Quantum master equation for collisional dynamics of massive particles with internal degrees of freedom

Andrea Smirne\textsuperscript{a,b} and Bassano Vacchini\textsuperscript{a,d}  
\textsuperscript{a}Università degli Studi di Milano, Dipartimento di Fisica, Via Celoria 16, I-20133 Milano, Italy  \textsuperscript{b}INFN, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy  

(Dated: October 27, 2010)

We address the microscopic derivation of a quantum master equation in Lindblad form for the dynamics of a massive test particle with internal degrees of freedom, interacting through collisions with a background ideal gas. When either internal or centre of mass degrees of freedom can be treated classically, previously established equations are obtained as special cases. If in an interferometric setup the internal degrees of freedom are not detected at the output, the equation can be recast in the form of a generalized Lindblad structure, which describes non-Markovian effects. The effect of internal degrees of freedom on centre of mass decoherence is considered in this framework.

PACS numbers: 03.65.Yz,05.20.Dd,03.75.-b,03.65.Ta

\textsuperscript{a}andrea.smirne@unimi.it  \textsuperscript{b}bassano.vacchini@mi.infn.it
I. INTRODUCTION

In recent times major advances in the experimental techniques have led to the realization of experiments in which quantum systems in a single particle regime are studied under their controlled interaction with some environment. A paradigmatic example in this context is given by the motion of a massive test particle in an interferometric setup, which gives rise to interference fringes as typical quantum signatures. When the coupling with the environment becomes of relevance, such interference fringes are gradually washed out, and a classical dynamics is eventually recovered. This phenomenon goes under the name of decoherence\[13\]. Its understanding and theoretical description require on the one hand a control over the environment, on the other hand a microscopic model for the interaction and the ensuing dynamics.

For the case of a tracer particle immersed in a dilute gas such a microscopic description has been obtained considering the centre of mass degrees of freedom only. The reduced dynamics is given by a master equation in Lindblad form which has been called quantum linear Boltzmann equation, since it provides the natural quantum counterpart of the classical linear Boltzmann equation (see \[14\] for a recent review and references therein). The microscopic input is given by the complex scattering amplitudes describing the collisions between gas and test particle, while the gas is characterized by its density and momentum distribution. In this paper we consider an extension of this result, which includes internal degrees of freedom of the tracer particle. The microscopic derivation is performed along the lines of a general strategy for the derivation of Markovian master equations, which relies on a scattering description of the interaction events \[5\]. Besides the gas properties, this approach takes as basic input the multichannel complex scattering amplitudes, which describe the influence of the internal states on the scattering events. Indeed, when the scattering cross section does not only depend on the relative motional state between tracer and gas particle, such an extension becomes mandatory in order to correctly describe the dynamics. According to the Markovian approximation, the obtained master equation is in Lindblad form. This derivation confirms the structure of the dissipative term, which has been heuristically obtained in \[6\], further determining the coherent contribution to the dynamics due to forward scattering. The latter becomes relevant in the determination of the index of refraction for matter waves. When either type of degrees of freedom can be described in classical terms, a Markovian quantum master equation is obtained. Such a result, corresponding to a classical treatment of the motional degrees of freedom, has been considered in \[7\]. In that context the name Bloch-Boltzmann equation was proposed for the equation, since for a two-level system an extension of the optical Bloch equations to include a Boltzmann-like collision term is obtained. In the same spirit, the name quantum Bloch-Boltzmann equation can be used to indicate a master equation, which gives a quantum description of both internal and centre of mass state.

An interesting situation appears when in the final detection the internal state of the test particle is not resolved at the output of the interferometer. In this case the internal degrees of freedom become part of the environment. Then a non-Markovian dynamics for the motional state appears, which can be described in terms of a coupled set of Lindblad equations for the unnormalized statistical operators corresponding to specific internal channels. This type of non-Markovian dynamics can be considered as a generalized non-Markovian Lindblad structure. It arises as a mean over a classical index, which can take place e.g. as a consequence of the interaction with a structured reservoir \[6\].\[8\]\[9\]. This situation is here considered in the study of the loss of visibility of the interference fringes in an interferometric setup. The ensuing decoherence effect is generally not described as an exponential loss of visibility depending on the strength of the interaction, as in the usual Markovian case.

The paper is organized as follows. In Sect. I[11] we consider the expression of the master equation, pointing to the main steps necessary for its derivation and putting into evidence the microscopic quantities determining its explicit form. A detailed microscopic derivation of the master equation is performed in Appendix [A]. The master equation is given both in terms of matrix elements of the statistical operator in the momentum and internal energy eigenstates basis, as well as an explicit operator expression, which makes its Lindblad structure manifest. This also allows to easily recover under suitable limits previously considered master equations, which describe either only one of the two kind of degrees of freedom or a hybrid quantum classical description of both. In Sect. III we show how the interplay between internal and motional states can influence the visibility in an interferometric setup for the study of decoherence, leading to a non-Markovian behaviour in the reduction of the visibility of the interference fringes.

II. THE MASTER EQUATION FOR A TEST PARTICLE WITH INTERNAL DEGREES OF FREEDOM

We first consider the key ingredients and steps which lead to obtain the master equation describing the collisional dynamics of a test particle immersed in a structureless background gas, keeping the internal degrees of freedom of the particle into account. The task of a full microscopic derivation will be accomplished in Appendix [A] relying on a method recently introduced for the derivation of Markovian master equations, which has been called monitoring approach \[5\]\[10\]\[12\]. In the monitoring approach the reduced dynamics of a system in contact with some environment
is obtained describing their interaction by means of scattering theory. The building blocks in such a formulation of the open system dynamics are therefore the S-matrix characterizing the single interaction events and the rate of collisions. Both quantities are given by operators on the tensor product Hilbert space of system and environment, which we shall denote by $S = 1 + iT$ and $\Gamma$ respectively. The operator nature of these quantities is crucial in order to keep the gas and test particle state into account in the dynamic description of the collisional interaction. The Markovian master equation for the reduced dynamics is obtained by assuming the various collisions as independent, so that their effect cumulates according to the state dependent scattering rate, and taking the trace over the environmental degrees of freedom.

A. Expression of the master equation

The formal expression of the master equation reads

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}[H, \rho] + \mathcal{L}\rho + \mathcal{R}\rho \quad (1)$$

where $H$ is the free Hamiltonian and $\rho$ is the statistical operator of the system. For the case at hand the free Hamiltonian of the system is given by

$$H = \frac{P^2}{2M} \otimes \sum_i \hbar \omega_i |i\rangle\langle i|, \quad (2)$$

where $P$ is the momentum operator of the test particle, $M$ its mass and $\{|i\rangle\}_{i=1,...,N}$ the basis of energy eigenstates in $\mathbb{C}^N$. The superoperators appearing at r.h.s. of Eq. (1) are defined according to

$$\mathcal{L}\rho = \text{Tr}_{\text{gas}} \left( T\Gamma^{1/2} [\rho \otimes \rho_{\text{gas}}] \Gamma^{1/2} T^\dagger \right)$$

$$-\frac{1}{2} \text{Tr}_{\text{gas}} \left( \Gamma^{1/2} T^\dagger T \Gamma^{1/2} [\rho \otimes \rho_{\text{gas}}] \right)$$

$$-\frac{1}{2} \text{Tr}_{\text{gas}} \left( [\rho \otimes \rho_{\text{gas}}] \Gamma^{1/2} T^\dagger T \Gamma^{1/2} \right) \quad (3)$$

and

$$\mathcal{R}\rho = i\text{Tr}_{\text{gas}} \left( \left[ \Gamma^{1/2} \text{Re} (T) \Gamma^{1/2}, \rho \otimes \rho_{\text{gas}} \right] \right) \quad (4)$$

respectively, where $\rho_{\text{gas}}$ is the single particle statistical operator describing the gas environment. Note that the operators $\mathcal{L}$ and $\mathcal{R}$ arise by acting with an operator in Lindblad form on a state of system plus gas in factorized form, further taking the partial trace with respect to the gas. While this operation is formally legitimate, and guarantees preservation of trace and Hermiticity of the statistical operator describing the test particle, it is generally not true that the resulting dynamics for the reduced system only is given by a master equation in Lindblad form, thus granting complete positivity and describing a well-defined Markovian dynamics. Indeed this step involves further approximations, which depend in a crucial way on details of system and interaction. It is well known that by taking the partial trace with respect to the unitary evolution of the overall system one can obtain a Markovian dynamics only if further hypotheses hold. This remains true for the case at hand, despite the fact that important approximations have already been introduced in replacing the Hamiltonian dynamics for system plus gas with a Lindblad operator only specified by $T$ and $\Gamma$. The actual proof that a Markovian dynamics applies to the situation of interest and the specific expression of the superoperators appearing in Eq. (1) is obtained through the microscopic calculations performed in Appendix A.

Relying on the results of Appendix A.1 we write the following expression for the contributions in Eq. (1) in the momentum and channel basis $\{|P, i\rangle\}$

$$\langle P, i | \mathcal{L}\rho | P', k \rangle = \sum_{j,l} \int dQ \left[ \langle P - Q, j | \rho | P' - Q, l \rangle M_{ik}^j (P, P'; Q) ight.$$

$$-\frac{1}{2} \langle P, j | \rho | P', k \rangle M_{ji}^l (P + Q, P + Q; Q)$$

$$-\frac{1}{2} \langle P, i | \rho | P', l \rangle M_{kj}^i (P' + Q, P' + Q; Q) \right] \quad (5)$$
where the complex rate functions $M_{ik}^{jl}(P, P'; Q)$ are given by

$$M_{ik}^{jl}(P, P'; Q) = \chi_{ik}^{jl} \int_{Q_\perp} dp \, L_{ij}(p, P - Q; Q) L_{kl}^{*}(p, P' - Q; Q),$$  

(6)

with the $\chi_{ik}^{jl}$ a notational shorthand to indicate that the contribution is different from zero only for $E_{ij} = E_{kl}$, where $E_{kj} = E_k - E_j$ denotes the difference in energy between internal states, while the $p$-integration is restricted to the plane $Q_\perp = \{ p \in \mathbb{R}^3 : p \cdot Q = 0 \}$. The functions $L_{ij}(p, P; Q)$ are defined according to

$$L_{ij}(p, P; Q) = \left( \frac{n_{\text{gas}} m}{m_\ast Q} \right)^{1/2} \mu \left( \frac{p_\perp + m}{M} P_\perp + \left( 1 + \frac{m}{M} \right) \frac{Q}{2} + \frac{E_{ij}}{Q^2/m_\ast} Q \right) \times f_{ij} \left( \text{rel}(p_\perp, P_\perp) - \frac{Q}{2} + \frac{E_{ij}}{Q^2/m_\ast} Q, \text{rel}(p_\perp, P_\perp) + \frac{Q}{2} + \frac{E_{ij}}{Q^2/m_\ast} Q \right),$$  

(7)

where $\mu(p)$ denotes the stationary gas distribution, $n_{\text{gas}}$ is the density of the gas, $m$ the mass of the gas particles, $m_\ast = m M/(M + m)$ the reduced mass, and $f_{ij}(p_f, p_i)$ denote the multichannel complex scattering amplitudes, which depend on the microscopic interaction potential and describe scattering from an initial momentum $p_i$ and internal state $j$ to a final state with momentum $p_f$ and internal state $k$. Moreover, $P_\perp$ and $P_\parallel$ indicate, respectively, the perpendicular and the parallel component of the momentum $P$ with respect to the vector $Q$; while $\text{rel}(p, P) \equiv (m_\ast/m) p - (m_\ast/M) P$ is the relative momentum between the gas particle momentum $p$ and the test particle momentum $P$. 

Exploiting these results we can easily write the master equation Eq. (1) directly in operator form. In fact using the functions $L_{ij}(p, P; Q)$ let us introduce the following family of jump operators

$$L_{Q, P, \varepsilon} = e^{i Q \cdot X / \hbar} \sum_{ij} L_{ij}(p, P; Q) \otimes \varepsilon_{ij},$$  

(8)

where $X$ and $P$ are position and momentum operators of the test particle, while the operators $\varepsilon_{ij} = |i\rangle \langle j|$ act on the internal degrees of freedom only, since $|i\rangle$ denotes the energy eigenstate with eigenvalue $\hbar \omega_i$, and the exponential factor describes momentum exchanges according to $\exp(i X \cdot Q / \hbar) |P\rangle = |P + Q\rangle$. Note that the functions $L_{ij}(p, P; Q)$ essentially depend on the scattering amplitudes and the momentum distribution of the gas, thus keeping into account all the details of the collisional interaction. These expressions appear operator-valued in the master equation, being evaluated for $P \rightarrow P$, so as to take into account the actual momentum of the colliding test particle. The incoherent contribution $\mathcal{L}$ in Eq. (1) finally reads

$$\mathcal{L} \rho = \sum_{\varepsilon} \int dQ \int_{Q_\perp} dp \left( L_{Q, P, \varepsilon} \rho L_{Q, P, \varepsilon}^\dagger - \frac{1}{2} \left\{ L_{Q, P, \varepsilon}^\dagger L_{Q, P, \varepsilon}, \rho \right\} \right).$$  

(9)

The superoperator $\mathcal{R}$ of Eq. (4) on its turn according to Appendix A1b amounts to the commutator with an effective Hamiltonian given by

$$H_n = -2\pi \hbar^2 n_{\text{gas}} / m_\ast \sum_{ij} \int d\rho_0 \mu(p_0) \text{Re} \left[ f_{ij} \left( \text{rel}(p_0, P), \text{rel}(p_0, P) \right) \right] \otimes \varepsilon_{ij}. $$  

(10)

Using the alternative expression given by Eq. (A18) to define $L_{ij}(p, P; Q)$, it is immediately seen that the incoherent term of this master equation confirms the result heuristically obtained in [8]. In the latter reference this equation has also been termed quantum Bloch-Boltzmann equation in that it provides a quantum description of both motional and internal degrees of freedom, thus extending the result of [13], where the centre of mass degrees of freedom were treated classically and the name Bloch-Boltzmann equation was used. These names should not confuse the reader. Indeed, only for the case of an atom in a two-level approximation undergoing a collisional dynamics this equation refers to an extension of the optical Bloch equations with a Boltzmann collision term.

### B. Limiting forms

As a compatibility check of the master equation derived in Appendix A, and in order to make contact with previous work, we will now show how in suitable limits it recovers already known equations. Since the equation describes the
quantum dynamics of a test particle with both internal and translational degrees of freedom immersed in a dilute gas, natural limiting situations appear considering a structureless test particle or an immobile system. These situations correspond to the quantum linear Boltzmann equation \[12\], and to the master equation for an immobile system interacting through collisions with a background gas \[9\] \[14\]. Another natural limit consists in a hybrid quantum classical description, in which either internal or centre of mass degrees of freedom are treated classically. The master equation corresponding to this last case has already been considered in \[13\]. A classical treatment of both kinds of degrees of freedom leads to the master equation for a classical Markov process, with a probability density depending on both a discrete and a continuous index.

a. Quantum linear Boltzmann equation If the internal degrees of freedom can be disregarded the sum in Eq. \(9\) has a single non vanishing contribution, so that instead of the multichannel scattering amplitudes \(f_{ij}(p_f, p_i)\) there is a single amplitude which can be indicated as \(f(p_f, p_i)\). The incoherent term in the master equation reduces to

\[
\mathcal{L} \rho = \int dQ \int_{Q_0} dp \left( L_{Q,P} \rho L_{Q,P}^\dagger - \frac{1}{2} \{ L_{Q,P}^\dagger L_{Q,P}, \rho \} \right) \tag{11}
\]

with

\[
L_{Q,P} = \frac{n_{\text{gas}} m}{m_s^2} e^{iQ x_0 / \hbar} \sqrt{\mu \left( p_\perp + \frac{m}{M} p_\parallel + \left( 1 + \frac{m}{M} \right) \frac{Q}{2} \right)} \times f \left( \text{rel} (p_\perp, p_\perp) - \frac{Q}{2}, \text{rel} (p_\perp, p_\perp) + \frac{Q}{2} \right),
\]

while the Hamiltonian term reads \(H_0 + H_n\), where \(H_0 = P^2 / 2M\) and

\[
H_n = -\frac{2\pi h^2 n_{\text{gas}}}{m_s^2} \int d\mu(p_0) \text{Re} \left[ f \left( \text{rel} (p_0, P), \text{rel} (p_0, P) \right) \right]
\]
takes into account the energy shift due to forward scattering. This result complies with the quantum linear Boltzmann equation obtained in \[12\], whose properties have been discussed in detail in \[4\].

b. Immobile tracer particle We now consider the opposite situation, corresponding to an infinitely massive test particle, so that the dynamics of the translational degrees of freedom can be neglected. To consider this limit it is convenient to come back to the expression Eq. \(5\) of the quantum master equation in terms of the complex rate functions \(M^j_{ik}(p, p'; Q)\), which in the limit \(M \to \infty\) when integrated over \(Q\) reduce to

\[
M^j_{ik} = \frac{n_{\text{gas}}}{m^2} \chi^j_{ik} \int dp_0 \delta \left( \frac{P^2 - P_0^2}{2m} + E_0 - E_j \right) f_{ij}(p, p_0) f^*_{kl}(p, p_0), \tag{12}
\]

where no dependence on the test particle's momentum is left. The matrix elements of the incoherent part of the quantum Bloch-Boltzmann equation are therefore now given by

\[
\bra{i} \mathcal{L} \rho \ket{k} = \sum_{jl} \left( \bra{j} \rho \ket{l} M^j_{ik} - \frac{1}{2} \bra{j} \rho \ket{l} M^j_{ii} - \frac{1}{2} \bra{i} \rho \ket{l} M^j_{ij} \right), \tag{13}
\]

while the coherent part corresponds to a effective Hamiltonian whose matrix elements in the energy eigenbasis are given by \(E^j_{ik} = -2\pi h^2 (n_{\text{gas}} / m_s) \chi^j_{ik} \int d\mu(p_0) \text{Re} \left[ f_{ij}(p_0, p_0) \right]\), thus confirming the result obtained in \[5\] for the case of a non degenerate Hamiltonian.

c. Quantum classical description The limiting expressions of the quantum Bloch-Boltzmann equation, obtained when either the internal or the translational degrees of freedom can be treated as a classical label, correspond to hybrid quantum classical descriptions, which naturally arise when decoherence affects on different time scales the two kind of degrees of freedom.

When the centre of mass degrees of freedom can be treated classically it is convenient to introduce the classical rates

\[
M^j_{ik}(P + Q; Q) := \frac{n_{\text{gas}}}{m^2} \chi^j_{ik} \int dp_0 \delta \left( \frac{P^2 - P_0^2}{2m} + E_0 - E_j \right) f_{ij}(p, p_0) f^*_{kl}(p, p_0), \tag{14}
\]

with \(M^j_{ik}(P, P'; Q)\) as in Eq. \(6\), so that the semiclassical Bloch-Boltzmann equation reads

\[
\frac{d}{dt} \rho(P) = \frac{1}{i\hbar} \left[ \sum_i \hbar \omega_i \ket{i} \bra{i} + H_n(P), \rho(P) \right] + \sum_{ijkl} \int dQ \left[ M^j_{ik}(P; Q) E_{ij} \rho(P - Q) E^*_{kl} - \frac{1}{2} M^j_{ik}(P + Q; Q) \left\{ E_{kl} E_{ij}, \rho(P) \right\} \right], \tag{15}
\]
where $H_n(P)$ is obtained from Eq. (10) with the replacement $P \rightarrow P$, and $\rho(P)$ denotes a collection of trace class operators in $\mathbb{C}^n$ normalized according to $\int dP \text{Tr}_{\mathbb{C}^n} \rho(P) = 1$.

If the classical approximation applies for the internal degrees of freedom the incoherent term of the master equation giving a quantum description of the translational dynamics only takes the form

$$
\frac{d}{dt} \rho_i = \frac{1}{i\hbar} \left[ \frac{p^2}{2M} + H_i^e, \rho_i \right] + \sum_j \int dQ \int_{\mathbb{P}^2} \frac{dp}{2\pi} \left\{ e^{iQ X/h} L_{ij} (p, P; Q) \rho_j L_{ij} (p, P; Q)^\dagger e^{-iQ X/h} \right\} - \frac{1}{2} \left\{ L_{ij} (p, P; Q)^\dagger L_{ij} (p, P; Q), \rho_i \right\}.
$$

Here

$$
H_i^e = -2\pi \hbar^2 n_{\text{gas}} \int dp_{0j} (p_0) \text{Re} \left[ f_{ij} (\text{rel} (p_0, P), \text{rel} (p_0, P)) \right],
$$

and $\rho_i$ denotes a collection of trace class operators in $L^2(\mathbb{R}^3)$ normalized according to $\sum_{i=1}^n \text{Tr}_{L^2(\mathbb{R}^3)} \rho_i = 1$.

For the case for which all the off-diagonal elements with respect to momentum and internal energy eigenvalues vanish, that is to say $\langle P, i | \rho | P', k \rangle = 0$ if $P \neq P'$ or $i \neq k$, the motional state of the test particle is fully characterized by the distribution of the diagonal terms $f_i (P) = \langle P, i | \rho | P, i \rangle$, which is a classical probability density obeying the classical Markovian master equation

$$
\frac{d}{dt} f_i (P) = \sum_j \int dQ f_j (P - Q) M_{ij}^{ii} (P; Q) - f_i (P) \sum_j \int dQ M_{ij}^{ii} (P + Q; Q),
$$

where the positive quantities $M_{ij}^{ii} (P; Q)$ defined in Eq. (14) can actually be interpreted as the transition rates from an initial momentum $P - Q$ and internal state $j$ to a final momentum $P$ and internal state $i$. This classical Markovian master equation provides the natural generalization of the classical linear Boltzmann equation to a particle with internal degrees of freedom [15].

### III. EFFECT OF INTERNAL DEGREES OF FREEDOM ON CENTRE OF MASS DECOHERENCE

The quantum linear Boltzmann equation has proven useful in the description of collisional decoherence, as well as in the evaluation of the index of refraction for matter waves [4, 12, 16–18]. We will now consider the effect of internal degrees of freedom, affecting the collisional interaction between massive test particle and background gas, on the visibility of the interference fringes in an interferometric setup. In particular we will show that the visibility can exhibit oscillations due to non-Markovian effects. The effect of the entanglement between internal and centre of mass degrees of freedom for the visibility of quantum interference experiments has already been considered in [19], in the absence however of decoherence effects.

#### A. Generalized Lindblad structure

The quantum master equation Eq. (1) is in Lindblad form: this means that the dynamics of the test particle is Markovian when both translational and internal degrees of freedom are described and detected. A different situation emerges if the translational or the internal degrees of freedom, although influencing the collisional dynamics, are not revealed during the measurement process. In this case they must be averaged out from the description of the system, by means of the partial trace, thus becoming part of the environment. As well known in the classical case, a non-Markovian dynamical regime becomes Markovian by suitably enlarging the set of degrees of freedom and vice-versa. Indeed, a unitary Markovian time evolution for both system and reservoir generally gives a non-Markovian reduced dynamics for the system, the degree of non-Markovianity of the description also depending on where we set the border between system and environment, which ultimately depends on the physical quantities actually measurable by the experimenter. A smaller set of observed degrees of freedom, with respect to those actually involved in the dynamics, can lead from a Markovian to a non-Markovian regime. A general mechanism describing this passage in quantum systems is presented in [4]: a Lindblad structure on a bipartite system can generate in the two reduced subsystems a generalized Lindblad structure, typically describing a non-Markovian dynamics.

In the situation we are considering, the bipartite system is formed by the translational and the internal degrees of freedom of the test particle. If the measurements at the output of the detector cannot probe the internal degrees of
freedom, the only experimentally accessible quantities are expectations or matrix elements of the statistical operator given by
\[ \rho(t) = \text{Tr}_C \{ \rho(t) \} = \sum_i \langle i | \rho(t) | i \rangle =: \sum_i \rho_i(t), \tag{19} \]

where \( \rho(t) \) is the statistical operator describing the full dynamics of the test particle. It is easy to see that, if the free Hamiltonian is non degenerate, the diagonal matrix elements in the energy basis with respect to the internal degrees of freedom of the master equation lead to Eq. \( \langle 16 \rangle \), i.e. a coupled system of equations for the collection \( \{ \rho_i(t) \} \), of trace class operators on \( L^2(\mathbb{R}^3) \). This system of equations has a generalized Lindblad structure \[ \langle 22 \rangle \], and therefore it can also describe highly non-Markovian dynamics for the statistical operator \( \rho(t) \) given by Eq. \( \langle 19 \rangle \). Indeed, there is generally no closed evolution equation for \( \rho(t) \), but from the knowledge of the initial collection \( \{ \rho_i(0) \} \), the generalized Lindblad structure allows to obtain the collection \( \{ \rho_i(t) \} \), at time \( t \) and therefore, through Eq. \( \langle 19 \rangle \), also \( \rho(t) \). In the next paragraph we are going to explicitly point out non-Markovian behaviour described by the generalized Lindblad structure, which express the effect of correlations between internal and translational degrees of freedom on the visibility of interference fringes for superpositions of motional states. A complementary situation has been considered in \[ \langle 20 \rangle \], where the effect of collisional decoherence on internal state superpositions of a cold gas has been studied in detail.

In typical interferometric experiments the test particle is much more massive than the particles of the background gas. The dependence on the momentum operator in the Lindblad operators describing the collisional dynamics and in the Hamiltonian part determining the energy shift can therefore be replaced by a fixed value \( P_0 \), which represents the initial momentum of the test particle entering the interferometer. Taking the diagonal matrix elements of the general form of the master equation given by Eq. \[ \langle 1 \rangle \] and specified by Eq. \[ \langle 10 \rangle \] and Eq. \[ \langle 9 \rangle \], assuming non degeneracy of the internal energy eigenvalues one finally obtains for the collection of operators \( \rho_i(t) = \langle i | \rho(t) | i \rangle \) the following coupled system of equations:
\[ \frac{d}{dt} \rho_i(t) = \frac{1}{i\hbar} \left[ P^2 \rho_i(t) \right] + \sum_j \left( \Gamma^{ij}_{P} \int dQ P^{ij}_{P}(Q) e^{iQ \cdot X / \hbar} \rho_j(t) e^{-iQ \cdot X / \hbar} - \Gamma^{ji}_{P} \rho_i(t) \right), \tag{20} \]

with \( P^{ij}_{P} \) and \( \Gamma^{ji}_{P} \) probability densities and transition rates defined by
\[ P^{ij}_{P}(Q) := \frac{M^{ij}_{ii}(P; Q)}{\int dQM^{ij}_{ii}(P + Q; Q)} \tag{21} \]
and
\[ \Gamma^{ij}_{P} := \int dQM^{ij}_{ii}(P + Q; Q). \tag{22} \]

respectively. Note that at variance with Eq. \[ \langle 16 \rangle \] we are now not assuming a classical dynamics for the internal degrees of freedom, we focus on the diagonal matrix elements of the internal states only since the latter are enough to determine the non-Markovian dynamics of the motional state according to Eq. \[ \langle 19 \rangle \]. The fact that the positive quantities \( M^{ij}_{ii}(P; Q) \) are transition rates implies that \( P^{ij}_{P}(Q) \) can be interpreted as the probability distribution function for a test particle with momentum \( P \) and internal energy eigenstate \( j \) to exchange a momentum \( Q \), and to go into the internal state \( i \) due to a collision with the gas. On the same footing, \( \Gamma^{ij}_{P} \) can be interpreted as the total transition rate for a test particle with momentum \( P \) and internal state \( j \) to go to a fixed final internal energy eigenstate \( i \).

**B. Explicit solutions in position representation**

We are now going to describe the visibility reduction predicted by the generalized Lindblad structure Eq. \[ \langle 20 \rangle \] obtained from the quantum Bloch-Boltzmann equation Eq. \[ \langle 1 \rangle \] in the limit of a very massive test particle.

To obtain the formula describing the fringes visibility in an explicit way we need to solve the equation of motion in the position representation. Starting from Eq. \[ \langle 20 \rangle \] and omitting for simplicity the explicit dependence on the classical label \( P_0 \) denoting the momentum of the test particle, we obtain
\[ \frac{d}{dt} \rho_i(X, X', t) = \frac{1}{i\hbar} (\Delta X - \Delta X') \rho_i(X, X', t) + \sum_j \left( \Gamma^{ij}_{P} \Phi^{ij}(X - X') \rho_j(X, X', t) - \Gamma^{ji}_{P} \rho_i(X, X', t) \right), \tag{23} \]
where \( \rho_i(X, X', t) \) denotes the matrix element \( \langle X | \rho_i(t) | X' \rangle \) and \( \Phi^{ij}(X - X') \) is the characteristic function of the probability density \( P^{ij}(Q) \) \[21\], i.e. its Fourier transform

\[
\Phi^{ij}(X - X') = \int dQ e^{i(X - X')Q/\hbar} p^{ij}(Q).
\]

(24)

We will now consider a few cases in which Eq. (23) can be solved analytically, so as to obtain an exact expression for the visibility, showing up different possible qualitative behaviour.

1. N-level system

When the collisions are purely elastic, so that they do not lead to transitions between different internal states, the scattering rates satisfy \( \Gamma^{ij} = \delta_{ij} \Gamma^{ii} \). This is the case when the energy exchanges involved in the single collisions are much smaller than the typical separation of the internal energy levels \[13\]. The equations for the different \( \rho_i \) then become uncoupled and take the form

\[
\frac{d}{dt} \rho_i(X, X', t) = \frac{1}{\hbar} (\Delta X - \Delta X') \rho_i(X, X', t) - \Gamma^{ii}(1 - \Phi^{ii}(X - X')) \rho_i(X, X', t).
\]

(25)

The latter equation can be conveniently solved introducing the function \[22\]

\[
\chi_i(\lambda, \mu, t) := \text{Tr} \left\{ \rho_i(t) e^{i(\lambda X + \mu P)/\hbar} \right\}
\]

(26)

where \( X \) and \( P \) as usual denote position and momentum operators of the test particle. In such a way Eq. (25) leads to

\[
\partial_t \chi_i(\lambda, \mu, t) = \left[ \frac{\lambda}{M} \cdot \partial_\mu - \Gamma^{ii}(1 - \Phi^{ii}(\mu)) \right] \chi_i(\lambda, \mu, t),
\]

(27)

which is an equation of first order solved by

\[
\chi_i(\lambda, \mu, t) = \chi_i^0(\lambda, \lambda t/M + \mu) e^{-\Gamma^{ii}(1 - \Phi^{ii}(\lambda t/M + \mu)) dt'},
\]

(28)

where the function \( \chi_i^0(\lambda, \lambda t/M + \mu) \) obeys the free equation \( \partial_t \chi_i(\lambda, \mu, t) = (\lambda/M) \cdot \partial_\mu \chi_i(\lambda, \mu, t) \). Inverting Eq. (26) by taking the Fourier transform with respect to \( \lambda \),

\[
\rho_i(X, X', t) = \int \frac{d\lambda}{(2\pi\hbar)^3} e^{-i\lambda(X + X')/2\hbar} \chi_i(t, \lambda, X - X'),
\]

(29)

we obtain the exact solution

\[
\rho_i(X, X', t) = \int \frac{d\lambda ds \, d\mu}{(2\pi\hbar)^3} e^{-i\lambda s/\hbar} e^{-\Gamma^{ii}(1 - \Phi^{ii}(\lambda t/M + X - X')) dt'} \rho_i^0(X + s, X' + s, t)
\]

expressed in terms of an integral of the freely evolved subcollections \( \rho_i^0(X, X', t) \) with a suitable kernel, where we have set

\[
\rho_i^0(X, X', t) = \int \frac{d\lambda}{(2\pi\hbar)^3} e^{-i\lambda(X + X')/2\hbar} \chi_i^0(\lambda, X - X', t)
\]

\[
= \langle X | \exp \left( - \frac{i P^2}{\hbar 2M} t \right) \rho_i(0) \exp \left( + \frac{i P^2}{\hbar 2M} t \right) | X' \rangle
\]

(30)

and \( \rho_i(0) = \langle i | \rho(0) | i \rangle \). The evolution of the statistical operator given by Eq. \[19\] is obtained summing the different \( \rho_i(X, X', t) \) over the discrete index \( i \). For an initial state given by a product state between the translational and the internal part, so that \( \rho_i(0) = p_i \rho(0) \), we finally obtain

\[
\rho(X, X', t) = \sum_i p_i \int \frac{d\lambda ds \, d\mu}{(2\pi\hbar)^3} e^{-i\lambda s/\hbar} e^{-\Gamma^{ii}(1 - \Phi^{ii}(\lambda t/M + X - X')) dt'} \rho^0(X + s, X' + s, t).
\]

(31)

This result reduces to the standard Markovian situation, when either only one \( p_i \) is different from zero (and therefore equal to one), or the rates are all equal. This limiting cases describes situations in which the initial state is in a specific internal state or the collisions do not depend on the internal state of the tracer particle.
2. Two-level system

For the case of a two-level system a natural situation corresponds to inelastic scattering taking place only when the test particle gets de-excited, so that only one of the two scattering rates is different from zero. This case can still be treated analytically. Assuming \( \Gamma^{21} = 0 \), the equation for \( \chi_2(t, \lambda, \mu) \) gets closed, and is solved by

\[
\chi_2(\lambda, \mu, t) = \int_0^t e^{-\Gamma^{12} t} e^{-\Gamma^{22} \int_0^t (1 - \Phi^{22}(\lambda(t')/M + \mu)) \, dt'} \chi_2(\lambda, \mu, t) \, dt' .
\]  

(32)

The equation for \( \chi_1(\lambda, \mu, t) \) then reads

\[
\partial_t \chi_1(\lambda, \mu, t) = \left[ \frac{\lambda}{M} \cdot \partial_\mu - \Gamma^{11} (1 - \Phi^{11}(\mu)) \right] \chi_1(\lambda, \mu, t) + \Gamma^{12} \Phi^{12}(\mu) \chi_2(\lambda, \mu, t)
\]

(33)

and its solution is given by

\[
\chi_1(\lambda, \mu, t) = e^{-\Gamma^{11} \int_0^t (1 - \Phi^{11}(\lambda(t'/M + \mu)) \, dt'} \chi_0(\lambda, \mu) + \Gamma^{12} \int_0^t \int_0^t e^{-\Gamma^{11} (\lambda(t'/M + \mu))} \chi_2(\lambda, \mu, t) \chi_2(\lambda, \mu, t') \, dt' \, dt''
\]

(34)

This formula explicitly shows that \( \chi_1(\lambda, \mu, t) \) depends on the function \( \chi_2(\lambda, \mu, \cdot) \) evaluated over the whole time interval between \( 0 \) and \( t \), a typical signature of non-Markovian dynamics. Assuming once again that the initial state is characterized by \( \rho_i(0) = \rho_i \), the statistical operator describing the translational degrees of freedom of the test particle is given at time \( t \) by the expression

\[
\rho(X, X', t) = \int \frac{d\lambda d\mu}{(2\pi\hbar)^3} e^{-i\lambda s/\hbar} \rho_0(X + s, X' + s, t) \left( p_2 e^{-\Gamma^{12} t} e^{-\Gamma^{22} \int_0^t (1 - \Phi^{22}(\lambda(t'/M + X + X')) \, dt'} + \Gamma^{12} \int_0^t \int_0^t e^{-\Gamma^{11} \lambda(t'/M + X + X')} \chi_2(\lambda, \mu, t) \chi_2(\lambda, \mu, t') \, dt' \, dt'' \right).
\]

(35)

C. Non-exponential visibility reduction

We can now explicitly present the visibility reduction predicted by the generalized Lindblad structure obtained from the quantum master equation for a test particle with internal degrees of freedom. Our aim is to obtain an exact expression for the loss of visibility in a double-slit arrangement as a function of the time of interaction with the environment, and to illustrate by means of example how the presence of the various scattering channels, corresponding to the different internal states, can actually lead to non-Markovian behaviours. In particular we will consider the situation of purely elastic collisions in full generality, also allowing for inelastic scattering in the case of a two-level system. While the experimental setting is always taken to be the same, the different number of internal degrees of freedom involved and the presence or absence of inelastic scattering events will lead to more or less marked non-exponential behaviours in the reduction of the visibility fringes.

1. Visibility formula

We first derive a formula for the visibility reduction in the case of a double-slit experiment in the far field approximation. A beam of particles moves towards a grating perpendicular to its direction of propagation, and with two identical slits separated by a distance \( d \), finally reaching a detector where the fringes of interference are observed. During the flight through the interferometer the beam particles interact through collisions with the environment in the background, thus undergoing decoherence. We consider an initial product state, so that in the notation of Eq. (20) one has \( \rho_i(0) = \rho_i \), where \( \rho(0) \) describes the translational degrees of freedom. If after the passage through the collimation slits the test particle is described by \( \rho_{sl} \), then the double-slit grating prepares the initial state

\[
\rho(0) = 2 \cos \left( \frac{P \cdot d}{2} \right) \rho_{sl} \cos \left( \frac{P \cdot d}{2} \right).
\]

(36)
where \( \rho \) and \( \hbar \) are the density matrix and the reduced Planck constant, respectively. For the denominator of Eq. (38) one can proceed in an analogous way, using the wavelength associated to the test particle and performing the integral over \( Y \) and \( \sigma \). The integrand is negligible if \( Y \gg t/M \), so that the integrand is negligible if \( Y \gg t/M \), we consider the quantity

\[
\mathcal{Y} = \frac{2}{\langle X - \frac{1}{2}d|u(t)[\rho_{0t}e^{-i\mathcal{H}d/h}]X - \frac{1}{2}d\rangle + \langle X + \frac{1}{2}d|u(t)[\rho_{0t}]X + \frac{1}{2}d\rangle},
\]

where now \( t \) is the time employed by the test particle to reach the detector. Indeed this result remains true for any translation-covariant time evolution.

For an initial factorized state of the test particle we can exploit Eq. (31) to obtain a closed formula for the time evolution operator \( u(t) \) depending on the initial internal state, i.e. on the coefficients \( p_i \) appearing in \( \rho_i(0) = p_i \rho(0) \): the numerator of Eq. (38) then reads

\[
2 \sum_i \rho_i \int \frac{d\lambda}{2\pi} e^{-i\lambda d/h} \langle X - \frac{1}{2}d + s|u_0(t)[\rho_{0t}e^{-i\mathcal{H}d/h}]|X - \frac{1}{2}d + s\rangle e^{i\lambda t} f_0(1 - \Phi^{ij}(X(t'-t)/M)) dt',
\]

where \( u_0(t) \) is the free evolution operator of the translational degrees of freedom, so that

\[
\langle X - \frac{1}{2}d + s|u_0(t)[\rho_{0t}]|X + \frac{1}{2}d + s\rangle = \langle X - \frac{1}{2}d + s|u_0(t)[\rho_{0t}]|X + \frac{1}{2}d + s\rangle.
\]

The latter expression can also be written

\[
\langle X - \frac{1}{2}d|u_0(t)[\rho_{0t}]|X + \frac{1}{2}d\rangle = \left( \frac{M}{t} \right)^3 e^{-iMd \cdot X/(ht)} \int \frac{dYdY'}{(2\pi\hbar)^3} e^{iM(Y^2 - Y'^2)/(2ht)} \times e^{-iM \cdot (Y - Y')/(ht)} e^{iMd \cdot (Y + Y')/(2ht)} |Y| \rho_{0t} |Y'\rangle,
\]

assuming due to symmetry \( \text{Tr}(X\rho_{0t}) = 0 \).

This formula enables us to implement the far field approximation. In fact, let \( \sigma \) be the width of the two slits, so that the integrand is negligible if \( \mathbf{Y} \) (and similarly for \( \mathbf{Y}' \)) takes values outside the support of \( \rho_{0t} \), then \( \mathbf{M}^2 \cdot (ht) \lesssim M \sigma^2 / (ht) \) and therefore for a time long enough such that \( \hbar t/M \gg \sigma^2 \) the first exponential can be disregarded. The same applies for the last exponential if \( \hbar t/M \gg \sigma d \). For times longer than \( \max \{ M\sigma^2/h, M\sigma d/h \} \), corresponding to the far field approximation, we get

\[
\langle X - \frac{1}{2}d|u_0(t)[\rho_{0t}]|X + \frac{1}{2}d\rangle \approx \left( \frac{M}{t} \right)^3 e^{-iMd \cdot X/(ht)} \bar{\rho}_{0t} \left( \frac{M}{t} \cdot X \right),
\]

where \( \bar{\rho}_{0t}(\cdot) \) is the distribution function for the momentum of the particle in the state \( \rho_{0t} \),

\[
\bar{\rho}_{0t} \left( \frac{M}{t} \cdot X \right) = \int \frac{dYdY'}{(2\pi\hbar)^3} e^{-iM \cdot (Y - Y')/(ht)} |Y| \rho_{0t} |Y'\rangle.
\]

The equivalence between the assumption \( \hbar t/M \gg \sigma^2 \) and the far field approximation \( L \gg \sigma^2 / \lambda \), where \( \lambda = \hbar / P_z \) is the wavelength associated to the test particle and \( L \) is the distance between grating and detector, is easily seen from the relation \( L = p_z t / M \), where \( p_z \) is the component along the \( z \) direction of the massive particle, assumed to be constant. Substituting Eq. (39) in the numerator of Eq. (38) and using the approximation \( \bar{\rho}_{0t}(M \cdot (X + s)/t) \approx \bar{\rho}_{0t}(M \cdot X / t) \) valid because of the localization of the state \( \rho_{0t} \), we can easily perform the integrals over \( s \) and \( \lambda \), further observing that \( \Phi^{ij}(0) = 1 \) for the normalization of \( P^{ij}(Q) \).
2. Non-exponential behaviours

The desired expression for the visibility in the absence of inelastic scattering and for an arbitrary number \( n \) of channels thus reads

\[
V = \left| \sum_{i=1}^{n} p_i e^{-\Gamma_{ii} \int_{0}^{t} \left(1 - \Phi_{ii}(d(s-1))\right) ds} \right|,
\]

where we recall that the probabilities \( p_i \) give the weight of the different internal states in the initial preparation. The dependence on \( t \) in this formula can be easily made explicit with the change of variable \( t'/t = s \), so that one has

\[
V = \left| \sum_{i=1}^{n} p_i e^{-\Gamma_{ii} \int_{0}^{1} \Phi_{ii}(d(s-1)) ds} \right|.
\]

FIG. 1. Plot of the visibility in a double-slit arrangement as a function of the interaction time with the environment, for the case of elastic scattering events only, according to Eq. (43) and with growing number of channels from left to right. The dashed lines represent the Markovian exponential decays occurring if a single elastic channel prevails on the others, the one with the highest and lowest decay rate corresponding to lower and upper line respectively. (a) Visibility for \( n = 2 \) elastic channels, according to the expression Eq. (46). It appears a non monotonic decay as a consequence of the interference between the contributions of the two different elastic channels. The coefficients \( \alpha_i \) and \( \beta_i \) defined in Eq. (45) are calculated for two Gaussian distributions \( P_{11}(Q) \) and \( P_{22}(Q) \) of the exchanged momenta. Taking \( d = d\hat{z} \) as direction of propagation inside the interferometer we only need to specify the mean and the variance of the exchanged momenta along this axis, respectively \( \mu_{ii} \) and \( \sigma_{ii} \), \( i = 1,2 \). The plot is for \( p_1 = p_2 = \frac{1}{2} \), while \( \Gamma_{11} = \Gamma_{22} = 10 \), \( d = 1 \), \( \sigma_{11} = \sigma_{22} = 0.1 \), \( \mu_{11} = -0.2 \), \( \mu_{22} = 0.3 \) in arbitrary units. (b) Visibility for \( n = 8 \) elastic channels according to the general expression Eq. (43). The characteristic functions \( \Phi_{ii} \) are calculated starting from Gaussian distributions, assuming equal rates \( \Gamma_{ii} = 10 \) and equal variances \( \sigma_{ii} = 0.1 \) in arbitrary units as in (a). The \( p_i \) are uniformly distributed and the means \( \mu_{ii} \) are equally spaced in the range from \(-0.2\) to \(0.3\).

From Eq. (43) one can easily see the difference between the Markovian situation, corresponding to \( n = 1 \), and the general case. If there is just one term in the sum, the modulus simply picks out the real part of the characteristic function in the exponential and Eq. (43) describes an exponential decay in time with a rate \( \Gamma \left(1 - \int_{0}^{1} \text{Re}\left\{\Phi(d(s-1))\right\} ds\right) \).

This can happen if only one internal energy state is populated in the initial preparation or the scattering events are actually independent on the internal state. If there are at least two terms, the modulus can generate oscillating terms as a consequence of the interference of the different phases arising since the functions \( \Phi_{ii} \) are in general complex valued. Even if the imaginary parts of the characteristic functions are zero, i.e. the distribution functions of the exchanged momenta are even, Eq. (43) can describe highly non-exponential behaviour. In this case in fact it reduces to

\[
V = \sum_{i=1}^{n} p_i e^{-\Gamma_{ii} \int_{0}^{1} \Phi_{ii}(d(s-1)) ds} t,
\]

i.e. the sum of different exponential functions. As shown in \( 0 \) this kind of relations can describe behaviour very different from the exponential one.
Let us consider in more detail the case of a two-level system. Introducing the notation

\[
\begin{align*}
\alpha^i &:= \text{Re} \int_0^1 \{ \Phi^{ii}(d(s-1)) \} \, ds \\
\beta^i &:= \text{Im} \int_0^1 \{ \Phi^{ii}(d(s-1)) \} \, ds,
\end{align*}
\]

the visibility reduction is explicitly given by

\[
V = \left[ p_1^2 e^{-2\Gamma^{11}(1-\alpha^1)t} + p_2^2 e^{-2\Gamma^{22}(1-\alpha^2)t} + 2p_1p_2 e^{-\Gamma^{11}(1-\alpha^1)t} e^{-\Gamma^{22}(1-\alpha^2)t} \cos \left[ \left( \Gamma^{11}\beta^1 - \Gamma^{22}\beta^2 \right)t \right] \right]^{1/2}.
\]

This formula describes a decrease modulated by the oscillations produced by the cosine function. To illustrate this behaviour in Fig. 1 a) we plot the visibility as a function of time, considering by means of example two Gaussian distributions. Note that the appearance of the oscillations depends on a non vanishing mean value for the distribution. Note also that the asymmetry in the single interaction channel being determined in this case by the direction of propagation.

The behaviour described by Eq. (43) for an n-level system is illustrated in Fig. 1 b), where we show how the increased number of levels can strongly suppress the oscillations and lead to a reduction of the visibility. The dashed lines represent the exponential decays pertaining to the Markovian situation arising if only one of the internal energy states is initially populated, the one with the highest or lowest decoherence rate corresponding to the lower or upper dashed line respectively. It appears that with growing n the interference between the contributions of the different channels to Eq. (43) rapidly determines a decay of the visibility sensibly faster than that occurring for the corresponding Markovian single-channel dynamics. Indeed in Fig. 1 left and right panel correspond to the same interaction strength but differ in the number of involved degrees of freedom, ranging to n = 2 to n = 8.

Relying on the results of Sect.III B 2 one can also obtain an expression of the visibility in the presence of inelastic scattering for a two-level system. Indeed starting from Eq. (55) and following the same procedure as above one comes to

\[
V = \left| e^{-\Gamma_{12}t} + \Gamma_{12} e^{-\Gamma_{11}t} \int_0^t \left( e^{-\Gamma_{12}t' e^{-\Gamma_{11}t_0'}} (1-\Phi^{11}(d(t(t'_0-t))/t)) dt' \Phi^{12} \left( d\frac{t'}{t} - t \right) \right) dt' \right|,
\]

where for simplicity \( p_2 = 1 \), and we have taken \( \Gamma^{22} = 0 \), so that the oscillations in the visibility cannot be traced back to interference among different components. An illustration of the behaviour of the visibility in this case has been plotted in Fig. 2, always assuming for the sake of generality a Gaussian distribution of momentum transfers. In this case the dashed line corresponds to the exponential Markovian decay occurring if only the elastic channel is involved in the dynamics. It immediately appears that a non monotonic behaviour in the loss of visibility is observed also in this case, due to the multiple time integration in Eq. (55).

We have here considered the visibility reduction as a function of time. However in typical interferometric experiments the time of flight is fixed, and it is more natural to study the loss of visibility as a function of the strength of the interaction with the environment. In collisional decoherence this depends on the number of collisions, directly proportional to the gas density or equivalently to its pressure for a fixed time of flight. As in the Markovian case, we can thus express the visibility as a function of the pressure of the background gas, which is the physical quantity directly tunable in actual experiments [16]. Introducing the effective cross section \( \sigma_{eff}(P_0, i) \) according to the relation

\[
\sum_j \int dQ M_{jj}^{ii}(P + Q; Q) = n_{gas} \frac{P}{M} \sigma_{eff}(P, i),
\]

where the l.h.s. denotes the classical loss term appearing in Eq. (18), one has for an ideal gas

\[
\Gamma_{P_0}^{ii} = n_{gas} \frac{P_0}{M} \sigma_{eff}(P_0, i) = \frac{p}{M k_B T} P_0 \sigma_{eff}(P_0, i),
\]

where \( p \) is the pressure of the gas and \( T \) its temperature. One can thus introduce a family of reference pressures depending on the initial internal state of the particle entering in the interferometer

\[
p_0^* = \frac{M k_B T}{P_0 \sigma_{eff}(P_0, i)}.
\]
FIG. 2. Plot of the visibility in a double-slit arrangement as a function of the interaction time with the environment, for the case in which one of the internal states also undergoes inelastic scattering, according to Eq. (47) with $n = 2$. It clearly appears a non-monotonic decay of the visibility as a consequence of the multiple time integration describing the contribution of the inelastic channel. The distributions of momentum transfers are assumed Gaussian, with $\sigma_{11} = 1, \sigma_{12} = 3, \mu_{11} = 1, \mu_{12} = 5$; moreover $\Gamma^{11} = 0.75$ and $\Gamma^{12} = 1.75$. The dashed line corresponds to the Markovian dynamics determined by the channel undergoing elastic scattering only.

where $t$ is the time of flight, so that Eq. (43) can equivalently be written as a function of the pressure in the interferometer

$$\Gamma^{ii} t = \frac{p}{p_0}.$$ (50)

This simply implies that the behaviour of the visibility as a function of time is equivalent to its behaviour with respect to the pressure and therefore the disturbance of the environment.

IV. CONCLUSIONS

We have derived the master equation describing the dynamics of a test particle with both translational and internal degrees of freedom, interacting through collisions with a low density background gas. This has been done building on the so-called monitoring approach [5], and confirms a previous heuristic argument put forward by one of us in [6]. The present microscopic derivation further allows to determine the energy shift. As we have checked, the result reduces to known equations in suitable limits: the quantum linear Boltzmann equation if the internal degrees of freedom are neglected [4], the master equation for an immobile test particle if the translational degrees of freedom are not relevant [5, 14], as well as quantum classical Markovian master equations if one or both kind of degrees of freedom can be treated as classical. Note that the natural bases in the derivation was given by momentum for the motional degrees of freedom and energy for the internal ones, the latter corresponding to the channel basis of scattering theory. In these cases different channels are only coupled through the collision term. If another internal basis can be of interest, also coherent tunnelling effects appear.

We have further focused on the situation in which the internal degrees of freedom, in spite of influencing the collisional scattering cross section, are not probed by the measuring apparatus and therefore have to be averaged out from the set of the observed dynamical variables, thus effectively becoming part of the environment. The equation obtained in this situation is no more of Lindblad type, but rather takes the form of a generalized Lindblad structure [6, 9]. It can therefore describe behaviour quite different from that characterizing a Markovian dynamics. Solving these equations in the position representation for an initial factorized state, we have obtained an explicit expression for the visibility reduction in interferometric experiments when internal degrees of freedom are involved. The behaviour of the visibility can indeed be quite different from the exponential decay corresponding to a Markovian dynamics, showing up e.g. oscillations and revivals.

The interplay between different degrees of freedom in a bipartite system is a natural source of non-Markovian behaviour, when either degrees of freedom cannot be controlled, thus acting as an environment. This scenario has here been studied in a concrete setting, assuming a factorized initial state, and describing the dynamics in terms of a generalized Lindblad structure. Such a choice of initial condition is relevant for the considered interferometric setting, it would be however of great importance to consider initially correlated states, which naturally appear when considering a non-Markovian dynamics in a strong coupling regime. It would further be of interest to study whether, instead
of using this generalized Lindblad structure, one can obtain a closed description of the reduced system dynamics in terms of a master equation with a memory kernel, at least in some simplified situations. We plan to address these topics in future research work.

V. ACKNOWLEDGMENTS

We are grateful to Ludovico Lanz for useful hints along the preparation of the manuscript. BV would also like to thank Klaus Hornberger for very helpful discussions and reading of the manuscript. This work was partially supported by MIUR under PRIN2008.

Appendix A: Microscopic derivation of the master equation

We here address the derivation of the Markovian master equation for the description of the dynamics of the test particle starting from the general expressions Eq. (3) and Eq. (4). The scattering and rate operators appearing in these equations are best expressed using the factorization of the total Hilbert space \( \mathcal{H}_{\text{tot}} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{gas}} = \mathcal{H}_{\text{cm}} \otimes \mathcal{H}_{\text{rel}} \), according to

\[
| P, i \rangle \langle P', j | \otimes | p \rangle \langle p' |_{\text{gas}} = | \text{rel}(p, P), i \rangle \langle \text{rel}(p', P'), j |_{\text{rel}} \otimes | P + p \rangle \langle P' + p' |_{\text{cm}},
\]

where the Hilbert space \( \mathbb{C}^n \) associated to the internal degrees of freedom is part of \( \mathcal{H}_{\text{rel}} \) \[27\], the notation is the same as in Sec. [1] In fact both these operators act in a trivial way on centre of mass coordinates: \( \Gamma = \mathcal{I}_{\text{cm}} \otimes \Gamma_0 \) and \( T = \mathcal{I}_{\text{cm}} \otimes T_0 \). The operator \( \Gamma_0 \) is given by

\[
\Gamma_0 = \frac{n_{\text{gas}}}{m_s} \sum_j | \text{rel}(p, P) \rangle \sigma_{\text{tot}}(\text{rel}(p, P), j) \otimes | j \rangle \langle j |,
\]

where \( \sigma_{\text{tot}}(\text{rel}(p, P), j) \) is the total cross section, depending on initial relative momentum and internal state. The relation \[27\]

\[
\langle p_f, k| \Gamma_0 | p_i, j \rangle = \frac{1}{2\pi \hbar m_s} \delta \left( \frac{p^2_f - p_i^2}{2m_s} + \epsilon_{kj} \right) f_{kj}(p_f, p_i)
\]

links the operator \( \Gamma_0 \) to the multichannel complex scattering amplitudes \( f_{kj}(p_f, p_i) \), referring to scattering from an initial momentum \( p_i \) and internal state \( j \) to a final state with momentum \( p_f \) and internal state \( k \). According to standard usage in scattering theory we call channels the asymptotically free internal energy eigenstates of the system. The differential cross section is given by \( \sigma_{kj}(p_f, p_i) = (|p_f|/|p_i|) |f_{kj}(p_f, p_i)|^2 \), so that the total cross section appearing in Eq. \[A2\] reads \( \sigma_{\text{tot}}(p_i, j) = \sum_k \int dp_f \sigma_{kj}(p_f, p_i) \).

1. Evaluation of the Lindblad structure in momentum and internal state basis

   a. Incoherent contribution

   We now first concentrate on the evaluation of the contribution given by Eq. \[3\], which under suitable approximations can be cast in Lindblad form, closely following \[12\], where the special case of a test particle without internal structure was dealt with. To this end we consider the matrix elements of \( \mathcal{L} \rho \) in the momentum and channel basis \( \{|P, i\}\) of the Hilbert space \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^n \) associated to the test particle. Denoting by \( \mu(p) \) the stationary gas momentum distribution and exploiting the relations Eq. \[A2\] and Eq. \[A3\] for the relevant operators we can express the result as is Eq. \[5\], where the complex rate functions

\[
M_{ij}^{\text{t}}(P, P'; Q) := \frac{(2\pi\hbar)^3}{|\Omega|} \int dp_0 \mu(p_0) \langle \text{rel}(p_0 - Q, P), i | \Gamma_0^{-1/2} | \text{rel}(p_0 - Q, P) \rangle_{\text{rel}}^\dagger \times \langle \text{rel}(p_0 - Q, P) | \Gamma_0^{-1/2} \Gamma_0^{-1/2} | \text{rel}(p_0 - Q, P') \rangle
\]

have been introduced, and \( |\Omega| \) denotes the volume in which the gas is confined. Note that \( M_{ij}^{\text{t}}(P, P'; Q) \) can be interpreted as classical rates for scattering of the test particle with momentum \( P - Q \) and internal energy eigenstate
to a final state with momentum $P$ and internal energy eigenstate $i$. The contributions at r.h.s. of Eq. (5) therefore play the role of quantum gain and loss term also depending on the internal degrees of freedom involved.

Relying on Eq. (5) we can now deal just with the complex rates $M_{ik}^{ji}(P, P'; Q)$ defined in Eq. (A4). To proceed, it is helpful to introduce the following functions of $p_0$

$$p_i = \text{rel}\left(p_0, \frac{P + P'}{2} - Q\right)$$

$$p_f = \text{rel}\left(p_0 - Q, \frac{P + P'}{2}\right),$$

(A5)

which denote the mean of the pairs of initial and final relative momenta appearing in $M_{ik}^{ji}(P, P'; Q)$ and which are related by $p_i - p_f = Q$. Introducing also $q = \text{rel}(0, (P - P')/2)$ the complex functions $M_{ik}^{ji}(P, P'; Q)$ can be expressed as an average over the gas distribution function $\rho$ of a complex density in the centre of mass frame

$$M_{ik}^{ji}(P, P'; Q) = \int dp_0 \rho(p_0) m_{ik}^{ji}(p_f, p_i; q)$$

(A6)

with

$$m_{ik}^{ji}(p_f, p_i; q) = \frac{(2\pi \hbar)^3}{|\Omega|} (p_f + q, i|T_0^{1/2}|p_i + q, j)$$

$$\times (p_i - q, l|\Gamma_0^{1/2}T_0^{1/2}|p_f - q, k).$$

(A7)

Evaluating this formula with the expressions Eq. (A2) and Eq. (A3) for $\Gamma_0$ and $T_0$ respectively, we obtain

$$m_{ik}^{ji}(p_f, p_i; q) = \frac{2\pi \hbar}{m_\sigma |\Omega|} \Gamma_0^{1/2}(p_i + q, j)\Gamma_0^{1/2}(p_i - q, j')$$

$$\times \delta \left(\frac{p_f^2 - p_i^2}{2m_\sigma} - \frac{(p_f - p_i) \cdot q}{m_\sigma} + \mathcal{E}_{kl}\right)$$

$$\times \delta \left(\frac{p_i^2 - p_f^2}{2m_\sigma} + \frac{(p_f - p_i) \cdot q}{m_\sigma} + \mathcal{E}_{ij}\right)$$

$$\times f_{ij}(p_f + q, p_i + q) f_{kl}(p_f - q, p_i - q),$$

(A8)

where $\Gamma_0(p_i, j) = n_{gas} |p| \sigma_{\text{tot}}(p, j)/m_\sigma$ is the eigenvalue of the operator $\Gamma_0$ relative to the state $|p, j\rangle$.

The expression given by Eq. (A8) looses its meaning in the infinite volume limit, due to appearance of the arbitrarily large normalization volume. This point has been extensively discussed in [12]. It is to be traced back to the fact that the operator $\Gamma$ in order to provide the actual rate of collisions should involve a projection on the subspace of incoming wave packets, which is not accounted for in Eq. (A7). To do this, we are going now to evaluate the operator $\Gamma$ on a properly modified state of the relative motion.

Before that, it is convenient to focus our attention on the two delta-functions appearing in Eq. (A8): employing the relation $\delta(a)\delta(b) = 2\delta(a + b)\delta(a - b)$, we can rewrite them as the product

$$1/2 \delta \left(\frac{p_f^2 - p_i^2}{2m_\sigma} + \mathcal{E}_{ij} + \mathcal{E}_{kl}\right) \delta \left(\frac{(p_f - p_i) \cdot q}{m_\sigma} + \frac{\mathcal{E}_{ij} - \mathcal{E}_{kl}}{2}\right).$$

These two constraints ensure that the scattering amplitudes appearing in Eq. (A8) are evaluated on shell. The function $m_{ik}^{ji}(p_f, p_i; q)$ gives a significant contribution to the integral in Eq. (A6) when the two energy differences are approximately equal, so that $\mathcal{E}_{ij} = \mathcal{E}_{kl}$, leading otherwise to rapidly oscillating phases, and this is actually a necessary condition in order to obtain a completely positive time evolution [13]. This implies in particular that integrating the generalized function $m_{ik}^{ji}(p_f, p_i; q)$ with a function $g(q)$, the contributions deriving from the parallel component of $q$ vanish

$$m_{ik}^{ji}(p_f, p_i; q) g(q) = m_{ik}^{ji}(p_f, p_i; q_\perp) g(q_\perp).$$

(A9)

We now therefore evaluate $m_{ik}^{ji}(p_f, p_i; q_\perp)$ with a properly modified state of relative motion, which takes into account the restriction of the expression to states which actually describe a colliding pair. To this end we write the complex rate $m_{ik}^{ji}$ as

$$m_{ik}^{ji}(p_f, p_i; q_\perp) = \langle p_f + q_\perp, i|T_0^{1/2} \exp \left(\frac{i \mathcal{E}_{ij} \cdot q_\perp}{\hbar}\right) \rho_i \otimes |j\rangle |l\rangle$$

$$\times \exp \left(\frac{i \mathcal{E}_{ij} \cdot q_\perp}{\hbar}\right) T_0^{1/2} T_0^{1/2} |p_f - q_\perp, k\rangle.$$
where $\rho_p$, denotes an improper state of relative motion

$$\rho_p = \frac{(2\pi\hbar)^3}{|\Omega|} |p_i\rangle\langle p_i|.$$  \hfill (A10)

Since the rate operator $\Gamma$ should have a vanishing expectation value for those states of the relative motion that are not of the incoming type, we make the replacement

$$\rho_p \otimes |j\rangle\langle l| \to \rho'_p \otimes |j\rangle\langle l| = \int_{\Lambda_p} \frac{dx_{|p_i|}}{|\Lambda_p|} \int_{\Sigma_{p_1}} \frac{dx_{|p_i|}}{|\Sigma_{p_1}|} \int dw \exp\left(-i \frac{x \cdot w}{\hbar}\right) \frac{1}{(2\pi\hbar m_*)^2} \frac{n_{gas}}{|p_i|} \times \left[ \frac{p_f^2 - p_i^2}{2m_*} - \frac{w^2}{8m_*} - \frac{q_\perp \cdot w}{2m_*} + \frac{E_{ij} + E_{kl}}{2} \right] \delta \left( p_i \cdot w \right) \frac{m_*}{|p_i|} (E_{kl} - E_{ij}) \times f_{ij} \left( p_f + q_\perp, p_i + q_\perp + \frac{w}{2} \right) f_{kl}^* \left( p_f - q_\perp, p_i - q_\perp - \frac{w}{2} \right) \times \sqrt{\frac{|p_i + q_\perp - \frac{w}{2}|}{|p_i - q_\perp - \frac{w}{2}|}} \sigma(p_i - q_\perp - \frac{w}{2}, l),$$

where we have exploited once again the relation $\delta(a)\delta(b) = 2\delta(a + b)\delta(a - b)$. We now first perform the integral over $w_{|p_i|}$, which denotes the component of $w$ parallel to $p_i$, thus evaluating the second delta-function at r.h.s. of the previous equation, so that the dependence on $x_{|p_i|}$ only appears in the term

$$\int_{\Lambda_p} \frac{dx_{|p_i|}}{|\Lambda_p|} \exp\left(-i \frac{x \cdot w}{\hbar} \sigma(p_i + q_\perp, l) (E_{ij} - E_{kl}) \right) \hat{p}_i \right).$$

The phase of the integrand varies very quickly for $(E_{ij} - E_{kl}) \gg \hbar/\Delta t$, where $\Delta t$ is the typical time elapsing between collisions, so that as already discussed its contribution vanishes unless $E_{ij} = E_{kl}$, corresponding to a rotating wave approximation, assuming a separation of time scales between internal and translational dynamics \cite{3}. Further considering the integral over $x_{\perp p_i}$ as an approximate expression for $\delta(w_{\perp p_i})$ we are led to

$$m^{|j|}_{ik} (p_f, p_i; q_\perp) = \frac{n_{gas}}{m_*^2} \chi^{|j|}_{ik} f_{ij} \left( p_f + q_\perp, p_i + q_\perp \right) \times f_{kl}^* \left( p_f - q_\perp, p_i - q_\perp \right) \delta \left( \frac{p_f^2 - p_i^2}{2m_*} + E_{ij} \right) \times \sqrt{\frac{|p_i + q_\perp|}{|p_i - q_\perp|}} \sigma(p_i + q_\perp, l) \sigma(p_i - q_\perp, l),$$

where the $\chi^{|j|}_{ik}$ act like a Kronecker’s delta factor, being defined according to

$$\chi^{|j|}_{ik} = \begin{cases} 1 & \text{if } E_{ij} = E_{kl} \\ 0 & \text{otherwise} \end{cases}.$$
In the last two terms we can disregard the dependence on \( q_\perp \) because we expect that a \( q_\perp \)-integration will average out the “far off-diagonal” contributions with large modulus \( |q_\perp| \), where the phases of the two scattering amplitudes are no longer synchronous. In conclusion, we have

\[
m^{ij}_{ik}(p_f, p_i; q_\perp) = \frac{\gamma_{\text{gas}}}{m^2} \delta^{ij} \left( \frac{p_f^2 - p_i^2}{2m_\ast} + E_i - E_j \right) M^{ij}_{\omega k}(p_f + q_\perp, p_i + q_\perp) f_{ki}^\dagger (p_f - q_\perp, p_i - q_\perp). \tag{A12}
\]

This relation determines the complex rate functions \( M^{ij}_{\omega k}(P, P'; Q) \) through Eq. \( (A6) \), and therefore the dissipative part of the master equation according to Eq. \( (5) \).

b. Energy shift

As a last step in the determination of the structure of the master equation we need to evaluate the contribution given by \( R \rho \). In the same notation as above, and within the same approximations, we directly obtain

\[
\langle P, i | R \rho | P', k \rangle = \langle P, i | i \text{ Tr}_{\text{gas}} \left[ \left( \frac{\Gamma}{2} \text{Re}(T) \right)^{\frac{1}{2}} \rho \otimes \rho_{\text{gas}} \right] \rangle |P', k \rangle
\]

\[
= \frac{1}{i \hbar} \sum_j \left( E^{ij}_{n}(P) \langle P, j | \rho | P', k \rangle - E^{jk}_{n}(P') \langle P, i | \rho | P', j \rangle \right), \tag{A13}
\]

with

\[
E^{ij}_{n}(P) = -2\pi \hbar^2 \gamma_{\text{gas}} \chi^{jk}_{\omega k} \int dp_0 \mu(p_0) \text{Re} \left[ f_{ij} (\text{rel} (p_0, P), \text{rel} (p_0, P')) \right]. \tag{A14}
\]

It is worth noting that for the case of a non-degenerate free internal Hamiltonian this formula reduces to

\[
\langle P, i | R \rho | P', k \rangle = \frac{1}{i \hbar} (E^{ii}_{n}(P) - E^{kk}_{n}(P')) \langle P, i | \rho | P', k \rangle
\]

with

\[
E^{ii}_{n}(P) = -2\pi \hbar^2 \gamma_{\text{gas}} \int dp_0 \mu(p_0) \text{Re} \left( f_{ii} (\text{rel} (p_0, P), \text{rel} (p_0, P)) \right). \tag{A15}
\]

2. Operator expression of the master equation

We now recast the master equation Eq. \( (1) \), whose matrix elements are given by Eq. \( (5) \) and Eq. \( (A13) \), in a way which allows to express it in a representation-independent form. The key point is to show that \( M^{ij}_{\omega k}(P, P'; Q) = \int dp_0 \mu(p_0)m^{ij}_{\omega k}(p_f, p_i; q) \) can be factorized into two terms, one depending on \( P \) and the other on \( P' \).

Changing the integration variable from \( p_0 \) to \( p_i \) and using the relations Eq. \( (A5) \) to obtain \( p_0 = p_i + (p_f + P) m/M + qm/m_\ast = p_i + (p_f + P) m/M - qm/m_\ast \), we have

\[
M^{ij}_{\omega k}(P, P'; Q) = \frac{m^3}{m^2} \gamma_{\text{gas}} \chi^{jk}_{\omega k} \int dp_0 \delta \left( \frac{p_f^2 - p_i^2}{2m_\ast} + \xi_{ij} \right) \mu^{1/2} \left( p_i + \frac{m}{M} (p_f + P) + \frac{m}{m_\ast} q_\perp \right) \times f_{ij} (p_f + q_\perp, p_i + q_\perp) f_{ki}^\dagger (p_f - q_\perp, p_i - q_\perp),
\]

where we replaced \( q \) by \( q_\perp \) in the arguments of \( \mu^{1/2} \), in accordance with Eq. \( (A9) \). Remembering that \( p_i - p_f = Q \) and \( q = m_\ast (P - P') / (2M) \), we consider the change of variable

\[
p_i \to \frac{m}{m_\ast} p_i + \frac{M}{2} (P_\perp + P_\perp') - \frac{m}{m_\ast} \frac{Q}{2} - \frac{\xi_{ij}}{Q^2/m}, \tag{A16}
\]

to obtain the desired factorization. If we further consider that the delta function \( \delta (p \cdot Q/m) \) restricts the \( p \)-integration to the plane \( Q_\perp = \{ p \in \mathbb{R}^3 : p \cdot Q = 0 \} \) we finally arrive at the expression Eq. \( (6) \), with \( L_{ij}(p, P; Q) \) as in Eq. \( (7) \).
which allows to obtain the operator expression of the master equation given by Eq. (9) and Eq. (10). If the gas
distribution function $\mu$ is given by a Maxwell-Boltzmann probability density $\mu_\beta(p) = 1/(\pi^{3/2}p_\beta^3)\exp(-p^2/p_\beta^2)$, where $p_\beta = \sqrt{2m/\beta}$ is the most probable momentum at temperature $T = 1/(k_\beta\beta)$, these functions can be expressed in
terms of the dynamic structure factor for a Maxwell-Boltzmann gas [28–29]

$$S_{MB}(Q, E) = \sqrt{\frac{\beta m}{2\pi Q}} \exp\left(-\frac{\beta}{8m}\frac{(Q^2 + 2mE)^2}{Q^2}\right).$$

(A17)

In fact, using the relation

$$\frac{m}{Q}\mu_\beta\left(p_\perp + \frac{m}{M}p_\parallel + \left(1 + \frac{m}{M}\right)\frac{Q}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m}\right) = \mu_\beta(p_\perp)S_{MB}(Q, E(Q, P) + \mathcal{E}_{ij}),$$

with $E(Q, P) = (P + Q)^2/2M - P^2/2M$ the energy transferred to the centre of mass in a collision changing the
momentum of the test particle from $P$ to $P + Q$. Eq. (7) can be written as [10]

$$L_{ij}(p, P; Q) = \sqrt{\frac{n_{gas}}{m^2\mu_\beta(p_\perp)}}S_{MB}(Q, E(Q, P) + \mathcal{E}_{ij})$$

$$\times f_{ij}\left(\text{rel}(p_\perp, P_\perp) - \frac{Q}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_\ast} Q, \text{rel}(p_\perp, P_\perp) + \frac{Q}{2} + \frac{\mathcal{E}_{ij}}{Q^2/m_\ast} Q\right).$$

(A18)

In the latter expression for the Lindblad operators, the dynamic structure factor appears evaluated for an energy
transfer corresponding to the sum of the contributions for centre of mass and internal state, as naturally expected. As
discussed in [4], the dynamic structure factor describes momentum and energy transferred to the test particle when
scattering off a macroscopic system, thus allowing for a more transparent physical understanding of the structure of
the Lindblad operators.

[1] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, Decoherence and the Appearance of a Classical
World in Quantum Theory, Springer, Berlin, 2nd edition, 2003.
[2] M. Schlosshauer, Decoherence and the Quantum-to-Classic Transition, Springer-Verlag, Berlin, 2007.
[3] H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, Oxford, 2007.
[4] B. Vacchini and K. Hornberger, Phys. Rep. 478, 71–120 (2009).
[5] K. Hornberger, EPL 77, 50007 (2007).
[6] B. Vacchini, Phys. Rev. A 78, 022112 (2008).
[7] S. Kryszewski and J. Czechowska, Phys. Rev. A 74, 02219 (2006).
[8] A. A. Budini, Phys. Rev. A 75, 053815 (2006).
[9] H.-P. Breuer, Phys. Rev. A 75, 022103 (2007).
[10] K. Hornberger, Phys. Rev. Lett. 966001 (2006).
[11] K. Hornberger, J. Phys.: Conf. Ser. 967, 012002 (2007).
[12] K. Hornberger and B. Vacchini, Phys. Rev. A 77, 022112 (2008).
[13] R. Alicki and S. Kryszewski, Phys. Rev. A 68, 013809 (2006).
[14] R. Dümcke, Commun. Math. Phys. 97, 331–359 (1985).
[15] R. F. Snider, Int. Rev. Phys. Chem. 17, 185–225 (1998).
[16] K. Hornberger, S. Uttenthaler, B. Brezger, L. Hackermüller, M. Arndt, and A. Zeilinger, Phys. Rev. Lett. 90, 160401
(2003).
[17] C. Champenois, M. Jacquey, S. Lepoutre, M. Buchner, G. Trene, and J. Vigue, Phys. Rev. A 77, 013621 (2008).
[18] M. Jacquey, M. Buchner, G. Trene, and J. Vigue, Phys. Rev. Lett. 98, 240405 (2007).
[19] M. Hillery, L. Mlodinow, and V. Buzek, Phys. Rev. A 71 (2005).
[20] C. J. Hemming and R. V. Krems, Phys. Rev. A 81, 052701 (2010).
[21] W. Feller, An introduction to probability theory and its applications. Vol. II, John Wiley & Sons Inc., New York, 1971.
[22] C. M. Savage and D. F. Walls, Phys. Rev. A 32, 2316–2323 (1985).
[23] K. Hornberger, Phys. Rev. A 73, 052102 (2006).
[24] B. Vacchini, Theoretical Foundations of Quantum Information Processing and Communication, edited by E. Bruening and
F. Petruccione, Lecture Notes in Physics 787, pages 39–77, Berlin, 2010, Springer.
[25] P. A. Alemany, J. Phys. A: Math. Gen. 30, 6587–6599 (1997).
[26] D. A. Kokorowski, A. D. Cronin, T. D. Roberts, and D. E. Pritchard, Phys. Rev. Lett. 86, 2191–2195 (2001).
[27] J. R. Taylor, Scattering Theory, John Wiley & Sons, New York, 1972.
[28] F. Schwabl, Advanced quantum mechanics, Springer, New York, 2003.
[29] L. Pitaevskii and S. Stringari, Bose-Einstein condensation, Oxford University Press, Oxford, 2003.