Driven collective quantum tunneling of ultracold atoms in engineered optical lattices

R. Khomeriki\textsuperscript{1,2}, S. Ruffo\textsuperscript{1} and S. Wimberger\textsuperscript{3}

\textsuperscript{1} Dipartimento di Energetica “S. Stecco” and CSDC, Università di Firenze and INFN - Via S. Marta 3, 50139 Firenze, Italy
\textsuperscript{2} Department of Exact and Natural Sciences, Tbilisi State University - 0128 Tbilisi, Georgia
\textsuperscript{3} CNR-INFM and Dipartimento di Fisica “E. Fermi”, Università degli Studi di Pisa - Largo Pontecorvo 3, 56127 Pisa, Italy

received 28 September 2006; accepted in final form 19 December 2006
published online 6 February 2007

PACS 03.75.Lm – Tunneling, Josephson effect, Bose-Einstein condensates in periodic potentials, solitons, vortices, and topological excitations
PACS 74.50.+r – Tunneling phenomena; point contacts, weak links, Josephson effects
PACS 75.45.+j – Macroscopic quantum phenomena in magnetic systems

Abstract – Collective quantum tunneling of a Bose-Einstein condensate between two parts of an optical lattice separated by an energy barrier is theoretically investigated. We show that by a pulsewise change of the barrier height, it is possible to switch between a tunneling regime and a self-trapped one. This property of the system is explained by effectively reducing the nonlinear dynamics of the system to that of a particle moving in a double square well potential. The analysis is performed for both attractive and repulsive interatomic forces, and emphasizes the experimental relevance of our findings.

Since the first experimental discovery [1] of Macroscopic Quantum Tunneling, this phenomenon has been usually associated with the tunneling between different states of the system with no reference to a spatial energy barrier [2–5]. In Josephson junctions (JJ) one observes a tunneling escape from a metastable state [1]. Afterwards, similar realizations have been discovered in completely different physical systems, such as liquid helium [6] and nanomagnets [7]. Besides that, there are also several examples where macroscopic objects, such as vortices [8,9] or fluxons [10] in JJs, or magnetic domain walls [11], tunnel through a spatial potential barrier. This latter particle tunneling effect is hereafter referred to as Collective Quantum Tunneling (CQT).

Recently [12,13], following earlier theoretical predictions [14,15], it has been found that a Bose-Einstein Condensate (BEC) trapped in a harmonic-well potential of mesoscopic length behaves like a single JJ: for nonzero initial imbalance of the number of atoms in different wells, Josephson oscillations are present in the system, i.e. the condensate tunnels back and forth through the barrier displaying a CQT regime. The only difference with respect to superconducting JJ’s is that, for large initial imbalance, the condensate is mainly trapped in one of the wells, producing what is called Macroscopic Self-Trapping (MST).

Our aim in this Letter is to suggest the experimental realization of a BEC in an optical lattice, which is engineered in such a way to mimic two weakly coupled chains of JJ’s (see fig. 1). Using such an experimental set-up, we demonstrate the feasibility of the efficient control of a switch between tunneling (CQT) and trapped (MST) states of the system. We show that our problem reduces to the Gross-Pitaevskii equation (GPE) [16] in a double square well, which displays very different properties from the previously considered double harmonic well potential [12,14,17–19]. Specifically, we show that for both attractive and repulsive nonlinearities the stationary solutions describing the CQT and MST regimes are characterized by very close energies in a wide range of values of the nonlinearity parameter. This property itself allows one to switch from the oscillatory tunneling regime to the trapped one and back via a simple pulselike adiabatic change of the energy barrier. Our results are broadly applicable and open the way to the experimental study of these phenomena in BEC dynamics.
We start from the following, one-dimensional Hamiltonian of a BEC in an optical lattice:

$$\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + \frac{2\hbar^2 a_s}{ma_\perp^2} |\psi|^2 \psi,$$  \hspace{1cm} (1)

where $m$ is the atomic mass, $a_s$ the scattering length ($a_s < 0$ corresponds to attractive atom-atom interactions and $a_s > 0$ to repulsive interactions) and $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the transversal oscillation length, which implicitly takes into account the real three-dimensionality of the system [20], $\omega_\perp$ being the transversal frequency of the trap. The optical lattice potential is

$$V(x) = v \cos^2(k_L x) \quad \text{for} \quad |k_L x| > \pi/2,$$

$$V(x) = (v + V_0) \cos^2(k_L x) \quad \text{for} \quad |k_L x| < \pi/2,$$  \hspace{1cm} (2)

where $k_L$ is the wave number of the laser beams that create the optical lattice and $V_0$ is the height of the additional spatial energy barrier placed in the middle of the optical lattice. Besides that, Dirichlet boundary conditions with $\psi(\pm L) = 0$ are chosen in order to describe the large confining barriers at both ends of the BEC. These boundary conditions could be realized experimentally by an additional optical lattice with larger amplitude and larger lattice constant, as shown in fig. 1.

Introducing a dimensionless length scale $\tilde{x} = 2k_L x$ and time $\tilde{t} = E_R t / \hbar$, where $E_R = 8E_B = 4\hbar^2 k_L^2 / m$ and $E_B$ is the recoil energy [21], we can rewrite eq. (1) as follows:

$$\tilde{t} \frac{\partial \Psi}{\partial \tilde{t}} = -\frac{1}{2} \frac{\partial^2 \Psi}{\partial \tilde{x}^2} + \tilde{V}(\tilde{x}) \Psi + \frac{g}{2} |\Psi|^2 \Psi,$$  \hspace{1cm} (3)

where the normalized wave function $\int |\Psi(\tilde{x}, \tilde{t})|^2 d\tilde{x} = 1$, is introduced [22]. The dimensionless potential $\tilde{V}$ still has the form (2) with the following dimensionless depths of the optical lattice

$$\tilde{v} = \frac{v}{E_B}, \quad \tilde{V}_0 = \frac{V_0}{E_B}, \quad g = \frac{Na_s}{k_L a_\perp^2},$$  \hspace{1cm} (4)

g being the dimensionless nonlinearity parameter and $N$ the number of atoms.

We have performed numerical simulations of eq. (3) with 12 wells (6 wells on each side of the barrier as presented in fig. 1) and the parameters $\tilde{v} = 0.25$ (in physical units this means that the depth of the optical lattice is $v = 2E_R$), $\tilde{V}_0 = 0.15$ and we fix the nonlinearity to the value $g = -0.025$, i.e., we choose attractive interactions. The dynamics is similar for repulsive interatomic forces (see the discussion below). The phenomenon we study in this letter does not depend significantly on the actual size of the system, if at least 3 lattice sites are present on each side of the barrier.

As seen from the top panel of fig. 2, the self-trapped state of the condensate persists until one applies the pulse-like time variation of the barrier displayed in the inset. After that action, the system makes a transition to the oscillating tunneling regime. The nature of these oscillations can be understood in terms of a two-mode approximation as was done in refs. [14,23], and in the limit of zero nonlinearity oscillations are Rabi-like. On the other hand, preparing the condensate in the oscillating tunneling regime (bottom graph in fig. 2) one can easily arrive at a self-trapping state by varying again the energy barrier in the middle as displayed in the inset. Let us mention that, as far as the energy of the barrier is changed adiabatically, the total energy of the condensate does not vary, i.e. the self-trapped and tunneling oscillatory regimes have the same energy. This is quite different from what happens in a double-harmonic-well potential [12,14,17–19]. The point is that, in the double-harmonic-well for $g < 0$, the asymmetric stationary solution is characterized by a smaller energy than the symmetric solution and this difference increases sharply with increasing nonlinearity. Hence, a significant energy injection is required in order to realize the transition between the two regimes; whilst in our case the transition is achieved by a simple pulsewise variation of the energy barrier. Below we argue that this happens because our case effectively reduces to the case of a double-square-well potential (see the inset of fig. 1 and the reduction procedure below) for which asymmetric and symmetric stationary solutions carry almost the same energies in a wide range of the nonlinearity parameter.

Now we proceed to reducing eq. (3) to a Discrete NonLinear Schrödinger equation (DNLS). We discretize it via a tight-binding approximation [24–26], representing the wave function $\Psi(\tilde{x}, \tilde{t})$ as

$$\Psi(\tilde{x}, \tilde{t}) = \sum_j \phi_j(\tilde{t}) \varphi_j(\tilde{x}),$$  \hspace{1cm} (5)

where $\varphi_j(\tilde{x})$ is a normalized isolated wave function in an optical lattice in the fully linear case $g = 0$ and could be
expressed in terms of Wannier functions (see, e.g., [27]).

For clarity, we use here its approximation for a harmonic trap centered at the points $r_j = j\pi((j + 1/2)/|j|)$ ($|j|$ varies from 1 to $n$). In the context of the evolution equation (3) $\varphi_j(\tilde{x})$ has the form

$$\varphi_j(\tilde{x}) = \left(\frac{\sqrt{\beta}}{\pi \sqrt{2}}\right)^{1/4} e^{-\sqrt{\beta}(\tilde{x} - r_j)^2/\pi},$$  \hspace{1cm} (6)

for $|j| \neq 1$, and one should substitute $\tilde{v}$ by $\tilde{v} + \tilde{V}_0$ in the above expression in order to get an approximate formula for the wave function for $|j| = 1$.

Assuming further that the overlap of the wave functions in neighboring sites is small, we get from (3) the following

DNLS equation for the sites $|j| \neq 1$:

$$i\hbar \frac{\partial \phi_j}{\partial t} = -Q(\phi_{j+1} + \phi_{j-1}) + U|\phi_j|^2\phi_j,$$  \hspace{1cm} (7)

while for $|j| = 1$ we have

$$i\hbar \frac{\partial \phi_{\pm 1}}{\partial t} = -Q\phi_{\pm 2} - Q_1\phi_{\mp 1} + U|\phi_{\pm 1}|^2\phi_{\pm 1},$$  \hspace{1cm} (8)

where we assume pinned boundary conditions. The constants $Q, Q_1, U$ and $U_1$ are easily computed from the following expressions ($(|j| \neq 0)$

$$Q = -\int \left[ \frac{\partial^2 \varphi_{j+1}}{\partial \tilde{x}^2} + \tilde{v} \varphi_{j+1} \cos^2 \frac{\tilde{x}}{2} \right] \tilde{x} d\tilde{x},$$

$$Q_1 = -\int \left[ \frac{\partial^2 \varphi_{j-1}}{\partial \tilde{x}^2} + (\tilde{v} + \tilde{V}_0) \varphi_{j-1} \cos^2 \frac{\tilde{x}}{2} \right] \tilde{x} d\tilde{x},$$

$$U = g \int \varphi_j^4 d\tilde{x} = U_1 = g \int \varphi_{\pm 1}^4 d\tilde{x}.$$  \hspace{1cm} (9)

In order to characterize the solutions of eqs. (7) and (8), we follow the same procedure used in ref. [23], which goes through a continuum approximation and leads to the transformation $j \rightarrow \tilde{x}$ again. Assuming that $\phi = \phi_1$ we finally arrive at

$$i\hbar \frac{\partial \phi}{Q \partial \tilde{t}} = -\frac{\partial^2 \phi}{\partial \tilde{x}^2} + W(\tilde{x})\phi + R|\phi|^2\phi,$$  \hspace{1cm} (10)

where $W(\tilde{x})$ is a double-square-well potential with a barrier height $w = 2(Q - Q_1)/Q$ and width $l = 1$, $\phi(\tilde{x}, \tilde{t})$ obeys pinned boundary conditions $\phi(\tilde{x} = \pm l/2) = 0$ (2L is the width of the double square well) and the nonlinearity parameter is given by $R = U/Q$.

Summarizing, we have reduced the initial problem, GPE with optical lattice and barrier potentials to a DNLS equation and then this latter again to a GPE with double-square-well potential. The reason for doing this, is to get rid of the optical lattice potential and to reduce our problem to the GPE with a double-square-well potential, for which one can easily find exact stationary solutions. They are sought as $\phi(\tilde{x}, \tilde{t}) = \Phi(\tilde{x}) \exp(-i\tilde{B}t)$ with a real-valued function $\Phi(\tilde{x})$ found in terms of Jacobi elliptic functions [28], in the case of attractive atom-atom interactions $R < 0$

$$-L < \tilde{x} < -l: \Phi = A \exp[\gamma_A(\tilde{x} + 2k_L L)] - \mathbb{K}(k_A), k_A],$$

$$l < \tilde{x} < L: \Phi = B \exp[\gamma_B(\tilde{x} - 2k_L L)] - \mathbb{K}(k_B), k_B],$$

$$-L < \tilde{x} < L: \Phi = C \exp[\gamma_C(\tilde{x} - 2k_L L)] - \mathbb{K}(k_C), k_C],$$  \hspace{1cm} (11)

with the parameters given in terms of the amplitudes by

$$\gamma_A^2 = \beta + \frac{A^2}{|R|}, \hspace{1cm} k_A^2 = \frac{A^2}{2(A^2 + |R|)},$$

$$\gamma_B^2 = \beta + \frac{B^2}{|R|}, \hspace{1cm} k_B^2 = \frac{B^2}{2(B^2 + |R|)},$$

$$\gamma_C^2 = w - \beta - \frac{C^2}{2|R|}, \hspace{1cm} k_C^2 = \frac{w - \beta - C^2}{w - \beta - C^2/2|R|}.$$
while, in the case of repulsive interactions, \( R > 0 \), one obtains the stationary solution written in the form

\[
\begin{align*}
-L < \tilde{x} < -l & : \Phi = A \, \text{sn}[\gamma_A(\tilde{x} + 2k_L L), k_A], \\
l < \tilde{x} < L & : \Phi = B \, \text{sn}[\gamma_B(\tilde{x} - 2k_L L), k_B], \\
-l < \tilde{x} < l & : \Phi = C/cn[\gamma_C(\tilde{x} - \tilde{x}_0), k_C],
\end{align*}
\]

where

\[
\begin{align*}
\gamma_A^2 &= \beta - \frac{A^2}{2|R|}, & k_A^2 &= \frac{A^2}{2|R|\beta - A^2}, \\
\gamma_B^2 &= \beta - \frac{B^2}{2|R|}, & k_B^2 &= \frac{B^2}{2|R|\beta - B^2}, \\
\gamma_C^2 &= 2 \left( w - \beta + \frac{C^2}{|R|} \right), & k_C^2 &= \frac{w - \beta + C^2/2|R|}{w - \beta + C^2/|R|}.
\end{align*}
\]

Here \( \mathcal{K} \) denotes the complete elliptic integral of the first kind, and, by construction, the above expressions verify the vanishing boundary values condition in \( \tilde{x} = \pm 2k_L L \).

The solutions are then given in terms of five parameters \((A, B, C, \beta, \tilde{x}_0)\), which are determined by the four continuity conditions in \( \tilde{x} = \pm l \) and the wave function normalization condition \( \int d\tilde{x} \Phi^2(\tilde{x}) = 1 \). In both cases, repulsive and attractive nonlinearities, one has a symmetric and an antisymmetric solution, and an additional asymmetric solution that appears above a given nonlinearity threshold value \(|g_t| \approx 0.018\). In fig. 3, we plot the profiles of the lowest energy symmetric (attractive case, upper panel) and antisymmetric (repulsive case, lower panel) solution, the asymmetric solutions for both the repulsive and the attractive case, and, in the insets, the relative energy differences \( \Delta E = 2(E_\alpha - E_\parallel)/E_\alpha + E_\parallel \) between the asymmetric \((E_\alpha)\) and the symmetric \((E_\parallel)\) case for attractive nonlinearities, while for repulsive nonlinearities \( E_\alpha \) is replaced by the energy of the asymmetric solution. Below \(|g_t| \) the energy of the asymmetric solution joins those of the symmetric and antisymmetric ones, making \( \Delta E = 0 \). As seen from the insets of both panels, these energies are very close in the nonlinearity range \( 0.018 < \left| g_t \right| < 0.03 \) (note that the numerical simulations presented in fig. 2 are made for \( g_t = -0.025 \)) and hence it is easy to switch from the tunneling regime to the self-trapped state and back again.

As mentioned above, for small nonlinearities only the symmetric and antisymmetric solution exist. For large nonlinearities, according to an intuitive guess, the symmetric solution has lower energy than the asymmetric one for repulsive nonlinearities (the system prefers to be equally distributed on different sides of the barrier), whilst the asymmetric solution is energetically preferable (the system prefers to occupy mostly one of the sides of the barrier) in the case of attractive atomic interactions. For large nonlinearities the behavior of the double square well coincides with that of the double harmonic well, for which we direct the reader to ref. [17], where the energetic comparison and stability analysis of the above-mentioned solutions are given in full detail. However, we remark once again that the behavior of the double-square-well potential differs from that of the double harmonic well [12,14,17–19] in the range of intermediate nonlinearities, where the energies of the symmetric and asymmetric solutions turn out to be very close to each other and the energetic ordering described for large nonlinearities is different.

In order to get an idea about a possible experimental realization of these effects, we choose \(^7\text{Li}\), which is characterized by an attractive atom-atom interaction [29]. First of all we note that the confining harmonic potential along the optical lattice does not change the observed switching effect if the characteristic longitudinal oscillation length \( a_\parallel = \sqrt{\hbar/m\omega_\parallel} \) of this potential (\( \omega_\parallel \) being

![Graph](image-url)
the longitudinal confining frequency) is much larger than system size. In particular, for $^7$Li this condition yields $\omega_l < 2\pi \times 30 \, \text{Hz}$. Then, with the realistic experimental parameters $\omega_l = 2\pi \times 30 \, \text{Hz}$, $a_s = -1.4 \, \text{nm}$, $k_L = 7.4 \cdot \mu \text{m}^{-1}$, from the formula for the nonlinearity $g = N a_s / a_s^2 k_L$, one gets that the total atom number needed to access values of $|g|$ around 0.025 is $N \approx 10000$. While in case of $^{23}$Na [30] with repulsive forces ($a_s = 4.9 \, \text{nm}$), the atom number should be $N \approx 1000$. Increasing the number of wells (system size) $n$ times, one should decrease the number of atoms $n^2$ times in order to observe the predicted effect. The optimal optical depth should be around $v = 0.25 E_B$ and barrier height $V_0 = 0.15 E_B$ (see eq. (2) for the definition of potential parameters). Increasing the potential barrier width the tunneling regime is suppressed and one should simultaneously decrease the potential barrier height in order to still observe the effect, keeping all the other parameters unchanged. The only restriction on the barrier width is that it should be much less than system size. Small fluctuations of the barrier parameters and position do not affect the predicted phenomenon. We note that in our numerical simulations, presented in fig. 1, the time is scaled in units of $\hbar^2/4 E_B$, where, let us remind, $E_B = 4\hbar^2 k_0^2 / \text{m}$.

To summarize, we predict the presence of a rich switching scenario from an oscillatory tunneling regime to a self-trapped one as a novel collective quantum tunneling (CQT) effect to be realized with ultracold atoms. We have shown that the problem effectively reduces to particle motion in a double-square-well potential, at variance with earlier studies dealing with double harmonic wells. This difference guarantees the possibility of a switch from an oscillatory tunneling to a self-trapped state via a pulse-like change of the central potential barrier. We have derived typical ranges of physical parameters, for both attractive and repulsive interatomic forces, in order to suggest a ready-to-implement experimental verification.

***

It is our pleasure to thank Oliver Morsch and Ennio Arimondo for useful discussions. SR acknowledges financial support by the PRIN05-MIUR grant on Dynamics and thermodynamics of systems with long-range interactions, R.Kh. by the Marie-Curie international incoming fellowship award (MIF2-CT-2006-021328) and USA CRDF Award # GEP2-2848-TB-06, and SW by the Alexander von Humboldt foundation (Feodor-Lynen Program).

REFERENCES

[1] Voss R. F. and Webb R. A., *Phys. Rev. Lett.*, **47** (1981) 265.
[2] Ueda M. and Leggett A. J., *Phys. Rev. Lett.*, **80** (1998) 1576.
[3] Yu Y., Han S., Chu Xi., Chu Shih-I. and Wang Zh., *Science*, **296** (2002) 889.
[4] Jin X. Y. et al., *Phys. Rev. Lett.*, **96** (2006) 177003.
[5] Barone A., Kureki G. and Kopman A. G., *Phys. Rev. Lett.*, **92** (2004) 200403.
[6] Hendry P. C., Lawson N. S., McClintock P. V. E. and Williams C. D. H., *Phys. Rev. Lett.*, **60** (1988) 604.
[7] Thomas L., Lioni F., Ballou R., Gatteschi D., Sessoli R. and Barbara B., *Nature*, **383** (1996) 145.
[8] Wallraff A. et al., *Nature*, **425** (2003) 155.
[9] Clarke J., *Nature*, **425** (2003) 133.
[10] Kim Ju H. and Moon K., *Phys. Rev. B*, **71** (2005) 104524.
[11] Brooke J., Rosenbaum T. F. and Aeppli G., *Nature*, **413** (2001) 610.
[12] Albiez M., Gati R., Folling J., Hunsmann S., Cristiani M. and Oberthaler M. K., *Phys. Rev. Lett.*, **95** (2005) 010402.
[13] Cataliotti F. S., Burger S., Fort C., Maddaloni P., Minardi F., Trombettoni A., Smerzi A. and Inguscio M., *Science*, **293** (2001) 843.
[14] Smerzi A., Fantoni S., Giovanazzi S. and Shenoy S. R., *Phys. Rev. Lett.*, **79** (1997) 4950; Raghavan S., Smerzi A., Fantoni S. and Shenoy S. R., *Phys. Rev. A*, **59** (1999) 620.
[15] Franzosi R., Penna V. and Zucchina R., *Int. J. Mod. Phys. B*, **14** (2000) 943.
[16] Pitaevskii L. P., *Sov. Phys. JETP*, **13** (1961) 451; Gross E. P., *Nuovo Cimento*, **20** (1961) 454; *J. Math. Phys.*, **4** (1963) 195.
[17] Montina A. and Arecchi F. T., *Phys. Rev. A*, **66** (2002) 013605.
[18] Kapitula T. and Kevrekidis P. G., *Nonlinearity*, **18** (2005) 2491; Kevrekidis P. G. et al., *Phys. Lett. A*, **340** (2005) 275.
[19] Anderlini M. et al., *J. Phys. B*, **39** (2006) S199.
[20] Bergeman T., Moore M. G. and Olshansii M., *Phys. Rev. Lett.*, **91** (2003) 163201.
[21] Bloch I., *J. Phys. B*, **38** (2005) S629; Morsch O. and Oberthaler M., *Rev. Mod. Phys.*, **78** (2006) 179.
[22] Carr L., Holland M. J. and Malomed B. A., *J. Phys. B*, **38** (2005) 3217; Wimbüser G., Schlagheck P. and Mannaelli R., *J. Phys. B*, **39** (2006) 729; Schlagheck P. and Paul T., *Phys. Rev. A*, **73** (2006) 023619.
[23] Komeriki R., Leon J. and Ruffo S., *Phys. Rev. Lett.*, **97** (2006) 143902.
[24] Smerzi A. and Trombettoni A., *Phys. Rev. A*, **68** (2003) 023613.
[25] Sukhorukov A. A., Neshev D., Krolikowski W. and Kivshar Y. S., *Phys. Rev. Lett.*, **92** (2004) 093901.
[26] Ablowitz M. J. and Musslimani Z. H., *Phys. Rev. Lett.*, **87** (2001) 254102; *Phys. Rev. E*, **65** (2002) 056618.
[27] Slater J. C., *Phys. Rev.*, **87** (1952) 807.
[28] Byrd P. F. and Friedman M. D., *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer, Berlin) 1954.
[29] Bradley C. C., Sackett C. A. and Hulet R. G., *Phys. Rev. Lett.*, **78** (1997) 985.
[30] Davis K. B. et al., *Phys. Rev. Lett.*, **75** (1995) 3969.