Numerical simulation of the coupled viscous Burgers equation using the Hermite formula and cubic B-spline basis functions

Muhammad Abdullah\(^1\), Muhammad Yaseen\(^1\) and Manuel de la Sen\(^2\)

\(^1\)Department of Mathematics, University of Sargodha, Pakistan
\(^2\)Institute of Research and Development of Processes, University of the Basque Country, 48940, Leioa (Bizkaia), Spain

E-mail: yaseen.yaqoob@uos.edu.pk

Received 10 September 2020, revised 28 September 2020
Accepted for publication 7 October 2020
Published 16 October 2020

Abstract

A numerical procedure dependent on the cubic B-spline and the Hermite formula is developed for the coupled viscous Burgers’ equation (CVBE). The method uses a combination of the Hermite formula and the cubic B-spline for discretization of the space dimension while the time dimension is approximated using the typical finite differences. A piecewise continuous sufficiently smooth function is obtained as a solution which allows to approximate solution at any location in the domain of interest. The scheme is tested for stability analysis and is proved to be unconditionally stable. Numerical experiments and comparison of outcomes reveal that the suggested scheme comes up with better accuracy and is extremely productive.

Keywords: Coupled viscous Burgers equation, Cubic B-spline, Hermite formula, Stability

(Some figures may appear in colour only in the online journal)

1. Introduction

Consider the CVBE,

\[
\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial z^2} + \eta \frac{\partial v}{\partial z} + \alpha \frac{\partial}{\partial z} (vw) = 0, \\
a \leq z \leq b, \quad t > 0,
\]

\[
\frac{\partial w}{\partial t} + \lambda \frac{\partial^2 w}{\partial z^2} + \zeta w \frac{\partial w}{\partial z} + \beta \frac{\partial}{\partial z} (vw) = 0, \\
a \leq z \leq b, \quad t > 0,
\]

with ICs,

\[
v(z, 0) = \phi_1(z), \quad w(z, 0) = \phi_2(z), \quad a \leq z \leq b,
\]

and the following BCs,

\[
\begin{aligned}
v(a, t) &= \psi_1(t), \quad v(b, t) = \psi_2(t), \\
w(a, t) &= \psi_3(t), \quad w(b, t) = \psi_4(t),
\end{aligned}
\]

\(t > 0\),

where \(\eta, \lambda, \zeta, \alpha, \beta\) are real constants and \(\alpha, \beta\) are arbitrary constants depending upon the system parameters. The given functions, \(\phi_1(z), \phi_2(z), \psi_1(t), \psi_2(t), \psi_3(t), \psi_4(t)\) are sufficiently smooth.

The CVBE was first used by Esipov [1] to study the model of polydispersive sedimentation. Numerous authors have investigated the coupled linear and nonlinear initial/boundary value problems. Nee and Duan [2] studied a coupled system of Burgers’ equations with zero Dirichlet boundary conditions and appropriate initial data. The harmonic differential quadrature finite differences coupled methodology [2] and conjugate channel approach [3] is available to deal with nonlinear coupled equations. A mesh free numerical procedure was proposed in [4] for the coupled non-linear PDEs. Khater et al [5] acquired the numerical solution of the VCBE utilizing the cubic-spline collocation method. Deghan et al [6] used Adomian Pade procedure to
deal with the coupled Burgers’ equation (CBE). Rashid and Ismail et al [7] used Fourier Pseudospectral technique to
discover the numerical solution of the CVBE. Abdoz and Soliman [8] used variational iteration method for solving the
CVBE. An explicit solution of the CVBE was obtained by
Kaya [9] using Adomian decomposition method. Soliman [10] presented a modified extended tanh-function method for
Burger like equations. Abazari and Borhanifar [11] obtained
of (1.1) are found as
\[
V(z, t) = \sum_{i=1}^{N-1} c_i(t) B_{3,i}(z), \quad W(z, t) = \sum_{i=1}^{N-1} d_i(t) B_{3,i}(z),
\]
where \(c_i(t)\) and \(d_i(t)\) are unknowns to be found and \(B_{3,i}(z)\) are cubic B-spline (CuBS) basis functions given by
\[
B_{3,i}(z) = \frac{1}{6h^3} \begin{cases}
(z - z_i)^3, & z \in [z_i, z_{i+1}]

h^3 + 3h^2(z - z_{i+1}) + 3h(z - z_{i+1})^2 - 3(z - z_{i+1})^3, & z \in [z_{i+1}, z_{i+2}]

h^3 + 3h^2(z_{i+3} - z) + 3h(z_{i+3} - z)^2 - 3(z_{i+3} - z)^3, & z \in [z_{i+2}, z_{i+3}]

(z_{i+4} - z)^3, & \text{Otherwise}
\end{cases}
\]

the numerical solution of the Burgers’ and CBE using the
differential transformation method. Further numerical solu-
tion of the CVBE using cubic B-spline functions is obtained
by Mittal and Arora [12]. Mokhtari et al [13] presented
generalized differential quadrature method for the Burgers’
equation. Srivastava et al [14–16] proposed various finite
difference methods for the two dimensional CVBE. Srivastava
et al [17] proposed an implicit logarithmic finite difference
method for the one-dimensional CBE. Higher order
trigonometric B-spline based algorithms were presented in
[18] to numerically study the coupled Burgers’ equation.

Stimulated by the success of the spline approach in
finding numerical solutions of partial differential equations,
we have used combination of the Hermite formula and cubic
B-spline for approximating the space derivative. This merger
has considerably augmented the accuracy of the scheme.
Another advantage is that the approximate solution is come
up as a smooth piecewise continuous function allowing one to
get solution at any wanted location in the domain.

The rest of the paper is assembled as follows. In
section 2, the numerical scheme is derived thoroughly. In
section 3, the stability of the scheme is talked about. Section 4
offers a contrast of our numerical consequences with the ones
presented earlier on. Section 5 sums up the outcomes of
this work.

2. The derivation of the scheme

Define \(\Delta t = \frac{t}{T}\) to be the time and \(h = \frac{b - a}{N}\) the step
sizes respectively, where \(M\) and \(N\) are positive integers. Let
\(t^n = n\Delta t\) \((0 \leq n \leq N)\), \(z_r = rh\) \((0 \leq r \leq N)\) be the part-
tion of time and space dimension. The spatial domain
\(a \leq z \leq b\) is evenly discretized into \(N\) subintervals \([z_i, z_{i+1}]\)
of equal length \(h\), where \(j = 0, 1, 2, \ldots, N - 1\), so that
\(a = z_0 < z_1 < \ldots < z_{N-1} < z_\text{N} = b\). The approximate solutions
\(V(z, t)\) and \(W(z, t)\) to the exact solutions \(v(z, t)\) and \(w(z, t)\)
Observe that just \(B_{3,i-1}(z), B_{3,i}(z)\) and \(B_{3,i+1}(z)\) are survived
at the grid point, \(z\) on account of the local support property
of CuBS. Accordingly, the approximations \(v^n_j\) and \(w^n_j\) at \(n^{th}\)
time level are given by
\[
v(z_j, t^n) = v^n_j = \sum_{i=1}^{i+1} c^n_i(t) B_{3,i}(z),
\]
\[
w(z_j, t^n) = w^n_j = \sum_{i=1}^{i+1} d^n_i(t) B_{3,i}(z).
\]
The unknowns \(c^n_i(t)\) and \(d^n_i(t)\) are perceived using the
collocation conditions on \(B_{3,i}(z)\) and the given initial and
boundary conditions. Thus the approximations \(\hat{v}^n_j\), \(\hat{w}^n_j\)
and necessary derivatives are obtained as
\[
\begin{align*}
\hat{v}_j^n &= a_1 c^n_{j-1} + a_2 c^n_j + a_3 c^n_{j+1}, \\
\hat{v}_j^n &= -b_1 c^n_{j-1} + b_2 c^n_{j+1}, \\
\hat{w}_j^n &= c^n_{j-1} + c^n_j + c^n_{j+1}, \\
\hat{w}_j^n &= a_1 d^n_{j-1} + a_2 d^n_j + a_3 d^n_{j+1}, \\
\hat{w}_j^n &= -b_1 d^n_{j-1} + b_2 d^n_{j+1}.
\end{align*}
\]
where
\[
\begin{align*}
\hat{v}_j^n &= a_1 c^n_{j-1} + a_2 c^n_j + a_3 c^n_{j+1}, \\
\hat{v}_j^n &= \frac{\hat{v}_j^n}{2} + \delta(\hat{v}_j^n) + \eta(\hat{v}_j^n) + \mu(\hat{v}_j^n) + \nu(\hat{v}_j^n)
\end{align*}
\]
\[
\begin{align*}
\hat{w}_j^n &= a_1 d^n_{j-1} + a_2 d^n_j + a_3 d^n_{j+1}, \\
\hat{w}_j^n &= \frac{\hat{w}_j^n}{2} + \delta(\hat{w}_j^n) + \eta(\hat{w}_j^n) + \mu(\hat{w}_j^n) + \nu(\hat{w}_j^n) + \omega(\hat{w}_j^n)
\end{align*}
\]
The Hermite Formula [19] is given by

\[
(w_{j-1}^{n} + 10(w_{j}^{n})^2 + (w_{j+1}^{n})) - \frac{12}{h^2}(v_{j-1}^{n} - 2v_{j}^{n} + v_{j+1}^{n}) = O(h^4).
\]  

(2.10)

Using (2.10) in (2.8) and (2.9), we obtain

\[
v_{j}^{n+1} + \frac{\Delta t}{2}\left(-\delta\left(w_{j}^{n+1}\right) + \zeta\left(w_{j}^{n+1}\right)\right) + \frac{12\delta}{10h^2}(v_{j-1}^{n} - 2v_{j}^{n} + v_{j+1}^{n}) + \frac{\Delta t}{2}\left(\alpha(w_{j}^{n+1}) + v_{j}^{n+1}\right) + \alpha\left(w_{j}^{n+1}\right) + v_{j}^{n+1} = w_{j}^{n} + \frac{\Delta \lambda}{2}(w_{j}^{n+1} + w_{j+1}^{n+1})
\]

and

\[
w_{j}^{n+1} + \frac{\Delta t}{2}\left(-\delta\left(w_{j}^{n+1}\right) + \zeta\left(w_{j}^{n+1}\right)\right) = \frac{12\delta}{10h^2}(v_{j-1}^{n} - 2v_{j}^{n} + v_{j+1}^{n}) + \frac{\Delta t}{2}\left(\alpha(w_{j}^{n+1}) + v_{j}^{n+1}\right) + \alpha\left(w_{j}^{n+1}\right) + v_{j}^{n+1} + \frac{\Delta \lambda}{2}(w_{j}^{n+1} + w_{j+1}^{n+1}) + \frac{\Delta t}{2}(\zeta\left(w_{j}^{n+1}\right) + w_{j}^{n+1})(w_{j}^{n+1}) + \beta((w_{j}^{n+1})(w_{j}^{n+1}))
\]

(2.11)

Now inserting (2.4) into (2.11) and (2.12), we obtain

\[
e_{00}c_{j-1}^{n+1} + e_{1}c_{j}^{n+1} + e_{2}c_{j}^{n+1} + e_{3}c_{j}^{n+1} + e_{4}c_{j}^{n+1} + e_{5}d_{j}^{n+1} + e_{6}d_{j}^{n+1} + e_{7}d_{j}^{n+1} = -e_{00}c_{j-2} + b_{11}c_{j-1}^{n+1} + b_{22}c_{j}^{n+1} + b_{11}c_{j+1}^{n+1} + b_{00}c_{j+2}^{n+1}
\]

(2.13)

and

\[
q_{00}d_{j}^{n+1} + q_{01}d_{j}^{n+1} + q_{02}d_{j}^{n+1} + q_{03}d_{j}^{n+1} + q_{11}d_{j}^{n+1} + q_{00}d_{j}^{n+1} + q_{11}d_{j}^{n+1} + q_{00}d_{j}^{n+1} + q_{00}d_{j}^{n+1} = -q_{00}d_{j-2} + c_{11}d_{j-1}^{n+1} + c_{22}d_{j}^{n+1} + c_{11}d_{j+1}^{n+1} + q_{00}d_{j+2}^{n+1}
\]

(2.14)

where \(j = 1, 2, 3, 4, \ldots N - 1,\) and

\[
e_{00} = \frac{12\delta \Delta t a_{1}}{20 h^2}, \quad q_{00} = \frac{12\lambda \Delta t c_{1}}{20 h^2},
\]

\[
e_{ij} = a_{1} + \frac{\Delta t}{2}(6\delta h^2(a_{1} - 2a_{1}) - \frac{\delta}{10}c_{1}^{2}) + \eta((w_{j}^{n+1}a_{1} + v_{j}^{n+1}b_{1} + \alpha((w_{j}^{n+1}a_{1} + w_{j}^{n+1}b_{1})
\]

\[
e_{ij} = a_{2} + \frac{\Delta t}{2}(6\delta h^2(a_{2} - 2a_{2}) - \frac{\delta}{10}c_{1}^{2}) + \eta((w_{j}^{n+1}a_{2} + \alpha(w_{j}^{n+1}a_{2})
\]

\[
e_{ij} = a_{1} + \frac{\Delta t}{2}(6\delta h^2(a_{3} - 2a_{3}) - \frac{\delta}{10}c_{1}^{2}) + \eta((w_{j}^{n+1}a_{3} + v_{j}^{n+1}b_{3} + \alpha((w_{j}^{n+1}a_{3} + w_{j}^{n+1}b_{3})
\]

\[
e_{ij} = a_{1} + \frac{\Delta t}{2}(6\delta h^2(a_{4} - 2a_{4}) - \frac{\delta}{10}c_{1}^{2}) + \eta((w_{j}^{n+1}a_{4} + v_{j}^{n+1}b_{4} + \alpha((w_{j}^{n+1}a_{4} + w_{j}^{n+1}b_{4})
\]

\[
q_{00} = \frac{12\lambda \Delta t c_{1}}{20 h^2},
\]

\[
q_{00} = \frac{12\lambda \Delta t c_{1}}{20 h^2},
\]
To obtain equations corresponding to \( j = 0 \) and \( j = N \), we substitute (2.4) into (2.8) and (2.9) to obtain

\[
\begin{align*}
q_{2j} &= a_2 + \frac{\Delta t}{2} \left( \frac{6 \lambda}{5h^2} (2a_1 - 2a_2) - \frac{\lambda}{10} c_1 + \zeta ((w_j \gamma_j a_2) + \beta((v_j \gamma_j a_2)) \right), \\
n_{2j} &= a_1 + \frac{\Delta t}{2} \left( \frac{6 \lambda}{5h^2} (a_2 - 2a_1) - \frac{\lambda}{10} c_2 + \zeta ((w_j \gamma_j a_1 + w_j^p b_1) + \beta((v_j \gamma_j a_1 + v_j^p b_1)) \right), \\
q_{sj} &= \frac{\Delta t}{2} \beta((w_j \gamma_j a_2), \\
n_{sj} &= \frac{\Delta t}{2} \beta((w_j^p b_1 + (w_j \gamma_j a_1), \\
\begin{align*}
b_{11} &= 1 + \frac{24 \delta \Delta t}{20h^2} a_1 + \frac{\delta \Delta t c_2}{20} - \frac{12 \delta \Delta t a_2}{20h^2}, \\
b_{22} &= 1 + \frac{24 \delta \Delta t}{20h^2} a_2 + \frac{\delta \Delta t c_2}{20} - \frac{12 \delta \Delta t a_1}{20h^2}, \\
c_{11} &= 1 + \frac{24 \lambda \Delta t}{20h^2} a_1 + \frac{\lambda \Delta t c_2}{20} - \frac{12 \lambda \Delta t a_2}{20h^2}, \\
c_{22} &= 1 + \frac{24 \lambda \Delta t}{20h^2} a_2 + \frac{\lambda \Delta t c_2}{20} - \frac{12 \lambda \Delta t a_1}{20h^2}, \\
\end{align*}
\]

and

\[
\begin{align*}
&g_{ij} c_j^{n+1} + g_{ij} c_j^{n+1} + \bar{g}_{ij} c_j^{n+1} + \bar{g}_{ij} c_j^{n+1} + \bar{g}_{ij} c_j^{n+1} + \bar{g}_{ij} c_j^{n+1} + \bar{g}_{ij} c_j^{n+1} + \bar{g}_{ij} c_j^{n+1} \\
&+ \bar{g}_{ij} c_j^{n+1} = b_{33} c_j^{n+1} + b_{44} c_j^{n+1} + b_{33} c_j^{n+1},
\end{align*}
\]

(2.15)

Figure 1. The numerical (circles, stars, triangles, diamonds) solutions, \( V \) (left figure) and \( W \) (right figure) and the exact (solid lines) solutions at varied times when \( N = 50, \Delta t = 0.001 \) for example 1.

Figure 2. The error profiles for \( v(z, t) \) (left) and \( w(z, t) \) (right) when \( N = 50, \Delta t = 0.001 \) and \( t = 1 \) for example 1.

Note that (2.13), (2.14), (2.15) and (2.16) comprise a system of \((2N + 2)\) linear equation in \((2N + 6)\) unknowns given by
Four additional equations are needed to obtain a consistent system. These equations can be extracted from the given boundary conditions. Now the resulting system can be uniquely solved using any Gaussian elimination based algorithm.

Initial State:
To start iterations, the initial vector $C^0 = (c^{0}_1, c^{0}_0, c^{1}_1, ..., c^{N+1}_1, d^{0}_1, d^{1}_1, ..., d^{N+1}_{N+1})$ is required which can be found by initial condition and boundary values of derivative of initial condition as

$v(z_j, 0) = \phi_1(z_j), \quad w(z_j, 0) = \phi_2(z_j), \quad \text{for } j = 0, 1, 2, ..., N.$

$v(z_j, 0) = \phi'_1(z_j), \quad w(z_j, 0) = \phi'_2(z_j), \quad \text{for } j = 0, N.$

The system produces an $(2N + 6) \times (2N + 6)$ matrix system. The required initial vector is unique solution of this system.

Figure 3. The approximate solution $V(z, t)$ (left figure) and the exact solution $v(z, t)$ (right figure) when $N = 50, \Delta t = 0.001$ for example 1.

Figure 4. The approximate solution $W(z, t)$ (left figure) and the exact solution $w(z, t)$ (right figure) when $N = 50, \Delta t = 0.001$ for example 1.

Figure 5. Error profile for $v$ (left figure) and $w$ (right figure) when $\Delta t = 0.01, N = 100$ and $t = 1$ for example 2.
Table 1. Comparison of errors for $\nu$ when $\Delta t = 0.001$ for Example 1.

| $z$ | Present | \text{L}-\text{LFDM} [17] | Mittal [12] | Rashid [7] |
|-----|---------|----------------|-------------|------------|
|     | $N = 200$ | $N = 400$ | $N = 200$ | $N = 400$ |
|     | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ |
| 0.1 | 2.6E-06 | 1.5E-06 | 6.5E-07 | 3.6E-07 | 9.1E-04 | 9.0E-04 | 9.9E-04 | 8.9E-04 | 8.2E-06 | 7.4E-06 | 2.1E-06 | 1.9E-06 | ... | ... |
| 0.5 | 8.8E-06 | 4.1E-06 | 2.2E-06 | 1.2E-06 | 9.9E-04 | 6.0E-04 | 9.8E-04 | 6.0E-04 | 2.5E-05 | 4.1E-05 | 1.0E-05 | 6.2E-06 | ... | ... |
| 1.0 | 1.1E-05 | 6.0E-06 | 2.6E-06 | 1.5E-06 | 9.8E-04 | 3.6E-04 | 9.7E-04 | 3.6E-04 | 3.0E-05 | 8.2E-05 | 2.0E-05 | 7.6E-06 | 2.9E-05 | 1.2E-05 |
| N  | Present | Mittal [12] | Present | Mittal [12] |
|----|---------|-------------|---------|-------------|
|    | $L\infty$ | Ratio | OC | $L\infty$ | Ratio | OC | $L\infty$ | Ratio | OC | $L\infty$ | Ratio | OC |
| 32 | 5.7762E-05 | ... | ... | 2.9104E-04 | ... | ... | 1.9362E-04 | ... | ... | 9.7478E-04 | ... | ... |
| 64 | 1.4504E-05 | 3.9824 | 1.994 | 7.2704E-05 | 4.0030 | 2.001 | 4.8614E-05 | 3.9828 | 1.994 | 2.4361E-04 | 4.0014 | 2.005 |
| 128| 3.6247E-06 | 4.0014 | 2.000 | 1.8178E-05 | 3.9996 | 1.999 | 1.2149E-05 | 4.0015 | 2.001 | 6.0896E-05 | 4.0004 | 2.001 |
| 256| 9.0082E-07 | 4.0239 | 2.009 | 4.5497E-05 | 3.9953 | 1.998 | 3.0192E-06 | 4.0239 | 2.009 | 1.5223E-05 | 4.0003 | 2.001 |
| 512| 2.1957E-07 | 4.1027 | 2.037 | 1.1430E-06 | 3.9806 | 1.993 | 7.3590E-07 | 4.1027 | 2.037 | 3.8052E-05 | 4.0006 | 2.002 |

Table 2. Maximum error and order of convergence (OC) for $v$ in Example 1.
3. Stability analysis

In this section, the stability of the proposed scheme (2.6) is proved which shows that the scheme is unconditionally stable for whole of the domain. For this purpose, we first change \( v_{zz}, w_{zz}, v_{wz} \) and \( w_{vz} \) to a linear terms by substituting \( v \) and \( w \) as a constant \( d_1 \) and \( d_2 \) as is done in Von Neumann method. The linearized form of (2.5) is given as

\[
\begin{align*}
    v_j^{n+1} &+ \frac{\Delta t}{2} (v_{zz})_j^{n+1} + \frac{\eta \Delta t}{2} d_1 (v_{zz})_j^{n+1} \\
    &+ \frac{\alpha \Delta t}{2} (d_1 (w_{zz})_j^{n+1} + d_2 (v_{wz})_j^{n+1}) \\
    &= v_j^n - \frac{\Delta t}{2} (v_{zz})_j^n - \frac{\eta \Delta t}{2} d_1 (v_{zz})_j^n \\
    &- \frac{\alpha \Delta t}{2} (d_1 (w_{zz})_j^n + d_2 (v_{wz})_j^n),
\end{align*}
\]

which on simplification by utilizing Hermite Formula [19] reduces to

\[
\begin{align*}
    v_j^{n+1} &+ \frac{\Delta t}{20} (v_{zz})_j^{n+1} + (v_{zz})_j^{n+1} \\
    &- 2v_j^{n+1} + v_j^{n+1} \\
    &+ \frac{\alpha \Delta t}{2} (d_1 (w_{zz})_j^{n+1} + d_2 (v_{wz})_j^{n+1}) \\
    &= v_j^n - \frac{\Delta t}{20} (v_{zz})_j^n + (v_{zz})_j^n \\
    &- 2v_j^n + v_j^n \\
    &+ \frac{\alpha \Delta t}{2} (d_1 (w_{zz})_j^n + d_2 (v_{wz})_j^n).
\end{align*}
\]

Using (2.4) in (3.2), we obtain

\[
\begin{align*}
    p_0 \xi_j^{n+1} + p_1 \xi_j^{n+1} + p_2 \xi_j^{n+1} + p_3 \xi_j^{n+1} + p_0 \xi_j^{n+1} \\
    - p_0 d_j^{n+1} + p_1 d_j^{n+1} = -p_0 c_j^{n+1} + p_2 c_j^{n+1} \\
    + p_0 d_j^{n+1} + p_1 d_j^{n+1} - p_0 c_j^{n+1} + p_2 d_j^{n+1} - p_0 d_j^{n+1}.
\end{align*}
\]
Table 3. Comparison of errors for $v$ when $\Delta t = 0.01$, $N = 100$ for example 2.

| $z$ | $\alpha$ | $\beta$ | $L_2$  | $L_\infty$ | $L_2$  | $L_\infty$ | $L_2$  | $L_\infty$ | $L_2$  | $L_\infty$ | $L_2$  |
|-----|-----------|---------|--------|------------|--------|------------|--------|------------|--------|------------|--------|
| 0.5 | 0.1       | 0.3     | 1.54E-04 | 4.19E-05  | 3.24E-05 | 9.62E-04 | 1.44E-03 | 4.3E-05   | 6.74E-04 | 4.17E-05   | 4.03E-04 | 2.64E-05 |
|     | 0.1       | 0.03    | 1.61E-04 | 4.39E-05  | 2.73E-05 | 4.31E-04 | 6.68E-04 | 4.58E-05 | 7.34E-04 | 4.59E-05   | 3.92E-04 | 2.62E-05 |
| 1.0 | 0.1       | 0.3     | 3.02E-04 | 8.28E-05  | 2.40E-05 | 1.15E-03 | 1.27E-03 | 8.70E-05 | 1.32E-03 | 8.26E-05   | 7.93E-04 | 5.20E-05 |
|     | 0.1       | 0.03    | 3.16E-04 | 8.69E-05  | 2.83E-05 | 1.27E-03 | 1.30E-03 | 9.16E-05 | 1.45E-03 | 9.18E-05   | 7.71E-04 | 5.16E-05 |

Students such as Rashid [7], Khater [5], Mittal [12], and Mokhtari [13] have significantly contributed to the field of numerical methods and computational physics.
Table 4. Comparison of errors for $w$ when $\Delta t = 0.01$, $N = 100$ for example 2.

| $z$ | $\alpha$ | $\beta$ | Present | Rashid [7] | Khater [5] | Mittal [12] | I-LFDM [17] | Mokhtari [13] |
|-----|---------|---------|---------|-----------|-----------|------------|-------------|--------------|
|     |         |         | $L_2$   | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ |
| 0.5 | 0.1     | 0.3     | 5.93E-05 | 2.18E-05 | 2.75E-05 | 3.33E-04 | 5.42E-04 | 4.99E-05 | 9.06E-04 | 1.48E-04 | 2.21E-04 | 1.04E-05 | 1.60E-03 | 3.80E-05 |
|     | 0.1     | 0.03    | 1.92E-04 | 4.10E-05 | 2.45E-04 | 1.15E-03 | 1.20E-03 | 1.81E-04 | 1.59E-04 | 5.73E-04 | 4.25E-04 | 3.07E-05 | 1.59E-03 | 1.85E-04 |
| 1.0 | 0.1     | 0.3     | 1.16E-04 | 4.21E-05 | 3.74E-05 | 1.16E-03 | 1.20E-03 | 9.92E-05 | 1.25E-03 | 4.77E-05 | 4.25E-04 | 1.98E-05 | 3.10E-03 | 7.58E-05 |
|     | 0.1     | 0.03    | 3.78E-04 | 9.94E-05 | 4.52E-04 | 1.64E-03 | 2.35E-03 | 3.62E-04 | 2.25E-03 | 3.62E-04 | 8.38E-04 | 6.10E-05 | 3.15E-03 | 3.67E-04 |
Table 5. Maximum values of $V$ and $W$ taking $\alpha = \beta = 10$ for example 3.

| $t$ | Present | I-LFDM [17] | Mittal [12] |
|-----|---------|-------------|-------------|
|     | Max. value of $V$ | At point $(z)$ | Max. value of $W$ | At point $(z)$ | Max. value of $V$ | At point $(z)$ | Max. value of $W$ | At point $(z)$ | Max. value of $V$ | At point $(z)$ | Max. value of $W$ | At point $(z)$ |
| 0.1 | 0.14454 | 0.58 | 0.14333 | 0.65 | 0.14047 | 0.54 | 0.15058 | 0.66 | 0.14456 | 0.58 | 0.14306 | 0.66 |
| 0.2 | 0.05241 | 0.55 | 0.04703 | 0.56 | 0.05455 | 0.52 | 0.05097 | 0.58 | 0.05237 | 0.54 | 0.04697 | 0.56 |
| 0.3 | 0.01934 | 0.52 | 0.01728 | 0.52 | 0.02117 | 0.52 | 0.01963 | 0.52 | 0.01932 | 0.52 | 0.01725 | 0.52 |
| 0.4 | 0.00720 | 0.51 | 0.00643 | 0.51 | 0.00829 | 0.48 | 0.00771 | 0.50 | 0.00718 | 0.50 | 0.00641 | 0.50 |
where \( j = 1, \ldots, N - 1 \) and
\[
\begin{align*}
p_0 &= \frac{12\Delta \delta t_1}{20h^2} - \frac{\delta \Delta t c_1}{20}, \\
p_1 &= \left[ 1 - \frac{24\Delta \delta t}{20h^2} \right] a_1 \\
     &+ \frac{\Delta t}{2} \left( \frac{12\delta a_2}{10h^2} - \frac{\delta c_2}{10} - (\rho a_1 + \alpha d_2) b_1 \right), \\
p_2 &= \left[ 1 - \frac{24\Delta \delta t}{20h^2} \right] a_2 \\
     &+ \frac{\Delta t}{2} \left( \frac{24\delta a_1}{10h^2} - \frac{2\delta c_1}{10} \right), \\
p_3 &= \left[ 1 - \frac{24\Delta \delta t}{20h^2} \right] a_1 \\
     &+ \frac{\Delta t}{2} \left( \frac{12\delta a_2}{10h^2} - \frac{\delta c_2}{10} + (\rho a_1 + \alpha d_2) b_1 \right), \\
p_4 &= \frac{\alpha \Delta t d_1}{2} b_1, \\
p_5 &= \left[ 1 + \frac{24\Delta \delta t}{20h^2} \right] a_1 \\
     &+ \frac{\Delta t}{2} \left( \frac{12\delta a_2}{10h^2} - \frac{\delta c_2}{10} - (\rho a_1 + \alpha d_2) b_1 \right).
\end{align*}
\]

Now substituting the Fourier modes \( e^{i \phi} = A \xi^o \exp(i \phi h) \), and \( d^o_i = B \xi^o \exp(i \phi h) \), where \( A \) and \( B \) are harmonics amplitudes, \( \xi \) is the element size, \( h \) is the element size and \( t = \sqrt{-1} \) in (3.3), we obtain
\[
\begin{align*}
p_0 &\xi A e^{i(\phi - j) h} + p_1 A e^{i(\phi + j) h} + p_2 A \xi^{*} e^{-i(\phi - j) h} \\
    + p_3 A \xi^{*} e^{-i(\phi + j) h} + p_4 A \xi^{*} e^{i(\phi - j) h} \\
    &+ p_5 B \xi^o e^{i(\phi - j) h} + p_6 B \xi^o e^{i(\phi + j) h} \\
    = &-p_0 A e^{-i(\phi - j) h} + p_1 A e^{i(\phi - j) h} + p_2 A \xi e^{i(\phi + j) h} \\
    + p_3 A \xi e^{-i(\phi - j) h} + p_4 A \xi e^{i(\phi + j) h} \\
    &+ p_5 B e^{i(\phi + j) h} - p_6 B e^{i(\phi + j) h}.
\end{align*}
\]

Dividing (3.4) by \( \xi^o \exp(i \phi h) \) and rearranging the equation, we obtain
\[
\begin{align*}
\xi &= -p_0 A e^{-2i \phi h} + p_1 A e^{-i \phi h} + p_2 A e^{i \phi h} - p_0 A e^{2i \phi h} + p_1 A e^{-i \phi h} + p_2 A e^{i \phi h} \\
    &+ p_3 B e^{i \phi h} - p_4 B e^{-i \phi h} - p_5 B e^{i \phi h}.
\end{align*}
\]

Using \( \cos(\phi h) = \frac{e^{i \phi h} + e^{-i \phi h}}{2} \) and \( \sin(\phi h) = \frac{e^{i \phi h} - e^{-i \phi h}}{2i} \) in equation (3.5) and simplifying, we obtain
\[
\xi = -2q_0 \cos 2\phi h + 2q_1 \cos \phi h \sin \phi h + 2q_2 \sin 2\phi h + 2q_3 \sin \phi h + q_4.
\]

Since \( \phi \in [-\pi, \pi] \), without loss of generality, we can assume that \( \phi = 0 \) so that (3.6) becomes
\[
\xi = -2q_0 + 2q_1 + q_3 \leq 1
\]

which proves (2.5) is unconditionally stable. Since (2.5) and (2.6) are symmetric in \( v \) and \( w \), similar results can be obtained from (2.6).

### 4. Numerical experiments and discussion

In this section the accuracy of the proposed scheme is verified by some test problems and is measured with two discrete \( L_2 \) and \( L_\infty \) error norms defined as
\[
L_2 = \sqrt{\sum_{j=0}^{N} (V(z_j, t_n) - V_j)^2}
\]

where, \( q_0 = A \left( \frac{12\Delta \delta t_1}{20h^2} - \frac{\delta \Delta t c_1}{20} \right) \).

\[
\begin{align*}
q_1 &= A \left( 1 + \frac{24\Delta \delta t}{20h^2} \right) a_1 + \frac{\delta \Delta t c_1}{20}, \\
q_2 &= A \left( \frac{12\Delta \delta t}{20h^2} \left( \frac{\delta c_2}{10} - (\rho a_1 + \alpha d_2) b_1 \right) \right), \\
q_3 &= A \left( \frac{12\Delta \delta t}{20h^2} a_2 + \frac{\delta \Delta t c_1}{10h^2} \right) a_2, \\
q_4 &= A \left( \frac{12\Delta \delta t}{20h^2} a_1 + \frac{\delta \Delta t c_1}{10h^2} \right) a_1, \\
q_5 &= A \left( \frac{12\Delta \delta t}{20h^2} a_2 + \frac{\delta \Delta t c_1}{10h^2} \right) a_2.
\end{align*}
\]

### Table 6. Maximum values of \( V \) and \( W \) when \( \alpha = \beta = 100 \) for example 3.

| \( t \) | \text{Max. value of } V \text{ at point } (z) | \text{Max. value of } W \text{ at point } (z) |
|---|---|---|
| 0.1 | 0.04175 | 0.46 | 0.05083 | 0.76 |
| 0.2 | 0.04179 | 0.53 | 0.01037 | 0.64 |
| 0.3 | 0.00535 | 0.54 | 0.00352 | 0.55 |
| 0.4 | 0.00198 | 0.51 | 0.01300 | 0.52 |
| \text{Mittal [12]} | | | | |
Table 7. Absolute error and OC for $V$ and $W$ when at $t = 0.1$ for example 3.

| $N$ | $V$ | Present | Mittal [12] | $W$ | Present | Mittal [12] |
|-----|-----|---------|-------------|-----|---------|-------------|
|     | $L_\infty$ | Ratio | OC | $L_\infty$ | Ratio | OC | $L_\infty$ | Ratio | OC |
| $\alpha = \beta = 100$ | | | | | | | | | |
| 50  | 0.001904 | … | … | 0.018812 | … | … | 0.001902 | … | … |
| 100 | 0.001890 | 1.0075 | 0.011 | 0.005508 | 3.4154 | 1.772 | 0.001889 | 1.0066 | 0.009 |
| 200 | 0.001253 | 1.5079 | 0.592 | 0.001649 | 3.3402 | 1.740 | 0.001253 | 1.5076 | 0.592 |
| $\alpha = \beta = 10$ | | | | | | | | | |
| 50  | 0.001821 | … | … | 0.016182 | … | … | 0.001822 | … | … |
| 100 | 0.001870 | 0.9739 | 0.038 | 0.004935 | 3.2790 | 1.713 | 0.001870 | 0.9747 | 0.037 |
| 200 | 0.001250 | 1.4963 | 0.581 | 0.001493 | 3.3045 | 1.724 | 0.001249 | 1.4965 | 0.582 |

and

\[ L_\infty = \|V - V_0\|_\infty = \max_j |V(z_j, t_0) - V_j^0|. \]

and the order of convergence is given by [12]

\[ R = \frac{\log (\text{Error}(N_1)/\text{Error}(N_2))}{\log (N_2/N_1)} \]

where \(\text{Error}(N_1)\) and \(\text{Error}(N_2)\) are \(L_\infty\) norms at \(N\) and \(2N\), respectively.

**Example 1.** Consider the CBE,

\[
\begin{align*}
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} - 2v\frac{\partial v}{\partial z} + \frac{\partial (vw)}{\partial z} &= 0 \\
\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial z^2} - 2w\frac{\partial w}{\partial z} + \frac{\partial (vw)}{\partial z} &= 0
\end{align*}
\]

(4.1)

with exact solution

\[ v(z, t) = w(z, t) = \exp(-t)\sin(z), \quad z \in [-\pi, \pi], \quad t > 0. \]

The ICs and BCs are extracted from the exact solution.

The scheme is applied to this problem to check its accuracy. In figure 1 the numerical and exact solutions at different times are compared with tremendous closeness. Figure 2 exhibits the absolute errors for \(v\) and \(w\) when \(N = 50, \Delta t = 0.001\) and \(t = 1\). Figures 3 and 4 plot the 3D contrast between the exact and approximate solutions. Tables 1 and 2 compare computed errors with the ones computed in [7, 12, 17].

The approximate solutions \(V(z, t)\) and \(W(z, t)\) when \(N = 20, \Delta t = 0.01\) and \(t = 1\) for example 1 are

Figure 8. The numerical solution \(V(z, t)\) (circles, stars, triangles, diamonds) and the exact (solid lines) solution when \(v(z, t)\) when \(\alpha = \beta = 10\) for example 3.

Figure 9. The approximate solutions \(V(z, t)\) (left figure) and \(W(z, t)\) (right figure) when \(\alpha = \beta = 10\) for example 3.
given by

\[
V(z, 1) = \begin{cases}
0.72114 + 1.42471 z + 0.57050 z^2 + 0.06050 z^3, & z \in \left[-\pi, -\frac{9\pi}{10}\right]
\vspace{0.5cm}
0.59802 + 1.29407 z + 0.52429 z^2 + 0.05505 z^3, & z \in \left[-\frac{9\pi}{10}, -\frac{4\pi}{5}\right]
\vspace{0.5cm}
0.41580 + 1.07657 z + 0.43775 z^2 + 0.04358 z^3, & z \in \left[-\frac{4\pi}{5}, -\frac{7\pi}{10}\right]
\end{cases}
\]

and

\[
W(z, 1) = \begin{cases}
0.72114 + 1.42471 z + 0.57050 z^2 + 0.06050 z^3, & z \in \left[-\pi, -\frac{9\pi}{10}\right]
\vspace{0.5cm}
0.59802 + 1.29407 z + 0.52429 z^2 + 0.05505 z^3, & z \in \left[-\frac{9\pi}{10}, -\frac{4\pi}{5}\right]
\vspace{0.5cm}
0.41580 + 1.07657 z + 0.43775 z^2 + 0.04358 z^3, & z \in \left[-\frac{4\pi}{5}, -\frac{7\pi}{10}\right]
\end{cases}
\]

Example 2. Consider the CVBE,

\[
\begin{align*}
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} + 2 \frac{\partial v}{\partial z} + \alpha \frac{\partial}{\partial z} (vw) &= 0 \\
\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial z^2} + 2 \frac{\partial w}{\partial z} + \beta \frac{\partial}{\partial z} (vw) &= 0
\end{align*}
\]

with exact solution

\[
\begin{align*}
v(z, t) &= a_0 \left(1 - 2A \left(\frac{2\alpha - 1}{4\alpha\beta - 1}\right) \tanh(A(z - 2At))\right), \\
w(z, t) &= a_1 \left(\frac{2\beta - 1}{2\alpha - 1} - 2A \left(\frac{2\alpha - 1}{4\alpha\beta - 1}\right) \tanh(A(z - 2At))\right),
\end{align*}
\]

where \(\alpha\) and \(\beta\) are arbitrary constant. The initial and boundary conditions correspond to the exact solution.
We utilize the proposed scheme to acquire numerical results. Figure 5 plots the error profiles when $\Delta t = 0.01$, $N = 100$ and $t = 1$ for $v$ and $w$. In figures 6 and 7 the 3D comparisons are shown between the exact and numerical solutions. An excellent comparison of error norms is tabulated in tables 3 and 4.

The approximate solutions $V(z, 0)$ and $W(z, 0)$ when $N = 20$, $\Delta t = 0.01$ and $t = 1$ for example 2 are given by

$$v(z, 0) = \begin{cases} \sin(2\pi z), & z \in [0, 0.5] \\ 0, & z \in (0.5, 1] \end{cases}$$

$$w(z, 0) = \begin{cases} 0, & z \in [0, 0.5] \\ -\sin(2\pi z), & z \in (0.5, 1] \end{cases}$$

and zero boundary conditions.

$$V(z, 1) = \begin{cases} 0.03906 - 0.00505z - 0.006036z^2 - 0.00001z^3, & z \in [-10, -9] \\ 0.05472 + 0.00017z + 0.00022z^2 + 9.64745 \times 10^{-6}z^3, & z \in [-9, -8] \\ 0.05022 - 0.00152z + 5.57184 \times 10^{-6}z^2 + 8.41878 \times 10^{-7}z^3, & z \in [-8, -7] \end{cases}$$

$$W(z, 1) = \begin{cases} 0.05018 - 0.00156z + 2.77605 \times 10^{-6}z^2 + 2.73915 \times 10^{-7}z^3, & z \in [7, 8] \\ 0.05297 - 0.00260z + 0.00013z^2 - 5.16165 \times 10^{-6}z^3, & z \in [8, 9] \\ 0.04503 + 0.00004z - 0.00016z^2 + 5.72000 \times 10^{-6}z^3, & z \in [9, 10], \end{cases}$$

$$v(z, 0) = \begin{cases} \sin(2\pi z), & z \in [0, 0.5] \\ 0, & z \in (0.5, 1] \end{cases}$$

$$w(z, 0) = \begin{cases} 0, & z \in [0, 0.5] \\ -\sin(2\pi z), & z \in (0.5, 1] \end{cases}$$

and zero boundary conditions.

$$V(z, 1) = \begin{cases} 0.04642 - 0.00549z - 0.00041z^2 - 0.00001z^3, & z \in [-10, -9] \\ 0.06403 + 0.00038z + 0.00024z^2 + 0.00001z^3, & z \in [-9, -8] \\ 0.05899 - 0.00151z + 6.55897 \times 10^{-6}z^2 + 9.02552 \times 10^{-7}z^3, & z \in [-8, -7] \end{cases}$$

$$W(z, 1) = \begin{cases} 0.05895 - 0.00156z + 3.08604 \times 10^{-6}z^2 + 2.40801 \times 10^{-7}z^3, & z \in [7, 8] \\ 0.06270 - 0.00297z + 0.00018z^2 - 7.09339 \times 10^{-6}z^3, & z \in [8, 9] \\ 0.05206 + 0.00058z - 0.00022z^2 + 7.50650 \times 10^{-6}z^3, & z \in [9, 10] \end{cases}$$

Example 3. Consider the CVBE,

$$\begin{align*}
\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial z^2} + \eta \frac{\partial v}{\partial z} + \alpha \frac{\partial}{\partial z} (vw) &= 0 \\
\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial z^2} + \zeta \frac{\partial w}{\partial z} + \beta \frac{\partial}{\partial z} (vw) &= 0
\end{align*}$$

(4.5)

Numerical calculations are performed using $\Delta t = 0.001$ and $N = 50$. The maximum values of $v$ and $w$ are computed and are compared with those presented in [12, 17]. The results are tabulated in tables 5 and 6. Table 7 compares absolute errors with the ones listed in [12]. In figure 8, a comparison between the exact and approximate solutions is shown for various values of $\alpha$, $\beta$, $\eta$ and $\zeta$. Figures 9 and 10 plots 3D approximate and exact solutions for various values of $\alpha$ and $\beta$. The approximate solutions $V(z, t)$ and $W(z, t)$ when $N = 20$, $\alpha = \beta = 10$, $\eta = \zeta = 2$, $\Delta t = 0.001$ and $t = 0.1$ for example 3 are given by

$$V(z, 0.1) = \begin{cases} -4.33681 \times 10^{-10} + 0.39175z - 0.01150z^2 - 0.45907z^3, & z \in \left[0, \frac{1}{20}\right] \\ 0.00001 + 0.39107z + 0.00205z^2 - 0.54943z^3, & z \in \left[\frac{1}{20}, \frac{1}{10}\right] \\ -0.00008 + 0.39388z - 0.02603z^2 - 0.45584z^3, & z \in \left[\frac{1}{10}, \frac{3}{20}\right] \\ \vdots \\vdots \\vdots \\vdots \\ -1.00316 + 4.06616z - 4.38102z^2 + 1.34406z^3, & z \in \left[\frac{17}{20}, \frac{9}{10}\right] \\ -1.67457 + 6.21421z - 6.76774z^2 + 2.22803z^3, & z \in \left[\frac{9}{10}, \frac{19}{20}\right] \\ -2.15512 + 7.73174z - 8.36514z^2 + 2.78852z^3, & z \in \left[\frac{19}{20}, 1\right]. \end{cases}$$
and

\[
W(z, 0.1) = \begin{cases}
(0.23199 + 0.000 42 z + 0.221 52 z^2), & z \in \left[ 0, \frac{1}{20} \right] \\
3.44165 \times 10^{-6} + 0.231 79 z + 0.004 55 z^2 + 0.193 99 z^3, & z \in \left[ \frac{1}{20}, \frac{1}{10} \right] \\
0.000069 + 0.229 83 z + 0.024 13 z^2 + 0.128 72 z^3, & z \in \left[ \frac{1}{10}, \frac{3}{20} \right] \\
-1.52107 + 5.632 51 z - 5.9566 z^2 + 1.843 99 z^3, & z \in \left[ \frac{17}{50}, \frac{9}{10} \right] \\
-2.32922 + 8.326 34 z - 8.949 74 z^2 + 2.952 56 z^3, & z \in \left[ \frac{9}{10}, \frac{19}{20} \right] \\
-2.78973 + 9.780 59 z - 10.4805 z^2 + 3.489 68 z^3, & z \in \left[ \frac{19}{20}, 1 \right].
\end{cases}
\]

5. Concluding remarks

This investigation presents a numerical procedure dependent on cubic B-spline and the Hermite formula for the CVBE. This technique uses the standard finite differences to discretize the time dimension while the space dimension is approximated using the Hermite formula and the cubic B-spline. The refinement of the scheme using the Hermite formula has appreciably increased the accuracy of the scheme. The stability of the scheme has been checked to affirm that it is unconditionally stable. Numerical and graphical comparisons reveal that the presented procedure is computationally better and effective. It is worthy of mention that the scheme can be tried to a variety of partial differential equations.

References

[1] Esipov S E 1995 Coupled Burgers’ equations: a model of polydispersive sedimentation Phys Rev E 52 3711–8
[2] Civalek O 2006 Harmonic differential quadrature-finite differences coupled approaches for geometrically nonlinear static and dynamic analysis of rectangular plates on elastic foundation J Sound Vib. 294 966–80
[3] Wei G W and Gu Y 2002 Conjugate filter approach for solving Burgers’ equation J Comput Appl Math. 149 439–56
[4] Islam S, Haq S and Uddin M 2009 A meshfree interpolation method for the numerical solution of the coupled nonlinear partial differential equations Eng And Bound Elem. 33 399–409
[5] Khatert A H, Temsah R S and Hassan M M 2008 A Chebyshev spectral collocation method for solving Burgers-type equations J Comput Appl Math. 222 333–50
[6] Deghan M, Hamidi A and Shakourifar M 2007 The solution of coupled Burgers’ equations using Adomian-Pade technique Appl Math Comput. 189 1034–47
[7] Rashid A and Ismail A I B 2009 A Fourier Pseudospectral method for solving coupled viscous Burgers equations Comp Methods Appl Math. 9 412–20
[8] Abdou M A and Soliman A A 2005 Variational iteration method for solving Burgers’s and coupled Burger’s equations J Comp Appl Math. 181 245–51
[9] Kaya D 2001 An explicit solution of coupled viscous Burgers’ equation by the decomposition method IJMMS. 27 675–80
[10] Soliman A A 2006 The modified extended tanh-function method for solving Burgers-type equations Physica A 361 394–404
[11] Abazari R and Borhanifar H 2010 Numerical study of the solution of the Burgers and coupled Burgers equations by a differential transformation method Comput. Math. Appl. 59 2711–22
[12] Mittal R C and Arora G 2011 Numerical solution of the coupled viscous Burgers’ equation Commun. Nonlinear Sci. Numer. Simulat. 16 1304–13
[13] Mokhtari R, Tooodar A S and Chegini N G 2011 Application of the generalized differential quadrature method in solving Burgers’ equations Commun. Theor. Phys. 56 1009–15
[14] Srivastava V K, Tamsir M, Bhardwaj U and Sanyasiraju Y V S S 2011 Crank-Nicolson scheme for numerical solutions of two-dimensional coupled Burgers’ equations Int J Eng Sci Res. 2 1–6
[15] Srivastava V K and Tamsir M 2012 Crank-Nicolson semi-implicit approach for numerical solutions of two-dimensional coupled nonlinear Burgers’ equations Int J Appl Mech Eng. 17 571–81
[16] Srivastava V K, Awasthi M K and Singh S 2013 An implicit logarithmic finite-difference technique for two dimensional coupled viscous Burgers’ equation AIP Adv. 3 1–9
[17] Srivastava V K, Tamsir M, Awasthi M K and Singh S 2014 One-dimensional coupled Burgers’ equation and its numerical solution by an implicit logarithmic finite-difference method AIP Adv. 4 1–10
[18] Onarcan A T and Hepson O E 2018 Higher order trigonometric B-spline algorithms to the solution of coupled Burgers’ equation AIP Conf Proc 1926 1–5
[19] Khader M M and Adel M M 2016 Numerical solution of fractional wave equation using an efficient class of FDM based on Hermite formula Adv Differ Equ 34 10
[20] Nee J and Duan J 1998 Limit set of trajectories of the coupled viscous Burgers’ equations Appl Math Lett. 11 57–61