A round trip from Caldirola to Bateman systems

J. Guerrero¹,², F.F. López-Ruiz², V. Aldaya² and F. Cossío²
¹Departamento de Matemática Aplicada, Campus de Espinardo, Universidad de Murcia, 30100 Murcia, Spain
²Instituto de Astrofísica de Andalucía, CSIC, Apartado Postal 3004, 18080 Granada, Spain
E-mail: juguerre@um.es

Abstract. For the quantum Caldirola-Kanai Hamiltonian, describing a quantum damped harmonic oscillator, a couple of constant of motion operators generating the Heisenberg algebra can be found. The inclusion in this algebra, in a unitary manner, of the standard time evolution generator $i\hbar \frac{\partial}{\partial t}$, which is not a constant of motion, requires a non-trivial extension of this basic algebra and the physical system itself, which now includes a new dual particle. This enlarged algebra, when exponentiated, leads to a group, named the Bateman group, which admits unitary representations with support in the Hilbert space of functions satisfying the Schrödinger equation associated with the quantum Bateman Hamiltonian, either as a second order differential operator as well as a first order one. The classical Bateman Hamiltonian describes a dual system of a damped (losing energy) particle and a dual (gaining energy) particle. The classical Bateman system has a solution submanifold containing the trajectories of the original system as a submanifold. When restricted to this submanifold, the Bateman dual classical Hamiltonian leads to the Caldirola-Kanai Hamiltonian for a single damped particle. This construction can also be done at the quantum level, and the Caldirola-Kanai Hamiltonian operator can be derived from the Bateman Hamiltonian operator when appropriate constraints are imposed.

1. Introduction

The interest in dissipative systems at the quantum level has remained constant since the early days of Quantum Mechanics. The difficulties in describing damping, which intuitively could be understood as a mesoscopic property, within the fundamental quantum framework, have motivated a huge amount of papers.

Applications of quantum dissipation abound. For example, in quantum optics, where the quantum theory of lasers and masers makes use of models including damping [1], or in the study of decoherence phenomena [2]. Some authors have modeled dissipation by means of the theory of open systems or the thermal bath approach, in which a damped system is considered to be a subsystem of a bigger one with infinite degrees of freedom [3][2]. However, damped systems are interesting in themselves as fundamental ones. In particular, the quantum damped harmonic oscillator, frequently described by the Caldirola-Kanai equation [1][6], has attracted much attention, as it could be considered one of the simplest and paradigmatic examples of dissipative system.

The description of the quantum damped harmonic oscillator by the Caldirola-Kanai model, which includes a time-dependent Hamiltonian, has been considered to have some flaws. For instance, uncertainty relations are not preserved under time evolution and could eventually...
be violated [6, 7]. Many considerations were made in this direction. For example, Dekker in [8] introduced complex variables and noise operators to tackle the problem, claiming that no dynamical description in terms of a Schrödinger wave function can be expected to exist. [9] proposed a non-linear Schrödinger-Langevin wave equation as the starting point in formulating the quantum theory.

Despite the problems for the Caldirola-Kanai model, many developments went ahead. Coherent states were calculated in [10] by finding creation and annihilation operators, built out of operators which commute with the Schrödinger equation. The corresponding number operator turns out to be an auxiliary, conserved operator, obviously different from the time-dependent Hamiltonian. This paper also defined the so-called loss energy states for the damped harmonic oscillator. The famous report by Dekker [11] provides a historical overview of some relevant results.

The analysis of damping from the symmetry point of view has proved to be especially fruitful. In a purely classical context, the symmetries of the equation of the damped harmonic oscillator with time-dependent parameters were found in [12]. Two comprehensive articles, [13, 14], are of special interest. In those papers the authors find, for the damped harmonic oscillator, finite-dimensional point symmetry groups for the corresponding Lagrangian (the un-extended Schrödinger group [15]) and the equations of motion ($SL(3, \mathbb{R})$) respectively, and an infinite contact one for the set of trajectories of the classical equation. They singularize a “non-conventional” Hamiltonian from those generators of the symmetry, recovering some results from [10]. Then, they conclude that the damped harmonic oscillator should not be claimed to be dissipative at all at the quantum level, as this true, “non-conventional” Hamiltonian is conserved, and should be related to an oscillator with variable frequency.

Many papers related to the Caldirola-Kanai model keep appearing, showing that the debate about fundamental quantum damping is far from being closed. We can mention [16], where the driven damped harmonic oscillator is analyzed, or the review [17]. Even the possible choices of classical Poisson structures and Hamiltonians, or generalizations to the non-commutative plane, have deserved attention as recently as in [18] and [19], respectively.

In fact, in [20] we provided a neat framework to study this model, based on a quantum generalization of the Arnold transformation [21]. The integrals of motion and symmetries were identified and exploited to calculate wave functions, basic operators and the exact time evolution operator.

Besides the Caldirola-Kanai model, the Bateman’s dual system appears as an alternative description of dissipation in the damped harmonic oscillator. In his original paper [26], Bateman looked for a variational principle for equations of motion with a friction term linear in velocity, but he allowed the presence of extra equations. This trick effectively doubles the number of degrees of freedom, introducing a time-reversed version of the original damped harmonic oscillator, which acts as an energy reservoir and could be considered as an effective description of a thermal bath. The Hamiltonian that describes this system was rediscovered by Feschbach and Tikhochinsky [27, 28, 11] and the corresponding quantum theory was immediately analyzed.

Some issues regarding the Bateman’s system arose. The Hamiltonian presents a set of complex eigenvalues of the energy (see [30] and references therein), and the vacuum of the theory decays with time. This last feature was treated in [29], where Celeghini et al. suggested that the quantum theory of the dual system could find a more natural framework in quantum field theory. On the other hand, in [30] the generalized eigenvectors corresponding to the complex eigenvalues are interpreted as resonant states.

Bateman’s dual system is still frequently discussed [31]. Many authors have considered this model as a good starting point for the formulation of the quantum theory of dissipation. One of the aims of this paper will be to show that the study of the symmetries of the Caldirola-Kanai
model leads to the Bateman’s dual system, thus to be considered as a natural starting point for the study of quantum dissipation.

Historically, Caldirola and Kanai derived their Hamiltonian from the Bateman one by means of time-dependent canonical transformations. In this work we are going to proceed in the opposite direction, deriving Bateman Hamiltonian from Caldirola-Kanai one by purely symmetry considerations.

Finally, we shall return back to Caldirola-Kanai by imposing constraints.

2. Caldirola-Kanai system

The Lagrangian for the Caldirola-Kanai system reads

\[ L_{CK} = e^{\gamma t} \left( \frac{1}{2} m \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \right), \]

and from this the classical equation of motion are derived:

\[ \ddot{x} + \gamma \dot{x} + \omega^2 x = 0. \]

Two independent solutions, from which all solution can be obtained by linear combinations with appropriate dimensional coefficients, are

\[ u_1(t) = e^{-\frac{\gamma}{2} t} \sin \Omega t, \quad u_2(t) = e^{-\frac{\gamma}{2} t} \cos \Omega t, \]

where

\[ \Omega = \sqrt{\omega^2 - \frac{\gamma^2}{4}}. \]

The Caldirola-Kanai Hamiltonian is \([4, 5]\):

\[ H_{CK}(t) = \frac{p^2}{2m} e^{-\gamma t} + \frac{1}{2} m \omega^2 x^2 e^{-\gamma t}, \]

and from this the quantum Hamiltonian can be obtained via canonical quantization, providing the Schrödinger equation for the Caldirola-Kanai system:

\[ i\hbar \frac{\partial}{\partial t} \phi(x,t) = \hat{H}_{CK}(t) \phi(x,t) \]

\[ \equiv \left[ -\frac{\hbar^2}{2m} e^{-\gamma t} \frac{\partial^2}{\partial x^2} + \frac{1}{2} m \omega^2 x^2 e^{-\gamma t} \right] \phi(x,t). \]

The Caldirola-Kanai Hamiltonian has been widely used to describe quantum dissipative systems, although it has many detractors, arguing that there are serious inconsistencies like that uncertainty relations are violated when \( t \to \infty \). However, this inconsistency seems to be associated with a confusion between canonical momentum and "physical" momentum (see, for instance, \([32]\)).

Let \( \mathcal{H}_0 = L^2(\mathbb{R}) \). Then any solution of the Schrödinger equation can be written as

\[ \phi(x,t) = \hat{U}(t) \phi(x), \]

where \( \phi(x) \in \mathcal{H}_0 \) and \( \hat{U}(t) \equiv \hat{U}(t,t_0) \) is the unitary time evolution operator associated with the Hamiltonian \( \hat{H}_{CK}(t) \), and \( t_0 \) is arbitrary (for instance \( t_0 = 0 \)). Denote by \( \mathcal{H}_t \) the Hilbert space
built up with solutions of the Schrödinger equation at time $t$. Then $\mathcal{H}_t$ is isomorphic to $\mathcal{H}_0$, and $\mathcal{H}_t = \hat{U}(t)\mathcal{H}_0$.

Note that $\hat{U}(t)$ does not constitute a one parameter unitary group of operators, since $\hat{H}_{CK}(t)$ does not commute with itself at different times, $[\hat{H}_{CK}(t_1), \hat{H}_{CK}(t_2)] \neq 0$. This makes the computation of $\hat{U}(t)$ more involved, and time-ordered exponential expansion or Magnus expansion must be used to compute it. See, however, [20] for an exact computation of $\hat{U}(t)$ as a product of exponential operators.

It should be stressed that

$$\mathcal{H}_t \neq \hat{\mathcal{H}}_t \equiv \left\{ \phi(x,t) \text{such that } \int_{-\infty}^{+\infty} |\phi(x,t)|^2 dx < \infty \right\},$$

(8)

where $\hat{\mathcal{H}}_t$ is the standard Hilbert space in quantum mechanics, usually identified with $L^2(\mathbb{R})$. In fact $\mathcal{H}_t \subset \hat{\mathcal{H}}_t$, and $\mathcal{H}_t$ can be seen as the quantum counterpart of the classical solution manifold in symplectic mechanics, made up with the constants of motion.

Symmetry (self-adjoint) operators, i.e. constants or integral of motion, preserve $\mathcal{H}_t$, while generic (self-adjoint) operators do not, only preserve $\hat{\mathcal{H}}_t$. The basic quantum operators, preserving $\mathcal{H}_t$ at each $t$, are given by (see [20]):

$$\hat{p}(t) = -i\hbar e^{-\frac{\gamma}{2\Omega}}(\cos(\Omega t) + \frac{\gamma}{2\Omega} \sin(\Omega t)) \frac{\partial}{\partial x} + m e^{\frac{\gamma}{2\Omega}} \frac{\omega^2}{\Omega} \sin(\Omega t) x,$$

$$\hat{x}(t) = e^{\frac{\gamma}{2\Omega}}(\cos(\Omega t) - \frac{\gamma}{2\Omega} \sin(\Omega t)) x + i\hbar e^{-\frac{\gamma}{2\Omega}} \sin(\Omega t) \frac{\partial}{\partial x}. $$

(9)

These operators match those found in [10] looking for integrals of motion commuting with the Schrödinger equation.

Note that

$$\hat{p}(t) = \hat{U}(t) \hat{p} \hat{U}(t)^\dagger$$

$$\hat{x}(t) = \hat{U}(t) \hat{x} \hat{U}(t)^\dagger,$$

(10)

where $\hat{p} \equiv \hat{p}(0) = -i\hbar \frac{\partial}{\partial x}$ and $\hat{x} \equiv \hat{x}(0) = x$.

Neither $\hat{H}_{CK}(t)$ nor $i\hbar \frac{\partial}{\partial x}$ make sense as operators acting on $\mathcal{H}_t$, only $i\hbar \frac{\partial}{\partial x} - \hat{H}_{CK}(t) \equiv 0$ does, while $\hat{p}(t)$ and $\hat{x}(t)$ act properly on $\mathcal{H}_t$. Notwithstanding this, $\hat{H}_{CK}(t)$ and $i\hbar \frac{\partial}{\partial x}$ are well defined operators on $\hat{\mathcal{H}}_t$, but they are not identical. The basic operators $\hat{x}(t)$, $\hat{p}(t)$, and the identity operator $\hat{I}$, close the Heisenberg algebra. There are also other operators acting on $\mathcal{H}_t$, which are quadratic in the basic operators, that close the Schrödinger (or Weyl-symplectic) algebra (see [20]).

3. Including time evolution as a symmetry

The generator of time evolution, $i\hbar \frac{\partial}{\partial t}$, nor $\hat{H}_{CK}(t)$ close under commutation with $\hat{x}(t)$ and $\hat{p}(t)$. We wonder if it is possible to incorporate them into the basic Lie algebra of operators, trying to close an enlarged Lie algebra acting on the (possibly enlarged) Hilbert space $\mathcal{H}_t$. The answer to this question is positive, but it requires a delicate analysis. The resulting enlarged algebra includes $\hat{X} \equiv \hat{x}(t)$, $\hat{P} \equiv \hat{p}(t)$, $\hat{H} \equiv i\hbar \frac{\partial}{\partial t}$ and four more generators (plus $\hat{I}$), denoted by $\hat{Q}, \hat{\Pi}, \hat{G}_1$ and $\hat{G}_2$.

The operators $\hat{Q}$ and $\hat{\Pi}$ (plus $\hat{I}$) expand a Heisenberg algebra, and $\hat{H}, \hat{G}_1$ and $\hat{G}_2$ expand a 2-D affine algebra (with $\hat{H}$ acting as dilations). However, in this realization $\hat{Q}$ and $\hat{\Pi}$ are not basic (this can be seen as an anomaly), and $\hat{H}$ and $\hat{G}_1$ are basic, resulting in time being a canonical variable. Clearly, this is not satisfactory, and an alternative description should be looked for.
A detailed study of the (projective) representations of the enlarged (7+1) dimensional Lie algebra shows that there are three relevant kinds of representations, describing systems with different degrees of freedom:

- A generic family with 3 degrees of freedom: \((\hat{X}, \hat{P}), (\hat{Q}, \hat{\Pi})\) and \((\hat{H}, \hat{G}_1)\), time being a canonical variable.
- An anomalous family with 2 degrees of freedom: \((\hat{X}, \hat{P})\) and \((\hat{H}, \hat{G}_1)\), time being a canonical variable (the one already described).
- A family with 2 degrees of freedom: \((\hat{X}, \hat{P})\) and \((\hat{Q}, \hat{\Pi})\).

Clearly, the interesting case is the third one, since it contains two degrees of freedom and time is not a canonical variable. Its algebra is given by:

\[
\begin{align*}
[\hat{X}, \hat{P}] &= i\hbar \hat{I} \\
[\hat{X}, \hat{Q}] &= -\frac{i\hbar}{m} \hat{G}_2 \\
[\hat{Q}, \hat{P}] &= i\hbar \hat{G}_1 - i\hbar \gamma \hat{G}_2 \\
[\hat{H}, \hat{X}] &= \frac{i\hbar}{m} \hat{\Pi} \\
[\hat{H}, \hat{Q}] &= \frac{i\hbar}{m} (\hat{P} + 2\hat{\Pi}) \\
[\hat{H}, \hat{\Pi}] &= i\hbar \gamma \hat{\Pi} \\
[\hat{H}, \hat{G}_1] &= -i\hbar \gamma \hat{G}_1 + 2i\hbar \omega^2 \hat{G}_2 \\
[\hat{H}, \hat{G}_2] &= -2i\hbar \hat{G}_1 + i\hbar \gamma \hat{G}_2.
\end{align*}
\]

In this case the operators \(\hat{G}_1\) and \(\hat{G}_2\) are gauge, and therefore are represented trivially.

The effective dimension of the algebra is 5 + 1: \((\hat{X}, \hat{P}), (\hat{Q}, \hat{\Pi}), \hat{H}\) and \(\hat{I}\).

\[
\begin{align*}
[\hat{X}, \hat{P}] &= i\hbar \hat{I} \\
[\hat{X}, \hat{Q}] &= 0 \\
[\hat{Q}, \hat{P}] &= 0 \\
[\hat{H}, \hat{X}] &= \frac{i\hbar}{m} \hat{\Pi} \\
[\hat{H}, \hat{Q}] &= \frac{i\hbar}{m} (\hat{P} + 2\hat{\Pi}) \\
[\hat{H}, \hat{\Pi}] &= i\hbar \omega^2 (\hat{X} - 2\hat{Q}) \\
[\hat{H}, \hat{\Pi}] &= -i\hbar \gamma \hat{\Pi}.
\end{align*}
\]

Here \(\hat{H}\) is not a basic operator, and can be written in terms of the basic ones in an irreducible representation:

\[
\hat{H} = -\frac{1}{m} \hat{\Pi} \hat{P} - \frac{\gamma}{2} (\hat{Q} \hat{\Pi} + \hat{\Pi} \hat{Q}) - \frac{\hat{\Pi}^2}{m} + m\omega^2 \hat{X} \hat{Q} - m\omega^2 \hat{Q}^2.
\]  

The classical version of the Hamiltonian is:

\[
H = -\frac{1}{m} \Pi \dot{P} - \gamma \dot{Q} \Pi - \frac{\Pi^2}{m} + m\omega^2 \dot{X} \dot{Q} - m\omega^2 \dot{Q}^2.
\]
4. Bateman system
This classical Hamiltonian can be transformed, using the linear, constant, canonical transformation:

\[
X = \frac{m \omega^2 y - (p_y + m \frac{\gamma}{2} x) i \Omega}{m \omega \sqrt{-\gamma i \Omega}},
\]
\[
P = \frac{\omega (p_x - m \frac{\gamma}{2} y + m x i \Omega)}{\sqrt{-\gamma i \Omega}},
\]
\[
Q = \frac{m \omega^2 y - (p_y - m \frac{\gamma}{2} x) i \Omega}{m \omega \sqrt{-\gamma i \Omega}},
\]
\[
\Pi = -\frac{\omega (p_x + m \frac{\gamma}{2} y + m x i \Omega)}{\sqrt{-\gamma i \Omega}},
\]

(15)

in to the Bateman dual Hamiltonian

\[
H_B = \frac{p_x p_y}{m} + \frac{\gamma}{2} (y p_y - x p_x) + m \Omega^2 x y ,
\]

(16)

that describes a damped particle \((x, p_x)\) and its time reversal \((y, p_y)\):

\[
\ddot{x} + \gamma \dot{x} + \omega^2 x = 0, \quad \ddot{y} - \gamma \dot{y} + \omega^2 y = 0.
\]

(17)

The quantum Bateman Hamiltonian is:

\[
\hat{H}_B = \frac{\hat{p}_x \hat{p}_y}{m} + \frac{\gamma}{2} (\hat{y} \hat{p}_y - \hat{x} \hat{p}_x) + m \Omega^2 \hat{x} \hat{y} ,
\]

(18)

and the Schrödinger equation for the Bateman system is given by\[1\]

\[
i \hbar \frac{\partial \phi(x, y, t)}{\partial t} = \left[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x \partial y} - i \hbar \frac{\gamma}{2} \left( y \frac{\partial}{\partial y} - x \frac{\partial}{\partial x} \right) + m \Omega^2 x y \right] \phi(x, y, t).
\]

(19)

The Bateman system is conservative, so our objective of including time evolution among the symmetries has been accomplished. \(H_B\) closes a 5+1 dimensional algebra with \((\hat{x}, \hat{p}_x)\) and \((\hat{y}, \hat{p}_y)\):

\[
[\hat{x}, \hat{p}_x] = i \hbar \hat{I}, \quad [\hat{y}, \hat{p}_y] = i \hbar \hat{I}, \quad [\hat{x}, \hat{y}] = 0, \quad [\hat{x}, \hat{p}_y] = 0, \quad [\hat{p}_x, \hat{p}_y] = 0,
\]
\[
[\hat{H}_B, \hat{x}] = \frac{i \hbar}{m} (-\hat{p}_y + m \frac{\gamma}{2} x), \quad [\hat{H}_B, \hat{p}_x] = i \hbar (-\frac{\gamma}{2} \hat{p}_x + m \Omega^2 \hat{y}),
\]
\[
[\hat{H}_B, \hat{y}] = \frac{i \hbar}{m} (-\hat{p}_x - m \frac{\gamma}{2} y), \quad [\hat{H}_B, \hat{p}_y] = i \hbar (\frac{\gamma}{2} \hat{p}_y + m \Omega^2 \hat{x}).
\]

(20)

However, it has been argued that the quantum Bateman system possesses inconsistencies, like complex eigenvalues and non-normalizable eigenstates. But Chruściński & Jurkowski\[30\] showed that \(\hat{H}_B\) has real, continuous spectrum, and that the complex eigenvalues are associated with resonances, which in last instance are the responsible of dissipation.

\[1\] The Bateman system admits an equivalent description in terms of a real, first order, Schrödinger equation (see\[35\].
5. Turning back to Caldirola-Kanai

Historically, Bateman first derived $H_B$, and later Caldirola and Kanai obtained $H_{CK}$ using time-dependent canonical transformations. Here we have gone the opposite way, started from $H_{CK}$ and derived $H_B$ closing a finite Lie algebra. Now we wonder if we can do the way back to the Caldirola-Kanai system. The answer, again, is positive, and can be achieved by using constraints. To know how to proceed, let us analyse first the classical case.

Classically, Bateman system and a pair of dual Caldirola-Kanai systems share the same second order equations of motion. If we impose them to share the first-order, Hamilton equations, the following constraint must be satisfied:

\[
y = \frac{\omega^2}{\Omega^2} e^{\gamma t} x + \frac{\gamma}{2m\Omega^2} p_x \\
p_y = e^{\gamma t} p_x + m\frac{\gamma}{2} x.
\]  

These constraints, although time dependent, preserve the equations of motion since they are equivalent to a relation among initials constants:

\[
y_0 = \frac{\omega^2}{\Omega^2} x_0 + \frac{\gamma}{2m\Omega^2} p_{x0} \\
p_{y0} = p_{x0} + m\frac{\gamma}{2} x_0.
\]  

These constraints can be seen to be of second order type, besides being time-dependent, therefore care should be taken when imposing them: Dirac theory for constraints can be used or we can embed the constraints in a time-dependent canonical transformation before applying them.

But we are interested in the quantum derivation. Therefore we try to impose the operator constraints:

\[
\hat{y} - \frac{\omega^2}{\Omega^2} \hat{x} - \frac{\gamma}{2m\Omega^2} \hat{p}_x = 0 \\
\hat{p}_y - \hat{p}_x - m\frac{\gamma}{2} \hat{x} = 0,
\]  

but only one of them can be imposed, since the operators at the lhs of the equations canonically commute: they are of second order type. At the quantum level, only one of them can be imposed, therefore we must select one of them. If we impose the constraint,

\[
\hat{y} = \frac{\omega^2}{\Omega^2} \hat{x} + \frac{\gamma}{2m\Omega^2} \hat{p}_x,
\]  

the Hilbert space reduces to those functions verifying:

\[
\phi(x,y,t) = e^{\int e^{-\gamma t'} e^{\gamma t'} \frac{\omega^2}{\Omega^2} (x' - x) + 2\left(e^{\gamma t'} + e^{-\gamma t'} - 1\right) e^{\gamma t'} \frac{\omega^2}{\Omega^2} x'} \psi(x',t),
\]  

where $x' = x + \frac{\Omega^2}{2\gamma^2} y e^{-\gamma t} \mu(t)$, and $\mu(t) = (2 - \frac{\gamma}{\Omega} \cot(\Omega t))$. The Schrödinger equation for the Bateman system reduces to:

\[
i\hbar \frac{\partial \psi(x',t)}{\partial t} = \left[ -\frac{\Omega^2 \hbar^2}{2m \omega^2} e^{-\gamma t} \mu(t) \frac{\partial^2}{\partial x'^2} - \frac{1}{2} i \hbar x' \Omega \mu(t) \frac{\partial}{\partial x'} + i \hbar \frac{\Omega^2}{\gamma} (\mu(t) - 2) \right] \psi(x',t).
\]  

When constraints are imposed, not all the operators acting on the original Hilbert space preserve the constrained Hilbert space. The notion of “good” (usually denoted gauge-independent
in constrained gauge theories) operators as those preserving the constrained Hilbert space naturally emerges.

In most of the cases “good” operators are characterized as those commuting with the constraints (see [40] for a detailed account of quantum constraints in a group-theoretical setting and a more general characterization of “good” operators). In this case they are:

\[ \hat{p}_x + \frac{2m\omega^2}{\gamma} \hat{x} \quad \hat{p}_y - \frac{2m\Omega^2}{\gamma} \hat{\bar{x}}. \]  

(27)

Note that \( \hat{H}_B \) (nor \( \imath \hbar \frac{\partial}{\partial \tau} \)) is not among the “good” operators since it does not preserve the constrained Hilbert space. Therefore, time invariance is lost in the process of going from the Bateman system to the Caldirola-Kanai system due to the very nature of the constraints imposed.

Now let us perform the transformation

\[ \psi(x', t) = e^{-\imath \frac{m\omega^2}{\hbar} x'^2 f(t)} g(t) \chi(\kappa, \tau), \]

(28)

where

\[ f(t) = -\frac{e^{\imath \tau}}{4\Omega\mu(t)^2} (\{ - (\gamma(2 + \cos(2\Omega t)) + 2\Omega \sin(2\Omega t)) \tau' - \gamma \mu(t) \tau'(t)^2 + \mu(t) \tau''(t) \}) \]  

(29)

\[ g(t) = e^{-\frac{\imath \tau}{4}} \left( -\frac{\tau'(t)}{\Omega \sin^2(\Omega t) \mu(t)} \right)^{1/4} \]

(30)

\[ \kappa = x' e^{\imath \frac{\tau}{2} (t - \tau)} \omega \sqrt{\tau'(t) / \mu(t)} \]

(31)

\[ \tau(t) = \frac{1}{\Omega} \text{ArcTan} \left[ \frac{A\tau'^2}{\mu(t)^2} \right], \quad A \in \mathbb{R} - \{0\}. \]

(32)

The Schrödinger equation finally transforms into:

\[ \imath \hbar \frac{\partial}{\partial \tau} \chi(\kappa, \tau) = \left[ -\frac{\hbar^2}{2m} e^{-\gamma \tau} \frac{\partial^2}{\partial \kappa^2} + \frac{1}{2} m\omega^2 \kappa^2 e^{\gamma \tau} \right] \chi(\kappa, \tau), \]

(33)

which is the Caldirola-Kanai equation in the variables \((\kappa, \tau)\). Even more, the two independent operators \([27]\) preserving the constrained Hilbert space turn, under the previous transformation, to the basic operators for the Caldirola-Kanai system \(\hat{x}(t)\) and \(\hat{\tilde{p}}(t)\). Therefore, we have recovered completely the Caldirola-Kanai system from the Bateman system by imposing one constraint.

It should be stressed that \( \tau'(0) = 0 \), therefore the time transformation is singular at the origin and there are two disconnected regions, one with \( t > 0 \) and other with \( t < 0 \). It also turns out that \( \text{sign}(\tau) = \text{sign}(A) \), therefore choosing appropriately the sign of \( A \) in each case we can map \( t > 0 \) to \( \tau > 0 \) and \( t < 0 \) to \( \tau < 0 \), respectively.

This kind of behavior coincides with the results of other authors (see [30]) where, starting with the Bateman system, they obtain two subspaces \( S^\pm \) for which the restriction of the one parameter group of unitary time-evolution operators \( \hat{U}(t) = e^{-\imath \frac{t}{\hbar} H_B} \) produces two semigroups of operators, for \( t < 0 \) and \( t > 0 \).

Therefore, starting from the quantum, conservative, Bateman system we have arrived to the quantum, time-dependent, Caldirola-Kanai system. All the process we have performed can be schematically showed as:
\[ t \in \mathbb{R} \quad \text{or} \quad t \in \mathbb{R}^- \]