G-TORSORS ON PERFECTOID SPACES

BEN HEUER

Abstract. For any rigid analytic group variety \( G \) over a non-archimedean field \( K \) over \( \mathbb{Q}_p \), we study \( G \)-torsors on adic spaces over \( K \) in the \( v \)-topology. Our main result is that on perfectoid spaces, \( G \)-torsors in the \( \acute{e}tale \) and \( v \)-topology are equivalent. This generalises the known cases of \( G = \mathbb{G}_a \) and \( G = \text{GL}_n \) due to Scholze and Kedlaya–Liu.

On a general adic space \( X \) over \( K \), where there can be more \( v \)-topological \( G \)-torsors than \( \acute{e}tale \)-ones, we show that for any open subgroup \( U \subseteq G \), any \( G \)-torsor on \( X \) admits a reduction of structure group to \( U \) \( \acute{e}tale \)-locally on \( X \). This has applications in the context of the \( p \)-adic Simpson correspondence: For example, we use it to show that on any adic space, generalised \( \mathbb{Q}_p \)-representations are equivalent to \( v \)-vector bundles.

1. Introduction

Let \( G \) be a smooth group scheme over a field \( K \), and let \( X \) be a \( K \)-scheme. Then by a well-known Theorem of Grothendieck [Gro68, Théorème 11.7, Remark 11.8], the functor

\[
\{ \text{\( G \)-torsors on }X_{\acute{e}t}\} \overset{\sim}{\longrightarrow} \{ \text{\( G \)-torsors on }X_{\text{fppf}}\}
\]

is an equivalence of categories. This article studies a similar functor in \( p \)-adic geometry:

Let \( K \) be a non-archimedean field over \( \mathbb{Q}_p \) and let \( X \) be an adic space over \( K \), considered as a diamond in the sense of Scholze [Sch22, §15]. Then \( X \) has an \( \acute{e}tale \) topology and a much finer \( v \)-topology [Sch22, §14]. Let \( G \) be any rigid analytic group variety over \( K \), not necessarily commutative, considered as a diamond. In this article, we study the functor

\[
\{ \text{\( G \)-torsors on }X_{\acute{e}t}\} \hookrightarrow \{ \text{\( G \)-torsors on }X_v\},
\]

where a \( G \)-torsor is a sheaf with a \( G \)-action that is locally isomorphic to \( G \) (see Section 3.3). This functor is always fully faithful. The question whether it is essentially surjective can be phrased in terms of the morphism of sites \( \nu : X_v \to X_{\acute{e}t} \) by asking whether \( R^1\nu_*G \) vanishes.

In this \( p \)-adic situation, the analogy to Grothendieck’s Theorem works best when \( X \) is a perfectoid space: Indeed, in this case, torsors under \( G = \mathbb{G}_a \) on \( X_{\acute{e}t} \) and \( X_v \) agree, namely Scholze proves that \( R\nu_*\mathbb{G}_a = \mathbb{G}_a \) [Sch22, Proposition 8.8]. By a Theorem of Kedlaya–Liu, also vector bundles on \( X_{\acute{e}t} \) and \( X_v \) agree [KL16, Theorem 3.5.8] (cf. [SW20, Lemma 17.1.8]), which is the case of \( G = \text{GL}_n \). This implies the case of linear algebraic \( G \) by the Tannakian formalism [SW20, §19.5]. Our main result is the following generalisation of all of these statements:

**Theorem 1.1.** Let \( X \) be a perfectoid space and let \( G \) be a rigid group over \( K \). The functor

\[
\{ \text{\( G \)-torsors on }X_{\acute{e}t}\} \overset{\sim}{\longrightarrow} \{ \text{\( G \)-torsors on }X_v\}
\]

is an equivalence of categories. If \( G \) is commutative, then we more generally have \( R\nu_*G = G \).

Our method is different to that of the aforementioned works, and also gives a new proof of Kedlaya–Liu’s Theorem. One interesting new case is \( G = \text{GL}_n(\mathcal{O}^+) \), which says that the categories of finite locally free \( \mathcal{O}^+ \)-modules on \( X_{\acute{e}t} \) and \( X_v \) agree.

As another application, recall that for any locally Noetherian adic space \( X \) over \( \mathbb{Q}_p \) (for example a rigid space), we have a hierarchy of topologies

\[
X_v \to X_{\text{qpro\acute{e}t}} \to X_{\text{pro\acute{e}t}} \to X_{\acute{e}t}.
\]

Here \( X_{\text{qpro\acute{e}t}} \) is the quasi-pro-\( \acute{e}tale \) site, which makes sense without the assumption that \( X \) is locally Noetherian. Since the three topologies on the left are locally perfectoid, we deduce:
Corollary 1.2. Let $X$ be any adic space over $K$, then the categories of $G$-torsors on $X_v$ and $X_{\text{pro\acute et}}$ (and $X_{\text{pro\acute et}}$ if $X$ is locally Noetherian) are equivalent.

1.1. $G$-torsors on rigid spaces. Let now $X$ be a rigid space, then it is known that there are in general many more $v$-topological $G$-torsors than there are étale ones: For example, if $K$ is a complete algebraically closed extension of $\mathbb{Q}_p$ and $X$ is smooth over $K$, then already for $G = \mathbb{G}_a$, a key step in Scholze’s construction of the Hodge–Tate spectral sequence is a canonical isomorphism on $X_\text{ét}$

$$R^1\nu_*\mathbb{G}_a = \Omega_X(-1)$$

where $\Omega_X(-1)$ is a Tate twist of the Kähler differentials on $X$ [Sch13b, Proposition 3.23].

Also for $G = \text{GL}_n$, it is known that étale vector bundles and $v$-vector bundles on $X$ are not the same, e.g. see [Heu22a] for the case of $G = \mathbb{G}_m$. The difference is closely related to the $p$-adic Simpson correspondence: As we explain in Section 2, there is an equivalence

\[
\{\text{finite locally free } \mathcal{O}^+/p^n\text{-modules on } X_v\} \xrightarrow{\sim} \{\text{generalised representations on } X\}
\]

where following Faltings [Fal05], generalised representations are compatible systems $(V_n)_{n\in\mathbb{N}}$ of finite locally free $\mathcal{O}^+/p^n$-modules $V_n$ on $X_\text{ét}$. The equivalence is similar in spirit to the equivalence between lisse $\mathbb{Q}$-sheaves on a scheme and locally free $\mathcal{O}$-modules on the pro-étale site in the sense of Bhatt–Scholze [BS15, §1]. The proof hinges on the following:

Proposition 1.3. We have an isomorphism $R\nu_*(\mathcal{O}^+/p^n) = \mathcal{O}^+/p^n$ (already before passing to the almost category) and $R^1\nu_*\text{GL}_n(\mathcal{O}^+/p^n) = 1$. In particular, we have an equivalence

$$\nu^*: \left\{\begin{array}{c}
\text{finite locally free} \\
\text{\mathcal{O}^+/p^n-modules on } X_\text{ét}
\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}
\text{finite locally free} \\
\text{\mathcal{O}^+/p^n-modules on } X_v
\end{array}\right\}.$$ 

One reason why we are interested in $v$-topological $G$-torsors under general rigid groups $G$ is for generalisations of the $p$-adic Simpson correspondence to more general non-abelian coefficients, as explored in [Heu22b]: These relate $v$-topological $G$-bundles to $G$-Higgs bundles. The relevance of Theorem 1.1 in this context is that it shows that $v$-topological $G$-bundles are “locally small”, namely they admit reductions of structure groups to small open subgroups.

1.2. Reduction of structure group. The technical heart of this article is the study of sheaves $F$ that “commute with tilde-limits”:

Definition 1.4. A sheaf $F$ on the big étale site of sousperfectoid spaces over $K$ is said to satisfy the approximation property if for any affinoid perfectoid tilde-limit $X \sim \lim_{i\in I} X_i$ of affinoid spaces $X_i$ such that $\lim_{i\in I} \mathcal{O}(X_i) \to \mathcal{O}(X)$ has dense image, $F(X) = \lim_{i\in I} F(X_i)$.

Examples of such sheaves include $\mathcal{O}^+/p^n$ and thus also $\text{GL}_n(\mathcal{O}^+/p^n)$. We show:

Theorem 1.5. Let $F$ be a sheaf of groups satisfying the approximation property. Then $F$ is already a $v$-sheaf and $R^1\nu_* F = 1$. If $F$ is a sheaf of abelian groups, then $R^1\nu_* F = F$.

The relevance to our study of $G$-torsors is then the following:

Proposition 1.6. Let $G$ be any rigid group. Let $U \subseteq G$ be any rigid open subgroup, not necessarily normal. Then $G/U$ satisfies the approximation property.

For example, both $\mathcal{O}^+/p^n$ and $\text{GL}_n(\mathcal{O}^+/p^n)$ are of this form. We use the Proposition to show that any $G$-torsor on $X$ admits a reduction of structure group to $U$ on an étale cover:

Theorem 1.7. Let $G$ be a rigid group over $K$ and let $U \subseteq G$ be a rigid open subgroup. Let $X$ be a sousperfectoid space over $K$ and let $\nu : X_v \to X_\text{ét}$ be the natural morphism of sites. Then the natural map

$$R^1\nu_* U \to R^1\nu_* G$$

is surjective. If $G$ is commutative, we more generally have $R^k\nu_* U = R^k\nu_* G$ for all $k \geq 1$. 

For non-commutative $G$, the map $R^1\nu_*U \to R^1\nu_*G$ is not in general an isomorphism.

The idea for the proof of Theorem 1.1 is now that by the theory of $p$-adic Lie groups, there is a large supply of open subgroups of $G$. If $G$ is commutative, then these are isomorphic to open subgroups of the Lie algebra of $G$ via the exponential, and one can deduce the result from the case of $\mathbb{G}_a$. In general, the exponential does not respect the group structure, but the relation to the Lie algebra suffices to trivialise $G$-torsors étale-locally by inductive lifting.

Conceptually speaking, the main aim of this work is to launch a systematic study of $G$-torsors on adic spaces with a view towards generalisations and reformulations of the $p$-adic Simpson correspondence. Apart from the above results, we therefore prove some further foundational results on $G$-torsors. We continue our study in [Heu22b], where based on the results of this article, we give an explicit description of the sheaf $R^1\nu_*G$ on smooth rigid spaces, and use this to construct analytic moduli spaces of $G$-torsors on rigid spaces.

Acknowledgements. We thank Johannes Anschütz, Gabriel Dospinescu, Arthur–César Le Bras, Ian Gleason, Lucas Mann, Peter Scholze, Annette Werner, Daxin Xu and Bogdan Zavyalov for very helpful conversations. This work was funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy – EXC-2047/1–390685813. The author was supported by DFG via the Leibniz-Preis of Peter Scholze.

Setup and Notation

Throughout let $p$ be a prime and $K$ a non-archimedean field of residue characteristic $p$. We fix a subring $K^+ \subseteq K$ of integral elements, which we often drop from notation. Let $\mathcal{O}_K$ be the ring of integers of $K$ and $\mathfrak{m}_K$ its maximal ideal. We also denote this by $\mathfrak{m}$ if $K$ is clear from context. For any $\epsilon > 0$, we denote by $p^\epsilon \mathfrak{m}_K$ the ideal $\{x \in K \text{ s.t. } |x| < p^{-\epsilon}\}$ of $\mathcal{O}_K$.

By an adic space over $K$ we mean an adic space over $\text{Spa}(K,K^+)$ in the sense of Huber. For any adic space $X$, we use the étale site $X_{\text{ét}}$ in the sense of Kedlaya–Liu [KL15, Definition 8.2.16]. By a rigid space over $K$ we mean an adic space locally of topologically finite type over $\text{Spa}(K,K^+)$. One example we use several times is the closed ball $B^d = \text{Spa}(K(T_1,\ldots,T_d))$. We use perfectoid spaces in the sense of [Sch12], as well as Scholze’s category of diamonds [Sch22]. Most adic spaces we consider throughout will be sousperfectoid in the sense of [SW20, §6.3][HK]: examples of such are smooth rigid spaces, perfectoid spaces, and products thereof. Moreover, we recall that for any sousperfectoid space $X$ over $K$, any object of the étale site $X_{\text{ét}}$ is again sousperfectoid, hence sheafy. We may therefore freely switch back and forth between $X$ and its associated locally spatial diamond $X^\diamond$ [Sch22, §15]. As we have fixed a structure map to $K$, we may regard $X^\diamond$ as a sheaf on the $v$-site $\text{Perf}_{K,v}$ of perfectoid spaces over $K$. Then diamondification identifies the étale sites and the structure sheaves [Sch22, Lemma 15.6] of $X$ and $X^\diamond$, where on the diamantine side we always mean the structure sheaf $\mathcal{O}$ on $X^\diamond$ induced from that on $\text{Perf}_{K,v}$.

We denote by $\text{Sous}_K$ the category of sousperfectoid spaces over $K$. We consider this as a site equipped either with the étale topology $\text{Sous}_{K,\text{ét}}$ or the $v$-topology $\text{Sous}_{K,v}$.

2. Generalised representations on rigid spaces

Before we discuss $G$-torsors for general rigid groups $G$, we study in this section the simpler case of $v$-vector bundles, that is, $\text{GL}_n$-torsors on $X_v$. While this simplifies the setting, the overall line of argument is already the same as in the general case. We begin by giving an alternative description of $v$-vector bundles that is of interest in the $p$-adic Simpson correspondence: The equivalence between $v$-vector bundles and Faltings’ generalised representations.
2.1. Generalised representations. We start by adapting generalised representations as defined by Faltings [Fal05, §2 and Theorem 3] from the algebraic setting of log schemes to an analytic setting: Throughout this section, let $K$ be any non-archimedean extension of $\mathbb{Q}_p$ and let $X$ be any adic space over $K$.

**Definition 2.1.**

1. A generalised representation on $X$ is a system $(M_k)_{k \in \mathbb{N}}$ of finite locally free $\mathcal{O}_X / p^k$ modules $M_k$ on $X_{\text{ét}}$ with isomorphisms $M_{k+1}/p^k \cong M_k$ for all $k$.
2. The isogeny category is the localisation of the category of generalised representations on $X$ at multiplication by $p$. A generalised $\mathbb{Q}_p$-representation on $X$ is a system of generalised representations $(M_U)_{U \in \mathcal{X}_1}$ on some étale cover $\mathcal{X}$ of $X$ with isomorphisms $M_{U_i \cap X \cup U_j} \to M_{U_j \cap U_i \cup X}$ in the isogeny category that satisfy the cocycle condition.

**Remark 2.2.** The name “generalised representation" stems from the following observation: Fix any base-point $x \in X(\overline{K})$ and consider the étale fundamental group $\pi_1(X) := \pi_1^\text{pro}(X, x)$, then to any semilinear continuous representation $\rho : \pi_1(X) \to \text{GL}(E)$ on a finite free $\mathcal{O}_K$-module $E$, we can associate a generalised representation: Namely, by continuity, the reduction $\rho_k : \pi_1(X) \to \text{GL}(\mathcal{O}_K / p^k)$ factors through a finite quotient of $\pi_1(X)$, corresponding to a finite étale cover $X' \to X$. We can then regard $\rho_k$ as a descent datum for a finite locally free $\mathcal{O}_X / p^k$-module $M_k$ on any $U \in X_{\text{ét}}$ to be

$$M_k(U) := \{ x \in E(U \times X / X') / p^k \mid \gamma^* x = \rho^{-1}(\gamma) x \text{ for all } \gamma \in \pi_1(X) \}.$$ 

The system $(M_k)_{k \in \mathbb{N}}$ then defines a generalised representation in the above sense.

The main goal of this short section is to prove the following equivalent characterisation:

**Proposition 2.3.** Let $X$ be an adic space over a non-archimedean field $K$ over $\mathbb{Q}_p$ such that every object of $X_{\text{ét}}$ is a sheafy adic space (e.g. $X$ can be any rigid spaces, or sousperfectoid).

1. The morphism of sites $\nu : X_{\text{pro}} \to X_{\text{ét}}$ induces a natural equivalence of categories

$$\{\text{finite locally free } \mathcal{O}_{X_{\text{pro}}}^+\text{-modules on } X_{\text{pro}}\} \to \{\text{generalised representations on } X\}$$

Moreover, these are equivalent to finite locally free $\mathcal{O}_X^+$-modules on $X_{\text{pro}^\text{ét}}$, and if $X$ is locally Noetherian also to finite locally free $\mathcal{O}_X^+$-modules on $X_{\text{pro}^\text{ ét}}$.

2. By localising at multiplication by $p$, this defines an equivalence of categories

$$\{\text{finite locally free } \mathcal{O}_{X_{\text{pro}}}\text{-modules on } X_{\text{pro}}\} \to \{\text{generalised } \mathbb{Q}_p\text{-representations on } X\}.$$ 

Moreover, these are equivalent to finite locally free $\mathcal{O}$-modules on $X_{\text{pro}^\text{ét}}$, and if $X$ is locally Noetherian also to finite locally free $\mathcal{O}$-modules on $X_{\text{pro}^\text{ ét}}$.

**Definition 2.4.** We also call a finite locally free $\mathcal{O}$-modules on $X_{\text{pro}}$ a $\nu$-vector bundle. We then call finite locally free $\mathcal{O}$-modules on $X_{\text{ét}}$ “étale vector bundles” to clarify the topology.

From this perspective, one could reverse-engineer Faltings’ definition of generalised $\mathbb{Q}_p$-representations by saying that they describe $\nu$-vector bundles purely in terms of $X_{\text{ét}}$.

**Remark 2.5.** The idea that generalised representations on a rigid space $X$ are vector bundles on $X_{\text{pro}^\text{ét}}$ is mentioned or implicit in other works: In the arithmetic setting of smooth rigid spaces over discretely valued fields, it is hinted at by Liu–Zhu [LZ17] (see Remark 2.6), and a related result appears in the work of Morrow–Tsuiji (cf [MT21, Theorem 5.7]) in a good reduction setting. In more algebraic geometric settings, related statements have been shown by Xu [Xu17, §3.29] and Yang–Zhu [YZ21, Lemma 2.10].

Proposition 2.3 now provides a result in the full generality of adic spaces over $K$, free from any choice of integral model, or additional assumptions on $X$ or $K$. We note that this analytic setting introduces some additional subtleties due to the difference between $\mathcal{O}_X^+/p^k$-modules on $X_{\text{ét}}$ and modules on the reduction mod $p^k$ of some given formal model.
The step to finite locally free $\mathcal{O}^+$-modules in the $\nu$-topology is new: It is relevant because $X_{\text{proét}}$ is only defined for locally Noetherian $X$, while the $\nu$-topology is available more generally, which is useful even for rigid spaces e.g. in the relative setting studied in [Hen22b].

The proof will take up the entire section as we will also discuss some related topics to prepare the general version for $G$-torsors. The basic idea is to compare locally free $\mathcal{O}^+/\mathcal{m}$-modules for various topologies. For this we can relax the setup of this section:

Let $K$ be any non-archimedean field, not necessarily over $\mathbb{Q}_p$, and let $\varpi \in \mathcal{O}_K$ be any pseudo-uniformiser of $K$. The following Proposition is the technical heart of this section:

**Proposition 2.6.** Let $X$ be an adic space over $K$ such that every object of its étale site $X_{\text{ét}}$ is sheafy. Then for $m = 0, 1$ the map

$$H^0_\nu(X, \text{GL}_n(\mathcal{O}^+/\varpi)) \to H^m_\nu(X, \text{GL}_n(\mathcal{O}^+/\varpi))$$

is an isomorphism for all $r \in \mathbb{N}$. Similarly for $\text{GL}_n(\mathcal{O}^+/\mathcal{m})$. In particular, the functor

$$\nu^* : \{\text{finite locally free } \mathcal{O}^+/\varpi\text{-modules on } X_{\text{ét}}\} \to \{\text{finite locally free } \mathcal{O}^+/\varpi\text{-modules on } X\}$$

is an equivalence of categories, where $\nu : X_{\text{ét}} \to X$ is the natural morphism.

**Proof.** To see that the functor is fully faithful, it suffices to see $\nu_* (\mathcal{O}^+/\varpi) = \mathcal{O}^+/\varpi$, i.e. $H^0_\varpi(X, \mathcal{O}^+/\varpi) = H^0_\nu(X, \mathcal{O}^+/\varpi)$ is an (honest, not just almost) isomorphism: Indeed, for any two finite locally free $\mathcal{O}^+/\varpi$-modules $E_1$ and $E_2$ on $X_{\text{ét}}$, we have $\mathcal{H}\text{om}(\nu^* E_1, \nu^* E_2) = \nu^* \mathcal{H}\text{om}(E_1, E_2)$, and locally where the locally free $\mathcal{O}^+/\varpi$-module $\mathcal{H}\text{om}(E_1, E_2)$ becomes trivial, the statement then follows.

Similarly, to see essential surjectivity, it suffices to see that $R^1 \nu_* \text{GL}_r(\mathcal{O}^+/\varpi) = 1$.

These statements, and their analogues for $\mathcal{O}^+/\mathcal{m}$, all follow by approximation arguments: We now axiomatise these and discuss them in detail in the following subsection.

### 2.2. Approximation of non-abelian sheaf cohomology

For the proof of Proposition 2.6, we will use ideas from [Sch13a, §3], [Sch22, Proposition 14.7], [MW22, §2], [Hen21b, Theorem 2.18]. We shall axiomatise the argument as we will apply it several times in the following section. Throughout this subsection, $K$ is any non-archimedean field. We begin by recalling some properties of affinoid tilde-limits of adic spaces:

**Definition 2.7 ([Hub96, (2.4.1)][SW13, §2.4]).** Let $(X_i)_{i \in I}$ be a cofiltered inverse system of affinoid adic spaces over $K$ and let $X$ be an affinoid adic space with compatible maps $X_i \to X$, for all $i \in I$. We write

$$X \sim \lim_{i \in I} X_i$$

if $|X| = \lim_{i \in I} |X_i|$ and if there is a cover of $X$ by affinoid opens $U$ for which the induced map $\lim_{i \in I} \mathcal{O}(\overline{U_i}) \to \mathcal{O}(U)$ has dense image, where the direct limit runs through all $i \in I$ and all affinoid opens $U_i \subseteq X_i$ through which $U \to X_i$ factors.

**Lemma 2.8.** In the situation of Definition 2.7, we have $X_\infty = \lim_{i \in I} X_i$ for the associated diamonds. For the associated quasi-compact quasi-separated étale sites, we have:

$$X_{\text{ét, qcqs}} = 2\lim_{i \in I} X_{i, \text{ét, qcqs}}$$

**Proof.** The first part is [SW13, Proposition 2.4.5], the second [Sch22, Proposition 11.23].

**Definition 2.9.** In the situation of Definition 2.7, if $X$ and all $X_i$ are affinoid, we write

$$X \approx \lim_{i \in I} X_i$$

if already $\lim_{i \in I} \mathcal{O}(X_i) \to \mathcal{O}(X)$ has dense image.
One reason why this stronger notion is useful is the following elementary observation:

**Lemma 2.10.** Suppose that we have cofiltered inverse systems \((X \to X_i)_{i \in I}\) as well as \((X_i \to X_{ij})_{j \in J_i}\) for all \(i \in I\) of affinoid adic spaces over \(K\) such that \(X \approx \varprojlim X_i\) and \(X_i \approx \varprojlim_{j \in J_i} X_{ij}\) for all \(i\). Then \(X \approx \varprojlim_{i \in I, j \in J_i} X_{ij}\).

**Proof.** The condition on topological spaces is clear, the one on global sections follows by sequential approximation of elements. \(\square\)

The crucial technical property of \(\text{GL}_n(O^+/\varpi)\) that we use to prove Proposition 2.6 is:

**Lemma 2.11.** Let \(X \approx \varprojlim_{i \in I} X_i\) be an affinoid perfectoid tilde-limit of affinoid adic spaces over \(K\). Then

\[
H^0_{\text{ét}}(X, \text{GL}_n(O^+/\varpi)) = \varprojlim_{i \in I} H^0_{\text{ét}}(X_i, \text{GL}_n(O^+/\varpi)).
\]

The analogous statement for \(\text{GL}_n(O^+/m\varpi)\) also holds.

**Proof.** Consider the Cartesian diagram of sheaves of sets

\[
\begin{array}{ccc}
\text{GL}_n(O^+/\varpi) & \longrightarrow & M_n(O^+/\varpi) \\
\downarrow \text{det} & & \downarrow \text{det} \\
(O^+/\varpi)^\times & \longrightarrow & O^+/\varpi.
\end{array}
\]

The statement holds for the bottom row by [Heng1a, Lemma 3.10]. It thus also holds for \(M_n(O^+/\varpi)\), and thus for the top left since \(H^0_{\text{ét}}\) preserves finite limits.

The statement for \(O^+/m\varpi = \varprojlim_{n \to 0} O^+/\varpi^{1+n}\) follows by taking colimits. \(\square\)

For later applications, we more generally consider sheaves satisfying an analogous property in a more general setting:

**Setup 2.12.** Throughout the rest of this subsection, let \(\mathcal{C}\) be a full subcategory of (sheafy) adic spaces over \(K\) satisfying the following conditions:

1. \(\mathcal{C}\) contains every perfectoid space \(X\) over \(K\) as well as \(X \times \mathbb{B}^d\) for any \(d \in \mathbb{N}\).
2. For any \(X \in \mathcal{C}\) and any étale morphism of pre-adic space \(Y \to X\) in the sense of [KL15, Definition 8.2.16], the pre-adic space \(Y\) is a sheafy adic space contained in \(\mathcal{C}\).

We endow \(\mathcal{C}\) with the étale topology. The latter requirement ensures that this is well-defined. For example, \(\mathcal{C}\) could be the big étale site \(\text{Sous}_{K,\text{ét}}\) or sousperfectoid spaces over \(K\).

**Definition 2.13.** Let \(F\) be a sheaf of sets on a category \(\mathcal{C}\) satisfying the above properties. We say that \(F\) satisfies the approximation property on \(\mathcal{C}\) if for any affinoid perfectoid tilde-limit \(X \approx \varprojlim_{i \in I} X_i\) of affinoid adic spaces over \(K\), we have \(F(X) = \varprojlim_{i \in I} F(X_i)\).

The goal of this subsection is to show:

**Proposition 2.14.** Let \(F\) be a sheaf of sets on \(\mathcal{C}\) satisfying the approximation property in the sense of Definition 2.13. Then:

1. \(F\) is already a \(v\)-sheaf.
2. If \(F\) is a sheaf of groups, then for any \(X\) in \(\mathcal{C}\), we have \(R^1\nu_* F = 1\) for \(\nu : X_v \to X_{\text{ét}}\).
3. If \(F\) is a sheaf of abelian groups, then \(R^m\nu_* F = 1\) for any \(m \geq 1\).

The application we have in mind for the purpose of this section is to show:

**Corollary 2.15.** Let \(X\) be any sousperfectoid space, then the map

\[
H^n_{\text{ét}}(X, O^+/\varpi) \to H^n_v(X, O^+/\varpi)
\]

is an (honest, not just almost) isomorphism for all \(n \geq 0\).
This is already non-trivial for \( n = 0 \) and \( X \) perfectoid, where almost acyclicity [Sch12, Theorem 6.3.(iv), Proposition 8.8] a priori only gives an almost isomorphism. For rigid \( X \), it is related to [Sch13a, Lemma 4.2], which implies the analogous result for \( X_{\text{pro-et}} \). For strictly totally disconnected \( X \), the Corollaries recovers [MW22, Proposition 2.13] by a similar proof.

**Proof of Proposition 2.14.** We start with a few technical Lemmas about the situation:

**Lemma 2.16.** Let \( F \) be a sheaf of (not necessarily abelian) groups on \( C \) satisfying the approximation property. Then for \( m = 0, 1 \) and any \( X \approx \lim_{i \in I} X_i \) as in Definition 2.13,

\[
H^m_\text{et}(X, F) = \lim_{i \in I} H^m_\text{et}(X_i, F).
\]

If \( F \) is abelian, this holds for any \( m \geq 0 \).

**Proof.** For \( m = 0 \) this holds by definition. For \( m \geq 1 \) it follows by a Čech argument: Every class in \( H^m_\text{et}(X, F) \) is trivialised by a standard-étale cover \( X' \) of \( X \). By Lemma 2.8, this cover arises via pullback from some \( X'_j \to X_i \). For \( j \geq i \) let \( X'_j := X_j \times_{X_i} X'_i \), then by [Heu21a, Lemma 3.13], we have a tilde-limit relation \( X' \approx \lim_{j \to \infty} X'_j \), so this fulfills the assumptions of the lemma and we therefore have \( H^m_\text{et}(X', F) = \lim_{i \in I} H^m_\text{et}(X'_j, F) \). The statement for \( H^m \) now follows by comparing Čech cohomology for the covers \( X' \to X \) and \( X'_j \to X' \).

**Lemma 2.17.** Any morphism \( f : Y \to X \) of affinoid perfectoid spaces over \( K \) arises as the tilde-limit \( Y \approx \lim_{i \in I} Y_i \) of a cofiltered inverse system of sousperfectoid spaces \( Y_i \subseteq \mathbb{B}^d \times X \to X \) that are rational open in unit balls over \( X \). If moreover \( X \) is strictly totally disconnected and \( f \) is a \( \nu \)-cover, then each morphism \( Y_i \to X \) admits a splitting.

**Proof.** This fact is used in the proof of [Sch22, Proposition 14.7]. For a proof, see [Heu21a, Proposition 3.17] and [Heu21b, Lemma 2.23] for the last sentence.

**Lemma 2.18.** Let \( X \) be any affinoid adic space over \( K \). Then:

1. There is an inverse system \( (X_i \to X)_{i \in I} \) of finite étale Galois covers with affinoid perfectoid tilde-limit \( X_{\infty} \approx \lim_{i \to \infty} X_i \).
2. There is an inverse system \( (\overline{X}_i \to X)_{i \in I} \) of surjective étale morphisms with strictly totally disconnected tilde-limit \( \overline{X} \approx \lim_{i \to \infty} \overline{X}_i \).

**Proof.** Part 1 is [Sch22, Lemma 15.3, Proposition 15.4] and goes back to Colmez [Col02, §4].

For part 2, we first recall that by [Sch22, Lemma 7.18], the perfectoid space \( X_{\infty} \) admits an affinoid pro-étale cover by a strictly totally disconnected space \( \overline{X} \). By [Sch22, Proposition 6.5], this means that there is a cofiltered inverse system of étale maps from affinoid perfectoid spaces \( (X_{\infty,l} \to X_{\infty})_{l \in L} \) such that \( \overline{X} \approx \lim_{l \to \infty} \overline{X}_{\infty,l} \). Using Lemma 2.8 and Lemma 2.10, these combine to give the desired system of étale morphisms over \( X \).

**Lemma 2.19.** Let \( Y \approx \lim_{i \in I} Y_i \to X \) be one of the affinoid perfectoid tilde-limits of Lemma 2.17 or Lemma 2.18. Then for any \( m \in \mathbb{N} \), the \( m \)-fold fibre product \( Y^{\times m} \) of \( Y \) with itself over \( X \) exists in the category of uniform adic spaces and is again affinoid perfectoid. Moreover, we still have

\[
Y_{/X}^{\times m} \approx \lim_{i \in I} Y_{i/X}^{\times m}.
\]

**Proof.** In the case of Lemma 2.17 where \( Y \to X \) is a morphism of affinoid perfectoid spaces, this can be seen exactly as in [BGH+22, Lemma 2.8], by approximation of simple tensors.

In the case of Lemma 2.18, the fibre product exists because we see inductively that for any \( k \geq 2 \) the projection \( Y_{/X}^{\times k} \to Y_{/X}^{\times (k-1)} \) is pro-étale over affinoid perfectoid, hence itself affinoid perfectoid. Moreover, for any \( 1 \leq k \leq m \), the map

\[
\lim_{i \to \infty} \mathcal{O}(Y_{/X}^{\times k} \times_{X} Y_{i/X}^{\times (m-k)}) \to \mathcal{O}(Y_{/X}^{\times m})
\]
has dense image as this is true for affinoid pro-étale maps of perfectoid spaces. Second, for any fixed $i$ also
\[
\lim_{j \in I} \mathcal{O}(Y_j \times_X Y_{j/i}(m-1)) \to \mathcal{O}(Y \times_X Y_{i/X}(m-1))
\]
has dense image by [Heu21a, Lemma 3.13]. The statement now follows from Lemma 2.10. □

We can now give the proof of Proposition 2.14: For the reader interested in an exposition of the argument in the special case of $F = \mathcal{O}^+/p^n$, we also refer to [Zav21, Appendix C].

For strictly totally disconnected $X$, part 1 is proved in [MW22, Lemma 2.12]. We repeat the argument here as it ties in well with the general case: Let $F_v := \nu_* \nu^* F$.

**Step 1:** $F \to \nu_* \nu^* F$ is injective for strictly totally disconnected $X$. Let $\beta \in F(X)$ be in the kernel of $F(X) \to F_v(X)$, then it is in the kernel of $F(X) \to F(Y)$ for some $\nu$-cover $Y \to X$. Applying the approximation property to the tilde-limit $Y \approx \lim Y_i$ from Lemma 2.17, we see that $F(Y) = \lim F(Y_i)$, so $\beta$ is already killed by some cover $Y_i \to X$. But Lemma 2.17 also says that the map $Y_i \to X$ has a section, so it follows that $\beta$ is trivial.

**Step 2:** $F \to \nu_* \nu^* F$ is injective for any $X$. Let $\tilde{X} \approx \lim X_i$ be the strictly totally disconnected pro-étale cover of $X$ from Lemma 2.18. Using that $\tilde{F}$ is an étale sheaf, then the approximation property, and finally step 1, we find that the following map is injective:
\[
F(X) \ni \lim_{\to} F(X_i) = F(\tilde{X}) \hookrightarrow F_v(\tilde{X}).
\]
As this factors through $F_v(X)$, this shows that $F(X) \to F_v(X)$ is injective.

**Step 3:** $F \to \nu_* \nu^* F$ is surjective for strictly totally disconnected $X$. Let $\alpha \in F_v(X)$, then there is an affinoid perfectoid $\nu$-cover $Y \to X$ such that $\alpha$ appears in $F(Y)$. By step 2 applied to $F(Y \times_X Y) \hookrightarrow F_v(Y \times_X Y)$, it thus lies in $\tilde{H}^0(Y \to X, F)$. By the same approximation $Y \approx \lim Y_i$ as in step 1, as well as Lemma 2.19 and the approximation property, we have $\tilde{H}^0(Y \to X, F) = \lim \tilde{H}^0(Y_i \to X, F)$. It follows that for $i$ large enough, $\alpha$ already lies in the image of $\tilde{H}^0(Y_i \to \tilde{X}, F) \to F_v(X)$.

We are not quite done yet because we do not require $Y_i \to X$ to be a cover in $\mathcal{C}$. To resolve this, observe that we can endow $\mathcal{C}$ with a “smooth” topology $\mathcal{C}_{sm}$, for which we define covers to be the surjective morphisms $U \to V$ that are given by a composition $U \to V \times \mathbb{B}^d \to V$ where the first map is étale and the second map is the projection. This indeed defines a site, and this “smooth” topology clearly refines the étale one. Restricting these topologies to $X$, we conclude that $\nu : X_{\nu} \to X_\mathcal{C}$ factors through a morphism of sites $\mu : X_{sm} \to X_\mathcal{C}$.

However, since $X$ is strictly totally disconnected, any cover of $X$ in $X_{sm}$ admits an étale refinement. For this reason, we see directly from the explicit definition of the sheafification in terms of the iteratively formed colimit of the 0-th Čech cohomology over all covers (see e.g. [dJ+23, 00W1]) that $\mu^* F(X) = F(X)$. Since $\tilde{H}^0(Y_i \to X, F) \to F_v(X)$ factors through $\tilde{H}^0(Y_i \to X, \mu^* F) = \mu^* F(X)$, this shows that $\alpha$ lies in the image of $F(X) \to F_v(X)$.

**Step 4:** $F \to \nu_* \nu^* F$ is an isomorphism for any adic space $X$. Using the cover from Lemma 2.18.2, we deduce as in step 2 from the approximation property that
\[
F_v(X) = \tilde{H}^0(\tilde{X} \to X, F_v) = \tilde{H}^0(\tilde{X} \to X, F) = \lim \tilde{H}^0(X_i \to X, F) = F(X),
\]
where the second equality uses step 3, as well as step 2 to see that $F(\tilde{X} \times_X \tilde{X}) \hookrightarrow F_v(\tilde{X} \times_X \tilde{X})$ is injective. The third inequality uses Lemma 2.19 to see $F(\tilde{X} \times_X \tilde{X}) = \lim F(X_i \times_X X_i)$.

We now prove the vanishing of $R^m \nu_* F$ for $m \geq 1$ by an intertwined induction on $m \geq 1$: the induction step is first carried out for strictly totally disconnected $X$, then for general $X$. 


Step 5: induction step to prove $R^m \nu_* F = 1$ for strictly totally disconnected $X$. As $X$ is strictly totally disconnected, it has trivial étale cohomology, so we need to show that $H^m_v(X, F) = 0$. Let $\alpha \in H^m_v(X, F)$. By locality of cohomology, there is an affinoid perfectoid $\nu$-cover $Y \to X$ such that $\alpha$ becomes trivial in $H^m_v(Y, F)$.

For the non-abelian case and $m = 1$, we now use the short exact sequence of pointed sets

$$H^1(Y \to X, F) \to H^1_\text{ét}(X, F) \to H^1_v(Y, F).$$

For the abelian case and $m \geq 1$, we more generally have the Čech-to-sheaf spectral sequence

$$H^k(Y \to X, H^j_\text{ét}(-, F)) \Rightarrow H^{k+j}(X, F)$$

for $m = k + j$. In either case, we use the inverse system from Lemma 2.17 and invoke Lemma 2.19 and Lemma 2.16. By induction hypothesis we have for $j < m$ that

$$H^k(Y \to X, H^j_\text{ét}(-, F)) = H^k(Y \to X, H^j_\nu(-, F)) = \lim_{\longrightarrow} H^k(Y_i \to X, H^j_\text{ét}(-, F)).$$

As $Y_i \to X$ is split, the last term vanishes for $k > 0$. Hence only the term for $k = 0, j = m$ contributes. This means that $H^m_v(X, F) \to H^m(X, F)$ has trivial kernel. Hence $\alpha = 0$.

Step 6: induction step to prove $R^m \nu_* F = 1$ for general $X$. Finally, we assume by induction hypothesis that $R^j \nu_* F = 1$ for any $1 \leq j < m$ and any $X$, as well as for $j = m$ if $X$ is totally disconnected, and deduce that $R^m \nu_* F = 1$. For this we again use the cover $\tilde{X} \to X$ from Lemma 2.18.2. As in step 5, we consider the short exact sequence of pointed sets, respectively the Čech-to-sheaf spectral sequence for abelian $F$

$$H^k(\tilde{X} \to X, H^j_\text{ét}(-, F)) \Rightarrow H^{k+j}(X, F)$$

for $k + j = m$. By induction hypothesis, we have

$$H^k(\tilde{X} \to X, H^j_\text{ét}(-, F)) = \lim_{\longrightarrow} H^k(X_i \to X, H^j_\text{ét}(-, F))$$

for all $k > 0$ and $j < m$, by Lemma 2.19 and Lemma 2.16. For $j = m, k = 0$, this equation also holds since by step 5 we have $H^m_v(\tilde{X}, F) = 0$ and $\lim_{\longrightarrow} H^m_\text{ét}(X_i, F) = H^m_\text{ét}(\tilde{X}, F) = 0$. By comparing to the Čech-to-sheaf spectral sequence for the étale cover $X_i \to X$,

$$H^k(X_i \to X, H^j_\text{ét}(-, F)) \Rightarrow H^{k+j}_\text{ét}(X, F),$$

we deduce that $H^m_{\text{ét}}(X, F) = H^m_v(X, F)$, as we wanted to see. □

This finishes the proof of Proposition 2.6.

2.3. Generalised representations on perfectoid spaces. With these preparations, we can now also prove an integral version of the following result of Kedlaya–Liu already mentioned in the introduction (see also [SW20, Lemma 17.1.8]):

**Theorem 2.20 ([KL16, Theorem 3.5.8]).** Let $X$ be a perfectoid space. Then the categories of vector bundles on $X_{\text{an}}$, on $X_{\text{ét}}$, on $X_{\text{pro}\text{-ét}}$, on $X_{\text{apro}\text{-ét}}$ and $X_v$, respectively, are all equivalent.

Our integral version is the following:

**Theorem 2.21.** Let $X$ be a perfectoid space. Then the categories of finite locally free $\mathcal{O}^\times$-modules on $X_{\text{ét}}$, on $X_{\text{pro}\text{-ét}}$, on $X_{\text{apro}\text{-ét}}$ and $X_v$, respectively, are all equivalent.

**Remark 2.22.** We suspect that they might not be equivalent to finite locally free $\mathcal{O}^\times$-modules on $X_{\text{an}}$, or even to finite projective $\mathcal{O}^\times(X)$-modules if $X$ is affinoid perfectoid.

**Remark 2.23.** It follows that finite locally free $\mathcal{O}^\times$-module on strictly totally disconnected spaces are trivial. Already for $\text{Spa}(\mathbb{C}_p)$, this becomes wrong in the almost category: For $c \in \mathbb{R} \setminus \mathbb{Q}$, the module $p^m$ becomes $\mathbb{Q}^\times$ on the $\nu$-cover $\text{Spa}(\mathbb{C}) \to \text{Spa}(\mathbb{C}_p)$, where $\mathbb{C}$ is any extension whose value group contains $c$. We thank Lucas Mann for pointing this out to us.
We will later prove a much more general version in Theorem 4.28, the Proposition being the case of \( G = \text{GL}_n(O^+) \). But for greater clarity in the much simpler case of \( G = \text{GL}_n(O^+) \), we first give them in this case and then later explain how to generalise. We first observe:

**Lemma 2.24.** Let \( X \) be an affinoid perfectoid space. Recall that \( \varpi \) is any pseudo-uniformiser.

1. Let \( V \) be a \( \varpi \)-torsionfree \( O^+ \)-module on \( X_v \), such that \( V = \lim_{\rightarrow} V/\varpi^k \). Assume that there is \( r \in \mathbb{N} \) such that \( V/\varpi^r \cong O^+/\varpi m \) as \( O^+ \)-modules. Then \( V \cong O^+ r \).

2. Let \( V \) be a locally free \( O^+ /\varpi \)-module. If \( V/m \) is free over \( O^+ /m \), then \( V \) is free.

**Proof.** Since \( V \) is \( \varpi \)-torsionfree, we have for any \( k > 0 \) a short exact sequence on \( X_v \)

\[
0 \to mV/\varpi^k m \xrightarrow{\varpi^k} V/\varpi^{k+1} m \to V/\varpi^k m \to 0.
\]

For part 1, assume that \( V/\varpi^k m \) is free of rank \( r \), and let \( v_1, \ldots, v_r \) be any basis of \( V/\varpi^k m(X) \) as a free \( O^+/\varpi^k m(X) \)-module. Consider the long exact sequence

\[
0 \to mV/\varpi^r m(X) \to V/\varpi^{r+1} m(X) \to V/\varpi^r m(X) \to H^1_r(X, mV/\varpi m).
\]

The last term vanishes since \( mV/\varpi m \cong mO^+ /\varpi m \) by assumption and \( H^1_r(X, O^+ /\varpi m) = 0 \) vanishes after tensoring with \( m \). This shows that we can lift the basis vectors to sections \( v_1', \ldots, v_r' \in V/\varpi^{r+1} m(X) \). Starting with \( k = 1 \), we can thus inductively define a map

\[
\phi_{k+1} : O^+ r/\varpi^{k+1} m \to V/\varpi^{k+1} m
\]

which reduces mod \( \varpi^k m \) to \( \phi_k \). By the 5-Lemma, \( \phi_{k+1} \) is an isomorphism. In the limit, due to the completeness assumption on \( V \), we obtain an isomorphism \( \phi = \lim_{\leftarrow} \phi_k : O^+ r \to V \).

To deduce part 2, observe that we can by the same argument lift any basis of \( V/m \) to sections of \( V/\varpi m \). This defines a map \( \phi : O^+ /\varpi m \to V/\varpi m \) that is an isomorphism mod \( m \). If we know a priori that \( V/\varpi m \) is finite locally free, we can consider \( \det \phi \in V/\varpi m \) which is an invertible section mod \( m \), thus invertible. This shows that \( \phi \) is an isomorphism. \( \square \)

**Proof of Theorem 2.21.** We clearly have a chain of fully faithful functors, so it suffices to prove that any locally free \( O^+ \)-module on \( X_v \) is already free étale-locally. For this we may assume that \( X \) is affinoid perfectoid. Then by Proposition 2.6 there is an étale cover \( X' \to X \) on which \( V/\varpi m \) becomes trivial. By Lemma 2.24, already \( V \) is trivial on \( X' \). \( \square \)

**Corollary 2.25.** Let \( X = \text{Spa}(K, K^+) \) where \( K \) is a perfectoid field, then any locally free \( O^+ -\)module on \( X_v \) is trivial.

**Proof.** Any étale covers of \( X \) is a disjoint union of \( \text{Spa}(L, L^+) \to \text{Spa}(K, K^+) \) where \( L|K \) is finite Galois and \( L^+ |K^+ \) is the integral closure, hence faithfully flat. Any \( \nu \)-vector bundle on \( X \) thus defines a finite projective \( K^+ \)-module. This is free as \( K^+ \) is a valuation ring. \( \square \)

As any adic space over \( K \) has a pro-étale perfectoid cover by Lemma 2.18, we deduce:

**Corollary 2.26.** Let \( X \) be any diamond, then the categories of finite locally free \( O^+ \)-modules on \( X_{\text{proét}} \) and \( X_v \) (and \( X_{\text{proét}} \) if \( X \) is locally Noetherian and char \( K = 0 \)) are equivalent.

We now return to the proof of Proposition 2.3. For the second part, we also need the following, which we will also generalise later in Proposition 4.8, by a different proof:

**Lemma 2.27.** Let \( X \) be an adic space over \( K \). Let \( V \) be a \( \nu \)-vector bundle on \( X \). Then étale-locally on \( X \), there is a finite locally free \( O^+ \)-module \( V^+ \) such that \( V^+ |_{\text{ét}} = V \).

**Proof.** We may assume that \( X \) is affinoid. By Lemma 2.18, there is then a pro-finite-étale affinoid perfectoid cover \( X_\infty \to X \) that is Galois for some profinite group \( N \). The pullback of \( V \) to \( X_\infty \) is étale-locally free by Theorem 2.20. By Lemma 2.8 we can replace \( X \) by some étale cover to assume that the pullback of \( V \) to \( X_\infty \) is trivial.
It follows that $V$ is associated to a descent datum on the trivial vector bundle on $X_\infty$, thus defines a class in the first term of the Cartan–Leray sequence (see [Hen22a, Proposition 2.8])

$$0 \to H^1_{\text{cts}}(\mathcal{N}, \text{GL}_n(\mathcal{O}(X_\infty))) \to H^1_\text{pro-\acute{e}t}(X, \text{GL}_n(\mathcal{O})) \to H^1_\text{pro-\acute{e}t}(X_\infty, \text{GL}_n(\mathcal{O})).$$

Let $\rho : N \to \text{GL}_n(\mathcal{O}(X_\infty))$ be any continuous 1-cocycle representing $V$. Let $N_0$ be the inverse image of the open subspace $\text{GL}_n(\mathcal{O}^+(X_\infty)) \subseteq \text{GL}_n(\mathcal{O}(X_\infty))$. Then by continuity of $\rho$, the subspace $N_0 \subseteq N$ is an open neighbourhood of the identity in $N$, so $N_0$ contains an open subgroup $N_1 \subseteq N$. Since $N/N_1$ is finite, this corresponds to a finite étale cover $f : X' \to X$, and the pullback of $V$ to $X'$ is defined by the 1-cocycle $\rho : N_1 \to \text{GL}_n(\mathcal{O}^+(X_\infty))$. By functoriality of the Cartan–Leray sequence this defines an element in $H^1_\text{pro-\acute{e}t}(X', \text{GL}_n(\mathcal{O}^+(X_\infty)))$ whose image in $\text{H}^1_\text{pro-\acute{e}t}(X', \text{GL}_n(\mathcal{O}))$ corresponds to the isomorphism class of $f^*V$. \hfill \Box

We can now prove the equivalence of $v$-vector bundles and generalised representations:

**Proof of Proposition 2.3.** The equivalence between the categories of finite locally free modules in various topologies is Corollary 2.26. We now first consider the functor in part 1: By Proposition 2.6, the $\mathcal{O}^+/p^n$-module $V/p^n$ is already locally free in the étale topology, so that $\nu_*(V/p^n)$ is a locally free $\mathcal{O}^+/p^n$-module on $X_\acute{e}t$. Thus the functor is well-defined.

It is fully faithful by Proposition 2.6 and because any system of compatible morphisms $V/p^n \to W/p^n$ on $X_\acute{e}t$ in the limit induces a unique $\mathcal{O}^+$-linear morphism $V \to W$.

It remains to see that the functor is essentially surjective. Let $(M_n)_{n \in \mathbb{N}}$ be a generalised representation and consider the $v$-sheaf $V := \varprojlim_{n \in \mathbb{N}} v^* M_n$. As the $v$-topology is replete in the sense of [BS15, §3.1] (see [Hen21b, Lemma 2.6]), we have $V/p^n = M_n$. To see that $V$ is finite locally free, let $X' \to X$ be an étale cover that trivialises $M_1$. Choose a pro-étale cover by an affinoid perfectoid $X'' \to X'$. Then by Lemma 2.24, the fact that $V$ is $p$-adically complete and $V/p$ is free on $X''$ implies that $V$ is a finite free $\mathcal{O}^+$-module on $X''$. This shows that $V$ is a finite locally free $\mathcal{O}^+$-module on $X_\acute{e}t$.

It thus remains to prove Proposition 2.3.2 about generalised $\mathbb{Q}_p$-representations. It is clear from Theorem 2.21 that the data in the definition of generalised $\mathbb{Q}_p$-representations translates into gluing data for vector bundles on $X_\text{pro-\acute{e}t}$. We thus have a fully faithful functor

$$\{\text{generalised } \mathbb{Q}_p\text{-representations on } X_\acute{e}t\} \to \{\text{finite locally free } \mathcal{O}\text{-modules on } X_\text{pro-\acute{e}t}\}.$$

This is essentially surjective if any finite locally free $\mathcal{O}$-modules on $X_\text{pro-\acute{e}t}$ comes from a finite locally free $\mathcal{O}^+$-module on an étale cover, which is guaranteed by Lemma 2.27. \hfill \Box

### 3. $G$-torsors for rigid groups $G$ on adic spaces

We now pass from vector bundles to torsors under any rigid analytic group $G$, the $p$-adic analogue of a complex Lie group. We start by discussing some background on rigid analytic group varieties, since non-commutative rigid groups are not so common in the literature. For this reason, and to provide a reference for our sequel articles, we discuss slightly more than is strictly necessary to prove the main result of this article.

From now on, we assume that $K$ is a perfectoid field over $\mathbb{Q}_p$. The characteristic 0 assumption is necessary in this context to obtain a $p$-adic exponential.

**Definition 3.1.** By a rigid analytic group variety, or just rigid group, we mean a group object $G$ in the category of adic spaces locally of topologically finite type over $\text{Spa}(K, K^+)$. We refer to [Far19, §1.2-1.3] for some background on rigid groups, some of which we recall below. Rigid groups have been studied primarily in the commutative case, e.g. in [Lüt95], but we do not assume $G$ to be commutative. We therefore write the group operation multiplicatively as $m : G \times G \to G$, and write $1 \in G$ for the identity. As before, we freely identify $G$ with its associated $v$-sheaf over $K$. This is harmless in characteristic 0 due to:
Lemma 3.2 ([Far19, Proposition 1]). Any rigid group $G$ is smooth. Moreover, there is a rigid open subspace $1 \in U \subseteq G$ for which there is an isomorphism $U \simto \mathbb{B}^d$ of rigid spaces.

Example 3.3. (1) For any algebraic group $G$ over $\text{Spec}(K)$, we get an associated rigid analytic group by analytification. Again, we often identify $G$ with its analytification as well as with the associated $\eta$-sheaf. If $G$ is affine, then the latter can explicitly be described in terms of the algebraic group as being the sheaf $G(\mathcal{O})$ on $\text{Perf}_K$ sending $(R, R^+) \mapsto G(R)$. We are particularly interested in the case $G = \text{GL}_n$. If $G$ is not affine, the description of the $\eta$-sheaf is still true after sheafification.

Two examples that we use frequently throughout are $G = \mathbb{G}_a$, which is the rigid affine line with its additive structure and represents $\mathcal{O}$, as well as $G = \mathbb{G}_m$, which is the rigid affine line punctured at the origin with its multiplicative structure and represents $\mathcal{O}^\times$.

More generally, for any finite dimensional vector space $W$ over $K$, we have the rigid group $W \otimes_K \mathbb{G}_a$. We call any rigid group of this form a rigid vector group.

(2) Let $\mathcal{G}$ be a smooth formal group scheme over $\text{Spf}(K^+)$, i.e. a group object in the category of formal schemes locally of topologically finite presentation over $\text{Spf}(K^+)$ that is smooth over $\text{Spf}(K^+)$. Then the adic generic fibre $\mathcal{G}_q^\text{ad} \to \text{Spa}(K, K^+)$ in the sense of [SW13, §2.2] is naturally a rigid group. We say that a rigid group $G$ over $K$ has good reduction if it arises in this way, i.e. if there is a smooth formal group scheme $\mathcal{G}$ over $\text{Spf}(K^+)$ whose adic generic fibre is isomorphic as a rigid group to $G$.

(3) Assume that $G$ is an algebraic group over $K$ that extends to a smooth algebraic group $G_{K^+}$ over $K^+$. For instance, by a Theorem of Chevalley–Demazure, such a model always exists if $G$ is a split connected reductive group, namely the Chevalley model [SGA3, XXV Corollaire 1.2]. Consider the $p$-adic completion $\mathcal{G}$ of $G_{K^+}$. Then the adic generic fibre of $\mathcal{G}$ is an open rigid subgroup $G^+ \subseteq G$ that has good reduction.

For $G = \mathbb{G}_a$ with its canonical extension to $K^+$, this construction yields the closed unit ball $\mathbb{G}_a^+ \subseteq \mathbb{G}_a$ which represents the $\eta$-sheaf $\mathcal{O}^+$. For $G = \text{GL}_n$, it recovers the rigid open subgroup $\text{GL}_n(\mathcal{O}^+) \subseteq \text{GL}_n(\mathcal{O})$ of integral matrices from the last section.

3.1. The correspondence between rigid groups and Lie algebras. For any rigid group $G$ over $(K, K^+)$, we denote by $\mathfrak{g} := \text{Lie } G := \ker(G(K[X]/X^2) \to G(K))$ the Lie algebra of $G$, defined exactly like for algebraic groups (see also [Far19, §1.2]). Its underlying $K$-vector space is the tangent space of $G$ at the identity, so $\dim \mathfrak{g} = \dim G$. Similarly as for algebraic groups, we can also regard $\mathfrak{g}$ as a rigid vector group over $K$, explicitly given by $\text{Lie } G \otimes_K \mathbb{G}_a$. As usual, we also consider this as a $\eta$-sheaf on $\text{Perf}_K$, explicitly given by any perfectoid $K$-algebra $(R, R^+)$ by $\mathfrak{g}(R) = \ker(G(R[X]/X^2) \to G(R))$. We shall therefore from now on write $\mathfrak{g}(K)$ when we mean the underlying $K$-vector space.

One application of the $\eta$-sheaf perspective is that it immediately shows that we have a rigid analytic adjoint action $\text{ad} : G \to \text{GL}(\mathfrak{g})$, a homomorphism of rigid groups sending $g \in G(R)$ to the $R$-linear automorphism of $\mathfrak{g}(R)$ induced on tangent spaces by the conjugation map $G \times_K R \to G \times_K R$ that sends $h \mapsto g^{-1}hg$. On tangent spaces, this induces a map $\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$ which defines the Lie bracket on $\mathfrak{g}$.

With these definitions, there is a $p$-adic analogue of the Lie group-Lie algebra correspondence in complex geometry, the main difference to the complex case being that one needs to account for the fact that in the $p$-adic setting there are many more open subgroups. While we do not know of a place in the literature where this is discussed in the present setting of rigid analytic groups, the construction is of course essentially classical as it follows immediately from the theory of $p$-adic Lie groups, as developed in [Bou72][Ser06][Sch11]:
Theorem 3.4 ([Ser06, II]). Sending $G \mapsto \text{Lie}(G)$ defines a functor
\[ \text{Lie} : \{ \text{rigid groups over } K \} \to \{ \text{fin. dim. Lie algebras over } K \}. \]

This becomes an equivalence of categories after localising the left hand side at the class of homomorphisms of rigid groups that are open immersions.

Moreover, for any homomorphism $f : G \to H$ of rigid groups for which the morphism $\text{Lie}(f) : \text{Lie}(G) \to \text{Lie}(H)$ is an isomorphism, there are rigid open subgroups $G_0 \subseteq G$ and $H_0 \subseteq H$ such that $f$ restricts to an isomorphism of rigid groups $G_0 \to H_0$.

Proof. Due to Lemma 3.2, there are open neighbourhoods of $1 \in G$ and $1 \in H$ inside of which rigid open subdiscs around $1$ with morphisms of rigid spaces between them are equivalent to open subdiscs of $G(K)$ and $H(K)$ centered at $1$ and analytic maps between them in the sense of $p$-adic Lie groups as defined in [Ser06, II, 2]. In particular, the category on the left becomes after localisation equivalent to Serre’s “analytic group chunks”.

Again by Lemma 3.2, the completion $\hat{G}$ of $G$ at $1$ is isomorphic as a formal scheme to $\text{Spf}(K[[X_1, \ldots, X_d]])$ and inherits the structure of a formal group law. This defines a functor $R : \{ d\text{-dim. rigid groups over } K \} \to \{ d\text{-dim. formal group laws over } K \}$.

On the other hand, sending a formal group law $H$ over $K$ to the tangent space $h$ at the origin defines an equivalence of categories [Ser06, II, 5, §6]
\[ S : \{ d\text{-dim. formal group laws over } K \} \to \{ d\text{-dim. Lie algebras over } K \}. \]

The composition is easily seen to coincide with $\text{Lie}$.

It thus suffices to prove the result for the functor $R$: That $R$ is essentially surjective follows from [Ser06, II, 4, §§, Theorem 1] (or [Sch11, Proposition 17.6]). It becomes full after the localisation by [Ser06, II, 5, §7, Theorem 1]. It is faithful because any morphism from a connected rigid group is determined on any open subgroup by Zariski-density.

The last sentence follows from [Ser06, II, 5, §7, Corollary 1.2]. For an alternative proof of this statement, see also [Far19, §1, Lemme 1.2]. □

3.2. The $p$-adic exponential of a rigid group $G$. For the rest of this section, we fix any rigid group $G$ over $(K, K^+)$. We now give a brief account of the $p$-adic exponential in the non-commutative setting of rigid groups: Exactly like for Theorem 3.4, this is essentially a translation of a classical result from the theory of $p$-adic Lie groups into the setting of rigid groups:

Proposition 3.5. Let $G$ be a rigid group over $K$ and $\mathfrak{g}$ its Lie algebra. Then there is an open $\mathcal{O}_K$-linear subgroup $\mathfrak{g}^\circ \subseteq \mathfrak{g}$ for which there is a unique open immersion of rigid spaces $\exp : \mathfrak{g}^\circ \to G$ with $\exp(0) = 1$ that induces the identity $\mathfrak{g} \to \mathfrak{g}$ on tangent spaces and makes the diagram

\[
\begin{array}{ccc}
\mathfrak{g}^\circ \times \mathfrak{g}^\circ & \xrightarrow{\exp} & G \times G \\
\downarrow \text{BCH} & & \downarrow m \\
\mathfrak{g}^\circ & \xrightarrow{\exp} & G
\end{array}
\]

commute, where BCH is the Baker–Campbell–Hausdorff formula

\[ \text{BCH}(x, y) = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x - y, [x, y]] + \ldots \]

It satisfies the following properties:

(1) For any open subgroup $\mathfrak{g}_1 \subseteq \mathfrak{g}^\circ$, the map $\exp : \mathfrak{g}_1 \to G$ is an isomorphism of rigid spaces onto an open subgroup of $G$ (but not necessarily a homomorphism).

(2) $\exp$ is functorial in $G$ (i.e. after shrinking $\mathfrak{g}^\circ$, the obvious diagram commutes).

Definition 3.6. In particular, part 1 says that $\exp : \mathfrak{g}^\circ \to G$ is an isomorphism onto an open subgroup $G^\circ \subseteq G$. We denote the inverse by $\log : G^\circ \to \mathfrak{g}^\circ$. 

Remark 3.7.  

(1) If $G$ is commutative, then $\text{BCH}(x, y) = x + y$ and the diagram says that $\text{exp}$ is an isomorphism of rigid groups. This case is discussed in [Far19, §1.5].

(2) The subgroup $g^*$ is in general not canonical: Already if $G = GL(W)$ for some finite dimensional $K$-vector space $W$, we need a basis to get a canonical $g^*$. In the following, we shall therefore fix a choice of $g^*$ and thus $G^0$, but this choice will be harmless as we will always be free to replace them by open subgroups. More canonically, one could consider the filtered system of all open subgroups on which $\text{exp}$ is defined.

(3) For $G = GL_n$, the exponential can be explicitly described by the usual formulas: We have $g = M_n$ and for any uniform Huber pair $(R, R^+)$ over $(K, K^+)$, the $p$-adic exponential and logarithm series define continuous homomorphisms

$$\exp : g^o := p^{\alpha_0}mM_n(R^+) \to 1 + p^{\alpha_0}mM_n(R^+), \quad x \mapsto \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\log : 1 + mM_n(R^+) \to M_n(R), \quad 1 + x \mapsto -\sum_{n=1}^{\infty} \frac{(x)^n}{n}$$

which are mutually inverse when restricted to the domain and codomain of the exponential, where we set $\alpha_0 \coloneqq 1/(p-1)$ if $p > 2$ and $\alpha_0 = 1/4$ otherwise. More generally, this describes $\exp$ for linear algebraic groups, or more generally any rigid group $G$ that admits an injective homomorphism $G \hookrightarrow GL_n$: The maps $\exp$ and $\log$ restrict to the closed subgroups $g^o := g \cap p^{\alpha_0}mM_n(O^+)$ and $G^0 := G \cap (1 + p^{\alpha_0}mM_n(O^+))$.

Proof. This is developed in [Bou72, II] and [Sch11, III, Corollary 18.19] for $p$-adic Lie groups instead of rigid groups, but since $p$-adic Lie groups are locally analytic, one can obtain from this a morphism of rigid groups on a neighbourhood of the identity. We sketch the proof:

We use the functor $S$ already mentioned in the proof of Theorem 3.4. Its inverse is given by sending a Lie algebra $\mathfrak{g}$ to the formal group law $F_k := U(\mathfrak{g})^*$, where $U(\mathfrak{g}^*)$ is the $K$-linear dual of the universal enveloping algebra with its natural co-algebra structure.

Applying this construction to $g$, the Theorem of Poincaré–Birkhoff–Witt identifies $U(\mathfrak{g})^*$ with the completed symmetric algebra $K[[\mathfrak{g}^*]]$ of $\mathfrak{g}^*$ [Sch11, Proposition 18.3]: This defines an isomorphism of formal schemes $\psi : F_k \to \widehat{G}_a \otimes g$. Then $\text{BCH}$ expresses the formal group law on $\widehat{G}_a \otimes g$ transported from $F_k$ via $\psi$, see [Sch11, Corollary 18.15].

The crucial calculation is now that $\text{BCH}$ converges on an open ball in $g$. Namely, choose any $K$-basis of $g$ and for $k \geq 0$ let $g_k^o$ be the $p^kM_n(O^+)$-span of the basis vectors, considered as an open ball. Then by [Sch11, Proposition 17.6], $\text{BCH}$ defines for $k \gg 0$ a function

$$\text{BCH} : g_k^o \times g_k^o \to g_k^o$$

that endows $g_k^o$ with the structure of a rigid group with Lie algebra $g$. Let us call this $g_k^o\text{BCH}$.

By the above equivalence of categories $S$, there is now a natural isomorphism between the formal groups associated to $g_k^o\text{BCH}$ and $g$ that is the identity on tangent spaces. By Theorem 3.4 and its proof, this is analytic in a neighbourhood of the identity, i.e. for $k \gg 0$ there is an open subgroup $G^0 \subseteq G$ for which there is an isomorphism of rigid groups $\exp : g_k^o\text{BCH} \cong G^0$. It is clear from the construction that this is functorial. \qed

Corollary 3.8. Any rigid group $G$ has a neighbourhood basis of the identity $(G_k)_{k \in \mathbb{N}}$ that consists of open subgroups $G_k \subseteq G$ whose underlying rigid space is isomorphic to the closed ball $\mathbb{B}^d$ of dimension $d = \dim G$. In particular, each $G_k$ has good reduction.

Proof. This is true for $g$ which is isomorphic as a rigid group to $G_k^o$. By Proposition 3.5.1 it then also holds for $G$. The group $G_k$ has good reduction because taking $O^+$ of $m : G_k \times G_k \to G_k$ defines a formal group scheme structure on the unit ball over $\text{Spf}(K^+)$.

One way to make exp explicit is to use the following consequence of Theorem 3.4:

Corollary 3.9. For any rigid group $G$, there exist $n > 0$ and an open subgroup $G^o \subseteq G$ that admits a homomorphism of rigid groups $G^0 \hookrightarrow GL_n$ that is a locally closed immersion.
Proof. By Ado’s Theorem, there is a faithful representation \( \text{Lie} G \to \text{Lie}(\text{GL}_n) \) for some \( n \).

By Theorem 3.4, it follows that this comes from a morphism \( \rho : G^0 \to \text{GL}_n \) for some open subgroup \( G^0 \subseteq G \). It follows from Proposition 3.5.2 that \( \rho \) is Zariski-closed.

In particular, one can then describe \( \exp \) on \( G_k \) using the explicit formulas in Remark 3.7.

We now record some more properties of \( \exp \) and \( \log \) that we do not need in this article, but that fit naturally in the discussion and that we will use in [Heu22b] in order to compute \( R^n \nu_* G \).

Lemma 3.10. Let \((R, R^+)\) be any \((K, K^+)\)-Banach algebra.

1. If \( A, B \in \mathfrak{g}^0(R) \) satisfy \( [A, B] = 0 \), then \( \exp(A) \) and \( \exp(B) \) commute in \( G(R) \), and
   \[ \exp(A + B) = \exp(A) \exp(B). \]

2. If \( g, h \in G^0(R) \) satisfy \( gh = hg \), then \( [\log(g), \log(h)] = 0 \) and
   \[ \log(gh) = \log(g) + \log(h). \]

3. Let \( g_1 \subseteq g_2 \subseteq g^0 \) and let \( G_1 \subseteq G_2 \subseteq G \) be the images under \( \exp \). If \( g \in G(R) \) is such that \( \text{ad}(g) \) sends \( g_1 \) into \( g_2 \), then conjugation by \( g \) sends \( G_1 \) into \( G_2 \) and for any \( A \in g_1(R) \) we have
   \[ \exp(\text{ad}(g)(A)) = g^{-1} \exp(A)g. \]

Proof. For the first part, if \( [A, B] = 0 \), then \( BCH(A, B) = A + B \), so this follows from the diagram in Proposition 3.5.

For part 3, we can reduce to the universal situation and replace \( R \) by a reduced affinoid \( K \)-algebra. Then we can check the statement on \( K \)-points. Here it follows from functoriality of \( \exp \) applied to the conjugation \( G \to G, h \mapsto g^{-1}hg \).

For part 2, we deduce from part 1 that \( \log([p^n]g) = p^n \log(g) \). Since it suffices to prove that \( p^n \log(g) \) and \( p^n \log(h) \) commute, we may therefore shrink \( G^0 \), and may thus assume by functoriality of \( \exp \) that the restriction of \( \text{ad} \) to \( G^0 \) fits into a commutative diagram

\[ \begin{array}{ccc}
G^0 & \xrightarrow{\text{ad}} & \text{GL}(g)^0 \\
\downarrow \log & & \downarrow \log \\
\mathfrak{g}^0 & \xrightarrow{\text{ad}} & \text{End}(g)^0.
\end{array} \]

By part 3, we know that \( \text{ad}(g)(\log h) = \log(g^{-1}hg) = \log h \). It follows that
\[ [\log(g), \log(h)] = \log(\text{ad}(g))(\log h) = - \sum_{n=1}^{\infty} \frac{(1-\text{ad}(g))^n}{n}(\log h) = 0, \]
where the first equality comes from the diagram since the bottom row defines \([-,-]\).

The displayed formula now follows from applying \( \log \) to the one from 1.

For \( G = \text{GL}_n \) with notation as in Remark 3.7, we slightly more generally have:

Lemma 3.11. Let \( A, B \in p^\infty \mathfrak{m} M_n(R^+) \) and \( N \in M_n(R) \). The following are equivalent:

1. \( AN = NB \)
2. \( \exp(A)N = N \exp(B) \)

Proof. (1) \( \Rightarrow \) (2) is clear from the formula for \( \exp \). To see (2) \( \Rightarrow \) (1), we can check the equality on points, which reduces us to the case of \( (R, R^+) = (K, K^+) \).

If \( N \) is invertible, then \( N^{-1} \exp(A)N = \exp(B) \in 1 + p^\infty \mathfrak{m} M_n(R^+) \) and the statement follows from applying \( \log \). In general, (b) implies that \( \exp(B) \) preserves \( \ker N \), which is equivalent to \( B \) preserving \( \ker N \). In particular, either defines an operator on \( \text{coim}(N) \).

Similarly \( \exp(A) \) preserves \( \text{im}(N) \), which is equivalent to \( A \) preserving \( \text{im}(N) \). It follows that we can reduce to the map \( N : \text{coim}(N) \to \text{im}(N) \), which is an isomorphism.
3.3. G-torsors in the étale and \(v\)-topology. The theme of this section is that we pass from vector bundles to \(G\)-torsors. There are several ways to define these, and we shall mainly use the following:

**Definition 3.12.** Let \(X\) be a sousperfectoid space and consider \(X_\tau\) for \(\tau\) one of ét or \(v\). Then by a \(G\)-torsor \(P\) on \(X_\tau\) we mean a cohomological \(G\)-torsor on the site \(X_\tau\) where \(G\) is regarded as a sheaf on \(X_\tau\) via analytification: Explicitly, \(P\) is a sheaf on \(X_\tau\) with a left action \(m : G \times P \to P\) by \(G = G(\mathcal{O})\) considered as a sheaf of \(X_\tau\) such that \(\tau\)-locally on \(X\), there is a \(G\)-equivariant isomorphism \(G \xrightarrow{\sim} P\). The morphisms of \(G\)-bundles are the \(G\)-equivariant morphisms of sheaves on \(X_\tau\).

We shall also refer to \(G\)-torsors on \(X_{\text{ét}}\) as étale \(G\)-torsors, and to \(G\)-torsors on \(X_v\) as \(v\)-\(G\)-torsors. In either case, we say that the torus is trivial if it is globally on \(X_\tau\) isomorphic to \(G\). It is clear from this definition that \(G\)-torsors on \(X_\tau\) up to isomorphism are classified by the non-abelian sheaf cohomology set

\[
H^1_\tau(X,G),
\]

with the distinguished point corresponding to the trivial torus.

**Remark 3.13.** For \(G = \text{GL}_m\), we recall that there is a natural functor from \(\text{GL}_n\)-torsors on \(X_\tau\) to \(\tau\)-vector bundles (i.e. finite \(\tau\)-locally free \(\mathcal{O}\)-modules) of rank \(n\), that is essentially surjective but not fully faithful: The morphisms of \(\text{GL}_n\)-torsors are precisely the isomorphisms of vector bundles. In particular, the difference is harmless as long as we are only concerned with isomorphisms, for example in the context of moduli stacks.

As usual, there is also a geometric perspective on \(G\)-torsors:

**Definition 3.14.** Let \(X\) be a locally spatial diamond over \(K\) (e.g. an adic space over \(K\)) and let \(\tau\) be one of ét or \(v\). Then a geometric \(G\)-torsor on \(X_\tau\) is a morphism of \(v\)-sheaves \(E \to X\) on \(\text{Perf}_K\) with a left \(G\)-action \(G \times E \to E\) over \(X\) such that there is a \(\tau\)-cover \(X' \to X\) of diamonds with a \(G\)-equivariant isomorphism \(G \times X' \xrightarrow{\sim} E \times X X'\) over \(X'\). The morphisms of geometric \(G\)-torsors are the \(G\)-equivariant morphisms of \(v\)-sheaves over \(X\).

**Remark 3.15.** While it may be true for an adic space \(X\) that geometric \(G\)-torsor on \(X_{\text{ét}}\) are represented by adic spaces, it is definitely not the case that geometric \(G\)-torsors on \(X_v\) are always represented by adic spaces. However, we will see in Corollary 4.29 that they are still always diamonds, which is not immediately obvious from the definition. In particular, one gets an equivalent definition if one takes \(E \to X\) to be a morphism of diamonds.

**Proposition 3.16.** The functor sending a geometric \(G\)-torsor \(E \to X\) to its \(v\)-sheaf of sections \(E \leftarrow X\) defines an equivalence of categories

\[
\{\text{geometric } G\text{-torsors on } X_\tau\} \xrightarrow{\sim} \{\text{cohomological } G\text{-torsors on } X_\tau\}.
\]

Any morphism in either category is an isomorphism. There is a natural fully faithful functor

\[
\{G\text{-torsors on } X_{\text{ét}}\} \hookrightarrow \{G\text{-torsors on } X_v\}
\]

given by sending a \(G\)-torsor on \(X_{\text{ét}}\) to the \(v\)-sheaf of sections of its associated geometric \(G\)-torsor. On isomorphism classes, this is given by the natural map \(H^1_{\text{ét}}(X,G) \to H^1_v(X,G)\).

As one can already see from the case of \(G = \mathbb{G}_m\), discussed in [Heu22a], this fully faithful functor is in general far from being an equivalence.

**Proof.** It is clear that both cohomological \(G\)-torsors on \(X_\tau\) and geometric \(G\)-torsors on \(X_\tau\) satisfy \(\tau\)-descent. It therefore suffices to see that the endomorphisms of the trivial object are identified via the functor: For the trivial cohomological \(G\)-torsor, it is clear that the \(G\)-equivariant morphisms \(G \to G\) over \(X\) correspond to \(g \in G(X)\) via the map \(h \mapsto hg\). Similarly, for the trivial geometric \(G\)-torsor \(G \times X\), any \(G\)-equivariant morphism \(\phi : G \times X \to G \times X\) of \(v\)-sheaves over \(X\) is uniquely determined by the map \(X \xrightarrow{1,\text{id}} G \times X \xrightarrow{\phi,\text{id}} G \times X \xrightarrow{\text{id},\text{id}} G\), hence the endomorphisms of the trivial geometric \(G\)-torsor are also identified with \(G(X)\). \(\square\)
Remark 3.17. If $G$ is a linear algebraic group, there is a third perspective on $G$-torsors: the Tannakian point of view. We sketch it here and refer to [SW20, Appendix to Lecture 19] for details: A Tannakian $G$-torsor on a sousperfectoid space $X$ is an exact tensor functor

$$P : \text{Rep}_G \rightarrow \text{Bun}_{X, \text{ét}}$$

where $\text{Rep}_G$ is the category of algebraic representations $G \rightarrow GL(V)$ of $G$ considered as an algebraic group scheme, and $\text{Bun}_{X, \text{ét}}$ is the category of étale vector bundles on $X$. By [SW20, Theorem 19.5.1], the categories of $G$-torsors and Tannakian $G$-torsors on $X_{\text{ét}}$ are equivalent via the functor that sends a $G$-torsor $P$ to the Tannakian $G$-torsor $V \rightarrow V \times^G P$.

One can deduce using Kedlaya Liu’s Theorem 2.20 that on a perfectoid space $X$, the categories of $G$-torsors on $X_{\text{ét}}$ and $G$-torsors on $X_{\text{ét}}$ are equivalent for linear algebraic $G$.

However, we do not know if the case of $G^+ := G(O^+)$ for linear algebraic groups over $K^+$ could be deduced in a similar way from that of $GL_n(O^+)$ discussed in Section 2: The Tannakian approach works via algebraisation, but for $G^+$-torsors it seems less clear when these come from $G^+$-torsors on $\text{Spec}(R^+)$, already for $G = GL_n$.

4. Reduction of structure group to open subgroups

In the past section, we have seen using the exponential that any rigid group $G$ has a neighbourhood basis of open subgroups. In this section, we study when $G$-torsors in the $v$-topology admit a reduction of structure group to open subgroups. We then deduce the Main Theorem.

4.1. Approximation for quotient sheaves. The key technical result of this section is a generalisation of the approximation property Lemma 2.11 for $GL_n(O^+ / v)$ to all rigid groups and all rigid open subgroups:

**Proposition 4.1.** Let $G$ be a rigid group and let $U \subseteq G$ be any rigid open subgroup (not necessarily normal). Then the sheaf of cosets $G/U$ on $\text{Sous}_{K, \text{ét}}$ satisfies the approximation property of Definition 2.13: For any affinoid perfectoid tilda limit $X = \lim_{i \in I} X_i$, we have

$$G/U(X) = \lim_{i \in I} G/U(X_i).$$

In particular, $G/U$ is already a $v$-sheaf on $\text{Sous}_{K, \text{ét}}$.

**Proof.** We consider the natural map

$$\lim_{i \in I} G(X_i)/U(X_i) \rightarrow G(X)/U(X).$$

Using Lemma 2.8, it makes sense to sheafify this map:

**Claim 4.2.** The map is injective after sheafification on $X_{\text{ét}, \text{qcqs}} = 2 \lim_{i \in I} X_{i, \text{ét}, \text{qcqs}}$.

**Proof.** If $g_1, g_2 \in G(X_i)$ have the same image, then $\delta = g_1^{-1} : g_2 \in G(X_i)$ lands in $U(X) \subseteq G(X)$. To deduce that $\delta$ lands in $U(X_j)$ for some $j \geq i$, we no longer need the group structure on $G$ and use the following argument that we learnt from [Sch12, Lemma 6.13.(iv)]:

**Claim 4.3.** Let $H$ be an affinoid adic space over $K$ and $V \subseteq H$ a rational subspace. Let $X_i \rightarrow H$ be a morphism such that $X \rightarrow X_i \rightarrow H$ factors through $V$. Then $X_j \rightarrow X_i \rightarrow H$ factors through $V$ for $j \gg i$.

**Proof.** For $j \geq i$, let $W_j$ be the pullback of $V$ along $X_j \rightarrow H$, then $W_j \subseteq X_j$ is still rational open, hence affinoid. The assumptions imply that $X \rightarrow X_j$ factors through $W_j$ and induce a homeomorphism $|X| \rightarrow \lim_{j \geq i} |W_j|$. We now use that $(|W_j|)_{j \geq i}$ is an inverse system of affinoid adic spaces, hence $(|W_j|)_{j \geq i}$ is an inverse system of spectral spaces with spectral transition maps. Since $\lim_{j \geq i} |X|/|W_j|$ is empty, it follows from [dJ+23, Tag 0A2W] that $X_j \backslash W_j$ is empty for some $j \geq i$. Hence $X_j = W_j$ and thus $X_j \rightarrow H$ factors through $V$. \qed
We apply the claim locally on $X$: Let $H$ be any affinoid open subspace of $G$ and cover $U \cap H$ by rational opens $V$. As $X_j$ is quasi-compact, finitely many such opens cover the image of $X_j \to U$. Replacing $X$ and the $X_j$ by the respective pullback of $V \subseteq H$, we may assume that $\delta$ lies in $V(X)$. The claim shows that already $\delta \in U(X_j)$ for some $j \geq i$. \hfill \square

It thus remains to prove that the map is surjective. For this we again localise on $G$: Let $\text{Spa}(A, A^+) = V \subseteq G$ be any affinoid open such that $(A, A^+)$ is of topologically finite type over $(K, K^+)$. By [Hub94, Lemma 3.3.(ii)], this means that there is a subring of definition $A_0 \subseteq A^+$ of topologically finite type (hence automatically of topologically finite presentation) over $K^+$ such that $A^+$ is the integral closure of $A_0$ in $A$. Write $X = \text{Spa}(R, R^+)$ and $X_i = \text{Spa}(R_i, R_i^+)$, then any point in $G(\mathbb{X}_\infty)$ that factors through $V$ defines a map $f : A_0 \to R^+$. By [Heu21a, Lemma 3.10], the assumption $X \approx \lim X_i$ implies $R^+/p^n = \lim R_i^+/p^n$ for all $n \in \mathbb{N}$. Since $A_0/p^n$ is of finite presentation over $K^+/p$, the reduction of $f$ mod $p^n$ factors through a morphism

$$A_0/p^n \to R_i^+/p^n.$$

for some $i \in I$. We now use:

**Lemma 4.4.** Let $A$ be a $p$-adically complete flat $K^+$-algebra such that $A[1/p]$ is a smooth affinoid $K$-algebra. Let us assume for simplicity that $\Omega^1_{A[1/p],K}$ is finite free. Then there is $t \geq 0$ such that for any $p$-adically complete $K^+$-algebra $R$ and any homomorphism of $K^+$-algebras $f : A_0 \to R$ with $f \equiv f_i$ mod $p^{n-t}$.

**Proof.** This is a statement about torsion in the cotangent complex: By [Ill71, III.2.2.4], there is for any $s \leq n \leq 2s$ in $\mathbb{N}$ and any morphism $f_n : A \to R/p^n$ an obstruction class

$$o_{n,s} \in \text{Ext}^s_{A/p^n}(\Omega_{A/p^n}^{\mathbb{Z}/p^n}, R/p^n, R/p^n)$$

that vanishes if and only $f_n$ lifts to a morphism $A \to R/p^{n+s}$.

**Claim 4.5.** There is $t \in \mathbb{N}$ independent of $s,n$ such that $\text{Ext}^s_{A/p^n}$ is $p^t$-torsion.

**Proof.** We first note that the inclusion $A \subseteq A[1/p]$ has bounded $p$-torsion cokernel. Similarly for $K^+ \to \mathcal{O}_K := K^\circ$. We may therefore without loss of generality replace $A$ by a ring of integral elements that is of topologically finite presentation over $\mathcal{O}_K$ and $R$ by $R \otimes_{K^+} \mathcal{O}_K$, so that we may assume that $K^+ = \mathcal{O}_K$. In this setting, we can use the analytic cotangent complex for formal schemes and rigid spaces introduced by Gabber–Ramero [GR03, §7]: This is a pseudo-coherent complex $L_{A/\mathcal{O}_K}^{an}$ of $A$-modules such that $H^0(L_{A/\mathcal{O}_K}^{an}) = A_{\mathcal{O}_K}$ and

$$L_{A/\mathcal{O}_K}^{an} \otimes_A A/p^k = L_{A/\mathcal{O}_K}^{an} \otimes_{A/p^k} A/p^k$$

for all $k \in \mathbb{N}$. Moreover,

$$L_{A/\mathcal{O}_K}^{an} \otimes_A A[p^t] = L_{A/\mathcal{O}_K}^{an} \otimes_{A/p^t} A[p^t]$$

due to the assumption that $A[1/p] \to K$ is smooth. Let $\Omega^+$ be any finite free $A$-sublattice of $\Omega_{A/K}$ that contains the image of $\Omega_{A/\mathcal{O}_K} \to \Omega_{A[p^t]/K}$. Then it follows that the cone $C := \text{cone}(h)$ of the canonical map $h : L_{A/\mathcal{O}_K}^{an} \to \Omega_{A/\mathcal{O}_K} \to \Omega^+$ is exact after inverting $p$. Thus by the same argument as in the proof of [Sch12, Proposition 6.10.(iii)], it follows that for all $i \leq 0$, the cohomology $H^i(C)$ is killed by $p^m$ for some $m$ depending on $i$. As $C$ is bounded above, we may therefore choose $t$ such that $H^i(C)$ is killed by $p^t$ for all $i \geq -2$.

We now first apply $\otimes_A A/p^k$ and then $\text{Ext}^s_{R/p^k}(-, R/p^k)$ to the distinguished triangle

$$C \to L_{A/\mathcal{O}_K}^{an} \to \Omega^+$$

and obtain an exact sequence

$$\text{Ext}^1(C \otimes_A R/p^k, R/p^k) \to \text{Ext}^1_{A/p^k}(\Omega^+ \otimes_A R/p^k, R/p^k).$$
The last term vanishes because $\Omega^+$ is a finite free $A$-module by construction. The first term is $p^t$-torsion because the second argument of $\text{Ext}^1$ is concentrated in degree 0, so only the first two terms of $C$ contribute, and the truncation $\tau_{\geq -2}C$ is killed by $p^t$.

Let now $s,n$ be such that $t < s \leq n < 2s$. It is clear from short exact sequences that

$$\text{Ext}^1_{n,s}(p^t, \text{Ext}^1_{n-t,s})$$

sends the obstruction class $o_{n,s}$ to $o_{n-t,s}$. By the claim, this morphism is zero. It follows that we may lift the reduction $f_{n-t} : A \xrightarrow{f} R/p^n \to R/p^{n-t}$ to a morphism $A \to R/p^{n+s-t}$. As $s > t$, we see inductively that we obtain a lift of $f_{n-t}$ to the complete $\mathcal{O}_K$-algebra $R$. □

Since we may after localising assume that $\Omega_{A/K}$ is finite free, the Lemma applies and setting $k = n - t$, we see that there is a sequence of indices $(i_k)_{k \in \mathbb{N}}$ and morphisms $f_k : A_0 \to R^n_{i_k}$ such that $f_k \equiv f \mod p^k$. Since the ring extension $A_0 \to A^+$ is integral with bounded cokernel, it is clear that $f_k$ extends to $f_k : A^+ \to R^n_{i_k}$, and that after re-indexing we may still assume that $f_k \equiv f \mod p^k$. After inverting $p$, the $f_k$ now define a sequence of points in $G(X_{i_k})$. Let $x_k \in G(X)$ be the images of these points under $G(X_{i_k}) \to G(X)$.

Claim 4.6. For $k \gg 0$, the difference $\delta_k := x^{-1} x_k$ lies in $U(X)$.

This will prove that

$$\lim_{\substack{i \in I \\text{finite}}} G(X_i) \to G(X)/U(X)$$

is surjective, which then finishes the proof of the approximation property. That $G/U$ is a $v$-sheaf then follows from Proposition 2.14.

Proof. Let $Z_k \subseteq X$ be the preimage of $U \subseteq G$ under the map $\delta_k : X \xrightarrow{\delta_k} G$. It is clear that $Z_k$ is open. It therefore remains to prove that each $z \in X$ is contained in $Z_k$ for some $k$. From this the claim follows by compactness of $X$.

Let $z \in X$ be any point, and let $\text{Spa}(C,C^+) \to X$ be any morphism where $(C,C^+)$ is a perfectoid field such that $z$ is in the image. The points $x$ and $x_k$ both define maps $\text{Spa}(C,C^+) \to X \to V$ that correspond to two homomorphisms of $K^+$-algebras

$$A^+ \to R^+ \to C^+.$$

By construction, these agree modulo $p^k$. We now consider the affinoid rigid space $V_C := V \times_{\text{Spa}(K)} \text{Spa}(C)$ over $\text{Spa}(C)$ and the topological space $V_C(C)$. Let $W_k \subseteq V_C(C)$ be the subspace of points whose associated morphisms $A^+ \hat{\otimes}_{K^+} C^+ \to C^+$ agree with $x \mod p^k$.

Claim 4.7. Any open neighbourhood of $x \in V_C(C)$ contains one of the $W_k$.

Proof. We reduce to the case of $B^d$, where the statement is clear: For this reduction, embed $V \subseteq B^d$ into some $d$-dimensional ball corresponding to a morphism $K^+(T_1, \ldots, T_n) \to A^+$ with bounded $p$-torsion cokernel, then also the compositions

$$K^+(T_1, \ldots, T_n) \to A^+ \to A^+/p^k \to C^+/p^k$$

agree mod $p^k$ for any two points in $W_k$. The claim follows since $V_C(C) \subseteq B^d(C)$ carries the subspace topology. □

For the base-change $U_C \subseteq G_C$ of $U \subseteq G$ to $C$, this now means that the open neighbourhood $(x \cdot U_C(C)) \cap V_C(C)$ of $x$ contains $W_k$ for $k \gg 0$. Thus $x_k \in W_k \subseteq x \cdot U_C(C)$, showing $\delta_k(z) = x^{-1} x_k \in U_C(C)$, whence $z \in Z_k$. □

This finishes the proof of Proposition 4.1. □
Proposition 4.8. Let $G$ be a rigid group over $K$ and let $U \subseteq G$ be a rigid open subgroup. Let $X$ be a sousperfectoid adic space over $K$ and let $\nu : X_\nu \to X_\et$ be the natural morphism of sites. Then the natural map

$$R^1\nu_* U \to R^1\nu_* G$$

is surjective. In other words, any $G$-torsor $E$ on $X_\nu$ admits a reduction of structure group to $U$ étale-locally on $X$, i.e. there is an étale cover $Y \to X$ and a $U$-torsor $F$ on $Y_\nu$ such that $F \times^U G = E \times_X Y$ as $G$-torsors. If $G$ is commutative, then we more generally have

$$R^k\nu_* U = R^k\nu_* G$$

and $R^k\nu_* (G/U) = 1$ for all $k \geq 1$. In particular, in this case $F$ is unique up to isomorphism.

Proof. We first observe that if $U \subseteq G$ is normal, then we can consider the short exact sequence

$$0 \to R^1\nu_* U \to R^1\nu_* G \to R^1\nu_* (G/U) \to 0$$

and the result follows from the fact that by Proposition 4.1, we may apply Proposition 2.14.2 to $G/U$ and conclude that $R^1\nu_* (G/U) = 1$. If $G$ is commutative, we more generally have $R^k\nu_* (G/U) = 1$ by Proposition 2.14.3, which gives the result in the commutative case.

In general, $G/U$ is only a sheaf of cosets, but we can still make sense of the change-of-fibre

$$E := E \times_G G/U$$

as a fibre bundle on $X_\nu$ with structure group $G$ and fibres $G/U$. For this we can show:

Lemma 4.9. The following are equivalent:

1. $E$ admits a reduction of structure group to $U$.
2. $E$ is isomorphic to $X \times G/U$.
3. $E$ admits a section $s \in E(X)$.

Proof. 1 $\Rightarrow$ 2 $\Rightarrow$ 3 are trivial. To see e $\Rightarrow$ 1, let $F \subseteq E$ be the subsheaf defined as the fibre product

$$F \longrightarrow E$$

$$X \longrightarrow \overline{E}.$$

in v-sheaves on $X_\nu$. Then $F$ is a reduction of structure group of $E$ to $U$. We first observe that the right-action of $U$ on $E$ leaves the reduction map $E \to \overline{E}$ invariant. It follows that the right $G$-action on $E$ restricts to a right $U$-action on $F$.

Let now $f : Y \to X$ be a $v$-cover by a strictly totally disconnected space on which $E$ admits a trivialisation $\gamma : Y \times_X E = Y \times G$. This induces an isomorphism $\overline{\gamma} : Y \times_X \overline{E} = Y \times G/U$ that we can use to regard $f^*s$ as a section of $G/U(Y)$. Using that $G/U(Y) = G(Y)/U(Y)$ since $Y$ is strictly totally disconnected, we can now compose $\gamma$ with an element in $G(Y)$ to ensure that $\overline{\gamma}(s) = 1$.

Then via $\gamma$, the pullback of the above diagram along $Y \to X$ is isomorphic to

$$Y \times_X F \longrightarrow Y \times G$$

$$Y \longrightarrow Y \times G/U.$$

It follows that we have a $U$-equivariant isomorphism

$$F \times_X Y \simto Y \times U$$

as the right hand side is also the fibre product. Thus $F$ is a $v$-topological $U$-torsor on $Y$.

Since the natural map $F \to E$ is $U$-equivariant by construction, it now follows that $F \times^U G \simto E$, hence $F$ is a reduction of structure group of $E$. □

It thus suffice to prove:
Lemma 4.10. There is an étale cover $Y \to X$ over which $\overline{E}$ admits a section.

Proof. We may without loss of generality replace $U$ by a smaller rigid open subgroup. By Corollary 3.8, we may therefore assume that the adic space underlying $U$ is isomorphic to the unit ball $\mathbb{B}^d$. In this case, we can argue similarly as in Proposition 2.14:

We first deal with the case that $X$ is strictly totally disconnected. Let $Y \to X$ be a $\nu$-cover by an affinoid perfectoid space on which there is an isomorphism $\alpha : G \times X \approx X \times E Y$. This induces a class $\psi \in G(Y \times_X Y)$ such that $\overline{E}$ can be described on any $Z \in X$, as the equaliser

$$\xymatrix{ \overline{E}(Z) \ar[r] & G/U(Y \times_X Z) \ar[r]_{\psi \pi_0^{-1}} & G/U(Y \times_X Y \times_X Z).}$$

Approximating $Y \to X$ by sousperfectoid spaces $Y_i \hookrightarrow \mathbb{B}^n \times X$ as in Lemma 2.17, and using that $Y_{X/X} \approx \lim_{\xymatrix{i \in I}} Y_{i/X}$ by Lemma 2.19, it follows by the approximation property for $G/U$ from Proposition 4.1 that

$$\overline{E}(Y) = \lim_{i \in I} \overline{E}(Y_i).$$

This shows that the composition

$$Y \xymatrix{\approx \ar[r]^{1\text{-id}} & G/U \times_X Y \ar[r]^\alpha & \overline{E} \times_X Y \ar[r] & \overline{E}}$$

factors through a morphism $y : Y_i \to \overline{E}$ for some $i$. Since $Y_i \to X$ is split, we thus obtain a $U$-invariant section $X \to \overline{E}$. This proves the Lemma for strictly totally disconnected $X$.

The general case follows from this by a similar approximation argument: Let $X$ be any sousperfectoid space and let $Y \to X$ be the quasi-pro-étale cover $Y \to X$ by a strictly totally disconnected space from Lemma 2.18, i.e. $Y \approx \lim_{i \in I} Y_i$ is a cofiltered tilde-limit of étale surjective maps $Y_i \to Y$. By the first part, there is an isomorphism $\alpha : G/U \times Y \approx \overline{E} \times_X Y$. Again by the approximation property, we have $\overline{E}(Y) = \lim_{i \in I} \overline{E}(Y_i)$. □

This finishes the proof of Proposition 4.8 □

As a further application, the strategy of this section also gives the following useful results:

Proposition 4.11. Let $G$ be a rigid group and let $U \subseteq G$ be any open subgroup. Let $X$ be a sousperfectoid adic space and let $E$ be a $G$-torsor on $X_U$. Let $Y \approx \lim_{i \in I} Y_i$ be a tilde-limit of sousperfectoid spaces with compatible maps $Y_i \to X$. If $E \times_X Y$ has a reduction of structure group to $U$, then there is $i \in I$ such that $E \times_X Y_i$ has a reduction of structure group to $U$.

Proof. By Lemma 4.9 and Lemma 4.10, there is a standard-étale cover $Z \to X$ over which there is an isomorphism $Z \times_X \overline{E} := Z \times G/U$. Using the cover $Z \times_X Y_i \to Y_i$, we thus see that we have an equaliser sequence

$$\overline{E}(Y_i) \to G/U(Z \times_X Y_i) \to G/U(Z \times_X Z \times_X Y_i),$$

compatible for varying $i$. Since we have $Z \times_X Y \approx \lim_{i \in I} Z \times_X Y_i$ by [Hen21a, Lemma 3.13], we can now use Proposition 4.1 to deduce that $\overline{E}(Y) = \lim_{i \in I} \overline{E}(Y_i)$. The statement now follows from Lemma 4.9. □

Corollary 4.12. Let $X$ be a smooth rigid or perfectoid space. Let $S$ be a perfectoid space. Let $E$ be a $G$-torsor on $(X \times S)_U$. Let $\eta : \text{Spa}(C,C^+) \to S$ be any morphism where $C$ is a perfectoid field and assume that the pullback of $E$ to $X \times \eta$ admits a reduction of structure group to $U$. Then there is an étale map $S' \to S$ whose open image contains im($\eta$) such that $E$ admits a reduction of structure group to $U$ over $X \times S'$.

Proof. We first assume that $S$ is strictly totally disconnected, then by Lemma 2.17 we can choose an approximation of $\eta$ of the form $\text{Spa}(C,C^+) \approx \lim S_i \to S$ where each $S_i \to S$ is split. Then $X \times \eta \approx \lim X \times S_i$, hence Proposition 4.11 shows that $V$ admits a reduction of structure group to $U$ on some $X \times S_i$. Since $S_i \to S$ is smooth, its image is an open in $S$,.
which is again strictly totally disconnected. Hence the map is split over its open image, so we obtain an open of \( S \) with the desired property.

We deduce the general case by considering the pro-étale strictly totally disconnected cover \( \bar{S} \approx S_i \rightarrow S \) of Lemma 2.18. The map \( \eta \) then factors through a map \( \tilde{\eta} : \text{Spa}(C,C^+) \rightarrow \bar{S} \). By the first part, there is an open \( W \subseteq \bar{S} \) such that \( E \) admits a reduction of structure group to \( U \) over \( X \times W \). After shrinking \( W \), we may without loss of generality assume that it comes via pullback from some open \( W_i \subseteq S_i \). Applying Proposition 4.11 once more, this shows that \( E \) becomes trivial over \( X \times W_i \) for some open \( W_i \subseteq S_i \).

4.2. \( p \)-adic integral subgroups for rigid groups of good reduction. In this subsection, we take a closer look at the structure of rigid open subgroups in the special case that \( G \) has good reduction. Throughout this subsection we fix a rigid group \( G \) of good reduction as well as a smooth formal model \( \mathcal{G} \) of \( G \) over \( \text{Spf}(K^+) \). The Lie algebra \( \mathfrak{g}^+ \) of \( \mathcal{G} \) is a finite projective \( K^+ \)-module, hence free, and we can consider it as a canonical open \( K^+ \)-lattice \( \mathfrak{g}^+ \subseteq \mathfrak{g} \). As before, we also consider these as \( v \)-sheaves on \( \kappa \) and write explicitly \( \mathfrak{g}^+(K) \) and \( \mathfrak{g}(K) \) for the underlying modules.

**Lemma 4.13.** The \( v \)-sheaf \( G \) on \( \text{Perf}_K \) is the analytic sheafification of the functor \( (R,R^+) \mapsto \mathcal{G}(R^+) \).

**Proof.** For any formal affine open \( \text{Spf} A \subseteq \mathcal{G} \) the adic generic fibre is by definition given by \( (\text{Spf} A_{\eta}^{\text{ad}} = \text{Spa}(A_{\frac{1}{p}}^+, A^+) \) where \( A^+ \) is the integral closure of \( A \) in \( A_{\frac{1}{p}}^+ \). Hence the points \( \mathcal{G}(R,R^+) \) correspond to the homomorphisms of \( K^+ \)-algebras \( A \rightarrow R^+ \).

We can use this to give a natural generalisation of the sheaves \( \text{GL}_n(O^+/p^k \mathfrak{m}) \):

**Definition 4.14.** For any \( 0 \leq k \in \log |K| \), consider the sheaf \( \mathcal{G}_k \) on the big étale site of \( \text{sousperfectoid spaces Sous}_{K,\text{ét}} \) defined by étale sheafification of the presheaf \( \text{Sous}_{K,\text{ét}} \)

\[
Y \mapsto \mathcal{G}(O^+/p^k \mathfrak{m}(Y)).
\]

We denote by \( G_k \subseteq G \) the kernel of the morphism \( G \rightarrow \mathcal{G}_k \).

For \( k = 0 \), the restriction to \( \text{Perf}_K \) of \( \mathcal{G}_k \) is the sheaf \( \mathcal{G}^\text{ad} \) studied in [Heu21b, §5].

**Lemma 4.15.** The following sequence on \( \text{Sous}_{K,\text{ét}} \) is short exact:

\[
0 \rightarrow G_k \rightarrow G \rightarrow \mathcal{G}_k \rightarrow 0.
\]

**Proof.** Left-exactness is clear by definition. It is right exact because \( \mathcal{G} \) is formally smooth over \( K^+ \), so for any \( k > 0 \) and any affinoid \( (S,S^+) \) in \( \text{Sous}_{K,\text{ét}} \), the map \( \mathcal{G}(S^+) \rightarrow \mathcal{G}(S^+/p^k \mathfrak{m}) \) is surjective. For \( k = 0 \), we observe that after passing to an analytic cover, any \( x \in \mathcal{G}(S^+/\mathfrak{m}) \) lifts to \( \mathcal{G}(S^+/p^\epsilon) \) for some \( \epsilon > 0 \) as \( \mathcal{G} \) is locally of topologically finite presentation.

We can describe \( G_k \) in more classical terms as follows: Let \( \mathcal{G} \) be the completion of \( G \) at the origin. This is a formal Lie group. Any local choice of generators of the sheaf of ideals defining the unit section in \( G \) induces an isomorphism of formal schemes

\[
\mathcal{G} \cong \text{Spf}(K^+[[T_1, \ldots, T_d]]),
\]

see [Mes72, p25-26]. In particular, on global sections, \( \mathcal{G} \) defines a formal group law over \( K^+ \).

**Lemma 4.16.**

1. \( G_0 \) is represented by the adic generic fibre \( \mathcal{G}_0^{\text{ad}} \) of \( \mathcal{G} \).

2. The natural morphism \( G_k \rightarrow G_0 \) is isomorphic to the adic generic fibre of the morphism of affine formal schemes defined on global sections by

\[
K^+[[T_1, \ldots, T_d]] \rightarrow K^+[[T_1', \ldots, T_d']] \quad T_i \mapsto p^k T_i'.
\]

In particular, \( G_k \subseteq G \) is represented by a normal rigid open subgroup whose underlying rigid space is isomorphic to an open ball.
The sheaf Proposition 4.17.

continuity, the images of the $T_i$ land in the topologically nilpotent elements $mR^+$. Thus $\varphi$ reduces mod $m$ to the unit section $A \to K^+/[T_1, \ldots, T_d]$ mod $K^+$. Conversely, if $A \to R^+$ factors mod $m$ through the unit section, it already does so mod $p^\epsilon$ for some $\epsilon > 0$ since $A$ is of topologically finite presentation. Thus for all $n \in \mathbb{N}$, the map $A \to R^+ \to R^+/p^n$ factors through $A/I^m$ for some $m$, and in the limit we get the desired map.

For part 2, we similarly observe that if a map $A \to R^+$ factors through $K^+/[T_1, \ldots, T_d]$, then it factors through the unit section after reducing mod $p^b$. Conversely, if it factors through the unit section mod $p^b$, then the induced morphism $K^+/[T_1, \ldots, T_d] \to R^+$ from the first part must send $T_i$ into $p^bR^+$ and thus factors through the displayed map.

That $G_k$ is normal can be proved on the level of sheaves, where it follows from the short exact sequence in Lemma 4.15.

Combined with our work in the previous sections, this shows:

Proposition 4.17. The sheaf $\overline{G}_k$ on $\text{Sous}_{K, \acute{e}t}$ is already a $v$-sheaf. Moreover,

$$R^1v_!\overline{G}_k = 1.$$  

If $G$ is commutative, we more generally have $Rv_!\overline{G}_k = \overline{G}_k$.

Proof. By Lemma 4.15 and Lemma 4.16, we know that $\overline{G}_k = G/G_k$ is the quotient of the rigid group $G$ by the normal open subgroup $G_k$. By Proposition 4.1, it follows that $\overline{G}_k$ is a $v$-sheaf with the desired properties. □

The system $(G_k)_{k \in \mathbb{N}}$ forms a neighbourhood basis of open subgroups and gives us a notion of completeness of $G$, generalising the isomorphism $\text{GL}_n(\mathcal{O}^+) = \varprojlim_k \text{GL}_n(\mathcal{O}^+/p^k\mathfrak{m})$:

Lemma 4.18. The natural map $G \to \varprojlim_k \overline{G}_k$ is an isomorphism of $v$-sheaves.

Proof. It is clear from Lemma 4.16 that $\varprojlim_k G_k = \cap_{k \in \mathbb{N}} G_k = 1$, so the map is injective. To see that it is surjective, suppose we are given a compatible system of elements $x_k \in G/G_k(R, R^+)$ for some perfectoid $(R, R^+)$. Using that the $v$-site is replete, we can inductively find a $v$-cover $(R, R^+) \to (S, S^+)$ such that the $x_k$ are in the image of $\mathcal{G}(S^+/p^k\mathfrak{m}) \to G/G_k(S, S^+)$ and form a compatible system of elements in $\mathcal{G}(S^+/p^k\mathfrak{m})$. On the level of formal schemes, this defines an element $x \in \mathcal{G}(S^+)$ whose image in $G(S, S^+)$ is a preimage of $(x_k)_{k \in \mathbb{N}}$. □

Remark 4.19. Due to the Lemma, one could now generalise Faltings’ notion of generalised representations by defining “generalised $G$-representations” to be compatible systems of $\mathcal{G}_k$-torsors on the étale site. Due to Proposition 4.17 and Lemma 4.18, one then sees exactly like in Section 2 that these are equivalent to $G$-torsors on $X_v$.

We can now identify the subgroups that appeared in the context of the exponential with the integral subgroups $G_k$ of this section in the context of good reduction:

Definition 4.20. For any $0 \leq k \in \log |K|$, we denote by $\mathfrak{g}_k^+ \subseteq \mathfrak{g}$ the subsheaf $p^k\mathfrak{mg}^+$, represented by an open rigid subgroup of $\mathfrak{g}$. We then set $\overline{\mathfrak{g}}_k^+ := \mathfrak{g}^+/\mathfrak{g}_k^+$ on $\text{Sous}_{K, \acute{e}t}$. Since $\mathfrak{g}^+$ is free, this is isomorphic as an $\mathcal{O}^+$-module to $(\mathcal{O}^+/p^k\mathfrak{m}\mathcal{O}^+)^d$ where $d = \dim G$.

Lemma 4.21. There is $\alpha \geq 1/(p-1)$ such that for any $\alpha < k \in \log |K|$, the exponential of $G$ is defined on $\mathfrak{g}_k^+$ and restricts to an isomorphism of rigid spaces

$$\exp : \mathfrak{g}_k^+ \xrightarrow{\sim} G_k.$$
Proof. As the exponential \( g^0 \to G \) induces the identity on Lie algebras, the associated morphism of formal Lie groups over \( K \) is of the form \( F : K[[T_1, \ldots, T_n]] \to K[[T_1, \ldots, T_n]] \) with \( F(T_i) = T_i + (\text{terms of higher degree}) \). As \( F \) converges on some disc centred at 0, this shows that after replacing \( T_i \) by \( T_i' \) on both sides via \( T_i \mapsto p^k T_i' \) for some \( k \gg 0 \), it restricts to \( F : K^+[T_1', \ldots, T_n'] \to K^+[T_1', \ldots, T_n'] \). Passing to the adic generic fibre, we see from Lemma 4.16.2 that the left hand side becomes \( \mathfrak{g}_k^+ \) and the right hand side becomes \( G_k \). \( \square \)

**Definition 4.22.** Recall that \( \alpha_0 := 1/(p-1) \) if \( p > 2 \) and \( \alpha_0 = 1/4 \) otherwise. We denote by \( \alpha \) the infimum of all \( k \geq \alpha_0 \) for which Lemma 4.21 holds. As before, we denote by \( \log \) the inverse of exp.

**Lemma 4.23.** For any \( k > \alpha \), the adjoint action of \( G_k \) preserves \( \mathfrak{g}_k^+ \).

*Proof.\* This follows from Lemma 4.21 and Lemma 3.10. \( \square \)

**Lemma 4.24.** For any \( \alpha < r < s \in \mathbb{Q} \) with \( s \leq 2r - \alpha_0 \in \mathbb{Q} \), the exponential induces an isomorphism of abelian sheaves on \( \text{Sous}_{K, \text{et}} \)

\[
\exp : \mathfrak{g}^+_r/\mathfrak{g}^+_s \xrightarrow{\sim} G_r/G_s.
\]

In fact, we already have an isomorphism \( \exp : \mathfrak{g}^+_r(X)/\mathfrak{g}^+_s(X) \xrightarrow{\sim} G_r(X)/G_s(X) \) for any \( X \in \text{Sous}_K \). We thus get a short exact sequence on \( \text{Sous}_{K, \text{et}} \)

\[
0 \to \mathfrak{g}^+_{s-r} \xrightarrow{\exp} \mathfrak{g}^+_s \to \mathfrak{g}^+_r \to 1.
\]

**Remark 4.25.** For \( G = \text{GL}_n \), this is the natural isomorphism

\[
M_n(p^r m \mathcal{O}^+/p^s m) \xrightarrow{\sim} 1 + p^r m M_n(\mathcal{O}^+)/1 + p^s m M_n(\mathcal{O}^+), \quad x \mapsto 1 + x
\]

which coincides with the exponential because the conditions on \( r \) and \( s \) ensure that

\[
x^2/2! + x^3/3! + \cdots \in p^r \mathcal{O}^+ \text{ for any } x \in p^r \mathcal{O}^+
\]

by the usual estimate \( v_p(x^n/n!) > nv_p(x) - \frac{n-1}{p-1} \). Similarly for linear algebraic groups.

*Proof.\* By the commutative diagram in Proposition 3.5, \( \text{BCH} \) defines on \( \mathfrak{g}_r^+ \) a rigid group structure isomorphic to \( G_r \). It suffice to prove that this agrees with the additive group structure on the quotient \( \mathfrak{g}_r^+/\mathfrak{g}_r^+ \). This is guaranteed by the following Lemma:

**Lemma 4.26.** For any \( r > \alpha \), let \( x, y \in \mathfrak{g}_r^+ \). Then \( \text{BCH}(x,y) \) converges in \( \mathfrak{g}_r^+ \) and

\[
\text{BCH}(x,y) \equiv x + y \mod \mathfrak{g}_s^+
\]

for any \( r < s < 2r - \alpha_0 \).

*Proof.\* This follows from the estimates in [Sch11, (26) and Proposition 17.6]: We have \( \text{BCH} = \sum_{n \geq 1} H_n \) where \( H_n(X,Y) \) is homogeneous of degree \( n \), and \( \|H_n\| \leq |p|^{-(n-1)\alpha_0} \). It follows that for \( n \geq 2 \), we have

\[
v_p(H_n(x,y)) > nr - (n-1)\alpha_0 = n(r - \alpha_0) + \alpha_0 \geq 2r - \alpha_0 \geq s
\]

where we define the valuation \( v_p \) on \( g \) with respect to \( g^+ \). Thus

\[
\text{BCH}(x,y) = x + y + H_2(x,y) + \cdots \in x + y + p^s m g^+
\]

is of the desired form. \( \square \)
4.3. G-torsors on perfectoid spaces. Given our technical preparations, we can now generalise the results on Section 2. First, by a generalisation of Lemma 2.24, small G-torsors on affinoid perfectoid $Z$ are trivial:

**Lemma 4.27.** Let $X$ be any perfectoid space with $H^1_f(X, m\mathcal{O}^+/p^r\mathfrak{m}) = 0$ for any $r > 0$, for example any affinoid perfectoid space, and let $k > \alpha$. Then a G-torsor $V$ on $X_v$ is trivial if and only if the associated $G_k$-torsor $V_k$ on $X_v$ is trivial. In particular,

$$H^1_f(X, G_k) = 1.$$ 

**Proof.** We set $r := k > \alpha > \alpha_0$ and let $s \in \mathbb{R}$ be such that $r < s < 2r - \alpha_0$. We then apply $H^1_f(X, -)$ to the short exact sequence of $\nu$-sheaves in Lemma 4.24. Since $\mathcal{G}^-_{s} = (m\mathcal{O}^+/p^s\mathfrak{m})^{\nu s}$, it follows from the assumption that $H^1_f(X, \mathcal{G}^-_{s}) = 0$. This shows that if $V_k$ is trivial, then so is $V_s$. Replacing $k$ by $s$ and arguing inductively, this shows that $V_s$ is trivial for any $s > k$. The same long exact sequence shows inductively that any element in $\mathcal{G}_k(X)$ can be lifted to $\mathcal{G}_s(X)$ for any $s > k$.

We then use that we have a Milnor exact sequence of pointed sets

$$1 \to R^1\lim_{s \to s} \mathcal{G}_s(X) \to H^1_f(X, G) \to \lim_{s \to s} H^1_f(X, \mathcal{G}_s).$$

Since we have just seen that $(\mathcal{G}_s(X))_{s \in \mathbb{R}}$ is a Mittag-Leffler system, the first term vanishes. This shows that an element of $H^1_f(X, G)$ vanishes if and only if its image in $\lim_{s \to s} H^1_f(X, \mathcal{G}_s)$ vanishes, if and only if its image in $H^1_f(X, \mathcal{G}_k)$ vanishes. This shows the first part.

By Lemma 4.18, we can also deduce that the map $G(X) \to \mathcal{G}_k(X)$ is surjective. The last sentence then follows from the long exact sequence of $H^1_f(X, -)$ on Lemma 4.15.

We can now prove the main theorem of this article, generalising Kedlaya–Liu’s Theorem 2.20 and Theorem 2.21, which were the cases of $G = \text{GL}_n$ and $G = \text{GL}_n(O^+)$:

**Theorem 4.28.** Let $X$ be a perfectoid space and let $G$ be any rigid group over $K$. Then the categories of $G$-torsors on $X_{\text{et}}$ and $G$-torsors on $X_v$ are equivalent.

**Proof.** By Corollary 3.8, there exists a rigid open subgroup $U \subseteq G$ which has good reduction. By Proposition 4.8, the map $R^1\nu_* U \to R^1\nu_* G$ is surjective, so it suffices to prove that $R^1\nu_* U = 1$. We have thus reduced to the case that $G$ has good reduction. By the same argument, we may then further replace $G$ by the open subgroup $G_k$ for $k > \alpha$. Then by Lemma 4.27, we have $R^1\nu_* G_k = 1$.

**Corollary 4.29.** Let $X$ be any adic space over $K$. Then the categories of $G$-torsors on $X_{\text{proet}}$ and $X_v$ are equivalent. Moreover, any geometric $G$-torsor on $X_v$ is a diamond.

**Proof.** The first part follows since by Lemma 2.18, the space $X$ admits a quasi-pro-étale cover by a perfectoid space. For the second part, let $E$ be a geometric $G$-torsor on $X_v$. The statement is étale-local on $X$, so we may assume that $X$ has a pro-finite-étale perfectoid Galois cover $X' \to X$ such that $E \times_X X' \cong G \times X'$. Then the projection $G \times X' \to E$ is a pro-finite-étale cover of $E$ by a sousperfectoid space. Hence $E$ is a diamond.

**References**

[BGH+22] C. Blakestad, D. Gvirtz, B. Heuer, D. Scholdrina, K. Shimizu, P. Wear, Z. Yao. Perfectoid covers of abelian varieties. *Math. Res. Lett.*, 29(3):631–662, 2022.

[Bour2] N. Bourbaki. *Éléments de mathématique. Fasc. XXXVII. Groupes et algèbres de Lie. Chapitre II: Algèbres de Lie libres. Chapitre III: Groupes de Lie. Actualités Scientifiques et Industrielles*, No. 1349. Hermann, Paris, 1972.

[BS15] B. Bhatt, P. Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015.

[Col02] P. Colmez. Espaces de Banach de dimension finie. *J. Inst. Math. Jussieu*, 1(3):331–439, 2002.

[dJ+23] A. J. de Jong, et al. The stacks project. 2023.

[Fal05] G. Faltings. A p-adic Simpson correspondence. *Adv. Math.*, 198(2):847–862, 2005.

[Far19] L. Fargues. Groupes analytiques rigides p-divisibles. *Math. Ann.*, 374(1-2):723–791, 2019.
