Multi-mode density matrices of light via amplitude and phase control

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Abstract

A new method is described for determining the quantum state of correlated multimode radiation by interfering the modes and measuring the statistics of the superimposed fields in four-port balanced homodyne detection. The full information on the $N$-mode quantum state is obtained by controlling both the relative amplitudes and the phases of the modes, which simplifies the reconstruction of density matrices to only $N + 1$ Fourier transforms. In particular, this method yields time-correlated multimode density matrices of optical pulses by superimposing the signal by a sequence of short local-oscillator pulses.

Recently, increasing interest has been devoted to the problem of reconstruction of the quantum state of optical fields from measurable data. The feasibility of reconstruction of single-mode density matrices was first demonstrated by Smithey et al. [1] using optical homodyne tomography. In the experiments they determined both the Wigner function and the density matrix in a field-strength basis. Later, the problem of direct reconstruction of the density matrix in a field-strength basis and the photon-number basis has been studied in Refs. [2] and [3], respectively. All these methods are based on the fact that in balanced homodyne four-port detection the field-strength distributions of a signal mode can be measured and knowledge of the field-strength distributions for all phases within a $\pi$-interval is equivalent to knowledge of the quantum state of the mode [4]. Whereas for determining the Wigner function a three-fold integral must be calculated, the direct reconstruction of the density matrix elements can be accomplished with two integrals. The method has first been extended to two-mode quantum states [5] and later on the general situation for reconstructing $N$-mode quantum states from $N$-fold joint difference-count distributions has been investigated.

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In these schemes the determinations of \( N \)-mode density matrices and phase space functions, respectively, require \( 2N \)- and \( 3N \)-fold integral transforms of the measured data.

In the present paper we propose a new method for determining \( N \)-mode density matrices \((N > 1)\) from the difference count distributions measurable in balanced homodyne four-port detection with pre-superimposed signal fields. Apart from the advantage that only two photodetectors are needed in that scheme, the reconstruction of the density matrices only requires \( N+1 \) rather than \( 2N \) integral transforms. The method does not only enable one to measure the (joint) density matrices of correlated \( N \)-mode fields very efficiently, but also yields \( N \)-mode density matrices of optical pulses in terms of time-localized nonmonochromatic modes. The basic quantities in our scheme are the distributions of the sum field strengths of two or more modes undergoing both amplitude and phase control. It is shown that these distributions yield the full information on the quantum state, provided that they are recorded as functions of the relative field amplitudes of the modes and for phases within \( \pi \)-intervals.

We will illustrate the principle by dealing with the two-mode case. An extension of the corresponding results to \( N \) modes \((N > 2)\) is straightforward. Consider two typical experimental schemes as given in Fig. 1. First, let us assume that two (correlated) modes can be used separately as input fields in an interferometer, such as a beam splitter. The superimposed light in one output channel of the interferometer is used as a signal mode in balanced four-port homodyne detection \([Fig. 1(a)]\). Second, two-mode density matrices of optical pulses can be measured \([Fig. 1(b)]\) by superimposing a signal pulse with a sequence of two short (strong) local-oscillator (LO) pulses and recording the (time-integrated) difference-count distributions \([7]\). The time-localized (nonmonochromatic) modes are defined by the short local-oscillator pulse envelopes \( f_1(t) \) and \( f_2(t) \), and the measured difference-count distributions yield the distributions of the sum field strengths of the two modes \([10]\). In both schemes, the relative amplitude of the two modes and their phases are controlled.

Consider a two-mode optical field with photon destruction operators \( \hat{a}_k \) \((k=1,2)\) and introduce

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Two possible schemes for reconstructing two-mode density matrices. (a) Two modes \((\hat{a}_1 \text{ and } \hat{a}_2)\) are mixed by a beam splitter and one of the interfering output modes \((\hat{b})\) is used as signal mode in balanced homodyning in order to measure the sum field strengths of the two modes (LO, strong local oscillator). (b) A signal pulse and a sequence of two short (strong) local-oscillator pulses with envelopes \( f_1(t) \) and \( f_2(t) \) are superimposed in balanced homodyne detection.}
\end{figure}
the scaled field-strength operators of the modes as

$$\hat{F}_k(\varphi_k) = |F| \left( \hat{a}_k e^{-i\varphi_k} + \hat{a}_k^\dagger e^{i\varphi_k} \right),$$

(1)

and $\hat{F}_k(\varphi_k)|\mathcal{F}_k,\varphi_k\rangle = \mathcal{F}_k|\mathcal{F}_k,\varphi_k\rangle$. The mode amplitude $|F|$ specifies the kinds of fields (electric, magnetic) under study, to compare with experiments it can be related to the shot noise \[\mathbb{B}\]. In the field strength basis the density matrix is given by \[\mathbb{B}\]

$$\langle \mathcal{F}_1 - \mathcal{F}_1', \mathcal{F}_2 - \mathcal{F}_2', \varphi_1, \varphi_2 | \hat{\mathcal{D}} | \mathcal{F}_1 + \mathcal{F}_1', \mathcal{F}_2 + \mathcal{F}_2', \varphi_1, \varphi_2 \rangle$$

$$= \left( \frac{1}{2\pi} \right)^2 \int dy_1 \int dy_2 \, e^{-i(y_1 \mathcal{F}_1 + y_2 \mathcal{F}_2)} \, \Psi(z_1, z_2, \psi_1, \psi_2),$$

(2)

where

$$z_k \equiv z_k(y_k, \mathcal{F}_k) = \sqrt{y_k^2 + \mathcal{F}_k^2}/|F|^2,$$

(3)

$$\psi_k \equiv \psi_k(y_k, \mathcal{F}_k) = \varphi_k - \text{arccot}[y_k|F|^2/\mathcal{F}_k^2]$$

(4)

($\mathcal{F}_k > 0$). The characteristic function $\Psi$ in Eq. (2) is closely related to the average of the two-mode coherent displacement operator and can be written as

$$\Psi(z_1, z_2, \psi_1, \psi_2) = \left\langle \exp \left[ iz_1 \hat{F}_1(\psi_1) + iz_2 \hat{F}_2(\psi_2) \right] \right\rangle.$$

(5)

We see that $\Psi$ as a function of the $z_k$ (for given $\psi_k$) is nothing but the characteristic function of the joint field-strength distribution

$$p_j(\mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2) \equiv \langle \mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2 | \hat{\mathcal{D}} | \mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2 \rangle$$

$$= \left( \frac{1}{2\pi} \right)^2 \int dz_1 \int dz_2 \, e^{-i(z_1 \mathcal{F}_1 + z_2 \mathcal{F}_2)} \, \Psi(z_1, z_2, \psi_1, \psi_2).$$

(6)

Thus, when the joint probability distributions of the two field strengths are measured for all values of the phases $\psi_k$ within $\pi$ intervals, two-fold Fourier transforms yield their characteristic functions and two more Fourier integrals must be performed to obtain the density matrix in Eq. (4).

Let us now suppose that in place of the above mentioned joint field-strength distributions the distributions $p_s(\mathcal{F}, \alpha, \psi_1, \psi_2)$ of the sum field strength

$$\hat{F} = \hat{F}_1(\psi_1) \cos \alpha + \hat{F}_2(\psi_2) \sin \alpha$$

(7)

are measured, where $\alpha \in (0, \frac{1}{2}\pi)$ controls the relative field amplitude. The probability distributions $p_s(\mathcal{F}, \alpha, \psi_1, \psi_2)$ and $p_j(\mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2)$ are related to each other as

$$p_s(\mathcal{F}, \alpha, \psi_1, \psi_2) = \int d\mathcal{F}_1 \int d\mathcal{F}_2 \, p_j(\mathcal{F}_1, \mathcal{F}_2, \psi_1, \psi_2) \, \delta(\mathcal{F} - \mathcal{F}_1 \cos \alpha - \mathcal{F}_2 \sin \alpha),$$

(8)

which together with Eq. \[\mathbb{B}\] implies that

$$p_s(\mathcal{F}, \alpha, \psi_1, \psi_2) = \frac{1}{2\pi} \int dz \, e^{-iz\mathcal{F}} \Psi(z \cos \alpha, z \sin \alpha, \psi_1, \psi_2),$$

(9)

and hence

$$\Psi(z_1, z_2, \psi_1, \psi_2) = \int d\mathcal{F} \, e^{iz\mathcal{F}} \, p_s(\mathcal{F}, \alpha, \psi_1, \psi_2),$$

(10)

where

$$z \equiv z(z_1, z_2) = \sqrt{z_1^2 + z_2^2},$$

(11)
\[ \alpha \equiv \alpha(z_1, z_2) = \arctan(z_2/z_1). \] (12)

This means that from measurements of the superimposed light for different \( \alpha \) we can “tomographically” reconstruct the joint characteristic function (10). Combining Eqs. (2) and (10) yields

\[
(\mathcal{F}_1 - \mathcal{F}_1', \mathcal{F}_2 - \mathcal{F}_2', \varphi_1, \varphi_2) \hat{\theta} |\mathcal{F}_1 + \mathcal{F}_1', \mathcal{F}_2 + \mathcal{F}_2', \varphi_1, \varphi_2) = \left( \frac{1}{2\pi} \right)^2 \int dy_1 \int dy_2 e^{-i(y_1 \mathcal{F}_1 + y_2 \mathcal{F}_2)} \int d\mathcal{F} e^{iy\mathcal{F}} p_s(\mathcal{F}, \beta, \psi_1, \psi_2),
\] (13)

where [according to Eqs. (2), (11), and (12)]

\[
y \equiv y(y_1, y_2, \mathcal{F}_1', \mathcal{F}_2') = \sqrt{y_1^2 + y_2^2 + (\mathcal{F}_1'^2 + \mathcal{F}_2'^2)/|\mathcal{F}|^4}, \quad (14)
\]

\[
\beta \equiv \beta(y_1, y_2, \mathcal{F}_1', \mathcal{F}_2') = \arctan \sqrt{\frac{y_2^2|\mathcal{F}|^4 + \mathcal{F}_2'^2}{y_1^2|\mathcal{F}|^4 + \mathcal{F}_1'^2}}, \quad (15)
\]

and \( \psi_k \) is given in Eq. (1). Equation (13) explicitly shows the feasibility of reconstructing the two-mode density matrix from the measured sum field-strength distributions [\( \psi_k \in (\varphi_k - \pi, \varphi_k), \beta \in (0, \frac{1}{2}\pi) \)].

Let us briefly comment on the effect of nonperfect detection. When the photodetectors have efficiencies \( \eta \) that are less than unity, the characteristic function of the (sum) field strength distribution in Eq. (13) is replaced according to [8]

\[
\int d\mathcal{F} e^{iy\mathcal{F}} p_s(\mathcal{F}, \beta, \psi_1, \psi_2) \rightarrow e^{y^2|\mathcal{F}|^2(1-\eta)/(2\eta)} \int d\mathcal{F} e^{iy\mathcal{F}} p_s(\mathcal{F}, \beta, \psi_1, \psi_2; \eta), \quad (16)
\]

where \( p_s(\mathcal{F}, \beta, \psi_1, \psi_2; \eta) \) denotes the sum field distribution measured with a detector of quantum efficiency \( \eta \). Consequently, Eq. (13) is generalized as

\[
(\mathcal{F}_1 - \mathcal{F}_1', \mathcal{F}_2 - \mathcal{F}_2', \varphi_1, \varphi_2) \hat{\theta} |\mathcal{F}_1 + \mathcal{F}_1', \mathcal{F}_2 + \mathcal{F}_2', \varphi_1, \varphi_2) = \left( \frac{1}{2\pi} \right)^2 \int dy_1 \int dy_2 e^{-i(y_1 \mathcal{F}_1 + y_2 \mathcal{F}_2)} e^{y^2|\mathcal{F}|^2(1-\eta)/(2\eta)} \int d\mathcal{F} e^{iy\mathcal{F}} p_s(\mathcal{F}, \beta, \psi_1, \psi_2; \eta),
\] (17)

Processing data from a real experiment requires a careful handling of the noise to avoid troubles arising from the exponential of \( y^2 \).

Let us return to the measurement schemes proposed above. Equation (17) yields the general relation of the measured distributions to the corresponding two-mode density matrices. In the first scheme [Fig. 3(a)] the difference count distributions directly yield the desired distributions \( p(\mathcal{F}, \beta, \psi_1, \psi_2; \eta) \) of the sum field of the two signal modes under study. The parameters \( \psi_1, \psi_2, \) and \( \beta \) are related to the input–output relations of the interferometer. Measurement of the difference-count distributions on a sufficiently dense grid of points \( (\psi_1, \psi_2, \beta) \), with \( \psi_k \in (\varphi_k - \pi, \varphi_k) \) and \( \beta \in (0, \frac{1}{2}\pi) \), is then equivalent to measurement of the quantum state of the two-mode field. In the second scheme [Fig. 3(b)], two-mode density matrices of optical pulses can be measured by superimposing the signal pulse with two short (strong) local oscillator (LO) pulses of envelopes \( f_1(t) \) and \( f_2(t) \). The desired information on the two-mode quantum state is obtained by recording the (time-integrated) difference-count distributions and performing the amplitude and phase control by controlling the relative strength and the positions of the LO pulses, for more details see [4]. Note that this method is not only of interest for studying pulses, it also yields insight in the time-dependent correlation properties of stationary fields at the level of the full quantum statistical information.

4
In certain cases of pulse measurements, when the signal pulse and the LO pulses are produced by different sources, it may become impossible to achieve complete phase control. However, the difference of the LO phases, \( \Delta \psi = \psi_2 - \psi_1 \), can be controlled and we measure the phase averaged probability distribution,

\[
\overline{p_s(F, \beta, \Delta \psi)} \equiv (2\pi)^{-1} \int d\psi_1 p_s(F, \beta, \psi_1, \psi_1 + \Delta \psi).
\]

Consequently, we can reconstruct the phase-averaged (with respect to \( \psi_1 \)) version of the density matrix in Eq. (17). Although losing some information, these averaged density matrices still contain interesting informations on the temporal correlations of the signal field.

It is straightforward to extend the method to measurements of \( N \)-mode density matrices \((N > 2)\). Again, we can measure the weighted sums of the field strengths in a homodyne setup with two detectors. The pre-superimposed signal can again be obtained using appropriate interferometric methods. In particular, in the scheme in Fig. (b) a train of \( N \) (strong) LO pulses must be used. The sum field strengths are now parameterized by \( N - 1 \) relative intensities of the \( N \) modes under consideration, and the characteristic functions of their probability densities are related to the characteristic functions of the joint probability densities by generalizing Eq. (3). Then the reconstruction of the \( N \)-mode density matrix can be accomplished with \( N + 1 \) Fourier integrals — one integral for obtaining the characteristic function of the density matrix from the measured data plus one integral per mode in order to determine the density matrix from its characteristic function.

In conclusion, we have shown that the reconstruction of multimode density matrices of light can be performed by measuring amplitude and phase controlled distributions of the sum fields. This simplifies the reconstruction of multimode density matrices by reducing both the number of photodetectors needed and the number of the desired integral transforms. Applications of the method for determining the quantum states of correlated field modes and of time-correlated multimode quantum states are discussed. The latter allows to perform time-dependent correlation measurements yielding the full quantum statistical information.

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