ON WEIGHTED CONDITIONS FOR THE ABSOLUTE CONVERGENCE OF FOURIER INTEGRALS

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Abstract. In this paper we obtain new sufficient conditions for representation of a function as an absolutely convergent Fourier integral. Unlike those known earlier, these conditions are given in terms of belonging to weighted spaces. Adding weights allows one to extend the range of application of such results to Fourier multipliers with unbounded derivatives.

1. Introduction

If, for \( n = 1, 2, \ldots \),

\[ f(x) = \int_{\mathbb{R}^n} g(t)e^{i(x,t)} dt, \quad g \in L_1(\mathbb{R}^n), \tag{1.1} \]

where \((x,t) = x_1 t_1 + \ldots + x_n t_n\), we say that \( f \) belongs to Wiener’s algebra \( W_0(\mathbb{R}^n) \), written \( f \in W_0(\mathbb{R}^n) \), with \( \|f\|_{W_0} = \|g\|_{L_1(\mathbb{R}^n)} \). Wiener’s algebra is an important class of functions and its in-depth study is motivated both by many points of interest in the topic itself and by its relations to other areas of analysis, such as Fourier multipliers or comparison of differential operators. The history, motivations and various conditions of belonging to Wiener’s algebra are overviewed in detail in a recent survey paper [13].

Of course, [13] summarized the long term studies of many mathematicians and gave a comprehensive picture of the subject. However, these studies are continuing, see, e.g., [9], [10], [11], [12]–[15], [30]. In these works the undertaken efforts have mainly been aimed at obtaining conditions of mixed type, in the sense that conditions are posed simultaneously on the function and its derivatives.

Naturally, certain such conditions were known earlier, see, e.g., [3], [16]. In the latter, the (multidimensional) Riesz fractional differentiation is defined by

\[ (-\Delta)^{\frac{\alpha}{2}} f = \mathcal{F}^{-1}|x|^\alpha \mathcal{F} f, \quad \alpha > 0, \]

where \( \mathcal{F} \) means the Fourier operator, while \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \) denotes the Laplace operator, and the result reads as follows (see [16]).

**Theorem A.** Let \( f \in L_2(\mathbb{R}^n) \), and \( (-\Delta)^{\frac{\alpha}{2}} f \in L_2(\mathbb{R}^n) \), \( \alpha > \frac{n}{2} \), then \( \mathcal{F} f \in L_1(\mathbb{R}^n) \).

To give further convenient formulations in the multivariate case, we need additional notations. Let \( \eta \) be an \( n \)-dimensional vector with the entries either 0 or 1 only. Here and in what follows, \( D^\eta f \) for \( \eta = 0 = (0,0,\ldots,0) \) or \( \eta = 1 = (1,1,\ldots,1) \) mean the
function itself and the mixed derivative in each variable, respectively, where

\[ D^n f(x) = \left( \prod_{j : \eta_j = 1} \frac{\partial}{\partial x_j} \right) f(x). \]

One of the multidimensional results we are going to generalize reads as follows (see [21]).

**Theorem B.** Let \( f \in L_1(\mathbb{R}^n) \). If all the mixed derivatives (in the distributional sense) \( D^n f(x) \in L_p(\mathbb{R}^n), \eta \neq 0 \), where \( 1 < p \leq 2 \), then \( f \in W_0(\mathbb{R}^n) \).

However, the main motivation to our work was given by the following recent result [11]: if \( f \in L^p(\mathbb{R}), 1 \leq p < \infty \), and \( f' \in L^q(\mathbb{R}), 1 < q < \infty \), for \( p \) and \( q \) such that

\[ \frac{1}{p} + \frac{1}{q} = 1, \]

then \( f \in W_0(\mathbb{R}); \) and if \( \frac{1}{p} + \frac{1}{q} < 1 \), then there is a function \( f \) such that \( f \in L^p(\mathbb{R}), f' \in L^q(\mathbb{R}) \) and \( f \not\in W_0(\mathbb{R}) \). Certainly, this result essentially generalizes many previous results, e.g., the one in [3], but we highlight it not only as a "landmark" but also since we shall pay much attention to the case where \( \frac{1}{p} + \frac{1}{q} = 1 \) and discuss why it does not ensure the belonging to \( W_0 \), except a unique special case \( p = q = 2 \) (see the proof of Proposition [6.3]). Despite the attraction of (1.2), one could see already in [11] the incompleteness of this condition. Indeed, it assumes not only the function to be "good" but the derivative to be "good" as well. However, related results on Fourier multipliers show that it is by no means necessary. The model case is delivered by the well-known multiplier (see [8], [26 Ch.4, 7.4], [5])

\[ m(x) = m_{\alpha, \beta}(x) = \rho(x) \frac{e^{i|x|^\alpha}}{|x|^\beta}, \]

where \( \rho \) is a \( C^\infty \) function on \( \mathbb{R}^n \), vanishing for \( |x| \leq 1 \) and equal to 1 if \( |x| \geq 2 \), with \( \alpha, \beta > 0 \). Recall that the Fourier multipliers are defined as follows. Let \( m : \mathbb{R}^n \to \mathbb{C} \) be an almost everywhere bounded measurable function (\( m \in L_\infty(\mathbb{R}^n) \)). Define on \( L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \) a linear operator \( \Lambda \) by means of the following identity for the Fourier transforms of functions \( f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \):

\[ \mathcal{F}(\Lambda f)(x) = m(x) \mathcal{F}f(x). \]

If a constant \( D > 0 \) exists such that for each \( f \in L_2(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \) there holds

\[ \| \Lambda f \|_{L_p(\mathbb{R}^n)} \leq D \| f \|_{L_p(\mathbb{R}^n)}, \]

then \( \Lambda \) is called a Fourier multiplier taking \( L_p(\mathbb{R}^n) \) into \( L_p(\mathbb{R}^n) \). This is written as \( m \in M_p \) and \( \| m \|_{M_p} = \| \Lambda \|_{L_p \to L_p} \). It is known (see [19]) that for \( n \geq 1 \) and \( \alpha \neq 1 \):

- if \( \frac{\beta}{\alpha} > \frac{n}{2} \), then \( m_{\alpha, \beta} \in M_1 \) (or \( m_{\alpha, \beta} \in M_\infty \)),
- if \( \frac{\beta}{\alpha} \leq \frac{n}{2} \), then \( m_{\alpha, \beta} \not\in M_1 \) (or \( m_{\alpha, \beta} \not\in M_\infty \)).

The instance \( \alpha \neq 1 \) is also considered in [19]; more details for the case \( \frac{\beta}{\alpha} = \frac{n}{2} \) can be found in [5], [18], and [20].
To prove certain cases in these and related estimates, the following recent refinement of (1.2) (see [9]) is more convenient.

**Theorem C.** Let \( 0 < q \leq \infty, 1 < r < \infty, s > \frac{n}{r}, \) and \( f \in C_0(\mathbb{R}^n) \). Suppose that either \( q = r = 2 \) or

\[
(1.4) \quad \left(1 - \frac{n}{2s}\right) \frac{1}{q} + \left(\frac{n}{2s}\right) \frac{1}{r} > \frac{1}{2}.
\]

If, in addition, \( f \in L_q(\mathbb{R}^n) \) and \((-\Delta)^{\frac{s}{2}} f \in L_r(\mathbb{R}^n)\), then \( f \in W_0(\mathbb{R}^n) \).

This theorem is sharp in the following sense: Let \( 1 < q, r < \infty, s \in \mathbb{N}, \) and \( s > \frac{n}{r} \). If we have

\[
(1.5) \quad \left(1 - \frac{n}{2s}\right) \frac{1}{q} + \left(\frac{n}{2s}\right) \frac{1}{r} < \frac{1}{2}
\]

instead of (1.4), then there is a function \( f \in C_0(\mathbb{R}^n) \) such that \( f \in L_q(\mathbb{R}^n) \) and \((-\Delta)^{\frac{s}{2}} f \in L_r(\mathbb{R}^n)\), but \( f \notin W_0(\mathbb{R}^n) \).

It is worth mentioning that we study the case of equality in (1.5) as well (see Proposition 6.3 below).

Theorem C, with \( s \in \mathbb{N} \) and \( \beta > ([n/2] + 1)\alpha - 1 \), allows one to prove the well-known fact about \( m_{\alpha, \beta} \) (cf. above): if \( \beta > \frac{n\alpha}{2} \), then \( m_{\alpha, \beta} \in W_0(\mathbb{R}^n) \). Note that the restriction \( \beta > ([n/2] + 1)\alpha - 1 \) appears since the derivatives of \( m_{\alpha, \beta} \) may be "bad", that is, unbounded.

A natural way out of such a situation is to advert to weighted spaces, in which the weight \( w \) is chosen so that the growth of the derivatives is dampened by means of the decay of the weight function. Our goal is to obtain multidimensional analogues of (1.2), or more precisely of Theorem C in the case of weighted spaces. New one-dimensional results follow as immediate consequences of our general one in the case \( n = 1 \).

To give an idea of our main results, we formulate here a representative one, a particular case of Corollary 3.4. In what follows

\[
(1.6) \quad w_\varepsilon(x) = \left(1 + |x|^2\right)^{\frac{\varepsilon}{2}}, \quad \varepsilon \in \mathbb{R}.
\]

**Proposition 1.1.** Let \( 0 < q < 2 < r < \infty, s > 0, \varepsilon > 0, \) and \( f \in C_0(\mathbb{R}^n) \). If (1.4) holds,

\[
f w_{(1 - \eta q)} \in L_q(\mathbb{R}^n), \quad \text{and} \quad (-\Delta)^{\frac{s}{2}} \left(f w_{\frac{x}{\eta r}}\right) \in L_r(\mathbb{R}^n), \quad \gamma = \frac{2 - q}{r - q},
\]

then \( f \in W_0(\mathbb{R}^n) \).

Further, one of the prominent and classical results on the interrelation between the function and its derivatives is the Gagliardo-Nirenberg inequality, which is of the form (in fact, of one of the various possible forms, see [7], [6], or [29])

\[
(1.7) \quad \|(-\Delta)^{\frac{s}{2}} f\|_{L_p} \leq C\|f\|_{L_q}^{1 - \delta}\|(-\Delta)^{\frac{s}{2}} f\|_{L_r}^{\delta},
\]

where \( 1 < p, q, r < \infty, \sigma, s \in \mathbb{R}, 0 \leq \delta \leq 1, \) and

\[
\frac{n}{p} - \tau = (1 - \delta)\frac{n}{q} + \delta \left(\frac{n}{r} - s\right), \quad \tau \leq \delta s.
\]

It turned out that for our aims it was beneficial to obtain weighted generalizations of (1.7), with natural and immediate applications to the absolute convergence of Fourier
integrals. For example, one such generalization, with weighted function \( w_\varepsilon(x) \), \( \varepsilon > 0 \), is as follows:

\[
\|(-\Delta)^{\frac{s}{2}}f\|_{L_p} \leq C \left\| \left| \frac{1}{w_\varepsilon} \right|_{L_q}^{1-\delta} \left\| (-\Delta)^{\frac{s}{2}} (f \frac{1}{w_\varepsilon}) \right\|_{L_r}^\delta, \]
\]

where \( \delta \) is as above, but \( 1 < q < p < r < \infty \), \( \tau, s \in \mathbb{R} \), \( 0 < \delta < 1 \), \( s - \frac{n}{r} \neq -\frac{n}{q} \), \( \varepsilon > 0 \), \( \gamma = \frac{p-q}{r-q} \), and \( \tau < \delta s \). However, since much is contained in [17], we omit this part in the present work.

As for the prospects, it is worth mentioning that possible directions of our interest are similar results in terms of the function and its derivatives belonging to other spaces, more balance between the radial and non-radial cases, and different methods for establishing the sharpness of the obtained results.

The outline of the paper is as follows. Since there are quite many different notations and technical tools, we feel it necessary and convenient to present them in the next section. In Section 3 we give our main results. Then Section 4 is devoted to auxiliary results. In Section 5 we prove our main result Theorem 3.2. Further, in Section 6 we discuss various corollaries and establish the sharpness of the obtained results, more precisely, of the most convenient of them Corollary 3.3.

2. Basic notations and definitions

Let \( \mathbb{R}^n \) be the \( n \)-dimensional Euclidean space with elements \( x = (x_1, \ldots, x_n) \), \( \xi = (\xi_1, \ldots, \xi_n) \) endowed with the scalar product \( (x, \xi) = x_1\xi_1 + \cdots + x_n\xi_n \) and norm \( |x| = (x, x)^{\frac{1}{2}} \). As usual, the space \( L_p(\mathbb{R}^n, w) \) consists of measurable functions \( f(x) \), \( x \in \mathbb{R}^n \), for which

\[
\|f\|_{L_p(w)} = \|f\|_{L_p(\mathbb{R}^n, w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \infty, \quad 0 < p < \infty,
\]

and

\[
\|f\|_{L_\infty(w)} = \|f\|_{L_\infty(\mathbb{R}^n, w)} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)| w(x) < \infty.
\]

If \( w_0(x) \equiv 1 \), we shall write \( \|f\|_p = \|f\|_{L_p(w_0)} \) and \( L_p = L_p(w_0) \). By \( C(\mathbb{R}^n) \), we denote the space of all bounded uniformly continuous functions on \( \mathbb{R}^n \). We will also deal with the following class of functions

\[
C_0(\mathbb{R}^n) = \left\{ f : f \in C(\mathbb{R}^n), \lim_{|x| \to \infty} f(x) = 0 \right\}.
\]

We will use standard notations for the space of tempered distributions \( \mathcal{S}'(\mathbb{R}^n) \) and for the corresponding space of test functions \( \mathcal{S}(\mathbb{R}^n) \). For \( f \in L_1(\mathbb{R}^n) \), we denote its Fourier transform in a standard manner

\[
\hat{f}(\xi) = \mathcal{F} f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-i\langle \xi, x \rangle} \, dx,
\]

and set also \( \mathcal{F}^{-1} f(\xi) = \mathcal{F} f(-\xi) \). In the sequel, we shall understand the operators \( \mathcal{F} \) and \( \mathcal{F}^{-1} \) in the distributional sense.

In this paper, the main theorems are formulated in terms of functions from the weighted Besov spaces \( B^s_{p,q}(\mathbb{R}^n, w) \). To define these spaces, let us consider a function
\( \varphi \in \mathcal{S}(\mathbb{R}^n) \) such that \( \text{supp } \varphi \subset \{ \xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2 \} \), \( \varphi(\xi) > 0 \) for \( 1/2 < |\xi| < 2 \) and
\[
\sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) = 1 \quad \text{if} \quad \xi \neq 0.
\]
We also introduce the functions \( \varphi_k \) and \( \psi \) by means of the relations
\[
\mathcal{F} \varphi_k(\xi) = \varphi(2^{-k}\xi) \quad \text{and} \quad \mathcal{F} \psi(\xi) = 1 - \sum_{k=1}^{\infty} \varphi(2^{-k}\xi).
\]

We begin with usual Besov spaces. Let \( s \in \mathbb{R}, 0 < p, q \leq \infty \). We will say that \( f \in \mathcal{S}'(\mathbb{R}^n) \) belongs to the (non-homogeneous) weighted Besov space \( B^s_{p,q}(\mathbb{R}^n, w) = B^s_{p,q}(w) \), if
\[
\|f\|_{B^s_{p,q}(w)} = \|\psi \ast f\|_{L^p(w)} + \left( \sum_{k=1}^{\infty} 2^{skq} \|\varphi_k \ast f\|_{L^p(w)}^q \right)^{\frac{1}{q}} < \infty,
\]
with standard modification for \( q = \infty \). If \( w_0(x) \equiv 1 \), then \( B^s_{p,q}(\mathbb{R}^n, w_0) = B^s_{p,q} \) are non-weighted Besov spaces.

Now, in order to define homogeneous Besov spaces, recall that
\[
\mathcal{H}(\mathbb{R}^n) = \{ \varphi \in \mathcal{S}(\mathbb{R}^n) : (D^\nu \varphi)(0) = 0 \quad \text{for all} \quad \nu \in \mathbb{N}^n \cup \{0\} \},
\]
where \( \mathcal{H}(\mathbb{R}^n) \) is the space of all continuous functionals on \( \mathcal{S}(\mathbb{R}^n) \). We will say that \( f \in \mathcal{H}(\mathbb{R}^n) \) belongs to the homogeneous Besov space \( B^s_{p,q}(w) \) if
\[
\|f\|_{B^s_{p,q}(w)} = \left( \sum_{k=-\infty}^{\infty} 2^{skq} \|\varphi_k \ast f\|_{L^p(w)}^q \right)^{\frac{1}{q}} < \infty,
\]
with standard modification for \( q = \infty \).

We shall also deal with weighted spaces of Bessel potentials \( H^s_p(\mathbb{R}^n, w) = H^s_p(w) \). Let \( 1 < p < \infty \) and \( s \in \mathbb{R} \). We will say that \( f \in \mathcal{H}(\mathbb{R}^n) \) belongs to the space \( H^s_p(w) \) if
\[
\|f\|_{H^s_p(w)} = \|(I - \Delta)^{\frac{s}{2}} f\|_{L^p(w)} < \infty.
\]

Now a discussion on weighted functions is in order. We shall deal with the so-called admissible weights.

**Definition 2.1.** We will say that a measurable function \( w : \mathbb{R}^n \mapsto \mathbb{R}_+ \) is an **admissible weight**, written \( w \in \mathcal{W}^n \), if
1) \( w \in C^\infty(\mathbb{R}^n) \);
2) For each multi-index \( \gamma \), there is a positive constant \( c_\gamma \) such that
\[
\left| \frac{\partial^{\gamma_1 + \ldots + \gamma_n}}{\partial x_1^{\gamma_1} \ldots \partial x_n^{\gamma_n}} w(x) \right| \leq c_\gamma w(x) \quad \text{for each} \quad x \in \mathbb{R}^n;
\]
3) There are constants \( c > 0 \) and \( \alpha \geq 0 \) such that
\[
0 < w(x) \leq cw(y)(1 + |x - y|^2)^{\frac{\alpha}{2}} \quad \text{for any} \quad x, y \in \mathbb{R}^n.
\]
Note that if \( w \in \mathcal{W}^n \) and \( w' \in \mathcal{W}^n \), then \( w^{-1} \in \mathcal{W}^n \) and \( ww' \in \mathcal{W}^n \). The functions
\[
w(x) = (1 + |x|^2)^{\frac{\alpha}{2}}, \quad v(x) = (1 + \log(1 + |x|^2))^\alpha, \quad \alpha \in \mathbb{R},
\]
are typical examples of admissible weights.
In the sequel, we shall denote by $C$ (or $C$ with indicated parameters) absolute positive constants (or constants depending only on the indicated parameters, respectively), while by $A$ certain specific finite constants. We also set $\frac{\infty}{\infty} = 1$ and $\frac{0}{0} = 0$.

3. Statement of the main results

The first result that we present cannot be called our main one only because it does not lead explicitly to the belonging of the considered function to Wiener’s algebra. It is an important tool for us rather than one of the intended goals. However, these Gagliardo-Nirenberg type inequalities for homogeneous Besov spaces are of similar nature and are of interest in their own right. Also, our main result that follows immediately will be more transparent.

Let in what follows

$$\gamma(p, q, r) = \begin{cases} \frac{p - q}{r - q}, & q < p; \\ \frac{q - p}{q - r}, & r < p. \end{cases}$$

Theorem 3.1. Let either $0 < q < p < r \leq \infty$ or $0 < r < p < q \leq \infty$, and let $\tau, \sigma, s \in \mathbb{R}$, $\sigma < s$, $0 < \eta < \infty$. Suppose that $u, v$ are measurable functions on $\mathbb{R}^n$ such that

(3.1) \[ 1 \leq u(x)v(x) \text{ for all } x \in \mathbb{R}^n, \]

and also

(3.2) \[ \sigma - \tau < n \left( \frac{1}{q} - \frac{1}{p} \right) \text{ and } 1 \leq u(x), \text{ if } 0 < q < p \]

and

(3.3) \[ n \left( \frac{1}{r} - \frac{1}{p} \right) < s - \tau \text{ and } 1 \leq v(x), \text{ if } 0 < r < p. \]

If

(3.4) \[ \frac{1 - \theta}{q} + \frac{\theta}{r} > \frac{1}{p}, \quad \theta = \frac{\tau - \sigma}{s - \sigma}, \]

and

$$f \in \dot{B}^\gamma_{q, \infty} (u^{1/(1-\gamma)}) \cap \dot{B}^s_{r, \infty} (v^{1/\gamma}), \quad \gamma = \gamma(p, q, r),$$

then $f \in \dot{B}^\tau_{p, \eta}$. If, in addition, $s - \frac{n}{r} \neq \sigma - \frac{n}{q}$, then

$$\|f\|_{\dot{B}^\tau_{p, \eta}} \leq C \|f\|_{\dot{B}^\gamma_{q, \infty} (u^{1/(1-\gamma)})}^{1-\delta} \|f\|_{\dot{B}^s_{r, \infty} (v^{1/\gamma})}^\delta,$$

where $\delta = \frac{\tau - \sigma + n(\frac{1}{q} - \frac{1}{p})}{s - \sigma + n(\frac{1}{q} - \frac{1}{r})}$ and $C$ is a constant independent of $f$.

The next theorem is our main result.
Theorem 3.2. Let either $0 < q < 2 < r \leq \infty$ or $0 < r < 2 < q \leq \infty$, $\sigma < s$, and let $f \in C_0(\mathbb{R}^n)$. Suppose that $u, v$ are measurable functions on $\mathbb{R}^n$ such that (3.1) holds, while (3.2), (3.3), and (3.4) are valid for $\tau = \frac{n}{2}$ and $p = 2$.

If

$$f \in \dot{B}^s_{q,\infty}(u^{1/(1-\gamma)}) \cap \dot{B}^s_{r,\infty}(v^{1/\gamma}), \quad \gamma = \gamma(2, q, r),$$

then $f \in W_0(\mathbb{R}^n)$. If, in addition, $s - \frac{n}{r} \neq -\frac{n}{q}$, then

$$\|f\|_{W_0} \leq C \|f\|^{1-\delta}_{\dot{B}^s_{q,\infty}(u^{1/(1-\gamma)})} \|f\|^\delta_{\dot{B}^s_{r,\infty}(v^{1/\gamma})},$$

where $\delta = \frac{n - \sigma}{s - \sigma + n(\frac{1}{q} - \frac{1}{r})}$ and $C$ is a constant independent of $f$.

As for the consequences, let us start with those for the weights from $W^n$. The following statement is less general but presents more effective sufficient conditions. By this we pass from homogeneous spaces, we have worked so far, to non-homogeneous Sobolev type spaces. This allows one to apply the well known embedding theorems.

Corollary 3.3. Let either $0 < q < 2 < r < \infty$ or $1 < r < 2 < q \leq \infty$, $s > 0$, and let $f \in C_0(\mathbb{R}^n)$. Suppose that $u, v \in W^n$ are such that (3.1) holds, while (3.2) and (3.3) are valid for $\tau = \frac{n}{2}$, $\sigma = 0$, and $p = 2$. If

$$(3.5) \quad \left(1 - \frac{n}{2s}\right) \frac{1}{q} + \left(\frac{n}{2s}\right) \frac{1}{r} > \frac{1}{2}$$

and

$$f \in L_{q}(u^{1/(1-\gamma)}) \cap H^s_{r}(v^{1/\gamma}), \quad \gamma = \gamma(2, q, r),$$

then $f \in W_0(\mathbb{R}^n)$. If, in addition, $s - \frac{n}{r} \neq -\frac{n}{q}$, then

$$\|f\|_{W_0} \leq C \|f\|^{1-\delta}_{L_{q}(u^{1/(1-\gamma)})} \|f\|^\delta_{H^s_{r}(v^{1/\gamma})},$$

where $\delta = \frac{n}{s + n(\frac{1}{q} - \frac{1}{r})}$ and $C$ is a constant independent of $f$.

Now, we return, in a sense, to homogeneity of the spaces. The following corollary is a direct (weighted) extension of (1.2).

Corollary 3.4. Let $0 < q < 2 < r < \infty$, $s > 0$, and let $f \in C_0(\mathbb{R}^n)$. Suppose that $u, v \in W^n$ are such that $1 \leq u(x)v(x)$ and $1 \leq u(x)$. If (3.5) holds, and

$$f u^{1/(1-\gamma)} \in L_q(\mathbb{R}^n) \text{ and } (-\Delta)^{\frac{s}{2}}(f v^{1/\gamma}) \in L_r(\mathbb{R}^n), \quad \gamma = \frac{2 - q}{r - q},$$

then $f \in W_0(\mathbb{R}^n)$.

Recall that if $f(x) = f_0(\rho)$, $\rho = |x|$, then using the well-known equality

$$\Delta f(x) = \frac{\partial^2 f_0(\rho)}{\partial \rho^2} + \frac{n - 1}{\rho} \frac{\partial f_0(\rho)}{\partial \rho},$$

we get the following version of Corollary 3.4 for radial functions.
Corollary 3.5. Let $0 < q < 2 < r < \infty$ and let $f \in C_0(\mathbb{R}^n)$. Suppose that $u, v \in W^n$ are such that $1 \leq u(x)v(x)$ and $1 \leq u(x)$, and, in addition, let $s$ be even and $f(x) = f_0(\rho)$, $u(x) = u_0(\rho)$, and $v(x) = v_0(\rho)$. If \([3.5]\) holds, and

$$
\|u_0\|_{\infty} \|v_0\|_{\infty} < w(x),
$$

then $f \in W_0(\mathbb{R}^n)$.

Remark 3.6. It is worth noting that Corollaries 3.3–3.5 hold true for a wider class of weights. In particular, for the class $W^s_0$ (see [25]), which differs from $W^n$ in the way that it suffices to claim in its definition the weaker condition

$$
0 < w(x) \leq CW(x-y)e^{d|x|}, \quad x,y \in \mathbb{R}^n,
$$

where $C > 0$ and $d > 0$ are some fixed constants, in place of 3) in Definition 2.1. For example, $W^s_0$ contains functions like

$$
w(x) = e^{\pm |x|^\beta}, \quad 0 < \beta \leq 1.
$$

4. Auxiliary results

If the weight is admissible, the following lemma, which relates usual and weighted Besov spaces, is of importance (see [4, p.156]).

Lemma 4.1. Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, and $w \in W^n$. Then the operator $f \mapsto w^{1/p^*}f$ ($p^* = p$, if $p < \infty$, and $p^* = 1$ for $p = \infty$) is an isomorphism $B^s_{p,q}(w^{1/p})$ on $B^s_{p,q}$. In particular, \(\|fu^{1/p^*}\|_{B^s_{p,q}}\) is an equivalent quasinorm on $B^s_{p,q}(w)$.

A similar lemma is valid for the spaces $H^s_p(\mathbb{R}^n, w)$, $w \in W^n$.

We shall make use of the embeddings for the Besov spaces, by presenting the next result which states that the spaces $B^{k}_{p,q}$, $W^{k}$, and $C^{k}$ are related in the following well-known way.

Lemma 4.2. Let $s > 0$, $k \in \mathbb{N} \cup \{0\}$, and $1 < p < \infty$. Then

$$
B^s_{p,1} \subset H^s_p \subset B^s_{p,\infty}
$$

and if $1 \leq p < \infty$, then

$$
B^k_{p,1} \subset W^k_p \subset B^k_{p,\infty}.
$$

Here and in what follows, $\subset$ means a continuous embedding. The proof of Lemma 4.2 can be found, e.g., in [28, Ch.2 and Ch.5] and [23, Ch.3 and Ch.11]. It readily follows from Lemma 4.1 that the above-mentioned embeddings are also valid for the weighted Besov spaces $B^s_{p,q}(w)$ provided, of course, that $w \in W^n$.

The following lemma is a simple corollary of Hölder’s inequality.

Lemma 4.3. Let $0 < q < p < r \leq \infty$, $\gamma = \frac{p}{q} - \frac{p}{r}$, and $u, v$ be positive measurable functions on $\mathbb{R}^n$ such that $u(x)v(x) \geq 1$, $x \in \mathbb{R}^n$. Then

$$
\|f\|_p \leq \|f\|_{L_q(u^{1/(1-\gamma)})}\|f\|_{L_r(v^{1/\gamma})}.
$$
5. Proof of the Main Results

Proof of Theorem 3.1. In order to prove that \( f \in \dot{B}^p_{r,q} \), it suffices to check that

\[
\sum_{k=-\infty}^{\infty} (2^{\eta k} \| \varphi_k \ast f \|_{L^p_x})^\eta = \sum_{k=0}^{\infty} + \sum_{k=-\infty}^{-1} = S_1 + S_2 < \infty.
\]

Let first \( 0 < q < p < r \leq \infty \).

Using the embedding \( \dot{B}^p_{r,q} \subset \dot{B}^p_{s,\infty} \), where \( s = \sigma - n(\frac{1}{q} - \frac{1}{p}) \) (see, for example, [1, 6.5.1]), we obtain

\[
S_2 = \sum_{k=1}^{\infty} (2^{-\eta k} \| \varphi_{-k} \ast f \|_{L^p_x})^\eta 
\]

\[
\leq \sup_{j \in \mathbb{Z}_+} (2^{-\sigma_1 j} \| \varphi_{-j} \ast f \|_{L^p_x})^\eta \sum_{k=1}^{\infty} 2^{-k(\sigma - \sigma_1)\eta} 
\]

\[
\leq C \| f \|^\eta_{\dot{B}^p_{r,q}} \leq C \| f \|^\eta_{\dot{B}^s_{r,\infty}} \leq C \| f \|^\eta_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})}.
\]

Let us proceed to the sum \( S_1 \). Denoting \( \lambda = \frac{r(p-q)}{p(r-q)} \) and applying Lemma 4.3, we get

\[
S_1 = \sum_{k=0}^{\infty} \left( 2^{\sigma j} \| \varphi_{-j} \ast f \|_{L^q(u^{1/(1-\gamma)})} \| \varphi_{-j} \ast f \|_{L^{2}(u^{1/(1-\gamma)})} \right)^{\eta(1-\lambda)}
\]

\[
\leq \left( \sup_{j \in \mathbb{Z}_+} 2^{\sigma j} \| \varphi_{-j} \ast f \|_{L^q(u^{1/(1-\gamma)})} \right)^{\eta(1-\lambda)} \sum_{k=0}^{\infty} 2^{\sigma(1-\lambda \sigma - \lambda \eta)k}
\]

\[
= \| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})} \| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})} \sum_{k=0}^{\infty} 2^{\sigma(1-\lambda \sigma - \lambda \eta)k}
\]

\[
\leq C \| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})} \| f \|^\eta_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})}.
\]

Hence, combining (5.2) and (5.3), we derive from (5.1) that

\[
\| f \|_{\dot{B}^p_{r,q}} \leq C \left\{ \| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})} + \| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})} \| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})} \right\}.
\]

It remains to pass to the product inequality, that is, to substitute \( f \rightarrow f(\varepsilon \cdot) \) and \( u \rightarrow u(\varepsilon \cdot) \), \( v \rightarrow v(\varepsilon \cdot) \), where

\[
\varepsilon = \left( \frac{\| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})}}{\| f \|_{\dot{B}^s_{r,\infty}(u^{1/(1-\gamma)})}} \right)^{\frac{1}{1-\sigma n(\frac{1}{q} - \frac{1}{p})}}.
\]

and to use the property of homogeneity for Fourier transform (see, for example, [28 3.4.1] and [21 7.2]).

The case \( 0 < r < p < q \leq \infty \) is considered in a similar manner.

Theorem 3.1 is proved. \( \square \)
Proof of Theorem 3.2. It is well known (see, e.g., [2], [22] or [23, p.119]) that if $f \in C_0 \cap \dot{B}^2_{2,1},$ then $f \in W_0(\mathbb{R}^n)$ and $\|f\|_{w_0} \leq C\|f\|_{\dot{B}^2_{2,1}},$ where $C$ is a constant independent of $f.$ Hence, the theorem obviously follows from Theorem 3.1. □

Proof of Corollary 3.3. This corollary can be easily obtained by using Lemma 4.1, Lemma 4.2, and Theorem 3.2 with $\sigma = 0.$ □

6. Certain consequences and sharpness

We will show both applicability and sharpness of the obtained above results by making use of the familiar power weight function $w_\varepsilon(x) = (1 + |x|^2)^{\varepsilon/2}.$

We first show that Corollary 3.3 can be applied to the study of the function $m_{\alpha,\beta}$ in the case where its derivatives are unbounded.

**Corollary 6.1.** Let $\alpha > 0, \beta > 0,$ and $\alpha \neq 1.$ If $\beta > \frac{n\alpha}{2},$ then $m_{\alpha,\beta} \in W_0(\mathbb{R}^n).$

**Proof.** It suffices to consider only the case $\beta > n.$ Indeed, if $\beta \leq n,$ then Corollary 6.1 follows from the non-weighted sufficient condition (see Corollary 3.3).

Observe first that condition $2\beta > n\alpha$ allows one to choose $\varepsilon > 0$ so that

$$\beta > n + \varepsilon \quad \text{and} \quad \beta - n(\alpha - 1) > -\varepsilon.$$ 

We then are able to choose $q,$ close enough to 1, and large enough $r$ such that

$$\beta - \frac{\varepsilon(r - q)}{q(r - 2)} > \frac{n}{q} \quad \text{and} \quad \beta - n(\alpha - 1) + \frac{\varepsilon(r - q)}{r(2 - q)} > \frac{n}{r},$$ 

which, in turn, means that

$$m_{\alpha,\beta} \in L_q \left( w_{-\varepsilon/2} \right) \cap H^n_{r} \left( w_{-\varepsilon/2} \right), \quad \gamma = \frac{2 - q}{r - q}.$$ 

It follows from inequalities (6.1) that

$$\left( \frac{1}{q} + \frac{1}{r} - 1 \right) \left( n + \frac{2\varepsilon(r - q)}{(2 - q)(r - 2)} \right) < 2\beta - n\alpha.$$ 

This allows us to choose, in addition, the parameters $q$ and $r$ so that

$$\frac{1}{q} + \frac{1}{r} > 1.$$ 

Therefore, by Corollary 3.3, we obtain $m_{\alpha,\beta} \in W_0(\mathbb{R}^n).$ □

The following notions and auxiliary statements are needed for proving the sharpness of the obtained results.

Further, let us denote

$$\mu_{\alpha,\beta}(x) = \frac{m_{\alpha,\beta}(x)}{\log |x|}$$

and

$$\nu_{\alpha}(x) = \mu_{\alpha, \frac{n\alpha}{2}}(x).$$
Lemma 6.2. Let $\beta > 0$.

(i) If $0 \leq \alpha < 1$, then
$$|\mathcal{F} \mu_{\alpha, \beta}(\xi)| \sim |\xi|^{-\frac{n-\beta}{1-\alpha}} (\log \frac{1}{|\xi|})^{-1}, \quad |\xi| \to 0.$$ 

(ii) If $\alpha > 1$, then
$$|\mathcal{F} \mu_{\alpha, \beta}(\xi)| \sim |\xi|^{-\frac{n-\beta+\alpha/2}{1-\alpha}} (\log |\xi|)^{-1}, \quad |\xi| \to \infty.$$ 

In the case $0 < \beta + \frac{n}{2} < \frac{1}{2}$, see the proof of this lemma in [18, Lemma 4]. The general case can be proved by using (modifying) the proofs of Lemmas 2.6 and 2.7 in [20].

We now show that Corollary 3.3 is sharp. First, the next proposition holds true.

Proposition 6.3. Let either $1 \leq q < 2 < r < \infty$ or $1 \leq r < 2 < q < \infty$, $s > \frac{n}{2}$, $s \in \mathbb{N}$. If

$$(6.2) \quad \left(1 - \frac{n}{2s}\right) \frac{1}{q} + \left(\frac{n}{2s}\right) \frac{1}{r} \leq \frac{1}{2},$$

then for each $\varepsilon > 0$ there is a function $f \in C_0(\mathbb{R}^n)$ such that either

$$f \in L_q \left(\frac{w_{\varepsilon}}{1-\gamma}\right) \cap H^s_r \left(\frac{w_{-\varepsilon}}{\gamma}\right), \quad \gamma = \frac{2 - q}{r - q}$$

or

$$f \in L_q \left(\frac{w_{-\varepsilon}}{1-\gamma}\right) \cap H^s_r \left(\frac{w_{\varepsilon}}{\gamma}\right), \quad \gamma = \frac{q - 2}{q - r},$$

but $f \notin W_0(\mathbb{R}^n)$.

Proof. Consider first the instance $1 \leq q < 2 < r < \infty$. Let $\alpha, \beta > 0$, $\alpha \neq 1$ be such that

$$(6.3) \quad \beta - \frac{\varepsilon}{1-\gamma} \geq \frac{n}{q}, \quad \beta - s(\alpha - 1) + \frac{\varepsilon}{\gamma} \geq \frac{n}{r}.$$ 

It follows from inequality (6.2) that

$$(6.4) \quad 2 \left(1 - \frac{n}{2s}\right) \frac{1}{q} + \left(\frac{n}{2s}\right) \frac{1}{r} - \frac{1}{2} \left(n + \frac{2\varepsilon(r - q)}{2q(r - 2)}\right) \leq 2\beta - n\alpha.$$ 

If the inequalities in (6.2)–(6.4) are strict, then for proving the proposition it suffices to consider the function $f = m_{\alpha, \beta}$. In particular, (6.4) implies that $\alpha > 0$, $\alpha \neq 1$, and $\beta > 0$ can be chosen in such a way that $2\beta - n\alpha \leq 0$, which leads to $f \notin W_0(\mathbb{R}^n)$.

If there are equalities in some of the (6.2)–(6.4), we take $f(x) = \nu_{\frac{2q}{q}}(x)$. It is easy to check that $f \in L_q \left(\frac{w_{\varepsilon}}{1-\gamma}\right) \cap H^s_r \left(\frac{w_{-\varepsilon}}{\gamma}\right)$ in this case. Further, since $1 \leq q < 2$, Lemma 6.2 (ii) implies

$$|\mathcal{F} f(\xi)| \sim (|\xi|^n \log |\xi|)^{-1}, \quad |\xi| \to \infty,$$

which gives $f \notin W_0(\mathbb{R}^n)$.

The proof is quite similar for $q > 2$. We just mention that in that case (i) of Lemma 6.2 should be used, which gives for $f$

$$|\mathcal{F} f(\xi)| \sim \left(|\xi|^n \log \frac{1}{|\xi|}\right)^{-1}, \quad |\xi| \to 0.$$ 

The proof is complete. \qed
We have shown that condition (3.5) on the parameters \( q \) and \( r \) in Corollary 3.3 is sharp. Let us show that conditions (3.2) and (3.3) in Corollary 3.3 are sharp as well. We restrict ourselves to condition (3.2).

**Proposition 6.4.** Let \( 1 \leq q < 2 < r < \infty, s > \frac{n}{2}, s \in \mathbb{N} \). If

\[
(1 - \frac{n}{2s}) \frac{1}{q} + \left( \frac{n}{2s} \right) \frac{1}{r} > \frac{1}{2},
\]

then for each \( \varepsilon > \frac{n(r-2)(2-q)}{2(r-q)} \) there is a function \( f \in C_0(\mathbb{R}^n) \) such that

\[
f \in L_q \left( w_{-\varepsilon} \right) \cap H^s_r \left( w_{\varepsilon} \right), \quad \gamma = \frac{2-q}{r-q},
\]

but \( f \notin W_0(\mathbb{R}^n) \).

**Proof.** One can use the function \( f(x) = m_{\alpha, \beta}(x) \) as in the proof of Proposition 6.3. Let \( \alpha, \beta > 0, \alpha \neq 1 \), be such that

\[
\beta - \frac{\varepsilon}{1-\gamma} > \frac{n}{q}, \quad \beta - s(\alpha - 1) + \frac{\varepsilon}{r} > \frac{n}{r}.
\]

Then it follows from inequality (6.5) that

\[
2 \left( \left( 1 - \frac{n}{2s} \right) \frac{1}{q} + \left( \frac{n}{2s} \right) \frac{1}{r} - \frac{1}{2} \right) \left( n - \frac{2\varepsilon(r-q)}{(2-q)(r-2)} \right) < 2\beta - n\alpha.
\]

It remains to make use of the fact that \( f \notin W_0(\mathbb{R}^n) \) if \( 2\beta \leq n\alpha \) and choose appropriate \( \alpha \) and \( \beta \). \qed

It is worth mentioning that Corollary 3.3 is, generally speaking, invalid for \( q = r = 2 \) and non-trivial weights, contrary to Beurling’s theorem in [3], which is valid for \( q = r = 2 \) and \( u = v \equiv 1 \).

**Proposition 6.5.** Let \( \varepsilon > -\frac{n}{2} \) and \( \varepsilon \neq 0 \). Then there is a function \( f \in C_0(\mathbb{R}^n) \) such that

\[
f \in L_2(w_\varepsilon) \cap H^s_2 \left( w_{-\varepsilon} \right),
\]

but \( f \notin W_0(\mathbb{R}^n) \).

**Proof.** It suffices to consider the function \( f(x) = \nu_{1+\frac{\varepsilon}{w}}(x) \), and apply Lemma 6.2. \qed

**Acknowledgements**

Both authors were the students of R.M. Trigub at different times. It is our pleasure to thank him for years of support, numerous discussions and for the store house we obtained from him. In addition, it is worth mentioning that the entire subject matter originated from Trigub’s conjecture (1.2).

The authors also thank A. Miyachi, S. Samko, and I. Verbitsky for valuable discussions.

This research has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 704030. The first author also acknowledges the support of the Gelbart Institute at the Department of Mathematics of Bar-Ilan University.
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