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Mass bound and thermodynamical behaviour of the charged BTZ Black Hole

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Abstract. The charged Bañados-Teitelboim-Zanelli (BTZ) black hole displays divergent boundary terms in the action, which lead to a divergent black hole mass. Once a finite, regularized, mass \( M \) is defined black hole states exist for arbitrarily negative values of \( M \), moreover there is no upper bound on the U(1) charge \( Q \). We show that these pathologies can be completely removed by using an alternative renormalization procedure for the black hole mass and defining a new mass \( M_0 \), which is the total energy inside the horizon. The new mass satisfies a BPS-like bound \( M_0 \geq \frac{\pi}{2} Q^2 \) and the heat capacity of the hole is positive. We also discuss the black hole thermodynamics that arises when \( M_0 \) is interpreted as the internal energy of the system. We show, using three independent approaches (black hole thermodynamics, Einstein equations, Euclidean action formulation) that \( M_0 \) satisfies the first law if a term describing the mechanical work done by the electrostatic pressure is introduced.

1. Introduction

It is well known that three-dimensional (3d) pure gravity has no propagating degrees of freedom. This feature makes it a suitable tool for analyzing the issue of quantum gravity and gauge theory in a rather simplified context. Moreover, 3d gravity with a negative cosmological constant can be expressed classically as a gauge theory, the action can be rewritten as a Chern-Simons action for an \( SO(2, 2) \) field composed by a spin connection and the vierbein. The discovery in 1992 of the existence of black hole solutions in 3d gravity with negative cosmological constant by Bañados, Teitelboim and Zanelli (BTZ) \([1, 2]\) made the subject of 3D gravity even more interesting. This black hole is asymptotically Anti-de Sitter and unlike higher dimensional counterparts, it has not a curvature singularity: this unusual feature is actually what one would expect, because AdS space has constant negative curvature everywhere. The black hole is obtained by orbifolding the AdS spacetime with a discrete subgroup of its \( SO(2, 2) \) isometry group. In this way the absence of curvature singularities is manifest and the resulting spacetime can be seen as a black hole because if one extends the metric past \( r = 0 \) to \( r < 0 \) the resulting manifold contains closed timelike lines \([2]\), so that the surface \( r = 0 \) is a singularity in the causal structure.

3d AdS gravity plays also a crucial role in the AdS/CFT correspondence. Much before Maldacena’s conjecture, it was pointed out by Brown and Henneaux \([4]\) that the diffeomorphisms preservning the asymptotic structure of AdS spacetime are generated by vector fields, which span a pair of Virasoro algebras. The conformal symmetry can be realized as

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canonical transformations and for an asymptotically AdS space one is forced to introduce boundary terms to the Hamiltonian and momentum constraints. In this way the Virasoro algebras acquire a central charge.

The BTZ black hole solution allows us to find the simplest realization of a 3d bulk gravity configurations that are holographically dual to thermal CFT states [4, 5]. One of the most striking successes of the $AdS_3/CFT_2$ correspondence is the exact computation of the Bekenstein-Hawking entropy of the BTZ black hole using the dual two-dimensional CFT [6]. Black holes in the 3d bulk can be viewed as excitations of the AdS background and these excitations are dual to thermal excitations of the boundary conformal field theory.

Another important issue concerns the investigation of supersymmetric solutions of 3d gravity. As shown in ref. [18], extreme dilaton black holes are solutions of $N = 4$ supergravity, and the cosmic censorship condition on the mass is equivalent to the BPS bound deriving from supersymmetry, requiring that the mass of the asymptotically flat spacetime be larger than or equal to the absolute value of all central charges. Coussaert and Henneaux have shown [19] that this is also true for AdS black holes, in particular the massless solution has 2 supersymmetries, the maximal number, and that it is the ground state of $(1,1)$ AdS supergravity with Ramond boundary conditions on the spinors. The extremal black hole has one supersymmetry, while generically there are no supersymmetries at all.

The BTZ solution has a $U(1)$ charged generalization [1, 7] with a different singularity structure. The 3d charged AdS black hole shows indeed a power-law curvature singularity, sharing this feature with its higher-dimensional counterparts, like the charged Reissner-Nordstrom solution in 4d AdS space. However, several pathologies make this black hole rather different from higher-dimensional ones. These pathologies would spoil any hope of exploiting the analogy between 3d and higher-dimensional charged black holes to investigate in a simplified context peculiar features of higher dimensional charged black holes, such as the existence of extremal black holes states of zero temperature and non-vanishing entropy.

The origin of these pathologies is related to the logarithmic divergence of the electrostatic potential of the hole, which can be traced back to the logarithmic behaviour of harmonic functions in two dimensions. This has some important consequences: a) When we vary the action we get divergent boundary terms, i.e we have a divergent black hole mass; b) Using a suitable renormalization procedure, we can define a mass $M$ of the solution, but we find that black hole state exist for arbitrarily negative values of $M$; c) At $M$ fixed there is no upper bound on the charge $Q$. Another, recently discovered, manifestation of this problematic behaviour, is the fact that the entropy function approach do not work for the extremal charged BTZ black hole [14].

These features make the charged BTZ black hole drastically different from the charged solutions in four and higher dimensions. In these latter cases the black hole mass satisfies a BPS bound $M \geq a^2 Q^2$, which guarantees that the mass is positive definite and that for a given mass the charge is bounded from above. The existence of these bounds is usually a consequence of the supersymmetry of the extremal black hole.

Recently, an alternative renormalization procedure leading to a finite value $M_0$ for the mass of the charged BTZ black hole, has been proposed [8, 9]. Physically, $M_0$ is the total energy (gravitational and electromagnetic) inside the black hole outer horizon. The identification of $M_0$ with the conserved charge associated with time-translations allows to reproduce microscopically the Bekenstein-Hawking entropy of the hole and to consider the charged BTZ geometry as a bridge between two AdS$_2$ geometries, a near horizon one ($AdS_2 \times S^1$), and an asymptotic one (linear dilaton AdS$_2$) [8]. In both cases, these can be realized as the chiral half of a 2D CFT. For the asymptotic CFT, the central charge is generated by the breaking of the $SL(2,\mathbb{R})$ isometry of the $AdS_2$ background due to the non constant dilaton, while the electromagnetic field only enters in the renormalization of the virasoro operator $L_0$. For the near horizon CFT, the central
charge is generated by the boundary conditions for the electromagnetic vector potential. For the asymptotic CFT one can compute the black hole entropy through Cardy’s formula, which matches the Bekenstein Hawking entropy.

In view of these results, one is led to ask if the use of the alternative renormalization procedure for the mass of Ref. [8, 9], allows also to cure the pathologies of the charged BTZ black hole. In this note we address this issue by reviewing the results of Ref. [24]. We will argue that all the problematic features of the charged BTZ black hole can be removed if one uses \( M_0 \) as the black hole mass. We will demonstrate that \( M_0 \) satisfies a BPS-like bound \( M_0 \geq \frac{\pi}{2} Q^2 \) and that for a black hole above extremality the heat capacity is positive and becomes zero in the extremal case. We also discuss the formulation of black hole thermodynamics when \( M_0 \) is interpreted as the internal energy of the thermodynamical system. We show, using three independent methods (black hole thermodynamics, Einstein equations, Euclidean action formulation) that \( M_0 \) satisfies the first law if a term describing the mechanical work done by the electrostatic pressure is introduced.

Recently, some new advances on this topics have been achieved in Ref. [23], through a computation of the microscopic entropy for the charged, rotating BTZ black hole by using the Cardy - Verlinde formula.

The structure of this proceeding is as follows. In Sect. 2 we briefly review the main features and pathologies of the charged BTZ black hole. In Sect. 3 we discuss in detail the two renormalization schemes, which can be used to get a finite black hole mass and show that the mass \( M_0 \) satisfies a BPS-like bound. In sect. 4 we discuss the thermodynamics of the charged BTZ black hole in the two cases when the internal energy of the system is identified with \( M \) or \( M_0 \). In Sect. 5 we present our conclusions.

2. The charged BTZ black hole

The charged BTZ black hole is a U(1) generalization of the uncharged BTZ black hole solution [1]. In this paper we consider the presence of a nonvanishing electric charge \( Q \) and zero angular momentum \( J \).

The action is

\[
I = \int d^3x \sqrt{-g} \left( \frac{R + 2\Lambda}{2\pi} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right),
\]

where \( F_{\mu\nu} \) is Maxwell tensor, \( \Lambda = 1/l^2 \) is the cosmological constant related to the AdS length \( l \) and we are using units such that three-dimensional (3d) Newton constant \( G \) is dimensionless, \( G = \frac{1}{8} \).

The solution for the electrically charged, non rotating black hole is given by [7, 9]

\[
ds^2 = -f(r)dt^2 + f^{-1}dr^2 + r^2d\varphi^2,
\]

\[0 \leq r < \infty, \ 0 \leq \varphi < 2\pi, \]

with metric function and Maxwell field

\[
f = \frac{r^2}{l^2} - M - \pi Q^2 \ln \frac{r}{w}, \quad F_{tr} = \frac{Q}{r}.
\]

\( M, Q, w \) are integration constants. We choose to keep the explicit dependence of the metric on the arbitrary length scale \( w \). However, since the solution depends only on two integration constants (which we will identify with the mass and the charge), this parameter \( w \) could be absorbed in a redefinition of \( M \). The above solution represents a 3d asymptotically AdS black hole, with a power-law singularity at \( r = 0 \), where \( R \sim \pi Q^2/r^2 \) and, generically, with an inner \((r_-)\) and outer \( r_+ \) horizon. \( M, Q \) could be naively seen as the black hole mass and charge,
respectively. $M$ can be easily expressed as a function of the charge and of the outer horizon radius,

$$M(r_+, Q, w) = \frac{r_+^2}{l^2} - \pi Q^2 \ln \left(\frac{r_+}{w}\right).$$

Whereas the interpretation of $Q$ as the black hole electric charge is straightforward, the same is not true for $M$.

In fact, as suggested by the same authors in Ref. [1], by varying the action (1), one finds a surface term which diverges in the limit $r \to \infty$:

$$(-\delta M - \pi \delta Q^2 \ln r) N(r),$$

where $N$ is the lapse function. The presence of the logarithmic divergent boundary term makes the black hole mass a poorly defined concept. This is the first pathology of the charged solution.

A second unpleasant feature emerges when one imposes a cosmic censorship condition, i.e the absence of naked singularities. The requirement that the singularity at $r = 0$ is shielded by an event horizon is equivalent to requiring that the metric function $f(r)$ evaluated on its minimum value, is equal or less than zero (corresponding respectively to one or two horizons). Introducing a function $\Delta(M, Q)$, the condition for the existence of the horizon(s) can be cast in the form

$$\Delta(M, Q) = f \left( r = r_{\text{min}} = \sqrt{\frac{\pi}{2} Q l} \right) = -M + \frac{\pi Q^2}{2} \left( 1 - \ln \frac{\pi Q^2}{2} \right) \leq 0,$$

Eq. (6) can be satisfied by arbitrarily negative values of the mass $M$, for $Q$ above a critical value $Q_0$. This can be immediately seen considering the $M - Q$ phase diagram shown in Fig (1).

![Figure 1](image)

**Figure 1.** Region in the phase space $M - Q$ where the black hole exists. The region of existence is the shaded region. Extremal black holes are in the boundary line between the shaded and the unshaded regions.

The presence of black hole states with mass values unbounded from below makes the system intrinsically unstable and the definition of thermodynamical ensembles problematic. In next section we show that the presence of black hole states with arbitrarily negative mass is not an intrinsic physical feature of the system, but instead it is an artifact due to the divergence of the boundary term.

### 3. Mass renormalization

We address the pathology related to the presence of a divergent boundary term through a renormalization procedure. First, we enclose the system in a circle of radius $r_0$. This allows to
rewrite the metric function in the form

\[ f(r) = -M_0(r_0, w) + \frac{r^2}{l^2} - \pi Q^2 \ln \left( \frac{r}{r_0} \right), \quad (7) \]

and to define a regularized mass as a function of \( r_0 \). This was also suggested by the authors of Ref. [7], but now we also have a dependence on the scale \( w \), which we will interpret as a running scale:

\[ M_0(r_0, w) = M + \pi Q^2 \ln \left( \frac{r}{r_0} \right) \quad (8) \]

\( M_0(r_0) \) is the total energy (gravitational and electromagnetic) inside a circle of radius \( r_0 \). In the limit \( r \to \infty \) one takes also \( r_0 \to \infty \), keeping the ratio \( r/r_0 = 1 \) [8]. One is left with two possible options: a) \( M \) is held fixed and the space-time metric is scale-dependent; b) The metric is \( w \)-invariant and \( M \) runs with \( w \).

Option a) is the renormalization procedure proposed in [7]. Eq. (8) is used to identify \( M \) as the total mass of the solution, an infinite constant is absorbed in \( M_0 \) and the logarithmic divergent term in Eq. (5) is removed. Apart from the drawback of having the metric (hence the positions of the horizons and the black hole entropy) running with \( w \), this procedure does not solve the stability problem, hence does not allow for a consistent interpretation of the charged BTZ black hole as a thermodynamical system.

In this paper we use the \( w \)-invariant renormalization prescription b) first proposed in Ref. [8]. To keep the metric function (7) unchanged as \( w \) runs, we hold \( M_0(r_0, w) \) fixed, whereas \( M \) changes with \( w \): \( w \to \lambda w, \quad M \to M + \pi Q^2 \ln \lambda \). The boundary term (5) becomes now

\[ \left( -\delta M_0 - \pi \delta Q^2 \ln \frac{r}{r_0} \right) N(r), \quad (9) \]

and in the limit \( r, r_0 \to \infty \) the divergent part is removed.

As a consequence of the \( w \)-invariance of \( f, M_0 \), we can arbitrarily choose \( w \) and \( M_0 \). Following [9] we choose to fix in terms of the AdS length \( w = l \), and \( r_0 \) to the horizon position \( r_0 = r_+ \). Thus we associate to every charged BTZ black hole solution (3) a finite mass given by

\[ M_0(r_+) = M + \pi Q^2 \ln \left( \frac{r_+}{l} \right) \quad (10) \]

The metric function becomes

\[ f(r, M_0) = -M_0 + \frac{r^2}{l^2} - \pi Q^2 \ln \frac{r}{r_+}. \quad (11) \]

Thus we can consider the \( w \)-invariant mass \( M_0(r_+) \) as the conserved charge associated with time translation invariance, instead of the mass \( M \). The renormalization prescription b) has further nice features. The renormalized mass \( M_0 \) depends only on the horizon position, is always positive-definite and shares with the uncharged BTZ black hole the mass/horizon-position dependence:

\[ M_0(r_+) = \frac{r_+^2}{l^2}. \quad (12) \]

Moreover, the identification of \( M_0 \) with the conserved charge associated with time-translation allows to reproduce exactly the Bekenstein-Hawking entropy of the charged BTZ black hole using a Cardy formula for the 2D dual CFT [9].
Let us now show that the use of $M_0$ as the physical mass of the system allows, at least in principle, to solve the instability problem. The new mass spectrum can be found by using Eq. (12) and expressing $\Delta$ of Eq. (6) as a function of $M_0$,

$$\Delta(M_0, Q) = -M_0 + \frac{\pi Q^2}{2} \left(1 - \ln \frac{\pi Q^2}{2M_0}\right),$$

(13)

Setting $\alpha = 2M_0/\pi Q^2$ the condition for the existence of the horizons, $\Delta \leq 0$ together with the relation $r_+ \geq \sqrt{\pi/2}Ql$, implies

$$M_0 \geq \frac{\pi}{2} Q^2,$$

(14)

and takes $\alpha - 1 \geq \ln \alpha$, which is always true.

Eq. (14) represents a BPS-like bound for the black hole mass. When the black hole is extremal, the bound is saturated and in Eq. (14) the equality holds. The extremal solution has mass $M_0 = (\pi/2)Q^2$, zero temperature and nonvanishing entropy $S = 2\sqrt{2\pi l}Q$ (see Eq. (15) below). All these features are shared by higher dimensional charged black holes. The quadratic form of the $M_0 - Q$ phase diagram, which results from Eq. (14), eliminates the negative, unbounded from below, tail present in Fig. 1. This result implies that the presence of black hole states with arbitrary negative mass is a consequence of identifying the energy of the system with the mass $M$. Moreover, they give a strong hint that the $M_0 = \pi Q^2/2$ extremal black hole could be a stable configuration. Obviously, in the context of our discussion stability is just a consequence of the validity of the cosmic censorship conjecture. A formal proof of the stability of this configuration would require a detailed analysis of the perturbation spectrum around the extremal black hole solution. Alternatively, stability can be proved by showing that the extremal background is supersymmetric, i.e. it allows for the existence of Killing spinors [18]. A detailed analysis of the stability of the extremal black hole is outside the aim of this paper. In the next sections we will show that using $M_0$ as black hole mass allows for a consistent formulation of the thermodynamics of the charged BTZ black hole.

4. The first law of thermodynamics for the charged BTZ black hole

The thermodynamical behavior of our charged BTZ black hole will depend on the identification of the black hole parameters in terms of thermodynamical variables. For the temperature $T$, the entropy $S$, the electric potential $\Phi$ (thought of as chemical potential) there is no ambiguity. $T, S, \Phi$ are given as usual in terms of respectively surface gravity, horizon area and time component of the vector potential,

$$T = \frac{1}{4\pi} \left(\frac{2r_+}{l^2} - \frac{\pi Q^2}{r_+}\right),$$

$$S = 4\pi r_+ = 4\pi l \sqrt{\pi Q^2 \ln \frac{r_+}{l}} + M,$$

$$\Phi = A_0(r_+) = -2\pi Q \ln \frac{r_+}{l}.$$  

(15)

On the other hand, for the internal energy $E$ of the thermodynamical system we have two possible choices: we can identify $E$ either with $M$ or with $M_0$. For $E = M$ the internal energy of the system is the total energy (gravitational and electrostatic) of the black hole and we expect the first principle to take the usual form, as was previously suggested in ref. [1]. In fact differentiating $M(r_+, Q)$ in Eq. (4) and making use of Eqs. (15) one easily obtains the first principle in the form

$$dM =TdS + \Phi dQ.$$  

(16)
The exact form $M(S, Q)$ can be easily determined. We have

$$M(S, Q) = \frac{S^2}{16\pi^2 r^2} - \pi Q^2 \ln \left( \frac{S}{4\pi l} \right) \quad (17)$$

Conversely, when $E = M_0$ the internal energy is identified with the energy of the black hole inside the radius $r_+$. In this case we expect that the presence of radial pressure gradients will give rise to additional terms in Eq. (16). The new form of the first principle can be obtained differentiating $M_0$ given in equation (10) and using Eq. (16). One has

$$dM_0 = TdS + \Phi dQ + dK, \quad (18)$$

where $K = \pi Q^2 \ln \left( \frac{r_+}{r_+} \right)$ is minus the electrostatic energy outside the horizon. The variation of $K$ cannot change the black hole entropy but represents mechanical work done by electrostatic pressure. We can compute $dK$ keeping the electrostatic potential $\Phi$ constant: this allows to express charge variation in terms of displacement of the horizon,

$$2\pi \ln \frac{r_+}{r} dQ = -\frac{2\pi Q}{r_+} dr_+. \quad (19)$$

Using Eq. (18) one finds

$$dM_0 = TdS + \Phi dQ - \frac{\pi Q^2}{r_+} dr_+. \quad (20)$$

The last term in the previous equation is the work done by the radial pressure

$$P_r = T_{rr} \quad (21)$$

generated by the electrostatic field ($T_{\mu\nu}$ is the stress-energy tensor for the Maxwell field). Explicit computation of the $T_{rr}$ gives

$$P_r (r_+) = -\frac{Q^2}{2r_+^2}. \quad (22)$$

Using this equation in (18) one obtains the first principle in the final form

$$dM_0 = TdS + \Phi dQ + P_r dA, \quad (23)$$

where $A = \pi r_+^2$ is the area inside the radius $r_+$. Notice that when $dA > 0$, the mechanical work $P_r dA$ in Eq. (23) is negative, i.e. it is done by the thermodynamical system. The pressure $P_r (r_+)$ goes to zero when $r_+ \to \infty$, in this situation $M = M_0$ and the two thermodynamical descriptions are equivalent.

The internal energy $M_0$ appearing in the first principle (23) at first sight seems to be a function of three independent extensive thermodynamical parameters $S, Q, A$; a simple calculation gives

$$M_0(S, Q, A) = \frac{S^2}{16\pi^2 r^2} - \pi Q^2 \ln \left( \frac{S}{4\pi l} \right) + \frac{\pi Q^2}{2} \ln \left( \frac{A}{\pi l^2} \right). \quad (24)$$

However, there are only two independent parameters because of the presence of a constraint. This constraint takes a different form for thermodynamical transformations at constant $\Phi$ or constant $Q$. In the first case the constraint takes the form (19), which can be also written in the form

$$\Phi dQ = -2P_r dA. \quad (25)$$
Conversely, keeping the charge \( Q \) constant the constraint takes the form

\[ Qd\Phi = -2AdP. \]  

(26)

It is also interesting to compute the thermal capacity of the black hole at constant charge as a function of \( M_0 \). We have

\[ C = T \frac{\partial S}{\partial T} \mid_{Q} = 4\pi l \sqrt{\frac{2M_0 - \pi Q^2}{2M_0 + \pi Q^2}}. \]  

(27)

The heat capacity is always positive when the black hole is above extremality, \( M_0 \geq \frac{\pi Q^2}{2} \), and becomes zero in the extremal case.

### 4.1. Derivation of the first law from Einstein’s equations

It is well known that black hole thermodynamics can be derived from the laws of black hole mechanics, i.e. it is codified in Einstein Equations. The first principle of thermodynamics for the charged BTZ black hole (23) can be derived from the \( rr \) component of Einstein’s equation,

\[ G_{rr} - \Lambda g_{rr} = \pi T r_r, \]  

(28)

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \) is the Einstein tensor. Evaluating the equations (28) on the horizon and using Eq. (21) one has

\[ \frac{f'(r_+)}{2r_+} - \frac{1}{l^2} = \pi P_r. \]  

(29)

Multiplying both terms of this equation by \( d(r_+^2) \) and using Eqs. (15) and (12) we obtain

\[ dM_0 = TdS - P_r dA. \]  

(30)

Using Eq. (25) we easily find that equation (30) is equivalent to the first law (23).

### 4.2. Euclidean action formulation

In this section we will show that the thermodynamics of the charged BTZ black hole described in the previous sections can be also derived using the Euclidean action formalism. In the Euclidean action approach to black hole thermodynamics, one can get Gibbs and Helmoltz free energies through analytic continuation of the action, with suitable boundary terms, in Euclidean space.

To compute the Euclidean action \( I_E \) we will follow the method of Bañados, Teitelboim e Zanelli [2], which uses the Hamiltonian version of the action (1). The bulk contribution is equal to zero and the Euclidean action is completely given by three surface terms. The first surface term must be added at infinity and is given by the mass of the solution times the periodicity of Killing time \( \beta = 1/T \). The other two surface terms make sure that the variational derivative of the action vanishes on the horizon.

If we use the regularization scheme a) of section 3 the boundary term at infinity is given by \( M \) and all together one has, for the Euclidean action:

\[ I_E = \beta M - 4\pi r_+ - \beta A_0(r_+)Q. \]  

(31)

The Gibbs free energy, \( G(T, \Phi) \) describing the system in the grand canonical ensemble, is given by \( G = TI_E \) and using Eq. (31) turns out to be, as expected, the Legendre transform of \( M \) with respect to \( S \) and \( Q \):

\[ G(T, \Phi) = M - TS - \Phi Q. \]  

(32)
The description of the thermodynamical system through \( G(T, \Phi) \) corresponds to the choice of \( M \) as the internal energy of the system. One can easily reproduce the entropy \( S \) and the charge \( Q \) as

\[
S = -\frac{\partial G}{\partial T}, \quad Q = -\frac{\partial G}{\partial \Phi}.
\]

If we use instead the renormalization scheme b) of section 3, the Euclidean action (31), hence Gibbs free energy do not change. The mass of the solution is now \( M_0 \) but the boundary term at infinity is still given by \( M \), the logarithmic term in Eq. (10) being an horizon contribution. From Eq. (10) it follows that \( M_0 \) can be written as

\[
M_0 = M - \frac{1}{2} Q \Phi (r_+).
\] (33)

Thus, the term needed to cancel the variational derivatives of the action on the horizon is now given by \(-4\pi r_+ - (1/2)\beta Q \Phi\). All these contributions sum up to the same result given in Eq. (31).

Corresponding to the choice of \( M_0 \) as the internal energy of the system, Gibbs free energy can be now expressed as a function of \( T, \Phi, P_r \). This can be done by first making use of Eq. (33) to write \( G \) of equation (32) in terms of \( M_0 \): \( G = M_0 - TS - \Phi Q + \frac{1}{2} \Phi dQ \), then differentiating, using the first law (23) and the constraints (25), (26). One obtains

\[
dG(T, \Phi, P) = -SdT - Qd\Phi - AdP_r.
\] (34)

5. Conclusions

We have shown that the renormalization of the mass of the charged BTZ black hole, in the prescription introduced by Cadoni, Melis, Setare [8, 9, 10], leads to a consistent thermodynamical description, once we introduce a pressure term in the first law generated by the electric field. Physically, the new mass is the total energy inside the horizon. Moreover, the new renormalized mass automatically cures the pathologies of the black hole, since it is positive definite for every value of the U(1) charge \( Q \). This suggests the presence of a bound for the new mass which resembles the BPS bound for a supersymmetric system; in particular this bound is saturated in the extremal case, a state with zero temperature and nonvanishing entropy, as one would expect. This bound also provides an upper bound for the charge. The thermal capacity of the hole is always positive and becomes zero at extremality. All these features make this black hole very similar to higher dimensional solutions such as Reissner Nordstrom in AdS_4.

Our results improve our understanding of charged black hole solution and could be also very useful in the AdS/CFT context. Similarly to the higher dimensional cases the BTZ black hole is the bridge between AdS_2 near-extremal, near-horizon and asymptotic AdS_3 geometries [9]. This feature could be very useful for understanding the nature of AdS_2 quantum gravity and in particular the microscopic entropy of extremal charged black holes [13, 12, 11, 10, 15, 16, 17].

The presence of the BPS-like bound (14), the vanishing of the temperature and of the thermal capacity in the extremal configuration strongly indicate that the extremal charged BTZ black hole is stable. An interesting application could be the investigation of the existence of a supersymmetric Killing spinor, which would automatically ensure that the extremal state is stable. Alternatively, a deeper understanding could be reached through a detailed analysis of the perturbation spectrum.

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