Optimization of network structure to random failures

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Abstract

Network’s resilience to the malfunction of its components has been of great concern. The goal of this work is to determine the network design guidelines, which maximizes the network efficiency while keeping the cost of the network (that is the average connectivity) constant. With a global optimization method, memory tabu search (MTS), we get the optimal network structure with the approximately best efficiency. We analyze the statistical characters of the network and find that a network with a small quantity of hub nodes, high degree of clustering may be much more resilient to perturbations than a random network and the optimal network is one kind of highly heterogeneous networks. The results strongly suggest that networks with higher efficiency are more robust to random failures. In addition, we propose a simple model to describe the statistical properties of the optimal network and investigate the synchronizability of this model.

Key words: Complex network; Network efficiency; Network resilience; Synchronization;
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1 Introduction

Complex networks arisen in natural and manmade systems play an essential role in modern society. Many real complex networks were found to be heterogeneous with power-law degree distributions: \( p(k) \sim k^{-\gamma} \), such as the Internet, metabolic networks, scientific citation networks, and so on [1,2]. Because of the ubiquity of scale-free networks in natural and manmade systems, the security of these networks, i.e., how well these networks work under failures or attacks, has been of great concern.

Recently, a great deal of attention has been devoted to the analysis of error and attack resilience of both artificially generated topologies and real world networks [3,4,5,6,7,8,9,10,11]. Also some researchers use the optimization approaches to improve the network’s robustness with percolation theory or information theory [12,13,14,15,16,17]. There are various ways in which nodes and links can be removed, and different networks exhibit diverse levels of resilience to such disturbances. It has been pointed out by a number of authors [3,4,5,6,7] that scale-free networks are resilient to random failures, while fragile to intentional attacks. That is, intentional attack on the largest degree (or betweeness) node will increase the average shortest path length greatly. While random networks show similar performance to random failures and intentional attacks.

The network robustness is usually measured by the average node-node distance, the size of the largest connected subgraph, or the average inverse geodestic length named efficiency as a function of the percentage of nodes removed. Efficiency has been introduced in the studies of small world networks [18] and used to evaluate how well a system works before and after the removal of a set of nodes [6].

The network structure and function strongly rely on the existence of paths between pairs of nodes. Different connectivity pattern between pairs of nodes makes the network different performance to attacks. Rewiring edges between different nodes to change the topological structure may improve the network’s function. As an example, consider the simple five nodes network shown in Fig. 1. The efficiency of Fig. 1(a) is equal to 8/25, while it is improved to 7/20 in Fig. 1(b) by rewiring. And we know that Fig. 1(b) is more robust than Fig. 1(a) to random failures. A natural question is addressed: how to optimize the robustness of a network when the cost of the network is given. That is, the number of links remains constant while the nodes connect in a different way. Should the network have any particular statistical characters? This question motivates us to use a heuristic approach to optimize the network’s function.

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by changing the network structure.

The paper is organized as follows: we firstly present MTS method in Section 2 and the numerical results are shown in Section 3. Then we construct a simple model to describe the optimal network and discuss one of the important dynamic processes happening on the network, synchronization, in Section 4. Finally, we give some insightful indications in Section 5.

2 The algorithm

Generally, a network can be described as an unweighted, undirected graph $G$. Such a graph can be presented by an adjacency binary matrix $A = \{a_{ij}\}$. $a_{ij} = 1$ if and only if there is an edge between node $i$ and $j$. Another concerned matrix $D = \{d_{ij}\}$, named distance matrix, consists of the elements denoting the shortest path length between any two different nodes. Then the efficiency $\varepsilon_{ij}$ between nodes $i$ and $j$ can be defined to be inversely proportional to the shortest distance: $\varepsilon_{ij} = 1/d_{ij}$ [18]. The global efficiency of the network is defined as the average of the efficiency over all couples of nodes.

$$E(G) = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \varepsilon_{ij} = \frac{1}{N(N-1)} \sum_{i \neq j \in G} \frac{1}{d_{ij}} \quad (1)$$

With the above robustness criterion in mind, we can define the optimization problem as follows:

$$\begin{align*}
\left\{ \begin{array}{l}
\max E(G) \\
s.t. \langle k \rangle = \text{const} \\
G \text{ is connected.}
\end{array} \right. 
\right. \quad (2)
\end{align*}$$
The above problem is a standard combinatorial optimization problem, for which we can derive good (though usually not perfect) solutions using one of the heuristic algorithms, tabu search, which is based on memory (MTS) [19]. MTS is described as follows:

**step 1:** Generate an initial random graph $G_0$ with $N$ nodes, $M$ edges. Set $G_t^* := G_0$; $t := 0$. Compute the efficiency $E$ of $G_t^*$ denoted by $E_{G_t^*}$.

**step 2:** If a prescribed terminal condition is satisfied, stop, otherwise random rewiring: specifically, a link connecting node $i$ and $j$ is randomly chosen and substituted with a link from $i$ to node $k$, not already connected to $i$, extracted with uniform probability among the nodes of the network, note the present network $G$ and the efficiency $E_G$.

**step 3:** If $E_G \geq E_{G_t^*}$, $G_{t+1}^* := G$, $G_{t+1} := G$; else if $E_G \geq E_{G_t}$, $G_{t+1} := G$; else if $G$ does not satisfy the tabu conditions, then $G_{t+1} := G$, else $G_{t+1} := G_t$. Go to **step 2**.

The following condition is used to determine if a move is tabu: $|E_{G_t} - E_G| > \delta$, which is the percentage improvement or destruction that will be accepted if the new move is accepted. Thus, the new graph at **step 2** is assumed tabu if the total change in the objective function is higher than a percentage $\delta$. In this paper, $\delta$ is a random number generated between 0.50 and 0.75. The terminal condition is that the present step is getting to the predefined maximal iteration steps.

**3 Numerical results**

Many real networks in nature and society share two generic properties: scale-free degree distribution and small-world effect (high clustering and short path length). Another important property of a network is the degree correlation of node $i$ and its neighbors. It is called assortative mixing if high-degree nodes are preferentially connected with other high-degree nodes, and disassortative mixing if high-degree nodes attach to low-degree nodes. Newman proposed a simple measure to describe the mixing pattern of nodes, which is a correlation function of the degrees. The empirically studied results show that almost all the social networks show assortive mixing pattern while other technological and biological networks are disassortative. The statistical properties are clearly described in Refs [20,21,22,23]. We start from a random graph with size $N = 300$ and $\langle k \rangle = 6.7$. The terminal condition is the maximal iteration step reaching 1000. It should be noted that for each step of the objective function being improved, we record the statistical properties of the present network. A typical run of statistical results are shown in Figs. 2 and 3.
Fig. 2. The characteristics of the optimal network evolving from a random network. (a) Efficiency of the network; (b) The average path length $L$; (c) The average clustering coefficient $C$; (d) The maximal degree of the network; (e) The degree correlation coefficient $r$. The network size is $N = 300$ and $\langle k \rangle = 6.7$.

Fig. 2 shows that with the increase of efficiency $E$, the average shortest path length $L$ becomes short and the maximal degree becomes large, indicating that hub nodes develop to be present with the evolving process. With the increase of the efficiency, the hub node develops to be the most important one to connect with almost other nodes in the network. For degree correlation coefficient $r$ (Fig2. (e)), it decreases in the whole process from zero to negative, which indicates that the nodes with high degree preferentially connect with the ones with low degree. Still, for the clustering coefficient $C$, it increases to a high value 0.6 and the network gets to be a highly clustering network. For the degree distribution, the cumulative degree distribution is shown in Fig. 3.

To check the optimal network’s tolerance to random failures, we show the efficiencies of both initial random network and the optimal network versus the fraction of removed nodes in Fig. 4. It can be clearly observed that compared with the initial random network, the optimal network’s robustness to errors is greatly improved.
Fig. 3. The cumulative degree distribution of the optimal network in log-linear scale. The network size is \( N = 300 \) and \( \langle k \rangle = 6.7 \).

Fig. 4. Efficiencies of the optimal network and the initial random network versus the fraction of removed nodes. The data are averaged over 10 independent runs of network size \( N = 300 \) and \( \langle k \rangle = 6.7 \).

4 Model and synchronization of the network

To provide a simple way to describe the properties of the optimal network, we consider to construct the network model directly. With a nongrowing network model it can be constructed in the following way.

(a) Start from a random network with \( N \) nodes, which can be implemented by rewiring edges of a regular graph with probability \( p = 1 \).

(b) Choose \( q \) nodes as hub nodes randomly from the whole network with equal probability.

(c) Add \( m \) edges randomly. One end point of the edge is selected randomly from the \( q \) hub nodes and the other is chosen randomly from the network.
Fig. 5. The statistical properties of the network model versus the parameter $m$. (a) Efficiency of the network; (b) The average path length $L$; (c) The average clustering coefficient $C$; (d) The maximal degree of the network; (e) The degree correlation coefficient $r$; (f) The eigenvalue ratio $R$ of the network, which is an important measure of network synchronizability. The network size is $N = 200$, $q = 1$, $\langle k \rangle = 5.2$. All data are averaged over 100 realizations.

In such a way, the network evolves to possess the statistical properties of the optimal network. This can be seen from Fig. 5. The primary goal of our simulation is to understand how the statistical properties of the network change with the process of adding edges. The construction of the model is similar to the two-layer model introduced by Nishikawa et. al in Ref [29]. The main difference is that the initial network in our model is a random network different from that of a regular network.

To show the effect of the parameter $q$, we also present simulation results versus $q$ in Fig. 6. With the increase of the parameter $q$, the network becomes less heterogeneous and more homogeneous, it’s natural to observe that both the efficiency of the network and the maximal degree reduce. So both the average path length $L$ and the correlation coefficient $r$ increase in the homogeneous phase compared with the values in the heterogeneous phase. Since the optimal network shows a strong heterogeneity, a small parameter of $q$ is reasonable. We know that most of the real world networks share the character of small-world effect and some degree of heterogeneity, so these networks are robust to random failures and they are also efficient in exchanging information.

Then we consider the synchronization of the network model, how does the
network’s synchronizability change with the adding of edges? Synchronization has been observed in diverse natural, social and biological systems. Consider a network consisting of $N$ identical oscillators coupled through the edges of the network. The dynamics of each individual oscillator is controlled by $\dot{x}_i = f(x_i)$ and $h(x_j)$ is the output function. Thus, the equations of motion are as follows:

$$\dot{x}_i = f(x_i) - \sigma \sum_{j=1}^{N} L_{ij} h(x_j),$$

where $\dot{x}_i = f(x_i)$ governs the dynamics of individual oscillator, $\sigma$ is the coupling strength, and $L = \{L_{ij}\}$ is the Laplacian matrix of the network. It has been shown that the eigenvalue ratio $R = \frac{\lambda_N}{\lambda_2}$ is an essential measure of the network synchronizability, the smaller the eigenvalue ratio $R$, the easier the network to synchronize [25]. The progress in the studies of the relationship between topological structure and synchronizability can be found in Refs [26,27,28,29,30,31,32,33].

To discuss the synchronizability of the network model, we show the eigenvalue ratio $R$ versus the number of adding edges $m$ and the number of adding hub nodes $q$ in Fig. 5 (f) and Fig. 6 (f) respectively. Note that in Fig. 5 (f), the case for $q = 1$ corresponds to the highest heterogeneity. The eigenvalue ratio $R$ increases with $m$ greatly, which means that the network becomes more difficult to synchronize with strong heterogeneity, even for short path length. In Fig. 6 (f), the network synchronizability is improved with the increase of $q$. These can all be explained as strong heterogeneity reduces network’s synchronizability,
which is consistent with the conclusion of Nishikawa et. al who has pointed out that networks with homogeneous distributions are more synchronizable than heterogeneous ones [29].

We can conclude that with the introduction of heterogeneity, though the network robustness to random failures and the efficiency of information exchange on the network are greatly improved, the network’s synchronizability is really reduced.

5 Conclusions

One ultimate goal of studies on complex networks is to understand the relationship between the network structure and its functions. To get the optimal strategies of a given function, we should evolve the structure with its function dynamically, which can be realized with optimization approaches. What special characters should the network have with a given function? This problem motivates us to explore the relationship between the structure properties and functions and then get some insightful conclusions.

By optimizing the network structure to improve the performance of the network resilience, we obtain the optimal network and do some statistics of the optimal network. We find that during the optimizing process, the average shortest path length $L$ becomes short. The increase of the maximal degree of the network indicates the hub nodes’ appearance. The degree correlation coefficient $r$ decreases and is always less than zero, which indicates that nodes with high degree preferentially connect with the low degree ones. The clustering coefficient $C$ increases in the whole process and arrives to a high level, then the network shows a high degree of clustering. As we all know that most of the real-world networks in social networks show high clustering, short path length and heterogeneity of degree distributions, which may indicate their good performance to random failures and high efficiency of information exchange. Then we present a nongrowing network model to try to describe the statistical properties of the optimal network and also analyze the synchronizability of the network. And we find that although the network’s robustness to random failures and the efficiency of information exchange are greatly improved (for the average distance of the network is small), the network’s synchronizability is really reduced for the network’s strong heterogeneity.

In summary, we try an alternative point of view to analyze the robustness of the network from its efficiency. By optimizing the network efficiency we find that a network with a small quantity of hub nodes, high degree of clustering may be much more resilient to perturbations. And the results strongly suggest that the network with higher efficiency are more robust to random failures,
though its synchronizability is being reduced greatly.

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