Coupling coefficient distribution in the doorway mechanism

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Abstract. In many-body and other systems, the physics situation often allows one to interpret certain, distinct states by means of a simple picture. In this interpretation, the distinct states are not eigenstates of the full Hamiltonian. Hence, there is an interaction that makes the distinct states act as doorways into background states which are modeled statistically. The crucial quantities are the overlaps between the eigenstates of the full Hamiltonian and the doorway states, that is, the coupling coefficients occurring in the expansion of true eigenstates in the simple model basis. Recently, the distribution of the maximum coupling coefficients was introduced as a new, highly sensitive statistical observable. In the particularly important regime of weak interactions, this distribution is very well approximated by the fidelity distribution, defined as the distribution of the overlap between the doorway states with interaction and without interaction. Using a random matrix model, we calculate the latter distribution exactly for regular and chaotic background states in the cases of preserved and fully broken time-reversal invariance. We also perform numerical simulations and find excellent agreement with our analytical results.

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1. Introduction

In open quantum systems, strength function phenomena [1] give structural information about the system itself and about the excitation mechanism. Here, we address statistical features of the doorway mechanism, which can be defined as follows: there are one or several somehow ‘distinct’ and ‘simple’ excitations whose amplitudes are spread over many ‘complicated’ states. In a many-body system, collective excitations are often distinct, because all, or large groups of, particles move in a coherent fashion. As compared to the complexity of the other, non-collective excitations, these states can be interpreted in the framework of a simple, typically semiclassical, picture. The distinct states act as ‘doorways’ to the background of the complicated states [1, 2]. Mostly, the statistical features of the latter are chaotic. The strength function has a Breit–Wigner shape, largely independent of the statistics of the background states. The width characterizing the Breit–Wigner strength function is referred to as spreading width [1].

The doorway mechanism is found in a rich variety of systems, comprising atoms and molecules [3], as well as atomic clusters, quantum dots and, more generally, mesoscopic systems [4]–[6]. Nuclear physics provides particularly beautiful and well-studied examples, such as Isobaric Analog States and multipole Giant Resonances [1], [7]–[10].

What is a suitable theoretical interpretation of the Breit–Wigner shape? Although the simple picture for the distinct excitations captures the main physics, it is important to realize that these states are not eigenstates of the real quantum Hamiltonian. Similarly, the statistical models for the background states do not describe eigenstates either. Thus, if we use the simple picture for the distinct states and the statistical model for the background states as a basis of the Hilbert space, there must be a non-vanishing interaction between these two classes of states. Rediagonalization then yields proper eigenstates of the model Hamiltonian. Averaging over
the background states, one obtains the local density of states around the energy of the distinct
state or states in the simple picture. The local density of states is once more of Lorentzian or
Breit–Wigner shape with a spreading width that is—depending on the particular situation—
closely related to or identical to the above-mentioned spreading width in the strength function.
It can be viewed as a measure for the quality of the simple picture describing the distinct states:
the smaller the spreading width, the closer is this picture to the physics reality.

The strength of the interaction between the two classes of states uniquely determines
the spreading width and, equivalently, the size of the overlap between the distinct state in
the simple picture and the true eigenstates of the model Hamiltonian. These overlaps are, of
course, the coupling coefficients when expanding the true eigenstates in the above-mentioned
basis. Recently, a new statistical observable was introduced: the distribution of the maximum
coupling coefficients [11]. The first two moments of this distribution had already been studied
in [12], but with assumptions not valid in our context. In [11], however, the full distribution is
addressed. Importantly, its shape sensitively depends on the interaction strength. Moreover, it is
an especially well-tailored measure to investigate weak interactions.

Here, we present exact results for the distribution of the coupling coefficients to a distinct
state in the framework of a random matrix model. In the particularly interesting regime of
weak interactions, this distribution coincides with the distribution of the maximum coupling
coefficients.

The present paper is organized as follows. After properly posing the problem in section 2,
we calculate the distribution exactly for regular and chaotic backgrounds, respectively, in
sections 3 and 4. We discuss our results in section 5.

2. Posing the problem

In section 2.1, we present the random matrix model for the doorway mechanism. We introduce
and define the distribution of the maximum coupling coefficient in section 2.2. In section 2.3,
we introduce and motivate an important approximation we made. Some further manipulations
useful for the following are described in section 2.4.

2.1. Doorway mechanism in a random matrix model

The model to be discussed here stems from nuclear physics [1] and is also often used in other
fields [13]. For the convenience of the reader and to define our notation, we compile its salient
features. As we are aiming at a random matrix model, it is convenient to choose from the
beginning a proper basis of the full Hilbert space such that we can represent the Hilbert space
operators by matrices. Introducing a cutoff, their dimension is finite. Eventually this cutoff effect
is removed by taking the matrix dimension to be infinity. We nevertheless use the Dirac notation
for the wave functions, even though they are finite-dimensional vectors.

The total Hamiltonian \( H \) consists of three parts: the Hamiltonian \( H_s \) for the \( K \) distinct
states that become the doorway states, the Hamiltonian \( H_b \) describing the \( N \) background states,
where \( N \) will eventually be taken to be infinity, and the interaction \( V \) coupling the two classes
of states. Hence, we have

\[
H = H_s + H_b + V = \sum_{j=1}^{K} E_{s_j} |s_j\rangle \langle s_j| + \sum_{\nu=1}^{N} E_{b_{\nu}} |b_{\nu}\rangle \langle b_{\nu}| + \sum_{j=1}^{K} \sum_{\nu=1}^{N} (V_{j\nu} |s_j\rangle \langle b_{\nu}| + h.c.). \quad (1)
\]
For the matrix elements of the interaction, we make the assumptions
\[ \langle s_j | V | s_k \rangle = \langle b_\nu | V | b_\mu \rangle = 0 \]
and \[ \langle b_\nu | V | s_j \rangle = V_{j\nu} \]
for any \( j, k, \mu, \nu \). Often, there is only one relevant doorway state, or the spacing between the doorway states is much larger than their spreading widths. We focus on these cases and consider only one doorway state by setting \( K = 1, |s_1 \rangle = |s \rangle \) and \( V_{j\nu} = V_{j} \).

The eigenvalues for the uncoupled Hamiltonians are
\[ H_s |s \rangle = E_s |s \rangle \]
and \[ H_b |b_\nu \rangle = E_\nu |b_\nu \rangle \].

Due to the interaction \( V \), the doorway state is not an eigenstate of the Hamiltonian \( H \). We denote the eigenstates of the full Hamiltonian by \( |n \rangle \). The eigenequation to be solved is
\[ H |n \rangle = E_n |n \rangle \].

Resembling the situation in most systems, we put the doorway state \( |s \rangle \) in the center of the background spectrum. It interacts with the surrounding \( N \) states. Without loss of generality, we may set \( E_s = 0 \).

The exact eigenstate of \( H \) that evolves from the doorway state in the presence of the interaction is referred to as \( |0 \rangle \). We expand the \( n \)th eigenstate of \( H \) in the basis spanned by \( |b_\nu \rangle \) and \( |s \rangle \) as
\[ |n \rangle = c_{ns}|s \rangle + \sum_{\nu=1}^{N} c_{n\nu}|b_\nu \rangle \]
where the coupling coefficient \( c_{ns} \) is the overlap between the doorway state \( |s \rangle \) in the non-exact picture for this distinct state and the \( n \)th exact eigenstate \( |n \rangle \) of the full Hamiltonian. We are interested in the statistical features of these coupling coefficients.

We have to solve our model for \( c_{ns} \). The action of the full Hamiltonian \( H \) on the eigenstate \( |n \rangle \) yields, on the one hand,
\[ H |n \rangle = \left( E_s c_{ns} + \sum_{\nu=1}^{N} V_{s\nu}^* c_{n\nu} \right) |s \rangle + \sum_{\nu=1}^{N} \left( c_{ns} V_{\nu} + E_\nu c_{n\nu} \right) |b_\nu \rangle \].

On the other hand, we have
\[ H |n \rangle = E_n c_{ns}|s \rangle + E_n \sum_{\nu=1}^{N} c_{n\nu}|b_\nu \rangle \].

Equating these two expressions, we find
\[ c_{n\nu} = \frac{V_{\nu}}{E_n - E_\nu} c_{ns} \]
such that
\[ |n \rangle = c_{ns} \left( |s \rangle + \sum_{\nu=1}^{N} \frac{V_{\nu}}{E_n - E_\nu} |b_\nu \rangle \right) \].

Using the normalization of \( |n \rangle \), we eventually arrive at
\[ |c_{ns}|^2 = \left( 1 + \sum_{\nu=1}^{N} \frac{|V_{\nu}|^2}{(E_n - E_\nu)^2} \right)^{-1}, \quad 0 \leq n \leq N, \]
which is the desired expression for \( c_{ns} \) in terms of the matrix elements of \( H \).

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2.2. Distribution of the maximum coupling coefficient

The new statistical observable introduced in [11] is the distribution of the maximum
\[ c_{\text{max}} = \max(|c_n|), \quad 0 \leq n \leq N \]  
(10)
of the overlaps between the eigenstates of the full Hamiltonian and the distinct state, that is, the doorway state \(|s\rangle\). In order to obtain it, we have to average in a suitable way over the interaction matrix elements and over the Hamiltonian modeling the background states. Hence, the distribution in question is given by
\[ p_{\text{max}}(c) = \langle \delta(c - c_{\text{max}}) \rangle. \]  
(11)
On the other hand, the distribution of overlap between the evolved doorway state and the unperturbed doorway state reads
\[ p_0(c) = \langle \delta(c - |c_0|) \rangle. \]  
(12)
The angular brackets denote an average over the interaction matrix elements and over the background Hamiltonian \(H_b\). If not stated otherwise, we assume that the interaction matrix elements are Gaussian distributed random variables. We have to distinguish two cases. The total Hamiltonian \(H\) can be time-reversal non-invariant or time-reversal invariant, where we disregard spin degrees of freedom. In the first case, labeled by the Dyson index \(\beta = 2\), the interaction matrix elements \(V_\nu\) are complex variables; in the second case, labeled \(\beta = 1\), they are real. Introducing the \(N\)-component vector \(V\), the corresponding distribution is
\[ P_i(V) = \left(\frac{\beta}{2\pi \nu^2}\right)^{\beta N/2} \exp\left(-\frac{\beta}{2\nu^2} V^\dagger V\right). \]  
(13)
The distribution \(P_i(V)\) is chosen such that \(\lambda\) is independent of \(\beta\). In addition to the behavior under time-reversal invariance, the statistical properties of the Hamiltonian \(H_b\), i.e. the presence or absence of eigenvalue correlations, are expected to affect the distribution \(p_0(c)\). Hence, we do not specify it yet.

As is well known from random matrix theory, the parameter governing the physics is
\[ \lambda = \frac{\sqrt{\langle V^\dagger V \rangle}}{\sqrt{ND}} = \frac{\nu}{D}, \]  
(14)
where \(D\) is the mean level spacing of the background states at the center of the band [1, 13].

2.3. Weak coupling approximation

The calculation of \(p_{\text{max}}(c)\) as defined in equation (11) is hampered by two obstacles. Firstly, the maximum function is very difficult to handle analytically. Secondly, formula (9), while being exact, is still a complicated implicit expression, since the eigenvalues of the full Hamiltonian \(E_n\) depend on the coupling coefficients \(V_\nu\).

We expect that for small coupling strength in most realizations, the fragmentation of the doorway state is weak, i.e. with high probability \(|c_0| \lesssim 1\) and \(|c_n| \gtrsim 0\) for \(n \neq 0\). Therefore, apart from very rare events, the overlap of the evolved doorway state with the unperturbed doorway state is identical with the maximal overlap. In this limit,
\[ p_{\text{max}}(c) \approx p_0(c) \]  
(15)
is expected.
To make this reasoning more quantitative and to overcome the second obstacle mentioned above, we proceed further by expanding the exact eigenvalues perturbatively in $V$,

$$E_n = E_{\nu(n)} + \frac{|V_{\nu(n)}|^2}{E_{\nu(n)}} - \sum_{\mu=1}^{N} \frac{|V_{\nu(n)}|^2|V_{\mu}|^2}{E_{\nu(n)}E_{\mu}} + \cdots \quad (1 \leq n \leq N),$$

$$E_0 = E_s - \sum_{v=1}^{N} \frac{|V_{v}|^2}{E_{v}} + \sum_{\mu=1}^{N} \sum_{v=1}^{N} \frac{|V_{v}|^2|V_{\mu}|^2}{E_{v}^2E_{\mu}},$$

where the eigenstate $|n\rangle$ of the full Hamiltonian to eigenvalue $E_n$ has evolved from the eigenstate $|b_{\nu(n)}\rangle$ of the unperturbed Hamiltonian by adiabatically switching on the perturbation. Plugging the expansion (16) into the exact expression (9), we see that, to zeroth order in $\lambda$, the sum in equation (9) is completely dominated by the term $\nu = \nu(n)$, which actually diverges such that $|c_{ns}|^2 \approx 0$ in zeroth order for all $n \neq 0$. The only overlap integral that remains finite is the overlap of the doorway state with itself $|c_0s|^2$. Here, we set $E_0 \approx E_s = 0$ and no divergence occurs.

In order to estimate the range of validity of the zeroth-order approximation, we expand $|c_{ns}|^2$ to second order in $\lambda$. We obtain

$$|c_{ns}|^2 \lesssim \frac{|V_{\nu(n)}|^2}{E_{\nu(n)}^2}, \quad 1 \leq n \leq N.$$

If we choose as $E_{\nu(n)}$ the unperturbed eigenvalue closest to $E_s = 0$ with a distance $D$, we might further estimate $|c_{ns}|^2 \lesssim \lambda^2$. We conclude that the zeroth-order approximation defined by

$$E_n \approx E_{\nu(n)}$$

is justified for $\lambda \ll 1$. In the following, we stick to this approximation, i.e. we keep only the leading order term in the perturbative expansion of equations (16) and (9). In this approximation we have essentially singled out $|c_{0b}|^2$ as the only non-vanishing—and thus inevitably maximum—overlap integral of the perturbed eigenstates with the doorway state. Therefore equation (15) is automatically implied.

The weak coupling approximation is certainly good for small interactions. Our numerical simulations will strongly corroborate this statement. Hence, we focus on the calculation of $p_0(c)$ in the form

$$p_0(c) = \left( \delta \left( c - \frac{1}{\sqrt{1 + \sum_{v=1}^{N} \frac{|V_{v}|^2}{E_{v}^2}}} \right) \right),$$

which can be treated analytically. In the sequel, we calculate the expressions (19) for generic choices of the Hamiltonian $H_b$ governing the dynamics of the background states. No further approximations will be made and the obtained results will be exact in the large $N$ limit.

2.4. Some further manipulations

Technically, it is more convenient to work out the probability density $Q(u)$ of the random variable

$$u = \frac{1}{c^2} = 1 + \sum_{v=1}^{N} \frac{|V_{v}|^2}{E_{v}^2},$$

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The relation between the two distributions reads
\[ p_0(c) = Q(u) \left| \frac{du}{dc} \right|_{u=1/c^2} = \frac{2}{c^3} Q(u) \left|_{u=1/c^2} \right. \tag{21} \]

Thus, once \( Q(u) \) is known, \( p_0(c) \) follows immediately.

We now use our statistical assumption that the interaction matrix elements \( V_\nu \) are Gaussian distributed. We write the distribution \( Q(u) \) in the form
\[ Q(u) = \int d[V] P_i(V) \left\{ \delta \left( u - 1 - \sum_{\nu=1}^{N} \frac{|V_\nu|^2}{E_\nu^2} \right) \right\}_N, \tag{22} \]

where \( d[V] \) is the product of the differentials of all independent variables in \( V \). The angular brackets with index \( N \) denote an average over the \( N \) background states, that is, over the Hamiltonian \( H_b \). For the calculation of the averages, it is helpful to write the distribution \( Q(u) \) as the Fourier transform
\[ Q(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk \exp(ik(u-1)) R(k) \tag{23} \]

with the characteristic function
\[ R(k) = \int d[V] P_i(V) \left\{ \exp \left( -ik \sum_{\nu=1}^{N} \frac{|V_\nu|^2}{E_\nu^2} \right) \right\}_N. \tag{24} \]

An alternative expression can be obtained by the rescaling \( V_\nu = y_\nu |E_\nu| \), where the vector \( y \) has real entries for \( \beta = 1 \) and complex ones for \( \beta = 2 \), respectively. This implies the change \( d[V] = \prod_{\nu} |E_\nu|^{\beta} |y| \) in the volume element, where \( d[y] \) is a product of the differentials of all independent entries of the vector \( y \). We obtain
\[ R(k) = \int d[y] \left\{ \prod_{\nu} |E_\nu|^{\beta} \exp \left( -ik \sum_{\nu=1}^{N} |y|^2 \right) P_i(Ey) \right\}_N \]
\[ = \sqrt{\frac{\beta}{2\pi v^2}} \int d[y] \exp(-ik y^\dagger y) \left| \det H_b \right|^{\beta} \exp \left( -\frac{\beta}{2v^2} y^\dagger H_b^2 y \right) \right\}_N. \tag{25} \]

In the first equation the expression \( Ey \) denotes the multiplication of the vector \( y \) with the diagonal matrix \( E = \text{diag} (E_1, \ldots, E_N) \). As a generating function, \( R \) is normalized to \( R(0) = 1 \).

### 3. Regular background

The doorway state is embedded into a regular background, if the eigenvalues \( E_\nu \) of \( H_b \) do not repel each other. The distribution of the background Hamiltonian then factorizes according to
\[ P_b(H_b) = \prod_{\nu=1}^{N} p_b(E_\nu). \tag{26} \]

In order to keep the discussion most general, we use for the interaction matrix elements a general factorizing distribution,
\[ P_i(V) = \prod_{\nu=1}^{N} p_i(V_\nu). \tag{27} \]
instead of the Gaussian distribution introduced before equation (13). We keep the assumption of statistical independence of the interaction matrix elements but relax the global orthogonal ($\beta = 1$) or unitary ($\beta = 2$) invariance of the interaction matrix elements, implicit in the measure defined in equation (13). For complex coupling matrix elements we assume in addition that the distribution $p_i$ is $U(1)$ invariant $p_i(V) = p_i(|V|)$. We assign to complex coupling matrix elements with this invariance the Dyson index $\beta = 2$ and to real coupling matrix elements the Dyson index $\beta = 1$.

A straightforward calculation reveals that the characteristic function (24) factorizes as well and becomes an $N^\text{th}$ power of a single integral,

$$R(k) = e^{N \ln r(k)},$$

$$r(k) = \int d^\beta[z] \ p_i(z) \ \int_{-\infty}^{+\infty} dE \ p_b(E) \ \exp\left(-\frac{ik|z|^2}{E^2}\right).$$

As we are interested in the local scale set by the mean level spacing $D$ of the background states, the distribution $p_b(c)$ should not be sensitive to the particular choice of the distribution $p_i$, as long as it does not contain scales competing with the mean level spacing $D$. The simplest choice is

$$p_b(E) = \frac{1}{\sqrt{N}} \begin{cases} 1, & |E| \leq \sqrt{N}/2, \\ 0, & |E| > \sqrt{N}/2, \end{cases}$$

where $D = 1/\sqrt{N}$ and $\sqrt{N} N D$ is the length of the background spectrum. The following calculation is similar to the one described in appendix B of [14]. We perform the integral over the background distribution in equation (29)

$$\int_{-\infty}^{+\infty} dE \ p_b(E) \ \exp\left(-\frac{ik|z|^2}{E^2}\right) = \frac{2}{\sqrt{N}} \int_{2/\sqrt{N}}^{\infty} du \ \frac{\exp(-ik|z|^2u^2)}{u^2}$$

$$= 1 - 2|z|\sqrt{\frac{i\pi k}{N}}.$$ (31)

In the second equation, we used an integral identity of the Fresnel type and the series expansion of the error function

$$\lim_{N \to \infty} \frac{id}{d\alpha} \int_{2/\sqrt{N}}^{\infty} du \ \frac{\exp(-iau^2)}{u^2} = \sqrt{\frac{\pi}{4ia}}.$$ (32)

We find for the characteristic function

$$R(k) = \exp(-2\sqrt{i\pi kNm_1}),$$

$$m_1 = \int d^\beta[z] \ p_i(z)|z|.$$ (34)

We observe that the distribution of the interaction matrix elements $p_i$ enters only via the expectation value $m_1$ as defined in equation (34) and not via the second moment $m_2 = v^2$. As pointed out after equation (14), $p_i$ is chosen such that the mean coupling strength, as defined through $v$, is independent of $\beta$. This means that $m_1$ in general is different for real and complex couplings. We write $m_1 = a_\beta v$, where now $a_\beta$ depends on the Dyson index and on the distribution $p_i$. For instance, for the Gaussian distribution we find

$$a_\beta^{(G)} = \begin{cases} \sqrt{\frac{2}{\pi}} \approx 0.80, & \beta = 1, \\ \sqrt{\frac{\pi}{4}} \approx 0.89, & \beta = 2. \end{cases}$$ (35)
Using the definition of $\lambda$ in equation (14), we finally find

$$R(k) = \exp(-2a_\beta \lambda \sqrt{\pi k}).$$

(36)

The reader can easily convince herself/himself that other reasonable choices for $p_b(E)$, such as a Gaussian distribution, yield the same functional form as in equation (36). The Fourier transform (23) results in

$$Q(u) = \frac{a_\beta \lambda}{(u - 1)^{3/2}} \exp\left(-\frac{(a_\beta \lambda)^2 \pi}{u - 1}\right).$$

(37)

Using the relation (21), we eventually arrive at

$$p_0(c) = \frac{2a_\beta \lambda}{(1 - c^2)^{3/2}} \exp\left(-\frac{(a_\beta \lambda)^2 \pi c^2}{1 - c^2}\right).$$

(38)

As anticipated, the interaction strength enters the distribution only via the dimensionless ratio $\lambda = v/D$. The distribution $p_i$ enters via the factor $a_\beta$ defined in equation (33). It is interesting to see that $a_\beta$ depends not only on the distribution but also on the symmetry factor $\beta$. This means that, for a regular background and for constant interaction strength $\lambda$, the distribution $p_0$ distinguishes between real interaction and complex interaction.

In figure 1, $p_0$ is plotted for three different values of the mean coupling strength $\lambda = 0.1$ (blue), 0.5 (green) and 2 (red). The difference between real and complex couplings increases for strong coupling $\lambda$.

Figure 1. Plot of the distribution function $p_0(c)$ for a regular background for real interaction matrix elements (dotted) and for complex matrix elements (full line) for three different values of the mean coupling strength $\lambda = 0.1$ (blue), 0.5 (green) and 2 (red). The difference between real and complex couplings increases for strong coupling $\lambda$. 

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4. Chaotic background

The dynamics of the background states is usually chaotic. The $N \times N$ Hamilton matrix $H_b$ modeling the background states has then to be chosen from a Gaussian random matrix ensemble. Again, we have to distinguish time-reversal invariant and non-invariant systems, that is, the cases labeled by the Dyson parameters $\beta = 1$ and $\beta = 2$, respectively. The Hamiltonian $H_b$ is from the Gaussian orthogonal ensemble (GOE) for $\beta = 1$ and from the Gaussian unitary ensemble (GUE) for $\beta = 2$ with variance $w^2$ of the diagonal elements

$$P_b(H_b) \sim \exp\left(-\frac{1}{2w^2} \text{tr} H_b^2\right).$$

As already said, $y$ is an $N$-component vector which has real or complex entries for $\beta = 1$ and $\beta = 2$, respectively.

In section 4.1, we reformulate the problem in terms of matrix invariants. This enables us to introduce a handy supermatrix model in section 4.2, which we then solve in section 4.3.

4.1. Reformulation in a rotation-invariant form

At first sight, the best way of tackling the problem seems to be to start from equation (22) and to introduce $y_v = V_v/|E_v|$. The $y$ integration is then simply an integration over an $N - 1$ sphere in a real or complex space. This leads to

$$Q(u) = \frac{\Omega_N}{(2v^2 \pi/\beta)^{-\beta N/2}} \langle | \text{det} H_b |^\beta e^{-u \text{tr} G H_b^2} \rangle_N,$$

where the $N \times N$ matrix

$$G = \text{diag}((u - 1)\beta/2v^2, 0, \ldots, 0)$$

has rank 1. Furthermore,

$$\Omega_N = \frac{2\sqrt{\pi}^{\beta N}}{\Gamma(\beta N/2)}$$

is the volume of the above-mentioned sphere. Unfortunately, the remaining ensemble average is only feasible for the GUE; we carry it out in appendix A. For the GOE, the calculation is hampered by the modulus of the determinant.

To address the GOE case and to have a method that is capable of handling both cases, GOE and GUE, in a unifying way, it turns out to be necessary to cast the ensemble average into an invariant form. To this end, we start from equation (25), perform the integration over the vector $y$ and obtain

$$R(k) = \langle \text{det}(H_b^2 + 2i v^2 k/\beta)^{-\beta/2} | \text{det} H_b |^\beta \rangle_N.$$
where $\tilde{H}$ is an $(N + 1) \times (N + 1)$ matrix and the ensemble average is over an $(N + 1) \times (N + 1)$-matrix GOE (GUE) ensemble. The derivation of equation (44), which is crucial for the calculation, is sketched in appendix B. On the right-hand side, the inconvenient modulus of the determinant has disappeared. The combinatorial factor $L_{N\beta}$ is given by

$$L_{N\beta} = \begin{cases} \frac{2\Gamma(1 + (N + 1)/2)}{\sqrt{\pi}(N + 1)} \overline{\beta}, & \beta = 1, \\ \frac{N!}{\beta}, & \beta = 2. \end{cases}$$

(45)

We express the determinant on the right-hand side of equation (44) as a Gaussian integral over a real ($\beta = 1$) or complex ($\beta = 2$) $N + 1$ vector $y$

$$\det(H_0^2 + 2iv^2k/\beta + \epsilon)^{-\beta/2} = \int \frac{dy}{\pi^{\beta/2}} \exp(-y^\dagger(H_0^2 + 2iv^2k/\beta + \epsilon)y).$$

(46)

We subsequently plug equations (43), (44) and (46) into equation (23) and write the integral over the $N + 1$ vector $y$ in radial coordinates. We find

$$Q(u) = \frac{L_{N\beta} \Omega_{N+1}}{2\pi^{\beta/2}} \left( \frac{\beta y^2}{2v^2\pi} \right)^{\beta/2} \lim_{\epsilon \to 0} \int_0^\infty dx \ x^{\beta(N+1)/2 - 1} \tilde{F}_{N+1} \left( 1 + \frac{\beta x y^2}{u^2} \right) \times \int_{-\infty}^\infty \frac{dk \ (\epsilon + ik)^{\beta/2}}{2\pi} e^{ik(u-1-x)},$$

(47)

where we introduced the function

$$\tilde{F}_{N+1}(g) = \sqrt{2\pi} w^2 \left( \text{tr} \delta(\tilde{H}) e^{-(1/2\nu^2)u \tilde{G} \tilde{H}^2} \right)_{N+1}$$

(48)

with the $(N + 1) \times (N + 1)$ matrix $\tilde{G} = \text{diag}(g - 1, 0, \ldots, 0)$. After combining the various constants, we arrive at

$$Q(u) = \frac{1}{\Gamma(\beta/2)} \left( \frac{\beta y^2}{2v^2} \right)^{\beta/2} \frac{d^{\beta/2}}{d(u - 1)^{\beta/2}} \int_0^\infty dx \ x^{\beta(N+1)/2 - 1} \tilde{F}_{N+1} \left( 1 + \frac{\beta x y^2}{u^2} \right) \delta(u - 1 - x),$$

(49)

where we formally introduced the fractional derivative

$$\frac{d^{\beta/2}}{dx^{\beta/2}} \delta(x) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_{-\infty}^\infty dk \ (\epsilon + ik)^{\beta/2} e^{ikx}.$$

(50)

We can evaluate the fractional distribution

$$\frac{d^{\beta/2}}{dx^{\beta/2}} \delta(x) = \frac{1}{\pi} \text{Re} \lim_{\epsilon \to 0} \frac{e^{i(\beta/4)\Gamma(\beta/2 + 1)}}{\epsilon - ix} \frac{\Gamma(\frac{\beta}{2} + 1)}{\Gamma(\frac{\beta}{2})}$$

$$= \frac{2}{\pi \beta} \frac{d}{dx} \text{Im} \lim_{\epsilon \to 0} \frac{e^{i(\beta/4)\Gamma(\beta/2 + 1)}}{\epsilon - ix} \frac{\Gamma(\frac{\beta}{2} + 1)}{\Gamma(\frac{\beta}{2})}$$

(51)

for arbitrary $\beta$. For $\beta = 1, 2$ we obtain

$$\frac{d^{\beta/2}}{dx^{\beta/2}} \delta(x) = \begin{cases} \frac{d}{dx} \frac{2\Gamma(3/2)}{\pi \sqrt{x}} \Theta(x), & \beta = 1, \\ \frac{d}{dx} \delta(x), & \beta = 2. \end{cases}$$

(52)

Here we see that the GOE case is more complicated than the GUE case.
For the GUE the fractional derivative disappears and we find without further problems

\[ Q(u) = \frac{d}{du} \left( \frac{\beta(u - 1)w^2}{2v^2} \right)^N \tilde{F}_{N+1} \left( 1 + \frac{\beta(u - 1)w^2}{v^2} \right). \]  

(53)

For the GOE we obtain an integral expression

\[ Q(u) = \frac{1}{\pi} \frac{d}{du} \left( \frac{\beta w^2}{2v^2} \right)^{N/2} \int_0^{(u-1)} \frac{x^{(N-1)/2}}{\sqrt{u - 1 - x}} \tilde{F}_{N+1} \left( 1 + \frac{\beta x w^2}{v^2} \right) dx. \]  

(54)

The remaining task is, in both cases (GOE and GUE), the calculation of \( \tilde{F}_{N+1}(g) \).

4.2. Mapping onto a supermatrix model

Using

\[ \text{tr} \delta(H) = \frac{1}{\pi} \text{Im} \frac{1}{H - i\epsilon} \]

\[ = \frac{1}{2\pi} \text{Im} \left. \frac{d}{dj} \text{det}(H + j) \right|_{j=0}, \]

(55)

the ensemble average \( \tilde{F}_{N+1}(g) \) defined in equation (48) can be expressed via standard techniques (see, for instance, [15]–[17]) as a supersymmetric matrix integral:

\[ \tilde{F}_{N+1}(g) = \sqrt{\frac{w}{2\pi}} \text{Im} \frac{d}{dj} \frac{1}{\sqrt{g}} \left( \frac{2}{1+g} \right)^{\beta N/2} \int d[\tau] \exp \left( -\frac{(1-g)^2}{8g w^2} \text{Str} \left( \tau + \frac{1+g}{1-g} J \right)^2 \right) \]

\[ \times \text{Sdet}^{-1}(\tau - J) \int d[\sigma] \exp \left( -\frac{1}{2w^2} \text{Str} \sigma^2 \right) \text{Sdet}^{-\beta N/2}(\sigma^2 + \tau) \]  

\[ \left| \text{Sdet}^{-1}(\tau - J) \int d[\sigma] \exp \left( -\frac{1}{2w^2} \text{Str} \sigma^2 \right) \text{Sdet}^{-\beta N/2}(\sigma^2 + \tau) \right|_{j=0} \].  

(56)

Here, \( \sigma \) and \( \tau \) are 2 × 2 (GUE) and 4 × 4 (GOE) supermatrices, respectively, of the form

\[
\begin{pmatrix}
  a_1 & \lambda_1^+ \\
  \lambda_1 & ia_2 \\
\end{pmatrix}, \quad \text{GUE},
\]

\[
\begin{pmatrix}
  a_1 & a_2 & \lambda_1^+ & -\lambda_1 \\
  a_2 & a_3 & \lambda_2^+ & -\lambda_2 \\
  \lambda_1 & \lambda_2 & ia_4 & 0 \\
  \lambda_1^+ & \lambda_2^+ & 0 & ia_4 \\
\end{pmatrix}, \quad \text{GOE}.
\]

(57)

The matrix entries in italic letters denote real commuting integration variables. Although not strictly necessary, we work with complex anticommuting integration variables, denoted by Greek letters. The volume elements \( d[\tau] \) and \( d[\sigma] \) are products of the differentials of all independent integration variables. The integration domain of the real commuting variables is the real axis. The matrix \( J \) is a 2 × 2 (GUE) or a 4 × 4 (GOE) diagonal supermatrix with entries \( J = \text{diag}(j, -j) \) (GUE) and \( J = \text{diag}(j, j, -j, -j) \) (GOE). Due to the broken rotation invariance of the original matrix model (1), the resulting supersymmetric representation (56) is a two-matrix model.

Since we are interested in the large \( N \) limit, it is sufficient to calculate the leading term of \( \tilde{F}_{N+1}(g) \) in an asymptotic expansion in \( 1/N \). This suggests the evaluation of the matrix integrals in equation (56) in a saddle-point approximation. However, the large parameter \( N \) only appears
in the integral over the matrix $\sigma$ and not in the $\tau$-integral. Therefore, we can only evaluate the $\sigma$-integral by a saddle-point approximation. The $\tau$-integral has to be performed exactly afterwards.

It is well known \[18\] that the $\sigma$-integral $K_N(\tau)$ yields for large $N$ in the saddle-point approximation

$$K_N(\tau) = \int d[\sigma] \exp\left(-\frac{1}{2w^2} \text{Str} \sigma^2\right) \text{Sdet}^{-\beta N/2}(\sigma^- + \tau) \approx \exp\left(-\frac{1}{2w^2} \text{Str} \tau^2 + \frac{i\beta \text{Str} \tau}{2D}\right) + O\left(\frac{1}{N}\right),$$

(58)

where $D = \sqrt{\frac{\beta \pi^2 w^2}{2N}}$ is the mean level spacing at the center of the band. However, this approximation is only valid if $\text{Str} \tau$ itself is of the order of the mean level spacing. Since the integration domain of $\tau$ is the whole real axis, this is not automatically guaranteed. A necessary condition is that the variance of the Gaussian in the second line of equation (56) is itself of the order of the mean level spacing, i.e. the $\tau$-integral in equation (56) is essentially localized to a small window of width $D$ around zero. Consequently, we require

$$\frac{(1-g)^2}{8gw^2} \approx N + O(1).$$

(59)

Therefore, $g$ should scale as $N$ for large $N$. The dimensionless coupling strength is given by $\lambda = \nu/D$. Since $\nu$ is of order one, we obtain for $g$ in the GUE case

$$g = 1 + \frac{2(u - 1)}{\pi^2 \lambda^2} N,$$

$$\approx \frac{2(u - 1)}{\pi^2 \lambda^2} N.$$ (60)

Fortunately, this is exactly the scaling behavior where we need to apply the saddle-point approximation. In the GOE case, equation (60) holds in any interval $\omega_c < x < (u - 1)$, where $\omega_c$ is an infrared cutoff in the integral, which is small compared to one but large compared to the mean level spacing. Therefore in the limit $N \to \infty$, equation (58) can be applied on the whole integration domain of the $x$-integral in equation (54). In conclusion, we can apply the approximation (58) in both the GUE and the GOE case.

4.3. Remaining matrix integration and final result

Plugging equation (58) into (56), we obtain, after a simple shift,

$$\tilde{F}_{N+1}(g) = \sqrt{\frac{w^2}{2\pi}} \text{Im} \frac{d}{dJ} \frac{1}{\sqrt{g}} \left(\frac{2}{1+g}\right)^{\beta N/2} \exp\left(\frac{2i\pi J}{D}\right) \times \int d[\tau] \exp\left(-\frac{g}{2w^2} \text{Str} \left(\tau - \frac{i\beta \pi w^2}{Dg} - J\right)^2\right) \text{Sdet}^{-\beta/2} \tau\bigg|_{J=0},$$

(61)

where we employed $g \approx N \gg 1$. Now the derivative with respect to the source term can be performed:

$$\tilde{F}_{N+1}(g) = \sqrt{\frac{2\pi w^2}{D}} \frac{1}{\sqrt{g}} \left(\frac{2}{1+g}\right)^{\beta N/2} \left(1 + D \sqrt{\frac{g}{\pi}} \text{Im}\left(\frac{1}{H + (\beta i \pi w^2/D \sqrt{g})}\right)\right),$$

(62)

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where we used the identity
\[ \left( \mathrm{tr} \left( \frac{1}{H + z} \right) \right)_N = \frac{1}{2 \pi} \int d\tau \left[ e^{-\left(\frac{1}{2}w^2\right)} \mathrm{Str} \left( \tau + e^{-J} \right)^2 \right] \left( -\beta N/2 \right)^{\frac{N}{2}} \left| \right|_{j=0}, \]
for a complex number \( z \) with negative imaginary part. The remaining average can be calculated by employing techniques of standard analysis. We use
\[ \mathrm{Im} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \ \frac{\exp(-x^2/2)}{x - ir} \right) = \mathrm{sgn}(r) \sqrt{\frac{\pi}{2}} \exp \left( \frac{r^2}{2} \right) \mathrm{erfc} \left( \sqrt{\frac{|r|}{2}} \right). \]
which holds for \( r \in \mathbb{R} \). We obtain
\[ \mathrm{Im} \left( \mathrm{tr} \left( \frac{1}{H + (\beta \pi w^2 / D \sqrt{g})} \right) \right) = \sqrt{\frac{\pi}{2w^2}} \exp \left( \frac{\beta N}{g} \right) \mathrm{erfc} \left( \sqrt{\frac{\beta N}{g}} \right). \]
Collecting everything, we arrive at
\[ \tilde{F}_{N+1}(g) = \sqrt{\frac{2\pi w^2}{g D^2}} \left( \frac{2}{1 + g} \right)^{\frac{N}{2}} \left( 1 + \frac{\beta}{2} \right) \sqrt{\frac{\pi g}{\beta N}} \exp \left( \frac{\beta N}{g} \right) \mathrm{erfc} \left( \sqrt{\frac{\beta N}{g}} \right). \]
This result simplifies considerably when we take into account the scaling behavior (60) of \( g \). In the large \( N \) limit we can write
\[ \lim_{N \to \infty} \left( \frac{\beta x w^2}{2u^2} \right)^{\frac{N}{2}} \tilde{F}_{N+1} \left( 1 + \frac{\beta x w^2}{u^2} \right) = \sqrt{\frac{2\pi \lambda^2}{\beta x}} \exp \left( -\frac{\beta (\pi \lambda)^2}{2x} \right) + \mathrm{erfc} \left( \sqrt{\frac{\beta (\pi \lambda)^2}{2x}} \right). \]
This can now be plugged into equation (49) to obtain expressions for \( Q(u) \) on the scale of the mean level spacing. For the GUE we find straightforwardly
\[ Q(u) = \sqrt{\frac{\pi \lambda^2}{4(u - 1)^3}} \exp \left( -\frac{\pi^2 \lambda^2}{(u - 1)} \right) \left( 1 + \frac{2\pi^2 \lambda^2}{u - 1} \right), \]
\[ p_0(c) = \sqrt{\frac{\pi \lambda^2}{(1 - c)^3}} \exp \left( -\frac{\pi^2 \lambda^2 c^2}{(1 - c^2)} \right) \left( 1 + \frac{2\pi^2 \lambda^2 c^2}{1 - c^2} \right). \]
For the GOE we are left with an integral expression for \( Q(u) \):
\[ Q(u) = \sqrt{\frac{\pi^3 \lambda^6}{2(u - 1)^5}} \int_0^1 dx \ \frac{1}{\sqrt{1 - x^2}} \ \exp \left( -\frac{\pi^2 \lambda^2}{2(u - 1)x} \right). \]
The integral can be evaluated further and expressed in terms of standard special functions. Finally, we arrive at
\[ Q(u) = \sqrt{\frac{\pi^3 \lambda^6}{8(u - 1)^5}} \exp \left( -\frac{\pi^2 \lambda^2}{4(u - 1)} \right) \left[ K_0 \left( \frac{\pi^2 \lambda^2}{4(u - 1)} \right) + K_1 \left( \frac{\pi^2 \lambda^2}{4(u - 1)} \right) \right], \]
\[ p_0(c) = \sqrt{\frac{\pi^3 \lambda^6 c^4}{2(1 - c)^5}} \exp \left( -\frac{\pi^2 \lambda^2 c^2}{4(1 - c^2)} \right) \left[ K_0 \left( \frac{\pi^2 \lambda^2 c^2}{4(1 - c^2)} \right) + K_1 \left( \frac{\pi^2 \lambda^2 c^2}{4(1 - c^2)} \right) \right], \]
where \( K_n \) is the modified Bessel function of the second kind of order \( n \). In figure 2, the distributions \( p_0(c) \) of equation (70) for the GOE (blue curves) and of equation (68) for the GUE (red curves) are plotted for the values \( \lambda = 0.1, 0.5 \) and 2 of the mean coupling strength \( \lambda \). We see that, for small \( \lambda \), there is only a minor difference between GUE and GOE backgrounds.
Figure 2. Figure showing the plots of the analytical results for the GOE (blue curves) and for the GUE (red curves) for three different coupling strengths ($\lambda = v/D$): $\lambda = 0.1$ (thick curves), $\lambda = 0.5$ (dashed curves) and $\lambda = 2$ (dotted curves).

Figure 3. Four figures showing the analytical results: (1) for a complex coupling of the doorway state to a GUE background (full red line), (2) for a real coupling to a GOE background (full green line), (3) for a complex coupling to a regular background (dashed blue line) and (4) for a real coupling to a regular background (full blue line) for four different coupling strengths ($\lambda = v/D$): $\lambda = 0.05$ (upper left panel), $\lambda = 0.1$ (upper right panel), $\lambda = 0.5$ (lower left panel) and $\lambda = 2$ (lower right panel).

4.4. Comparison

In figure 3, the distribution function of the overlap integral $|\langle 0 | s \rangle|$ of the evolved doorway state with the unperturbed doorway state is plotted for four different coupling strengths $\lambda = 0.05$, 0.1, 0.5 and 2 and for all types of couplings and background complexities considered. These are
Figure 4. Comparison of the analytical curves (dashed lines) obtained from equation (70) with Monte Carlo simulations (full lines) for $\lambda = 0.1$ (red), $\lambda = 0.5$ (green) and $\lambda = 2$ (blue) for a GOE background and real coupling coefficients.

(i) complex coupling to a regular background (dashed blue line), (ii) real coupling to a regular background (full blue line), (iii) complex coupling to a GUE background (full red line) and (iv) real coupling to a GOE background (full green line). As a general trend, mixing with the background is strongest for a complex coupling to a GUE background and weakest for real coupling to a Poissonian background. However, the difference in the distributions for different background complexities is rather small. This suggests a certain degree of universality of the curves. One might choose other ensembles for the background Hamiltonian, such as semi-Poisson [19] or transition ensembles. However, we expect that for these ensembles that lie between the two extreme cases, GUE and Poissonian, their corresponding distributions will also lie in the channel between the full red line (GUE) and the full blue line (Poissonian with real coupling). For the most interesting case of small $\lambda$, this channel is small.

On the other hand, it is seen from figure 3 that the distributions are highly sensitive with respect to a change in the coupling strength $\lambda$.

In figure 4, we compare the curves for $p_0(c = |\langle 0|s\rangle|)$ obtained from the analytical results (in this case from equation (70)) with Monte Carlo simulations in the case of a real coupling to a GOE background. The figure shows $p_0(c = |\langle 0|s\rangle|)$ for three values of the coupling strength, $\lambda = 0.1$, 0.5 and 2. We see fairly good agreement for all three values, even for the strong coupling value $\lambda = 2$. This shows that the approximation of section 2.3 is justified far beyond the perturbative regime.

5. Discussion

The distribution of the maximum coupling coefficients in the doorway mechanism has been introduced as a new statistical observable. These coupling coefficients, that is, the overlaps between the eigenstates of the full Hamiltonian and the doorway state, are not always available. However, in situations where they are accessible, this distribution provides a highly sensitive measure for the interaction strength. Of particular interest is the regime of weak interactions. In this regime, the distribution of the maximum coupling coefficients is very well approximated by...
the distribution of the overlap between the evolved doorway state and the unperturbed doorway state. While calculating the former seems unfeasible at present, we calculated the latter exactly for regular and chaotic background states in the cases of preserved and fully broken time-reversal invariance. We performed our calculations in the framework of random matrix theory, which is well known to provide reliable models for regular and chaotic systems. We also carried out numerical simulations that fully confirm our analytical results.

Our exact calculations are of general interest for matrix models. We have managed to reformulate a problem with the breaking of rotation invariance in the space of $N \times N$ random matrices in terms of a rotation invariant problem involving $(N+1) \times (N+1)$ random matrices. This made it possible to map the matrix model in ordinary space onto a matrix model in superspace, which we solved by a saddle-point approximation in the limit of infinite level number.

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Appendix A. Gaussian unitary ensemble (GUE) background with finite level number

We define the average

$$\langle \ldots \rangle_{g^N} = A_N \int d[H] \langle \ldots \rangle e^{-\left(1/2w^2\right) trG^xH^2},$$

(A.1)

where the diagonal $N \times N$ matrix $G^x$ is defined as $G^x = \text{diag}(g', 1, \ldots, 1)$. Here, $g'$ is related to the parameters of the main text as $g' = 1 + 2(u - 1)w^2/v^2$. The integration is over the set of all Hermitian $N \times N$ matrices, i.e. over the GUE ensemble. The normalization $A_N$ is chosen such that $\langle 1 \rangle_{g^N} = 1$. The task is to calculate the average

$$F_N(g') = \langle \det H^2 \rangle_{g^N}. \quad (A.2)$$

A Laplace expansion of the determinant yields

$$F_N(g') = \sum_{\omega, \omega' \in S_N} (-1)^{\text{sgn}(\omega) + \text{sgn}(\omega')} \prod_{n=1}^{N} H_{n\omega(n)} H_{\omega'(n)n} \langle H_{n\omega(n)} H_{\omega'(n)n} \rangle_{g^N},$$

(A.3)

where $S_N$ is the permutation group. Obviously, only terms $\omega = \omega'$ contribute

$$F_N(g') = \sum_{\omega \in S_N} \left( \prod_{n=1}^{N} |H_{n\omega(n)}|^2 \right)_{g^N}. \quad (A.4)$$

It is useful to expand the remaining sum in cycles involving the index 1. For indices $k_n > 1$ and $k_n \neq k_m$, we define the cycles $C_n^{(1)}$ as

$$C_n^{(1)}(k_1, \ldots, k_n) = |H_{1k_1}|^2 |H_{k_1k_2}|^2 \cdots |H_{k_{n-1}k_n}|^2 |H_{k_n1}|^2,$$

$$C_0^{(1)} = |H_{11}|^2.$$  

(A.5)
Since the indices $1, k_1, \ldots, k_n$ do not appear in the remainder of the product, we can integrate over the remainder separately. This yields

$$F_N(g') = \sum_{n=0}^{N-1} \sum_{k_1 \neq k_2 \ldots k_n} \langle C_n^{(1)}(k_1, \ldots, k_n) \rangle_{g,n} F_{N-n-1}(1).$$\hspace{1cm} (A.6)

The average over the cycles is simple as well. Only terms involving the index 1 yield a factor different from $w^2$. We obtain

$$\langle C_0^{(1)} \rangle = \left( \frac{2}{g' + 1} \right)^{N-1} \frac{w^2}{g'^{3/2}},$$

$$\langle C_1^{(1)}(k_1) \rangle = \left( \frac{2}{g' + 1} \right)^{N-1} \frac{2w^4}{\sqrt{g'}},$$

$$\langle C_n^{(1)}(k_1, \ldots, k_n) \rangle = \left( \frac{2}{g' + 1} \right)^{N+1} \frac{w^{2(n+1)}}{\sqrt{g'}} , \ n > 1.$$\hspace{1cm} (A.7)

The averages are independent of the indices $k_n$. The sum over the indices yields the combinatorial factor $(N - 1)!/(N - 1 - n)!$. Altogether, we obtain

$$F_N(g') = \frac{1}{\sqrt{g'}} \left( \frac{2}{g' + 1} \right)^{N-1} \left( \frac{w^2}{g'} F_{N-1}(1) + 2(N - 1)w^4 \left( \frac{2}{g' + 1} \right)^2 F_{N-2}(1) \right) + \left( \frac{2}{g' + 1} \right)^{2(N-1)} \sum_{n=2}^{N-1} \frac{w^{2(n+1)}}{(N - 1 - n)!} \frac{(N - 1)!}{F_{N-n-1}(1)}.$$\hspace{1cm} (A.8)

Evaluating this equation for $g' = 1$ allows us to replace the sum. After some further simple manipulations, we finally obtain

$$F_N(g') = \frac{1}{\sqrt{g'}} \left( \frac{2}{g' + 1} \right)^{N+1} \left( F_N(1) + \frac{w^2(g' - 1)^2}{4g'} F_{N-1}(1) \right),$$\hspace{1cm} (A.9)

which is almost our final result. The remaining task is to evaluate the $g'$-independent constant $F_N(1)$. This is facilitated by the observation that

$$F_N(1) = \sqrt{\pi} (\beta w^2)^N N! K_{N+1}(0, 0),$$\hspace{1cm} (A.10)

where $K_{N+1}(x, y) = \sum_{n=0}^{N} \phi_n(x)\phi(y)$ is the standard GUE kernel as defined in equation (6.2.10) of Mehta’s book [20]. The

$$\phi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} \exp(-x^2/2) H_n(x)$$\hspace{1cm} (A.11)

are oscillator wave functions and $H_n$ is the $n$th Hermite polynomial. The constant $K_N(0, 0)$ can also be evaluated:

$$K_N(0, 0) = \frac{1}{\sqrt{\pi}} \frac{N!}{2^{N-1}} \left[ \frac{((N - 1)/2)!}{((N - 2)/2)!} \right]^{2} N \text{ odd}, \left[ \frac{((N - 1)/2)!}{((N - 2)/2)!} \right]^{2} N \text{ even.}$$\hspace{1cm} (A.12)

Of course, $K_N(0, 0)$ is the inverse level spacing at the center of the SC. Therefore, $\lim_{N \to \infty} K_N(0, 0)/\sqrt{2N} = 1/\pi$. We obtain our final result

$$F_N(g') = (2w^2)^N N! K_{N+1}(0, 0) \sqrt{\frac{\pi}{g'}} \left( \frac{2}{g' + 1} \right)^{N+1} \left( \frac{(g' - 1)^2}{4g' N} + c_N \right),$$\hspace{1cm} (A.13)
where

\[ c_N = \begin{cases} 
1 & \text{if } N \text{ odd,} \\
1 + 1/N & \text{if } N \text{ even.} 
\end{cases} \quad (A.14) \]

The even–odd difference disappears in the large \( N \) limit. In the following, we set \( c_N = 1 \). Using equation (40), we find for \( \tilde{Q}(u) \)

\[ Q(u) = \left( \frac{u^2}{v^2} \right)^N K_{N+1}(0, 0) (u - 1)^{N-1} \sqrt{\frac{\pi}{g'}} \left( \frac{2}{g' + 1} \right)^{N+1} \left( \frac{(g' - 1)^2}{4g'} + N \right). \quad (A.15) \]

This exact result can be compared with equation (53) in order to find a differential equation for \( \tilde{F}_{N+1}(g') \):

\[ \left( N + (g' - 1) \frac{d}{dg'} \right) \tilde{F}_{N+1}(g') = K_{N}(0, 0) \sqrt{\frac{\pi}{g'}} \left( \frac{2}{g' + 1} \right)^{N+1} \left( \frac{(g' - 1)^2}{4g'} + N \right). \quad (A.16) \]

This differential equation can easily be solved. However, the solution

\[ \tilde{F}_{N+1}(g') = \sqrt{\frac{\pi}{g'}} \left( \frac{2}{g' + 1} \right)^N K_{N+1}(0, 0) \left( 1 + \sqrt{g'} \left( \frac{g' + 1}{g' - 1} \right)^N \int_{g'}^{\rho(g')} \frac{dx}{\sqrt{x}(x + 1)^{N+1}} \right) \quad (A.17) \]

is highly complicated. It discourages any attempt to calculate the matrix integral of equation (56) for finite \( N \) in the GOE case. To make contact with the results obtained in the main text, we introduce the function

\[ \rho(g') = \sqrt{\frac{2Ng'}{(1-g')^2}}. \quad (A.18) \]

With a change to variables \( y = \rho(x) \) in the integral, we can write

\[ \tilde{F}_{N+1}(g') = \sqrt{\frac{\pi}{g'}} \left( \frac{2}{g' + 1} \right)^N K_{N+1}(0, 0) \times \left( 1 + \sqrt{\frac{2g'}{N}} \left( 1 + \frac{2\rho^2(g')}{N} \right)^{N/2} \int_{\rho(g')}^{\rho(x)} dy \left( 1 + \frac{2y^2}{N} \right)^{-1} \right), \quad (A.19) \]

which coincides with equation (66) in the large \( N \) limit and for \( \beta = 2 \).

**Appendix B. Derivation of equation (44)**

We define the function

\[ G_N(z) = \langle \text{det}(H^2 + z)^{-\beta/2} \rangle_N, \quad (B.1) \]

where \( H \) is an \( N \times N \) GOE or GUE random matrix and the angular brackets denote the corresponding GOE or GUE average. \( G_N(z) \) is an analytic function in the cut complex plane \( \mathbb{C} \setminus \mathbb{R} \). In this appendix, we prove the identity

\[ G_N(z) = L_{N\beta} z^{\beta/2} \left\{ \text{tr} \delta(\tilde{H}) \text{det}(\tilde{H}^2 + z)^{-\beta/2} \right\}_{N+1}, \quad (B.2) \]

where \( \tilde{H} \) is an \( (N+1) \times (N+1) \) GOE or GUE random matrix. The constant \( L_{N\beta} \) is given in equation (45). We write the rhs of equation (B.2) in angle eigenvalue coordinates \( \tilde{H} \rightarrow U^{-1} \tilde{E} U \),
where $\tilde{E}$ is an $N+1 \times N+1$ diagonal matrix of the eigenvalues $\tilde{E}_i$ of $\tilde{H}$. Since the average is over an invariant function, the integral over the diagonalizing group is trivial. The average on the rhs can now be written as

$$\text{lhs} = C_{(N+1),\beta} L_{N\beta} \zeta^{\beta/2} \int d[\tilde{E}] \left\{ \delta(\tilde{E}) \prod_{i=1}^{N+1} (\tilde{E}_i^2 + z)^{-\beta/2} |\Delta_{N+1}(\tilde{E})|^{\beta} \right\} \exp \left( -\frac{1}{2w^2} \sum_{i=1}^{N+1} \tilde{E}_i^2 \right).$$

(B.3)

The power of the Vandermonde determinant $\Delta_N(x) = \prod_{i<j}(x_i - x_j)$ arises as Jacobian from the coordinate transformation. The constant $C_{(N+1),\beta}$ arising from the group integration can be found in Mehta’s book [20]. Now the integral over the $\delta$-distribution can be performed:

$$\text{lhs} = (N+1)C_{(N+1),\beta} L_{N\beta} \int d[E] |E_i|^{\beta} (E_i^2 + z)^{-\beta/2} |\Delta_N(E)|^{\beta} \exp \left( -\frac{1}{2w^2} \sum_{i=1}^{N} E_i^2 \right).$$

(B.4)

We see that the resulting integral can be written as a GOE (GUE) average over $N \times N$ matrices. This is indicated by using $E_i$ instead of $\tilde{E}_i$ as integration variables. We go back to Cartesian coordinates $U^{-1} E U \rightarrow H$ and find

$$\text{lhs} = \frac{(N+1)C_{(N+1),\beta}}{C_{N,\beta}} L_{N\beta} \langle |\det H|^{\beta} (H^2 + z)^{-\beta/2} \rangle_N$$

$$= \frac{(N+1)C_{(N+1),\beta}}{C_{N,\beta}} L_{N\beta} G_N(z).$$

(B.5)

This is the desired identity with $L_{N\beta} = C_{N,\beta}/(N+1)C_{N+1,\beta}$. Equation (44) is obtained for $z = \epsilon + 2iu^2k/\beta$.

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