1. Introduction and main results.

It is an old and difficult question in differential geometry if there exists a "best" or canonical metric on a smooth manifold, which makes a manifold "most symmetric". Standard examples are round spheres and flat tori, where the word "best" means constant curvature. If there are no assumptions made about a manifold then there is a high chance that there are no reasonable "best" metrics, which was informally explained by Gromov in [2]. Usually one assumes the existence of some geometric or algebraic structure on the manifold and considers only a class of metrics compatible with the structure.

Classical examples include Kahler metrics on compact complex manifolds, left- or biinvariant metrics on Lie groups or metrics compatible with a conformal structure. And canonical metric should ideally be uniquely defined by the structure.

One of the first well-known examples of canonical metrics is of course a hyperbolic metric on a compact Riemann surface of genus $g \geq 2$. In case of Kahler manifolds the Yau's proof of Calabi conjecture provides the existence of a distinguished metric in the same Kahler class as the initial one [6].

For a compact smooth manifold by the solution to the Yamabe problem, achieved in works of Trudinger [5], Aubin [1] and Schoen[4], each conformal structure on a compact manifold supports a metric of a constant scalar curvature. However this metric is in general not unique in the case of positive scalar curvature.

In a more recent work Habermann and Jost [3] construct canonical metrics in a conformal class using Green function of the Yamabe operator. Their construction requires local conformal flatness of a class if the dimension of the manifold is greater than 3.

In this paper we construct canonical metrics in a given conformal class for a $2n$-dimensional oriented compact smooth manifold $M$, with non-trivial $n$-th de Rham cohomology and some natural non-degeneracy assumption on the conformal class.
We use Hodge theory of harmonic forms and the key point, which makes the construction very explicit is a well-known observation that $n$-dimensional harmonic forms of a $2n$-dimensional manifold remain harmonic under conformal change of a metric.

We define a functional $E$ on the space of all Riemannian metrics invariant under the natural action of the group $Diff$, which we call a Harmonic Energy. Informally speaking the functional $E$ measures the failure of a wedge product of two harmonic forms to be harmonic. Then we prove that inside a given conformal class there exists a unique normalized metric minimizing $E$.

Moreover we obtain an explicit formula for the extremal metric in terms of the initial metric representing the conformal class and an orthonormal basis of harmonic $n$-forms. We also explicitly compute the critical Harmonic Energy and observe that corresponding value can be defined for any conformal class without any non-degeneracy assumptions and so may serve as a conformal invariant of any closed oriented smooth $2n$-manifold.

In the next chapter we apply our construction to Riemann surfaces, thus producing a natural family of metrics on them.

From now and further by manifold we mean a closed oriented smooth manifold.

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1.1. Hodge theory.

Here we briefly discuss the basics of the Hodge theory and prove a lemma which will be in use later.

Let $V$ be an oriented $d$-dimensional Euclidean space. Then the inner product produces a natural isomorphism $V = V^*$, which naturally extends to the graded isomorphism of Grassman algebras $\bigwedge^k V = \bigwedge^k V^*$. The latter isomorphism provides a geometric interpretation of the space of $k$-forms $\Omega^k(V)$. Namely, for any oriented $k$-plane $P$, generated by the ordered set of orthonormal vectors $e_1, \cdots, e_k$ we define a so-called decomposable $k$-form $\omega_P$ as follows. For a set of vectors $v_1, \cdots, v_k$, the value $\omega_P(v_1, \cdots, v_k)$ equals to the algebraic $k$-volume of the projection of the parallelepiped $<v_1, \cdots, v_k>$ onto $P$. In terms of the isomorphism above one can check the identity $\omega_P = e_1 \wedge \cdots \wedge e_k$. Any other $k$-form $\omega$ is a linear combination of decomposable $k$-forms.

The inner product of two decomposable $k$-forms $\omega_P, \eta_Q$ is defined
as a Jacobian of the orthogonal projection of oriented $k$-plane $P$ into oriented $k$-plane $Q$ and then extended by linearity to $\Omega^k(V)$.

The Hodge Star operator $\ast : \Omega^k \to \Omega^{d-k}$ is also defined in terms of decomposable forms and then extended by linearity on $\Omega^k(V)$. For any positively oriented orthonormal basis $e_1, \cdots, e_d$ we put by definition

$$*(e_1 \wedge \cdots \wedge e_k) = e_{k+1} \wedge \cdots \wedge e_d.$$ 

Now we consider a $d$-dimensional Riemannian manifold $(M, g)$ and naturally define the Hodge Star operator $\Omega^k(M) \to \Omega^{d-k}(M)$ as the Riemannian metric $g$ defines inner product on the tangent space $T_x M$ at each point $x \in M$.

We then define the $L^2$- inner product of two $k$-forms $\omega, \eta \in \Omega^k(M)$ as $$(\omega, \eta)_{L^2} = \int (\omega, \eta) Vol_g$$ where $Vol_g$ is a Riemannian volume form associated to $g$ and we have a well-known equality $$(\omega, \eta)_{L^2} = \int \omega \wedge * \eta.$$ 

For a differential $d : \Omega^{k-1} \to \Omega^k$ there exists a formal adjoint with respect to the inner product defined above and given by the formula

$$\delta : \Omega^k \to \Omega^{k-1}, \delta = (-1)^k *^{-1} d*.$$ 

The Hodge Laplacian is defined as

$$\Delta : \Omega^k(M) \to \Omega^k(M)$$

$$\Delta = d\delta + \delta d.$$ 

The form $\omega$ is called harmonic if $\Delta \omega = 0$ and it is easy to check that $\Delta \omega = 0$ iff $d\omega = 0, d^* \omega = 0$.

Let $H^k(M)$ be the space of harmonic $k$-forms. Since all harmonic forms are closed we have a natural map $\Phi : H^k(M) \to H^k(M)$, where $H^k(M)$ is a de Rham cohomology of $M$ and a celebrated theorem of Hodge asserts that $\Phi$ is an isomorphism.

**Lemma 1.1.** Let $V$ be a $2n$-dimensional oriented space with an inner product $g$ and let $*_{g} : \Omega^n(V) \to \Omega^n(V)$ be a corresponding Hodge Star operator acting on $n$-forms. Let also $k$ be a positive constant. Then $*_{g} = *_{kg}$.

**Proof.** Obvious from the definition of the Hodge Star. Q.E.D.

**Corollary 1.2.** Let $M$ be a $2n$-dimensional manifold, $g$ be a Riemannian metric, $\rho \in C^\infty(M), \rho > 0$ and $\omega, \eta$ be a pair of harmonic $n$-forms with respect to $g$. Then $\omega, \eta$ are harmonic with respect to $\rho g$ and $$(\omega, \eta)_{L^2}^g = (\omega, \eta)_{L^2}^{\rho g}.$$ 

**Proof.** The first statement follows from the Lemma 1.1. and the fact that harmonic forms $\omega$ satisfy $d\omega = 0, d^* \omega = 0$.

For the second statement the harmonicity of forms is not essential and it follows from the Lemma 1.1. and a formula mentioned before
\[(\omega, \eta)_{L^2} = \int \omega \wedge *\eta. \quad \text{Q.E.D.}\]

2. Harmonic Energy.

Let \(M\) be a 2\(n\)-dimensional manifold and \(g\) be a Riemannian metric on it. Then all spaces \(\Omega^k(M)\) inherit an \(L^2\)-inner product defined above and the \(n\)-th cohomology space \(H^n(M)\) becomes a Euclidean space via the Hodge isomorphism \(\Phi\). Then \(H^n(M) \otimes H^n(M)\) naturally becomes a Euclidean space and if \(e_1 \cdot \cdot \cdot e_p\) is an orthonormal basis for \(H^n(M)\) then \(e_i \otimes e_j, 1 \leq i, j \leq p\) is an orthonormal basis for \(H^n(M) \otimes H^n(M)\).

For cohomology classes \([\omega], [\eta]\) and \([\omega \wedge \eta]\) let \(\omega_H, \eta_H, (\omega \wedge \eta)_H\) be the corresponding harmonic representatives and let \(\Omega^{2n}(M)\) be a completion of \(\Omega^{2n}(M)\) with respect to the \(L^2\)-norm. Then a linear operator \(A: H^n(M) \otimes H^n(M) \to \Omega^{2n}(M)\) is defined by:

\[A([\omega] \otimes [\eta]) = \omega_H \wedge \eta_H - (\omega \wedge \eta)_H.\]

\(A\) measures the failure of a wedge product of two harmonic forms to be harmonic. It is a linear operator from Euclidean to a Hilbert space and has a natural norm \(|A|\), given by the formula \(|A|^2 = Tr(A^*A)\). Any choice of orthonormal basis \(e_1, \cdot \cdot \cdot , e_p \in H^n(M)\) allows us to write an explicit formula:

\[|A|^2 = \sum A(e_i \otimes e_j) \cdot A(e_i \otimes e_j)\]

where \(\omega \cdot \eta\) denotes the \(L^2\)-product of \(2n\)-forms \(\omega, \eta\) inside \(\Omega^{2n}(M)\).

**Definition 1.** Let \(g\) be a Riemannian metric on \(M\). Then its Harmonic Energy is

\[E(g) = |A|^2\]

**Definition 2.** Let \(g\) be a metric on \(M^{2n}\), such that \(\int Vol_g = 1\). Then its Normalized Conformal Class is by definition:

\[C(g) = \{\rho g | \int \rho^n Vol_g = 1, \rho \in C^\infty(M), \rho > 0\}\]

which is just a set of conformally equivalent metrics with total volume equal to one.

**MAIN THEOREM.** Let \((M^{2n}, g_0)\) be a Riemannian manifold such that for any \(x \in M\) there exists a form \(\omega \in \mathcal{H}^n(M), \omega(x) \neq 0\). Then there exists a unique metric \(g \in C(g_0)\) minimizing \(E\).
Remark. The assumption of the theorem of course immediately implies $H^n(M) \neq 0$.

PROOF.

Let us pick a set of harmonic forms $\xi_1, \cdots, \xi_p$ which form an orthonormal basis in $H^n(M)$ for any metric from $C(g_0)$. This is a crucial ingredient of the proof and such a choice is possible because of the Corollary 1.2.

Now we consider any metric $g = \rho g_0 \in C(g_0)$ and calculate $E(g)$ using that $g$-harmonic 2n-forms are proportional to the Vol$g$:

$$E(g) = \sum A(\xi_i \otimes \xi_j) \cdot A(\xi_i \otimes \xi_j) = \sum \int A(\xi_i \otimes \xi_j) \wedge \ast A(\xi_i \otimes \xi_j) = \sum \int (\xi_i \wedge \xi_j - (\int \xi_i \wedge \xi_j) Vol_g) \wedge \ast (\xi_i \wedge \xi_j - (\int \xi_i \wedge \xi_j) Vol_g)$$

Let us now introduce the following notations: let $f_{ij}$ be smooth functions, defined by $\xi_i \wedge \xi_j = f_{ij} Vol_0$ and $c_{ij}$ be constants, defined by $c_{ij} = \int \xi_i \wedge \xi_j$.

Using that $Vol_g = \rho^n g_0$, we can rewrite $E(g) = \sum \int (f_{ij} \rho^{-n} - c_{ij})^2 \rho^n Vol_0$. Opening brackets we obtain $E(g) = \int f^2 \rho^{-n} Vol_0 - C^2$ where $f = \sqrt{\sum f^2_{ij}}$ and $C = \sqrt{\sum c^2_{ij}}$.

Now let us prove by contradiction that $f(x) > 0$ for any $x \in M$. Indeed $f(x) = 0 \implies f_{ij}(x) = 0 \implies \omega \wedge \eta(x) = 0$ for any $\omega, \eta \in H^n(M)$ as $i, j$ run through the basis of $H^n(M)$. But then for any $\omega \in H^n(M)$, $(\omega, \omega)_x V ol = \omega \wedge \ast \omega(x) = 0$ which implies $\omega(x) = 0$ and contradicts the theorem assumption.

To find a minimum of $E(g)$ among all metrics $g \in C(g_0)$ we use a form of integral Cauchi inequality on the functions $f, \rho$:

$$\int f^2 \rho^{-n} Vol_0 \cdot \int \rho^n Vol_0 \geq (\int f Vol_0)^2$$

which immediately implies that the extremal is $g = \rho g_0$ with $\rho = (\int f Vol_0)^1/n$ and minimal $E(g) = (\int f Vol_0)^2 - C^2$. Q.E.D.

Remark 1. Using explicit formulas from the proof one can easily check that the minimizing metric and critical harmonic energy are indeed independent on the initial metric $g_0$ we started with.

Remark 2. As the expression for the critical energy $E(g)$ is independent on the initial metric $g_0$ one can define the harmonic energy of a given conformal class on any compact smooth oriented manifold $M^{2n}$ dropping the assumption of the theorem, which is only required to guarantee the non-degeneracy of the critical metric.
3. **Canonical metrics on Riemann surfaces.**

Consider a closed oriented surface $M$ of genus $g \geq 2$. The conformal classes of metrics on $M$ are in natural one-to-one correspondence with complex structures on it. Let us fix a conformal class on $M$. Then harmonic 1-forms for such a conformal class are precisely the real parts of abelian differentials for the corresponding complex structure.

As it is well known that for any $x \in M$ there exists an abelian differential $w$, $w(x) \neq 0$ we have that either $\Re(w) \neq 0$ or $\Im(w) = \Re(-iw) \neq 0$ which means that natural conformal class of any Riemann surface satisfies the assumptions of the main theorem.

So we produced a canonical metric for any Riemann surface.

4. **Open questions.**

1. It seems to be an interesting question to describe the properties of the metric on the Riemann surface, defined above, and in particular to see if its curvature is negative on $M$.

2. A manifold is called formal if there exists a metric such that the wedge product of any two harmonic forms is harmonic. It is not hard to show that any closed surface of genus $g \geq 2$ is not formal which implies that the value of critical Harmonic Energy is a positive smooth function $E_g$ on the moduli space $\mathcal{M}_g$ of complex curves of genus $g$.

As $\mathcal{M}_g$ is not compact it is an interesting question if $E_g$ is strictly positive on $\mathcal{M}_g$ which can be reformulated as if $M$ is formal ”at infinity” of the moduli space.
Bibliography

[1] Th. Aubin, Nonlinear analysis on manifolds. Monge-Ampere equations, Springer, 1982.

[2] Misha Gromov, Spaces and questions, 1999.

[3] L. Habermann, J. Jost, Green functions and conformal geometry, J. Differential Geometry, 52(1999)

[4] R.M. Schoen. Conformal deformation of a Riemannian metric to constant scalar curvature, J. Differential Geometry. 20(1984) 479-495

[5] N. Trudinger. Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa, 22(1968) 265-274.

[6] S.T. Yau, On the Ricci curvature of a compact Kahler manifold and the complex Monge-Ampere equation. I, Comm. Pure Appl. Math. 31 (1978) 339-411.