Primordial non-Gaussianity in multi-scalar slow-roll inflation

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Abstract. We analyse the non-Gaussianity for primordial curvature perturbations generated in a multi-scalar slow-roll inflation model including the model with non-separable potential by making use of $\delta N$ formalism. Many authors have investigated the possibility of large non-Gaussianity for the models with separable potential, and they have found that the non-linear parameter, $f_{NL}$, is suppressed by the slow-roll parameters. We show that for the non-separable models $f_{NL}$ is given by the product of a factor which is suppressed by the slow-roll parameters and a possible enhancement factor which is given by exponentials of quantities of O(1).

Keywords: cosmological perturbation theory, inflation, physics of the early universe

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1. Introduction

Inflation has been widely recognized as a standard mechanism for generating primordial density perturbations which seed the structure formation of the universe and the cosmic microwave background (CMB) anisotropies. In the simplest single-field inflationary universe scenario, primordial fluctuations are created by vacuum fluctuations of the inflaton. However, in constructing realistic inflation models based on supergravity, it seems more natural to consider that the energy scale of inflation is much lower and that the scalar field may have multi-components during inflation. The discrimination of the simplest single-field inflation model from the other low energy inflation models will be most clearly done by the future observation of CMB B-mode polarization [1]–[3]. The simplest single-field model predicts the high energy scale of inflation. Thus the amplitude of the tensor perturbation is large and is expected to be observed soon. In contrast, in the case of low energy inflation models, the tensor perturbation is negligibly small. Therefore no primordial tensor perturbation will be detected.

Recently, the non-linearity (non-Gaussianity) of the primordial perturbations also has been a focus of constant attention by many authors [4]–[16]. The main reason for attracting much attention is that meaningful measurement of this quantity, which brings us valuable information about the dynamics of inflation if detected, will become observationally available in the near future. In order to parametrize the amount of non-Gaussianity of primordial perturbations, a non-linear parameter, $f_{NL}$, is commonly used which is related to the bispectrum of the curvature perturbation [4].

Meanwhile, in the single-field slow-roll inflation it is found that $f_{NL}$ is suppressed by slow-roll parameters to an undetectable level [6,12]. But, for example, in the curvaton scenario [17,18], it is predicted by many authors [19]–[22] that there is a possibility of large...
non-Gaussianity enough to be detectable by future experiments, such as PLANCK [1], which is expected to detect the non-linear parameter if \( |f_{\text{NL}}| \gtrsim 5 \) [4]. In the curvaton scenario, primordial curvature perturbations are sourced by isocurvature perturbations related to the vacuum fluctuations of a light scalar field (other than inflaton), called a curvaton, which is an energetically subdominant component during inflation. As the energy density of the universe drops after inflation, the fraction of this component becomes significant. Then, through the process that the curvaton decays into radiation after inflation, the curvaton isocurvature perturbations are converted into curvature (adiabatic) perturbations. The curvaton scenario predicts a nearly scale-invariant spectrum as in the case of the standard inflation scenario, but a large value of \( f_{\text{NL}} \) is possible in this scenario.

For the multi-scalar field inflation, however, the possibility of the generation of primordial non-Gaussianity has been studied mostly for the models with the separable potential within the slow-roll approximation [23]–[27]. (The only exceptions are references [28]. We will mention the relation of our present paper to these references in section 4.) For such models with separable potential, it was predicted that \( f_{\text{NL}} \) is suppressed by slow-roll parameters as long as slow-roll conditions are satisfied.

In this paper, we analyse the primordial non-Gaussianity in multi-scalar field inflation models without specifying the explicit form of the potential. What we assume is just the slow-roll conditions. To make the derivation of the formula for \( f_{\text{NL}} \) transparent, we take advantage of the \( \delta N \) formalism [29] extended to the non-linear regime [12,30].

In section 2.1 we briefly review the power spectrum and the bispectrum which is related to the two- and three-point correlation functions of the curvature perturbations, respectively, and we define the non-linear parameter, \( f_{\text{NL}} \). In section 2.2 we review how the non-linear parameter can be described in the \( \delta N \) formalism which was proposed in [12]. In section 3 we show how one can derive a formula for \( f_{\text{NL}} \) in terms of the potential of the scalar field in the slow-roll approximation. We also discuss the possibility of generation of a large amplitude of the primordial non-Gaussianity in multi-scalar slow-roll inflation. We give a summary in section 4.

2. \( \delta N \) formalism and non-linear parameter \( f_{\text{NL}} \)

2.1. Power spectrum and bispectrum

In this subsection, we briefly review the power spectrum and bispectrum of curvature perturbations and define the non-linear parameter, \( f_{\text{NL}} \), following [4,12,13].

As a gauge-invariant perturbation variable, we choose the curvature perturbation on a uniform density hypersurface, \( \zeta \). If the perturbation is pure Gaussian, its statistical properties are characterized by its power spectrum, \( P_\zeta \), defined by

\[
\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle \equiv \delta^{(3)}(\mathbf{k}_1 + \mathbf{k}_2) \frac{2\pi^2}{k_1^3} P_\zeta(k_1),
\]

where \( \zeta_{\mathbf{k}} \) represents a Fourier component given by

\[
\zeta_{\mathbf{k}}(t) = \frac{1}{(2\pi)^{3/2}} \int d^3 x \zeta(t, \mathbf{x}) \exp(-i\mathbf{k} \cdot \mathbf{x}).
\]

In order to constrain the non-Gaussianity of the curvature perturbation, \( \zeta \), from CMB observations, the non-linear parameter, \( f_{\text{NL}} \), has been commonly used. This non-linear
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parameter expresses the departure of ζ from a pure Gaussian random variable given in the form [4]

$$\zeta(x) = \zeta_G(x) - \frac{3}{5} f_{NL} \zeta_G^2(x),$$  \hspace{1cm} (2)

where ζ_G satisfies Gaussian statistics. Neglecting the quadratic or higher-order terms in f_{NL}, the power spectrum of ζ is identical to that of ζ_G, i.e. P_ζ = P_G, while the three-point correlation function is affected by the non-linear part, −\frac{6}{5} f_{NL} \zeta_G^2. The three-point correlation function is characterized by the bispectrum, B, defined by

$$\langle \zeta_{k_1} \zeta_{k_2} \zeta_{k_3} \rangle \equiv \delta^{(3)} (k_1 + k_2 + k_3) B_\zeta(k_1, k_2, k_3).$$  \hspace{1cm} (3)

When the statistics of ζ is defined by equation (2), the bispectrum B is expressed as

$$B_\zeta(k_1, k_2, k_3) = -\frac{6}{5} \frac{f_{NL}}{(2\pi)^3/k^3} \left[ P_\zeta(k_1) P_\zeta(k_2) + P_\zeta(k_2) P_\zeta(k_3) + P_\zeta(k_3) P_\zeta(k_1) \right],$$  \hspace{1cm} (4)

where \( P_\zeta(k) = 2\pi^2 P_\zeta(k) / k^3 \).

2.2. δN formalism

In this subsection we briefly review the δN formalism and show a simple formula for f_{NL}, following [29, 30] and [12].

In this paper we consider a minimally coupled D-component scalar field whose action is given by

$$S_{\text{fields}} = -\int d^4x \sqrt{-g} \left[ \frac{1}{2} g^\mu\nu \partial_\mu \phi^I \partial_\nu \phi^J + V(\phi) \right],$$

where \( V(\phi) \) represents the potential of the scalar field. The background e-folding number between an initial hypersurface at \( t = t_* \) and a final hypersurface at \( t = t_c \) is defined by

$$N \equiv \int H \, dt.$$

Here, we assume that the time derivative of \( \phi^I(t) \) is not independent of \( \phi^I(t) \) as in the case of standard slow-roll inflation, and then we can regard N as a function of the homogeneous background field configuration \( \phi^I(t_*) \) on the initial hypersurface at \( t = t_* \) and \( \phi^I(t_c) \) on the final hypersurface at \( t = t_c \):

$$N = N(\phi^I(t_*), \phi^I(t_c)).$$

Let us take \( t_* \) to be a certain time soon after the relevant length scale crossed the horizon scale, and \( t_c \) to be a time when the complete convergence of background trajectories in the phase space of the D-component scalar field has occurred. At \( t > t_c \) the history of the universe is labelled by a single parameter and only the adiabatic modes remain. (We have neglected the case in which isocurvature perturbations persist until later.) Then, it is well known that the curvature perturbation on a uniform density hypersurface, ζ, becomes constant in time on super-horizon scales. Thus, in the estimation of the spectrum, what we need is only the final value of the curvature perturbation \( \zeta(t_c) \). Based on the δN formalism, ζ evaluated at \( t = t_c \) is given by \( \delta N(t_c, \phi^I(t_*)) \), where we substitute \( \phi^I(t_c) \) with \( t_c \) because we have taken \( t_c \) to be a time when the background trajectories have converged.

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The relation between $\zeta$ and $\delta N$ up to the linear order is given in [29], and its non-linear extension is given in [30] (see also [10]). Up to second order the relation becomes

$$\zeta(t_c) \simeq \delta N(t, \phi^I) = N^I_t \delta \phi^I_t + \frac{1}{2} N^I_{tJ} \delta \phi^I_t \delta \phi^J_t,$$

where $\simeq$ means the use of the super-horizon approximation and $\delta \phi^I_t$ represent the field perturbations on the initial flat hypersurface at $t = t_*$. We have also defined $N^I_t = N_I(t_*)$ and $N^I_{tJ} = N_{IJ}(t_*)$ with

$$N^I_t = \left. \frac{\partial N(t, \phi^I)}{\partial \phi^I} \right|_{\phi^I = \phi^I(t)},$$

$$N^I_{tJ} = \left. \frac{\partial^2 N(t, \phi^I)}{\partial \phi^I \partial \phi^J} \right|_{\phi^I = \phi^I(t)}.$$

For the multi-field case ($D \geq 2$), the form of equation (5) differs from that of equation (2) because of the contribution of the isocurvature modes. Nevertheless, to the leading order, the bispectrum $B$ is expressed in the form (4) in terms of a single non-linear parameter $f_{NL}$, which is given by [12, 15, 23, 24]

$$\frac{6}{5} f_{NL} \simeq \frac{N^I_t N^J_t N^I_{tJ}}{(N^K_t N^K_t)^2},$$

where the indices are lowered and raised by using Kronecker’s delta like $N^I_t = \delta^{IJ} N^J_t$.

Here we have assumed that the field perturbation on the initial flat hypersurface, $\delta \phi^I_t$, is Gaussian\(^3\), and we have neglected the logarithmic term appearing in [12] because this term is suppressed by the power spectrum of curvature perturbation, $P_\zeta \sim 10^{-10}$.

### 3. Non-linear parameter in multi-scalar slow-roll inflation

#### 3.1. Background equations in slow-roll regime

Using the background $e$-folding number as the time coordinate, the background equation is obtained as

$$\frac{d^2}{dN^2} \phi^I + \left(3 + \frac{1}{H} \frac{dH}{dN}\right) \frac{d\phi^I}{dN} + \frac{V^I}{H^2} = 0,$$

where $V^I = \delta^I(J)(\partial V/\partial \phi^J)$. The background Friedmann equation is given by

$$H^2 = \frac{1}{3} \left( \frac{1}{2} H^2 \frac{d\phi^I}{dN} \frac{d\phi^I}{dN} + V(\phi) \right).$$

\(^3\) In [7], the authors have calculated the contribution of the non-Gaussianity of the field perturbations on the initial flat hypersurface, $\delta \phi^I_t$, and they concluded that the initial non-Gaussianity will not be large when the slow-roll conditions are satisfied. The non-linear parameter defined by equation (4) is given by a sum of the contribution of the super-horizon evolution given by equation (6) and the contribution of the initial non-Gaussianity of $\delta \phi^I_t$ [12, 15, 23, 24].
We define the slow-roll parameters in terms of the potential of the scalar field as
\[ \epsilon \equiv \frac{1}{2} \frac{V''}{V'}, \quad \eta_{IJ} \equiv \frac{V_{IJ}}{V}. \]
Hereinafter, we assume \( \epsilon \ll 1 \) and \( |\eta_{IJ}| \ll 1 \), which we call ‘relaxed’ slow-roll conditions (RSRC)\(^4\). Under these conditions, the background equations reduce to
\[
\frac{d\delta\phi}{dN} \simeq -\frac{V'}{V},
\]
\[
H^2 \simeq \frac{1}{3} V(\phi).
\]

In most cases, the complete convergence of background trajectories in phase space of the \( D \)-component scalar field occurs after the RSRC are invalidated. Under the present approximation, therefore, we cannot evaluate \( \zeta(t_c) \), the curvature perturbation on a uniform density hypersurface after the convergence of trajectories. In this paper, we concentrate on the non-Gaussianity of the curvature perturbation generated during the RSRC phase. For this purpose, we introduce \( t_f \), a time at which the RSRC are still satisfied. We divide the process of evaluating \( \zeta(t_c) \) into two parts: (i) evaluation of \( N(t_c, \phi^f(t_f)) \), the e-folding number to reach \( \phi^f(t_c) \) starting with \( \phi^f = \phi^f(t_f) \), and (ii) expressing \( \delta \phi^f(t_c) \) in terms of \( \delta \phi^f \), where we expand the scalar field as \( \phi^f = \phi^f + \delta \phi^f \). As we use the e-folding number itself as the time coordinate, \( N + N(N_c, \phi^f(N)) \), by definition, is equal to \( N_c \) and is constant independent of \( N \). Therefore one can say
\[
\zeta(N_c) \simeq \delta N(N_c, \phi^f(N_c)) \approx \delta \phi^f + \frac{1}{2} N^T_{IJ} \delta \phi^f \delta \phi^f, \quad (9)
\]
where \( \delta \phi^f = \delta \phi^f(N_f), \delta N(N_c, \phi^f(N_f)) = N(N_c, \phi^f(N_f)) - N(N_c, \phi^f(N_c)), N^T_f = N_I(N_f) \) and \( N^T_{IJ} = N_{IJ}(N_f) \). Our formulation does not apply for the curvature perturbation generated after the RSRC are violated. Postponing the evaluation of this part to the future, we study how one can express \( \delta \phi^f \) in terms of \( \delta \phi^f \) to the second-order perturbation. In the succeeding subsection, assuming that \( N^T_f \) and \( N^T_{IJ} \) are given, we calculate the non-linear parameter, \( f_{NL} \).

### 3.2. Analytic formula for the non-linear parameter

In order to obtain the non-linear parameter, \( f_{NL} \), introduced in equation (6), we need to evaluate \( N^T_I \) and \( N^T_{IJ} \). Once we obtain the relation between \( \delta \phi^f \) and \( \delta \phi^f \), one can express \( N^T_I \) and \( N^T_{IJ} \) by using \( N^T_f \) and \( N^T_{IJ} \) from the comparison of equations (5) and (9). Thus, we first solve the evolution of \( \delta \phi^f(N) \) to the second order.

The scalar field is expanded up to second order as
\[
\phi^f \equiv \phi^f + \delta \phi^f + \frac{1}{2} \delta \phi^f + \cdots.
\]

\(^4\) In the ‘standard’ slow-roll approximation, one assumes that \( \epsilon \approx |\eta_{IJ}| \ll 1 \). Here, we do not assume the relation between the order of \( \epsilon \) and that of \( \eta_{IJ} \).
Taking the variation of equation (7), we obtain
\[
\frac{d}{dN} \delta^{(1)}_\phi(N) = \delta^{(1)}_\phi(N) P^{(1)}_{IJ}(N),
\]
(10)
\[
\frac{d}{dN} \delta^{(2)}_\phi(N) = \delta^{(2)}_\phi(N) P^{(2)}_{IJ}(N) + \delta^{(1)}_\phi(N) \delta^{(1)}_\phi(N) Q^{(1)}_{IKJ}(N),
\]
(11)
with
\[
P^{(1)}_{IJ}(N) = \left[ -\frac{V^{(1)}_{I}}{V} + \frac{V^{(1)}_{J}}{V^2} \right]_{\phi = \phi(N)},
\]
\[
Q^{(1)}_{IKJ}(N) = \left[ -\frac{V^{(1)}_{IK}}{V} + \frac{V^{(1)}_{IJK}}{V^2} + \frac{V^{(1)}_{IJKV}}{V^2} - 2 \frac{V^{(1)}_{IJKV}}{V^3} \right]_{\phi = \phi(N)}.\]

Let us consider the conditions under which we can use equations (10) and (11). Differentiating equation (10) with respect to \(N\), we have
\[
\frac{d}{dN} \left( \frac{d}{dN} \delta^{(1)}_\phi(N) \right) = \frac{d}{dN} \left( \delta^{(1)}_\phi(N) P^{(1)}_{IJ}(N) \right) = \delta^{(1)}_\phi(N) \left( P^{(2)}_{IJ} \right) - \delta^{(1)}_\phi(N) \delta^{(1)}_\phi(N) Q^{(1)}_{IKJ}(N) \frac{V^K}{V}.
\]
(12)
If we consider the minus of equation (12) as the corrections to the rhs of equation (10), we naively give an estimate for the correction to \(\delta^{(1)}_\phi\)
\[
\Delta \left( \delta^{(1)}_\phi \right) \simeq \int \! dN \delta^{(1)}_\phi P^{(1)}_{IJ} + \int \! dN \delta^{(1)}_\phi \left( P^{(2)}_{IJ} \right) - \int \! dN \delta^{(1)}_\phi Q^{(1)}_{IKJ} \frac{V^K}{V}.
\]
(13)
From this expression, in order that equation (10) is a good approximation, the conditions
\[
\left| \int \! dN \left( P^{(2)}_{IJ} \right) \right| \ll 1,
\]
(14)
and
\[
\left| \int \! dN \left( Q^{(1)}_{IKJ} \frac{V^K}{V} \right) \right| \ll 1,
\]
(15)
must be satisfied. If we define a small parameter \(\xi\) by
\[
\left| \frac{V^K}{V} \right| \equiv O(\xi),
\]
we can estimate the duration of inflation measured in \(N\) as
\[
\int \! dN \simeq \int \! dV \frac{dV}{dN}^{-1} \sim O(\xi^{-2}).
\]
Then, roughly speaking, the conditions (14) and (15) reduce to
\[
\left| P^{(1)}_{IJ} \right| \ll O(\xi), \quad \left| Q^{(1)}_{IKJ} \right| \ll O(\xi).
\]
(16)
Differentiating equation (11) with respect to $N$, we also have
\[
\frac{d}{dN} \left( \frac{d}{dN} \delta \phi^I J \right) = \frac{d}{dN} \left( \delta \phi^I J \right) P^I J + \delta \phi^I J \frac{d}{dN} P^I J + 2 \frac{d}{dN} \left( \delta \phi^I J \right) \delta \phi^K \delta \phi^J K Q^I J K + \delta \phi^I J \delta \phi^K \frac{d}{d\phi^K} \left( Q^I J K \right) \left( -\frac{V^L}{V} \right).
\]
In the same way, the condition in which we can neglect the contribution of the second derivative becomes
\[
\left| \frac{d}{d\phi^K} \left( Q^I J K \right) \right| \xi \ll \left| Q^I J K \right| .
\tag{17}
\]
Thus, in order to use the approximate equations, equations (10) and (11), the required conditions are
\[
\left| \frac{V^I}{V} \right| = O(\xi),
\tag{18}
\]
\[
\left| \frac{V^I J}{V} \right| \sim \left| \frac{V^I J K}{V} \right| \ll O(\xi),
\tag{19}
\]
and equation (17) for a small parameter $\xi$, and we redefine RSRC by these required conditions.

These equations are $D$-component coupled differential equations. In general, we cannot solve equations (10) and (11) analytically. We give a formal solution of equation (10) as
\[
\delta \phi^I J(N) = \Lambda^I J(N, N_{*}) \delta \phi^I_{*},
\tag{20}
\]
and
\[
\Lambda^I J(N, N_{*}) = \left[ T \exp \left( \int_{N_{*}}^{N} P(N'') \, dN'' \right) \right] J K L M (N').
\tag{21}
\]
where $T$ means the time-ordered product. We also give the formal solution of equation (11) as
\[
\delta \phi^I (N) = \Lambda^I J(N, N_{*}) \int_{N_{*}}^{N} dN' \left[ \Lambda(N', N_{*})^{-1} \right] J K L M (N').
\tag{22}
\]
Substituting these solutions to equation (9), we obtain
\[
\zeta(N_{*}) = \Lambda^I J_i \left( \Lambda^I J_i(N_i, N_{*}) \delta \phi^I_{*} + \frac{1}{2} \int_{N_{*}}^{N_i} dN' \Lambda^I J_i(N_i, N') Q^I J K L (N') \times \Lambda^K M (N', N_{*}) \Lambda^L N (N', N_{*}) \delta \phi^M_{*} \delta \phi^N_{*} \right) + \frac{1}{2} N^I J_i \Lambda^I J_i(N_i, N_{*}) \Lambda^J L_i (N_i, N_{*}) \delta \phi^K_{*} \delta \phi^L_{*}.
\]
Comparing this expression with equation (5), we find that $N^I_i$ is expressed as
\[
N^I_i = N^I J_i \Lambda^J L_i (N_i, N_{*}).
\]
Since the above relation should hold for arbitrary $N_*$, we also have

$$N_f(N) = N_f^I \Lambda^I(N_f, N).$$

With the aid of this relation, $N_{IJ}^*$ is expressed as

$$N_{IJ}^* = N_J^K \Lambda^K_I(N_f, N_*) \Lambda^L_J(N_f, N_*) + \int_{N_*}^{N_f} dN' N_K(N') Q^K_L(N') \Lambda^L_J(N_f, N_*) \Lambda^M_J(N_f, N_*) + \int_{N_*}^{N_f} dN' N_J(N') Q^K_L(N') \Theta^K(N_f, N_*) \Theta^L(N_f, N_*) \Lambda^L_J(N_f, N_*) \Lambda^M_J(N_f, N_*) .$$

Substituting the above relations into equation (6), and using $\langle \phi^*_I \phi^*_J \rangle \propto \delta^{IJ}$, we finally obtain a very concise formula for the non-linear parameter

$$-6 f_{NL} = (N_f^I N_f^J)^{-2} \left( N_{JK}^I \Theta^K(N_f, N_*) \Theta^L(N_f, N_*) + \int_{N_*}^{N_f} dN' N_J(N') Q^K_L(N') \Theta^K(N_f, N_*) \Theta^L(N_f, N_*) \right),$$

where we have introduced a new vector

$$\Theta^I(N) \equiv \Lambda^I_J(N, N_*) N_*^J .$$

Equation (23) is the main result of this paper. If we directly evaluate the formula (6) for the non-linear parameter $f_{NL}$, we need to calculate $(\phi^*_I \phi^*_J)$ as functions of $\phi^*_I$. Namely, we need to compute the coefficients $\phi^*_I J K(N_f)$ defined by $\phi^*_I(N_f) = \phi^*_I J K(N_f) \phi^*_J \phi^*_K$. However, our final expression (23) does not require to compute the evolution of such a quantity that has three indices. Instead, we only have to deal with vector-like quantities $N_I(N_f)$ and $\Theta^I(N_f)$, which are obtained by solving

$$\frac{d}{dN} N_I(N_f) = -P^I J (N_f) N_J(N_f),$$

$$\frac{d}{dN} \Theta^I(N_f) = P^I J (N_f) \Theta^J(N_f).$$

The boundary condition for $N_I(N_f)$ is given at $N = N_f$ and that for $\Theta^I(N_f)$ is given by

$$\Theta^I(N_*) = N^J(N_*) .$$

When a specific model is concerned, one can numerically evaluate $f_{NL}$ by using the above formula rather easily. We first numerically integrate $N_I(N_f)$ backwards in time until the initial time $N_*$, then the initial condition for $\Theta^I(N_f)$ is given by (26). Solving equation (25), we obtain $\Theta^I(N_f)$. Finally, substituting these results into the formula (23) and integrating over $N'$, one obtains $f_{NL}$.

### 3.3. Non-linearity generated until $N = N_f$

Here we evaluate $\zeta(N_f)$, the curvature perturbation on a uniform density hypersurface evaluated at $N = N_f$, where the background trajectories have not completely converged yet. Therefore $\zeta$ evolves after $N_f$ until $N_*$, obviously. In this subsection, we roughly estimate the non-Gaussianity of the curvature perturbations generated until $N = N_f$. Neglecting the later evolution of the curvature perturbations, we obtain a simple analytic
formula for the non-linear parameter, $f_{NL}$, written in terms of the potential of the scalar field.

It has been shown that perturbation of the background trajectories $\delta \phi^I(N_f)$ can be interpreted as the perturbation in a particular gauge, which we call the $N$-constant gauge [30]. Furthermore, it has also been shown that, under the assumption that one can neglect the purely decaying mode contribution, $N$-constant gauge is equivalent to the flat slicing. Therefore $\delta \phi^I(N_f)$ can be recognized as the field perturbation on the flat slicing at $N = N_f$. In the slow-roll regime, the uniform energy density hypersurface is approximately the same as the $V = \text{constant}$ hypersurface, since $\rho = V + O(\epsilon)$. Then, $\zeta(N_f)$ is evaluated by the time shift $\delta N$ necessary to transform to the $V = \text{constant}$ hypersurface from the flat slicing. This leads to the relation

$$V(\phi(N_f + \delta N)) = V\left(\phi(N_f)\right).$$

From this equation, we can obtain the relation between $\delta N = \zeta(N_f)$ and $\delta \phi^I$. Up to second order, equation (27) can be expanded as

$$\left(\frac{V_{IJ}V^I V^J}{V^2} - \frac{P_{IJ} V^I V^J}{V}\right) \delta N^2 - 2 \left(\frac{V^I V_J}{V} + V_{IJ} \frac{\delta \phi^I}{V} - V_I \frac{d}{dN} \frac{\delta \phi^I}{V}\right) \delta N$$

$$+ 2V_I \delta \phi^I_N + V_J \delta \phi^J_N + V_{IJ} \frac{\delta \phi^I}{V} \frac{\delta \phi^J}{V} \bigg|_{\phi = \phi_f} = 0,$$

where we have used the equation of motion for $\phi^I(N)$, equation (7), and its time derivative

$$\frac{d^2}{dN^2} \phi^I = -P_{IJ} \frac{V^J}{V}.$$

Solving equation (28) for $\delta N$ up to second order in $\delta \phi$, we have

$$\zeta(N_f) \approx \delta N = \frac{V}{V^I V_J} \left[V_J \delta \phi^I_N + \frac{1}{2} V_J \delta \phi^J_N + \frac{1}{2} U_{MN} \delta \phi^M_N \delta \phi^N_N \right]_{\phi = \phi_f},$$

with

$$U_{MN} \equiv V_{MN} + 2 \frac{V_K V^K V^L V_M V_N}{(V^L V^J)^2} + \frac{V_M V_N}{V} - 4 \frac{V_K (V^K V^L V_M V_N)}{V^J V^J},$$

where we have used equation (10).

From this expression, we find that one can apply the formula obtained in the preceding subsection with the identification

$$N_f^I = \left(\frac{V}{V^I V_J}\right) V_I \bigg|_{\phi = \phi_f},$$

$$N_{IJ} = \frac{1}{2} \left(\frac{V}{V^K V_K}\right) U_{IJ} \bigg|_{\phi = \phi_f}.$$
In the present case the analytic formula for the non-linear parameter, \( f_{NL} \), can be written down more explicitly as

\[
-\frac{6}{5} f_{NL} = 2 \left[ \epsilon + \frac{n_{IJ}}{2V^K V^L} (2V^I V^J - 4V^I \hat{\Theta}^J + \hat{\Theta}^I \hat{\Theta}^J) \right]_{\phi = \phi_f}^{(0)} + (N^I N^J)^{-2} \int_{N_i}^{N_f} dN N_I(N) Q^I_{JK}(N) \Theta^J(N) \Theta^K(N),
\]

with

\[
\hat{\Theta}^I = (N^I N^J)^{-1} V \Theta^I.
\]

Here we have used the relation \( \hat{\Theta}^I V_I = V^I V_I \), which holds at \( \phi^I = \phi_f^{(0)} \). In the single-field case this result, equation (30), corresponds to the well-known simple formula given in [13] (see appendix A). Under the condition that \( N \)-constant hypersurface is identical to \( V = \) constant hypersurface, it can be shown that \( \zeta(N_f) \) becomes independent of \( N_f \), and so is \( f_{NL} \) (see appendix B).

We discuss here the rough order estimate of the above expression. Here we assume equations (17)–(19) for the order of magnitude of the derivative of the potential. Namely, \( V_{IJ}/V \ll O(\xi) \) and \( V_{IJK}/V \ll O(\xi^2) \). The duration of inflation measured in \( N \) will be estimated by \( V(dV/dN)^{-1} = O(\xi^{-2}) \). Further, we assume that all components of \( \Lambda^I_{JJK} \) do not become much larger than unity. Then, \( N_f \) and \( \Theta_f \) are estimated as \( O(\xi^{-1}) \) since they are roughly the same order as \( V/V_I \). \( \hat{\Theta} \) is of \( O(V \xi) \) and thence of \( O(V^I) \). This rough estimate of the order of magnitude indicates that \( f_{NL} \) is smaller than \( O(1) \) within the range of validity of our present approximation. The large non-Gaussianity \( (f_{NL} \geq 1) \) is not likely to be generated even in the case of multi-scalar inflation with non-separable potential in the standard slow-roll approximation, where \( V_{IJ}/V = O(\epsilon) \) and \( V_{IJK}/V = O(\epsilon^{3/2}) \), \( f_{NL} \) is definitely suppressed by the slow-roll parameter, \( \epsilon \). Nevertheless, a little loophole exists in the above estimate. We assumed that \( \Lambda^I_{JJK} \) is always of \( O(1) \). However, since the exponent in equation (21) can be \( O(1) \), \( \Lambda^I_{JJK} \) is not guaranteed to stay of \( O(1) \).

4. Discussion and conclusion

We have studied the primordial non-Gaussianity in \( D \)-component scalar-field inflation models for arbitrary potential under the slow-roll approximation by making use of the \( \delta N \) formalism. The obtained formula (23) has non-local terms, the terms expressed as an integral over \( N' \) and the terms containing \( \Theta^I \), which implicitly contains partial information of \( \Lambda \sim T \exp \left( \int dN' P \right) \). Here \( \Lambda \) and \( P \) are a \( D \times D \) matrix and \( T \) represents the time-ordered product. The explicit form of \( P^I_{J} \) is given in (10). In [28], a similar expression for the primordial non-Gaussianity has also been derived. However, it is a remarkable simplification that our final expression (23) is expressed only using \( D \) vector quantities, \( N_I \) and \( \Theta^I \). Thus, we think that our improved formula will be a powerful tool for computing the non-linear parameter systematically for a wide variety of models of the multi-scalar field inflation. We also obtained a little more concise formula for the non-linear parameter (30) for \( \zeta \) evaluated at a time during the slow-roll inflation, assuming that the later evolution of \( \zeta \) can be neglected.
Our formula is valid when $\eta_{IJ} \equiv V_{IJ}/V \ll O(\xi)$ and $V_{IJK}/V \ll O(\xi)$, where we defined $\xi$ by $V_{II}/V = O(\xi)$. In this case, we find that $f_{NL}$ is smaller than $O(1)$, under the assumption that $N_I$ and $\Theta^I$ stays of $O(\xi^{-1})$, even in the case of multi-scalar inflation with non-separable potential. We also find that $f_{NL}$ is suppressed by the slow-roll parameter, $\epsilon$, in the standard slow-roll inflation, where $\eta_{IJ} = O(\epsilon)$ and $V_{IJK}/V = O(\epsilon^{3/2})$. Under this assumption, primordial non-Gaussianity does not become large enough to be detectable by future satellite missions for the cosmic microwave background ($f_{NL} \geq 1$), such as PLANCK.

However, it is not clear if this assumption is guaranteed in general. In the standard slow-roll inflation, the exponent in the expression of $\Lambda$ also becomes $O(1)$. Hence, $\Lambda$ itself can be much larger than unity. As a future work, we will investigate various possibilities of generating large non-Gaussianity in multi-scalar inflation by constructing explicit models.

Another possibility to generate detectable non-Gaussianity is to relax the conditions, $\eta_{IJ} \ll O(\xi)$ and $V_{IJK}/V \ll O(\xi)$. Observations currently constrain the magnitudes of the first and second derivatives of the potential in a certain direction in the phase space at the horizon crossing time, but the derivatives in other directions might be larger. Moreover, present observations do not constrain the form of the potential after the horizon crossing time.

In this paper we imposed the conditions that the derivatives of potential in all directions in field space are sufficiently small. We can relax these conditions still keeping all the observational constraints satisfied. We will also analyse such possibilities in a future work by extending our formalism to non-slow-roll cases.

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Appendix A. Single-field slow-roll case

In the single-field slow-roll case, the power spectrum is given by

$$P_\zeta = N_\phi^2 P_* = \left( \frac{V \Lambda}{V_\phi} \right)^2 \left( \frac{H}{2\pi} \right)^2.$$

In this case, using the slow-roll parameters we can write $\Lambda$ as

$$\Lambda = \exp \left[ \int_{N_*}^{N} dN' \left( 2\epsilon - \eta \right) \right].$$

Moreover, we have

$$\frac{d}{dN} V(N) = -2\epsilon V(N),$$

$$\frac{d}{dN} V_\phi(N) = -\eta V_\phi(N).$$
Using these equations, we obtain $V(N) = V^* \exp\left(-2 \int_{N_c}^{N} \epsilon \, dN'\right)$ and $V_\phi(N) = V_\phi^* \exp\left(-\int_{N_c}^{N} \eta \, dN'\right)$. Then the power spectrum evaluated at the horizon crossing time becomes

$$P_\zeta = \left(\frac{V_\phi}{V_\phi^*}\right) \left(\frac{H_*}{2\pi}\right)^2 = \left(\frac{H_*^2}{2\pi \phi_*}\right)^2.$$  

This is consistent with a standard formula.

In the single-field slow-roll case, the non-linear parameter, $f_{NL}$, is simply given by

$$-\frac{6}{5} f_{NL} = \frac{N_{\phi\phi}}{N^2 \phi},$$

where $N_\phi = \partial N/\partial \phi$. Since the e-folding number is given by

$$N = -\int_{\phi_*}^{\phi} \frac{V}{V_\phi} \, d\phi,$$

we obtain

$$N_\phi = \frac{V}{V_\phi}, \quad N_{\phi\phi} = 1 - \frac{VV_{\phi\phi}}{V_\phi^2}.$$  

Thus we have a simple formula as [13]

$$-\frac{6}{5} f_{NL} = \frac{V_{\phi\phi}^2(N)}{V^2(N)} - \frac{V_{\phi\phi}(N)}{V(N)} + \frac{V_\phi}{V\Lambda} \int_0^N dN' Q(N') \Lambda(N')$$

$$= 2\epsilon(N) - \eta(N) + \int_0^N dN' \left[ \frac{d\eta}{dN'} - 2 \frac{d\epsilon}{dN'} \right]$$

$$= 2\epsilon_* - \eta_*.$$

This agrees with the formula given by equation (A.1).

**Appendix B. Constancy of $\delta N$**

In section 3.3, we have evaluated $\zeta(N_I)$, the curvature perturbation on a uniform density hypersurface evaluated at $N = N_I$. If $N(\phi, N_c) = \text{constant surface}$ is identical to $V$-constant surface at $N = N_I$, we have $\zeta(N_I) = \zeta(N_c)$. If it is always the case that $N$-constant surfaces in the configuration space are identical to $V$-constant ones around $\phi^I = \phi^I(N_I)$, we have $\zeta(N) = \zeta(N_c)$ for any $N$ close to $N_I$. Hence, $\zeta(N)$ becomes independent of $N$.

This can be directly verified as follows. In this case, the derivative of $\zeta(N_I)$ given in (29) with respect to $N_I$:

$$\dot{\zeta}(N) = \left(\dot{N}_I + N_J P^J_I\right) \left(\delta \phi^{I(1)} + \frac{1}{2} \delta \phi^{I(2)}\right) + \frac{1}{2} \left[ \dot{N}_{IJ} + Q^K_{IJ} N_K + N_{IK} P^K_J \right]$$

$$+ N_{JK} P^K_J \delta \phi^{I(1)} \delta \phi^{I(1)},$$

(B.1)
will be constant independent of $N$, where we have replaced $N_I$ with $N$. The change rate of $V$ must be constant on $N$-constant surface in the present case. This condition can be written as

$$\frac{d}{d\phi^j} \left( \frac{V'^2}{V} \right) \left( \delta^j_K - \frac{V'^2 V_K}{V'^2} \right) = 0,$$

which is further rewritten as

$$V_{IJ} V'^I V^J V_K - V'^2 V^I V_{IK} = 0. \quad \text{(B.2)}$$

Using the identity $N_I \dot{\phi}^I = -1$ and $N_I \propto V_I$ which immediately follows from the fact that $N$-constant surfaces and $V$-constant surfaces are identical, we find that

$$N_I = \left( \frac{V}{V'^2} \right) V_I.$$

Then, we have

$$\dot{N}_I + N_J P^J_I = \frac{2}{V'^2} \left( V_{JK} V'^J V^K V_I - V_I K V^K \right) = 0,$$

where we used the condition (B.2). Differentiation of the above equality gives

$$0 = \frac{d}{d\phi^K} \left( \dot{N}_I + N_J P^J_I \right)$$

$$= N_{IJK} \frac{d\phi^J}{dN} + N_{IJ} \frac{d}{d\phi^K} \dot{\phi}^J + N_J P^J_I + N_J Q^J_{IK}$$

$$= \left( \dot{N}_{IK} + N_{IJK} P^K_{J} + N_{IK} P^I_J + Q^J_{IK} N_J \right). \quad \text{(B.3)}$$

From these relations, we can explicitly see that $\zeta(N)$ is independent of $N$ when $N$-constant surfaces agree with $V$-constant surfaces. When this condition is satisfied, equation (30) is slightly simplified. By using equation (B.2), the first two terms in round brackets can be unified into one term.

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