Multiple zeta star values on 3–2–1 indices

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Received: 14 September 2018 / Accepted: 6 August 2022 / Published online: 10 October 2022
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Abstract
In 2008, Muneta found explicit evaluation of the multiple zeta star value \( \zeta^*([3, 1]^d) \), and in 2013, Yamamoto proved a sum formula for multiple zeta star values on 3–2–1 indices. In this paper, we provide another way of deriving the formulas mentioned above. It is based on our previous work on generating functions for multiple zeta star values and also on constructions of generating functions for restricted sums of alternating Euler sums. As a result, the formulas obtained are simpler and computationally more effective than the known ones. Moreover, we give explicit evaluations of \( \zeta^*([2]^m, 3, [2]^m, 1]^d) \) and \( \zeta^*([2]^m, 3, [2]^m, 1]^d, [2]^{m+1}) \), which are new and have not appeared in the literature before.

Keywords Multiple zeta star value · Multiple zeta value · Generating function · Sum formula · Alternating interpolated zeta value

Mathematics Subject Classification 11M32 · 05A15 · 30B10

1 Introduction

The multiple zeta value (MZV) and the multiple zeta star value (MZSV) are defined by

\[
\zeta(s_1, \ldots, s_r) = \sum_{k_1 > \cdots > k_r \geq 1} \frac{1}{k_1^{s_1} \cdots k_r^{s_r}},
\]

1 Introduction

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\[
\zeta(s_1, \ldots, s_r) = \sum_{k_1 > \cdots > k_r \geq 1} \frac{1}{k_1^{s_1} \cdots k_r^{s_r}},
\]
\[ \zeta^*(s_1, \ldots, s_r) = \sum_{k_1 \geq \cdots \geq k_r \geq 1} \frac{1}{k_1^{s_1} \cdots k_r^{s_r}} \]  

(2)

for any multi-index \( s = (s_1, \ldots, s_r) \in \mathbb{N}^r \) with \( s_1 > 1 \). By convention, the number \( |s| := s_1 + \cdots + s_r \) is called the weight, and \( l(s) := r \) the depth (or length) of the multiple zeta (star) value. The two types of zeta values are expressible in terms of each other via the relations

\[ \zeta^*(s) = \sum_{p} \zeta(p) \quad \text{and} \quad \zeta(s) = \sum_{p} (-1)^{\sigma(p)} \zeta^*(p), \]

where \( p \) runs through all the indices of the form \((s_1 \circ s_2 \circ \cdots \circ s_r)\) with “\( \circ \)” being either the symbol “,” or the sign “+,” and the exponent \( \sigma(p) \) is the number of signs “+” in \( p \). This implies that the two \( \mathbb{Q} \)-vector spaces spanned by the MZVs and by the MZSVs coincide.

Special values of (1) and (2) have recently been the subject of a lot of experimentation and several conjectures, many of which were suggested by numerical calculations. The simplest precise evaluations of multiple zeta values are given by

\[ \zeta(\{2\}^d) = \frac{\pi^{2d}}{(2d + 1)!} \quad \text{and} \quad \zeta^*(\{2\}^d) = (-1)^{d+1}(2^{2d} - 2) \frac{B_{2d}}{(2d)!} \pi^{2d}, \]

(3)

where \( B_{2d} \) is a classical Bernoulli number. Here by \( \{a_1, \ldots, a_k\}^d \) we mean \( d \) successive repetitions of the sequence \( \{a_1, \ldots, a_k\} \). The formulas easily follow from the infinite product of the sine function and Laurent series expansions of functions \( \sin(\pi z)/\pi z \) and \( \pi z/\sin(\pi z) \), respectively. Formulas in (3) were further extended by many authors (see, for example, [1, 8, 14]) to

\[ \zeta(\{2m\}^d) = C_{m,d} \pi^{2md} \quad \text{and} \quad \zeta^*(\{2m\}^d) = C^*_{m,d} \pi^{2md}, \]

(4)

with explicitly given rational constants \( C_{m,d} \) and \( C^*_{m,d} \).

Another example of arbitrary depth evaluation includes

\[ \zeta(\{3,1\}^d) = \frac{2\pi^{4d}}{(4d + 2)!}, \]

(5)

which was conjectured by Zagier [16] and first proved by Borwein et al. [3] by using generating functions, namely, by showing that

\[ \sum_{d=0}^{\infty} \zeta(\{3,1\}^d) z^{4d} = \frac{\sin(\frac{1}{2}(1+i)\pi z)}{\frac{1}{2}(1+i)\pi z} \frac{\sin(\frac{1}{2}(1-i)\pi z)}{\frac{1}{2}(1-i)\pi z}. \]

Later, a purely combinatorial proof was given in [2] based on shuffle properties of iterated integrals. A multiple zeta star version of formula (5) was proved by Muneta [14]
\[ \zeta^*([3, 1]^d) = \pi^{4d} \sum_{j=0}^{d} \frac{2}{(4j + 2)!} \sum_{n_0 + n_1 = 2(d-j)}^{n_0, n_1 \geq 0} (-1)^{n_1} \frac{(2^{2n_0} - 2)B_{2n_0}}{(2n_0)!} \frac{(2^{2n_1} - 2)B_{2n_1}}{(2n_1)!} \]

(6)

with the help of the identity

\[ \zeta^*([3, 1]^d) = \sum_{j=0}^{d} \zeta([3, 1]^j) \zeta^*([4]^{d-j}), \]

(7)

which after substitution of known formulas (4) and (5) implies (6). Muneta’s proof of (7) uses algebraic and combinatorial properties of the harmonic (shuffle) product.

A further generalization of (5) to a sum of multiple zeta values obtained by inserting blocks of twos of constant total length in the argument string \([3, 1]^d\),

\[ Z(d, n) := \sum_{a_1, \ldots, a_{2d+1} \geq 0} \zeta([2]^{a_1}, 3, [2]^{a_2}, 1, \ldots, 3, [2]^{a_{2d}}, 1, [2]^{a_{2d+1}}), \]

was given by Bowman and Bradley [4]

\[ Z(d, n) = \left(\frac{n + 2d}{n}\right) \frac{\pi^{2n+4d}}{(2d + 1)(2n + 4d + 1)!}. \]

(8)

Its counterpart for multiple zeta star values

\[ Z^*(d, n) := \sum_{a_1, \ldots, a_{2d+1} = n} \zeta^*([2]^{a_1}, 3, [2]^{a_2}, 1, \ldots, 3, [2]^{a_{2d}}, 1, [2]^{a_{2d+1}}) \]

was first proved in the form of the inclusion \(Z^*(d, n) \in \mathbb{Q}_{>0} \pi^{4d+2n}\) by Kondo et al. [13] and then explicitly by Yamamoto [15] by using generating series of truncated multiple zeta sums. In fact, Yamamoto first proved an identity expressing \(Z^*(d, n)\) in terms of \(Z(\cdot, \cdot)\),

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\[ Z^*(d, n) = \sum_{2m+k+u=2d, j+l+v=n} (-1)^{j+k} \binom{k+l}{k} \binom{u+v}{u} Z(m, j)\zeta^*((2)^{k+l})\zeta^*((2)^{u+v}) \]

and then by applying Bowman–Bradley formula (8) and formulas (3), obtained an explicit evaluation for \( Z^*(d, n) \),

\[ Z^*(d, n) = \pi^{4d+2n} \sum_{2m+k+u=2d, j+l+v=n} (-1)^{j+k} \binom{k+l}{k} \binom{u+v}{u} \binom{2m+j}{2m+j} \times \frac{\beta_{k+l} \beta_{u+v}}{(2m+1)(4m+2j+1)!}, \] \hspace{1cm} (9)

where

\[ \beta_r = (2^{2r} - 2) \frac{(-1)^{r-1} B_{2r}}{(2r)!}. \]

In this paper, we provide another way to explicitly evaluate the values of \( \zeta^*((3, 1)^d) \) and \( Z^*(d, n) \), which is self-contained and does not use corresponding results on multiple zeta values. It is based on generating functions for multiple zeta star values from our paper [10] and generating functions for restricted sums of alternating Euler sums.

There is a lot of work on restricted sum formulas for multiple zeta values [7, 8, 11, 17] that served as a source of inspiration for us. As a result, the formulas obtained are simpler than those in (6), (9), and computationally more effective.

**Theorem 1.1** For any non-negative integer \( d \), we have

\[ \zeta^*((3, 1)^d) = 4\pi^{4d} \cdot \sum_{k=0}^{2d} (-1)^k (4^{k+1} - 1) \frac{B_{2k+2}}{(2k+2)!} \frac{B_{4d-2k}}{(4d-2k)!}, \] \hspace{1cm} (10)

\[ \zeta^*((3, 1)^d, 2) = 4\pi^{4d+2} \cdot \sum_{k=0}^{2d+1} (-1)^k (4^{k+1} - 1) \frac{B_{2k+2}}{(2k+2)!} \frac{B_{4d+2-2k}}{(4d+2-2k)!}. \]

For example, if Bernoulli numbers are tabulated (given), the computational complexity of formula (6) is \( O(d^2) \), while for (10), it is linear, \( O(d) \). Formulas in (10) follow from the generating function

\[ \sum_{d=0}^{\infty} \zeta^*((3, 1)^d) z^{4d} - \sum_{d=0}^{\infty} \zeta^*((3, 1)^d, 2) z^{4d+2} = \tanh(\pi z/2) \cdot \cot(\pi z/2) \]

that will be proved in Sect. 2 (see Theorem 2.3). The second formula in (10) is a more direct version of the conjectural formula

\[ (2d + 1)\zeta^*((3, 1)^d, 2) = \sum_{j+k=d} \zeta^*((3, 1)^j)\zeta^*((2)^{2k+1}) \]
given in [12, Conjecture 4.5B] and then proved by Zhao [18, Theorem 6.1]. Zhao’s proof is based on ‘2-3-2-1’ and ‘2-3-2-1-2’ formulas [18, Theorems 4.8, 4.10]

\[
\zeta^*(\{2\}^{b_1}, 3, \{2\}^{a_1}, 1, \ldots, \{2\}^{b_d}, 3, \{2\}^{a_d}, 1) = \sum_{p=(2b_1+2)\circ(2a_1+2)\circ\cdots\circ(2b_d+2)\circ(2a_d+2)} 2^l(p) \zeta(p) \tag{11}
\]

and

\[
\zeta^*(\{2\}^{b_1}, 3, \{2\}^{a_1}, 1, \ldots, \{2\}^{b_d}, 3, \{2\}^{a_d}, 1, \{2\}') = -\sum_{p=(2b_1+2)\circ(2a_1+2)\circ\cdots\circ(2b_d+2)\circ(2a_d+2)\circ(2\ell)} 2^l(p) \zeta(p), \tag{12}
\]

where \( t \geq 1, a_j, b_j \) are non-negative integers, and \( p \) runs through all indices of the specified form with “\( \circ \)” being either the comma “,” or the O-plus “\( \oplus \)” defined by

\[
a \oplus c = a \oplus c = a + c, \quad a \oplus c = a \oplus c = a + c, \quad a, c \in \mathbb{N}_0.
\]

For the definition of signed numbers, see the beginning of Sect. 2.

We also provide explicit evaluation of the sum \( Z^*(d, n) \) and its two sub-sums depending on whether \( a_{2d+1} \) is zero or not. Let

\[
Z^*_0(d, n) = \sum_{a_1 + \cdots + a_{2d} = n \atop a_1, \ldots, a_{2d} \geq 0} \zeta^*(\{2\}^{a_1}, 3, \{2\}^{a_2}, 1, \ldots, 3, \{2\}^{a_{2d}}, 1) \tag{13}
\]

and

\[
Z^*_1(d, n) = \sum_{a_1 + \cdots + a_{2d+1} = n-1 \atop a_1, \ldots, a_{2d+1} \geq 0} \zeta^*(\{2\}^{a_1}, 3, \{2\}^{a_2}, 1, \ldots, 3, \{2\}^{a_{2d}}, 1, \{2\}^{a_{2d+1}+1}), \tag{14}
\]

then

\[
Z^*(d, n) = Z^*_0(d, n) + Z^*_1(d, n). \tag{15}
\]

**Theorem 1.2** For any positive integer \( d \) and a non-negative integer \( n \), we have

\[
Z^*_0(d, n) = 4\pi^{2n+4d} \sum_{k=0}^{n+2d} \sum_{r=0}^{n} (-1)^{n+k+r} \binom{k}{r} \left(\begin{array}{c} n+2d-k \\ n-r \end{array}\right) \times \frac{B_{2k+2}}{(2k+2)!} \frac{B_{2n+4d-2k}}{(2n+4d-2k)!},
\]

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\[ Z^*(d, n) = 4\pi^{2n+4d} \sum_{k=0}^{n+2d} \sum_{r=0}^{n} (-1)^{n+k+r} (4^{k+1} - 1) \binom{k+1}{r} \binom{n+2d-k}{n-r} \times \frac{B_{2k+2}}{(2k+2)!} \frac{B_{2n+4d-2k}}{(2n+4d-2k)!}. \]

and for \( n \geq 1, \)

\[ Z^*_1(d, n) = 4\pi^{2n+4d} \sum_{k=0}^{n+2d} \sum_{r=0}^{n-1} (-1)^{n+k+r+1} (4^{k+1} - 1) \binom{k}{r} \binom{n+2d-k}{n-1-r} \frac{B_{2k+2}}{(2k+2)!} \times \frac{B_{2n+4d-2k}}{(2n+4d-2k)!}. \]

Note that the case \( n = 1 \) for \( Z^*_0(d, n) \) was conjectured by Imatomi et al. [12, Conjecture 4.5C] as

\[ Z^*_0(d, 1) = \sum_{j+k=d-1} \zeta^*([3, 1]^j, 2) \zeta^*([2]^{2k+2}) \]

and then proved by Zhao [18, Theorem 6.1] using (11) and (12).

Moreover, we find explicit formulas for

\[ \zeta^*([\{2\}^m, 3, \{2\}^m, 1]^d) \quad \text{and} \quad \zeta^*([\{2\}^m, 3, \{2\}^m, 1, \{2\}^{m+1}], \]

which seem to be new and have not appeared in the literature before. We prove that these numbers are rational multiples of \( \pi^{4(m+1)d} \) and \( \pi^{2(m+1)(2d+1)} \), respectively, and give explicit evaluations of the rational factors in two ways. The first is based on computation of product of generating functions, while the second uses properties of Bell polynomials.

**Theorem 1.3** For any positive integer \( d \) and a non-negative integer \( m \),

\[ \zeta^*([\{2\}^m, 3, \{2\}^m, 1]^d) = 4^{m+1} \pi^{4d(m+1)} \times \sum_{n_0+\ldots+n_m=2(m+1)d} \exp \left( \frac{2\pi i}{m+1} \sum_{k=0}^{m} kn_k \right) \times \prod_{k=0}^{m} \left( \frac{n_k}{4^{l_k+1} - 1} \frac{B_{2l_k+2}}{(2l_k+2)!} \frac{B_{2n_k-2l_k}}{(2n_k-2l_k)!} e^{\frac{\pi il_k}{m+1}} \right) \]
and
\[
\zeta^*\left(\{2\}^m, 3, \{2\}^m, 1\right)^d, (2)^{m+1} = (-1)^m 4^{m+1} \pi^{4d+2(m+1)} \\
\times \sum_{n_0 + \cdots + n_m = (m+1)(2d+1)} \exp\left(\frac{2\pi i m}{m+1} \sum_{k=0}^m k n_k\right) \\
\times \prod_{k=0}^m \left(\sum_{l_k=0}^{n_k} (4^{l_k+1} - 1) \frac{B_{2l_k+2}}{(2l_k+2)!} \frac{B_{2n_k-2l_k}}{(2n_k-2l_k)!} e^{\frac{\pi i l_k}{m+1}}\right).
\]

**Theorem 1.4** For any positive integers \(d\) and \(m\),
\[
\zeta^*\left(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1\right)^d \\
= \pi^{4md} \sum_{k_1+3k_2+\cdots+(2d+1)k_d=2d} \prod_{j=1}^d \frac{1}{k_j!} \left(\frac{(2 - 2^{2m}(2j-1)) B_{2m(2j-1)}}{(2j-1) \cdot (2m(2j-1))!}\right)^{k_j},
\]
where the sum is taken over all non-negative integers \(k_1, \ldots, k_d\) satisfying \(\sum_{j=1}^d (2j-1)k_j = 2d\), and
\[
\zeta^*\left(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1\right)^d, (2)^{m} = \pi^{2m(2d+1)} \\
\times \sum_{k_1+3k_2+\cdots+(2d+1)k_{d+1}=2d+1} \prod_{j=1}^{d+1} \frac{1}{k_j!} \left(\frac{(2 - 2^{2m}(2j-1)) B_{2m(2j-1)}}{(2j-1) \cdot (2m(2j-1))!}\right)^{k_j},
\]
where the sum is over all non-negative integers \(k_1, \ldots, k_{d+1}\) such that \(\sum_{j=1}^{d+1} (2j-1)k_j = 2d + 1\).

For small values of \(d\), \(d = 1, 2\), we easily get the following explicit formulas.

**Corollary 1.4.1** For each positive integer \(m\),
\[
\zeta^*\left(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1\right) = 2\pi^{4m} \left(\frac{1 - 2^{2m-1}}{(2m)!}\right)^2 B_{2m},
\]
\[
\zeta^*\left(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1, (2)^{m}\right) = \pi^{6m} \left(\frac{(2^{2m} - 2)^3 B_{2m}^3}{2 \cdot (2m)!^3} + \frac{(2^{6m} - 2) B_{6m}}{6m!}\right)\cdot
\]
\[
\zeta^*\left(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1, (2)^{m-1}, 3, (2)^{m-1}, 1\right) = (2 - 2^{2m}) B_{2m} \pi^{8m} \left(\frac{1 - 2^{2m-1}}{(2m)!^3} B_{2m}^3 + \frac{(2 - 2^{6m}) B_{6m}}{(6m)!}\right).
\]
Note that an MZV version of Theorem 1.3 or Theorem 1.4 is open. A related conjecture was formulated by Borwein et al. [1],

\[
\zeta(\{2\}^m, \{3, \{2\}^m, 1, \{2\}^m \}) = 2(m + 1) \cdot \pi^{4(m+1)d+2m} \frac{1}{(2m+1)(2d+1)!},
\]

and it still remains unresolved. A non-explicit version of (16),

\[
\zeta(\{2\}^m, \{3, \{2\}^m, 1, \{2\}^m \}) = \in Q \cdot \pi^{4(m+1)d+2m},
\]

was proved by Charlton [5] by using motivic zeta values.

We use an alternative representation of Zhao’s formulas given in terms of generating functions in our paper [10]. This allows us to derive closed form evaluations for some special generating functions appeared in Theorems 2.3, 3.2, and 4.3. This also provides a unified approach for proving all three types of identities mentioned above.

**Remark** Note that the referee mentioned that by (11), (12) the values \(\zeta^*(\{\{2\}^m, 3, \{2\}^m, 1\})\) and \(\zeta^*(\{\{2\}^m, 3, \{2\}^m, 1\}, \{2\}^m+1)\) can be written in terms of alternating interpolated \(\zeta\) values

\[
\zeta^t(s_1, \ldots, s_r) = \sum_{\mathbf{p}=s_1 \circ \cdots \circ s_r} t^o(\mathbf{p}) \zeta(\mathbf{p}),
\]

where each “o” is either the comma “,” or the O-plus “⊕,” as

\[
\zeta^*(\{\{2\}^m, 3, \{2\}^m, 1\}) = 2^{2d} \zeta^{\frac{1}{2}}((\overline{2m+2})^{2d})
\]

and

\[
\zeta^*(\{\{2\}^m, 3, \{2\}^m, 1\}, \{2\}^m+1) = -2^{2d+1} \zeta^{\frac{1}{2}}((\overline{2m+2})^{2d+1}).
\]

These formulas are equivalent to representations (21) and (22) in the text. Moreover,

\[
Z^*_0(d, n) = 2^{2d} \sum_{a_1+\cdots+a_{2d}=n \atop a_1, \ldots, a_{2d} \geq 0} \zeta^{\frac{1}{2}}(2a_1+2, 2a_2+2, \ldots, 2a_{2d}+2)
\]

and

\[
Z^*_1(d, n) = -2^{2d+1} \sum_{a_1+\cdots+a_{2d+1}=n-1 \atop a_1, \ldots, a_{2d+1} \geq 0} \zeta^{\frac{1}{2}}(2a_1+2, 2a_2+2, \ldots, 2a_{2d+1}+2).
\]

In this connection, the results of Theorems 1.2, 1.3, and 1.4 can be reformulated in terms of corresponding formulas and sum formulas for alternating interpolated \(\zeta\) values. For some other interesting applications of Zhao’s formulas [18] the reader is referred to [6].

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The paper is organized as follows. In Sect. 2, we derive generating functions for restricted sums of alternating Euler sums and prove Theorem 1.1. In Sect. 3, we find generating functions for the sequences \( \zeta^*(\{2\}^m, 3, \{2\}^m, 1)^d \) and \( \zeta^*(\{2\}^m, 3, \{2\}^m, 1)^d, \{2\}^m+1 \), and prove Theorems 1.3 and 1.4. In Sect. 4, we prove the sum formulas for multiple zeta star values on 3–2–1 indices.

2 Explicit evaluation of \( \zeta^*(\{3, 1\}^d) \) and \( \zeta^*(\{3, 1\}^d, 2) \)

We consider alternating Euler sums (or alternating multiple zeta values) defined by

\[
\zeta(s; \varepsilon) = \zeta(s_1, \ldots, s_r; \varepsilon_1, \ldots, \varepsilon_r) = \sum_{k_1 > k_2 > \cdots > k_r \geq 1} \varepsilon_{k_1} \cdots \varepsilon_{k_r} k_1^{s_1} \cdots k_r^{s_r}
\]

for all positive integers \( s_1, \ldots, s_r \) and \( \varepsilon_j \in \{-1, 1\}, j = 1, \ldots, r \), with \( (s_1, \varepsilon_1) \neq (1, 1) \) in order for the series to converge. We will also use the signed number notation for the multi-index \( s \) associated with \( \zeta(s; \varepsilon) \) by writing \( s_j \) for the \( j \)-th component of \( s \) if \( \varepsilon_j = 1 \), and \( \overline{s}_j \) if \( \varepsilon_j = -1 \). For example,

\[
\zeta(\overline{s}_1, s_2, s_3, s_4, s_5) = \zeta(s_1, s_2, s_3, s_4, s_5; -1, 1, 1, -1, -1).
\]

As for the multiple zeta (star) values, we call the number \( |s| = s_1 + \cdots + s_r \) the weight, and \( r \) the depth (or length) of the alternating Euler sum \( \zeta(s; \varepsilon) \). For a special type of alternating zeta values of the form

\[
\zeta(ms_1, ms_2, \ldots, ms_r; (-1)^{s_1}, (-1)^{s_2}, \ldots, (-1)^{s_r}),
\]

with positive integers \( s_1, \ldots, s_r \), we define the sum of all such values of depth \( r \) and weight \( mn \) with arguments multiples of \( m \) by

\[
A(m, n, r) = \sum_{s_1 + \cdots + s_r = n} \zeta(ms_1, ms_2, \ldots, ms_r; (-1)^{s_1}, (-1)^{s_2}, \ldots, (-1)^{s_r}). \tag{17}
\]

We will need the following theorem, the proof of which was inspired by [8, Theorem 5.1].

**Theorem 2.1** For \( \lambda \in \mathbb{R}, z \in \mathbb{C}, |z| < 1, \) we have

\[
\sum_{n=0}^{\infty} \left( \sum_{r=1}^{n} A(m, n, r)(1 + \lambda)^r \right) z^{mn} = \prod_{k=1}^{\infty} \left( 1 + \lambda \frac{(-1)^k z^m}{k^m} \right) \left( 1 - \frac{(-1)^k z^m}{k^m} \right)^{-1}.
\]

\( \tag{18} \)
Proof Let the right-hand side of (18) be \( F(\lambda, z) \). Then we have

\[
F(\lambda, z) = \prod_{k=1}^{\infty} \left[ \left( 1 + \lambda \frac{(-1)^k z^m}{k^m} \right) \cdot \sum_{s=0}^{\infty} \frac{(-1)^k z^m}{k^m(s+1)} \right]
\]

\[
= \prod_{k=1}^{\infty} \left[ \sum_{s=0}^{\infty} \frac{(-1)^k z^m}{k^m} + \lambda \sum_{s=0}^{\infty} \frac{(-1)^k z^m(s+1)}{k^m(s+1)} \right]
\]

\[
= \prod_{k=1}^{\infty} \left[ \sum_{s=0}^{\infty} \frac{(-1)^k z^m}{k^m} + \lambda \sum_{s=1}^{\infty} \frac{(-1)^k z^m}{k^m} \right]
\]

\[
= \prod_{k=1}^{\infty} \left[ 1 + (\lambda + 1) \sum_{s=1}^{\infty} \frac{(-1)^k z^m}{k^m} \right]
\]

\[
= 1 + \sum_{r \geq 1 \atop s_1, \ldots, s_r \geq 1} (\lambda + 1)^r \sum_{k_1 > \cdots > k_r \geq 1} \frac{(-1)^{k_1 s_1} \cdots (-1)^{k_r s_r}}{k_1^{m s_1} \cdots k_r^{m s_r}} \cdot z^{m(s_1 + \cdots + s_r)}
\]

\[
= 1 + \sum_{n=1}^{\infty} z^{mn} \left( \sum_{r=1}^{n} (\lambda + 1)^r A(m, n, r) \right),
\]

and the statement follows. \( \square \)

To express the values of \( \zeta^\star([3, 1]^d) \) and \( \zeta^\star([3, 1]^d, 2) \) in terms of (17), we will use [10, Theorems 1.3 and 1.9].

**Theorem 2.2** For any positive integer \( d \), we have

\[
\begin{align*}
\zeta^\star([3, 1]^d) &= \sum_{r=1}^{2d} 2^r \cdot A(2, 2d, r), \\
\zeta^\star([3, 1]^d, 2) &= -\sum_{r=1}^{2d+1} 2^r \cdot A(2, 2d + 1, r).
\end{align*}
\]

Proof From [10, Theorem 1.3], we have

\[
\zeta^\star([3, 1]^d) = \sum_{k_1 \geq \cdots \geq k_{2d} \geq 1} \prod_{j=1}^{2d} \frac{(-1)^{k_j} A(k_{j-1}, k_j)}{k_j^2},
\]

where \( k_0 = 0 \) and

\[
\Delta(a, b) = \begin{cases} 
0, & \text{if } a = b; \\
1, & \text{else}.
\end{cases}
\]
Rewriting it in the form

\[ \zeta^*(\{3,1\}^d) = 2 \sum_{k_1 \geq k_2 \geq \cdots \geq k_{2d} \geq 1} \frac{(-1)^{k_1 + k_2 + \cdots + k_{2d}}}{k_1^2 k_2^2 \cdots k_{2d}^2}, \]

and then grouping powers of two, we get

\[ \zeta^*(\{3,1\}^d) = \sum_{r=1}^{2d} 2^r \sum_{s_1 + \cdots + s_r = 2d, k_1 > k_2 > \cdots > k_r \geq 1} \frac{(-1)^{s_1 k_1} (-1)^{s_2 k_2} \cdots (-1)^{s_r k_r}}{k_1^{2s_1} k_2^{2s_2} \cdots k_r^{2s_r}}, \]

which is exactly the required formula.

Similarly, from [10, Theorem 1.9], we have

\[ \zeta^*(\{3,1\}^d,2) = -\sum_{k_0 \geq k_1 \geq \cdots \geq k_{2d} \geq 1} \prod_{j=0}^{2d} \frac{(-1)^{k_j} 2^\Delta(k_{j-1},k_j)}{k_j^2}, \]

where \( k_{-1} = 0 \). Regrouping in powers of two, we obtain

\[
\begin{align*}
\zeta^*(&\{3,1\}^d,2) \\
&= -2 \sum_{k_0 \geq k_1 \geq \cdots \geq k_{2d} \geq 1} \frac{(-1)^{k_0+k_1+\cdots+k_{2d}}}{k_0^2 k_1^2 \cdots k_{2d}^2} \cdot 2^{\sum_{j=1}^{2d} \Delta(k_{j-1},k_j)} \\
&= -\sum_{r=1}^{2d+1} 2^r \sum_{s_1 + \cdots + s_r = 2d+1, k_1 > k_2 > \cdots > k_r \geq 1} \frac{(-1)^{s_1 k_1} (-1)^{s_2 k_2} \cdots (-1)^{s_r k_r}}{k_1^{2s_1} k_2^{2s_2} \cdots k_r^{2s_r}} \\
&= -\sum_{r=1}^{2d} 2^r \cdot A(2,2d+1,r),
\end{align*}
\]

and the theorem follows.

**Theorem 2.3** For \( z \in \mathbb{C}, |z| < 1 \), we have the following generating function:

\[
\sum_{d=0}^{\infty} \zeta^*\{3,1\}^d z^{2d} - \sum_{d=0}^{\infty} \zeta^*\{3,1\}^d,2 z^{2d+2} = \text{tanh}(\pi z/2) \cdot \cot(\pi z/2).
\]
Proof By Theorems 2.1 and 2.2, we have

\[
\sum_{d=0}^{\infty} \xi^*([3, 1]^d) z^{4d} - \sum_{d=0}^{\infty} \xi^*([3, 1]^d, 2) z^{4d+2} = \sum_{d=0}^{\infty} z^{4d} \left( \sum_{r=1}^{2d} 2^r \cdot A(2, 2d, r) \right) + \sum_{d=0}^{\infty} z^{4d+2} \left( \sum_{r=1}^{2d+1} 2^r \cdot A(2, 2d+1, r) \right) = \sum_{n=0}^{\infty} 2^{2n} \left( \sum_{r=1}^{n} 2^r \cdot A(2, n, r) \right) = \prod_{k=1}^{\infty} \left( 1 + \frac{(-1)^k z^2}{k^2} \right) \left( 1 - \frac{(-1)^k z^2}{k^2} \right)^{-1}.
\]

Now evaluate the infinite products to see that

\[
\prod_{k=1}^{\infty} \left( 1 + \frac{(-1)^k z^2}{k^2} \right) = \prod_{j=1}^{\infty} \left( 1 + \frac{z^2}{(2j)^2} \right) = \prod_{j=1}^{\infty} \left( 1 - \frac{z^2}{(2j-1)^2} \right) = \frac{\sin(\pi iz/2)}{\pi iz/2} \cdot \sin(\pi z) \cdot \frac{\pi z/2}{\sin(\pi z/2)} = \frac{\sinh(\pi z/2)}{\pi z/2} \cdot \cos(\pi z/2).
\]

Similarly, we have

\[
\prod_{k=1}^{\infty} \left( 1 - \frac{(-1)^k z^2}{k^2} \right)^{-1} = \frac{\pi iz/2}{\sinh(\pi iz/2)} \cdot \frac{1}{\cos(\pi iz/2)} = \frac{\pi z/2}{\sin(\pi z/2) \cosh(\pi z/2)}.
\]

Therefore, the resulting product is

\[
\frac{\sinh(\pi z/2) \cdot \cos(\pi z/2)}{\pi z/2} = \frac{\pi z/2}{\sin(\pi z/2) \cosh(\pi z/2)} = \tanh(\pi z/2) \cdot \cot(\pi z/2).
\]

Clearly, the definition of the Bernoulli numbers, which are rational numbers given by the generating function

\[
\frac{t}{e^t - 1} = \sum_{s=0}^{\infty} B_s \frac{t^s}{s!}.
\]

\(\square\)
The first few values are
\[
B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \ldots,
\]
and \(B_{2k+1} = 0\) for all integers \(k \geq 1\). The values of the Riemann zeta function at even positive integers can be easily evaluated in terms of \(\pi\) and Bernoulli numbers:
\[
\zeta(2k) = (-1)^{k+1} \frac{B_{2k}}{2(2k)!} (2\pi)^{2k}, \quad k = 0, 1, 2, \ldots, \tag{19}
\]
where by convention, \(\zeta(0) = -1/2\).

### 2.1 Proof of Theorem 1.1

**Proof** The formulas easily follow from Theorem 2.3. Using the well-known power series expansions
\[
\tanh(\pi z/2) = \frac{4}{\pi z} \sum_{n=1}^{\infty} (-1)^{n+1} (4^n - 1) \zeta(2n) \frac{z^{2n}}{4^n}, \quad |z| < 1, \tag{20}
\]
\[
\cot(\pi z/2) = -\frac{4}{\pi z} \sum_{n=0}^{\infty} \zeta(2n) \frac{z^{2n}}{4^n}, \quad |z| < 1,
\]
and multiplying the series, we get
\[
\tanh(\pi z/2) \cdot \cot(\pi z/2) = \frac{4}{\pi^2} \sum_{n=0}^{\infty} \sum_{n_1=0}^{\infty} (-1)^{n_1+1} (4^{n_1+1} - 1) \zeta(2n_1+2) \zeta(2n_2) \frac{z^{2(n_1+n_2)}}{4^{n_1+n_2}}
\]
\[
\sum_{n_1+n_2=n} \sum_{n_1,n_2 \geq 0} (-1)^{n_1+1} (4^{n_1+1} - 1) \zeta(2n_1+2) \zeta(2n_2).
\]

Now by Theorem 2.3, extracting coefficients of \(z^{4d}\) and \(z^{4d+2}\), we obtain
\[
\zeta^\star([3, 1]^d) = \frac{1}{4^{2d-1}\pi^2} \cdot \sum_{k=0}^{2d} (-1)^{k+1} (4^{k+1} - 1) \zeta(2k+2) \zeta(4d-2k),
\]
\[
\zeta^\star([3, 1]^d, 2) = \frac{1}{4^{2d}\pi^2} \cdot \sum_{k=0}^{2d+1} (-1)^{k} (4^{k+1} - 1) \zeta(2k+2) \zeta(4d+2-2k).
\]

Finally, substituting expression (19) in terms of \(\pi\) and Bernoulli numbers, we get the desired formulas. \(\square\)
Corollary 2.3.1 For any non-negative integer $d$,\[
\zeta^\star((3, 1)^d) = \sum_{k=0}^{2d} \zeta((\overline{2})^k) \zeta^\star((\overline{2})^{2d-k}),
\]
\[
\zeta^\star((3, 1)^d, 2) = - \sum_{k=0}^{2d+1} \zeta((\overline{2})^k) \zeta^\star((\overline{2})^{2d+1-k}).
\]

Proof From the proof of Theorem 2.3, we have
\[
F(z) := \sum_{d=0}^{\infty} \zeta^\star([3, 1]^d) z^{4d} - \sum_{d=0}^{\infty} \zeta^\star([3, 1]^d, 2) z^{4d+2}
\]
\[
= \prod_{k=1}^{\infty} \left(1 + \frac{(-1)^k z^2}{k^2}\right) \left(1 - \frac{(-1)^k z^2}{k^2}\right)^{-1}.
\]
Since the first infinite product is the generating function for the sequence $\zeta((\overline{2})^k)$, and the second one is the generating function for the sequence $\zeta^\star((\overline{2})^k)$, we have
\[
\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^k z^2}{k^2}\right) = \sum_{k=0}^{\infty} \zeta((\overline{2})^k) z^{2k},
\]
\[
\prod_{k=1}^{\infty} \left(1 - \frac{(-1)^k z^2}{k^2}\right)^{-1} = \sum_{k=0}^{\infty} \zeta^\star((\overline{2})^k) z^{2k}.
\]
Thus,
\[
F(z) = \sum_{k=0}^{\infty} \zeta((\overline{2})^k) z^{2k} \cdot \sum_{l=0}^{\infty} \zeta^\star((\overline{2})^l) z^{2l} = \sum_{n=0}^{\infty} \sum_{k+l=n} z^{2n} \zeta((\overline{2})^k) \zeta^\star((\overline{2})^l),
\]
and the formulas follow by comparing coefficients of powers of $z$. \qed

3 Explicit evaluation of $\zeta^\star(\{2\}^m, 3, \{2\}^m, 1)^d$ and $\zeta^\star(\{2\}^m, 3, \{2\}^m, 1)^d, \{2\}^{m+1})$

In this section, we show that Theorem 2.2 is easily generalizable, which allows us to obtain explicit expressions for multiple zeta star values from the title.

Theorem 3.1 For any non-negative integers $d$ and $m$,\[
\zeta^\star(\{2\}^m, 3, \{2\}^m, 1)^d) = \sum_{r=1}^{2d} 2^r \cdot A(2m + 2, 2d, r),
\]

\(\square\) Springer
\[ \zeta^*(\{2^m\}, 3, \{2^m\}, 1^d, \{2^m+1\}) = - \sum_{r=1}^{2d+1} 2^r \cdot A(2m + 2, 2d + 1, r). \]

**Proof** From [10, Theorem 1.9], we have

\[ \zeta^*(\{2^m\}, 3, \{2^m\}, 1^d) = \sum_{k_1 \geq \cdots \geq k_{2d} \geq 1} \frac{(-1)^{k_1} 2^{\Delta(k_{j-1}, k_j)}}{k_j^{2m+2}}, \quad (21) \]

where \( k_{-1} = 0 \). Grouping in powers of 2, we obtain

\[
\begin{align*}
\zeta^*(\{2^m\}, 3, \{2^m\}, 1^d) &= 2 \sum_{k_1 \geq \cdots \geq k_{2d} \geq 1} \frac{(-1)^{k_1 + \cdots + k_{2d}} 2^{\sum_{j=2}^{2d} \Delta(k_{j-1}, k_j)}}{k_1^{2m+2} \cdots k_{2d}^{2m+2}} \\
&= \sum_{r=1}^{2d} 2^r \sum_{s_1 + \cdots + s_r = 2d} \sum_{k_1 \geq \cdots \geq k_r \geq 1} \frac{(-1)^{k_1 s_1 \ldots (-1)^{k_r s_r}}}{k_1^{(2m+2)s_1} \cdots k_r^{(2m+2)s_r}} \\
&= \sum_{r=1}^{2d} 2^r \cdot A(2m + 2, 2d, r).
\end{align*}
\]

Similarly, from [10, Theorem 1.9], we have

\[
\zeta^*(\{2^m\}, 3, \{2^m\}, 1^d, \{2^m+1\}) = -2 \sum_{k_0 \geq k_1 \geq \cdots \geq k_{2d} \geq 1} \frac{(-1)^{k_0 + k_1 + \cdots + k_{2d}} 2^{\sum_{j=1}^{2d} \Delta(k_{j-1}, k_j)}}{k_0^{2m+2} k_1^{2m+2} \cdots k_{2d}^{2m+2}} \quad (22)
\]

\[
\begin{align*}
&= -2 \sum_{r=1}^{2d+1} 2^r \cdot A(2m + 2, 2d + 1, r).
\end{align*}
\]

\( \square \)

**Theorem 3.2** For \(|z| < 1\) and any non-negative integer \(m\), we have

\[
\begin{align*}
&\sum_{d=0}^{\infty} \zeta^*(\{2^m\}, 3, \{2^m\}, 1^d) z^{4(m+1)d} \\
&\quad - \sum_{d=0}^{\infty} \zeta^*(\{2^m\}, 3, \{2^m\}, 1^d, \{2^m+1\}) z^{2(m+1)(2d+1)} \\
&= -i \prod_{k=0}^{m} \tan \left( \frac{\pi z}{2} e^{\frac{\pi i (2k+1)}{2m+1}} \right) \cot \left( \frac{\pi z}{2} e^{\frac{\pi i k}{m+1}} \right).
\end{align*}
\]
Proof By Theorems 3.1 and 2.1, for any positive integer \( m \), we have

\[
\sum_{d=0}^{\infty} \xi^*([2]^{m-1}, 3, [2]^{m-1}, 1)^d z^{4md} \\
- \sum_{d=0}^{\infty} \xi^*([2]^{m-1}, 3, [2]^{m-1}, 1)^d, [2]^{m}) z^{2m(2d+1)} \\
= \sum_{d=0}^{\infty} z^{4md} \sum_{r=1}^{2d} 2^r \cdot A(2m, 2d, r) + \sum_{d=0}^{\infty} z^{2m(2d+1)} \sum_{r=1}^{2d+1} 2^r \cdot A(2m, 2d + 1, r) \\
= \sum_{n=0}^{\infty} z^{2mn} \sum_{r=1}^{n} 2^r \cdot A(2m, n, r) = \prod_{k=1}^{\infty} \left( 1 + \frac{(-1)^k \zeta^{2m}}{k^{2m}} \right) \left( 1 - \frac{(-1)^k \zeta^{2m}}{k^{2m}} \right)^{-1}.
\]

Further factoring gives

\[
\prod_{k=1}^{\infty} \left( 1 + \frac{(-1)^k \zeta^{2m}}{k^{2m}} \right) \\
= \prod_{j=1}^{\infty} \left( 1 + \frac{\zeta^{2m}}{(2j)^{2m}} \right) \left( 1 - \frac{\zeta^{2m}}{(2j - 1)^{2m}} \right) \\
= \prod_{j=1}^{\infty} \left( 1 + \frac{\zeta^{2m}}{(2j)^{2m}} \right) \left( 1 - \frac{\zeta^{2m}}{j^{2m}} \right) \left( 1 - \frac{\zeta^{2m}}{(2j)^{2m}} \right)^{-1} \\
= \prod_{j=1}^{\infty} \prod_{k=0}^{m-1} \left( 1 - e^{-\pi i (2k+1)/m} \frac{\zeta^2}{4j^2} \right) \left( 1 - e^{\pi i k/m} \frac{\zeta^2}{j^2} \right) \left( 1 - e^{\pi i k/m} \frac{\zeta^2}{4j^2} \right)^{-1}.
\]

Similarly,

\[
\prod_{k=1}^{\infty} \left( 1 - \frac{(-1)^k \zeta^{2m}}{k^{2m}} \right)^{-1} \\
= \prod_{j=1}^{\infty} \left( 1 - \frac{\zeta^{2m}}{(2j)^{2m}} \right)^{-1} \left( 1 + \frac{\zeta^{2m}}{(2j - 1)^{2m}} \right)^{-1} \\
= \prod_{j=1}^{\infty} \left( 1 - \frac{\zeta^{2m}}{(2j)^{2m}} \right)^{-1} \left( 1 + \frac{\zeta^{2m}}{j^{2m}} \right)^{-1} \left( 1 + \frac{\zeta^{2m}}{(2j)^{2m}} \right) \\
= \prod_{j=1}^{\infty} \prod_{k=0}^{m-1} \left( 1 - e^{\pi i (2k+1)/m} \frac{\zeta^2}{4j^2} \right)^{-1} \left( 1 - e^{\pi i k/m} \frac{\zeta^2}{j^2} \right)^{-1} \left( 1 - e^{\pi i k/m} \frac{\zeta^2}{4j^2} \right).
\]
Hence, the right-hand side of (23) becomes

\[
\text{RHS} = \prod_{j=1}^{\infty} \prod_{k=0}^{m-1} \left( 1 - e^{\frac{2\pi i}{m} \frac{z^2}{4j^2}} \right)^{-2} \left( 1 - e^{\frac{-\pi (2k+1)}{m} \frac{z^2}{4j^2}} \right)^2 \\
\times \left( 1 - e^{\frac{-\pi (2k+1)}{m} \frac{z^2}{j^2}} \right)^{-1} \left( 1 - e^{\frac{2\pi i}{m} \frac{z^2}{j^2}} \right).
\]

Using the infinite product for the sine function, we get

\[
\text{RHS} = \prod_{k=0}^{m-1} \left( \sin \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right) \right)^{-2} \left( \sin \left( \frac{\pi z}{2} e^{\frac{-\pi (2k+1)}{2m}} \right) \right)^2 \\
\times \left( \sin \left( \frac{\pi z}{2} e^{\frac{-\pi (2k+1)}{2m}} \right) \right)^{-1} \left( \sin \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right) \right)
\]

\[
= -i \prod_{k=0}^{m-1} \left[ \sin \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right) \right]^{-2} \left( \sin \left( \frac{\pi z}{2} e^{\frac{-\pi (2k+1)}{2m}} \right) \cos \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right) \right)^2
\]

\[
= -i \prod_{k=0}^{m-1} \frac{\sin \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right) \cos \left( \frac{\pi z}{2} e^{\frac{-\pi (2k+1)}{2m}} \right)}{\sin \left( \frac{\pi z}{2} e^{\frac{-\pi (2k+1)}{2m}} \right) \cos \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right)}
\]

\[
= -i \prod_{k=0}^{m-1} \tan \left( \frac{\pi z}{2} e^{\frac{\pi i}{m}} \right) \cot \left( \frac{\pi z}{2} e^{\frac{-\pi (2k+1)}{2m}} \right).
\]

Now replacing \( m \) by \( m + 1 \), we conclude the proof. \( \square \)

From (23) we easily get the following formulas.

**Corollary 3.2.1** For any non-negative integers \( d, m \),

\[
\zeta^* \left( \{ 2 \}^m, 3, \{ 2 \}^m, 1 \right) = \sum_{k=0}^{2d} \zeta \left( (2m+2)^k \right) \zeta^* \left( (2m+2)^{2d-k} \right),
\]

\[
\zeta^* \left( \{ 2 \}^m, 3, \{ 2 \}^m, 1 \right) = -\sum_{k=0}^{2d+1} \zeta \left( (2m+2)^k \right) \zeta^* \left( (2m+2)^{2d+1-k} \right).
\]
3.1 Proof of Theorem 1.3

**Proof** By Theorem 3.2, the formulas follow from the power series expansion of the product

\[ \Pi := -i \prod_{k=0}^{m} \tan \left( \frac{\pi z}{2} e^{\frac{\pi i (2k+1)}{2(m+1)}} \right) \cot \left( \frac{\pi z}{2} e^{\frac{\pi i k}{m+1}} \right). \]

Using the fact that \( \tan(z) = -i \tanh(iz) \) and applying formulas (20), we get

\[
\begin{align*}
\tan \left( \frac{\pi z}{2} e^{\frac{\pi i (2k+1)}{2(m+1)}} \right) \cdot \cot \left( \frac{\pi z}{2} e^{\frac{\pi i k}{m+1}} \right) &= -\frac{4}{\pi^2} e^{\frac{\pi i (2k+1)}{2(m+1)}} \sum_{n=0}^{\infty} \frac{4^{n+1} - 1}{4^n} \zeta(2n+2) e^{\frac{\pi i (2k+1)n}{m+1}} \zeta(2n) \cdot \sum_{n=0}^{\infty} \frac{\zeta(2n)}{4^n} e^{\frac{2\pi i kn}{m+1}} e^{2n} \\
&= -\frac{4}{\pi^2} e^{\frac{\pi i (2k+1)}{2(m+1)}} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{4^n} e^{\frac{2\pi i kn}{m+1}} \sum_{l=0}^{n} (4^{l+1} - 1) \zeta(2l+2) \zeta(2n-2l) e^{\frac{\pi il}{m+1}}.
\end{align*}
\]

Therefore, expanding the product, we have

\[
\begin{align*}
\Pi &= -i \left( -4 \right)^{m+1} \frac{\pi^2}{\pi^2 (2m+1)} e^{\frac{\pi i (2k+1)}{2}} \prod_{k=0}^{m} \left( \sum_{n=0}^{\infty} \frac{\zeta(2n)}{4^n} e^{\frac{2\pi i kn}{m+1}} \sum_{l=0}^{n} (4^{l+1} - 1) \zeta(2l+2) \zeta(2n-2l) e^{\frac{\pi il}{m+1}} \right) \\
&= \frac{(-1)^{m+1} 4^{m+1}}{\pi^2 (2m+1)} \sum_{n_0=0}^{\infty} \cdots \sum_{n_m=0}^{\infty} \frac{\zeta(2(n_0 + \cdots + n_m))}{4^{n_0 + \cdots + n_m}} \exp \left( \frac{2\pi i}{m+1} \sum_{k=0}^{m} k n_k \right) \times \prod_{k=0}^{m} \sum_{l_k=0}^{n_k} (4^{l_k+1} - 1) \zeta(2l_k + 2) \zeta(2n_k - 2l_k) e^{\frac{\pi i l_k}{m+1}} \\
&= \frac{(-1)^{m+1} 4^{m+1}}{\pi^2 (2m+1)} \sum_{n=0}^{\infty} \frac{\zeta(2n)}{4^n} \sum_{n_0 + \cdots + n_m = n} \sum_{n_0, \ldots, n_m \geq 0} \exp \left( \frac{2\pi i}{m+1} \sum_{k=0}^{m} k n_k \right) \times \prod_{k=0}^{m} \sum_{l_k=0}^{n_k} (4^{l_k+1} - 1) \zeta(2l_k + 2) \zeta(2n_k - 2l_k) e^{\frac{\pi i l_k}{m+1}}.
\end{align*}
\]

Substituting formulas (19), we obtain

\[
\begin{align*}
\Pi &= 4^{m+1} \sum_{n=0}^{\infty} (-1)^n \pi^{2n} \zeta(2n) \sum_{n_0 + \cdots + n_m = n} \sum_{n_0, \ldots, n_m \geq 0} e^{\frac{2\pi i}{m+1} \sum_{k=0}^{m} k n_k} \times \prod_{k=0}^{m} \sum_{l_k=0}^{n_k} (4^{l_k+1} - 1) e^{\frac{\pi i l_k}{m+1}} \frac{B_{2l_k+2}}{(2l_k+2)!} \frac{B_{2n_k-2l_k}}{(2n_k-2l_k)!}.
\end{align*}
\]
Now comparing coefficients of powers of \( z \) for \( n = 2d(m+1) \) and \( n = (2d+1)(m+1) \), by Theorem 3.2, we get the required formulas.

We can give alternative evaluations of \( \zeta^\star \left( \{[2]^m, 3, [2]^m, 1 \}^d \right) \), \( \zeta^\star \left( \{[2]^m, 3, [2]^m, 1 \}^d \right) \) by using Bell polynomials. The modified Bell polynomials \( P_n(x_1, \ldots, x_n) \) are defined by the generating function \[8, 9\]

\[
\exp \left( \sum_{k=1}^{\infty} x_k \frac{z^k}{k} \right) = \sum_{n=0}^{\infty} P_n(x_1, \ldots, x_n) z^n,
\]

where \( P_0 = 1 \) and for \( n \geq 1 \),

\[
P_n(x_1, \ldots, x_n) = \sum_{k_1+2k_2+\cdots+nk_n=n} \frac{1}{k_1!k_2!\cdots k_n!} (\frac{x_1}{1})^{k_1} (\frac{x_2}{2})^{k_2} \cdots (\frac{x_n}{n})^{k_n}.
\]

### 3.2 Proof of Theorem 1.4

**Proof** From (23), we have

\[
\sum_{d=0}^{\infty} \zeta^\star \left( \{[2]^{m-1}, 3, [2]^{m-1}, 1 \}^d \right)z^{4md}
\]

\[
- \sum_{d=0}^{\infty} \zeta^\star \left( \{[2]^{m-1}, 3, [2]^{m-1}, 1 \}^d , [2]^m \right)z^{2m(2d+1)}
\]

\[
= \exp \left( \sum_{k=1}^{\infty} \log \left( 1 + \frac{(-1)^k z^{2m}}{k^{2m}} \right) \right) - \sum_{k=1}^{\infty} \log \left( 1 - \frac{(-1)^k z^{2m}}{k^{2m}} \right)
\]

\[
= \exp \left( \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^{kn} z^{2mn}}{k^{2mn}} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{kn} z^{2mn}}{n \cdot k^{2mn}} \right)
\]

\[
= \exp \left( \sum_{n=1}^{\infty} (1 - (-1)^n) \cdot \zeta(2mn; (-1)^n) \cdot \frac{z^{2mn}}{n} \right) = \sum_{n=0}^{\infty} P_n(x_1, \ldots, x_n)z^{2mn},
\]

where

\[
x_k = (1 - (-1)^k) \cdot \zeta(2mk; (-1)^k) = \begin{cases} 0, & \text{if } k \text{ is even;} \\ 2\zeta(2mk), & \text{if } k \text{ is odd.} \end{cases}
\]

Comparing coefficients of powers of \( z \) on both sides, we obtain

\[
\zeta^\star \left( \{[2]^{m-1}, 3, [2]^{m-1}, 1 \}^d \right) = P_{2d}(x_1, \ldots, x_{2d})
\]
and
\[ \zeta^\star(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1)^d, \{2\}^m) = P_{2d+1}(x_1, \ldots, x_{2d+1}). \]

Expanding the above expression, we get
\[ \zeta^\star(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1)^d = P_{2d}(2\zeta(2m), 0, 2\zeta(6m), 0, \ldots, 2\zeta(2m(2d-1)), 0) \]
\[ \sum_{k_1+k_3+\cdots+(2d-1)k_{2d-1}=2d, k_1, k_3, \ldots, k_{2d-1} \geq 0} \frac{1}{k_1!k_3! \cdots k_{2d-1}!} \left( \frac{2\zeta(2m)}{1} \right)^{k_1} \left( \frac{2\zeta(6m)}{3} \right)^{k_3} \cdots \]
\[ \times \left( \frac{2\zeta(2m(2d-1))}{2d-1} \right)^{k_{2d-1}}. \]

Similarly,
\[ \zeta^\star(\{2\}^{m-1}, 3, \{2\}^{m-1}, 1)^d, \{2\}^m) \]
\[ = P_{2d+1}(2\zeta(2m), 0, 2\zeta(6m), 0, \ldots, 0, 2\zeta(2m(2d+1))) \]
\[ \sum_{k_1+k_3+\cdots+(2d+1)k_{2d+1}=2d+1, k_1, k_3, \ldots, k_{2d+1} \geq 0} \frac{1}{k_1!k_3! \cdots k_{2d+1}!} \left( \frac{2\zeta(2m)}{1} \right)^{k_1} \left( \frac{2\zeta(6m)}{3} \right)^{k_3} \cdots \]
\[ \times \left( \frac{2\zeta(2m(2d+1))}{2d-1} \right)^{k_{2d+1}}. \]

Finally, using the formula
\[ \zeta(s) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^s} = (2^{1-s} - 1)\zeta(s) \]
and applying representations (19) in terms of \( \pi \) and Bernoulli numbers, we get the desired formulas after replacing \( k_{2j-1} \) by \( k_j \) for each \( j \). \( \square \)

4 Sum formulas for multiple zeta star values on 3–2–1 indices

The purpose of this section is to find explicit evaluation of the sum
\[ Z^\star(d, n) = \sum_{a_1+\cdots+a_{2d+1}=n} \zeta^\star(\{2\}^{a_1}, 3, \{2\}^{a_2}, 1, \ldots, 3, \{2\}^{a_{2d}}, 1, \{2\}^{a_{2d+1}}) \]
by applying generating functions from [10]. We will split the above sum into two parts $Z^*_0(d, n)$ and $Z^*_1(d, n)$ defined in (13) and (14), and then evaluate each of these sub-sums separately.

**Theorem 4.1** For any positive integer $d$ and any complex $z$ with $|z| < 1$, we have

\[
\sum_{n=0}^{\infty} Z^*_0(d, n)z^{2n} = \sum_{r=1}^{2d} 2^r \cdot A_z(2, 2d, r),
\]

\[
\sum_{n=0}^{\infty} Z^*_1(d, n + 1)z^{2n} = -\sum_{r=1}^{2d+1} 2^r \cdot A_z(2, 2d + 1, r),
\]

where

\[
A_z(m, n, r) = \sum_{s_1 + \cdots + s_r = n} \sum_{k_1 \geq \cdots \geq k_r \geq 1} \frac{(-1)^{k_1 s_1} \cdots (-1)^{k_r s_r}}{(k_1^m - z^m)s_1 \cdots (k_r^m - z^m)s_r}.
\]

**Proof** From [10, Theorem 1.3], we have

\[
\sum \xi^*([2]^{a_1}, 3, [2]^{a_2}, 1, \ldots, 3, [2]^{a_{2d}}, 1)z^{2a_1} \cdots z^{2a_{2d}}
\]

\[
= \sum_{k_1 \geq \cdots \geq k_{2d} \geq 1} \prod_{j=1}^{2d} \frac{(-1)^{k_j} 2 \Delta(k_{j-1}, k_j)}{k_j^2 - z^2},
\]

where $k_0 = 0$. Putting $z_1 = \cdots = z_{2d} = z$, we get

\[
\sum_{n=0}^{\infty} z^{2n} \sum_{a_1 + \cdots + a_{2d} = n} \xi^*([2]^{a_1}, 3, [2]^{a_2}, 1, \ldots, 3, [2]^{a_{2d}}, 1)
\]

\[
= \sum_{k_1 \geq \cdots \geq k_{2d} \geq 1} \prod_{j=1}^{2d} \frac{(-1)^{k_j} 2 \Delta(k_{j-1}, k_j)}{k_j^2 - z^2}.
\]

Grouping the terms in powers of 2 in the last sum, we obtain

\[
\sum_{n=0}^{\infty} Z^*_0(d, n)z^{2n} = \sum_{r=1}^{2d} 2^r \sum_{s_1 + \cdots + s_r = 2d} \sum_{k_1 \geq \cdots \geq k_r \geq 1} \frac{(-1)^{s_1 k_1} \cdots (-1)^{s_r k_r}}{(k_1^2 - z^2)s_1 \cdots (k_r^2 - z^2)s_r}
\]

\[
= \sum_{r=1}^{2d} 2^r \cdot A_z(2, 2d, r),
\]

and the first formula is proved.
Similarly, for evaluating $\mathcal{Z}_1^*(d, n)$, by [10, Theorem 1.7], we have

$$
\sum_{a_1, \ldots, a_{2d+1} \geq 0} \zeta^*([2]a_1, 3, [2]a_2, 1, \ldots, 3, [2]a_{2d}, 1, [2]a_{2d+1}+1) \zeta_1^{2a_1} \ldots \zeta_{2d+1}^{2a_{2d+1}}
$$

$$
= - \sum_{k_1 \geq \cdots \geq k_{2d+1} \geq 1} \prod_{j=1}^{2d+1} \frac{(-1)^{k_j} 2^{\Delta(k_j-1, k_j)}}{k_j^2 - z_j^2},
$$

where $k_0 = 0$. Putting $z_1 = \cdots = z_{2d+1} = z$, we obtain

$$
\sum_{n=0}^{\infty} z^{2n} \sum_{a_1, \ldots, a_{2d+1} = n} \zeta^*([2]a_1, [2]a_2, 1, \ldots, [2]a_{2d}, 1, [2]a_{2d+1}+1)
$$

$$
= - \sum_{k_1 \geq \cdots \geq k_{2d+1} \geq 1} \prod_{j=1}^{2d+1} \frac{(-1)^{k_j} 2^{\Delta(k_j-1, k_j)}}{k_j^2 - z^2}.
$$

Grouping the terms in powers of 2, we get

$$
\sum_{n=0}^{\infty} \mathcal{Z}_1^*(d, n+1) z^{2n} = - \sum_{r=1}^{2d+1} 2^r \sum_{s_1 + \cdots + s_r = 2d+1} \sum_{k_1 \geq \cdots \geq k_r \geq 1} \prod_{j=1}^{r} \frac{(-1)^{s_j k_j}}{(k_j^2 - z^2)^{s_j}}
$$

$$
= - \sum_{r=1}^{2d+1} 2^r \cdot A_z(2, 2d + 1, r),
$$

and the theorem follows.

**Theorem 4.2** For any integer $m$, real $\lambda$, and complex $w, z$ with $|w| < 1$, $|z| < 1$, we have

$$
\sum_{n=0}^{\infty} \left( \sum_{r=1}^{n} A_z(m, n, r)(1 + \lambda)^r \right) w^{mn}
$$

$$
= \prod_{k=1}^{\infty} \left[ \left( 1 + \frac{\lambda(-1)^k w^m}{k^m - z^m} \right) \left( 1 - \frac{(-1)^k w^m}{k^m - z^m} \right)^{-1} \right].
$$

(24)
Proof Let the right-hand side of (24) be $F(\lambda, z, w)$. Then we have

$$
F(\lambda, z, w) = \prod_{k=1}^{\infty} \left[ \left( 1 + \frac{\lambda (-1)^{k} w^{m}}{k^{m} - z^{m}} \right) \cdot \sum_{s=0}^{\infty} \frac{(-1)^{k} w^{m} w^{s}}{(k^{m} - z^{m})^{s}} \right]
$$

$$
= \prod_{k=1}^{\infty} \left[ \sum_{s=0}^{\infty} \frac{(-1)^{k} w^{m} w^{s}}{(k^{m} - z^{m})^{s}} + \lambda \sum_{s=0}^{\infty} \frac{(-1)^{k(s+1)} w^{m(s+1)}}{(k^{m} - z^{m})^{s+1}} \right]
$$

$$
= \prod_{k=1}^{\infty} \left[ \sum_{s=0}^{\infty} \frac{(-1)^{k} w^{m} w^{s}}{(k^{m} - z^{m})^{s}} + \lambda \sum_{s=1}^{\infty} \frac{(-1)^{k} w^{m} w^{s}}{(k^{m} - z^{m})^{s}} \right]
$$

$$
= \prod_{k=1}^{\infty} \left[ 1 + (\lambda + 1) \sum_{s=1}^{\infty} \frac{(-1)^{k} w^{m} w^{s}}{(k^{m} - z^{m})^{s}} \right]
$$

$$
= 1 + \sum_{r \geq 1} \sum_{s_{1}, \ldots, s_{r} \geq 1} (\lambda + 1)^{r} \sum_{k_{1} > \cdots > k_{r} \geq 1} \frac{(-1)^{k_{1} s_{1}} \cdots (-1)^{k_{r} s_{r}}}{(k_{1}^{m} - z^{m})^{s_{1}} \cdots (k_{r}^{m} - z^{m})^{s_{r}}} \cdot w^{m(s_{1} + \cdots + s_{r})}
$$

$$
= 1 + \sum_{n=1}^{\infty} w^{mn} \left( \sum_{r=1}^{n} (\lambda + 1)^{r} A_{2}(m, n, r) \right),
$$

and the theorem follows. \(\square\)

Theorem 4.3 Let $z, w \in \mathbb{C}$, $|z| < 1$, $|w| < 1$. Then

$$
\sum_{d=0}^{\infty} \sum_{n=0}^{\infty} Z_{0}(d, n) z^{2n} w^{4d} - \sum_{d=0}^{\infty} \sum_{n=0}^{\infty} Z_{1}(d, n) z^{2n-2} w^{4d+2}
$$

$$
= \frac{\sqrt{z^{2} + w^{2}}}{\sqrt{z^{2} - w^{2}}} \tan \left( \frac{\pi \sqrt{z^{2} - w^{2}}}{2} \right) \cot \left( \frac{\pi \sqrt{z^{2} + w^{2}}}{2} \right).
$$

(25)

Proof By Theorems 4.1 and 4.2, we have

$$
\sum_{d=0}^{\infty} \sum_{n=0}^{\infty} Z_{0}(d, n) z^{2n} w^{4d} - \sum_{d=0}^{\infty} \sum_{n=1}^{\infty} Z_{1}(d, n) z^{2n-2} w^{4d+2}
$$

$$
= \sum_{d=0}^{\infty} \left( \sum_{r=1}^{2d} A_{2}(2d, r) \cdot 2^{r} \right) w^{4d} + \sum_{d=0}^{\infty} \left( \sum_{r=1}^{2d+1} A_{2}(2d + 1, r) \cdot 2^{r} \right) w^{4d+2}
$$

(26)

$$
= \sum_{n=0}^{\infty} \left( \sum_{r=1}^{n} A_{2}(2n, r) \cdot 2^{r} \right) w^{2n}
$$

$$
= \prod_{k=1}^{\infty} \left[ 1 + \frac{(-1)^{k} w^{2}}{k^{2} - z^{2}} \right] \left( 1 - \frac{(-1)^{k} w^{2}}{k^{2} - z^{2}} \right)^{-1}.
$$
Hence, the next step is to evaluate the infinite product in (26). Using the infinite product expansion for the sine function, we have

\[
\prod_{k=1}^{\infty} \left(1 + \frac{(-1)^k w^2}{k^2 - z^2}\right) = \prod_{k=1}^{\infty} \left(1 - \frac{z^2 - (-1)^k w^2}{k^2}\right) \left(1 - \frac{z^2}{k^2}\right)^{-1} = \prod_{k=1}^{\infty} \left(1 - \frac{z^2 - w^2}{4k^2}\right) \left(1 - \frac{z^2 + w^2}{2k^2(2k - 1)}\right) \left(1 - \frac{z^2}{k^2}\right)^{-1} \left(1 - \frac{z^2}{k^2}\right)^{-1} = \frac{\sin(\pi \sqrt{z^2 - w^2}/2)}{\pi \sqrt{z^2 - w^2}/2} \cdot \frac{\sin(\pi \sqrt{z^2 + w^2}/2)}{\pi \sqrt{z^2 + w^2}/2} \cdot \frac{\pi \sqrt{z^2 + w^2}/2}{\sin(\pi z)} \cdot \cos(\pi \sqrt{z^2 + w^2}/2).
\]

Similarly, we have

\[
\prod_{k=1}^{\infty} \left(1 - \frac{(-1)^k w^2}{k^2 - z^2}\right)^{-1} = \frac{\sin(\pi z)}{\pi z} \cdot \frac{\pi \sqrt{z^2 + w^2}/2}{\sin(\pi \sqrt{z^2 + w^2}/2)} \cdot \cos(\pi \sqrt{z^2 + w^2}/2).
\]

Finally, combining both products, we get the required formula. □

### 4.1 Proof of Theorem 1.2

**Proof** The formulas easily follow from Theorem 4.3. Expanding trigonometric functions into power series of \( w \) and \( z \), we have

\[
\frac{1}{\sqrt{z^2 - w^2}} \tan \left(\frac{\pi \sqrt{z^2 - w^2}}{2}\right) = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{4k+1 - 1}{4^k} \frac{\zeta(2k + 2)(z^2 - w^2)^k}{\zeta(2k + 2)(z^2 - w^2)^k} = \frac{1}{\pi} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^l \frac{4k+1 - 1}{4^k} \frac{\zeta(2k + 2) \binom{k}{l} z^{2(k-l)}}{z^{2(k-l)}} = \frac{1}{\pi} \sum_{l=0}^{\infty} \sum_{r=0}^{\infty} (-1)^l \frac{4^{l+r+1} - 1}{4^{l+r}} \frac{\zeta(2l + 2r + 2)}{z^{2l + 2r + 2}}.
\]
In the same way,

\[
\sqrt{z^2 + w^2} \cot \left( \frac{\pi \sqrt{z^2 + w^2}}{2} \right) = -\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\zeta(2m)}{4^m} (z^2 + w^2)^m
\]

\[
= -\frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\zeta(2m)}{4^m} \sum_{s=0}^{m} \left( \frac{m}{s} \right) w^{2s} z^{2(m-s)}
\]

\[
= -\frac{4}{\pi} \sum_{s=0}^{\infty} w^{2s} \sum_{m=s}^{\infty} \left( \frac{m}{s} \right) \frac{\zeta(2m)}{4^m} z^{2(m-s)}
\]

\[
= -\frac{4}{\pi} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} w^{2s} z^{2t} \left( \frac{s+t}{s} \right) \frac{\zeta(2s + 2t)}{4^{s+t}}.
\]

After multiplying the series, the right-hand side of (25) becomes

\[
\text{RHS} = -\frac{4}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} (-1)^l w^{2(l+s)} z^{2(r+t)} \left( \frac{l+r}{l} \right) \left( \frac{s+t}{s} \right) \frac{4^{l+r+1} - 1}{4^{l+r+s+t}}
\]

\[
\times \zeta(2l + 2r + 2) \zeta(2s + 2t)
\]

\[
= -\frac{4}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{s=0}^{\infty} \sum_{r=0}^{n} (-1)^l w^{2m} z^{2n} \left( \frac{l+r}{l} \right) \left( \frac{m+n-l-r}{m-l} \right) \frac{4^{l+r+1} - 1}{4^{m+n}}
\]

\[
\times \zeta(2l + 2r + 2) \zeta(2m + 2n - 2(l + r))
\]

\[
= -\frac{4}{\pi^2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{m+n} \sum_{r=0}^{n} (-1)^{k+r} w^{2m} z^{2n} \left( \frac{k}{r} \right) \left( \frac{m+n-k}{n-r} \right) \frac{4^{k+1} - 1}{4^{m+n}}
\]

\[
\times \zeta(2k + 2) \zeta(2m + 2n - 2k).
\]

Extracting coefficients of powers \(z^{2n} w^{4d}\) and \(z^{2n-2} w^{4d+2}\), we get

\[
Z_0^*(d, n) = -\frac{1}{4^{2d+n-1} \pi^2} \sum_{k=0}^{n+2d} \sum_{r=0}^{n} (-1)^{k+r} (4^{k+1} - 1) \left( \frac{k}{r} \right) \left( \frac{2d + n - k}{n - r} \right)
\]

\[
\times \zeta(2k + 2) \zeta(4d + 2n - 2k)
\]

and

\[
Z_1^*(d, n) = \frac{1}{4^{2d+n-1} \pi^2} \sum_{k=0}^{n+2d} \sum_{r=0}^{n-1} (-1)^{k+r} (4^{k+1} - 1) \left( \frac{k}{r} \right) \left( \frac{2d + n - k}{n - 1 - r} \right)
\]

\[
\times \zeta(2k + 2) \zeta(4d + 2n - 2k).
\]
Finally, substituting formulas (19) in terms of $\pi$ and Bernoulli numbers, we get the first and the third formulas. The formula for $Z^*(d, n)$ follows from (15) and the observation

$$\sum_{r=0}^{n} (-1)^r \binom{k}{r} \binom{n + 2d - k}{n - r} + \sum_{r=0}^{n-1} (-1)^{r+1} \binom{k}{r} \binom{n + 2d - k}{n - 1 - r}$$

$$= \sum_{r=0}^{n} (-1)^r \binom{k}{r} \binom{n + 2d - k}{n - r} + \sum_{r=1}^{n} (-1)^r \binom{k}{r - 1} \binom{n + 2d - k}{n - r}$$

$$= \binom{n + 2d - k}{n} + \sum_{r=1}^{n} (-1)^r \binom{n + 2d - k}{n - r} \left( \binom{k}{r} + \binom{k}{r-1} \right)$$

$$= \sum_{r=0}^{n} (-1)^r \binom{n + 2d - k}{n - r} \binom{k}{r + 1}.$$ 

\[ \square \]

**Acknowledgements** The authors would like to thank the anonymous referee for careful reading of the manuscript and useful comments.

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