A unified convergence analysis for the fractional diffusion equation driven by fractional Gaussian noise with Hurst index $H \in (0, 1)$

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Abstract Here, we provide a unified framework for numerical analysis of stochastic nonlinear fractional diffusion equation driven by fractional Gaussian noise with Hurst index $H \in (0, 1)$. A novel estimate of the second moment of the stochastic integral with respect to fractional Brownian motion is constructed, which greatly contributes to the regularity analyses of the solution in time and space for $H \in (0, 1)$. Then we use spectral Galerkin method and backward Euler convolution quadrature to discretize the fractional Laplacian and Riemann-Liouville fractional derivative, respectively. The sharp error estimates of the built numerical scheme are also obtained. Finally, the extensive numerical experiments verify the theoretical results.

Keywords Stochastic nonlinear fractional diffusion equation · Fractional Gaussian noise · The unified regularity analysis · Sharp error estimate

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1 Introduction

In this paper, we numerically solve the stochastic nonlinear fractional diffusion equation driven by fractional Gaussian noise with Hurst index $H \in (0, 1)$, i.e.,

$$
\begin{align*}
\begin{cases}
\partial_t u + \partial_1^{1-\alpha} A^s u = f(u) + W_H^Q & \text{in } D, \ t \in (0, T], \\
u(\cdot, 0) = 0 & \text{in } D, \\
u = 0 & \text{on } \partial D, \ t \in (0, T],
\end{cases}
\end{align*}
$$

where $D \subset \mathbb{R}^d \ (d = 1, 2, 3)$ is a convex polygonal domain; $A^s$ with $s \in (0, 1)$ is fractional Laplacian defined by

$$
A^s u = \sum_{k=1}^{\infty} \lambda_k^s (u, \phi_k) \phi_k,
$$

and we let $A = -\Delta$ with a zero Dirichlet boundary condition, which has non-decreasing eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ and $L^2$-norm normalized eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$; $f(u)$ is a nonlinear term and we assume that for $u, v \in H$,

$$
\|f(u)\|_H \leq C (1 + \|u\|_H),
$$

$$
\|f(u) - f(v)\|_H \leq C \|u - v\|_H,
$$

with $C$ being a positive constant; and $\partial_1^{1-\alpha}$ with $\alpha \in (0, 1)$ is the Riemann-Liouville fractional derivative, whose definition is \[22\]

$$
\partial_1^{1-\alpha} u = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^t (t - \xi)^{\alpha-1} u(\xi)d\xi.
$$

Here $W_H^Q$ with $H \in (0, 1)$ is the fractional Gaussian process on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ defined by

$$
W_H^Q = \sum_{k=1}^{\infty} \sqrt{\Lambda_k} \phi_k W_k^H,
$$

where $W_k^H$ is one dimensional fBm and $Q$ is a self-adjoint, non-negative, linear operator and its eigenfunctions are same with $A$, and $\{\Lambda_k\}_{k=1}^{\infty}$ are the corresponding eigenvalues of $Q$. Then $A^{s-\rho} Q^1/2$ with $\rho$ being a real number is a Hilbert-Schmidt operator on $\mathbb{H} = L^2(D)$.

Now, we present a brief introduction to \[1\]. Let $D$ be a bounded domain, $B(t)$ be a Brownian motion with $B(0) \in D$, and $\tau_D = \inf\{t > 0 : B(t) \notin D\}$. Denote $T_t$ as a $s$-stable subordinator. Let

$$
X(t) = \begin{cases}
B(T_t), & T_t < \tau_D, \\
\Theta, & T_t \geq \tau_D
\end{cases}
$$

with $\Theta$ being a coffin state, implying to subordinate a killed Brownian motion (when first leaving the domain $D$). The infinitesimal generator of $X(t)$ is $A^s$ \[24\]. Further do the time change to $X(t)$ by the inverse $\alpha$-stable subordinator,
the Fokker-Planck equation of which is $\partial_t u + \partial_t^{1-\alpha} A^s u = 0$, describing the competition between superdiffusion and subdiffusion. If the population of the particles is also being affected by the external source term depending on the density of the particles and the external fractional Gaussian noise, the Fokker-Planck equation Eq. (1) is reached. Most of the time, the influence of external noise \cite{20,23} on a system is unavoidable. Fractional Gaussian noise is one of the most popular external noises, the Hurst index $H$ of which describes the long-range dependence of the fractional Gaussian process \cite{4}. To be specific, the fractional Gaussian process with $H \in (0,1/2)$ and $H \in (1/2,1)$ can be used to model the phenomena with a short memory and a long memory, respectively. From the viewpoint of mathematics, the Hurst index reflects the Hölder property of fractional Brownian motion’s trajectory.

In the past few decades, there have been some numerical discussions for the stochastic PDEs driven by fractional Gaussian noise with the index $H \in (1/2,1)$ or $H = 1/2$ \cite{3,15,16,25,26,27}. In addition, \cite{6,7} use the finite element method to solve the PDE driven by spatial fractional Gaussian noise with an index $H \in (0,1/2)$, where some special Green functions and Itô isometry are used to provide the regularity of the solution, but these techniques can not reflect the influence of the temporal fractional Gaussian noise with $H \in (0,1/2)$ on the regularity of the mild solution and there are hardly researches for the temporal fractional Gaussian noise with $H \in (0,1/2)$. To try to fill the gap, a unified argument for $H \in (0,1)$ is proposed in this paper. A key step of the analysis is to give a novel estimate for $H \in (0,1)$ by the Itô isometry and the equivalence of different fractional Sobolev spaces, i.e.,

\[
E \left( \int_0^T g(r) dW^H(r) \right)^2 \leq C \| \partial_t^{1-2H} g \|^2_{L^2([0,T])}, \quad (4)
\]

which makes $E \left( \int_0^T g(r) dW^H(r) \right)^2$ be bounded by a convolution of $g$ instead of the multiple integral of $g$. Thanks to \cite{4}, we obtain the sharp regularity estimates of the mild solution in time and space for \cite{4} with $H \in (0,1)$ by operator theory approach. Then the full discretization is built by the spectral Galerkin method in space and backward Euler convolution quadrature in time, respectively; the optimal error estimates are provided by transforming the solutions of discrete schemes into a convolution form and using regularity estimates of the mild solution and \cite{4}. It must be emphasized that, because of the nonlinear term $f(u)$ and the operator $\partial_t^{1-2\alpha}$, the derivation of temporal error estimate is not an easy task and some new skills based on Laplace transform need to be introduced; for the details, see Section 4. At the same time, the corresponding error estimates can show the influence of $H$, $s$, and $\alpha$ on convergence rates. Finally, the numerical examples validate the effectiveness of the numerical scheme. To the best of our knowledge, this is the first work on strong convergence analysis for the stochastic PDE driven by fractional Gaussian noise with $H \in (0,1)$. 
The rest of the paper is organized as follows. In Section 2, we first provide some preliminaries and useful lemmas, and then study the spatial regularity and temporal regularity of the mild solution of Eq. (1). We apply spectral Galerkin method to discretize fractional Laplacian and derive optimal error estimate for semidiscrete scheme in Section 3. In Section 4, the backward Euler convolution quadrature method is used to discretize the time fractional derivative and the temporal error estimate of the numerical scheme is also obtained. We provide some numerical examples to validate the theoretically predicted convergence order in Section 5. The paper is concluded with some discussions in the last section. Throughout the paper, we denote by $C$ a generic positive constant, whose value may differ at different occurrences.

2 Regularity of the solution

2.1 Preliminaries

We first recall some definitions and properties on fractional Sobolev spaces [1, 2, 8]. Let $D \subset \mathbb{R}^d$. The fractional Sobolev space $H^s(D)$ with $s \in (0, 1)$ can be defined by

$$H^s(D) = \left\{ w \in H^s(D) : |w|_{H^s(D)} = \int_D \int_D \frac{(w(x) - w(y))^2}{|x - y|^{d+2s}} dxdy < \infty \right\},$$

whose norm can be written as $\| \cdot \|_{H^s(D)} = \| \cdot \|_{L^2(D)} + | \cdot |_{H^s(D)}$. The functions in $H^s(\mathbb{R}^d)$ with support in $\bar{D}$ consist in $H^s_0(D) = \left\{ w \in H^s(\mathbb{R}^d), \text{supp } w \subset \bar{D} \right\}$, with norm

$$\| w \|_{H^s_0(D)} = \| w \|_{L^2(D)} + c_{d,s} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(w(x) - w(y))^2}{|x - y|^{d+2s}} dxdy,$$

which can also be expressed as

$$\| w \|_{H^s_0(D)} = \| (1 + |\omega|^2)^{s/2} \mathcal{F}(w)(\omega) \|_{L^2(\mathbb{R}^d)},$$

with $\mathcal{F}(w)$ being the Fourier transform of $w$ and $c_{d,s} = \frac{2^{2s} \Gamma(d/2 + s)}{\pi^{d/2} \Gamma(1 - s)}$.

**Remark 1** According to [1], $H^s(D)$ coincides with $H^s_0(D)$ when $s \in (0, 1)$. In what follows, we provide some preliminary facts on Hilbert-Schmidt operator, which can refer to [12, 21]. Let $\mathcal{L}(U; V)$ be the Banach space consisting of all bounded linear operators $U \to V$, where $U$ and $V$ are two separable Hilbert spaces and their norms and inner products are denoted by $\| \cdot \|_U$, $\| \cdot \|_V$, and $(\cdot, \cdot)_U$ and $(\cdot, \cdot)_V$, respectively. Let $\mathcal{L}_2(U; V) (\subset \mathcal{L}(U; V))$ consist of all Hilbert-Schmidt operators, whose norm and inner product are defined by

$$\| T \|^2_{\mathcal{L}_2(U; V)} = \sum_{j \in \mathbb{N}^*} \| T \mu_j \|^2_V, \quad (S, T)_{\mathcal{L}_2(U; V)} = \sum_{j \in \mathbb{N}^*} (S \mu_j, T \mu_j)_V, \quad S, T \in \mathcal{L}_2(U; V),$$

where $\mu_j$ are the eigenfunctions of $-\Delta$ in $D$. We also denote $\mathcal{L}_2(D)$ as $\mathcal{L}_2^D(U; V)$.
where \( \{\mu_j\}_{j\in\mathbb{N}} \) are the orthonormal bases in \( \mathbb{U} \) and the above definitions are independent of the specific choice of orthonormal bases. Let \( \mathbb{H} = L^2(D) \) with inner product \( \langle \cdot, \cdot \rangle \) and covariance operator \( Q \) be a self-adjoint, nonnegative linear operator on \( \mathbb{H} \). Below we use the notation \( \sim \) for taking Laplace transform; \( \mathbb{E} \) denotes expectation; let \( \epsilon > 0 \) be arbitrarily small number and abbreviate \( \| \cdot \|_{\mathcal{L}(\mathbb{H})} \) as \( \| \cdot \| \).

Finally, we provide the definitions of the sectors and contour that will be used in the following proofs. For \( \kappa > 0 \) and \( \pi/2 < \theta < \pi \), we define sectors
\[
\Sigma_\theta = \{ z \in \mathbb{C} : z \neq 0, |\arg z| \leq \theta \}, \quad \Sigma_{\theta,\kappa} = \{ z \in \mathbb{C} : |z| > \kappa, |\arg z| \leq \theta \},
\]
and the contour \( \Gamma_{\theta,\kappa} \) by
\[
\Gamma_{\theta,\kappa} = \{ re^{i\theta} : r \geq \kappa \} \cup \{ \kappa e^{i\psi} : |\psi| \leq \theta \} \cup \{ re^{i\theta} : r \geq \kappa \},
\]
where the circular arc is oriented counterclockwise and the two rays are oriented with an increasing imaginary part and \( i^2 = -1 \).

2.2 Some useful lemmas

Here, we provide a few technical lemmas, which can help to get the regularity of the solution of Eq. (1). Let’s first recall the definitions of left- and right-sided Riemann-Liouville fractional integrals and derivatives.

**Definition 1** ([22]) The left- and right-sided Riemann-Liouville fractional integrals of order \( \alpha (\alpha > 0) \) are defined by
\[
x^\alpha_{-} u = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} u(\xi)d\xi, \quad x^\alpha_{+} u = \frac{1}{\Gamma(\alpha)} \int_x^b (\xi-x)^{\alpha-1} u(\xi)d\xi
\]
with \( a, b \in \mathbb{R} \). The Fourier transforms of \( -\infty \partial_{-x}^{-\alpha} u \) and \( x^\alpha_{-} u \) are
\[
\mathcal{F}(-\infty \partial_{-x}^{-\alpha} u)(\omega) = (i\omega)^{-\alpha} \mathcal{F}(u)(\omega), \quad \mathcal{F}(x^\alpha_{-} u)(\omega) = (-i\omega)^{-\alpha} \mathcal{F}(u)(\omega).
\]

**Definition 2** ([22]) The left- and right-sided Riemann-Liouville fractional derivatives of order \( \alpha (\alpha \in (0, 1)) \) are defined by
\[
x^\alpha_{+} u = \partial_x x^\alpha_{-} u = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_a^x (x-\xi)^{-\alpha} u(\xi)d\xi, \quad x^\alpha_{-} u = \partial_x x^\alpha_{+} u = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_x^b (\xi-x)^{-\alpha} u(\xi)d\xi
\]
with \( a, b \in \mathbb{R} \). The Fourier transforms of \( -\infty \partial_{x}^{\alpha} u \) and \( x^\alpha_{+} u \) can be written as
\[
\mathcal{F}(-\infty \partial_{x}^{\alpha} u)(\omega) = (i\omega)^{\alpha} \mathcal{F}(u)(\omega), \quad \mathcal{F}(x^\alpha_{+} u)(\omega) = (-i\omega)^{\alpha} \mathcal{F}(u)(\omega).
\]
According to [9], for \( u \in H^s(a, b) \) with \( a, b \in \mathbb{R} \) and \( \text{supp } u \subset (a, b) \), there exist two positive constants \( C_1 \) and \( C_2 \) such that

\[
\begin{align*}
C_1 \| \partial_x^s u \|_{L^2(a, b)} &\leq |u|_{H^s(a, b)} \leq C_2 \| \partial_x^s u \|_{L^2(a, b)}, \\
C_1 \| \partial_b^s u \|_{L^2(a, b)} &\leq |u|_{H^s(a, b)} \leq C_2 \| \partial_b^s u \|_{L^2(a, b)}.
\end{align*}
\tag{5}
\]

Then we provide a lemma on left- and right-sided Riemann-Liouville fractional integrals.

**Lemma 1** For \( u \in L^2(\mathbb{R}) \), \( \nu \in (0, \frac{1}{2}) \) and \( \text{supp } u \subset (a, b) \) with \( a, b \in \mathbb{R} \), we have

\[
\int_a^b \partial_x^{-\nu} u \cdot \partial_b^{-\nu} u \, dx = \frac{1}{2\Gamma(2\nu)} \int_a^b \int_a^b u(\xi) u(\eta) |\eta - \xi|^{2\nu-1} d\eta d\xi.
\]

and

\[
\int_a^b \int_a^b u(\xi) u(\eta) |\eta - \xi|^{2\nu-1} d\eta d\xi \leq C_1 \| \partial_x^{-\nu} u \|_{L^2(a, b)}^2;
\]

\[
\int_a^b \int_a^b u(\xi) u(\eta) |\eta - \xi|^{2\nu-1} d\eta d\xi \leq C_2 \| \partial_b^{-\nu} u \|_{L^2(a, b)}^2.
\]

**Proof** For \( \text{supp } u \subset (a, b) \), there holds

\[
\begin{align*}
\int_a^b \partial_x^{-\nu} u \cdot \partial_b^{-\nu} u \, dx &= \frac{1}{2\Gamma(\nu)^2} \int_a^b \int_a^b (x - \xi)^{\nu-1} u(\xi) \, d\xi \int_x^b (\eta - x)^{\nu-1} u(\eta) \, d\eta d\xi \\
&= \frac{1}{2\Gamma(\nu)^2} \int_a^b \int_a^b \int_x^b (x - \xi)^{\nu-1} (\eta - x)^{\nu-1} u(\xi) u(\eta) \, d\eta d\xi d\xi \\
&= \frac{1}{2\Gamma(\nu)^2} \int_a^b \int_a^b \int_x^b (x - \xi)^{\nu-1} (\eta - x)^{\nu-1} u(\xi) u(\eta) \, d\eta d\xi d\xi \\
&= \frac{1}{2\Gamma(\nu)^2} \int_a^b \int_a^b \int_x^b (x - \xi)^{\nu-1} (\eta - x)^{\nu-1} \, d\xi u(\xi) u(\eta) \, d\eta d\xi \\
&= \frac{1}{2\Gamma(2\nu)} \int_a^b \int_a^b u(\xi) u(\eta) |\eta - \xi|^{2\nu-1} d\eta d\xi \leq |\eta - \xi|^{2\nu-1} d\eta d\xi.
\end{align*}
\]

For \( \text{supp } u \subset (a, b) \) and \( \nu \in (0, \frac{1}{2}) \), Parseval’s equality leads to

\[
\int_a^b \partial_x^{-\nu} u \cdot \partial_b^{-\nu} u \, dx = \int_{-\infty}^\infty \partial_x^{-\nu} u \cdot \partial_b^{-\nu} u \, dx = \cos(\nu\pi) \| \omega^{-\nu} \mathcal{F}(u) \|_{L^2(\mathbb{R})}^2.
\tag{6}
\]
Moreover, for \( \text{supp} \ u \subset (a, b) \) and \( \nu \in (0, \frac{1}{2}) \), by the Cauchy-Schwarz inequality, one has

\[
\int_a^b a \partial_x^{-\nu} u \partial_b^{-\nu} u dx \leq \frac{C}{\varepsilon_1} \|a \partial_x^{-\nu} u\|_{L^2(a, b)}^2 + \varepsilon_1 \|a \partial_x^{-\nu} u\|_{L^2(\mathbb{R})}^2
\]

\[
\leq \frac{C}{\varepsilon_1} \|a \partial_x^{-\nu} u\|_{L^2(a, b)}^2 + \varepsilon_1 \|\omega^{-\nu} F(u)\|_{L^2(\mathbb{R})}^2.
\]

Combining (6) and taking a suitable \( \varepsilon_1 \), we can get the first desired result and the second one follows by similar arguments.

For one-dimensional fBm, the following facts hold.

**Lemma 2** ([5, 6, 7]) For \( H \in (0, 1/2) \) and \( g_1(t), g_2(t) \in H^{1-2H}_0([0, T]) \), we have

\[
E \left[ \int_0^T g_1(r) dW^H(r) \int_0^T g_2(r) dW^H(r) \right]
= \frac{1}{2} H(1 - 2H) \int \int (g_1(r_1) - g_1(r_2))(g_2(r_1) - g_2(r_2)) \frac{1}{|r_1 - r_2|^{2-2H}} dr_1 dr_2,
\]

(7)

where \( W^H \) means one-dimensional fBm.

**Lemma 3** ([12, 21]) For \( H \in (1/2, 1) \) and \( g_1(t), g_2(t) \in L^2([0, T]) \), there holds

\[
E \left[ \int_0^T g_1(r) dW^H(r) \int_0^T g_2(r) dW^H(r) \right]
= H(2H - 1) \int \int_0^T g_1(r_1)g_2(r_2)|r_1 - r_2|^{2H-2} dr_1 dr_2,
\]

(8)

where \( W^H \) means one-dimensional fBm.

Furthermore, we provide a lemma which plays a key role in this paper.

**Lemma 4** Let \( g \in L^2([0, T]) \), \( \partial_t^{1-2H} g \in L^2([0, T]) \) and \( H \in (0, 1) \). Then we have

\[
E \left( \int_0^T g(r) dW^H(r) \int_0^T g(r) dW^H(r) \right) \leq C \left\| \partial_t^{1-2H} g \right\|_{L^2([0, T])}^2,
\]

\[
E \left( \int_0^T g(T - r) dW^H(r) \int_0^T g(T - r) dW^H(r) \right) \leq C \left\| \partial_t^{1-2H} g \right\|_{L^2([0, T])}^2.
\]
Proof When $H = \frac{1}{2}$, the desired results hold directly. As for $H \in (0, \frac{1}{2})$, using Lemma 2, Eq. (5), and Remark 1, one has

$$
\mathbb{E} \left( \int_0^T g(r) dW^H(r) \int_0^T g(r) dW^H(r) \right)
= \frac{1}{2} H(1 - 2H) \int_\mathbb{R} \int_\mathbb{R} \frac{(\hat{g}(r_1) - \hat{g}(r_2))^2}{|r_1 - r_2|^{2-2H}} dr_1 dr_2
\leq C \|g\|^2_{H \frac{1-2H}{H} (0,T)}
\leq C \left\| \partial_t^{\frac{1-2H}{2}} g \right\|^2_{L^2(0,T)}.
$$

where $\hat{g}$ is the zero extension of $g$. As for $H \in (\frac{1}{2}, 1)$, Lemmas 1 and 3 give

$$
\mathbb{E} \left( \int_0^T g(r) dW^H(r) \int_0^T g(r) dW^H(r) \right)
= H(2H - 1) \int_0^T \int_0^T \frac{g(r_1)g(r_2)}{|r_1 - r_2|^{2-2H}} dr_1 dr_2
\leq C \left\| \partial_t^{\frac{1-2H}{2}} g \right\|^2_{L^2(0,T)}.
$$

As for the second estimate, it can be got similarly.

Remark 2 Different from the skills used in [3,15,16,25,26,27], with the help of Lemma 4, we can obtain the corresponding regularity and error estimates for $H \in (0, 1)$ by Laplace transform and operator theory approach.

Lemma 5 ([13,14]) Let $D$ be a bounded domain in $\mathbb{R}^d$ ($d = 1, 2, 3$) and $\lambda_k$ the $k$-th eigenvalue of the Dirichlet boundary problem for the Laplace operator $A = -\Delta$ in $D$. Then, for all $k \geq 1$,

$$
\lambda_k \geq \frac{C_d d^{2/d}}{d + 2} |D|^{-2/d},
$$

where $C_d = (2\pi)^{d/2} B_d^{-2/d}$, $|D|$ is the volume of $D$, and $B_d$ means the volume of the unit $d$-dimensional ball.

2.3 A priori estimate of the solution

In this subsection, we provide the regularity of the solution of Eq. (1) in time and space. Firstly, we denote operator $\mathcal{R}(t)$ as

$$
\mathcal{R}(t) = \frac{1}{2\pi i} \int_{\Gamma_{\alpha,\infty}} e^{zt} z^{\alpha-1} (z^\alpha + A^\alpha)^{-1} dz
$$
and it satisfies
\[ \| A^{\beta s} \tilde{R}(z) \| \leq C |z|^{\beta \alpha - 1} \quad \forall z \in \Sigma_\theta \text{ and } \beta \in [0, 1]. \] (9)

Also, we can rewrite \( R(t) \) as
\[ R(t)u = \sum_{k=1}^{\infty} E_k(t)(u, \phi_k)\phi_k, \]
where \( \{E_k(t)\}_{k=1}^{\infty} \) are defined by
\[ E_k(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{zt} z^{\alpha - 1}(z^\alpha + \lambda_k s)^{-1} \, dz, \]
and the following estimate holds
\[ |\lambda_k^{s\beta} \tilde{E}_k(z)| \leq C |z|^{\beta \alpha - 1} \quad \forall z \in \Sigma_\theta \text{ and } \beta \in [0, 1]. \] (10)

Thus by taking Laplace and inverse Laplace transforms, the mild solution of Eq. (1) can be written as
\[ u = \int_0^t R(t-r)f(u(r)) \, dr + \int_0^t R(t-r)dW^H_Q(r) \]
\[ = \int_0^t R(t-r)f(u(r)) \, dr + \sum_{k=1}^{\infty} \int_0^t \sqrt{A_k}E_k(t-r)\phi_k dW^H_k(r). \] (11)

Then we can obtain the following spatial regularity of \( u \).

**Theorem 1** Let \( u \) be the mild solution of Eq. (1) and \( \|A^{-\rho}Q^{1/2}\|_{L_2} < \infty \) with \( \rho \in [0, \min(\frac{sH}{\alpha}, s+\epsilon)] \) and \( \alpha \in (0, 1) \). Then the mild solution \( u \) satisfies
\[ \mathbb{E} \|A^\sigma u\|_H^2 \leq C, \]
where \( \sigma \in [0, \min(s-\rho, \frac{sH}{\alpha}-\rho-\epsilon)] \).

**Proof** Simple calculations give
\[
\mathbb{E} \|A^\sigma u\|_H^2 \leq 2 \mathbb{E} \left\| \int_0^t A^\sigma R(t-r)f(u(r)) \, dr \right\|_H^2 + 2 \mathbb{E} \left\| \int_0^t A^\sigma R(t-r)dW^H_Q(r) \right\|_H^2 \\
\leq I + II.
\]

As for \( I \), the resolvent estimate (9), assumption (2), and the Cauchy-Schwarz inequality lead to
\[
I \leq C \mathbb{E} \left( \int_0^t (t-r)^{-\sigma \alpha/s}(1 + \|u(r)\|_H) \, dr \right)^2 \\
\leq C \mathbb{E} \left( \int_0^t (t-r)^{-2\sigma \alpha/s+1-\epsilon}(1 + \|u(r)\|_H^2) \, dr \right) \\
\leq C + C \int_0^t (t-r)^{-2\sigma \alpha/s+1-\epsilon}\mathbb{E}\|u(r)\|_H^2 \, dr,
\]
where we need to require $-2\sigma\alpha/s + 1 > -1$, i.e., $\sigma < \frac{\alpha}{s}$.

As for $II$, using (11), Lemma 4, and $\|A^{-\rho}Q^{1/2}\|_{L_2} < \infty$, we have

\[
II \leq C \sum_{k=1}^{\infty} \mathbb{E} \left( \int_0^t \lambda_k^\sigma \sqrt{\Lambda_k} E_k(t-r) \phi_k dW^H_k(r), \int_0^t \lambda_k^\sigma \sqrt{\Lambda_k} E_k(t-r) \phi_k dW^H_k(r) \right) \\
\leq C \sum_{k=1}^{\infty} \int_0^t |\lambda_k^\sigma \sqrt{\Lambda_k} E_k(r)|^2 dr \\
\leq C \sup_{k \in \mathbb{N}} \int_0^t |\lambda_k^{\sigma+\rho} \partial_r^{-\frac{1-2H}{2}} E_k(r)|^2 dr.
\]

By the definition of $E_k(r)$ and resolvent estimate (10), one can get

\[
\int_0^t |\lambda_k^{\sigma+\rho} \partial_r^{-\frac{1-2H}{2}} E_k(r)|^2 dr \\
\leq C \int_0^t \left( \int_{\Gamma_{r,s}} |e^{zr}| \lambda_k^{\sigma+\rho} z^{-\frac{1-2H}{2}} \tilde{E}_k(z) dz \right)^2 dr \\
\leq C \int_0^t \left( \int_{\Gamma_{r,s}} |e^{zr}| |z|^{-\frac{1-2H}{2}+(\sigma+\rho)\alpha/s-1} \tilde{E}_k(z) dz \right)^2 dr \\
\leq C \int_0^t z^{2H-1-2(\sigma+\rho)\alpha/s} dr.
\]

To preserve the boundedness of $E\|A^\sigma u\|_{L_2}^2$, we require $2H - 1 - 2(\sigma + \rho)\alpha/s > -1$, which leads to $\sigma < \frac{4H}{\alpha} - \rho$. Combining the estimates of $I$ and $II$ and the Grönwall inequality, the desired result is obtained.

In the following, we provide the Hölder regularity of the mild solution $u$.

**Theorem 2** Let $u$ be the mild solution of Eq. (1) and $\|A^{-\rho}Q^{1/2}\|_{L_2} < \infty$ with $\rho \in [0, \frac{4H}{\alpha}) \cap [0, s]$ and $\alpha \in (0, 1)$. Then we have

\[
\mathbb{E} \left\| \frac{u(t) - u(t-\tau)}{\tau^{\gamma}} \right\|_{L_2}^2 \leq C,
\]

where $\gamma \in \left[0, H - \frac{\alpha}{s}\right)$. 

Proof We first divide $E \left\| \frac{u(t) - u(t - \tau)}{\tau^\gamma} \right\|^2_H$ into two parts

$$
E \left\| \frac{u(t) - u(t - \tau)}{\tau^\gamma} \right\|^2_H \\
\leq E \left\| \int_0^t \mathcal{R}(t - r)f(u(r))dr - \int_0^{t-\tau} \mathcal{R}(t - \tau - r)f(u(r))dr \right\|^2_H \\
+ E \left\| \int_0^t \mathcal{R}(t - r)W_H^2(r) - \int_0^{t-\tau} \mathcal{R}(t - \tau - r)W_H^2(r) \right\|^2_H \\
\leq I + II.
$$

As for $I$, we have

$$
I \leq E \left\| \int_0^{t-\tau} (\mathcal{R}(t - r) - \mathcal{R}(t - r - \tau))f(u(r))dr \right\|^2_H \\
+ E \left\| \int_0^{t-\tau} \mathcal{R}(t - r)f(u(r))dr \right\|^2_H \\
\leq I_1 + I_2.
$$

For $I_1$, the fact $|z^{1-\gamma}| \leq C|z|^\gamma$ with $z \in \Gamma_{\theta,\kappa}$ and $\gamma \in [0, 1]$ [10] and Eq. (2) give

$$
I_1 \leq C \left\| \int_0^{t-\tau} \int_{\Gamma_{\theta,\kappa}} e^{z(t-r)} - e^{z(t-r-\tau)} \mathcal{R}(z)dz f(u(r))dr \right\|^2_H \\
\leq C \left( \int_0^{t-\tau} \int_{\Gamma_{\theta,\kappa}} |e^{z(t-r)}| |z|^\gamma-1 d\gamma \right) \|f(u(r))\|_H dr \\
\leq C \left( \int_0^{t-\tau} (t - \tau - r)^{-\gamma} (1 + \|u(r)\|_H) dr \right)^2 \\
\leq C,
$$

where we need to require $\gamma \in [0, 1)$. Similarly, $I_2$,

$$
I_2 \leq C \left( \int_0^{t-\tau} \|f(u(r))\|_H^2 dr \right)^2 \\
\leq C \tau^{1-2\gamma} \int_{t-\tau}^t \|f(u(r))\|_H^2 dr \\
\leq C,
$$

where we need to require $\gamma \in [0, 1)$. 
As for $II$, there are

\[ II \leq C \mathbb{E} \left\| \sum_{k=1}^{\infty} \left( \int_0^{t-\tau} \sqrt{\Lambda_k} E_k(t-r) \phi_k dW_k^H(r) - \int_0^{t-\tau} \sqrt{\Lambda_k} E_k(t-\tau-r) \phi_k dW_k^H(r) \right) \right\|_H^2 \]

\[ \leq C \mathbb{E} \left\| \sum_{k=1}^{\infty} \int_0^{t-\tau} \sqrt{\Lambda_k} [E_k(t-r) - E_k(t-\tau-r)] \phi_k dW_k^H(r) \right\|_H^2 \]

\[ + C \mathbb{E} \left\| \sum_{k=1}^{\infty} \int_{t-\tau}^{t} \sqrt{\Lambda_k} E_k(t-r) \phi_k dW_k^H(r) \right\|_H^2 \]

\[ \leq II_1 + II_2. \]

For $II_1$, $\|A^{-\rho}Q^{1/2}\| < \infty$ and the definition of $E_k$ lead to

\[ II_1 \leq C \sum_{k=1}^{\infty} \int_0^{t-\tau} \left| \sqrt{\Lambda_k} \phi_k \frac{1-2H}{\tau^\gamma} (E_k(r+\tau) - E_k(r)) \right|^2 \, dr \]

\[ \leq C \sup_{k \in \mathbb{N}} \int_0^{t-\tau} \left| \int_{\Gamma_{\theta,\kappa}} e^{zr} e^{2\tau r \gamma} - \frac{1}{\tau^\gamma} \lambda_k^\theta \frac{2-2H}{\gamma} \bar{E}_k(z) \, dz \right|^2 \, dr. \]

Combining (10) and the fact $|e^{zr}| \leq C |z|^\gamma$ with $z \in \Gamma_{\theta,\kappa}$ and $\gamma \in [0, 1]$, one has

\[ \int_0^{t-\tau} \left( \int_{\Gamma_{\theta,\kappa}} \left| e^{zr} \frac{1-2H}{\tau^\gamma} \lambda_k^\theta \frac{2-2H}{\gamma} \bar{E}_k(z) \right|^2 \, dz \right)^2 \, dr \]

\[ \leq C \int_0^{t} \left( \int_{\Gamma_{\theta,\kappa}} |e^{zr}| \frac{1-2H+\gamma+\rho\alpha/s-1}{\gamma} |dz| \right)^2 \, dr \]

\[ \leq C \int_0^{t} e^{2H-1-2\gamma-2\rho\alpha/s} \, dr. \]

To preserve the boundedness of $II_1$, $\gamma$ should satisfy $2H-1-2\gamma-2\rho\alpha/s > -1$, which yields $\gamma < H - \frac{\rho\alpha}{s}$.

As for $II_2$, when $H = 1/2$, the Itô isometry shows

\[ II_2 \leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_0^{t-\tau} |\sqrt{\Lambda_k} E_k(t-r)|^2 \, dr \]

\[ \leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_0^{\tau} |\sqrt{\Lambda_k} E_k(r)|^2 \, dr; \]
when \( H \in (1/2, 1) \), by Lemmas 1 and 3 we deduce

\[
\begin{align*}
II_2 &\leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{t-\tau}^{t} \int_{t-\tau}^{t} \frac{\sqrt{A_k} E_k(t-r_1) \sqrt{A_k} E_k(t-r_2)}{|r_1 - r_2|^{2-2H}} dr_1 dr_2 \\
&\leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{0}^{\tau} \int_{0}^{\tau} \frac{\sqrt{A_k} E_k(r_1) \sqrt{A_k} E_k(r_2)}{|r_1 - r_2|^{2-2H}} dr_1 dr_2 \\
&\leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{0}^{\tau} \int_{0}^{\tau} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(r) dr_1 dr_2
\end{align*}
\]

when \( H \in (0, 1/2) \), by Lemma 2 and Remark 1 one can get

\[
II_2 \leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{0}^{\tau} \int_{0}^{\tau} \frac{(\sqrt{A_k} E_k(t-r_1) - \sqrt{A_k} E_k(t-r_2))^2}{|r_1 - r_2|^{2-2H}} dr_1 dr_2 \\
\leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{0}^{\tau} \int_{0}^{\tau} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(r) dr_1 dr_2 \\
\leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{0}^{\tau} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(r) dr,
\]

where \( \chi_{[a,b]} \) is the characteristic function on \([a, b]\). According to \( \| A^{-\rho} Q^{1/2} \|_{L^2} < \infty \), we have

\[
II_2 \leq C \tau^{-2\gamma} \sum_{k=1}^{\infty} \int_{0}^{\tau} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(r) dr \\
\leq C \tau^{-2\gamma} \sup_{k \in \mathbb{N}^*} \int_{0}^{\tau} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(r) dr.
\]

By (10), one gets

\[
\int_{0}^{\tau} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(r) dr \\
\leq C \int_{0}^{\tau} \left( \int_{\Gamma_0} e^{zT} \sqrt{A_k} \partial_{\tau}^{1-2H} E_k(z) dz \right)^2 dr \\
\leq C \int_{0}^{\tau} \left( \int_{\Gamma_0} e^{zT} |z|^{1-2H - \rho \alpha/s - 1} dz \right)^2 dr \\
\leq C \int_{0}^{\tau} r^{2H-1-2\rho \alpha/s} dr \\
\leq C \tau^{2H-2\rho \alpha/s},
\]

which implies \( \gamma < H - \rho \alpha/s \). Thus the proof is completed.
3 Spatial discretization and error analysis

In this section, we use a spectral Galerkin method to discretize fractional Laplacian and the error estimate is built. We first introduce a finite dimensional subspace of $\mathbb{H}$ by $\mathbb{H}_N = \text{span}\{\phi_1, \phi_2, \ldots, \phi_N\}$ with $N \in \mathbb{N}^*$ and define the projection operator $P_N : \mathbb{H} \rightarrow \mathbb{H}_N$ by

$$P_N u = \sum_{i=1}^{N} (u, \phi_i) \phi_i \quad \forall u \in \mathbb{H}.$$ 

Define $A_N^s : \mathbb{H}_N \rightarrow \mathbb{H}_N$ as

$$(A_N^s u_N, v_N) = (A^s u_N, v_N) \quad \forall u_N, v_N \in \mathbb{H}_N.$$ 

It is easy to verify that

$$A_N^s u_N = A_N^s P_N u_N = P_N A_N^s u_N = \sum_{k=1}^{N} \lambda_k^s (u_N, \phi_k) \phi_k.$$ 

Thus the spectral Galerkin semidiscrete scheme of (1) can be written as: find $u_N(t) \in \mathbb{H}_N$ satisfying

$$\begin{cases}
\partial_t u_N + \partial_t^{1-\alpha} A_N^s u_N = P_N f(u_N) + P_N W^H_Q H Q t \in (0, T], \\
u_N(0) = 0.
\end{cases} \quad (12)$$

Taking Laplace transform and inverse Laplace transform gives

$$u_N = \int_0^t \mathcal{R}_N(t-r)P_N f(u_N(r))dr + \int_0^t \mathcal{R}_N(t-r)P_N dW^H_Q (r),$$

where

$$\mathcal{R}_N(t) = \frac{1}{2\pi i} \int_{\Gamma_{s, \infty}} e^{zt} z^{\alpha-1} (z^\alpha + A_N^s)^{-1} dz.$$ 

By the definitions of $A_N^s$ and $P_N$, we rewrite $u_N$ as

$$u_N = \int_0^t \mathcal{R}_N(t-r)P_N f(u_N(r))dr + \sum_{k=1}^{N} \int_0^t \sqrt{\lambda_k} E_k(t-r) \phi_k dW^H_Q (r). \quad (13)$$

Similar to the proofs of Theorems 1 and 2, one can get the following two estimates on $u_N$.

**Theorem 3** Let $u_N$ be the mild solution of Eq. (12) and $\|A^{-\rho}Q^{1/2}\|_{\mathcal{L}_2} < \infty$ with $\rho \in \left[0, \min\left(\frac{sH}{\alpha}, s + \epsilon\right)\right]$ and $\alpha \in (0, 1)$. Then the mild solution $u_N$ satisfies

$$\mathbb{E}\|A^\sigma u_N\|_{\mathbb{H}}^2 \leq C,$$

where $\sigma \in \left[0, \min\left(s - \rho, \frac{sH}{\alpha} - \rho - \epsilon\right)\right]$. 

Theorem 4 Let $u_N$ be the mild solution of Eq. (12) and $\|A^{-\rho}Q^{1/2}\|_{L_2} < \infty$ with $\rho \in [0, \frac{H}{\alpha}) \cap [0, s]$ and $\alpha \in (0, 1)$. Then we have

$$E \left\| \frac{u_N(t) - u_N(t - \tau)}{\tau^{\gamma}} \right\|_H^2 \leq C,$$

where $\gamma \in [0, H - \frac{\mu}{s})$.

Now we provide the spatial error estimate.

Theorem 5 Let $u$ and $u_N$ be the solutions of (1) and (12), respectively. Assuming $\|A^{-\rho}Q^{1/2}\|_{L_2} < \infty$ with $\rho \in \left[0, \min\left(\frac{H}{\alpha}, s + \epsilon\right)\right)$ and $\alpha \in (0, 1)$, we have

$$\left( E \|u - u_N\|_H^2 \right)^{1/2} \leq C(N + 1)^{-2\sigma/d},$$

where $\sigma \in \left(0, \min(s - \rho, \frac{H}{\alpha} - \rho - \epsilon)\right] \text{ and } d \text{ is the dimension of } \Omega.$

Proof By Eqs. (11) and (13), there is

$$E \|u - u_N\|_H^2 \leq C E \left\| \int_0^t \mathcal{R}(t - r)f(u(r)) - \mathcal{R}_N(t - r)P_N f(u_N(r))dr \right\|_H^2$$

$$+ C E \left\| \sum_{k=N+1}^{\infty} \int_0^t \sqrt{A_k} E_k(t - r) \phi_k dW_k^H(r) \right\|_H^2$$

$$\leq I + II.$$

For $I$, we split it into two parts

$$I \leq C E \left\| \int_0^t \mathcal{R}(t - r)(f(u(r)) - f(u_N(r)))dr \right\|_H^2$$

$$+ C E \left\| \int_0^t (\mathcal{R}(t - r) - \mathcal{R}_N(t - r)P_N) f(u_N(r))dr \right\|_H^2$$

$$\leq I_1 + I_2.$$

The resolvent estimate (9) gives

$$I_1 \leq C E \left( \int_0^t \|f(u(r)) - f(u_N(r))\|_H dr \right)^2$$

$$\leq C \int_0^t E \|u(r) - u_N(r)\|_H^2 dr.$$
By the definitions of $R_N$, $P_N$, and Theorem 3, one has

\[
I_2 \leq CE \sum_{k=N+1}^{\infty} \left( \int_0^t E_k(t-r)(f(u_N(r)), \phi_k) \, dr \right)^2 \\
\leq CE \sum_{k=N+1}^{\infty} \lambda_k^{-2\sigma} \left( \int_0^t E_k(t-r) \lambda_k^\sigma (f(u_N(r)), \phi_k) \, dr \right)^2 \\
\leq C \sup_{k \geq N+1} \lambda_k^{-2\sigma} \int_0^t (t-r)^{1-\epsilon} \lambda_k^\sigma E_k(t-r) \, dr \\
\leq C \sup_{k \geq N+1} \lambda_k^{-2\sigma},
\]

where $\sigma < \frac{\alpha}{2}$. As for $II$, one can get

\[
II \leq C \sum_{k=N+1}^{\infty} \int_0^t \left| \sqrt{A_k} \partial_r^{1-2H} E_k(r) \right|^2 \, dr \\
\leq C \sup_{k \geq N+1} \int_0^t \left| \lambda_k^\sigma \partial_r^{1-2H} E_k(r) \right|^2 \, dr,
\]

where we use Lemma 4 and $\| A^{-\rho} Q^{1/2} \|_{L_2} < \infty$. Then Eq. (10) gives

\[
\int_0^t \left| \lambda_k^\sigma \partial_r^{1-2H} E_k(r) \right|^2 \, dr \\
\leq C \int_0^t \int_{T_{i,n}} \left| e^{zr} \lambda_k^\sigma \partial_r^{1-2H} \tilde{E}_k(z) \right|^2 \, dz \, dr \\
\leq C \lambda_k^{-2\sigma} \int_0^t \left( \int_{T_{i,n}} |e^{zr}| \left| \partial_r^{1-2H} \tilde{E}_k(z) \right| \, dz \right)^2 \, dr \\
\leq C \lambda_k^{-2\sigma} \int_0^t \left( \int_{T_{i,n}} |e^{zr}| \left| \partial_r^{1-2H+\sigma+\rho}/s-1 \right| \, dz \right)^2 \, dr \\
\leq C \lambda_k^{-2\sigma} \int_0^t r^{2H-1-2(\sigma+\rho)/s} \, dr,
\]

where we need to require $2H - 1 - 2(\sigma+\rho)/s > -1$, i.e., $\sigma < \frac{2H}{s} - \rho$. Thus, by the Grönwall inequality and Lemma 5, the desired result is obtained.

4 Time discretization and error analysis

Here backward Euler convolution quadrature [17,18,19] is used to discretize the Riemann-Liouville fractional derivative and the corresponding error estimate is also provided. Let the time step size $\tau = T/L$ with $L \in \mathbb{N}^*$, $t_i = i\tau$, ...
\( i = 0, 1, \ldots, L \), and \( 0 = t_0 < t_1 < \cdots < t_L = T \). Introduce the operator \( \tilde{\partial}_t \) as

\[
\tilde{\partial}_t u(t) = \begin{cases} 
0 & t = t_0, \\
\frac{u(t_j) - u(t_{j-1})}{\tau} & t \in (t_{j-1}, t_j]. 
\end{cases}
\]  
(14)

Using backward Euler method to discretize the corresponding temporal operator, the fully-discrete scheme of (1) can be written as

\[
\frac{u_N^n - u_N^{n-1}}{\tau} + \sum_{i=0}^{n-1} d_i^{(1-\alpha)} A_s^i u_N^{n-i} = P_N f(u_N^{n-1}) + P_N \tilde{\partial}_t W^H(t_n),
\]  
(15)

where

\[
(\delta_t(\zeta))^\alpha = \sum_{i=0}^{\infty} d_i^{(\alpha)} \zeta^i, \quad \delta_t(\zeta) = \frac{1 - \zeta}{\tau}.
\]

Denote \( \bar{F}(t) \) as

\[
\bar{F}(t) = \begin{cases} 
0 & t = t_0, \\
f(u_N^{t-1}) & t \in (t_{j-1}, t_j], 
\end{cases}
\]

and \( F(t) = f(u_N(t)) \). In the following, we abbreviate \( P_N F(t) \) and \( P_N \bar{F}(t) \) as \( F_N \) and \( \bar{F}_N \). With the help of the facts [10]

\[
\sum_{n=1}^{\infty} \partial_t W^H(t_n)e^{-zt_n} = \frac{z}{e^{z\tau} - 1} \tilde{\partial}_t W^H(t_n), \quad \sum_{n=1}^{\infty} \bar{F}_N(t_n)e^{-zt_n} = \frac{z}{e^{z\tau} - 1} \bar{F}_N(z)
\]

and doing simple calculations, we can get

\[
u_N^n = \int_0^{t_n} \bar{R}_N(t_n - \tau) \bar{F}_N(\tau) d\tau + \sum_{k=1}^{N} \int_0^{t_n} \sqrt{A_k} \bar{E}_k(t_n - \tau) \phi_k \tilde{\partial}_t W^H_k(\tau) d\tau,
\]

(16)

where

\[
\bar{R}_N(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{z\tau} (\delta_t(e^{-z\tau}))^{\alpha-1} ((\delta_t(e^{-z\tau}))^\alpha + A_k)^{-1} \frac{z\tau}{e^{z\tau} - 1} dz
\]

and

\[
\bar{E}_k(t) = \frac{1}{2\pi i} \int_{\Gamma_{\theta,\kappa}} e^{z\tau} (\delta_t(e^{-z\tau}))^{\alpha-1} ((\delta_t(e^{-z\tau}))^\alpha + \lambda_k)^{-1} \frac{z\tau}{e^{z\tau} - 1} dz.
\]

Here \( \Gamma_{\theta,\kappa} = \{ z \in \mathbb{C} : \kappa \leq |z| \leq \frac{\pi}{\sin(\theta)} \}, |\arg z| = \theta \} \cup \{ z \in \mathbb{C} : |z| = \kappa, |\arg z| \leq \theta \}. \)

To estimate \( \mathbb{E}\|u_N(t_n) - u_N^N\|^2_H \), the following lemma is needed.
Lemma 6 (10) Let the given $\alpha \in (0,1)$ and $\theta \in (\frac{\pi}{2}, \arccot (-\frac{2}{3}))$, where \arccot means the inverse function of cot, and a fixed $\xi \in (0,1)$. Then, when $z$ lies in the region enclosed by $\Gamma_\theta^\tau = \{ z = -\ln(1/\tau) + iy : y \in \mathbb{R} \text{ and } |y| \leq \pi/\tau \}$, $\Gamma_\theta^\tau$, and the two lines $\mathbb{R} \pm i\pi/\tau$, whenever $0 < \kappa \leq \min(1/T, -\ln(\xi)/\tau)$, $\delta_\tau(e^{-z\tau})$ and $(\delta_\tau(e^{-z\tau}) + A)^{-1}$ are both analytic. Furthermore, we have

$$
\delta_\tau(e^{-z\tau}) \in \Sigma_\theta \quad \forall z \in \Gamma_\theta^\tau,
$$

$$
C_0|z| \leq |\delta_\tau(e^{-z\tau})| \leq C_1|z| \quad \forall z \in \Gamma_\theta^\tau,
$$

$$
|\delta_\tau(e^{-z\tau}) - z| \leq C\tau|z|^2 \quad \forall z \in \Gamma_\theta^\tau,
$$

$$
|\delta_\tau(e^{-z\tau})^\alpha - z^\alpha| \leq C\tau|z|^{\alpha+1} \quad \forall z \in \Gamma_\theta^\tau,
$$

where $\kappa \in (0, \min(1/T, -\ln(\xi)/\tau))$ and the constants $C_0$, $C_1$, and $C$ are independent of $\tau$.

Introduce $G_k$ as

$$
G_k(t_{i-1}) = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \tilde{E}_k(r) dr \quad t \in [t_{i-1}, t_i),
$$

and denote $G_k(t_i)$ as $G_{k,i}$. According to the definition of $\tilde{E}_k(r)$, we have

$$
G_{k,i-1} = \frac{1}{\tau} \int_{t_{i-1}}^{t_i} \frac{1}{2\pi i} \int_{\Gamma_\theta^\tau} e^{z\tau} \tilde{E}_k(z) dz dr
$$

$$
= \frac{1}{2\pi i} \int_{\Gamma_\theta^\tau} \frac{e^{zt_{i-1}} - e^{zt_i}}{zt} \tilde{E}_k(z) dz
$$

$$
= \frac{1}{2\pi i} \int_{\Gamma_\theta^\tau} e^{zt_{i-1}} (\delta_\tau(e^{-z\tau}))^{\alpha-1} ((\delta_\tau(e^{-z\tau}))^{\alpha} + \lambda_k^\alpha)^{-1} dz.
$$

Simple calculations lead to

$$
\sum_{i=0}^{\infty} G_{k,i} \zeta^i = \frac{1}{\tau} (\delta_\tau(\zeta))^{\alpha-1} ((\delta_\tau(\zeta))^{\alpha} + \lambda_k^\alpha)^{-1}.
$$

Thus the Laplace transform of $G_k$ can be written as

$$
\tilde{G}_k(z) = \int_0^\infty e^{-zt} G_k(t) dt = \sum_{i=0}^{\infty} G_{k,i} \int_{t_i}^{t_{i+1}} e^{-zt} dt
$$

$$
= \sum_{i=0}^{\infty} G_{k,i} e^{-zt_i} \frac{1 - e^{-zT}}{z} = \frac{1}{z} (\delta_\tau(e^{-z\tau}))^\alpha ((\delta_\tau(e^{-z\tau}))^{\alpha} + \lambda_k^\alpha)^{-1}.
$$

Then we provide an estimate of $\tilde{G}_k(z)$. (18)
Lemma 7 Let $G_k$ be defined in \((17)\). For $\tau < \tau^*$ (the value of $\tau^*$ depends on $\lambda_k$), one has
\[
|\lambda_k^{\beta} G_k(z)| \leq \begin{cases} 
C|z|^{\beta \alpha - 1} e^{\beta |z| \tau} & z \in \Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^\tau, \\
C|z|^{\beta \alpha - 1} & z \in \Gamma_{\theta, \kappa}^\tau,
\end{cases}
\]
where $\beta \in [0, 1]$, $s \in (0, 1)$, and $\alpha \in (0, 1)$.

Proof When $z \in \Gamma_{\theta, \kappa}^\tau$, the desired estimate can be got by Eq. \((10)\) and Lemma 6. As for $z = |z| e^{i \theta} \in \Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^\tau$, we consider $\tau \delta_{\tau}(e^{-z \tau})$ first. Simple calculations give
\[
|\tau \delta_{\tau}(e^{-z \tau})| = \left|1 - e^{-z \tau} \right| = \left|1 - e^{-|z| \cos(\theta) \tau} e^{-|z| \sin(\theta) \tau} \right|.
\]
Since $z \in \Gamma_{\theta, \kappa} \setminus \Gamma_{\theta, \kappa}^\tau$, one has $|z| \geq \pi \tau \sin(\theta)$. Choosing a suitable $\theta \in \left(\pi^2, \pi\right)$ satisfying $\cot(\theta) \pi \leq -1$, we have
\[
|\tau \delta_{\tau}(e^{-z \tau})| \geq e - 1,
\]
which yields
\[
|\delta_{\tau}(e^{-z \tau})| \geq \frac{e - 1}{\tau}.
\]
Let $\tau$ be small enough to satisfy $\left(\frac{e - 1}{\tau}\right)^\alpha > 2 \lambda_k^\alpha$. Then it has
\[
\left|\lambda_k^{\beta} \left(\delta_{\tau}(e^{-z \tau})\right)^\alpha + \lambda_k^\alpha \right| \leq C|\delta_{\tau}(e^{-z \tau})|^{(\beta - 1)\alpha}.
\]
Combining the definition of $G_k$, one can get
\[
|\tilde{G}_k| \leq C|z|^{-1} |\delta_{\tau}(e^{-z \tau})|^{\beta \alpha}.
\]
According to the fact $\delta_{\tau}(e^{-z \tau}) \leq |z| \sum_{k=1}^\infty \frac{|z|^k}{k!} \leq |z| |z|^{\tau}$, $\forall z \in \Gamma_{\theta, \kappa} [11]$, the desired result is reached.

In the rest of paper, we take $\kappa \leq \frac{\pi}{\tau \sin(\theta)}$. Then we provide the temporal error estimate.

Theorem 6 Let $u_N(t_n)$ and $u_N^\kappa$ be the solutions of Eqs. \((12)\) and \((15)\), respectively, and $\|A^{-\rho} Q^{1/2} \|_{L_2} < \infty$ with $\rho \in [0, \frac{1}{\alpha}) \cap [0, s]$ and $\alpha \in (0, 1)$. Then there holds
\[
\left(\mathbb{E}\|u_N(t_n) - u_N^\kappa\|_{\mathbb{R}}^2\right)^{1/2} \leq C\tau^{H - \frac{\rho}{\alpha} - \epsilon}.
\]
Proof Using (13) and (16) and taking the expectation of \( \| u_N(t_n - u^n_N) \|_H^2 \) yield

\[
\begin{align*}
\mathbb{E}\| u_N(t_n - u^n_N) \|^2_H & = \mathbb{E} \left\| \int_0^{t_n} \mathcal{R}_N(t_n - r)F_N(r) - \bar{\mathcal{R}}_N(t_n - r)\bar{F}_N(r)dr \right\|^2_H \\
& + \mathbb{E} \left\| \sum_{k=1}^{N} \left( \int_0^{t_n} \sqrt{\Lambda_k}E_k(t_n - r)\phi_k dW^H_k(r) \\
- \int_0^{t_n} \sqrt{\Lambda_k}E\bar{E}_k(t_n - r)\phi_k dW^H_k(r)dr \right) \right\|^2_H \\
& \leq \mathbb{E} \left\| \int_0^{t_n} \mathcal{R}_N(t_n - r)F_N(r) - \bar{\mathcal{R}}_N(t_n - r)\bar{F}_N(r)dr \right\|^2_H \\
& + \mathbb{E} \left\| \int_0^{t_n} \sum_{k=1}^{N} \sqrt{\Lambda_k}(E_k(t_n - r) - \bar{E}_k(t_n - r))\phi_k dW^H_k(r) \right\|^2_H \\
& + \mathbb{E} \left\| \int_0^{t_n} \sum_{k=1}^{N} \sqrt{\Lambda_k}E\bar{E}_k(t_n - r)\phi_k (dW^H_k(r) - \bar{\theta}_K W^H_k(r)dr) \right\|^2_H \\
= \vartheta_1 + \vartheta_2 + \vartheta_3.
\end{align*}
\]

For \( \vartheta_1 \), one can split it as

\[
\begin{align*}
\vartheta_1 & \leq \mathbb{E} \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \mathcal{R}_N(t_n - r)(F_N(r) - F_N(t_{i-1}))dr \right\|^2_H \\
& + \mathbb{E} \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} (\mathcal{R}_N(t_n - r) - \bar{\mathcal{R}}_N(t_n - r))F_N(t_{i-1})dr \right\|^2_H \\
& + \mathbb{E} \left\| \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \bar{\mathcal{R}}_N(t_n - r)(F_N(t_{i-1}) - \bar{F}_N(t_i))dr \right\|^2_H \\
= \vartheta_{1,1} + \vartheta_{1,2} + \vartheta_{1,3}.
\end{align*}
\]

By the assumption (2) and Theorem 4 there holds

\[
\vartheta_{1,1} \leq C\tau^{2H - \frac{2m}{2m - 2}}.
\]
As for $\vartheta_{1,2}$, one can get
\[
\vartheta_{1,2} \leq C E \sum_{i=1}^{n} \int_{t_{i-1}}^{t_{i}} (t_{n} - r)^{1-\epsilon} \| \mathcal{R}_{N}(t_{n} - r) - \mathcal{R}_{N}(t_{n} - r) \|^{2} \| F_{N}(t_{i}) \|_{H}^{2} dr \\
\leq C \int_{0}^{t_{n}} (t_{n} - r)^{1-\epsilon} \| \mathcal{R}_{N}(t_{n} - r) - \mathcal{R}_{N}(t_{n} - r) \|^{2} dr \\
\leq C \int_{0}^{t_{n}} r^{1-\epsilon} \left( \int_{\Gamma_{\omega,\kappa} \setminus \Gamma_{\theta,\kappa}^{*}} |e^{zr}| \| \mathcal{R}_{N}(z) \| |dz| \right)^{2} dr \\
+ C \int_{0}^{t_{n}} r^{1-\epsilon} \left( \int_{\Gamma_{\theta,\kappa}^{*}} |e^{zr}| \| \mathcal{R}_{N}(z) - \mathcal{R}_{N}(z) \| |dz| \right)^{2} dr \\
= \vartheta_{1,2,1} + \vartheta_{1,2,2}.
\]

As for $\vartheta_{1,2,1}$, there holds
\[
\vartheta_{1,2,1} \leq C \int_{0}^{t_{n}} r^{1-\epsilon} \left( \int_{\Gamma_{\omega,\kappa} \setminus \Gamma_{\theta,\kappa}^{*}} |e^{zr}| |z|^{-1} |dz| \right)^{2} dr \\
\leq C \tau^{2-2\epsilon} \int_{0}^{t_{n}} r^{1-\epsilon} \left( \int_{\Gamma_{\omega,\kappa} \setminus \Gamma_{\theta,\kappa}^{*}} |e^{zr}| |z|^{-\epsilon} |dz| \right)^{2} dr \\
\leq C \tau^{2-2\epsilon}.
\]

Similarly, one has
\[
\vartheta_{1,2,2} \leq C \tau^{2} \int_{0}^{t_{n}} r^{1-\epsilon} \left( \int_{\Gamma_{\theta,\kappa}^{*}} |e^{zr}| |dz| \right)^{2} dr \\
\leq C \tau^{2} \int_{0}^{t_{n}} r^{1-\epsilon} \int_{\Gamma_{\theta,\kappa}^{*}} |e^{2zr}| |z|^{1-2\epsilon} |dz| \int_{\Gamma_{\theta,\kappa}^{*}} |z|^{-1+2\epsilon} |dz| dr \\
\leq C \tau^{2-2\epsilon}.
\]

As for $\vartheta_{1,3}$, one can get
\[
\vartheta_{1,3} \leq C \tau \sum_{i=1}^{n-1} \| u_{N}(t_{i}) - u_{N}^{i} \|_{H}^{2}.
\]

For $\vartheta_{2}$, we have
\[
\vartheta_{2} \leq C \sum_{k=1}^{N} \int_{0}^{t_{n}} \left( \sqrt{A_{k_{0}} \partial_{r}^{ \frac{1-2M}{2} } (E_{k}(r) - \bar{E}_{k}(r))} \right)^{2} dr \\
\leq C \sup_{1 \leq k \leq N} \int_{0}^{t_{n}} \left( \sqrt{A_{k_{0}} \partial_{r}^{ \frac{1-2M}{2} } (E_{k}(r) - \bar{E}_{k}(r))} \right)^{2} dr.
\]
Then we split the following formula into two parts
\[
\int_0^{t_n} \left( \lambda_k^0 \partial_r \frac{1-2H}{2} (E_k(r) - \bar{E}_k(r)) \right)^2 dr
\]
\[
\leq \int_0^{t_n} \left| \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\tau,\kappa}} e^{2\tau} \lambda_k^0 \frac{1-2H}{2} \bar{E}_k(z) dz \right|^2 dr
\]
\[
+ \int_0^{t_n} \left| \int_{\Gamma_{\tau,\kappa}} e^{2\tau} \lambda_k^0 \frac{1-2H}{2} (\bar{E}_k(z) - \bar{E}_k(z)) dz \right|^2 dr
\]
\[= \vartheta_{2,1} + \vartheta_{2,2}.
\]

From (10), it follows that
\[
\vartheta_{2,1} \leq C \int_0^{t_n} \left( \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\tau,\kappa}} |e^{2\tau}||z|^{1-2H+\frac{\alpha s}{r}} |dz| \right)^2 dr
\]
\[\leq C \tau^{2H-\frac{2\alpha s}{r}} \int_0^{t_n} \int_{\Gamma_{\theta,\kappa} \setminus \Gamma_{\tau,\kappa}} |e^{2\tau}||z|^{-\epsilon} |dz| dr
\]
\[\leq C \tau^{2H-\frac{2\alpha s}{r}}.
\]

Combining Lemma 6 and (10) leads to
\[
\vartheta_{2,2} \leq C \tau^2 \int_0^{t_n} \left( \int_{\Gamma_{\tau,\kappa}} |e^{2\tau}||z|^{1-2H+\frac{2\alpha s}{r}} |dz| \right)^2 dr
\]
\[\leq C \tau^2 \int_0^{t_n} \int_{\Gamma_{\tau,\kappa}} |e^{2\tau}||z|^{1-2H+\frac{2\alpha s}{r}} |dz| |dz| \int_{\Gamma_{\tau,\kappa}} |dz|
\]
\[\leq C \tau^{2H-\frac{2\alpha s}{r}}.
\]

Lemma 4 gives
\[
\vartheta_3 \leq C E \left\| \int_0^{t_n} \sum_{k=1}^{N} \sqrt{\lambda_k} \left( \bar{E}_k(t_n - r) - \frac{1}{r} \sum_{i=1}^{r} \chi(t_{i-1},t_i)(r) \int_{t_i}^{t} \bar{E}_k(t_n - \xi) d\xi \right) \phi_k dW^H_k(r) \right\|^2_{H}
\]
\[\leq C \sum_{k=1}^{N} \int_0^{t_n} \left| \sqrt{\lambda_k^0} \partial_r \frac{1-2H}{2} (\bar{E}_k(r) - G_k(r)) \right|^2 dr
\]
\[\leq C \sup_{1 \leq k \leq N} \int_0^{t_n} \left| \lambda_k^0 \partial_r \frac{1-2H}{2} (\bar{E}_k(r) - G_k(r)) \right|^2 dr.
\]
As for $\vartheta_1$. Simple calculations give
\[
0 \leq C \int_{t_1}^{t_n} \left| \lambda_k^0 \vartheta_1 \left( E_k(r) - G_k(r) \right) \right|^2 \, dr
\]

Divide $\vartheta_3$ into three parts
\[
\int_0^{t_n} \left| \lambda_k^0 \partial_r \left( E_k(r) - G_k(r) \right) \right|^2 \, dr
\]
\[
\leq C \int_{t_1}^{t_n} \int_{\Gamma_{\beta,x}} \left( e^{\vartheta r} \lambda_k^0 \lambda_k^0 \frac{1-2H}{2} \tilde{E}_k(z) \right) \, dz \, dr
\]
\[
+ \int_0^{t_1} \left| \lambda_k^0 \partial_r \left( E_k(r) - G_k(r) \right) \right|^2 \, dr
\]
\[
= \vartheta_{3,1} + \vartheta_{3,2} + \vartheta_{3,3}.
\]

Eq. (10) gives
\[
\vartheta_{3,1} \leq C r^2 \int_{t_1}^{t_n} \int_{\Gamma_{\beta,x} \setminus \Gamma_{\beta,x}^r} e^{\vartheta r} \lambda_k^0 \lambda_k^0 \frac{1-2H}{2} \tilde{E}_k(z) \, dz \, dr
\]
\[
\leq C r^2 \int_{t_1}^{t_n} \int_{\Gamma_{\beta,x} \setminus \Gamma_{\beta,x}^r} \left| e^{2\vartheta r} \right| \left| z \right|^{1-2H+2\alpha} \, dz \, dr
\]
\[
\leq C r^{2H-2\alpha + \frac{1}{2}}.
\]

As for $\vartheta_{3,2}$, using Lemma 4 one has
\[
\vartheta_{3,2} \leq C \int_{t_1}^{t_n} \left( \int_{\Gamma_{\beta,x} \setminus \Gamma_{\beta,x}^r} \left| e^{\vartheta r} \left| z \right|^{\alpha} \right| \left| z \right|^{1-2H+2\alpha - 1} \, dz \right)^2 \, dr
\]
\[
\leq C r^{2H-2\alpha - \epsilon} \int_{t_1}^{t_n} \int_{\Gamma_{\beta,x} \setminus \Gamma_{\beta,x}^r} \left| e^{2\vartheta r} \left| z \right|^{\alpha} \right| \left| z \right|^{-\epsilon} \, dz \, dr
\]
\[
\leq C r^{2H-2\alpha + \frac{1}{2}}.
\]

As for $\vartheta_{3,3}$, we consider $\lambda_k^0 \left( \tilde{E}_k(r_1) - \tilde{E}_k(r_2) \right)$ with $|r_1 - r_2| \leq \tau$ and $r_1, r_2 > 0$ first. Simple calculations give
\[
\left| \lambda_k^0 \left( \tilde{E}_k(r_1) - \tilde{E}_k(r_2) \right) \right|
\]
\[
\leq C \left| \int_{\Gamma_{\beta,x}^r} \left( e^{\vartheta r_1} - e^{\vartheta r_2} \right) \lambda_k^0 \tilde{E}_k(z) \, dz \right|
\]
\[
\leq C r^\gamma \int_{\Gamma_{\beta,x}^r} \left| e^{\vartheta r_1 + \tau} \right| \left| z \right|^{\alpha s - 1 + \gamma} \, dz
\]
\[
\leq C r^\gamma (r_1 + \tau)^{-\gamma - \alpha s}.
\]
where $\gamma \in [0, 1 - \rho \alpha/s)$. Thus when $H = 1/2$, the mean value theorem gives

$$\vartheta_{3,3} \leq C r^{1-2\rho \alpha/s}.$$ 

Similarly, when $H \in (1/2, 1)$, we have

$$\vartheta_{3,3} \leq \int_0^{t_1} \left( \lambda_k^p \int_0^r (r - r_1)^{\frac{r-1}{2}} \left| \tilde{E}_k(r_1) - \frac{1}{\tau} \int_0^{t_1} \tilde{E}_k(\xi) d\xi \right| dr_1 \right)^2 dr \leq C r^{2H-2\rho \alpha/s}. $$

As for $H \in (0, 1/2)$, one has

$$\vartheta_{3,3} \leq \int_0^{t_1} \left( \lambda_k^p \int_0^r (r - r_1)^{\frac{r-1}{2}} \left| \tilde{E}_k(r_1) - \frac{1}{\tau} \int_0^{t_1} \tilde{E}_k(\xi) d\xi \right| dr_1 \right)^2 dr$$

$$\leq \int_0^{t_1} \lambda_k^p \int_0^r (r - r_1)^{\frac{r-1}{2}} \partial_{r_1} \left( \tilde{E}_k(r_1) - \frac{1}{\tau} \int_0^{t_1} \tilde{E}_k(\xi) d\xi \right) dr_1 \right)^2 dr$$

$$+ \int_0^{t_1} \lambda_k^p \partial_{r_1} \int_0^r (r - r_1)^{\frac{r-1}{2}} \left( \tilde{E}_k(0) - \frac{1}{\tau} \int_0^{t_1} \tilde{E}_k(\xi) d\xi \right) dr_1 \right)^2 dr$$

$$ \leq C r^{2H-2\rho \alpha/s}, $$

where we use that for $\gamma \in [0, 1]$,

$$|\lambda_k^p \partial_{r_1} (\tilde{E}_k(r_1) - \tilde{E}_k(r_1 + \tau))|$$

$$\leq C \left| \partial_{r_1} \int_{\Gamma_{r_{1+\tau}}} e^{z(r_1+\tau)} (1 - e^{z\tau}) \lambda_k^p \tilde{E}_k(z) dz \right|$$

$$\leq C \tau^{\gamma} \int_{\Gamma_{r_{1+\tau}}} |e^{z(r_1+\tau)}| |z^{\rho \alpha/s + \gamma}| dz$$

$$\leq C \tau^{\gamma} (r_1 + \tau)^{-\gamma - \rho \alpha/s - 1}. $$

Collecting the above estimates and using the discrete Grönewall inequality, the desired result is reached.

### 5 Numerical experiments

In this section, we provide some numerical examples to verify the theoretical results. Here we take $Q$’s eigenvalues $\Lambda_k = k^m$, $k = 1, 2, \cdots$. According to the assumption $\|A^{-\rho}Q^{1/2}\|\mathcal{L}_2 < \infty$ and Lemma 3, we have that $\rho$ is approximately equal to $\frac{1}{4 + m}d$.

In the numerical experiments, we consider the equation

$$\begin{cases}
\partial_t u + \partial_t^{1-\alpha} A^s u = \sin(u) + \dot{W}^H_Q & \text{in } D, \ t \in (0, T], \\
u(\cdot, 0) = 0 & \text{in } D, \\
u = 0 & \text{on } \partial D, \ t \in (0, T],
\end{cases}$$

(19)
where $D = (0, 1)$ and $T = 0.01$. We take 100 trajectories to calculate the solution of Eq. (19). Since the exact solution of (19) is unknown, the spatial errors and temporal errors can be measured by

$$e_N = \left( \frac{1}{100} \sum_{i=1}^{100} \| u_N^L(\omega_i) - u_{2N}^L(\omega_i) \|_H^2 \right)^{1/2},$$

$$e_\tau = \left( \frac{1}{100} \sum_{i=1}^{100} \| u_\tau(\omega_i) - u_{\tau/2}(\omega_i) \|_H^2 \right)^{1/2},$$

where $u_N^L(\omega_i)$ ($u_\tau(\omega_i)$) means the numerical solution of $u$ at time $t_L$ with $N$ spectral bases (step size $\tau$) and sample $\omega_i$; we can respectively calculate the spatial and temporal convergence rates by

$$\text{Rate} = \frac{\ln(e_N/e_{2N})}{\ln(2)}, \quad \text{Rate} = \frac{\ln(e_\tau/e_{\tau/2})}{\ln(2)}.$$
Table 2: Temporal errors and convergence rates with $H = 0.5$

| $\alpha$, $s$ \ $\mathcal{T}/\tau$ | 32 | 64 | 128 | 256 | Rate       |
|----------------------------------|-----|----|-----|-----|-----------|
| 0 (0.5,0.7)                     | 5.669E-02 | 4.250E-03 | 3.378E-03 | 2.653E-03 | 0.343 (0.321) |
| 0 (0.6,0.7)                     | 5.345E-02 | 6.729E-03 | 5.387E-03 | 4.109E-03 | 0.363 (0.286) |
| -0.5 (0.3,0.5)                  | 6.51E-02   | 8.89E-04  | 6.239E-04 | 4.543E-04 | 0.482 (0.425)  |
| -0.5 (0.5,0.5)                  | 6.12E-02   | 1.43E-03  | 9.607E-04 | 6.371E-04 | 0.603 (0.375)  |
| -1 (0.3,0.4)                    | 7.071E-02 | 4.388E-04 | 2.913E-04 | 1.861E-04 | 0.609 (0.5)    |
| -1 (0.3,0.7)                    | 7.071E-02 | 5.48E-04  | 3.746E-04 | 2.441E-04 | 0.569 (0.5)    |

Table 3: Temporal errors and convergence rates with $H = 0.8$

| $\alpha$, $s$ \ $\mathcal{T}/\tau$ | 32 | 64 | 128 | 256 | Rate       |
|----------------------------------|-----|----|-----|-----|-----------|
| 0 (0.5,0.7)                     | 7.883E-02 | 1.856E-04 | 1.189E-04 | 7.688E-05 | 0.646 (0.621) |
| 0 (0.6,0.7)                     | 7.653E-02 | 3.032E-04 | 1.929E-04 | 1.218E-04 | 0.665 (0.586) |
| -0.5 (0.3,0.5)                  | 8.515E-02 | 4.13E-04  | 2.403E-04 | 1.330E-05 | 0.804 (0.725) |
| -0.5 (0.5,0.5)                  | 8.216E-02 | 6.741E-05 | 3.781E-05 | 2.072E-05 | 0.863 (0.675) |
| -1 (0.3,0.7)                    | 8.944E-02 | 2.742E-05 | 1.405E-05 | 8.038E-06 | 0.876 (0.8)    |
| -1 (0.5,0.5)                    | 8.944E-02 | 2.48E-05  | 1.272E-05 | 6.399E-06 | 0.970 (0.8)    |

Table 4: Spatial errors and convergence rates with $H = 0.3$

| $\alpha$, $s$, $N$ \ $m$ | 8 | 16 | 32 | 64 | Rate       |
|--------------------------|---|----|----|----|-----------|
| 0 (0.3,0.7)              | 9.487E-02 | 4.303E-02 | 2.849E-02 | 1.596E-02 | 0.743 (0.9)  |
| 0 (0.6,0.7)              | 4.72E-02   | 1.587E-01 | 1.511E-01 | 1.273E-01 | 0.173 (0.2)  |
| -0.5 (0.3,0.4)           | 7.416E-02 | 6.858E-02 | 4.717E-02 | 3.210E-02 | 0.555 (0.55) |
| -0.5 (0.3,0.7)           | 1.072E-01 | 1.984E-02 | 8.381E-03 | 3.588E-03 | 1.261 (1.15) |
| -1 (0.3,0.4)             | 8.944E-02 | 2.689E-02 | 1.341E-02 | 6.787E-03 | 1.014 (0.8)  |
| -1 (0.3,0.7)             | 1.183E-01 | 7.179E-03 | 2.437E-03 | 8.080E-04 | 1.667 (1.4)  |

Table 5: Spatial errors and convergence rates with $H = 0.5$

| $\alpha$, $s$, $N$ \ $m$ | 8 | 16 | 32 | 64 | Rate       |
|--------------------------|---|----|----|----|-----------|
| 0 (0.3,0.7)              | 9.487E-02 | 1.308E-02 | 8.56E-03  | 4.80E-03  | 0.780 (0.900) |
| 0 (0.6,0.7)              | 8.165E-02 | 5.509E-02 | 3.97E-02  | 2.787E-02 | 0.518 (0.667) |
| -0.5 (0.3,0.4)           | 7.416E-02 | 2.416E-02 | 1.704E-02 | 1.143E-02 | 0.612 (0.550) |
| -0.5 (0.6,0.4)           | 6.455E-02 | 5.299E-02 | 4.490E-02 | 3.519E-02 | 0.351 (0.417) |
| -1 (0.3,0.4)             | 8.944E-02 | 9.550E-03 | 5.066E-03 | 2.397E-03 | 1.062 (0.800) |
| -1 (0.6,0.4)             | 8.165E-02 | 2.283E-02 | 1.371E-02 | 7.445E-03 | 0.857 (0.667) |
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Table 6

| $m$ | $(\alpha, s), N$ | 8   | 16   | 32   | 64   | Rate |
|-----|------------------|-----|------|------|------|------|
| 0   | (0.3, 0.4)       | 5.47E-02 | 1.49E-02 | 1.40E-02 | 1.26E-02 | 0.155 (0.3) |
|     | (0.5, 0.7)       | 9.48E-02 | 3.25E-03 | 2.02E-03 | 1.10E-03 | 0.803 (0.9) |
| -0.5| (0.3, 0.4)       | 7.41E-02 | 5.71E-03 | 4.25E-03 | 2.60E-03 | 0.595 (0.55) |
|     | (0.6, 0.7)       | 1.07E-01 | 5.06E-03 | 2.35E-03 | 9.96E-04 | 1.207 (1.15) |
| -1  | (0.6, 0.4)       | 8.94E-02 | 5.31E-03 | 3.24E-03 | 1.73E-03 | 0.862 (0.8) |
|     | (0.6, 0.7)       | 1.18E-01 | 1.96E-03 | 7.34E-04 | 2.27E-04 | 1.646 (1.4) |

The regularity estimates of mild solution in time and space are developed based on a novel estimate of the second moment of stochastic integral of fBm. The fully discrete scheme constructed by spectral Galerkin method and backward Euler convolution quadrature is proposed and optimal error estimates are obtained. The theoretical results are also verified by numerical experiments.

References

1. Acosta, G., Bersetche, F.M., Borthagaray, J.P.: Finite element approximations for fractional evolution problems. Fract. Calc. Appl. Anal. 22, 767–794 (2019)
2. Acosta, G., Borthagaray, J.P.: A fractional Laplace equation: regularity of solutions and finite element approximations. SIAM J. Numer. Anal. 55, 472–495 (2017)
3. Arezoomandan, M., Soheili, A.R.: Spectral collocation method for stochastic partial differential equations with fractional Brownian motion. J. Comput. Appl. Math. 389, 113369 (2021)
4. Banna, O.: Fractional Brownian Motion: Approximations and Projections. John Wiley and Sons Inc, Hoboken NJ (2019)
5. Bardina, X., Jolis, M.: Multiple fractional integral with Hurst parameter less than 1/2. Stochastic Process. Appl. 116, 463–479 (2006)
6. Cao, Y., Hong, J., Liu, Z.: Approximating stochastic evolution equations with additive white and rough noises. SIAM J. Numer. Anal. 55, 1958–1981 (2017)
7. Cao, Y., Hong, J., Liu, Z.: Finite element approximations for second-order stochastic differential equation driven by fractional Brownian motion. IMA J. Numer. Anal. 38, 184–197 (2018)
8. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker’s guide to the fractional Sobolev spaces. Bull. Sci. Math. 136, 521–573 (2012)
9. Ervin, V.J., Roop, J.P.: Variational formulation for the stationary fractional advection dispersion equation. Numer. Methods Partial Differential Equations 22, 558–576 (2006)
10. Gunzburger, M., Li, B., Wang, J.: Sharp convergence rates of time discretization for stochastic time-fractional PDEs subject to additive space-time white noise. Math. Comp. 88, 1715–1741 (2018)
11. Jin, B., Zhou, Z.: Incomplete iterative solution of subdiffusion. Numer. Math. 145, 693–725 (2020)
12. Kloeden, P.E., Platen, E.: Numerical Solution of Stochastic Differential Equations. Springer-Verlag, Berlin and New York (1992)
13. Laptev, A.: Dirichlet and Neumann eigenvalue problems on domains in Euclidean spaces. J. Funct. Anal. 151, 531–545 (1997)
14. Li, P., Yau, S.T.: On the Schrödinger equation and the eigenvalue problem. Comm. Math. Phys. 88, 309–318 (1983)
15. Li, Y., Wang, Y., Deng, W.: Galerkin finite element approximations for stochastic space-time fractional wave equations. SIAM J. Numer. Anal. 55, 9175–9202 (2017)
16. Liu, X., Deng, W.: Higher order approximation for stochastic space fractional wave equation forced by an additive space-time Gaussian noise. J. Sci. Comput. 87, 11 (2021)
17. Lubich, C.: Convolution quadrature and discretized operational calculus. I. Numer. Math. 52, 129–145 (1988)
18. Lubich, C.: Convolution quadrature and discretized operational calculus. II. Numer. Math. 52, 413–425 (1988)
19. Lubich, C., Sloan, I.H., Thomée, V.: Nonsmooth data error estimates for approximations of an evolution equation with a positive-type memory term. Math. Comp. 65, 1–18 (1996)
20. Mandelbrot, B.B., van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. 10, 422–437 (1968)
21. Mishura, I.S.: Stochastic Calculus for Fractional Brownian Motion and Related Processes. Springer, Berlin (2008)
22. Podlubny, I.: Fractional Differential Equations. Academic, San Diego and London (1999)
23. Simonsen, I.: Measuring anti-correlations in the nordic electricity spot market by wavelets. Phys. A 322, 597–606 (2003)
24. Song, R., Vondraček, Z.: Potential theory of subordinate killed Brownian motion in a domain. Probab. Theory Relat. Fields. 125, 578–592 (2003)
25. Wang, X., Qi, R., Jiang, F.: Sharp mean-square regularity results for SPDEs with fractional noise and optimal convergence rates for the numerical approximations. BIT 57, 557–585 (2017)
26. Wu, X., Yan, Y., Yan, Y.: An analysis of the L1 scheme for stochastic subdiffusion problem driven by integrated space-time white noise. Appl. Numer. Math. 157, 69–87 (2020)
27. Yan, L., Yin, X.: Optimal error estimates for fractional stochastic partial differential equation with fractional Brownian motion. Discrete Contin. Dyn. Syst. Ser. B 24, 615–635 (2019)