Geometrical control of transverse electromagnetic wave propagation in nonuniform microwave superconducting transmission lines

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It is well known that photonic structures with subwavelength features can be homogenized and thus be accurately approximated by homogeneous yet spatially dispersive structures. This idea is here applied to nonuniform superconducting transmission lines with subwavelength nonuniformities, i.e. subcentimeter features in the microwave regime. A closed form expression is found for the equivalent characteristic impedance and propagation constant of a uniform transmission line that can accurately model the transverse electromagnetic (TEM) propagation within nonuniform superconducting transmission lines with subwavelength inhomogeneities. It is shown that electromagnetic wave propagation, in particular the group velocity of propagating TEM waves, can be well controlled by introduction of subwavelength (subcentimeter) geometrical variations within the line. This ability is of great importance in design of matching stages. In our numerical examples, quasi-TEM superconducting microstrip lines with a superconductive strip of varying width are considered and the group velocity of the quasi-TEM modes is controlled by duly varying the line width. The obtained results are validated by being compared against the well-known technique of stair-case approximation whereby the considered nonuniform line is sliced into several sections each being small enough to be considered uniform.

1. Introduction

Homogenization of photonic structures with subwavelength inhomogeneities by using the static field approximation is now a well-known approach to devise novel structures and to control electromagnetic propagation [1-4]. Here, the analogy of quasi-TEM Maxwell's equations and the telegrapher's equations is employed and thus a novel homogenization technique is proposed to model nonuniform microwave superconducting transmission lines with a surrogate uniform transmission line, whose electromagnetic dispersion is geometrically controlled by introducing subwavelength nonuniformities. The insignificance of dispersion in uniform microwave superconducting transmission lines is a further motivation to geometrically control quasi-TEM electromagnetic wave propagation. The controllable dispersion achieved in this manner then finds interesting applications, e.g. in impedance matching and in microwave pulse shaping.

In this manuscript, the phenomenological two-fluid model [5] is first employed to describe the general behavior of high temperature superconductors at microwave frequencies and then the telegrapher's equations for a nonuniform superconducting microstrip line are written. Assuming subwavelength variations for the width of the strip-line, approximate yet accurate enough closed form expressions are afterwards provided for the voltage and current distribution along the line. It is shown that the obtained distribution function of the voltage and current along the nonuniform line with subwavelength nonuniformities resembles the distribution function of the voltage and current along a uniform transmission line whose characteristic impedance and effective propagation constant are weighted averages of the characteristic impedance and effective propagation constant in the original line, respectively. Having such a surrogate uniform transmission line, the group velocity of the original nonuniform transmission line is straightforwardly derived in terms of the geometrical parameters of the nonuniform line and is later used to design an adiabatic matching stage.

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2. Formulation
The schematic of a typical nonuniform high-temperature superconducting microstrip transmission line is shown in the figure 1. The width of the strip-line, \( w(z) \), is longitudinally varying and alters the electromagnetic behaviour of the transmission line. The thickness of the strip-line, \( t \), the thickness of the dielectric substrate, \( d \), and its width, \( w_g \), are all on the other hand fixed and non-varying.

![Figure 1. (a) the cross sectional view and (b) top view of a nonuniform high-temperature superconducting microstrip transmission line](image)

2.1. Telegrapher's equations
In the framework of the empirical two-fluid (ETF) model proposed by Vendik et al. [5], the complex conductivity of the superconductor film at temperature \( T \) can be written down as [6],

\[
\sigma = \sigma_1 - j \sigma_2, \tag{1}
\]

where

\[
\sigma_1 = \sigma_n(T)(T/T_c)^\eta, \tag{2}
\]

\[
\sigma_2 = \frac{1}{\mu_0 \lambda_L(T)}, \tag{3}
\]

and

\[
\sigma_n(T) = \sigma_n(T_c)[(T/T_c)^{\eta-1} + \Lambda[1-(T/T_c)^\eta]], \tag{4}
\]

In these equations, \( T_c \) denotes the critical temperature, \( \Lambda \) is a fitting parameter, \( \eta \) is a factor ranging between 1.5 and 2.5, and \( \lambda_L(T) \) is the London penetration depth [5,6]:

\[
\lambda_L(T) = \lambda_L(0)\sqrt[\eta]{1-(T/T_c)^\eta}, \tag{5}
\]

where \( \lambda_L(0) \) in the SI system is

\[
\lambda_L(0) = 0.12 \times 10^{-6} \exp(1.27 - 0.5\eta), \tag{6}
\]

Using the complex conductivity of the superconductor film and combing Maxwell's and London's equations, the surface impedance of the strip-line at the angular frequency \( \omega \) is found as [6]

\[
Z_s = R_s + jX_s = \left(\frac{j \omega \mu_0}{\sigma}\right)\coth\left[j \omega \mu_0 \sigma \left(\frac{1}{\lambda_L} + \frac{t/\lambda_L}{\sinh(t/\lambda_L)}\right)\right], \tag{7}
\]

Now, assuming that the working temperature \( T \) is not too close to the critical temperature \( T_c \), the surface resistance, \( R_s \), and the surface reactance, \( X_s \), can be expressed as [6]

\[
R_s = \frac{1}{2} \omega^2 \mu_0 \sigma \lambda_L^2 \coth(t/\lambda_L) + \frac{t/\lambda_L}{\sinh(t/\lambda_L)} = R_s' \omega^2, \tag{8}
\]

\[
X_s = \omega \mu_0 \lambda_L \coth(t/\lambda_L) = X_s' \omega, \tag{9}
\]
By following the approach in [7] and under the assumption of $W_g \gg W$, the per-unit-length resistance $R(z)$, inductance $L(z)$, capacitance $C(z)$, and conductance $G(z)$ of the line are,

$$R(z) = gR(z), \quad (10a)$$

$$L(z) = \frac{X_o}{\omega W(z)K(z)} + \frac{\mu_o d}{W(z)K(z)}, \quad (10b)$$

$$C(z) = \frac{\varepsilon_{r}\varepsilon_{o}W(z)}{d}K(z), \quad (10c)$$

$$G(z) = \frac{\omega \varepsilon_{r}\varepsilon_{o}}{d} \tan \frac{\omega W(z)}{d}K(z), \quad (10d)$$

where $K$ and $g$ denote the fringe factor that can be found by using the conformal mapping technique presented in [8]. These factors are necessary to consider the narrowness of the strip-line and read as follows [7]

$$K(z) = \begin{cases} \left( \frac{1}{2\pi} \ln \left( \frac{8d + W(z)}{W(z) + 4d} \right) \right)^{-1} \frac{d}{W(z)} & W(z) \leq d \\ \frac{W(z)}{d} + 2.42 - 0.44 \frac{d}{W(z)} + \left( 1 - \frac{d}{W(z)} \right)^{6} \frac{d}{W(z)} & W(z) \geq d \end{cases}, \quad (11)$$

$$g(z) = \begin{cases} \frac{1}{2\pi W(z)} \left( 2 + \frac{W(z)}{d} \right)^{32 - \left( \frac{W(z)}{d} \right)^{2}} & W(z) \leq d \\ \frac{W(z) + 2d}{W^{2}(z)K^{2}(z)} \left[ 1 + \frac{0.44}{\left( \frac{W(z)}{d} \right)^{2}} + \left( \frac{1 - \frac{d}{W(z)}}{\left( \frac{W(z)}{d} \right)^{2}} \right)^{5} \right] & W(z) \geq d \end{cases}, \quad (12)$$

For quasi-TEM waves, the telegrapher's equations for voltage, $V(z)$, and current, $I(z)$, distributions can thus be cast in the following matrix form:

$$\frac{dU(z)}{dz} = W(z)U(z), \quad (13)$$

where

$$U(z) = \begin{bmatrix} V(z) \\ I(z) \end{bmatrix},$$

$$W(z) = \begin{bmatrix} 0 & \gamma(z)Z_{ch}(z) \\ \gamma(z)Z_{ch}(z) & 0 \end{bmatrix}, \quad (15)$$

$\gamma$ stands for the effective propagation constant;

$$\gamma(z) = \sqrt{\left( R(z) + j\omega L(z) \right) \left( G(z) + j\omega C(z) \right)}, \quad (16)$$

and $Z_{ch}$ denotes the characteristic impedance of the nonuniform line;

$$Z_{ch}(z) = \sqrt{\frac{R(z) + j\omega L(z)}{G(z) + j\omega C(z)}}, \quad (17)$$
2.2. Analytical solution

The solution to the set of coupled differential equations (13) is usually written as

$$
\begin{bmatrix}
V(z) \\
I(z)
\end{bmatrix} = Q_{0\to z} \begin{bmatrix}
V(0) \\
I(0)
\end{bmatrix},
$$

(18)

where $Q_{0\to z}$, being referred to as the transfer matrix from 0 to $z$, is to be calculated numerically. In analogy with the photonic case [1], the following approximate expression can be proposed for the transfer matrix:

$$
Q_{0\to z} = \exp \left[ \int_0^z W(z) \, dz \right] = \exp \left\{ \int_0^z \begin{bmatrix}
0 & m_{12}(z) \\
m_{21}(z) & 0
\end{bmatrix} \, dz \right\},
$$

(19)

or

$$
Q_{0\to z} = \begin{bmatrix}
\cosh(\sqrt{m_{12}(z)m_{21}(z)}) & -\frac{m_{12}(z)}{\sqrt{m_{21}(z)}} \sinh(\sqrt{m_{12}(z)m_{21}(z)}) \\
-\frac{m_{12}(z)}{\sqrt{m_{21}(z)}} \sinh(\sqrt{m_{12}(z)m_{21}(z)}) & \cosh(\sqrt{m_{12}(z)m_{21}(z)})
\end{bmatrix},
$$

(20)

where

$$
m_{12}(z) = \int_0^z \gamma(z')Z_{ch}(z') \, dz',
$$

(21)

$$
m_{21}(z) = \int_0^z \frac{\gamma(z')}{Z_{ch}(z')} \, dz',
$$

(22)

This approximate expression is quite accurate whenever the maximum frequency of the spatial Fourier transform of the elements in the $W(z)$ matrix is smaller than the wavelength of the electromagnetic wave, i.e. whenever the scale of spatial variations and nonuniformities along the line is subwavelength. On the other hand, the transfer matrix of a uniform transmission line having constant characteristic impedance, $Z_0$, and constant effective index of propagation, $\gamma_0$, reads as;

$$
Q_{0\to z} = \begin{bmatrix}
\cosh(\gamma_0 z) & -Z_0 \sinh(\gamma_0 z) \\
-\frac{1}{Z_0} \sinh(\gamma_0 z) & \cosh(\gamma_0 z)
\end{bmatrix},
$$

(23)

Comparing the transfer matrix of the nonuniform transmission line of length $L$ using (20) and the transfer matrix of a uniform transmission line of length $L$ using (23), a surrogate uniform transmission line can now be proposed to model wave propagation along the original nonuniform line. The equivalent characteristic impedance, $Z_{eff}$, and the equivalent effective index of propagation, $\gamma_{eff}$, of such a surrogate line are;

$$
Z_{eff} = \sqrt{\frac{m_{12}(L)}{m_{21}(L)}} \sqrt{\int_0^L \frac{\gamma(z)Z_{ch}(z) \, dz}{Z_{ch}(z) \, dz}},
$$

(24)

$$
\gamma_{eff} = \frac{1}{L} \int_0^L \gamma(z)Z_{ch}(z) \, dz \int_0^L \frac{\gamma(z)}{Z_{ch}(z)} \, dz,
$$

(25)

The group velocity of a propagating wave along the nonuniform line, $v_g$, can then be easily obtained;

$$
v_g = \frac{d\omega}{d\beta},
$$

(26)

where $\beta$ stands for the imaginary part of the effective index of propagation $\gamma_{eff}$. 
Assuming a lossless dielectric substrate (\( \tan \delta = 0 \)), substituting (16) and (17) in (24) and (25), and after some algebraic manipulations, the equivalent characteristic impedance, \( Z_{\text{eff}} \), and the equivalent imaginary part of the effective index of propagation, namely phase constant, \( \beta_{\text{eff}} \), of the surrogate line can be written as:

\[
Z_{\text{eff}} = \frac{A \omega + j C}{jB},
\]

\[
\beta_{\text{eff}} = \omega \sqrt{\frac{1}{2} CB \sqrt{1 + \left( \frac{\omega A}{C} \right)^2 + 1}},
\]

where

\[
A = \frac{1}{L} \int_0^L g(z) R' \, dz,
\]

\[
B = \frac{1}{L} \int_0^L W(z) \frac{\rho_{\text{eff}}}{d} K(z) \, dz,
\]

\[
C = \frac{1}{L} \int_0^L X_e' + \mu_d d \, dz,
\]

These expressions clearly witness the fact that subwavelength geometrical nonuniformities can control the characteristic impedance and group velocity of the quasi-TEM wave propagation along microstrip lines.

3. Numerical Examples

In this section, we consider a superconducting microstrip line whose width is a sawtooth function of \( z \), propagation direction. For a sawtooth microstrip line of length \( 2l \), schematically shown in the figure 2, the strip width function, \( W(z) \), can be written in terms of \( a, b, \) and \( l \), all of them shown in the figure.

\[
W(z) = \begin{cases} 
    az + b & 0 < z < l \\
    -az + b + 2a & l < z < 2l
\end{cases}
\]

The parameters \( A, B \) and \( C \) defined in (29)-(31) are then;

\[
A = \frac{1}{2l} \left( \int_0^l g(z) R' \, dz + \int_l^{2l} g(z) R' \, dz \right),
\]

\[
B = \frac{1}{2l} \left( \int_0^l (a\omega + b) \frac{\rho_{\text{eff}}}{d} K(z) \, dz + \int_l^{2l} (-a\omega + b + 2a\elld) \frac{\rho_{\text{eff}}}{d} K(z) \, dz \right),
\]

\[
C = \frac{1}{2l} \left( \int_0^l X_e' + \mu_d d \, dz + \int_l^{2l} X_e' + \mu_d d \, dz \right),
\]

Figure 2. Microstrip line having a sawtooth strip.
and thereby the equivalent characteristic impedance, $Z_{eq}$, and also the equivalent effective phase constant, $\beta_{eq}$, can be straightforwardly calculated.

In accordance with the figure 3, sawtooth microstrip lines of length 2l can be employed to adiabatically match two different uniform transmission lines with two different group velocities. As an example, a particular case is here considered, where the thickness of the dielectric substrate is $d = 500 \mu m$, the thickness of the superconducting strip is $t = 0.4 \mu m$, $\varepsilon_r = 23.5$, $\mu_r = 1$, $A = 6.5$, $\eta = 1.5$, $\sigma(T) = 1.8 \times 10^6 \text{ s/m}$, $T_c = 92k$, $T = 77k$, $\lambda = 3 \text{ cm}$, and the angular frequency is $\omega = 2\pi \times 10^{10}$. The group velocity of the uniform transmission line having a strip width of $w = 1.5 \text{ mm}$ on the left is $v_{g,1} = 7.2 \times 10^7 \text{ m/s}$, and the group velocity of the uniform transmission line having a strip width of $w_r = 2 \text{ cm}$ on the right is $v_{g,2} = 6.4 \times 10^7 \text{ m/s}$. These two different lines with their different group velocities are adiabatically matched together by using 4, 8, and 10 sawtooth microstrip lines each one slowly decreasing the group velocity of quasi-TEM waves propagating from left to right. If left unmatched, 48% of the incident power injected from the transmission line on the left side will be reflected back.

First, four nonuniform sawtooth microstrip lines with the following group velocities and geometrical parameters $v_{g,1} = 7.04 \times 10^7 \text{ m/s}$ ($a_1 = 0.3$, $b_1 = 2 \text{ mm}$, $l_1 = 3.75 \text{ mm}$), $v_{g,2} = 6.88 \times 10^7 \text{ m/s}$ ($a_2 = 0.3$, $b_2 = 3.5 \text{ mm}$, $l_2 = 3.75 \text{ mm}$), $v_{g,3} = 6.72 \times 10^7 \text{ m/s}$ ($a_3 = 0.3$, $b_3 = 4.5 \text{ mm}$, $l_3 = 3.75 \text{ mm}$), and $v_{g,4} = 6.56 \times 10^7 \text{ m/s}$ ($a_4 = 0.3$, $b_4 = 8 \text{ mm}$, $l_4 = 3.75 \text{ mm}$), are inserted between the above-mentioned uniform transmission lines. In this case, 0.361 of the incident power is reflected back. Then, eight nonuniform sawtooth microstrip lines with group velocities $v_{g,1} = 7.11 \times 10^7 \text{ m/s}$ ($a_1 = 0.4$, $b_1 = 1.9 \text{ mm}$, $l_1 = 1.87 \text{ mm}$), $v_{g,2} = 7.02 \times 10^7 \text{ m/s}$ ($a_2 = 0.4$, $b_2 = 2 \text{ mm}$, $l_2 = 1.87 \text{ mm}$), $v_{g,3} = 6.93 \times 10^7 \text{ m/s}$ ($a_3 = 0.4$, $b_3 = 2.5 \text{ mm}$, $l_3 = 1.87 \text{ mm}$), $v_{g,4} = 6.84 \times 10^7 \text{ m/s}$ ($a_4 = 0.4$, $b_4 = 2.8 \text{ mm}$, $l_4 = 1.87 \text{ mm}$), $v_{g,5} = 6.75 \times 10^7 \text{ m/s}$ ($a_5 = 0.4$, $b_5 = 4.2 \text{ mm}$, $l_5 = 1.87 \text{ mm}$), $v_{g,6} = 6.66 \times 10^7 \text{ m/s}$ ($a_6 = 0.4$, $b_6 = 5.9 \text{ mm}$, $l_6 = 1.87 \text{ mm}$), $v_{g,7} = 6.57 \times 10^7 \text{ m/s}$ ($a_7 = 0.4$, $b_7 = 7.8 \text{ mm}$, $l_7 = 1.87 \text{ mm}$), and $v_{g,8} = 6.48 \times 10^7 \text{ m/s}$ ($a_8 = 0.4$, $b_8 = 11.5 \text{ mm}$, $l_8 = 1.87 \text{ mm}$) are then used and the back reflection is decreased to 0.166 of the incident power. Finally, ten nonuniform sawtooth microstrip lines with group velocities $v_{g,1} = 7.12 \times 10^7 \text{ m/s}$ ($a_1 = 0.5$, $b_1 = 1.88 \text{ mm}$, $l_1 = 1.5 \text{ mm}$), $v_{g,2} = 7.05 \times 10^7 \text{ m/s}$ ($a_2 = 0.5$, $b_2 = 2 \text{ mm}$, $l_2 = 1.5 \text{ mm}$), $v_{g,3} = 6.98 \times 10^7 \text{ m/s}$ ($a_3 = 0.5$, $b_3 = 2.1 \text{ mm}$, $l_3 = 1.5 \text{ mm}$), $v_{g,4} = 6.91 \times 10^7 \text{ m/s}$ ($a_4 = 0.5$, $b_4 = 2.6 \text{ mm}$, $l_4 = 1.5 \text{ mm}$), $v_{g,5} = 6.83 \times 10^7 \text{ m/s}$ ($a_5 = 0.5$, $b_5 = 3.3 \text{ mm}$, $l_5 = 1.5 \text{ mm}$), $v_{g,6} = 6.76 \times 10^7 \text{ m/s}$ ($a_6 = 0.5$, $b_6 = 4 \text{ mm}$, $l_6 = 1.5 \text{ mm}$), $v_{g,7} = 6.69 \times 10^7 \text{ m/s}$ ($a_7 = 0.5$, $b_7 = 5 \text{ mm}$, $l_7 = 1.5 \text{ mm}$), and $v_{g,8} = 6.62 \times 10^7 \text{ m/s}$ ($a_8 = 0.5$, $b_8$
The back reflection is further decreased to 0.111. As expected, the more gradual variation of group velocity between successive lines results in smaller back reflection of incident power. It should be noticed that even though the approximate expression are used to design the matching elements, the exact numerical solution of the telegrapher's equations (13) is used to simulate the designed structure and calculate the power reflection.

The bandwidths of the adiabatic matching stages having ten and eight sawtooth microstrip lines are also demonstrated in the figure 4, where the reflection of incident power is plotted versus frequency. This figure shows that the designed matching stage is working over a wide bandwidth (8-12 GHz). Where again, the exact numerical solution of the telegrapher's equations (13) is used to extract the bandwidth.

![Figure 4. Reflection of the incident power in an adiabatic matching stage with eight (solid) and ten (dashed) sawtooth microstrip lines](image)

### 4. Conclusion

In this paper, a nonuniform superconducting microstrip transmission line with subwavelength nonuniformity is modelled by a surrogate uniform transmission line whose characteristics are analytically derived. This modelling is quite useful in geometrical control of quasi-TEM wave propagation along microwave superconducting lines. As an example, an adiabatic matching stage is proposed and designed to match two different uniform transmission lines using nonuniform transmission lines with sawtooth strip. Although, the design stage is made by using the proposed model, the obtained results are calculated by following the exact numerical solution of the telegrapher's equations governing the quasi-TEM wave propagation.

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