Differentially Private Federated Learning via Inexact ADMM with Multiple Local Updates

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Abstract—Differential privacy (DP) techniques can be applied to the federated learning model to statistically guarantee data privacy against inference attacks to communication among the learning agents. While ensuring strong data privacy, however, the DP techniques hinder achieving a greater learning performance. In this paper we develop a DP inexact alternating direction method of multipliers algorithm with multiple local updates for federated learning, where a sequence of convex subproblems is solved with the objective perturbation by random noises generated from a Laplace distribution. We show that our algorithm provides \(\bar{\epsilon}\)-DP for every iteration, where \(\bar{\epsilon}\) is a privacy budget controlled by the user. We also present convergence analyses of the proposed algorithm. Using MNIST and FEMNIST datasets for the image classification, we demonstrate that our algorithm reduces the testing error by at most 31\% compared with the existing DP algorithm, while achieving the same level of data privacy. The numerical experiment also shows that our algorithm converges faster than the existing algorithm.

Index Terms—Differential privacy, federated learning, inexact alternating direction method of multipliers, multiple local updates, convergence analyses.

I. INTRODUCTION

In this work we propose a privacy-preserving algorithm for training a federated learning (FL) model [1], namely, a machine learning (ML) model that aims to learn global model parameters without collecting locally stored data from agents to a central server. The proposed algorithm is based on an inexact alternating direction method of multipliers (IADMM) that solves a sequence of subproblems whose objective functions are perturbed by injecting some random noises for ensuring differential privacy (DP) on the distributed data. We show that the proposed algorithm provides more accurate solutions compared with the state-of-the-art DP algorithm [2] while both algorithms provide the same level of data privacy. As a result, the proposed algorithm can mitigate a trade-off between data privacy and solution accuracy (i.e., learning performance in the context of ML), which is one of the main challenges in developing DP algorithms, as described in [3].

Developing highly accurate privacy-preserving algorithms can enhance the practical uses of FL in applications with sensitive data (e.g., electronic health records [4] and mobile device data [5]) because a greater learning performance can be achieved while preserving privacy on the sensitive data exposed to be leaked during a training process. Furthermore, it would allow a stronger differential privacy budget to FL. Because of the importance of FL, incorporating privacy-preserving techniques into optimization algorithms for solving the FL models has been studied extensively [6], [2], [7], [8].

Related Work. The empirical risk minimization (ERM) model used for learning parameters in supervised ML is often vulnerable to adversarial attacks [9], a situation that motivates the application of privacy-preserving techniques (e.g., DP [10] and homomorphic encryption [11]) to protect data. Among these techniques, DP has been widely used in the ML community and is especially useful for protecting data against inference attacks [12].

Formally, DP is a privacy-preserving technique that randomizes the output of a query such that any single data point cannot be inferred by an adversary that can reverse-engineer the randomized output. Depending on where to inject noises to randomize the output, DP can be categorized by input [13], [14], output [10], [15], and objective [15], [16] perturbation methods. Compared with the input perturbation, which directly perturbs input data by adding random noises, \textit{output perturbation} and \textit{objective perturbation} methods provide a randomized output of an optimization problem by injecting random noises into its true output and objective function, respectively. In [15], the authors propose a differentially private ERM that utilizes the output and objective perturbation methods to ensure DP on data. Also, Abadi et al. [17] apply the output perturbation to stochastic gradient descent (SGD) in order to ensure DP on data for every iteration of the algorithm. The privacy-preserving technique in our work is the \textit{objective perturbation} method. For details of differentially private ML, we refer readers to [18], [16], [19].

Within the context of FL, various optimization algo-
gorithms have been developed for solving the distributed ERM model in a communication-efficient manner. For example, FedAvg in [5] reduces the number of communication rounds by allowing each agent to conduct multiple local updates via SGD while a central server performs model averaging for a global update. Another example is FedProx [20], constructed by adding a proximal function to the objective function of the local model considered in FedAvg, resulting in better learning performance. Recently, the authors in [21] develop a communication-efficient ADMM by enhancing local computation whereas the vanilla ADMM conducts a single local computation per communication round. Even though these algorithms can reduce the number of communication rounds to mitigate the chance of data leakage, they do not guarantee data privacy during a training process, preventing their practical uses. Readers interested in details of FL should see [22], [23], [24].

In order to preserve privacy on data used for the FL model, various DP algorithms have been proposed in the literature, where the output and objective perturbations are incorporated for ensuring DP (see [25], [7], [8], [6], [2]). For example, the intermediate model parameters and/or gradients computed for every iteration of the FedAvg-type and FedProx-type algorithms are perturbed for guaranteeing DP as in [8] and [7], respectively, which can be seen as the output perturbation. Also, in [6], the primal and dual variables computed for every iteration of the vanilla ADMM algorithm are perturbed, which can be seen as the output and objective perturbations, respectively. Zhang and Zhu [6] compare the two perturbation methods, as Chaudhuri et al. [15] did under the general ML setting, and show that the objective perturbation can provide more accurate solutions compared with the output perturbation. The use of the objective perturbation is somewhat limited, however, because it requires the objective function to be twice differentiable and strongly convex whereas the twice differentiability restriction can be relaxed to the once differentiability for the output perturbation. In [2], the authors incorporate the output perturbation into IADMM that utilizes the first-order approximation with a proximal function to the objective function of the local model parameter. This local model parameter is connected to a central server. Each agent maintains a local model parameter vector, \( w \), and a training dataset \( \mathcal{I} \), where \( \mathcal{I} \) has \( m \) data features, \( \mathcal{I} \) is a K-dimensional data feature, and \( y_{pi} \in \mathbb{R}^K \) is a K-dimensional data label. We consider a distributed ERM problem given by

\[
\min_{w \in \mathcal{W}} \sum_{p=1}^{P} \left\{ \frac{1}{I_p} \sum_{i=1}^{I_p} l(w; x_{pi}, y_{pi}) + \beta \frac{1}{P} r(w) \right\},
\]

(1)

where \( w \in \mathbb{R}^{J \times K} \) is a global model parameter vector, \( \mathcal{W} \) is a compact convex set, \( l(\cdot) \) is a convex loss function, \( r(\cdot) \) is a convex regularizer function, \( \beta > 0 \) is a regularizer parameter, and \( I := \sum_{p=1}^{P} I_p \).

By introducing a local model parameter \( z_{pi} \in \mathbb{R}^{J \times K} \) defined for every agent \( p \in [P] \), we can rewrite (1) as

\[
\min_{w, \{z_{pi}\}_{p=1}^{P} \in \mathcal{W}} \sum_{p=1}^{P} f_{p}(w, z_{pi})
\]

(2a)
where

\[ f_p(z_p) := \frac{1}{T} \sum_{t=1}^{T} l(z_p; x_{pt}, y_{pt}) + \beta \frac{1}{P} r(z_p). \]  

(2c)

Since (2) is a convex optimization problem, it can be expressed by the equivalent Lagrangian dual problem:

\[
\max_{\lambda_p} \min_{\lambda_p} \left\{ \sum_{p=1}^{P} \{ f_p(z_p) + \langle \lambda_p, w - z_p \rangle \} \right\}, \tag{3}
\]

where \( \lambda_p \in \mathbb{R}^{T \times K} \) is a dual vector associated with constraints (2b).

**ADMM.** ADMM is an iterative optimization algorithm that can find an optimal solution of (3) in an augmented Lagrangian form. More specifically, for every \( t \in [T] \), where \( T \) is the number of iterations, it updates

\[ (w^t, z^t, \lambda^t) \rightarrow (w^{t+1}, z^{t+1}, \lambda^{t+1}) \]

by solving the following subproblems sequentially:

\[ w^{t+1} \leftarrow \arg \min_{w} \sum_{p=1}^{P} \{ f_p(z_p) + \frac{\rho_t}{2} \| w - z_p \|^2 \}, \tag{4a} \]

\[ z^{t+1}_p \leftarrow \arg \min_{z_p \in \mathcal{W}} f_p(z_p) - \langle \lambda^t_p, z_p \rangle + \frac{\rho_t}{2} \| w^{t+1} - z_p \|^2, \quad \forall p \in [P], \tag{4b} \]

\[ \lambda^{t+1}_p \leftarrow \lambda^t_p + \rho_t (w^{t+1} - z^{t+1}_p), \quad \forall p \in [P], \tag{4c} \]

where \( \rho_t > 0 \) is a hyperparameter that may be fine-tuned for better performance.

**Inexact ADMM.** The subproblem (4b) does not need to be solved exactly in each iteration to guarantee the overall convergence. In [2], (4b) is replaced with the following inexact subproblem:

\[ z^{t+1}_p \leftarrow \arg \min_{z_p \in \mathcal{W}} \langle f_p'(z_p^t), z_p \rangle + \frac{1}{2\eta_t} \| z_p - z^t_p \|^2 + \frac{\eta_t}{2} \| w^{t+1} - z_p \|^2 + \frac{1}{\rho_t} \| z^t_p - z_p \|^2, \tag{5} \]

which is obtained by (i) replacing the convex function \( f_p(z_p) \) in (4b) with its lower approximation \( f_p(z_p) := f_p(z^t_p) + f_p'(z^t_p) \cdot (z_p - z^t_p) \), where \( f_p'(z^t_p) \) is a subgradient of \( f_p \) at \( z^t_p \), and (ii) adding a proximal term \( \frac{1}{\eta_t} \| z_p - z^t_p \|^2 \) with a proximity parameter \( \eta_t > 0 \) that controls the proximity of the constraint. Note that the proximal term is used for finding a new solution \( z^{t+1}_p \) from \( z^t_p \) computed from the previous iteration. Thus, the proximal term takes data as inputs. A formal definition follows.

**III. Differentially Private IADMM with Multiple Local Updates**

We generalize the IADMM algorithm by introducing multiple local updates and differential privacy techniques. The proposed algorithm aims to (i) improve learning performance by introducing multiple local updates and (ii) protect data privacy against adversaries that can infer the locally stored data by reverse-engineering the local model parameters communicated during the training process. We present the privacy and convergence analyses of the proposed algorithm in Section III-A and III-B, respectively.

**Multiple Local Updates.** We introduce the multiple local updates in IADMM, namely, solving (5) multiple times, to improve communication efficiency. In other words, for every \( e \in [E] \), where \( E \) is the number of local updates, we solve

\[
\hat{z}^{t+1, e}_p \leftarrow \arg \min_{z_p \in \mathcal{W}} \langle f_p'(z^{t+1, e}_p), z_p \rangle + \frac{1}{2\eta_t} \| z_p - z^{t+1, e}_p \|^2 + \frac{\eta_t}{2} \| w^{t+1} - z_p \|^2 + \frac{1}{\rho_t} \| z^{t+1, e}_p - z_p \|^2.
\]

This is different from the existing work [21] that considers both multiple local primal and dual updates, namely, solving (5) and (4c) multiple times per iteration, resulting in communicating not only local model parameters but also dual information. In contrast, our approach does not require communicating dual information and hence reduces the communication burden. This point will be made clearer when describing Algorithm 1.

**DP via Objective Perturbation.** We propose to perturb the objective function of the constrained subproblem (6) by adding some random noise for ensuring differential privacy. DP is a data privacy preservation technique that aims to protect data by randomizing outputs of a function that takes data as inputs. A formal definition follows.

**Definition 1. (Definition 3 in [15])** A randomized function \( A \) provides \( \epsilon \)-DP if for any two datasets \( D \) and \( D' \) that differ in a single entry and for any set \( S \),

\[
\left| \ln \left( \frac{P(A(D) \in S)}{P(A(D') \in S)} \right) \right| \leq \epsilon,
\]

where \( A(D) \) (resp. \( A(D') \)) is the randomized output of \( A \) on input \( D \) (resp. \( D' \)).
The definition implies that as $\bar{\varepsilon}$ decreases, it becomes harder to distinguish the two datasets $\mathcal{D}$ and $\mathcal{D}'$ by analyzing the randomized outputs, thus providing stronger data privacy.

We aim to construct the randomized function $A$ satisfying (7) by introducing some calibrated random noise into the objective function of the subproblem (6) to protect data in an $\varepsilon$-DP manner. To this end, we add an affine function $\frac{1}{2\rho} \| \bar{\varepsilon} z_p \|^2 - \langle \bar{\varepsilon} z_p, z_p \rangle + \frac{1}{2\rho} (\lambda_p^* - \bar{\varepsilon} z_p)\|z_p\|^2$ to (6), resulting in

$$z_{p,e}^{t+1} \leftarrow \arg \min_{z_p \in \mathcal{W}} \langle f'_p(z_{p,e}^{t}), z_p \rangle + \frac{1}{2\rho} \| z_p - z_{p,e}^{t} \|^2 + \frac{\rho}{2} \| u^{t+1} - z_p + \frac{1}{\rho} (\lambda_p^* - \bar{\varepsilon} z_p) \|^2,$$

where $\bar{\varepsilon} z_p$ is a noise vector sampled from a Laplace distribution with zero mean and a scale parameter $\bar{\Delta}_p \bar{\varepsilon}$ whose probability density function (pdf) is given by

$$\text{Lap}(\bar{\varepsilon} z_p; 0, \bar{\Delta}_p \bar{\varepsilon}) := \frac{\bar{\varepsilon}}{2 \bar{\Delta}_p \bar{\varepsilon}} \exp \left(- \frac{\| \bar{\varepsilon} z_p \|_1}{\bar{\Delta}_p \bar{\varepsilon}} \right), \tag{9a}$$

where $\bar{\varepsilon} > 0,$

$$\bar{\Delta}_p := \max_{D_p \in \mathcal{D}_p} \| f'_p(z_{p,e}; D_p) - f'_p(z_{p,e}; D'_p) \|_1, \tag{9b}$$

$\mathcal{D}_p :=$ a collection of datasets differing a single entry from a given dataset $D_p$. \tag{9c}

Note that (8) with $\bar{\varepsilon} z_p = 0$ is equal to (6). We use $f'_p(z_{p,e}; D_p)$ and $f'_p(z_{p,e})$ interchangeably, where $D_p$ is a given dataset.

**DP-IADMM.** In Algorithm 1, we present the proposed DP-IADMM with multiple local updates. We describe the steps of the algorithm as follows. The computation at the central server is described in lines 1–9, while the local computation for each agent $p$ is described in lines 11–24. In lines 2–3, the initial points are sent from the server to all agents. In lines 5–6, the global parameter $w^{t+1}$ is computed and sent to the local agents. In lines 15–22, the local agent $p$ receives $w^{t+1}$ from the server, conducts local updates for $E$ times, and sends the resulting local model parameter $z_{p,e}^{t+1}$ to the server. Note that $z_{p,e}^{t+1}$ is a randomized output: it is perturbed by injecting random noise to the objective function of (8). The dual updates are performed at the server and the local agents individually as in line 8 and in line 23, respectively. Note that those dual updates are identical since the initial points at the server and the local agents are the same.

The benefits of Algorithm 1 include that (i) the quality of the solution can be improved via the multiple local updates that could result in reducing the total number of iterations, (ii) the amount of communication is reduced by excluding the communication of the dual information, and (iii) $\bar{\varepsilon}$-DP on data is guaranteed for any communication rounds, which will be proved in the next section.

### Algorithm 1 DP-IADMM with multiple local updates.

1. **(Server):**
   1. Initialize $\lambda_1, \ldots, \lambda_P, z_1, \ldots, z_P$.
   2. Send $\lambda_p, z_p$ to all agent $p \in [P]$ (to line 12).
   3. for $t \in [T]$ do
      4. $w^{t+1} \leftarrow \frac{1}{P} \sum_{p=1}^{P} (z_{p}^{t} - \frac{1}{P} \lambda_p^*)$.
      5. Send $w^{t+1}$ to all agents (to line 15).
   7. Receive $z_p^{t+1}$ from all agents (from line 22).
   8. $\lambda_{p}^{t+1} \leftarrow \lambda_p^* + \rho'(w^{t+1} - z_{p}^{t+1})$ for all $p \in [P]$.
   9. end for
10. for $p \in [P]$ do
    11. Receive $\lambda_{p}, Z^{t+1}$ from the server (from line 3).
    12. Initialize $z_p^{0,E+1} = z_p$.
    13. for $t \in [T]$ do
        14. Receive $w^{t+1}$ from the server (from line 6).
        15. Set $z_p^{t+1} \leftarrow Z_p^{t+1} - E^{t+1}$.
        16. for $e \in [E]$ do
            17. Sample $\bar{\varepsilon} z_p^{t+1}$ from (9).
            18. Compute $z_p^{t,e+1}$ by solving (8).
        19. end for
        20. $z_p^{t+1} \leftarrow \frac{1}{P} \sum_{e=1}^{E} z_p^{t,e+1}$.
        21. Send $z_p^{t+1}$ to the server (to line 7).
        22. $\lambda_{p}^{t+1} \leftarrow \lambda_p^* + \rho'(w^{t+1} - z_{p}^{t+1})$.
    23. end for
12. end for

### A. Privacy Analysis

In this section we show that $\bar{\varepsilon}$-DP in Definition 1 is guaranteed for any iteration of Algorithm 1. To this end, using the following lemma, we show that the constrained subproblem (8) provides $\bar{\varepsilon}$-DP.

**Lemma 1. (Theorem 1 in [16])** Let $A$ be a randomized algorithm induced by the random noise $\xi$ that provides output $\phi(\mathcal{D}, \xi)$ for all $\ell \in [L]$ and satisfies a pointwise convergence condition, namely, $\lim_{n \to \infty} \phi(\mathcal{D}, \xi) = \phi(\mathcal{D}, \xi)$, then $A$ is also $\bar{\varepsilon}$-DP.

For the rest of this section we fix $t \in [T], e \in [E]$, and $p \in [P]$. For ease of exposition, we denote the objective function of (8), which is strongly convex, by

$$G_{p}^{t,e}(z_p) := \langle f'_p(z_{p,e}^{t}), z_p \rangle + \frac{1}{2\rho} \| z_p - z_{p,e}^{t} \|^2$$

$$+ \frac{\rho}{2} \| u^{t+1} - z_p + \frac{1}{\rho} (\lambda_p^* - \bar{\varepsilon} z_p) \|^2 \tag{10}$$

and the feasible region of (8) by

$$\mathcal{W} = \{ z_p \in \mathbb{R}^{d \times K} : h_m(z_p) \leq 0, \forall m \in [M] \},$$

where $h_m(z_p)$ is the $m$th constraint.

By applying Lemma 1 to Algorithm 1, we obtain the privacy guarantee for Algorithm 1. This completes the proof of Lemma 1.
where $h_m$ is convex and twice continuously differentiable and $M$ is the total number of inequalities.

By utilizing an indicator function $I_W(z_p)$ that outputs zero if $z_p \in W$ and $\infty$ otherwise, (8) can be expressed by the following problem:

$$\min_{z_p \in \mathbb{R}^{J \times K}} G_{t,e}^\ell(z_p) + I_W(z_p).$$

We note that the indicator function can be approximated by the following function:

$$g_t(z_p) := \sum_{m=1}^{M} \ln(1 + e^{h_m(z_p)}),$$

where $\ell > 0$. Increasing $\ell$ enforces the feasibility, namely, $h_m(z_p) \leq 0$, resulting in $g_t(z_p) \to 0$. It is similar to the logarithmic barrier function (LBF), namely $- (1/\ell) \sum_{m=1}^{M} \ln(-h_m(z_p))$, in that the approximation becomes closer to the indicator function as $\ell \to \infty$. However, the penalty function $g_t$ is different from LBF in that the domain of $z_p$ is not restricted. By replacing the indicator function with the penalty function in (11), we construct the following unconstrained problem:

$$z_{p}^{t,e+1} \leftarrow \text{arg min}_{z_p \in \mathbb{R}^{J \times K}} G_{t,e}^\ell(z_p) + g_t(z_p),$$

where the objective function is strongly convex because $g_t$ is convex over all domains and $G_{t,e}^\ell$ is strongly convex. Therefore, $z_{p}^{t,e+1}$ is the unique optimal solution. We first show that (12) satisfies the pointwise convergence condition and provides $\bar{\epsilon}$-DP as in Propositions 1 and 2, respectively.

**Proposition 1.** For fixed $t, e, p, \ell$, and $\rho$, $\lim_{\ell \to \infty} z_{p}^{t,e+1} = z_{p}^{t,e+1}$, where $z_{p}^{t,e+1}$ and $z_{p}^{t,e+1}$ are from (8) and (12), respectively.

**Proof.** See Appendix A. □

**Proposition 2.** For fixed $t, e, p, \ell$, and the dataset $D_p$, we denote by $z_{p}^{t,e+1}(D_p)$ the optimal solution of (12). It provides $\bar{\epsilon}$-DP that satisfies

$$\left| \ln \left( \frac{P(z_{p}^{t,e+1}(D_p) \in S)}{P(z_{p}^{t,e+1}(D_p) \in \bar{\epsilon})} \right) \right| \leq \bar{\epsilon},$$

for all $S \subset \mathbb{R}^{J \times K}$ and $D_p \in \bar{D}_p$, where $\bar{D}_p$ is from (9c). The result in C3 requires additional assumptions.

**Remark 1.** Theorem 1 shows that $\bar{\epsilon}$-DP is guaranteed for every iteration of Algorithm 1. This result can be extended by introducing the existing composition theorem in [3] to ensure $\bar{\epsilon}$-DP for the entire process of the algorithm.

**B. Convergence Analysis**

In this section we show that a sequence of iterates generated by Algorithm 1 convergences to an optimal solution of (2) in expectation under the following assumptions.

**Assumption 1.**

(i) $\rho^t$ in (8) satisfies $\rho^1 \leq \rho^2 \leq \ldots \leq \rho^T \leq \rho^\text{max}$.

(ii) $\exists \gamma > 0: \gamma \geq 2\|\lambda^*\|$, where $\lambda^*$ is a dual optimal.

(iii) $f_p$ in (2c) is $H$-Lipschitz over a set $W$ with respect to the Euclidean norm.

Assumption 1 is typically used for the convergence analysis of ADMM and IADMM (see Chapter 15 of [27]). We adopt the assumptions because IADMM is a special case of Algorithm 1 by setting $E = 1$ and $\bar{\epsilon}_{p}^{t,e} = 0$.

Based on Assumption 1 (iii) used for bounding subgradients, we define the following bounds (see Appendix C for details):

$$U_1 := \max_{u,v \in W, p \in [p]} \|f_p(u; D_p)\|,$$

$$U_2 := \max_{u,v \in W} \|u - v\|,$$

$$U_3 := \max_{u,v \in W, p \in [p], D_p \in \bar{D}_p} \|f_p(u; D_p) - f_p(u; D_p')\|.$$
Then we have

\[ f_2(\rho_{\text{max}} + L/E) + (\gamma + \|\lambda^1\|)^2/\rho^1 \]

(16a)

where

\[ R^t(\sqrt{T}, \bar{\epsilon}) := \frac{2PJKU_3^2 + U_3^2/(2E)}{\bar{\epsilon}\sqrt{T}}, \]

(16b)

and

\[ z^{(T)} := \frac{1}{T} \sum_{t=1}^{T} z^{t,e+1}. \]

(18)

Proof. See Appendix D. \(\square\)

According to Theorem 3.60 in [27], the inequality (16a) derived under Assumption 1 implies that the rate of convergence in expectation is \(O(1/(\bar{\epsilon}\sqrt{T}))\) for \(\bar{\epsilon} \in (0, \infty)\), while in a nonprivate setting it is \(O(1/T)\) because \(R^t(\sqrt{T}, \bar{\epsilon})\) in (16b) is zero when \(\bar{\epsilon} = \infty\).

**Theorem 3.** Suppose that Assumption 1 holds, \(f_p\) (defined in (2c)) is a nonsmooth convex function, and

\[ \eta^t = 1/\sqrt{T}, \forall t. \]

(19)

Then we have

\[ E[f(z^{(T)}) - f(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\|] \leq R^{\text{NS}}(\sqrt{T}, \bar{\epsilon}) \]

(20a)

where

\[ R^{\text{NS}}(\sqrt{T}, \bar{\epsilon}) := \frac{2PJKU_3^2/\bar{\epsilon}^2 + PU_1^2 + U_3^2/(2E)}{\sqrt{T}} \]

(20b)

\(U_1, U_2, U_3\) are from (14), \(z^*\) is an optimal solution of (2), \(F(z), A, w^{(T)}\) are from (17), and

\[ z^{(T)} := \frac{1}{T} \sum_{t=1}^{T} \sum_{e=1}^{E} z^{t,e}. \]

(21)

Proof. See Appendix E. \(\square\)

The inequality (20a) derived under Assumption 1 implies that the rate of convergence in expectation is \(O(1/(\bar{\epsilon}\sqrt{T}))\) for \(\bar{\epsilon} \in (0, \infty)\), while in a nonprivate setting it is \(O(1/\sqrt{T})\) because \(R^{\text{NS}}(\sqrt{T}, \bar{\epsilon})\) in (20b) is \((PU_1^2 + U_3^2/(2E))/\sqrt{T}\) when \(\bar{\epsilon} = \infty\).

**Theorem 4.** Suppose that Assumptions 1 and 2 hold, \(f_p\) (defined in (2c)) is \(\alpha\)-strongly convex, and

\[ \eta^t = 2/(\alpha(t + 2)), \forall t. \]

(22)

Then we have

\[ E[f(z^{(T)}) - f(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\|] \leq \]

(23)

where \(U_1, U_2, U_3\) are from (14), \(z^*\) is an optimal solution, \(F(z), A\) are from (17), and

\[ w^{(T)} := \frac{2}{t+1} \sum_{t=1}^{T} tw^{t+1}, \]

\[ z^{(T)} := \frac{2}{(t+1)} \sum_{t=1}^{T} t(\frac{1}{t} E_{\sum_{e=1}^{E} z^{t,e}}). \]

Proof. See Appendix F. \(\square\)

The inequality (23) derived under Assumptions 1 and 2 implies that the rate of convergence in expectation is \(O(1/(\bar{\epsilon}^2T))\) for \(\bar{\epsilon} \in (0, \infty)\), while in a nonprivate setting it is \(O(1/T)\) when \(\bar{\epsilon} = \infty\).

**Corollary 1.** (Effect of the multiple local update)

Increasing the number \(E\) of local updates decreases the values on the right-hand side of (16a), (20a), and (23). This implies that the gap between \(F(z^{(T)})\) and \(F(z^*)\) can become smaller by increasing \(E\) for fixed \(T\). This may result in greater learning performance by introducing the multiple local updates, which will be numerically demonstrated in Section IV.

**IV. NUMERICAL EXPERIMENTS**

In this section we compare Algorithm 1 with the state of the art in [2] as a baseline algorithm. The algorithm in [2] has demonstrated more accurate solutions than the other existing DP algorithms, such as DP-SGD [17], DP-ADMM with the output perturbation method (Algorithm 2 in [2]), and DP-ADMM with the objective perturbation method [6] (see Figure 6 in [2]). Note that as a DP technique, the output perturbation method is used in the baseline algorithm in [2], whereas the objective perturbation method is used in our algorithm.

We implemented the algorithms in Python and ran the experiments on Swings, a 6-node GPU computing cluster at Argonne National Laboratory. Each node of Swings has 8 NVIDIA A100 40 GB GPUs, as well as 128 CPU cores. The implementation is available at https://github.com/APPFL/DPL-1ADMM-Classification.git.

**Algorithms.** We denote

- the baseline algorithm in [2] by \text{OutP},
- Algorithm 1 with \(E = 1\) by \text{ObjP}, and
- Algorithm 1 with \(E = 10\) by \text{ObjPM}.

Note that \text{OutP} and \text{ObjP} are equivalent in a nonprivate setting.

**FL Model.** We consider a multiclass logistic regression model (see Appendix G for details).

**Datasets.** We consider two publicly available datasets for image classification: MNIST [29] and FEMNIST [30]. For the MNIST dataset, we split the 60,000 training data points over \(P = 10\) agents, each of which is
assigned to have the same number of independent and identically distributed (IID) dataset. For the FEMNIST dataset, we follow the preprocess procedure\(^1\) to sample 5% of the entire 805,263 data points in a non-IID manner, resulting in 36,708 training samples distributed over \(P = 195\) agents.

**Parameters.** Under the multiclass logistic regression model, we can compute \(\Delta_{p,e}^t\) in (9b) as

\[
\Delta_{p,e}^t = \max_{i^* \in [P]} \sum_{j=1}^K \left| \frac{1}{T} \langle h_k(z_{p,e}^t; x_{p,e}) - y_{p,e}^t \rangle \right|.
\]

Note that \(\Delta_{p,e}^t / \bar{\epsilon}\) is proportional to the standard deviation of the Laplace distribution in (9a), thus controlling the errors on average (solid line) with the \(2\)\(^0\) over \(m\) manner, resulting in 36,708 training samples distributed 5% of the entire 805,263 data points in a non-IID manner, resulting in 36,708 training samples distributed over \(P = 195\) agents.

In the experiments, we consider various \(\bar{\epsilon} \in \{0.05, 0.1, 1, 5\}\), where stronger data privacy is achieved with smaller \(\bar{\epsilon}\).

We emphasize that the baseline algorithm Out\(P\) guarantees \((\bar{\epsilon}, \bar{\delta})\)-DP, which provides stronger privacy as \(\delta > 0\) decreases for fixed \(\bar{\epsilon}\), but still weaker than \(\bar{\epsilon}\)-DP. In the experiments, we set \(\delta = 10^{-6}\) for Out\(P\). In addition, we set the regularization parameter \(\beta\) in (2c) by \(\beta = 10^{-6}\), as in [2].

The parameter \(\rho^t\) in Assumption 1 affects the learning performance because it controls the proximity of the local model parameters from the global model parameters. For all algorithms, we set \(\rho^t = \rho^t(\bar{t})\) given by

\[
\rho^t = \min \{10^3, c_1(1.2)^{[t/T_c]} + c_2/\bar{t}, \forall t \in [T]\},
\]

where (i) \(c_1 = 2, c_2 = 5\), and \(T_c = 10000\) for MNIST and (ii) \(c_1 = 0.005, c_2 = 0.05\), and \(T_c = 2000\) for FEMNIST, which are chosen based on the justifications described in Appendix H. Note that the chosen parameter \(\rho^t\) is nondecreasing and bounded above, thus satisfying Assumption 1 (i).

**A. Comparison of Testing Errors**

Using the MNIST and FEMNIST datasets, we compare testing errors produced by Out\(P\), Obj\(P\), and Obj\(PM\) under various \(\bar{\epsilon}\). We note that the testing errors produced by a nonprivate IADMM (i.e., Algorithm 1 with \(\bar{\epsilon} = \infty\)) with the multiclass logistic regression model on MNIST and FEMNIST are 9.1% and 37.27%, respectively.

For each dataset and a given \(\bar{\epsilon}\), we collect the testing errors for 10 runs, each of which has different realizations of the random noises, but all of which guarantee the \(\bar{\epsilon}\)-DP on data. In Figure 1 we report the testing errors on average (solid line) with the 20- and 80-percentile confidence bounds (shaded) for every iteration \(t \in [20000]\). The subfigures on the top and bottom rows are the testing error results for MNIST and FEMNIST, respectively.

In what follows, we present some observations from the figures and their implications.

- The testing errors of all algorithms increase as \(\bar{\epsilon}\) decreases (i.e., stronger data privacy). This indicates the trade-off between data privacy and learning performance, well known in the literature on DP algorithms [3].
- The testing errors of Obj\(P\) are lower than those of Out\(P\). This result is consistent with the findings in [15], [6] that the better performance of the objective perturbation than the output perturbation is guaranteed with higher probability.
- The testing errors of Obj\(PM\) are lowest, demonstrating the effectiveness of the multiple local updates presented in Corollary 1. When \(\bar{\epsilon} = 1\), Obj\(PM\) produces testing errors close to those of the nonprivate IADMM while the other algorithms do not. This result implies that Obj\(PM\) can mitigate the trade-off between data privacy and learning performance.
- When \(\bar{\epsilon} = 0.05\), among the 10 runs from the MNIST dataset, the best testing error of Obj\(PM\) is 11.74% while that of Out\(P\) is 21.79%, a 10.05% improvement.
- When \(\bar{\epsilon} = 0.05\), among the 10 runs from the FEMNIST dataset, the best testing error of Obj\(PM\) is 59.42% while that of Out\(P\) is 91.05%, a 31.63% improvement.

In Figure 2, for every algorithm and \(\bar{\epsilon}\), we report the best testing error among the 10 instances, which showcases the outperformance of Obj\(PM\).

**B. Comparison of Random Noises**

The random noises to Out\(P\) are generated by the Gaussian mechanism with decreasing variance as in [2] and injected into the output of the subproblem, whereas the noises to our algorithms are generated by the Laplacian mechanism and injected into the objective function of the subproblem. To compare the two different mechanisms in terms of the magnitude of noises generated, we compute the following average noise magnitude:

\[
\frac{1}{PJK} \sum_{p=1}^P \sum_{j=1}^J \sum_{k=1}^K |\xi_{pjk}^t|, \forall t \in [T],
\]

where \(\xi_{pjk}^t\) is a realization of random noise \(\tilde{\xi}_{pjk}^t\).

In Figure 3, using the same instances as in Section IV-A, we show that the average noise magnitudes of all the algorithms increase as \(\bar{\epsilon}\) decreases, achieving stronger data privacy. For fixed \(\bar{\epsilon}\), the average noise magnitudes of our algorithms Obj\(PM\) and Obj\(P\) are greater than those of Out\(P\) while the testing errors of our algorithms are less than those of Out\(P\). These results imply that the performance of our algorithms is less sensitive to random perturbation than that of Out\(P\), even with a larger magnitude of noises for stronger \(\bar{\epsilon}\)-DP.

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\(^1\)https://github.com/TalwalkarLab/leaf/tree/master/data/femnist
Fig. 1: Testing errors for every iteration under various $\bar{\epsilon}$ (top: MNIST; bottom: FEMNIST).

Fig. 2: Best testing errors of the three algorithms under various $\bar{\epsilon}$.

Fig. 3: Average noise magnitudes for every iteration (top: MNIST; bottom: FEMNIST).

V. CONCLUSION

We incorporated the objective perturbation and multiple local updates into an IADMM algorithm for training the FL model while ensuring data privacy during the training process. The proposed DP-IADMM algorithm iteratively solves a sequence of subproblems whose objective functions are randomly perturbed by noises sampled from a calibrated Laplace distribution to ensure $\bar{\epsilon}$-DP. We showed that the rate of convergence in expectation for the proposed Algorithm 1 is $O(1/\sqrt{T})$ for both a smooth and a nonsmooth convex setting and $O(1/T)$ for a strongly convex setting. The outperformance of the proposed algorithm was numerically demonstrated with the MNIST and FEMNIST datasets.

We note that the performance of the proposed DP algorithm can be further improved by lowering the magnitude of noises required for ensuring the same level of data privacy (see Figure 3 showing that our algorithm requires larger noises). By improving the performance further, we expect that the proposed DP algorithm can be utilized for learning from larger decentralized datasets with more features and classes.

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APPENDIX A

PROOF OF PROPOSITION 1

We aim to show that, as $\ell$ increases, $z_{p\ell}^{t,e+1}$ converges to $z_p^{t,e+1}$, where $z_p^{t,e+1}$ (resp., $z_{p\ell}^{t,e+1}$) is the optimal solution of an optimization problem in (8) (resp., (12)).

Suppose that $z_{p\ell}^{t,e+1}$ converges to $\tilde{z} \neq z_p^{t,e+1}$ as $\ell$ increases. Consider $\zeta := \|\tilde{z} - z_p^{t,e+1}\|/2$. Since $z_{p\ell}^{t,e+1}$ converges to $\tilde{z}$, there exists $\ell' > 0$ such that $\|\tilde{z} - z_{p\ell}^{t,e+1}\| < \zeta$ for all $\ell \geq \ell'$. By the triangle inequality, we have

\[
\|z_{p\ell}^{t,e+1} - z_p^{t,e+1}\| \geq \|\tilde{z} - z_p^{t,e+1}\| - \|\tilde{z} - z_{p\ell}^{t,e+1}\| > 2\zeta - \zeta = \zeta, \quad \forall \ell \geq \ell'.
\]  

(25a)

Since $G_p^{t,e}$ is strongly convex with a constant $\mu > 0$, we have

\[
G_p^{t,e}(z_{p\ell}^{t,e+1}) - G_p^{t,e}(z_p^{t,e+1}) \geq \frac{\mu^2}{2}||z_{p\ell}^{t,e+1} - z_p^{t,e+1}\|^2 > \frac{\mu^2}{2}, \quad \forall \ell \geq \ell'.
\]  

(25b)

where the last inequality holds by (25a). By adding $g(\tilde{z}) \geq 0$ to the left-hand side of (25b), we derive the following inequality:

\[
\{G_p^{t,e}(z_{p\ell}^{t,e+1}) + g(\tilde{z})\} - G_p^{t,e}(z_p^{t,e+1}) > \frac{\mu^2}{2}, \quad \forall \ell \geq \ell'.
\]  

(25c)

To see the contradiction, consider $\epsilon \in (0, \frac{\mu^2}{2})$. The continuity of $G_p^{t,e} : \mathcal{W} \mapsto \mathbb{R}$ at $z_p^{t,e+1}$ implies that for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $z \in \mathcal{W}$:

\[
z \in B_{\delta}(z_p^{t,e+1}) := \{z \in \mathbb{R}^{J \times K} : \|z - z_p^{t,e+1}\| < \delta\} \Rightarrow G_p^{t,e}(z) - G_p^{t,e}(z_p^{t,e+1}) < \epsilon.
\]  

(25d)

Consider $\tilde{z} \in B_{\delta}(z_p^{t,e+1}) \cap \text{relint}(\mathcal{W})$, where relint indicates the relative interior. Since $h_m(\tilde{z}) < 0$ for all $m \in [M]$, $g(\tilde{z})$ goes to zero as $\ell$ increases. Hence, there exists $\ell'' > 0$ such that

\[
g(\tilde{z}) = \sum_{m=1}^M \ln(1 + e^{\ell''h_m}(\tilde{z})) < \epsilon, \quad \forall \ell \geq \ell''.
\]  

(25e)

For all $\ell \geq \ell''$, we derive the following inequalities:

\[
G_p^{t,e}(z_{p\ell}^{t,e+1}) + g(\tilde{z}) \leq G_p^{t,e}(\tilde{z}) + \epsilon < G_p^{t,e}(z_p^{t,e+1}) + \epsilon < G_p^{t,e}(z_{p\ell}^{t,e+1}) + 2\epsilon,
\]  

(25f)

where the first inequality holds because $z_{p\ell}^{t,e+1}$ is the optimal solution of (12), the second inequality holds by (25e), and the last inequality holds by (25d). Therefore, we have

\[
\{G_p^{t,e}(z_{p\ell}^{t,e+1}) + g(\tilde{z})\} - G_p^{t,e}(z_p^{t,e+1}) < \frac{2\epsilon}{2}, \quad \forall \ell \geq \ell''.
\]  

(25g)

Therefore, for all $\ell \geq \max\{\ell', \ell''\}$, (25c) and (25g) contradict. This completes the proof.

APPENDIX B

PROOF OF PROPOSITION 2

It suffices to show that the following is true:

\[
e^{-\tilde{\epsilon}} \text{pdf}(z_p^{t,e+1}(D_p') = \psi) \leq \text{pdf}(z_p^{t,e+1}(D_p) = \psi) \leq e^{\tilde{\epsilon}} \text{pdf}(z_p^{t,e+1}(D_p') = \psi), \quad \forall \psi \in \mathbb{R}^{J \times K},
\]  

(26a)

where pdf represents a probability density function.

Consider $\psi \in \mathbb{R}^{J \times K}$. If we have $z_p^{t,e+1}(D_p) = \psi$, then $\psi$ is the unique minimizer of (12) because the objective function in (12) is strongly convex. From the optimality condition of (12), we derive

\[
\tilde{\xi}_p^{t,e} = - f'(z_p^{t,e}; D_p) + \rho(t^{e+1} - t) + \lambda_p - \nabla g(\psi) - \frac{\lambda_p}{\mu_p}(\psi - z_p^{t,e}),
\]  

(26b)

where $\nabla g(\psi) = \sum_{m=1}^M \frac{\sum_{t \in T_m(v)} \mu_t h_m(v)}{\mu} \nabla h_m(v)$. Note that the mapping from $\psi$ to $\tilde{\xi}_p^{t,e}$ via (26b) is injective. Also the mapping is surjective because for $\tilde{\xi}_p^{t,e}$, there exists $\psi$ (i.e., the unique minimizer of (12)) such that (26b) holds. Therefore, the relation between $\psi$ and $\tilde{\xi}_p^{t,e}$ is bijective, which enables us to utilize the inverse function theorem (Theorem 17.2 in [31]), namely,

\[
\text{pdf}(z_p^{t,e+1}(D_p) = \psi) \cdot \det(\nabla \tilde{\xi}_p^{t,e}(\psi; D_p)) = \text{Lap}(\tilde{\xi}_p^{t,e}(\psi; D_p); 0, \Delta_p^{t,e}/\ell),
\]  

(26c)
where \( \text{det} \) represents a determinant of a matrix, \( \text{Lap} \) is from (9a), and \( \nabla \bar{\xi}_p^{t,e}(\psi; D_p) \) represents a Jacobian matrix of the mapping from \( \psi \) to \( \bar{\xi}_p^{t,e} \) in (26b), namely,

\[
\nabla \bar{\xi}_p^{t,e}(\psi; D_p) = (-\rho' - 1/\eta')I_{JK} - \nabla \left( \frac{M}{1 + e^{\ell_m}} \nabla \ell_m(\psi) \right),
\]

(26d)

where \( I_{JK} \) is an identity matrix of \( JK \times JK \) dimensions. Since the Jacobian matrix is not affected by the dataset, we have

\[
\nabla \bar{\xi}_p^{t,e}(\psi; D_p) = \nabla \bar{\xi}_p^{t,e}(\psi; D'_p).
\]

(26e)

Based on (26c) and (26e), we derive the following inequalities:

\[
\frac{\text{pdf}(z_{p,t}^{t,e+1}(D_p) = \psi)}{\text{pdf}(z_{p,t}^{t,e}(D_p) = \psi)} \leq \frac{\text{Lap}(\bar{\xi}_p^{t,e}(\psi; D_p))}{\text{det}(\nabla \bar{\xi}_p^{t,e}(\psi; D'_p))} \leq \frac{\text{Lap}(\bar{\xi}_p^{t,e}(\psi; D_p))}{\text{det}(\nabla \bar{\xi}_p^{t,e}(\psi; D'_p))} \leq \exp(\frac{\bar{\xi}_p^{t,e}(\psi; D'_p) - \bar{\xi}_p^{t,e}(\psi; D_p)}{\Delta_{JK}^{t,e}}) \leq \exp(\epsilon / \Delta_{JK}^{t,e})
\]

(26f)

where \( \exp \) represents the exponential function. Similarly, one can derive a lower bound in (26a). Integrating \( \psi \) in (26a) over \( S \) yields (13). This completes the proof.

**APPENDIX C**

**Existence of \( U_1, U_2, \) and \( U_3 \) in (14)**

(Existence of \( U_2 \)) \( U_2 \) is well defined because the objective function \( \|u - v\| \) is continuous and the feasible region \( \mathcal{W} \) is compact.

(Existence of \( U_1 \)) The necessary and sufficient condition of Assumption 1 (iii) is that, for all \( u \in \mathcal{W} \) and \( v \in \partial f_p(u), \|v\|_* \leq H \), where \( \| \cdot \|_* \) is the dual norm. As the dual norm of the Euclidean norm is the Euclidean norm, we have \( \|f_p(u)\| \leq H \). Since the objective function, which is a maximum of finite continuous functions, is continuous and \( \mathcal{W} \) is compact, \( U_3 \) is well defined.

(Existence of \( U_3 \)) From the norm inequality, we have

\[
\|f_p(u; D_p) - f_p(u; D'_p)\| \leq \sqrt{JK} \|f_p(u; D_p) - f_p(u; D'_p)\|_2
\]

\[
\leq \sqrt{JK} \{\|f_p(u; D_p)\|_2 + \|f_p(u; D'_p)\|_2\} \leq 2H\sqrt{JK}, \quad \forall u \in \mathcal{W},
\]

where the last inequality holds by Assumption 1 (iii). Therefore, \( U_3 \) is well defined.

**APPENDIX D**

**Proof of Theorem 2**

**A. Preliminaries**

First, we note that for any symmetric matrix \( A \),

\[
(a - b)^\top A(c - d) = \frac{1}{2} \{\|a - d\|_A^2 - \|a - c\|_A^2 - \|c - b\|_A^2 - \|d - b\|_A^2\},
\]

(27)

where \( a, b, c, \) and \( d \) are vectors of the same size.

Second, we define \( \lambda_p^t := \lambda_p^t + \rho'(w^{t+1} - z_p^t) \) for fixed \( t \in [T] \) and \( p \in [P] \). From the optimality condition of (4a), namely, \( \sum_{p=1}^P \lambda_p^t + \rho'(w^{t+1} - z_p^t) = \sum_{p=1}^P \lambda_p^t = 0 \), we have

\[
\sum_{p=1}^P (\lambda_p^t, w^{t+1} - w) = 0, \quad \forall w.
\]

(28)
B. Inequality derivation for a fixed iteration $t$ and $e$.

First, for a given $p \in [P]$, the optimality condition of (8) is given by

$$\langle \nabla f_p(z_{t,e}^p) - \{\lambda_p + \rho'(w_{t+1} - \zeta_{t,e+1})\}, z_{t,e}^p, z_{p}^t, z_{t,e+1}^p - z_p \rangle \leq \frac{1}{\eta_t^2} \langle z_{t,e}^p, z_{t,e+1}^p - z_p \rangle, \forall z_p \in \mathcal{W}. $$

By defining $\lambda_{p}^{t,e+1} := \lambda_p + \rho'(w_{t+1} - \zeta_{t,e+1})$ for the “A” term and applying (27) on the “B” term from the above inequalities, we have

$$\langle \nabla f_p(z_{t,e}^p) - \lambda_{p}^{t,e+1} + \tilde{\xi}_{t,e}^{p}, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle \leq \frac{1}{2\eta_t^2} \left( \|z_{t,e}^p - z_{t,e+1}^p\|^2 - \|z_p - z_{t,e}^p - z_{t,e+1}^p\|^2 \right). \quad (29)$$

Second, by adding a term $\langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle$ to the subgradient inequality $f_p(z_{t,e}^p) - f_p(z_p) \leq \langle \nabla f_p(z_{t,e}^p), z_{t,e}^p - z_p \rangle$ for all $z_p$, we derive

$$f_p(z_{t,e}^p) - f_p(z_p) = \langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle + \langle \nabla f_p(z_{t,e}^p) - \lambda_{p}^{t,e+1} + \tilde{\xi}_{t,e}^{p}, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle \leq \langle \nabla f_p(z_{t,e}^p) - \lambda_{p}^{t,e+1} + \tilde{\xi}_{t,e}^{p}, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle \leq \langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle + \langle \nabla f_p(z_{t,e}^p) + \tilde{\xi}_{t,e}^{p}, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle.$$ 

Since the “C” term from the above inequalities can be written as

$$\langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle = \langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle + \langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle,$$ 

we obtain

$$f_p(z_{t,e}^p) - f_p(z_p) = \langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle + \langle \nabla f_p(z_{t,e}^p) + \tilde{\xi}_{t,e}^{p}, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle + \frac{1}{2\eta_t^2} \left( \|z_p - z_{t,e}^p\|^2 - \|z_{t,e}^p - z_{t,e+1}^p\|^2 \right). \quad (30)$$

Third, we derive from the “D” term in (30) that

$$\langle \nabla f_p(z_{t,e}^p) + \tilde{\xi}_{t,e}^{p}, z_{t,e}^p, z_{t,e+1}^p \rangle = \langle \nabla f_p(z_{t,e}^p), z_{t,e}^p, z_{t,e+1}^p \rangle \leq \langle \nabla f_p(z_{t,e}^p), z_{t,e}^p, z_{t,e+1}^p \rangle \leq \left\{ \frac{1}{2(1/\eta_t^2 - L)} \|z_{t,e}^p\|^2 + \frac{1}{2\eta_t^2} \|z_{t,e+1}^p - z_{t,e}^p\|^2 \right\},$$

where $1/\eta_t^2 - L > 0$ by the construction of $\eta_t$ in (15). Therefore, we derive from (30) the following inequalities:

$$f_p(z_{t,e}^p) - f_p(z_p) = \langle \lambda_{p}^{t,e+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p \rangle + \langle \nabla f_p(z_{t,e}^p), z_{t,e}^p, z_{t,e+1}^p \rangle \leq \left\{ \frac{1}{2(1/\eta_t^2 - L)} \|z_{t,e}^p\|^2 + \frac{1}{2\eta_t^2} \left( \|z_p - z_{t,e}^p\|^2 - \|z_{t,e}^p - z_{t,e+1}^p\|^2 \right) \right\}, \forall z_p \in \mathcal{W}. \quad (31)$$

C. Inequality derivation for a fixed iteration $t$.

Summing (31) over all $e \in [E]$ and dividing the resulting inequalities by $E$, we obtain

$$\frac{1}{E} \sum_{e=1}^{E} f_p(z_{t,e}^p) - f_p(z_p) = \langle \lambda_{p}^{t+1} - \lambda_p, z_{t,e}^p, z_{t,e+1}^p - z_p \rangle + \langle \nabla f_p(z_{t,e}^p), z_{t,e}^p, z_{t,e+1}^p \rangle \leq \left\{ \frac{1}{2(1/\eta_t^2 - L)} \|z_{t,e}^p\|^2 + \frac{1}{2\eta_t^2} \left( \|z_p - z_{t,e}^p\|^2 - \|z_{t,e}^p - z_{t,e+1}^p\|^2 \right) \right\}, \forall z_p \in \mathcal{W}. \quad (32)$$
The “$\mathcal{E}$” term from (32) is non-positive because
\[
\frac{1}{E} \sum_{p=1}^E \sum_{e=1}^E (z_{p,e}^{t+1} - z_{p}^{t+1} + z_{p}^{t,e+1} - z_{p}^{t+1}) = \frac{1}{E^2} \sum_{e=1}^E \sum_{e'=1}^E \sum_{e'=1}^E (z_{p,e'}^{t+1} - z_{p}^{t+1} + z_{p}^{t,e'+1} - z_{p}^{t+1}) = \frac{1}{E^2} \sum_{e=1}^E \sum_{e'=1}^E (z_{p,e}^{t+1} - z_{p}^{t+1}, z_{p}^{t,e'} - z_{p}^{t+1}) + (z_{p,e}^{t+1} - z_{p}^{t+1}, z_{p}^{t,e'} - z_{p}^{t+1}) = 1/E^2 \sum_{e=1}^E \sum_{e'=1}^E \langle z_{p,e}^{t+1} - z_{p}^{t+1}, z_{p}^{t,e'} - z_{p}^{t+1} \rangle \leq \frac{1}{E^2} \sum_{e=1}^E \sum_{e'=1}^E \|z_{p,e}^{t+1} - z_{p}^{t+1}\|^2 \leq 0.
\] (33)

Summing the inequalities resulting from (32) and (33) over $p \in [P]$, we have
\[
\sum_{p=1}^P \left[ \frac{1}{E} \sum_{e=1}^E \langle f_p(z_{p,e}^{t+1}) - f_p(z_p) - (\lambda_p^{t+1} - z_p^{t+1} - z_p) \rangle \right] \leq \sum_{p=1}^P \left[ \frac{1}{E} \sum_{e=1}^E \left( \frac{\|\tilde{\xi}_p,e\|^2}{2(1/\eta_t - L)} + \frac{1}{2\eta_t} (\|z_p^{t+1} - z_p^{t,e}\|^2 - \|z_p - z_p^{t,e}\|^2) \right) + (\tilde{\xi}_p,e, z_p^{t+1} - z_p^{t,e}) \right].
\] (34)

For ease of exposition, we introduce the following notation:
\[
z := [z_1^T, \ldots, z_p^T]^T, \quad \lambda := [\lambda_1^T, \ldots, \lambda_p^T]^T, \quad \tilde{\lambda} := [\tilde{\lambda}_1^T, \ldots, \tilde{\lambda}_p^T]^T,
\]
\[
x := [w^T, z]^T, \quad \tilde{x} := [w^{t+1}, z^{t+1}]^T, \quad x^* := [w^*, z]^T, \quad A := \begin{bmatrix} I_J & 0 \\ 0 & -A_{pJ} \end{bmatrix}, \quad G := \begin{bmatrix} 0 & 0 & A^T \\ 0 & 0 & -A_{pJ} \end{bmatrix},
\]
\[
x^{(T)} := \frac{1}{T} \sum_{t=1}^T \tilde{x}^t, \quad w^{(T)} := \frac{1}{T} \sum_{t=1}^T w^{t+1}, \quad z^{(T)} := \frac{1}{T} \sum_{t=1}^T z^{t,e+1}, \quad \lambda^{(T)} := \frac{1}{T} \sum_{t=1}^T \tilde{\lambda}^t,
\]
\[
A^T \tilde{\lambda} = \sum_{p=1}^P \tilde{\lambda}_p^t, \quad F(z) := \sum_{p=1}^P f_p(z_p), \quad \tilde{\xi}^{t,e} := [(\tilde{\xi}_1^{t,e})^T, \ldots, (\tilde{\xi}_p^{t,e})^T]^T.
\]

Based on the above notation as well as (28) and (34), we derive $\text{LHS}^f(w^*, z^*) \leq \text{RHS}^f(z^*)$ at optimal $w^*$ and $\{(z_p^*)_{p=1}^P \in \mathcal{W}$, where
\[
\text{LHS}^f(w^*, z^*) := \frac{1}{E} \sum_{e=1}^E F(z_{p,e}^{t+1}) - F(z) - \langle \lambda^{t+1} + z^{t+1} - z^* \rangle + \langle A^T \tilde{\lambda}, w^{t+1} - w^* \rangle,
\] (36a)
\[
\text{RHS}^f(z^*) := \frac{1}{E} \sum_{e=1}^E \left( \frac{\|\tilde{\xi}_{p,e}^t\|^2}{2(1/\eta_t - L)} + \frac{1}{2\eta_t} (\|z_*^{t+1} - z^{t,e}\|^2 - \|z^* - z^{t,e+1}\|^2) \right) + (\tilde{\xi}_{p,e}, z^* - z^{t,e+1})\right).
\] (36b)

D. Lower bound on $\text{LHS}^f(w^*, z^*)$.

Recall that
\[
\lambda^{t+1} = \lambda^t + \rho_t (A w^{t+1} - z^{t+1}), \quad \tilde{\lambda}^t = \lambda^t + \rho_t (A w^{t+1} - z^t).
\] (37)

By utilizing (37), we rewrite $\text{LHS}^f(w^*, z^*)$ in (36a) as follows:
\[
\text{LHS}^f(w^*, z^*) = \frac{1}{E} \sum_{e=1}^E F(z_{p,e}^{t+1}) - F(z) + \begin{bmatrix} w^{t+1} - w^* \\ z^{t+1} - z^* \\ \lambda^t \end{bmatrix} \begin{bmatrix} \tilde{\lambda}^t \\ -A w^{t+1} + z^{t+1} \end{bmatrix} + \begin{bmatrix} A^T \tilde{\lambda} \\ -\tilde{\lambda}^t \\ -A w^{t+1} + z^{t+1} \end{bmatrix} - \begin{bmatrix} 0 \\ \rho_t (z^* - z^{t+1}) \\ (\lambda^t - \lambda^{t+1})/\rho_t \end{bmatrix}.
\] (38a)

The third term in (38a) can be written as
\[
\langle \begin{bmatrix} w^{t+1} - w^* \\ z^{t+1} - z^* \\ \lambda^t \end{bmatrix}, \begin{bmatrix} A^T \tilde{\lambda} \\ -\tilde{\lambda}^t \\ -A w^{t+1} + z^{t+1} \end{bmatrix} \rangle = \langle \tilde{x}^t - x^*, G \tilde{x}^t \rangle = \langle \tilde{x}^t - x^*, G(\tilde{x}^t - x^*) \rangle + \langle \tilde{x}^t - x^*, G x^* \rangle = 0 \text{ as } G \text{ is skew-symmetric}.
\]
\[
= (\hat{z}^t - x^*, Gx^*),
\]

Based on (27), the last term in (38a) can be written as

\[
\begin{bmatrix}
w^* - w^{t+1}
v^* - v^{t+1}
\end{bmatrix}
\begin{bmatrix}
\rho^t(z^t - z^{t+1})
\lambda - \lambda^t
\end{bmatrix}
= \frac{\rho^t}{2} \left( \|z^* - z^{t+1}\|^2 - \|z^* - z^t\|^2 + \|z^{t+1} - z^t\|^2 \right) + \frac{1}{2\rho^t} \left( \|\lambda - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 + \|\lambda^t - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 \right)
\geq 0
\]

(38c)

Therefore, we have

\[
\text{LHS}(w^*, z^*) \geq \frac{1}{E} \sum_{t=1}^{E} \sum_{e=1}^{E} F(z_{e,t}, z_{e,t+1}) - F(z^*) + \langle \hat{z}^t - x^*, Gx^* \rangle
+ \frac{\rho^t}{2} \left( \|z^* - z^{t+1}\|^2 - \|z^* - z^t\|^2 \right) + \frac{1}{2\rho^t} \left( \|\lambda - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 \right).
\]

(38d)

E. Lower bound on \( \text{LHS}(w^*, z^*) := \frac{1}{T} \sum_{t=1}^{T} \text{LHS}_t(w^*, z^*). \)

Summing (38d) over \( t \in [T] \) and dividing the resulting inequality by \( T \), we have

\[
\begin{align*}
\text{LHS}(w^*, z^*) & \geq \frac{1}{T E} \sum_{t=1}^{T} \sum_{e=1}^{E} F(z_{e,t}, z_{e,t+1}) - F(z^*) + \langle \hat{z}^t - x^*, Gx^* \rangle \\
& \geq F(z^{(T)}) \text{ as } F \text{ is convex} \\
& \geq \frac{1}{T} \left\{ \sum_{t=1}^{T} \rho^t \left( \|z^* - z^{t+1}\|^2 - \|z^* - z^t\|^2 \right) + \sum_{t=1}^{T} \frac{1}{2\rho^t} \left( \|\lambda - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 \right) \right\}. 
\end{align*}
\]

(39)

The “F” term in (39) can be written as

\[
\langle x^{(T)} - x^*, Gx \rangle = \langle Aw^{(T)} - z^{(T)} - Aw^* + z^*, \lambda \rangle - \langle \lambda^{(T)} - \lambda, Aw^* - z^* \rangle = \langle \lambda, Aw^{(T)} - z^{(T)} \rangle.
\]

The “G” term in (39) can be written as

\[
\sum_{t=1}^{T} \rho^t \left( \|z^* - z^{t+1}\|^2 - \|z^* - z^t\|^2 \right) = \frac{\rho^1}{2} \|z^* - z^1\|^2 + \sum_{t=2}^{T} \left( \frac{\rho^{t-1} - \rho^t}{2} \right) \|z^* - z^t\|^2 + \frac{\rho^T}{2} \|z^* - z^{T+1}\|^2
\geq \frac{U_2^2 \rho^T}{2} \geq 0.
\]

The “H” term in (39) can be written as

\[
\sum_{t=1}^{T} \frac{1}{2\rho^t} \left( \|\lambda - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 \right) = - \frac{1}{2\rho^t} \|\lambda - \lambda^1\|^2 + \sum_{t=2}^{T} \left( \frac{1}{2\rho^t} - \frac{1}{2\rho^t} \right) \|\lambda - \lambda^t\|^2 + \frac{1}{2\rho^T} \|\lambda - \lambda^{T+1}\|^2
\geq 0.
\]

Therefore, we derive

\[
\text{LHS}(w^*, z^*) \geq F(z^{(T)}) - F(z^*) + \langle \lambda, Aw^{(T)} - z^{(T)} \rangle - \frac{1}{T} \left( \frac{U_2^2 \rho_{\max}}{2} + \frac{1}{2\rho^t} \|\lambda - \lambda^1\|^2 \right).
\]

(40)
Since this inequality holds for any \( \lambda \), we select \( \lambda \) that maximizes the right-hand side of (40) subject to a ball centered at zero with the radius \( \gamma \):

\[
\begin{align*}
\text{max}_{\lambda:||\lambda|| \leq \gamma} \langle \lambda, Aw^{(T)} - z^{(T')} \rangle &= \gamma \| Aw^{(T)} - z^{(T')} \|, \\
\text{max}_{\lambda:||\lambda|| \leq \gamma} ||\lambda - \lambda^1||^2 &= \| \lambda^1 \|^2 + \text{max}_{\lambda:||\lambda|| \leq \gamma} \{ ||\lambda||^2 - 2 \langle \lambda, \lambda^1 \rangle \} \leq (\gamma + ||\lambda^1||)^2. 
\end{align*}
\]  

(41a)  

(41b)

Based on (40) and (41), we derive

\[
\text{LHS}(w^*, z^*) \geq F(z^{(T)}) - F(z^*) + \gamma \| Aw^{(T)} - z^{(T')} \| - \frac{U_2^2 \rho_{\text{max}}^2 + (\gamma + ||\lambda^1||)^2 / \rho}{2T}.
\]  

(42)

F. Upper bound on \( \text{RHS}(z^*) := \frac{1}{\bar{T}} \sum_{t=1}^{T} \text{RHS}^t(z^*) \).

It follows from (36b) that

\[
\text{RHS}(z^*) = \frac{1}{TE} \left[ \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{\| \xi^{t,e} \|^2}{2(1/\eta^t - L)} + \langle \xi^{t,e}, z^* - z^{t,e} \rangle \right\} + \sum_{t=1}^{T} \sum_{e=1}^{E} \frac{1}{2\eta^t} \left( \|z^* - z^{t,e} - z^* - z^{t,e+1}||^2 \right) \right].
\]

The “\( -1 \)” term from the above can be written as

\[
\sum_{t=1}^{T} \sum_{e=1}^{E} \frac{1}{2\eta^t} \left( \|z^* - z^{t,e} - z^* - z^{t,e+1}||^2 \right) = \sum_{t=1}^{T} \frac{1}{2\eta^t} \left( \|z^* - z^{t,1}||^2 - \|z^* - z^{t,E+1}||^2 \right)
\]

\[
= \frac{1}{2\eta^t} \|z^* - z^{1,1}||^2 + \sum_{t=2}^{T} \frac{1}{2\eta^t - 1} \|z^* - z^{t,1}||^2 - \frac{1}{2\eta^t} \|z^* - z^{t,E+1}||^2 \leq \frac{U_2^2}{2\eta^t}.
\]

Therefore, we have

\[
\text{RHS}(z^*) \leq \frac{U_2^2}{2TE\eta^t} + \frac{1}{TE} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{\| \xi^{t,e} \|^2}{2(1/\eta^t - L)} + \langle \xi^{t,e}, z^* - z^{t,e} \rangle \right\}. 
\]  

(43)

G. Taking expectation.

By taking expectation on the inequality derived from (42) and (43), we have

\[
\mathbb{E}\left[ F(z^{(T)}) - F(z^*) + \gamma \| Aw^{(T)} - z^{(T')} \| \right] \leq \frac{U_2^2 \rho_{\text{max}}^2 + (\gamma + ||\lambda^1||)^2 / \rho}{2T} + \frac{U_2^2}{2TE\eta^t} + \frac{1}{TE} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{1}{2(1/\eta^t - L)} \sum_{p=1}^{P} \mathbb{E}(\| \xi_p^{t,e} \|^2) \right\}.
\]

where

\[
\mathbb{U}(\epsilon) := 2JKU_2^2 / \epsilon^2 \geq \sum_{j=1}^{J} \sum_{k=1}^{K} 2(\Delta_p^{j,k})^2 / \epsilon^2 = \sum_{j=1}^{J} \sum_{k=1}^{K} \mathbb{E}(\| \xi_p^{j,k} \|^2) = \mathbb{E}(\| \xi_p \|^2) 
\]  

(44)

By noting that

\[
\frac{U_2^2}{2TE\eta^t} = \frac{U_2^2L}{2TE} + \frac{U_2^2}{2\bar{\epsilon}E\sqrt{t}}.
\]

\[
\sum_{t=1}^{T} \sum_{e=1}^{E} \frac{1}{2(1/\eta^t - L)} = \bar{\epsilon} \sum_{t=1}^{T} \frac{1}{\sqrt{t} + \sqrt{t - 1}} = E \bar{\epsilon} \sum_{t=1}^{T} (\sqrt{t} - \sqrt{t - 1}) = E \bar{\epsilon} \sqrt{T},
\]

we have

\[
\mathbb{E}\left[ F(z^{(T)}) - F(z^*) + \gamma \| Aw^{(T)} - z^{(T')} \| \right] \leq \frac{U_2^2(\rho_{\text{max}} + L/E) + (\gamma + ||\lambda^1||)^2 / \rho_3}{2T} + \frac{2PJKU_2^2 + U_2^2/(2E)}{\epsilon \sqrt{T}}. 
\]

This completes the proof.
Appendix E
Proof of Theorem 3

The proof in this section is similar to that in Appendix D except that
1) the $L$-smoothness of $f_p$ can no longer be applied to the “D” term in (30) when deriving an upper bound of the term in a non-smooth setting and
2) the definition of $(\tilde{x}^t, z^{(T)})$ is different from that in (35) of Appendix D.

A. Inequality derivation for a fixed iteration $t$ and $e$.

Applying Young’s inequality on the “D” term in (30) yields

$$f_p(z_{p}^{t,e}) - f_p(z_p) - \langle \lambda_p^{t+1}, z_{p}^{t,e+1} - z_p \rangle \leq \rho^t \langle \lambda_{p}^{t+1} - z_p^{t,e+1}, z_{p}^{t,e+1} - z_p \rangle +$$

$$\eta^t \| f_p'(z_{p}^{t,e}) + \tilde{\xi}_{p}^{t,e} \|^{2} + \frac{1}{2\eta^t} \left( \| z_p - z_{p}^{t,e} \|^{2} - \| z_p - z_{p}^{t,e+1} \|^2 \right) + \langle \tilde{\xi}_{p}^{t,e}, z_p - z_{p}^{t,e} \rangle, \forall z_p \in \mathcal{W}. \quad (45)$$

B. Inequality derivation for a fixed iteration $t$.

Summing (45) over all $e \in [E]$ and dividing the resulting inequalities by $E$, we get

$$\frac{1}{E} \sum_{e=1}^{E} f_p(z_{p}^{t,e}) - f_p(z_p) - \langle \lambda_p^{t+1}, \sum_{e=1}^{E} z_{p}^{t,e+1} - z_p \rangle \leq \rho^t \frac{1}{E} \sum_{e=1}^{E} \langle \lambda_{p}^{t+1} - z_p^{t,e+1}, \sum_{e=1}^{E} z_{p}^{t,e+1} - z_p \rangle +$$

$$\frac{1}{E} \sum_{e=1}^{E} \left\{ \frac{\eta^t}{2} \| f_p'(z_{p}^{t,e}) + \tilde{\xi}_{p}^{t,e} \|^{2} + \frac{1}{2\eta^t} \left( \| z_p - z_{p}^{t,e} \|^{2} - \| z_p - z_{p}^{t,e+1} \|^2 \right) + \langle \tilde{\xi}_{p}^{t,e}, z_p - z_{p}^{t,e} \rangle \right\}. \quad (46)$$

Summing the inequalities (46) over $p \in [P]$, we have

$$\frac{1}{E} \sum_{e=1}^{E} \left[ \sum_{e=1}^{E} f_p(z_{p}^{t,e}) - f_p(z_p) - \langle \lambda_p^{t+1}, \sum_{e=1}^{E} z_{p}^{t,e+1} - z_p \rangle \right] \leq$$

$$\frac{1}{E} \sum_{e=1}^{E} \left\{ \frac{\eta^t}{2} \| f_p'(z_{p}^{t,e}) + \tilde{\xi}_{p}^{t,e} \|^{2} + \frac{1}{2\eta^t} \left( \| z_p - z_{p}^{t,e} \|^{2} - \| z_p - z_{p}^{t,e+1} \|^2 \right) + \langle \tilde{\xi}_{p}^{t,e}, z_p - z_{p}^{t,e} \rangle \right\}. \quad (47)$$

For ease of exposition, we introduce $z, \lambda, \tilde{\lambda}, x, A, G, x^{(T)}, w^{(T)}, \lambda^{(T)}, A^{\top} \tilde{\lambda}^{T}, F(z), \tilde{\xi}^{t,e}$ defined in (35) with modifications of the following notation:

$$\tilde{x}^t := \begin{bmatrix} w^{t+1} \\ z^t \\ \tilde{\lambda}^t \end{bmatrix}, \quad z^{(T)} := \frac{1}{T} \sum_{t=1}^{T} \sum_{e=1}^{E} z_{p}^{t,e}. \quad (48a)$$

We also define

$$f'(z) := \left[ f_1'(z_1)^T, \ldots, f_{P}'(z_{p})^T \right]^T. \quad (48b)$$

Based on this notation as well as (28) and (47), we derive $LHS'(w^*, z^*) \leq RHS'(z^*)$ at optimal $w^*$ and $\{z_p^*\}_{p=1}^{P} \in \mathcal{W}$, where

$$LHS'(w^*, z^*) := \frac{1}{E} \sum_{e=1}^{E} \sum_{e=1}^{E} F(z_{p}^{t,e}) - F(z^*) - \langle \lambda_p^{t+1}, z_{p}^{t,e+1} - z^* \rangle + \langle A^{\top} \tilde{\lambda}^t, w^{t+1} - w^* \rangle, \quad (49a)$$

$$RHS'(z^*) := \frac{1}{E} \sum_{e=1}^{E} \left\{ \frac{\eta^t}{2} \| f'(z_{p}^{t,e}) + \tilde{\xi}_{p}^{t,e} \|^{2} + \frac{1}{2\eta^t} \left( \| z^* - z_{p}^{t,e} \|^{2} - \| z^* - z_{p}^{t,e+1} \|^2 \right) + \langle \tilde{\xi}_{p}^{t,e}, z^* - z_{p}^{t,e} \rangle \right\}. \quad (49b)$$
C. Lower bound on LHS$^i(w^*, z^*)$.

By following the steps in Appendix D-D, one can derive inequalities similar to (38d), as follows:

\[
\text{LHS}^i(w^*, z^*) \geq \frac{1}{TE} \sum_{t=1}^{T} \sum_{e=1}^{E} \left[ F(z_t^e) - F(z^*) + (\gamma^T x^* - G x^*) - (\lambda, z_t^{t+1} - z^t) \right] \\
+ \frac{\rho^1}{2} \left( \|z^* - z_t^{t+1}\|^2 - \|z^* - z_t^t\|^2 \right) + \frac{1}{2\rho^1} \left( \|\lambda - \lambda_t^{t+1}\|^2 - \|\lambda - \lambda_t^t\|^2 \right).
\]

(50)

Note that the “J” term in (50) does not exist in (38d) because the definition of $\tilde{\xi}$ in (48) is different from that in (35).

D. Lower bound on LHS$^i(w^*, z^*) := \frac{1}{T} \sum_{t=1}^{T} \text{LHS}^i(w^*, z^*)$.

Summing (50) over $t \in [T]$ and dividing the resulting inequality by $T$, we have

\[
\text{LHS}(w^*, z^*) \geq \frac{1}{TE} \sum_{t=1}^{T} \sum_{e=1}^{E} \left[ F(z_t^e) - F(z^*) + (\gamma^T x^* - G x^*) - \frac{1}{T} (\lambda, z_t^{t+1} - z^t) \right] \\
\geq F(z(T)) - F(z^*) + \langle \gamma^T x^* - G x^* \rangle - \frac{1}{T} \left( \|\lambda\| \|z_t^{t+1} - z^t\| \right) \\
+ \frac{1}{T} \left( \sum_{t=1}^{T} \frac{\rho^1}{2} \left( \|z^* - z_t^{t+1}\|^2 - \|z^* - z_t^t\|^2 \right) + \sum_{t=1}^{T} \frac{1}{2\rho^1} \left( \|\lambda - \lambda_t^{t+1}\|^2 - \|\lambda - \lambda_t^t\|^2 \right) \right).
\]

(51)

The “K” term in (51) can be written as

\[
- \frac{1}{T} \langle \lambda, z_t^{t+1} - z^t \rangle \geq - \frac{1}{T} \|\lambda\| \|z_t^{t+1} - z^t\| \geq -\|\lambda\|U_2.
\]

Therefore, we have

\[
\text{LHS}(w^*, z^*) \geq F(z(T)) - F(z^*) + \langle \gamma^T x^* - G x^* \rangle - \frac{1}{T} \left( \|\lambda\| U_2 + \frac{U_2^2 \rho_{\max}}{2} + \frac{1}{2\rho^1} \|\lambda - \lambda^1\|^2 \right).
\]

Based on (41) and $\max_{\lambda:||\lambda||}\leq \gamma \|\lambda\| \leq \gamma U_2$, we have

\[
\text{LHS}(w^*, z^*) \geq F(z(T)) - F(z^*) + \langle \gamma^T x^* \rangle - \frac{U_2^2 \rho_{\max}}{2} + \langle \gamma + \|\lambda^1\|^2 \rangle / \rho^1 + 2\gamma U_2.
\]

(52)

E. Upper bound on RHS$^i(z^*) := \frac{1}{T} \sum_{t=1}^{T} \text{RHS}^i(z^*)$.

By following the steps in Appendix D-F, we obtain

\[
\text{RHS}(z^*) \leq \frac{U_2^2}{2TE\eta^T} + \frac{1}{TE} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{\eta^T}{2} \left( f'(z_t^e) + \hat{\xi}^e_t \right) + \|\xi^e_t - z_t^e\|^2 \right\}.
\]

(53)

F. Taking expectation.

By taking expectation on the inequality derived from (52) and (53), we have

\[
\mathbb{E} \left[ F(z(T)) - F(z^*) + \langle \gamma^T x^* \rangle - \frac{U_2^2 \rho_{\max}}{2} + \langle \gamma + \|\lambda^1\|^2 \rangle / \rho^1 + 2\gamma U_2 \right] \\
+ \frac{1}{2TE\eta^T} + \frac{1}{TE} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{\eta^T}{2} \left( \mathbb{E}[f'(z_t^e) + \hat{\xi}_t^e] + \|\xi_t^e - z_t^e\|^2 \right) \right\} \leq 0
\]

where $\bar{U}(\hat{\epsilon})$ is from (44). We note that

\[
\frac{U_2^2}{2TE\eta^T} = \frac{U_2^2/(2E)}{\sqrt{T}},
\]
\[
\sum_{k=1}^{T} \sum_{t=1}^{E} \eta_k^t = E \sum_{t=1}^{T} \frac{1}{2\sqrt{t}} \leq E \sum_{t=1}^{T} \frac{1}{\sqrt{t} + \sqrt{t-1}} = E \sum_{t=1}^{T} (\sqrt{t} - \sqrt{t-1}) = E\sqrt{T}.
\]

Therefore, we have
\[
\mathbb{E}\left[ F(z^{(T)}) - F(z^*) + \gamma\|A_w(T) - z^{(T)}\| \right] \leq \frac{U_2^2 \rho^{\max} + (\gamma + \|\lambda^1\|)^2 / \rho^1 + 2\gamma U_2 + 2PKU_2^2 / \bar{e} + PU_2^2 + U_2^2 / (2E)}{\sqrt{T}}.
\]

This completes the proof.

**APPENDIX F**

**PROOF OF THEOREM 4**

The proof in this section is similar to that in Appendix E except that

1) the \(\alpha\)-strong convexity of \(f_p\) is utilized to tighten the right-hand side of inequality (45) and
2) the definition of \(x^{(T)}, w^{(T)}, z^{(T)}, \lambda^{(T)}\) is modified to the following:

\[
x^{(T)} := \frac{2}{T(T+1)} \sum_{t=1}^{T} T \tilde{x}^t, \quad w^{(T)} := \frac{2}{T(T+1)} \sum_{t=1}^{T} Tw^{t+1},
\]

\[
z^{(T)} := \frac{2}{T(T+1)} \sum_{t=1}^{T} \left[ \frac{1}{E} \sum_{e=1}^{E} z_t^e \right], \quad \lambda^{(T)} := \frac{2}{T(T+1)} \sum_{t=1}^{T} T \lambda^t.
\]

**A. Inequality derivation for a fixed iteration \(t\) and \(e\).**

For a given \(p \in [P]\), it follows from the \(\alpha\)-strong convexity of the function \(f_p\) that

\[
f_p(z_p^t, e) - f_p(z_p) \leq \langle f'_p(z_p^t, e), z_p^t - z_p \rangle - \frac{\alpha}{2} \|z_p - z_p^t\|_p^2.
\]

By utilizing (55) and (45), we obtain

\[
f_p(z_p^t) - f_p(z_p) - \langle \lambda_p^{t+1} - z_p^{t+1}, z_p^t - z_p \rangle \leq \rho^t \langle z_p^t - z_p^{t+1}, z_p^{t+1} - z_p \rangle - \frac{\alpha}{2} \|z_p - z_p^t\|_p^2 + \frac{\eta^t}{2} \|f_p'(z_p^{t, e})\|_p^2 + \frac{1}{2\eta^t} \left( \|z_p - z_p^{t, e}\|_p^2 - \|z_p - z_p^{t, e+1}\|_p^2 \right) + \langle \tilde{f}_p^{t, e}, z_p - z_p^{t, e} \rangle, \quad \forall z_p \in \mathcal{W}.
\]

Note that compared with (45), the inequalities (56) have an additional term \(-\frac{\alpha}{2} \|z_p - z_p^t\|_p^2\).

**B. Inequality derivation for a fixed iteration \(t\).**

Following the steps to derive (47) in Appendix E-B, we derive the following from (56):

\[
\sum_{p=1}^{P} \left[ \frac{1}{E} \sum_{e=1}^{E} f_p(z_p^t, e) - f_p(z_p) - \langle \lambda_p^{t+1} - z_p^{t+1}, z_p^t - z_p \rangle \right] \leq \sum_{p=1}^{P} \left[ \frac{\eta^t}{2} \|f_p'(z_p^{t, e})\|_p^2 + \frac{1}{2\eta^t} \left( \|z_p - z_p^{t, e}\|_p^2 - \|z_p - z_p^{t, e+1}\|_p^2 \right) + \langle \tilde{f}_p^{t, e}, z_p - z_p^{t, e} \rangle \right].
\]

For ease of exposition, we introduce \(z, \lambda, \tilde{x}, x, x^*, A, G, A^T \tilde{x}, F(z), \tilde{f}_t^e\) as defined in (35), \(\tilde{x}^t, f'_t(z)\) as defined in (48), and the definition (54). Based on this notation as well as (28) and (57), we derive \(\text{LHS}^{(t)}(w^*, z^*) \leq \text{RHS}^{(t)}(z^*)\) at optimal \(w^*\) and \(\{z_p^t\}_{p=1}^{P} \in \mathcal{W}\), where

\[
\text{LHS}^{(t)}(w^*, z^*) := \frac{1}{E} \sum_{e=1}^{E} F(z^{t, e}) - F(z^*) - \langle \lambda^{t+1}, z^{t+1} - z^* \rangle + \langle A^T \tilde{x}^t, w^{t+1} - w^* \rangle,
\]

\[
\text{RHS}^{(t)}(z^*) := \frac{1}{E} \sum_{e=1}^{E} \left[ \frac{\eta^t}{2} \|f'_t(z^{t, e})\|_p^2 + \left( \frac{1}{2\eta^t} - \frac{\alpha}{2} \right) \|z^* - z^{t, e}\|_p^2 - \frac{1}{2\eta^t} \|z^* - z^{t, e+1}\|_p^2 + \langle \tilde{f}_p^{t, e}, z^* - z^{t, e} \rangle \right].
\]
C. Lower bound on \( \text{LHS}(w^*, z^*) := \frac{2}{T(T+1)} \sum_{t=1}^{T} t \ LHS^t(w^*, z^*) \).

By utilizing (50), namely, a lower bound on (58a), we have

\[
\text{LHS}(w^*, z^*) \geq \frac{2}{T(T+1)} \sum_{t=1}^{T} t \left( \frac{1}{E} \sum_{e=1}^{E} F(z^{t,e}) \right) - F(z^*) - \langle z^{(T)} - x^*, Gx^* \rangle - \frac{2}{T(T+1)} \left( \lambda, \sum_{t=1}^{T} t(z^{t+1} - z^t) \right) \\
\geq \text{F}(z^{(T)}) \quad \text{as F is convex}
\]

\[
+ \frac{2}{T(T+1)} \left\{ \sum_{t=1}^{T} t \rho^t \left( \|z^t - z^{t+1}\|^2 - \|z^t - z^T\|^2 \right) + \sum_{t=1}^{T} t \left( \frac{1}{2\rho^t} \left( \|\lambda - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 \right) \right\}. \quad (59)
\]

The “L” term in (59) can be written as

\[
\langle \lambda, \sum_{t=1}^{T} t(z^{t+1} - z^t) \rangle = \langle \lambda, \sum_{t=1}^{T} (z^{T+1} - z^t) \rangle \leq \sum_{t=1}^{T} \|\lambda\|z^{T+1} - z^t \leq \sum_{t=1}^{T} \|\lambda\|U_2 = TU_2\|\lambda\|.
\]

The “M” term in (59) can be written as

\[
\sum_{t=1}^{T} t \rho^t \left( \|z^t - z^{t+1}\|^2 - \|z^t - z^T\|^2 \right) = -\frac{\rho^1}{2} \|z^1 - z\|^2 + \sum_{t=2}^{T} \left( \frac{(t-1)\rho^{t-1} - t\rho^t}{2} \right) \|z^t - z^T\|^2 + \frac{T\rho^T}{2} \|z^T - z^T\|^2 \\
\geq -\frac{\rho^1 U^2_2}{2} + \sum_{t=2}^{T} \left( \frac{(t-1)\rho^{t-1} - t\rho^t}{2} \right) U^2_2 = -\frac{U^2_2 T\rho^T}{2} \geq -\frac{U^2_2 T\rho_{\text{max}}}{2}.
\]

The “N” term in (59) can be written as

\[
\sum_{t=1}^{T} t \rho^t \left( \|\lambda - \lambda^{t+1}\|^2 - \|\lambda - \lambda^t\|^2 \right) \geq -\frac{1}{2\rho^1} \|\lambda - \lambda^1\|^2 + \sum_{t=2}^{T} \left( \frac{t-1}{2\rho^{t-1}} - \frac{t}{2\rho^t} \right) \|\lambda - \lambda^t\|^2.
\]

Therefore, we have

\[
\text{LHS}(w^*, z^*) \geq F(z^{(T)}) - F(z^*) - \langle \lambda, Aw^{(T)} - z^{(T)} \rangle - \frac{2U_2\|\lambda\|}{T+1} - \frac{U_2^2 \rho_{\text{max}}}{T+1} \\
+ \frac{2}{T(T+1)} \left( -\frac{1}{2\rho^1} \|\lambda - \lambda^1\|^2 + \sum_{t=2}^{T} \left( \frac{t-1}{2\rho^{t-1}} - \frac{t}{2\rho^t} \right) \|\lambda - \lambda^t\|^2 \right). \quad (60)
\]

In addition to (41), by Assumption 2 (i), we have

\[
\max_{\|\lambda\|\|\lambda\|\leq \gamma} \|\lambda - \lambda^t\|^2 = \|\lambda^t\|^2 + \max_{\|\lambda\|\|\lambda\|\leq \gamma} \{ \|\lambda\|^2 - 2\langle \lambda, \lambda^t \rangle \} \leq 4\gamma^2.
\]

By utilizing this to derive a lower bound of the last term in (60), we have

\[
-\frac{1}{2\rho^1} \|\lambda - \lambda^1\|^2 + \sum_{t=2}^{T} \left( \frac{t-1}{2\rho^{t-1}} - \frac{t}{2\rho^t} \right) \|\lambda - \lambda^t\|^2 \geq -\frac{T}{2\rho^1} 4\gamma^2 \geq -\frac{T}{2\rho^1} 4\gamma^2.
\]

Therefore, we have

\[
\text{LHS}(w^*, z^*) \geq F(z^{(T)}) - F(z^*) + \gamma \|Aw^{(T)} - z^{(T)}\| - \frac{2U_2\gamma + U_2^2 \rho_{\text{max}} + 4\gamma^2}{T+1}. \quad (61)
\]
D. Upper bound on RHS$(z^*) := \frac{2}{T(T+1)} \sum_{t=1}^{T} t \text{ RHS}^t(z^*)$.

It follows from (58b) that

\[
\text{RHS}(z^*) = \frac{2}{T(T+1)} \sum_{t=1}^{T} t \left( \frac{1}{E} \sum_{e=1}^{E} \left\{ \eta_t \frac{1}{2} \| f'(z^t,e) + \xi_t,e \|^2 \right. \right.
\]
\[
\left. + \left( \frac{1}{2\eta_t} - \frac{\alpha}{2} \right) \| z^* - z^t,e \|^2 - \frac{1}{2\eta_t} \| z^* - z^{t,e+1} \|^2 + \langle \xi_t,e, z^* - z^t,e \rangle \right) \right\}.
\]

Note that

- \( \eta_t = 2/(\alpha(t + 2)) \),

- \( \sum_{t=1}^{T} \sum_{e=1}^{E} t \eta_t \| f'(z^t,e) + \xi_t,e \|^2 = \sum_{t=1}^{T} \frac{t}{\alpha(t + 2)} \| f'(z^t,e) + \xi_t,e \|^2 \),

- \( \sum_{t=1}^{T} \sum_{e=1}^{E} t \left\{ \frac{1}{2\eta_t} \| z^* - z^t,e \|^2 - \frac{1}{2\eta_t} \| z^* - z^{t,e+1} \|^2 \right\} = \frac{\alpha}{4} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ t^2 \| z^* - z^{t,e} \|^2 - (t^2 + 2t) \| z^* - z^{t,e+1} \|^2 \right\} \)

\[
\leq \frac{\alpha}{4} \sum_{t=1}^{T} \left\{ t^2 \| z^* - z^{t,e} \|^2 - t(t + 2) \| z^* - z^{t,e+1} \|^2 \right\}
\]

\[
= \frac{\alpha}{4} \left\{ \| z^* - z^{1,e} \|^2 + \sum_{t=2}^{T} t^2 \| z^* - z^{t,e} \|^2 - \sum_{t=2}^{T} t(t + 2) \| z^* - z^{t,e+1} \|^2 \right\} \]

\[
\leq \frac{\alpha}{4} \left\{ \| z^* - z^{1,e} \|^2 + \sum_{t=2}^{T} \| z^* - z^{t-1,e+1} \|^2 \right\} \leq \frac{\alpha}{4} T U_2^2.
\]

Therefore, we have

\[
\text{RHS}(z^*) \leq \frac{\alpha U_2^2 / E}{2(T+1)} + \frac{2}{ET(T+1)} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{t}{\alpha(t+2)} \sum_{p=1}^{P} \| f'_p(z^t,p) + \xi_t,e \|^2 + \| f'_p(z^t,p) \|^2 + t(t+2) \| z^* - z^{t,e} \|^2 \right\}.
\]

E. Taking expectation.

By taking expectation on the inequality derived from (61) and (62), we have

\[
\mathbb{E} \left[ F(z^{(T)}) - F(z^*) + \gamma \| Au^{(T)} - z^{(T)} \| \right] \leq 2U_2 \gamma + U_2^2 \rho_{\text{max}} + 4\gamma^2 / \rho^1 + \alpha U_2^2 / (2E)
\]
\[
+ \frac{2}{ET(T+1)} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{t}{\alpha(t+2)} \sum_{p=1}^{P} \mathbb{E} \left[ \| f'_p(z^t,p,e) + \xi_t,e \|^2 \right] + \| f'_p(z^t,p,e) \|^2 \right\} \leq U_2^2 \mathcal{U}(\bar{c})
\]
\[
\leq 2U_2 \gamma + U_2^2 \rho_{\text{max}} + 4\gamma^2 / \rho^1 + \alpha U_2^2 / (2E) + \frac{2}{ET(T+1)} \sum_{t=1}^{T} \sum_{e=1}^{E} \left\{ \frac{t + 2}{\alpha} \sum_{p=1}^{P} \mathbb{E} \left[ \| u^2 + \mathcal{U}(\bar{c}) \right] \right\} \leq U_2^2 \mathcal{U}(\bar{c}) / \alpha,
\]

where \( \mathcal{U}(\bar{c}) \) is from (44). This completes the proof.
APPENDIX G
MULTICLASS LOGISTIC REGRESSION MODEL

The multiclass logistic regression model considered in this paper is (1) with
\[ \ell(w; x_{pi}, y_{pi}) := - \sum_{k=1}^{K} y_{pi} \ln(h_k(w; x_{pi})), \forall p \in [P], \forall i \in [I_p], \]
\[ h_k(w; x_{pi}) := \frac{\exp(\sum_{j=1}^{J} x_{pij} w_{jk})}{\sum_{k'=1}^{K} \exp(\sum_{j=1}^{J} x_{pij} w_{jk'})}, \forall p \in [P], \forall i \in [I_p], \forall k \in [K], \]
\[ r(w) := \sum_{j=1}^{J} \sum_{k=1}^{K} w_{jk}^2, \]
\[ f_p(w) = -\frac{1}{T_c} \sum_{i=1}^{I_p} \sum_{k=1}^{K} \{ y_{pic} \ln(h_k(w; x_{pi})) \} + \beta \frac{2}{T_c} \sum_{j=1}^{J} \sum_{k=1}^{K} w_{jk}^2, \forall p \in [P], \]
\[ \nabla_w f_p(w) = \frac{1}{T_c} \sum_{i=1}^{I_p} x_{pij} (h_k(w; x_{pi}) - y_{pic}) + 2\beta w_{jk}, \forall p \in [P], \forall j \in [J], \forall k \in [K]. \] (63)

APPENDIX H
CHOICE OF THE PENALTY PARAMETER \( \rho^t \)

We test various \( \rho^t \) for our algorithms and set it as \( \hat{\rho}^t \) in (24) with (i) \( c_1 = 2, c_2 = 5, \) and \( T_c = 10000 \) for MNIST and (ii) \( c_1 = 0.005, c_2 = 0.05, \) and \( T_c = 2000 \) for FEMNIST.

Since these parameter settings may not lead \( \text{OutP} \) to its best performance, we test various \( \rho^t \) for \( \text{OutP} \) using a set of static parameters, \( \rho^t \in \{0.1, 1, 10\} \) for all \( t \in [T], \) where \( \rho^t = 0.1 \) is chosen in [2], and dynamic parameters \( \rho^t \in \{\hat{\rho}^t, \hat{\rho}^t/100\} \), where \( \hat{\rho}^t \) is from (24). In Figure 4 we report the testing errors of \( \text{OutP} \) using MNIST and FEMNIST under various \( \rho^t \) and \( \bar{\epsilon}. \) The results imply that the performance of \( \text{OutP} \) is not greatly affected by the choice of \( \rho^t \), but \( \bar{\epsilon}. \) Hence, for all algorithms, we use \( \hat{\rho}^t \) in (24).

Fig. 4: Testing errors of \( \text{OutP} \) using MNIST (top) and FEMNIST (bottom).