Quantum coherence generating power, maximally abelian subalgebras, and Grassmannian Geometry

Paolo Zanardi and Lorenzo Campos Venuti
Department of Physics and Astronomy, and Center for Quantum Information Science & Technology, University of Southern California, Los Angeles, CA 90089-0484

We establish a direct connection between the power of a unitary map in $d$-dimensions ($d < \infty$) to generate quantum coherence and the geometry of the set $\mathcal{M}_d$ of maximally abelian subalgebras (of the quantum system full operator algebra). This set can be seen as a topologically non-trivial subset of the Grassmannian over linear operators. The natural distance over the Grassmannian induces a metric structure on $\mathcal{M}_d$ which quantifies the lack of commutativity between the pairs of subalgebras. Given a maximally abelian subalgebra one can define, on physical grounds, an associated measure of quantum coherence. We show that the average quantum coherence generated by a unitary map acting on a uniform ensemble of quantum states in the algebra (the so-called coherence generating power of the map) is proportional to the distance between a pair of maximally abelian subalgebras in $\mathcal{M}_d$ connected by the unitary transformation itself. By embedding the Grassmannian into a projective space one can pull-back the standard Fubini-Study metric on $\mathcal{M}_d$ and define in this way novel geometrical measures of quantum coherence generating power. We also briefly discuss the associated differential metric structures.

I. INTRODUCTION

The last few years have witnessed a renewed and strong interest in the quantitative theory of quantum coherence\textsuperscript{1–4}. This has been partly motivated by the key role that this concept plays in quantum information processing\textsuperscript{5}, quantum metrology\textsuperscript{6}, quantum thermodynamics\textsuperscript{7,8} and even in the so-called field of quantum biology\textsuperscript{9}. A related natural question concerns with the ability of a quantum operation to generate quantum coherence. Different approaches have been explored in the literature to quantify the coherence generating power (CGP) of quantum operations\textsuperscript{10–12}. For a thoughtful and comprehensive review of the current efforts on the theory of quantum coherence and CGP see Ref.\textsuperscript{13}. Also, in\textsuperscript{14} one can find the most recent updates and progress on the resource theory of coherence, states and beyond.

The goal of this paper is to develop some mathematical aspects of the approach to CGP for finite-dimensional quantum unital operations introduced in\textsuperscript{15,16}. This is a probabilistic approach that builds on top of an analog strategy in the context of entanglement theory\textsuperscript{17}. We shall unveil the underlying geometrical and algebraic structures to the CGP measures for unitary maps defined in\textsuperscript{15,16}. More precisely, we will show how the formalism there introduced can be interpreted and generalized in terms of metric structures over the space of maximally Abelian subalgebras (MASA) of the algebra of operators (the latter being endowed with the Hilbert-Schmidt scalar product). The space of MASAs can be seen as a topologically non-trivial subset of the Grassmannian manifold of $d$–dimensional ($d$ being the Hilbert space dimension) subspaces of of the full operator algebra and thereby inherits the Grassmannian metric structure.

Quite remarkably the quantitative notion of CGP introduced in\textsuperscript{15}, on purely physical grounds, turns out to be exactly proportional to the induced distance over the space of MASAs. This distance, in turn, will be shown to quantitatively measure the lack of commutativity between pairs of MASAs. Finally by exploiting standard embeddings of the Grassmannian into projective spaces we will show how to introduce novel measures of CGP for unitaries as well as to unveil the deep geometrical origin of known ones.

In Sect. II we introduce the basic elements of the formalism, maximally abelian algebras and quantum coherence, and discuss their elementary properties. In Sect. III we establish the connection between CGP measures and the geometry of the Grassmannian over linear operators. In Sect IV we briefly analyze the associated differential metric structure. Sect. V contains the conclusions.
II. QUANTUM COHERENCE AND MAXIMALLY ABELIAN SUBALGEBRAS

Let $\mathcal{H} \cong \mathbb{C}^d, (d < \infty)$ be the complex Hilbert space associated to a $d$-dimensional quantum system. The algebra of Linear operators $L(\mathcal{H})$ is equipped with the Hilbert-Schmidt scalar product $(A,B) := \text{tr}(A^\dagger B)$ and $\|X\|_2 := \sqrt{\langle X,X \rangle}$. In the following, when $L(\mathcal{H})$ is thought of as an Hilbert space itself with respect to this scalar product, it will be denoted by $\mathcal{H}_{HS} \cong \mathbb{C}^d$.

In the physical literature the notion of quantum coherence is usually formulated in relation to some distinguished orthonormal basis in the Hilbert space of the quantum system. However, in this paper we find convenient to use a slightly more abstract, approach. We start by providing a few basic definitions and associated elementary facts.

**Definition 0–** A family of orthogonal projectors $B := \{\Pi_i\}_{i=1}^m \subset L(\mathcal{H})$ is called an orthogonal resolution of the identity (ORI) if a) $\Pi_i \Pi_j = \delta_{ij} \Pi_j$, b) $\sum_{j=1}^m \Pi_j = \mathbb{1}$, c) $\Pi_i = \Pi_j$, $\forall j = 1,2,\ldots,m$. If all the projectors are rank one ($\Rightarrow H$ has a natural partial order ($\leq$) with respect to this scalar product, it will be denoted by $\mathcal{H}_{HS} \cong \mathbb{C}^d$.

Let us next consider one of the main objects of this paper: the algebra of operators which are diagonal in the representation associated to any frame in the equivalence class.

**Definition 1–** Given an MORI $B = \{\Pi_i\}_{i=1}^m$ we define the associated $d$-dimensional abelian subalgebra (ASA) of $L(\mathcal{H})$ by: $\mathcal{A}_B := \langle \sum_{j=1}^d \lambda_j \Pi_j / (\lambda_j)_{j=1}^d \in \mathbb{C}^d \rangle \subset \mathcal{H}_{HS}$. The map

$$D_B : \mathcal{H}_{HS} \rightarrow \mathcal{H}_{HS}/X \mapsto \sum_{j=1}^m \Pi_j X \Pi_j$$

is an orthogonal projection (in $\mathcal{H}_{HS}$) whose range is $\mathcal{A}_B$.

Clearly $\mathcal{A}_B$ is closed under hermitean conjugation. At the physical level the algebra projection $D_B$ is the measurement map associated to the MORI $B$ and it is a completely positive (CP) trace-preserving unital map i.e., $D_B(\mathbb{1}) = \mathbb{1}$. Crucially, the spaces $\mathcal{A}_B$ are maximal ASAs (MASA) in the sense that they are not a proper subalgebras of any other abelian one. This basically follows from the fact that the map $B \mapsto \mathcal{A}_B$ between ORIs and subalgebras of $L(\mathcal{H})$ is a morphism of partially ordered sets i.e., $B \leq B' \Rightarrow \mathcal{A}_B \subset \mathcal{A}_{B'}$.

**Proposition 1–** Let $\mathcal{M}_d$ denote the family of MASAs over $\mathcal{H}_{HS}$.

i) The correspondence $B \mapsto \mathcal{A}_B$ is a bijection between the set of all MORIs and $\mathcal{M}_d$.

ii) The action $U(d) \times \mathcal{M}_d \rightarrow \mathcal{M}_d$: $(U, \mathcal{A}) \mapsto U(\mathcal{A}) := \{U(X) := UXU^\dagger / X \in \mathcal{A}\}$ is transitive. Moreover,

$$\mathcal{M}_d \cong \frac{X_d}{S_d}, \quad X_d = \frac{U(d)}{U(1)^d}$$

where $S_d$ denotes the permutation group of $d$-objects.

**Proof–** i) We have to show that a) If $B$ is a MORI then $\mathcal{A}_B \in \mathcal{M}_d$ and b) if $\mathcal{A} \in \mathcal{M}_d$ then there exists a MORI $B$ such that $\mathcal{A} = \mathcal{A}_B$. Moreover the correspondence $B \mapsto \mathcal{A}_B$ is one-to-one.

Let $B = \{\Pi_i = |i\rangle\langle i|\}_{i=1}^d$ and suppose $\mathcal{A}_B \subset \mathcal{A}$ where $\mathcal{A}$ is an abelian subalgebra of $L(\mathcal{H})$. If $X \in \mathcal{A}$ then $[X,A] = 0 \forall A \in \mathcal{A}$. In particular $A \Pi_i = \Pi_i A \forall i$ whence $A |i\rangle = |i\rangle |i\rangle |i\rangle$. This shows that $A$ is $B$-diagonal i.e., $A \in \mathcal{A}_B \forall A \in \mathcal{A}$ and therefore $\mathcal{A} = \mathcal{A}_B$. b) Suppose $\mathcal{A} \subset L(\mathcal{H})$ is a MASAs. Any
A ∈ A can be written as sum of an hermitian and an anti-hermitean commuting parts. Moreover since all A’s commute there exists an ORI \{Q_j\} such that A = ∑_j α_j Q_j (joint diagonal form for all elements of A). Now, all the Q_j’s have to be one-dimensional i.e., \{Q_j\} has to be a MORI. In fact, if it were not so it would exist at least one j_0 such that Q_j_0 is higher-dimensional and therefore there it would exist a S ∈ L(H) which is non-diagonal but still commutes with all Q_j’s (and therefore with all elements in A). For example one may consider a unitary map S which acts like the identity everywhere but on the range of Q_j_0 where it is a non-trivial unitary. The algebra generated by A and S is still abelian and strictly contains A. This shows that, unless all the Q_j are rank one the algebra A is not a MASA. In conclusion if A is a MASA then it is generated by a MORI i.e., the map B → A_B is surjective.

Finally let us assume A_B = A_B. This implies in particular that \prod_i = ∑_j p_j \prod_j (\forall i). The spectrum of the LHS of this identity is \{0, 1\} while the one of the RHS is \{p_j\}_j=1^d. Therefore \prod_j ∈ \{0, 1\} because they form a probability distribution, \forall i there exists a j = j(i) ∈ \{1, . . . , d\} such that \prod_i = \prod_j(j(i)). This shows that the elements of B are just a permutation of those of B i.e., B = B. This amounts to prove that B → A_B is surjective.

\[ \text{l)} \text{ Since } M_d \text{ is, as a set, the same as the set of all MORIs we will focus on the structure of the latter. Let us consider the set } X_d = \{(\prod_i)_i=1^d\} \text{ of all ordered } d \text{-tuples of projectors } \prod_i \text{’s such that } (\prod_i)_i=1^d \text{ is a MORI. By defining the } S_d \text{-action on } X_d \text{ by } (\sigma, (\prod_i)_i=1^d) → (\prod_i Π(\sigma)) \text{ clearly the set of MORIs is nothing but } X_d/S_d. \text{ Now, if } x := (\prod_i)_i=1^d, \bar{x} = (\prod_i Π(\sigma)) \text{ then } U = ∑_i^d |i\rangle \langle i| (U^*)^{-1} \text{ maps one into the other by } x → U \cdot x := (U\prod U^*)^{-1} \text{ maps one into the other by } x → U \cdot x := (U\prod U^*)^{-1}. \text{ This means that } U(d) \text{ acts transitively over } X_d \text{ as well as on } M_d \text{ (forgetting the order). On the other hand the stabilizer of } x \text{ is given by } (\prod_i Π_i \chi_i, \chi_i ∈ U(1), i = 1, . . . , d) \equiv U(1)^d. \text{ Then } X_d = U(d)/U(1)^d \text{ follows from the standard identification of the } U(d)-\text{homogeneous space } X_d \text{ with the coset space obtained by dividing the group } U(d) \text{ by the stabilizer subgroup. This concludes the proof.} \]

The space X_d in Eq. (2) can be seen as the compact, simply-connected manifold of orthogonal full-flags. This implies that M_d is topologically non-trivial as its fundamental group is isomorphic to S_d. Indeed π_1(M_d) = π_1(\{X_d\}) ∼= S_d.

Having introduced the basic algebraic and geometrical objects of our formalism we now turn to physical concepts.

**Definition 2**—Given the MORI B = \{\prod_i\}_i=1^d we define:

\[ \text{a)} \text{ The } B\text{-incoherent states as the set of quantum states in } \mathcal{A}_B \text{ i.e., } I_B := \{∑_i=1^d p_j \prod_j / p_j ≥ 0, ∑_j=1^d p_j = 1\} \subset \mathcal{A}_B \in M_d. \]

\[ \text{b)} \text{ Given the quantum state } ρ \text{ we define its } B\text{-coherence by } c_B(ρ) := \inf_{X ∈ A_B} ||ρ - X||_2 = ||ρ - D_B(ρ)||_2 = ||Q_B(ρ)||_2. \]

\[ \text{c)} \text{ A unital CP-map } T : \mathcal{H}_{HS} → \mathcal{H}_{HS} \text{ is called incoherent iff } [T, D_B] = 0. \]

A couple of comments are here in order. 1) The definition above relies on the Hilbert-Schmidt norm || • ||_2 this, on the one hand, leads to a somewhat simplified theory of quantum coherence as naturally restricts the set of allowed operations to unital ones (for which the Hilbert-Schmidt norm is not increasing). On the other hand the simpler properties of || • ||_2 allows one to obtain a wealth of rigorous analytical that can be hardly obtained by more information-theoretic motivated choices e.g., the trace norm || • ||_1. 2) We also note that our definition of incoherent operations above falls in the class of dephasing-covariant incoherent operation as per the categorification of Ref. (see Table II therein).

The set I_B is clearly a \((d-1)\)-dimensional simplex. The first equality in Eq. (3) stems from the fact that A_B is a (closed) linear subspace of H_HS and that D_B is the corresponding orthogonal projection on it. This equality also shows that c_B(ρ) = inf_{X ∈ A_B} ||ρ - X||_2^2. The second equality simply defines the complementary projection Q_B := 1 - D_B. Notice that, from c) above, an incoherent map T is such that T(\mathcal{A}_B) ⊂ \mathcal{A}_B. The latter condition, which can be written as D_B T D_B = T D_B, is a weaker notion of incoherence coinciding with c) for normal maps T.\[15\]
Next we show that Eq. (3) defines a good coherence measure for unital maps and that it can also be seen as quantitative measure of the lack of commutativity between the state \( \rho \) and \( \mathcal{A}_B \in \mathcal{M}_d \).

**Proposition 2**—i) The map \( \rho \mapsto c_B(\rho) \) over quantum states \( \rho \) defined by Eq. (3) defines a good coherence measure i.e., \( c_B(\rho) = 0 \) if \( \rho \in I_B \) and \( c_B(T(\rho)) \leq c_B(\rho) \) for \( T \) incoherent.

ii) Let \( B = \{ \Pi_i \}_{i=1}^d \) be a MORI and \( \rho \) a quantum state, then one has \( c_B(\rho) = \frac{1}{2} \sum_{j=1}^d \| \Pi_j, \rho \|_2^2 \).

*Proof.—* i) This was proved in [14], we report the proof here for completeness. First, from (3) and the definition of \( I_B \) one has \( c_B(\rho) = 0 \Leftrightarrow Q_B(\rho) = 0 \Leftrightarrow \rho = D_B(\rho) \Leftrightarrow \rho \in I_B \). Second, \( c_B(T(\rho)) = \| Q_B(\rho) \|_2^2 = \| T(\rho) \|_2^2 \leq \| Q_B(\rho) \|_2^2 = c_B(\rho) \). Here we have used that for incoherent maps \( [T, Q_B] = 0 \) and that the Hilbert-Schmidt is monotonic under unital maps \( T \) i.e., \( \| T(X) \|_2 \leq \| X \|_2 \), (\( X \in \mathcal{H}_{HS} \)).

ii) A simple computation shows:

\[
\| \Pi_j, \rho \|_2^2 = \| \Pi_j \rho - \rho \Pi_j \|_2^2 = tr (\Pi_j \rho^2 \Pi_j + \rho \Pi_j \rho - \Pi_j \rho \Pi_j \rho - \rho \Pi_j \rho \Pi_j) = 2 \left[ tr(\rho^2 \Pi_j) - tr(\Pi_j \rho \Pi_j \rho) \right].
\]

Summing over \( i \) one obtains \( \sum_{i=1}^d \| \Pi_i, \rho \|_2^2 = \| \Pi \rho - \rho \Pi \|_2^2 = 2 \left( \| \rho \|_2^2 - \| Q_B(\rho) \|_2^2 = 2 \| Q_B(\rho) \|_2^2 = 2 c_B(\rho) \right). \)

This result shows that the geometric notion of distance, the algebraic one of non-commutativity and the physical one of quantum coherence are tightly tied together at the level of a single quantum state \( \rho \). In the following we will demonstrate that this connection holds at the level of the coherence generating power of unitary maps and pairs of MASAs.

### III. COHERENCE POWER AND GRASSMANNIAN GEOMETRY

Once \( B \)-coherence is defined one can ask the question about the ability of a unital CP-map to generate it. Here we will follow the probabilistic approach advocated in Ref. [15]. The idea is that the coherence generating power (CGP) of a map \( T \) is the average coherence as quantified by (4) – generated by \( T \) acting on a uniform ensemble of incoherent states. More precisely, let us now consider the uniform probability measure over \( I_B \) and denote by \( \mathbf{E}_{\text{unit}, I_B} [\bullet] \) the corresponding expectation.

**Definition 3**—Given the unital CP-map \( T : \mathcal{H}_{HS} \to \mathcal{H}_{HS} \) we define its \( B \)-coherence generating power (CGP) by

\[
C_B(T) := \mathbf{E}_{\text{unit}, I_B} [c_B(T(\rho))]
\]

This approach to CGP is based on probabilistic averages as opposed to optimizations over set of states and/or protocols see e.g., [4]. Clearly, this choice makes harder to envisage a direct operational and information-theoretic meaning of (4). However, this strategy, along with the nice algebraic properties of the Hilbert-Schmidt norm, allows one to find explicit analytical results for arbitrary (unital) maps and dimensions. In fact, in Ref. [15] we have proven the following fact

**Proposition 3**—Let \( B = \{ \Pi_i := \langle i | i \rangle \}_{i=1}^d \) be a MORI and \( T \) a unital CP over \( \mathcal{H}_{HS} \) then \( C_B(T) = N_d \sum_{j=1}^d \| Q_B(\Pi_j) \|_2^2 \) \( N_d^{-1} := d(d + 1) \). In particular for unitary CP-maps \( U(X) := UXU^\dagger \) \((U \in U(d))\) one has

\[
C_B(U) = N_d (d - \sum_{j=1}^d \| \langle i | U | j \rangle \|_2^2)
\]

*Proof.*—See Prop. 4 in Ref. [15] \( \square \)

It is now important to observe that \( \mathcal{M}_d \) is a subset of the Grassmannian of \( d \)-dimensional linear subspaces \( W \)’s

\[
\mathcal{G}_d(\mathcal{H}_{HS}) := \{ W \subset \mathcal{H}_{HS} / \dim W = d \} \supset \mathcal{M}_d
\]
This is a differentiable manifold with (real) dimension $d^2(d^2 - 2)$. Now we would like to show that the CGP has an underlying origin at the level of the geometry of $M_d$. The first step is to observe that, since MASAs belong to the Grassmannian, the set $M_d$ inherits the metric structure of the latter.

**Definition 4**—Let $\mathcal{A}_B, \mathcal{A}_B \in M_d$ we define a metric structure over $M_d$ by

\[ D(\mathcal{A}_B, \mathcal{A}_B) := \|D_B - D_B\|_{HS}. \]  

where $\| \cdot \|_{HS}$ denotes the Hilbert-Schmidt norm over $L(\mathcal{H}_{HS})$.

It is a well-known fact that the distance between subspaces in a Grassmanian can be taken to be the (Hilbert-Schmidt) distance between the corresponding orthogonal projections; Eq. (7) is just the particular case for elements of $M_d \subset G_d(\mathcal{H}_{HS})$. Notice that the distance (7) is invariant under the $U(d)$ action over $M_d$ i.e., $D(U(\mathcal{A}_B), U(\mathcal{A}_B)) = D(\mathcal{A}_B, \mathcal{A}_B)^{20}$.

One can now establish a direct connection between the distance (7) between MASAs and the, apparently totally unrelated, CGP of unitaries $[5]$. The following proposition contains some of the key conceptual as well as technical results of this paper.

**Proposition 4**—i) Let $U$ be a unitary CP-map, then

\[ C_B(U) = \frac{N_d}{2} D^2(\mathcal{A}_B, U(\mathcal{A}_B)) \]  

ii) $D^2(\mathcal{A}_B, \mathcal{A}_B) = \sum_{i,j=1}^d ||[\Pi_i, \Pi_j]^2||^2$

**Proof**—i) We first show how to compute traces over $L(\mathcal{H}_{HS})$ for maps of the form $\mathcal{F} : X \mapsto AXB$ where $A, B \in L(\mathcal{H})$. By definition $\text{Tr}(\mathcal{F}) = \sum_{s=1}^d \langle X_s, \mathcal{F}(X_s) \rangle$ where $\{X_i\}_{i=1}^d$ is an orthonormal basis in $\mathcal{H}_{HS}$. If $\{|i\rangle \}_{i=1}^d$ is an orthonormal basis of $\mathcal{H}$ let us pick $X_s = |i\rangle\langle j|$, $s = (l, m)$, $(l, m = 1, \ldots, d)$. Whence, $\text{Tr}(\mathcal{F}) = \sum_{l,m=1}^d \text{tr}(|i\rangle\langle l|)(|l\rangle\langle m|)B) = (\sum_{l,m=1}^d |\langle l|A|m\rangle|^2)(\sum_{l,m=1}^d |\langle l|B|m\rangle|^2) = \text{tr}(A)\text{tr}(B)$.

Let $\mathcal{A}_B, \mathcal{A}_B \in M_d$ associated to the MORIs $B = \{\Pi = |i\rangle\langle i|\}_{i=1}^d$ and $\tilde{B} = \{\Pi = |\tilde{i}\rangle\langle \tilde{i}|\}_{i=1}^d$ respectively. Now $D^2(\mathcal{A}_B, \mathcal{A}_B) = \|D_B - D_B\|_{HS}^2 = \|D_B\|_{HS}^2 + \|D_B\|_{HS}^2 - 2\text{Tr}(D_BD_B)$.

The first term can be written as $\text{Tr}(D_B) = \text{Tr}(\sum_{i,j=1}^d \Pi_i \Pi_j \bullet \Pi_j \Pi_i) = \sum_{i,j=1}^d \text{Tr}(\Pi_i \Pi_j)^2 = \sum_{i,j=1}^d \delta_{ij} \text{Tr}(\Pi_j)^2 = d$. The same is true for the second term. Let us now turn to the last term $\text{Tr}(D_BD_B) = \sum_{i,j=1}^d \text{Tr}(\Pi_i \Pi_j \bullet \Pi_j \Pi_i) = \sum_{i,j=1}^d \text{Tr}(\Pi_i \Pi_j)^2 = \sum_{i,j=1}^d \text{Tr}(\Pi_i)^2 = \sum_{i,j=1}^d |\langle i|j\rangle|^4$. Adding the three terms one gets

\[ D^2(\mathcal{A}_B, \mathcal{A}_B) = 2(d - \sum_{i,j=1}^d |\langle i|j\rangle|^4). \]  

Now set $|j\rangle := U|j\rangle$ in the last equation and compare with Eq. (5).

ii) It is a direct computation. $\sum_{i,j=1}^d ||[\Pi_i, \Pi_j]^2||^2 = \sum_{i,j=1}^d ||[\Pi_i, \Pi_j - \tilde{\Pi}_i, \Pi_j - \Pi_j, \Pi_i, \Pi_j - \Pi_i, \Pi_j, \Pi_i, \Pi_j]||^2 = 2 \sum_{i,j=1}^d (\text{tr}(\Pi_i \Pi_j - \Pi_j, \Pi_i, \Pi_j - \Pi_i, \Pi_j, \Pi_i, \Pi_j)) = 2 \sum_{i,j=1}^d (\text{tr}(\Pi_i \Pi_j) - \text{tr}(\Pi_j, \Pi_i, \Pi_j)) = 2(d - \sum_{i,j=1}^d |\langle i|j\rangle|^4)$. Comparing with Eq. (9) concludes the proof.

An alternative proof can be obtained by setting in Prop. 2 $\rho = \sum_{k=1}^d p_k U(\Pi_k)$, expanding the commutators norms and using $\mathcal{E}_{\text{unit}, I}[p_k p_k] = N_d(1 + \delta_{ik})^{13}$. □

Eq. (5) allows one to immediately and elegantly derive several properties of the CGP of unitaries $[5]$. First, the only unitaries with zero CGP are those which fix $\mathcal{A}_B$ i.e., the incoherent ones [see Def. 2 c)]. Second, if $W$ is $B$-incoherent because of the unitary invariance of the distance (7) one has that $D(\mathcal{A}_B, WU(\mathcal{A}_B)) = D(W(\mathcal{A}_B), WU(\mathcal{A}_B)) = D(\mathcal{A}_B, U(\mathcal{A}_B))$. Now Eq. (8), implies $C_B(U) = C_B(WU)$. Namely, the CGP of a map is invariant under post-processing by incoherent unitaries$^{15}$. Invariance under pre-processing by incoherent maps is trivial from (8). Third, from $D(U(\mathcal{A}_B), \mathcal{A}_B) = D(U(\mathcal{A}_B), U(U(\mathcal{A}_B))) = D(U(\mathcal{A}_B), \mathcal{A}_B) = D(U(\mathcal{A}_B), U^*(\mathcal{A}_B))$ one gets $C_B(U) = C_B(U^*)$. The CGP of a unitary is equal to the one of its inverse.

At the conceptual level these results demonstrate that the physical concept of CGP, the metric structure of the Grassmannian $G_d(\mathcal{H}_{HS})$ (more precisely of $M_d$) and quantum non-commutativity are profoundly connected to each other. In words: the $B$-coherence generating power of a unitary map $U$ is proportional to the Grassmannian distance between the input $B$-diagonal algebra $\mathcal{A}_B$ and
its image under $\mathcal{U}$. This distance, in turn, can be quantitatively identified with the lack of commutativity between these two algebras. It is important to stress that, in the light of the results of Grassmannian metric and non-commutativity are endowed with a physical as well as operational meaning.

In passing we mention that relation (8) suggests a straightforward path to extend the notion of CGP to infinite dimensions. Indeed one can replace the Hilbert-Schmidt norm in Eq. (7) by any unitary invariant norm for CP-maps and then define the CGP of a unitary map as the corresponding distance between $\mathcal{A}_B$ and $\mathcal{U}(\mathcal{A}_B)$. However, for $d = \infty$ the characterization of the set of MASAs is a much more challenging task and it lies beyond the scope of this paper.

Another appealing feature of the framework here discussed is that it also allows one to introduce novel measures for CGP of unitaries with an underlying geometrical meaning. To this aim it is useful to introduce one more definition.

**Definition 5**—Given a pair of ordered MORIs $B_\prec \triangleq (\Pi_i = |i\rangle\langle i|)_{i=1}^d$, $B_\preceq \triangleq (\tilde{\Pi}_i = \tilde{|i\rangle\langle i|})_{i=1}^d \in X_d$ we define the associated $d \times d$ non-negative overlap matrix by $\hat{O}_{\prec}(B_\prec, B_\preceq) \triangleq \langle \Pi_i, \tilde{\Pi}_j \rangle = (\langle i | j \rangle)^2 \geq 0$, $(i, j = 1, \ldots, d)$. In particular if $B_\prec$ and $B_\preceq$ are connected by the unitary $U$ i.e., $B_\preceq = B U_{\prec} \triangleq (\mathcal{U}(\Pi_i))_{i=1}^d$ we define $\hat{X}_B(U) \triangleq \hat{O}(B_\preceq, B U_{\prec}) = |\langle i | U j \rangle|^2$, $(i, j = 1, \ldots, d)$

We first notice that $\hat{O}$ is doubly-stochastic for any pair $(B_\prec, B_\preceq)$. Indeed, summing over $j$ one finds $\sum_{j=1}^d \hat{O}_{\prec}(B_\prec, B_\preceq) = \sum_{j=1}^d \langle \Pi_i, \tilde{\Pi}_j \rangle = \langle \Pi_i, 1 \rangle = \text{tr} \Pi_i = 1, (\forall i)$. The same result is obtained by summing over $i$. Let us remind the reader that, from the Proof of i) of **Prop 1**, the set $X_d$ of unordered MORIs is acted upon by $S_d$ via $(\Pi_i)_{i=1}^d \mapsto (\Pi_{\sigma(i)})_{i=1}^d$ and that a MORI in $M_d$ is just an $S_d$-equivalence class $[(\Pi_i)]_{i=1}^d$ [see Eq. (2)]. The next proposition shows the other properties of the overlap matrix and how it can be used to define novel metric structures over $M_d$ as well as CGP measures for unitaries.

**Proposition 5.—** i) The real-valued functions over $X_d \times X_d$ defined by $||\hat{O}(B_\prec, B_\preceq)||_2$ and $|\det \hat{O}(B_\prec, B_\preceq)|$ depend only on the $S_d$-equivalence classes $B = [B_\prec]$ and $B = [B_\preceq]$ i.e., they are functions over $M_d \times M_d$. Moreover

$$1 \leq ||\hat{O}(B_\prec, B_\preceq)||_2^2 = \text{Tr}(D_B D_{\tilde{B}}) \leq d, \quad |\det \hat{O}(B_\prec, B_\preceq)| = 1 \Leftrightarrow B = B_\preceq.$$  \hspace{1cm} (10)

ii) The function $D_{FS} : M_d \times M_d \rightarrow \mathbb{R}^+$ given by

$$D_{FS}(\mathcal{A}_B, \mathcal{A}_{B'}) := \cos^{-1}(|\det \hat{O}(B_\prec, B_\preceq)|), \hspace{2cm} (11)$$

where $B_\prec$ and $B_\preceq$ are any ordered MORIs in the $S_d$-equivalence classes $B = [B_\prec]$ and $B = [B_\preceq]$ respectively, defines a unitary invariant metric over $M_d$.

iii) The following functions define good CGP measures.

$$C_B(U) := D_{FS}(\mathcal{A}_B, \mathcal{U}(\mathcal{A}_{B})) = \cos^{-1}(|\det \hat{X}_B(U)|) \hspace{2cm} (12)$$

$$\varphi_B(U) := -\frac{1}{d} \ln |\det \hat{X}_B(U)| \hspace{2cm} (13)$$

where $B_\prec$ is any ordered MORI in the $S_d$-equivalence class $B = [B_\prec]$.

**Proof.—** i) It can easily checked that if one reorders the elements in $B_\prec$ and $B_\preceq$ the overlap matrix transforms according to $\hat{O} \mapsto Q O P^T$ where $P$ and $Q$ are unitary permutation matrices. From which one immediately obtains the first part of i). The first equality in Eq. (10) reads $\sum_{i=1}^d |\langle i | j \rangle|^2 = \text{Tr}(D_{\tilde{B}} D_B)$ which has been already proven in the proof of **Prop 4** (see lines above Eq. (9)). The range indicated follows from the fact that this norm is the sum of the purities of $d$ probability vectors in $d$-dimensions (see below). Let us now turn to the second equality in Eq. (10). Let $\hat{O} = W O_D V^\dagger$ be a Singular Value Decomposition of $\hat{O}$ with $W$ and $V$ unitaries and $O_D = \text{diag}(\lambda_1, \ldots, \lambda_d)$ the diagonal matrix of the singular values of $\hat{O}$. One has that $|\det(\hat{O})| = |\det(O_D)| = \prod_{i=1}^d \lambda_i$. The squares of the $\lambda_i$’s on the other hand are the eigenvalues of the doubly-stochastic matrix $\hat{O}^T \hat{O}$ whence $0 \leq \lambda_i \leq 1, (\forall i)$. It follows that $|\det(\hat{O})| = 1$ iff $\lambda_i = 1 (\forall i)$. This in turn is equivalent to $||\hat{O}||_2^2 = \text{tr}(\hat{O}^T \hat{O}) = \sum_{i,j=1}^d \hat{O}_{ij}^2 = \sum_{i=1}^d \lambda_i^2 = d$. Since $\hat{O} := (\hat{O}_{ij})_{j=1}^d$ (from double-stochasticity) are probability...
vectors \( \forall i \) the former equality is possible iff \( \sum_{j=1}^{d} \hat{O}_{ij} = 1 \) (\( \forall i \)) i.e., all the \( \hat{O}_{ij} \)’s are pure. This means that \( \forall i \neq j = \sigma(i) \) such that \( \hat{O}_{ij} = \delta_{j \sigma(i)} \). Moreover since \( \sum_{j=1}^{d} \hat{O}_{j \sigma(i)} = 1 \) one sees that \( \sigma \) must be in \( S_d \). In summary \( |\det(\hat{O})| = 1 \) if \( \exists \sigma \in S_d \) such that \( \hat{O}_{ij} = \delta_{j \sigma(i)} \) this amounts to say that \( \Pi_i = \Pi_{\sigma(i)} \), for some permutation \( \sigma \) and \( \forall i \) i.e., \( B = \hat{B} \).

ii) In order to show that (11) defines a distance function over \( \mathcal{M}_d \) we resort to the well-known Plücker embedding. For MASAs \( \mathcal{A}_B \) with \( B = \{ \Pi_i = |i\rangle\langle i| \}_{i=1}^{d} \) this embedding takes the form

\[
\psi: \mathcal{M}_d \to \mathbf{P}(\bigwedge_{i=1}^{d} \mathcal{H}_{FS}) / \mathcal{A}_B \mapsto [\bigwedge_{i=1}^{d} \Pi_i],
\]

where \( \bigwedge_{i=1}^{d} \Pi_i := \frac{1}{d!} \sum_{\sigma \in S_d} (-1)^{|\sigma|} \otimes_{i=1}^{d} \Pi_{\sigma(i)} \), and \([\bullet]\) denotes the projective equivalence class. The standard (\( \bigwedge_{i=1}^{d} \Pi_i \)) of \( \mathcal{A}_B \) is nothing but the pull-back of the Fubini-Study metric via the Plücker embedding (14).

Unitary invariance of the metric (11) stems from the fact that the overlap matrix associated with \( \mathcal{U}(\mathcal{A}_B) \) and \( \mathcal{U}(\mathcal{A}_B) \) is given by \( \hat{O}(BU_{\xi}, BU_{\zeta})_{i,j} = \langle \mathcal{U}(\Pi_i), \mathcal{U}(\Pi_j) \rangle = \langle \Pi_i, \Pi_j \rangle = \hat{O}(B_{\xi}, B_{\zeta})_{i,j}, \) where we have used unitary invariance of the Hilbert-Schmidt scalar product.

iii) Finally, \( \tilde{C}_B(U) := D_{FS}(\mathcal{A}_B, \mathcal{U}(\mathcal{A}_B)) = 0 \) iff \( \mathcal{A}_B = \mathcal{U}(\mathcal{A}_B) \) iff \( U \) is incoherent. Moreover, if \( W \) is incoherent \( \tilde{C}_B(WU) := D_{FS}(\mathcal{A}_B, W\mathcal{U}(\mathcal{A}_B)) = D_{FS}(WU(\mathcal{A}_B), WU(\mathcal{A}_B)) = D_{FS}(\mathcal{A}_B, \mathcal{U}(\mathcal{A}_B)) = D_{FS}(U) \), where we have used unitary invariance of \( D_{FS} \). This shows that \( \tilde{C}_B \) is a good CGP measure for unitaries (15).

Turning to \( \varphi_B \) in (12) we see that \( \varphi_B(U) = 0 \) iff \( |\det \hat{X}_{B_{\xi}}(U)| = 1 \) that from the above is equivalent to \( |\det \hat{O}(B_{\xi}, BU_{\zeta})| = 1 \Rightarrow B = BU \) namely \( U \) is incoherent. Since for incoherent \( W \)’s one has \( \hat{X}_{B_{\xi}}(WU) = Q_W \hat{X}_{B_{\xi}} \) (where \( Q_W \) is a permutation matrix depending on \( W \)) one finds \( |\det \hat{X}_{B_{\xi}}(WU)| = |\det \hat{X}_{B_{\xi}}(U)| \) which shows invariance under post-processing by incoherent \( W \’s \).

In view of Eq. (9), and Def. 5 one can write \( D^{\ell}(\mathcal{A}_B, \mathcal{A}_B) = 2(d-|\det \hat{O}(B_{\xi}, B_{\zeta})|) \) where \( B_{\xi} \) and \( \hat{B}_{\zeta} \) are any ordered MORIs in the \( S_d \)-equivalence class of \( B = \{ \Pi_i \}_{i=1}^{d} \) and \( \hat{\Pi}_i = \{ \Pi_{\sigma(i)} \}_{i=1}^{d} \). Eq. (10) now shows that the maximum distance between MASAs is given by \( \sqrt{2(d-1)} \) and it is achieved when the overlap matrix is given by the Van der Waerden’s matrix i.e., \( \langle \Pi_i, \Pi_j \rangle = 1/d, \langle \mathcal{U}(\Pi_i), \mathcal{U}(\Pi_j) \rangle = 1/d \). In this case the MORIs \( B \) and \( \hat{B} \) correspond to mutually unbiased bases and the unitary connecting them, because of (5), has maximum CGP.

It is also worthwhile stressing that Eq. (15) shows that the modulus of the determinant of the overlap matrix \( \hat{O}(B_{\xi}, BU_{\zeta}) = \hat{X}_{B_{\xi}}(U) \) as a natural interpretation as fidelity between the input and output MASAs \( \mathcal{A}_B \) and \( \mathcal{U}(\mathcal{A}_B) \).

The second measure in Eq. (12) was introduced in (ii) in Prop. 9] here we see that it is rooted in the geometry of the Grassmannian seen as sub-variety of the projective space \( \mathbf{P}(\bigwedge_{i=1}^{d} \mathcal{H}_{FS}) \). If \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) and \( B_a = \{ \Pi_{\alpha \beta} \}_{\alpha=1}^{d_1} =: \dim \mathcal{H}_\alpha \) (\( \alpha = 1, 2 \)) is a MORI over \( \mathcal{H}_\alpha \) (\( \alpha = 1, 2 \)) one can define a product MORI by \( B = \{ \Pi_1 \otimes \Pi_2 \}_{i=1}^{d_1 \times d_2} \) where \( i = 1, \ldots, d_2, j = 1, \ldots, d_1 \). From (13) (see also Ref. (8)) one finds \( \varphi_B(U_1 \otimes U_2) = \varphi_{B_1}(U_1) + \varphi_{B_2}(U_2) \) where \( U_i \in \mathcal{U}(\mathcal{H}_i), (i = 1, 2) \) i.e., the measure \( \varphi_B \) is additive (25).

IV. DIFFERENTIAL GEOMETRY OF COHERENCE POWER

We now move to consider a differentiable metric structure. This is done in terms of the natural Riemannian metric over the Grassmannian \( ds^2 = D(P, P + dP)^2 = Tr(dP^2) \). In view of the result (5) this metric will have the physical interpretation as the CGP of the unitary associated with an
infinitesimal change of the MORI. For example if $B$ is the MORI associated to (a non-degenerate) Hamiltonian $H$ a perturbation $H \rightarrow H + \delta V = H'$ will induce a change to an infinitesimally close one $B'$. In view of Eq. (8) the distance between the corresponding MASAs would then measure the CGP of the (infinitesimal) adiabatic intertwiner $\delta W_{ad}$ between the eigenstate systems of $H$ and $H'$ i.e., $C_B(\delta W_{ad}) = \frac{1}{4} d^2$.

**Proposition 6—** If $B = \langle \Pi_1 \rangle = |\langle \rho |i \rangle |^2$ then

$$d^2 = \text{Tr}(d\mathcal{D}B)^2 = 4 \sum_{i=1}^d \chi_i, \quad \chi_i := \langle d|\rho|i \rangle - \langle |\rho|d\rangle^2,$$

(16)

Moreover $d^2 = \frac{1}{4} ds^2$.

**Proof—**

Let us write differentials as $dX = X dt$ then $(ds/dt)^2 = \text{Tr}(d\mathcal{D}B)^2$. One has $\mathcal{D}B = \sum_{i=1}^d (\Pi_1 \cdot \Pi_i + \Pi_i \cdot \Pi_1 )$, whence $(d\mathcal{D}B)^2 = \sum_{i,j=1}^d \left( \Pi_j . (\Pi_i \cdot \Pi_j + \Pi_j \cdot \Pi_i ) + \Pi_j (\Pi_i \cdot \Pi_i + \Pi_i \cdot \Pi_i ) \right)$. Now, Tr$(d\mathcal{D}B)^2 = 2 \sum_{i=1}^d \text{tr}(\Pi_i)^2 + \sum_{i,j=1}^d \text{tr}(\Pi_i\Pi_j)\text{tr}(\Pi_i\Pi_j)$. From orthonormality follows tr$(\Pi_i\Pi_j) = \text{tr}(\Pi_i\Pi_j) = 0$ (for $i, j$), therefore $d\mathcal{D}B)^2 = 2 \sum_{i=1}^d \text{tr}(\Pi_i)^2$. By writing $\Pi_i = |\langle \rho |i \rangle$ and differentiating, a standard calculation shows that $\parallel \Pi_i \parallel^2 = \langle \Pi_i | \Pi_i \rangle^2 = \chi_i/dt^2$ therefore $(ds/dt)^2 = 4 \sum_{i=1}^d (\chi_i/dt^2 - |\langle \rho |i \rangle|^2)$. Reabsorbing the $dt$ factors on the RHS one finds (16).

To see that $d^2 = ds^2$ has the same expression as $d^2$ we use the fact $\text{det}(1 + \Phi\Phi^T) = 1 + \text{tr} \Phi^T \Phi + \cdots$. Expanding $\Pi_i$ near $|\langle \rho |i \rangle$ one finds $\langle \Phi\Phi^T \rangle_i = \langle \Pi_i, d\Pi_i \rangle + \frac{1}{2} \langle \Pi_i, d^2\Pi_i \rangle$. Taking the trace one has $\text{Tr}(\langle \Phi\Phi^T \rangle_i) = \frac{1}{2} \text{Tr}(\langle \Pi_i, d\Pi_i \rangle) = -\frac{1}{2} \sum_{i=1}^d \text{Tr}(d\Pi_i, d\Pi_i)$. Here we have used that $\Pi_i, d\Pi_i, d^2\Pi_i = 0$ and $\sum_{i=1}^d \Pi_i, d\Pi_i = -\sum_{i=1}^d \Pi_i, d\Pi_i \Pi_i$ (obtained by differentiating and adding the identities $\Pi_i^2 = \Pi_i$). As in the above $|d\Pi_i|^2 = 2\chi_i$. Now one has that $d^2 = \cos^{-1}(1 - \sum_{i=1}^d \chi_i)$ and the claim is obtained by expanding the cosine.

The $\chi_i$’s in Eq. (16) are projective space metrics associated to the $|\langle \rho |i \rangle$’s. When the latter are Hamiltonian eigenstates the $\chi_i$’s are known as fidelity susceptibility. The metric (16) is a sum of projective space ones. This reflects the fact that locally (see the numerator of Eq. (3)) the set of MASAs is the full-flag manifold $U(d)/U(1)^d$ which is the set of ordered tuples $(\Pi_i)_{i=1}^d$. The latter can be seen as a subvariety of $G_1(H)^d = P(H)^d$ by the obvious embedding.

Physically, the ground state susceptibility $\chi_0$ plays a key role in the differential geometric approach to quantum phase transitions (QPT) started in Ref. 29. The idea is that when $\chi_0$, which depends of the parameters defining the Hamiltonian, shows some singularity in the thermodynamical limit or a super extensive (for local Hamiltonians) behaviour for finite-size systems a QPT is occurring at that point in the parameter space.29

From this perspective Eqs. (8) and (16) are intriguing as they comprise information about all eigenstates. It is therefore tempting to wonder whether these geometric quantities, which are quantifying quantum coherence power at the same time, can be exploited to study phase transitions in which a radical change is occurring at the level of whole Hamiltonian eigenstate system e.g., many-body localization.

V. ONE QUBIT

In order to illustrate the general results proved in this paper we consider explicitly the qubit case i.e., $d = 2$. In this case

$$X_2 = \frac{U(2)}{U(1) \times U(1)} \equiv \frac{SU(2)}{U(1)} \equiv \text{CP}^1 \equiv S^2 \Rightarrow \mathcal{M}_2 \equiv \frac{S^2}{Z_2},$$

(17)

where we used Eq. (2) and $S_2 \cong \mathbb{Z}_2$. This has a simple geometrical interpretation since MORIs (and therefore MASAs) in two-dimensions have the form $B = \langle \Pi_0 \rangle$ where $\Pi_0 := \frac{1}{2} (1 + \alpha \cdot n \cdot \sigma)$, $(\alpha = \pm)$ the $\sigma_{\alpha}$’s are the standard Pauli matrices and $n = (n_x, n_y, n_z) \in S^2$. Thus it is clear that $n$ is
identified with \(-n\) as they both correspond to the same MORI. This simple example also shows that the set of MASAs may have non-trivial topology: loops in \(M_2\) fall in two topologically distinct categories, the trivial (non-trivial) which corresponds to \((\Pi_1, \Pi_2) \mapsto (\Pi_1, \Pi_2) \mapsto (\Pi_2, \Pi_1))\) i.e., \(\pi_1(M_2) \cong \mathbb{Z}_2\).

If \(\tilde{B} = (\frac{1}{2}(1 + \alpha \n \cdot \hat{n})\sigma)\), one can easily check that the overlap matrix is given by \(\hat{O}(B, \tilde{B}) = \frac{1}{2}(1 + \alpha \beta \n \cdot \hat{n})\), whose spectrum is \(\{1, \n \cdot \hat{n}\}\) and therefore \(\text{det} \hat{O}(B, \tilde{B}) = \n \cdot \hat{n}\), and \(\|\hat{O}(B, \tilde{B})\|^2_2 = 1 + (\n \cdot \hat{n})^2\). Whence

\[
D^2(\mathcal{A}_B, \mathcal{A}_{\tilde{B}}) = 2(d - \|\hat{O}(B, \tilde{B})\|^2_2) = 2(1 - (\n \cdot \hat{n})^2) = 2 \sin^2 \psi, \tag{18}
\]

where \(\psi := \cos^{-1}(\n \cdot \hat{n})\). From Eq. (18) we clearly see that the MASAs corresponding to \(n\) and \(-n\) are identified thus confirming that globally \(M_2 \cong S^2/\mathbb{Z}_2\) as given by Eq. (17). On the other hand \(n \perp \hat{n}\) correspond to maximally far apart MASAs.

Now we consider the commutators \(\{\Pi_\alpha, \Pi_\beta\} = \frac{i}{\hbar} (\n \times \hat{n}) \cdot \sigma\). From which \(\sum_{\alpha, \beta = \pm} ||\Pi_\alpha, \Pi_\beta||^2_2 = 2\|n \times \hat{n}\|^2 = 2 \sin^2 \psi\). Comparing this last Eq. with (18) confirms ii) of Prop. 4. If \(n = (0, 0, 1)\) and \(B = BU\) with \(U = a|0\rangle\langle 0| + a^*|1\rangle\langle 1| - b^*|0\rangle\langle 1| + b|1\rangle\langle 0|, (a = \cos(\theta/2), b = e^{i\phi} \sin(\theta/2))\). Then \(\hat{n} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \Rightarrow \psi = \cos^{-1}(\n \cdot \hat{n}) = \theta\). From Prop. 4 and (18) one gets\(^{15}\)

\[
C_B(U) = \frac{N_d}{2} D^2(\mathcal{A}_B, \mathcal{A}_{\tilde{B}}) = \frac{1}{6} \sin^2 \theta.
\]

Maximum CGP is attained by all \(U\)'s with \(\theta = \pi/2\) irrespective of \(\varphi\) as the corresponding \(\hat{n}\)'s are equidistant from \(n = (0, 0, 1)\).

Furthermore, from \(\text{det} \hat{O}(B, \tilde{B}) = \n \cdot \hat{n} = \cos \psi\), it follows immediately from Eq. (11) that \(D_{FS}(\mathcal{A}_B, \mathcal{A}_{\tilde{B}}) = \cos^{-1}|\omega|\psi\) which is given immediately by \(\psi\) for \(\psi \in [0, \pi/2]\) and by \(\pi - \psi\) for \(\psi \in [\pi/2, \pi]\).

Finally, from \(ds^2 = 2\sum_{\alpha = \pm} ||d\Pi_\alpha||^2_2 = 2\sum_{\sigma = \pm} \|\alpha \sigma \cdot \sigma||^2_2 = 2 \|d\n\|^2\) one sees that the CGM metric is proportional to the standard euclidean metric of \(S^2\).

VI. CONCLUSIONS

In this paper we have unveiled a deep connection between the notion of quantum coherence generating power of unitary operations (introduced on purely physical grounds) and the geometry of the Grassmannian of subspaces of the algebra of linear operators. Given a maximal orthogonal resolution of the identity \(B\) in the Hilbert space \(\mathcal{H} \cong \mathbb{C}^d\) of a quantum system one can consider the \(d\)-dimensional algebra \(\mathcal{A}_B\) generated by \(B\). This is a maximal abelian subalgebra (MASA) of the full operator algebra \(L(\mathcal{H})\) which is closed under hermitean conjugation. The set of all MASAs is a topologically non-trivial subset of the Grassmannian of \(d\)-dimensional subspaces of the Hilbert-Schmidt space \(L(\mathcal{H}) \cong \mathbb{C}^{d\times d}\). Given a unitary map \(\mathcal{U}\) we have shown that its coherence generating power with respect \(B\) is proportional to the Grassmannian distance, as well as the lack of commutativity, between the MASAs \(\mathcal{A}_B\) and \(\mathcal{U}(\mathcal{A}_B)\). By embedding the set of MASAs into the projectivation of the \(d\)-th exterior power of Hilbert-Schmidt space one can pull-back the standard Fubini-Study metric and obtain novel coherence power measures endowed by a natural geometrical interpretation.

ACKNOWLEDGMENTS

This work was partially supported by the ARO MURI grant W911NF-11-1-0268. We thank G. Stiliaris for useful discussion and for insisting on the "geometrical approach".

---

1 T. Baumgratz, M. Cramer, and M. B. Plenio, Phys. Rev. Lett. 113, 140401 (2014).
2 F. Levi and F. Mintert, New J. Phys. 16, 033007 (2014).
3 D. Girolami, Phys. Rev. Lett. 113, 170401 (2014).
4 Y. Yao, G. H. Dong, X. Xiao and C. P. Sun, Scientific Reports 6, 32010 (2016).
5 M. A. Nielsen and I. Chuang, Quantum Computation and Quantum Information, Cambridge University Press (2000).
6 I. Marvian and R. W. Spekkens, Phys. Rev. A 94, 052324 (2016). I. Marvian, R. W. Spekkens and P. Zanardi, Phys. Rev. A 93, 052331 (2016).
In fact the projection associated to $P$ is a $d$-dimensional linear space (with $d = \sum_{\alpha \in S} \parallel\tilde{D}_\alpha\parallel_{HS}^2$) such that $\parallel\tilde{D}_\alpha\parallel_{HS} = \sum_{\alpha \in S} \parallel\tilde{D}_\alpha\parallel_{HS}$. But $A = B$ from maximality. Therefore $B \subset A$. The converse inclusion is obtained in the same way and therefore $A = B$.

If $W$ is a $n$-dimensional linear space (with $n > d > 0$) we denote by $\wedge_{i=1}^{d} W$ the total anti-symmetric part of $W^\otimes d$. By $P(W)$ we denote the projective space associated with $W$.

P. Zanardi, N. Paunkovic Phys. Rev. E 74, 031123 (2007); P. Zanardi, P. Giorda, M. Cozzini Phys. Rev. Lett. 99, 100603 (2007); L. Campos Venuti, P. Zanardi Phys. Rev. Lett. 99, 095701 (2007).

J. Schwinger, Proc. Natl. Acad. Sci. U.S.A. 46, 570 (1960); I.D. Ivanovic, J. Phys. A. 14, 3241 (1981).

Additivity is a direct consequence of the property $X_B(U_1 \otimes U_2) = X_B(U_1) \otimes X_B(U_2)$ and $\det(A \otimes B) = \det(A)^d \det(B)^d$.

L. Campos Venuti et al, Quantum coherence and many-body localization (unpublished).