Hurwitz numbers and matrix integrals labeled with chord diagrams

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Abstract. We consider products of complex random matrices from independent complex Ginibre ensembles. The products include complex random matrices $Z_i, Z_i^\dagger, i = 1, \ldots, n$, and $2n$ sources (these are the complex matrices $C_i, C_i^*, i = 1, \ldots, n,$ which play the role of parameters). We consider collections of products $X_1, \ldots, X_F,$ constrained by the property, that each of the matrices of the set \{\(Z_iC_i, Z_i^\dagger C_i^*, i = 1, \ldots, n\)\} is included only once on the product $X = X_1 \cdots X_F$. It can be represented graphically as a collection of $F$ polygons with a total number of edges $2n$, and the polygon with number $a$ encodes the order of the matrices in $X_a$. The matrices $Z_i$ and $Z_i^\dagger$ are distributed along the edges of this collection of polygons, and the sources are distributed at their vertices. The calculation of the expected values involves pairing the matrices $Z_i$ and $Z_i^\dagger$. There is a standard procedure for constructing a 2D surface by pairwise gluing edges of polygons, this procedure results to a ribbon graph embedded in the surface $\Sigma_{e^*}$ of some Euler characteristic $e^*$ (this graph also known as embedded graph or fatgraph). We propose a matrix model that generates spectral correlation functions for matrices $X_a, a = 1, \ldots, F$ in the Ginibre ensembles, which we call the matrix integral, labeled network chord diagram. We show that the spectral correlation functions generate Hurwitz numbers $H_{e^*}$ that enumerate nonequivalent branched coverings of $\Sigma_{e^*}$. The role of sources is the generation of ramification profiles in branch points which are assigned to the vertices of the ribbon graph drawn on the base surface $\Sigma_{e^*}$. The role of coupling constants of our model is to generate ramification profiles in $F$ additional branch points assigned to the faces of the ribbon graph (the faces of the ‘triangulated’ $\Sigma_{e^*}$). The Hurwitz numbers for Klein surfaces can also be obtained by a small modification of the model. To do this, we pair any of the source matrices (in that case presenting a hole on $\Sigma_{e^*}$) with the tau function, which we call Mobius one. The presented matrix models generate Hurwitz numbers for any given Euler characteristic of the base surface $e^*$ and for any given set of ramification profiles.

Mathematics Subject Classification (2010). 05A15, 14N10, 17B80, 35Q51, 35Q53, 35Q55, 37K20, 37K30.
Keywords. Hurwitz number, Schur functions, Klein surface, independent complex Ginibre ensembles, products of complex random matrices, Euler characteristic of network chord diagrams, ribbon graph, gluing of polygons, discrete beta-ensembles.

1. Introduction

Hurwitz numbers count $d$-sheeted branched covers of Riemann surfaces of a given genus (we will denote it $g^*$), see for instance [40] for a review. The direct analogue exits also for the case of Klein surfaces, see [8], [9]. A number of facts is known for the topic of Hurwitz numbers and matrix integrals, see [43], [5], [1], [38], [6], [7], [24], [26], [53], [14], [16], [54], [32], [12]. On the other hand, last few years products of random matrices were in the focus of studies for applications in quantum chaos and in information theory [1], [2], [3], [68], [69]. The relation of these two topics was considered in [66], [59]. In [66] it was shown that the partition function of the matrix model generating spectral correlators for the product of complex matrices is the Toda lattice [71] (see also [70] for the overview) tau function of the type introduced in [60] (earlier appeared in different form in [39]). Let us note that the relation of a number of matrix models with Hurwitz numbers follows directly from comparing results of [23] and of [65] (see also [29]). In [59] special products of complex matrices from independent Ginibre ensembles were considered to generate Hurwitz numbers in case the Euler characteristic of the base surface is less than 2. Here, we develop [59] for the case of any given product of complex matrices which is suitable to encode by chord diagrams. We show that the related matrix integral generates a discrete $\beta$-ensemble (where the Euler characteristic of the base surface plays the role of $\beta$), thus, instead of integration over $nN^2$ complex variables we get summation over $N$ variables (this may be compared to [64] and [4]), see formulae (35),(50),(55).

The present work does not deal with the study of Hurwitz numbers in the framework of integrable systems which was started in the pioneer works of Okounkov [56, 57] and later in the article by Goulden and Jackson [23] which was further developed in many papers$^1$

A brief summary of the present work is presented in the Abstract.

$^1$ see [48], [49], [5], [6], [29], [53], [30], [72], [13], [13], [55], [54] (see also reviews [32] and [36]), where the hypergeometric Toda lattice tau functions were used to enumerate covers of the Riemann sphere, and also [33], [51] where hypergeometric BKP tau functions were used to enumerate covers of the real projective plane $\mathbb{RP}^2$. 


2. Preliminaries

2.1. Hurwitz numbers

The geometric definition of Hurwitz numbers can be found in the Appendix A. Here we give combinatorial definition.

**Orientable case.** Consider symmetric group $S_d$ and the equation

$$A_1 \cdots A_k \prod_{j=1}^{g^*} a_j b_j a_j^{-1} b_j^{-1} = 1$$

where $a_j, b_j, A_i \in S_d, j = 1, \ldots, g^*, i = 1, \ldots, k$ and where each $A_i$ belongs to a given cycle class $C_{\Delta^i}$. Then the number of the solutions of this equation over the order of the symmetric group

$$H_{2-2g^*}(\Delta^1, \ldots, \Delta^k) = \frac{1}{d!} \{a_1, \ldots, a_{g^*}, b_1, \ldots, b_{g^*}, A_1, \ldots, A_k \in S_d | A_i \in C_{\Delta^i} \}$$

is called Hurwitz number. These numbers admit geometrical interpretation. In short the Hurwitz number enumerate branched $d$-sheeted covers of a Riemann surface $\Sigma_{g^*}$ of genus $g^*$ by (not necessarily connected) Riemann surfaces with given ramification profiles $\Delta^i, i = 1, \ldots, k$ at each of $k$ critical points, details may be found in the Appendix. The genus $g$ of a cover $\Sigma_g$ is defined with the help of the Riemann-Hurwitz relation

$$e^* = d e - \sum_{i=1}^{k} (d - \ell(\Delta^i))$$

where $\ell(\Delta^i)$ is the length of the partition $\Delta^i$ and where $e^*$ and $e$ are Euler characteristics respectively of the base and of the cover (respectively equal to $2 - 2g^*$ and to $2 - 2g$). In the geometric interpretation equation (1) results from the homomorphism of the fundamental group of the (base) Riemann surface $\Sigma_{g^*}$ to the symmetric group which acts on the numbered $d$ shives of the cover. The path around a critical points, say, $z_i \in \Sigma_{g^*}$ maps to the product of the cyclic permutations related to the ramification profile $\Delta^i$.

**Non-orientable case.** The enumeration problem of counting of branched $d$-sheeted coverings of Klein surfaces of the Euler characteristic $e^* = 2 - g^*$ by other Klein (or Riemann) surfaces may be reduced to the counting of the number of the solutions of the equation

$$A_1 \cdots A_k \prod_{j=1}^{g^*} R_j^2 = 1$$

where $R_j, A_i \in S_d$ and where each $A_i \in C_{\Delta^i}$, where $\Delta^i, i = 1, \ldots, k$ are the set of given ramification profiles in the critical points. Similarly to the

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2It is important that in this consideration only isolated critical points are admissible.
Hurwitz number may be defined as the number of the solutions of (4) over the order of the permutation group:

\[ H_{2-g^*}(\Delta^1, \ldots, \Delta^k) = \# \frac{1}{d!} \{ R_1, \ldots, R_{g^*}, A_1, \ldots, A_k \in S_d | A_i \in C_{\Delta^i} \} \quad (5) \]

For instance, take \( k = 1, \Delta^1 = (1^3) \) and \( g^* = 1 \) (the number \( 2 - g^* = 1 \) is the Euler characteristic of the real projective plane \( \mathbb{R}P^2 \) and the number \( H_1((1^d)) \) counts \( d \)-sheeted unbranched covers of \( \mathbb{R}P^2 \), see the Appendix A devoted to the geometrical definition of Hurwitz numbers). Then the number of solutions of the equation \( R_1^2 = 1 \) in \( S_d, d = 3 \) is equal to 4 (three transpositions and the identity element). \( H_1((1^3)) = \frac{4}{3!} = \frac{2}{3} \) (compare to Example 4 in Appendix A).

**Mednykh formula.** It was found in the papers of A. Mednykh [44], Mednykh and Pozdnyakova [45] (and also in [20]) that in both orientable and non-orientable cases there is the unique formula for Hurwitz numbers in terms of characters of the symmetric group. It depends on the Euler characteristic of the base surface \( e^* \) and the set of ramification profiles \( \Delta^i \) in critical points as follows:

\[ H_{e^*}(\Delta^1, \ldots, \Delta^k) = \sum_{\lambda} \left( \frac{\dim \lambda}{d!} \right)^{e^*} \varphi_{\lambda}(\Delta^1) \cdots \varphi_{\lambda}(\Delta^k) \quad (6) \]

Here \( \varphi_{\lambda}(\Delta^i) = |C_{\Delta^i}| \frac{\chi_{\lambda}(\Delta^i)}{\dim \lambda} \) where \( \chi_{\lambda}(\Delta^i) \) is the character of the irreducible representation of \( S_d \) labelled by the partition \( \lambda \) and evaluated at the cycle class labelled by the partition \( \Delta^i \), \( \dim \lambda = \chi_{\lambda}(1^d) \) is the dimension of this representation and \( |C_{\Delta^i}| \) is the cardinality of the cycle class \( \Delta^i \).

At last, let us introduce the following notation

\[ H_{e^*}^E (\lambda^1, \ldots, \lambda^m, k + m) = \sum_{\Delta^1, \ldots, \Delta^k} H_{e^*}(\lambda^1, \ldots, \lambda^m, \Delta^1, \ldots, \Delta^k) \quad (7) \]

where the summation range is constrained by the Riemann-Hurwitz condition (6): \( d(e^* - k - m) + \sum_{i=1}^m \ell(\lambda^i) + \sum_{i=1}^k \ell(\Delta^k) = E \) which denotes the sum of all Hurwitz numbers that enumerate \( d \)-sheeted covers of the Euler characteristic \( e^* \) with at most \( k \) branch points on the base surface with Euler characteristic \( e^* \), and one ramification profile is fixed as \( \lambda \).

**2.2. Network of chord diagrams and its genus**

There a number of studies of the so-called chord diagrams, for some review see [40]. I will present this topic in a way that is convenient for our purposes.

Consider \( F \) circles (loops), each of which is divided into an even number of clockwise directed arcs of alternating color: black and white. The arcs can be drawn with arrows, respectively black or white. In the future (in the Section 3) we will associate black arcs with matrices from the Ginibre ensemble (alternatively: from a circular ensemble in Subsection 3.5), and
white arcs with source matrices (free parameters of our model). The total number of black (white) arcs is a given fixed number $2n$. Note that more often than black arcs, the edges of a polygon are considered naturally they are separated from each other by vertices instead of white arrows. (All figures in the form of circles (loops) and polygons, we consider up to homeomorphisms thus, do not distinguish polygons and circles with arcs).

Each black arc has a single partner among the other black arcs that can belong to either the same or different loops. We associate these partners with the lines. In the Section 3, these partners will be hermitian conjugate matrices, and the lines indicate the pairing in the statistical ensemble. We call the lines connecting arcs belonging to one loop, chords and lines connecting arcs belonging to different loops, links.

A connected set of the loops discribed above together with chords and links we will call a network chord diagram or simply a network for the sake of brevity.

Let us describe the procedure which may be called ”cutting and joining” loops of the network by contracting chords and links:

- We contract a chord and get two loops, where we preserve the order of the arrows
- We contract a link and naturally unify two loops into one, also preserving the order of the arrows.

Let us remove in $n$ steps all the links and chords in any order. In the end, we get a set consisting of $V$ loops without chords, which are not connected by links. The number of these loops does not depend on of the order in which we carry out these actions, see below the Lemma 1.

We denote $\tilde{g}^*$ the number of links which we contract along the cutting-and-joining procedure. Let us note that we get the following relation

$$V = F + n - 2\tilde{g}^*$$

Indeed, in the beginning we have $F$ loops. Each cutting action adds one loop and each joining action removes one loop.

Next, let us introduce the number $E^* := F - n + V$ and the number $g^*$ related to $E^*$ via $E^* = 2 - 2g^*$. We get $E^* = 2F - 2\tilde{g}^*$ and $g^* = F - 1 + \tilde{g}^*$.

The meaning of $E^*$ and of $g^*$ will be clear from the following consideration:

We describe this process in more detail from a different point of view (as the creation of the so-called ribbon graph (also known as the fatgraph and the embedded graph)):

- When we contract a chord, we attract together two black chord-partners (glue together with rubber glue between) so that the beginning of one arrow corresponds to the end of the arrow-partner. One can see it as the strip bounded by oppositely directed arrows which becomes the (“rubber made”) first edge of the ribbon graph (the same: of the embedded graph, of the fatgraph). Thus, we divide the loop into two ones, keeping the order of all the remaining arrows. Note that in each loop obtained, we
get more white arcs in comparison with black ones. Notice that we do not tear the chain of arrows-arcs of the initial loop and can make a roundtrip following the arrows in its original order and the part of this roundtrip belong to the boundary of the new edge (the edge of future ribbon graph). The interior of the initial loop turn into the interiors of the new loops and of the new fat edge.

- When we contract a link we glue arcs-partners that belong to different loops again in the way that the beginning of one arrow corresponds to the end of the arrow-partner. In this case we also get an edge of the ribbon graph as a strip bounded by to oppositely directed black arrows.

- Finally, we glue all pairs the black arcs and get the so-called embedded (ribbon) graph (see, for example, [40]) the vertices of which in our case consist of loops (or, if you like, polygons). This graph consists of strips and vertices and can be placed on a Riemann surface (for instance, see [40]). One calls the genus $g^*$ and the Euler characteristic $E^*$ to the original system of loops, chords and links (and also the genus and the Euler characteristic of the ribbon graph) genus and the Euler characteristic of this Riemann surface: $g^*(\Gamma) = g^*(\tilde{\Gamma})$, $E^*(\Gamma) = E^*(\tilde{\Gamma})$. It is defined as $E^* = V - n + F$, where $V$ is the number of vertices, $n$ is the number of edges and $F$ is the number of faces (domains homeomorphic to a disk and bounded by edges of a graph). The ribbon graph performs a "triangulation" of the Riemann surface.

- More about the vertices: if we forget about the edges of the ribbon graph, we’ll see a system of $V$ loops, each of which consists of white arcs (arrows pointing clockwise). If we regard it as a polygon, replacing the arcs with edges, then from each vertex of such a polygon, the edges of the ribbon graph are emitted. Each arrow of the white loop follows the black arrow of the border of the strip emerging from the vertex of the polygon that preserves the original order of the arrows. Therefore, ”chord diagrams without chords”, mentioned above, as an end result of the cutting and joining procedure should be considered as the vertices of the ribbon graphs.

- Let us number the pairs of white arrows that directly follow the black arrow-partners and assign symbols $C_i, C_i^*$ for each pair, $i = 1, \ldots, n$. Let’s go around a given loop-vertex and enter the word attached to this vertex that we will compose as the product of the symbols from the set $\{C_i, C_i^*, i = 1, \ldots, n\}$ in the order in accordance with the order of the white arrows on the loop-vertex (we define the product of the symbols up to cyclic permutations). We get $V$ words attached to the vertices $V$ of the ribbon graph. The length of each word is equal to the number of edges of the ribbon graph going from this vertex.

Thus, cutting-and-joining procedure results in the creation of the ribbon graph from a network chord diagram. If we denote the network $\Gamma$ and the ribbon graph $\tilde{\Gamma}$ then the cutting-and-join procedure may be symbolically
written as
\[ \Gamma \rightarrow \tilde{\Gamma} \]
The network may be characterized by the data \( D_\Gamma \) which are the number of
faces \( F \), the number of edges \( n \), the number of vertices \( V \) of the ribbon graph
and also the set of words \( \tilde{C}_1, \ldots, \tilde{C}_V \).

What we get. When we approach a given vertex following the boundary
of the edge of the ribbon graph along a chosen black arrow, we encounter
a white arrow on boundary of the loop-vertex. We follow it and move to
another edge of the ribbon graph, which is black arrow that followed the
white on the original loop. Following this black arrow, we move on to the
next white arrow, which is the boundary of another loop-vertex. So we can
have a round trip according to the chosen initial loop. As we see, indeed, the
number of faces of the ribbon graph is equal to the number of initial loops
and the number of edges is the number of pairs of black arrows. Then the
Euler characteristic of the ribbon graph is completely defined by the number
of it’s vertices.

**Lemma 1.**

1. There exists \( n! \) way to contract all chord and links. The number
   \( \tilde{g}^* \) does not depend on the way
2. The number of the vertices of the ribbon graph is equal to \( V = F + n - 2\tilde{g}^* \). Therefore, the Euler characteristic \( F - n + V =: 2 - 2g^* \) of the
   network chord diagram is equal to \( 2F - 2\tilde{g}^* \) (and the genus \( g^* \) is equal
to \( F - 1 + \tilde{g}^* \)).

Thus, for a network which consists of a single loop with chords \( \tilde{g}^* = g^* \).
The first item will be proven at the end of Section [3].
The proof of the second item is as follows. First, let us make a

**Remark.** One can transform a given network to a minor network by replacing
1) a given neighboring white–black–white arrows by a single white arrow 2) doing the same with the black partner of the chosen black above arrows.
This is the procedure of forgetting of a black pair. Then, one can recollect it
and insert the pair back.

One chose the order to perform the creation of the ribbon graph by
numbering of black pairs. Gluing the first pair he forgets about all other
black arrows replacing all of them as explained above. He gets one edge and
two white arrows which form either a single, or two white loop-vertices. This
is the simplest ribbon graph. Then he recollect the second black pair and
gets the second ribbon graph. Thus one gets the sequence of ribbon graphs
defined by the numeration of the steps.

\[ \Gamma \rightarrow \Gamma_1 \rightarrow \Gamma_{1,2} \rightarrow \cdots \rightarrow \Gamma_{1,\ldots,n} = \tilde{\Gamma} \quad \text{(9)} \]

One can chose another consequence of steps which is obtaines by the re-
enumaration of \( 1, \ldots, n \rightarrow \sigma(1), \ldots, \sigma(n) \), \( \sigma \in S_n \):

\[ \Gamma \rightarrow \Gamma_{\sigma(1)} \rightarrow \Gamma_{\sigma(1),\sigma(2)} \rightarrow \cdots \rightarrow \Gamma_{\sigma(1),\ldots,\sigma(n)} = \tilde{\Gamma} \quad \text{(10)} \]
There exists \( n! \) paths to achieve \( \tilde{\Gamma} \) and there are \( \frac{n!}{k!(n-k)!} \) different \( \Gamma \) with \( k \) subscripts.

Having this remark in mind we see that each cutting step (contraction of the chord) results in adding of 1 edge to the ribbon graph and also of 1 loop-vertex. While each joining step (contraction of the link) results in adding of 1 edge and removing of 1 vertex. We have \( F \) vertices in the beginning and \( n \) steps to create the final ribbon graph. Therefore, at the end we get \( V = F + n - \tilde{g}^* \) vertices.

This Lemma together with Lemma 2 is important.

As is well known after the papers of Kazakov, Bresin [15], Migdal and Gross [25] (see [40] for a review which emphasizes mathematical aspects), the ribbon graphs can be listed using models of Hermitian matrices. In our case (see Section 3) the ribbon graph will initially be specified by the choice of the matrix model. Thus, for each Feynman graph of the one-matrix model we assign the matrix model labeled by this graph.

2.3. Random matrices. Complex Ginibre ensemble

**Complex Ginibre ensembles.** On this subject there is an extensive literature, for instance see [1–3, 68, 69].

We will consider integrals over \( N \times N \) complex matrices \( Z_1, \ldots, Z_n \) where the measure is defined as

\[
d\Omega(Z_1, \ldots, Z_n) = \prod_{\alpha=1}^{n} d\mu(Z_\alpha) = c_N^n \prod_{\alpha=1}^{n} \prod_{i,j=1}^{N} d\Re(Z_\alpha)_{ij} d\Im(Z_\alpha)_{ij} e^{-N|\langle Z_\alpha \rangle_{ij}|^2} \tag{11}
\]

where the integration range is \( \mathbb{C}^{N^2} \times \ldots \times \mathbb{C}^{N^2} \) and where \( c_N^n \) is the normalization constant defined via \( \int d\Omega(Z_1, \ldots, Z_n) = 1. \)

We treat this measure as the probability measure. The related ensemble is called the ensemble of \( n \) independent complex Ginibre ensembles. The expectation of a quantity \( f \) which depends on entries of the matrices \( Z_1, \ldots, Z_n \) is defined by

\[
\mathbb{E}_{n,N}(f) = \int f(Z_1, \ldots, Z_n) d\Omega(Z_1, \ldots, Z_n).
\]

The subscript \( n \) reminds that the expectation is estimated in the product of \( n \) independent Ginibre ensembles, and the second subscript, \( N \), - that the Gauss measure is not chosen as \( e^{-\text{tr} ZZ^\dagger} \), but in the form \( e^{-N \text{tr} ZZ^\dagger} \).

**Spectral correlation functions.** For any given matrix \( X \) and a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) we introduce the following notations

\[
p(X) = (\text{tr } X, \text{tr } X^2, \text{tr } X^3, \ldots) \tag{12}
\]

\[
p_\lambda(X) = \text{tr } X^{\lambda_1} \text{tr } X^{\lambda_2} \ldots \text{tr } X^{\lambda_\ell} \tag{13}
\]

Each \( \text{tr } X^{\lambda_i} \) is the Newton sum \( \sum_{a=1}^{N} x_a^{\lambda_i} \) of the eigenvalues \( x_a, a = 1, \ldots, N \) of the matrix \( X \).
We are interested in the spectral correlation functions $E_{n,N}(p_{\lambda^1}(X_1) \cdots p_{\lambda^m}(X_m))$ where $X_i$, $i = 1, \ldots, m$ is a set of matrices and $\lambda^i = (\lambda^i_1, \lambda^i_2, \ldots)$, $i = 1, \ldots, m$ is a set of given partitions.

Let us introduce the notations $p = (p_1, p_2, \ldots)$ which is the semi-infinite set of parameters and $V(X, p) = \sum_{n>0} x^n s(n)(p)$

Then it is well-known that

$$e^{N \text{tr} V(X, p)} = \sum_{\Delta} \frac{1}{z_{\Delta} N^{\ell(\Delta)}} p_{\Delta}(X) p_{\Delta} \quad (15)$$

where the sum ranges over all partitions $\Delta = (\delta_1, \delta_2, \ldots, \delta_k)$, $\delta_k > 0$, $k = 0, 1, 3, \ldots$ and $\ell(\Delta)$ denotes the length of the partition $\Delta$, i.e. the number of the non-vanishing parts of $\Delta$. The notations are as follows: $p_{\Delta} = p_{\delta_1} p_{\delta_2} \cdots$, and $z_{\Delta} = \prod_{i=1}^{\infty} i^{m_i} m_i!$ where $m_i$ is the number of parts $i$ which occur in the partition $\Delta$. For instance, for the partition $\Delta = (5, 5, 2, 1, 1)$ we get $z_{\Delta} = 5^2 \times 2! \times 2 \times 1! \times 1^2 \times 2! = 200$.

**Remark.** Let us note that the generation function of the spectral invariants may be chosen as

$$\mathbb{E} \left( e^{N \text{tr} V(X, p^{(1)})} \cdots e^{N \text{tr} V(X_m, p^{(m)})} \right) \quad (16)$$

Indeed, with the help of (15) the Taylor series in parameters $p_k^{(i)}$ yields the mentioned spectral correlation functions.

### 2.4. Hypergeometric tau functions

**Schur functions.** In what follows we need polynomials in many variables called functions of Schur labeled by partitions [42]. First, we introduce the so-called elementary Schur functions $s_{(n)}$, labeled by partitions $(n)$ with one part equal to $\lambda_1 = n$, which are defined as follows:

$$e^V(x, p) = \sum_{n \geq 0} x^n s_{(n)}(p)$$

In particular, $s_{(0)}(p) = 1$, $s_{(1)}(p) = p_1$, $s_{(2)}(p) = \frac{1}{2}(p_1^2 + p_2)$.

Schur function $s_{\lambda}$ labeled by a given partition $\lambda = (\lambda_1, \ldots, \lambda_N)$ is defined in terms of the elementary ones by

$$s_{\lambda}(p) = \det \left( s_{(\lambda_i - i+j)}(p) \right)_{i,j} \quad (17)$$

We shall write the Schur function also as the function of matrix argument which we write as a capital letter say $X$ having in mind that it is $s_{\lambda}(X) := s_{\lambda}(p(X))$ where $p(X) = (p_1(X), p_2(X), \ldots)$ with $p_n(X) = \text{tr} X^n$.

3Throughout the paper the upper index of the parts partitions is not a power but a label which indexes different partitions.
If $x_1, \ldots, x_N$ are the eigenvalues of the $N \times N$ matrix $X$ then $s_\lambda(X)$ is the symmetric homogenous polynomial in eigenvalues and can be written as

$$s_\lambda(X) = \frac{\det (x_j^{N+i-j-1})}{\det (x_j^{N-j})}$$ \hspace{1cm} (18)

The formula known as Cauchi-Littlewood relation is very useful

$$e^{N \text{tr} V(X,p)} = \sum_\lambda s_\lambda(X)s_\lambda(Np)$$ \hspace{1cm} (19)

where the sum ranges over all partitions whose length (the number of non-vanishing parts) does not exceed $N$, and $Np := (Np_1, Np_2, Np_3, \ldots)$.

**Degree and Euler characteristic.** For each ratio of Schur functions labeled with the same partition, we assign the degree $deg$ as follows

$$deg \left( \prod_i (s_\lambda(A_i))^{d_i} \right) = \sum_i d_i$$ \hspace{1cm} (20)

As follows from the Mednykh formulas (1) and (4), sums over all $\lambda$ of such expressions can be used to generate the Hurwitz numbers, where the degree gives the Euler characteristic of the base surface.

**Content product.** For a given number $x$ and a given Young diagram $\lambda$ the content product is defined as the product

$$(x)_\lambda := \prod_{(i,j) \in \lambda} (x + j - i)$$ \hspace{1cm} (21)

The number $j - i$, which is the distance of the node with coordinates $(i, j)$ to the main diagonal of the Young diagram $\lambda$ is called the content of the node. For one-row $\lambda$, the content product is the Pochhammer symbol $(a)_\lambda^1$. For a given function of one variable $r$, we define the generalized content product (the generalized Pochhammer symbol) as

$$r_\lambda(x) = \prod_{(i,j) \in \lambda} r(x + j - i)$$ \hspace{1cm} (22)

The content product plays an important role in the representation theory of the symmetric groups. It was used in [60] to define certain family of tau functions which we called hypergeometric tau functions.

**Example.** The example of the content product may be constructed purely in terms of the Schur functions: if we choose

$$r(x) = \prod_i \left( \frac{1-q_i t_i x}{1-t_i x} \right)^{d_i}$$ \hspace{1cm} (23)

where $q_i, t_i, d_i$ are parameters, we obtain

$$r_\lambda(x) = \prod_i \left( \frac{s_\lambda(p(q_i, t_i))}{s_\lambda(p(0, t_i))} \right)^{d_i}$$ \hspace{1cm} (24)
One can degenerate (23) to the rational function and obtain

\[ r_\lambda(x) = \prod_{i=1}^{p} (a_i)^{\lambda} \prod_{i=1}^{q} (b_i)^{\lambda} \]

Above we used the following special notations:

\[ p_\infty = (1, 0, 0, \ldots), \quad p(a) = (a, a, a, \ldots), \quad p_m(q, t) = \frac{1 - q^m}{1 - t^m} \]

Actually, any reasonable content product can be interpolated by expressions (25). Because of this, the degree of content products always vanishes.

Hypergeometric tau functions of the Toda lattice and two-component KP hierarchy. The function

\[ \tau_r(x, p, p^*) := \sum_\lambda r_\lambda(x)s_\lambda(p)s_\lambda(p^*) \]

solves an infinite number of compatible equations of differential equations, separately, in the variables p (KP hierarchy), separately in variables p* (second KP hierarchy) and also in the variable x which is supposed to be a discrete variable. It was introduced and analyzed in detail in [60], but, in fact, it appeared earlier in [39] in a different way without the usage of content product. This family of tau functions has numerous applications, some of them are mentioned in the Appendices to to [60] and to [62]. The well-known hypergeometric functions in one variable (the Gauss one, basic ones, the so-called generalized ones) together with certain hypergeometric functions of matrix argument (for instance Milne’s hypergeometric function) are examples of (27).

We will write also \( \tau_r(x, p, X) \) having in mind that the Schur function in (27) is written as a matrix. For instance, if we select the content product as in example (25), and if we choose the matrix X to be 1 × 1 matrix and p to be (1, 0, 0, \ldots), we obtain the so-called generalized hypergeometric function

\[ _pF_q(\{a_i\}, \{b_i\}, X) \]

Let us note that we can write the argument of the tau function not as \( p = (p_1, p_2, \ldots) \) but as \( Np = (Np_1, Np_2, \ldots) \). In this case the variables \( Np_i, i > 0 \) play the role of the higher times [33]. This replacement turns out to be suitable in \( N \rightarrow \infty \) limit. It was used, say, in [55] in the study of Hurwitz numbers generated by the model of normal matrices. It is also suitable for us in view of the choice of Gauss measure in the Ginibre ensembles in form presented by (11).

The simplest (and the main for our purposes) example is the case \( r \) identically is equal to one. Such tau function will be denoted \( \tau_1 \). It does not

\[ ^4 \text{In what follows, this leads to the fact that in generation functions the content products do not affect the Euler characteristic of the base surface, but only affect ramification type of the covering map.} \]
depend on $x$:
\[
\tau_1(x, X, Np) = e^{N \text{tr} V(X, p)} = e^{N \sum_{i=1}^{N} \sum_{m>0} \frac{1}{m} x_i^m p_m} = \sum_{\lambda} s_{\lambda}(X) s_{\lambda}(Np)
\]  
(28)

where $x_1, \ldots, x_N$ are eigenvalues of $X$, in addition, for such tau function we have (15).

Remark. Let us specify the set of variables $p = (p_1, p_2, \ldots)$ in formula (19) as follows:
\[
p_m = p_m(\{d_i, x_i\}) := -\sum_{i=1}^{L} d_i x_i^m
\]  
(29)

Then,
\[
e^{N \text{tr} V(X, p)} = \prod_{i=1}^{N} \det (1 - x_i X)^{Nd_i}
\]  
(30)

If all $Nd_i$ are natural numbers, (30) is a polynomial function of entries of $X$; the right hand sides of (19) and of (15) have a finite number of terms. In this case, as follows from the properties Schur functions, see (74) in Appendix B $s_{\lambda}(Np) = 0$ if $\lambda_1 > N \sum_i d_i$ and tau function (27) is also a polynomial.

**Hypergeometric tau function of the BKP hierarchy.** The expression
\[
\tau_r^B(M, x, p) := \sum_{\ell(\lambda) \leq M} r_\lambda(x)s_\lambda(p)
\]  
(31)

is also a tau function but now it is a tau function of the hierarchy introduced in [34], which authors called the "fermionic" BKP hierarchy and we call the "large" BKP hierarchy (to make difference with the BKP hierarchy invented in [33]). Tau function (31) appeared in [63]. The simplest (and most important for us) example is again the case where $r$ is identically equal to 1 and $M = \infty$:
\[
\tau_1(X) = \sum_{\lambda} s_\lambda(X) =
\]  
(32)

\[
e^{\sum_{m>0} \frac{1}{m} (\text{tr} X)^{2m} + \sum_{m\text{ odd}} \frac{1}{m} \text{tr} X^m} = \prod_{i=1}^{N} (1 - x_i)^{-1} \prod_{i<j}^{N} (1 - x_i x_j)^{-1}
\]

where $x_1, \ldots, x_N$ are eigenvalues of $X$.

Notice that
\[
\deg (\tau_r) = 2, \quad \deg (\tau_r^B) = 1
\]  
(33)

3. **Products of complex and random matrices and certain sums related to chord diagrams and Hurwitz numbers**

The expression
\[
Z_1Z_2 \cdots Z_{n-1}Z_n Z^\dagger_n Z^\dagger_{n-1} \cdots Z^\dagger_2 Z^\dagger_1
\]
where random matrices $Z_i, i = 1, \ldots, n$ belong to $n$ independent complex Ginibre ensembles was the object of study in numerous papers (in particular,
in relation to quantum chaos and to transmission problems see [1], [2], [3], [68], [69], [10], in relation to Hurwitz numbers see [14], [53] in relation to tau functions see [66]).

We want to consider modifications of this product, namely, let us:

- add constant (the "source") matrices between random ones: \( Z_i \to Z_i C_i, \) \( Z_i^\dagger \to Z_i^\dagger C_i^* \)
- permute the order in the product in an arbitrary way which we encode by a chord diagram
- factorize this product into \( F \) factors and introduce network chord diagram to encode it

### 3.1. The model of complex matrices labeled by a network

Consider a set of \( N \times N \) matrices \( \{Z_1 C_1, Z_1^\dagger C_1^*, Z_2 C_2, Z_2^\dagger C_2^*, \ldots, Z_n C_n, Z_n^\dagger C_n^* \} \), where \( C_1, \ldots, C_n, C_1^*, \ldots, C_n^* \) are given complex matrices (source matrices) and each of \( Z_i, i = 1, \ldots, n \) belongs to the \( i \)-th complex Ginibre ensemble. Here and below, the dag denotes Hermitian conjugation, and \( C_i^* \) is unrelated to \( C_i \). Notice that each matrix from Ginibre ensemble is multiplied from the right by the source matrix with the same number. The order in the Thus, we consider a product of \( 2n \) matrices where the matrices \( Z_i C_i \) and \( Z_i^\dagger C_i^* \) enter in a given order. Each of the written above \( 2n \) matrices enters the product only once, and this condition is important in what follows. We denote this product \( X \). Each possible product \( X \) can be presented graphically as a loop with \( 2n \) black directed arcs and \( 2n \) white directed arcs as we explained in Subsection 2.2, black arrows are related to random matrices and white arrows are related to the source matrices. Each pair of hermitian conjugate random matrices is associated by the chord. This is the case of the single loop (the chord diagram), that is \( F = 1 \) as it explained in Subsection 2.2. The general case related to the network of chord diagrams is obtained by splitting this product into factors (sub-products) \( X = X_1 \cdots X_F \) in a way that the source matrices are nearest right neighbours of each \( Z_i \) and to each \( Z_i^\dagger \) as it was before, and we also ask the obtained network to be connected.

Thus, we have a given network, say \( \Gamma \), which defines matrix products in \( X_1, \ldots, X_F \) and related ribbon graph \( \tilde{\Gamma} \) equipped with data \( D_\Gamma \), namely, the number of faces \( F \), the number of edges \( n \), the number of vertices \( V \) (and the Euler characteristic \( \chi \) equal to \( F - n + V \)) and the set of words \( \tilde{C}_1, \ldots, \tilde{C}_V \). Then

**Theorem 1.** Consider the set of tau functions \( \Gamma \):

\[
\tau_{r(1)} \left( x, X_1, Np^{(1)} \right), \ldots, \tau_{r(F)} \left( x, X_F, Np^{(F)} \right)
\]

defined by the set of given functions \( r^{(1)}, \ldots, r^{(F)} \), which depend on the matrix products \( X_1, \ldots, X_F \) described by the network \( \Gamma \) with data \( F, n, V \) and
Consider the expectation value of the product of these tau functions in $n$ independent Ginibre ensembles.

$$
\mathbb{E}_{n,N} \left( \prod_{a=1}^{F} \tau_{r(a)}(x, X_a, Np^{(a)}) \right) = \sum_{\ell(\lambda) \leq N} r_{\lambda}(x) \left( s_{\lambda}(Np_\infty) \right)^{-n} \prod_{a=1}^{F} s_{\lambda}(Np^{(a)}) \prod_{a=1}^{V} s_{\lambda}(\tilde{C}_a) \tag{35}
$$

where $r_{\lambda}(x)$ is the content product \((22)\) where $r = \prod_{a=1}^{F} r^{(a)}$, where $p_\infty := (1, 0, 0, \ldots)$, and $p^{(a)} = (p_1^{(a)}, p_2^{(a)}, \ldots)$, $a = 1, \ldots, F$ are sets of parameters.

**Remark.** Remark 1. Note that for general values of the parameters $p^{(a)}$ both (34) and (35) diverge. However, there are open domains of these variables (parameterized by numbers $NL, Nd_1, \ldots, Nd_L \in \mathbb{N}$ and $x_1, \ldots, x_L \in \mathbb{C}$ from (29)), where both (34) and (35) are finite.

Notice that if we choose the function $r$ to be in form \((22)\) the series (35) is written only in terms of the Schur functions. (35) generalizes the Hurwitz generating series suggested in [6].

In certain cases the integral of tau functions (34) is a tau function itself, however these cases are related to $E^* = 2, 1$ see for instance, Examples 2, 3, 4 in this Subsection.

**Remark.** Remark 2. Notice that the degree of the product of the tau functions in (34) is equal to $2F$, while the degree of (35) is $F - n + V =: 2F - 2\bar{g}^*$ where $\bar{g}^* \geq 0$.

**Corollary 1.** For $F = 1$ and $r = 1$ case, we get

$$
\mathbb{E}_{n,N} \left( e^{N \text{tr} \text{tr}(Xp)} \right) = \sum_{\lambda} (s_{\lambda}(Np_\infty))^{-n} s_{\lambda}(Np) \prod_{i=1}^{V} s_{\lambda}(\tilde{C}_i) \tag{36}
$$

In particular, if all sources are $N \times N$ identity matrices we get

$$
\mathbb{E}_{n,N} \left( e^{N \text{tr} \text{tr}(Xp)} \right) = \sum_{\lambda} (s_{\lambda}(Np_\infty))^{-n} s_{\lambda}(Np) (s_{\lambda}(\mathbb{I}_N))^V \tag{37}
$$

where $s_{\lambda}(\mathbb{I}_N) = (N)_\lambda s_{\lambda}(p_\infty)$, for the notation $(N)_\lambda$ see (27).

**Example 1.** Consider the product $X = Z_1C_1Z_2C_2Z_1^\dagger C_1^* Z_2^\dagger C_2^*$ which is related to the chord diagram with two intersecting chords. As we can find in this case $F = 1$, $n = 2$, $V = 1$ (so, $E^* = 0$ which is related to the torus) and we get a single word equal to $C_2C_1^* C_1^*$ (Thus, we get 4 edges of the ribbon graph coming from the single vertex). In case all source matrices were $\mathbb{I}_N$, we obtain $\sum_{\lambda} (N)_\lambda s_{\lambda}(Np) (s_{\lambda}(Np_\infty))^{-1}$ in the right hand side of (37).
Example 2 Consider $X = Z_1 C_1 Z_2 C_2 \cdots Z_n C_n (Z_n^* C_n^* \cdots Z_2^* C_2^* Z_1^*)^5$. This is an example of a chord diagram where chords do not intersect. It is easy to show that in such case we always have $e^* = 2$. The set of words consists of $V = n + 1$ matrices: $C_n$, $C_1^*$ and of $C_i C_{i+1}^*$, $i = 1, \ldots, n - 1$. The ribbon graph is the linear tree graph. In case all source matrices are identity ones (therefore, $X$ is Hermitian), we get $\sum_\lambda \left((N \lambda)^{n+1}\right) s_\lambda(N \mathbf{p}) s_\lambda(N \mathbf{p}_\infty)$ in the right hand side of (37) that is tau function (27) where $r(x) = x^{n+1}$, and this case was carefully studied, in particular see [1], [2], [3].

Example 3. Consider $X = (Z_1 C_1 Z_1^* C_1^+) \cdots (Z_n C_n Z_n^* C_n^+)$. This is another example of chord diagram where chords do not intersect. The number of vertices is equal to $n + 1$. The words are $C_1, C_2, \ldots, C_n$ and $C_1^* C_2^* \cdots C_n^*$ (thus, $n$ edges of the ribbon graph come out of the single nontrivial vertex. This is a star-like ribbon graph drawn on the Riemann sphere). In case all source matrices are identity ones, $X$ is the product of positive Hermitian matrices. In that case we get the same answer $\sum_\lambda \left((N \lambda)^{N+1}\right) s_\lambda(N \mathbf{p}) s_\lambda(N \mathbf{p}_\infty)$ in the right hand side of (37) as in the previous Example.

Other examples of the $F = 1$, in particular related to the case $e^*$ may be found in the previous work [67].

Now consider the case where $r = 1$ with sets of faces $F > 1$.

Corollary 2.

$$E_{m,N} \left(e^{N \text{ tr } \mathbf{v}(X_1 \mathbf{p}^1)} \cdots e^{N \text{ tr } \mathbf{v}(X_F \mathbf{p}^F)}\right) = \sum_\lambda (s_\lambda(N \mathbf{p}_\infty))^{-n} \prod_{i=1}^F s_\lambda(N \mathbf{p}^{(i)}) \prod_{i=1}^V s_\lambda(C_i)$$  \hspace{1cm} (38)

In particular, if all source matrices are equal to $\mathbb{I}_N$ we get

$$E_{m,N} \left(e^{N \text{ tr } \mathbf{v}(X_1 \mathbf{p}^1)} \cdots e^{N \text{ tr } \mathbf{v}(X_F \mathbf{p}^F)}\right) = \sum_\lambda (s_\lambda(\mathbb{I}_N))^{V} (s_\lambda(N \mathbf{p}_\infty))^{-n} \prod_{i=1}^F s_\lambda(N \mathbf{p}^{(i)})$$  \hspace{1cm} (39)

Let us notice that if $N = 1$, the matrices commute, and the answer does not depend on the order in the product that defines the number $V$. And we see that it is the case because each $s_\lambda(\mathbb{I}_N) = 1$.

Thus, the number of the factors $s_\lambda(C_i)$ is the number of vertices, the number of factors $s_\lambda(N \mathbf{p}_\infty)$ is the number of edges, and the number of factors $s_\lambda(N \mathbf{p}^{(i)})$ is the number of faces of the ribbon graph. The formula (38) is nice. In [61] we appreciate the expression of hypergeometric tau functions written only in terms of the Schur functions, which obtained if we use the content product (25).

By (15) (choosing only $|\lambda| = 1$ terms in the right hand side of (38)), we get

---

5 In case where $C_1, C_n$ are Hermitian and $C_i = C_{i+1}^t$, $C_i^* = C_i^t$, $i = 1, \ldots, n - 1$ the matrix $X$ is Hermitian and it is the only case of Hermitian $X$. 

---
Corollary 3.

\[ E_{n,N}(\text{tr} X_1 \cdots \text{tr} X_F) = N^{-n} \prod_{i=1}^{V} \text{tr} \tilde{C}_i \]  

(40)

In case all sources are identity \( N \times N \) matrices, we obtain

\[ E_{n,N}(\text{tr} X_1 \cdots \text{tr} X_F) = N^{V-n} = N^{e^* - F} \]  

(41)

It follows from this Corollary then the expectation value in the right hand side of (40) grows with \( N \) only in case \( e^* = 2 \) (Riemann sphere) and \( F = 1 \). Otherwise the right hand side of (41) vanishes if \( N \to \infty \).

From this Corollary it follows that the expectation on the right-hand side of (40) grows together with \( N \) only in the case \( e^* = 2 \) (the Riemann sphere) and \( F = 1 \).

**Example 4.** Take \( F = 2 \) and \( X_1 = Z_1 C_1 \ldots Z_n C_n, \) \( X_2 = Z_1^\dagger C_n^* \ldots Z_n^\dagger C_1^* \).

As one can see in this case \( V = n \) (thus, \( e^* = 2 \)) and the words are \( C_i C_i^*, \ i = 1, \ldots, n \). We have two faces (regions delimited by the graph). The right hand side of (38) is equal to

\[
\sum_\lambda s_\lambda(N P^{(1)}) s_\lambda(N P^{(2)}) \prod_{i=1}^{n} \frac{s_\lambda(C_i C_i^*)}{s_\lambda(N P_\infty)} , \quad C_{n+1} := C_1^*
\]

In case all source matrices were \( \mathbb{I}_N \) it is equal to \( \sum_\lambda ((N)\lambda)^n s_\lambda(N P^{(1)}) s_\lambda(N P^{(2)}) \) which is tau function (27) with \( r(x) = x^n \).

**Example 5.** Take \( F = n \) and \( X_a = Z_a C_a Z_{a+1}^\dagger C_{a+1}^* \), \( 1 \leq a < n \) and \( X_F = Z_n C_n Z_1^\dagger C_1^* \) (a closed chain). We obtain two vertices (so, \( e^* = 2 \)) and two words \( \tilde{C}_1 = C_1 C_2 \cdots C_n, \tilde{C}_2 = C_n^* C_{n-1}^* \cdots C_1^* \).

In case all source matrices were \( \mathbb{I}_N \), we get that the right hand side of (38) is equal to

\[
\sum_\lambda ((N)\lambda)^2 (s_\lambda(N P_\infty))^{2-n} \prod_{a=1}^{n} s_\lambda(N P^{(a)}).
\]

It can be identified with the tau function (27), if we fix each set \( P^{(a)}, a = 1, \ldots, n \) to be in the form (26), with the exception of the selected two that we will interpret as higher times of the two-component KP hierarchy.

**About certain sums.** Consider the sum

\[ Y = \sum_{i=1}^{n} \left( Z_i C_i + Z_i^\dagger C_i^* \right) \]  

(42)

where matrices \( Z_i, i = 1, \ldots, n \) belong to \( n \) independent complex Ginibre ensembles, and complex matrices \( C_i, i = 1, \ldots, \), plays the role of sources. Let us split the sum \( Y \) into the sum of \( v \) terms \( Y = Y_1 + \cdots + Y_F \). Denote \( k_i \) the number of terms in \( Y_i \), and denote \( J_i \) the collection of all matrices from the set \( \{ Z_a C_a, Z_a^\dagger C_a^*, \ a = 1, \ldots, n \} \) that enter \( Y_i \). We have \( k_1 + \cdots + k_F = 2n \). For instance, \( Y_1 = Z_1 C_1 + Z_2 C_2 + Z_n^\dagger C_n^* \) and \( Y_2 = Y - Y_1 \); then, \( k_1 = 3, k_2 = 2n - 3 \)

We denote the subset of matrices which enter \( Y_i \) by \( J_i \).

Let us rescale \( C_i \to a_i^{-1} C_i \). Consider

\[ E_{n,N}(\text{tr} Y_1^{m_1} \cdots \text{tr} Y_F^{m_F}) = \text{Pol}(a^{-1}) \]  

(43)
where $m_i \leq k_i$ and $m_1 + \cdots + m_F \leq 2n$. The right hand side of expression (43) is written to notify that it is a polynomial in $a_1^{-1}, \ldots, a_n^{-1}$. Monomials which are multilinear in $a_1^{-1}, \ldots, a_n^{-1}$ may be evaluated with the help of relation (10). Indeed, thanks to the summation in the right hand side of (42), the left hand side of (43) is the sum of many terms which are monomials bilinear in random matrices $Z_i$ and $Z_i^\dagger$. Each monomial obtained in this way may be written as $\text{tr} X_1 \text{tr} X_2 \cdots \text{tr} X_F$, where each $X_i$ is a product of matrices $Z_{ij} C_{ij}$ and $Z_{kj}^\dagger C_{kj}^*$ from the subset of matrices $J_j$ which enter $Y_j$. To apply (10) one needs the requirement that each of $\{Z_i C_i, Z_i^\dagger, i = 1, \ldots, n\}$ enters the product $X_1 \cdots X_F$ at most once. We get it by picking up residuum terms in the right hand side of (43) which is a polynomial in $a_i^{-1}, i = 1, \ldots, n$. We obtain

$$\text{res}_{a_1} \cdots \text{res}_{a_n} E_{n, N} (\text{tr} Y_1^{m_1} \cdots \text{tr} Y_F^{m_F}) = N^{-n} \sum_{\Gamma} \prod_{i=1}^V \text{tr} \tilde{C}_i(\Gamma)$$

where $\Sigma_{\Gamma}$ denotes the sum over $k_1! \cdots k_F!$ networks of chord diagrams with $F$ loops which are obtained by all permutations of endpoints of chord and links along each of loops (which encode all permutations of the matrices in the sets $J_i$), where diagrams obtained by cyclic permutation along loops give the same contribution. For the case when all source matrices are identity ones, in $N \to \infty$ limit the main contribution proportional to $N^{2-F}$ give diagrams with $e^* = 2$ (see (11)), and the main term is equal to $c_2(F) N^{2-F} k_1 \cdots k_F$, where $c_{e^*}(F)$ is the number of chord diagrams with $F$ faces and the Euler characteristic equal to $e^*$.

### 3.2. Hurwitz numbers

Starting from [56], expressions containing sums over $\lambda$ each term of which consists products of the Schur functions labeled with the same partition were used to generate Hurwitz numbers, see for instance [5], [4], [6], [30], [54]. One can assign the ‘Euler characteristic’ to such sums [67], by assigning $\text{deg}$ equal to 1 to each Schur function and getting the degree of ratios of the Schur functions.

The present case is characterized by the fact that, firstly, Euler’s characteristic can be any integer not exceeding 2, and secondly, an amazing coincidence of the Euler characteristic of the base surface for Hurwitz numbers and the Euler characteristics of the network chord diagram (in case of orientable base surface).

By Corollary 2 we obtain

**Theorem 2.** For a given set of partitions $\mu^1 = (\mu_1^1, \mu_2^1, \ldots), \cdots, \mu^F = (\mu_1^F, \mu_2^F, \ldots)$ the spectral correlation functions generates Hurwitz numbers as follows:

$$E_{n, N}(p_{\mu^1}(X_1) \cdots p_{\mu^F}(X_F)) \prod_{a=1}^F \frac{1}{z_{\mu^a}}$$

\[ \delta(\mu^1, \ldots, \Delta^V) N^{-nd} \sum_{\Delta^1, \ldots, \Delta^V} H_{F-n+V} (\mu^1, \ldots, \mu^F, \Delta^1, \ldots, \Delta^V) \prod_{i=1}^{V} p_{\Delta^i}(\tilde{C}_i) \]

where \( \delta(\mu^1, \ldots, \Delta^V) = 1 \) if \(|\mu^1| = \cdots = |\mu^F| = |\Delta^1| = \cdots = |\Delta^V| = d \) and vanishes otherwise. Here \( V \) is the number of vertices of the ribbon graph, \( n \) is the number of edges, \( F \) is the number of faces.

In particular, if all sources matrices are equal to \( I_N \) we get

\[ N^\ell(\mu^1) + \cdots + \ell(\mu^F) \prod_{a=1}^{F} \frac{1}{z_{\mu^a}} = \sum_{E} H_{F-n+V}^E (\mu^1, \ldots \mu^F; \mu^F, \Delta^1, \ldots, \Delta^V) N^E \]

where the summation range is \( \sum_{a=1}^{F} \ell(\mu^a) - nd \leq E \leq Vd + \sum_{a=1}^{F} \ell(\mu^a) - nd \).

In particular,

**Corollary 4.** We have

\[ E_{n,N}(p_{\mu}(X)) := E_{n,N}(\text{tr } X^{\mu_1} \cdots \text{tr } X^{\mu_\ell}) \]

\[ = z_{\mu} N^{-nd} \sum_{\Delta^1, \ldots, \Delta^V} H_{F-n+V} (\mu, \Delta^1, \ldots, \Delta^V) \prod_{i=1}^{V} p_{\Delta^i}(\tilde{C}_i) \]

where \( V = n + 1 - 2g^* = E^* + n - F \). In particular, if all \( \tilde{C}_i = I_N \), we get

\[ E_{n,N}(p_{\mu}(X)) := E(\text{tr } X^{\mu_1} \cdots \text{tr } X^{\mu_\ell}) \]

\[ = z_{\mu} N^{-nd} \sum_{g} H_{2-2g^*}^2 (\mu; V+1) N^V \]

where \( V = n + 1 - 2g^* \), and where \( H_{2-2g^*}^2 (\mu; V) \) is the Hurwitz number counting \( d = |\mu| \)-sheeted covers of Riemann surface of genus \( g^* \) by Riemann surfaces of genus \( g \) with at most \( V+1 \) critical points (see \( \#1 \)) for the notation \( H_{2-2g^*}^2 \).

Thus, the expectation in the r. h. s. of \( (47) \) is expressed in terms of the Hurwitz numbers which enumerate \( d \)-sheeted coverings of Riemann surfaces of Euler characteristic \( 2 - 2g^* \) with \( n + 1 - g^* \) branch points with profiles \( \mu, \Delta^1, \ldots, \Delta^{n-2g^*} \) where \( d = |\mu| = |\Delta^1| = \cdots = |\Delta^{n-2g^*}| \).

We get

**3.3. Non-orientable case. Hurwitz numbers for Klein surfaces.**

To get Hurwitz numbers as expectation values of spectral function we use the “Meubius” tau function \( (32) \):

\[ \tau_1^B(Z) := \sum_{\lambda} s_{\lambda}(Z) = \prod_{i<j} (1 - z_i z_j)^{-1} \prod_{i=1}^{N} (1 - z_i)^{-1} \]
where \( z_i, i = 1, \ldots, N \) are eigenvalues of \( Z \). This trick was done in [53] and [59]. This tau function was pointed out in [63] as the simplest example of the BKP tau function.

Straightforward generalization of 1 reads as

**Theorem 3.** Under conditions of Theorem 1 we have

\[
E_{n,N} \left( \prod_{a=1}^{F-e} \tau_{r(a)}(x, X_a, Np^{(a)}) \right) \left( \prod_{a=F-e+1}^{F} \tau_{r(a)}(x, X_a) \right) = (49)
\]

\[
\sum_{i(\lambda) \leq N} r_{\lambda}(x) (s_{\lambda}(NP_{\infty}))^{-n} \prod_{i=1}^{F-e} s_{\lambda}(NP^{(a)}) \prod_{b=1}^{V} s_{\lambda}(\tilde{C}_b) (50)
\]

where \( F-e > 0 \) and where

\[
r_{\lambda}(x) := \prod_{(i,j) \in \lambda} r(x + j - i)
\]

where each \( \tau_{r(a)} \) is defined by (31) and where \( r = \prod_{a=1}^{F} r^{(a)} \). The degree of the (50) is equal to \( F-e-n+V \) is equal to \( e^* - e \) where \( e^* \) is the Euler characteristic of the network chord diagram.

We need \( F-e > 0 \) to have a non-empty set of parameters \( p^{(a)} \) to provide the convergency of the expectation value (see Remark 1 after Theorem 1).

One can interpret the degree \( e^* - e \) of the (50) as follows. The faces \( X_1, \ldots, X_{F-e} \) related to the tau functions \( \tau_{r(a)}, a = 1, \ldots, F-e \) (let us call them punctured one) are treated as before. The faces \( X_{F-e+1}, \ldots, X_F \) related to functions \( \tau_{r(B)} \) should be interpreted as holes glued by Mobius sheets. Insertion of each Mobius sheet diminishes the Euler characteristic of the base surface by 1. This rule sounds more like mnemonic since there is yet no direct connection of the series of the ratios of the Schur functions to the topology of surfaces.

In certain cases the expression (50) is a tau function, see Examples 3’ and 4’ below, however these cases are related to \( e^* = 1 \).

Take \( r = 1 \) below. The analogues of Examples 3 and 4 may be chosen as

**Example 4’.** Take \( F = 2, e = 1 \) and \( X_1 = Z_1C_1 \ldots Z_nC_n, X_2 = Z_1^*C_1^* \ldots Z_n^*C_n^* \). As one can see in this case \( V = n \) (thus, \( e^* = F - 1 - n + V = 1 \)) and the words are \( C_iC_i^* \), \( i = 1, \ldots, n \). The right hand side of (50) is equal to

\[
\sum_{\lambda} s_{\lambda}(NP^{(1)})^{-n} s_{\lambda}(C_iC_{i+1}^*) \frac{s_{\lambda}(C_iC_{i+1}^*)}{s_{\lambda}(NP_{\infty})}, \quad C_{n+1}^* := C_1^*
\]

In case all source matrices were \( I_N \), it is equal to \( \sum_{\lambda} ((N)\lambda)^n s_{\lambda}(NP^{(1)}) \) which is the BKP tau function (31) with \( r(x) = x^n \).

**Example 5’.** Take \( F = n \) and \( X_a = Z_aC_aZ_{a+1}^*C_{a+1}^* \), \( 1 \leq a < n \) and \( X_F = Z_nC_nZ_1^*C_1^* \) (a closed chain). We obtain two vertices (so, \( e^* = 2 \) and
two words $\tilde{C_1} = C_1 C_2 \cdots C_n$, $\tilde{C_2} = C_n^* C_{n-1} \cdots C_1^*$. In case all source matrices were $I_N$, we get that the right hand side of (38) is equal to

$$\sum_\lambda ((N)\lambda)^2 (s_\lambda(Np_\infty))^{1+e-n} \prod_{a=1}^{n-e} s_\lambda(Np(a)).$$

It can be identified with the tau function (27), if we fix each set $p(a), a = 1, \ldots, n$ to be in the form (26), with the exception of the selected one that we will interpret as higher times of the BKP hierarchy.

**Hurwitz numbers.** We get the following generation functions of Hurwitz numbers of Klein surfaces:

**Theorem 4.** We have

$$E_{n,N}(p_1(X_1) \cdots p_F(X_F) \tau^B(X_{F-e+1}) \cdots \tau^B(X_F)) \prod_{a=1}^{F-e} \frac{1}{z_\mu^a} =$$

$$\delta(\mu_1, \ldots, \Delta^V)N^{-nd} \sum_{\Delta^1, \ldots, \Delta^V} H_{e^{F-e}}(\mu_1, \ldots, \mu^{F-e}, \Delta^1, \ldots, \Delta^V) \prod_{i=1}^V p_{\Delta^i}(\tilde{C}_i)$$

(51)

where $\delta(\mu_1, \ldots, \Delta^V) = 1$ if $|\mu_1| = \cdots = |\mu^{F-e}| = |\Delta^1| = \cdots = |\Delta^V|$ and vanishes otherwise. Here $V$ is the number of vertices of the ribbon graph (fat-graph) obtained from the original network, $F-e$ is the number of punctured faces.

### 3.4. Discrete ensembles, $\beta$-ensembles (not finished)

Sums in the right hand sides of (55) and more generally of (50) may be treated as discrete ensembles which generalize known ensembles which can be related to $e^*$ series in the Schur functions [39] and [64].

**$\beta$-ensemble.** The matrix models labeled with networks may written as discrete $\beta$-ensembles if we fix parameters $p(a)$ with the help (26) that means that we study expectation value of products of powers of determinant $s$ (and one of this power should be a natural number, see Remark 1 after Theorem 1). This topic will be developed in a more detailed version, now, let me explain the idea. One need to use relations

$$s_\lambda(Np(d,a)) = a^{\lambda_1}(-Nd)_\lambda s_\lambda(p_\infty), \quad s_\lambda(Np_\infty) = N^{\lambda_1} \prod_{i<j} (h_i - h_j) \prod_{i=1}^N h_i!$$

(52)

where $h_i = \lambda_i - i + N$ are shifted parts of $\lambda$ and the notation $(-d)_\lambda$ was defined in [21]. Let $Nd_1 = NL > 0$ is integer. Notice that $(-NL)_\lambda$ vanishes for $\lambda_1 > NL$. For $Nd_i$ that are not natural numbers, we use

$$(-Nd_i)_\lambda = \prod_{j=1}^{N-1} (-Nd_i - j)^{N-j+1} \prod_{j=1}^N \frac{\Gamma(h_j + 1 - N - Nd_i)}{\Gamma(-Nd_i)}$$

where $\delta = N/Nd_1$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$.
Then, choosing any $e$ within $0 \leq e \leq F - 1$, we get
\[
\mathbb{E}_{n,N} \left( \det (1 - a_1 X_1)^N \prod_{i=2}^{F-1-e} (1 - a_i X_i)^{Nd_i} \prod_{i=F-e+1}^{F} \tau_1(X_i) \right) =
\]
\[
= \frac{c_N}{N!} \sum_{h_1, \ldots, h_N \geq 0} \prod_{a<b}^N |h_a - h_b|^{F-n+V-e} \prod_{j=1}^{N} \frac{a_j h_j \prod_{i=1}^{F-e-1} a_i^{h_j} \Gamma(h_j + 1 - N - N d_i)}{(\Gamma(h_j + 1))^{F-n+V} \Gamma(NL + N - h_j)}
\]
where $\Sigma'$ means that all $h_i$, $i = 1, \ldots, N$ are different (therefore the Vandermond product does not vanish), and where $c_N = c_N(\{a_i, d_i\}) = \prod_{i=1}^{F-e} a_i \prod_{j=1}^{N} (-Nd_i - j)^{N \cdot (-Nd_i - j)}$

We intentionally separate the case $Nd_1 = NL$ to avoid possible divergence in the summation, with $L$ be a natural number the right hand side. It is a finite sum with the summation range $0 \leq h_i \leq NL + N, \ i = 1, \ldots, N$.

One could write down the equation for the equilibrium Young diagram related to the discrete 2D Coulomb gas on the semiline (in case $\beta = 1, 2$) or, 2D 'gravitational’ gas on the semiline in case $\beta < 0$.

**Coupled, or, Kontsevich-type ensembles.** It may be available to fix $p^{(2)}$ in different way as $p^{(2)} = p^{(2)}(Y)$ where $Y_{ij} = \delta_{i,j} \exp y_i$, $i = 1, \ldots, N$ (see [12] for the notation). The matrix $Y$ plays the role of an additional source matrix similar to the role of external matrix in the coupled matrix model. (One can still take any of $Nd_i$ to be a natural number in case the sum is divergent). Instead of (53) we get
\[
\mathbb{E}_{n,N} \left( \det (1 - Y \otimes X_1)^{-N} \prod_{i=2}^{F-1-e} (1 - a_i X_i)^{Nd_i} \prod_{i=F-e+1}^{F} \tau_1(X_i) \right) =
\]
\[
= \frac{\tilde{c}_N}{N!} \sum_{h_1, \ldots, h_N \geq 0} \prod_{a<b}^N (h_a - h_b)^{F-n+V-e-1} \prod_{j=1}^{N} \frac{a_j h_j \prod_{i=1}^{F-e-1} a_i^{h_j} \Gamma(h_j + 1 - N - N d_i)}{(\Gamma(h_j + 1))^{F-n+V} \exp(-Ny_j h_j)}
\]
where $\tilde{c}_N = c_N \prod_{i<j}(e^{Ny_i} - e^{Ny_j})$, compare to the similar replacement in [39] and [63].

### 3.5. Products of unitary matrices.
If we replace $n$ independent complex Ginibre ensembles by $n$ independent circular $\beta = 2$ ensembles, namely, if each $N \times N$ matrix $Z_i$ is replaced by an $N \times N$ unitary matrix $U_i$, and, respectively, each $Z_i^{\dagger}$ is replaced by $U_i^{\dagger}$, and the sources matrices $C_i, C_i^{\ast}$ are unitary (or, more general, matrices diagonalizable by unitary transform) then we get the same Theorems where $s_\lambda(Np_\infty)$ is replaced by $s_\lambda(I_N)$, where $I_N$ is $N \times N$ identity matrix. We also get certain versions of Theorems however formulations of these ones needs more space (see for instance cases related to $E^{\ast}$ in [30] and $E^{\ast} = 1$ in [54]).

For instance, the analogue of the Corollary reads as...
Proposition 1. Consider the product $X = X_1 \cdots X_F$ where each matrix from the set $\{U_i C_i, U_i^\dagger C_i^*, i = 1, \ldots, n\}$ enters as a factor to the product $X$ only once. Denote the genus of the related chord diagram $g^*$, and related words $\tilde{C}_i$, $i = 1, \ldots, V$, the number of edges is $n$, the number of vertices is $V$, and the genus $g^*$ is defined by $2 - 2g^* = V - n + F$. Then we have

$$E_{n,N} \left( e^{\text{tr} V(X_1.p^1)} \cdots e^{\text{tr} V(X_F.p^F)} \right) = \sum_{\lambda} (s_\lambda(\mathbb{I}_N))^{-n} \prod_{i=1}^{F} s_\lambda(Np^{(i)}) \prod_{i=1}^{V} s_\lambda(\tilde{C}_i)$$

(55)

In particular, if all source matrices are equal to $\mathbb{I}_N$ and $p^{(i)} = p^{(i)}(d_i, a_i)$ (namely, $p_m^{(i)} = -d_i a_m^{(i)}$, $m > 0$) we get

$$E_{n,N} \left( \prod_{i=1}^{F} \det (1 - a_i X_i)^{N d_i} \right) = \sum_{\lambda} (s_\lambda(\mathbb{I}_N))^{V-n+F} \prod_{i=1}^{F} \frac{(N d_i)_\lambda}{(N)_\lambda}$$

(56)

3.6. The sketch of proofs.

First, we know how to evaluate the integrals with the Schur function via Lemma used in [65] and [53, 54] (for instance see [42] for the derivation).

Lemma 2. Let $A$ and $B$ be normal matrices (i.e. matrices diagonalizable by unitary transformations). Then

$$\int_{U(N)} s_\lambda(AUBU^{-1})d_\lambda U = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(\mathbb{I}_N)},$$

(57)

For $A, B \in \text{GL}(N)$ we have

$$\int_{U(N)} s_\mu(AU)s_\lambda(U^{-1}B)d_\lambda U = \frac{s_\lambda(AB)}{s_\lambda(\mathbb{I}_N)} \delta_{\mu,\lambda}.$$

(58)

Below $p_\infty = (1, 0, 0, \ldots)$.

$$\int_{C^{N/2}} s_\lambda(AZBZ^+)e^{-N \text{tr} ZZ^+} \prod_{i,j=1}^{N} d^2 Z_{ij} = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(N p_\infty)}$$

(59)

and

$$\int_{C^{N/2}} s_\mu(AZ)s_\lambda(Z^+B)e^{-N \text{tr} ZZ^+} \prod_{i,j=1}^{N} d^2 Z_{ij} = \frac{s_\lambda(AB)}{s_\lambda(N p_\infty)} \delta_{\mu,\lambda}.$$  

(60)

These relations are used for step-by-step integration (Gaussian in the case of complex matrices).

As we can see, these relations perform the procedure of cutting and joining loops in a network of chord diagrams, and also create edges of ribbon graph (each edge is a coupled pair of conjugate random matrices). Namely, the equation (59) performs the splitting of the loop $AZBZ^\dagger$ into two loops, $A$ and $B$, for complex Ginibre ensembles (the resulting equation (57) splits the loop $AUBU^\dagger$ for circular ensembles), and equation (60) performs the union of two loops $A$ and $B$ for complex Ginibre ensembles (and the equation 58).
does the same for circular ensembles). Every time we apply some of the relations (58)-(60), we get the factor (the ”propagator” of the edge of the ribbon graph), which is \( \frac{1}{s_\lambda(N p_\infty)} \) in the case of complex Ginibre ensemble and \( \frac{1}{s_\lambda(\tilde{\pi} N)} \) in the circular case.

In this way we prove Theorem 1 and Theorem 3 and their analogues for the circular ensembles.

Then, Theorems 2 and 4 follows, respectively, from Theorems 1 and Theorem 3, and by the usage of the Mednykh formula (6) and the characteristic map relation [42]:

\[
s_\lambda(N p) = \frac{\dim \lambda}{d!} \sum_{\Delta \mid |\Delta| = |\lambda|} \varphi_\lambda(\Delta) p_\Delta N^{\ell(\Delta)}
\]

where \( \ell(\Delta) \) is the length of the partition \( \Delta \), where \( p_\Delta = p_{\Delta_1} \cdots p_{\Delta_\ell} \) and where the summation ranges over all partitions \( \Delta = (\Delta_1, \ldots, \Delta_\ell) \) whose weight coinsides with the weight of \( \lambda \): \( |\lambda| = |\Delta| = d \).

Here

\[
\dim \lambda = d!s_\lambda(p_\infty), \quad p_\infty = (1, 0, 0, \ldots)
\]

is the dimension of the irreducible representation of the symmetric group \( S_d \).

We imply that \( \varphi_\lambda(\Delta) = 0 \) if \( |\Delta| \neq |\lambda| \).

Acknowledgements

This work was done during my visits to Belowezie, Bialystok and to Anger university. I am grateful to director of the Institute of Mathematics in Bialystok Anatol Odzijewicz and to Prof. Vladimir Roubtsov in Anger university for their kind hospitality. The work has been funded by RFBR grant 18-01-00273a and the RAS Program “Fundamental problems of nonlinear mechanics” and by the Russian Academic Excellence Project ‘5-100’. I thank Borot, A.Mudrov, S.Lando and M.Kazarian for their remarks which allow me to compare my results with combinatorial problems already appeared in the well-known model of Hermitian matrices, and also to E. Strakhov who drew my attention to the works on quantum chaos devoted to the products of random matrices and for fruitful discussions. I thankful to S.Natanzon, A.Mironov and J.Harnad for numerous discussions on Hurwitz numbers.

References

1. G. Akemann, J. R. Ipsen, M. Kieburg, Products of Rectangular Random Matrices: Singular Values and Progressive Scattering, arXiv:1307.7560
2. G. Akemann, T. Checinski, M. Kieburg, Spectral correlation functions of the sum of two independent complex Wishart matrices with unequal covariances, arXiv:1502.01667
3. G. Akemann, E. Strahov, Hard edge limit of the product of two strongly coupled random matrices, arXiv:1511.09410
4. Alexandrov, A.: Matrix models for random partitions. Nucl. Phys. B 851, 620-650 (2011)
5. A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, Integrability of Hurwitz Partition Functions. I. Summary, J.Phys.A: Math Theor. 45(2012) 045209, arXiv: 1103.4100
6. A. Alexandrov, A. Mironov, A. Morozov and S. Natanzon, On KP-integrable Hurwitz functions, JHEP 11(2014) 080, arXiv: 1405.1395
7. A. Alexandrov and A. V. Zabrodin Free fermions and tau-functions, J. Geom. Phys. 67 (2013) pp. 37-80; arXiv:1212.6049
8. A. A. Alexeevski and S. M. Natanzon, Noncommutative two-dimensional field theories and Hurwitz numbers for real algebraic curves, Selecta Math. N.S. v.12 (2006), n.3, pp. 307-377, arXiv:math/0202164
9. A. V. Alekseevskii and S. M. Natanzon, The algebra of bipartite graphs and Hurwitz numbers of seamed surfaces, Izvestiya Mathematics 72:4 (2008) pp. 627-646
10. G. Alfano, “Products of Ginibre and deterministic matrices in the analysis of correlated multiantenna channels” https://www2.physik.uni-bielefeld.de
11. N. L. Alling and N. Greenleaf, Foundation of the theory of Klein surfaces, Springer-Verlag, 1971, Lecture Notes in Math. v. 219
12. J. Ambjorn and L. Chekhov The matrix model for hypergeometric Hurwitz number, Theoret. and Math. Phys., 1 81:3 (2014), 1486-1498; arXiv:1409.3553
13. S.R. Carrell, “The Non-Orientable Map Asymptotics Constant ps”, arXiv:1406.1760
14. J. Ambjorn and L. O. Chekhov, The matrix model for dessins d’enfants, Ann. Inst. Henri Poincare D, 1:3 (2014), 337-361; arXiv:1404.4240
15. E. Brezin and V. Kazakov, Exactly solvable field theories of closed strings, Phys Lett B236 pp 144-150 (1990);
16. L. O. Chekhov, The Harer-Zagier recursion for an irregular spectral curve, J. Geom. Phys., 110 (2016), 30-43, arXiv: 1512.09278
17. R. Dijkgraaf, Mirror symmetry and elliptic curves, The Moduli Space of Curves, R. Dijkgraaf, C. Faber, G. van der Geer (editors), Progress in Mathematics, 129, Birkhauser, 1995.
18. P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin and L. Spitz, Polynomiality of Hurwitz numbers, Bouchard-Marino conjecture, and a new proof of the ELSV formula, arXiv:1307.4729
19. T. Ekedahl, S. K. Lando, V. Shapiro and A. Vainshtein, On Hurwitz numbers and Hodge integrals, C.R. Acad. Sci. Paris Ser. I. Math. Vol. 146, N2, pp. 1175-1180 (1999)
20. Gareth A. Jones, Enumeration of Homomorphisms and Surface-Coverings, Quart. J. Math. Oxford (2), 46 (1995), pp. 485-507
21. A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, “Matrix models of two-dimensional gravity and Toda theory”, Nuclear Physics B 357 (2-3), 565-618 (1992)
22. I. P. Goulden and D. M. Jackson, “The combinatorial relationship between trees, cacti and certain connection coefficients for the symmetric group”, European J. Combin. 13 (1992) pp. 357-365
23. I. P. Goulden and D. M. Jackson, *The KP hierarchy, branched covers, and triangulations*, Advances in Mathematics, 219 pp. 932-951, 2008

24. I. P. Goulden, M. Guay-Paquet and J. Novak, *Monotone Hurwitz numbers and HCIZ integral*, Ann. Math. Blaise Pascal 21 pp. 71-99 (2014)

25. D. Gross and A. Migdal, *Nonperturbative two-dimensional quantum gravity*, Phys. Rev. Lett. 64 p 127 (1990)

26. M. Guay-Paquet and J. Harnad, *2D Toda tau-functions as combinatorial generating functions*, Letters in Mathematical Physics 105, pp. 827-852 (2015)

27. J. Harnad and A. Yu. Orlov, *Scalar product of symmetric functions and matrix integrals*, Theoretical and mathematical physics 137 (3), pp. 1676-1690 (2003)

28. J. Harnad and A. Yu. Orlov, *Matrix integrals as Borel sums of Schur function expansions*, Symmetry and Perturbation Theory 2002, Cala Gonone (Sardinia), May 1-26, pp. (2002). Proceedings, pp. 116-123 (World Scientific, Singapore, eds. S. Abenda, G. Gaeta); [arXiv:nlin/0209035](http://arxiv.org/abs/nlin/0209035)

29. J. Harnad and A. Yu. Orlov, *Fermionic construction of partition functions for two matrix models and perturbative Schur functions expansions*, J. Phys. A 39, pp. 8783-8809 (2006)

30. J. Harnad and A. Yu. Orlov, *Hypergeometric τ-functions, Hurwitz numbers and enumeration of paths*, Commun. Math. Phys. 338 (2015) pp. 267-284 arxiv: math-ph/1407.7800

31. J. Harnad, *Multispecies quantum Hurwitz numbers*, SIGMA 11, 097 (2015); [arXiv:1410.8817](http://arxiv.org/abs/1410.8817)

32. J. Harnad, *Weighted Hurwitz numbers and hypergeometric τ-functions: an overview*, AMS Proc. Symp. Pure Math. 93 (2016) pp. 289-333 ; [arXiv:1504.03408](http://arxiv.org/abs/1504.03408)

33. M. Jimbo and T. Miwa, “Solitons and infinite dimensional Lie algebras”, Publ. RIMS Kyoto Univ. 19, pp. 943–1001 (1983)

34. V. Kac and J. van de Leur, *The Geometry of Spinors and the Multicomponent BKP and DKP Hierarchies*, CRM Proceedings and Lecture Notes 14 (1998) pp. 159-202

35. V. A. Kazakov, M. Staudacher, T. Wynter, *Character Expansion Methods for Matrix Models of Dually Weighted Graphs*, Commun.Math.Phys. 177 (1996) 451-468; [arXiv:hep-th/9502132](http://arxiv.org/abs/hep-th/9502132)

36. M. Kazarian and S. Lando, *Combinatorial solutions to integrable hierarchies*, Uspekhi Mat. Nauk 70 (2015), no. 3(423), pp. 77-106. English translation: 2015 Russ. Math. Surv. 70, pp. 453-482; [arXiv:1512.07172](http://arxiv.org/abs/1512.07172)

37. M. E. Kazarian and S. K. Lando, *An algebro-geometric proof of Witten’s conjecture*, J. Amer. Math. Soc. 20:4 (2007), pp. 1079-1089

38. M. Kazarian and P. Zograph, *Virasoro constraints and topological recursion for Grothendieck’s dessin counting*, arxiv1406.5976

39. S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, *Generalized Kazakov-Migdal-Kontsevich Model: group theory aspects*, International Journal of Mod Phys A10 (1995) p.2015

40. S. K. Lando, A. K. Zvonkin *Graphs on Surfaces and their Applications, Encyclopedia of Mathematical Sciences, Volume 141, with appendix by D. Zagier, Springer, N.Y. (2004).*
41. J. W. van de Leur, *Matrix Integrals and Geometry of Spinors*, J. of Nonlinear Math. Phys. 8, pp. 288-311 (2001)
42. I.G. Macdonald, Symmetric Functions and Hall Polynomials, Clarendon Press, Oxford, (1995).
43. R. de Mello Koch and S. Ramgoolam, *From Matrix Models and quantum fields to Hurwitz space and the absolute Galois group*, arXiv: 1002.1634
44. A. D. Mednykh, *Determination of the number of nonequivalent covering over a compact Riemann surface*, Soviet Math. Dokl., 19(1978), pp. 318-320
45. A. D. Mednykh and G. G. Pozdnyakova, *The number of nonequivalent covering over a compact nonorientable surface*, Sibirs. Mat. Zh, 27(1986), +1, pp. 123-131.199
46. M. L. Mehta “Random Matrices”, 3nd edition (Elsevier, Academic, San Diego CA, 2004)
47. M.Mineev-Weinstein, P.Wiegmann, A.Zabrodin, *Integrable Structure of Interface Dynamics*, Phys. Rev. Lett. 84 (2000) 5106-5109
48. A. D. Mironov, A. Yu. Morozov and S. M. Natanzon, *Complect set of cut-and-join operators in the Hurwitz-Kontsevich theory*, Theor. and Math.Phys. 166:1,(2011), pp.1-22; [arXiv:0904.4227]
49. A. D. Mironov, A. Yu. Morozov and S. M. Natanzon, *Algebra of differential operators associated with Young diagrams*, J.Geom.and Phys. n.62(2012), pp. 148-155
50. S. M. Natanzon, *Klein surfaces*, Russian Math.Surv., 45:6(1990), pp. 53-108.
51. S. M. Natanzon, *Moduli of Riemann surfaces, real algebraic curves and their superanalogs*, Translations of Math. Monograph, AMS, Vol.225 (2004), 160 p.
52. S. M. Natanzon, *Simple Hurwitz numbers of a disk*, Funk. Analysis ant its applications, v.44 (2010), n1, pp. 44-58
53. S. M. Natanzon and A. Yu. Orlov, *Hurwitz numbers and BKP hierarchy*, arXiv:1407.832
54. S. M. Natanzon and A. Yu. Orlov, *BKP and projective Hurwitz numbers*, Letters in Mathematical Physics, 107(6), 1065-1109 (2017); [arXiv:1501.01283]
55. S. M. Natanzon and A. Zabrodin, *Toda hierarchy, Hurwitz numbers and conformal dynamics*, Int. Math. Res. Notices 2015 (2015) 2082-2110
56. A. Okounkov, *Toda equations for Hurwitz numbers*, Math. Res. Lett. 7, pp. 447-453 (2000). See also arxivmath-004128.
57. A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz theory and completed cycles*, Annals of Math 163 p.517 (2006); arxiv [math.AG/0204305]
58. A. Yu. Orlov, *Soliton theory, symmetric functions and matrix integrals*, Acta Applicandae Mathematica 86 (1-2), pp. 131-158 (2005)
59. A.Yu.Orlov *Hurwitz numbers and products of random matrices*, Theoretical and Mathematical Physics 193(3) pp 1282-1323 (2017); [arXiv:1701.02296]
60. A. Yu. Orlov and D. Scherbin, *Fermionic representation for basic hypergeometric functions related to Schur polynomials*, arXiv preprint [nlin/0001001]
61. A. Yu. Orlov and D. Scherbin, *Hypergeometric solutions of soliton equations*, Theoretical and Mathematical Physics 128 (1), pp. 906-926 (2001)
Appendix A. Counting of branched covers

In this section the Euler characteristic of the base surface is denoted $e$.

Let us consider a connected compact surface without boundary $\Omega$ and a branched covering $f : \Sigma \to \Omega$ by a connected or non-connected surface $\Sigma$. We will consider a covering $f$ of the degree $d$. It means that the preimage $f^{-1}(z)$ consists of $d$ points $z \in \Omega$ except some finite number of points. This points are called critical values of $f$.

Consider the preimage $f^{-1}(z) = \{p_1, \ldots, p_\ell\}$ of $z \in \Omega$. Denote by $\delta_i$ the degree of $f$ at $p_i$. It means that in the neighborhood of $p_i$ the function $f$ is homeomorphic to $x \mapsto x^{\delta_i}$. The set $\Delta = (\delta_1, \ldots, \delta_\ell)$ is the partition of $d$, that is called topological type of $z$.

For a partition $\Delta$ of a number $d = |\Delta|$ denote by $\ell(\Delta)$ the number of the non-vanishing parts ($|\Delta|$ and $\ell(\Delta)$ are called the weight and the length of $\Delta$, respectively). We denote a partition and its Young diagram by the same letter. Denote by $(\delta_1, \ldots, \delta_\ell)$ the Young diagram with rows of length $\delta_1, \ldots, \delta_\ell$ and corresponding partition of $d = \sum \delta_i$.

Fix now points $z_1, \ldots, z_k$ and partitions $\Delta^{(1)}, \ldots, \Delta^{(k)}$ of $d$. Denote by $\widetilde{C}_{\Omega(z_1, \ldots, z_k)}(d; \Delta^{(1)}, \ldots, \Delta^{(k)})$
the set of all branched covering \( f : \Sigma \to \Omega \) with critical points \( z_1, \ldots, z_k \) of topological types \( \Delta^{(1)}, \ldots, \Delta^{(k)} \).

Coverings \( f_1 : \Sigma_1 \to \Omega \) and \( f_2 : \Sigma_2 \to \Omega \) are called isomorphic if there exists an homeomorphism \( \varphi : \Sigma_1 \to \Sigma_2 \) such that \( f_1 = f_2 \varphi \). Denote by \( \text{Aut}(f) \) the order of the group of automorphisms of the covering \( f \). Isomorphic coverings have isomorphic groups of automorphisms of degree \( |\text{Aut}(f)| \).

Consider now the set \( C_{\Omega(z_1, \ldots, z_k)}(d; \Delta^{(1)}, \ldots, \Delta^{(k)}) \) of isomorphic classes in \( \widetilde{C}_{\Omega(z_1, \ldots, z_k)}(d; \Delta^{(1)}, \ldots, \Delta^{(k)}) \). This is a finite set. The sum

\[
H_E(\Delta^{(1)}, \ldots, \Delta^{(k)}) = \sum_{f \in C_{\Omega(z_1, \ldots, z_k)}(d; \Delta^{(1)}, \ldots, \Delta^{(k)})} \frac{1}{|\text{Aut}(f)|}, \quad (63)
\]
don’t depend on the location of the points \( z_1 \ldots, z_k \) and is called Hurwitz number. Here \( k \) denotes the number of the branch points, and \( E \) is the Euler characteristic of the base surface.

In case it will not produce a confusion we admit ‘trivial’ profiles \( (1^d) \) among \( \Delta^1, \ldots, \Delta^k \) in (63) keeping the notation \( H_E(\Delta^{(1)}, \ldots, \Delta^{(k)}) \) though the number of critical points now is less than \( k \).

In case we count only connected covers \( \Sigma \) we get the connected Hurwitz numbers \( H^{\text{con}}_E(\Delta^{(1)}, \ldots, \Delta^{(k)}) \).

The Hurwitz numbers arise in different fields of mathematics: from algebraic geometry to integrable systems. A special interest in this topic arose after the papers [17] and [19] (see [37] and [40] for a review). They are well studied for orientable \( \Omega \). In this case the Hurwitz number coincides with the weighted number of holomorphic branched coverings of a Riemann surface \( \Omega \) by other Riemann surfaces, having critical points \( z_1, \ldots, z_k \in \Omega \) of the topological types \( \Delta^{(1)}, \ldots, \Delta^{(k)} \) respectively. The well known isomorphism between Riemann surfaces and complex algebraic curves gives the interpretation of the Hurwitz numbers as the numbers of morphisms of complex algebraic curves.

Similarly, the Hurwitz number for a non-orientable surface \( \Omega \) coincides with the weighted number of the dianalytic branched coverings of the Klein surface without boundary by another Klein surface and coincides with the weighted number of morphisms of real algebraic curves without real points [11][50][51]. An extension of the theory to all Klein surfaces and all real algebraic curves leads to Hurwitz numbers for surfaces with boundaries may be found in [8][52].

Riemann-Hurwitz formula related the Euler characteristic of the base surface \( E^* \) and the Euler characteristic of the \( d \)-sheeted cover \( E \) as follows:

\[
E = dE^* + \sum_{i=1}^{k} \left( \ell(\Delta^{(i)}) - d \right) = 0 \quad (64)
\]

where the sum ranges over all branch points \( z_i, i = 1, 2, \ldots \) with ramification profiles given by partitions \( \Delta^i, i = 1, 2, \ldots \) respectively, and \( \ell(\Delta^{(i)}) \) denotes...
the length of the partition $\Delta^{(i)}$ which is equal to the number of the preimages $f^{-1}(z_i)$ of the point $z_i$.

Example 1. Let $f : \Sigma \to \mathbb{CP}^1$ be a covering without critical points. Then, each $d$-sheeted cover is the disjoint union of $d$ Riemann spheres: $\mathbb{CP}^1 \sqcup \cdots \sqcup \mathbb{CP}^1$, then $|\text{Aut} f| = d!$ and $H_2((1^d)) = \frac{1}{d!}$. The same answer one gets from Mednykh formula (6).

Example 2. Let $f : \Sigma \to \mathbb{CP}^1$ be a $d$-sheeted covering with two critical points with the profiles $\Delta^{(1)} = \Delta^{(2)} = (d)$. (One may think of $f = x^d$). Then $H_2((d), (d)) = \frac{1}{d}$. Let us note that $\Sigma$ is connected in this case (therefore $H_2((d), (d)) = H_2^{\text{con}}((d), (d))$) and its Euler characteristic $\chi = 2$.

Example 3. The generating function for the Hurwitz numbers $H_2((d), (d))$ from the previous Example may be written as

\[ F(h^{-1}p^{(1)}, h^{-1}p^{(2)}) := h^{-2} \sum_{d > 0} H_2^{\text{con}}((d), (d)) p_d^{(1)} p_d^{(2)} = h^{-2} \sum_{d > 0} \frac{1}{d} p_d^{(1)} p_d^{(2)} \]

Here $p^{(i)} = (p_1^{(i)}, p_2^{(i)}, \ldots), i = 1, 2$ are two sets of formal parameters. The powers of the auxiliary parameter $\frac{1}{h}$ count the Euler characteristic of the cover $\Sigma$ which is 2 in our example. Then thanks to the known general statement about the link between generating functions of “connected” and “disconnected” Hurwitz numbers (see for instance [40]) one can write down the generating function for the Hurwitz numbers for covers with two critical points, $H_2(\Delta^{(1)}, \Delta^{(2)})$, as follows:

\[ \tau(h^{-1}p^{(1)}, h^{-1}p^{(2)}) = e^{F(h^{-1}p^{(1)}, h^{-1}p^{(2)})} = e^{h^{-2} \sum_{d > 0} \frac{1}{d} p_d^{(1)} p_d^{(2)}} = \sum_{d \geq 0} \sum_{\Delta^{(1)}, \Delta^{(2)}} H_2(\Delta^{(1)}, \Delta^{(2)}) h^{-\ell(\Delta^{(1)}) - \ell(\Delta^{(2)})} \prod_{i=1}^2 p^{(i)} \Delta^{(i)} \Delta^{(i)} (65) \]

where $p^{(i)} := p_1^{(i)} p_2^{(i)} p_3^{(i)} \cdots, i = 1, 2$ and where $\ell(\Delta^{(1)})$ and $\ell(\Delta^{(2)})$ in agreement with (64) where we put $k = 2$. From (65) it follows that the profiles of both critical points coincide, otherwise the Hurwitz number vanishes. Let us denote this profile $\Delta$, and $|\Delta| = d$ and from the last equality we get

\[ H_2(\Delta, \Delta) = \frac{1}{z_\Delta} \]

Here

\[ z_\Delta = \prod_{i=1}^\infty i^{m_i} m_i! \]

(66)

where $m_i$ denotes the number of parts equal to $i$ of the partition $\Delta$ (then the partition $\Delta$ is often denoted by $(1^{m_1} 2^{m_2} \cdots)$).

Example 4. Let $f : \Sigma \to \mathbb{RP}^2$ be a covering without critical points. Then, if $\Sigma$ is connected, then $\Sigma = \mathbb{RP}^2$, deg $f = 1$ or $\Sigma = S^2$, deg $f = 2$. Next, if $d = 3$, then $\Sigma = \mathbb{RP}^2 \sqcup \mathbb{RP}^2 \sqcup \mathbb{RP}^2$ or $\Sigma = \mathbb{RP}^2 \sqcup S^2$. Thus, $H_2(\Delta^{(1)}) = \frac{1}{3} + \frac{1}{3} = 2$. The same answer one obtains from the combinatorial definition. Indeed, the equation $R^2 = 1$ has 4 solutions in $S_3$. 
Example 5. Let \( f : \Sigma \to \mathbb{R}P^2 \) be a covering with a single critical point with profile \( \Delta \), and \( \Sigma \) is connected. Note that due to (64) the Euler characteristic of \( \Sigma \) is \( E' = \ell(\Delta) \). (One may think of \( f = z^d \) defined in the unit disc where we identify \( z \) and \(-z\) if \(|z| = 1\). In case we cover the Riemann sphere by the Riemann sphere \( z \to z^m \) we get two critical points with the same profiles. However we cover \( \mathbb{R}P^2 \) by the Riemann sphere, then we have the composition of the mapping \( z \to z^m \) on the Riemann sphere and the factorization by antipodal involution \( z \to -z \). Thus we have the ramifications profile \((m, m)\) at the single critical point 0 of \( \mathbb{R}P^2 \). The automorphism group is the dihedral group of the order \( 2m \) which consists of rotations on \( \frac{2\pi}{m} \) and antipodal involution \( z \to -\frac{1}{z} \). Thus we get that

\[
H^\text{con}_1(2m; (m, m)) = \frac{1}{2m}
\]

From (64) we see that \( 1 = \ell(\Delta) \) in this case. Now let us cover \( \mathbb{R}P^2 \) by \( \mathbb{R}P^2 \) via \( z \to z^d \). From (64) we see that \( \ell(\Delta) = 1 \). For even \( d \) we have the critical point 0, in addition each point of the unit circle \(|z| = 1\) is critical (a folding), while from the beginning we restrict our consideration only on isolated critical points. For odd \( d = 2m - 1 \) there is the single critical point 0, the automorphism group consists of rotations on the angle \( \frac{2\pi}{2m-1} \). Thus, in this case

\[
H^\text{con}_1(2m - 1; (2m - 1)) = \frac{1}{2m-1}
\]

Example 6. The generating series of the connected Hurwitz numbers with a single critical point from the previous Example is

\[
F(h^{-1}p) = \frac{1}{h^2} \sum_{m>0} p_m^2 H^\text{con}_1(2m; (m, m)) + \frac{1}{h} \sum_{m>0} p_{2m-1} H^\text{con}_1(2m - 1; (2m - 1))
\]

where \( H^\text{con}_1 \) describes \( d \)-sheeted covering either by the Riemann sphere \((d = 2m)\) or by the projective plane \((d = 2m - 1)\). We get the generating function for Hurwitz numbers with a single critical point

\[
\tau(h^{-1}p) = e^{F(h^{-1}p)} = e^{\frac{1}{h} \sum_{m>0} \frac{1}{2m} p_m^2 + \frac{1}{h} \sum_{m \text{ odd}} \frac{1}{m} p_m} = \sum_{d>0} \sum_{|\Delta| = d} h^{-\ell(\Delta)} p_\Delta H_1(d; \Delta)
\]

Then \( H_1(d; \Delta) \) is the Hurwitz number describing \( d \)-sheeted covering of \( \mathbb{R}P^2 \) with a single branch point of type \( \Delta = (d_1, \ldots, d_l) \), \(|\Delta| = d\) by a (not necessarily connected) Klein surface of Euler characteristic \( E' = \ell(\Delta) \). For instance, for \( d = 3 \), \( E' = 1 \) we get \( H_1(\Delta) = \frac{1}{6} \delta_{\Delta, (3^2, 1)} \). For unbranched coverings (that is for \( \Delta = (1^d) \)) we get the generating formula \( e^{\frac{1}{2} \sum_{m \geq 0} c d H_1 (d; (1^d))} \).

One can also get the answers considered in the examples by the usage of the Mednykh formula (6).
Corollaries of the Mednykh-Pozdnyakova Character Formula [54]. It follows from the paper [17] by Dijkgraaf that the Hurwitz numbers for closed orientable surfaces form a 2D topological field theory. An extension of this result to the case of Klein surfaces (thus to orientable and non-orientable surfaces) was found in Theorem 5.2 of [8], (see also Corollary 3.2 in [9]). On the other hand, the Mednykh-Pozdnyakova formula describes the Hurwitz numbers in terms of characters of the symmetric groups. One can interpret the axioms of the Klein topological field theory [8] for Hurwitz numbers in terms of characters of symmetric groups.

Lemma 3.

\[
H_{E+E_1}(\Delta^{(1)}, \ldots, \Delta^{(k+k_1)}) = \sum_\Delta \frac{d!}{|C_\Delta|} H_{E+1}(\Delta^{(1)}, \ldots, \Delta^{(k)}, \Delta) H_{E_1+1}(\Delta, \Delta^{(k+1)}, \ldots, \Delta^{(k_1)}).
\]

In particular,

\[
H_{E-1}(\Delta^{(1)}, \ldots, \Delta^{(k)}) = \sum_\Delta H_E(\Delta^{(1)}, \ldots, \Delta^{(k)}, \Delta) \chi(\Delta),
\]

where \(\chi(\Delta) = d! H_1(\Delta)/|C_\Delta|\) are rational numbers explicitly defined in the following way by a partition \(\Delta\):

\[
\chi(\Delta) = \sum_{|\lambda|=|\Delta|} \chi_\lambda(\Delta) = \left[ \prod_{i>0, \text{even}} e^\frac{i^2}{2} \frac{\partial^2}{\partial p_i^2} \cdot p_i^{m_i} \prod_{i>0, \text{odd}} e^\frac{i^2}{2} \frac{\partial^2}{\partial p_i^2} + \frac{\partial}{\partial p_i} \cdot p_i^{m_i} \right]_{p=0}
\]

and \(\chi_\lambda(\Delta)\) is the character of the representation \(\lambda\) of the symmetric group \(S_d, d=|\lambda|\), evaluated on the cycle class \(\Delta = (1^{m_1}2^{m_2}\ldots)\).

As a corollary we get that the Hurwitz numbers of the projective plane may be obtained from the Hurwitz numbers of the Riemann sphere, while the Hurwitz numbers of the torus and the Klein bottle may be obtained from the Hurwitz numbers of the projective plane.

On combinatorial approach. The study of the homomorphisms between the fundamental group of the base Riemann sufrace of genus \(g^*\) (the Euler characteristic is ressectively \(E = 2 - 2g^*\)) with \(k\) marked points and the symmetric group in the context of the counting of the non-equivalent \(d\)-sheeted covering with given profiles \(\Delta_i, i = 1, \ldots, k\) results to the equation (1) (for instance, for the details, see Appendix A written by Zagier for the Russian edition of [40] or works [20, 44]).

For instance, Example 3 considered above counts non-equivalent solutions to the equation \(A_1 A_2 = 1\) with given cycle classes \(C_{\Delta^1}\) and \(C_{\Delta^2}\). Solutions of this equation consist of all elements of class \(C_{\Delta^1}\) and inverse elements, so \(\Delta^2 = \Delta^1 := \Delta\). The number of elements of any class \(C_\Delta\) (the cardinality of \(|C_\Delta|\)) divided by \(|\Delta|!\) is \(\frac{1}{\bar{z}_\Delta}\) as we got in the Example 3.

For Klein surfaces (see [45], [20]) instead of (1) we get (11).
In (4), \(g^*\) is the so-called genus of non-orientable surface which is related to its Euler characteristic \(e^*\) as \(e = 2 - g^*\). For the projective plane \((e^* = 1)\) we have \(g^* = 1\), for the Klein bottle \((e^* = 1)\) \(g^* = 2\).

Consider unbranched coverings \((k = 0)\) of the torus (equation (1) where \(g = 1\)), of the projective plane and the Klein bottle (equation (4) respectively \(g^* = 0\) and \(g^* = 1\)). For the real projective plane we have \(g^* = 1\) in (4) only one \(R_0 = ab\). If we treat the projective plane as the unit disk with identified opposit points of the boarder \(|z| = 1\), then \(R_0\) is related to the path from \(z\) to \(-z\). For the Klein bottle \((g = 2\) in (4)) there are \(R_0 = ba^{-1}\) and \(R_1 = a\).

**Appendix B. Partitions and Schur functions**

Let us recall that the characters of the unitary group \(\mathbb{U}(N)\) are labeled by partitions and coincide with the so-called Schur functions \([42]\). A partition \(\lambda = (\lambda_1, \ldots, \lambda_n)\) is a set of nonnegative integers \(\lambda_i\) which are called parts of \(\lambda\) and which are ordered as \(\lambda_i \geq \lambda_{i+1}\). The number of non-vanishing parts of \(\lambda\) is called the length of the partition \(\lambda\), and will be denoted by \(\ell(\lambda)\). The number \(|\lambda| = \sum \lambda_i\) is called the weight of \(\lambda\). The set of all partitions will be denoted by \(\mathbb{P}\).

The Schur function labelled by \(\lambda\) may be defined as the following function in variables \(x = (x_1, \ldots, x_N)\):

\[
s_\lambda(x) = \frac{\det [x_j^{\lambda_i+i-N}]}{\det [x_j^{-i+N}]}_{i,j}
\]

(71)

in case \(\ell(\lambda) \leq N\) and vanishes otherwise. One can see that \(s_\lambda(x)\) is a symmetric homogeneous polynomial of degree \(|\lambda|\) in the variables \(x_1, \ldots, x_N\), and \(\deg x_i = 1, i = 1, \ldots, N\).

**Remark.** In case the set \(x\) is the set of eigenvalues of a matrix \(X\), we also write \(s_\lambda(X)\) instead of \(s_\lambda(x)\).

There is a different definition of the Schur function as quasi-homogeneous non-symmetric polynomial of degree \(|\lambda|\) in other variables, the so-called power sums, \(p = (p_1, p_2, \ldots)\), where \(\deg p_m = m\).

For this purpose let us introduce

\[s_{\{h\}}(p) = \det[s_{(h_i+j-N)}(p)]_{i,j},\]

where \(\{h\}\) is any set of \(N\) integers, and where the Schur functions \(s_{(i)}\) are defined by \(e^{\sum_{m>0} \frac{1}{m}p_m z^m} = \sum_{m \geq 0} s_{(i)}(p) z^i\). If we put \(h_i = \lambda_i - i + N\), where \(N\) is not less than the length of the partition \(\lambda\), then

\[s_\lambda(p) = s_{\{h\}}(p)\]

(72)
The Schur functions defined by (71) and by (72) are equal, \( s_\lambda(p) = s_\lambda(x) \), provided the variables \( p \) and \( x \) are related by the power sums relation

\[
p_m = \sum_{i=1}^{N} x_i^m
\]

(73)

In case the argument of \( s_\lambda \) is written as a non-capital fat letter the definition (72), and we imply the definition (71) in case the argument is not fat and non-capital letter, and in case the argument is capital letter which denotes a matrix, then it implies the definition (71) with \( x = (x_1, \ldots, x_N) \) being the eigenvalues.

It may be easily checked that

\[
s_\lambda(p) = (-1)^{|\lambda|} s_{\lambda^{tr}}(-p)
\]

(74)

where \( \lambda^{tr} \) is the partition conjugated to \( \lambda \) (in [42] it is denoted by \( \lambda^* \)). The Young diagram of the conjugated partition is obtained by the transposition of the Young diagram of \( \lambda \) with respect to its main diagonal. One gets \( \lambda_1 = \ell(\lambda^{tr}) \). And then it follows that for \( L \times L \) matrix \( X \) the Schur function \( s_\lambda(-p(X)) \) vanishes if \( \lambda_1 > L \).

**Appendix C. More about tau functions**

The product over all nodes of the Young diagram \( \lambda \) is called content product and plays a certain role in the representation theory of symmetric groups (in the context of Hurwitz numbers see, for instance [23], [26]). These \( \tau_r \) parametrized by the choice of the function \( r \) form the family of the Toda lattice (TL) tau functions where the sets \( p(X) \) and \( p \) play the role of the so-called higher times and the (discrete) variable \( x \) plays the role of the site number in the lattice. The content product can be viewed as the generalized Pochhammer symbol related to Young diagrams. That’s why such family of tau functions [60] were called hypergeometric ones. Let us also note that TL hypergeometric functions generate Hurwitz numbers themselves, in this case the base surface is Riemann sphere (\( e^* = 2 \)), see [23], [6], [30], [54].

In addition, there exist numerous representations of TL hypergeometric tau functions in form of matrix integrals, see, for instance [65].

This \( \tau_r^B \) is called hypergeometric tau function [63] of the “large” BKP hierarchy, (BKP hierarchy introduced in [34]). Similar to the Toda lattice case, \( \tau_r^B \) generates Hurwitz numbers, however in this case \( e^* = 1 \), see [53], [54].

**Appendix D. Matrix integrals as generating functions of Hurwitz numbers from [53, 54]**

Hurwitz numbers can be generated by series in the Schur functions. In turn, series in the Schur functions can be generated as perturbation series of various matrix models. Let us note that the very first papers devoted to the
perturbation series of certain matrix models in terms of the Schur functions was [35].

In case the base surface is \( \mathbb{CP}^1 \) the set of examples of matrix integrals generating Hurwitz numbers were studied in works [6, 12, 14, 38, 40, 43, 72]. One can show that the perturbation series in coupling constants of these integrals (Feynman graphs) may be related to TL (KP and two-component KP) hypergeometric tau functions. It actually means that these series generate Hurwitz numbers with at most two arbitrary profiles (An arbitrary profile corresponds to a certain term in the perturbation series in the coupling constants which are higher times. The TL and 2-KP hierarchies there are two independent sets of higher times which yeilds two critical points for Hurwitz numbers).

Here, very briefly, we will write down few generating series for the \( \mathbb{RP}^2 \) Hurwitz numbers. These series may be not tau functions themselves but may be presented as integrals of tau functions of matrix argument. (The matrix argument, which we denote by a capital letter, say \( X \), means that the power sum variables \( p \) are specified as \( p_i = \text{tr} X^i, \ i > 0 \). Then instead of \( s_\lambda(p) \), \( \tau(p) \) we write \( s_\lambda(X) \) and \( \tau(X) \)). If a matrix integral in examples below is a BKP tau function then it generates Hurwitz numbers with a single arbitrary profile and all other are subjects of restrictions identical to those in \( \mathbb{CP}^1 \) case mentioned above. In all examples \( V(x, p) := \sum_{m>0} \frac{1}{m} x^m p_m \). We also recall the notation \( p_\infty = (1, 0, 0, \ldots) \). We also recall that numbers \( H_E(d; \ldots) \) are Hurwitz numbers only in case \( d \leq N \), \( N \) is the size of matrices.

For more details of the \( \mathbb{RP}^2 \) case see [53]. New development in [53] with respect to the consideration in [65] is the usage of products of matrices. Here we shall consider a few examples. All examples include the simplest BKP tau function, of matrix argument \( X \) written down in (32) as the part of the integration measure. Other integrands are the simplest KP tau functions \( \tau_1^{2KP}(X, p) := e^{\text{tr} V(X, p)} \) where the parameters \( p \) may be called coupling constants. The perturbation series in coupling constants are expressed as sums of the Schur functions over partitions and are similar to the series we considered in the previous sections.

**Example B1.** The projective analog of Okounkov’s generating series for double Hurwitz series as a model of normal matrices. From the equality

\[
(2\pi\zeta_1^{-1})^{\frac{1}{2}} e^{\frac{(n_0c_0)^2}{2\zeta_1}} e^{\zeta_0 n c + \frac{1}{2} \zeta_1 c^2} = \int_\mathbb{R} e^{x_i n_0 c_0 + (c x_i - \frac{1}{2} x_i^2) \zeta_1} d x_i,
\]

in a similar way as was done in [64] using \( \varphi_\lambda(\Gamma) = \sum_{(i, j) \in \lambda} (j - i) \), one can derive

\[
e^{n(|\lambda| n_0 c_0) \varphi_1(\lambda(\Gamma)) \delta_{\lambda, \mu}} = \kappa \int s_\lambda(M) s_\mu(M^\dagger) \det \left( M M^\dagger \right)^{n_0 c_0} e^{-\frac{1}{2} \zeta_1 \text{tr} (\log(M M^\dagger))^2} d M
\]
where $k$ is unimportant multiplier, where $M$ is a normal matrix with eigenvalues $z_1, \ldots, z_N$ and $\log |z_i| = x_i$, and where (see [47])

$$dM = d_* U \prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^N d^2 z_i.$$

Then the $\mathbb{RP}^2$ analogue of Okounkov’s generating series may be presented as the following integral ([56]) may be written

$$\sum_{\lambda, \ell(\lambda) \leq N} e^{n|\lambda|\zeta_0 + \zeta_1 \varphi_\lambda(\Gamma)} s_\lambda(p) = K \int e^{\text{tr} V(M, p)} e^{\zeta_0 \text{tr} \log (MM^\dagger)} - \frac{1}{2} \zeta_1 (\text{tr} \log (MM^\dagger))^2 \tau_1^B (M^\dagger) dM \quad (75)$$

Recall that in the work [56] there were studied Hurwitz numbers with an arbitrary number of simple branch points and two arbitrary profiles. In our analog, describing the coverings of the projective plane, an arbitrary profile only one, because, unlike the Toda lattice, the hierarchy of BKP has only one set of (continuous) higher times.

A similar representation of the Okounkov $\mathbb{CP}^1$ was earlier presented in [7].

Below we use the following notations

- $d_* U$ is the normalized Haar measure on $\mathbb{U}(N)$: $\int_{\mathbb{U}(N)} d_* U = 1$
- $Z$ is a complex matrix

$$d\Omega(Z, Z^\dagger) = \pi^{-n^2} e^{-\text{tr}(ZZ^\dagger)} \prod_{i,j=1}^N d\Re Z_{ij} d\Im Z_{ij}$$

- Let $M$ be a Hermitian matrix the measure is defined

$$dM = \prod_{i\leq j} d\Re M_{ij} \prod_{i<j} d\Im M$$

It is known [42]

$$\int s_\lambda(Z) s_\mu(Z^\dagger) d\Omega(Z, Z^\dagger) = (N)_\lambda \delta_{\lambda,\mu} \quad (76)$$

where $(N)_\lambda := \prod_{(i,j) \in \lambda} (N + j - i)$ is the Pochhammer symbol related to $\lambda$. A similar relation was used in [58], [29], [65], [9], [64], for models of Hermitian, complex and normal matrices.

By $I_N$ we shall denote the $N \times N$ identity matrix. We recall that

$$s_\lambda(I_N) = (N)_\lambda s_\lambda(p_\infty), \quad s_\lambda(p_\infty) = \frac{\dim \lambda}{d!}, \quad d = |\lambda|.$$
**Example B2. Three branch points.** The generating function for $\mathbb{R}P^2$ Hurwitz numbers with three ramification points, having three arbitrary profiles:

$$
\sum_{\lambda, \ell(\lambda) \leq N} \frac{s_\lambda(p(1)) s_\lambda(\Lambda) s_\lambda(p(2))}{(s_\lambda(p_\infty))^2} \\
= \int \tau_1^B(Z_1 \Lambda Z_2) \prod_{i=1,2} e^{\text{tr} v(Z_i^+, p^{(i)})} d\Omega(Z_i, Z_i^+) 
$$

If $p^{(2)} = p(q, t)$ with any given parameters $q, t$, and $\Lambda = \mathbb{I}_N$ then (77) is the hypergeometric BKP tau function.

**Example B3. 'Projective' Hermitian two-matrix model.** The following integral

$$
\int \tau_1^B(cM_2) e^{\text{tr} v(M_1, p) + \text{tr}(M_1 M_2)} dM_1 dM_2 = \sum_{\lambda} c^{\lambda|}(N)_\lambda s_\lambda(p) 
$$

where $M_1, M_2$ are Hermitian matrices is an example of the hypergeometric BKP tau function.

**Example B4. Unitary matrices.** Generating series for projective Hurwitz numbers with arbitrary profiles in $n$ branch points and restricted profiles in other points:

$$
\int e^{\text{tr}(cU_1^+...U_{n+m}^+) \left( \prod_{i=n+1}^{n+m} \tau_1^B(U_i) d_* U_i \right)} \left( \prod_{i=1}^{n} \tau_{\text{KP}}^B(U_i, p^{(i)}) d_* U_i \right) \\
= \sum_{d \geq 0} c^d (d!)^{1-m} \sum_{\lambda, |\lambda| = d, \ell(\lambda) \leq N} \left( \frac{\dim \lambda}{d!} \right)^{2-m} \left( \frac{s_\lambda(\mathbb{I}_N)}{\dim \lambda} \right)^{1-m-n} \prod_{i=1}^{n} \frac{s_\lambda(p^{(i)})}{\dim \lambda} \tag{78}
$$

Here $p^{(i)}$ are parameters. This series generate certain linear combination of Hurwitz numbers for base surfaces with Euler characteristic $2 - m$, $m \geq 0$. In case $n = 1$ this BKP tau function may be viewed as an analogue of the generating function of the so-called non-connected Bousquet-Melou-Schaeffer numbers (see Example 2.16 in [37]). In case $n = m = 1$ we obtain the following BKP tau function

$$
\int \tau_1^B(U_2) e^{\text{tr} v(U_1, p) + \text{tr}(cU_1^+ U_2^+)} d_* U_1 d_* U_2 = \sum_{\lambda} c^{\lambda|} \frac{s_\lambda(p)}{(N)_\lambda} 
$$
Example B5. **Integrals over complex matrices.** A pair of examples. An analogue of Belyi curves generating function [72], [14] is as follows:

\[
\sum_{\ell = 1}^{\Delta^{(1)} \ldots \Delta^{(n+1)} = 1} c^\ell H_\ell(d; \Delta^{(1)}, \ldots, \Delta^{(n+1)}) \prod_{i=1}^{n} \mathbf{P}_\Delta^{(i)} = \sum_{\lambda} c|\lambda| \frac{d|\lambda|(N)_{\lambda}}{\left(\dim \lambda\right)^{m-2}} \prod_{i=1}^{n} s_\lambda(\mathbf{P}_i) = \int e^{\text{tr}(cZ_1^\dagger \cdots Z_n^\dagger)} \left( \prod_{i=n+1}^{n+m} \tau_1^B(Z_i)d\Omega(Z_i, Z_i^\dagger) \right) \times \left( \prod_{i=1}^{n} \tau_1^{\text{KP}}(Z_i, \mathbf{P}_i)d\Omega(Z_i, Z_i^\dagger) \right)
\]

(79)

where \(E = 2 - m\) is the Euler characteristic of the base surface.

The series in the following example generates the projective Hurwitz numbers themselves where to get rid of the factor \((N)_{\lambda}\) in the sum over partitions we use mixed integration over \(U(N)\) and over complex matrices:

\[
\sum_{\Delta^{(1)} \ldots \Delta^{(n)}} c^\ell H_\ell(d; \Delta^{(1)}, \ldots, \Delta^{(n)}) \prod_{i=1}^{n} \mathbf{P}_\Delta^{(i)} = \sum_{\lambda, \ell(\lambda) \leq N} c|\lambda| \frac{\dim \lambda}{d!} \prod_{i=1}^{n} s_\lambda(\mathbf{P}_i) = \int \tau_1^{\text{KP}}(cU^\dagger Z_1^\dagger \cdots Z_n^\dagger, \mathbf{P}^{(n)})d\Omega(U, \mathbf{P}^{(n)}) \prod_{i=1}^{n-1} \tau_1^{\text{KP}}(Z_i, \mathbf{P}_i)d\Omega(Z_i, Z_i^\dagger)
\]

(80)

Here \(Z, Z_i, i = 1, \ldots, n - 1\) are complex \(N \times N\) matrices and \(U \in U(N)\). As in the previous examples one can specify all sets \(\mathbf{P}_i = \mathbf{P}(q_i, t_i), i = 1, \ldots, n\) except a single one which in this case has the meaning of the BKP higher times.

**Appendix E. The unitary ensemble as an example of a tensor model and Hurwitz numbers**

**E.1. One matrix model and combinatorics of graphs**

Let me recall some facts about Dyson-Wigner unitary ensemble and one-matrix model. The probability measure on the space of \(N \times N\) Hermitian matrices is defined as

\[
d\nu_N(h) = c_N \prod_{i > j} e^{-(\mathbb{R}h_{ij})^2 - (\mathbb{I}h_{ij})^2} d\mathbb{R}h_{ij}d\mathbb{I}h_{ij} \prod_{i=1}^{N} dh_{ii}
\]

(81)
see [46], the constant $c_N$ is chosen from the condition $\int d\nu_N(h) = 1$ where one integrates over the space of $N \times N$ Hermitian matrices. The expectation value for the Dyson-Wigner ensemble is defined as

$$\mathbb{E}^{DW}_N(f) = \int f(h) d\nu_N(h)$$

The famous pioneer works of Kazakov, Brezin [15], Migdal and Gross [25] relates this model to the theory of the two-dimensional quantum gravity and combinatorial models of Riemann surfaces on the one hand and to the Painlevé equation to the other hand. The relation to the Virasoro constrained tau functions of the Toda lattice was worked out in [21].

Here we review the combinatorial aspects of this model in very short. For details I send the reader to the bright review of this topic in [40]. Consider the following expectation value

$$\mathbb{E}^{DW}_N(\text{tr} h^{\lambda_1} \cdots \text{tr} h^{\lambda_\ell}) =: \mathbb{E}^{DW}_N(p_{\lambda}(h)) \quad (82)$$

where $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ is a partition of length $\ell$ (it means that $\lambda_\ell > 0$). One can check that this expectation value vanishes if the weight $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ if the partition $\lambda$ is odd. Let $|\lambda| = 2n$. has the following meaning. Let us consider $\ell$ polygons with ressectively $\lambda_1$, $\lambda_2$, $\ldots$, $\lambda_\ell$ edges. We imply that the polygons are (say, clockwise) oriented. Each edge is linked with a single edge. Let us connect such pairs by a line - as we did it before in subsection 3. We will call these lines which connect edges of the same polygon chords, and lines which connect different polygon links. One can glue all edges connected (either by chord or by link) in the pairwise way, identifying the end of one edge with the beginning of the other one (we remember that polygons are oriented).

The central statement is that the expectation (82) counts the number of the ways one can glue the polygons, see for instance Chapter 3.3. called ”Matrix Integrals for Multi-face maps” in [40] for the best review. Each way of gluing yields the model of orientable two-dimentional surface $\Sigma_{g^*}$ of genus $g^*$ and the ribbon graph with $n$ edges and with $v = n - \ell + g^*$ vertices.

The expectations (82) are generated by the famous one-matrix model, introduced in [15]:

$$\mathbb{E}^{DW}_N(e^N \text{tr} V(h,p)) = \sum_{\lambda} \frac{1}{z_\lambda^{\ell(\lambda)}} \mathbb{E}^{DW}_N(p_{\lambda}(h)) p_\lambda \quad (83)$$

where $p = (p_1, p_2, \ldots)$ are parameters (the coupling constants).\footnote{In the original model all $p_i = 0$ except $p_2$ and $p_3$, the infinite set of parameters - Toda lattice higher times - was introduced in [21].}

To get the statement one need to do the following steps

1. to write down each trace, say, $\text{tr} h^k$ as $S_k = h_{i_1,i_2} h_{i_2,i_3} \cdots h_{i_k,i_1}$ where we imply the summation over repeated indices. We assign a $k$- polygon to each trace $\text{tr} h^k$, thus, we get $\ell$ polygons respectively of sizes $\lambda_1, \ldots, \lambda_\ell$. Each term in the sum $S_k$ is labeled by a given set $i_1, \ldots, i_k$ which labels vertices
of the polygon in, say, anti-clockwise direction, while the edge between the vertices \( i_a, i_{a+1} \) are assigned to the entry \( h_{i_a, i_{a+1}} \).

(2) Consider \( E^D_N(S_{\lambda_1} \cdots S_{\lambda_\ell}) \) and take Gauss integrals of each term in the sum over all variables. Then, only these terms contribute whose all \( k \) factors meet their pair. One uses the chord diagrams to denote the Wick’s pairing of the entries. Each chord connects a pair of either of \( h_{ij} \) and \( h_{ji} \) where \( N \geq i > j \), or the pairing of \( h_{ii} \) with itself where \( i = 1, \ldots, N \). The pairing means gluing of the sides of polygons. One gets the oriented two-dimensional surface of a genus which is the genus of the chord diagram \( g^* \).

(3) the result of the Gauss integration of each monomial term of the product \( S_{\lambda_1} \cdots S_{\lambda_\ell} \) is equal either 1 or 0. Thus, the whole sum \( (83) \) is equal to the number of possible chord diagrams up to the weight of the automorphism group of each chord diagram (not to count it twice or more times).

E.2. A tensor model based on the one-matrix model

Consider the \( N \times N \) matrix \( h \) with noncommuting entries. In our case, one can think of the Hermitian matrix \( H \) (of the size \( L \times L \) where \( L = NM \)) splitted into blocks of the size \( N \times N \). Then, each entry may be labeled by 4 indices \( h_{i,j}^{a,b} \) where \( i, j = 1, \ldots, N \) and \( a, b = 1, \ldots, M \). Let us consider Hermitian \( H \). Then \( h_{ij} = h_{ji}^\dagger \). Let us introduce axillary complex matrices \( Z_i, i = 1, \ldots, N \) such that \( h_{ii} = \frac{1}{\sqrt2} \left( Z_i + Z_i^\dagger \right) \) (of cause, such matrices are not defined in the unique way) and introduce \( y_i := \frac{1}{\sqrt2} (Z_i - Z_i^\dagger) \) which is Hermitian.

Consider Dyson-Wigner unitary ensemble of the \( L \times L \) matrices \( H \). The probability measure can be written as

\[
d\nu_L(H) = \prod_{i>j}^N d\mu(h_{ij}) \prod_{i=1}^N d\nu(h_{ii}) \int \prod_{i=1}^N d\nu_N(y_i) = \prod_{i>j}^N d\mu(h_{ij}) \prod_{i=1}^N d\mu(Z_i)
\]

where the measures \( d\mu \) and \( d\nu \) are defined respectively by \( (11) \) and by \( (81) \).

On the other hand, it is the model of \( \frac{1}{2} N(N+1) \) independent complex Ginibre ensembles without sources (all sources are identity matrices). This is the ensembles of complex matrices \( \{h_{i,j}, N \geq i > j\} \) and \( \{Z_i, i = 1, \ldots, N\} \).

We should keep in mind that the set of matrices \( Z_i, i = 1, \ldots, N \) enters into

\[
E_L \left( \text{tr}_3 H^{\lambda_1} \cdots \text{tr}_3 H^{\lambda_\ell} \right)
\]

slightly differentely.

Let us consider the one-matrix model based on \( L \times L \) matrices which is known to be the Virasoro constraint 1D Toda lattice (which is also a special KP, 2-KP and also 2D Toda lattice tau function):

\[
\tau_L(p) = E^D_L \left( e^{t \text{tr} V(H,p)} \right) = \sum_{\lambda} (L)_{\lambda} s_\lambda(p)s_\lambda(0, 1, 0, \ldots)
\]

Let us consider the same products of \( S_k = h_{i_1,i_2}h_{i_2,i_3} \cdots h_{i_k,i_1} \) but now
At last, let us note that the measure $d\nu_L(H)$ can be treated as the measure of the simple tensor model written as

$$
\begin{align*}
    d\omega(h) &= e^{-\sum_{a,b=1,\ldots,M}^{N \geq i \geq j} h_{i,j}^{a,b} h_{j,i}^{b,a}} \\
    &\times \prod_{i,j,N \geq i \geq j} d\Re h_{i,j}^{a,b} d\Im h_{i,j}^{a,b} \prod_{i=1,\ldots,N} d\Re h_{i,i}^{a,b} d\Im h_{i,i}^{a,b} \prod_{i,a,M \geq a > b} d\Re h_{i,i}^{a,a} d\Im h_{i,i}^{a,a}
\end{align*}
$$

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