Universal quasi-equilibrium state of self-gravitating systems not depending on dimensions

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Abstract. We study a universality of self-gravitating systems in the quasi-equilibrium state. It is shown numerically that the two dimensional self-gravitating system in the quasi-equilibrium state has the same kind of density profile as the three dimensional one. We develop a model to describe this universal quasi-equilibrium state by using a special Langevin equation with a distinctive random noise to self-gravitating systems. We find that the density profile derived theoretically is consistent well with results of observations and simulations.

1. Introduction
There are many astronomical objects in which a number of elements are organized mainly by gravity. They are called self-gravitating systems (SGSs). It is known that, before the thermal equilibrium state, there is another stable state for SGS which we call quasi-equilibrium state (QES) [1]. Recent observations suggest that a density profile of SGS in QES has a universality not depending on dimensions. Globular clusters consisting of about hundreds of thousands of stars, a good instance of three dimensional SGS, are well-known to have a spherically symmetric number density. We shall point to a filamentary structure of molecular clouds as an example of two dimensional SGS. The Herschel Space Observatory have revealed the 27 filamentary structures in IC 5146 which is a reflection nebula in Cygnus and the number density of molecular clouds have a cylindrical symmetry around the axis of filament [2]. Therefore, the self-gravity generated by molecular clouds can be regarded as the two dimensional SGS: By using the mass density of molecular clouds $\rho_{mc}(r)$ and the gravitational potential per unit mass $\phi_{mc}(r)$ where $r$ means a distance from the axis, the Poisson equation becomes

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi_{mc}(r)}{dr} \right) = 4\pi G \rho_{mc}(r),$$

which is formally equivalent to the Poisson equation of two dimensional SGS as discussed later. These SGSs of different dimensions have the following universal density profile in QES: The number density is constant around the center of SGS and decrease in a power law at large radius. Thus, the number density $N(r)$ can be approximated well by

$$N(r) \simeq \frac{N(0)}{\left(1 + r^2/a^2\right)^\kappa}$$

(2)
where $r$ is the distance from the center of SGS and $a$ is called a core radius. With regard to molecular clouds and globular clusters, $0.75 < \kappa < 1.25$ [2] and $\kappa$ is around $3/2$ [1], respectively. Therefore, a new theory explaining the physical mechanism behind the universality is necessary. The author has developed a model to unveil the mechanism in three dimensional SGS [3, 4]. So, the aim of this paper is to describe QES universally by extending the dimension to $\mu$ which is 2 or 3.

2. Two dimensional gravity and N-body simulations of two dimensional SGS

2.1. Two dimensional gravity

The two dimensional gravitational potential per unit mass $\phi$ generated by mass source $\rho$ distributing circular-symmetrically satisfies the following Poisson equation in the two dimensional space.

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d\phi(r)}{dr} \right\} = 4\pi G' \rho(r). \quad (3)$$

This Poisson equation is formally equivalent to one governing molecular clouds which are distributed axisymmetrically.

When the mass source is a mass point with a mass $m$ existing in the origin, $\rho(r) = m\delta(0)/\pi r$. Then, the Poisson equation is altered to

$$\frac{1}{r} \frac{d}{dr} \left\{ r \frac{d\phi(r)}{dr} \right\} = \frac{4mG'\delta(0)}{r}. \quad (4)$$

Therefore, the gravity per unit mass from the mass point in the two dimensional space is unveiled to be

$$-\phi'(r) = - \frac{2G'm}{r}. \quad (5)$$

2.2. N-body simulations

In order to investigate QES of two dimensional SGS, we followed the time evolution of $10^4$-body system in the two dimensional space interacting by Eq. (5) numerically. A polytrope solution with a polytrope index $n$ was adopted as the initial condition. Generally, the solution in the three dimensional space is well-known [1]. Here, we have extended it to the two dimensional space: The distribution function $f(r, v)$ in phase space can be shown as

$$f(r, v) \propto \begin{cases} \mathcal{E}^{n-1} & (\mathcal{E} < 0) \\ 0 & (\mathcal{E} \geq 0) \end{cases} \left( \mathcal{E} \equiv \phi(R) - \left\{ \frac{1}{2} v^2 + \phi(r) \right\} \right) \quad (6)$$

where $R$ means a radius of the system. Then, we run the N-body simulation by varying an initial virial ratio $V$ which is one of the kinetic energy to the gravity potential of the system [5]. Specifically, when $V = 0$, it is found that the number density in QES has a universality and the density around the center of the system can be fitted well by Eq. (2). Figure 1 denotes that the best-fit $\kappa$ which is the index in Eq. (2) ranges from 0.8 to 1.1. The results have a good consistency with the observations of molecular clouds [2].

3. Fokker-Planck model

In the previous section, the number density of the two dimensional SGS in QES can be represented well by Eq. (2) with $0.8 < \kappa < 1.1$. On the other hand, the number density of most globular clusters, a good example of the three dimensional SGS, is well-known to be fitted by Eq. (2) with $\kappa \approx 3/2$ (see Refs. in [3]). Furthermore, we have reported that the same density
profile is obtained through N-body simulations of the three dimensional SGS [3, 4]. Therefore, we can conclude that the SGS has the universal density profile not depending on dimensions.

Here, let us describe this density profile universally by a special Fokker-Planck equation in the $\mu$ dimensional space. Before constructing the Fokker-Planck equation, we shall model forces influencing an element of the system. That is, we shall begin by constructing a Langevin equation.

We assume that the frictional force $-m\gamma \dot{r}^{\mu}(t)$ and the random noise with constant intensity $\sqrt{2D\xi^{\mu}(t)}$ which are essential for a many-body system to reach the thermal equilibrium state act on the element, where $m$ is a mass of the element, $r^{\mu}$ is a position vector, and $\xi^{\mu}(t)$ means Gaussian-white noises. The index represents that the vector is in the $\mu$ dimensional space. The element is also influenced by a mean gravitational force $-F^{\mu}(r)$ along the radial direction of the system which is derived by differentiating $m\varphi^{\mu}(r)$:

$$-F^{\mu}(r) = -m\partial_{r}^{\mu}\varphi^{\mu}(r)$$

where $\varphi^{\mu}(r)$ is the mean gravitational potential in $\mu$ dimensional space. However, this is just a mean gravity. It is natural to consider that the element actually is influenced by a fluctuating gravity around the mean value. This means that another noise which prevents the system from reaching the thermal equilibrium state simply is added to the ordinary Langevin equation, and so this system goes to another stable state, that is, QES. Therefore, we can consider the noise distinctive to SGS.

If assuming the intensity of the noise to be constant, we can obtain the following Langevin equation in the over-damped limit:

$$m\gamma \dot{r}^{\mu}(t) = -F^{\mu}(r) \left\{ 1 + \sqrt{2\epsilon}\eta(t) \right\} e^{\mu}_{r} - \frac{\epsilon}{2m\gamma} F^{\mu}(r)^{2} e^{\mu}_{r} + \sqrt{2D\xi^{\mu}(t)} ,$$

where $e^{\mu}_{r}$ is a unit vector along the radial direction in $\mu$ dimensional space. The second term on the right hand side of the above equation is a correction term in order to regard products as the Storatonovich product [6].

The corresponding Fokker-Planck equation is given by

$$\frac{\partial}{\partial t} P^{\mu}(r, t) = \frac{D}{(m\gamma)^2} \left\{ \frac{\partial^2}{\partial r^2} + \frac{\mu - 1}{r} \frac{\partial}{\partial r} \right\} P^{\mu}(r, t) + \frac{1}{m\gamma} \frac{1}{r} F^{\mu}(r) P^{\mu}(r, t)
+ \frac{\epsilon}{(m\gamma)^2} \left\{ \frac{\partial^2}{\partial r^2} + \frac{\mu - 1}{r} \frac{\partial}{\partial r} + \frac{\mu - 1}{r^{\mu-1}} \frac{\partial}{\partial r} r^{\mu-2} \right\} F^{\mu}(r)^{2} P^{\mu}(r, t) .$$

We shall treat the system as a circular ($\mu = 2$) or a spherical symmetric one ($\mu = 3$) including $N$ elements. So, PDF $P^{\mu}$ is a function of the distance from the origin $r \equiv |r^{\mu}|$ where the index $\mu$ are abbreviated for simplify. Note that a relation among PDF $P^{\mu}$, the number density $N^{\mu}$ and the mass density $\rho^{\mu}$ is as follows: $P^{\mu} = N^{\mu}/N = \rho^{\mu}/(mN)$.

![Figure 1.](image-url) (color online) The optimum values of $\kappa$ in Eq. (2) for fitting the number densities vs the initial polytope index $n$.  

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Let us use $P^\mu(r,t) = J^\mu(r)P^\mu(r,t)$ instead of $P^\mu$, where $J^\mu$ means Jacobian determinant\(^1\).

$$\frac{\partial}{\partial t} P^\mu(r,t) = \frac{D}{(m\gamma)^2} \left\{ \frac{\partial^2}{\partial r^2} - \frac{\partial}{\partial r} \frac{\mu - 1}{r} \right\} P^\mu(r,t) + \frac{\epsilon}{(m\gamma)^2} \frac{\partial^2}{\partial r^2} F^\mu(r)P^\mu(r,t) + \frac{1}{m\gamma} \frac{\partial}{\partial r} F^\mu(r)P^\mu(r,t)$$

When the system reaches QES, $\partial_t P^\mu_{qe}(r) = 0$. Here, we integrate the Fokker-Planck equation by $r$. Owing to use $P^\mu_{qe}$, the integration becomes easier:

$$\left\{ \frac{D}{(m\gamma)^2} + \frac{\epsilon F^\mu(r)^2}{(m\gamma)^2} \right\} P^\mu_{qe}(r) - \left\{ \frac{D}{(m\gamma)^2} \frac{\mu - 1}{r} - 2\epsilon F^\mu(r)F^\mu(r) + \frac{F^\mu(r)}{m\gamma} \right\} P^\mu_{qe}(r) = \text{const.}$$

Since $P^\mu_{qe}(0)$ and $P^\mu_{qe}(0)$ are bounded, $P^\mu_{qe}(0) = P^\mu_{qe}'(0) = 0$. Therefore, the constant of the right hand side of Eq. (10) becomes 0:

$$P^\mu_{qe}(r) = -\frac{r F^\mu(r) \left\{ m\gamma + 2\epsilon F^\mu(r) \right\} - (\mu - 1)D}{r \left\{ D + \epsilon F^\mu(r)^2 \right\}} P^\mu_{qe}(r).$$

If using the number density in QES $N^\mu_{qe} = (\mathcal{N}^\mu_{qe} / J^\mu)$, we can obtain

$$N^\mu_{qe}(r) = -\frac{r F^\mu(r) \left\{ m\gamma + 2\epsilon F^\mu(r) \right\} + (\mu - 1)D}{r \left\{ D + \epsilon F^\mu(r)^2 \right\}} N^\mu_{qe}(r).$$

By substituting the gravitational potential per unit mass $\phi^\mu(r) = \frac{1}{m} \int dr F^\mu(r)$ into the Poisson equation $\triangle \phi^\mu = 4\pi G^\mu \rho^\mu_{qe}$, an equation governing $F^\mu$ can be obtained as follows:

$$F^\mu(r) + \frac{\mu - 1}{r} F^\mu(r) = 4\pi G^\mu m(N^\mu_{qe}(r)) = 4\pi G^\mu m^2 N^\mu_{qe}(r),$$

where $\rho^\mu_{qe} = mN^\mu_{qe} = mNP^\mu_{qe} = mNP^\mu / J^\mu$ and $G^\mu$ relates quantities in the previous sections as $G^2 = G'$ and $G^3 = G$.

Here, we nondimensionalize these equations by using the following units of length and force:

$[\text{length}] = \sqrt{\frac{2(\mu + 2)\mu D}{4\pi G^\mu m^2 N^\mu_{qe}(0)}} \{ \mu^2 r^2 + 8\pi(\mu^2 + 4\mu + 2) \}, [\text{force}] = \sqrt{\frac{8\pi(\mu^2 + 2)DG^\mu m^2 N^\mu_{qe}(0)}{\mu^2 r^2 + 8\pi(\mu^2 + 4\mu + 2)}} \{ \mu^2 r^2 + 8\pi(\mu^2 + 4\mu + 2) \}.$

Then, equations (12) and (13) are altered to

$$N^\mu_{qe}(\bar{r}) = -\frac{2(\mu + 2)\bar{r} F^\mu(\bar{r}) \left\{ 1 + 2q F^\mu(\bar{r}) \right\} + 2(\mu - 1)(\mu + 1)q F^\mu(\bar{r})^2}{\bar{r} \left\{ \mu + 2(\mu^2 + 4\mu + 2)q + 2(\mu + 2)q F^\mu(\bar{r})^2 \right\}} N^\mu_{qe}(\bar{r}),$$

and

$$F^\mu(\bar{r}) + \frac{\mu - 1}{\bar{r}} F^\mu(\bar{r}) = \mu N^\mu_{qe}(\bar{r}),$$

where $q = 4\pi\epsilon G^\mu m^2 N^\mu_{qe}(0) / (\mu m\gamma)$ and variables with overbars denote dimensionless. We should solve these equations with boundary condition $N^\mu_{qe}(0) = 1$.

4. Results

The numerical solutions for $\mu = 2$ and 3 are shown in Fig. 2 by changing $q$. For comparison with observations, $1/(1 + r^2)$ and $1/(1 + r^2)^{3/2}$ are also plotted by a dashed curve in Fig. 2 (a) and (b), respectively. The dashed curves are typical best-fit ones for densities of molecular clouds or globular clusters. We can see that the best-fit curves are derived from our model by varying $q$ appropriately.

\(^1\) For $\mu = 2$ and 3, $J^2 = 2\pi r$ and $J^3 = 4\pi r^2$, respectively: We can depict it by $J^\mu = 2^{\mu - 1}\pi r^{\mu - 1}$. 

Figure 2. (color online) Numerical solutions of Eqs. (14) and (15) for $\mu = 2$ (a) and $\mu = 3$ (b). As the curve changes from the left to the right in (a), $q$ gets larger from 0 to 0.24 in steps of 0.04. On the other hand, $q$ gets larger from 0 to 0.12 in steps of 0.02 in (b). The (red) dashed curve in (a) and (b) means $1/(1 + \bar{r}^2)$ and $1/(1 + \bar{r}^2)^{3/2}$, respectively.

Finally, we examine the range of the index $\kappa$. Owing to adopt Eqs. (12) and (13) as the units of length and force, the number density around the center of the system in QES can be depicted by

$$\bar{N}^\mu(\bar{r}) \simeq \frac{1}{(1 + \bar{r}^2)\kappa(q)},$$

(16)

where $\kappa$ is a function of $q$:

$$\kappa(q) = \frac{(\mu + 2)\{1 + (\mu + 1)q\}}{\mu + 2(\mu^2 + 4\mu + 2)q}.$$  

(17)

From this equation, because of $q \geq 0$, one can see the range of $\kappa$ as follows:

$$\frac{(\mu + 1)(\mu + 2)}{2(\mu^2 + 4\mu + 2)} < \kappa \leq \frac{\mu + 2}{\mu}.$$  

(18)

Therefore, with regard to $\mu = 2$ and $\mu = 3$, $\frac{3}{7} < \kappa \leq 2$ and $\frac{10}{23} < \kappa \leq \frac{5}{3}$, respectively. Both ranges include the observed indexes.

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