Improved quantum annealer performance from oscillating transverse fields

Eliot Kapit
Department of Physics and Engineering Physics, Tulane University, New Orleans, LA 70118

Quantum annealing is a promising application of quantum hardware for solving hard classical optimization problems. The runtime of the quantum annealing algorithm, in absence of noise or other effects such as the constructive interference of multiple diabatic crossings, and at constant adiabatic evolution rate, is proportional to the inverse minimum gap squared. In this article, we show that for a large class of problem Hamiltonians, one can improve in the runtime of a quantum annealer (relative to minimum gap squared scaling) by adding local oscillating fields, which are not amenable to efficient classical simulation. For many hard $N$-qubit problems these fields can act to reduce the difficulty exponent of the problem, providing a polynomial runtime improvement. We argue that the resulting speedup should be robust against local qubit energy fluctuations, in contrast to variable-rate annealing, which is not. We consider two classes of hard first order transition (the Grover problem and $N$-spin transitions between polarized semiclassical states), and provide analytical arguments and numerical evidence to support our claims. The oscillating fields themselves can be added through current flux-qubit based hardware by simply incorporating oscillating electric and magnetic lines, and could thus be implemented immediately.

INTRODUCTION

Quantum annealing [1-5] is one of the most promising near-term applications of quantum hardware to real-world problems. An analog quantum computing scheme, quantum annealers solve hard optimization problems through adiabatically interpolating the Hamiltonian of a cluster of spins (typically, flux qubits [6-8]) from a simple paramagnet along $x$ to a complex spin glass along $z$. Since all terms in the evolution are implemented physically with energy scales large compared to the environmental temperature, quantum annealing can succeed, and achieve measurable quantum entanglement, without error correction [9-19], though performance can be improved by tailoring the problem Hamiltonian to include redundant encoding inspired by error correction protocols [20-24].

However, proving the existence of a quantum speedup from quantum annealing has proved difficult [13, 25-31]. Current hardware implements a stoquastic [32] Hamiltonian, where all off-diagonal terms in the computational basis are real and negative, and this in turn means that much of their operation can be simulated classically with quantum Monte Carlo [31, 33-35]. The problem of adiabatically finding the ground state through evolution of stoquastic Hamiltonians is referred to as StoqAQC. Whether or not the flux qubit hardware (subject as it is to realistic noise) displays superior scaling in problem size $N$ than classically simulated quantum annealing (SQA) is currently disputed [19, 34, 36-40], and the best classical algorithms are similarly competitive with or even exceed the performance of the d-Wave system [41, 42]. More generally, the large-$N$ scaling exponents of quantum annealing schemes against NP-complete problems such as 3SAT are understood only through extrapolation from direct simulation of relatively small problem sizes and may not hold at larger $N$. Non-stoquastic Hamiltonian terms render QMC simulation impossible and are widely regarded as a promising route to achieving a quantum speedup [43-47], though for realistic problems of interest the actual mechanism for such a speedup, and guidelines for how to choose terms to obtain it, are not well understood.

For some highly symmetric “unstructured” problems such as Grover’s search algorithm [48, 51], StoqAQC can recover the full square root quantum speedup by varying the evolution rate to scale as the inverse of the instantaneous gap. However, translating this prediction to demonstrate real world advantages has been stymied by two challenges. First, for general optimization problems the instantaneous gap is not known a priori, and measuring it to the required exponential precision near a first order transition is generally just as difficult as solving the problem itself. Second, in any real implementation local energy fluctuations (such as $1/f$ flux noise) “blur” the trajectory, erasing variable-rate annealing’s advantage.

For the Grover problem (and other “scrambled cost function” problems related to it), evolving the system with a constant schedule loses the quantum speedup [38-40], and for two important and paradigmatic types of first order transitions, allow the true ground state to be found.

In this article, we propose to modify the StoqAQC algorithm through the inclusion of oscillating fields. This new scheme, which we call RFQA (with the implied double meaning of random field quantum annealing and radio frequency quantum annealing), modifies the local transverse field Hamiltonian by independently oscillating the magnitudes (a strategy we refer to as RFQA-M) and/or directions in the $x-y$ plane (referred to as RFQA-D), of each transverse field term. Through analytical arguments and numerical simulation, we will show that these fields dramatically change the evolution of the system, and for two important and paradigmatic types of first order transitions, allow the true ground state to be found.
much more quickly than in constant schedule StoqAQC. These transitions are the paramagnet to Grover state transition of the Grover problem, and transitions between distinct “bit string” states (configurations where each spin has a fixed $z$ eigenvalue, up to small corrections from the transverse field), which model tunneling between spin glass minima in realistic large-$N$ problems.

In particular, we show that RFQA is capable of recovering a quantum speedup in the Grover problem even with a constant annealing schedule, though the predicted scaling of $2^{N/2}$ falls short of the provably optimal $2^{N/4}$. Based on these results, we expect RFQA to outperform StoqAQC for a wide range of hard optimization problems. In contrast to StoqAQC, simulating the oscillating fields classically requires direct simulation of real-time quantum dynamics, which has a computational cost that scales as $2^N$ in both time and memory. Any performance boost obtained through RFQA thus cannot be duplicated in classical simulation. Finally, the oscillating fields themselves can be implemented in current flux qubit hardware through simple oscillating electric or magnetic fields applied to each qubit, and thus are not difficult to engineer.

The remainder of this article is structured as follows. We first discuss Landau-Zener tunneling in the presence of multiple oscillating fields, and show that the average transition rate between states scales as the sum of the squared Rabi frequencies of all the oscillating terms in the appropriate frequency range. We then define the RFQA-D and RFQA-M methods, and sketch their parameters of operation and conditions under which they will succeed. We will show that RFQA-D is capable of yielding a quantum speedup for the Grover problem, but fails to accelerate $N$-spin tunneling between global minima deep in a spin glass phase. In contrast, we find that RFQA-M provides no advantage in the paramagnet to Grover state transition, but provides a clear performance boost for tunneling between semiclassical bit string ground states.

**LANDAU-ZENER TRANSITIONS IN OSCILLATING FIELDS**

Our first goal to establish that the average rate of mixing in a Landau-Zener-like sweep across $N$ oscillating fields scales as the sum of the squared Rabi frequencies of all $N$ tones. To do so, we reconsider the LZ transition from the perspective of Fermi’s Golden rule, and consider the minimum gap $\Delta_{\text{min}} = 2\Delta$ as a perturbation which causes decay from $|0\rangle$ to $|1\rangle$, with energy transferred into an environment with a Lorentzian density of states peaked about $\epsilon = 0$ with narrow, fictitious width $\Gamma'$ (which we can take to zero later). We assume that we sweep from $z$ bias $\epsilon = -W/2$ to $+W/2$ in a time $t_f$. Assuming the sweep is quick enough that we can linearize the transition probability, and that the linewidth is narrow compared to the range of the energy sweep, $\Gamma' \ll W$ we obtain

$$P_1(t_f) \simeq \int_0^{t_f} dt \, |\Omega|^2 \frac{\Gamma'}{\Gamma'/2 + W^2} \left( \frac{t}{t_f} - \frac{1}{2} \right)^2 \approx \frac{4 |\Omega|^2 \arctan \left( \frac{W}{\Gamma'} t_f \right)}{W} t_f$$

If we time average this result we obtain the mean transition rate $\bar{\Gamma}_{10} = 2\pi |\Omega|^2 / W$; exponentiating this recovers the Landau-Zener result.

Now imagine that instead of a simple $H(t) = \epsilon(t) \sigma_z / 2 + \Omega \sigma^x$, we instead have a more complex oscillatory driving element:

$$H(t) = \epsilon(t) \frac{\sigma_z}{2} + \left( 2 \sum_{i=1}^{N} \Omega_i \cos (2\pi \omega_i t + \phi_i) \right) \sigma^x$$

For a single tone ($N = 1$), one can recover the adiabatic result [1] by a simple rotating frame transformation. But for larger $N$, the same Fermi’s Golden rule argument applies, provided that the frequencies $\omega_i$ are well-separated compared to the amplitudes $\Omega_i$ and that all the $\omega_i$ are contained within the energetic range $W$ swept through in a time $t_f$. Taking into account that the rate of transition from $|0\rangle$ to $|1\rangle$ is the same as the rate to be driven back from $|1\rangle$ to $|0\rangle$, assuming that the system begins in state $|0\rangle$ at $t = 0$, we arrive at a final excitation probability $P_1(t_f)$ given by:

$$P_1(t_f) = 0.5 \left( 1 - \exp \left(-4\pi \Gamma_T t_f \right) \right) \frac{\sum_{i=1}^{N} |\Omega_i|^2}{W}$$

This matches [1] for short ramp times, and is also valid in the large $t_f$ limit, though unlike the adiabatic LZ problem the long time asymptotic state is an incoherent mixture of $|0\rangle$ and $|1\rangle$ with equal probability. Tones $\omega_i$ which do not lie in the energetic range $W$ do not contribute to the transition probability. We thus conclude that the mixing rate between states for a single spin in a slowly varying $z$ field, subject to $N$ weak transverse oscillating fields, scales as the sum of the squared Rabi frequencies of all tones, demonstrated numerically in FIG. 1.

**With longitudinal noise**

As observed previously [15, 56, 57], longitudinal noise, in the form of a randomly fluctuating energy difference between the two states, can frustrate the transition and reduce the success probability. First, in the presence of random oscillatory terms, even arbitrarily long evolution times will not produce a unit success probability, since the noise can induce diabatic transitions between the two states. Consequently, unless the noise amplitude...
is weak compared to the minimum gap, the maximum success probability will be at most 1/2, as in the previous case where the transverse field oscillated. Second, the previous treatment can easily account for energetic fluctuations by replacing the fictitious Lorentzian width \( \Gamma' \) with the average magnitude \( W_r \) of the energetic fluctuations. For \( W_r \ll W \), this will have negligible effect (aside from reducing the maximum success probability to 1/2), but as \( W_r \) reaches or exceeds \( W \), the arctan function decreases as \( W/W_r \), leading to an overall solution rate \( \Gamma_T \propto \Delta^2_{\min}/\max \{ W, W_r \} \).

Now, in a multi-qubit system the effect of noise may be significantly worse, as noise which is sufficiently high frequency and/or high amplitude can create local excitations out of the two-state manifold. However, if the noise is low enough in frequency and the local excitation gap (the energy cost to flip a single spin) is constant these events are exponentially suppressed, and the effect of noise is simply to reduce the solution rate by a factor \( 1/W_r \), where \( W_r \propto N^{1/2} \) for a generic \( N \)-qubit system with independent noise sources, depending on the type of transition. This does not change the difficulty exponent of problems where the time to solution increases exponentially in \( N \), again provided that local excitations do not occur.

THE RFQA METHOD AND MECHANISM FOR A QUANTUM SPEEDUP

The RFQA method leaves the problem Hamiltonian alone while modifying the driver Hamiltonian \( H_0 \to H_{D/M} (t) \), where

\[
H_D (t) = -\kappa \sum_{i=1}^{N} \cos [\bar{\alpha}_i \sin 2\pi f_i t] \sigma_i^x + \sin [\bar{\alpha}_i \sin 2\pi f_i t] \sigma_i^y
\]

and

\[
H_M (t) = -\kappa \sum_{i=1}^{N} (1 + \bar{\alpha}_i \sin (2\pi f_i t)) \sigma_i^z \quad (4)
\]

Here, the \( \bar{\alpha}_i \) are dimensionless numbers \( \pm \bar{\alpha} \) and the \( f_i \) are frequencies. By default, we randomize the signs of the \( \bar{\alpha}_i \) and randomly draw each \( f_i \) from the range \( f_i \in \{ f_{\min}, f_{\max} \} \), where \( f_{\min} \) and \( f_{\max} \) are frequencies chosen based on the problem class (typically, both \( f_{\min} \) and \( f_{\max} \) will scale as \( 1/N \)). This box distribution is somewhat arbitrary and chosen for convenience; as we shall see, the precise details of the distribution are generally unimportant in what follows. In RFQA-M the amplitudes and frequencies \( \bar{\alpha}_i \) and \( f_i \) may be chosen randomly or synchronized based on knowledge of the problem structure; it is not possible to achieve \( \bar{\alpha}_i \geq 1 \).

We will show later that for a flux qubit based implementation, the oscillation between \( x \) and \( y \) can be introduced by simply adding a fluctuating electric field to each qubit, for example through capacitive coupling to a microwave line. The amplitude oscillations in RFQA-M can be obtained through oscillating magnetic fields applied to flux qubits with split junctions, adjusting the driving terms applied to a microwave line. The amplitude oscillations in RFQA-M can be introduced simultaneously (and at different frequencies), though we will not consider that situation in this work.

As in StoqAQC, in RFQA the system begins in the ground state of \( H_{D/M} \) (with all oscillating fields turned off). We then simultaneously evolve \( H (t) = (1 - s (t)) H_{D/M} (t) + s (t) H_P \), where \( s (0) = 0 \) and \( s (T_f) = 1 \), and ramp the oscillating fields up to a finite amplitude; this strength should be reached well before expected phase transitions. For this paper we will assume a constant annealing schedule of \( s (t) = t/T_f \), though of course some benefit can still be gained by slowing down in the vicinity of phase transitions, if the locations of the transitions are known.

As one might anticipate from the Landau Zener model of the previous section, the addition of oscillating terms dramatically changes the evolution, and will allow us to find the solution more quickly than in the StoqAQC, at the cost of having a maximum success probability of 1/2 rather than 1 (even as \( T_f \to \infty \)). For an \( N \)-qubit problem, one would naively expect a reduction in runtime proportional to \( N \), since there are \( N \) tones applied to the system. Given that commercial quantum annealing hardware with over two thousand qubits already exists at the time of this writing (the D-Wave Systems model 2000Q), even linear scaling with \( N \) is significant. Remarkably however, higher order processes arising from
coherent quantum many-body dynamics lead to much more substantial improvements for a broad class of problems.

The basic mechanism for the quantum speedup in RFQA is depicted in FIG. 2. Imagine we have $N$ spins driven with oscillating fields with $N$ distinct frequencies $f_i$. For many important problems, the single spin matrix element between competing ground states $|G\rangle$ and $|E\rangle$ from a $\sigma^x_i$ or $\sigma^y_i$ operation scales identically with $N$ when compared to the minimum gap; we will prove this scaling explicitly for the Grover problem and tunneling between bit string state minima later in this work, and conjecture that it is a relatively generic feature of first order quantum phase transitions. This scaling, combined with the arguments of the previous section, immediately suggests a factor of $2N/C$ (where $C$ is a problem and amplitude dependent constant) improvement in the time to solution compared to constant schedule StoqAQC; the factor of 2 comes from both positive frequencies acting before the avoided crossing and negative frequencies acting after it.

Now consider the effect of two spins driven in combination at frequencies $f_1$ and $f_2$, and let the energy $\omega_G - \omega_E$ cross $f_1 + f_2$. In generic problems there will be a resonant process where the ground state $|G\rangle$ absorbs two photons from the oscillating spins to mix with $|E\rangle$, and a simple examination of perturbative corrections suggests that the scaling with $N$ of the Rabi frequency $\Omega_{12}$ should be identical to that of $\Omega_1$, albeit with a reduced prefactor as it arises at second order in the drive amplitudes $\alpha_i$. However, there are $2^2 \binom{N}{2}$ such terms, so at large $N$ this contribution will dominate the linear order term. One can similarly extend the analysis to combinations of $m$ tones, of which there are $2^m \binom{N}{m}$ distinct combinations, all of which will contribute to the solution rate in Eq. 3! We thus conclude that $N$ spin driving produces an exponential number of distinct driven transitions (of which a handful are sketched in FIG. 2), all of which can accelerate transitions in hard optimization problems relative to constant schedule StoqAQC. Further, as argued below, the resulting potential disorder that may arise from randomly varying the transverse field is at most linear in $N$, the resulting potential disorder that may arise from constant schedule StoqAQC. Further, as argued below, the resulting potential disorder that may arise from randomly varying the transverse field is at most linear in $N$; and if the few-spin matrix elements decay to zero much more quickly than $\Delta_{\min}$ itself, for both problem classes we consider in this work, the scaling of the matrix elements and $\Delta_{\min}$ is identical, and indeed, we have never encountered a problem where the $N$-scaling of the two quantities significantly differs. However, such models may be possible, and if the few-spin matrix elements decay to zero much more quickly than $\Delta_{\min}$, RFQA methods will likely not be effective.

Mitigating heating

Before moving onto our concrete results for an RFQA speedup, we would like to address one potential objection head on: the issue of the oscillating fields heating the system. Off-resonant heating, where diabatic transitions induced by the drive fields create physical local excitations (as opposed to virtual excitations that contribute to perturbative multi-spin resonances), is indeed a serious concern for this method, as excitations out of the two-level manifold can potentially place the system in a many-excitation continuum, driving it far from the ground state. Obviously, if the system is driven into higher excited bands the probability of returning to the true ground state could become negligible, so we must take care to keep this from occurring.

One option is to compensate for this with a cold bath,
though the resulting incoherent tunnelings may interfere with the coherent many-body transitions induced by the drive fields and degrade performance. A better solution is to lower the frequencies with increasing $N$. Let us assume for simplicity that the runtime in absence of such unwanted excitations scales as $e^{cN}$, and let $\omega = \omega_0/N^{1+\delta}$, where $\delta$ is a small positive number. To see why this is sufficient to prevent off-resonant heating, let us consider driving a single spin, over a single oscillatory cycle. We can qualitatively model the evolution as a Landau-Zener tunneling model, with a “velocity” $v = \alpha k \omega / k$ parametrizing the rate of evolution between the states at $t = 0$ and $t = \pi / \omega$. Here $k$ is a proportionality constant dependent on the details of the model and driving protocol; we choose this form as the “distance” in Hamiltonian parameter space is proportional to the product of the transverse field strength and oscillation amplitude, traversed at a rate proportional to $\omega$. We also include in our Landau-Zener model a “transverse field” $\Delta_{\text{local}}$ equal to the energy difference between the many-body ground state and the local excitation created by a diabatic transition, which we assume to be $O(1)$. The mean transition probability of creating a local excitation in a single cycle $P_E$ can be found from the Landau-Zener formula, and scales as:

$$P_E \propto e^{-\frac{k^2 \Delta_{\text{local}}^2}{\alpha k \omega}} = e^{-E_0 N^{1+\delta}} \quad (5)$$

We see immediately from this that the error probability per cycle decays exponentially in $\frac{k^2 \Delta_{\text{local}}^2}{\alpha k \omega}$, and thus, to avoid off-resonant heating in a single cycle, we must alter the magnitude or direction of the transverse field adiabatically compared to $1/\Delta_{\text{local}}$, and not $1/\Delta_{\text{min}}$, the true many-body energy gap at the avoided crossing. Provided we are in this adiabatic regime, the per-cycle error rate will decay exponentially with polynomial decreases in $\omega$.

To complete the analysis, we note that there are a total of $e^{cN}/N^{1+\delta}$ cycles in the total evolution, and $N$ spins being driven so the probability of not creating a diabatic error at any time in the evolution is approximately:

$$P_{NE} \simeq (1 - NP_E)^{e^{cN}/N^{1+\delta}} \simeq 1 - \frac{e^{cN-dN^{1+\delta}}}{N^d} \quad (6)$$

The error rate thus vanishes as $N \to \infty$; if the problem runtime is given by a stretched exponential $e^{cN}$, then we simply let $\omega = \omega_0/N^{1+\delta}$. This analysis assumes that the average local excitation gap $\Delta_{\text{local}}$ is constant at large $N$; one can reduce the frequency even more quickly to compensate for this so long as $\Delta_{\text{local}}$ is at least inverse polynomial in $N$.

Of course, a physical implementation of the oscillating fields may introduce secondary heating concerns not captured in the spin model of driver and problem Hamiltonians. For example, oscillating electromagnetic fields in a flux qubit implementation can heat the environment around the qubits themselves, increasing thermal noise and unwanted excitations. Such problems are beyond the scope of this work, though they could pose important classical cooling challenges in a physical realization of RFQA; we simply argue that at the level of the spin Hamiltonian, potential off-resonant heating issues can be easily addressed RFQA method by polynomially decreasing the frequencies of the applied tones as the problem size increases.

**RFQA-D AND THE GROVER PROBLEM**

To demonstrate the power of oscillating fields in quantum annealing, we will apply the RFQA-D formulation to the Grover problem [38][51]. In the AQC context, the Grover oracle Hamiltonian is a projector onto a single (unknown) bit string $|G\rangle$, with

$$H_G = -\frac{N}{2} |G\rangle \langle G| \quad (7)$$

We let $\kappa = 1/2$, choose our annealing schedule to be $H(t) = (1 - s(t)) H_0 + s(t) H_G$, and let $|0\rangle$ be the paramagnetic ground state of $H_0$. We will use $H_0 = -\frac{1}{2} \sum_{i=1}^N \sigma_i^x$ in diagonalizing the system; it will be replaced with $H_D$ in RFQA. The problem difficulty is controlled by a single first order paramagnet to spin glass transition at $s_c = 0.5 + O(1/N)$. The minimum classical number of queries to solve this problem is $O(2^N)$, and the optimality of Grover’s algorithm [39] ensures that, formulated as we have here, quantum methods cannot solve it in less than $O(2^{N/2}/N)$ time. This square root speedup can be arrived at through adiabatically annealing with a variable rate schedule, or by diabatically jumping to the transition point $s_c$ and waiting a time $T \propto \Delta_{\text{min}}^{-1} \propto (2^{N/2}/N)$ for the states to be mixed; in contrast, naive evolution at a constant rate requires $T \propto \Delta_{\text{min}}^{-2}/W \propto 2^N/N$, eliminating the quantum speedup. Explicitly,

$$\Delta_{\text{min}} \approx (1 - s_c) N 2^{-N/2}, \quad (8)$$

where $s_c \approx 1/2$; this result will be derived below. The diabatic jump speedup can be recovered in RFQA by halting the evolution at some $s$ near the transition where the energy gap is known exactly, and driving the system at that precisely that frequency.

However, both variable rate annealing and diabatic jumps require a detailed knowledge of the energy spectrum to exponential precision, something that is extremely difficult in more general problems and impossible in current analog quantum hardware, where $1/f$ noise causes constant random drift in the energy of competing ground states. So in applying RFQA-D to the Grover oracle we will simply compare the performance of constant annealing schedules, with and without oscillating fields. Though it has no realistic analog implementation,
FIG. 2: Basic mechanism of RFQA. In (a), we schematically depict some of the transitions induced by driving three spins \( \{i, j, k\} \) with oscillating fields as the system evolves across a transition. As the system evolves with the annealing parameter \( s \), the energy difference \( E_1 - E_0 \) of the two crossing states passes a series of resonant transitions induced by the driving, causing the two states to rapidly mix. Due to quantum many-body effects, additional resonant transitions occur at combinations of frequencies from multiple driven spins; only two-tone transitions are shown here, though combinations of tones up to \( N \)th order will contribute to the total transition rate in (3). For the two classes of transitions considered in this work, the strength of an \( n \)th order resonant tone combination decreases exponentially in \( n \); however, there are \( 2^n \binom{N}{n} \) such terms. When the contributions at all orders are summed, for problems characterized by a hard first order transition \( \Delta_{\text{min}} \propto e^{-cN} \) the net effect is to reduce the difficulty exponent, providing a polynomial quantum speedup over cases where the transverse fields do not oscillate. In (b), we depict an example hardware implementation of our scheme. For flux qubit based quantum annealers, the magnitude of the transverse field can be adjusted by threading an external flux \( \Phi(t) \) through the qubit’s SQUID loop, implementing the RFQA-M method. Likewise, the direction of the transverse field in the \( x-y \) plane can be oscillated through an applied voltage \( V(t) \), implementing RFQA-D.

we study this model because of its simplicity and its empirical value, as it is the simplest case of a larger class of scrambled cost function and random energy models \([60–62]\), which are computationally hard quantum spin glasses. Many problems with a first order paramagnet to spin glass transition are phenomenologically similar to the Grover problem, so general methods for speeding up the time to solution in Grover should be widely applicable.

The simplicity of the problem Hamiltonian allows us to predict the scaling of matrix elements analytically, though due to the paramagnetic structure the analysis is more complex than in the original formulation of this problem, where \( H_0 \) was a simple projector onto the paramagnetic ground state and did not have local spin structure \([49]\). To find an analytic solution, we consider a basis consisting of the Grover state \( |G\rangle \) and all the paramagnetic states with fewer than \( N^* < N \) local excitations; the choice of sufficiently large \( N^* \) is arbitrary as higher energy states have increasingly small contributions to the low energy physics (see Eq. 11 below), and we choose it to avoid concerns about working in an overcomplete basis. Throughout, we will perform all calculations to leading order in \( 2^{-N/2} \) and drop all terms which decay as \( 2^{-N} \).

We let \( |0\rangle \) be the paramagnetic ground state, and denote the excited states by \( |i\rangle \equiv \sigma_z^i|0\rangle \), \( |ij\rangle \equiv \sigma_z^i\sigma_z^j|0\rangle \), \( |ijk\rangle \equiv \sigma_z^i\sigma_z^j\sigma_z^k|0\rangle \), and so on. We first orthogonalize the set of states \( \{|G\rangle, |0\rangle, |i\rangle, |ij\rangle, \ldots \} \) through a Gram-
Schmidt process. Under this mapping

\[ |G\rangle \rightarrow |G\rangle, \]
\[ |0\rangle \rightarrow |0\rangle - \frac{1}{2N^{2/2}} |G\rangle, \]
\[ |i\rangle \rightarrow |i\rangle - \frac{c_i}{2N^{2/2}} |G\rangle + O(2^{-N}), \]
\[ |ij\rangle \rightarrow |ij\rangle - \frac{c_ic_j}{2N^{2/2}} |G\rangle + O(2^{-N}) \ldots \]

where we define \( c_i \equiv -(G|\sigma^z_i|G) = \pm 1 \). In this new basis, all states are orthogonal.

We now wish to diagonalize \( H(s) = (1-s) H_0 + s H_P \).

We first note that in absence of mixing between \( |G\rangle \) and the paramagnetic states,

\[ E_0 \approx -(1-s) \frac{N}{2}, \quad E_G \approx -s \frac{N}{2} - \frac{(1-s)^2}{2}. \]

We further observe:

\[ \langle G| H_0 |0\rangle = -\frac{N}{2N^{2/2+1}}, \quad \langle G| H_0 |i\rangle = -\frac{(N-2)c_i}{2N^{2/2+1}}, \]
\[ \langle G| H_0 |ij\rangle = -\frac{(N-4)c_ic_j}{2N^{2/2+1}}, \ldots \]
\[ \langle 0| H_0 |0\rangle = -\frac{N}{2} + O(2^{-N}), \]
\[ \langle 0| H_0 |i\rangle = -\frac{N}{2} + O(2^{-N}), \ldots \]
\[ \langle i| H_0 |0\rangle = \langle i| H_0 |j\rangle = \langle i| H_0 |k\rangle \ldots = O(2^{-N}), \]
\[ \langle i| H_P |0\rangle = \langle i| H_P |j\rangle = \langle i| H_P |k\rangle \ldots = O(2^{-N}), \]
\[ \langle G| H_P |0\rangle = \langle G| H_P |i\rangle = \langle G| H_P |ij\rangle \ldots = 0. \]

With these relations in hand, we can readily diagonalize \( H(s) \) and predict matrix elements. We are particularly interested in the limit where \( E_0 - E_G \) is polynomially, but not exponentially, small in \( N \), e.g. when \( s \) is near the transition at \( s_c = 1/2 + O(1/N) \). Working to leading order in \( 2^{-N/2} \), we find new basis states

\[ |0'\rangle = |0\rangle + 2^{-\frac{N}{2}} \frac{(1-s)N}{2(E_0 - E_G)} |G\rangle, \]
\[ |i'\rangle = |i\rangle + 2^{-\frac{N}{2}} \frac{(1-s)(N-2)c_i}{2(E_0 + (1-s) - E_G)} |G\rangle, \]
\[ |ij'\rangle = |ij\rangle + 2^{-\frac{N}{2}} \frac{(1-s)(N-4)c_ic_j}{2(E_0 + 2(1-s) - E_G)} |G\rangle, \]
\[ |G'\rangle = |G\rangle - 2^{-\frac{N}{2}} \left( \frac{(1-s)N}{2(E_0 - E_G)} \right) |0\rangle \]
\[ -2^{-\frac{N}{2}} \sum_{i=1}^{N} \frac{(1-s)(N-2)c_i}{2(E_0 + (1-s) - E_G)} |i\rangle \]
\[ -2^{-\frac{N}{2}} \sum_{i<j}^{N} \frac{(1-s)(N-4)c_ic_j}{2(E_0 + 2(1-s) - E_G)} |ij\rangle + \ldots \]

These expressions neglect corrections which scale as \( 2^{-N} \).

From the corrections to \( |0\rangle \) and \( |G\rangle \), we can readily read off the minimum gap, \( \Delta_{\text{min}} = 2\Omega_0 = (1-s_c) N 2^{-N/2} \).

We will now compute the resonant matrix elements for the mixing of \(|0'\rangle \) and \(|G'\rangle \) through the oscillating \( y \) fields.

Let us first consider the transition rate \( \Omega_1 \) for the mixing of \(|0'\rangle \) and \(|G'\rangle \) through an oscillating field Hamiltonian driving a single spin as \( \alpha \frac{1}{2} \sigma^y |G\rangle \sin(2\pi f_t t) \). From \([11]\), with \( f_t = E_G - E_0 \) we can immediately read off the driving amplitude \( \Omega_1 \) as

\[ \Omega_1 \approx \frac{1}{4} \langle G'| \sigma^y |0\rangle \approx \frac{(1-s)^2(N - 2) 2^{-\frac{N}{2}}}{8(E_0 + (1-s) - E_G)} \approx \frac{\alpha}{4} (1-s_c)(N - 2) 2^{-N/2}. \]

Taking the limit of \( (E_G - E_0) \rightarrow 0 \) and \( s \rightarrow s_c \) yields the rate of the second line. We take this limit as we expect the drive frequencies inducing these transitions to decrease polynomially with \( N \) (remaining large compared to \( \Delta_{\text{min}} \), which decays exponentially), for reasons explained below. The denominator thus reduces to powers of \((1-s)\), which is constant as \( N \) increases.

Now imagine we drive two spins at amplitudes \( \alpha \) and frequencies \( f_1 \) and \( f_2 \). If \( |f_1 \pm f_2| \approx E_G - E_0 \) the system will be resonantly driven between \(|0'\rangle \) and \(|G'\rangle \) through a two-spin process, where one spin absorbs an off-resonant photon, virtually exciting it into the \( |i\rangle \) manifold, and the second spin then absorbs a second photon, promoting it to \(|G'\rangle \) through the component of \( |G'\rangle \) along \( |ij\rangle \). Noting that combinatorics will provide a factor of two increase (from the order in which photons are absorbed),

\[ \Omega_2 \approx \frac{\alpha^2 (1-s)^3 (N-4) 2^{-\frac{N}{2}}}{8(1-s - f_{1/2}) (E_0 + 2(1-s) - E_G)} \approx \frac{(\alpha^2)^2 (1-s_c) (N-4) 2^{-N/2}}{2}. \]

Again, the rate in the second line is in the limit \( (E_G - E_0) \rightarrow 0 \) and \( s \rightarrow s_c \).

We can further extend these results to three spins, driven at frequencies \( f_{1/2} \). The same arguments yield

\[ \Omega_3 \approx (\alpha^3) (1-s_c) (N-6) 2^{-N/2} \]

as the factor of 6 from combinatorics balances the denominator of \( 6(1-s)^3 \). Extending this result to \( m \) spins, we finally conclude:

\[ \Omega_m \approx (\alpha^m (1-s)^m) \frac{m(1-s)(N-2m) 2^{-\frac{N}{2}+1}}{4m} (E_0 + m(1-s) - E_G) \approx \frac{\alpha^m}{4m} (1-s_c)(N-2m) 2^{-\frac{N}{2}+1}. \]

We have numerically verified this scaling in FIG. 3(a). To do so, we directly simulated driven evolution in a reduced basis, with \( \alpha = 1 \) (corresponding to a one-photon driving amplitude of 0.88, as explained below).
To predict the maximum performance of RFQA-D we must find the optimal value of $\bar{\alpha}$ (which we call $\bar{\alpha}_m$). To find $\bar{\alpha}_m$, we note that the tones that contribute to the sum of driven transitions come out of the expansion of $\sin (\bar{\alpha} \sin (2\pi f_i t))$. Taking into account higher order terms,

$$\sin (\bar{\alpha} \sin (2\pi f_i t)) = g_1 e^{2\pi i f_i t} + g_3 e^{6\pi i f_i t} + ... + \text{H.c.}(15)$$

All of these terms contribute to the driven many-photon transitions, though in practice terms at fifth order and higher are negligible. We can therefore find $\bar{\alpha}_m$ by maximizing $g_1 \bar{\alpha}^2 + g_3 \bar{\alpha}^3$, since both terms enter quadratically into the sum of many-photon transitions; in what follows, $g_1 \equiv 2\bar{\alpha}$. We arrive at an effective maximum drive amplitude

$$\alpha_m \simeq 1.18,$$

at a raw drive strength $\bar{\alpha} \simeq 0.59\pi$. Values of $\bar{\alpha}$ larger than this are counterproductive, as they produce weaker coefficients at low orders and increase the possibility of generating off-resonant excitations. Plugging $\alpha = \alpha_m$ into the sum of transition rates predicted below thus allows us to estimate the optimal performance of RFQA-D for a given problem.

We now return to the Grover problem, and sum the contributions of all the terms at each order which will contribute to the multifrequency LZ prediction of Eq. [3] At zeroth order, we the minimum gap itself, with $\Omega_0 \simeq (1 - s_c) N 2^{-N/2}/2 = \Delta_{\text{min}}/2$. At first order we have a total of $2N$ contributions, $N$ from positive frequencies ahead of the transition and $N$ from negative frequencies after the minimum gap has been crossed. At second order we have $2^2 \binom{N}{2}$ independent terms, as each contribution of two frequencies (positive or negative) drives an independent transition between the two states. At third order we have $2^3 \binom{N}{3}$, and so on; summing all contributions as in Eq. [3] results in a total solution rate of

$$\Gamma_T \simeq \frac{(1 - s_c)^2 N^2}{2^N W} \sum_{n=0}^{N} \frac{(\alpha_m^2/8)^n}{(1 - 2N)^2 \binom{N}{n}}$$

$$\simeq \frac{N \alpha_m^2/2N + (1 - \alpha_m^2/8)^2}{(1 + \alpha_m^2/8)^2} e^{-\left(\log 2 - \frac{\alpha_m^2}{8}\right) N}. (17)$$

In the last step we used $W \simeq N$ and $s_c \simeq 1/2$. Plugging in $\alpha = \alpha_m$ from [16], we arrive at an average time to solution of $2^{0.747N}$, which is obviously worse than the optimal runtime of $2^{N/2}$ achievable by variable rate annealing. Nonetheless, it still represents a quantum speedup, and one which requires no detailed knowledge of the instantaneous gap. Based on the arguments earlier in this work, it will thus be at least somewhat resilient to energy fluctuations from $1/f$ noise. While the Grover oracle is not a realistic model for analog quantum annealing, we take these results as suggestive that RFQA can provide a useful speedup near difficult paramagnet to spin glass transitions as well.
Details and analysis

Before moving on to bit string state transitions, it is worth making three further points of interest here. First, for both the Grover problem (see Eq. 12) and the spin glass transition, the scaling of the site-averaged single-spin matrix element \( \langle \tilde{O} \rangle \equiv \langle \langle E \rangle \sigma^x_i | y \rangle \rangle_i \), with \( N \) near the transition (where the gap is small but large compared to \( \Delta_{\text{min}} \)) is identical to the scaling of the minimum gap itself. We conjecture that this behavior is generic near first order transitions in relevant problems (and indeed, we have never found a model where the single spin matrix element and minimum gap diverge from each other as \( N \) increases), though we offer no proof of this conjecture and one could likely design a model where this is not the case. That said, in cases where this conjecture holds, this suggests that, at large enough \( N \), even with no oscillating fields at all, constant schedule annealing with a cold bath has greater computational power than constant schedule annealing in a closed system, by a factor of \( N \left( g / \kappa \right)^2 \), where \( g \) is the system-bath coupling strength (see also [63, 64]). However, because the cold bath does not induce coherent multi-spin transitions the way that a combination of oscillating tones can, these predictions suggest it cannot reduce the difficulty exponent of the problem through relaxation processes alone.

Second, the RFQA-M method (amplitude oscillations in the transverse field, described below) will fail to produce a quantum speedup in this problem. This is because locally varying the magnitude of a \( \sigma^x_i \) field cannot create an excitation (real or virtual) in the paramagnet Hamiltonian (this is true up to irrelevant corrections that scale as \( 2^{-N} \) when the Grover Hamiltonian is included). Consequently, there is no mechanism to resonantly combine tones beyond 1st order, and the exponential proliferation of tone combinations depicted in FIG. 2 will not occur. However, will see in the next section that RFQA-D fails for transitions between bit string states, indicating that neither scheme is a “black box” that accelerates all first order phase transitions in AQC.

Finally, in contrast to the RFQA-M protocol studied in the next section, we do not include direct numerical simulations of annealing the Grover problem under RFQA-D in this work. This stems from the constraint that the \( f_i \) must be low enough to avoid populating off-resonant single-spin excitations, detailed in the previous section concerning mitigating heating. When compared to the modest single spin excitation energy (\( \Delta_{\text{local}} \approx 0.5 \) near \( s_c \)), \( \Delta_{\text{min}} \) only becomes dramatically smaller at fairly large \( N \); even for \( N = 14, \Delta_{\text{local}} / \Delta_{\text{min}} < 10 \). This means that we must consider larger \( N \) to be in a regime where off-resonant excitations do not erase any performance advantage, and at such large \( N \) the exponential cost of directly integrating the Schrodinger equation, for long times and with enough random samples to obtain good statistics, is significant. Likewise, the runtimes must be relatively long to tease out the advantage of RFQA-D since one must adequately sample the drive tones, which is impossible unless \( t_f \) is large compared to \( 1 / f_i \). However, direct verification of the driven matrix elements (Eq. 14 shown in FIG. 3), combined with the fact that oscillations in other spins which are far off resonant from a particular tone combination do not appreciably interfere with the evolution (demonstrated indirectly below), allows us to be confident that the annealing speedup in Eq. 17 should be correct in the large-\( N \) limit.

An alternative protocol for solving the Grover problem with RFQA-D

To overcome some of the shortcomings listed above we consider a different protocol for demonstrating the RFQA-D speedup in the Grover problem. In comparison to constant schedule StoqAQC this method has a runtime which is longer by an \( N \)-independent prefactor; this longer runtime will allow us to better sample low-frequency drive tones (reducing heating issues), and in return it can clearly display the superior \( N \)-scaling of RFQA-D when oscillating fields are introduced, at small enough \( N \) to be amenable to classical simulation. The method, which could be applicable in other AQC contexts, works as follows:

(i) Initialize the system in the paramagnetic ground state of \( H_0 = -\sum_{i=1}^{N} \sigma^x_i / 2 \).

(ii) Pick a value of \( s \in \{ s_{\text{min}}, s_{\text{max}} \} \), where \( s_{\text{min}} \) and \( s_{\text{max}} \) define an \( N \)-independent range that includes the transition point \( s_c \). We choose a random \( s \) as in more general problems we do not expect to know the location of the transition, and in any realistic analog implementation it would be obscured by random longitudinal noise. Choosing \( s \) exactly in noise-free evolution recovers the full Grover speedup. Instantaneously jump to this point; that is, begin evolving the system under \( H \left( s \right) \) defined at the beginning of this section.

(iii) Wait a time \( t_f = C / \Delta_{\text{min}} \), where \( C \) is some \( O \left( 1 \right) \) constant, and then measure the state. The probability of finding the system in the solution state \( |G \rangle \) depends on the energy difference \( \epsilon \) between \( |G \rangle \) and \( |0\rangle \), with \( P \left( \epsilon \right) \approx \frac{\Delta_{\text{min}}^2}{\Delta_{\text{max}}^2} \left( 1 + \frac{\Delta_{\text{max}}^2}{\Delta_{\text{min}}^2} \right) \). Averaging this over the energetic range \( W \) produces \( \langle P \rangle \propto \Delta_{\text{min}} / W \), and since we must wait a time \( C / \Delta_{\text{min}} \) for each guess, the total runtime of the algorithm scales as \( W / \Delta_{\text{min}}^2 \). If the range \( \{ s_{\text{min}}, s_{\text{max}} \} \) has constant width then \( W \propto N \) and the time to solution scales as \( 2^N / N \), identical to StoqAQC up to a constant prefactor.

If we modify steps (ii) and (iii) to include the oscillating fields of RFQA-D, an exponentially growing number of distinct transitions contribute to \( P \left( \epsilon \right) \), so that the probability of finding the solution is increased relative to
evolving under a static Hamiltonian. A similar analysis to that of the previous subsection shows that $P(\epsilon)$ is enhanced by the same factor as in Eq. \[17\] and consequently this method should see the same RFQA speedup that we derived for constant schedule annealing.

Our numerical simulations verify this prediction (FIG. 4). For $N$ running from 11 to 17, we simulated evolution under random jumping in the range $s \in \{0.38, 0.58\}$, chosen to be roughly symmetric about $s_r \approx 0.48$. We compared evolution under a fixed $H(s)$ with no oscillating fields, to evolution under RFQA-D, with each spin oscillating with bare amplitude $\bar{a} = \pm \alpha_m$ and $f_i$ chosen randomly from the range $\{1.2/N, 2.16/N\}$. The system was evolved until $t_f = 2.16/\Delta_{\text{min}}$, and the fields were ramped up in amplitude from $t = 0$ with a hyperbolic tangent profile to reduce the number of off-resonant excitations generated by driving. We found the average success probability numerically with 800 random $s$ values (and frequency combinations for RFQA-D) per data point. As shown in FIG. 4, $P(N)$ scales numerically as $2^{-0.48N}$ for evolving under the simple combination of $H_0$ and $H_P$, but its best fit decay of $2^{-0.26N}$ under RFQA-D is much more gradual, proving the quantum advantage of the RFQA technique. The resulting time to solution of $2^{0.76N}/N$ is very close to the analytical prediction made in Eq. [17].

Having demonstrated both analytically and numerically that RFQA is capable of producing a quantum speedup, we now turn to the second formulation of our method, RFQA-M, where the magnitude of the transverse field oscillates locally. Though this protocol was not applicable to the Grover problem, it is capable of yielding a provable quantum speedup for $N$-spin first-order transitions between nearly classical “bit string” ground states. We will first outline the phenomenological details of these transitions (and their relevance as a model for hard spin glass problems), then show through a very similar calculation that the exponential proliferation of tone combinations from local oscillations accelerates the transition between states and reduces the difficulty exponent.

**RFQA-M AND BIT STRING TRANSITIONS**

We consider a quantum spin glass, with Hamiltonian

$$H_{SG} = \sum_{i<j} J_{ij} \sigma_i^z \sigma_j^z + \sum_i \beta_i \sigma_i^x.$$  \[18\]

Here, the $J_{ij}$ couple every pair of qubits, with an overall strength that scales as $1/N$. In the limit of all $\beta_i \to 0$ and in the presence of a transverse field of strength $\kappa$, with $\kappa$ small enough that the system is in the spin glass phase, the system will have a spectrum comprised of pairs of states with all $N$ spins flipped along $z$, with exponentially small splittings due to the $\kappa$ terms. When $\kappa = 0$, finding the ground state of the most general case of this problem (without restrictions on connectivity or the distribution of $J_{ij}$ and $\beta_i$) is classically NP-hard.

A key bottleneck mechanism in finding the ground state of these models with AQC was recently identified $[52, 55]$. Namely, deep within the spin glass phase at sufficiently large $N$, we expect to find additional avoided crossings between spin glass minima (which are fixed configurations in the $z$ basis, up to small transverse field corrections) as $\kappa$ is reduced to zero. Deep in the spin glass phase, the low-lying eigenstates of the system are nearly classical, accurately described by a single $z$-spin configuration with small corrections due to the transverse field. We refer to these states as bit string states in this work, due to their semiclassical nature. It is important to note however that the higher order corrections are not ignorable, as they create avoided crossings that allow for quantum tunneling between states.

An intuitive mechanism for such crossings can be derived in second order perturbation theory, though they arise non-perturbatively as well. Imagine that when $\kappa = 0$, there is a local minimum state $|E\rangle$ which is sep-
arated from the true ground state $|G\rangle$ by $M$ spin flips, with an $O(1)$ energy difference between the states. If we turn on a finite transverse field $\kappa$, the energies of $|G\rangle$ and $|E\rangle$ are modified by

$$\Delta E_{G/E} \simeq -\kappa^2 \sum_{i=1}^{N} \frac{1}{\delta_{G/E}}.$$

(19)

$$\delta_{\psi} \equiv \langle \psi | \sigma_i^z H_{SG} \sigma_i^z | \psi \rangle - \langle \psi | H_{SG} | \psi \rangle.$$

Generically $\delta_G \neq \delta_E$ (though we expect the differences to be small), so if the energy difference between $|G\rangle$ and $|E\rangle$ is small enough, the transverse field $\kappa$ can push the energy of $|G\rangle$ above that of $|E\rangle$, leading to an $M$-spin tunneling event between $|G\rangle$ and $|E\rangle$ at some critical $\kappa_c$. This is a first-order quantum phase transition and should have a gap exponentially small in $M$ in general cases, so can act as a bottleneck even when the earlier paramagnet to spin glass transition is second-order and has a much larger gap. Given that a “clustering” phase, where the system supports many ground states and near-ground state local minima, separated from each other by an extensive number of spin flips, is phenomenologically observed in fully connected NP-complete problems [65–67] we expect this bottleneck to occur in a wide range of important practical problems. Transitions between bit string states also set the difficulty of other AQC protocols, such as the one outlined in [68].

Since these transitions only occur in the large-$N$ limit it is not possible to simulate their full quantum dynamics on a classical computer. To study the effect of RFQA on such transitions, we instead consider a simplified picture of $N$ spin tunneling. We imagine that our annealing protocol begins in one of the two ground state minima at constant $\kappa$, with the $\beta$ terms small but not equal to zero, and then slowly ramps the $\beta$ values to opposite signs so that the system ends in the other minimum through an $N$-spin tunneling process if the ramp is sufficiently slow. This phenomenologically approximates the influence of the varying $\delta_{G/E}$ in [19]. For simplicity we will assume all the $\beta$ terms are aligned along the direction of one minimum, thus ensuring that there is no competition with other states during the ramp.

Ground state energy gap

To illustrate the mechanism by which RFQA-M speeds transitions between nearly classical ground states, we must first determine the minimum energy gap, and to do so, it suffices to compute the $N$-spin tunneling matrix element. To do so, we let $\{ |n\rangle \}$ be the set of states which differ from the ground state by $n$ spin flips. For simplicity, we assume that there are no low-energy sequences of spin flips between states, or if there are, that combinatorial advantage of summing over all possible sequences dominates them. We then further define:

$$\epsilon_n \equiv \langle |n| H_{SG} |n\rangle - \langle G | H_{SG} | G \rangle ; \quad \chi_p \equiv \prod_{m=1}^{P} \epsilon_m$$

(20)

Here the average in $\epsilon_n$ is taken over all states which are $n$ spin flips away from the minimum, and we expect $\epsilon_n$ to be peaked around $m = N/2$ and roughly symmetric about that point. Formally, the minimum gap is a sum over all sequences of spin flips which join the two minima, which contains $N!$ terms and is thus generally prohibitive to compute in practice. However, under the assumption above that there are no low-energy paths, we can approximate the sum by replacing all the energy terms in the denominators by their averages $\epsilon_n$ [40], and conclude

$$\Delta_{\min} \simeq N! \prod_{i=1}^{N} \frac{\kappa_i}{\chi_{N-1}}.$$

(21)

Here we have allowed the transverse field $\kappa_i$ to vary from site to site; in many schemes it is uniform. For generic problems, $\Delta_{\min} \propto e^{-cN}$, since the denominator scales as $\chi_{N-1} \sim \delta^{N-1} (\Delta)^2$, for some constant energy scale $\delta$. This equation [21] is qualitatively accurate for the massively connected problems we consider in this work.

We also note that this form for $\Delta_{\min}$ should be shared by more complex transitions such as non-perturbative crossings or perturbative crossings dominated by particular low energy paths (such as a transition between ferromagnetic ground states of an Ising chain induced by varying a longitudinal field), at least in the limit where all $\kappa_i$ are small. The reason for this is that if all $N$ spins must flip and the $\kappa_i$ are the only transverse terms in the Hamiltonian, the gap between the states vanishes by symmetry if any of the $\kappa_i$ are taken to zero. Thus, the full expression for the gap must be of the form

$$\Delta_{\min} = \left( \prod_{i=1}^{N} \kappa_i \right) \times f (\kappa_1, ..., \kappa_N),$$

(22)

where $f$ is a function of all the $\kappa$ terms (and the parameters in $H_{SG}$) that does not necessarily vanish as $\kappa_i \to 0$. Higher order corrections from this function will modify the effective driving amplitude, as explained below.

Oscillating fields and speedups

We now consider the $N$-spin transition between spin glass minima $|G\rangle$ and $|E\rangle$, parametrized by an annealing parameter $s$ which controls a bias field along $z$ (and not the magnitude of a transverse field) as argued above. Provided that we are deep in the spin glass phase, near to the transition so that the energy difference between $|G\rangle$ and $|E\rangle$ is small compared to the single-spin excitation gap, we can approximate the dynamics near the
transition as that of a two-level system, with

$$ H(s,t) = U(s) (|G\rangle \langle G| - |E\rangle \langle E|) + \frac{\Delta}{2} (|G\rangle \langle E| + \text{H.c.}), $$

where $\Delta$ is given by [21]. We now introduce the oscillating terms, modifying $\Delta \rightarrow \Delta(t)$ through locally varying the transverse fields:

$$ \Delta(t) \simeq N! \prod_{i=1}^{N} \kappa_i (1 + \bar{\alpha}_i \sin 2\pi f_i t) \chi^{N-1} \tag{23} $$

$$ \simeq \Delta_{\text{min}} \prod_{i=1}^{N} (1 + \bar{\alpha}_i \sin 2\pi f_i t). \tag{24} $$

Now, in real systems, the scaling of the minimum gap with variation of a single transverse field $\kappa_i \rightarrow (1 + v) \kappa_i$ does not exactly yield $\Delta \rightarrow (1 + v) \Delta$, due to higher order corrections beyond $N$th order perturbation theory. While reducing $\kappa_i$ ($v$ negative) typically produces a nearly linear change in $\Delta$, as $\kappa_i$ is increased the corresponding increase in $\Delta$ typically lags $(1 + v)$, in turn changing both the magnitude of the oscillating term and the mean value as $v$ sinusoidally oscillates. The result is:

$$ \Delta(t) \simeq \Delta_{\text{min}} \prod_{i=1}^{N} (M_0 + \alpha_i \sin 2\pi f_i t) \tag{24} $$

Here, $M_0 < 1$ (by a small amount that tends to increase with increasing $\kappa$), and $\alpha_i$ similarly lags the bare amplitude $\bar{\alpha}_i$.

We can predict the total solution rate $\Gamma_T$ from an oscillatory gap of the form [24] using the same arguments that predicted the scaling of RFQA-D in the previous section. Namely, we expand [24] as a power series in $\alpha$ and sum the contributions at all frequency combinations. Taking into account the factors of 1/2 that arise from breaking the sines and cosines into exponentials, we obtain

$$ \Gamma_T \simeq \frac{\Delta_{\text{min}}^2}{W} \times M_0^N \sum_{n=0}^{N} \frac{(\alpha^2)}{2}^n \binom{N}{n} \tag{25} $$

$$ \simeq \frac{\Delta_{\text{min}}^2 M_0^N (1 + \frac{\alpha^2}{2})^N}{W}. $$

As in the Grover case of RFQA-D, the sum in [25] grows exponentially with $N$. Were this exponential to meet or even exceed the decay of $\Delta_{\text{min}}^2$ with $N$, the computational complexity of the problem would change. Since the maximum achievable $\alpha$ and $M_0$ both decrease as $\kappa$ increases, whether or not this is possible is a subtle question and we will not attempt to answer it. Undoubtedly, even if it is possible for RFQA to change exponential to polynomial difficulty scaling for a particular class of problems, it should not be possible for all problems. We note also that static, random variation of the transverse field strengths has been argued to provide meaningful speedups for hard optimization problems [69]. Somewhat remarkably, RFQA speedup in [25] occurs in spite of the fact that the mean value of the minimum gap is smaller than the $\alpha = 0$ case, due to the asymmetry that produces $M_0 < 1$.

In generic problems, varying the transverse field creates local potential disorder along $z$ as well, through the mechanism sketched in [19]. These effects can be modeled through fluctuating $z$ fields at each site, and for the protocol sketched in this subsection (where the tones are chosen with randomized frequencies and phases), we do not expect them to significantly limit performance, since they will create an energetic uncertainty $W_r \propto \sqrt{N}$. As argued earlier, this will at most reduce the solution rate by a factor of $\sqrt{N}$, which is modest compared to the exponential increase that comes from summing combinations of tones.

To numerically verify Eq. [25] through a simple example, we considered transitions between ferromagnetic ground states of an all-to-all connected model as we vary the value of a longitudinal bias field:

$$ H = -\frac{1}{N} \left( \sum_{i=1}^{N} \sigma_i^z \right)^2 + \frac{1 - 2s}{2N} \sum_{i=1}^{N} \sigma_i^z - \kappa \sum_{i=1}^{N} \sigma_i^z \tag{26} $$

Here, $s$ evolves from 0 to 1 at a time $t_f$ in the annealing process, at a constant rate so that $s(t) = t/t_f$. To implement RFQA-M we modified the transverse field term by:

$$ \kappa \sum_{i=1}^{N} \sigma_i^z \rightarrow \kappa \sum_{i=1}^{N} (1 + \alpha_i \sin 2\pi f_i t) \sigma_i^z, \tag{27} $$

with the $f_i$ randomly drawn from a uniform range and $\alpha_i$ randomly chosen as $\pm \alpha$.

We probed the advantage of RFQA-M through direct numerical simulation with $N$ running from 4 to 10. In one set of simulations, we worked at fixed $\kappa = 0.5$ ($M_0 \simeq 0.97, \alpha \simeq 0.845$); in this case the RFQA speedup increases exponentially with $N$, corresponding to a decrease in the difficulty exponent of the problem. We numerically simulated evolution of the full system Hamiltonian with $\alpha = 0.9$ and randomly chosen frequencies, for 30 values of $t_f$ running from 30 to 600, averaged over 400 random tone sets per data point. In the other set, we allowed $\kappa$ to increase from 0.192 to 0.646 so that the minimum gap $\Delta \simeq 0.00177$ remained constant as increasing $N$, to probe how the effect of $M_0$ and $a$ scaling with increasing $\kappa$ reduces the advantage of RFQA. Effective parameters $\{M_0, \alpha\}$ varied from $\{0.992, 0.885\}$ to $\{0.956, 0.81\}$ as $N$ increased from 4 to 10, illustrating the general observation that both fall as $\kappa$ increases. In both cases we chose such small gap points so that the full advantage of RFQA would be most easily visible; as
mentioned earlier, if the gap is so large that the sinusoids can only oscillate a few times during the evolution \((t_f \sim 1/f_i)\) then the driven transitions will not be adequately sampled to contribute to a speedup. Results are shown in FIG. 5, dots correspond to numerically fitting the time-dependent solution probability, and curves plot the analytical prediction \((25)\), showing good quantitative agreement. For fixed transverse field the RFQA advantage grows exponentially in \(N\), demonstrating a reduction of the difficulty exponent of the problem.

**Failure of RFQA-D for these transitions**

It is important to note that unlike the Grover problem, the RFQA-D method of locally oscillating the directions of the transverse field terms will not produce a quantum speedup for transitions between bit string states. This is because the tunneling term in the two-level approximation above scales as the product of all the \(\sigma^z\) matrix elements, and thus its phase rotates as the product of the phase rotations of all the spins. This is in contrast to the Grover problem, where individual spins or groups of spins could create independent transitions. Given this multiplication of phases even a handful of tones will rapidly merge into a single large oscillation combining many frequencies, with average amplitude far beyond the optimal value calculated in the RFQA-D section. This means that individual frequency components end up with extremely weak amplitudes, and the proliferation of equal-strength tone combinations central to generating the RFQA speedup will not occur. In problems where both paramagnet to spin glass transitions and transitions between semiclassical bit string minima act as bottlenecks, both methods would likely need to be employed during different stages of the evolution to ensure a quantum speedup throughout.

**Phase locking tones**

An intriguing possibility to further improve performance in RFQA-M (though not RFQA-D) is to synchronize the frequencies and phases of drive fields, so that oscillations in the transverse fields of different spins interfere constructively, boosting the strength of the driving term. For example, driving \(P\) spins at the same frequency produces

\[
\Delta (t) \simeq \Delta_{\min} \left( M_0 + \alpha \sin 2\pi ft \right)^P. \tag{28}
\]

When we sum the squared effective amplitudes of the driven terms in \((28)\) at the relevant frequency contributions the total contribution to \(\Gamma_T\) scales as \(\left( (M_0 + \alpha)^P - M_0 \right)^2 / \sqrt{P}\) (found by numerically Fourier transforming Eq. \((28)\), which is potentially a much more substantial boost to performance. Driving \(k\) groups of \(P\) spins produces additional resonances from frequency combination, and leads to an expression similar in form to \((25)\).

However, one must be cautious in synchronizing too many tones, as the time-dependent potential disorder created by the oscillations also grows with \(P\) and \(\alpha\). While in the case of many randomly drawn frequencies this can simply be modeled as a line broadening of the energy difference between the two states (and thus increases the time to solution by a factor of \(N\) at most, without altering the difficulty exponent), in the case of large oscillations at frequencies commensurate with the driving this is no longer the case, and the advantage of RFQA can fall dramatically. For example, if we adopt the standard annealing procedure of simultaneously ramping the transverse field to zero while ramping the spin glass Hamiltonian from zero to its maximum value, oscillating all the transverse field amplitudes in phase is equivalent to combining a sinusoidally varying offset in the annealing parameter \(s(t)\) itself with a global modulation of the Hamiltonian energy scale that is irrelevant to the dynamics. Such oscillations do not sample the transition more efficiently than a simple linear ramp and do not accelerate finding solutions.

Optimal performance would likely be reached by choosing \(k\) distinct frequencies for the drive fields distributed across the spins, with \(1 < k < N\) and \(k\) likely problem-dependent. Determining the optimal number of frequencies, and how such constructions might interact with embedding schemes \(70, 71\) or quantum annealing correction protocols that represent a single logical qubit with
many physical qubits, are issues beyond the scope of this paper.

Transverse couplers

The performance of either flavor of RFQA could likely be improved further through the inclusion of transverse coupling terms to the Hamiltonian and oscillating their magnitudes and/or directions. Realizing transverse coupler terms, of the form $\sigma^x \sigma^x$ or $\sigma^y \sigma^y$, is an important near-term goal for quantum annealing. If these terms were included in the Hamiltonian, oscillating their magnitudes and/or directions would contribute additional sources to the proliferation of multi-photon transitions just as the single spin fields do. Since the performance boost predicted in Eqs. (17,25) is exponential in the number of tones, oscillating couplers could provide further dramatic performance increases. That said, including transverse couplers in the Hamiltonian makes the problem significantly more complex, and analytical predictions of the performance would likely be difficult to formulate. While we will not attempt such an analysis here, it could be a fruitful direction for future research.

FLUX QUBIT IMPLEMENTATION

Both RFQA-D and RFQA-M are straightforward to implement in superconducting flux qubit architectures. A flux qubit consists of a loop of superconducting wire interrupted by a SQUID. Including couplings to a external voltage difference $V_e$ in the plane of the qubit, and external magnetic fluxes $\Phi_S$ and $\Phi_L$, which thread through the SQUID loop and full qubit loop, respectively, the flux qubit Hamiltonian in the phase $\phi$ basis is:

$$H = 4E_C \left( i \frac{\partial}{\partial \phi} + \frac{CV_e}{2e} \right)^2 + \frac{E_L}{2} \phi^2 - E_J \left( \cos(\phi + \Phi_S + \Phi_L) + \cos(\phi + \Phi_L) \right).$$

For $\Phi_L \approx \pi$ the system has two ground states with $\langle \phi \rangle = \pm \phi_A$ (for some $\phi_A$ that minimizes the potential terms), corresponding to clockwise and counter-clockwise persistent currents. There is a large nonlinearity that separates these two states from higher excited states, which are generally ignored in the AQC context.

At $\Phi_L = \pi$ the energy of the symmetric and anti-symmetric superpositions have a tunneling energy splitting $\Delta$ which decreases exponentially in $E_J/E_C$. Biasing $\Phi_L$ away from $\pi$ splits the degeneracy between the two states, and adjusting $\Phi_S$ lowers the tunneling barrier by reducing the effective $E_J$. Similarly, a static offset charge $CV_e/2e$ can be eliminated through a gauge transformation $|\psi\rangle \rightarrow e^{i CV_e/2e \phi} |\psi\rangle$; however, if $V_e$ varies in time it can induce transitions between the two states. If we choose a reduced basis such that the two persistent current states are eigenvalues of $\sigma^z$ the flux qubit Hamiltonian becomes:

$$H \approx \Delta (\Phi_S) \left( \cos[cV_e] \sigma^z + \sin[cV_e] \sigma^y \right) + U \left( \Phi_L \right) \sigma^z,$$

Where $c$ is a proportionality constant. From this Hamiltonian we can easily see that the amplitude oscillations of RFQA-M can be induced through oscillating $\Phi_S$ and the directional oscillations in RFQA-D can be induced through applied voltages. Both terms can be controlled through external current and voltage sources as in FIG.2(b).

Of course, one can perform AQC in a variety of quantum computing and simulation platforms. In most cases one could similarly implement RFQA, though the physical details of the implementation would obviously be very different; all of the predictions in this work are agnostic to the underlying physical system.

CONCLUSIONS

In this article we have proposed a novel mechanism, which we call RFQA, for obtaining a quantum speedup in quantum annealing. We have shown that adding local oscillating fields in the magnitude and/or direction of the transverse fields produces an exponential proliferation of weak transitions, the sum of which reduces the difficulty exponent near hard quantum first order transitions. By polynomially reducing the frequency range as the problem size grows, off-resonant excitations can be exponentially suppressed, without impeding the ability of the oscillating fields to more rapidly mix competing ground states near first order avoided crossings. We were able to prove the quantum speedup from RFQA analytically and numerically for two paradigmatic types of transition. Given that these transitions are common bottlenecks in hard optimization problems, that the requisite oscillations can be induced in flux qubits through electric and magnetic field lines, and that the RFQA method should not lose its advantage in the presence of realistic local potential fluctuations, we expect that it could be broadly useful for near-term quantum hardware.

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