Path Integral Solution by Sum Over Perturbation Series

De-Hone Lin *

Department of Physics, National Tsing Hua University
Hsinchu 30043, Taiwan
(July 6, 2021)

Abstract

A method for calculating the relativistic path integral solution via sum over perturbation series is given. As an application the exact path integral solution of the relativistic Aharonov-Bohm-Coulomb system is obtained by the method. Different from the earlier treatment based on the space-time transformation and infinite multiple-valued trasformation of Kustaanheimo-Stiefel in order to perform path integral, the method developed in this contribution involves only the explicit form of a simple Green’s function and an explicit path integral is avoided.

PACS: 02.30.Mv; 03.65.-w

*e-mail:d793314@phys.nthu.edu.tw
I. INTRODUCTION

Based on the perturbation expansion of path integral formulation, Feynman firstly introduce his famous diagram technique to give a neat interpretation of the terms in the perturbation series and calculate the quantities of quantum electrodynamics order by order [1,2]. Over the last five decades, Feynman’s method has been successfully applied to diverse areas of physics and achieved many accomplishments [3]. Nevertheless, the exact result of summing the perturbation series is still the aim of seeking because of many physical effects in which non-perturbative exact result plays the pivot’s role. In this contribution, a method for calculating the relativistic path integral is given in which the exact results only involve the computation of some kind of moments $Q^n$ over the Feynman measure and summing them in accordance with the Feynman-Kac type formula. So clear and neat is the method that it provides us not only with an alternative approach but a completely diverse viewpoint for treating physical problems. As an application, we apply the formula to calculate the path integral solution of the relativistic Aharonov-Bohm-Coulomb (A-B-C) system. It turns out that the method presented in this paper is neat due to the avoidance of space-time and (Kustaanheimo-Stiefel) K-S transformation in directly performing path integral [4]. The A-B-C case can serves as the prototype for the treatment of arbitrary problems via summing the perturbation series.

II. PATH INTEGRAL SOLUTION BY SUMMING THE PERTURBATION SERIES

The starting point is the path integral representation for the Green’s function of a relativistic particle in external electromagnetic fields [4,5]:

$$G(x_b, x_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dS \int \mathcal{D}\rho(\lambda) \Phi[\rho(\lambda)] \int \mathcal{D}^Dx(\lambda) \exp \left\{ -A_E[x, \dot{x}] / \hbar \right\} \rho(0)$$

with the action
\[ A_E [\mathbf{x}, \mathbf{\dot{x}}] = \int_{\lambda_a}^{\lambda_b} d\lambda \left[ \frac{m}{2\rho(\lambda)} \dot{x}^2 (\lambda) - i(\varepsilon/c) \mathbf{A}(\mathbf{x}) \cdot \mathbf{\dot{x}}(\lambda) - \rho(\lambda) \frac{(E - V(\mathbf{x}))^2}{2mc^2} + \rho(\lambda) \frac{mc^2}{2} \right], \]

(2.2)

where \( S \) is defined as

\[ S = \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda), \]

(2.3)

in which \( \rho(\lambda) \) is an arbitrary dimensionless fluctuating scale variable, \( \rho(0) \) is the terminal point of the function \( \rho(\lambda) \), and \( \Phi[\rho(\lambda)] \) is some convenient gauge-fixing functional \([4–6]\). The only condition on \( \Phi[\rho(\lambda)] \) is that

\[ \int \mathcal{D} \rho(\lambda) \Phi [\rho(\lambda)] = 1. \]

(2.4)

\( \hbar/mc \) is the well-known Compton wave length of a particle of mass \( m \), \( \mathbf{A}(\mathbf{x}) \) and \( V(\mathbf{x}) \) stand for the vector and scalar potential of the systems, respectively. \( E \) is the system energy, and \( \mathbf{x} \) is the spatial part of the \((D+1)\) vector \( x^\mu = (\mathbf{x}, \tau) \).

The functional integral for \( \mathbf{x} \) in representation of Eq. (2.1) can be interpreted as the expectation value of the real functional \( \exp \left\{ -\frac{1}{\hbar} \int_{\lambda_a}^{\lambda_b} d\lambda \beta \rho(\lambda) V(\mathbf{x}(\lambda)) \right\} \) over the measure \( K_0(x_b, x_a; \lambda_b - \lambda_a) = \int \mathcal{D}^D x(\lambda) e^{-\frac{1}{\hbar} \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda) \left[ \frac{m}{2\rho(\lambda)} \dot{x}^2 (\lambda) - i \frac{\varepsilon}{c} \mathbf{A}(\mathbf{x}) \cdot \mathbf{\dot{x}}(\lambda) - \rho(\lambda) \frac{V(\mathbf{x})^2}{2mc^2} \right]} \),

(2.5)

and the entire Green’s function reduces to the following formula

\[ G(x_b, x_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dS \int \mathcal{D} \rho(\lambda) \Phi [\rho(\lambda)] e^{-\frac{1}{\hbar} \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda) \mathcal{E}} \]

\[ \times \left\langle \exp \left\{ -\frac{1}{\hbar} \int_{\lambda_a}^{\lambda_b} d\lambda \beta \rho(\lambda) V(\mathbf{x}(\lambda)) \right\} \right\rangle \rho(0) \]

(2.6)

in which \( \mathcal{E} = (m^2c^4 - E^2)/2mc^2 \), \( \beta = E/mc^2 \) with the notation \( \langle \star \rangle \) standing for the expectation value of the moment \( \star \) over the measure \( K_0(x_b, x_a; \lambda_b - \lambda_a) \). Eq. (2.6) forms the basis for studying the relativistic potential problems by the Feynman-Kac type formula. Although we have chosen the term \( V(\mathbf{x}(\lambda)) \) to expansion, it has the aesthetic appeal on choosing convenient one according to which the most suitable term is expanded for calculation.
Expanding the potential $V(x)$ in Eq. (2.6) into a power series and interchanging the order of integration and summation, we have

\[
G(x_b, x_a; E) = \frac{i\hbar}{2mc} \int_0^\infty dS \int D\rho \Phi(\rho) e^{-\frac{i}{\hbar} \int_{x_a}^{x_b} d\rho(x) e^{i E t}}
\]

\[
\times \sum_{n=0}^\infty \left( \frac{-\beta}{\hbar} \right)^n \left\langle \left( \int_{x_a}^{x_b} d\rho(x) V(x) \right)^n \right\rangle \rho(0).
\] 

We see that the calculation of path integral now turns into the computation of the expectation value of moments $Q^n$ ($Q = \int_{x_a}^{x_b} d\rho V(x)$) over the Feynman measure and summing them in accordance with the Feynman-Kac type formula. Ordering the $\lambda$ as $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \lambda_b$ and denoting $x(\lambda_i) = x_i$, the perturbation series in Eq. (2.7) explicitly turns into

\[
\sum_{n=0}^\infty \frac{(-\beta/\hbar)^n}{n!} \left\langle \left( \int_{x_a}^{x_b} d\rho(x) V(x) \right)^n \right\rangle = K_0(x_b, x_a; \lambda_b - \lambda_a)
\]

\[
+ \sum_{n=1}^\infty \left( \frac{\beta}{\hbar} \right)^n \int_{x_a}^{x_b} d\lambda_n \int_{\lambda_n}^{\lambda_a} d\lambda_1 \cdots \frac{\rho_i}{r_i} d\chi_i.
\]

where $\lambda_0 = \lambda_a$, $\lambda_{n+1} = \lambda_b$, $x_{n+1} = x_b$, and $x_0 = x_a$.

As an application of Eq. (2.7), let’s apply it to the relativistic A-B-C system in three dimensions. In this case, we have the vector and scalar potentials

\[
A(x) = 2g \frac{-y\hat{e}_x + x\hat{e}_y}{x^2 + y^2}, \quad V(r) = -\frac{e^2}{r},
\] 

where $\hat{e}_{x,y}$ stands for the unit vector along the $x, y$ axis, respectively. The perturbative expansion in Eq. (2.8) becomes

\[
\sum_{n=0}^\infty \frac{(\beta e^2/\hbar)^n}{n!} \left\langle \left( \int_{x_a}^{x_b} d\rho(x) \frac{1}{r} \right)^n \right\rangle = K_0(x_b, x_a; \lambda_b - \lambda_a)
\]

\[
+ \sum_{n=1}^\infty \left( \frac{\beta e^2}{\hbar} \right)^n \int_{x_a}^{x_b} d\lambda_n \int_{\lambda_n}^{\lambda_a} d\lambda_1 \cdots \frac{\rho_i}{r_i} d\chi_i.
\]
The corresponding amplitude \( K_0 \) takes the form
\[
K_0(x_b, x_a; \lambda_b - \lambda_a) = \int D^3 x e^{-\frac{i}{\hbar} \int_{\lambda_a}^{\lambda_b} d\lambda \left[ \frac{m}{2 \rho(\lambda)} \frac{\dot{x}^2(\lambda)}{2} - i \frac{e}{c} A(x) \cdot \dot{x}(\lambda) - \rho(\lambda) \frac{\hbar^2 \alpha^2}{2 m} \right]},
\]
where \( \alpha = e^2/\hbar c \) is the fine structure constant. We now choose \( \Phi[\rho] = \delta[\rho - 1] \) to fix the value of \( \rho(\lambda) \) to unity. The path integral in Eq. (2.7) becomes
\[
G(x_b, x_a; E) = \frac{i \hbar}{2mc} \int_0^\infty dS e^{-\frac{\hbar S}{2}} \left\{ K_0(x_b, x_a; S)
\right.
\]
\[
+ \sum_{n=1}^{\infty} \left( \frac{\beta e^2}{\hbar} \right)^n \int_{\lambda_a}^{\lambda_b} d\lambda_n \int_{\lambda_a}^{\lambda_{n-1}} d\lambda_n \cdots \int_{\lambda_a}^{\lambda_2} d\lambda_1 \int \left[ \prod_{j=0}^{n} K_0(x_{j+1}, x_j; \lambda_{j+1} - \lambda_j) \right] \left[ \prod_{i=1}^{n} \frac{d x_i}{r_i} \right] \right\}.
\]

We observe that the integration over \( S \) is a Laplace transformation. Because of the convolution property of the Laplace transformation, we obtain
\[
G(x_b, x_a; E) = \frac{i \hbar}{2mc}
\]
\[
\times \left\{ G_0(x_b, x_a; \mathcal{E}) + \sum_{n=1}^{\infty} \left( \frac{\beta e^2}{\hbar} \right)^n \int \left[ \prod_{j=0}^{n} G_0(x_{j+1}, x_j; \mathcal{E}) \right] \left[ \prod_{i=1}^{n} \frac{d x_i}{r_i} \right] \right\} \quad (2.13)
\]
with \( G_0(x_b, x_a; \mathcal{E}) \) is the Laplace transformation of pseudopropagator \( K_0(x_b, x_a; \lambda_b - \lambda_a) \).

Let’s first analyze the influence of A-B effect on the \( G_0(x_b, x_a; \mathcal{E}) \). Introducing the azimuthal angle around the A-B tube
\[
\varphi(x) = \arctan(y/x),
\]
the components of the vector potential can be expressed as
\[
A_i = 2g \partial_i \varphi(x).
\]

The associated magnetic field lines are confined to an infinitely thin tube along the z-axis:
\[
B_3 = 2g \epsilon_{3ij} \partial_i \partial_j \varphi(x) = 4\pi g \delta(x_\perp),
\]

(2.16)
where $x_\perp$ stands for the transverse vector $x_\perp = (x, y)$. Note that the derivatives in front of $\varphi(x)$ commute everywhere, except at the origin where Stokes’ theorem yields

$$\int d^2x (\partial_x \partial_y - \partial_y \partial_x) \varphi(x) = \oint d\varphi = 2\pi. \tag{2.17}$$

The magnetic flux through the tube is defined by the integral

$$\Omega = \int d^2xB_3. \tag{2.18}$$

This shows that the coupling constant $g$ is related to the magnetic flux by

$$g = \frac{\Omega}{4\pi}. \tag{2.19}$$

Inserting $A_i = 2g\partial_i\varphi(x)$ into the action of Eq. (2.11), the magnetic interaction takes the form

$$A_{mag} = i\hbar\beta_0 \int_0^S d\lambda \dot{\varphi}(\lambda), \tag{2.20}$$

where $\varphi(\lambda) = \varphi(x(\lambda))$, $\dot{\varphi} = d\varphi/d\lambda$, and $\beta_0$ is the dimensionless number

$$\beta_0 = -\frac{2\epsilon g}{\hbar c}. \tag{2.21}$$

The minus sign is a matter of convention. Since the particle orbits are present at all times, their worldlines in spacetime can be considered as being closed at infinity, and the integral

$$k = \frac{1}{2\pi} \int_0^S d\lambda \dot{\varphi}(\lambda) \tag{2.22}$$

is the topological invariant with integer values of the winding number $k$. The magnetic interaction is therefore purely topological, its value being

$$A_{mag} = i\hbar\beta_0 2k\pi. \tag{2.23}$$

The influence of A-B effect in the Green’s function $G_0(x_b, x_a; \mathcal{E})$ is as follows. In the lacking of A-B effect, the Green’s function

$$G_0(x_{j+1}, x_j; \mathcal{E}) = \frac{m}{\hbar (r_{j+1}r_j)^{1/2}} \sum_{l=0}^{\infty} \sum_{k=-l}^{l} g_l^{(0)}(r_{j+1}, r_j; \mathcal{E}) Y_l^k(\mathbf{x}_{j+1}) Y_{-l}^k(\mathbf{x}_j), \tag{2.24}$$
where \( Y_{lk}(\hat{x}) \) is the 3-dimensional spherical harmonics \( Y_{lk}(\hat{x}) \) and the \( g_l^{(0)} \) is the radial Green’s function of a particle moving in a centrifugal potential given by \[ [7,8] \]

\[
\int_0^\infty \frac{dS}{S} e^{-\frac{r_j}{\hbar} S} e^{-m(r_{j+1}^2 + r_j^2)/2\hbar S} I \sqrt{l(l+1) - \alpha^2} \left( \frac{m r_{j+1} r_j}{\hbar S} \right). \tag{2.25}
\]

The notation \( I \) denotes the modified Bessel function. With the following formulas [p.166, p.210, p.212, [4]],

\[
\begin{aligned}
P_{\mu}(x) &= \frac{(1+x)^{-\mu/2}(1-x)^{-\mu/2}}{\Gamma(1-\mu)} F\left(-\nu, 1+\nu; 1-\mu; \frac{1-x}{2}\right), \\
P_n^{(\alpha,\beta)}(x) &= \frac{\Gamma(1+n+\alpha)}{\Gamma(n+1)} F\left(-n, \alpha + \beta + n + 1; \alpha + 1; \frac{1-x}{2}\right), \tag{2.26}
\end{aligned}
\]

where \( P_{\mu}(x) \), \( P_n^{(\alpha,\beta)}(x) \) are the associated Legendre polynomial and Jacobi function and \( F \) the hypergeometric function, it is not difficult to prove the following result

\[
P_l^k(\cos \theta) = (-1)^k \frac{\Gamma(1+k+l)}{\Gamma(1+l)} (\cos \theta/2 \sin \theta/2)^k P_{l-k}^{(k,k)}(\cos \theta). \tag{2.27}
\]

The angular part of Eq. (2.24) turns into

\[
\sum_{k=-l}^l Y_{lk}(\hat{x}_{j+1}) Y_{lk}^*(\hat{x}_j) = \sum_{k=-l}^l \frac{2l+1}{4\pi} \frac{\Gamma(1+l-k)}{\Gamma(1+l+k)} P_l^k(\cos \theta_{j+1}) P_l^k(\cos \theta_j) e^{ik(\varphi_{j+1} - \varphi_j)}
\]

\[
= \sum_{k=-l}^l \left[ \frac{2l+1}{4\pi} \frac{\Gamma(1+l-k)}{\Gamma(1+l+k)} \right] (\cos \theta_{j+1}/2 \cos \theta_j/2 \sin \theta_{j+1}/2 \sin \theta_j/2)^k
\]

\[
\times P_{l-k}^{(k,k)}(\cos \theta_{j+1}) P_{l-k}^{(k,k)}(\cos \theta_j) e^{ik(\varphi_{j+1} - \varphi_j)}. \tag{2.28}
\]

To go further, let’s change the variable \( l \) by defining \( l - k = q \) into \( q \). It is easily to fine that the Green’s function of Eq. (2.24) becomes

\[
G_0(x_{j+1}, x_j; \mathcal{E}) = \frac{m}{\hbar(r_{j+1} r_j)^{1/2}} \sum_{q=0}^\infty \sum_{k=-\infty}^\infty g_q^{(0)}(r_{j+1}, r_j; \mathcal{E})
\]

\[
\times \left[ \frac{2(q+k) + 1}{4\pi} \frac{\Gamma(1+q) \Gamma(1+q+2k)}{\Gamma^2(1+q+k)} \right] e^{ik(\varphi_{j+1} - \varphi_j)}
\]

\[
\times (\cos \theta_{j+1}/2 \cos \theta_j/2 \sin \theta_{j+1}/2 \sin \theta_j/2)^k P_q^{(k,k)}(\cos \theta_{j+1}) P_q^{(k,k)}(\cos \theta_j). \tag{2.29}
\]
with \( g_{q+k}^{(0)} \) being the radial Green’s function

\[
\int_0^\infty \frac{dS}{S} e^{-\frac{q}{\hbar} S} e^{-m(r_j^2 + r_j^2)/4\hbar S} I_{(q+k+1/2)^2 - \alpha^2} \left( \frac{m r_j + 1}{\hbar} \right). \tag{2.30}
\]

Let us invoke the Poisson’s summation formula [p.469 [9]]

\[
\sum_{k=-\infty}^{\infty} f(k) = \int_{-\infty}^{\infty} dy \sum_{n=-\infty}^{\infty} e^{2\pi nyi} f(y). \tag{2.31}
\]

The entire Green’s function \( G_0(x_b, x_a; \mathcal{E}) \) containing the A-B effect becomes

\[
G_0(x_{j+1}, x_j; \mathcal{E}) = \frac{m}{\hbar (r_{j+1} r_j)^{1/2}} \sum_{q=0}^{\infty} \sum_{k=-\infty}^{\infty} g_{q+k}^{(0)}(r_{j+1}, r_j; \mathcal{E})
\times \left[ \frac{2(q + z) + 1}{4\pi} \Gamma(1 + q) \Gamma(1 + q + 2z) \right] e^{i(z - \beta_k)(\varphi_{j+1} + 2k\pi - \varphi_j)}
\times (\cos \theta_{j+1}/2 \cos \theta_j/2 \sin \theta_{j+1}/2 \sin \theta_j/2)^z P_q^{(z,z)}(\cos \theta_{j+1}) P_q^{(z,z)}(\cos \theta_j). \tag{2.32}
\]

The sum over all \( k \) in Eq. (2.32) forces \( z \) to be equal to \( \beta_0 \) modulo an arbitrary integral number leading to

\[
G_0(x_{j+1}, x_j; \mathcal{E}) = \frac{m}{\hbar (r_{j+1} r_j)^{1/2}} \sum_{q=0}^{\infty} \sum_{k=-\infty}^{\infty} g_{q+k}^{(0)}(r_{j+1}, r_j; \mathcal{E})
\times \left[ \frac{2(q + |k + \beta_0|) + 1}{4\pi} \Gamma(1 + q) \Gamma(1 + q + 2|k + \beta_0|) \right] e^{i|k + \beta_0|(\varphi_{j+1} - \varphi_j)}
\times (\cos \theta_{j+1}/2 \cos \theta_j/2 \sin \theta_{j+1}/2 \sin \theta_j/2)^{|k + \beta_0|}
\times P_q^{(|k + \beta_0|, |k + \beta_0|)}(\cos \theta_{j+1}) P_q^{(|k + \beta_0|, |k + \beta_0|)}(\cos \theta_j) \tag{2.33}
\]

with

\[
g_{q+k}^{(0)}(r_{j+1}, r_j; \mathcal{E})
= \int_0^\infty \frac{dS}{S} e^{-\frac{q}{\hbar} S} e^{-m(r_{j+1}^2 + r_j^2)/4\hbar S} I_{(q+k+1/2)^2 - 4\alpha^2/2} \left( \frac{m r_{j+1} r_j}{\hbar S} \right). \tag{2.34}
\]
Using the orthogonality relations of Jacobi polynomials \([p212, 9]\),

\[
\int_{-1}^{1} dx \, (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha, \beta)}(x) \, P_m^{(\alpha, \beta)}(x) = \frac{2^{\alpha+\beta+1} \Gamma(\alpha+n+1) \Gamma(\beta+n+1)}{\alpha+\beta+2n+1 \, n! \Gamma(\alpha+\beta+n+1)} \delta_{m,n},
\]

we perform the intermediate angular part of Eq. (2.13), it yields

\[
G(x_b, x_a; E) = \frac{i h}{2mc} \sum_{q=0}^{\infty} \sum_{k=-\infty}^{\infty} G_{q,k+|\beta_0|}(r_b, r_a; E)
\times \left[ \frac{2(q + |k + \beta_0|) + 1}{4\pi} \right]^{1/2} \Gamma^2(1 + q + |k + \beta_0|) e^{i k(\varphi_b - \varphi_a)}
\times \left( \cos \theta_b/2 \cos \theta_a/2 \sin \theta_b/2 \sin \theta_a/2 \right)^{|k+|\beta_0|} P_q^{(k+|\beta_0|,|\beta_0|)}(\cos \theta_b) \, P_q^{(k+|\beta_0|,|\beta_0|)}(\cos \theta_a).
\]

The pure radial amplitude \(G_{n,k+|\beta_0|}(r_b, r_a; E)\) has the form

\[
G_{q,k+|\beta_0|}(r_b, r_a; E) = \frac{m}{h} \frac{1}{(r_br_a)^{1/2}} \sum_{n=0}^{\infty} \left( \frac{m \beta e^2}{h^2} \right)^n g_{q+k+|\beta_0|}^{(n)}(r_b, r_a; E)
\]

with \(g_{q+k+|\beta_0|}^{(n)}\) given by

\[
g_{q+k+|\beta_0|}^{(n)}(r_b, r_a; E) = \int_0^\infty \cdots \int_0^\infty \left[ \prod_{i=0}^{n} g_{q+k+|\beta_0|}^{(0)}(r_{j+1}, r_j; E) \right] \prod dr_i.
\]

To obtain the explicit result of \(g_{q+k+|\beta_0|}^{(n)}\), we note that \([10]\)

\[
\int_0^\infty \frac{dS}{S} e^{-\frac{\kappa}{S}} \, e^{-m(r_b^2+r_a^2)/2hS} I_\rho \left( \frac{m r_b r_a}{h} \right)
= 2 \int_0^\infty dz \frac{1}{\sinh z} e^{-\kappa(r_b+r_a)\coth z} I_{2\rho} \left( \frac{2\kappa \sqrt{r_b r_a}}{\sinh z} \right)
\]

with \(\kappa = \sqrt{m^2c^4 - E^2}/hc\). With the help of the integral formula \([9]\),

\[
\int_0^\infty dr \, e^{-r^2/a} I_\nu(\lambda r) I_\nu(\xi r) = \frac{a}{2} e^{a(\lambda^2+\xi^2)/4} I_\nu \left( a \xi \lambda / 2 \right),
\]

we obtain the result
\( g_{q+|k+\beta|}^{(1)}(r_b, r_a; \mathcal{E}) = \int_0^\infty g_{q+|k+\beta|}^{(0)}(r_b, r; \mathcal{E}) g_{q+|k+\beta|}^{(0)}(r, r_a; \mathcal{E}) dr \)
\[
= \frac{2^2}{\kappa} \int_0^\infty z h(z) dz,
\tag{2.41}
\]

where the function \( h(z) \) is defined as
\[
h(z) = \frac{1}{\sinh z} e^{-\kappa(r_b + r_a) \coth z} I^\frac{2}{\sqrt{2(q+|k+\beta|)+1}} \left( \frac{2\kappa \sqrt{r_b r_a}}{\sinh z} \right).
\tag{2.42}
\]

The expression for \( g_{q+|k+\beta|}^{(n)}(r_b, r_a; \mathcal{E}) \) can be obtained by induction with respect to \( n \), and is given by
\[
g_{q+|k+\beta|}^{(n)}(r_b, r_a; \mathcal{E}) = \frac{2^{n+1}}{n!} \kappa^n \int_0^\infty z^n h(z) dz.
\tag{2.43}
\]

Inserting the expression in Eq. (2.37), we obtain
\[
G_{q,|k+\beta|}(r_b, r_a; \mathcal{E}) = \frac{2}{m} \frac{m}{2} \int_0^\infty dze^{\frac{2m\beta}{N^2}z} \frac{1}{\sinh z} e^{-\kappa(r_b + r_a) \coth z} I^\frac{2}{\sqrt{2(q+|k+\beta|)+1}} \left( \frac{2\kappa \sqrt{r_b r_a}}{\sinh z} \right).
\tag{2.44}
\]

The integration can be done by the formula [e.g. ch. 9 \cite{7}]
\[
\int_0^\infty dy \frac{e^{2\nu y}}{\sinh y} \exp \left[ -\frac{t}{2} (\zeta_a + \zeta_b) \coth y \right] I_\mu \left( \frac{t \sqrt{\zeta_b \zeta_a}}{\sinh y} \right)
\]
\[
= \frac{\Gamma \left( (1 + \mu) / 2 - \nu \right)}{t \sqrt{\zeta_b \zeta_a} \Gamma (\mu + 1)} W_{\nu,\mu/2} (t \zeta_b) M_{\nu,\mu/2} (t \zeta_a),
\tag{2.45}
\]

where \( M_{\mu,\nu} \) and \( W_{\mu,\nu} \) are the Whittaker functions and the range of validity is given by
\[
\zeta_b > \zeta_a > 0, \\
\text{Re}[(1 + \mu)/2 - \nu] > 0, \\
\text{Re}(t) > 0, | \arg t | < \pi.
\]

We complete the integration and obtain
\[
G_{q,|k+\beta|}(r_b, r_a; \mathcal{E}) = \frac{1}{(r_b r_a) \sqrt{m^2 c^4 - \mathcal{E}^2}}
\]
\( m c \)
\[
\times \frac{\Gamma \left( \frac{1}{2} + \sqrt{\frac{2(q + |k + \beta_0|) + 1}{4} - 4\alpha^2 / 2 - E\alpha/\sqrt{m^2c^4 - E^2}} \right)}{\Gamma \left( 1 + \sqrt{\frac{2(q + |k + \beta_0|) + 1}{4} - 4\alpha^2} \right)}
\]

\[
\times W_{E\alpha/\sqrt{m^2c^4 - E^2}, \sqrt{[2(q +|k + \beta_0|) + 1]^2 - 4\alpha^2 / 2}} \left( \frac{2}{\hbar c} \sqrt{m^2c^4 - E^2} r_b \right)
\]

\[
\times M_{E\alpha/\sqrt{m^2c^4 - E^2}, \sqrt{[2(q + |k + \beta_0|) + 1]^2 - 4\alpha^2 / 2}} \left( \frac{2}{\hbar c} \sqrt{m^2c^4 - E^2} r_a \right).
\] (2.46)

The entire solution of path integral becomes

\[
G(x_b, x_a; E) = \frac{i\hbar}{2mc} \frac{mc}{4\pi r_b r_a \sqrt{m^2c^4 - E^2}}
\]

\[
\times \sum_{q=0}^{\infty} \sum_{k=-\infty}^{\infty} \left\{ \frac{\Gamma \left( \frac{1}{2} + \sqrt{\frac{2(q + |k + \beta_0|) + 1}{4} - 4\alpha^2 / 2 - E\alpha/\sqrt{m^2c^4 - E^2}} \right)}{\Gamma \left( 1 + \sqrt{\frac{2(q + |k + \beta_0|) + 1}{4} - 4\alpha^2} \right)} \right\}
\]

\[
\times W_{E\alpha/\sqrt{m^2c^4 - E^2}, \sqrt{[2(q +|k + \beta_0|) + 1]^2 - 4\alpha^2 / 2}} \left( \frac{2}{\hbar c} \sqrt{m^2c^4 - E^2} r_b \right)
\]

\[
\times M_{E\alpha/\sqrt{m^2c^4 - E^2}, \sqrt{[2(q + |k + \beta_0|) + 1]^2 - 4\alpha^2 / 2}} \left( \frac{2}{\hbar c} \sqrt{m^2c^4 - E^2} r_a \right).
\]

\[
\times \left\{ \frac{\Gamma (1 + q) \Gamma (1 + q + 2|k + \beta_0|) [2(q + |k + \beta_0|) + 1]}{\Gamma^2 (1 + q + |k + \beta_0|)} \right\}
\]

\[
\times e^{ik(\varphi_b - \varphi_a)} (\cos \theta_b/2 \cos \theta_a/2 \sin \theta_b/2 \sin \theta_a/2)^{|k + \beta_0|}
\]

\[
\times P_q^{(|k + \beta_0|, |k + \beta_0|)} (\cos \theta_b) P_q^{(|k + \beta_0|, |k + \beta_0|)} (\cos \theta_a).
\] (2.47)

This result is given in Refs. [4], and p.304 [8] in the first time where the same result must invoke the complicate space-time and multi-valued K-S transformations to perform the path integral. In present paper, this procedures is avoided and can be applied to arbitrary potential problems.
III. CONCLUDING REMARKS

In the paper, a method for calculating the relativistic path integral involved essentially the computation of the expectation value of convenient moments $Q^n$, such as $Q = \int_{\lambda_a}^{\lambda_b} d\lambda \rho(\lambda)V(x)$ if we expands the term in action potential term, over the Feynman measure and summing them in accordance with the Feynman-Kac type formula is given. As an realization, the path integral solution of relativistic A-B-C system is given. Different from the former treatment in Ref. [4], where the same problem must invoke the complicated space-time and the multi-valued K-S transformations to perform path integral, the merits of the method used in the paper is that it involves only the explicit form of some known Green’s function and explicit path integral is avoided. The A-B-C system can serves as the prototype for the treatment of arbitrary problems via summing the perturbation series. It is our hope that our studies would help to achieve the ultimate goal of obtaining a comprehensive and complete solutions in perturbation series based on the path integral of quantum mechanics and quantum field theory, including quantum gravity and cosmology.
REFERENCES

[1] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw Hill, New York, 1965.

[2] R. P. Feynman, *Quantum Electrodynamics*, Benjamin, New York, 1962.

[3] M. Gell-mann, Dick Feynman-The Guy in the Office Down the Hall. *Physics Today* **43** 48 (1989).

[4] D. H. Lin, J. Math. Phys. **40** 1246 (1999).

[5] H. Kleinert, Phys. Lett. **A 212** 15 (1996).

[6] D. H. Lin, J. Phys. **A 31** 4785 (1998).

[7] H. Kleinert, *Path Integrals in Quantum Mechanics, Statistics and Polymer Physics*, Singapore, World Scientific (1990).

[8] C. Grosche and F. Steiner, *Handbook of Feynman Path Integrals*, Springer Tracts in Modern Physics, Vol. **145**, Springer, Berlin 1998.

[9] W. Magnus, F. Oberhettinger, R.P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, Springer, Berlin, 1966.

[10] D. H. Lin, J. Phys. **A 31** 7577 (1998).