Closed Kac–Rice type formulas on Riemannian manifolds

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Abstract

We establish a few formulas that compute the volume of the zero-set (or nodal set) of a function on a compact Riemannian manifold as integrals of functionals of the function and its derivatives.

1 Introduction

In their study of the volume of the zero-sets (or nodal sets) of Gaussian random fields on \((\mathbb{R}/\mathbb{Z})^n\), the authors of [2] established a few formulas (of “Kac–Rice type”) that compute the volume of the zero-set of a real-valued function \(f\) on \((\mathbb{R}/\mathbb{Z})^n\) as an integral of a functional of \(f\) and its derivatives. In particular, they gave a general formula in the one-dimensional case and a few specific formulas in higher dimension. In this note, we establish a general formula for functions on compact Riemannian manifolds.

The importance of these formulas for the applications studied in [2], compared to existing Kac–Rice formulas, stems from the fact that they are in “closed form” as opposed to being limits of an integral depending on a parameter. Also, in order to apply techniques of the Malliavin calculus to these formulas, one needs the integrands to be Lipschitz continuous functionals of \(f\) and its derivatives, a requirement met by Formula (15). The only difference between (15) and [2, Prop. 7] is the additional term involving the Ricci curvature, \(|f|\eta_f^{-3}\text{Ric}(\nabla f, \nabla f)\), and this term is in the required domain of the Malliavin calculus by the same proof as [2, Lem. 2 p. 26]. Therefore, [2, Thm. 1] holds on any compact Riemannian manifold. Similarly, Formula (10) shows that the extra boundary terms are not problematic, so [2, Thm. 1] holds on any compact Riemannian manifold with corners, a generalization which includes [2, Thm. 2] as a special case.

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2 Statement of the results

Let \((M, g)\) be a compact Riemannian manifold. If \(f \in C^2(M, \mathbb{R})\), we denote by \(\sigma_f : M \to \{-1, 0, +1\}\) its sign and we set

\[
\eta_f := \sqrt{f^2 + \|\nabla f\|^2}.
\]

(1)

If \(f\) is nondegenerate, that is, \(f(x) = 0\) implies \(df(x) \neq 0\), then \(\eta_f \in C^1(M, \mathbb{R}_{>0})\). We denote the zero-set of \(f\) by

\[
Z_f := f^{-1}(0).
\]

(2)
If $f$ is nondegenerate, then $Z_f$ is a compact Riemannian submanifold of $M$ of class $C^2$ and codimension 1. The remaining notation should be clear and will be further explained in the next section. Our results can be summarized as follows.

**Theorem.** Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold. Let $f \in C^2(M, \mathbb{R})$ be nondegenerate.

1. Let $F \in C^1(TM, TM)$ be such that $F(u) \sim \frac{u}{|u|}$ and $\text{div} \left( F \circ \frac{\nabla f}{f} \right) \in L^1(M)$. Then,

$$\text{vol}(Z_f) = -\frac{1}{2} \int_M \left( \text{div} \left( F \circ \frac{\nabla f}{f} \right) \right) \text{vol}_M.$$

(3)

2. Let $G \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ be such that $G(x) \sim_+ x^{-1}$ and $\left( G \circ \frac{\nabla f}{f} \right) \in L^1(M)$. Then,

$$\text{vol}(Z_f) = \frac{1}{2} \int_M \left( G - \frac{\nabla f}{f^2} \right) \left( \left\| \nabla f \right\|^2 - \Delta f \right) + \sigma_f \left( G' - \frac{\nabla f}{f^2} \right) \left( \left\| \nabla f \right\|^3 - \frac{\text{Hess}(f)(\nabla f, \nabla f)}{2\left\| \nabla f \right\|^2} \right) \text{vol}_M.$$

(4)

3. Let $g \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ be such that $\lim_{x \to +\infty} g(x) = 1$ and $g' \in L^1(\mathbb{R})$. Then,

$$\text{vol}(Z_f) = \frac{1}{2} \int_M \left( \sigma_f \left( g \circ \frac{\nabla f}{|f|} \right) \left( f \left\| \nabla f \right\|^2 + \text{Hess}(f)(\nabla f, \nabla f) - \eta_f^2 \Delta f \right) + \frac{\left\| \nabla f \right\|^2}{f^2\eta_f} \left( g' \circ \frac{\nabla f}{|f|} \right) \left( \left\| \nabla f \right\|^2 - \frac{f\text{Hess}(f)(\nabla f, \nabla f)}{\left\| \nabla f \right\|^2} \right) \right) \text{vol}_M.$$

(5)

Let $g \in C^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ be such that $\lim_{x \to +\infty} g(x) = 1$ and $g' \in L^1(\mathbb{R})$ and $g$ is twice differentiable at 0. Then,

$$\text{vol}(Z_f) = \frac{1}{2} \int_M \left( \sigma_f \left( g \circ \frac{\nabla f}{|f|} \right) \left( \text{Hess}(f)(\nabla f, \nabla f) - \eta_f^2 \Delta f \right) + \left( g' \circ \frac{\nabla f}{|f|} \right) \left( \left\| \nabla f \right\|^2 - \frac{f\text{Hess}(f)(\nabla f, \nabla f)}{\left\| \nabla f \right\|^2} \right) \right) \text{vol}_M.$$

(6)

4. One has

$$\text{vol}(Z_f) = \frac{1}{2} \int_M \left( f \left\| \nabla f \right\|^2 + \text{Hess}(f)(\nabla f, \nabla f) - \eta_f^2 \Delta f \right) \text{vol}_M$$

(7)

and

$$\text{vol}(Z_f) = \frac{1}{2} \int_M \left( \left\| \nabla f \right\|^{-1} \left( \text{arctan} \circ \frac{\nabla f}{f} \right) \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{\left\| \nabla f \right\|^2} - \Delta f \right) + \frac{\eta_f^{-2}}{\left\| \nabla f \right\|^2} \left( \left\| \nabla f \right\|^2 - \frac{f\text{Hess}(f)(\nabla f, \nabla f)}{\left\| \nabla f \right\|^2} \right) \right) \text{vol}_M.$$

(8)
\[
\text{vol}(\mathcal{Z}_f) = \frac{1}{2} \int_M \left( \|\nabla f\|^{-1} \left( \tanh \circ \frac{\|\nabla f\|}{f} \right) \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{\|\nabla f\|^2} - \nabla f \right) + \left( \cosh \circ \frac{\|\nabla f\|}{f} \right)^{-2} \left( \frac{\|\nabla f\|^2}{f^2} - \frac{\text{Hess}(f)(\nabla f, \nabla f)}{f\|\nabla f\|^2} \right) \right) \text{vol}_M. \tag{9}
\]

Remarks.

1. By “\(F(u) \sim \frac{u}{\|u\|}\)”, we mean that \(\lim_{\|u\| \to +\infty} d\left(F(u), \frac{u}{\|u\|}\right) = 0\), where \(d\) is the distance on \(TM\) induced by the Riemannian metric of \(M\) (or any distance, since \(M\) is compact and \(\frac{u}{\|u\|}\) has unit norm).

2. Since \(\mathcal{Z}_f\) is negligible and \(\text{div} \left( F \circ \nabla f \right)\) is defined on \(M \setminus \mathcal{Z}_f\), it makes sense to require “\(\text{div} \left( F \circ \nabla f \right) \in L^1(M)\)” and similarly for Item 2.

3. In the last three formulas, all terms of the integrands are bounded on \(M\) and continuous on \(M \setminus \mathcal{Z}_f\). Indeed, the Hessian expressions are quadratic in \(\|\nabla f\|\), the arctan and tanh expressions are linear in \(\|\nabla f\|\) when \(\|\nabla f\|\) is small, and the cosh expression is exponentially small in \(|f|\) when \(|f|\) is small. However, not all terms need be continuous on \(M\). This problem is treated in the next section.

4. The cases considered in [2] correspond to \(M = (\mathbb{R}/\mathbb{Z})^n\) with the standard flat metric. In particular, Formula (7) is essentially [2, Prop. 5] (in the case of \((\mathbb{R}/\mathbb{Z})^n\) with the standard flat metric).

5. In dimension 1, the general formula (3) reduces to [2, Prop. 2], and in that case, only the condition \(\lim_{x \to \pm \infty} F(x) = \pm 1\) is required if one considers the integral as an improper Lebesgue integral. Similarly, Formula (7) reduces to [2, Prop. 1] and Formula (8) to [2, Cor. 1 of Prop. 2].

6. It is possible to extend these results to compact Riemannian manifolds with corners, assuming that \(\mathcal{Z}_f\) intersects \(\partial M\) transversely. The proof goes along the same lines as below, using a divergence theorem on compact manifolds with corners. Boundary terms will appear in the formulas. For instance, Formula (3) becomes

\[
\text{vol}(\mathcal{Z}_f) = \frac{1}{2} \left( \int_{\partial M} \left( F \circ \frac{\nabla f}{f} , v \right) \text{vol}_{\partial M} - \int_M \left( \text{div} \left( F \circ \frac{\nabla f}{f} \right) \right) \text{vol}_M \right). \tag{10}
\]

This formula reduces in dimension 1 to [2, Prop. 3].

7. Some degenerate functions are considered in [2, Prop. 4]. For instance, the 1-dimensional version of Formula (8) holds for any \(f \in C^2(\mathbb{R}/\mathbb{Z}, \mathbb{R})\) such that for any \(x \in \mathbb{R}/\mathbb{Z}\) there exists \(r \in \mathbb{N}\) such that \(f^{(r)}(x)\) exists and is nonzero.

3 Proof of the results

Let \((M, g)\) be a compact \(n\)-dimensional Riemannian manifold with boundary. Its metric will also be denoted by \(\langle -, - \rangle\) and the associated norm by \(\| - \|\). We denote by \(\nabla^{\text{lc}}\) its Levi-Civita...
connection. Let $\text{vol}_M$ be the Riemannian density on $M$ and $\text{vol}_{\partial M}$ be the induced density on $\partial M$. Let $v \in \Gamma(TM|_{\partial M})$ be the unit outward normal vectorfield. The symbol vol will denote the volume of $(n - 1)$-dimensional submanifolds.

The gradient of a function $f \in C^1(M, \mathbb{R})$ is defined by $\nabla f := (d f)^2$. The divergence of a $C^1$-vectorfield $X \in \mathfrak{X}(M)$ is defined by $\mathcal{L}_X \text{vol}_M = (\text{div} X) \text{vol}_M$ where $\mathcal{L}$ denotes the Lie derivative. The Laplacian of $C^2$-functions is defined by $\triangle := \text{div} \circ \nabla$. The Hessian of a function $f \in C^2(M, \mathbb{R})$ is defined by $\text{Hess} f := \nabla^{\mathcal{L}} c d f = (\nabla^{\mathcal{L}} \nabla f, -)$.

Our basic tool is:

**Divergence theorem.** Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold with boundary. Let $X \in \mathfrak{X}(M)$ be a continuous vectorfield on $M$ which is of class $C^1$ on $\text{int } M$ and such that $\text{div } X \in L^1(\text{int } M)$. Then,

$$\int_M (\text{div } X) \text{vol}_M = \int_{\partial M} \langle X, v \rangle \text{vol}_{\partial M}. \quad (11)$$

**Proof.** If $X$ is of class $C^1$ on $M$, then this is the standard divergence theorem. Else, we consider the geodesic flow from the boundary of $M$ along the unit normal vectorfield. For $\varepsilon > 0$ small enough, set $\theta_\varepsilon: \partial M \to M, x \mapsto \exp(x, -\varepsilon v_x)$ and set $M_\varepsilon := M \setminus \bigcup_{k \in [0, \varepsilon]} \theta_\varepsilon(\partial M)$. For $\varepsilon$ small enough, $M_\varepsilon$ is a compact submanifold with boundary of $M$, and $\theta_\varepsilon$ is a diffeomorphism onto $\partial M_\varepsilon$. Applying the standard divergence theorem on $M_\varepsilon$, one obtains $\int_{M_\varepsilon} (\text{div } X) \text{vol}_M = \int_{\partial M_\varepsilon} \langle X, v \rangle \text{vol}_{\partial M_\varepsilon}$.

When $\varepsilon \to 0$, the left-hand side converges to $\int_M (\text{div } X) \text{vol}_M$ by Lebesgue’s dominated convergence theorem, since $\text{div } X \in L^1(\text{int } M)$. The right-hand side is equal, by change of variable, to $\int_{\partial M} (\theta_\varepsilon^* X, v)(\det \theta_\varepsilon) \text{vol}_{\partial M}$, which converges to $\int_{\partial M} \langle X, v \rangle \text{vol}_{\partial M}$ since the integrand is uniformly convergent and $\partial M$ is compact. \hfill $\Box$

We can now prove the main theorem.

**Proof of the main theorem.** Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold (without boundary). Let $f \in C^2(M, \mathbb{R})$ be a nondegenerate function on $M$. Its zero-set $Z_f$ is a compact Riemannian submanifold of dimension $n - 1$ of class $C^2$. It has finitely many connected components, say $Z_1, \ldots, Z_d$. Let $(A_j)_{j \in J}$ be the connected components of $M \setminus Z_f$. They are open submanifolds of $M$ and there are at most $|\pi_0(M)| + 2d$ of them.

**Proof of 1.** Let $F \in C^1(TM, TM)$ be as in the statement. The vectorfield $F \circ \nabla f/f$ is defined on $M \setminus Z_f$. For all $j \in J$, its restriction to $A_j$ extends continuously to $\overline{A_j} = A_j \cup \bigcup_{k \in b(j)} Z_k$ and is equal to the unit inward normal vectorfield on the boundary. The divergence theorem applied to this extension on $\overline{A_j}$ gives

$$\int_{A_j} \text{div} \left( F \circ \frac{\nabla f}{f} \right) \text{vol}_M = \int_{\partial A_j} \left( F \circ \frac{\nabla f}{f}, v \right) \text{vol}_{\partial A_j} = \int_{\partial A_j} (-1) \text{vol}_{\partial A_j} = - \sum_{k \in b(j)} \text{vol}(Z_k).$$

Therefore, summing over $j \in J$, one obtains

$$\int_M \text{div} \left( F \circ \frac{\nabla f}{f} \right) \text{vol}_M = -2 \text{vol}(Z_f) \quad (12)$$

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the factor 2 coming from the fact that each $Z_i$ borders exactly two $A_j$’s.

**Proof of 2.** We apply (1.) to a function $F$ that is radial. Namely, let $G$ be as in the statement and apply (1.) to $F := (G \circ \| - \|) \cdot \text{id}_{TM}$. The Leibniz rule for the divergence gives

$$\text{div} \left( F \circ \frac{\nabla f}{f} \right) = \left( G \circ \frac{\| \nabla f \|}{|f|} \right) \text{div} \frac{\nabla f}{f} + \left\langle \nabla \left( \frac{1}{f} G \circ \frac{\| \nabla f \|}{|f|} \right), \nabla f \right\rangle.$$

One has $\nabla |f|^{-1} = -f^{-2} \nabla |f|$ and $d\|\nabla f\|^2 = 2(\nabla^L \nabla f, \nabla f) = 2 \nabla f \text{Hess}(f)$, so $\nabla \|\nabla f\|^2 = 2 \nabla^L \nabla f \nabla f$, so $\nabla \|\nabla f\| = \nabla^L \nabla f \nabla f \|\nabla f\|^{-1}$, so

$$\left\langle \nabla \|\nabla f\|, \nabla f \right\rangle = \text{Hess}(f)(\nabla f, \nabla f) \|\nabla f\|^{-1}.$$

Therefore,

$$\text{div} \left( F \circ \frac{\nabla f}{f} \right) = \left( G \circ \frac{\| \nabla f \|}{|f|} \right) \left( \frac{\nabla f}{f} - \| \nabla f \|^2 \right) + \sigma_f \left( G' \circ \frac{\| \nabla f \|}{|f|} \right) \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{f^2 \| \nabla f \|} - \frac{\| \nabla f \|^3}{f^3} \right)$$

which yields the result.

**Proof of 3.** Let $g$ be as in the first part of the statement. Apply (2.) to the function $G$ defined by $G(x) := \frac{g(x)}{\sqrt{1+x^2}}$. One has $G \circ \| \nabla f \| = \left( g \circ \frac{\| \nabla f \|}{\| f \|} \right) \| f \|$ and $G' \circ \| \nabla f \| = \left( g' \circ \frac{\| \nabla f \|}{\| f \|} \right) \| f \| - \left( g \circ \frac{\| \nabla f \|}{\| f \|} \right) \frac{\| \nabla f \|^2}{\| f \|^2}$. Therefore,

$$\text{div} \left( F \circ \frac{\nabla f}{f} \right) = \sigma_f \left( g \circ \frac{\| \nabla f \|}{\| f \|} \right) \left( \frac{f}{\eta_f} \left( \frac{\nabla f}{f} - \| \nabla f \|^2 \right) - \frac{f^2 \| \nabla f \|}{\eta_f} \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{f^2 \| \nabla f \|} - \frac{\| \nabla f \|^3}{f^3} \right) + \left( g' \circ \frac{\| \nabla f \|}{\| f \|} \right) \frac{1}{\eta_f} \left( \frac{\text{Hess}(f)(\nabla f, \nabla f)}{f \| \nabla f \|} - \frac{\| \nabla f \|^3}{f^2} \right) \right)$$

which yields the result after simplification. As for integrability, in local coordinates adapted to the boundary, one has $f(x) \sim_0 A/x_0$, so it is sufficient to prove that $g'(x^{-1})x^{-2}$ is locally integrable at 0. By change of variable, this amounts to the integrability of $g'$ at $+\infty$.

Let $g$ be as in the second part of the statement. Apply (2.) to the function $G$ defined by $G(x) := \frac{g(x)}{x}$. One has $G \circ \| \nabla f \| = \left( g \circ \frac{\| \nabla f \|}{\| f \|} \right) \| f \|$ and $G' \circ \| \nabla f \| = \left( g' \circ \frac{\| \nabla f \|}{\| f \|} \right) \| f \| -
\((g \circ \frac{\|\nabla f\|}{|f|})^2\). Therefore,

\[
\text{div} \left( F \circ \frac{\nabla f}{f} \right) = \sigma_f \left( g \circ \frac{\|\nabla f\|}{|f|} \right) \left( \frac{f}{\|\nabla f\|} \left( \frac{\Delta f}{f} - \frac{\|\nabla f\|^2}{f^2} \right) \right) - \frac{f^2}{\|\nabla f\|^2} \left( \frac{\text{Hess}(f)(\nabla f, \nabla f) - \|\nabla f\|^3}{f^2 \|\nabla f\|} - \frac{\|\nabla f\|^2}{f^3} \right) +
\]

\[
\left( g' \circ \frac{\|\nabla f\|}{|f|} \right) \left( \frac{\text{Hess}(f)(\nabla f, \nabla f) - \|\nabla f\|^2}{f \|\nabla f\|^2} - \frac{\|\nabla f\|^2}{f^2} \right)
\]

which yields the result after simplification. The integrability condition is the same as for the first case.

**Proof of 4.** The three formulas are obtained respectively by letting \(g = 1\) in the first case of (3) and letting \(g = \frac{2}{\pi} \arctan\) and \(g = \tanh\) in the second case of (3). \(\square\)

### 4 Obtaining continuous integrands

As remarked above, the only non-continuous terms of the integrands are of the form

\[
\sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 \right)
\]

with \(h \in C^1(M, \mathbb{R})\), namely, \(h = (g \circ \|\nabla f\|)\eta_f^{-3}\) and \(h = (g \circ \|\nabla f\|)\|\nabla f\|^{-3}\). This is dealt with in [2] (in the case of Equation (7) on the flat torus) using an integration by parts. The same method extends to compact Riemannian manifolds as follows. One has

\[
\text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 = \langle \nabla f, \nabla^{LC} \nabla f - (\Delta f) \nabla f \rangle
\]

Therefore,

\[
\sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 \right) = \langle \nabla |f|, h (\nabla^{LC} \nabla f - (\Delta f) \nabla f) \rangle.
\]

We temporarily assume that \(f\) is of class \(C^3\) and we use the fact that \(\text{div}(|f|h(\nabla^{LC} \nabla f - (\Delta f) \nabla f))\) has a vanishing integral on \(M\) (by the standard divergence theorem). Therefore,

\[
\int_M \sigma_f h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\Delta f)\|\nabla f\|^2 \right) \text{vol}_M =
\]

\[
\int_M |f| \text{div}(h((\Delta f) \nabla f - \nabla^{LC} \nabla f)) \text{vol}_M.
\]

One has \(\text{div}((\Delta f) \nabla f) = (\Delta f)^2 + \langle \nabla \Delta f, \nabla f \rangle\). The Bochner formula yields

\[
\text{div} (\nabla^{LC} \nabla f) = \text{div} \left( \frac{1}{2} \nabla \|\nabla f\|^2 \right)
\]

\[
= \frac{1}{2} \Delta \|\nabla f\|^2
\]

\[
= \langle \nabla \Delta f, \nabla f \rangle + \|\text{Hess} f\|^2 + \text{Ric}(\nabla f, \nabla f)
\]
where the norm of the Hessian is the Hilbert–Schmidt norm. Therefore, the third derivatives cancel out. Since \(C^2(M, \mathbb{R})\) is dense in \(C^3(M, \mathbb{R})\) for the \(C^2\)-topology (see for instance [1]) and the involved quantities are continuous in this topology, one has, for any \(f\) of class \(C^2\),

\[
\int_M \sigma_h \left( \text{Hess}(f)(\nabla f, \nabla f) - (\triangle f)\|\nabla f\|^2 \right) \text{vol}_M = \\
\int_M |f| \left( \|\nabla f\|^2 - \|\text{Hess} f\|^2 - \text{Ric}(\nabla f, \nabla f) \right) + \\
\langle \nabla h, (\triangle f)\nabla f - \nabla^{\text{LC}} f \nabla f \rangle \text{vol}_M. \tag{14}
\]

For example, one has \(\nabla \eta_f^{-3} = -3\eta_f^{-5} (f \nabla f + \nabla^{\text{LC}} \nabla f \nabla f)\), so Formula (7) becomes

\[
\text{vol}(Z_f) = \frac{1}{2} \int_M \left( \frac{|f|}{\eta_f^3} \left( \|\nabla f\|^2 - |f| \triangle f + (\triangle f)^2 - \|\text{Hess} f\|^2 - \text{Ric}(\nabla f, \nabla f) \right) + \\
3\eta_f^{-5} \left( f \text{Hess}(f)(\nabla f, \nabla f) + \text{Hess}(f)(\nabla f, \nabla^{\text{LC}} f \nabla f) - \\
(\triangle f)^2 (f \|\nabla f\|^2 + \text{Hess}(f)(\nabla f, \nabla f)) \right) \right) \text{vol}_M. \tag{15}
\]

This is the version of [2, Prop. 7] for compact Riemannian manifolds.

These formulas can also be written in terms of the tracefree Hessian. Recall that \(\text{Hess}^0(f) = \text{Hess}(f) - \frac{\triangle f}{n} \text{id}\). One easily sees that a tracefree linear map is Hilbert–Schmidt-orthogonal to the identity, so \(\|\text{Hess} f\|^2 = \frac{(\triangle f)^2}{n} + \|\text{Hess}^0 f\|^2\).

References

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