A new class of solutions to the WDVV equation

Olaf Lechtenfeld\textsuperscript{a} and Kirill Polovnikov\textsuperscript{b}

\textsuperscript{a} Institut f"ur Theoretische Physik, Leibniz Universit"at Hannover, 30167 Hannover, Germany
\textsuperscript{b} Laboratory of Mathematical Physics, Tomsk Polytechnic University, 634050 Tomsk, Russia

Abstract

The known prepotential solutions $F$ to the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation are parametrized by a set \{\alpha\} of covectors. This set may be taken to be indecomposable, since $F_{\{\alpha\} \oplus \{\beta\}} = F_{\{\alpha\}} + F_{\{\beta\}}$. We couple mutually orthogonal covector sets by adding so-called radial terms to the standard form of $F$. The resulting reducible covector set yields a new type of irreducible solution to the WDVV equation.
1 Introduction

The Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation

\[ F_i F_k^{-1} F_j = F_j F_k^{-1} F_i \quad \text{with} \quad i, j, k = 1, \ldots, n \]  

(1.1)

is a nonlinear constraint on a set of \( n \times n \) matrices \( F_i \) whose entries are the third derivatives of a function \( F(x^1, \ldots, x^n) \) called the prepotential,

\[ (F_i)_{pq} = \partial_i \partial_p \partial_q F \]  

(1.2)

(we abbreviate \( \partial_i = \frac{\partial}{\partial x^i} \)). This equation was first introduced in the context of two dimensional topological field theory \([1, 2]\) and \( \mathcal{N}=2 \) SUSY Yang-Mills theory \([3]\) and was intensively studied during the following years. In 1999, it was shown \([4]\) that one can construct solutions by taking the ansatz \([3]\)

\[ F = -\frac{1}{2} \sum_{\alpha} f_\alpha (\alpha \cdot x)^2 \ln |\alpha \cdot x| , \]  

(1.3)

where \( \{\alpha\} \) is the (positive) root system of some simple Lie algebra of rank \( n \). This ansatz defines a constant metric given by the matrix

\[ G = -x^i F_i = \sum_{\alpha} f_\alpha \alpha \otimes \alpha . \]  

(1.4)

Shortly thereafter, it was proved \([5]\) that certain deformations of such root systems still solve \((1.1)\), and the list of possible collections of covectors \( \{\alpha\} \) was extended to the so-called \( \mathbb{V} \)-systems \([6]\).

In 2005, an interesting connection between the WDVV equation and \( \mathcal{N}=4 \) superconformal mechanics was discovered \([7]\). In this context, a slightly different but equivalent formulation of the WDVV equation appeared, namely

\[ [F_i, F_j] = 0 \quad \leftrightarrow \quad (\partial_i \partial_k \partial_p F)(\partial_j \partial_l \partial_p F) = (\partial_j \partial_k \partial_p F)(\partial_i \partial_l \partial_p F) , \]  

(1.5)

supplemented by the homogeneity condition

\[ -x^i F_i = 1 \quad \leftrightarrow \quad -x^i \partial_i \partial_j \partial_k F = \delta_{jk} . \]  

(1.6)

The latter picks a Euclidean metric in \( \mathbb{R}^n \) and determines a linear change of coordinates which relates the two formulations within the ansatz \([1, 3]\) by mapping \( G \mapsto 1 \) \([8]\). We also write \( x_j = \delta_{jk} x^k = x^j \).

Rather recently, it was observed \([9]\) that the ansatz \((1.3)\) can be successfully extended by adding a ‘radial term’ \([10]\),

\[ F = -\frac{1}{2} \sum_{\alpha} f_\alpha (\alpha \cdot x)^2 \ln |\alpha \cdot x| - \frac{1}{2} f_R R^2 \ln R^2 , \]  

(1.7)

where \( \{\alpha\} \) is some \( \mathbb{V} \)-system and \( R \) denotes the radial coordinate in \( \mathbb{R}^n \),

\[ R^2 := \sum_{i=1}^n x_i^2 \quad \text{so that} \quad G(x, x) = \sum_{\alpha} f_\alpha (\alpha \cdot x)^2 = (1-2f_R) R^2 . \]  

(1.8)

The two types of term in \((1.7)\) are extremal cases of a general ansatz employing arbitrary quadratic forms. For \( n>2 \) only two choices for \( f_R \) are compatible with the WDVV equation, namely \( f_R = 0 \) and \( f_R = 1 \), which are related by flipping the sign of the metric \( G \) via \( f_\alpha \mapsto -f_\alpha \).

In this Letter we further generalize the ansatz \((1.7)\) for the prepotential \( F \) and construct novel solutions to the WDVV equation.

2 Relative radial terms

Let us consider a reducible covector system \( L = L_1 \oplus L_2 \oplus \ldots \oplus L_M \), which is a direct sum of \( M \) irreducible covector collections with dimensions \( n_1, n_2, \ldots, n_M \), respectively. Each subsystem \( L_I \) \((I = 1, \ldots, M)\) is chosen to solve the WDVV equation via the ansatz \((1.3)\) for \( \alpha \in L_I \). Furthermore, we work in the ‘Euclidean coordinates’ conforming to \((1.6)\). The prototypical example is the root system of a non-simple but semi-simple Lie algebra. According to the covector set decomposition, we split the index set

\[ \{1, 2, \ldots, n\} = \{1, \ldots, n_1\} \cup \{n_1+1, \ldots, n_1+n_2\} \cup \cdots \cup \{n-n_M+1, \ldots, n\} = S_1 \cup S_2 \cup \cdots \cup S_M \]  

(2.1)

\footnote{The covector \( \alpha \) evaluates on \( x = (x^1, \ldots, x^n) \) via \( \alpha(x) = \alpha_i x^i =: \alpha \cdot x \). Positive coefficients \( f_\alpha \) may be absorbed in \( \alpha \).}
and introduce the notation
\[ X_i^I = \begin{cases} x_i & \text{if } i \in S_I \\ 0 & \text{otherwise} \end{cases}, \quad X_i^{IJ} = \begin{cases} x_i & \text{if } i \in S_{IJ} \\ 0 & \text{otherwise} \end{cases} \quad \text{etc. for } S_{IJ} = S_I \cup S_J \quad \text{etc.}, \quad (2.2) \]

and likewise for \( \delta_{ij}^I, \delta_{ij}^{IJ} = \delta_{ij}^I + \delta_{ij}^J \) etc. An important concept is that of \textit{relative} radial coordinates
\[ r_i^2 = \sum_{i \in S_I} x_i^2 = x^I X_i^I, \quad r_{I,J}^2 = r_i^2 + r_j^2 = \sum_{i \in S_{IJ}} x_i^2 = x^{IJ} X_i^{IJ}, \quad \ldots, \quad R^2 = \sum_i x_i^2 \quad (2.3) \]
for the subspaces \( S_I, \quad L_{IJ} = L_I \oplus L_J, \quad L_{IJK} = L_I \oplus L_J \oplus L_K \) etc., all the way up to \( L \).

The key idea is to couple the mutually orthogonal components of this reducible covector system by adding to the ansatz \( \boxed{1.3} \) not only the overall radial term as in \( \boxed{1.7} \) but also all possible \textit{relative} radial terms,
\[ F = -\frac{1}{2} \sum_{\alpha \in L} f_{\alpha} (\alpha \cdot x)^2 \ln |\alpha \cdot x| - \frac{1}{2} \sum_{I=1}^{M} f_{r_I} r_I^2 \ln r_I^2 - \frac{1}{2} \sum_{I < J} F_{r_{I,J}} r_{I,J}^2 \ln r_{I,J}^2 - \ldots - \frac{1}{2} \sum_{I < J} f_{r_{R}} R^2 \ln R^2 \quad (2.4) \]
defining a hierarchy \( f_{r_I} \subset f_{r_{I,J}} \subset f_{r_{I,J,K}} \ldots \subset f_{R} \) of radial couplings. We point out that this ansatz is no longer decomposable.

In order to verify \( \boxed{2.4} \), we have to compute the WDVV coefficients
\[ \partial_\alpha \partial_\beta \partial_\kappa F = - \sum_{I=1}^{M} \sum_{\alpha \in L_I} f_{\alpha} \frac{\alpha_i \alpha_j \alpha_k}{(\alpha \cdot x)} - \sum_{I=1}^{M} f_{r_I} \left( \frac{x^I_i \delta_{ij}^I + x^I_j \delta_{ij}^I + x^I_k \delta_{ij}^I}{r_I^2} - 2 \frac{x^I_i x^I_j x^I_k}{r_I^4} \right) - \sum_{I < J} f_{r_{I,J}} \left( \frac{x^I_i \delta_{ij}^{IJ} + x^I_j \delta_{ij}^{IJ} + x^I_k \delta_{ij}^{IJ}}{r_{I,J}^2} - 2 \frac{x^I_i x^I_j x^I_k}{r_{I,J}^4} \right) - \ldots - \sum_{I < J} f_{r_{R}} \left( \frac{x_i \delta_{ij}^R + x_j \delta_{ij}^R + x_k \delta_{ij}^R}{R^2} - 2 \frac{x_i x_j x_k}{R^4} \right). \quad (2.5) \]
Contracting with \( x^I \) should reproduce \( \boxed{2.6} \). Taking into account \( \boxed{2.3} \), one obtains a system of equations,
\[ \sum_{\alpha \in L_I} f_{\alpha} \alpha_i \alpha_j + 2 \delta_{ij}^I \left( f_{r_I} + \sum_{J \neq I} f_{r_{I,J}} + \sum_{J, K \neq I} f_{r_{I,J,K}} + \ldots + f_{R} \right) = \delta_{ij}^I. \quad (2.6) \]
Since all covector collections \( L_I \) are \( \vee \)-systems, we must have\footnote{This is not true for \( n_I \leq 2 \), because then \( f_{R} \) may take any value since the WDVV equation is empty.}
\[ \sum_{\alpha \in L_I} f_{\alpha} \alpha_i \alpha_j = \epsilon_I \delta_{ij}^I \quad \text{with} \quad \epsilon_I \in \{+1,-1\}, \quad (2.7) \]
hence
\[ f_{r_I} + \sum_{J \neq I} f_{r_{I,J}} + \sum_{J, K \neq I} f_{r_{I,J,K}} + \ldots + f_{R} = \begin{cases} 0 & \text{for } \epsilon_I = +1 \\ 1 & \text{for } \epsilon_I = -1 \end{cases}. \quad (2.8) \]
This takes care of the homogeneity condition \( \boxed{1.6} \). We come to the WDVV equation \( \boxed{1.5} \), which reads
\[ \frac{1}{2} \sum_{\alpha, \beta \in L} f_{\alpha} f_{\beta} \frac{\alpha \wedge \beta}{(\alpha \cdot x)^2} (\alpha \wedge \beta) \otimes x + \sum_{I=1}^{M} 4 f_{r_I} \left( 1 - f_{r_I} - 2 \left( \sum_{J \neq I} f_{r_{I,J}} + \ldots + f_{R} \right) \right) \frac{T_I^I}{r_I^2} \quad (2.9) \]
\[ + \sum_{I < J} 4 f_{r_{I,J}} \left( 1 - f_{r_{I,J}} - 2 \left( \sum_{K \neq I, J} f_{r_{I,J,K}} + \ldots + f_{R} \right) \right) \frac{T_I^J}{r_{I,J}^2} + \ldots + 4 f_{R} \left( 1 - f_{R} \right) \frac{T_R}{R^2} = 0 \]
with
\[(\alpha \wedge \beta)^{\otimes 2}_{ijkl} = (\alpha_i \beta_j - \alpha_j \beta_i)(\alpha_k \beta_l - \alpha_l \beta_k),\]
\[T_{ijkl}^I = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \delta_{ik} \hat{X}_j^I \hat{X}_l^I + \delta_{il} \hat{X}_j^I \hat{X}_k^I - \delta_{jk} \hat{X}_j^I \hat{X}_l^I + \delta_{jl} \hat{X}_j^I \hat{X}_k^I + \delta_{jk} \hat{X}_i^I \hat{X}_l^I,\]
\[T_{ijkl}^{IJ} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \delta_{ik} \hat{X}_j^{IJ} \hat{X}_l^{IJ} + \delta_{il} \hat{X}_j^{IJ} \hat{X}_k^{IJ} - \delta_{jk} \hat{X}_j^{IJ} \hat{X}_l^{IJ} + \delta_{jl} \hat{X}_j^{IJ} \hat{X}_k^{IJ} + \delta_{jk} \hat{X}_i^{IJ} \hat{X}_l^{IJ},\] (2.10)
\[
\cdots
\]
\[T_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} - \delta_{ik} \hat{x}_j \hat{x}_l + \delta_{il} \hat{x}_j \hat{x}_k - \delta_{jk} \hat{x}_j \hat{x}_k + \delta_{jl} \hat{x}_i \hat{x}_l,\]
where
\[\hat{X}_i^I = \frac{X_i^I}{r_I}, \quad \hat{X}^{IJ} = \frac{X^{IJ}}{r_{IJ}}, \cdots, \quad \hat{x}_i = \frac{x_i}{r}.\] (2.11)

Projecting onto the different (independent) poles in (2.10), the WDVV equation requires that
\[
\sum_{\alpha,\beta \in L_i} f_{\alpha} f_{\beta} \frac{\alpha \otimes \beta}{\alpha \cdot \beta \cdot \delta} (\alpha \wedge \beta)^{\otimes 2} = 0,
\]
\[f_{r_{II}} \left\{ 1 - f_{r_{I}} - 2 \left( \sum_{J(I)} f_{r_{IJ}} + \ldots + f_{R} \right) \right\} = 0,\] (2.12)
\[f_{r_{IJ}} \left\{ 1 - f_{r_{II}} - 2 \left( \sum_{K(I,J)} f_{r_{IKJ}} + \ldots + f_{R} \right) \right\} = 0,\]
\[\cdots\]
\[f_{R} \left\{ 1 - f_{R} \right\} = 0.\]

Staring for a while at (2.12) while taking into account (2.18), one realizes that the only admissible radial couplings are
\[f_{r_{II}}, f_{r_{IJ}}, f_{r_{IK}}, \ldots, f_{R} \in \{0, +1, -1\} \quad \text{but} \quad f_{R} \neq -1.\] (2.13)

The solutions are best described by a sequential procedure, starting from the ‘radial-free’ configuration $f_{r_{II}} = f_{r_{IJ}} = \ldots = f_{R} = 0 \ \forall I, J, \ldots$, corresponding to $\epsilon_I = +1 \ \forall I$. Now, let us turn on some radial couplings of the first hierarchy level, $f_{r_{II}} = +1$ for some values of $I$, which flips the signs of the corresponding $\epsilon_I$. On the next level, we may now switch on further radial couplings $f_{IJ}$, but only if they do not overlap. For each nonzero $f_{IJ}$ we must flip the signs of the corresponding $\epsilon_I$ and $\epsilon_J$ as well as those of $f_{I}$ and $f_{J}$. Continuing this scheme, we eventually arrive at the highest level, where activating $f_{R}$ will flip all signs in the hierarchy. In this way, a multitude of possible new WDVV solutions is generated.

3 Examples

To illustrate the new possibilities for WDVV solutions, let us consider the semi-simple Lie algebra $A_1 \oplus A_2$. Our generalized ansatz (2.4) for this case ($M=2$) reads

\[F = -\frac{1}{2} f_0 (x_1 + x_2 + x_3)^2 \ln |x_1 + x_2 + x_3| - \frac{1}{2} f f_3 \sum_{i<j} (x_i - x_j)^2 \ln |x_i - x_j| - \frac{1}{2} f r^2 \ln r^2 - \frac{1}{2} f R^2 \ln R^2\]

with
\[r^2 = \frac{2}{3} (x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3) \quad \text{and} \quad R^2 = x_1^2 + x_2^2 + x_3^2.\] (3.1)

The last term in (3.1) couples the center of mass (the $A_1$ part) to the relative motion (the $A_2$ part). Since both subsystems are at most two-dimensional, the WDVV equation is empty, being a consequence of the homogeneity condition (1.1). Hence, we only have to fulfill (2.6), which yields

\[3 f_0 + 2 f_R = 1 \quad \text{and} \quad 3 f + 3 f_R + 2 f_R = 1 \quad \Rightarrow \quad f_R = \begin{cases} 0 & \Rightarrow & f_0 = f + f_R = +\frac{1}{3} \\ 1 & \Rightarrow & f_0 = f + f_R = -\frac{1}{3} \end{cases},\] (3.2)

giving us a one-parameter family of solutions.
To see the power of the WDVV equation, we present a more generic $M=3$ example, based on the Lie algebra $D_3 \oplus D_3 \oplus D_4$:

\[
F = -\frac{1}{2} f_1 \left( \sum_{1 \leq i < j \leq 3} (x_i + x_j)^2 \ln |x_i + x_j| + \sum_{1 \leq i < j \leq 3} (x_i - x_j)^2 \ln |x_i - x_j| \right) + \\
-\frac{1}{2} f_2 \left( \sum_{4 \leq i < j \leq 6} (x_i + x_j)^2 \ln |x_i + x_j| + \sum_{4 \leq i < j \leq 6} (x_i - x_j)^2 \ln |x_i - x_j| \right) + \\
-\frac{1}{2} f_3 \left( \sum_{7 \leq i < j \leq 9} (x_i + x_j)^2 \ln |x_i + x_j| + \sum_{7 \leq i < j \leq 9} (x_i - x_j)^2 \ln |x_i - x_j| \right) + \\
-\frac{1}{2} f_{r_1} r_1^2 \ln r_1^2 - \frac{1}{2} f_{r_2} r_2^2 \ln r_2^2 - \frac{1}{2} f_{r_3} r_3^2 \ln r_3^2 + \\
-\frac{1}{2} f_{r_{12}} r_{12}^2 \ln r_{12}^2 - \frac{1}{2} f_{r_{13}} r_{13}^2 \ln r_{13}^2 - \frac{1}{2} f_{r_{23}} r_{23}^2 \ln r_{23}^2 - \frac{1}{2} f_R R^2 \ln R^2 ,
\]

where $f_I = \frac{1}{3} \epsilon_I$ and

\[
r_1^2 = \sum_{i=1}^{3} x_i^2 , \quad r_2^2 = \sum_{i=4}^{6} x_i^2 , \quad r_3^2 = \sum_{i=7}^{9} x_i^2 , \quad r_{ij}^2 = r_i^2 + r_j^2 , \quad R^2 = r_1^2 + r_2^2 + r_3^2 = \sum_{i=1}^{9} x_i^2 .
\]

We list here (up to permutations) all possible radial coupling configurations:

| $\epsilon_1$ | $\epsilon_2$ | $\epsilon_3$ | $f_{r_1}$ | $f_{r_2}$ | $f_{r_3}$ | $f_{r_{12}}$ | $f_{r_{13}}$ | $f_{r_{23}}$ | $f_R$ |
|---|---|---|---|---|---|---|---|---|---|
| $+1$ | $+1$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $-1$ | $+1$ | $+1$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $-1$ | $-1$ | $+1$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $-1$ | $-1$ | $-1$ | $+1$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $+1$ | $-1$ | $+1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $+1$ | $+1$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $-1$ | $-1$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $+1$ | $-1$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |

We thank S. Krivonos for collaboration at various stages of this project. K.P. is grateful to the Institut für Theoretische Physik at the Leibniz Universität Hannover for hospitality. The research was supported by RF Presidential grant NS-2553.2008.2, RFBR grant 09-02-00078 and the Dynasty Foundation.

References

[1] E. Witten, Nucl. Phys. B 340 (1990) 281.
[2] R. Dijkgraaf, H. Verlinde, E. Verlinde, Nucl. Phys. B 352 (1991) 59.
[3] A. Marshakov, A. Mironov, A. Morozov, Phys. Lett. B 389 (1996) 43 [hep-th/9607109].
[4] R. Martini, P.K.H. Gragert, J. Nonlin. Math. Phys. 6 (1999) 1.
[5] A.P. Veselov, Phys. Lett. A 261 (1999) 297 [hep-th/9902142].
[6] A.P. Veselov, in: Integrability: The Seiberg-Witten and Whitham Equations, eds. H.W. Braden, I.M. Krichever, Gordon and Breach, 2000, p. 125 [hep-th/0105020].
[7] S. Bellucci, A. Galajinsky, E. Latini, Phys. Rev. D 71 (2005) 044023 [hep-th/0411232].
[8] O. Lechtenfeld, in: Problems of Modern Theoretical Physics,
ed. V. Epp, Tomsk State Pedagogical University Press, 2008, p. 256 [arXiv:0805.3245].
[9] A. Galajinsky, O. Lechtenfeld, K. Polovnikov, JHEP 0903 (2009) 113 [arXiv:0802.4386].