Research Article

Method for Studying the Multisoliton Solutions of the Korteweg-de Vries Type Equations

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We present a new approach to find travelling wave solutions for the Korteweg-de Vries type equations, which allows extending the class of known soliton solutions. Also we propose method for studying the multisoliton solutions of the Korteweg-de Vries type equations.

1. Introduction

In recent years the investigation of separated waves plays an important role in many applied scientific fields. Travelling wave solutions can describe various phenomena in fluid mechanics, hydrodynamics, optics, plasma physics, solid state physics, biology, meteorology, and other fields. Pay attention that separated waves often occur at the boundaries of dynamic environments with different physical characteristics, such as “water-air” (in this case we consider the shallow water equations), and the limits of stratified fluids, on the verge of “gas-vacuum” (in [1] it is considered thin gas disks that rotate in a gravitational field) and on the border of crust and mantle (Moho surface).

Many models were proposed to describe the physical phenomena of separated waves existence and a variety of methods were proposed to construct the exact and approximate solutions to nonlinear equations. It is well known the KdV equations describe the unidirectional propagation of shallow water waves and a number of generalizations; for example,

\[ u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_{xxxx} = 0 \] (is given by [2]),

\[ u_t + \beta u u_x + u_{xxx} + u_{xxxx} = 0, \quad u_t + a u^2 u_x + b u_x u_{xx} + c u u_{xxx} + d u_{xxxx} = 0 \] (is given by [3]),

\[ u_t = u_{xxxx} + 10 u u_{xxx} + 20 u_x u_{xxx} + 30 u^2 u_x \] (is given by [4]),

\[ u_t + u u_x + u_{xxx} = -au_{xx} - \beta u_{xxxx} \] (equation Kuramoto-Sivashinsky, KdV [5]),

\[ u_t + u u_x + u_{xxx} - u_{xxxx} = 0 \] (the Kawahara equation [6]),

\[ u_t + v u u_x + \mu u_{xx} + au_x + \gamma u_{xxxx} = 0, \quad u_t + u u_x + \alpha u_{xx} + \beta u_{xxx} + \gamma u_{xxxx} = 0 \] (KdV-Burgers-Kuramoto [7]).

The KdV equations extended to several physical problems such as long internal waves in a density stratified ocean and acoustic waves on a crystal lattice. In recent years many efficient methods of finding the traveling wave solutions were developed such as Infinite Series method [8], Backlund transformation method [9], Darboux transformation [10], tanh method [11, 12], extended tanh function method [4], modified and extended tanh function method [13], the generalized hyperbolic function [14], the variable separation method, first integral method, and exp-function method [15].

A number of papers were devoted to the problems of the asymptotic solutions of the Korteweg-de Vries equation [16].

In this paper, we propose a new technique of finding the PDE’s traveling wave solutions which is based on the T-transformations.
2. New Solution of KdV Equation Based on the T-Representation

Let us consider a general nonlinear differential equation in the form

\[
F\left(t, x, \frac{\partial u(x, t)}{\partial t}, \frac{\partial u(x, t)}{\partial x}, \frac{\partial^2 u(x, t)}{\partial x^2}, \ldots, \frac{\partial^n u(x, t)}{\partial x^n}\right) = 0,
\]

where \( u \) is a dependent variable, \( x, t \) are independent variables, and \( F(\cdot) \) is a polynomial function concerning indicated variables. According to well-known travelling wave approach we unite the independent variables \( x \) and \( t \) into one particular wave variable \( \xi = x - vt \), where \( v \) is speed of the wave. Then (1) is converted to ODE. But then we have problem which is how to find separated wave solution.

Let \( \bar{x}(t) \) be function, determining the point of maximum wave disturbance. Define a function that describes the shape of the wave in the form \( \exp(-g(x - \bar{x}(t))) \), where \( g(\cdot) \) is sufficiently smooth function, which satisfies conditions

\[
g(x) \geq 0, \\
g(0) = 0, \\
g^{(i)}(0) = 0, \quad i = 1, 3.
\]

Obviously that function \( \gamma \exp(-g(x - \bar{x}(t))/\epsilon) \) can describe a positive perturbation of any shape; \( \gamma \) is amplitude parameter; \( \epsilon \) is a parameter that determines the location of disturbance. In the simplest case function \( g(\cdot) \) is measure defined on the set of intervals in \( R^1 \).

Let \( \gamma \) be a function, \( \gamma = \gamma(t) \). Let us consider the solution which we are looking for in the next form:

\[
u(x, t) = \gamma(t) \exp\left(-\frac{g(x - \bar{x}(t))}{\epsilon}\right). \tag{3}
\]

It is easy to build some kind of solution (3) that would be general for the system of differential equations in partial derivatives:

\[
F_1\left(t, x_1, \ldots, x_n; u^{(1)}_1, u^{(1)}_x, \ldots, u^{(m)}_1, u^{(m)}_x, \ldots\right) = 0, \quad i = 1, k.
\]

We expand the solution in the form

\[
\left(u^{(1)}, u^{(2)}, \ldots, u^{(m)}\right)^T = \left(\gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t)\right)^T W(x_1, x_2, \ldots, x_n),
\]

where \( W(x_1, x_2, \ldots, x_n) = \exp(-\mu(x_1, x_2, \ldots, x_n))\), \( \mu(a, b) \) is function of measures defined on the set of intervals \( [a \cdot b], a, b \in R^1 \), \( \mu(\alpha, \beta) = 0 \), \( \alpha = \beta \), \( \gamma_1(t), \gamma_2(t), \ldots, \gamma_m(t) \) are amplitude functions, \( \epsilon \) is a parameter that determines the location of disturbance, and \( \bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_m(t) \) are functions that determine the trajectory of perturbation.

Representation (3) makes it possible to find new approach to research the multisoliton solution. In the case of small or large parameter \( \epsilon \) we can investigate some infinitesimal properties of travelling wave solution. Let us consider an example, the well known Korteweg-de Vries equation in the form

\[
u_t + 6\nu \nu_x + \nu_{xxx} = 0. \tag{6}
\]

We can write the derivatives of the function \( u(x, t) \):

\[
u_x = -\frac{g'(x - \bar{x}(t))}{\epsilon} u = \eta u, \\
u_{xx} = \eta_x u + \eta u_x = \eta_\epsilon u + \eta^2 u = (\eta_x + \eta^2) u, \\
u_{xxx} = (\eta_{xx} + 2\eta\eta_x) u + (\eta_x + \eta^2) \eta u = (\eta_{xx} + 3\eta\eta_x + \eta^3) u, \\
u_t = \left(\frac{\gamma'}{\gamma}(t) + \frac{g'(x - \bar{x}(t))}{\epsilon} \bar{x}'(t)\right) u = \left(\frac{\gamma'}{\gamma}(t) - \eta \bar{x}'(t)\right) u.
\]

Substituting derivatives into (5) we obtain the equation

\[
\left(\frac{\gamma'}{\gamma}(t) - \eta \bar{x}'(t)\right) u + 6\nu \nu u + (\eta_{xx} + 3\eta\eta_x + \eta^3) u = 0.
\]

Let \( x - \bar{x}(t) = 0 \). Then, based on the properties of functions \( g(\cdot) \) from (8) we obtain

\[
\frac{\gamma'}{\gamma}(t) u = 0.
\]

Therefore we can formulate proposition.

**Proposition 1.** All travelling wave solutions of the KdV equation have constant amplitude.

From (8) we obtain

\[
-\eta \bar{x}'(t) u + 6\nu \nu u + (\eta_{xx} + 3\eta\eta_x + \eta^3) u = 0. \tag{10}
\]

Equation (10) defines a general condition for functions \( g(\cdot) \in G \) and \( \bar{x}(t) \) at representation (3). Formulate the following statement.

**Proposition 2.** Infinitely wide soliton (2) \( \lim_{x \to -\infty} \nu(x, t) = \gamma \) is a trivial solution of the KdV equation.
This property follows from the analysis of the boundaries of the form
\[ \lim_{\epsilon \to \infty} \left( \frac{g'(x - \bar{x}(t))}{\epsilon} \bar{x}_t(t) - 6 \frac{g'(x - \bar{x}(t))}{\epsilon} u + \left( \frac{-g'''(x - \bar{x}(t))}{\epsilon} \right) \right) u = 0. \]

(11)

Proposition 3. Infinitely narrow soliton (2) \((\lim_{\epsilon \to 0^+} u(x, t) = 0)\) is a solution of the KdV.

This proposition follows from the limits
\[ \lim_{\epsilon \to 0^+} \left( \frac{g'(x - \bar{x}(t))}{\epsilon} \bar{x}_t(t) - 6 \frac{g'(x - \bar{x}(t))}{\epsilon} u + \left( \frac{-g'''(x - \bar{x}(t))}{\epsilon} \right) \right) u = 0. \]

(12)

The last limit follows from these considerations:
\[ \lim_{\epsilon \to 0^+} \lim_{\epsilon \to 0^+} \exp\left(\frac{-y^2}{\epsilon}\right) = \lim_{\epsilon \to 0^+} \exp\left(\frac{-y^2}{\epsilon}\right) = \lim_{\epsilon \to 0^+} \exp\left(\frac{-y^2}{\epsilon}\right) = 0, \]

(13)

\[ \lim_{\epsilon \to 0^+} \lim_{x \to +\infty} \exp\left(\frac{-y^2}{\epsilon}\right) \cdot x = \lim_{x \to +\infty} \exp\left(\frac{-y^2}{\epsilon}\right) = 0. \]

(14)

It is obvious that similar properties have the solutions of other KdV-type equations that admit a trivial solution. At the same time, special KdV with variable coefficients species \(u_t + \alpha(t)u_{tt} + k\alpha(t)u_{xxx} = F(t)\) (given by [15]) does not have the appropriate properties.

Let \(\bar{x}(x, t) = vt\). Then (10) can be expressed in the form
\[ \frac{g'(x - vt)}{\epsilon} v - 6 \frac{g'(x - vt)}{\epsilon} e^{-g(x - vt)/\epsilon} \]
\[ + \left( \frac{-g'''(x - vt)}{\epsilon} + 3 \frac{g''(x - vt)}{\epsilon} g'(x - vt) \right) \]
\[ - \left( \frac{g'(x - vt)}{\epsilon} \right)^3 = 0. \]

(15)

Let \(g(\cdot)/\epsilon = \tilde{g}(\cdot), x - vt = \gamma\). Then we obtain
\[ \tilde{g}''(\gamma) - 3\tilde{g}''(\gamma) \tilde{g}'(\gamma) + (\tilde{g}'(\gamma))^2 - v\tilde{g}'(\gamma) \]
\[ + 6\gamma \tilde{g}'(\gamma) e^{-g(\gamma)} = 0. \]

(16)

Let \(\tilde{g}(\gamma) = p(\gamma)\). Then
\[ \tilde{g}''(\gamma) = pp'; \]
\[ \tilde{g}'''(\gamma) = \left(p''p + (p')^2\right)p; \]
\[ \left(p''p + (p')^2\right) - 3pp' + (p)^2 - v + 6\gamma e^{-g} = 0. \]

Let \(w = p^2\). Therefore
\[ w' = 2pp'; \]
\[ w'' = 2\left(p'\right)^2 + 2pp''; \]
\[ w''' = 4p''p'' + 2p'p'' + 2pp''; \]
\[ \frac{w'''}{2} - \frac{3w'}{2} + w = v - 6\gamma e^{-g}. \]

(18)

We obtain linear equation. The partial solution can be obtained in the form
\[ w = k + le^{-g}; \]
\[ \frac{le^{-g}}{2} + \frac{3le^{-g}}{2} + k + le^{-g} = v - 6\gamma e^{-g}, \]
\[ k = v, l = -2\gamma. \]

(19)

The general solution is \(w = v - 2\gamma e^{-g} + C_1 e^{2\gamma} + C_2 e^{\gamma}\).

Therefore \(p = \pm \sqrt{v - 2\gamma e^{-g} + C_1 e^{2\gamma} + C_2 e^{\gamma}}\) and \(\tilde{g}(\gamma) = \pm \sqrt{v - 2\gamma e^{-g} + C_1 e^{2\gamma} + C_2 e^{\gamma}}\).

We obtain the condition
\[ v - 2\gamma e^{-g} + C_1 e^{2\gamma} + C_2 e^{\gamma} \geq 0, \]
\[ v - 2\gamma + C_1 + C_2 = 0. \]

(20)

Under the condition of (2), we have \(C_1 \geq 0\).
Finally we can define the function $g(y)$ as solution of Koshi problem:

$$g'(y) = \text{sign}(y) \sqrt{\nu - 2\psi e^{-g} + C_1 e^{2g} + C_2 e^g},$$

$$g(0) = 0. \quad (21)$$

The well known solution of KdV equation can be shown in the form

$$u(x,t) = 2\chi^2 ch^{-2}(\chi(x - 4\chi^2 t - \varphi)),$$  

where $\chi, \varphi$ are parameters.

Let us prove that (22) is a partial case of (3). Let $C_1 = C_2 = 0$. Then from (10) we obtain $\nu = 2\psi$. Problem (21) can be shown in the form

$$g'(y) = \sqrt{\nu - ve^{-g}},$$

$$g(0) = 0. \quad (23)$$

Taking into account (22) and (2), we get $2\chi^2 ch^{-2}(\chi(x - 4\chi^2 t - \varphi)) = \nu \exp(-g(x - vt - \varphi))/2$.

Therefore $g(x - vt - \varphi) = 2\ln ch(\chi(x - vt - \varphi))$.

Let $x - vt - \varphi = y$. Then

$$g(y) = 2\ln ch(\chi y). \quad (24)$$

Let $\nu = 4\chi^2$. It is easy to prove that (24) is solution of (23).

Differentiating $g(y)$ we obtain $g'(y) = 2\chi sh(\chi y)/ch(\chi y)$.

Substituting (24) into second part of (23), we get

$$\sqrt{\nu - ve^{-g}} = 2\chi \sqrt{1 - e^{-2\ln ch(\chi y)}} = 2\chi \sqrt{1 - ch^2(\chi y)}$$

$$= 2\chi \sqrt{\frac{(ch^2(\chi y) - 1)}{ch^2(\chi y)}} = 2\chi \sqrt{\frac{sh^2(\chi y)}{ch^2(\chi y)}}$$

$$= \frac{2\chi sh(\chi y)}{ch(\chi y)} = g'(y). \quad (25)$$

The proof is completed.

3. Method for Studying the Interaction of Solitons

Let us consider the double-soliton solution. Let $u_i(x, t) = y_i \exp(-g_i(x - \bar{x}_i(t))/|\epsilon_i|), i = 1, 2$, exact solutions of the KdV equation. Obviously, the function $u_1(x, t) + u_2(x, t)$ is a solution of the KdV equation in the region $\Omega_1 = \{x, t) : x = \bar{x}_1(t) \lor x = \bar{x}_2(t)\}$. Therefore, we consider the following generalization. Let the parameters determining the amplitudes be functions of time, $y_i = y_i(t)$.

Substituting the sum $u_1(x, t) + u_2(x, t)$ into (6), we obtain

$$(y_1') + (u_2)_1 + 6(\nu + u_2) (u_1 + u_2) + u_1 + u_3$$

$$+ u_2 + u_3 = 0,$$

$$y'_1(t) \exp(-g_1(x - \bar{x}_1(t))/\epsilon_1) + g_1'(x - \bar{x}_1(t)) \bar{x}_1''(t)$$

$$\cdot \frac{u_1}{\epsilon_1} + g_1'(x - \bar{x}_1(t)) \frac{\bar{x}_1''(t)}{\epsilon_1} + g_2'(x - \bar{x}_2(t)) \bar{x}_2'(t)$$

$$\cdot \frac{u_2}{\epsilon_2} + 6 \frac{u_1 + u_2}{\epsilon_1}$$

$$+ \left(\frac{\bar{x}_1''(t)}{\epsilon_1} + \frac{\bar{x}_2''(t)}{\epsilon_2}\right) u_2 = 0. \quad (26)$$

Let $d(t) = \bar{x}_2(t) - \bar{x}_1(t)$. If $x = \bar{x}_1(t)$ and $x = \bar{x}_2(t)$ from (26) yield

$$y'_1(t) + y'_2(t) \exp(-g_2(-d(t))/\epsilon_2) + g'_2(-d(t))$$

$$\cdot \bar{x}_2'(t) y_2(t) \exp(-g_2(-d(t))/\epsilon_2) \frac{\bar{x}_2''(t)}{\epsilon_2} + 6 \frac{y_1(t)}{\epsilon_1}$$

$$+ y_2(t) \exp(-g_2(-d(t))/\epsilon_2) \left(-g'_2(-d(t)) \frac{u_2}{\epsilon_2}\right)$$

$$+ \left(-g''_2(-d(t))/\epsilon_2 + 3g'_2(-d(t)) \frac{u_2}{\epsilon_2}\right) \frac{\bar{x}_2''(t)}{\epsilon_2}$$

$$\cdot \frac{u_1}{\epsilon_1} + \left(-g''_2(-d(t))/\epsilon_2 + 3g'_2(-d(t)) \frac{u_2}{\epsilon_2}\right) \frac{\bar{x}_2''(t)}{\epsilon_2}$$

$$= 0,$$

$$y'_1(t) \exp(-g_1(d(t))/\epsilon_1) + y'_2(t) + g'_1(d(t)) \bar{x}_1'(t)$$

$$\cdot y_1(t) \exp(-g_1(d(t))/\epsilon_1) / \epsilon_1$$

$$+ 6 \frac{y_1(t)}{\epsilon_1} \exp(-g_1(d(t))/\epsilon_1) + y_2(t)$$

$$+ 6(\nu + u_2) (u_1 + u_2) (u_1 + u_2) + u_1 + u_3$$

$$+ u_2 + u_3 = 0.$$
\[
\begin{align*}
\gamma_1'(t) &= \left(-g_2'(-d(t))\gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right) / 6 + \left(-g_2''(-d(t))/\varepsilon_2 + 3g_1'(d(t))g_2''(-d(t))/\varepsilon_2^2 \right) \\
&+ \left(g_1'(d(t))/\varepsilon_1 \right) \gamma_2(t) \exp\left(-g_1(d(t))/\varepsilon_1\right) / \varepsilon_1) = 0,
\end{align*}
\]

then from (27) we get

\[
\gamma_1'(t) = \left(-g_2'(-d(t))\gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
\varepsilon_2 - 6 \left(y_1(t) + y_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) \right)
\]

\[
\varepsilon_2y_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
\cdot \left(-g_1'(d(t)) \gamma_1(t) \exp\left(-g_1(d(t))/\varepsilon_1\right) / \varepsilon_1 \right)
\]

\[
\cdot \left(g_1''(d(t))/\varepsilon_1 - 3g_1'(d(t))g_2''(d(t))/\varepsilon_1^2 \right)
\]

\[
\cdot \left(g_1'(d(t))/\varepsilon_1 \right) \gamma_1(t) \exp\left(-g_1(d(t))/\varepsilon_1\right) / \varepsilon_1) \right)
\]

\[
- \exp\left(-g_2(-d(t))/\varepsilon_2\right) / (1 - \exp\left(-g_1(d(t))/\varepsilon_1\right)
\]

\[
y_2'(t) = -g_1'(d(t))\gamma_1(t) \exp\left(-g_1(d(t))/\varepsilon_1\right) / \varepsilon_1 \right)
\]

\[
- 6 \left(\gamma_1(t) + \gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
\cdot \left(-g_2''(-d(t)) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
- \exp\left(-g_1(d(t))/\varepsilon_1\right) \right)
\]

\[
\cdot \left(-g_2(-d(t))/\varepsilon_2 \right)
\]

\[
\cdot \left(g_2''(-d(t))/\varepsilon_2 - 3g_2'(-d(t))g_2''(-d(t))/\varepsilon_2^2 \right)
\]

\[
\cdot \left(g_2'(-d(t))/\varepsilon_2 \right) \gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2
\]

Writing the obvious initial conditions \(y_1(0) = y_1\) and \(y_2(0) = y_2\), we get the Cauchy problem that can be solved with known velocities \(\tilde{x}_1(t), \tilde{x}_2(t)\). Thus, we find exact solutions of the KdV equation in the region \(\Omega_1 = \{(x, t) : x = \tilde{x}_1(t) \vee x = \tilde{x}_2(t)\} \). Let \(\tilde{x}_i(t) = cy_i(t), c_i = 2, i = 1, 2\). For the well known KdV equation solution the rate is equal to the double amplitude. Then from (27) we obtain the Cauchy problem:

\[
x_1(t) = 2y_1(t),
\]

\[
x_2(t) = 2y_2(t),
\]

\[
y_1'(t) = \left(-g_2'(-d(t)) \gamma_1(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
- 6 \left(\gamma_1(t) + \gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
\cdot \left(-g_2''(-d(t)) \gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]

\[
- \gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / (1 - \exp\left(-g_1(d(t))/\varepsilon_1\right)
\]

\[
y_2'(t) = -g_1'(d(t))\gamma_1(t) \exp\left(-g_1(d(t))/\varepsilon_1\right) / \varepsilon_1 \right)
\]

\[
- 6 \left(\gamma_1(t) + \gamma_2(t) \exp\left(-g_2(-d(t))/\varepsilon_2\right) / \varepsilon_2 \right)
\]
In Figure 1 we show a graphical illustration of appropriate solutions.

Obviously, the proposed solution in the form \( u_1(x,t) + u_2(x,t) \) is approximate and the KdV equation is satisfied only in the area \( \Omega_1 \). For further clarification, consider new functions in the forms \( u_1^*(x,t) = y_1^*(t) \exp(-g_1(x - \bar{x}_1(t)) / \epsilon_1) \) and \( u_2^*(x,t) = y_2^*(t) \exp(-g_2(x - \bar{x}_2(t)) / \epsilon_2) \), where \( \alpha, \beta \) are some parameters and inserts to the KdV equation. Let \( \bar{u}_i(x,t) = y_i^*(t) \exp(-g_i(x) / \epsilon_i) \). To simplify record superscript amplitude functions will lower. Using a similar approach, if \( x = \bar{x}_i(t) + \alpha \) we get

\[
\begin{align*}
\gamma_1'(t) \exp(-g_1(\alpha) / \epsilon_1) + & \gamma_2'(t) \exp(-g_2(-d(t)) / \epsilon_2) \\
+ & g_1'(\alpha) \bar{x}_1'(t) \bar{u}_1(\alpha, t) / \epsilon_1 + g_2'(-d(t)) \bar{x}_2'(t) \bar{u}_2(t) / \epsilon_2 \\
\cdot & \exp(-g_2(-d(t)) / \epsilon_2) / \epsilon_2 \\
+ & 6(\gamma_1(t) \exp(-g_1(\alpha) / \epsilon_1) \\
+ & \gamma_2(t) \exp(-g_2(-d(t)) / \epsilon_2)) \\
\cdot & (-g_1'(\alpha) \bar{u}_1(\alpha, t) / \epsilon_1 \\
- & g_2'(-d(t)) \bar{u}_2(-d(t), t) / \epsilon_2) + \left(-g_1''(\alpha) / \epsilon_1 \right. \\
+ & 3g_1'(\alpha) g_1''(\alpha) / \epsilon_1 - \left(g_1'(\alpha)^3 / \epsilon_1^3 \right) \bar{u}_1(\alpha, t) \\
+ & \left( -g_2''(-d(t)) / \epsilon_2 + 3g_2'(-d(t)) g_2''(-d(t)) / \epsilon_2^2 \\
- & \left( g_2'(-d(t))^3 / \epsilon_2^3 \right) \gamma_2(t) \exp(-g_2(-d(t)) / \epsilon_2) \\
= & 0,
\end{align*}
\]

where \( d(t) = \bar{x}_2(t) - \bar{x}_1(t) - \alpha \).

Similarly \( x = \bar{x}_2(t) + \beta \) and \( d_1(t) = \bar{x}_2(t) + \beta - \bar{x}_1(t) \):

\[
\begin{align*}
\gamma_1'(t) \exp(-g_1(d_1(t)) / \epsilon_1) + & \gamma_2'(t) \exp(-g_2(\beta) / \epsilon_2) \\
+ & g_1'(d_1(t)) \bar{x}_1'(t) u_1(d_1(t), t) / \epsilon_1 + g_2'(\beta) \bar{x}_2'(t) u_2(\beta, t) / \epsilon_2 \\
\cdot & u_2(\beta, t) / \epsilon_2 + 6(\gamma_1(t) \exp(-g_1(d_1(t)) / \epsilon_1) \\
+ & \gamma_2(t) \exp(-g_2(\beta) / \epsilon_2)) \\
\cdot & \left(-g_1'(d_1(t)) \gamma_1(t) \exp(-g_1(d_1(t)) / \epsilon_1) / \epsilon_1 \\
- & g_2'(\beta) \bar{u}_2(\beta, t) / \epsilon_2 + \left( -g_1''(\beta) / \epsilon_2 \right. \\
+ & 3g_1'(\beta) g_1''(\beta) / \epsilon_2^2 - \left(g_1'(\beta)^3 / \epsilon_2^3 \right) \bar{u}_2(\beta, t) \\
+ & \left( -g_2''(d_1(t)) / \epsilon_1 + 3g_2'(d_1(t)) g_2''(d_1(t)) / \epsilon_1^2 \\
- & \left( g_1'(d_1(t))^3 / \epsilon_1^3 \right) \gamma_1(t) \exp(-g_1(d_1(t)) / \epsilon_1) \\
= & 0.
\end{align*}
\]
Hence we obtain the system of equations:

\[
\gamma'(t) = \left( -\exp\left( -g_2(\beta)/\varepsilon_2 \right) \left( g_1'(\alpha) \bar{x}_1'(t) \bar{u}_1(\alpha, t) / \varepsilon_1 + g_2'(d(t)) \bar{x}_2'(t) y_2(t) \exp\left( -g_2(-d(t)) / \varepsilon_2 \right) / \varepsilon_2 + 6(\gamma(t)\right)
\]

\[
\cdot \exp\left( -g_1(\alpha) / \varepsilon_1 \right) + y_2(t) \exp\left( -g_2(-d(t)) / \varepsilon_2 \right) \left( -g_1'(\alpha) \bar{u}_1(\alpha, t) / \varepsilon_1 - g_2'(d(t)) \bar{u}_2(\alpha, t) / \varepsilon_2 \right) + \left( -g_2''(\beta) / \varepsilon_2 \right) / \varepsilon_2^2
\]

\[
+ 3g_2'(\beta) / \varepsilon_2 \left( g_2'(\beta) / \varepsilon_2^2 \right) u_2(\beta, t) / \varepsilon_2^2 + \left( -g_2''(\beta) / \varepsilon_2 \right) / \varepsilon_2^2 + 3g_2'(\beta) / \varepsilon_2^2
\]

\[
\cdot \exp\left( -g_2(-d(t)) / \varepsilon_2 \right) / \varepsilon_2^3 \left( g_2'(\beta) / \varepsilon_2^3 \right) \bar{u}_2(\beta, t) / \varepsilon_2^3 + 3g_2'(\beta) / \varepsilon_2^3
\]

\[
\cdot \exp\left( -g_2(-d(t)) / \varepsilon_2 \right) / \varepsilon_2^3 \left( g_2'(\beta) / \varepsilon_2^3 \right) \bar{u}_2(\beta, t) / \varepsilon_2^3 + 3g_2'(\beta) / \varepsilon_2^3
\]

\[
+ g_2'(d(t)) / \varepsilon_2 \left( g_2'(d(t)) / \varepsilon_2^2 - \left( g_2'(d(t)) / \varepsilon_2^3 \right) u_2(\beta, t) / \varepsilon_2^3 \right) y_2(t) \exp\left( -g_2(-d(t)) / \varepsilon_2 \right) / \varepsilon_2^3
\]

Writing the obvious initial conditions, \( y_1(0) = y_1^0 \exp(-g_1(\alpha)/\varepsilon_1) \) and \( y_2(0) = y_2^0 \exp(-g_2(\beta)/\varepsilon_2) \), we get new Cauchy problem and can build the exact solution in the area \( \Omega_{a\beta} = \{(x, t) : x = \bar{x}_1(t) + \alpha, x = \bar{x}_2(t) + \beta\} \).

Besides, we can consider the other laws of wave motion. Suppose

\[
\bar{x}_1(t) = c_1 y_1(t) t + x_1^0
\]

For this case we construct similar Cauchy problem for the amplitude functions and specification. An example of corresponding solutions is shown in Figure 5.

The \( N \)-soliton solutions for any \( N > 2 \) can be always constructed in a similar way (Figure 6).

4. Numerical Results

Let us consider some numerical results for special case of the functions \( g(y) \) and \( u(x, t) \). Let \( C_1 = 0.9, C_2 = -0.901, v = 0.201, \) and \( \gamma = 0.1 \). These parameters satisfied (10). In Figure 1 we plot 2D graphs of the function \( g(y) \), in Figure 2 \( u(x, 50) \), in Figure 3 classical soliton solution (22), and in Figure 4 amplitude functions, solution of system (29).
Let us consider an example of soliton interaction. According to approach given in Section 3, define the functions $g_1(y), g_2(y)$. Let $g_1(y) = 2\ln(\cosh(\sqrt{v_1/4}y))$ and $g_2(y) = 2\ln(\cosh(\sqrt{v_2/4}y))$, $v_1 = 0.3$, $v_2 = 0.1$. Solving problem (29) with initial conditions $\gamma_1(0) = 0.15$, $\gamma_2(0) = 0.05$, $\tilde{x}_1(0) = 0$, and $\tilde{x}_2(0) = 20$, we can construct function $u_1(x,t) + u_2(x,t)$. 
In case (33) we get a similar system of equations and for initial conditions 
\( \gamma_1(0) = 0.15, \gamma_2(0) = 0.05, \tilde{x}_1(0) = 0, \) and 
\( \tilde{x}_2(0) = 30, \) see Figure 7.

5. Conclusions

In this paper, we propose a new technique of finding the PDE’s traveling wave solutions; this technique is based on the T-transformations. Using T-representation method we find a new class of KdV solution and prove that well known solution (22) is a partial case of representation (3).

The proposed method can be applied to find solutions of other differential equations in partial derivatives in the form of solitary waves and can be useful for the investigation of multisoliton solutions. In order to obtain the resultant equation for the amplitudes of perturbations representations (1) or their combinations should be used. We get the resulting equation only for the maximum of perturbation and ignore the exact wave profiles. Wave profile can be found in the simplest cases, in particular for one-soliton dynamics. This approach is effective for equations of shallow water and others.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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