Pareto optimal structures producing resonances of minimal decay under $L^1$-type constraints

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Abstract

Optimization of resonances associated with 1-D wave equations in inhomogeneous media is studied under the constraint $\|B\|_1 \leq m$ on the nonnegative function $B \in L^1(0, \ell)$ that represents the medium’s structure. From the Physics and Optimization points of view, it convenient to generalize the problem replacing $B$ by a nonnegative measure $dM$ and imposing on $dM$ the condition that its total mass is $\leq m$. The problem is to design for a given frequency $\alpha \in \mathbb{R}$ a medium that generates a resonance $\omega$ on the line $\alpha + i\mathbb{R}$ with a minimal possible decay rate $|\text{Im}\, \omega|$. Such resonances are said to be of minimal decay and form a Pareto frontier. We show that corresponding optimal measures consist of finite number of point masses, and that this result yields non-existence of optimizers for the problem over the set of absolutely continuous measures $B(x)dx$. Then we derive restrictions on optimal point masses and their positions. These restrictions are strong enough to calculate optimal $dM$ if the optimal resonance $\omega$, the first point mass $m_1$, and one more geometric parameter are known. This reduces the original infinitely-dimensional problem to optimization over four real parameters. For small frequencies, we explicitly find the Pareto set and the corresponding optimal measures $dM$. The technique of the paper is based on the two-parameter perturbation method and the notion of local boundary point. The latter is introduced as a generalization of local extrema to vector optimization problems.

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### 1 Introduction

Wave equations equipped with damping or radiation boundary conditions are used to model open resonators. When separation of variables is possible, the Fourier decomposition leads to a non-self-adjoint spectral problem that has a spectral parameter both in the equation and in the boundary conditions. This parameter is usually called a quasi-(normal) eigenvalue or a resonance.

Quasi-eigenvalues considered in this paper are the values of the spectral parameter \( \omega \in \mathbb{C} \setminus \{0\} \) such that the problem
\[
-\partial^2_x y(x) = \omega^2 B(x) y(x), \quad \partial_x y(0) = 0, \quad y(\ell) = -i \partial_x y(\ell)/\omega, \tag{1.1}
\]
has a nontrivial solution \( y \) on the interval \( I = [0, \ell] \). A natural generalization of this problem
\[
-\frac{d^2}{dMdx} y(x) = \omega^2 y(x), \tag{1.2}
\]
\[
\partial_x y(0) = 0, \tag{1.3}
\]
\[
y(\ell) + \frac{i}{\omega} \partial_x^+ y(\ell) = 0, \tag{1.4}
\]
involves the Krein-Feller differential expression $\frac{d^2}{dMdx}$; see [17, 9] and Section 2.1. It corresponds to the change of the nonnegative function $B$ in (1.1) to a nonnegative measure $dM$. When $dM = Bdx$ with the density $B \in L^1(0, \ell)$, problem (1.2)-(1.4) turns into (1.1).

These models are relevant to many physical systems with 1-D, multilayered, or radially symmetric structures. These include vibrations of a damped string with the mass distribution $dM$ (or the density $B$) [2, 23, 24, 8, 33, 34], the Regge problem [13, 34], and standing EM waves in an open optical cavity with a symmetric 1-D structure [27, 28, 16, 37, 20]. In the latter case, the cavity is often called a 1-D photonic crystal and is described by the relative permittivity function $B$. Very kindred problems arise in connection with resonances or scattering poles of Schrödinger equation [12, 15, 11], standing acoustic waves [38], and gravitational radiation from a star [27, 28].

In Physics, problem (1.2)-(1.4) usually involves a measure $dM$ of the form $B(x)dx + \sum_{j=1}^n m_j \delta(x-a_j)dx$. Here $B(x)dx$ is an absolutely continuous part of $dM$, and $m_j \delta(x-a_j)dx$ is a point mass $m_j$ positioned at $x = a_j$. In Optical Engineering, the term $m_j \delta(x-a_j)dx$ corresponds to a thin layer of high relative permittivity forming a partially transmitting dielectric mirror [25, 27, 28].

The above models involve either damping, or leakage of energy into surrounding medium. For standing waves associated with (1.2)-(1.4), this leads to exponential decay in time and to the fact that quasi-eigenvalues $\omega$ lie in the open lower half-plane $\mathbb{C}_-$. The decay rate $Dr(\omega, dM)$ of standing waves associated with $\omega$ and $dM$ equals the minus imaginary part of $\omega$ and is always positive. The real part $Re\omega$ is the frequency of oscillations. The set of quasi-eigenvalues associated with the problem (1.2)-(1.4) is denoted by $\Sigma(dM)$. The associated eigenfunctions are called (quasi-)modes (metastable states, in the context of Quantum Mechanics).

The recent engineering progress in design of resonators with small and high decay rate [1, 40, 30] attracted considerable current interest to numerical aspects of emerging optimization problems [16, 21, 4, 31, 37, 32].

This paper is devoted to optimization of an individual quasi-eigenvalue under total mass constraints on the coefficient $B$ or $dM$ and, in particular, to minimization of the decay rate $|Im\omega|$. Optimization is considered over the set of measures (admissible family)

$$A_M := \{dM : dM \text{ is a nonnegative Borel measure such that } \int_{0-}^\ell dM \leq m\}, \quad m > 0,$$

and (as a by-product) over

$$A_1 := \{B(x)dx \in A_M : B \in L^1(0, \ell)\} \text{ and } A_{\text{disc}} := \left\{ \sum_{j=1}^n m_j \delta(x-a_j)dx \in A_M : n \in \mathbb{N} \right\}.$$

Note that numerical methods were used in [39] and, recently, in [16, Problem Opt_{area}] to consider very kindred optimization problems (with slightly different Physics backgrounds). From Spectral Theory point of view, the optimization over the above sets can be considered as an attempt to obtain sharp estimates on quasi-eigenvalues in terms of $\|B\|_1$ or the norm $\|dM\|$ of $dM$ (such estimates are a classical topic in Mathematical Physics [14]).

Our main attention is directed to rigorous derivation of structural theorems for optimizers. The motivation is that an accurate numerical discretization requires some understanding of optimizers’ structure. The structural optimization for quasi-eigenvalues and, more generally, infinite-dimensional non-self-adjoint spectral problems is not adequately developed.

Short reviews of applied and numerical studies relevant to quasi-eigenvalue optimization can be found in [31] and the monograph [37]. Most of the analytical literature on non-self-adjoint spectral
optimization is concerned with optimization of spectral abscissa for damped systems. This problem involves simultaneous optimization of all eigenvalues and, in infinite-dimensional case, becomes very difficult and quite different [7, 8, 6]. Though the finite-dimensional theory is more developed (e.g. [5, 10]), the structural study of optimizers is not a simple question [10].

The essential difficulty for the standard critical point argument arises when eigenvalues are not differentiable with respect to (w.r.t.) the coefficients. Such effects usually appears when eigenvalue is non-simple (degenerate). In this case, perturbations are studied with the use of Puiseux series and Vishik-Lyusternic-Lidskii theory. Relevant references and connections with the optimization of spectral abscissa can be found in [6] (where an explicit example of a non-differentiable splitting for a multidimensional dissipative wave equation is given), in [5, 10] (matrix case), and in the authors papers [18, 20] concerning optimization of an individual quasi-eigenvalue under side constraints on $B$. Examples related to degeneration and merging of 1-D quasi-eigenvalues are known in Physics and Spectral Theory, see [28, 19] and the discussion in Section 3.2. It follows from results of Section 3.4 that non-semi-simple quasi-eigenvalues of (1.2)-(1.4) are always non-differentiable w.r.t. $dM$, see (3.17) and Example 3.3. According to [6, 28], these degeneracy and non-differentiability issues cannot be discarded as mathematical exotic and can have experimental consequences (e.g. diverging laser quantum noise [26]).

Another difficulty arises since the standard proof of existence of optimizers relies on compactness arguments [22] and is applicable only to weakly* compact admissible families. The latter is the case for $A_M$, but not for $A_1$ since the space $L^1(0,1)$ has no predual. In fact, one of the results of this paper states that there are no minimizers for the decay rate over $A_1$ (see Theorem 5.3 for the precise statement). On the other hand, the adaptation of compactness arguments to dissipative problems uses an additional restriction $\alpha \in [\alpha_1, \alpha_2]$ on the real part of the spectral parameter [15, 39, 18]. As a result, the optimizer is not necessarily a critical point of the decay rate functional $\text{Dr}(\omega, dM)$. This makes it difficult to use the standard one-parameter perturbation theory in the study of optimizers’ structures, see the discussions in [18, 20].

In the author’s paper [18], the two-parameter perturbation approach was developed to overcome the above obstacles in the study of optimization over the family

$$A_\infty := \{ B(x)dx \in A_M : B \in L^\infty(0,\ell) \text{ and } c_1 \leq B(x) \leq c_2 \text{ a.e.}, \ 0 \leq c_1 < c_2 \}.$$

It was shown that structures of minimal decay are extreme points of $A_\infty$ and, moreover, are piecewise constant functions taking only extreme possible values. Each optimal $B$ is related to one of corresponding optimal modes $y$ via $B(x) = c_1 + (c_2 - c_1) \chi_{C_+}(y^2(x))$, where $\chi_{C_+}(z) := 1$ for $z \in C_+$, and $\chi_{C_+}(z) := 0$ for $z \in C \setminus C_+$. This leads to the conclusion that optimal $\omega$ are eigenvalues of nonlinear equation

$$y'' + \omega^2 y \left[ c_1 + (c_2 - c_1) \chi_{C_+}(y^2) \right] = 0$$

equipped with the boundary conditions (1.3)-(1.4).

The goals of the present paper require further refinement of the two-parameter perturbation method and the essential use of convex analysis and geometric arguments. Following [18], Section 2 gives the rigorous statement of the optimization problem in the framework of vector (Pareto) optimization. Roughly speaking, a quasi-eigenvalue $\omega$ itself is considered as an $\mathbb{R}^2$-valued cost function depending on $dM$. (Actually, the formalization of $\omega(dM)$ leads to the set-valued map $dM \mapsto \Sigma(dM)$. It is multivalued even locally due to the coalescence and splitting issues.) Quasi-eigenvalues of minimal decay are defined as the points of the Pareto set. Note that this provides sharp bounds on quasi-eigenvalues in terms of $\|dM\|$ when the Pareto set is found (see Theorem 7.2).

Section 3 contains basic properties and definitions concerning quasi-eigenvalues, as well as preparative results including: analyticity and derivatives of the characteristic determinant $F$, discussion
of multiplicity and one-parameter perturbation formulae for quasi-eigenvalues, and an example of a degenerate and non-differentiable quasi-eigenvalue.

In Section 4, we introduce the notion of local boundary point for images of set-valued maps. Local boundary points are introduced as an extension of the notion of local extrema to vector optimization problems with two criteria (these are the frequency \( \alpha = \text{Re} \omega \) and the decay rate \( \beta = -\text{Im} \omega \)). Alternatively, the set of local boundary point can be considered as the local version of the Pareto set. Theorem 4.1 reduces the study of local boundary points to convex analysis of the \( x \)-trajectory of the mode. The proof is based on two-parameter perturbations of Lemma 4.3. This lemma deals with a zero surface of an analytic function of three complex variables in a neighbourhood of a degenerate root. The proof of Lemma 4.3 refines that of [18, Lemma 3.6] with the use of homotopy arguments.

The rest of the paper is essentially concerned with the study of local boundary point of the set of admissible quasi-eigenvalues \( \Sigma[A_M] := \bigcup_{dM \in A_M} \Sigma(dM) \). The analysis of local boundary points is convenient since they are independent of the choice of the function of \( \omega \) that has to be optimized. In particular, for each frequency \( \alpha \), the quasi-eigenvalue of minimal decay rate for \( \alpha \) is a local boundary point of \( \Sigma[A_M] \).

It is easy to see that optimizers over \( A_M \) are not necessarily extreme points of \( A_M \). Indeed, all quasi-eigenvalues produced by the extreme points \( m\delta(x - x_n)dx \) belong to the circle \( \{ z \in \mathbb{C} : |z + i/m| = 1/m \} \) and so can not be optimal for high frequencies. Theorem 5.2 states that optimal measures belongs to finite-dimensional faces of \( A_M \). In other words, they consist of finite number of point masses. An example of an optimizer that is not an extreme point of \( A_M \) is given in Section 8. So the optimization over \( A_M \) is equivalent to the same problem over \( A_{\text{disc}} \). The absence of optimizers over \( A_1 \) (Theorem 5.3) is obtained as a by-product of this study.

In Sections 5.2 and 6, we derive restrictions on the point masses and their positions. These restrictions are strong enough to calculate optimal \( dM \) if the optimal quasi-eigenvalue \( \omega \), the first point mass \( m_1 \), and one more geometric parameter are known. In Section 7, this allows us to explicitly find the Pareto frontier and corresponding optimal measures \( dM \) for small frequencies.

The optimization problems of this paper can be considered as problems with two constraints, one for the total mass and the other for the length of the interval \( I = [0, \ell] \). In Section 8, we show that at least one of these constraints is achieved by every optimizer.

**Notation.** We use the convention that a sum equals zero if the lower index exceeds the upper.

The following sets of real and complex numbers are used: open half-lines \( \mathbb{R}_\pm = \{ x \in \mathbb{R} : \pm x > 0 \} \), open half-planes \( \mathbb{C}_\pm = \{ z \in \mathbb{C} : \pm \text{Im} z > 0 \} \), open discs \( D_\epsilon(\zeta) := \{ z \in \mathbb{C} : |z - \zeta| < \epsilon \} \) with the center at \( \zeta \) and radius \( \epsilon \), the unit circle \( T := \{ z \in \mathbb{C} : |z| = 1 \} \), the infinite sector (without the origin \( z = 0 \))

\[
\text{Sec}[\xi_1, \xi_2] := \{ z \in \mathbb{C} \setminus 0 : \arg z = \xi \ (\mod 2\pi) \} \text{ for certain } \xi \in [\xi_1, \xi_2], \quad \xi_1 \leq \xi_2, \quad \xi_{1,2} \in \mathbb{R},
\]

(1.7)

closed line-segments \([z_1, z_2]\) between endpoints \( z_{1,2} \in \mathbb{C} \), and line-segments with excluded endpoints \((z_1, z_2) := [z_1, z_2] \setminus \{z_1, z_2\}\). A line segment is called degenerate if its endpoints coincide. \( Q_I, Q_{II}, Q_{III}, \) and \( Q_{IV} \) are the open quadrants in \( \mathbb{C} \).

Let \( S \) be a subset of a linear space \( U \) over \( \mathbb{C} \) (including the case \( U = \mathbb{C} \)). For \( u_{0,1} \in U \) and \( z \in \mathbb{C} \),

\[
zS + u_0 := \{ zu + u_0 : u \in S \} \quad \text{and} \quad [u_0, u_1] := \{ (1 - \lambda)u_0 + \lambda u_1 : \lambda \in [0, 1] \}.
\]

The closure of a set \( S \) (in the norm or the Euclidean topology) is denoted by \( \overline{S} \), in particular, \( \overline{\mathbb{R}}_\pm = (-\infty, 0], \overline{\mathbb{R}}_+ = [0, +\infty) \), and \( \mathbb{R}_+^N = [0, +\infty)^N \) is the set of \( N \)-tuples of nonnegative numbers. \( \text{int} S \) and \( \text{bd} S \) are the sets of interior and boundary points of \( S \), respectively.
For basic definitions of convex analysis we refer to [36]. The convex hull of $S$ is denoted by conv $S$. The convex cone generated by the set $S$ (the set of all nonnegative linear combinations of elements of $S$) is denoted by cone $S$.

For open balls in a normed space $U$, we use $B_ε(u_0) = B_ε(u_0; U) := \{u \in U : \|u - u_0\|_U < ε\}$.

By $\mathcal{M}_C$ and $\| \cdot \|$ we denote the Banach space of complex Borel measures on $I = [0, \ell]$ and the corresponding norm. $\mathcal{M}_+$ is the cone of nonnegative measures in $\mathcal{M}_C$. The set of nonnegative measures in the closed unit ball is denoted by $\mathcal{B}^{+}_C := \{dM \in \mathcal{M}_+ : \|dM\| \leq 1\}$.

For $dM \in \mathcal{M}_C$ and a Borel set $S$, define the projection $pr_S dV$ of $dV$ to $S$ as the measure that coincides with $dV$ on $S$ and coincides with the zero-measure on $R \setminus S$. The topological support $\text{supp} dM$ of $dM$ is the smallest closed set $S$ such that $\|\text{pr}_{R \setminus S} dM\| = 0$. By $\int_{a,x}^{b,x} f(x) dM(x)$ the corresponding Lebesgue–Stieltjes integrals over the intervals $(a,b)$, $(a,b]$, $[a,b]$, and $[a,b)$ are denoted. The notation $0dx$ ($1dx$) means the zero measure on $I$ (resp., the Lebesgue measure on $I$).

$L^p(0,\ell)$ are the Lebesgue spaces of complex valued functions and

$$W^{k,p}[0,\ell] := \{y \in L^p(0,\ell) : \partial^j_y \in L^p(0,\ell), \ 1 \leq j \leq k\}$$

are Sobolev spaces with standard norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{W^{k,p}}$. The space of continuous complex valued functions with the uniform norm is denoted by $C[0,\ell]$.

For a function $f$ defined on a set $S$, $f[S]$ is the image of $S$. By $\partial_x f$, $\partial_{x'} f$, etc. denote (ordinary or partial) derivatives with respect to (w.r.t.) $x$, $z$, etc.

We write $z_1^{[n]} \simeq z_2^{[n]}$ as $n \to \infty$ if the sequences $z_1^{[n]} / z_2^{[n]}$ and $z_2^{[n]} / z_1^{[n]}$ are bounded for $n$ large enough.

Sometimes, the complex plane $C$ is considered as a real linear space $R^2$ with the scalar product

$$\langle z_1, z_2 \rangle_C := \text{Re} z_1 \text{Re} z_2 + \text{Im} z_1 \text{Im} z_2, \quad z_{1,2} \in C.$$ (1.8)

## 2 The statement of optimization problem

### 2.1 Quasi-eigenvalues of Krein strings

Let $\mathcal{M}_+$ be the set of bounded nonnegative Borel measures on the interval $I = [0,\ell]$ of a finite positive length $\ell$. In the settings of this paper, a (Krein) string is the interval $[0,\ell]$ carrying a dispersed mass, which is represented by a measure $dM \in \mathcal{M}_+$.

With the string one can associate the quasi-eigenvalue problem (1.2)–(1.4). To define the left- and right-hand derivatives $\partial_x^- y(0)$ and $\partial_x^+ y(\ell)$ in (1.3) and (1.4), it is convenient to assume that

$$dM \text{ is continued to } (-\infty,0) \text{ and } (\ell, +\infty) \text{ by the zero measure} \quad (2.1)$$

and $y$ satisfies (1.2) on $R$ (see [17, 2, 9] for details). (2.2)

**Remark 2.1.** Actually, each locally bounded Borel continuation of $dM$ to a vicinity of $[0,\ell]$ provides the same values of $\partial_x^- y(0)$ and $\partial_x^+ y(\ell)$. The continuation of $dM$ to $(\ell, \ell_1]$ by the Lebesgue measure was used in [24] since with such a special continuation the change of $\ell$ to $\ell_1$ in (1.4) saves the positions of quasi-eigenvalues.

Assuming that $dM$ is extended by (2.1), we define

$$a_s(dM) := \sup \{x \in (-\infty,\ell) : \text{ the restriction of } dM(s) \text{ on } (-\infty, x) \text{ is the zero measure} \}. \quad (2.3)$$
The eigenvalue problem (1.2)-(1.4) corresponds to free transverse harmonic oscillations of a string with the left end \( x = 0 \) sliding without friction and the damped right end \( x = \ell \). The damping force is proportional to the velocity of motion. A simple way to understand (1.2)-(1.4) is to rewrite the problem in the integral equation form

\[
y(x) = y(0) - \omega^2 \int_0^x \, dt \int_{0-}^{t-} y(s) \, dM(s),
\]

\[
y(\ell) - i\omega \int_{0-}^{\ell+} y(s) \, dM(s) = 0.
\]

In this paper, the boundary conditions and the length \( \ell \) of the interval \( \mathcal{I} \) are fixed. So the string is completely determined by the measure \( dM \). Therefore we will speak about the string \( dM \).

Eigen-parameters \( \omega \in \mathbb{C} \setminus \{0\} \) such that (1.2) has a nontrivial continuous solution \( y \) will be called quasi-eigenvalues of \( dM \). (Recall that a solution \( y \) is said to be nontrivial if \( y \neq 0 \) on \([0, \ell]\)). The real part \( \alpha = \text{Re} \omega \) of the quasi-eigenvalue \( \omega \) characterizes the frequency of oscillations corresponding to the mode \( y \). For simplicity, \( \alpha \) will be called the frequency of \( \omega \) (actually, the angular frequency of the oscillations equals \(|\alpha|\)). The minus imaginary part \( \beta = -\text{Im} \omega \) is always positive and characterizes the rate of decay of the oscillations.

The set of quasi-eigenvalues \( \omega \) of a string \( dM \) is denoted by \( \Sigma(dM) \). It is known that \( \Sigma(dM) \subset \mathbb{C}_- \), that quasi-eigenvalues are isolated, and that \( \infty \) is their only possible accumulation point [23, 24] (see also [8]). The case \( \omega = 0 \) is mathematically and physically special and is usually excluded from the considerations, see the explanations in [8] and Section 3.1.

**Example 2.1.** If \( dM = 1dx \) (the Lebesgue measure on \([0, \ell]\)) or \( dM = 0dx \) (the zero measure), then \( \Sigma(dM) \) is empty.

### 2.2 Minimal decay rate

The aim of the paper is to study optimization of quasi-eigenvalues of strings with constrained total mass and fixed length \( \ell \) of the interval \( \mathcal{I} \). We consider optimization over the admissible families \( \mathcal{A}_1 \) and \( \mathcal{A}_M \) defined by (1.5)-(1.6) with a constant \( m > 0 \).

In the following, the admissible family \( \mathcal{A} \) is either \( \mathcal{A}_1 \), or \( \mathcal{A}_M \), if not said otherwise. The strings in \( \mathcal{A} \) are called admissible. We say that a complex number \( \omega \) is an admissible quasi-eigenvalue if it belongs to the set \( \Sigma[\mathcal{A}] := \bigcup_{dM \in \mathcal{A}} \Sigma(dM) \), and say that \( \alpha \in \mathbb{R} \) is an admissible frequency if \( \alpha = \text{Re} \omega \) for some admissible quasi-eigenvalue \( \omega \).

**Definition 2.1** ([18]). Let \( \alpha \) be an admissible frequency for an admissible family \( \mathcal{A} \).

(i) The minimal decay rate \( \beta_{\min}(\alpha) = \beta_{\min}(\alpha; \mathcal{A}) \) for the frequency \( \alpha \) is defined by

\[
\beta_{\min}(\alpha) := \inf\{\beta \in \mathbb{R} : \alpha - i\beta \in \Sigma[\mathcal{A}]\}.
\]

(ii) If \( \omega = \alpha - i\beta_{\min}(\alpha) \) is a quasi-eigenvalue of certain admissible string \( dM \in \mathcal{A} \) (i.e., the infimum is achieved), we say that \( \omega \) and \( dM \) are of minimal decay for the frequency \( \alpha \) (over the admissible family \( \mathcal{A} \)).

### 3 Properties of quasi-eigenvalues, modes, and related maps

When a number \( z \) or a string \( dM \) is fixed we will write simply \( \varphi(x) \) or \( \varphi(x, z) \) instead of \( \varphi(x, z; dM_0) \) and will use the same shortening for the number \( a_* \) defined by (2.3).
3.1 Complex Krein strings and the characteristic determinant $F$

It is convenient for us to generalize problem (1.2)–(1.4) and the definition of quasi-eigenvalues to measures $dM$ from the class $\mathbb{M}_C$ of complex Borel measures on $[0, \ell]$. Recall that $\mathbb{M}_C$ is a complex Banach space with the norm $\|dM\| := \int_{0^-}^{\ell^+} |dM|$, where $|dM|$ denotes the total variation measure of $dM$.

Assuming that $dM$ is extended according to (2.1), we denote by $\varphi(x) = \varphi(x, z; dM)$ and $\psi(x) = \psi(x, z; dM)$ the solutions of $\frac{d^2}{dx^2} y(x) = -z^2 y(x)$ on $\mathbb{R}$ satisfying

$$\varphi(0, z; dM) = \partial_x^0 \psi(0, z; dM) = 1, \quad \partial_x^0 \varphi(0, z; dM) = \psi(0, z; dM) = 0.$$

The function $\varphi$ is absolutely continuous in each bounded interval and have the left and right derivatives $\partial_x^\pm \varphi$ at every $x \in \mathbb{R}$. Moreover, $\lim_{x \to x_0^+} \partial_x^\pm \varphi(x) = \partial_x^\pm \varphi(x_0)$, $\lim_{x \to x_0^-} \partial_x^\pm \varphi(x) = \partial_x^- \varphi(x_0)$. The same holds for the function $\psi$, see [17] for details.

Obviously, $\varphi(x, z; dM)$ is a unique solution to the integral equation

$$y(x) = 1 - z^2 \int_{0^-}^x (x-s) y(s) \, dM(s), \quad 0 \leq x \leq \ell. \quad (3.1)$$

The following lemma is a rigorous form for the integral reformulation of the quasi-eigenvalue problem.

**Lemma 3.1** (see e.g. [2]). A number $\omega \in \mathbb{C}$ belongs to $\Sigma(dM)$ if and only if there exists nontrivial $y(x) \in C[0, \ell]$ satisfying (3.1) with $z = \omega$ and (2.5).

Note that in the integral settings there is no need to exclude separately the case $\omega = 0$. When $\omega = 0$, it is easy to see that (2.4), (2.5) has no nontrivial solutions, see the proof of [18, Lemma 2.7].

Consider the functional

$$F(z; dM) := \varphi(\ell, z; dM) - iz \int_{0^-}^{\ell^+} \varphi(s, z; dM) \, dM(s), \quad (3.2)$$

produced by (2.4). When $z \neq 0$, one has

$$F(z; dM) = \varphi(\ell, z; dM) + \frac{i}{z} \partial_x^+ \varphi(\ell, z; dM).$$

From Lemma 3.1 or directly from the statement of the quasi-eigenvalue problem we see that $\omega \in \Sigma(dM)$ exactly when $F(\omega; dM) = 0$.

We say that a map $G : U_1 \to U_2$ between normed spaces $U_{1,2}$ is bounded-to-bounded if the set $G[S]$ in $U_2$ is bounded for every bounded set $S$ in $U_1$. Basic facts about analytic maps on Banach spaces can be found in [35].

**Lemma 3.2.** (i) The functional $F(z; dM)$ is analytic on $\mathbb{C} \times \mathbb{M}_C$. In particular, $\Sigma(dM)$ is the set of zeroes of the entire function $F(\cdot) = F(\cdot; dM)$.

(ii) The map $(z, dM) \mapsto \varphi(\cdot, z; dM)$ is bounded-to-bounded and analytic from $\mathbb{C} \times \mathbb{M}_C$ to $W_C^{1, \infty} [0, \ell]$. Its Maclaurin series is given by

$$\varphi(x, z; dM) = 1 - \varphi_1(x; dM)z^2 + \varphi_2(x; dM)z^4 - \varphi_3(x; dM)z^6 + \ldots, \quad (3.3)$$

$$\varphi_0(x; dM) \equiv 1, \quad \varphi_j(x; dM) = \int_{0^-}^x (x-s) \varphi_{j-1}(s; dM) dM(s), \quad j \in \mathbb{N}. \quad (3.4)$$
Proof. (i) follows immediately from (ii). Let us prove (ii). Consider (3.3) as a $C[0,\ell]$-valued series. One can see that $|\varphi_j(x; dM)| \leq \varphi_j(x; |dM|) \leq \varphi_j(\ell; |dM|)$ for all $x \in [0,\ell]$. By [17, Section 2] (see also [9, Exercise 5.4.2]), one has

$$\varphi_j(\ell; |dM|) \leq \frac{(2\ell||dM||_M)^j}{(2j)!}, \quad j \in \mathbb{N}. \quad (3.5)$$

Hence, the series (3.3) converge uniformly on every bounded set of $\mathbb{C} \times M_\mathbb{C}$. So (3.3) defines an analytic map from $\mathbb{C} \times M_\mathbb{C}$ to $C[0,\ell]$. This map is also bounded-to-bounded due to (3.5). Since (3.3) satisfies (3.1), series (3.3) is the Maclaurin series of $\varphi$. By (3.1),

$$\partial^+_x \varphi(x) = -z^2 \int_{0-}^{x\pm} \varphi(x) dM(s), \quad x \in (0,\ell], \quad (3.6)$$

and

$$\partial_x \varphi(x) = \partial^+_x \varphi(x) = \partial^-_x \varphi(x) \text{ a.e. on } (0,\ell). \quad (3.7)$$

This implies $\partial_x \varphi \in L^\infty(0,\ell)$. Plugging (3.3) into (3.6)-(3.7), one gets an $L^\infty$-valued Maclaurin series for $\partial_x \varphi$. Due to (3.5), this series converges uniformly on every bounded set of $\mathbb{C} \times M_\mathbb{C}$ to a bounded-to-bounded map from $\mathbb{C} \times M_\mathbb{C}$ to $L^\infty(0,\ell)$. This proves (ii). \hfill $\square$

For Sturm-Liouville equations, formal expansions of such type were already known to Hermann Weyl, while the analyticity of fundamental solution w.r.t. the coefficients of the equation was emphasized and intensively used in [35].

### 3.2 Multiplicities and examples of degenerate quasi-eigenvalues

It is obvious that all modes $y(\cdot)$ corresponding to $\omega \in \Sigma(dM)$ are equal to $\varphi(\cdot, \omega; dM)$ up to a multiplication by a constant. So the geometric multiplicity of any quasi-eigenvalue equals 1. In the following, the multiplicity of a quasi-eigenvalue means its algebraic multiplicity.

**Definition 3.1.** The multiplicity of a quasi-eigenvalue is its multiplicity as a zero of the entire function $F(\cdot; dM)$. A quasi-eigenvalue is called simple if its multiplicity is 1.

This definition was used in [2, 23, 24] for nonnegative $dM$ and is a natural extension of the classical M.V. Keldysh definition of multiplicity for eigenvalue problems with an eigen-parameter in boundary conditions [29]. Indeed, when $dM = Bdx$, the function $F(z) = F(z; dM)$ is equal to the characteristic determinant [29] of (1.2)-(1.4) up to a multiplication on a nonzero constant.

Since $F(0; \cdot) \equiv 1$, each quasi-eigenvalue has a finite multiplicity. There exist strings $dM \in M_+$ with multiple quasi-eigenvalues (i.e., quasi-eigenvalues of multiplicity $\geq 2$). A simple example that fits the settings of the present paper was given recently in [19, Remark 2.1] (see formula (3.9) below). For slightly different classes of strings, the existence of multiple quasi-eigenvalues can be obtained from the results on the direct spectral problem for quasi-eigenvalues [24, Theorem 3.1], [13], and [34, Theorem 4.1] and was explicitly noticed in [28] (the last paper provides an example with the boundary condition $y(0) = 0$ instead of (1.3)).

**Example 3.3 ([19]).** Let $dM$ be the string consisting of a mass $m_0 > 0$ placed at a point $x_0 \in [0,\ell]$, i.e., writing with Dirac’s $\delta$-function $dM = m_0 \delta(x - x_0) dx$. Then the set of quasi-eigenvalues of $dM$ has the following description:

$$\Sigma(dM) = \begin{cases} \{-im_0^{-1}\} & \text{if } x_0 = \ell, \\ -i \frac{1}{2(\ell - x_0)} \pm \sqrt{\frac{1}{m_0(\ell - x_0)^2} - \frac{1}{4(\ell - x_0)^2}} & \text{if } x_0 < \ell. \end{cases} \quad (3.8)$$

$$\Sigma(dM) = \begin{cases} \{-im_0^{-1}\} & \text{if } x_0 = \ell, \\ -i \frac{1}{2(\ell - x_0)} \pm \sqrt{\frac{1}{m_0(\ell - x_0)^2} - \frac{1}{4(\ell - x_0)^2}} & \text{if } x_0 < \ell. \end{cases} \quad (3.9)$$
These formulae take into account the multiplicities. That is, when \( 4(\ell - x_0) = m_0 \), (3.9) means that \( \frac{-1}{2(\ell - x_0)} \) is a quasi-eigenvalue of multiplicity 2. In the other cases, each quasi-eigenvalue is simple.

### 3.3 Trajectories of \( \varphi \)-solutions

**Lemma 3.4.** (i) \( \varphi(x, z, dM) = 1 \) for all \( x \in [0, a_*] \).

(ii) If \( dM \) is a nonnegative measure, \( z^2 \notin \mathbb{R} \), and \( x \in (a_*, \ell) \), then

\[
\text{Im} z^2 \text{Im} \left[ \varphi(x, z) \partial_x^\pm \varphi(x, z) \right] < 0,
\]

in particular, \( \varphi(x) \neq 0 \) and \( \partial_x^\pm \varphi(x) \neq 0 \). (Recall that \( a_* = a_*(dM) \) is defined by (2.3).)

Statement (i) is obvious. Statement (ii) follows immediately from the well-known particular case of the Lagrange identity \( \text{Im} \frac{\varphi(s, z)}{\sqrt{\lambda}} | \varphi(s, z) |^2 dM = - \text{Im} \left[ \varphi(x, z) \partial_x^\pm \varphi(x, z) \right] \). The details can be found in [17, Section 2] (note that there is an obvious misprint in [17, formula (2.25)], namely, the last part of the equality has to be with minus sign). From another point of view, (3.10) is a reformulation of the fact that the Titchmarsh-Weyl m-coefficients \( \frac{\varphi(x, \sqrt{\lambda})}{\partial_x^\pm \varphi(x, \sqrt{\lambda})} \) (associated with problem (1.2), (1.3) on the interval \([0, x]\) and depending on \( \lambda = z^2 \)) are non-degenerate (R)-functions (Nevanlinna functions) in the terminology of [17]. One more interpretation of this fact in terms of the trajectory of \( \varphi(\cdot, z) \) is given in the next lemma.

**Lemma 3.5.** Let \( dM \) be a nonnegative measure and \( z^2 \notin \mathbb{R} \). Denote by

\[ \text{arg}_x \varphi(x) \]

the continuous in \( x \) branch of \( \text{arg} \varphi(x) \) fixed by \( \text{arg}_x \varphi(0) = 0 \).

Then:

in the case \( z^2 \in \mathbb{C}_- \), \( \text{arg}_x \varphi(x) \) is strictly increasing on \( (a_*, \ell) \); \hspace{1cm} (3.11)

in the case \( z^2 \in \mathbb{C}_+ \), \( \text{arg}_x \varphi(x) \) is strictly decreasing on \( (a_*, \ell) \).

(Note that \( \text{arg}_x \varphi(x) \) is well-defined due to Lemma 3.4 and the equality \( \varphi(0) = 1 \).)

The following statement is obvious from the integral equation (3.1).

**Lemma 3.6.** Let \( (x_0, x_1) \cap \text{supp} \ dM = \emptyset \). Then the point \( \varphi(x) \) with \( x \) running through the interval \( (x_0, x_1) \) either moves along the ray \( \{ \varphi(x_0) + s \partial_x^\pm \varphi(x_0) : s > 0 \} \) with the constant speed \( \partial_x^\pm \varphi(x_0) \), or stays at \( \varphi(x_0) \) (in the case \( \partial_x^\pm \varphi(x_0) = 0 \)).

### 3.4 Quasi-eigenvalues’ perturbations and the derivatives of \( F \).

Note that if \( \omega \) is a quasi-eigenvalue of \( dM_0 \), then \( \omega \neq 0 \) (since \( F(0; dM_0) = 1 \)) and \( \varphi(\ell, \omega; dM_0) \neq 0 \) (otherwise (1.4) implies \( \partial_x^\pm \varphi(\ell) = 0 \) and, in turn, \( \varphi(x) \equiv 0 \) on \([0, \ell]\)).

**Proposition 3.7.** Let \( \omega \in \Sigma(dM_0) \) and \( \varphi(x) = \varphi(x, \omega; dM_0) \). At the point \( (\omega, dM_0) \in \mathbb{C} \times M_\mathbb{C} \), the derivative in \( z \) of the functional \( F(z; dM) \) is given by

\[
\partial_z F(\omega; dM_0) = -\frac{\varphi(\ell)}{\omega} + \frac{2i}{\varphi(\ell)} \int_{0^-}^{\ell^+} \varphi^2(s) \ dM_0(s),
\]

and the directional derivative of \( F \) w.r.t. \( dM \) in a direction \( dV \in M_\mathbb{C} \) equals

\[
\frac{\partial F(\omega; dM_0)}{\partial M} (dV) = -\frac{i \omega}{\varphi(\ell)} \int_{0^-}^{\ell^+} \varphi^2(s) \ dV(s).
\]
\textbf{Proof.} Put } \psi(x) := \psi(x, \omega; dM_0). \text{ The formulae }
\begin{align*}
\partial_\ell F(\omega; dM_0) &= -\frac{\varphi(\ell)}{\omega} + 2 [\omega \psi(\ell) + i \partial_x \psi(\ell)] \int_{0^-}^{\ell^+} \varphi^2(s) dM_0(s), \\
\frac{\partial F(\omega; dM_0)}{\partial M}(dV) &= -\omega \left[ \omega \psi(\ell) + i \partial_x \psi(\ell) \right] \int_{0^-}^{\ell^+} \varphi^2(s) dV(s),
\end{align*}
\text{can be obtained in the lines of the proof of [18, Lemma 3.2] with the change of the usual variation of parameters method to its integral equation version [3, Sec. 11] (see also [17]). To obtain (3.12) and (3.13), it remains to note that }
\omega \psi(\ell) + i \partial_x \psi(\ell) = i / \varphi(\ell). \tag{3.14}
\text{The last identity easily follows from } \varphi(\ell) = -\frac{i}{\omega} \partial_\ell^+ \varphi(\ell) \text{ and the constancy of the Wronskian }
\begin{vmatrix}
\varphi(x) & \psi(x) \\
\partial_\ell^+ \varphi(x) & \partial_\ell^+ \psi(x)
\end{vmatrix} = 1. \tag{3.15}
\]

In the rest of this subsection we assume that \( \omega \) is a quasi-eigenvalue of algebraic multiplicity \( r \in \mathbb{N} \) of a nonnegative string \( dM_0 \). Proposition 3.7 leads to several statements given below. Though we do not use them directly, they stand behind crucial points of the subsequent sections.

The Fréchet derivative \( \frac{\partial F(\omega; dM_0)}{\partial M} \) is nonzero. The directional derivative \( \frac{\partial F(\omega; dM_0)}{\partial M}(dV) \) is nonzero whenever the direction \( dV \) is an atom measure \( \delta(x - x_0)dx \) with \( x_0 \in \mathcal{I} \) (more generally, whenever nonnegative and nonzero \( dV \) is supported on a small enough interval).

\textbf{Lemma 3.8.} The quasi-eigenvalue \( \omega \) is non-simple exactly when \( \varphi^2(\ell) = 2i \omega \int_{0^-}^{\ell^+} \varphi^2(s) dM_0(s) \).

When \( \omega \) is a simple quasi-eigenvalue of \( dM_0 \), the implicit function theorem for analytic maps implies that there exists a functional \( \Omega \) analytic in a certain neighborhood \( W \) of \( dM_0 \) such that \( \Omega(dM_0) = \omega_0 \) and \( \Omega(dM) \in \Sigma(dM) \) for all \( dM \in W \). The derivative \( \frac{\partial \Omega(dM_0)}{\partial M}(dV) \) of \( \Omega \) at \( dM_0 \) in a direction \( dV \in M_\mathbb{C} \) is given by
\[ \frac{\partial \Omega(dM_0)}{\partial M}(dV) = \frac{\omega^2 \int_{0^-}^{\ell^+} \varphi^2(s, \omega; dM_0) dV(s)}{2 \omega \int_{0^-}^{\ell^+} \varphi^2(s, \omega_0; dM_0) dM_0(s) + i \varphi^2(\ell, \omega_0; dM_0)}. \tag{3.16} \]

\textbf{Remark 3.1.} For absolutely continuous \( dM \) and \( dV \) and the boundary condition \( y(0) = 0 \), the analogues of Lemma 3.8, formula (3.16), as well as higher order corrections for \( \Omega \) were obtained in Physics papers [27], [28] (with slightly more intuitive arguments).

When \( \omega \) is a multiple quasi-eigenvalue (i.e., \( r \geq 2 \)) and \( \frac{\partial F(\omega; dM_0)}{\partial M}(dV) \neq 0 \), it is possible to obtain from the Weierstrass preparation theorem that there exist nonempty open discs \( \mathbb{D}_\delta(0), \mathbb{D}_\varepsilon(\omega) \), and a convergent in \( \mathbb{D}_\delta(0) \) \( r \)-valued Puiseux series
\[ \Omega(\zeta) = \omega + \sum_{n=1}^{\infty} c_n \zeta^{j/r} \text{ with } c_1 = \left[ \frac{\Gamma(r) \omega \int_{0^-}^{\ell^+} \varphi^2(s, \omega) dV(s)}{\varphi(\ell, \omega) \partial_\ell^+ F(\omega; dM_0)} \right]^{1/r} \neq 0 \tag{3.17} \]
such that for each \( \zeta \in \mathbb{D}_\delta(0) \), the \( r \) values of \( \Omega(\zeta) \) give all the quasi-eigenvalues of \( dM_0 + \zeta dV \) in \( \mathbb{D}_\varepsilon(\omega) \) and all these \( r \) quasi-eigenvalues are distinct and simple (for details, see the proof of [18, Proposition 3.5] and also Lemma 9.1 below). When \( r = 1 \), (3.17) turns into (3.16).
4 Local boundary points and two-parameter perturbations

The quasi-eigenvalues of $dM$ are zeroes of the entire function $F(\cdot) = F(\cdot; dM)$. If $\omega$ is a boundary point of the set of admissible quasi-eigenvalues $\Sigma[\mathcal{A}]$, then perturbations of $\omega_0$ as a zero of $F(\cdot; dM)$ under small changes of $dM$ inside the admissible family $\mathcal{A}$ cannot cover a neighborhood of $\omega$. We want to use this fact to derive strong restriction on the structure of strings that produce quasi-eigenvalues on the boundary of $\Sigma[\mathcal{A}]$ (and, in particular, on the structure of strings of minimal decay).

This approach requires the study of multi-parameter perturbations of zeroes of analytic functions of three complex variables.

Let $U$ be a topological space. Consider a functional $G : \mathbb{C} \times U \to \mathbb{C}$. For $u \in U$, denote by

$$
\Sigma_G(u) := \{ z \in \mathbb{C} : G(z; u) = 0 \}
$$

the set of zeroes of the function $G(\cdot, u)$. Then $\Sigma_G$ is a set-valued map from $U$ to $\mathbb{C}$. We will systematically use only one specific definition of the theory of set-valued maps. For a subset $S$ of $U$, the image $\Sigma_G[S]$ of $S$ is defined by $\Sigma_G[S] := \bigcup_{u \in S} \Sigma_G(u)$.

**Definition 4.1.** Let $S$ be a set in a topological space $U$. Let $u \in S$ and $z \in \Sigma_G(u)$. Then:

(i) the complex number $z$ is called a $u$-local boundary point of the image $\Sigma_G[S]$ if $z$ is a boundary point of $\Sigma_G[S \cap W]$ for a certain open neighborhood $W$ of $u$.

(ii) $z$ is called a $u$-local interior point $\Sigma_G[S]$ if $z$ is an interior point of the set $\Sigma_G[S \cap W]$ for any open neighborhood $W$ of $u$.

If $u \in S$, then a complex number $z \in \Sigma_G(u)$ is a $u$-local interior point of $\Sigma_G(S)$ exactly when it is not a $u$-local boundary point of $\Sigma_G[S]$. The above definition depends not only on the point $z$ and the set $\Sigma_G[S]$, but also on the choice of $u$, $S$, $G$, and a topology in $U$. Our main (but not only) uses of the above definition will involve $\mathcal{M}_C$ with the norm or weak* topology as a topological space $U$ and the functional $F$ defined by (3.2). The set-valued map $dM \mapsto \Sigma(dM)$ that associates a string $dM$ with the set of the string’s quasi-eigenvalues can be identified with the map $\Sigma_F$, see Lemma 3.2 (i). So $\Sigma(dM) = \Sigma_F(dM)$ and $\Sigma[\mathcal{A}] = \Sigma_F[\mathcal{A}]$.

**Definition 4.2.** Let $\mathcal{M}_C$ equipped with the norm topology (the weak* topology) be taken as the topological space $U$. Let $dM \in \mathcal{A}$ and $\omega \in \Sigma(dM)$. Then $\omega$ is called a strongly (resp., weakly*) $dM$-local boundary point of $\Sigma[\mathcal{A}]$ if $\omega$ is a $dM$-local boundary point of $\Sigma_F[\mathcal{A}]$ in the sense of Definition 4.1.

When $U$ is a Banach space and $G : \mathbb{C} \times U \to \mathbb{C}$ is analytic, we denote by

$$
\frac{\partial G(z; u_0)}{\partial u}(u_1) := \lim_{\zeta \to 0} \frac{G(z; u_0 + \zeta u_1) - G(z; u_0)}{\zeta},
$$

the directional derivative of $G(z; \cdot)$ along the vector $u_1 \in U$ at the point $(z, u_0) \in \mathbb{C} \times U$. By $\frac{\partial G(z; u_0)}{\partial u}$ the corresponding Fréchet derivative in $u$ is denoted. The Fréchet derivative $\frac{\partial G(z; u_0)}{\partial u}$ is a linear functional on $U$ and $\frac{\partial G(z; u_0)}{\partial u}[S]$ defines the image of a set $S \subset U$ under this functional.

The following theorem is one of our main technical tools.

**Theorem 4.1.** Let $U$ be a Banach space and $G(\cdot; \cdot)$ be an analytic functional on $\mathbb{C} \times U$. Assume that
(i) \( S \) is a convex subset of \( U \),
(ii) \( z_0 \in \mathbb{C} \) and \( u_0 \in S \) are such that \( G(z_0, u_0) = 0 \),
(iii) \( \frac{\partial G(z_0; u_0)}{\partial u}(u_0) \) is an interior point of the set of directional derivatives \( \frac{\partial G(z_0; u_0)}{\partial u}[S] \). Then \( z_0 \) is a strongly \( u_0 \)-local interior point of \( \Sigma_G[S] \).

Theorem 4.1 can be easily obtained from the two following statements. The first of them gives several equivalent reformulations of the condition (iii) and is an easy exercise in Convex Analysis.

**Proposition 4.2.** Under assumptions (i) and (ii) of Theorem 4.1, condition (iii) is equivalent to each of the following conditions:

(iii') The origin 0 is an interior point of the set \( \frac{\partial G(z_0; u_0)}{\partial u}[S - u_0] \).

(iii'') 0 is an interior point of cone \( \frac{\partial G(z_0; u_0)}{\partial u}[S - u_0] \).

(iii''') cone \( \frac{\partial G(z_0; u_0)}{\partial u}[S - u_0] = \mathbb{C} \).

Here and below cone \( W \) is the (nonnegative) convex cone generated by a set \( W \).

The second one, Lemma 4.3, is the technical core of Theorem 4.1. Essentially, it describes local structure of the sets covered by the zeroes of \( G(\cdot; u) \) under two-parameter perturbations of \( u \).

In Lemma 4.3 and its proof, a 2-tuple \( \zeta = (\zeta_1, \zeta_2) \) belongs to \( \mathbb{C}^2 \) and \( Q(z; \zeta) = Q(z, \zeta_1, \zeta_2) \) is a function of three complex variables analytic in a neighborhood of the origin \( 0 = (0, 0, 0) \). As before, \( \Sigma_Q(\zeta) \) is the set of zeroes of \( Q(\cdot; \zeta) \) and \( \Sigma_Q[S] := \bigcup_{\zeta \in S} \Sigma_Q(\zeta) \) for any set \( S \subset \mathbb{C}^2 \). We use (real) triangles \( T_\delta := \{\zeta = (\zeta_1, \zeta_2) : \zeta_1, \zeta_2 \in \mathbb{R}_+ \text{ and } \zeta_1 + \zeta_2 < \delta\} \).

**Lemma 4.3.** Let \( Q(z, \zeta_1, \zeta_2) \) be a function of three complex variables analytic in a neighborhood of the origin \( 0 \). Assume that 0 is an \( r \)-fold zero \((1 \leq r < \infty)\) of the function \( Q(\cdot, 0, 0) \). Denoting

\[
\eta_j := -\frac{r! \partial^2_{\zeta_j} Q(0)}{\partial^2_{\zeta} Q(0)}, \quad j = 1, 2, \tag{4.1}
\]

assume that

\[
\eta_1 \neq 0, \quad \eta_2 \neq 0 \quad \text{and} \quad \arg \eta_2 = \arg \eta_1 + \xi_0 \pmod{2\pi} \quad \text{with certain } \xi_0 \in (0, \pi). \tag{4.2}
\]

Then for small enough and positive \( \delta_1 \) and \( \delta_2 \), there exists \( \varepsilon > 0 \) such that

\[
D_\varepsilon(0) \cap \text{Sec}[\arg \sqrt{\eta_1} + \delta_2, \arg \sqrt{\eta_2} - \delta_2] \subset \Sigma_Q[T_{\delta_1}].
\]

Here \( \sqrt{\zeta} \) and \( \arg z \) are arbitrary branches of the corresponding multi-functions continuous on the infinite sectors \( \text{Sec}[\arg \eta_1, \arg \eta_1 + \xi_0] \) and \( \text{Sec}[\arg \sqrt{\eta_1}, \arg \sqrt{\eta_1} + \xi_0/r] \), respectively.

The proof of Lemma 4.3 is given in Appendix. It is an essential refinement of that of [18, Lemma 3.6]. With the use of continuous curve of zeroes produced in [18, Lemma 3.6], we construct a loop and show that this loop can be homotopically shrunken into origin. As a result, the projections of zeroes cover the desired set.
5 Existence, non-existence, and discrete strings

The next proposition easily follows from Lemmas 3.1, 3.2 (ii), Example 3.3, and standard weak* compactness arguments [22, 15] (the scheme of the proof is the same as in [18]).

Proposition 5.1. (i) The set $\Sigma[A_M]$ is closed.

(ii) For each frequency $\alpha \in \mathbb{R}$, there exists a string of minimal decay over $A_M$.

In the sequel, we fix a string $dM_0$ and its quasi-eigenvalue $\omega$ and then show that various assumptions on $\omega$ impose strong restrictions on $dM_0$ and the corresponding mode

$$
\Phi(x) = \varphi(x, \omega; dM_0).
$$

This section is devoted to two following theorems.

Theorem 5.2. Let a string $dM_0$ be of minimal decay over $A_M$ for a frequency $\alpha \in \mathbb{R}$. Then $dM_0$ consists of a finite number of point masses, i.e., $dM_0 = \sum_{j=1}^{n} m_j \delta(x - a_j)dx$ with $n \in \mathbb{N}$, positive $m_j$, and distinct increasingly ordered $a_j \in I$.

Remark 5.1. Theorem 5.2 will be further refined in Sections 6.1 and 8.1, where additional relations connecting $a_j, m_j$, and $\Phi$ will be derived and an iterative procedure for calculation of $m_j$ and $a_j$ will be given.

The proof of the theorem is given in Subsection 5.3. Essentially, it is based on the study of $dM_0 \in A_M$ and $\omega \in \Sigma[A_M]$ such that

$$
\omega \text{ is a strongly } dM_0\text{-local boundary point of } \Sigma[A_M],
$$

(5.1)

see Subsection 5.2. These subsection contains a description of specific properties of the corresponding mode $\Phi$ and their connections with the structure of $dM_0$. Then, Theorem 5.2 is obtained with the use of the obvious implications

$$
dM_0 \text{ and } \omega \text{ are of minimal decay for a frequency } \alpha \text{ (over } A) \implies \omega \text{ is a weakly* } dM_0\text{-local boundary point of } \Sigma[A_M] \implies \omega \text{ is a strongly } dM_0\text{-local boundary point of } \Sigma[A_M].
$$

(5.2) (5.3) (5.4)

As a by-product of the study of optimization over $A_M$, we obtain the following result concerning optimization over the admissible family $A_1$.

Theorem 5.3. There are no strings of minimal decay over $A_1$ (for any frequency $\alpha \in \mathbb{R}$).

This result is proved in Subsection 5.3.

Recall that $a_* = a_*(dM_0)$ equals $\min \supp dM$ when $dM$ is nonzero, otherwise $a_* := \ell$. The trajectories associated with the functions $\Phi(\cdot)$ and $\Phi^2(\cdot) := (\Phi(\cdot))^2$ are denoted by

$$
\Phi[I] := \{ \Phi(x) : x \in [0, \ell] \} \text{ and } \Phi^2[I] := \{ \Phi^2(x) : x \in [0, \ell] \}.
$$

In the sequel, points of $\mathbb{C}$ are perceived both as complex numbers and as $\mathbb{R}^2$-vectors. In particular, hyperplanes in $\mathbb{C}$ are lines. By $\langle z_1, z_2 \rangle_C := \text{Re } z_1 \text{ Re } z_2 + \text{Im } z_1 \text{ Im } z_2$ we denote the $\mathbb{R}^2$ scalar product of two complex numbers. Note that, for every $c \in \mathbb{C}$,

$$
\langle cz_1, cz_2 \rangle_C = |c|^2 \langle z_1, z_2 \rangle_C.
$$

(5.5)

In this and subsequent sections, by $\sqrt{z}$ and $\text{arg}_0 z$ we denote continuous in $\mathbb{C} \setminus \mathbb{R}_-$ branches of $z^{1/2}$ and $\text{arg } z$, resp., fixed by $\sqrt{T} = 1$ and $\text{arg}_0 1 = 0$. For $z \in \mathbb{R}_-$, put $\text{arg}_0 z = -\pi$. 


5.1 Lemmas on directions producing extremal derivatives.

Denote
\[ \overline{B}^1_1 := \{dM \in \mathbb{M}_1 : \|dM\| \leq 1\} \quad \text{and} \quad \overline{B}_{1,ac}^+ := \{dM \in \overline{B}^1_1 : dM = B(x)dx \text{ with } B \in L^1\}. \]

Since \( A_M = m\overline{B}^1_1 \), Proposition 3.7 shows that the set of the directional derivatives \( \frac{\partial F(\omega, dM)}{\partial M} \) (dV) with dV running through the admissible family \( A_M \) is the image of \( \overline{B}^1_1 \) under the functional \( \left\langle -\frac{m}{\Phi(\ell)} \Phi^2, \cdot \right\rangle \) (defined on \( \mathbb{M} \)). Boundary points of this image will play a special role due to Theorem 4.1.

With a function \( y \in C[0, \ell] \) and a set \( W \subset \mathbb{M} \), we associate the set of complex numbers
\[ \langle y, W \rangle = \left\{ \int_{0-}^{t+} y(s) \, dV(s) : dV \in W \right\}. \]

Note that \( \langle y, W \rangle \) is convex whenever \( W \) is convex and, in particular, when \( W = \overline{B}^1_1 \) or \( W = \overline{B}_{1,ac}^+ \).

**Lemma 5.4.** Let \( W = \overline{B}^1_1 \) or \( W = \overline{B}_{1,ac}^+ \). Suppose \( dV \in W \), \( \zeta = \langle y, dV \rangle \), and that \( \zeta \) is an extreme point of \( \langle y, W \rangle \). Then \( y|\text{supp} \, dV| = \{y(x) : x \in \text{supp} \, dV\} \) is a subset of \( \{\zeta\} \), i.e., \( y|\text{supp} \, dV| \) either consists of the single point \( \zeta \), or is empty. If, additionally, \( \|dV\| < 1 \), then \( \zeta = 0 \).

**Proof.** Step 1. Let us prove the lemma for the case when \( y \) is real-valued. Obviously,
\[ \left( \min_{x \in I} y(x), \max_{x \in I} y(x) \right) \cup \{0\} \subset \langle y, W \rangle \subset \text{conv}\{\min_{x \in I} y(x), \max_{x \in I} y(x), 0\}. \]

Assume that \( \min y \cdot \max y \leq 0 \). Since \( \zeta \) is extreme, we see that either \( \zeta = \min y \), or \( \zeta = \max y \). To be specific, assume \( \zeta = \max y \). If \( \|\text{pr}_{\{x : y(x) < \zeta\}} dV\| > 0 \), then \( \zeta = \int y dV < \zeta \), a contradiction. So \( y(x) = \zeta \) for all \( x \in \text{supp} \, dV \), and \( \zeta = \zeta \|dV\| \). The latter implies that \( \zeta = 0 \) in the case \( \|dV\| < 1 \).

These arguments work also in the case \( \zeta \neq 0 \). Assume now that \( \min y \cdot \max y > 0 \) and \( \zeta = 0 \). Then \( dV = 0 \, dx \) and \( \text{supp} \, dV = \emptyset \).

**Step 2. The general case.** Since \( \zeta \) is extreme, there exists a supporting line \( L \) to \( \langle y, W \rangle \) at \( \zeta \). Let us write \( L \) in the form \( \{z \in \mathbb{C} : \langle z - \zeta, p\rangle_C = 0\} \) with certain \( p \in \mathbb{C} \setminus \{0\} \). Applying Step 1 to the real-valued function \( y_1(x) := \langle y(x), p\rangle_C \), one can show that \( y|\text{supp} \, dV| \subset L \) and, moreover, that \( 0 \in L \) in the case \( \|dV\| < 1 \).

Since \( \zeta \) is an extreme point of \( S_1 := L \cap \langle y, W \rangle \), we see that
\[ S_1 \text{ lies in one of the two rays } \zeta \pm ip\mathbb{R}_+. \] (5.6)

In the case \( \|dV\| = 1 \), this immediately implies that \( \|\text{pr}_{\{y(x) \neq \zeta\}} dV\| = 0 \) and \( y|\text{supp} \, dV| = \{\zeta\} \).

In the case \( \|dV\| < 1 \), we see that the line-segment \( [0, \|dV\|^{-1}\zeta] \) is a subset of \( S_1 \). Since \( \zeta \) is extreme, \( \zeta = 0 \). Now (5.6) easily implies \( y|\text{supp} \, dV| \subset \{\zeta\} \).

It is essential for the rest of the paper that \( \langle y, \overline{B}^1_1 \rangle \) contains the trajectory \( y[L] = \{y(x) : x \in [0, \ell]\} \) and the origin \( 0 \). The following sharpening of this obvious fact is not crucial, but makes many of subsequent proofs more transparent.

**Lemma 5.5.** \( \langle y, \overline{B}^1_1 \rangle = \text{conv} (y[L] \cup \{0\}) \).
Proof. By the Helly selection and convergence theorems, the set \( \langle y, B_1^+ \rangle \) is compact and convex. So, by the Minkowski theorem, \( \langle y, B_1^+ \rangle \) is a convex hull of the set \( \text{Ext} \) of its extreme points. Let us show that \( \text{Ext} \subset y[\mathbb{Z}] \cup \{0\} \). Indeed, assume that a nonzero point \( z \) is extreme and \( z = \langle y, dV \rangle \) with \( dV \in B_1^+ \). By Lemma 5.4, \( \|dV\| = 1 \) and \( y[\text{supp } dV] = \{z\} \). Taking arbitrary \( x_0 \in \text{supp } dV \) one can see that \( z = \langle y, \delta(x-x_0)dx \rangle = y(x_0) \in y[\mathbb{Z}] \).

Lemma 5.6. Let \( W = B_1^+ \) or \( W = B_1^{+\text{ac}} \). Assume that \( dV \in W \) and \( \zeta := \langle y, dV \rangle \) is a boundary point of the set \( \langle y, W \rangle \). Then there exists a supporting line (hyperplane) \( L \) to the set \( \langle y, W \rangle \) at the point \( \zeta \). For every such \( L \) the following assertions hold:

(i) \( y[\text{supp } dV] \subset L \),

(ii) if additionally \( \|dV\| < 1 \), then \( 0 \in L \).

Proof. The existence of a supporting line follows immediately from \( \zeta \in \text{bd}\langle y, W \rangle \). Assertions (i) and (ii) can be easily obtained from the real-valued case of Lemma 5.4 and the arguments of Step 2 of its proof.

Now we apply this lemma to strongly local boundary points. Assume that \( \mathbb{A} = \mathbb{A}_M \) or \( \mathbb{A} = \mathbb{A}_1 \) and \( dM_0 \in \mathbb{A} \). The set \( \langle \Phi^2, m^{-1}\mathbb{A} \rangle \) consists of the complex numbers \( \langle \Phi^2, m^{-1}dM \rangle := \frac{1}{m} \int_{0}^{1} \Phi^2(s) dM(s) \) produced by admissible strings \( dM \in \mathbb{A} \). On the other side, by (3.13),

\[
\langle \Phi^2, m^{-1}dM \rangle = \frac{C_0}{m} \frac{\partial F(\omega, dM_0)}{\partial M} (dM), \quad \text{where } C_0 := -\frac{\Phi(\ell)}{i\omega} \text{ is a nonzero constant.} \tag{5.7}
\]

So \( \langle \Phi^2, m^{-1}\mathbb{A} \rangle \) is the image of the admissible family \( \mathbb{A} \) under the Fréchet derivative \( \frac{\partial F(\omega, dM_0)}{\partial M} \) additionally scaled and rotated by the multiplication on \( C_0/m \).

The point

\[
z_0 := \langle \Phi^2, m^{-1}dM_0 \rangle \text{ can be written as } z_0 = \frac{C_0}{m} \frac{\partial F(\omega, dM_0)}{\partial M} (dM_0). \tag{5.8}
\]

It belongs to \( \langle \Phi^2, m^{-1}\mathbb{A} \rangle \) and plays a special role due to Theorem 4.1.

Proposition 5.7. Let \( \mathbb{A} = \mathbb{A}_M \) or \( \mathbb{A} = \mathbb{A}_1 \). Let \( dM_0 \in \mathbb{A} \) and let \( \omega \) be a strongly \( dM_0 \)-local boundary point of \( \Sigma[\mathbb{A}] \). Then \( z_0 \) is a boundary point of \( \langle \Phi^2, m^{-1}\mathbb{A} \rangle \), and for every supporting line \( L \) to the set \( \langle \Phi^2, m^{-1}\mathbb{A} \rangle \) at \( z_0 \), the following assertions hold:

(i) \( L \) contains the image \( \Phi^2[\text{supp } dM_0] \),

(ii) \( 1 \in L \), i.e., the line passes through the point \( z = 1 \),

(iii) if additionally \( \|dM_0\| < m \), then \( L = \mathbb{R} \).

Proof. The assertion that \( z_0 \) is a boundary point of \( \langle \Phi^2, m^{-1}\mathbb{A} \rangle \) follows from Theorem 4.1 applied to the functional \( F \) and the set \( \mathbb{A} \). Statement (i) follows from Lemma 5.6. To prove (ii), note that \( a_\ast \in \text{supp } dM_0 \) and \( \Phi(a_\ast) = 1 \). Now (iii) follows from (ii) and Lemma 5.6.

5.2 Strings producing strongly local boundary points

In this subsection, we assume that \( \omega \) is a strongly \( dM_0 \)-local boundary point of \( \Sigma[\mathbb{A}_M] \) and \( \Re \omega \neq 0 \). Without narrowing generality, the last assumption can be replaced by \( \Re \omega > 0 \).
Remark 5.2. Indeed, $\Sigma[A_M]$ is symmetric w.r.t. $i\mathbb{R}$, as well as $\Sigma(dM)$ for $dM \in \mathbb{M}_+$. Thus, if $\omega = \alpha - i\beta$ is a strongly (weakly*) $dM_0$-local boundary point of $\Sigma[A_M]$ then $-\omega = -\alpha - i\beta$ is so. Note that according to Definition 4.2, assumption (5.1) yields $\omega \in \Sigma(dM_0)$.

Recalling that $\Sigma[A_M] \subset \mathbb{C}_-$, we see that $\omega$ belongs to the quadrant $\mathbb{Q}_{IV}$. So $\xi_0 := \arg_0(\omega^2)$ belongs to $(0, \pi)$. (5.9)

Under these assumptions the measure $dM_0$ is nonzero and $a_\ast = \min \supp dM_0 < \ell$. (5.10)

This follows from Examples 2.1, 3.3, and the fact that $dM_0$ produces $\omega \not\in i\mathbb{R}$.

With $dM_0$ and $\omega$, we associate the set of complex numbers

$$S_0 := \langle \Phi^2, \bar{\mathbb{R}}_1 \rangle = \left\{ \int_{0^-}^{\ell^+} \Phi^2(s) \ dV(s) : dV \in \mathbb{M}_+ \text{ and } \|dV\| \leq 1 \right\}. \tag{5.11}$$

The set $S_0$ is a convex and, by Lemma 5.5,

$$S_0 = \text{conv} \left( \Phi^2[I] \cup \{0\} \right). \tag{5.12}$$

Obviously, the point $z_0$ defined by (5.8) belongs to $S_0$.

5.2.1 Hyperbolic "billiard"

Assume additionally that there exists a supporting line $L$ to $S_0$ at $z_0$ such that $0 \not\in L$. (5.13)

We introduce the following parametrization of $L$. Let $p_0$ be the point of $L$ closest to 0. Since $p_0 \neq 0$, we see that $p := \frac{p_0}{\|p_0\|}$ is a normal unit vector to $L$. Proposition 5.7 (ii) yields $1 \in L$. So $L = 1 + ip\mathbb{R} = \{1 + ips : s \in \mathbb{R}\}$. The line $L$ divides $\mathbb{C} \setminus L$ into two open half-planes

$$H_0 = H_0(p) := \{z \in \mathbb{C} : \langle z - 1, p \rangle_\mathbb{C} < 0\} \quad \text{and} \quad \mathbb{C} \setminus \overline{H_0}. \tag{5.14}$$

Note that $0 \in H_0$ since $\text{Re} \ p = \langle 1, p \rangle_\mathbb{C} > 0$ and $H_0 = \{z \in \mathbb{C} : \langle z, p \rangle_\mathbb{C} < \text{Re} \ p\}$.

The preimage of $L$ under the map $z \mapsto z^2$ is a rectangular hyperbola Hyp consisting of two branches Hyp$^+$ and Hyp$^-$,

$$\text{Hyp}^\pm := \{\pm \sqrt{1 + ips} : s \in \mathbb{R}\}. \tag{5.15}$$

Clearly, $\pm 1 \in \text{Hyp}^\pm$. The hyperbola Hyp divides $\mathbb{C} \setminus \text{Hyp}$ into three connected components. By $\text{Hcmp}_0$ we denote the connected component containing 0, $\text{Hcmp}_0 = \{\zeta \in \mathbb{C} : \langle \zeta^2, p \rangle_\mathbb{C} < \text{Re} \ p\}$. The two other components

$$\text{Hcmp}_\pm = \{\zeta \in \mathbb{C} : \pm \text{Re} \ \zeta > 0 \text{ and } \langle \zeta^2, p \rangle_\mathbb{C} > \text{Re} \ p\} \text{ are convex.} \tag{5.16}$$

Since $L$ is a supporting line and $0 \in S_0$, we see that

$$S_0 \subset \overline{H_0}. \tag{5.17}$$

This, (5.12), and the fact that $\text{Hcmp}_0$ is the preimage of $H_0$ under $z \mapsto z^2$ imply

$$\Phi^2[I] \subset \overline{H_0} \text{ and } \Phi[I] \subset \overline{\text{Hcmp}_0}. \tag{5.18}$$
Differentiating parameterizations (5.15), we see that, for the point \( \zeta = \pm \sqrt{1 + ips} \in \text{Hyp} \),
\[
\pm ip \sqrt{1 + ips}
\]
is a tangent vector to \( \text{Hyp} \) at \( \zeta \),
and \[
\pm p \sqrt{1 + ips}
\]
is a normal to \( \text{Hyp} \) at \( \zeta \) pointing toward \( \text{Hcmp}_{\pm} \).

(5.19)

**Proposition 5.8.** Suppose \( \text{Re} \omega > 0 \), (5.1), and (5.13). Then \( \|dM_0\| = m \) and there exists a finite set of real numbers \( \{a_j\}_{j=1}^{n+1} \) with the following properties:

(i) \( a_1 = a_1, \quad a_j < a_{j+1} \leq \ell \) for \( 1 \leq j \leq n-1 \), and \( a_{n+1} = \ell \),
(ii) supp \( dM_0 = \{a_j\}_{j=1}^{n} \),
(iii) The trajectory \( \Phi[I] \) is a piecewise linear path consisting of the (closed) line-segments \( [\Phi(a_j), \Phi(a_{j+1})] \), \( j = 1, \ldots, n \), lying in the closure \( \text{Hcmp}_0 \).
(iv) For \( 1 \leq j \leq n \), the endpoints \( \Phi(a_j) \) of these line-segments lies on the hyperbola \( \text{Hyp} \). Moreover,
\[
\Phi(a_j) \in \text{Hyp}^+ \text{ if } j \text{ is odd}, \quad \text{and } \Phi(a_j) \in \text{Hyp}^- \text{ if } j \text{ is even}.
\]
(v) If \( n \geq 2 \) and real numbers \( s_j \) (\( j = 1, \ldots, n \)) are such that \( \Phi^2(a_j) = 1 + ips_j \), then
\[
s_1 = 0; \quad s_j > s_{j+1} \quad \text{and} \quad s_j > \frac{\langle \omega^2, p \rangle_{\mathbb{C}}}{\text{Im} \omega^2} \quad \text{for } 1 \leq j \leq n - 1.
\]
(vi) If \( a_n < \ell \), then \( s_n \geq \frac{\langle \omega^2, p \rangle_{\mathbb{C}}}{\text{Im} \omega^2} \).
(Note that (i) allows \( a_n = a_{n+1} = \ell \). According to (i)-(ii), \( a_n = a_{n+1} = \ell \) \( \iff \ell \in \text{supp} dM_0 \).

5.2.2 The proof of Proposition 5.8

**Lemma 5.9.** (i) \( \langle \omega^2, p \rangle_{\mathbb{C}} \geq 0 \)
(ii) \( a_1 \) is an isolated point of \( \text{supp} \ dM_0 \).

**Proof.** Recall that \( \xi_0 = \text{arg} \langle -\omega^2 \rangle \) and, by Lemma 3.5 and (5.9),
\[
\text{arg} \ \Phi(x) \text{ is strictly increasing on } [a_1, \ell].
\]

(5.21)

For small enough \( \varepsilon > 0 \), there exists \( \delta = \delta(\varepsilon) > 0 \) such that \( |\Phi(x) - 1| < \varepsilon \) and \( \Phi(x) \in \operatorname{Sec}[0, \varepsilon] \) for all \( x \in (a_1, a_1 + \delta) \). Combining this and \( \Phi(a_1) = 1 \) with the integral equation (3.1) (\( \Phi \) satisfies (3.1) with \( z = \omega \)), it is easy to see that
\[
\Phi(x) - 1 \in \operatorname{Sec}[\xi_0, \xi_0 + \varepsilon] \cap \mathbb{D}(0) \quad \text{for all } x \in (a_1, a_1 + \delta).
\]

(5.22)

Let us prove (i) by *reductio ad absurdum*. Assume \( \langle -\omega^2, p \rangle_{\mathbb{C}} > 0 \). Note that \( p \) is a normal to \( \text{Hyp} \) at 1 pointing toward \( \text{Hcmp}_{\pm} \), see e.g. (5.19). This, \( \langle -\omega^2, p \rangle_{\mathbb{C}} > 0 \), and (5.22) imply that, for small enough \( \varepsilon > 0 \),
\[
\Phi(x) \in \text{Hcmp}_{\pm} \quad \text{for all } x \in (a_1, a_1 + \delta(\varepsilon)).
\]
The latter contradicts (5.18) since \( a_1 < \ell \).

(ii) Since \( \text{Re} p > 0 \) and \( \text{Im} (-\omega^2) > 0 \), we see that \( \xi := \xi_0 - \text{arg} p \in (-\pi/2, 3\pi/2) \). Statement (i) implies \( \xi \in [\pi/2, 3\pi/2] \). Combining this, (5.22), and taking \( \varepsilon \) small enough we see that
\[
0 < |\Phi(x) - 1| < \varepsilon \quad \text{and} \quad \langle \Phi(x) - 1, p \rangle_{\mathbb{C}} \leq 0 \quad \text{for } x \in (a_1, a_1 + \delta(\varepsilon)).
\]
For sufficiently small \( \varepsilon \), the set \( \{z : 0 < |z - 1| < \varepsilon \}, \langle z - 1, p \rangle_{\mathbb{C}} \leq 0 \) lies in \( \text{Hcmp}_0 \).

Taking such \( \varepsilon \), we ensure \( \Phi(x) \in \text{Hcmp}_0 \). So \( \Phi^2(x) \notin L \) for all \( (a_1, a_1 + \delta(\varepsilon)) \). By Proposition 5.7, \( \text{supp} dM_0 \cap (a_1, a_1 + \delta(\varepsilon)) = \emptyset \). This completes the proof. \( \square \)
Let $m_{(x)}$ be the mass of the point $x$ w.r.t. the measure $dM_0$.

**Lemma 5.10.** Assume that $x_0, x_1 \in \text{supp } dM_0$ and $t_0 \in \mathbb{R}$ are such that

\[
x_0 < x_1, \quad m_{(x_0)} > 0, \quad (x_0, x_1) \cap \text{supp } dM_0 = \emptyset, \quad \text{and } \Phi(x_0) = \pm \sqrt{1 + i pt_0} \quad \text{(in particular, } \Phi(x_0) \in \text{Hyp}^\pm).\]

Then:

(i) $\{\Phi(x) : x \in [x_0, x_1]\}$ is a non-degenerate closed line-segment $[\Phi(x_0), \Phi(x_1)]$. Moreover,

\[
\text{for all } x \in (x_0, x_1), \quad \partial_x \Phi(x) = \partial^+ \Phi(x_0) = \partial^- \Phi(x_1) = : v_{(x_0, x_1)}.
\]

(ii) There exists $t_1 \in \mathbb{R}$ such that $\Phi(x_1) = \mp \sqrt{1 + i pt_1}$ (this means that $\Phi(x_1) \in \text{Hyp}^-$ if $\Phi(x_0) \in \text{Hyp}^+$, and vise versa).

(iii) $(\Phi(x_0), \Phi(x_1)) \subset \text{Hcmp}_0$ (recall that $(z_1, z_2) := [z_1, z_2] \setminus \{z_1, z_2\}$).

(iv) $\left\langle \partial^- \Phi(x_0), \frac{\mp p}{\sqrt{1 + i pt_0}} \right\rangle \geq 0$ and $\left\langle \partial^- \Phi(x_1), \frac{\mp p}{\sqrt{1 + i pt_1}} \right\rangle > 0$.

(v) $t_0 > \frac{\langle \omega^2, p \rangle_C}{\Im \omega^2}$ and $t_0 > t_1$

(vi) If additionally $x_1 < \ell$, then $t_1 > \frac{\langle \omega^2, p \rangle_C}{\Im \omega^2}$.

(vii) $x_1$ is an isolated point of $\text{supp } dM_0$ (in particular, $m_{(x_1)} > 0$).

(viii) $x_1 - x_0 > \frac{2|p|^{1/2}}{|v_{(x_0, x_1)}|} \geq \frac{2|p|^{1/2}}{\|\partial_x \Phi\|_{L^\infty}}$

(Recall that $\partial_x \Phi(x)$ exists for a.a. $x \in [0, \ell]$ and is in $L^\infty$ due to (3.7) and Lemma 3.2 (ii).)

**Proof.** (i) follows from Lemmas 3.6 and 3.4.

(ii) Since $x_0, x_1 \in \text{supp } dM_0$, Proposition 5.7 yields $\Phi(x_0), \Phi(x_1) \in \text{Hyp}$. Assume that $\Phi(x_0)$ and $\Phi(x_1)$ lie on the same branch of $\text{Hyp}$, say $\text{Hyp}^+$. Then the line-segment $(\Phi(x_0), \Phi(x_1))$ belongs to $\text{Hcmp}_+$. This follows from (5.16) and the fact that $\text{Hyp}_+$ do not contain line-segments. However, $(\Phi(x_0), \Phi(x_1)) \subset \text{Hcmp}_0$ according to (5.18), a contradiction.

(iii) Due to (5.18), it is enough to prove that $\Phi(x) \not\subseteq \text{Hyp}$ for all $x \in (x_0, x_1)$. The latter easily follows from the arguments of the previous step.

(iv) By (1.3), $\left\langle \partial_x \Phi(x_0), \frac{\pm p}{\sqrt{1 + i pt_0}} \right\rangle \geq 0$ if $x_0 = 0$. Assume $x_0 > 0$ and $\left\langle \partial^- \Phi(x_0), \frac{\mp p}{\sqrt{1 + i pt_0}} \right\rangle < 0$.

This and (5.19) yield that $\Phi(x) \in \text{Hcmp}_+$ for $x < x_0$. The latter contradicts (5.18).

The same arguments show that $\left\langle \partial^- \Phi(x_1), \frac{\mp p}{\sqrt{1 + i pt_1}} \right\rangle \geq 0$. Assume that $\left\langle \partial^- \Phi(x_1), \frac{\mp p}{\sqrt{1 + i pt_1}} \right\rangle = 0$.

Then statement (i) implies that the line-segment $[\Phi(x_0), \Phi(x_1)]$ lies on the tangent line $L_1$ to $\text{Hyp}$ at the point $\Phi(x_1)$. The intersection of the tangent line $L_1$ and $\text{Hyp}$ consists of exactly one point. This contradicts (i). Thus, $\left\langle \partial^- \Phi(x_1), \frac{\mp p}{\sqrt{1 + i pt_1}} \right\rangle > 0$.

(v) By (3.6),

\[
\partial^+ \Phi(x) = \partial^- \Phi(x) - \omega^2 \Phi(x)m_{(x)}.
\]

Let us show that

\[
\left\langle -\omega^2 \Phi(x_0), \frac{\pm p}{\sqrt{1 + i pt_0}} \right\rangle < 0.
\]

Assume converse. Then (5.26), $m_{(x_0)} > 0$, and statement (iv) imply $\left\langle \partial^+ \Phi(x_0), \frac{\pm p}{\sqrt{1 + i pt_0}} \right\rangle \geq 0$.

The sign $>$ in the last inequality leads to a contradiction due to (5.19) and (5.18), see the proof of
(iv). So \( \langle \partial_x^+ \Phi(x_0), \frac{\mp p}{\sqrt{1 + ip t_0}} \rangle_C = 0 \) and, by (i), the line-segment \([\Phi(x_0), \Phi(x_1)]\) lies on the tangent line to Hyp at \( \Phi(x_0) \). This contradicts the fact that \( \Phi(x_1) \) belongs to the other branch of Hyp, see the end of the proof of (iv). Thus, (5.27) is proved.

Rewriting (5.27) with the use of (5.24), we get \( 0 < \langle \omega^2(\pm 1)\sqrt{1 + ip t_0}, \frac{\mp p}{\sqrt{1 + ip t_0}} \rangle_C \). Using (5.5) and \(|p| = 1\), let us modify this inequality:

\[
0 < \langle \omega^2(1 + ip t_0), p \rangle_C = \langle \omega^2, p \rangle_C + t_0|p|^2 \langle \omega^2, 1 \rangle_C = \langle \omega^2, p \rangle_C + t_0 \text{Im}(-\omega^2). \tag{5.28}
\]

This yields \( t_0 > \frac{\langle \omega^2, p \rangle_C}{\text{Im} \omega^2} \).

The assertion \( t_0 > t_1 \) is geometrically obvious if one takes (5.21) and the central symmetry of Hyp into account.

(vi) Using (5.26) and \( \langle \partial_x^- \Phi(x_1), \frac{\mp p}{\sqrt{1 + ip t_1}} \rangle_C > 0 \) from statement (iv), one can apply the arguments of the previous step to prove

\[
\langle -\omega^2 \Phi(x_1), \frac{\mp p}{\sqrt{1 + ip t_1}} \rangle_C < 0, \tag{5.29}
\]

and then \( t_1 > \frac{\langle \omega^2, p \rangle_C}{\text{Im} \omega^2} \).

(vii) If \( x_1 = \ell \), the statement is obvious from (5.23).

Assume \( x_1 < \ell \). Adjusting the proof of Lemma 5.9 (ii), it is possible to show that

\[
\Phi^2(x) \not\in L \text{ for sufficiently small positive } x - x_1. \tag{5.30}
\]

Then Proposition 5.7 gives the desired result.

(viii) The distance between \( \text{Hyp}_+ \) and \( \text{Hyp}_- \) equals \( 2|p_0|^{1/2} \). So statement (i) implies (viii).

Lemma 5.11. Let \( x_0 = \max \text{ supp } dM_0 \) and assume \( x_0 < \ell \). Put \( x_1 = \ell \). Then:

(i) statement (i) of Lemma 5.10 is valid for the part \( \{ \Phi(x) : x \in [x_0, \ell] \} \) of the trajectory \( \Phi[I] \).

(ii) \( (\Phi(x_0), \Phi(\ell)) \subset \text{Hemp}_p \).

The lemma can be easily obtained by the arguments of the proof of Lemma 5.10.

Now we are able to prove Proposition 5.8. The equality \( \|dM_0\| = m \) follows immediately from 0 \( \not\in L \) and Proposition 5.7 (iii).

Consider the case \( \text{supp } dM_0 = \{a_*\} \). Putting \( n = 1 \), \( a_1 = a_* \), and \( a_2 = \ell \), we ensure that (i)-(iii) follows from Lemma 5.11. Note that statements (iv) and (v) involve only \( a_1 \) and \( s_1 \). This makes them trivial. Lemma 5.9 (i) and \( \text{Im} \omega^2 < 0 \) yield \( s_1 = 0 \geq \frac{\langle \omega^2, p \rangle_C}{\text{Im} \omega^2} \). This gives (vi).

Consider the case \( \text{supp } dM_0 \setminus \{a_*\} \not= \emptyset \). In this case, there exists \( a_2 := \inf (\text{supp } dM_0 \setminus \{a_1\}) \), where \( a_1 := a_* \) as before. By Lemma 5.9 (ii), \( a_2 > a_1 \). So Lemma 5.11 is applicable to the interval \([a_1, a_2]\) and yields that \( a_2 \) is an isolated point of \( \text{supp } dM_0 \). If \( a_2 = \max \text{ supp } dM_0 \), we apply Lemma 5.11. Otherwise, there exists \( a_3 := \min (\text{supp } dM_0 \setminus \{a_2\}) \) and \( a_3 > a_2 \). Continuing this process, we obtain a set of point \( \{a_j\}_{j=1}^n \), \( n \leq \infty \), such that the support of \( dM_0 \) in the interval \([0, \sup_j a_j]\) consists of the points \( a_j \) and \( \sup_j a_j \). Lemma 5.10 (viii) implies \( a_{j+1} - a_j > \frac{2|p_0|^{1/2}}{\|p_0 \Phi[I]\}_{\infty} \). So the lengths of the intervals \([a_j, a_{j+1}]\) (with \( j < n \)) are separated from 0. Thus, \( n < \infty \) and this immediately yields \( \text{supp } dM_0 = \{a_j\}_{j=1}^n \). The rest of statements of the proposition easily follows from Lemmas 5.10 and 5.11. This completes the proof.
5.2.3 The case when the hyperbola degenerates

Assume now that

\[ \text{the real line } \mathbb{R} \text{ is a supporting line to the set } S_0 \text{ at } z_0. \] (5.31)

Proposition 5.7 implies that (5.31) is fulfilled whenever \( \|dM_0\| < m \).

From the point of view of the settings of Section 5.2.1, the assumption (5.31) means that \( p = -i \), \( H_0 = H_0(-i) = \mathbb{C}_+ \), and the hyperbola \( \text{Hyp} \) degenerates into the union of coordinate axes \( \mathbb{R} \cup i\mathbb{R} \). The set \( \text{Hcmp}_0 \) degenerates into the union \( \mathbb{Q}_{I} \cup \mathbb{Q}_{III} \) of the closures of the first and the third quadrants. The fact that \( \text{Hcmp}_0 \setminus \{0\} \) is not path-connected essentially simplifies the description of \( \text{supp } dM_0 \) and the trajectory \( \Phi[I] \) in comparison with Proposition 5.8. In particular, \( a_* = a_1 < a_2 = \ell \) (in the settings of statements (i)-(ii) of Proposition 5.8).

**Proposition 5.12.** Let (5.1), (5.31) be fulfilled and \( \text{Re } \omega > 0 \). Then \( a_* < \ell \) and the following statements hold:

(i) Either \( \text{supp } dM_0 = \{a_*\} \), or \( \text{supp } dM_0 = \{a_*, \ell\} \).

(ii) The trajectory \( \Phi[I] \) is the closed line-segment \([1, \Phi(\ell)]\) (recall that \( \Phi(a_*) = 1 \)).

(iii) \( \text{Re } \Phi(\ell) \geq 0 \) and \( \text{Im } \Phi(\ell) > 0 \). Moreover, \( \text{Re } \Phi(\ell) = 0 \) in the case \( \ell \in \text{supp } dM_0 \).

**Proof.** By (5.10), \( a_* < \ell \). It follows from (3.11), that \( \Phi(x) \) and \( \Phi^2(x) \) are in \( \mathbb{C}_+ \) for \( x \) greater than \( a_* \) and sufficiently close to \( a_* \). This, (5.12), (5.31), and Lemma 3.4 (ii) yields that \( \Phi[I] \subset \mathbb{C}_+ \setminus \{0\} \) and \( \Phi[I] \subset \mathbb{Q}_I \setminus \{0\} \). In particular, \( 0 \leq \text{Re } \Phi(\ell) \). Applying (3.11) again, we see that \( 0 < \text{Im } \Phi(x) \) for all \( x \in (a_*, \ell] \), and \( 0 < \text{Re } \Phi(x) \) for all \( x \in (a_*, \ell) \). This and Proposition 5.7 (i) prove statements (i) and (iii). Statement (ii) follows from statement (i) and Lemma 3.6.

5.3 Proofs of Theorems 5.2 and 5.3

When \( \text{Re } \omega \neq 0 \), Theorem 5.2 follows immediately from Propositions 5.8 and 5.12. Consider the special case \( \text{Re } \omega = 0 \).

**Lemma 5.13** ([19], see also [8] and Section 4.1 in [24]). Let \( dM \in \mathbb{M}_+ \) and \( (-i)\beta \in \Sigma(dM) \). Then \( \beta \geq \|dM\|^{-1} \). The equality holds if and only if \( dM \) consists of a single point mass placed at \( \ell \).

This lemma easily yields the following proposition.

**Proposition 5.14.** \( \omega = -im^{-1} \) is the quasi-eigenvalue of minimal decay for the zero frequency \( (\alpha = 0) \) over \( \mathbb{A}_M \). The corresponding string of minimal decay is unique and equals \( dM_0 = m\delta(x - \ell) \).

This completes the proof of Theorem 5.2.

Now we consider optimization over \( \mathbb{A} = \mathbb{A}_1 \) and prove Theorem 5.3. First, we show that

there is no a string of minimal decay rate for the zero frequency (over \( \mathbb{A}_1 \)). (5.32)
Proposition 5.15. Let \( x_0 \in [0, \ell) \), \( B(x) = 0 \) for \( x \in [0, x_0] \) and \( B(x) = c \) for \( x \in (x_0, \ell] \), where \( c > 1 \) is a constant. Then the sequence \( \omega_k = -i \frac{1}{2(\ell - x_0)\sqrt{c}} \ln \frac{\sqrt{c} + 1}{\sqrt{c} - 1} + \frac{k\pi}{(\ell - x_0)\sqrt{c}} \), \( k \in \mathbb{Z} \), forms the set of quasi-eigenvalues of the string \( B(x)dx \).

The proposition can be obtained by direct computation, see e.g. [8].

Taking \( B(x) \) introduced above with \( c \) going to \(+\infty\) and \( x_0 \) such that \( (x_0 - \ell)c = m \), one can see that \( \Re \omega_0 = 0 \) (and so \( \alpha = 0 \) is an admissible frequency) and that \( \omega_0 \) tends to \(-i)m^{-1} \). On the other side, Lemma 5.13 implies that \(-i)m^{-1} \) is not a quasi-eigenvalue for any string in \( A_1 \). This proves (5.32). Note that the above arguments provide an optimizing sequence (see [19]) for \( \alpha = 0 \).

In the case when the frequency \( \alpha \neq 0 \), Theorem 5.3 follows immediately from the following result.

Proposition 5.16. Assume that \( \Re \omega \neq 0 \) and \( \omega \) is a quasi-eigenvalue of \( \mathrm{d}M_0 \in A_1 \). Then \( \omega \) is a strongly \( \mathrm{d}M_0 \)-local interior point of \( \Sigma(A_1) \). In particular, the set \( \Sigma(A_1) \setminus \mathbb{R} \) is open.

Proof. Assume that \( \omega \) is strongly \( \mathrm{d}M_0 \)-local boundary point of \( \Sigma(A_1) \). Then \( a_* = a_*(\mathrm{d}M_0) \) is an isolated point of \( \text{supp} \, \mathrm{d}M_0 \). This fact follows from the proofs of Lemma 5.9 and Proposition 5.12 (due to Proposition 5.7, arguments of these proofs work without changes for the case of the admissible family \( A_1 \)). Hence the measure \( \mathrm{d}M_0 \) is not absolutely continuous and so \( \mathrm{d}M_0 \not\in A_1 \), a contradiction. Thus, each \( \omega \in \Sigma[A_1] \setminus \mathbb{R} \) is a local interior point (for every string in \( A_1 \) that produce the quasi-eigenvalue \( \omega \)). This yields \( \omega \in \text{int}(\Sigma[A_1] \setminus \mathbb{R}) \). \( \square \)

6. Weakly* local boundary and the "reflection law"

Propositions 5.8 and 5.12 leave too much freedom in the choice of masses \( m_j \) and the position \( a_1 \) of the first point mass. In this section, we study \( \mathrm{d}M_0 \in A_M \) and \( \omega \in Q_{IV} \) (i.e, \( \Re \omega > 0 \), \( \Im \omega < 0 \)) under the stronger assumption that

\[
\omega \text{ is a weakly* } \mathrm{d}M_0 \text{-local boundary point of } \Sigma[A_M]. \tag{6.1}
\]

We show that \( m_j, a_j, \) and \( \omega \) are connected by additional relations. Then, the results on weakly* local boundary points can be immediately applied to strings of minimal decay due to the implication \((5.2) \Rightarrow (5.3)\). Note that, from the point of view of the minimal decay, only the case \( \Re \omega > 0 \) is interesting since the case \( \Re \omega = 0 \) is described by Proposition 5.14 (see also Remark 5.2).

6.1 The "reflection" law and a position of the first point mass

Assumption (6.1), the implication \((5.3) \Rightarrow (5.4)\), and Propositions 5.8, 5.12 imply that

\[
\mathrm{d}M_0 = \sum_{j=1}^{n} m_j \delta(x - a_j)dx \tag{6.2}
\]

with \( n \in \mathbb{N} \), positive masses \( m_j \), and distinct increasingly ordered positions \( a_j \), and also connect \( a_j \) with supporting lines \( L \) of Propositions 5.8 and 5.12. This constitute the beginning of the following theorem. The essentially new part of the theorem consists of statements (ii)-(iii).

In the sequel, \( n \) is always the number of points in \( \text{supp} \, \mathrm{d}M_0 \). As before, \( \Phi(x) := \varphi(x, \omega; \mathrm{d}M_0) \) and \( a_* = a_*(\mathrm{d}M_0) = \min \text{supp} \, \mathrm{d}M_0 = a_1 \).
Theorem 6.1. Let $\alpha = \Re \omega > 0$. Assume that $\omega$ and $dM_0$ satisfy (6.1) (in particular, the latter holds if $\omega$ and $dM_0$ are of minimal decay for $\alpha$). Then $dM_0$ has the form (6.2) with $n \in \mathbb{N}$, $m_j > 0$, and distinct increasingly ordered $a_j$. Moreover, $a_j$, $m_j$, and the mode $\Phi$ are connected by the following statements:

(i) There exists a line $L \subset \mathbb{C}$ passing through $z_0 = m^{-1} \int_{0^{-}}^{t^+} \Phi^2(x) dM_0(x)$ and there exists a unit normal $p$ to $L$ such that:

(i.a) The trajectory $\Phi[Z]$ is a piecewise linear path lying in

$$
\text{Hcmp}_0 := \{ \zeta \in \mathbb{C} : \langle \zeta^2 - 1, p \rangle_\mathbb{C} \leq 0 \}
$$

and consisting of the closed line-segments $[\Phi(a_j), \Phi(a_{j+1})]$, $j = 1, \ldots, n$. Here $a_{n+1} := \ell$ (note that $a_{n+1} \notin \text{supp} dM_0$ whenever $a_n < \ell$).

(i.b) In the case $a_n < \ell$, the formula $\{ a_j \}_{j=1}^n = \{ x \in [a_s, \ell] : \Phi^2(x) \in L \}$ holds. In the case $a_n = \ell$, the formula $\{ a_j \}_{j=1}^n = \{ x \in [a_s, \ell] : \Phi^2(x) \in L \}$ takes place. Note that, in both the cases, $1 = \Phi(a_1) \in L$ and so $L = 1 + ip\mathbb{R}$.

(ii) When $n \geq 2$, the line $L$ and the unit normal $p$ are uniquely determined by assertions (i.a)-(i.b) and the following statements hold:

(ii.a) $a_1 = 0$

(ii.b) Let the real numbers $s_j$ ($j = 1, \ldots, n$) be defined by the equalities $\Phi^2(a_j) = 1 + ips_j$. Then $0 = s_1 > s_2 > \cdots > s_n$.

(ii.c) When $j \geq 2$ and $a_j < \ell$,

$$
m_j = \frac{\langle \partial_x^- \Phi^2(a_j), p \rangle_\mathbb{C}}{\langle \omega^2 \Phi^2(a_j), p \rangle_\mathbb{C}} = -\frac{\langle \partial_x^+ \Phi^2(a_j), p \rangle_\mathbb{C}}{\langle \omega^2 \Phi^2(a_j), p \rangle_\mathbb{C}}. \quad (6.3)
$$

(ii.d) When $a_n = \ell$,

$$
m_n = \frac{\partial_x^- \Phi(a_n)}{\omega^2 \Phi(a_n)} - \frac{i}{\omega}. \quad (6.4)
$$

(iii) If $\|dM_0\| < m$, then $L = \mathbb{R}$, $n \leq 2$, and $a_1 = 0$.

Proof of statement (i) of Theorem 6.1. Only (i.b) needs an additional argument. It was proved that $\Phi^2[\text{supp} dM_0] \subset L$. Assertion (i.b) means that $\bar{x} \in [a_s, \ell]$ and $\Phi^2(\bar{x}) \in L$ imply $\bar{x} \in \text{supp} dM_0$. Let us prove this by \textit{reductio ad absurdum}. Assume $\bar{x} \notin \text{supp} dM_0$. Then $\bar{x} \in (a_j, a_{j+1})$ for certain $1 \leq j \leq n$. By (ia), $\Phi(\bar{x}) \in (\Phi(a_j), \Phi(a_{j+1}))$. On the other side, $\Phi(\bar{x}) \in \text{Hyp}$. So the line passing through $\Phi(a_j)$ and $\Phi(a_{j+1})$ is a tangent line to $\text{Hyp}$ at $\Phi(\bar{x})$. On the other side, it intersects $\text{Hyp}$ at least twice (at $\Phi(a_j)$ and $\Phi(\bar{x})$), a contradiction. \hfill $\square$

Remark 6.1. Actually, it is proved that (i) is valid if (6.1) is replaced by the weaker assumption (5.1).

Statements (ii) and (iii) will be proved in Subsection 6.5.

It follows from (5.26) that

$$
\partial_x^+ \Phi(a_j) - \partial_x^- \Phi(a_j) = -\omega^2 \Phi(a_j)m_j. \quad (6.5)
$$

In the case when $j \geq 2$ and $a_j < \ell$, the proof of (ii.c) shows that

$$
\langle \partial_x^+ \Phi^2(a_j), p \rangle_\mathbb{C} = -\langle \partial_x^- \Phi^2(a_j), p \rangle_\mathbb{C}. \quad (6.6)
$$

The combination of the last equality and (6.5) can be interpreted as \textit{a nonstandard reflection law} for the hyperbolic billiard of Section 5.2.1.
6.2 Weakly*-continuous coordinates on $N$-dimensional faces

Let $N \in \mathbb{N}$. With two $N$-tuples $b = (b_1, b_2, \ldots, b_N) \in \mathbb{R}^N$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{R}^N$ we associate the measure $d\Delta_{b,\mu} := \sum_{j=1}^{N} \mu_j \delta(x - b_j)dx$. Then (3.1) implies

$$\varphi(x, z; d\Delta_{b,\mu}) = 1 - z^2 \sum_{b_j < x} (x - b_j) \varphi(b_j, z; d\Delta_{b,\mu}) \mu_j.$$  \hspace{1cm} (6.7)

We say that $b$ is a tuple of ordered positions if

$$0 \leq b_1 \leq b_2 \leq \cdots \leq b_N \leq \ell.$$ \hspace{1cm} (6.8)

The set of tuples of ordered positions is denoted by $\mathbb{P}_N$. It is convex.

The following lemma concerns analyticity of $F(z; d\Delta_{b,\mu})$ in $b_j$ and $\mu_j$.

Lemma 6.2. For each $N \in \mathbb{N}$, there exists polynomials $F_{\text{disc}}(z; b, \mu)$, $\varphi_1(z; b, \mu)$, $\ldots$, $\varphi_N(z; b, \mu)$, in the $2N + 1$ variables $z$, $b_1$, $\ldots$, $b_N$, $\mu_1$, $\ldots$, $\mu_N$, such that for any $b \in \mathbb{P}_N$ the equalities

$$F(z; d\Delta_{b,\mu}) = F_{\text{disc}}(z; b, \mu) \quad \text{and} \quad \varphi(b_j, z; d\Delta_{b,\mu}) = \varphi_j(z; b, \mu), \quad j = 1, \ldots, N$$

hold (note that the polynomials depend also on $N$).

Proof. Assume (6.8). Then (6.7) takes the form

$$\varphi(b_j, z; d\Delta_{b,\mu}) = 1 - z^2 \sum_{k=1}^{j-1} (x - b_j) \varphi_k(z; b, \mu) \mu_j.$$  \hspace{1cm} \text{(6.8)}

In particular, $\varphi(b_1, z; d\Delta_{b,\mu}) = 1 =: \varphi_1(z; b, \mu)$ and one can inductively show that $\varphi(b_k, z; d\Delta_{b,\mu})$ are polynomials in $z$, $b_j$, and $\mu_j$ for all $k = 1, \ldots, N$. We denote them by $\varphi_k(z; b, \mu)$. Next, (6.7) and (6.8) yield that $\varphi(\ell, z; d\Delta_{b,\mu})$ is a polynomial. It follows from (6.7) (or directly from (3.6)) that

$$\partial^+_x \varphi(x, z; d\Delta_{b,\mu}) = -z^2 \sum_{b_j < x} (x - b_j) \varphi(b_j, z; d\Delta_{b,\mu}) \mu_j.$$  \hspace{1cm} \text{(6.9)}

Thus, $z^{-1} \partial^+_x \varphi(\ell, z; d\Delta_{b,\mu})$ and, in turn, $F(z; d\Delta_{b,\mu})$ are also polynomials.

Assume that $b^0 = (b^0_j)_{j=1}^N \in \mathbb{P}_N$ is fixed. A tuple $v = (v_j) \in \mathbb{R}^N$ is a tuple of $b^0$-order preserving velocities if there exists $\varepsilon > 0$ such that the $t$-dependent tuple

$$b(t) = (b_j(t)) := (b^0_j + tv_j)$$ \hspace{1cm} (6.9)

belongs to $\mathbb{P}_N$ for all $t \in [0, \varepsilon]$. The set of tuples of $b^0$-order preserving velocities is denoted by $\mathbb{V} = \mathbb{V}(b^0)$. Obviously,

$$\mathbb{V}(b^0) = \text{cone}(\mathbb{P}_N - b^0).$$

Fixing $\mu$ and taking $z = \omega$, consider the function $F(\omega; d\Delta_{b(t),\mu})$ with $b(t)$ as in (6.9). This function depends only on $t$. Its right derivative taken at $t = 0$ is a functional of the velocities tuple $v$. It will be denoted by

$$D^+_{\varphi^0,\mu}(v) := [\partial^+_t F(\omega; d\Delta_{b(t),\mu})]_{t=0}.$$
According to Lemma 6.2 this derivative exists (at least) for \( v \in \mathcal{V}(b^0) \). The set
\[
D^+[\mathcal{V}] = D^+_{\varphi, \mu}[\mathcal{V}(b^0)] := \{D^+_{\varphi, \mu}(v) : v \in \mathcal{V}(b^0)\}
\]
of such derivatives is a convex cone in \( \mathbb{C} \).

One can see from (6.2) that for every \( N_0 \geq n \) there exists a tuple of ordered positions \( b^0 \in \mathbb{P}_{N_0} \) and a tuple of nonnegative numbers \( \mu^0 \in \mathbb{R}^+_0 N_0 \) such that \( \text{d}M_0 = \text{d}\Delta_{\varphi, \mu^0} \).

**Proposition 6.3.** Assume \( \text{Re} \omega > 0 \) and (6.1). Let the set \( S_0 \) and the complex numbers \( C_0, z_0 \) be defined as in (5.11), (5.7), (5.8). Let \( b^0 \in \mathbb{P}_{N_0} \) and \( \mu^0 \in \mathbb{R}^+_0 N_0 \) be such that \( \text{d}M_0 = \text{d}\Delta_{\varphi, \mu^0} \). Then
\[
0 \text{ is a boundary point of the convex cone } \text{Cone}_0 := \text{cone} \left( C_0D^+[\mathcal{V}] \cup [S_0 - z_0] \right)
\]
generated by the convex sets \( C_0D^+[\mathcal{V}] = \left\{ C_0D^+_{\varphi, \mu^0}(v) : v \in \mathcal{V}(b^0) \right\} \) and \( S_0 - z_0 \).

### 6.3 Proof of Proposition 6.3

It is enough to prove that \( 0 \in \text{bd} S_*, \) where
\[
S_* := \text{conv} \left( \frac{C_0}{m} D^+[\mathcal{V}] \cup [S_0 - z_0] \right). \tag{6.10}
\]

This fact can be easily obtained from the following three lemmas.

For a tuple \( b = (b_j) \in \mathbb{R}^N \), we denote by \( \Phi^2[b] \) the set consisting of \( \Phi^2(b_1), \ldots, \Phi^2(b_N) \).

**Lemma 6.4.** Assume \( 0 \in \text{int} S_* \). Then there exist \( N \geq N_0, b^1 \in \mathbb{P}_N \) and \( \mu^1 \in \mathbb{R}^+_N \) such that \( \text{d}M_0 = \text{d}\Delta_{b^1, \mu^1} \) and \( 0 \in \text{int} S_2, \) where
\[
S_2 := \text{conv} \left( \frac{C_0}{m} D^+_{\varphi, \mu^1}([\mathcal{V}(b^1)] \cup S_1) \right) \text{ and } S_1 := \text{conv} \left( \Phi^2[b^1] \cup \{0\} \right) - z_0. \tag{6.11}
\]

**Proof.** Since \( 0 \in \text{int} S_* \), it is possible to take \( \zeta_1, \zeta_2, \zeta_3 \in S_* \) such that \( 0 \in \text{int} \text{conv}\{\zeta_1, \zeta_2, \zeta_3\} \). By (6.10) and (5.12), \( \zeta_j \) can be represented as convex combinations
\[
\zeta_j = \lambda_j^{[0]} \frac{C_0}{m} D^+_{\varphi, \mu^0}(v^j) - \lambda_j^{[0]} z_0 + \sum_{k=1}^{n_j} \lambda_j^{[k]} [\Phi^2(x_{j,k}) - z_0] \tag{6.12}
\]
with certain \( v^j \in \mathcal{V}(b^0), n_j \geq 0, x_{j,k} \in \mathcal{I} \) and certain \( \lambda_j^{[k]} \geq 0 \) satisfying \( \sum_{k=1}^{n_j} \lambda_j^{[k]} = 1 \). Let us construct a new tuple \( b^1 \in \mathbb{P}_{N+n_1+n_2+n_3} \) adjoining \( (x_{j,k})_{j=1,2,3, k=1}^{n_1,n_2,n_3} \) to \( b^0 = (b^0)_{i=1}^{N_1} \) and then sorting the obtained tuple in the increasing order. The associated tuple \( \mu^1 \in \mathbb{R}^+_{N+n_1+n_2+n_3} \) is constructed by insertion of 0 into the tuple \( \mu^0 = (\mu^0_j)_{j=1}^{N_1} \) at the places where \( x_{j,k} \) were inserted into \( b^1 \). So
\[
\text{d}\Delta_{b^1, \mu^1} = \text{d}\Delta_{\varphi, \mu^0} = \text{d}M_0.
\]
Since any movements of zero point masses \( \mu^1 \) at \( b^1 = x_{j,k} \) do not influence the measure \( \text{d}\Delta_{b^1, \mu^1} \), it is easy to see that
\[
D^+_{\varphi, \mu^1}[\mathcal{V}(b^1)] = D^+_{\varphi, \mu^0}[\mathcal{V}(b^0)].
\]
This, (6.12), and \( 0 \in \text{int} \text{conv}\{\zeta_1, \zeta_2, \zeta_3\} \) yield that 0 is an interior point of \( S_2 \).

Denote by \( A^\text{disc}_N \) the set of \( (b, \mu) \in \mathbb{P}_N \times \mathbb{R}^+_N \) such that \( \text{d}\Delta_{b, \mu} \in A_M \). Clearly, \( A^\text{disc}_N \) is convex. Let \( F^\text{disc}_N \) be the polynomial defined in Lemma 6.2.
Lemma 6.5. Let \((b, \mu) \in A^\text{disc}_N\) be such that \(dM_0 = d\Delta_{b,\mu}\). Assume that \(\omega\) is a (strongly) \((b, \mu)\)-local interior point of \(\Sigma_{F_{\text{disc}}}[A^\text{disc}_N]\). Then \(\omega\) is a weakly* \(dM_0\)-local interior point of \(\Sigma[A_M]\).

Proof. Let \(W\) be a neighborhood of \(dM_0\) in the weak* topology \(T_{ws}\). Take any \(W_1 \subset W\) from the standard base of \(T_{ws}\)-neighborhoods of \(dM_0\). The latter means that there exist \(\varepsilon > 0\) and a finite family of functions \(y_j \in C[0, \ell], j = 1, \ldots, k\), such that \(W_1 = \{dM : |(y_j, dM - dM_0)| < \varepsilon, j = 1, \ldots, k\}\). Clearly, there exists a neighborhood \(W_E\) of \((b, \mu)\) (in the Euclidean topology of \(\mathbb{R}^{2N}\)) with the property

\[
\{d\Delta_{b,\mu} : (b^1, \mu^1) \in W_E\} \subset W_1.
\]

This and Lemma 6.2 yield \(\Sigma_{F_{\text{disc}}}[A^\text{disc}_N \cap W_E] \subset \Sigma[A_M \cap W]\). By the assumption of the lemma, \(\omega \in \text{int} \Sigma_{F_{\text{disc}}}[A^\text{disc}_N \cap W_E]\). Thus, \(\omega\) is an interior point of \(\Sigma[A_M \cap W]\). \(\square\)

Lemma 6.6. Assume that \((b^1, \mu^1) \in A^\text{disc}_N\), \(dM_0 = d\Delta_{b^1,\mu^1}\), and \(0 \in S_2\), where \(S_2\) is defined by (6.11). Then \(\omega\) is a \((b^1, \mu^1)\)-local interior point of \(\Sigma_{F_{\text{disc}}}[A^\text{disc}_N]\).

Proof. By Theorem 4.1 and Proposition 4.2, it is enough to show that assumption (iii”) of Proposition 4.2 holds with \(G = F_{\text{disc}}\) and \(S = A^\text{disc}_N\).

To this end, we extend the polynomial \(F_{\text{disc}}\) to complex \(b_j\) and \(\mu_j\). Then \(F_{\text{disc}}\) is an analytic functional of \((z; b, \mu) \in \mathbb{C} \times \mathbb{C}^N \times \mathbb{C}^N\). Denote by \(\partial F_{\text{disc}}(v)\) and \(\partial F_{\text{disc}}(\tilde{\mu})\) the directional derivatives in \(b\) and \(\mu\), resp., at the point \((\omega; b^1, \mu^1)\). We consider these derivatives as linear functionals of \(v \in \mathbb{C}^N\) and \(\tilde{\mu} \in \mathbb{C}^N\), respectively.

Obviously,

\[
\frac{\partial F_{\text{disc}}(\tilde{\mu})}{\partial \mu} = \frac{\partial F(\omega; \Delta_{b^1,\mu^1})}{\partial M}(\Delta_{b^1,\tilde{\mu}}) = C_0^{-1} \sum_{j=1}^{N} \Phi^2(b^1_j)\mu_j.
\]

Hence the set \(S_1\) defined in (6.11) can be expressed as \(S_1 = \left\{ \frac{c_0}{m} \frac{\partial F_{\text{disc}}(\tilde{\mu})}{\partial \mu} : (b^1, \mu^1 + \tilde{\mu}) \in A^\text{disc}_N \right\}\). It follows from \(\mathbb{V}(b^0) = \text{cone}(\mathbb{P}_N - b^0)\) that \(\text{cone}\left(\frac{\partial F_{\text{disc}}(\mathbb{P}_N - b^0)}{\partial \mu}\right) = D^+[\mathbb{V}(b^0)]\).

Summarizing, we see that \(0 \in \text{int} S_2\) implies that \(0\) is an interior point of the convex cone \(S_3\) generated by the union of the sets \(\frac{c_0}{m} \frac{\partial F_{\text{disc}}}{\partial b}(\mathbb{P}_N - b^0)\) and \(\frac{c_0}{m} \left\{ \frac{\partial F_{\text{disc}}(\tilde{\mu})}{\partial \mu} : (b^1, \mu^1 + \tilde{\mu}) \in A^\text{disc}_N \right\}\) (actually, this means that \(S_3 = \mathbb{C}\)). So condition (iii”) of Proposition 4.2 is fulfilled. Thus, Theorem 4.1 yields that \(\omega\) is a \((b^1, \mu^1)\)-local interior point of \(\Sigma_{F_{\text{disc}}}[A^\text{disc}_N]\). \(\square\)

6.4 Movements of point masses

Proposition 6.7. Assume that \(b^0 \in \mathbb{P}_N\), \(v \in \mathbb{V}(b^0)\), and \(1 \leq k \leq N\) are such that

\[
v_j = 0 \quad \text{and} \quad b^0_j \neq b^0_k \quad \text{for all} \quad j \neq k.
\]

Assume that \(\mu \in \mathbb{R}^N\) and \(F(\omega; \Delta_{b^0,\mu}) = 0\). Let \(b(t) = b^0 + tv\) and \(\varphi(x) = \varphi(x, \omega; \Delta_{b^0,\mu})\). Then the right \(t\)-derivative \(D^+ := \left[ \partial_t^+ F(\omega, \Delta_{b(t),\mu}) \right]_{t=0}\) is given by

\[
D^+ = -\frac{\text{Im} \varphi(t)}{\varphi(t)} \left[ \varphi_x(b^0_k) + \varphi_x(b^0_k) \right] v_k \mu_k.
\]

Proof. Let \(\Phi(x, t) := \varphi(x, \omega; \Delta_{b(t),\mu})\). Then

\[
\Phi(x, t) = 1 - \omega^2 \sum_{j \neq x} \left[ x - b_j(t) \right] \Phi(b_j(t), t) \mu_j,
\]

\[
\partial_t^+ \Phi(x, t) = -\omega^2 \sum_{j \neq x} \Phi(b_j(t), t) \mu_j.
\]
Put $\Phi_j(t) := \Phi(b_j(t), t)$, $\Phi_j^+(t) := -\omega^2 \sum_{i=1}^j \Phi_i(t) \mu_i$. For $t \in (0, t_0)$ with $t_0 > 0$ small enough, we have $b(t) \in \mathbb{P}_N$ and

$$\Phi_j(t) = 1 - \omega^2 \sum_{i=1}^{j-1} (b_j(t) - b_i(t)) \Phi_i(t) \mu_i. \tag{6.17}$$

Differentiating (6.17) and (6.15) in $t$ with the use of (6.13), one gets

$$\begin{align*}
\partial_t^+ \Phi_j(0) &= 0 \quad \text{if } j < k, \\
\partial_t^+ \Phi_k(0) &= -v_k \omega^2 \sum_{i=1}^{k-1} \Phi_i(0) \mu_i, \tag{6.18} \\
\partial_t^+ \Phi(x, 0) &= -\omega^2 \sum_{i=k}^{j} (x - b_i^0) \partial_t^+ \Phi_i(0) \mu_i + \omega^2 v_k \varphi(b_k^0) \mu_k \quad \text{if } x > b_k^0. \tag{6.19}
\end{align*}$$

The last equality shows that, for $x \in [b_k^0, +\infty)$, the function $\theta(x) := \lim_{x \to +\infty} \partial_t^+ \Phi(x, 0)$ is a unique solution to the integral equation

$$y(x) = c_0 + c_1 (x - b_k^0) - \omega^2 \int_{b_k^0}^x (x - s) y(s) d\Delta_{\varphi, \mu} \quad \text{on the interval } x \in [b_k^0, +\infty) \tag{6.20}$$

with $c_0 = \omega^2 \varphi(b_k^0) v_k \mu_k$ and $c_1 = -\omega^2 \partial_t^+ \Phi_k(0) \mu_k$.

Note that $\theta(x) = \partial_t^+ \Phi(x, 0)$ when $x > b_k^0$, and $\theta(b_k^0) = \partial_t^+ \Phi(b_k^0, 0)$ when $v_k \leq 0$. In the case $b_k^0 = \ell$, the inequality $v_k \leq 0$ holds due to $v \in \mathbb{V}(b_k^0)$.

Since the functions $\varphi(x)$ and $\psi(x) := \psi(x, \omega; d\Delta_{\varphi, \mu})$ are also solutions of (6.20) with the ”initial data” $(c_0, c_1)$ equal to $(\varphi(b_k^0), \partial_x^+ \varphi(b_k^0))$ and $(\psi(b_k^0), \partial_x^+ \psi(b_k^0))$, resp., the theory of Volterra integral equations (see e.g. [3]) states that

$$\theta(x) = c_2 \varphi(x) + c_3 \psi(x), \quad \text{where } c_{2,3} \text{ are certain constants.} \tag{6.21}$$

By definition, $D^+ = \partial_t^+ \Phi(\ell, 0) + \frac{i}{\omega} \partial_x^+ \partial_t^+ \Phi(\ell, 0)$. Taking $\partial_t^+$ of (6.16) and $\partial_x^+$ of (6.19) and (6.20), we see that $\partial_x^+ \theta(\ell) = \partial_t^+ \partial_t^+ \Phi(\ell, 0) = \partial_t^+ \partial_x^+ \Phi(\ell, 0)$. So $D^+ = \theta(\ell) + \frac{i}{\omega} \partial_x^+ \theta(\ell)$. Using (6.21) and the equalities $\varphi(\ell) + \frac{i}{\omega} \partial_x^+ \varphi(\ell) = 0$ and (3.14), we get

$$D^+ = c_3 \left[ \psi(\ell) + \frac{i}{\omega} \partial_t^+ \psi(\ell) \right] = \frac{ic_3}{\omega \varphi(\ell)}. \tag{6.22}$$

The constants $c_2$ and $c_3$ can be expressed via $c_0$ and $c_1$ using (6.21) and (3.15) for $x = b_k^0$. For $c_3$, one gets

$$c_3 = \left| \begin{array}{cc}
\varphi(b_k^0) & c_0 \\
\partial_x^+ \varphi(b_k^0) & c_1
\end{array} \right| = -\omega^2 \varphi(b_k^0) \left| \begin{array}{cc}
1 & c_0 \\
\partial_x^+ \varphi(b_k^0) & \partial_t^+ \Phi_k(0) \mu_k
\end{array} \right| = -\omega^2 \varphi(b_k^0) I_1,$$

where $I_1 = \partial_t^+ \Phi_k(0) \mu_k + \partial_x^+ \varphi(b_k^0) v_k \mu_k$. Using (6.18) and the equality $\partial_x^+ \varphi(b_k^0) = -\omega^2 \sum_{i=1}^{k-1} \Phi_i(0) \mu_i$, one gets $I_1 = \left[ \partial_x^+ \varphi(b_k^0) + \partial_t^+ \varphi(b_k^0) \right] v_k \mu_k$. Combining this with (6.22), we complete the proof. \qed
6.5 Proof of statements (ii)-(iii) of Theorem 6.1

First, we consider the case $n = 2$ and prove statement (ii). Propositions 5.8 (v) and 5.12 imply that the set $\{\Phi(a_j)\}_{j=1}^n$ consists of $n$ distinct points. This yields the uniqueness of the line $L$ satisfying (i.a)-(i.b) and the fact that $L$ is the unique supporting line to the set $S_0$ at $z_0 = m^{-1} \sum \Phi^2(a_j)m_j$.

Statement (ii.b) follows from Propositions 5.8 (v) and 5.12. It follows from (ii.b) that $\{\Phi^2(a_j)\}_{j=1}^n \subset [\Phi^2(a_1), \Phi^2(a_n)]$. In turn, this yields

$$z_0 \in (\Phi^2(a_1), \Phi^2(a_n)).$$

(6.23)

Indeed, when $\|dM_0\| = m$, one can see that $z_0$ is a convex combination of $\Phi^2(a_j)$ with nonzero coefficients $m_j/m$. So (6.23) follows directly from $\{\Phi^2(a_j)\}_{j=1}^n \subset [\Phi^2(a_1), \Phi^2(a_n)]$. Consider the case $\|dM_0\| < m$. By Proposition 5.12, one has $L = \mathbb{R}$ and

$$\Phi^2(a_1) = 1 > 0, \quad \Phi^2(a_2) = \Phi^2(\ell) < 0.$$  

(6.24)

Since $m_1/m$ and $m_2/m$ are less than 1, we again get (6.23).

It is easy to get from (6.23) and (i.b) that

$$\overline{H_0} - z_0 = \text{cone}(S_0 - z_0),$$

(6.25)

where $H_0$ is the closed half-space defined in Sections 5.2.1 and 5.2.3. Indeed, (i.b) implies that the points of $\Phi^2 \left[ I \setminus \{a_j\}_{j=1}^n \right]$ do not belong to $\text{bd} \, H_0 = L$. By (5.18), these points are in $H_0$. So the inclusion (5.17) yields $S_0 \cap H_0 \neq \emptyset$. This, (6.23), and $[\Phi^2(a_1), \Phi^2(a_n)] \subset S_0 \subset \overline{H_0}$ imply (6.25).

Writing (6.2) with the use of the tuples of positions $a := (a_j)_{j=1}^n \in \mathbb{P}_n$ and masses $\mu = (m_j)_{j=1}^n \in \mathbb{R}_+^n$, we see that $dM_0 = d\Delta_{a,\mu}$. Combining (6.25) with Proposition 6.3 (applied to $\mu^0 = \mu$, $b^0 = a$ and $V = V(a)$), we see that 0 is a boundary point of $\text{Cone}_1 := \text{cone} \left( C_0 D^+ [V] \cup \overline{H_0} - z_0 \right)$. So $\text{Cone}_1 \neq \emptyset$ and

$$C_0 D^+[V] \subset \overline{H_0} - z_0.$$  

(6.26)

Let us prove (ii.a). Assume that $a_1 > 0$. Then $(v_j)_{j=1}^n$ is a tuple of $a$-order preserving velocities whenever $v_j = 0$ for all $j \geq 2$ and $v_1 < 0$ (i.e., we can move $a_1$ slightly back keeping the tuple of positions ordered). Put $v_1 = -1$. It follows from $\Phi(a_1) = 1$, $\partial_x^+ \Phi(a_1) = 0$, $\partial_x^- \Phi(a_1) = -\omega^2 m_1$, and Proposition 6.7 that

$$C_0 D^+_{a,\mu}(v) = -\partial_x^+ \Phi(a_1) m_1 = \omega^2 m_1^2.$$  

From $n \geq 2$, Proposition 5.8 (v), and Proposition 5.12, we see that $0 = s_1 > \langle \omega^2, p \rangle_C$. Since $\text{Im} \, \omega^2 < 0$, this gives $\langle \omega^2, p \rangle_C > 0$. The latter is equivalent to $\omega^2 \not\in \overline{H_0} - z_0$. So $C_0 D^+_{a,\mu}(v) \not\in \overline{H_0} - z_0$. This contradicts (6.26).

Let us prove (ii.c). Define the tuple $v = (v_j)_{j=1}^n$ by $v_k = 0$ for $k \neq j$ and $v_j = 1$. Since $a_{j-1} < a_j < \ell$, we see that $\pm v$ are tuples of $a$-order preserving velocities. Proposition 6.7 and (3.10) yield

$$C_0 D^+_{a,\mu}(v) = \Phi(a_j) [\partial_x^+ \Phi(a_j) + \partial_x^+ \Phi(a_j)] m_1 \neq 0.$$  

Since $D^+_{a,\mu}(-v) = -D^+_{a,\mu}(v)$, the inclusion (6.26) implies that $\pm C_0 D^+_{a,\mu}(v)$ are parallel to $L$. This can be written as (6.6). Combining (6.6) and (6.5), one gets $\langle \partial_x^+ \Phi(a_j), p \rangle_C = m_j \langle \omega^2 \Phi(a_j), p \rangle_C$, and then (6.3).

Statement (ii.d) follows from (6.5) and (1.4).

Finally, consider the case $\|dM_0\| < m$ and prove statement (iii). By Propositions 5.7 and 5.12, $\mathbb{R}$ is the only supporting line to $S_0$ at $z_0$ and $n \leq 2$. Moreover, $z_0 \in (\infty, 1)$. Indeed, in the
case \( n = 2 \), the latter follows from (6.24). In the case \( n = 1 \), from \( m_1 = \|dM_0\| < m \) and \( \Phi(a_1) = 1 \). Since every line \( L \) satisfying (i) contains \( z_0 \) and 1, we see that \( L = \mathbb{R} \).

It is easy to show that \( \text{cone}(S_0 - z_0) = \overline{C_+} \). Indeed, in the case \( n = 1 \), this follows from \([0, 1] \subset S_0 \) and \( z_0 \in (0, 1) \). The case \( n = 2 \) has been considered in the proof of (ii), see (6.25).

Assume \( a_1 > 0 \). Then, similarly to the proof of (ii.a), one can show that \( \omega^2 m_1 \in C_0 D^+[\nu] \). This and \( \omega^2 \in \mathbb{C} \) yield that \( 0 \in \text{int \, Cone}_0 \). The latter contradicts Proposition 6.3.

\section{Calculation of optimizers for small frequencies}

Recall that, when \( dM_0 \) has the form (6.2), \( n \) stands for the number of points in \( \text{supp \, dM_0} \).

Proposition 5.1 and the implication (5.2) \( \Rightarrow \) (5.3) yield the following statement.

**Proposition 7.1.** Let \( \text{loc-bd}_{\nu^*} \Sigma[A_M] \) be the set of \( \omega \) that satisfy (6.1) for certain \( dM_0 \in A_M \). Then

\[
\beta_{\min}(\alpha; A_M) = \min \{-\text{Im} \, \omega : \omega \in \text{loc-bd}_{\nu^*} \Sigma[A_M] \text{ and } \text{Re} \, \omega = \alpha\}
\]

(\text{the minimum exists for each } \alpha \in \mathbb{R}).

Let \( \beta_1 : \mathbb{R} \to \mathbb{R}_+ \) be an even function defined by the equalities

\[
\beta_1(0) := m^{-1}, \quad \beta_1(\alpha) := \frac{1}{2\ell} \quad \text{when } \alpha \neq 0 \text{ and } \alpha^2 \geq \frac{1}{m\ell} - \frac{1}{4\ell^2},
\]

and (in the case \( m < 4\ell \)) \( \beta_1(\alpha) := m^{-1} + \sqrt{m^2 - \alpha^2} \) when \( 0 < \alpha^2 < \frac{1}{m\ell} - \frac{1}{4\ell^2} \).

**Theorem 7.2.** Let \( -(m\ell)^{-1/2} \leq \alpha \leq (m\ell)^{-1/2} \). Then:

(i) \( \beta_{\min}(\alpha; A_M) = \beta_1(\alpha) \)

(ii) There exists a unique string \( dM^{[\alpha]} \) of minimal decay for the frequency \( \alpha \).

(iii) \( dM^{[\alpha]} = m_1 \delta(x - a_1)dx \) with \( m_1 \) and \( a_1 \) given by

\[
m_1 = m, \quad a_1 = \ell \quad \text{when } \alpha = 0;
\]

\[
m_1 = m, \quad a_1 = \ell - \frac{1}{2m^{-1} + 2(m^{-2} - \alpha^2)^{1/2}} \quad \text{when } 0 < \alpha^2 < \frac{1}{m\ell} - \frac{1}{4\ell^2};
\]

\[
m_1 = \ell \frac{1}{(4\ell)^{-1} + \alpha^2}, \quad a_1 = 0 \quad \text{when } \alpha \neq 0 \text{ and } \alpha^2 \geq \frac{1}{m\ell} - \frac{1}{4\ell^2}. \tag{7.1}
\]

The proof is given in the Subsection 7.2.

\subsection{The case of two supporting lines}

**Lemma 7.3.** Let (5.1) be fulfilled and \( \text{Re} \, \omega > 0 \). Then the following statements are equivalent:

(i) There exist two distinct supporting lines to the set \( S_0 \) at \( z_0 \).

(ii) \( dM_0 = m\delta(x - a_1)dx \) and \( \text{arg}_0(\Phi^2(\ell) - 1) \neq \text{arg}_0 \omega^2 \).

**Proof.** From Proposition 5.7, we see that there exists at least one supporting line to \( S_0 \) at \( z_0 \), that every such a line contains \( \{\Phi(a_j)\}_{j=1}^n \), and, in particular, contains \( 1 = \Phi(a_1) \). So every supporting line has the form \( L(p) := 1 + ip\mathbb{R} \), where \( p \) is a univ normal to \( L(p) \). Let \( \mathcal{P} \) be the set of \( p \in \mathbb{T} \) such that \( S_0 \subset \overline{H_0(p)} \) and \( z_0 \in L(p) \) (see Sections 5.2.1 and 5.2.3 for the definition of \( H_0(p) \)). So \( \{L_p\}_{p \in \mathcal{P}} \) is the set of all supporting lines to \( S_0 \) at \( z_0 \). Clearly, \( \mathcal{P} \neq \emptyset \) and \( \mathcal{P} \subset \{e^{i\xi} : \xi \in [-\pi/2, \pi/2]\} \) (since \( 0 \in S_0 \subset \overline{H_0(p)} \)).
Step 1: statement (i) \( \Rightarrow \) \( n=1 \) and \( m_1 = m \). Suppose \( n \geq 2 \). Since the points \( \Phi^2(a_1) \) and \( \Phi^2(a_2) \) are distinct and belong to \( L(p) \) for each \( p \in \mathcal{P} \), we see that there exists only one supporting line to \( S_0 \) at \( z_0 \), a contradiction. So \( n = 1 \). Suppose \( m_1 < m \). By Proposition 5.7 (iii), \( \mathbb{R} \) is the only supporting line to \( S_0 \) at \( z_0 \), a contradiction.

Step 2: restrictions on \( \mathcal{P} \) in the case when \( n = 1 \) and \( m_1 = m \). Suppose \( n = 1 \) and \( m_1 = m \). In this case, \( z_0 = 1 \). In particular, \( z_0 \in L(p) \) is satisfied for every \( p \in \mathbb{T} \). Hence,

\[
p \in \mathcal{P} \text{ if and only if } \arg_0 p \in [-\pi/2, \pi/2] \text{ and } (\Phi^2(x) - 1, p)_\mathbb{C} \leq 0 \text{ for all } x \in [a_1, \ell]. \tag{7.2}
\]

Since \( n = 1 \), we see that \( \Phi[\mathcal{T}] = \{1 - \omega^2m_1(x - a_1) : x \in [a_1, \ell - a_1]\} \). It follows easily from \( \text{Im} \omega < 0 \) and \( 0 \notin \Phi[\mathcal{T}] \) that, for \( x \in (a_1, \ell) \), there exists a continuous strictly increasing branch \( \xi_1(x) \) of the multifunction \( \arg(\Phi^2(x) - 1) \) singled out by \( \lim_{x \to a_1^+} \xi_1(x) = \arg_0 (\omega^2) \). When \( x = a_1 \), we define \( \xi_1(a_1) := \arg_0 (-\omega^2) \). It follows from (7.2) that

\[
p \in \mathcal{P} \text{ if and only if } \arg_0 p \in [-\pi/2, \pi/2) \text{ and } [\xi_1(a_1), \xi_1(\ell)] \subset [\arg_0 p + \pi/2, \arg_0 p + 3\pi/2]. \tag{7.3}
\]

Step 3: (i) \( \Rightarrow \) (ii) If \( \xi_1(\ell) = \xi_1(a_1) + \pi \), (7.3) yields that \( \mathcal{P} \) consists of one number, and so a supporting line is unique, a contradiction.

Step 4: (ii) \( \Rightarrow \) (i) Since \( \mathcal{P} \neq \emptyset \), (7.3) implies \( \xi_1(\ell) \leq \xi_1(a_1) + \pi \). Combining this with (ii), we see that \( \xi_1(\ell) < \xi_1(a_1) + \pi \). The latter, (7.3), and \( \xi_1(a_1) = \text{Im}(\omega^2) > 0 \) easily imply that there exist infinitely many \( p \in \mathbb{T} \) satisfying the right-hand side of the equivalence (7.3). \( \square \)

### 7.2 Proof of Theorem 7.2 and the case of single point mass

**Lemma 7.4.** Let (6.1) be fulfilled and \( \text{Re} \omega > 0 \). Assume that \( n = 1 \) and that there exists only one supporting line \( L \) to the set \( S_0 \) at \( z_0 \). Then \( L = \mathbb{R} \), \( m_1 < m \), and \( a_1 = 0 \).

**Proof.** To prove \( L = \mathbb{R} \), assume converse. It follows from Proposition 5.7, Lemma 7.3 and the lemma’s proof that \( m_1 = m \) and \( \Phi^2(\ell) - 1 = s\omega^2 \) for certain \( s > 0 \). Since \( \Phi^2(\ell) - 1 = \omega^2m_1(\ell - a_1)^2 - 2\omega^2m(\ell - a_1) \), we have \( s = \omega^2m^2(\ell - a_1)^2 - 2m(\ell - a_1) \). By (5.10) , \( a_1 < \ell \) and so \( \text{Im} \omega^2 = 0 \). This contradicts \( \omega \in \mathbb{Q}[\mathbb{V}] \).

Let us prove \( m_1 < m \) and \( a_1 = 0 \). If \( m_1 = m \), then Lemma 7.3 yields \( \Phi^2(\ell) - 1 \in \omega^2\mathbb{R}_+ \) and, in turn, \( \text{Im} \Phi^2(\ell) < 0 \). The latter contradicts Proposition 5.12. So \( m_1 < m \). Theorem 6.2 (iii) yields \( a_1 = 0 \). \( \square \)

The following proposition essentially describes the strings consisting of a single point mass and producing weakly* local boundary points of \( \Sigma[A_M] \) with nonzero frequency.

**Proposition 7.5.** Suppose (6.1), \( \text{Re} \omega > 0 \), and \( n = 1 \). Then \( m_1 < 4\ell \) and (exactly) one of the following two assertions holds:

(i) \( m_1 = m < 4(\ell - a_1) \) and \( \omega = -\frac{i}{2(\ell - a_1)} + \sqrt{\frac{1}{m(\ell - a_1)} - \frac{1}{4(\ell - a_1)^2}} \).

(ii) \( m_1 < m, a_1 = 0 \), and \( \omega = -\frac{i}{2\ell} + \sqrt{\frac{1}{m_1\ell} - \frac{1}{4\ell^2}} \).

**Proof.** Formula (3.9) and \( \text{Re} \omega > 0 \) easily imply \( m_1 < 4\ell \) and, in combination with Lemmas 7.3-7.4, the rest of the statement. \( \square \)

**Proposition 7.6.** Suppose (5.1), \( \text{Re} \omega > 0 \), and \( n \geq 2 \). Then \( \text{Re} \omega^2 \geq \frac{1}{m_1(a_2 - a_1)} \geq \frac{1}{m_1} > 0 \).
Propositions 7.5 and 7.6 allows one to describe all $\omega$ and $dM_0$ satisfying (6.1), $Re\omega \neq 0$, and $Re\omega^2 < \frac{1}{m\ell}$. Straightforward calculations and Propositions 5.14, 7.1 complete the proof of Theorem 7.2.

8 Concluding remarks

1. Reduction to a problem with four real parameters. Let $Re\omega > 0$. Assume that $\omega$ is of minimal decay for $Re\omega$ (or, more generally, satisfies (6.1) with a certain $dM_0$), but $dM_0$ and $\Phi(x) = \varphi(x, \omega; dM_0)$ are unknown. Then the quasi-eigenvalue $\omega$, the first mass $m_1 \in \mathbb{R}_+$, and the normal $p \in \{e^{\xi} : \xi \in [-\pi/2, \pi/2]\}$ from Theorem 6.1 completely determine the mode $\Phi$ and the string $dM_0$, which can be calculated via the following procedure.

Case 1. If $\omega$ belongs to the circle $\mathbb{T}_{1/m}(-i/m) = \{z \in \mathbb{C} : |z + i/m| = 1/m\}$, then $m_1 = m$ and the case (i) of Proposition 7.5 takes place (this follows from [19, Lemma 4.1]). So $n = 1$ and $a_1$ can be recovered from the formula for $\omega$, i.e., $a_1 = \ell + \frac{1}{2Im\omega}$. This gives $dM_0 = m\delta(x - a_1)dx$ (and, in turn, gives $\Phi$ if one needs it).

Case 2. If $\omega \notin \mathbb{T}_{1/m}(-i/m)$, then Theorem 6.1 (ii.a) and Proposition 7.5 imply that $a_1 = 0$. Since $m_1$, $\omega$, and $p$ are known, one can find the first line segment of the trajectory of $\Phi$, positions $a_2$, and masses $m_2$ can be continued inductively with the use of assertions (i) and (ii) of Theorem 6.1.

If Ray, intersects Hyp at a point $1 - \omega^2m_1\gamma$ with $\gamma > 0$, there are three cases. In the case $\gamma > \ell$, one can see that $n = 1$ and again $dM_0 = m_1\delta(x)dx$. When $\gamma = \ell$, we have either $n = 2$, $a_2 = \ell$, and $m_2 > 0$ given by (6.4), or $n = 1$ and $dM_0 = m_1\delta(x)dx$ (if (6.4) gives 0 for $m_2$). In the third case $\gamma < \ell$, one has $n \geq 2$, $a_2 = \gamma$, and the second mass $m_2$ can be found via (6.3). Then (in the third case) the procedure of finding of line segments of the trajectory of $\Phi$, positions $a_j$, and masses $m_j$ can be continued inductively with the use of assertions (i) and (ii) of Theorem 6.1.

Thus, the four parameters, $Re\omega$, $Im\omega$, $m_1$, and $\xi = arg_0 p$, completely determine $\Phi$ and $dM_0$.

2. Optimizers over $\mathcal{A}_M$ that are not extreme points of $\Sigma[\mathcal{A}_M]$. In the case when $a \neq 0$ and $\frac{1}{m\ell} - \frac{1}{4\ell^2} < a^2 \leq \frac{1}{m\ell}$, formula (7.1) and Proposition 7.5 imply that the unique string $dM^{[a]}$ of minimal decay for the frequency $a$ has the form $m_1\delta(x)dx$ with $0 < m_1 < m$. So $dM^{[a]}$ is not an extreme point of $\varphi_{\mathcal{A}_M}$ (but it belongs to the 2-D face $[0d, \delta(x)dx]$ of $\mathcal{A}_M$).

3. Weakly* local boundary points of $\Sigma[\mathcal{A}_M]$ that are not of minimal decay over $\mathcal{A}_M$. Let $m < 2\ell$ and $0 < a_1 \leq \ell - \frac{m\ell}{4\ell - m}$. Then $\omega = -\frac{i}{2\ell(1-a_1)} + \sqrt{\frac{1}{m(1-a_1)} - \frac{1}{4(1-a_1)^2}}$ is a boundary point of $\Sigma[\mathcal{A}_M]$, but is not a quasi-eigenvalue of minimal decay for corresponding frequency $a = \sqrt{\frac{1}{m(1-a_1)} - \frac{1}{4(1-a_1)^2}}$.

Indeed, $\omega \in bd\Sigma[\mathcal{A}_M]$ follows from [19, Lemma 4.1]. On the other side, $\omega$ is not a quasi-eigenvalue of minimal decay for $a$ due to the fact that the case (ii) of Proposition 7.5 gives the smaller value $\frac{1}{2\ell}$ for the decay rate.

Obviously, $\omega$ is weakly* $dM_0$-local boundary point of $\Sigma[\mathcal{A}_M]$ for the associated string $dM_0 = m\delta(x - a_1)dx$.

4. Optimizers reach one of the constraints as it is shown by the following result.
Corollary 8.1. Let $dM_0$ be of minimal decay for a certain frequency. Then at least one of the two following statements hold true: (i) $\|dM_0\| = m$, (ii) $a_*(dM_0) = 0$. (In other words, $dM_0$ has maximal possible mass or $dM_0$ has maximal possible length.)

This fact immediately follows from the combination of Theorem 6.1 with Propositions 7.5 and 5.14.

It is interesting to compare Corollary 8.1 with numerical [21], [16, the problem Opt_max] and analytical [18, 19] results on the quasi-eigenvalue optimization with quite different constraints on coefficients. These results also lead to the conclusion that at least one of the constraints is reached for optimizers.

9 Appendix: the proof of Lemma 4.3

We use several lemmas from [18]. Lemma 9.1 is [18, Lemma 3.4], Lemmas 9.2 and 9.3 can be easily extracted from the proof of [18, Lemma 3.6].

Lemma 9.1 ([18]). Let $P(z, \tau) = z^r + h_1(\tau)z^{r-1} + \cdots + h_r(\tau)$ be a monic polynomial in $z \in \mathbb{C}$ with coefficients $h_j$ analytic in the complex variable $\tau$ for $\tau \in \mathbb{D}_\delta(0)$ (with $\delta > 0$). Suppose

$$h_j(0) = 0 \quad \text{for } j = 1, \ldots, r, \quad \text{and } h_r'(0) \neq 0.$$ 

Then for $\tau$ close enough to 0 there exist exactly $r$ distinct roots of $P(\cdot, \tau) = 0$ and these roots are given by an $r$-valued analytic function $Z(\tau)$ that admits a Puiseux series representation

$$Z(\tau) = \sum_{j=1}^{\infty} c_j \tau^{j/r} \quad \text{with the leading coefficient } c_1 = \sqrt{-h_r'(0)} \neq 0 \quad (9.1)$$

(here $\sqrt{\cdot}$ is an arbitrary branch of the multi-function $\tau^{1/r}$, and $c_j \in \mathbb{C}$ are constants).

It is not an essential restriction to assume that $\sqrt{\cdot}$ and arg $z$ are continuous in $\mathbb{C} \setminus \mathbb{R}_-$ and fixed by $\sqrt{1} = 1$, arg $1 = 0$, and that the complex numbers $\eta_j$ defined by (4.1) satisfy

$$\arg \sqrt{\eta_j} = (-1)^j \frac{\xi_0}{2r}, \quad j = 1,2. \quad (9.2)$$

By the Weierstrass preparation theorem, $Q(z; \zeta) = P(z; \zeta)R(z; \zeta)$ in a certain nonempty polydisc $\mathbb{D}_{\varepsilon_1}(0) \times \mathbb{D}_{\delta_1}(0) \times \mathbb{D}_{\delta_1}(0)$ with a Weierstrass polynomial

$$P(z; \zeta) = z^r + q_1(\zeta)z^{r-1} + \cdots + q_r(\zeta), \quad (9.3)$$

and functions $R$ and $q_j$ such that:

- the function $R$ is analytic and has no zeroes in $\mathbb{D}_{\varepsilon_1}(0) \times \mathbb{D}_{\delta_1}(0) \times \mathbb{D}_{\delta_1}(0)$,
- the coefficients $q_j$ of the Weierstrass polynomial $P$ are analytic in $\mathbb{D}_{\delta_1}(0) \times \mathbb{D}_{\delta_1}(0)$,
- $q_j(0,0) = 0$ for all $j = 1, \ldots, r$. 

Then

\[ Q \text{ and } P \text{ have the same zeroes in } \mathbb{D}_{\delta_1}(0) \times \mathbb{D}_{\delta_1}(0) \times \mathbb{D}_{\delta_1}(0). \quad (9.4) \]

Let \( 2\delta_2 < \xi_0/r \). Then there exist \( \theta_1, \theta_2 \) such that \( 0 < \theta_1 < \theta_2 < 1 \) and

\[ \arg \sqrt[\theta_1]{\eta_1} < \arg \sqrt[\theta_2]{\eta_1} + \delta_2 < \arg \sqrt[\theta_2]{\eta_2} - \delta_2 < \arg \sqrt[\theta_2]{\eta_2}, \text{ where } \eta_j := (1 - \theta_j)\eta_1 + \theta_j\eta_2. \quad (9.5) \]

Let us define the (real) triangles

\[ T_{\delta}[\theta_1, \theta_2] := \{ ([1 - \theta]c, \theta c) \subset \mathbb{C}^2 : c \subset (0, \delta], \theta \subset [\theta_1, \theta_2] \}. \]

So \( T_{\delta}[\theta_1, \theta_2] \subset T_{\delta} \). The following property of \( T_{\delta}[\theta_1, \theta_2] \) is essential for the next lemma: if a sequence \( \{\zeta^{[n]}\}_{n=1}^{\infty} \subset T_{\delta}[\theta_1, \theta_2] \) tends to \( (0, 0) \), then

\[ \zeta_1^{[n]} \simeq \zeta_2^{[n]} \simeq |\zeta^{[n]}| \text{ as } n \to \infty. \quad (9.6) \]

Here and below by \( \zeta^{[n]} \) we denote the pair \( (\zeta_1^{[n]}, \zeta_2^{[n]}) \subset \mathbb{C}^2 \).

**Lemma 9.2** ([18]). Assume that there exist sequences \( z^{[n]}, \zeta_1^{[n]} \) and \( \zeta_2^{[n]} \) such that

(i) \( \zeta^{[n]} = (\zeta_1^{[n]}, \zeta_2^{[n]}) \) belong to \( T_{\delta}[\theta_1, \theta_2] \) and tend to \( (0, 0) \) as \( n \to \infty \),

(ii) \( P(z^{[n]}, \zeta^{[n]}) = 0 \) for all \( n \in \mathbb{N} \).

Then \( z^{[n]} \to 0 \). Moreover,

\[ (z^{[n]})^r = \left( \zeta_1^{[n]}\eta_1 + \zeta_2^{[n]}\eta_2 \right) [1 + o(1)] \text{ and } |z^{[n]}| \simeq |\zeta^{[n]}|^{1/r} \text{ as } n \to \infty. \quad (9.7) \]

We include the proof of Lemma 9.2 since it explains the definition of \( \eta_{1,2} \), and assumption (4.2). To prove the lemma, it is enough to note that (ii) and (9.3) yields that \( z^{[n]} = o(1) \) as \( n \to \infty \). Plugging this back to (ii), one gets

\[ (z^{[n]})^r = -q_r(\zeta^{[n]}) + q_{r-1}(\zeta^{[n]})o(1) + \cdots + q_1(\zeta^{[n]})o(1). \]

So (9.6) and \( q_j(0, 0) = 0 \) imply

\[ (z^{[n]})^r = -\zeta_1^{[n]}\partial_{\zeta_1}q_r(0, 0) - \zeta_2^{[n]}\partial_{\zeta_2}q_r(0, 0) + o(|\zeta^{[n]}|). \]

The differentiation of the equality \( Q = PR \) and formula (9.3) easily gives \( \partial_{\zeta_j}q_r(0, 0) = -\eta_j, j = 1, 2 \). Thus, (9.7) follows from (9.6) and assumption (4.2).

**Lemma 9.3** ([18]). Let \( 0 < \theta_1 < \theta_2 < 1 \). There exists \( \epsilon \in (0, \delta_1) \) with the following property: if \( P(z, \zeta) = 0 \) and \( \zeta \in T_{\epsilon}[\theta_1, \theta_2] \), then \( z \) is a simple zero of the polynomial \( P(\cdot, \zeta) \).

Let \( \epsilon \) be as in Lemma 9.3. It is essential for the next lemma that \( (0, 0) \notin T_{\epsilon}[\theta_1, \theta_2] \). So there exist open neighborhoods (in the sense of \( \mathbb{C}^2 \)) of \( T_{\epsilon}[\theta_1, \theta_2] \) that do not contain the origin \( (0, 0) \).

**Lemma 9.4.** There exist an open simply connected set \( W \subset \mathbb{C}^2 \) and an analytic in \( W \) function \( Z_1(\zeta_1, \zeta_2) \) with the following properties:

(i) \( T_{\epsilon}[\theta_1, \theta_2] \subset W \),

(ii) for each \( \zeta \in T_{\epsilon}[\theta_1, \theta_2] \), the number \( Z_1(\zeta) = Z_1(\zeta_1, \zeta_2) \) is a zero of \( P(\cdot, \zeta_1, \zeta_2) \) and

\[ Z_1(\zeta) = \sqrt{\zeta_1\eta_1 + \zeta_2\eta_2 + o(\zeta)} \text{ as } \zeta \to 0, \quad (9.8) \]
(iii) for every $\theta \in \left[\theta_1, \theta_2\right]$ there exists $\epsilon_1(\theta) > 0$ such that the series representation

$$Z_1([1 - \theta]\tau, \theta\tau) = \sum_{j=1}^{\infty} c_j(\theta) \left(\sqrt[1/r]{\tau}\right)^j \text{ with the leading coefficient } c_1(\theta) = \sqrt[1/r]{(1 - \theta)\eta_1 + \theta\eta_2} \quad (9.9)$$

holds for $\tau \in [0, \epsilon_1(\theta)]$.

Proof. Applying Lemma 9.1 to the function $\tilde{P}(z, \tau) := P(z, (1 - \theta)\tau, \theta\tau)$ with complex variables $z$ and $\tau$ and a fixed parameter $\theta \in \left[\theta_1, \theta_2\right]$, one can produce the $r$-valued Puiseux series for the zeroes $Z([1 - \theta]\tau, \theta\tau)$ of $P(\cdot, (1 - \theta)\tau, \theta\tau)$:

$$Z ([1 - \theta]\tau, \theta\tau) = \sum_{j=1}^{\infty} c_j(\theta)\tau^{j/r}, \text{ with } c_1(\theta) = \sqrt[1/r]{(1 - \theta)\eta_1 + \theta\eta_2}. \quad (9.10)$$

Choosing $\theta_0 \in \left[\theta_1, \theta_2\right]$ and placing the branch $\sqrt[1/r]{\tau}$ (chosen at the beginning of this subsection) instead of the multi-function $\tau^{1/r}$ in (9.10), we obtain a branch $Z_1([1 - \theta_0]\tau, \theta_0\tau)$ of the multi-function $Z([1 - \theta_0]\tau, \theta_0\tau)$. For $\tau \in [0, \epsilon_1(\theta_0)]$ with $\epsilon_1(\theta_0) > 0$ small enough, the series converges and indeed gives a zero of $P(\cdot, (1 - \theta_0)\tau, \theta_0\tau)$.

Since $T_\epsilon[\theta_1, \theta_2]$ is simply connected, Lemma 9.3 and the implicit function theorem for simple zeroes imply that $Z_1$ can be extended from the line-segment

$$\{ ([1 - \theta_0]\tau, \theta_0\tau) \in \mathbb{C}^2 : \tau \in [0, \epsilon_1(\theta)] \}$$

to an analytic function $Z_1(\zeta_1, \zeta_2)$ in a certain open simply connected neighborhood $W$ of $T_\epsilon[\theta_1, \theta_2]$ saving the property $P(Z_1(\zeta); \zeta) = 0$. If such a neighborhood $W$ is fixed, the extension is unique. (Note that $T_\epsilon[\theta_1, \theta_2]$ does not contain the origin and, in the case $r > 1$, the neighborhood $U$ also must not contain the origin).

Lemma 9.2 implies that $Z_1(\zeta_1, \zeta_2) \to 0$ as $\zeta \to 0$ in $T_\epsilon[\theta_1, \theta_2]$. That is, $Z_1$ is continuous in the closure $\overline{T_\epsilon[\theta_1, \theta_2]}$. (Note that $Z_1(0, 0) = 0$ due to (9.10) and that $\overline{T_\epsilon[\theta_1, \theta_2]}$ consists of $(0, 0)$ and $T_\epsilon[\theta_1, \theta_2]$). Similar to the proof of Lemma 9.2, one can show that

$$Z_1^*(\zeta) = \zeta_1\eta_1 + \zeta_2\eta_2 + f(\zeta), \quad (9.11)$$

where $f$ is analytic in $U$ and $f(\zeta) = o(\zeta)$ as $\zeta \to 0$ in $T_\epsilon[\theta_1, \theta_2]$. So it is possible to take $\epsilon_2 < \epsilon$ such that $\zeta \in T_{\epsilon_2}[\theta_1, \theta_2]$ implies

$$Z_1^*(\zeta) \in \text{Sec}[\text{arg}\eta_1, \text{arg}\eta_2] \quad \text{and, in particular, } Z_1(\zeta) \neq 0 \quad (9.12)$$

(note that, by definition (1.7), the sector $\text{Sec}[\text{arg}\eta_1, \text{arg}\eta_2]$ does not contain 0). From this and the expression for $c_1(\theta_0)$ in (9.10), we get

$$\sqrt[r]{Z_1^*(\zeta)} = Z_1(\zeta) \quad (9.13)$$

for all $\zeta = ([1 - \theta_0]\tau, \theta_0\tau)$ with $0 < \tau < \min\{\epsilon_2, \epsilon_1(\theta_0)\}$.

Now (9.8) follows from (9.11) and the fact that formula (9.13) survives the above considered analytic extension to $T_{\epsilon_2}[\theta_1, \theta_2]$. Indeed, assume that for certain $\zeta \in T_{\epsilon_2}[\theta_1, \theta_2]$ and a certain nontrivial $r$-th root of unity $1^{1/r}$ ($1^{1/r} \neq 1$) the equality $Z_1(\zeta) = 1^{1/r} \sqrt[r]{Z_1^*(\zeta)}$ holds. Then (9.12) and the standard argument concerning simultaneously closed and open subsets of a connected set imply that $Z_1$ is discontinuous at some point $\tilde{\zeta} \in T_{\epsilon_2}[\theta_1, \theta_2]$, a contradiction.
For every $\theta \in [\theta_1, \theta_2]$, there exists $\epsilon_1(\theta) > 0$ and a branch $\sqrt[\alpha]{\tau}$ of $\tau^{1/r}$ such that the function $Z_1([1 - \theta]\tau_1, \theta_1\tau_2)$ admits the series representation (9.10) for $\tau \in [0, \epsilon_1(\theta)]$ with $\langle \cdot \rangle^{1/r}$ replaced by the branch $\sqrt[\alpha]{\tau}$. (In other words, for a fixed $\theta$ formula (9.10) holds with a fixed branch $\sqrt[\alpha]{\tau}$ of $\tau^{1/r}$; note that for $\theta = \theta_0$ this fact holds by definition of $Z_1$). Indeed, for $\tau \leq \epsilon_1(\theta)$ with $\epsilon_1(\theta)$ small enough, Lemma 9.1 implies that all the roots of $P(z, \tau)$ are distinct and are produced by the $r$-valued series (9.10). Assume for a moment that for two different numbers $\tau_1, \tau_2 \in (0, \epsilon_1(\theta)]$, the value of $Z_1([1 - \theta]\tau_1, \theta_1\tau_2)$ is given by (9.10) with different branches of $\langle \cdot \rangle^{1/r}$. Then standard arguments imply that $Z_1([1 - \theta]\tau_1, \theta_1\tau_2)$ is not continuous on $(0, \epsilon_1(\theta)]$. This contradicts the definition of $Z_1$.

Finally, comparing asymptotics of (9.8) with that of the first term in the series (9.10), we see that $Z_1([1 - \theta]\tau_1, \theta_1\tau_2)$ corresponds to the branch $\sqrt[\alpha]{\tau}$ (i.e., $\sqrt[\alpha]{\tau}$ = $\sqrt[\alpha]{\tau}$ for all $\theta \in [\theta_1, \theta_2]$).

Lemma 9.5. For $\varepsilon > 0$ small enough, the image $Z_1(T_{\delta_1}[\theta_1, \theta_2])$ of $T_{\delta_1}[\theta_1, \theta_2]$ contains the set

$$S_1 := \mathbb{D}_\varepsilon(0) \cap \operatorname{Sec}[\arg \sqrt[\eta_1] + \delta_2, \arg \sqrt[\eta_2] - \delta_2].$$

Proof. For $\tau > 0$, consider loops (closed paths) $\gamma_\tau : [0, 1] \to \mathbb{R}^2 \subset \mathbb{C}^2$ that are parameterizations of the boundaries of the triangles $T_\tau[\theta_1, \theta_2]$ and have the following property: the map $\gamma(s, \tau) = \gamma_\tau(s)$ defined on $[0, 1] \times [0, \nu]$ is a homotopy of $\gamma_0$ into the loop $\gamma_0 \equiv 0$ ($\nu$ is arbitrary positive, $\gamma_0$ is defined by $\gamma_0(s) = 0$ for all $s \in [0, 1]$, and so the loop $\gamma_0$ degenerates to the origin $(0, 0)$). To be specific, put

$$\gamma_\tau(s) := \begin{cases} t_1(s) (1 - \theta_1, \theta_1) & \text{for } s \in [0, 1/3] \\
\tau (1 - t_2(s), t_2(s)) & \text{for } s \in [1/3, 2/3] \\
t_3(s) (1 - \theta_2, \theta_2) & \text{for } s \in [2/3, 1] \end{cases}$$

with $t_1(s) := 3s$, $t_2(s) := (\theta_2 - \theta_1)(3s - 1) + \theta_1$, $t_3(s) := 3\tau(1 - s) - 1$.

$Z_1(\gamma_\tau)$ are loops in $\mathbb{C}$ (these loops are rectifiable in the sense that $\partial Z_1(\gamma_\tau(\cdot)) \in L^1_{\mathbb{C}}[0, 1]$, the loop $Z_1(\gamma_0) \equiv 0$ is reduced to the point $0$). For any fixed $\nu \geq 0$, the map $Z_1(\gamma(s, \tau)) : [0, 1] \times [0, \nu] \to \mathbb{C}$ is a loop homotopy of $Z_1(\gamma_0)$ to $0$.

By inequalities (9.5) and Lemma 9.4 (see also the proof of Lemma 9.4), there exists $\nu \in (0, \delta_1]$ such that

- for $\zeta \in T_{\nu}[\theta_1, \theta_2]$,

$$Z_1(\zeta) \neq 0 \text{ and, moreover, } Z_1(\zeta) \in \operatorname{Sec}[\arg \sqrt[\eta_1], \arg \sqrt[\eta_2]]; \quad (9.14)$$

- for $\tau \in (0, \nu]$,

$$\arg Z_1([1 - \theta_1]\tau, \theta_1\tau) < \arg \sqrt[\eta_1] + \delta_2 < \arg \sqrt[\eta_2] - \delta_2 < \arg Z_1([1 - \theta_2]\tau, \theta_2\tau). \quad (9.15)$$

Take $\varepsilon$ such that

$$0 < \varepsilon < \min_{\theta \in [\theta_1, \theta_2]} |Z_1([1 - \theta]\nu, \theta\nu)|. \quad (9.16)$$

Then for arbitrary $z_0 \in S_1$ the index of the point $z_0$ w.r.t. the loop $Z_1(\gamma_\nu)$ equals 1,

$$\operatorname{ind}[z_0; Z_1(\gamma_\nu)] = 1. \quad (9.17)$$

Indeed, by (9.14), (9.15), and (9.16), the increments of the argument of $Z_1(\gamma_\nu(s))$ on each of the intervals $s \in [0, 1/3]$, $s \in [1/3, 2/3]$, and $s \in [2/3, 1]$ are positive, but less than $\pi$, $2\pi$, and $\pi$, respectively.

It follows from (9.17) that $z_0$ belongs to the image $S_2 := Z_1(T_{\nu}[\theta_1, \theta_2])$. In fact, assume that $z_0 \notin S_2$. Then $z_0$ does not belong to the curve (image of the loop) $Z_1(\gamma_\tau)$ for every $\tau \in [0, \nu]$. So $Z_1(\gamma(s, \tau))$ is a homotopy of $Z_1(\gamma_\nu)$ into $Z_1(\gamma_0) \equiv 0$ in the domain $\mathbb{C} \setminus \{z_0\}$. Thus, $\operatorname{ind}[z_0; Z_1(\gamma_\nu)] = \operatorname{ind}(z_0; Z_1[\gamma_0]) = 0$. This contradicts (9.17). \qed
Summarizing, we see from Lemma 9.4 that $P(Z_1(\zeta); \zeta) = 0$ and from (9.4) that $Q(Z_1(\zeta); \zeta) = 0$ for all $\zeta \in T_{\delta_1}[\theta_1, \theta_2]$. Thus, $Z_1[T_{\delta_1}[\theta_1, \theta_2]] \subset \Sigma_Q[T_{\delta_1}[\theta_1, \theta_2]] \subset \Sigma_Q[T_{\delta_1}]$. Lemma 9.5 completes the proof.

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