BOUNDS ON BILINEAR SUMS OF KLOOSTERMAN SUMS

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Abstract. We use some elementary arguments to obtain a new bound on bilinear sums with weighted Kloosterman sums which complements those recently obtained by E. Kowalski, P. Michel and W. Sawin (2020).

1. Introduction

Let $p$ be a prime number and let $\mathbb{F}_p^\times$ denote the multiplicative group of the finite field $\mathbb{F}_p$ of $p$ elements. We always assume that $\mathbb{F}_p$ is represented by the elements $\{0, 1, \ldots, p - 1\}$.

We now define $r$-multidimensional Kloosterman sums

$$K_{r,p}(n) = \frac{1}{p^{(r-1)/2}} \sum_{x_1, \ldots, x_r \in \mathbb{F}_p^\times, \atop x_1 \cdots x_r \equiv n \mod p} e_p(x_1 + \cdots + x_r),$$

where $e_p(t) = \exp(2\pi it/p)$ for all $t \in \mathbb{R}$. These are also called hyper-Kloosterman sums. Such sums can be interpreted as the inverse Mellin transformation of powers of Gauss sum, thus can be used to study the distribution of Gauss sums. By the classical bound of Deligne [5] we know that

$$K_{r,p}(n) \leq r. \tag{1.1}$$

Here we continue to study cancellations between such sums. In particular, we consider the bilinear Type-I sums

$$\mathcal{J}_{r,p}(\alpha; M, N) = \sum_{m \in M} \sum_{n \in N} \alpha_m K_{r,p}(mn)$$

with two sets $M, N \subseteq \mathbb{F}_p^\times$ and complex weights $\alpha = (\alpha_m)_{m \in M}$, and also more general Type-II sums

$$\mathcal{J}_{r,p}(\alpha, \beta; M, N) = \sum_{m \in M} \sum_{n \in N} \alpha_m \beta_n K_{r,p}(mn)$$

with complex weights $\alpha = (\alpha_m)_{m \in M}$ and $\beta = (\beta_n)_{n \in N}$.

2010 Mathematics Subject Classification. 11L05, 11T23.

Key words and phrases. Kloosterman sums, bilinear sums, primes.
The case when one or both of the sets $M$ and $N$ are intervals of $M$ and $N$ consecutive integers

$$
I = \{ A + 1, \ldots, A + M \} \subseteq \mathbb{F}_p^\times \quad \text{and} \quad J = \{ B + 1, \ldots, B + N \} \subseteq \mathbb{F}_p^\times
$$

with some integers $A, B, M, N$ has received most of attention.

In the classical case of $r = 2$, when $N = J$, for Type-I sums the strongest bound in the range when $M, N = p^{1/2 + o(1)}$, which crucial for applications to moments of various $L$-functions, is given in [14], see also [3, 4, 6, 10, 11, 15–17]. For example, one of the applications of this bound in an improvement in [14, Theorem 3.1] of the error term $p^{-1/68+o(1)}$ of Blomer, Fouvry, Kowalski, Michel and Milićević [3, Theorem 1.2] in the asymptotic formula for mixed moments of $L$-series associated with Hecke eigenforms down to $p^{-1/64+o(1)}$.

On the other hand, if both sets are intervals, $M = I$ and $N = J$ the widest range in which there exists a nontrivial bound, for any $r \geq 2$ is due to Kowalski, Michel and Sawin [9], see also [6, 8] for previous results. For example, the bound of [9, Theorem 1.2] improves (1.1) for $M, N \geq p^{3/8+\varepsilon}$ for any fixed $\varepsilon > 0$, while [9, Theorem 1.3] does so for $M, N \geq p^{1/3+\varepsilon}$. In fact, in both [9, Theorem 1.2 and 1.3] the set $M$ can be more general than an interval, however some restrictions of the size of the elements of $M$ are still necessary.

We also notice the method of Shkredov [13], which is based on additive combinatorics and thus also has a potential to be extended to general sets.

Since the case of an arbitrary set $M$ without any restrictions is of independent interest, and also has some applications to the average values of the divisor function in arithmetic progressions [7], Banks and Shparlinski [2, Theorem 2.4] have extended [9, Theorem 1.3] to arbitrary sets $M$ and the weights $|\alpha_m| \leq 1$, $m \in M$, and shown that in this case fixed even integer $\ell \geq 1$, we have

$$
|\mathcal{S}_{r,p}(\alpha; I, J)| \leq MN \left( N^{-1/2\ell} + M^{-1/8\ell} N^{-1/\ell} p^{3/8\ell+1/2\ell^2} \\
+ M^{-1/2\ell} N^{-1/\ell} p^{1/2\ell+1/2\ell^2} + N^{-3/2\ell} p^{1/2\ell+1/2\ell^2} \right) p^{o(1)}.
$$

In particular, the bound of [2, Theorem 2.4] remains nontrivial in the same range $M, N \geq p^{1/3+\varepsilon}$ as [9, Theorem 1.3] as for an arbitrary set $M$.

**Remark 1.1.** We note that comparing with [9, Theorem 4.3] the above bound requires $\ell$ to be even. It seems that this condition is actually missing in the formulation of [9, Theorem 4.3], as the conclusion that
\[ \gamma \geq 1/2 \] at the very end of [9, Section 4], where it appeals to [9, Theorem 4.5] requires the inequality \[ \left( (\ell - 1)/2 \right) \geq \ell/2 \], which fails for odd \( \ell \). This is, certainly inconsequential for [9, Theorem 1.3] (we also note that [9, Theorems 1.2 and 4.1] are not effected).

Here we obtain analogous results for \( \mathcal{S}_{\alpha, \beta; \mathcal{M}, \mathcal{J}} \).

2. General notation

We define the norms
\[
\| \alpha \|_\infty = \max_{m \in \mathcal{I}} |\alpha_m| \quad \text{and} \quad \| \alpha \|_\sigma = \left( \sum_{m \in \mathcal{I}} |\alpha_m|^\sigma \right)^{1/\sigma},
\]
where \( \sigma > 0 \) and similarly for other sequences.

Since the symbol \( \overline{x} \) is reserved for the modular inverse of \( x \in \mathbb{F}_p^\times \), we use \( \overline{z} \) to denote the complex conjugate of \( z \in \mathbb{C} \).

Throughout the paper, as usual \( A \ll B \) is equivalent to the inequality \( |A| \leq cB \) with some constant \( c > 0 \), which occasionally, where obvious, may depend on the integer parameter \( \nu \geq 1 \), and is absolute otherwise. The letter \( p \) always denotes a prime number.

For a positive \( A \) also write \( a \sim A \) to denote that \( a \) is in the dyadic interval \( A \leq a \leq 2A \).

3. Main results

We remark that when the weights \( \alpha \) satisfy \( |\alpha_m| \leq 1, m \in \mathcal{M}, \) then by (1.1) and the Cauchy inequality we have
\[
|\mathcal{S}_{\alpha, \beta; \mathcal{M}, \mathcal{N}}| \leq \| \beta \|_2 MN^{1/2},
\]
where
\[
\| \beta \|_2 = \left( \sum_{n \in \mathcal{J}} |\beta_n|^2 \right)^{1/2}.
\]

**Theorem 3.1.** Consider complex weights \( \alpha \) and \( \beta \) with \( |\alpha_m| \leq 1 \). Then for any fixed integer \( \ell \geq 2 \), for set \( \mathcal{M} \subseteq \mathbb{F}_p^\times \) of cardinality \( M \) and an interval \( \mathcal{N} = \{B+1, \ldots, B+N\} \subseteq \mathbb{F}_p^\times \) of length \( p > N \geq p^{3/2\ell} \), we have
\[
|\mathcal{S}_{\alpha, \beta; \mathcal{M}, \mathcal{N}}| \leq \| \beta \|_2 MN^{1/2} \Delta p^{o(1)}
\]
where
\[
\Delta = M^{-1/2} + M^{-1/4\ell} N^{-1/4\ell} p^{1/8\ell} + M^{-5/16\ell} N^{-1/2\ell} p^{5/16\ell + 3/8\ell^2} + M^{-1/2\ell} N^{-1/2\ell} p^{3/8\ell + 3/8\ell^2} + M^{-1/4\ell} N^{-3/4\ell} p^{3/8\ell + 3/4\ell^2}.
\]
Remark 3.2. Our bound in Theorem 3.1 is weaker than the bound in [9, Theorem 4.1], but it applies in larger generality without any restrictions on $M^+ = \max\{m : m \in \mathcal{M}\}$, which is given in [9].

In particular, taking a sufficiently large values of $\ell$, after simple calculations we derive from Theorem 3.1:

Corollary 3.3. Let $\varepsilon > 0$ be fixed. Consider complex weights $\alpha$ and $\beta$ with $|\alpha_m|, |\beta_n| \leq 1$. Then for any set $\mathcal{M} \subseteq \mathbb{F}_p^\times$ of cardinality $M$ and an interval $\mathcal{N} = \{B + 1, \ldots, B + N\} \subseteq \mathbb{F}_p^\times$ of length $N$ with

$$M, N \geq p^\varepsilon, \quad M^5 N^8 \geq p^{5+\varepsilon}, \quad MN \geq p^{3/4+\varepsilon}, \quad M^2 N^6 \geq p^{3+\varepsilon},$$

we have

$$|\mathcal{S}_{r,p}(\alpha, \beta; \mathcal{M}, \mathcal{N})| \ll MN p^{-\eta}$$

where $\eta > 0$ depends only on $\varepsilon$.

4. Comparison

First, we note that it appears that the exponent $1/2 - 3/4\ell$ in the condition

$$N \leq \frac{1}{2} q^{1/2 - 3/4\ell}$$

of [9, Theorem 4.1] can be replaced with $1/2 + 3/4\ell$. Now, to see that Corollary 3.3 improves on the range of [9, Theorem 4.1] we choose $N = \lfloor p^{3/4+\delta} \rfloor$ for some small $\delta > 0$. As we have mentioned, in order to apply [9, Theorem 4.1] one needs to choose $\ell$ with

$$N \leq \frac{1}{2} q^{1/2 + 3/4\ell}$$

which leaves only one option $\ell = 2$. It is now easy to see that [9, Theorem 4.1] is nontrivial only under the condition

$$\frac{p^{3/4+3/8}}{MN} < 1,$$

which means that it only applies to sets of cardinality $M > p^{3/8-\delta}$. On the other hand, for the same choice $N = \lfloor p^{3/4+\delta} \rfloor$, Corollary 3.3 requires only $M \geq p^\varepsilon$ for fixed $\varepsilon > 0$.

5. Possible Applications

As an example of an application of Corollary 3.3 we note the bound

$$\sum_{m=1}^{M} \sum_{n=1}^{N} \alpha_m \beta_n K_{r,p}(f(m)n) \ll MN p^{-\eta},$$
with a non-constant polynomial \( f \in \mathbb{F}_p[X] \) which holds under the same conditions as in Corollary 3.3 and which is unaccessible via the results of [8, 9] unless \( M \) is small and \( N \) a rather narrow range. Namely for the bound of [9, Theorem 4.1] to apply to the above sum, one needs \( f \) is a fixed polynomial, defined over \( \mathbb{Z}[X] \) and with bounded coefficients and then one also needs the following rather stringent conditions

\[
(p^{1+\varepsilon}/M)^{1/2} \leq N \leq p^{1+\varepsilon}/M^d.
\]

where \( d = \deg f \). Bilinear sums of this type with quadratic polynomials of the form \( f(X) = (X + A)^2 \) (for which our result is still new) appears in applications to the distribution of squarefree numbers in arithmetic progressions [12], see also [8, Section 1.5.1].

6. Mixed additive energy of intervals and arbitrary sets

Given a prime \( p \), an integer \( H \in [1, p) \), and an arbitrary set \( \mathcal{M} \subseteq \mathbb{F}_p^\times \), where \( \mathbb{F}_p \) is the finite field with \( p \) elements, let \( J(H, \mathcal{M}) \) denote the number of solution of the congruence

\[
xm \equiv ym \pmod{p},
\]

for \( x, y \in [1, H) \) and \( m, n \in \mathcal{M} \). We need the following result of Banks and Shparlinski [2, Theorem 2.1]

**Lemma 6.1.** For an integer \( H \geq 1 \) and \( \mathcal{M} \subseteq \mathbb{F}_p^\times \), of cardinality \( M \), the following holds,

\[
J(H, M) \ll H^2 M^2 p^{-1} + \begin{cases} HMp^{o(1)}, & \text{if } H \geq p^{2/3}; \\ HMT^{4/7}p^{-1+o(1)} + M^2, & \text{if } H < p^{2/3}, M \geq p^{1/3}; \\ HMp^{o(1)} + M^2, & \text{if } H < p^{2/3}, M < p^{1/3}. \end{cases}
\]

We note that in the last case \( H < p^{2/3} \) and \( M < p^{1/3} \) of Lemma 6.1 the leading term \( H^2 M^2 p^{-1} \) never dominates and can be omitted.

7. Proof of Theorem 3.1

First applying the Cauchy inequality we take the norm of \( \{\beta_m\} \) outside and obtain

\[
|\mathcal{S}_{r,p}(\alpha, \beta; \mathcal{M}, \mathcal{N})|^2 \leq \|\beta\|^2_2(\|\alpha\|^2_2 N + \mathcal{G}) \leq \|\beta\|^2_2 (MN + \mathcal{G}),
\]

where

\[
\mathcal{G} = \sum_{m_1, m_2 \in \mathcal{M}, m_1 \neq m_2} \alpha_{m_1} \bar{\alpha}_{m_2} \sum_{n \in \mathcal{N}} K_{r,p}(m_1 n) K_{r,p}(m_2 n).
\]
We now consider all variables to be elements of $\mathbb{F}_p$ and so we use the language of equations instead of congruences modulo $p$.

Consider two integer parameters $A$ and $B$ such that $2AB \leq N$. Then we follow a strategy followed in [9, Section 4.1]. In particular, for some complex weight $\eta = (\eta_b)_{b \sim B}$ with $|\eta_b| = 1$, to have

\begin{equation}
S \ll \log p \frac{\log p}{AB} \sum_{s, t, u \in \mathbb{F}_p^\times} \nu(s, t, u) \left| \sum_{b \sim B} \eta_b K_{r, p}(s(u + b)) \overline{K_{r, p}(t(u + b))} \right|,
\end{equation}

where

$$\nu(s, t, u) = \sum_{m_1, m_2 \in M, m_1 \neq m_2, \overline{a} \sim A} |\alpha_{m_1} \alpha_{m_2}|.$$

Now, writing

$$\nu(s, t, u) = \nu(s, t, u)^{(\ell-1)/\ell} (\nu(s, t, u)^2)^{1/2\ell}$$

and then using the Hölder inequality in (7.2) with weights $\ell/(\ell - 1)$, $2\ell$ and $2\ell$, we have

\begin{equation}
|\mathcal{S}|^{2\ell} \leq \frac{1}{(AB)^2} R_1^{2\ell - 2} R_2 S^\rho(1),
\end{equation}

where

$$R_1 = \sum_{s, t, u \in \mathbb{F}_p^\times} \nu(s, t, u) \quad \text{and} \quad R_2 = \sum_{s, t, u \in \mathbb{F}_p^\times} \nu(s, t, u)^2,$$

and

$$S = \sum_{s, t, u \in \mathbb{F}_p^\times} \left| \sum_{b \sim B} \eta_b K_{r, p}(s(u + b)) \overline{K_{r, p}(t(u + b))} \right|^{2\ell}.$$

Trivially, we have

$$\sum_{s, t, u \in \mathbb{F}_p^\times} \nu(s, t, u) \ll AM^2N.$$

Consider the set $\mathcal{A}$, which contains integers $a$ such that $a \sim A$. Now using the fact that $|\alpha_m| \leq 1$, we have

$$R_2 \leq \left\{ (a_1, a_2, m_1, m_2, m_3, m_4, n_1, n_2) \in \mathbb{A}^2 \times \mathcal{M}^4 \times \mathbb{N}^2 : \right. \left. a_1 n_1 = a_2 n_2, a_1 m_1 = a_2 m_2, a_1 m_3 = a_2 m_4 \right\}.$$
Now if we fix one of \( J(2A, \mathcal{M}) \) solutions, \((a_1, a_2, m_1, m_2) \in \mathcal{A}^2 \times \mathcal{M}^2 \) to \( a_1 m_1 = a_2 m_2 \), then there are at most \( MN \) many solutions \((m_3, m_4, n_1, n_2) \in \mathcal{M}^2 \times \mathcal{N}^2 \) to
\[
\overline{a_1} n_1 = \overline{a_2} n_2 \quad \text{and} \quad a_1 m_3 = a_2 m_4.
\]
Hence the above bound becomes,
\[
R_2 \ll MN J(2A, \mathcal{M}).
\]
Now to bound \( S \), we use a result of Kowalski, Michel and Sawin \cite[Equation (4.7)]{9}, which together with the bounds of \cite[Theorem 4.5]{9} gives
\[
S \ll p^3 B^\ell + p^2 B^{2\ell - [(\ell - 1)/2]} + p^{3/2} B^{2\ell}.
\]
Choose \( B = [0.25 p^{3/2 \ell}] \). This gives
\[
S \ll p^3 B^\ell + p^2 B^{2\ell - [(\ell - 1)/2]}.
\]
We now observe that for \( \ell \geq 2 \) we have
\[
[(\ell - 1)/2] \geq \ell/3.
\]
Hence for the above choice of \( B \) we have
\[
p^2 B^{2\ell - [(\ell - 1)/2]} \leq p^2 B^{5\ell/3} = p^{9/2} = p^3 B^\ell.
\]
Therefore,
\[
S \ll p^3 B^\ell.
\]
and from (7.3), we have
\[
|\mathcal{S}|^{2\ell} \leq \frac{1}{(AB)^{2\ell}} (AM^2 N)^{2\ell - 2} J(2A, \mathcal{M}) M N B^\ell p^{3+o(1)}
\]
\[
= A^{-2} B^{-\ell} M^{4\ell - 3} N^{2\ell - 1} J(2A, \mathcal{M}) p^{3+o(1)}.
\]
Now we apply Lemma 6.1 for the bound for \( J(2A, M) \) and get
\[
|\mathcal{S}| \leq A^{-1/\ell} M^{2 - 3/2\ell} N^{1 - 1/2\ell}
\]
\[
\left( A^2 M^2 p^{-1} + AM^{7/4} p^{-1/4} + AM + M^2 \right)^{1/2\ell} p^{3/4 + o(1)}.
\]
Choosing \( A = [0.5 NB^{-1}] \), so that
\[
N p^{-3/2\ell} \ll A \ll N p^{-3/2\ell},
\]
we now derive
\[
|\mathcal{S}| \leq M^2 N \Gamma^{1/2\ell} p^{o(1)},
\]
where
\[
\Gamma \leq A^{-2}M^{-3}N^{-1} \left( A^2M^2p^{-1} + AM^{7/4}p^{-1/4} + AM + M^2 \right) p^{3/2}
\]
\[
= M^{-1}N^{-1}p^{1/2} + A^{-1}M^{-5/4}N^{-1}p^{5/4}
+ A^{-1}M^{-2}N^{-1}p^{3/2} + A^{-2}M^{-1}N^{-1}p^{3/2}
\]
\[
= M^{-1}N^{-1}p^{1/2} + M^{-5/4}N^{-2}p^{5/4+3/2\ell}
+ M^{-2}N^{-2}p^{3/2+3/2\ell} + M^{-1}N^{-3}p^{3/2+3/\ell}.
\]
Inserting this bound in \((7.1)\), after simple calculations, we complete the proof.

8. Comments

It is easy to see that under the Generalised Riemann Hypothesis (GRH), the bound of Lemma 6.1 can be improves as
\[
J(H, M) = H^2M^2p^{-1} + O \left( HMp^{o(1)} \right),
\]
see \[2, Equation (1.4)]\). In turn, this implies that under the GRH, one can recover all main results of \[9\] for arbitrary sets without any restrictions on
\[
M^+ = \max \{m : m \in \mathcal{M} \},
\]
which are imposed in \[9\].

We also use the opportunity to note that using the results of \[1, Theorem 1\], one can extend \[9, Theorems 4.1 and 4.3\] to intervals \(\mathcal{N}\) in an arbitrary position rather than just at the origin.

On the other hand, the method of \[8\] uses completing technique and thus it is not clear how to extend it to arbitrary sets \(\mathcal{M}\).

Acknowledgement

During the preparation of this work N.B. was supported by the post-doctoral fellowship, Harish-Chandra Research Institute, Prayagraj, India, and I.E.S. by the Australian Research Council Grant DP170100786 and by the Natural Science Foundation of China Grant 11871317.

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