On Cross Hyperoperatorial Migration of Properties, Related to Natural Number Division Operator

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Abstract

In the article integer divisibility properties and related prime factors natural number representation concepts have been defined over the whole infinite hyperoperation hierarchy. The definitions have been made across and above of unique arithmetic operations, composing this hierarchy (addition, multiplication, exponentiation, tetration and so on). It allows the habitual concepts of "prime factor", "multiplier", "divider", "natural number factors representation" etc., to be associated mainly with the each of those operations. As analogy of multiplication-based Fundamental Theorem of Arithmetic (FTA), an exponentiation-based theorem is formulated. The theorem states that any natural number $M$ can be uniquely represented as a tower-like exponentiation: $M = a_n \uparrow (a_{n-1} \uparrow (\ldots (a_2 \uparrow a_1) \ldots))$, where $a_i \neq 1 (i = 1, \ldots, n)$ are primitive in some sense (related to the exponentiation operation), exponentiation components, following one by one in some unique order and named in the article as biprimes.

1 Introduction

The English mathematician R.L. Goodstein is well known as the author of an interesting and original approach to constructive mathematical analysis. This approach differs significantly in the general idea and nature of the basic concepts from the approaches used by other mathematicians. Goodstein’s approach is closely related to the calculus of equality that he introduced, which is an axiomatic fragment of the theory of recursive arithmetic functions, which have some important advantages. In particular, he used only such algorithms that, by their definition, always finish work. Functions specifically designed for this purpose, such as the Goodstein hyperoperator sequence, extend the basic arithmetic operations well beyond exponentiation. Goodstein’s theoretical constructions are presented in [1-3]. These works contain a systematic and detailed research of the Goodstein calculus of primitive-recursive equalities and research of some modifications of this calculus. With the Goodstein sequence of hyperoperations, the method of denoting very large integers, introduced by Donald Knuth in 1976 [4], so called Knuth’s up-arrow notation, is related. This article is devoted to some problems arising in the theory of Goodstein.
We will start with some auxiliary definitions and denotations.

Let \( \mathbb{N} \) be the set of all natural integers, \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \), and \( H_r(a, x) \) be a hyperoperator of rank \( r(a, x, r \in \mathbb{N}_0) \), defined by following recursive schema:

\[
\begin{align*}
H_0(a, x) &= a + x,
H_{r+1}(a, 0) &= 1, 
H_{r+1}(a, 1) &= a, 
H_{r+1}(a, x + 1) &= H_r(a, H_{r+1}(a, x)), \quad x \geq 1.
\end{align*}
\]

(1)

From this definition follows, that for natural numbers, the notation of hyperoperator generalizes and extends the notation of arithmetic addition operator. And it allows building, on addition operation basis, the infinite sequence of other homogeneously defined arithmetic operations of arbitrary higher ranks (as a members of the sequence they can be called hyperoperations). The structure we have chosen for definition (1) corresponds to the following beginning of rank-ordered hierarchy of these operations:

\[
H_0(a, x) = a + x, 
H_1(a, x) = a \times x, 
H_2(a, x) = a^x, \ldots
\]

(2)

Inverse function \( H_r^{-1}(a, y) \) for any element \( H_r(a, x) \) of sequence (2) can be defined in such a way that \( H_r^{-1}(a, H_r(a, x)) = x \). Then, for example, assuming \( r = 0, 1, 2, \) we get conventional arithmetic operations: subtraction \( (y - a) \), integer division \( (y \div a) \) and computing discrete logarithm \( (\log_a y) \), correspondently.

**Definition 1.** At being given a natural number sequence (vector) \( \varpi = (v_1, \ldots, v_n) \) and any \( r \in \mathbb{N}_0 \), we will call \( r \)-multiplier (or, multiplier of rank \( r \)) of vector \( \varpi \) an operator \( \Pi^{(r/)} \), represented by the following formula:

\[
\Pi^{(r/)} \varpi = \Pi^{(r/)}_{i=1} v_i = H_r(v_n, H_r(v_{n-1}, \ldots, H_r(v_2, H_r(v_1, Sgn(r))) \ldots)).
\]

(3)

Talking about the right part of the formula (3) and assuming \( r = 0, 1, 2, \ldots \), we will also be calling it \( r \)-multiplication (thus, generalizing by this term concepts of addition, multiplication, exponentiation and so on).

The significance of the \( r \)-multiplication notion is that it allows considering hyperoperatorial operations for operand’s vectors of an arbitrary none zero length. It should be noted also, that operands binding in formula (3) is carried out from right to left. That’s essential in cases when \( r > 1 \) and associative property may not be true (in hyperoperator hierarchy (2) it’s always true for addition and multiplication only).

Examples:

\[
\begin{align*}
\Pi^{(0/)}_{i=1} v_i &= H_0(v_1, Sgn(0)) = v_1 + 0 = v_1, \\
\Pi^{(1/)}_{i=1,3} v_i &= H_1(v_3, H_1(v_2, H_1(v_1, Sgn(1)))) = v_3 \times (v_2 \times (v_1 \times 1)) = v_3 \times (v_2 \times v_1), \\
\Pi^{(2/)}_{i=1,3} v_i &= H_2(v_3, H_2(v_2, H_2(v_1, Sgn(2)))) = v_3 \uparrow (v_2 \uparrow (v_1 \uparrow 1)) = v_3^{(v_2^{v_1})}.
\end{align*}
\]

2 The rank’s cross over divisibility

**Definition 2.** Let \( r \in \mathbb{N}_0 \) be the rank of some hyperoperator \( H_r(a, x) \). Any natural number \( M > Sgn(r) \) is called \( r \)-decomposable (\( r \)-compound) if the equation

\[
H_r(a, x) = M
\]

(4)

has a solution \((a_0, x_0)\) over \( \mathbb{N} \), where \( a_0, x_0 > Sgn(r) \).
Otherwise (that is, if there are no such solutions of that equation), we will call given \( M \) \( r \)-prime.

Particularly, in cases when \( r = 0, 1, 2 \), we can talk about additive, multiplicative and exponential decomposability (or, on the contrary, quantity to be prime) of number \( M \), correspondently. In addition, for convenience, instead of term ”2-prime” we will often use in this article a new special term — ”biprime”.

When \( r = 1 \) equation (4) is specified as \( a \times x = M, (M, a, x > 1) \), and the notion of multiplicative decomposability (or, on the contrary, of quantity to be 1-prime) coincides with usual notion of decomposability (or, on the contrary, of quantity to be prime) of given number \( M \).

For \( r = 0 \) we will get from (4) the equation \( a + x = M, (M, a, x > 0) \), which corresponds to trivial statement about additive decomposability of any natural number \( M \) (the only 0-prime in this case is 1).

Notion of \( r \)-decomposability of given natural number \( M > Sgn(r) \), particularly, in cases \( r = 0, 1, 2 \), is equal by sense to possibility of representation of this number, correspondently, as sum, product, power of two non trivial (for production and exponentiation – not equal to 1) natural numbers.

The only 0–prime number, as already noted, is 1; number 6, for example, is a biprime (equation \( x^y = 6 \) has the trivial solution \( (6,1) \) only), while the number 9 can serve as an example of a bi-decomposable number (since \( 3^2 = 9 \)).

Incidentally, we also note that any prime number is biprime (the converse statement is obviously not true).

**Definition 3.** For a given hiperoperator’s rank \( r \), two different natural numbers \( a, b \) we will call \( r \)-coprimes if there isn’t exist a natural \( d > Sgn(r) \) such that system of equations:

\[
\begin{align*}
H_r(d, x) &= a, \\
H_r(d, y) &= b, \\
x, y &\geq Sgn(r)
\end{align*}
\]

has at least one solution \((x_0, y_0)\) over \( \mathbb{N} \) with respect to the variables \( x, y \).

If \( a, b \) are not \( r \)-coprimes, that is, if there exists a natural number \( d > Sgn(r) \) such that system (5) is resolvable with respect to the variables \( x, y \), then the number \( d \) will be called common \( r \)-divisor of \( a \) and \( b \).

At last, if \( d \) is common \( r \)-divisor of \( a \) and \( b \), wherein \((x_0, Sgn(r))\) is a solution of (5) with respect of \( x, y \) (in that case \( d = b \)), then \( d \) is called \( r \)-divisor of \( a \); \( x_0 \) is called quotient of \( r \)-divisor \( a \) by \( d \) (\( r \)-quotient) and so on.

For example, number 3 is 0−, 1−, 2− divisor of numbers 5, 6, 9, correspondently (since \( 3 + 2 = 5, 3 \times 2 = 6, 3^2 = 9 \)).

The concepts of \( r \)-remainder, greatest common \( r \)-divisor, common \( r \)-multiple and some others can be defined in a similarly manner, by extending of well known notions, terms and, possibly, some statements, associated with multiplying and divisibility (that the same, 1-multiplying and 1-divisibility) of natural numbers to the case of their \( r \)-multiplying and \( r \)-divisibility, where \( r \neq 1 \).

No doubt that among all categories listed above, the most interesting one is the category of statements. In fact, it would be interesting to know what well known theorems on the divisibility of natural numbers remain valid (or, at least, can be transformed into similar in meaning their analogues) when going over to hyperoperator ranks different from the multiplicative one.
The rest of this paper is devoted to the first step in the direction of possible consider-
ations of this issue. There we will try the Fundamental Theorem of Arithmetic (FTA) to
variant of an exponential decomposition of natural numbers (that is, we will translate the
statement of FTA from the hyperoperator’s rank \( r = 1 \) to the rank \( r = 2 \)).

We will conclude the section with the following general remark, concerning the
system of notation we have adopted.

Talking about the properties of \( r \)-operators, everywhere, if possible, we will be using
the standard mathematical notation, adding upper \( /r/ \) marks, where they are needed, to
conventional mathematical designations. Thus, for instance, on the base of using of symbol
“|” as a designator of predicate “divides”, the previous examples of \( r \)-divisibility can be
written, correspondently, in the form of statements: \( 3|/0/5, 3|/1/6, 3|/2/9. \)

3 The exponential decomposability

Following the generalized hyperoperator paradigm of the consideration of basic arith-
etic operations, adopted in this article, as an example of possible cross rank boundary
migration of not only concepts, but also facts and theorems, related to natural numbers
divisibility, let’s ask ourselves the question:

If, in accordance with the FTA, any natural number greater than 1 either is a prime
itself or can be represented as the product of prime factors, so that such a representation is
unique, up to (except for) the order of the factors, then what can be said about the existence
of an analogous representation of any natural number as the bi-product of biprime factors?

The answer to this question is given by the following theorem.

Theorem 1. Theorem (on bi-decomposability). Any natural number \( M \) greater than
1 either is a biprime itself or can be uniquely represented as a tower-like exponentiation:
\( M = a_n \uparrow (a_{n-1} \uparrow (\ldots (a_2 \uparrow a_1) \ldots)), \) where \( a_i \neq 1(i = 1, \ldots, n) \) – are following one by one
in some unique order biprime exponentiation components.

Postponing for a while the proof of this theorem, we will first formulate and prove
the two lemmas.

Lemma 1. Let \( D \) is a bi-compound natural integer. Then there are exist \( d_0 \)-biprime and
\( D_0 \in \mathbb{N}(D_0 > 1) \) such, that:
\[
D = d_0^{D_0}. \tag{6}
\]

Proof of Lemma 1. So far as \( D \) is a bi-compound natural integer, it can be represented as:
\[
D = d_1^{D_1}. \tag{7}
\]

where \( d_1, D_1 \in \mathbb{N}, d_1 > 1, D_1 > 1 \). Here two cases are possible:

1) Number \( d_1 \) is biprime. Then in the formula (6) we set \( d_0 = d_1, D_0 = D_1 \) and
stop. For this case the lemma is proved.

2) Number \( d_1 \) is bi-compound. Then for some \( c_2 \in \mathbb{N} (c_2 > 1) \) we have \( d_1 = d_2^{c_2}, \)
where \( 1 < d_2 < d_1. \)

Substituting this expression for \( d_1 \) in (7) we obtain:
\[
D = d_2^{D_1} = d_2^{D_2}. \tag{8}
\]
If $d_2$ is biprime then in (6) we assume $d_0 = d_2$, $D_0 = D_2$, and lemma is proved. Otherwise there again can be found $c_3 \in \mathbb{N}$ $(c_3 > 1)$ and $1 < d_3 < d_2$, such that $d_2 = d_3^3$. Having substituted $d_2$ in (8), we will obtain $D = d_3^{2D_3} = d_3^{3D_3}$.

Continuing this process further and constructing a sequence of decreasing but staying greater than 1 elements $d_1$, $d_2$, . . . , we will necessarily stop and come to some element $d_s$, which turns out to be biprime.

The final result of this process is the exponent $D = d_s^{2D_s−1} = d_1^{c_1c_2...c_sD_1} = d_s^{D_s}$, where is $d_s$ biprime, $D_s > 1$. Supposing at the exponent $d_0 = d_s$, $D_0 = D_s$ we will obtain from it the expression (6).

The lemma 1 is proved for this case too. □

Lemma 2. For arbitrarily chosen different biprimes $a$, $b$, the exponential Diophantine equation $a^x = b^y$ is not solvable with respect to $x, y$.

Proof of Lemma 2. In the conditions of the above restrictions on $a$ and $b$, let us admit the opposite, that is, assume that the equation $a^x = b^y$ has at least one solution $(x_0, y_0)$. We have:

\[ a^{x_0} = b^{y_0} = C. \]  

Let $C = \prod_{i=1}^n p_i^{c_i}$, where $p_i$ — different prime numbers, and $c_i (c_i \in \mathbb{N})$ are their powers. Then, taking in consideration 9, we can write down:

\[ a = \prod_{i=1}^n p_i^{\alpha_i}, \quad b = \prod_{i=1}^n p_i^{\beta_i}, \]  
\[ \Rightarrow a^{x_0} = \prod_{i=1}^n p_i^{c_i x_0}, \quad b^{y_0} = \prod_{i=1}^n p_i^{\beta_i y_0}, \]  
\[ \Rightarrow \alpha_i x_0 = \beta_i y_0, \quad (i = 1, 2, ..., n). \]  

After denoting the least common multiple of numbers $x_0, y_0$ as $I = \text{LCM}(x_0, y_0)$, from (11, 12) can be deduced: $\alpha_i x_0 = I k_i$, $\beta_i y_0 = I k_i$, where $I = q x_0$, $I = r y_0$ and $q, r$, $k_i \in \mathbb{N}$.

That is, we have: $\alpha_i x_0 = q x_0 k_i \Rightarrow \alpha_i = q k_i; \beta_i y_0 = r y_0 k_i \Rightarrow \beta_i = r k_i$.

Substituting the last expressions for $\alpha_i$ and $\beta_i$ into (10) gives:

\[ a = \prod_{i=1}^n p_i^{q k_i}, \quad b = \prod_{i=1}^n p_i^{r k_i}. \]  

At last, denoting $d = \prod_{i=1}^n p_i^{k_i}$, from the formula (13), we deduce $a = d^q$, $b = d^r$, which contradicts the initial assumption that $a, b$ are biprimes. This contradiction proves the lemma 2. □

Proof of the Theorem 1. Let $M > 1$ be a natural number mentioned in the hypothesis of the Theorem 1. Here we are going to show that there exists a bi-factorization of $M$ into a sequence of biprime components.

If $M$ is biprime, then our goal is trivially reached. Otherwise, by lemma 1, we can write: $M = M_0 = a_1^{M_1}$, where $a_1$ is biprime, $M_1 \in \mathbb{N}$, $1 < M_1 < M_0$.

If $M_1$ is biprime, then the required sequence of biprime components is found. Otherwise, we will write down again: $M_1 = a_2^{M_2}$, where $a_2$ is biprime, $M_2 \in \mathbb{N}$, $1 < M_2 < M_1$.

Continuing this process further, we obtain a decreasing sequence of natural numbers $M_0 > M_1 > M_2 > ...$, which, because of its boundedness below, at some point breaks off on the element $M_s$ such that $M_s > 1$ and $M_s$ is non bi-compound. Thus $M_s$ turns out to be biprime. Renaming $M_s = a_s$, we get the desired bi-factorization:

\[ M = a_1 \uparrow (a_2 \uparrow (\ldots (a_{s−1} \uparrow a_s) \ldots)). \]  

(14)
or, the same in common notation (3):

$$M = \Pi_{i=s,1} H_2(a_1, H_2(a_2, \ldots, H_2(a_{s-1}, H_2(a_s, Sgn(2)))) \ldots)).$$

(15)

2. Now we will show the uniqueness of bi-factorization (14). Suppose it’s not unique, that is, can be found at least two different sequences of primes \((a_1, a_2, \ldots, a_{s_1})\) and \((b_1, b_2, \ldots, b_{s_2})\) such, that:

$$M = a_1 \uparrow (a_2 \uparrow (\ldots (a_{s_1-1} \uparrow a_{s_1}) \ldots)) = b_1 \uparrow (b_2 \uparrow (\ldots (b_{s_2-1} \uparrow b_{s_2}) \ldots)).$$

(16)

Let \(t = \min(s_1, s_2)\). The following cases may be occurred:

A) \(\exists e (1 \leq e \leq t, a_e \neq b_e)\) and \(\forall i (i = 1, e-1, a_i = b_i)\). In this case we obtain from (16) either the equality \(a_e^{M_1} = b_e^{M_2}\) \((a_e, b_e\) are biprimes, \(M_1 > 1, M_2 > 1)\), which, by the hypothesis of lemma 2, is impossible, or one of the two equalities: \(a_e^M = b_e\), or \(b_e^M = a_e\) \((a_e, b_e\) are biprimes, \(M > 1)\). The last two equalities, due to the fact that a biprime cannot coincide with a bi-compound number, are obviously also not possible.

B) \(a_i = b_i, (i = 1, t)\). In this case, from (16) we obtain a numerical equality of the form:

$$a_1 \uparrow (a_2 \uparrow (\ldots (a_t \uparrow M) \ldots)) = a_1 \uparrow (a_2 \uparrow (\ldots (a_{t-1} \uparrow a_t) \ldots)), \text{ where } M > 1.$$

(17)

But the last equality is, obviously, also not true (value of expression in the left part is greater than value of expression in the right one).

Thus, the equality (16) is possible only if the expression of its left part exactly coincide with the expression of its right part. The proof of the uniqueness of bi-factorization and the theorem as a whole are completed.

\[ \square \]

4 Conclusion, Open Problems

In this article we have considered the possibility of expanding the system of concepts based on the multiplication and division of natural numbers, in such a way as it would cover the entire infinite hierarchy of hyperoperatorial arithmetic operations. As an example and object for the first step, illustrating the author’s approach to such an extension, a hyperoperator of exponentiation was chosen.

Basic notions for this case are introduced: a bi-product and biprime ones. In particular, the latter is understood to mean a natural number that cannot be represented as the degree of two nontrivial (not equal to 1) natural numbers.

For an arithmetical hyperoperator of exponentiation, as an analog of the Fundamental Theorem of Arithmetic, a theorem on exponential (tower-like) decomposability of natural numbers is formulated and proved. The theorem states that any natural number (not equal to 1) is either biprime or can be uniquely represented as a bi-product of following in some order biprimes.

From the theorem, in particular, it follows (in the article this fact is stated and proved in the form of a lemma), that for arbitrarily chosen, different biprimes \(a, b\), the exponential Diophantine equation \(a \uparrow x = b \uparrow y\) is not solvable with respect to the \(x, y\).

There are two main directions on which this study could be continued.

1. Further development of the theory of bi-decomposability and biprime numbers; search for possible applications of this theory.
2. Extending of system of those the classical concepts, facts and statements related
to the multiplication and divisibility of natural numbers that admit their natural interpre-
tation throughout the hierarchy of hyperoperatorial arithmetic operations.

In connection with the second of the two directions of research mentioned above,
it would be interesting, in particular, the issue of the validity of the following generalized
version of the proved in this article theorem (here we will express this version in the form of
a hypothesis which it seems to be correct).

Hypothesis (on $r$–decomposability). For a given natural $r \neq 1$, any natural number
$M > 1$ is either $r$–simple in itself or it can be uniquely represented as an $r$–product of
$r$–prime components that follow in a certain order.

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