RENORMALIZATION AND $\alpha$-LIMIT SET FOR EXPANDING LORENZ MAPS

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Abstract. We show that there is a bijection between the renormalizations and proper completely invariant closed sets of expanding Lorenz map, which enable us to distinguish periodic and non-periodic renormalizations. Based on the properties of periodic orbit of minimal period, the minimal completely invariant closed set is constructed. Topological characterizations of the renormalizations and $\alpha$-limit sets are obtained via consecutive renormalizations. Some properties of periodic renormalizations are collected in Appendix.

1. Introduction

Lorenz equations is a system of ordinary differential equations in $\mathbb{R}^3$ which has been enormous influential in Dynamics, providing inspiration for the definition of a variety of examples including the geometric models and Hénon maps [31]. The Lorenz maps we study are a simplified model for two-dimensional return maps associated to the flow of the Lorenz equations.

Numerically studies of the Lorenz equations led Lorenz to emphasize the importance of sensitive dependence of initial conditions—an essential factor of unpredictability in many systems. The simulations for an open neighborhood suggest that almost all points in phase space approach to a strange attractor—the Lorenz attractor. Afraimovic, Bykov and Sil’nikov [1] and Guckenheimer and Williams [14] introduced a geometric model that is an abstraction of the numerically-observed features possessed by solution to Lorenz equations. Tucker [29, 30] proved the geometric model is valid, so the Lorenz equations define a geometric Lorenz flow. Luzzatto, Melbourne and Paccaut showed that such a Lorenz attractor is mixing [17].

A Lorenz map on $I = [a, b]$ is an interval map $f : I \to I$ such that for some $c \in (a, b)$ we have

(i) $f$ is strictly increasing on $[a, c)$ and on $(c, b]$;
(ii) $\lim_{x \to c} f(x) = b$, $\lim_{x \to c} f(x) = a$.

If, in addition, $f$ satisfies the topological expanding condition

(iii) The pre-images set $C = \cup_{n \geq 0} f^{-n}(c)$ of $c$ is dense in $I$,
then $f$ is said to be an expanding Lorenz map. [12, 13, 15].

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As was mentioned in [18], maps with discontinuities are extremely natural and important, arising for example in billiards or as return maps for flows with equilibrium points, and very often in modeling and applications. Lorenz map admits a discontinuity $c$. It is convenient to leave $f(c)$ undefined, and regard $c$ as two points, $c+$ and $c-$, $f(c+) = a$ and $f(c-) = b$ from the definition. Lorenz map plays an important role in the study of the global dynamics of families of vector fields near homoclinic bifurcations, see [21, 22, 26, 30, 31] and references therein. The expanding condition follows from [13, 15, 19], which is weaker than many other conditions used in [6, 14, 26] etc.

Renormalization is a central concept in contemporary dynamics. The idea is to study the small-scale structure of a class of dynamical systems by means of a renormalization operator $R$ acting on the systems in this class. This operator is constructed as a rescaled return map, where the specific definition depends essentially on the class of systems. The idea of renormalization for Lorenz map was introduced in studying simplified models of Lorenz attractor, apparently firstly in Primer [23] and Parry [25] (cf. [10]). The renormalization operator in Lorenz map family, is the first return map of the original map to a smaller interval around the discontinuity, rescaled to the original size. Glendinning and Sparrow [13] presented a comprehensive study of the renormalization by investigating the kneading invariants of expanding Lorenz map.

**Definition 1.** A Lorenz map $f : I \to I$ is said to be renormalizable if there is a proper subinterval $[u, v] \ni c$ and integers $\ell, r > 1$ such that the map $g : [u, v] \to [u, v]$ defined by

$$g(x) = \begin{cases} f^\ell(x) & x \in [u, c), \\ f^r(x) & x \in (c, v], \end{cases}$$

is itself a Lorenz map on $[u, v]$. The interval $[u, v]$ is called the renormalization interval.

If $f$ is not renormalizable, it is said to be prime.

The renormalization map $g$ is the first return map of $f$ on the renormalization interval $[u, v]$ (cf. [20]). Let $f$ be a renormalizable Lorenz map. $f$ may have different renormalizations (cf. [13, 20]). A renormalization $g = (f^\ell, f^r)$ of $f$ is said to be minimal if for any other renormalization $(f^{\ell'}, f^{r'})$ of $f$ we have $\ell' \geq \ell$ and $r' \geq r$ (cf. [13, 16, 20, 27] etc.).

It is not an easy problem to determine whether $f$ is renormalizable or not. In fact, it is impossible to check if $f$ is prime or not in finite steps, because $\ell$ and $r$ in (1) may be large.

In this paper we will investigate the renormalization and $\alpha$-limit set of expanding Lorenz map. The non-expanding case is more suitable to state in terms of kneading theory, which is relegated to another paper. The key observation is that one can renormalize expanding Lorenz map via its *proper completely invariant closed set*, which turns out to be an $\alpha$-limit set of some periodic points. For given expanding Lorenz map $f$, there is a one-to-one correspondence between the renormalizations and the proper completely invariant closed sets of $f$ (Theorem A). Then we characterize (Theorem B) the renormalizability of $f$ by constructing the minimal completely invariant closed set $D$, which is just the $\alpha$-limit set of the periodic orbit with minimal period. Since the minimal completely invariant closed set corresponds to the minimal renormalization of $f$, one can define the renormalization operator $R$
on the space of (expanding) Lorenz maps: $Rf$ is the minimal renormalization of $f$. Using the consecutive actions of renormalization operator, we characterize the $\alpha$-limit set of each point (Theorem C).

The paper is organized as follows. We state our main results in Section 2, and establish the correspondence between proper completely invariant closed set and renormalization of expanding Lorenz map in Section 3. In Section 4, we study the renormalizability of expanding Lorenz map by constructing the minimal completely invariant closed set. We characterize the $\alpha$-limit sets via consecutive renormalizations in Section 5. At last, we collect some facts about periodic renormalization in Appendix.

2. Main results

For any nonempty open interval $U \subseteq I$, put

$$N(U) = \min \{n \geq 0 : \exists z \in U \text{ such that } f^n(z) = c\}. \tag{2}$$

By the definition of $N(U)$, we have $c \in f^{N(U)}(U)$, $N(U) \leq N(V)$ if $V \subseteq U$, and

$$N(f^i(U)) = N(U) - i, \quad i = 0, 1, \cdots, N(U). \tag{3}$$

In fact, $N(U)$ is the maximal integer such that $f^{N(U)}$ is continuous on $U$. We can regard $N(U)$ as the index of continuity for the interval $U$. There exists a unique $z \in U$ such that $f^{N(U)}(z) = c$ because $f^{N(U)-1}$ is continuous and strictly increasing on $U$. If $f$ is expanding, $N(U) < \infty$ for all open interval $U$.

$A \subseteq I$, $A'$ represents for the derived set of $A$, that is, the accumulation point set of $A, A'' = (A')', A^n = (A^{n-1})', n = 1, 2, \cdots$. $x \in I$, $\text{orb}(x)$ is the orbit with initial value $x$, $\text{orb}(A) := \bigcup_{x \in A} \text{orb}(x) = \bigcup_{n \geq 0} f^n(A)$.

Recall that a subset $E$ of $I$ is completely invariant under $f$ if $f(E) = f^{-1}(E) = E$,

and it is proper if $E \neq I$.

**Theorem A.** Let $f$ be an expanding Lorenz map. There is a one-to-one correspondence between the renormalizations and proper completely invariant closed sets of $f$. More precisely, suppose $E$ is a proper completely invariant closed set of $f$, put

$$e_- = \sup\{x \in E : x < c\}, \quad e_+ = \inf\{x \in E : x > c\}, \tag{4}$$

and

$$\ell = N((e_-, c)), \quad r = N((c, e_+)). \tag{5}$$

Then

$$f^\ell(e_-) = e_-, \quad f^r(e_+) = e_+$$

and the following map

$$R_E f(x) = \begin{cases} f^\ell(x) & x \in [f^r(c+), c] \\ f^r(x) & x \in (c, f^\ell(c-)] \end{cases} \tag{6}$$

is a renormalization of $f$.

On the other hand, if $g$ is a renormalization of $f$, then there exists a unique proper completely invariant closed set $B$ such that $R_B f = g$. 

A remarkable property of proper completely invariant closed set is illustrated by (5): the two closest points to $c$, from the left and right, are periodic. This property is essential for us to obtain a renormalization. In their study on the renormalization theory of expanding Lorenz map via kneading invariant, Glendinning and Sparrow [13] proposed a combinatorial proof for the existence of such two periodic points.

**Definition 2.** Suppose $E$ is a proper completely invariant closed set of expanding Lorenz map $f$. The renormalization $R_{E}f$ defined by (6) in Theorem A is called the renormalization associated with $E$. And $E$ is called the repelling set associated to the renormalization $R_{E}$. The interval $(e^{-}, e^{+})$, with endpoints $e^{+}$ and $e^{-}$ defined in (4), is called the critical interval of $E$ and $R_{E}$.

**Definition 3.** A renormalization is said to be periodic if the endpoints of its critical interval belong to the same periodic orbit.

The periodic renormalization is interesting because $\beta$-transformation $T_{\beta,\alpha}(x) = \beta x + \alpha \mod 1$, $1 < \beta \leq 2$, $0 \leq \alpha < 1$ can only be periodically renormalized (see [8, 11], and Appendix for details). This kind of renormalization was studied by Alsedà and Falcò [3], Malkin [19]. It was called phase locking renormalization by Alsedà and Falcò in [3] because it appears naturally in Lorenz map whose rotational interval degenerates to a rational point. As we shall see in Theorem B and Theorem C, the periodic renormalization corresponds to completely invariant closed set with isolated points, while non-periodic renormalization corresponds to Cantor set. It is easy to check if the minimal renormalization is periodic or not (see Appendix).

By Theorem A, a possible way to characterize the renormalizability is to look for the minimal completely invariant closed set $D$ of $f$, in the sense that $D \subseteq E$ for each completely invariant closed set $E$ of $f$. If we can find a minimal completely invariant closed set $D$ of $f$, then $f$ is renormalizable if and only if $D \neq I$. The construction of minimal completely invariant closed set seems difficult, because we do not even know whether a Lorenz map always admits such a minimal completely invariant closed set or not.

The construction of minimal completely invariant closed set is closely related to the locally eventually onto (l.e.o.) property of $f$.

**Definition 4.** Let $f$ be an expanding Lorenz map on $I$. A closed set $B \subseteq I$ is said to be locally eventually onto (l.e.o.) under $f$, if for each open interval $U$ with $U \cap B \neq \emptyset$, there exists integer $m$ such that $\bigcup_{i=0}^{m} f^{i}(U) = I$. And $f$ is l.e.o. if $I$ itself is l.e.o. under $f$.

We shall construct the minimal completely invariant closed set for expanding Lorenz map $f$ by choosing some periodic point $p \in I$ and showing that the $\alpha$-limit set of $p$, $\alpha(p)$, is indeed the minimal completely invariant closed set. The periodic orbit with minimal period is important in constructing the minimal completely invariant closed set. It relates naturally to the so called primary cycle which was used to characterize the renormalization of $\beta$-transformation [11]. We begin with the minimal period $\kappa$ of periodic points of expanding Lorenz map $f$ (see Lemma 3). Then we show (see Lemma 5) that the periodic orbit $O$ with minimal period is unique, and the $\kappa$-periodic orbit $O$ is l.e.o. under $f$. Based on the locally eventually onto property of $O$, we can prove that the $\alpha$-limit set of each $\kappa$-periodic point is the minimal completely invariant closed set.
Theorem B. Let \( f \) be an expanding Lorenz map with minimal period \( \kappa \), \( 1 < \kappa < \infty \), \( O \) be the unique \( \kappa \)-periodic orbit, and \( D = \bigcup_{n \geq 0} f^{-n}(O) \). Then we have the following statements:

1. \( D \) is the minimal completely invariant closed set of \( f \).
2. \( f \) is renormalizable if and only if \( D \neq \emptyset \). If \( f \) is renormalizable, then \( R_D \), the renormalization associated to \( D \), is the minimal renormalization of \( f \).
3. The following trichotomy holds:
   - \( D = I \) if and only if \( f \) is prime;
   - \( D = O \) if and only if \( R_D \) is periodic;
   - \( D \) is a Cantor set if and only if \( R_D \) is not periodic.

It is easy to see the cases \( \kappa = 1 \) and \( \kappa = \infty \) are prime. Theorem B describes the renormalizability of expanding Lorenz map completely.

It follows from Theorem B that when \( 1 < \kappa(f) < \infty \), \( f \) is prime if and only if \( D = I \). Since \( O \) is l.e.o. under \( f \) implies \( D = \bigcup_{n \geq 0} f^{-n}(O) \) is l.e.o. under \( f \), \( f \) is prime if and only if \( f \) is l.e.o.

The dynamics of prime expanding Lorenz map \( f \) is well understood: \( f \) is prime if and only if it is l.e.o. (see Corollary 1). The l.e.o. property is an ideal topological property, which implies transitivity.

Glendinning and Sparrow defined the l.e.o. property of \( f \) in more strict sense. Our definition of l.e.o. reduces to their definition when \( \kappa \leq 2 \)(see Proposition 1 in Section 4.3).

According to Theorem B, the minimal renormalization of renormalizable expanding Lorenz map always exists. We can define a renormalization operator \( R \) from the set of renormalizable expanding Lorenz maps to the set of expanding Lorenz maps (cf. [12], [13]). For each renormalizable expanding Lorenz map, we define \( Rf \) to be the minimal renormalization map of \( f \). For \( n > 1 \), \( R^n f = R(R^{n-1}f) \) if \( R^{n-1}f \) is renormalizable. And \( f \) is \( m \) \((0 \leq m \leq \infty) \) times renormalizable if the renormalization process can proceed \( m \) times exactly. For \( 0 < i \leq m \), \( R^i f \) is the \( i \)th renormalization of \( f \). Formally, we denote \( R^0 f := f \) as the 0th renormalization, whose renormalization interval is denoted by \([a_0, b_0] := [a, b] \). The consecutive renormalization process can be used to characterize all the \( \alpha \)-limit sets and obtain a canonical decomposition of the nonwandering set of expanding Lorenz map.

Remember that the \( \alpha \)-limit set \( \alpha(x) \) of a point \( x \in I \) under \( f \) is defined as

\[
\alpha(x) = \bigcap_{n \geq 0} \bigcup_{k \geq n} \{f^{-k}(x)\}.
\]

\( \alpha \)-limit set is important in understanding homoclinic behavior in dynamics. It is often relates to homeomorphism because the inverse \( \{f^{-1}(x)\} \) is only one point. For endomorphism \( f \), the \( \alpha \)-limit set is more difficult to understand than \( \omega \)-limit set in general, because \( \{f^{-k}(x)\} \) is more complex than \( \{f^k(x)\} \). It seems that \( \alpha \)-limit set is "difficult" to describe. But \( f \) may not have too many different \( \alpha \)-limit sets because the \( \alpha \)-limit set is "large" in some sense. We have the following unexpected result.

Theorem C. Let \( f \) be an \( m \) \((0 \leq m \leq \infty) \) renormalizable expanding Lorenz map, \([a_i, b_i] \)(0 \leq i \leq m) be the renormalization interval of the \( i \)th renormalization \( R^i f \), and \( \text{orb}([a_i, b_i]) = \bigcup_{n \geq 0} f^n([a_i, b_i]) \). Then we have:
(1) \( f \) admits \( m \) proper \( \alpha \)-limit sets which can be ordered as
\[
\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I.
\]
(2) \( E_i \) is a Cantor set if the \( i \)th renormalization is not periodic, and \( E'_i = E_{i-1} \)
if the \( i \)th renormalization is periodic.
(3) For \( 0 < i \leq m \), \( \alpha(x) = E_i \) if and only if \( x \in \text{orb}([a_{i-1}, b_{i-1}]) \setminus \text{orb}([a_i, b_i]) \),
and \( \alpha(x) = I \) if and only if \( x \in A := \bigcap_{i=0}^{m} \text{orb}([a_i, b_i]) \).

By Theorem C, we know that expanding Lorenz map admits a cluster of \( \alpha \)-limit sets, and we can determine the \( \alpha \)-limit set of each point. Note that \( A \) is the attractor of \( f \): \( A = I \) if \( m = 0 \), \( A = \text{orb}([a_m, b_m]) \) if \( m < \infty \), and \( A = \bigcap_{i=0}^{\infty} \text{orb}([a_i, b_i]) \) is a Cantor set if \( m = \infty \) (see Theorem D). So \( f \) is prime implies that \( \alpha(x) = I \), \( \forall x \in I \). Since \( I \) is the largest \( \alpha \)-limit set, \( f \) admits exactly \( m + 1 \) different \( \alpha \)-limit sets.

Remember that the depth of \( A \) is the minimal integer \( n \) such that the \( n \)th derived set \( A^{(n)} \) is empty (cf. [5], p. 33). An interesting consequence of Theorem C appears when all the renormalizations of \( f \) are periodic. In this case, Theorem C implies that, the \( i \)th derived set of \( E_k \) is \( E_{k-i} \): \( (E_k)^i = E_{k-i} \) for \( 0 \leq i \leq k \leq m < \infty \). We can construct closed sets with given depth in a dynamical way.

The proof of Theorem C is based on the 1-1 correspondence between the \( \alpha \)-limit sets and completely invariant closed sets: Each \( \alpha \)-limit set is a completely invariant closed set (cf. Lemma 1), and each completely invariant closed set is the \( \alpha \)-limit set for some periodic point (cf. Lemma 6). Using the same ideas, we characterize the \( \alpha \)-limit sets of a unimodal map without homterval in [7].

3. Completely invariant closed set and renormalization

Let \( f \) be an expanding Lorenz map. A set \( E \subseteq I \) is said to be completely invariant under \( f \), if
\[
f(E) = E = f^{-1}(E).
\]
A completely invariant set \( E \) is proper if \( E \neq I \). Since \( f \) is surjective, \( E \) is completely invariant is equivalent to \( E \) is backward and forward invariant \((f^{-1}(E) \subseteq E) \) and \( (f(E) \subseteq E) \).

Lemma 1 collects some useful facts of completely invariant closed set.

**Lemma 1.** Let \( f \) be an expanding Lorenz map.

1. A completely invariant closed set \( E \) is proper if and only if \( c \notin E \);
2. Any proper completely invariant closed set is nowhere dense.
3. The derived set of proper completely invariant closed set is also completely invariant.
4. \( \forall x \in I, \alpha(x) \) is a completely invariant closed set of \( f \).
5. If \( p \) is periodic, then \( \alpha(p) = \bigcup_{n \geq 0} f^{-n}(p) \).
(6) If $E$ is a completely invariant closed set of $f$, then for $A \subseteq I$, we have

\[
  f^{-1}(A \cap E) = f^{-1}(A) \cap E, \quad f(A \cap E) = f(A) \cap E.
\]

Proof. 1, It is necessary to prove $c \in E$ implies that $E = I$. By the invariance of $E$ under $f^{-1}$, $c \in E$ implies that $f^{-n}(c) \subseteq E$. So $\bigcup_{n \geq 0} f^{-n}(c) \subseteq E$, which implies that $E \supseteq \bigcup_{n \geq 0} f^{-n}(c) = I$.

2, If $E$ contains some interval $U$, then $c \in f^{N(U)}(U) \subseteq E$ because $E$ is invariant under $f$, we obtain a contradiction. So $E$ contains no interval.

3, Suppose $E$ is a proper completely invariant closed set. It follows that $c \notin E$. So both $f$ and $f^{-1}$ are continuous at each point of $x \in E$, which implies that $E'$ is backward invariant and forward invariant.

4, $x \in I$, $\alpha(x) = \bigcap_{n \geq 0} \bigcup_{k \geq n} \{f^{-k}(x)\}$. For each $n \in N$, $\bigcup_{k \geq n} \{f^{-k}(x)\}$ is invariant under $f^{-1}$, it follows

\[
  f^{-1}(\alpha(x)) \subseteq \alpha(x).
\]

Remember $y \in \alpha(x)$ is equivalent to the fact that there exists a sequence $\{x_k\} \subseteq I$ and an increasing sequence $\{n_k\} \subseteq N$ such that $f^{n_k}(x_k) = x$ and $x_k \to y$ as $k \to \infty$. Assume $y \in \alpha(x)$, we have $f(y) \in \alpha(x)$ if $y$ is not the discontinuity $c$. If $y = c$ we consider $c$ as two points $c^+$ and $c^-$. It is easy to see $c \in \alpha(x)$ implies $f(c^+) = 0 \in \alpha(x)$, and $c^- \in \alpha(x)$ implies $f(c^-) = 1 \in \alpha(x)$. So we conclude

\[
  f(\alpha(x)) \subseteq \alpha(x).
\]

5, If $p$ is periodic with period $m$, then $p \in f^{-km}(p)$ for all $k \in N$, which implies that $p \in \alpha(p)$. Since $\alpha(p)$ is completely invariant, we know that $f^{-n}(p) \subseteq \alpha(p)$. We have $\alpha(p) \subseteq \bigcup_{n \geq 0} f^{-n}(p)$. The converse inclusion $\alpha(p) \subseteq \bigcup_{n \geq 0} f^{-n}(p)$ is trivial.

6, Since $E$ is completely invariant, it follows $f^{-1}(A \cap E) = f^{-1}(A) \cap f^{-1}(E) = f^{-1}(A) \cap E$. The first equality holds.

The inclusion $f(A \cap E) \subseteq f(A) \cap E$ is trivial. To prove the converse inclusion, suppose $x \in f(A) \cap E$, $x \in f(A)$ implies that $f(y) = x$ for some $y \in A$. $x \in E$ implies $\{f^{-1}(x)\} \subseteq E$. As a result, one gets $y \in E$. Hence, $y \in A \cap E$, which implies $f(A) \cap E \subseteq f(A \cap E)$. The second equality follows.

For expanding Lorenz map $f$, Lemma[1] indicates that each completely invariant closed set containing $c$ is trivial. It is possible that all the completely invariant closed set of $f$ is trivial. If this is the case, $f$ is prime because $\alpha(x) = I$ for all $x \in I$.

Lemma 2. Let $f$ be an expanding Lorenz map, $E$ be a proper completely invariant closed set of $f$, $J_E = (c_-, e_+)$ be the critical interval of $E$. $N((e_-, c)) = t$, $N((c, e_+)) = r$, $[u, v] = [f^t(c^+), f^t(c^-)]$. Then

\[
  I \setminus E = \bigcup_{n \geq 0} f^{-n}(J_E) = \bigcup_{n \geq 0} f^{-n}([u, v]).
\]

Proof. Since $E$ is completely invariant, we have $E \cap \bigcup_{n \geq 0} f^{-n}(J_E) = \emptyset$, which indicates $\bigcup_{n \geq 0} f^{-n}(J_E) \subseteq I \setminus E$.

$x \in I \setminus E$, there exists an open interval $U$ such that $x \in U \subset I \setminus E$ because $I \setminus E$ is open. Furthermore, we can assume that $U$ is the maximal open interval containing $x$ which belongs to $I \setminus E$. Since $f$ is expanding, $N(U) < \infty$, and $c \in f^{N(U)}(U)$. It follows that $f^{N(U)}(U) \subseteq J_E$ because $f^{N(U)}(U) \cap E = \emptyset$. The maximality of $U$
indicates that \( f^{N(U)}(U) = J_E \). As a result, \( f^{N(U)}(x) \in J_E \), i.e., \( x \in f^{-N(U)}(J_E) \).

Hence, \( I \setminus E \subseteq \bigcup_{n \geq 0} f^{-n}(J_E) \). We have proved \( I \setminus E = \bigcup_{n \geq 0} f^{-n}(J_E) \).

Since \( E \) is a completely invariant closed set, we have \((u, v) \subseteq J_E \). It follows that \( \bigcup_{n \geq 0} f^{-n}(J_E) \supseteq \bigcup_{n \geq 0} f^{-n}((u, v)) \).

\( \forall x \in (e_-, c) \), put \( \ell_x = N((e_-, x)) \). We get \( f^{\ell_x}(x) \in (c, v) \) by the complete invariance of \( E \). So we conclude \((e_-, c) \subseteq \bigcup_{n \geq 0} f^{-n}((u, v)) \). By the same argument, we can obtain \((c, e_+) \subseteq \bigcup_{n \geq 0} f^{-n}((u, v)) \). So \( J_E = (e_-, e_+) \subseteq \bigcup_{n \geq 0} f^{-n}((u, v)) \). Since \( \bigcup_{n \geq 0} f^{-n}((u, v)) \) is backward invariant, we have
\[
\bigcup_{n \geq 0} f^{-n}(J_E) \subseteq \bigcup_{n \geq 0} f^{-n}((u, v)).
\]

This completes the proof of \( \Box \).

3.1. Proof of Theorem A. Suppose \( E \) is a proper completely invariant closed set of \( f \). \( e_+, e_-, \ell \) and \( r \) are defined as in the statement of Theorem A. By Lemma \[ e_-, e_+, \ell \) and \( r \) are well defined and \( e_- < c < e_+ \).

At first, we prove \( f^\ell(e_-) = e_- \).

By the definition of \( \ell \), \( f^\ell \) is continuous and monotone on \((e_-, c) \). Put \( z \) be the unique point in \((e_-, c) \) such that \( f^\ell(z) = c \). Since \( E \) is completely invariant, we conclude that \( f^\ell(e_-) = c \). In fact, if \( f^\ell(e_-) > z \) and then \( e_- < f^\ell(e_-) < f^\ell(c) = c \), which contradicts to the maximality of \( e_- \) because \( f^\ell(e_-) \in E \cap (e_-, c) \). If \( f^\ell(e_-) < c \) and \( e_- \), there must be some point \( y \in (e_-, c) \) such that \( f^\ell(y) = e_- \), which contradicts also to the maximality of \( e_- \) and the complete invariance of \( E \) under \( f \).

Similarly, we can prove \( f^\ell(e_+) = e_+ \).

Since \( E \) is completely invariant, we conclude
\[
f^\ell((e_-, c)) = (e_-, f^\ell(c-)) \subseteq (e_-, e_+).
\]

If, on the contrary, \( f^\ell(c-) > c_+ \), then there exists \( z \in (e_-, c) \) such that \( f^\ell(z) = e_+ \), which implies \( z \in E \) because \( E \) is completely invariant. This contradicts to the minimality of \( e_+ \).

Similarly,
\[
f^\ell((c, e_+)) \subseteq (e_-, e_+).
\]

It follows that the map \( R_E f \) defined in Theorem A is a renormalization of \( f \).

Now we prove the second statement. Suppose \( g = (f^m, f^n) \) is a renormalization map of \( f \) with renormalization interval \([u, v] := [f^k(c_+), f^m(c_-)] \). Put
\[
F_g = \{ x \in I, \text{orb}(x) \cap (u, v) \neq \emptyset \},
\]
\[
J_g = \{ x \in I, \text{orb}(x) \cap (u, v) = \emptyset \}.
\]

Since \( F_g = \bigcup_{n \geq 0} f^{-n}((u, v)) \), \( F_g \) is a completely invariant open set. And \( J_g = I \setminus F_g \) is a completely invariant closed set of \( f \). \( R_{J_g} = g \) follows from Lemma \[2 \).

The proof of Theorem A is complete.

\[ \Box \]

4. Minimal completely invariant closed set

Applying Theorem A, the renormalizability problem of expanding Lorenz map reduces to check whether it admits a proper completely invariant closed set. In this section, we shall construct the minimal completely invariant closed set of \( f \). We begin with the minimal period of the periodic orbits of \( f \), and show that the periodic orbit \( O \) with minimal period of \( f \) is unique. Then we conclude that periodic orbit
O has the locally eventually onto (l.e.o.) property, which enables us to show that the \(\alpha\)-limit set \(D := \alpha(O) = \bigcup_{n \geq 0} f^{-n}(O)\) is the minimal completely invariant closed set of \(f\). By Theorem A, \(f\) is renormalizable if and only if \(D \neq \emptyset\). Based on the structure of \(D\), we can prove Theorem B. Using Theorem B, we can obtain two Propositions about the l.e.o. property and trivial renormalization of \(f\).

4.1. Periodic orbit with minimal period.

In this subsection, we will show that the periodic orbit with minimal period is very special because it relatives to the minimal completely invariant closed set.

The period of periodic points of Lorenz map was well studied by Alsedà et al in [4]. It is shown that a Lorenz map is asymptotically periodic if and only if the derived set \(C'(f)\) of \(C(f) = \bigcup_{n \geq 0} f^{-n}(c)\) is countable [9]. The following Lemma determines the minimal period \(\kappa\) of periodic points of expanding \(f\) via the preimages of \(c\).

Lemma 3. Suppose \(f\) is an expanding Lorenz map on \([a, b]\) without fixed point. The minimal period of \(f\) is equal to \(\kappa = m + 2\), where

\[
m = \min \{i \geq 0 : f^{-i}(c) \in [f(a), f(b)]\}.
\]

Proof. We prove the result by two steps: we first prove that \(f\) has \((m + 2)\)-periodic point, then we show that \(f\) has no periodic point with period less than \(m + 2\).

Notice that \(f^{-m}(c) \in [f(a), f(b)]\) and \(x\) admits two preimages if and only if \(x \in [f(a), f(b)]\). Let \(c_{m+1}\) and \(c'_{m+1}\) with \(c_{m+1} < c'_{m+1}\) be the two preimages of \(f^{-m}(c)\). The set \(f^{-i}(c)\) for \(i = 0, 1, \ldots, m\) is a singleton. Denote \(c_i := f^{-i}(c)\), \(i = 0, 1, \ldots, m\). Let \(Q_1 \in (a, c)\) and \(Q_2 \in (c, b)\) be the points such that \(f(Q_1) = f(b)\) and \(f(Q_2) = f(a)\). See Figure 1 for an intuitive picture of \(m = 2\).

Since \(m\) is the smallest integer such that \(f^{-m}(c) \in [f(a), f(b)]\), we have

\[
c_{m+1} \leq Q_1 < c_i < Q_2 \leq c'_{m+1} \quad (0 \leq i \leq m).
\]

Let \(c_m\) be the minimal point in \(\{c, c_1, \ldots, c_m\}\). For interval \([c_{m+1}, c_m]\), by (11), we obtain that

\[
[c_{m+1}, c_m] \xrightarrow{f_0} [c_{m+1-i_0}, c] \xrightarrow{f} [c_{m-i_0}, b] \supseteq [c_{m-i_0}, c'_{m+1}]
\]

\[
[c_{m-i_0}, c'_{m+1}] \xrightarrow{f^{-i_0}} [c, c_{i_0+1}] \xrightarrow{f} [a, c_{i_0}] \supseteq [c_{m+1}, c_m],
\]

which implies that

\[
[c_{m+1}, c_m] \subseteq f^{m+2}([c_{m+1}, c_m]).
\]

So, \(f\) has an \(m + 2\)-periodic point in \([c_{m+1}, c_m]\).

Fix \(1 < j < m + 2\). We shall prove that \(f\) admits no \(j\)-periodic point. Put \(c_1 = \min\{c, c_1, \ldots, c_m\}\), \(c_r = \max\{c, c_1, \ldots, c_m\}\).

Claim: \(f\) can not have \(j\)-periodic points in \((a, c_\ell)\) and \((c_r, b)\).

By the selection of \(m\), we get \(N((a, c_\ell)) > m\). So \(f^j\) is continuous and monotone on \((a, c_\ell)\). It is easy to see \(f^j(a) > a\). If \(f^j\) admits a fixed point \(x^*\) in \((a, c_\ell)\), then

\[
a < f^j(a) < f^{2j}(a) < \cdots < f^{nj}(a) < x^*, \quad n > 0.
\]

So \(\{f^{nj}(a)\}_n\) approaches to a fixed point of \(f^j\) as \(n \to \infty\), which is impossible because expanding Lorenz map does not admits attractive periodic orbit.

Similarly, if \(f^j\) admits a fixed point in \((c_\ell, b)\), then \(\{f^{nj}(b)\}_n\) will converge to a fixed point of \(f^j\), which contradicts to \(f\) is expanding.
Now, for any open interval $J$ with both endpoints in $\{c, c_1, \ldots, c_m\}$ and $J \cap \{c, c_1, \ldots, c_m\} = \emptyset$, we know that $N(J) > m$, and at least one of the following cases hold:

- $f^j(J) \cap J = \emptyset$;
- $f^i(J) \subseteq ((a, c_\ell) \cup (c_r, b))$ for some $1 < i \leq j$.

It follows that $f$ admits no $j$-periodic point in $J$. \qed

**Remark 1.** For $m$ defined in (10), it is interesting to note when $f^{-m}(c)$ is happen to be one of the endpoints of $[f(a), f(b)]$. If $f^{-m}(c) = f(a)$, then $c_+$, as well as $a$, is a periodic point with period $m + 2$. If $f^{-m}(c) = f(b)$, then $f^{m+2}(c) = c$.

Let $P_L$ be the largest $\kappa$-periodic point in $[a, c)$ and $P_R$ be the smallest $\kappa$-periodic point in $(c, b]$.

**Lemma 4.** Put $L_1 = (P_L, c)$, $R_1 = (c, P_R)$. We have

$$N(L_1) = N(R_1) = \kappa.$$  \hspace{1cm} (12)

*Proof.* We only prove $N(L_1) = \kappa$ by showing both $N(L_1) < \kappa$ and $N(L_1) > \kappa$ are impossible.

Suppose that $N(L_1) < \kappa$. We have $f^{N(L_1)}(P_L) \neq P_L$. By the definition of $N(L_1)$, there exists $z \in L_1$ such that $f^{N(L_1)}(z) = c$. Since $P_L$ is the largest $\kappa$-periodic point of $f$ in $[0, c)$ and $f^{N(L_1)}(P_L)$ is a $\kappa$-periodic point, we must have

$$f^{N(L_1)}(P_L) < P_L.$$
\(N(L_1) \leq \kappa\) implies that \(f^{N(L_1)}(L)\) is increasing on \([P_L, c]\). For the interval \((P_L, z)\), it follows that

\[
f^{N(L_1)}(L) = [f^{N(L_1)}(P_L), c) \supseteq [P_L, z].
\]

So there exists \(P_* \in (P_L, z)\) such that \(f^{N(L_1)}(P_*) = P_*\) by the continuity of \(f^{N(L_1)}\) on \((P_L, z)\). Hence \(P_*\) is a periodic point of \(f\) with period \(N(L_1) \leq \kappa\), which contradicts to the minimality of \(L\).

Assume \(N(L_1) > \kappa\). It follows from \((2)\) that \(f^c\) is continuous and increasing on \(L_1 = [P_L, c]\). We have to exclude two cases: \(f^c(c-) > c\) and \(f^c(c-) < c\), which imply that \(N(L_1) > \kappa\) is also impossible.

If \(f^c(c-) > c\), there exists \(z \in (P_L, c) = L_1\) such \(f^c(z) = c\), which contradicts to the minimality of \(N(L_1)\).

If \(f^c(c-) < c\), by the monotone property of \(f^c\) on \([P_L, c]\), we obtain a decreasing sequence \(\{f^{\alpha n}(c-)\}\) with lower bound \(P_L\). Hence, \(f^{-n}(c) \cap [P_L, c] = \emptyset\), which contradicts to the fact that \(f\) is expanding.

\[\square\]

**Lemma 5.** Suppose that \(f\) is an expanding Lorenz map, and \(1 < \kappa < \infty\) is the smallest period of the periodic points of \(f\). Then

\(i)\) \(f\) admits a unique \(\kappa\)-periodic orbit;

\(ii)\) We have

\[
\bigcup_{i=0}^{\kappa-1} f^i([P_L, P_R]) = I;
\]

\(iii)\) For any open interval \(U\) containing a \(\kappa\) periodic point, there exists positive integer \(n\) such that

\[
\bigcup_{i=0}^{n} f^i(U) = I.
\]

**Proof.** \(i\) Suppose that \(f\) has two distinct \(\kappa\)-periodic orbits \(\text{orb}(P_L)\) and \(\text{orb}(Q_L)\), where \(P_L\) and \(Q_L\) are the maximal points in \(L\) of these two periodic orbits respectively. Without loss of generality, we can suppose \(P_L\) is the largest \(\kappa\)-periodic point in \(L\). Put \(L_1 = (P_L, c)\) and \(L_2 = (Q_L, c)\).

By Lemma 3 \(L_1 \subset L_2\) we know that \(N(L_2) \leq N(L_1) = \kappa\). If \(N(L_2) = \kappa_1 < \kappa\), there exists a point \(z \in (Q_L, P_L)\) such that \(f^{\kappa_1}(z) = c\). Since \(f^{\kappa_1}(Q_L) < c\), it follows that \(f^{\kappa_1}(Q_L) \leq Q_L\) according to the choice of \(Q_L\). So we have

\[
f^{\kappa_1}(Q_L, z) = (f^{\kappa_1}(Q_L), c) \supset (Q_L, z),
\]

which implies that \(f\) admits an \(\kappa_1\)-periodic point in \((Q_L, P_L)\). We obtain a contradiction because \(\kappa\) is the minimal period of periodic points. So we conclude that \(N(L_2) = \kappa\) and \(f^\kappa\) is continuous on \(L_2\).

Consider the action of \(f^\kappa\) on the interval \([Q_L, P_L]\), we have

\[
(f^{\kappa n}(Q_L, P_L) = [Q_L, P_L],
\]

which contradicts to the fact \(f\) is expanding.

So \(f\) admits a unique \(\kappa\)-periodic orbit.

\(ii)\) By the proof of \(i)\), we get \(f^\kappa([P_L, c]) \supset [P_L, c]\) and \(f^\kappa((c, P_R]) \supset (c, P_R]\). Observe that

\[
f([P_L, c]) = [f(P_L), b), \quad f^2([P_L, c]) = [f^2(P_L), f(b))
\]
Assume that set \( D \) prove that

\[
\kappa
\]

Since \( f \) is expanding implies \( f(a) \leq f(b) \), we conclude

\[
\frac{d}{dt} \left( f^2(P_L), P_R \right) \supseteq \left[ f^2(P_L), f^2(P_R) \right].
\]

It follows

\[
f^i([P_L, P_R]) \supseteq \left[ f^i(P_L), f^i(P_R) \right] \quad \text{for} \quad i = 2, \cdots, \kappa.
\]

Hence,

\[
\bigcup_{i=0}^{\kappa-1} f^i([P_L, P_R]) = I.
\]

iii) Put \( U = (x, y) \). Without loss of generality, we only consider the case \( P_L \in U \) because some iterates of \( U \) contains \( P_L \). Let \( N((P_L, y)) = i \), \( N((x, P_L)) = j \). Since

\[
f^i([P_L, y]) \supseteq [P_L, c], \quad f^j((x, P_L)) \supseteq (c, P_R).
\]

The conclusion follows from ii).

The third statement in the Lemma \([\text{III}]\) implies that the \( \kappa \)-periodic orbit \( O \) is l.c.e.o. under \( f \). This result holds trivially for \( D = \bigcup_{n \geq 0} f^{-n}(O) \). This is why the \( \alpha \)-limit set \( D \) of the \( \kappa \)-periodic orbit is so important in describing the renormalization of expanding Lorenz map.

4.2. Proof of Theorem B. According to Lemma \([\text{III}]\) \( f \) admits a unique periodic orbit \( O \) with period \( \kappa \). We denote \( D := \alpha(O) = \bigcup_{n \geq 0} f^{-n}(O) \) as the \( \alpha \)-limit set of the \( \kappa \)-periodic orbit of \( f \).

(1) By Lemma \([\text{II}]\) we know that \( D \) is a completely invariant closed set. We shall prove that \( D \) is minimal.

Suppose \( E \) is a completely invariant closed set. We have two cases:

Case 1: \( E \cap (P_L, P_R) \neq \emptyset \).

In this case, we can suppose that \( (P_L, c) \cap E \neq \emptyset \) without loss of generality. Assume that \( y \in (P_L, c) \cap E \). By Lemma \([\text{I}]\) we know that \( f^\kappa \) is continuous on \( (P_L, c) \) and \( f^\kappa((P_L, c)) \supseteq (P_L, c) \). So there exists \( y_1 \in (P_L, c) \cap E \) and \( y_1 < y \) such that \( f^\kappa(y_1) = y \). Similarly, we can obtain a decreasing sequence \( \{y_n\} \subset (P_L, c) \cap E \) such that \( f^k(y_{n+1}) = y_n, n = 1, 2, \cdots \) and \( \lim_{n \to \infty} y_n = P_L \). So \( P_L \in E \) because \( E \) is closed. Hence, \( \bigcup_{n \geq 0} f^{-n}(O) \subset E \) because \( E \) is backward invariant. \( E \) is closed implies that \( D \subseteq E \).

Case 2: \( E \cap (P_L, P_R) = \emptyset \).

By Lemma \([\text{I}]\) ii) we know that \( \bigcup_{i=0}^{\kappa-1} f^i([P_L, P_R]) = I \). So \( [P_L, P_R] \cap E \neq \emptyset \). The assumption \( E \cap (P_L, P_R) = \emptyset \) indicates \( [P_L, P_R] \cap E = \{P_L, P_R\} \), which implies that \( D = E = O \).

The proof of the minimality of \( D \) is complete.

(2) By Theorem A, \( f \) is renormalizable if and only if \( f \) admits a proper completely invariant closed set. Since \( D \) is the minimal completely invariant closed set, we know that \( f \) is renormalizable is equivalent to \( D \neq I \).

If \( D \neq I \), according to Theorem A, we know that \( R_D \) is a renormalization of \( f \), where

\[
R_D f(x) = \begin{cases} 
  f^\ell(x) & x \in [f^\ell(c+), c) \\
  f^r(x) & x \in (c, f^r(c-)].
\end{cases}
\]
and
\[
\ell = N([d_-, c)) \quad d_- = \sup\{x < c : x \in D\}, \\
r = N((c, d_+)) \quad d_+ = \inf\{x > c : x \in D\}.
\]

Assume \(g = (f^\ell, f^r)\) is a renormalization of \(f\) with renormalization interval \([u, v]\). By Theorem A there exists a completely invariant closed set
\[
E = \{x \in I : orb(x) \cap (u, v) = \emptyset\}
\]
such that \(g = R_E\), and
\[
\ell' = N((e_-, c)) \quad e_- = \sup\{x < c : x \in E\}, \\
r' = N((c, e_+)) \quad e_+ = \inf\{x > c : x \in E\}.
\]

The minimality of \(D\) indicates \(d_- \leq e_- < c < e_+ \leq d_+\), which implies that \(\ell \leq \ell' \) and \(r \leq r'\).

So \(R_D\) is the minimal renormalization.

(3) In order to describe the structure of \(D\), we can consider the following three cases, which cover all possible cases.

- **Case A:** \(c \in D\),
- **Case B:** \(c \notin D\) and \(D \cap (P_L, P_R) = \emptyset\),
- **Case C:** \(c \notin D\) and \(D \cap (P_L, P_R) \neq \emptyset\).

**Case A:** If \(c \in D\), the complete invariancy of \(D\), together with Lemma 1 implies \(D = I\), which is equivalent to \(f\) is prime.

**Case B:** If \(c \notin D\) and \(D \cap (P_L, P_R) = \emptyset\), it follows from the proof of Case 2 that \(D = O\). In this case, one can check easily that \(d_- = P_L\) and \(d_+ = P_R\) in the definition of \(R_D\). By Lemma 4 we know that \(N((P_L, c)) = N((c, P_R)) = \kappa\). It follows \(R_D\) is periodic.

Conversely, assume that the minimal renormalization \(R_D\) is periodic. Follows from the definition of \(R_D\), we know that the renormalization interval of \(R_D\) is \((f^\kappa(c+), f^\kappa(c-)) \subseteq (P_L, P_R)\) and the critical interval of \(R_D\) is \((P_L, P_R)\). Consider the critical interval \((P_L, P_R)\), it follows \(D \cap (P_L, P_R) = \emptyset\). So we get \(D = O\) as in Case 2 of the proof of (1).

**Case C:** If \(c \notin D\) and \(D \cap (P_L, P_R) \neq \emptyset\), it is necessary to prove \(D\) is a Cantor set, the equivalence between \(D\) is a Cantor set and \(R_D\) is not periodic is obvious.

By Lemma 1 \(c \notin D\) implies \(D\) is nowhere dense. Now we show that \(D\) is perfect, i.e., \(D = D'\).

Since \(D = O\) is equivalent to
\[
f^\kappa((P_L, P_R)) = (P_L, P_R),
\]
\(D \cap (P_L, P_R) \neq \emptyset\) implies \(f^\kappa((P_L, P_R)) = (P_L, P_R)\) is not true. Without loss of generality, we can suppose that \(f^\kappa(c+) < P_L\). Then there exists \(y_1 \in (c, P_R)\) such that \(f^\kappa(y_1) = P_L\). And there exists \(y_2 \in (c, P_R)\) and \(y_2 > y_1\) such that \(f^\kappa(y_2) = y_1\), i.e., \(f^{2\kappa}(y_2) = P_L\).

Repeat the above arguments, we can obtain an increasing sequence \(\{y_n\}\) in \((c, P_R)\) such that \(f^{n\kappa}(y_n) = P_L\) and \(y_n \to P_R\) as \(n \to \infty\). Since \(\{y_n\}\) are preimages of \(P_L\), we know that \(\{y_n\} \subset D\). It follows \(P_R\) is a limit point of \(D\), i.e., \(P_R \in D'\). By Lemma 1 we know that \(D'\) is backward invariant, so \(\cup_{n \geq 0} f^{-n}(P_R) \subset D'\). Therefore, \(D \subset D'\), \(D\) is perfect.

The proof of Theorem B is complete.
4.3. The locally eventually onto property. By Definition 4 a Lorenz map is locally eventually onto (l.e.o) if for any open interval $U$, there exists positive integer $n$ depending on $U$, such that $\bigcup_{i=0}^{n} f^i(U) = I$.

Corollary 1. Let $f$ be an expanding Lorenz map on $[a, b]$ with $\kappa < \infty$. Then $f$ is l.e.o. if and only if it is prime.

Proof. If $1 < \kappa(f) < \infty$, by Lemma 5 $f$ is l.e.o. if and only if $D = I$. By Theorem B, $f$ is prime if and only if $D = I$. So $f$ is l.e.o. is equivalent to it is prime.

If $\kappa(f) = 1$, it is easy to see that $f$ is prime, and $f$ is l.e.o., because either $a$ or $b$ is fixed.

Glendinning and Sparrow described the locally eventually onto (l.e.o.) property as follows: $f$ is said to be l.e.o. if for each open interval $U$, there exists subintervals $U_1, U_2$ of $U$, and positive integers $n_1, n_2$ such that $f^{n_1}$ and $f^{n_2}$ map $U_1$ and $U_2$ homeomorphically to $(a, c)$ and $(c, b)$, respectively. The following proposition relates two definitions of l.e.o.

Proposition 1. Our definition of l.e.o. coincides with which of Glendinning and Sparrow in [13] when $\kappa \leq 2$.

Proof. It is necessary to show that our definition of l.e.o. reduces to Glendinning and Sparrow’s definition when $\kappa \leq 2$. The converse is trivial.

Now suppose $f$ is prime and $\kappa \leq 2$. There are two cases: $\kappa = 1$ and $\kappa = 2$.

If $\kappa = 1$, at least one of the following holds:

$$f(a) = a \quad \text{and} \quad f(b) = b.$$ 

Without loss of generality, we suppose $f(a) = a$. For any open interval $U = (x, y)$, let $z_0$ be the point in $U$ such that $f^{N(U)}(z_0) = c$, and $z_1$ be the point in $(z_0, y) \subset U$ such that $f^{N((z_0, y))}(z_1) = c$. By the definition of Lorenz map and $f(a) = a$ we obtain

$$f^{N((z_0, y)) + 1}(z_0, z_1) = (a, b).$$

So there exists positive integers $n$ and a subinterval $V \subseteq U$ such that $f^n$ maps $V$ to $(a, b)$ homeomorphically, which implies that $f$ is locally eventually onto.

For the case $\kappa = 2$. Suppose $f$ is prime, let $P_L < c < P_R$ be the 2–periodic points. By Lemma 4 we know that $N((P_L, c)) = N((c, P_R)) = 2$, so $N((P_R, b)) = 1$ because $f((P_L, c)) = (P_R, b)$. Let $x_1$ be the point in $(P_R, b)$ such that $f^2(x_1) = c$, $y_1$ be the point in $(P_R, b)$ such that $f(y_1) = c$. Consider the interval $J_1 = (x_1, y_1)$, one can check that $f^2(J_1) = (c, b) \supset J_1$. There exists an subinterval $J_2 \subset J_1$ so that $f^2(J_2) = J_1$. So we can obtain a sequence of nested intervals $\{J_n\}$, $J_n = (x_n, y_n)$ satisfy:

$$J_{n+1} \subset J_n, \quad f^2(J_{n+1}) = J_n, \quad f^{2n}(J_n) = (c, b), \quad n = 1, 2, \ldots .$$

Since $\{x_n\}$ and $\{y_n\}$ are monotone and $f$ is expanding, the length of $|J_n| \to 0$ as $n \to \infty$.

Now we prove that $f$ is l.e.o. in the sense of Glendinning and Sparrow. It is necessary to check the l.e.o. conditions for intervals containing $P_R$, because $f$ is prime implies that any open interval contains a subinterval which can be mapped homeomorphically to an open interval containing $P_R$. For any open interval $F$ containing $P_R$, we can find subinterval $J_i \subset F$, which implies that $f^{2i}$ maps $J_i$ homeomorphically to $(c, b)$ by the construction of $\{J_n\}$. Furthermore, $(c, b)$
contains an interval $(c, y_1)$, which can be mapped by $f$ homeomorphically to $(a, c)$.
So $J_i$ contains a subinterval $(x_i, z_i)$ such that $f^{2i+1}(x_i, z_i) = (a, c)$. Hence, $f$ is l.e.o. in the sense of Glendinning and Sparrow.

\[ \square \]

Remark 2. The exact formulation of l.e.o. varies in the literatures. For the definition we use, we mention the following:

1. The l.e.o. property is just the strongly transitive property in Parry [24].
2. It agrees with the one in [13], when $\kappa \leq 2$;
3. $f$ is prime if and only if $f$ is l.e.o.;
4. The l.e.o. property of expanding Lorenz map comes from the l.e.o. property of the periodic orbit with minimal period. According to Lemma 6 for expanding Lorenz map $f$, the minimal completely invariant closed set $D$ of $f$ admits the l.e.o. property: for each open interval $U$ satisfying $U \cap D \neq \emptyset$, there exists integer $n > 0$ so that $\bigcup_{i=0}^{n} f^i(U) = I$. As a result, $f$ is l.e.o. if and only if $D = I$.

5. Consecutive renormalizations: characterization of $\alpha$-limit set

Thanks to Theorem B, the minimal renormalization of renormalizable expanding Lorenz map always exists. We can define a renormalization operator $R$ from the set of renormalizable expanding Lorenz maps to the set of expanding Lorenz maps. For each renormalization expanding Lorenz map, $Rf := R_D f$, where $D$ is the minimal proper completely invariant closed set of $f$. Obviously, $Rf$ is also expanding. If $Rf$ is renormalizable, we can obtain $R^2 f := R(Rf)$. In this way, we define $R^n f$ as the minimal renormalization of $R^{n-1} f$ if $R^{n-1} f$ is renormalizable. If the renormalization process can proceed $m$ times, we say that $f$ is $m (0 \leq m \leq \infty)$ times renormalizable. If $f$ is $m$-renormalizable, then $\{R^i f\}_{i=1}^m$ are all the renormalizations of $f$. We call $R^i f$ the $i$th renormalization of $f$. The process of consecutive renormalizations can be used to characterize all the $\alpha$-limit sets and nonwandering set of expanding Lorenz map.

5.1. $\alpha$-limit set.

Lemma 6. Let $f$ be an expanding Lorenz map. Each proper completely invariant closed set of $f$ is an $\alpha$-limit set.

Proof. Suppose $f$ is $m$-renormalizable ($0 \leq m \leq \infty$), with renormalization intervals $[a_i, b_i], i = 1, \cdots, m$. There are $m$ proper completely invariant closed sets for $f$,

\[ E_i = \{ x : \text{orb}(x) \cap (a_i, b_i) = \emptyset \}, \quad i = 1, \cdots, m. \]

We have

\[ E_1 \subset E_2 \subset \cdots \subset E_m \]

because $[a_i, b_i] \supset [a_{i+1}, b_{i+1}], 0 < i < m$.

Now we prove that $E_i$ is an $\alpha$-limit set of $f$ for $0 < i \leq m$. Put $e^i_- = \sup\{ x \in E_i : x < c \}$. According to Theorem A we know that $e^i_-$ is periodic. By Lemma [1] $\alpha(e^i_-)$ is indeed a completely invariant closed set, and $e^i_- \in \alpha(e^i_-)$. We must have $\alpha(e^i_-) = E_k$ for some $k = 1, 2, \cdots, m$, because $f$ admits exact $m$ proper completely invariant closed sets.

Since

\[ (e^{i-1}_-, e^i_-) \supset (a_{i-1}, b_{i-1}) \supset (e^i_-, e^i_+) \supset (a_i, b_i) \supset (e^{i+1}_-, e^{i+1}_+) \supset (a_{i+1}, b_{i+1}), \]

...
by the definition of $E_i$ and $E_{i+1}$, we know that $e_i^+ \notin E_{i-1}$ and $e_i^{i+1} \in E_{i-1}\setminus E_i$.

Observe that $e_i^- \in \alpha(e_i^+)$ and $e_i^- \notin E_{i-1}$ indicate that $k \geq i$, and $e_i^{i+1} \in E_{i+1}\setminus E_i$ implies $k < i + 1$, we conclude that $k = i$, i.e., $\alpha(e_i^+) = E_i$. Hence, $E_i$ is an $\alpha$-limit set.

5.2. Proof of Theorem C. (1) By Lemma 1 we know that each $\alpha$-limit set is completely invariant. And by Lemma 2 each completely invariant set is an $\alpha$-limit set. So completely invariant closed set and $\alpha$-limit set of $f$ are the same thing in different names. If $f$ is $m$-renormalizable, then $f$ has exact $m$ proper $\alpha$-limit sets. Follows from the proof of Lemma 3 all the $\alpha$-limit sets are $\{E_i\}_{i=1}^m$ defined in (13), and

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I.$$  

(2) At first we prove that if the $i$th ($0 < i \leq m < \infty$) renormalization is periodic, then $E_i' = E_{i-1}$.

Suppose $g = R_i^{-1}f$. $g$ is an expanding Lorenz map on $[a_{i-1}, b_{i-1}]$ with discontinuity $c$. Denote $\kappa_1$ as the minimal period of periodic points of $g$, $O_1$ as the $\kappa_1$-periodic orbit of $g$, and $P_L^i$ and $P_R^i$ are two adjacent $\kappa_1$-periodic point of $g$ with $P_L^i < c < P_R^i$. By Lemma 1 and the proof of Lemma 3 we know that $E_i = \bigcup_{n \geq 0} f^{-n}(P_L^i)$.

Put

$$e_i^{-1} = \sup\{x \in E_{i-1}, x < c\}, \quad e_i^{i-1} = \inf\{x \in E_{i-1}, x > c\},$$

$$\ell' = N((e_i^{-1}, c)), \quad r' = N((c, e_i^{i-1})).$$

According to the definition of minimal renormalization, we have $e_i^{-1} < a_{i-1} \leq P_L^i < c < P_R^i \leq b_{i-1} < e_i^{i-1}$.

By assumption, the minimal renormalization of $g$ is periodic, it follows from Theorem B that the minimal completely invariant closed set of $g$ is $O_1$, which implies $P_R^i$ is an isolated point of $E_i$. So $E_i' \neq E_i$.

Observe that $f^{\ell'}((e_i^{-1}, c)) = (e_i^{-1}, b_{i-1})$, there exists a decreasing sequence \{\{x_n\} in $E_{i-1} \cap (e_i^{-1}, c)$ such that

$$f^{\ell'}(x_1) = P_R^i, \quad f^{\ell'}(x_{n+1}) = x_n, \quad n = 1, 2, \cdots$$

and $x_n \to e_i^{-1}$ as $n \to \infty$. So $e_i^{-1} \in E_i'$. By Lemma 1 we know $E_i'$ is also a completely invariant closed set, we have

$$E_{i-1} = \bigcup_{n \geq 0} f^{-n}(e_i^{-1}) \subseteq E_i' \neq E_i.$$

It follows $E_i' = E_{i-1}$.

Now we show that if the $i$th renormalization $R_i^f$ is not periodic, then $E_i$ a Cantor set. From the proof of first part, we know that $E_i = \bigcup_{n \geq 0} f^{-n}(P_L^i)$. Since the $i$th renormalization is not periodic, the minimal completely invariant closed set of $R_i^{-1}f$ is a Cantor set. So $E_i$ admits no isolated point in $[a_{i-1}, b_{i-1}]$. $E_i' \cap [a_{i-1}, b_{i-1}] \neq \emptyset$, which implies that $E_i' = E_i$.

(3) Now we are ready to characterize the $\alpha$-limit set of every point in $I$. At first, we describe the set $\{x \in I, \alpha(x) = D\}$, where $D$ is the minimal completely invariant closed set of $f$. 

is the union of finite closed intervals, and orb renormalized finite times. Suppose a α the minimal renormalization R f = R D f := (f ℓ, f r). It follows that

\[ \text{orb}([a_1, b_1]) = \bigcup_{n \geq 0} f^n([a_1, b_1]) = \left( \bigcup_{n=0}^{r-1} f^n([a_1, c]) \right) \bigcup \left( \bigcup_{n=0}^{r-1} f^n([c, b_1]) \right) \]

is the union of finite closed intervals, and orb([a_1, b_1]) is forward invariant under f.

Since D is the minimal completely invariant closed of f, by Lemma 6 D is also the minimal α-limit set of f. So α(x) ⊆ D for all x ∈ I.

Let D1 be the minimal completely invariant closed set of the minimal renormalization R D f. It follows that D1 ∩ D = ∅, and D1 ⊆ E2. If x /∈ orb([a_1, b_1]), then f −n(x) ∩ orb([a_1, b_1]) = ∅ because orb([a_1, b_1]) is forward invariant under f. So α(x) is disjoint with the interior of orb([a_1, b_1]), which indicates α(x) ∩ D1 = ∅.

Hence, α(x) /∈ E2, i.e., α(x) = D = E1.

On the other hand, by the minimality of D1, α(x, R D f) = D1 for all x ∈ [a_1, b_1]. For x ∈ [a_1, b_1], since orb(x, R D f) = orb(x, f) ∩ [a_1, b_1], we see that α(x) ⊆ α(x, R D f). So α(x) ∩ D1 = ∅, which implies that α(x) /∈ D for x ∈ [a_1, b_1]. Notice that f(x) ∈ α(f(x)), we conclude α(x) /∈ D for all x ∈ orb([a_1, b_1]).

The proof of the Claim is complete.

For 0 ≤ i ≤ m, we denote [a_i, b_i] as the renormalization interval of the ith renormalization R^i f, and D_i as the minimal completely invariant closed set of R^i f.

By the Claim we know that α(x) = E_i if and only if

\[ x \in I \backslash \text{orb}([a_1, b_1]) = \text{orb}([a_0, b_0]) \backslash \text{orb}([a_1, b_1]). \]

For the case i = 2 ≤ m, we consider the map R f := R D f on [a_1, b_1]. According to the Claim, we obtain that α(x, R f) = D_1 if and only if x /∈ orb([a_2, b_2]). It follows that α(x) = E_2 if and only if

\[ x \in \text{orb}([a_1, b_1]) \backslash \text{orb}([a_2, b_2]). \]

Repeat the above arguments, we conclude α(x) = E_i if and only if

\[ x \in \text{orb}([a_{i-1}, b_{i-1}]) \backslash \text{orb}([a_i, b_i]) \quad \text{for} \quad 0 < i \leq m. \]

If m < ∞, R_m f is prime on [a_m, b_m], α(x, R_m f) = [a_m, b_m] for all x ∈ [a_m, b_m]. By Lemma 1 the completely invariant closed set containing [a_m, b_m] ⊃ c is I, we conclude that α(x) = I for all x ∈ orb([a_m, b_m]).

For the case m = ∞, put A = \bigcap_{i \geq m} \text{orb}([a_i, b_i]), it is known that A := \overline{\text{orb}(c+)} = \overline{\text{orb}(c−)} (cf. 13 16), which is a Cantor set. Since c ∈ A, the completely invariant closed set containing A is I. As a result, α(x) = I for all x ∈ A. \hfill □

5.2.1. Example: α-limit set with given depth.

We can use Theorem C to construct countable α-limit set with given depth.

Consider the piecewise linear symmetric Lorenz map: 1 < a ≤ 2,

\[ f_a(x) = \begin{cases} \frac{a}{2}x + \frac{1}{2} & x \in [0, \frac{1}{2}] \\ \frac{a}{2}x - \frac{1}{2} & x \in (\frac{1}{2}, 1]. \end{cases} \tag{16} \]

According to Glendinning 11 and Palmer 22, f_a can only be periodically renormalized finite times. Suppose a ∈ (2^{−(m+1)}, 2^{−m}], Parry 25 proved that
Let $f_a$ be (periodically) renormalized $m$ times. In this case, by Theorem A and Theorem C, $f_a$ has exact $m$ different $\alpha$-limit sets. Let $p_i$ be one of the $2^i$-periodic point of $f_a$,

$$E_i = \bigcup_{n \geq 0} f_a^{-n}(p_i), \quad i = 1, \cdots, m.$$ 

Then $\{E_i\}_{i=1}^m$ is the cluster of $\alpha$-limit sets of $f_a$. Moreover, according to Theorem C, $E_n^{(i)} = E_{n-i}, i = 1, 2, \cdots, m$. So $E_m$ is a countable closed set and the $m$th derived set $E_m^{(m)}$ is empty. The depth of $E_m$ is $m$.

5.3. Nonwandering set.

The following Lemma 7 indicates that the dynamics on the minimal completely invariant closed $D$ is indecomposable.

**Lemma 7.** Let $f$ be an expanding Lorenz map with $1 \leq \kappa < \infty$, $D$ be its minimal completely invariant closed set. Then $f : D \to D$ is l.e.o., and $\Omega(f|_D) = D$.

**Proof.** By Theorem B, there are three cases: $D = O$, $D = I$ and $O \subset D \subset I$, where $O$ is the unique $\kappa$-periodic orbit. If $D = O$ or $D = I$, applying Theorem B, it is easy to see $f : D \to D$ is l.e.o., and $\Omega(f|_D) = D$.

For the case $O \subset D \subset I$, we know that $D$ is a completely invariant Cantor set of $f$. We shall prove that $f : D \to D$ is l.e.o.. Suppose $A$ is an open set of $D$ (in the induced topology from $I$), there exists an open set $U$ of $I$ such that $A = U \cap D$. By the l.e.o. property of $O$, there is positive integer $N$ such that $\bigcup_{n=0}^N f^n(U) = I$. It follows from Lemma 7 that $f^n(U \cap D) = f^n(U) \cap D$. We have

$$\bigcup_{n=0}^N f^n(A) = \bigcup_{n=0}^N f^n(D \cap U) = \left( \bigcup_{n=0}^N f^n(U) \right) \cap D = D,$$

which implies that $f : D \to D$ is l.e.o.. As a result, $\Omega(f|_D) = D$. \hfill \Box

Now we prove the nonwandering set decomposition of expanding Lorenz map. As mentioned before, Glendinning and Sparrow [13] gave a decomposition based on kneading theory.

**Proposition 2.** Let $f$ be an $m$-renormalizable ($0 \leq m \leq \infty$) expanding Lorenz map and

$$\emptyset = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset I$$

be all the $\alpha$-limit sets of $f$, $I_i = [a_i, b_i]$ be the renormalization interval of the $i$th renormalization $R^i f$, and $D_i$ is the minimal completely invariant closed set of $R^i f$.

Then there is a canonical decomposition of the nonwandering set $\Omega(f)$ of $f$ into $m$-invariant closed set $\Omega_i$ ($i = 1, \cdots, m$) and an attractor $A$

$$\Omega(f) = \bigcup_{i=1}^m \Omega_i \cup A. \quad (17)$$

This decomposition has the following properties:

1. $\Omega_i := \Omega_i \cap \text{orb}(I_{i-1}) = \text{orb}(D_{i-1})$, $1 \leq i \leq m$ and $f|_{\Omega_i}$ is l.e.o.. $\Omega_i$ is either a periodic orbit or a Cantor set depending on whether the renormalization $R^i f$ is periodic or not.
(2) A is the attractor of $f$: $\omega(x) \subseteq A$ for $x \notin E_\infty := \bigcup_{i \geq 0} E_i$. $f|_A$ is l.e.o.

Moreover, $A = \bigcap_{i=0}^m \text{orb}(\ell, b_i)$: $A = I$ if $m = 0$, $A$ is a finite union of closed intervals if $0 < m < \infty$, and $A$ is a Cantor set if $m = \infty$. In the last case, $\omega(x) = A$ for $x \notin E_\infty$.

Based on their renormalization theory on kneading invariants, Glendinning and Sparrow \cite{13} obtained the nonwandering set decomposition \cite{17}. Our proof of the decomposition is independent of kneading theory. We obtain the exact expression of $\Omega_1$, and emphasize that $\Omega_1$ is indecomposable: $f|_{\Omega_1}$ is l.e.o.

Proof. If $m = 0$, $f$ is prime. By Theorem B and Theorem C, we know that $f$ is l.e.o., $A = I = \Omega(f)$, and $\alpha(x) = I, \forall x \in I$.

Now suppose $m > 0$, i.e., $f$ is renormalizable. By Theorem A and Theorem C, all the completely invariant closed sets of $f$ are:

$$E_{m+1} = I$$ is just a notation when $m = \infty$. We can decompose $I = E_{m+1}$ as follows:

$$I = (E_1 \setminus E_0) \cup (E_2 \setminus E_1) \cup \cdots \cup (E_m \setminus E_{m-1}) \cup (E_{m+1} \setminus E_m).$$

Since $E_i$ is completely invariant, $E_i \setminus E_{i-1}$ ($i = 1, \ldots, m$) and $I \setminus E_m$ are invariant under $f$. It follows that

$$\Omega(f) = \bigcap_{i=1}^{m+1} (E_i \setminus E_{i-1}) = \bigcup_{i=1}^{m+1} \{\Omega(f) \cap (E_i \setminus E_{i-1})\} := \bigcup_{i=1}^{m} \Omega_i \cup A$$

where $\Omega_i = \Omega(f) \cap (E_i \setminus E_{i-1})$ and $A = \Omega(f) \cap (I \setminus E_m)$.

For $0 < i \leq m$, the $(i-1)$th renormalization of $f$ is

$$R_{i-1}f(x) = \begin{cases} f_{i-1}^{\ell_{i-1}}(x) & x \in [a_{i-1}, c) \\ f_{i-1}^{r_{i-1}}(x) & x \in (c, b_{i-1}] \end{cases}.$$  

$D_{i-1}$ is the minimal completely invariant closed set of $R_{i-1}f$.

In what follows, we only show that

$$\Omega_i = \text{orb}(D_{i-1}) = E_i \cap \text{orb}([a_{i-1}, b_{i-1}])$$

in three steps. By Lemma 7 we know that $f|_{\Omega_i}$ is l.e.o.. For the proof of remain parts, see \cite{13} or \cite{16}.

**Step 1:** $\text{orb}(D_{i-1}) = E_i \cap \text{orb}([a_{i-1}, b_{i-1}])$.  

By the definitions of $E_i$, $E_{i-1}$ and $D_{i-1}$, $D_{i-1} \subseteq E_i \cap [a_{i-1}, b_{i-1}]$. On the other hand, $x \in [a_{i-1}, b_{i-1}] \setminus E_i$ indicates $\text{orb}(R_{i-1}f, x) \cap [a_{i}, b_{i}] \neq \emptyset$. By Lemma 2 $x \notin D_{i-1}$. We obtain $[a_{i-1}, b_{i-1}] \setminus D_{i-1} \subseteq [a_{i-1}, b_{i-1}]$, which implies $D_{i-1} \supseteq E_i \cap [a_{i-1}, b_{i-1}]$. By Lemma 4 we get the desired equality.

**Step 2:** $\text{orb}(D_{i-1}) \subseteq \Omega_i := \Omega(f) \cap (E_i \setminus E_{i-1})$.

By definitions, $\text{orb}(D_{i-1}) \subseteq E_i \setminus E_{i-1}$.

By Lemma 7 we know that $R_{i-1}f|_{D_{i-1}}$ is l.e.o., and $\Omega(R_{i-1}f|_{D_{i-1}}) = D_{i-1}$. Since $R_{i-1}f$ is the first return map of $f$ on the renormalization interval $I_{i-1} := [a_{i-1}, b_{i-1}]$, we have

$$\text{orb}(x, R_{i-1}f) = \text{orb}(x, f) \cap I_{i-1}, \quad \forall x \in I_{i-1}.$$
It follows that $D_{i-1} \subset \Omega(R^{i-1}f) \subset \Omega(f)$, and $\text{orb}(D_{i-1}) \subset \Omega(f)$ because $\Omega(f)$ is invariant under $f$.

**Step 3:** $\text{orb}([a_{i-1}, b_{i-1}]) \supset \Omega_i := \Omega(f) \cap (E_i \setminus E_{i-1})$.

It is necessary to show that any point in $E_i \setminus \text{orb}([a_{i-1}, b_{i-1}])$ is wandering. Suppose $x \in E_i$, and $x \notin (E_{i-1} \cup \text{orb}([a_{i-1}, b_{i-1}])$. $x \notin E_{i-1}$ implies the orbit of $x$ will go into $(a_{i-1}, b_{i-1})$, and stay in the forward invariant closed set $\text{orb}([a_{i-1}, b_{i-1}])$ forever. So $x$ is wandering because $x \notin \text{orb}([a_{i-1}, b_{i-1}])$. $\square$

**Appendix: Periodic renormalization**

We collect some facts for periodic renormalization in this appendix.

Let $f$ be an expanding Lorenz map on $[a, b]$, $\kappa$ is the minimal period of $f$, $O$ is the unique $\kappa$-periodic orbit, $P_L$ is the largest $\kappa$-periodic point in $[a, c]$ and $P_R$ be the smallest $\kappa$-periodic point in $(c, b]$, $D$ is the minimal completely invariant closed set of $f$. If $f$ is $m$-renormalizable, $0 \leq i \leq m$, $R_i$ is the $i$th renormalization of $f$, and $E_i$ is the completely invariant closed set corresponds to $R_i f$, $\Omega_i = E_i \cap \Omega(f)$.

1. The minimal renormalization $Rf$ is periodic if and only if $D = O$ (Theorem B).
2. $Rf$ is periodic if and only if
   \[
   [f^\kappa(c+), f^\kappa(c-)] \subseteq [P_L, P_R].
   \]
3. One can check if $Rf$ is periodic or not in following steps:
   - Find the minimal period $\kappa$ of $f$ by considering the preimages of $c$, see Lemma \[3\]
   - Find the $\kappa$-periodic orbit;
   - Check if the inclusion \[19\] holds or not.
4. $Rf$ is periodic if and only if the rotational interval of $f$ is degenerated to a rational point $\frac{p}{q}$. \[4, 19\]
5. $R_i f$ is periodic if and only if $E_i$ admits isolated points.
6. $R_i f$ is periodic if and only if $\Omega_i$ is consists of a periodic orbit.
7. $R_i f$ is periodic if and only if the topological entropy $h(f|_{\Omega_i}) = 0$.
8. If the first $k(k \leq m)$ renormalizations are all periodic, then $E_i = E_k^{(k-i)}$, $i = 0, 1, \ldots, k$. The depth of $E_i$ is $i$.
9. Suppose $f$ is a $\beta$-transformation. $f$ can only be renormalized periodically, i.e., each renormalization of $f$ is periodic. Since $\beta$-transformation is finitely renormalizable, one can obtain all of the renormalizations of $f$ in finite steps.
10. An expanding Lorenz map $f$ is conjugated to a $\beta$-transformation if and only if $f$ is finitely renormalizable and each renormalization of $f$ is periodic \[11, 23\].
11. A piecewise linear Lorenz map that expand on average is conjugate to $\beta$-transformation \[8\].
12. For $1 < a \leq 2$, put
   \[
   f_a(x) = \begin{cases} 
   ax + 1 - \frac{1}{a} & x \in [0, \frac{1}{a}) \\
   a(x - \frac{1}{2}) & x \in (\frac{1}{2}, 1].
   \end{cases}
   \]
   If $a \in (2^{2^{-(m+1)}}, 2^{-m}]$, then $f_a$ is $m$-renormalizable \[25\].
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