On the pseudo hermiticity and q-deformation

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Abstract. The non hermiticity of the Hamiltonian is shown to be generated from the q-deformation of the algebra. As an example the q-deformed harmonic oscillator is investigated within the leading order approximation in the pseudo Hermitian perturbation theory. A generalized form of the metric is constructed. The spectrum as well as eigenstates of the system are derived explicitly. Some applications and illustrations in q-deformed quantum statistical mechanics are also discussed.

1. Introduction
Generalized non-Hermitian Hamiltonian quantum mechanics as proposed by Bender et al [1]-[3] are shown to be useful to describe consistently many physical systems namely the dissipative ones. During the last few years ago, a huge and remarkable class of non-Hermitian quantum Hamiltonians exhibiting an entire real spectrum have been revived [1]-[11]. Furthermore, quantum groups and deformed Heisenberg algebras have been the subject of intensive investigations leading to a very important applications in various branches of physics and chemistry. This work is devoted to the study of a quantum system described by a pseudo Hermitian Hamiltonian which is the q-deformed harmonic oscillator which is shown to have real spectra[12]-[16]. We construct first the metric using perturbation theory in the first approximation with respect to the q-deformation parameter. A new scalar product is defined and the eigenstates as well as eigenvalues of the system. In section 2 we give a brief review of the model and related mathematical formalism. In section 3, applications to q-deformed statistical mechanics of a canonical system are investigated. Finally, in section 4, we draw our conclusions.

2. Brief review of the model
The creation and annihilation operators $a$ and $a^\dagger$ of the q-deformed harmonic oscillator satisfy the following q-deformed commutation relation (Heisenberg algebra) [12]

$$\hat{a}\hat{a}^\dagger - q\hat{a}^\dagger\hat{a} = [\hat{a},\hat{a}^\dagger]_q = 1$$

(1)

where $q$ is a deformation parameter which in our case is taken to be real and small $q=1+\epsilon$ ($\epsilon \ll 1$). In terms of the ordinary creation and annihilation operators $a^\dagger$ and $a$ one can show that in the leading first order approximation of the perturbation theory with respect to the parameter $\epsilon$ (LFOA) the operators $\hat{a}$ and $\hat{a}^\dagger$ have the following expressions:
Here $\hat{X}_q$ and $\hat{P}_q$ are the q-deformed position and momentum respectively having as a representation:

\[
\hat{X}_q = x + \frac{\varepsilon}{6}(x^3 - xp^2 + ix^2p - x + ip^3 + pxp + p^2x) \tag{4}
\]

and

\[
\hat{P}_q = p \tag{5}
\]

Now, let us consider a q-deformed harmonic oscillator which has the following form:

\[
\hat{H}_q = \hat{P}_q^2 + \hat{X}_q^2 \tag{6}
\]

Straightforward simplifications lead to:

\[
\hat{H}_q = H_0 + \frac{\varepsilon}{6}H_1 + O(\varepsilon^2) \tag{7}
\]

where $H_0$ and $H_1$ are the non perturbed and perturbed Hamiltonians respectively given by:

\[
H_0 = p^2 + x^2
\]

and

\[
H_1 = 2x^4 - x^2 - 3 + 3p^2 + 2ix^3p + 2ix^2p^2 - 8xp \tag{8}
\]

The adjoint of the previous Hamiltonian takes the form:

\[
\hat{H}_q^* = p^2 + x^2 + \frac{\varepsilon}{6}(2x^4 - 7x^2 + 1 - 2ix^3p - 3p^2 + 2x^2p^2 - 2ip^3) \tag{9}
\]

Notice that $\hat{H}_q$ is not Hermitian.

2.1. The metric

It is believed so far as it was shown in many papers that the metric $\eta$ is always a function of the coordinates and is unique. In this work, we will show that not only the metric can be a operator but not unique as well. It is very important to notice that in the literature the metric has been derived in many ways. In ref. (.14.), Fring et al have derived the metric from the position operator $X$ through the definition $X^* \eta = \eta^{-1}X$ instead of that using the Hamiltonian operator $H^* \eta = \eta^{-1}H$, leading to completely different results. Throughout this paper we will adopt the second approach. It is worth to mention that the mapping from the non Hermitian Hamiltonian $H$ to an isospectral Hermitian Hamiltonian $h$, is realized via a positive-definite metric $\rho$ as

\[
h = \rho H \rho^{-1} \tag{10}
\]

where

\[
\rho = \eta \eta^+ \tag{11}
\]

To determine the expression of the metric $\eta$ using the LFOA, one writes first

\[
\eta = \eta^+ = 1 + (\varepsilon/\beta)\eta^{(1)} \tag{12}
\]

and than it is easy to show that

\[
H_1^+ - H_1 = [\eta^{(1)}, H_0] \tag{13}
\]

Now if we write $\eta^{(1)}$ in the form:
where $A(x)$, $B(x)$ and $C(x)$ are function of the position operator. Straight forward but tedious calculations give the following system of differential equations:

$$C' = 2x$$

$$2B' + C'' = 6$$

$$2xB + 2C + A'' = -6x - 4$$

and

$$2A' + B'' + 4Cx = -4x^3 + 8x$$

Here prime and double prime stand for the first and second derivatives respectively. The solution of these differential equations leads to:

$$\eta^{(1)} = [-x^4 + (2 - \alpha)x^2 + \delta] + 2x \frac{d}{dx} + (x^2 + \alpha) \frac{d^2}{dx^2}$$

with $\alpha$ and $\delta$ are integration constants. Notice that $\eta^{(1)}$ is not unique as it was pointed out before.

Regarding the hermitian isospectral Hamiltonian $h$, it is easy to show that:

$$h = p^2 + x^2 + \frac{\epsilon}{8} (2x^3 - 4x^2 - 1 + 4ipx + 2p^2x^2) = h^+$$

The most important thing in this works is that we can generalize the obtained results to any Hamiltonian of the form $\hat{H}_q = \hat{p}_q^2 + \hat{x}_q^2$.

2.2. Eigenfunctions and eigenvalues

In order to get the energy spectrum $E_n$ and the corresponding eigen functions $\phi_n(x)$ let us start from eq.(7) using the stationary LFOA.. In fact, if we write:

$$\eta = \eta_0 + \epsilon \eta_1 + O(\epsilon^2)$$

$$E_n = E_n^0 + \frac{\epsilon}{8} E_n^1 + O(\epsilon^2)$$

and

$$|\Phi_n > = |\Phi_n^0 > + \frac{\epsilon}{8} |\Phi_n^1 > + O(\epsilon^2)$$

where

$$E_n^0 = \frac{\langle \Phi_n^0 | \eta_0 | \Phi_n^0 \rangle}{\langle \Phi_n^0 | \eta_0 | \Phi_n^0 \rangle}$$

With $E_n^0$ are the eigenstates of $\hat{H}_0$ that is:

$$\hat{H}_0 |\Phi_n^0 > = E_n^0 |\Phi_n^0 >$$

$$E_n^1 = \sum_k \frac{\langle \Phi_n^0 | \eta_0 R_k | \Phi_n^0 \rangle}{\langle \Phi_n^0 | \eta_0 | \Phi_n^0 \rangle}$$

and

$$|\Phi_n^1 > = -\frac{1}{2} \epsilon < \Phi_n^0 | \eta_1 | \Phi_n^0 > |\Phi_n^0 > + \sum_k \frac{\langle \Phi_n^0 | \eta_0 R_k | \Phi_n^0 \rangle}{E_n^0 - E_k^0} |\Phi_k^0 >$$

Here in the case of our interest $\eta_0 = 1$.Then, it is easy to show that the energy spectrum reads:

$$E_n^0 = 2n + 1$$
Similarly, in the configuration space, the eigenfunction $\phi_n(x)$ is such that:

$$E_n^1 = 4n^2$$  \hspace{1cm} (26)

and

$$\phi_n^1(x) = \frac{1}{\sqrt{n}} \frac{1}{\sqrt{2}} \frac{1}{\pi^{\frac{1}{2}}} e^{-\frac{x^2}{2}} H_n(x)$$  \hspace{1cm} (27)

where $n$ is a positive integer. $H_n(x)$ are the Hermite polynomials. Notice that The eigenvalues of $H$ are the same as $H$ but the eigenvectors that is:

$$E_n(H^+) = E_n(H) = 2n + 1 + \frac{\varepsilon}{8}(4n^2) + O(\varepsilon^2)$$  \hspace{1cm} (29)

and

$$I_{\phi_n}(H^+) = |\phi_n^+| > |\phi_n^+| + \frac{\varepsilon}{8} \left(2n+1\right) \sqrt{(n+1)(n+2)} |\phi_n^+| + O(\varepsilon^2)$$  \hspace{1cm} (30)

Thus, we have shown explicitly that the q-deformed harmonic oscillator Hamiltonian of eq.(6) is pseudo hermitian with the metric $\eta$ given by eq.(12).

3. Q-deformed quantum statistical mechanics

In this section and as an illustration of the q-deformation we investigate some of the thermodynamical properties of the q-deformed harmonic oscillator. In fact, for a canonical ensemble the quantum deformed partition function $Q_N^q$ for N particles are shown to have the following expression:

$$Q_N^q = Tr e^{-\beta \hat{H}} = \sum_{\psi\in|\psi\rangle} \frac{\langle \psi | e^{-\beta \hat{H}} | \psi \rangle}{\langle \psi | \psi \rangle}$$  \hspace{1cm} (31)

where $\beta = \frac{1}{k_{B}T}$ and $k_{B}$ is the Boltzmann constant. Using the LFOA, straightforward but tedious calculations give!

$$Q_N^q(\beta) = \frac{1}{(2\sinh\beta)^N} \left[ 1 - N\varepsilon \frac{\beta^2(\cosh2\beta - \sinh2\beta+3)}{16\sinh^2\beta} \right]$$  \hspace{1cm} (32)

Thus, the q-deformed Helmholtz function and internal energy read

$$F_q = \frac{1}{\beta} N [\ln(2\sinh\beta) + \varepsilon \frac{\beta^2(\cosh2\beta - \sinh2\beta+3)}{16\sinh^2\beta}]$$  \hspace{1cm} (33)

and

$$U_q(\beta) = N(\coth\beta) + \frac{\varepsilon}{32\sinh\beta} [1 - 16\beta \cos h \beta + (5 + 4\beta) \sin h \beta - \cos h 3\beta + \sin h 3\beta]$$  \hspace{1cm} (34)

The corresponding expression of the q-deformed heat capacity at a constant volume $C_V$ is:

$$C_V = Nk \frac{\beta^2}{\sinh^2\beta} \left[ 1 - \varepsilon \frac{(0\beta - 1)^2(1+4\beta) \sin h \beta - (4\beta) \sin h 2\beta}{8\sinh^2\beta} \right]$$  \hspace{1cm} (35)
Notice from the figure that for a fixed value of $\beta$, the maximum of $\frac{C_{V}}{Nk}$ is a decreasing function of $\varepsilon$. Of course, for $\varepsilon = 0$ and as it is expected, when $\beta \to 0$, $\frac{C_{V}}{Nk} \to 1$ (ordinary classical perfect gas) and as $\beta \to \infty$, $\frac{C_{V}}{Nk} \to 0$ however if $\varepsilon \neq 0$, the maximum is shifted towards the larger values of $\beta$. Also, if $\varepsilon$ is positive and increases the heat capacity decreases. Then, we deduce that $\varepsilon$ is an order parameter related to a sort of the dissipation of the system. Moreover, the figure illustrates the dependence of $\frac{C_{V}}{Nk}$ as a function of $\beta$. Notice that if $\varepsilon \neq 0$, this function increases than decreases smoothly passing by a maximum. Furthermore, the minimum value of $\beta$ (lower bound) depends strongly on the value of the $\varepsilon$ order parameter. It is worth to mention that the quantity $\frac{C_{V}}{Nk}$ can not be negative otherwise the convergence of the perturbation theory is violated.

4. Conclusion
In the present paper, we have investigated the q-deformed harmonic oscillator H and show that it is pseudo hermitian in the LFOA with the appropriate $\eta$ metric in the mathematical formalism of the stationary perturbation theory. Moreover, the generalized form of the corresponding metric has been constructed explicitly and shown to be an operator and not unique. The Hamiltonian and its conjugate have the same real energy spectrum but completely different eigenstates. Finally using the convergence of the perturbation theory, bounds on the q deformed parameter can be obtained. As an application we have considered the thermodynamics of the q-deformed harmonic oscillator and obtained some of the Thermo-dynamical functions the heat capacity at a constant volume. The latter has been shown to depend strongly on the deformation parameter.

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