Momentum interferences of a freely expanding Bose-Einstein condensate in 1D due to interatomic interaction change

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The dynamics of Bose-Einstein condensates has been much studied, both experimentally and theoretically, for determining the properties of the condensate and its transport behavior in waveguides or free space with potential applications in nonlinear atom optics, atom chips and interferometry. Fully free (three dimensional) or dimensionally constrained expansions have in particular been subjected to close scrutiny to obtain, from time series of cloud images and the appropriate theoretical models, information on the condensate and/or its confining potentials [1]. In reduced dimensions, the expansions have also been examined to characterize and identify different dynamical regimes (the mean-field dominated Thomas-Fermi limit, quasi-condensates and the dilute Tonks-Girardeau gas of impenetrable bosons [2, 3]), or to investigate statistical behavior in a partial release of the condensate into a box of finite size [4]. An effectively one dimensional (1D) Bose gas may be realized experimentally by a tight confinement of the atomic cloud in two (radial) dimensions and a weak confinement in the axial direction so that radial motion is constrained to the ground transversal state. Expansions are quite generally described and observed in coordinate space but very interesting phenomena occur in momentum space [5, 6]. Also interferences, which show the wave nature of the condensate and may form the basis of metrological applications, are usually apparent in coordinate space, between independent condensates [7] or as a self-interference [8], but, as in our present case, they may be genuinely momentum space effects, possibly with an indirect and less obvious spatial manifestation. (Other striking example of quantum momentum-space interference phenomenon for an ordinary, non-condensate, one-particle wavefunction colliding with a barrier has been discussed recently [9].) The expansions may be manipulated in different ways, e.g. by controlling the time dependence and shape of the external trapping potentials or by varying the interatomic interaction using a magnetically tunable Feshbach resonance [10] or an optically induced Feshbach resonance [11].

In this paper we shall take advantage of these control possibilities and study a quantum interference effect originating from a change of the interatomic interaction strength. The preparation of the condensate may be carried out in a 1D harmonic trap with negligible interatomic interaction strength such that the ground state is approximately Gaussian. Removing the trap under these conditions would preserve the momentum distribution, but if the interatomic interaction is immediately increased as the trap is removed, the momentum distribution evolution changes dramatically and in a highly non-classical way: a time dependent interference occurs consisting of an orderly, one-by-one increase of the number of peaks in the momentum distribution, see Fig. 1 during the early stage of the 1D expansion, which is described accurately by the Thomas-Fermi (TF) model. In the same period of time of the snapshots shown, the spatial profile has barely evolved from the initial profile. Classical mechanics only explains the global broadening of the momentum distribution due to the release of (mean field) interatomic potential energy, notice the motion of the outer peaks in Fig. 1 but not the oscillatory pattern, which will require a quantum interference analysis. A way to make the momentum interference visible in coordinate space is to switch off the interatomic interaction again so that the subsequent free flight maps the momentum peaks into spatial ones.

Let us now discuss the details. Assume that an effectively 1D Bose-Einstein condensate is prepared in a harmonic trap. The condensate wave function is the ground state of the 1D (stationary) Gross-Pitaevskii equation. We shall assume first that the initial interatomic interaction is zero (this will be relaxed later on). Then the Gross-Pitaevskii equation becomes a linear Schrödinger equation so that the ground state condensate wave func-
a realistic switching procedure needing a finite time will be

Note that in the TF regime the spatial density remains

momentum space distribution changes substantially: more
and more peaks are created as time increases, see Fig. 4
with $\psi(t,v) = \sqrt{\frac{m}{2\pi\hbar}} \int dv \psi(t,v) \exp\left(-i\frac{m}{\hbar}x\right)$. The peak
creation can be also seen in Fig. 2 (solid lines), where the
velocities of the peaks versus time are plotted forming a
characteristic structure where bifurcations and creation
of a new central peak follow each other.

To understand this effect, we shall approximate Eq.

If we neglect the kinetic energy (TF regime) we get

The solution is $\psi_{TF}(t,v) = \psi_0(x) \exp\left(-\frac{i}{2} g_1 |\psi_0(x)|^2\right)$

or, in momentum space,

$$\psi_{TF}(t,v) = \sqrt{\frac{m}{2\pi\hbar}} \int dv \exp\left[-\frac{m\omega_x}{\hbar} x^2 - i\frac{vm}{\hbar} x\right].$$

Note that in the TF regime the spatial density remains
unchanged. The results for Eq. 4 are also plotted in
Figs. 1 and 2. There is a good agreement between the
exact result and TF, and both show the same interference
behavior. It is clear that the nonlinearity is playing a key
role in the effect.

We may compare the quantum dynamics with a similar
classical one. Let us assume an ensemble of classical
particles where the probability density of initial positions
and velocities is given by $\rho_0(x,v) = |\psi_0(x)|^2 |\psi_0(v)|^2$, with
$\psi_0(v)$ given by Eq. 1. The probability density is then evolved with the classical Liouville equation with the Hamiltonian $H = \frac{\hbar g_1}{2} \int dv' \rho_0(x,v')$, without
kinetic term in analogy to the quantum Hamiltonian of
the TF regime, see Eq. 3. The solution is given by

FIG. 1: Wave function in momentum space; exact result
$|\psi(t,v)|^2$ (lines), TF approximation $|\psi_{TF}(t,v)|^2$ (crosses),
classical result $P(t,v)$ (dashed lines); (a) $t=0$, (b) $t=0.2$ ms,
(c) $t=0.4$ ms, (d) $t=0.6$ ms.

FIG. 2: Velocities of the peaks of the wave function $|\psi(t,v)|^2$:
g_0 = 0 (solid lines), g_0 = 0.00234 cm/s (dashed lines), g_0 =
0.0234 cm/s (dotted lines); TF approximation $|\psi_{TF}(t,v)|^2$
circles). In this and similar figures we always apply a thresh-
old level such that the wave function is assumed to be zero if
$|\psi(x)|^2 < 0.001 \times \max_{x'} |\psi(x')|^2$. 

$\psi_0(x) = \sqrt{\frac{m\omega_x}{\pi\hbar}} \exp\left(-\frac{m\omega_x}{2\hbar} x^2\right), \tag{1}$

where $\omega_x$ is the axial (angular) frequency. For the rest
of the paper we choose $m =$ mass(Na) and $\omega_x = 5/$s. At
time $t = 0$ the confining trap is switched off and the inter-
atomic interaction is switched on immediately. (A more
realistic switching procedure needing a finite time will be
discussed below.) The time evolution is now described by
the time-dependent Gross-Pitaevskii equation,

$$i\hbar \frac{\partial \psi(t,x)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(t,x)}{\partial x^2} + \frac{\hbar}{2} g_1 |\psi(t,x)|^2 \psi(t,x), \tag{2}$$

where $g_1$ is the effective 1D coupling parameter. We
are examining short times $t < t_c := 1/(200 \omega_x) = 1$ ms
(the factor 1/200 is arbitrary, it should be true that$t \ll 1/\omega_x$) for which the absolute square of the Gaus-
sian in Eq. 1 under the free evolution (i.e. Eq. 2
with $g_1 = 0$) is nearly not changing in coordinate space.
Even with $g_1 = 234.4$ cm/s and for times $t < 1$ ms,
$|\psi(t,x)|^2$ is nearly not changing. In contrast, the mo-
ρ_t(x,v) = ρ₀(x,v + αt) with α = h/2m∂/∂x ∫ dv ρ₀(x,v).

P(t,v) := ∫ dx ρ_t(x,v) is also plotted in Fig. 3 The classical picture provides, approximately, the outer peaks moving outwards with time because of the release of potential energy, but not the central ones, which are therefore associated with quantum interference. Let us examine this interference.

Defining dimensionless quantities ζ = √mωₓℏx, τ = √mωₓℏg₁t, and κ = √m/ℏωₓv, we write Eq. (4) as

$$\tilde{\psi}_{TF}(\tau, \kappa) = \frac{1}{2\sqrt{\pi}m\omega_x \hbar} \exp \left( -\frac{\kappa^2}{2} - i\tau \left( \frac{1}{2\sqrt{\pi}}e^{-\zeta^2} + \frac{\kappa}{\tau} \zeta \right) \right).$$ (5)

An important characteristic time is the value τ₀ when the second maximum of $$|\tilde{\psi}_{TF}(\tau₀, \kappa = 0)|^2$$ in the TF approximation appears (the first maximum is at τ = 0), see Fig. 3. This sets the scale of the oscillations and the value is found numerically,

$$\tau₀ \approx 27.703 \quad \Rightarrow \quad t₀ \approx \frac{27.703}{g₁} \sqrt{\frac{\hbar}{m\omega_x}}.$$ This time t₀ should be much smaller than the critical time t_c when the kinetic energy starts to influence. Therefore we get a necessary condition for g₁,

$$t_c = \frac{1}{200 \omega_x} > t₀ \approx \frac{27.703}{g₁} \sqrt{\frac{\hbar}{m\omega_x}} \Rightarrow g₁ \geq 27.703 \times 200 \sqrt{\frac{\omega_x}{m}} \approx 65.12 \text{ cm/s}.$$ We will simplify further Eq. (5) by means of the stationary phase method. If κ/τ is constant and τ → ∞ then the main contribution to the integral (6) comes from the values ζ which fulfill

$$\frac{\partial}{\partial \zeta} \left( \frac{1}{2\sqrt{\pi}}e^{-\zeta^2} + \frac{\kappa}{\tau} \zeta \right) = 0 \Rightarrow -\zeta e^{-\zeta^2} + \frac{\kappa}{\tau} = 0.$$ (7)

If 0 < |κ/τ| < exp(−1/2)/√2π there are two solutions ζ₀, ζ₁ with 0 < |ζ₀| < 1/√2 < |ζ₁|. Physically these are two positions in which the force exerted on the atom is equal because the Gaussian profile of the potential changes from concave-down around the center, to concave-up in the tails. Thus these two positions contribute to the same momentum and the corresponding amplitudes will interfere quantum mechanically. The stationary phase approximation of Eq. (6) is

$$\tilde{\psi}_s(\tau, \kappa) = \frac{1}{\sqrt{\pi}m\omega_x \hbar} \exp \left( -\frac{\kappa^2}{2} - i\tau \left( \frac{1}{2\sqrt{\pi}}e^{-\zeta^2} \right) \right) \times \left\{ \frac{\exp \left[ -i\tau\left( \frac{\kappa^2}{2\sqrt{\pi}} + \frac{\kappa}{\tau} \zeta_0 \right) \right]}{\sqrt{1 - 2\zeta_0^2}} - i \frac{\exp \left[ -i\tau\left( \frac{\kappa^2}{2\sqrt{\pi}} + \frac{\kappa}{\tau} \zeta_1 \right) \right]}{\sqrt{2\zeta_1^2 - 1}} \right\}.$$ (6)

Fig. 4 shows also the peaks of $$|\tilde{\psi}_s(\tau, \kappa)|^2$$ resulting from using Eq. (6) in the range 0 < |κ/τ| < exp(−1/2)/√2π. Except for the outer peaks and the κ = 0 line, Eq. (6) gives the correct peak behavior. The opening of the cone of the effect can be approximated by the definition of a critical κᵣ(τ) or vₜ(τ) to make the interference possible,

$$\frac{\kappa_c}{\tau} = \exp \left( -1/2 \right) \frac{\sqrt{2\pi}}{\sqrt{2\zeta_0^2}} \Rightarrow \kappa_c(\tau) = \frac{\exp \left( -1/2 \right) \sqrt{2\zeta_0^2}}{\tau},$$

or vₜ(τ) = exp(−1/2)ωₓg₁τ/√2π.

The interference pattern in momentum space can be seen in coordinate space if the interaction is switched off at a given time tₐff. Then $$|\tilde{\psi}(v, t)|^2$$ is not changing and therefore the position and number of peaks is not changing. On the other hand, the different peaks separate in coordinate space. Therefore for different times tₐff a different peak pattern will appear in coordinate space (see Fig. 4). Only the central peak cannot be seen.
\(\Delta t\) even of the order of \(t_0\). For a time dependent coupling parameter given by

\[
g(t) = g_1 \times \begin{cases} 
0 & : t \leq 0 \text{ or } t \geq t_{\text{off}} \\
1 & : \Delta t \leq t \leq t_{\text{off}} - \Delta t \\
f(t) & : 0 < t < \Delta t \\
f(t_{\text{off}} - t) & : t_{\text{off}} - \Delta t < t < t_{\text{off}} 
\end{cases}, \quad (7)
\]

with \(f(t) = (t/\Delta t)^2(3-2t/\Delta t)\), the result for \(\Delta t = 0.1\) ms is shown in Fig. 5; the effect is not changing qualitatively, only the positions of the peaks are squeezed.

We shall also examine the stability with respect to a non-zero value of the coupling constant used to prepare the initial state, \(g_0\). The initial ground state is then no longer a Gaussian and can only be calculated numerically. The effect survives as long as \(g_0 \ll g_1\) but the interference pattern is again squeezed, see Fig. 2.

Summarizing we have examined the short-time behavior of the evolution of a Bose-Einstein condensate in 1D when the interatomic interaction is negligible for the preparation in the harmonic trap, and strongly increased when the potential trapping is removed. We have found a quantum interference effect in momentum space originating from the interatomic interaction change. The momentum distribution expands due to the release of mean field energy and the number of peaks increases with time because of the interference of two positions in coordinate space contributing to the same momentum or velocity. The effect is stable in a parameter range and could be observed with current technology.

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**References**

[1] W. Ketterle, D.S. Durfee, and D.M. Stamper-Kurn, In Bose-Einstein condensation in atomic gases, edited by M. Inguscio, S. Stringari and C.E. Wieman (IOS Press, Amsterdam, 1999) pp. 67-176.

[2] P. Öhberg and L. Santos, Phys. Rev. Lett. 89, 240402 (2002).

[3] S. Dettmer, D. Hellweg, P. Ryytty, J.J. Arlt, W. Ertmer, K. Sengstock, D.S. Petrov and G.V. Shlyapnikov, H. Kreutzmann, L. Santos, and M. Lewenstein, Phys. Rev. Lett. 87, 160406 (2001).

[4] P. Villain and M. Lewenstein, Phys. Rev. A 62, 043601 (2000).

[5] J. Stenger, S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. 82, 4569 (1999).

[6] L. Pitaevskii and S. Stringari, Phys. Rev. Lett. 83, 4237 (1999).

[7] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Science 275, 637 (1997).

[8] K. Bongs, S. Burger, G. Birkl, K. Sengstock, W. Ertmer, K. Rzazewski, A. Sanpera, and M. Lewenstein, Phys. Rev. Lett. 83, 3577 (1999).

[9] A. Pérez, S. Brouard, and J. G. Muga, Phys. Rev. A 64, 012710 (2001).

[10] S. Inouye, M.R. Andrews, J. Stenger, H.-J. Miesner, D.M. Stamper-Kurn, and W. Ketterle, Nature 392, 151 (1998).

[11] M. Theis, G. Thalhammer, K. Winkler, M. Hellwig, G. Ruff, G. Grimm, and J. Hecker Denschlag, Phys. Rev. Lett. 93, 123001 (2004).