The Variational Methods of Quantum Systems in Holomorphic Representation

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Abstract. We show that variational method for harmonic oscillator in Bargmann representation breaks down since we always end up with one of the exact eigenvalues of Hamiltonian using any normalized monomials in the complex plane. Using the holomorphic representation of pure spin systems, we also encounter the same result found earlier for harmonic oscillators in Bargmann representation. We conclude that the exact energy eigenvalues of these systems are unavoidable starting from any normalized monomials of the complex phase-space coordinate. Since \( \left\{ \frac{z^n}{\sqrt{n!}} \right\} \) form an orthonormal basis, one could in principle expand any complex function \( f(z) \) in powers of \( z \) and use this expansion as energy eigenstates. Such expansion is valid since the magnitude of \( z \) carry no physical information only their non-negative exponents \( n \) are physically important. These results may lead to new selection principle regarding the quantized energy spectrum of physical systems and enhance our understanding of quantum world.

1. Introduction

Variational methods in quantum mechanics can be used to find the approximate ground states energies starting from a good trial wave function with suitable boundary conditions. This method has been used successfully in many quantum systems such as harmonic oscillator, hydrogen and helium atoms \([1, 2]\). However for more complicated systems such as spin chains, the direct usage of this method is not well-established. Here we try to test the ability of using the variational method for spin chains namely the XXX model in the holomorphic representation. However our method can be used in more general systems such as the XYZ and XXZ models. Spin chains play important role in physics and mathematics. Due to their complexity since the Hilbert space for spin chain of length \( N \) is of dimensions \( 2^N \times 2^N \) which is very large when \( N \) is large. It has been a lot of efforts in studying their energy spectrum especially the integrable spin chains \([3, 4, 5, 6]\).

In this work, we show that energy eigenvalues for pure quantum systems are unavoidable starting from any normalized monomial \( f(z_j) \) of the complex phase space variable \( z_j = \hat{Q}_j + i \hat{P}_j \) where \( \hat{Q}_j \) and \( \hat{P}_j \) represent the coordinate and momentum operators respectively. Here by pure quantum systems, we mean any quantum system with quantized energy eigenvalues (i.e. can be written as multiple of \( \hbar \)).
For example, harmonic oscillator under electric field perturbation is not pure if the electric field varies continuously.

One interesting formulation of quantum mechanics is the holomorphic representation introduced by Bargmann and Segal in 60s\cite{7,8}. The main idea behind this construction is to represent the wave functions as monomials in the complex plane i.e. \( \psi_n(z) = \frac{z^n}{\sqrt{n!}} \), not as matrices. In this case, the raising and lowering operators are \( a^\dagger = z \) and \( a = \frac{\partial}{\partial z} \) respectively since for any given arbitrary complex function \( f(z) \) we have

\[
\left[ \frac{\partial}{\partial z}, z \right] f = \frac{\partial}{\partial z} (zf) - z \frac{\partial f}{\partial z} = f
\]

Thus \( \left[ \frac{\partial}{\partial z}, z \right] = 1 \) has the same structure similar to the commutator of \( a \) and \( a^\dagger \). This assignment of raising and lowering operators was originally proposed by Fock\cite{9}.

Moreover, the series \( \psi_n(z) = \sum c_n z^n \sqrt{n!} \) converges uniformly in any compact domain of the complex \( z \)-plane since \( \sum_{n=0}^{\infty} |c_n|^2 = 1 \)\cite{10,11}. Possible applications of Bargmann representation in various physical systems can be found in \cite{12,13,14,15,16}.

The organization of this paper goes as follows, first we give an overview of the holomorphic representation of quantum systems in Sec. 2. Then we try to define the variational method on Bargman spaces and prove the breakdown of this method for harmonic oscillators and spin chains written with respect to the complex phase-space variables \( z_i \). Finally we discuss the consequences of our findings.

2. The Holomorphic Representation of Quantum Systems

The Bargmann or Segal-Bargmann spaces in other references \( \mathcal{H}L(\mathbb{C}^d, \mu) \) are the spaces of holomorphic functions with Gaussian integration measure \( \mu = \pi^{-d} e^{-|z|^2} \) such that the inner product endowed with this space is \cite{7,8,17,11}

\[
\langle f | g \rangle_\mu = (\pi)^{-d} \int_{\mathbb{C}^d} \overline{f}(z) g(z) e^{-|z|^2} dz,
\]

where \( |z|^2 = |z_1|^2 + \cdots + |z_d|^2 \).

Any entire analytic function \( f(z) \) in \( \mathcal{H}L(\mathbb{C}^d, \mu) \) obeys the following square-integrability condition

\[
||f||^2 := \langle f | f \rangle_\mu = (\pi)^{-d} \int_{\mathbb{C}^d} |f(z)|^2 e^{-|z|^2} dz < \infty
\]

where \( dz \) is the 2\( d \)-dimensional Lebesgue measure in \( \mathbb{C}^d \). The Bargmann space \( \mathcal{H}L^2(\mathbb{C}^d, \mu) \) is in fact a Hilbert space as shown by Bargmann in \cite{7}. Using the inner product defined in \cite{4} we can prove that both \( \overline{z} \) and \( \frac{\partial}{\partial z} \) have the same effect i.e. \( \langle \frac{\partial f}{\partial z}, g \rangle_\mu = \langle f, zg \rangle_\mu \)\cite{17}. We define the inverse Segal-Bargmann transform which maps each function of the holomorphic variables \( z \) and \( \frac{\partial}{\partial z} \) with its relation written in coordinate representation as \cite{17}

\[
F(x) = \int_{\mathbb{C}} \exp[-(\overline{z} \cdot z - 2 \sqrt{2} \overline{z} \cdot x + x \cdot x)/2] f(z) e^{-|z|^2} dz.
\]
The monomials \( \{z^n/\sqrt{n!}\} \) form an orthonormal basis \( \mathcal{H}_L(\mathbb{C}^d, \mu) \) since
\[
\int \frac{dz \, d\bar{z}}{\pi} \exp[-z \bar{z}] z^n \bar{z}^m = n! \delta_{mn}.
\]

The harmonic oscillator Hamiltonian operator \( \hat{H} = \hbar \omega (\hat{a}^{\dagger} \hat{a} + \frac{1}{2}) \) assumes the following form in the Bargmann representation \[18\]
\[
\hat{H} = \hbar \omega \left( z \frac{d}{dz} + \frac{1}{2} \right)
\]
and the corresponding energy eigenvalues are
\[
\hat{H}|n\rangle = \hbar \omega \left( z \frac{d}{dz} + \frac{1}{2} \right) \frac{z^n}{\sqrt{n!}} = \hbar \omega \left( n + \frac{1}{2} \right) \frac{z^n}{\sqrt{n!}} = \hbar \omega \left( n + \frac{1}{2} \right)|n\rangle.
\]
which should be the same regardless of the representation. Its ground state wave function in the Bargmann representation is simply \( \psi_0(z) = 1 \) while in coordinate representation it is \( \psi_0(x) = (\frac{m \omega}{2\pi})^{1/4} e^{-m \omega x^2/2\hbar} \). Generally, the wave functions of quantum harmonic oscillator in Bargmann representation \( \psi_n(z) = \frac{z^n}{\sqrt{n!}} \) correspond to \( \psi_n(x) = (\frac{m \omega}{2\pi})^{1/4} e^{-m \omega x^2/2\hbar} H_n(\sqrt{m \omega} x) \) where \( H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n}(e^{-y^2}) \) is the Hermite polynomials in coordinate representation.

For a given entire analytic function \( f_{\alpha, \beta}(z, w) = \frac{z^n}{\sqrt{\alpha! \beta!}} \), we define the angular momentum operators in the Bargmann space \( \mathcal{H}_L(\mathbb{C}^2, \mu) \) as
\[
\hat{J}_1 = \frac{\hbar}{2} \left( z \frac{\partial}{\partial w} + w \frac{\partial}{\partial z} \right),
\]
\[
\hat{J}_2 = \frac{\hbar}{2i} \left( z \frac{\partial}{\partial w} - w \frac{\partial}{\partial z} \right),
\]
\[
\hat{J}_3 = \frac{\hbar}{2} \left( z \frac{\partial}{\partial z} - w \frac{\partial}{\partial w} \right).
\]
These operators belong to the \( SU(2) \) Lie algebra and obey the following commutation relations \( [\hat{J}_i, \hat{J}_j] = i\hbar \varepsilon_{ijk} \hat{J}_k \) since the only non-trivial commutators between \( z, w \) and their partial derivatives are \( [\frac{\partial}{\partial z}, z] = [\frac{\partial}{\partial w}, w] = 1 \). \( \varepsilon_{ijk} \) is the well-known Levi-Civita totally antisymmetric symbol. With this construction, we can express the non-Hermitian raising and lowering operators as \( \hat{J}_+ = \hat{J}_1 + i\hat{J}_2 = \frac{\hbar}{2} \frac{\partial}{\partial w} \) and \( \hat{J}_- = \hat{J}_1 - i\hat{J}_2 = \hbar w \frac{\partial}{\partial z} \).

The inner product in the two-dimensional Bargmann space is
\[
\langle f_{\alpha', \beta'}(z, w) | f_{\alpha, \beta}(z, w) \rangle = \int dz \, d\bar{z} \, dw \, d\bar{w} \exp[-\bar{z}z - \bar{w}w] f_{\alpha', \beta'}(z, w) f_{\alpha, \beta}(z, w)
\]
\[
= \pi^2 \alpha! \beta! \delta_{\alpha', \alpha} \delta_{\beta', \beta}
\]
Let us explicitly calculate the expectation values of the operators $\hat{J}_3$ (or $\hat{J}_z$ as called in most of references), $\hat{J}^2$ and $\hat{J}_\pm$ using the basis $f_{\alpha,\beta}(z,w)$, we find

$$
\langle f'_{\alpha',\beta'}|\hat{J}_3|f_{\alpha,\beta}\rangle_{\mu} = \hbar \left(\alpha - \beta\right) \delta_{\alpha',\alpha} \delta_{\beta',\beta},
$$

$$
\langle f'_{\alpha',\beta'}|\hat{J}^2|f_{\alpha,\beta}\rangle_{\mu} = \hbar^2 \left(\alpha + \beta\right) \left(\frac{\alpha + \beta}{2}\right) \delta_{\alpha',\alpha} \delta_{\beta',\beta},
$$

$$
\langle f'_{\alpha',\beta'}|\hat{J}_+|f_{\alpha,\beta}\rangle_{\mu} = \hbar \sqrt{(\alpha + 1)^2} \delta_{\alpha',\alpha+1} \delta_{\beta',\beta-1},
$$

$$
\langle f'_{\alpha',\beta'}|\hat{J}_-|f_{\alpha,\beta}\rangle_{\mu} = \hbar \sqrt{(\beta + 1)^2} \delta_{\alpha',\alpha-1} \delta_{\beta',\beta+1}.
$$

In contrast to the standard treatment of angular momentum mentioned for example in [19], we can directly compute the expectation values of operators $\hat{J}_{1,2}$ without invoking $\hat{J}_\pm$ and $\hat{J}_3$ in our calculations. We obtain

$$
\langle f'_{\alpha',\beta'}|\hat{J}_1|f_{\alpha,\beta}\rangle_{\mu} = \frac{\hbar}{2} \left(\sqrt{(\alpha + 1)^2} \delta_{\alpha',\alpha+1} \delta_{\beta',\beta-1} + \sqrt{(\beta + 1)^2} \delta_{\alpha',\alpha-1} \delta_{\beta',\beta+1}\right),
$$

$$
\langle f'_{\alpha',\beta'}|\hat{J}_2|f_{\alpha,\beta}\rangle_{\mu} = \frac{\hbar}{2i} \left(\sqrt{(\alpha + 1)^2} \delta_{\alpha',\alpha+1} \delta_{\beta',\beta-1} - \sqrt{(\beta + 1)^2} \delta_{\alpha',\alpha-1} \delta_{\beta',\beta+1}\right).
$$

Comparing $\hat{J}_3$ and $\hat{J}^2$ with the standard relations written in $|jm\rangle$ basis i.e. $\hat{J}_z|jm\rangle = m\hbar|jm\rangle$ and $\hat{J}^2|jm\rangle = \hbar^2(j+1)|jm\rangle$, we find

$$
\langle f'_{\alpha',\beta'}|\hat{J}_1|f_{\alpha,\beta}\rangle_{\mu} = \frac{\hbar}{2} \left(\alpha + \beta\right), \quad m = \frac{\alpha - \beta}{2}.
$$

The addition of quantum angular momentum using holomorphic representation can be done in a systematic way by adding monomials without worrying about the dimensionality of tensor product of single-particle angular momentum subspaces.

For simplicity we consider a two-particle system. The total angular momentum operator is

$$
\hat{J} = \hat{J}_1 + \hat{J}_2
$$

where 1 and 2 labels the two particles. we assume the joint two-particle system basis to be $|f_{\alpha_1,\alpha_2,\beta_1,\beta_2}\rangle = \frac{z^{\alpha_1} \bar{z}^{\alpha_2} w^{\beta_1} \bar{w}^{\beta_2}}{\sqrt{\alpha_1! \alpha_2! \beta_1! \beta_2!}}$, then the operators $\hat{J}^2$ and $\hat{J}_3$ are

$$
\hat{J}^2|f_{\alpha_1,\alpha_2,\beta_1,\beta_2}\rangle = \left(\hat{J}_1^2 + \hat{J}_2^2\right)|f_{\alpha_1,\alpha_2,\beta_1,\beta_2}\rangle = \hat{J}_1^2|f_{\alpha_1,\beta_1}\rangle + \hat{J}_2^2|f_{\alpha_2,\beta_2}\rangle = \hbar^2 \left(\frac{\alpha_1 + \beta_1}{2}\right) \left(\frac{\alpha_1 + \beta_1}{2} + 1\right) |f_{\alpha_1,\beta_1}\rangle + \hbar^2 \left(\frac{\alpha_2 + \beta_2}{2}\right) \left(\frac{\alpha_2 + \beta_2}{2} + 1\right) |f_{\alpha_2,\beta_2}\rangle = \hbar^2 \left(\frac{\alpha + \beta}{2}\right) \left(\frac{\alpha + \beta}{2} + 1\right) |f_{\alpha,\beta}\rangle,
$$

$$
\hat{J}_3|f_{\alpha_1,\alpha_2,\beta_1,\beta_2}\rangle = \hbar \left(\frac{\alpha - \beta}{2}\right) |f_{\alpha_1,\alpha_2,\beta_1,\beta_2}\rangle,
$$

as required.
where $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$. The generalization to $N$-particle system is very natural in the analytical approach. The previous relations \[20\] and \[21\] generalize to
\begin{equation}
\hat{J}_2 |f_{\alpha_1 \ldots \alpha_N, \beta_1 \ldots \beta_N}\rangle = \hbar \left( \frac{\alpha + \beta}{2} \right) \left( \frac{\alpha + \beta}{2} + 1 \right) |f_{\alpha_1 \ldots \alpha_N, \beta_1 \ldots \beta_N}\rangle,
\end{equation}
\begin{equation}
\hat{J}_3 |f_{\alpha_1 \ldots \alpha_N, \beta_1 \ldots \beta_N}\rangle = \hbar \left( \frac{\alpha - \beta}{2} \right) |f_{\alpha_1 \ldots \alpha_N, \beta_1 \ldots \beta_N}\rangle,
\end{equation}
where here $\alpha = \alpha_1 + \cdots + \alpha_N$ and $\beta = \beta_1 + \cdots + \beta_N$.

Having established the angular momentum operators and their main addition relations in the holomorphic representation, it is a straightforward procedure to reproduce all known formulas related to quantum angular momentum such as the Clebsch-Gordon coefficients, Wigner 3-$j$, 6-$j$ and Racah symbols by means of analytical functions and their partial derivatives\[20\].

3. Variational Methods in Bargmann Spaces $\mathcal{H}L(\mathbb{C}^d, \mu)$

The variational method in quantum mechanics can be written for any normalized trial wave function $\psi(x_i)$ as\[1\]
\begin{equation}
E_g \leq \langle \psi | \hat{H} | \psi \rangle = \langle \hat{H} \rangle.
\end{equation}
This method assures that for any test wave function, the Hamiltonian expectation value is equal or greater than the true ground-state energy $E_g$. Practically the variational method algorithm follows the following steps
- Input trial wavefunction $\phi(a_i)$ (with suitable boundary conditions)
- Compute normalization constant $N^2 = \langle \phi | \phi \rangle$
- Compute unnormalized kinetic energy $\langle T \rangle = \frac{\hbar^2}{2m} \langle |\nabla|^2 \phi \rangle$
- Compute unnormalized potential energy $\langle V \rangle = \langle \phi | V | \phi \rangle$
- Differentiate with respect to variational parameters $J_{a_i} = \frac{d}{da_i} \left( \frac{\langle T \rangle + \langle V \rangle}{N^2} \right)$
- Solve $J_{a_i} = 0$ for all $a_i$
- Substitute optimal values of $a_i$ into $\phi$
- Compute $\frac{\langle T \rangle + \langle V \rangle}{N^2}$ using optimized wavefunction.

In holomorphic representation, the variational method can be written for arbitrary normalized wavefunction $\psi(z_i)$ as
\begin{equation}
E_g \leq \langle \psi(z_i) | \hat{H}(z_i, \frac{\partial}{\partial z_i}) | \psi(z_i) \rangle_{\mu} = \langle \hat{H} \rangle_{\mu},
\end{equation}
or
\begin{equation}
E_g \leq \frac{\langle \psi(z_i) | \hat{H}(z_i, \frac{\partial}{\partial z_i}) | \psi(z_i) \rangle_{\mu}}{\langle \psi(z_i) | \psi(z_i) \rangle_{\mu}},
\end{equation}
for non-normalized test wave function $\psi(z_i)$.

**Theorem:** the condition $J_{a_i} = 0$ always give $a_i = 0$ for any trial wavefunction written with respect to the phase-space coordinates $z_i$. 
Proof: for trial wavefunctions of the form $\psi(z_i) = \sum \alpha_i z_i^n$, the normalization condition enforces $a_i = \frac{1}{\sqrt{n_i!}}$ and the corresponding average energy is

$$\langle \hat{H}_i \rangle = \sum_i \alpha_i^2 \left( n_i + \frac{1}{2} \right) \langle z_i^n | z_i^n \rangle$$

(27)

Then the condition $J \alpha_i = \frac{d\langle \hat{H}_i \rangle}{d\alpha_i} = 0$ cannot be satisfied unless $\alpha_i = 0$ and that is contradiction. For more complicated trial wavefunctions of any shape $f(z_i)$ the same fact holds since we can make series expansion of any analytic $f(z_i)$ in complex plane and reduce our problem to some sort of orthonormal basis written as monomials in this complex plane. The series expansion is valid since $f(z_i)$ is analytic and the magnitude of $z_i$ is not physically important since they are basis, only their non-negative exponents are physically measurable and gives the energy levels.

For example, consider the trial wave function $\alpha z$ for first excited state of harmonic oscillator where $\alpha$ is an arbitrary real constant. The normalization of complex monomials enforces the condition $\alpha = 1$ while $\frac{d\langle \hat{H}_i \rangle}{d\alpha_i} = 0$ imposes the condition $\alpha = 0$ thus we end up with contradiction. The same situation happen to be present for any test complex monomial of any power. We conclude that applying any normalized complex monomial of power $n$ in the Hamiltonian always give one of the exact energy eigenvalues of harmonic oscillator.

For spin systems, we consider the following $XXX$ spin chain as model example in our analysis.

$$\hat{H} = J \sum_i \left( \hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \hat{S}_i^z \hat{S}_{i+1}^z \right)$$

(28)

where $i$ is the site index. Using the holomorphic representation of spin operators, the $XXX$ spin chain Hamiltonian becomes

$$\hat{H} = \frac{J}{4} \sum_i \left( z_i z_{i+1} \frac{\partial^2}{\partial z_i \partial z_{i+1}} + w_i w_{i+1} \frac{\partial^2}{\partial w_i \partial w_{i+1}} \right)$$

(29)

Due to the orthonormality condition, the last two terms of (29) will give zero expectation values and thus have no measurable effects on the eigenvalues of Hamiltonian. The Hamiltonian operator (29) reduces to

$$\hat{H} = \frac{J}{4} \sum_i (\hat{n}_i \hat{n}_{i+1})$$

(30)

where $\hat{n}_i = z_i \frac{\partial}{\partial z_i} + w_i \frac{\partial}{\partial w_i} = \hat{n}_{z_i} + \hat{n}_{w_i}$ is the total number operator written in Bargmann representation and equals to the number operator of two un-coupled harmonic oscillators. Thus the $XXX$ spin chain Hamiltonian can be factorized to four un-coupled harmonic oscillators at site index $i$ and $i+1$ on the spin lattice. For spin-$1/2$ chains with $m = 1/2$ (all spins are spin-up), the equation (30) reduces to $\hat{H} = \frac{J}{4} \sum_i \hat{n}_{z_i} \hat{n}_{z_{i+1}}$ since the exponents $\beta_i = 0$ for all $i$ (this can be seen easily...
from\textsuperscript{[18]}. Analogously for spin-1/2 chains with $m = -1/2$ (all spins are spin-down), the equation \textsuperscript{[30]} reduces to $\hat{H} = \frac{\hbar^2}{2} \sum_i \hat{n}_i \hat{n}_{i+1}$ since the exponents $\alpha_i = 0$ for all $i$. Therefore any $XXX$ spin-1/2 chain can be factorized into two un-coupled harmonic oscillators at sites $i$ and $i+1$. The factorization of $XXX$ spin chain in \textsuperscript{[30]} may not be the case for other models such as the $XYZ$ or $XXZ$ spin chains. For illustration purposes, we test the variational method for spin-1/2 $XXX$ chain composed of spin-up particles. We take the first excited state wavefunction to be $\psi = \psi(z_i) \cdot \psi(z_{i+1}) = \gamma_i \gamma_{i+1} z_i z_{i+1}$ where $\gamma_i$'s are real constants. The normalization of this trial wavefunction is
\begin{equation}
\langle \psi | \psi \rangle = \langle \psi(z_i) | \psi(z_i) \rangle \langle \psi(z_{i+1}) | \psi(z_{i+1}) \rangle = 1,
\end{equation}
where
\begin{equation}
\langle \psi(z_i) | \psi(z_i) \rangle = \gamma_i^2 \int \frac{dz_i d\bar{z}_i}{\pi} e^{-z_i \bar{z}_i} z_i = 1.
\end{equation}
The normalization condition gives $\gamma_i = 1$. Similarly, $\langle \psi(z_{i+1}) | \psi(z_{i+1}) \rangle = 1$ gives $\gamma_{i+1} = 1$. If we simply start with unnormalized trial wavefunction $\psi = \gamma_i \gamma_{i+1} z_i z_{i+1}$, then the expectation value of the Hamiltonian operator reads
\begin{equation}
\langle \psi | \hat{H} | \psi \rangle = \frac{\hbar^2 J \gamma_i \gamma_{i+1}}{4}.
\end{equation}
The condition $\frac{\partial \hat{H}}{\partial \gamma_i} = 0$ (or $\frac{\partial \hat{H}}{\partial \gamma_{i+1}} = 0$) gives $\gamma_{i+1} = 0$ (or $\gamma_i = 0$) and this is a contradiction. The same situation happens to be present in all spin systems using the holomorphic representation mentioned earlier in this work. We conclude that starting from any normalized monomial we always encounter one of the exact energy eigenvalues. Thus the variational method fails in the case of pure quantum mechanical systems with discrete energy spectrum such as harmonic oscillators, spin chains using complex normalized monomials as trial wave functions. If we start with trial wave functions with complex $\gamma_i$'s constants, the final result will be the same since $\gamma_i \gamma_{i+1} = \gamma_i^2$ is always real.

Since $\{z_i^n\}$ forms an orthogonal basis, their magnitude carry no information only the exponents $n$ are physically important. Thus, we can impose conditions on $z_i$ so that we can safely make Taylor expansion of the function $f(z_i)$ valid and use it as energy eigenstates. Many famous functions in mathematics give series expansion in term of monomials $z_i$ and their powers \textsuperscript{[21]}. Thus we may extend the energy configuration space to include these as possible solutions. For example, consider the series expansion of exponential function $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{\phi(z)}{\sqrt{n!}} = \sum_{n=0}^{\infty} \frac{\phi(z)}{\sqrt{n!}}$ as possible energy eigenstates where $\gamma(z) = \frac{e^z}{\sqrt{\pi}}$ and $\phi(z) = z^n / n!$. Then the expectation value $\sum_{n=0}^{\infty} n! |\phi(z)|^2 \frac{d}{dz} \left( \frac{d}{dz} + \frac{1}{2} \right) |\phi(z)|$ gives the energy eigenvalues of the harmonic oscillator $E = (\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \ldots)$ as expected. Similarly, considering the series expansion of the functions $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ and $\cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}$ give $E = (\frac{3}{2} + \frac{5}{2} + \frac{7}{2} + \ldots)$ and $E = (\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \ldots)$ respectively using proper normalization constants (namely multiplying the expectation value of the Hamiltonian by $2n!$ for $\cosh(z)$ and $(2n+1)!$ for $\sinh(z)$). One could consider other functions and combinations. For example, the series expansion of $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots$ or $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots$. The expectation value of the Hamiltonian with proper normalization gives the same
overall result as previous series expansion of sinh\((z)\) and cosh\((z)\) respectively. The minus sign in front of some monomials can be absorbed automatically and has no effect on the energy eigenvalues, consider for example the state \(|\xi(z)\rangle = -z\),

\begin{equation}
\hat{H}|\xi\rangle = -\hbar\omega \left( z\frac{d}{dz} + \frac{1}{2} \right) z = \frac{3}{2}\hbar\omega(-z) = \frac{3}{2}\hbar\omega|\xi\rangle
\end{equation}

However complex monomials of the form \(z^{-n}\) give negative energies. Thus they are excluded from the possible solutions of physical systems due to the fact that negative energies at large scale violate the second law of thermodynamics and may lead to un-stability in the vacuum [22].

We realize that the energy configuration space is much larger than what we expected in the holomorphic representation. Any polynomial of complex variable \(z\), would give some sort of energy eigenvalues up to some normalization factors. The only issue to address is that the energy eigenvalue should be eventually positive and coincides with the value found in the coordinate representation for the same energy level \(n\). It is important to mention that we are only interested in the exponents \(n\) of normalized complex monomials not the value of \(z\) itself since they are only basis and may assume any arbitrary value in principle. The infinitely possible ways for obtaining each energy eigenvalue in the holomorphic representation starting from normalized setup corresponds to one way in the coordinate representation. Thus nature imposes some mechanism for selecting the energy outcomes at quantum level.

To summarize, we studied the possibility of applying the variational methods for pure quantum systems such as harmonic oscillators and spin chains in holomorphic representation. Interestingly we found that this method can not be used and blows up. The energy eigenvalues are always exact starting from any normalized monomials in the complex plane. These findings give strong endorsement of the uniqueness of energy eigenvalues the quantum system can have and extend our understanding of quantum world.

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Conflict Of Interest
The authors have no conflicts to disclose.

Data Availability Statement
The data that supports the findings of this study are available within the article.

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