Boundary behaviour of Hurwitz schemes

by

Torsten Ekedahl.

The purpose of this paper is to get a convenient way of computing the topological types of degenerations of covers of the projective line. This leads immediately to the theory of Hurwitz schemes. Hurwitz schemes are normally thought of as the moduli spaces of maps of curves to $\mathbb{P}^1$ with prescribed degree and local behaviour. We will rather consider the Galois hull of the mapping and thus consider curves together with an action of a finite group with only trivial invariants on 1-cohomology. If the group then is provided with a transitive permutation representation, a curve with a map to $\mathbb{P}^1$ is obtained and the local behaviour is easily described in terms of the (conjugacy classes of) stabiliser subgroups of points. Now, if the curve and the group action varies in a family the family of curves has, after a generically finite base extension, a stable extension to a proper base and the group action automatically extends. If we look at boundary points we will then get an action of our group $G$ on a stable curve. What this note actually does is to provide a description of such actions in terms of homomorphisms from a kind of fundamental group of the quotient curve into $G$, a description which is completely analogous to the well known case of a smooth curve. A direct compactification of the Hurwitz scheme was provided by Harris and Mumford in [Ha-Mu] and it is clear that one could extract a description of the possible topological types from their construction. The present paper differs from such an approach mainly in emphasis. Here we will take the point of view of actions of finite groups on stable curves, which leads, I believe, to an easier way of understanding the topological types of the degenerations. If one tries to make the approach chosen in this paper into a definition of a moduli problem one will also get a problem slightly different from the one of Harris and Mumford, showing that the two possible approaches are not formally equivalent. The moduli point of view will however not be important to us. We will in this paper take the algebraic point of view. Over the complex numbers an analytic approach would, of course, be just as possible. We will from time to time indicate the changes necessary in an analytic setting. The same goes for the change from the viewpoint of stacks, which is that taken here, to orbifolds.

0. Preliminaries

We will need to compare the cohomology of a curve and a stable reduction of it. Normally this would be done using vanishing cycles. However, here we will be only interested in rather crude properties of cohomology (essentially its character under a finite group action) so we will give a, for us, more convenient description.

Remark: The end result is of course well known irrespective of if one uses vanishing cycles or not, only that we need to be careful about identifications as we will have a group acting on the curve and thus on cohomology.

Recall that for a curve $C$ (which to us will be reduced and proper over an algebraically closed field) we have both the sheaf of Kähler 1-forms $\Omega^1_C$ and the dualising sheaf $\omega_C$. We further have a trace map $\Omega^1_C \to \omega_C$. Composing this trace map with the total differential we get a complex of sheaves $\mathcal{O}_C \to \omega_C$, which we will call $\omega_C$. As $C$ is Cohen-Macaulay it is clear that the cohomology of this complex is self-dual just as in the smooth case. Furthermore, the spectral sequence abutting to the hypercohomology of this complex degenerates for obvious reasons showing that $H^0(C, \omega_C) = H^0(C, \mathcal{O}_C)$, $H^1(C, \omega_C)$ has $H^0(C, \omega_C)$ as a subspace with quotient $H^1(\mathcal{O}_C)$ and $H^2(C, \omega_C) = H^1(C, \omega_C)$. We will, for obvious reasons, call this hypercohomology the de-Rham cohomology of $C$.

In particular if $C$ varies in a flat family of curves, then we get a relative complex $\omega_{C/S}$ whose higher direct images to $S$ will be flat sheaves commuting with base change. The descriptions of the hypercohomology given will also be true in a family.
Remark:  i) Note that in general these higher direct images will not have integrable connections, this seems to be the major difference from the smooth case. We will only be using the fact that the character of a finite group is constant in a (connected) family and we do not need an integrable connection for that.

ii) As we are really only interested in the character on cohomology for a smooth curve described in terms of the action of the group on a degeneration we could also have argued that the virtual character on the smooth curve is the virtual character on the cohomology of the structure sheaf plus its complex conjugate and then only used that the cohomology of the structure sheaf behaves well in families. The definition given here of de Rham-cohomology could however be of independent interest. (When all singularities are nodes it is the log-de Rham complex associated to a log-structure of the curve.)

Let us now concentrate on the case which will interest us, namely when $C$ has only ordinary double points. Let us agree to use the term *branch at a node* to denote one of the points above the node in the normalisation of the curve (the standard terminology being to let it denote one of local irreducible components at the node, there is a natural and canonical bijection between these and our branches). We will then compare the cohomology of $\omega$ for $C$ and its normalisation $\pi: C' \to C$. To begin with we have a map $\mathcal{O}_C \to \mathcal{O}_{C'}$ whose cokernel is the sum over the double points of $C$ of copies of the base field. Note, however, that that copy is not canonically isomorphic to the base field but rather isomorphic to the free vector space on the two branches divided by the sum of the basis elements. Thus an automorphism that switches the two branches acts on the copy by multiplication by $-1$. Dualising this we get a map $\omega_{C'} \to \omega_C$. The cokernel may again be identified with the 2-dimensional vector space based on the branches divided by the sum of the base vectors. Taking long exact sequences of cohomology gives a subspace of $H^1(C, \mathcal{O}_C)$ isomorphic to the cokernel of the quotient map $\mathcal{O}_{C'} \to \mathcal{O}_{C'}/\mathcal{O}_C$ whose quotient is isomorphic to $H^1(C', \mathcal{O}_{C'})$. The description of $H^0(C, \omega_C)$ is dual.

Remark: In the case of the cokernel of $\omega_{C'} \to \omega_C$ the map to the 1-dimensional vector space may be seen clearly by identifying the sheaf pullback of $\omega_C$ to $C'$ with 1-forms on $C'$ with at most simple poles at the branches and for which the sum of the residues at the branches of a fibre is zero. This also makes it clear that an automorphism switching the two branches acts by multiplication by $-1$.

We are now almost at the point where we can summarise this dévissage of the cohomology. However, as has already been mentioned, we will be interested only in the action of a finite group and in fact, as far as cohomology is concerned we will only care about the virtual character for the group. It is clear that the dual graph and its cohomology should appear in that dévissage. As may already be apparent from the discussion above some care has to be taken in how to define the dual graph. We will need to adopt the definition of [Se:2.1.1] of a graph.

**Definition 0.1.**  i) A graph consists of a vertex set $V$, a set $E$ of oriented edges, a map

$$V \to E \times E$$

$$v \mapsto (o(v), t(v)),$$

and an involution $\overline{\cdot}: V \to V$ such that $o(\overline{v}) = t(v)$ and $\overline{v} \neq v$ for all vertices $v$. The vertices $o(v)$ and $t(v)$ will be called the source and the sink of $v$ respectively. The equivalence classes under $\overline{\cdot}$ of $V$ will be called the edges of the graph. Automorphisms of a graph are permutations of $E$ and $V$ commuting with $o$, $t$ and $\overline{\cdot}$. If $e$ is an oriented edge $\overline{e}$ will be called its opposite edge.

ii) A generalised graph consists of the same data and conditions as a graph with the exception that $\overline{\cdot}$ is not assumed to be fixpoint free.

**Remark:** Our graphs will mainly be dual graphs of stable curves which will always fulfill the conditions for the definition of a graph. However, when a finite group is acting on a stable curve we will be interested in the quotient of its dual graph by the group and such a quotient will, in general, only be a generalised graph. The reason why we make the distinction is that the discussion of the 1-chains would not be completely true for a generalised graph (an edge equal to its own opposite would give an element of order 2 in the group of 1-chains).
The chain complex of a graph is defined as follows. We let the 0-chains be the ordinary 0-chains but let the 1-chains be the free abelian group on oriented 1-simplices divided by the subgroup generated by the elements which are the sum of an edge and its opposite. Note that this 1-chain group is isomorphic to the free group on the edges of the graph but not canonically so (or maybe canonically but not naturally isomorphic). In particular an automorphism that fixes an edge but permutes the oriented edges of that edge will act by multiplication by \(-1\) on the corresponding element of the 1-chains. Representation-theoretically, if a group acts on the graph we see that the 0-chains are isomorphic to the permutation representation on vertices, whereas the representation on 1-chains is the sum over the orbits of the group on edges of the representations induced from signum characters. By the latter I mean the following. Pick a representing edge of an orbit and consider its stabiliser. Each element of the stabiliser will either fix the two oriented edges or permute them. If we assign \(1\) resp. \(-1\) to the element we get a character of the stabiliser which we will call the signum character. We can then induce this character to a representation of the whole group. We will call an orbit where the signum character is trivial an orientable orbit and in the opposite case we will speak about an unorientable orbit, in the orientable case the representation is of course just the permutation representation. Note now that if the group acts on the curve then that an automorphism fixes an edge means exactly that it fixes the corresponding double point and that it fixes the edge’s two vertices means that it fixes the two branches. Thus we see that the space of global sections of \(\mathcal{O}_C\) is (canonically) isomorphic with the 0-chains (more naturally 0-cochains) with scalars extended to the base field and that the global sections of \(\pi_*\mathcal{O}_{C'}/\mathcal{O}_C\) is (canonically) isomorphic with the 1-chains again with scalars extended to the base field. A moment’s thought will also convince the reader (though we will not need it) that the map induced on global sections from the quotient map equals the coboundary map on cochains. Hence by the dévissage above we see that \(H^1(C,\mathcal{O}_C)\) contains a subspace isomorphic to the first cohomology space of the dual graph (with coefficients in the base field) with quotient space equal to \(H^1(C',\mathcal{O}_{C'})\). There is also a dual description for \(H^0(C,\omega_C)\). We summarise this in the following, where by the virtual character of some cohomology we, naturally, mean the alternating sum of the character of the individual cohomology groups.

**Proposition 0.2.** Let \(C\) be a curve with only ordinary double points and \(C'\) its normalisation. Assume \(C\) is acted upon by a finite group \(G\).

i) The virtual character of the cohomology on the de-Rham cohomology of \(C\) is equal to the sum of that of the de-Rham cohomology of \(C'\) and 2 times the virtual character of the cohomology of the dual graph.

ii) The virtual character of the cohomology of the dual graph is equal to the difference of the permutation character of the vertices minus the sum over the orbits of \(G\) on the edges of the representations induced from the signum representation.

**Proof:** The two statements follow directly from the previous discussion.

When we later want to discuss the description of a stable curve with an action by a group \(G\) we will need to describe this fundamental group in terms of graphs of groups (cf. [Se:4.4.8]). However, it turns out that the notion we need is a slight extension of this notion to generalised graphs.

**Definition 0.3.** i) A (generalised) graph of groups consists of a generalised graph and the association of a group \(G_e\) to each vertex of the graph, a group \(H_e\) to each edge of the graph, a group homomorphism \(a \mapsto a^y; H_y \to G_{t(y)}\) for each edge \(y\) and an isomorphism \(\bar{\cdot}; G_y \to G_{\bar{y}}\) whose square is the identity.

ii) The fundamental group of a generalised graph of groups is the same as that of [Se:p. 42] with the modification that (using the notation of (loc. cit.)) \(ya^y\bar{y} = \bar{a}^y\).

**Remark:** When the generalised graph is a graph one can use the isomorphisms \(G_y \to G_{\bar{y}}\) to identify \(G_y\) and \(G_{\bar{y}}\) and then arrive at the notion defined by Serre (except that Serre assumes that \(a \mapsto a^y\) is injective, on the one hand that condition will not be necessary for our purposes, on the other hand it will always be fulfilled in our applications).

The more precise way in which graphs of groups come up in our calculations is through the use of the van Kampen theorem. As we will have occasion to apply this theorem to the case where the intersection is not connected, we will quickly recall the results in that case as it may be less known.
than the case of connected intersection. We are also forced to formulate the result for algebraic stacks (orbifolds) as we are going to apply it in that generality.

**Definition-Lemma 0.4.** Let $U_1$ and $U_2$ be an open covering of a connected algebraic stack $X$. Assume that an étale cover of $U_1$, one of $U_2$ together with an isomorphism of their respective restrictions to the fibre product of $U_1$ and $U_2$ over $X$ comes from a unique cover $X$. Then the fundamental group of $X$ (wrt some base point) is isomorphic to the fundamental group of the following graph of groups:

Take one vertex for each connected component of $U_1$ and one for each connected component of $U_2$. Attach to these vertices the fundamental group of the corresponding component. For each connected component $V$ of $U_1 \cap U_2$ and components $U'_i$ of $U_i$, $i = 1, 2$ which both contain $V$ one defines a pair of oriented arrows from $U'_1$ to $U'_2$ and vice versa. Attach to these edges the fundamental group of $V$ and the morphism from that fundamental group to the fundamental group of $U'_i$ induced by inclusion.

**Remark:** The condition would seem to be always verified. However, note that, by necessity an open covering means an open étale cover (as we are dealing with stacks). Hence one would in general also need descent conditions on the “intersection” (i.e. fibre product) of the $U_i$ with themselves.

**PROOF:** The proof becomes more understandable if one passes to fundamental groupoids instead of fundamental groups (which also explains why base points do not play any role in the formulation of the result). To do that we define the notion of map from a graph of groups to a groupoid (in analogy with the definition of [Se:1.1.1]) as a map taking vertices to objects, a homomorphism from the group at a vertex to the automorphism group of the corresponding object and a map from edges to morphisms the source object to the sink object, fulfilling the obvious relations. Two maps which are related by an equivalence of groupoids are then appropriately identified so as to give rise to the notion of 2-limit of a graph of groups. It is then an easy exercise to show that the fundamental group of a graph of groups is a 2-limit in the category of groupoids. The point of considering groupoids instead of groups is that the proper formulation of the van Kampen theorem when the intersection is non-connected is that the fundamental groupoid of the union is the 2-limit of the fundamental groupoid of the parts with gluing of the intersection (this follows for instance directly from the description of the fundamental groupoid as constructed from the category of étale coverings). Gathering together these strands gives the result.

1. Group quotients of stable curves.

As our first step we will study the quotient of a stable curve by a finite group of automorphisms and then discuss what topological data are needed to recover the original curve from its quotient. We begin by recalling the following observation from [Ka-Ma] which shows that we do not have to worry about whether we want to take a quotient of a family or a member of that family.

**Lemma 1.1.** Let $X \to S$ be a map and $G$ a finite group acting on $X$ over $S$. If the order of $G$ is invertible in $O_S$ taking the quotient by $G$ commutes with base change.

**PROOF:** See [Ka-Ma:A7.1.3.4].

**Remark:** We will only be interested in the case when the order of the group is indeed invertible so that from now on the order of a group acting on a curve will, unless otherwise mentioned, be supposed to be invertible in the base.

This means that in order to study the properties of a quotient of a stable family of curves we should often be able to reduce to the base being a field. Note that if we have a curve $C$ with a finite group $G$ acting on it and $x$ is an ordinary double point on $C$ then the stabiliser has the following structure. First there is a subgroup, $G'_x$, of index at most 2 of the elements which preserve the two branches. That subgroup acts on the tangent spaces of the two points of the normalisation lying over $x$, giving rise to two characters of that group. We will call these two characters the characters of the branches.

**Definition 1.2.** Let $C$ be a curve, $G$ a finite group acting on it and $x$ an ordinary double point of $C$. The action of $G$ is said to be admissible at $x$ if the two character of the branches are inverses to
each other and the square of any element of the stabiliser \( G_x \) of \( x \) exchanging the two branches at \( x \) acts trivially in a neighbourhood of \( x \).

**Remark:** We will see later that the action is admissible at a point iff that point is equivariantly smoothable.

It will be convenient to have the following description of admissibility which is almost purely group theoretical.

**Definition-Lemma 1.3.** Let \( C \) be a curve, \( G \) a finite group acting on it and \( x \) an ordinary double point of \( C \). Let \( H_x \) be the stabiliser of \( x \) modulo those elements which act trivially on a neighbourhood of \( x \). Then the action of \( G \) is admissible at \( x \) iff either

1) \( H_x \) is dihedral of order greater than 4

or

\( H_x \) is dihedral of order 4 or 2 with some element exchanging branches and the characters of the branches being equal.

or

2) \( H_x \) is cyclic and the two characters of the branches are each other’s inverses.

In these two cases we will say that \( x \) is a dihedral resp. cyclic node (for the given action of \( G \)).

**Remark:** The formulation of the condition in i) is of course complicated by the fact that the “dihedral” groups of order 2 and 4 differ from the other dihedral groups in that they are not really identifiable as such.

**Proof:** If there are no elements of \( H_x \) exchanging the branches then the two characters give a faithful representation of the whole group. If they are inverses to each other then that means that the group is cyclic. This leads to case ii) where the converse is obvious. If there is an \( \sigma \) element of \( H_x \) exchanging the branches then such an element has order 2 (if the point is admissible) and the subgroup \( H'_x \) of index 2 of \( H_x \) is cyclic as in the first part. Furthermore, \( \sigma \) exchanges the branches and thus exchanges their characters. As they are faithful and inverses to each other conjugation by \( \sigma \) has to invert any element of \( H'_x \) and thus \( H_x \) is dihedral. Conversely, where we may assume that the order of \( H_x \) is greater than 4, if \( H_x \) is dihedral then there must be an element exchanging the branches because the subgroup of \( H_x \) fixing the branches is abelian. For the same reason this element must be outside the cyclic subgroup of index 2. As conjugation by any such element acts by multiplication by \(-1\) on the cyclic subgroup, the two characters of the branches are inverses to each other.

The first structure result on quotients that we will obtain is a description of the singularities of it; as we will see all singularities will again be nodes. Note that as we have already concluded that quotients commute with base change the conclusion is that if we have a family of curves all of whose singularities are nodes then the same is true of a quotient by a finite group acting admissibly.

**Proposition 1.4.** Let \( C \) be a curve all of whose singularities are nodes and \( G \) a finite group of automorphisms of \( C \) such that the action is admissible at all nodes. Then \( C/G \) has only nodes as singularities and the singularities are the points below the cyclic nodes of \( C \).

**Proof:** The quotient of a smooth point is normal and thus again smooth. At a node we may localise and complete and then, as the order of \( G \) is invertible in the base field, we may choose a \( G_x \)-invariant complement to the square of the maximal ideal in the maximal ideal; \( m = V \oplus m^2 \).

When \( x \) is cyclic \( V \) is the direct sum of the two characters of the branches and when \( V \) is dihedral it is irreducible. In any case we may choose an isomorphism of the completed local ring with \( k[[x, y]]/(xy) \) in a way such that up to multiplication by scalars \( x \) and \( y \) are either fixed or permuted by elements of \( G_x \). In the cyclic case they are always fixed and if the order of the characters of the branches is \( n \) then the fixed ring is \( k[[x^n, y^n]]/(x^n y^n) \) which shows that the quotient has a nodal singularity. In the dihedral case we may first take invariants under the subgroup of index 2. As we have seen that fixed ring has a nodal singularity so that we are reduced to the case when the \( G_x \) has order two and the involution permutes \( x \) and \( y \). Then the ring of invariants is simply \( k[[x + y]] \) which is a regular ring.

If we now want to emulate the description of actions of groups on smooth curves in terms of the quotient and topological data on it to reconstruct the ramified cover, we are faced with the problem
that quotients of dihedral points are smooth yet we evidently need to treat them differently from those lying below smooth points. What is happening here is that the quotient should really be thought of as an algebraic stack with the points below dihedral points being non-scheme points. Furthermore, at a singular point of the base, a quotient map will in general not be étale nor will any map which is étale in a punctured neighbourhood be accepted. Again the necessary restrictions on the local behaviour can be formulated in terms of gluing in a non-scheme point at a singular point and as we have already decided to take the plunge into algebraic stacks it seems reasonable to adopt that approach for those points as well. Finally, for reasons of symmetry it seems reasonable to treat ordinary ramification points in the same manner though that is clearly not traditional, nor is it necessary.

Remark:  i) As an apology to readers who feel ill at ease with algebraic stacks I would offer that I started with the ambition to simplify their lives by avoiding the use of stacks but eventually found that approach too cumbersome to feel able to uphold that ambition.

   ii) Depending on one’s choice of education one may, of course, be used to talk of orbifolds rather than algebraic stacks. For readers with this tendency I would like to point out a few differences.

   a) Usually orbifolds are assumed to be smooth, that is, modeled on actions by groups on smooth manifolds. We will have occasion to consider those which are not (and thus maybe should be called “orbispaces” instead).

   b) Usually, orbifolds are modeled on quotients by finite groups, here we will need to use infinite groups.

   c) Here we are not only considering things topologically but also algebraically-analytically, in fact we will really only do things algebraically even though the changes to make it work in the complex analytic category are trivial.

To prepare for the definition of the appropriate algebraic stacks we will define how to glue in the desired special points. As we are only considering group actions where the order of the group is invertible in the base field we will in the same vein only consider tame coverings. Let $C$ be a curve and $c \in C$ a smooth point on $C$. Let $R$ be the (strict) Henselisation of the local ring of $C$ at $c$. Let $\tilde{R}$ be the extension of $R$ obtained by adjoining all the roots, of order prime to the characteristic of the base field, of a uniformiser of $R$ (this of course is the maximal tamely ramified extension of $R$).

The disjoint union, $D$, of $\text{Spec} R$ and $C \setminus c$ maps to $C$. If we take the fibre product over $C$ of this disjoint union then outside of $c$ it is an étale equivalence relation. We will modify that fibre product at $c$ so that it together with $D$ forms an étale groupoid. In fact, $D$ has a component $\tilde{P}$ which is the fibre product of $\text{Spec} \tilde{R}$ with itself over $\mathcal{O}_{C,c}$. This in turn is a scheme over the fibre product of $\text{Spec} R$ with itself over $\mathcal{O}_{C,c}$. This latter scheme is the disjoint union of schemes some of which maps isomorphically to $\text{Spec} R$ through any of the projections (and which are parametrised by the Galois group of the residue field of $\mathcal{O}_{C,c}$) and spectra of fields. As these latter maps into $C \setminus c$ we may ignore them wanting only to modify the fibre product at $c$. For each of the components of the former type, the inverse image of $\tilde{P}$ over it is isomorphic to the fibre product $Q$ with itself of $\text{Spec} \tilde{R}$ over $\text{Spec} R$ and so we may consider instead that case and then simply transfer the modifications to $\tilde{P}$. Now, we have an action of the Galois group $\mathcal{G}$ of the fraction field of $\tilde{R}$ over $R$ (which is isomorphic to $\hat{\mathbb{Z}}$ with the component $\mathbb{Z}_p$ removed when the base field has positive characteristic) and hence we have a map $\mathcal{G} \times \text{Spec} \tilde{R} \to Q$. This is an isomorphism over the fraction field of $R$ and our modification consists exactly of replacing $Q$ by $\mathcal{G} \times \text{Spec} \tilde{R}$. That we in this fashion do indeed get an étale groupoid is easily checked. This defines an algebraic stack $C_{\text{cyc}}^\text{et}$. It has an open subset isomorphic to $C \setminus c$ and the complement consists of a closed point. We will call this process gluing in a cyclic point at $c$. It can evidently be repeated so that we can glue in cyclic points at a finite set of points.

Remark: Intuitively, we have taken a small neighbourhood of $c$, taken the maximal tamely ramified cover of the punctured neighbourhood, added a point to that cover lying over $c$ and then glued in the stack quotient by the Galois group of the cover, where the ordinary quotient is just the original neighbourhood. That intuitive picture was somewhat hidden both by not making the assumption that the residue field of $c$ should be algebraically closed and by the fact that an étale neighbourhood is not a subset. If one works in an analytic setting instead then the truth is indeed very close to this intuition. Indeed, adding a point to the universal cover of a punctured neighbourhood of $c$ is exactly the procedure of adding a cusp at $i\infty$ to the upper half plane. Thus taking the stack quotient of the
upper half plane with cusps added at rational points of $\mathbb{P}^1(\mathbb{R})$ by $PSL(2, \mathbb{Z})$ is the same thing as compactifying the moduli stack of elliptic curves and then gluing in a cyclic point at $\infty$.

We have also set up things so that a covering $D \to C$ which is tamely ramified over $c$ extends uniquely to a cover of $C_{c^{\text{cycl}}}^{\text{et}}$ which is étale over $c$ and, conversely, a cover of $C_{c^{\text{cycl}}}^{\text{et}}$ étale over $c$ gives a covering $D \to C$ which is tamely ramified over $c$.

We will now slightly modify this construction so as to make it apply in the other two situations mentioned above. First we consider again a curve $C$ with a smooth point $c$. We may again also consider $R$, the strict henselisation of the local ring at $c$, and $\hat{R}$, the maximal tamely ramified extension of $R$. We now consider the direct product of two copies of $\hat{R}$ as a ring and the subring of that product consisting of the pairs $(r, s)$ for which $r$ and $s$ are congruent modulo the maximal ideal of $\hat{R}$. This new ring $\hat{R}$ has an action of the Galois group of (the fraction field of) $\hat{R}$ over (the fraction field of) $R$, which acts the same way on both factors of the direct product. We also have the automorphism of $\hat{R}$ which exchanges the two factors. Together with the automorphisms coming from the Galois group it generates a dihedral group. We now continue the construction exactly as in the previous case using $\hat{R}$ instead of $\hat{R}$. We will call this process \textit{gluing in a dihedral point at $c$}. It can evidently be repeated so that we can glue in dihedral points at a finite set of points (as well as some dihedral and some cyclic points).

Finally, we will describe how to glue in a cyclic point at an ordinary node. Let us first recall some facts about algebraic fundamental groups and “paths”. Thus if $R$ is a discrete valuation ring the tame fundamental group of its fraction field is canonically identified with $\mathbb{Z}(-1)[1/p]$, the dual of the Tate module of roots of unity in an algebraic closure of the fraction field; $p$ being the characteristic exponent of the fraction field. In particular this group only depends on the choice of an algebraic closure of the residue field of $R$. This means that if we have an ordinary double point and if we consider the (strict) Henselisation of of its local then ring, then the tame fundamental group of its two fraction fields are canonically identified.

\begin{remark}
We haven’t mentioned complex analytic analogues of our constructions but they are clearly possible. In the analytic case a choice of orientation of $C$ identifies the local fundamental group with $\mathbb{Z}$ and thus the fundamental groups of the two components of a small punctured neighbourhood of an ordinary double point are also canonically identified and the gluing in of a cyclic point at an ordinary node can be done.

We start now with a point $c$ which is supposed to be an ordinary double point. Again we consider $R$, the henselisation of the local ring. Its total ring of fraction is the product of two fields. For each of those we take the maximal tamely ramified extension. The normalisation of $R$ in the product of these two fields is itself the product of two copies of the same local ring. We then let $\hat{R}$ be the subring of that ring of pairs $(r, s)$ for which $r$ is congruent to $s$ modulo the maximal ideal. Again the Galois group of the maximal tamely ramified extension acts on $\hat{R}$, where we let the action by given by the natural action on one of the copies and the inverse of the natural action on the other copy. We then continue as before.
\end{remark}

\begin{remark}
As was mentioned above these constructions are introduced to describe certain coverings. If we repeat this motivation together with some more concrete facts we get the following.

i) For a smooth cyclic point any covering of the curve extends uniquely to a covering of the stack which is \textit{étale} at the cyclic point.

ii) For a cyclic ordinary double point any covering of the curve outside of the point extends uniquely to a covering of the stack which is \textit{étale} at the cyclic point iff the ramification indices at the two branches are the same. Hence for instance $s^2 = x, t^3 = y$ does not so extend over $xy = 0$.

iii) For a dihedral point the situation is different. Just as in case ii) a covering may not extend to an \textit{étale} covering of the stack. The main difference is that the extension, if it exists, is not necessarily unique. As we will see the following happens. If the covering is given by a permutation representation an extension to an \textit{étale} covering of the stack will involve an extension of the cyclic ramification groups at the point to dihedral groups.

\begin{definition}
i) A pointed curve is a curve with only nodes as singularities together with two disjoint finite sets of smooth points. The points belonging to these two sets will be called cyclic resp. dihedral points.
\end{definition}
ii) Let \((C, S_c, S_d)\) be a pointed curve, where \(S_c\) is the set of cyclic and \(S_d\) the set of dihedral points. The pointed stack associated it is obtained from \(C\) by gluing in smooth cyclic points at \(S_c\), dihedral points at \(S_d\) and cyclic points at the nodes of \(C\). (Note that the pointed curve can be recovered from the pointed stack as the algebraic space associated to the stack, with the sets \(S_c\) and \(S_d\) as the appropriate non-spatial points on the stack.)

\[\text{iii)} \text{ If } (C, S_c, S_d) \text{ is a pointed curve then the dual (generalised) graph of it is defined as follows. The vertices are the irreducible components of } C, \text{ for each branch of } C \text{ at a node one defines an oriented edge ending at the component on which the branch lies and starting at the component on which the other branch at the node lies (these components may of course be the same). The edge inverse to this is the one constructed from the other branch at the node. For each point of } S_d \text{ one defines an edge which equals its own opposite starting and ending at the component on which the point lies.}\

\[\text{iv)} \text{ If } (C, S_c, S_d) \text{ is a pointed curve then the dual (generalised) graph of groups is the graph of groups whose underlying graph is the dual graph of the pointed curve. The group associated to a vertex is the fundamental group of the corresponding irreducible component of } C \text{ with the union of its cyclic, dihedral and singular points removed. The group associated to an edge corresponding to a branch at a node is the tame local fundamental group at the branch on the normalisation of } C. \text{ The homomorphism into the fundamental group at the component on which the node lies is induced by the inclusion and the isomorphism with the tame local fundamental at the opposite branch is given by the inverse of the canonical identification. The group associated to an edge corresponding to a dihedral point is the tame local fundamental group, the homomorphism into the fundamental group of the component on which it lies is induced by the natural inclusion and the involution of the edge group given by the fact that the edge equals its opposite is multiplication by } -1.\]

Remark: To make the definition in iv) completely unambiguous one would have to choose paths and basepoints appropriately. The result, however, is independent up to isomorphism of these choices.

If we want to compute the fundamental group of one of our pointed stacks it will of course be necessary to know when it is connected. The answer contains no surprises.

Lemma 1.6. Let \((C, S_c, S_d)\) be a pointed curve for which \(C\) is connected. Then the associated pointed stack is connected.

Proof: Let us recall that to every algebraic stack we may construct its associated algebraic space which is universal for maps of the stack into algebraic spaces. Practically by construction the algebraic space associated to the pointed stack associated to \((C, S_c, S_d)\) is \(C\) itself. As the associated algebraic space functor commutes with disjoint unions the lemma follows.

We are now ready to compute the fundamental group of a (connected) pointed stack associated to a pointed curve. As a point of terminology let us agree to call, for a given abelian group \(A\), the semidirect product of \(A\) by a group of order 2, where the non-trivial element acts by multiplication by \(-1\), the dihedral group based on \(A\). We are also interested in the geometric fundamental group only so we assume that the base field is algebraically closed.

Theorem 1.7. Let \((C, S_c, S_d)\) be a pointed connected curve. Then the fundamental group of the associated pointed stack \(C''\) is isomorphic to the fundamental group of its dual graph.

Remark: i) I leave to the reader to write down a presentation for this group.

ii) More important than a presentation is the resulting description of étale covers with structure a finite group \(G\); a homomorphism from the fundamental group of each component of \(C'\) to \(G\) such that the composed map from the local fundamental group of the two branches of a singular point of \(C\) are inverses to each other, together with a choice, for each dihedral point, of an element of order 1 or 2 of \(G\) which normalises the image of the local fundamental group and acts by inversion on each element of that image.

iii) It is clear from the description that when \(S_d\) is non-empty then the map on fundamental groups induced from the inclusion of \(C'\) in \(C''\) is not surjective. This is not altogether surprising as \(C''\) is not normal.

iv) No mention of a basepoint was made in the statement of the theorem, nor of paths from it to points close to the special points which are needed to identify the local fundamental groups with
specific subgroups of the global one. The reason for this is that up to isomorphism the constructions are independent of such choices.

v) No explicit mention of which category in which the groups were to be placed was made. The reason for this is that while we explicitly work in the algebraic category and hence the groups are implicitly forced to be profinite, we could have worked in the analytic category and then the theorem, interpreted in the category of ordinary groups, would still be true.

**Proof:** Let us consider the covering of $C''$, given by $C'$ and the disjoint union of the henselisations of all the points of $S_c$, $S_d$ as well as the singularities of $C$ (with cyclic or dihedral points glued in). Let us assume that we may indeed apply van Kampen’s theorem to this covering. Then by (0.4) we get a graph $vK$ of groups whose fundamental group is the searched for fundamental group. It then remains to show that that graph of groups has the same fundamental group as the dual graph $D$ of groups of the pointed curve. This can be done by considering each special point separately. A smooth cyclic point makes $vK$ differ from $D$ by one vertex and one edge from that vertex to the vertex representing the component on which the point lies. The mapping from the edge group to the extra vertex group is an isomorphism so that the vertex and edge may be removed from the graph of groups without affecting the fundamental group. A dihedral point $d$ is responsible for one vertex and one pair of edges between that vertex and the vertex representing the component on which the point lies. The group at the extra vertex is the dihedral group based on the tame fundamental group at the dihedral point and the edge group is the tame fundamental itself. Hence this vertex and the two edges represent (cf. [Sep. 49]) an amalgamation of the dihedral group and the fundamental group of the graph of groups with the vertex and edges removed along the tame local fundamental group. The graph of groups $D$ differs from $vK$ by replacing the vertex and edges associated to $d$ by a single edge which is its own composite. Such an edge also represents the same amalgamation. Finally a node gives rise to two vertices and four edges on $vK$ and one edge on $D$. The verification that we again get the same contribution to the fundamental groups is left to the reader.

We are left with the verification of the conditions of (0.4). This amounts to proving the following statement.

Suppose $D$ is a (not necessarily proper or connected) curve and $c$ is either a smooth point or a node on it and we glue in a cyclic or dihedral point to get $D'$. Let $U_1$ be the complement of $d$ (in $D'$) and let $U_2$ be the the pullback of the stack to the Henselisation of the local ring at $d$ in $D$. Then if we have one étale cover over $U_1$, one étale cover over $U_2$ and an isomorphism of their restrictions to the fibre product of $U_1$ and $U_2$ this set of data comes from a unique étale cover of $D'$.

If we take the disjoint union, $U$, of $U_1$ and $U_2$ then general descent theory tells us that an étale cover of this union together with descent data for the morphism to $D'$ comes from a unique cover of $D'$. To understand these descent data we need to understand the fibre product of $U$ with itself over $D'$. It consists of 3 pieces; the fibre product of $U_1$ with itself, the fibre product of $U_2$ with itself and the fibre product of $U_1$ and $U_2$. By assumption we have descent data on the fibre product of $U_1$ and $U_2$ and what remains to be shown is that there is a unique way of extending these descent data to the other 2 pieces. The fibre product of $U_1$ with itself is equal to $U_1$ as $U_1$ is an ordinary open subset. Hence that piece takes care of itself. To understand the fibre product of $U_2$ with itself we first consider the fibre product of the Henselisation $\hat{R}$ of the local ring at $d$ on $D$ over the local ring $R$ at $d$. As $\hat{R}$ is Henselian this fibre product is a disjoint union of open subsets of $\text{Spec} \hat{R}$. The components isomorphic to $\text{Spec} \hat{R}$ correspond to the spectrum of the tensor product of two copies of the residue field of $\hat{R}$ over the residue field of $R$, but as the latter residue field is algebraically closed this spectrum consists of one point. Hence in the product of $\text{Spec} \hat{R}$ with itself over $\text{Spec} R$ the complement of the diagonal is a disjoint union of copies of the spectrum of the fraction field of $\hat{R}$. Thus the mapping of this union into $D'$ factors over the fibre product of $U_1$ and $U_2$ and thus has descent data coming from the given one, whereas the diagonal has the obvious descent data. This proves the result.

**Remark:** The verification of the descent condition in the corresponding analytic situation is completely trivial, there we can let $U_2$ be a small disc with a cyclic or dihedral point glued in and then the fibre product of it with itself is equivalent with $U_2$ itself. The argument given in the algebraic case simply confirms that for many practical purposes the étale topology behaves as the
2. Deformation theory.

We start by giving some general facts on equivariant deformations of curves. Here the assumption that the order of the group is invertible in the base field will play an essential role.

**Proposition 2.1.** Let $G$ be a finite group acting on a stable curve $C$ with only ordinary double points as singularities and assume that the order of $G$ is invertible in the base field $k$. Then the deformation problem of equivariant deformations is formally smooth at $(C,G)$ with tangent space the $G$-invariants on $\text{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C)$. Furthermore, the forgetful map to the deformation problem of non-equivariant deformations is unramified.

**Proof:** We start by some general observations on equivariant deformations of curves. (For the ordinary deformation theory of stable curves we refer to [De-Mu:31].) If we consider a curve $C$ with only ordinary double points as singularities then deformations are unobstructed and the liftings of deformations over a small extension with base field $k$ is a torsor over $\text{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \otimes V$. All this is natural for automorphisms of $C$ and thus the set of liftings of deformations over a small extension as above is a $G$-torsor over the $G$-module $\text{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \otimes V$. As multiplication on $\text{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \otimes V$ by the order of $G$ is bijective all $G$-torsors are trivial and liftings always exist over small extensions and thus over all nilimmersions. This shows that equivariant deformations are unobstructed and the moduli stack is smooth. Furthermore, as the tangent space at a point is a subspace of that for non-equivariant deformations the forgetful map is unramified.

As our goal is to determine the possible equivariant (stable) degenerations of a smooth equivariant family of curves it makes sense to try to determine when an equivariant stable curve has an equivariant smoothing. This is what we will do now. If we work in local coordinates so that an ordinary double point has ring of functions $k[x, y]/(xy)$ then a miniversal deformation is given by $k[x, y][t]/(xy - t)$. We notice that we may choose $x$ and $y$ so that an element of stabiliser either have them as eigenvectors or permute them up to multiplication by a scalar. This makes the following result plausible.

**Proposition 2.2.** Let $G$ be a finite group acting on a stable curve $C$ with only ordinary double points as singularities and assume that the order of $G$ is invertible in the base field $k$. There exists a deformation of the pair $(C,G)$ smoothing a singularity of $C$ iff the action of $G$ is admissible at the point. Consequently, all singularities are equivariantly smoothable iff they are all admissible.

**Proof:** We first note that $G$ will permute the singularities and that for each orbit of them under $G$ that orbit is in bijection with the cosets of $G$ with respect to the stabiliser of a point in the orbit. We then recall the situation in the non-equivariant case. The tangent space to deformations of the local singularity is equal to the local $\text{Ext}$-group, $\text{Ext}^1_{\mathcal{O}_{C,x}}(\Omega^1_{C,x}, \mathcal{O}_C)$ and the map which takes a global deformation to the local one is just the map from the global $\text{Ext}^1$ to the local one. As we have seen, the tangent space to the global equivariant problem is just the invariants under $G$ and the same is true about the local equivariant problem (the only difference in proof from the global case is that there are infinitesimal automorphisms but as the group of them has multiplication by the order of $G$ invertible no problem is caused). More precisely, the sum of the local $\text{Ext}^1$’s in an orbit under $G$ is stable under $G$ and forms the induced representation of the action of a stabiliser of one of the points of the orbit on that local $\text{Ext}^1$. The same argument shows that equivariant deformations in the local case are unobstructed as well. As the order of $G$ is invertible in $k$, the restriction map from global to the sum of local $\text{Ext}$-groups is surjective also on $G$-invariants. The invariants on the sum of $\text{Ext}^1$’s over one orbit is the invariants of the stabiliser of one point on that local $\text{Ext}^1$ and those invariants project onto the invariants of the stabiliser of any point in that orbit on the local $\text{Ext}^1$. This implies immediately that the singularity $x$ is smoothable iff $G_x$ acts trivially on the 1-dimensional space $\text{Ext}^1_{\mathcal{O}_{C,x}}(\Omega^1_{C,x}, \mathcal{O}_C)$. We should thus aim to prove that a singularity is admissible iff $G_x$ acts trivially on the local $\text{Ext}^1$-group. The consequence in the proposition then follows immediately. Now, a miniversal deformation of an ordinary double point is given by $\text{Specf } k[[x, y, t]]/(xy - t) \rightarrow \text{Specf } k[[t]]$. This shows that the tangent space to the local deformation problem may be described invariantly as the tensor product of the tangent spaces of the two points of the desingularisation lying over $x$. This
means that an automorphism of the singularity which fixes the branches will act on the tangent space by multiplication by the product of the character of the two branches and hence for all those elements to act trivially the two characters have to be inverses of each other. On the other hand, let us consider an automorphism \( \sigma \) exchanging the two branches, choose a non-zero tangent vector of the tangent space at one of the points over \( x \) as basis for its tangent space and let its transform under \( \sigma \) be the basis element for the tangent space of the other point. Having done this we see that \( \sigma \) will act on the tangent space of the deformation by multiplication by the scalar by which the square of \( \sigma \) acts on any of the tangent spaces of the points lying above \( x \). This means that that square acts trivially on the local ring at \( x \) and hence in a neighbourhood of \( x \).

**Example:** Let us consider the action of the icosahedral group, which is isomorphic to \( A_5 \), on \( \mathbb{P}^1 \). There is one orbit of length 12 whose stabiliser is the Sylow 5-group of order 5. As the normaliser of that subgroup is of order 10 there is an invariant equivalence relation on the 12 points of the orbit such that each class contains 2 elements. Identifying each such pair gives us a curve, of (arithmetic) genus 6, with nodes on which the icosahedral group still acts. Each stabiliser of a node is a normaliser of a Sylow 5-group and is dihedral so that the action of the group is admissible. Thus there exists a smooth genus 5 curve with an action of \( A_5 \). Using later results of this article we will be able to see that by taking the quotient of such a curve by \( A_5 \) we get \( \mathbb{P}^1 \) and the quotient map is ramified at 4 points on \( \mathbb{P}^1 \). It is also easy to figure out the ramification behaviour for this map. In this way we get an algebraic proof for the existence of a particular ramified covering of \( \mathbb{P}^1 \) (whose existence, of course is ensured by Riemann’s existence theorem).

**Remark:** It should be emphasized that this is not a viable strategy for giving an algebraic proof of Riemann’s existence theorem in general; the irreducible components of a degeneration will in general be more complicated than just a rational curve.

We will also need to know what happens with the action of a stabiliser under smoothing. The result is given in the following lemma.

**Lemma 2.3.** Let the finite group \( G \) act on a curve \( C \). Let \( c \in C \) be an admissible singularity, assume that the stabiliser of \( c \) acts faithfully on \( c \) and let \( C \to B \) be an equivariant deformation of \( C \) with \( b \in B \) corresponding to \( C \). We will say that a geometric fibre \( C' \) of \( C \to B \) over a generalisation \( b' \in B \) of \( b \) for which \( c \) is not the specialisation of a singularity of \( C' \) is a smoothing of \( C \). A point on \( C' \) of a having \( c \) as a specialisation will be referred to as a point specialising to \( c \).

i) If \( c \) is a cyclic node then there will be no point with a non-trivial stabiliser specialising to \( c \).

ii) If \( c \) is a dihedral node then there will be 4 or 2 orbits of points on a smoothing specialising to \( c \) whose stabilisers are dihedral involutions, depending on whether the order of the stabiliser is divisible by 4 or not. No other points with non-trivial stabilisers specialise to \( c \).

**Proof:** We first want to show that the statement is local around \( c \). Indeed, for i) this is obvious and for ii) one needs to prove that if two neighbouring fixpoints are conjugate they are conjugate under the action of the stabiliser of \( c \). This however is clear as an element taking one fixpoint to another has to take the irreducible component of the fixpoint locus at the first point to the irreducible component of the fixpoint locus at the other point. As both of these components pass through \( c \) the element has to be in the stabiliser. Thus we may localise around \( c \) and we may also pass to a miniversal deformation. Hence we are reduced to looking at an admissible action on \( k[[x, y, t]]/(xy - t) \) over \( k[[t]] \) and we may also assume that an element of \( G \) either acts by multiplying \( x \) by a scalar and by multiplying \( y \) by the inverse of that scalar or permutes acts by \( x \mapsto \lambda y \) and \( y \mapsto \lambda^{-1} x \) for some non-zero \( \lambda \). If \( g \in G \) acts on \( x \) by multiplication by \( \lambda \neq 1 \), then the fixpoint locus of \( g \) is defined by the ideal generated by \( gx - x = (\lambda - 1)x \) and \( gy - y = (\lambda^{-1} - 1)y \). As \( \lambda \) is different from 1 this equals the ideal generated by \( x \) and \( y \) and thus the fixpoint locus equals \( c \). This proves i). If \( g \) instead takes \( x \) to \( \lambda y \) and \( y \) to \( \lambda^{-1} x \) then the fixpoint locus is defined by \( \lambda y - x \) and \( \lambda^{-1} x - y \) i.e. by \( \lambda y - x \). Dividing out \( k[[x, y, t]]/(xy - t) \) by this relation gives \( k[[x, y, t]]/((\lambda y^2 - t) \) which is finite flat of degree 2 over \( k[[t]] \) and thus gives 2 fixpoints for neighbouring values of \( t \). The dihedral involutions form 2 or 1 conjugacy classes depending on whether the order of the dihedral group is divisible by 4 or not. This gives 4 resp. 2 orbits of fixpoints. □
3. On the boundary of the Hurwitz scheme

In this section we will start tying things together. We begin with a result which shows that our constructions do indeed capture the whole boundary of smooth curves with an action of a finite group.

**Definition 3.1.** For a finite group $G$ and an integer $g > 1$ we denote by $\mathcal{M}_g(G)$ the moduli stack of smooth, connected curves of genus $g$ together with an action of $G$. We also denote by $\overline{\mathcal{M}}_g(G)$ the moduli stack of stable, connected curves of (arithmetic) genus $g$ together with an admissible action of $G$ on the curve.

**Remark:** In general, $\mathcal{M}_g(G)$ is of course not connected even if one would restrict to only actions which are faithful (which would be a reasonable thing to do).

We then get the following result.

**Theorem 3.2.** $\overline{\mathcal{M}}_g(G)$ is a smooth and proper algebraic stack with $\mathcal{M}_g(G)$ as a dense open substack and the forgetful map $\overline{\mathcal{M}}_g(G) \rightarrow \mathcal{M}_g$ is unramified.

**Proof:** The fact that this stack is formally smooth follows directly from (2.1) and that $\mathcal{M}_g(G)$ is a dense open subset follows from (2.2). That the stack is of finite type follows directly from the fact that $\mathcal{M}_g$ is and the finiteness of the automorphism group of a stable curve of genus $> 1$. To show properness we use the valuative criterion. Hence we assume that we have a stable curve with an action by $G$ over the fraction field $K$ of a discrete valuation ring $R$. By the stable reduction theorem after possibly extending $K$ we may assume that there is a unique extension of the curve to a flat family over $R$ with stable fibres. By the uniqueness the action of $G$ will extend to this family and as the generic fibre is dense in the total space this extension is unique. This shows properness. The fact that the forgetful map is unramified follows from (2.1).

Having the results of the two previous sections in mind it should be clear that the data given by an admissible action of a group $G$ on a curve with only nodes as singularities is very closely related to the data given by a pointed curve and étale covers of the associated stack. The precise relation is as follows.

**Theorem 3.3.** Let the group $G$ act admissibly on the curve $C$ with only nodes as singularities. Let $C' = \overline{\mathcal{M}}_g(G)$ be the algebraic stack obtained from $C$ by gluing in cyclic nodal points at all the nodes and smooth cyclic points at all smooth points of $C$ having a non-trivial stabiliser group under the action of $G$. The action of $G$ on $C$ extends (uniquely) to an action of $G$ on $C'$. Let $C''$ be the stack quotient of $C'$ by $G$ (so that the quotient map $C' \rightarrow C''$ is étale). Let $C_1$ be the quotient of $C$ under $G$ (as an ordinary scheme) and let $C_1''$ be the stack obtained from $C_1$ by gluing in smooth cyclic points at all points of $C_1$ below a smooth point with non-trivial $G$-stabiliser, dihedral points at smooth points below a node on $C$ and nodal cyclic points at nodes of $C_1$. Then there natural isomorphism between $C''$ and $C_1''$ with their non-scheme points removed extends (uniquely) to an isomorphism between $C''$ and $C_1''$. In particular, $C$ is the scheme associated to the étale $G$-cover $C'$ of $C_1''$.

**Proof:** As the morphism is defined outside of the special points only the existence of an extension needs to be proved and that is a question local around the singularities of $C$ and the points of $C$ with non-trivial $G$-stabilisers. From this one immediately reduces down to something local at points of $C$ under the action of the $G$-stabiliser of the point. Hence we may consider the strict Henselisation $\hat{R}$ of a point and an admissible action of a group on that point. In all cases we have to compare the result obtained by first gluing in a non-scheme point and then taking the stack quotient by the group and that obtained by first taking the scheme quotient and then gluing in a non-scheme point. Let us first consider the case of a smooth point with a (necessarily) cyclic stabiliser. Gluing in a cyclic point means passing to the normalisation $\hat{R}$ of $R$ in the maximal tamely ramified extension and taking the stack quotient by the Galois group of that extension. Then taking a further stack quotient by a cyclic group $G$ is clearly the same thing as considering $\hat{R}$ the normalisation of the maximal tamely ramified extension acting on the normalisation of these invariants. This shows the result for the action on a smooth point. The case of the action of a cyclic group on a node is completely similar. The case
of a dihedral group acting on the node is slightly different. To get that case one needs to see that the following two constructions give the same result. 1’st construction: Start with a strictly Henselian local ring with an ordinary node as closed point. Then consider the subring of the normalisation in the product of the maximal tamely ramified extensions of the two fraction fields consisting of pairs \((r, s)\) congruent to each other modulo the respective maximal ideals (as in the construction of the gluing in of nodal cyclic point above). 2’nd construction: Consider an action of a dihedral group acting admissibly on the Henselisation of a node and take the invariant ring which is a regular ring. Then take the normalisation of this ring in the maximal tamely ramified extension and consider the subring of the product of this ring with itself consisting of the pairs \((r, s)\) which are equal modulo the respective maximal ideals (as in the construction of the gluing in of a dihedral point). Now, we may first take invariants under the normal cyclic subgroup to reduce to the case of a dihedral group of order 2. Then it is clear that the normalisations of the constructed rings are the same and hence so are the rings themselves being defined as subrings of their normalisations by identical conditions.

An immediate consequence of this result is the principal technical result of this paper. To formulate it let us agree to the following definition.

**Definition 3.4.** Let a finite group \(G\) act admissibly on a curve \(C\) with only nodes as singularities. We give the quotient curve \(C/G\) the structure of a pointed curve by letting the cyclic points be the smooth points below smooth points with non-trivial \(G\)-stabiliser and the dihedral points lying below nodal points. We will call this pointed curve the pointed curve quotient of \(C\) by \(G\).

This definition then, naturally, leads to the following corollary of the theorem.

**Corollary 3.5.** Let \((C, S_c, S_d)\) be a pointed curve with \(S_c\) the set of cyclic points and \(S_d\) the set of dihedral points. For a finite group \(G\) there is a natural 1-1 correspondence between admissible actions of \(G\) on a curve with only nodes as singularities together with an isomorphism of the pointed quotient with \((C, S_c, S_d)\) and étale \(G\)-covers of the pointed stack associated to \((C, S_c, S_d)\). In particular such actions correspond to conjugacy classes of homomorphisms of the fundamental group of the pointed stack to \(G\).

**Proof:** That one gets an étale cover of the pointed stack is the content of the theorem. To go from an étale cover of the stack to a curve one simply looks at the algebraic space associated to the stack cover. 

**Remark:** The explicit consequence of this corollary combined with (1.7), namely a concrete description of actions of \(G\) on curves with only nodes as singularities in terms of the pointed curve quotient, can be formulated without ever mentioning algebraic stacks. The reason for using algebraic stacks is that a lot of tedious verifications are avoided by referring instead to the general fact that the category of étale coverings of a connected stack form a Galois category. Let me mention however in concrete terms how one constructs the curve cover associated to a representation of the fundamental \(\pi\) group of the generalised graph of groups associated to a pointed curve \((C, S_c, S_d)\) in a group \(G\). First because the fundamental group of each irreducible component of \(C\) minus the union of the singularities of \(C, S_c\) and \(S_d\) maps into \(\pi\), we get a ramified \(G\)-cover of such a component. The amalgamation condition for the two branches of a node of \(C\) implies that there exists a \(G\)-equivariant identification of the points of the respective ramified covers over the two branches. One may thus equivariantly pair these points off to from nodes mapping down to the appropriate node of \(C\). Similarly, the dihedral involution given by the amalgamation at a dihedral point gives a pairing off of points above a dihedral point to construct nodes above it.

If we now go back to the motivating example of the Hurwitz scheme of ramified covers of the projective line and their descriptions in terms of such covers which are Galois we see that we have indeed obtained a description of possible degenerations of such covers: We start with a family of covers of the line, we form the family of Galois covers and then (after possibly extending the base) extend it to a complete family of stable curves with the Galois group \(G\) acting on it. We then finally take the quotient of this family by the stabiliser of a point in the permutation representation of \(G\) given by the original family which gives us a compactification of the original family. Compactifications of Hurwitz families have, of course, been considered previously (for instance in [Ha-Muj]) and then they
are done directly in terms of the covering of the projective line. We will now make a few comments on the comparisons of the approach given in this article and these more direct approaches.

**Remark:** In the preceding paragraph, as well as in the paragraphs to follow the precise relations between moduli problems are glossed over. The aim of the present article is to give a convenient description of the topological types of degenerations rather than to describe a suitable moduli problem. Thus we are primarily interested in individual curves rather than families of curves. The precise relations between moduli problems are therefore left to the interested reader. (Some of the differences are expounded upon in what follows.)

Let us now go on to make a rough comparison of the present approach with that of [Ha-Mu]. Thus consider a stable curve $C$ with a (faithful) action of the finite group $G$ such that the quotient of $C$ by $G$ is of genus 0. We may consider the fixpoint loci of non-trivial elements of $G$ and then throw away the parts corresponding the branch preserving actions on singular points. We then consider the quotient of these loci as a subscheme on the quotient of $C$ by $G$. By construction the union of them is a subscheme with support in the smooth part and it follows from the calculation done in the proof of (2.3) complemented by a similar calculation for a cyclic point that this subscheme is of length 2 at a dihedral point and of length 1 at a smooth. By the calculations of (2.3) and a similar calculation at a smooth cyclic point this construction behaves well in families so that the quotient of $C$ by $G$ naturally comes equipped with the structure of a relative divisor with support in its smooth part and with local multiplicity at most 2. On the other hand in the setup of [Ha-Mu] a genus 0 curve with a relative divisor supported in the smooth part which has everywhere local multiplicity 1 is used as a starting point. The general relation between these two structures would need some elaboration but let me explain it in a “miniversal” situation. We therefore assume that we have a 1-dimensional family of stable curves with $G$-action, a point $c$ on a fibre $D$ which is a dihedral node on that fibre (and for simplicity assume it to be the only one) and such that the divisor of fixed points on the quotient of the family by $G$ has two irreducible components at the image $d$ of $c$ both of them intersecting the fibre to which $d$ belongs as well as the other component transversally. (This situation can be obtained by a quadratic base change from a miniversal deformation, which follows from the calculations of the proof of (2.3).) We then blow up the point $d$. This adds a projective line to $C$, giving the new fibre $C'$, meeting one other component in a smooth point. The divisor of fixpoints will meet this new component in 2 points both of which lie only on that component. Thus $C'$ is a genus 0 curve with a relative divisor supported in the smooth part and which has everywhere local multiplicity 1. It is also a pointed curve whose cyclic points are the cyclic points of $C$ plus the 2 points on the new component and with no dihedral points. Its fundamental group is obtained from that, $\Pi$ say, of the pointed curve which is $C$ but with $d$ as a cyclic point by amalgamating the local fundamental group at $d$ with the local fundamental group at one point of $\mathbb{P}^1$ minus 3 points. The fundamental group of $C$ is the amalgamation of $\Pi$ with a dihedral group based on the local fundamental group at $d$ along that local fundamental group. There is an obvious group homomorphism from the fundamental group of $\mathbb{P}^1$ minus 3 points to the dihedral group based on one of its local fundamental groups taking generators of the two other local fundamental groups to dihedral involutions. This morphism glues to a surjective map from the fundamental group $\Pi_1$ of the pointed curve $C'$ to that of the pointed curve $C$, $\Pi_2$ say. Now if think of the action of $G$ on $D$ as a group homomorphism from $\Pi_2$ to $G$ we may compose that with the surjection from $\Pi_1$ to $\Pi_2$ and hence get a $G$-covering $D'$ of $C'$. The curve $D'$ maps equivariantly to $D$ and may be thought of as being obtained from $D$ by replacing each dihedral node by a projective line which meets the two branches of the node in 0 and $\infty$ and for which the stabiliser in $G$ of the node acts in the standard way on the projective line (with the dihedral involutions exchanging 0 and $\infty$). In the situation of Harris and Mumford we also have a transitive permutation representation of $G$ and by dividing $D$ by the stabiliser of a point in that representation we get a curve $E$ which is an admissible covering of $C'$ (except that to get only simple ramification of this covering one needs some conditions on the permutation representation, however Harris and Mumfords construction immediately generalises to the case when this is not assumed). Thus very roughly speaking the Harris-Mumford construction avoids dihedral stabilisers by blowing up those points (all the stabilisers on $D'$ are cyclic).
4. Codimension 1 and some examples

We will now discuss the simplest possible degenerations, those of codimension 1 in the moduli space and look at a few examples. We saw in our study of the deformation theory of admissible actions that any equivariant deformation of the singularities of a curve could be realised by a global equivariant deformation. This means that we can always equivariantly smooth one orbit of singularities while keeping the singularities in other orbits. This means that one may smooth all but one orbit and then the rest of the special points will be cyclic. To simplify matters let us assume that we are in the case motivating us, namely that the quotient curve is of genus 0. Then it will either be just the projective line with 1 dihedral and a certain number, $n$, say, (at least 2 to make it stable) of cyclic points, or it will have as irreducible components 2 projective lines meeting in a node and then a certain number, $n_i$, $i = 1, 2$ say, of cyclic points on each component (at least 2 on each to make the curve stable). In the first case the fundamental group will have generators $\sigma, x_0, x_1, \ldots, x_n$ with relations $\sigma^2 = e$, $\sigma x_0 \sigma = x_0^{-1}$ and $x_0 x_1 \cdots x_n = e$ and in the second case generators $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ with relations $x_1 x_2 \cdots x_n = e$, $y_1 y_2 \cdots y_n = e$ and $x_n y_1 = e$. An equivalent, but sometimes more convenient, presentation in the dihedral case is obtained by setting $\tau := \sigma x_0$ and then we get generators $\sigma, \tau, x_1, \ldots, x_n$ with relations $\sigma^2 = \tau^2 = e$ and $\sigma \tau x_1 \cdots x_n = e$. In the dihedral case, we will have use for the following observation. If we consider the normalisation of the pointed stack associated to the pointed curve consisting of the line with the chosen dihedral point and $n$ chosen cyclic points then the map on fundamental groups induced by the normalisation map is just the inclusion of the group generated by $x_0, x_1, \ldots, x_n$ into the one generated by $\sigma, x_0, x_1, \ldots, x_n$. Thus if we have a finite group $G$ and elements $s, g_0, g_1, \ldots, g_n$ with $s^2 = e$, $sg_0 s = g_0^{-1}$ and $g_0 g_1 \cdots g_n = e$ and let $C$ be the corresponding curve with $G$-action whose quotient is the projective line then the normalisation of $C$ as a (ramified) $G$-covering of the line is classified by the elements $g_0, g_1, \ldots, g_n$ thought of as decribing a $G$-covering of the line unramified outside of the given dihedral and cyclic points.

Let us now consider our previous example of the icosahedral group acting on the projective line in this light. Thus we pick elements $g_0, g_1$ and $g_2$ of order 5, 2 and 3 respectively whose product is one in $A_5$. This gives a $A_5$-covering of $P^1$ ramified at 0, 1 and $\infty$. Now the normaliser of a Sylow 5-subgroup of $A_5$ is dihedral so we may pick elements $s$ and $t$ of it of order 2 s.t. $st = g_0$. Now consider the pointed curve which is $P^1$ with 0 as a dihedral point and 1 and $\infty$ as cyclic points. The collection $s, t, g_1, g_2$ gives an étale $A_5$-cover of the associated stack whose normalisation is the $A_5$-cover constructed using $g_0, g_1, g_2$. This cover is the one considered in the previous example. Now, when smoothing it we get a covering of $P^1$ ramified in 4 points corresponding to the tuple $s, t, g_1, g_2$. Using our results on the dévissage of the action of the group on cohomology we get a quick way of computing the character of the action of $A_5$ on these smooth genus 6 curves; it is simply 2 times the non-trivial character of degree 1 of the subgroup of $A_5$ of order 10 induced up to $A_5$. (This character can of course be computed also from the ramification behaviour and Lefschetz fixpoint formula.)

The same example can also be obtained by dividing $s, t, g_1, g_2$ in the groups $s, t, (st)^{-1} = g_0^{-1}$ and $st = g_0, g_1, g_2$ which gives a cover of two crossing $P^1$'s with two cyclic points on each component, the covering of one component is the icosahedral, the covering of the other is given by the action of a dihedral group of order 10 on $P^1$.

A similar example is obtained by choosing elements of order 7, 2, and 3 in $PSL(2, F_7)$ whose product is the identity and then use that the normaliser of the Sylow 7-group is dihedral of order 14. Our starting curve - a $PSL(2, F_7)$-cover of $P^1$ ramified at 3 points with ramification groups of order 7, 2, and 3, is then the Klein curve of genus 3. In the end we get a $PSL(2, F_7)$-covering of genus 15 of $P^1$ ramified in 4 points with ramification groups of order 2, 2, 2, and 3.

Bibliography

[De-Mu]: P. DELIGNE, D. MUMFORD, On the irreducibility of the moduli space of curves, *Publ. I.H.E.S.* 36 (1969), 75 – 110.

[Ha-Mu]: J. HARRIS, D. MUMFORD, On the Kodaira dimension of the moduli space of curves, *Inv. math.* 67 (1982), 23 – 86.
[Ka-Ma]: N. Katz, B. Mazur, *Arithmetic moduli of elliptic curves*, Princeton, Princeton Univ. Press, 1985.

[Se]: J.-P. Serre, *Trees*, Berlin, Springer Verlag, 1980.