Convolutional Neural Bandit: Provable Algorithm for Visual-aware Advertising

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Abstract

Online advertising is ubiquitous in web business. Image displaying is considered as one of the most commonly used formats to interact with customers. Contextual multi-armed bandit has shown success in the application of advertising to solve the exploration-exploitation dilemma existed in the recommendation procedure. Inspired by the visual-aware advertising, in this paper, we propose a contextual bandit algorithm, where the convolutional neural network (CNN) is utilized to learn the reward function along with an upper confidence bound (UCB) for exploration. We also prove a near-optimal regret bound $\tilde{O}(\sqrt{T})$ when the network is over-parameterized and establish strong connections with convolutional neural tangent kernel (CNTK). Finally, we evaluate the empirical performance of the proposed algorithm and show that it outperforms other state-of-the-art UCB-based bandit algorithms on real-world image data sets.

1 Introduction

Online display advertising plays an indispensable role in online business to deliver information to customers via various channels, e.g., e-commerce and news platforms. Image ads are considered as one of the most prevalent formats to catch the attention of potential customers. To maximize the Click-Through Rate (CTR), it is crucial to choose the most appealing image among candidates and display it on spot. For example, on Amazon, only one image of the product is allowed to display on the first-level page although multiple images are provided. Such a scenario can be easily found on other advertising platforms. On the other hand, the exploration-exploitation dilemma also exists in visual advertising, as all the candidates should be displayed to customers for exploring new knowledge.

The contextual Multi-Armed Bandit (MAB) can naturally formulate the procedure and has shown success in online advertising [Li et al., 2010; Wang et al., 2021; Li et al., 2016a; Ban and He, 2021]. In the standard bandit setting, suppose there are $n$ arms (images) in a round, each of which is represented by a feature vector or matrix, the learner needs to pull an arm and then observe the reward (CTR). The goal of this problem is to maximize the expected rewards of played rounds.

The contextual MAB has been studied for decades [Abbasi-Yadkori et al., 2011; Valko et al., 2013a; Ban et al., 2021]. By imposing linear assumptions on the reward function, a line of works [Abbasi-Yadkori et al., 2011; Gentile et al., 2014; Ban and He, 2021] have achieved promising results in both theory and practice, where the expected reward is assumed to be linear with respect to the arm vector. However, this assumption may fail in real-world applications [Valko et al., 2013b]. To learn the non-linear reward function, recent works have embedded deep neural networks in the contextual bandits. [Zhang et al., 2020] and [Zhou et al., 2020] both used a fully-connected neural network to learn the reward function; the former adopted the Thompson sampling and the latter adopted the Upper Confidence Bound (UCB) strategies for exploration. To further improve the performance specifically for visual recommendation, [Wang et al., 2021] used Convolutional Neural Networks (CNN) to train embeddings of images and then fed it into the Thompson sampling bandit based on
the framework of Riquelme et al. [2018]. However, these two approaches did not provide analysis regarding the regret bound.

In this paper, we focus on visual advertising and propose to use convolutional neural networks to learn the reward function with UCB-based exploration in contextual bandits. Note that Wang et al. [2021] used CNN to train the embeddings for arms instead of learning the reward function. Inspired by recent advances on the convergence theory in over-parameterized neural networks [Du et al., 2019; Allen-Zhu et al., 2019], we build strong connections among the network function, Convolutional Neural Tangent Kernel (CNTK) [Jacot et al., 2018; Arora et al., 2019], and ridge regression to prove an upper confidence bound and the regret bound. To the best of our knowledge, this is the first near-optimal regret bound for convolutional neural networks in bandits. To summarize, our key contributions are as follows:

(1) We propose a new algorithm, CNN-UCB, modeled as a convolutional neural network along with a provable upper confidence bound. It focuses on capturing the visual pattern of image input.

(2) Under standard assumptions, we provide a regret upper bound that can reduce to $O\left(\sqrt{T O \left(\log O \left(T\right)\right)}\right)$. In contrast, state-of-the-art UCB-based neural bandit [Zhou et al., 2020] has the regret bound of $O\left(\sqrt{T O \left(\log O \left(T\right)\right)} \cdot \left(\sqrt{O \left(\log O \left(T\right)\right)} + O \left(T\right) \cdot \left(1 - O \left(T^{-1}\right)\right)^k\right)\right)$, where $k$ is the number of rounds. To eliminate the error caused by $O \left(\log O \left(T\right)\right)$, they need an additional assumption on $k$ that needs to be extremely large, while CNN-UCB does not have this concern.

(3) We conduct experiments on real-world image data sets and show that CNN-UCB achieves significant improvements on the visual advertising problem.

The rest of the paper is organized as follows. After briefly introducing the related work in Section 2, we formally present the problem definition and the proposed algorithm in Sections 3 and 4, respectively. The main theorems and the proof workflow are introduced in Sections 5 and 6. In the end, we show the empirical results in Section 7 before concluding the paper in Section 8. The proofs are attached in Appendix.

2 Related Work

**Contextual bandits.** The most studied literature is the linear contextual bandit [Li et al., 2010; Abbasi-Yadkori et al., 2011; Valko et al., 2013a], where the reward function is governed by the product of the arm feature vector and an unknown parameter. By the UCB exploration, many algorithms [Abbasi-Yadkori et al., 2011; Li et al., 2016a] can achieve the near-optimal $O\left(\sqrt{T}\right)$ regret bound. To break the linear assumption, Filippi et al. [2010] dealt with a composition of linear and non-linear function; Bubeck et al. [2011] assumed it to have a Lipschitz property in a metric space; Valko et al. [2013b] embedded the reward function into Reproducing Kernel Hilbert Space. Above assumptions all can be thought of as special cases in the problem we study. To improve bandit’s performance on online recommendation, other variants of bandit have been studied such as clustering of bandits [Gentile et al., 2014; Li et al., 2016a; Ban and He, 2021], Multi-facet Bandits [Ban et al., 2021], and outlier arm detection [Zhuang et al., 2017; Ban and He, 2020].

**Neural Bandits.** Thanks to the representation power, deep neural networks (DNN) have been adapted to bandits. Riquelme et al. [2018] used L-layer DNN to learn a representation for each arm and applies Thompson sampling on the low-dimension embeddings for exploration. Wang et al. [2021] extended the above framework to visual advertising by using CNN to train arm embeddings. However, they did not provide the regret analysis. Zhang et al. [2020] first introduced a provable neural bandit algorithm in Thompson sampling where a fully-connected network learns the reward function. The most relevant work [Zhou et al., 2020] used the fully-connected neural network to learn the reward function with UCB exploration and provided a regret bound. However, CNN-UCB is different from this work from many perspectives: First, CNN-UCB targets on image data as CNN usually outperforms fully-connected network on visual recommendation according to recent advances in computer vision [Simonyan and Zisserman, 2014; Chen et al., 2016]; Second, Zhou et al. [2020] built the regret analysis on the recent advances in fully-connected network [Allen-Zhu et al., 2019; Cao and Gu, 2019; Arora et al., 2019], which can not directly apply to CNN. Instead, we prove these intermediate lemmas in this paper such as the relation with CNTK; Third, these two papers have
different assumptions on activation functions, where CNN-UCB uses the Lipschitz-smooth function contrary to ReLU function used in Zhou et al. [2020]. Fourth, CNN-UCB achieves a better regret bound than the one in Zhou et al. [2020].

3 Problem Definition

Notations. Given the number of rounds \( T \), we denote by \([T]\) the sequence \( \{1, \ldots, T\} \) and \( \{x_t\}_{t=1}^T \) the sequence \( \{x_{t,1}, \ldots, x_T\} \). We use \( \|v\|_2 \) to denote the Euclidean norm for a vector \( v \) and \( \|W\|_F \) denote the spectral and Frobenius norm for a matrix \( W \). We use \( \langle \cdot, \cdot \rangle \) to denote the standard Euclidean inner product between two vectors or two matrices. We use \( C \) with subscripts to denote the constants and \( \Psi \) with subscripts to denote the intermediate results. We use standard notations \( \mathcal{O}(\cdot) \) and \( \Omega(\cdot) \) to hide constants and \( \mathcal{N}(\mu, 1) \) to represents the Gaussian distribution of mean \( \mu \) and variance 1.

In standard stochastic MAB, a learner is faced with \( n \) arms in each round \( t \in [T] \), where each arm is represented by a vector or matrix \( x_{t,i}, i \in [n] \). The learner aims to select an arm to maximize the reward of each round. The reward is assumed to be governed by a function with respect to \( x_{t,i} \)

\[
r_{t,i} = f^*(x_{t,i}) + \xi_t,
\]

where \( f^* \) is a linear or non-linear reward function satisfying \( 0 \leq f^*(x_{t,i}) \leq 1 \) and \( \xi_t \) is the noise drawn from \( \mathcal{N}(0, \cdot) \). For brevity, we use \( x_t \) to denote the selected arm and \( r_t \) to represent the received reward in round \( t \). Following the standard evaluation of bandits, the regret of \( T \) rounds is defined as

\[
R_T = \mathbb{E} \left[ \sum_{t=1}^{T} (r_t^* - r_t) \right] = \sum_{t=1}^{T} (f^*(x_t^*) - f^*(x_t))
\]

where \( x_t^* = \arg \max_{i \in [n]} f^*(x_{t,i}) \). Our goal is to design a pulling algorithm to minimize \( R_T \).

4 Proposed Algorithm

Motivated by the applications of visual advertising, we consider each arm as an image represented by a matrix \( x_{t,i} \in \mathbb{R}^{c \times p} \), where \( c \) is the number of input channels and \( p \) is the number of pixels. We denote by \( f(x; \theta) \) a CNN model to learn the reward function \( f^* \), consisting of \( L \) convolutional layers and one subsequent fully-connected layer. For any convolutional layer \( l \in [L] \), layer \( l \) has the same number of channels \( m \). For simplicity, we use the standard zero paddings and set stride size as 1 to ensure the output of each layer has the same size, following the setting of [Arora et al. [2019], Du et al. [2019]]. Let \( h_l^i \) be the output of layer \( l \) and thus \( h_l \in \mathbb{R}^{m \times p} \). For convenience, we may use \( h_0^i \) to denote the input \( x_{t,i} \).

To represent the convolutional operation of layer \( l \), we use an operator \( \phi(\cdot) \) to divide the input \( h_l^{i-1} \) into \( p \) patches, where each patch has the size \( qm \). \( m \) is the number of channels of last layer and \( q \) is considered as the size of kernel (assume all the kernels have the same size for simplicity of analysis). For example, give a \( 2 \times 2 \) kernel, then \( q \) should set as 4. \( \phi(h_l^{i-1}) \in \mathbb{R}^{qm \times p} \) is figured as following:

\[
\begin{pmatrix}
  h_{0,0:3}^{l-1} & \cdots & h_{0,p-1:p+2}^{l-1} \\
  \vdots & \ddots & \vdots \\
  h_{m-1,0:3}^{l-1} & \cdots & h_{m-1,p-1:p+2}^{l-1}
\end{pmatrix}.
\]

In accordance, we have the kernel weight matrix \( W_l^i \in \mathbb{R}^{m \times qm} \). Then, the convolutional operation of layer \( l \) can be naturally represented by \( W_l^i \phi(h_l^{i-1}) \). Therefore, the output of layer \( l \in [L] \) is defined as

\[
h_l = \frac{1}{\sqrt{qm}} \sigma(W_l^i \phi(x_{t,1})), \quad W_l^i \in \mathbb{R}^{m \times qc}
\]

\[
h_l = \frac{1}{\sqrt{qm}} \sigma(W_l \phi_l(h_l^{i-1})), \quad W_l^i \in \mathbb{R}^{m \times qm}, \quad 2 \leq l \leq L,
\]

where \( \sigma \) is the activation function. Finally, with a fully-connected layer \( L + 1 \), the output is defined as

\[
f(x_{t,i}; \theta) = \langle W^{L+1}, h_L^i \rangle / \sqrt{m},
\]
Algorithm 1 CNN-UCB

Input: $f, T, \eta, k, \lambda$

1: Initialize $\theta_0 : (W^1, \ldots, W^{L+1}) \sim \mathcal{N}(0, 1)$ and $W^{L+1} \sim \mathcal{N}(0, 1/m)$
2: $A_0 = \lambda I$
3: for each $t \in [T]$ do
4: Observe $n$ arms $\{x_{t,1}, \ldots, x_{t,n}\}$
5: $\theta_t = \arg \max_{\theta \in \Theta} \left( f(x_{t,i}; \theta) + \Psi_1 \|g(x_{t,i}; \theta)\|_2 / \sqrt{m} A_{t-1}^{-1} + \Psi_2 + \Psi_3 \right)$ (defined in Theorem 4)
6: Play $x_t$ and observe reward $r_t$
7: $A_t = A_{t-1} + g(x_t; \theta_t) g(x_t; \theta_t)^T / m$
8: $\theta_t = \text{GradientDescent}(\theta_0, \{x_{t,i}\}_{i=1}^t, \{r_i\}_{i=1}^t)$
9: end for
10: procedure GRADIENTDESCENT($\theta_0, \{x_{t,i}\}_{i=1}^t, \{r_i\}_{i=1}^t$)
11: $\theta^{(0)} = \theta_0$
12: for $i \in \{1, \ldots, k\}$ do
13: $\theta^{(i+1)} = \theta^{(i)} - \eta \nabla \theta^{(i)} \mathcal{L} (\{x_{t,i}\}_{i=1}^t, \{r_i\}_{i=1}^t)$
14: end for
15: Return $\theta^{(k)}$
16: end procedure

where the vector $\theta = (\text{vec}(W^1)^T, \ldots, \text{vec}(W^{L+1})^T)^T \in \mathbb{R}^d$ represents the learned parameters. We denote by $g(x_{t,i}; \theta) = \nabla f(x_{t,i}; \theta) \in \mathbb{R}^d$ the gradient of the neural network $f$.

After $t$ rounds, we have $t$ selected arms $\{x_{t,i}\}_{i=1}^t = \{x_1, \ldots, x_t\}$ and $t$ received rewards $\{r_i\}_{i=1}^t = \{r_1, \ldots, r_t\}$. Thus, to learn $f^*$, the learning of $\theta$ can be transform into the solution of the following minimization problem by gradient descent:

$$\min_{\theta} \mathcal{L} (\{x_{t,i}\}_{i=1}^t, \{r_i\}_{i=1}^t) = \frac{1}{2} \sum_{i=1}^t (f(x_i; \theta) - r_i)^2$$

where $\mathcal{L}$ is the quadratic loss function.

To solve the exploitation and exploration dilemma, we use the UCB-based strategy. First, we define a confidence interval for $f$:

$$\mathbb{P} \left[ |f(x_{t,i}; \theta) - f^*(x_{t,i})| > \text{UCB}(x_{t,i}) \right] < \delta,$$

where UCB$(x_{t,i})$ is defined in Theorem 4 and $\delta$ usually is a small constant. Then, in each round $t$, the arm is determined by

$$x_t = \arg \max_{i \in [n]} (f(x_{t,i}; \theta) + \text{UCB}(x_{t,i}))$$

Algorithm 4 depicts the workflow of CNN-UCB. First, we initialize all parameters, where each entry of $W^l$ is drawn from $\mathcal{N}(0, 1)$ for $l \in [L]$ and each entry of $W^{L+1}$ is drawn from $\mathcal{N}(0, 1/m)$. When observing a set of $n$ arms, CNN-UCB chooses the arm using the UCB-based strategy. After receiving the reward, CNN-UCB conducts the gradient descent to update $\theta$ with the new collected training pairs $\{x_t\}_{i=1}^t$ and $\{r_i\}_{i=1}^t$.

5 Main Theorems

In this section, we introduce two main theorems, the upper confidence bound of CNN function and the regret analysis of CNN-UCB, inspired by recent advances in convergence theory of ultra-wide networks \cite{Du et al. 2019, Allen-Zhu et al. 2019, Arora et al. 2019} and analysis in the linear contextual bandit \cite{Abbasi-Yadkori et al. 2011}.

Our analysis is based on the Lipschitz and Smooth activation function which holds for many activation functions such as Sigmoid and Softplus. The following condition is to guarantee the stability of training process and build connections with CNTK.
With above UCB, we provide the following regret bound of CNN-UCB.

**Theorem 2.** In the round \( f \) and confidence bound: with probability at least \( 1 - \delta \), for any \( x \in \mathbb{R}^{c \times p} \) satisfying \( \|x\|_F = 1 \), it has

\[
 f^*(x) = (g(x; \theta_i), \theta_i^* - \theta_0).
\]

This lemma shows that \( f^* \) can be represented by a linear function with respect to the gradient \( g(x; \theta_i) \). Note that \( \theta_i^* \) is introduced for the sake of analysis rather than the ground-truth parameters. Suppose there exist a constant \( \bar{S} \) such that \( \|\theta_i^* - \theta_0\|_2 \leq \bar{S}/\sqrt{m} \), where \( 1/\sqrt{m} \) is for the sake of analysis. As \( \bar{S}/\sqrt{m} \) is fixed to every arm in round \( f \), it does not affect the exploration performance of UCB and thus the complexity of regret bound.

**Theorem 1.** In the round \( t + 1 \), given a set of arms \( \{x_i\}_{i=1}^T \) and a set of rewards \( \{r_i\}_{i=1}^T \), let \( f(x; \theta) \) be the convolutional neural network defined in Section 4. Then, there exist constants \( C_0 > 0 \), \( 1 < C_1, C_2 < 2 \) such that if

\[
 m \geq \max \left\{ t^4 \mathcal{O}\left((C_1 \mu)^L\right)e^{\mathcal{O}\left(C_1 \mu \sqrt{\bar{S}}\right)} + C_2, \right\} \frac{\lambda C_0}{1 - \delta}
\]

\[
 \eta \leq (m\lambda + 1)^{-1}, \quad \|\theta_i^* - \theta_0\|_2 \leq \bar{S}/\sqrt{m}
\]

with probability at least \( 1 - \delta \), for any \( x \in \mathbb{R}^{c \times p} \) satisfying \( \|x\|_F = 1 \), we have the following upper confidence bound:

\[
 |f^*(x) - f(x; \theta_i)| \leq \Psi_1 |g(x; \theta_i)|/\sqrt{m} \left| A_i^{-1} + \Psi_2 + \Psi_3 \right|
\]

where

\[
 \Psi_1 = \left( \sqrt{\log \left( \frac{\det(A_i)}{\det(A)} \right)} - 2 \log \delta + \sqrt{\lambda \bar{S}} \right)
\]

\[
 \Psi_2 = \left( \sqrt{L + 1} \left( \sqrt{\bar{p}}(C_1 \mu \sqrt{\bar{q}})^L/m + \sqrt{q} \Psi_{L,(k')}((C_1 \mu)^L + 2) \right) \right) \left[ \frac{\bar{A}_1 + \bar{A}_2 + \bar{A}_3}{m \lambda} + \sqrt{t/m} \right]
\]

\[
 + \frac{t(L + 1)}{m} \left( 2 \sqrt{\bar{p}}(C_1 \mu \sqrt{\bar{q}})^L/m + \sqrt{q} \Psi_{L,(k')}((C_1 \mu)^L + 2) \right) \sqrt{\Psi_{L,(k')}((C_1 \mu)^L + 2)}
\]

\[
 \Psi_3 = \left\{ \frac{2 \Psi_{L,(k')}((C_1 \mu)^L + 1)w (L - 1)}{2 \left( \Psi_{L,(k')}((C_1 \mu)^L + 1) \right)} / \sqrt{m} \right\}
\]

and

\[
 \bar{A}_1 = t \sqrt{2q(L + 1)} \Psi_{L,(k')}((C_1 \mu)^L + 2)
\]

\[
 \bar{A}_2 = t \sqrt{(L + 1)} \sqrt{p} (C_1 \mu \sqrt{q})^L \cdot \sqrt{t} \left( \Psi_{L,(k')}((C_1 \mu)^L + 1) \right) m^{-3/2}
\]

\[
 \bar{A}_3 = \lambda \sqrt{L + 1}w/\sqrt{m} \quad \Psi_{L,(k')} = \frac{\mu w (2 \mu C_1 \sqrt{q})^L - 1}{m(2 \mu C_1 \sqrt{q} - 1)}
\]

\[
 w = \frac{2t \sqrt{2\mu} e^{C_1(L - 1) \sqrt{q} + C_2 ((C_1 \mu)^L + 1)}}{C_0}
\]

With above UCB, we provide the following regret bound of CNN-UCB.

**Theorem 2.** After \( T \) rounds, given the set of arms \( \{x_i\}_{i=1}^T \) and the set of rewards \( \{r_i\}_{i=1}^T \), let \( f(x; \theta) \) be the convolutional neural network defined in Section 4. Then, there exist constants \( C_0 > 0 \),
1 < C_1, C_2 < 2 such that if
\[ m \geq \max \left\{ \frac{T^4 \mathcal{O} ((C_1 \mu)^L) e^{O(C_1 L \sqrt{\eta}) + C_2)}{\lambda C_0}, \Omega \left[ \log \left( \frac{LT}{\delta} \right) \right] \right\} \]
\[ \eta \leq (m \lambda + 1)^{-1}, \quad \| \theta^*_t - \theta_0 \|_2 \leq \tilde{S}/\sqrt{m}, \forall t \in [T], \]
then with probability at least 1 - \delta, the regret of CNN-UCB is upper bounded by
\[ R_T \leq \sqrt{2Td \log(1 + T/\lambda) + 1} \left( d \log(1 + T/\lambda) + 2 \log(1/\delta) + 1 + \sqrt{\lambda S} \right) + 2. \]
where
\[ d = \frac{\log \det (I + G_0^T G_0/\lambda)}{\log(1 + T \lambda)} \quad \text{and} \quad G = (g(x_1; \theta_0), \ldots, g(x_T; \theta_0)) \in \mathbb{R}^{d \times T}. \]
\( d \) is the effective dimension first defined in [Valko et al., 2013b] to analyze the kernel bandits and then applied by [Zhou et al., 2020] to measure the diminishing rate of NTK [Jacot et al., 2018]. We adapt it to the CNTK [Arora et al., 2019; Yang, 2019] to alleviate the predicament caused by the blowing up of parameters.

The above regret bound can be reduced to the time complexity of \( \mathcal{O} \left( \sqrt{T \log \log \left( (T - 1)^k \right)} \right) \), which is the same as the state-of-the-art regret bound in linear contextual bandits [Abbasi-Yadkori et al., 2011] while we do not impose any assumption on the reward function. The most relevant work [Zhou et al., 2020], instead, achieves the regret bound of \( \mathcal{O} \left( \sqrt{T \log \log \left( (T - 1)^k \right)} \right) \cdot \left( \sqrt{\log \log \left( C^L \right)} + \Omega(1) \cdot (1 - O(T^{-1}))^k \right) \). The term \( O(T) \cdot (1 - O(T^{-1}))^k \) can result in the exploding of error unless the number of iterations \( k \) is extremely large.

## 6 Proof of Main Theorems

In this section, we present the sketch of the proof of Theorem 1 and 2 with important lemmas. The proof roadmap is different from the [Zhou et al., 2020] from many aspects. In addition to the architecture difference between CNN and fully-connected networks, we focus on the lipschitz-smooth activation function while their proof build on the ReLU function. To achieve the regret bound, we require \( m \geq \mathcal{O}(T^4 \mu^L) \) while they require \( \mathcal{O}(T^7 L^2) \). In practice, \( T \) usually is much larger than \( L \).

For example, commonly used CNN models such as VGG (L=16) [Simonyan and Zisserman, 2014] are not extremely deep. Instead, \( T > 1 \times 10^4 \) in the experiments of many bandit papers [Zhou et al., 2020; Li et al., 2010, 2016b].

### Equivalence to Convolutional Neural Tangent Kernel

First, we introduce the lemma to bridge the network function and CNTK. Neural Tangent Kernel (NTK) [Jacot et al., 2018; Allen-Zhu et al., 2019; Arora et al., 2019] usually is defined as the feature space formed by the gradient at random initialization. Therefore, given any two arms \( x_1, x_2 \), CNTK is defined as
\[ K_{\text{CNTK}}(x_1, x_2) = \langle g(x_1; \theta_0), g(x_2; \theta_0) \rangle. \]

Given an arm \( x \), the CNTK objective is defined as
\[ f_{\text{CNTK}}(x; \theta_t - \theta_0) = \langle g(x; \theta_0), \theta_t - \theta_0 \rangle. \]

[Jacot et al., 2018] prove that, for a fully-connected network, the dynamic NTKs are identical to NTK at initialization when \( m \) is infinite, because \( \lim_{m \to \infty} \| \theta_t - \theta \|_2 = 0 \). We present the following lemma to show that the same results hold for CNTK.

**Lemma 6.1.** In a round \( t \), with probability at least \( 1 - \delta \), if \( m \) satisfies Condition Eq. (1), we have
\[ \text{(1)} \| g(x; \theta_t) - g(x; \theta_0) \|_2 \leq \sqrt{q(L+1)\Psi_{L, (k')}((C_1 \mu)^L + 2)} \]
\[ = \sqrt{q(L+1)\Psi_{L, (k')}((C_1 \mu)^L + 2)} \leq m \sqrt{\Psi_{L, (k')}((C_1 \mu)^L + 2)} \]
\[ \text{(2)} \| g(x_1; \theta_t), g(x_2; \theta_t) \| - K_{\text{CNTK}}(x_1, x_2) \]
\[ \leq (L+1) \sqrt{q(C_1 \mu \sqrt{\tilde{S}})^L/m + \sqrt{q\Psi_{L, (k')}((C_1 \mu)^L + 2)}} \]
\[ \text{(3)} f(x; \theta_t) - f_{\text{CNTK}}(x; \theta_t - \theta_0) \]
\[ \leq \sqrt{\Psi_{L, (k')}((C_1 \mu)^L + 2)} \]
\[ \frac{(L+1)(2\sqrt{q(C_1 \mu \sqrt{\tilde{S}})^L/m + \sqrt{q\Psi_{L, (k')}((C_1 \mu)^L + 2)}} \sqrt{q\Psi_{L, (k')}((C_1 \mu)^L + 2)} + \sqrt{w((L+1)(C_1 \mu)^L + 1)}) / \sqrt{m}.} \]
The above lemma shows that \( f(x; \theta_t) \to f^\text{CNTK}(x; \theta_t - \theta_0) \) when \( m \to \infty \).

Next, we present the following lemma to connect CNTK with ridge regression. According to the linear bandit [Abbasi-Yadkori et al. 2011], the estimation for \( \theta \) by standard ridge regression is defined as

\[
\hat{\theta}_t = A_t^{-1}b_t \quad b_t = \sum_{i=1}^{t} r_i g(x_i; \theta_t) / \sqrt{m}.
\]

**Lemma 6.2.** In round \( t \), with probability at least \( 1 - \delta \), if \( m, \eta \) satisfy Condition Eq. [\ref{eq:cond}], we have

\[
\| \theta_t - \theta_0 - \hat{\theta}_t / \sqrt{m} \|_2 \leq \frac{A_1 + A_2 + A_3}{m\lambda} + \sqrt{\frac{t}{m\lambda}} + \frac{t(L+1)}{m} \left( 2\sqrt{\log(C_1 \mu \sqrt{q})} L / m + \sqrt{q} \Psi_{L,\lambda}(\eta) ((C_1 \mu)^L + 2) \right) \sqrt{q} \Psi_{L,\lambda}(\eta) ((C_1 \mu)^L + 2)
\]

Based on the above lemmas, we can bound \( |f^\text{CNTK}(x; \theta_t - \theta_0) - \langle g(x; \theta_0), \hat{\theta}_t / \sqrt{m} \rangle| \). With the above lemmas, we can easily prove Theorems 1 and 2.

**Proof of Theorem 1.** Given an arm \( x_t \), based on the Lemma [5.1] we have

\[
|f^* (x_t) - f(x_t; \theta_t)| \leq \left| \langle g(x_t; \theta_t) / \sqrt{m}, \sqrt{m} \theta_t \rangle - \langle g(x_t, \theta_0) / \sqrt{m}, \hat{\theta}_t \rangle \right| + \left| f(x_t; \theta_t) - \langle g(x_t; \theta_t) / \sqrt{m}, \hat{\theta}_t \rangle \right|
\]

Then, based on the Theorem 2 in [Abbasi-Yadkori et al. 2011], we have

\[
\left| \langle g(x_t; \theta_t) / \sqrt{m}, \sqrt{m} \theta_t \rangle - \langle g(x_t; \theta_0) / \sqrt{m}, \hat{\theta}_t \rangle \right| \leq \left( \sqrt{\log \frac{\det(A_T)}{\det(\lambda \Gamma)}} - 2 \log \delta + \lambda^{1/2} S \right) \| g(x_t; \theta_t) / \sqrt{m} \|_{A_t^{-1}} = \Psi_1 \| g(x_t; \theta_t) / \sqrt{m} \|_{A_t^{-1}}
\]

Next, we have

\[
\left| f(x_t; \theta_t) - \langle g(x_t; \theta_t) / \sqrt{m}, \hat{\theta}_t \rangle \right| \leq \| f(x_t; \theta_t) - \langle g(x_t; \theta_t), (\theta_t - \theta_0) \rangle \| + \left| \langle g(x_t; \theta_t), (\theta_t - \theta_0) \rangle - \langle g(x_t; \theta_0), (\theta_t - \theta_0) \rangle \right| + \left| \langle g(x_t; \theta_t), (\theta_t - \theta_0 - \hat{\theta}_t / \sqrt{m}) \rangle \right|
\]

By a variant of Lemma [6.1] we can prove \( I_1 \leq \Psi_3 \). For \( I_2 \), we have

\[
\left| \langle g(x_t; \theta_t), (\theta_t - \theta_0 - \hat{\theta}_t / \sqrt{m}) \rangle \right| \leq \left( \| g(x_t; \theta_0) \|_2 + \| g(x_t; \theta_t) - g(x_t; \theta_0) \|_2 \right) \| \theta_t - \theta_0 - \hat{\theta}_t / \sqrt{m} \|_2 \leq \Psi_2.
\]

Adding everything together, the proof is completed:

\[
|f^* (x_t) - f(x_t; \theta_t)| \leq \Psi_1 \| g(x_t; \theta_t) / \sqrt{m} \|_{A_t^{-1}} + \Psi_2 + \Psi_3.
\]

**Lemma 6.3.** With probability at least \( 1 - \delta \), if \( m, \eta \) satisfy Condition Eq. [\ref{eq:cond}], \( \log \left( \frac{A_T}{\Gamma} \right) \leq \bar{d} \log (1 + T / \lambda) + 1 \).

**Proof of Theorem 2** For a round \( t \), with probability at least \( 1 - \delta \), its regret is

\[
R_t = f^* (x_t^*) - f^* (x_t) \leq \| f^* (x_t^*) - f(x_t^*; \theta_t) \| + f(x_t^*; \theta_t) - f^* (x_t) \leq \text{UCB}(x_t^*) + f(x_t^*; \theta_t) - f^* (x_t) \leq \text{UCB}(x_t) + f(x_t^*; \theta_t) - f^* (x_t) \leq 2 \text{UCB}(x_t)
\]

where the third inequality is as the result of pulling criteria of CNN-UCB satisfying

\[
\text{UCB}(x_t^*) + f(x_t^*; \theta_t) \leq \text{UCB}(x_t) + f(x_t^*; \theta_t).
\]
Therefore, for $T$ rounds, we have

$$R_T = \sum_{t=1}^{T} R_t \leq 2 \sum_{t=1}^{T} \text{UCB}(x_t)$$

$$= 2 \sum_{t=1}^{T} \left( \Psi_1 \|g(x_t; \theta_t)\|_{\frac{1}{m} \Lambda_{t-1}} + \Psi_2 + \Psi_3 \right)$$

$$\leq 2 \Psi_1 \sqrt{T \sum_{t=1}^{T} \|g(x_t; \theta_t)\|_{\frac{1}{m} \Lambda_{t-1}} + 2 \sum_{t=1}^{T} \Psi_2 + 2 \sum_{t=1}^{T} \Psi_3}$$

where the last inequality is based on the Lemma 11 in [Abbasi-Yadkori et al., 2011]. For $I_2$, applying Lemma 6.3 we have

$$I_2 \leq \sqrt{2T \bar{d} \log(1 + T/\lambda)} + 1.$$ 

For $I_1$, applying Lemma 6.3 again, we have

$$2I_1 \leq 2 \left( \sqrt{d \log(1 + T/\lambda) + 2 \log(1/\delta)} + 1 + \sqrt{\bar{S}} \right).$$

For $I_3$ and $I_4$, as the choice of $m$, we have

$$2 \sum_{t=1}^{T} \Psi_2 \leq 1, \quad 2 \sum_{t=1}^{T} \Psi_3 \leq 1.$$ 

Therefore, adding everything together, we have

$$R_T \leq 2 \sqrt{2T \bar{d} \log(1 + T/\lambda) + 1} \left( \sqrt{d \log(1 + T/\lambda) + 2 \log(1/\delta)} + 1 + \sqrt{\bar{S}} \right) + 2.$$

### 7 Experiments

In this section, we evaluate the empirical performance of CNN-UCB compared with strong baselines on image data sets.

**Data sets.** We choose three well-known image data sets: Mnist [LeCun et al., 1998], NotMnist, and Cifar-10 [Krizhevsky et al., 2009]. All of them are 10-class classification data sets. Following the evaluation setting of existing works [Zhou et al., 2020; Valko et al., 2013a; Deshmukh et al., 2017], transform the classification into bandit problem. Consider an image $x \in \mathbb{R}^{c \times p}$, we aim to classify it from 10 classes. Then, in each round, 10 arms is presented to the learner, formed by 10 tensors in sequence $x_1 = (x, 0, \ldots, 0), x_2 = (0, x, \ldots, 0), \ldots, x_{10} = (0, 0, \ldots, x) \in \mathbb{R}^{10 \times c \times p}$, matching the 10 classes. The reward is defined as 1 if the index of selected arm equals the index of $x'$ ground-truth class; Otherwise, the reward is 0. For example, an image with number "6" belonging to the 7-th class on Mnist data set will be transformed into 10 arms in a round and the reward will be 1 if selecting the 7-th arm; Otherwise, the reward is 0.

**Baselines.** We choose three UCB-based competing algorithms. (1) NeuUCB [Zhou et al., 2020] uses a fully-connected neural network to learn reward function with the UCB exploration strategy. (2) KerUCB [Valko et al., 2013a] uses a predefined kernel matrix to learn reward function coming with a UCB exploration strategy. (4) LinUCB [Li et al., 2010] assumes the reward is a product of arm feature vector and an unknown parameter and then uses ridge regression to do the estimation and UCB-based exploration.

**Configuration.** We construct CNN-UCB by 3 convolutional layers combined with one fully-connected layers. For each convolutional layer, we set 20 channels and the kernel size is set as $4 \times 4$. For NeuUCB, we use a 4-layer fully-connected neural network and the width of each
layer is set as 100. For above two algorithms, we use the Sigmoid activation function and set $\lambda = 1, \eta = 0.001, \delta = 0.1$. The number of iterations $k$ is set as $t$ when $t \leq 200$ and $k = 100$ if $t > 200$. For KerUCB, we use the radial basis function kernel and stop adding contexts after 500 rounds, following setting of Gaussian process in [Zhou et al., 2020; Riquelme et al., 2018]. For LinUCB, we use the same $\delta, \lambda$ as above. All experiments are repeated 5 times and report the averaged results for comparison.

**Results.** Figure 1 shows the regret comparison for all algorithms on three data sets. CNN-UCB achieves the best performance across all data sets as expected, because CNN can capture the visual pattern of image to exploit the past knowledge and our derived UCB can explore these new arms to gain new information. Among baselines, NeuUCB obtains the lowest regret, outperforming LinUCB and KerUCB, thanks to the representation power of neural network. But it still is much worse than CNN-UCB. For instance, on Mnist data set, in the first 300 rounds, cumulative regret received by CNN-UCB and NeuUCB are very similar, as both of them are exploring to collect training samples. After around 300 rounds, the accuracy of CNN-UCB for choosing arm starts being much better than Neural UCB, because they have enough training samples and the superiority of CNN is reflected on classifying images. For KerUCB, it shows that a simple kernel like radial basis function has the limitation to learn complicated reward functions. Unfortunately, LinUCB obtains the worst performance because the linear-reward assumption and a large number of input dimensions make it hard to estimate the reward function accurately.

**8 Conclusion**

In this paper, we propose a contextual bandit algorithm, CNN-UCB, which uses the CNN to learn the reward function with an UCB-based exploration. We also achieve an near-optimal regret bound $O(\sqrt{T})$ built on the connections among network function, CNTK, and ridge regression. CNN-UCB has direct applications on visual-aware advertising and outperforms state-of-the-art baselines on real-world image data sets.
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Appendix

In the appendix, we provide the detail proofs of Lemma 6.1 and 6.2. For reproducibility, the code has been released[1]

Proof of Lemma 5.1. Given an arm $x_t$ and its ground-truth reward $f^*(x_t)$, combined with previous pair \{$(x_t;\theta_t)$\}, we define $G_t = [g(x_1;\theta_t),\ldots,g(x_t;\theta_t)] \in \mathbb{R}^{d \times t}$ and $\bar{f} = [f^*(x_1),\ldots,f^*(x_t)] \in \mathbb{R}^t$.

Suppose the single value decomposition of $G_t$ is $\Sigma \Sigma^\top$, where $\Sigma \in \mathbb{R}^{t \times t}$, and $\Sigma^\top \in \mathbb{R}^{t \times t}$. Thus, there exist $\theta^*_t = \theta(0) + U \Sigma^{-1} T^\top \bar{f}^*$ such that $G^*_t(\theta^*_t - \theta(0)) = T \Sigma U \Sigma^{-1} T^\top \bar{f}^* = \bar{f}^*$.

This indicates for any $x \in \{x_i\}_{i=1}^t$, $(g(x;\theta_t),\theta^*_t - \theta_0) = f^*(x)$. Next, easy to have $\|\theta^*_t - \theta_0\|_2^2 = f^* T \Sigma^{-1} U \Sigma^{-1} T^\top \bar{f}^* = f^* (G^*_t G_t)^{-1} f^*$

Suppose $G^*_t G_t \geq \lambda_1 I$, then we have $\|\theta^*_t - \theta_0\|_2^2 \leq \frac{1}{\lambda_1} f^* T \bar{f}^*$.

Prove of Lemma 6.3.

Define $G_t = [g(x_1;\theta_t),\ldots,g(x_t;\theta_t)] \in \mathbb{R}^{p \times t}$.

\[
\begin{align*}
\log \left( \frac{A_T}{\lambda I} \right) &= \log \det \left( I + \sum_{i=1}^T g(x_i;\theta_t)g(x_i;\theta_t)^\top / \lambda \right) \\
&= \log \det (I + G^*_t G_t / \lambda) \\
&= \log \det (I + G^*_0 G_0 / \lambda + (G^*_t G_t - G^*_0 G_0) / \lambda) \\
&\leq \log \det (I + G^*_0 G_0 / \lambda + (I + G^*_0 G_0 / \lambda)^{-1} (G^*_t G_t - G^*_0 G_0) / \lambda) \\
&\leq \log \det (I + G^*_0 G_0 / \lambda + \|G^*_t G_t - G^*_0 G_0\|_F / \lambda) \\
&\leq \log \det (I + G^*_0 G_0 / \lambda + \sqrt{\Psi G_0} / \lambda) \\
&\leq \lambda \log (1 + T \lambda) + 1
\end{align*}
\]

where the first inequality is due to the concavity of log det and the last inequality is because of the definition of $d$ and the choice of $m$. The third inequality is because $I + G^*_0 G_0 / \lambda \geq I$. The fourth inequality is because of Lemma[8.1]

Lemma 8.1. With probability at least $1 - \delta$, if $m, \eta$ satisfy Condition Eq. (7), we have

\[
\begin{align*}
(1) \|A_t\|_2 &\leq \lambda + T(L + 1) \left( \sqrt{p}((C_1 \mu \sqrt{\eta})^L / m + \sqrt{q} \Psi_{L,k'}((C_1 \mu)^L + 2))^2 / m \\
(2) \|G^*_t G_t - G^*_0 G_0\|_F &\leq \frac{\Psi G_0}{m} \\
&= \frac{T(L + 1)}{m} \left( 2 \sqrt{p}((C_1 \mu \sqrt{\eta})^L / m + \sqrt{q} \Psi_{L,k'}((C_1 \mu)^L + 2)) \sqrt{q} \Psi_{L,k'}((C_1 \mu)^L + 2) \right)
\end{align*}
\]

Proof 8.1. For (1) we have

\[
\begin{align*}
\|A_t\|_2 &= \|\lambda I + \sum_{i=1}^t g(x_i;\theta_t)g(x_i;\theta_t)^\top / \lambda \|_2 \\
&\leq \lambda + \sum_{i=1}^t \|g(x_i;\theta_t)\|_2^2 / \lambda \\
&\leq \lambda + t(L + 1) \left( \sqrt{p}((C_1 \mu \sqrt{\eta})^L / m + \sqrt{q} \Psi_{L,k'}((C_1 \mu)^L + 2))^2 / m \right)
\end{align*}
\]

[1]https://anonymous.4open.science/r/CNN-UCB-C208/
where the last inequality is because of Lemma 9.6. For (2) we have

$$
\|G^T G - G^T_0 G_0\|_F = \frac{1}{m} \sum_{i} \sum_{j} \langle g(x_i; \theta_i), g(x_j; \theta_i) \rangle - \langle g(x_i; \theta_0), g(x_j; \theta_0) \rangle)^2
$$

$$
\leq \frac{1}{m} \sum_{i} \sum_{j} \left( \|g(x_i; \theta_i)\|_2 + \|g(x_j; \theta_0)\|_2 \right)^2 \|g(x_j; \theta_i) - g(x_j; \theta_0)\|_2^2
$$

$$
\leq \frac{1}{m} \sum_{i} \sum_{j} \left( \sqrt{L + 1} \left( 2\sqrt{\Psi_{L,(k')}((C_1\mu)^L + 2)} \right) \right)^2 \|g(x_j; \theta_i) - g(x_j; \theta_0)\|_2^2
$$

$$
= \frac{T(L + 1)}{m} \left( 2\sqrt{\Psi_{L,(k')}((C_1\mu)^L + 2)} \right)^2 \right)^2 \|g(x_j; \theta_i) - g(x_j; \theta_0)\|_2^2
$$

where second inequality is based on Lemma 9.6. The proof is completed.

9 Proof of Lemma 6.1

By the definition of convolutional operation $\phi(\cdot)$, we have

$$
\|h^{l-1}\|_F \leq \|\phi(h^{l-1})\|_F \leq \sqrt{q} \|h^{l-1}\|_F
$$

$$
\|\phi(h^{l-1}) - \phi(h^{l-1})\|_F \leq \sqrt{q} \|h^{l-1} - h^{l-1}\|_F.
$$

For any matrix $h^{l-1} \in \mathbb{R}^{m \times p}$, for the sake of presentation, the operator $\phi(\cdot)$ is represented by

$$
\phi(h^{l-1}) = h^{l-1} \circ K \in \mathbb{R}^{mq \times p}.
$$

For each $l \in 1, \ldots, L$, the activation function $\sigma(\cdot)$ is represented by

$$
\sigma(W^l \phi(h^{l-1})) = D^l \circ (W^l \phi(h^{l-1}))
$$

where $D^l \in \mathbb{R}^{m \times p}$ and

$$
\begin{cases}
\sigma^l([h^l \phi(h^l)]_{0,0}), & \ldots, \\
\sigma^l([h^l \phi(h^l)]_{0,p-1}), & \ldots, \\
\sigma^l([h^l \phi(h^l)]_{m-1,0}), & \ldots, \\
\sigma^l([h^l \phi(h^l)]_{m-1,p-1})
\end{cases}
$$

Therefore, given the set $\{x_i\}_{i=1}^t$ and $\{r_i\}_{i=1}^t$, the formula of the gradient of CNN $f$ with respect to one layer $W^l$ is derived as:

$$
\frac{\partial L(\theta)}{\partial W^l} = \sum_{i=1}^{t} (f(x_i; \theta) - r_i) \phi(h^{l-1}) W^{L+1} \left( L_{j=l+1}^{L} \bigcirc D^j W^j (h^{l-1} \circ K) \right) \circ D^l.
$$

We omit the subscript $i$ for the brevity.

9.1 Lemmas

Lemma 9.0 [Theorem 7.1 in Du et al. (2019) In a round $t$, at $k$-th iteration, let $F^{(k)}_t = \left( f(x_1; \theta^{(k)}), \ldots, f(x_t; \theta^{(k)}) \right)$ and $R_t = (r_1, \ldots, r_t)^T$. There exists a constant $C_0$, such that for any $k' \in [k]$, it has

$$
\|R_t - F^{(k')}_t\|_2^2 \leq (1 - C_0 \eta)^k' \|R_t - F^{(0)}_t\|_2^2,
$$

where $C_0 \eta < 1$.

Therefore, we have

$$
\|R_t - F^{(k')}_t\|_2 \leq (1 - \lambda \eta)^{k/2} \|R_t - F^{(0)}_t\|_2 \leq \|R_t - F^{(0)}_t\|_2 \leq \sqrt{2t},
$$

where the last inequality is because $r_1 \leq 1, \forall t$ and the $f(x_i; \theta^{(0)}) \leq 1, \forall x_i$ by lemma 9.1 and choice of $m$.
Lemma 9.1. Over the randomness of $\theta_k$, with probability at least $1 - O(tL)e^{-\Omega(m)}$, there exist constants $1 < C_1, C_2 < 2$, such that

$$\forall l \in 1, \ldots, L, \|h_l^{(0)}\|_F \leq (C_1\mu)^L, \|W_l^{(0)}\|_2 \leq C_1\sqrt{m},$$

$$f(x; \theta^{(0)}) \leq \frac{(C_1\mu)^LC_2}{\sqrt{m}}, \|W^{L+1,(0)}\|_2 \leq C_2.$$  

Lemma 9.2. In a round $t$, at $k$-th iteration of gradient descent, assuming $h_l^{(k)} \leq (C_1\mu)^L + \Psi_{L,(k')}$ with probability at least $1 - O(tL)e^{-\Omega(m)}$, we have

$$\|W_l^{(k)} - W_l^{(0)}\|_F \leq \frac{w}{\sqrt{m}}$$

where

$$w = \frac{2t\sqrt{2}\mu^L C_1^2 (L-1)\sqrt{q} + C_2((C_1\mu)^L + \Psi_{L,(k')})}{C_0}.$$  

Lemma 9.3. In a round $t$, at $k$-th iteration of gradient descent, suppose $\|W_l^{(k)} - W_l^{(0)}\|_F \leq w/\sqrt{m}, \forall l \in [T]$, with probability at least $1 - O(tL)e^{-\Omega(m)}$, we have

$$\forall l \in [L], \|h_l^{(k)} - h_l^{(0)}\|_F \leq \frac{\mu w ((2\mu C_1\sqrt{q})^L - 1)}{m(2\mu C_1\sqrt{q} - 1)} = \Psi_{L,(k')}$$

$$\forall l \in [L], \|h_l^{(k)}\|_F \leq (C_1\mu)^L + \Psi_{L,(k')}.$$  

Definition 9.1. For any layer $1 \leq l \leq L - 1$, we define

$$\tilde{h}_l = m^{-\frac{1}{2}} qm - \frac{L-1}{\sum_{j=1}^L} W^{L+1, T} \left( \prod_{j=l+1}^L \circ D^j (W^j \circ K) \right) \circ D^l$$

Therefore, $\nabla_{W_l} f(x; \theta)$ can be represented by

$$\nabla_{W_l} f(x; \theta) = \phi(\tilde{h}_l^{-1}) \tilde{h}_l.$$  

Lemma 9.4. In a round $t$, at $k$-th iteration, with probability at least $1 - O(tL)e^{-\Omega(m)}$, for $l \in [L]$, we have

$$\|\tilde{h}_l^{(0)}\|_F \leq \frac{\mu \sqrt{p}}{m} (\mu C_1 \sqrt{q})^{L-1}$$

$$\|\tilde{h}_l^{(k)} - \tilde{h}_l^{(0)}\|_F \leq \frac{\mu w ((2\mu C_1 \sqrt{q})^L - 1)}{m(2\mu C_1 \sqrt{q} - 1)} = \Psi_{L,(k')}$$

$$\|\tilde{h}_l^{(k)}\|_F \leq \frac{\mu \sqrt{p}}{m} (\mu C_1 \sqrt{q})^{L-1} + \Psi_{L,(k')}.$$  

Lemma 9.5. In a round $t$, at $k$-th iteration, with probability at least $1 - O(tL)e^{-\Omega(m)}$, for $l \in [L]$, we have

$$\forall 1 \leq l \leq L + 1, \|\nabla_{W_l} f(x; \theta^{(0)})\|_F \leq \sqrt{p} (C_1\mu \sqrt{q})^L / m,$$

$$\|\nabla_{W_{l+1}} f(x; \theta^{(k)}) - \nabla_{W_{l+1}} f(x; \theta^{(0)})\|_F \leq \Psi_{L,(k')} \sqrt{q}$$

$$\forall l \in [L], \|\nabla_{W_l} f(x; \theta^{(k)}) - \nabla_{W_l} f(x; \theta^{(0)})\|_F \leq \Psi_{L,(k')} \sqrt{q} ((C_1\mu)^L + 2)$$  

Lemma 9.6. In a round $t$, at $k$-th iteration of gradient descent, with probability at least $1 - O(tL)e^{-\Omega(m)}$, we have

$$(1) \|g(x; \theta^{(0)})\|_2 \leq \sqrt{(L + 1)} \sqrt{p} (C_1\mu \sqrt{q})^L / m$$

$$(2) \|g(x; \theta^{(k)}) - g(x; \theta^{(0)})\|_2 \leq \sqrt{q(L + 1)} \Psi_{L,(k')} ((C_1\mu)^L + 2)$$

$$(3) \|g(x; \theta^{(k)})\|_2 \leq \sqrt{L + 1} \left( \sqrt{p} (C_1\mu \sqrt{q})^L / m + \sqrt{q} \Psi_{L,(k')} ((C_1\mu)^L + 2) \right)$$

Lemma 9.7. In a round $t$, at $k$-th iteration, with probability at least $1 - O(tL)e^{-\Omega(m)}$, we have

$$|f(x; \theta^{(k)}) - \left<g(x; \theta^{(0)}), \theta^{(k)} - \theta^{(0)}\right>|$$

$$\leq \left< \Psi_{L,(k')} (C_2 + 1) + (C_1\mu)^L C_2 + \sqrt{q} w((L - 1)(C_1\mu)^L + 1) \right> / \sqrt{m}.$$
Lemma 9.8. In a round $t$, at $k$-th, iteration, with probability at least $1 - O(tL)e^{-\Omega(m)}$, we have

$$|f(x; \theta^{(k)}) - g(x; \theta^{(k)}, \theta^{(k)} - \theta^{(0)})| \leq \left\{ C_2(\Psi_{L,(k')} + (C_1\mu)^L) + \sqrt{q}(1 + \Psi_{L,(k')})w \left[(L - 1)(\Psi_{L,(k')} + (C_1\mu)^L) + 1\right]\right\} / \sqrt{m}$$

Lemma 9.6 and 9.7 equals the Lemma 6.1 and thus the proof is completed. Lemma 9.8 is the variant of Lemma 6.1 used in the proof of Theorem 1.

9.2 Proofs

Prove 9.1. According to Vershynin [2010] and Lemma G.2 in Du et al. [2019], with probability at least $1 - e^{-\left(c' - \sqrt{\frac{q}{2}}\right)^2 m}$, there exists a constant $c'$, for $W^l \in \mathbb{R}^{m \times q m}$, such that

$$\|W^l\|_2 \leq c' \sqrt{m}$$

where $c' > \sqrt{q} + 1$. Thus, we can derive

$$\|W^l\|_2 \leq C_1 \sqrt{q m}, \forall l \in [L]$$

with probability at least $1 - O(L)e^{-\Omega(m)}$ and $1 < C_1 < 2$. For $W^{L+1} \in \mathbb{R}^{m \times P}$, applying Lemma in Vershynin [2010] again, with probability at least $1 - e^{-\left(c' - \sqrt{\frac{q}{2}}\right)^2 m}$ we have

$$\sqrt{m}\|W^l\|_2 \leq C_2 \sqrt{m},$$

where $1 < C_2 < 2$ because $m > p$.

For $l \in [L]$, we have

$$h^l = \frac{1}{\sqrt{q m}} \sigma(W^l \phi (h^{l-1})) \leq \frac{\mu}{\sqrt{q m}}\|W^l\|_2\|\phi (h^{l-1})\|_F$$

$$\leq C_1\mu \|h^{l-1}\|_F$$

where $\phi$ is a linear operator. Thus, we have

$$h^l \leq \frac{\mu}{\sqrt{q m}}\|W^l\|_2\|\phi (h^{l-1})\|_F \leq C_1\mu \|x\|_F \leq C_1\mu$$

Therefore, we have $h^l \leq (C_1\mu)^L$. Then apply the union bound with $L, t$.

Prove 9.2. The induction hypothesis is $\|W^{l,(k)} - W^{l,(0)}\|_F \leq w/\sqrt{m}$. To bound the gradient of one lay, we need the following claims:

$$\prod_{j=l+1}^{L} \|D^{j,(k)} \circ (W^{j,(k)} \circ K)\|_2$$

$$\leq \prod_{j=l+1}^{L} \mu\|W^{j,(k)} \circ K\|_2 \leq \prod_{j=l+1}^{L} \mu\sqrt{q}\|W^{j,(k)}\|_2$$

$$\leq \prod_{j=l+1}^{L} \mu\sqrt{q} \left(\|W^{j,(0)}\|_2 + \|W^{j,(k)} - W^{j,(0)}\|_F\right)$$

$$\leq \mu^{L-l} \frac{1}{L-q + w/m} \left((C_1\sqrt{q} + w/m)^{L-l} m^{-l+1}\right)$$

In $(k+1)$-th iteration, for $1 \leq l \leq L$, we have
Thus, for $1 \leq l \leq L$, we have

$$\|W^{l,(k+1)} - W^{l,(k)}\|_F \leq \sum_{i=0}^{k} (1 - C_0\eta)^{k/2} m^{-1} \eta \sqrt{t} \|R_t - F_t^{(0)}\|_{2} \|h^{L,(k)}\|_F$$

$$\leq \frac{2}{C_0} m^{-1} \sqrt{t} \|R_t - F_t^{(0)}\|_{2} \|h^{L,(k)}\|_F \leq \frac{w}{\sqrt{m}}.$$

For the layer $L + 1$, we have,

$$\|W^{L+1,(k+1)} - W^{L+1,(0)}\|_F$$

$$\leq \|W^{L+1,(k+1)} - W^{L+1,(k)}\|_F + \|W^{L+1,(k)} - W^{L+1,(0)}\|_F$$

$$\leq \sum_{i=0}^{k} (1 - C_0\eta)^{k/2} m^{-1} \eta \sqrt{t} \|R_t - F_t^{(0)}\|_{2} \|h^{L,(k)}\|_F$$

$$\leq \frac{2}{C_0} m^{-1} \sqrt{t} \|R_t - F_t^{(0)}\|_{2} \|h^{L,(k)}\|_F \leq \frac{w}{\sqrt{m}}.$$

The proof is completed.
We prove this lemma by induction. The induction hypothesis is \( \|h_l^{(k)} - h_l^{(0)}\|_F \leq \frac{\mu w}{m} g(l) \).

\[
\|h_l^{(k)} - h_l^{(0)}\|_F = \frac{1}{\sqrt{qm}} \left\| D^l \odot \left( W_l^{(k)} \phi \left( h_l^{(1,k)} \right) \right) - D^l \odot \left( W_l^{(0)} \phi \left( h_l^{(1,0)} \right) \right) \right\|_F \\
\leq \frac{1}{\sqrt{qm}} \mu \left( \| W_l^{(k)} \phi \left( h_l^{(1,k)} \right) - W_l^{(0)} \phi \left( h_l^{(1,0)} \right) \|_F + \| W_l^{(k)} \phi \left( h_l^{(1,0)} \right) - W_l^{(0)} \phi \left( h_l^{(1,0)} \right) \|_F \right) \\
= \frac{1}{\sqrt{qm}} \| W_l^{(k)} \left( \phi \left( h_l^{(1,k)} \right) - \phi \left( h_l^{(1,0)} \right) \right) \|_F + \frac{1}{\sqrt{qm}} \mu \| \left( W_l^{(k)} - W_l^{(0)} \right) \phi \left( h_l^{(1,0)} \right) \|_F \\
\leq \frac{1}{\sqrt{m}} \mu \left( \| W_l^{(0)} \|_2 + \| W_l^{(k)} - W_l^{(0)} \|_F \right) \cdot \| h_l^{(1,k)} - h_l^{(1,0)} \|_F \\
+ \frac{1}{\sqrt{m}} \mu \| h_l^{(1,0)} \|_F \| W_l^{(k)} - W_l^{(0)} \|_F \\
\leq \mu \left( C_1 \sqrt{q + w/m} \| h_l^{(1,k)} - h_l^{(1,0)} \|_F + \mu w/m \right) \\
\leq \mu \left( C_1 \sqrt{q + w/m} \mu w/m g(l - 1) + \mu w/m \right) \\
= \mu w/m \left( C_1 \sqrt{q} g(l - 1) + \mu w/m g(l - 1) + 1 \right) \\
\leq \mu w/m \left( 2 \mu C_1 \sqrt{q} g(l - 1) + 1 \right) \\
= \mu w/m \left( 2 \mu C_1 \sqrt{q} g(1) \right) \\
= \mu w/m \left( 2 \mu C_1 \sqrt{q} g(1) \right)
\]

Then, for \( l = 1 \), we have

\[
\| h_1^{(k)} - h_1^{(0)} \|_F = \frac{\mu}{\sqrt{qm}} \left\| W_1^{(k)} \phi (x) - W_1^{(0)} \phi (x) \right\|_F \\
\leq \frac{\mu}{\sqrt{m}} \left( \| W_1^{(k)} - W_1^{(0)} \|_F \| x \|_F \right) \\
\leq \frac{\mu w}{m} \mu w/m g(1),
\]

where \( g(1) = 1 \). By calculation, we have

\[
\forall l \in [L], \| h_l^{(k)} - h_l^{(0)} \|_F \leq \frac{\mu w \left( 2 \mu C_1 \sqrt{q} \right) L - 1}{m(2 \mu C_1 \sqrt{q} - 1)} = \Psi_{L,(k')}
\]

\[
\forall l \in [L], \| h_l^{(k)} \|_F \leq \| h_l^{(0)} \|_F + \| h_l^{(k)} - h_l^{(0)} \|_F \leq \mu L + \Psi_{L,(k')}
\]

The proof is completed.

**Prove 9.4.** Define \( D^l = D^l - D^l \) To show the results, we need the following claims.

\[
\| W_l^{(1,k)} \odot K \odot D^l - W_l^{(1,0)} \odot K \odot D^l \|_F \\
\leq \mu \| W_l^{(1,k)} \odot K - W_l^{(1,0)} \odot K \|_F \\
\leq \mu \sqrt{q} \| W_l^{(1,k)} - W_l^{(1,0)} \|_F \\
\leq \frac{\mu w}{m} \sqrt{qm}
\]
Then, we have
\[
\| \hat{h}^{L,(k)} - \tilde{h}^{L,(0)} \|_F
= m^{-1/2} (qm)^{-L+1/2} \| W^{L+1,(k)} \left( \prod_{j=l+1}^{L} \circ D^{j,(k)} W^{j,(0)} \circ K \right) D^l \]
\[
- W^{L+1,(0)} \left( \prod_{j=l+1}^{L} \circ D^{j,(k)} W^{j,(0)} \circ K \right) D^l \|_F
\leq \frac{1}{\sqrt{qm}} \| \hat{h}^{L+1,(k)} W^{l+1,(k)} \circ K \circ D^l - \tilde{h}^{L+1,(0)} W^{l+1,(0)} \circ K \circ D^l \|_F
\]
\[
+ \frac{1}{\sqrt{qm}} \| \tilde{h}^{L+1,(0)} W^{l+1,(0)} \circ K \circ D^l - \tilde{h}^{L+1,(0)} W^{l+1,(0)} \circ K \circ D^l \|_F
\leq \frac{\mu}{\sqrt{qm}} \| \hat{h}^{L+1,(k)} - \tilde{h}^{L+1,(0)} \|_F \| W^{l+1,(k)} \circ K \|_2
\]
\[
+ \frac{1}{\sqrt{qm}} \| \tilde{h}^{L+1,(0)} \|_F \| W^{l+1,(0)} \circ K \circ D^l - W^{l+1,(0)} \circ K \circ D^l \|_2
\]
\[
\leq \mu (C_1 \sqrt{q} + w/m) \| \hat{h}^{L+1,(k)} - \tilde{h}^{L+1,(0)} \|_F + \mu w/m
\]
\[
\leq \mu(C_1 \sqrt{q} + w/m) \frac{LW}{m} g(l + 1) + \mu w/m
\]
\[
\leq \frac{\mu w}{m} (2\mu C_1 \sqrt{q}g(l + 1) + 1)
\]
\[
\leq \frac{\mu w}{m} g(L)
\]

For the \( L \)-th layer, we have
\[
\| \hat{h}^{L,(k)} - \tilde{h}^{L,(0)} \|_F
\leq m^{-1/2} \frac{1}{\sqrt{qm}} \| W^{L+1,(k)} \circ D^l - W^{L+1,(0)} \circ D^l \|_F
\]
\[
\leq m^{-1/2} \frac{\mu}{\sqrt{qm}} \| W^{L+1,(k)} - W^{L+1,(0)} \|_F
\]
\[
\leq \frac{\mu w}{m} \frac{1}{\sqrt{qm}} \leq \frac{\mu w}{m} g(L)
\]

where \( g(L) = 1 \). Therefore, by calculation, we have
\[
\| \hat{h}^{l,(k)} - \tilde{h}^{l,(0)} \|_F \leq \frac{\mu w}{m} \left( \frac{(2\mu C_1 \sqrt{q})^L}{2} - 1 \right) = \Psi_{L,(k')}
\]

For \( \tilde{h}^{l,(0)} \), we have
\[
\| \tilde{h}^{l,(0)} \|_F \leq m^{-1/2} qm^{-L+1/2} \| W^{l+1,(0)} \left( \prod_{j=l+1}^{L} \circ D^{j,(0)} W^{j,(0)} \circ K \right) \circ D^l \|_F
\]
\[
\leq m^{-1/2} qm^{-L+1/2} \mu \| W^{l+1,(0)} \left( \prod_{j=l+1}^{L} \circ D^{j,(0)} W^{j,(0)} \circ K \right) \circ D^{l+1,(0)} \|_F \| W^{l+1,(0)} \circ K \|_2
\]
\[
\leq \mu C_1 \sqrt{q} \| \tilde{h}^{l+1,(0)} \|_F
\]

For the layer \( L \), we have
\[
\| \tilde{h}^{L,(0)} \|_F = \frac{1}{\sqrt{qm}} \| W^{L+1,(0)} \circ D^L \|_F \leq \frac{\mu w}{\sqrt{qm}}.
\]
By calculation, we have
\[ \| \tilde{h}_l^{(0)} \|_F \leq \frac{\mu \sqrt{p}}{m} (\mu C_1 \sqrt{q})^{L-1} \leq 1. \]

The proof is completed.

**Prove 9.5.** By Lemma 9.1, 9.3, 9.4 for \( l \in [L] \), we have
\[ \| \nabla_{\mathbf{W}_l} f(\mathbf{x}; \theta^{(0)}) \|_F = \| \tilde{h}_l^{(0)} \phi (h_l^{(1,0)}) \|_F \leq \sqrt{q} \| \tilde{h}_l^{(0)} \|_F \| h_l^{(1,0)} \|_F \leq \sqrt{p} (C_1 \mu \sqrt{q})^{L}/m. \]

For \( L + 1 \) layer, we have \( \| \nabla_{\mathbf{W}_{L+1}} f(\mathbf{x}; \theta^{(0)}) \|_F = \| h_{L+1}^{(0)} \|_F \leq 1. \)

\[ \| \nabla_{\mathbf{W}_l} f(\mathbf{x}; \theta^{(k)}) - \nabla_{\mathbf{W}_l} f(\mathbf{x}; \theta^{(0)}) \|_F = \| \tilde{h}_l^{(0)} \phi (h_l^{(1,k)}) - \tilde{h}_l^{(0)} \phi (h_l^{(1,0)}) \|_F \leq \left( \sqrt{q} \Psi_{L,(k')}(C_1 \mu)^L + \Psi_{L,(k')} \right) \Psi_{L,(k') \sqrt{q}}. \]

This proof is completed.

**Prove 9.6.** For (1), we have
\[ \| g(\mathbf{x}; \theta^{(0)}) \|_2 = \left( \sum_{l=1}^{L+1} \| \nabla_{\mathbf{W}_l} f(\mathbf{x}; \theta^{(0)}) \|_F^2 \right)^{1/2} \leq \sqrt{(L+1)} \sqrt{p} (C_1 \mu \sqrt{q})^{L}/m. \]

For (2), we have
\[ \| g(\mathbf{x}; \theta^{(k)}) - g(\mathbf{x}; \theta^{(0)}) \|_2 = \left( \sum_{l=1}^{L+1} \| \nabla_{\mathbf{W}_l} f(\mathbf{x}; \theta^{(k)}) - \nabla_{\mathbf{W}_l} f(\mathbf{x}; \theta^{(0)}) \|_F^2 \right)^{1/2} \leq \sqrt{q(L+1)} \Psi_{L,(k')}(C_1 \mu)^{L+2}. \]

For (3), we have
\[ \| g(\mathbf{x}; \theta^{(k)}) \|_2 \leq \sqrt{L+1} \left( \sqrt{p} (C_1 \mu \sqrt{q})^L/m + \sqrt{q} \Psi_{L,(k')}(C_1 \mu)^{L+2} \right). \]

With Lemma 9.5, the proof is completed.

**Prove 9.7.** By Lemma 9.1, 9.2, 9.3, 9.4 we have

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\[ |f(x; \theta^{(k)}) - \left \langle g(x; \theta^{(0)}), \theta^{(k)} - \theta^{(0)} \right \rangle | = | \langle h^{L,(k)}, W^{L+1,(k)} \rangle / \sqrt{m} - \langle h^{L,(0)}, (W^{L+1,(k)} - W^{L+1,(0)}) \rangle / \sqrt{m} \]
\[ - \sum_{l=1}^{L} \tilde{h}_{l}^{(k)}(W_{l}^{(k)} - W_{l}^{(0)}) \phi \left( h_{l-1}^{(0)} \right) | \]
\[ = \left | \langle h^{L,(k)} - h^{L,(0)}, W^{L+1,(k)} \rangle / \sqrt{m} - \langle h^{L,(0)}, W^{L+1,(0)} \rangle / \sqrt{m} - \sum_{l=1}^{L} \tilde{h}_{l}^{(0)}(W_{l}^{(k)} - W_{l}^{(0)}) \phi \left( h_{l-1}^{(0)} \right) \right | \]
\[ \leq \| h^{L,(k)} - h^{L,(0)} \|_{F} \| W^{L+1,(k)} \|_{2} / \sqrt{m} + \| h^{L,(0)} \|_{F} \| W^{L+1,(0)} \|_{2} / \sqrt{m} \]
\[ + \sum_{l=1}^{L} \| \tilde{h}_{l}^{(0)} \|_{F} \| W_{l}^{(k)} - W_{l}^{(0)} \|_{F} \phi \left( h_{l-1}^{(0)} \right) \|_{F} \]
\[ \leq \Psi_{L,(k')} (C_{2} + 1) / \sqrt{m} + ((C_{1}\mu)^{L} C_{2}) / \sqrt{m} + \sqrt{q} w((L - 1)(C_{1}\mu)^{L} + 1) / \sqrt{m} \]
\[ = \left \{ \Psi_{L,(k')} (C_{2} + 1) + (C_{1}\mu)^{L} C_{2} + \sqrt{q} w((L - 1)(C_{1}\mu)^{L} + 1) \right \} / \sqrt{m}. \]

The proof is completed.

**Prove 9.8.** By Lemma 9.1, 9.2, 9.3, 9.4 we have

\[ |f(x; \theta^{(k)}) - \left \langle g(x; \theta^{(0)}), \theta^{(k)} - \theta^{(0)} \right \rangle | \]
\[ = | \langle h^{L+1,(k)}, W^{L+1,(k)} \rangle / \sqrt{m} - \langle h^{L+1,(0)}, (W^{L+1,(k)} - W^{L+1,(0)}) \rangle / \sqrt{m} \]
\[ + \sum_{l=1}^{L} \tilde{h}_{l}^{(k)}(W_{l}^{(k)} - W_{l}^{(0)}) \phi \left( h_{l-1}^{(0)} \right) | \]
\[ = \left | \langle W^{L+1,(0)}, h^{L,(k)} \rangle / \sqrt{m} - \sum_{l=1}^{L} \tilde{h}_{l}^{(k)}(W_{l}^{(k)} - W_{l}^{(0)}) \phi \left( h_{l-1}^{(0)} \right) \right | \]
\[ \leq \| W^{L+1,(0)} \|_{2} \| h^{L,(k)} \|_{F} / \sqrt{m} + \sum_{l=1}^{L} \| \tilde{h}_{l}^{(k)} \|_{F} \| W_{l}^{(k)} - W_{l}^{(0)} \|_{F} \sqrt{q} \| h_{l-1}^{(0)} \|_{F} \]
\[ \leq C_{2} (\Psi_{L,(k')} + (C_{1}\mu)^{L}) / \sqrt{m} + \sqrt{q} (1 + \Psi_{L,(k')}) w \left \{ (L - 1) (\Psi_{L,(k')} + (C_{1}\mu)^{L}) + 1 \right \} / \sqrt{m} \]
\[ = \left \{ C_{2} (\Psi_{L,(k')} + (C_{1}\mu)^{L}) + \sqrt{q} (1 + \Psi_{L,(k')}) w \left \{ (L - 1) (\Psi_{L,(k')} + (C_{1}\mu)^{L}) + 1 \right \} \right \} / \sqrt{m}. \]

The proof is completed.

10 Proof of Lemma 6.2

**Definition 10.1.** Given the context vectors \( \{x_{i}\}_{i=1}^{t} \) and the rewards \( \{r_{i}\}_{i=1}^{t} \), then we define the estimation \( \hat{\theta}_{0} \) via ridge regression:

\[ A_{0} = \lambda I + \sum_{i=1}^{t} g(x_{i}; \theta^{(0)}) g(x_{i}; \theta^{(0)})^{T} / m \]
\[ b_{0} = \sum_{i=1}^{t} r_{i} g(x_{i}; \theta^{(0)}) / \sqrt{m} \]
\[ \hat{\theta}_{0} = A_{0}^{-1} b_{0} \]
Definition 10.2.

\[ G^{(k)} = \left(g(x_1; \theta^{(k)}), \ldots, g(x_t; \theta^{(k)})\right) \]
\[ G^{(0)} = \left(g(x_1; \theta^{(0)}), \ldots, g(x_t; \theta^{(0)})\right) \]
\[ f^{(k)} = \left(f(x_1; \theta^{(k)}), \ldots, f(x_t; \theta^{(k)})\right)^T \]
\[ r = (r_1, \ldots, r_t)^T \]
\[ \theta^{(k+1)} = \theta^{(k)} - \eta \left[ G^{(k)}(f^{(k)} - r) \right] \]

Inspired by Lemma B.2 in [Zhou et al. 2020], we define the auxiliary sequence following:

\[ \bar{\theta}^{(0)} = \theta^{(0)}, \quad \bar{\theta}^{(k+1)} = \bar{\theta}^{(k)} - \eta \left[ G^{(0)} \left( [G^{(0)}]^T (\theta^{(k)} - \bar{\theta}^{(0)}) - r \right) + m\lambda (\bar{\theta}^{(k)} - \bar{\theta}^{(0)}) \right] \]

Lemma 10.1. In a round \( t \), at \( k \)-th iteration of gradient descent, with probability at least \( 1 - O(pL)e^{-1/m} \), we have

\[ (1) \| G^{(0)} \|_F \leq \sqrt{t(L + 1)\sqrt{p}(C_1\mu\sqrt{q})^L/m} \]
\[ (2) \| G^{(k)} - G^{(0)} \|_F \leq \sqrt{tq(L + 1)\Psi_{L,(k')}(C_1\mu)^L + 2} \]
\[ (2) \| f^k - G^{(0)}^T (\theta^{(k)} - \theta^{(0)}) \|_2 \leq \sqrt{t\Psi_{L,(k')}(C_2 + 1) + (C_1\mu)^L C_2 + \sqrt{q}w((L - 1)(C_1\mu)^L + 1) / \sqrt{m}}. \]

Lemma 10.2. In a round \( t \), at \( k \)-th iteration of gradient descent, with probability at least \( 1 - O(pL)e^{-1/m} \), we have

\[ (1) \| \bar{\theta}^{(k)} - \theta^{(0)} - \bar{\theta}^{(0)} / \sqrt{m} \|_2 \leq \sqrt{t} \]
\[ (2) \| \bar{\theta} - \bar{\theta}^{(0)} \|_F \leq \frac{t(L + 1)}{m} \left( 2\sqrt{p(C_1\mu\sqrt{q})^L} + \sqrt{q}\Psi_{L,(k')}(C_1\mu)^L + 2 \right) \sqrt{\Psi_{L,(k')}(C_1\mu)^L + 2}. \]

Proof of Lemma 6.2

\[ \| \theta^{(k+1)} - \theta^{(k+1)} \|_2 = \| \theta^{(k)} - \theta^{(k)} - \eta \left[ G^{(0)} \left( [G^{(0)}]^T (\theta^{(k)} - \theta^{(k)} - \bar{\theta}^{(k)} - \bar{\theta}^{(0)}) - r \right) + m\lambda (\theta^{(k)} - \theta^{(k)} - \bar{\theta}^{(k)} - \bar{\theta}^{(0)}) \right] \]
\[ + \eta \left[ G^{(k)}(f^{(k)} - r) \right] \| \]
\[ \leq \| (1 - \eta m\lambda) (\theta^{(k)} - \bar{\theta}^{(k)}) + \eta(G^{(k)} - G^{(0)})(f^{(k)} - r) - \eta G^{(0)} \left( f^{(k)} - [G^{(0)}]^T (\theta^{(k)} - \theta^{(0)}) \right) \]
\[ - \eta G^{(0)} G^{(0)}^T (\theta^{(k)} - \bar{\theta}^{(k)}) \|_2 \]
\[ \leq \eta \| G^{(k)} - G^{(0)}(f^{(k)} - r) \|_2 + \eta \| G^{(0)} \|_2 \| f^{(k)} - [G^{(0)}]^T (\theta^{(k)} - \theta^{(0)}) \|_2 \]
\[ + \eta \lambda \| \theta^{(k)} - \theta^{(0)} \|_2 + \left\| I - \eta(m\lambda I + G^{(0)} G^{(0)}^T) \right\|_2 \| \theta^{(k)} - \bar{\theta}^{(k)} \|_2 \]
\[ = A_1 + A_2 + A_3 + A_4 \]

For \( A_1 \), by Lemma [10.1] we have

\[ A_1 \leq \eta \| G^{(k)} - G^{(0)} \|_2 \| f^{(k)} - r \|_2 \leq \eta \sqrt{tq(L + 1)\Psi_{L,(k')}(C_1\mu)^L + 2} \sqrt{2t} = \eta \tilde{A}_1 \]

For \( A_2 \), by Lemma [10.1] we have

\[ A_2 \leq \eta \sqrt{(L + 1)\sqrt{p}(C_1\mu\sqrt{q})^L} \cdot \sqrt{t} \left( \Psi_{L,(k')}(C_2 + 1) + (C_1\mu)^L C_2 + \sqrt{q}w((L - 1)(C_1\mu)^L + 1) \right) m^{-3/2} = \eta \tilde{A}_2. \]

For \( A_3 \), by Lemma [10.1] and Lemma [9.7] we have

\[ A_3 \leq (1 - \eta m\lambda) \| \theta^{(k)} - \bar{\theta}^{(k)} \|_2 \| \tilde{A}_1 \| \| \tilde{A}_2 \| \]
because $I - \eta(m\lambda I + G^{(0)}G^{(0)\top}) \preceq I - \eta(m\lambda I + \left(\sqrt{t(L+1)}\sqrt{p(C_1\mu\sqrt{q})^L/m}\right)^2) \preceq I$ with the choice of $m$ and $\eta$.

For $A_3$, by Lemma 9.2 we have

$$\eta\lambda\|\theta^{(k)} - \theta^{(0)}\|_2 = \eta\lambda \sum_{l=1}^{L+1} \|W_l^{(k)} - W_l^{(0)}\|_2^2 = \eta\lambda\sqrt{L} + \text{I} w/\sqrt{m} = \eta A_3.$$  

Therefore, adding everything together, we have

$$\|\theta^{(k+1)} - \tilde{\theta}^{(k+1)}\|_2 \leq (1 - \eta m\lambda)\|\theta^{(k)} - \tilde{\theta}^{(k)}\|_2 + \eta A_1 + \eta A_2 + \eta A_3.$$  

As $\|\theta^{(0)} - \tilde{\theta}^{(0)}\| = 0$, by induction, we have

$$\|\theta^{(k)} - \tilde{\theta}^{(k)}\|_2 \leq \frac{\eta A_1 + \eta A_2 + \eta A_3}{\eta m\lambda} \leq \frac{A_1 + A_2 + A_3}{m\lambda}.$$  

By Lemma 10.2 we have

$$\|\theta^{(k)} - \theta^{(0)} - \tilde{\theta}/\sqrt{m}\|_2 = \|\theta^{(k)} - \tilde{\theta}^{(k)} - \theta^{(0)}\| \leq \frac{\sqrt{t(L+1)\sqrt{p(C_1\mu\sqrt{q})^L/m}} + \sqrt{t\Psi L,(k')}((C_1\mu)^L+2)}{\sqrt{m}}.$$  

The proof is completed.

### 10.1 Proofs

**Proof 10.1.** By Lemma 9.5, 9.7 we have

For (1), we have

$$\|G^{(0)}\|_F \leq \sqrt{\sum_{i=1}^{t} \|g(x_i; \theta^{(0)})\|_2^2} \leq \sqrt{t(L+1)\sqrt{p(C_1\mu\sqrt{q})^L/m}}.$$  

For (2), we have

$$\|G^{(k)} - G^{(0)}\|_F = \sqrt{\sum_{i=1}^{t} \|g(x_i; \theta^{(k)}) - g(x_i; \theta^{(0)})\|_2^2} \leq \sqrt{tq(L+1)\Psi L,(k')}((C_1\mu)^L+2).$$  

For (3), we have

$$\|f^k - G^{(0)\top}(\theta^{(k)} - \theta^{(0)})\|_2 = \sqrt{\sum_{i=1}^{t} \|f(x_i; \theta^{(k)}) - \langle g(x_i; \theta^{(0)), \theta^{(k)} - \theta^{(0)}\rangle\|_2^2} \leq \sqrt{t} (\Psi L,(k')(C_2+1) + (C_1\mu)^L C_2 + \sqrt{q}(L+1)(C_1\mu)^L + 1)) / \sqrt{m}.$$  

**Proof 10.2.** The sequence of $\tilde{\theta}^{(k)}$ is updates by using gradient descent on the loss function:

$$\min_{\tilde{\theta}} L(\tilde{\theta}) = \frac{1}{2}\|G^{(0)}\|_F (\tilde{\theta} - \theta^{(0)}) - r\|^2_2 + \frac{m\lambda}{2} \|\tilde{\theta} - \theta^{(0)}\|^2_2.$$  

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By standard results of gradient descent on ridge regression, $\tilde{\theta}^{(k)}$ converges to $\theta^{(0)} + \hat{\theta}_0 / \sqrt{m}$. Therefore, we have
\[
\|\tilde{\theta}^{(k)} - \theta^{(0)} - \hat{\theta}_0 / \sqrt{m}\|_2^2 \leq \frac{1}{m\lambda} \left( L(\theta^{(0)}) - L(\theta^{(0)} + \hat{\theta}_0 / \sqrt{m}) \right)
\]
\[
\leq \frac{2(1 - \eta m\lambda)^k}{m\lambda} L(\theta^{(0)})
\]
\[
= \frac{2(1 - \eta m\lambda)^k}{m\lambda} \|\hat{r}\|_2^2
\]
\[
\leq \frac{t(1 - \eta m\lambda)^k}{m\lambda} \leq \frac{t}{m\lambda}
\]

For (2), by Lemma 10.1, we have
\[
\|\hat{\theta}_t - \hat{\theta}_0\|_F = \sum_{i=1}^{t} \|g(x_i; \theta_t)g(x_i; \theta_t)^T - g(x_i; \theta_0)g(x_i; \theta_0)^T\|_F / m
\]
\[
\leq \sum_{i=1}^{t} \|g(x_i; \theta_t)g(x_i; \theta_t)^T - g(x_i; \theta_0)g(x_i; \theta_0)^T\|_F / m
\]
\[
+ \sum_{i=1}^{t} \|g(x_i; \theta_0)g(x_i; \theta_t)^T - g(x_i; \theta_0)g(x_i; \theta_0)^T\|_F / m
\]
\[
\leq \sum_{i=1}^{t} (\|g(x_i; \theta_t)\|_2 + \|g(x_i; \theta_0)\|_2) \|g(x_i; \theta_t) - g(x_i; \theta_0)\|_2 / m
\]
\[
\leq \sum_{i=1}^{t} (2\|g(x_i; \theta_t)\|_2 + \|g(x_i; \theta_t) - g(x_i; \theta_0)\|_2) \|g(x_i; \theta_t) - g(x_i; \theta_0)\|_2
\]
\[
\leq \frac{t(L + 1)}{m} \left( 2\sqrt{p(C_1\mu \sqrt{q})^L / m + \sqrt{\Psi L_{, (k')((C_1\mu)^L + 2)}} \sqrt{\Psi L_{, (k')((C_1\mu)^L + 2)}} \right)
\]

The proof is completed.