stability analysis of the cosmological models with bounce and Genesis in beyond Horndeski theory. In particular, Sec. 2.2 discusses the no-go theorem for Horndeski theory as well as ways to circumvent it; Sec. 2.3 considers the nature of $\gamma$-crossing and specifies its role in bouncing and Genesis solutions. In Sec. 2.4 we briefly describe the reconstruction procedure for obtaining completely stable solutions with bounce and Genesis in beyond Horndeski theories and revisit explicit examples of these models suggested in Refs. [60, 64, 65]. In Sec. 3 we discuss the relation of Horndeski theories with their extensions via disformal transformation and show, in the covariant formalism, that beyond Horndeski theories with completely stable non-singular solutions are related to Horndeski theories via field redefinition which is inevitably singular. We conclude in Sec. 4.

2 Stability of non-singular cosmological scenarios in beyond Horndeski theory

This section reviews the existing results on the construction of cosmological solutions and their stability analysis in Horndeski theories and beyond. In what follows we make use of notations introduced in Ref. [66], which have been later adopted in Refs. [60, 64, 65].
2.1 Lagrangian and stability conditions

The general form of beyond Horndeski Lagrangian reads (metric signature $ (+, -, -,-)$):

$$ S = \int d^4 x \sqrt{-g} \left( \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_\text{BH} \right), \quad (6) $$

$$ \mathcal{L}_2 = F(\pi, X), \quad (7) $$

$$ \mathcal{L}_3 = K(\pi, X) \Box \pi, \quad (8) $$

$$ \mathcal{L}_4 = \pi g_{\mu \nu} \pi_{, \mu \nu} + 2 \mathcal{G}_4X(\pi, X) \left[ (\Box \pi)^2 - \pi_{, \mu \nu} \pi^{\mu \nu} \right], \quad (9) $$

$$ \mathcal{L}_5 = \pi \mathcal{G}_5 X \left[ (\Box \pi)^3 - 3 (\Box \pi) \pi_{, \mu \nu} \pi^{\mu \nu} + 2 \pi_{, \mu \nu} \pi^{\mu \rho} \pi_{, \rho \nu} \right] + \frac{1}{3} \mathcal{G}_5 \pi_{, \mu \nu} \pi_{, \rho \nu} + \frac{1}{2} \mathcal{G}_5 (\pi, X) \epsilon_{\mu \nu \rho \sigma} \epsilon^{\mu \nu \rho \sigma}, \quad (10) $$

where $\pi$ is the scalar (Galileon) field, $X = \pi_{, \mu \nu} \pi_{, \mu \nu}$, $\pi_{, \mu} = \partial_{\mu} \pi$, $\pi_{, \mu \nu} = \nabla_{\nu} \nabla_{\mu} \pi$, $\Box \pi = \pi_{, \mu \nu} \pi^{\mu \nu}$, $G_{4X} = \partial G_4 / \partial X$, etc.; $R$ in eq. (9) and $G^{\mu \nu}$ in eq. (10) denote Ricci scalar and Einstein tensor, respectively. The terms (7) – (10) describe Horndeski theory and involve 4 independent functions $F(\pi, X)$, $K(\pi, X)$, $G_4(\pi, X)$ and $G_5(\pi, X)$. The functions $F_4(\pi, X)$ and $F_5(\pi, X)$ in eq. (11) are characteristic of beyond Horndeski theory. Let us note that the action (6) already contains the gravitational part (see eqs. (9) and (10)): the Einstein–Hilbert action is restored by setting $G_4(\pi, X) = 1/2 \kappa$ and $G_5(\pi, X) = 0$, where $\kappa = 8 \pi G$ and $G$ is the gravitational constant. The Lagrangian for cubic Horndeski theory, which was mentioned above and extensively used in recent works, reads

$$ \mathcal{L}_{\text{cub}} = - \frac{1}{2 \kappa} R + \mathcal{L}_2 + \mathcal{L}_3. \quad (12) $$

By adding terms (9) and (10) to the Lagrangian (12), one obtains quartic and quintic Horndeski theories, respectively.

Clearly, the Lagrangian of the general Horndeski theory, i.e., the theory with $F_4 = F_5 = 0$, contains the second derivatives of both the Galileon field $\pi$ and metrics. Generally, these second derivatives cannot be removed by integration by parts. Nevertheless, all field equations are differential equations of the second order at most. Beyond Horndeski theories with $F_4 \neq 0$ and/or $F_5 \neq 0$ do not have this property; however, as we mentioned in Introduction, these theories propagate the same number of DOF as the general Horndeski theories, i.e., two tensor and one scalar modes. The same is true for even more general classes of DHOST and U-degenerate theories, whose Lagrangians are not given in this review, see Refs. [30, 31]: it is sufficient for our purposes to consider theories with the Lagrangian (6). Moreover, to make the formulas that follow more concise we take

$$ G_5 = 0, \quad F_5 = 0. $$

There is nothing fundamentally new for our studies in the general case with $G_5 \neq 0, F_5 \neq 0$, while the formulas become cumbersome.

We consider the cosmological models described by spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) metric:

$$ ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j. \quad (13) $$
In our setup, the background Galileon field is homogeneous, \( \pi = \pi(t) \). In this case the independent field equations for the theory with the action (6) read:

\[
\delta g^{00} : \quad F - 2F_X X - 6HK_X \pi + K_\pi X + 6H^2G_4 + 6HG_4 \ddot{\pi} \\
-24H^2X(G_{4X} + G_{4X}X) + 12HG_{4\pi}X \ddot{\pi} - 6H^2X^2(5F_4 + 2F_{4X}X) = 0 ,
\]

\[
\delta g^{ii} : \quad F - X(2K_X \ddot{\pi} + K_\pi) + 2(3H^2 + 2\dot{H})G_4 - 12H^2G_{4X}X \\
-8HG_{4X}X - 8HG_{4\pi} \ddot{\pi} - 16HG_{4X}X \dddot{\pi} \\
+2(\dddot{\pi} + 2H \ddot{\pi})G_{4\pi} + 4XG_{4\pi}X(\ddot{\pi} - 2H \ddot{\pi}) + 2XG_{4\pi} \\
-2F_4X(3H^2X + 2\dot{H}X + 8H \ddot{\pi}) - 8HF_{4X}X^2 \dddot{\pi} - 4HF_4X^2 \dddot{\pi} = 0
\]

where \( H = \dot{a}/a \) is the Hubble parameter. The field equation obtained by varying the action (6) over \( \pi \) is a linear combination of eqs. (14), (15) and their derivatives.

The central issue for cosmological models is their stability under linearized inhomogeneous perturbations. In the linearized theory we study both metric perturbations and scalar field perturbations \( \pi \). Let us introduce the following notations for the metric components, which include both background and linearized perturbations:

\[
g_{00} = 1 + 2\alpha, \quad g_{0i} = -\partial_i \beta, \quad g_{ij} = -a^2 \left( 2\zeta \delta_{ij} + h_{ij}^T \right),
\]

where \( \alpha, \beta \) and \( \zeta \) are scalar perturbations, \( h_{ij}^T \) stand for tensor modes, which are traceless \( (h_{ii}^T = 0) \) and transverse \( (\partial_i h_{ij}^T = 0) \). Note that there are no non-vanishing vector perturbations in the scalar-tensor theories in question, and that we have partly used gauge freedom in the parametrization (16). The perturbation about the homogeneous background Galileon field \( \pi_c \) is denoted by \( \chi \):

\[
\pi \rightarrow \pi_c(t) + \chi(t,r).
\]

Generally, the linearized theory is invariant under infinitesimal coordinate transformations of the following form:

\[
x^\mu \rightarrow x^\mu + \xi^\mu,
\]

where \( \xi^\mu \) are infinitesimal parameters. Part of this gauge freedom has been already used in eq. (16). The residual gauge freedom is parametrized by the gauge function \( \xi^0 \). In terms of the parametrization (16) and (17), the transformation law for the scalar modes reads:

\[
\chi \rightarrow \chi + \xi^0 \ddot{\pi}, \quad \alpha \rightarrow \alpha + \dddot{\xi}^0, \quad \beta \rightarrow \beta - \xi^0, \quad \zeta \rightarrow \zeta + \xi^0 \frac{\dot{\alpha}}{a}.
\]

We fix the residual gauge freedom by setting \( \chi = 0 \) (unitary gauge), so that the only non-trivial modes in the scalar sector are \( \alpha, \beta \) and \( \zeta \). Then the quadratic action for perturbations in theory (6) has the following form:

\[
S = \int dt d^3 x a^3 \left[ \left( \frac{\mathcal{G}_T}{8} \left( \ddot{h}_{ij}^T \right)^2 - \frac{\mathcal{F}_T}{8a^2} \left( \partial_k h_{ij}^T \right)^2 \right) + \left( -3\mathcal{G}_T \dot{\zeta}^2 + \mathcal{F}_T \frac{(\nabla \zeta)^2}{a^2} \right) -2(\mathcal{G}_T + \mathcal{P} \ddot{\pi}) \alpha \frac{\Delta \zeta}{a^2} + 2\mathcal{G}_T \frac{\Delta \beta}{a^2} + 6\Theta \alpha \dot{\zeta} - 2\Theta \alpha \frac{\Delta \beta}{a^2} + \Sigma \alpha^2 \right],
\]
where \((\nabla \zeta)^2 = \delta^{ij}\partial_i \zeta \partial_j \zeta\), \(\Delta = \delta^{ij}\partial_i \partial_j\), and coefficients \(G_T, D, F_T, \Theta\) and \(\Sigma\) are expressed in terms of the Lagrangian functions:

\[
G_T = 2G_4 - 4G_{4X}X - 2F_4X^2,
\]

\[
D = 2F_4X\dot{\pi},
\]

\[
F_T = 2G_4,
\]

\[
\Theta = -K_XX\dot{\pi} + 2G_4H - 8HG_{4X}X - 8HG_{4XX}X^2 + G_4\dot{\pi} + 2G_{4\pi}X\dot{\pi}
\]

\[
-10HF_4X^2 - 4HF_4XX^3,
\]

\[
\Sigma = F_X + 2F_{XX}X^2 + 12HK_XX\dot{\pi} + 6HK_{XX}X^2 - K_{XX}X^2 - 6HG_{4\pi}\dot{\pi}
\]

\[
-30HG_{4\pi}X\dot{\pi} - 12HG_{4XX}X^2 - 90H^2F_4X^2 + 78H^2F_{4XX}X^3 + 12H^2F_{4XXX}X^4
\]

We note that fixing the gauge directly in the quadratic action (20) (rather than in the field equations) is legitimate since the Galileon field equation follows from eqs. (14), (15) (see Ref. [66] for discussion and Ref. [31] for details).

Due to the structure of the quadratic action (20), both \(\alpha\) and \(\beta\) are non-dynamical. Varying the action (20) with respect to \(\alpha\) and \(\beta\), one obtains two constraint equations:

\[
\frac{\Delta \beta}{a^2} = \frac{1}{\Theta} \left( 3\Theta \dot{\zeta} - (G_T + D\dot{\pi}) \frac{\Delta \zeta}{a^2} + \Sigma \alpha \right),
\]

\[
\alpha = \frac{G_T \dot{\zeta}}{\Theta}.
\]

By utilizing the constraints (26) and (27), one recasts the action (20) in terms of dynamical DOF only:

\[
S = \int \! dt \! d^3 x \! a^3 \left[ \frac{G_T}{8a^2} \left( h_{ij}^T \right)^2 - \frac{F_T}{8a^2} \left( \partial_i h_{ij}^T \right)^2 + G_S \dot{\zeta}^2 - F_S \frac{(\nabla \zeta)^2}{a^2} \right],
\]

where the following notations are introduced:

\[
G_S = \frac{\Sigma G_T^2}{\Theta^2} + 3G_T,
\]

\[
F_S = \frac{1}{a} \frac{d\zeta}{dt} - F_T,
\]

\[
\xi = \frac{a (G_T + D\dot{\pi}) G_T}{\Theta}.
\]

The quadratic action (28) describes one scalar (\(\zeta\)) and two tensor (\(h_{ij}^T\)) DOF. The propagation speed squared of the scalar and tensor perturbations reads, respectively:

\[
c_T^2 = \frac{F_T}{G_T}, \quad c_S^2 = \frac{F_S}{G_S}.
\]

Let us comment on the main types of instabilities, which can possibly arise in the quadratic action (28). In the case of homogeneous and isotropic background, the coefficients \(G_T, F_T, G_S\) and
and \( F_S \) are functions of time. The most dangerous instabilities are those arising in the high energy regime, i.e., when the characteristic scales of temporal and spatial variations of \( \zeta \) and \( h_{ij}^T \) are considerably smaller than that of the homogeneous background. These are the instabilities that we consider in this review. In the high energy approximation, the coefficients \( G_{S,T} \) and \( F_{S,T} \) can be treated as time-independent at relevant time intervals. Then the following situations are possible (the notations \( G_{S,T}, F_{S,T} \) refer to pairs of coefficients \( G_S, F_S \) or \( G_T, F_T \)):

1. Gradient instabilities (exponential growth of perturbations):
   \[
   G_{S,T} > 0, \quad F_{S,T} < 0, \quad \text{or} \quad G_{S,T} < 0, \quad F_{S,T} > 0.
   \]

2. Ghosts (catastrophic instability of vacuum state, see Ref. [13] for discussion):
   \[
   G_{S,T} < 0, \quad F_{S,T} < 0.
   \]  

3. Stable solution:
   \[
   G_{S,T} > 0, \quad F_{S,T} > 0.
   \]

Let us note that due to the form of the action (28) in the unitary gauge, the tachyonic instabilities do not develop in the system.

Hence, according to eq. (34), the absence of ghost and gradient instabilities about a homogeneous background solution implies the following restrictions on the coefficients in the quadratic action (28):

\[
G_T \geq F_T > \epsilon > 0, \quad G_S \geq F_S > \epsilon > 0.
\]

Hereafter \( \epsilon \) denotes a positive constant, whose actual value is irrelevant for our reasoning, so it may be different in different formulas below. This constant is introduced in eq. (35) to avoid the situations when \( G_{S,T}, F_{S,T} \to 0 \), which, at least naively, corresponds to strong coupling regime \(^2\). The inequalities (35) also ensure that both scalar and tensor perturbations propagate at the speed of light at most.

As we alluded to above, it was shown in Refs. [53, 55] that it is impossible to satisfy the constraints (35) over the entire evolution in Horndeski theories with \( F_4(\pi, X) = F_5(\pi, X) = 0 \) (see eq. (11)). This is precisely the no-go theorem which states that in the general Horndeski theory there are no completely stable bouncing and Genesis cosmologies. We discuss this no-go theorem in the next subsection in order to clarify how the stability conditions for cosmological solutions get modified in the presence of terms with \( F_4(\pi, X) \) (and \( F_5(\pi, X) \)) in the Lagrangian.

### 2.2 No-go theorem in Horndeski theory

The no-go theorem in Horndeski theory is obtained by the stability analysis of cosmological scenarios, under the assumption that the scale factor \( a \) is bounded from below by a positive constant,

\(^2\)We do not consider the special case of ghost condensate [67].
which ensures the geodesic completeness. The theorem is based on the requirement of absence of gradient instabilities (see eq. (30)):

$$\frac{d\xi}{dt} = a (F_s + F_T) > \epsilon > 0.$$  

(36)

According to eq. (36), the coefficient

$$\xi = \frac{aG_T^2}{\Theta}$$  

(37)

has to be a monotonously growing function of time. Note that the definition of $\xi$ in eq. (37) is valid only in Horndeski theory where $D = 0$ (cf. eq. (31)). It follows from the constraint (36), which must hold at any moment of time, and eq. (35) that $\xi \to -\infty$ as $t \to -\infty$ and $\xi \to +\infty$ as $t \to +\infty$, and, hence, $\xi$ necessarily crosses zero at some moment(s) of time. The latter fact is true irrespectively of whether or not the coefficient $\Theta$ vanishes at some moment(s) of time. Let us note that $\xi$ behaves as described above in beyond Horndeski theories as well (i.e., when $D \neq 0$ and $\xi$ is defined by eq. (31)), since the condition (36) holds for both Horndeski and beyond Horndeski subclasses. However, it follows from the definition (37) that in Horndeski theory, $\xi$ cannot behave in the way it is supposed to: since $a > 0$ and $G_T > \epsilon > 0$, the only way $\xi$ can cross zero is when $\Theta \to \infty$, which in turn corresponds to a singularity in the classical solution. Thus, there are no completely stable bouncing and Genesis models in Horndeski theory. This result still holds when $G_5 \neq 0$ (but $F_4 = F_5 = 0$).

A comment is in order on attempts to evade the no-go theorem within the general Horndeski subclass [55, 43]. One of the possible options is to make zero-crossing of $\xi$ happen due to simultaneously vanishing $\Theta$ and $G_T$, i.e., $\Theta(t_*) = 0$ and $G_T(t_*) = 0$, which violates the conditions (35). This option not only implies fine-tuning, but also faces the problem of strong coupling in the tensor sector (see eq. (28)). Another way to evade the no-go theorem is to partly give up the restrictions on the asymptotic behaviour of the theory, i.e., allow $F_{S,T} \to 0$ as $t \to -\infty$ and/or $t \to +\infty$. This case is potentially problematic because of naive strong coupling in the asymptotic past and/or asymptotic future.

The situation in beyond Horndeski theory is fundamentally different: the definition of $\xi$, eq. (31), involves $D \neq 0$ due to the function $F_4(\pi, X)$ (and $F_5(\pi, X)$). While the coefficient $G_T$ is still responsible for stability in the tensor sector and has to be always positive, the combination $(G_T + D\dot{\pi})$ is unconstrained and can take any values, including zero and negative ones. It is due to this coefficient $D$ that $\xi$ can monotonously grow and cross zero at some moment of time; the no-go theorem no longer holds. Therefore, the form of the stability conditions (35) in beyond Horndeski theories points towards the opportunity to construct bouncing and Genesis solutions free of ghost and gradient instabilities during entire evolution.

### 2.3 $\gamma$-crossing

Before we move on to explicit examples of bouncing and Genesis solutions in beyond Horndeski theory, let us discuss the possible behaviour of the coefficient $\Theta$ in eq. (31) and, in particular, the so-called $\gamma$-crossing phenomenon, meaning $\Theta = 0$ (in Refs. [43, 62, 63], where the issue was
originally addressed, the notations differ and the coefficient $\Theta$ is denoted by $\gamma$, which explains the terminology). In Ref. [63] it was shown that $\gamma$-crossing occurs at the change of the branch of the solution to eq. (14), considered as quadratic equation for the Hubble parameter. It was mentioned above that $\gamma$-crossing does not help circumvent the no-go theorem in Horndeski theory. Instead, this phenomenon plays a crucial role in determining the asymptotic behaviour of a beyond Horndeski theory as $t \to \pm \infty$. Namely, if we require that the beyond Horndeski theory which admits the non-singular cosmological solution in question, tends to GR with, say, a conventional massless scalar field in both asymptotic past and asymptotic future, the function $\Theta(t)$ must cross zero at some moment $t_* \in (-\infty, +\infty)$. Indeed, these asymptotics imply that $F_4 \to 0$ as $t \to \pm \infty$ and, consequently, $D \to 0$ as $t \to \pm \infty$ (see eq. (22)). As we argued above, $\xi < 0$ as $t \to -\infty$ and $\xi > 0$ as $t \to +\infty$; together with $D \to 0$ as $t \to \pm \infty$ this means that $\Theta < 0$ as $t \to -\infty$ and $\Theta > 0$ as $t \to +\infty$, which proves that the coefficient $\Theta$ vanishes at some finite moment of time $t_*$. However, the expressions for $G_S$ and $F_S$ in eqs. (29), (30) show singular behaviour of both coefficients at the moment of $\gamma$-crossing, $\Theta = 0$. At a glance this appears unacceptable. This was the reason for requiring that $\Theta$ is always positive when constructing one of the first completely stable bouncing solutions in Ref. [60]. In full accordance with the discussion above, the forbidden $\gamma$-crossing made it impossible to design a bouncing model whose asymptotics as $t \to \pm \infty$ are both described by GR, so gravity in the solution of Ref. [60] significantly differs from GR in the asymptotic past. But, interestingly, the same expressions (29), (30) indicate that the dispersion relation $c^2_S = F_S/G_S$ remains finite at $\gamma$-crossing, which in turn suggests that the situation is in fact not pathological. And, indeed, it was shown in Ref. [62] that the equations for perturbations in the Newtonian gauge do not exhibit singularities when $\Theta = 0$. Later similar calculations have been carried out in the unitary gauge [64], and it has been found that the solution for the scalar DOF $\zeta$ is regular at $\gamma$-crossing. Therefore, it has been proven that $\gamma$-crossing is acceptable. The latter observation made it possible to construct cosmological solutions with bounce and Genesis, whose asymptotics are simple as $t \to \pm \infty$ so that the theory reduces there to GR + a conventional massless scalar field [64, 65].

In the following subsection we review the explicit examples of models with completely stable bounce and Genesis in beyond Horndeski theories [60, 64, 65]. We highlight the way the no-go theorem is evaded in each of these solutions and describe their specific features.

### 2.4 Completely stable models with bounce and Genesis: examples

We do not go into details of the construction of solutions, which are described in Refs. [60, 64, 65]. Instead, we focus on the main ideas and results.

One way to design the models in question is to employ the reconstruction method, which was extensively used in the previous works, e.g., in Refs. [42, 53]. The general strategy is to find the Lagrangian functions $F, K, G_4, F_4$ in eqs. (7) – (11) such that the theory with the Lagrangian (6) admits the desired solution (we still take $G_5 = F_5 = 0$). In the first place, by making use of field

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3 Let us also mention the discussion in Ref. [63] of the solutions of Ref. [42].
redefinition it is always possible to choose the linear Galileon background
\[ \pi_c(t) = t. \] (38)

Then \( X_c = g^{\mu\nu} \partial_\mu \pi_c \partial_\nu \pi_c = 1. \) Now, the field equations (14), (15) and stability conditions (35) involve functions of time \( F(\pi_c, X_c) = F(t, 1), F_X(\pi_c, X_c) = (\partial F/\partial X)(t, 1) \equiv F_X(t, 1), \) etc., which are independent of each other (while, for instance, the function \( G_{4,\pi} \) equals \( \dot{G}_4(t, 1) \)). The aim is to select these functions for the explicitly specified Hubble parameter \( H(t) \). This selection should satisfy the following requirements: (i) the field equations (14), (15) should hold; (ii) the solution must be stable, i.e., the stability conditions (35) with the coefficients (21)–(25) should be satisfied. Clearly, these requirements do not uniquely determine all functions \( F(t, 1), F_X(t, 1), \ldots, F_{4XX}(t, 1) \) entering eqs.(14), (15), (35): there are only two equations, while the stability conditions (35) are inequalities rather than equations. Hence, the reconstruction we discuss has high degree of arbitrariness, and some of the functions are chosen on simplicity basis.

We also impose the constraint on the asymptotic behaviour of the theory as \( t \to \pm \infty \), which is not an obligatory requirement but rather a matter of choice. Namely, below we arrange the solutions in such a way that in the asymptotic future (and in the asymptotic past in cases 2.4.2 and 2.4.3) the beyond Horndeski theory tends to GR with a conventional massless scalar field. Let us recall that the massless scalar field minimally coupled to gravity has the equation of state \( p = \rho \), so that the spatially flat solution in GR has the following form:
\[ a(t) \propto |t|^{1/3}, \quad H(t) = \frac{1}{3t}, \]
while the canonical scalar field behaves as \( \phi_c(t) = \pm \sqrt{\frac{2}{3}} \ln |t| \). In view of eq. (38), it is related to \( \pi \) by
\[ \pi = \exp \left( \sqrt{\frac{3}{2}} \phi \right). \]
This must hold in the corresponding asymptotic. Here and in what follows we set
\[ \kappa = 8\pi G = 1. \]

2.4.1 Cosmological bounce with an exotic contraction stage

One of the first examples of a completely stable bouncing solution with an explicitly constructed beyond Horndeski Lagrangian is given in Ref. [60]. This solution has a characteristic feature of forbidden \( \gamma \)-crossing, which, as we discussed above, makes it impossible to have a completely stable solution with GR asymptotics both as \( t \to +\infty \) and \( t \to -\infty \). So, we set \( \Theta > 0 \) at all times and require that the theory reduces to GR + massless scalar field only in the future asymptotics \( t \to +\infty \).

Within the reconstruction approach, the scale factor and hence the Hubble parameter are chosen at one’s will. In this model, a simple choice is made:
\[ H(t) = \frac{t}{3(\tau^2 + t^2)}, \quad a(t) = (\tau^2 + t^2)^{\frac{1}{2}}, \] (39)
so that the bounce occurs at \( t = 0 \); \( \tau \) is a parameter which defines the duration of the bouncing epoch (in what follows we set \( \tau = 10 \) for definiteness), while the asymptotic behaviour \( H(t)\big|_{t\to+\infty} \to (3t)^{-1} \) agrees with the required property of the theory as \( t \to +\infty \). According to the reconstruction procedure, we choose part of the Lagrangian functions in such a way that the stability conditions (35) and asymptotic constraints are satisfied, and then find the rest of functions from the background equations of motion with \( H(t) \) given by eq. (39).

![Figure 1](image.png)

Figure 1: (a) The plots of \( \xi \), \((\mathcal{G}_T + \mathcal{D}\dot{\pi})\) and \( \Theta \) for the model of Ref. [60]: \( \xi \) crosses zero at \( t \approx -1.039 \) due to the behaviour of \((\mathcal{G}_T + \mathcal{D}\dot{\pi})\); \( \Theta \) is always positive, i.e., there is no \( \gamma \)-crossing. (b) Sound speeds squared for the scalar and tensor modes: \( c_S^2 \to 0.006, c_T^2 \to 0.18 \) as \( t \to -\infty \); \( c_S^2, c_T^2 \to 1 \) as \( t \to +\infty \). (c) The coefficients \( \mathcal{G}_S \) and \( \mathcal{F}_S \); both are finite as \( t \to -\infty \): \( \mathcal{F}_S \to 0.193 \). (d) The coefficients \( \mathcal{G}_T \) and \( \mathcal{F}_T \).

Since in this scenario \( \Theta > 0 \) at any time, the no-go theorem is circumvented by making a judicial choice for the function \( F_4(\pi, X) \), which determines the behaviour of \( \mathcal{D} \) in eq. (31): we have \((\mathcal{G}_T + \mathcal{D}\dot{\pi}) < 0 \) as \( t \to -\infty \) and \((\mathcal{G}_T + \mathcal{D}\dot{\pi}) > 0 \) as \( t \to +\infty \). In Fig. 1 (a) we plot \( \xi \), \((\mathcal{G}_T + \mathcal{D}\dot{\pi})\) and \( \Theta \) in this scenario to illustrate that the key coefficient \( \xi \) entering the would-be no-go theorem is indeed a monotonously growing function and that it crosses zero together with \((\mathcal{G}_T + \mathcal{D}\dot{\pi})\). As shown in Fig. 1 (c) and (d), coefficients \( \mathcal{G}_T, \mathcal{F}_T, \mathcal{G}_S \) and \( \mathcal{F}_S \) in the quadratic action (28) are
positive at all times, hence there are neither ghosts nor gradient instabilities.

The sound speeds squared in the scalar and tensor sectors are shown in Fig. 1 (b). Both speeds are always positive and tend to the speed of light as $t \to +\infty$ in full accordance with the required asymptotic behaviour described by GR with a massless scalar field.

Finally, let us give the asymptotic form of the Lagrangian as $t \to \pm \infty$. As required, at late times, the Lagrangian has a simple form of GR with a massless scalar field:

$$L|_{t=+\infty} = -\frac{1}{2} R + \frac{1}{3} \frac{\left( \partial \pi \right)^2}{\pi^2} = -\frac{1}{2} R + \frac{1}{2} \left( \partial \phi \right)^2. \quad (40)$$

At early times the bouncing model with no $\gamma$-crossing is described by the Lagrangian of beyond Horndeski type:

$$L|_{t=-\infty} = C_0 \cdot \frac{1}{\pi^2} + \left( \frac{1}{3} + C_1 \right) \frac{\left( \partial \pi \right)^2}{\pi^2} + C_2 \frac{\left( \partial \pi \right)^4}{\pi^2} + 2 \frac{\left( \partial \pi \right)^2}{\pi} \Box \pi - \frac{1}{16} \frac{\left( \partial \pi \right)^2 R}{\Box \pi} + \frac{1}{8} \left[ \Box \pi - \nabla_{\mu \nu \pi} \nabla_{\mu \nu \pi} \right] + \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} \epsilon^{\mu' \nu' \rho' \sigma'} \nabla_{\mu \pi} \nabla_{\mu' \pi} \nabla_{\nu' \pi} \nabla_{\rho' \pi}, \quad (41)$$

where $C_0 = 2.43$, $C_1 = -5.53$ and $C_2 = 1.06$ are model dependent constants. The theory (41) does not reduce to GR, in full accordance with the absence of $\gamma$-crossing.

### 2.4.2 Cosmological bounce with $\gamma$-crossing and two simple asymptotics

The bouncing model with $\gamma$-crossing [64] is a modification of the scenario discussed above. The main difference between the two constructions is that in the present case, there is $\gamma$-crossing happening at some moment of time. This allows to construct the solution with both asymptotics $t \to \pm \infty$ described by the Lagrangian (40).

The Hubble parameter in this scenario coincides with that in the previous model, see eq. (39). Since we aim to have GR in both asymptotics $t \to \pm \infty$, for the sake of simplicity we choose $G_4(\pi, X)$ and $F_4(\pi, X)$ in the Lagrangian in such a way that $G_T = F_T = 1$ during entire evolution. Thus, there are no instabilities in the tensor sector, while the gravitational waves always propagate at the speed of light, $c_T^2 = 1$.

As discussed in Sec. 2.3, $\gamma$-crossing (sign change of $\Theta$) enables one to choose the Lagrangian function $F_4$ in such a way that $D|_{t \to \pm \infty} \to 0$ (as before, $G_5 = F_5 = 0$) and at the same time satisfy the inequality $\dot{\xi} > \epsilon > 0$ in eq. (36) and ensure that the scalar sector is free of gradient instabilities. According to Fig. 2, the function $(G_T + D\dot{\pi})$ crosses zero twice, while $(G_T + D\dot{\pi})|_{t \to \pm \infty} \to 1$, in full agreement with the choice $G_T = 1$ at all times in this scenario. As before, $\xi$ vanishes simultaneously with $(G_T + D\dot{\pi})$. Unlike in the previous model, the reason for the negative sign of $\xi$ at $t \to -\infty$ is negative $\Theta$: since the theory reduces to GR, we have $\Theta \to H$ as $t \to -\infty$. For completeness we illustrate the case of fine-tuned solution in Fig. 2 (right panel): despite $\gamma$-crossing, $\xi$ is finite at all times because $(G_T + D\dot{\pi})$ touches zero at the moment when $\Theta = 0$.

To confirm the stability of the scalar sector, we show the functions $G_S$ and $F_S$ in Fig. 3: both coefficients are positive and diverge at the moment of $\gamma$-crossing, while their ratio is finite and strictly positive, in line with eqs. (29) and (30).
Figure 2: The plots of $\xi$, $(G_T + D\pi)$ and $\Theta$ for the model of Ref. [64] (left panel): $\xi$ crosses zero twice due to behaviour of $(G_T + D\pi)$; $\Theta$ crosses zero and changes sign. The right panel illustrates the fine-tuned case: $\xi$ remains finite at $\gamma$-crossing.

Figure 3: The coefficients $G_S$ and $F_S$ (left panel): both are positive and diverge at the moment of $\gamma$-crossing, while their ratio is finite, strictly positive and equals to the sound speed squared of the scalar mode (right panel); $\min(c_S^2) \simeq 0.001$.

2.4.3 Genesis and its modifications in Horndeski theory and beyond

In the same paper [53], where the no-go theorem was initially proven for the cubic Horndeski theory (12), a modified version of the Genesis scenario evading the no-go theorem in the same cubic subclass was suggested: the scale factor, instead of staying asymptotically constant, tends to zero as $t \to -\infty$ (but in such a way that the space-time curvature also vanishes). This enables one to avoid gradient instabilities throughout entire evolution. The price to pay for complete stability, however, is geodesic incompleteness of the solution as $t \to -\infty$. Unlike the original Genesis scenario of Ref. [4] with $H \propto (-t)^{-3}$, in the modified version the Hubble parameter and scale factor during the Genesis-like stage have the following time-dependence:

$$H = -\frac{h}{t}, \quad a(t) \propto \frac{1}{(-t)^h}, \quad h = \text{const}, \quad h > 1, \quad t < 0,$$

(42)
while the energy density \( \rho \) and effective pressure \( p \) behave as \( t^{-2} \) as \( t \to -\infty \) (within the original scenario, \( \rho \propto t^{-6}, \ p \propto t^{-4} \)).

Later on, the analog of the modified Genesis was constructed in beyond Horndeski theory in Ref. [60]. The Hubble parameter and the scale factor were chosen as follows:

\[
H(t) = \frac{1}{3\sqrt{\tau^2 + t^2}}, \quad a(t) = \left[ t + \sqrt{\tau^2 + t^2} \right]^{\frac{1}{2}},
\]

where \( \tau \) is a characteristic time scale. The essentially new property here is sufficiently slow evolution of the scale factor, \( a(t) \propto |t|^{-1/3} \) as \( t \to -\infty \), and hence geodesic completeness. The Genesis-like solution (43) evades the no-go theorem in a similar way as in the above bouncing solution without \( \gamma \)-crossing. Here \( \gamma \)-crossing is also forbidden, so the asymptotic behaviour of the theory as \( t \to -\infty \) corresponds to a substantially modified gravity of beyond Horndeski type. Like in the geodesically incomplete case (42), the theory transforms, as \( t \to +\infty \), to GR + massless scalar field (40).

Finally, in Ref. [65] a completely stable Genesis scenario with simple form of both asymptotics was constructed within beyond Horndeski theory. At early times the theory coincides with the original Genesis [5]:

\[
t \to -\infty : \quad H = \frac{f^3}{4\Lambda^3} \left( 1 + \frac{\alpha}{3} \right) \frac{|t|}{|t|^3}, \quad a(t) = 1 + \frac{f^3}{8\Lambda^3} \left( 1 + \frac{\alpha}{3} \right) \frac{|t|}{|t|^2}, \quad (44)
\]

where \( \Lambda, f \) and \( \alpha \) are the same parameters as in Ref. [5]. During the Genesis epoch the Lagrangian belongs to the cubic subclass of Horndeski theories:

\[
L_{t \to -\infty} = -\frac{1}{2} R - \frac{3f^3}{4\Lambda^3} \left( 1 + \alpha \right) \frac{X}{\pi^4} + \frac{3f^3}{4\Lambda^3} \left( 1 + \frac{\alpha}{3} \right) \frac{X^2}{\pi^4} - \frac{f^3}{2\Lambda^3} \frac{X}{\pi^3} \Box \pi, \quad (45)
\]

and upon field redefinition \( \phi = f \cdot \log \left( -\sqrt{\frac{3f}{2\Lambda^3}} \frac{1}{\pi} \right) \), its Lagrangian coincides with that in Ref. [5]. Due to \( \gamma \)-crossing, the late time asymptotics of the theory is GR: the Lagrangian transforms, as \( t \to +\infty \), to the standard form (40), while \( H = (3t)^{-1} \).

The Hubble parameter in this model reads

\[
H(t) = \left[ \left( 4 \frac{\Lambda^3}{f^3} \cdot \frac{1 - \tanh(t/\tau)}{2(1+\alpha/3)} + 3 \cdot \frac{1 + \tanh(t/\tau)}{2} \right) \sqrt{2\tau^2 + t^2} \right]^{-1}. \quad (46)
\]

The reconstruction procedure is analogous to that used in the bouncing case with \( \gamma \)-crossing: \( G_T = F_T = 1 \) at all times, so that tensor modes always propagate at the speed of light (\( c_T^2 = 1 \)); the behaviour of the key functions entering the would-be no-go argument, namely, \( \xi, (G_T + D\pi) \) and \( \Theta \), is similar to that shown in Fig. 2 (left panel). Time-dependence of the coefficients \( G_S \) and \( F_S \) responsible for stability of the scalar sector, is shown in Fig. 4, where \( G_S|_{t \to +\infty} \to 3, F_S|_{t \to +\infty} \to 3, \) and \( c_S^2|_{t \to +\infty} \to 1. \) At early times \( (t \to -\infty) \) we have \( G_S|_{t \to -\infty} \propto |t|^2 \) and \( F_S|_{t \to -\infty} \propto |t|^2 \), which is a distinguishing feature of the Genesis scenario, while \( c_S^2|_{t \to -\infty} < 1. \)
Figure 4: The coefficients $G_S$ and $F_S$ in the Genesis model (46) (left panel): both are positive and diverge at the moment of $\gamma$-crossing, while their ratio is finite, strictly positive and equals to the sound speed squared of scalar mode (right panel); $\min(c_S^2) \simeq 0.02$.

3 Disformal transformation and no-go theorem

It was mentioned in Introduction that some of the subclasses of DHOST and U-degenerate theories are related to the Horndeski theories [29, 30, 34, 35] by the invertible disformal transformation (5) of metric [32], which is a generalization of the standard conformal transformation $\bar{g}_{\mu\nu} = \Omega^2(\pi)g_{\mu\nu}$. Let us recall several features of disformal transformations. Disformal transformations (5), with $\Omega^2(\pi)$ and $\Gamma(\pi)$ being the functions of the scalar field $\pi$ only, do not extend the Horndeski subclass $\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$ to a wider one. The invertible transformations with $\Omega^2(\pi, X) = 1$ and arbitrary $\Gamma(\pi, X)$ extend the Horndeski theory $\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4$ to its generalization with $F_4(\pi, X) \neq 0$ [22, 33]. Conversely, the Lagrangian

$$\mathcal{L}(F, K, G_4, F_4) = F(\pi, X) + K(\pi, X)\Box \pi - G_4(\pi, X) R + (2G_4X(\pi, X) - F_4(\pi, X) X)\left[ (\Box \pi)^2 - \pi_{\mu\nu}\pi^{\mu\nu} \right] + 2F_4(\pi, X) \left[ \pi^{\mu\nu}\pi_{\mu\rho}\pi^{\nu}\pi - \pi^{\mu\nu}\pi_{\mu\lambda}\pi^{\nu\lambda}\pi_{\rho\lambda} \right],$$

(47)

which admits the stable solutions discussed in the previous section, can be transformed by the disformal transformation to the form $\bar{\mathcal{L}} = \mathcal{L}(\bar{F}, \bar{K}, \bar{G}_4)$, i.e., to the "non-extended" Horndeski theory (modulo a subtlety that we are about to discuss).

One may wonder whether there is a contradiction between the existence of completely stable solutions in beyond Horndeski theories (47) and the no-go theorem valid in any Horndeski theory with $F_4(\pi, X) = 0$, given that these theories are apparently related by field redefinition. The resolution of this "paradox" was given within the EFT approach in Ref. [59]: the pertinent disformal transformation is singular right at the moment when $\xi$ in eq. (31) crosses zero. The latter fact was established in Ref. [59] by analyzing the effective action for perturbations written in the most general form. This section aims at obtaining this result in the covariant framework.

Let us consider the disformal transformation

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Gamma_4(\pi, X)\partial_\mu \pi \partial_\nu \pi,$$

(48)
which converts the Lagrangian (47) to the Horndeski type:

$$\mathcal{L}_4[G_4] + \mathcal{L}_4[F_4] = \tilde{\mathcal{L}}_4[\tilde{G}_4].$$  (49)

The equation for the function $\Gamma_4(\pi, X)$, which implements the transformation (49), was found within the covariant approach in Refs. [24, 35]:

$$\Gamma_{4X} = \frac{F_4}{G_4 - 2G_{4X}X - F_4X^2}. \quad (50)$$

The relation between the new function $\tilde{G}_4$ in Horndeski theory $\tilde{\mathcal{L}}_4$ and the original $G_4$ in beyond Horndeski theory $\mathcal{L}_4$ was established as well:

$$\tilde{G}_4(\pi, \tilde{X}) = G_4(\pi, X) \frac{X}{\sqrt{1 + X\Gamma_4}}, \quad \tilde{X} = \frac{X}{\sqrt{1 + X\Gamma_4}}. \quad (51)$$

Let us demonstrate that the transformed function $\tilde{G}_4\tilde{X}$ which enters the Lagrangian $\tilde{\mathcal{L}}_4$ becomes singular when $\xi$ crosses zero together with $(G_T + D\dot{\pi})$ in eq. (31).

The relation between the functions $\tilde{G}_4\tilde{X}$ and $G_4X$, follows from the transformation rules (51) and reads:

$$\tilde{G}_4\tilde{X} = \frac{\partial G_4}{\partial \tilde{X}} = \frac{\sqrt{1 + X\Gamma_4}}{1 - X^2\Gamma_{4X}} \left( G_4(1 + X\Gamma_4) - \frac{1}{2}G_4(\Gamma_4 + X\Gamma_{4X}) \right). \quad (52)$$

Let us recast $\Gamma_{4X}$ in eq. (50) in terms of notations in eqs. (21) and (22):

$$\Gamma_{4X} = \frac{D\dot{\pi}}{X^2(G_T + 2D\dot{\pi})}. \quad (53)$$

We now substitute the expression for $\Gamma_{4X}$ in eq. (53) into the denominator factor in eq. (52):

$$\frac{1}{1 - X^2\Gamma_{4X}} = \frac{(G_T + 2D\dot{\pi})}{G_T + D\dot{\pi}}. \quad (54)$$

It follows from eq. (54) that the denominator of the transformation (52) crosses zero right at the moment when $\xi = 0$, see eq. (31), so that $\tilde{G}_4\tilde{X}$ is divergent at that moment of time. According to the discussion in Sec. 2.2, one evades the no-go theorem by going beyond Horndeski and having $D \neq 0$. This enables one to satisfy the requirements for $\xi$ given by eq. (36). The latter imply that $G_T + D\dot{\pi}$ vanishes at some moment(s) of time. Therefore, the beyond Horndeski theories which admit completely stable non-singular solutions, are disformally “related” to the Horndeski theories by singular transformations. So there is, in fact, no contradiction between the existence of completely stable solutions and the no-go theorem in apparently disformally related theories.

4 Conclusion.

In this mini-review we have briefly discussed the recent studies of non-singular cosmological scenarios and their stability in beyond Horndeski theory.
We have described specific examples of bouncing Universe and Genesis models in beyond Horndeski theory, which are free of ghost and gradient instabilities during entire evolution from $t \to -\infty$ to $t \to +\infty$. A nice feature of some of these models is the simple form of the theory in the asymptotics $t \to \pm\infty$, which is GR with a conventional massless scalar field. The advantage of having the asymptotic behaviour described by GR becomes clear when attempting to construct reasonably realistic bouncing and Genesis scenarios. In particular, the models make use of the Galileon property of safe NEC/NCC violation, crucial for the bounce and Genesis, while they avoid dealing with exotic matter away from the NEC/NCC-violating regime. Although the analysis of phenomenological prospects of these cosmological scenarios is beyond the scope of this review, we think this is a promising field of research.

A particular topic we have addressed is the disformal relation between beyond Horndeski and Horndeski theories. Our motivation here was to collect results obtained in the covariant formalism so far. The question about the consistency of the no-go theorem in Horndeski theory, on the one hand, and the existence of completely stable solutions in beyond Horndeski theory, on the other, in view of their relation by field redefinition, has been often raised even after the appearance of Ref. [59]. We consider our confirmation, in the covariant formalism, of the results of Ref. [59] a useful addition which fully resolves the issue.

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