Fundamentals of Poisson Lie Groups
with Application to the Classical Double

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Abstract

We give a constructive account of the fundamental ingredients of Poisson Lie theory as the basis for a description of the classical double group \( D \). The double of a group \( G \) has a pointwise decomposition \( D \sim G \times G^* \), where \( G \) and \( G^* \) are Lie subgroups generated by dual Lie algebras which form a Lie bialgebra. The double is an example of a factorisable Poisson Lie group, in the sense of Reshetikhin and Semenov-Tian-Shansky [1], and usually the study of its Poisson structures is developed only in the case when the subgroup \( G \) is itself factorisable. We give an explicit description of the Poisson Lie structure of the double without invoking this assumption. This is achieved by a direct calculation, in infinitesimal form, of the dressing actions of the subgroups on each other, and provides a new and general derivation of the Poisson Lie structure on the group \( G^* \). For the example of the double of SU(2), the symplectic leaves of the Poisson Lie structures on SU(2) and SU(2)* are displayed.
I. Introduction

A Poisson Lie group is a Lie group with a compatible Poisson structure. Here compatibility refers to group multiplication, which is required to be a Poisson map from the product manifold $G \times G$ to the group $G$. Consequently the Poisson structure is termed multiplicative. Much of the motivation for the study of such a structure is provided by the dressing transformation group of a classical integrable system. This is the symmetry group of the system in the Lax pair formalism and it carries a natural Poisson structure which is multiplicative. A quantised integrable system has a quantum group symmetry and so Poisson Lie groups also generate interest as the classical counterparts of quantum groups.

Associated to any Poisson Lie group $G$ is a Poisson Lie group $G^*$ and also a Lie bialgebra. The groups $G$ and $G^*$ form a dual Poisson Lie pair, so termed because their Lie algebras, $\mathcal{G}$ and $\mathcal{G}^*$ respectively, are dual to each other as vector spaces. The linearisation of the Poisson structure on one group produces the Lie algebra structure of the dual. The condition which fixes the Poisson structures to be multiplicative also ensures that these dual Lie algebras satisfy the defining property of a Lie bialgebra. This is because a multiplicative Poisson structure and a Lie bialgebra are both defined in terms of a 1-cocycle.

In many cases, and notably for semisimple Lie groups, the 1-cocycle is constructed from a solution of a Yang-Baxter type equation, denoted $r \in \mathcal{G} \otimes \mathcal{G}$ and called an r-matrix. Particular interest is shown when a Poisson Lie group is linked with a quasitriangular r-matrix, that is, an r-matrix which is a non-antisymmetric solution of the classical Yang-Baxter equation. Such an r-matrix can be used to construct an isomorphism between the universal enveloping algebras of the dual Lie algebras $\mathcal{G}$ and $\mathcal{G}^*$ in such a way as to pick out a unique factorisation property of the group $G$. Hence the group and the Lie bialgebra associated to a quasitriangular r-matrix are both described as factorisable. An important class of factorisable groups is provided generically by the double group. This is constructed from any dual Poisson Lie pair so as to have a pointwise factorisation into the product of these groups. An example is the group $\text{SL}(2, \mathbb{C})$ which is the double of $\text{SU}(2)$. Any element of $\text{SL}(2, \mathbb{C})$ factors uniquely into the product of an $\text{SU}(2)$ and an $\text{SU}(2)^*$ group element; here the factorisation property simply provides a restatement of the Iwasawa decomposition.

The double group provides a natural setting for the study of a dual Poisson Lie pair, since it
incorporates both groups together. This is reflected in much of the early literature. Drinfel’d [2] was the first to explicitly write down the structure of a Poisson Lie group, relating it to solutions of the modified Yang-Baxter equation and also to Lie bialgebras, from which he constructed the double Lie algebra. Semenov-Tian-Shansky [3] established that the Poisson Lie structures of $G$ and $G^*$ could be obtained by reduction from a symplectic structure on the double group, and also gave the multiplicative Poisson structure for $G^*$ in the case when $G$ is factorisable. Importantly, this paper also presented dressing transformations as Poisson Lie actions, and showed that their orbits give the symplectic leaves of the structures on $G$ and $G^*$. The term factorisable was actually coined in a later work with Reshetikhin [1], where it was used for Lie bialgebras but with the factorisation at group level explained using the double as the principal example. Lu and Weinstein [4] noticed that the Iwasawa decomposition [5] is a concrete realisation of factorisation of the double group, whilst in an independently motivated approach Majid [6] derived this result. In an as yet unfinished line of work, Kosmann-Schwarzbach and Magri [7] have put the double Lie algebra in the context of twilled extensions, as well as relating the Yang-Baxter equations to conditions on a Schouten curvature. A worked example of the general formalism of a dual Poisson Lie pair has been given by Babelon and Bernard [8] for a Toda system.

The Poisson Lie structure of the double group has been well documented in these papers, yet much of the presentation has been at a formal level only. It is straightforward to write down the multiplicative Poisson tensor since the $r$-matrix of the double is well known. However, to actually write down the corresponding Poisson brackets requires a calculation, at the infinitesimal level, of the actions of the subgroups $G$ and $G^*$ on each other. These actions are the abstract description of dressing transformations. Semenov-Tian-Shansky [9] briefly indicated how they can be inferred by considering the group $G$ to be factorisable, and most explicit presentations of the Poisson brackets of the double concentrate on such a scenario. This, however, restricts the Poisson Lie pairs that are described. In particular, the important example of the dual pair constructed from the maximal compact subgroup of a complex semisimple Lie group is not addressed.

With this in mind, we provide an alternative approach to the description of the double group without restricting to the class of Poisson Lie dual pairs generated by a factorisable group. We construct a new isomorphism between the enveloping algebras of a dual Poisson Lie
pair which is more generally applicable than the factorisation isomorphism. This facilitates a direct calculation of the dressing transformations required to display the structure of the double group. The result is a simple working form for the Poisson structure of the double group, from Poisson tensor to Poisson brackets, and a completely explicit general presentation of the Poisson Lie structure on the group $G^\ast$.

We give an introduction to Lie bialgebras and then construct specific examples. The double-sided nature of Lie bialgebras is made explicit. We describe the usual r-matrix approach to the defining cocycle condition which is used when constructing a Lie bialgebra from a semisimple Lie algebra. However, we also describe how to start with the dual non-semisimple Lie algebra and satisfy the cocycle condition. Within the parallel development of a complex semisimple Lie algebra and its maximal compact real subalgebra we describe the factorisation isomorphism of a Poisson Lie dual pair, applicable in the former case only, and show that there is an alternative isomorphism which is valid in both cases. After a description of the double Lie algebra and multiplicative Poisson structures this new isomorphism is applied to the calculation of the Poisson Lie structure of the double group. We conclude with the example of the Poisson Lie structures on SU(2) and SU(2)$^\ast$ worked through in coordinates, and display their symplectic leaves.

Our analysis looks only at the local properties of Poisson Lie groups and does not address their global geometry.

II. Lie Bialgebras

A Lie bialgebra is constructed from a Lie algebra by endowing its dual vector space with a Lie bracket. This is done subject to a particular compatibility condition, which is usually satisfied using the solution of a Yang-Baxter type equation – an r-matrix. An alternative method, applicable to the duals of non-semisimple Lie algebras, is given in the next section. Consider a Lie algebra $\mathcal{G}$, its dual vector space $\mathcal{G}^\ast$, and a pairing between them denoted by $< t^a, t_b > = \delta^a_b$ for basis vectors $t^a \in \mathcal{G}^\ast$, $t_a \in \mathcal{G}$. We can define a Lie bracket $[~,~]$ on $\mathcal{G}^\ast$ via a map, the Lie cobracket

$$\delta : \mathcal{G} \rightarrow \mathcal{G} \wedge \mathcal{G},$$

$$< [\xi, \zeta], X > = < \xi \otimes \zeta, \delta(X) > , \quad \xi, \zeta \in \mathcal{G}^\ast , \quad X \in \mathcal{G}.$$
Here and elsewhere we use capital Roman letters $X, Y, Z \ldots$ to denote elements of $G$ and Greek letters $\xi, \zeta, \chi \ldots$ to denote elements of $G^*$. The structure constants for $G^*$ and $G$ are

$$\left[ t^a, t^b \right]_* = C^{ab}_c t^c, \quad \left[ t^a, t_b \right] = f^{c}_{ab} t^c.$$

The pair $(G, G^*)$ constitutes a Lie bialgebra when the Lie brackets satisfy a compatibility condition. This condition was introduced by Drinfel’d as the infinitesimal version of the algebra map condition on a coproduct, and requires that the Lie cobracket of $G^*$ should be represented as a 1-cocycle on $G$, with respect to the adjoint representation. That is

$$\delta([X,Y]) = X.\delta(Y) - Y.\delta(X), \quad X, Y \in G,$$

where $X.(Y \land Z) = [X \otimes 1 + 1 \otimes X, Y \land Z]$. It is important to note that this condition is symmetric between $G$ and $G^*$, as can be seen, for instance, by writing it in terms of structure constants. Hence it also ensures that the Lie cobracket of $G$ is a 1-cocycle on $G^*$, and so $(G^*, G)$ is also a Lie bialgebra.

The easiest way to satisfy (1), and the only way for semisimple $G$, is to fix $\delta$ to be the coboundary of an element $r = r^{ab} t^a \otimes t^b \in G \otimes G$, considered as a 0-cochain, so that

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r], \quad X \in G,$$

$$< [\xi, \zeta], X > = < \xi \otimes \zeta, [X \otimes 1 + 1 \otimes X, r] >, \quad \xi, \zeta \in G^*.$$

Antisymmetry of the Lie cobracket requires

$$[X \otimes 1 + 1 \otimes X, r + \tau(r)] = 0,$$

where $\tau$ is the permutation operator, $\tau(X \otimes Y) = Y \otimes X$, so that only the antisymmetric part of $r$ contributes to the structure of $G^*$.

The Jacobi identity is satisfied provided

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, B_r] = 0,$$

where

$$B_r = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in G \otimes G \otimes G.$$
in which, for example, \( r^{ab}t_a \otimes 1 \otimes t_b \) is denoted by \( r_{13} \). The ad-invariance of \( B_r \), condition (4), is known as the modified Yang-Baxter equation (MYBE) for \( r \). The more restrictive condition \( B_r = 0 \) is called the classical Yang-Baxter equation (CYBE).

If we decompose \( r \) into symmetric and antisymmetric parts, \( r = s + a, \tau(s) = s, \tau(a) = -a \), then we can use (3) to write (4) as a requirement on the antisymmetric part of \( r \) only. From (3) it follows that \( B_s \) is ad-invariant, and that \( B_r = B_s + B_a \), so that \( B_r \) is ad-invariant if and only if \( B_a \) is ad-invariant.

The structure constants of \( \mathcal{G}^* \) are

\[
C_{\alpha \beta}^{\gamma} = f_{\beta \alpha}^{b} r^{ad} + f_{\alpha \beta}^{a} r^{db}.
\]  

Again it should be recognised that the symmetric part of \( r \) does not contribute to this expression. For semisimple \( \mathcal{G} \) expression (6) leads to a non-semisimple dual Lie algebra.

In section III we indicate how, under certain quite broad conditions, this equation can be inverted to give the structure constants of \( \mathcal{G} \) in terms of those on \( \mathcal{G}^* \) - see (8).

We can consider \( r \) to be a linear operator

\[
r : \mathcal{G}^* \rightarrow \mathcal{G},
\]

\[
r(\xi) = \langle \xi \otimes \text{id}, r \rangle, \quad \xi \in \mathcal{G}^*,
\]

with transpose

\[
r^* : \mathcal{G}^* \rightarrow \mathcal{G},
\]

\[
r^*(\xi) = \langle \xi \otimes \text{id}, \tau(r) \rangle = \langle \text{id} \otimes \xi, r \rangle.
\]

This allows us to write the bracket on \( \mathcal{G}^* \) as

\[
[\xi, \zeta]_* = -\ad^* r(\xi).\zeta - \ad^* r^*(\zeta).\xi,
\]

where we use the convention \( \langle \ad^* X.\xi, Y \rangle = \langle \xi, \ad X.Y \rangle = \langle \xi, [X, Y] \rangle \).

**III. Group Factorisation**

Following the discussion in section II we see that it is possible to specify a Lie bialgebra in terms of a solution of the MYBE; the structure constants of \( \mathcal{G}^* \) are given by (6). In this
section we shall implement this for two generic examples; a complex, semisimple Lie algebra \( \mathcal{G} \), and its real, maximal, compact subalgebra \( \mathcal{K} \). This incorporates the useful cases of \( \text{sl}(2,\mathbb{C}) \) and \( \text{su}(2) \). The practical distinction between these two examples is that \( \mathcal{G} \otimes \mathcal{G} \) admits a non-antisymmetric solution of the classical Yang-Baxter equation, i.e., a quasitriangular r-matrix, but \( \mathcal{K} \otimes \mathcal{K} \) does not. In both instances the second rank Casimir element \( t_a \otimes t_a \), which defines a natural vector space isomorphism between \( \mathcal{G}^* (\mathcal{K}^*) \) and \( \mathcal{G} (\mathcal{K}) \), can be defined to be an isomorphism of Lie algebra structures, equipping the vector space \( \mathcal{G} (\mathcal{K}) \) with a new Lie bracket. We describe the extension of this Lie algebra isomorphism to the enveloping algebras, as given by Reshetikhin and Semenov-Tian-Shansky [3], which produces a unique factorisation of the group associated to \( \mathcal{G} \). We also give a new extension of the same Lie algebra isomorphism in a way which is valid for both \( \mathcal{G} \) and \( \mathcal{K} \). This alternative shall prove useful when discussing the Poisson structure of the double group.

To facilitate a straightforward comparison of the analysis for \( \mathcal{K} \) and \( \mathcal{G} \) we shall consider expressions in the basis

\[
\begin{align*}
t_a &= iH^r, \quad V^\alpha, W^\alpha \\
V^\alpha &= \frac{i}{\sqrt{2}}(E^\alpha + E^{-\alpha}), \quad W^\alpha = \frac{1}{\sqrt{2}}(E^\alpha - E^{-\alpha}) \quad (\alpha > 0)
\end{align*}
\]

where \([H^r, E^\alpha] = \alpha^r E^\alpha, [E^\alpha, E^{-\alpha}] = \alpha^r H^r\). This basis is suitable for both \( \mathcal{K} \), over \( \mathbb{R} \), and \( \mathcal{G} \), over \( \mathbb{C} \).

Choosing \( \text{Trace}(t_a t_b) = y \delta_{ab}, \ y \) dependent upon the representation, we work with a non-degenerate, invariant inner product denoted \(( , ) = y^{-1}\text{Trace}( . , )\). Taken with the square of the longest root to be unity this is intrinsic to \( \mathcal{G} \) and so independent of any choice of basis or representation [10].

A general solution of the MYBE for \( \mathcal{G} \) takes the form [11]

\[
r = \lambda \sum_{\alpha > 0} (V^\alpha \otimes W^\alpha - W^\alpha \otimes V^\alpha) + \mu t_a \otimes t_a .
\]

If \( \lambda = \pm i\mu \) we have a solution of the CYBE, but to be admissible as an r-matrix for \( \mathcal{K} \) we must have \( \lambda \) and \( \mu \) both real. This leads us to choose the following r-matrices for \( \mathcal{K} \) and \( \mathcal{G} \) respectively

\[
\begin{align*}
k_+ &= \frac{1}{2}(I + r_0) \in \mathcal{K} \otimes \mathcal{K} , \\
r_+ &= \frac{1}{2}(-iI + r_0) \in \mathcal{G} \otimes \mathcal{G} ,
\end{align*}
\]
where \( r_0 = \sum_{\alpha>0} (V^\alpha \otimes W^\alpha - W^\alpha \otimes V^\alpha) \) and \( I = t_a \otimes t_a \), the second rank Casimir. It should be noted that the complex nature of \( \mathcal{G} \) means that \( ir_+ \) is also an acceptable r-matrix.

We emphasise that

\[
B_{r_\pm} = 0, \quad B_{k_\pm} = \frac{1}{2} B_{r_0} = \frac{1}{2} B_I = \frac{1}{2} f_{ab} t_a \otimes t_b \otimes t_c,
\]

where we denote \( \tau(r_+) = -r_- \) and \( \tau(k_+) = -k_- \). In particular, the conditions imposed on r-matrices in [9] are satisfied by \( ir_+ \) for \( \mathcal{G} \), but are not satisfied by \( k_+ \) for \( \mathcal{K} \).

We remark that if the antisymmetric part of an r-matrix satisfies the MYBE in the fashion given by (7), then it is possible to invert the relation between the structure constants of \( \mathcal{G}^* \) and \( \mathcal{G} \), given in (6). That is, we can construct the 1-cocycle on \( \mathcal{G}^* \) which corresponds to the usual Lie bracket on \( \mathcal{G} \). It is given by

\[
< \xi, [X, Y] > = < \bar{\delta}(\xi), X \otimes Y > ,
\]

\[
\bar{\delta}(\xi) = ( < [t^a, t^b]_*, r_0(\xi) > + < [t^b, \xi]_*, r_0(t^a) > + < [\xi, t^a]_*, r_0(t^b) > ) t^a \otimes t^b ,
\]

so that

\[
f_{ab}^c = C_{ab}^{cd} r_0^{cd} + C_{ad}^{bc} r_0^{ad} + C_{ca}^{bd} r_0^{bd} .
\]

This allows for a double-sided understanding of the structure of the Lie bialgebra in this case.

The Lie cobrackets for \( \mathcal{K}^* \) and \( \mathcal{G}^* \) are

\[
\delta_{\mathcal{K}}(X) = [X \otimes 1 + 1 \otimes X, k_{\pm}] = \frac{1}{2} [X \otimes 1 + 1 \otimes X, r_0] ,
\]

\[
\delta_{\mathcal{G}}(X) = [X \otimes 1 + 1 \otimes X, r_{\pm}] = \frac{1}{2} [X \otimes 1 + 1 \otimes X, r_0] .
\]

These are the same, by design, so that \( \mathcal{G}^* \) is the complexification of \( \mathcal{K}^* \), as with their dual Lie algebras.

We may regard \( I = t_a \otimes t_a \) as defining the vector space isomorphism

\[
I : \mathcal{G}^* \rightarrow \mathcal{G} ,
\]

\[
I : t^a \mapsto < t^a \otimes id, I > = t_a .
\]

Upon restriction to \( \mathcal{K}^* \), that is, over \( \mathbb{R} \), we also clearly have \( I : \mathcal{K}^* \rightarrow \mathcal{K} \).
In addition,

\[ (X, Y) = < I^{-1}(X), Y > = < I^{-1}(Y), X > . \]

This allows us to consider \( I \) to be a Lie algebra isomorphism, defining a second Lie algebra structure, denoted by \( [,]_R \), on the vector space \( \mathcal{G} \) via

\[ < [I^{-1}(X), I^{-1}(Y)]_*, Z > = < I^{-1}[X, Y]_R, Z > = ([X, Y]_R, Z) . \]

Using (6) we can write this second Lie bracket as

\[ [X, Y]_R = [R_\pm X, Y] + [X, R_\mp Y] = \frac{1}{2} ([R_0 X, Y] + [X, R_0 Y]) , \]

where we set \( R_\pm \equiv r_\pm \circ I^{-1} \). Again, if we replace \( r_\pm \) by \( k_\pm \) this will reduce to the same form, so it is equally valid as a new bracket on the vector space \( \mathcal{K} \).

We signify with a subscript \( R \) a vector space with this new Lie bracket, so that

\[ I: \mathcal{G}^* \to \mathcal{G}_R \]

is a Lie algebra isomorphism, and over the real field \( I: \mathcal{K}^* \to \mathcal{K}_R \).

Since \( I = i(r_+ - r_-) = k_+ - k_- \), any element of \( \mathcal{G} \) or \( \mathcal{K} \) admits a unique decomposition

\[ \mathcal{G} \ni X = I(\xi) = iX_+ - iX_- , \quad X_\pm = r_\pm(\xi) , \]
\[ \mathcal{K} \ni Y = I(\zeta) = Y_+ - Y_- , \quad Y_\pm = k_\pm(\zeta) . \]

However, there is a further property unique to \( r_\pm \) as solutions of the CYBE. The expression (5) for \( B_r \) can be rewritten

\[ < I^{-1}(X) \otimes I^{-1}(Y) \otimes \text{id}, B_r > = [RX, RY] - R([X, RY] - [R^T X, Y]) . \]

Hence, it follows that

\[ [R_\pm X, R_\pm Y] = R_\pm ([X, Y]_R) , \]
\[ [r_\pm(\xi), r_\pm(\zeta)] = r_\pm([\xi, \zeta]_*) . \]

That is, \( r_\pm \) are Lie algebra homomorphisms. In these circumstances the Lie bialgebra \( (\mathcal{G}, \mathcal{G}^*) \) is said to be factorisable, terminology which will become clearer when we investigate the associated group.
Thus, turning our attention to the corresponding universal enveloping algebras, we have natural extensions of $r_\pm$ as algebra maps,

\[
\tilde{r}_\pm : U(G^*) \to U(G) , \\
\tilde{r}_\pm(1*) = 1 , \\
\tilde{r}_\pm(\xi \zeta) = r_\pm(\xi)r_\pm(\zeta) , \quad \text{etc.}
\] (11)

The map $I' = -iI$, which is equivalent to the isomorphism $I$ when acting on a complex Lie algebra, can be extended [3] in the form

\[
\tilde{I}' = m \circ (\text{id} \otimes S) \circ (\tilde{r}_+ \otimes \tilde{r}_-) \circ \Delta : U(G^*) \to U(G_R) ,
\]

where $\Delta$ is the usual coproduct, $\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi$, and $S$ the usual antipode, $S(\xi) = -\xi$, extended as algebra and anti-algebra maps respectively. Denoting the product in $U(G_R)$ by a $\ast$, this produces

\[
X \ast Y = R_+X.Y - Y.R_-X = \frac{1}{2}(R_0X.Y - Y.R_0X - iXY - iYX) \quad (12)
\]

Applied at the level of the group, within a suitable completion of the enveloping algebra, this indicates that group elements have a unique factorisation

\[
\tilde{I}' : G^* \to G_R , \\
\exp\{r_+(\xi)\}, \exp\{-r_-(\xi)\} = g_+g_-^{-1} = g . \quad (13)
\]

This factorisation of group elements is valid in a region around the identity. For a study of the conditions required for this factorisation to be a global property see Lu’s thesis [12].

The group $G_R$ has the same elements as $G$ but a different product law, given by

\[
g \ast h = g_+h_+(g_-h_-)^{-1} = g_+hg_-^{-1} . \quad (14)
\]

The map $\tilde{I}'$ cannot be constructed from $k_\pm$, because they are not Lie algebra isomorphisms, and so it is necessarily complex, as seen in (12) for example. Hence it does not map $U(K^*)$ into the real algebra $U(K_R)$. However, this extension of $I$ as an algebra map is not unique, the only requirement is that $U(G_R)$ be defined such that $\tilde{I}(\xi\zeta) - \tilde{I}(\zeta\xi) = [I(\xi), I(\zeta)]_R$. 

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It shall prove useful when discussing the Poisson structure of the double group to have an extension which does map real algebras to real algebras. This can be achieved in a straightforward manner by defining

\[
\tilde{I} : U(\mathcal{G}^*) \rightarrow U(\mathcal{G}_R), \\
\tilde{I}(1_*) = 1_R, \\
\tilde{I}(\xi_\zeta \chi \ldots) = i(\tilde{r}_+ - \tilde{r}_-)(\xi_\zeta \chi \ldots). 
\] (15)

This leads to

\[
\tilde{I}(\xi_\zeta) = i\tilde{r}_+(\xi_\zeta) - i\tilde{r}_-(\xi_\zeta), \\
= i\tilde{r}_+(\xi)\tilde{r}_+(\zeta) - i\tilde{r}_-(\xi)\tilde{r}_-(\zeta), \\
= \frac{1}{2}\{I(\xi)r_0(\zeta) + r_0(\xi)I(\zeta)\}, \\
= \tilde{I}(\xi) * \tilde{I}(\zeta), 
\] (16)

so that, in this instance, the product in \(U(\mathcal{G}_R)\) is

\[
X * Y = \frac{1}{2}(X.R_0Y + R_0X.Y) 
\] (17)

As required, \(X * Y - Y * X = [X, Y]_R\), but now the product in \(U(\mathcal{G}_R)\) is real so that it is equally valid for \(U(\mathcal{K}_R)\), but without replacing \(r_\pm\) with \(k_\pm\) at any stage. At the group level we find that

\[
\tilde{I} : e^\xi \mapsto e^{\tilde{I}(\xi)} = 1 + i[\exp(r_+(\xi)) - \exp(r_-(\xi))] = 1 + i(g_+ - g_-) = g. 
\]

The product law is equivalent to that in the previous case, that is

\[
g * h = 1 + i(g_+h_+ - g_-h_-). 
\]

**IV. The Double**

The double of a Lie algebra can be used to construct a factorisable Lie bialgebra and is an object of considerable interest. We give a description of its properties as a prelude to examining the Poisson Lie structure of the associated group.
To motivate the construction of the double we observe that, with the decomposition of vectors \( X = X_+ - X_- \) the Lie bracket (9) on \( G_R \) can be written

\[
[X, Y]_R = [X_+, Y_+] - [X_-, Y_-].
\]

This bracket resembles the direct sum of two Lie algebras. Writing \( G = G_+ + G_- \) as a direct sum of vector spaces then \( G_\pm \) are Lie subalgebras if \([X_\pm, Y_\pm] \in G_\pm \) for all \( X_\pm, Y_\pm \in G_\pm \). In this case \( G_R = G_+ \oplus G_{-opp} \) as a direct sum of Lie algebras, where \( G_{-opp} \) signifies \( G_- \) with the opposite Lie bracket. Also, if the Lie bialgebra \((G, G^*)\) is factorisable, \( \pm R_\pm \) are projection operators from \( G_R \) on to \( G_\pm \), since by (10)

\[
[X_\pm, Y_\pm] = [R_\pm X, R_\pm Y] = R_\pm ([X, Y]_R) = R_\pm ([X_+, Y_+] - [X_-, Y_-])
\]

The double \( D \) is constructed so that \( G_+, G_- \) are dual Lie algebras. That is, the double is a Lie algebra containing \( G \) and \( G^* \) as Lie subalgebras,

\[
D = G + G^*,
\]

and \( G \) and \( G^* \) form a Lie bialgebra \((G, G^*)\). It has a natural inner product

\[
(X, \xi, Y, \zeta) = \zeta(X) + \xi(Y), \quad X, Y \in G, \quad \xi, \zeta \in G^*,
\]

which, when required to be ad-invariant, fixes the “mixed” commutator to be

\[
[(X, 0), (0, \xi)] \equiv [X, \xi] = ad^* \xi.X - ad^* X.\xi,
\]

\[
[t^a, t^b] = C^c_{\alpha \beta} \cdot t^c - f^b_{\alpha \gamma} t^c.
\]

This can be derived straightforwardly by considering \([X, \xi] \) contracted separately with \((Y, 0)\) and \((0, \zeta)\). The Jacobi identity is satisfied precisely because of the Lie bialgebra compatibility condition (1).

We note that \( G, G^* \) are isotropic subalgebras with respect to this inner product, ie.

\[
<(X, 0), (X, 0)> = 0 \quad \forall (X, 0) \in G \subset D,
\]

\[
<(0, \xi), (0, \xi)> = 0 \quad \forall (0, \xi) \in G^* \subset D.
\]
Clearly, requiring $G_\pm$ to be isotropic is equivalent to requiring them to be dual, and consequently there is a one-to-one correspondence between Manin triples and the double of a Lie algebra (see for example [4]).

By our opening argument the r-matrices for $D$, which we distinguish by a bar from those for $G$, are just projection operators

$$\bar{r}_+ = (0, t^a) \otimes (t_a, 0), \quad \bar{r}_- = -\tau(\bar{r}_+) = - (t_a, 0) \otimes (0, t^a),$$

$$\bar{\bar{r}}_+: D^* \to G \subset D, \quad \bar{\bar{r}}_+: (\xi, X) \mapsto <(\xi, X) \otimes id, (0, t^a) \otimes (t_a, 0) > (X, 0),$$

$$\bar{\bar{r}}_-: D^* \to G^* \subset D, \quad \bar{\bar{r}}_-: (\xi, X) \mapsto -<(\xi, X) \otimes id, (t_a, 0) \otimes (0, t^a) > (0, \xi), \quad (19)$$

where $(\xi, X) \in D^*$. We do not explicitly distinguish the pairing between $D$ and $D^*$ from the inner product on $D$, it should be clear in the usage which one is required.

It is straightforward to confirm that $\bar{r}_\pm$ satisfy the CYBE, as well as condition (3), hence the pair $(D, D^*)$ is indeed a factorisable Lie bialgebra. The Lie algebra isomorphism $I_D = (\bar{r}_+ - \bar{r}_-)$ acts in the natural manner

$$I_D: D^* \to D_R,$$

$$I_D: (\xi, X) \mapsto (X, \xi),$$

where, by design,

$$D^* = G^{* \text{opp}} \oplus G,$$

$$D_R = G \oplus G^{* \text{opp}}, \quad (20)$$

as the direct sum of Lie algebras.

If the double is constructed from a Lie bialgebra $(G, G^*)$ which is itself factorisable then we can go further in its description. Clearly we can map

$$D = G + G^* \to G + G_R, \quad (X, \xi) \mapsto (X, I(\xi)),$$

where $I = (r_+ - r_-): G^* \to G_R$, and $r_\pm$ are the r-matrices for $(G, G^*)$. However, we also have the linear isomorphism

$$J: D \to D' = G \oplus G,$$

$$J: (X, \xi) \mapsto (X, X) + (r_+ \xi, r_- \xi).$$
The algebras $\mathcal{G}$ and $\mathcal{G}_R (\equiv \mathcal{G}^*)$ are embedded in $\mathcal{D}'$ by
\[
\mathcal{G} \hookrightarrow \mathcal{D}' , \quad X \mapsto (X, X) , \\
\mathcal{G}_R \hookrightarrow \mathcal{D}' , \quad Y = I(\xi) \mapsto (R_+ Y, R_- Y) = (r_+ \xi, r_- \xi).
\]
Since any element of $\mathcal{D}'$ can be written as a unique sum of elements in $\mathcal{G}$ and $\mathcal{G}_R$, we have
\[
\mathcal{D}' = \mathcal{G} + \mathcal{G}_R = \mathcal{G} \oplus \mathcal{G} . \tag{21}
\]
The inner product on $\mathcal{D}'$ is
\[
< (X, Y), (X', Y') >= \text{tr}(XX') - \text{tr}(YY') ,
\]
which is obviously ad-invariant, and also ensures that the subalgebras $\mathcal{G}$ and $\mathcal{G}_R$ are isotropic. The r-matrices for $\mathcal{D}'$ are just the images of those in $\mathcal{D}$, eg.
\[
\bar{r}'_+ = (J \otimes J) \bar{r}_+ = \sum_a (r_+ t^a, r_- t^a) \otimes (t_a, t_a).
\]
They project on to $\mathcal{G} \subset \mathcal{D}'$ and $\mathcal{G}_R \subset \mathcal{D}'$;
\[
\bar{r}'_+: \mathcal{D}'^* \to \mathcal{G} \subset \mathcal{D}' , \quad \bar{r}'_+: J^{*-1}(\xi, X) \mapsto (X, X) , \\
\bar{r}'_-: \mathcal{D}'^* \to \mathcal{G}_R \subset \mathcal{D}' , \quad \bar{r}'_-: J^{*-1}(\xi, X) \mapsto (r_+ \xi, r_- \xi),
\]
where we identify an element of $\mathcal{D}'^*$ as
\[
J^{*-1}: \mathcal{D}^* \to \mathcal{D}'^* , \\
J^{*-1}: (\xi, X) \mapsto (I^{-1} X, I^{-1} X) + (I^{-1} r_+ \xi, I^{-1} r_- \xi).
\]
To elaborate on the working here we develop
\[
< J^{*-1}(\xi, X) \otimes id, \bar{r}'_+ >=< (I^{-1} X, I^{-1} X) + (I^{-1} r_+ \xi, I^{-1} r_- \xi), (r_+ t^a, r_- t^a) > (t_a, t_a) = \{< I^{-1} X, r_+ t_a > - < I^{-1} X, r_- t_a > \} (t_a, t_a) =< I^{-1} X, t_a > (t_a, t_a) = (X, X) ,
\]
which agrees with above. Similarly it can be confirmed that
\[
\mathcal{D}'^* = \mathcal{G}_{opp}^* \oplus \mathcal{G}_R^* \\
\mathcal{D}'_R = \mathcal{G} \oplus \mathcal{G}_{R_{opp}}.
\]
V. Poisson Lie Groups

As a final precursor to investigating the Poisson structure of the double group we give a
general explanation of multiplicative Poisson structures. We make the connection with Lie
bialgebras by identifying a Lie bialgebra as the linearisation of a Poisson Lie structure at
the identity of the group.

A Poisson manifold is a manifold $M$ with a Poisson bracket structure, ie. a Lie algebra
structure $\{ , \}_M$ on the vector space $C^\infty(M)$, such that

$$\{ f_1, f_2 f_3 \}_M = f_3 \{ f_1, f_2 \}_M + f_2 \{ f_1, f_3 \}_M , \quad f_i \in C^\infty(M) .$$

The product of two Poisson manifolds $(M, \{ , \}_M)$ and $(N, \{ , \}_N)$ is the Poisson manifold
$(M \times N, \{ , \}_{M \times N})$ where

$$\{ f_1, f_2 \}_{M \times N}(m, n) = \{ f_1^m, f_2^m \}_N(n) + \{ f_1^n, f_2^n \}_M(m) , \quad f_i \in C^\infty(M \times N) ,$$

and $f_i^m$ denotes the smooth function on $N$ obtained from $f_i \in C^\infty(M \times N)$ by keeping the
$m \in M$ component fixed, etc.

A Poisson morphism $\phi$ is a smooth map from a Poisson manifold $(M, \{ , \}_M)$ to the Poisson
manifold $(N, \{ , \}_N)$ which satisfies

$$\phi: M \to N , \quad \phi^* \{ f_1, f_2 \}_N = \{ \phi^* f_1, \phi^* f_2 \}_M , \quad f_i \in C^\infty(N) .$$

This is to be understood as $\{ f_1, f_2 \}_N \circ \phi = \{ f_1 \circ \phi, f_2 \circ \phi \}_M$.

We can now define a Poisson Lie group as a Lie group $G$ with a Poisson structure $\{ , \}_G$
such that group multiplication is a Poisson morphism from $(G \times G, \{ , \}_{G \times G})$, equipped
with the product structure, to $(G, \{ , \}_G)$.

Using $L_g, R_g$ to denote left and right multiplication by $g \in G$ we can write property (22)
as

$$\{ f_1, f_2 \}_G(gh) = \{ f_1 \circ L_g, f_2 \circ L_g \}_G(h) + \{ f_1 \circ R_h, f_2 \circ R_h \}_G(g) , \quad (23)$$

for $g, h \in G$ and $f_i \in C^\infty(G)$.

In terms of a Poisson bivector $\Lambda \in \Gamma(TG \wedge TG)$ defined by

$$\Lambda(df_1, df_2) = \{ f_1, f_2 \}_G ,$$
i.e. $< df_1(g) \otimes df_2(g), \Lambda(g) >= \Lambda(df_1, df_2)(g) = \{ f_1, f_2 \}_G(g) ,$

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the Poisson morphism condition becomes
\[
\Lambda(gh) = L_{g*}\Lambda(h) + R_{h*}\Lambda(g) = [L_{g*}\Lambda](gh) + [R_{h*}\Lambda](gh),
\]
where \( L_{g*}, R_{g*} \) denote left and right translation lifted to the tangent space.

Fixing \( g = h = e \), the identity element of \( G \), we find \( \Lambda(e) = \Lambda(e) + \Lambda(e) \). Thus \( \Lambda(e) = 0 \); and we see that the Poisson structure cannot be symplectic since it is degenerate at the identity.

The Poisson morphism condition can be satisfied by constructing the Poisson tensor from solutions of the MYBE. Consider
\[
\{ f_1, f_2 \}(g) = \langle d^L f_1(g) \otimes d^L f_2(g), r > + \langle d^R f_1(g) \otimes d^R f_2(g), r' > \quad (24)
\]
where \( r, r' \in G \otimes G \), \( G \) being the Lie algebra corresponding to \( G \), and the left and right differentials are given by \( d^L f(g) = L_g^* df(g) = d(f \circ L_g)(e) \) etc., where \( L_g^*, R_g^* \) are the pullbacks of left and right translation.

In tensor notation (24) reads
\[
\{ g \otimes g \} = g \otimes g . r + r'.g \otimes g \quad (25)
\]
Decomposing \( r \) and \( r' \) into symmetric and antisymmetric parts
\[
r = s + a \quad r' = s' + a'
\]
\[
\tau(s) = s \quad \tau(s') = s' \quad \tau(a) = -a \quad \tau(a') = -a'
\]
then antisymmetry of the Poisson bracket requires that
\[
g \otimes g . s + s'.g \otimes g = 0 \quad (26)
\]
so we see that only the antisymmetric parts of \( r, r' \) contribute to the Poisson structure.

Using (26) the Jacobi identity becomes the condition
\[
\{(g \otimes g) \otimes g \} + \text{cyclic} = (a_{12}a_{13} + a_{12}a_{23} + a_{23}a_{21} + a_{23}a_{31} + a_{31}a_{32} + a_{31}a_{12})g \otimes g \otimes g + g \otimes g \otimes g(a'_{13}a'_{12} + a'_{23}a'_{12} + a'_{21}a'_{23} + a'_{31}a'_{23} + a'_{31}a'_{32} + a'_{32}a'_{31} + a'_{12}a'_{13})
\]
\[
= B_a.g \otimes g \otimes g - g \otimes g \otimes g . B_{a'},
\]
\[
= 0.
\quad (27)
\]
Hence we see that the Jacobi identity is satisfied if both \( a, a' \) are equivalently normalised solutions of the MYBE, so that \( B_a = B_{a'} = \frac{1}{2} f^d_{bc} t_b \otimes t_c \otimes t_d \) for instance, or if they are both solutions of the CYBE, \( B_a = B_{a'} = 0 \). We can solve (26) by putting \( s = -s' \) and requiring \( s \) to be ad-invariant, in which case, by previous arguments, we see that this combines with (27) to make (25) a Poisson bracket if \( r \) and \( r' \) are solutions of the MYBE. Therefore, given that (25) is a Poisson bracket on \( G \) when \( r \) and \( r' \) are solutions of the MYBE with equal but opposite ad-invariant symmetric parts, we can investigate under what conditions it satisfies the multiplicative property (23). Comparing

\[
\{ f_1, f_2 \}(gh) = < d^L f_1(gh) \otimes d^L f_2(gh), r > + < d^R f_1(gh) \otimes d^R f_2(gh), r' > \\
= < d(f_1 \circ L_g)(h) \otimes d(f_2 \circ L_g)(h), L_h \ast r > \\
+ < d(f_1 \circ R_h)(g) \otimes d(f_2 \circ R_h)(g), R_g \ast r' >
\]

with the sum of the two terms

\[
\{ f_1 \circ L_g, f_2 \circ L_g \}(h) = < d(f_1 \circ L_g)(h) \otimes d(f_2 \circ L_g)(h), L_h \ast r + R_h \ast r' > \\
\{ f_1 \circ R_h, f_2 \circ R_h \}(g) = < d(f_1 \circ R_h)(g) \otimes d(f_2 \circ R_h)(g), L_g \ast r + R_g \ast r' > ,
\]

we see we have Poisson morphisms

\[
G_{(r, r')} \times G_{(r, -r)} \to G_{(r, r')} \\
G_{(-r', r')} \times G_{(r, r')} \to G_{(r, r')}
\]

Here \( G_{(r, r')} \) means \( G \) taken with the Poisson structure (25), so that the first Poisson morphism reads \( \{ f_1, f_2 \}_{(r, r')}(gh) = \{ f_1 \circ R_h, f_2 \circ R_h \}_{(r, r')}(g) + \{ f_1 \circ L_g, f_2 \circ L_g \}_{(r, -r)}(h) \). In particular, with \( r' = -r \) we have

\[
G_{(r, -r)} \times G_{(r, -r)} \to G_{(r, -r)},
\]

so that

\[
\{ f_1, f_2 \} = < d^L f_1 \otimes d^L f_2 - d^R f_1 \otimes d^R f_2, r > \tag{28}
\]

is a Poisson Lie structure on the group \( G \).

There is a direct correspondence between Poisson Lie groups and Lie bialgebras. The vector space \( \mathcal{G}^* \) has a Lie bracket defined on it via

\[
[\xi_1, \xi_2]_* = d\{ f_1, f_2 \}(e) \equiv d_e \{ f_1, f_2 \} \tag{29}
\]
where $\xi_i = d_e f_i \in G^*$. If we denote by $\Lambda_r$ the Poisson bivector corresponding to the Poisson Lie structure of $G_{(r,-r)}$, then by defining

$$\eta: G \to \mathcal{G} \wedge \mathcal{G},$$

$$\eta(g) = R_g^{-1} \Lambda_r = \text{Ad}_g \otimes \text{Ad}_g, r - r,$$

we can write the Poisson bracket on $G$ in the form

$$\{f_1, f_2\} = \langle d^R f_1 \otimes d^R f_2, \eta \rangle,$$

and the Lie bracket on $G^*$ as

$$[\xi_1, \xi_2]_* = \eta^*(\xi_1 \otimes \xi_2),$$

where $\eta^*$ is the transpose of the tangent map associated with $\eta$. Also

$$\eta(gh) = R_{(gh)^{-1}} \Lambda_r = \text{Ad}_g \otimes \text{Ad}_g, \eta(h) + \eta(g),$$

so that $\eta$ is a 1-cocycle on $G$ with respect to the adjoint representation of $G$, and

$$\langle [\xi_1, \xi_2], X \rangle = \langle \eta^*(\xi_1 \otimes \xi_2), X \rangle,$$

$$= \langle \xi_1 \otimes \xi_2, \eta(X) \rangle,$$

$$= \langle \xi_1 \otimes \xi_2, [X \otimes 1 + 1 \otimes X, r] \rangle.$$

This is exactly the Lie bialgebra structure given in (2), here obtained by differentiating the 1-cocycle on $G$ which defines its multiplicative Poisson structure. Hence, $(G, \mathcal{G}^*)$ is referred to as the tangent Lie bialgebra of $G$.

**VI. Poisson Lie Structure of the Double Group**

We now have all the background material necessary to discuss the multiplicative Poisson structure of the double group. We provide an analysis which, in a similar vein to section III, treats on an equal footing the construction of the double group from either a complex semisimple Lie group or its real maximal subgroup. Previous treatments have not given an explicit description of the latter case.
For ease of presentation we write

\[ \mathcal{D} = \mathcal{G} + \mathcal{G}^* \]

where \( \mathcal{G} \) is considered to be either a complex semisimple Lie algebra or its real maximal compact subalgebra. That is, here \( \mathcal{G} \) denotes either the Lie algebra \( \mathcal{G} \) or the Lie algebra \( \mathcal{K} \) of section III. Hence the r-matrices \( r_{\pm} \in \mathcal{G} \otimes \mathcal{G} \) are not necessarily quasitriangular and the Lie bialgebra \( (\mathcal{G}^*, \mathcal{G}) \) is not necessarily factorisable.

From (28) we have that the Poisson Lie structure is constructed from an r-matrix, and this r-matrix was explicitly given for the double in (19). Therefore, the Poisson Lie structure of the double group can be written

\[ \{f_1, f_2\} = < d^L f_1 \otimes d^L f_2 - d^R f_1 \otimes d^R f_2, (0, t^a) \otimes (t_a, 0) > \quad (30) \]

for \( f_i \in C^\infty(D) \).

By construction, the double Lie algebra leads to a factorisable Lie bialgebra \( (\mathcal{D}, \mathcal{D}) \), where we identify \( \mathcal{D}^* = \mathcal{D} \). So, by (13), we have a unique pointwise decomposition of the associated group

\[ D \sim G \times G^* , \quad (31) \]

\[ D \ni x = g.\Omega , \quad g \in G , \quad \Omega \in G^* , \]

in a region around the identity. This factorisation, because of its particular ordering, leads to what is known as a dressing action of \( G^* (G) \) on \( G (G^*) \). That is, we naturally reorder

\[ \Omega.g = g^\Omega.\Omega^g , \quad \Omega, \Omega^g \in G^*, \quad g, g^\Omega \in G , \]

\( g^\Omega \) dependent upon \( \Omega \), and \( \Omega^g \) dependent upon \( g \). Fixing \( g \) and varying \( \Omega \), and vice versa, we obtain the dressing transformations

\[ G^* \times G \to G , \quad g \mapsto g^\Omega , \]
\[ G^* \times G \to G^* , \quad \Omega \mapsto \Omega^g . \]

The importance of dressing transformations is that they are Poisson actions [3].

The subgroups \( G \subset D \) and \( G^* \subset D \) are Poisson Lie subgroups. It is natural to write the Poisson structure (30) in terms of functions of these subgroups. To do this we note that
$L_{g*}(0, t^a) = (0, L_{g*}t^a)$ and $R_{\Omega*}(t_a, 0) = (R_{\Omega*}t_a, 0)$, because they agree with the ordering given in (31). However,

\[
R_{g*}(0, t^a) \quad \text{and} \quad L_{\Omega*}(t_a, 0)
\]

are infinitesimal dressing transformations which need to be calculated. In [9] a method was briefly indicated when the bialgebra $(G, G^*)$ is factorisable, and we give details of this approach in an appendix. However, we display here an alternative calculation which does not use this assumption.

We take a direct approach in which we iteratively reorder the product and then calculate the commutators produced using the mixed commutator of the double Lie algebra, relation (18). Finally, we rewrite the structure constants of $G^*$ in terms of the structure constants of $G$ and the components of $r$, using (6).

\[
R_{g*}(0, t^a) \sim t^a g = t^a e^X = t^a(1 + X + \frac{1}{2!} X^2 + \ldots),
\]

\[
= (1 + X + \frac{1}{2!} X^2 + \ldots) t^a + (1 + X + \ldots) [t^a, X] + \frac{1}{2!} [[t^a, X], X] + \ldots,
\]

\[
= e^X \left( t^a + X^b (f_{bc} t^c - C_{bc} t_e) + \frac{1}{2!} X^b X^d (f_{bc} f_{de} t^e - C_{d} t_e) - C_{bc} f_{de} t^e + \ldots \right),
\]

\[
= e^X \left( t^a + X^b f_{bc} t^c + \frac{1}{2!} X^b X^d f_{bc} f_{de} t^e + \ldots \right.
\]

\[
- X^b (f_{bd} r_{ad} + f_{bd} r_{dc}) t^c + \frac{1}{2!} X^b X^d (f_{bc} f_{de} r_{af} - f_{bc} f_{de} r_{fe}) t^e + \ldots \right).
\]

(32)

Introducing the adjoint matrix $L(g)^{ab} \equiv L^{ab}$ defined by $L_{g*} R_{g-1*} t_a = g t_a g^{-1} = L_{g*} t_b$, it is straightforward to write (32) in the closed form

\[
R_{g*}(0, t^a) = L_{g*} \left( t_b (r^{bc} L^{ac} - r^{ca} L^{cb}), L^{ab} t^b \right),
\]

\[
= (L_{g*} t_b (r^{bc} L^{ac} - r^{ca} L^{cb}), L^{ab} L_{g*} t^b).
\]

(33)
The second transformation is manipulated similarly,

\[ L_{\Omega^+}(t_a, 0) \sim \Omega t_a = e^\xi t_a = (1 + \xi + \frac{1}{2!}\xi^2 + \ldots) t_a , \]

= \[ t_a(1 + \xi + \frac{1}{2!}\xi^2 + \ldots) + [\xi, t_a](1 + \xi + \ldots) + \frac{1}{2!} [\xi, [\xi, t_a]] + \ldots , \]

= \[ (\chi^{-1})^{ab} R^{\Omega^+} t_b + \xi b J_{ac} t^c(1 + \xi + \ldots) + \frac{1}{2!} \xi b \xi c (f_{ab} C_{cd}^{de} - f_{de} C_{ab}^{bd}) t^e + \ldots , \]

(34)

where \( \chi(\Omega)^{ab} \equiv \chi^{ab} \) is the adjoint matrix for \( G^* \). However, to put this expression into closed form we refer back to the isomorphism \( \bar{I} \) of enveloping algebras that we introduced in section III. Using its defining relations (16) we can write

\[ f_{aa} C_{cd}^{de} - f_{de} C_{ab}^{bd} = (ad)_{ae} \circ \bar{I}(t^b t^c) , \]

(35)

where \( ad(t_a)_{bc} = f_{a}^{b c} \) signifies the adjoint representation of \( G \). Hence, we see that (34) can be written

\[ L_{\Omega^+}(t_a, 0) = \left( (\chi^{-1})^{ab} R^{\Omega^+} t_b , L^{ab} \circ \bar{I}(\Omega^{-1}) R^{\Omega^+} t_b - R^{\Omega^+} t^a \right) , \]

\[ = \left( (\chi^{-1})^{ab} R^{\Omega^+} t_b , i L^{ab} \circ (\bar{r}^-) (\Omega^{-1}) R^{\Omega^+} t^b \right) , \]

\[ = \left( (\chi^{-1})^{ab} R^{\Omega^+} t_b , i [L(\Omega_{\pm}^{-1})^{ab} - L(\Omega_{\pm}^{-1})^{ab} R^{\Omega^+} t^b \right) . \]

(36)

Here \( \bar{r}_{\pm} \) are the extensions as algebra maps of the r-matrices of the complex semisimple Lie algebra, as defined in (11). Hence, we see the appearance of the r-matrices of the complex semisimple Lie algebra even when we start out with the real compact subalgebra \( K \) in the construction of the double.

We can now write the Poisson structure on the double in terms of functions of its subgroups \( G \) and \( G^* \). For functions of \( g \in G \subset D \) we have

\[ \{ g \otimes g \}(g, \Omega) = \{ g \otimes g \}(g) \]

= \[ < (dg, 0) \otimes (dg, 0) , L^{ab}(0, t^a) \otimes L^{ab}(t_a, 0) - R^{ab}(0, t^a) \otimes R^{ab}(t_a, 0) > , \]

= \[ < (dg, 0) \otimes (dg, 0) , (0, L^{ab} t^a) \otimes (L^{ab} t^a, 0) > \]

= \[ - < (dg, 0) \otimes (dg, 0) , L^{ab} t^a \{ r_{bc} L^{ac} - r_{ca} L^{cb} \} , L^{ab} L^{bc} t^b \otimes (R^{ab} t_a, 0) > , \]

= \[ < dg \otimes dg , (r_{ab} - L^{ac} L^{bd} r_{cd}) R^{ab} t_a \otimes R^{ab} t_b > , \]

= \[ [r_{\pm} , g \otimes g] , \]

(37)
where we use $L^{ba} = L(g^{-1})^{ab}$ since $G$ is semisimple, and we make note of (3) to see that the result is valid for $r_+$ and $r_-$. This well-known result is just the structure of $G_{(r_-, r_+)}$, up to a sign and is called the Sklyanin bracket [13] on $G$.

For functions of $\Omega \in G^* \subset D$ we find

$$\{\Omega \otimes \Omega\}(g.\Omega) = \{\Omega \otimes \Omega\}(\Omega),$$

$$= i(L(\Omega^{-1})^{ab} - L(\Omega^{-1})^{ab})(\Omega t^a \otimes t^b\Omega).$$

To verify that this is indeed a Poisson Lie structure on $G^*$ note that (35) can be written

$$(ad)_{ac} \circ \tilde{I}(t^b t^c) = ad_*(t^b)_{ad} ad(t_c)_{de} - ad(t_b)_{ad} ad_*(t^c)_{ed},$$

$$= ad_*(t^b)_{ad} ad_{de} \circ \tilde{I}(t^c) - ad_{ad} \circ \tilde{I}(t^b) ad_*(t^c)_{ed},$$

where $ad_*(t^a)_{bc} = C^{ac}_b$ signifies the adjoint representation of $G^*$. It follows directly that

$$L^{ab} \circ \tilde{I}(\Omega'^{-1}\Omega^{-1}) - \delta^{ab} = \chi(\Omega'^{-1})^{ac}(L^{cb} \circ \tilde{I}(\Omega^{-1}) - \delta^{cb}) - \chi(\Omega)^{bc}(L^{ac} \circ \tilde{I}(\Omega'^{-1}) - \delta^{ac}).$$

This allows us to write

$$\{\Omega\Omega' \otimes \Omega\Omega'\} = \{\Omega \otimes \Omega\}\{\Omega' \otimes \Omega'\} + (\Omega \otimes \Omega)\{\Omega' \otimes \Omega'\}$$

which is exactly the multiplicative property (23), as required. As we have seen in section V, this reflects that

$$\eta_{G^*} : G^* \rightarrow G^* \wedge G^*$$

$$\eta_{G^*}(\Omega) = \chi^{ac}(L(\Omega^{-1})^{cb} - L(\Omega^{-1})^{cb})t^a \otimes t^b$$

is a 1-cocycle on $G^*$.

The mixed bracket gives

$$\{g \otimes \Omega\}(g.\Omega) = <(dg, 0)(g.\Omega) \otimes (0, d\Omega)(\Omega), (0, L_{g.\Omega^*}t^a) \otimes L_{\Omega^*}(t_a, 0)>$$

$$- <(dg, 0)(g) \otimes (0, d\Omega)(g.\Omega), R_{g^*}(0, t^a) \otimes (R_{g^*}t_a, 0)>,$$

$$= 0.\quad(39)$$

If we recall (29) we know that the Lie brackets on $D^*$ are obtained by differentiating the Poisson Lie structure on $D$, and by (20) we require $D^*$ to be a direct sum of Lie algebras, so
naturally the mixed bracket must vanish. It is straightforward to differentiate the brackets (37) and (38) at the identity, and to confirm that they lead to $D^* = G^*_{opp} \oplus G$.

In the case that $(G, G^*)$ is a factorisable Lie bialgebra it is straightforward to arrive at the usual presentation of results. For our purposes it is sufficient to consider

$$D = G + G_R, \quad D \sim G \times G_R,$$

$$D \ni x = g.\Omega, \quad g \in G, \quad \Omega = \Omega^+ \Omega^- \in G_R.$$

Since we are using factorised group elements the only difference from above is that we should use the appropriate product in the enveloping algebra, given by (12) and (14). Hence, we have directly from (38)

$$\{\Omega \otimes \Omega\}(g.\Omega) = \{\Omega \otimes \Omega\}(\Omega),$$

$$= i(L(\Omega^+_1)^{ab} - L(\Omega^-_1)^{ab})(\Omega \ast t_a \otimes t_b \ast \Omega),$$

$$= i(L(\Omega^+_1)^{ab} - L(\Omega^-_1)^{ab})(\Omega_+ t_a \Omega^-_1 \otimes (r_+ [t^a] \Omega - \Omega r_- [t^a])),$$

$$= i(r_+ \Omega \otimes \Omega + \Omega \otimes \Omega, r_- - \Omega \otimes 1 \ast \Omega = 1 \otimes \Omega, r_- \Omega \otimes 1).$$

The final expanded form is how this Poisson bracket is commonly presented, but it should be remembered that this Poisson bracket is multiplicative only with respect to the product on $G_R$.

VII. The Double of SU(2)

In order to illustrate the constructions described previously we work through the example of the double of SU(2). The Poisson Lie structure is given in terms of coordinates on SU(2) and SU(2)$^*$, and the symplectic leaves for these groups are displayed. It is confirmed that linearisation of the Poisson brackets leads to the correct Lie bialgebra structure.

The Pauli matrices provide a basis for su(2) in the fundamental representation. Normalised according to our convention in section III they are

$$t_a = \frac{i}{2} \sigma_a, \quad t_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad t_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$
where $\epsilon_{abc}$ is the totally antisymmetric Levi-Civita tensor, and the inner product is given by

$$(t_a, t_b) = -2 \text{Trace}(t_a t_b).$$

The r-matrix $r_+ = r^{ab} t_a \otimes t_b$, denoted $k_+$ in section III, has components

$$r^{ab} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (40)$$

This allows us to specify the structure constants of the dual Lie algebra $\text{su}(2)^*$ as

$$C_{c}^{ab} = f_{cd}^{\ b} r^{ad} + f_{cd}^{\ a} r^{db}, \quad C_1^{13} = -\frac{1}{2} = C_2^{23}. \quad (41)$$

A basis for $\text{su}(2)^*$ is then

$$t^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix}, \quad t^3 = \frac{1}{4} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (42)$$

with the pairing between the dual vector spaces being given by

$$< t^a, t_b > = 4 \text{Im Trace}(t^a t_b).$$

A group element of $D$, the double of $\text{SU}(2)$, is the product of group elements of $\text{SU}(2)$ and $\text{SU}(2)^*$,

$$g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in \text{SU}(2), \quad a\bar{a} + b\bar{b} = 1,$$

$$\Omega = \begin{pmatrix} u & v + i w \\ 0 & u^{-1} \end{pmatrix} \in \text{SU}(2)^*, \quad u, v, w \in \mathbb{R}, \quad (43)$$

$$x = g \Omega = \begin{pmatrix} au & a(v + i w) + bu^{-1} \\ -\bar{b}u & \bar{a}u^{-1} - \bar{b}(v + i w) \end{pmatrix} \in D \simeq \text{SL}(2, \mathbb{C}),$$

where we make the natural identification of $D$ with $\text{SL}(2, \mathbb{C})$.

The Poisson Lie structure on $\text{SU}(2)$, given by

$$\{ f_1, f_2 \}(g) = < d^L f_1 \otimes d^L f_2 - d^R f_1 \otimes d^R f_2, r_+ >,$$
reduces to
\[
\{a, \bar{a}\} = -\frac{i}{2} b \bar{b}, \quad \{a, b\} = \frac{i}{4} ab, \quad \{a, \bar{b}\} = \frac{i}{4} ac, \quad \{b, \bar{b}\} = 0. \tag{44}
\]
In terms of Euler angles we have
\[
g = e^{i\alpha \sigma_3} e^{i\beta \sigma_2} e^{i\gamma \sigma_3} \in SU(2),
\{\alpha, \sin \beta\} = \frac{1}{8} \sin \beta, \quad \{\alpha, \gamma\} = 0, \quad \{\gamma, \sin \beta\} = \frac{1}{8} \sin \beta.
\tag{45}
\]
To verify that these relations generate the structure of \(su(2)^*\) when linearised, note that
the basis elements of \(su(2)^*\) follow from
\[
g = e^{\lambda^a t^a} \in SU(2), \quad d_e \lambda^a = t^a \in su(2)^* \tag{46}
\]
We have
\[
\tilde{\lambda}^1 = \sin(\alpha - \gamma) \sin \beta, \quad \tilde{\lambda}^2 = \cos(\alpha - \gamma) \sin \beta, \quad \tilde{\lambda}^3 = \sin(\alpha + \gamma) \cos \beta,
\]
where \(\tilde{\lambda}^i = (\lambda^i / \theta) \sin(\theta/2)\), and \(\theta^2 = \sum_i (\lambda^i)^2\). For these variables the Poisson brackets are
\[
\{\tilde{\lambda}^1, \tilde{\lambda}^2\} = 0, \quad \{\tilde{\lambda}^1, \tilde{\lambda}^3\} = -\frac{1}{4} \cos(\theta/2) \tilde{\lambda}^1, \quad \{\tilde{\lambda}^2, \tilde{\lambda}^3\} = -\frac{1}{4} \cos(\theta/2) \tilde{\lambda}^2.
\]
If we implement (29), that is, linearise the Poisson structure, we obtain
\[
[d_e \lambda^1, d_e \lambda^2]_* = 0, \quad [d_e \lambda^1, d_e \lambda^3]_* = -\frac{1}{2} d_e \lambda^1, \quad [d_e \lambda^2, d_e \lambda^3]_* = -\frac{1}{2} d_e \lambda^2, \tag{47}
\]
where we have used \(d_e \lambda^i = 2d_e \tilde{\lambda}^i\). These brackets give rise to the structure constants in (41), as required.
We remark that it is straightforward to display the symplectic leaves of this Poisson structure on \(SU(2)\). It is clear that \(\alpha - \gamma\) is a Casimir, so the symplectic leaves are specified by \(\alpha - \gamma = c\), a constant. Hence we have an \(S^1\) of two-dimensional leaves given by
\[
g = e^{i\alpha \sigma_3} e^{i\beta \sigma_2} e^{i\gamma \sigma_3} e^{ic \sigma_3} \sim \begin{pmatrix} e^{i\alpha} \cos \beta & e^{ic} \sin \beta \\ -e^{-ic} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix}, \tag{48}
\]
except for $\beta = 0$, for which we have an $S^1$ of zero-dimensional leaves

$$g = e^{i\sigma_3} = \begin{pmatrix} e^{ic} & 0 \\ 0 & e^{-ic} \end{pmatrix}.$$  \hfill (49)

More general work on the symplectic leaves of compact Poisson Lie groups can be found in [14] and in references therein.

The Poisson structure on $SU(2)^*$ is given by

$$\{f_1, f_2\} = \langle d^R f_1 \otimes d^R f_2, i\chi^{ac}(L(\Omega^+_1)^{cb} - L(\Omega^-_1)^{cb})t^a \otimes t^b \rangle.$$  \hfill (49)

Since, by (43), $\Omega$ is dependent only on two variables, one real and one complex, we need to consider

$$\{\Omega \otimes \bar{\Omega}\} = i\chi^{ac}(L(\Omega^+_1)^{cb} - L(\Omega^-_1)^{cb})t^a \Omega \otimes \bar{t}^b \bar{\Omega},$$

where a bar indicates complex conjugation, in order to calculate all the relations. As before, the basis elements of the dual Lie algebra, in this case $su(2)$, come from

$$\Omega = e^{\mu_a t^a} = \begin{pmatrix} e^{\mu_3/4} & 2 \frac{(\mu_1 - i\mu_2)}{\mu_3} \sinh \mu_3/4 \\ 0 & e^{-\mu_3/4} \end{pmatrix}, \quad d_e \mu_a = t_a \in su(2).$$ \hfill (51)

We find

$$\Omega_+ = \begin{pmatrix} e^{\mu_3/4} & 2 \frac{\mu_-}{\mu_3} \sinh \mu_3/4 \\ 0 & e^{-\mu_3/4} \end{pmatrix}, \quad \Omega_- = \begin{pmatrix} e^{-\mu_3/4} & 0 \\ -2 \frac{\mu_+}{\mu_3} \sinh \mu_3/4 & e^{\mu_3/4} \end{pmatrix},$$

where $\Omega_+, \Omega_- \in SL(2, \mathbb{C})$, and consequently

$$\chi^{1c}(L(\Omega^+_1)^{c2} - L(\Omega^-_1)^{c2}) = 2ie^{\mu_3/2}(\sinh \mu_3/2 + 2 \frac{\mu_+ - \mu_-}{\mu_3} \sinh^2 \mu_3/4),$$

$$\chi^{1c}(L(\Omega^+_1)^{c3} - L(\Omega^-_1)^{c3}) = -4ie^{\mu_3/4} \frac{\mu_2}{\mu_3} \sinh \mu_3/4,$$

$$\chi^{2c}(L(\Omega^+_1)^{c3} - L(\Omega^-_1)^{c3}) = 4ie^{\mu_3/4} \frac{\mu_1}{\mu_3} \sinh \mu_3/4,$$

where $\mu_\pm = \mu_1 \pm i\mu_2$. It follows that (50) gives rise to the relations

$$\{\mu_3, \mu_+\} = i\mu_+, \quad \{\mu_3, \mu_-\} = -i\mu_-, \quad \{\mu_+, \mu_-\} = \frac{1}{2} \mu_3 \coth \mu_3/4 + 2i\mu_+\mu_- \left(\frac{1}{4} \coth \mu_3/4 - \frac{1}{\mu_3}\right).$$  \hfill (52)
Linearisation produces

\[ [d_e \mu_3, d_e \mu_+] = id_e \mu_+ , \quad [d_e \mu_3, d_e \mu_-] = id_e \mu_- , \quad [d_e \mu_+, d_e \mu_-] = 2id_e \cos \mu_3 , \]

which is the structure of su(2) written in terms of the generators \( t_1 \pm it_2 , t_3 \).

It is not difficult to find that \( \frac{1}{2} \cosh(\mu_3/2) + (\mu_+\mu_-/\mu_3^2) \sinh^2(\mu_3/4) \) is a Casimir for the Poisson structure of SU(2)*. Therefore, we have two-dimensional symplectic leaves given by

\[\Omega = \begin{pmatrix} e^x & z e^{iy} \\ 0 & e^{-x} \end{pmatrix}, \quad e^{2x} + e^{-2x} + z^2 = c^2 \]

(53)

where \( x, y, z \in \mathbb{R} \), and \( c \) is a real constant, and the zero-dimensional leaf

\[\Omega = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

(54)

VIII. Concluding Remark

We have given a detailed account of the Poisson Lie structure of the double of a semisimple group, developing all the basic elements required. This structure has been displayed completely explicitly and its usage made straightforward.

Closely associated to the Poisson Lie structure of the double is a second Poisson structure given by

\[ \{ f_1, f_2 \} = -< d^L f_1 \otimes d^L f_2, \bar{r} > - < d^R f_1 \otimes d^R f_2, \tau(\bar{r}) > \]

For the double as developed in section VI this leads to the relations

\[ \{ g \otimes g \} = [ g \otimes g, r_\pm ] , \]

\[ \{ \Omega \otimes \Omega \} = i(L(\Omega^{-1})^{ab} - L(\Omega^{-1})^{ba})(\Omega t^a \otimes t^b \Omega) . \]

\[ \{ g \otimes \Omega \} = -gt_a \otimes t^a \Omega . \]

This structure is symplectic for the double group considered to be connected and simply connected [2]. It should be interesting to extend our analysis to give an algebraic description of this symplectic structure. A full geometric treatment has been given by Alekseev and Malkin [15].
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Appendix - Alternative Calculation of Dressing Transformations

We give here an explanation of how the infinitesimal dressing transformations calculated in section VI may be obtained by an alternative method, applicable in the particular case of the double constructed from a factorisable Lie bialgebra. We elaborate on brief details given in [9].

For a factorisable Lie bialgebra \((G, G^*)\) it was shown in section IV how the double can be considered in the form

\[
\mathcal{D} = G + G^* \to \mathcal{D}' = G \oplus G = G + G_R ,
\]

\[
\mathcal{D} \ni (t_a, 0) \mapsto (t_a, t_a) \in G \subset \mathcal{D}' ,
\]

\[
\mathcal{D} \ni G^* \ni (0, t^a) \mapsto \equiv (t^+_a, t^-_a) \equiv (t^+_a, t^-_a) \equiv G_R \subset \mathcal{D}' .
\]

For convenience we consider here \(r_+ - r_- = I\) which differs by an unimportant factor of \(i\) to the convention used in sections III and VI.

The corresponding double group is

\[
D' = G \times G \sim G \times G_R ,
\]

\[
D' \ni x = (g, g)(\Omega_+, \Omega_-) , \quad g \in G , \quad \Omega = \Omega_+\Omega_-^{-1} \in G_R . \tag{A.1}
\]

The infinitesimal dressing transformations we want to calculate become

\[
R_{g^*}(0, t^a) \mapsto R_{(g, g)^*}(t^+_a, t^-_a) = L_{(g, g)^*} \text{Ad}_{(g^{-1}, g^{-1})}(t^+_a, t^-_a) ,
\]

\[
L_{\Omega^*}(t_a, 0) \mapsto L_{(\Omega_+, \Omega_-)^*}(t_a, t_a) = R_{(\Omega_+, \Omega_-)^*} \text{Ad}_{(\Omega_+, \Omega_-)}(t_a, t_a) , \tag{A.2}
\]

where \(\text{Ad}\) signifies the adjoint representation of the double group.

We have

\[
(L(g)^{ba} t_a, 0) \mapsto \text{Ad}_{(g, g)}(t_a, t_a) = (gt_a g^{-1}, gt_a g^{-1}) ,
\]

\[
(0, X(\Omega)^{ba} t^a) \mapsto \text{Ad}_{(\Omega_+, \Omega_-)}(t^+_a, t^-_a) = (\Omega_+ t^+_a \Omega_-^{-1}, \Omega_- t^-_a \Omega_-^{-1}) , \tag{A.3}
\]
and also
\[
\text{Ad}_{(g^{-1}, g^{-1})}(t_a^+, t_a^-) = (g^{-1}t_a^+ g, g^{-1}t_a^- g),
\]
\[
\text{Ad}_{(\Omega_+, \Omega_-)}(t_a, t_a) = (\Omega_+ t_a \Omega_-^{-1}, \Omega_- t_a \Omega_+^{-1}).
\] (A.4)

These need to be rewritten in terms of elements of \( \mathcal{G}, \mathcal{G}_R \subset \mathcal{D}' \) and this can be done essentially by observation.

We see that
\[
\text{Ad}_{(g, g)}(t_a, t_a) = (g^{-1}t_a^+ g - (g^{-1}t_a g)^+, g^{-1}t_a^- g - (g^{-1}t_a g)^-) + ((g^{-1}t_a g)^+, (g^{-1}t_a g)^-),
\]
where
\[
g^{-1}t_a^+ g - (g^{-1}t_a g)^+ = g^{-1}t_a^- g - (g^{-1}t_a g)^- = (r^{bc} L^{ac} - r^{ca} L^{cb}) t_b.
\] (A.5)

Hence, we deduce that (A.5) is the image of
\[
\text{Ad}_g(0, t^a) = ((r^{bc} L^{ac} - r^{ca} L^{cb}) t_b, L(g^{-1})^{ba} t^b)
\] (A.6)

which leads directly to (33).

Similarly, we can write
\[
\text{Ad}_{(\Omega_+, \Omega_-)}(t_a, t_a) = (\Omega_+ t_a \Omega_-^{-1} - \Omega_+ t_a^- \Omega_-^+, \Omega_- t_a^+ \Omega_-^+ - \Omega_- t_a^- \Omega_-^+ + (\Omega_+ t_a^+ \Omega_-^+ - \Omega_- t_a^+ \Omega_-^-, \Omega_+ t_a^- \Omega_-^- - \Omega_- t_a^- \Omega_-^-). \] (A.7)

The second term can be written
\[
(\Omega_+ t_a^+ \Omega_-^+ - \Omega_- t_a^+ \Omega_-^-, \Omega_+ t_a^- \Omega_-^- - \Omega_- t_a^- \Omega_-^-) = (L(\Omega_+)^{ba} - L(\Omega_-)^{ba})(t_a^+, t_a^-)
\]
whilst for the first term of (A.7) we use (A.4) to note that
\[
X^{ba} t_b = \Omega_+ t_a^+ \Omega_-^- - \Omega_- t_a^- \Omega_-^+ \iff X^{ab} = L(\Omega_+)^{ac} r^{cb} + L(\Omega_-)^{ac} r^{cb}.
\]

It follows that we can write (A.7) as
\[
\text{Ad}_{(\Omega_+, \Omega_-)}(t_a, t_a) = X(\Omega_-)^{ab}(t_b, t_b) + (L(\Omega_+)^{ba} - L(\Omega_-)^{ba})(t_a^+, t_a^-) \] (A.8)

which is clearly the image of
\[
\text{Ad}_\Omega(t_a, 0) = (X(\Omega_-)^{ab} t_b, [L(\Omega_+)^{ba} - L(\Omega_-)^{ba}] t^b) \] (A.9)
This leads directly to (36), up to the factor of $i$ mentioned above.

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