MINIMAL ENTROPY CONDITIONS FOR SCALAR HYPERBOLIC
CONSERVATION LAWS WITH GENERAL CONVEX FLUXES

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To Costas Dafermos on the Occasion of his 80th Birthday
with Admiration and Affection

Abstract. We are concerned with the minimal entropy conditions for one-dimensional scalar hyperbolic conservation laws with general convex and genuinely nonlinear flux functions (not necessarily strictly convex). For such scalar hyperbolic conservation laws, we prove that a single convex entropy-entropy flux pair \((\eta(u), q(u))\) with \(\eta(u)\) of genuine nonlinearity is sufficient to single out the unique entropy solution from a broad class of weak solutions in \(L^\infty_{loc}\) that satisfy the inequality: \(\eta(u)_t + q(u)_x \leq \mu\) in the distributional sense for some Radon measure \(\mu\). Furthermore, we extend this result to the class of weak solutions in \(L^p_{loc}\) and show that the optimal index \(p\) depends on the growth rates of the flux function \(f(u)\) and any fixed entropy function \(\eta(u)\) at infinity. The proofs are based on the equivalence between the entropy solutions of one-dimensional scalar hyperbolic conservation laws and the viscosity solutions of the corresponding Hamilton-Jacobi equations, as well as the bilinear form and commutator estimates as used similarly in the compensated compactness theory.

1. Introduction

We are concerned with the minimal entropy conditions for one-dimensional scalar hyperbolic conservation laws with general convex and genuinely nonlinear flux functions (not necessarily strictly convex):

\[
u_t + f(u)_x = 0 \quad \text{for } (t, x) \in \mathbb{R}^+ \times \mathbb{R} := (0, \infty) \times (-\infty, \infty),
\]

\[
u|_{t=0} = u_0(x),
\]

where the initial data function \(u_0 \in L^p_{loc}\) for \(p \geq 1\). A flux function \(f(u) \in \text{Lip}_{loc}(\mathbb{R})\) is called convex and genuinely nonlinear if \(f(u)\) satisfies that

\[f'(u) \text{ is strictly increasing on } \mathbb{R}.
\]

This is equivalent to saying that \(f(u)\) is convex and satisfies that there is no interval in \(\mathbb{R}\) in which \(f(u)\) is affine, which is a generalized notion of classical genuine nonlinearity. In general, such a flux function \(f(u)\) allows itself nonstrictly convex.

It is well known that, due to the nonlinearity of the flux function \(f(u)\), no matter how smooth the initial data function \(u_0(x)\) is, the solution may form shock waves generically in a finite time. Thus, the solution should be understood in a weak sense, which means that the solution as a function with suitable integrability solves equation \((1.1)\) in the distributional sense. In general, the weak solutions are not unique, so we need some entropy conditions to characterize the unique entropy solution from the weak solutions.

As shown by Oleinik [14], for equation \((1.1)\) with uniformly convex flux \(f(u) \in C^2(\mathbb{R})\), the entropy condition, so called Condition (E), is sufficient to single out a unique weak solution (physically relevant) among all possible weak solutions. Condition (E) is the Oleinik’s one-sided

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inequality for the solution $u(t, x)$: For any $x_2 > x_1$ and $t > 0$, 
\[
\frac{u(t, x_2) - u(t, x_1)}{x_2 - x_1} \leq \frac{1}{ct},
\]
where $c := \inf\{f''(u) : u \in \mathbb{R}\} > 0$. Condition (1.4) implies the regularizing effect that the initial data in $L^\infty$ are regularized to $BV_{\text{loc}}$ instantaneously for the solutions. This condition also yields many fine properties such as the regularity of solutions, the decay rates, and the convergence of approximation schemes, among others; see Dafermos [7] and Lax [12].

For one-dimensional scalar hyperbolic conservation laws with general fluxes, not necessarily convex (even for the multi-dimensional case), a general method for enforcing the uniqueness of solutions in $L^\infty$ was established by Kruzkov [9], by following the earlier results for solutions in $BV_{\text{loc}}$ by Conway-Smoller [6] and Vol’pert [17]. Besides the existence of weak solutions in $L^\infty$, Kruzkov also proved the uniqueness by the so-called Kruzkov’s entropy condition: An entropy solution is a weak solution $u \in L^\infty$ satisfying that 
\[
\eta(u) + q(u)_x \leq 0 \quad \text{ in } D',
\]
where $(\eta(u), q(u)) \in \{(\eta_k(u), q_k(u))\}_{k \in \mathbb{R}}$, the family of which is defined by 
\[
\eta_k(u) := |u - k|, \quad q_k(u) := \text{sgn}(u - k)(f(u) - f(k)).
\]
It is equivalent to say that (1.5) holds for all convex entropy-entropy flux pair $(\eta(u), q(u))$ with convex function $\eta(u)$ and 
\[
q'(u) = \eta'(u)f'(u) \quad \text{a.e. } u \in \mathbb{R}.
\]

When the flux function is uniformly convex, the two entropy conditions (1.4) and (1.5) are equivalent, which implies that the entropy solutions characterized by Oleinik’s condition (E) coincide with the Kruzkov entropy solutions. In 1989, Kruzkov [10] posed an important open question on whether only one single convex entropy $\eta(u)$ satisfying (1.5) can enforce the uniqueness of the solution, which is called the Minimal Entropy Conditions in De Lellis-Otto-Westdickenberg [11]. In view of the lackness of convex entropies for hyperbolic systems of conservation laws, the question of Minimal Entropy Conditions becomes important for the mathematical theory of hyperbolic conservation laws.

Panov [15] first gave a positive answer to this question by proving that the solution $u \in L^\infty$ satisfying (1.1) and (1.5), with a flux function $f(u)$ and an entropy function $\eta(u)$ that are both uniformly convex, is the unique entropy solution in Oleinik’s sense, or equivalently Kruzkov’s sense. This result was also proved by De Lellis-Otto-Westdickenberg [11], in which they further proved that, for the Burgers flux: $f(u) = \frac{1}{2}u^2$ and the special convex entropy $\eta(u) = \frac{1}{2}u^2$, the weak solution $u \in L^1_{\text{loc}}(\Omega)$ satisfying the Minimal Entropy Condition:
\[
\left(\frac{1}{2}u^2\right)_t + \left(\frac{1}{3}u^3\right)_x \leq \mu \quad \text{in } D'(\Omega),
\]
for some non-negative Radon measure $\mu$ with $\lim_{r \to 0} \frac{\mu(B_r)}{r} = 0$, must be the entropy solution of the Burgers equation. For the Kruzkov-type estimates via the entropy inequalities bounded by Random measures, also see Bouchut-Perthame [2].

One of the motivations of this paper is that the uniform convexity is a strong restriction on both the flux function $f(u)$ and the entropy function $\eta(u)$, since most of convex functions $g(u)$ such as those with $g''(u) = 0$ possessing only isolated roots and/or with an asymptotic line, saying $g(u) = \frac{1}{2n}u^{2n}$ with $n > 1$ and $g(u) = e^{ku}$, are not uniformly convex. Another motivation is based on the conjecture by De Lellis-Otto-Westdickenberg [11] that the result for the Burgers equation with the minimal entropy condition (1.7) in [11] could be generalized to allowing for different strictly convex flux functions and entropy functions.

The purpose of this paper is to give a positive answer to the question of Minimal Entropy Conditions by generalizing the previous results in [11, 15] in three aspects:

(i) The flux function $f(u) \in \text{Lip}_{\text{loc}}(\mathbb{R})$ is required to be only convex with genuine nonlinearity in the sense of (1.3) (not necessarily strictly convex).
A function \( h \) (Viscosity solutions)

**Definition 2.2**

almost everywhere,

\[
\eta(u)_t + q(u)_x \leq \mu \quad \text{in } D'(\Omega),
\]

(1.8)

for the flux function \( f(u) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) and any fixed entropy function \( \eta(u) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) satisfying (1.3) only (i.e., convex with genuine nonlinearity), instead of the requirement of uniform convexity. In fact, condition (2.5) in §2 on the nonnegative Radon measure \( \mu \) is optimal in the sense presented in Remark 2.6 below.

(iii) The weak solution \( u(t,x) \) is required to be in \( L^p_{\text{loc}} \) only, instead of \( u \in L^\infty \) or \( u \in L^4_{\text{loc}} \), where the index \( p \) depends on the growth rates of the flux function \( f(u) \) and any fixed entropy function \( \eta(u) \) at infinity.

As in Panov [15] and De Lellis-Otto-Westdickenberg [11], our proof is also based on the relation between one-dimensional scalar hyperbolic conservation laws (1.1) and the definition of viscosity solutions of Hamilton-Jacobi equations:

\[
h_t + f(h_x) = 0 \quad \text{for } (t, x) \in [0, \infty) \times \mathbb{R}.
\]

(1.9)

Formally, (1.1) and (1.9) would be equivalent via the relation \( h_x = u \). It follows from the existence and uniqueness theory for viscosity solutions for (1.9), first introduced by Crandall-Lions in [2] (also see Lions [13]), that, when the Hamiltonian function \( f(u) \) is strictly convex, \( h \) is a viscosity solution of (1.9) if and only if \( u = h_x \) is an entropy solution of (1.1). For the flux function \( f(u) \) satisfying (1.3) only (i.e., convex with genuine nonlinearity, but not necessarily strictly convex), it can be shown that, if \( h \) is a viscosity solution of (1.9), which can be obtained by the vanishing viscosity method, then \( h_x = u \) is an entropy solution of (1.1). In this paper, we prove that the weak solution \( u \) satisfying the minimal entropy condition (1.8) implies that \( h \), with \( h_x = u \), is actually a viscosity solution of (1.9), which implies that \( u \) must be an entropy solution satisfying the entropy inequality (1.5). Our analysis is highly motivated by the arguments in De Lellis-Otto-Westdickenberg [11] (also see Ambrosio-Lecumberry-Riviere [1]); in particular, the proofs are based on the bilinear form and commutator estimates, for which similar arguments have been used in the compensated compactness theory (cf. [4, 10]).

This paper is organized as follows: In §2, we first introduce some basic concepts and then present the main theorems of this paper. In §3, we prove several lemmas on the averages of functions for the subsequent development. In §4, we complete the proof of Theorem 2.3 for the case that \( u \in L^\infty_{\text{loc}} \). In §5, we prove Theorem 2.4 for the case that \( u \in L^p_{\text{loc}} \) under the condition of the growth rates of the flux function \( f(u) \) and any fixed entropy function \( \eta(u) \) at infinity.

2. Basic Concepts and Main Theorems

In this section, we first introduce the definition of entropy solutions of scalar hyperbolic conservation laws (1.1) and the definition of viscosity solutions of Hamilton-Jacobi equations (1.9), respectively. Then we state the main theorems of this paper.

**Definition 2.1** (Entropy solutions). Let \( f(u) \in W_{\text{loc}}^{1,\infty}(\mathbb{R}) \) and \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \). A function \( u \in L^1_{\text{loc}}(\Omega) \) is called an entropy solution of the scalar hyperbolic conservation law (1.1) if, for any convex entropy-entropy flux pair \( (\eta(u), q(u)) \in W_{\text{loc}}^{1,\infty}(\mathbb{R}) \) satisfying \( q'(u) = \eta'(u)f'(u) \) almost everywhere, \( u \) satisfies (1.1) in \( D'(\Omega) \) and

\[
\eta(u)_t + q(u)_x \leq 0 \quad \text{in } D'(\Omega).
\]

(2.1)

**Definition 2.2** (Viscosity solutions). A function \( h \in C(\Omega) \) is called a viscosity solution of the Hamilton-Jacobi equation (1.9) if \( h \) satisfies that, for any \( (t, x) \in \Omega \),

(i) For any \( C^1 \) function \( \varphi \) such that \( h - \varphi \) has a local maximum at \((t, x)\),

\[
\varphi_t(t, x) + f(\varphi_x(t, x)) \leq 0.
\]

(2.2)

(ii) For any \( C^1 \) function \( \varphi \) such that \( h - \varphi \) has a local minimum at \((t, x)\),

\[
\varphi_t(t, x) + f(\varphi_x(t, x)) \geq 0.
\]

(2.3)

A function \( h \) is called a viscosity subsolution (resp. supersolution) if (i) (resp. (ii)) holds.
Our first main theorem is for the case that \( u \in L^\infty_{\text{loc}} \).

**Theorem 2.3** \((u \in L^\infty_{\text{loc}})\). Let the flux function \( f(u) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) and any fixed entropy function \( \eta(u) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) be convex and genuinely nonlinear in the sense of (1.3) (not necessarily strictly convex). Assume that \( u \in L^\infty(\Omega) \) for an open set \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \) satisfies (1.1) in \( D'(\Omega) \) and

\[
\eta(u)_t + q(u)_x \leq \mu \quad \text{in} \ D'(\Omega) \tag{2.4}
\]

for some non-negative Radon measure \( \mu \) satisfying

\[
\lim_{r \to 0} \frac{1}{r} \mu(B_r(t, x)) = 0 \quad \text{for every} \ (t, x) \in \Omega. \tag{2.5}
\]

Then the function \( h \) with \( h_x = u \) and \( h_t = -f(u) \) is a viscosity solution of the Hamilton-Jacobi equation (1.9), and \( u \in L^\infty_{\text{loc}}(\Omega) \) is an entropy solution of the conservation law (1.1) in \( \Omega \).

For the case that \( u \in L^p_{\text{loc}} \) with \( p \geq 1 \), we need the condition on the growth rates of the flux function \( f(u) \) and any fixed entropy function \( \eta(u) \) at infinity: There exist some constants \( M_1, M_2 > 0 \) and \( \alpha, \beta > 0 \) such that

\[
\lim_{u \to \pm \infty} \frac{f'(u)}{\text{sgn}(u)|u|^\alpha} = M_1, \quad \lim_{u \to \pm \infty} \frac{\eta'(u)}{\text{sgn}(u)|u|^\beta} = M_2. \tag{2.6}
\]

**Theorem 2.4** \((u \in L^p_{\text{loc}})\). Let the flux function \( f(u) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) and any fixed entropy function \( \eta(u) \in \text{Lip}_{\text{loc}}(\mathbb{R}) \) be convex and genuinely nonlinear in the sense of (1.3) and satisfy the growth rates in (2.6). Assume that \( u \in L^p_{\text{loc}}(\Omega) \) for an open set \( \Omega \subset \mathbb{R}^+ \times \mathbb{R} \) with

\[
p \geq \alpha + \beta + 2 \tag{2.7}
\]

satisfies (1.1) and (2.4) with (2.5) in \( D'(\Omega) \). Then the function \( h \) with \( h_x = u \) and \( h_t = -f(u) \) is a viscosity solution of the Hamilton-Jacobi equation (1.9) which is H"older continuous in \((t, x)\), and \( u \in L^p_{\text{loc}}(\Omega) \) is an entropy solution of the scalar hyperbolic conservation law (1.1) in \( \Omega \).

**Remark 2.5.** In Theorem 2.4, the condition on the growth rates (2.6) can be replaced by

\[
f'(u) \simeq \pm M_1^\pm |u|^{\alpha \pm}, \quad \eta'(u) \simeq \pm M_2^\pm |u|^{\beta \pm} \quad \text{as} \ u \to \pm \infty, \tag{2.8}
\]

with the index \( p \) in (2.7) replaced by

\[
p \geq \max\{\alpha_+, \alpha_-\} + \max\{\beta_+, \beta_-\} + 2, \tag{2.9}
\]

where the constants \( M_1^\pm, M_2^\pm \) and \( \alpha_+, \beta_\pm \) are all positive numbers.

**Remark 2.6.** Condition (2.5) on the nonnegative Radon measure is optimal in the sense that (2.4) may not enforce the uniqueness if there exists \( c_0 > 0 \) such that, for small \( r > 0 \),

\[
\mu(B_r(\bar{t}, \bar{x})) \geq c_0 r \quad \text{for some points} \ (\bar{t}, \bar{x}) \in \Omega. \tag{2.10}
\]

In fact, for the Riemann problem of (1.1) with initial data \( u_0(x) = u_{\pm} \) (with \( u_{+} > u_{-} \)) for \( \pm (x - x_0) > 0 \), (2.4) with (2.10) admits two weak solutions: one is clearly a rarefaction wave, and the other is an under-compressive shock \( S \) (passing through \( \Omega \)) defined by

\[
u(t, x) = \begin{cases} 
  u_- & \text{if} \ x < x_0 + \sigma_0 t, \\
  u_+ & \text{if} \ x > x_0 + \sigma_0 t,
\end{cases} \tag{2.11}
\]

for sufficient small \( u_{+} - u_{-} > 0 \), where the shock speed \( \sigma_0 \) is determined by \( \sigma_0 = \frac{|f(u)_+|_{\infty}}{|u_{\pm}|_1} \).

In order to show that the under-compressive shock \( u(t, x) \) in (2.11) satisfies (2.4) with (2.10), it suffices to check that \( \mu_+ := \eta(u)_t + q(u)_x \leq \mu \) holds for the points \((\bar{t}, \bar{x}) \in \Omega \cap S\). Let \( u_{+} - u_{-} > 0 \) be sufficient small such that

\[
0 < \int_{u_-}^{u_+} \eta'(s) (f'(s) - \sigma_0) \, ds \leq \frac{c_0}{2}, \tag{2.12}
\]
where, by (3.7) later, the left hand always holds for \( u_+ > u_- \). According to (2.11)–(2.12), for sufficient small \( r > 0 \), we have
\[
\mu_{\eta}(B_r(\bar{t}, \bar{x})) = \int_{B_r(\bar{t}, \bar{x})} (\eta(u)_t + q(u)_x) \, dt \, dx
\]
\[
= \int_{B_r(\bar{t}, \bar{x}) \cap S} (\sigma_0(\eta(u_-) - \eta(u_+)) - (q(u_-) - q(u_-))) \, dt \, dx
\]
\[
= 2r \int_{u_-}^{u_+} \eta'(s)(f'(s) - \sigma_0) \, ds \leq c_0r \leq \mu(B_r(\bar{t}, \bar{x})),
\]
as desired.

**Remark 2.7.** Our proofs depend highly on the generalized genuine nonlinearity of the flux functions \( f(u) \), i.e., \( f'(u) \) is strictly increasing. As shown in Dafermos [8], both the regularity and large-time behavior of entropy solutions of the scalar hyperbolic conservation laws without convexity are highly relative to the quantity of \( f'(u(t, x)) \), instead of the quantity of the solution \( u(t, x) \) itself. For the case of the flux function \( f(u) \) without convexity, a single convex entropy function \( \eta(u) \) may not be sufficient to enforce the uniqueness; instead of which the minimal number of convex entropy functions \( \eta(u) \) needed to enforce the uniqueness may depend on the number of the inflection points of the flux function \( f(u) \), or equivalently the number of the maximum intervals on each of which the function \( f'(u) \), the first derivative of the flux function \( f(u) \), is monotone.

As mentioned above, our proofs are based on the equivalence between the entropy solutions of the one-dimensional scalar hyperbolic conservation laws (1.1) and the viscosity solutions of the corresponding Hamilton-Jacobi equations (1.9).

In fact, this can be seen via the vanishing viscosity method: Let \( h(t, x) \) be the unique viscosity solution of (1.9) with the Cauchy initial data:
\[
h|_{t=0} = h_0(x). \tag{2.13}
\]
Then it can be proved that \( h(t, x) \) can be regarded as the limit function of the viscosity approximate solution sequence \( h^\varepsilon \) when \( \varepsilon \to 0 \), as proved in Crandall-Lions [3], where \( h^\varepsilon \) is the unique solution of the Cauchy problem:
\[
h_t^\varepsilon + f(h_x^\varepsilon) = \varepsilon h_{xx}^\varepsilon \tag{2.14}
\]
with the Cauchy initial data (2.13) for each fixed \( \varepsilon > 0 \). Furthermore, \( v^\varepsilon := h_x^\varepsilon \) solves
\[
v_t^\varepsilon + f(v^\varepsilon)_x = \varepsilon v_{xx}^\varepsilon, \tag{2.15}
\]
and \( v^\varepsilon \to v = h_x \), where \( v \) is the unique entropy solution of (1.1)–(1.2) with \( v(0, x) = h_0'(x) \).

On the other hand, if a function \( \bar{h} \) defined by \( \bar{h}_x = u \) and \( \bar{h}_t = -f(u) \) as in Theorem 2.3 (resp., Theorem 2.4) is a viscosity solution of (1.1) and (2.13), by the uniqueness of viscosity solutions, we conclude that
\[
h = \bar{h}, \quad h_x \stackrel{a.e.}{=} \bar{h}_x.
\]
Thus, the function \( u \) in Theorem 2.3 (resp., Theorem 2.4) satisfies that
\[
u = \bar{h}_x \stackrel{a.e.}{=} h_x = v,
\]
which means that the function \( u \) of Theorem 2.3 (resp., Theorem 2.4) is the unique entropy solution of the scalar hyperbolic conservation law (1.1). In a similar way, if \( u(t, x) \) is the unique entropy solution of the scalar hyperbolic conservation law (1.1) with \( u_0(x) = h_0(x) \), then we can prove that the function \( h(t, x) \) defined by \( h_x = u \) and \( h_t = -f(u) \) satisfying (2.13) is the viscosity solution of (1.9) and (2.13).

**Remark 2.8.** According to Cao-Chen-Yang [3], for a convex flux function \( f(u) \) with genuine nonlinearity in the sense of (1.3), \( h = h(t, x) \) defined by \( h_x = u \) and \( h_t = -f(u) \) is Lipschitz continuous. Furthermore, \( h_x^\pm(t, x) = u^\pm(t, x) \) are well-defined pointwise with
\[
h_x^+(t, x) = u^+(t, x) \leq u^-(t, x) = h_x^-(t, x). \tag{2.16}
\]
Notice that \( \text{(2.16)} \) is equivalent to the following shock admissibility condition:

\[
\langle q(u^+) - q(u^-) \rangle - \langle q(u^+) - q(u^-) \rangle \geq 0,
\]

for any convex entropy-entropy pair \((\eta(u), q(u))\); see Dafermos [7]. This implies that \( u \) is actually the entropy solution of the scalar hyperbolic conservation law \( \text{(1.1)} \).

Therefore, to complete the proof of Theorems 2.3 (resp., Theorem 2.4), it suffices to prove that \( h \) defined by \( h_x = u \) and \( h_t = -f(u) \) as in Theorem 2.3 (resp., Theorem 2.4) is a viscosity solution of \( \text{(1.9)} \) for \( u \in L^\infty_{\text{loc}} \) (resp., \( u \in L^p_{\text{loc}} \)).

3. Averages of Functions

In order to prove the main theorems, we need Proposition 3.2 below on the averages of functions, which generalizes the Proposition 3.2 in De Lellis-Otto-Westdickenberg [11].

**Definition 3.1** (Averages of functions). Assume that \( \mu \) is a probability measure on \( \mathbb{R} \). For every vector-valued map \( w \in L^1(\mathbb{R}, \mu) \), set

\[
\langle w(u) \rangle := \int_{\Omega} w(u) \, d\mu(u).
\]

Let \( f(u), \eta(u) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \) and denote \( q(u) := \int_0^u \eta'(s)f'(s) \, ds \). When \( \mu \) is compactly supported, we define the bilinear form:

\[
B(f, \eta) := \langle (-f(u), u) \cdot (\eta(u), q(u)) \rangle - \langle (-f(u), u) \cdot (\eta(u), q(u)) \rangle
= \langle u\eta(u) \rangle - \langle u \rangle \langle q(u) \rangle - \langle \eta(u)f(u) \rangle - \langle \eta(u)\rangle \langle f(u) \rangle.
\]

(3.2)

When \( \mu \) is of noncompact support, we define \( B(f, \eta) \) whenever the functions in (3.2) are all \( \mu \)-summable.

Then we have the following property of the bilinear form in (3.2):

**Proposition 3.2.** Let \( f(u), \eta(u) \in W^{1,\infty}_{\text{loc}}(\mathbb{R}) \) and \( q(u) := \int_0^u \eta'(s)f'(s) \, ds \).

(i) If \( \tilde{f}(u) = f(u) - (au + b) \) and \( \tilde{\eta}(u) = \eta(u) - (cu + d) \), then

\[
B(f, \eta) = B(\tilde{f}, \tilde{\eta}) = B(f, \tilde{\eta}) = B(\tilde{f}, \tilde{\eta}).
\]

(3.3)

(ii) If \( f(u) \) and \( \eta(u) \) are both convex and genuinely nonlinear, then

\[
B(f, \eta) \geq \langle \eta(u) - \eta(\langle u \rangle) \rangle \langle f(u) - f(\langle u \rangle) \rangle \geq 0.
\]

(3.4)

**Proof.** We now give the proof for the two cases respectively.

(i) Since \( \tilde{f}(u) = f(u) - (au + b) \),

\[
\tilde{q}(u) = \int_0^u \eta'(s)f'(s) \, ds = \int_0^u \eta'(s)(f'(s) - a) \, ds = q(u) - a\eta(u).
\]

(3.5)

Then, by the definition of \( B(f, \eta) \) in (3.2), we have

\[
B(f, \eta) = B(\tilde{f}(u) + au + b, \eta(u))
= \langle u(\tilde{q}(u) + a\eta(u)) \rangle - \langle \eta(u)\tilde{f}(u) \rangle - a\langle u\eta(u) \rangle - b\langle \eta(u) \rangle
+ \langle \tilde{f}(u) \rangle \langle \eta(u) \rangle + a\langle u \rangle \langle \eta(u) \rangle + b\langle \eta(u) \rangle - \langle u \rangle \langle \tilde{q}(u) \rangle - a\langle u \rangle \langle \eta(u) \rangle
= \langle u\tilde{q}(u) \rangle - \langle \eta(u)\tilde{f}(u) \rangle + \langle \tilde{f}(u) \rangle \langle \eta(u) \rangle - \langle u \rangle \langle \tilde{q}(u) \rangle
= B(\tilde{f}, \tilde{\eta}).
\]

Similarly, for \( \tilde{\eta}(u) = \eta(u) - (cu + d) \), we have

\[
B(f, \eta) = B(f, \tilde{\eta}) = B(\tilde{f}, \tilde{\eta}).
\]
Remark 3.3. Thus, we conclude that
c
where
ζ

there exists a unique
f
Since
h
is a viscosity solution.

For the term
\langle P \rangle, P = 0 \text{ when } u = \langle u \rangle. \text{ When } u \neq \langle u \rangle, \text{ by the strictly increasing of } f'(u), \text{ there exists a unique } \zeta \text{ such that }
c(\zeta) := \frac{f(u) - f(\langle u \rangle)}{u - \langle u \rangle} \in [f'(\zeta_0), f'(\zeta_+)].

Since \( f'(u) \) and \( \eta'(u) \) are both strictly increasing, we see that, if \( u > \langle u \rangle \), then
\[ P = (u - \langle u \rangle) \int_u^u \eta'(s) (f'(s) - c(\zeta)) \, ds \]
\[ = (u - \langle u \rangle) \left( \int_\zeta^u \eta'(s) (f'(s) - c(\zeta)) \, ds - \int_\zeta^\zeta \eta'(s) (c(\zeta) - f'(s)) \, ds \right) \]
\[ > (u - \langle u \rangle) \left( \int_\zeta^u d(\zeta) (f'(s) - c(\zeta)) \, ds - \int_\zeta^\zeta d(\zeta) (c(\zeta) - f'(s)) \, ds \right) \]
\[ = (u - \langle u \rangle) d(\zeta) \int_u^u (f'(s) - c(\zeta)) \, ds \]
\[ = 0, \] (3.7)

where \( d(\zeta) \in [\eta'(\zeta_0), \eta'(\zeta_+)] \). Similarly, if \( u < \langle u \rangle \), then
\[ P = (\langle u \rangle - u) \int_u^u \eta'(s) (f'(s) - c(\zeta)) \, ds > 0. \]

Thus, we conclude that \( P \geq 0 \) so that
\( \langle P \rangle \geq 0. \) (3.8)

For the term \( Q \), since \( f(u) \) and \( \eta(u) \) are both convex, we use the Jensen inequality to see that
\[ \langle f(u) \rangle - f(\langle u \rangle) \geq 0, \quad \langle \eta(u) \rangle - \eta(\langle u \rangle) \geq 0. \] (3.9)

Combining (3.6)–(3.9) together, we conclude that
\[ B(f, \eta) = \langle P \rangle + Q \geq \langle \eta(u) \rangle \langle f(u) - f(\langle u \rangle) \rangle \geq 0, \]
as desired. □

Remark 3.3. If \( f(u), \eta(u) \in C^2(\mathbb{R}) \) are both uniformly convex with \( c_1 := \inf f''(u) > 0 \) and \( c_2 := \inf \eta''(u) > 0 \), in view of (3.3), choose \( \tilde{f}(u) \) and \( \tilde{\eta}(u) \) such that \( \tilde{f}(\langle u \rangle) = f'(\langle u \rangle) = 0 \) and \( \tilde{\eta}(\langle u \rangle) = \eta'(\langle u \rangle) = 0 \). By a simple calculation, we have
\[ B(f, \eta) = B(\tilde{f}, \tilde{\eta}) \geq \langle \tilde{\eta}(u) \rangle \langle \tilde{f}(u) \rangle \geq \frac{c_1 c_2}{4} |u - \langle u \rangle|^2 \geq \frac{c_1 c_2}{4} |u - \langle u \rangle|^4. \] (3.10)

4. Proof of Theorem 2.3

In Theorem 2.3, since \( u \in L^\infty_{\text{loc}} \) is assumed, then \( h \) is a local Lipschitz function. We now prove that \( h \) is both a viscosity subsolution and supersolution. Therefore, by Definition 2.2, \( h \) is a viscosity solution.
4.1. Viscosity subsolution. By the definition of \( h \) in Theorem 2.3,

\[
h_t = -f(u) = -f(h_x) \quad \text{a.e. in } \Omega.
\] (4.1)

Let \( \zeta \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}) \) be non-negative with \( \int_{\mathbb{R}^+ \times \mathbb{R}} \zeta \, dt \, dx = 1 \), and set

\[
\zeta_h(t, x) = \frac{1}{\epsilon^2} \zeta(\frac{t}{\epsilon}, \frac{x}{\epsilon}).
\]

Since \( f(u) \) is convex, by the Jensen inequality, we see that

\[
0 = (h_t + f(h_x)) \ast \zeta_h \geq h_t \ast \zeta_h + f(h_x \ast \zeta_h) = (h \ast \zeta) + \hat{f}(h \ast \zeta). 
\]

Hence, \( h \ast \zeta \) is a classical subsolution so that it is also a viscosity subsolution (see Lions [5, Corollary I.6]). Since \( h \) is continuous, \( h \ast \zeta \) converges locally uniform to \( h \) as \( \epsilon \) tends to 0. Thus, \( h \) is also a viscosity subsolution, by the stability result in Lions [5, Theorem I.1].

4.2. Viscosity supersolution. We divide the proof into five steps.

1. To prove that \( h \) is a viscosity supersolution, we need to show that, if \( \varphi \) is a \( C^1 \) function such that \( h - \varphi \) has a minimum at some point \( (t, x) \in \Omega \), then \( (\varphi_t + f(\varphi_x))(t, x) \geq 0 \).

Without loss of generality, we may assume that \( (t, x) = (0, 0) \) and \( (h - \varphi)(0, 0) = 0 \). Then it suffices to show

\[
(\varphi_t + f(\varphi_x))(0, 0) = 0. \tag{4.2}
\]

2. To simplify the notation, we use \( p(\epsilon, \delta) \lesssim k(\epsilon, \delta) \) to denote that there exist some positive constants \( C \) and \( c \) such that

\[
p(\epsilon, \delta) \leq Ck(\epsilon, \delta) \quad \text{for } |(\epsilon, \delta)| \leq c. \tag{4.3}
\]

Let \( \varphi_\epsilon = \varphi - \epsilon \langle t, x \rangle \) with \( \epsilon \in (0, 1) \). Then \( h - \varphi_\epsilon \) has a strict minimum at \( (0, 0) \). We define \( \Omega_{\epsilon, \delta} \) for \( \delta > 0 \) as

\[
\Omega_{\epsilon, \delta} := \{(t, x) : (h - \varphi_\epsilon)(t, x) < \delta \}. \tag{4.4}
\]

Since \( h - \varphi_\epsilon \) has a strict minimum at \( (0, 0) \), from (4.4), we have

\[
h - \varphi_\epsilon = (h - \varphi)(t, x) + \epsilon \langle t, x \rangle \geq (h - \varphi)(0, 0) + \epsilon \langle t, x \rangle = \epsilon \langle t, x \rangle,
\]

which infers that

\[
\Omega_{\epsilon, \delta} \subset \{(t, x) : \epsilon \langle t, x \rangle < \delta \} = B_r(0, 0) \quad \text{for } r := \frac{\delta}{\epsilon}. \tag{4.5}
\]

Denote \( (\cdot)_{\epsilon, \delta} \) by

\[
(u)_{\epsilon, \delta} := \int_{\Omega_{\epsilon, \delta}} u(t, x) \, dt \, dx = \frac{1}{|\Omega_{\epsilon, \delta}|} \int_{\Omega_{\epsilon, \delta}} u(t, x) \, dt \, dx. \tag{4.6}
\]

Then we observe that

\[
\langle (\varphi_t, \varphi_x) \rangle_{\epsilon, \delta} = \langle (-f(u), u) \rangle_{\epsilon, \delta}. \tag{4.7}
\]

Indeed, this can be seen from the definition of \( h \) in Theorem 2.3 that

\[
\langle (-f(u), u) \rangle_{\epsilon, \delta} - \langle (\varphi_t, \varphi_x) \rangle_{\epsilon, \delta} = \langle ((h - \varphi_\epsilon)_t, (h - \varphi_\epsilon)_x) \rangle_{\epsilon, \delta}. \tag{4.8}
\]

Then, from (4.4) and (4.5), we have

\[
\langle (h - \varphi_\epsilon)_t \rangle_{\epsilon, \delta} = \frac{1}{|\Omega_{\epsilon, \delta}|} \int_{\Omega_{\epsilon, \delta}} (h - \varphi_\epsilon)_t \, dt \, dx = \frac{1}{|\Omega_{\epsilon, \delta}|} \int_{\mathbb{R}^2} \left( \min\{h - \varphi_\epsilon - \delta, 0\} \right)_t \, dt \, dx = 0. \tag{4.9}
\]

Similarly, we obtain that \( \langle (h - \varphi_\epsilon)_x \rangle_{\epsilon, \delta} = 0 \). This infers that (4.7) is true.

3. We will prove (4.2) as follows: We will first show in Step 4 below that

\[
0 \leq \langle \eta(u) - \eta((u)_{\epsilon, \delta}) \rangle_{\epsilon, \delta} f(u) - f((u)_{\epsilon, \delta}) \rangle_{\epsilon, \delta} \leq B_{\epsilon, \delta}(f, \eta) \lesssim g(\epsilon, \delta), \tag{4.10}
\]

where \( B_{\epsilon, \delta}(f, \eta) \) is defined by replacing \( (\cdot) \) in (3.2) by \( (\cdot)_{\epsilon, \delta} \) in (4.6) and \( g(\epsilon, \delta) \) is given by

\[
g(\epsilon, \delta) := \frac{1}{\delta} \mu(B_0(0, 0)) + \frac{\delta}{\epsilon} + \epsilon. \tag{4.11}
\]
From (4.5), \( \delta = r \varepsilon \). Then (4.11) implies
\[
\lim_{\varepsilon \to 0} \lim_{r_k \to 0} g(\varepsilon, r_k \varepsilon) = \lim_{\varepsilon \to 0} \lim_{r_k \to 0} \left( \frac{1}{r_k} \mu(B_{r_k}(0, 0))\varepsilon^{-1} + r_k + \varepsilon \right) = 0,
\]
(4.12)
where \( \{r_k\} \) is the subsequence of \( r \) in (2.5) such that
\[
\lim_{r_k \to 0} \frac{1}{r_k} \mu(B_{r_k}(0, 0)) = 0.
\]
(4.13)
Then, with (4.10)–(4.12), we will show in Step 5 below that
\[
\lim_{\varepsilon \to 0} \lim_{r_k \to 0} |\langle f(u) - f(\langle u \rangle_{\varepsilon, r_k \varepsilon}) \rangle_{\varepsilon, r_k \varepsilon}| = 0.
\]
(4.14)
Combining (4.5) and (4.7) with (4.10)–(4.14) together, we have
\[
|\langle \varphi_t + f(\varphi_x) \rangle(0, 0)| \lesssim |\langle \varphi_t \rangle_{\varepsilon, r_k \varepsilon} + f(\langle \varphi_x \rangle_{\varepsilon, r_k \varepsilon})| + r_k
\]
\[
\lesssim |\langle \varphi_t \rangle_{\varepsilon, r_k \varepsilon} + f(\langle \varphi_x \rangle_{\varepsilon, r_k \varepsilon})| + \varepsilon + r_k
\]
\[
= | - \langle f(u) \rangle_{\varepsilon, r_k \varepsilon} + f(\langle u \rangle_{\varepsilon, r_k \varepsilon})| + \varepsilon + r_k
\]
\[
= |\langle f(u) - f(\langle u \rangle_{\varepsilon, r_k \varepsilon}) \rangle_{\varepsilon, r_k \varepsilon}| + \varepsilon + r_k.
\]
(4.15)
Using (4.14), by letting first \( r_k \) and then \( \varepsilon \) go to 0 in (4.15), we conclude (4.2) as desired, which means that \( h \) is a viscosity supersolution.

4. We now prove (4.10) with \( g(\varepsilon, \delta) \) given by (4.11). We first notice that
\[
\delta^2 \lesssim |\Omega_{\varepsilon, \delta}| \quad \text{for small } \delta > 0.
\]
(4.16)
Since \( u \in L_\infty \) and \( \varphi \) is smooth, we use (4.5) and (4.7) to see that
\[
\langle |u| \rangle_{\varepsilon, \delta} \lesssim 1, \quad \langle |f(u)| \rangle_{\varepsilon, \delta} \lesssim 1, \quad \langle |\varphi_t| \rangle_{\varepsilon, \delta} \lesssim 1, \quad \langle |\varphi_x| \rangle_{\varepsilon, \delta} \lesssim 1.
\]
(4.17)
It follows from the definition of \( h \) in Theorem 2.4 that
\[
\langle |h_t| \rangle_{\varepsilon, \delta} \lesssim 1, \quad \langle |h_x| \rangle_{\varepsilon, \delta} \lesssim 1.
\]
(4.18)
Then, from (4.17)–(4.18), we obtain
\[
\langle |((h - \varphi_x)_t, (h - \varphi_x)_x) \rangle \rangle_{\varepsilon, \delta} \lesssim \langle |(h_t, h_x)| \rangle_{\varepsilon, \delta} + \langle |(\varphi_t, \varphi_x)| \rangle_{\varepsilon, \delta} \lesssim 1.
\]
(4.19)
From (4.19), we have
\[
\int_{\mathbb{R}^2} \left| \left( \min\{h - \varphi_x - \delta, 0\}\right)_t, \left( \min\{h - \varphi_x - \delta, 0\}\right)_x \right| \, dtdx
\]
\[
= \int_{\Omega_{\varepsilon, \delta}} \left| \left( (h - \varphi_x)_t, (h - \varphi_x)_x \right) \right| \, dtdx \lesssim |\Omega_{\varepsilon, \delta}|.
\]
(4.20)
Then, by the Sobolev inequality,
\[
\left( \int_{\mathbb{R}^2} \left( \min\{h - \varphi_x - \delta, 0\}\right)^2 \, dtdx \right)^{\frac{1}{2}} \lesssim |\Omega_{\varepsilon, \delta}|,
\]
(4.21)
which, due to the Hölder inequality, implies
\[
- \int_{\mathbb{R}^2} \min\{h - \varphi_x - \delta, 0\} \, dtdx \leq |\Omega_{\varepsilon, \delta}|^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \left( \min\{h - \varphi_x - \delta, 0\}\right)^2 \, dtdx \right)^{\frac{1}{2}} \lesssim |\Omega_{\varepsilon, \delta}|^{\frac{3}{2}}.
\]
(4.22)
Denote
\[
I(\delta) := - \int_{\mathbb{R}^2} \min\{h - \varphi_x - \delta, 0\} \, dtdx = \int_0^\delta |\Omega_{\varepsilon, \sigma}| \, d\sigma.
\]
(4.23)
Then, from (4.22), we obtain the following differential inequality:
\[
I(\delta) \lesssim |\Omega_{\varepsilon, \delta}|^{\frac{3}{2}} = \left( \frac{d}{d\delta} I(\delta) \right)^{\frac{3}{2}}.
\]
(4.24)
Since $I(\delta) > 0$ for $\delta > 0$, it follows from (4.23) that $1 \lesssim \frac{4}{\delta^3} (I(\delta))^{\frac{1}{4}}$, which infers $\delta^3 \lesssim I(\delta)$. Noticing that $|\Omega_{\varepsilon,\delta}|$ is a non-decreasing function of $\delta$, we have
\[
\delta^2 \lesssim \frac{1}{\delta} I(\delta) = \frac{1}{\delta} \int_0^\delta |\Omega_{\varepsilon,\sigma}| \, d\sigma \lesssim |\Omega_{\varepsilon,\delta}|.
\]

We now estimate $B_{\varepsilon,\delta}(f, \eta)$ in (4.10). In fact, from (3.4) and (4.7)–(4.8), we have
\[
B_{\varepsilon,\delta}(f, \eta) = \langle ((-f(u), u) \cdot (\eta(u), q(u))) \rangle_{\varepsilon,\delta} - \langle ((-f(u), u) \cdot (\eta(u), q(u))) \rangle_{\varepsilon,\delta}
\]
\[
+ \langle ((\varphi_{\varepsilon,\varphi_{\varepsilon}}) \cdot (\eta(u), q(u))) \rangle_{\varepsilon,\delta} - \langle ((\varphi_{\varepsilon,\varphi_{\varepsilon}}) \cdot (\eta(u), q(u))) \rangle_{\varepsilon,\delta}
\]
\[
\leq \langle ((h - \varphi_{\varepsilon,x}) \cdot (\eta(u), q(u))) \rangle_{\varepsilon,\delta}
\]
\[
+ \sup_{\Omega_{\varepsilon,\delta}} |(\varphi_{\varepsilon,\varphi_{\varepsilon}}) - (\varphi_{\varepsilon,\varphi_{\varepsilon}})|_{\varepsilon,\delta} |(\eta(u), q(u))|_{\varepsilon,\delta}
\]
\[
=: I_1(\varepsilon, \delta) + I_2(\varepsilon, \delta).
\]

For the term $I_1(\varepsilon, \delta)$, it follows from (2.4), (3.5), and (4.16) that
\[
I_1(\varepsilon, \delta) = \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\Omega_{\varepsilon,\delta}} \langle (h - \varphi_{\varepsilon,x}) \cdot (\eta(u), q(u)) \rangle \, dt \, dx
\]
\[
= \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbb{R}^2} \langle (\min\{h - \varphi_{\varepsilon} - \delta, 0\}) \cdot (\eta(u), q(u)) \rangle \, dt \, dx
\]
\[
= \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbb{R}^2} \langle -\min\{h - \varphi_{\varepsilon} - \delta, 0\} \rangle \cdot (\eta(u) + q(u)) \rangle \, dt \, dx
\]
\[
\leq \frac{1}{|\Omega_{\varepsilon,\delta}|} \int_{\mathbb{R}^2} \max\{\delta - (h - \varphi_{\varepsilon}), 0\} \, d\mu
\]
\[
\leq \frac{\delta}{|\Omega_{\varepsilon,\delta}|} \mu(\Omega_{\varepsilon,\delta}) \lesssim \frac{1}{\delta} \mu(B_{\varepsilon}(0, 0)).
\]

For the term $I_2(\varepsilon, \delta)$, it follows from $u \in L_{\text{loc}}^\infty$ that $\langle |\eta(u)|\rangle_{\varepsilon,\delta} \lesssim 1$ and $\langle |q(u)|\rangle_{\varepsilon,\delta} \lesssim 1$. Then, according to the definition of $I_2(\varepsilon, \delta)$ in (4.25), we have
\[
I_2(\varepsilon, \delta) \lesssim \sup_{\Omega_{\varepsilon,\delta}} |(\varphi_{\varepsilon,\varphi_{\varepsilon}}) - (\varphi_{\varepsilon,\varphi_{\varepsilon}})|_{\varepsilon,\delta} \leq \text{osc}_{\Omega_{\varepsilon,\delta}}(\varphi_{\varepsilon,\varphi_{\varepsilon}}) \leq \frac{1}{\varepsilon} \varphi_{\varepsilon,\varepsilon} + 2\varepsilon \lesssim \frac{\delta}{\varepsilon} + \varepsilon.
\]

According to (4.25)–(4.27), we have
\[
B_{\varepsilon,\delta}(f, \eta) \lesssim \frac{1}{\delta} \mu(B_{\varepsilon}(0, 0)) + \frac{\delta}{\varepsilon} + \varepsilon,
\]
which, by combining (3.4), implies (4.10) with $g(\varepsilon, \delta)$ given by (4.11).

5. We now prove (4.14). Notice that $\Omega_{\varepsilon,\delta} \subset B_{\varepsilon}(0, 0)$ is uniformly bounded in $(\varepsilon, \delta)$ with $\delta \leq \varepsilon$. Since $f(u) \in \text{Lip}_{\text{loc}}(\mathbb{R})$ and $u \in L_{\text{loc}}^\infty$, then for $\delta \leq \varepsilon$, there exists a constant $C > 0$ independent on $\varepsilon$ and $\delta$ such that
\[
|\langle f(u) \rangle_{\varepsilon,\delta} - f(\langle u \rangle_{\varepsilon,\delta})| \leq C |\langle u - \langle u \rangle_{\varepsilon,\delta} \rangle|_{\varepsilon,\delta}.
\]
This means that, in order to prove (4.14), it suffices to show
\[
\lim_{\varepsilon \to 0} \lim_{r \to 0} \langle |u - \langle u \rangle_{\varepsilon,r \varepsilon}| \rangle_{\varepsilon,r \varepsilon} = 0.
\]

In fact, for any fixed $\sigma > 0$, we have
\[
\frac{1}{|\Omega_{\varepsilon,r \varepsilon}|} \int_{\Omega_{\varepsilon,r \varepsilon} \cap \{\langle u - \langle u \rangle_{\varepsilon,r \varepsilon} \rangle \leq \sigma\}} |u - \langle u \rangle_{\varepsilon,r \varepsilon}| \, dx \leq \frac{1}{|\Omega_{\varepsilon,r \varepsilon}|} \int_{\Omega_{\varepsilon,r \varepsilon} \cap \{\langle u - \langle u \rangle_{\varepsilon,r \varepsilon} \rangle \leq \sigma\}} \sigma \, dx \leq \sigma.
\]
Thus, if we can prove that, for any fixed $\sigma > 0$,
\[
\lim_{\varepsilon \to 0} \lim_{r_k \to 0} \frac{1}{|\Omega_{\varepsilon, r_k}|} \int_{\Omega_{\varepsilon, r_k} \cap \{|u - \langle u \rangle_{\varepsilon, r_k} > \sigma\}} |u - \langle u \rangle_{\varepsilon, r_k}| \, dt \, dx = 0,
\]
then it follows from (4.31) that, for any fixed $\sigma > 0$,
\[
\lim_{\varepsilon \to 0} \lim_{r_k \to 0} \langle |u - \langle u \rangle_{\varepsilon, r_k}| \rangle_{\varepsilon, r_k} \leq \sigma,
\]
which yields (4.30) directly.

We now prove (4.32) by contradiction. If (4.32) does not hold, then there exist $\sigma_0 > 0$ and $c_0 > 0$ such that
\[
\lim_{\varepsilon \to 0} \lim_{r_k \to 0} \frac{1}{|\Omega_{\varepsilon, r_k}|} \int_{\Omega_{\varepsilon, r_k} \cap \{|u - \langle u \rangle_{\varepsilon, r_k} > \sigma_0\}} |u - \langle u \rangle_{\varepsilon, r_k}| \, dt \, dx \geq 2c_0 > 0,
\]
which is equivalent to say that there exists a subsequence $(\varepsilon_j, r_{k_j})$ such that, for small $\varepsilon_j$ and $r_{k_j}$,
\[
\lim_{j \to \infty} \frac{1}{|\Omega_{\varepsilon_j, r_{k_j}}|} \int_{\Omega_{\varepsilon_j, r_{k_j}} \cap \{|u - \langle u \rangle_{\varepsilon_j, r_{k_j}} > \sigma_0\}} |u - \langle u \rangle_{\varepsilon_j, r_{k_j}}| \, dt \, dx \geq c_0 > 0.
\]
According to Proposition 3.2 if we choose $\tilde{f}(u)$ and $\tilde{\eta}(u)$ as
\[
\begin{align*}
\tilde{f}(u) &= f(u) - f((u)_{\varepsilon, \delta}) - f'(((u)_{\varepsilon, \delta} + 0)(u - \langle u \rangle_{\varepsilon, \delta}), \\
\tilde{\eta}(u) &= \eta(u) - \eta((u)_{\varepsilon, \delta}) - \eta'((u)_{\varepsilon, \delta} + 0)(u - \langle u \rangle_{\varepsilon, \delta}),
\end{align*}
\]
then, from (3.3)–(3.4) and (4.10), we have
\[
0 \leq \langle \tilde{\eta}(u) \rangle_{\varepsilon, \delta} \tilde{f}(u) \leq B_{\varepsilon, \delta}(f, \eta) = B_{\varepsilon, \delta}(f, \eta) \leq g(\varepsilon, \delta).
\]
Since $f'(u)$ and $\eta'(u)$ are both strictly increasing, by the definition of $\tilde{f}(u)$ and $\tilde{\eta}(u)$ in (4.35), $\tilde{f}(u)$ is strictly increasing with $\tilde{f}((u)_{\varepsilon, \delta}) = 0$ and $(u - \langle u \rangle_{\varepsilon, \delta})\tilde{f}(u) > 0$ for $u \neq (u)_{\varepsilon, \delta}$, and $\tilde{\eta}(u)$ is strictly increasing with $\tilde{\eta}((u)_{\varepsilon, \delta}) = 0$ and $(u - \langle u \rangle_{\varepsilon, \delta})\tilde{\eta}(u) > 0$ for $u \neq (u)_{\varepsilon, \delta}$. Thus, we have
\[
\tilde{f}(u) = \tilde{f}(u) - \tilde{f}((u)_{\varepsilon, \delta}) = \int_0^1 |\tilde{f}'((u)_{\varepsilon, \delta} + \theta(u - \langle u \rangle_{\varepsilon, \delta}))| \, d\theta \, |u - \langle u \rangle_{\varepsilon, \delta}|,
\]
\[
\tilde{\eta}(u) = \tilde{\eta}(u) - \tilde{\eta}((u)_{\varepsilon, \delta}) = \int_0^1 |\tilde{\eta}'((u)_{\varepsilon, \delta} + \theta(u - \langle u \rangle_{\varepsilon, \delta}))| \, d\theta \, |u - \langle u \rangle_{\varepsilon, \delta}|.
\]
Furthermore, if $u > (u)_{\varepsilon, \delta} + \sigma_0$,
\[
\tilde{f}'((u)_{\varepsilon, \delta} + \theta(u - \langle u \rangle_{\varepsilon, \delta})) > \tilde{f}'((u)_{\varepsilon, \delta} + \theta\sigma_0) > 0 \quad \text{for } \theta \in (0, 1],
\]
and, if $u < (u)_{\varepsilon, \delta} - \sigma_0$,
\[
\tilde{f}'((u)_{\varepsilon, \delta} + \theta(u - \langle u \rangle_{\varepsilon, \delta})) < \tilde{f}'((u)_{\varepsilon, \delta} - \theta\sigma_0) < 0 \quad \text{for } \theta \in (0, 1].
\]
According to (4.37), if $u > (u)_{\varepsilon, \delta} + \sigma_0$, then
\[
\tilde{f}(u) \geq \int_0^1 |\tilde{f}'((u)_{\varepsilon, \delta} + \theta\sigma_0)| \, d\theta \, |u - \langle u \rangle_{\varepsilon, \delta}| = \frac{1}{\sigma_0} \tilde{f}((u)_{\varepsilon, \delta} + \sigma_0) |u - \langle u \rangle_{\varepsilon, \delta}|,
\]
and, if $u < (u)_{\varepsilon, \delta} - \sigma_0$, then
\[
\tilde{f}(u) \geq \int_0^1 |\tilde{f}'((u)_{\varepsilon, \delta} - \theta\sigma_0)| \, d\theta \, |u - \langle u \rangle_{\varepsilon, \delta}| = \frac{1}{\sigma_0} \tilde{f}((u)_{\varepsilon, \delta} - \sigma_0) |u - \langle u \rangle_{\varepsilon, \delta}|.
\]
On the other hand, according to (4.35), we have
\[
\tilde{f}((u)_{\varepsilon, \delta} \pm \sigma_0) = f((u)_{\varepsilon, \delta} \pm \sigma_0) - f((u)_{\varepsilon, \delta}) - f'((u)_{\varepsilon, \delta} + 0)(\pm \sigma_0)
\]
\[
= \int_0^1 f'((u)_{\varepsilon, \delta} \pm \theta\sigma_0) \, d\theta \, (\pm \sigma_0) - f'((u)_{\varepsilon, \delta} + 0)(\pm \sigma_0)
\]
\[
= \sigma_0 \int_0^1 |f'((u)_{\varepsilon, \delta} \pm \theta\sigma_0) - f'((u)_{\varepsilon, \delta} + 0)| \, d\theta.
\]
Therefore, from (4.31) and (4.39)–(4.40), we have
\[
(\tilde{f}(u))_{\varepsilon_j, r_k \varepsilon_j} \geq \frac{1}{|\Omega_{\varepsilon_j, r_k \varepsilon_j}|} \int_{\Omega_{\varepsilon_j, r_k \varepsilon_j}} \{ |u - \langle u \rangle_{\varepsilon_j, r_k \varepsilon_j}| > \sigma_0 \} \tilde{f}(u) \, dt \, dx
\]
\[
\geq \frac{1}{\sigma_0} \min \{ \tilde{f}(\langle u \rangle_{\varepsilon_j, r_k \varepsilon_j} \pm \sigma_0) \} 
\times \frac{1}{|\Omega_{\varepsilon_j, r_k \varepsilon_j}|} \int_{\Omega_{\varepsilon_j, r_k \varepsilon_j}} \{ |u - \langle u \rangle_{\varepsilon_j, r_k \varepsilon_j}| > \sigma_0 \} \, dt \, dx
\]
\[
\geq \frac{c_0}{\sigma_0} \min \{ \tilde{f}(\langle u \rangle_{\varepsilon_j, r_k \varepsilon_j} \pm \sigma_0) \}. \quad (4.42)
\]
Similarly, for $\tilde{\eta}(u)$ as in (4.38),
\[
(\tilde{\eta}(u))_{\varepsilon_j, r_k \varepsilon_j} \geq \frac{c_0}{\sigma_0} \min \{ \tilde{\eta}(\langle u \rangle_{\varepsilon_j, r_k \varepsilon_j} \pm \sigma_0) \}, \quad (4.43)
\]
and $\tilde{\eta}(\langle u \rangle_{\varepsilon, \delta} \pm \sigma_0)$ satisfies
\[
\tilde{\eta}(\langle u \rangle_{\varepsilon, \delta} \pm \sigma_0) = \sigma_0 \int_0^1 |\tilde{\eta}'(\langle u \rangle_{\varepsilon, \delta} \pm \theta \sigma_0) - \eta'(\langle u \rangle_{\varepsilon, \delta} \pm \theta \sigma_0)| \, d\theta. \quad (4.44)
\]
Combining (4.42) with (4.43), we obtain
\[
(\tilde{\eta}(u))_{\varepsilon_j, r_k \varepsilon_j} \tilde{f}(u))_{\varepsilon_j, r_k \varepsilon_j} \geq \left( \frac{c_0}{\sigma_0} \right)^2 \min \{ \tilde{f}(\langle u \rangle_{\varepsilon_j, r_k \varepsilon_j} \pm \sigma_0) \} \min \{ \tilde{\eta}(\langle u \rangle_{\varepsilon_j, r_k \varepsilon_j} \pm \sigma_0) \}. \quad (4.45)
\]
In order to pass the limits in (4.45), we choose a subsequence (still denoted by $\varepsilon_j, r_k$) in (4.31) such that $\lim \lim_{\varepsilon_j \to 0, r_k \to 0} \langle u \rangle_{\varepsilon_j, r_k \varepsilon_j} =: \bar{u}$ and the limits of $\langle \tilde{\eta}(u) \rangle_{\varepsilon_j, r_k \varepsilon_j}$ and $\tilde{f}(u))_{\varepsilon_j, r_k \varepsilon_j}$ exist. Since $f'(u)$ and $\eta'(u)$ are both strictly increasing, it follows from (4.41), (4.44), and (4.45) that
\[
\lim_{\varepsilon_j \to 0, r_k \to 0} \lim_{\varepsilon_j \to 0, r_k \to 0} \langle \tilde{\eta}(u) \rangle_{\varepsilon_j, r_k \varepsilon_j} \tilde{f}(u))_{\varepsilon_j, r_k \varepsilon_j} 
\geq \left( \frac{c_0}{\sigma_0} \right)^2 \min \{ \tilde{f}(\bar{u} \pm \sigma_0) \} \min \{ \tilde{\eta}(\bar{u} \pm \sigma_0) \}
\geq c_0^2 \min \{ |f'(\bar{u} \pm \theta \sigma_0) - f'(\bar{u} + 0)| \} \min \{ |\eta'(\bar{u} \pm \theta \sigma_0) - \eta'(\bar{u} + 0)| \}
\geq 0 \quad (4.46)
\]
for some $\theta_1, \theta_2 \in (0, 1)$.
Restricting to the subsequence $(\varepsilon_j, r_k)$, it is clear that (4.46) contradicts to (4.36) with (4.12). We conclude that (4.32) holds, so does (4.30).

5. Proof of Theorem 2.4

In Theorem 2.4 we assume that $u \in L^p_{\text{loc}}$ with $p \geq \alpha + \beta + 2$ as in (2.7). The growth rates of the flux function $f(u)$ and the entropy function $\eta(u)$ at infinity are given by (2.4), which implies that
\[
f(u) \simeq \frac{1}{\alpha + 1} M_1 |u|^{\alpha + 1}, \quad \eta(u) \simeq \frac{1}{\beta + 1} M_2 |u|^{\beta + 1} \quad \text{as } u \to \pm \infty. \quad (5.1)
\]

5.1. Hölder continuity of $h$. By the definition of $h$ in Theorem 2.4 and (5.1):
\[h \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}), \quad h_x = u \in L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}), \quad h_t = -f(u) \in L^1_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}).\]

We claim that $h$ is Hölder continuous in $(t, x)$ on any compact subset in $\mathbb{R}^+ \times \mathbb{R}$. The proof is divided into five steps.
1. Testing with a cut-off function \( \phi \) in \( x \), we obtain from (2.4) that \( \int \phi \eta(u) \, dx \) is locally bounded in \( t \), i.e.,

\[
\eta(u) \in L^\infty_l(\mathbb{R}^+_t, L^1_{loc}(\mathbb{R}^+_x)).
\] (5.2)

Then, using (5.1), we have

\[
u \in L^\infty_l(\mathbb{R}^+_t, L^{\beta+1}_{loc}(\mathbb{R}^+_x)).
\] (5.3)

Since \( h_x = u \), it follows from (5.3) that, for any \( t \in [t_1, t_2] \subset \mathbb{R}^+_t \),

\[
[h(t, \cdot)]_{C^{\alpha,\gamma}_l(\mathbb{R}^+_x)} \lesssim \|u\|_{L^{\beta+1}_{loc}(\mathbb{R}^+_x)} < \infty,
\] (5.4)

where \( \gamma_1 > 0 \) is defined by

\[
\gamma_1 = 1 - \frac{1}{\beta + 1} = \frac{\beta}{\beta + 1}.
\] (5.5)

This implies that \( h \) is Hölder continuous in \( x \), local uniformly in \( t \), i.e.,

\[
h \in L^\infty_l(\mathbb{R}^+_t, C^{\alpha,\gamma}_l(\mathbb{R}^+_x)).
\] (5.6)

2. For the case that \( \alpha \leq \beta \), the Sobolev inequality directly implies that, for any \( t' > t \) and \( x' > x \),

\[
\int_{t}^{t'} \int_{x}^{x'} |u|^{\alpha+1} \, d\tau \, d\xi \leq \left( \int_{t}^{t'} \int_{x}^{x'} |u|^{\beta+1} \, d\tau \, d\xi \right)^{\frac{\alpha+1}{\beta+1}} \left( \int_{t}^{t'} \int_{x}^{x'} (t'(t'(x') - x))^{-\frac{1}{1+\alpha}} \right)^{\frac{\beta+1}{\alpha+1}}.
\] (5.7)

For the case that \( \alpha > \beta \), by choosing \( \beta' \) such that \( 1 + \beta' = \frac{(\beta+1)^2}{\alpha+1} \), we have

\[
\beta' < \beta, \quad \frac{\alpha - \beta'}{\beta - \beta'} (\beta + 1) = \alpha + \beta + 2 \leq p.
\]

Since \( u \in L^p_{loc} \), by the Sobolev inequality, we obtain that, for any \( t' > t \) and \( x' > x \),

\[
\int_{t}^{t'} \int_{x}^{x'} |u|^{\alpha+1} \, d\tau \, d\xi \leq \left( \int_{t}^{t'} \int_{x}^{x'} |u|^{\beta+1} \, d\tau \, d\xi \right)^{\frac{\alpha+1}{\beta+1}} \left( \int_{t}^{t'} \int_{x}^{x'} |u|^{\beta+1} \, d\tau \, d\xi \right)^{\frac{\beta+1}{\alpha+1}}
\]

\[
= \left( \int_{t}^{t'} \int_{x}^{x'} |u|^{\beta+1} \, d\tau \, d\xi \right)^{\frac{\alpha+1}{\beta+1}} \left( \int_{t}^{t'} \int_{x}^{x'} |u|^{\alpha+1} \, d\tau \, d\xi \right)^{\frac{\beta+1}{\alpha+1}}.
\] (5.8)

3. We now show that

\[
h \in C^{\alpha,\gamma_2}_l(\mathbb{R}^+_t, L^1_{loc}(\mathbb{R}^+_x)),
\] (5.9)

where \( \gamma_2 \) is defined by

\[
\gamma_2 = \begin{cases} 
1 & \text{if } \alpha \leq \beta, \\
\frac{\beta+1}{\alpha+1} & \text{if } \alpha > \beta.
\end{cases}
\] (5.10)

Fixed \( t_2 > t_1 \) and \( x_2 > x_1 \). Since \( h_t = -f(u) \), it follows from (5.1) that, for any \( t, t' \in [t_1, t_2] \) with \( t' > t \),

\[
\left| \int_{x_1}^{x_2} h(t', x) \, dx - \int_{x_1}^{x_2} h(t, x) \, dx \right| \leq \int_{x_1}^{x_2} |h(t', x) - h(t, x)| \, dx \\
\leq \int_{t}^{t'} \int_{x_1}^{x_2} |f(u)| \, d\tau \, dx \\
\lesssim \int_{t}^{t'} \int_{x_1}^{x_2} |u|^\alpha \, d\tau \, dx.
\] (5.11)
For the case that $\alpha \leq \beta$, from (5.3), (5.7), and (5.11), we have
\[
\left| \int_{x_1}^{x_2} h(t', x) \, dx - \int_{x_1}^{x_2} h(t, x) \, dx \right| \lesssim \left( \int_{t'}^{t+} \int_{x_1}^{x_2} |u|^{\beta+1} \, dx \, d\tau \right)^{\frac{\alpha+1}{\alpha+1}} \left( \int_{t'}^{t+} \int_{x_1}^{x_2} |u|^{\alpha+\beta+2} \, dx \, d\tau \right)^{\frac{\beta-\alpha}{\alpha+1}} \\
\leq \max_{[t_1, t_2]} \left\{ \|u\|_{L^{\beta+1}([t_1, t_2], x)} \right\} \left( t' - t \right)^{\frac{\beta-\alpha}{\alpha+1}} \\
\lesssim t' - t. \tag{5.12}
\]
For the case that $\alpha > \beta$, since $u \in L^p_{\text{loc}}$ with $p \geq \alpha + \beta + 2$, it follows from (5.3), (5.8), and (5.11) that
\[
\left| \int_{x_1}^{x_2} h(t', x) \, dx - \int_{x_1}^{x_2} h(t, x) \, dx \right| \\
\lesssim \left( \int_{t'}^{t+} \int_{x_1}^{x_2} |u|^{\beta+1} \, dx \, d\tau \right)^{\frac{\beta+1}{\alpha+1}} \left( \int_{t'}^{t+} \int_{x_1}^{x_2} |u|^{\alpha+\beta+2} \, dx \, d\tau \right)^{\frac{\alpha-\beta}{\alpha+1}} \\
\leq \left( \int_{t'}^{t+} \int_{x_1}^{x_2} |u|^{\beta+1} \, dx \, d\tau \right)^{\frac{\beta+1}{\alpha+1}} \left( \int_{t_1}^{t_2} \int_{x_1}^{x_2} |u|^{\alpha+\beta+2} \, dx \, d\tau \right)^{\frac{\alpha-\beta}{\alpha+1}} \\
\lesssim \left( \int_{t'}^{t+} \int_{x_1}^{x_2} |u|^{\beta+1} \, dx \, d\tau \right)^{\frac{\beta+1}{\alpha+1}} \\
\leq \max_{[t_1, t_2]} \left\{ \|u\|_{L^{\beta+1}([t_1, t_2], x)} \right\} \left( t' - t \right)^{\frac{\beta+1}{\alpha+1}} \\
\lesssim (t' - t)^{\frac{\beta+1}{\alpha+1}}. \tag{5.13}
\]
4. By interpolation, (5.6) and (5.9) imply that
\[
h \in L^\infty_{\text{loc}}(\mathbb{R}^n, C^0_{\text{loc}}(\mathbb{R}^n_t)), \tag{5.14}
\]
where $\gamma_3$ is defined by
\[
\gamma_3 = \begin{cases} 
\frac{\beta}{\alpha+1} & \text{if } \alpha \leq \beta, \\
\frac{\beta(\beta+1)}{(\alpha+1)(2\beta+1)} & \text{if } \alpha > \beta.
\end{cases} \tag{5.15}
\]
In fact, let $\zeta \in C^\infty_0(\mathbb{R}_x)$ be non-negative with $\text{spt} \, \zeta \subset (-1, 1)$ and $\int_{\mathbb{R}} \zeta \, dx = 1$. Set $\zeta_\varepsilon := \frac{1}{\varepsilon} \zeta(\frac{x}{\varepsilon})$, and let $*$ denote the convolution in the $x-$variable. Then we have
\[
|h(t', x) - h(t, x)| \leq |h(t', x) - (h * \zeta_\varepsilon)(t', x)| + |(h * \zeta_\varepsilon)(t', x) - (h * \zeta_\varepsilon)(t, x)| \\
\quad + |h(t, x) - (h * \zeta_\varepsilon)(t, x)| \\
\leq \varepsilon^{\gamma_1} \sup_{y, |y-x| \leq \varepsilon} |h(t', x) - h(t', y)| + \sup_{\mathbb{R}} |\zeta| \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |h(t', y) - h(t, y)| \, dy \\
\quad + \varepsilon^{\gamma_2} \sup_{y, |y-x| \leq \varepsilon} |h(t, x) - h(t, y)| \\
\lesssim \varepsilon^{\gamma_1} + \varepsilon^{\gamma_2} |t' - t|^2. \tag{5.16}
\]
Choosing $\varepsilon = |t' - t|^\sigma$ with $\sigma = \frac{\beta+2}{\alpha+1}$ if $\alpha \leq \beta$ and $\sigma = \frac{(\beta+2)^2}{(\alpha+1)(2\beta+1)}$ if $\alpha > \beta$, then $\gamma_1 \sigma = \gamma_2 - \sigma = \gamma_3$. Thus, (5.16) implies (5.14) with $\gamma_3$ defined by (5.15).

5. For any bounded $\mathcal{K} \subseteq \mathbb{R}^+ \times \mathbb{R}$, we have
\[
\sup_{(t,x),(t',x') \in \mathcal{K}} \frac{|h(t', x') - h(t, x)|}{|t' - t|^{\gamma_3} + |x' - x|^{\gamma_1}} < \infty, \tag{5.17}
\]
where $\gamma_1$ and $\gamma_3$ are defined by (5.3) and (5.15), respectively.
In fact, according to (5.6) and (5.14), for any \((t, x), (t', x') \in \mathcal{K} \subset [t_1, t_2] \times [x_1, x_2],\)

\[
\frac{|h(t', x') - h(t, x)|}{|t' - t|^\gamma_1 + |x' - x|^\gamma_1} \leq \frac{|h(t', x') - h(t', x)|}{|t' - t|^\gamma_3 + |x' - x|^\gamma_3} + \frac{|h(t', x) - h(t, x)|}{|t' - t|^\gamma_3 + |x' - x|^\gamma_3}
\]

\[
\leq \text{ess sup}_{[t_1, t_2]}[h(t', \cdot)]_{C^0, \gamma_1}([x_1, x_2]) + \text{ess sup}_{[x_1, x_2]}[h(\cdot, x)]_{C^0, \gamma_3}([t_1, t_2]) < \infty.
\]

### 5.2. Viscosity solution

Similar to §4.1, it can be checked that \(h\) is a viscosity subsolution of (1.9).

Now we show that \(h\) is also a viscosity supersolution of (1.9). That is to say, if \(\varphi\) is a \(C^1\) function such that \(h - \varphi\) has a minimum at \((0,0)\) with \((h - \varphi)(0,0) = 0\), then

\[\varphi + f(\varphi_x)(0,0) = 0.\]  

(5.18)

Following the same arguments as Step 2 in §4.2, we let \(\varphi_\varepsilon = \varphi - \varepsilon|\cdot|(t, x)\) with \(\varepsilon \in (0, 1)\) such that \(h - \varphi_\varepsilon\) has a strict minimum at \((0,0)\), and \(\Omega_{\varepsilon, \delta}\) and \(\langle \cdot \rangle_{\varepsilon, \delta}\) are defined by (4.10) and (5.2), respectively. Then (1.5) and (4.10) all hold.

We divide the remaining proof into three steps.

1. According to (1.25), in order to complete the estimate of \(B_{\varepsilon, \delta}(f, \eta)\) as Step 4 in §4.2, we need the estimates of \(I_1(\varepsilon, \delta)\) in (4.20) and \(I_2(\varepsilon, \delta)\) in (4.22). For the term \(I_1(\varepsilon, \delta)\), we only need to show the last step of (4.20), which is directly implied by

\[\delta^2 \lesssim |\Omega_{\varepsilon, \delta}| \quad \text{for small } \delta > 0.\]  

(5.19)

By the same arguments for \(\Omega_{\varepsilon, \delta}\) as Step 4 in §4.2, the key point for proving (5.19) is to establish (4.18). For this purpose, it suffices to show that

\[|\langle u \rangle_{\varepsilon, \delta}| \lesssim 1, \quad |\langle f(u) \rangle_{\varepsilon, \delta}| \lesssim 1.\]

Since \(|\langle \varphi_{\varepsilon t} \rangle_{\varepsilon, \delta}| \lesssim 1\) and \(|\langle \varphi_{\varepsilon x} \rangle_{\varepsilon, \delta}| \lesssim 1\), by (1.7), we have

\[|\langle u \rangle_{\varepsilon, \delta}| \lesssim 1, \quad |\langle f(u) \rangle_{\varepsilon, \delta}| \lesssim 1.\]  

(5.20)

According to (5.1), we have

\[|\langle u \rangle_{\varepsilon, \delta}| \lesssim |\langle u^{\alpha + 1} \rangle_{\varepsilon, \delta}| \lesssim |\langle f(u) \rangle_{\varepsilon, \delta}| \lesssim |\langle f(u) \rangle_{\varepsilon, \delta}| \lesssim 1,\]  

(5.21)

which yields that (4.18) holds. Thus, by the same argument for (4.20), it follows from (2.4) and (5.19) that

\[I_1(\varepsilon, \delta) \leq \frac{1}{|\Omega_{\varepsilon, \delta}|} \int \max\{\delta - (h - \varphi_\varepsilon), 0\} \, d\mu \leq \frac{\delta}{|\Omega_{\varepsilon, \delta}|} \mu(\Omega_{\varepsilon, \delta}) \lesssim \frac{1}{\delta} \mu(B_{\varepsilon}^r(0,0)).\]  

(5.22)

For the term \(I_2(\varepsilon, \delta)\), we only need to establish the additional estimate of \(|\langle \eta(u) \rangle_{\varepsilon, r_k \varepsilon}\) and \(|\langle q(u) \rangle_{\varepsilon, r_k \varepsilon}\), where \(\{r_k\}\) is the subsequence in (4.13). From (2.6), we obtain that, as \(u \to \pm \infty,\)

\[f'(u) \simeq M_1 \text{sgn}(u)|u|^{\alpha}, \quad \eta'(u) \simeq M_2 \text{sgn}(u)|u|^{\beta}.\]  

(5.23)

Then it is direct to check that

\[q(u) = \int_{0}^{u} \eta'(s)f'(s)ds \simeq \frac{M_1M_2}{\alpha + 1} \text{sgn}(u)|u|^{|\alpha + \beta + 1|} \quad \text{as } u \to \pm \infty,\]  

(5.24)

so that

\[(-f(u), u) \cdot \langle \eta(u), q(u) \rangle \simeq \frac{\alpha \beta M_1M_2}{(\alpha + 1)(\beta + 1)(\alpha + \beta + 1)} |u|^{\alpha + \beta + 2} \quad \text{as } u \to \pm \infty.\]  

(5.25)
According to the definition of $I_1(\varepsilon, \delta)$ in (4.24), from (5.22)–(5.25), we have
$$
\langle |u|^{\alpha+\beta+2} \rangle \lesssim \langle -f(u), q(u) \rangle \varepsilon. 
$$
from (5.22)–(5.25), we have
$$
\langle |u|^{\alpha+\beta+2} \rangle \lesssim \langle -f(u), q(u) \rangle \varepsilon. 
$$
From (5.22)–(5.25), we have
$$
\langle |u|^{\alpha+\beta+2} \rangle \lesssim \langle -f(u), q(u) \rangle \varepsilon. 
$$
for some $\tau \in (0, \frac{1}{2})$.

Therefore, the estimate of $I_2(\varepsilon, \delta)$ in (4.27) becomes
$$
I_2(\varepsilon, \delta) = \sup_{\varepsilon, \delta, \varepsilon, \delta} \langle |u|^{\alpha+\beta+2} \rangle \lesssim \langle -f(u), q(u) \rangle \varepsilon. 
$$
Using (5.22) and (5.29), similar to (4.28), we have
$$
B_{\varepsilon, \delta}(f, \eta) \leq I_1(\varepsilon, \delta) + I_2(\varepsilon, \delta) \lesssim \frac{1}{\varepsilon} \mu(B_{\varepsilon, \delta}(0, 0)) \varepsilon^{-1} + r_k \varepsilon^{-1+\tau} + \varepsilon^\tau. 
$$
Then it follows from (3.3) that
$$
0 \leq \langle \eta(u) - \eta(\langle u \rangle) \rangle \varepsilon \leq \langle -f(u), q(u) \rangle \varepsilon. 
$$
On the other hand, by the definition of $\tilde{g}(\varepsilon, r_k)$ in (5.30) and (4.13), we have
$$
\lim_{\varepsilon \to 0, r_k \to 0} \tilde{g}(\varepsilon, r_k) = 0. 
$$
2. We now show that
$$
\lim_{\varepsilon \to 0, r_k \to 0} \langle f(u) - f(\langle u \rangle) \rangle \varepsilon = 0. 
$$
In fact, there exist two cases related to the growth rates of the flux function $f(u)$ and the entropy function $\eta(u)$ at infinity. From (5.1) and (5.20), we obtain that, as $u \to \pm \infty$,
$$
f(u) - f(\langle u \rangle) \sim |u - \langle u \rangle|^{\alpha+1}, \quad \eta(u) - \eta(\langle u \rangle) \sim |u - \langle u \rangle|^{\beta+1}. 
$$
For the case that $\alpha \leq \beta$, by the Sobolev inequality, we have
$$
\langle |u - \langle u \rangle|^{\alpha+1} \rangle \lesssim \left( \langle |u - \langle u \rangle|^{\beta+1} \rangle \right)^{\frac{\alpha+1}{\beta+1}}. 
$$
Then, according to (5.31)–(5.32) and (5.31), we have
$$
\lim_{\varepsilon \to 0, r_k \to 0} \langle |u - \langle u \rangle|^{\alpha+1} \rangle \varepsilon = 0, 
$$
which, by (5.34), implies (5.33).

For the case that $\alpha > \beta$, by choosing $\beta'$ such that $1 + \beta' = \frac{(\beta+1)^2}{\alpha+1}$, we have
$$
\beta' < \beta, \quad \frac{\alpha - \beta'}{\beta - \beta'}(\beta + 1) = \alpha + \beta + 2 \leq p. 
$$
Since \( u \in L^p_{\text{loc}} \), and \( \Omega_{\varepsilon,\delta} \subset B_{\frac{1}{2}}(0,0) \) is uniformly bounded in \((\varepsilon, \delta)\) with \( \delta \leq \varepsilon \), by the Sobolev inequality, we have
\[
\langle |u - \langle u \rangle_{\varepsilon,\delta}|^{\alpha+1} \rangle_{\varepsilon,\delta} \leq \langle |u - \langle u \rangle_{\varepsilon,\delta}|^{\beta+1} \rangle_{\varepsilon,\delta}^{\frac{\beta+1}{\alpha+1}} \langle |u - \langle u \rangle_{\varepsilon,\delta}|^{\alpha-\beta} \rangle_{\varepsilon,\delta}^{\frac{\alpha-\beta}{\alpha+1}}
\]
\[
= \langle |u - \langle u \rangle_{\varepsilon,\delta}|^{\beta+1} \rangle_{\varepsilon,\delta}^{\frac{\beta+1}{\alpha+1}} \langle |u - \langle u \rangle_{\varepsilon,\delta}|^{\alpha+2} \rangle_{\varepsilon,\delta}^{\frac{\alpha-\beta}{\alpha+1}}
\]
\[
\lesssim \langle |u - \langle u \rangle_{\varepsilon,\delta}|^{\beta+1} \rangle_{\varepsilon,\delta}^{\frac{\beta+1}{\alpha+1}}.
\]

Thus, using (5.31)–(5.32) and (5.34), we also obtain (5.35), so that (5.33) holds.

3. Finally, from (4.3) and (1.7), we conclude
\[
\langle (\varphi_t + f(\varphi_x))(0,0) \rangle \lesssim \langle (\varphi_t)_{\varepsilon,r_k} + f((\varphi_x)_{\varepsilon,r_k}) \rangle + r_k \\
\lesssim \langle (\varphi_t)_{\varepsilon,r_k} + f((\varphi_x)_{\varepsilon,r_k}) \rangle + \varepsilon + r_k \\
= \langle (f(u))_{\varepsilon,r_k} + f((u)_{\varepsilon,r_k}) \rangle + \varepsilon + r_k \\
= \langle (f(u) - f((u)_{\varepsilon,r_k}))_{\varepsilon,r_k} \rangle + \varepsilon + r_k.
\]

Using (5.33), by letting first \( r_k \) and then \( \varepsilon \) go to 0 in (5.36), we obtain
\[
(\varphi_t + f(\varphi_x))(0,0) = 0,
\]
as desired, which means that \( h \) is also a viscosity supersolution.

This completes the proof of Theorem 2.4.

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