On Some Integral means

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Abstract. Harmonic, Geometric, Arithmetic, Heronian and Contra-harmonic means have been studied by many mathematicians. In 2003, H. Eves studied these means from geometrical point of view and established some of the inequalities between them in using a circle and its radius. In 1961, E. Beckenback and R. Bellman introduced several inequalities corresponding to means. In this paper, we will introduce the concept of mean functions and integral means and give bounds on some of these mean functions and integral means.

1. Introduction

In their book of inequalities, Beckenback and Bellman established several inequalities between arithmetic, harmonic and contra-harmonic means[2]. These means are defined in the following paragraph, based on the original text by Eve[1]. Let \(a, b > 0\) and \(a \neq b\). Putting together the results from works of several mathematicians, in particular Taneja established that \(\max\{a, b\} > C > r > g > A > Hn > G > H > \min\{a, b\}\) in [3] and [4], where \(C = \frac{a^2 + b^2}{a + b}\) is contraharmonic mean, \(r = \sqrt{\frac{a^2 + b^2}{2}}\) is root square mean, \(g = \frac{2(a^2 + ab + b^2)}{3(a + b)}\) is gravitational mean (also called centroidal mean), \(A = \frac{a + b}{2}\) is arithmetic mean, \(Hn = \frac{a + b + \sqrt{ab}}{3}\) is Heronian mean, \(G = \sqrt{ab}\) is geometric mean and \(H = \frac{2ab}{a + b}\) is harmonic mean of \(a\) and \(b\).

In this paper we introduce the notion of a mean function and utilize it to define some integral means of \(a\) and \(b\) and then we establish some inequalities corresponding to those mean functions and integral means.

2. Definitions and Main Theorems

All the means that appear in this paper are functions \(F\) with conditions \(a\) and \(b\) satisfied:

a) \(F : \mathbb{R}^2_+ \to \mathbb{R}_+\), where \(\min\{x, y\} \leq F(x, y) \leq \max\{x, y\}\), Provided that \((x, y) \in \mathbb{R}^2_+\).

2010 Mathematics Subject Classification. 26E60; 26D15.

Key words and phrases. Mean function, integral mean, harmonic mean, arithmetic complimentary.

Communicated by ...
b) \( F(x, y) = F(y, x) \), such that \((x, y) \in \mathbb{R}_+^2 \). Consequently, \( F(x, x) = x \) where \( x \in \mathbb{R}_+ \).

We say \( F \) is a mean function when the two above conditions are satisfied. All throughout the paper we are assuming that \( a, b > 0 \) and without loss of generality can assume \( b \geq a \), by symmetry.

**Definition 2.1.** Let \( M \) be a mean of \( a \) and \( b \). We define \( M_A := 2A - M \) to be \( A \)-complementary (arithmetic complementary) of \( M \).

It is obvious that \( M_A \) and \( M_G \) are means of \( a \) and \( b \).

**Theorem 2.1.** Let \( M \in \mathcal{R}(\mathbb{R}_+^2) \) be a mean function. Then

(i) 
\[
\mathcal{I}_M := \mathcal{I}_M(a, b) := \begin{cases} 
\frac{1}{(b-a)^2} \int_a^b \int_a^x M(x, y) \, dxdy, & \text{if } a \neq b, \\
a, & \text{if } a = b 
\end{cases}
\]

is a mean of \( a \) and \( b \),

(ii) \( \mathcal{J}_M := \mathcal{J}_M(a, b) := 3\mathcal{I}_M(a, b) - 2A(a, b) \) is a mean of \( a \) and \( b \) and finally

(iii) \( \frac{2}{3}A < \mathcal{I}_M < \frac{4}{3}A \).

**Proof.** Let \( b > a \).

(i):

\[
\min\{a, b\} = a < \frac{2a + b}{3} = \frac{1}{(b-a)^2} \int_a^b \int_a^x y \, dxdy + \frac{1}{(b-a)^2} \int_a^b \int_x^b x \, dydx = \frac{1}{(b-a)^2} \int_a^b \int_a^b \min\{x, y\} \, dxdy \leq \mathcal{I}_M(a, b) \leq \frac{1}{(b-a)^2} \int_a^b \int_a^b \max\{x, y\} \, dxdy = \frac{1}{(b-a)^2} \int_a^b \int_a^x y \, dydx + \frac{1}{(b-a)^2} \int_a^b \int_x^b x \, dydx = \frac{a + 2b}{3} < b = \max\{a, b\}.
\]

Also, it is obvious that \( \mathcal{I}_M \) is symmetric.

(ii):

By proof of (i), we have \( \frac{2a + b}{3} \leq \mathcal{I}_M(a, b) \leq \frac{a + 2b}{3} \). So,

\[
a \leq 3\mathcal{I}_M(a, b) - (a + b) \leq b.
\]

(iii):

\[
\frac{2}{3} < \frac{2(a + b)}{3(a + b)} \leq \frac{\mathcal{I}_M(a, b)}{A(a, b)} \leq \frac{2(a + 2b)}{3(a + b)} < \frac{4}{3},
\]

multiplying by \( A(a, b) \) we get the result. \( \square \)
Proposition 2.1. Let $M, M_1$ and $M_2$ be mean functions and $\lambda \in \mathbb{R}$. Then

(i) $M_1 > M_2 \Rightarrow \mathcal{I}_{M_1}(a, b) > \mathcal{I}_{M_2}(a, b)$ and $\mathcal{J}_{M_1}(a, b) > \mathcal{J}_{M_2}(a, b)$, $a \neq b$,

(ii) $\mathcal{I}_{\lambda M_1(\cdot \lambda) M_2(\cdot \lambda)} = \lambda \mathcal{I}_{M_1} + (1 - \lambda) \mathcal{I}_{M_2}$, if $\lambda M_1 + (1 - \lambda) M_2$ is a mean function.

In particular, $\mathcal{I}_{M_a} = (\mathcal{I}_M)_a$ and $\mathcal{I}_{\mathcal{J}_M} = \mathcal{J}_\mathcal{I}_M$.

Proof. Proof is easily done by straightforward calculations. $\square$

Here are some examples where the above proposition is used:

Let $a \neq b$.

Example 2.1.

$$\mathcal{I}_A(a, b) = \frac{1}{(b - a)^2} \int_a^b \int_a^b \frac{1}{2} (x + y) \, dxdy = \frac{1}{2(b - a)^2} \int_a^b \left[ \frac{x^2}{2} + xy \right]_a^b \, dy$$

$$= \frac{1}{4(b - a)^2} \left[ y(b^2 - a^2) + y^2(b - a) \right]_a^b = \frac{b + a}{2} = A(a, b).$$

Example 2.2.

$$\mathcal{I}_G(a, b) = \frac{1}{(b - a)^2} \int_a^b \int_a^b \sqrt{xy} \, dxdy = \left( \frac{1}{b - a} \int_a^b \sqrt{t} \, dt \right)^2 = \left( \frac{2(b^3 - a^3)}{3(b - a)} \right)^2$$

$$= g^2(\sqrt{a}, \sqrt{b}).$$

Example 2.3.

$$\mathcal{I}_H(a, b) = \frac{2}{(b - a)^2} \int_a^b \int_a^b \frac{xy}{x + y} \, dxdy = \frac{2}{(b - a)^2} \int_a^b \left[ xy - y^2 \ln(x + y) \right]_a^b \, dy$$

$$= \frac{2}{3(b - a)^2} \left[ y^2(b - a) + y(b^2 - a^2) - (y^3 + b^3) \ln(y + b) + (y^3 + a^3) \ln(y + a) \right]_a^b = \frac{4}{3} \left( 2A(a, b) + \frac{1}{(b - a)^2} \left( a^3 \ln \frac{A(a, b)}{a} + b^3 \ln \frac{A(a, b)}{b} \right) \right), \quad a \neq b.$$

Example 2.4. Let $a < b$, then:

$$\sqrt{2}(b - a)^2 \mathcal{I}_r(a, b) = \int_a^b \int_a^b \sqrt{x^2 + y^2} \, dxdy$$

$$= \int_{\tan^{-1} \frac{a}{b}}^{\frac{\pi}{2}} \int_{\frac{a \cos \theta}{\sin \theta}}^{b \cos \theta} \rho^2 \, dpd\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{a \cos \theta}{\sin \theta}}^{b \cos \theta} \rho^2 \, dpd\theta.$$

Double integrals in the above expression are easily calculated and the final result is:

$$\mathcal{I}_r(a, b) = \frac{1}{3\sqrt{2}(b - a)^2} \left( (\sqrt{2} + \ln(1 + \sqrt{2}))(a^3 + b^3) - a^3 \ln \left( \frac{b + \sqrt{a^2 + b^2}}{a} \right) \right) - ...$$
$$b^3 \ln \left( \frac{a + \sqrt{a^2 + b^2}}{b} \right) - 2ab\sqrt{a^2 + b^2}, \quad a \neq b.$$
Therefore,

\[ 3I_r(a, b) - 2A(a, b) > 2A(a, b) - I_H(a, b) > A(a, b), \quad a \neq b. \]

In other words,

\[ J_r(a, b) > I_C(a, b) > A(a, b), \quad a \neq b. \]

Similarly,

\[ A(a, b) > I_G(a, b) > J_G(a, b), \quad a \neq b, \]

and

\[ I_C(a, b) + I_G(a, b) > 2A(a, b) > I_r(a, b) + I_H(a, b), \quad a \neq b \]

**Proposition 2.3.** If \( a \neq b \), then

(i) \( \frac{8}{9} A < I_G < A \),

(ii) \( \frac{8(1 - \ln 2)}{3} A < I_H < A \),

(iii) \( A < I_C < \frac{2(-1+4 \ln 2)}{3} A \),

(iv) \( \frac{26}{27} A < I_Hn < A \),

(v) \( A < I_g < \frac{4(1+2 \ln 2)}{9} A \),

(vi) \( A < J_r < \frac{2+\sqrt{2} \ln(1+\sqrt{2})}{3} A \),

(vii) \( \frac{2}{9} A < J_G < A \),

(viii) \( 2(3 - 4 \ln 2) A < J_H < A \),

(ix) \( A < J_C < 4(-1+2 \ln 2)A \),

(x) \( \frac{8}{9} A < J_Hn < A \),

(xi) \( A < J_g < \frac{2(-1+4 \ln 2)}{3} A \),

(xii) \( A < J_r < \sqrt{2} (\ln(1+\sqrt{2}))A \),

where \( 1, \frac{8}{9}, \frac{8(1 - \ln 2)}{3}, \frac{2(-1+4 \ln 2)}{3}, \frac{26}{27}, \frac{4(1+2 \ln 2)}{9}, \frac{2+\sqrt{2} \ln(1+\sqrt{2})}{3}, \frac{2}{9}, 2(3 - 4 \ln 2), 4(-1+2 \ln 2), \) and \( \sqrt{2} \ln(1+\sqrt{2}) \) are the best possible bounds we found for the inequalities between the integral means and the mean functions.
Proof. (i): If we take $b = at^2$, $t > 1$, then the following will be concluded:
\[ f_1(t) := \frac{8(t^2 + t + 1)^2}{(t^2 + 1)(t + 1)^2} = \frac{I_3(a, b)}{A(a, b)}. \]
Taking the derivative, we get: $f'_1(t) = \frac{16at^2 - 1}{8(t^2 + 1)(t + 1)^2} < 0$. Therefore, $f_1$ is strictly decreasing. So, $\lim_{t \to \infty} f_1(t) = \frac{8}{9} < f_1(t) < \lim_{t \to 1^+} f_1(t) = 1$, $t > 1$.
(ii): If we take $b = at$, $t > 1$, then we will have $f_2(t) := \frac{3}{4}(1 + \frac{(t^3 + 1)\ln t + t^2 \ln t}{(t-1)^2(t+1)^2}) = \frac{I_{2}(a, b)}{A(a, b)}$.
\[ f'_2(t) = \frac{-8f_2(t)}{3(t^2 + t + 1)^2}, \]
where $f_3(t) := (t + 1)^3\ln \frac{t + 1}{2} - (t^3 + 3t^2)\ln t + t^3 - t^2 - t + 1$.
\[ f'_3(t) = 3(t + 1)^2\ln \frac{t + 1}{2} - 3(t^2 + 2t)\ln t + 3t^2 - 3t. \]
\[ f''_3(t) = 6(t + 1)\ln \frac{t + 1}{2} - 6(t + 1)\ln t + 6 - \frac{6}{t}. \]
\[ f'''_3(t) = \frac{6}{t^2(t + 1)} > 0. \]
Therefore, $f''_3$ is strictly increasing, so $f''_3(t) > \lim_{t \to 1^+} f''_3(t) = 0$, $t > 1$. Consequently, $f''_3(t)$ is strictly increasing, hence $f'_3(t) > \lim_{t \to 1^+} f'_3(t) = 0$, $t > 1$. Therefore, $f'_3(t) > \lim_{t \to 1^+} f'_3(t) = 0$, $t > 1$. Thus, $f_3$ is strictly increasing, hence $f_3(t) > \lim_{t \to 1^+} f_3(t) = 0$, $t > 1$. Consequently, $f'_3(t) < 0$, $t > 1$. Therefore, $f_3(t)$ is strictly decreasing.
(iii): $a \neq b \Rightarrow \frac{8(1 - \ln 2)}{3} < \frac{I_2(a, b)}{A(a, b)} < 1 \Rightarrow \frac{2(1 - 4\ln 2)}{3} = 2 - \frac{8(1 - \ln 2)}{3} > 2 - \frac{I_2(a, b)}{A(a, b)} > \frac{I_2(a, b)}{A(a, b)} > 2 - 1 = 1$.
(iv): $a \neq b \Rightarrow \frac{3}{4} \frac{a}{A(a, b)} < \frac{I_3(a, b)}{A(a, b)} < 3/4 \Rightarrow \frac{3}{8} \frac{(1 - \ln 2)}{9} > \frac{3}{4} \frac{a}{A(a, b)} = \frac{I_3(a, b)}{A(a, b)} > \frac{3}{4} < \frac{3}{4} = 1$.
(vi):
\[ f_4(t) := \frac{\sqrt{3}}{3} \left( k(t^3 + 1) - t^3 \ln \frac{1 + \sqrt{1 + t^2}}{t + 1} \right) = \frac{I_4(a, b)}{A(a, b)}, \]
where $a = bt$, $0 < t < 1$, and $k := \sqrt{2} + \ln(1 + \sqrt{2})$. The we have:
\[ f'_4(t) = \frac{\sqrt{2}f_4(t)}{3(t^2 + 1)(t + 1)^3}, \]
where
\[ f_5(t) := (3t^2 + 2t + 3)\sqrt{1 + t^2} + t^2(t + 3)\ln \frac{1 + \sqrt{1 + t^2}}{t} + ... \]
\[ (3t + 1)\ln(t + \sqrt{1 + t^2}) - k(t + 1)^3. \]
Therefore,
\[ f'_5(t) = (3t^2 + 6t)\ln \frac{1 + \sqrt{1 + t^2}}{t} + 3\ln(t + \sqrt{1 + t^2}) + (9t + 3)\sqrt{1 + t^2} - 3k(t + 1)^2, \]
\[ f''_5(t) = (6t + 6)\ln \frac{1 + \sqrt{1 + t^2}}{t} + \frac{(18t^2 + 6)}{\sqrt{1 + t^2}} - 6k(t + 1), \]
\[ f'''_5(t) = 6\left( \frac{3(1 - 4t^2) + 4t^2 - t - 1}{(t^2 + 1)^2} \right) + \frac{1 + \sqrt{1 + t^2}}{t} - k. \]
and

\[ f_5'''(t) = \frac{6(1 - t + 8t^2 - t^3 + t^4)}{t^2(t^2 + 1)^2}. \]

Since \( t \in (0, 1) \), so \( 1 - t + 8t^2 - t^3 + t^4 > (1 - t) + (8 - t + t^2)t^2 > 0 \). Hence, \( f_5''' > 0 \) on \((0, 1)\).

Consequently, \( f_5'' \) will be strictly increasing. Therefore, \( f_5''(t) = \lim_{t \to 1^-} f_5''(t) = 6(\sqrt{2} + \ln(1 + \sqrt{2}) - k) = 0 \), for \( 0 < t < 1 \). Thus, \( f_5'' \) is strictly decreasing.

Hence, \( f_5'(t) > \lim_{t \to 1^-} f_5'(t) = 12 \ln(1 + \sqrt{2}) + 12\sqrt{2} - 12k = 0 \), for \( 0 < t < 1 \). So, \( f_5' \) will be strictly increasing. Therefore, \( f_5(t) < \lim_{t \to 1^-} f_5(t) = 12\ln(1 + \sqrt{2}) + 12\sqrt{2} - 8k = 0 \), on \((0, 1)\) and consequently \( f_4' < 0 \) on \((0, 1)\). Thus, \( f_4 \) will be strictly decreasing on \((0, 1)\). Therefore, \( 1 = \lim_{t \to 1^-} < \)

By (i), (ii), (iii), (iv), (v) and (vi), (vii), (viii), (ix), (x), (xi) and (xii) are straightforward.

**Theorem 2.2.** Let \( M \in \mathcal{R}(\mathbb{R}_+^2) \) be a mean function and \( \varphi : [\alpha_0, \beta_0] \to [0, \infty) \) be an integrable function; that is, \( \varphi \in \mathcal{R}([\alpha_0, \beta_0]) \), where \( \alpha_0, \beta_0 \in \mathbb{R} \) and \( \alpha_0 < \beta_0 \). Besides, \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a Lipschitz function with the constant of 1; that is

\[ |\varphi(x) - \varphi(y)| \leq |x - y|, \quad (x, y) \in \mathbb{R}_+^2. \]

Then \( S_{M, \varphi, \psi} := S_{M, \varphi, \psi}(a, b) \), defined in the following way:

\[ S_{M, \varphi, \psi} := -\varphi + A(a, b) - A(\varphi(a), \varphi(b)) + \frac{1}{\beta_0 - \alpha_0} \int_{\alpha_0}^{\beta_0} M(\varphi(t) + \varphi(a), \varphi(t) + \varphi(b)) \, dt \]

is a mean of \( a \) and \( b \), where

\[ \varphi := \frac{1}{\beta_0 - \alpha_0} \int_{\alpha_0}^{\beta_0} \varphi(t) \, dt. \]

**Proof.**

\[
\min\{a, b\} = A(a, b) - \frac{|a - b|}{2} \leq A(a, b) - \frac{\varphi(a) - \varphi(b)}{2} = A(a, b) - A(\varphi(a), \varphi(b)) + \min\{\varphi(a), \varphi(b)\} \leq S_{M, \varphi, \psi} \leq A(a, b) - A(\varphi(a), \varphi(b)) + \max\{\varphi(a), \varphi(b)\} = A(a, b) + \frac{|\varphi(a) - \varphi(b)|}{2} \leq A(a, b) + \frac{|a - b|}{2} = \max\{a, b\}.
\]

Also, it is obvious that \( S_{M, \varphi, \psi} \) is symmetric. \( \square \)

**Remark 2.1.** Since \( \varphi \in \mathcal{R}([\alpha_0, \beta_0]) \leftrightarrow \varphi \circ \eta \in \mathcal{R}([0, 1]) \),

where \( \eta(t) := (\beta_0 - \alpha_0)t + \alpha_0. \) So, without loss of generality, we can assume \( \alpha_0 = 0 \) and \( \beta_0 = 1. \)
In particular, mean function.

\[ S_M,\varphi,\psi(a, b) = A(a, b). \]

Remark 2.3. If \( \varphi = c \geq 0 \) (c is a constant), then

\[ S_{M,c,\psi} = S_{M,c,\psi}(a, b) = -c + M(c + \psi(a), c + \psi(b)) - A(\psi(a), \psi(b)) + A(a, b). \]

Remark 2.4.

\[ (S_{M,c,\psi})_A = (S_{M,c,\psi})_A(a, b) = c - M(c + \psi(a), c + \psi(b)) + A(\psi(a), \psi(b)) + A(a, b). \]

Proposition 2.4. Let \( M, M_1 \) and \( M_2 \) be mean functions and \( \lambda \in \mathbb{R} \). Then

(i) \( S_{A,\varphi,\psi} = A \),

(ii) \( M_1 > M_2 \Rightarrow S_{M_1,\varphi,\psi} > S_{M_2,\varphi,\psi}, \ a \neq b \),

In particular, \( M > (\langle < \rangle A \Rightarrow S_{M,\varphi,\psi} > (\langle < \rangle A \right. \)

(iii) \( S_{\lambda M_1 + (1 - \lambda)M_2,\varphi,\psi} = \lambda S_{M_1,\varphi,\psi} + (1 - \lambda)S_{M_2,\varphi,\psi} \), if \( \lambda M_1 + (1 - \lambda)M_2 \) is a mean function.

In particular, \( S_{A,\varphi,\psi} = (S_{M,\varphi,\psi})_A \).

Proof. It is straightforward by direct calculations. \( \square \)

Example 2.9. Let \( M = G, \ \varphi(t) = c \geq 0 \) (c is a constant) and \( \psi(t) = \frac{t - \sin t}{2} \), then

\[ N_c := N_c(a, b) := S_{G,c,\psi}(a, b) = A(a, b) - \frac{1}{4} \left( \sqrt{2c + a - \sin a} - \sqrt{2c + b - \sin b} \right)^2. \]

In particular,

\[ N_0 = N_0(a, b) = A(a, b) - \frac{1}{4} \left( \sqrt{a - \sin a} - \sqrt{b - \sin b} \right)^2. \]

We can see

\[ \langle N_c \rangle_A = (N_c)_A(a, b) = A(a, b) + \frac{1}{4} \left( \sqrt{2c + a - \sin a} - \sqrt{2c + b - \sin b} \right)^2. \]

In particular,

\[ (N_0)_A = (N_0)_A(a, b) = A(a, b) + \frac{1}{4} \left( \sqrt{a - \sin a} - \sqrt{b - \sin b} \right)^2. \]

Example 2.10. Let \( M = G, \ \varphi(t) = c \geq 0 \) (c is a constant) and \( \psi(t) = \ln(t^2 + 1) \), then

\[ L_c := L_c(a, b) := S_{G,c,\psi}(a, b) = -c + A(a, b) - \ln(\sqrt{(a^2 + 1)(b^2 + 1)}) + \ldots \]

\[ \sqrt{(c + \ln(a^2 + 1))(c + \ln(b^2 + 1))}. \]

In particular,

\[ L_0 = L_0(a, b) = A(a, b) - \ln(\sqrt{(a^2 + 1)(b^2 + 1)}) + \sqrt{(\ln(a^2 + 1))(\ln(b^2 + 1))}. \]
Also we can see
\[
(L_c)_A = (L_c)_A(a, b) = c + A(a, b) + \ln(\sqrt{(a^2 + 1)(b^2 + 1)}) - \ldots
\]
\[
\sqrt{(c + \ln(a^2 + 1))(c + \ln(b^2 + 1))}.
\]

In particular,
\[
(L_0)_A = (L_0)_A(a, b) = A(a, b) + \ln(\sqrt{(a^2 + 1)(b^2 + 1)}) - \sqrt{(\ln(a^2 + 1))(\ln(b^2 + 1))}.
\]

**Example 2.11.** Let \( M = H \), \( \varphi(t) = Id(t) = t \), then
\[
J_\psi := J_\psi(a, b) := S_{H, Id, \psi}(a, b) = A(a, b) - \left(\frac{\psi(a) - \psi(b)}{2}\right)^2 \ln \left(1 + \frac{1}{A(\psi(a), \psi(b))}\right).
\]

In particular,
\[
J_{Id} = J_{Id}(a, b) = A(a, b) - \left(\frac{a - b}{2}\right)^2 \ln \left(1 + \frac{1}{A(a, b)}\right).
\]

We can see
\[
(J_\psi)_A = (J_\psi)_A(a, b) = A(a, b) + \left(\frac{\psi(a) - \psi(b)}{2}\right)^2 \ln \left(1 + \frac{1}{A(\psi(a), \psi(b))}\right).
\]

In particular,
\[
(J_{Id})_A = (J_{Id})_A(a, b) = A(a, b) + \left(\frac{a - b}{2}\right)^2 \ln \left(1 + \frac{1}{A(a, b)}\right).
\]

**Example 2.12.** Let \( M = G \) and \( \psi(t) = Id(t) = t \), then
\[
I_\varphi := I_\varphi(a, b) := S_{G, \varphi, Id}(a, b) = -\varphi + \int_0^1 \sqrt{(\varphi(t) + a)(\varphi(t) + b)} dt.
\]

In particular,
\[
I_{Id} = I_{Id}(a, b) = -\frac{1}{2} + F_1(a + 1, b + 1) - F_1(a, b) - F_2(a + 1, b + 1) + F_2(a, b),
\]

where
\[
F_1(x, y) := \frac{1}{4}(x + y)\sqrt{xy}, \quad F_2(x, y) := \frac{1}{4}(x - y)^2 \ln \frac{\sqrt{x} + \sqrt{y}}{\sqrt{2}}.
\]

We can see
\[
(I_\varphi)_A = (I_\varphi)_A(a, b) = a + b + \varphi - \int_0^1 \sqrt{(\varphi(t) + a)(\varphi(t) + b)} dt.
\]

In particular,
\[
(I_{Id})_A = (I_{Id})_A(a, b) = a + b + \frac{1}{2} - F_1(a + 1, b + 1) - F_1(a, b) + F_2(a + 1, b + 1) - F_2(a, b).
\]

If \( a \neq b \), by proposition 4 (ii), we will have
\[
N_c < A, \quad L_c < A, \quad J_\psi < A, \quad I_\varphi < A.
\]
Proposition 2.5. If \( a \neq b \), then \( I_\varphi > G \), for every \( \varphi \), whose support is positive measure; equivalently, there exists \( S \subseteq [0,1] \) with \( |S| > 0 \), such that \( \varphi(t) > 0 \) for every \( t \in S \).

Proof. Let \( a \neq b \). We know that

\[
I_\varphi > G
\]
is equivalent to

\[
\int_0^1 \left( \sqrt{(\varphi(t) + a)(\varphi(t) + b)} - \varphi(t) - G(a,b) \right) \, dt > 0.
\]

Let us start our argument by working with the integrand:

\[
\sqrt{(\varphi(t) + a)(\varphi(t) + b)} - \varphi(t) - G(a,b) \geq 0
\]
This results in:

\[
(\sqrt{a} - \sqrt{b})^2 \varphi(t) > 0.
\]

Thus,

\[
\sqrt{(\varphi(t) + a)(\varphi(t) + b)} - \varphi(t) - G(a,b) \geq 0, \quad t \in [0,1]
\]

and

\[
\sqrt{(\varphi(t) + a)(\varphi(t) + b)} - \varphi(t) - G(a,b) > 0, \quad t \in S.
\]

By (2) and (2), we will have

\[
\int_0^1 \left( \sqrt{(\varphi(t) + a)(\varphi(t) + b)} - \varphi(t) - G(a,b) \right) \, dt > 0.
\]

Hence, \( I_\varphi > G \). □

Proposition 2.6. If \( a \neq b \), then

(i) \( J_{Id} > H \),

(ii) neither \( J_{Id} > G \) nor \( J_{Id} < G \),

(iii) neither \( L_0 < G \) nor \( L_0 > H \),

(iv) neither \( N_0 < G \) nor \( N_0 > H \).

Proof. Let \( a \neq b \).

(i): \( J_{Id} > H \iff \frac{(a-b)^2}{2(a^2+b^2)} > \frac{(a-b)^2}{4} \iff \frac{1}{2(a^2+b^2)} > \ln(1 + \frac{2}{a+b}) \sqrt{.}

(ii) One counter example is:

\( J_{Id}(0.5,1) < 0.6971 < 0.7071 < G(0.5,1) \). On the other hand, \( J_{Id}(0.5,0.2) > 0.31962 > 0.31623 > G(0.5,0.2) \).

(iii) A counter examples would be: \( L_0(0.1,0.2) > 0.14516 > 0.1443 > G(0.1,0.2) \).

On the other hand, we have:

\( L_0(4.1754412,4.175399) - H(4.1754412,4.175399) < -10^{-9} < 0 \).

(iv) Here is a counter example:

\( N_0(0.5,0.2) > 0.34713 > 0.31623 > G(0.5,0.2) \).

On the other hand, \( N_0(4.1,4.100000001) - H(4.1,4.100000001) < -10^{-19} < 0 \). □
Theorem 2.3. Let \( M \in \mathcal{R}(\mathbb{R}_+^2) \) be a mean function and \( \varphi : [\alpha_0, \beta_0] \to \mathbb{R}_+ \) be an integrable function; that is, \( \varphi \in \mathcal{R}([\alpha_0, \beta_0]) \), where \( \alpha_0, \beta_0 \in \mathbb{R} \) and \( \alpha_0 < \beta_0 \). Also, \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a Lipschitz function with the constant of 1.

Then \( \mathcal{P}_{M,\varphi,\psi}(a,b) \), defined in the following way:

\[
\mathcal{P}_{M,\varphi,\psi}(a,b) := A(a,b) - A(\psi(a), \psi(b)) + \frac{1}{(\beta_0 - \alpha_0)\varphi} \int_{\alpha_0}^{\beta_0} M(\varphi(t)\psi(a), \varphi(t)\psi(b)) \, dt
\]

is a mean of \( a \) and \( b \).

Proof.

\[
\min\{a, b\} = A(a, b) - \frac{|a - b|}{2} \leq A(a, b) - \frac{\psi(a) - \psi(b)}{2} = A(a, b) - A(\psi(a), \psi(b)) + \min\{\psi(a), \psi(b)\} \leq \mathcal{P}_{M,\varphi,\psi}(a,b) \leq \ldots
\]

\[
A(a, b) - A(\psi(a), \psi(b)) + \max\{\psi(a), \psi(b)\} = A(a, b) + \frac{\psi(a) - \psi(b)}{2}
\]

\[
\leq A(a, b) + \frac{|a - b|}{2} = \max\{a, b\}.
\]

Also, it is obvious that \( \mathcal{P}_{M,\varphi,\psi} \) is symmetric. \( \square \)

Remark 2.5. Since \( \varphi \in \mathcal{R}([\alpha_0, \beta_0]) \Leftrightarrow \varphi \circ \eta \in \mathcal{R}([0, 1]) \), where \( \eta(t) := (\beta_0 - \alpha_0)t + \alpha_0 \). So, without loss of generality, we can assume \( \alpha_0 = 0 \) and \( \beta_0 = 1 \).

Remark 2.6. If \( \psi(a) = \psi(b) \), then \( \mathcal{P}_{M,\varphi,\psi}(a,b) = A(a,b) \).

Remark 2.7. If \( \varphi = c > 0 \) (\( c \) is a constant), then

\[
\mathcal{P}_{M,c,\psi} = \mathcal{P}_{M,c,\psi}(a,b) = \frac{1}{c} M(c\psi(a), c\psi(b)) - A(\psi(a), \psi(b)) + A(a, b).
\]

Remark 2.8.

\[
(\mathcal{P}_{M,c,\psi})_A = (\mathcal{P}_{M,c,\psi})(a,b) = -\frac{1}{c} M(c\psi(a), c\psi(b)) + A(\psi(a), \psi(b)) + A(a, b).
\]

Remark 2.9. If \( M \) is a homogeneous function of order 1; that is

\[
M(xz, yz) = zM(x, y), \quad \forall x, y, z \in \mathbb{R}_+,
\]

then \( \mathcal{P}_{M,\varphi,\psi}(a,b) = A(a, b) - A(\psi(a), \psi(b)) + M(\psi(a), \psi(b)) \).

Remark 2.10. \( C, r, g, A, H_n, G \) and \( H \) are homogeneous mean functions of order 1.

Proposition 2.7. Let \( M, M_1 \) and \( M_2 \) be mean functions and \( \lambda \in \mathbb{R} \). Then

(i) The following three equations hold:

\[
\mathcal{P}_{A,\varphi,\psi}(a,b) = A(a,b),
\]

\[
\mathcal{P}_{G,\varphi,\psi}(a,b) = A(a,b) - A(\psi(a), \psi(b)) + G(\psi(a), \psi(b)) = A(a,b) - \frac{1}{2}(\sqrt{\psi(a)} - \sqrt{\psi(b)})^2,
\]

\[
(\mathcal{P}_{G,\varphi,\psi})_A(a,b) = A(a,b) + \frac{1}{2}(\sqrt{\psi(a)} - \sqrt{\psi(b)})^2.
\]
(ii) \( M_1 > M_2 \Rightarrow \mathcal{P}_{M_1, \varphi, \psi} > \mathcal{P}_{M_2, \varphi, \psi}, \; a \neq b, \)

In particular, \( M > (\langle \rangle)A \Rightarrow \mathcal{P}_{M, \varphi, \psi} > (\langle \rangle)A, \)

(iii) \( \mathcal{P}_{\lambda M_1 + (1-\lambda)M_2, \varphi, \psi} = \lambda \mathcal{P}_{M_1, \varphi, \psi} + (1-\lambda)\mathcal{P}_{M_2, \varphi, \psi}, \) if \( \lambda M_1 + (1-\lambda)M_2 \) is a mean function.

In particular, \( \mathcal{P}_{M, \varphi, \psi} = (\mathcal{P}_{M, \varphi, \psi})_A. \)

Proof. It is straightforward. \qed

Here are some examples where the above proposition is used:

Example 2.13. Let \( M = G \) and \( \psi(t) = \frac{t - \sin t}{2}, \) then
\[
\mathcal{P}_{G, \varphi, \hat{\psi}} = N_0.
\]

Example 2.14. Let \( M = G \) and \( \tilde{\psi}(t) = \ln(t^2 + 1), \) then
\[
\mathcal{P}_{G, \varphi, \tilde{\psi}} = L_0.
\]

Example 2.15. Let \( M = H, \; \varphi(t) = \text{Id}(t) = t, \) then
\[
\mathcal{P}_{H, \text{Id}, \psi}(a, b) = A(a, b) - A(\psi(a), \psi(b)) + H(\psi(a), \psi(b)) = A(a, b) - \frac{(\psi(a) - \psi(b))^2}{2(\psi(a) + \psi(b))}.
\]

In particular,
\[
\mathcal{P}_{H, \text{Id}, \text{Id}}(a, b) = H(a, b).
\]

We can see
\[
\mathcal{P}_{H, \text{Id}, \psi}(a, b) < S_{H, \text{Id}, \psi}(a, b), \quad a \neq b.
\]

Let \( M \) be a mean function. We define
\[
\hat{S}_M := \hat{S}_M(a, b) := \int_0^\pi M(a \sin \theta, b \cos \theta) \, d\theta.
\]

We can easily see \( \hat{S}_M(a, b) = \hat{S}_M(b, a). \) Also,
\[
\hat{S}_M(a, b) \leq \int_0^\pi \left( \frac{a \sin \theta + b \cos \theta}{2} + \frac{|a \sin \theta - b \cos \theta|}{2} \right) \, d\theta =
\]
\[
A(a, b) + \frac{1}{2} \int_0^{\tan^{-1} \frac{b}{a}} (b \cos \theta - a \sin \theta) \, d\theta + \frac{1}{2} \int_{\tan^{-1} \frac{b}{a}}^\pi (-b \cos \theta + a \sin \theta) \, d\theta =
\]
\[
A(a, b) + \frac{1}{2} \int_0^{\tan^{-1} \frac{b}{a}} \frac{b \cos \theta - a \sin \theta}{\sqrt{a^2 + b^2}} \, d\theta - \frac{1}{2} \int_0^{\tan^{-1} \frac{b}{a}} \frac{|a \sin \theta - b \cos \theta|}{\sqrt{a^2 + b^2}} \, d\theta =
\]
\[
A(a, b) - \frac{1}{2} \int_0^{\tan^{-1} \frac{b}{a}} (-b \cos \theta + a \sin \theta) \, d\theta + \frac{1}{2} \int_{\tan^{-1} \frac{b}{a}}^\pi (-b \cos \theta + a \sin \theta) \, d\theta =
\]
\[
2A(a, b) - \sqrt{a^2 + b^2}.
\]
Thus, by (2) and (2), we have
\[(2.5)\]
\[2\mathcal{A}(a,b) - \sqrt{a^2 + b^2} \leq \hat{S}_M \leq \sqrt{a^2 + b^2}.\]
From (2.5), if we take \(\xi := \xi(a,b) > 0\) and \(\zeta := \zeta(a,b)\), such that
\[(2.6)\]
\[\min\{a,b\} \leq \frac{(a+b)\xi}{\sqrt{a^2 + b^2}} - \xi + \zeta \leq \frac{\xi}{\sqrt{a^2 + b^2}} \hat{S}_M + \zeta \leq \xi + \zeta \leq \max\{a,b\},\]
then we will have
\[(2.7)\]
\[0 < \xi \leq \frac{|a-b|\sqrt{a^2 + b^2}}{2\sqrt{a^2 + b^2} - (a+b)}\]
and
\[(2.8)\]
\[\min\{a,b\} + (1 - \frac{a+b}{\sqrt{a^2 + b^2}})\xi \leq \zeta \leq \max\{a,b\} - \xi.\]
By (2.8), (2.9) and (2.8), we infer
\[(2.9)\]
\[S_{M,\xi,\zeta} := S_{M,\xi,\zeta}(a,b) := \frac{\xi(a,b)}{\sqrt{a^2 + b^2}} \hat{S}_M(a,b) + \zeta(a,b)\]
is a mean of \(a\) and \(b\).
For example, if we take \(\xi := |a-b|\) and \(\zeta := \frac{1}{2} \left( \min\{a,b\} + (1 - \frac{a+b}{\sqrt{a^2 + b^2}})|a-b| + \max\{a,b\} - |a-b| \right) = \mathcal{A}(a,b) - \frac{|a^2-b^2|}{2\sqrt{a^2 + b^2}},\) then from (2.9)
\[(2.10)\]
\[S_M := S_M(a,b) := \mathcal{A}(a,b) - \frac{|a^2-b^2|}{2\sqrt{a^2 + b^2}} + \frac{|a-b|}{\sqrt{a^2 + b^2}} \int_{0}^{\frac{\pi}{2}} M(a \sin \theta, b \cos \theta) \, d\theta\]
is a mean of \(a\) and \(b\). Thus, we will have the following theorem

**Theorem 5** Let \(M\) is a mean function. Then \(S_M\) which is defined by (2.10), is a mean function.

**Proposition 2.8.** Let \(M, M_1\) and \(M_2\) be mean functions and \(\lambda \in \mathbb{R}\). Then
\[(i)\]
\[S_{\lambda M_1 + (1-\lambda)M_2} = \lambda S_{M_1} + (1-\lambda)S_{M_2}, \text{ if } \lambda M_1 + (1-\lambda)M_2 \text{ is a mean function.}\]
Specially, \(S_{\mathcal{A}} = (S_{\mathcal{M}})_{\mathcal{A}}.\)
\[(ii)\]
\[M_1 > M_2 \Rightarrow S_{M_1} > S_{M_2}, \quad a \neq b,\]
specially, \(M > (\langle \rangle A) \Rightarrow S_{M} > (\langle \rangle A).\)

**Proof.** is straightforward. \(\square\)

**Example 2.16.**
\[S_{\mathcal{A}} = A, \quad S_{\mathcal{G}}(a,b) = A(a,b) + \frac{G(a,b)|a-b|^2(\frac{4}{7})}{\sqrt{\pi(a^2 + b^2)}} - \frac{|a^2-b^2|}{2\sqrt{a^2 + b^2}},\]
\[ \Gamma\left(\frac{3}{4}\right) \approx 1.225416702. \]

**Example 2.17.**

\[ S_H(a, b) = A(a, b) - \frac{|a^2 - b^2|(a^2 + b^2 - 4ab)}{2(a^2 + b^2)^\frac{3}{2}} = \ldots \]

\[ \frac{4a^2b^2|a - b|}{(a^2 + b^2)^2} \ln \frac{a + b + \sqrt{a^2 + b^2}}{\sqrt{2ab}}. \]

**Example 2.18.** By proposition 2.8 (i)

\[ S_g = A(a, b) + \frac{|a^2 - b^2|(a^2 + b^2 - 4ab)}{6(a^2 + b^2)^\frac{3}{2}} + \ldots \]

\[ \frac{4a^2b^2|a - b|}{3(a^2 + b^2)^2} \ln \frac{a + b + \sqrt{a^2 + b^2}}{\sqrt{2ab}} \]

and

**Example 2.19.**

\[ S_C = A(a, b) + \frac{|a^2 - b^2|(a^2 + b^2 - 4ab)}{2(a^2 + b^2)^\frac{3}{2}} + \ldots \]

\[ \frac{4a^2b^2|a - b|}{(a^2 + b^2)^2} \ln \frac{a + b + \sqrt{a^2 + b^2}}{\sqrt{2ab}}. \]

By proposition 2.8 (ii), for \( a \neq b \)

\[ \left( \frac{\sqrt{2(a^2 + b^2)}}{|a - b|} \right) S_C - \frac{a + b}{\sqrt{2}} \left( \frac{\sqrt{a^2 + b^2}}{|a - b|} - 1 \right) \]

\[ > \int_0^{\frac{\pi}{2}} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} \, d\theta > \]

\[ \left( \frac{\sqrt{2(a^2 + b^2)}}{|a - b|} \right) S_g - \frac{a + b}{\sqrt{2}} \left( \frac{\sqrt{a^2 + b^2}}{|a - b|} - 1 \right) \]

and

\[ S_C(a, b) > S_g(a, b) > A(a, b) > S_C(a, b) > S_H(a, b), \quad a \neq b. \]

Specially, \( A(3, 4) > S_C(3, 4) > S_H(3, 4) \), which we infer

\[ \frac{7}{12} > \frac{\Gamma^2\left(\frac{3}{4}\right)}{\sqrt{3\pi}} > \frac{140 - 48 \ln 6}{125}. \]

**Theorem 2.4.** Let \( M_1, M_2 \in R( R_+^2) \) be mean functions. Then

\[ \mathcal{T}_{M_1, M_2} := \mathcal{T}_{M_1, M_2}(a, b) := \begin{cases} \frac{1}{b-a} \int_a^b M_1(M_2(a, b), x) \, dx, & a \neq b, \\ a, & a = b \end{cases} \]

is a mean of \( a \) and \( b \).
Proof. Let \( b > a \) and \( x \in [a, b] \). We have

\[
\frac{1}{b-a} \int_a^b M_1(M_2(a, b), x) \, dx \leq \frac{1}{b-a} \int_a^b \max \{ M_2(a, b), x \} \, dx = \\
\frac{1}{b-a} \int_a^b M_2(a, b) \, dx + \frac{1}{b-a} \int_a^b x \, dx = \ldots
\]

\[
\frac{1}{2(b-a)} (b^2 + M_2^2(a, b) - 2aM_2(a, b)).
\]

If we take \( m_1(t) := b^2 + t^2 - 2at, \ t \in [a, b], \) then \( m_1 \) will be increasing on \([a, b]\). So, \( m_1(t) \leq m_1(b) = 2b(b-a), \) for \( t \in [a, b]. \) Hence, from (2), we will get

\[
\frac{1}{b-a} \int_a^b M_1(M_2(a, b), x) \, dx \leq b.
\]

Similarly,

\[
\frac{1}{b-a} \int_a^b M_1(M_2(a, b), x) \, dx \geq \frac{1}{b-a} \int_a^b \min \{ M_2(a, b), x \} \, dx = \\
\frac{1}{b-a} \int_a^b x \, dx + \frac{1}{b-a} \int_a^b M_2(a, b) \, dx = \ldots
\]

\[
\frac{1}{2(b-a)} (-a^2 - M_2^2(a, b) + bM_2(a, b)).
\]

If we take \( m_2(t) := -a^2 - t^2 + 2bt, \ t \in [a, b], \) then \( m_2 \) will be increasing on \([a, b]\). So, \( m_2(t) \geq m_2(a) = 2a(b-a), \) for \( t \in [a, b]. \) Therefore, from (2), we will get

\[
\frac{1}{b-a} \int_a^b M_1(M_2(a, b), x) \, dx \geq a.
\]

Also, it is obvious that \( T_{M_1, M_2} \) is symmetric. \( \square \)

Proposition 2.9. Let \( M_1, M_1', M_2 \) and \( M_2' \) be mean functions and \( \lambda \in \mathbb{R} \). Then

(i) \( M_1 > M'_1 \Rightarrow T_{M_1, M_2}(a, b) > T_{M_1, M_2}(a, b), \ a \neq b, \)

(ii) If \( M_1 \) is strictly increasing and \( M_2 > M'_2, \) then

\( T_{M_1, M_2}(a, b) > T_{M_1, M_2}(a, b), \ a \neq b, \)

(iii) \( T_{\lambda M_1 + (1-\lambda)M_1', M_2} = \lambda T_{M_1, M_2} + (1-\lambda) T_{M_1', M_2}, \)

if \( \lambda M_1 + (1-\lambda)M'_1 \) is a mean function.

Specially, \( T_{M_1, A} = 2T_{A, M_2} - T_{M_1, M_2}. \)

Proof. is straightforward. \( \square \)
Some Examples  Let $M$ be a mean function.

(1) \[ T_{A,M}(a,b) = A(M(a,b), A(a,b)). \]

(2) \[ T_{G,M}(a,b) = G(M(a,b), g^2(\sqrt{a}, \sqrt{b})). \]

(3) \[ T_{H,M}(a,b) = 2M(a,b) \left(1 - \frac{M(a,b)}{b - a} \ln \frac{b + M(a,b)}{a + M(a,b)}\right), \quad a \neq b. \]

(4) \[ T_{r,M}(a,b) = \frac{1}{2\sqrt{2}} \left(\frac{(a + b)(a^2 + b^2 + M^2(a,b))}{(b - a) \ln \frac{b + \sqrt{b^2 + M^2(a,b)}}{a + \sqrt{a^2 + M^2(a,b)}}}\right), \quad a \neq b. \]

By proposition 2.9, we will get
\[ T_{r,M}(a,b) > T_{A,M}(a,b) > T_{G,M}(a,b) > T_{H,M}(a,b), \quad a \neq b. \]

Also, by proposition 2.9(ii) and note 12, we will have
\[ T_{A,A}(a,b) > T_{A,G}(a,b) > T_{A,H}(a,b), \quad a \neq b, \]
\[ T_{G,A}(a,b) > T_{G,G}(a,b) > T_{G,H}(a,b), \quad a \neq b \]
and
\[ T_{H,A}(a,b) > T_{H,G}(a,b) > T_{H,H}(a,b), \quad a \neq b. \]

Besides, by proposition 2.9(iii), we will have
\[ T_{H,M} = \frac{4}{3} T_{A,M} + \frac{2}{3} T_{G,M}, \]
\[ T_{G,M} = \frac{4}{3} T_{A,M} - \frac{1}{3} T_{H,M} \]
and
\[ T_{(M_1)A,M_2} = 2T_{A,M_2} - T_{M_1,M_2}, \]
if $M, M_1$ and $M_2$ are mean functions.

References

1. H. Eves, Means Appearing in Geometrical Figures, Mathematics Magazine 76(4) (2003), 292-294.
2. E. Beckenbach, R. Bellman, An Introduction to Inequalities, Random House, Inc., New York, 1961.
3. Inder J. Taneja, Refinement of Inequalities among Means, Journal of Combinatorics, Information and Systems Sciences 31 (2006), 357-378.
4. Inder J. Taneja, Inequalities having seven means and proportionality relations, arXiv:1203.2288