A LARGE $N$ CHIRAL TRANSITION ON A PLAQUETTE

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Abstract

We construct a model of a chiral transition using the well known large $N$ transition in two dimensional $U(N)$ lattice gauge theory. Restricting the model to a single plaquette, we introduce Grassmann variables on the corners of the plaquette with the natural phase factors of staggered fermions and couple them to the $U(N)$ link variables. The classical theory has a continuous chiral symmetry which is broken at strong couplings, but is restored for weak couplings in the $N \rightarrow \infty$ limit.

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I. INTRODUCTION

Chiral transitions can occur in (non-Abelian) gauge theories coupled to fermions. These transitions can be continuous or discontinuous depending on the specific dynamics of the model. If the up and down quark masses were exactly zero, we expect the real world to undergo such a transition at a finite temperature. It is further believed that the order of chiral transitions crucially depends on the number of quark flavors \[1\]. If the strange quark was sufficiently light, the phase transition in the real world could be first order. These possibilities have created a lot of interest in studying chiral transitions in lattice simulations \[2\].

Our theoretical understanding of such transitions is essentially based on universality arguments \[3\], which have worked quite well for many condensed matter systems. The present evidence from lattice simulations supports the predictions of universality. However, recently there have been some concerns \[4\] regarding the validity of such arguments for fermionic systems. Further, even if universality arguments are valid, there exist dynamics specific to the gauge fields that can be important near the transition and cannot be determined in the universal model. For example, it is well known that anomalous axial symmetry breaking effects can play an important role in determining the order of the transition \[4\]. Recent lattice simulations \[4\] have suggested a rapid evolution in the anomalous symmetry breaking effects close to the chiral transition. Thus it is useful to understand the dynamics of the gauge fields that are responsible for chiral symmetry restoration. As a step in this direction it can be useful to study simplified models, that have some non-trivial gauge dynamics.

Chiral symmetry breaking in gauge theories is closely connected to the small eigenvalue density of the Dirac operator. Recently some progress has been made in understanding the spectrum of small eigenvalues of the Dirac operator in QCD \[5\]. The models that have been studied are the so called random matrix models. Such models are motivated by the chiral symmetries of the Dirac operator. It is possible that these models incorporate some dynamics of gauge fields at a crude level. When such models are extended \[5\] to include finite temperature effects, one obtains a chiral transition. An interesting feature of the random
matrix models is that the dynamical variable is typically an $N \times N$ matrix. The transitions are obtained in the thermodynamic limit which involves taking $N$ to infinity. In this sense, these can be regarded as large $N$ transitions of a matrix model.

A question that immediately strikes is whether there exist other known large $N$ transitions that can be interpreted as a chiral transition with an appropriate extension. In this article we show that one can construct a chiral transition based on the large $N$ transition discovered by Gross and Witten [8] and Wadia [9] independently. They showed that the two dimensional $U(N)$ lattice gauge theory with the Wilson gauge action, has a large $N$ transition essentially due to the non-analytic behavior of a single plaquette integral. Consequently the pure $U(N)$ lattice gauge theory with the Wilson gauge action has a third order phase transition separating strong and weak couplings. In the present work we show that, if we consider only a single plaquette, it is possible to include fermionic variables on the corners of the plaquette and turn the transition into a chiral transition. In the next two sections we define the theory more precisely and solve it in the large $N$ limit, showing the existence of the chiral transition. In the final section we make some observations on the results obtained.

II. THE ONE PLAQUETTE THEORY

The theory we are considering is the well known one plaquette model, which consists of four $N \times N$ unitary matrices living on the sides of a square (plaquette). To this we add Grassmann $N$-vectors, $\chi_n, \bar{\chi}_n$ associated to the corners $n = 1, 2, 3, 4$. We will refer to them as fermionic variables as opposed to the unitary matrices that will be referred to as gauge (or link) variables. The unitary matrix that connects the adjacent corners $n$ and $m$ will be represented as $U_{nm}$. The theory is described by the action

$$S[U, \chi, \bar{\chi}] = -\frac{1}{g^2} Tr(U_p + U_p^\dagger) - \bar{\chi}_n(D_{nm} + m\delta_{nm})\chi_m,$$

where $U_p = U_{12}U_{23}U_{43}U_{14}^\dagger$. Note that the sign in front of the gauge action is natural from lattice QCD. The operator $D$ couples the fermionic variables on adjacent cites. We will refer to $D$ as the Dirac operator in the present model. The non-zero components of the
operator $D_{nm}$ are given by $D_{12} = \eta_{12} U^{12}$, $D_{23} = \eta_{23} U^{23}$, $D_{34} = \eta_{43} U^{43\dagger}$, $D_{41} = \eta_{14} U^{14\dagger}$, with the anti-hermitian property that $D_{nm} = -(D_{nm})^\dagger$. Thus in the matrix notation we have

$$D = \begin{pmatrix}
0 & \eta_{12} U^{12} & 0 & -\eta_{14} U^{14} \\
-\eta_{12} U^{12\dagger} & 0 & \eta_{23} U^{23} & 0 \\
0 & -\eta_{23} U^{23\dagger} & 0 & \eta_{43} U^{43\dagger} \\
\eta_{14} U^{14\dagger} & 0 & -\eta_{43} U^{43\dagger} & 0
\end{pmatrix}. \quad (2)$$

The factors $\eta_{12}, \eta_{23}, \eta_{43}$ and $\eta_{14}$ could be thought of as extra couplings in the theory. In order to fix these couplings we use the analogy with staggered fermions. Such factors occur naturally in the staggered fermion formulation \[10\] as phase factors which are remnants of gamma matrices. In two dimension one encounters phase factors denoted by $\eta_\mu(x)$, where $\mu = 1, 2$ represents the direction and $x = (x_1, x_2)$ the lattice site. For example one can chose $\eta_1(x) = 1$ and $\eta_2 = (-1)^{x_1}$. The well known staggered fermion action is then given by

$$S_{\text{staggered}} = \sum_x \chi(x) \eta_\mu(x) \left(U_{x,x+\mu} \chi(x + \mu) - U_{x-\mu,x}^\dagger \chi(x - \mu)\right). \quad (3)$$

In the present model since we have a plaquette, we can think of it as an object in two dimension. Restricting the staggered fermion action eq.(3) to a single plaquette the factors $\eta_{12}, \eta_{23}, \eta_{43}$ and $\eta_{14}$ can be naturally associated with $\eta_\mu(x)$. Using this connection we obtain one choice of the factors to be

$$\eta_{12} = \eta_{43} = \eta_{14} = -\eta_{23} = 1. \quad (4)$$

We will see that with the definition of the plaquette action given in eq.(1), the above choice of the $\eta$ factors is necessary for the existence of the chiral transition.

We can now define the partition function of the theory by the usual integral over the gauge and fermionic variables

$$Z = \int [dU] [d\chi] [d\bar{\chi}] \exp(-S[U, \chi, \bar{\chi}]). \quad (5)$$

\[1\] Other equivalent choices would lead to the same results.
It is obvious by construction that the theory is gauge invariant in the the usual sense. Further the model also has a chiral symmetry, given by
\[
\chi_n \rightarrow \exp(-i\theta(-1)^n)\chi_n, \quad \overline{\chi} \rightarrow \overline{\chi} \exp(-i\theta(-1)^n)
\] (6)
in the chiral limit \( m \rightarrow 0 \). This is the remnant of the chiral symmetry usually present in a staggered fermion formulation. This suggests that the chiral order parameter, \( \langle \overline{\chi}\chi \rangle \), defined by
\[
\langle \overline{\chi}\chi \rangle = \frac{1}{Z} \int [dU][d\chi][d\overline{\chi}] \frac{1}{N} \overline{\chi}_n \chi_n \exp(-S[U,\chi,\overline{\chi}]),
\] (7)
would vanish in the chiral limit. This will certainly be true for finite \( N \). However, as we will show below in the large \( N \) limit, the theory breaks chiral symmetry at strong couplings. Further, the well known large \( N \) transition of the pure gauge theory restores the chiral symmetry in the weak coupling.

III. LARGE N SOLUTION

We will follow the work Gross and Witten \[8\] closely\footnote{with minor differences in notation} and calculate the chiral condensate defined in eq.(7) in the large \( N \) limit. It is important to recognize that the large \( N \) limit is not an approximation here, but is necessary for the transition. It is obvious that the above partition function will not show non-analytic properties when \( N \) is finite. Further as usual the chiral limit must be considered only after the large \( N \) limit is taken. Hence we are naturally lead to study the large \( N \) solution to the theory.

Since the theory is gauge invariant we can go to a convenient gauge before doing the integrals. We will fix the gauge such that \( U^{23} = U^{14} = U^{43} = 1 \). This naturally imposes the constraint that \( U^{12} = U_p \). Using the properties of the Haar measure we obtain,
\[
\langle \overline{\chi}\chi \rangle = \frac{1}{Z} \int [dU_p] \frac{1}{N} Tr \left( \frac{1}{D+m} \right) \exp \left( \frac{1}{g^2} Tr(U_p + U_p^+) + Tr \log [D+m] \right),
\] (8a)
\[
Z = \int [dU_p] \exp \left( \frac{1}{g^2} Tr(U_p + U_p^+) + Tr \log [D+m] \right).
\] (8b)
The matrix $D$ is now given by

$$
D = \begin{pmatrix}
0 & U_p & 0 & -1 \\
-U_p^\dagger & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
\end{pmatrix},
$$

(9)

where it is important to remember that each element in the above $4 \times 4$ matrix is an $N \times N$ matrix. The integrand in eq.(8a) and eq.(8b) only depends on the eigenvalues, $\alpha_i, i = 1, ..., N$ of $U_p$. This is obviously true for the part of the integrand that depends only on the trace of $U_p$. To see that this is also true for the remaining terms that depend on the matrix $D$, note that only eigenvalues of $D$ enter the integrand. It is easy to compute the eigenvalues of $D$. Firstly note that the only $N \times N$ matrix in $D$ defined in eq.(9), that is not a multiple of unity is $U_p$. Thus using the matrix $T \delta_{n,m}$, where $T$ is the $N \times N$ matrix such that $(T^\dagger U_p T)_{ij} = \delta_{ij} e^{i\alpha_i}; \alpha_i \in (-\pi, \pi]$, it is possible to block diagonalize $D$ into $4 \times 4$ matrices, $D^i$, each of which is associated with one eigenvalue $\alpha_i$ and is given by

$$
D^i = \begin{pmatrix}
0 & \exp(i\alpha_i) & 0 & -1 \\
-\exp(-i\alpha_i) & 0 & -1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & -1 & 0 \\
\end{pmatrix}.
$$

(10)

Thus the eigenvalues of $D$ turn out to be

$$
\lambda_j^+ = \pm i \sqrt{2(1 + \sin \frac{\alpha_j}{2})}, \quad \lambda_j^- = \pm i \sqrt{2(1 - \sin \frac{\alpha_j}{2})}, \quad j = 1, 2, ..., N.
$$

(11)

Note that, due to the anti-hermitian nature of $D$ and the chiral symmetry discussed above, the eigenvalues are imaginary and come in pairs of opposite sign. Clearly if $\alpha_i = \pi \lambda_i^- = 0$ and $D^i$ has zero eigenvalues. This connection between $\alpha = \pi$ and zero eigenvalues of $D$ plays an important role in the existence of the chiral transition being discussed. The Haar measure simplifies to $[dU_p] = \text{const} \ dT \prod_i d\alpha_i \Delta^2(\alpha_i)$, where

$$
\Delta^2(\alpha_i) = \prod_{i<j} \sin^2 \left| \frac{\alpha_i - \alpha_j}{2} \right|.
$$

(12)
Hence the integral over $T$ and various constants drop from expectation values such as the chiral condensate. Further in the large $N$ limit, the integral is dominated by a single $U_p$ with a specific eigenvalue distribution. This distribution can be obtained using the saddle point condition,

$$\frac{2}{g^2} \sin \alpha_j + \frac{4 \sin \alpha_j}{m^4 + 4m^2 + 4 \cos^2(\frac{\alpha_j}{2})} = \sum_{k \neq j} \cot \left| \frac{\alpha_j - \alpha_k}{2} \right|.$$  \hspace{1cm} (13)

Except for the second term on the left hand side, this is exactly the same equation as in [8]. The extra term comes from the extra fermionic determinant in the present model. To get nontrivial results we take the $N \to \infty$ limit while keeping $G = Ng^2$ fixed. In this limit the extra fermionic piece plays no role since it is suppressed by a power of $N$, a familiar effect related to the usual suppression of fermions in the large $N_c$ limit. Thus in the large $N$ limit the saddle point equation, eq.(13), reduces to a continuum version that was studied in [8], namely

$$\frac{2}{G} \sin \alpha(x) = P \int_0^1 dy \ cot \left| \frac{\alpha(x) - \alpha(y)}{2} \right|,$$  \hspace{1cm} (14)

where $\alpha(x)$ is a nondecreasing function in the range $0 \leq x \leq 1$ with $\alpha_i = \alpha(i/N)$. This equation can be solved by introducing the eigenvalue density $\rho_{U_p}(\alpha) = dx/d\alpha$ which is nonzero in the interval $|\alpha| \leq \alpha_c$, $\alpha_c \leq \pi$. We have introduced the subscript $U_p$ to distinguish this eigenvalue density from the one to be introduced shortly, namely the eigenvalue density of the operator $D$. It is the latter that plays an interesting role in the calculation of the chiral condensate. Borrowing the results from [8] we have

$$\rho_{U_p}(\alpha) = \frac{1}{2\pi} \left[ 1 + \frac{2}{G} \cos \alpha \right], \ |\alpha| \leq \alpha_c, \ G \geq 2,$$  \hspace{1cm} (15a)

$$\rho_{U_p}(\alpha) = \frac{2}{G\pi} \cos(\alpha/2) \left[ \sin^2(\alpha/2) - \sin^2(\alpha/2) \right]^{1/2}, \ |\alpha| \leq \alpha_c, \ G \leq 2,$$  \hspace{1cm} (15b)

where $\alpha_c$ is equal to $\pi$ when $G \geq 2$ and is given by $G/2 = \sin^2(\alpha_c/2)$ for $G \leq 2$.

We can use these results now to calculate the chiral condensate. In the large $N$ limit one just has to evaluate $\frac{1}{N} Tr[1/(D + m)]$ in the background of $U_p$ whose eigenvalue distribution is given by eq.(13). We obtain
\[
\langle \bar{\chi}\chi \rangle = \int_{-\alpha_c}^{\alpha_c} d\alpha \ \rho_{\nu_\mu}(\alpha) \left( \frac{2m}{\lambda^+(\alpha)^2 + m^2} + \frac{2m}{\lambda^-(\alpha)^2 + m^2} \right),
\]
where the values of \(D\), namely \(\lambda^+(\alpha)\) and \(\lambda^-(\alpha)\) are given by eq.(11) by just dropping the subscript \(j\).

The reason for the chiral transition is now clear. Since we need a non-zero density of small eigenvalues of \(D\), we need to understand the values of \(\alpha\) for which one obtains small eigenvalues of \(D\). Note that \(\lambda^+(\alpha)\) is always non-zero, while \(\lambda^-(\alpha)\) can become zero when \(\alpha = \pi\). Thus for the chiral condensate to be non-zero in the chiral limit, we need values of \(\alpha\) close to \(\alpha = \pi\) to contribute. For strong coupling, i.e. \(G > 2\) this is clearly possible. However for weak coupling, i.e. \(G \leq 2\), one sees that \(\rho_{\nu_\mu}(\alpha) = 0\) for \(\alpha = \pi\), which then suggests the possibility of the vanishing of chiral condensate.

We define the eigenvalue density for the operator \(D\) to be \(\rho(\lambda)\). It is easy to show that \(\rho(\lambda) = \rho(\alpha)(|d\alpha/d\lambda| + |d\alpha/d\lambda^-|)\), where the first term is evaluated at \(|\lambda^+| = \lambda\) and the second term is evaluated at \(|\lambda^-| = \lambda\)

\[
\rho(\lambda) = \frac{4}{\pi} \begin{cases} 
\left(1 + \frac{2}{G}[1 - 2(1 - \lambda^2/2)^2]\right)/\sqrt{4 - \lambda^2} & 0 \leq \lambda \leq 2, \quad G \geq 2, \\
\frac{2\lambda^2}{G}\sqrt{(G/2 - (1 - \lambda^2/2)^2)} & \lambda_1 \leq \lambda \leq \lambda_2, \quad G \leq 2, \\
0 & \text{otherwise,}
\end{cases}
\]

where \(\lambda_1 = \sqrt{2(1 - \sqrt{G/2})}\) and \(\lambda_2 = \sqrt{2(1 + \sqrt{G/2})}\). In figure 1 we plot \(\rho(\lambda)\) for a few values of \(G\) to illustrate the situation. Using the definition of \(\rho(\lambda)\) we can write the chiral condensate to be

\[
\langle \bar{\chi}\chi \rangle = \int_0^2 d\lambda \ \rho(\lambda) \left( \frac{2m}{\lambda^2 + m^2} \right).
\]

In the chiral limit we obtain \(\langle \bar{\chi}\chi \rangle = \pi \rho(0)\). We see that for \(G \geq 2\), \(\langle \bar{\chi}\chi \rangle \sim (G - 2)\), and for \(G \leq 2\), \(\langle \bar{\chi}\chi \rangle = 0\). Further, when \(G = 2\), we obtain \(\langle \bar{\chi}\chi \rangle \sim m\). If we naively read off the critical exponents we find \(\beta = 1\) and \(\delta = 1\), quite different from mean field. However, since we do not know of any universality arguments applicable to the model, we need not take the exponents too seriously. Further the underlying transition is a third order transition. 

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IV. CONCLUSIONS

Some time ago it was shown by Eguchi and Kawai [11] that in the large $N$ limit, $U(N)$ lattice gauge theory in $d$ dimensions can be reduced to a model on a $d$ dimensional hypercube. If this could be done in the presence of staggered fermions one would obtain a model on the $d$ dimensional hypercube. In the present work we have formulated a simple model on the plaquette by hand. Thus it would be interesting to know if similar results can be obtained from the full 2-d theory in the spirit of the Eguchi-Kawai model where the final theory lives on the plaquette. Moreover we know, the underlying phase transition, responsible for the chiral transition just studied, is an artifact of the Wilson action used for the plaquette variable. The phase transition can be eliminated if one uses the heat kernel action [12]. This limits the applicability of the results obtained here. However, some qualitative features of a chiral transition can be motivated using this example.

On a crude level, the results obtained here seem similar to the results of [7]. First, the transition studied here is driven by the gauge dynamics. The fermion determinant does not play any role; remember that in the large $N$ limit the fermion determinant dropped out of the saddle point equation eq.(13). In other words, the chiral transition occurs even in the quenched theory. Next, the density of eigenvalues of the Dirac operator develops a gap for small eigenvalues above the transition (see figure 1). One might wonder if these are generic features of chiral transitions involving gauge fields. In QCD, it is well known that the number of flavors plays an important role in determining the order of the transition. For example, it is believed that for one flavor there is no chiral transition [1], due to the presence of the anomaly. This suggests that there will not be any transition in the quenched case, since the absence of the fermion determinant is likely to enhance the small eigenvalues of the Dirac operator and enhance the effects of chiral symmetry breaking. If this is true, we can expect the chiral transition to be driven mainly by the fermion determinant. It is through the fermion determinant that the number of flavors enters the problem. Thus the present model and the random matrix models do not reflect these expected features of the QCD chiral phase transition.
Another interesting observation is the role of the staggered fermion phases defined in eq.(4). Notice that in the weak coupling phase the gap in $\rho(\lambda)$ for small $\lambda$ is related to the gap in the eigenvalue density $\rho_{U_p}(\alpha)$ at $\alpha = \pi$. This can be traced back to the zero eigenvalues of $D^i$, defined in eq.(10), which exist when $\alpha_i = \pi$ as discussed in section III. If we had chosen all the signs in eq.(4) to be positive, the zero eigenvalues of $D^i$ would be related to $\alpha_i = 0$. Since this would not effect the saddle point equation, the density $\rho_{U_p}(\alpha)$ would not change. However, since the eigenvalues of $D$ would be related to $\alpha$ differently, we would have $\rho(\lambda = 0) \neq 0$ for any $G$. The chiral transition would disappear. A similar effect can be seen by changing the sign of the gauge action. Of course we have used the natural sign of the Wilson gauge action that is necessary for the continuum limit to emerge from a lattice action. Thus the staggered fermion phases and the natural sign of the Wilson action in eq.(1) have conspired together to produce the chiral transition.

Finally, it is interesting to know if one could extend such a calculation to the more interesting large $N$ phase transitions of Douglas and Kazakov [13], which unlike the transition studied here is a continuum transition. Since the topology of space-time plays an interesting role in this continuum transition, including fermionic variables becomes non-trivial.

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FIG. 1. This plot of the eigenvalue density of $D$ defined in eq.(17) for various values of the coupling $G$. At finite $N$, $D$ has $4N$ eigenvalues. In the large $N$ limit these can be naturally be described by the density given above which is normalized to 4. $G = 2$ is the critical coupling.