Electron in the Einstein-Weyl space

S. C. Tiwari

Institute of Natural Philosophy
C/o 1 Kusum Kutir, Mahamanapuri
Varanasi 221005, India

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Abstract

The classical unified theory of Weyl is revisited. The possibility of stable extended electron model in the Einstein-Weyl space is suggested.

1 Introduction

The Kaluza-Klein unified theory is well known amongst physicists since the advent of higher (> 4) dimensional theories - specially inspired by superstring theories, however Weyl’s was the first unified theory of gravitation and electromagnetism [1]. Weyl’s theory was rejected on physical grounds by Einstein, Eddington and Pauli, and Weyl himself abandoned it later in the light of the developments in quantum theory: thus the noncompact group of homothetic transformations in the original gauge group was replaced by the compact circle group of phase transformations. It seems very few mathematicians have been interested in Weyl geometry, see remarks in [2], however Dirac in an attempt to revive this theory in 1973 [3] noted that, “Weyl’s theory remains as the outstanding one, unrivalled by its simplicity and beauty”. I believe there are good reasons to explore Weyl geometry further; (1) influence of Fichtean philosophy on Weyl’s thinking [4] and his idea of continuum that, ‘a manifold is continuous if the points are joined together in such a way that it is impossible to single out a point just for itself, but always only together with a vaguely delimited surrounding halo, with a neighbourhood’
suggest that the study of Weyl geometry may throw light on some fundamental questions in geometry, and (2) even if one considers quantum theory to be the fundamental physical theory, there is a natural generalization of complex Hilbert space of quantum states incorporating Weyl’s vector length holonomy such that the lengths of the complex vectors in the Hilbert space are allowed to change under parallel transport, see [5]. Indeed, as Weyl’s quantum principle of 1929 [6] & [7] provided ground work for gauge field theories, it also suggests the generalization from the invariant scale factor of quantum mechanics to spaces with a varying scale factor.

A special case of Weyl’s theory, namely Einstein-Weyl manifolds have been extensively studied, see [2] for a recent review. An Einstein-Weyl space is defined to be the Weyl space with an extra condition on the curvature, namely the Einstein-Weyl equation, see Equation (21) below. It is well known that the question of the inertial mass of the electron and the stability of the extended structure could not be addressed in the unified theory of Weyl. In this paper we explore the Einstein-Weyl space to get hints for an electron model in a classical geometrical framework. Since some of the definitions and notations are not very familiar in mathematics literature, we summarize them in the next section. In section 3, the action functional is constructed, and the field equations are derived using the variational principle. Concluding remarks constitute the last section.

2 Rudiments of Weyl geometry

In the Riemannian geometry, the relative lengths of two vectors at arbitrary distant points can be compared, therefore for a true infinitesimal geometry it is natural to enlarge the general coordinate transformations postulating gauge transformation. Levi-Civita parallelism was used by Weyl to affect this generalization such that besides a metric

\[ ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \]

there was a linear ground form \( A_\mu dx^\mu \). We confine ourselves to four dimensions, and adopt the notations and sign-index conventions as given by Dirac [3] or Eddington [8] with a slight change that \( A_\mu \) is used for \( k_\mu \). Gauge transformation is defined as

\[ ds \rightarrow ds' = \lambda ds \]

\[ A_\mu \rightarrow A'_\mu = A_\mu + (\ln \lambda), \mu. \]
For ordinary derivatives comma (,) is used, and covariant derivative is denoted by Colon (:). A vector under parallel transport from point $x^\mu$ to point $x^\mu + dx^\mu$ gets a length change given by

$$\delta l = l A_\mu dx^\mu.$$  \hspace{1cm} (4)

Total change in length round a small closed loop is determined by the distance curvature,

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu}. \hspace{1cm} (5)$$

Generalized tensors in Weyl space are denoted by $^*T$. Tensors which get multiplied by $\lambda^n$ under the gauge change (2) are called co-tensors of power $n$, and in-tensors are gauge-invariant ($n = 0$). It can be seen that $g_{\mu\nu}$ is a co-tensor of power 2 and $\sqrt{-g}$ has power 4. One can define co-covariant or Weyl derivative for any co-tensor. To give an example, let $S$ be a co-scalar of power $n$ then its co-covariant derivative is

$$S_{^*\mu} = S_\mu - nA_\mu S. \hspace{1cm} (6)$$

A gauge-invariant affine connection can be obtained to be as

$$^*_\tau^{\alpha}_{\mu\nu} = \tau^{\alpha}_{\mu\nu} - g^{\alpha}_{\mu}\Lambda_\nu - g^{\alpha}_{\nu}\Lambda_\mu + g_{\mu\nu}\Lambda_\alpha. \hspace{1cm} (7)$$

Finally we give the expressions for the generalized Ricci tensor and scalar curvature

$$^*_R^{\mu\nu} = R^{\mu\nu} - 2F^{\mu\nu} - (A_{\mu,\nu} + A_{\nu,\mu}) - g_{\mu\nu}\Lambda_\alpha - 2A_\mu A_\nu + 2g_{\mu\nu}A_\alpha A_\alpha \hspace{1cm} (8)$$

and

$$^*_R = R - 6A_{\mu;\mu} + 6A_\mu A^\mu. \hspace{1cm} (9)$$

Unlike $R_{\mu\nu}$, which is symmetric in $\mu$, $\nu$ in the Riemann space, the corresponding tensor $^*_R^{\mu\nu}$ contains an antisymmetric part in the Weyl space. Dirac, in his paper [3], makes it symmetric by adding few terms, while Eddington [8] retains the complete expression given above i.e. Equation (8). It seems the difference between the symmetric and antisymmetric Ricci tensors in Weyl space may be related to the two kinds of affine connection [9]. Recall that the covariant derivatives of covariant and contravariant vectors in the Riemann space are different [9]. Proceeding in the same way, the generalised affine connections obtained from the co-covariant derivatives of the covariant and contravariant vectors are different due to the difference in the Weyl powers. One of them leads to the symmetric Ricci tensor, while the other one gives Equation (8).
3 Derivation of Einstein-Weyl equations

Let us return to the main objective of this paper. Following our earlier work \cite{10} we present an action functional demanding that the action must be in-invariant. Let us write the action integral

\[ S = \int W \sqrt{-g} d^4x. \]  \hfill (10)

Since \( \sqrt{-g} \) is a co-invariant of power 4, W must be a co-scalar of power -4. Noting that \( *R \) is a co-scalar of power -2, one may assume \( W = *R^2 \); this was the choice of Weyl \cite{11}. However, this does not give the Einstein-Weyl equation. A natural assumption preserving the simplicity of the action for Einstein’s field equation in Riemann space is that \( W \) is proportional to \( *R \).

Thus introducing a co-scalar \( \chi \) of power -2, it is proposed that

\[ W = \xi *R + 2C\xi^2. \]  \hfill (11)

Here \( C \) is an arbitrary real number. To derive the Euler-Lagrange equations of motion, we perform infinitesimal variations in \( g_{\mu\nu} \) and \( A_\mu \) in the action

\[ S_{E-W} = \int (\xi *R + 2C\xi^2) \sqrt{-g} d^4x \]  \hfill (12)

and set \( \delta S_{E-W} \) equal to zero (remember that the variations vanish at and near the boundary of the space-time region). Straightforward calculations give

\[ \xi \delta (R\sqrt{-g}) = [\xi(\frac{1}{2}g_{\mu\nu}R - R_{\mu\nu}) + g^{\mu\nu}\xi^\alpha_\alpha - \xi^{\mu\nu}]\sqrt{-g}\delta g_{\mu\nu}, \]  \hfill (13)

\[ \xi \delta (A^\alpha_\alpha\sqrt{-g}) = (\xi^\mu A^\nu - \frac{1}{2}\xi_\alpha A^\alpha g_{\mu\nu})\sqrt{-g}\delta g_{\mu\nu} - \xi^{\mu}\sqrt{-g}\delta A_\mu \]  \hfill (14)

and

\[ \xi \delta (A_\alpha A^\alpha\sqrt{-g}) = \xi(-A^\mu A^\nu + \frac{1}{2}A_\alpha A^\alpha g^{\mu\nu})\sqrt{-g}\delta g_{\mu\nu} + 2\xi A^\mu \sqrt{-g}\delta A_\mu. \]  \hfill (15)

Substituting \( *R \) from Equation (9) in Equation (12) and using the variational principle we get the following field equations

\[ C\xi^2 g^{\mu\nu} + \xi(\frac{1}{2}g^{\mu\nu} R - R^{\mu\nu}) + g^{\mu\nu}\xi^\alpha_\alpha - 6\xi^\mu A^\nu + 3\xi_\alpha A^\alpha g^{\mu\nu} = 0 \]  \hfill (16)
and
\[ \xi^\mu + 2\xi A^\mu = 0. \] (17)

Using Equation (17), we can rewrite Equation (16) in the form
\[ R^{\mu\nu} = g^{\mu\nu} \Lambda + 2A^\mu A^\nu + 2A^{\mu\nu}, \] (18)

where
\[ \Lambda = C\xi + \frac{1}{2} R + A_\alpha A^\alpha - 2A^{\alpha,\alpha}. \] (19)

Variation of \( \xi \) in the action, Equation (12), gives
\[ \xi = -\frac{R}{4C}. \] (20)

Equation (20) is not an independent one; contraction of Equation (18) gives Equation (20). Substituting \( \xi \) in Equation (18), and rewriting the term \( 2A^{\mu\nu} \) in symmetric and skew-symmetric parts, we obtain the Einstein-Weyl equation
\[ *S^{\mu\nu} = \frac{1}{4}*R g^{\mu\nu}. \] (21)

Here \( *S^{\mu\nu} \) is the symmetric part of the Ricci tensor
\[ *R^{\mu\nu} = *S^{\mu\nu} - 2F^{\mu\nu}. \] (22)

Evidently
\[ F^{\mu\nu} = 0 \] (23)
in view of the rearranged form of \( 2A^{\mu\nu} \) used in Equation (18)
\[ 2A^{\mu\nu} = A^{\mu\nu} + A^{\nu\mu} - F^{\mu\nu}. \] (24)

From Equation (23) it is clear that \( A_\mu \) is locally a gradient. This result is consistent with the contracted Bianchi identity for a Weyl space \[ \square \], see also \[ \square \]. In general
\[ (*S^{\mu\nu} - \frac{1}{2}*R g^{\mu\nu})_{;\nu} + F^{\mu\nu}_{;\nu} = 0. \] (25)

In 4 dimensions, the Weyl derivative of \( F^{\mu\nu} \) is an ordinary covariant divergence, and for the E-W space use can be made of Equation (21) to reduce the Equation (25) in to the form
\[ F^{\mu\nu}_{;\nu} = \frac{1}{4}(*R^\mu + 2*S A^\mu). \] (26)
Substituting Equation (20) and Equation (23) in Equation (26), we get back Equation (17).

This derivation leads to the Einstein-Weyl equations, but for a restricted class of locally conformal Einstein spaces. Examples are known, see [2], such that $F_{\mu \nu}$, is non-vanishing. E-W Equations (21) do not contain an energy-momentum tensor for the field $F_{\mu \nu}$. Therefore this result, namely that the natural E-W space to be a locally conformal Einstein space, seems significant. Is it possible to have a non-zero $F_{\mu \nu}$ together with the E-W equations? The Bianchi identity Equation (26) indicates that r.h.s. of this equation can be interpreted as a source current density for the Maxwell field. Thus, Equation (17) can be generalized to

$$\xi^{\mu} + 2\xi A^{\mu} = J^{\mu}. \quad (27)$$

For a consistent derivation of the field equations, the action function, Equation (11) should be changed to

$$W = \xi^{*} R + 2\xi^{2} + p F^{\mu \nu} F_{\mu \nu}. \quad (28)$$

Here $p$ is an arbitrary real number, and $C$ is set equal to 1. Variation in $g_{\mu \nu}$ contributes an additional term $2pE^{\mu \nu}$, with the electromagnetic energy-momentum tensor being

$$E^{\mu \nu} = \frac{1}{4} g^{\mu \nu} F^{\lambda \sigma} F_{\lambda \sigma} - F^{\mu \lambda} F_{\nu \lambda}. \quad (29)$$

Varying $A_{\mu}$, instead of Equation (17) we get

$$F^{\mu \nu}_{\mu \nu} = \frac{3}{2p}(\xi^{\mu} + 2\xi A^{\mu}). \quad (30)$$

In order to identify Equation (30) with the Bianchi identity, Equation (26), for the E-W space, we assume $p = -3/2$. The co-scalar field $\xi$ determines the current density 4-vector, and Equation (27) can be used to arrive at the following equations:

$$\xi^{2} g^{\mu \nu} + \frac{1}{2} \xi^{*} R g^{\mu \nu} - \xi^{*} S^{\mu \nu} + B1 + B2 = 0, \quad (31)$$

$$B1 = \xi F^{\mu \nu} - \frac{1}{2}[J^{\mu ; \nu} - J^{\nu ; \mu}] - 4[J^{\mu} A^{\nu} - J^{\nu} A^{\mu}] \quad (32)$$
and
\[ B_2 = -3E^{\mu\nu} - \frac{1}{2}[J^{\mu;\nu} + J^{\nu;\mu}] - 2[J^\mu A^\nu + J^\nu A^\mu] + J_\alpha A^\alpha g^{\mu\nu}. \] (33)

Since \( B_1 \) and \( B_2 \) are trace-less, contraction of Equation (31) gives Equation (20) with \( C = 1 \). The skew-symmetric part, \( B_1 \), must vanish. It is further postulated that the symmetric part \( B_2 \) is also zero. Equation (31) then reduces to the E-W Equation (21).

At first, assuming \( B_1 \) and \( B_2 \) to be zero may look arbitrary, but returning to the fundamental unresolved problem of the stability of a purely electromagnetic model of electron we gain a new insight for analysing Equation (32) and Equation (33). To simplify the problem, let us consider the E-W space for \( \xi = \) a constant. The current continuity equation gives the Lorentz gauge condition as
\[ A^{\mu;\mu} = 0, \] (34)
and Equation (20) becomes
\[ R + 6A_\mu A^\mu = \text{constant}. \] (35)
Since \( J^\mu = 2\xi A^\mu \), \( B_1 \) is identically zero. Equation (33) shows that for \( B_2 \) to be zero, the electromagnetic energy momentum tensor must balance with a stress tensor of charge density in which the constant \( \xi \) appears. The implication that \( B_2 = 0 \) is reminiscent of the Poincare stress for an electron model [13] and the problem of inertial mass and electromagnetic energy in a Weyl space discussed by Weyl [1] and Eddington [8].

4 Conclusion

A natural Einstein-Weyl space having pure gauge fields \( A_\mu \) is shown to follow from a simple action principle. A non-vanishing distance curvature (or the Maxwell field) and the contracted Bianchi identity in the Weyl space indicate a non-trivial generalisation postulating current-density four-vector. The general field Equation (31), is shown to admit a special class of space, namely the E-W space if certain conditions (i.e. \( B_1 = B_2 = 0 \)) are imposed. Physical arguments emanating from the stability of the extended model of the electron seem to justify these conditions. However, construction of an actual physical model of the electron remains as an exciting possibility. We conclude this paper by asking: "Is the interior of electron an E-W space?"
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