Approximate Analytical Pricing of European Options under the Non-affine Stochastic Volatility Model

Yuhang Yao, Youfa Sun* and Xingyin Liang
School of Economics and Commerce, Guangdong University of Technology, China

*Corresponding author email: y.f.sun@gdut.edu.cn

Abstract. The non-affine stochastic volatility model has attracted increasing attention in recent years, due to its excellent performance in describing the nonlinear characteristics of asset price path. However, the fact that there is no close-form of option pricing formula under this model, restricts its use badly. To remove this restriction, two approximate analytical option pricing approaches are proposed in this paper, that is, the piecewise first-order Taylor expansion method and the perturbation-based asymptotic expansion method of implied volatility. The first method is used to derive an approximate characteristic function, then based on which the Fourier-Cosine expansion method calculates the European option price. In the second way, implied volatility of the European option price is asymptotically expanded around non-affine term of volatility. Compared with the existing methods in literature, numerical experiments show that the first method has higher accuracy, while the second method is of more practical significance. In addition, the accuracy of the first method is generally higher than that of the second method in most cases, while the second method performs better when the volatility is extremely small.

1. Introduction
The Heston square-root stochastic volatility model has been widely used since it can well characterize the time variability, agglomeration and bar effect of asset price fluctuations[1]. However, a number of empirical results show that it still has defects in characterizing the real market. For instance, Andersen finds that the nonlinear characteristics of the underlying asset path can not be described under the assumption that the form of fluctuation process is square root [2]. Hence some literatures try to relax this assumption and propose a non-affine stochastic volatility model such as Chacko[3]. Kaeck’s empirical results prove that the non-affine model has better pricing performance than the stochastic volatility model with jump[4].

However, there is no close form for option pricing under the non-affine stochastic volatility model, which limits its application in real investment. Though numerical approaches such as Monte Carlo[5] and finite difference[6] work, they are usually inefficient. Hence, the approximation approach has been popular in recent literatures[7]~[11]. Among these literatures, there are some outstanding. Lewis derives a second order asymptotic expansion formula of option price around volatility of volatility[12]. Chacko uses first-order Taylor expansion to transform the non-affine process into affine process[3]. Inspired by Chacko’s method, Wu simplifies the derivation of characteristic function and then obtains the option prices using Fast Fourier Transform[13]. Sun, et al. proposes the quasi-closed option pricing formula with Fourier-Cosine pricing method based on first-order Taylor expansion method[14]. However, the first-order perturbation method mentioned in Sun, et al (2020) is restricted by its approximation accuracy of the characteristic function, while the asymptotic expansion method in Lewis(2000) has a minimum requirement for the volatility of volatility, which usually don’t match the actual situation.
In order to improve the approximation accuracy, we propose a piecewise first-order Taylor expansion method to reduce the error with the original function. Moreover, to break through the minimum requirement, we expand the implied volatility around the non-affine term of the volatility, which makes our algorithm widely applicable.

2. Non-affine Stochastic Volatility Model
Under the risk neutral measure, Chacko\cite{3} gives the form of non-affine stochastic volatility model

\[
\begin{align*}
    dS_t &= rS_t dt + \sqrt{\nu_t} S_t dW_{1,t} \\
    d\nu_t &= \kappa(\theta - \nu_t) dt + \sigma \nu_t^\gamma dW_{2,t}
\end{align*}
\]

where \( r \) denotes risk-free interest rate, \( \kappa \) indicates mean reversion speed, \( \theta \) represents long-run mean, \( \sigma \) is volatility of volatility, \( \gamma \) is non-affine coefficient, and the instantaneous correlation between \( dW_{1,t} \) and \( dW_{2,t} \) is \( \rho \).

3. Fourier-Cosine Pricing Method Based on Piecewise First-order Taylor Expansion
In the Fourier-Cosine pricing method, the probability density function of yield to maturity of the underlying asset is approximated by Fourier cosine series, and then the coefficient in series expansion is expressed by characteristic function\cite{15}. The formula for European call options is written as

\[
\begin{align*}
    \text{Call}^{(j)}(X,t_0) &\approx e^{-rT} \sum_{n=0}^{N} \Re\left[ e^{-i\pi j \frac{\alpha'}{\beta'} \pi / 2} f\left( \frac{iu \pi - \alpha'}{\beta' - \alpha} X \right) \right] G_u
\end{align*}
\]

where \( X = \ln \left( \frac{S_t}{K} \right), y = \ln \left( \frac{S_t}{K} \right), G_u = \frac{2}{b' - a} \int_{u}^{\infty} \text{Call}^{(j)}(y,T) \cos \left( \frac{iu \pi - \alpha'}{\beta' - \alpha} y \right) dy \), \( S_t \) denote underlying asset prices at the current moment and maturity, respectively. \( K \) represents the exercise price. \( \tau = T - t_0 \) is the remaining time between current moment and maturity, \( \Re(\cdot) \) means taking the real part, \( f(\cdot) \) is the conditional characteristic function of the random variable \( y \), and the symbol ' \) represents half of the first term of the series expansion. The \( \alpha'/b' \) value is \( c_{1/2} + \sqrt{c_2 + \cdots} \). \( c_n \) is the \( n \)-th order cumulant random variable \( y \).

Then the conditional characteristic function of the logarithmic price of the underlying asset is defined as \( f(X,v;r;\varphi) = E( e^{i\varphi X} | X = \ln S_t, v = v) \), where \( E(\cdot) \) represents expectation under the risk neutrality measure, \( \varphi \) is the frequency, \( i = \sqrt{-1} \). The following partial differential equation satisfied by the conditional characteristic function is derived from Kolmogorov backward equation\cite{3}

\[
\begin{align*}
    \frac{1}{2} \frac{\partial^2 f}{\partial x^2} + \rho \sigma \nu^{\gamma/2} \frac{\partial f}{\partial \nu} + \frac{1}{2} \nu^{\gamma/2} \frac{\partial^2 f}{\partial \nu^2} + \left( r - \frac{1}{2} \nu \right) \frac{\partial f}{\partial x} + \kappa(\theta - \nu) \frac{\partial f}{\partial \nu} - \frac{\partial f}{\partial \tau} = 0
\end{align*}
\]

Using the first-order Taylor expansion at the \( v = 0.5\vartheta, \theta, 2\vartheta \), the curves of \( \nu^{\gamma/2}, \nu \) are approximated by three lines in corresponding intervals.

Take the intersection points on the lines as the dividing points. Since the lines dependent on the different curves in the same intervals have different demarcation points and the curve of \( \nu^{\gamma/2} \) is steeper than the curve of \( \nu \), we make demarcation points of \( \nu^{\gamma/2} \) equal to demarcation points of \( \nu \). Taylor expansion formulas are expressed as
\[
\begin{align*}
\frac{\partial B}{\partial \tau} &= \frac{1}{2} \sigma^2 \gamma \left( \frac{\theta}{2} \right)^{\gamma-1} B^2 + \left( i \phi \sigma \frac{\gamma+1}{2} \right) \left( \frac{\theta}{2} \right)^{\gamma-1} - \kappa \right) B + \frac{1}{2} i \phi (i \phi - 1) \\
\frac{\partial A}{\partial \tau} &= \frac{1}{2} \sigma^2 \gamma \left( \frac{\theta}{2} \right)^{\gamma-1} (1-\gamma) B^2 + \left( i \phi \sigma \frac{1-\gamma}{2} \right) \left( \frac{\theta}{2} \right)^{\gamma-1} + \kappa \theta \right) B + i \phi \varphi
\end{align*}
\]

where

\[
c' = \frac{\left( \frac{\theta}{2} \right)^\gamma - \theta^\gamma}{\theta^{\gamma-1} - \left( \frac{\theta}{2} \right)^{\gamma-1}} \gamma
d' = \frac{\left( \theta - (2\theta)^\gamma \right)(1-\gamma)}{\left(2\theta\right)^{\gamma-1} - \theta^{\gamma-1}} \gamma
\]

Refer to Chacko[3], define the conditional characteristic function as

\[
f(\phi, \tau; X, v, \mu) = \exp \left[ i \phi X + A(\tau) + B(\tau) v \right]
\]

Substitute conditional characteristic function into the equation (3). According to the undetermined coefficient method, coefficients in front of the \(v\) terms are set equal to zero, and the Riccati equations in three different intervals can be obtained, which can be solved by Mathematica. We only show the Riccati equations in the interval \(v \in [0, u]\) as follow, and the other equation has a similar form.

4. Asymptotic Expansion Pricing Method Based on Non-affine Term of Volatility

To avoid solving complex characteristic function of the non-affine stochastic volatility model, we provide another approximate analytical pricing method. The implied volatility of European options is asymptotically expanded around non-affine terms of volatility. Referring to Lewis' fundamental transformation method[12], European call options can be written as:

\[
Call^{(2)}(S, v, \tau) = S - \frac{Ke^{-r\tau}}{2\pi} \int_{-\infty}^{1+i\epsilon} e^{-ik} \frac{\tilde{H}(k, v, \tau)}{k^2 - ik} dk
\]

where \(S\) is the underlying asset price, \(K\) is option exercise price, \(\tilde{H}(k, V, \tau)\) satisfies:

\[
\frac{\partial \tilde{H}}{\partial \tau} = \frac{1}{2} \sigma^2 \frac{\partial^2 \tilde{H}}{\partial \nu^2} + \left( \kappa (\theta - \nu) - ik \rho \nu \sqrt{\nu} \right) \frac{\partial \tilde{H}}{\partial \nu} - \frac{k^2 - ik}{2} \nu \tilde{H}
\]

Boundary condition is \(\tilde{H}(k, v, 0)\), and \(\tilde{H}(k, v, \tau)\) can be divided into two parts

\[
\tilde{H}(k, v, \tau) = H(k, v, \tau) \cdot h(k, v, \tau)
\]

where \(H(k, v, \tau)\) is the solution of equation (9) while \(\sigma = 0\), \(h(k, v, \tau)\) is the solution of equation (9) when \(\sigma \neq 0\). Define non-affine term of volatility as \(\nu^2 = \eta\), \(h(k, v, \tau)\) can be approximated by
\[ h(k, v, \tau) \approx 1 + \eta h^{(1)}(k, v, \tau) + \eta^2 h^{(2)}(k, v, \tau) \quad (11) \]

Substitute \( H(k, v, \tau) \) into equation (9), and all coefficients in front of \( \eta \) terms are set equal to zero. After obtaining specific formulas of \( h(k, v, \tau) \), substitute \( H(k, v, \tau) \) into equation (8). Finally, the implied volatility expressions of European call options can be expressed as

\[ V^{(2)}_{imp} \approx V(v, \tau) + \eta J^{(1)}(v, \tau) \mathcal{Q}^{(1,0)} + \eta^2 \left[ \frac{1}{2} J^{(1)}(v, \tau) \mathcal{Q}^{(2,0)} + J^{(2)}(v, \tau) \mathcal{Q}^{(2,2)} + J^{(3)}(v, \tau) \mathcal{Q}^{(3,2)} - \frac{1}{2} (J^{(0)}(v, \tau))^2 \mathcal{Q}^{(2,0)} \right] \quad (12) \]

where

- \( V(v, \tau) = \theta + (v - \theta) \left[ 1 - \frac{1}{\kappa \tau} \right], X = \ln \left( \frac{S}{K e^{-\kappa \tau}} \right), Y = V(v, \tau) \)
- \( Z(s, v) = \theta + e^{-\kappa \tau} (v - \theta), R(v) = \kappa(\theta - v), \chi(v) = \rho \sqrt{v} \)
- \( \zeta(v, s) = \frac{Z(r, v) - V}{b(v)}, \xi(v, s) = \exp \left[ -\int_0^s \frac{R(Z(v, \lambda))d\lambda}{Z(v, \lambda)} \right] \)
- \( J^{(0)}(v, \tau) = \int_0^s \sigma^2 \xi(v, s)^2 \chi(Z(v, s), \tau - s)ds \)
- \( J^{(1)}(v, \tau) = \int_0^s \sigma^2 \xi(v, s)^2 \chi(Z(v, s), \tau - s)ds \)
- \( J^{(2)}(v, \tau) = \int_0^s \sigma^2 \xi(v, s)^2 \chi(Z(v, s), \tau - s) \left[ J^{(1)}(Z(v, s), \tau - s) \cdot \chi + \frac{\partial J^{(0)}(Z(v, s), \tau - s)}{\partial v} \right]ds \)
- \( J^{(3)}(v, \tau) = \int_0^s \sigma^2 \xi(v, s)^2 \chi(Z(v, s), \tau - s) \cdot J^{(1)}(Z(v, s), \tau - s)ds \)

\[ \mathcal{Q}^{(2,0)} = \tau \left[ \frac{1}{2} X^2 - \frac{1}{2Y} - \frac{1}{8} \right] \]

\[ \mathcal{Q}^{(1,0)} = \frac{1 - X}{2Y} - \frac{1}{2Y} \]

\[ \mathcal{Q}^{(1,2)} = \frac{X^2}{Y} - \frac{X}{4Y} (4 - Y) \]

\[ \mathcal{Q}^{(2,2)} = \tau \left[ \frac{1}{2} X^2 - \frac{1}{2Y} - \frac{1}{8Y} + \frac{1}{32Y} (12 + Y) + \frac{1}{32Y} (48 - Y^2) \right] \quad (15) \]

5. Numerical Experiment

The numerical experiment in this section is divided into three parts. Some initial parameters are selected by reference to the estimated results of parameters in the literature\(^{[13][14]}\).

Firstly, we compare effectiveness of the improved Fourier-Cosine method with Sun’s direct first-order Taylor expansion method\(^{[14]}\). Given initial parameters: \( S = 100, K = 100, \gamma = 2, r = 0.005, \tau = 0.4, \kappa = 2, \theta = 0.14, \sigma = 0.8, \rho = -1 \), we set \( v \in [0.001, 0.02] \) and \( v \in [0.5, 0.8] \), which may demonstrate the distinct difference between the formula in direct first-order Taylor expansion and the original function.

Secondly, we compare performances of the improved asymptotic expansion method with Lewis’s method\(^{[12]}\). Small parameter should be set close to minimal or original function should be remarkably larger for the reason that the asymptotic expansion method based on the perturbation method works well in such situations. Since the small parameters we choose for the improved asymptotic expansion method are different from those chosen for Lewis’s method, some of initial parameters should differ from those used in different methods according to property of methods. Therefore, given the same initial parameters: \( S = 100, K = 100, \gamma = 2, r = 0.005, \theta = 0.14, \rho = -1 \), we set up four experiments, two of which is used to test the performance of improved asymptotic expansion method with parameter
settings: \( \tau=0.4, \kappa=0.03, v \in [0.001,0.02], \sigma \in [0.001,0.2], \) and the others is for Lewis’s method with \( \tau=0.02, \kappa=1, v \in [0.1,0.8], \sigma \in [0.01,0.8] \).

Finally, we compare improved asymptotic expansion method and the improved Fourier-Cosine method. Because the property that the former method is suitable to minimum volatility, we give initial parameters: \( S=100, K=100, \gamma=2, r=0.005, \tau=0.4, \kappa=0.03, \theta=0.14, \sigma=0.3, \rho=-1, v \in [0.001,0.01] \).

Since there is no theoretical value for options under the non-affine stochastic model, we refer the Sun’s well-designed Monte Carlo method\(^{[14]}\).

Figure 1. Comparison of Fourier-Cosine methods before and after improvement

Figure 2. Comparison of asymptotic expansion method based on non-affine term of volatility and volatility of volatility
Figure 3. Comparison of two improved methods

Figure 1 shows that the approximate option price obtained by the improved Fourier-Cosine method are superior to the Sun's method whatever volatility values. Figure 2 shows that option price given by the improved asymptotic expansion method is very close to the reference value when volatility is extremely small. The influence of volatility of volatility on this method is unstable. On the contrary, when the volatility is very large or the volatility of volatility is very small, the approximate values by Lewis’s method is very close to the reference value. Figure 3 shows that the improved Fourier-Cosine pricing method outperforms the improved asymptotic expansion pricing method in most cases. However, reference values cannot be completely approximated by Fourier-Cosine method, for the reason it not only simplifies the characteristic function, but also truncates the integral interval and expansion series. This defect can be solved under the improved asymptotic expansion method in the special situation that the volatility is minimal.

6. Conclusion

In this paper, we make improvements to the famous Fourier-Cosine pricing method and asymptotic expansion pricing method under non-affine stochastic volatility model. Numerical results show that in general the improved Fourier-Cosine method outperforms all methods shown in this paper, but it cannot completely approximate reference values. The asymptotic expansion pricing method can effectively overcome the defects of the improved Fourier-Cosine method when volatility is minimal.

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