Equidistribution of negative statistics and quotients of Coxeter groups of type $B$ and $D$

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Abstract

We generalize some identities and $q$-identities previously known for the symmetric group to Coxeter groups of type $B$ and $D$. The extended results include theorems of Foata and Schützenberger, Gessel, and Roselle on various distributions of inversion number, major index, and descent number. In order to show our results we provide characterizations of the systems of minimal coset representatives of Coxeter groups of type $B$ and $D$.

1 Introduction

A well known theorem of MacMahon [17] shows that the length function and the major index are equidistributed over the symmetric group $S_n$. We recall that the length of a permutation $\sigma \in S_n$ is given by the number of inversions, denoted $\text{inv}(\sigma) := |\{(i, j) \mid i < j, \sigma(i) > \sigma(j)\}|$, and the major index of $\sigma$ is the sum of all its descents. More precisely,

$$\text{maj}(\sigma) := \sum_{i \in \text{Des}(\sigma)} i,$$

where $\text{Des}(\sigma) := \{i \in [n - 1] \mid \sigma(i) > \sigma(i + 1)\}$. Foata gave a bijection proof of this equidistribution theorem in [9]. He studied further his bijection and together with Schützenberger derived the two following results [13]. The first one is a refinement of MacMahon’s theorem, asserting the equidistribution of major index and number of inversions over descent classes.

**Theorem 1.1** (Foata-Schützenberger). Let $M = \{m_1, \ldots, m_t\} \subseteq \{1, \ldots, n - 1\}$. Then

$$\sum_{\sigma \in S_n | \text{Des}(\sigma^{-1}) = M} q^{\text{maj}(\sigma)} = \sum_{\sigma \in S_n | \text{Des}(\sigma^{-1}) = M} q^{\text{inv}(\sigma)}$$

* Dedicated to the memory of my friend and colleague Giulio Minervini.
The second one concerns the symmetry of the distribution of the major index and the inversion number over the symmetric group.

**Theorem 1.2** (Foata-Schützenberger). The pairs of statistics \( (\text{maj}, \text{inv}) \) and \( (\text{inv}, \text{maj}) \) have the same distribution on \( S_n \), namely

\[
S_n(t, q) := \sum_{\sigma \in S_n} t^{\text{maj}(\sigma)} q^{\text{inv}(\sigma)} = \sum_{\sigma \in S_n} t^{\text{inv}(\sigma)} q^{\text{maj}(\sigma)}.
\]

Theorem 1.2 has been extensively studied and generalized in many ways in the last three decades. Nevertheless, it still receives a lot of attention as shown by two recent papers of Hivert, Novelli, and Thibon [16], and of Adin, Brenti, and Roichman [3], where a multivariate generalization and an extension to the hyperoctahedral group of it are provided. In the latter paper, the problem of finding an analogue of this Foata-Schützenberger theorem for the Coxeter groups of type \( D \) is proposed [3, Problem 5.6].

In this paper we answer this question. Actually, we show that the negative major indices \( "nmaj" \), introduced in [2] on Coxeter groups of type \( B \), and \( "dmaj" \), defined in [6] on Coxeter groups of type \( D \), give generalizations of the first and second Foata-Schützenberger identities to \( B_n \) and \( D_n \). In our analysis we derive nice relations among quotients, or sets of minimal coset representatives, of \( B_n \) and \( D_n \) that are interesting in their own. Explicit maps between these quotients are shown, and used to compute some generating functions.

Finally, we use our results, and the negative descent numbers, to give generalizations to \( B_n \) and \( D_n \) of two classical \( q \)-identities. The first one, due to Roselle [19] (see also Rawlings [18, (2.4)]), is the generating function of the inversion number and major index over the symmetric group: for undefined notation see next section.

**Theorem 1.3** (Roselle).

\[
\sum_{n \geq 0} S_n(t, q) \frac{u^n}{(t; t)_n(q; q)_n} = \frac{1}{(u; t, q)_{\infty, \infty}},
\]

where \( S_0(t, q) = 1 \). The second one is the trivariate distribution of inversion number, major index, and number of descents, due to Gessel [15, Theorem 8.4], (see also [14]).

**Theorem 1.4** (Gessel).

\[
\sum_{n \geq 0} \frac{u^n}{[n]! q^n} \frac{\sum_{\sigma \in S_n} t^{\text{maj}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{des}(\sigma)}}{(t; q)_n^{n+1}} = \sum_{k \geq 0} p^k e[u]_q e[stu]_q \cdots e[t^k u]_q.
\]

## 2 Preliminaries and notation

In this section we give some definitions, notation and results that will be used in the rest of this work. For \( n \in \mathbb{N} \) we let \( [n] := \{1, 2, \ldots, n\} \) (where \( [0] := \emptyset \)). Given \( n, m \in \mathbb{Z}, \ n \leq m, \)
we let \([n, m] := \{n, n + 1, \ldots, m\}\). We let \(\mathbb{P} := \{1, 2, 3, \ldots\}\). The cardinality of a set \(A\) will be denoted by \(|A|\) and we let \(\binom{[n]}{S} := \{S \subseteq [n] \mid |S| = 2\}\). Given a set \(A\), we denote \(A_{<} := \{a_1, a_2, \ldots\}\) where \(a_1 < a_2 < \ldots\).

For our study we need notation for \(q\)-analogs of the factorial, binomial coefficient, and multinomial coefficient. These are defined by the following expressions

\[
[n]_q := 1 + q + q^2 + \ldots + q^{n-1}; \quad [n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q;
\]

\[
\binom{n}{m}_q := \frac{[n]!_q}{[m]!_q[n-m]!_q}; \quad \binom{n}{m_1, m_2, \ldots, m_t}_q := \frac{[n]!_q}{[m_1]!_q[m_2]!_q \cdots [m_t]!_q}.
\]

As usual we let

\[
(a; q)_0 := 1, \quad (a; q)_n := (1-a)(1-aq) \cdots (1-aq^{n-1}), \quad (a; q)_\infty := \prod_{n \geq 1} (1-aq^{n-1}).
\]

Moreover, for \(r, s \in \mathbb{N}\) we let

\[
(a; t, q)_{r,s} := \begin{cases} 1 & \text{if } r \text{ or } s \text{ are zero} \\ \prod_{1 \leq i \leq r} \prod_{1 \leq j \leq s} (1-at^{i-1}q^{j-1}) & \text{if } r, s \geq 1 \end{cases},
\]

and

\[
(a; t, q)_{\infty, \infty} := \prod_{i \geq 1} \prod_{j \geq 1} (1-at^{i-1}q^{j-1}).
\]

Finally,

\[
e[u]_q := \sum_{n \geq 0} \frac{u^n}{[n]!_q},
\]

is the \(q\)-analogue of the exponential function. The following \(q\)-binomial theorem is well known (see e.g. [4])

**Theorem 2.1.**

\[
(-xq; q)_n = \sum_{m=0}^{n} \binom{n}{m}_q q^{\binom{m+1}{2}} x^m.
\]

### 2.1 Coxeter groups of type \(B\) and \(D\)

We denote by \(B_n\) the group of all bijections \(\beta\) of the set \([-n, n] \setminus \{0\}\) onto itself such that

\[
\beta(-i) = -\beta(i)
\]

for all \(i \in [-n, n] \setminus \{0\}\), with composition as the group operation. This group is usually known as the group of signed permutations on \([n]\), or as the hyperoctahedral group of rank
n. If $\beta \in B_n$ then we write $\beta = [\beta(1), \ldots, \beta(n)]$ and we call this the window notation of $\beta$. As set of generators for $B_n$ we take $S_B := \{s_1^B, \ldots, s_{n-1}^B, s_0^B\}$ where for $i \in [n-1]$  
\[s_i^B := [1, \ldots, i-1, i+1, i, i+2, \ldots, n]\]
and $s_0^B := [-1, 2, \ldots, n]$. It is well known that $(B_n, S_B)$ is a Coxeter system of type $B$ (see e.g., [8, §8.1]).

![Figure 1: The Dynkin diagram of $B_n$](image)

To give an explicit combinatorial description of the length function $\ell_B$ of $B_n$ with respect to $S_B$, we need the following statistics. For $\beta \in B_n$ we let

\[N_1(\beta) := |\{i \in [n] \mid \beta(i) < 0\}|, \text{ and} \]
\[N_2(\beta) := \left| \left\{ \{i, j\} \in \left(\mathbb{Z}/2\right) \mid \beta(i) + \beta(j) < 0 \right\} \right|.\]

Note that, if $\beta \in B_n$,

\[N_1(\beta) + N_2(\beta) = -\sum_{\{i \in [n] \mid \beta(i) < 0\}} \beta(i). \quad (1)\]

For example if $\beta = [-3, 1, -6, 2, -4, -5] \in B_6$ then $N_1(\beta) = 4$, and $N_2(\beta) = 14$.

The following characterizations of the length function, and of the right descent set of $\beta \in B_n$ are well known [8].

**Proposition 2.2.** Let $\beta \in B_n$. Then

\[\ell_B(\beta) = \text{inv}(\beta) + N_1(\beta) + N_2(\beta), \text{ and} \]
\[\text{Des}_B(\beta) = \{i \in [0, n-1] \mid \beta(i) > \beta(i+1)\},\]

where $\beta(0) := 0$.

We denote by $D_n$ the subgroup of $B_n$ consisting of all the signed permutations having an even number of negative entries in their window notation, more precisely

\[D_n := \{\gamma \in B_n \mid N_1(\gamma) \equiv 0 \pmod{2}\}.\]

It is usually called the even-signed permutation group. As a set of generators for $D_n$ we take $S_D := \{s_0^D, s_1^D, \ldots, s_{n-1}^D\}$ where for $i \in [n-1]$

\[s_i^D := s_i^B\] and $s_0^D := [-2, -1, 3, \ldots, n]$.

There is a well known direct combinatorial way to compute the length, and the right descent set of $\gamma \in D_n$, (see, e.g., [8, §8.2]).
Proposition 2.3. Let $\gamma \in D_n$. Then

\[
\ell_D(\gamma) = \text{inv}(\gamma) + N_2(\gamma), \quad \text{and}
\]
\[
\text{Des}_D(\gamma) = \{ i \in [0, n-1] \mid \gamma(i) > \gamma(i+1) \},
\]

where $\gamma(0) := -\gamma(2)$.

2.2 Negative statistics

In [2], Adin, Brenti and Roichman introduced the following statistics on $B_n$. For $\beta \in B_n$ let

\[ \text{NDes}(\beta) := \text{Des}(\beta) \biguplus \{ -\beta(i) \mid \beta(i) < 0 \}, \]

and define

\[ \text{nmaj}(\beta) := \sum_{i \in \text{NDes}(\beta)} i, \quad \text{and} \quad \text{ndes}(\beta) := |\text{NDes}(\beta)|. \]

It follows from (1) that

\[
\text{nmaj}(\beta) = \text{maj}(\beta) + N_1(\beta) + N_2(\beta), \quad \text{and} \quad (2)
\]

\[
\text{ndes}(\beta) = \text{des}(\beta) + N_1(\beta). \quad (3)
\]

For the element $\beta = [-3, 1, -6, 2, -4, -5] \in B_6$, $\text{nmaj}(\beta) = 29$, and $\text{ndes}(\beta) = 7$.

In [6], a notion of descent multiset for $\gamma \in D_n$ is introduced

\[ \text{DDes}(\gamma) := \text{Des}(\gamma) \biguplus \{ -\gamma(i) - 1 \mid \gamma(i) < 0 \} \setminus \{0\}, \]

and the following statistics are defined

\[ \text{dmaj}(\gamma) := \sum_{i \in \text{DDes}(\gamma)} i, \quad \text{and} \quad \text{ddes}(\gamma) := |\text{DDes}(\gamma)|. \]

It easily follows that

\[
\text{dmaj}(\gamma) = \text{maj}(\gamma) + N_2(\gamma), \quad \text{and} \quad (4)
\]

\[
\text{ddes}(\gamma) = \text{des}(\gamma) + N_1(\gamma) + \epsilon(\gamma), \quad (5)
\]
where
\[ \epsilon(\gamma) := \begin{cases} -1 & \text{if } 1 \not\in \gamma([n]) \\ 0 & \text{if } 1 \in \gamma([n]) \end{cases}. \]  
(6)

For example if \( \gamma = [-4, 1, 3, -5, -2, -6] \in D_6 \) then \( \text{dmaj}(\gamma) = 21 \), and \( \text{ddes}(\gamma) = 5 \).

The statistics \( \text{nmaj} \) and \( \text{dmaj} \) are usually called negative major indices; \( \text{ndes} \) and \( \text{ddes} \) negative descent numbers for \( B_n \) and \( D_n \), respectively. The negative major indices are Mahonian statistics, namely they are equidistributed with the length over the group,

\[ \sum_{\beta \in B_n} q^{\text{nmaj}(\beta)} = \sum_{\beta \in B_n} q^{\ell_B(\gamma)}, \quad \text{and} \quad \sum_{\gamma \in D_n} q^{\text{dmaj}(\gamma)} = \sum_{\gamma \in D_n} q^{\ell_D(\gamma)}. \]

The pairs \( (\text{ndes}, \text{nmaj}) \) and \( (\text{ddes}, \text{dmaj}) \) give generalizations to \( B_n \) and \( D_n \) of a famous identity of Carlitz, see [2, Theorem 3.2], and [6, Theorem 3.4].

2.3 Quotients of Coxeter groups

To show some of the next results we will need of the following decomposition that comes from the general theory of Coxeter group. We refer the reader to [8] for any undefined notation.

Let \((W, S)\) be a Coxeter system, for \( J \subseteq S \) we let \( W_J \) be the parabolic subgroup of \( W \) generated by \( J \), and

\[ W^J := \{ w \in W \mid \ell(ws) > \ell(w) \text{ for all } s \in J \}, \]

the set of minimal left coset representatives of \( W_J \), or the (right) quotient. The quotient \( W^J \) is a poset according to the Bruhat order. The following is well known (see [8, \S 2.4]).

**Proposition 2.4.** Let \((W, S)\) be a Coxeter system, and let \( J \subseteq S \). Then:

i) Every \( w \in W \) has a unique factorization \( w = w^J w_J \) such that \( w^J \in W^J \) and \( w_J \in W_J \).

ii) For this factorization \( \ell(w) = \ell(w^J) + \ell(w_J) \).

As a first application of this decomposition to the groups \( B_n \) (and \( D_n \)), let us consider the parabolic subgroup generated by \( J := S_B \setminus \{s_0^B\} \). In this case, by looking at the Dynkin diagram in Figure 1 we obtain that \( B_J = S_n \). Moreover it is not hard to see that

\[ B^J := B^J_n = \{ u \in B_n \mid u(1) < u(2) < \ldots < u(n) \}. \]  
(7)

Hence from Proposition 2.4 we get

\[ B_n = \biguplus_{\sigma \in S_n} \{ u\sigma \mid u \in B^J \}, \]  
(8)
where \( \sqcup \) denotes disjoint union. Note that in the case \( D_n \), for \( J := S_D \setminus \{s_0^{B}\} \), a similar decomposition holds,

\[
D_n = \biguplus_{\sigma \in S_n} \{u\sigma \mid u \in D^J\},
\]

where once again \( D_J = S_n \), and \( D^J = \{u \in D_n \mid u(1) < u(2) < \ldots < u(n)\} \).

**Remark 2.5.** The construction or right quotient can be mirrored, by considering *left* descents. Let \( J \subseteq S \). A *left quotient* of \( W \) is defined by

\[
\mathcal{J}W := \{w \in W \mid \ell(sw) > \ell(w) \text{ for all } s \in J\}.
\]

Proposition 2.4 holds for left quotients too, but the factorization in i) becomes \( w = w_J \cdot \mathcal{J}w \), with \( \mathcal{J}w \in \mathcal{J}W \). Left and right quotients are isomorphic posets, by means of the inversion map. In the next section, we will work with subsets of \( B_n \) and \( D_n \) that are left quotients. They are called descent classes for reasons that will be immediately clear.

### 3 Combinatorial description of descent classes

Let us fix a subset of descents \( M := \{m_1, m_2, \ldots, m_t\}_< \subseteq [0, n - 1] \). The set

\[
B(M) := \{\beta \in B_n \mid \text{Des}_B(\beta^{-1}) \subseteq M\},
\]

is usually called a *B-descent class*. Note that this set is nothing but a left quotient of \( B_n \). More precisely, it is the one corresponding to the subset \( J = S \setminus \bar{M} \), where \( \bar{M} := \{s_i \mid i \in M\} \). The following result can be found in [3, Lemma 4.1].

**Lemma 3.1.** Let \( \beta \in B_n \), and \( M = \{m_1, \ldots, m_t\}_< \subseteq [0, n - 1] \). Let \( m_{t+1} := n \). Then \( \text{Des}_B(\beta^{-1}) \subseteq M \) if and only if there exist (unique) integers \( r_1, \ldots, r_t \) satisfying \( m_i \leq r_i \leq m_{i+1} \) for all \( i \), and such that \( \beta \) is a shuffle of the following increasing sequences:

\[
(1, 2, \ldots, m_1),
(-r_1, -r_1 + 1, \ldots, -(m_1 + 1)),
(r_1 + 1, r_1 + 2, \ldots, m_2),
\]

\[
\vdots
\]

\[
(-r_t, -r_t + 1, \ldots, -(m_t + 1)),
(r_t + 1, r_t + 2, \ldots, n).
\]

Some of these sequences may be empty, if \( r_i = m_i \) or \( r_i = m_{i+1} \) for some \( i \), or if \( m_i = 0 \).

The following one is an explicit description of *D-descent classes*

\[
D(M) := \{\gamma \in D_n \mid \text{Des}_D(\gamma^{-1}) \subseteq M\}.
\]
Lemma 3.2. Let $\gamma \in D_n$, and $M = \{m_1, \ldots, m_t\} \subseteq [0, n - 1]$. Let $m_{t+1} := n$. Then $\text{Des}_D(\gamma^{-1}) \subseteq M$ if and only if there exist (unique) integers $r_1, \ldots, r_t$ satisfying $m_i \leq r_i \leq m_{i+1}$ for all $i$, and such that $\gamma$ is a shuffle of the following increasing sequences. There are three cases, and six possible “blocks” of sequences.

1) If $0 \in M$: ($m_1 = 0$)

\[
\begin{align*}
(-r_1, -r_1 + 1, \ldots, -2, -1), \\
(r_1 + 1, r_1 + 2, \ldots, m_2), \\
&\vdots \\
&(-r_t, -r_t + 1, \ldots, -(m_t + 1)), \\
&\begin{cases} \\
(m_1, m_1) & \text{if } r_1 = 2 \\
(r_1 + 1, r_1 + 2, \ldots, m_2) & \text{if } r_1 = 1 \text{ and } m_1 = 2 \\
&\vdots \\
&(r_t + 1, r_t + 2, \ldots, n)
\end{cases}
\end{align*}
\]

with $\sum_{i=1}^{t} (r_i - m_i) \equiv 0 \pmod{2}$.

2) If $0, 1 \notin M$: (note $m_1 \geq 2$)

\[
\begin{align*}
(1, 2, \ldots, m_1), & \quad (-1, 2, \ldots, m_1) \\
(-r_1, -r_1 + 1, \ldots, -(m_1 + 1), & \quad (-r_1, -r_1 + 1, \ldots, -(m_1 + 1)) \\
(r_1 + 1, r_1 + 2, \ldots, m_2), & \quad (r_1 + 1, r_1 + 2, \ldots, m_2) \\
&\vdots \\
&(-r_t, -r_t + 1, \ldots, -(m_t + 1)), \\
&\begin{cases} \\
(m_1, m_1) & \text{if } r_1 = 2 \\
(r_1 + 1, r_1 + 2, \ldots, m_2) & \text{if } r_1 = 1 \text{ and } m_1 = 2 \\
&\vdots \\
&(r_t + 1, r_t + 2, \ldots, n)
\end{cases}
\end{align*}
\]

with $\sum_{i=1}^{t} (r_i - m_i) \equiv 0 \pmod{2}; \quad \sum_{i=1}^{t} (r_i - m_i) \equiv 1 \pmod{2}$.

3) If $0 \notin M$ and $1 \in M$: (note $m_2 \geq 2$, and $r_1 \geq 2$)

\[
\begin{align*}
(1) & \quad (-r_1, \ldots, -2, 1) \\
(2, 3, \ldots, m_2), & \quad (-1, 2, 3, \ldots, m_2) \\
(-r_2, -r_2 + 1, \ldots, -(m_2 + 1), & \quad (-r_2, -r_2 + 1, \ldots, -(m_2 + 1)) \\
(r_2 + 1, r_2 + 2, \ldots, m_3), & \quad (r_2 + 1, r_2 + 2, \ldots, m_3) \\
&\vdots \\
&(-r_t, -r_t + 1, \ldots, -(m_t + 1)), \\
&\begin{cases} \\
(m_1, m_1) & \text{if } r_1 = 2 \\
(r_1 + 1, r_1 + 2, \ldots, m_2) & \text{if } r_1 = 1 \text{ and } m_1 = 2 \\
&\vdots \\
&(r_t + 1, r_t + 2, \ldots, n)
\end{cases}
\end{align*}
\]

with $\sum_{i=2}^{t} (r_i - m_i) \equiv 0 \pmod{2}; \quad \sum_{i=2}^{t} (r_i - m_i) \equiv 1 \pmod{2}; \quad \sum_{i=1}^{t} (r_i - m_i) \equiv 0 \pmod{2}$.

Some of these sequences may be empty, if $r_i = m_i$ or $r_i = m_{i+1}$ for some $i$, or if $m_i = 0$. 

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Proof. The only difference with respect to the $B_n$ case is for the 0, 1 descents. They depend on the relative positions of $\pm 1$ and $\pm 2$ in the window notation of $\gamma$. The following are the $D$-descent classes of all elements of $B_2$. We have that

$$
\begin{align*}
D(\emptyset) &= \{[1, 2], [-1, 2]\} \\
D(\{0\}) &= \{[2, -1], [-2, -1]\} \\
D(\{1\}) &= \{[-2, 1], [2, 1]\} \\
D(\{0, 1\}) &= \{[1, -2], [-1, -2]\}.
\end{align*}
$$

From this, the parity conditions $\sum_{i=1}^{t} (r_i - m_i) \equiv 0 \text{ or } \equiv 1 \pmod{2}$, and Lemma 3.1 the result follows.

Remark 3.3. Let us fix a subset of descents $M := \{m_1, m_2, \ldots, m_t\}$. Consider the decompositions of $B_n$ and $D_n$ given by Proposition 2.4 by using left quotients. Recall that $|B_n| = 2^n n!$ and that $|D_n| = 2^{n-1} n!$. By looking at the Dynkin diagrams in Figure 1 and Figure 2, it is easy to derive the following equalities.

- If $0 \in M$, then $|B(M)| = 2 \cdot |D(M)|$;
- If $0, 1 \not\in M$, then $|B(M)| = |D(M)|$;
- If $0 \not\in M$, and $1 \in M$, then $|B(M)| = m_2 \cdot |D(M)|$.

Now we make explicit these equalities by showing relations between $D$ and $B$ left quotients.

**Proposition 3.4.** Let $0 \in M$. Then

i) $B(M)$ splits into the disjoint union

$$B(M) = D(M) \cup \bar{D}(M),$$

where $\bar{D}(M) := \{\bar{\gamma} = (-\gamma(1), \gamma(2), \ldots, \gamma(n)) \mid \gamma \in D(M)\} = \{\gamma \cdot s_0^B \mid \gamma \in D(M)\}$.

ii) Moreover

$$\sum_{\beta \in B(M)} q^\ell D(\beta) = 2 \cdot \sum_{\gamma \in D(M)} q^\ell D(\gamma).$$

Proof. Let $\gamma \in D(M)$. By Lemma 3.2 $\gamma$ is a shuffle of the sequences in [12], and so it can also be obtained as a shuffle of the sequences in [10]. Hence $\gamma \in B(M)$. Now, let us change the sign to the first entry of $\gamma$, by getting $\bar{\gamma}$. We are changing the sign of $-r_i$, or of $r_i + 1$ for $i \in [t]$, in one of the sequences in [12]. Note that this operation does not create a new B-descent for $\bar{\gamma}$. Hence $\bar{\gamma} \in B(M) \setminus D(M)$.

More precisely, $\bar{\gamma}$ can be obtained by shuffling the same sequences that give $\gamma$ where the twos involving $r_i$ are replaced either by $(-r_i + 1, \ldots, -(m_i + 1))$ and $(r_i, r_i + 1, \ldots, m_i)$, or by $(-r_i - 1, \ldots, -(m_i + 1))$ and
(r_i + 2, \ldots, m_i), depending if it is the sign of –r_i, or of r_i + 1, that changes. All those sequences belong to (10). So i) follows by Remark 3.3.

Now, it is easy to see that for all \( \gamma \in D(M) \), one has \( \ell_D(\gamma) = \ell_D(\bar{\gamma}) \). To see that, suppose \( \gamma(1) > 0 \). Then

\[
\text{inv}(\bar{\gamma}) = \text{inv}(\gamma) - (\gamma(1) - 1) \quad \text{and} \quad N_2(\bar{\gamma}) = N_2(\gamma) + (\gamma(1) - 1),
\]

and so the length \( \ell_D \) is stable. If \( \gamma(1) < 0 \) a similar computations holds, hence ii) follows.

Note that the two subsets \( D(M) \) and \( \bar{D}(M) \) are not isomorphic as posets, when they are considered as sub-posets of \( (B(M), \prec_B) \), where \( \prec_B \) denote the B-Bruhat order. An example is given for \( n = 3 \) and \( M = \{0, 2\} \).

**Proposition 3.5.** Let \( 0, 1 \not\in M \). Then

i) The map \( \varphi : B(M) \to D(M) \) defined by

\[
\beta \mapsto \begin{cases} 
\beta, & \text{if } \beta \in D_n; \\
{s_0^B} \cdot \beta, & \text{otherwise},
\end{cases}
\]

is a bijection.

ii) Moreover

\[
\sum_{\beta \in B(M)} q^{\ell_D(\beta)} = \sum_{\gamma \in D(M)} q^{\ell_D(\gamma)}. 
\]

**Proof.** Let \( \beta \in B(M) \), it is a shuffle of the sequences in \( (10) \). If \( \beta \in D_n \), then it is also a shuffle of the sequences in the first block of \( (13) \). Hence \( \beta \in D(M) \). Now suppose that \( \beta \not\in D_n \). Since \( 0 \not\in \text{Des}_B(\beta^{-1}) \), then \( 1 \in \beta[n] \). By multiplying on the left by \( s_0^B \), we change the sign of 1, and so the parity of \( \beta \). Hence \( s_0^B \cdot \beta \in D_n \). Actually, we obtain an element which is a shuffle of the sequences in second block of \( (13) \). From Remark 3.3 i) follows.

Since \( N_2(\gamma) = N_2(\varphi(\gamma)) \) and \( \text{inv}(\gamma) = \text{inv}(\varphi(\gamma)) \), one has \( \ell_D(\gamma) = \ell_D(\varphi(\gamma)) \), and so ii) follows.

The map \( \varphi \) is not a poset isomorphism between \( (B(M), \prec_B) \) and \( (D(M), \prec_D) \), where \( \prec_B \) and \( \prec_D \) denote the corresponding Bruhat orders. When \( n = 3 \) and \( M = \{2\} \), \( B(M) \) is a chain, while in \( D(M) \) there are two elements not comparable.

**Proposition 3.6.** Let \( 0 \not\in M \), and \( 1 \in M \). Then

i) \( B(M) \) splits as the disjoint union of the following \( m_2 \) subsets

\[
B(M) = D_1(M) \uplus D_{12}(M) \uplus \ldots \uplus D_{12\ldots m_2}(M).
\]

Each \( D_{1\ldots i}(M) \) is in bijection with \( D(M) \), and it is recursively defined as follows:
1) $D_1(M)$ is obtained by shuffling the sequences defining $D(M)$ where $-1$ (if present) is replaced with $1$.

2) For each $i \geq 2$, $D_{12 \ldots i}(M)$ is obtained by shuffling the sequences defining $D_{12 \ldots i-1}(M)$ where:

♣ $1$ and $\pm i$ are switched if they are in the same sequence;

♣ $i$ is replaced by $-i$, otherwise. This case happens when $1$ is at the beginning of a sequence of type $1,-(i-1),\ldots,-2$, and $i$ is the initial value of the sequence $(i, i+1, \ldots, m_2)$.

ii) Moreover

$$\sum_{\beta \in B(M)} q^{\ell_D(\beta)} = [m_2]_q \cdot \sum_{\gamma \in D(M)} q^{\ell_D(\gamma)}.$$ 

Before writing down the proof let us consider an example.

**Example 3.7.** Consider $n = 4$ and $M = \{1, 3\}$. Then $D(M)$ is given by the shuffles of the following blocks of increasing sequences (written in column).

$$D(M) = \left\{ \begin{array}{cccc}
(1) & (-1, 2, 3) & (-2, 1) & (3, 1, -2) \\
(2, 3); & (-4); & (3); & (4) \\
(4); & (4) \\
\end{array} \right\}$$

Then $B(M)$ splits as disjoint union of the following three subsets:

$$D_1(M) = \left\{ \begin{array}{cccc}
(1) & (1, 2, 3) & (-2, 1) & (-3, 2, 1) \\
(2, 3); & (-4); & (3); & (4) \\
(4); & (4) \\
\end{array} \right\}$$

$$D_{12}(M) = \left\{ \begin{array}{cccc}
(-2) & (2, 1, 3) & (1, -2) & (3, 1, -2) \\
(1, 3); & (-4); & (3); & (4) \\
(4); & (4) \\
\end{array} \right\}$$

$$D_{123}(M) = \left\{ \begin{array}{cccc}
(-2) & (2, 3, 1) & (-3, 2) & (1, -3, 2) \\
(3, 1); & (-4); & (1); & (4) \\
(4); & (4) \\
\end{array} \right\}.$$ 

**Proof.** The transformations defining $D_{1 \ldots i}(M)$ involve only the first two sequences of the three blocks of (14). It is easy to see that $D_{1 \ldots i}(M) \subseteq B(M)$ for all $i \in [m_2]$, and that $D_{1 \ldots i}(M)$ and $D_{1 \ldots j}(M)$ are disjoint if $i \neq j$. Hence the decomposition in $i$) follows from Remark 3.3.
Since changing \(-1\) into 1 in a signed permutation \(\gamma\) affects neither \(\text{inv}(\gamma)\) nor \(N_2(\gamma)\), it follows that

\[
\sum_{\gamma \in D(M)} q^{\ell_D(\gamma)} = \sum_{\gamma \in D_1(M)} q^{\ell_D(\gamma)}.
\]

Now let us show that for all \(i \geq 2\)

\[
\sum_{\gamma \in D_{1\ldots i-1}(M)} q^{\ell_D(\gamma)} = q \sum_{\gamma \in D_{1\ldots i-1}(M)} q^{\ell_D(\gamma)}.
\]

Let \(\gamma \in D_{1\ldots i-1}(M)\). Consider the block in [14] whose a particular shuffle gives \(\gamma\).

If 1 and \(\pm i\) are in the same sequence, it can be either of the form \((\ldots, 1, \ldots, m_2)\), or of the form \((-r_1, \ldots, -i, 1 \ldots, -2)\). Now consider the shuffle giving \(\gamma\), where 1 has been switched with \(\pm i\). We get a new element \(\tilde{\gamma} \in D_{1\ldots i}(M)\). It is clear that \(\tilde{\gamma}\) has one more inversion with respect to \(\gamma\), and so the \(D\)-length go up by 1. In fact, all other sequences in the block (whose shuffle gives \(\gamma\)) are made by elements that are either all bigger or all smaller of both 1 and \(\pm i\). Hence the difference between \(\text{inv}(\gamma)\) and \(\text{inv}(\tilde{\gamma})\) depends only on the relative positions of 1 and \(\pm i\) within the same sequence.

Suppose that 1 and \(i\) are not in the same sequence. This means that 1 is at the beginning of the sequence \((1, -(i-1), \ldots, -2)\) and \(i\) is at the beginning of the sequence \((i, i+1, \ldots, m_2)\). So \(\tilde{\gamma} \in D_{1\ldots i}(M)\), the element corresponding to \(\gamma\) after the switch, is obtained by shuffling a block that contains the following two sequences

\((-i, -(i-1), \ldots, -2)\) and \((1, i+1, \ldots, m_2)\).

Once again all other sequences of the block are made by elements that are either all smaller or bigger of both 1 and \(i\). The difference between the values of \(\text{inv}(\tilde{\gamma})\) and \(\text{inv}(\gamma)\) depends only on the relative positions of 1 and \(i\). Hence \(\tilde{\gamma}\) loses \(i-2\) inversions with respect to \(\gamma\) (the ones given by the 1 at the beginning of the sequence), and \(N_2(\tilde{\gamma}) = N_2(\gamma) + (i - 1)\) thanks to \(-i\). So \(\ell_D(\tilde{\gamma}) = \ell_D(\gamma) + 1\).

\section{Equidistribution over descent classes}

In this section we show generalizations of Theorem 1.1 to Coxeter groups of type \(B\) and \(D\). We need the following classical result; see [14, Theorem 3.1], and [20, Example 2.2.5] for a proof.

\textbf{Theorem 4.1.} Let \(n \in \mathbb{P}\) and \(M = \{m_1, m_2, \ldots, m_t\} < \subseteq [n-1]\). Then

\[
\sum_{\{\sigma \in S_n | \text{Des}(\sigma^{-1}) \subseteq M\}} q^{\text{maj}(\sigma)} = \sum_{\{\sigma \in S_n | \text{Des}(\sigma^{-1}) \subseteq M\}} q^{\text{inv}(\sigma)} = \left[\begin{array}{c} n \\ m_1, m_2 - m_1, \ldots, n - m_t \end{array}\right]_q.
\]
Proof. Let \( r \sum \) Lemma 4.4. Let \( \sum \) Proof. Let us denote by Sh introduced by Adin and Roichman in \[1\].
The second equality and the sum have been computed in \[3\]. The symbol \( \text{fmaj} \) denote the elements in the shuffled sequences. From this, and the definitions of \( \text{nmaj}(\beta) \) and of \( \ell_B(\beta) = \text{inv}(\beta) + N_1(\beta) + N_2(\beta) \), the first equality in \[13\] follows. The second equality and the sum have been computed in \[3\]. The symbol \( \text{fmaj} \) denote the flag-major index introduced by Adin and Roichman in \[1\].

By the Principle of Inclusion-Exclusion we obtain

\[
q^n \text{maj}(\beta) = \sum_{\beta \in \text{Sh}(r_1, \ldots, r_t)} \sum_{\beta \in \text{Sh}(r_1, \ldots, r_t)} q^{\text{maj}(\beta)} = \left[\begin{array}{c} n \\ m_1, m_2 - m_1, \ldots, n - m_t \end{array}\right] \cdot \prod_{i=m_1+1}^{n} (1 + q^i). \tag{15}
\]

In fact inversion number and major index of a shuffle depend only on the order of the elements in the shuffled sequences. From this, and the definitions of \( \text{nmaj}(\beta) = \text{maj}(\beta) + N_1(\beta) + N_2(\beta) \) and of \( \ell_B(\beta) = \text{inv}(\beta) + N_1(\beta) + N_2(\beta) \), the first equality in \[13\] follows. The second equality and the sum have been computed in \[3\]. The symbol \( \text{fmaj} \) denote the flag-major index introduced by Adin and Roichman in \[1\].

By the Principle of Inclusion-Exclusion we obtain

\[
\sum_{\{\beta \in B_n | \text{Des}_B(\beta^{-1}) \subseteq M\}} q^{\text{nmaj}(\beta)} = \sum_{\{\beta \in B_n | \text{Des}_B(\beta^{-1}) = M\}} q^{\ell_B(\beta)} = \sum_{\{\beta \in B_n | \text{Des}_B(\beta^{-1}) = M\}} q^{\text{fmaj}(\beta)}.
\]

The following lemma will be useful in the computation of our main result Theorem 4.5.

Lemma 4.4. Let \( n \in \mathbb{P} \) and \( M = \{m_1, m_2, \ldots, m_t\} \subseteq [0, n-1] \). Then

\[
\sum_{\{\beta \in B_n | \text{Des}_B(\beta^{-1}) \subseteq M\}} q^{\ell_D(\beta)} = \left[\begin{array}{c} n \\ m_1, m_2 - m_1, \ldots, n - m_t \end{array}\right] \cdot \prod_{i=m_1+1}^{n} (1 + q^i).
\]

Proof. Let \( \beta \in B(M) \). Recall that \( \ell_D(\beta) = \ell_B(\beta) - N_1(\beta) \), and that \( \ell_B(\beta) = \text{inv}(\beta) + \sum_{\beta(i) < 0} |\beta(i)| \). Note that \( \beta(i) < 0 \) if and only if there exists a \( j \) such that \( m_j + 1 \leq |\beta(i)| \leq r_j \). Therefore

\[
\sum_{\beta(i) < 0} |\beta(i)| = \sum_{i=1}^{t} (m_i + 1) + \ldots + r_i
\]

\[
= \sum_{i=1}^{t} \left[ (r_i - m_i)m_i + \frac{(r_i - m_i)(r_i - m_i + 1)}{2} \right]
\]

\[
= \sum_{i=1}^{t} \frac{1}{2}(r_i - m_i)(r_i + m_i + 1).
\]
Moreover \( N_1(\beta) = \sum_{i=1}^{t} (r_i - m_i) \), and so

\[
\ell_D(\beta) = \text{inv}(\beta) + \sum_{i=1}^{t} \frac{1}{2} (r_i - m_i)(r_i + m_i + 1) - (r_i - m_i)
\]

\[
= \text{inv}(\beta) + \sum_{i=1}^{t} \left( \frac{r_i - m_i + 1}{2} \right) + (r_i - m_i)(m_i - 1)
\]

Hence by (16)

\[
\sum_{\beta \in B(M)} q^{\ell_D(\beta)} = \sum_{r_1, \ldots, r_t} \sum_{\beta \in \text{Sh}(r_1, \ldots, r_t)} q^{\text{inv}(\beta)} q^{\sum_{i=1}^{t} \left( \frac{r_i - m_i + 1}{2} \right) + (r_i - m_i)}
\]

\[
= \sum_{r_1, \ldots, r_t} \left[ m_1, m_2 - m_1, \ldots, n - r_t \right] \cdot \prod_{i=1}^{m_{i+1}} \sum_{r_i = m_i}^{n} \left[ r_i - m_i \right] \cdot q^{\sum_{i=1}^{t} \left( \frac{r_i - m_i + 1}{2} \right) + (r_i - m_i)(m_i - 1)}
\]

\[
= \left[ m_1, m_2 - m_1, \ldots, m_{i+1} \right] \cdot \prod_{i=1}^{m_{i+1}-1} \prod_{j=m_i}^{n} (1 + q^j)
\]

\[
(17)
\]

where the sum runs over \( m_i \leq r_i \leq m_{i+1} \), and (17) is obtained by applying the \( q \)-binomial Theorem 2.1 with \( x = q^{(m_i-1)} \).

\[ \square \]

**Theorem 4.5.** Let \( n \in \mathbb{P} \) and \( M = \{ m_1, m_2, \ldots, m_t \} < [0, n - 1] \). Then

\[
\sum_{\gamma \in D(M)} q^{\text{dmax}(\gamma)} = \sum_{\gamma \in D(M)} q^{\ell_D(\gamma)}
\]

\[
= \left\{ \begin{array}{ll}
\left[ m_1, m_2 - m_1, \ldots, n - m_t \right] \cdot \prod_{i=1}^{n-1} (1 + q^i) & \text{if } 0 \in M; \\
\left[ m_1, m_2 - m_1, \ldots, n - m_t \right] \cdot \prod_{i=m_1}^{n-1} (1 + q^i) & \text{if } 0, 1 \not\in M; \\
\left[ m_1, m_2 - m_1, \ldots, n - m_t \right] \cdot \prod_{i=m_1}^{n-1} \left( 1 + q^i \right) & \text{if } 0 \not\in M, \text{ and } 1 \in M.
\end{array} \right.
\]

**Proof.** Once again the first equality follows from (16) and the definitions of \( \text{dmax} \) and \( \ell_D \). The computation of the sum is now an easy application of Lemma 4.4 together with Propositions 3.4, 3.5, and 3.6.

As corollary we obtain the desired generalization.
Corollary 4.6.

\[ \sum_{\gamma \in D_n|\text{Des}_D(\gamma^{-1})=M} q^{d\text{maj}(\gamma)} = \sum_{\gamma \in D_n|\text{Des}_D(\gamma^{-1})=M} q^{\ell_D(\gamma)}. \]

**Remark 4.7.** If we replace \(\text{Des}_B\) with the usual descent set \(\text{Des}\), Corollary 4.3 is still valid. It easily follows from Theorem 4.2 since \(\text{Des}(\beta^{-1}) \subseteq M\) if and only of \(\text{Des}_B(\beta^{-1}) \subseteq M \cup \{0\}\). Analogously, by replacing \(\text{Des}_D\) with \(\text{Des}\), Corollary 4.6 holds for the Coxeter group of type \(D\).

The two corollaries are not true if as descent set one choose \(N\text{Des}\) for \(B_n\) and \(D\text{Des}\) for \(D_n\).

5 Symmetry of the joint distribution

In this section we find generalizations of Foata-Schützenberger Theorem 1.2, Roselle Theorem 1.3 and Gessel Theorem 1.4.

The following is an easy computation.

**Lemma 5.1.** Let \(n \in \mathbb{P}\). Then

\[ \sum_{u \in B^J} p^{N_1(u)} q^{N_1(u)+N_2(u)} = \sum_{S \subseteq [n]} p^{|S|} q^{\sum_{i \in S} i} = \prod_{i=1}^{n} (1 + pq^i) = (-pq;q)_n. \]

Moreover

\[ \sum_{u \in D^J} p^{N_1(u)+\epsilon(u)} q^{N_2(u)} = \sum_{S \subseteq [n-1]} p^{|S|} q^{\sum_{i \in S} i} = \prod_{i=1}^{n-1} (1 + pq^i) = (-pq;q)_{n-1}. \]

**Proposition 5.2.** The distribution of \((n\text{maj}, \ell_B)\) over \(B_n\) is symmetric, namely

\[ B_n(t,q) := \sum_{\beta \in B_n} t^{n\text{maj}(\beta)} q^{\ell_B(\beta)} = \sum_{\beta \in B_n} q^{|\beta|} t^{n\text{maj}(\beta)}. \]

**Proof.** Let consider the decomposition (8) of \(B_n\). Let \(u \in B^J\) (or \(D^J\)) and \(\sigma \in S_n\). Then the following equalities hold

\[ \text{maj}(u\sigma) = \text{maj}(\sigma) \text{ and } \text{inv}(u\sigma) = \text{inv}(u). \]

Moreover

\[ N_1(u\sigma) = N_1(u) \text{ and } N_2(u\sigma) = N_2(u). \]
Then from Theorem 1.2 it follows
\[ \sum_{\beta \in B_n} t^{\ell_B(\beta)} q^{\nmaj(\gamma)} = \sum_{u \in B^J} \sum_{\sigma \in S_n} t^{\inv(\sigma)+N_1(\sigma)+N_2(\sigma)} q^{\maj(\sigma)+N_1(\sigma)+N_2(\sigma)} \]
\[ = \sum_{u \in B^J} t^{N_1(u)+N_2(u)} q^{N_1(u)+N_2(u)} \sum_{\sigma \in S_n} t^{\inv(\sigma)} q^{\maj(\sigma)} \]
\[ = \sum_{u \in B^J} t^{N_1(u)+N_2(u)} q^{N_1(u)+N_2(u)} \sum_{\sigma \in S_n} t^{\maj(\sigma)} q^{\inv(\sigma)} \]
\[ = \sum_{u \in B^J} \sum_{\sigma \in S_n} t^{\maj(\sigma)+N_1(\sigma)+N_2(\sigma)} q^{\inv(\sigma)+N_1(\sigma)+N_2(\sigma)} \]
\[ = \sum_{\beta \in B_n} t^{\maj(\beta)} q^{\ell_B(\beta)}. \]

The analogous result holds for \( D_n \). The proof is very similar to that of \( B_n \) and is left to the reader.

**Proposition 5.3.** The pair of statistics \((\dmaj, \ell_D)\) is symmetric, namely
\[ D_n(t, q) := \sum_{\gamma \in D_n} t^{\dmaj(\gamma)} q^{\ell_D(\gamma)} = \sum_{\gamma \in D_n} t^{\ell_D(\gamma)} q^{\dmaj(\gamma)}. \]

Note that, the flag-major index and the \( D\)-major index \([7]\) do not share with \( \nmaj \) and \( \dmaj \) this symmetric distribution property.

The following identities are generalizations of Theorem 1.3 of Roselle to \( B_n \) and \( D_n \). They easily follow from the proof of Proposition 5.2, Lemma 5.1 and from Theorem 1.3.

**Proposition 5.4** (Roselle Identities for \( B_n \) and \( D_n \)).
\[ \sum_{n \geq 0} B_n(t, q) \frac{u^n}{(t; t)_n (q; q)_n (-qt; qt)_n} = \frac{1}{(u; t, q)_{\infty, \infty}}, \quad (B_0(t, q) := 0); \]
\[ 1 + \sum_{n \geq 1} D_n(t, q) \frac{u^n}{(t; t)_n (q; q)_n (-qt; qt)_{n-1}} = \frac{1}{(u; t, q)_{\infty, \infty}}. \]

Similarly the following identities, which generalize Gessel formula, follow from the proof of Proposition 5.2, Lemma 5.1 and Theorem 1.3.

**Proposition 5.5** (Gessel Identities for \( B_n \) and \( D_n \)).
\[ \sum_{n \geq 0} \frac{u^n}{n!} \sum_{\beta \in B_n} t^{\maj(\sigma)} q^{\ell_B(\beta)} p^{\des(\beta)} \]
\[ = \sum_{k \geq 0} p^k e[u] q e[tu] \cdots e[t^ku] q; \]
\[ \frac{1}{1-t} + \sum_{n \geq 1} \frac{u^n}{n!} \sum_{\gamma \in D_n} t^{\maj(\gamma)} q^{\ell_D(\gamma)} p^{\des(\gamma)} \]
\[ = \sum_{k \geq 0} p^k e[u] q e[tu] \cdots e[t^ku] q. \]
6 Concluding remarks

As we mentioned along the paper, there exists another family of statistics, the flag-statistics, defined on Coxeter groups of type $B$, $D$ (see [1] and [7]), and more generally on complex reflection groups [5]. Several generating functions involving flag-statistics have already been computed. In particular, we refer to the series of papers of Foata and Han [10] [11] [12], for a complete overview on the argument.

We remark that among the series computed, none involve a combination of flag-statistics and length. This is why we conclude the paper with the following interesting proposal.

**Problem 6.1.** What kind of identities, generalizing the ones of Roselle and Gessel, might be obtained by using flag-statistics?

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