Stochastic Optimization with Non-stationary Noise

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Abstract

We investigate stochastic optimization problems under relaxed assumptions on the distribution of noise that are motivated by empirical observations in neural network training. Standard results on optimal convergence rates for stochastic optimization assume either there exists a uniform bound on the moments of the gradient noise, or that the noise decays as the algorithm progresses. These assumptions do not match the empirical behavior of optimization algorithms used in neural network training where the noise level in stochastic gradients could even increase with time. We address this behavior by studying convergence rates of stochastic gradient methods subject to changing second moment (or variance) of the stochastic oracle as the iterations progress. When the variation in the noise is known, we show that it is always beneficial to adapt the step-size and exploit the noise variability. When the noise statistics are unknown, we obtain similar improvements by developing an online estimator of the noise level, thereby recovering close variants of RMSPProp [33]. Consequently, our results reveal an important scenario where adaptive stepsize methods outperform SGD.

1 Introduction

Stochastic gradient descent (SGD) is one of the most popular optimization methods in machine learning because of its computational efficiency compared to traditional full gradient methods. Great progress has been made in understanding the performance of SGD under different smoothness and convexity conditions [1, 3, 8, 12, 13, 28, 29]. These results show that with a fixed step size, SGD can achieve the minimax optimal convergence rate for both convex and nonconvex optimization problems, provided the gradient noise is uniformly bounded.

Yet, despite the theoretical minimax optimality of SGD, adaptive gradient methods [9, 18, 33] have become the methods of choice for training deep neural networks, and have received a surge of attention recently [2, 7, 14, 19–24, 31, 34–40]. Instead of using fixed stepsizes, these methods construct their stepsizes adaptively using the current and past gradients. But despite advances in the literature on adaptivity, theoretical understanding of the benefits of adaptation is still quite limited.

We provide a different perspective on understanding the benefits of adaptivity by considering it in the context of non-stationary gradient noise, i.e., the noise intensity varies with iteration. Surprisingly, this setting is rarely studied, even for SGD. To our knowledge, this is the first work to formally study stochastic gradient methods in this varying noise scenario. Our main goal is to show that:

Adaptive step-sizes can guarantee faster rates than SGD when the noise is non-stationary.

We focus on this goal based on several empirical observations (Section 2), which lead us to model the noise of stochastic gradient oracles via the following more realistic assumptions:

\[ \mathbb{E}[\|g(x_k)\|^2] = m_k^2, \quad \text{or} \quad \mathbb{E}[\|g(x_k) - \nabla f(x_k)\|^2] = \sigma_k^2, \quad (1) \]

where \( g(x_k) \) is the stochastic gradient and \( \nabla f(x_k) \) the true gradient at iteration \( k \). The second moments \( \{m_k\}_{k \in \mathbb{N}} \) and variances \( \{\sigma_k^2\}_{k \in \mathbb{N}} \) are independent of the algorithm.

Assumption (1) relaxes the standard assumption (on SGD) that uniformly bounds the variance, and helps model gradient methods that operate with iteration dependent noise intensity. It is intuitive
Figure 1: We empirically evaluate the second moment (in blue) and variance (in orange) of stochastic gradients during the training of neural networks. We observe that the magnitude of these quantities changes significantly as iteration count increases, ranging from 10 times (ResNet) to $10^3$ times (Transformer). This phenomenon motivates us to consider a setting with non-stationary noise.

that one should prefer smaller stepsizes when the noise is large and vice versa. Thus, under non-stationarity, an ideal algorithm should adapt its stepsizes according to the parameters $m_k$ or $\sigma_k$, suggesting a potential benefit of using adaptive stepsizes.

**Contributions.** The primary contribution of our paper is to show that a stochastic optimization method with adaptive stepsizes can achieve a faster rate of convergence (by a factor that is polynomial-in-$T$) than fixed-step SGD. We first analyze an idealized setting where the noise intensities are known, using it to illustrate how to select noise dependent stepsizes that are provably more effective (Theorem 1). Next, we study the case with unknown noise, where we show under an appropriate smoothness assumption on the noise variation that a variant of RMSProp \cite{33} can achieve the idealized convergence rate (Theorem 3). Remarkably, this variant does not require the noise levels. Finally, we generalize our results to nonconvex settings (Theorems 12 and 13).

2 Motivating observation: nonstationary noise in neural network training

Neural network training involves optimizing an empirical risk minimization problem of the form $\min_x f(x) := \frac{1}{n} \sum_{i=1}^{n} f_i(x)$, where each $f_i$ represents the loss function with respect to the $i$-th data or minibatch. Stochastic methods optimize this objective randomly sampling an incremental gradient $\nabla f_i$ at each iteration and using it as an unbiased estimate of the full gradient. The noise intensity of this stochastic gradient is measured by its second moments or variances, defined as,

1. **Second moment:** $m^2(x) = \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x) \|^2$;

2. **Variance:** $\sigma^2(x) = \frac{1}{n} \sum_{i=1}^{n} \| \nabla f_i(x) - \nabla f(x) \|^2$, where $\nabla f(x)$ is the full gradient.

To illustrate how these quantities evolve over iterations, we empirically evaluate them along three popular tasks of neural network training: ResNet18 training on Cifar10 dataset for image classification\(^1\), LSTM training on PTB dataset for language modelling\(^2\); transformer training on WMT16 en-de for language translation\(^3\). The results are shown in Figure 1, where both the second moments and variances are evaluated using the default training procedure of the original code.

On one hand, the variation of the second moment/variance has a very different shape in each of the considered tasks. In the CIFAR experiment, the noise intensity is quite steady after the first iteration, indicating a fast convergence of the training model. In LSTM training, the noise level increases and converges to a threshold. While, in training Transformers, the noise level increases very fast at the early epochs, then reaches a maximum, and turns down gradually.

On the other hand, the preferred optimization algorithms in these tasks are also different. For CIFAR10, SGD with momentum is the most popular choice. While for language models, adaptive methods such as Adam or RMSProp are the rule of thumb. This discrepancy is usually taken as granted, based on empirical validation; and little theoretical understanding of it exists in the literature.

Based on the observations made in Figure 1, a natural candidate emerges to explain this discrepancy in the choice of algorithms: the performance of different stochastic algorithms used varies according

\(^1\)Code source for CIFAR10 https://github.com/kuangliu/pytorch-cifar
\(^2\)Code source for LSTM https://github.com/salesforce/awd-lstm-lm
\(^3\)Code source for Transformer https://github.com/jadore801120/attention-is-all-you-need-pytorch
the characteristics of gradient noise encountered during training. Despite this behavior, noise level modeling has drawn surprisingly limited attention in prior art. Reference [26] studies convergence of SGD assuming each component function is convex and smooth; extensions to the variation of the full covariance matrix are in [11]. A more fine-grained assumption is that the variances grow with the gradient norm as $\sigma^2 + c\|\nabla f(x)\|^2$, or grow with the suboptimality $\sigma^2 + c\|x - x^*\|^2$ [5, 17, 30].

Unfortunately, these known assumptions fail to express the variation of noise observed in Figure 1. Indeed, the norm of the full gradient, represented as the difference between the orange and the blue line, is significantly smaller compared to the noise level. This suggests that noise variation is not due to the gradient norm, but due to some implicit properties of the objective function. The limitations of the existing assumptions motivates us to introduce the following assumption on the noise:

**Assumption 1 (non-stationary noise oracle).** The stochasticity of the problem is governed by a sequence of second moments $\{m_k\}_{k \in \mathbb{N}}$ or variances $\{\sigma_k\}_{k \in \mathbb{N}}$, such that, at the $k$th iteration, the gradient oracle returns an unbiased gradient $g(x_k)$ such that $\mathbb{E}[g(x_k)] = \nabla f(x_k)$, and either

(a) with second moment $\mathbb{E}[\|g(x_k)\|^2] = m_k^2$; or

(b) with variance $\mathbb{E}[\|g(x_k) - \nabla f(x_k)\|^2] = \sigma_k^2$.

By introducing time dependent second moments and variance, we aim to understand how the variation of noise influences the convergence rate of optimization algorithms. Under our assumption, the change of the noise level is decoupled from its location, meaning that the parameters $m_k$ or $\sigma_k$ only depend on the iteration number $k$, but do not depend on the specific location where the gradient is evaluated. This assumption holds for example when the noise is additive to the gradient, namely $g(x_k) \sim \nabla f(x_k) + \mathcal{N}(0, \sigma_k^2)$. Even though this iterate independence of noise may seem restricted, it is already more relaxed than the standard assumption on SGD that requires the noise to be uniformly upper bound by a fixed constant. Thus, our relaxed assumption helps us take the first step toward our goal: **characterize the convergence rate of adaptive algorithms under non-stationary noise.**

To avoid redundancy, we present our results mainly based on the second moment parameters $m_k$ and defer the discussion on $\sigma_k$ to Section 5. One reason that we prioritize the second moment than the variance is to draw a connection with the well-known adaptive method RMSProp [33].

## 3 The benefit of adaptivity under nonstationary noise

In this section, we investigate the influence of nonstationary noise in an idealized setting where the noise parameters $m_k$ are known. To simplify the presentation, we will first focus on the convex setting. Similar results also hold for nonconvex problems and are noted later in Section 5.

Let $f$ be convex and differentiable. We consider the problem $\min_x f(x)$, where the gradient is given by the nonstationary noise oracle satisfying Assumption 1. We assume that the optimum is attained at $x^*$ and we denote $f^*$ the minimum of the objective. We are interested in studying the convergence rate of a stochastic algorithm with update rule

$$x_{k+1} = x_k - \eta_k g(x_k),$$

where $\eta_k$ are stepsizes that are oblivious of the iterates $\{x_k\}_{k \in \mathbb{N}}$.

**Theorem 1.** Under Assumption 1 (a), the weighted average $\mathbf{x}_T = (\sum_{k=1}^T \eta_k x_k) / (\sum_{k=1}^T \eta_k)$ of the iterates obtained by the update rule (2) satisfies the suboptimality bound

$$\mathbb{E}[f(\mathbf{x}_T) - f^*] \leq \frac{\|x_1 - x^*\|^2 + \sum_{k=1}^T \eta_k^2 m_k^2}{\sum_{k=1}^T \eta_k}.$$

The theorem follows from standard analysis, yet it leads to valuable observations explained below.

**Corollary 2.** Let $R = \|x_1 - x^*\|$. We have the following two convergence rate bounds for SGD:

1. **SGD with constant stepsize:** if $\eta_k = \eta = \frac{R}{\sqrt{\sum_{k=1}^T m_k^2}}$, then

$$\mathbb{E}[f(\mathbf{x}_T) - f^*] \leq 2R \frac{\sqrt{\sum_{k=1}^T m_k^2}}{T} = \frac{2R}{\sqrt{T}} \cdot \sqrt{\sum_{k=1}^T m_k^2}.$$

(constant baseline)
2. **SGD with idealized stepsize:** If \( \eta_k = \frac{R}{\sqrt{T m_k}} \), then

\[
\mathbb{E}[f(\mathbb{E}_T) - f^*] \leq \frac{2R\sqrt{T}}{\sum_{k=1}^{T} m_k} = \frac{2R}{\sqrt{T}} \cdot \frac{T}{\sum_{k=1}^{T} m_k},
\]

(idealized baseline)

To facilitate comparison, in Corollary 2 we have normalized the convergence rates with respect to the conventional rate \( 2R/\sqrt{T} \). The constant baseline has an additional factor depending on the average of the \( m_k^2 \) value, and the remaining factor in idealized baseline is inversely proportional to the average of \( 1/m_k \). In particular, from Jensen’s inequality \( \mathbb{E}[X]^{-2} \leq \mathbb{E}[X^{-2}] \), we have

\[
\left( \frac{1}{T} \sum_{k=1}^{T} \frac{1}{m_k} \right)^{-2} \leq \frac{1}{T} \sum_{k=1}^{T} m_k^2,
\]

implying that the idealized baseline is always better than the constant baseline. This observation is rather expected, as the stepsizes are adapted to the noise in an idealized way. It is worth noticing that the constant baseline also benefits from explicit knowledge on \( m_k \), replacing the upper bound \( M \) by the average value \( \sqrt{\sum m_k^2/T} \).

If the actual value of \( m_k \) is unavailable, but an upper bound \( M \) such that \( m_k \leq M \) is known, then replacing all the \( m_k \) values by \( M \) in both algorithms recovers the standard \( R M/\sqrt{T} \) result [27].

To further illustrate the difference in the convergences rate, we consider the following synthetic noise model, mimicking our observations in the training of Transformer (see Figure 1(c)).

**Example 1.** Consider the following piece-wise linear noise model with \( \gamma = 5(1 - T^{-\alpha})/T \).

\[
m_k = \begin{cases} 
\frac{1}{T} & \text{if } k \in [1, T]; \\
\gamma(k - \frac{2T}{3}) + 1 & \text{if } k \in \left( \frac{2T}{3}, \frac{4T}{3} \right]; \\
1 & \text{if } k \in \left( \frac{4T}{3}, \frac{5T}{3} \right]; \\
\gamma\left(\frac{3T}{4} - k \right) + 1 & \text{if } k \in \left( \frac{3T}{4}, \frac{5T}{4} \right]; \\
\frac{1}{T} & \text{if } k \in \left( \frac{5T}{4}, T \right].
\end{cases}
\]

For the above \( m_k \), the maximum noise level \( M \) is 1, and the minimum level is \( 1/T^\alpha \), inducing a large ratio of order \( T^\alpha \). Following the bounds developed in Corollary 2, the performance of the constant baseline maintains the standard \( O(1/\sqrt{T}) \) convergence rate, while as the idealized baseline converges at \( O(1/T^{1+\alpha}) \). Hence a nontrivial acceleration of order \( T^\alpha \) is obtained by using the idealized stepsizes, and this acceleration can be arbitrarily large as \( \alpha \) increases.

This example is encouraging, showing that the speedup due to adaptive stepsizes can be polynomial in the number of iterations, especially when the ratio between the maximum and the minimum noise level is large. However, explicit knowledge on \( m_k \) is required to implement these idealized stepsizes, which is unrealistic. The goal in the rest of the paper is to show that approximating the moment bound in an online fashion can also achieve convergence rate comparable with the idealized setting.

### 4 Adaptive methods: Online estimation of noise

From now on, we assume that the moment bounds \( m_k \) are not given. To address the non-stationarity, we estimate the noise intensity based on an exponential moving average, a technique commonly used in adaptive methods. More precisely, the moment estimator \( \hat{m}_k \) is constructed recursively as

\[
\hat{m}_{k+1}^2 = \beta \hat{m}_k^2 + (1 - \beta) \| g_k \|^2,
\]

(ExpMvAvg)

where \( g_k \) is the \( k \)-th stochastic gradient and \( \beta \) is the decay parameter. Then we choose the stepsizes inversely proportional to \( \hat{m}_{k+1} \), leading to Algorithm 1.

Indeed, Algorithm 1 could be viewed as a “norm” version of RMSProp [33]: while in RMSProp the exponential moving average is performed coordinate-wise, we use the full norm of \( g_k \) to update the moment estimator \( \hat{m}_{k+1} \). Such a simplification via a full norm variant has also been used in the uniformly bounded noise setting [19–21, 34]—we leave the more advanced coordinate-wise version as a topic of future research. Another important component in the stepsize is the correction constant \( m \), shown in the denominator of the stepsize. This constant provides a safety threshold when \( \hat{m}_k \)
underestimates $m_k$, which is commonly used in the practical implementation of adaptive methods, and even beyond, in reinforcement learning as a so-called exploration bonus [4, 15, 32].

To show the convergence of the algorithm, we need to impose a regularity assumption on the sequence of noise intensities. Otherwise, previous estimate may not provide any information of the next one.

**Assumption 2.** We assume that an upper bound $M$ on $m_k$ is given, i.e. $\max_k m_k \leq M$ such that

(a) The fourth moment of $g_k$ is bounded by $M^4$, namely, $\mathbb{E}[(\|g_k\|^2 - m_k^2)^2] \leq M^4$, $\forall k$.

(b) The total variation on $m_k$ is bounded by

$$\sum_k |m_k^2 - m_{k+1}^2| \leq D^2 = 4M^2.$$  \hfill (4)

The fourth moment assumption ensures concentration of $\|g_k\|^2$, which is necessary to provide a finite-sample analysis. In particular, this is satisfied when $\|g_k\|$ follows a $m_k$ sub-Gaussian distribution. The other assumption is less straightforward, and deserves a broader discussion here. A key aspect of the bounded variation is to avoid infinite oscillation, such as the pathological setting where $m_{2k} = 1$ and $m_{2k+1} = M$, in which case the total variation scales with the number of iterations $T$. We emphasize that the choice of the constant 4 is not special, and it can be easily replaced by any arbitrary constant, i.e., $D^2 = \Omega(M^2)$. When $m_k$ is increasing in the first half and decreasing in the second half, as in the Transformer and Example 1, the total variation bound in (13) is satisfied. More generally, one could allow $K$ piece-wise monotone fragment by changing the constant to $K^2$.

With the above assumptions, we are now ready to present our convergence analysis.

**Theorem 3.** Under Assumptions 1, 2, and $T$ large enough such that $2T^{-1/9} \ln(T)^{1/3} \leq 1$, with probability at least $1/2$, the iterates generated by Algorithm 1 using parameters $\beta = 1 - 2T^{-2/3}$, $m = 2MT^{-1/9} \ln(T)^{1/3}$, $c = \frac{R}{\sqrt{T}}$ satisfy

$$f(\hat{x}_T) - f^* \leq \frac{2R}{\sqrt{T}} \sum_{k=1}^{\frac{4T}{\ln(T)^{1/3}}} \frac{1}{m_k + m}.$$  

**Remark 4.** Our result directly implies a $1 - \delta$ high probability style convergence rate, by restarting it $2 \log(1/\delta)$ times. An additional $\log(1/\delta)$ dependency will be introduced in the complexity, as in standard high probability results [10, 16, 21, 27].

The key to prove the theorem is to effectively bound the estimation error $|\hat{m}_k^2 - m_k^2|$ relying on concentration, and on bounded variation in Assumption 2. In particular, the choice of the decay parameter $\beta$ is critical, determining how fast the contribution of past gradients decays. Because of the non-stationarity in noise, the online estimator $\hat{m}_k$ is biased. The proposed choice of $\beta$ carefully balances the bias error and the variance error, leading to sublinear regret, see Appendix B.

Due to the correction constant $m$, the obtained convergence rate inversely depends on $\sum_{k=1}^{T} \frac{1}{m_k + m}$, instead of the idealized dependency $\sum_{k=1}^{T} \frac{1}{m_k}$. This additional term makes the comparison less straightforward and we now provide different scenarios to better understand it.

### 4.1 Discussion of the convergence rate

To illustrate the difference between convergence rates, we first consider the synthetic noise model introduced in Example 1. The detailed comparison is presented in Table 1, where we observe two regimes regarding the exponent $\alpha$:
\[
\begin{array}{ccc}
0 \leq \alpha \leq \frac{1}{9} & O\left(T^{-\frac{1}{2}}\right) & O\left(T^{-\frac{14+2\alpha}{9}}\right) \\
\frac{1}{9} < \alpha & O\left(T^{-\frac{1}{2}}\right) & O\left(T^{-\frac{14}{18}}\right)
\end{array}
\]

Table 1: Comparison of the convergence rate under the noise example 1.

- When \(0 \leq \alpha \leq \frac{1}{9}\), the rate of the adaptive algorithm matches (idealized baseline) up to logarithmic dependency, and is \(T^{\alpha}\) better than the (constant baseline).
- When \(\frac{1}{9} < \alpha\), the adaptive convergence rate no longer matches the (idealized baseline). Nevertheless, it is always \(T^{\frac{1}{9}}\) faster than the (constant baseline).

In both cases, the adaptive method achieves a non-trivial improvement, polynomial in \(T\), compared to the (constant baseline). Even though the improvement \(T^{\frac{1}{9}}\) might seem insignificant, it is the first result showing a plausible non-trivial advantage of adaptive methods over SGD under nonstationary-noise. Further, note that the adaptive convergence rate does not always match the (idealized baseline) when \(\alpha\) is large. Such a discrepancy comes from the correction term \(m\), which makes the stepsize more conservative than it should be, especially when \(m_k\) is small.

The condition in Corollary 6 is strictly weaker than the condition in Corollary 5, which means even though an adaptive method may not match the idealized baseline, it could still be non-trivially better than the constant baseline. This case happens e.g., when \(\alpha > \frac{1}{9}\) in Table 1, where the adaptive method is \(O(T^{\frac{1}{9}})\) faster than the constant baseline. Indeed, \(O(T^{\frac{1}{9}})\) is the maximum improvement one can expect according to our current analysis.

**Corollary 5.** If the ratio \(M/(\min_k m_k) \leq T^{\frac{1}{9}}\), then adaptive method converges in the same order as the (idealized baseline), up to logarithmic dependency.

This result is remarkable since the adaptive method does not require any knowledge of \(m_k\) values, and yet it achieves the idealized rate. In other words, the exponential moving average estimator successfully adapts to the variation in noise, allowing faster convergence than constant stepizes.

**Corollary 6.** Let \(m_{avg}^2 = \sum m_k^2 / T\) be the average second moment. If \(M/m_{avg} \leq T^{\frac{1}{9}}\), then adaptive method is no slower than the (constant baseline), up to logarithmic dependency.

The condition in Corollary 6 is strictly weaker than the condition in Corollary 5, which means even though an adaptive method may not match the idealized baseline, it could still be non-trivially better than the constant baseline. This case happens e.g., when \(\alpha > \frac{1}{9}\) in Table 1, where the adaptive method is \(O(T^{\frac{1}{9}})\) faster than the constant baseline. Indeed, \(O(T^{\frac{1}{9}})\) is the maximum improvement one can expect according to our current analysis.

**Corollary 7.** Recall that \(M\) is an upper bound on \(m_k\), i.e., \(\max m_k \leq M\). Therefore

1. The convergence rate of the constant baseline is no slower than \(O(2RM/\sqrt{T})\).
2. The convergence rate of the adaptive method is no faster than \(\hat{O}(2RM/T^{\frac{1}{4}+\frac{1}{9}})\).

The order of maximum improvement \(O(T^{\frac{1}{9}})\) is determined by the specific choice of \(m\) in Theorem 3, which is chosen to be \(\hat{O}(MT^{-\frac{1}{9}})\). Indeed, the correction term is helpful when the estimator \(\hat{m}_k\) underestimates the true value \(m_k\), avoiding the singularity at zero. Hence, the choice of \(m\) is related to the average deviation between \(\hat{m}_k\) and \(m_k\). Under a stronger concentration assumption, we can strengthen the maximum improvement to \(O(T^{\frac{1}{9}})\), as shown in Appendix D.

The noise model in Example 1 provides a favorable scenario where the maximum improvement is attained. However, in some scenarios, the convergence rate of an adaptive method can be slower than the constant baseline.

**Adversarial scenario.** If \(m_k = 1/T^\alpha\) for all \(i \in [1, T]\) except at \(T/2\) it takes the value \(m_{T/2} = 1\) with \(\alpha > 1/9\), then the convergence rate of both constant and idealized baselines are \(O(T^{-\frac{1}{9}})\), while the adaptive method only converges in \(\hat{O}(T^{-\frac{1+2\alpha}{9}})\). The subtle change at iteration \(T/2\) amplifies the exponential moving average estimator and requires a non-negligible period to get back to the constant level. It is clear that the estimator becomes less meaningful under such a subtle change.

Overall, it is hard to provide a complete characterization of the variation in noise. In Corollary 5 and 6, we show that when the ratio between the maximum and the minimum/average second moment is not growing too fast, adaptive methods do improve upon SGD.
5 Extensions of Thm 3

In this section, we discuss several extensions to Thm 3. The results are nontrivial but the analysis is almost the same. Hence we defer the exact statements and proofs to appendices.

Addressing the variance oracle So far, we have focused on the noise oracle based on the second moment \(m_k\) and made the connection with existing adaptive methods. However, there is some unnaturalness underlying the assumption on \(m_k\). Indeed, it is hard to argue that \(m_k\) is iterate independent since \(m_k^2 = \sigma_k^2 + \|\nabla f(x_k)\|^2\). Even though the influence of \(\|\nabla f(x_k)\|^2\) might be minor when the variance \(\sigma_k^2\) is high (e.g. as in Figure 1), it is still changing \(m_k\). In contrast, the variance \(\sigma_k\) is an intrinsic quantity coming from the noise model, which could be iterate independent. Hence the variance oracle is theoretically more sound. We now present the necessary modifications in order to adapt to the variance oracle.

First, in order to estimate the variance, we need to query two stochastic gradients \(g_k\) and \(g'_k\) at the same iterate, then we construct the estimator following the recursion
\[
\hat{\sigma}_{k+1}^2 = \beta \hat{\sigma}_k^2 + (1 - \beta)\|g_k - g'_k\|^2.
\]

Second, the smoothness condition on \(f\) is required, i.e., \(L\)-Lipschitzness on the gradient of \(f\). In this case, it is necessary to ensure that the step-size being not larger than \(1/2L\). This translates to an additional constraint on the correcting constant \(m\). More precisely, the stepsize is given by
\[
\eta_k = \frac{e}{\sigma_k + m} \quad \text{with} \quad m \geq 2cL.
\]

Remark that the \(L\)-smoothness condition is not required in the second moment oracle. This is why the second moment assumption is usually imposed in the non-differentiable setting (see Section 6.1 of [6]). A thorough algorithm for the variance oracle is provided in Algorithm 2. The convergence results are essentially the same by substituting \(m_k\) to \(\sigma_k\), we refer to Appendix G for more details.

Extension to nonconvex setting We also provide an extension of our analysis to non-convex smooth setting. In which case, we characterize the convergence with respect to the gradient norm \(\|\nabla f(x_k)\|^2\), i.e. convergence to stationary point. The conclusion are very similar to the one in the convex setting and the results (Thm 12, Thm 13) are deferred to Appendix E.

Variants on stepsizes To go beyond the second moment of noise, one could apply an estimator of the form \(\hat{m}_{k+1}^p = \beta \hat{m}_k^p + (1 - \beta)\|g_k\|^p\) when the \(p\)-th moment of the gradient is bounded. This allows stepsize of the shape \(\eta_k \propto 1/(\hat{m}_k^p + m^p)^{1/p}\) as in Adamax [18].

Online estimator Note that we have chosen the exponential moving average estimator since it is the most popular choice in practice. One could apply other estimator to approximate the noise level. The log factor in Lemma 8 is mainly due to the non-uniform accumulation of the error created by the exponential moving average. An alternative is to apply a uniform-averaging estimator of type \(\hat{m}_{k+1} = \sum_{i=1}^W \hat{m}_{k-i}/W\), where \(W\) represents a predefined window size. In this case, log factor in Lemma 8 could be removed. However, memory proportional to \(W\) is required to implement this uniform estimator. Hence, we choose the exponential moving average for simple implementation.

6 Experiments

In this section, we describe two sets of experiments that verify the faster convergence of Algorithm 1 against the vanilla SGD. The first experiment solves a linear regression problem with injected noise. The second experiment trains an AWD-LSTM model [25] on PTB dataset for language modeling.

6.1 Synthetic experiments

In the synthetic experiment, we generate a random linear regression dataset using the sklearn library. We design the stochastic oracle as full gradient with injected Gaussian noise, whose coordinate-wise standard deviation \(\sigma\) is shown in the left figure of Fig 2. We then run the four algorithms discussed in this work: standard baseline, idealized baseline, Alg 1 and Alg 2. We finetune the step sizes for each algorithm by grid-searching among \(10^k\), where \(k\) is an integer. We repeat the experiment for 10 runs and show the average training trajectory as well as the function suboptimality in Fig 2. We observe that the performance is ranked as follows: idealized baseline, Alg 2, Alg 1 and standard baseline.

\footnote{scikit-learn.org/stable/modules/generated/sklearn.datasets.make_regression.html}
Figure 2: **Left:** The injected noise level over iterations. **Middle:** Average loss trajectory over 10 runs for four different algorithms: standard baseline, idealized baseline, Alg 1 and Alg 2. The curve (idealized vs standard) confirms that adopting step sizes inverse to the noise level lead to faster convergence and less variations. **Right:** Average and standard deviation of function suboptimality. The values are normalized by the average MSE of the standard baseline.

Figure 3: The training loss and validation loss for LSTM Language modelling from [25]. The baseline is provided by the authors. Algorithm 1 is described in Alg 1.

### 6.2 Neural Network training

We demonstrate how the proposed algorithm performs in real-world neural network training. We implement our algorithm into the AWD-LSTM codebase described in [25]. The original codebase trains the network using clipped gradient descent followed by an average SGD (ASGD) algorithm to prevent overfitting. As generalization error is beyond our discussion, we focus on the first phase (which takes about 200 epochs) by removing the ASGD training part. We see from Figure 3 that our proposed algorithm can achieve similar performance as the finetuned clipped SGD baseline provided by [25]. This confirms that Alg 1 is a practical algorithm. However, further explorations on more state of art architectures is required to conclude its effectiveness.

### 7 Conclusions

This paper discusses convergence rates of stochastic gradient methods in an empirically motivated setting where the noise level is changing over iterations. We show that under mild assumptions, one can achieve faster convergence than the fixed step SGD by a factor that is polynomial in number of iterations, by applying online noise estimation and adaptive step sizes. Our analysis, therefore provides one explanation for the recent success of adaptive methods in neural network training.

There is much more to be done along the line of non-stationary stochastic optimization. Under our current analysis, there is a gap between the adaptive method and the idealized method when the noise variation is large (see second row in Table 1). A natural question to ask is whether one could reduce this gap, or alternatively, is there any threshold preventing the adaptive method from getting arbitrarily close to the idealized baseline? Moreover, could one attain further acceleration by combining momentum or coordinate-wise update techniques? Answering these questions would provide more insight and lead to a better understanding of adaptive methods.

Perhaps a more fundamental question is regarding the iterate dependency: the setting where the moments $m_k$ or the variance $\sigma_k$ are functions of the current update $x_k$, not just of the iteration index $k$. Significant effort needs to be spent to address this additional correlation under appropriate regularity conditions. We believe our work lays the foundation to address this challenging research problem.

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A Proof of Theorem 1

Proof. The iterate suboptimality have the following relationship:
\[ \|x_{k+1} - x^*\|^2 = \|x_k - \eta_k g_k - x^*\|^2 = \|x_k - x^*\|^2 - 2\eta_k \langle g_k, x_k - x^* \rangle + \eta_k^2 \|g_k\|^2. \]
Rearrange and take expectation with respect to \( g_k \) we have
\[ 2\eta_k (f(x_k) - f^*) \leq 2\eta_k \langle \nabla f(x_k), x_k - x^* \rangle \]
\[ \leq \mathbb{E}\|x_{k+1} - x^*\|^2 - \mathbb{E}\|x_k - x^*\|^2 + \eta_k^2 m_k^2 \]
Sum over \( k \) and take expectation we get
\[ \mathbb{E}\left( \sum_{k=1}^{T} 2\eta_k (f(x_k) - f^*) \right) \leq \|x_1 - x^*\|^2 + \sum_{k=1}^{T} \eta_k^2 m_k^2 \]
Then from convexity, we have
\[ \mathbb{E}[f(\mathbb{x}_T) - f^*] \leq \frac{\|x_1 - x^*\|^2 + \sum_{k=1}^{T} \eta_k^2 m_k^2}{\sum_{k=1}^{T} \eta_k} \]
where \( \mathbb{x}_T = (\sum_{i=1}^{T} \eta_i x_i)/(\sum_{i=1}^{T} \eta_i) \). Corollary 2 follows from specifying the particular choices of the stepsizes.

B Key Lemma

Lemma 8. Under Assumptions 2, taking \( \beta = 1 - 2T^{-2/3} \), the total estimation error of the \( \hat{m}_k^2 \) based on (ExpMvAvg) is bounded by:
\[ \mathbb{E}\left[ \sum_{k=1}^{T} |\hat{m}_k^2 - m_k^2| \right] \leq 2(D^2 + M^2)T^{2/3} \ln(T^{2/3}) \]

Proof. On a high level, we decouple the error in a bias term and a variance term. We use the total variation assumption to bound the bias term, and use the exponential moving average to reduce variance. Then we pick \( \beta \) to balance the two terms.
From triangle inequality, we have
\[ \sum_{k=0}^{T} \mathbb{E}\left[ |\hat{m}_k^2 - m_k^2| \right] \leq \sum_{k=1}^{T} \mathbb{E}\left[ |\hat{m}_k^2 - \mathbb{E}[\hat{m}_k^2]| + |\mathbb{E}[\hat{m}_k^2] - m_k^2| \right] \]
(5)
We first bound the bias term. By definition of \( \hat{m}_k \), we have
\[ \mathbb{E}[\hat{m}_k^2] - m_k^2 = \beta \mathbb{E}[\hat{m}_{k-1}^2] + (1 - \beta) m_{k-1}^2 - m_k^2 \]
\[ = \beta (\mathbb{E}[\hat{m}_{k-1}^2] - m_{k-1}^2) + m_{k-1}^2 - m_k^2 \]
Hence by recursion,
\[ \mathbb{E}[\hat{m}_k^2] - m_k^2 = \beta^{k-1}(\mathbb{E}[\hat{m}_1^2] - m_1^2) + \beta^{k-2}(m_1^2 - m_2^2) + \cdots + (m_{k-1}^2 - m_k^2) \]
Therefore, the bias term could be bounded by
\[ \sum_{k=1}^{T} \mathbb{E}[|\hat{m}_k^2 - m_k^2|] \leq \sum_{k=1}^{T} \sum_{j=1}^{k-1} \beta^{k-1-j} |m_j^2 - m_{j+1}^2| \]
\[ = \sum_{k=1}^{T-1} |m_k^2 - m_{k+1}^2| \sum_{j=0}^{T-1-k} \beta^j \]
\[ \leq \frac{1}{1 - \beta} \sum_{k=1}^{T-1} |m_k^2 - m_{k+1}^2| \]
\[ \leq \frac{D^2}{1 - \beta} \quad \text{(From Assumption (2))} \]
The first inequality follows by triangle inequality. The third inequality uses the geometric sum over $\beta$. To bound the variance term, we remark that

$$\hat{m}_k^2 = (1 - \beta)g_{k-1}^2 + (1 - \beta)\beta g_{k-2}^2 + \cdots + (1 - \beta)\beta^{k-2}g_1^2 + \beta^{k-1}g_0^2.$$ 

Hence from independence of the gradients, we have

$$\mathbb{E}[|\hat{m}_k^2 - \mathbb{E}[\hat{m}_k^2]|] \leq \sqrt{\text{Var}[\hat{m}_k^2]}$$

$$= \sqrt{\text{Var}[(1 - \beta)g_{k-1}^2] + \text{Var}[(1 - \beta)\beta g_{k-2}^2] + \cdots + \text{Var}[(1 - \beta)\beta^{k-2}g_1^2] + \text{Var}[\beta^{k-1}g_0^2]}$$

$$\leq \sqrt{(1 - \beta)^2 + (1 - \beta)^2\beta^2 + \cdots + (1 - \beta)^2\beta^{2(k-2)} + \beta^{2(k-1)}M^2},$$

where $M^2$ is an upperbound on the variance. The first inequality follows by Jensen’s inequality. The second equality uses independence of $g_i$ given $g_1, ..., g_{i-1}$. The last inequality follows by assumption 2.

We distinguish two cases, when $k$ is small, we simply bound the coefficient by 1, i.e.

$$\sqrt{(1 - \beta)^2 + (1 - \beta)^2\beta^2 + \cdots + (1 - \beta)^2\beta^{2(k-2)} + \beta^{2(k-1)}} \leq 1$$

When $k$ is large such that $k \geq 1 + \gamma$, with $\gamma = \frac{1}{2(1 - \beta)} \ln(\frac{1}{1 - \beta})$, we have $\beta^{2(k-1)} \leq 1 - \beta$, thus

$$\sqrt{(1 - \beta)^2 + (1 - \beta)^2\beta^2 + \cdots + (1 - \beta)^2\beta^{2(k-2)} + \beta^{2(k-1)}}$$

$$\leq \sqrt{\frac{(1 - \beta)^2}{1 - \beta^2} + \beta^{2(k-1)}}$$

$$\leq \sqrt{\frac{(1 - \beta)^2}{1 - \beta^2} + (1 - \beta)}$$

$$\leq \sqrt{2(1 - \beta)}$$

The second inequality follows by $k \geq 1 + \gamma$, with $\gamma = \frac{1}{2(1 - \beta)} \ln(\frac{1}{1 - \beta})$. Therefore, when $k \geq 1 + \gamma$,

$$\mathbb{E}[|\hat{m}_k^2 - \mathbb{E}[\hat{m}_k^2]|] \leq \sqrt{2(1 - \beta)}M$$

Therefore, substitute in the above equation into the

$$\sum_{k=1}^{T} \mathbb{E}[|\hat{m}_k^2 - \mathbb{E}[\hat{m}_k^2]|] = \sum_{k=1}^{T} \mathbb{E}[|\hat{m}_k^2 - \mathbb{E}[\hat{m}_k^2]|] + \sum_{k=\gamma+1}^{T} \mathbb{E}[|\hat{m}_k^2 - \mathbb{E}[\hat{m}_k^2]|]$$

$$\leq (\gamma + (T - \gamma)\sqrt{2(1 - \beta)})M^2$$

Summing up the variance term and the bias term yields,

$$\sum_{k=0}^{T} \mathbb{E}[|\hat{m}_k^2 - m_k^2|] \leq \frac{D^2}{1 - \beta} + (\gamma + (T - \gamma)\sqrt{2(1 - \beta)})M^2$$

(6)

Taking $\beta = 1 - T^{-2/3}/2$ yields,

$$\sum_{k=0}^{T} \mathbb{E}[|\hat{m}_k^2 - m_k^2|] \leq 2(D^2 + M^2)T^{2/3} \ln(T^{2/3})$$

(7)
C  Proof of Theorem 3

On a high level, the difference between the adaptive stepsize and the idealized stepsize mainly depends on the estimation error \( \| \hat{m}_k - m_k \| \), which has a sublinear regret according to Lemma B. Then we carefully integrate this regret bound to control the derivation from the idealized algorithm, reaching the conclusion.

\textbf{Proof.} By the update rule of \( x_{k+1} \), we have,
\[
\| x_{k+1} - x^* \|^2 = \| x_k - \eta_k g_k - x^* \|^2 = \| x_k - x^* \|^2 - 2\eta_k \langle g_k, x_k - x^* \rangle + \eta_k^2 \| g_k \|^2.
\]
Noting that the stepsize \( \eta_k \) is independent of \( g_k \), taking expectation with respect to \( g_k \) conditional on the past iterates lead to
\[
2\eta_k (f(x_k) - f^*) \leq 2\eta_k \langle \nabla f(x_k), x_k - x^* \rangle = -\mathbb{E} \| x_{k+1} - x^* \|^2 | x_k, \cdots, x_1 | + \| x_k - x^* \|^2 + \eta_k^2 m_k^2.
\]
Recall that \( R = \| x_1 - x^* \| \), taking expectation and sum over iterations \( k \), we get
\[
\mathbb{E} [2(\sum_{k=1}^{T} \eta_k) (f(\pi_T) - f^*))] \leq R^2 + \mathbb{E} [\sum_{k=1}^{T} \eta_k^2 m_k^2].
\]
Hence by Markov’s inequality, with probability at least 3/4,
\[
2(\sum_{k=1}^{T} \eta_k) (f(\pi_T) - f^*)) \leq 4\mathbb{E} [2(\sum_{k=1}^{T} \eta_k) (f(\pi_T) - f^*))] \leq 4(R^2 + \mathbb{E} [\sum_{k=1}^{T} \eta_k^2 m_k^2]). \tag{8}
\]
Now we can upper bound the right hand side, indeed
\[
\sum_{k=1}^{T} \mathbb{E} [\eta_k^2 m_k^2] = c^2 \sum_{k=1}^{T} \mathbb{E} \left[ \frac{m_k^2}{(\hat{m}_k + m)^2} \right]
\leq c^2 \left( \sum_{k=1}^{T} \mathbb{E} \left[ \frac{m_k^2 - m_k^2}{(\hat{m}_k + m)^2} \right] + \sum_{k=1}^{T} \mathbb{E} \left[ \frac{\hat{m}_k^2}{(\hat{m}_k + m)^2} \right] \right)
\leq c^2 \left( \frac{1}{m^2} \sum_{k=1}^{T} \mathbb{E} [m_k^2 - \hat{m}_k^2] + T \right)
\leq c^2 \left( \frac{(M^2 + D^2)T^2/3 \ln(T^2/3)}{m^2} + T \right) \leq 3c^2 T \tag{9}
\]
The last inequality follows by the choice on \( m \) that
\[
\frac{M^2}{m^2} \leq \frac{1}{4} \frac{T^{1/3}}{\ln(T)}, \quad D^2 \leq 4M^2
\]
Hence, from Eq. (8), we have with probability at least 3/4,
\[
2(\sum_{k=1}^{T} \eta_k) (f(\pi_T) - f^*)) \leq 4(R^2 + 3c^2 T) \tag{10}
\]
Next, by denoting \( (x)_+ = \max(x, 0) \), we lower bound the left hand side,
\[
\frac{1}{c} \sum_{k} \eta_k = \sum \frac{1}{\hat{m}_k + m}
= \sum \frac{1}{m_k + m} + \sum \left( \frac{1}{\hat{m}_k + m} - \frac{1}{m_k + m} \right)
\geq \sum \frac{1}{m_k + m} - \sum (m_k + m)(\hat{m}_k - m_k)_+\]
\[
= \sum \frac{1}{m_k + m} - \sum \frac{(\hat{m}_k - m_k)_+}{\sqrt{m_k + m} \cdot m^{3/2}}
\geq \sum \frac{1}{m_k + m} - \sum \frac{1}{2} \left( \frac{(\hat{m}_k - m_k)_+^2}{m^3} + \frac{1}{m_k + m} \right)
= \frac{1}{2} \sum \frac{1}{m_k + m} - \frac{1}{2m^2} \sum (\hat{m}_k - m_k)_+^2 \tag{11}
\]
Finally, by Markov’s inequality, with probability $3/4$
\[
\sum (\hat{m}_k - m_k)^2 \leq 4E[\sum (\hat{m}_k - m_k)^2] \leq 4E[\sum (\hat{m}_k - m_k)^2] \leq 8(D^2 + M^2)T^{2/3} \ln(T^{2/3}).
\]
Following the choice of $m = 8MT^{-\frac{1}{3}} \ln(T)^{1/3}$ and $T$ large enough such that $\ln(T) \leq T^{1/3}$, we have
\[
\frac{1}{2m^3} \sum_{k=1}^T (\hat{m}_k - m_k)^2 \leq \frac{T}{4(M + m)} \leq \frac{1}{4} \sum_{k=1}^T \frac{1}{m_k + m}
\]
Consequently, together with (10) and (11), we know that with probability at least $1 - \frac{1}{4} - \frac{1}{4} = 1/2$,
\[
f(\bar{x}_T) - f^* \leq \frac{3\epsilon^2 T + R^2}{2} \leq \frac{2R}{\sqrt{T}} \sum_k \frac{8T}{(m_k + m)},
\]
where the last inequality follows by setting $\epsilon = \frac{R}{\sqrt{T}}$.

\[\square\]

**Remark 9.** For more general choices of stepsizes $\eta_k = \frac{1}{(m_k^2 + m^p)^{1/p}}$, the upper bound in Eq.(9) holds exactly as in the above proof, and the lower bound in Eq.(11) follows from
\[
\sum \eta_k = \sum \frac{1}{(m_k^p + m^p)^{1/p}} = \sum \frac{1}{(m_k^p + m^p)^{1/p}} + \sum \frac{1}{(m_k^p + m^p)^{1/p}} (\frac{1}{m_k^p + m^p})^{1/p} - \frac{1}{m_k^p + m^p})^{1/p}
\]
\[
\geq \sum (\frac{1}{m_k^p + m^p})^{1/p} - \sum (\frac{1}{m_k^p + m^p})^{1/p} (\frac{1}{m_k^p + m^p})^{1/p}
\]
\[
\geq \sum \frac{1}{2} \sum (\frac{1}{m_k^p + m^p})^{1/p} - \frac{1}{2m^3} \sum \frac{1}{m_k^p + m^p})^{2/p}
\]

\section{D Proof with concentrate noise}

In this section, we add additional constraints on noise concentrations.

**Assumption 3.** The expected absolute value is not very different from the square root of the second moment, i.e.
\[
2E[\|g(x)\|] \geq \sqrt{E[\|g(x)\|^2]}.
\]
The constant “2” in the assumption above is arbitrary and can be increased to any fixed constant. The above assumption is satisfied if $g(x)$ follows Gaussian distribution. It is also satisfied if for some fixed constant $\gamma$, $p[\|g(x)\| \geq r] \leq \gamma E[\|g(x)\|^3] r^{-4}$, for all $r \geq \gamma E[\|g(x)\|]$.

We assume that the total variation on the first moment is bounded.

**Assumption 4.** We denote $\lambda_k = E[\|g_k\|], and assume that an upper bound $M$ such that
\[\square\]
(a) The second moment of $g_k$ is bounded by $M^2$, namely, $E[\|g_k\|^2] \leq M^2, \forall k$.
(b) The total variation on the first moment $\lambda_k$ is bounded by
\[
\sum_k |\lambda_k - \lambda_{k+1}| \leq D = 2M.
\]
With a better concentration of the online estimator, we could allow a less conservative correction under assumptions 1, 3, 4, with Theorem 10.

Theorem 10. Under assumptions 1, 3, 4, with $m = 8MT^{-1/6}(\ln(T))^{1/2}$, $c = \frac{R}{\sqrt{m}}$, and $T$ large enough such that $8 \ln(T) \leq T^{1/3}$, Algorithm 1 with update rule (14) achieves convergence rate

$$f(x_T) - f^* \leq \frac{2R}{\sqrt{T}} \frac{3T}{\sum_k \frac{1}{(m_k + m)}}.$$

With a better concentration of the online estimator, we could allow a less conservative correction constant $m$, in the order of $MT^{-\frac{1}{6}}$. It is this parameter that controls the maximum attainable improvement compared to the constant baseline. Indeed, we again consider the noise example 1, given in Table 2. In this case, the adaptive method can obtain an improvement of order $T^{\frac{1}{6}}$ compared to the constant baseline, while as previously only $T^{\frac{1}{2}}$ is achievable.

The proof of the result follows a similar routine as the proof of Theorem 3. We start by presenting an equivalent lemma of Lemma 8.

**Lemma 11.** Under assumption 4, we can achieve the following bound on total estimation error.

$$\mathbb{E}\left[\sum_{k=1}^{T} |\hat{m}_k - \lambda_k|\right] \leq 2(D + M)T^{2/3} \ln(T^{2/3})$$

**Proof.** The proof is the same as the proof of Lemma 8, by replacing the second $m_k^2$ by the first moment $\lambda_k = \mathbb{E}[||g_k||]$.

**Proof of Theorem 10.** By Assumption 3, we can use first moment of $g_k$ to bound the second moment. Hence, Eq. (8) implies that with probability at least 3/4,

$$2\sum_{k=1}^{T} \eta_k (f(x_T) - f^*) \leq 4(R^2 + \mathbb{E}[4 \sum_{k=1}^{T} \eta_k^2 \lambda_k^2]).$$

(16)

Now we upper bound the right hand side, indeed

$$\sum_{k=1}^{T} \mathbb{E}[\eta_k^2 \lambda_k^2] = c^2 \sum_{k=1}^{T} \mathbb{E} \left[ \frac{\lambda_k^2}{(m_k + m)^2} \right]$$

\leq c^2 \left( \sum_{k=1}^{T} \mathbb{E} \left[ \frac{\lambda_k^2 - \hat{m}_k^2}{(m_k + m)^2} \right] + \sum_{k=1}^{T} \mathbb{E} \left[ \frac{\hat{m}_k^2}{(m_k + m)^2} \right] \right)

\leq c^2 \left( \sum_{k=1}^{T} \mathbb{E} \left[ \frac{(\lambda_k - \hat{m}_k)(\lambda_k + \hat{m}_k)}{(m_k + m)^2} \right] + T \right)

\leq c^2 \left( \frac{2M}{m^2} \sum_{k=1}^{T} \mathbb{E} [||\lambda_k - \hat{m}_k||] + T \right)

\leq c^2 \left( \frac{4(M + D)MT^{2/3} \ln(T^{2/3})}{m^2} + T \right) \leq 2c^2T
We make the following smoothness assumptions on which our techniques build upon.

Algorithm 2 Variance Adaptive SGD \((x_1, T, c, m)\)

1: Initialize \(\hat{\sigma}_1 = \|g_1 - g_1'\|^2\), where \(g_1, g_1'\) are two independent stochastic gradients at \(x_1\).
2: for \(k = 1, 2, ..., T\) do
3: Query two independent stochastic gradient \(g_k, g_k'\) at \(x_k\).
4: Update \(x_{k+1} = x_k - \eta_k(g_k + g_k')/2\) with \(\eta_k = \frac{c}{\sigma_k + m}\) and \(m \geq 2cL\).
5: Update \(\hat{\sigma}_k^2 + 1 = \beta \hat{\sigma}_k^2 + (1 - \beta)\|g_k - g_k'\|^2\)
6: end for
7: return \(\pi_T = \sum_{i=1}^{T} \eta_i x_i / (\sum_{i=1}^{T} \eta_i)\).

Hence by Markov inequality, with probability at least 3/4,

\[
2(\sum_{k=0}^{T-1} \eta_k)(f(x_I) - f^*) \leq 4\mathbb{E}[2(\sum_{k=0}^{T-1} \eta_k)(f(x_I) - f^*)] \leq 4(R^2 + 2c^2T)
\]

Next, we lower bound the left hand side,

\[
\frac{1}{c} \sum \eta_k = \sum \frac{1}{\hat{m}_k + m}
\]

\[
= \sum \frac{1}{\lambda_k + m} - \sum \left( \frac{1}{\hat{m}_k + m} - \frac{1}{\lambda_k + m} \right)
\]

\[
\geq \sum \frac{1}{\lambda_k + m} - \sum \frac{(\hat{m}_k - \lambda_k)}{\hat{m}_k + m)(\lambda_k + m)}
\]

\[
\geq \sum \frac{1}{\lambda_k + m} - \frac{1}{m^2} \sum (\hat{m}_k - \lambda_k)
\]

By Markov’s inequality and Lemma 11, with probability 3/4, we have

\[
\sum (\hat{m}_k - \lambda_k)^2_+ \leq 4\mathbb{E}[\sum |\hat{m}_k - \lambda_k|] \leq 8(D + M)T^{2/3} \ln(T^{2/3}).
\]

Following the choice of \(m = 8MT^{-\frac{2}{3}} \ln(T)^{\frac{2}{3}}\) and \(T\) large enough such that \(8 \ln(T) \leq T^{1/3}\), we have

\[
\frac{1}{m^2} \mathbb{E}[|\hat{m}_k - \lambda_k|] \leq \frac{T}{2(M + m)} \leq \frac{1}{2} \sum \frac{1}{\lambda_k + m}
\]

Consequently, we know that with probability at least \(1 - \frac{1}{4} - \frac{1}{4} = 1/2\),

\[
f(\pi_T) - f^* \leq \frac{2c^2T + R^2}{\sum_k \frac{1}{2(\lambda_k + m)}} \leq \frac{2R}{\sqrt{T}} \cdot \frac{3T}{\sum_k \frac{1}{(m_k + m)}},
\]

by setting \(c = \frac{R}{\sqrt{T}}\) and the fact that \(\lambda_k \leq m_k\).

\[\square\]

E Smooth nonconvex optimization

Our analysis above happens in a convex setting, but the estimation error bound in Theorem 8 does not depend on convexity. Therefore, without any modification to the algorithm, all the results have a natural nonconvex generalization. However, these analysis will result in convergence rates that depend on \(\mathbb{E}[\|g(x_k)\|^2]\), while careful analysis for SGD can result in dependence only on \(\mathbb{E}[\|g(x_k) - \nabla f(x_k)\|^2]\). Though according to experiments in Figure 1 these two quantities are not very different, this may not be satisfactory from a theoretical perspective.

Therefore, in this section, we show how the original algorithm can be modified to utilize the smoothness assumption and achieve convergence rates that only depend on \(\mathbb{E}[\|g(x_k) - \nabla f(x_k)\|^2]\). In neural network training, this is not the recommended algorithm because: (i) the improvement from \(\mathbb{E}[\|g(x_k)\|^2]\) to \(\mathbb{E}[\|g(x_k) - \nabla f(x_k)\|^2]\) is marginal; (ii) it requires two gradient evaluations per iteration.

We make the following smoothness assumptions on which our techniques build upon.
Assumption 5. The function is $L$-smooth, i.e. for any $x, y$, $\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|$. Additionally, we assume that the noise does not go to zero.

Assumption 6. For any $k$, $\sigma_k \geq 2L \sqrt{f(x_0) - f(x^*)} / \sqrt{T}$.

As long as the noise doesn’t go to zero, the above inequality always hold for $T$ large enough. This assumption help us focus on the case when noise (instead of the shape of the deterministic function) dominates the convergence rate and facilitate clean comparison. The need for Assumption 6 does not result from nonconvexity but is coupled with exploiting smoothness. With the above assumptions,

Theorem 12. Denote $\sigma_k^2 = \mathbb{E}[\| g_k - \nabla f(x_k) \|^2] = m_k^2 - \| f(x_k) \|^2$. Under the assumptions 1 and 5, if we run SGD with a noise-independent step size, with probability 1/2

1. $\eta_k = \sqrt{\frac{f(x_0) - f^*}{L \sum_k \sigma_k^2}} \| \nabla f(x_T) \| \leq \frac{2 \sqrt{(f(x_0) - f^*) L \sum_k \sigma_k^2}}{T}.$

2. $\eta_k = \sqrt{\frac{f(x_0) - f^*}{\sigma_k}} \| \nabla f(x_T) \| \leq \frac{2 \sqrt{(f(x_0) - f^*) L T}}{T \sum_k \sigma_k^2}.$

As before, we refer to the first step size choice as the standard baseline, and the second step size choice as the oracle baseline.

Note that in the convergence rate above, the first term converges at the rate $O(T^{-1/2})$, while the second term converges at the rate $O(T^{-1})$ (which results from the step size choice $\frac{1}{T}$). Therefore, unless the noise goes to zero, the first term would dominate. If we compare the first terms of the standard and the oracle baselines, we can see that the oracle baseline is always better, just as in the remarks for Theorem 3.

We then modify our algorithm to make use of the smoothness assumption. Particularly, we keep the assumption on the variation of the variance.

Assumption 7. We assume an upper bound on $\sigma_k^2 = \mathbb{E}[\| g_k - \nabla f(x_k) \|^2]$, i.e. $\max_k \sigma_k \leq M$. We also assume that the total variation in $\sigma_k$ is bounded, i.e. $\sum_k |\sigma_k^2 - \sigma_{k+1}^2| \leq D^2 = 4M^2$.

With the above assumptions, algorithm 2 achieve the following rate.

Theorem 13. Under the assumptions 1, 5, 7 and $T \geq 2000$, algorithm 2 with $m = 2MT^{-1/9}\ln(T)^{1/3} + 2L$, $c = \frac{K}{\sqrt{T}}$, achieves with probability 1/2, $\| \nabla f(x_T) \| \leq \frac{s \sqrt{(f(x_0) - f^*) T}}{\sum_k ((m_k + 5m)} T}$.

The above theorem is almost the same as Theorem 3, and hence all the remarks for Theorem 3 also applies in the nonconvex case.

F Proof of Theorem 12

Proof.

$f(x_{k+1}) \leq f(x_k) - \eta_k \langle g_k, \nabla f(x_k) \rangle + \frac{L\eta_k^2}{2} \| g_k \|^2$

Rearrange and take expectation with respect to $g_k$, we get

$\eta_k \frac{\| \nabla f(x_k) \|^2}{2} \leq f(x_k) - f(x_{k+1}) + \frac{L}{2} \eta_k^2 (\| \nabla f(x_k) \|^2 + \mathbb{E}[\| g_k - \nabla f(x_k) \|^2]).$

By Assumption 6, we know that $\eta_k \leq \frac{1}{2L}$. Hence,

$\frac{\eta_k}{2} \| \nabla f(x_k) \|^2 \leq f(x_k) - f(x_{k+1}) + \frac{L}{2} \eta_k^2 \mathbb{E}[\| g_k - \nabla f(x_k) \|^2].$

Sum over $k$ and take expectation,

$\mathbb{E}[\sum_{k=1}^{T} \eta_k \| \nabla f(x_k) \|^2] \leq f(x_0) - f(x^*) + \frac{L}{2} \sum_k \eta_k^2 \sigma_k^2.$
Denote \( I \) as the random variable such that \( \mathbb{P}(I = i) \propto \eta_i \). We know
\[
\mathbb{E}(\sum_{k=1}^{T} \eta_k)\|\nabla f(x_i)\|^2 \leq f(x_0) - f(x^* + \frac{L}{2} \sum_k \eta_k^2 \sigma_k^2.
\]

Let \( \eta_k = \eta = \sqrt{\frac{f(x_0) - f^*}{L \sum_k \sigma_k^2}} \), we have that
\[
\mathbb{E}(\|\nabla f(x_i)\|^2) \leq \frac{\sqrt{(f(x_0) - f^*)L \sum_k \sigma_k^2}}{T}.
\]
By Markov inequality we know that with probability 0.5
\[
\|\nabla f(x_i)\|^2 \leq \frac{2\sqrt{(f(x_0) - f^*)L \sum_k \sigma_k^2}}{T}.
\]
(18)

On the other hand, if we let \( \eta_k = \frac{1}{\sqrt{T \sigma_k / \sqrt{f(x_0) - f(x^*)}}} \), we have that
\[
f(x_0) - f(x^*) + \frac{L}{2} \sum_k \eta_k^2 \sigma_k^2 \leq f(x_0) - f(x^*) + \frac{L}{2} \sum_k \frac{f(x_0) - f(x^*)}{T \sigma_k^2} \sigma_k^2
\]
\[
\leq 2(f(x_0) - f(x^*))
\]
Hence we know that
\[
\mathbb{E}(\|\nabla f(x_i)\|^2) \leq \frac{\sqrt{(f(x_0) - f^*)LT}}{\sum_k \sigma_k^{-1}}.
\]
By Markov inequality we know that with probability 0.5
\[
\|\nabla f(x_i)\|^2 \leq \frac{2\sqrt{(f(x_0) - f^*)LT}}{\sum_k \sigma_k^{-1}}.
\]
(19)

\[\square\]

\section{G Proof of Theorem 13}

We start by presenting an equivalent theorem of Theorem 8 below.

\textbf{Theorem 14.} Under assumption 7, we can achieve the following bound on total estimation error using the estimator (ExpMvAvg):
\[
\mathbb{E}(\sum_{k=1}^{T} \eta_k)\|\nabla f(x_i)\|^2 \leq 2(D^2 + M^2)T^{2/3} \ln(T^{2/3})
\]
where \( \sigma_k = \mathbb{E}(\|g_k - g_k\|) \).

\textbf{Proof.} This follows by exactly the same proof as Theorem 8 and the fact that \( \mathbb{E}(\|g - g'\|^2) = 2\mathbb{E}(\|g - \nabla f\|^2) \). \[\square\]

\textbf{Proof of Theorem 13.} We combine the proofs for Theorem 12 and Theorem 3. By Taylor expansion,
\[
f(x_{k+1}) \leq f(x_k) - \eta_k (g_k, \nabla f(x_k)) + \frac{L}{2} \eta_k^2 \|g_k\|^2
\]
Rearrange and take expectation with respect to \( g_k \), we get
\[
\frac{\eta_k}{2} \|\nabla f(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{L}{2} \eta_k^2 (\|\nabla f(x_k)\|^2 + \mathbb{E}(\|g_k - \nabla f(x_k)\|^2)).
\]
By the fact that \( \eta_k \leq \frac{1}{\sqrt{T}} \), we know
\[
\frac{\eta_k}{2} \|\nabla f(x_k)\|^2 \leq f(x_k) - f(x_{k+1}) + \frac{L}{2} \eta_k^2 \mathbb{E}(\|g_k - \nabla f(x_k)\|^2).
\]
Sum over $k$ and take expectation,
\[
\mathbb{E}[\sum_{k=1}^{T} \eta_k \|\nabla f(x_k)\|^2] \leq f(x_0) - f(x^*) + \frac{L}{2} \mathbb{E}[\sum_{k=1}^{T} \eta_k^2 \sigma_k^2].
\]

Denote $I$ as the random variable such that $\mathbb{P}(I = i) \propto \eta_i$. We know
\[
\mathbb{E}[\sum_{k=1}^{T} \eta_k \|\nabla f(x_l)\|^2] \leq f(x_0) - f(x^*) + \frac{L}{2} \sum_{k=1}^{T} \mathbb{E}[\eta_k^2 \sigma_k^2].
\]

The last inequality follows by the choice that $\mathbb{P}(I = i) \propto \eta_i$. We know
\[
\mathbb{E}[\sum_{k=1}^{T} \eta_k \|\nabla f(x_l)\|^2] \leq f(x_0) - f(x^*) + \frac{L}{2} \sum_{k=1}^{T} \mathbb{E}[\eta_k^2 \sigma_k^2].
\]

The rest of the proof is exactly the same as the proof for Theorem 3. We can upper bound the right hand side, indeed
\[
\sum_{k=0}^{T-1} \mathbb{E}[\eta_k^2 \sigma_k^2] = c^2 \sum_{k=0}^{T-1} \mathbb{E}\left[\frac{\sigma_k^2}{(\hat{\sigma}_k + \sigma)^2}\right]
\leq c^2 \left( \sum_{k=0}^{T-1} \mathbb{E}\left[\frac{\sigma_k^2 - \hat{\sigma}_k^2}{(\hat{\sigma}_k + \sigma)^2}\right] + \sum_{k=0}^{T-1} \mathbb{E}\left[\frac{\hat{\sigma}_k^2}{(\hat{\sigma}_k + \sigma)^2}\right] \right)
\leq c^2 \left( \frac{1}{\sigma^2} \sum_{k=0}^{T-1} \mathbb{E}\left[|\sigma_k^2 - \hat{\sigma}_k^2|\right] + T \right)
\leq c^2 \left( \frac{(M^2 + D^2)T^{2/3} \ln(T^{2/3})}{\sigma^2} + T \right) \leq 3c^2T
\]

The last inequality follows by the fact that
\[
\frac{M^2}{\sigma^2} \leq \frac{1}{4} \frac{T^{1/3}}{\ln(T)}, \quad D^2 \leq 4M^2
\]

Hence by Markov inequality, with probability at least $5/6$,
\[
2(\sum_{k=1}^{T} \eta_k \|\nabla f(x_l)\|^2) \leq 6\mathbb{E}[2(\sum_{k=0}^{T-1} \eta_k)(f(x_l) - f(x^*))] \leq 6(f(x_0) - f(x^*) + 3c^2T)
\]

Next, we lower bound the left hand side,
\[
\sum \eta_k = c \sum \frac{1}{\hat{\sigma}_k + m}
= c \sum \frac{1}{\sigma_k + 5m} + \sum \left( \frac{1}{\hat{\sigma}_k + m} - \frac{1}{\sigma_k + 5m} \right)
= c \sum \frac{1}{\sigma_k + 5m} - c \sum \left( \frac{\hat{\sigma}_k - \sigma_k - 4m}{(\sigma_k + m)(\sigma_k + 5m)} \right)
\leq c \sum \frac{1}{\sigma_k + 5m} - c \sum \left( \frac{\hat{\sigma}_k - \sigma_k - 4m}{(\hat{\sigma}_k + m)(\sigma_k + 5m)} \right)
\]

We note that
\[
\mathbb{E}[\sum \left( \frac{\hat{\sigma}_k - \sigma_k - 4m}{(\hat{\sigma}_k + m)(\sigma_k + 5m)} \right)] = \mathbb{E}\left[\sum \left( \frac{\hat{\sigma}_k^2 - \sigma_k^2}{(\hat{\sigma}_k + m)(\sigma_k + 5m)} \right) \right]
\leq \mathbb{E}\left[\sum \left( \frac{\hat{\sigma}_k^2 - \sigma_k^2}{(\hat{\sigma}_k + m)(\sigma_k + 5m)} \right) \right]
\leq \mathbb{E}\left[\sum \left( \frac{\hat{\sigma}_k^2 - \sigma_k^2}{20m^3} \right) \right] \leq \frac{T^{2/3} \ln(T^{2/3})(D^2 + M^2)}{20m^3} \leq \frac{T}{160M}
\]

The last inequality follows by the choice that $m/M = 2T^{-1/9}(\ln(T))^{1/3}$. Then by Markov inequality, we know that with probability $2/3$,
\[
\sum \left( \frac{\hat{\sigma}_k - m_k - 2m}{(\hat{\sigma}_k + m)(\sigma_k + 5m)} \right) \leq \frac{T}{160M} \leq \sum_k \frac{1}{2(\sigma_k + 5m)}
\]
The last inequality follows by the fact that $T \geq 2000 \implies m \leq M$. Hence with probability $2/3,$

$$\sum_{k} \eta_k \geq \sum_{k} \frac{c}{(\sigma_k + 5m)} - \sum_{k} \frac{c}{2(\sigma_k + 5m)} = \sum_{k} \frac{c}{2(\sigma_k + 5m)} \quad (20)$$

Consequently, we know that with probability at least $1 - \frac{1}{6} - \frac{1}{3} = 1/2,$

$$f(x_I) - f^* \leq \frac{3c^2T + f(x_0) - f(x^*)}{\sum_k \frac{c}{2(\sigma_k + 5m)}} \quad (21)$$

The result follows by setting $c = \frac{\sqrt{f(x_0) - f(x^*)}}{\sqrt{T}}$. 

$\square$