Quartification On An Orbifold

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We investigate quartification models in five dimensions, with the fifth dimension forming an $S^1/Z_2 \times Z_2'$ orbifold. The orbifold construction is combined with a boundary Higgs sector to break the quartified gauge group directly to a group $H \subset SU(3)^4$ which is operative at the electroweak scale. We consider $H = G_{SM} \otimes SU(2)_\ell$ and $H = G_{SM}$, where $G_{SM}$ is the standard model gauge group, and find that unification occurs only when the remnant leptonic colour symmetry $SU(2)_\ell$ remains unbroken. Furthermore, the demands of a realistic low energy fermion spectrum specify a unique symmetry breaking route for the unifying case of $H = G_{SM} \otimes SU(2)_\ell$. We contrast this with four dimensional quartification models where unification may be achieved via a number of different symmetry breaking routes both with and without the remnant $SU(2)_\ell$ symmetry. The boundary Higgs sector of our model may be decoupled to achieve a Higgsless limit and we show that the electroweak Higgs doublet may be identified as the fifth component of a higher dimensional gauge field.

I. INTRODUCTION

The notion of quartification is predicated upon the hypothesis of quark-lepton universality at high energies. Quark-lepton universality is typically implemented by the imposition of a discrete interchange symmetry between quarks and leptons, a demand which further requires the introduction of a leptonic colour gauge group $SU(3)_\ell$. This leads one to the so called quark-lepton symmetric model \[1, 2, 3, 4, 5, 6, 7, 8\] wherein the known leptons are identified as one member of a generalised $SU(3)_\ell$ lepton triplet.

One of the primary goals of modern theoretical particle physics is to identify extensions to the standard model (SM), with some of the most appealing extensions being grand unified theories (GUTs). Adding a leptonic colour group to the SM clearly renders the traditional approaches to unification, namely $SU(5)$ and $SO(10)$, inapplicable. It was the desire to uncover a unified theory capable of accommodating the notion of quark-lepton universality which motivated the construction of the quartification model \[9, 10\]. Borrowing from the notion of trinification \[11, 12\], which postulates the gauge group $G_T = SU(3)_c \otimes SU(3)_L \otimes SU(3)_R$, the quartification model posits the gauge group $SU(3)^4 \equiv G_T \otimes SU(3)_\ell$ with a cyclic $Z_4$ symmetry permuting the group factors. The discrete groups ensure a single coupling constant at the unification scale and thus, a crucial feature of GUTs, namely the unification of the gauge coupling parameters, can ensue. Whilst in its original implementation only partial unification was achieved \[9\], it was subsequently demonstrated that the modification of the exotic lepton mass spectrum permitted full unification \[10\]. This motivated a recent study of quartification models with intermediate symmetry breaking scales \[16, 17\] where it was shown that unification could be achieved via a number of different symmetry breaking routes.

In \[16, 17\] a subset of schemes which allowed for unification and maintained phenomenological consistency were uncovered. Although representing an improvement on the original quantified theories of Refs. \[9\] and \[10\], these models exhibited some of the unsatisfactory features common to conventional GUTs, resulting mostly from the Higgs sector. A large number of scalar multiplets were required to achieve a realistic model and any potential describing their self-interactions would contain a plethora of unknown parameters. Furthermore, the vacuum expectation values (VEVs) and mass spectra of the fields was not predicated by the theory, forcing a variety of assumptions on this sector.

Orbifold compactifications, however, uncover a new arena by which to explore GUTs. The compactification process provides

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a geometrical origin for the symmetry breaking, allowing for the removal of the scalar sectors which complicate conventional 4d unified theories. The ability to obtain a consistent effective field theory on these constructions has motivated the use of orbifolds to probe grand unified theories \[14, 15\]. Even the SM Higgs field has a natural origin in orbifold models, with gauge-Higgs unified theories identifying the Higgs doublet as higher dimensional components of the gauge fields \[19\].

This type of hybrid model was recently explored in a supersymmetric trinification theory by Carone and Conroy \[14\]. They placed the trinification gauge supermultiplets on a five dimensional orbifold, and localised the full trinified matter content onto a brane. The orbifold action reduced the symmetry down to a subgroup on this subspace. This symmetry was then broken to the SM gauge group by a boundary scalar sector which decoupled from the low-energy theory. Additionally, the SM Higgs doublets were recovered as remnant zero modes of the gauge fields.

The realisation of a GUT that is not reliant upon a scalar sector and its associated problems provides the motivation to explore quartification in this orbifold context. We follow a similar approach to Carone and Conroy \[14\], considering a five dimensional quartification model and studying the prospects for unification within this framework. Interestingly we find that unification is only achieved when the remnant leptonic colour group \(SU(2)_\ell \subset SU(3)_\ell\) remains unbroken. Furthermore the demands of a phenomenologically acceptable fermion spectrum specifies the choice of orbifold symmetry reduction on the SM matter brane and thus we arrive at a unique five dimensional quartification model which achieves unification.

The layout of the paper is as follows. Section II provides the reader with a brief summary of the quartification framework. In Section III we consider the higher dimensional symmetry breaking necessary to reduce \(SU(3)^4\) to an acceptable low energy subgroup. The symmetry breaking occurs in two distinct ways, via both orbifolding and the introduction of a boundary Higgs sector (which we ultimately decouple from the theory), and we detail each of these mechanisms. In Section IV we consider fermion mass in the model and demonstrate that the exotic fermions all obtain GUT scale masses whilst the SM fermions remain massless until electroweak symmetry breaking occurs. Section V covers the issue of gauge coupling unification within the model and we contrast our five dimensional model with conventional approaches in Section VII. We conclude in Section VIII.

II. QUARTIFICATION PRELIMINARIES

In this section we briefly surmise the conventional quartification framework. For more details refer to \[10, 16, 17\]. The quartification gauge group is

\[
G_A = SU(3)_q \otimes SU(3)_L \otimes SU(3)_\ell \otimes SU(3)_R.
\]

A \(Z_4\) symmetry which cyclicly permutes the gauge groups as per \(q \to L \to \ell \to R \to q\) is also imposed to ensure a single gauge coupling constant. The fermion assignment is anomaly-free, with each family contained within a left-handed \(36\) of Eq. \(2.1\)

\[
36 = (1, 1, 3, \overline{3}) \oplus (\overline{3}, 1, 1, 3) \oplus (3, \overline{3}, 1, 1) \oplus (1, 3, \overline{3}, 1)
\]

\(q\) denotes the left-handed quarks, \(\ell\) the left-handed leptons and \(q^c\) and \(\ell^c\) the left-handed anti-quarks and anti-leptons respectively.

These are represented by \(3 \times 3\) matrices, with the first family having the form

\[
q \sim \begin{pmatrix} d & u & h \\ d & u & h \\ d & u & h \end{pmatrix}, \quad q^c \sim \begin{pmatrix} d^c & d^c & d^c \\ u^c & u^c & u^c \\ h^c & h^c & h^c \end{pmatrix},
\]

\[
\ell \sim \begin{pmatrix} x_1 & x_2 & \nu \\ y_1 & y_2 & e \\ z_1 & z_2 & N \end{pmatrix}, \quad \ell^c \sim \begin{pmatrix} x_1^c & y_1^c & z_1^c \\ x_2^c & y_2^c & z_2^c \\ \nu^c & e^c & N^c \end{pmatrix}.
\]

The quark multiplets contain exotic quark colour triplets \(h\) and \(h^c\), and exotic fermions are also required to fill the lepton representations. The SM leptons have partners \(x_i, x_i^c, y_i, y_i^c, z_i\) and \(z_i^c, i = 1, 2\) which transform as \(SU(2)_\ell\) doublets, and there are two exotic singlets \(N\) and \(N^c\). Under \(H = G_{SM} \otimes SU(2)_\ell\), the generator of hypercharge is

\[
Y = \frac{1}{\sqrt{3}} (T_{3L} + T_{3L} + T_{3R}) + T_{3R}
\]
where the $T$’s refer to the Gell-Mann generators. In this framework, the exotic particles have the electric charges $Q(x_1, y_1, z_i) = (1/2, -1/2, 1/2)$, $Q(N) = 0$, and $Q(h) = -2/3$. In the case where the leptonic colour symmetry is entirely broken $H = G_{SM}$, the $x_1, x_1^c, y_2, y_2^c, z_1$ and $z_1^c$ leptons become neutral.

The Higgs sector required to reproduce acceptable low energy phenomenology in 4d quantification models [10, 16, 17] consists of two different 36’s of $G_4$ which are labelled as per

$$
\Phi_{\alpha} \sim (1, 3, 1, \mathbf{3}), \quad \Phi_{\rho} \sim (\mathbf{3}, 1, 3, 1), \quad \Phi_{\chi} \sim (1, 3, 1, \mathbf{3}), \quad \Phi_{\delta} \sim (3, 1, \mathbf{3}, 1),
$$

$$
\Phi_{\ell} \sim (1, 3, \mathbf{3}, 1), \quad \Phi_{\rho^{c}} \sim (1, 1, 3, \mathbf{3}), \quad \Phi_{q^{c}} \sim (\mathbf{3}, 1, 1, 3), \quad \Phi_{\eta} \sim (3, \mathbf{3}, 1, 1).
$$

These fields are closed under the $Z_4$ symmetry and generate realistic fermion masses and mixings. We shall not require the two 36’s of scalars to accomplish the necessary symmetry breaking in the 5d model, relying instead on the orbifold geometry. The subsequent absence of scalars with non-trivial $SU(3)_Q$ quantum numbers removes issues of proton stability in this higher dimensional context.

### III. GAUGE SYMMETRY BREAKING FRAMEWORK

#### A. Orbifold Symmetry Breaking

We consider a pure quantification gauge theory defined in a five dimensional spacetime, with the extra dimensional coordinate labelled $y$. This extra dimension is compactified on an $S^1/Z_2 \times Z'_2$ orbifold. This procedure constrains the spacetime geometry as per

$$
y \to y + 2\pi R, \quad y \to -y, \quad y' \to -y',
$$

where $y' = y + \pi R/2$. These identifications have the effect of reducing the physical region to the interval $y \in [0, \pi R/2]$ with two fixed points located at $y = 0$ and $y = \pi R/2$. These points are geometrical singularities and are thus chosen as the location of 4d branes. For simplicity, we shall consider all matter fields to be localised on the $y = \pi R/2$ brane, which we shall refer to as the matter brane, while the brane at $y = 0$ remains “hidden”.

The orbifold action also has a definition on the space of gauge fields which freely propagate in the bulk. The quantification gauge fields are denoted by

$$
A^M(x^\mu, y) = A^M_a(x^\mu, y) + A^M_L(x^\mu, y) + A^M_\ell(x^\mu, y) + A^M_R(x^\mu, y),
$$

$$
= A^M_a T^a + A^M_L T^a + A^M_\ell T^a + A^M_R T^a
$$

where $a = 1, 2, \ldots, 8$ is the gauge index and $M$ is the Lorentz index $M = \mu, 5$. We define $P$ and $P'$ to be $3 \times 3$ matrix representations of the orbifold actions $Z_2$ and $Z'_2$ respectively. To maintain gauge invariance under these projections, the gauge fields are required to have the transformations

$$
A_\mu(x^\mu, y) \to A_\mu(x^\mu, -y) = PA_\mu(x^\mu, y)P^{-1},
$$

$$
A_5(x^\mu, y) \to A_5(x^\mu, -y) = -PA_5(x^\mu, y)P^{-1},
$$

$$
A_\mu(x^\mu, y') \to A_\mu(x^\mu, -y') = P' A_\mu(x^\mu, y')P''^{-1},
$$

$$
A_5(x^\mu, y') \to A_5(x^\mu, -y') = -P' A_5(x^\mu, y')P''^{-1}.
$$

Given that $P$ and $P'$ define a representation of reflection symmetries, their eigenvalues are $\pm 1$, and thus we can express these matrices in diagonal form, with a freedom in the parity choice of the entries. The exact nature of these actions then completely determines the gauge symmetry which remains unbroken at the fixed points. Unless $P$ is the identity matrix, not all the gauge fields will commute with the orbifold action. These fields are projected off the brane, and thus only a subset of the 5d gauge theory is manifest at the fixed points. Ideally, one would desire the matter brane to respect only the SM gauge group; however, this is not directly possible here via orbifolding. The $Z_2 \times Z'_2$ actions are abelian and commute with the diagonal quantification generators. Subsequently, the rank of the quantification group must be preserved on the matter brane. This means that breaking
unwanted $SU(3)$ factors has the trade-off of retaining the spurious $U(1)$ subgroups. Thus one must invoke a mechanism in tandem to orbifolding in order to accomplish the breaking to $G_{SM}$.

We shall choose to have Higgs fields $\chi_i$ localised on $y = \pi R/2$ to instigate the rank-reducing breaking, giving the general symmetry breaking framework

$$F \equiv SU(3)^4 \xrightarrow{\text{orbifold}} G^{\text{Higgs}} \quad H \equiv G_{SM} \otimes SU(2)_\ell.$$  \hspace{1cm} (3.5)

This type of hybrid model has been explored recently in the context of $SO(10)$ [18] and the aforementioned $SU(3)^3$ [13, 14] orbifold GUTs. We shall consider the case where there is a residual $SU(3)$ indistinguishability issues surface between the SM leptons and their liptonic partners.

The towers of $2^n M$ towers $nM$ of fields possess massless zero modes, and hence the massless sector is restricted to a subset of the full 5d theory. The group $G$ is determined by the desire to reproduce the correct SM matter content on the brane at the $y = \pi R/2$ fixed point. It turns out that the only feasible choice is to consider the orbifold returning the symmetry $G_{\pi R/2} \equiv SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$. All other options do not return a favourable low energy theory. For example, we require the $SU(3)_R$ symmetry to generate realistic quark masses, and the $SU(2)_L$ symmetry must be realised so as to return a SM Higgs.

The full leptonic colour symmetry $SU(3)_\ell$ also needs to be respected on the matter brane. If this symmetry was broken, then indistinguishability issues surface between the SM leptons and their liptonic partners.

The orbifold action can be decomposed as $(P, P') = (P_q \oplus P_L \oplus P_t \oplus P_R, P'_q \oplus P'_L \oplus P'_t \oplus P'_R)$, where

$$P_q = \text{diag}(1, 1, 1), \quad P_L = \text{diag}(1, 1, -1), \quad P_t = \text{diag}(1, 1, -1), \quad P_R = \text{diag}(1, 1, -1),$$

$$P'_q = \text{diag}(1, 1, 1), \quad P'_L = \text{diag}(1, 1, -1), \quad P'_t = \text{diag}(1, 1, -1), \quad P'_R = \text{diag}(1, 1, -1).$$ \hspace{1cm} (3.6)

Under this action, the parity assignments of the vector components of the gauge multiplets are

$$A^\mu_q = \begin{pmatrix} (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \end{pmatrix}, \quad A^\mu_L = \begin{pmatrix} (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \end{pmatrix},$$

$$A^\mu_t = \begin{pmatrix} (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \end{pmatrix}, \quad A^\mu_R = \begin{pmatrix} (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \\ (+, +) & (+, +) & (+, +) \end{pmatrix}.$$ \hspace{1cm} (3.7)

The wavefunctions of these component fields have the Kaluza-Klein decompositions

$$A^\mu_+(x', y) = \frac{1}{\sqrt{2\pi R}} A^\mu_0(x') + \frac{1}{\sqrt{2\pi R}} \sum_{n=1}^\infty A^\mu_{2n}(x') \cos \left( \frac{2ny}{R} \right),$$

$$A^\mu_-(x', y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=0}^\infty A^\mu_{2n+1}(x') \sin \left( \frac{(2n + 1)y}{R} \right),$$

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$$A^\mu_-(x', y) = \frac{1}{\sqrt{2\pi R}} \sum_{n=0}^\infty A^\mu_{2n+2}(x') \sin \left( \frac{(2n + 2)y}{R} \right).$$ \hspace{1cm} (3.8-3.11)

The towers of 4d fields $A^\mu_{2n}(x')$, $A^\mu_{2n+1}(x')$, $A^\mu_{2n+1}(x')$, $A^\mu_{2n+2}(x')$ upon compactification acquire KK masses $2nM_c$, $(2n + 1)M_c$, $(2n + 1)M_c$, $(2n + 2)M_c$ respectively where $M_c \equiv 1/R$ is defined to be the compactification scale. Only the $(+, +)$ fields possess massless zero modes, and hence the massless sector is restricted to a subset of the full 5d theory. The towers $(-, +)$ and $(+, -)$ have degenerate mass spectra, as do the towers $(-, -)$ and $(+, +)$ excluding the zero mode.

We can now discern the nature of the theory on the branes. In general, only those fields which are odd under $P(P')$ vanish at $y = 0 (y' = 0)$. Thus, one can see that at $y = 0$, the gauge symmetry $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3$ is respected, while at the fixed point $y = \pi R/2$ the symmetry is $SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L$. The cumulative
effect of the orbifold compactification is the reduction of the overall symmetry to the intersection of these two subgroups, \( G \equiv G_\| = G = SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3 \). Clearly further symmetry breaking is required to firstly reproduce, then break, the SM.

As evident in Eq. 3.5 the scalar components \( A_i^5 \) necessarily have opposite parities to their vector counterparts allowing one to determine the scalar content at the fixed points. The transformation of \( A_i^5 \) importantly reveals a scalar doublet which has a massless zero mode at the 4d level, and this possesses the appropriate hypercharge to be identified as a SM Higgs doublet. This type of identification is referred to as gauge-Higgs unification and has been employed extensively as an origin for the EW Higgs in orbifold GUTs \([19]\). Attention to this sector will be given in Section IV B.

### B. Brane Breaking

After the orbifold compactification, the symmetry of the 4d effective theory is \( SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3 \), while on the matter brane the gauge symmetry is \( G_{\pi R/2} \equiv SU(3)_q \otimes SU(2)_L \otimes SU(3)_\ell \otimes SU(3)_R \otimes U(1)_L \). The usual approach to breaking this remaining symmetry is to implement boundary conditions on the compactified space for the gauge fields \([21]\). This method is more generalised than the orbifold mechanism, leading to a greater set of symmetry breaking opportunities.

The structure of the boundary conditions are realised in a UV completion of our GUT theory in which localised scalar fields \( \chi_i \) prescribe the breaking. As the fields are localised on the matter brane where only the reduced \( G_{\pi R/2} \) symmetry is operative, the fields are housed in incomplete quantification multiplets, with transformation properties

\[
\chi_{1,2} \sim (1,1,1,3), \quad \chi_{3} \sim (1,1,3,1), \quad \chi_{4} \sim (1,1,3,\overline{3}).
\]

The three fields \( \chi_{1,2,3} \) carry \( U(1)_L \) charge \(-2/\sqrt{3}\) and are necessary for the \( SU(2)_R \otimes U(1)^3 \rightarrow U(1)_Y \) breaking. The final Higgs field, \( \chi_4 \) is neutral under \( U(1)_L \). Its VEVs do not increase the symmetry breaking but they are important contributors to fermion mass generation. As will be detailed later, we shall be considering the limit in which this sector decouples entirely from the brane, and so the addition of these scalars does not pose a complication.

The effect of these fields can be ascertained from the kinetic sector of our action. The 5d action assumes the form

\[
S \supset \int d^5x \left( -\frac{1}{4} F_{\mu \nu}^a F^{a \mu \nu} - \frac{1}{2} F_{5 \mu \nu}^\alpha F^{\alpha 5 \mu \nu} + \left( D_\mu \chi_i \right)^\dagger D^\mu \chi_i - V(\chi) \right) \delta(y - \pi R/2),
\]

and the relevant terms to consider are

\[
\int_0^{\pi R/2} dy \left( -\frac{1}{2} \partial_5 A_\mu(x^\nu, y) \partial^5 A^\mu(x^\nu, y) + \frac{g_5^2}{4} \langle \chi \rangle^\dagger \langle \chi \rangle A^\mu(x^\nu, y) A_\mu(x^\nu, y) \delta(y - \pi R/2) \right).
\]

\( g_5 \) is the five-dimensional gauge coupling constant defined by \( g_5 \equiv g_4 \sqrt{\frac{\pi R}{2}} \) and has mass dimension \(-1/2\). Variation in \( A^\mu(x^\nu, y) \) gives, after integration over the extra dimension, the surface terms

\[
-\partial_5 A_\mu(x^\nu, y) \big|_{y=0}, \quad \left( -\partial_5 A_\mu(x^\nu, y) + \frac{g_5^2}{2} \langle \chi \rangle^\dagger \langle \chi \rangle A_\mu(x^\nu, y) \right) \delta A^\mu(x^\nu, y) \big|_{y=\pi R/2}
\]

which must vanish. Making the definition

\[
V \equiv \frac{g_5^2}{2} \langle \chi \rangle^\dagger \langle \chi \rangle,
\]

we obtain the boundary conditions

\[
\partial_5 A_\mu(x^\nu, 0) = 0, \quad \partial_5 A_\mu(x^\nu, \pi R/2) = V A_\mu(x^\nu, \pi R/2),
\]

which illustrate the constraints imposed on the gauge fields by the boundary Higgs sector. \( V \), having dimensions of mass, reflects the order parameter of the symmetry breaking. If \( V = 0 \), Eq. 3.18 reduces to Neumann boundary conditions and returns the
usual orbifold behaviour. If $V \neq 0$, then the zero mode for $A_{\mu}$ is no longer massless, and one has gauge boson mass terms localised at the matter brane as we shall now clarify.

These boundary conditions affect only the fields which are even at $y = \pi R/2$, with the towers corresponding to the parities $(\pm, -)$ disappearing on the matter brane. The non-vanishing fields $A_{\mu}^{(+, +)}$ have the generic profile

$$A_\mu(x^\nu, y) = N_n \cos(M_n y) A^{(n)}_\mu(x^\nu)$$

with KK mass $M_n$. Enforcement of Eq. $3.18$ appreciably modifies the wavefunctions of these gauge bosons, with their KK masses shifted by an amount governed by

$$M_n \tan(M_n \pi R/2) = -V.$$  (3.20)

For large values of $V$, Eq. $3.20$ shows that the KK tower has a mass spectrum approximated by

$$M_n \approx M_c (2n + 1) \left(1 + \frac{M_c}{\pi V} + \ldots\right), \quad n = 0, 1, 2, \ldots$$ (3.21)

giving a tower with the lowest-lying states $M_c$, $3M_c$, $5M_c$, etc. This represents an offset of $M_c$ relative to the $V = 0$ tower, with the field no longer retaining a massless zero mode. As $V \to \infty$, the brane localised mass terms for the gauge fields become larger and their wavefunctions are eventually expelled from the brane. A similar mass shift can be induced by the boundary conditions on the $(-, +)$ fields.

From Eq. $3.16$ the association of $V$ with the VEVs of the boundary scalar sector implies that the limit $V \to \infty$ is attained when $\langle \chi \rangle \to \infty$. However, when the VEVs of the Higgs fields are taken to infinity, the shift in the KK masses of the gauge fields is finite, giving the exotic gauge fields masses dependent only upon the compactification scale $M_c$. Consequently, these fields remain as ingredients in the effective theory while the boundary Higgs sector decouples entirely, and we can view our reduced symmetry theory in an effective Higgsless limit. Interestingly, in this limit also, the high-energy behaviour of the massive gauge boson scattering remains unspoilt as shown in $[21]$.

Depending on the exact nature of $V$, a shift can be induced in the KK towers of the gauge fields generating a greater symmetry breaking than the orbifold compactification. The gauge fields are decomposed into one of six possible KK towers depending on the orbifold and boundary conditions, and we need to choose the latter such that the $G_{\pi R/2}/H$ gauge fields are exiled from the brane, while those corresponding to the $H$ symmetry remain unperturbed. Given that the structure of the breaking parameter $V$ is predetermined by the Higgs fields, the desired boundary constraints can be satisfied with our scalar fields acquiring VEVs of the form

$$\langle \chi_1 \rangle = \begin{pmatrix} 0 \\ 0 \\ v_1 \end{pmatrix}, \quad \langle \chi_2 \rangle = \begin{pmatrix} v_2 \\ 0 \\ v_3 \end{pmatrix},$$

$$\langle \chi_3 \rangle = \begin{pmatrix} 0 \\ 0 \\ v_4 \end{pmatrix}, \quad \langle \chi_4 \rangle = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ v_5 & v_6 \end{pmatrix}.$$ (3.22)

The fields $\chi_{1,2,3}$ and $\chi_4$ have a $U(1)_L$ charge of $-2/\sqrt{3}$ and $0$ respectively, where these assignments and the VEV structure are motivated by the embedding of the fields in the conventional quantification models $[17]$.

These fields transform trivially under $SU(3)_c \otimes SU(2)_L$ and thus the gauge fields defining these symmetries remain unaffected by the scalars. The $SU(3)_c \otimes SU(3)_c$ gauge fields, however, are subject to the constraint

$$\partial_\nu A_{\mu}^i(x^\nu, \pi R/2) = V_{ij} A_{\mu}^j(x^\nu, \pi R/2)$$  (3.24)

where $V_{ij}$ is a matrix in the space of these fields. For the non-abelian components this can be decomposed into the form

$$\partial_\nu A_{\mu}^a(x^\nu, \pi R/2) = 0, \quad a = 1, 2, 3,$$  (3.25)

$$\partial_\nu A_{\mu}^{\dot{a}}(x^\nu, \pi R/2) = V_{ij} A_{\mu}^i(x^\nu, \pi R/2), \quad \dot{a} = 4, 5, 6, 7,$$  (3.26)

$$\partial_\nu A_{\mu}^{\dot{a}}(x^\nu, \pi R/2) = V_{ij} A_{\mu}^i(x^\nu, \pi R/2), \quad \dot{a} = 1, 2, 4, 5, 6, 7,$$  (3.27)
where the $a$’s refer to the unbroken group indices and $\hat{a}$ the broken indices. The $V_{\ell,R}$ entries have the general form $\sum_i c_i g_i^2 v_i^2$, for some constant $c_i$, with their precise values determined by the dynamics of the UV completed theory.

The explicit form of $A^\mu_Y$ was determined in Ref. [15] for a model described by the trinification gauge group, with the entries

$$A^\mu_Y = \left( \frac{2}{\sqrt{3}} (A^5_L + A_\ell^\mu + A^\mu_R) + 2 A^3_R \right)^\mu$$

(3.28)

and this must remain massless in the low-energy theory. Subsequently, this linear combination must correspond to the sole zero eigenvalue of the relevant subset of the generalised $V$ matrix of Eq. 3.23 defined by

$$\partial_5 \hat{A}^\mu_i (x^\nu, \pi R/2) = V_{ij} \hat{A}^\mu_j, \quad i, j = 1, 2, 3, 4$$

(3.29)

where

$$\hat{A}^\mu \equiv \left( A^8_{i R} (x^\nu, \pi R/2) \ A^8_{i \ell} (x^\nu, \pi R/2) \ A^3_R (x^\nu, \pi R/2) \ A^8_{R \ell} (x^\nu, \pi R/2) \right)^T.$$  

(3.30)

The explicit form of $V_{ij}$ was determined in Ref. [15] for a model described by the trinification gauge group, with the entries functions of $g_i^2 v_i^2$ as defined by the VEV structure. This parametrisation generalises to the quartification case, and reveals a zero eigenvalue for the vector $1/\sqrt{3} \{1, 1, \sqrt{3}, 1\}$, and non-zero eigenvalues for the remaining three eigenvectors, denoted as $A^8_{X_1}, A^8_{X_2}$ and $A^8_{X_3}$. Consequently, the $(+, +)$ towers corresponding to these latter three physical fields have their mass spectra shifted by $M_\ell$ and they are expelled from the low-energy theory.

The effect of the implementation of these boundary conditions on the 32 quartification gauge fields is summarised in Table I, where the fields have been listed in terms of their $G_{\pi R/2}$ representations. The mass spectra shown incorporates any induced shifts due to non-trivial values of $V$, from which it can be deciphered that only the vector fields respecting $G_{SM} \otimes SU(2)_L$ possess a massless mode, with the other $(+, +)$ towers misaligned. The new mass spectra of these shifted towers is now degenerate with the $(-, +)$ towers. Furthermore, the only scalar that has a massless mode transforms as an $SU(2)_L$ doublet (and its conjugate). This mode has a component that can not be gauged away, giving us a physical electroweak Higgs field.

| Fields          | Representation | KK tower | Mass          |
|-----------------|----------------|----------|---------------|
| $A^a_{R \ell}$, $a = 1, 2, ..., 8$ | $(8, 1, 1, 1)$ | $(+, +)$ | $2 n M_c$     |
| $A^a_{R \ell}$, $a = 1, 2, ..., 8$ | $(8, 1, 1, 1)$ | $(-, -)$ | $2 (n+1) M_c$ |
| $A^a_{L \ell}$, $a = 1, 2, 3$ | $(1, 3, 1, 1)$ | $(+, +)$ | $2 n M_c$     |
| $A^a_{L \ell}$, $a = 1, 2, 3$ | $(1, 3, 1, 1)$ | $(-, -)$ | $2 (n+1) M_c$ |
| $A^a_{V \ell}$, $a = 4, 5, 6, 7$ | $(1, 2, 1, 1) + (1, 2, 1, 1)$ | $(-, -)$ | $2 (n+1) M_c$ |
| $A^a_{V \ell}$, $a = 4, 5, 6, 7$ | $(1, 2, 1, 1) + (1, 2, 1, 1)$ | $(+, +)$ | $2 n M_c$     |
| $A^a_{R \ell \ell}$, $a = 1, 2, 3$ | $(1, 1, 3, 1)$ | $(+, +, V = 0)$ | $2 n M_c$     |
| $A^a_{R \ell \ell}$, $a = 1, 2, 3$ | $(1, 1, 3, 1)$ | $(-, -)$ | $2 (n+1) M_c$ |
| $A^a_{V \ell \ell}$, $a = 4, 5, 6, 7$ | $(1, 1, 2, 1) + (1, 1, 2, 1)$ | $(+, -)$ | $2 (n+1) M_c$ |
| $A^a_{V \ell \ell}$, $a = 4, 5, 6, 7$ | $(1, 1, 2, 1) + (1, 1, 2, 1)$ | $(-, +)$ | $2 (n+1) M_c$ |
| $A^a_{R \ell \ell}$, $a = 1, 2$ | $\subset (1, 1, 1, 3)$ | $(+, +, V \rightarrow \infty)$ | $2 (n+1) M_c$ |
| $A^a_{R \ell \ell}$, $a = 1, 2$ | $\subset (1, 1, 1, 3)$ | $(-, -)$ | $2 (n+1) M_c$ |
| $A^a_{V \ell \ell}$, $a = 4, 5, 6, 7$ | $(1, 1, 1, 2) + (1, 1, 1, 2)$ | $(-, +)$ | $2 (n+1) M_c$ |
| $A^a_{V \ell \ell}$, $a = 4, 5, 6, 7$ | $(1, 1, 1, 2) + (1, 1, 1, 2)$ | $(+, -)$ | $2 (n+1) M_c$ |

| $A^a_{V \ell}$ | $(1, 1, 1, 1)$ | $(+, +, V = 0)$ | $2 n M_c$     |
| $A^a_{X_1}$    | $(1, 1, 1, 1)$ | $(+, +, V \rightarrow \infty)$ | $2 (n+1) M_c$ |
| $A^a_{X_2}$    | $(1, 1, 1, 1)$ | $(+, +, V \rightarrow \infty)$ | $2 (n+1) M_c$ |
| $A^a_{X_3}$    | $(1, 1, 1, 1)$ | $(+, +, V \rightarrow \infty)$ | $2 (n+1) M_c$ |

TABLE I: The decomposition of the 32 gauge fields into their respective KK towers at the fixed point $y = \pi R/2$. Their representations are given in terms of the brane symmetry before the additional breaking via the localised Higgs sector. The towers $(\pm, -)$ have wavefunctions which are odd on this brane, and thus are vanishing.
To summarise, the orbifold compactification alone reduces the gauge symmetry on the two fixed points to
\[ G_0 = SU(3)_q \otimes SU(2)_L \otimes SU(2)_R \otimes U(1)^3; \quad (3.31) \]
\[ G_{\pi R/2} = SU(3)_q \otimes SU(2)_L \otimes SU(3)_{\ell} \otimes SU(3)_R \otimes U(1)_L, \quad (3.32) \]
which reduces the overall symmetry of the low-energy 4d theory to \( SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(2)_R \otimes U(1)^3 \). The boundary scalar sector then instigates the further breaking
\[ G_{\pi R/2} \rightarrow SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes U(1)_Y \] returning the desired \( G_{SM} \otimes SU(2)_\ell \) symmetry operative at the EM level.

IV. FERMION MASSES

A. GUT scale masses

We now introduce matter into our scheme. Due to the identification of the Higgs field as a remnant zero mode arising from the extra dimensional component of the gauge fields, if the fermions were bulk fields, their coupling with the Higgs would be universal. Thus, it would prove difficult to recover the detailed structure of Yukawa couplings in the low-energy theory. As an aside we note that it may be interesting to study a higher dimensional quartification model employing split fermions. The high degree of symmetry present in the quartification model can be utilised to motivate the fermion localisation pattern of a split fermion model [22], though presumably a different symmetry-breaking mechanism to that employed here would be required. In this work we shall assume that the fermions are localised on the brane where they are housed in representations that need only respect the \( SU(3)_q \otimes SU(3)_{\ell} \otimes SU(3)_R \otimes U(1)_L \) symmetry.

Although blind to the entire \( SU(3)^4 \) symmetry, we consider the full fermion content of minimal quartification so that we can achieve anomaly cancellation. The exotic content, it transpires, will become massive through couplings with the boundary Higgs field. The fermion representations under the brane symmetry are defined as follows:
\[ q_1 \sim (3, 2, 1, 1) \left( \frac{1}{\sqrt{3}} \right), \quad q_2 \sim (3, 1, 1, 1) \left( -\frac{2}{\sqrt{3}} \right), \quad q^c \sim (\bar{3}, 1, 1, 3)(0), \quad (4.1) \]
\[ \ell_1 \sim (1, 2, \bar{3}, 1) \left( -\frac{1}{\sqrt{3}} \right), \quad \ell_2 \sim (1, 1, \bar{3}, 1) \left( \frac{2}{\sqrt{3}} \right), \quad \ell^c \sim (1, 1, 3, \bar{3})(0), \quad (4.2) \]
whose matrix representations can be deduced from Eq. 3.3.

The boundary Higgs fields \( \chi_i \) couple to the localised fermions giving six invariant Yukawa interaction terms
\[ Y_1 q^c q_2 \chi_1, \quad Y_2 q^c q_2 \chi_2, \quad Y_3 \ell_1 \ell_1 \chi_3, \quad Y_4 \ell_2 \ell^c \chi_1, \quad Y_5 \ell_2 \ell^c \chi_2, \quad Y_6 \ell^c \ell^c \chi_4. \quad (4.3) \]

Eq. 4.3 gives the quark mass terms
\[ (Y_1 v_1 + Y_2 v_3) h^c h + Y_2 v_2 d^c h, \quad (4.4) \]
revealing mixing between the \( h^c \) and \( d^c \), giving only one linear combination that is massless. This we identify to be the physical left-handed \( d \) antiquark. Similarly, the exotic leptonic components all mix. These mixings are sufficient to generate GUT scale Dirac masses for all but one exotic lepton, leaving only the SM fields and the right-handed neutrino as the low-energy massless fermion spectrum. As these mass terms are proportional to \( v_i \), in the decoupling limit \( v_i \rightarrow \infty \), all the exotic fermions are removed from the low-energy theory along with the \( \chi_i \) fields.

B. Electroweak Scale Masses And The Higgs

In addition to the fermion spectrum, we have an electroweak scalar doublet arising from the fields \( A_L^{\alpha} \) which has a zero mode on the brane. We denote this doublet as \( \Phi \sim (1, 2, 1, 1)(-1) \). What remains now is to induce electroweak symmetry breaking
and to generate the correct Yukawa structure of these fields with the fermions. However, given that the SM Higgs is a remnant of the higher dimensional gauge sector, its dynamics are still dictated by the bulk theory. Local couplings between $\Phi$ and the fermion fields are restricted by the higher dimensional gauge invariance. This can be seen explicitly by looking at the gauge transformation parameters. A general $SU(3)^4$ gauge transformation is defined by

$$U = e^{i [\eta(x^\mu) \xi(y)]^a_j T_a^j}, \quad j = q, L, \ell, R$$  \hspace{1cm} (4.5)$$

where we have assumed a separable form for the gauge parameters. At the fixed points the $y$-dependent parameters corresponding to the broken generators satisfy $\xi^a_j = 0$ and $\partial_y \xi^a_j |_{y_j} \neq 0$. A remnant of the bulk symmetry at $y = \pi R/2$, however, is the condition $A^5_{i\dot{a}} \to A^5_{i\dot{a}} + \partial_y \xi^i_\dot{a}$. This shift is broken by any Yukawa couplings directly involving $\Phi$. As a result, the only gauge invariant interactions between the brane fermions and the SM Higgs field are non-local, involving Wilson lines.

The Wilson line operator is defined as

$$W = \mathcal{P} \exp \left( i \int_{y_i}^{y_f} dy \Phi \right) |_{R},$$  \hspace{1cm} (4.6)$$

in the representation $R$ where $\mathcal{P}$ is the path-ordered product, and $y_i$ and $y_f$ are fixed points of the orbifold. If $y_1 = y_2 = \pi R/2$, this operator is linear under the gauge transformations and thus can couple to the fermion fields. Its non-locality means that the Higgs potential generated at the quantum level is also non-local and thus insensitive to the UV physics. We assume that the generation of the Wilson lines is due to some dynamics in the UV completion of the theory. Irrespective of their origin, these operators are natural entities to consider, satisfying invariance under both the gauge symmetry and the orbifold projections.

The non-local Lagrangian has the general form

$$\mathcal{L}^{NL} = \lambda f_1 |_{y=y_f} W \chi |_{y=y_f} \chi + \lambda' f_2 |_{y=y_f} \bar{W} \chi |_{y=y_f} \chi + \ldots$$  \hspace{1cm} (4.7)$$

where $f_1, f_2, \ldots$ are localised fermion bilinears, and each term is suppressed by the appropriate power of the fundamental scale $\Lambda$. If the Higgs field acquires a non-trivial VEV, then its non-local interaction with the brane fields generates the effective 4d local Yukawa terms

$$\frac{1}{\lambda} \lambda_1 \chi_1 q_1 q^c \Phi, \quad \frac{1}{\lambda} \lambda_2 \chi_2 q_1 q^c \Phi, \quad \frac{1}{\lambda} \lambda_3 \chi_1 \chi_2 q_1 q^c \Phi$$

$$\frac{1}{\lambda} \lambda_4 \chi_1 \ell_1 \ell_2 \Phi, \quad \frac{1}{\lambda} \lambda_5 \chi_1 \ell_1 \ell_2 \Phi, \quad \frac{1}{\lambda} \lambda_6 \chi_2 \chi_4 \ell_1 \ell_2 \Phi, \quad \frac{1}{\lambda} \lambda_7 \chi_1 \ell_1 \ell^c \Phi$$

These terms arise after the boundary Higgs sector has developed its VEV structure. As commented in Ref. [14], these terms remain at the electroweak order as $\chi/\Lambda \sim O(1)$ in the decoupling limit $v_1 \to \Lambda \to \infty$, with their interactions dictated only by $G_{SM} \otimes SU(2)_L$ not the UV physics.

These couplings return electroweak scale Dirac masses to the SM quarks and GUT scale masses to the exotic quarks, with mixing between the $d$ and $h$ quarks heavily suppressed. Analysis of the leptonic sector reveals that the electron does not mix at all with the exotic particles, and the mixing between the neutrinos and exotic neutral singlets is significantly suppressed. There are GUT scale Dirac masses for all the exotic particles, while the electron and neutrino acquire electroweak scale Dirac masses. We shall not concern ourselves with naturally generating see-saw suppressed Majorana masses for the neutrinos. We suspect that the inclusion of gauge singlet fermions with Majorana masses at the unification scale will invoke a seesaw style suppression of the electroweak scale Dirac masses, as occurs in 4d models [16]. However we shall settle for highly tuned Dirac Yukawa coupling constants in what follows. Nevertheless, the Yukawa interactions that are induced on the brane by the Wilson line operators yield an appealing fermion mass spectrum. We note that as our construct is non-supersymmetric the usual fine tuning is required to stabilise the electroweak scale relative to the unification scale.

V. GAUGE COUPLING UNIFICATION

We turn now to calculating the unification scale of our scheme. To evaluate the evolution of the gauge coupling constants, we need to ascertain how exactly the presence of the extra dimension affects the RGEs. We essentially have two mass scales to consider; the compactification scale which characterises the size of the extra dimension, and the cut-off or unification scale $\Lambda$. 
Corrections due to the physics in the UV regime arise in the form of brane localised operators. These respect the $G_{SM} \otimes SU(2)_{\ell}$ symmetry, and even if set to zero at tree-level can be generated by radiative corrections. The effects of these operators, however, can be tamed, and much effort has been expended on ensuring that their presence does not destroy the higher dimensional unification [23, 24, 25]. The zero mode coupling in the effective 4d theory is obtained by integrating the 5d action for the gauge fields, and this estimates the value at the cut-off to be

$$\frac{1}{g_s^2(\Lambda)} = \frac{2 \pi R}{g_5^2(\Lambda)} + \frac{1}{g_4^2(\Lambda)}. \quad (5.1)$$

g_5$ is the $SU(3)^4$-invariant coupling parameter while $g_4$ is the dimensionless coupling constant arising from these brane localised operators. If the volume factor of the extra dimension is sufficiently large then the sensitivity to these brane corrections is diluted by the bulk contribution [23]. If $g_4^2 = \zeta/\Lambda$, $g_5^2 = \zeta_0$ then the suppression is given by $(\zeta/\zeta_0)(1/\Lambda \pi R)$. So as long as the extra dimension is sufficiently large in extent, the volume factor $\Lambda \pi R$ is sufficient to negate the brane modification of $g_4(\Lambda)$.

What remains is to determine the threshold contributions which arise at the compactification scale $M_c$. At energies below $M_c$, the extra dimension is not observable. The zero-modes of the KK gauge boson towers and brane localised fermions define our effective theory is the 5-dimensional GUT with symmetry group $SU(3)^4$. The coupling constants proceeds via the usual renormalisation group equations

$$\frac{1}{\alpha_i(M_c)} = \frac{1}{\alpha_i(M_{EW})} + \frac{b_i}{2 \pi} \ln \frac{M_{EW}}{M_c}. \quad (5.2)$$

$\alpha_i = g_i^2/4\pi$ and the $b_i$ are the beta coefficients which enumerate both the number and type of particles which contribute to the evolution.

Above $M_c$ the extra dimension is manifest through the appearance of the infinite towers of KK modes with increasing mass, and our effective theory is the 5-dimensional GUT with symmetry group $SU(3)^4$. The towers, however, do not have a universal contribution to the running of the coupling constants. This is in part because they do not fill complete $SU(3)^4$ representations, and also due to the misalignment of the towers resulting from the brane breaking. At $M_c$ the $G = SU(3)_q \otimes SU(2)_L \otimes SU(3)_{\ell} \otimes SU(3)_{R} \otimes U(1)$ symmetry is valid, with the modes corresponding to the $G/H$ fields starting to appear. These states contribute at each $(2n + 1) M_c$ level. As the evolution proceeds from $M_c$ to $2M_c$ the full quartification symmetry emerges with the $SU(3)^4/G$ states inputting at the $(2n + 2) M_c$ energy levels. At each $n$-th level, new excitation modes contribute until the couplings merge at $M_{GUT}$. It has been shown that if the difference on the runnings $\delta_i(\mu) \equiv \alpha_i^{-1}(\mu) - \alpha_j^{-1}(\mu)$ is considered then these KK modes dominate the evolution above $M_c$, with the differential running logarithmically sensitive to the cut-off as per [23, 24, 25]

$$\delta_i(M_{GUT}) = \delta_i(M_{EW}) + \frac{(\beta_i - \beta_j)}{2 \pi} \ln \frac{M_{EW}}{M_c} - \frac{1}{2 \pi} \Delta_i(M_{GUT}), \quad (5.3)$$

where

$$\Delta_i(M_{GUT}) = (\beta_i - \beta_j) \ln \frac{M_{GUT}}{M_c} + (\gamma_i - \gamma_j) \sum_{n=0}^{N_k} \ln \frac{M_{GUT}}{(2n + 2) M_c}.$$

(5.4)

Here, $\beta_i$ are the zero-modes beta coefficients, $\gamma_i$ are the beta coefficients of the modes with mass $(2n + 1) M_c$ and $\eta_i$ those of the modes with mass $(2n + 2) M_c$. These factors compensate for the varying contributions of the KK levels. $N_k$ is the level at which the KK towers are truncated, with $(2N_k + 1) M_c \lesssim M_{GUT}$. We choose to explore the running with respect to the evolution of $\alpha_Y$, taking $j = Y$.

Table lists the beta coefficients for all the KK modes, where the gauge bosons have been re-labelled in terms of their $G_{SM} \otimes SU(2)_{\ell}$ representations. The multiplet $A_{SM}^{i}$ consists of the $G_{SM} \otimes SU(2)_{\ell}$ gauge fields, while the EW Higgs is contained in $A_{(1,2,1)(\pm 1)}$. The remaining two multiplets represent exotic fields contained in $G/H$. These arise from the $A_{R}^{\mu}$ fields aligned along the $T^4$, $T^5$ directions and those corresponding to $A_{R}^{\mu=1,2}$ whose KK masses have received a shift.
Given that the exotic states begin to emerge at $M_c$, the compactification scale has the lower bound $M_c \gtrsim 1 \text{ TeV}$. If unification could result at this compactification energy, then there would be a promising spectrum of exotic particles within reach of the LHC. Unfortunately the three SM coupling parameters do not unify with a low $M_c$. The lowest value of the compactification scale which is consistent with approximate unification is $M_c \sim 10^{10} \text{ GeV}$, with unification occurring at $M_{GUT} \sim 10^{12} \text{ GeV}$. As we increase the compactification scale further, the energy interval between $M_c$ and $M_{GUT}$ decreases slightly, and fewer KK modes contribute to the running. The most favourable unification scenario occurs for $M_c \sim 2 \times 10^{14} \text{ GeV}$. Here we have $\sim 50$ KK states for each tower in the summation with the coupling constants intersecting at $M_{GUT} \sim 1.9 \times 10^{16} \text{ GeV}$ as shown in Fig. I. In this case, the leptonic colour coupling parameter must be $\alpha_\ell \sim 0.19$ at the electroweak level, surpassing the electroweak scale value of the strong coupling constant. We reiterate that the differential running has been defined relative to the hypercharge coupling constant so that
\[
\delta_{q,L,\ell}(\mu) = \alpha_{q,L,\ell}^{-1}(\mu) - \alpha_Y^{-1}(\mu).
\] (5.5)

To complete the unification analysis we determine the largest possible compactification scale which allows for unification. As $M_c$ increases in energy, necessarily so too does the unification scale. Thus the upper bound on $M_c$ will result when the GUT scale is identified as the 5d Planck scale. This is attained for $M_c \sim 10^{16} \text{ GeV}$ which gives unification at $M_{GUT} \sim 8.2 \times 10^{17} \text{ GeV}$. To summarise we find that the relation $\delta_i(M_{GUT}) = 0$, $\forall i = q, L, \ell, Y$ is satisfied only for a compactification scale of order $10^{10} \text{ GeV} < M_c < 10^{16} \text{ GeV}$, with the corresponding unification scale lying in the range $10^{12} \text{ GeV} < M_{GUT} < 8 \times 10^{17} \text{ GeV}$. Within these bounds, the amalgamation of the strong and electroweak couplings constrains the value of the leptonic colour coupling parameter to be $0.08 \lesssim \alpha_\ell(M_{EW}) \lesssim 0.25$ at the electroweak scale.

VI. A MODEL WITHOUT $SU(2)_\ell$

We now briefly comment on 5d quartified models which do not have a residual leptonic $SU(2)_\ell$ symmetry, i.e. $H = G_{SM}$. There exists only a limited approach to extend upon the previous scheme in which this symmetry remained in the low energy theory. We require the breaking of the leptonic symmetry to be achieved by a combination of the orbifolding and brane localised scalar fields.

To obtain a consistent low-energy model the orbifold projections must be
\[
P_q = \text{diag}(1,1,1), \quad P_L = \text{diag}(1,1,-1), \quad P_\ell = \text{diag}(1,1,1), \quad P_R = \text{diag}(1,1,-1),
\]
\[
P_q' = \text{diag}(1,1,1), \quad P_L' = \text{diag}(1,1,-1), \quad P_\ell' = \text{diag}(1,1,-1), \quad P_R' = \text{diag}(1,1,1),
\] (6.1)
which break the quartification symmetry to $SU(3)_q \otimes SU(2)_L \otimes SU(2)_\ell \otimes SU(3)_R \otimes U(1)_L \otimes U(1)_\ell$ on the matter brane. As the quarks are singlets under leptonic colour, we still require the full $SU(3)_R$ symmetry on the brane and the fields $\chi_1$ and $\chi_2$ to generate mass terms. Hence, our quark sector will be identical to that of the previous model. To instigate the breaking to the SM gauge group we need also the additional boundary scalar fields

\[ \chi_3 \sim (1, 1, 1, 3)(0, -2/\sqrt{3}), \quad \chi_4 \sim (1, 1, 2, 1)(-2/\sqrt{3}, -1/\sqrt{3}), \quad \chi_5 \sim (1, 1, 2, 3)(0, 1/\sqrt{3}) \]  

with the VEV patterns

\[ \langle \chi_3 \rangle = \begin{pmatrix} 0 \\ 0 \\ v_4 \end{pmatrix}, \quad \langle \chi_4 \rangle = \begin{pmatrix} v_5 \\ 0 \end{pmatrix} \]

\[ \langle \chi_5 \rangle = \begin{pmatrix} v_6 & 0 & v_7 \\ 0 & 0 & v_8 \end{pmatrix} \]

We now have only four charged lepton fields per family and they all mix. The Yukawa interaction terms generated locally through $\chi_1 - \chi_4$ and those induced non-locally with $\Phi$ impart an electroweak scale Dirac mass to the electron, muon and tauon particles. The remaining three eigenvalues per family are all of GUT scale. The neutral particles again have Dirac masses, with three of these arising from the shifted $(+, +)$ towers of $A^\mu_u$ and $A^\mu_d$.

Above the compactification scale, the excitation modes of the SM gauge fields and the multiplet containing the SM Higgs field appear at each $(2n + 2)M_c$ level. Four exotic singlets with hypercharge $Y = \pm 2$ also emerge at the $(2n + 1)M_c$ energy levels, with three of these arising from the shifted $(+, +)$ towers of $A^\mu_u$ and $A^\mu_d$.

We have investigated the prospects for unification with this new symmetry framework. We find that the fewer states contributing at each $(2n + 1)M_c$ level fail to sway the coupling constants to unify in a phenomenologically consistent fashion. The couplings converge only when $M_c \geq 10^{16}$GeV. However, for compactification scales this large, the unification always occurs at energies greater than the 5d Planck scale and so cannot result. It is interesting that unification within our 5d framework requires $SU(2)_\ell$ to remain as an exact symmetry.

**VII. COMPARISON WITH CONVENTIONAL MODELS**

We have shown that unification can be achieved within a five dimensional quartification model. Before concluding we shall contrast the 5d construct with the conventional quartification models of [16, 17]. Our intention is to draw the readers attention to the advantages and disadvantages of each approach.
Let us first consider the case of unbroken $SU(2)_c$. In the 4d case unification could be achieved with four distinct models, each differing in their symmetry breaking pattern. No predictions regarding the favourability of the distinct models was possible with all of these models permitting a flexible range of symmetry breaking scales consistent with unification. However intermediate symmetry breaking scales were required, otherwise unification mandates the inclusion of 14-37 SM Higgs doublets. The models with intermediate breaking scales required eight distinct scalar $SU(3)^4$ multiplets (related by the cyclic $Z_4$ symmetry), and these naturally returned seven SM Higgs doublets. Furthermore, this results in complicated Higgs potentials with many arbitrary parameters. These parameters must be tuned to generate the desired stages of symmetry breaking, with further assumptions required to ensure that the masses of the scalars were consistent with unification. The GUT scale was found to lie in the range $10^{12} - 10^{18}$ GeV and proton decay inducing scalars were required to be of order the GUT scale to ensure proton longevity. Finally we note that several of the 4d models predicted that the $SU(2)_R W$ and $Z'$ bosons should be order TeV and are thus observable at the LHC.

We contrast this with the 5d case. Here the demands of a realistic SM fermion spectrum restricted the intermediate symmetry group on the matter brane $G_{\pi R/2}$, resulting in just one feasible choice, namely $G_{\pi R/2} \equiv SU(3)_q \otimes SU(2)_L \otimes SU(3)_c \otimes SU(3)_R \otimes U(1)_L$. We have shown that unification can be achieved via this route, occurring for a range of compactification scales, $10^{10} \lesssim M_c \lesssim 10^{16}$ GeV. Only one SM Higgs doublet arose in the 5d construct and this was adequate to achieve unification. The reduced symmetry operative on the matter brane meant that brane localised scalars need not fill out entire $SU(3)^4$ representations. Thus no coloured scalars were required and the proton was found to be stable. This has the advantage of permitting unification at low scales (relative to typical 4d unification scales). The 5d framework naturally motivates the intermediate mass scales necessary for unification. These are introduced in a somewhat ad hoc way in 4d constructs, but here they arise as KK excitations of bulk fields. Thus all intermediate mass scales are set by the one parameter, $M_c$, and the inclusion of intermediate mass scales doesn’t imply an increase in parameter numbers. As we consider the Higgsless limit the complications which arise in the Higgs sector of 4d models essentially disappear. However, no new phenomenology appears in our model until the scale $M_c$ and given the large $M_c$ values required to achieve unification this prohibits the observation of any new particles at the LHC. This is similar to the 5d trinification models of [14] which reproduce the minimal supersymmetric SM at low energies but predict no additional phenomenology.

The 4d models with broken $SU(2)_c$ found to unify in [17] possessed essentially the same features as those outlined above for the unbroken $SU(2)_c$ case and thus provide no real advantage over these models. Given that unification cannot be achieved without a remnant $SU(2)_c$ symmetry in our 5d model we shall not comment further on these.

VIII. CONCLUSION

We have investigated five dimensional quartification models where the symmetry breaking is achieved by a combination of orbifold compactification and the introduction of a boundary Higgs sector. We have shown that the SM Higgs doublet may be identified as the fifth component of a higher dimensional gauge field. This forces matter to be brane localised, with the SM Yukawa structure arising from fermion couplings with Wilson line operators. The models may be considered in a Higgsless limit wherein all gauge fields corresponding to generators broken above the electroweak scale have their mass set by the compactification scale. As in 4d models, intermediate mass scales were required to ensure unification. However only one arbitrary scale, namely the compactification scale, was required, with the embedding of the quartification model in a higher dimensional framework naturally introducing a new class of threshold corrections to the running coupling constants, corresponding to KK excitations of bulk gauge fields. Surprisingly we found that a unique symmetry breaking pattern consistent with both unification and phenomenological demands emerged from our framework. This required $SU(2)_c$ to remain as an exact low-energy symmetry and differed markedly from the 4d case where multiple symmetry breaking routes consistent with unification have been found for both broken and unbroken $SU(2)_c$ models. Importantly the higher dimensional model alleviates the complications arising from the Higgs sector of 4d models by rendering this sector supplementary.
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