Analysis of weighted $\ell_1$-minimization for model-based compressed sensing

Sidhant Misra and Pablo A. Parrilo

Abstract

Model-based compressed sensing refers to compressed sensing with extra structure about the underlying sparse signal known a priori. Recent work has demonstrated that both for deterministic and probabilistic models imposed on the signal, this extra information can be successfully exploited to enhance recovery performance. In particular, weighted $\ell_1$-minimization with suitable choice of weights has been shown to improve performance at least for a simple class of probabilistic models. In this paper, we consider a more general and natural class of probabilistic models where the underlying probabilities associated with the indices of the sparse signal have a continuously varying nature. We prove that when the measurements are obtained using a matrix with i.i.d Gaussian entries, weighted $\ell_1$-minimization with weights that have a similar continuously varying nature successfully recovers the sparse signal from its measurements with overwhelming probability. It is known that standard $\ell_1$-minimization with uniform weights can recover sparse signals up to a known sparsity level, or expected sparsity level in case of a probabilistic signal model. With suitable choice of weights which are chosen based on our signal model, we show that weighted $\ell_1$-minimization can recover signals beyond the sparsity level achievable by standard $\ell_1$-minimization.

I. INTRODUCTION

Compressed Sensing has emerged as a modern alternative to traditional undersampling for compressible signals. Previously, the most common way to view recovery of signals from samples was based on the Nyquist criterion. According to the Nyquist criterion, a band-limited signal has to be sampled at a rate at least twice its bandwidth to allow exact recovery. In this case the band-limitedness of the signal is the extra known information that allows us to reconstruct the signal from compressed measurements. In compressed sensing the additional structure considered is that the signal is sparse with respect to a certain known basis. As opposed to sampling at Nyquist rate and subsequently compressing, we now obtain linear measurements of the signal and the compression and measurement steps are now combined by obtaining much smaller number of linear measurements than what would be in general required to reconstruct the signal from its measurements.

After fixing the basis with respect to which the signal is sparse, the process of obtaining the measurements can be written as $y = Ax$, where, $y \in \mathbb{R}^m$ is the vector of measurements, $x \in \mathbb{R}^n$ is the signal and $A \in \mathbb{R}^{m \times n}$ represents the $m$ linear functionals acting on the signal $x$. We call $A$ the measurement matrix. The signal $x$ is considered to have at most $k$ non-zero components and we are typically interested in the scenario where $k$ is much smaller than $n$. Compressed Sensing revolves around the fact that for sparse signals, the number of such linear measurements needed to reconstruct the signal can be significantly smaller than the ambient dimension of the signal itself. The reconstruction problem can be formulated as finding the sparsest solution $x$ satisfying the constraints imposed by the linear measurements $y$. This can

Laboratory of Information and Decision Systems, Department of Electrical Engineering and Computer Science, MIT, Cambridge, MA, 02139, email- sidhant@mit.edu, parrilo@mit.edu
be represented by

$$\min ||x||_0 \quad \text{subject to} \quad y = Ax.$$  

This problem is inherently combinatorial in nature and is in general a NP-hard problem. This is because for a certain value of the size of the support of x given by k, one needs to search through all \( \binom{n}{k} \) possible supports of the signal. Seminal work by Candés and Tao in [1] and Donoho in [2] show that under certain conditions on the measurement matrix A, \( \ell_1 \) norm minimization, which can be recast as a linear program, can recover the signal from its measurements. Additionally, a random matrix with i.i.d. Gaussian entries with mean zero satisfies the required condition with overwhelming probability. Linear programming is known to have polynomial time complexity and the above mentioned result tells us that for a large class of measurement matrices A we can solve an otherwise NP-hard combinatorial problem in polynomial time. Subsequently, iterative methods based on a greedy approach were formulated which recover sparse signals from their measurements by obtaining an increasingly accurate approximation to the actual signal in each iteration. Examples of these include CoSaMP [3] and IHT [4]. The compressed sensing framework has also been generalized to the problem of recovering low rank matrices from compressed linear measurements represented by linear matrix equations [5] [6].

Most of the earlier literature on Compressed Sensing focused on the case where the only constraints on the signal x are those imposed by its measurements y. On the other hand it is natural to consider the case where besides sparsity, there is certain additional information on the structure of the underlying signal. This is true in several applications, examples of which include encoding natural images (JPEG), MRI and DNA microarrays [7], [8]. This leads us to Model Based Compressed Sensing where the aim is to devise recovery methods specific to the signal model at hand. Furthermore, one would also want to quantify the possible benefits it has over the standard method (e.g. lesser number of required measurements for the same level of sparsity of the signal). This has been explored in some recent papers. The authors in [9] analyzed a deterministic signal model, were the support of the underlying signal is constrained to belong to a given known set. This defines a subset \( \mathcal{M} \) of the set of all k-sparse signals, which is now the set of allowable signals. This results in an additional constraint on the original reconstruction problem.

$$\min ||x||_0 \quad \text{subject to} \quad y = Ax, \quad x \in \mathcal{M}.$$  

It was shown that an intuitive modification to the CoSaMP or IHT method succeeds in suitably exploiting the information about the model. The key property defined in [1], known as the Restricted Isometry Property was adapted in [9] to a model based setting. With this, it was shown that results similar to [1] can be obtained for model-based signal recovery.

As opposed to this, a probabilistic model, i.e. a Bayesian setting, was considered in [10]. Under this model there are certain known probabilities associated with the components of the signal x. Specifically, \( p_i, i = 1, 2, \ldots, n \) with \( 0 \leq p_i \leq 1 \) are such that

$$P(x_i \text{ is non-zero}) = p_i, \quad i = 1, 2, \ldots, n.$$  

The deterministic version of the same model called the “nonuniform sparse model” was considered in [11]. For this, the use of weighted \( \ell_1 \)-minimization was suggested, given by

$$\min ||x||_{w,1} \quad \text{subject to} \quad y = Ax,$$
where \( \|x\|_{w,1} = \sum_{i=1}^{n} w_i |x_i| \) denotes the weighted \( \ell_1 \) norm of \( x \). The quantities \( w_i, i = 1, 2, \ldots, n \) are some positive scalars. Similar to \([2, 12]\), ideas based on high dimensional polytope geometry were used to provide sufficient conditions under which weighted \( \ell_1 \)-minimization recovers the sparse signal. This method was introduced earlier in \([13]\) where it was used to analyze the robustness of \( \ell_1 \)-minimization in recovering sparse signals from noisy measurements. The specific model considered in \([10]\) can be described as follows. Consider a partition of the indices 1 to \( n \) into two disjoint sets \( T_1 \) and \( T_2 \). Let \( p_i = P_1, i \in T_1 \) and \( p_i = P_2, i \in T_2 \). As a natural choice choose the weights in the weighted \( \ell_1 \)-minimization as \( w_i = W_1, i \in T_1 \) and \( w_i = W_2, i \in T_2 \). The main result of \([10]\) is that under certain conditions on \( \frac{W_2}{W_1}, P_1 \) and \( A \), weighted \( \ell_1 \)-minimization can recover a signal drawn from the above model with overwhelming probability.

In this paper, we extend this approach to a more general class of signal models. In our model, the probabilities \( p_i \) as described above are assumed to vary in a “continuous” manner across the indices. The specific way in which the probabilities vary is governed by a known shape function \( p(.) \) which is continuous, non-negative and non-increasing. For a signal drawn from this model, we propose the use of weighted \( \ell_1 \)-minimization to reconstruct it from its compressed linear measurements. In addition, to match the signal model, we propose that the weights be chosen according to a continuous, non-negative and non-decreasing shape function \( f(.) \). We prove that under certain conditions on \( p(.), f(.) \) and the measurement matrix \( A \), we can reconstruct the signal perfectly with overwhelming probability. We also suggest a way to select the shape function \( f(.) \) for the weights based on \( p(.) \). Our results are also applicable when the functions \( p(.) \) and \( f(.) \) are piece-wise continuous and the results in \([10]\) can be obtained from ours as a special case.

The rest of the paper is organized as follows. In Section II, we introduce the basic notation. We formulate the exact questions that we set out to answer in this paper and state our main theorem. In Section III we focus on how weighted \( \ell_1 \)-minimization behaves by restricting our attention to special class of signals that are particularly suitable for the ease of analysis. We later show that the methods generalize to other simple classes and is all we need to establish the main result of this paper. In Section IV we describe the key features of a typical signal drawn from our model. We also prove a suitable large deviation result for the probability that a signal drawn from our model lies outside this typical set. In Section V we provide numerical computations to demonstrate the results we derive. We also provide simulations to reinforce our results. We then conclude the paper with possibly interesting questions that may be explored in the future.

II. Problem Formulation

A. Notation and parameters

We denote scalars by lower case letters (e.g. \( c \)), vectors by bold lower case letters (e.g. \( x \)), matrices with bold upper case letters (e.g. \( A \)). Probability of an event \( E \) is denoted by \( \Pr(E) \). The \( i^{th} \) standard unit vector in \( \mathbb{R}^n \) is denoted by \( e_i = (0, 0, \ldots, 1, \ldots, 0)^T \), where the “1” is located in the \( i^{th} \) position.

The underlying sparse signal is represented by \( x \in \mathbb{R}^n \), the measurement matrix by \( A \in \mathbb{R}^{m \times n} \). The vector of observations is denoted by \( y \) and is obtained through linear measurements of \( x \) given by \( y = Ax \). Typically we would need \( n \) linear measurements to be able to recover the signal. The scalar \( \alpha = \frac{m}{n} \) determines how many measurements we have as a fraction of \( n \). We call this the compression ratio.

B. Model of the Sparse Signal

Let \( p : [0, 1] \to [0, 1] \) be a continuous monotonically non-increasing function. We call \( p(.) \) the probability shape function and the reason for this name will become clear from the description below. The support of
the signal $x$ is decided by the outcome of $n$ independent Bernoulli random variables. In particular, if $E_i$ denotes the event that $i \in \text{Supp}(x)$, then the events $E_i$ for $i = 1, \ldots, n$ are independent and $P(E_i) = p(\frac{i}{n})$. Although we assume throughout the paper that $p(.)$ is continuous, our result generalizes to piecewise continuous functions as well. Let us denote by $k$ the cardinality of $\text{Supp}(x)$. Note that under the above model $k$ is a random variable that can take any value from 0 to $n$. The expected value of $k$ is given by $\sum_{i=1}^{n} p \left( \frac{i}{n} \right)$. We denote by $\delta$ the expected fractional sparsity given by $\delta = \frac{1}{n} E[k] = \frac{1}{n} \sum_{i=1}^{n} p \left( \frac{i}{n} \right)$.

As is standard in Compressed Sensing literature we assume that the entries of the measurement matrix $A$ are i.i.d. Gaussian random variables with mean zero and variance one. The measurements are obtained as $y = Ax$.

### C. Weighted $\ell_1$-minimization

When the fractional sparsity $\delta$ is much smaller than one, a signal sampled from the model described above is sparse. Hence, it is possible to recover the signal from its measurements $y$ by $\ell_1$-minimization which is formulated as

$$\min \quad ||x||_1$$

subject to $\quad Ax = y$.

However this does not exploit the extra information available from the knowledge of the priors. Instead, we use weighted $\ell_1$-minimization to recover the sparse signal which is captured by the following optimization problem:

$$\min \quad ||x||_{w,1}$$

subject to $\quad Ax = y$.

where $w \in \mathbb{R}^n$ is a vector of positive weights and $||x||_{w,1} = \sum_{i=1}^{n} w_i |x_i|$ refers to the weighted $\ell_1$ norm of $x$, for a given weight vector $w$. The weight vector $w$ plays a central role in determining whether $\ell_1$ successfully recovers the sparse signal $x$. Intuitively, $w$ should be chosen in a certain way depending on $p(.)$ so as to obtain the best performance (although at this point we have not precisely defined the meaning of this). Keeping in mind the structure of $p_i$, for $i = 1, \ldots, n$, we suggest using weights $w_i$, for $i = 1, \ldots, n$ which have a similar structure. Formally, let $f : [0, 1] \rightarrow \mathbb{R}^+$ be a non-negative non-decreasing continuous function. Then we choose the weights as $w_i = f \left( \frac{i}{n} \right)$. We call $f(.)$ the weights shape function.

### D. Problem statement

In this paper we try to answer the following two questions:

- Given the problem parameters (size of the matrix defined by $m, n$), the functions $p(.)$ and $f(.)$, does weighted $\ell_1$-minimization in (1) recover the underlying sparse signal $x$ with high probability?
- Given a probability shape function $p(.)$, and a family of weight shape functions $\mathcal{W}$, how to choose a function $f(.) \in \mathcal{W}$ that has the best performance guarantees?

We give an answer in the affirmative to the first question, given that the functions $p(.)$ and $f(.)$ satisfy certain specified conditions. This is contained in the main result of this paper which is

**Theorem 1.** Let the probability shape function $p(.)$ and the weight shape function $f(.)$ be given. Let $E$ be the event that weighted $\ell_1$-minimization described in (1) fails to recover the correct sparse vector $x$. There exists a quantity $\psi_{\text{tot}}(p, f)$ which can be computed explicitly as described in Section 3 of such that whenever $\psi_{\text{tot}}(p, f) < 0$ the probability of failure $P(E)$ of weighted $\ell_1$-minimization decays exponentially
with respect to \( n \). More precisely, if for some \( \epsilon > 0 \) we have \( \psi_{\text{tol}}(p, f) \leq -\epsilon \), then there exists a constant \( c(\epsilon) > 0 \) such that for large enough \( n \), the probability of failure satisfies \( P(E) \leq e^{-nc(\epsilon)} \).

To answer the second question, we first define the measure of performance. For a given value of \( \alpha = \frac{\beta}{q} \), let \( \delta \) be the maximum value of \( \delta \) for which weighted \( \ell_1 \)-minimization can be guaranteed to succeed with high probability. For a given family of weight functions \( W \), the weight function \( f(.) \) that has the highest \( \delta \) is said to have the best performance. In this paper we will describe a method to choose a weight function in \( W \) and demonstrate the method in Section V for a linearly parameterized family.

### III. Analysis of Weighted \( \ell_1 \)-Minimization

In this section, we analyze the performance of weighted \( \ell_1 \)-minimization with weights specified by \( f(.) \) on a certain special class of sparse signals \( x \). As we will see in Section IV, we can easily generalize the analysis to the signals drawn from our signal model.

Recall that the failure event \( E \) from Theorem I is defined as the event that weighted \( \ell_1 \)-minimization fails to recover the correct sparse vector \( x \). We call the probability of this event \( P(E) \) as the probability of failure. Without loss of generality we can assume that \(|x|_{w,1} = 1\), that is \( x \) lies on a \( k - 1 \)-dimensional face of the weighted cross-polytope

\[
\mathcal{P} \triangleq \{ x \in \mathbb{R}^n \text{ s.t. } |x|_{w,1} \leq 1 \}
\]

which is the weighted \( \ell_1 \)-ball in \( n \) dimensions. The specific face of \( \mathcal{P} \) on which \( x \) lies is determined by the support of \( x \). The probability of failure \( P(E) \) can be written as

\[
P(E) = \sum_{F \in \mathcal{F}} P(x \in F)P(E|x \in F),
\]

where \( \mathcal{F} \) is the set of all faces of the polytope \( \mathcal{P} \). Then, as shown in [10], the event \( \{ E|x \in F \} \) is precisely the event that there exists a non-zero \( u \in \text{null}(A) \) such that \(|x + u|_{w,1} \leq |x|_{w,1} \) conditioned on \( \{ x \in F \} \). Also since \( A \) has i.i.d. Gaussian entries, sampling from the null space of \( A \) is equivalent to sampling a subspace uniformly from the Grassmann manifold \( Gr(n-m)(n) \). So conditioned on \( \{ x \in F \} \) the event \( \{ E|x \in F \} \) is same as the event that a uniformly chosen \((n-m)\)-dimensional subspace shifted to \( x \) intersects the polytope \( \mathcal{P} \) at a point other than \( x \). The probability of the above event is also called the complementary Grassmann Angle for the face \( F \) with respect to the polytope \( \mathcal{P} \) under the Grassmann manifold \( Gr(n-m)(n) \). Based on work by Santaló [14] and McMullen [15] the Complementary Grassmann Angle can be expressed explicitly as the sum of products of internal and external angles.

\[
P(E|x \in F) = 2 \sum_{s \geq 0} \sum_{G \in J(m+1+2s)} \beta(F,G)\gamma(G,P)
\]

where \( \beta(F,G) \) and \( \gamma(G,P) \) are the internal and external angles and \( J(r) \) is the set of all \( r \)-dimensional faces of \( \mathcal{P} \). The definitions of internal and external angles can be found in [10]. We include them here for completeness.

- The **internal angle** \( \beta(F,G) \) is the fraction of the volume of the unit ball covered by the cone obtained by observing the face \( G \) from any point in face \( F \). The quantity \( \beta(F,G) \) is defined to be zero if \( F \) is not contained in \( G \) and is defined to be one if \( F = G \).
- The **external angle** \( \gamma(G,P) \) is defined to be the fraction of the volume of the unit ball covered by the cone formed by the outward normals to the hyperplanes supporting \( \mathcal{P} \) at the face \( G \). If \( G = \mathcal{P} \) then \( \gamma(G,P) \) is defined to be one.
In this section we describe a method to obtain upper bounds for $\mathbb{P}(E|x \in F)$ by finding upper bounds on the internal and external angles described above. We first analyze $\mathbb{P}(E|x \in F)$ for the “simplest” class of faces. We denote by $F_0^k$, the face whose vertices are given by $\frac{1}{w_1}e_1, \frac{1}{w_2}e_2, \ldots, \frac{1}{w_k}e_k$. Thus the vertices are defined by the first $k$ indices and we call this face the “leading” $k-1$-dimensional face of $\mathcal{P}$. We will spend much of this section developing bounds for $\mathbb{P}(E|x \in F_0^k)$ for such “leading” faces. Then in Section IV, we describe the typical set of our signal model and show that for the purposes of bounding $\mathbb{P}(E)$ it is sufficient to consider a certain special class of faces. The bounds we develop in this section for “leading” faces can be easily generalized to faces belonging to this special class.

A. Angle Exponents for leading faces

We define the family of leading faces $\mathcal{F}_1$ as the set of all faces whose vertices are given by $\frac{1}{w_1}e_1, \frac{1}{w_2}e_2, \ldots, \frac{1}{w_k}e_k$ for some $k$. In this section we will establish the following result which bounds the internal and external angles related to a leading face of $\mathcal{P}$. We will then use this result in the next section to provide an upper bound on $\mathbb{P}(E|x \in F)$ for $F \in \mathcal{F}_1$.

**Theorem 2.** Let $\frac{1}{\pi} = \delta$ and $\frac{1}{\pi} = \tau$ with $\tau > \delta$. Let $G_0^l$ be the face whose vertices are given by $\frac{1}{w_1}e_1, \frac{1}{w_2}e_2, \ldots, \frac{1}{w_l}e_l$ and let $F_0^k \subset G_0^l$ be the face whose vertices are given by $\frac{1}{w_1}e_1, \frac{1}{w_2}e_2, \ldots, \frac{1}{w_k}e_k$. Then there exist quantities $\bar{\psi}_{\text{int}}(\tau)$ and $\bar{\psi}_{\text{ext}}(\delta, \tau)$ which can be computed explicitly as described in Section III-C2 and Section III-C3 respectively, such that for any $\epsilon > 0$, there exist integers $n_0(\epsilon)$ and $n_1(\epsilon)$ satisfying

- $n^{-1} \log(\beta(F_0^k, G_0^l)) < \bar{\psi}_{\text{int}}(\delta, \tau) + \epsilon$, for all $n > n_0(\epsilon)$,
- $n^{-1} \log(\gamma(G_0^l, P)) < \bar{\psi}_{\text{ext}}(\tau) + \epsilon$, for all $n > n_1(\epsilon)$.

The quantities $\bar{\psi}_{\text{int}}(\delta, \tau)$ and $\bar{\psi}_{\text{ext}}(\tau)$ are called the optimized internal angle exponent and the external angle exponent respectively.

Whenever any of the angle exponents described above is negative, the corresponding angle decays at an exponential rate with respect to the ambient dimension $n$. We will prove Theorem 2 over the following two subsections by finding the optimized internal and external angle exponents that satisfy the requirements of the theorem.

1) **Internal Angle Exponent:** In this subsection we find the optimized internal angle exponent $\bar{\psi}_{\text{int}}(\delta, \tau)$ that satisfies the conditions described in Theorem 2. We begin by stating the following result from [10] which provides the expression for the internal angle of a face $F$ with respect to a face $G$ in terms of the weights associated with their vertices. Subsequently, we find an asymptotic upper bound on the exponent of the internal angle using that expression. In what follows, we denote by $HN(0, \sigma^2)$ the distribution of a half normal random variable obtained by taking the absolute value of a $N(0, \sigma^2)$ distributed random variable.

**Lemma 1.** [10] Define $\sigma_{i,j} = \sum_{p=i}^{j} w_p^2$. Let $Y_0 \sim N(0, \frac{1}{2})$ be a normal random variable and $Y_p \sim HN(0, \frac{w_p^2}{2\sigma_{i,k}^2})$ for $p = 1, 2, \ldots, l-k$ be independent half normal distributed random variables that are independent of $Y_0$. Define the random variable $Z \triangleq Y_0 - \sum_{p=1}^{l-k} Y_p$. Then,

$$\beta(F_0^k, G_0^l) = \frac{\sqrt{\pi}}{2^{l-k}} \sqrt{\frac{\sigma_{1,l}}{\sigma_{k,l}}} p_Z(0),$$

where $p_Z(\cdot)$ is the density function of $Z$. 
We now proceed to derive an upper bound for the quantity $p_Z(0)$. Much of the analysis is along the lines of [2]. Let the random variable $S$ be defined as $S = \sum_{p=1}^{l-k} Y_p$, so, $Z = Y_0 - S$ and using the convolution integral, the density function of $Z$ can be written as

$$p_Z(0) = \int_{-\infty}^{\infty} p_{Y_0}(-v)p_S(v)dv = 2 \int_{0}^{\infty} v p_{Y_0}(v)F_S(v)dv,$$

where $F_S(v)$ is the cumulative distribution function of $S$. Let $\mu_S$ be the mean of the random variable $S$. Then,

$$p_Z(0) = 2 \int_{0}^{\mu_S} v p_{Y_0}(v)F_S(v)dv + 2 \int_{\mu_S}^{\infty} v p_{Y_0}(v)F_S(v)dv.$$

As in [2], the second term satisfies $II < \int_{\mu_S}^{\infty} 2vp_{Y_0}(v)dv = e^{-\mu_S^2}$. As we will see later in this section $\mu_S^2 \sim cn$ for some $c > 0$ and hence $II \sim e^{-cn^2}$. Since we are interested in computing the asymptotic exponent of $p_Z(0)$, we can ignore $II$ from this computation. To bound $I$, we use $F_S(v) \leq \exp(-\lambda_S^*(v))$, where $\lambda_S^*(v)$ is the rate function (convex conjugate of the characteristic function $\lambda_S(v)$) of the random variable $S$. So we get

$$I \leq \frac{2}{\sqrt{\pi}} \int_{0}^{s_k+1,l} ve^{-s_k+1,l\lambda_S^*(v)}dv.$$

For ease of notation we define the following quantities.

$$s_{k+1,l} \triangleq \sum_{p=1}^{l-k} w_p$$
$$\lambda(s) \triangleq \frac{1}{2}s^2 + \log(2\Phi(s))$$
$$\lambda_0(s) \triangleq \frac{1}{s_{k+1,l}} \sum_{p=1}^{l-k} \lambda(w_{p+k}s)$$
$$\lambda_0^*(y) \triangleq \max_{s} sy - \lambda_0(s).$$

Here $\lambda(s)$ is the characteristic function of the standard half normal random variable and $\Phi(x)$ is the standard normal distribution function. Using the above definition, we can express the relation between $\lambda_S^*(y)$ and $\lambda_0^*(y)$ as

$$\lambda_S^*(y) = \frac{1}{s_{k+1,l}} \lambda_S^* \left( \frac{s_{k+1,l}y}{\sqrt{\sigma_{1,k}}} \right)$$

We compute

$$\mu_S = \sum_{p=1}^{l-k} EY_p = \sqrt{\frac{2}{\pi}} \frac{s_{k+1,l}}{\sqrt{\sigma_{1,k}}}$$

Changing variables in (3) by substituting $v = \frac{s_{k+1,l}y}{\sqrt{\sigma_{1,k}}}$, we get

$$I \leq \frac{2}{\sqrt{\pi}} \int_{0}^{\frac{s_{k+1,l}}{\sqrt{\sigma_{1,k}}}} y \exp\left[-\frac{s_{k+1,l}^2}{2\sigma_{1,k}} y^2 - s_{k+1,l}\lambda_0^*(y)\right]dy.$$
Now, as $w_i = f \left( \frac{i}{n} \right)$, we have

$$s_{k+1,l} = \sum_{i=k+1}^{l} f \left( \frac{i}{n} \right) = n \left( \int_{\delta}^{\tau} f(x)dx + o(1) \right).$$

Define $c_0(\delta, \tau) \triangleq \int_{\delta}^{\tau} f(x)dx$. This gives us $s_{k+1,l} = n(c_0(\delta, \tau) + o(1))$. Similarly,

$$\sigma_{1,k} = \sum_{i=1}^{k} w_i^2 = \sum_{i=1}^{k} f^2 \left( \frac{i}{n} \right) = n(\int_{0}^{\delta} f^2(x)dx + o(1)) = n(c_1(\delta) + o(1)),$$

where $c_1(\delta)$ is defined as $c_1(\delta) \triangleq \int_{0}^{\delta} f^2(x)dx$. This gives us

$$\frac{s_{k+1,l}^2}{2\sigma_{k+1,l}} y^2 + s_{k+1,l} \lambda_0'(y) = n \left( \frac{c_0}{2c_1} y^2 + c_0 \lambda_0'(y) + o(1) \right) = n(y(\eta(y) + o(1)).$$

where $\eta(y)$ is defined as $\eta(y) \triangleq \left( \frac{c_0}{2c_1} y^2 + c_0 \lambda_0'(y) \right)$. Using Laplace’s method to bound $I$ as in [2], we get

$$I \leq R_n e^{-n\eta(y^*)}, \quad (4)$$

where $n^{-1} \log(R_n) = o(1)$, and $y^*$ is the minimizer of the convex function $\eta(y)$. Let

$$\lambda_0'(y^*) = \max_s sy^* - c_0(s) = s^* y^* - \lambda_0(s^*).$$

Then the maximizing $s^*$ satisfies $\lambda_0'(s^*) = y^*$. From convex duality, we have $\lambda_0'(y^*) = s^*$. The minimizing $y^*$ of $\eta(y)$ satisfies

$$\frac{c_0}{c_1} y^* + c_0 \lambda_0'(y^*) = 0$$

$$\implies \frac{c_0}{c_1} y^* + c_0 s^* = 0. \quad (5)$$

This gives

$$\lambda_0'(s^*) = -\frac{c_1}{c_0} s^*. \quad (6)$$

First we approximate $\lambda_0(s)$ as follows

$$\lambda_0(s) = \frac{1}{s_{k+1,l}} \sum_{p=1}^{l-k} \lambda(w_{p+k}s) = \frac{1}{nc_0} \sum_{p=k+1}^{l} \lambda \left( f \left( \frac{p}{n} \right) s \right) + o(1) = \frac{1}{c_0} \int_{\delta}^{\tau} \lambda(s f(x))dx + o(1).$$

From the above we obtain

$$\frac{d}{ds} \lambda_0(s) = \frac{1}{c_0} \int_{\delta}^{\tau} f(x) \lambda'(s f(x))dx + o(1).$$

Combining this with equation (6), we can determine $s^*$ up to a $o(1)$ error by finding the solution to the equation

$$\int_{\delta}^{\tau} f(x) \lambda'(s f(x))dx + c_1 s^* = 0. \quad (7)$$
We define the internal angle exponent as
\[ \psi_{\text{int}}(\beta, \tau, y) \triangleq - (\tau - \delta) \log 2 - \eta(y) \] (8)
and the optimized internal angle exponent as
\[ \bar{\psi}_{\text{int}}(\beta, \tau) \triangleq \max_y \psi_{\text{int}}(\beta, \tau, y) = - (\tau - \delta) \log 2 - \eta(y^*) \] (9)
where \( y^* \) is determined through \( s^* \) from equation (5) and \( s^* \) is determined by solving the equation (7).

From inequality (4) and Lemma 1 we see that the function \( \bar{\psi}_{\text{int}}(\beta, \tau) \) satisfies the conditions described in the first part of Theorem 2.

2) External Angle Exponent: In this subsection we find the optimized external angle exponent \( \bar{\psi}_{\text{ext}}(\tau) \) that satisfies the statement of Theorem 2. We proceed similar to the previous section and begin by stating the following lemma from [10] which provides the expression for the external angle of a face \( G \) with respect to the polytope \( P \) in terms of the weights associated with its vertices. We then find an upper bound on the asymptotic exponent of the external angle.

Lemma 2. [10] The external angle \( \gamma(G^l_0, P) \) is given by
\[ \gamma(G^l_0, P) = \pi \frac{n-l+1}{2} 2^{n-l} \int_0^\infty e^{-x^2} \prod_{p=l+1}^n \left( \int_0^\infty e^{-y^2} dy \right) dx. \]

To simplify the expression for the external angle, we define the standard error function as
\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \]
and rewrite the external angle as
\[ \gamma(G^l_0, P) = \sqrt{\frac{\sigma_1, l}{\pi}} \int_0^\infty e^{-\sigma_1, lx^2} \prod_{i=l+1}^n \text{erf}(w_i x) dx. \]

Similar to the method used in the previous section
\[ \sigma_1, l = \sum_{i=1}^l w_i^2 = n \left( \int_0^\tau f^2(x) dx + o(1) \right) = nc_2(\tau) + o(1). \]
where \( c_2(\tau) \) is defined as \( c_2(\tau) \triangleq \int_0^\tau f^2(x) dx \). Substituting this we have
\[ \gamma(G^l_0, P) = \sqrt{\frac{\sigma_1, l}{\pi}} \int_0^\infty \exp \left[ - n \left( c_2 x^2 - \frac{1}{n} \sum_{i=l+1}^n \log(\text{erf}(w_i x)) \right) \right] dx \\
= \sqrt{\frac{\sigma_1, l}{\pi}} \int_0^\infty \exp[-n\zeta(x)] dx. \]
where \( \zeta(x) \) is defined as \( \zeta(x) \triangleq (c_2x^2 - \frac{1}{n} \sum_{i=l+1}^{n} \log(\text{erf}(w_ix))) \). Again using Laplace’s method we get

\[
\gamma(G^0_0, P) \leq R_n \exp[-n\zeta(x^*)],
\]

(12)

where \( x^* \) is the minimizer of \( \zeta(x) \) and \( n^{-1}\log(R_n) = o(1) \). The minimizing \( x^* \) satisfies \( 2c_2x^* = G'(x^*) \), where

\[
G_0(x) \triangleq \frac{1}{n} \sum_{i=l+1}^{n} \log(\text{erf}(w_ix)).
\]

We first approximate \( G_0(x) \) as follows:

\[
G_0(x) = \frac{1}{n} \sum_{i=l+1}^{n} \log(\text{erf}(w_ix)) = \frac{1}{n} \sum_{i=l+1}^{n} \log(\text{erf}(f(i/n)x)) = \int_{\tau}^{1} \log(\text{erf}(xf(y))) \, dy + o(1).
\]

So the minimizing \( x^* \) can be computed up to an error of \( o(1) \) by solving the equation

\[
2c_2x^* = \int_{\tau}^{1} \frac{f(y) \text{erf}'(xf(y))}{\text{erf}(xf(y))} \, dy.
\]

(13)

We define the external angle exponent as

\[
\bar{\psi}_{\text{ext}}(\tau, x) \triangleq -\zeta(x)
\]

(14)

and the optimized external angle exponent as

\[
\bar{\psi}_{\text{ext}}(\tau) \triangleq \max_x \bar{\psi}_{\text{ext}}(\tau, x) = -\zeta(x^*) = - \left( c_2 x^*^2 - \int_{\tau}^{1} \log(\text{erf}(x^* f(y))) \, dy \right).
\]

(15)

where \( x^* \) can be obtained by solving equation (13).

From (12) it is clear that this function satisfies the conditions in the second part of Theorem 2. This completes the proof of both parts of the theorem.

**B. Recovery Threshold for the family of leading faces**

In this section we use the bounds on the asymptotic exponents of the internal and external angle from Theorem 2 to find an upper bound for \( P(E | x \in F) \) for \( F \in F_1 \). The main result of this section is the following theorem.

**Theorem 3.** Let \( F = F^k_0 \) for \( k = \delta n \), \( F_\delta \in F_1 \) be the face with \( k = \delta n \). There exists a function \( \bar{\psi}_{\text{tot}}(\delta) \) which we call the total exponent of \( F^k_0 \) such that given \( \epsilon > 0 \), there exists \( n(\epsilon) \) such that for all \( n > n(\epsilon) \),

\[
\frac{1}{n} \log(P(E | x \in F^k_0)) < -n\bar{\psi}_{\text{tot}}(\delta) + \epsilon.
\]

The function \( \bar{\psi}_{\text{tot}}(\delta) \) can be computed explicitly as described in (16).

Using the decomposition equation (2) and the fact that \( \beta(F, G) \) is non-zero only if \( F \subseteq G \), we get

\[
P(E | x \in F) \leq 2 \sum_{(l > m)} \sum_{(G \supseteq F, G \in J(l))} \beta(F, G) \gamma(G, P).
\]

Recall that \( J(r) \) is the set of all \( r \)-dimensional faces of \( P \).

To proceed, we will need the following useful lemma (proof can be found in the appendix).
Lemma 3. Among all \((l - 1)\)-dimensional faces \(G\) of \(\mathcal{P}\) satisfying \(F^k_0 \subseteq G\), the face that maximizes \(\beta(F^k_0, G)\) and \(\gamma(G, \mathcal{P})\) is the one with \(\frac{1}{w_1} e_1, \frac{1}{w_2} e_2, \ldots, \frac{1}{w_l} e_l\) as its vertices.

Using Lemma 3

\[
P(E|\mathbf{x} \in F^k_0) \leq \sum_{l=m+1}^{n} \binom{n-k}{l-k} 2^{l-k} \beta(F^k_0, G^l_0) \gamma(G^l_0, \mathcal{P})
\]

\[
\leq (n - m) \max_{\{l : l > m\}} \binom{n-k}{l-k} 2^{l-k} \beta(F^k_0, G^l_0) \gamma(G^l_0, \mathcal{P})
\]

Using \(\tau = \frac{1}{n}\), \(\delta = \frac{k}{n}\) and Stirling’s approximation for factorials, it can be shown that

\[
\frac{1}{n} \log(1 - \delta) \rightarrow (1 - \delta) H\left(\frac{\tau - \delta}{1 - \delta}\right),
\]

where \(H(x)\) is the entropy function with base \(e\) defined by \(H(x) = -x \log(x) - (1 - x) \log(1 - x)\).

Define the combinatorial exponent \(\psi_{com}(\delta, \tau)\) as

\[
\psi_{com}(\delta, \tau) \triangleq (1 - \delta) H\left(\frac{\tau - \delta}{1 - \delta}\right) + (\tau - \delta) \log 2.
\]

So, we conclude,

\[
\frac{1}{n} \log(P(E|\mathbf{x} \in F^k_0)) \leq \max_{\{\tau : \tau \geq \alpha\}} \bar{\psi}_{com}(\delta, \tau) + \bar{\psi}_{int}(\delta, \tau) + \bar{\psi}_{ext}(\tau) + o(1).
\]

The function \(\bar{\psi}_{tot}(\delta)\) defined by

\[
\bar{\psi}_{tot}(\delta) \triangleq \max_{\{\tau : \tau \geq \alpha\}} \bar{\psi}_{com}(\delta, \tau) + \bar{\psi}_{int}(\delta, \tau) + \bar{\psi}_{ext}(\tau)
\]

satisfies the condition given in Theorem 3 and hence establishes the theorem.

C. Obtaining tighter exponents

In Section II-B we developed a method that provides an upper bound on \(P(E|\mathbf{x} \in F^k_0)\) in terms of internal and external angle exponents. For a given value of \(l\), we bounded \(\beta(F^k_0, G)\) and \(\gamma(G, \mathcal{P})\) for each \(G \in J(l)\) by \(\beta(F^k_0, G^l_0) \gamma(G^l_0, \mathcal{P})\) by using Lemma 3. This bounding step is generally rather overly conservative because it does not take into account the variation of the function \(f(.)\) over the complete interval \([0, 1]\). The quality of bound is especially poor for choices of function \(f(.)\) which are rapidly increasing. To improve on the bound, we can use a simple technique which allows us to take into account the variation of \(f(u)\) with respect to \(u\) more accurately over the whole interval.

Divide the set of indices \(k + 1, \ldots, n\) into two parts with \(T_1 \triangleq \{k + 1, \ldots, \frac{n+k}{2}\}\) and \(T_2 \triangleq \{\frac{n+k}{2} + 1, \ldots, n\}\). For a particular \(l\), let \(G\) have \(l_1\) vertices in \(T_1\) and \(l_2\) vertices in \(T_2\). Using Lemma 3 among all faces \(G\) with the values of \(l_1\) and \(l_2\) specified as above, the choice that maximizes \(\beta(F^k_0, G)\) and \(\gamma(G, \mathcal{P})\) is the face \(G\) with vertices given by the indices \(1, \ldots, l_1, \frac{n+k}{2} + 1, \ldots, \frac{n+k}{2} + l_2\). Using this we get a revised upper bound on \(P(E|\mathbf{x} \in F^k_0)\)

\[
P(E|\mathbf{x} \in F^k_0) \leq \sum_{l=m+1}^{n} \sum_{l_1+l_2=l} \binom{n-k}{l_1} \binom{n-k}{l_2} \beta(F^k_0, G) \gamma(G, \mathcal{P})
\]

\[
\leq (n - m) \max_{l_1+l_2 \geq m-k} \binom{n-k}{l_1} \binom{n-k}{l_2} \beta(F^k_0, G^l_0) \gamma(G^l_0, \mathcal{P}),
\]

where \(\beta(F^k_0, G)\) and \(\gamma(G, \mathcal{P})\) are the face's internal and external angle exponents.
Define $\gamma_1 = \frac{l_1}{n}$ and $\gamma_2 = \frac{l_2}{n}$. Then the revised bound on the exponent of $P(E| x \in F_0^n)$ can be obtained as

$$
\frac{1}{n} \log(P(E| x \in F)) \leq \max_{\gamma_1+\gamma_2 \geq \alpha - \delta} \left( \frac{1 - \delta}{2} H \left( \frac{2 \gamma_1}{1 - \delta} \right) + \frac{1 - \delta}{2} H \left( \frac{2 \gamma_2}{1 - \delta} \right) + \psi_{\text{int}}(\delta, \gamma_1, \gamma_2) + \psi_{\text{ext}}(\gamma_1, \gamma_2) + o(1) \right)
$$

where we define $\psi_{\text{com}}(\delta, \gamma_1, \gamma_2) \triangleq \frac{1 - \delta}{2} H \left( \frac{2 \gamma_1}{1 - \delta} \right) + \frac{1 - \delta}{2} H \left( \frac{2 \gamma_2}{1 - \delta} \right)$ and call it the combinatorial exponent. The expressions for $\psi_{\text{int}}(\delta, \gamma_1, \gamma_2)$ and $\psi_{\text{ext}}(\gamma_1, \gamma_2)$ are can be obtained by using the methods of the previous subsection (we will give the exact expressions shortly). The function

$$
\bar{\psi}_{\text{tot}}(\delta) \triangleq \max_{\gamma_1+\gamma_2 \geq \alpha - \delta} \psi_{\text{com}}(\delta, \gamma_1, \gamma_2) + \psi_{\text{int}}(\delta, \gamma_1, \gamma_2) + \psi_{\text{ext}}(\gamma_1, \gamma_2)
$$

now satisfies the role of the total exponent in Theorem 3.

We can repeat the above argument for any $r$ by dividing the indices denoted by $k + 1, \ldots, n$ into $r$ parts. Define the quantities $\gamma_i$, $i = 1, \ldots, r$ as $\gamma_i \triangleq \frac{l_i}{n}$ where $l_i$ is the number of indices of face $G$ in the $i^{th}$ interval. Also define $h_i \triangleq r \gamma_i$ and let $h = (h_1, \ldots, h_r)^t$. The total exponent is now given by

$$
\bar{\psi}_{\text{tot}} = \max_h \bar{\psi}_{\text{com}}(h) + \bar{\psi}_{\text{int}}(h) + \bar{\psi}_{\text{ext}}(h)
$$

subject to

$$
\frac{1}{r} \sum_{i=1}^{r} h_i \geq \alpha - \delta,
$$

where

$$
\bar{\psi}_{\text{int}}(h) = \max_y \psi_{\text{int}}(h, y), \quad \text{analogous to equation (9)} \quad \text{and}
$$

$$
\bar{\psi}_{\text{ext}}(h) = \max_x \psi_{\text{ext}}(h, x), \quad \text{analogous to equation (14)}.
$$

The dependence of the exponent functions on other variables has been suppressed for compactness.

So, the total exponent can be obtained by the following maximization:

$$
\bar{\psi}_{\text{tot}} = \max_{h, x, y} \psi_{\text{com}}(h) + \psi_{\text{int}}(h, y) + \psi_{\text{ext}}(h, x) \quad (16)
$$

subject to

$$
\frac{1}{r} \sum_{i=1}^{r} h_i \geq \alpha - \delta.
$$

We now compute the expressions for each of the functions appearing in the above maximization. For the subsequent derivation, we define $f_i \triangleq f \left( \delta + \frac{(i-1)(1-\delta)}{r} \right)$.

**1) Combinatorial Exponent:** The combinatorial exponent is given by

$$
\psi_{\text{com}} = \frac{1}{r} \sum_{i=1}^{r} (1 - \delta) H \left( \frac{r \gamma_i}{1 - \delta} \right) + \left( \sum_{i=1}^{r} \gamma_i - \delta \right) \log 2
$$

$$
= \frac{1 - \delta}{r} \sum_{i=1}^{r} H \left( \frac{h_i}{1 - \delta} \right) + \left( \sum_{i=1}^{r} \frac{r h_i}{1 - \delta} \right) \log 2.
$$
For the internal and external angle exponents, in addition to obtaining their expressions, we will also bound them suitably by analytically simpler expressions so that the optimization problem described in (16) becomes more tractable.

2) **External Angle Exponent:** We start with

\[ \zeta(x) = c_2 x^2 - G_0(x), \]

where

\[ c_2 = \int_0^\delta f^2(u)du + r \sum_{i=1}^{r} \int_{\delta + \frac{(i-1)(1-\delta)}{r}}^{\delta + \frac{i(1-\delta)}{r}} f^2(u)du \]

\[ G_0(x) = \int_\delta^1 \log(\text{erf}(xf(u)))du - \frac{1}{r} \sum_{i=1}^{r} \int_{\delta + \frac{(i-1)(1-\delta)}{r}}^{\delta + \frac{i(1-\delta)}{r}} \log(\text{erf}(xf(u)))du. \]

As \( f(.) \) is an increasing function, the integral appearing above in the expression of \( c_2 \) can be bound by its left Riemann sum.

\[ c_2 \geq \int_0^\delta f^2(u)du + \frac{1}{r} \sum_{i=1}^{r} f_i h_i \triangleq \bar{c}_2. \]

Similarly,

\[ G_0(x) \leq \int_\delta^1 \log(\text{erf}(xf(u)))du - \frac{1}{r} \sum_{i=1}^{r} \log(\text{erf}(xf_i)) \triangleq \bar{G}_0(x). \]

So,

\[ \zeta(x) \geq \bar{c}_2 x^2 - \bar{G}_0(x) \triangleq \bar{\zeta}(x). \]

Combining we obtain a simplified expression for the external angle exponent as

\[ \psi_{\text{ext}}(x) = -\bar{\zeta}(x) = - \left( \bar{c}_2 x^2 - \bar{G}_0(x) \right). \]

The optimized external angle exponent is then given by

\[ \bar{\psi}_{\text{ext}} = \max_x \psi_{\text{ext}}(x). \]

3) **Internal Angle Exponent:** We start with

\[ \eta(y) = \frac{c_0^2}{2c_1} y^2 + c_0 \lambda_0^*(y) \]

where

\[ c_0 = \sum_{i=1}^{r} \int_{\delta + \frac{(i-1)(1-\delta)}{r}}^{\delta + \frac{i(1-\delta)}{r}} f(u)du \geq \frac{1}{r} \sum_{i=1}^{r} f_i h_i \triangleq \bar{c}_0. \]

\( \lambda_0^*(y) \) is the convex conjugate of \( \lambda_0(.) \) and is given by,

\[ \lambda_0^*(y) = \max_s sy - \lambda_0(s). \]
We are interested in only in the region $s \leq 0$. In this region we have,

$$\lambda_0(s) = \frac{1}{c_0} \sum_{i=1}^{r} \int_{\delta}^{(i-1)(\delta - 1)} \lambda(sf(u))du \leq \frac{1}{r\epsilon_0} \sum_{i=1}^{r} h_i \lambda(sf_i) \triangleq \lambda_0(s),$$

which follows from the fact that $s \leq 0$ and $\lambda(u)$ is an increasing function of $u$. This gives

$$\max_s sy - \lambda_0(s) \geq \max_s sy - \lambda_0(s),$$

and hence

$$\lambda_0^*(y) = \max_s sy - \lambda_0(s).$$

So, we conclude

$$\eta(y) \geq \frac{\epsilon_0^2}{2c_1} y^2 + \bar{c}_0 \bar{\lambda}_0^*(y) \triangleq \bar{\eta}(y).$$

From the above we obtain a simplified expression for the internal angle exponent as

$$\psi_{int}(y) = -(r \sum_{i=1}^{r} h_i - \delta) \log 2 - \bar{\eta}(y) = -(r \sum_{i=1}^{r} h_i - \delta) \log 2 - \left(\frac{\epsilon_0^2}{2c_1} y^2 + \bar{c}_0 \bar{\lambda}_0^*(y)\right).$$

The optimized internal angle exponent is then obtained as

$$\bar{\psi}_{int} = \max_y \psi_{int}(y).$$

To show that the maximization with respect to $h$ in (16) can be performed efficiently, we will show that $\psi_{tot}(h, x, y)$ is a concave function of $h$ for fixed value of $x$ and $y$. This follows from the following lemma. The proof can be found in the appendix.

**Lemma 4.** The combinatorial, internal and external angle exponents are concave functions of $h$ for fixed values of $x$ and $y$.

The parameter $r$ which governs the number of evaluation points we use to compute the angle exponents gives us a method to control the accuracy of the computed exponents. If we can obtain a value $\delta$ such that for all $\delta \leq \bar{\delta}$ we have $\bar{\psi}_{tot}(\delta) < 0$, then Theorem 3 guarantees that whenever $\delta \leq \bar{\delta}$, weighted $\ell_1$-minimization recovers the corresponding sparse signal with overwhelming probability. We call this $\bar{\delta}$ the guaranteed bound on recoverable sparsity levels $\delta$. Increasing $r$ results in a tighter guaranteed bound but at the expense of an increased cost in performing the optimization in (16). In Figure 1, we show via simulation how this bound improves with the parameter $r$. For this, we fix a compression ratio $\alpha = \frac{m}{n} = 0.5$. The weights $w_i$ are chosen as $w_i = 1 + \frac{1}{i}$ which corresponds to the weight function $f(u) = 1 + u$. Note how the bound improves with increasing value of $r$ but saturates quickly. This indicates that a moderately large value of $r$ (e.g. $r = 30$ from the figure) will suffice for accurate enough angle exponent computations.

**IV. PROOF OF THEOREM 1**

In this section, we establish the main result of this paper which is Theorem 1. We do this in two steps. We first characterize the properties of a typical signal drawn from our model and provide a large deviation result on the probability that a signal drawn from our model does not satisfy this property. For signals
Fig. 1: Guaranteed bound on recoverable $\delta$ vs $r$ for compression ratio $\alpha = 0.5$ and weight function $f(u) = 1 + u$ computed using the methods of Section III. As $r$ increases the computed bound also increases indicating an improvement in the tightness of the bound. Also since the improvement saturates fairly fast, we can use a moderately large value of $r$ to obtain accurate enough angle exponents. The value $r = 30$ in the figure represents such a choice.

which have this typical property, we then use the analysis method of Section III to find the exponent $\psi_{tot}(p, f)$ in the statement of Theorem 1. We start by writing

$$ P(E) = \sum_{F \in F} P(x \in F) P(E \mid x \in F). $$

To analyze this expression we will further split the sum into faces belonging to different “classes” which we describe below. Divide the set of indices from $1, \ldots, n$ into $r$ equal parts with $I_i$ denoting the $i^{th}$ interval of indices. Any face $F$ of the skewed cross polytope $P$ can be fully specified by the index of its vertices (up to the signs of the vertices). For a given face $F$, let the set of indices representing the face $F$ be denoted by $I(F)$. For a given $k = (k_1, \ldots, k_r)$ denote by $F_r(k)$, the set of all faces of $P$ with $|I(F) \cap I_i| = k_i$. Also let $g_i = k_i \frac{r}{n}$. Recall that we denote by $E$ the failure event, i.e. the event that the weighted $\ell_1$-minimization does not produce the correct solution. Then we have

$$ P(E) = \sum_{k} \sum_{F \in F_r(k)} P(x \in F) P(E \mid x \in F). $$

(17)
We can just consider one representative among the $2^\sum_{k_i}$ faces created by the different sign patterns. This is because, by the symmetry of the problem, all the faces have the same probabilities $P(x \in F)$ and $P(E|\mathbf{x} \in F)$. For simplicity we always choose this representative as the face that lies in the first orthant.

$$P(x \in F) = \left( \prod_{i=1}^r \prod_{j \in I(F) \cap I_i} \frac{p(j/n)}{1 - p(j/n)} \right) \times \prod_{i=1}^n (1 - p(j/n)).$$

The function $\frac{p}{1-p}$ is an increasing function of $x$ for $x \in (0,1)$. So,

$$P(x \in F) \leq \left( \prod_{i=1}^r \left( \frac{p(j/n)}{1 - p(j/n)} \right)^{k_i} \right) \prod_{i=1}^n (1 - p(j/n)).$$

Denote the right hand side of the above inequality be $P(k)$, which means for $x \in F_r(k)$ we have $P(x \in F) \leq P(k)$. Define

$$P(x \in F_r(k)) \triangleq P(x \in F \text{ for some } F \in F_r(k)).$$

Then

$$P(x \in F_r(k)) \leq \sum_{F \in F_r(k)} P(x \in F) \leq \left( \prod_{i=1}^r \left( \frac{n}{j/n} \right) \right) P(k).$$

By Lemma 3, among all faces $F \in F_r(k)$, the one which maximizes $P(E|x \in F)$ is the one obtained by stacking all the indices to the right of each interval. We denote this maximum probability by $P(E|k)$. Combining the above we get

$$P(E) \leq \sum_k P(x \in F_r(k))P(E|k). \tag{18}$$

As the function $p(u)$ is monotonically decreasing, $P(x \in F_r(k))$ is not same for all $k$. We now proceed to show that $P(x \in F_r(k))$ is significant only when $x$ takes values in a “typical” set. For values of $k$ outside the typical set $P(x \in F_r(k))$ is exponentially small in $n$ and can be ignored in the sum in (18). For $k$ in the typical set, we will use the methods of Section III to bound $P(E|k)$ and hence $P(E)$.

The following Lemma provides bounds for $P(x \in F_r(k))$ which will motivate the definition of typicality that follows.

**Lemma 5.** Let $D(q||p)$ denote the Kullback-Leibler distance between two Bernoulli random variables with probability of success given by $q$ and $p$ respectively. Define $\tilde{p}_i = r \int_{\tilde{p}}^{\frac{i+1}{r}} p(u)du$. If we have $\frac{1}{r} \sum_{i=1}^r D(g_i||\tilde{p}_i) > \epsilon$, then there exists $n_0$, such that for all $n > n_0$,

$$P(x \in F_r(k)) < e^{-b(\epsilon)n},$$

where $b(\epsilon)$ is a positive constant.

**Proof.** We have

$$P(x \in F_r(k)) = \left( \prod_{i=1}^r \left( \frac{\tilde{p}_i}{n} \right) \right) P(k).$$
Letting $g$ denote the number of indices of face $F$, now further divide each of the intervals into $r$ equal parts thus forming a total of $tr$ intervals. Let $g_x = \frac{1}{r} \sum_{j=t}^{tr} g_t$ denote the number of indices of face $F$ in the $(tr)^{th}$ interval counted according to the new partition. Summing up the number of indices in all the intervals of length $\frac{1}{r}$ contained in an interval of length $\frac{1}{r}$ gives $g_x = \frac{1}{r} \sum_{j=t}^{tr} g_t$. Also let $p_{r,j} = p \left( \frac{j}{r} \right)$. The calculations just carried out above can be repeated for this new partition to give

\[
\lim_{n \to \infty} \frac{1}{n} \log(P(x \in F_r(k))) \leq -\frac{1}{tr} \sum_{i=1}^{t} \sum_{j=1}^{t} D(g_{t,i+j}||p_{t,i+j}) + \Delta(tr).
\]

We now bound the term $\sum_{j=1}^{t} D(g_{t,i+j}||p_{t,i+j})$.

\[
\sum_{j=1}^{t} D(g_{t,i+j}||p_{t,i+j}) = \sum_{j=1}^{t} g_{t,i+j} \log \left( \frac{g_{t,i+j}}{p_{t,i+j}} \right) + \sum_{i=1}^{r} \sum_{j=1}^{t} (1 - g_{t,i+j}) \log \left( \frac{1 - g_{t,i+j}}{1 - p_{t,i+j}} \right).
\]

Using the log-sum inequality, we obtain,

\[
\frac{1}{t} \sum_{j=1}^{t} D(g_{t,i+j}||p_{t,i+j}) \geq \left( \frac{\sum_{j=1}^{t} g_{t,i+j}}{t} \right) \log \left( \frac{\sum_{j=1}^{t} g_{t,i+j}}{\sum_{j=1}^{t} p_{t,i+j}} \right) + \left( \frac{\sum_{j=1}^{t} 1 - g_{t,i+j}}{t} \right) \log \left( \frac{\sum_{i=1}^{r} 1 - g_{t,i+j}}{\sum_{i=1}^{r} 1 - p_{t,i+j}} \right).
\]

\[
= g_x \log \left( \frac{g_x}{p_{t,t}} \right) + (1 - g_x) \log \left( \frac{1 - g_x}{1 - p_{t,t}} \right)
\]

\[
=D(g_x||p_{t,t}),
\]
where, $\bar{p}_{i,t} = \frac{1}{t} \sum_{j=1}^{t} p_{i,t+i,j}$. Using this,
\[
\lim_{n \to \infty} \frac{1}{n} \log(\mathbb{P}(x \in \mathcal{F}_r(k))) \leq -\frac{1}{r} \sum_{i=1}^{r} D(g_i || \bar{p}_i) + \Delta(tr).
\]
Since this is true for every $t$, we let $t \to \infty$ to get
\[
\lim_{n \to \infty} \frac{1}{n} \log(\mathbb{P}(x \in \mathcal{F}_r(k))) \leq -\frac{1}{r} \sum_{i=1}^{r} D(g_i || \bar{p}_i).
\]
So, if the condition of the lemma is satisfied, then
\[
\lim_{n \to \infty} \frac{1}{n} \log(\mathbb{P}(x \in \mathcal{F}_r(k))) \leq -\frac{\epsilon}{r}.
\]
and the claim in the lemma then follows.

Lemma 5 motivates the following natural definition of epsilon typicality.

**Definition 1.** Given $\epsilon > 0$, define $k = (k_1, \ldots, k_r)$ to be $\epsilon$-typical if $\frac{1}{r} \sum_{i=1}^{r} D(g_i || \bar{p}_i) \leq \epsilon$.

Using this definition of $\epsilon$-typicality, we can now bound the probability of failure $\mathbb{P}(E)$ by using (18) as
\[
\mathbb{P}(E) \leq \sum_{k \text{ is } \epsilon \text{-typical}} \mathbb{P}(x \in \mathcal{F}_r(k)) \mathbb{P}(E|k) + \sum_{k \text{ not } \epsilon \text{-typical}} \mathbb{P}(x \in \mathcal{F}_r(k)) \mathbb{P}(E|k).
\]
For a fixed value of $r$, the number of possible values of $k_1, k_2, \ldots, k_r$ is bounded by $n(n + r - 1)^{r-1}$. Since $\mathbb{P}(E|k) \leq 1$, the second term $II$ can be bounded as
\[
II \leq n(n + r - 1)^{r-1} e^{-c(\epsilon)n},
\]
for some positive constant $c(\epsilon)$. As $\lim_{n \to \infty} \frac{1}{n} \log \left(n(n + r - 1)^{r-1}\right) = 0$, there exists $n_0$, such that for all $n > n_0$,
\[
II \leq e^{-c_0(\epsilon)n},
\]
for some positive constant $c_0(\epsilon)$. This allows us to only consider the first term $I$ for $k$ which are $\epsilon$-typical. Among all $k$ $\epsilon$-typical, let $F_{\epsilon,p}$ be the face which maximizes the probability $\mathbb{P}(E|x \in F)$. Then,
\[
I \leq \mathbb{P}(E|x \in F_{\epsilon,p}).
\]
This gives
\[
\lim_{n \to \infty} \frac{1}{n} \log(I) \leq \lim_{n \to \infty} \frac{1}{n} \log(\mathbb{P}(E|x \in F_{\epsilon,p})).
\]
Also, since we can choose $\epsilon$ to be as small as we want, we can essentially just consider the case with $\epsilon = 0$. This gives us $k = (k_1, \ldots, k_r)$ with $k_i = \bar{p}_i$. This defines a fixed face $F_\bar{p}$ which we call the typical face of $\mathcal{P}$ and we only need to bound the term $\mathbb{P}(E|x \in F_\bar{p})$ which can be done by a simple generalization of the methods in Section III. The corresponding expressions for the optimized internal and external angle exponents can be found in the Appendix.
In this section, we compute the bounds using the techniques developed in this paper for a specific probability function $p(.)$ and weight function $f(.)$. In particular we choose $p(.)$ to be a linear function whose slope is governed by a parameter $c$.

$$p(u) = \delta - c(u - 0.5), \quad u \in [0, 1].$$

(19)

The expected level of sparsity of signals drawn from this model is

$$\int_0^1 p(u) du = \delta.$$  

The value of $c$ governs how tilted the signal model is. Larger values of $c$ results in a higher tilt which means a random signal drawn from this model will have most of its non-zero entries concentrated at the beginning. To recover signals drawn from this model we use weighted $\ell_1$-minimization with choice of weights defined by the weight function

$$f(u) = 1 + \rho u, \quad u \in [0, 1].$$

(20)

The value of the constant $\rho$ controls the variation of the weights used. As a general guideline, we need to choose higher values of $\rho$ for higher values of $c$. Later in this section we describe a method to make this choice based on the theoretical bounds computed in this paper.

Before proceeding to evaluation of the performance of weighted $\ell_1$-minimization in the problem specified by the above choice of functions, we will first present theoretical bounds and simulations related to the behavior of the so-called family of leading faces $F_1$, when the corresponding cross-polytope is described by the function $f(.)$. This is contained in Section V-A. The performance evaluation of weighted $\ell_1$-minimization for the model described above is delayed till Section V-B.

### A. Behavior of the leading family of faces $F_1$ under random projection.

Recall that, for a given set of weights, the weighted $\ell_1$-ball is the cross polytope in $n$ dimensions whose vertices in the first orthant are given by $\frac{\delta}{\bar{\psi}_{tot}(\delta)}$, for some $k$. We call this face $F_0^k$. In Section III-B, we developed sufficient conditions when weighted $\ell_1$-minimization recovers a signal $x$ whose support is the first $k$ indices with overwhelming probability whenever $k = \delta n$. We define the quantity

$$\bar{\delta} \triangleq \max \delta$$

subject to $\bar{\psi}_{tot}(\delta) \leq 0$,

where $\bar{\psi}_{tot}(\delta)$ is the total exponent for the face $F_0^k$ with $k = \delta n$ as defined in (16). From the above definition and Theorem 1 it can be concluded that whenever $\delta < \bar{\delta}$, we are guaranteed to be able to recover the corresponding sparse signal with overwhelming probability. We call the quantity $\bar{\delta}$ to be the guaranteed bound on recoverable sparsity levels $\delta$. In view of our choice of weights, which is described by the function $f(u) = 1 + \rho u$ with $\rho \geq 0$, higher values of $\rho$ correspond to more steeply varying weights. Intuitively, one may expect that higher values of $\rho$, will make the weighted $\ell_1$ norm cost function favor non-zero entries in the first few indices which may allow the threshold $\bar{\delta}$ to be larger.

We will show that the quantity $\bar{\delta}$ follows an increasing trend as described above. To demonstrate this for a certain choice of the parameters, we fix the compression ratio $\frac{m}{n} = 0.5$ and compute the bound using the methods developed in Section III-B. Based on Figure 1 we choose $r = 30$ as a reasonable value for the accuracy parameter in our computations. Figure 2 shows the dependence of this threshold on the value of $\rho$ which governs the weight function $f(.)$. The value of the bound at $\rho = 0$ corresponds to the case when $F_0^{\delta n}$ is a face of the regular $\ell_1$ ball. This is the threshold for $\delta$ below which standard $\ell_1$-minimization succeeds in recovering a signal with sparsity level $\delta$ with overwhelming probability. As expected this value matches the value reported earlier in [2].
Fig. 2: Guaranteed bound on recoverable $\delta$ vs $\rho$ for the “leading face” $F_0^{\delta n}$, computed using the methods of Section [I]-[B] for $r = 30$

To evaluate the accuracy of the bound, we then compare the values of the threshold predicted by the guaranteed bound to that obtained empirically through simulations for two different values of the parameter $\rho$. For this, we set $m = 200, n = 400$, and obtain the fraction of times $x \in F_0^{\delta n}$ failed to be recovered correctly via weighted $\ell_1$-minimization from a total of 500 iterations. This is done by randomly generating a vector $x$ for each iteration with support $1, \ldots, k$ and using weighted $\ell_1$-minimization to recover that $x$ from its measurements given by $y = Ax$. Figure 3a and Figure 3b show this plot for $\rho = 0$ and $\rho = 1$ respectively. The vertical lines in the plots (marked A and B respectively) denote the guaranteed bounds corresponding to the value of $\rho$ in Figure 2. The simulations show a rapid fall in the empirical value of $\mathbb{P}(E| x \in F_0^{\delta n})$ around the theoretical guaranteed bound as we decrease the value of $\delta$. This indicates that the guaranteed bounds developed are fairly tight.

B. Performance of weighted $\ell_1$-minimization.

In this subsection, we compute the theoretical bound on recoverable sparsity levels using the methods of this paper. We use the probability function and weight function in (19) and (20) respectively. The choice of $\rho$ plays an important role in the success of weighted $\ell_1$-minimization and it would be of interest to be able to obtain the value of $\rho$ for which one gets best performance. One way to estimate the effect of $\rho$ is to compute the guaranteed bound $\bar{\delta}$ as suggested in Section [V] and observe the trend. We can then pick the value of $\rho$ which maximizes $\bar{\delta}$.

To demonstrate this via computations we fix the ratio $\alpha = \frac{m}{n} = 0.5$ and compute the guaranteed bound on recoverable $\delta$ (which denotes the expected fraction of non-zero components of the signal) using the methods developed in Section [V]. The accuracy parameter $r$ is fixed at 60. Figure 4 shows the dependence of $\bar{\delta}$ on the values of $\rho$ for three different values of $c$. The curves suggest that for larger values of $c$, which correspond to more rapidly decaying probabilities, the value of $\rho = \rho^*(c)$ which maximizes $\bar{\delta}$ is also higher. At the same time, the value of $\bar{\delta}$ evaluated at $\rho = \rho^*(c)$ also increases with increasing $c$. This
TABLE I: $c$ vs $\rho^*(c)$ using theoretical guaranteed bounds with $r = 60$ (Figure 4).

| $c$  | 0   | 0.16 | 0.26 | 0.36 |
|------|-----|------|------|------|
| $\rho$ | 0   | 0.6  | 1.0  | 1.8  |

suggests that rapidly decaying probabilities allow us to recover less sparse signals by using an appropriate weighted $\ell_1$-minimization.

To provide evidence that weighted $\ell_1$-minimization indeed improves performance, we conduct simulations to compare standard and weighted $\ell_1$-minimization. We fix the value of $\delta$ to be 0.185 which is known to be greater than the recoverable threshold for standard $\ell_1$-minimization. We then explore the effect of choosing different values of the model parameter $c$ in (19) on recoverability. We sample random signals with supports generated by the distribution imposed by $p(.)$. We then utilize the curves computed in Figure 4 to make the best choice of $\rho$ (see Table I) and use weighted $\ell_1$-minimization corresponding to this $\rho$ to recover the generated signal from its measurements. We compute the fraction of the experiments for which this method fails to recover the correct signal over 500 experiments. The values of $m$ and $n$ are chosen to be 500 and 1000 respectively. To compare the performance of weighted $\ell_1$-minimization to standard $\ell_1$-minimization, we repeat the same procedure but use standard $\ell_1$-minimization to recover the signal. Figure 5 compares the values generated by each method. Notice how the performance of the standard $\ell_1$-minimization method remains more or less invariant with increasing $c$. This shows that standard $\ell_1$-minimization fails to exploit the extra information present because of the knowledge of $c$ (i.e. the decaying nature of the probabilities) and its performance depends only on the value of $\delta$, the expected fractional level of sparsity and is insensitive to the tilt of the model given by $c$. On the other hand, the performance of weighted $\ell_1$-minimization improves with $c$. 
Fig. 4: Guaranteed bound on recoverable $\delta$ vs $\rho$ for $\frac{m}{n} = 0.5$ computed using the methods of this paper, for $c = 0.16$, $c = 0.26$ and $c = 0.36$. The parameter $r$ is fixed at 60.

VI. CONCLUSION

In this paper we analyzed sparse signal recovery via weighted $\ell_1$-minimization for a special class of probabilistic signal model, namely when the weights are uniform samples of a continuous function. We leveraged the techniques developed in [2] and [10] to provide sufficient conditions under which weighted $\ell_1$-minimization succeeds in recovering the sparse signal with overwhelming probability. In the process, we also provided conditions under which certain special class of faces of the skewed cross-polytope get “swallowed” under random projections.

A question central to the weighted $\ell_1$-minimization based approach is the choice of optimal weights. The authors in [11] were able to answer this question for the simple class of models they considered. In this paper, we were able to do the same provided that we restrict our search to a certain family of weight functions. An interesting question to pursue in future work would be to characterize this family of functions in terms of the function $p(u)$ that specifies the prior probabilities.

APPENDIX A

PROOF OF LEMMAS 3 AND 4

A. Proof of Lemma 3

Let $G_0$ be the face whose vertices are given by $\frac{1}{w_1}e_1, \ldots, \frac{1}{w_k}e_k, \frac{1}{w_{k+1}}e_{k+1}, \ldots, \frac{1}{w_l}e_l$. Let $G$ be any face other than $G_0$ whose vertices are given by $\frac{1}{w_1}e_1, \ldots, \frac{1}{w_k}e_k, \frac{1}{w_{n_1}}e_{n_1}, \ldots, \frac{1}{w_{n_l}}e_{n_l}$. Consider forming a sequence of faces $G^0, G^1, \ldots, G^{l-k}$, where $G^0 = G$, and $G^{i+1}$ is obtained from $G^i$ by swapping the
vertices $w_{n_{l-i+1}}$ and $w_{l-i+1}$. Since $w_{n_{l-i+1}} \geq w_{l-i+1}$, the expression in Lemma \ref{lem:external_angle} for the external angle increases at each step. Hence, $\gamma(G_0, P) \geq \gamma(G, P)$.

For a fixed value of $l$, the exponent for the internal angle is only affected by the term $p_Z(0)$ in the expression for internal angle exponent in Lemma \ref{lem:internal_angle}. Also

$$p_Z(0) = 2 \int_0^\infty v p_Y(v) F_S(v) dv.$$  

Following the same procedure as above for generating the sequence of faces $G^i$, it can be seen that at each step the variance of some $Y_p$ is decreased while keeping the other $Y_i$s unchanged. Thus $F_S(v)$ in the above expression for $p_Z(0)$ increases at each step. Thus, the exponent for $\beta(F_0, G_0)$ is greater than or equal to that of $\beta(F_0, G)$.

**B. Proof of Lemma \ref{lem:combinatorial_exponent}**

We rewrite the expression for the combinatorial exponent from Section \ref{sec:combinatorial_exponent}:

$$\psi_{com}(h) = \frac{1}{r} - \delta \sum_{i=1}^r H \left( \frac{h_i}{1 - \delta} \right) + (\sum_{i=1}^r rh_i - \delta) \log 2.$$  

The concavity of this function follows from the fact that the standard entropy function $H(.)$ is a concave function. From its expression in Section \ref{sec:external_angle} we observe that the external angle exponent is a linear
function of \( h \) and hence a concave function. The concavity of the internal angle exponent is slightly more involved and we spend the rest of this section in proving it.

The internal angle exponent as computed in Section III-C3 is given by

\[
\psi_{\text{int}}(h) = -\left( \sum_{i=1}^{r} rh_i - \delta \right) \log 2 - \left( \frac{\bar{c}_0}{2c_1} y^2 + \bar{c}_0 \lambda_0^*(y) \right).
\]

The quantity \( \bar{c}_0 = \bar{c}_0(h) = \sum_i f_i h_i \) is a linear function of \( h \). So \( \bar{c}_0(h) = \sum_{i=1}^{r} f_i h_i \) is a convex quadratic function of \( h \). Therefore it suffices to prove that \( F(h) \) is convex in \( h \).

A PPENDIX

ANGLE EXPONENTS FOR THE TYPICAL FACE \( F_p \)

Divide the interval \([0, 1]\) into \( r \) equally spaced intervals. Let the face \( F_p \) in consideration have \( n \delta_i \) indices in the \( i^{th} \) interval. Also, let \( g_i = r \delta_i \). The asymptotic exponents for this face can be obtained easily by a straight-forward generalization of the procedure described in Section III-A. We give the final expressions for the combinatorial, internal and external angle exponents for a given value of \( g = (g_1, g_2, \ldots, g_r)^T \).

In what follows, we use \( f_i = f(i r) \).

1) Combinatorial Exponent:

\[
\psi_{\text{com}}(h) = \frac{1}{r} \sum_{i=1}^{r} (1 - g_i) H \left( \frac{h_i}{1 - g_i} \right) + \sum_{i=1}^{r} (h_i - g_i),
\]

where \( H(.) \) is the binary entropy function with base \( e \).

2) Internal Angle Exponent: The negative of the internal angle exponent is given by

\[
\psi_{\text{int}}(h, y) = -r \sum_{i=1}^{r} (h_i - g_i) - \left( \frac{\bar{c}_0^2}{2c_1} y^2 + c_0 \lambda_0^*(y) \right),
\]

where

\[
c_0 = \frac{1}{r} \sum_{i=1}^{r} f_i h_i,
\]

\[
c_1 = \frac{1}{r} \sum_{i=1}^{r} f_i^2 g_i,
\]

\[
\lambda_0^*(y) = \max_s sy - \lambda_0(s),
\]

\[
\lambda_0(s) = \frac{1}{r} \sum_{i=1}^{r} \frac{1}{c_0} \lambda(s f_i)(h_i).
\]

Here \( \lambda(u) = \frac{u^2}{2} + \log(2\Phi(u)) \) is the characteristic function of the standard half-normal distribution.
3) **External Angle Exponent:** The negative of the external angle exponent is given by
\[
\psi_{\text{ext}}(h, x) = -\left(c_2 x^2 - \log(G_0(x))\right),
\]
where,
\[
c_2 = \frac{1}{r} \sum_{i=1}^{r} f_i^2 (g_i + h_i), \quad \text{and}
\]
\[
\log(G_0(x)) = \int_0^1 \log(\text{erf}(x f(u))) du - \frac{1}{r} \sum_{i=1}^{r} \log(\text{erf}(x f_i))(h_i + g_i).
\]

4) **Total Exponent:** Combining the exponents we define the total exponent as
\[
\psi_{\text{tot}} = \max_{h, x, y} \psi_{\text{com}} + \psi_{\text{int}}(h, y) + \psi_{\text{ext}}(h, x)
\]
subject to
\[
\frac{1}{r} \sum_{i=1}^{r} h_i \geq \alpha - \delta,
\]
\[
0 \leq h_i \leq 1 - g_i,
\]
where \(\delta = \frac{1}{r} \sum_{i=1}^{r} g_i\). From Theorem 3, the total exponent satisfies
\[
\frac{1}{n} \log(P(E|\mathbf{x} \in F)) \leq \bar{\psi}_{\text{tot}} + o(1).
\]
So as long as the quantity \(\bar{\psi}_{\text{tot}} < 0\), weighted \(\ell_1\)-minimization succeeds in recovering the sparse signal with an exponentially small probability of failure.

REFERENCES

[1] E. Candès and T. Tao, “Decoding by linear programming,” IEEE Trans. on Information Theory, vol. 51, no. 12, pp. 4203–4215, December 2005.

[2] D. Donoho, “High dimensional centrally symmetric polytopes with neighborliness proportional to dimension,” Discrete and Computational Geometry, vol. 102, no. 27, pp. 617–652, 2006.

[3] D. Needell and J. A. Tropp, “CoSaMP: Iterative signal recovery from incomplete and inaccurate samples,” Appl. Comput. Harmon. Anal., vol. 26, no. 3, pp. 301–321, 2009.

[4] T. Blumensath and M. E. Davies, “Iterative hard thresholding for compressed sensing,” Applied and Computational Harmonic Analysis, vol. 27, no. 3, pp. 265–274, 2009.

[5] B. Recht, M. Fazel, and P. Parrilo, “Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization,” SIAM Review, vol. 52, no. 3, pp. 471–501, 2010.

[6] B. Recht, W. Xu, and B. Hassibi, “Necessary and sufficient conditions for success of the nuclear norm heuristic for rank minimization,” Mathematical Programming, Series B, vol. 125, no. 3, pp. 265–274, 2010.

[7] W. Dai, M. Seikth, O. Milenkovic, and R. Baraniuk, “Compressive sensing DNA Microarrays,” EURASIP Journal on Bioinformatics and Systems Biology, 2009.

[8] H. Vikalo, F. Parvaresh, S. Misra, and B. Hassibi, “Recovering sparse signals using sparse measurement matrices in compressed DNA microarrays,” IEEE Trans. on Signal Processing, special issue on genomic signal processing, vol. 2, no. 3, June 2008.

[9] R. G. Baraniuk, V. Cevher, M. F. Duarte, and C. Hegde, “Model based compressive sensing,” IEEE Trans. on Information Theory, vol. 56, pp. 1982–2001, April 2010.

[10] W. Xu, “Compressive sensing for sparse approximations: Construction, algorithms and analysis,” Ph.D. dissertation, California Institute of Technology, Department of Electrical Engineering, 2010.

[11] M. A. Khajehnejad, W. Xu, A. S. Avestimehr, and B. Hassibi, “Analyzing weighted \(\ell_1\) minimization for sparse recovery with nonuniform sparse models,” IEEE Trans. on Signal Processing, vol. 59, no. 5, pp. 1985–2001, 2011.

[12] D. Donoho and J. Tanner, “Neighborliness of randomly-projected simplices in high dimensions,” Proc. National Academy of Sciences, vol. 102, no. 27, 2005.

[13] W. Xu and B. Hassibi, “Compressed sensing over the Grassmann manifold: A unified analytical framework,” Allerton Conference, 2008.
[14] L. A. Santaló, “Geometría integral en espacios de curvatura constante.” Rep. Argentina Publ. Com. Nac. Energía Atómica, Ser. Mat 1, No. 1, 1952.

[15] P. McMullen, “Non-linear angle-sum relations for polyhedral cones and polytopes,” Math. Proc. Cambridge Philos. Soc., vol. 78, no. 2, pp. 247–261, 1975.