FINITE SYNTOMIC TOPOLOGY AND ALGEBRAIC COBORDISM OF NON-UNITAL ALGEBRAS

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Abstract. Elmanto, Hoyois, Khan, Sosnilo, and Yakerson [EHK’21] and [EHK’20] invented that the algebraic cobordism is the sphere spectrum of the $\mathbb{P}^1$-stable homotopy category of framed motivic spectra with finite syntomic correspondence. Inspired by their works and Dwyer–Kan [DK80b]’s hammock localization, we consider the localization of the stable $\infty$-category of motivic spectra by zero-section stable finite syntomic surjective morphisms. This paper results that the localization functor is $\mathbb{A}^1$-homotopy equivalent to the finite syntomic hyper-sheafification, and the algebraic cobordism is weakly equivalent to the motivic sphere spectrum after the localization (or the hyper-sheafification). Furthermore, on the finite syntomic topology, we prove the tilting equivalence between the algebraic cobordisms for non-unital integral perfectoid algebras.

1. Introduction

Let $S$ be a scheme. Voevodsky [Voe98] first introduced $\mathbb{P}^1$-stable homotopy category $\text{SH}(S)$, being an algebraic analogy of stable homotopy category, where $\mathbb{P}^1$ denotes the pointed projective line. The algebraic cobordism $\text{MGL}$ analog to the complex cobordism. Gepner and Snaith [GS09] and Panin, Pimenov, and Röndigs [PPR09] proved that the algebraic cobordism is the universal object of oriented motivic spectra as the complex cobordism is the universal oriented cohomology theory.

Voevodsky; Garkusha and Panin [GP21] constructed another $\mathbb{P}^1$-stable homotopy category $\text{SH}^f(S)$ by framed correspondence, being categorical equivalent to $\text{SH}(S)$ [EHK’21].

After coming Framed correspondence, due to Elmanto, Hoyois, Khan, Sosnilo, and Yakerson [EHK’20], the algebraic cobordism is the sphere spectrum of the stable $\infty$-category $\text{SH}^\text{FSyn}(S)$ of framed motivic spectra with finite syntomic correspondence.

This paper is inspired by the work [EHK’20] and Dwyer–Kan [DK80b]’s hammock localization of $\infty$-category, and will clarify that the stable $\infty$-category of motivic spectra with finite syntomic topology realize the algebraic cobordism as the unit object of the monoidal structure.

In this paper, we fix a base scheme $S$ and consider $\mathbb{A}^1$-homotopy theory on the Grothendieck site finite syntomic topology (it is an fppf topology finer than étale topology) of the category of

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schemes of finite presentation over $S$. Our finite syntomic $\mathbb{A}^1$-homotopy theory provides the following result: Finite syntomic hyper-sheafification is $\mathbb{A}^1$-homotopy equivalent to the localization by zero-section stable finite syntomic surjective morphisms (Theorem 3.8). Furthermore, motivic spectra of finite syntomic hypersheaves are oriented.

Remark 1.1. The relation the stable $\infty$-category of finite syntomic motivic spectra $\operatorname{MSp}^{\text{FSyn}}$ between those $\mathbb{P}^1_+$-stable homotopy categories $\operatorname{SH}^{\text{FSyn}}(S)$, $\operatorname{SH}^{\text{FSyn}}(S)$, and $\operatorname{SH}^{\text{FSyn}}(S)$ in [EHK+20] is represented as the following diagram:

$$
\begin{array}{c}
\text{SH}(S) \\
\downarrow \sim \\
\uparrow \\
\text{SH}_f(S) \\
\end{array} \\
\begin{array}{c}
\text{Ho}(\operatorname{MSp}) \\
\downarrow L_{\text{FSyn}} \\
\downarrow \sim \\
\text{Ho}(\operatorname{L}_{\text{FSyn}}^{\text{MSp}}) \\
\end{array} \\
\begin{array}{c}
\text{Ho}(\operatorname{MSp}^{\text{FSyn}}) \\
\downarrow \sim \\
\text{Ho}(\operatorname{L}_{\text{FSyn}}^{\text{MSp}}) \\
\end{array}
$$

where $(−)_\text{FSyn}$ is the induced functor by finite syntomic hyper-sheafification, $L_{\text{FSyn}}$ is the localization by the family of zero-section stable finite syntomic surjective morphisms, and $L_{\text{FSyn}}^{\text{MSp}}$ denote the functor localizing finite syntomic correspondences $X \leftarrow Z \rightarrow Y$, here $f$ is finite syntomic surjective, by the following equivalence relation

\[
\begin{array}{c}
f \downarrow Z \\
X \downarrow Z \downarrow g \\
W \\
Y \uparrow \uparrow f' \uparrow \uparrow \uparrow \\
\end{array}
\]

where all of $f$, $f'$, and $g$ are finite syntomic surjective, and $g$ is also zero-section stable.

As an application of finite syntomic motivic spectra, the algebraic cobordism can be equivalent to the motivic sphere spectrum after finite syntomic hyper-sheafification. Furthermore, for non-unital algebras, including non-unital perfectoid algebras, we obtain the tilting equivalence of algebraic cobordism.

This paper is organized as follows. Section 2 recalls the definition and property of model categories and hammock localization of model categories to study finite syntomic motivic spaces and spectra. In section 3, we introduce $\mathbb{A}^1$-homotopy theory on the finite syntomic site $\text{Sch}_S^{\text{fp}}$, where $\text{Sch}_S^{\text{fp}}$ denotes the small category of $S$-schemes of finite presentation. Finite syntomic topology and prove that equivalence between finite syntomic hyper-sheafification and localization by zero-section stable finite syntomic coverings in the $\mathbb{A}^1$-homotopy theory. Section 4 proves that an initial object of the stable $\infty$-category of finite syntomic motivic spectra coincides with the (finite syntomic) algebraic cobordism. Section 5, we prove the tilting equivalence for algebraic cobordisms of non-unital perfectoid algebras (Theorem 5.12).
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2. A short preliminary of model categories

In this section, we recall a short preliminary of model category, in particular, Dwyer and Kan’s hammock localization.

2.1. Definition of model category.

**Definition 2.1.** A model category is a category $M$ with three classes of morphisms $W_M$, $C_M$, and $F_M$, such that the following properties hold:

- MC1 The category $M$ admits all small limits and colimits.
- MC2 The class $W_M$ has the 2-out-of-3 property.
- MC3 The all of three classes $W_M$, $C_M$, and $F_M$ contain all isomorphisms and are closed under all retracts.
- MC4 The class $F_M$ has the right lifting property for all morphisms in $C_M = W_M$, and the class $F_M = W_M$ has the right lifting property for all morphisms in $C_M$.
- MC5 The couples $(C_M = W_M, F_M)$ and $(C_M, F_M = W_M)$ are weak functorial factorization systems.

We say that a morphism in $W_M$, $C_M$, and $F_M$ is called a weak equivalence, a cofibration, and a fibration. In addition, we say that a morphism in the class $C_M = W_M$ and $F_M = W_M$ is respectively called a trivial cofibration and a trivial fibration.

**Definition 2.2.** An adjunction $F : M ⇄ N : G$ between model categories is called a Quillen adjunction if $F$ and $G$ preserve the factorization systems in the axiom MC5 in Definition 2.1. Moreover, if the Quillen adjunction $F : M ⇄ N : G$ induces categorical equivalences between their homotopy categories (See [Qui67, Chapter 1]), it is called a Quillen equivalence of model categories. Then $F$ is called a left Quillen equivalence and $G$ a right Quillen equivalence.

**Definition 2.3.** A model category is left proper if the class of weak equivalences is closed under cobase change by cofibrations. Dually, we say that a model category is right proper if the class of weak equivalences is closed under base change by fibrations.

**Example 2.4.** The category $\text{Set}_\Delta$ of simplicial sets has a model structure described as follows:

- (C) A cofibration is a monomorphism of simplicial sets.
- (F) A fibration is a Kan fibration of simplicial sets.
A morphism \( f : X \to Y \) of simplicial sets is a weak equivalence if the induced map \(|f| : |X| \to |Y|\) of the geometric realizations is a homotopy equivalence of topological spaces.

This model structure of \( \text{Set}_\Delta \) is called Kan–Quillen model structure, being both left and right proper. For example, we can refer to proof of the properness in [GJ09, Chapter II.9].

**Definition 2.5.** Let \( \mathbf{M} \) be a \( \text{Set}_\Delta \)-enriched model category. The model structure of \( \mathbf{M} \) is **simplicial** if \( \mathbf{M} \) is tensored and cotensored, and the tensor product \( - \otimes - : \text{Set}_\Delta \times \mathbf{M} \to \mathbf{M} \) is a left Quillen bifunctor.

**Definition 2.6.** Let \( H \) be a collection of morphisms in a model category \( \mathbf{M} \). Let \( \overline{H} \) denote the set of morphisms in \( \mathbf{M} \) have the right lifting property for all morphisms of \( H \). Similarly, we let \( H^\perp \) denote the set of morphisms in \( \mathbf{M} \) with the left lifting property of all morphisms in \( H \). The set \( (\overline{H})^\perp \) is called the **weakly saturated class** generated by \( H \).

**Definition 2.7.** Let \( \mathbf{M} \) be a locally presentable model category. Let \( \mathbf{W}_\mathbf{M} \) be the class of weak equivalences in \( \mathbf{M} \) and \( \mathbf{C}_\mathbf{M} \) the class of cofibrations in \( \mathbf{M} \). We say that \( \mathbf{M} \) is **combinatorial** if there exist two small sets \( I \) and \( J \) such that the classes \( \mathbf{C}_\mathbf{M} \) and \( \mathbf{C}_\mathbf{M} \cap \mathbf{W}_\mathbf{M} \) are weakly saturated classes of morphisms generated by \( I \) and \( J \), respectively. We say that a combinatorial model category \( \mathbf{M} \) is **tractable** if \( I \) can be chosen cofibrant domains.

If \( \mathbf{M} \) is a model category with the property that every object is cofibrant, then \( \mathbf{M} \) is tractable.

**Example 2.8.** The model category \( \text{Set}_\Delta \) is combinatorial. The collection of cofibrations is generated by morphisms which form \( \partial \Delta^n \hookrightarrow \Delta^n \) (\( n \geq 0 \)) and the collection of trivial cofibrations is generated by morphisms that form \( \Lambda_i^n \hookrightarrow \Delta^n \) (\( 0 \leq i \leq n, n \geq 0 \)). (See e.g. [GJ09, Chapter II.9].)

It is known that any left proper combinatorial simplicial model category has Bousfield localization which is described as the followings:

**Definition 2.9.** Let \( \mathbf{M} \) be a left proper combinatorial simplicial model category. Let \( H \) be a collection of cofibrations. We say that a fibrant object of \( X \in \mathbf{M} \) is \( H \)-local if for any morphism \( f : Y \to Y' \) in \( H \), the induced map
\[
\map{f^*} : \text{Map}_\mathbf{M}(Y', X) \to \text{Map}_\mathbf{M}(Y, X)
\]
is a trivial Kan fibration of simplicial sets. A morphism of \( f : Y \to Y' \) is an \( H \)-weak equivalence if for any \( H \)-local object \( X \), the induced map
\[
\map{f^*} : \text{Map}_\mathbf{M}(Y', X) \to \text{Map}_\mathbf{M}(Y, X)
\]
is a trivial Kan fibration.
Proposition 2.10. Let $\mathcal{M}$ be a left proper combinatorial simplicial model category with a collection of cofibrations $H$. Let $L^H_\mathcal{M}$ denote the category whose underlying category is the same of $\mathcal{M}$. The model structure of $L^H_\mathcal{M}$ is defined as follows:

(C) The collection of cofibrations of $\mathcal{M}$ is the same of $\mathcal{M}$.
(W) The collection of weak equivalences of $\mathcal{M}$ is the collection of $H$-weak equivalences.
(F) The collection of fibrations is the collection of morphisms that have the right lifting property for all morphisms, which are both cofibrations and $H$-weak equivalences.

Then $L^H_\mathcal{M}$ is a left proper combinatorial simplicial model category. Furthermore, the functor $L^H : \mathcal{M} \to \mathcal{M}$ induced by the identity functor on the underlying category is a left Quillen functor of simplicial model categories.

proof. See [Lur09, p.904, Section Appendix A.3.7].

The model category $L^H_\mathcal{M}$ is said to be the Bousfield localization of $\mathcal{M}$ by $H$.

2.2. Hammock localization of simplicial model categories. We explain the hammock localization [DK80b]: Let $\mathcal{C}$ be a category and $H$ a small subcategory containing all the identity morphism. The hammock localization $\mathcal{L}(\mathcal{C}, H)$ is a simplicial category with the same object of $\mathcal{C}$ and with the set of morphisms $\text{Hom}_{\mathcal{L}(\mathcal{C}, H)}(X, Y)$ which is a simplicial set whose $m$-simplices are “hammocks of width $m$” form $X$ to $Y$, being commutative diagrams

![Hammock Diagram](https://via.placeholder.com/150)

where all going-to-left and vertical morphisms belong to $H$. In the case that $H$ is closed under base changes, the hammock localization is equivalent to the Verdier localization. Indeed, the limit of the zig-zag

$$X \leftarrow K_{0,1} \rightarrow K_{0,2} \cdots \rightarrow K_{0,n-1} \rightarrow Y$$

determines the roof $X \leftarrow K \rightarrow Y$, where the left arrow belongs to $H$ and $K$ is the limit of the diagram $(K_{i,j})$.

Theorem 2.11 ([DK83], Proposition 3.2). Let $\mathcal{M}$ be a left proper combinatorial simplicial model category and let $\mathcal{M}^\circ$ denote the subcategory spanned by fibrant-cofibrant objects. Then
the canonical functor

\[ \mathsf{M}^\circ \to \mathcal{L}(\mathsf{M}, \mathsf{W}_\mathsf{M}) \]

is a Dwyer-Kan equivalence of simplicial categories. Therefore the induced functor \( N_\Delta(\mathsf{M}^\circ) \to N_\Delta(\mathcal{L}(\mathsf{M}^\circ, \mathsf{W}_\mathsf{M} \cap \mathsf{M}^\circ)) \) is a categorical equivalence between \( \infty \)-categories, where \( N_\Delta : \mathbf{Cat} \to \mathbf{Set} \) is the simplicial nerve functor in [Lur09, p.20, Section 1.1.5], being a right adjoint to the Cordier nerve \( \mathcal{C} : \mathbf{Cat} \to \mathbf{Set} \).

**Corollary 2.12** ([DK83], Corollary 4.7, Proposition 4.8). Let \( H \) be a class of cofibrations of \( \mathsf{M} \) and \( L_H \mathsf{M} \) the Bousfield localization. Then the canonical functor \( L_H \mathsf{M} \to \mathcal{L}(\mathsf{M}, H) \) induces a categorical equivalence \( N_\Delta(L_H \mathsf{M}^\circ) \to N_\Delta(\mathcal{L}(\mathsf{M}, H)^\circ) \) of \( \infty \)-categories, where \( \mathcal{L}(\mathsf{M}, H)^\circ \) the fibrant replacement in the model category \( \mathbf{Cat}_\Delta \) whose model structure is Dwyer–Kan–Bergner model structure in [Ber07] and [Lur09, p.865, Theorem A.3.2.24.].

### 2.3. Projective and injective model structures.

We recall the definition of the projective model structure:

**Lemma 2.13** ([Bar10], p.21 Definition 2.11 and Lemma 2.12). Let \( \mathsf{M} \) be a combinatorial model category and \( \mathsf{C} \) a locally presentable model category. Given an adjunction

\[ E : \mathsf{M} \rightleftarrows \mathsf{C} : F, \]

we will define a model structure on \( \mathsf{C} \) by the following:

- **(F)** A morphism \( f : X \to Y \) in \( \mathsf{C} \) is a fibration if \( F(f) : F(X) \to F(Y) \) is a fibration in the model category \( \mathsf{M} \).

- **(W)** A morphism \( f : X \to Y \) in \( \mathsf{C} \) is a weak equivalence if \( F(f) : F(X) \to F(Y) \) is a weak equivalence in the model category \( \mathsf{M} \).

- **(WF)** A morphism \( f : X \to Y \) in \( \mathsf{C} \) is a trivial fibration if \( F(f) : F(X) \to F(Y) \) is a trivial fibration in the model category \( \mathsf{M} \).

- **(C)** A morphism \( f : X \to Y \) in \( \mathsf{C} \) is a cofibration if it has the right lifting property with respect to all trivial fibrations.

Assume that in \( \mathsf{C} \), transfinite compositions and push-outs of trivial cofibrations of \( \mathsf{C} \) are weak equivalences. Then the locally presentable category \( \mathsf{C} \) is a combinatorial model category, and it is a tractable model category if \( \mathsf{M} \) is. Furthermore, the above adjunction is a Quillen adjunction.

We say that the model structure of \( \mathsf{C} \) is the projective model structure induced by \( E \).

**Example 2.14.** Let \( \mathcal{C} \) be a category and \( \mathsf{M} \) a model category. The diagonal functor \( \Delta_\mathcal{C} : \mathsf{M} \to \mathsf{M}^\mathcal{C} \) has a right adjoint \( \prod_\mathcal{C} \). The model structure of \( \mathsf{M}^\mathcal{C} \) is said to be induced by \( \mathsf{M} \).

Next we recall the injective model structure of \( \mathsf{M}^\mathcal{C} \) in Example 2.14.
**Definition 2.15.** A morphism \( f : F \to G \) in \( \mathbf{M}^C \) is an injective cofibration or injective weak equivalence if \( f(X) : F(X) \to G(X) \) is a cofibration or weak equivalence in \( \mathbf{M} \) for any \( X \in \mathcal{C} \). Injective cofibration and injective weak equivalence in \( \mathbf{M}^C \) determine a model structure called the injective model structure.

**Theorem 2.16** ([Bar10], Theorem 2.14, Theorem 2.16, and Proposition 2.17). Let \( \mathbf{M} \) be a combinatorial model category and \( \mathcal{C} \) a small category. If the model structure of \( \mathbf{M} \) is left or right proper, then so is the projective model structure or the injective model structure of \( \mathbf{M}^C \). \( \square \)

3. **Motivic spectra with the finite syntomic topology**

This section defines motivic spaces and motivic spectra by following Jardine’s book [Jar15].

To introduce finite syntomic topology, we use a small category \( \mathbf{Sch}^f \) of schemes of finite presentation, instead of smooth schemes. For any category \( \mathcal{C} \), we will abbreviate the category of simplicial object of \( \mathcal{C} \) as \( \mathcal{C}^\Delta = \text{Fun}(\Delta^{op}, \mathcal{C}) \).

3.1. **Finite syntomic topology.** We recall finite syntomic topology, which is a Grothendieck topology finer than étale site and courser than fppf (flat and locally of finite presentation) topology.

**Definition 3.1.** A morphism \( f : Y \to X \) of schemes is finite syntomic if \( f \) is finite, flat, locally of finite presentation, and the relative cotangent complex \( L_{X/Y} \) has tor-amplitude in \([-1, 0]\). A finite syntomic topology is generated by étale topology and families \( \{ Y \to X \} \) containing a single finite syntomic surjective morphism.

By definition, a finite syntomic covering is an fppf covering, and any étale, Nisnevich, or Zariski covering is a finite syntomic covering. A finite syntomic hypercover \( \pi : Y_* \to X \) is a simplicial object of the category of schemes over \( X \), satisfying the following properties:

- The argumentation \( \pi : Y_0 \to X \) is a finite syntomic covering.
- For each \( n \geq 1 \), \( Y_n \to (\cosk_{n-1} Y_*)_n \) is a finite syntomic covering, where the coskelton functor \( \cosk_k : \Delta^{op} \to \mathbf{Sch}_X \) is the right Kan extension along the inclusion \( i_k : \Delta_{\leq k} \to \Delta \).

Here, \( \Delta_{\leq k} \) denote the full subcategory of \( \Delta \) spanned by those objects \([m]\) for \( 0 \leq m \leq k \).

Due to Stack-project [Sta22], finite syntomic morphisms are locally represented as pull-back of the universal finite syntomic morphisms:

**Proposition 3.2** (c.f. [Sta22], Section 0FKX). Let \( f : Y \to X \) be a finite syntomic morphism. Then for any \( x \in X \) there exists an integer \( d \geq 0 \) and a commutative diagram

\[
\begin{array}{ccc}
Y & \xleftarrow{f} & V \\
\phantom{f} & \searrow & \downarrow_{\pi d} \\
X & \xleftarrow{g} & U \\
\phantom{f} & \searrow & \downarrow_{\eta d} \\
& \phantom{f} & \text{Spec}\mathbb{Z}
\end{array}
\]
with the following properties

- $U \subset X$ is an open neighborhood of $x$ and $V = f^{-1}(U)$,
- $\pi_d : V_d \to U_d$ is the “universal finite syntomic” morphism of rank $d \geq 0$ in \cite[Section 0FKX, Example 49.11.2, Lemma 49.11.3]{Sta22},
- both $U_d$ and $V_d$ are smooth over $\text{Spec} \mathbb{Z}$ (See \cite[Section 0FKX, Lemma 49.11.4, Example 49.11.6]{Sta22}),
- where the middle square is Cartesian (See \cite[Section 0FKX, Lemma 49.11.7]{Sta22}).

We say that a simplicial sheaf $E$ satisfies finite syntomic descent if $E$ is a injective fibrant fibrant object of $\text{Pre}(\text{Sch}^{fp}_S)_\Lambda$ and the induced map

$$E(\pi^*) : E(X) \to E(|Y|) = \lim_{\leftarrow} E(Y_n)$$

is a weak equivalence of simplicial sets for any hypercover $\pi : Y_\bullet \to X$ of $X \in \text{Sch}_S$. Here the functor $|−| : \text{Pre}(\text{Sch}^{fp}_S)_\Lambda \to \text{Pre}(\text{Sch}^{fp}_S)_\Lambda$ is the geometric realization of simplicial objects.

3.2. **Finite syntomic motivic spaces and spectra.** Let $\text{Shv}^{\text{FSyn}}(\text{Sch}^{fp}_S)_\Lambda$ denote the full subcategory of $\text{Pre}(\text{Sch}^{fp}_S)_\Lambda$ spanned by finite syntomic hyper-sheaves. The simplicial category $\text{Shv}^{\text{FSyn}}(\text{Sch}^{fp}_S)_\Lambda$ has a model structure: A simplicial sheaf $X$ is said to be motivic fibrant if $X$ satisfies finite syntomic descent for any hypercovering, and has a right lifting property with respect to the morphism $(j, f) : (\mathbb{A}^1 \times A) \cup_B \mathbb{A}^1 \times B$ arising from a cofibration $j : A \to B$ and an $S$-rational point $f : * \to \mathbb{A}^1$, where $*$ is a final object. A map $f : X \to Y$ in $\text{Shv}^{\text{FSyn}}(\text{Sch}^{fp}_S)_\Lambda$ is a motivic equivalence if the pull-back

$$f^* : \text{Map}_{\text{Shv}(\text{Sm}_S)}(Y, Z) \to \text{Map}_{\text{Shv}(\text{Sm}_S)}(X, Z)$$

is a weak homotopy equivalence of simplicial sets for each motivic fibrant object $Z$. Let $\text{MS}^{\text{FSyn}}$ denote the full subcategory of $\text{Shv}^{\text{FSyn}}(\text{Sch}^{fp}_S)_\Lambda$ spanned by motivic fibrant objects, becoming an $\infty$-category. We call an object of $\text{MS}^{\text{FSyn}}$ a finite syntomic motivic space.

The category $\text{Spt}_{\mathbb{P}^1_+}(\text{Sch}^{fp}_S)$ of $\mathbb{P}^1_+$-spectra objects of $\text{Shv}^{\text{FSyn}}(\text{Sch}^{fp}_S)_\Lambda$ has a canonical model structure called $\mathbb{P}^1_+$-stable model structure (See \cite[Section 10.3]{Jar15}), which is Bousfield localization of strict model structure \cite[p.362, Proposition 10.15]{Jar15}. Let $\text{MSp}$ denote the full subcategory of $\text{Spt}_{\mathbb{P}^1_+}^{\text{FSy}}(\text{Sch}^{fp}_S)$ spanned by $\mathbb{F}^1$-stable fibrant object, and we say that $\text{MSp}$ is the $\infty$-category of motivic spectra.

3.3. **Fixing an equivalence** $i_0 : \{0\} \to \mathbb{A}^n$. Let $K$ be a simplicial set. We recall the covariant model structure of the over category $\text{Set}_{\Lambda/K}$:

**Definition 3.3** (\cite[pp.68–69, Definition 2.1.4.5 and Lemma 2.1.4.6]{Lur09}). Let $K$ be a simplicial set. A morphism $f : X \to Y$ in $\text{Set}_{\Lambda/K}$ is

- a covariant cofibration if it is a monomorphism.
• a covariant trivial cofibration is a morphism which is in the smallest weakly saturated
class generated by the collection of anodyne $K$-morphisms $\Lambda^n_i \to \Delta^n$ ($0 \leq i < n$) for
each $n \geq 0$.

The model structure of $\text{Set}_{\Lambda/K}$ is called the **covariant model structure**, being also left proper
combinatorial simplicial [Lur09] p.70, proposition 2.1.4.8. A fibrant object $f : X \to K$ is said to be a **left fibration**.

In the model category $\text{Set}_{\Lambda/K}$, for any simplicial set $X$ over $K$, the inclusion $i_{0,X} : X \simeq [0] \times X \to \Delta^1 \times X$ is a covariant trivial cofibration. The covariant model structure is a Bausfield localization of the Joyal model structure [Joy02] of $\text{Set}_{\Lambda/K}$ by those anodyne morphisms $\Lambda^n_0 \rightarrow \Delta^n$ ($n \geq 1$).

**Example 3.4.** We mainly are interested in the case $K = \Delta^n$ ($n \geq 1$). More simply, the inclusion $[0] \rightarrow \Delta^1$ is not a covariant fibration and the other $[1] \rightarrow \Delta^1$ a covariant fibration. The point $[1]$ in $\Delta^1$ is weakly initial in $\text{Set}_{\Lambda/\Delta^1}$. Since $[1] \rightarrow \Delta^1$ is a left fibration and $[0] \rightarrow \Delta^1$ a covariant equivalence, the base-change $\emptyset \rightarrow [1]$ of $[0] \rightarrow \Delta^1$ is an equivalence in $\text{Set}_{\Lambda/\Delta^1}$. Note that $i_*([0]) \amalg i_*([1])$ is a homotopy pushout. Therefore one has a chain of equivalences

$$i_*([\partial\Delta^1]) = i_*([0] \amalg [1]) = i_*([0]) \amalg i_*([1]) = i_*([0]) \amalg i_*([1]) \approx i_*([0]) \amalg i_*([1]),$$

canceling out the other point $[1]$ by the covariant model structure.

**Definition 3.5.** For any $n \geq 0$, the initial point $i : [0] \rightarrow \Delta^n$ and the projection $\pi : \Delta^n \rightarrow [0]$ induce Quillen adjunctions

$$i_* : \text{Set}_{\Delta} \rightleftarrows \text{Set}_{\Delta/\Delta^n} : i^* \text{ and } \pi_* : \text{Set}_{\Delta/\Delta^n} \rightleftarrows \text{Set}_{\Delta} : \pi^*$$

of model categories. A simplicial set $X$ is **zero-section stable** if the canonical morphism $\pi \circ i \circ \eta : X \rightarrow \pi_*(i_*(X))$ induced by the unit map $\eta : X \rightarrow (i^* \circ \pi^*)((\pi_* \circ i_*)(X))$ is a weak equivalence.

The canonical morphism $\pi \circ i \circ \eta : X \rightarrow \pi_*(i_*(X))$ corresponds to $\pi \circ i : (\pi \circ i)^{-1}(X) \rightarrow X$ via the adjunction $((\pi \circ i)_*, (\pi \circ i)^*)$.

Let $\mathbb{A}_n^*$ denote the cosimplicial motivic space defined by $\mathbb{A}_n^*([n]) = \mathbb{A}_n^\infty$ for each $n \geq 0$. Then the singular functor $\text{Hom}_{\mathbb{A}}(\mathbb{A}_n^*, -) : \text{MS} \rightarrow \text{MS}$ admits a left adjoint. We We say that $X$ is locally zero-section stable if the simplicial sheaf $\text{Hom}_{\mathbb{A}}(\mathbb{A}_n^*, X)$ is stalk-wise zero-section stable.

**Proposition 3.6.** Let $\text{MS}_0$ denote the full subcategory spanned by locally zero-section stable spaces. Composition of the embedding $i_0 : [0] \rightarrow \mathbb{A}_\infty$ and the projection $\pi : \mathbb{A}_\infty = \lim_{\rightarrow} \mathbb{A}_n \rightarrow [0]$ induces a localization functor

$$Z_0 = \lim_{n \geq 1} (\pi_* \circ i_*) \circ (\pi_* \circ i_*) \circ \cdots \circ (\pi_* \circ i_*) : \text{MS} \rightarrow \text{MS}_0$$

of $\infty$-categories.
**proof.** This is straightforward.

The following is a crucial property of the finite syntomic $\mathbb{A}^1$-homotopy theory.

**Proposition 3.7.** Let $f : Y \to X$ be a finite syntomic surjective morphism. Assume that $Y$ is locally zero-section stable. Then the pullback $f^* : L_{FSyn}(X) \to L_{FSyn}(Y)$ is a (finite syntomic) motivic equivalence, where $L_{FSyn}(T)$ denote the finite syntomic hyper-sheafification of $\text{Map}_S(-, T)$ for any $S$-scheme $T$.

**proof.** By the argument of the proof of [Sta22, Section 068E, Lemma 37.58.8], we may assume that the finite syntomic morphism $f : Y \to X$ factors as $f : Y \to \mathbb{A}^n_X \to X$ for some integer $n \geq 0$, where $j$ is a Koszul regular closed immersion of virtual dimension zero and $\pi$ is the canonical projection. We fix a weak equivalence $i_0 : \{0\} \to \mathbb{A}^n_S$. Then we have a factorization

$$Y \times_{\mathbb{A}^n_S} Y \to Y \times_{\mathbb{A}^n_S} \mathbb{A}^n_Y = Y \times_X Y \to Y \times_{\mathbb{A}^n_S} Y.$$

The canonical projection $Y \times \pi : Y \times_X Y = Y \times_{\mathbb{A}^n_S} \mathbb{A}^n_Y \to Y \times_{\mathbb{A}^n_S} Y$ and, by the assumption of $Y$, the above composition are weak equivalences. Therefore, the first morphism $Y \times i_0 : Y \times_{\mathbb{A}^n_S} Y \to Y \times_X Y$ is a trivial cofibration. We obtain that the morphism of $\check{\text{C}}$ech nerves:

$$\check{\mathcal{C}}_{\mathbb{A}^n_X}(Y)_\bullet \to \check{\mathcal{C}}_X(Y)_\bullet,$$

is also a trivial cofibration for the projective model structure, where

$$\check{\mathcal{C}}_{\mathbb{A}^n_X}(Y)_m = \overline{Y \times_{\mathbb{A}^n_S} \cdots \times_{\mathbb{A}^n_S} Y}$$

and $\check{\mathcal{C}}_X(Y)_m = \overline{Y \times_X \cdots \times_X Y}$

for each $m \geq 0$. Note that $i : Y \to \mathbb{A}^n_X$ is a monomorphism and $Y \times_{\mathbb{A}^n_X} Y = Y$. Then $\check{\mathcal{C}}_{\mathbb{A}^n_X}(Y)_\bullet$ is a constant nerve valued $Y$. Hence; we get a zig-zag of weak equivalences

$$X \leftarrow |\check{\mathcal{C}}_X(Y)_\bullet| \leftarrow |\check{\mathcal{C}}_{\mathbb{A}^n_X}(Y)_\bullet| \to Y,$$

where $|\cdot|$ denotes the geometric realization defined as the (homotopy) colimit.

In the framework of $\mathbb{A}^1$-homotopy theory, we obtain that finite syntomic hyper-sheafification is equivalent to the Bousfield localization by locally zero-section stable finite syntomic surjective morphisms:

**Theorem 3.8.** Let $S$ be a scheme and $\text{MS}$ the stable $\infty$-category of motivic spaces of the Nisnevich site $\text{Sch}_S^{op}$. The finite syntomic hyper-sheafification functor $(\cdot)_{FSyn} : \text{MS} \to \text{MS}^{FSyn}$ in $\text{MS}$ is equivalent to the Bousfield localization

$$L_{FSyn} : \text{MS} \to \text{MS}$$

by the collection of locally zero-section stable finite syntomic surjective morphisms.
proof. The functor $L_{FSyn}$ induces a functor $L_{FSyn}MS \to MS^{FSyn}$. It is sufficient to prove that the forgetful functor $MS^{FSyn} \to L_{FSyn}MS$ is essentially surjective. Let $X$ be a motivic space and $f : Y \to X$ a finite syntomic surjective morphism. We may assume that $X = S$ and $\Delta_X^0 = X$, being a final object. Then $X$ is zero-section stable and $f : Y \to X$ factors through the zero-section stable scheme $Z_0(Y)$. Since $\pi \circ i : Y \to \mathbb{A}^n_Y \to Y$ is finite syntomic for each $n \geq 0$, $Y \to Z_0(Y)$ is ind-finite syntomic. Therefore $Z_0(Y) \to X$ is ind-finite syntomic, being it is a motivic equivalence.

Note that $\pi \circ i : Y \to (\pi \circ i)_*(Y)$ is an epimorphism. The coimage, which is the colimit of the Čech nerve $\check{C}_{(\pi \circ i)_*(Y)}$, is equivalent to the target $(\pi \circ i)_*(Y)$, implying that $|\check{C}_{Z_0(Y)}(Y)| \simeq Z_0(Y)$. Thus, the covering $f : Y \to X$ is refined by $Z_0(Y) \to X$ and $|Y| \simeq |Z_0(Y)| \simeq X$. □

Corollary 3.9. Any finite syntomic motivic spectrum $E$ is zero-finite syntomic local. That is, the localization $L_{FSyn} : MSp \to MSp$ by the family of zero-section stable finite syntomic surjective morphisms induces a categorical equivalence of stable $\infty$-categories $L_{FSyn}(MSp) \to MSp^{FSyn}$. □

4. The (finite syntomic) motivic spectra of algebraic cobordism

In this section, we prove that finite syntomic hyper-sheafification of motivic sphere spectrum realizes the algebraic cobordism.

4.1. Finite syntomic invariant property and Thom equivalence. Let $X$ be a compact motivic space and $f : V(\mathcal{E}) \to X$ a vector bundle on $X$ of dimension $n$, where $\mathcal{E}$ is locally free sheaf on $X$ of rank $n$. The Thom space of $\mathcal{E}$ is the quotient $V(\mathcal{E})/(V(\mathcal{E}) \setminus Z)$, where $Z$ denotes the image of the zero-section. We refer to the pointed motivic space $\text{Thom}_X(\mathcal{E}) = V(\mathcal{E})/(V(\mathcal{E}) \setminus Z)$ as the Thom space of $\mathcal{E}$.

We recall the following Nisnevich purity of smooth schemes:

Theorem 4.1 ([MV99] Theorem 2.23). Let $i : Z \to X$ be a closed immersion of smooth $S$-schemes. Assume that $S$ is a Noetherian scheme. Then one has an $\mathbb{A}^1$-weak equivalence:

$$X/(X \setminus Z) \simeq \text{Thom}_Z(N_{X,Z}),$$

where $N_{X,Z}$ is the normal bundle. □

Definition 4.2. A motivic spectrum $E$ is preserving Thom equivalence if there is a weak equivalence $E(\text{Thom}_X(\mathcal{E})) \simeq E(\Sigma^n_X)$ for any pointed motivic space $X$, vector bundle $\mathcal{E}$ on $X$ of rank $n$, and $n \geq 0$. Here $\Sigma^n_X = (\mathbb{P}^1)^{\wedge n} \wedge X$.

Proposition 4.3. Assume that $S$ is a Noetherian scheme and let $F$ be a motivic spectrum satisfying that

$$F(f) : F(X) \to F(Y)$$
is an equivalence for any zero-section stable finite syntomic morphism \( f : Y \to X \). Then \( F \) is preserving Thom equivalence on smooth \( S \)-schemes.

**proof.** Let \( \pi : V(\mathcal{E}) \to X \) be a vector bundle of rank \( n \) over a smooth \( S \)-scheme. Then the zero-section \( Y \) is finite syntomic over \( X \) by definition. Therefore, by Lemma [72] we may assume that there exists a closed immersion \( i : Y \to \mathbb{A}^n_X \), which is locally complete intersection. Generally, the normal bundle \( N_{\mathcal{E}/X} \) is isomorphic to \( \mathcal{E} \). Therefore, by the Nisnevich purity \( V(\mathcal{E})/(V(\mathcal{E}) \setminus Y) = \mathbb{A}^n_X/(\mathbb{A}^n_X \setminus Y) \) and the assumption of \( F \), we obtain that \( F(\mathbb{A}^n_X/(\mathbb{A}^n_X \setminus Y)) \cong F(\mathbb{A}^n_X/(\mathbb{A}^n_X \setminus X)) \) and \( F(\mathbb{A}^n_X/((\mathbb{A}^n_X \setminus \{Y\})) \cong F(V(\mathcal{E})/(V(\mathcal{E}) \setminus Y)) \). \( \square \)

4.2. **Functors representing algebraic cobordism.** Due to [EHK+21] and [EHK+20], algebraic cobordism \( MGL \) can be represented by an ind-smooth scheme. To obtain the continuity of \( MGL \), we recall the following.

**Definition 4.4.** Let \( S \) be a scheme and \( X \) an \( S \)-scheme. We define (recall) the following functors \( h^i_{FSyn}(X) \), \( Hilb^{FSyn}(X/S) \), and \( FSyn \) from the category of schemes to one of groupoids:

1. For any scheme \( T \), \( Hilb^{FSyn}(X/S)(T) \) is an \( \infty \)-groupoid whose points are closed sub-schemes of \( X \times_S T \) such that those schemes are finite syntomic over \( T \).
2. For any scheme \( T \), \( FSyn(X/T) \) is an \( \infty \)-groupoid whose points are finite syntomic \( T \)-schemes.

**Theorem 4.5 [EHK+20] Lemma 3.5.1.** The forgetful morphism \( Hilb^{FSyn}(\mathbb{A}^{\infty}_S/S) \to FSyn \) is a universally \( \mathbb{A}^1 \)-equivalence on affine schemes. \( \square \)

**Theorem 4.6 [EHK+21] Lemma 5.1.3.** The scheme \( Hilb^{FSyn}(\mathbb{A}^n_S/S) \) is smooth over \( S \) for each \( n \geq 1 \). \( \square \)

**Theorem 4.7 [EHK+20] Theorem 3.4.1.** Let \( S \) be a scheme. In the equivalence between \( SH(S) \) and \( SH^\infty(S) \), there is an equivalence of motivic \( \mathbb{B} \)-rings \( MGL \cong \Sigma^n_{\infty}FSyn \). \( \square \)

**Corollary 4.8.** The motivic \( \mathbb{B} \)-ring algebraic cobordism \( MGL : CRing \to Sp \) preserves filtered colimit of commutative rings, where \( CRing \) denotes the category of commutative rings and \( Sp \) the stable \( \infty \)-category of spectra.

**proof.** Let \((A_\alpha)\) be a filtered inductive system of unital commutative rings. by those results mentioned above, one has a chain of weak equivalences:

\[
\begin{align*}
\text{Map}(\lim \text{Spec}A_\alpha, MGL) \cong \lim \text{Map}(\lim \text{Spec}A_\alpha, \Sigma^n_{\infty}Hilb^{FSyn}(\mathbb{A}^{\infty}_S/S)) \\
\cong \lim_{\alpha} \text{Map}(\lim \text{Spec}A_\alpha, \Sigma^n_{\infty}Hilb^{FSyn}(\mathbb{A}^{\infty}_S/S)) \cong \lim_{\alpha} \text{Map}(\text{Spec}A_\alpha, \Sigma^n_{\infty}Hilb^{FSyn}(\mathbb{A}^{\infty}_S/S)) \\
\cong \lim_{\alpha} \text{Map}(\text{Spec}A_\alpha, \Sigma^n_{\infty}Hilb^{FSyn}(\mathbb{A}^{\infty}_S/S)) \cong \lim_{\alpha} \text{Map}(\text{Spec}A_\alpha, MGL). 
\end{align*}
\]
We consider the class of morphism of $H$ is the class of morphisms locally generated by the family of universal finite syntomic morphisms $(\pi_d : V_d \to U_d)_{d \geq 0}$ in Proposition 3.2 (or equivalently, let $H$ be the class of the regular closed immersions (cofibrations) $i_d : V_d \to \mathbb{A}^d_{U_d}$ $(d \geq 0)$, where $\pi_d = p_d \circ i_d : V_d \to \mathbb{A}^d_{U_d} \to U_d$ in Proposition 3.2). Then the $\infty$-category admits a localization functor $L_{FSyn} : MSp \to MSp$ of finite syntomic morphisms. We can regard as the localization $L_{FSyn}MSp$ is obtained by the hammock localization $\text{Stab}_{\mathcal{P}}(\mathcal{C}(\text{Shv}^{FSyn}(\mathcal{S}ch_S)^r, H))$.

Write $h_X = \text{Hom}_S(\cdot, X)$ for any $S$-scheme $X$. By definition of $FSyn$ and considering the hammock localization, one has an equivalence: $L_{FSyn}(h_S)(X) \simeq L_{FSyn}(FSyn_S)(X)$. Therefore $\Sigma_\infty^c L_{FSyn}(FSyn_S)$ is an initial object of $MSp^{FSyn}$. Furthermore, the finite syntomic motivic sphere spectrum $L_{FSyn}(h_S)(X)$ has monoidal structure and is oriented by Proposition 4.3. Hence the finite syntomic sheafification $MGL \to MGL^{FSyn}$ factors through $L_{FSyn}(h_S)$. This implies $MGL^{FSyn} \simeq L_{FSyn}(h_S)$ Therefore, those spectra $L_{FSyn}(h_S)$ and $MGL$ are weakly equivalent in the stable $\infty$-category of finite syntomic motivic spectra.

**Corollary 4.9.** For an arbitrary scheme $S$, the algebraic cobordism $MGL$ is a unit object of the stable monoidal $\infty$-category $MSp^{FSyn}$. □

**Remark 4.10.** By Theorem 4.7, one has a chain of equivalences $MGL^{FSyn} \simeq L_{FSyn}(\Sigma_\infty^{c} h_S) \simeq L_{FSyn}(h_S)$ via the diagram in Remark 1.1.

5. **Algebraic cobordism of non-unital algebra**

As an application of the theory of finite syntomic motivic spectra, we consider algebraic cobordism of non-unital algebras and prove the tilting equivalence of non-unital perfectoid algebras. From this section, rings have not necessarily a multiplicative unit.

5.1. **Preliminary of homotopy algebra.**

**Proposition 5.1.** Let $A$ be a $V$-algebra and $I$ an ideal which is contained in the Jacobson radical of $A$. Write $A_n = A/I^n$ for each $n \geq 1$, and $A_\infty = \lim A_n$. Then

$$\lim : 2 - \lim \text{PMod}_{A_n} \to \text{PMod}_{A_\infty}$$

is a categorical equivalence. Furthermore, the quasi-inverse is the functor $P_\infty \mapsto (P_\infty \otimes_{A_\infty} A_n)$ for any $A_\infty$-module $P_\infty$.

**proof.** Let $(P_n)$ be an inverse system of finitely generated projective $A_n$-modules. Since the canonical map : $A'_n \to (\lim A'_n) \otimes_{A_\infty} A_n$ is an isomorphism for each $n \geq 1$ and $r \geq 0$, $P_n \to (\lim P_n) \otimes_{A_\infty} A_n$ is an isomorphism for each $n \geq 1$. Then $(P_n) \to ((\lim P_n) \otimes_{A_\infty} A_n)$ is an isomorphism of inverse systems.

Conversely, let $P_\infty$ be a finitely generated projective $A_\infty$-module. Then we show that $P_\infty \to \lim P_\infty \otimes_{A_\infty} A_n$ is an isomorphism. Since $P_\infty$ is canonically isomorphic to the second dual $P_\infty^{\vee \vee}$,
\[ P_\infty \otimes A_n \cong \text{Hom}_{A_n}(P_{\infty}, A_n). \] Therefore one has \( \lim P_\infty \otimes A_n \cong \lim \text{Hom}_{A_n}(P_{\infty}, A_n) \cong \text{Hom}_{A_n}(P_{\infty}, A_\infty) \cong P_\infty. \) \qed

A complex \( E \) of \( A \)-modules is \textit{perfect} if there exist a complex \( P \) of finitely generated projective \( A \)-modules such that \( E \) and \( P \) are quasi-isomorphic. Let \( \text{Perf}(A) \) denote the stable \( \infty \)-category of perfect \( A \)-complexes.

**Proposition 5.2.** Let \( A \) be a \( V \)-algebra and \( I \) an ideal which is contained in the Jacobson radical of \( A \). For \( n \geq 1 \), set \( A_n = A/I^n \), and \( A_\infty = \lim A_n \). Then the canonical adjunction \((- \otimes_A A_n) : D(A_\infty) \rightleftarrows 2 - \lim D(A_n) : \mathbb{R} \lim_n \) induces a categorical equivalence

\[ \mathbb{R} \lim_n : 2 - \lim \text{Perf}(A_n) \rightarrow \text{Perf}(A_\infty) \]

between stable \( \infty \)-categories.

**proof.** Let \( E \) be a perfect \( A_\infty \)-complex. Since \( E \) is a dualizable object of the derived category \( D(A_\infty) \), \( E \otimes_{A_\infty} (-) \) preserves all small limits. Therefore the canonical morphism \( E \rightarrow \mathbb{R} \lim \mathbb{R} \lim_n (E \otimes_{A_n} A_n) \) is a quasi-isomorphism. Hence \( E \rightarrow \mathbb{R} \lim \mathbb{R} \lim_n (E \otimes_{A_n} A_n) \)

Conversely, let \( (E_n) \) be an inverse system of complex of finitely generated projective \( A_n \)-modules. Since \( A_\infty \rightarrow A_n \) is surjective for each \( n \geq 1 \), \( \lim E_n \rightarrow E_n \) is also surjective for each \( n \geq 1 \). Therefore, clearly, the canonical morphism \( \lim E_n \otimes_{A_n} A_n \rightarrow E_n \) is a quasi-isomorphism for each \( n \geq 1 \). Hence the morphism \((\lim E_n) \otimes_{A_\infty} A_n \rightarrow (E_n)\) is an inverse system of quasi-isomorphisms.

Let \( \sigma_{[a,b]} E \) denote the stupid truncation of \( E \) of degree \( a \) to \( B \):

\[ \sigma_{[a,b]} E : \cdots \rightarrow 0 \rightarrow E_a \rightarrow E_{a-1} \rightarrow \cdots E_b \rightarrow 0 \rightarrow 0 \rightarrow \cdots \]

Considering the fiber sequences \( \sigma_0 E \rightarrow E \rightarrow \sigma_{<0} E \) and \( \sigma_0 E^{\vee \vee} \rightarrow E^{\vee \vee} \rightarrow \sigma_{<0} E^{\vee \vee} \), the problem can be reduced the case of projective modules Proposition 5.1 by induction on the length of complexes. Hence \( E \rightarrow E^{\vee \vee} \) is a quasi-isomorphism. \qed

**Theorem 5.3.** Let \( (A_n)_{n \geq 1} \) be an inverse system of commutative rings and write \( A = \lim A_n \). Assume that \( \lim : 2 - \lim \text{PMod}_{A_n} \rightarrow \text{PMod}_A \) is a categorical equivalence. The functor \((- \otimes_A A_n)_{n \geq 1} : \text{Alg}^\text{FSyn}_{A_n} \rightarrow 2 - \lim \text{Alg}^\text{FSyn}_{A_n} \) is a categorical equivalence.

**proof.** Let \( (B_n) \) be an inverse system of finite syntomic \( A_n \)-algebras. Then, by the assumption, \( B = \lim A_n \) is finite, projective \( A \)-algebra. We show that the relative cotangent complex \( L_{B/A} \) is perfect and tor-amplitude in \([-1, 0]\). Since \( B \rightarrow B_n \) is unramified, the sequence \( A \rightarrow B \rightarrow B_n \) induces a quasi-isomorphism \( L_{B/A} \otimes_B B_n \rightarrow L_{B_n/A_n} \). Similarly, the sequence \( A \rightarrow A_n \rightarrow B_n \) induces a quasi-isomorphism \( L_{B_n/A_n} \rightarrow L_{B_n/A_n} \). Therefore \( L_{B/A} \otimes_B B_n \rightarrow L_{B_n/A_n} \) is a quasi-isomorphism. By the Milnor exact sequence

\[ 0 \rightarrow \lim H_{*+1}(L_{B_n/A_n}) \rightarrow H_*(\lim L_{B_n/A_n}) \rightarrow \lim H_*(L_{B_n/A_n}) \rightarrow 0, \]
the projective limit \( \lim_{\leftarrow} (L_{B/A} \otimes_B B_n) \) is tor-amplitude in \([-1, 0] \). Since each \( B_n \) is finitely generated projective \( A \)-module, \( B \otimes_A A_n \to B_n \) is an isomorphism for each \( n \). Then \( L_{B/A} \otimes_A A_n \to L_{B/A} \otimes_B (B \otimes_A A_n) \to L_{B/A} \otimes_B B_n \) is a composition of quasi-isomorphisms for each \( n \). By Proposition 5.2, \( L_{B/A} \to \lim_{\leftarrow} L_{B_n/A_n} \) is a quasi-isomorphism, and it is a perfect complex of \( B \)-modules. □

5.2. **Algebraic cobordism of nilpotent algebras.** Let \( A \) be a commutative ring. Then the direct sum \( \mathbb{Z} \oplus A \) has a canonical unital ring structure defined by

\[(m, a) \cdot (n, b) = (mn, na + mb + ab)\]

for \( m, n \in \mathbb{Z} \) and \( a, b \in A \).

Let \( \text{CRng} \) denote the category of commutative rings and \( \text{Cring} \) the category of commutative unital rings. Then the functor \( \mathbb{Z} \oplus - : \text{Cring} \) induces a categorical equivalence between \( \text{CRng} \) and the category of augmented commutative rings \( \text{Cring}_{/\mathbb{Z}} \). The right adjoint of \( \mathbb{Z} \oplus - \) is the augmented ideal functor \( \text{Ker} (- \to \mathbb{Z}) : \text{Cring}_{/\mathbb{Z}} \to \text{Cring} \), becoming the quasi-inverse of \( \mathbb{Z} \oplus - \).

**Definition 5.4.** Let \( f : A \to B \) be a homomorphism of commutative rings. Then the relative cotangent complex \( L_{B/A} \) is defined to be the cotangent complex \( L_{\mathbb{Z}\oplus B/\mathbb{Z}\oplus A} \) of the induced unital ring homomorphism.

**Proposition 5.5.** Let \( 0 \to A' \to A \to A'' \to 0 \) be an exact sequence of commutative rings. Then the functor \( L : \text{CRng} \to D(\text{Mod}_{\mathbb{Z}}) \) induces a distinguished triangle:

\[ L_{A'} \to L_A \to L_{A''} \to L_{A'}[1]. \]

**proof.** Since the functor \( \mathbb{Z} \oplus - \) preserves all small colimits, one has a canonical isomorphism \( \mathbb{Z} \oplus A'' \cong (\mathbb{Z} \oplus A) \oplus_{\mathbb{Z}\oplus A'} \mathbb{Z} \). The cotangent complex functor \( L : \text{CAlg}_{/\mathbb{Z}} \to \text{Mod}_{\mathbb{Z}} \) of stable \( \infty \)-categories preserves all small colimits. Hence, \( L_{A'} \to L_A \to L_{A''} \) is a cofiber sequence. □

Let \( V \) be a commutative ring and \( \omega \in V \) a non-zero divisor. For \( n \geq 0 \), set \( A_n = A/\omega^{n+1}A \).

Write \( \overline{A} = \omega A/\omega^2 A \) and consider an exact sequence

\[(5.1) \quad 0 \to \overline{A} \xrightarrow{\omega^{n-1}} A_n \to A_{n-1} \to 0.\]

We remark that the map multiplication by \( \omega^{n-1} : \overline{A} \to A_n \) is a ring homomorphism. Indeed, one has \( \omega^{n-1} \cdot \omega^2 = 0 = \omega^{2n} \) in \( A_n \) for \( n \geq 1 \).

**Definition 5.6.** Let \( K \) be a complete non-Archimedian non-discrete valuation field of rank 1, and \( K^0 \) denote the subring of powerbounded elements. We say that \( K \) is a **perfectoid field** if the Frobenius \( \Phi : K^0/p \to K^0/p \) is surjective, where \( p \) is a positive prime integer which is equal to the characteristic of the residue field of \( K^0 \).

This section fixes a perfectoid field \( K \) whose valuation ring \( K^0 \) is mixed characteristic \((0, p)\).

We put \( V = K^0 \) and \( m = K^{00} \), where \( K^{00} = \{ x \in K \mid |x| < 1 \} \) is the maximal ideal of \( K^0 \). Then
\( m \) is the set of topologically nilpotent elements, being idempotent. We fix a pseudouniformizer \( \omega \in V \) with \(|p| \leq |\omega| < 1\).

**Definition 5.7.** An integral perfectoid \( V \)-algebra is an \( \omega \)-adic complete flat \( V \)-algebra \( A \) on which Frobenius induces an isomorphism \( \Phi : A/\omega^{1+1} A \to A/\omega A \). For any \( V \)-algebra \( B \), \( B^\nu \) denotes the tilting \( \lim \leftarrow_{\omega \to \omega^p} B/\omega B \) of \( B \).

**Proposition 5.8.** Let \( K \) be a perfectoid field with the valuation ring \( V \) whose residue field is of characteristic \( p > 0 \). Let \( \omega \) be a pseudouniformizer and \( m = \lim m \geq 1 \omega^{-m} V \). Let \( A \) be an integral perfectoid \( V \)-algebra and write \( A_n = A/\omega^{n+1} A \). Then \( \mathbb{Z} \oplus m \otimes_V A \) is an ind-finite syntomic \( \mathbb{Z} \oplus m \otimes_V A \)-algebra for \( n \geq 1 \).

**proof.** We prove that \( \mathbb{Z} \oplus m \otimes_V A \) is finite syntomic over \( \mathbb{Z} \oplus m \otimes_V A \). The injection \( \mathbb{Z} \oplus m \otimes_V A \to \mathbb{Z} \oplus m \otimes_V A \) is induced by the inductive system of ring homomorphisms:

\[
\omega^{n-1} \cdot \text{Id} : \omega^{1+1+1} A/\omega^{2+1+1} A \to \omega^{1+1+1} A/\omega^{n+1+1+1} A,
\]

which is finite syntomic for each \( m \geq 1 \). That is, the ring \( \omega^{1+1} A/\omega^{n+1+1} A \) is finite syntomic over the image \( (\omega^{1+1})^{n+1} A/\omega^{n+1+1} A \). Hence the morphism \( \mathbb{Z} \oplus m \otimes_V A \to \mathbb{Z} \oplus m \otimes_V A \) is ind-finite syntomic. \( \Box \)

### 5.3. Finite syntomic algebraic cobordism of non-unital algebras.

**Proposition 5.9.** Let \( K \) be a perfectoid field with the valuation ring \( V \) whose residue field is of characteristic \( p > 0 \). Let \( \omega \) be a pseudouniformizer and \( m = \lim m \geq 1 \omega^{-m} V \). Then the injection \( \mathbb{Z} \oplus m \otimes_V A \) induces a weak equivalence

\[
\text{MGL}(\mathbb{Z} \oplus m \otimes_V A) \to \text{MGL}(\mathbb{Z}_0(\mathbb{Z} \oplus m \otimes_V A_n))
\]

of finite syntomic algebraic cobordisms for each \( n \geq 1 \), where \( \mathbb{Z}_0(\mathbb{A}) = \mathbb{Z}_0(\text{Spec} A) \) denotes the zero-section stabilization.

**proof.** Since

\[
\text{MGL}(\mathbb{Z} \oplus \omega^{1+1+1} A/\omega^{2+1+1} A) \to \text{MGL}(\mathbb{Z}_0(\mathbb{Z} \oplus \omega^{1+1+1} A/\omega^{n+1+1+1} A))
\]

is a weakly equivalence for each \( m \geq 1 \), the assertion follows from Corollary 4.8. \( \Box \)

Let \( V \) be a commutative ring and \( m \) an idempotent ideal. Assume that \( m \) is a flat \( V \)-module. For any \( V \)-algebra, the algebraic cobordism \( \text{MGL}(m \otimes_V A) \) of \( m \otimes_V A \) is defined to be the homotopy fiber of \( \text{MGL}(\mathbb{Z} \oplus m \otimes_V A) \to \text{MGL}(\mathbb{Z}) \). Clearly, one has a weak equivalence: \( \text{MGL}(m \otimes_V A) \otimes \text{MGL}(\mathbb{Z}) \approx \text{MGL}(\mathbb{Z} \oplus m \otimes_V A) \).
Lemma 5.10 (c.f. [Ste12] p.257, Lemma 5.6.). Let \((A_n)_{n \geq 1}\) be an inverse system of commutative rings. For any motivic spectrum \(E\), there is a Milnor sequence:

\[
0 \to \lim \uparrow \Ext^{q-1}(\Spec A_n, E) \to \Ext^q(\Spec A, E) \to \lim \Ext^q(\Spec A_n, E) \to 0
\]

for \(q \in \mathbb{Z}\).

**proof.** Let \(\phi_n : A_{n+1} \to A_n\) denote the transition map for \(n \geq 1\). Then the distinguished triangle

\[
\bigoplus_n \Spec A_n \xrightarrow{id-\phi_n} \bigoplus_n \Spec A_n \to \lim_n \Spec A_n \to \bigoplus_n S^1 \wedge \Spec A_n
\]

induces a long exact sequence

\[
\cdots \to \prod_n \Ext^q(S^1 \wedge \Spec A_n, E) \to \Ext^q(\lim_n \Spec A_n, E) \to \prod_n \Ext^q(\Spec A_n, E) \to \cdots .
\]

This long exact sequence splits into the following short exact sequences:

\[
0 \to \lim \uparrow \Ext^{q-1}(\Spec A_n, E) \to \Ext^q(\lim_n \Spec A_n, E) \to \lim \Ext^q(\Spec A_n, E) \to 0.
\]

\[\square\]

Given a diagram \(B \leftarrow A \to C\) of commutative non-unital rings, \(B \Box A C\) denotes the colimit of non-unital algebras: it is defined to be the kernel of the augmentation \((\mathbb{Z} \oplus B) \otimes_{\mathbb{Z}[A]} (\mathbb{Z} \oplus C) \to \mathbb{Z}\). By definition, the induced map \(\mathbb{Z} \oplus (B \Box A C) \to (\mathbb{Z} \oplus B) \otimes_{\mathbb{Z}[A]} (\mathbb{Z} \oplus C)\) is clearly an isomorphism.

**Theorem 5.11.** Let \(V\) be a mixed characteristic perfectoid valuation ring with the unit, and \(V^b\) denote the tilting. For any perfectoid \(V\)-algebra \(A\), we abbreviate as \(A_n = m \otimes_V A/\omega^{n+1}A\), \(\overline{A} = \mathfrak{m} \otimes_V A/\omega^2A\), and \(A_n^b = m^b \otimes_V A^b/((\omega^b)^{n+1}A^b)\). Then the induced maps

\[
\MGL(\lim_n A_n) \to \MGL(\lim(\lim_n \Box \overline{A}^b_n))) \leftrightarrow \MGL(\lim_n A_n^b)
\]

are weak equivalences, where \(Z_0\) denotes the zero-section stabilization.

**proof.** Consider the diagrams

\[
\begin{array}{cccc}
\mathbb{R} \lim S^1 \wedge \MGL(A_n) & \to & \MGL(\lim_n A_n) & \to & \mathbb{R} \lim \MGL(A_n) \\
\| & & & & \\
\mathbb{R} \lim S^1 \wedge \MGL(\lim(\lim_n \Box \overline{A}^b_n))) & \to & \MGL(\lim(\lim_n \Box \overline{A}^b_n))) & \to & \mathbb{R} \lim \MGL(\lim(\lim_n \Box \overline{A}^b_n))) \\
\| & & & & \\
\mathbb{R} \lim S^1 \wedge \MGL(A_n^b) & \to & \MGL(\lim_n A_n^b) & \to & \mathbb{R} \lim \MGL(A_n^b),
\end{array}
\]

where the horizontal arrows are homotopy fiber sequences by Lemma 5.10 and the left and right vertical morphisms are weak equivalences by Proposition 5.8. Therefore the middles are weak equivalences. \[\square\]
Note that $m$ is flat. By Theorem 5.3 and Theorem 5.11, we obtain the main theorem of this section:

**Theorem 5.12.** Let $V$ be a mixed characteristic perfectoid commutative valuation ring with unit and $m$ the maximal ideal. For any integral perfectoid $V$-algebra $A$, the spectra of algebraic cobordisms $\text{MGL}(m \otimes V A)$ and $\text{MGL}(m^p \otimes V A^p)$ are weakly equivalence. □

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