STRONG CONNECTIONS AND THE RELATIVE CHERN-GALOIS CHARACTER FOR CORINGS

GABRIELLA BÖHM AND TOMASZ BRZEZIŃSKI

Abstract. The Chern-Galois theory is developed for corings or coalgebras over non-commutative rings. As the first step the notion of an entwined extension as an extension of algebras within a bijective entwining structure over a non-commutative ring is introduced. A strong connection for an entwined extension is defined and it is shown to be closely related to the Galois property and to the equivariant projectivity of the extension. A generalisation of the Doi theorem on total integrals in the framework of entwining structures over a non-commutative ring is obtained, and the bearing of strong connections on properties such as faithful flatness or relative injectivity is revealed. A family of morphisms between the $K_0$-group of the category of finitely generated projective comodules of a coring and even relative cyclic homology groups of the base algebra of an entwined extension with a strong connection is constructed. This is termed a relative Chern-Galois character. Explicit examples include the computation of a Chern-Galois character of depth 2 Frobenius split (or separable) extensions over a separable algebra $R$. Finitely generated and projective modules are associated to an entwined extension with a strong connection, the explicit form of idempotents is derived, the corresponding (relative) Chern characters are computed, and their connection with the relative Chern-Galois character is explained.

1. Introduction

The Chern-Connes pairing is one of the most important and powerful tools of non-commutative differential geometry [18]. In a recent paper [12] a formalism has been developed connecting a certain class of coalgebra-Galois extensions or non-commutative principal bundles [13] [11] with the K-theoretic aspects of associated modules or (modules of sections of) non-commutative vector bundles. The key tool in this formalism is the notion of a strong connection introduced for Hopf-Galois extensions in [22] and extended to the coalgebra-Galois case in [14], while the bridge between coalgebra-Galois extensions and the K-theory is provided by a Chern-Galois character.

In a series of papers [26], [27] L. Kadison has shown that certain depth 2 extensions of algebras fit naturally into a framework of generalised Galois-type extensions in which a Hopf algebra or, more generally, a coalgebra, is replaced by a Hopf algebroid or, more generally, by a coring. The key here is the crucial discovery made in [28] that to any depth 2 extension one can associate two bialgebroids which (co)act on the total algebra of the extension. Geometrically, these new families of Galois-type extensions can be understood as (principal) bundles in which a standard fibre is a groupoid (rather than a group as in classical principal bundles). From the algebraic point of view, as in the case of coalgebra-Galois extensions, these Hopf algebroid extensions are given in terms of Galois corings. The Galois theory for Hopf algebroids has been
developed and generalised further in [6] within a framework of entwining structures over non-commutative rings, introduced earlier in [4]. Galois extensions for Frobenius Hopf algebroids were studied in [1] using the language of double algebras.

The Chern-Galois theory is developed in [12] within the framework of bijective entwining structures over a field. The aim of the present paper is to develop the relative Chern-Galois theory (relative in the sense of Hochschild) within bijective entwining structures over non-commutative rings, and then to illustrate that the theory thus developed is perfectly suited for describing examples coming from depth 2 extensions. The outline and main results of the paper are as follows.

We begin in Section 2 with a description of preliminary results. The topics covered include \( R \)-rings and corings, entwining structures over non-commutative rings and their entwined modules, and the relative cyclic homology. In particular we introduce the notion of an entwined extension in Definition 2.2 and construct a new example of an entwining structure over a ring in Example 2.3 and Remark 2.4. In Section 3 we introduce the notion of a strong \( T \)-connection in a bijective entwining structure over a non-commutative ring – the first main object of studies of the present paper. Theorem 3.7 contains the first main result of the article: it relates the existence of a strong connection to the Galois property and equivariant projectivity of an entwined extension. The notion of equivariant projectivity is introduced earlier in Section 3 and is related to the existence of connections through a generalised Cuntz-Quillen Theorem 3.3. We illustrate the notion of a strong connection on a number of examples. In particular, we construct an example of a strong connection in a depth 2 Frobenius split (or, dually, separable) extension over a separable algebra \( R \).

Section 4 is devoted to a special case of entwined extensions in which the coaction is given by a grouplike element in a coring. In particular we extend Doi’s theorem on total integrals in Proposition 4.2. We also describe how the existence of strong \( T \)-connections is related to the direct summand property, faithful flatness and relative injectivity of an entwined extension.

The main results of the paper are contained in Section 5. First in Theorem 5.4, for any \( T \)-flat entwined extension \( B \subseteq A \) with a strong \( T \)-connection \( \ell_T \) and a finitely generated and projective comodule of the structure coring \( C \), we construct a family of even cycles in the \( T \)-relative cyclic complex of \( B \). These cycles give rise to a family of maps from the Grothendieck group of isomorphism classes of finitely generated and projective \( C \)-comodules to even \( T \)-relative cyclic homology of \( B \). This family that a priori depends on the choice of a strong connection, is termed a relative Chern-Galois character. We compute explicit forms of the relative Chern-Galois character for cleft Hopf algebroid extensions and for depth 2 Frobenius split extensions. We then prove that, with additional assumptions, the value of the relative Chern-Galois character is equal to the value of the relative Chern character of \( B \). This is done by associating a finitely generated (relative) projective \( B \)-module to an entwined extension with a strong \( T \)-connection in Theorem 5.8 and calculating its idempotent in Theorem 5.10 and the corresponding relative Chern character in Lemma 5.13. In this case the relative Chern-Galois character does not depend on the choice of a strong \( T \)-connection by the cyclic-homology arguments.
2. Entwined extensions and relative cyclic homology

We work over an associative commutative ring $k$ with unit. All algebras are assumed to be associative and with a unit. Throughout, $R$ denotes a $k$-algebra and $C$ denotes an $R$-coring with a coproduct $\Delta_C$ and a counit $\varepsilon_C$. $M_R$ denotes the category of right $R$-modules, $M^C_R$ denotes the category of right $C$-comodules, etc. The notational conventions for Hom-functors are $\text{Hom}_{R-}(\cdot, \cdot)$ for right $R$-modules, $\text{Hom}_{R,R-}(\cdot, \cdot)$ for left $R$-modules, $\text{Hom}_{R,R}(\cdot, \cdot)$ for $R$-bimodules, and similarly but with upper indices for comodules of a coring. Actions and coactions are denoted by $\phi$ with the position of the index indicating the side on which a ring or coring acts or coacts, for example $\phi_M$ denotes the right action of a ring on a module $M$, and $^{A}\phi$ denotes a left coaction of a coring on a comodule $A$ etc.

2.1. $R$-rings. An $R$-ring $A$ is a monoid in the monoidal category of bimodules for a $k$-algebra $R$. That is, $A$ is an $R$-$R$ bimodule equipped with an $R$-$R$ bilinear associative multiplication map $\mu_A : A \otimes R A \to A$ and an $R$-$R$ bilinear unit map for the multiplication, $\eta_A : R \to A$.

Composing $\mu_A$ with the canonical epimorphism $A_k \otimes A \to A \otimes A$, one obtains a $k$-algebra structure on $A$ with unit element $1_A := \eta_A(1_R)$. Conversely, the $R$-ring structure is determined by the $k$-algebra structure of $A$ and the unit map $\eta_A$ via the usual coequaliser construction.

Since any module for the $k$-algebra $A$ is in particular an $R$-module (via the algebra homomorphism $\eta_A$), the notions of modules for the $R$-ring $A$, and for the corresponding $k$-algebra, coincide.

The left regular module for the base algebra $R$ extends to a left $A$-module if and only if there exists a left augmentation map $\rho_R$, i.e. a left $R$-linear retraction of $\eta_A$ satisfying $\rho_R(a a') = \rho_R(a \eta_A(R \rho_R(a')))$, for all $a, a' \in A$.

The invariants of a left $A$-module $M$ with respect to the left augmentation $\rho_R$ are defined as elements $m \in M$ for which $a m = \eta_A(R \rho_R(a))m$ for all $a \in A$. The $k$-module of invariants in $M$ is isomorphic to $\text{Hom}_A(R, M)$.

An $R$-ring $A$ is called separable if the multiplication map $\mu_A : A \otimes R A \to A$ is a split epi in the category of $A$-$A$ bimodules. It is equivalent to the existence of an $R$-$R$ bilinear map $\zeta : R \to A \otimes R A$ such that

$$\quad (\mu_A \otimes R) \circ (A \otimes A) = (A \otimes R \mu_A) \circ (\zeta \otimes R A), \quad \mu_A \circ \zeta = \eta_A.$$ 

The element $\zeta(1_R)$ of $A \otimes R A$ is called a separability idempotent. If $A$ is a separable $R$-ring such that the unit map $\eta_A$ is injective, then $A$ is called a separable extension of $R$.

2.2. $R$-corings. An $R$-coring $C$ is a comonoid in the monoidal category of bimodules for a $k$-algebra $R$. That is, $C$ is an $R$-$R$ bimodule equipped with an $R$-$R$ bilinear coassociative comultiplication map $\Delta_C : C \to C \otimes R C$ and an $R$-$R$ bilinear counit map $\varepsilon_C : C \to R$.

A right comodule for an $R$-coring $C$ is a pair $(M, \phi^M)$ consisting of a right $R$-module $M$ and a right $R$-linear coassociative and counital coaction $\phi_M : M \to M \otimes R C$ of $C$ on $M$. The morphisms of right $C$-comodules are right $R$-linear maps which are compatible with the coactions. In accordance with our general conventions, the
category of right $C$-comodules is denoted by $M^C$. Left comodules for $R$-corings and their morphisms are defined analogously, and their category is denoted by $C^M$.

In explicit calculations we often use Sweedler’s notation for coproduct $\Delta_C(c) = \sum c(1)_e \otimes c(2)_e$, right coaction $\varrho^M(m) = \sum m_{[0]}_e \otimes m_{[1]}_e \in M \otimes e^R$, and left coaction $\varrho_N(n) = \sum n_{[0]}_e \otimes n_{[0]}_e \in C \otimes e^N$.

We make notational difference between left and right coactions (note different types of brackets used), since later on we deal with $R$-$R$ bimodules which are left and right $C$-comodules, but not necessarily $C$-$C$ bicomodules.

Given a $k$-algebra $B$, $B^M_C$ denotes the category of $B$-$R$ bimodules that are at the same time right $C$-comodules with a left $B$-linear coaction. For any $M \in B^M_C$, the right $R$-module $B \mathop{\otimes} R M$ is a right $C$-comodule with the natural coaction $\varrho^{B \mathop{\otimes} R M} = B \mathop{\otimes} \varrho^M$, and the left $B$-multiplication $M \mathop{\otimes} R M \to M$ is a right $C$-comodule map.

By [9], Lemma 5.1, the right regular module for the base algebra $R$ extends to a right $C$-comodule if and only if the left regular $R$-module extends to a left $C$-comodule. These properties are equivalent to the existence of a grouplike element in $C$, that is an element $e$ such that $\Delta_C(e) = e \otimes e$ and $\varepsilon_C(e) = 1_R$. In terms of a grouplike element $e$ the right and left coactions are given by

$$\varrho^R : R \to C, \quad r \mapsto er, \quad \text{and} \quad R \varrho : R \to C, \quad r \mapsto re.$$  

The coinvariants of a right $C$-comodule $M$ with respect to a grouplike element $e \in C$ are defined as elements $m \in M$ such that $\varrho^M(m) = m \otimes e$. The $k$-module of coinvariants in $M$ is isomorphic to $\text{Hom}^{-C}(R, M)$ [16, 28.4]. The coinvariants of left $C$-comodules with respect to a grouplike element are defined analogously. In particular, the coinvariants of $R$ as a left, and as a right $C$-comodule coincide, and form a unital $k$-subalgebra $B$ of $R$ [16, 28.5 (1)]. An $R$-coring $C$ with a grouplike element $e$ is called a Galois coring, provided the canonical map

$$\text{can}_R : R \mathop{\otimes} R \to C, \quad r \mathop{\otimes} r' \mapsto rer'$$

is bijective. In this case $\text{can}_R$ establishes an $R$-coring isomorphism between $C$ and the canonical Sweedler coring $R \mathop{\otimes} R_R$.

For an $R$-coring $C$, the left $R$-dual $^*C : = \text{Hom}_{R-}(C, R)$ is an $R$-ring with multiplication

$$(ff')(c) = \sum f'(c(1)) f(c(2)), \quad \text{for all } f, f' \in ^*C, \quad c \in C.$$  

The unit map $R \to ^*C$ is given by $r \mapsto [c \mapsto \varepsilon_C(cr)]$. Any right $C$-comodule $M$ with coaction $\varrho^M(m) = \sum m_{[0]}_e \otimes m_{[1]}_e$ determines a right module for $^*C$ via

$$mf = \sum m_{[0]}_e f(m_{[1]}_e), \quad \text{for all } m \in M, \quad f \in ^*C.$$  

Since $C$-colinear maps are $^*C$-linear, there is a faithful functor $M^C \to M_{^*C}$. It is an isomorphism if and only if $C$ is a finitely generated and projective left $R$-module [16, 19.5-6].

An $R$-coring $C$ is called a coseparable coring if the coproduct $\Delta_C : C \to C \mathop{\otimes} C$ is a split monic in the category of $C$-$C$ bicomodules. By [16, 26.1], this is equivalent to the existence of a cointegral, i.e. an $R$-$R$ bilinear map $\delta : C \mathop{\otimes} R C \to R$, such that

$$(C \mathop{\otimes} \delta) \circ (\Delta_C \mathop{\otimes} C) = (\delta \mathop{\otimes} R C) \circ (C \mathop{\otimes} \Delta_C), \quad \delta \circ \Delta_C = \varepsilon_C.$$
2.3. **R-entwining structures and entwined modules.** A right entwining structure $(A, C, \psi)_R$ over a $k$-algebra $R$ consists of an $R$-ring $A$, an $R$-coring $C$ and an $R$-$R$ bilinear map $\psi : C \otimes_R A \to A \otimes_C C$ satisfying the following conditions

\[
\begin{align*}
\psi \circ (C \otimes_R \mu_A) &= (\mu_A \otimes_R C) \circ (A \otimes_R \psi) \circ (\psi \otimes_R A), \\
\psi \circ (C \otimes_R \eta_A) &= \eta_A \otimes_R C, \\
(A \otimes_R \Delta_C) \circ \psi &= (\psi \otimes_R C) \circ (C \otimes_R \psi) \circ (\Delta_C \otimes_R A), \\
(A \otimes_R \varepsilon_C) \circ \psi &= \varepsilon_C \otimes_R A.
\end{align*}
\]

A right entwined module for a right entwining structure $(A, C, \psi)_R$ is a right $A$-module and a right $C$-comodule $M$ such that

\[
\varrho^M \circ \varrho_M = (\varrho_M \otimes_R C) \circ (M \otimes_R \psi) \circ (\varrho^M \otimes_R A).
\]

The morphisms of entwined modules are right $A$-linear right $C$-colinear maps. The category of right entwined modules for a right entwining structure $(A, C, \psi)_R$ is denoted by $\mathcal{M}_A^C(\psi)$.

Applying the construction in [11, Example 4.5], to a right entwining structure $(A, C, \psi)_R$ one can associate an $A$-coring $(A \otimes_R C)_\psi$ as follows. $(A \otimes_R C)_\psi = A \otimes C$ with the obvious left $A$-module structure. The right $A$-module structure is given by

\[
(a \otimes c)a' = a\psi(c \otimes a'), \quad \text{for all } a \otimes c \in A \otimes_R C, \; a' \in A.
\]

The coproduct is $A \otimes_R \Delta_C$ and the counit is $A \otimes_R \varepsilon_C$. Using the same line of argument as in [11, Proposition 2.2], one checks that right comodules for the $A$-coring $(A \otimes_R C)_\psi$ can be identified with right entwined modules for the entwining structure $(A, C, \psi)_R$ via the identification of $M \otimes_R (A \otimes R C)$ with $M \otimes C$ for any object $M$ in $\mathcal{M}^{(A \otimes R C)}_R$.

Conversely, if $A \otimes_R C$ is an $A$-coring with obvious left multiplication, coproduct $A \otimes_R \Delta_C$ and counit $A \otimes_R \varepsilon_C$, then the map

\[
\psi : C \otimes_R A \to A \otimes_R C, \quad c \otimes a \mapsto (1 \otimes_R c)a
\]

gives rise to a right entwining structure $(A, C, \psi)_R$ over $R$, provided that

\[
(2.3) \quad \varrho_{A \otimes_R C} \circ (A \otimes_R C \otimes_R \eta_A) = A \otimes_R \varrho_C,
\]

where $\varrho_{A \otimes_R C}$ is the right $A$-product in $A \otimes R C$ and $\varrho_C$ is the right $R$-product in $C$. The condition $(2.3)$ is necessary and sufficient to ensure that the map $\psi$ in $(2.2)$ is right $R$-linear.

If $A$ is an entwined module for a right entwining structure $(A, C, \psi)_R$ with the right regular action and a $C$-coaction $\varrho^A$, then $\varrho^A(1_A)$ is a grouplike element in $(A \otimes_R C)_\psi$.

For an entwined module $M$, the coinvariants with respect to $\varrho^A(1_A)$ are denoted by $M^{\co C}$.

**Left entwining structures** over a $k$-algebra $R$ are defined in a symmetric way. A left entwining structure $\mathcal{L}(A, C, \psi)$ consists of an $R$-ring $A$, an $R$-coring $C$ and an $R$-$R$ bilinear map $\psi : A \otimes_R C \to C \otimes_R A$ satisfying compatibility conditions, analogous to the right entwining case. Analogously, one defines the category $\mathcal{L}(\mathcal{M}(\psi))$ of left entwined modules. To a left entwining structure $\mathcal{L}(A, C, \psi)$ one can associate an $A$-coring $(C \otimes_R A)_\psi$, whose left comodules can be identified with left entwined modules. In
case $A$ is itself a left entwined module with the left regular action, the coinvariants of any left entwined module $M$ with respect to the grouplike element $A\varphi(1_A) \in (C \otimes_R A)_{\psi}$ are denoted by $\text{co} C M$.

An example of a left entwining structure over $R$ can be constructed by the straightforward generalisation of \[\text{Theorem 2.7}\].

**Example 2.1.** Consider an $R$-coring $D$, and an $R$-ring $A$ that is also a left $D$-comodule with the coaction $A\varphi$. Let

$$B = \{ b \in A \mid \forall a \in A, \ A\varphi(ab) = A\varphi(a)b \}$$

and suppose that the (left canonical) map

$$\text{can}_l : A \otimes_R A \rightarrow D \otimes_R A,$$

$$a \otimes a' \mapsto A\varphi(a)a'$$

is bijective. Then $R(A, D, \text{can}_l)$ is a left entwining structure with the entwining map

$$\psi_{\text{can}} : A \otimes_R D \rightarrow D \otimes_R A,$$

$$a \otimes d \mapsto \text{can}_l(a \text{ can}_l^{-1}(d \otimes 1_A)).$$

The corresponding coring $(D \otimes_R A)_{\psi_{\text{can}}}$ is a Galois $A$-coring with respect to the grouplike element $A\varphi(1_A)$, hence $A$ is a left entwined module over $R(A, D, \psi_{\text{can}})$, and $B = \text{co} D A$.

A right entwining structure $(A, C, \psi)_{\ R}$ is said to be bijective if $\psi$ is a bijection. In this case $R(A, C, \psi^{-1})$ is a left entwining structure and $\psi$ is an isomorphism of corings $(C \otimes_R A)_{\psi^{-1}} \rightarrow (A \otimes_R C)_{\psi}$.

2.4. **Entwined extensions.** In view of Sections 2.2 and 2.3, the following statements about a bijective right entwining structure $(A, C, \psi)_{\ R}$ are equivalent.

(a) $A \in \mathcal{M}_R^C(\psi)$ with the right regular $A$-action and the coaction $\varphi^A : A \rightarrow A \otimes C$;

(b) $A$ is a right comodule for the $A$-coring $(A \otimes_C)_{\psi}$ with the right regular $A$-action and the coaction $\varphi^A : A \rightarrow A \otimes (A \otimes C) \simeq A \otimes C$;

(c) $\varphi^A(1_A)$ is a grouplike element in the $A$-coring $(A \otimes_C)_{\psi}$;

(d) $A$ is a left comodule for the $A$-coring $(A \otimes_C)_{\psi}$ with the left regular $A$-action and the coaction

$$A \rightarrow (A \otimes_C) \otimes A \simeq A \otimes C$$

$$a \mapsto a \varphi^A(1_A);$$

(e) $\psi^{-1}(\varphi^A(1_A))$ is a grouplike element in the $A$-coring $(C \otimes A)_{\psi^{-1}}$;

(f) $A$ is a right comodule for the $A$-coring $(C \otimes A)_{\psi^{-1}}$ with the right regular $A$-action and the coaction

$$A \rightarrow A \otimes (C \otimes A) \simeq C \otimes A$$

$$a \mapsto \psi^{-1}(\varphi^A(1_A))a \equiv \psi^{-1}(\varphi^A(a));$$

(g) $A$ is a left comodule for the $A$-coring $(C \otimes A)_{\psi^{-1}}$ with the left regular $A$-action and the coaction

$$A \rightarrow (C \otimes A) \otimes A \simeq C \otimes A$$

$$a \mapsto \psi^{-1}(a \varphi^A(1_A));$$

(h) $A \in \mathcal{M}_R(\psi^{-1})$ with the left regular $A$-action and the coaction

$$A\varphi : A \rightarrow C \otimes_R A$$

$$a \mapsto \psi^{-1}(a \varphi^A(1_A)).$$
Suppose that \((A,C,\psi)\) is a bijective right entwining structure such that the equivalent conditions (a)-(h) hold. The coinvariants of \(A\) as a right entwining module for the right entwining structure \((A,C,\psi)\), and as a left entwined module for the left entwining structure \(R(A,C,\psi^{-1})\) are the elements of the same \(k\)-subalgebra of \(A\),

\[
B = A^{coC} = \{ b \in A \mid g^A(b) = bg^A(1_A) \} = \{ \sum a_i \otimes \tilde{a}_i \in A \otimes \tilde{A} \mid a_i \in A, \quad \text{coprod}(a_i) = b \} = \{ b \in A \mid g^A(b) = \frac{bg^A(1_A)}{b} \} = \{ b \in A \mid \forall a \in A, \quad A(g(ab)) = A(g(a)b) \}.
\]

**Definition 2.2.** Let \((A,C,\psi)\) be a bijective right entwining structure. An algebra extension \(B \subseteq A\) is called an *entwined extension* provided the equivalent conditions (a)-(h) are satisfied and \(B\) is the \(k\)-subalgebra of coinvariants.

If, in a bijective right entwining structure \((A,C,\psi)\) over \(R\), the coring \(C\) possesses a grouplike element \(e\), then \(1_A \otimes e\) is a grouplike element in the \(A\)-coring \((A \otimes C)\).

Hence the conditions (a)-(h) hold. In particular, the right \((A,C,\psi)\), and the left \(R(A,C,\psi^{-1})\) coactions on \(A\) come out as, for \(a \in A\),

\[
\varphi^A(a) = \psi(e \otimes a) \quad \text{and} \quad \lambda^A(b) = \psi^{-1}(a \otimes e).
\]

The subalgebra \(B = A^{coC}\) of coinvariants in \(A\) comes out as

\[
B = \{ b \in A \mid g^A(b) = b \otimes e \} \equiv \{ b \in A \mid \lambda^A(b) = e \otimes b \}.
\]

In this situation we say that the entwined extension \(B \subseteq A\) is given by a grouplike element \(e \in C\).

The following example shows that to an extension of algebras \(B \subseteq A\) of the type described in Example 2.1 one can associate not only a left entwining structure over \(R\), but also a right entwining structure over \(B\), provided that the extension is faithfully flat.

**Example 2.3.** Let \(R\) be a \(k\)-algebra, \(D\) and \(B \subseteq A\) as in Example 2.1. For all \(d \in D\), write \(\varpi(d) = \text{can}^{-1}(d \otimes 1_A)\). Notice that \(\varpi\) is left \(R\)-linear by the left \(R\)-linearity of \(\lambda^A\) and it is also right \(R\)-linear since \(A\) is an \(R\)-ring and \(\text{can}\) is right \(A\)-linear. Suppose that \(A\) is a faithfully flat right \(B\)-module. Let \(C\) be the associated Ehresmann \(B\)-coring (cf. [16, 34.13]), i.e. \(C\) is a \(B\)-\(B\) subbimodule of \(A \otimes A\),

\[
C = \{ \sum a_i \otimes \tilde{a}_i \in A \otimes A \mid \sum a_i \otimes \varpi(\tilde{a}_i(-1)) \otimes \tilde{a}_i(0) = \sum 1_A \otimes a_i \otimes \tilde{a}_i \},
\]

with the coproduct and the counit,

\[
\Delta_C = \sum a_i \otimes \tilde{a}_i = \sum a_i \otimes \varpi(\tilde{a}_i(-1)) \otimes \tilde{a}_i(0), \quad \varepsilon_C = \sum a_i \otimes \tilde{a}_i = \sum a_i \tilde{a}_i.
\]

Then \((A,C,\psi)_B\) is a right entwining structure over \(B\), where

\[
\psi : C \otimes A_B \rightarrow A \otimes C_B, \quad \sum a_i \otimes \tilde{a}_i \otimes a \mapsto \sum a_i \otimes \varpi(\tilde{a}_i(-1)) \otimes \tilde{a}_i(0),
\]

is the entwining map.

**Check.** Recall from [16, 34.13] that \(A \otimes A\) is a left entwined module for the left entwining structure \(R(A,D,\psi_{\text{can}})\), constructed in Example 2.1. Equivalently, it is a left comodule for the Galois \(A\)-coring \((D \otimes A)_{\psi_{\text{can}}}\) with coaction \(a \otimes \tilde{a} \mapsto \sum \psi_{\text{can}}(a \otimes \tilde{a}_i(-1)) \otimes \tilde{a}_i(0)\). Note that \(C = coD(A \otimes A)\).
Since $A$ is a faithfully flat right $B$-module, the functors
\[
\coD(-) : A^{op}M(\psi_{can}) \rightarrow B^* \quad \text{and} \quad A \otimes -: B M \rightarrow A^{op}M(\psi_{can})
\]
are inverse equivalences by the [left-comodule version of] the Galois co-ring structure theorem [9, Theorem 5.6]. Hence the unit of the adjunction
\[
\theta_{A \otimes A} : A \otimes A \rightarrow A \otimes C,
\]
is an isomorphism of $A$-$B$ bimodules with the inverse
\[
\theta_{A \otimes A}^{-1} : A \otimes C \rightarrow A \otimes A,
\]
Note that the map $\theta_{A \otimes A}$ is a right $B$-module map by the definition of $B$ as left $D$-coinvariants in $A$. The bimodule $A \otimes A$ has the canonical Sweedler $A$-co-ring structure. Using this structure and the isomorphism $\theta_{A \otimes A}$, one can induce a unique $A$-co-ring structure on $A \otimes C$, such that $\theta_{A \otimes A}$ is an isomorphism of $A$-co-rings. A result of this one finds that the coproduct and counit in $A \otimes C$ come out as
\[
\Delta_{A \otimes C} := (\theta_{A \otimes A} \otimes \theta_{A \otimes A}) \circ \Delta_{A \otimes A} \circ \theta_{A \otimes A}^{-1} = A \otimes \Delta_C
\]
and
\[
\varepsilon_{A \otimes C} := \varepsilon_{A \otimes A} \circ \theta_{A \otimes A}^{-1} = A \otimes \varepsilon_C.
\]
Note that the fact that $\theta_{A \otimes A}$ is a right $B$-module map implies that the compatibility condition (2.3) between the original and induced $B$-actions on $A \otimes C$ is satisfied. The computed form of the coproduct and counit in $A \otimes C$ thus implies that $A \otimes C$ is a co-ring associated to a right entwining structure over $B$. In particular, the entwining map $\psi$ is induced from the right $A$-multiplication in $A \otimes C$ as in (2.2), i.e.
\[
\psi(1 \otimes a) = (1 \otimes a = \theta_{A \otimes A}^{-1} \theta_{A \otimes A} (1 \otimes c) a),
\]
and has the asserted form. ✓

One can investigate further the entwining structure described in Example 2.3. The following remark summarises basic properties of this structure.

**Remark 2.4.** The notation and hypotheses of Example 2.3 are assumed.

(1) If the canonical left entwining structure $\rho^A$ in Example 2.1 is bijective, then so is the induced right entwining structure $(A, C, \psi)_B$, provided $A$ is a faithfully flat left $B$-module. In this case $A$ is a right $D$-comodule (cf. equivalent conditions (a)-(h)), and the inverse of $\psi$ comes out explicitly as
\[
\psi^{-1} : A \otimes B \rightarrow C \otimes B,
\]
where
\[
\sum a_{(0)} \otimes a_{(1)} := \psi^{-1}_{can}(\rho_{can}^A(1_A) a)
\]
is the right $D$-coaction on $A$ induced by the inverse of the entwining map $\psi_{can}$.

(2) $A$ is a right entwined module for $(A, C, \psi)_B$ with the coaction
\[
\varrho^A : A \rightarrow A \otimes B,
\]
where
\[
\sum a_{(-1)} \otimes a_{(0)} = \rho^A(a).
\]
i.e. \( \sum \varpi(1_{A(-1)}) \otimes 1_A(0) \) is a grouplike element in the associated \( A \)-coring \( (A \otimes C)_\psi \). The coinvariants with respect to this element come out as
\[
T = \{ s \in A \mid \sum_s r \otimes 1_A = 1_A \otimes s \}. 
\]
If \( A \) is a faithfully flat left or right \( R \)-module, then \( T = R \). By the construction, \( (A \otimes C)_\psi \cong A \otimes A \) as corings, hence in this case \( (A \otimes C)_\psi \) is a Galois \( A \)-coring.

Therefore, if \( A \) is a faithfully flat left or right \( R \)-module, starting from a faithfully flat left “coring-Galois extension” \( B \subseteq A \) for an \( R \)-coring \( D \), i.e. from a Galois \( A \)-coring \( (D \otimes A)_\psi \), we construct the associated Ehresmann \( B \)-coring \( C \), and conclude that \( (A \otimes C)_\psi \) is a Galois \( A \)-coring, i.e. there is a right “coring-Galois extension” \( R \subseteq A \) for the \( B \)-coring \( C \). This suggests an interesting generalisation of and, perhaps, more symmetric approach to bi-Galois extensions (cf. [21, Definition 2.6], [32, Definition 3.4]). It is shown in [32] that to any faithfully flat Hopf-Galois object one can associate a bi-Galois extension. This construction is restricted to Hopf-Galois objects as only in this case the associated Ehresmann coring is a Hopf algebra. To obtain more general and more symmetric situation one needs to replace coalgebras (or Hopf algebras) over the same ring by corings over different rings. Examples of this situation, when \( A \) is a comodule algebra for two bialgebroids \( D \) and \( C \), are described in terms of \( R \)-\( B \) torsors in [23].

(3) In view of (1) and (2), if the canonical left entwining structure \( _RA(D, \psi) \) in Example 2.5 is bijective and \( A \) is a faithfully flat left \( B \)-module, then \( T \subseteq A \) is an entwined extension in \( (A, C, \psi)_B \).

A rich source of examples of entwining structures over rings and entwined extensions is provided by \( \times_R \)-bialgebroids or bialgebroids introduced in [31] and [30], and Hopf algebroids introduced in [8]. We refer to [16, Section 31] and [28, Section 2] for a review of left and right bialgebroids, respectively, and to [5] for more information on Hopf algebroids.

**Example 2.5.** (1) Let \((H, R, s, t, \gamma, \pi)\) be a right bialgebroid. This means that \( H \) is the total algebra, \( R \) is the base algebra, \( s \) is the source and \( t \) is the target map, \( \gamma \) is the coproduct and \( \pi \) is the counit. Let \( A \) be a right comodule algebra with coaction \( \varphi^A(a) = \sum a_{(0)} \otimes a_{(1)} \). Since the unit of \( H \) is a grouplike element, \( A \) is an entwined module for the entwining structure over \( R \) consisting of the \( R \)-ring \( A \), the \( R \)-coring \((H, \gamma, \pi)\) and the entwining map
\[
(2.7) \quad \psi : H \otimes A \to A \otimes H, \quad h \otimes a \mapsto \sum a_{(0)} \otimes h a_{(1)}. 
\]

(2) Recall from [5] that a Hopf algebroid \( H \) consists of a left bialgebroid \( H_L = (H, L, s_L, t_L, \gamma_L, \pi_L) \), right bialgebroid \( H_R = (H, R, s_R, t_R, \gamma_R, \pi_R) \) and a map \( S : H \to H \), called an antipode, satisfying a number of compatibility conditions. Let \( A \) be a right \( H \)-comodule algebra in the sense of [7] (arXiv version), Definition 2.8). This means that \( A \) is both a right \( H_R \)-comodule algebra and a right \( H_L \)-comodule algebra, such that the \( H_R \)-coaction \( a \mapsto \sum a_{(0)} \otimes a_{(1)} \) is \( H_L \)-colinear and the \( H_L \)-coaction \( a \mapsto \sum a_{(0)} \otimes a_{(1)} \) is \( H_R \)-colinear.

Under the additional assumption that the antipode is bijective, the entwining map \((2.7)\) was shown in [6, Lemma 4.1] to be bijective with the inverse
\[
\psi^{-1} : A \otimes R \to H \otimes A, \quad a \otimes h \mapsto \sum h S^{-1}(a_{(1)}) \otimes a_{(0)}. 
\]
Hence in this case \( A \) is an entwined extension of its \( \mathcal{H}_R \)-coinvariants (which are, in turn, the same as the \( \mathcal{H}_L \)-coinvariants, see \([7\text{ (arXiv version), Proposition 2.4)}\)].

(3) An extension \( B \subseteq A \) of \( k \)-algebras is called a depth 2 (or D2, for short) extension if \( A \otimes_B A \) is a direct summand in a finite direct sum of copies of \( A \) both as an \( A-B \) bimodule and as a \( B-A \) bimodule. By \([28\text{ Lemma 3.7}])\), this is equivalent to the existence of right and left D2 quasi-bases. A right D2 quasi-basis consists of finite sets \( \{\gamma_j\}_{j \in J} \) in \( \text{End}_{B,K}(A) \) and \( \{\sum_{m \in M_j} c^j_m \otimes c^j_m\}_{j \in J} \) in the commutant \((A \otimes_B A)^B \) of \( B \) in the \( A-A \) bimodule \( A \otimes_B A \), satisfying

\[
\sum_{j \in J, m \in M_j} a^j_m(c^j_m) \otimes c^j_m = a \otimes a',
\]

for all \( a \otimes a' \in A \otimes_B A \). A left D2 quasi-basis is defined symmetrically. Examples of D2 extensions include Hopf-Galois extensions with finitely generated and projective Hopf algebras, centrally projective extensions and H-separable extensions (cf. \([28\text{ for more details)}]\).

It is shown in \([28\text{ Sections 4 and 5}])\) that a depth 2 extension \( B \subseteq A \) determines a dual pair of finitely generated projective bialgebroids, with total algebras \((A \otimes_B A)^B \) (with multiplication inherited from \( A^{\text{op}} \otimes_B A \)) and \( \text{End}_{B,K}(A) \) (with multiplication given by composition), respectively. The base algebra is in both cases \( R \), the commutant of \( B \) in \( A \).

\( A \) is a right \((A \otimes_B A)^B \)-comodule algebra, and if \( A \) is a balanced right \( B \)-module (e.g. if \( A \) is a right \( B \)-generator), then the coinvariants coincide with \( B \). The corresponding (right) entwining map, described in (1), is given in terms of a right D2 quasi-basis as a map

\[
(A \otimes_B A)^B \otimes_R A \rightarrow A \otimes_R (A \otimes_B A)^B, \quad \sum_n (x_n \otimes x'_n) \otimes_R a \mapsto \sum_{j \in J, m \in M_j} \gamma_j(a) \otimes (c^j_m x_n \otimes x'_n c^j_m).
\]

As a consequence of the D2 property, this entwining map is bijective. The inverse has a similar form in terms of the left D2 quasi-basis. In particular, a balanced depth 2 extension is an entwined extension.

Similarly, \( \mathcal{E} : = \text{End}_{B-}(A) \) is a left \( \text{End}_{B,K}(A) \)-comodule algebra. The subalgebra of coinvariants consists of the multiplications with the elements of \( A \) on the right. This comodule algebra determines a bijective left entwining structure.

(4) In the case when a D2 extension \( B \subseteq A \) is also a Frobenius extension, the bialgebroids \((A \otimes_B A)^B \) and \( \text{End}_{B,K}(A) \) are Hopf algebras with two-sided non-degenerate integrals, and hence with bijective antipodes (cf. \([8\text{ [3]}\]). In this case the inverse of the entwining map has a simple form as in part (2).

2.5. Connections and relative cyclic homology. The concepts of a connection and cyclic homology were introduced by A. Connes in \([13\text{ ]})\). The notion of a relative cyclic homology was introduced by L. Kadison in \([24\text{ 25}])\).

Given a \( T \)-ring \( B \) with product \( \mu_B \), one defines a differential graded algebra \( \Omega B \) of \( T \)-relative differential forms on \( B \) (cf. \([19\text{ Section 2}])\), by

\[
\Omega^n B = \Omega^1 B \otimes_B \Omega^1 B \otimes_B \cdots \otimes_B \Omega^1 B \quad (n\text{-times}),
\]

where \( \Omega^1 B = \ker \mu_B \subseteq B \otimes_B B \). The differential is the \( T-T \) bilinear map \( d : B \rightarrow \Omega^1 B \), \( b \mapsto 1_B \otimes b - b \otimes 1_B \), and is uniquely extended as a graded differential to the whole
of $\Omega B$. Let $M$ be a left $B$-module. A $k$-linear map $\nabla_T : M \to \Omega^1 B \otimes M \simeq \Omega^1 B M$ is called a $T$-connection, provided it satisfies the Leibniz rule, i.e. for all $b \in B$ and $m \in M$,

$$\nabla_T(bm) = d(b) \otimes m + b\nabla_T(m).$$

Notice that a $T$-connection is necessarily left $T$-linear. There is a close relationship between $T$-connections in $M$ and the $T$-relative projectivity of $M$ revealed by the Cuntz-Quillen Theorem [19, Proposition 8.2]: a $T$-connection in $M$ exists if and only if there is a left $B$-linear section of the multiplication map $B \otimes M \to M$.

For any $T$-ring $B$ one defines an $(n + 1)$-fold circular tensor product $B \tilde{\otimes}_{T(n+1)}$ as the $T$-$T$ bimodule $B \tilde{\otimes}_{T(n+1)}$ factored by the submodule generated by $b_0 \otimes b_1 \otimes \ldots \otimes b_n - t b_0 \otimes b_1 \otimes \ldots \otimes b_n$ (see [29, 1.2.11]). Note that $B \otimes_T 1 = B/[B,T]$. On such a circular tensor product it is possible to define the cyclic operators

$$\tau_n : B \tilde{\otimes}_{T(n+1)} \to B \tilde{\otimes}_{T(n+1)}, \quad b_0 \otimes b_1 \otimes \ldots \otimes b_n \mapsto (-1)^n b_n \otimes b_0 \otimes \ldots \otimes b_{n-1}.$$  

The relative cyclic homology of a $T$-ring $B$ is then defined as the total homology of the following bicomplex

$$\cdots \to B \tilde{\otimes}_{T3} \to B \tilde{\otimes}_{T2} \to B \tilde{\otimes}_{T1} \to B/[B,T] \to \cdots$$

where $\tilde{\tau}_n = B \tilde{\otimes}_{T(n+1)} - \tau_n$, $N_n = \sum_{i=0}^{n} (\tau_n)^i$, and

$$\partial'_n(b_0 \otimes b_1 \otimes \ldots \otimes b_n) = \sum_{i=0}^{n-1} (-1)^i b_0 \otimes b_1 \otimes \ldots \otimes b_i b_{i+1} \otimes \ldots \otimes b_n,$$

$$\partial_n = \partial'_n + (-1)^n b_n b_0 \otimes b_1 \otimes \ldots \otimes b_{n-1}.$$  

The relative cyclic homology of $B$ is denoted by $HC_*(B[T])$.

The (canonical) surjections $\lambda_n : B \tilde{\otimes}_{k(n+1)} \to B \tilde{\otimes}_{T(n+1)}$ induce the epimorphism $\lambda_* : HC_*(B) \to HC_*(B[T])$, where $HC_*(B)$ is the usual cyclic homology of the $k$-algebra $B$. If $T$ is a separable $k$-algebra, then $\lambda_*$ is an isomorphism [24].

3. Strong connections over non-commutative rings

Throughout the paper we are dealing with left or right modules with a compatible additional structure (e.g., left modules which are also right comodules, bimodules, etc.), and we are interested in properties which are respected by this additional structure. Thus we are led to the following
Definition 3.1. An object $M \in _B M^C$ is called a $T$-relative right $C$-equivariantly projective left $B$-module or a $(C,T)$-projective left $B$-module for a $k$-algebra $T$ provided that $B$ is a right $T$-module and $M$ is a $T$-$R$ bimodule such that the $C$-coaction is left $T$-linear, the left $B$-action is $T$-balanced (i.e. factors through $B \otimes M$), and there exists a left $B$-module, right $C$-comodule section of the $B$-multiplication map $B \otimes M \to M$.

A $T$-relative left $C$-equivariantly projective right $B$-module is defined in an analogous way.

A $B-B'$ bimodule $M$ is called a $T$-relative right $B'$-equivariantly projective left $B$-module or a $(B',T)$-projective left $B$-module for a $k$-algebra $T$ provided that $B$ is a right $T$-module and $M$ is a $T$-$B'$ bimodule such that the left $B$-action is $T$-balanced and there exists a $B-B'$ bilinear section of the $B$-multiplication map $B \otimes M \to M$.

A $T$-relative left $B$-equivariantly projective right $B'$-module is defined in a similar way.

Note that in Definition 3.1 it is not assumed that $B$ is a $T$-ring (but, when $B$ is a $T$-ring, then a left $B$-action on $M$ factors through $B \otimes M$). On the other hand, if there exists a $(C,T)$-projective and faithful left $B$-module (e.g. a $B$-ring which is a $(C,T)$-projective left $B$-module), then $B$ is a $T$-ring: the map $T \to B$, $t \mapsto 1_B t$ is an algebra homomorphism and the right action of $T$ on $B$ coincides with that induced by the above map, i.e. $bt = b(1_B t)$.

It is checked in the standard way that if $B$ is a $T$-ring, then the $(C,T)$-projectivity of an object $M$ in $_B M^C$ is equivalent to the property that for any epimorphism $p \in \text{Hom}_{_B C}(X,Y)$, which splits in $_T M^C$, and any morphism $f \in \text{Hom}_{_B C}(M,Y)$ there exists a morphism $g \in \text{Hom}_{_B C}(M,X)$ such that $p \circ g = f$. Analogously, the $(B',T)$-projectivity of a $B-B'$ bimodule $M$ is equivalent to the property that for any epimorphism $p \in \text{Hom}_{B,B'}(X,Y)$, which splits as a $T-B'$ bimodule map, and every $B-B'$ bimodule map $f : M \to Y$ there exists a $B-B'$ bimodule map $g : M \to X$ such that $p \circ g = f$.

Any object $M \in _B M^C$ is a $B^*C$ bimodule, where $^*C$ is the left dual ring. If $M$ is a $(C,T)$-projective left $B$-module, then $M$ is a $(^*C,T)$-projective left $B$-module. Furthermore, since forgetful functors $_B M^C \to _B M_R$ and $_B M_R \to _B M$ preserve retraction, a $(C,T)$-projective left $B$-module is $(R,T)$-projective, and an $(R,T)$-projective left $B$-module is $T$-relative projective. Clearly, a $T$-relative projective left $B$-module that is projective as a left $T$-module is a projective left $B$-module.

Since separable functors reflect retraction, part (1) of the following Proposition 3.2 (which is a straightforward generalisation of [17, 3.2 Theorem 27]) shows that in the case of a separable $k$-algebra $R$, any $B-R$ bimodule, which is a $T$-relative projective left $B$-module such that the right $R$-action is left $T$-linear, is also $(R,T)$-projective. Similarly, part (2) of Proposition 3.2 (which is a straightforward generalisation of [9, Theorem 3.5], [16, 26.1]) shows that in the case of a coseparable coring $C$, any object $M \in _B M^C$, which is an $(R,T)$-projective left $B$-module such that the right $C$-coaction is left $T$-linear, is also $(C,T)$-projective.

Proposition 3.2. (1) The forgetful functor $_B M_R \to _B M$ is separable for any $k$-algebra $B$ if and only if $R$ is a separable $k$-algebra.

(2) The forgetful functor $_B M^C \to _B M_R$ is separable for any $k$-algebra $B$ if and only if $C$ is a coseparable coring.
Proof. (1) Suppose that the forgetful functor $B \mathbb{M}^C \to B \mathbb{M}_R$ is separable for all $k$-algebras $B$. In particular it is a separable functor for $B = k$, and then $R$ is a separable $k$-algebra (cf. [17, 3.2 Theorem 27]).

Conversely, suppose that $R$ is a separable $k$-algebra and construct a functorial retraction $\Phi$ of the functorial morphism $\text{Hom}_{B,R}(-, -) \to \text{Hom}_{B,-}(B(-), B(-))$, $u \mapsto u$ as follows. For any $B$-$R$ bimodules $M$, $N$ and a left $B$-module map $f : M \to N$,

$$\Phi(f) = \varrho_N \circ (f \otimes_R) \circ (\varrho_M \otimes_k) \circ (M \otimes \zeta),$$

where $\zeta : k \to R \otimes_R$ is a $k$-linear map satisfying conditions (2.1).

(2) If the forgetful functor is separable for all $k$-algebras $B$, then it is separable for $B = k$ and hence $C$ is a coseparable coring by [9, Theorem 3.5] or [16, 26.1].

The proof of the converse is completely analogous to the proof in [9, Theorem 3.5] or [16, 3.29]. Explicitly, a functorial retraction $\Phi$ of the functorial morphism $\text{Hom}_{B,R}(-, -) \to \text{Hom}_{B,R}(B(-), B(-))$, $u \mapsto u$ is given by the map, associating to a $B$-$R$ bimodule map $f : M \to N$ the $B$-$linear$ $C$-colinear map

$$\Phi(f) = (N \otimes_R \delta) \circ (\varrho_N \otimes_R) \circ (f \otimes_C) \circ \varrho_M,$$

where $M$ and $N$ are objects in $B \mathbb{M}^C$ and $\delta$ is a cointegral for $C$. □

Finally, let us note that if $B$ is a separable $T$-ring, then any object in $B \mathbb{M}^C$ is a $(C,T)$-projective left $B$-module. Indeed, a left $B$-linear, right $C$-colinear splitting of the left $B$-multiplication $B \mathbb{M} \to M$ can be constructed in terms of a separability idempotent $\sum_{i \in L} e_i \otimes f_i \in B \mathbb{M}$ as $m \mapsto \sum_{i \in L} e_i \otimes f_i m$.

Following the same line of argument as in [19], one easily establishes a relationship between $(C,T)$-projectivity and the existence of $T$-connections. For any $M \in B \mathbb{M}^C$, $\Omega^1 \mathbb{M} = \Omega^1 \mathbb{M} \to \Omega^1 \mathbb{M}$ is a right $C$-comodule with the natural coaction $\varrho^1 \mathbb{M} = \Omega^1 \mathbb{M} : \Omega^1 \mathbb{M} \to \Omega^1 \mathbb{M} \otimes C$.

**Theorem 3.3 (Cuntz-Quillen).** Let $B$ be a $T$-ring and $C$ an $R$-coring. An object $M \in B \mathbb{M}^C$ is a $(C,T)$-projective left $B$-module if and only if there exists a right $C$-colinear $T$-connection $\nabla_T : M \to \Omega^1 \mathbb{M}$.

Proof. Note that $\Omega^1 \mathbb{M}$ can be identified with the kernel of the $B$-product $\mathbb{M} \varrho : B \mathbb{M} \to M$, by the commutativity of the following diagram:

$$
\begin{array}{ccc}
0 & \to & \Omega^1 \mathbb{M} & \xrightarrow{\nu_1} & B \mathbb{M} \\
\downarrow & & \downarrow \pi & & \downarrow & & \downarrow \pi \\
0 & \to & \ker \mathbb{M} \varrho & \xrightarrow{\nu_2} & B \mathbb{M} \\
\end{array}
$$

where $\nu_1$, $\nu_2$ are obvious inclusions both split by $\pi : b \mathbb{T} \mapsto b \mathbb{T} - 1 \otimes bm$.

Suppose that $M$ is a $(C,T)$-projective left $B$-module, and let $\sigma_T : M \to B \mathbb{M}$ be a left $B$-linear, right $C$-colinear section of the $B$-multiplication map $\mathbb{M} \varrho$. Then

$$\nabla_T : M \to \Omega^1 \mathbb{M}, \quad m \mapsto 1_b \mathbb{T} m - \sigma_T(m)$$
is a right $C$-colinear $T$-connection. Conversely, given a $T$-connection $\nabla_T$, define
\[ \sigma_T : M \to B \otimes M, \quad m \mapsto 1_B \otimes m - \nabla_T(m). \]
Using the Leibniz rule for $\nabla_T$ one easily checks that $\sigma_T$ is left $B$-linear. It obviously splits the product (as $\nabla_T(M) \subseteq \ker M \varrho$) and is right $C$-colinear as a difference of right $C$-comodule maps. 

Using arguments similar to these in the proof of Theorem 3.3 one easily proves that a $B$-$B'$ bimodule $M$, where $B$ is a $T$-ring, is a $(B',T)$-projective left $B$-module if and only if there exists a right $B'$-linear connection $\nabla_T : M \to \Omega^1 B \otimes M$.

The following definition introduces the main object studied in the present paper.

**Definition 3.4.** Let $(A,C,\psi)_R$ be a bijective right entwining structure over $R$ and $B \subseteq A$ an entwined extension. Let $T$ be a $k$-subalgebra of $B$. View $A \otimes A$ as a right $C$-comodule via $A \otimes \varrho_A$ and as a left $C$-comodule via $A \otimes \varrho_A$, where $\varrho_A$ is the right and $A \varrho$ is the left $C$-coaction on $A$, related by (2.5). A left and right $C$-comodule map $\ell_T : C \to A \otimes A$ is called a **strong $T$-connection in $(A,C,\psi)_R$** iff, for all $c \in C$,
\[ \tilde{\text{can}}_T(\ell_T(c)) = 1_A \otimes c, \]
where $\tilde{\text{can}}_T : A \otimes A \to A \otimes_R C$, $a \otimes a' \mapsto a \varrho_A(a')$.

**Lemma 3.5.** Let $(A,C,\psi)_R$ be a bijective right entwining structure over $R$ and $B \subseteq A$ an entwined extension. Let $T' \subseteq T \subseteq B$ be $k$-subalgebras. If $A$ is a $(C,T')$-projective left or right $T$-module and there exists a strong $T$-connection in $(A,C,\psi)_R$, then there exists a strong $T'$-connection in $(A,C,\psi)_R$.

**Proof.** Suppose that $A$ is a $(C,T')$-projective left $T$-module, and let $\xi$ be a left $T$-module right $C$-comodule splitting of the multiplication map $T \otimes A \to A$. Define a right $C$-comodule map
\[ \ell_{T'} = (A \otimes \xi) \circ \ell_T : C \to A \otimes_{T'} A, \]
where the canonical isomorphism $A \otimes T \to A$ is implicitly used (here and also in the computations below). Since $A \varrho$ is a right $T$-module map, the canonical isomorphism $A \otimes T \to A$ is a left $C$-comodule map. Thus $\ell_{T'}$ is a left $C$-comodule map. Using the right $C$-colinearity of $\xi$ (in the second equality) and the fact that $\xi$ is a section of the product $T \otimes A \to A$ (in the third), we can compute
\[ \tilde{\text{can}}_{T'} \circ \ell_{T'} = (\mu_A \otimes C) \circ (A \otimes \varrho_A) \circ (A \otimes \xi) \circ \ell_T = (\mu_A \otimes C) \circ (A \otimes \varrho_A) \circ \ell_T = \tilde{\text{can}}_T \circ \ell_T, \]
(note that $\mu_A$ denotes the product of $A$ both as a $T$- and as a $T'$-ring). Thus we conclude that, for all $c \in C$, $\tilde{\text{can}}_{T'}(\ell_{T'}(c)) = 1_A \otimes c$, i.e. $\ell_{T'}$ is a strong $T'$-connection in $(A,C,\psi)_R$. The $(C,T')$-projective right $T$-module case is proven in a similar way. \(\square\)

**Remark 3.6.** In [10, Theorem 4.4] it has been shown for a general class of comodules of corings that the canonical map is an isomorphism (of corings) provided that it is a split epimorphism of comodules. Originally, [10, Theorem 4.4] was formulated for base algebras over fields. The proof was extended to base algebras over commutative
rings \( k \) in \cite[Theorem 2.1]{15}. Notice, however, that in the proof of \cite[Theorem 2.1]{15} the commutativity of \( k \) does not play any role. Hence, by repeating the arguments there, one can prove the following (cf. \cite[5.9]{35}).

Let \( D \) be an \( A \)-coring with a grouplike element \( e \). Denote the corresponding coinvariants of \( A \) by \( B \). Let \( T \) be a \( k \)-subalgebra of \( B \). Suppose that the obvious inclusion \( B \otimes A \to \text{Hom}^D(A, A \otimes A) \) \( \simeq \{ \sum \alpha_i \otimes \delta_i \in A \otimes A \mid \sum \alpha_e \otimes \delta_i = \sum a_i \otimes \delta_i \} \), \( b \otimes a \mapsto b \otimes a \), is an isomorphism. Then \( D \) is a Galois \( A \)-coring if

\[
\overline{\text{can}}_T : A \otimes \overline{A} \to D, \quad a \otimes a' \mapsto a \rho(A)(a') = e a a',
\]

is a split epimorphism of left \( D \)-comodules.

In light of the bijective correspondence between \( T \)-connections and \( (C,T) \)-projective modules described in Theorem \cite[3.3]{3.3}, the following theorem, which is the main result of this section, justifies the use of the term "\( T \)-connection" in Definition \cite[3.4]{3.4}.

Note that, for an entwined extension \( B \subseteq A \) in a bijective right entwining structure \( (A,C,\psi)_R \) and a subalgebra \( T \) of \( B \), \( A \otimes A \) is a right entwined module for \( (A,C,\psi)_R \) with the coaction \( A \otimes \rho \otimes 1 \) and a left entwined module for \( R(A,C,\psi^{-1}) \) with the coaction \( A \rho \otimes 1 \), where \( A \rho \) is related to \( \rho \) as in (2.3), and with obvious \( A \)-multiplications.

**Theorem 3.7.** Let \( (A,C,\psi)_R \) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension. Let \( T \) be a \( k \)-subalgebra of \( B \). Consider the following statements.

(a) There exists a strong \( T \)-connection in \( (A,C,\psi)_R \).

(b) \( (A \otimes C)_\psi \) is a Galois \( A \)-coring and \( A \) is a \( (C,T) \)-projective left \( B \)-module.

(c) \( (C \otimes A)_{\psi^{-1}} \) is a Galois \( A \)-coring and \( A \) is a \( (C,T) \)-projective right \( B \)-module.

Then

(1) The statement (b) implies (a).

(2) The statement (c) implies (a).

(3) If the obvious inclusion

\[
B \otimes A \to \text{co}(A \otimes A), \quad b \otimes a \mapsto b \otimes a,
\]

is an isomorphism, then (a) implies (b).

(4) If the obvious inclusion

\[
A \otimes B \to (A \otimes A)^{\text{co-c}}, \quad a \otimes b \mapsto a \otimes b,
\]

is an isomorphism, then (a) implies (c).

**Proof.** (1) We show that if \( (A \otimes C)_\psi \) is a Galois \( A \)-coring, then there exists a strong \( B \)-connection in the bijective right entwining structure \( (A,C,\psi)_R \). Then the claim follows by Lemma \cite[3.5]{3.5}.

If \( (A \otimes C)_\psi \) is a Galois \( A \)-coring, then there exists an \( A \)-coring inverse \( \text{can}_A^{-1} \) of the canonical map \( \text{can}_A : A \otimes A \to A \otimes C, a \otimes a' \mapsto a \rho(A)(a') \). Let

\[
\varpi : C \to A \otimes A, \quad c \mapsto \text{can}_A^{-1}(1_A \otimes c),
\]

be the translation map and write \( \varpi(c) = \sum c[1] \otimes c[2] \). Since \( \text{can}_A^{-1} \) is a coring map, it is, in particular, a morphism of right \( (A \otimes C)_\psi \)-comodules, hence of right \( C \)-comodules.
Therefore, also $\varpi$ is a morphism of right $C$-comodules. Furthermore, $\text{can}_{A}^{-1}$ is a morphism of left $A$-modules. Since $A$ is an $R$-ring, this implies that $\varpi$ is a left $R$-module map. To exploit further the $A$-coring map property of $\text{can}_{A}^{-1}$, start from the equality

$$\sum \text{can}_{A}^{-1}(1_{A} \otimes c_{(1)}) \otimes \text{can}_{A}^{-1}(1_{A} \otimes c_{(2)}) = \sum c^{[1]} \otimes 1_{A} \otimes 1_{A} \otimes c^{[2]},$$

and apply $(\psi^{-1} \otimes A \otimes A) \circ (\text{can}_{A} \otimes A \otimes A)$ to arrive at the left $C$-colinearity of $\varpi$ (note that the left $C$-coaction $A^{\mathcal{C}}$ is right $B$-linear). Thus $\varpi$ is a strong $B$-connection, hence Lemma 3.5 implies that there is a strong $T$-connection in $(A, C, \psi)^{R}$.

(3) Let $\ell_{T} : C \to A \otimes A$ be a strong $T$-connection. For all $c \in C$, write $\ell_{T}(c) = \sum c^{(1)} \otimes c^{(2)}$ and define

$$\sigma_{T} : A \to A \otimes A, \quad a \mapsto \sum a_{[0]}' \ell_{T}(a_{[1]}) = \sum a_{[0]}' a_{[1]}^{(1)} \otimes a_{[1]}^{(2)}.$$

For all $a \in A$,

$$\left(A^{\mathcal{B}} \otimes A\right) \circ \sigma_{T}(a) = \sum A^{\mathcal{B}}(a_{[0]}' a_{[1]}^{(1)}) \otimes a_{[1]}^{(2)}$$

$$= \sum \psi^{-1}(a_{[0]} \otimes a_{[1]}^{(1)})(-1) a_{[1]}^{(1)} \otimes a_{[1]}^{(2)}$$

$$= \sum \psi^{-1}(a_{[0]}' \otimes a_{[1]}^{(1)}) a_{[1]}^{(2)} \otimes a_{[1]}^{(2)}$$

$$= \sum \psi^{-1}(\sigma_{T}(1_{A})) a_{[0]}' a_{[1]}^{(1)} \otimes a_{[1]}^{(2)} = A^{\mathcal{B}}(1_{A}) \sigma_{T}(a),$$

where the second equality follows by the fact that $A$ is a left entwined module and the third one follows by the left colinearity of $\ell_{T}$. The penultimate equality is a consequence of the right $A$-linearity of the right coaction $\psi^{-1} \circ A^{\mathcal{B}}$ of the $A$-coring $(C \otimes A) \psi^{-1}$ on $A$. Thus, for all $a \in A$, $\sigma_{T}(a)$ is a coinvariant of the left $(A \otimes C)\psi$-comodule $A \otimes A$, hence,

$$\sigma_{T} : A \to B \otimes A.$$  

Since $\ell_{T}$ is a right $C$-colinear map, so is $\sigma_{T}$. Furthermore, $\sigma_{T}$ is a left $B$-linear map, as, by the definition of $B$, $\sigma^{A}$ is a left $B$-linear map. The splitting property of $\ell_{T}$, $\text{can}_{T}(\ell_{T}(c)) = 1_{A} \otimes c$, implies that, for all $c \in C$, $\sum c^{(1)} c^{(2)} = 1_{A} \varepsilon_{C}(c)$, hence

$$\mu_{A}(\sigma_{T}(a)) = \sum a_{[0]}' a_{[1]}^{(1)} a_{[1]}^{(2)} = \sum a_{[0]}' \varepsilon_{C}(a_{[1]}) = a.$$  

Thus we conclude that $\sigma_{T}$ is a left $B$-linear right $C$-colinear section of the multiplication map $B \otimes A \to A$.

It remains to show that the right canonical map $\text{can}_{A}$ is an isomorphism of $k$-modules. To this end we study first the properties of the lifted canonical map $\text{can}_{T}$. Since $A$ is a left entwined module for the left entwining structure $R(A, C, \psi^{-1})$ with the coaction $A^{\mathcal{B}}$, $A \otimes A$ is a left $(A \otimes C)\psi$-comodule with the coaction $a \otimes a' \mapsto \sum a_{1}A_{[0]} \otimes a_{1}A_{[1]} \otimes a'$. On the other hand, $A \otimes C$ is a left $(A \otimes C)\psi$-comodule via the regular coaction (the coproduct), $a \otimes c \mapsto \sum a \otimes c_{(1)} \otimes 1_{A} \otimes c_{(2)}$. Define a left $A$-module map (hence also a left $R$-module map, as $A$ is an $R$-ring)

$$\kappa : A \otimes C \to A \otimes A, \quad a \otimes c \mapsto a \ell_{T}(c).$$
Since the lifted canonical map \( \widetilde{\mathrm{can}}_T \) is left \( A \)-linear and, for all \( c \in C \), \( \widetilde{\mathrm{can}}_T(\ell_T(c)) = 1_A \otimes c \), one immediately finds that \( \kappa \) is a section of \( \widetilde{\mathrm{can}}_T \). We claim that \( \kappa \) is a left \( (A \otimes C)_\psi \)-comodule map. In view of the definitions of \( \kappa \), the \( (A \otimes C)_\psi \)-coactions and the right action of \( A \) on \( (A \otimes C)_\psi \), this is equivalent to that statement that for all \( a \in A \), \( c \in C \),

\[
\sum a \psi(c_1(1) \otimes c_2(1)) \otimes c_2(2) = \sum ac_1(1) 1_A[0] \otimes 1_A[1] \otimes c(2).
\]

Using the left \( C \)-colinearity of \( \ell_T \) and the form (2.5) of the left \( C \)-coaction \( A \psi \) we can compute

\[
\sum a \psi(c_1(1) \otimes c_2(1)) \otimes c_2(2) = \sum a \psi(c^{(1)} - 1(1) \otimes c^{(1)}(0)) \otimes c(2)
\]

\[
= \sum a \psi(c_{\psi}^{-1}(c^{(1)} 1_A[0] \otimes 1_A[1])) \otimes c(2)
\]

\[
= \sum ac_1(1) 1_A[0] \otimes 1_A[1] \otimes c(2),
\]

as required. Thus \( \kappa \) is a left \( (A \otimes C)_\psi \)-comodule section of \( \widetilde{\mathrm{can}}_T \). Finally note that

\[
\mathrm{Hom}(A \otimes_C)_\psi^{-1}(A, A \otimes_T A) \simeq \mathrm{Hom}(C \otimes_A)_\psi^{-1}(A, A \otimes_A A) \simeq \mathrm{coC}(A \otimes_T A) = B \otimes_T A,
\]

where the last equality follows by assumption (3). Hence, the \( T \)-module version of [10, Theorem 4.4] adapted to this situation in Remark 3.6 implies that \( (A \otimes C)_\psi \) is a Galois coring.

Assertions (2) and (4) follow by the left-right symmetry. \( \square \)

The assumptions of part (3) (resp. (4)) in Theorem 3.7 are automatically satisfied, provided \( A \) is a flat left (resp. right) \( T \)-module. Hence Theorem 3.7 leads to the following

**Corollary 3.8.** Let \( (A, C, \psi)_R \) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension. Let \( T \) be a \( k \)-subalgebra of \( B \). If \( A \) is a flat left (resp. right) \( T \)-module, then the following statements are equivalent.

1. There exists a strong \( T \)-connection in \( (A, C, \psi)_R \).
2. \( (A \otimes C)_\psi \) is a Galois \( A \)-coring and \( A \) is a \( (C, T) \)-projective left (resp. right) \( B \)-module.

Repeating the arguments in [10, Theorem 4.3] one proves also

**Corollary 3.9.** Let \( (A, C, \psi)_R \) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension. Let \( T \) be a \( k \)-subalgebra of \( B \). Suppose that

(a) There exists a strong \( T \)-connection in \( (A, C, \psi)_R \);
(b) \( A \) is a flat left (resp. right) \( B \)-module;
(c) \( A \) is a faithfully flat left (resp. right) \( T \)-module.

Then \( A \) is a faithfully flat left (resp. right) \( B \)-module.

Note that if \( A \) is a projective left (resp. right) \( T \)-module, then, in view of Theorem 3.7 (3) (resp. (4)), the assumption (a) in Corollary 3.9 implies that \( A \) is a projective (hence in particular flat) left (resp. right) \( B \)-module.

We conclude this section by constructing strong connections in some examples.
Example 3.10. Consider an entwining structure $(A, C, \psi)_B$ constructed in Example 2.3. Assume that $A$ is faithfully flat as a left $B$-module and that the canonical left entwining map $\psi_{\text{can}}$ is bijective, hence $(A, C, \psi)_B$ is a bijective entwining structure by Remark 2.4 (1). The canonical inclusion map

$$\ell_R : C \to A \otimes_R A, \quad \sum_i a_i \otimes_R \tilde{a}_i \mapsto \sum_i a_i \otimes_R \tilde{a}_i$$

is a strong $R$-connection. In particular, by Corollary 3.8, if $A$ is a flat left or right $R$-module, then $(A \otimes_R C)_\psi$ is a Galois coring.

Let $T$ be a $k$-subalgebra of $R$. If $A$ is a $(C, T)$-projective left $R$-module, then, by Lemma 3.5, a strong $T$-connection in $(A, C, \psi)_R$ can be constructed in terms of a left $R$-linear, right $C$-colinear section $\sigma_T : A \to R \otimes_T A$ of the $R$-multiplication map,

$$\ell_T : C \to A \otimes_T A, \quad \sum_i a_i \otimes_R \tilde{a}_i \mapsto \sum_i a_i (\eta_A \otimes_T A)(\sigma_T(\tilde{a}_i)).$$

In particular, if $R$ is a separable $k$-algebra and $\sigma_k : A \to R \otimes_k A$ is determined by a separability idempotent $\sum_{i \in L} e_i \otimes f_i$, then the strong $k$-connection comes out as

$$\ell(\sum_i a_i \otimes_R \tilde{a}_i) = \sum_{i,l} a_i \eta_A(e_l) \otimes \eta_A(f_l) \tilde{a}_i.$$ 

Example 3.11. Let $\mathcal{H}$ be a Hopf algebroid as in Example 2.5 (2). Let $A$ be a right $\mathcal{H}$-comodule algebra (cf. Example 2.5 (2)), and let $B$ be the $\mathcal{H}_R$-ccoinvariants of $A$. The extension $B \subseteq A$ is called a clef $\mathcal{H}$-extension if

(a) In addition to being an $R$-ring (with unit morphism $\eta_R$), $A$ is also an $L$-ring (with unit morphism $\eta_L$) such that $B$ is an $L$-subring of $A$.

(b) There exists a left $L$-linear right $\mathcal{H}_R$-colinear map $j : H \to A$ that is convolution invertible in the sense that there exists a left $R$-linear right $L$-linear map $\tilde{j} : H \to A$ such that

$$\mu_A \circ (j \otimes_R \tilde{j}) \circ \gamma_R = \eta_L \circ \pi_L, \quad \mu_A \circ (\tilde{j} \otimes_L j) \circ \gamma_L = \eta_R \circ \pi_R.$$ 

It turns out that the $\mathcal{H}$-clef property of an extension $B \subseteq A$ is sufficient and necessary for the coring $(A \otimes_R H)_\psi$, corresponding to the entwining structure in Example 2.5 (1), to be a Galois $A$-coring and $A$ to be isomorphic to $B \otimes_H$ both as left $B$-modules and as right $\mathcal{H}_R$-comodules (normal basis property).

Let $B \subseteq A$ be a clef extension for a Hopf algebroid $\mathcal{H} = (\mathcal{H}_L, \mathcal{H}_R, S)$. Suppose that the antipode $S$ is bijective, hence there exists a bijective right entwining structure $(A, \mathcal{H}_R, \psi)_R$ over $R$ as in Example 2.5 (1-2). Since $(A \otimes_H)_\psi$ is a Galois $A$-coring, Theorem 3.7 (1) implies that a sufficient condition for the existence of a strong $T$-connection in $(A, \mathcal{H}_R, \psi)_R$ for a $k$-subalgebra $T$ of $B$ is the $(\mathcal{H}_R, T)$-projectivity of the left $B$-module $A$. For an $\mathcal{H}$-clef extension $B \subseteq A$, there is an isomorphism of $k$-modules

$$\text{Hom}_{B-\mathcal{H}_R}(A, B \otimes_T A) \simeq \text{Hom}_{L-L}(H, B \otimes_B),$$

hence sections of the $B$-multiplication map $B \otimes T A \to A$ in $B \mathcal{M}^{\mathcal{H}_R}$ are in bijective correspondence with $L-L$ bimodule maps $f_T : H \to B \otimes_B$, such that $\mu_B \circ f_T = \eta_L \circ \pi_L$. In terms of $f_T$, a strong $T$-connection, given by formula (3.1), is

$$\ell_T := (\mu_A \otimes T \mu_A) \circ (\tilde{j} \otimes L f_T \otimes j) \circ (\gamma_T \otimes L H) \circ \gamma_L.$$
In particular, an \( L-L \) bimodule map \( f_L : H \to B \otimes_L B \), satisfying \( \mu_B \circ f_L = \eta_L \circ \pi_L \) is given by
\[
  h \mapsto \eta_L(\pi_L(h)) \otimes 1_B = 1_B \otimes \eta_L(\pi_L(h)).
\]
The resulting strong \( L \)-connection is \( \ell_L = (\tilde{j} \otimes \tilde{j}) \circ \gamma_L \).

If \( L \) is a separable \( k \)-algebra, then any separability idempotent \( \sum_i e_i \otimes f_i \) determines a strong \( k \)-connection via \( f(h) := \sum_i \eta_L(\pi_L(h)e_i) \otimes \eta_L(f_i) \) for \( h \in H \).

**Example 3.12.** Let \( B \subseteq A \) be a depth 2 balanced extension of \( k \)-algebras. The \( R \)-coring \( A \otimes_R (A \otimes_R A) \) and the \( \mathcal{E} \)-coring \( \text{End}_{B,B}(A) \otimes_R \mathcal{E} \), corresponding to the bijective entwining structures in Example 2.5 (3), were shown to be Galois corings in [26], [27].

Let \( T \) be a subalgebra of \( B \) such that \( A \) is a \( T \)-relative projective left \( B \)-module. Then it follows by Proposition 3.2 and Theorem 3.7 (1) that if \( R \) is a separable \( k \)-algebra and \( (A \otimes A)^R \) and \( \text{End}_{B,B}(A) \) are coseparable \( R \)-corings, respectively, then there exist strong \( k \)-connections in the bijective entwining structures in Example 2.5 (3). As it turns out, \( (A \otimes A)^R \) is a coseparable \( R \)-coring if the D2 extension \( B \subseteq A \) is split and \( \text{End}_{B,B}(A) \) is coseparable if the extension \( B \subseteq A \) is separable.

Motivated by forthcoming Example 5.7, we focus on the case when the depth 2 balanced extension \( B \subseteq A \) is also a Frobenius extension. In this case \((A \otimes A)^R\) is coseparable if and only if the D2 Frobenius extension \( B \subseteq A \) is split, and \( \text{End}_{B,B}(A) \) is coseparable if and only if the D2 Frobenius extension \( B \subseteq A \) is separable. Since in the latter case \( A^{\text{op}} \subseteq \mathcal{E} \) (where the inclusion is given by the left multiplication) is a D2 Frobenius split extension and the bialgebroids \((\mathcal{E} \otimes \mathcal{E})^{A^{\text{op}}} \) and \( \text{End}_{B,B}(A) \) are anti-isomorphic, we consider the split case only. Fix

(a) a Frobenius system \( \{ \omega, \sum_k K u_k \otimes v_k \} \) for the extension \( B \subseteq A \);
(b) a right D2 quasi-basis \( \{ \gamma_j \sum_{m \in M_j} c^j_m \otimes c^j_m \} \) for the extension \( B \subseteq A \);
(c) a \( B-B \) bilinear map \( \varphi \) of the extension \( B \subseteq A \);
(d) a separability idempotent \( \sum_{i \in L} e_i \otimes f_i \) for the \( k \)-algebra \( R \).

A strong \( k \)-connection is explicitly constructed as the map
\[
  (A \otimes A)^R \to A \otimes A, \quad a \otimes a' \mapsto \sum_{l \in L, k \in K} \omega(\gamma_j(a') \otimes v_k) \otimes \varphi(v_k u_l d^j_m) c^j_m.
\]

4. The Relative Injectivity and Group-like Elements

In this section we derive conditions for an entwined extension to be a relative injective \( C \)-comodule. We begin with the following simple generalisation of [33, Remark 4.2].

**Proposition 4.1.** Let \((A, C, \psi)_R \) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension such that \((A \otimes C)_R \) is a Galois \( A \)-coring. If the extension \( B \subseteq A \) splits as a right (resp. left) \( B \)-module, then \( A \) is a relative injective right (resp. left) \( C \)-comodule.

**Proof.** Since the right \( C \)-coaction on \( A \) is left \( B \)-linear, the assumption that \( B \) is a direct summand of \( A \) as a right \( B \)-module implies that \( A \simeq B \otimes A \) is a direct summand of \( A \otimes A \) as a right \( C \)-comodule. By the Galois property, \( A \otimes A \) is isomorphic to \( A \otimes C \), in particular as a right \( C \)-comodule, hence \( A \) is a direct summand of the relative
injective right $C$-comodule $A \otimes C$. Therefore, it is a relative injective $C$-comodule. The left-sided claim is proven analogously using the left-right symmetry of bijective entwining structures. □

The assumptions of Proposition 4.1 take a particularly natural form in the case of entwined extensions given by a grouplike element $e \in C$. In particular, one can derive the following generalisation of Doi’s theorem on total integrals [20, (1.6) Theorem].

**Proposition 4.2.** Let $(A, C, ψ)_R$ be a bijective right entwining structure over $R$ and let $e \in C$ be a grouplike element. The right $C$-comodule $A$ with the coaction given by the first of equations (2.6) is $R$-relative injective if and only if there exists a right $C$-comodule map $j : C \to A$, such that $j(e) = 1_A$. The left $C$-comodule with the coaction given by the second of equations (2.6) is $R$-relative injective if and only if there exists a left $C$-comodule map $\tilde{j} : C \to A$, such that $\tilde{j}(e) = 1_A$.

**Proof.** The $R$-relative injectivity of the right $C$-comodule $A$ is equivalent to the existence of a right $C$-colinear retraction $h$ of the coaction $g^A$ in (2.6), see [16, 18.18]. We show that the $k$-module of comodule maps $j$, such that $j(e) = 1_A$, is a (non-zero) direct summand in the $k$-module of $C$-colinear retractions $h$ of $g^A$.

To any right $C$-colinear retraction $h$ of $g^A$ associate the map

$$j : C \to A, \quad c \mapsto h(1_A \otimes c).$$

The map $j$ is right $C$-colinear by the colinearity of $h$. Since $g^A(1_A) = 1_A \otimes e$, $j$ is also normalised, as

$$j(e) = h(1_A \otimes e) = h(g^A(1_A)) = 1_A.$$

Conversely, given an $e$-normalised right $C$-comodule map $j$, define the map

$$h = \mu_A \circ (j \otimes A) \circ ψ^{-1} : A \otimes C \to A.$$

The map $h$ is right $C$-colinear since

$$g^A \circ h = g^A \circ \mu_A \circ (j \otimes A) \circ ψ^{-1}$$

$$= (\mu_A \otimes C) \circ (A \otimes ψ) \circ (g^A \otimes A) \circ (j \otimes A) \circ ψ^{-1}$$

$$= (\mu_A \otimes C) \circ (A \otimes ψ) \circ (j \otimes C \otimes A) \circ (\Delta_C \otimes A) \circ ψ^{-1}$$

$$= [\mu_A \circ (j \otimes A) \circ ψ^{-1} \otimes C] \circ (A \otimes \Delta_C) = (h \otimes C) \circ (A \otimes \Delta_C),$$

where the second equality follows by the fact that $A$ is a right entwined module, the third one by the right $C$-colinearity of $j$, and the penultimate one by the definition of entwining structures. Since $j(e) = 1_A$, the map (4.2) is a retraction of $g^A$.

Starting with a normalised $C$-comodule map $j : C \to A$, and associating first the map (4.2) to it and then the map (4.1) to the result, we obtain the comodule map

$$c \mapsto \mu_A[(j \otimes A)(ψ^{-1}(1_A \otimes c))] = \mu_A(j(c) \otimes 1_A) = j(c),$$

where the first equality follows from the definition of entwining structures.

Similar arguments apply to left $C$-comodule $A$ with the coaction $g^A$ in (2.6). □

As an immediate application of the above results, one concludes that in the case of a D2 extension $B \subseteq A$ in Example 2.5 (3), $E = \text{End}_{B-}(A)$ is an $R$-relative injective left comodule of $\text{End}_{B,B}(A)$. Indeed, recall from Example 3.12 that the $E$-coring $\text{End}_{B,B}(A) \otimes_R E$, corresponding to the bijective entwining structure in Example 2.5 (3),
is a Galois coring. What is more, the extension $A^{op} \subseteq E$ is split in the category of left $A^{op}$-modules by the map $\alpha \mapsto \alpha(1_A)$, hence the relative injectivity of the comodule in question follows by Proposition 4.1. (The obvious inclusion $\text{End}_{B,B}(A) \subseteq \text{End}_{B-}(A)$ is a required left colinear map $\tilde{j}$ as in Proposition 4.2.) Furthermore, if a D2 extension $B \subseteq A$ is split (by a map $\varphi : A \rightarrow B$, say) in the category of right $B$-modules, then also $A$ is a relative injective right $(A \otimes A)^B$-comodule (as $j = \varphi \otimes A : (A \otimes A)^B \rightarrow A$ is a normalised right comodule map).

The derivation of sufficient conditions for assumptions of Proposition 4.1 is the subject of the remainder of the present section. All these conditions turn out to be closely related to the existence of strong connections.

**Lemma 4.3.** Let $(A,C,\psi)_R$ be a bijective right entwining structure over $R$ and let $B \subseteq A$ be an entwined extension given by a grouplike element $e \in C$. Let $T$ be a $k$-subalgebra of $B$. Suppose that $A$ is a flat left (resp. right) $T$-module and $B$ is a flat right (resp. left) $T$-module. Then

1. If there exists a left $B$-linear, right $C$-colinear section $\sigma_T$ of the multiplication map $B \otimes A \rightarrow A$ (resp. a right $B$-linear, left $C$-colinear section $\tilde{\sigma}_T$ of the multiplication map $A \otimes B \rightarrow A$), then $\sigma_T(1_A) \in B \otimes B$ (resp. $\tilde{\sigma}_T(1_A) \in B \otimes B$).

2. If there exists a strong $T$-connection $\ell_T$ in $(A,C,\psi)_R$, then $\ell_T(e) \in B \otimes B$.

**Proof.** (1) Let $\sigma_T$ be a left $B$-linear, right $C$-colinear section of the multiplication map $B \otimes A \rightarrow A$. By the right $C$-colinearity of $\sigma_T$ and $\sigma^A(1_A) = 1_A \otimes e$,

$$(B \otimes \sigma^A)((\sigma_T(1_A))) = (\sigma_T \otimes C)(\sigma^A(1_A)) = \sigma_T(1_A) \otimes e,$$

that is, $\sigma_T(1_A) \in (B \otimes A)^{coC}$. Since $B$ is a flat right $T$-module, $(B \otimes A)^{coC} = B \otimes B$, hence $\sigma_T(1_A) \in B \otimes B$, as required. The claim about $\tilde{\sigma}_T$ follows by the left-right symmetry of bijective entwining structures.

(2) Recall from the proof of Theorem 3.7 (3) that, for any strong $T$-connection $\ell_T$, there exists a section $\sigma_T \in \text{Hom}^{-C}_{B-}(A,B \otimes A)$ of the multiplication map such that $\ell_T$ is related to the translation map $\varpi$ as $\ell_T = (A \otimes \sigma_T) \circ \varpi$. Since $\sigma^A(1_A) = 1_A \otimes e$, the canonical map $\text{can}_A$ is normalised so that $\text{can}_A(1_A \otimes 1_A) = 1_A \otimes e$. Then the translation map is also normalised, $\varpi(e) = \text{can}_A^{-1}(1 \otimes e) = 1_A \otimes 1_A$. Therefore, $\ell_T(e) = (A \otimes \sigma_T)(\varpi(e)) = \sigma_T(1_A)$, which is an element of $B \otimes B$ by (1). $\square$

**Proposition 4.4.** Let $(A,C,\psi)_R$ be a bijective right entwining structure over $R$ and let $B \subseteq A$ be an entwined extension given by a grouplike element $e \in C$. Let $T$ be a $k$-subalgebra of $B$. Suppose that

1. There exists a strong $T$-connection in $(A,C,\psi)_R$;
2. $A$ is a flat left (resp. right) $T$-module;
3. $B$ is a flat right (resp. left) $T$-module;
4. $B$ is a direct summand of $A$ as a left (resp. right) $T$-module.

Then $B$ is a direct summand of $A$ as a left (resp. right) $T$-module.

**Proof.** Consider the assumptions without parenthesis. By Theorem 3.7 (3), there exists a left $B$-linear section $\sigma_T$ of the multiplication map $B \otimes A \rightarrow A$. In terms of
\( \sigma_T \) and a left \( T \)-module splitting \( f \) of the canonical monomorphism \( B \to A \), a left \( B \)-linear splitting can be constructed as
\[
\phi : = \mu_B \circ (B \otimes f) \circ \sigma_T.
\]
Since \( \phi \) is a composite of left \( B \)-linear maps, it is left \( B \)-linear, and it satisfies
\[
\phi(1_A) = \mu_B[(B \otimes f)(\sigma_T(1_A))] = \mu_A(\sigma_T(1_A)) = 1_A,
\]
where the second equality follows by Lemma 4.3 (1).

Similarly, if the assumptions in parenthesis hold, a right \( B \)-linear splitting of the canonical monomorphism \( B \to A \) is given by
\[
\tilde{\phi} = \mu_B \circ (f \otimes B) \circ \tilde{\sigma}_T,
\]
where \( \tilde{\sigma}_T \) is a right \( B \)-linear section of the multiplication map \( A \otimes B \to A \), the existence of which follows by Theorem 3.7 (4), and \( f \) is a right \( T \)-module section of the canonical inclusion. \( \Box \)

In case \( A \) is a projective left (resp. right) \( T \)-module, the \((C,T)\)-projectivity of \( A \) as a left (resp. right) \( B \)-module implies its projectivity. This leads to the following

**Corollary 4.5.** Let \((A,C,\psi)_R\) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension given by a grouplike element \( e \in C \). Let \( T \) be a \( k \)-subalgebra of \( B \). Suppose that

1. there exists a strong \( T \)-connection in \((A,C,\psi)_R\);
2. \( A \) is a projective left (resp. right) \( T \)-module;
3. \( B \) is a flat right (resp. left) \( T \)-module;
4. \( B \) is a direct summand of \( A \) as a left (resp. right) \( T \)-module.

Then \( A \) is faithfully flat as a left (resp. right) \( B \)-module.

**Proof.** By Corollary 3.8 \( A \) is a \( T \)-relative projective left (resp. right) \( B \)-module, hence it is projective as left (resp. right) \( B \)-module by assumption (b). By Proposition 4.4, \( B \) is a direct summand of \( A \) as a left (resp. right) \( B \)-module. Then \( A \) is a generator in \( BM \) (resp. \( M_B \)), which implies the claim. \( \Box \)

**Corollary 4.6.** Let \((A,C,\psi)_R\) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension given by a grouplike element \( e \in C \). Let \( T \) be a \( k \)-subalgebra of \( B \). Suppose that

1. there exists a strong \( T \)-connection in \((A,C,\psi)_R\);
2. \( A \) is a flat left (resp. right) \( T \)-module;
3. \( B \) is a flat right (resp. left) \( T \)-module;
4. \( B \) is a direct summand of \( A \) as a left (resp. right) \( T \)-module.

Then \( A \) is \( R \)-relative injective as a left (resp. right) \( C \)-comodule.

**Proof.** By Proposition 4.4, \( B \) is a direct summand of \( A \) as a left (resp. right) \( B \)-module and by Corollary 3.8, \((A \otimes_C C)\psi\) is a Galois \( A \)-coring. Hence the assertion follows by Proposition 4.1. \( \Box \)

5. **The relative Chern-Galois character. Associated modules**

The aim of this section is, given an entwined extension \( B \subseteq A \) with a strong \( T \)-connection, to associate an Abelian group map from the Grothendieck group of a coring to the even \( T \)-relative cyclic homology of \( B \). A family of such maps is called a relative Chern-Galois character.
The Grothendieck group $K_0(C)$ of an $R$-coring $C$ is defined as the Abelian group of equivalence classes of left (or right) $C$-comodules that are finitely generated and projective as $R$-modules. The addition is induced from direct sum of such comodules.

The following lemma shows that – just as one can associate an idempotent matrix of elements of $B$ to a finitely generated and projective module of an algebra $B$ – finitely generated and projective comodules of a coring $C$ can be characterised in terms of a ‘co-idempotent’ matrix of elements of $C$.

**Lemma 5.1.** Let $W$ be a left comodule of an $R$-coring $C$ and suppose that $W$ is a finitely generated and projective $R$-module. Let $w = \{w_i \in W, \chi_i \in \mathbb{K}^W\}_{i \in I}$ be a finite dual basis of $W$. Define

$$e_{ij} = (C \otimes \chi_j)\left(W \varrho(w_i)\right) = \sum_{j \in I} w_{i(-1)} \chi_j(w_{i(0)}), \quad i, j \in I.$$  

Then, for all $i, j \in I$,

1. $W \varrho(w_i) = \sum_{j \in I} e_{ij} \otimes w_j$;
2. $e_{ij} = \sum_{k \in I} \chi_k(w_i)e_{kj} = \sum_{k \in I} e_{ik}\chi_j(w_k)$;
3. $\Delta_C(e_{ij}) = \sum_{k \in I} e_{ik} \otimes e_{kj}$.

**Proof.** (1) By the dual basis property,

$$W \varrho(w_i) = \sum_{j \in I} w_{i(-1)} \otimes w_{i(0)} = \sum_{j \in I} w_{i(-1)} \chi_j(w_{i(0)}) \otimes w_j = \sum_{j \in I} e_{ij} \otimes w_j.$$  

(2) Since the coaction $W \varrho$ is left $R$-linear, we can compute

$$W \varrho(w_i) = \sum_{k \in I} W \varrho(\chi_k(w_i)w_k) = \sum_{k \in I} \chi_k(w_i)w_{k(-1)} \otimes w_{k(0)},$$

where the first equality follows by the dual basis property. Apply $C \otimes \chi_j$ to obtain the first assertion. The equality $e_{ij} = \sum_{k \in I} e_{ik}\chi_j(w_k)$ follows by (1).

(3) The coassociativity of $W \varrho$ and property (1) imply that

$$\sum_{k \in I} \Delta_C(e_{ik}) \otimes w_k = \sum_{k, l} e_{ik} \otimes e_{kl} \otimes w_l.$$  

Apply $C \otimes C \otimes \chi_j$, use the $R$-linearity of $\Delta_C$ and (2) to obtain the assertion. \(\square\)

In fact Lemma 5.1 has a converse, which allows one to reconstruct a left $C$-comodule $W$ that is finitely generated and projective as an $R$-module from a finite matrix $e = (e_{ij})_{i, j \in I}$ of elements in $C$ such that $\Delta_C(e_{ij}) = \sum_{k \in I} e_{ik} \otimes e_{kj}$. Indeed, the counit property implies that the matrix $p = (p_{ij} = \varepsilon_C(e_{ij}))_{i, j \in I}$ is an idempotent matrix, hence $W = R^{(I)} p$ is a finitely generated and projective left $R$-module. The left $C$-coaction $W \varrho : W \rightarrow C \otimes R W$ is then defined by

$$(\sum_{i \in I} r_i p_{ij})_{j \in I} \mapsto \sum_{i, k \in I} r_i e_{ik} \otimes (p_{kj})_{j \in I} \simeq (\sum_{i \in I} r_i e_{ij})_{j \in I}.$$  

Thus, rather than specifying a left $C$-comodule $W$, we can equally well specify a matrix of elements in $C$ that satisfy conditions of Lemma 5.1 (3). Note further that such a matrix also determines a right $C$-comodule that is finitely generated and projective as a right $R$-module. This reflects the duality between finitely generated and projective left and right comodules.
Definition 5.2. Let \((A, C, \psi)_R\) be a bijective right entwining structure over \(R\) and let \(B \subseteq A\) be an entwined extension. Let \(T\) be a \(k\)-subalgebra of \(B\). Define the map
\[
v_T : A/[A, T] \to (A\otimes C)_\psi/[A\otimes C, T],
\]
as the projection of the \(T\)-\(T\)-bimodule map
\[
A \to (A\otimes C)_\psi, \quad a \mapsto \varrho^A(a) - a1_{[0]}\otimes 1_{[1]}.
\]
Note that, for all \(b \in B\), \(v_T([b]) = 0\).

We say that the entwined extension \(B \subseteq A\) is \(T\)-flat if \(B\) and \(A\) are flat as left and right \(T\)-modules and the map
\[
(5.1) \quad B/[B, T] \to \ker v_T, \quad [b] \mapsto [b],
\]
is an isomorphism.

If \(T = k\), then the map \((5.1)\) is always an isomorphism. Hence, in this case, \(B \subseteq A\) is \(T\)-flat, provided \(A\) and \(B\) are flat \(T\)-modules. More generally, the map \((5.1)\) is an isomorphism if the functor
\[
T\text{M}_T \to \text{M}_k, \quad M \mapsto M/[M, T],
\]
preserves monics (e.g. if \(T\) is a separable \(k\)-algebra).

Lemma 5.3. Let \((A, C, \psi)_R\) be a bijective right entwining structure over \(R\) and let \(B \subseteq A\) be an entwined extension. Let \(T\) be a \(k\)-subalgebra of \(B\) such that \(A\) is a flat left and right \(T\)-module. For any strong \(T\)-connection \(\ell_T : C \to A \otimes_T A\) in \((A, C, \psi)_R\), write \(\ell_T(c) = \sum c^{(1)} \otimes c^{(2)}\), for all \(c \in C\). Then
\[
\sum \ell_T(c^{(1)})\ell_T(c^{(2)}) = \sum c^{(1)} \otimes c^{(1)} c^{(2)} \otimes c^{(2)} \in A \otimes_T B \otimes_T A.
\]

Proof. Since \(T\) is a \(k\)-subalgebra of \(B\) and \(B\) is a \(k\)-subalgebra of \(A\), the coaction \(\varrho^A : A \to (A\otimes C)_\psi\) being left \(B\)-linear is left \(T\)-linear and being right \(A\)-linear is right \(T\)-linear. Compute, for all \(c \in C\),
\[
(A \otimes g^A \otimes A)(\sum \ell_T(c^{(1)})\ell_T(c^{(2)})) = \sum c^{(1)} \otimes c^{(1)} [0] \psi(c^{(1)}(2)_1 \otimes c^{(2)}(1)) \otimes c^{(2)}(2) = \sum c^{(1)} \otimes c^{(1)}(2) \psi(c^{(2)}(1)) c^{(3)(1)} \otimes c^{(3)(2)} = \sum c^{(1)} \otimes c^{(1)}(2) \psi(c^{(2)}(1) (-1) \otimes c^{(2)}(1)(0)) \otimes c^{(2)}(2) = \sum c^{(1)} \otimes c^{(1)}(2) c^{(2)}(1) A[0] \otimes 1_{A[1]} \otimes c^{(2)(2)},
\]
where the first equality follows by the fact that \(A\) is a right entwined module, the second and the third equalities follow by the \(C\)-colinearity of \(\ell_T\), and the final equality is a consequence of the definition \((2.5)\) of the left \(C\)-coaction on \(A\). Since \(A\) is flat as a left and as a right \(T\)-module, the assertion follows. \(\square\)

Theorem 5.4. Let \((A, C, \psi)_R\) be a bijective right entwining structure over \(R\) and let \(B \subseteq A\) be a \(T\)-flat entwined extension. Suppose that there exists a strong \(T\)-connection \(\ell_T : C \to A \otimes_T A\) in \((A, C, \psi)_R\), and write \(\ell_T(c) = c^{(1)} \otimes c^{(2)}\). For any finite matrix
\( \boldsymbol{e} = (e_{ij})_{i,j \in I} \) of elements of \( C \) such that \( \Delta_C(e_{ij}) = \sum_{k \in I} e_{ik} \otimes e_{kj} \) and for any non-negative integer \( n \), define the following element of the circular tensor product

\[
\tilde{\text{chg}}_n^T(\boldsymbol{e}) = \sum_{i_1, \ldots, i_{n+1} \in I} e_{i_1 i_2} (2) \ell_T(e_{i_2 i_3}) \ell_T(e_{i_3 i_4}) \cdots \ell_T(e_{i_n i_{n+1}}) \ell_T(e_{i_{n+1} i_1}) e_{i_1 i_2} (1)
\]

Then there is a family of maps of Abelian groups

\[
\text{chg}_n^T : K_0(C) \to HC_{2n}(B[T]), \quad [W] \mapsto \left[ \bigoplus_{l=0}^{2n} (-1)^{[l/2]} \frac{l!}{[l/2]!} \tilde{\text{chg}}_l(\boldsymbol{e}) \right],
\]

where \( \boldsymbol{e} \) corresponds to a left \( C \)-comodule \( W \) as in Lemma 5.1 and \( [x] \) is the integer value of \( x \). The family \( \text{chg}_n^T \) is termed a relative Chern-Galois character of the \( T \)-flat \( (A, C, \psi)_R \)-entwined extension \( B \subseteq A \) with a strong \( T \)-connection \( \ell_T \).

**Proof.** Lemma 5.3 together with the assumption that \( B \subseteq A \) is a \( T \)-flat extension implies that, for all \( n \in \mathbb{N} \), \( \text{chg}_n(\boldsymbol{e}) \in B^{\otimes_T(n+1)} \). Repeating the same steps as in the proof of Lemma 5.3 one finds that \( u^T([\sum_{i \in I} e_{ii} (2) e_{ii} (1)]) = 0 \). Since \( B \subseteq A \) is a \( T \)-flat extension, this implies that \( \text{chg}_0(\boldsymbol{e}) \in B/[B, T] \).

As \( \ell_T \) is a strong \( T \)-connection, \( \sum c^{(1)} c^{(2)} = \eta_A(\varepsilon_C(c)) \). Note further that the \( \text{chg}_n(\boldsymbol{e}) \) are invariant under the cyclic operator (modulo sign). These two facts allow one to derive the following relations (cf. Section 2.3 for the definitions of the maps involved):

\[
N_n(\text{chg}_n(\boldsymbol{e})) = (n+1)\text{chg}_n(\boldsymbol{e}), \quad \partial_n(\text{chg}_n(\boldsymbol{e})) = \text{chg}_{n-1}(\boldsymbol{e})
\]

if \( n \) is even, and

\[
\partial'_n(\text{chg}_n(\boldsymbol{e})) = \text{chg}_{n-1}(\boldsymbol{e}), \quad \tilde{\tau}_n(\text{chg}_n(\boldsymbol{e})) = 2\text{chg}_n(\boldsymbol{e})
\]

if \( n \) is odd (and 0 otherwise). These relations imply in turn that, for any non-negative integer \( n \), \( \bigoplus_{l=0}^{2n} (-1)^{[l/2]} \frac{l!}{[l/2]!} \text{chg}_l(\boldsymbol{e}) \) is a cycle in the \( T \)-relative cyclic complex of \( B \).

Next we prove that the \( \text{chg}_n(\boldsymbol{e}) \) do not depend on the choice of a dual basis. In addition to a basis \( \mathbf{w} = \{ w_i \in W, \chi_i \in \ast W \}_{i \in I} \) that leads to the matrix \( \mathbf{e} \), take a different dual basis \( \tilde{\mathbf{w}} = \{ \tilde{w}_k \in W, \tilde{\chi}_k \in \ast W \}_{k \in \tilde{I}} \), and let \( \tilde{\mathbf{e}} = (\tilde{e}_{kl})_{k,l \in \tilde{I}} \) be as in Lemma 5.1 corresponding to basis \( \tilde{\mathbf{w}} \). Then, for all \( i \in I \),

\[
\sum_{\tilde{v} \in \tilde{I}} e_{i\tilde{v}} \otimes w_{\tilde{v}} = Wg(w_i) = \sum_{k \in \tilde{I}} Wg(\tilde{\chi}_k(w_i) \tilde{w}_k) = \sum_{k,l \in \tilde{I}} \tilde{\chi}_k(w_i) \tilde{e}_{kl} \otimes \tilde{w}_l.
\]

Applying \( C \otimes \chi_j \) and using Lemma 5.1 (2), we obtain, for all \( i, j \in I \),

\[
e_{ij} = \sum_{k,l \in \tilde{I}} \tilde{\chi}_k(w_i) \tilde{e}_{kl} \chi_j(\tilde{w}_l).
\]
With this relation between the $e_{ij}$ and the $\tilde{e}_{kl}$ at hand, and using the fact that a strong $T$-connection is both left and right $T$-linear we can compute, for all $i, j \in I$,
\[
\sum_{p \in I} \ell_T(e_{ip})\ell_T(e_{pj}) = \sum_{p \in I, k, l, m, q \in \overline{I}} \tilde{\chi}_k(w_i)\tilde{\chi}_l(w_j)\chi_m(w_p)\chi_q(w_q) = \sum_{k, l, m, q \in \overline{I}} \tilde{\chi}_k(w_i)\chi_m(w_p)\tilde{\chi}_l(w_j)\chi_q(w_q),
\]
where the second equality follows by the dual basis property. Using the left $R$-linearity of $\ell_T$ and Lemma 5.1 (2) for the $\tilde{e}_{kl}$, this can be rewritten further as
\[
\sum_{p \in I} \ell_T(e_{ip})\ell_T(e_{pj}) = \sum_{k, l, m, q \in \overline{I}} \tilde{\chi}_k(w_i)\ell_T(\tilde{e}_{kl})\ell_T(\tilde{\chi}_m(w_l))\chi_j(w_q) = \sum_{k, l, q \in \overline{I}} \tilde{\chi}_k(w_i)\ell_T(\tilde{e}_{kl})\ell_T(\tilde{\chi}_m(w_l))\chi_j(w_q).
\]
Using this equality sufficiently many times we obtain, for all $n \in \mathbb{N}$,
\[
\widehat{\text{ch}_n}(e) = \sum_{i_1, i_2, \ldots, i_{n+1}, j \in \overline{I}} \tilde{e}_{i_1i_2}(2)\ell_T(\tilde{e}_{i_3i_4})\ell_T(\tilde{e}_{i_5i_6})\cdots\ell_T(\tilde{e}_{i_{n+1}i_1})\chi_i(w_i)\tilde{\chi}_i(w_i)\tilde{e}_{i_{n+2}}(1) = \sum_{i, j \in \overline{I}} \tilde{\chi}_i(w_i)\ell_T(\tilde{e}_{ij})\ell_T(\tilde{\chi}_j(w_j))\chi_j(w_j) = \widehat{\text{ch}_0}(e),
\]
where we used the dual basis property, the right $R$-linearity of $\ell_T$ and by Lemma 5.1 (2). Similarly, for the zeroth coefficient,
\[
\widehat{\text{ch}_0}(e) = \sum_{i, j \in \overline{I}} \tilde{\chi}_i(w_i)\ell_T(\tilde{e}_{ij})\ell_T(\tilde{\chi}_j(w_j)) = \ell_T(\tilde{\chi}_i(w_i)) = \chi_i \circ \widehat{\varphi}^{-1} \in \widehat{W},
\]
where the last equality follows by the $R$-linearity of $\ell_T$ and by Lemma 5.1 (2). Thus we conclude that the $\widehat{\text{ch}_n}(e)$ do not depend on the choice of a dual basis of $W$.

Suppose that there is a left $C$-comodule isomorphism $\varphi : W \rightarrow \widehat{W}$. As the $\widehat{\text{ch}_n}(e)$ do not depend on the choice of a dual basis, we can choose the dual basis $\{\overline{w}_i = \varphi(w_i) \in \widehat{W}, \overline{\chi}_i = \chi_i \circ \varphi^{-1} \in \widehat{W} \}_{i \in I}$ in $\widehat{W}$. Then
\[
\hat{e}_{ij} = \sum_{k} \overline{w}_{i(1)}\overline{\chi}_j(\overline{w}_{i(0)}) = \sum_{k} \varphi(w_i)\overline{\chi}_j(\varphi^{-1}(\varphi(w_i))) = \sum_{k} w_{i(1)}\chi_j(w_{i(0)}) = e_{ij}.
\]
The second equality follows by the left colinearity of $\varphi$. Thus the components $\hat{\text{ch}_n}(\overline{e})$ of the relative Chern-Galois cycle for the matrix $\overline{e}$ corresponding to $\overline{W}$ coincide with $\widehat{\text{ch}_n}(e)$. All this proves that the maps $\text{ch}_T$ are well-defined. To prove that these are group morphisms, note that if $W = W^1 \oplus W^2$, then the matrix $e = (e_{ij})_{i, j \in I}$ is a direct sum $e = e^1 \oplus e^2$ of corresponding matrices for $W^1$ and $W^2$. The explicit form of the relative Chern-Galois cycle then immediately implies that $\text{ch}_T(e) = \text{ch}_T(e^1) + \text{ch}_T(e^2)$, hence $\text{ch}_T([W^1 \oplus W^2]) = \text{ch}_T([W^1]) + \text{ch}_T([W^2])$, as required. $\square$
In case $T = k$, the components of the relative Chern-Galois characters are denoted by $\tilde{\mathrm{ch}}_{n,1}$ and $\tilde{\mathrm{ch}}_{n,2}$. We now describe Chern-Galois characters for examples of entwined extensions constructed in Section 3.

**Example 5.5.** Consider a strong $T$-connection $\ell_T$ in a bijective entwining structure $(A,C,\psi)_B$ described in Example 3.10. Assume that the extension $A^{\otimes C} \subseteq A$ is $T$-flat. Let $e = (e_{ij})_{i,j \in I}$ be a finite matrix of elements of $C$ such that the condition (3) in Lemma 5.1 is satisfied. Write $e_{ij} := \sum_{m \in M_{ij}} a_{ij}^{(m)} \otimes \tilde{a}_{ij}^{(m)}$ and $\sigma_T(a) = \sum a^{(1)}_1 \otimes a^{(2)}_T$ for the components of a left $R$-linear, right $C$-colinear section of the $R$-multiplication map $R \otimes A \to A$ that defines $\ell_T$. Then

$$\tilde{\mathrm{ch}}^T_n(e) = \sum \tilde{a}_{i_1 i_2}^{(m_1)}(2) a_{i_2 i_3}^{(m_2)}(m_2) \eta_A(\tilde{a}_{i_2 i_3}^{(m_2)}(1)) \hat{\otimes}^T \tilde{a}_{i_2 i_3}^{(m_3)}(4) a_{i_3 i_4}^{(m_3)}(3) \hat{\otimes}^T \cdots \hat{\otimes}^T \eta_A(\tilde{a}_{i_2 i_3}^{(m_2)}(1)),$$

where summation is over the indices: $i_1, \ldots, i_{n+1} \in I$, $m_1 \in M_{i_1 i_2}, \ldots, m_{n+1} \in M_{i_{n+1} i_1}$ and over the components of the map $\sigma_T$.

In particular, if $R$ is a separable $k$-algebra and the strong $k$-connection $\ell$ is induced by a separability idempotent $\sum_i \tilde{f}_i \otimes f_i$, then

$$\tilde{\mathrm{ch}}^T_n(e) = \sum \eta_A(f_{i_1}) \tilde{a}_{i_1 i_2}^{(m_1)}(2) a_{i_2 i_3}^{(m_2)}(m_2) \eta_A(e_{i_2}) \hat{\otimes}^T \eta_A(f_{i_1}) \tilde{a}_{i_2 i_3}^{(m_3)}(3) \eta_A(e_{i_3}) \hat{\otimes}^T \cdots \eta_A(f_{i_1}) \tilde{a}_{i_{n+1} i_1}^{(m_{n+1})} a_{i_1 i_2}^{(m_1)} \eta_A(e_{i_1}),$$

where summation is over the indices: $i_1, \ldots, i_{n+1} \in I$, $m_1 \in M_{i_1 i_2}, \ldots, m_{n+1} \in M_{i_{n+1} i_1}$ and $f_{i_1}, \ldots, f_{i_{n+1}} \in L$.

**Example 5.6.** Let $H$ be a Hopf algebroid with a bijective antipode and let $B \subseteq A$ be a left $H$-extension as in Example 3.11. Suppose that there exists a strong $T$-connection $\gamma_T$ in the bijective entwining structure in Example 2.3 (1-2) and assume that the extension $B \subseteq A$ is $T$-flat.

Let $e = (e_{ij})_{i,j \in I}$ be a finite matrix of elements of $H$ such that the condition (3) in Lemma 5.1 is satisfied for the coproduct $\gamma_R$ of the right bialgebroid $H$ in $\mathcal{H}$, and set $c := \sum_{i \in I} e_{ii}$. For a map $f_T$ in Example 3.11 write $f_T(h) = \sum h^{(1)} \otimes h^{(2)}$. Then

$$\tilde{\mathrm{ch}}^L_n(e) = \sum c_{n+3-t}^{(2)} c_{n+4-t}^{(1)} \hat{\otimes}^L \cdots \hat{\otimes}^L c_{n-1}^{(2)} \hat{\otimes}^T c_{n+2-t}^{(1)} \hat{\otimes}^L c_{n+2-t}^{(2)} c_{n+3-t}^{(1)} \hat{\otimes}^T \cdots \hat{\otimes}^T c_{n+2-t}^{(2)} c_{n+3-t}^{(1)},$$

for any $t = 1, \ldots, n+1$, where $\sum c_{1} \otimes \cdots \otimes c_{n+3}$ stands for the action of the coproduct $\gamma_L$ iterated $n+2$ times on $c$.

In the particular example of $\ell_L = (\hat{j} \otimes j) \circ \gamma_L$ we obtain

$$\tilde{\mathrm{ch}}^L_n(e) = \sum 1_A \hat{\otimes} L \cdots 1_A \hat{\otimes} L j(c_{(2)}) \hat{\otimes}^L 1_A \hat{\otimes} L \cdots \hat{\otimes}^L 1_A.$$

In contrast to cleft Hopf algebra extensions, cleft Hopf algebroid extensions provide non-trivial relative Chern-Galois characters.
**Example 5.7.** Let $B \subseteq A$ be a balanced depth 2 split Frobenius extension of $k$-algebras such that the commutant $R$ of $B$ in $A$ is a separable $k$-algebra. Suppose that $A$ is a flat $k$-module. Let $W$ be a left comodule for the right bialgebroid $(A \otimes B)_{B}$ that is finitely generated and projective as a left $R$-module and let $e = (e_{ij})_{i,j \in I}$ be the corresponding matrix of elements of $(A \otimes B)_{B}$ (cf. Lemma 5.1). Take a finite dual basis $w = \{w_{i} \in W, \chi_{i} \in \ast W\}_{i \in I}$ and introduce the element

$$
\sum_{q} x_{q} \otimes x'_{q} = \sum_{i \in I} ((A \otimes B)_{B} \otimes R)^{\chi_{i}} W_{q}(w_{i}) \equiv \sum_{i \in I} e_{ii}
$$

of $(A \otimes B)_{B}$. Fix the data as in (a)-(d) of Example 3.12. By Lemma 5.1 (3) and the form of the coproduct in the bialgebroid $(A \otimes B)_{B}$, the following identity holds in $(A \otimes B)_{B} \otimes_{R} (A \otimes B)_{B}$.

$$
\sum_{j \in J, m \in M_{j}, q} (x_{q} \otimes \gamma_{j}(x'_{q})) \otimes (c_{m} \otimes c_{j} m) = \sum_{j \in J, m \in M_{j}, q} (c_{m} \otimes c_{j} m) \otimes (x_{q} \otimes \gamma_{j}(x'_{q})).
$$

Hence, substituting the strong connection in Example 3.12 in the definition of the $(k$-relative) Chern-Galois cycle of $e$ in Theorem 5.4, its components come out as

$$
\text{chg}_{n}(e) = \sum_{k \subseteq K, i \subseteq L, q} \varphi(v_{k}f_{i}x_{q}) \omega(x'_{q}e_{ij}u_{k}) \text{ and, for } n \in \mathbb{N},
$$

$$
\text{chg}_{n}(e) = \sum_{p=0}^{n-2} (\otimes_{p=1}^{t} \varphi(v_{k_{p}}f_{i_{p}}c_{j_{p} m_{p}}) \omega(\gamma_{j_{p+1}(c_{j_{p} m_{p}}) e_{l_{p+1}} u_{k_{p+1}}})) \otimes \varphi(v_{k_{t-1}}f_{i_{t-1}} c_{j_{t-1} m_{t-1}}) \omega(c_{j_{t-1} m_{t-1}} e_{l_{t}} u_{k_{t}}) \otimes \varphi(v_{k_{t+1}} f_{i_{t+1}} x_{q}) \omega(\gamma_{j_{t+1}}(c_{j_{t+1}}) e_{l_{t+1}} u_{k_{t+1}}) \otimes (\otimes_{p=t+1}^{n+1} \varphi(v_{k_{p}} f_{i_{p}} c_{j_{p-1} m_{p-1}}) \omega(\gamma_{j_{p}}(c_{j_{p-1} m_{p-1}}) e_{l_{p+1}} u_{k_{p+1}}))
$$

for any $t = 1, \ldots, n+1$, where the indices of $k$ and $l$ are understood modulo $n+1$ and the indices of $j$ and $m$ are understood modulo $n$. The summation is over the indices: $k_{1}, \ldots, k_{n+1} \subseteq K, l_{1}, \ldots, l_{n+1} \subseteq L, j_{1}, \ldots, j_{n} \subseteq J, m_{1} \subseteq M_{j_{1}}, \ldots, m_{n} \subseteq M_{j_{n}}$ and $q$.

Noting that $HC_{0}(B) = B/[B, B]$, and using the above explicit expression for the components of the relative Chern-Galois cycle, one immediately finds that the zeroth component of the relative Chern-Galois character has the following simple form

$$
\text{chg}_{0}(e) = \sum_{q} [\varphi(x'_{q} x_{q})] \in B/[B, B].
$$

The relative Chern-Galois character defined in Theorem 5.4 is ‘relative’, because it depends on $T$ and also on the choice of a strong connection. The rest of this section is devoted to finding sufficient conditions for $\text{chg}_{*}^{T}$ to be independent of $\ell_{T}$. The basic idea for this is to associate a finitely generated and projective $B$-module to a given entwined extension and a comodule, and then to reveal a connection between the relative Chern-Galois character and the relative Chern character of $B$.

Note that, given an entwined extension $B \subseteq A$ in a right entwining structure $(A, C, \psi)_{R}$, and a left $C$-comodule $W$, the cotensor product $A \square_{C} W$ is a left $B$-module with the natural product $b \otimes \sum_{i} a_{i} \otimes w_{i} \mapsto \sum_{i} b a_{i} \otimes w_{i}$. We will study when this module is (relatively) projective.

**Theorem 5.8.** Let $(A, C, \psi)_{R}$ be a bijective right entwining structure over $R$ and let $B \subseteq A$ be an entwined extension. Let $T$ be a $k$-subalgebra of $B$. Suppose that $A$ is a projective right $T$-module, a flat left $T$-module and a faithfully flat right $B$-module. If there exists a strong $T$-connection in $(A, C, \psi)_{R}$, then, for any left $C$-comodule $W$
that is finitely generated as a left \( R \)-module, \( \Gamma = A \square_C W \), is a finitely generated and \( T \)-relative projective left \( B \)-module.

**Proof.** First we show that \( \Gamma \) is a finitely generated left \( B \)-module. Since \( A \) is a flat right \( B \)-module, there is an isomorphism of left \( A \)-modules, \( A \otimes (A \square_C W) \cong (A \otimes A) \square_C W \). In view of the flatness of \( A \) as a left or as a right \( T \)-module, the canonical map \( \text{can}_A : A \otimes A \to A \otimes C \) is an isomorphism of right \( C \)-comodules by Corollary 3.8. Since \( - \square_C W : M^C \to M_k \) is a covariant functor (cf. [16, 21.3]), the map \( \text{can}_A \square_C W \) is an isomorphism, so that the above isomorphism can be extended to

\[
A \otimes (A \square_C W) \cong (A \otimes A) \square_C W \cong (A \otimes C) \square_C W.
\]

Finally, note that \( (A \otimes C) \square_C W \cong A \otimes W \) (cf. [16, 21.4-5]), hence there is an isomorphism of left \( A \)-modules

\[
A \otimes \Gamma \cong A \otimes W.
\]

Since \( W \) is a finitely generated left \( R \)-module, \( A \otimes W \) is a finitely generated left \( A \)-module, hence also \( A \otimes \Gamma \) is a finitely generated left \( A \)-module. Since \( A \) is a faithfully flat right \( B \)-module, [2, Ch. 1 § 3 Prop. 11] implies that \( \Gamma \) is a finitely generated left \( B \)-module.

Since \( A \) is a projective right \( T \)-module by assumption and a \( (C, T) \)-projective right \( B \)-module by Theorem 3.7 (4), it follows that \( A \) is a projective right \( B \)-module. As it is also a faithfully flat right \( B \)-module, the right regular \( B \)-module is a direct summand of \( A \) (cf. [31, 2.11.29]). In particular the right \( T \)-module \( B \) is a direct summand of the flat \( T \)-module \( A \), hence also \( B \) is a flat right \( T \)-module.

Since all the assumptions of Theorem 3.7 (3) are satisfied, there exists a left \( B \)-module right \( C \)-comodule section \( \sigma_T : A \to B \otimes_T A \) of the product, and we can define the left \( B \)-module map

\[
\tilde{\sigma} = \sigma_T \square_C W : \Gamma = A \square_C W \to (B \otimes_T A) \square_C W \cong B \otimes_T (A \square_C W) = B \otimes_T \Gamma.
\]

The last isomorphism follows by the flatness of \( B \) as a right \( T \)-module. The map \( \tilde{\sigma} \) clearly is a section of the \( B \)-multiplication in \( \Gamma \) (as \( \sigma_T \) is a section of the multiplication map \( B \otimes_T A \to A \)), hence \( \Gamma \) is a \( T \)-relative projective left \( B \)-module. \( \square \)

**Remark 5.9.** In view of Corollary 3.9 the assumption of Theorem 5.8 that \( A \) is a faithfully flat right \( B \)-module is satisfied provided \( A \) is a faithfully flat right \( T \)-module. Furthermore, if the \( C \)-coaction on \( A \) is given by a grouplike element \( e \in C \), then \( A \) is a faithfully flat right \( B \)-module provided \( B \) is a flat left \( T \)-module and a right \( T \)-module direct summand of \( A \) by Corollary 4.5 (These assumptions hold e.g. in Example 5.7). Obviously, all these assumptions are satisfied in case \( T \) is equal to a ground field \( k \).

In relation to the Chern-Galois character, the case of a comodule \( W \) that is not only finitely generated but also projective as a left \( R \)-module is of particular interest. In this case, the \( B \)-module \( \Gamma \) is projective under weakened assumptions on the \( T \)-module \( A \) and the explicit form of an idempotent can be worked out.

Recall from [30] (cf. [16, 42.10]) that a right \( T \)-module \( M \) is called a **locally projective module** if every finitely generated submodule of \( M \) is projective, i.e. if, for any finitely generated submodule \( X \) of \( M \), there exist elements \( x_1, \ldots, x_n \in M \) and \( \xi_1, \ldots, \xi_n \in M^* \) such that, for all \( x \in X \), \( x = \sum_{i=1}^n x_i \xi_i(x) \). In particular, any locally projective module is a flat module (cf. [16, 42.11]). The following theorem gives an explicit
form of an idempotent for $\Gamma$ and thus asserts that $\Gamma$ is a finitely generated projective $B$-module.

**Theorem 5.10.** Let $(A, C, \psi)_R$ be a bijective right entwining structure over $R$ and let $B \subseteq A$ be an entwined extension. Let $T$ be a $k$-subalgebra of $B$ and suppose that the extension $B \subseteq A$ splits as a $B$-$T$ bimodule. Suppose furthermore that $A$ is a locally projective right $T$-module, a flat left $T$-module and that there exists a strong $T$-connection $\ell_T : C \to A \otimes_R A$ in $(A, C, \psi)_R$. Let $W$ be a left $C$-comodule that is finitely generated and projective as a left $R$-module with a finite dual basis $w = \{w_i \in W, \chi_i \in \text{Hom}_R(W, W)\}_{i \in I}$, and let $e_{ij} \in C$ be as in Lemma 5.7. Let $X$ be a right $T$-submodule of $A$ finitely generated by $e_{ij}^{(1)\nu} , i, j \in I$, where $\ell_T(e_{ij}) = \sum_{\nu=1}^n e_{ij}^{(1)\nu} \otimes e_{ij}^{(2)\nu}$. Choose a finite set $x = \{x_p \in A, \xi_p \in \text{Hom}_T(A, T)\}_{p \in P}$ such that, for all $x \in X$, $x = \sum_{p \in P} x_p \xi_p(x)$, and define the family of right $C$-comodule maps

$$\ell_p = (\xi_p \otimes A) \circ \ell_T : C \to A, \quad p \in P.$$  

Choose a $B$-$T$ bimodule retraction $\phi : A \to B$ of the inclusion $B \subseteq A$ and define the finite matrix of elements in $B$,

$$E = (E(i,p),(j,q))_{(i,p),(j,q) \in I \times P}, \quad E(i,p),(j,q) = \phi(\ell_p(e_{ij})x_q).$$

The matrix $E$ is an idempotent and the left $B$-module $\Gamma = A \square_C W$ is isomorphic to $B^{(I \times P)}E$.

**Remark 5.11.** In view of Corollary 3.9 in the case when $T$ is equal to the ground ring $k$, the assumptions of Theorem 5.10 that $A$ is a locally projective $k$-module and $B$ is a direct summand in $A$ as a left $B$-module are satisfied provided $A$ is a faithfully flat and projective $k$-module. Furthermore, by Proposition 4.4 if the $C$-coaction in $A$ is given by a grouplike element $e \in C$ and $A$ is a locally projective $k$-module, then $B$ is a direct summand in $A$ as a left $B$-module provided $B$ is a $k$-direct summand of $A$. Obviously, all these assumptions are satisfied in case $k$ is a field.

The assumptions of Theorem 5.10 hold in Example 5.7 provided that $A$ is a locally projective $k$-module.

Note also that if, in addition to the assumptions of Theorem 5.10, the map (5.1) is an epimorphism, then $B \subseteq A$ is a $T$-flat extension.

The proof of the theorem uses the following lemma formulated within the notation and assumptions of Theorem 5.10.

**Lemma 5.12.** For all $i \in I$ and $p \in P$, let $\gamma_{ip} = \sum_{j \in I} \ell_p(e_{ij}) \otimes w_j \in A \otimes_R W$. Then

1. $\gamma_{ip} \in \Gamma$;
2. $\sum_{q \in P} E(i,p),(j,q) \gamma_{jq} = \gamma_{ip}$.

**Proof.** (1) This is proven by the following explicit computation

$$\langle g^A \otimes_R W \rangle(\gamma_{ip}) = \sum_{j \in I} \xi_p(e_{ij}^{(1)}) e_{ij}^{(2)}[0] \otimes e_{ij}^{(2)}[1] \otimes w_j$$

$$= \sum_{j, k \in I} \xi_p(e_{ik}^{(1)}) e_{ik}^{(2)} \otimes e_{kj} \otimes w_j$$

$$= \sum_{k \in I} \ell_p(e_{ik}) \otimes \sum_{j \in I} e_{kj} \otimes w_j = (A \otimes_R W) \langle g \rangle(\gamma_{ip}),$$
where the first equality follows by the left $T$-linearity of $\varphi^4$ and the second equality follows by the right $C$-colinearity of the strong $T$-connection $\ell_T$ and by Lemma 5.1 (3).

(2) By the definition of the set $x$, for all $c \in C$,

\[(5.2) \quad \sum_{p \in P} x_p \otimes \ell_T(p)(c) = \ell_T(c).\]

Compute

\[
\sum_{j \in I, q \in P} E_{i,p}(j,q) \gamma_{jq} = \sum_{j,k \in I, q \in P} \phi(\ell_p(e_{ij})x_q)\ell_q(e_{jk}) \otimes w_k
\]

where the second equality follows by the $T-T$ bilinearity of $\phi$ and (5.2). The third equality follows by Lemma 5.1 (3) combined with Lemma 5.3 and the fact that $\phi$ restricted to $B$ is the identity map. The penultimate equality is a direct consequence of the definition of a strong $T$-connection. This concludes the proof that $E$ is an idempotent matrix.

Proof of Theorem 5.10. We first show that $E$ is an idempotent matrix. This is proven by the following direct computation, for all $i, k \in I$ and $p, r \in P$,

\[
\sum_{j \in I, q \in P} E_{i,p}(j,q) E_{j,q}(k,r) = \sum_{j \in I, q \in P} \phi(\ell_p(e_{ij})x_q)\phi(\ell_q(e_{jk})x_r)
\]

where the second equality follows by the $T-T$ bilinearity of $\phi$ and (5.2), the third one is a consequence of the left $B$-linearity of $\phi$. The fourth equality follows by Lemma 5.1 (3) combined with Lemma 5.3 and the fact that $\phi$ restricted to $B$ is the identity map, and the final equality is a consequence of the definition of a strong $T$-connection. This concludes the proof that $E$ is an idempotent matrix.

Consider the left $B$-module map

$$\Theta : B^{(I \times P)}E \rightarrow \Gamma,$$

where the last equality follows by Lemma 5.12 (2). The map $\Theta$ has its range in $\Gamma$ by Lemma 5.12 (1). We first show that $\Theta$ is an injective map. Suppose that
\[ \sum_{i \in I, p \in P} b_{ip} \gamma_{ip} = 0. \] This implies that, for all \( j \in I, \)
\[
0 = \sum_{i, k \in I, p \in P} b_{ip} \ell_p(e_{ik}) \chi_j(w_k) = \sum_{i, k \in I, p \in P} b_{ip} \ell_p(e_{ik} \chi_j(w_k)) = \sum_{i \in I, p \in P} b_{ip} \ell_p(e_{ij}),
\]
where we used that \( \ell_p \) is a right \( C \)-comodule map, hence a right \( R \)-module map, and Lemma 5.1 (2). Thus, in particular, for all \( j \in I \) and \( q \in P, \sum_{i \in I, p \in P} b_{ip} \ell_p(e_{ij}) x_q = 0, \) hence, as \( \phi \) is left \( B \)-linear,
\[
0 = \sum_{i \in I, p \in P} b_{ip} \phi(\ell_p(e_{ij}) x_q) = \sum_{i \in I, p \in P} b_{ip} \gamma_{(i,p)(j,q)}.
\]
Therefore, \( \Theta \) is a left \( B \)-module monomorphism. To prove that \( \Theta \) is an epimorphism we first take any \( \sum_n a_n \otimes v^n \in \Gamma \) and compute
\[
\sum_{i, k \in I, p \in P, n} a_n \chi_i(v^n) x_p \otimes \ell_p(e_{ik}) \otimes w_k = \sum_{i,k \in I,n} a_n \chi_i(v^n) \ell_T(e_{ik}) \otimes w_k \\
= \sum_{i,k \in I,n} a_n \ell_T(\chi_i(v^n) e_{ik}) \otimes w_k = \sum_{i \in I,n} a_n \ell_T((\chi_i(v^n) w_i)(-1)) \otimes (\chi_i(v^n) w_i)_0) \\
= \sum_{n} a_n \ell_T(v^n(-1)) \otimes v^n(0) = \sum_{n} a_n \ell_T(a_n(1)) \otimes v^n,
\]
where the first equality follows by (5.2), the second by the \( R \)-linearity of \( \ell_T \) and the third by Lemma 5.1 (1). Then the dual basis property is used, and finally, the fact that \( \sum_n a_n \otimes v^n \) is in the cotensor product is employed. Recall from the proof of Theorem 3.7 that the map \( a \mapsto \sum a_0 \ell_T(a_1) \in B \otimes A \rightarrow A \), so that, applying \((\mu_A \otimes W) \circ (\phi \otimes A \otimes W)\) to the equality just derived, we obtain
\[
\sum_{n} a_n \otimes v^n = \sum_{i \in I, p \in P, n} \phi(a_n \chi_i(v^n) x_p) \gamma_{ip}.
\]
This shows that \( \sum_n a_n \otimes v^n \in \text{Im} \Theta \), hence \( \Theta \) is an epimorphism. Therefore, \( \Theta \) is a required isomorphism of left \( B \)-modules. \( \square \)

With every finite idempotent matrix \( F = (f_{ij}) \) with entries from a \( T \)-ring \( B \) one associates a family of elements in the circular tensor product (cf. Section 2.5)
\[
(5.3) \quad \tilde{\chi}_{T_n}(F) = \sum_{i_1, \ldots, i_{n+1}} f_{i_1 i_2} \otimes f_{i_2 i_3} \otimes \cdots \otimes f_{i_{n+1} i_1} \in B \hat{\otimes} (n+1),
\]
\( n \in \mathbb{N} \cup \{0\} \). Since \( F \) is an idempotent and \( \tilde{\chi}_{T_n}(F) \) is invariant under the cyclic operator (modulo sign), one easily finds that \( \Theta_{2n} = (-1)^{[l/2]} \frac{n}{[l/2]!} \tilde{\chi}_{T_l}(F) \) is a \( 2n \)-cycle in the total relative cycle homology complex (cf. the proof that \( \text{ch}_{\hat{\otimes} 2n}(e) \) is an even cycle). The corresponding family of homology classes
\[
(5.4) \quad \text{ch}_{2n}^T(F) := \left[ \bigoplus_{l=0}^{2n} (-1)^{[l/2]} \frac{l!}{[l/2]!} \tilde{\chi}_{T_l}(F) \right] \in HC_{2n}(B|T)
\]
is called a \( T \)-relative Chern character of \( B \). Each of the elements \( \tilde{\chi}_{T_n}(F) \) defined by equation (5.3) is called the \( n \)-th component of the \( T \)-relative Chern cycle associated to \( F \). Note that the relative Chern character \( \text{ch}_{2n}^T(F) \) is related to the Chern character
\( \text{ch}_{2n}(F) \) (cf. [29] Section 8.3), by \( \text{ch}^T_{2n}(F) = \lambda \circ \text{ch}_{2n}(F) \), where \( \lambda : \text{HC}_*(B) \to \text{HC}_*(B/T) \) is the canonical surjection. Hence the relative Chern character defines a family of Abelian group morphisms \( K_0(B) \to \text{HC}_{2n}(B/T) \).

**Lemma 5.13.** With the notation and assumptions of Theorem 5.10, the components of the \( T \)-relative Chern cycle associated to \( E \) come out as \( \tilde{\text{ch}}^T_n(E) = [\sum_{i \in I} \phi(e_{ii}^{(2)}e_{ii}^{(1)})] \) for all \( n \in \mathbb{N} \),

\[
\tilde{\text{ch}}^T_n(E) = \sum_{i_1, \ldots, i_{n+1} \in I} e_{i_1i_2}^{(2)}e_{i_2i_3}^{(2)}e_{i_3i_4}^{(1)} \cdots e_{i_{n+1}i_1}^{(1)}.
\]

**Proof.** The explicit form of \( \tilde{\text{ch}}^T_n(E) \) is computed directly from the definition of \( E \) and from equation (5.3). For the zeroth component,

\[
\tilde{\text{ch}}^T_0(E) = [\sum_{i \in I, p \in P} \phi(\ell(p)(e_{ii})x_p)] = [\sum_{i \in I, p \in P} \phi(e_{ii}^{(2)}x_p\ell(p)(e_{ii}^{(1)}))] = [\sum_{i \in I} \phi(e_{ii}^{(2)}e_{ii}^{(1)})],
\]

where the second equality follows by the definition of \( A/[A,T] \) and the \( T \)-bilinearity of \( \phi \). With the help of (5.3), Lemma 5.1 (3) and Lemma 5.3, and also using the fact that \( \phi \) is a \( T-T \) bimodule map and, if restricted to \( B \), it is the identity map, we can compute, for all \( n > 0 \),

\[
\tilde{\text{ch}}^T_n(E) = \sum_{i_1, \ldots, i_{n+1} \in I, p_1, \ldots, p_{n+1} \in P} E_{(i_1,p_1)}(i_2,p_2) \cdots \hat{\otimes}_T E_{(i_{n+1},p_{n+1})}(i_{n+1},p_{n+1}) \hat{\otimes}_T E_{(i_1,p_1)}(i_2,p_2) \cdots \hat{\otimes}_T E_{(i_{n+1},p_{n+1})}(i_{n+1},p_{n+1})
\]

\[
= \sum_{i_1, \ldots, i_{n+1} \in I, p_1, \ldots, p_{n+1} \in P} \phi(\ell(p_1(e_{i_1i_2})x_{p_2}) \cdots \hat{\otimes}_T \phi(\ell(p_n(e_{i_{n+1}}x_{p_{n+1}}))x_{p_{n+1}}) \hat{\otimes}_T \phi(\ell(p_{n+1}(e_{i_{n+1}i_1})x_{p_1})x_{p_1})
\]

\[
= \sum_{i_1, \ldots, i_{n+1} \in I} e_{i_1i_2}^{(2)}e_{i_2i_3}^{(1)} \cdots e_{i_{n+1}i_1}^{(1)}.
\]

This completes the proof. \( \square \)

If the map (5.1) is an isomorphism, then the components of the Chern character associated to \( E \) are equal to the components of the Chern-Galois character. This observation leads to the following

**Theorem 5.14.** Let \( (A,C,\psi)_R \) be a bijective right entwining structure over \( R \) and let \( B \subseteq A \) be an entwined extension with a strong \( T \)-connection, where \( T \) is a \( k \)-subalgebra of \( B \). Suppose that the extension \( B \subseteq A \) splits as a \( B-T \) bimodule, that \( A \) is a locally projective right \( T \)-module, a flat left \( T \)-module and that the map (5.1) is an epimorphism. Then the relative Chern-Galois character \( \text{ch}_{2n}^T : K_0(C) \to \text{HC}_{2n}(B/T) \) does not depend on the choice of a strong connection.

**Proof.** If \( \Gamma = A \otimes_C W \), isomorphic \( C \)-comodules \( W \) lead to isomorphic \( B \)-modules \( \Gamma \). As cotensor product respects direct sums, the assignment \( W \mapsto \Gamma \) descends to an
Abelian map of Grothendieck groups. Thus, in view of Lemma 5.13, any component of the relative Chern-Galois character can be understood as a composition
\[
\begin{array}{ccc}
\text{chg}_{2n}^T : & K_0(C) & \longrightarrow K_0(B) \xrightarrow{\text{ch}_{2n}} HC_{2n}(B) \xrightarrow{\lambda_*} HC_{2n}(B|T).
\end{array}
\]
The first map does not depend on the choice of \(\ell_T\) as the definition of \(\Gamma\) is independent of the choice of a strong connection. The second map is independent of \(\ell_T\), since the Chern character is independent of the choice of an idempotent by \([29,\ \text{Theorem 8.3.4}]\). The canonical epimorphism \(\lambda_* : HC_*(B) \to HC_*(B|T)\) (cf. Section 2.3) is obviously independent of \(\ell_T\). \(\square\)

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