Dynamic Regret of Convex and Smooth Functions

Peng Zhao
Yu-Jie Zhang
Lijun Zhang
Zhi-Hua Zhou

National Key Laboratory for Novel Software Technology
Nanjing University, Nanjing 210023, China

Abstract

We investigate online convex optimization in non-stationary environments and choose the dynamic regret as the performance measure, defined as the difference between cumulative loss incurred by the online algorithm and that of any feasible comparator sequence. Let $T$ be the time horizon and $P_T$ be the path-length that essentially reflects the non-stationarity of environments, the state-of-the-art dynamic regret is $O(\sqrt{T(1+P_T)})$. Although this bound is proved to be minimax optimal for convex functions, in this paper, we demonstrate that it is possible to further enhance the dynamic regret by exploiting the smoothness condition. Specifically, we propose novel online algorithms that are capable of leveraging smoothness and replace the dependence on $T$ in the dynamic regret by problem-dependent quantities: the variation in gradients of loss functions, and the cumulative loss of the comparator sequence. These quantities are at most $O(T)$ while could be much smaller in benign environments. Therefore, our results are adaptive to the intrinsic difficulty of the problem, since the bounds are tighter than existing results for easy problems and meanwhile guarantee the same rate in the worst case.

1. Introduction

In many real-world applications, data are inherently accumulated over time, and thus it is of great importance to develop a learning system that updates in an online fashion. Online Convex Optimization (OCO) is a powerful paradigm for learning in such a circumstance, which can be regarded as an iterative game between a player and an adversary (Zinkevich, 2003). At iteration $t$, the player selects a decision $x_t$ from a convex set $\mathcal{X}$ and the adversary reveals a convex function $f_t : \mathcal{X} \to \mathbb{R}$. The player subsequently suffers an instantaneous loss $f_t(x_t)$. The performance measure is the (static) regret (Zinkevich, 2003),

$$S\text{-Regret}_T = \sum_{t=1}^{T} f_t(x_t) - \min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_t(x),$$

which is the difference between cumulative loss incurred by the online algorithm and that of the best decision in hindsight. The rationale behind such a metric is that the best fixed decision in hindsight is reasonably good over all the iterations. However, this is too optimistic and may not hold in changing environments, where data are evolving and the optimal decision is drifting over time. To address this limitation, dynamic regret is proposed.
to compete with changing comparators \( u_1, \ldots, u_T \in \mathcal{X} \),

\[
    \text{D-Regret}_T(u_1, \ldots, u_T) = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t),
\]

which draws considerable attention recently (Besbes et al., 2015; Jadabaie et al., 2015; Mokhtari et al., 2016; Yang et al., 2016; Zhang et al., 2017, 2018; Auer et al., 2019; Baby and Wang, 2019; Zhao et al., 2020). The measure is also called the universal dynamic regret, in the sense that it holds against any comparator sequence. Note that static regret (1) can be viewed as its special form by setting comparators as the fixed best decision in hindsight. Moreover, a variant appeared frequently in the literature is the worst-case dynamic regret defined as

\[
    \text{D-Regret}_T(x^*_1, \ldots, x^*_T) = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*_t),
\]

which specializes (2) by setting \( u_t = x_t^* \in \arg\min_{x \in \mathcal{X}} f_t(x) \). However, the worst-case dynamic regret is often too pessimistic, whereas the universal one is more adaptive to the non-stationary environments.

There are many studies on the worst-case dynamic regret (Besbes et al., 2015; Jadabaie et al., 2015; Yang et al., 2016; Zhang et al., 2017; Baby and Wang, 2019), but few results are known for the universal dynamic regret. Zinkevich (2003) shows that online gradient descent (OGD) achieves an \( O(\sqrt{T}(1 + P_T)) \) universal dynamic regret, where \( P_T = \sum_{t=1}^{T} \|u_t - u_{t-1}\|_2 \) is the path-length of comparators \( u_1, \ldots, u_T \). Nevertheless, there exists a large gap between this upper bound and the \( \Omega(\sqrt{T}(1 + P_T)) \) lower bound established recently by Zhang et al. (2018), who further propose a novel online algorithm, attaining an \( O(\sqrt{T}(1 + P_T)) \) universal dynamic regret, and thereby close the gap.

Although the rate is minimax optimal for convex functions, we would like to design algorithms with more adaptive bounds, replacing the dependence on \( T \) by certain problem-dependent quantities that are \( O(T) \) in the worst case while could be much smaller in benign environments (i.e., easy problems). In the study of static regret, we can attain such bounds when additional curvature like smoothness is presented, including the small-loss bounds (Srebro et al., 2010) and the gradient-variation bounds (Chiang et al., 2012). Thus, a natural question arises whether it is possible to leverage smoothness to achieve more adaptive universal dynamic regret?

**Our results.** In this paper, we provide an affirmative answer by designing algorithms with problem-dependent dynamic regret bounds. Specifically, we focus on the following two adaptive quantities: the gradient variation of online functions \( V_T \), and the cumulative loss of the comparator sequence \( F_T \)

\[
    V_T = \sum_{t=2}^{T} \sup_{x \in \mathcal{X}} \|\nabla f_{t-1}(x) - \nabla f_t(x)\|_2^2, \quad \text{and} \quad F_T = \sum_{t=1}^{T} f_t(u_t).
\]

We propose a novel online approach for convex and smooth functions, named Smoothness-aware online learning with dynamic regret (abbreviated as Sword). There are three versions, including Sword_{\text{var}}, Sword_{\text{small}}, and Sword_{\text{best}}. All of them enjoy the problem-dependent dynamic regret bounds.
• Sword\textsubscript{var} enjoys a gradient-variation bound of $\mathcal{O}(\sqrt{(1 + P_T + V_T)(1 + P_T)})$;
• Sword\textsubscript{small} enjoys a small-loss bound of $\mathcal{O}(\sqrt{(1 + P_T + F_T)(1 + P_T)})$;
• Sword\textsubscript{best} enjoys a best-of-both-worlds bound of $\mathcal{O}(\sqrt{(1 + P_T + \min\{V_T, F_T\})(1 + P_T)})$.

Comparing to the minimax rate of $\mathcal{O}(\sqrt{T(1 + P_T)})$, our bounds replace the dependence on $T$ by the problem-dependent quantity $P_T + \min\{V_T, F_T\}$. Since the quantity is at most $\mathcal{O}(T)$, our bounds become much tighter when the problem is easy (for example when $P_T$ and $V_T/F_T$ are sublinear in $T$), and meanwhile safeguard the same guarantee in the worst case. Furthermore, one may wonder whether it is possible to replace $T$ by $\min\{V_T, F_T\}$ only. We prove that the term of $P_T$ is unavoidable in general, by an $\Omega(P_T)$ lower bound argument for dynamic regret of convex and smooth functions.

Technical contributions. We highlight challenges and technical contributions of this paper. First, we note that there exist studies showing that the worst-case dynamic regret can benefit from smoothness (Mokhtari et al., 2016; Yang et al., 2016; Zhang et al., 2017). However, their analyses do not apply to our case, since we cannot exploit the optimality condition of comparators $u_1, \ldots, u_T$, in stark contrast with the worst-case dynamic regret analysis. Therefore, we adopt the meta-expert framework to hedge the non-stationarity while keeping the adaptivity. We can use variants of OGD as the expert-algorithm to exploit smoothness, but it is difficult to design an appropriate meta-algorithm. Existing meta-algorithms and their variants either lead to problem-independent regret bounds or introduce terms that are incompatible to the desired problem-dependent quantity. To address the difficulty, we adopt the technique of optimistic online learning (Rakhlin and Sridharan, 2013; Syrgkanis et al., 2015), in particular OptimisticHedge, to design novel meta-algorithms.

For Sword\textsubscript{var}, we apply OptimisticHedge with carefully designed optimism, which allows us to exploit the negative term in the regret analysis of OptimisticHedge (Syrgkanis et al., 2015). In this way, the meta-regret only depends on the gradient variation. The construction of the special optimism is the most challenging part of our paper. For Sword\textsubscript{small}, the design of meta-algorithm is simple, and we directly use the vanilla Hedge, which can be treated as OptimisticHedge with null optimism. Finally, for Sword\textsubscript{best}, we still employ OptimisticHedge as the meta-algorithm, but introduce a parallel meta-algorithm to learn the best optimism to ensure a best-of-both-worlds dynamic regret guarantee.

2. Related Work

We present a brief review of static and dynamic regret minimization for online convex optimization.

2.1 Static Regret

Static regret has been extensively studied in online convex optimization. Let $T$ be the time horizon and $d$ be the dimension, there exist online algorithms with static regret bounded by $\mathcal{O}(\sqrt{T})$, $\mathcal{O}(d \log T)$, and $\mathcal{O}(\log T)$ for convex, exponentially concave, and strongly convex functions, respectively (Zinkevich, 2003; Hazan et al., 2007). These results are proved to be minimax optimal (Abernethy et al., 2008). More results can be found in the seminal books (Shalev-Shwartz, 2012; Hazan, 2016) and reference therein.
In addition to exploiting convexity of functions, there are studies improving static regret by incorporating smoothness, whose main proposal is to replace the dependence on $T$ by problem-dependent quantities. Such problem-dependent bounds enjoy much benign properties, in particular, they can safeguard the worst-case minimax rate yet can be much tighter in easy problem instances. In the literature, there are two kinds of such bounds, small-loss bounds (Srebro et al., 2010) and gradient variation bounds (Chiang et al., 2012).

Small-loss bounds are first introduced in the context of prediction with expert advice (Littlestone and Warmuth, 1994; Freund and Schapire, 1997), which replace the dependence on $T$ by cumulative loss of the best expert. Later, Srebro et al. (2010) show that in the online convex optimization setting, OGD can achieve an $O(\sqrt{T})$ small-loss regret bound when the function is convex and smooth, where $F_T^*$ is the cumulative loss of the best decision in hindsight, namely, $F_T^* = \sum_{t=1}^T f_t(x^*)$ with $x^*$ chosen as the offline minimizer.

Gradient variation bounds are introduced by Chiang et al. (2012), rooting in the development of second-order bounds for prediction with expert advice (Cesa-Bianchi et al., 2005) and online convex optimization (Hazan and Kale, 2008). For convex and smooth functions, Chiang et al. (2012) establish an $O(\sqrt{V_T})$ static regret bound, where $V_T = \sum_{t=2}^T \sup_{x \in X} |\nabla f_{t-1}(x) - \nabla f_t(x)|^2$ is the gradient variation. Gradient-variation bounds are particularly favored in slowly changing environments in which the online functions evolve gradually.

### 2.2 Dynamic Regret

Dynamic regret enforces the player to compete with time-varying comparators, and thus is particularly favored in online learning in non-stationary environments. The notion of dynamic regret is also referred to as tracking regret or shifting regret in the settings of prediction with expert advice (Herbster and Warmuth, 1998, 2001). It is known that in the worst case, sublinear dynamic regret is not attainable unless imposing certain regularities on the comparator sequence or the function sequence (Besbes et al., 2015; Jadbabaie et al., 2015). The path-length $P_T = \sum_{t=1}^T \|u_{t-1} - u_t\|_2$ is introduced by Zinkevich (2003). Other regularities include squared path-length $S_T = \sum_{t=2}^T \|u_{t-1} - u_t\|^2_2$ (Zhang et al., 2017), and function variation $V_T^f = \sum_{t=2}^T \sup_{x \in X} |f_{t-1}(x) - f_t(x)|$ (Besbes et al., 2015).

There are two kinds of dynamic regret in previous studies. The universal dynamic regret (2) aims to compare with any feasible comparator sequence, while the worst-case dynamic regret specifies the comparator sequence to be the sequence of minimizers of online functions. In the following, we present related works respectively. Notice that we will use notations of $P_T$ and $S_T$ for path-length and squared path-length of the sequence $\{u_t\}_{t=1, \ldots, T}$, while $P_T^*$ and $S_T^*$ for that of the sequence $\{x_t^*\}_{t=1, \ldots, T}$ where $x_t^*$ is the minimizer of the online function $f_t$.

**Universal dynamic regret.** The seminal work of Zinkevich (2003) demonstrates that the online gradient descent (OGD) actually enjoys an $O(\sqrt{T(1 + P_T)})$ universal dynamic regret. Nevertheless, the results is far from the $\Omega(\sqrt{T(1 + P_T)})$ lower bound established recently by Zhang et al. (2018), who further close the gap by proposing a novel online algorithm that attains an optimal rate of $O(\sqrt{T(1 + P_T)})$ for convex functions (Zhang et al., 2018). Our work improve the minimax rate of $O(\sqrt{T(1 + P_T)})$ to problem-dependent regret guarantees by further exploiting the smoothness condition.
**Worst-case dynamic regret.** More efforts of the dynamic regret analysis are devoted to studying the worst-case dynamic regret. Yang et al. (2016) prove that OGD enjoys an $O(\sqrt{T(1+P_T^*)})$ worst-case dynamic regret bound for convex functions when the path-length $P_T^*$ is known. For strongly convex and smooth functions, (Mokhtari et al., 2016) show that an $O(P_T^*)$ dynamic regret bound is achievable, and Zhang et al. (2017) further propose the online multiple gradient descent algorithm and prove that the algorithm enjoys an $O(\min\{P_T^*, S_T^*, V_T^f\})$ regret bound, which is recently enhanced to $O(\min\{P_T^*, S_T^*, V_T^f, V_{fT}^f\})$ by an improved analysis (Zhao and Zhang, 2020). Yang et al. (2016) further show that $O(P_T^*)$ rate is attainable for convex and smooth functions, provided that all the minimizers $x_t^*$’s lie in the interior of the domain $\mathcal{X}$. The above results use the path-length (or squared path-length) as the regularity, which is in terms of the trajectory of comparator sequence. In another line of research, they instead use the variation with respect to the function values as the regularity. Specifically, Besbes et al. (2015) show that OGD with a restarting strategy attains an $O(T^{2/3}\sqrt{V_T^f})$ regret for convex functions when the function variation $V_T^f$ is available, which is recently improved to $O(T^{1/3}\sqrt{V_T^f})$ for 1-dim square loss (Baby and Wang, 2019).

### 3. Gradient-Variation and Small-Loss Bounds

We first list assumptions used in the paper, then propose online algorithms with gradient-variation and small-loss dynamic regret respectively, and next present the lower bound. At the end of this section, we present two concrete examples to illustrate the significance of the obtained problem-dependent bounds.

#### 3.1 Assumptions

We introduce the following common assumptions that might be used in the theorems.

**Assumption 1.** The gradients are bounded by $G$, i.e., $\|\nabla f_t(x)\|_2 \leq G$, for all $x \in \mathcal{X}$ and $t \in [T]$.

**Assumption 2.** The domain $\mathcal{X}$ contains the origin $0$, and $\|x - x'\|_2 \leq D$ for any $x, x' \in \mathcal{X}$.

**Assumption 3.** All the online functions are $L$-smooth, i.e., for any $x, x' \in \mathcal{X}$ and $t \in [T]$,

$$\|\nabla f_t(x) - \nabla f_t(x')\|_2 \leq L\|x - x'\|_2.$$  \hspace{1cm} (5)

**Assumption 4.** All the online functions are non-negative.

Meanwhile, we treat double logarithmic factors in $T$ as a constant, following previous studies (Adamskiy et al., 2012; Luo and Schapire, 2015).

#### 3.2 Gradient-Variation Bound

We design an approach in a meta-expert framework, and prove its gradient-variation dynamic regret.
3.2.1 Expert-Algorithm

In the study of static regret, Chiang et al. (2012) propose the following online extra-gradient descent (OEGD) algorithm, and show that the algorithm enjoys gradient-variation static regret bound. The OEGD algorithm performs the following update:

\[
\begin{align*}
\hat{x}_{t+1} &= \Pi_{\mathcal{X}} [\hat{x}_t - \eta \nabla f_t(x_t)], \\
x_{t+1} &= \Pi_{\mathcal{X}} [\hat{x}_{t+1} - \eta \nabla f_t(\hat{x}_{t+1})],
\end{align*}
\]

where \(x_t, \hat{x}_t \in \mathcal{X} \), \(\eta > 0\) is the step size, and \(\Pi_{\mathcal{X}}[\cdot]\) denotes the projection onto the nearest point in \(\mathcal{X}\). For convex and smooth functions, Chiang et al. (2012) prove that OEGD achieves an \(O(\sqrt{T})\) static regret. In this paper, we further demonstrate that OEGD also enjoys the dynamic regret.

**Theorem 1.** Under Assumptions 1, 2, and 3, by choosing \(\eta \leq \frac{1}{4T}\), OEGD (6) satisfies

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq \frac{D^2 + 2DP_T}{2\eta} + \eta V_T + GD = O\left(\frac{1 + P_T}{\eta} + \eta V_T\right).
\]

for any comparator sequence \(u_1, \ldots, u_T \in \mathcal{X}\).

Theorem 1 demonstrates that it is crucial to tune the step size to balance non-stationarity (path-length \(P_T\)) and adaptivity (gradient variation \(V_T\)). Notice that the optimal tuning \(\eta^* = \sqrt{(D^2 + 2DP_T)/(2V_T)}\) requires the prior information of \(P_T\) and \(V_T\) that are generally unavailable. We emphasize that \(V_T\) is empirically computable, while \(P_T\) remains unknown even after all iterations due to the fact that the comparator sequence is unknown and can be chosen arbitrarily as long as it is feasible. Therefore, the doubling trick can only remove the dependence on the unknown \(V_T\) term but not \(P_T\).

To handle the uncertainty, we adopt the meta-expert framework to hedge the non-stationarity while keeping the adaptivity, inspired by the recent advance in learning with multiple learning rates (Gaillard et al., 2014; van Erven and Koolen, 2016; Zhang et al., 2018). Concretely, we first construct a pool of candidate step sizes to discretize value range of the optimal step size, and then initialize multiple experts simultaneously, denoted by \(E_1, \ldots, E_N\). Each expert \(E_i\) returns its prediction \(x_{t,i}\) by running OEGD (6) with a step size \(\eta_i\) from the pool. Finally, predictions of all the experts are combined by a meta-algorithm as the final output \(x_t\) to track the best expert. From the procedure, we observe that the dynamic regret can be decomposed as,

\[
\text{D-Regret}_T = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) = \sum_{t=1}^{T} f_t(x_t) - f_t(x_{t,i}) + \sum_{t=1}^{T} f_t(x_{t,i}) - f_t(u_t),
\]

where \(\{x_t\}_{t=1, \ldots, T}\) denotes the final output sequence, and \(\{x_{t,i}\}_{t=1, \ldots, T}\) is the prediction sequence of expert \(E_i\). The first part is the difference between cumulative loss of final output sequence and that of prediction sequence of expert \(E_i\), which is introduced by the meta-algorithm and thus named as meta-regret; the second part is the dynamic regret of expert \(E_i\) and therefore named as expert-regret.

The expert-algorithm is set as OEGD (6), and Theorem 1 upper bounds the expert-regret. The main difficulty lies in the design and analysis of an appropriate meta-algorithm.
3.2.2 Meta-Algorithm

Formally, there are \( N \) experts and expert \( E_i \) predicts \( x_{t,i} \) at iteration \( t \), the meta-algorithm is required to produce \( x_t = \sum_{i=1}^{N} p_{t,i} x_{t,i} \), a weighted combination of expert predictions, where \( p_{t} \in \Delta_N \) is the weight vector. It is natural to use Hedge (Freund and Schapire, 1997) for weight update in order to track the best expert.

In order to be compatible to the gradient-variation expert-regret, the meta-algorithm is required to incur a problem-dependent meta-regret of order \( O(\sqrt{V_T \ln N}) \). However, the meta-algorithms used in existing studies (van Erven and Koolen, 2016; Zhang et al., 2018) cannot satisfy the requirements. For example, the vanilla Hedge (multiplicative weights update) suffers from an \( O(\sqrt{T \ln N}) \) meta-regret, which is problem-independent and thus not suitable for us. To this end, we design a a novel variant of Hedge by leveraging the technique of optimistic online learning with carefully designed optimism, specifically for our problem.

The optimistic online learning is developed by Rakhlin and Sridharan (2013) and further expanded by Syrgkanis et al. (2015). For the prediction with expert advice setting, they consider that at the beginning of iteration \( (t+1) \), in addition to the loss vector \( \ell_t \in \mathbb{R}^N \) returned by the experts, the learner can receive a vector \( m_{t+1} \in \mathbb{R}^N \) called optimism. The authors propose the OptimisticHedge algorithm (Rakhlin and Sridharan, 2013; Syrgkanis et al., 2015), which updates the weight vector \( p_{t+1} \in \Delta_N \) by

\[
p_{t+1,i} \propto \exp\left(-\varepsilon \left( \sum_{s=1}^{t} \ell_{s,i} + m_{t+1,i} \right) \right), \quad \forall i \in [N].
\]

Syrgkanis et al. (2015) prove the following regret guarantee for OptimisticHedge.

**Lemma 1** (Syrgkanis et al., 2015, Theorem 19). The meta-regret of OptimisticHedge is upper bounded by

\[
\sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \ell_{t,i} \leq \frac{2 + \ln N}{\varepsilon} + \varepsilon D_{\infty} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2,
\]

which holds for any expert \( i \in [N] \). Besides, \( D_{\infty} = \sum_{t=1}^{T} \|\ell_t - m_t\|_{\infty}^2 \) measures the adaptivity.

By proper learning rate tuning, OptimisticHedge enjoys an \( O(\sqrt{D_{\infty} \ln N}) \) meta-regret.

The optimistic online learning is very powerful for designing adaptive methods, in that the adaptivity \( D_{\infty} \) in Lemma 1 is very general and can be specialized flexibly with different configurations of the feedback loss \( \ell_t \) and optimism \( m_t \). Based on the OptimisticHedge, we propose VariationHedge, the meta-algorithm for Sword\textsubscript{var}, by specializing OptimisticHedge as follows:

- the feedback loss \( \ell_t \) is set as the linearized surrogate loss, namely, \( \ell_{t,i} = \langle \nabla f_t(x_t), x_{t,i} \rangle \);
- the optimism \( m_t \) is set with a careful design: for each \( i \in [N] \)

\[
m_{t,i} = \langle \nabla f_{t-1}(\bar{x}_t), x_{t,i} \rangle, \quad \text{where} \quad \bar{x}_t = \sum_{i=1}^{N} p_{t-1,i} x_{t,i}.
\]
Algorithm 1 Sword\textsubscript{var}: Meta-algorithm (VariationHedge)

Input: step size pool \( \mathcal{H}_{\text{var}} = \{\eta_i\}_{i=1}^{N} \) as specified in (11); learning rate \( \varepsilon \)
1: Initialization: let \( \mathbf{x}_1 \) be any point in \( \mathcal{X} \), and set \( p_{0,i} = 1/N \) for \( \forall i \in [N] \)
2: for \( t = 1 \) to \( T \) do
3: \( \text{Receive } \mathbf{x}_{t+1,i} \) from expert \( \mathcal{E}_i (\eta_t) \)
4: Update weight \( p_{t+1,i} \) by (10)
5: Predict \( \mathbf{x}_{t+1} = \sum_{i=1}^{N} p_{t+1,i} \mathbf{x}_{t+1,i} \)
6: end for

Algorithm 2 Sword\textsubscript{var}: Expert-algorithm (OEGD)

Input: step size \( \eta_i \)
1: Let \( \hat{\mathbf{x}}_{1,i} \) be any point in \( \mathcal{X} \)
2: for \( t = 1 \) to \( T \) do
3: \( \hat{\mathbf{x}}_{t+1,i} = \Pi_\mathcal{X} \left[ \mathbf{x}_{t,i} - \eta_t \nabla f_t(\mathbf{x}_{t,i}) \right] \)
4: \( \mathbf{x}_{t+1,i} = \Pi_\mathcal{X} \left[ \hat{\mathbf{x}}_{t+1,i} - \eta_t \nabla f_t(\hat{\mathbf{x}}_{t+1,i}) \right] \)
5: Send \( \mathbf{x}_{t+1,i} \) to meta-algorithm
6: end for

So the meta-algorithm of Sword\textsubscript{var} (namely, VariationHedge) updates the weight by

\[
P_{t+1,i} \propto \exp \left( -\varepsilon \sum_{s=1}^{t} \langle \nabla f_s(\mathbf{x}_s), \mathbf{x}_{s,i} \rangle + \langle \nabla f_t(\hat{\mathbf{x}}_{t+1,i}), \mathbf{x}_{t+1,i} \rangle \right), \quad \forall i \in [N]. \quad (10)
\]

Algorithm 1 summarizes detailed procedures of the meta-algorithm, which in conjunction with the expert-algorithm of Algorithm 2 yields the Sword\textsubscript{var} algorithm.

Remark 1. The design of optimism in (9) (in particular, \( \bar{\mathbf{x}}_t \)) is crucial, and is the most challenging part in this work. The key idea is to use the negative term in the regret of OptimisticHedge, as shown in (8), to convert the adaptive quantity \( D_\infty \) to the desired gradient variation \( V_T \). Indeed,

\[
\| \ell_t - \mathbf{m}_t \|_\infty^2 \overset{(9)}{=} \max_{i \in [N]} \langle \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\bar{\mathbf{x}}_t), \mathbf{x}_{t,i} \rangle^2 \\
\leq D^2 \| \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\bar{\mathbf{x}}_t) \|_2^2 \\
\leq 2D^2 (\| \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_t) \|_2^2 + \| \nabla f_{t-1}(\mathbf{x}_t) - \nabla f_{t-1}(\bar{\mathbf{x}}_t) \|_2^2) \\
\leq 2D^2 \sup_{\mathbf{x} \in \mathcal{X}} \| \nabla f_t(\mathbf{x}) - \nabla f_{t-1}(\mathbf{x}) \|_2^2 + 2D^2 L^2 \| \mathbf{x}_t - \bar{\mathbf{x}}_t \|_2^2
\]

where the last step makes use of smoothness. Therefore, \( D_\infty \) can be upper bounded by the gradient variation \( V_T \) and the summation of \( \| \mathbf{x}_t - \bar{\mathbf{x}}_t \|_2^2 \). The latter one can be further expanded as

\[
\| \mathbf{x}_t - \bar{\mathbf{x}}_t \|_2^2 = \left\| \sum_{i=1}^{N} (p_{t,i} - p_{t-1,i}) \mathbf{x}_{t,i} \right\|_2^2 \\
\leq \left( \sum_{i=1}^{N} \| p_{t,i} - p_{t-1,i} \|_2 \| \mathbf{x}_{t,i} \|_2 \right)^2 \\
\leq D^2 \| \mathbf{p}_t - \mathbf{p}_{t-1} \|_1^2
\]

which can be eliminated by the negative term in (8), with a suitable setting of the learning rate \( \varepsilon \).
3.2.3 Regret Guarantees

We prove that the meta-regret of VariationHedge is $O(\sqrt{V_T \ln N})$, compatible to the expert-regret.

**Theorem 2.** Under Assumptions 1, 2, and 3, by setting the learning rate optimally as $\varepsilon = \min\{\sqrt{1/(8D^4L^2)}, \sqrt{(2+\ln N)/(2DV_T^2)}\}$, the meta-regret of VariationHedge is at most

$$\text{meta-regret} \leq 2D\sqrt{2V_T(2+\ln N)} + 4\sqrt{2D^2L(2+\ln N)} = O(\sqrt{V_T \ln N}).$$

Note that the dependence on $V_T$ in the optimal learning rate tuning can be removed by the doubling trick. Furthermore, actually we can set the optimal learning rate of the meta-algorithm with $\bar{V}_T = \sum_{t=1}^T \|\nabla f_t(x_t) - \nabla f_{t-1}(\bar{x}_t)\|_2^2$ instead of the original gradient variation $V_T$ via a more refined analysis. The quantity $\bar{V}_T$ can be regarded as an empirical approximation of $V_T$, and it can be calculated directly without involving the inner problem solving of $\sup_{x \in X} \|\nabla f(x) - \nabla f_{t-1}(x)\|_2^2$ in order to evaluate $V_T$. Thereby, we can perform the doubling trick by monitoring $\bar{V}_T$ with much less computational efforts.

Combining Theorem 1 (expert-regret) and Theorem 2 (meta-regret), we have the following dynamic regret bound.

**Theorem 3.** Under Assumptions 1, 2, and 3, setting the pool of candidate step sizes $H_{\text{var}}$ as

$$H_{\text{var}} = \left\{ \eta_i = 2^{i-1} \sqrt{\frac{D^2}{2GT}}, i \in [N_1] \right\},$$

where $N_1 = \lceil 2^{-2}\log_2(GT/(8D^2L^2)) \rceil + 1$. For any comparator sequence $u_1, \ldots, u_T \in \mathcal{X}$, Sword$_{\text{var}}$ (Algorithms 1 and 2) satisfies

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u_t) \leq O\left(\sqrt{(1 + P_T + V_T)(1 + P_T)}\right).$$

**Remark 2.** Compared with the existing $O(\sqrt{T(1 + P_T)})$ dynamic regret (Zhang et al., 2018), our result is more adaptive in the sense that it replaces $T$ by the problem-dependent quantity $P_T + V_T$. Therefore, the bound will be much tighter in easy problems, for example when both $V_T$ and $P_T$ are $o(T)$. Meanwhile, it safeguards the same minimax rate, since both quantities are at most $O(T)$. We finally mention that the $P_T$ term of the problem-dependent quantity is unavoidable in general, because an $\Omega(P_T)$ dynamic regret for convex and smooth functions is necessary as shown later in Theorem 6.

3.3 Small-Loss Bound

In this part, we turn to another problem-dependent quantity, cumulative loss of the comparator sequence, and prove the small-loss dynamic regret. We start from the online gradient descent (OGD),

$$x_{t+1} = \Pi_\mathcal{X}[x_t - \eta \nabla f_t(x_t)].$$

Srebro et al. (2010) prove that OGD achieves an $O(\sqrt{F_T^*})$ static regret, where $F_T^* = \sum_{t=1}^T f_t(x^*)$ is the cumulative loss of the comparator benchmark $x^*$. For the dynamic
regret, since the benchmark is changing, a natural replacement is the cumulative loss of the comparator sequence \( u_1, \ldots, u_T \), namely \( F_T = \sum_{t=1}^{T} f_t(u_t) \). We show that OGD enjoys such a small-loss dynamic regret.

**Theorem 4.** Under Assumptions 2, 3, and 4, by choosing any step size \( \eta \leq \frac{1}{4T} \), OGD satisfies

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq \frac{7D^2 + 4DP_T}{4\eta(1 - 2\eta L)} + \frac{2\eta L}{1 - 2\eta L} \sum_{t=1}^{T} f_t(u_t) = O\left(\frac{1 + P_T}{\eta} + \eta F_T\right)
\]

for any comparator sequence \( u_1, \ldots, u_T \in \mathcal{X} \).

Similar to Sword_var, the step size needs to balance between non-stationarity (\( P_T \)) and adaptivity (\( F_T \), this time). Notice that the optimal tuning depends on \( P_T \) and \( F_T \), both of which are unknown even after all \( T \) iterations. Therefore, we again compensate the lack of this information via the meta-expert framework to hedge the non-stationarity while keeping the adaptivity. The expert-algorithm is set as OGD. The meta-algorithm is required to suffer a small-loss meta-regret of order \( O(\sqrt{F_T \ln N}) \). We discover that vanilla Hedge with linearized surrogate loss is qualified, which updates the weight by

\[
p_{t+1, i} \propto \exp\left(-\varepsilon \sum_{s=1}^{t} \langle \nabla f_t(x_s), x_s^i \rangle\right), \quad \forall i \in [N].
\]

Notice that vanilla Hedge can be treated as OptimisticHedge with null optimism, i.e., \( m_{t+1} = 0 \). Therefore, by Lemma 1 we know that its meta-regret is of order \( O(\sqrt{D \ln N}) \) and

\[
D_\infty = \sum_{t=1}^{T} \max_{i \in [N]} \langle \nabla f_t(x_t), x_t^i \rangle \leq D^2 \sum_{t=1}^{T} \| \nabla f_t(x_t) \|_2^2 \leq 4D^2 L \sum_{t=1}^{T} f_t(x_t),
\]

where the last inequality follows from the self-bounding property of smooth functions (Srebro et al., 2010, Lemma 3.1). As a result, the meta-regret is now \( O(\sqrt{F_T \ln N}) \), where \( F_T^x = \sum_{t=1}^{T} f_t(x_t) \) is the cumulative loss of decisions. Note that the term \( F_T^x \) can be further processed to the desired small-loss quantity \( F_T = \sum_{t=1}^{T} f_t(u_t) \), the cumulative loss of comparators. We will present details in Appendix B.2.

To summarize, Sword_small chooses OGD (12) as the expert-algorithm, and uses the vanilla Hedge with linearized loss (13) as the meta-algorithm. The theorem below shows that the algorithm enjoys the small-loss dynamic regret bound.

**Theorem 5.** Under Assumptions 2, 3, and 4, setting the pool of candidate step sizes \( \mathcal{H}_{\text{small}} \) as

\[
\mathcal{H}_{\text{small}} = \left\{ \eta_i = 2^{i-1} \sqrt{\frac{7D}{8LGT}}, i \in [N_2] \right\},
\]

where \( N_2 = \lceil 2^{-1} \log_2(\frac{GT}{(14DL)}) \rceil + 1 \). Setting the learning rate of meta-algorithm optimally as \( \varepsilon = \sqrt{\frac{2 + \ln N}{(D^2F_T^x)}} \), then for any comparator sequence \( u_1, \ldots, u_T \in \mathcal{X} \), Sword_small satisfies

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq O(\sqrt{(1 + P_T + F_T)(1 + P_T)}).
\]
Note that the optimal learning rate tuning requires the knowledge of $F_T^\mathcal{A}$, which can be easily removed by doubling trick or self-confident tuning (Auer et al., 2002), since it is empirically evaluable at each iteration.

### 3.4 Lower Bound

We here present the lower bound for dynamic regret of convex and smooth functions.

**Theorem 6.** For any online algorithm $\mathcal{A}$, there always exists a sequence of convex and smooth functions $f_1, \ldots, f_T$ and a sequence of comparator decisions $u_1, \ldots, u_T$, such that

$$
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) = \Omega(P_T(u_1, \ldots, u_T)).
$$

Comparing to the minimax dynamic regret of $O(\sqrt{T(1+P_T)})$, the gradient-variation and small-loss bounds essentially replace the dependence on $T$ by the problem-dependent quantity $P_T + V_T$ or $P_T + F_T$. The lower bound implies that one should not expect to replace $T$ by $V_T$ or $F_T$ only, because an $\Omega(P_T)$ dynamic regret is necessary, which demonstrates the tightness of the obtained dynamic regret bounds.

### 3.5 Significance of Problem-Dependent Bounds

In this part, we justify the significance of our problem-dependent dynamic regret bounds. Specifically, we will present two concrete instances to demonstrate that it is possible to achieve a constant dynamic regret bound instead of the minimax rate $O(\sqrt{T(1+P_T)})$ by exploiting the problem’s structure.

We consider the quadratic loss function of the form $f_t(x) = \frac{1}{2}(a_t \cdot x - b_t)^2$, where $a_t \neq 0$ and $x \in \mathcal{X} = [-1, 1]$. Clearly, the function $f_t : \mathbb{R} \mapsto \mathbb{R}$ is convex and smooth. Denote by $T$ the time horizon. The coefficients $a_t$ and $b_t$ will be specified below in each instance.

**Instance 1** ($V_T \ll F_T$). Let the time horizon $T = 2K + 1$ be an odd with $K > 2$. We set the coefficients $a_t = 0.5 - \frac{t-1}{T}$ and $b_t = 1$ for all $t \in [T]$.

We set the comparator $u_t$ to be the minimizer of $f_t$, i.e., $u_t = x_t^* = \arg\min_{x \in \mathcal{X}} f_t(x)$. Clearly, $u_t = 1$ for $t \in [K+1]$, and $u_t = -1$ for $t = K + 2, \ldots, T$. Therefore, we have

$$
V_T = \sum_{t=2}^{T} \sup_{x \in \mathcal{X}} \left| (a_{t-1}^2 - a_t^2)x - (a_{t-1} - a_t) \right|^2 \sum_{t=2}^{T} \sup_{x \in \mathcal{X}} \left| \left( \frac{T - 2t + 3}{T^2} \right) \cdot x - \frac{1}{T} \right|^2
$$

$$
= \sum_{t=2}^{K+2} \left( \frac{2T - (2t - 3)}{T^2} \right)^2 + \sum_{t=K+3}^{T} \left( \frac{2t - 3}{T^2} \right)^2 \leq \sum_{t=2}^{T} \left( \frac{2}{T} \right)^2 = O(1).
$$

$$
F_T = \sum_{t=1}^{T} \frac{1}{2} (a_t u_t - b_t)^2 = \sum_{t=1}^{K+1} \frac{1}{2} \left( 0.5 - \frac{t-1}{T} - 1 \right)^2 + \sum_{t=K+2}^{T} \frac{1}{2} \left( -0.5 + \frac{t-1}{T} - 1 \right)^2 = \Theta(T).
$$

We can observe that $V_T \leq O(1)$ is significantly smaller than $F_T = \Theta(T)$ (as well as the problem-independent quantity $T$) in this instance. Meanwhile, the path-length term
$P_T = \mathcal{O}(1)$. As a result, the minimax dynamic regret bound is $\mathcal{O}(\sqrt{T(1+P_T)}) = \mathcal{O}(\sqrt{T})$; the small-loss bound is $\mathcal{O}(\sqrt{(1+P_T + F_T)(1 + P_T)}) = \mathcal{O}(\sqrt{T})$; and the gradient-variation bound is $\mathcal{O}(\sqrt{(1+P_T + V_T)(1 + P_T)}) = \mathcal{O}(1)$. In other words, by exploiting the problem’s structure, our approach (Sword$_{\text{var}}$) can enjoy a constant dynamic regret in this scenario.

**Instance 2** ($F_T \ll V_T$). Let the time horizon $T = 2K$ be an even. During the first half iterations, $(a_t, b_t)$ is set as $(1, 1)$ on odd rounds and $(0.5, 0.5)$ on even rounds. During the remaining iterations, $(a_t, b_t)$ is set as $(1, -1)$ on odd rounds and $(0.5, -0.5)$ on even rounds.

We set the comparator $u_t$ to be the minimizer of $f_t$, i.e., $u_t = x_t^* = \arg \min_{x \in \mathcal{X}} f_t(x)$. Clearly, $u_t = 1$ for $t \in [K]$, and $u_t = -1$ for $t = K + 1, \ldots, T$. Therefore, we have

$$V_T = \sum_{t=2}^{T} \sup_{x \in \mathcal{X}} \left\{(a_{t-1}^2 - a_t^2)x - (a_{t-1}b_{t-1} - a_tb_t)\right\}^2 = \mathcal{O}(T), \quad F_T = 0.$$

We can see that $F_T = 0$ is considerably smaller than $V_T = \mathcal{O}(T)$ (as well as the problem-independent quantity $T$) in this scenario. Meanwhile, the path-length term $P_T = \mathcal{O}(1)$. As a result, the minimax dynamic regret bound is $\mathcal{O}(\sqrt{T(1+P_T)}) = \mathcal{O}(\sqrt{T})$; the gradient-variation bound is $\mathcal{O}(\sqrt{(1+P_T + V_T)(1 + P_T)}) = \mathcal{O}(\sqrt{T})$; and the small-loss bound is $\mathcal{O}(\sqrt{(1+P_T + F_T)(1 + P_T)}) = \mathcal{O}(1)$. In other words, by exploiting the problem’s structure, our approach (Sword$_{\text{small}}$) can enjoy a constant dynamic regret in this scenario.

### 4. Best-of-Both-Worlds Bound

In the last section, we propose Sword$_{\text{var}}$ and Sword$_{\text{small}}$ that achieve gradient-variation and small-loss bounds respectively. Due to different problem-dependent quantities are involved, these two bounds are generally incomparable and are favored in different scenarios, as demonstrated by the concrete examples in Section 3.5. Therefore, it is natural to ask for a best-of-both-worlds guarantee: the regret of the minimum of variation and small-loss bounds.

To this end, we require a meta-algorithm that enjoys both kinds of adaptivity to combine all the experts, with an $\mathcal{O}(\sqrt{\min\{V_T, F_T\}} \ln N)$ meta-regret. Based on the observation that both VariationHedge and vanilla Hedge are essentially special cases of OptimisticHedge with different configurations of optimism, we adapt the OptimisticHedge to be the meta-algorithm for Sword$_{\text{best}}$, where a parallel meta-algorithm is introduced to learn the best optimism for OptimisticHedge to ensure the best-of-both-worlds meta-regret. We describe the expert-algorithm and meta-algorithm as follows.

**Expert-algorithm.** We aggregate the experts of Sword$_{\text{var}}$ and Sword$_{\text{small}}$, and thus there are $N = N_1 + N_2$ experts in total. The step size of each expert is set according to the pool $\mathcal{H} = \mathcal{H}_{\text{var}} \cup \mathcal{H}_{\text{small}}$, and the first $N_1$ experts run the OEGD algorithm (6) with the step size chosen from $\mathcal{H}_{\text{var}}$, and the other $N_2$ experts perform the OGD algorithm (12) with step size specified by $\mathcal{H}_{\text{small}}$. At iteration $t$, the final output is a weighted combination of predictions returned by the expert-algorithms, namely,

$$x_t = \sum_{i=1}^{N_1} p_{t,i} x^\varphi_{t,i} + \sum_{i=N_1+1}^{N_1+N_2} p_{t,i} x^\phi_{t,i}, \quad (17)$$

where $p_t \in \Delta_{N_1+N_2}$ is the weight vector. It remains to specify the meta-algorithm for weight update.
Table 1: Summary of expert-algorithms and meta-algorithms as well as different optimism used in the proposed algorithms (including three variants of Sword).

| Method    | Expert | Meta          | Optimism          |
|-----------|--------|---------------|-------------------|
| Sword_{var} | OEGD  | VariationHedge | by (9)            |
| Sword_{small} | OGD   | vanilla Hedge  | $m_{t+1} = 0$     |
| Sword_{best} | OEGD & OGD | OptimisticHedge | by (19), (22)    |

**Meta-algorithm.** We adopt the OptimisticHedge algorithm along with the linearized surrogate loss as the meta-algorithm, where the weight vector $p_{t+1} \in \Delta_{N_1+N_2}$ is updated according to

$$p_{t+1,i} \propto \exp \left( -\varepsilon \left( \sum_{s=1}^{t} \langle \nabla f_t(x_s), x_{s,i} \rangle + m_{t+1,i} \right) \right),$$

where the optimism $m_{t+1} \in \mathbb{R}^{N_1+N_2}$. In order to facilitate the meta-algorithm with both kinds of adaptivity ($V_T$ and $F_T$), it is crucial to design best-of-both-worlds optimism.

We set the optimism $m_{t+1}$ in the following way: for each $i \in [N_1 + N_2]$

$$m_{t+1,i} = \langle M_{t+1}, x_{t+1,i} \rangle,$$  

where $M_{t+1} \in \mathbb{R}^d$ is called the optimistic vector, $x_{t,i} = x_{t,i}^v$ for $i = 1, \ldots, N_1$ and $x_{t,i} = x_{t,i}^s$ for $i = N_1 + 1, \ldots, N_1 + N_2$. Therefore, we are left with the task of determining the term of $M_{t+1}$ in (19). Inspired by the seminal work of Rakhlin and Sridharan (2013), we treat the problem of selecting the sequence of optimistic vectors as another online learning problem.

Specifically, consider the following learning scenario of **prediction with two expert advice**. At the beginning of iteration $(t+1)$, we receive two optimistic vectors $M_{t+1}^v, M_{t+1}^s \in \mathbb{R}^d$, based on which the algorithm determines the optimistic vector $M_{t+1} \in \mathbb{R}^d$ for Sword_{best}. Then the online function $f_{t+1}$ is revealed, and we subsequently observe the loss of $d_{t+1}(M_{t+1}^v)$ and $d_{t+1}(M_{t+1}^s)$, where $d_{t+1}(M) = \|\nabla f_{t+1}(x_{t+1}) - M\|_2^2$. In above, the vectors of $M_{t+1}^v$ and $M_{t+1}^s$ are

$$M_{t+1}^v = \nabla f_{t+1}(\bar{x}_{t+1}), \quad M_{t+1}^s = 0,$$

where $\bar{x}_{t+1}$ is the instrumental output. Similar to the construction of (9), it is designed as

$$\bar{x}_{t+1} = \sum_{i=1}^{N_1} p_{t,i} x_{t+1,i}^v + \sum_{i=N_1+1}^{N_1+N_2} p_{t,i} x_{t+1,i}^s.$$  

Notice that the function $d_1 : \mathbb{R}^d \rightarrow \mathbb{R}$ is 2-strongly convex with respect to the $\|\cdot\|_2$-norm, we thus choose Hedge of strongly convex functions (Cesa-Bianchi and Lugosi, 2006, Chapter 3.3) as the parallel meta-algorithm for updating,

$$M_{t+1} = \beta_{t+1} M_{t+1}^v + (1 - \beta_{t+1}) M_{t+1}^s,$$  

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Algorithm 3 Sword\textsubscript{best}: Meta-algorithm (OptimisticHedge)

Input: step size pool \( H = \{\eta_i\}_{i=1}^N \) as specified in (24); learning rate \( \varepsilon \)

1: Initialization: let \( x_1 \) be any point in \( X \); set \( N = N_1 + N_2 \) and \( p_{0,i} = 1/N \) for \( \forall i \in [N] \)
2: for \( t = 1 \) to \( T \) do
3:   Receive the prediction \( x_{t+1,i} \) from expert \( E_i \)
4:   Set \( M_{t+1}^v \) and \( M_{t+1}^s \) by (20) and (21)
5:   Update the weight \( \beta_{t+1} \) by (23)
6:   Obtain the optimism \( M_{t+1} \) (22)
7:   Update the weight \( p_{t+1,i} \) by (18) and (19)
8: end for
9: Output the prediction \( x_{t+1} = \sum_{i=1}^N p_{t+1,i} x_{t+1,i} \)

Algorithm 4 Sword\textsubscript{best}: Expert-algorithm (OEGD & OGD)

Input: step size \( \eta_i \)

1: Let \( \hat{x}_{1,i} \) be any point in \( X \)
2: for \( t = 1 \) to \( T \) do
3:   if \( i \in \{1, \ldots, N_1\} \) then
4:     \( \hat{x}_{t+1,i} = \Pi_X [\hat{x}_{t,i} - \eta_i \nabla f_t(x_{t,i})] \)
5:     \( x_{t+1,i} = \Pi_X [\hat{x}_{t+1,i} - \eta_i \nabla f_t(\hat{x}_{t+1,i})] \)
6:   else
7:     \( x_{t+1,i} = \Pi_X [x_{t,i} - \eta_i \nabla f_t(x_{t,i})] \)
8: end if
9: Send the prediction \( x_{t+1,i} \) to meta-algorithm
10: end for

where the weight \( \beta_{t+1} \in [0, 1] \) for learning optimistic vectors is updated by

\[
\beta_{t+1} = \frac{\exp(-2D^v_t)}{\exp(-2D^v_t) + \exp(-2D^s_t)}
\]  

(23)

with \( D^v_t = \sum_{\tau=1}^t d_{\tau}(M^v_{\tau}) \) and \( D^s_t = \sum_{\tau=1}^t d_{\tau}(M^s_{\tau}) \).

Algorithm 3 summarizes the meta-algorithm of Sword\textsubscript{best}. In the last two columns of Table 1, we present comparisons of the meta-algorithms and optimism designed for different methods.

Regret Analysis. Recall that the meta-regret of OptimisticHedge is of order \( O(\sqrt{D_\infty \ln N}) \). From the setting of surrogate loss (18) and optimism (19), we have

\[
D_\infty = \sum_{t=1}^T \max_{i \in [N]} \langle (\nabla f_t(x_t) - M_t, x_{t,i}) \rangle^2 \leq D^2 \sum_{t=1}^T \| \nabla f_t(x_t) - M_t \|_2^2.
\]
Besides, the regret analysis of Hedge for strongly convex functions (Cesa-Bianchi and Lugosi, 2006, Proposition 3.1) implies
\[ \sum_{t=1}^{T} \| \nabla f_t(x_t) - M_t \|^2_2 = \sum_{t=1}^{T} d_t(M_t) \leq \min \{ \bar{V}_T, \bar{F}_T \} + \frac{\ln 2}{2}, \]
where \( \bar{V}_T = \sum_{t=1}^{T} \| \nabla f_t(x_t) - \nabla f_{t-1} (\bar{x}_t) \|^2_2 \) and \( \bar{F}_T = \sum_{t=1}^{T} \| \nabla f_t(x_t) \|^2_2 \). The two terms can be further converted to the desired gradient variation \( V_T \) and small loss \( F_T \), by exploiting smoothness and expert-regret analysis. We can thus ensure the following meta-regret bound.

**Theorem 7.** By setting the learning rate optimally as \( \varepsilon = \min \{ 1/(8D^4L^2), \varepsilon^* \} \), the meta-algorithm of Sword\(_{\text{best}}\) (an adaptation of OptimisticHedge in (18)) satisfies
\[ \text{meta-regret} \leq 2D \sqrt{(2 + \ln N)(\min \{ 2V_T, \bar{F}_T \} + \ln 2) + 4\sqrt{2}D^2L(2 + \ln N)} \]
where \( \varepsilon^* = \sqrt{(2 + \ln N)/(2D \min \{ 2V_T, \bar{F}_T \} + 2D^2 \ln 2)} \).

Because \( V_T \) and \( \bar{F}_T \) are both empirically observable, we can easily get rid of their dependence in the optimal learning rate tuning. Also see the discussion below Theorem 2 on replacing the original gradient variation \( V_T \) by its empirical approximation \( \tilde{V}_T \) to save computational costs. Besides, the \( \bar{F}_T \) term of meta-regret will be converted to the desired small-loss quantity \( F_T \) in the final regret bound. Combining above meta-regret analysis and expert-regret analysis of OEGD and OGD algorithms, we can finally achieve the best of both worlds.

**Theorem 8.** Under Assumptions 1, 2, 3, and 4, setting the pool of candidate step sizes as
\[ \mathcal{H} = \mathcal{H}_{\text{var}} \cup \mathcal{H}_{\text{small}}, \quad (24) \]
the dynamic regret of Sword\(_{\text{best}}\) is upper bounded by
\[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq O(\sqrt{(1 + P_T + \min \{ V_T, F_T \}) (1 + P_T)}). \]

**Remark 3.** The dynamic regret bound in Theorem 8 achieves a minimum of gradient-variation and small-loss bounds, and therefore combines their advantages and enjoys both kinds of adaptivity. In particular, let us revisit the two instances presented in Section 3.5. Actually, we can now run a single algorithm (Sword\(_{\text{best}}\)) to achieve constant dynamic regret in both instances.

5. Conclusion

In this paper, we exploit smoothness to enhance the dynamic regret, with the aim to replace the time horizon \( T \) in the state-of-the-art \( O(\sqrt{T(1 + P_T)}) \) bound by problem-dependent quantities that are at most \( O(T) \) but can be much smaller in easy problems. We achieve this goal by proposing two meta-expert algorithms: Sword\(_{\text{var}}\) which attains a variation bound of order \( O(\sqrt{(1 + P_T + V_T)(1 + P_T)}) \), and Sword\(_{\text{small}}\) which enjoys a small-loss bound of
order $O(\sqrt{(1 + P_T + F_T)(1 + P_T)})$. Here, $V_T$ measures the variation in gradients and $F_T$ is the cumulative loss of the comparator sequence. They are at most $O(T)$ yet could be very small when the problem is easy, and thus reflect the difficulty of the problem instance. As a result, our bounds improve the minimax rate of universal dynamic regret by exploiting smoothness. Furthermore, we design Sword$_{\text{best}}$ to combine advantages of both variation and small-loss algorithms and achieve a best-of-both-worlds bound of order $O(\sqrt{(1 + P_T + \min\{V_T, F_T\})(1 + P_T)})$. Our dynamic regret bounds are universal in the sense that they hold against any feasible comparator sequence, and thus the algorithms are more adaptive to the non-stationary environments. We finally present the lower bound for dynamic regret of convex and smooth functions, showing the tightness of our obtained upper bounds. In the future, we will investigate the possibility of exploiting other function curvatures, such as strong convexity or exp-concavity, into the analysis of the universal dynamic regret.

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A. Proof of Gradient-Variation Bound

In this section, we provide proofs of gradient-variation bounds, including analysis of the expert-algorithm and meta-algorithm, as well as the proof of overall dynamic regret.

A.1 Analysis of Expert-Algorithm (Online Extra-Gradient Descent)

In this part, we analyze the expert-algorithm of Sword_car, namely, the online extra-gradient descent.

We first restate the gradient-variation static regret proved by Chiang et al. (2012) as follows.

**Theorem 9.** Under Assumptions 1, 2, and 3, by choosing $\eta \leq \frac{1}{4T}$, for any $x \in \mathcal{X}$, OEGD (6) satisfies

$$
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x) \leq \frac{2}{\eta} + 2\eta \sum_{t=2}^{T} \sup_{x \in \mathcal{X}} \|\nabla f_{t-1}(x) - \nabla f_t(x)\|^2_2 + GD = O\left(\frac{1}{\eta} + \eta V_T\right).
$$

Therefore, by choosing $\eta = \min\{1/(4L), 1/\sqrt{T}\}$, OEGD achieves an $O(\sqrt{V_T})$ static regret. Note that the unpleasant dependence on $V_T$ can be eliminated by the doubling trick (Cesa-Bianchi et al., 1997), because the gradient variation $V_T$ is empirically evaluable at each iteration.

Recall that the static regret is a special case of the universal dynamic regret by setting comparators as the best decision in hindsight, namely, $u_1 = u_2 = \ldots = u_T = x^* \in \arg\min_{x \in \mathcal{X}} \sum_{t=1}^{T} f_t(x)$. It is clear that the dynamic regret bound in Theorem 1 recovers the static regret bound in Theorem 9. Therefore, it is sufficient for us to prove the dynamic regret bound, which is presented as follows.

**Proof of Theorem 1.** From the update procedure in (6) and by employing Lemma 7, we have

$$
\langle x_{t+1} - u_t, \eta \nabla f_t(x_t) \rangle \leq \frac{1}{2} \|u_t - \hat{x}_t\|^2_2 - \frac{1}{2} \|u_t - \hat{x}_{t+1}\|^2_2 - \frac{1}{2} \|\hat{x}_{t+1} - \hat{x}_t\|^2_2 \quad (25)
$$

$$
\langle x_t - \hat{x}_{t+1}, \eta \nabla f_{t-1}(\hat{x}_t) \rangle \leq \frac{1}{2} \|\hat{x}_t - \hat{x}_{t+1}\|^2_2 - \frac{1}{2} \|x_t - \hat{x}_{t+1}\|^2_2 - \frac{1}{2} \|x_t - \hat{x}_t\|^2_2 \quad (26)
$$

Notice that the instantaneous dynamic regret can be decomposed as

$$
f_t(x_t) - f_t(u_t) = \langle \nabla f_t(x_t), x_t - u_t \rangle = \langle \nabla f_t(x_t) - \nabla f_{t-1}(\hat{x}_t), x_t - \hat{x}_{t+1} \rangle + \langle \nabla f_{t-1}(\hat{x}_t), x_t - \hat{x}_{t+1} \rangle + \langle \nabla f_t(x_t), \hat{x}_{t+1} - u_t \rangle.
$$

Each term can be upper bounded by,

- **term (a)**
  $$
term (a) \leq \|\nabla f_t(x_t) - \nabla f_{t-1}(\hat{x}_t)\|_2 \|x_t - \hat{x}_{t+1}\|_2 \quad (27)
$$

- **term (b)**
  $$
term (b) \leq \frac{1}{2\eta} (\|\hat{x}_t - \hat{x}_{t+1}\|^2_2 - \|x_t - \hat{x}_{t+1}\|^2_2 - \|x_t - \hat{x}_t\|^2_2) \quad (28)
$$

- **term (c)**
  $$
term (c) \leq \frac{1}{2\eta} (\|u_t - \hat{x}_t\|^2_2 - \|u_t - \hat{x}_{t+1}\|^2_2 - \|\hat{x}_{t+1} - \hat{x}_t\|^2_2) \quad (29)
$$

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where (27) holds due to the Hölder inequality, (28) and (29) are obtained by a rearrangement of (25) and (26). So we can combine all three inequalities above and get

\[
f_t(x_t) - f_t(u_t) \leq \frac{\eta}{2} \|\nabla f_t(x_t) - \nabla f_{t-1}(\tilde{x}_{t})\|_2^2 + \frac{1}{2\eta} \|x_t - \tilde{x}_{t+1}\|_2^2 \\
+ \frac{1}{2\eta} \left( \|u_t - \tilde{x}_{t}\|_2^2 - \|u_t - \tilde{x}_{t+1}\|_2^2 - \|\tilde{x}_{t+1} - x_t\|_2^2 - \|x_t - \tilde{x}_{t}\|_2^2 \right),
\]

where we make use of the fact that \(ab \leq \frac{a^2}{2\eta} + \frac{b^2}{2}\) holds for any \(a, b \geq 0\) and \(\eta > 0\).

Summing the above inequality over all iterations, we can bound the dynamic regret as follows,

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq f_1(x_1) - f_1(u_1) + \sum_{t=2}^{T} \frac{\eta}{2} \|\nabla f_t(x_t) - \nabla f_{t-1}(\tilde{x}_{t})\|_2^2 + \frac{1}{2\eta} \|x_t - \tilde{x}_{t}\|_2^2 - \|u_t - \tilde{x}_{t+1}\|_2^2 - \|x_t - \tilde{x}_{t}\|_2^2 \\
\leq GD + \frac{\eta}{2} \sum_{t=2}^{T} \|\nabla f_t(x_t) - \nabla f_{t-1}(\tilde{x}_{t})\|_2^2 + \frac{1}{2\eta} \sum_{t=2}^{T} \|x_t - \tilde{x}_{t}\|_2^2 - \|u_t - \tilde{x}_{t+1}\|_2^2 - \|x_t - \tilde{x}_{t}\|_2^2.
\]

We exploit smoothness to bound term (i),

\[
\text{term (i)} = \frac{\eta}{2} \sum_{t=2}^{T} \|\nabla f_t(x_t) - \nabla f_{t-1}(\tilde{x}_{t})\|_2^2 \\
\leq \frac{\eta}{2} \sum_{t=2}^{T} 2(\|\nabla f_t(x_t) - \nabla f_{t}(\tilde{x}_{t})\|_2^2 + \|\nabla f_{t}(\tilde{x}_{t}) - \nabla f_{t-1}(\tilde{x}_{t})\|_2^2) \\
\overset{(5)}{\leq} \eta \sum_{t=2}^{T} \left( L^2 \|x_t - \tilde{x}_{t}\|_2^2 + \sup_{x \in \mathcal{X}} \|\nabla f_t(x) - \nabla f_{t-1}(x)\|_2^2 \right) \\
= \eta L^2 \sum_{t=2}^{T} \|x_t - \tilde{x}_{t}\|_2^2 + \eta V_T.
\]

It suffices to bound term (ii),

\[
\text{term (ii)} = \frac{1}{2\eta} \sum_{t=2}^{T} (\|u_t - \tilde{x}_{t}\|_2^2 - \|u_t - \tilde{x}_{t+1}\|_2^2) \\
= \frac{1}{2\eta} \|u_1 - \tilde{x}_2\|_2^2 + \frac{1}{2\eta} \sum_{t=2}^{T} (\|u_t - \tilde{x}_{t}\|_2^2 - \|u_{t-1} - \tilde{x}_{t}\|_2^2) \\
\leq \frac{D^2}{2\eta} + \frac{1}{2\eta} \sum_{t=2}^{T} \|u_t - \tilde{x}_{t} + u_{t-1} - \tilde{x}_{t}\|_2 \|u_t - u_{t-1}\|_2
\]

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\[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq GD + (\eta L^2 - \frac{1}{2\eta}) \sum_{t=2}^{T} \|x_t - \tilde{x}_t\|_2^2 + \eta V_T + \frac{D^2}{2\eta} + \frac{D}{\eta} \sum_{t=2}^{T} \|u_t - u_{t-1}\|_2 \]  

(30)

where the last step makes use of the condition \( \eta \leq 1/(4L) \). This completes the proof. \( \square \)

A.2 Analysis of Meta-Algorithm (VariationHedge)

In this part, we analyze the meta-algorithm of Sword\textsubscript{var}, i.e., VariationHedge. We first present a general meta-regret bound of VariationHedge in Theorem 10, which holds for any learning rate \( \varepsilon > 0 \). Then, we prove Theorem 2 as a consequence by choosing a proper learning rate.

Let us restate the weight update procedure of VariationHedge. From (10), we know that VariationHedge updates the weight \( p_{t+1} \in \Delta_N \) as follows,

\[ p_{t+1,i} = \frac{\exp \left( -\varepsilon \left( \sum_{s=1}^{t} \langle \nabla f_s(x_s), x_s,i \rangle + \langle \nabla f_t(\tilde{x}_{t+1}), x_{t+1,i} \rangle \right) \right)}{\sum_{i=1}^{N} \exp \left( -\varepsilon \left( \sum_{s=1}^{t} \langle \nabla f_s(x_s), x_s,i \rangle + \langle \nabla f_t(\tilde{x}_{t+1}), x_{t+1,i} \rangle \right) \right)} \]  

(31)

and the instrumental output \( \tilde{x}_{t+1} \) is carefully designed as

\[ \tilde{x}_{t+1} = \sum_{i=1}^{N} p_{t,i} x_{t+1,i}. \]  

(32)

The motivation of the design has been illustrated in Remark 1. Note that VariationHedge is actually a specialization of OptimisticHedge by setting the linearized surrogate loss \( \ell_{t,i} = \langle \nabla f_t(x_t), x_t,i \rangle \) and optimism \( m_{t+1,i} = \langle \nabla f_t(\tilde{x}_{t+1}), x_{t+1,i} \rangle \). Therefore, by Lemma 1 and the setting of the instrumental output \( \tilde{x}_{t+1} \), we have the following result regarding its meta-regret.

\textbf{Theorem 10.} Under Assumptions 1, 2 and 3, the meta-regret of the VariationHedge satisfies

\[ \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq \frac{2 + \ln N}{\varepsilon} + 2\varepsilon D^2 V_T + \left( 2\varepsilon D^4 L^2 - \frac{1}{4\varepsilon} \right) \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2 + O(1), \]  

(33)

which holds for any expert \( i \in [N] \).
Proof of Theorem 10. By convexity, we know that the dynamic regret with respect to the original loss function is bounded by that with respect to the linearized surrogate loss, namely,
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq \sum_{t=1}^{T} \langle \nabla f_t(x_t), x_t - x_{t,i} \rangle = \sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i}.
\]
Since VariationHedge is a variant of OptimisticHedge by assigning the feedback loss of expert $E_i$ as $\ell_{t,i} = \langle \nabla f_t(x_t), x_{t,i} \rangle$ and the optimism as $m_{t+1,i} = \langle \nabla f_t(x_{t+1}), x_{t+1,i} \rangle$, Lemma 1 implies
\[
\sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i} \\
\leq \varepsilon \sum_{t=1}^{T} \| \ell_t - m_t \|_\infty^2 + \frac{2 + \ln N}{\varepsilon} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2
\]
\[
= \varepsilon \sum_{t=1}^{T} \left( \max_{i \in [N]} \langle \nabla f_t(x_t) - \nabla f_{t-1}(x_t), x_{t,i} \rangle \right)^2 + \frac{2 + \ln N}{\varepsilon} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2
\]
\[
\leq \varepsilon D^2 \sum_{t=1}^{T} \| \nabla f_t(x_t) - \nabla f_{t-1}(\bar{x}_t) \|_2^2 + \frac{2 + \ln N}{\varepsilon} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2
\]
\[
\leq 2\varepsilon D^2 \sum_{t=1}^{T} \left( \| \nabla f_t(x_t) - \nabla f_{t-1}(x_t) \|_2^2 + \| \nabla f_{t-1}(x_t) - \nabla f_{t-1}(\bar{x}_t) \|_2^2 \right)
\]
\[
+ \frac{2 + \ln N}{\varepsilon} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2
\]
\[
\leq 2\varepsilon D^2 \sum_{t=1}^{T} \sup_{x \in X} \| \nabla f_t(x) - \nabla f_{t-1}(x) \|_2^2 + 2\varepsilon D^2 L^2 \sum_{t=1}^{T} \| x_t - \bar{x}_t \|_2^2
\]
\[
+ \frac{2 + \ln N}{\varepsilon} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2
\]
\[
\leq 2\varepsilon D^2 V_T + 2\varepsilon D^2 L^2 \sum_{t=1}^{T} \| x_t - \bar{x}_t \|_2^2 + \frac{2 + \ln N}{\varepsilon} - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2 + O(1),
\]
where the second inequality holds due to Jensen’s inequality and Assumption 2, and (34) holds due to the smoothness. Notice that the extra $O(1)$ term appears in (35) due to that the definition of gradient variation $V_T$ begins from the index of 2. We will keep the notation of $O(1)$ without presenting the detailed values, as the constant will not affect the regret order.

We now focus on the last two terms. Indeed,
\[
2\varepsilon D^2 L^2 \sum_{t=1}^{T} \| x_t - \bar{x}_t \|_2^2 - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \| p_t - p_{t-1} \|_1^2
\]
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq \frac{2 + \ln N}{\varepsilon} + 2\varepsilon D^2 V_T + \left(2\varepsilon D^4 L^2 - \frac{1}{4\varepsilon}\right) \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2 + \mathcal{O}(1).
\]

Since \(\varepsilon \leq \sqrt{1/(8D^4 L^2)}\), we have \((2\varepsilon D^4 L^2 - 1/(4\varepsilon)) \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2 < 0\). Therefore the meta-regret of VariationHedge is bounded by,

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq \frac{2 + \ln N}{\varepsilon} + 2\varepsilon D^2 V_T + \mathcal{O}(1).
\]

Let \(\varepsilon_\ast = \sqrt{2 + \ln N / 2D^2 V_T}\) and \(\varepsilon_0 = \sqrt{1 / 8D^4 L^2}\), we set the learning rate as \(\varepsilon = \min\{\varepsilon_0, \varepsilon_\ast\}\). We consider the following two cases:

- when \(\varepsilon_\ast \leq \varepsilon_0\), the meta-regret is at most
  \[
  \sum_{t=1}^{T} f_t(x_t) - f_t(x_{t,i}) \leq \frac{2 + \ln N}{\varepsilon_0} + 2\varepsilon_0 D^2 V_T = 2\sqrt{2D^2(2 + \ln N)V_T}.
  \]

- when \(\varepsilon_\ast \geq \varepsilon_0\), the meta-regret is bounded by
  \[
  \sum_{t=1}^{T} f_t(x_t) - f_t(x_{t,i}) \leq \frac{2 + \ln N}{\varepsilon_0} + 2\varepsilon_0 D^2 V_T \leq 4\sqrt{2D^2 L(2 + \ln N)},
  \]

where the last inequality makes use of the condition of \(\varepsilon_\ast \geq \varepsilon_0\).
Hence, taking the two cases into account, the meta-regret is bounded by
\[
\sum_{t=1}^{T} f_t(x_t) - f_t(x_{t,i}) \leq 2D \sqrt{2V_T(2 + \ln N)} + 4\sqrt{2}D^2L(2 + \ln N) + \mathcal{O}(1)
\]
\[
= \mathcal{O}\left(\sqrt{(\ln N + V_T) \ln N}\right),
\]
which completes the proof. 

A.3 Proof of Gradient-Variation Dynamic Regret Bounds (Theorem 3)

Proof of Theorem 3. Notice that the dynamic regret can be decomposed into the following two parts
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) + \sum_{t=1}^{T} f_t(x_{t,i}) - \sum_{t=1}^{T} f_t(u_t),
\]
which holds for any expert index \(i \in [N]\). In above, \(\{x_t\}_{t=1,...,T}\) denotes the final output sequence, and \(\{x_{t,i}\}_{t=1,...,T}\) is the prediction sequence of expert \(E_i\). The first part is the difference between cumulative loss of final output sequence and that of prediction sequence of expert \(E_i\), which is introduced by the meta-algorithm and thus named as meta-regret; the second part is the dynamic regret of expert \(E_i\) and therefore named as expert-regret.

In the following, we upper bound these two terms respectively.

Upper bound of meta-regret. Recall that in Sword_{var}, the final decision \(x_t\) at iteration \(t\) is a weighted combination of predictions returned from the expert-algorithms, and the weight is updated by the meta-algorithm (VariationHedge). Therefore, we can apply Theorem 2 to track any expert \(i \in [N]\) and obtain the upper bound of the meta-regret,
\[
\text{meta-regret} = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq 2D \sqrt{2V_T(2 + \ln N)} + 4\sqrt{2}D^2L(2 + \ln N).
\]

Upper bound of expert-regret. To make the bound in (37) tight, we find the expert \(k \in [N]\) with the smallest expert-regret. In other words, we need to identify the nearly optimal step size.

Recall that the optimal step size is \(\eta^* = \min\{\frac{1}{4L}, \sqrt{(D^2 + 2DP_T)/(2V_T)}\}\). Meanwhile, \(V_T = \sum_{t=2}^{T} \sup_{x \in X} \|\nabla f_t(x) - \nabla f_{t-1}(x)\|_2^2 \leq 4G^2T\) due to Assumption 1 and Assumption 2. Consequently, the possible minimal and maximal values of the optimal step size are
\[
\eta_{\min} = \sqrt{\frac{D^2}{8G^2T}}, \quad \eta_{\max} = \frac{1}{4L}.
\]

By the construction of the candidate step size pool \(\mathcal{H}_{var}\), we know that the step size therein is monotonically increasing with respect to the index, in particular,
\[
\eta_1 = \sqrt{\frac{D^2}{8G^2T}} = \eta_{\min}, \quad \text{and} \quad \eta_N \leq \frac{1}{4L} = \eta_{\max}.
\]
Therefore, we confirm that there exists an integer $k \in [N]$ such that $\eta_k \leq \eta^* \leq \eta_{k+1} = 2\eta_k$. The gap between the cumulative loss of final decisions and that of expert $k$ can be upper bounded as follows,

$$\text{expert-regret} = \sum_{t=1}^{T} f_t(x_{t,k}) - \sum_{t=1}^{T} f_t(u_t)$$

\(30\) \[ D^2 + 2DP_T \leq \frac{D^2 + 2DP_T}{2\eta_k} + \eta_k V_T + GD \]

\(30\) \[ \leq \frac{D^2 + 2DP_T}{\eta^*} + \eta^* V_T + GD \] \(40\)

\(30\) \[ \leq 3\sqrt{V_T(D^2 + 2DP_T) + 6L(D^2 + 2DP_T) + GD} \] \(41\)

\(30\) \[ \leq 3\sqrt{2(V_T + 4L^2D^2 + 8L^2DP_T)(D^2 + 2DP_T) + GD} \] \(42\)

where \(40\) holds due to $\eta_k \leq \eta^* \leq 2\eta_k$, and \(42\) follows from $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$, $\forall a, b > 0$. Meanwhile, \(41\) holds by noticing that the optimal step size $\eta^*$ is either $\sqrt{(D^2 + 2DP_T)/(2V_T)}$ or $\frac{1}{16}$, and therefore

- when $\eta^* = \sqrt{(D^2 + 2DP_T)/(2V_T)}$, R.H.S of \(40\) $= \frac{3}{2} \sqrt{2V_T(D^2 + 2DP_T)} + GD$.
- when $\eta^* = \frac{1}{16}$, R.H.S of \(40\) $= 4L(D^2 + 2DP_T) + \frac{1}{16} V_T + GD \leq 6L(D^2 + 2DP_T) + GD$, where the last inequality holds due to $1/(4L) \leq \sqrt{(D^2 + 2DP_T)/(2V_T)}$ in this case.

We sum over the upper bounds of two conditions and obtain \(41\).

**Upper bound of dynamic regret.** Combining \(38\) and \(42\), we obtain

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t)
\]

\(37\) \[ \frac{1}{\eta_k} \sum_{t=1}^{T} f_t(x_{t,k}) - \sum_{t=1}^{T} f_t(x_{t,k}) + \sum_{t=1}^{T} f_t(x_{t,k}) - \sum_{t=1}^{T} f_t(u_t) \]

\(38\) \(42\) \[ \leq 2D\sqrt{2V_T(2 + \ln N)} + 4\sqrt{2D^2L(2 + \ln N)} \\
+ 3\sqrt{2(V_T + 4L^2D^2 + 8L^2DP_T)(D^2 + 2DP_T) + GD} \\
\leq 3\sqrt{4(V_T + 4L^2D^2 + 8L^2DP_T)(D^2 + 2DP_T) + 4DV_T(2 + \ln N)} \\
+ 4\sqrt{2D^2L(2 + \ln N) + GD} \\
\leq 6\sqrt{(3 + \ln N)V_T + 4L^2D^2 + 8L^2DP_T)(D^2 + 2DP_T) + 4DV_T(2 + \ln N) + GD} \\
= O\left(\sqrt{(1 + P_T + V_T)(1 + P_T)}\right).
\]

The derivation uses the inequality of $\sqrt{a} + \sqrt{b} \leq \sqrt{2(a+b)}$, $\forall a, b > 0$. Meanwhile, we treat the double logarithmic factor in $T$ as a constant, following previous studies (Adamskiy et al., 2012; Luo and Schapire, 2015). We remark that the bound is the universal dynamic regret in that it holds for any sequence of comparators. \(\square\)
B. Proof of Small-Loss Bounds

In this section, we provide proofs of small-loss bounds, including analysis of the expert-algorithm and meta-algorithm, as well as the proof of overall dynamic regret.

B.1 Analysis of Expert-Algorithm (Online Gradient Descent)

In this part, we analyze the expert-algorithm of the Sword\textsubscript{var} algorithm, namely, the online gradient descent. We will present the proof of the small-loss dynamic regret bound (Srebro et al., 2010, Theorem 2). Before that, in the following we first restate the small-loss static regret bound (Srebro et al., 2010, Theorem 2) as well as its proof.

**Theorem 11** (Theorem 2 of Srebro et al. (2010)). Under Assumptions 2, 3, and 4, by choosing any step size \( \eta \leq \frac{1}{4L} \), OGD satisfies

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x^*) \leq \frac{D^2}{2\eta(1-2\eta L)} + \frac{2\eta L}{(1-2\eta L)} \sum_{t=1}^{T} f_t(x^*) = O\left( \frac{1}{\eta} + \eta F_T^* \right)
\]

for any \( x^* \in X \), where \( F_T^* = \sum_{t=1}^{T} f_t(x^*) \) is the cumulative loss of the comparator benchmark \( x^* \).

Theorem 11 indicates an \( O(\sqrt{T}) \) regret bound with a proper choice of step size, which is tighter than the minimax rate of \( O(\sqrt{T}) \) when the cumulative loss is small.

**Proof of Theorem 11.** First, notice that Assumptions 4 and 3 imply \( f_t(\cdot) \) is nonnegative and \( L \)-smooth. From the self-bounding property of smooth functions (Srebro et al., 2010), as shown in Lemma 4, we have

\[
\| \nabla f_t(x) \|^2 \leq 4L f_t(x), \quad \forall x \in X.
\]

Define \( x_{t+1} = x_t - \eta \nabla f_t(x_t) \). For any \( x \in X \), we have

\[
f_t(x_t) - f_t(x) \leq \langle \nabla f_t(x_t), x_t - x \rangle = \frac{1}{\eta}(x_t - x, x_{t+1} - x_t - x)
\]

\[
= \frac{1}{2\eta} \left( \| x_t - x \|^2 - \| x_{t+1} - x \|^2 + \| x_t - x_{t+1} \|^2 \right)
\]

\[
= \frac{1}{2\eta} \left( \| x_t - x \|^2 - \| x_{t+1} - x \|^2 + \frac{\eta}{2} \| \nabla f_t(x_t) \|^2 \right)
\]

\[
\leq \frac{1}{2\eta} \left( \| x_t - x \|^2 - \| x_{t+1} - x \|^2 \right) + 2\eta L f_t(x_t)
\]

(44)

Summing the above inequality over all iterations, we have

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x) \leq \frac{1}{2\eta} \| x_1 - x \|^2 + 2\eta L \sum_{t=1}^{T} f_t(x_t) \leq \frac{D^2}{2\eta} + 2\eta L \sum_{t=1}^{T} f_t(x_t)
\]

which implies

\[
(1 - 2\eta L) \left( \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x) \right) \leq \frac{D^2}{2\eta} + 2\eta L \sum_{t=1}^{T} f_t(x).
\]
We complete the proof by dividing both sides by $(1 - 2\eta L)$, as the step size satisfies $\eta \leq 1/(4L)$.

Proof of Theorem 4. Let $x'_{t+1} = x_t - \eta \nabla f_t(x_t)$. Following the standard analysis, we have

$$f_t(x_t) - f_t(u_t) \leq \langle \nabla f_t(x_t), x_t - u_t \rangle = \frac{1}{\eta} \langle x_t - x'_{t+1}, x_t - u_t \rangle$$

$$= \frac{1}{2\eta} \left( \|x_t - u_t\|^2_2 - \|x'_{t+1} - u_t\|^2_2 + \|x_t - x'_{t+1}\|^2_2 \right)$$

$$= \frac{1}{2\eta} \left( \|x_t - u_t\|^2_2 - \|x'_{t+1} - u_t\|^2_2 \right) + \frac{\eta}{2} \|\nabla f_t(x_t)\|^2_2$$

$$\leq \frac{1}{2\eta} \left( \|x_t - u_t\|^2_2 - \|x_{t+1} - u_t\|^2_2 \right) + 2\eta L f_t(x_t)$$

$$= \frac{1}{2\eta} \left( \|x_t\|^2_2 - \|x_{t+1}\|^2_2 \right) + \frac{1}{\eta} (x_{t+1} - x_t)^\top u_t + 2\eta L f_t(x_t).$$

Summing the above inequality over all iterations, we have

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u_t) \leq \frac{1}{2\eta} \|x_1\|^2_2 + \frac{1}{\eta} \sum_{t=1}^T (x_{t+1} - x_t)^\top u_t + 2\eta L \sum_{t=1}^T f_t(x_t)$$

$$= \frac{1}{2\eta} \|x_1\|^2_2 + \frac{1}{\eta} (x_{T+1}^\top u_T - x_1^\top u_1) + \frac{1}{\eta} \sum_{t=2}^T (u_{t-1} - u_t)^\top x_t + 2\eta L \sum_{t=1}^T f_t(x_t)$$

$$\leq \frac{7D^2}{4\eta} + \frac{D}{\eta} \sum_{t=2}^T \|u_{t-1} - u_t\|^2_2 + 2\eta L \sum_{t=1}^T f_t(x_t)$$

(46)

where the last step makes use of the boundedness of the domain $X$, more precisely

$$\|x_1\|^2_2 = \|x_1 - 0\|^2_2 \leq D^2,$$

$$x_{T+1}^\top u_T \leq \|x_{T+1}\|^2_2 \|u_T\|_2 \leq D^2;$$

$$-x_1^\top u_1 \leq \frac{1}{4} \|x_1\|^2_2 \leq \frac{1}{4} D^2;$$

$$(u_{t-1} - u_t)^\top x_t \leq \|u_{t-1} - u_t\|^2_2 \|x_t\|_2 \leq D \|u_{t-1} - u_t\|^2_2.$$

We complete the proof by simplifying (46).

B.2 Analysis of Meta-Algorithm (vanilla Hedge)

In this part, we analyze the meta-algorithm of Sword_{small}, i.e., the vanilla Hedge with linearized surrogate loss. Notice that the vanilla Hedge can be treated as a special case of OptimisticHedge by setting $\ell_{t,i} = \langle \nabla f_t(x_t), x_t, i \rangle$ and $m_{t,i} = 0$ for all $i \in [N]$. Therefore, we will prove the following meta-regret bound based on Lemma 1 and the smoothness of the loss function.
Theorem 12. Under Assumptions 1, 2, 3 and 4, by setting the learning rate optimally as 
\[ \varepsilon = \sqrt{(2 + \ln N)/(D^2 F_T)} \], the meta-regret of the vanilla Hedge satisfies,
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq 4LD^2(2 + \ln N) + \sqrt{L(2 + \ln N)F_T},
\]
(47)
where \( \bar{F}_T = \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2 \) is the cumulative gradient norm, and \( F_T^i = \sum_{t=1}^{T} f_t(x_{t,i}) \) is the cumulative loss of expert \( \mathcal{E}_i \). The result holds for any \( i \in [N] \).

Proof of Theorem 12. Similar to the argument in the proof of Theorem 10, the dynamic regret with respect to the original loss is bounded by that with respect to the surrogate loss, namely,
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq \sum_{t=1}^{T} \langle \nabla f_t(x_t), x_t - x_{t,i} \rangle = \sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i},
\]
where \( \ell_{t,i} = \langle \nabla f_t(x_{t,i}), x_{t,i} \rangle \) and \( p_t \) is updated by (7) by setting \( m_{t+1,i} = 0 \). Since the vanilla Hedge with surrogate loss function can be seen as a special OptimisticHedge, Lemma 1 implies
\[
\sum_{t=1}^{T} \langle p_t, \ell_t \rangle - \sum_{t=1}^{T} \ell_{t,i}
\]
\[
\leq \frac{2 + \ln N}{\varepsilon} + \varepsilon \sum_{t=1}^{T} \|\ell_t - m_t\|_\infty^2 - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2
\]
\[
= \frac{2 + \ln N}{\varepsilon} + \varepsilon \sum_{t=1}^{T} \left( \max_{i \in [N]} \langle \nabla f_t(x_t), x_{t,i} \rangle \right)^2 - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2
\]
\[
\leq \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2
\]
where the last inequality makes use of the Jensen’s inequality and drops the negative term.

By setting the learning rate as \( \varepsilon = \sqrt{(2 + \ln N)/(D^2 F_T)} \), the meta-regret is upper bounded by
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq 2D \sqrt{(2 + \ln N) \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2} \leq 4D \sqrt{L(2 + \ln N) \sum_{t=1}^{T} f_t(x_t)}.
\]
The last inequality makes use of the self-bounding property of non-negative and smooth functions (Lemma 4), which states that for any non-negative \( L \)-smooth functions \( f \), we have \( \|\nabla f_t(x)\|_2^2 \leq 4L f_t(x) \). We mention that the optimal learning rate tuning depends on the unknown cumulative gradient norm \( \bar{F}_T = \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2 \), and issue can be easily addressed by the doubling trick (Cesa-Bianchi et al., 1997) or the self-confident tuning (Auer et al., 2002).
Furthermore, the right hand side is the cumulative loss of decisions returned by the meta-algorithm, which can be further converted to the cumulative loss of decisions returned by expert $E_i$. The conversion can be achieved by applying Lemma 5, which shows that $x - y \leq \sqrt{ax}$ implies $x - y \leq a + \sqrt{ay}$, for any $x, y, a \in \mathbb{R}^+$. Since all loss functions are non-negative, we have
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq 16LD^2(2 + \ln N) + 4D\sqrt{L(2 + \ln N)F_T^i},
\]
which completes the proof.

**B.3 Proof of Small-Loss Dynamic Regret Bounds (Theorem 5)**

Proof of Theorem 5. The proof is analogous to that of Theorem 3, where the dynamic regret is decomposed into the following two parts
\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) = \underbrace{\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i})}_{\text{meta-regret}} + \underbrace{\sum_{t=1}^{T} f_t(x_{t,i}) - \sum_{t=1}^{T} f_t(u_t)}_{\text{expert-regret}},
\]

**Upper bound of meta-regret.** According to Theorem 12, the meta-regret of Sword$_{small}$ is bounded by
\[
\text{meta-regret} = \sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(x_{t,i}) \leq 16LD^2(2 + \ln N) + 4D\sqrt{L(2 + \ln N)F_T^i},
\]
where $F_T^i = \sum_{t=1}^{T} f_t(x_{t,i})$ is the cumulative loss of expert $E_i$.

**Upper bound of expert-regret.** Similar to the argument in Section A.3, we identify that the optimal step size is $\eta^* = \min \{1/(4L), \sqrt{(7D^2 + 4DP_T)/(8LF_T)} \}$. Meanwhile, $F_T = \sum_{t=1}^{T} f_t(u_t) \leq GDT$ due to Assumption 1 and Assumption 2. As a result, the possible minimal and maximal values of the optimal step size are
\[
\eta_{\text{min}} = \sqrt{\frac{7D}{8LGT}}, \quad \eta_{\text{max}} = \frac{1}{4L}.
\]

By the construction of the candidate step size pool $\mathcal{H}_{\text{small}}$ in (15), we know that the step size therein is monotonically increasing with respect to the index, and $\eta_1 = \sqrt{\frac{7D}{8LGT}} = \eta_{\text{min}}$, $\eta_N \leq \frac{1}{4L} = \eta_{\text{max}}$. Therefore, we confirm that there exists an integer $k \in [N]$ such that $\eta_k \leq \eta^* \leq \eta_{k+1} = 2\eta_k$.

We proceed to upper bound the expert-regret for the expert $k$ as follows.
\[
\text{expert-regret} = \sum_{t=1}^{T} f_t(x_{t,k}) - \sum_{t=1}^{T} f_t(u_t) \quad \overset{(46)}{\leq} \quad \frac{7D^2 + 4DP_T}{4\eta_k(1 - 2\eta_k L)} + \frac{2\eta_k L}{1 - 2\eta_k L} F_T
\]

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\[ \leq \frac{7D^2 + 4DP_T}{2\eta_k} + 4\eta_kLF_T \]  
\[ \leq \frac{7D^2 + 4DP_T}{\eta^*} + 4\eta^*LF_T \]  
\[ \leq 3\sqrt{(2LF_T)(7D^2 + 4DP_T)} + 6L(7D^2 + 4DP_T) \]  
\[ \leq 6\sqrt{L(F_T + 7D^2 + 4DP_T)(7D^2 + 4DP_T)} \]  

where (51) uses the fact that \( \eta_k \leq \frac{1}{2L} \), (52) holds due to \( \eta_k \leq \eta^* \leq 2\eta_k \), and (54) follows because of \( \sqrt{a} + \sqrt{b} \leq \sqrt{2(a + b)} \), \( \forall a, b > 0 \). Meanwhile, (53) holds by noticing that the optimal step size \( \eta^* \) is either \( 1/(4L) \) or \( \sqrt{(7D^2 + 4DP_T)/(8LF_T)} \), and therefore

- when \( \eta^* = \sqrt{(7D^2 + 4DP_T)/(8LF_T)} \), R.H.S of (52) = \( 3\sqrt{(2LF_T)(7D^2 + 4DP_T)} \).
- when \( \eta^* = 1/(4L) \), R.H.S of (52) = \( 4L(7D^2 + 4DP_T) + F_T \leq 6L(7D^2 + 4DP_T) \), where the last inequality holds due to \( 1/(4L) \leq \sqrt{(7D^2 + 4DP_T)/(8LF_T)} \) in this case.

We sum over the upper bounds of two conditions and obtain (53).

**Upper bound of dynamic regret.** Combining (49) and (54), we get

\[
\sum_{t=1}^{T} f_t(x_t) - \sum_{t=1}^{T} f_t(u_t) \leq 16LD^2(2 + \ln N) + 4D\sqrt{L(2 + \ln N)F_T} + 6\sqrt{L(F_T + 7D^2 + 4DP_T)(7D^2 + 4DP_T)}
\]

\[
\leq 16LD^2(2 + \ln N) + 4D\sqrt{L(2 + \ln N)(F_T + 6\sqrt{L(F_T + 7D^2 + 4DP_T)(7D^2 + 4DP_T)})} + 6\sqrt{L(F_T + 7D^2 + 4DP_T)(7D^2 + 4DP_T)}
\]

\[
\leq (6 + 4D\sqrt{6L(2 + \ln N)})\sqrt{F_T} + \sqrt{L(F_T + 7D^2 + 4DP_T)(7D^2 + 4DP_T))} + 16LD^2(2 + \ln N)
\]

\[
\leq (6 + 4D\sqrt{6L(2 + \ln N)})(\sqrt{14D^2L + 8DLP_T} + 2(L + 1)F_T)(7D^2 + 4DP_T)) + 16LD^2(2 + \ln N)
\]

\[
= \mathcal{O}\left(\sqrt{(1 + P_T + F_T)(1 + P_T)}\right)
\]

The last two inequalities follow from \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \leq \sqrt{2(a + b)} \), \( \forall a, b > 0 \). Meanwhile, double logarithmic factors in \( T \) are treated as a constant, following previous studies (Adamskiy et al., 2012; Luo and Schapire, 2015). This completes the proof. \( \square \)

**C. Proof of Best-of-Both-Worlds Bounds**

In this section, we provide the regret analysis of the best-of-both-worlds bounds. Specifically, we prove the meta-regret (Theorem 7) and overall dynamic regret (Theorem 8).
Proof of Theorem 7. Since the meta-algorithm used in Sword\textsubscript{best} is a specific configuration of the OptimisticHedge algorithm, we can apply Lemma 1 to upper bound the meta-regret by

\[
\sum_{t=1}^{T} f_t(x_t) - f_t(x_{t,i}) \leq \frac{2 + \ln N}{\varepsilon} + \varepsilon \sum_{t=1}^{T} \|\ell_t - m_t\|_2^2 - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2.
\]

(8)

\[
(18) \leq \frac{2 + \ln N}{\varepsilon} + \varepsilon \sum_{t=1}^{T} \max_{i \in [N]} ((\nabla f_t(x_t) - M_t, x_{t,i}))^2 - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2.
\]

\[
\leq \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \sum_{t=1}^{T} \|\nabla f_t(x_t) - M_t\|_2^2 - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2.
\]

(55)

On the other hand, by noticing that the online function \(d_t\) is strongly convex and exploiting the regret guarantee of Hedge (Cesa-Bianchi and Lugosi, 2006, Proposition 3.1), we have

\[
\sum_{t=1}^{T} d_t(M_t) \leq \min\left\{ \sum_{t=1}^{T} d_t(M_t^v), \sum_{t=1}^{T} d_t(M_t^s) \right\} + \frac{\ln 2}{2},
\]

which implies

\[
\sum_{t=1}^{T} \|\nabla f_t(x_t) - M_t\|_2^2 \leq \min\left\{ \sum_{t=1}^{T} \|\nabla f_t(x_t) - \nabla f_{t-1}(x_t)\|_2^2, \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2 \right\} + \frac{\ln 2}{2}.
\]

(56)

Combining (55) and (56), we immediately have

\[
\text{meta-regret} \leq \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \left( \min\{\hat{V}_T, \hat{F}_T\} + \frac{\ln 2}{2} \right) - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2 = \min\{A_T, B_T\}
\]

where \(\hat{V}_T = \sum_{t=1}^{T} \|\nabla f_t(x_t) - \nabla f_{t-1}(x_t)\|_2^2\) and \(\hat{F}_T = \sum_{t=1}^{T} \|\nabla f_t(x_t)\|_2^2\). Besides, \(A_T\) and \(B_T\) are defined as follows.

\[
A_T = \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \left( \hat{V}_T + \frac{\ln 2}{2} \right) - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2,
\]

\[
B_T = \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \left( \hat{F}_T + \frac{\ln 2}{2} \right) - \frac{1}{4\varepsilon} \sum_{t=2}^{T} \|p_t - p_{t-1}\|_1^2.
\]

Notice that the above terms are essentially the meta-regret of gradient-variation and small-loss bounds, up to constant factors. Therefore, we can make use of their meta-regret analysis to bound the meta-regret of Sword\textsubscript{best}. Specifically, by applying the analysis of Theorem 10, we know that

\[
A_T \leq \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \left( 2V_T + \frac{\ln 2}{2} \right).
\]

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holds if the learning rate satisfies $\varepsilon \leq \sqrt{1/(8D^4L^2)}$. Under such circumstances, the meta-regret can be further bounded by

$$\text{meta-regret} \leq \min \{A_T, B_T\} \leq \frac{2 + \ln N}{\varepsilon} + \varepsilon D^2 \left( \min \{2V_T, \bar{F}_T\} + \ln 2 \right).$$

Therefore, we set the learning rate as $\varepsilon = \min \{\varepsilon_0, \varepsilon^*\}$, where

$$\varepsilon_0 = \sqrt{1/(8D^4L^2)}, \quad \text{and} \quad \varepsilon^* = \sqrt{(2 + \ln N)/(D^2 \min \{2V_T, \bar{F}_T\} + D^2 \ln 2)}.$$

We bound the meta-regret by considering two cases.

- When $\varepsilon^* \leq \varepsilon_0$, the meta-regret is bounded by
  $$(2 + \ln N)/\varepsilon^* + \varepsilon^* D^2 \left( \min \{2V_T, \bar{F}_T\} + \ln 2 \right) = 2D\sqrt{(2 + \ln N)(\min \{2V_T, \bar{F}_T\} + \ln 2)}.$$

- When $\varepsilon^* \geq \varepsilon_0$, the meta-regret is bounded by
  $$\frac{2 + \ln N}{\varepsilon_0} + 2\varepsilon_0 D^2 V_T \leq 4\sqrt{2D^2L(2 + \ln N)},$$

where the last inequality makes use of the condition of $\varepsilon^* \geq \varepsilon_0$. Hence, taking the two cases into account, the meta-regret is bounded by

$$\text{meta-regret} \leq 2D\sqrt{(2 + \ln N)(\min \{2V_T, \bar{F}_T\} + \ln 2) + 4\sqrt{2D^2L(2 + \ln N)}} = O\left(\sqrt{(1 + \ln N + \min \{V_T, \bar{F}_T\}) \ln N}\right),$$

which completes the proof.

**Proof of Theorem 8.** Notice that the dynamic regret can be decomposed into the following two parts

$$\sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u_t) = \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(x_{t,i}) + \sum_{t=1}^T f_t(x_{t,i}) - \sum_{t=1}^T f_t(u_t),$$

meta-regret \hspace{1cm} expert-regret

Since the Sword_{best} algorithm maintains $N_1 + N_2$ experts, where the first $N_1$ experts run OEGD and the other $N_2$ experts perform OGD. Therefore, the expert-regret can be upper bounded by the minimum of the expert-regret of variation and small-loss algorithms. Meanwhile, in Theorem 7, we have proved that the meta-regret of Sword_{best} also achieves a minimum of the meta-regret of variation and small-loss algorithms. Combining the expert-regret and meta-regret analysis, we thus confirm that Sword_{best} attains a best-of-both-worlds dynamic regret bound.
D. Proof of Lower Bound

Proof. The theorem is proved by the probabilistic method.

For iterations $t = 1, \ldots, T$, we randomly sample a convex and smooth function $f_t : \mathbb{R}^d \mapsto \mathbb{R}$ from the distribution $P$. More specifically, we construct the function $f_t$ as follows

$$f_t(x) = \|x - \sigma \varepsilon_t\|^2_2,$$

where $\sigma > 0$ and $\varepsilon_t \in \mathbb{R}^d$ is a random vector with components sampled independently from the Rademacher distribution, that is, $\varepsilon_t(i) = 1$ or $-1$ with equal probability of 50%.

We further set the comparator $u_t = x^*_t \in \text{arg min}_{x \in X} f_t(x) = \sigma \varepsilon_t$. Denote by $x_t$ the decision returned by any deterministic online algorithm $A$. Then the expected dynamic regret is defined as

$$\mathbb{E}[D-\text{Regret}_T] = \mathbb{E}\left[\sum_{t=1}^T f_t(x_t) - f_t(u_t)\right].$$

In the following we show that $\mathbb{E}[D-\text{Regret}_T] \geq \mathbb{E}[P_T(u_1, \ldots, u_T)]$. On one hand,

$$\mathbb{E}[D-\text{Regret}_T] = \mathbb{E}\left[\sum_{t=1}^T f_t(x_t) - f_t(u_t)\right] = \sum_{t=1}^T \mathbb{E}[\|x_t - \sigma \varepsilon_t\|^2_2]$$

$$= \sum_{t=1}^T \mathbb{E}[\|x_t\|^2_2] + 2\sigma \langle x_t, \varepsilon_t \rangle + \sigma^2 \|\varepsilon_t\|^2_2 \geq dT\sigma^2.$$

On the other hand, we have

$$\mathbb{E}[P_T(u_1, \ldots, u_T)] = \sigma \cdot \sum_{t=2}^T \mathbb{E}[\|\varepsilon_t - \varepsilon_{t-1}\|_2] = \sigma \cdot \sum_{t=2}^T \mathbb{E}\left[\sqrt{\sum_{i=1}^d \delta_t^2(i)}\right] \leq 2\sqrt{dT}\sigma.$$

By choosing $\sigma \geq 2/\sqrt{d}$, we can ensure that $\mathbb{E}[D-\text{Regret}_T] \geq \mathbb{E}[P_T(u_1, \ldots, u_T)]$. We note that the choice of $\sigma$ might lead to a violation of the assumption of domain boundedness, which can be easily fixed by the rescaling. So the probabilistic argument implies that for any algorithm $A$ there exists a sequence of online functions $f_1, \ldots, f_T$ such that

$$D-\text{Regret}_T \geq P_T(u_1, \ldots, u_T),$$

which concludes the proof.

E. Proof of Lemma 1

Lemma 1 guarantees the regret bound of OptimisticHedge, which is originally proved by Syrgkanis et al. (in (Syrgkanis et al., 2015, Theorem 19)). For self-containedness, we present its proof and adapt to our notations. Before showing the proof, we need to introduce two related lemmas.

The first one is on the property of strongly convex functions (Nesterov, 2018).
Lemma 2. If $F : \mathcal{X} \mapsto \mathbb{R}$ is a $\lambda$-strongly convex function with respect to a norm $\| \cdot \|$ and $x_* = \arg \min_{x \in \mathcal{X}} F(x)$, then for any $x \in \mathcal{X}$, we have

$$F(x) \geq F(x_*) + \frac{\lambda}{2} \| x - x_* \|^2. \quad (57)$$

Proof. According to the definition of strongly convex function, we have $F(x) \geq F(x_*) + \langle \nabla F(x_*), x - x_* \rangle + \frac{\lambda}{2} \| x - x_* \|^2$. Besides, by the first order condition of convex functions, we have $\langle \nabla F(x_*), x - x_* \rangle \geq 0$. We complete the proof by combining these two inequalities. □

The second lemma is due to Syrgkanis et al. (2015), which exploits the stability of the Follow the Regularized Leader (FTRL) algorithm. The FTRL algorithm updates the decision $x_t$ in the form of

$$x_t = \arg \min_{x \in \mathcal{X}} \varepsilon \langle L_t, x \rangle + R(x),$$

where the regularizer $R : \mathcal{X} \mapsto \mathbb{R}$ is strongly convex.

Lemma 3. If $x_* = \arg \min_{x \in \mathcal{X}} \varepsilon \langle x, L \rangle + R(x)$ and $x'_* = \arg \min_{x \in \mathcal{X}} \varepsilon \langle x, L' \rangle + R(x)$ for a $\lambda$-strongly convex regularizer $R : \mathcal{X} \mapsto \mathbb{R}$ with respect to a norm $\| \cdot \|$ and some $L \in \mathbb{R}^d$ and $L' \in \mathbb{R}^d$. Then we have

$$\lambda \| x_* - x'_* \| \leq \varepsilon \| L - L' \|_* \quad (58)$$

where $\| \cdot \|_*$ is the dual norm of $\| \cdot \|$. Proof. Define $F(x) = \varepsilon \langle x, L \rangle + R(x)$ and $F'(x) = \varepsilon \langle x, L' \rangle + R(x)$. We can see that $F(x)$ and $F'(x)$ are all $\lambda$-strongly convex functions. So according Lemma 2, we have,

$$F(x'_*) \geq F(x_*) + \frac{\lambda}{2} \| x'_* - x_* \|^2 \quad (59)$$

and

$$F'(x_*) \geq F'(x'_*) + \frac{\lambda}{2} \| x'_* - x_* \|^2 \quad (60)$$

Combining (59) and (60), we have

$$\varepsilon \langle x'_* - x_*, L - L' \rangle \geq \lambda \| x'_* - x_* \|^2.$$  

By Cauchy Schwartz inequality, we have

$$\varepsilon \| x'_* - x \| \cdot \| L - L' \|_* \geq \varepsilon \langle x'_* - x_*, L - L' \rangle \geq \lambda \| x'_* - x_* \|^2,$$

which completes the proof by rearranging the term. □

We prove Lemma 1 based on the above two lemmas.

Proof of Lemma 1. First, we can see that the update procedure of OptimisticHedge

$$p_{t+1,i} \propto \exp (-\varepsilon (L_{t,i} + m_{t+1,i})), \quad \forall i \in [N]$$

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is essentially the solution of the optimization problem
\[
p_t = \arg\min_{p \in \Delta_N} \varepsilon \langle L_t, p \rangle + \mathcal{R}(p),
\]
where \(\mathcal{R}(p) = \sum_{i \in [N]} p_i \ln p_i\) is a 1-strongly convex function with respect to \(\| \cdot \|_1\). Thus, to prove Lemma 1, it is sufficient to analyze the property of update procedure (61). Actually, we can prove a more general result that for any comparator \(q \in \Delta_N\), the regret of the decision is bounded as
\[
\sum_{t=1}^{T} \langle \ell_t, p_t - q \rangle \leq \frac{\ln N + \mathcal{R}(q)}{\varepsilon} + \varepsilon D_\infty - \frac{1}{2\varepsilon} \sum_{t=1}^{T} (\| p_t - p'_t \|_1^2 + \| p_t - p'_{t-1} \|_1^2),
\]
where Lemma 1 holds by setting \(p = e_i\), the zero vector except that the \(i\)-th entry equals 1, and \(\mathcal{R}(q) < 0\) for all \(q \in \Delta_N\). The last term can be further bounded as
\[
\frac{1}{2\varepsilon} \sum_{t=1}^{T} (\| p_t - p'_t \|_1^2 + \| p_t - p'_t \|_1^2) \leq \frac{1}{2\varepsilon} \sum_{t=1}^{T} (\| p_t - p'_t \|_1^2 + \| p_{t+1} - p'_t \|_1^2) \geq \frac{1}{2}\sum_{t=1}^{T} \| p_{t+1} - p_t \|_1^2 - 2.
\]
The last inequality follows from the fact \((a + b)^2 \leq 2a^2 + 2b^2\) and the triangle inequality.

We now proceed to prove (62), where the regret of the decision can be decomposed as
\[
\sum_{t=1}^{T} \langle \ell_t, p_t - q \rangle = \sum_{t=1}^{T} \langle \ell_t - m_t, p_t - p'_t \rangle + \sum_{t=1}^{T} \langle m_t, p_t - p'_t \rangle + \sum_{t=1}^{T} \langle \ell_t, p'_t - q \rangle,
\]

where
\[
p_t = \arg\min_{p \in \Delta_N} \varepsilon \langle \ell_t + m_t, p \rangle + \mathcal{R}(p)
\]
and
\[
p'_t = \arg\min_{p \in \Delta_N} \varepsilon \langle \ell_t, p \rangle + \mathcal{R}(p).
\]
According to Lemma 3, since \(\mathcal{R}(\cdot)\) is 1-strongly convex with respect to \(\| \cdot \|_1\)-strongly, we have
\[
\text{term (a)} = \sum_{t=1}^{T} \langle \ell_t - m_t, p_t - p'_t \rangle \leq \sum_{t=1}^{T} \| p_t - p'_t \|_1 \cdot \| \ell_t - m_t \|_\infty \quad \text{(by Jensen’s Inequality)} \leq \varepsilon \sum_{t=1}^{T} \| \ell_t - m_t \|_\infty^2 = \varepsilon D_\infty.
\]
Then we only need to prove the following result,

\[
\sum_{t=1}^{T} \langle m_t, p_t - p'_t \rangle + \sum_{t=1}^{T} \langle \ell_t, p'_t - q \rangle \leq \frac{\ln N + \mathcal{R}(q)}{\varepsilon} - \frac{1}{2\varepsilon} \sum_{t=1}^{T} \left( \|p_t - p'_t\|^2 + \|p_t - p'_{t-1}\|^2 \right). \tag{63}
\]

It turns out that the above inequality can be proved by induction: the base case (when \( T = 0 \)) holds apparently because of \( \mathcal{R}(q) > -\ln N \). Suppose (63) holds at iteration \( T \), we show that this inequality is also satisfied at iteration \( T + 1 \) for all \( q \in \Delta_N \).

Denoting \( A_T = \frac{1}{T} \sum_{t=1}^{T} \|p_t - p'_t\|^2 \), we have

\[
\sum_{t=1}^{T+1} \langle m_t, p_t - p'_t \rangle + \sum_{t=1}^{T+1} \langle \ell_t, p'_t \rangle \\
\leq \langle m_{T+1}, p_{T+1} - p'_{T+1} \rangle + \langle \ell_{T+1}, p'_{T+1} \rangle + \frac{\ln N + \mathcal{R}(p'_{T+1}) - A_T}{\varepsilon} + \sum_{t=1}^{T} \langle \ell_t, p'_t \rangle \\
\leq \langle m_{T+1}, p_{T+1} - p'_{T+1} \rangle + \langle \ell_{T+1}, p'_{T+1} \rangle + \frac{\ln N + \mathcal{R}(p_{T+1}) - A_T - \frac{1}{2} \|p_{T+1} - p'_{T+1}\|^2}{\varepsilon} + \sum_{t=1}^{T} \langle \ell_t, p_{T+1} \rangle \\
= \langle \ell_{T+1} - m_{T+1}, p'_{T+1} \rangle + \frac{\ln N + \mathcal{R}(p'_{T+1}) - A_{T+1}}{\varepsilon} + \sum_{t=1}^{T} \langle \ell_t + m_{T+1}, p'_{T+1} \rangle \\
\leq \langle \ell_{T+1} - m_{T+1}, p'_{T+1} \rangle + \frac{\ln N + \mathcal{R}(q) - A_{T+1}}{\varepsilon} + \sum_{t=1}^{T+1} \langle \ell_t, p'_{T+1} \rangle.
\]

The first inequality holds by the induction assumption and setting \( q = p'_{T+1} \). The second inequality holds by (57) and that \( F_T(p) = \varepsilon \sum_{t=1}^{T} \langle \ell_t, p \rangle + \mathcal{R}(p) \) is \( 1 \)-strongly convex with respect to \( \| \cdot \|_1 \) as well as \( p'_{T+1} = \arg \min_{p \in \Delta_N} \mathcal{F}_T(p) \). The third inequality holds by the same argument as the second one and that \( p_{T+1} = \arg \min_{p \in \Delta_N} \varepsilon \sum_{t=1}^{T} \langle \ell_t + m_{T+1}, p \rangle + \mathcal{R}(p) \). The last inequality holds by the fact that \( p'_{T+1} = \arg \min_{p \in \Delta_N} \varepsilon \sum_{t=1}^{T} \langle \ell_{T+1}, p \rangle + \mathcal{R}(p) \).

\[ \square \]

**F. Technical Lemmas**

In this part, we present several technical lemmas used in the proofs. First, we introduce the self-bounding property of smooth functions (Srebro et al., 2010, Lemma 3.1), which is crucial and frequently used in proving problem-dependent bounds for convex and smooth functions.

**Lemma 4.** For an \( L \)-smooth and nonnegative function \( f : \mathcal{W} \mapsto \mathbb{R}_+ \),

\[ \| \nabla f(w) \|_2 \leq \sqrt{4Lf(w)}, \quad \forall w \in \mathcal{W}. \]
Lemma 5 (Lemma 19 of Shalev-Shwartz (2007)). For any $x, y, a \in \mathbb{R}_+$ that satisfy $x - y \leq \sqrt{ax}$,

$$x - y \leq a + \sqrt{ay}.$$  \hfill (64)

Based on Lemma 5, we have the following result.

Lemma 6. For any $x, y, a, b \in \mathbb{R}_+$ that satisfy $x - y \leq \sqrt{ax} + b$,

$$x - y \leq a + b + \sqrt{ay + ab}.$$  \hfill (65)

The following lemma is shown to be useful in analyzing the gradient descent algorithm.

Lemma 7. Let $\mathcal{X}$ be a convex set in a Banach space $\mathcal{B}$. Then, any update of the form $x^* = \Pi_{\mathcal{X}}[c - \nabla]$ satisfies the following inequality

$$\langle x^* - u, \nabla \rangle \leq \frac{1}{2}\|c - u\|^2 - \frac{1}{2}\|x^* - u\|^2 - \frac{1}{2}\|x^* - c\|^2$$  \hfill (66)

for any $u \in \mathcal{X}$.

Proof. It is equivalent to prove the following inequality

$$\langle u - x^*, (c - \nabla) - x^* \rangle \leq 0.$$  \hfill (67)

We consider two cases by noting that $x^* = \Pi_{\mathcal{X}}[c - \nabla]$:

1. $c - \nabla \in \mathcal{X}$: $\langle u - x^*, (c - \nabla) - x^* \rangle = 0$ clearly satisfies (67);

2. $c - \nabla \notin \mathcal{X}$: the Pythagorean theorem (Hazan, 2016, Theorem 2.1) implies (67).

This ends the proof. \hfill \Box