Noise-synchronizability of opinion dynamics

Wei Su * Ge Chen† Yongguang Yu ‡ Xueqiao Wang §

Abstract

With the analysis of noise-induced synchronization of opinion dynamics with bounded confidence (BC), a natural and fundamental question is what opinion structures can be synchronized by noise. In the traditional Hegselmann-Krause (HK) model, each agent examines the opinion values of all the other ones and then choose neighbors to update its own opinion according to the BC scheme. In reality, people are more likely to interchange opinions with only some individuals, resulting in a predetermined local discourse relationship as introduced by the DeGroot model. In this paper, we consider an opinion dynamics that combines the schemes of BC and local discourse topology and investigate its synchronization induced by noise. The new model endows the heterogeneous HK model with a time-varying discourse topology. With the proposed definition of noise-synchronizability, it is shown that the compound noisy model is almost surely noise-synchronizable if and only if the time-varying discourse graph is uniformly jointly connected, taking the noise-induced synchronization of the classical heterogeneous HK model as a special case. As a natural implication, the result for the first time builds the equivalence between the connectivity of discourse graph and the beneficial effect of noise for opinion consensus.

Keywords: Noise-synchronizability, bounded confidence, discourse topology, opinion dynamics

1 Introduction

Recently analysis on the beneficial role noise plays in opinion consensus is growing. It was first found by some pioneering simulation studies that noise could enhance the consensus of opinion dynamics in some situations [1, 6]. Very recently, a theoretical analysis of noise-induced consensus was well established for the widely known Hegselmann-Krause (HK) opinion dynamics [7]. It is proved that for any initial conditions the homogeneous HK model will almost surely achieve an approximate consensus (quasi-consensus) in the presence of noise and a critical noise strength is given. Later, using a HK-type truth seeking model, noise is proved to be able to drive the whole group to achieve the truth [8]; in [8], a noise-based control strategy which needs intervene only one agent is shown to eliminate the disagreement of a divisive HK system in finite time.

*School of Automation and Electrical Engineering, University of Science and Technology Beijing & Key Laboratory of Knowledge Automation for Industrial Processes, Ministry of Education, Beijing 100083, China, suwei@amss.ac.cn
†National Center for Mathematics and Interdisciplinary Sciences & Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China, chenge@amss.ac.cn
‡School of Science, Beijing Jiaotong University, Beijing 100044, China, ygyu@bjtu.edu.cn
§Beijing key laboratory of information service engineering, Beijing Union University, Beijing 100101, China, ldxueqiao@buu.edu.cn
These analytical findings are all based on the HK dynamics, whose structure arises from the typical bounded confidence (BC) mechanism of opinion dynamics. Then a natural and fundamental question is with what structure an opinion dynamics can be synchronized by noise. Solution of this problem will benefit much in exploring noise-based control theory of opinion dynamics. There are mainly two classes of opinion models: one is BC-based model such as Deffuant-Weisbuch model [10] and Hegselmann-Krause model [11]; the other is topology-based model, such as DeGroot model [12] and Friedkin-Johnsen model [13]. The BC mechanism says that each agent has a bounded confidence and only takes opinions within its confidence into account for updating; the topology-based mechanism preassigns an interaction relationship of opinions (no matter time-varying or time invariant). While the theoretical analysis shows that the BC structure can be synchronized by noise, we still need to investigate whether the noise-induced synchronization can occur with topology-based mechanism. The theory of robust consensus of noisy multi-agent systems implies that only the topology-based structure does not have the property of noise-induced synchronization, i.e., a system of non-consensus cannot be synchronized by noise. So we need to investigate what happens when introducing the topology-based mechanism into the BC mechanism. From another side, study of the models that combine BC and topology-based mechanism possesses more practical significance for opinion dynamics. In the HK model, each agent examines all the other agents and then chooses the neighbors to update its own opinion; in reality however, one is more likely to exchange opinions with only some individuals from time to time, resulting in a time-varying local discourse topology behind the BC mechanism.

In this paper, we study a noisy opinion model that endows the heterogeneous HK model with a time-varying local discourse topology. We introduce the concept of noise-synchronizability for the compound model, that is, from any initial state the noise with proper strength could drive the model to reach synchronization almost surely. The main result of this paper is proving that the compound model is noise-synchronizable if and only if the time-varying discourse graph is uniformly jointly connected. The concept of “joint connectivity” is core in the consensus theory of time-varying multi-agent systems [14] where it is revealed that the consensus of agents is determined by the joint connectivity of the time-varying graph. However in the analysis of noise-induced consensus of the traditional HK model in [7], no connectivity of evolution graph is needed, seeming that noise-synchronizability is independent of connectivity. The conclusion in this paper shows that, though the connectivity of evolution graph is not needed, some another implicit connectivity, i.e., discourse graph, is still necessary. In the original HK model, the discourse graph is a complete graph, and hence connected. This paper for the first time establishes the equivalence between noise-synchronizability and connectivity of discourse graph. The conclusion takes the noise-induced consensus of the heterogeneous HK model as a special case, and implies that it is the BC and uniformly jointly connected discourse graph together that account for the noise-synchronizability of opinion dynamics.

The rest of this paper is organized as follows: Section 2 introduces the compound model; Section 3 gives the main result about the sufficient and necessary condition of noise-synchronizability; Section 4 presents some simulation studies to illustrate the theoretical conclusion and Section 5 concludes the paper.

2 Model

Compared to the original HK model where each agent can talk with any other one, here we assume in addition a time-varying graph of discourse for the agents. Let \( \mathcal{V} = \{1, 2, \ldots, n\} \) be the set of \( n \) agents, \( x_i(t) \in [0, 1], i \in \mathcal{V}, t \geq 0 \) be the opinion value of agent \( i \) at \( t \) where 0 and 1
are the boundary opinions, and $\epsilon_i \in (0, 1]$ be the heterogeneous confidence threshold of $i$. Denote $G(t) = \{V, E(t)\}$ as the undirected graph of discourse at $t$, where the node set $V$ is the agents and the edge set $E(t) = \{(i, j) : \text{nodes } i \text{ and } j \text{ are linked at } t\}$ is the time-varying discourse relationship between agents. A graph $G(t)$ is called a connected graph if for any $i \neq j$, there are edges $(i, i_1), (i_1, i_2), \ldots, (i_k, j)$ in $E(t)$; and $G(t)$ is called a complete graph if $(i, j) \in E(t)$ for any $i, j \in V$.

The proposed compound model with noise has dynamics

$$x_i(t + 1) = \begin{cases} 
1, & x^*_i(t) > 1 \\
x^*_i(t), & x^*_i(t) \in [0, 1], \forall i \in V, t \geq 0, \\
0, & x^*_i(t) < 0 
\end{cases} \tag{2.1}$$

where

$$x^*_i(t) = |\mathcal{N}(i, t)|^{-1} \sum_{j \in \mathcal{N}(i, t)} x_j(t) + \xi_i(t + 1) \tag{2.2}$$

and

$$\mathcal{N}(i, t) = \{j \in V \mid |x_j(t) - x_i(t)| \leq \epsilon_i, (i, j) \in E(t)\}. \tag{2.3}$$

Here, $\mathcal{N}(i, t)$ is the neighbor set of agent $i$ at $t$ and determined by the bounded confidence and the discourse graph together. $\xi(t), i \in V, t > 0$ are the independent and identically distributed (i.i.d.) random noise with $E\xi_1(1) = 0, E\xi_2^2(1) > 0$ and $|\xi_i(t)| \leq \delta$ a.s. where $\delta > 0$ can be viewed as noise strength. If $G(t)$ is always a complete graph, the system \((2.1)-(2.3)\) degenerates to the original heterogeneous HK model; while if $\epsilon_i \equiv 1$, the system \((2.1)-(2.3)\) becomes a DeGroot model.

For the deterministic case of model \((2.1)-(2.3)\) ($\delta = 0$), its convergence has not been solved (even the convergence of heterogeneous HK model is still an open problem \([13, 16]\)). Simulations and case studies can easily show that the deterministic model is asymptotically convergent to a state of consensus or more often disagreement (see Fig. 1). If introducing noise into the model, it can be found that sometime the disagreement disappears but in some cases the disagreement keeps all the time. In the following, we will investigate when the system can be synchronized by noise.

### 3 Main results

In this section, we will prove the main result about the noise-induced synchronization of model \((2.1)-(2.3)\). In the following, we denote $x(t) = [x_1(t), \ldots, x_n(t)]$ and $\xi(t) = [\xi_1(t), \ldots, \xi_n(t)]$. To begin with, some definitions are needed.

**Definition 3.1.** A graph sequence $\{G(t), t \geq 0\}$ is said to be uniformly jointly connected if there is a positive number $q \geq 0$ such that the union of $G(t), \ldots, G(t + q)$ is connected for all $t \geq 0$.

**Definition 3.2.** Denote $\xi = \min_i \epsilon_i$, then the system \((2.1)-(2.3)\) is said to be noise-synchronizable if for all $x(0) \in [0, 1]^n$ and $\xi \in (0, 1)$ there is a $\delta > 0$ such that $\lim_{t \to \infty} d_Y(t) \leq \xi$.

**Definition 3.3.** is modified from the quasi-consensus ($\lim_{t \to \infty} d_Y(t) \leq \xi$) in \([7]\), which implies that the opinions of all agents are close enough that they are neighbors to each other once they have a talk and hence share a same updated opinion without noise. In this definition, we only consider the case of $\xi < 1$, since the definition is trivial when $\xi = 1$. With the definitions, we have
Theorem 3.3. The system (2.1)–(2.3) is almost surely noise-synchronizable if and only if \( \{ \mathcal{G}(t), t \geq 0 \} \) is uniformly jointly connected.

**Remark 3.4.** In the consensus theory of traditional multi-agent systems, joint connectivity of the interaction graph is an essential condition for the consensus or robust consensus of the system. However, the joint connectivity of the discourse graph of model (2.1)–(2.3) does not imply the joint connectivity of interaction graph of opinions, since even when two agents can talk with each other, their opinions will not interact if the opinion difference exceeds their confidence thresholds.

To prove Theorem 3.3, some lemmas are needed:

**Lemma 3.5.** \([17]\) Let \( A = (a_{ij}) \in R^{n \times n} \) be a stochastic matrix and denote \( d(y) = \max_{i,j} |y_i - y_j| \) for a vector \( y \in R^n \), then \( d(Ay) \leq \tau(A)d(y) \) where \( \tau(A) = \frac{1}{2} \max_{i,j} \sum_{k=1}^{n} |a_{ik} - a_{jk}| \).

The next lemma is one of the key lemmas in proving the issues of noise-induced order, which roughly states that when a random walk has a uniform positive probability to enter a region within a finite time, then it will almost surely enter the region.

**Lemma 3.6.** Let \( \{ x_t, t \geq 1 \} \) be a random walk on \( R^n \), and \( \{ T_i : \Omega \to \mathbb{N}^+, i \geq 1 \} \) be a sequence of random variables. For \( D \subset R^n \) denote \( T = \inf_{t \geq 1} \{ t : x_t \in D \} \) and \( \bar{D} = R^n - D \). If for any \( T_i, i \geq 1 \) there is a constant \( 0 \leq p \leq 1 \) such that \( P \{ x_{T_i+1} \in D | \bigcap_{k \leq i} \{ x_{T_k} \in \bar{D} \} \} \geq p \), then \( P \{ T < \infty \} = 1 \).

**Proof:** It can be checked that
\[
P(T = \infty) = P \left\{ \bigcap_{t=1}^{\infty} \{ x_t \in \bar{D} \} \right\} \\
\leq P \left\{ \bigcap_{i=1}^{\infty} \{ x_{T_i} \in \bar{D} \} \right\} = \lim_{m \to \infty} P \left\{ \bigcap_{i=1}^{m} \{ x_{T_i} \in \bar{D} \} \right\} \\
= \lim_{m \to \infty} \prod_{i=1}^{m-1} P \left\{ x_{T_{i+1}} \in \bar{D} \bigg| \bigcap_{k \leq i} \{ x_{T_k} \in \bar{D} \} \right\} P \{ x_{T_1} \in \bar{D} \} \\
< \lim_{m \to \infty} (1 - p)^m = 0,
\]
implying \( P \{ T < \infty \} = 1 \). \(\square\)

**Lemma 3.7.** Suppose the discourse graph \( \{ \mathcal{G}(t), t \geq 0 \} \) of the system (2.1)–(2.3) is jointly connected. If there exists a constant \( \bar{d} \in [\xi/2, \xi] \) and a finite time \( 0 \leq T < \infty \) such that \( d_{\mathcal{G}}(T) \leq \bar{d} \), then there exists a constant \( \delta > 0 \) such that on \( \{ T < \infty \} \), we have \( d_{\mathcal{G}}(t) \leq \xi \) for all \( t > T \) and \( 0 < \delta \leq \bar{d} \).

**Proof:** For simplicity of description, suppose \( T = 0 \) a.s. Denote \( D = \{ y \in R^n : d(y) \leq \xi \} \), then \( x(0) \in D \). Let \( A(t) \) be the adjacency matrix of graph \( \mathcal{G}(t) \) with \( x(t) \in D \) and denote \( \Phi(t, t+h) = A(t) \cdots A(t+h), t \geq 0, h \geq 1 \). \( A(t) \) is determined by the discourse topology and the BC mechanism together. With \( A(t) \), model (2.2) can be rewritten as \( x^*(t) = A(t)x(t) + \xi(t+1) \).
When \( x(t) \in D \), all agents are within the confidence region of each other, and the neighbors are totally determined by the undirected discourse relationship. It can be checked that \( A(t) \) satisfies the following properties:
(I) $A(t)$ is a stochastic matrix, whose diagonal entries is positive with a lower bound $\frac{1}{n}$.

(II) $A(t)$ can be presented as $A(t) = I - D^{-1}(t)L(t)$, where $D(t)$ is a diagonal matrix with all positive diagonal entries, $L(t)$ is a symmetric matrix. Moreover, the operator norms of $D^{-1}(t)$ and $L(t)$ have an upper bound uniformly in $t$;

(III) The non-zero entries of $A(t)$ have a uniform low bound, i.e. $\min_{A_{i,j}(t)>0} A_{i,j} \geq \beta > 0, t \geq 0$.

Since $\{G(t), t \geq 0\}$ is uniformly jointly connected, there exists a constant $q \geq 0$ such that the unions of $G(t), \ldots, G(t + q)$ for all $t \geq 0$ are connected and hence $\Phi(t, t + q)$ is ergodic \cite{14, 18} for $t \geq 0$. Notice $A(t)$ satisfies (I)-(III), then by the proof of (18) in \cite{19}, there is constants $l \geq f(n)q$ when $f(n)$ is a constant determined by $n$ and $0 < \alpha < 1$ such that

$$\tau(\Phi(t, t + l)) \leq \alpha$$

(3.1) for all $t \geq 0$ (see Appendix for a detailed proof). Since $\Phi(t, t + l)$ is a stochastic matrix, by Lemma 3.3, it has

$$d(\Phi(0,l)(x(0))) \leq ad(x(0)) = ad.$$  

(3.2)

Let $\bar{d} = \min\left\{\frac{d}{a}, \frac{(1-a)d}{2}\right\}$ and for given $x(0)$ denote $A'(t)$ as the adjacency matrix of $G(t)$ for $t \geq 0$. Notice that $A'(t), t \geq 0$ are stochastic matrices, then by (2.1) and (2.2), it has

$$d_{\nu}(t) \leq d + 2\delta t \leq \xi, \quad 0 \leq t \leq l.$$  

(3.3)

This implies $A(t) = A'(t)$ for $0 \leq t \leq l$. Then by (2.1), (2.3), it has

$$x(l) = A(l)x(l - 1) + \xi(l)$$

$$= \Phi(0,l)x(0) + \Phi(1,l)\xi(1) + \ldots + \Phi(l-1,l)\xi(l-1) + \xi(l).$$  

(3.4)

By (3.2) and (3.4), it has

$$d_{\nu}(l) = d(x(l)) \leq \alpha d + ld(\xi(1)) \leq \alpha d + 2l\bar{d} \leq \bar{d}.$$  

(3.5)

Noticing (3.3) and (3.5) and repeating the above process, we obtain the conclusion. \hfill $\square$

**Sufficiency of Theorem 3.3:** If $G(t)$ is uniformly jointly connected, by Lemma 3.7, we only need to prove there exists a finite time $T$ such that $d_{\nu}(T) \leq \frac{\xi}{2}$ for any initial state $x(0) \in R^n$ and $\epsilon_j \in (0, 1]$ where $\bar{d}$ is given in the proof of Lemma 3.7. Since $E\xi_1(1) = 0, E\xi_1^2(1) > 0$ and $\xi_i(t) \leq 0$ are connected, there exist constants $0 < a \leq \bar{d}, 0 < p < 1$ such that

$$P\{\xi_1(1) > a\} \geq p, \quad P\{\xi_1(1) < -a\} \geq p.$$  

(3.6)

If $d_{\nu}(0) \leq \frac{\xi}{2}$, by Lemma 3.7, the conclusion holds. Otherwise, suppose $d_{\nu}(0) > \frac{\xi}{2}$ and consider the following noise protocol: for all $i \in V, t \geq 0$,

$$\begin{align*}
\xi_i(t + 1) & \in [a, \delta], \text{if } \min_{j \in V} x_j(t) \leq \bar{x}_i(t) \leq \min_{j \in V} x_j(t) + \frac{d_{\nu}(t)}{2}; \\
\xi_i(t + 1) & \in [-a, -\bar{a}], \text{if } \min_{j \in V} x_j(t) + \frac{d_{\nu}(t)}{2} < \bar{x}_i(t) \leq \max_{j \in V} x_j(t),
\end{align*}$$  

(3.7)

where $\bar{x}_i(t) = |N(i, x(t))|^{-1} \sum_{j \in N(i, x(t))} x_j(t)$. Since $A(t)$ is a stochastic matrix, it is easy to check that for $t \geq 0$,

$$\min_{i \in V} x_i(t) \leq \min_{i \in V} \bar{x}_i(t) \leq \max_{i \in V} x_i(t),$$

$$\min_{i \in V} x_i(t) \leq \max_{i \in V} \bar{x}_i(t) \leq \max_{i \in V} x_i(t).$$  

(3.8)
Then under the protocol (3.7), by (3.8), it has
\[ d_{\mathcal{V}}(t) \leq (\max_{i \in \mathcal{V}} \bar{x}_i(t) - a) - (\min_{i \in \mathcal{V}} \bar{x}_i(t) + a) \]
\[ \leq \max_{i \in \mathcal{V}} x_i(t) - \min_{i \in \mathcal{V}} x_i(t) - 2a \]
\[ = d_{\mathcal{V}}(0) - 2a. \] (3.9)

Let \( L = \left\lceil \frac{d_{\mathcal{V}}(0) - \epsilon/2}{2a} \right\rceil \), then by the independence of \( \{\xi_i(t), i \in \mathcal{V}, t \geq 1\} \) and (3.6), it has
\[ P\{d_{\mathcal{V}}(L) \leq \frac{\epsilon}{2}\} \geq P\{\text{protocol (3.7) occurs } L \text{ times}\} \geq p^{nL} > 0. \] (3.10)

Then
\[ P\{d_{\mathcal{V}}(L) > \frac{\epsilon}{2}\} < 1 - p^{nL} < 1. \] (3.11)

Denote events for \( m \geq 0 \)
\[ E_0 = \Omega, \]
\[ E_m = \{\omega : d_{\mathcal{V}}(t) > \frac{\epsilon}{2}, (m - 1)L < t \leq mL\}. \] (3.12)

Since \( x(0) \) is arbitrarily given, by (3.11), we can get for \( m \geq 1 \) that
\[ P\left\{ E_m \left| \bigcap_{j<m} E_j \right. \right\} \leq P\left\{ d_{\mathcal{V}}(mL) > \frac{\epsilon}{2} \left| \bigcap_{j<m} E_j \right. \right\} \leq 1 - p^{nL} < 1. \] (3.13)

Denote \( D = \{y \in \mathbb{R}^n : d(y) \leq \frac{\epsilon}{2}\}, T_k = kL \), then by (3.11) and Lemmas 3.6 and 3.7 the conclusion holds. \( \square \)

Sufficiency of Theorem 3.3 implies the quasi-consensus of the heterogeneous HK model. Actually, let \( \mathcal{G}(t) \) be a constant complete graph, which is uniformly jointly connected, and hence
\[ \lim_{t \to \infty} d_{\mathcal{V}}(t) \leq \frac{\epsilon}{2}. \]

Necessity of Theorem 3.3: We only need to prove that when \( \mathcal{G}(t) \) is not uniformly jointly connected, then for any \( \delta > 0 \), it has \( \limsup_{t \to \infty} d_{\mathcal{V}}(t) = 1 \) a.s., i.e.
\[ P\left\{ \bigcup_{T=0}^{\infty} \{d_{\mathcal{V}}(t) < 1, t > T\} \right\} = 0. \]

If \( \mathcal{G}(t) \) is not uniformly jointly connected, then for any \( T \geq 0 \) and \( l > 0 \), there exists a moment \( t_l \geq T \) such that the union of \( \mathcal{G}(t_l + 1), \ldots, \mathcal{V}(t_l + l) \) is not connected. Without loss of generality, suppose \( \mathcal{G}(t_l + 1), \ldots, \mathcal{G}(t_l + l) \) consist of two disjointly connected components, i.e., for \( t_l + 1 \leq t \leq t_l + l, \mathcal{G}_1(t) = \{V_1, \mathcal{E}_1(t)\} \) and \( \mathcal{G}_2(t) = \{V_2, \mathcal{E}_2(t)\} \) where \( V_1 \cup V_2 = \mathcal{V}, V_1 \cap V_2 = \emptyset \). Take \( l > \frac{1}{a} \) where \( a \) is given in (3.6), then consider the following noise protocol: for \( t_l + 1 \leq t \leq t_l + l, \)
\[
\begin{cases}
  \xi_i(t + 1) \in [a, \delta], & \text{for } i \in V_1; \\
  \xi_i(t + 1) \in [-\delta, -a], & \text{for } i \in V_2.
\end{cases}
\] (3.14)

By (3.8), it has under the protocol (3.14) that
\[
\begin{align*}
  \min_{i \in V_1} x_i(t + 1) & \geq \min_{i \in V_1} x_i(t) + a, \\
  \max_{i \in V_2} x_i(t + 1) & \leq \max_{i \in V_2} x_i(t) - a.
\end{align*}
\] (3.15)
Denote $L = \lceil \frac{1}{a} \rceil$, then by (3.9), it has under the protocol (3.14) 
\[
\min_{i \in V_1} x_i(t_L + L) = 1, \quad \max_{i \in V_2} x_i(t_L + L) = 0,
\]
implying $d_V(t_L + L) = 1$. By (3.6) and the independence of $\{\xi_i(t), i \in V, t \geq 0\}$, it has 
\[
P\{d_V(t_L + L) = 1\} \geq P\{\text{protocol (3.14) occurs } L \text{ times}\} \geq p^n L, \tag{3.16}
\]
then by (3.16), 
\[
P\{d_V(t_L + L) < 1\} < 1 - p^n L < 1. \tag{3.17}
\]
Since $x(0)$ is arbitrarily given, repeating the above procedure obtaining (3.17), it has for $m \geq 1$ and $l > mL$ 
\[
P\{d_V(t_L + mL) < 1\} < 1 - p^n L < 1. \tag{3.18}
\]
Then by (3.18) and the independence of $\xi_i(t), i \in V, t \geq 1$, 
\[
P\left\{ \bigcap_{t=t_1+1}^{t_1+l} \{d_V(t) < 1\} \right\} \leq P\left\{ \bigcap_{m=1}^{\lceil \frac{l}{L} \rceil} \bigcap_{k<m} d_V(t_L + kL) < 1 \right\} 
= \prod_{m=2}^{\lceil \frac{l}{L} \rceil} P\{d_V(t_L + mL) < 1 \mid \bigcap_{k<m} d_V(t_L + kL) < 1 \} \cdot P\{d_V(t_L + L) < 1\} 
= \prod_{m=2}^{\lceil \frac{l}{L} \rceil} P\{d_V(t_L + mL) < 1\} \cdot P\{d_V(t_L + L) < 1\} 
< (1 - p^n L)^{\lceil \frac{l}{L} \rceil - 1}, \tag{3.19}
\]
subsequently by (3.19), it has 
\[
P\{d_V(t) < 1, t > T\} = P\left\{ \bigcap_{t=T+1}^{\infty} \{d_V(t) < 1\} \right\} 
\leq \lim_{l \to \infty} P\left\{ \bigcap_{t=t_1+1}^{t_1+l} \{d_V(t) < 1\} \right\} 
\leq \lim_{l \to \infty} (1 - p^n L)^{\lceil \frac{l}{L} \rceil - 1} = 0.
\]
This completes the proof. \qed

4 Simulations

In this section, we will present some simulations to illustrate the conclusions. We first consider $n = 40$ agents, whose initial opinion values and heterogeneous confidence thresholds are randomly generated on $[0, 1]$. For simplicity, we take a constant discourse topology which is connected, i.e., divide $V$ into four subgroups $V_i, i = 1, 2, 3, 4$, and in each subgroup there are 10 agents who can talk with each other. For each subgroup $V_i, i = 1, 2, 3$, there is an agent who
Figure 1: Opinion evolution of a noise-free system with 40 agents whose discourse topology is constant and connected. The initial opinion value $x(0)$ and the heterogeneous confidence thresholds $\epsilon_i$ are generated by a uniform distribution on $[0, 1]$. It is shown that the system converges to a divisive state.
can talk with an agent in the next subgroup. Under this discourse topology, it is shown in Fig. 1 that the system converges to a divisive state without noise. Then we introduce a noise into the system, and Fig. 2 shows that the system is finally synchronized.

Next we show that when the discourse graph is not connected, the system cannot keep synchronization with noise. For a better illustration, we only need to show that a system with an identical initial state will get separated with noise when the discourse topology is not connected. Consider $n = 20$ agents whose initial states are 0.5, and confidence thresholds are randomly generated on $[0, 1]$. The group is divided equally into two subgroups. All the agents in one subgroup can talk with each other, but can not talk with agents in the other subgroup. Then Fig. 3 shows that the system gets separated at some moment.

5 Conclusions

In this paper, we investigated the noise-synchronizability of opinion dynamics that combines the BC scheme and local discourse topology. A compound model was proposed by endowing the heterogeneous HK model with a time-varying discourse topology. It is proved that the compound model is noise-synchronizable if and only if the time-varying discourse graph is uniformly jointly connected. The result takes the noise-induced consensus of the heterogeneous HK model as a special case. The result for the first time builds the equivalence between the noise-synchronizability and the joint connectivity of discourse graph of opinion dynamics. Moreover, the investigation of the opinion structures with noise-synchronizability will pave the way for exploring the noise-based control theory of opinion dynamics.
Figure 3: Opinion evolution of system of 20 agents whose discourse topology is not connected. The initial opinion values $x(0)$ are 0.5 and the confidence thresholds $\epsilon_i$ are randomly generated on $[0, 1]$. It is shown that the system gets separated at some moment with noise.

Appendix

Proof of (3.1): For a stochastic matrix $A$, denote $\lambda(A) = \min_{i,j} \sum_{k=1}^{n} \min\{A_{ik}, A_{jk}\}$, then it has $\tau(A) = 1 - \lambda(A)$ by [17]. Since $\Phi(t, t + q)$ is ergodic for all $t \geq 0$, by [18], it has a constant $l \geq f(n)q$ where $f(n)$ is a constant determined only by $n$ such that $\Phi(t, t + l)$ is scrambling matrix, which implies $\lambda(\Phi(t, t + l)) \geq 0$. Since all non-zero entries of $A(t)$ is no less than $\frac{1}{n}$, it can be calculated that all non-zero entries of $\Phi(t, t + l)$ are no less than $\frac{1}{nl}$, hence $\lambda(\Phi(t, t + l)) \geq \frac{1}{nl}$. In consequence, $\tau(\Phi(t, t + l)) \leq 1 - \frac{1}{nl} = \alpha < 1$.

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