Flatness-based MPC for underactuated surface vessels in confined areas

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Abstract: A two-phase model predictive controller (MPC) is proposed for underactuated surface vessel operation in confined environments. For general driving maneuvers (phase one) the ship’s geometry is not considered explicitly while in more restricted areas (stage two) which occur, e.g., in mooring maneuvers, the ship’s geometry is approximated to ensure collision avoidance. To remove the dynamical constraint in the problem setup, the differential flatness of the fully actuated system is exploited and the flat outputs are parameterized using B-spline functions. Underactuated behavior is retained by means of inequality constraints that are imposed on the non-controllable input. In an effort to solve the MPC, a static nonlinear optimization problem is formulated and feasibility w.r.t. obstacles and actuator constraints is ensured at collocation points. Static obstacles are considered as constructive solid geometry functions in the MPC which also takes into account disturbances induced by wind.

Keywords: Surface vessel, optimal control, flatness, model predictive control, constrained environment, ship motion control, autonomous vehicle, docking.

1. INTRODUCTION

Recent years show an advancing interest in the field of autonomous vessels. This is due to the variety of challenging applications where autonomous systems can be advantageous to humanly-operated vessels but also because of the task to solve the arising complex problems that involve environmental disturbances and nonlinear vessel dynamics, see Streng and Kuipers (2020).

Along with classical path-following scenarios using PID controllers as shown in Barslett et al. (2018), more advanced approaches such as Lyapunov-based methods involving, e.g., passivity and backstepping techniques were applied in Fossen et al. (2002); Breivik and Fossen (2004); Do and Pan (2006, 2009); Fossen (2011). Furthermore, exact feedback linearization and differential flatness were exploited in Agrawal and Sira-Ramirez (2004); de Aquino Lima-averde Filho and Fortaleza (2013); Paliotta et al. (2018). In general, the mentioned approaches are not able to handle input and state constraints. To deal with such issues, a third branch has emerged which utilizes optimization-based techniques, see, e.g., Bitar et al. (2018, 2019); Lekkas et al. (2016). Essentially, optimization-based methods aim to minimize a cost functional depending on the control inputs subject to the system dynamics and additional equality and inequality constraints. Methods to solve this optimal control problem (OCP) can be characterized as indirect or direct, where the former leads to a two-point boundary value problem and the latter directly minimizes the cost functional by suitable discretization.

While nonlinear and optimization-based techniques constitute independent methods, their combination can lead to increased performance and reduced complexity. This combined approach goes back to Agrawal and Fuá (1998) and was further extended to the class of differentially flat systems, e.g. in Milam et al. (2000). Herein, the so-called flat outputs are parameterized with B-spline functions to obtain an OCP, where the constraint imposed by the system dynamics is implicitly fulfilled. Therefore, this constraint can be omitted in the problem setup. Subsequent discretization in time transfers the OCP to a static optimization problem (direct method). This approach has already been used for fuel optimization in hybrid electric drives and trajectory generation for quadrocopters, see Abel and Joševski (2015) and Abel et al. (2016), respectively.

In this contribution, the combined flatness and optimization approach is extended and applied to an underactuated surface vessel model. In Agrawal and Sira-Ramírez (2004) the flatness of the considered model is verified under restrictive assumptions on the model parameters. Moreover, the resulting flat state and input parameterizations contain several singularities, which severely restrict its applicability. To address this in the following the so-called defect elimination method is used as suggested, e.g., in Oldenburg and Marquardt (2002). Utilizing this approach, the underactuated dynamics is achieved by means of the singularity free flat parameterization obtained for a fully actuated vessel model. This comes at the cost of an additional equality constraint that must be imposed on the parameterized, non-controllable input. For practical reasons, however, this equality constraint is replaced by two inequality constraints. This approach is evaluated for driving maneuvers in confined environments including mooring based on closed-loop MPC involving disturbances induced by wind. Herein, the maneuver is separated into two phases. The first phase will be referred to as the driving phase where the ship geometry is not explicitly
considered to evaluate obstacle collisions. Subsequently, the second phase will be referred to as the mooring phase, where the ship geometry is approximated to ensure obstacle avoidance for the entire ship hull.

The paper is organized as follows. The vessel model is introduced in Section 2 together with its flat state and input parameterization. Section 3 describes the general input parameterization. Section 3 describes the general form of an OCP and introduces the used approach for obstacle modeling with constructive solid geometry (CSG) functions. Additionally, the flatness-based direct solution method is described by briefly introducing the main properties of B-spline functions and formulating their connection to flat outputs. To account for wind-induced disturbances, the extension to MPC is proposed in Section 4. Subsequently, a two-phase MPC is presented together with its flat state and properties of B-spline functions and formulating their connection to flat outputs. To account for wind-induced disturbances, the extension to MPC is proposed in Section 4. Subsequently, a two-phase MPC is presented together with short remarks on the used disturbance model which is assumed to be unknown to the MPC. Finally, Section 5 shows simulation results and the paper closes with some conclusions in Section 6.

2. SURFACE VESSEL MODEL

Assuming that the vessel operates in calm sea conditions, e.g., in harbor areas or near shore shipping applications, roll, pitch and heave velocities can be neglected. This results in a three degrees of freedom (3DOF) description of a surface vessel for which two sets of coordinates are required. The first set \( \eta = [x \ y \ \psi]^T \) describes the vessel location and pose in the North-East-Down (NED) frame with origin \( O_r \), where \( x \) corresponds to the north and \( y \) to the east coordinate. The third component \( \psi \) describes the vessel orientation w.r.t. the north axis. This set of coordinates is a reference frame for the second set of coordinates \( \nu = [u \ v \ r]^T \) which represents the vessels surge and sway velocities as well as its yaw rate in a body-fixed coordinate frame, respectively. These relations can be observed in Fig. 1.

\[ x = f(x) + Bu + \tau_w, \quad t > 0, \quad x(0) = x_0 \] (7a)

where

\[
\begin{align*}
R(\psi) &= \begin{bmatrix}
\cos(\psi) & -\sin(\psi) & 0 \\
\sin(\psi) & \cos(\psi) & 0 \\
0 & 0 & 1
\end{bmatrix} \\
M &= \begin{bmatrix}
m_{11} & 0 & 0 \\
0 & m_{22} & m_{23} \\
0 & m_{32} & m_{33}
\end{bmatrix}
\end{align*}
\] (2)

is the rotation matrix and

\[
\begin{align*}
C(\nu) &= -C(\nu)^T \\
D(\nu) &= -\begin{bmatrix}
X_u & X_{\nu|u|}\nu \\
0 & Y_v & Y_{\nu|v|}\nu \\
0 & N_r & N_{\nu|r}\nu
\end{bmatrix}
\end{align*}
\] (3)

combines linear damping terms \( X_u, Y_v, N_r, N_{\nu|r} \) and nonlinear second order modulus model terms \( X_{\nu|u|}, Y_{\nu|v|}, N_{\nu|r} \). For an unducted surface vessel it holds that the effect of the control input \( \tau_c = [\tau_u \ \tau_r]^T \) is applied with the actuator configuration matrix

\[
B_r = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] (6)

The vector \( \tau_w \) describes wind-induced disturbances. For a compact notation, the state vector \( x = [\eta^T \ \nu^T]^T \in \mathbb{R}^n \), where \( n = 6 \) is the number of states, and input vector \( u = [\tau_c \ \tau_w]^T \in \mathbb{R}^m \), where \( m = 2 \) is the number of inputs, are defined such that (1) can be rewritten in nonlinear input-affine form

\[ \dot{x} = f(x) + Bu + \tau_w, \quad t > 0, \quad x(0) = x_0 \] (7a)

where

\[
\begin{align*}
f(x) &= \begin{bmatrix}
R(\psi)\nu \\
-M^{-1}(C(\nu) + D(\nu))\nu
\end{bmatrix} \\
B &= \begin{bmatrix}0^{(3 \times m)}\end{bmatrix} \\
\tau_w &= \begin{bmatrix}0^{(3 \times 1)}\end{bmatrix} \\
\end{align*}
\] (7b)

2.2 Differential flatness

In the following, the differential flatness of the vessel model is shown. Theoretical background concerning differential flatness is provided in, e.g., Fließ et al. (1995); Rothfuß (1997); Fließ et al. (1999). The flat parameterization of the underactuated vessel model shows several singularities, see Agrawal and Sira-Ramirez (2004). Therefore, a fully actuated model with \( u = \tau_c = [\tau_u \ \tau_r \ \tau_w]^T \) and \( B = [0^{(m \times 3)} \ {M^{-1}B'_r}^T]^T \) where \( B'_r = I^{(3 \times 3)} = diag(1, 1, 1) \) is assumed. Furthermore, the disturbance term in (7a) is neglected so that \( \tau_w = 0 \). Choosing the flat output \( z = \)}
3.1 Obstacle modeling method for the OCP using B-splines is discussed. Furthermore, a flatness-based solution representing arbitrary shapes. These can be included to account confined environments for mooring maneuvers. In the following, CSG functions are discussed which can control of the vessel is required while also taking into account the flat parameterization of the fully actuated vessel.

The vessel parameters are taken from Do and Pan (2006) and are provided in Table 1. The terms $\theta_{\tau_0}$, $\theta_{\tau_r}$, and $\theta_x$ are provided in Appendix A. It becomes apparent that no singularities arise in (8).

To recover the original underactuated vessel dynamics from the flat parameterization of the fully actuated vessel it is necessary to impose the constraint

$$\theta_x = 0,$$  (9)

which induces an ODE in the components of $z$. In principle, this ODE can be interpreted as the internal dynamics, see, e.g., the analysis in Rothfuss et al. (1996). For the considered OCP (9) is included by means of two inequality constraints to be fulfilled in terms of the decision variables.

2.3 Model parameters

The vessel parameters are taken from Do and Pan (2006) for a model ship and are summarized in Tab. 1. Therein, $L_S$ and $W_S$ are the vessel length and width, respectively. The inputs are constrained according to

$$-5 \text{ N} \leq \tau_u \leq 5 \text{ N},$$

$$-0.2 \text{ N m} \leq \tau_r \leq 0.2 \text{ N m}.$$  (10a, 10b)

| Table 1. Vessel parameters |
|-----------------------------|
| Mass matrix | Damping matrix | Vessel |
| linear | nonlinear | |
| $M_{11}$ | 25.80 | $X_u$ = 12.0 | $X_{12}$ = 2.1 | $L_S$ = 1.20 m |
| $M_{22}$ | 33.80 | $Y_c$ = -17.0 | $Y_{12}$ = -4.5 | $W_S$ = 0.35 m |
| $M_{23}$ | 6.20 | $N_r$ = -0.2 | $N_{12}$ = -0.1 | $m$ = 17.00 kg |
| $M_{33}$ | 2.76 | $N_r$ = -0.5 | $d_x$ | $d_y$ |

3. FLATNESS-BASED OPTIMAL CONTROL

The aim for the desired approach is to generate trajectories while also considering actuator constraints. In other words, a combined trajectory-generation and motion control of the vessel is required while also taking into account confined environments for mooring maneuvers. In the following, CSG functions are discussed which can represent arbitrary shapes. These can be included to an OCP formulation. Furthermore, a flatness-based solution method for the OCP using B-splines is discussed.

3.1 Obstacle modeling

For obstacles of arbitrary shapes, CSG functions are used, see Ricci (1973). These are based on geometric primitive functions $f^p(x)$ such as ellipsoids, lines, and triangles. In order to describe the surface $S$ of a shape mathematically, a function of the form

$$f^S(x) \leq 1$$  (11)
can be formulated which combines several primitive shapes using the maximum operator, i.e.

$$f^S(x) = \max \{ f_1^p(x), \ldots, f_l^p(x) \},$$  (12)

where $l$ is the number of primitive functions used to define the shape. Since the gradient of the maximum operator is not smooth the approximation

$$\max \{ f_1^p(x), \ldots, f_l^p(x) \} \approx \left( f_1^p(x)^p + \ldots + f_l^p(x)^p \right)^{\frac{1}{p}}$$  (13)
is used, where the approximation quality increases with increasing $p \in \mathbb{N}$. In the following scenarios, rectangles are used to reflect confined areas. A rectangle can be constructed from two shifted and rotated parabolas, so that

$$f^S_{\text{rect}}(x|\tau) = \left( \frac{\cos(\alpha)(x - \bar{x}) + \sin(\alpha)(y - \bar{y})}{d_x} \right)^{2p} + \left( \frac{-\sin(\alpha)(x - \bar{x}) + \cos(\alpha)(y - \bar{y})}{d_y} \right)^{2p} \right)^{\frac{1}{p}},$$  (14)

where the elements of $r = [\bar{x}, \bar{y}, d_x, d_y, \alpha, p]^T$ describe the center position, length, width, orientation, and approximation quality parameter in the reference frame.

3.2 Problem formulation

In the following, the OCP for the considered system is expressed with

$$\min_u J(u) = \varphi(t_\text{f}, x(t_\text{f}))$$  (15a)

s.t.

$$\dot{x} = f(x) + Bu, \quad t > 0, \quad x(0) = x_0$$  (15b)

$$g(t, x(t)) = 0$$  (15c)

$$h(x) \leq 0$$  (15d)

$$u^- \leq u \leq u^+,$$  (15e)

where $J(u)$ represents the cost functional in Mayer form that is to be minimized, $t_\text{f}$ is the final time, (15b) denotes the ODE constraint imposed by the system dynamics with initial condition $x(0) = x_0$. Furthermore, terminal path constraints are included with (15c), and state constraints imposed by obstacles are formulated with (15d). Herein, $h(x)$ is obtained by rearranging (11) and including (14) which yields $h_i(x) = 1 - f^S_{\text{rect}, i}(x)$, $i = 1, \ldots, q$, where $q$ is the number of rectangular obstacles. Input constraints are expressed using (15e), where $u^-$, and $u^+$ denote the lower and upper input bounds, respectively.

3.3 Flatness-based solution using B-splines

The ODE constraint (15b) is implicitly fulfilled by the flat parameterization (8) of the system. Therefore, the differential flatness of the vessel system can be exploited when the OCP is formulated in flat coordinates thereby eliminating the ODE constraint. Since the problem is still an infinite-dimensional it is convenient to parameterize the
\begin{align}
\hat{z}_j(t, \mathbf{p}_j) &= \sum_{i=0}^{N_j} B_{i,j}(t) p_{i,j}, \quad t \in [0, t_f], \quad j = 1, \ldots, m, \\
B_{i,0}(t) &= \begin{cases} 1, & \text{for } t \in [u_i, u_i+1) \\ 0, & \text{else} \end{cases}, \\
B_{i,j}(t) &= \left\{ \begin{array}{l}
\frac{t-u_{i,j}}{u_{i,j+1}-u_i} B_{i,j-1}(t) \\
\frac{u_{i,j+1}-t}{u_{i,j+1}-u_{i+1}} B_{i+1,j-1}(t)
\end{array} \right.,
\end{align}

for the \( j \)th component of the flat output \( z \). Herein, \( B_{i,j}(t) \) are basis functions of order \( D_j \) and the vector \( \mathbf{p}_j = [p_{0,j} \ldots p_{N_j,j}]^T \) summarizes the individual \( N_j \) weights. In general, the ability to approximate complex function behavior is improved as \( N_j \) is increased. Using B-spline functions the basis functions can be calculated recursively using the Cox-DeBoor scheme, see Piegl and Tiller (2013), i.e.

\begin{align}
B_{i,0}(t) &= \begin{cases} 1, & \text{for } t \in [u_i, u_i+1) \\ 0, & \text{else} \end{cases}, \\
B_{i,j}(t) &= \left\{ \begin{array}{l}
\frac{t-u_{i,j}}{u_{i,j+1}-u_i} B_{i,j-1}(t) \\
\frac{u_{i,j+1}-t}{u_{i,j+1}-u_{i+1}} B_{i+1,j-1}(t)
\end{array} \right.,
\end{align}

In the recursion formula it can be seen that the time horizon \( t \in [0, t_f] \) is separated using a so-called knot vector

\begin{align}
\mathbf{u}_j = [u_{0,j} \ldots u_{M_j,j}]^T, \quad j = 1, \ldots, m.
\end{align}

At the knot points, the curve segments are joined to form the B-spline function. As can be seen from the recursion, the \( i \)th basis function \( B_{i,j}(t) \) that is weighted with \( p_{i,j} \) for the \( j \)th flat output is nonzero on the interval \( t \in [u_{i,j}, u_{i,j+1}) \). Thus, choosing

\begin{align}
\mathbf{u}_j = [0 \ldots 0 t_f t_f \ldots t_f]^T,
\end{align}

results in \( B_{k,0}(0) = 0 \) for \( k < D_j \) and only \( B_{D_j,0}(0) = 1 \), so that

\begin{align}
\hat{z}_j(0, \mathbf{p}_j) = p_{0,j}.
\end{align}

Similarly, this choice of the knot vector yields \( \hat{z}_j(t_f, \mathbf{p}_j) = p_{N_j,j} \). In this way, initial and final values (of the flat outputs) are parameterized using the control points \( p_{0,j} \) and \( p_{N_j,j} \), respectively. The parameter \( M_j \) in \( \mathbf{u} \) can be determined with \( M_j = D_j + N_j + 1 \). The flat parameterization requires derivatives of the flat outputs up to order \( \beta \). The \( k \)th order derivative of a B-spline function is given by

\begin{align}
\hat{z}_j^{(k)}(t, \mathbf{p}_j) &= \sum_{i=0}^{N_j} B_{i,j}^{(k)}(t) p_{i,j}, \quad t \in [0, t_f], \quad j = 1, \ldots, m,
\end{align}

where

\begin{align}
B_{i,j}^{(k)}(t) &= \left\{ \begin{array}{l}
\frac{t-u_{i,j}}{u_{i,j+1}-u_i} B_{i,j-1}^{(k-1)}(t) \\
\frac{u_{i,j+1}-t}{u_{i,j+1}-u_{i+1}} B_{i+1,j-1}^{(k-1)}(t)
\end{array} \right.,
\end{align}

\( \beta \) is the air density, \( V_{abs} \) is the absolute wind velocity, \( V_{abs} \sim N(\mu, \sigma) \) and an absolute wind direction \( \beta_w \sim W(\kappa, \lambda) \), where \( \kappa \) and \( \lambda \) are shape and scale parameters of the Weibull distribution. With this, the forces and torque applied to the vessel can be calculated with

\begin{align}
\tau_w = \frac{1}{2} \rho (V_{rel})^2 \begin{bmatrix}
C_X A_f \\
C_Y A_f \\
C_N A_l L_S
\end{bmatrix},
\end{align}

where \( \rho \) is the air density, \( V_{rel} \) is the relative wind velocity which, together with the coefficients \( C_X, C_Y \), and \( C_N \), depends on the absolute wind direction \( \beta_w \) and speed \( V_{w, abs} \). The parameters \( L_S, A_f \) and \( A_l \) are vessel length, projected frontal and lateral areas, respectively.
5. SIMULATION RESULTS

Simulation results are generated in MATLAB using CasADi with IPOPT as NLP solver, see Andersson et al. (2018) and Wächter (2002), respectively. The underactuated vessel dynamics using the flat parameterization of the fully actuated system is retained by taking into account (9) which for numerical purposes is approximated by
\[-\epsilon \leq \theta_{\tau} \leq \epsilon, \tag{29}\]
for \(\epsilon \ll 1\). For the simulation, only the solutions of \(\theta_{\tau}, i\) and \(\theta_{\tau}, r\) are applied to the underactuated model.

**Remark 1.** Setting \(\epsilon = 0\) would result in \(N + 1\) equality constraints which reduces the number of free decision variables in the NLP potentially rendering it unsolvable. Choosing \(\epsilon > 0\) avoids this issue.

Initial and terminal (desired) states are chosen to be
\[
\begin{align*}
\mathbf{x}_0 &= \begin{bmatrix} 3.5 & 2 & \pi/2 & 0 & 0 & 0 \end{bmatrix}^T, \tag{30a} \\
\mathbf{x}_f &= \begin{bmatrix} 2.4 & 18 & 0 & 0 & 0 \end{bmatrix}^T. \tag{30b}
\end{align*}
\]
Further, the switching point is chosen to be
\[
R_s = \frac{t_{\text{hor}} \sqrt{\frac{u_0^2}{\mu^2} + v_{\text{max}}^2}}{\pi}, \tag{31}\]
where \(u_{\text{max}} = 0.38 \text{m/s}, v_{\text{max}} \approx 0 \text{m/s}\) describe the maximum surge and sway velocity of the vessel, respectively. The fixed time horizon is set to \(t_{\text{hor}} = 15 \text{s}\) in the driving phase. The MPC horizon is shifted each iteration for \(t_{\text{MPC}} = 1 \text{s}\). Additionally, four obstacles are considered where \(h_i(x), i = 1, 2, 3\) are relevant for the mooring maneuver and \(h_i(x)\) affects the driving maneuver. Feasibility w.r.t. constraints is ensured at \(N + 1 = 200\) collocation points. Additional scenario parameters are summarized in Tab. 2. The top view of the path, orientation, initial and final position, as well as the switching point are shown in Fig. 2a. It can be seen that there is no collision with any obstacle. Figure 2b shows the inputs with constraints marked using dashed-red lines which are satisfied for all times. The remainder of states is shown in Fig. 2c together with the switching time \(t_s = 31 \text{s}\). Sudden changes in the inputs can be explained by numerical issues and disturbances which could push the vessel into the obstacles resulting in feasibility issues for the NLP solver. This could be avoided using soft constraints as described in Scokaert and Rawlings (1999).

6. CONCLUSION

In this paper a flatness-based MPC for an underactuated nonlinear surface vessel model is introduced. The fully actuated system is shown to be differentially flat so that the ODE constraint in the OCP can be removed. The flat outputs are parameterized using B-spline functions. A discretization in time of the OCP in flat coordinates allows the formulation of a NLP which can be solved numerically. Underactuated vessel dynamics are retained using inequality constraints imposed on the non-controllable input and obstacles are included to the OCP using CSG functions which can approximate arbitrary shapes. The concept is evaluated in a two-phase simulation scenario resulting in different OCP formulations. Future work focuses on real-time feasibility which can be achieved by approximating the highest-order derivative of each flat output and subsequent integration thus avoiding recursive computation of basis functions as shown in Oldenburg and Marquardt (2002). Further work also focuses on soft constraints and extending the concept to include collision avoidance regulations (COLREGS).

Appendix A. INPUT PARAMETRIZATION

The terms arising in (8b) read
\[ \theta_{r_u} = -X_{\nu}^i \left( \sin(z_3) \dot{z}_2 + \cos(z_3) \dot{z}_1 \right) \]
\[ \cdot \left[ \sin(z_3) \dot{z}_2 + \cos(z_3) \dot{z}_1 \right] + m_{111} \sin(z_3) \dot{z}_2 \]
\[ - \left[ \left( m_{22} - m_{11} \right) \cos(z_3) \dot{z}_2 + X_0 \sin(z_3) \right] \dot{z}_2 \]
\[ - \left[ \left( m_{11} - m_{22} \right) \sin(z_3) \dot{z}_3 + X_0 \cos(z_3) \right] \dot{z}_1 \]
\[ - \frac{1}{2} \left( m_{23} + m_{32} \right) \dot{z}_3 + m_{111} \cos(z_3) \dot{z}_1, \]  
(A.1a)

\[ \theta_{r_e} = Y_{\nu}^i \left( \sin(z_3) \dot{z}_1 - \cos(z_3) \dot{z}_2 \right) \]
\[ \cdot \left[ \cos(z_3) \dot{z}_2 - \sin(z_3) \dot{z}_1 \right] + M_{22} \cos(z_3) \dot{z}_2 \]
\[ + \left[ \left( m_{11} - m_{22} \right) \sin(z_3) \dot{z}_3 - Y_0 \cos(z_3) \right] \dot{z}_2 \]  
\[ + \left[ \left( m_{11} - m_{22} \right) \cos(z_3) \dot{z}_3 + Y_0 \sin(z_3) \right] \dot{z}_1 \]
\[ + m_{23} \dot{z}_3 - Y_r \dot{z}_3 - m_{22} \sin(z_3) \dot{z}_1, \]  
(A.1b)

\[ \theta_{r_r} = m_{32} \cos(z_3) \dot{z}_2 + m_{31} \dot{z}_3 - N_{r,r} \dot{z}_3 \dot{z}_3 - N_r \dot{z}_3 \]
\[ + \left[ \left( m_{22} - m_{11} \right) \sin(z_3) \cos(z_3) \right] \dot{z}_2 \]
\[ + \left[ \left( m_{11} - m_{22} \right) \sin^2(z_3) - \cos^2(z_3) \right] \dot{z}_1 \]
\[ + \frac{1}{2} \left( m_{23} - m_{32} \right) \sin(z_3) \dot{z}_3 - N_r \cos(z_3) \dot{z}_2 \]
\[ + \left[ \left( m_{11} - m_{22} \right) \sin(z_3) \cos(z_3) \right] \dot{z}_1 \]
\[ + \frac{1}{2} \left( m_{23} - m_{32} \cos(z_3) \dot{z}_3 + N_r \sin(z_3) \right] \dot{z}_1 \]
\[ - m_{32} \sin(z_3) \dot{z}_1. \]  
(A.1c)

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