Near \textbf{NP}-Completeness for Detecting $p$-adic Rational Roots in One Variable

Martin Avendaño
TAMU 3368
Mathematics Dept.
College Station, TX 77843-3368, USA
mavendar@yahoo.com.ar

Ashraf Ibrahim*
TAMU 3368
Mathematics Dept.
College Station, TX 77843-3368, USA
albrahim@math.tamu.edu

J. Maurice Rojas*
TAMU 3368
Mathematics Dept.
College Station, TX 77843-3368, USA
rojas@math.tamu.edu

Korben Rusek*
TAMU 3368
Mathematics Dept.
College Station, TX 77843-3368, USA
korben@rusek.org

\textbf{ABSTRACT}

We show that deciding whether a sparse univariate polynomial has a $p$-adic rational root can be done in \textbf{NP} for most inputs. We also prove a polynomial-time upper bound for trinomials with suitably generic $p$-adic Newton polygon. We thus improve the best previous complexity upper bound of \textbf{EXPTIME}. We also prove an unconditional complexity lower bound of \textbf{NP}-hardness with respect to randomized reductions for general univariate polynomials. The best previous lower bound assumed an unproved hypothesis on the distribution of primes in arithmetic progression. We also discuss how our results complement analogous results over the real numbers.

\section{1. INTRODUCTION}

The fields $\mathbb{R}$ and $\mathbb{Q}_p$ (the reals and the $p$-adic rationals) bear more in common than just that they are both metric: increasingly, complexity results for one field have inspired and motivated analogous results in the other (see, e.g., [Coh69, DvdD88] and the pair of works [Kho91 and Roj04]). We continue this theme by transposing recent algorithmic results for sparse polynomials over the real numbers [BRS09] to the $p$-adic rationals, sharpening the underlying complexity bounds along the way (see Theorem 1.3 below).

More precisely, for any commutative ring $R$ with multiplicative identity, we let FEAS$_R$ — the $R$-feasibility problem (a.k.a. Hilbert’s Tenth Problem over $R$ [DLPvG00]) — denote the problem of deciding whether an input polynomial system $F \subseteq \bigcup_{k,n \in \mathbb{N}} [\mathbb{Z}[x_1, \ldots, x_n]]^k$ has a root in $R^n$. (The underlying input size is clarified in Definition 1.1 below.) Observe that FEAS$_R$, FEAS$_Q$, and \{FEAS$_p$\}$_p$ a prime power are central problems respectively in algorithmic real algebraic geometry, algorithmic number theory, and cryptography.

\footnote{Partially supported by NSF individual grant DMS-0915245 and NSF CAREER grant DMS-0349309. Rojas was also partially supported by Sandia National Laboratories.}

\section{1.1 The Ultrametric Side: Relevance and Results}

Algorithmic results over the $p$-adics are central in many computational areas: polynomial time factoring algorithms over $\mathbb{Q}[x_1]$, computational complexity [Roj02, Pap95], studying prime ideals in number fields [Coh94, Ch. 4 & 6], elliptic curve cryptography [Lau04], and the computation of zeta functions [CDV06]. Also, much work has gone into using $p$-adic methods to algorithmically detect rational points on algebraic plane curves via variations of the Hasse Principle ([see, e.g., C-T98] [Poo01b] [Poo06]).

However, our knowledge of the complexity of deciding the existence of solutions for \textbf{sparse} polynomial equations over $\mathbb{Q}_p$ is surprisingly coarse: good bounds for the number of solutions over $\mathbb{Q}_p$ in one variable weren’t even known until the late 1990s [Len99]. So we focus on precise complexity bounds for one variable.

\begin{definition}
Let $f(x) := \sum_{i=1}^{m} c_i x_i^{n_i} \in \mathbb{Z}[x_1, \ldots, x_n]$ where $x_i^{n_i} := x_1^{n_1} \cdots x_n^{n_n}$, $c_i \neq 0$ for all $i$, and the $a_i$ are pairwise distinct. We call such an $f$ an $n$-variate \textbf{m}-nomial.

Let us also define 

\begin{align*}
\text{size}(f) &:= \sum_{i=1}^{m} \log_2 \left( (2 + |c_i|)(2 + |a_{i1}|) \cdots (2 + |a_{i,n_i}|) \right) \\
\text{and, for any } F &:= (f_1, \ldots, f_k) \in (\mathbb{Z}[x_1, \ldots, x_n])^k, \text{ we}
\end{align*}

\end{definition}

1 If $F(x_1, \ldots, x_n) = 0$ is any polynomial equation and $\mathbb{Z}_K$ is its zero set in $\mathbb{K}^n$, then the Hasse Principle is the assumption that $|\mathbb{Z}_\mathbb{K}|$, $\mathbb{Z}_K \neq 0$, and $\mathbb{Q}_p \neq 0$ for all primes $p$ implies $\mathbb{Z}_p \neq 0$ as well. The Hasse Principle is a theorem when $\mathbb{Z}_\mathbb{K}$ is a quadric hypersurface or a curve of genus zero, but fails in subtle ways already for curves of genus one (see, e.g., [Poo01a]).
define \( \text{size}(F) := \sum_{i=1}^{\deg f} \text{size}(f_i) \). Finally, we let \( \mathcal{F}_{i,m} \) denote the subset of \( \mathbb{Z}[x_1, \ldots, x_n] \) consisting of polynomials with exactly \( m \) monomial terms.

For instance, \( \text{size}(1 + cx_1^{99} + x_1^4) = \Theta(\log(c) + \log(d)) \). So the degree, \( \deg f \), of a polynomial \( f \) can sometimes be exponential in its size. Note also that \( \mathbb{Z}[x_1] \) is the disjoint union \( \bigcup_{m \geq 0} \mathcal{F}_{1,m} \).

**Definition 1.2.** Let \( \text{FEAS}_{\text{primes}} \) denote the problem of deciding, for an input polynomial system \( F \in \bigcup_{k \in \mathbb{N}} \mathbb{Z}[x_1, \ldots, x_n]^k \) and an input prime \( p \), whether \( F \) has a root in \( \mathbb{Q}_p \). Also let \( \mathbb{P} \cap \mathbb{N} \) denote the set of primes and, when \( I \) is a family of such pairs \( (F, p) \), we let \( \text{FEAS}_{\text{primes}}(I) \) denote the restriction of \( \text{FEAS}_{\text{primes}} \) to inputs in \( I \). The underlying input sizes for \( \text{FEAS}_{\text{primes}} \) and \( \text{FEAS}_{\text{primes}}(I) \) shall be \( \text{size}(F) := \text{size}(F) + \log p \) (cf. Definition 1.1). Finally, let \( (\mathbb{Z} \times (\mathbb{N} \cup \{0\}))^\infty \) denote the set of all infinite sequences of pairs \( ((a_i, c_i))_{i=1}^\infty \) with \( c_i = a_i = 0 \) for \( i \) sufficiently large.

**Remark 1.3.** Note that \( \mathbb{Z}[x_1] \) admits a natural embedding into \( ((\mathbb{Z} \times (\mathbb{N} \cup \{0\}))^\infty) \) by considering coefficient-exponent pairs in order of increasing exponents, e.g., \( a + bx^{10} + x^{2001} \mapsto ((a, 0), (b, 99), (1, 2001), (0, 0), (0, 0), \ldots) \).

While there are now randomized algorithms for factoring \( f \in \mathbb{Z}[x_1] \) over \( \mathbb{Q}_p[x_1] \) with expected complexity polynomial in \( \text{size}_p(f) + \deg(f) \) (see also [Ch91]), no such algorithms are known to have complexity polynomial in \( \text{size}_p(f) \) alone. Our main theorem below shows that such algorithms are hard to find because their existence is essentially equivalent to the \( P = \text{NP} \) problem. Moreover, we obtain new sub-cases of \( \text{FEAS}_{\text{primes}}(\mathbb{Z}[x_1] \times \mathbb{P}) \) lying in \( P \).

**Theorem 1.4.**
1. \( \text{FEAS}_{\text{primes}}(\mathcal{F}_{1,k} \times \mathbb{P}) \in P \) for \( k \in \{0, 1, 2\} \).
2. For any \( f(x_1) = c_1 + c_3 x_1^2 + c_5 x_1^3 \in \mathbb{Z}[x_1] \) with the points \( \{(0, \text{ord}_p(c_1)), (a_2, \text{ord}_p(c_2)), (a_3, \text{ord}_p(c_3))\} \) non-collinear, and \( p \) not dividing \( a_2, a_3, \) or \( a_3 - a_2 \), we can decide the existence of a root in \( \mathbb{Q}_p \) for \( f \) in \( P \).
3. There is a countable union of algebraic hypersurfaces \( E \subseteq \mathbb{Z}[x_1] \times \mathbb{P} \), with natural density 0, such that \( \text{FEAS}_{\text{primes}}((\mathbb{Z}[x_1] \times \mathbb{P}) \setminus E) \in \text{NP} \). Furthermore, we can decide in \( P \) whether an \( f \in \mathcal{F}_{1,3} \) also lies in \( E \).
4. If \( \text{FEAS}_{\text{primes}}(\mathbb{Z}[x_1] \times \mathbb{P}) \in \text{ZPP} \), then \( \text{NP} \subseteq \text{ZPP} \).
5. If the Wagstaff Conjecture is true, then \( \text{FEAS}_{\text{primes}}(\mathbb{Z}[x_1]) \in P \implies P = \text{NP} \), i.e., we can strengthen Assertion (4) above.

**Remark 1.5.** The Wagstaff Conjecture, dating back to 1979 (see, e.g., [BS96], Conj. 8.5.10, pg. 224)), is the assertion that the least prime congruent to \( k \mod N \) is \( O(\varphi(N) \log^2 N) \), where \( \varphi(N) \) is the number of integers in \( \{1, \ldots, N\} \) relatively prime to \( N \). Such a bound is significantly stronger than the known implications of the Generalized Riemann Hypothesis (GRH).

While the real analogue of Assertion (1) is known (and easy), the stronger real analogue \( \text{FEAS}_s(\mathcal{F}_{1,3}) \in P \) to Assertion (2) was unknown until [BS96] Thm. 1.3. We hope to strengthen Assertion (2) to \( \text{FEAS}_{\text{primes}}(\mathcal{F}_{1,3} \times \mathbb{P}) \in P \) in future work. In fact, we can obtain polynomial complexity already for more inputs in \( \mathcal{F}_{1,3} \times \mathbb{P} \) than stated above, and this is clarified in Section 3.

Note that \( \mathbb{Q}_p \) is uncountable and thus, unlike \( \text{FEAS}_{\text{primes}} \), \( \mathbb{Q}_p \) does not admit an obvious succinct certificate. Indeed, while it has been known since the late 1990’s that \( \text{FEAS}_{\text{primes}}(\mathbb{Q}_p) \in \text{EXPTIME} \) relative to our notion of input size \( \text{MW96} \), we are unaware of any earlier algorithms yielding \( \text{FEAS}_{\text{primes}}(\mathbb{Z}[x_1, \ldots, x_n] \times \mathbb{P}) \in \text{NP} \) for any fixed \( n \).

**Example 1.6.** Let \( T \) denote the family of pairs \( (f, p) \in \mathbb{Z}[x_1] \times \mathbb{P} \) with \( f(x_1) = a + bx_1^4 + cx_1^{17} + x_1^3 \) and let \( T^* := T \setminus E \). Then there is a sparse \( \mathbb{Q}_p \times \mathbb{P} \) such that \( f \in T^* \) \( \iff \) \( f \not\in \mathbb{Q}_p \). By Theorem 1.4, \( \text{FEAS}_{\text{primes}}(T^*) \in \text{NP} \), and Corollary 1.7 tells us that for large coefficients, \( T^* \) occupies almost all of \( T \). In particular, letting \( \mathcal{T}(H) \) denote those pairs \( (f, p) \) for \( T \) (resp. \( T^* \)) with \( |a|, |b|, |c|, p \leq H \), we have \( \mathcal{T}(H) \cup \mathcal{T}(H)^c = \{1 \leq H\} \). For instance, one can check via Maple that \( (3^9 + 11^2 + 2x_1^3)^{-1} \in T^* \) for all but 352 primes \( p \).

The exceptions in Assertion (3) appear to be due to the presence of *ill-conditioned* polynomials: \( f \) having a root \( c \) with the \( (p, \text{adic}) \)-norm of \( f(c) \) very small — a phenomenon of approximation present in complete fields like \( \mathbb{R} \), \( \mathbb{C} \), and \( \mathbb{Q}_p \). Curiously, the real analogue of Assertion (3) remains unknown [BR90] Sec. 1.2.

As for lower bounds, while it is not hard to show that the full problem \( \text{FEAS}_{\text{primes}} \) is \( \text{NP} \)-hard from scratch, the least \( m \) making \( \text{FEAS}_{\text{primes}}(\mathbb{Z}[x_1, \ldots, x_n] \times \mathbb{P}) \) \( \text{NP} \)-hard appears not to have been known unconditionally. In particular, a weaker version of Assertion (4) was found recently, but only under the truth of an unproved hypothesis on the distribution of primes in arithmetic progressions [RojeTha, Main Thm.]. Assertion (4) thus also provides an interesting contrast to earlier work of W. H. Lenstra, Jr. and Lenstra, who showed that one can actually find all low degree factors of a sparse polynomial (over \( \mathbb{Q}[x_1] \) as opposed to \( \mathbb{Q}_p[x_1] \)) in polynomial time.

### 1.2 Random Primes and Tropical Tricks

The key to proving our lower bound results (Assertions 4 and 5 of Theorem 1.3) is an efficient reduction from a problem discovered to be \( \text{NP} \)-hard by David Alan Plaisted: deciding whether a sparse univariate polynomial vanishes at a complex \( \mathbb{C} \) root of unity [Pla84, RojeTha]. Reducing from this problem to its analogue over \( \mathbb{Q}_p \) is straightforward, provided \( \mathbb{Q}_p \) contains a cyclic subgroup of order \( D \) where \( D \) has sufficiently many distinct prime divisors. We thus need to consider the factorization of \( p-1 \), which in turn leads us to primes congruent to \( 1 \mod \) certain integers.

While efficiently constructing random primes in arbitrary arithmetic progressions remains a famous open problem, we can now at least efficiently build random primes \( p \) such that \( 2^k \).
p is moderately sized but p−1 has many prime factors. We use the notation [j]:={1,...,j} for any j∈N.

**Theorem 1.7.** For any δ > 0, a failure probability ε ∈ (0,1/2), and n ∈ N, we can find — within O(\((n/ε)^{\frac{1}{3}+δ}+(n\log(n)+\log 2)^{\frac{1}{2}+δ}\)) randomized bit operations — a sequence P = \(\{p_n\}_{n=1}^n\) of consecutive primes and a positive integer c such that
\[
\log(c)\log\left(\prod_{i=1}^n p_i\right) = \Theta(n\log(n)+\log(ε/c))
\]
and, with probability ≥ 1 − ε, the number \(p:=1+c\prod_{i=1}^n p_i\) is prime.

Theorem 1.7 and its proof are inspired in large part by an algorithm of von zur Gathen, Karpinski, and Szpankowski \[vzGK96\] Algorithm following Fact 4.9. In particular, they used an intricate random sampling technique \[vzGK96\] Thm. 4.10 to show, in our notation, that the enumerative analogue of FEAS\(_{\text{primes}}\)(\(\mathbb{Z}[x_1, x_2]\)) is \#P-hard \[vzGK96\] Thm. 4.11. Note in particular that neither of Theorem 4.10 of \[vzGK96\] or Theorem 1.7 above implies the other.

Our harder upper bound results (Assertions (2) and (3) of Theorem 1.4) will follow from an arithmetic analogue of toric deformations. Here, this simply means that we find ways to reduce problems involving general \(f ∈ \mathbb{Z}[x_1]\) to similar problems involving binomials. As a warm-up, let us recall that the convex hull of any subset \(S ⊆ \mathbb{R}^2\) is the smallest convex set containing \(S\). Also, an edge of a polygon \(P ⊂ \mathbb{R}^2\) is called lower iff it has an inner normal with positive last coordinate, and the lower hull of \(P\) is simply the union of all its lower edges.

**Lemma 1.8.** (See, e.g., \[Rob08\] Ch. 6, sec. 1.6.) Given any polynomial \(f(x_1):=\sum_{i=1}^m c_i x_1^i ∈ \mathbb{Z}[x_1]\), we define its p-adic Newton polygon, \(\text{Newt}_p(f)\), to be the convex hull of the points \(\{(a_i, \text{ord}_p c_i) | i ∈ \{1,...,m\}\}\). Then the number of roots of \(f\) in \(\mathbb{C}_p\) with valuation \(v\), counting multiplicities, is exactly the horizontal length of the lower face of \(\text{Newt}_p(f)\) with inner normal \((v, 1)\).

**Example 1.9.** For the polynomial \(f(x_1):=243x^6 - 3646x^5 + 18240x^4 - 35310x^3 + 29305x^2 - 8868x + 36\), the polygon \(\text{Newt}_3(f)\) can easily be verified to resemble the following illustration:

Note in particular that there are exactly 3 lower edges, and their respective horizontal lengths and inner normals are 2, 3, 1, and (1, 1), (0, 1), and (−5, 1). Lemma 1.8 then tells us that \(f\) has exactly 6 roots in \(\mathbb{C}_3\); 2 with 2-adic valuation 1, 3 with 3-adic valuation 0, and 1 with 3-adic valuation −5. Indeed, one can check that the roots of \(f\) are exactly 6, 1, and \(\frac{1}{243}\), with respective multiplicities 2, 3, and 1.

The binomial associated to summing the terms of \(f\) corresponding to the vertices of a lower edge of \(\text{Newt}_p(f)\) containing no other point of the form \((a, \text{ord}_p c_a)\) in its interior is called a lower binomial.

**Lemma 1.10.** Suppose \(f(x_1) = c_1 + c_2 x_1^2 + c_3 x_1^3 ∈ \mathbb{Z}[x]\), the points \(\{(0, \text{ord}_p(c_1)), (a_2, \text{ord}_p(c_2)), (a_3, \text{ord}_p(c_3))\}\) are non-collinear, and \(p\) is a prime not dividing \(a_2, a_3,\) or \(a_3 − a_2\). Then the number of roots of \(f\) in \(\mathbb{Q}_p\) is exactly \(\#\text{lower edges of the polygon} of the p-adic lower binomials of \(f\) in \(\mathbb{Q}_p\).

Our last lemma follows easily (taking direct limits) from a more general result \(\[Al09\] Thm. 4.5) relating the number of roots of \(f\) with the number of roots of its lower binomials over \(\mathbb{Z}/p^N\mathbb{Z}\) for \(N\) sufficiently large.

Our main results are proved in Section 3 after the development of some additional theory below.

### 2. BACKGROUND AND ANCILLARY RESULTS

Our lower bounds will follow from a common chain of reductions, so we will begin by reviewing the fundamental problem from which we reduce. We then show how to efficiently construct random primes \(p\) such that \(p−1\) has many prime factors in Section 2.2 and conclude with some quantitative results for transferring complexity results over \(\mathbb{C}\) to \(\mathbb{Q}_p\) in Section 2.3.

#### 2.1 Roots of Unity and NP-Completeness

Recall that any Boolean expression of one of the following forms:

\((\lor) y_i \lor y_j \lor y_k, \neg y_i \lor y_j \lor y_k, y_i \lor \neg y_j \lor y_k, \neg y_i \lor y_j \lor \neg y_k, \neg y_i \lor \neg y_j \lor y_k, \neg y_i \lor \neg y_j \lor \neg y_k, \) for \(i, j, k \in [3n]\),

is a 3CNFSAT clause. Let us first refine slightly Plaisted’s elegant reduction from 3CNFSAT to feasibility testing for univariate polynomial systems over the complex numbers \(\[Pla84\] Sec. 3, pp. 127–129].

**Definition 2.1.** Letting \(P := (p_1, ..., p_n)\) denote any strictly increasing sequence of primes, let us inductively define a semigroup homomorphism \(P \rightarrow \text{the Plaisted morphism with respect to} P \rightarrow \) from certain Boolean expressions in the variables \(y_1, ..., y_n\) to \(\mathbb{Z}[x]\), as follows

- For \(\phi = (\lor) y_i \lor y_j \lor y_k\), \(P(\phi) := 0\),
- For \(\phi = (\land) y_i \land y_j \land y_k\), \(P(\phi) := x_i^{P(y_i)/p_i−1}\),
- For \(\phi = (\neg) y_i \lor \neg y_i\), \(P(\phi) := (x_i^{P(y_i)} − 1)/P(\phi)\), for any Boolean expression \(B\) for which \(P(B)\) has already been defined,
- For \(\phi = (B_1 \lor B_2) := \text{lcm}(P(B_1), P(B_2))\), for any Boolean expressions \(B_1\) and \(B_2\) for which \(P(B_1)\) and \(P(B_2)\) have already been defined.

**Lemma 2.2.** \(\[Pla84\] Sec. 3, pp. 127–129]\) Suppose \(P = (p_i)_{i=1}^n\) is an increasing sequence of primes with \(\log(p_i) = O(k^n)\) for some constant \(k\). Then, for all \(n \in \mathbb{N}\) and any clause \(C\) of the form \((\lor)\), we have \(\text{size}(P(C)) = n\) in \(\mathbb{N}\). In particular, \(P\) can be evaluated at any such \(C\) in time polynomial in \(n\). Furthermore, if \(K\) is any field possessing \(D_P\) distinct \(P\)-roots of unity, then a 3CNFSAT instance \(B(y) := C_1(y) \land \ldots \land C_N(y)\) has a satisfying assignment iff the univariate polynomial system \(F_p := (P(C_1), ..., P(C_h))\) has a root \(ζ \in K\) satisfying \(ζ^{P(\phi)} = 1\).
2.2 Randomization to Avoid Riemann Hypotheses

The result below allows us to prove Theorem 1.7 and further tailor Plaisted’s clever reduction to our purposes. We let \( \pi(x) \) the number of primes \( \leq x \), and let \( \pi(x; M, 1) \) denote the number of primes \( \leq x \) that are congruent to 1 mod \( M \).

AGP Theorem. (very special case of [AGP94, Thm. 2.1, pg. 712]) There exist \( x_0 > 0 \) and an \( \ell \in \mathbb{N} \) such that for each \( x \geq x_0 \), there is a subset \( \mathcal{E}(x) \subset \mathbb{N} \) of finite cardinality \( \ell \) with the following property: If \( M \in \mathbb{N} \) satisfies \( M \leq x^{2/5} \) and a \( \ell M \) for all \( a \in \mathcal{E}(x) \) then \( \pi(x; M, 1) \geq \frac{\pi(x)}{\log x} \).

For those familiar with [AGP94, Thm. 2.1, pg. 712], the result above follows immediately upon specializing the parameters there as follows:

\[(A, \varepsilon, \delta, y, a) = (49/20, 1/2, 2/425, x, 1)\]

(see also [vzGKS96, Fact 4.9].)

The AGP Theorem enables us to construct random primes from certain arithmetic progressions with high probability. An additional ingredient that will prove useful is the famous recent AKS algorithm for deterministic polynomial-time primality checking [AKS02]. Consider now the following algorithm.

Algorithm 2.3.

Input: A constant \( \delta > 0 \), a failure probability \( \varepsilon \in (0, 1/2) \), a positive integer \( n \), and the constants \( x_0 \) and \( \ell \) from the AGP Theorem.

Output: An increasing sequence \( P = (p_j)_{j=1}^n \) of primes such that \( \log p = O(n \log(n) + \log(1/\varepsilon)) \) and, with probability \( 1 - \varepsilon \),

\[
p := 1 + c \prod_{i=1}^n p_i \text{ is prime. In particular, the output always gives a true declaration as to the primality of } p.
\]

Description:

0. Let \( L := \lceil 2/\varepsilon \rceil \ell \) and compute the first \( nL \) primes \( p_1, \ldots, p_{nL} \) in increasing order.

1. Define (but do not compute) \( M_j := \prod_{k=(j-1)n + 1}^{jn} p_k \) for any \( j \in \mathbb{N} \). Then compute \( M_j, M_i \) for a uniformly random \( i \in [L] \), and \( x := \max \left \{ x_0, 17, 1 + M_{\ell/2}^2 \right \} \).

2. Compute \( K := \lceil (x-1)/M_j \rceil \) and \( J := \lceil 2 \log(2/x) \log x \rceil \).

3. Pick uniformly random \( c \in [K] \) until one either has \( p := 1 + c M_j \text{ prime, or one has } J \text{ such numbers that are each composite (using primality checks via the AKS algorithm along the way).} \)

4. If a prime \( p \) was found then output

\[1 + c \prod_{j=1}^n p_j \text{ is a prime that works!} \]

and stop. Otherwise, stop and output

"I have failed to find a suitable prime. Please forgive me."

Remark 2.4. In our algorithm above, it suffices to find integer approximations to the underlying logarithms and square-roots. In particular, we restrict to algorithms that can compute the \( \log L \) most significant bits of \( \log L \), and the \( 1/2 \log_2 L \) most significant bits of \( \sqrt{L} \) using

\[O(\log L)(\log \log L) \log \log \log L)\]

bit operations. Arithmetic-Geometric Mean Iteration and (suitably tailored) Newton Iteration are algorithms that respectively satisfy our requirements (see, e.g., [Ber03] for a detailed description).

Proof of Theorem 1.7. It clearly suffices to prove that Algorithm 2.3 is correct, has a success probability that is at least \( 1 - \varepsilon \), and works within

\[O\left(\left(\frac{1}{\varepsilon}\right)^{5/2} + O(n \log(n) + \log(1/\varepsilon))^{7+5/2}\right)\]

randomized bit operations, for any \( \delta > 0 \). These assertions are proved directly below.

Proving Correctness and the Success Probability Bound for Algorithm 2.3. First observe that \( M_1, \ldots, M_L \) are relatively prime. So at most \( \ell \) of the \( M_i \) will be divisible by elements of \( \mathcal{E}(x) \). Note also that \( K \geq 1 \) and \( 1 + c M_i \leq 1 + K M_j \leq 1 + \{(x-1)/M_i\} M_i = x \) for all \( i \in [L] \) and \( c \in [K] \).

Since \( x \geq x_0 \) and \( x^{2/5} > (x-1)^{2/5} \geq (M_5^{2/5})^{2/5} = M_j \) for all \( j \in [L] \), the AGP Theorem implies that with probability \( \geq 1 - \frac{1}{5} \) (since \( i \in [\lceil 2/\varepsilon \rceil \ell] \) is uniformly random), the arithmetic progression \( \{1 + M_i, \ldots, 1 + K M_j\} \) contains at least \( \frac{\pi(x)}{\pi(n)} \geq \frac{1}{\log x} \) primes. In which case, the proportion of numbers in \( \{1 + M_i, \ldots, 1 + K M_j\} \) that are prime is \( \frac{\pi(x)}{2 K M_j} > \frac{\pi(x)}{2 K M_j} > \frac{1}{\lceil 2 \log x \rceil} \) since \( \pi(x) > x/\log x \) for all \( x \geq 17 \) [BS96, Thm. 8.1, pg. 233]. So let us now assume that \( x \) is fixed and \( M_i \) is not divisible by any element of \( \mathcal{E}(x) \).

Recalling the inequality \( (1 - \frac{1}{5})^{\delta} \leq e^{-\varepsilon} \) (valid for all \( c \geq 0 \) and \( t \geq 1 \)), we then see that the AGP Theorem implies that the probability of not finding a prime of the form \( p = 1 + c M_i \) after picking \( J \) uniformly random \( c \in [K] \) is \( \left(1 - \frac{1}{\lceil 2 \log x \rceil}\right)^{\delta} \leq \left(1 - \frac{1}{\log e}\right)^{\delta} \leq e^{-\varepsilon} = \frac{1}{2} \).

In summary, with probability \( \geq 1 - \frac{1}{5} - \frac{1}{5} - \frac{1}{2} = 1 - \varepsilon \), Algorithm 2.3 picks an \( i \) with \( M_i \) not divisible by any element of \( \mathcal{E}(x) \) and a \( c \) such that \( p := 1 + c M_i \) is prime. In particular, we clearly have that \( \log p = O(\log(1 + K M_j)) = O(n \log(n) + \log(s/\varepsilon)) \).

Complexity Analysis of Algorithm 2.3. Let \( L' := n L \) and, for the remainder of our proof, let \( p_i \) denote the \( i \)-th prime. Since \( L' \geq 6, p_{L'} \leq L'(\log L') + \log L' \) by [BS96, Thm. 8.8.4, pg. 233]. Recall that the primes in \([L']\) can be listed simply by deleting all multiples of 2 in \([L']\), then deleting all multiples of 3 in \([L']\), and so on until one reaches multiples of \( \sqrt{L} \). (This is the classic sieve of Eratosthenes.) Recall also that one can multiply an integer in \([\mu]\) and an integer \([\nu]\) within \( O(\log \mu)(\log \log \nu)(\log \log \nu) + (\log \nu)(\log \mu)(\log \log \mu) \) bit operations (see, e.g., [BS96, Table 3.1, pg. 49]). So let us define the function \( \lambda(a) := (\log a) \log \log a \).

Step 1: This step consists of \( n - 1 \) multiplications of primes with \( O(\log L') \) bits (resulting in \( M_i \), which has \( O(n \log L') \) bits), multiplication of a small power of \( M_i \) by a square root of \( M_i \), division by an integer with \( O(\log L') \) bits, a constant number of additions of integers of comparable size, and the generation of \( O(\log L) \) random bits. Employing Remark 2.4 along the way, we thus arrive routinely at an estimate of...
O(n²(log(L') + log(1/ε) λ(n log L')) for the total number of bit operations needed for Step 1.

Step 2: Similar to our analysis of Step 1, we see that Step 2 has bit complexity

\[ O(n log(L') + log(1/ε) λ(n log L')). \]

Step 3: This is our most costly step here: We require \( O(\log K) = O(n log(L') + log(1/ε)) \) random bits and \( J = O(\log x) = O(n log(L') + log(1/ε)) \) primality tests on integers with \( O(log(1 + c M)) = O(n log(L') + log(1/ε)) \) bits. By an improved version of the AKS primality testing algorithm [AKS02, LP05] (which takes \( O(N^{\delta}) \) bit operations to test an \( N \) bit integer for primality), Step 3 can then clearly be done within

\[ O((n log(L') + log(1/ε))^{7+\delta}) \]

bit operations, and the generation of \( O(n log(L') + log(1/ε)) \) random bits.

Step 4: This step clearly takes time on the order of the number of output bits, which is just \( O(n log(n) + log(1/ε)) \) as already observed earlier.

Conclusion: We thus see that Step 0 and Step 3 dominate the complexity of our algorithm, and we are left with an overall randomized complexity bound of

\[ O\left(L^{3/2} log^3(L') + (n log(L') + log(1/ε))^{7+\delta}\right) \]

\[ = O\left((n^{3/2} log^3(n/ε) + (n log(n) + log(1/ε))^{7+\delta}\right) \]

\[ = O\left((n^{3/2} log^3(n/ε) + (n log(n) + log(1/ε))^{7+\delta}\right) \]

randomized bit operations.

2.3 Transferring from Complex Numbers to p-adics

Proposition 2.5. Given any \( f_1, \ldots, f_k \in \mathbb{Z}[x_1] \) with maximum coefficient absolute value \( H \), let \( d_i := \deg f_i \) and \( \bar{f}(x) := x^{d_i} f_i(x) (1/x_1) + \cdots + x^k f_k(x) (1/x_1) \). Then \( f_1 = \cdots = f_k = 0 \) has a root on the complex unit circle if \( \bar{f} \) has a root on the complex unit circle. In particular, if \( f_i \in \mathbb{F}_{1, \mu} \) for some \( \mu \leq m \), then \( \bar{f} \in \mathbb{F}_{1, \mu} \) and \( \bar{f} \) has maximum coefficient bit-size \( O(\log(kmH)) \).

Proposition follows easily upon observing that \( f_i(x_1) f_i(1/x_1) = |f_i(x_1)|^2 \) for all \( i \in [k] \) and any \( x_1 \in \mathbb{C} \) with \( |x_1| = 1 \).

Lemma 2.6. (See, e.g., [AKS94, Ch. 12, Sec. 1, pp. 397-402].) Suppose \( f(x_1) = a_0 + \cdots + a_d x_1^d \) and \( g(x_1) = b_0 + \cdots + b_{d'} x_1^{d'} \) are polynomials with indeterminate coefficients. Define their Sylvester matrix to be the \((d + d') \times (d + d') \) matrix

\[
S_{d,d'}(f,g) := \begin{bmatrix}
    a_0 & a_d & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \vdots & \vdots \\
    0 & 0 & a_0 & \cdots & a_d \\
    b_0 & b_{d'} & 0 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & 0 & b_0 & \cdots & b_{d'} \\
\end{bmatrix}
\]

and their Sylvester resultant to be \( R_{d,d'}(f,g) := \det S_{d,d'}(f,g) \). Then, assuming \( f, g \in K[x_1] \) for some field \( K \) and \( a_0, b_0 \neq 0 \), we have that \( f = g = 0 \) has a root in the algebraic closure of \( K \) if \( R_{d,d'}(f,g) = 0 \). Finally, if we assume further that \( f \) and \( g \) have complex coefficients of absolute value \( \leq H \), and \( f \) (resp. \( g \)) has exactly \( m \) (resp. \( m' \)) monomial terms, then \( |R_{d,d'}(f,g)| \leq m^{d'/2} m'^{d'/2} H^{d+d'} \).

The last part of Lemma 2.6 follows easily from Hadamard’s Inequality (see, e.g., [Mig82, Thm. 1, pg. 259]).

Lemma 2.7. Suppose \( D \in \mathbb{N} \) and \( f \in \mathbb{Z}[x_1] \{0\} \) has degree \( d \), exactly \( m \) monomial terms, and maximum coefficient absolute value \( H \). Also let \( p \) be any prime congruent to \( 1 \mod D \). Then \( f \) vanishes at a complex \( D \)-root of unity \( \Leftrightarrow f \) vanishes at a \( D \)-root of unity in \( \mathbb{Q}_p \).

Remark 2.8. Note that \( x_1^2 + x_1 + 1 \) vanishes at a \( 3 \)-rd root of unity in \( \mathbb{C} \), but has no roots at all in \( \mathbb{F}_2 \) or \( \mathbb{Q}_2 \). Hence our congruence assumption on \( p \) in Lemma 2.7.

Proof of Lemma 2.7. First note that by our assumption on \( p, \mathbb{Q}_p \) has distinct \( D \)-roots of unity. This follows easily from Hensel’s Lemma (cf. the Appendix) and \( \mathbb{F}_p \), having \( D \)-distinct \( D \)-roots of unity. Since \( \mathbb{Z} \to \mathbb{Q}_p \) and \( \mathbb{Q}_p \) contains all \( D \)-roots of unity by construction, the equivalence then follows directly from Lemma 2.6.

2.4 A Remark on Natural Density

Let us now introduce the \( \mathcal{A} \)-discriminant and clarify how often our \( p \)-adic speed-ups hold for inputs with bounded coefficients.

Definition 2.9. Write any \( f \in \mathbb{C}[x_1] \) as \( f(x_1) = \sum_{i=0}^{m} c_i x_1^i \) with \( 0 \leq a_1 < \cdots < a_m \). Let \( \mathcal{A} = \{a_1, \ldots, a_m\} \), and following the notation of Lemma 2.7, we then define \( D(A) \) to be

\[ \mathcal{R}_{(a_m-a_0, a_m-a_2)} \left( x, \frac{\partial (x_1)}{\partial x_1} \right) \left/ x^{a_2-1} \right/ c_m \]

to be the \( \mathcal{A} \)-discriminant of \( f \) (see also [GRZ94, Ch. 12, pp. 403-408]). Finally, if \( c_i \neq 0 \) for all \( i \), then we call \( \text{Supp}(f) := \{a_1, \ldots, a_m\} \) the support of \( f \).

Corollary 2.10. For any subset \( A \subset \mathbb{N} \cup \{0\} \) of cardinality \( m \), let \( \mathcal{I}_A \) denote the family of pairs \( (p, f) \in \mathbb{Z}[x_1] \times \mathbb{F}_p \) with \( f(x) = \sum_{i=0}^{m} c_i x_1^i \) and let \( \mathcal{I}_A \) denote the subset of \( \mathcal{I}_A \) consisting of those pairs \( (p, f) \) with \( p / D(A) \). Also let \( \mathcal{I}_A(H) \) (resp. \( \mathcal{I}_A(H) \)) denote those pairs \( (p, f) \in \mathcal{I}_A \) (resp. \( \mathcal{I}_A \)) where \( |c_i| \leq H \) for all \( i \) and \( p \leq H \). Then

\[ \# \mathcal{I}_A(H) \geq \left( 1 - \frac{m - 1}{H} \right) \left( 1 - \frac{d \log(H)}{H} \right) \]

Our corollary above follows easily from our proof of Assertion (3) of Theorem 1.4 via an application of Lemma 2.6 and the Schwartz-Zippel Lemma [Sch80], and is not used in any of our proofs.

3. THE PROOF OF THEOREM 1.4

Assertion (1): \( \text{FEAS}_{\text{primes}}(\mathcal{F}_m, \mathbb{P}) \in \mathbb{P} \) for \( m \leq 2 \):

First note that the case \( m = 1 \) is trivial: such a univariate monomial has no roots in \( \mathbb{Q}_p \) iff it is a nonzero constant.

So let us now assume \( m = 2 \).

Next, we can easily reduce to the special case \( f(x) = x^d - \alpha \) with \( \alpha \in \mathbb{Q} \), since we can divide any input by a suitable monomial term, and arithmetic over \( \mathbb{Q} \) is doable in polynomial time. The case \( \alpha = 0 \) always results in the root 0, so let us also assume \( \alpha \neq 0 \).

Clearly then, any \( p \)-adic root \( \zeta \) of \( x^d - \alpha \) satisfies \( \text{ord}_p \zeta = \text{ord}_\alpha \).

Since we can compute \( \text{ord}_p \alpha \) and reductions of integers mod \( d \) in polynomial-time
Let $k := \ord_d \alpha$, note that $f'(x) = dx^{k-1}$ and thus $\ord_d f(C(\zeta)) = \ord_d (d) + (d - 1)\ord_d \alpha = k$. So by Hensel's Lemma (cf. the Appendix), it suffices to decide whether the mod $p'$ reduction of $f$ has a root in $(\mathbb{Z}/p')^\times$, for $\ell = 1 + 2k$. Note in particular that size($p'$) $= O(\log(p)\ord_d(d)) = O(\log(p)\log(d)/\log(p) = O(\log(d))$ which is linear in our notion of input size. By Lemma 5.2 of the Appendix, we can then clearly decide whether $x^\alpha - \alpha$ has a root in $(\mathbb{Z}/p')^\times$ within $P$ (via a single fast exponentiation), provided $p' \notin \{8, 16, 32, \ldots\}$.

To dispose of the remaining cases $p' \notin \{8, 16, 32, \ldots\}$, first note that we can replace $d$ by its reduction mod $2^{d-2}$ since every element of $(\mathbb{Z}/2^d)^\times$ has order dividing $2^d-2$, and this reduction can certainly be computed in polynomial-time. Let us then define $d := 2^d - 2$ where $2^{d'}$ and $h \in \{0, \ldots, \ell - 3\}$, and compute $d' := 1/d' \mod 2^{d-2}$. Clearly then, $x^\alpha - \alpha$ has a root in $(\mathbb{Z}/2^d)^\times$ iff $x^{2^h - \alpha'}$ has a root in $(\mathbb{Z}/2^d)^\times$, where $\alpha' := \alpha d'$. (Since exponentiation by any odd power is an automorphism of $(\mathbb{Z}/2^d)^\times$.) Note also that $\alpha'$, $d'$, and $d''$ clearly can be computed in polynomial time.

Since $x^{2^h - \alpha'}$ always has a root in $(\mathbb{Z}/2^d)^\times$ when $h = 0$, we can then restrict our root search to the cyclic subgroup $\{1, 5^2, 5^3, 5^6, \ldots, 5^{m-2}-2\}$ when $h \geq 1$ and $\alpha'$ is a square (since there can be no roots when $h \geq 1$ and $\alpha'$ is not a square). Furthermore, we see that $x^{2^h - \alpha'}$ can have no roots in $(\mathbb{Z}/2^d)^\times$ if $\ord_d \alpha'$ is odd. So, by rescaling $x$, we can assume further that $\ord_d \alpha' = 0$, and thus that $\alpha'$ is odd. Now an odd $\alpha'$ is a square in $(\mathbb{Z}/2^d)^\times$ iff $\alpha' \equiv 1 \mod 8$ [HS90 Ex. 38, p. 192], and this can clearly be checked in $P$. So we can at last decide the existence of a root in $\mathbb{Q}_p$ for $x^\alpha - \alpha$ in $P$. Simply combine fast exponentiation with Assertion 3 of Lemma 5.2 again, applied to $x^{2^h - \alpha'}$ over the cyclic group $\{1, 5^2, 5^3, 5^6, \ldots, 5^{m-2}-2\}$.

(Statement 2): \textbf{FEAS}_{primes}(\mathcal{F}_{1,3} \times P) \in \textbf{NP}$ for non-flat Newt(p)($f$): First note that $x \in \mathbb{Q}_p \cap \mathbb{Z}_p \iff 4 \in \mathbb{Z}_p$. Letting $f'(x) := x^{\deg(f)} f(1/p)$ denote the reciprocal polynomial of $f$, note that the set of $p$-adic rational roots of $f$ is simply the union of the $p$-adic integer roots of $f$ and the reciprocals of the $p$-adic integer roots of $f'$. So we need only show we can detect roots in $\mathbb{Z}_p$ in $P$.

As stated, Assertion 2 then follows directly from Lemma 1.10.

So let us now concentrate on extending polynomiality to some of our exceptional inputs: Writing $f(x) = c_1 + c_2 x^{c_2} + c_3 x^{c_3}$ as before, let us consider the special case where $f \in \mathcal{F}_{1,3}$ has a degenerate root in $\mathbb{C}_p$ and $\gcd(c_2, a_2) = 1$. Note that we now allow $p$ to divide any number from $\{a_2, a_3, a_2 - a_3\}$. (It is easily checked that the collinearity condition fails for such polynomials since their $p$-adic Newton polygons are line segments.) The $\{a_2, a_3, a_2 - a_3\}$-discriminant of $f$ then turns out to be $\Delta := (a_2 - a_3)^{a_2 - a_3 - 2a_2^{a_2 - a_2} - (a_2 - a_3)^{a_2 - a_2} (\text{see, e.g., [CK92]} \text{ Prop. 1.8, p. 274})).$ In particular, while one can certainly evaluate $\Delta$ with a small number of arithmetic operations, the bit-size of $\Delta$ can be quite large. However, we can nevertheless efficiently decide whether $\Delta$ vanishes for integer $c_1$ via gcd-free bases (see, e.g., [BRS00 Sec. 2.4]). Thus, we can at least check whether $f$ has a degenerate root in $\mathbb{C}_p$ in $P$.
(Assertion (4): \( \text{FEAS}_{\text{prime}}(Z[x_1] \times P) \) is NP-hard under ZPP-reductions): We will prove a (ZPP) randomized polynomial-time reduction from 3CNFSAT to \( \text{FEAS}_{\text{prime}}(Z[x_1] \times P) \), making use of the intermediate input families \( \{Z[x_1]\}^k \mid k \in \mathbb{N} \) and \( Z[x_1] \times \{x_1^D - 1 \mid D \in \mathbb{N}\} \) along the way.

Toward this end, suppose \( B(y) := C_1(y) \wedge \cdots \wedge C_k(y) \) is any 3CNFSAT instance. The polynomial system \( (P_1, \ldots, P_r) \) for \( P \) the first \( n \) primes (employing Lemma 2.2), then clearly yields the implication \( \text{FEAS}_C(\{Z[x_1]\}^k \mid k \in \mathbb{N}) \in P \implies P = \text{NP} \). Composing this reduction with Proposition 2.5 we then immediately obtain the implication \( \text{FEAS}_C(Z[x_1] \times \{x_1^D - 1 \mid D \in \mathbb{N}\}) \in P \implies P = \text{NP} \).

At this point, we need only find a means of transferring \( \text{C} \to \text{Q} \). This we do by preceding our reductions above by a judicious (possibly new) choice of \( P \). In particular, by applying Theorem 1.7 with \( \varepsilon = 1/3 \) (cf. Lemma 2.7) we immediately obtain the implication \( \text{FEAS}_C((Z[x_1] \times \{x_1^D - 1 \mid D \in \mathbb{N}\}) \times P) \in \text{ZPP} \implies \text{NP} \subseteq \text{ZPP} \).

To conclude, observe that any root \((x, y) \in \mathbb{Q}_p^2 \setminus \{(0, 0)\}\) of the quadratic form \( x^2 - py^2 \) must satisfy \( 2\text{ord}_p x = 1 + 2\text{ord}_p y \), an impossibility. Thus the only \( p \)-adic rational root of \( x^2 - py^2 \) is \((0, 0)\) and we easily obtain a polynomial-time reduction from \( \text{FEAS}_C(Z[x_1] \times \{x_1^D - 1 \mid D \in \mathbb{N}\}) \times P) \) to \( \text{FEAS}_C(Z[x_1] \times P) \), simply map any instance \((f(x_1), x_1^D - 1, p)\) of the former problem to \((f(x_1)^2 - (x_1^D - 1)p, p, p)\). So we are done.

(Assertion (5): \( \text{FEAS}_{\text{prime}}(Z[x_1] \times P) \) is NP-hard, assuming Wagstaff’s Conjecture): If we also have the truth of the Wagstaff Conjecture then we simply repeat our last proof, replacing our AGP Theorem-based algorithm with a simple brute-force search. This maintains polynomial complexity, but with the added advantage of completely avoiding randomization. 

Acknowledgements

The authors would like to thank David Alan Plaisted for his kind encouragement, and Eric Bach, Sidney W. Graham, and Igor Shparlinski for many helpful comments on primes in arithmetic progression. We also thank Matt Papanikolas for valuable \( p \)-adic discussions. Finally, we thank an anonymous referee for insightful comments that greatly helped clarify our presentation.

4. REFERENCES

[AKS02] Agrawal, Manindra; Kayal, Neeraj; and Saxena, Nitin, “PRIMES is in P,” Ann. of Math. (2) 160 (2004). no. 2, pp. 781–793.

[AGP94] Alford, W. R.; Granville, Andrew; and Pomerance, Carl, “There are Infinitely Many Carmichael Numbers,” Ann. of Math. (2) 139 (1994), no. 3, pp. 703–722.

[AI09] Avendaño, Martin and Ibrahim, Ashraf, “Ultrametric Root Counting,” submitted for publication, also available as Math ArXiv preprint 0901.3393v3.

[BM88] Babai, László and Moran, Shlomo, “Arthur-Merlin Games: A Randomized Proof System and a Hierarchy of Complexity Classes,” Journal of Computer and System Sciences, 36:254–276, 1988.

[BS96] Bach, Eric and Shallit, Jeff, Algorithmic Number Theory, Vol. I: Efficient Algorithms, MIT Press, Cambridge, MA, 1996.

[Ber03] Bernstein, Daniel J., “Computing Logarithm Intervals with the Arithmetic-Geometric Mean Iterations.” available from

[BR09] Bihan, Frederic; Rojas, J. Maurice; Stella, Case E., “Faster Real Feasibility via Circuit Discriminants.” proceedings of International Symposium on Symbolic and Algebraic Computation (ISSAC 2009, July 28-31, Seoul, Korea), pp. 39-46, ACM Press, 2009.

[CG00] Cantor, David G. and Gordon, Daniel M., “Factoring polynomials over \( p \)-adic fields,” Algorithmic number theory (Leiden, 2000), pp. 185–208, Lecture Notes in Comput. Sci., 1838, Springer, Berlin, 2000.

[CDV06] Castrick, Wouter; Denef, Jan; and Vercauteren, Frederik, “Computing Zeta Functions of Nondegenerate Curves,” International Mathematics Research Papers, vol. 2006, article ID 72017, 2006.

[Ch91] Chistov, Alexander L., “Efficient Factoring of \( f \) Polynomials over Local Fields and its Applications,” in I. Satake, editor, Proc. 1990 International Congress of Mathematicians, pp. 1509–1519, Springer-Verlag, 1991.

[Coh94] Cohen, Henri, A course in computational algebraic number theory, Graduate Texts in Mathematics, 138, Springer-Verlag, Berlin, 1993.

[Coh69] Cohen, Paul J., “Decision procedures for real and \( p \)-adic fields,” Comm. Pure Appl. Math. 22 (1969), pp. 131–151.

[C-T98] Colliot-Thélène, Jean-Louis, “The Hasse principle in a pencil of algebraic varieties,” Number theory (Tiruchirapalli, 1996), pp. 19–39, Contemp. Math., 210, Amer. Math. Soc., Providence, RI, 1998.

[DvDD88] Denef, Jan and van den Dries, Lou, “\( p \)-adic and Real Subanalytic Sets,” Annals of Mathematics (2) 128 (1988), no. 1, pp. 79–138.

[DLpG00] Hilbert’s Tenth Problem: Relations with Arithmetic and Algebraic Geometry, Papers from a workshop held at Ghent University, Ghent, November 2–5, 1999. Edited by Jan Denef, Leonard Lipshitz, Thanases Pheidas and Jan Van Geel. Contemporary Mathematics, 270, American Mathematical Society, Providence, RI, 2000.

[Ede87] Edelsbrunner, Herbert, Algorithms in combinatorial geometry, EATCS Monographs on Theoretical Computer Science, 10, Springer-Verlag, Berlin, 1987.

[GG79] Garey, Michael R. and Johnson, David S., Computers and Intractability: A Guide to the Theory of NP-Completeness, A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, Calif., 1979.
5. APPENDIX: ADDITIONAL BACKGROUND

Let us first recall briefly the following complexity classes (see also [Pap95] for a good textbook treatment):

- **P** The family of decision problems which can be done within time polynomial in the input size.
- **ZPP** The family of decision problems admitting a randomized polynomial-time algorithm giving a correct answer, or a report of failure, the latter occurring with probability $\leq \frac{1}{2}$.
- **NP** The family of decision problems where a "Yes" answer can be certified within time polynomial in the input size.

EXPTIME The family of decision problems solvable within time exponential in the input size.

The classical Hensel's Lemma can be phrased as follows.

**Lemma 5.1.** Suppose $f \in \mathbb{Z}_p[x]$ and $\zeta_0 \in \mathbb{Z}_p$ satisfies $f(\zeta_0) \equiv 0 \pmod{p}$ and $\operatorname{ord}_p f'(\zeta_0) < \frac{1}{2}$. Then there is a root $\zeta \in \mathbb{Z}_p$ of $f$ with $\zeta \equiv \zeta_0 \pmod{p^{\operatorname{ord}_p f'(\zeta_0)}}$ and $\operatorname{ord}_p f'(\zeta) = \operatorname{ord}_p f(\zeta)$.

The final tool we will need is a standard lemma on binomial equations over certain finite groups. Recall that for any ring $R$, we denote its unit group by $R^\times$.

**Lemma 5.2.** (See, e.g., [BS90] Thm. 5.7.2 & Thm. 5.6.2, pg. 169). Given any cyclic group $G$, $a \in G$, and an integer $d$,

1. the equation $x^d = a$ has a solution $a \in G$.
2. the order of a divides $\gcd(d, \#G)$.
3. $a^{\#G / \gcd(d, \#G)} = 1$.

Also, $F_q^*$ is cyclic for any prime power $q$, and $(\mathbb{Z}/p^e \mathbb{Z})^* \cong \mathbb{F}_q^*$ is cyclic for any $(p, \ell)$ with $p$ an odd prime or $\ell \leq 2$. Finally, for $\ell \geq 3$, $(\mathbb{Z}/2^\ell \mathbb{Z})^* \cong \{\pm 1, \pm 5, \pm 5^2, \ldots, \pm 5^{2^{\ell-2}} \pmod{2^\ell}\}$.

\[\text{Note that the underlying polynomial depends only on the problem in question (e.g., matrix inversion, shortest path finding, primality detection) and not the particular instance of the problem.}\]