A Gröbner-bases approach to syndrome-based fast Chase decoding of Reed–Solomon codes

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Abstract

We present a simple syndrome-based fast Chase decoding algorithm for Reed–Solomon (RS) codes. Such an algorithm was initially presented by Wu (IEEE Trans. IT, Jan. 2012), building on properties of the Berlekamp–Massey (BM) algorithm. Wu devised a fast polynomial-update algorithm to construct the error-locator polynomial (ELP) as the solution of a certain linear-feedback shift register (LFSR) synthesis problem. This results in a conceptually complicated algorithm, divided into 8 subtly different cases. Moreover, Wu’s polynomial-update algorithm is not immediately suitable for working with vectors of evaluations. Therefore, complicated modifications were required in order to achieve a true “one-pass” Chase decoding algorithm, that is, a Chase decoding algorithm requiring $O(n)$ operations per modified coordinate, where $n$ is the RS code length.

The main result of the current paper is a conceptually simple syndrome-based fast Chase decoding of RS codes. Instead of developing a theory from scratch, we use the well-established theory of Gröbner bases for modules over $\mathbb{F}_q[X]$ (where $\mathbb{F}_q$ is the finite field of $q$ elements, for $q$ a prime power). The basic observation is that instead of Wu’s LFSR synthesis problem, it is much simpler to consider “the right” minimization problem over a module. The solution to this minimization problem is a simple polynomial-update algorithm that avoids syndrome updates and works seamlessly with vectors of evaluations. As a result, we obtain a conceptually simple algorithm for one-pass Chase decoding of RS codes. Our algorithm is general enough to work with any algorithm that finds a Gröbner basis for the solution module of the key equation as the initial algorithm (including the Euclidean algorithm), and it is not tied only to the BM algorithm.

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1 Introduction

1.1 Motivation and known results

The subject of decoding Reed–Solomon (RS) codes beyond half the minimum distance has been extensively studied over the years. The breakthrough work of Guruswami and Sudan [11] (following the original work of Sudan [25]) presented interpolation-based hard-decision (HD) list decoding of RS codes up to the so-called Johnson radius in polynomial time. Wu [26] presented an even more efficient HD list decoding algorithm for decoding RS codes up to the Johnson radius. Kötter and Vardy [15] extended the Guruswami–Sudan algorithm to take channel reliability information into account, thus presenting a polynomial-time soft-decision (SD) decoding algorithm for RS codes.

Before [15], it seems reasonable to say that the main SD decoding algorithms for block codes with an efficient HD decoder in general, and for RS codes in particular, were the generalized minimum distance (GMD) decoding of Forney [10], and the Chase decoding algorithms [6]. GMD decoding consists of repeated applications of errors-and-erasures decoding, while successively erasing an even number of the least reliable coordinates.

In Chase decoding, there is some pre-determined list of test error patterns on the \( \eta \) least reliable coordinates for some small \( \eta \) (typically, \( \eta \leq \lfloor d/2 \rfloor \), where \( d \) is the minimum Hamming distance of the code). For example, this list may consist of all possible non-zero vectors, all vectors of a low enough weight, a pre-defined number of random vectors, etc.. The decoder successively runs on error patterns from the list. Each such error pattern is subtracted from the received word, and the result is fed to an HD decoder. If the HD decoder succeeds, then its output is saved into the output list of the decoder.

Informally, the list of test error patterns in Chase decoding should be a good covering code for all likely error patterns on the least reliable coordinates (see [19]). At least heuristically, this suggests that in order to achieve a substantial gain over HD decoding, the number of test error patterns should grow exponentially with \( d \).

Despite this exponential nature of Chase decoding, for high-rate codes of moderate length, it is known to have a better complexity/performance tradeoff than other algebraic SD decoding algorithms, including the Kötter–Vardy algorithm (see, e.g., [27]). For this reason, Chase decoding of RS codes is still of great interest. The idea behind fast Chase decoding algorithms is to share computations between HD decodings of different test error patterns.\(^3\)

\(^3\)This idea is at the heart of all fast Chase decoding algorithms, including [5], [27], and...
For example, if a new error pattern differs from the previous one in one additional non-zero coordinate, it seems plausible that there is no need to run a full HD decoding algorithm for the new error pattern, and that intermediate results from the previous HD decoding can be used in order to reduce the complexity of the new HD decoding.

It is well-known that HD decoding of RS codes has complexity \( O(dn) \) (where \( n \) is the length of the RS code), and that this complexity is governed by the exhaustive root search, rather than by the algorithm for finding the error-locator polynomial (ELP) (such as the Berlekamp–Massey (BM) algorithm), which has a complexity of \( O(d^2) \).

In [27], Wu defines a one-pass Chase decoding algorithm as a Chase algorithm that has the following properties: (1) For any test error pattern \( z \) of (Hamming) weight \( w \), there is some sequence \( z_1, \ldots, z_{w-1} \) in the list of test error patterns, such that for all \( i \), the weight of \( z_i \) equals \( i \), and such that \( \text{supp}(z_1) \subset \cdots \subset \text{supp}(z_{w-1}) \subset \text{supp}(z) \) \(^3\) (2) The algorithm produces decoding results for all the sequence \( z_1, \ldots, z_{w-1}, z \) in a complexity of \( O(wn) \) finite field operations. In particular, if \( w = O(d) \), then the complexity for decoding the subset \( \{z_1, \ldots, z_{w-1}, z\} \) is \( O(dn) \), just like HD decoding. Put differently, the complexity is \( O(n) \) per each additional modified coordinate. Note that in a naive application of Chase decoding, the complexity of decoding the above sequence is \( O(dn) \) per each additional modified coordinate.

Before [27], there have been several one-pass and “almost” one-pass Chase decoding algorithms for BCH and RS codes, where by “almost” we mean that some of these algorithms satisfied the above complexity requirement only for producing the ELP, but not for the essential following exhaustive root search. These algorithms include the low-complexity interpolation-based algorithm of Bellorado and Kavčič [5] for RS codes (based on the Guruswami–Sudan algorithm), and the algorithm of Kamiya [13] for binary BCH codes, based on the Welch–Berlekamp algorithm. Also, in the context of [5], Zhu et al. [32] introduced an efficient method for backward interpolation, which enables to cancel the interpolation in one point. This allows [32] to order the \( 2^w \) test vectors according to adjacent vertices of the binary hypercube (Gray code), thus avoiding the need to save \( 2^{w-1} \) intermediate interpolation results on the decoding tree of [5]. For a thorough literature review on fast Chase decoding algorithms before [27], we refer to [27].

Focusing on RS codes and considering the algorithm of [5], we note that this algorithm works in the “time domain,” i.e., on the received vector itself, rather than on the syndrome. As noted in [5] p. 946, in the context of fast

\(^2\)For \( b := (b_1, \ldots, b_n) \), \( \text{supp}(b) := \{i | b_i \neq 0\} \) is the support of \( b \).
Chase decoding, it is somewhat easier to work directly on the received vector rather than on the syndrome, because the syndromes of similar test error patterns are very far from similar.

For decoding high-rate codes, it is typically beneficial to replace the long received vector by the short syndrome once and for all before the decoding begins. In his important paper [27], Wu introduced a true one-pass Chase decoding algorithm based on the BM algorithm. Thus, Wu introduced a solution both for the problem of handling the exhaustive root searches while maintaining a complexity of $O(n)$ per modified coordinate, and for the need for a syndrome-based algorithm.

After Wu’s work, [31] proposed a backward step for Wu’s algorithm for binary BCH codes. Additional time-domain Chase decoding algorithms for binary BCH codes were developed, e.g., in [30]. Also, for RS codes, time-domain Chase decoding algorithms based on basis reduction for univariate polynomial modules were presented in [29] and the references therein. Inspired by [21] and [28], the fast Chase algorithm of [29] decreases the average complexity and latency over that of [3], while maintaining the worst-case complexity. It should be noted that for the setup of [3], the worst-case complexity of [3] and [29] is $O(2^n \cdot (n - k)^2)$, which is $O(2^n \cdot n^2)$ if the asymptotic rate is $<1$, while that of a true one-pass Chase decoding algorithm is $O(2^n \cdot n)$, as it requires $O(n)$ operations per edge of the decoding tree, for a maximum of $2^n - 1$ visited edges.

1.2 Our results

We use the well-established tool of Gröbner bases for modules over $\mathbb{F}_q[X]$ to derive an algorithm for syndrome-based fast Chase decoding of RS codes. The main observation is that instead of Wu’s LFSR synthesis problem, it is much simpler to consider “the right” minimization problem over a module. This minimization problem can be solved by adopting Kötter’s Gröbner basis algorithm, in the general form appearing in [16, Sect. VII.C].

- We present a clean and simple polynomial-update algorithm for fast Chase decoding, namely, Algorithm A of Section 4.1. This algorithm is considerably simpler than Algorithm 1 of [27], which is divided into 8 intricately different cases. Besides of the obvious benefit of having a clear and short algorithm and the theoretical interest of finding further connections between decoding algorithms and Gröbner bases, there is also a practical benefit in a simply-presented algorithm, being easier to implement and debug.
As opposed to Algorithm 1 of [27], our polynomial-update algorithm (Algorithm A) is automatically suited for working with vectors of evaluations, and it is easily converted into Algorithm B, which has the required \(O(n)\) complexity per modified coordinate. Again, Algorithm B is considerably cleaner and simpler than Algorithm 2 of [27], which, besides of being long and including 8 different cases, requires the introduction of auxiliary polynomials without a clear meaning.

As opposed to the algorithms of [27], Algorithms A and B of the current paper are not tied to the BM algorithm as the initial HD decoding algorithm, and can practically work with any of the existing syndrome-based HD decoding algorithms. In some detail, Algorithms A and B can be initiated with any algorithm that finds a Gröbner basis for the solution module of the key equation (for an appropriate monomial ordering). As shown by Fitzpatrick [9], practically any of the existing syndrome-based HD decoding algorithms can be put in this form, including the Euclidean algorithm.

On the practical side, we present Algorithm C, which is a variant of Algorithm A that runs on low-degree polynomials and has a lower complexity than Algorithm 1 of [27].

1.3 Organization

Section 2 includes the notation used throughout the paper, some basic definitions, and a review of required known results on algebraic decoding of (generalized) RS codes. Wu’s idea of fast Chase decoding on a tree is also recalled in this section.

The new minimization problem over an \(\mathbb{F}_q[X]\)-module and its relation to fast Chase decoding are presented in Section 3 which is the heart of the paper. The minimization problem is translated into an application of Kötter’s Gröbner basis algorithm in Section 4. The polynomial update algorithm is presented in Subsection 4.1, and the true one-pass Chase decoding algorithm, working with vectors of evaluations, is presented in Subsection 4.2. Section 4 is concluded by Subsection 4.3 which presents an overall high-level description of the entire decoding process. Finally, Section 5 includes some conclusions and open questions.

The paper includes two appendices, containing some interesting supplemental results. In Appendix A which may be considered as the counterpart of [27, Lemma 5 (ii)], we consider a certain interesting case that is not required for the algorithms of Section 4, and show that even in this case,
the ELP can be extracted from the output of the polynomial-update algorithm. Appendix \[3\] includes some practical simplifications of Algorithm A: a method for avoiding the need to work with two pairs of polynomials, so that it is possible to work with just two scalar polynomials, a heuristic stopping condition for (almost) avoiding unnecessary exhaustive root searches, and a method that uses a transformation that significantly reduces the degrees of the updated polynomials, and results in the low-complexity Algorithm C.

2 Preliminaries

2.1 Generalized Reed–Solomon codes

Let \( q \) be a prime power, and let \( \mathbb{F}_q \) be the finite field of \( q \) elements. We will consider a primitive generalized Reed–Solomon (GRS) code, \( C \), of length \( n := q - 1 \) and minimum Hamming distance \( d \in \mathbb{N}^* \), \( d \geq 2 \). In detail, let \( \tilde{a} = (\tilde{a}_0, \ldots, \tilde{a}_{n-1}) \in (\mathbb{F}_q)^n \) be a vector of non-zero elements (where \( \mathbb{F}_q^* := \mathbb{F}_q \setminus \{0\} \)). For a vector \( f = (f_0, f_1, \ldots, f_{n-1}) \in \mathbb{F}_q^n \), let \( f(X) := f_0 + f_1X + \cdots + f_{n-1}X^{n-1} \in \mathbb{F}_q[X] \). Now \( C \subseteq \mathbb{F}_q^n \) is defined as the set of all vectors \( f \in \mathbb{F}_q^n \) for which \( \tilde{a}(X) \odot f(X) \) has roots \( 1, \lambda, \ldots, \lambda^{d-2} \) for some fixed primitive \( \lambda \in \mathbb{F}_q \), where \( (\cdot \odot \cdot) \) stands for coefficient-wise multiplication of polynomials.\(^4\) We note that when \( (\tilde{a}_0, \ldots, \tilde{a}_{n-1}) = (1, \ldots, 1) \), \( C \) is a Reed–Solomon code.

To recall the key equation [23, Sec. 6.3], suppose that a codeword \( x \in C \) is transmitted, and the received word is \( y := x + e \) for some error vector \( e \in \mathbb{F}_q^n \). For \( j \in \{0, \ldots, d-2\} \), let \( S_j = S_j^{(y)} := (\tilde{a} \odot y)(\lambda^j) \). The syndrome polynomial associated with \( y \) is \( S^{(y)}(X) := S_0 + S_1X + \cdots + S_{d-2}X^{d-2} \). By the definition of the GRS code, the same syndrome polynomial is associated with \( e \).

If \( v \in \mathbb{F}_q^n \) is such that \( v(X) = X^i \) for some \( i \in \{0, \ldots, n - 1\} \), then \( S_j^{(v)} = (\tilde{a} \odot v)(\lambda^j) = \tilde{a}_i(\lambda^j)^i \), so that

\[
S^{(v)}(X) = \tilde{a}_i(1 + \lambda^iX + \cdots + (\lambda^i)^{d-2}X^{d-2}) \equiv \frac{\tilde{a}_i}{1 - \lambda^iX} \mod (X^{d-1}). \tag{1}
\]

So, if the error locators are some distinct elements \( \alpha_1, \ldots, \alpha_e \in \mathbb{F}_q^* \) (where \( e \in \{1, \ldots, n\} \) is the number of errors) and the corresponding error values

\(^3\)Since the most general GRS code (e.g., [23, Sec. 5.1]) may be obtained by shortening a primitive GRS code, there is no loss of generality in considering only primitive GRS codes.

\(^4\)For \( f(X) = \sum_{i=0}^{r} f_iX^i \) and \( g(X) = \sum_{i=0}^{s} g_iX^i \), let \( m := \min\{r, s\} \), and define \( f(X) \odot g(X) := \sum_{i=0}^{m} f_ig_iX^i \).
are $\beta_1, \ldots, \beta_\varepsilon \in F_q^*$, then

$$S^{(y)}(X) = S^{(e)}(X) \equiv \sum_{i=1}^{\varepsilon} \frac{\beta_i a_i}{1 - \alpha_i X} \mod (X^{d-1}),$$

(2)

where $a_i := \tilde{a}_{i'}$ for the $i' \in \{0, \ldots, n - 1\}$ with $\alpha_i = \lambda^{i'}$.

Defining the error-locator polynomial (ELP), $\sigma(X) \in F_q[X]$, by

$$\sigma(X) := \prod_{i=1}^{\varepsilon} (1 - \alpha_i X),$$

and the error-evaluator polynomial (EEP), $\omega(X) \in F_q[X]$, by

$$\omega(X) := \sum_{i=1}^{\varepsilon} \beta_i a_i \prod_{j \neq i} (1 - \alpha_j X),$$

it follows from (2) that

$$\omega \equiv S^{(y)} \sigma \mod (X^{d-1}).$$

(3)

Equation (3) is the so-called key equation.

Another useful relation is Forney’s formula (see, e.g., [23, Sec. 6.5]), which states that for all $i \in \{1, \ldots, \varepsilon\}$,

$$\beta_i a_i \sigma'(\alpha_i^{-1}) = -\alpha_i w(\alpha_i^{-1}),$$

(4)

where for a polynomial $f(X)$, $f'(X)$ stands for its formal derivative.

Let

$$M_0 = M_0(S^{(y)}) := \{(u, v) \in F_q[X]^2 \mid u \equiv S^{(y)} v \mod (X^{d-1})\}$$

be the solution module of the key equation.\(^5\) Next, we would like to recall that if the number of errors in $y$ is up to $t := \lfloor (d - 1)/2 \rfloor$, then $(\omega, \sigma)$ is a minimal element in $M_0$ for an appropriate monomial ordering on $F_q[X]^2$

For background on monomial orderings and Gröbner bases for modules, see, e.g., [8, Sec. 5.2] for the general case, and [9] for the special case of submodules of $K[X]^2$ (for $K$ a field), which is mostly sufficient for the current paper. Recall that for $\ell \in N$, a monomial in $K[X]^\ell+1$ is a vector of the form $\mathbf{m} := X^i \cdot \mathbf{u}_j$ for some $i \in N$, and some $j \in \{0, \ldots, \ell\}$, where $\mathbf{u}_j = \begin{cases} The reason for the subscript “0” in $M_0$ will become apparent later, when we define modules $M_r$ for each $r$ in Definition 3.1.\end{cases}$

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some list-decoding applications, it is convenient to identify

\[(0, \ldots, 0, 1, 0, \ldots, 0),\]

and where the 1 sits in the \(j\)-th position (counting from 0). In such a case, we will say that \(m\) contains the \(j\)-th unit vector.

The monomial ordering of the following definition is the special case of the ordering \(<_r\) of \([9]\) corresponding to \(r = -1\). If a pair \((f(X), g(X))\) is regarded as the bivariate polynomial \(f(X) + Yg(X)\), then this ordering is also the \((1, -1)\)-weighted-lex ordering with \(Y > X\).

**Definition 2.1.** Define the following monomial ordering, \(<\), on \(\mathbb{F}_q[X]^2\):

\((X^i, 0) < (X^j, 0)\) iff \(i < j\), \((0, X^i) < (0, X^j)\) iff \(i < j\), while \((0, 0) < (0, X^j)\) iff \(i \leq j - 1\).

Unless noted otherwise, \(\text{LM}(u, v)\) will stand for the leading monomial of \((u, v)\) with respect to the above monomial ordering, \(<\). Also, a “Gröbner basis” will stand for a Gröbner basis with respect to \(<\). Finally, \(d_H(\cdot, \cdot)\) will stand for the Hamming distance.

The following proposition is a special case of \([9\), Thm. 3.2]. We include its simple and standard proof for completeness.

**Proposition 2.2.** Using the above notation, suppose that \(d_H(y, x) \leq t\). Let \((u, v) \in M_0(S^{(y)}) \setminus \{(0, 0)\}\) satisfy \(\text{LM}(u, v) \leq \text{LM}(\omega, \sigma)\). Then there exists some \(c \in \mathbb{F}_q^*\) such that \((u, v) = c \cdot (\omega, \sigma)\). Hence, \((\omega, \sigma)\) is the unique minimal element \((u, v)\) in \(M_0\) with \(v(0) = 1\).

**Proof.** First, we claim that if there exist \((\tilde{u}, \tilde{v}), (u, v) \in M_0(S^{(y)})\) and \(d_1, d_2 \in \mathbb{N}\) with \(d_1 + d_2 < d - 1\), \(\gcd(\tilde{u}, \tilde{v}) = 1\), \(\deg(u), \deg(\tilde{u}) \leq d_1\), and \(\deg(v), \deg(\tilde{v}) \leq d_2\), then there exists a polynomial \(f \in \mathbb{F}_q[X]\) such that \((u, v) = f \cdot (\tilde{u}, \tilde{v})\). To see this, note that from \(u \equiv S^{(y)}v \mod (X^{d-1})\) and \(\tilde{u} \equiv S^{(y)}\tilde{v} \mod (X^{d-1})\), we get \(uv \equiv \tilde{u}\tilde{v} \mod (X^{d-1})\). In view of the above degree constraints, the last congruence implies \(\tilde{u}\tilde{v} = \tilde{u}v\). Since \(\gcd(\tilde{u}, \tilde{v}) = 1\), we must have \(\tilde{u}|u, \tilde{v}|v\), and \(u/\tilde{u} = v/\tilde{v}\). This establishes the claim.

Now let \((u, v) \in M_0(S^{(y)})\), and note that \(\gcd(\omega, \sigma) = 1\). If \(\deg(v) > t \geq \deg(\sigma)\), then clearly \(\text{LM}(u, v) > \text{LM}(\omega, \sigma) = (0, X^{\deg(\sigma)})\). Similarly, if \(\deg(u) > t - 1 \geq \deg(\sigma) - 1\), then \(\text{LM}(u, v) > \text{LM}(\omega, \sigma)\). Hence, we may assume w.l.o.g. that \(\deg(v) \leq t\) and \(\deg(u) \leq t - 1\). The above claim then shows that \((u, v) = f \cdot (\omega, \sigma)\) for some \(f \in \mathbb{F}_q[X]\). If \(\text{LM}(u, v) \leq \text{LM}(\omega, \sigma)\), this implies that \(f\) is a constant, as required. This also shows that \(\text{LM}(u, v) = \text{LM}(\omega, \sigma)\).

It will also be useful to recall that the uniqueness in the previous proposition is an instance of a more general result.

\[6\text{The reason for labeling coordinates with } 0, 1, \ldots \text{ rather than with } 1, 2, \ldots \text{ is that in some list-decoding applications, it is convenient to identify } K[X]^{n+1} \text{ with the polynomials in } K[X, Y] \text{ with } Y\text{-degree at most } \ell, \text{ by mapping } (f_0(X), \ldots, f_{\ell}(X)) \text{ to } \sum_{j=0}^{\ell} f_j(X)Y^j.\]
Proposition 2.3. For a field $K$ and for $\ell \in \mathbb{N}^*$, let $\prec$ be any monomial ordering on $K[X]^\ell$, and let $M \subseteq K[X]^\ell$ be any $K[X]$-submodule. Suppose that both $f := (f_1(X), \ldots, f_\ell(X)) \in M \setminus \{0\}$ and $g := (g_1(X), \ldots, g_\ell(X)) \in M \setminus \{0\}$ have the minimal leading monomial in $M \setminus \{0\}$. Then there exists $c \in K^*$ such that $f = c \cdot g$.

**Proof.** Suppose not. Since $\text{LM}(f) = \text{LM}(g)$, there exists a constant $c \in K^*$ such that the leading monomial cancels in $h := f - c g$. By assumption, $h \neq 0$, and $\text{LM}(h) \prec \text{LM}(f)$ – a contradiction.

2.2 Kötter’s Gröbner-basis iteration

Let us now recall the general form of Kötter’s iteration [14], [20], as presented by McEliece [16, Sect. VII.C].

Let $K$ be a field. For $\ell \in \mathbb{N}^*$ and for a $K[X]$-submodule $M$ of $K[X]^\ell+1$ with $\text{rank}(M) = \ell+1$, suppose that we have a Gröbner basis $G = \{g_0, \ldots, g_\ell\}$ for $M$ with respect to some monomial ordering $\prec$ on $K[X]^\ell+1$. In such a case, the leading monomials of the $g_j$ must contain distinct unit vectors, and we may therefore assume w.l.o.g. that the leading monomial of $g_j$ contains the $j$-th unit vector, for all $j \in \{0, \ldots, \ell\}$ (where coordinates of vectors are indexed by $0, \ldots, \ell$).

Now let $D : K[X]^\ell+1 \to K$ be a non-zero linear functional that satisfies the following property:

$\text{MOD} \ M^+ := M \cap \text{ker}(D)$ is a $K[X]$-module.

The purpose of Kötter’s iteration is to convert the $(\ell+1)$-element Gröbner basis $G$ of $M$ to an $(\ell+1)$-element Gröbner basis $G^+ = \{g_0^+, \ldots, g_\ell^+\}$ of $M^+$.

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7 We have learned from Johan Rosenkilde that [3], [4], which predated [14], already presented algorithms similar to, and more general than Kötter’s iteration (see also [12], Sec. 2.6). For problems related to modules of vectors of univariate polynomials, algorithms for computing the shifted (weak or canonical) Popov form of $K[X]$-matrices have the lowest asymptotic complexity in some cases – see, e.g., [22] and the references therein for the case of simultaneous Hermite–Padé approximation, and [18, Sec. 1.3.4] for the connection to Gröbner bases. However, for the fast Chase decoding algorithms considered in this paper, we currently do not know if such methods will turn out to be more efficient than Kötter’s iteration.

8 For otherwise, the leading monomial of two basis vectors would contain the same unit vector, so that the leading monomial of one vector divides the leading monomial of the other vector. In such a case, we may discard one of the basis vectors and remain with a Gröbner basis, which is, in particular, a set of generators. So, we end up with a set of less than $\ell + 1$ generators for a free module of rank $\ell + 1$ – a contradiction (see, e.g., Ex. 11 on p. 32 of [1]).

9 Where in this subsection, “Gröbner basis” means a Gröbner basis with respect to $\prec$. 


while maintaining the property that \( \text{LM}(\mathbf{g}^+_j) \) contains the \( j \)-th unit vector for all \( j \in \{0, \ldots, \ell\} \).

The following is a pseudo-code describing Kötter’s iteration.

**Kötter’s iteration without inversions**

**Input** A Gröbner basis \( G = \{ \mathbf{g}_0, \ldots, \mathbf{g}_\ell \} \) for the submodule \( M \subseteq \mathbb{F}_q[X]^{\ell+1} \), with \( \text{LM}(\mathbf{g}_j) \) containing the \( j \)-th unit vector for all \( j \).

**Output** A Gröbner basis \( G^+ = \{ \mathbf{g}^+_0, \ldots, \mathbf{g}^+_\ell \} \) for \( M^+ \) with \( \text{LM}(\mathbf{g}^+_j) \) containing the \( j \)-th unit vector for all \( j \) (assuming MOD holds).

**Algorithm**

- For \( j = 0, \ldots, \ell \), calculate \( \Delta_j := D(\mathbf{g}_j) \)
  - Set \( J := \{ j \in \{0, \ldots, \ell\} \mid \Delta_j \neq 0 \} \)
  - For \( j \in \{0, \ldots, \ell\} \setminus J \),
    - Set \( \mathbf{g}^+_j := \mathbf{g}_j \)
  - Let \( j^* \in J \) be such that \( \text{LM}(\mathbf{g}_{j^*}) = \min_{j \in J} \{ \text{LM}(\mathbf{g}_j) \} \) /* the leading monomials are distinct, and so \( j^* \) is unique */
  - For \( j \in J \)
    - If \( j \neq j^* \)
      * Set \( \mathbf{g}^+_j := \Delta_{j^*} \mathbf{g}_j - \Delta_j \mathbf{g}_{j^*} \)
    - Else /* \( j = j^* \) */
      * Set \( \mathbf{g}^+_j := \Delta_{j^*} \mathbf{X} \mathbf{g}_{j^*} - D(\mathbf{X} \mathbf{g}_{j^*}) \cdot \mathbf{g}_{j^*} \)
      /(*) = (\Delta_{j^*} \mathbf{X} - D(\mathbf{X} \mathbf{g}_{j^*})) \mathbf{g}_{j^*} */

Note that for clarity of presentation, we have introduced a whole new set of variables \( \{ \mathbf{g}^+_j \} \), although this is not really necessary.

**Proposition 2.4.** At the end of Kötter’s iteration, it holds that \( G^+ = \{ \mathbf{g}^+_0, \ldots, \mathbf{g}^+_\ell \} \) is a Gröbner basis for \( M^+ \) and for all \( j \), \( \text{LM}(\mathbf{g}_j^+) \) contains the \( j \)-th unit vector.

For a proof, see [16, Sec. VII.C].

### 2.3 Fast Chase decoding on a tree

In the Chase-II decoding algorithm [6, p. 173] for decoding a binary code of minimum distance \( d \), all possible \( 2^{\lfloor d/2 \rfloor} \) error patterns on the \( \lfloor d/2 \rfloor \) least
reliable coordinates are tested (i.e., subtracted from the received word). For each tested error pattern, bounded distance decoding is performed, resulting in a list of up to $2^{\lceil d/2 \rceil}$ candidate codewords. Finally, if the list is not empty, then the most likely codeword is chosen from the list.

For GRS codes over $\mathbb{F}_q$, the type of Chase algorithm considered in the current paper is the following variant of the Chase-II algorithm. First, we assume a memoryless channel, e.g., as in [15, Sec. III]. As in [6], we assume that the decoder has probabilistic reliability information on the received symbols. The $\eta$ least reliable coordinates are identified for some pre-defined and (loosely speaking) small $\eta \in \mathbb{N}^*$. Let $\alpha_1, \ldots, \alpha_\eta$ be these least reliable coordinates (where as usual, coordinates are labeled by elements of $\mathbb{F}_q^*$), and put $I := \{\alpha_1, \ldots, \alpha_\eta\}$.

Fix some $\mu \in \{1, \ldots, q\}$ and for each $i \in \{1, \ldots, \eta\}$, let $A'_i \subset \mathbb{F}_q$ be a subset of $\mu$ most probable choices for the $\alpha_i$-th code symbol given the $\alpha_i$-th received coordinate. Let $a^*$ be a symbol in $A'_i$ with the highest probability given the $\alpha_i$-th received coordinate, and set $A_i := \{a - a^* \mid a \in A'_i\}$. Hence $a^*$ is the hard-decision (HD) input to the decoder at coordinate $\alpha_i$ (an entry of the vector $y$ of Subsection 2.1), while $A_i$ is a corresponding set of $\mu$ most probable errors given the received symbol.

Finally, fix some $r_{\text{max}} \in \{1, \ldots, \eta\}$. The Chase decoding considered in the current paper runs over all test error patterns on $I$ that are taken from $A_1 \times \cdots \times A_\eta$ and have a Hamming weight of up to $r_{\text{max}}$. For each such error pattern, the algorithm performs (the equivalent of) bounded distance decoding. Note that when $r_{\text{max}} = \eta$, the test error patterns are all the vectors in $A_1 \times \cdots \times A_\eta$.

Let $B$ be the set of vectors of Hamming weight at most $r_{\text{max}}$ in $A_1 \times \cdots \times A_\eta$. As in [27], a directed tree $T = T(\eta, I, r_{\text{max}}, A_1, \ldots, A_\eta)$ of depth $r_{\text{max}}$ is constructed in the following way. The root is the all-zero vector, and for all $r \in \{1, \ldots, r_{\text{max}}\}$, the vertices at depth $r$ are the vectors in $B$ of weight $r$.

To define the edges of $T$, for each $r \geq 1$ and for each vertex $\mathbf{b} = (\beta_1, \ldots, \beta_\eta)$ at depth $r$ with non-zero entries at coordinates $i_1, \ldots, i_r$, we pick a single vertex $\mathbf{b}' = (\beta'_1, \ldots, \beta'_\eta)$ at depth $r - 1$ that is equal to $\mathbf{b}$ on all coordinates, except for one $i_\ell \in \{1, \ldots, r\}$, for which $\beta'_i = 0$. Note that given $\mathbf{b}$, there are $r$ distinct ways to choose $\mathbf{b}'$, and we simply fix one such choice of $\mathbf{b}'$ for each $\mathbf{b}$. Now the edges of $T$ are exactly all such pairs $(\mathbf{b}', \mathbf{b})$ (see Figure 1 for an example).

\footnote{Here, by bounded distance decoding for a code of minimum distance $d$, we mean a decoding algorithm that returns the unique codeword of distance up to $(d - 1)/2$ from the received word (if exists), or declares failure otherwise.}

\footnote{In the language of [15], we look for $\mu$ largest values in the $\alpha_i$-th column of the reliability matrix. For example, in [5], $\mu = 2$.}
Note that the edge \((\beta', \beta)\) defined above corresponds to adding exactly one additional modified coordinate, namely, coordinate \(\alpha_{i_r}\), in which the assumed error value is \(\beta_{i_r}\). Hence, the edge \((\beta', \beta)\) can be identified with the pair \((\beta', (\alpha_{i_r}, \beta_{i_r}))\). Similarly, a path from the root to a vertex at depth \(r \geq 1\) (and hence the vertex itself) can be identified with a sequence \(((\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_r}, \beta_{i_r})) \in ((\mathbb{F}_q)^2)^r\) for which the \(\alpha_{i_r}\)'s are distinct.

Figure 1: Example of the tree \(T\) for \(r_{\text{max}} = \eta = 5\), \(A_1 = \cdots = A_5 = \{0, 1\}\). The vertices at each depth \(r \in \{0, \ldots, r_{\text{max}} = 5\}\) are the vectors of weight \(r\), and the edges are chosen such that for each vertex at depth \(r \geq 1\), we connect exactly one vertex from depth \(r - 1\) that is obtained by transforming one non-zero value to zero. There are \(r\) different ways to do this, but we simply fix one of them. For example, in the figure, 11011 at depth 4 is connected to 01011 at depth 3. Alternatively, it could be connected to either one of 10011, 11001, 11010.

The main ingredient of Wu’s fast Chase algorithm, as well as of the algorithm of the current paper, is an efficient algorithm for updating the ELP (and additional polynomials) for adding a single modified coordinate \(\alpha_{i_r}\) and the corresponding error value, \(\beta_{i_r}\). The tree \(T\) is then traversed depth first, saving intermediate results on vertices whose out degree is larger than 1, and applying the polynomial-update algorithm on the edges. Because the tree is traversed depth first and has depth \(r_{\text{max}}\), there is a need to save at most
$r_{\text{max}}$ vertex calculations at each time (one for each depth). See ahead for details.

#### 3 The minimization problem for fast Chase decoding

Wu’s LFSR minimization problem $A[\sigma_i]$ [27] p. 112 is defined over an $\mathbb{F}_q$-vector space of pairs of polynomials that in general is not an $\mathbb{F}_q[X]$-module. The key observation is that using Forney’s formula, Wu’s minimization problem can be replaced by a minimization problem over an $\mathbb{F}_q[X]$-module.

**Remark.** For simplicity, we will assume from this point on that $d$ is odd, so that $d = 2t + 1$. It is straightforward to modify the following derivation for the case of even $d$.

**Definition 3.1.** For $r \in \{0, \ldots, n\}$, distinct $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q^*$, and $\beta_1, \ldots, \beta_r \in \mathbb{F}_q^*$ (not necessarily distinct), let

\[ M_r = M_r(S(y), \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r) \]

be the set of all pairs $(u, v) \in \mathbb{F}_q[X]^2$ satisfying the following conditions:

1. $u \equiv S(y)v \mod (X^d - 1)$

2. $\forall j \in \{1, \ldots, r\}$, $v(\alpha_j^{-1}) = 0$ and $\beta_j a_j v'(\alpha_j^{-1}) = -\alpha_j u(\alpha_j^{-1})$ (with $a_j := \tilde{a}_{j'}$ for the $j'$ with $\alpha_j = \lambda^{j'}$).

The possibility of using Kötter’s iteration as an alternative to Wu’s method follows almost immediately from the following theorem.

**Theorem 3.2.**

1. For all $r$, $M_r$ is an $\mathbb{F}_q[X]$-module.

2. With the terminology of the previous section, if $d_H(y, x) \leq t + r$, $\alpha_1, \ldots, \alpha_r$ are error locations and $\beta_1, \ldots, \beta_r$ are the corresponding error values, then $(\omega, \sigma) \in M_r$ and

\[ \text{LM}(\omega, \sigma) = \min \{ \text{LM}(u, v) | (u, v) \in M_r \setminus \{0\} \} \]

**Proof.** 1. $M_r$ is clearly an $\mathbb{F}_q$-vector space. For $f(X) \in \mathbb{F}_q[X]$ and $(u, v) \in M_r$, we would like to show that $f \cdot (u, v) \in M_r$. Clearly, $(fu, fv)$ satisfies the

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\[ ^{12}\text{We thank I. Tamo for pointing this out.} \]
required congruence, and also \( fv \) has the required roots. It remains to verify that for all \( j \), 
\[
\beta_j a_j (fv)'(\alpha_j^{-1}) = -\alpha_j (fu)(\alpha_j^{-1}).
\]
Now,
\[
(fv)'(\alpha_j^{-1}) = (f'v)(\alpha_j^{-1}) + (fv')(\alpha_j^{-1}) = (fv')(\alpha_j^{-1})
\]
\[
= f(\alpha_j^{-1}) \cdot -\frac{\alpha_j}{\beta_j a_j} u(\alpha_j^{-1}) = -\frac{\alpha_j}{\beta_j a_j} (fu)(\alpha_j^{-1}),
\]
where in the second equality we used \( v(\alpha_j^{-1}) = 0 \) and in the third equality we used \( \beta_j a_j v'(\alpha_j^{-1}) = -\alpha_j u(\alpha_j^{-1}) \) (note that \( \beta_j a_j \neq 0 \)).

2. First, \((\omega, \sigma) \in M_r\) by the key equation \((3)\) and Forney’s formula \((4)\). The proof of minimality is by induction on \( r \). For \( r = 0 \), the assertion is just Proposition \(2.2\). Suppose that \( r \geq 1 \), and the assertion holds for \( r - 1 \). Let \( \hat{\gamma} \) be obtained from \( \gamma \) by subtracting \( \beta_r \) from coordinate \( \alpha_r \). Let \( \tilde{\sigma} := \sigma/(1 - \alpha_r X) \) (the error locator for \( \hat{\gamma} \)) and let \( \tilde{\omega} \) be the error evaluator for \( \hat{\gamma} \). By the induction hypothesis,
\[
\text{LM}(\tilde{\omega}, \tilde{\sigma}) = \min \{ \text{LM}(u, v)|(u, v) \in \hat{M}_{r-1} \},
\]
with
\[
\hat{M}_{r-1} := M_{r-1}(S(\hat{\gamma}), \alpha_1, \ldots, \alpha_{r-1}, \beta_1, \ldots, \beta_{r-1}).
\]

To continue, we will need a lemma.

**Lemma.** For \((u, v) \in M_r\), write \( \hat{\gamma} := v/(1 - \alpha_r X) \) and put \( h := u - \beta_r a_r \hat{\gamma} \).
Then \((1 - \alpha_r X) h(X)\). Moreover, writing \( \hat{h} := h/(1 - \alpha_r X) \), the map \( \psi: (u, v) \mapsto (\hat{h}, \hat{\gamma}) \) maps \( M_r \) into \( \hat{M}_{r-1} \), and satisfies \( \psi(\omega, \sigma) = (\tilde{\omega}, \tilde{\sigma}) \).

**Proof of Lemma.** Since \( v = (1 - \alpha_r X) \hat{\gamma} \), we have \( v' = -\alpha_r \hat{\gamma} + (1 - \alpha_r X) \hat{\gamma}' \), and therefore \( \hat{\gamma}'(\alpha_r^{-1}) = -\alpha_r \hat{\gamma}(\alpha_r^{-1}) \).
Hence,
\[
h(\alpha_r^{-1}) = u(\alpha_r^{-1}) - \beta_r a_r \hat{\gamma}(\alpha_r^{-1})
\]
\[
= -\frac{\beta_r a_r}{\alpha_r} \hat{\gamma}'(\alpha_r^{-1}) - \beta_r a_r \hat{\gamma}(\alpha_r^{-1}) = 0,
\]
which proves the first assertion.

For the second assertion, note first that by \((1)\),
\[
S(\hat{\gamma}) \equiv S(\gamma) - \frac{\beta_r a_r}{1 - \alpha_r X} \mod (X^{d-1}),
\]
and therefore
\[
S(\hat{\gamma}) \hat{\gamma} \equiv S(\gamma) \hat{\gamma} - \frac{\beta_r a_r}{1 - \alpha_r X} \hat{\gamma}
\]
\[
= \frac{1}{1 - \alpha_r X}(S(\gamma) v - \beta_r a_r \hat{\gamma})
\]
\[
\equiv \frac{1}{1 - \alpha_r X}(u - \beta_r a_r \hat{\gamma}) = \hat{h},
\]

14
(where “≡” stands for congruence modulo $X^{d-1}$), which implies that $(\tilde{h}, \tilde{v})$

satisfies the required congruence relation in the definition of $\tilde{M}_{r-1}$. Also,

clearly $\tilde{v}(\alpha_j^{-1}) = 0$ for all $j \in \{1, \ldots, r-1\}$. Finally, using $\nu' = -\alpha_r \tilde{v} + (1 - \alpha_r X) \tilde{v}'$ again, we see that for all $j \in \{1, \ldots, r-1\}$,

$$\tilde{v}'(\alpha_j^{-1}) = \frac{\nu'(\alpha_j^{-1})}{1 - \alpha_r \alpha_j} = -\frac{\alpha_j}{\beta_j a_j} \cdot \frac{u(\alpha_j^{-1})}{1 - \alpha_r \alpha_j} = -\frac{\alpha_j}{\beta_j a_j} \cdot \tilde{h}(\alpha_j^{-1}).$$

This proves that $\psi$ maps $M_r$ into $\tilde{M}_{r-1}$.

Finally, we have $\psi(\omega, \sigma) = (\tilde{h}, \tilde{\sigma})$ with $\tilde{h} = (\omega - \beta_r a_r \tilde{\sigma})/(1 - \alpha_r X)$, and it remains to verify that $\tilde{h} = \tilde{\omega}$. Let $\alpha_1', \ldots, \alpha_z' \in \mathbb{F}_q^*$ be some enumeration of all error locators, let $\beta_1', \ldots, \beta_z' \in \mathbb{F}_q^*$ be the corresponding error values, and let $a_1', \ldots, a_z'$ be the corresponding entries of the vector $\tilde{a}$. Assume w.l.o.g. that $\alpha_z' = \alpha_r$ (and hence $\beta_z' = \beta_r$ and $a_z' = a_r$). Then

$$\tilde{h} = \frac{\omega - \beta_r a_r \tilde{\sigma}}{1 - \alpha_r X}$$

$$= \frac{1}{1 - \alpha_z' X} \left( \sum_{i=1}^z \beta_i' a_i' \prod_{j=1,j \neq i}^z (1 - \alpha_j' X) - \beta_z' a_z' \prod_{j=1}^{z-1} (1 - \alpha_j' X) \right)$$

$$= \frac{1}{1 - \alpha_z' X} \left( \sum_{i=1}^{z-1} \beta_i' a_i' \prod_{j=1,j \neq i}^{z-1} (1 - \alpha_j' X) \right)$$

$$= \sum_{i=1}^{z-1} \beta_i' a_i' \prod_{j=1,j \neq i}^{z-1} (1 - \alpha_j' X) = \tilde{\omega}.$$

We can now return to the proof of part 2 of the theorem. If $(u, v) \in M_r$ and $v = c \cdot \sigma$

for some $c \in \mathbb{F}_q^*$, then we must have $\text{LM}(u, v) \geq (0, X^{\deg(\sigma)}) = \text{LM}(\omega, \sigma).$ Let us therefore take $(u, v) \in M_r \setminus \{0\}$ with $v \neq c \sigma$ for all $c \in \mathbb{F}_q^*$. Then also $\psi(u, v) \neq c(\tilde{\omega}, \tilde{\sigma})$ for all $c \in \mathbb{F}_q^*$, and hence, by the induction hypothesis, the lemma, and Proposition 2.3,

$$\text{LM}(\psi(u, v)) > \text{LM}(\tilde{\omega}, \tilde{\sigma}) = (0, X^{\deg(\sigma) - 1}).$$

(6)

If the leading monomial of $\psi(u, v)$ is of the form $(0, X^\ell)$ for some $\ell$, then $\text{LM}(\psi(u, v)) = (0, X^{\deg(v) - 1})$, and (6) implies $\deg(v) > \deg(\sigma)$, so that certainly $\text{LM}(u, v) > \text{LM}(\omega, \sigma)$.

---

13 Note that if the total number of errors is at most $d - 1$, then it is clear from the above

that $\tilde{h} = \tilde{\omega}$, as both are congruent to $S(\tilde{\omega}, \tilde{\sigma})$ modulo $X^{d-1}$ and have a degree $\leq d - 2$. However, the following proof does not require this assumption.
Suppose therefore that \( \text{LM}(\psi(u, v)) \) is of the form \((X^\ell, 0)\) for some \( \ell \), that is, \( \text{LM}(\psi(u, v)) = (X^{\deg(h)-1}, 0) \). In this case, (5) implies that \( \deg(h) - 1 > \deg(\sigma) - 2 \), that is, \( \deg(h) \geq \deg(\sigma) \). But since \( h = u - \beta_r \tilde{v} \), this implies that at least one of \( u \) and \( \tilde{v} \) must have a degree that is at least as large as \( \deg(\sigma) \). Now, if \( \deg(u) \geq \deg(\sigma) \), that is, if \( \deg(u) > \deg(\sigma) - 1 \), then \( \text{LM}(u, v) > \text{LM}(\omega, \sigma) = (0, X^{\deg(\sigma)}) \). Similarly, if \( \deg(\tilde{v}) \geq \deg(\sigma) \), then \( \deg(v) > \deg(\sigma) \), and again \( \text{LM}(u, v) > \text{LM}(\omega, \sigma) \). This completes the proof of Theorem 3.2.

When moving from \( M_r := M_r(S(y), \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r) \) to \( M_{r+1} := M_{r+1}(S(y), \alpha_1, \ldots, \alpha_{r+1}, \beta_1, \ldots, \beta_{r+1}) \), two additional functionals must be zeroed. It was already proved in the theorem that each \( M_r \) is an \( \mathbb{F}_q[X] \)-module. Also, the intersection of \( M_r \) with the set of pairs \((u, v)\) for which \( v(\alpha_r^{-1}) = 0 \) is clearly an \( \mathbb{F}_q[X] \)-module. Hence, if each “root condition” comes before the corresponding “derivative condition,” we may use Kötter’s iteration twice in order to move from a Gröbner basis for \( M_r \) to a Gröbner basis for \( M_{r+1} \).

A detailed description of the application of Kötter’s iteration for moving from \( M_r \) to \( M_{r+1} \) appears in the following subsection. This is the algorithm carried out on the edges of the tree \( T \) of Section 2.3.

For initiating the fast Chase algorithm on the root of \( T \), we need a Gröbner basis for \( M_0 \). Several algorithms for achieving this goal appear in [9]. In particular, Algorithm 4.3 of [9] is the Euclidean algorithm, while Algorithm 4.7 of [9] is similar in nature to the BM algorithm.

In fact, we remark that the BM algorithm itself can be used to obtain a Gröbner basis for \( M_0 \). Informally, after running the BM algorithm, two pairs of polynomials are obtained from the two polynomials updated during the algorithm, and then at most one additional leading monomial cancellation is required for obtaining the desired Gröbner basis. Since the proof is rather technical and this is outside the main scope of the current paper, we will not elaborate on this issue.

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14 Note that by Proposition 2.2 any algorithm that finds a Gröbner basis for \( M_0 \) can also be used for bounded-distance decoding.

15 The stopping condition of [9, Alg. 4.3] assures that throughout its run, the leading monomials of both processed pairs of polynomials contain \((1, 0)\). Hence, the division algorithm used in the algorithm effectively divides the two scalar polynomials on the first coordinate, and performs the same calculations as the Euclidean algorithm.
4 Algorithms

4.1 The basic algorithm on an edge of the decoding tree

Using the terminology of Section 2.2 in the current context we have $\ell = 1$, and, as already mentioned, we have two types of Kötter iterations: one for a root condition, and the other for a derivative condition. For convenience, we will use here a version of Kötter’s iteration that includes inversions. In this version, the right-hand sides of the update rules are both divided by $\Delta_j$ (multiplication of elements by non-zero constants obviously takes a Gröbner basis to a Gröbner basis).

In the $r$-th root iteration, the linear functional $D$ of Kötter’s iteration acts on a pair $(u, v)$ as $D(u, v) = v(\alpha_r^{-1})$, and hence on $X \cdot (u, v)$ as $D(X \cdot (u, v)) = \alpha_r^{-1}D(u, v)$. In the $r$-th derivative iteration (which must come after the $r$-th root iteration), we have

$$D(u, v) = \beta_r a_r v'(\alpha_r^{-1}) + \alpha_r u(\alpha_r^{-1}),$$

and therefore also

$$D(X \cdot (u, v)) = \beta_r a_r (Xv)'(\alpha_r^{-1}) + \alpha_r (Xu)(\alpha_r^{-1}) = \beta_r a_r \alpha_r^{-1}v'(\alpha_r^{-1}) + u(\alpha_r^{-1}) = \alpha_r^{-1}D(u, v),$$

where in the second equality we used $(Xv)' = Xv' + v$ and $v(\alpha_r^{-1}) = 0$. So, for both types of iterations, we have $D(X \cdot (u, v))/D(u, v) = \alpha_r^{-1}$ if $D(u, v) \neq 0$. Hence, the iteration corresponding to a single location $\alpha_r$ has the following form.

Note that the above root and derivative iterations correspond to the values root, der (resp.) of the variable $\tau$ in Algorithm A.

**Algorithm A: Kötter’s iteration for adjoining error location $\alpha_r$**

**Input**

- A Gröbner basis $G = \{g_0 = (g_{00}, g_{01}), g_1 = (g_{10}, g_{11})\}$ for $M_{r-1}(S(y), \alpha_1, \ldots, \alpha_{r-1}, \beta_1, \ldots, \beta_{r-1})$, with $\text{LM}(g_j)$ containing the $j$-th unit vector for $j \in \{0, 1\}$

- The next error location, $\alpha_r$, and the corresponding error value, $\beta_r$

**Output**

A Gröbner basis $G^+ = \{g_0^+ = (g_{00}^+, g_{01}^+), g_1^+ = (g_{10}^+, g_{11}^+)\}$ for $M_r(S(y), \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r)$ with $\text{LM}(g_j^+)$ containing the $j$-th unit vector for $j \in \{0, 1\}$
Algorithm

• For type = root, der
  
  – If type = der,
    * For \( j = 0, 1 \), set \( g_j := g_j^+ \) /* init: output of root iter. */
  – For \( j = 0, 1 \), calculate
    \[
    \Delta_j := \begin{cases} 
      g_{j1}(\alpha_r^{-1}) & \text{if type = root} \\
      \beta_r g_j^1(\alpha_r^{-1}) + \alpha_r g_{j0}(\alpha_r^{-1}) & \text{if type = der}
    \end{cases}
    \]
  – Set \( J := \{ j \in \{0, 1\} | \Delta_j \neq 0 \} \)
  – For \( j \in \{0, 1\} \setminus J \), set \( g_j^+ := g_j \)
  – Let \( j^* \in J \) be such that \( \text{LM}(g_{j^*}) = \min_{j \in J} \{\text{LM}(g_j)\} \)
  – For \( j \in J \)
    * If \( j \neq j^* \)
      · Set \( g_j^+ := g_j - \frac{\Delta_j}{\Delta_j^*} g_{j^*} \)
    * Else /* \( j = j^* */
      · Set \( g_{j^*}^+ := (X - \alpha_r^{-1})g_{j^*} \).

Again, for clarity of presentation, we have introduced a whole new set of variables \( \{ g_j^+ \} \), although this is not really necessary.

If a successive application of the algorithm down the path from the root to a vertex \( ((\alpha_{i1}, \beta_{i1}), \ldots, (\alpha_{ir-1}, \beta_{ir-1})) \) of \( T \) results in Gröbner basis for \( M_{r-1}(S(y), \alpha_{i1}, \ldots, \alpha_{ir-1}, \beta_{i1}, \ldots, \beta_{ir-1}) \), then an additional application on the edge \((\alpha_{ir}, \beta_{ir})\) will result in a Gröbner basis for \( M_r(S(y), \alpha_{i1}, \ldots, \alpha_{ir}, \beta_{i1}, \ldots, \beta_{ir}) \).

It therefore follows from Theorem 3.2 that if a vertex

\[
\mathbf{v} := ((\alpha_{i1}, \beta_{i1}), \ldots, (\alpha_{ir}, \beta_{ir}))
\]

of \( T \) is a “direct hit,” in the sense that \( \alpha_{i1}, \ldots, \alpha_{ir} \) are indeed error locations with respective error values \( \beta_{i1}, \ldots, \beta_{ir} \), and if \( \varepsilon \leq t + r \), then the second element of the Gröbner basis on \( \mathbf{v} \) is \( c \cdot (\omega, \sigma) \) for some non-zero \( c \).

While not necessary for the correctness of the algorithm, it is of interest to consider the case where, although the tested error pattern is not a direct hit, the difference between the number of correct indices and incorrect indices is at least \( \varepsilon - t \). For this case, see Appendix A.

Two faster versions of Algorithm A appear in Appendix [B] in the first, two polynomials (rather than two pairs of polynomials) are maintained, and in the second, which is even more efficient, the algorithm works with low-degree polynomials.

\[\text{Recall that LM}(\omega, \sigma) \text{ contains the unit vector } (0, 1).\]
Remark 4.1. At a first glance, it may seem that the need to use two stages (root and derivative iterations) comes at the cost of doubling the complexity in comparison to [27, Alg. 1]. However, this is not the case: As shown ahead in Appendix B for the variant of Algorithm A described in Section B.1, the complexity of Wu’s algorithm is lower only by a factor about $5/6$ (or $10/11$ in characteristic $2$) than Algorithm A. See also Remark 4.2 ahead for Algorithm B of the following section. We also note that Algorithm C of Section B.3, which is another variant of Algorithm A, has a lower complexity than [27, Alg. 1].

4.2 Working with vectors of evaluations

As already mentioned, to achieve a complexity of $O(n)$ per modified symbol, one can use Kötter’s method of updating vectors of evaluations. Whereas in [27] this requires a complicated modification of the original algorithm in order to avoid syndrome updates, it is straightforward to modify Algorithm A to an “evaluated” version.

In Algorithm B below, for a fixed primitive element $\lambda \in \mathbb{F}_q^*$ we let $\lambda := (1, \lambda^{-1}, \lambda^{-2}, \ldots, \lambda^{-(q-2)})$. Also, for a polynomial $f \in \mathbb{F}_q[X]$, we let $f(\lambda) := (f(1), f(\lambda^{-1}), f(\lambda^{-2}), \ldots, f(\lambda^{-(q-2)}))$. Finally, in the algorithm below, $-\odot-$ stands for component-wise multiplication of vectors, that is,

$$(v_1, v_2, \ldots, v_\ell) \odot (u_1, u_2, \ldots, u_\ell) := (v_1u_1, v_2u_2, \ldots, v_\ell u_\ell)$$

(where $\ell \in \mathbb{N}^*$ and the $u_i, v_i$ are taken from some ring).

Note that the algorithm requires tracing the evaluation vectors of the four polynomials implicit in the Gröbner basis, as well as the evaluation vectors of the formal derivatives of two of these four polynomials.

Algorithm B: Adjoining error location $\alpha_r$ for vectors of evaluations, complexity $O(n)$

Input

- For a Gröbner basis $G = \{g_0 = (g_{00}, g_{01}), g_1 = (g_{10}, g_{11})\}$ for $M_{r-1}(S^{(y)}, \alpha_1, \ldots, \alpha_{r-1}, \beta_1, \ldots, \beta_{r-1})$ with $\text{LM}(g_j)$ containing the $j$-th unit vector for $j \in \{0,1\}$, the input includes the following data:

  $\gamma_j := (v_{j0}, v_{j1}, v_{j2}, m_j) := (g_{j0}(\lambda), g_{j1}(\lambda), g'_{j1}(\lambda), \text{LM}(g_j)), j = 0, 1$

- The next error location, $\alpha_r$, and the corresponding error value, $\beta_r$
Output For some Gröbner basis $G^+ = \{g_0^+, (g_0^0, g_0^1), g_1^+ = (g_1^0, g_1^1)\}$ for $M_r(S^\omega), \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r$ with $\text{LM}(g_j^+)$ containing the $j$-th unit vector for $j \in \{0, 1\}$, the output consists of the following data:

$$\gamma_j^+ := (v_{j0}^+, v_{j1}^+, v_{j2}^+, m_j^+) := (g_j^0(\lambda), g_j^1(\lambda), (g_j^1(\lambda), \text{LM}(g_j^+)), j = 0, 1$$

Algorithm

- For type = root, der
  - If type = der,
    * For $j = 0, 1$, set $\gamma_j := \gamma_j^+ /*$ init: output of root iter. */
  - For $j = 0, 1$, calculate (using appropriate entries of $v_{j0}, v_{j1}, v_{j2}$)
    $\Delta_j := \begin{cases} g_{j1}(\alpha_r^{-1}) & \text{if type=root} \\ \beta_r \alpha_r g_{j1}'(\alpha_r^{-1}) + \alpha_r g_{0j}(\alpha_r^{-1}) & \text{if type=der} \end{cases}$
  - Set $J := \{j \in \{0, 1\} | \Delta_j \neq 0\}$
  - For $j \in \{0, 1\} \setminus J$, set $\gamma_j^+ := \gamma_j$
  - Let $j^* \in J$ be such that $m_{j^*} = \min_{j \in J} \{m_j\}$
  - For $j \in J$
    * If $j \neq j^*$
      - For $i = 0, 1, 2$, set $v_{ji}^+ := v_{ji} - \frac{\Delta_j}{\Delta_j^*} v_{j^*i}$
      - Set $m_j^+ := m_j$ and put $\gamma_j^+ := (v_{j0}^+, v_{j1}^+, v_{j2}^+, m_j^+)$
    * Else /* $j = j^*$ */
      - For $i = 0, 1$, set
        $$v_{ji}^+ := (\lambda - (\alpha_r^{-1}, \alpha_r^{-1}, \ldots, \alpha_r^{-1})) \odot v_{ji}$$
      - Set /* using $[(X - \alpha_r^{-1})g_{j1}]' = (X - \alpha_r^{-1})g_{j1} + g_{j1}$ */
        $$v_{j2}^+ := (\lambda - (\alpha_r^{-1}, \alpha_r^{-1}, \ldots, \alpha_r^{-1})) \odot v_{j2} + v_{j1}$$
      - Set $m_j^+ := X \cdot m_j$ and put $\gamma_j^+ := (v_{j0}^+, v_{j1}^+, v_{j2}^+, m_j^+)$

Remark 4.2. 1. Algorithm B maintains a total of 6 evaluation vectors, and its complexity is dominated by a total of $2 \cdot 6 = 12$ per-coordinate multiplications of evaluation vectors. Hence, the total complexity on one edge is $12n$ finite-field multiplications.

2. As opposed to [27] Alg. 1], in [27] Alg. 2] there is an explicit equivalent to the two stages (root and derivative) of Algorithm B: in each
application of [27, Alg. 2], there is one stage called “2) Updating”, followed by a stage called “3) Converting”. Just like Algorithm B, [27, Alg. 2] maintains 6 evaluation vectors, where in each of the Updating and Converting stages only 4 of them are updated. However, the complexity depends on the total number of per-coordinate multiplications of an evaluation vector, and not on the number of updated vectors. This number of per-coordinate multiplications depends on the case in the Updating stage. For example, for Case 3, it seems that there are 8 distinct per-coordinate multiplications in the Updating stage, followed by 4 per-coordinate multiplications in the Converting stage. This gives a total of 12 such multiplications, just as in Algorithm B. On the other hand, the number of per-coordinates multiplications appears to be higher for Case 8 of the Updating stage. All-in-all, it seems to be fair to say that the two algorithms have a similar complexity.

4.3 High-level description of the decoding algorithm

Let us now describe the high-level flow of the decoding algorithm.

1. Perform bounded-distance HD decoding. If this decoding finds a codeword within Hamming distance \( t \) from the received word, output this codeword and exit. Otherwise, proceed to the fast Chase decoding algorithm.

2. Find a Gröbner basis \( \{ \mathbf{g}_0 = (g_{00}, g_{01}), \mathbf{g}_1 = (g_{10}, g_{11}) \} \) for \( M_0 \) with \( \text{LM}(\mathbf{g}_0) \) containing \((1, 0)\) and \( \text{LM}(\mathbf{g}_1) \) containing \((0, 1)\). As shown in [9], this can be done with an equivalent of any of the standard bounded-distance HD decoding algorithms, and can also be used for HD decoding in Step 1.

3. Calculate the derivatives \( g'_{01}, g'_{11} \), and evaluate polynomials to obtain

\[
\gamma_j := (g_{j0}(\lambda), g_{j1}(\lambda), g'_{j1}(\lambda), \text{LM}(\mathbf{g}_j)), j = 0, 1.
\]

Store \( \gamma_0, \gamma_1 \) in the memory for depth 0.

4. Using reliability information, identify a set \( I = \{\alpha_1, \ldots, \alpha_\eta\} \) of \( \eta \) least reliable coordinates. For each \( \alpha \in I \), find \( A_\alpha \), the set of \( \mu \) most probable HD errors for the \( \alpha \)-th coordinate given the \( \alpha \)-th received symbol. Together with the pre-defined depth, \( r_{\text{max}} \), this completely determines the tree \( T \) of Section 2.3.

5. Traverse the tree \( T \) depth first. When visiting an edge \((u', u)\) between a vertex \( u' \) at depth \( r-1 \) and a vertex \( u \) at depth \( r \):

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• Perform Algorithm B, taking the inputs $\gamma_0, \gamma_1$ from the memory for depth $r - 1$.
• Store the outputs $\gamma_0^+, \gamma_1^+$ in the memory for depth $r$.
• If the following conditions hold:
  - $\mathbf{v}_{1,1}^+$ has exactly $t + r$ zero entries, and
  - $\mathbf{m}_i^+ = (0, X^{t+r})$ (this is equivalent to $\deg(g_{1,1}^+) = t + r$, as $\text{LM}(g_{1}^+)$ contains $(0, 1)$)

then:
  - Letting $i_1, \ldots, i_{t+r}$ be the indices of zero entries of $\mathbf{v}_{1,1}^+$ (counting indices from 0), let the error locators be $\alpha_j := \lambda^{i_j}, j = 1, \ldots, t + r$. Calculate the corresponding error values using appropriate entries of $\mathbf{v}_{1,0}^+$ (evaluation vector of $g_{10}^+$) and $\mathbf{v}_{1,2}^+$ (evaluation vector of $(g_{11}^+)'$) by the Forney formula (4):

$$\beta_j := -\frac{\alpha_j g_{10}^+(\alpha_j^{-1})}{a_j (g_{11}^+)'(\alpha_j^{-1})}, j = 1, \ldots, t + r$$

  - If all the $\beta_j$ are non-zero, add the resulting error to a list of potential errors.

Note that error vectors added to the list in the above flow must have the same syndrome as the received word, as follows from the following proposition

**Proposition 4.3.** Suppose that $\tilde{\sigma} \in \mathbb{F}_q[X]$ is separable, splits in $\mathbb{F}_q$, and satisfies $\tilde{\sigma}(0) \neq 0$. Suppose also that $\tilde{\omega} \in \mathbb{F}_q[X]$ satisfies $\deg(\tilde{\omega}) < \deg(\tilde{\sigma})$ and $\tilde{\omega} \equiv S(\omega)\tilde{\sigma} \mod (X^{d-1})$. Let $\tilde{e}$ be the vector with support $\{\alpha^{-1}|\tilde{\sigma}(\alpha) = 0\}$ and corresponding non-zero entries obtained by Forney’s formula (4) with $\tilde{\sigma}$ and $\tilde{\omega}$. Then $S(\tilde{e})(X) = S(\omega)(X)$.

**Proof.** By dividing both $\tilde{\sigma}$ and $\tilde{\omega}$ by $\tilde{\sigma}(0)$, we may assume w.l.o.g. that $\tilde{\sigma}(0) = 1$. Note first that the EEP related to $\tilde{e}$ is indeed $\tilde{w}$: If $\tilde{w}$ is the EEP related to $\tilde{e}$, then by Forney’s formula (4), $\tilde{w} - \tilde{\omega}$ has $\deg(\tilde{\sigma})$ roots. By the degree assumption in the proposition, $\deg(\tilde{w} - \tilde{\omega}) < \deg(\tilde{\sigma})$, which implies that $\tilde{w} - \tilde{\omega}$ is the zero polynomial.

Writing “$\equiv$” for congruence modulo $(X^{d-1})$, it holds that

$$S(\tilde{w})\tilde{\sigma} \equiv \tilde{w} \equiv S(\omega)\tilde{\tilde{\sigma}},$$

\[\text{Note that in the expression for } \beta_j, \text{ the denominator is non-zero because } g_{11}^+ \text{ is separable by the above assumptions.}\]

\[\text{We thank M. Twitte for pointing out this observation.}\]
where the first congruence follows from the key equation, while the second congruence holds by assumption. Hence $X^{d-1} | (S^{(\tilde{e})} - S^{(y)})$. As $\deg(S^{(\tilde{e})} - S^{(y)}) \leq d - 2$, this completes the proof. □

5 Conclusions and open questions

We presented a conceptually simple fast Chase decoding algorithm for RS codes, building on the theory of Gröbner bases for $\mathbb{F}_q[X]$-modules. Working with “the right” minimization problem in an $\mathbb{F}_q[X]$-module results in a considerably simplified polynomial-update algorithm, which is also automatically suited to working with vectors of evaluations. Our algorithms are not tied to the BM algorithm for HD initialization, and practically any syndrome-based HD decoding algorithm can be used for this purpose.

It should be noted that both Algorithm A and B can be easily converted to a fast GMD algorithm, by simply omitting the derivative iteration. For Algorithm B, this means that there is no longer a need to maintain the vectors of evaluations of the derivatives. Moreover, a fast application of combinations of GMD and Chase decoding can be obtained in this way.

We conclude with some open questions:

- Any Chase decoding algorithm for GRS codes is automatically also a Chase decoding algorithm for their subfield-subcodes, the alternate codes, which include BCH codes as a special case. However, in [27], the polynomial-update algorithm for binary BCH codes is simpler than that of the corresponding RS codes. Is there a way to further simplify Algorithm A of the current paper for the case of binary BCH codes?\(^{19}\)

- Interestingly, Algorithms A and B remain valid also when the total number of errors is $\geq d - 1$, as long as the conditions of Theorem 3.2 are satisfied. Can this be of any practical value in some cases? Note that while the output list size grows exponentially beyond $d - 1$, this can be solved by adding a small number of CRC bits, or even without CRC bits, when the RS code is part of a generalized concatenated code [17, Sec. 18.8.2].

- Is it possible to further reduce the complexity by using fast algorithms for basis reduction of polynomial matrices, e.g., as in [18], [22] and

\(^{19}\)We note that in a companion work [24], using a completely different method, some of the authors have devised a syndrome-based Chase decoding algorithm for binary BCH codes that is both conceptually simple and updates polynomials of a lower degree than those of Algorithm 5 of [27]. However, for completeness, it is still an interesting question whether the current algorithm can be further simplified in the case of binary BCH codes.
the references therein, or by using fast algorithms for structured linear algebra, e.g., as in [7]?

Appendix

A  The case of enough correct modifications

In this appendix, we consider the case mentioned near the end of Section 4.1, that is, the case where, although the tested error pattern is not a direct hit, the difference between the number of correct indices and incorrect indices is at least $\varepsilon - t$. The main result is Proposition A.2, which shows that in this case, the outputs of Algorithm A can still be used for finding the correct transmitted codeword.

We begin with a remark that will be useful in the proof of Proposition A.2.

Remark A.1. For distinct $\alpha_1, \ldots, \alpha_r \in \mathbb{F}_q^*$ and for $\beta_1, \ldots, \beta_r \in \mathbb{F}_q^*$, let

$$ M_{r+\frac{1}{2}} := M_r(S^{(y)}, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r) \cap \{(u, v) | v(\alpha_r^{-1}) = 0\}. $$

Taking $v(X) := \prod_{i=1}^{r}(1 - \alpha_i^X)$, and letting $u_0 := S^{(y)} \cdot v$, there exists a polynomial $f(X) \in \mathbb{F}_q[X]$ such that, setting $u_f := u_0 + X^{d-1} \cdot f$, the $r$ “derivative equations” from part 2 of Definition 3.1 are satisfied for $(u_f, v)$ (this is just an interpolation problem for $f$, and it obviously has a solution for $f$ of high enough degree). For such a choice of $f$, clearly $(u_f, v) \in M_r \setminus M_{r+\frac{1}{2}}$.

Similarly, taking now $v(X) := \prod_{i=1}^{r+1}(1 - \alpha_i^X)$, and letting again $u_0 := S^{(y)} \cdot v$, there exists a polynomial $f(X) \in \mathbb{F}_q[X]$ such that, setting $u_f := u_0 + X^{d-1} \cdot f$, the $r$ equations from part 2 of Definition 3.1 are satisfied for $(u_f, v)$, while $\beta_r + \alpha_{r+1}v'(\alpha_r^{-1}) \neq \alpha_{r+1}u_f(\alpha_r^{-1})$ (again, this is an interpolation problem for $f$, now with a lot of freedom in the choice of $f(\alpha_r^{-1})$). For such a choice of $f$, clearly $(u_f, v) \in M_{r+\frac{1}{2}} \setminus M_{r+1}$.

We conclude that

$$ M_r \supseteq M_{r+\frac{1}{2}} \supseteq M_{r+1}. $$  \hspace{1cm} (7)

Proposition A.2. Consider a vertex $v = ((\alpha_i, \beta_i), \ldots, (\alpha_r, \beta_r))$ of $T$. Let

$$ S := \{\ell \in \{1, \ldots, r\} | \text{the error value at } \alpha_{i_{\ell}} \text{ is not } \beta_{i_{\ell}}\}, $$

and let

$$ S_1 := \{\ell \in \{1, \ldots, r\} | \alpha_{i_{\ell}} \text{ is not an error location} \} \subseteq S. $$

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Finally, let $S_2 := S \setminus S_1$. For any vertex $u$ of the tree $T$, let $\{g^+_0(u), g^+_1(u)\}$ be the Gröbner basis calculated inductively from the root to the vertex $u$ by applying Algorithm A on the edges. Then if $r - |S| \geq \varepsilon - t + |S_1|$, then it holds that
\[ \text{LM}(g^+_1(u)) < \text{LM}(g^+_0(u)), \]
and
\[ g^+_1(u) = c \cdot (\omega, \sigma) \cdot \prod_{\ell \in S_2} (X - \alpha_{i_\ell}^{-1}) \cdot \prod_{\ell \in S_1} (X - \alpha_{i_\ell}^{-1})^2 \quad (8) \]
for some $c \in \mathbb{F}_q^*$. Hence, writing $g^+_1(u) = (g^+_{10}, g^+_{11})$,
\[ \frac{g^+_{11}(X)}{\prod_{\ell = 1} (X - \alpha_{i_\ell}^{-1})} = \frac{c \cdot \sigma(X)}{\prod_{\ell \not\in S} (X - \alpha_{i_\ell}^{-1})} \cdot \prod_{\ell \in S_1} (X - \alpha_{i_\ell}^{-1}). \quad (9) \]

**Remark.** Equation (9) means that $g^+_{11}(X)/\prod_{\ell = 1} (X - \alpha_{i_\ell}^{-1})$ is an “effective ELP” corresponding to the modification sequence in $u$: Correct modifications are canceled out from $\sigma$, wrong modification at correct locations have no effect, while modification at locations without errors effectively add error locations.

**Proof of Proposition A.2.** Observe that $|S_1|$ is the number of wrongly-modified correct coordinates for $u$, while $r - |S|$ is the number of correctly-modified erroneous coordinates. Write $\delta := |S|$, $\delta_1 := |S_1|$. Modifying the order of the pairs defining $u$ does not change the corresponding module
\[ M_r(S(u), \alpha_{i_1}, \ldots, \alpha_{i_r}, \beta_{i_1}, \ldots, \beta_{i_r}), \]
and hence also does not change the unique minimal element (by Proposition 2.3). We may therefore assume w.l.o.g. that $S = \{r - \delta + 1, \ldots, r\}$, and that $S_1 = \{r - \delta_1 + 1, \ldots, r\}$. Hence, if $\ell$ is one of the $\delta - \delta_1$ smallest elements of $S$, then $\alpha_{i_\ell}$ is an error location and $\beta_{i_\ell}$ is not the corresponding error value. Similarly, if $\ell$ is one of the $\delta_1$ largest elements of $S$, then $\alpha_{i_\ell}$ is not an error location.

The idea of the proof is to trace the updates in Algorithm A, and (loosely speaking) to show that for $\ell \in S_2$, $g^+_1$ is multiplied once by $(X - \alpha_{i_\ell}^{-1})$, while for $\ell \in S_1$, $g^+_1$ is multiplied twice by $(X - \alpha_{i_\ell}^{-1})$.

By assumption, the first $r - \delta$ pairs in $u$ are correct error locations and corresponding values. As we also assume that $r - \delta - \delta_1 \geq \varepsilon - t$, it follows from Theorem 3.2 that when moving from the root of $T$ to the vertex $u' := ((\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_r-\delta-\delta_1}, \beta_{i_r-\delta-\delta_1}))$ at depth $r - \delta - \delta_1$, we have $g^+_1(u') = c \cdot (\omega, \sigma)$ for some $c \neq 0$. Moreover,
\[ \text{LM}(g^+_1(u')) < \text{LM}(g^+_0(u')). \quad (10) \]
Now, for the next $\delta_1$ edges on the path from $v'$ to $v$, we still have correctly-modified coordinates. Hence, in Algorithm A, $\Delta_1 = 0$ for both the root and derivative iterations, and only $g_0$ might be modified in all of the corresponding $2\delta_1$ root and derivative iterations. Moreover, $g_0$ is indeed modified in each and every one of the iterations, for otherwise the Gröbner basis would be unchanged, and hence the generated module would be unchanged, contradicting (7). Hence, writing $v'' := \left( (\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_r-\delta}, \beta_{i_r-\delta}) \right)$, we have

\[
(g_0^+(v''), g_1^+(v'')) = \left( g_0^+(v') \cdot \prod_{\ell = r - \delta - \delta_1}^{r - \delta} (X - \alpha_{i_{\ell}}^{-1})^2, g_1^+(v') \right). \tag{11}
\]

It is now left to consider the last $\delta$ applications of Algorithm A, on the path from $v''$ to $v$. Write $v_{r-\delta} = v'', v_{r-\delta+1}, \ldots, v_r = v$ for the consecutive vertices on the path from $v''$ to $v$. We first prove by induction that for all $\ell' \in \{r - \delta, \ldots, r - \delta_1\}$,

\[
\text{LM}(g_1^+(v_{\ell'})) < \text{LM}(g_0^+(v_{\ell'})), \tag{12}
\]

and

\[
g_1^+(v_{\ell'}) = g_1^+(v') \cdot \prod_{\ell = r - \delta + 1}^{\ell'} (X - \alpha_{i_{\ell}}^{-1})
\]

\[
= c \cdot (\omega, \sigma) \cdot \prod_{\ell = r - \delta + 1}^{\ell'} (X - \alpha_{i_{\ell}}^{-1}). \tag{13}
\]

The basis of induction, for $\ell' = r - \delta$ (where the product on the right of (13) is empty), follows from (11) and (10). For the step, assume that $\ell' \in \{r - \delta + 1, \ldots, r - \delta_1\}$, and that (12), (13) hold for $\ell' - 1$. As $\alpha_{i_{\ell'}}$ is an error location, it follows from the induction hypothesis that in the root iteration of Algorithm A, $\Delta_1 = 0$, and consequently, $g_1^+ = g_1$. We claim that in the derivative iteration, $\Delta_1 \neq 0$. For this, let $\beta$ be the correct error value for the (correct) error location $\alpha_{i_{\ell'}}$. Write

\[
M := M_{r-\delta+1}(S(y), \alpha_{i_1}, \ldots, \alpha_{i_{r-\delta}}, \alpha_{i_{\ell'}}, \beta_{i_1}, \ldots, \beta_{i_{r-\delta}}, \beta).
\]

Then clearly $(\omega, \sigma) \in M$, and since by the induction hypothesis $g_1^+(v_{\ell'-1})$ is obtained by multiplying $(\omega, \sigma)$ by a scalar polynomial, $g_1^+(v_{\ell'-1})$ is also in the module $M$. Hence

\[
\beta \cdot a_{i_{\ell'}}[g_1^+|_{v_{\ell'-1}}]'(\alpha_{i_{\ell'}}^{-1}) + a_{i_{\ell'}}[g_0^+|_{v_{\ell'-1}}](\alpha_{i_{\ell'}}^{-1}) = 0, \tag{14}
\]

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where for $i \in \{0, 1\}$ and for a vertex $u$ of $T$, we write $g_i^+(u) = (g_{i0}^+(u), g_{i1}^+(u))$. Since it can be verified by the induction hypothesis that \( [g_{i1}^+(v_{e-1})]'(\alpha_{i'_{e'}}^{-1}) \neq 0 \)

it follows that replacing $\beta$ by $\beta_{i'_{e'}} \neq \beta$ on the left-hand side of (11) will result in a non-zero value. This completes the proof that $\Delta_1 \neq 0$ on the derivative iteration, and hence, using the induction hypothesis for (12), (13), proves (13) for the induction step.

For (12), note that since in the root iteration $g_i^+ = g_i^+(v_{e-1})$ (that is, $g_i^+ = g_i$), it follows from (7) that

$$g_i^+ = g_i^0(v_{e-1}) \cdot (X - \alpha_i^{-1}).$$

Hence, the induction hypothesis implies that $\text{LM}(g_i^0) > X \text{LM}(g_i^+)$, and therefore after the derivative iteration it necessarily holds that $\text{LM}(g_i^0) > \text{LM}(g_i^+)$. This completes the induction step for (12).

Using (11) and (15), and noting that by the above it holds that $\text{LM}(g_i^+) = \text{LM}(g_i^0)$ in the derivative iteration, it also follows by induction that for all $\ell' \in \{r - \delta + 1, \ldots, r - \delta_1\}$,

$$\text{LM}(g_i^+(v_{e})) = \text{LM}(g_i^+(v')) \cdot \prod_{\ell=r-\delta+1}^{r-\delta} (X - \alpha_i^{-1})^2 \cdot \prod_{\ell=r-\delta+1}^{\ell'} (X - \alpha_i^{-1}),$$

(16)

where we have used $\text{LM}(\text{f} \cdot h) = X^{\text{deg}(\text{f})} \cdot \text{LM}(h)$ for $f \in \mathbb{F}_q[X]$ and $h \in \mathbb{F}_q[X]^2$.

To complete the proof, we will prove by induction that for all $\ell' \in \{r - \delta_1, \ldots, r\}$, $\text{LM}(g_i^+(v_{e})) < \text{LM}(g_i^0(v_{e}))$ and

$$g_i^+(v_{e}) = c \cdot (\omega, \sigma) \cdot \prod_{\ell=r-\delta+1}^{r-\delta_1} (X - \alpha_i^{-1}) \cdot \prod_{\ell=r-\delta+1}^{\ell'} (X - \alpha_i^{-1})^2.$$  

(17)

The basis of the induction, for $\ell' = r - \delta_1$, follows from (12), (13).

To continue, recall that in both the root and the derivative iterations of Algorithm A, if the leading monomial $m$ of one of the pairs is changed, then it is changed to $Xm$. Hence, it follows from substituting $\ell' = r - \delta_1$ in (13), (16) and the fact that $\text{LM}(g_i^+(v')) < \text{LM}(g_i^+(v_{e}))$, that for all $\ell' \in \{r - \delta_1 + 1, \ldots, r\}$, it holds that $\text{LM}(g_i^+(v')) < \text{LM}(g_i^0(v_{e}))$ for both the root and derivative iterations of Algorithm A.\(^{22}\)

Hence, for $\ell' \in \{r - \delta_1, \ldots, r - 1\}$ there are only three possible ways in which $g_i^+(v_{e'})$ can be updated to $g_i^+(v_{e'+1})$: (1) $g_i^+(v_{e'+1}) = g_i^+(v_{e'})$, (2) $g_i^+(v_{e'+1}) = g_i^+(v_{e'}) \cdot (X - \alpha_i^{-1})$, or (3) $g_i^+(v_{e'+1}) = g_i^+(v_{e'}) \cdot (X - \alpha_i^{-1})^2$.\(^{23}\)

\(^{22}\)Note that the induction hypothesis implies that $g_{i1}^+(v_{e'-1}) = f(X) \cdot \sigma(X)$ for some $f$ with $f(\alpha_i^{-1}) \neq 0$.

\(^{23}\)In detail, note that $\text{LM}(g_i^+(v_{r-\delta_1})) = X^{2\delta_1 + \delta - \delta_i} \cdot \text{LM}(g_i^0(v'))$, while $\text{LM}(g_i^+(v_{r-\delta_1})) = X^{2\delta_1 + \delta - \delta_i}$.
For the induction step of the proof of (17), assume that $\ell' \in \{r - \delta_1, \ldots, r - 1\}$, and that (17) holds for $\ell'$. Considering options (1)–(3) above, it is sufficient to prove that when moving from $v_{\ell'}$ to $v_{\ell'+1}$, it holds that $\Delta_1 \neq 0$ for both the root and derivative iterations of Algorithm A.

As $\alpha_{i'_{\ell'+1}}$ is not an error location, it follows from the induction hypothesis that $\alpha_{i'_{\ell'+1}}$ is not a root of $g_{11}^+(v_{\ell'})$, and therefore $\Delta_1 \neq 0$ in the root iteration. Hence, at the end of the root iteration, we have

$$g_1^+ = g_1^+(v_{\ell'}) \cdot (X - \alpha_{i'_{\ell'+1}})^{-1}. \quad (18)$$

Therefore,

$$[g_{11}^+]'(\alpha_{i'_{\ell'+1}}^{-1}) = [g_{11}^+(v_{\ell'})](\alpha_{i'_{\ell'+1}}^{-1}) \neq 0,$$

where the last inequality follows again from the induction hypothesis. Also, it follows from (18) that $g_{10}^+(\alpha_{i'_{\ell'+1}}^{-1}) = 0$, and finally that $\Delta_1 \neq 0$ in the derivative iteration, as required.

## B Simplifications for Algorithm A

### B.1 Moving from two pairs of polynomials to two polynomials

In Algorithm A, two pairs of polynomials have to be maintained, rather than just two polynomials. In the above form, the algorithm will work even if $\varepsilon \geq 2t$, where $\varepsilon$ is the total number of errors. However, as we shall now see, if $\varepsilon \leq 2t - 1$, then there is no need to maintain the first coordinate of the Gröbner basis.

In order to omit the first entry in each pair, we have to consider the following questions:

1. How can we efficiently calculate $g_{j0}(\alpha_{r-1}^{-1})$ ($j \in \{0, 1\}$) when only $g_{j1}$ is available?

2. How can we find $\text{LM}(g_0)$ without maintaining $g_{00}$ (recall that the leading monomial of $g_0$ is on the left)?

$X^{\delta-\delta_1} \cdot \text{LM}(g_1^+(v'))$. Hence, for all $\ell' \in \{r - \delta_1 + 1, \ldots, r\}$, we have

$$\text{LM}(g_1^+(v_{\ell'})) \leq X^{2(\ell'-(r-\delta_1))} X^{\delta-\delta_1} \cdot \text{LM}(g_1^+(v')) \leq X^{\delta + \delta_1} \cdot \text{LM}(g_1^+(v')) \quad \text{(substituting } \ell' = r) \leq \text{LM}(g_0^+(v_{r-\delta_1})) \leq \text{LM}(g_0^+(v_{\ell'})).$$
The answer to the second question is almost trivial: Introduce a variable \(d_0\) to track the degree of \(g_{00}\). Whenever \(j^* = 0\), increase \(d_0\) by 1, and in all other cases keep \(d_0\) unchanged (note that when \(0 \in J\) but \(0 \neq j^*\), \(\text{LM}(g_{0j}) = \text{LM}(g_0)\), which justifies keeping \(d_0\) unchanged). Now \(\text{LM}(g_0) = (X^{d_0}, 0)\).

So, let us turn to the first question. We know that for all \(r \in \mathbb{Z}\) and all \((u, v) \in M_r(S^{(w)}, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r)\), we have \(u \equiv S^{(w)} v \mod (X^{2r})\), and hence one can calculate \(u(\alpha_1^{-1})\) directly from \(v\) if \(\deg(u) \leq 2t - 1\) (see ahead).

Now, as \(M\) is generated as an \(F\)-module by \(\{X^{di}, 0\rangle, \langle S^{(w)}(X), 1\rangle\}\), we have \(\deg(g_{0}) \leq 2t - 1\) and \(\deg(g_{20}) \leq 2t - 1\) for all Kötter's iterations involved in fast Chase decoding, assuming the hypotheses of Theorem 3.2 hold.

We will first need a small modification of the first part of [2, Prop. 2]. To keep this paper self-contained, we will also include the proof. From this point on, we will say that a monomial in \(F_q[X]^2\) is on the left if it contains the unit vector \((1, 0)\), and on the right if it contains the unit vector \((0, 1)\).

**Proposition B.1** ([2]). Let \(\{h_0 = (h_{00}, h_{01}), h_1 = (h_{10}, h_{11})\}\) be a Gröbner basis for \(M_0\) with respect to the monomial ordering \(<\), and suppose that the leading monomial of \(h_0\) is on the left, while the leading monomial of \(h_1\) is on the right. Then \(\deg(h_{00}(X)) + \deg(h_{11}(X)) = 2t\).

**Proof.** Since \((S^{(w)}, 1)\) is in the \(F_q[X]\)-span of \(\{h_0, h_1\}\), it follows that \(1 \in (h_{01}, h_{11})\), and hence that \(h_{01}\) and \(h_{11}\) are relatively prime. Now suppose that \(\alpha(X), \beta(X) \in F_q[X]\) are such that \(\alpha(X)h_0 - \beta(X)h_1 = (\gamma(X), 0)\) for some \(\gamma(X)\). Then \(\alpha(X)h_{01}(X) = \beta(X)h_{11}(X)\), and because \(\gcd(h_{01}, h_{11}) = 1\), this implies that \(h_{11}(X)|\alpha(X), h_{01}(X)|\beta(X)\),

\[
\frac{\alpha(X)}{h_{11}(X)} = \frac{\beta(X)}{h_{01}(X)}
\]

and these two equal rational functions are in fact a polynomial in \(F_q[X]\). Write \(r(X) \in F_q[X]\) for this polynomial. Let \(\pi_0 : F_q[X]^2 \to F_q[X]\) be the projection to the first coordinate. Now, the second coordinate of the vector

\[
f := h_{11}(X)h_0 - h_{01}(X)h_1 \in M_0
\]

is 0. Also, for any \(\alpha(X), \beta(X)\) as above, it follows from the definition of \(r(X)\) that

\[
\alpha(X)h_0 - \beta(X)h_1 = r(X) \cdot f.
\]

This shows that \(\pi_0(f)\) has the lowest degree in \(\pi_0(M_0 \cap (F_q[X] \times \{0\}))\).

Now, as \(M_0\) is generated as an \(F_q[X]\)-module by \(\{(X^{2t}, 0), (S^{(w)}(X), 1)\}\), we
know that this lowest degree is 2t. Hence \( \deg(\pi_0(f)) = 2t \). Now,

\[
\deg(\pi_0(f)) = \deg (h_{11}(X)h_{00}(X) - h_{01}(X)h_{10}(X)) = \deg (h_{11}(X)h_{00}(X)),
\]

because by assumption \( \deg(h_{11}) \geq \deg(h_{10}) + 1 \) and \( \deg(h_{00}) > \deg(h_{01}) - 1 \), so that \( \deg(h_{11}h_{00}) > \deg(h_{01}h_{10}) \).

With Proposition B.1 we can now prove that for all iterations of Kötter’s algorithm, \( \deg(g_{10}) \leq 2t - 1 \) and \( \deg(g_{20}) \leq 2t - 1 \) when \( \varepsilon \leq 2t - 1 \). Before the proof, it will be useful to introduce some additional notation.

**Definition B.2.** For \( i = 1, \ldots, r, j \in \{0,1\} \), and \( \tau \in \{\text{root, der}\} \) write \( g_j(i; \tau) = (g_{j0}(i; \tau), g_{j1}(i; \tau)) \) and \( g^+_j(i; \tau) = (g^+_{j0}(i; \tau), g^+_{j1}(i; \tau)) \) for the values in the root iteration \( (\tau = \text{root}) \) or the derivative iteration \( (\tau = \text{der}) \) of Algorithm A corresponding to adjoining error location \( \alpha_i \). By convention, \( \{g_{j0}(1; \text{root}), g_{j1}(1; \text{root})\} \) is a Gröbner basis for \( M_0 \) with \( \text{LM}(g_{j0}(1; \text{root})) \) on the left and \( \text{LM}(g_{j1}(1; \text{root})) \) on the right. Note that for all \( i, g_j(i, \text{der}) = g^+_j(i, \text{root}) (j = 1, 2) \), and for all \( i \geq 2, g_j(i, \text{root}) = g^+_j(i-1, \text{der}) (j = 1, 2) \).

**Proposition B.3.** Suppose that the condition in part 2 of Theorem 3.2 holds, and that the total number \( \varepsilon \) of errors is exactly \( t + r \). Then for all \( i \in \{1, \ldots, r\} \), all \( j \in \{0,1\} \) and all \( \tau \in \{\text{root, der}\} \), \( \deg(g_{j0}^+(i; \tau)) \leq \varepsilon \) and \( \deg(g_{j1}^+(i; \tau)) \leq \varepsilon \).

*Proof.* By Theorem 3.2, \( (\omega, \sigma) = c \cdot g^+_1(r; \text{der}) \) for some \( c \in F_q^* \) (as the leading monomial of \( (\omega, \sigma) \) is on the right). Note that for all \( i, j, \) and \( \tau \), we have \( \text{LM}(g_j^+(i; \tau)) \geq \text{LM}(g_j(i; \tau)) \), and so for all \( i \) and \( \tau \), we must have \( \text{LM}(g_j(i; \tau)) \leq \text{LM}(\omega, \sigma) = (0, X^\varepsilon) \). In particular, \( \deg(g_{j1}(i; \tau)) \leq \varepsilon \) and \( \deg(g_{j0}(i; \tau)) \leq \varepsilon - 1 \). The same argument applies also to \( g^+_{j0}(i; \tau) \) and \( g^+_{j1}(i; \tau) \).

Turning to \( g_0(i; \tau) \), note that for all \( i \) and \( \tau \), \( \text{LM}(g_0^+(i; \tau)) > \text{LM}(g_0(i; \tau)) \) for at most one \( j \in \{0,1\} \). Also, for \( j \in \{0,1\} \) and for each \( i \) and \( \tau \) with \( \text{LM}(g_j^+(i; \tau)) > \text{LM}(g_j(i; \tau)) \), we have \( \text{LM}(g_j^+(i; \tau)) = X\text{LM}(g_j(i; \tau)) \). Since the degree of the second coordinate of \( g_j^+(i; \tau) \) (the coordinate containing the leading monomial) must increase from \( \deg(g_{11}(1; \text{root})) \) for \( i = 1 \) and \( \tau = \text{root} \) to \( \deg(\sigma) = \varepsilon \) for \( i = r \) and \( \tau = \text{der} \), we see that

\[
|\{(i, \tau) | \text{LM}(g_j^+(i; \tau)) > \text{LM}(g_j(i; \tau))\}| = \varepsilon - \deg(g_{11}(1; \text{root})),
\]

and therefore

\[
|\{(i, \tau) | \text{LM}(g_{j0}^+(i; \tau)) > \text{LM}(g_0(i; \tau))\}| \leq 2r - (\varepsilon - \deg(g_{11}(1; \text{root}))) = \deg(g_{11}(1; \text{root})) + r - t.
\]  

\(^{\text{22}}\)Actually, by (7) we can replace “≤” by “=” in the following equation.
Hence, for all $i$ and $\tau$,

\[
\deg(g_0^+(i; \tau)) \leq \deg(g_0^+(r; \text{der})) \quad \text{(LM (on the left) does not decrease)}
\]

\[
\leq \deg(g_0^0(1; \text{root})) + \deg(g_{11}(1; \text{root})) + r - t = t + r = \varepsilon \quad \text{(by Proposition B.1)}.
\]

Finally, since the leading monomial of $g_0^+(i; \tau)$ is on the left, we must have

\[
\deg(g_0^1(i; \tau)) - 1 < \deg(g_0^0(i; \tau)) \leq \varepsilon,
\]

which proves that $\deg(g_0^1(i; \tau)) \leq \varepsilon$.

Using Proposition B.3, we can calculate $g_{j0}(\alpha^{-1})$ in Algorithm A while maintaining only the right polynomials $g_{j1}$ ($j \in \{0, 1\}$). We shall now describe an efficient $O(t)$ method for calculating this evaluation.

For a polynomial $v(X) \in \mathbb{F}_q[X]$, assume that $\delta := \deg(v) \leq \varepsilon \leq 2t - 1$, and write $v(X) = v_0 + v_1X + \cdots + v_{2t-1}X^{2t-1}$. For short, write $S(X) = S_0 + S_1X + \cdots + S_{2t-1}X^{2t-1} := S(v)(X)$. Then for $\beta \in \mathbb{F}_q$, $(Sv \mod (X^{2t}))(\beta)$ can be expressed as

\[
S_0v_0 + (S_0v_1 + S_1v_0)\beta + (S_0v_2 + S_1v_1 + S_2v_0)\beta^2 + \cdots + (S_0v_{2t-1} + S_1v_{2t-2} + S_2v_{2t-3} + \cdots + S_{2t-1}v_0)\beta^{2t-1}.
\]

(19)

For $j \in \{0, \ldots, 2t-1\}$, let $A_j(v, \beta)$ be the sum over the $j$-th column of (19). Then

\[
A_j(v, \beta) = S_j\beta^j(v_0 + v_1\beta + \cdots + v_{2t-1-j}\beta^{2t-1-j}).
\]

If $2t - 1 - j \geq \delta (= \deg(v))$, then $A_j(v, \beta) = S_j\beta^jv(\beta)$. Hence, if $v(\beta) = 0$ (which we will assume from this point on, considering the previous root iteration of Algorithm A), then

\[
(Sv \mod (X^{2t}))(\beta) = \sum_{j=0}^{2t-1} A_j(v, \beta) = \sum_{j=2t-\delta}^{2t-1} A_j(v, \beta).
\]

(20)

The sum on the right-hand side of (20) may be calculated recursively. For this, let

\[
\tilde{A}_j(v, \beta) := \beta^j \sum_{i=0}^{2t-1-j} v_i\beta^i,
\]

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so that \( A_j(v, \beta) = S_j \hat{A}_j(v, \beta) \). Then \( \hat{A}_{2t-\delta-1} = 0 \) (as \( v(\beta) = 0 \)), and for all \( j \in \{2t-\delta-1, \ldots, 2t-2\} \), \( \hat{A}_{j+1}(v, \beta) = \beta \hat{A}_j(v, \beta) - \beta^{2t}v_{2t-1-j} \), that is,

\[
\frac{\hat{A}_{j+1}(v, \beta)}{\beta^{2t}} = \beta \cdot \frac{\hat{A}_j(v, \beta)}{\beta^{2t}} - v_{2t-1-j}.
\] (21)

Calculating \( \beta^{2t} \) takes \( O(\log_2(2t)) \) squarings and multiplications. In fact, this can be calculated once, before starting the depth-first search in the tree, for all non-reliable coordinates (not just for those corresponding to a particular vertex). After that, each one of the \( \delta \) iterations of (21) in the calculation of the sum \( \text{(20)} \) requires 2 finite-field multiplications: one for moving from \( \hat{A}_j(v, \beta)/\beta^{2t} \) to \( \hat{A}_{j+1}(v, \beta)/\beta^{2t} \), and one for multiplying by \( S_{j+1} \) before adding to an accumulated sum. Then, after the calculation of the accumulated sum, one additional multiplication by \( \beta^{2t} \) is required. We conclude that calculating \( (Sv \mod (X^{2t}))(\beta) \) requires a total of \( 2\delta + 1 \) finite-field multiplications (recall that \( \delta = \deg(v) \)).

For comparing the complexity with \[27\text{ Alg. 1}]\), let us now estimate the total number of finite-field multiplications required for performing the above variant of Algorithm A. For this purpose, for \( \tau \in \{\text{root, der}\} \), let \( 2\partial(r; \tau) \) be an upper bound on the sum of the degrees of \( g^+_0(r; \tau) \) and \( g^+_{11}(r; \tau) \).

The following proposition proves that we may take \( \partial(r; \text{root}) = t + r - 1/2 \) and \( \partial(r; \text{der}) = t + r \).

**Proposition B.4.** For all edge connecting a vertex at depth \( r - 1 \) to a vertex at depth \( r \),

\[
\deg(g^+_0(r; \tau)) + \deg(g^+_{11}(r; \tau)) \leq \begin{cases} 
2t + 2r - 1 & \text{if } \tau = \text{root} \\
2t + 2r & \text{if } \tau = \text{der}.
\end{cases}
\]

Consequently, since \( \deg(g^+_0(r; \tau)) \leq \deg(g^+_0(r; \tau)) \) for all \( \tau \)[24] it also holds that

\[
\frac{1}{2} \left( \deg(g^+_0(r; \tau)) + \deg(g^+_{11}(r; \tau)) \right) \leq \begin{cases} 
t + r - 1/2 & \text{if } \tau = \text{root} \\
t + r & \text{if } \tau = \text{der}.
\end{cases}
\]

**Proof.** By Proposition B.1 the assertion holds for \( r = 0 \) (with an obvious convention in this case). Now, for each root and derivative iteration, the leading monomial increases for exactly one value of \( j \) (namely \( j = j^* \)), for which it is multiplied by \( X \). Since for all \( i, \tau \), the leading monomial of \( g^+_0(i; \tau) \) is \( (X^{\deg(g^+_0(i; \tau))}, 0) \) while the leading monomial of \( g^+_{11}(i; \tau) \) is \( (0, X^{\deg(g^+_{11}(i; \tau))}) \), and since there is a total of either \( 2r - 1 \) (or \( 2r \)) root and derivative iterations for \( \tau = \text{root} \) (resp., \( \tau = \text{der} \)), the assertion follows. \( \square \)

[24] Recall that the leading monomial of \( g^+_0(r; \tau) \) is on the left.
In the following complexity estimation for the number of multiplications on an edge connecting a vertex at depth \( r - 1 \) to a vertex at depth \( r \), we assume that all involved discrepancies are non-zero. In the other cases, which are typically rare, the complexity is lower.

- **In the root iteration:**
  
  - **Evaluation:** For \( j = 0, 1 \), we have to calculate \( g_{j1}(\alpha_r^{-1}) \). Hence, we have two substitutions in polynomials whose sum of degrees is at most \( 2\partial(r - 1; \text{der}) \), which requires a total of at most \( 2\partial(r - 1; \text{der}) \) multiplications.

  - **Multiplication of a polynomial by a constant:** For \( j \neq j^* \), we have to calculate the constant \( \Delta_j / \Delta_{j^*} \), which requires a single multiplication (assuming that we have a table for calculating inverses), and to multiply two polynomials whose sum of degrees is at most \( 2\partial(r - 1; \text{der}) \) by a constant. This requires a total of \( 1 + 2\partial(r - 1; \text{der}) + 2 = 2\partial(r - 1; \text{der}) + 3 \) multiplications (the “+2” accounts for the fact that a polynomial of degree \( d \) has \( d + 1 \) coefficients).

- **In the derivative iteration:**
  
  - **Evaluation:** For \( j = 0, 1 \), we have to calculate \( g'_{j1}(\alpha_r^{-1}) \). In general, this requires at most \( 2\partial(r; \text{root}) - 2 \) multiplications (in characteristic 2, only up to \( \partial(r; \text{root}) \) multiplications are required). We then have to calculate \( \beta_\alpha r \) and multiply the two evaluation results by \( \beta_\alpha r \), which adds 3 multiplications. Finally, we have to calculate \( g_{j0}(\alpha_r^{-1}) \) (using the above method) and multiply by a constant for \( j = 0, 1 \), requiring at most \( 2(2\partial(r; \text{root}) + 1 + 1) \) multiplications. Hence, the overall number of multiplications for evaluation in the derivative step is \( 6\partial(r; \text{root}) + 5 \) in general (or \( 5\partial(r; \text{root}) + 7 \) in characteristic 2).

---

\(^{24}\)Using Horner’s method.

\(^{25}\)Actually, in the current form of Algorithm A, we multiply the same polynomial \( g_{j^*1}(r - 1; \text{der}) \) twice by a constant, instead of multiplying the two polynomials \( g_{j1}^+(r - 1; \text{der}), g_{j^*1}^+(r - 1; \text{der}) \) whose sum of degrees was bounded in Proposition 3.4. However, this can be resolved by changing the update rule of Algorithm A for \( j \neq j^* \) into \( g_j^+ := \frac{\Delta_j}{\Delta_{j^*}} g_j - g_{j^*} \). A similar remark is relevant also for the complexity analysis for Algorithm C ahead.

\(^{26}\)In characteristic 2, for all polynomial \( u(X) \), there exists a polynomial \( f(X) \) such that \( u'(X) = f(X)^2 \) with \( \deg(f) \leq \lfloor \deg(u)/2 \rfloor - 1 \). Moreover, the coefficients of \( f \) are obtained as the square roots of the odd coefficients of \( u \), and the square root calculation amounts to a cyclic shift when elements are represented according to a normal basis over \( \mathbb{F}_2 \).
- **Multiplication of a polynomial by a constant**: This is the same as in the root iteration: a total of at most \(2\partial(r;\text{root}) + 3\) multiplications.

Summing up, we obtain that the total number of multiplications is at most

\[
M_A := 2\partial(r - 1;\text{der}) + 2\partial(r - 1;\text{der}) + 3 + 6\partial(r;\text{root}) + 5 + 2\partial(r;\text{root}) + 3
\]

\[
= 4\partial(r - 1;\text{der}) + 8\partial(r;\text{root}) + 11 = 2(2t + 2r - 2) + 4(2t + 2r - 1) + 11
\]

\[
= 12t + 12r + 3
\]

in general, while in characteristic 2, the number of multiplications is at most

\[
M_A - \partial(r;\text{root}) + 2 = 12t + 12r + 3 - (t + r - 1/2) + 2
\]

\[
= 11t + 11r + 5.5.
\]

Note that the fraction appears because this is just a bound, but since the number of multiplications is an integer, it is bounded by

\[
M'_A := 11t + 11r + 5
\]

in characteristic 2.

Next, we would like to make a similar calculation for Wu’s polynomial update algorithm, [27, Alg. 1]. We note that the complexities of Cases 3–8 in Step 3 of [27, Alg. 1] are similar, and we will assume any one of these cases, in analogy to the above assumption that no discrepancy is zero in our algorithm. To make concrete statements, we will focus on Case 3 as a representative for all of these cases. Similarly to the above, we let \(\partial\) be an upper bound on the degree for all involved polynomials (before update) on an edge connecting a vertex at depth \(r - 1\) to a vertex at depth \(r\). While an exact account of the polynomial degrees in Wu’s algorithm is outside the scope of the current paper, it seems reasonable to assume that we may take \(\partial = t + r - 1\) for Wu’s algorithm.

- **Direct Evaluation in Step 2**: There are two evaluations of polynomials of degree up to \(\partial\) (in the calculations of \(\Lambda_i\) and \(B_i\)), plus 3 additional multiplications: one for calculating \(y_i\alpha_i\), and two for multiplying evaluation results by \(\alpha_iy_i\), resulting in a total of \(2\partial + 3\) multiplications.

- **Evaluation by the recursions [27, Eq. (23),(24)] in Step 2**: It seems that these recursions for calculating \(\Omega_i\), \(\Theta_i\) serve the same purpose as a calculation explained above for Algorithm A: to calculate
We will therefore assume that each of these calculations requires \(2\partial+1\) multiplications, for a total of \(2(2\partial+1) = 4\partial+2\) multiplications.

- **Additional multiplications in Step 2:** In the two last lines of Step 2, there are 5 additional multiplications.

- **Multiplication of a polynomial by a constant in Case 3 of Step 3:** There are up to \(3(\partial+1)\) multiplications coming from multiplying a polynomial of degree up to \(\partial\) by a scalar, plus 3 additional multiplications (calculating \(\alpha_i^{-1}\Psi_i\), and multiplying its inverse by two constants) for calculating the relevant scalars, resulting in \(3\partial+6\) multiplications.

- **Syndrome update in Step 1:** There are \(t+r\) syndrome entries to update, each requiring a single multiplication.

Summing up, we obtain that the total number of multiplications in Wu’s algorithm on an edge between depth \(r-1\) and depth \(r\) is at most

\[
M_{Wu} := 2\partial + 3 + 4\partial + 2 + 5 + 3\partial + 6 + t + r \\
= 9\partial + 16 + t + r \\
= 9(t + r - 1) + 16 + t + r = 10t + 10r + 7.
\]

multiplications. Comparing \(M_{Wu}\) to \(M_A\) in the general case and to \(M’_A\) in the case of characteristic 2, we see that Wu’s algorithm has a somewhat lower complexity, by a factor of about \(5/6\) in general, or \(10/11\) in characteristic 2. However, in Section B.3, we will present yet an additional variant of Algorithm A (namely, Algorithm C), that has a lower complexity than Wu’s algorithm.

It should be noted that the above complexity comparison does not account for exhaustive root searches, as it is reasonable to assume that the probability of falsely meeting the stopping criterion of Section B.2 ahead is similar to the corresponding probability for Wu’s stopping criterion.

**B.2 A heuristic stopping criterion**

To reduce the number of required exhaustive root searches for the ELP in Algorithm A, it is useful to introduce a heuristic stopping criterion, which determines whether or not an exhaustive root search is required. Such a stopping criterion must never miss the correct ELP, but is allowed to falsely trigger an exhaustive root search with a low probability.

In [27, Sec. V], Wu introduced such a heuristic criterion for his algorithm, based on an LFSR-length variable. For Algorithm A, it is possible to obtain
a similar criterion based on the discrepancy $\Delta_1$. Using the terminology of Section 2.3 suppose that the total number of errors is $t + r$, and that there are $r + 1$ errors on $I$, for some $r < r_{\text{max}}$. Then by Theorem 3.2, the correct EEP $\omega$ and ELP $\sigma$ will appear (up to a multiplicative scaler) as the pair $g_1^+$ both for some vertex $v$ at depth $r + 1$ and for its parent $u$ at depth $r$.

Moreover, on the edge connecting $u$ to $v$, we must have $\Delta_1 = 0$, both for the root iteration and for the derivative iteration, by Forney’s formula (4). Hence, demanding that $\Delta_1 = 0$ for both the root iteration and the derivative iteration will never miss the true ELP under the above assumptions.

Special care should be taken for the case considered in Appendix A, as one can verify that $\Delta_1 = 0$ twice also if a correct error location, $\alpha_{r+1}$, is encountered after the condition of Proposition A.2 holds (we omit the proof). While Proposition A.2 can be used to restore the correct ELP also in such a case, this is outside the scope of the current paper. Here, we will only specify a method to avoid a useless exhaustive evaluation in these cases.

Observing (8), we see that for the case considered in Proposition A.2, the estimated ELP and its derivative have at least one common root in $\{\alpha_1, \ldots, \alpha_r\}$. Hence, to avoid an unnecessary exhaustive root search in such a case, one can first evaluate the estimated ELP and its derivative on $\{\alpha_1, \ldots, \alpha_r\}$, and then check that there are no common roots. Note that in case of a direct hit, this condition does hold, as the ELP is separable.

To conclude, the stopping criterion now has the following form:

1. $\Delta_1 = 0$ for both the root and derivative iterations on the edge connecting $u$ and $v$, and

2. The estimated ELP, $g_{11}(X)$, and its derivative have no common roots on the $r$ locations corresponding to the vertex $u$.

Note that Condition 2 should be checked only if Condition 1 holds, and hence rarely. In other cases that there is no need to perform exhaustive evaluation, it seems reasonable to heuristically assume that the probability that $\Delta_1 = 0$ for both the root and the derivative iterations is about $1/q^2$, and hence small.

B.3 Working with low-degree polynomials

In this section, we will show that by using an appropriate transformation, one can work with polynomials whose degrees typically grow from 0 to $2r_{\text{max}}$, instead of typically growing from $t$ to $t + r_{\text{max}}$ (see ahead for a detailed complexity comparison with [27, Alg. 1]).

\footnote{We thank A. Dor for pointing this out.}
Until this point, we have used only the monomial ordering $<_1$ of \[9\]. In this section, we will use the general definition of Fitzpatrick’s monomial ordering, which we shall now recall. Define a monomial ordering $<_w$ on $\mathbb{F}_q[X]^2$ as follows: $(X^{j_1}, 0) <_w (X^{j_2}, 0)$ iff $j_1 < j_2$, $(0, X^{j_1}) <_w (0, X^{j_2})$ iff $j_1 < j_2$, and $(X^{j_1}, 0) <_w (0, X^{j_2})$ iff $j_1 \leq j_2 + w$. Note again that this is a monomial ordering even when $w$ is not positive. We will write $\text{LM}_w(u, v)$ for the leading monomial of $(u, v)$ with respect to $<_w$.

Let $\{h_0 = (h_{00}, h_{01}), h_1 = (h_{10}, h_{11})\}$ be a Gröbner basis for $M_0$ with respect to the monomial ordering $<_1$ such that the leading monomial of $h_0$ is on the left, while the leading monomial of $h_1$ is on the right. Since $\{h_0, h_1\}$ is also a free-module basis, every element $(u, v) \in M_0$ can be written as $(u, v) = f_0(X)h_0 + f_1(X)h_1$ for a unique pair $(f_0(X), f_1(X))$, and the map

$$
\mu: M_0 \longrightarrow \mathbb{F}_q[X]^2
$$

$$(u, v) \longmapsto (f_0, f_1)
$$

is an isomorphism of $\mathbb{F}_q[X]$-modules.

Note that for all $r$, $M_r \subseteq M_0$ is a submodule, and let

$$N_r = N_r(S^{(w)}, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r) := \mu(M_r(S^{(w)}, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r))$$

be the $\mu$-image of $M_r$ in $\mathbb{F}_q[X]^2$. For obvious reasons, we call $N_r$ the **module of coefficient polynomials** of $M_r$. Then by writing a typical element $(u, v) \in M_r$ as $(u, v) = f_0(X)h_0 + f_1(X)h_1$ and substituting in the constraints in the definition of $M_r$, we immediately obtain the following characterization of $N_r$.

**Proposition B.5.** It holds that $N_r$ is the set of all pairs $(f_0, f_1) \in \mathbb{F}_q[X]^2$ that satisfy the following condition:

- $\forall i \in \{1, \ldots, r\}$,
  - $b_{0i}f_0(\alpha_i^{-1}) + b_{1i}f_1(\alpha_i^{-1}) = 0$
  - $b_{0i}f'_0(\alpha_i^{-1}) + c_{0i}f_0(\alpha_i^{-1}) + b_{1i}f'_1(\alpha_i^{-1}) + c_{1i}f_1(\alpha_i^{-1}) = 0$, where
    $$
    b_{0i} := h_{0i}(\alpha_i^{-1}), \quad b_{1i} := h_{1i}(\alpha_i^{-1}),
    $$
    $$
    c_{0i} := h'_{0i}(\alpha_i^{-1}) + \frac{\alpha_i}{\beta_i a_i}h_{00}(\alpha_i^{-1}),
    $$
    $$
    c_{1i} := h'_{1i}(\alpha_i^{-1}) + \frac{\alpha_i}{\beta_i a_i}h_{10}(\alpha_i^{-1}),
    $$
    $$
    (\text{with } a_i := \bar{a}_i \text{ for the } i' \text{ with } \alpha_i = \lambda').
    $$

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Proof. The conditions in the definition of $M_r$ (as a sub-module of $M_0$) translate to the following conditions, both for all $i$:

\[(f_0h_{01} + f_1h_{11})(\alpha_i^{-1}) = 0,\]

and

\[\beta_i a_i (f_0h_{01} + f_1h_{11})' (\alpha_i^{-1}) + \alpha_i (f_0h_{00} + f_1h_{10}) (\alpha_i^{-1}) = 0.\]

Dividing the second equation by $\beta_i a_i$, expanding, and re-arranging terms, the proposition follows.

Note that it follows immediately from Theorem 3.2 that for all $r$, $N_r$ is a module, since it is a homomorphic image of a module. It also follows that the “intermediate” module, obtained by intersecting $N_r$ only with the first constraint for $i = r + 1$, is a module, again, as a homomorphic image of a module.

To use the minimality assertion of Theorem 3.2 also for the coefficient-polynomial modules $N_r$, we have the following proposition.

**Proposition B.6.** Let $w := \deg(h_{11}) - \deg(h_{00}) - 1$. Then for all $(u_1, v_1), (u_2, v_2) \in M_0$, it holds that

\[\text{LM}_{-1}(u_1, v_1) <_{-1} \text{LM}_{-1}(u_2, v_2) \iff \text{LM}_w(\mu(u_1, v_1)) <_w \text{LM}_w(\mu(u_1, v_1))\]

Proof. For $(u, v) \in M_0$ write $(u, v) = f_0h_0 + f_1h_1$. With respect to $<_{-1}$, the leading monomial of $f_0h_0$ is $(X^{\deg(f_0)+\deg(h_{00})}, 0)$, while the leading monomial of $f_1h_1$ is $(0, X^{\deg(f_1)+\deg(h_{11})})$ (we have used the fact that the leading monomial of $h_0$ is on the left and the leading monomial of $h_1$ is on the right). Hence

\[\text{LM}_{-1}(u, v) = \begin{cases} (X^{\deg(f_0)+\deg(h_{00})}, 0) & \text{if } \deg(f_0) + \deg(h_{00}) \geq \deg(f_1) + \deg(h_{11}) \\ (0, X^{\deg(f_1)+\deg(h_{11})}) & \text{if } \deg(f_0) + \deg(h_{00}) \leq \deg(f_1) + \deg(h_{11}) - 1 \end{cases}\]

It follows that

\[\text{LM}_{-1}(u, v) = \begin{cases} (X^{\deg(f_0)+\deg(h_{00})}, 0) & \text{if } \deg(f_0) \geq \deg(f_1) + w + 1 \\ (0, X^{\deg(f_1)+\deg(h_{11})}) & \text{if } \deg(f_0) \leq \deg(f_1) + w. \end{cases}\]

We conclude that

\[\text{LM}_{-1}(u, v) = (X^{\deg(f_0)+\deg(h_{00})}, 0) \iff \text{LM}_w(f_0, f_1) = (X^{\deg(f_0)}, 0)\]
and

\[ \text{LM}_{\text{w}}(f_0, f_1) = (0, X^{\deg(f_1)}) \quad \leftrightarrow \quad \text{LM}_{\text{w}}(f_0, f_1) = (0, X^{\deg(f_1)}) \]  

(23)

and these are all the possible cases. The assertion now follows by considering four possible cases of whether the \(<_{-1}\)-leading monomial of \((u_1, v_1), (u_2, v_2)\) is on the left/right, and \((22), (23)\).

We therefore obtain the following corollary to Theorem 3.2:

**Corollary B.7.** Write \(\omega, \sigma = f_0 h_0 + f_1 h_1\). If \(d_H(y, x) \leq t + r\), \(\alpha_1, \ldots, \alpha_r\) are error locations and \(\beta_1, \ldots, \beta_r\) are the corresponding error values, then \((f_0, f_1) \in N_r\) and

\[ \text{LM}_{\text{w}}(f_0, f_1) = \min \{ \text{LM}_{\text{w}}(g_0, g_1) | (g_0, g_1) \in N_r \setminus \{0\} \}. \]

The corollary shows that we can work directly with coefficient polynomials, using Kötter’s iteration with respect to the monomial ordering \(<_w\). Also, the heuristic stopping criterion of Appendix B.2 works just as before, because the discrepancies in \(N_r\) are zero iff the corresponding discrepancies in \(M_r\) are zero.

The resulting application of Kötter’s iteration is listed below in Algorithm C. Note that the coefficients \(b_0r, b_1r, c_0r, c_1r\) appearing in the calculation of \(\Delta_j\) are defined in Proposition B.5. Note also that all required evaluations of \(h_{00}, h_{01}, h_{10}, h_{11}\) and their derivatives can be pre-computed once for all non-reliable coordinates.

**Algorithm C: Kötter’s iteration for coefficient vectors**

**Input**

- A Gröbner basis \(G = \{ f_0 = (f_{00}, f_{01}), f_1 = (f_{10}, f_{11}) \}\) for \(N_{r-1}(S(y), \alpha_1, \ldots, \alpha_{r-1}, \beta_1, \ldots, \beta_{r-1})\), with \(\text{LM}_{\text{w}}(f_j)\) containing the \(j\)-th unit vector for \(j \in \{0, 1\}\), where \(w := \deg(h_{11}) - \deg(h_{00}) - 1\)

- The next error location, \(\alpha_r\), and the corresponding error value, \(\beta_r\)

**Output**

A Gröbner basis \(G^+ = \{ f_0^+ = (f_{00}^+, f_{01}^+), f_1^+ = (f_{10}^+, f_{11}^+) \}\) for \(N_r(S(y), \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_r)\) with \(\text{LM}_{\text{w}}(f_j^+)\) containing the \(j\)-th unit vector for \(j \in \{0, 1\}\)

**Algorithm**

- For type = root, der
  - If type = der,
    * For \(j = 0, 1\), set \(f_j := f_j^+ /\) init: output of root iter. */
For $j = 0, 1$, calculate

$$
\Delta_j := \begin{cases} 
  b_{0r}f_{j0}(\alpha_r^{-1}) + b_{1r}f_{j1}(\alpha_r^{-1}) & \text{if type=\text{root}} \\
  b_{0r}f'_{j0}(\alpha_r^{-1}) + c_{0r}f_{j0}(\alpha_r^{-1}) + b_{1r}f'_{j1}(\alpha_r^{-1}) + c_{1r}f_{j1}(\alpha_r^{-1}) & \text{if type=\text{der}}
\end{cases}
$$

/* $b_{0r}, b_{1r}, c_{0r}, c_{1r}$ defined in Prop. [B.5] */

Set $J := \{j \in \{0, 1\} | \Delta_j \neq 0\}$

For $j \in \{0, 1\} \setminus J$, set $f_j^+ := f_j$

Let $j^* \in J$ be such that $\text{LM}_w(f_{j^*}) = \min_{j \in J} \{\text{LM}_w(f_j)\}$

For $j \in J$

* If $j \neq j^*$
  * Set $f_j^+ := f_j - \frac{\Delta_j}{\Delta_j^*}f_{j^*}$

* Else /* $j = j^*$ */
  * Set $f_j^+ := (X - \alpha_r^{-1})f_{j^*}$.

Remark B.8. The validity of the update $f_j^+ := (X - \alpha_r^{-1})f_{j^*}$ for both the root and derivative iterations can be proved as follows. First, it can be verified directly that for both the root and derivative iterations, $D(Xf_j) = \alpha_r^{-1}D(f_j)$ (as done above for Algorithm A), where $D$ is the linear functional of Kötter’s iteration for the respective iteration. A simpler way to prove this is as follows. In the definition of $N_r$, we have implicitly defined functionals (for the respective iterations) $D': \mathbb{F}_q[X]^2 \to \mathbb{F}_q$ by setting, for all $f \in \mathbb{F}_q[X]^2$, $D'(f) := D(\mu^{-1}(f))$ for $D$ of Subsection [4.1]. Hence

$$
D'(Xf) = D(\mu^{-1}(Xf)) = D(X\mu^{-1}(f)) = \alpha_r^{-1}D(\mu^{-1}(f)) = \alpha_r^{-1}D'(f),
$$

where the second equality follows from the $\mathbb{F}_q[X]$-linearity of $\mu^{-1}$, and the third equality follows from what we have already proved for Algorithm A (where we assume, as before, that the derivative iteration comes after the root iteration).

The version of fast Chase decoding using Algorithm C is initiated on the root of the tree $T$ with the Gröbner basis $\{(1, 0), (0, 1)\}$ for $N_0 = \mathbb{F}_q[X]^2$. When the heuristic stopping condition of Appendix [B.2] holds, one can perform exhaustive substitution in one of two ways, which we shall now describe.

For short, in the following we let $f = (f_0, f_1)$ be the pair with the minimum $<_w$-leading monomial from $\{f_0^+, f_1^+\}$ in the derivative iteration for adjoining $\alpha_r$. 

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1. Re-construct an estimated ELP (up to a non-zero multiplicative constant) as \( \hat{\sigma}(X) = f_0(X)h_{01}(X) + f_1(X)h_{11}(X) \) and evaluate.

- For the method for ruling out indirect hits of Appendix B.2, we can then readily calculate the derivative \( \hat{\sigma}'(X) \) and evaluate it.

2. Calculate and store in advance the evaluations \( \{h_{01}(z^{-1})\}_{z \in \mathbb{F}_q^*}, \{h_{11}(z^{-1})\}_{z \in \mathbb{F}_q^*} \).

Now only the low-degree polynomials \( f_0, f_1 \) need to be evaluated for calculating the evaluations \( \hat{\sigma}(z^{-1}) = f_0(z^{-1})h_{01}(z^{-1}) + f_1(z^{-1})h_{11}(z^{-1}) \) for all \( z \in \mathbb{F}_q^* \).

- For the method for ruling out indirect hits of Appendix B.2, we can also calculate and store in advance the evaluations of derivatives \( \{h'_{01}(z^{-1})\}_{z \in \mathbb{F}_q^*}, \{h'_{11}(z^{-1})\}_{z \in \mathbb{F}_q^*} \), and then calculate

\[
\hat{\sigma}'(z^{-1}) = f'_0(z^{-1})h_{01}(z^{-1}) + f_0(z^{-1})h'_{01}(z^{-1}) + f'_1(z^{-1})h_{11}(z^{-1}) + f_1(z^{-1})h'_{11}(z^{-1}).
\]

To bound the complexity of Algorithm C, we will need the following proposition, in which we shall use a notation similar to that of Definition B.2 for Algorithm C instead of Algorithm A, where "g" is replaced by "f" throughout.

**Proposition B.9.** When Algorithm C is applied on an edge connecting a vertex at depth \( r - 1 \) to an edge at depth \( r \) (\( r \geq 1 \)), we have

\[
\deg(f^+_{00}(r; \tau)) + \deg(f^+_{11}(r; \tau)) \leq \begin{cases} 2r - 1 & \text{if } \tau = \text{root} \\ 2r & \text{if } \tau = \text{der}, \end{cases}
\]

and

\[
\max\{\deg(f^+_{01}(r; \tau)), 0\} + \max\{\deg(f^+_{10}(r; \tau)), 0\} \leq \begin{cases} 2r - 2 & \text{if } \tau = \text{root} \\ 2r - 1 & \text{if } \tau = \text{der}. \end{cases}
\]

**Remark.** Note that the usage of \( \max\{\deg(\cdot), 0\} \) means that we sum only over the degrees of non-zero polynomials.

**Proof.** Recalling that the algorithm is initiated with the Gröbner basis \( \{(1, 0), (0, 1)\} \) on the root of the decoding tree, (24) follows by induction, as in each root and derivative iteration at most one leading monomial is increased, and the increased leading monomial is multiplied by \( X \).
Since $\text{LM}_w(f_0^+)\text{ is on the left and }\text{LM}_w(f_1^+)$ is on the right, we have

$$\deg(f_{01}(r; \tau)) < \deg(f_{00}(r; \tau)) - w$$

and

$$\deg(f_{10}(r; \tau)) \leq \deg(f_{11}(r; \tau)) + w.$$ 

Summing the last two inequalities (and using (24)) proves (25) for the case where the two involved polynomials are non-zero. Also, if both involved polynomials are zero, then there is nothing to prove. It therefore remains to consider the case where one of the polynomials is zero and the other is non-zero.

For this case, we will prove that for all $r \geq 1$,

$$\deg(f_{01}(r; \tau)) \leq \begin{cases} 2r - 2 & \text{if } \tau = \text{root} \\ 2r - 1 & \text{if } \tau = \text{der} \end{cases}$$

(26)

(a similar proof works also for $f_{10}(r; \tau)$). For $r = 1$, (26) can be verified directly by checking 4 options of $j^*$ in the root and derivative iterations.

Assume by induction that $r \geq 2$ and that (26) holds for $r - 1$. For $\tau = \text{root}$, there are 3 options to consider: If $\Delta_0 = 0$ (no update), then (26) obviously holds for $\tau = \text{root}$ and $r$. If $\Delta_0 \neq 0$ and $j^* = 0$, then $\deg(f_{01}(r; \text{root})) = \deg(f_{01}(r-1; \text{der}))+1$, and again (26) holds for $\tau = \text{root}$ and $r$. Finally, if $\Delta_0 \neq 0$ and $j^* = 1$, then considering the update rule in this case and (24) for $r - 1$ and $\tau = \text{der}$, it follows again that (26) holds for $\tau = \text{root}$ and $r$. So, the induction hypothesis implies (26) for $r$ and $\tau = \text{root}$. Applying the same arguments again, it can be shown that (26) holds also for $r$ and $\tau = \text{der}$.

Let us now proceed to bounding the complexity of Algorithm C on an edge connecting a vertex at depth $r - 1$ to a vertex at depth $r$ (for $r \geq 1$).

- **In the root iteration:**
  - **Evaluation:** Running over $j = 0, 1$ we eventually have to evaluate once each of $f_{00}(r - 1; \text{der})$, $f_{11}^+(r - 1; \text{der})$, $f_{01}(r - 1; \text{der})$, and $f_{10}(r - 1; \text{der})$. Using Proposition 13.9, the required number of multiplications is at most $2(r - 1) + 2(r - 1) - 1 = 4r - 5$. Also, for each of $j = 0, 1$, we have 2 multiplications by a scalar (an overall of 4 such multiplications), for a total of $4r - 1$ multiplications.
- **Multiplication of a polynomial by a constant**: For $j \neq j^*$, we have to calculate the constant $\Delta_j/\Delta_{j^*}$, which requires a single multiplication (assuming, as before, that we have a table for calculating inverses), and to multiply 4 polynomials whose sum of degrees is at most $4r - 5$ by a constant (recall Footnote 25). This requires a total of $1 + 4r - 5 + 4 = 4r$ multiplications (the “+4” accounts for the fact that a polynomial of degree $d$ has $d + 1$ coefficients).

- **In the derivative iteration**:
  
  - **Evaluation**: Running over $j = 0, 1$, we eventually have to evaluate once each of $f_{00}^j(r; \text{root})$, $f_{11}^j(r; \text{root})$, $f_{01}^j(r; \text{root})$, $f_{10}^j(r; \text{root})$, and their derivatives. Taking the worst case assumption of characteristic $\neq 2$ and using Proposition B.9 again, this requires at most 
    \[(2r - 1) + (2r - 2) + (2r - 1 - 2) + (2r - 2 - 2) = 8r - 10\]
  multiplications. There are also 4 multiplications for calculating $c_{0r}$, $c_{1r}$, and 8 additional multiplications after the substitutions, giving a total of at most $8r + 2$ multiplications.
  
  - **Multiplication of a polynomial by a constant**: Similarly to the root iteration (but now with $r$ instead of $r - 1$ and \text{root} instead of \text{der} in the bound of Proposition B.9), this gives a total of at most $1 + (2r - 1) + (2r - 2) + 4 = 4r + 2$ multiplications.

Summing all the above bounds, we obtain that the total number of multiplications for moving from depth $r - 1$ to depth $r$ with algorithm $C$ is at most

\[M_C = 4r - 1 + 4r + 8r + 2 + 4r + 2 = 20r + 3.\]

Comparing this with $M_{\text{Wu}} = 10(t + r) + 7$ calculated in Subsection B.1, we see that the complexity of Algorithm $C$ is lower for each $r \leq r_{\text{max}}$ when $r_{\text{max}} < t$, since $2r < t + r$. Note that as before, this complexity calculation does not account for the (heuristically, rare) unrequired exhaustive root searches.

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