RG Flows in the $D$-Series of Minimal CFTs

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Using results of the thermodynamic Bethe Ansatz approach and conformal perturbation theory we argue that the $\phi_{1,3}$-perturbation of a unitary minimal $(1 + 1)$-dimensional conformal field theory (CFT) in the $D$-series of modular invariant partition functions induces a renormalization group (RG) flow to the next-lower model in the $D$-series. An exception is the first model in the series, the 3-state Potts CFT, which under the $\mathbb{Z}_2$-even $\phi_{1,3}$-perturbation flows to the tricritical Ising CFT, the second model in the $A$-series. We present arguments that in the $A$-series flow corresponding to this exceptional case, interpolating between the tetracritical and the tricritical Ising CFT, the IR fixed point is approached from “exactly the opposite direction”. Our results indicate how (most of) the relevant conformal fields evolve from the UV to the IR CFT.

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1. Introduction

Among the few aspects of quantum field theories (QFTs) often amenable to analytical study is their behaviour at large and/or short distances. The asymptotic behaviour in these regimes is governed by (possibly trivial) fixed points of the renormalization group (RG), and RG-improved perturbation theory can provide important insights into the full non-perturbative behaviour of the theory.

A particularly interesting situation arises if both the ultraviolet (UV) and infrared (IR) fixed points of a QFT are nontrivial. This is interesting, from a QFT point of view because RG flows between such fixed points provide some understanding of the topology of the space of QFTs, and from a statistical mechanics viewpoint because it yields examples of the cross-over between different universality classes of critical phenomena. One would like to understand such RG flows in detail, e.g. not just between which theories the flow interpolates but also how specific operators evolve from the UV to the IR.

In 1+1 dimensions (where the subject may have implications also for the non-perturbative formulation of string theory) it is possible to study some of these questions explicitly. The best known examples of RG flows between non-trivial fixed points are those \(^1\) induced by \(\varphi_{1,3}\)-perturbations of the unitary minimal models \(^4\) of central charge

\[ c_m = 1 - \frac{6}{m(m+1)} \]

with diagonal modular invariant partition functions (MIPFs) \(^3\). Here we study the \(\varphi_{1,3}\)-induced flows between the unitary minimal models with non-diagonal MIPFs.

Denote a unitary minimal model of central charge \(c_m\) and given MIPF in the ADE classification \(^3\) by \(X_m\), with \(m = 3, 4, 5, \ldots\) for \(X = A\), \(m = 5, 6, 7, \ldots\) for \(X = D\), and \(m = 11, 12, 17, 18, 29, 30\) for \(X = E\). Recall that unlike the \(A\)-models, the \(D\)- and \(E\)-models have non-diagonal MIPFs. Also, except for the \(E\)-models with \(m = 11, 17, 29\) all the models contain \(\varphi_{1,3}\), which is the least relevant (spinless, primary) field.\(^3\) Hence, except for the latter three models, one can define the perturbed CFTs \(X^{(\pm)}_m\) via the (euclidean) action

\[ A_{X^{(\pm)}_m} = A_{X_m} + \lambda \int d^2x \, \varphi_{1,3}(x) \, , \]  

where

\(^1\) \(D_5\), the 3-state Potts CFT \(^3\), is special in that its spectrum contains two copies of \(\varphi_{1,3}\) that can be taken to be even \((\varphi_{1,3}^+\)\) and odd \((\varphi_{1,3}^-)\) with respect to the global \(\mathbb{Z}_2\) symmetry of the model; by “the \(\varphi_{1,3}\) perturbation” in this case we will always mean the \(\mathbb{Z}_2\)-even perturbation.
where the superscripts \((\pm)\) refer to the sign of \(\lambda\). The theories \(X_m^{(\pm)}\) are integrable non-scale-invariant theories, the mass scale being proportional to \(\lambda^{1/y_m}\) where \(y_m = \frac{4}{m+1}\) is the RG eigenvalue of the field \(\phi_{1,3}\) (its scaling dimension is \(d_{1,3}^{(m)} = 2 - y_m\)). Since under repeated fusions \(\phi_{1,3}\) closes on itself, modulo irrelevant operators and the identity, renomalization of the UV divergences in the perturbation theory based on (1.1) will not generate additional counterterms (the counterterm corresponding to the identity does, however, affect the perturbative expansions we will be interested in — see sect. 2).

Perturbations in opposite directions of the same CFT may lead to completely different non-scale-invariant QFTs. \(A_m^{(-)}\), for example, is believed to be a theory of massive kinks \(\mathbb{1}\). On the other hand, \(A_m^{(+)} (m \geq 4)\) is believed to be massless and flow to \(A_{m-1}\) in the IR, as strongly suggested \(\mathbb{1} \mathbb{2}\) by the LG approach together with perturbative RG analysis (the latter applicable when \(m \gg 1\)).\(\mathbb{2}\) We will present concrete evidence that the theories \(D_m^{(+)}\) describe the flows

\[
D_m \to \begin{cases} A_4 & \text{if } m = 5 \\ D_{m-1} & \text{if } m \geq 6 \end{cases}
\tag{1.2}
\]

the UV (IR) limits corresponding to \(\lambda = 0 (\infty)\), respectively. It is also natural to conjecture that \(E_m^{(+)}\), \(m = 12, 18, 30\), describe the flows \(E_m \to E_{m-1}\) (see sect. 3.4).

The two techniques traditionally used to study these and other (conjectured) RG flows are Landau-Ginzburg (LG) analysis \(\mathbb{10} \mathbb{12} \mathbb{11} \mathbb{13}\) and RG-improved perturbative calculations \(\mathbb{1} \mathbb{2} \mathbb{11} \mathbb{13}\). Although LG descriptions involving two scalar (real) fields have been proposed \(\mathbb{3} \mathbb{14}\) for \(D_m\) and \(E_m\), they do not seem to be useful when analyzing \(\phi_{1,3}\)-induced RG flows. This is not surprising since the LG models involve two strongly interacting fields, even for large \(m\). On the other hand, RG-improved perturbative calculations \(\mathbb{1} \mathbb{2}\) can be performed for any \(X_m^{(+)}\) provided \(m \gg 1\) (which excludes the \(E\)-models from consideration), so that the IR fixed point is close to the UV one. This will be discussed in sect. 3.3.

We will also provide evidence for (1.2) — in particular for small \(m\) — along quite different lines. This is possible due to some recent developments in the study of the finite-volume spectrum of non-scale-invariant integrable QFTs in 1 + 1 dimensions. The thermodynamic Bethe Ansatz (TBA) technique allows one to obtain non-linear integral

\[^2\] The existence of these flows was first alluded to by Huse \(\mathbb{9}\), who noticed a cross-over between different critical behaviours in regime IV of the integrable RSOS lattice models underlying the QFTs \(A_m^{(+)}\).
equations for the exact ground state energy \( E_0(R) \) of such a QFT in finite volume \( R \), given its factorizable \(^{15}\) \( S \)-matrix. The small \( R \) behaviour of \( E_0(R) \) contains information about the UV CFT, which can be compared with predictions of conformal perturbation theory (CPT). This approach has been extensively used in the last two years as a means of checking the purely massive scattering theories conjectured to describe the on-shell behaviour of certain perturbed CFTs (see \(^{16}\) for a brief review of all this, \(^{17}\) \(^{18}\) \(^{19}\) for the details in the case of perturbed CFTs with diagonal \( S \)-matrices, and \(^{20}\) for the first analysis of a non-diagonal scattering theory).

Recently \(^{21}\) Al. Zamolodchikov proposed an explicit scattering theory of massless particles for the non-scale-invariant QFT \( A_4^{(+)} \), presumably \(^{11}\) interpolating between the tricritical and the critical Ising CFT. Comparison of the solutions of the TBA equations with CPT around the UV and IR CFTs leaves little doubt that the conjectured \( S \)-matrix is correct. In \(^{21}\) it was further conjectured that \( E_0(R) \) in \( A_m^{(+)} \) for \( m > 4 \) is given by a certain generalization (see sect. 2) of the TBA equations for \( A_4^{(+)} \). It was checked that the conjectured TBA equations give the expected UV and IR central charges. Nevertheless, since for \( m > 4 \) the TBA equations have simply been guessed (there are not even any conjectures for the \( S \)-matrices of \( A_m^{(+)} \) for \( m > 4 \)), further evidence for the correctness of the equations is required.

In \(^{22}\) we provided such evidence for \( m = 5, 6, 7 \). Furthermore, we proposed TBA-like equations for certain finite-volume excitation energies in \( A_m^{(\pm)} \) for \( m \) even. For \( A_m^{(\pm)} \) these equations explicitly demonstrate the flow of the UV conformal fields \( \phi_{2,2} \) and \( \phi_{\frac{m}{2}, \frac{m}{2}} \) to fields with the same Kac indices in \( A_{m-1} \), in agreement with expectations based on LG and perturbative RG analyses. These results provide further strong (and independent) support for the existence of the flows concerned, and show the usefulness of the study of finite-volume spectrum in the context of RG flows in general.

In fact, the “TBA approach” has led the authors of \(^{23}\) to conjecture new RG flows that has not been proposed before, namely between the \( Z_N \)-parafermion CFTs \(^{24}\) \((N \geq 3)\) and the minimal models \( A_{N+1} \). The conjectures are based on equations for \( E_0(R) \) of the perturbed parafermion models (the perturbation is by a certain field that breaks the \( Z_N \) symmetry, leaving only a global \( Z_2 \) symmetry in the resulting theory). Although the

\(^3\) The difficulty in performing the numerical analysis of the TBA equations, required for the comparison with CPT, increases rapidly with \( m \).
equations have been basically obtained by guesswork and it is desirable to come up with further evidence that they are correct, previous experience suggests to take them seriously. In the particular case \( N = 3 \), where the perturbed \( \mathbb{Z}_3 \)-parafermion model is just the \( \phi_{1,3} \) perturbed 3-state Potts CFT \( D_5^{(+)} \), the equations for \( E_0(R) \) are the same as the ones proposed in [21] for the ground state energy in \( A_5^{(+)} \), corresponding to the flow \( A_5 \rightarrow A_4 \).

More generally, the ground state energy in \( X_m^{(\pm)} \) does not depend on the MIPF of the unperturbed theory; this is predicted by UV-CPT (see sect. 3.2). Assuming the existence of the flows within the \( A \)-series, it is therefore clear that for all \( m \geq 5 \) \( D_m^{(+)} \) describes a flow of \( D_m \) to \( X_{m-1} \), but with what MIPF? Our answer is eq. (1.2).

In sect. 2 we will describe in more detail known TBA and CPT results, leading to our conjecture (1.2). In sect. 3 new evidence from UV and IR CPT as well as RG-improved CPT is presented. We also discuss in some detail the special case of \( D_5^{(+)} \), arguing that from an IR-CPT point of view, \( i.e. \) considering it as a \( \phi_{3,1} \)-perturbation \( (+ \) higher corrections) of \( A_4 \), it is just the sign of the \( \phi_{3,1} \)-coupling that distinguishes it from \( A_5^{(+)} \). In sect. 4 we briefly discuss our results and methods, concluding with an outlook on some open questions. Some observations regarding the appearance of logarithmic terms in CPT expansions are described in the appendix.

2. TBA and CPT results

We now describe in more detail the results of the TBA approach for the theories under consideration. Define the scaling function \( e(r) = (2\pi)^{-1} RE(\lambda, R) \) corresponding to a generic excitation energy \( E(\lambda, R) \) in a one-parameter \( (\lambda, \) of a specific sign) family of perturbed CFTs. Here \( R \) is the circumference of the cylinder on which the theory is defined (periodic boundary conditions are assumed), and \( r = MR \) is a dimensionless scaling parameter, \( M \) being the mass scale of the theory. \( M \) is usually chosen in a way that is natural from the point of view of the off-critical theory, \( e.g. \) the infinite-volume mass of the lightest particle if the the theory is purely massive. It turns out that such a choice is also convenient for the TBA equations below. \( M \) is related to \( \lambda \) through

\[
|\lambda| = \kappa M^y ,
\]  

where \( \kappa > 0 \) is a numerical constant \( (y \) is the RG eigenvalue of the perturbing field). \( \kappa \) has been determined in many perturbed CFTs numerically, and in a few cases also analytically (cf. the appendix).
At present, all the TBA-like equations known or conjectured to give the exact scaling functions of certain excitations in certain integrable theories are of the “universal” form

$$e(r) = -\frac{r}{4\pi^2} \sum_{a=1}^{N} \int_{-\infty}^{\infty} d\theta \nu_a(\theta) \ln \left( 1 + t_a e^{-\epsilon_a(\theta)} \right),$$  \hspace{0.5cm} (2.2)

where the $r$-dependent functions $\epsilon_a(\theta)$ satisfy the equations

$$\epsilon_a(\theta) = r\nu_a(\theta) - \sum_{b=1}^{N} \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} K_{ab}(\theta - \theta') \ln \left( 1 + t_b e^{-\epsilon_b(\theta')} \right).$$  \hspace{0.5cm} (2.3)

Here the $t_a, a = 1, \ldots, N$ (that will be collectively referred to as the “type” $t$ of the TBA system) are certain roots of unity, the kernel $K$ is a symmetric matrix whose elements are even functions of $\theta$, exponentially decaying as $|\theta| \to \infty$, and the $\nu_a(\theta)$ are of the form

$$\nu_a(\theta) \in \left\{ \hat{m}_a \cosh \theta, \, \frac{1}{2}\hat{m}_a e^{\pm \theta} \right\}$$  \hspace{0.5cm} (2.4)

where the $\hat{m}_a$ are some non-negative dimensionless parameters (if non-zero, they are mass ratios in the theory).

Ground state scaling functions $e_0(r)$ always correspond to the “trivial” type choice $t = (1, \ldots, 1)$. That other choices of type may yield excitations $e(r)$ in certain theories was first proposed in [25][22][26]. Fendley [27] then showed that when the theory has some global discrete symmetry and is described by a diagonal $S$-matrix, the $e(r)$ corresponding to nontrivial types are the ground state energies of some sectors of the theory with “twisted” boundary conditions. These $e(r)$ coincide with scaling functions of excited states in the theory (or some “orbifolled” [28] version of it) with periodic boundary conditions. With the notable exception of the massless theory $A_4^{(+)}$, the procedure works only for the spontaneously broken symmetry phase of the theory, and the excited states obtained become degenerate with the ground state in infinite volume. It is not clear at present how the approach of [27] can be extended to justify the applicability of the “change of type” prescription in theories with non-diagonal $S$-matrices.

We will not elaborate here on all the specific choices of $N, t_a, \nu_a(\theta)$, and $K_{ab}(\theta)$ in (2.2)–(2.3) that are known, or conjectured, to give rise to certain scaling functions in perturbed CFTs. Rather, we will specialize to the cases we are interested in here, namely the theories $X_m^{(\pm)}$. Let $e(X_m^{(\pm)}(p,q)|r)$ denote the scaling function of the energy eigenstate in $X_m^{(\pm)}$ whose UV limit is created by the spinless primary field $\phi_{p,q}$ in the CFT $X_m$. This means [29][3], in particular, that $e(X_m^{(\pm)}(p,q)|0) = d_{p,q}^{(m)} - \frac{c}{12}$ where
$d^{(m)}_{p,q} = \frac{(p(m+1) - qm)^2 - 1}{2m(m+1)}$ is the scaling dimension of $\phi_{p,q}$ and $c_m$ is the central charge of $X_m$. Recall \cite{3} that in the CFT $A_m$ all the primary fields are spinless and each $\phi_{p,q}$ with $1 \leq q \leq p \leq m - 1$ appears exactly once.\footnote{Alternatively, we can allow all pairs $(p,q)$ with $1 \leq p \leq m - 1$ and $1 \leq q \leq m$, identifying $(p,q) \equiv (m - p, m + 1 - q)$.} The $D_m$ and $E_m$ models do not contain all of the above spinless primary fields. Instead, some primary fields $\phi_{p,q;q',q''}$ (with different left and right Kac label pairs) with nonzero spin appear, and some spinless fields are doubled (the precise field content of the $D_m$ models will be given in sect. 3.1). The ground state always corresponds to $(p,q) = (1,1) \equiv (m-1,m)$.

The equations for the ground state scaling function $e(A_1^{(-)}, (1,1)|r)$ proposed in \cite{20} are given by \cite{22} with $N = m - 2$, all $t_a = 1$, $\nu_1(\theta) = \cosh \theta$, $\nu_{a>1}(\theta) \equiv 0$, and $K(\theta) = I^{(m-2)}/\cosh \theta$ where $I^{(N)}$ is the incidence matrix of the simple Lie algebra $A_N$, i.e. $I^{(N)}_{ab} = \delta_{a,b-1} + \delta_{a,b+1}$ for $a,b = 1, \ldots, N$. The equations for $e(A^{(\pm)}_m, (1,1)|r)$ differ from the above only by the choice of $\nu_1(\theta)$ and $\nu_{m-2}(\theta)$, which are taken to be $e^\theta$ and $e^{-\theta}$, respectively. In \cite{22} we then conjectured that for $m$ even the equations for $e(A^{(\pm)}_m, (2,2)|r)$ and $e(A^{(\pm)}_m, (\frac{m}{2}, \frac{m}{2})|r)$ are obtained from those for $e(A^{(\pm)}_m, (1,1)|r)$ simply by changing the type to $t = (1, \ldots, 1, t^{p-1}_{\phi} = -1, t^{q-1}_{\phi} = -1, 1, \ldots, 1)$ and $t = (-1, \ldots, -1)$, respectively (the particular case $e(A^{(-)}_4, (2,2)|r)$ of this conjecture appeared also in \cite{24}).

In all cases the assignment of the Kac labels to the solutions of the corresponding TBA-like equations was justified by an analytic calculation of their $r \to 0$ limits. Further strong support for the conjecture that the solutions give the correct scaling functions was provided by numerical calculations and comparison with CPT in the cases $m = 4, 5, 6, 7$ \cite{22} (such analysis of the ground state for $m = 4$ was first performed in \cite{20} \cite{21}). Specifically, CPT based on an action of the form \cite{14} predicts the small $r$ expansion of the scaling functions

$$e(r) = e(0) + \sum_{n=1}^{\infty} a_n r^{ny} + (\text{term(s) nonanalytic in } r^y, \text{ possibly}), \quad (2.5)$$

where $y$ is the RG eigenvalue of the perturbing field and the \textit{CPT coefficients} $a_n$ can be written as integrated critical correlators (see eq. (3.6) below). In all cases of integrable relevant perturbations of nontrivial CFTs studied so far, the possible non-analyticity in $r^y$ turns out to be given by a single term. This term is either proportional to $r^2$, if 2 is not an integral multiple of $y$, or to $r^2 \ln r$ otherwise. An $r^2 \ln r$ term arises when integrated
correlators leading to some \(a_n\) (namely with \(n = 2/y\)) still diverge after regularization of their UV divergences — required whenever \(y \leq 1\) — by means of analytic continuation in \(y\). Both of these terms can be understood as arising from the RG mixing of the perturbing field with the identity operator. The precise coefficients of these terms are determined by a renormalization condition, say \(\lim_{R \to \infty} E_0(R) = 0\), which is automatically enforced in the TBA calculation. No matter what the form of the non-analytic term is, it is clear that it is the same for all scaling functions in a given model. It has been determined analytically from the TBA equations in many cases. (On the other hand, the part of the small \(r\) expansion (2.5) which is regular in \(r^y\) cannot be derived analytically within the TBA approach by present methods. See [30], however, for arguments making its appearance plausible starting from the TBA equations.)

An important consequence of the CPT analysis is that the scaling functions \(e^{(\pm)}(r)\) in the perturbed theory with \(\lambda = \pm|\lambda|\), respectively, should be related by analytic continuation \(r^y \to -r^y\), up to the above non-analytic term. This condition, implying the relation

\[
a_n^{(+)} = (-1)^n a_n^{(-)}
\]

(2.6)

between the corresponding CPT coefficients, already imposes strong constraints on scaling functions as conjectured within the TBA approach. The ultimate tests are made by comparing the explicit values of the (regularized) CPT coefficients, eq. (3.6), with the ones obtained from the TBA results. Unfortunately, it is technically very difficult in general to carry out the CPT calculations using (3.6) for more than the first one or two leading coefficients, due to the lack of convenient representations for the critical correlators involved; within the TBA approach, on the other hand, until now it has been possible to obtain the expansion coefficients only numerically (to very high accuracy, for the leading ones). Though limited, these computations proved sufficient to provide highly nontrivial consistency checks on the CPT and TBA results, and as a byproduct accurate values for \(\kappa\) of (2.1) were obtained (Table 1 summarizes the results [20] [21] [22] for the theories relevant to this paper; interestingly, the choice of mass scales in \(A_m^{(\pm)}\) that we implicitly made by taking the nonzero \(\nu_a(\theta)\) in the TBA systems to be exactly \(\cosh \theta\) and \(\frac{1}{2}e^{\pm \theta}\), respectively, leads to the same \(\kappa_m\) [21] [22] in the two theories within the numerical accuracy).

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5 See, however, the generically more powerful method of calculating the \(a_n\) (numerically) based on Hamiltonian CPT, which is discussed in [22].
Table 1: \( \kappa \), eq. (2.1), in the first few models \( A_m^{(\pm)} \). The number in parenthesis is the estimated error in the last digit given. (See (A.9) for a conjectured exact expression for \( \kappa_5 \).)

| \( m \) | \( \kappa_m \) |
|-------|-------|
| 3     | \( \frac{1}{2\pi} \) |
| 4     | 0.148695516112(3) |
| 5     | 0.130234474(2)    |
| 6     | 0.11334655(2)     |
| 7     | 0.099267(2)       |

The analysis of the IR limit \( r \to \infty \) showed [21][22] that \( e(A_m^{(\pm)}, (p,p)|\infty) = d_{p,p}^{(m-1)} - \frac{c_{m-1}}{12} \) (with \( p = 1 \) for all \( m \geq 4 \), as well as \( p = 2, \frac{m}{2} \) for even \( m \geq 4 \)), exhibiting the flows of the fields \( \phi_{p,p} \) in question in \( A_m \) to the fields with the same Kac labels in \( A_{m-1} \). These flows of fields are expected from the LG and perturbative RG analyses of \( A_m^{(\pm)} \). More generally, these methods predict [1][2] for any \( m \geq 4 \)

\[
\begin{align*}
\phi_{p,p} & \to \phi_{p,p} & \text{for } p = 1, \ldots, m-2 \\
\phi_{m-1,m-1} & \to \phi_{2,1} \\
\phi_{p+1,p} & \to \phi_{p+2,p+1} & \text{for } p = 1, \ldots, m-4 ,
\end{align*}
\]  

and in addition the perturbing field \( \phi_{1,3} \) is expected [1] to flow to the irrelevant field \( \phi_{3,1} \) (\( T\bar{T} \) in the \( m = 4 \) case [11]) in \( A_{m-1} \). The latter observation lets one hope that it is possible to describe the large \( r \) behaviour of \( A_m^{(\pm)} \) using IR-CPT, i.e. perturbation theory around the IR CFT \( A_{m-1} \) with the irrelevant perturbation \( \phi_{3,1} \) (presumably together with an infinite series of more irrelevant fields). The large \( r \) expansions of scaling functions obtained using such IR-CPT are expected to be only asymptotic, in contrast to the UV-CPT expansions around \( r = 0 \), which most probably have a nonzero radius of convergence [7][13]. Nevertheless, successful comparison [21][22] of the leading terms in these asymptotic expansions with the large \( r \) behaviour of the conjectured exact results from the TBA approach provides further evidence to the correctness of the latter.

In [22] we noticed (but at that time did not understand) the following interesting fact concerning the TBA systems obtained from those of the ground state energy \( e(A_m^{(\pm)}, (1,1)|r) \) (see above) with \( m \geq 5 \) odd by just changing the type to \( t = (-1, \ldots, -1) \).
Denote the corresponding solutions (temporarily) by \( e^{(\pm)}(m|r) \). Analytic calculation of the UV limit gives
\[
e^{(\pm)}(m|0) = d^{(m)}_{m+1, m+1} - \frac{c_{m+1}}{12}.
\]
This suggests to identify \( e^{(\pm)}(m|r) \) as \( e(A^{(\pm)}_m, (\frac{m+1}{2}, \frac{m+1}{2})|r) \). However, for the ‘+’-case this identification turns out to contradict the LG prediction of (2.7), since
\[
e^{(+)}(m|\infty) = d^{(m-1)}_{m-1, m-1} - \frac{c_{m-1}}{12}.
\]
Furthermore, numerically solving the equations for \( e^{(\pm)}(m|r) \) for \( m = 5, 7 \) we determined the small \( r^y_m \) expansion coefficients \( a^{(\pm)}_n (m) \) (see eq. (2.5)) which we present in Table 2.

| \( n \) | \( a^{(+)}_n (5) \) | \( a^{(-)}_n (5) \) | \( a^{(+)}_n (7) \) | \( a^{(-)}_n (7) \) |
|---|---|---|---|---|
| 1 | -0.056643444(3) | -0.056643449(4) | -0.042641(1) | -0.042640(1) |
| 2 | 0.0484328(4) | -0.0012546(5) | 0.03918(1) | -0.01743(2) |
| 3 | -0.020718(4) | 0.01173(1) | -0.0366(2) | 0.0100(5) |
| 4 | 0.00140(4) | 0.0006(1) | 0.017(2) | 0.008(2) |

**Table 2:** The first few coefficients in the (regular part of the) small \( r \) expansion of the functions \( e^{(\pm)}(m|r), m = 5, 7 \), conjectured to describe scaling functions of the excitations specified in eq. (2.9).

We here propose an “explanation” for the above observations. We conjecture that for \( m \) odd the correct identification of the above functions \( e^{(\pm)}(m|r) \) obtained in the TBA approach (namely the solutions of (2.2)–(2.3) with \( N = m - 2, t = (-1, \ldots, -1) \), \( K(\theta) = I^{(m-2)}/\cosh \theta \), and \( \nu(\theta) = (\cosh \theta, 0, \ldots, 0) \) for \( e^{(-)}(m|r) \), \( \nu(\theta) = (\frac{1}{2}e^\theta, 0, \ldots, 0, \frac{1}{2}e^{-\theta}) \) for \( e^{(+)}(m|r) \)) correspond to the following scaling functions:
\[
e^{(-)}(m|r) = e(A^{-}_m, (\frac{m+1}{2}, \frac{m+1}{2})|r)
\]
\[
e^{(+)}(m|r) = e(D^{(+)}_m, (\frac{m+1}{2}, \frac{m+1}{2})|r).
\]
[The superscript ‘−’ on the Kac label \((m+\frac{1}{2}, m+\frac{1}{2})\) on the second line indicates that the corresponding UV conformal state is created by the \(\mathbb{Z}_2\)-odd spinless primary field \(\phi_{m+1, m+1}^-\) in \(D_m\). This is crucial when \(m \equiv 1 \pmod{4}\), in which case \(D_m\) contains two copies of \(\phi_{m+1, m+1}^-\).]

In the following we will present evidence for the conjecture (2.9). If true, one concludes from (2.8) and the fact that the perturbing field does not break the \(\mathbb{Z}_2\) symmetry, that

\[
\begin{align*}
\hat{\phi}^{-}_{3, 3} &\to \phi_{2, 2} \quad \text{in} \quad D_5^{(+)} : D_5 \to A_4 \\
\hat{\phi}^{-}_{m+1, m+1} &\to \phi_{m-1, m-1}^- \quad \text{in} \quad D_m^{(+)} : D_m \to D_{m-1} \quad (m = 7, 11, 15, \ldots) \\
\hat{\phi}^{-}_{m+1, m+1} &\to \phi_{m-1, m-1}^- \quad \text{in} \quad D_m^{(+)} : D_m \to X_{m-1} \quad (m = 9, 13, 17, \ldots),
\end{align*}
\]

which is our concrete evidence for (1.2) when \(m\) is odd. (Using IR-CPT we will argue in sect. 3.4 that \(X_{m-1}\) in the last line of (2.10) is in fact \(D_{m-1}\). See also sect. 3.3.) Based on CPT arguments, we also claim that for the \(\mathbb{Z}_2\)-even spinless primary fields \(\phi_{p, q}\) in the model \(D_m\), one has

\[
e(D_m^{(\pm)}, (p, q)|r) = e(A_m^{(\pm)}, (p, q)|r) \quad (\phi_{p, q} \text{ even}).
\]

Similarly, for the \(\mathbb{Z}_2\)-odd fields \(\phi_{m, q}^-\) \((q = 1, \ldots, \frac{m}{2})\) in \(D_m\) with \(m\) even

\[
e(D_m^{(\pm)}, (\frac{m}{2}, q)|r) = e(A_m^{(\pm)}, (\frac{m}{2}, q)|r) \quad (m = 6, 8, 10, \ldots).\]

Assuming the LG/perturbative-RG predictions (2.7) (we do not have any TBA results for most of the scaling functions involved) we conclude that the fields in \(D_m\) corresponding to the scaling functions in (2.11)–(2.12) flow under the \(\phi_{1, 3}\)-perturbation to the fields (possibly non-primary) with the same Kac labels as in the \(A\)-flows, with the \(\mathbb{Z}_2\) symmetry resolving possible ambiguities due to doubling of fields in the \(D\)-models. To complete the description of flows of relevant fields in \(D_m\) that remain relevant in the IR CFT \(D_{m-1}\), it is natural to guess that \(\hat{\phi}^{-}_{m+3, m+1} \to \phi_{m-1, m-3}^-\) for \(m = 5, 9, 13, \ldots\), with the ‘−’ superscript moved over to the IR field when \(m = 7, 11, 15, \ldots\). These observations, and the fact that they provide support to our main claim (1.2), will become clearer once we discuss the \(D\)-models and their perturbations in more detail.
3. CPT analysis of the theories $D_m^{(±)}$

3.1. The CFTs $D_m$

The field content of the models $D_m$ is encoded in their MIPFs, written in terms of Virasoro characters \[[@5][@31][@6]. We have to distinguish between four cases:

(i) $m = 4\rho + 1$ ($\rho \geq 1$):

$$ Z = \sum_{p=1}^{4\rho} \sum_{q \text{ odd}=1}^{2\rho-1} |\chi_{p,q}|^2 + \sum_{p=1}^{2\rho} |\chi_{p,2\rho+1}|^2 + \sum_{p=1}^{2\rho} |\chi_{p,2\rho+3-q}|^2 \quad (3.1) $$

(ii) $m = 4\rho + 2$ ($\rho \geq 1$):

$$ Z = \sum_{p \text{ odd}=1}^{2\rho-1} \sum_{q=1}^{4\rho+2} |\chi_{p,q}|^2 + \sum_{q=1}^{2\rho+1} |\chi_{2\rho+1,q}|^2 + \sum_{q=1}^{2\rho+1} |\chi_{2\rho+3-q}|^2 \quad (3.2) $$

(iii) $m = 4\rho - 1$ ($\rho \geq 2$):

$$ Z = \sum_{p=1}^{4\rho-2} \sum_{q \text{ odd}=1}^{2\rho-1} |\chi_{p,q}|^2 + \sum_{p=1}^{2\rho-1} |\chi_{p,2\rho}|^2 + \sum_{q=1}^{4\rho-2} \sum_{q \text{ even}=2}^{2\rho-2} \chi_{p,q} \chi_{p,4\rho-q}^* \quad (3.3) $$

(iv) $m = 4\rho$ ($\rho \geq 2$):

$$ Z = \sum_{p \text{ odd}=1}^{2\rho-1} \sum_{q=1}^{4\rho} |\chi_{p,q}|^2 + \sum_{q=1}^{2\rho} |\chi_{2\rho,q}|^2 + \sum_{q=1}^{4\rho} \sum_{q \text{ even}=2}^{2\rho-2} \chi_{p,q} \chi_{p,4\rho-q}^* \quad (3.4) $$

This (not always economic) way of writing the MIPFs makes clear the following facts. Each term $|\chi_{p,q}|^2$ in $Z$ corresponds to (the conformal family of) a spinless primary field $\phi_{p,q}$ in the model, whereas a term $\chi_{p,q}\chi_{p,4\rho-q}^*$ always with $(p,q) \not\equiv (\overline{p},\overline{q})$ different from $(p,q)$.
corresponds to a primary field $\phi_{p,q,\overline{p},\overline{q}}$ with nonzero spin. In addition, the fields corresponding to the terms on the first (second) line in each case are $\mathbb{Z}_2$-even(odd) fields. Note that the models $A_m$ and $D_m (m \geq 5)$ have the same $\mathbb{Z}_2$-even primary fields. On the other hand, the $\mathbb{Z}_2$-odd (spinless) primary fields in $A_m$ are “replaced” by primary fields with nonzero spin in $D_m$, which also contains $[m/2]$ “extra” odd primary fields which are spinless (and lead to a doubling of fields iff $m \equiv 1$ or 2 (mod 4)). In particular, all the relevant primary fields in $D_m$ are spinless and there are $m - 2$ such even fields (excluding the trivial identity field $\phi_{1,1}$) and two odd ones; in $A_m$, on the other hand, there are $2(m - 2)$ nontrivial relevant (spinless, primary) fields, half of them even and half odd.

An important ingredient in the CPT analysis below is the structure of the operator algebra of the models $D_m$. The fusion rules for these models were given explicitly in [31]. They are essentially the $A$-series fusion rules (applied separately to the left and right Kac label pairs $(p,q)$ and $(\overline{p},\overline{q})$ appearing in (3.1)–(3.4)) intersected with the $\mathbb{Z}_2$-symmetry selection rules. On the other hand, the problem of finding the (nonvanishing) operator product expansion coefficients (OPECs) is much more involved [32] than in the corresponding $A$-models. This is the case for CFTs with non-diagonal MIPFs in general [33]. The main subtlety lies in the signs of certain OPECs in the non-diagonal models. What is clear [32] [33], however, is that if the fusion of two spinless primary fields in the non-diagonal model gives rise to primary fields that are all spinless, then the OPECs involved are identical to those in the diagonal model [7] For $D_m$, it is known [32] that the nonvanishing OPECs (of primary fields of definite $\mathbb{Z}_2$ parity) satisfy

$$\left(C^{(D_m)}_{(p_1,q_1,\overline{p}_1,\overline{q}_1)(p_2,q_2,\overline{p}_2,\overline{q}_2)(p_3,q_3,\overline{p}_3,\overline{q}_3)}\right)^2 = C^{(A_m)}_{(p_1,q_1)(p_2,q_2)(p_3,q_3)} C^{(A_m)}_{(\overline{p}_1,\overline{q}_1)(\overline{p}_2,\overline{q}_2)(\overline{p}_3,\overline{q}_3)}, \quad (3.5)$$

6 A notable exception is the 3-state Potts model $D_5$ that contains two relevant \((d = \frac{9}{5})\) primary fields with nonzero spin: $\phi_{2,1;3,1}$ and $\phi_{3,1;2,1}$.

7 More generally, if the correlator of some fields in conformal families whose ancestors are all spinless primary fields gets contributions from intermediate fields (through “factorization”) all of whose ancestors are again spinless, then this correlator is the same as in the corresponding CFT with diagonal MIPF. This follows from the uniqueness of the solution to the monodromy problem [33].
where the OPECs on the right-hand side are those of the primary fields in $A_m$. Now recall that the primary fields in $A_m$ are normalized, up to signs, by requiring that
\[ \langle \phi_{p_1,q_1}(\infty,\infty) \phi_{p_2,q_2}(0,0) \rangle = \delta_{p_1,p_2} \delta_{q_1,q_2} \] (here $1 \leq q_i \leq p_i \leq m-1$). The signs can be further fixed so that all the OPECs in $A_m$ are non-negative. Hence eq. (3.5) gives us the OPECs in $D_m$ up to signs, which turn out to be crucial for the discussion below.

3.2. UV-CPT for $D_m^{(\pm)}$

We are now ready to use CPT to study the theories $D_m^{(\pm)}$, trying to support the validity of eqs. (2.9)–(2.12) that are our evidence for the main claim (1.2). For the scaling functions $e(X_m^{(\pm)},(p,q)|r)$, CPT based on (1.1) gives the small $r$ expansion (2.5) with the CPT coefficients

\[ a_n(X_m^{(\pm)},(p,q)) = -(2\pi)^{1-y_m} (\mp \kappa_m)^n \times \int \prod_{j=1}^{n-1} \frac{d^2z_j}{(2\pi|z_j|)^y_m} \langle \phi_{p,q}(\infty,\infty) \phi_{1,3}(1,1) \prod_{j=1}^{n-1} \phi_{1,3}(z_j,\bar{z}_j) \phi_{p,q}(0,0) \rangle_{\text{conn}} \]  

(3.6)

The correlators here are connected (with respect to the “in- and out-states” created by $\phi_{p,q}$) $(n+2)$-point functions in the CFT $X_m$ on the plane. On the r.h.s. we suppressed the factor $\langle \phi_{p,q}(\infty,\infty) \phi_{p,q}(0,0) \rangle^{-1}$, which is set to 1. In particular, we will assume this standard CFT normalization for the combinations $\phi_{p,q}^{\pm}$ of definite $Z_2$-parity in the relevant D-models. Since in the perturbations of these latter models (like in all the other $A^{(\pm)}$ and $D^{(\pm)}$ theories) the $Z_2$ charge is a good quantum number also away from criticality, it makes sense to consider states in the perturbed theory whose UV limits correspond to $\phi_{p,q}^{+}$ and $\phi_{p,q}^{-}$. The problem of (UV and IR) regularization of such integrated correlators was discussed at length in [22] (see also the appendix).

Of particular importance to us will be the first CPT coefficient, which is never divergent, given simply in terms of conformal OPECs by

\[ a_1(X_m^{(\pm)},(p,q)) = \pm (2\pi)^{1-y_m} \kappa_m C^{(X_m)}_{(p,q)(1,3)(p,q)} \]  

(3.7)

Note that it always vanishes for the ground state (in the unitary theories under consideration) corresponding to $(p,q) = (1,1)$.

Let us now prove eq. (2.11). This equation is a consequence of the fact that all spinfull primary fields in the $D$-models are $Z_2$-odd and therefore cannot appear in the fusion of
two \( \mathbb{Z}_2 \)-even fields. It is therefore clear (cf. footnote 7) that the CPT coefficients involved are identical, including signs, in \( D_m^{(\pm)} \) and \( A_m^{(\pm)} \). Analytic continuation of the small \( r \) CPT expansion to all \( r \) proves (2.11). Actually we see from the above argument, which did not use the fact that the \( \mathbb{Z}_2 \)-even fields \( \phi_{p,q} \) involved are primary, that eq. (2.11) generalizes to the statement that the whole \( \mathbb{Z}_2 \)-even sectors of the finite-volume spectra of \( D_m^{(\pm)} \) and \( A_m^{(\pm)} \) are identical.

Similarly, (2.12) follows from the fact that in the fusion (in \( D_m, m \) even)

\[
[\phi_{1,3}] \times [\phi_{-\frac{m}{2},q}] = [\phi_{-\frac{m}{2},q-2}] + [\phi_{-\frac{m}{2},q}] + [\phi_{-\frac{m}{2},q+2}]
\]

(3.8)

no primary fields of nonzero spin can appear on the r.h.s. (there are no such fields for which the first entries of the left and right Kac label pairs are both \( \frac{m}{2} \)). Again, this implies that the relevant CPT coefficients in \( A_m^{(\pm)} \) and \( D_m^{(\pm)} \) are identical (this is true also for all the scaling functions corresponding to descendants of the primary fields \( \phi_{-\frac{m}{2},q} \)); the fact that \( \phi_{-\frac{m}{2},q} \) is \( \mathbb{Z}_2 \)-even in \( A_m \) if \( m \equiv 2 \) (mod 4) does not matter. Note that in this latter case (2.11)–(2.12) imply, in particular, that the \( \mathbb{Z}_2 \)-doublet \( \phi_{-\frac{m}{2},q}^{\pm} \) in \( D_m \) flows to the doublet with the same Kac label in \( D_{m-1} \).

Now to (2.9)–(2.10), our most interesting claim. Note first of all that in the conjectured flow \( \phi_{\frac{m+1}{2},\frac{m+1}{2}} \rightarrow \phi_{\frac{m-1}{2},\frac{m-1}{2}} \) (with appropriate ‘−’ superscripts as in (2.10)) in \( D_m \rightarrow D_{m-1}, m \) odd, the field becomes more relevant during the flow, i.e. its scaling dimension decreases:

\[
d^{(m-1)}_{\frac{m-1}{2},\frac{m-1}{2}} - d^{(m)}_{\frac{m+1}{2},\frac{m+1}{2}} = -\frac{1}{4m} \frac{m^2 + 3}{m^2 - 1} < 0 .
\]

(3.9)

This is in contrast to all the known flows of fields in the \( A \)-series, where fields become less relevant (the fact that this feature of the \( A \)-series flows, reminiscent of the decrease of the central charge along unitary RG trajectories [1], is not universal, was already noted in [11]).

The qualitative difference can be seen already in the first order of the small \( r \) expansion of the relevant scaling function: Remember our empirical observation concerning the first expansion coefficients of the “scaling functions” \( e^{(\pm)}(m|r) \) of the type \( t = (-1,\ldots,-1) \) TBA systems with \( m \) odd. Assuming (2.9) we can now state it equivalently as

\[
a_1(D_m^{(+)},(\frac{m+1}{2},\frac{m+1}{2}))^- = + a_1(A_m^{(-)},(\frac{m+1}{2},\frac{m+1}{2})) \quad (m \text{ odd})
\]

\[
= - a_1(A_m^{(+)},(\frac{m+1}{2},\frac{m+1}{2})),
\]

(3.10)
where the ‘−’ superscript on the l.h.s. can be ignored in the case \( m \equiv 3 \pmod 4 \). Actually we found [22] the first equality (the second is just a particular case of (2.10)) only for \( m = 5, 7 \), but in the meantime also have checked it for \( m = 9, 11 \), and believe that it holds for all odd \( m \).

What is the origin of this difference between the flows in the \( D \) and the \( A \) series from the point of view of CPT? We claim that for \( m \equiv 3 \pmod 4 \) the crucial fact, proving (3.10), is

\[
C^D_{(m+1, m+1)}(1, 3)(m+1, m+1) = -C^A_{(m+1, m+1)}(1, 3)(m+1, m+1), \quad m \equiv 3 \pmod 4. \tag{3.11}
\]

The crucial sign difference is consistent with the fact that now spinful fields appear in the fusion \([\phi_{1,3}] \times [\phi_{m+1, m+1}]\). Note that sign differences between OPECs like the one in (3.11) have a less trivial effect on higher CPT coefficients, so that for \( n > 1 \) we do not expect any simple relations like (3.10). This is consistent with our numerical results — cf. Table 2.

For \( m \equiv 1 \pmod 4 \), on the other hand, the crucial fact is, we claim,

\[
C^D_{(m+1, m+1)}(1, 3)(m+1, m+1) = -C^D_{(m+1, m+1)+1, 3}(m+1, m+1), \quad m \equiv 1 \pmod 4. \tag{3.12}
\]

This allows the scaling dimension to decrease in the flow \( \phi_{m+1, m+1} \to \phi_{m+1, m+1} \) in \( D_{m+1}^+ \), whereas in the flow \( \phi_{m+1, m+1}^+ \to \phi_{m+1, m+1}^\pm \) the scaling dimension increases, as in the corresponding \( A \)-series flow.

For \( m = 5 \) we can use the \( \mathbb{Z}_3 \) symmetry of the 3-state Potts model \( D_5 \) to easily prove the first equality in (3.12) (note that the second equality holds in general, according to our remarks in sect. 3.1). Under this \( \mathbb{Z}_3 \)-symmetry \( \phi_{3,3} \equiv \frac{1}{\sqrt{2}}(\phi_{3,3}^+ + i\phi_{3,3}^-) \) and \( \phi_{3,3}^* \equiv \frac{1}{\sqrt{2}}(\phi_{3,3}^+ - i\phi_{3,3}^-) \) form a doublet of oppositely charged fields. Recall that the perturbation is by \( \phi_{1,3}^+ = \frac{1}{\sqrt{2}}(\phi_{1,3} + \phi_{1,3}^\star) \), which implies

\[
C^D_{(3, 3)\pm}(1, 3)^\pm = \pm \frac{1}{2\sqrt{2}} C^D_{\phi_{3,3}^\star \phi_{3,3}^*, \phi_{1,3} + \phi_{1,3}^\star, \phi_{3,3} \pm \phi_{3,3}^*} = \pm \frac{1}{\sqrt{2}} C^D_{\phi_{3,3}, \phi_{1,3}, \phi_{3,3}^*}, \tag{3.13}
\]

8 Unless we misunderstand the results of [32], our (3.11) disagrees with them when \( m \equiv 3 \pmod 8 \). The same remark applies to our (3.12), now with \( m \equiv 1 \pmod 8 \).
where we used the linearity of the OPECs and in the last step also the conservation of the \( Z_3 \) charge and the \( Z_2 \) symmetry (we choose to assign the same \( Z_3 \) charge to \( \phi_{3,3} \) and \( \phi_{1,3} \)). This proves (3.12) for \( m = 5 \).

[It is interesting to explicitly see how different the consequences of the \( Z_3 \) symmetry are in the theory \( D_6(\pm) \). (\( D_6 \), the tricritical 3-state Potts CFT \[3\], and \( D_5 \) are the only \( Z_3 \)-symmetric models among the \( X_m \).) First, note that there is a single \( Z_3 \)- (and \( Z_2 \)-)neutral field \( \phi_{1,3} \) in \( D_6 \), in contrast to \( D_5 \), so that both the \( Z_2 \) and the \( Z_3 \) symmetries are preserved in the perturbed theories \( D_6(\pm) \). (This fact by itself shows that \( D_6(+) \) cannot possibly flow to \( A_5 \), as \( A_5 \) is not \( Z_3 \)-symmetric.) Now consider the first CPT coefficients relevant for the flow of the doublet \( \phi_{3,3}^{\pm,3} \rightarrow \phi_{3,3}^{\pm,3} \) in \( D_6(+) \). Here, in contradistinction to (3.13), conservation of the \( Z_3 \) charge gives

\[
|C^{(D_6)}_{\phi_{3,3}, \phi_{1,3}, \phi_{3,3}, \phi_{3,3}}| = \frac{1}{2} |C^{(D_6)}_{\phi_{3,3}^*, \phi_{1,3}, \phi_{3,3}, \phi_{3,3}^*}| , \quad (3.14)
\]

implying that \( a_1(D_6^{(+)}, (3,3)^+) = a_1(D_6^{(+)}, (3,3)^-) \) as expected from other considerations discussed earlier.]

3.3. RG-improved CPT

To gain direct information about the IR fixed point from standard UV-CPT on the cylinder, as discussed in the last subsection, would require analytic continuation of the small \( r \) expansion to large \( r \). This is generically not possible in practice. However, one can sometimes directly “see” the IR fixed point by using the RG to (partially) sum up perturbation theory. This assumes, of course, that one knows the \( \beta \)-function quite accurately up to its IR zero. In the present context this restricts the applicability of this approach to large \( m \), where the IR fixed point is close to the UV one (as measured by the renormalized coupling, not the bare coupling \( \lambda \) of sect. 3.2 in terms of which the IR fixed point is at \( \lambda = \infty \)). This method has been discussed in detail in the literature \[1\][2][13], so there is no need to review it. We just summarize the results relevant for us.

Consider the RG flow induced by perturbing a CFT of central charge \( c \) by a slightly relevant \( (y \ll 1) \) operator \( \phi \), which up to irrelevant operators closes on itself and the identity operator under repeated fusions. The central charge \( c' \) at the new IR fixed point (attained by choosing the appropriate sign for the perturbing term) then satisfies

\[
c' - c = -\frac{y^3}{(C_{\phi\phi\phi})^2} + O(y^4) . \quad (3.15)
\]
For a field \( \phi_\alpha \) which to leading order does not mix with any other field (like the fields \( \phi_{p,p} \) on the “diagonal” of the Kac table of a minimal model) the change in scaling dimension is

\[
d'_{\alpha} - d_{\alpha} = \frac{2C_{\phi_\alpha \phi_\alpha}}{C_{\phi_\phi}} y + \mathcal{O}(y^2) .
\]

(3.16)

In the case we are interested in \( \phi = \phi_{1,3} \), \( y = \frac{4}{m+1} \), \( C_{\phi_\phi} = \frac{4}{\sqrt{3}} + \mathcal{O}(\frac{1}{m}) \) [1][2], so that \( c' - c = -\frac{12}{m} + \mathcal{O}(\frac{1}{m}) \), consistent with a flow \( X_m \to X_{m-1} \). For \( \phi_\alpha = \phi_{p,p} \) with \( p \leq \frac{m}{2} \) odd the OPECs \( C_{\phi_\alpha \phi_\alpha} = \frac{p^2 - 1}{2\sqrt{3m^2}} + \mathcal{O}(\frac{p^2}{m^3}) \) [1] are identical in \( A_m \) and \( D_m \), showing the existence of flows \( \phi_{p,p} \to \phi_{p,p} \) in \( X_m \to X_{m-1} \). These flows do not allow us to directly distinguish between, say, \( D_m \to D_{m-1} \) and \( D_m \to A_{m-1} \) (although it is clearly hard to see where the fields \( \phi_{p,p} \), \( p \) even, in \( A_{m-1} \) should come from!).

For \( m \equiv 3 \) (mod 4), however, the flow of the \( \mathbb{Z}_2 \)-odd field \( \phi_\alpha = \phi_{m+1, m+1} \) does enable us to directly distinguish the two possibilities for flows starting from \( D_m \). Depending on the sign of \( C_{\phi_\alpha \phi_\alpha} = \pm(\frac{1}{8\sqrt{3}} + \mathcal{O}(\frac{1}{m})) \) we now have

\[
d'_{\alpha} - d_{\alpha} = \pm \frac{1}{4m} + \mathcal{O}(\frac{1}{m^2}) ,
\]

(3.17)

indicating the flow of scaling dimensions \( d^{(m)}_{m+1, m+1} \to d^{(m-1)}_{m+1, m+1} \) or \( d^{(m)}_{m+1, m+1} \to d^{(m-1)}_{m-1, m-1} \), respectively. According to our claim (3.11) the second possibility holds for \( D_m^{(+)}, \) which is only possible if \( D_m \to D_{m-1} \) (because of the \( \mathbb{Z}_2 \) symmetry, as there is a single field \( \phi_{m-1, m-1} \) — which is \( \mathbb{Z}_2 \)-even — in \( A_{m-1} \)).

For \( m \equiv 1 \) (mod 4) the analogous sign difference (3.12) again indicates the flow \( \phi^{(+)\pm}_{m+1, m+1} \to \phi^{(+)\pm}_{m-1, m-1} \) in \( D_m^{(+)}. \) But now, unfortunately, this flow alone does not allow us to decisively conclude that the IR CFT is \( D_{m-1} \). In fact, for \( m = 5 \) we know this cannot be the case. To show that \( D_m \to A_{m-1} \) is not possible for \( m = 9, 13, 17, \ldots \) requires the study of flows of other fields, not on the “diagonal” of the Kac table, but we will not pursue such studies here. Intuitively, the fact that there are about half as many relevant fields in \( D_m \) than in \( A_{m-1} \), for \( m \gg 1 \), makes the flow between these two theories highly implausible. A more explicit version of this argument will be offered in sect. 3.4.

For \( m \) even the OPEC \( C_{\phi_\alpha \phi_\alpha}, \phi_\alpha = \phi^{(+)\pm}_{\frac{m}{2}, \frac{m}{2}} \), in \( D_m \) (with superscripts ‘\( \pm \)’ for \( m \equiv 2 \) (mod 4)) is identical to the corresponding one in \( A_m \). Therefore \( \phi^{(+)\pm}_{\frac{m}{2}, \frac{m}{2}} \in D_m, m \equiv 2 \) (mod 4), must flow to fields \( \phi^{(+)\pm}_{\frac{m-2}{2}, \frac{m-2}{2}} \) in \( X_{m-1} \), which is only possible if \( X_{m-1} = D_{m-1} \). For \( m \equiv 0 \) (mod 4) there is only one (\( \mathbb{Z}_2 \)-odd) field \( \phi^{(+)\pm}_{\frac{m}{2}, \frac{m}{2}} \), and so this argument cannot be applied. As in the case of \( m \equiv 1 \) (mod 4), it is necessary to study the flow of other fields to conclude within the RG-improved CPT approach that \( D_m \to A_{m-1} \) is not possible.
3.4. **IR-CPT for \(D_m^{(+)})**

We finally discuss what can be learned about the RG flows we are considering from CPT around the IR fixed point. The basic idea is that the action

\[
A_{A_{m-1}} + \int d^2x \left[ g \phi_{3,1}(x) + \ldots \right], \quad m \geq 5 \tag{3.18}
\]

can be used to study perturbatively the IR asymptotics of the theory \(A_m^{(+)})\), in particular \([21][22]\) the large \(r\) behaviour of the energy scaling functions in this theory. The leading nontrivial term in these scaling functions will generically be proportional to \(r^{-\frac{4}{m-1}}\), since \(d_{3,1}^{(m-1)} = 2 + \frac{4}{m-1}\). The ‘\ldots’ in the integrand refers to a presumably infinite series of fields more irrelevant than \(\phi_{3,1}\). (Note that the perturbation theory we are considering here corresponds to a non-renormalizable interaction in standard Lagrangian QFT.) Given the first (or perhaps more, see below) terms in \(3.18\), the infinite and finite parts of the counterterms written as ‘\ldots’ are determined, in principle, by two conditions: a) That they make \(E_0(R)\), say, finite order by order in perturbation theory, and b) that the presumably asymptotic large \(r\) series is consistent with the TBA and/or the (analytic continuation of the small \(r\)) UV-CPT series. In \([21][22]\) it was shown that the first few terms (independent of ‘\ldots’) in the large \(r\) expansion of certain scaled energies for \(m = 4, 5, 6, 7\), calculated from \(3.18\) as in UV-CPT, agree with those obtained in the TBA approach.

We will now examine what \(3.18\) and its generalization to the \(D\)-flows can teach us. For \(m \geq 6\) we should consider, according to \([1.2] and [2.11]\), the action \(3.18\) with \(A_{m-1}\) replaced by \(D_{m-1}\) (and \(\phi_{3,1}\) taken to be \(\phi_{3,1}^{+}\) when \(m = 7\); note that in this case the IR perturbation breaks the \(Z_3\) symmetry of \(D_6\), which explains the accidental — from the point of view of UV-CPT — appearance of this symmetry at the IR limit of the theory \(D_7^{(+)})\). The comparison between the “backward flows” induced in the \(D\)-series and those in the \(A\)-series essentially parallels the previous discussion of UV-CPT. Again, the conclusions are that flows of certain fields are the same in the two series (provided the IR perturbation is identical), whereas for other flows (e.g. \(2.10\)) the difference is consistent with differences in the operator algebras of the models \(D_{m-1}\) and \(A_{m-1}\).

9 A similar situation occurs in the flows of \(A_5\) and \(D_5\) to \(A_4\), the (superconformal) tricritical Ising CFT. There the IR perturbation \(\phi_{3,1}\), the bottom rather than the top component of (the determinant of) the super-stress-energy tensor, explicitly breaks the supersymmetry.
The most interesting case is $m = 5$, on which we now elaborate. Here IR-CPT seems to present us with a little puzzle: If we are correct in our claim that $A_4$ is the common IR limit of both $A_5^{(+)}$ and $D_5^{(+)}$, and moreover that the IR fixed point is approached in both cases along the direction of $\phi_{3,1}$, what is the difference in the IR perturbations of $A_4$ that leads to the two different “backward flows” to $A_5$ and $D_5$? Recall that by UV-CPT we concluded that the $\mathbb{Z}_2$-even sector of the spectrum of the two theories is identical, so that the IR limits of UV fields in the conformal families of $\phi_{p,1}$ ($p = 1, 2, 3, 4$) and $\phi_{p,3}^+$ ($p = 1, 3$) in $D_5$ and $A_5$ (the superscript ‘+’ being redundant in the latter) is the same. In particular, by (2.7) we have for the relevant $\mathbb{Z}_2$-even UV fields $\phi_{1,1} \rightarrow \phi_{1,1}, \phi_{3,3}^+ \rightarrow \phi_{3,3}, \phi_{2,1} \rightarrow \phi_{1,3},$ and $\phi_{1,3}^+ \rightarrow \phi_{3,1}$. However, the $\mathbb{Z}_2$-odd sector is necessarily different: Our main TBA result (2.3) implies that $\phi_{3,3}^- \rightarrow \phi_{2,2}$ in $D_5^{(+)}$ whereas (2.7) predicts $\phi_{2,2} \rightarrow \phi_{2,2}$ in $A_5^{(+)}$; (2.7) also predicts $\phi_{1,2} \rightarrow \phi_{2,1}$ in $A_5^{(+)}$, and our guess for the “corresponding” flow in $D_5^{(+)}$ is $\phi_{1,3}^- \rightarrow \phi_{2,1}$. We also believe that all the other $\mathbb{Z}_2$-odd fields in $D_5$, including the primary spinfull ones, flow to descendants in the families $[\phi_{2,2}]$ and $[\phi_{2,1}]$ in $A_4$, the UV origin of these families in $A_5$ being $[\phi_{p,2}]$ ($p = 1, 2, 3, 4$).

Trying to understand the difference in the IR perturbations leading to $D_5^{(+)}$ and $A_5^{(+)}$, the first thought that comes to mind is to blame the unspecified ‘…’ for this difference. However, this would imply that the $\mathcal{O}(r^{-1})$ corrections to the IR limits of all the energy scaling functions in the two theories are the same, whereas based on the flows described above one would expect this only for the $\mathbb{Z}_2$-even sector! We would like to suggest that actually the IR perturbations are already different in the first $\phi_{3,1}$ term, the difference being the sign of the coupling $g$. [Note that this sign difference is also the only way to avoid the following potential contradiction: If the leading IR perturbation were the same, then conditions a) and b) mentioned earlier and the equality of $E_0(R)$ in $A_5^{(+)}$ and $D_5^{(+)}$ would imply that these are described by exactly the same IR perturbation of $A_4$, i.e. are the same theories, which is wrong. There is however one caveat, namely, if (3.18) contains terms of the form $g^\alpha \int d^2 x \tilde{\phi}(x), \alpha > 1$, which are distinct for $A_5^{(+)}$ and $D_5^{(+)}$, and $\tilde{\phi}$ is such that it does not appear in the (repeated) fusion of $\phi_{3,1}$ with itself. We consider the existence of such terms unlikely, but do not really have an argument to exclude them.]

Consider, in fact, the perturbative expansion of scaling functions based on (3.18) for $m = 5$ with the ‘…’ ignored, namely $A_{A_4} + g \int \phi_{3,1}$. The resulting CPT coefficients are of the form (3.6) with $\phi_{1,3}$ replaced by $\phi_{3,1}$, $y_m$ by $2 - d_{3,1}^{(4)} = -1$, and $\kappa_m$ by $\kappa_{ir}$. Now as already noted in [22], in $A_4$ $\phi_{3,1}$ is the same as $\phi_{1,4}$, the latter being a $\tilde{\mathbb{Z}}_2$-odd member of the $(1, q)$-operator subalgebra of the model. (This $\tilde{\mathbb{Z}}_2$ symmetry, corresponding to the...
self-duality of the tricritical Ising lattice model, has nothing to do with the $\mathbb{Z}_2$ symmetry discussed so far in the paper, which corresponds to “spin reversal”. In particular, the $(1, q)$-operator subalgebra of $A_4$, containing both $\hat{Z}_2$-even and $\hat{Z}_2$-odd fields, constitutes the whole $\mathbb{Z}_2$-even sector of the model.) Therefore all correlators $\langle \phi \prod_{i=1}^{n} \phi_{i,4} \phi \rangle$ with a $\mathbb{Z}_2$-even field $\phi$ (i.e. of definite $\hat{Z}_2$-charge) and $n$ odd vanish in $A_4$, and so do the corresponding CPT coefficients. As a result, the whole perturbative expansion of scaling functions corresponding to the $\mathbb{Z}_2$-even IR fields is independent of the sign of $g$, and is in fact in powers of $r^{-2}$ (modulo an $r^{-2} \ln r$ term, see below). On the other hand, for the $\mathbb{Z}_2$-odd IR fields in the conformal families of $\phi_{2,2}$ and $\phi_{2,1}$ (not belonging to the $(1, q)$ subalgebra, i.e. not being $\hat{Z}_2$ eigenstates) the sign of $g$ does matter.

Of course the above arguments have to be generalized to include additional IR perturbing fields. [One may try to exploit our observations regarding the finite-volume spectra of the two theories $X_5^{(+)}$ when trying to determine these additional perturbations. For example, allowed perturbing fields are all the spinless descendants of $\phi_{3,1}$, whose RG eigenvalues are necessarily odd negative integers, and all spinless descendants of the $\hat{Z}_2$-even identity field, whose RG eigenvalues are even negative integers.] Still, they are very suggestive and make us believe that in the (conjectured) RG flows from $A_5$ and $D_5$ to the common IR fixed point $A_4$, the latter is approached from exactly opposite directions.

An important lesson of the above discussion is the following. For $m \geq 6$ the $n$-point functions of the IR-perturbing field $\phi_{3,1}$ in $X_{m-1}$ do not vanish for all odd $n$. Hence, consistency with the UV-CPT observation that the $\mathbb{Z}_2$-even sectors of $A_m^{(+)}$ and $D_m^{(+)}$ have the same finite-volume spectrum, seems to require that the IR perturbation of $X_{m-1}$ leading to $A_m^{(+)}$ and $D_m^{(+)}$ is the same, also in sign. Therefore, $A_m^{(+)} : A_m \rightarrow A_{m-1}$ and $A_m^{(+)} \neq D_m^{(+)}$ clearly imply that $D_m$ flows to $D_{m-1}$, not $A_{m-1}$, for $m \geq 6$. By the same argument, the CFT at the IR limit of $E_m^{(+)}$, $m = 12, 18, 30$, must be $E_{m-1}$.

To conclude this section we briefly describe results of the TBA approach relevant to the IR behaviour of $X_5^{(+)}$. In this case, the energy scaling functions for which integral equations have been conjectured are $e(A_5^{(+)}, (1, 1)|r) = e(D_5^{(+)}, (1, 1)|r)$ and $e(D_5^{(+)}, (3, 3)^{-}|r)$. Solving numerically the TBA equations, we found \[22\] for $r \gg 1$

$$e(X_5^{(+)}, (1, 1)|r) = -\frac{7}{120} - 0.02723(2) \, r^{-2} \ln r + 0.0173(2) \, r^{-2} + \ldots$$  \hspace{1cm} (3.19)

for the ground state, and for the excitation describing the flow $\phi_{3,3} \rightarrow \phi_{2,2}$ in $D_5^{(+)}$ we estimate

$$e(D_5^{(+)}, (3, 3)^{-}|r) = \frac{1}{60} + 0.03333(1) \, r^{-1} + \ldots.$$  \hspace{1cm} (3.20)
These fits are based on numerical results for the scaling functions evaluated at $200 \leq r \leq 600$.

We should emphasize that in general it is much more difficult to perform the fits of the TBA scaling functions at large $r$, leading in our case to (3.19)–(3.20), than those of the small $r$ dependence leading to UV-CPT coefficients (Table 2, for instance). We were therefore able to determine with some confidence only the leading (asymptotic) expansion terms indicated in (3.19)–(3.20). Still, the basic features predicted by IR-CPT based on (3.18) are already noticeable, namely the expansion in powers of $r^{-1}$. The presence of the $r^{-2}\ln r$ term in (3.19) indicates the divergence of the second CPT coefficient (multiplying $r^{-2}$) even after its UV regularization via analytic continuation in $y_{\text{IR}} = 2 - d_{3,1}^{(4)}$. As in UV-CPT, the same non-analytic in $r^{y_{\text{IR}}}$ term is expected to appear in the large $r$ expansions of all the scaling functions in the model, in particular in (3.20). Unfortunately, the accuracy of our numerical results for $e(D_{5}^{(+)}, (3, 3)|r \gg 1)$ does not allow us to test this prediction.

There is however one intriguing observation we can make based on (3.19)–(3.20), which can be taken (with a grain of salt) as a consistency check of these results. First, using (3.7) with $C_{(2,2)(3,1)(2,2)}^{(A_{4})} = \frac{1}{30}$ [34] we extract from the $\mathcal{O}(r^{-1})$ term in (3.20) the value $\kappa_{\text{IR}} = 14\pi^{-2} \cdot 0.03333(1) = 0.04728(2)$. We now extend the observations made in the appendix regarding logarithmic terms in UV-CPT to the case at hand, using the recipe (A.4) to treat the logarithmic divergence of the leading IR-CPT expansion term of the ground state. The divergence is now due to the simple pole at $y = -1$ in eq. (A.1), whose residue is $R_{0} = -\pi^{4}\kappa_{\text{IR}}^{2}/16$. Eq. (A.4) then expresses the coefficient of the $r^{-2}\ln r$ term in (3.19) as $B_{\text{IR}} = \alpha R_{0}$, and from the numerically obtained values of $\kappa_{\text{IR}}$ and $B = -0.02723(2)$ we compute $\alpha = 2.001(3)$. After reading the appendix, the reader will hopefully believe that in fact $\alpha = 2$, which leads to the (conjectured) exact relation

$$B_{\text{IR}} = -\frac{\pi^{4}\kappa_{\text{IR}}^{2}}{8}. \quad (3.21)$$

4. Discussion

We have presented evidence for the existence of RG flows between members of the $D$-series of minimal unitary CFTs. These flows are induced by perturbation of the UV CFT by the least relevant (spinless) field in the model, a perturbation that preserves the integrability of the theory away from criticality. We have studied the explicit flow of scaling dimensions from the UV to the IR CFT for various operators. For the RG flows $D_{5} \to A_{4}$
and $A_5 \to A_4$ we have presented strong evidence that they approach $A_4$ from exactly opposite directions.

If there is one general lesson to be learned from this and other related studies, we think it is this: In trying to understand integrable QFTs interpolating between different RG fixed points, the study of the finite-volume spectrum can be a very powerful tool. The small- and large-volume behaviour of the spectrum gives information about the UV and IR fixed points. It can be studied with a variety of analytical and numerical techniques, like the TBA, UV- and IR-CPT (as well as the “truncated conformal space approach” which we have not utilized here). Admittedly, “TBA” integral equations (in particular for excited states) often involve some guesswork, but the structural regularities observed in the known TBA equations make such guesses quite natural, in many cases. The TBA approach is non-perturbative and the equations for excited states show quantitatively how certain fields flow from the UV all the way to the IR; we think this is at least as interesting as the LG analyses (which in many cases are only qualitative, at best) and the perturbative RG calculations which only apply if the theory is “close” to a model that can be expressed in terms of (compactified, perhaps) free fields.

We would like to comment on an interesting consequence of (2.9) regarding the “staircase model” of Al. Zamolodchikov. This model is given by a one-parameter ($\theta_0$) family of diagonal $S$-matrix theories (the status of the underlying QFTs is not clear) describing the scattering of a single particle. The TBA analysis of the model shows that as the parameter $\theta_0$ gets larger, the ground state energy scaling function approximates better and better all the functions $e(A_m^{(+)}|r)$, $m = 3, 4, 5, \ldots$, “stringed together” (see for the precise meaning of this statement). Therefore in the limit $\theta_0 \to \infty$ the model describes, in some sense, the whole RG trajectory of flows between the unitary minimal models starting at $m = \infty$ ($c_{uv} = 1$) and going all the way down to the Ising CFT $m = 3$ and finally to the trivial massive IR fixed point ($c_{ir} = 0$).

In [27] the TBA equations for the lowest energy level of the model in the sector of antiperiodic boundary conditions were considered. Using the results of [22] the corresponding scaling function was seen to approximate (as $\theta_0 \to \infty$) the whole set of $e(A_m^{(+)}|r)$ for $m$ even and $e^{(+)}(m|r)$ for $m$ odd, stringed together (in an alternating pattern). As was pointed out in [27], this does not approximate a sequence of excitations in the $A$-series flow. But now having the new insight (2.9), our interpretation is that the sequence of excitations involved is in the $D$-series, describing the flow of spin fields $\ldots \to \phi_{2\rho+1,2\rho+1} \to \phi_{2\rho,2\rho} \to \phi_{2\rho,2\rho} \to \phi_{2\rho-1,2\rho-1} \to \phi_{2\rho-1,2\rho-1} \to \ldots \to \phi_{3,3} \to \phi_{2,2} \to \phi_{2,2} \quad \text{in} \quad \ldots \to D_{4\rho+1} \to
$D_{4\rho} \to D_{4\rho-1} \to D_{4\rho-2} \to D_{4\rho-3} \to \ldots \to D_5 \to A_4 \to A_3$, respectively, and finally in the last step the relevant excitation is $e(A_3^{(-)}, (2,2)|r)$ that becomes degenerate with the ground state $e(A_3^{(+)}, (1,1)|r) = e(A_3^{(-)}, (1,1)|r)$ in the IR limit $r \to \infty$.

The significance of this observation is not yet clear. In particular, does it indicate that the staircase model actually interpolates (asymptotically) between all the CFTs in the $D$-series rather than in the $A$-series (recall that the finite-volume ground state energy is the same in the $A$- and $D$-flows)? Or, does it indicate the existence of an “orbifolded staircase model” approximating the $D$-flow, with the modified TBA equations discussed above describing an excitation in this model? Obviously, to answer these questions a better understanding of the staircase model itself is required.

Related to this, the reader might wonder why the simplest possible integral equation for an excited state in $X_{m}^{(+)}$, namely the type $t = (-1, \ldots, -1)$ system discussed in sect. 2, describes an excitation in $D_{m}^{(+)}$ and not $A_{m}^{(+)}$ when $m$ is odd (for $m$ even the excitation energies in question coincide in the theories $A_{m}^{(+)}$ and $D_{m}^{(+)}$). We do not have a satisfactory explanation of this at present, it is basically an “empirical” observation. However, we note that in the study \cite{28} of off-critical orbifolds of the lattice models related to $A_{m}^{(\pm)}$ a fundamental difference was observed between the cases of $m$ even and $m$ odd\cite{10} (specifically, applying the “orbifolding” procedure \cite{28} to the lattice models of $A_{m}^{(\pm)}$ leads to a model in the $D$-series, namely $D_{\frac{m+3}{2}}^{(\pm)}$, only when $m \geq 5$ is odd). We think that this work, together with \cite{27}, might be helpful in understanding the issue.

**Note added**

While putting the final touches on this paper we received \cite{39}, which presents a perturbative argument for flows between minimal models high up ($m \gg 1$) in the $D$-series. It is based on \cite{10}, where the difference of the torus partition functions of $A_{m}^{(\pm)}$ at its UV and IR fixed points is calculated to leading order in $1/m$. The fact that flows in the $D$-series can be distinguished from those in the $A$-series already to leading order, turns out to follow essentially from the fact (cf. our sect. 3) that there are half as many spinless primary fields in $D_{m}$ than in $A_{m}$, for large $m$. To leading order, at least, it is not possible to follow the flow of individual operators using this method.

\footnote{10 We thank P. Fendley for this comment.}
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Appendix A. Logarithmic terms in CPT

We present some observations regarding logarithmic terms in the small $r$ expansion of energy scaling functions in certain perturbed CFTs. As mentioned in sect. 2, the appearance of such terms is indicated within the framework of CPT by the divergence, even after analytic continuation in $y$ (the RG eigenvalue of the perturbing field), of certain CPT coefficients in the perturbative expansion in powers of $r^y$. It seems that CPT is completely useless in such cases, in particular there is no way to predict the form of the resulting non-analyticity in $r^y$ in the true non-perturbative answer. However, by considering cases where exact non-perturbative information is available, we empirically arrive at a recipe which — though incomplete, as we shall see — allows one to obtain some interesting (conjectured) results.

Consider a generic perturbed (unitary) CFT defined by an action of the form (1.1), with a single perturbing field $\phi$ of RG eigenvalue $0 < y < 2$ replacing $\phi_{1,3}$ there. (We assume that $\phi$ is the most relevant field — except for the identity — in some subalgebra of the operator algebra of the model, so that no nontrivial fields have to be added [7] as counterterms to the perturbation when renormalizing the theory.) For the ground state energy scaling function $e_0(r)$ one can obtain the two leading CPT coefficients analytically from (3.6) (see e.g. [22]). Namely, in the CPT expansion $e_0(r) = -\frac{c}{12} + \sum_{n=1}^{\infty} a_n r^{ny}$ we have $a_1 = 0$ (cf. (3.7)) and

$$a_2(y) = -\frac{1}{4}(2\pi)^2(1-y)^2 \kappa^2 \gamma^2(1 - \frac{y}{2}) \gamma(y-1),$$  \hspace{1cm} (A.1)

$$a_3(y) = \frac{1}{48}(2\pi)^3(1-y)^3 \kappa^3 \mathcal{C}_{\phi\phi} \gamma^3(\frac{2-y}{4}) \gamma(\frac{3y-2}{4}).$$  \hspace{1cm} (A.2)

Here $\gamma(s) = \Gamma(s)/\Gamma(1-s)$ and $\mathcal{C}_{\phi\phi}$ is an operator product coefficient. (In order not to complicate the notation we considered only the perturbation with positive $\lambda$.)

Eqs. (A.1) and (A.2) actually give the analytic continuation in $y$ of the corresponding integrals (3.6), which converge only when $1 < y < 2$ and $\frac{2}{3} < y < 2$, respectively.
Known non-perturbative results of the TBA approach [20][21][22] lead one to conclude that these analytic continuations (and analogous ones for higher CPT coefficients that we are unable to compute explicitly) provide a consistent renormalization scheme — not only UV regularization — for all \( y \in (0, 2) \) different from \( \frac{2}{N} \), where \( N \) is a positive integer. (We are not aware of a rigorous proof of this fact, though it is highly plausible, since this kind of renormalization scheme is essentially equivalent to dimensional regularization with minimal subtraction in ordinary QFT.)

But what if \( y = \frac{2}{N}, \ N = 2, 3, \ldots \)? Then the analytically continued \( a_N \) has a simple pole at \( \frac{2}{N} \) (\( a_n \) with \( n < N \) are finite after analytic continuation and those with \( n > N \) are convergent to begin with — see [19][22] and references therein for detailed discussion). The simple pole indicates that if one introduces a UV cutoff \( \varepsilon \) to regularize the divergence, then the integral for \( a_N \) diverges logarithmically when \( \varepsilon \to 0 \), unlike powerlike divergences in \( \varepsilon \) that are encountered in \( a_n \) with \( n < N \). There are two cases, the theories \( A_3^{(\pm)} \) and \( A_5^{(\pm)} \) where \( y = \frac{2}{N} \) with \( N = 2, 3 \), respectively, in which in order to obtain the known exact TBA results the following recipe can be used: Expand (the analytically continued) \( a_N \) in Laurent series around \( \frac{2}{N} \)

\[
a_N(y) = \sum_{k=-1}^{\infty} R_{k+1} \left( y - \frac{2}{N} \right)^k ; \quad (A.3)
\]

then in the CPT expansion of \( e_0(r) \) replace

\[
a_N r^{ny} = a_N r^2 \quad \text{with} \quad (\alpha R_0 \ln \beta r + R_1) \ r^2 \quad (A.4)
\]

where \( \alpha, \beta \) are certain (real) constants.

The remarkable fact is that \( \alpha \) comes out to be a “nice” rational number — see below.\(^{11}\) Moreover, the recipe is supposed to be “universal” in the sense that one has to use the

\(^{11}\) This is also the case in a similar recipe that was proposed by Dotsenko [37] for treating divergences in the CPT expansion of the spin-spin correlation function in the so-called Ising field theory (IFT) \( A_3^{(\pm)} \) on the plane. There the scaling variable analogous to our \( r \) is proportional to the separation of the fields rather than the volume of space. A major difference between Dotsenko’s problem and ours is that his recipe is meant to cure IR divergences that are present in \( \text{infinitely many orders of CPT} \), whereas we deal here with UV divergences of only \( \text{finitely many terms} \) in the perturbative expansion. This is manifested in the non-perturbative result [38] for the problem in [37] by the existence of infinitely many power-log terms in the expansion, whereas in the finite-volume energies there is (apparently) at most a single such term.
same $\alpha$ and $\beta$ when treating the logarithmically divergent $N$-th CPT coefficients for all the energy scaling functions in the given model. This requires that we keep the $R_1$ term in (A.4), which could have been absorbed in $\beta$ as long as only the ground state is concerned, allowing for different coefficients of $r^2$ in different scaling functions (note, though, that the coefficient of the $r^2 \ln r$ term is the same in all of them, as follows from the fact that the "strength" of a possible UV divergence in (3.4) is independent of the in- and out-fields $\phi_{p,q}$).

We first demonstrate the use of our recipe for the IFT $A_3^{(\pm)}$. In this case the complete finite-volume spectrum is known analytically (see sect. 6 of [19] and references therein). For the ground state scaling function, in particular,

$$e_0(r) = -\frac{1}{24} - \frac{\kappa^2}{2} r^2 \ln r + \frac{\kappa^2}{2} \ln \left( \pi e^{\frac{1}{2} - \gamma_E} \right) r^2 + O(r^4),$$  \hspace{1cm} (A.5)

where $\gamma_E = 0.577215 \ldots$ is Euler's constant and $\kappa = \frac{1}{2\pi}$. ($\kappa$ is exactly known from the Lagrangian formulation of $A_3^{(+)}$ as obtained from a free massive Majorana fermion through a "GSO projection" [22], since the perturbation $\lambda \phi_{1,3}$ leading to $A_3^{(+)}$ is equivalent there to the fermion mass term $\frac{M}{2\pi} \bar{\psi} \psi$, in conventional complex notation.) Using this exact result we conclude from (A.1) and (A.4) that

$$\alpha = 2, \quad \beta = \frac{4}{3} e^{\gamma_E - \frac{1}{2}}$$  \hspace{1cm} (A.6)

in IFT. We verified that our recipe gives the correct $r^2$ term also in the scaling function corresponding to the field $TT$ of the UV Ising CFT (the integration of the relevant correlator can be performed analytically using results of sect. 4.2 of [22]).

We turn to the ground state in $A_5^{(+)}$, for which Al. Zamolodchikov evaluated the coefficient of the $r^2 \ln r$ term analytically from the TBA equations he proposed [21]. Together with our numerical results [22], the leading expansion terms obtained in the TBA approach read (recall that $y = \frac{2}{3}$ here)

$$e_0(r) = -\frac{1}{15} + 0.016781684(2) r^{4/3} + \frac{1}{12\pi^2} r^2 \ln r - 0.00926790(5) r^2 + O(r^{8/3}).$$  \hspace{1cm} (A.7)

Now comparing $a_2$ here with the CPT prediction (A.4) we obtained $\kappa_5 = 0.130234474(2)$ of Table 1 (actually this value is deduced from our slightly more accurate results for $a_2$ in $A_5^{(-)}$). Applying our recipe (A.4) to (A.2) (in our case $C_{\phi\phi\phi} = C_{(1,3)(1,3)(1,3)} = \frac{2\sqrt{2}}{3}$ [24]), we then obtain for the coefficient of the $r^2 \ln r$ term

$$\frac{1}{12\pi^2} = \frac{2\pi \sqrt{2}}{27} \gamma^3 \left( \frac{1}{3} \right) \alpha \kappa_5^3.$$  \hspace{1cm} (A.8)
Using the numerical value for \( \kappa_5 \) this relation gives \( \alpha = 1.49999998(6) \), which according to the “principle of nice numbers” is nothing but \( \frac{3}{2} \). The numerical value for \( \beta \), which can be obtained using the \( a_3 \) that we found, has not yet been illuminating to us.

Assuming the validity of (A.4) with \( \alpha = \frac{3}{2} \), we can invert the argument and obtain a new result. Namely the exact expression for \( \kappa_5 \) that we presented (just alluding to the “derivation” given here) in [22]:

\[
\kappa_5 = \frac{18^{1/6} \gamma \left( \frac{2}{3} \right)}{2\pi} = 0.13023447336 \ldots .
\] (A.9)

The same exact expression has been also given recently in [23] \(^{12}\). There the authors study certain integrable perturbations of the \( Z_N \)-parafermion CFTs, the case \( N = 3 \) corresponding to \( D_5^{(\pm)} \) where \( \kappa \) is same as in \( A_5^{(\pm)} \). Though conceptually different, their analysis also resorts in the last step to some (ad-hoc) prescription for obtaining a finite number out of a logarithmically divergent integral — see eq. (32) there.

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\(^{12}\) The \( \kappa_5 \) read off from eq. (33) of [23] differs from ours by a factor of \( \sqrt{2} \) due to a different normalization of the perturbing field.
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