HOLOMORPHIC FOLIATIONS TANGENT TO LEVI-FLAT SUBSETS

JANE BRETAS & ARTURO FERNÁNDEZ-PÉREZ & ROGÉRIO MOL

Abstract. We study Segre varieties associated to Levi-flat subsets in complex manifolds and apply them to establish local and global results on the integration of tangent holomorphic foliations.

1. Introduction

Let $H \subset M$ be a real analytic hypersurface, where $M$ is a complex manifold of dimension $C$ $M = N$. Let $H_{\text{reg}}$ denote its regular part, that is, the collection of all points near which $H$ is a manifold of maximal dimension. For each $p \in H_{\text{reg}}$, there is a unique complex hyperplane $L_p$ contained in the tangent space $T_pH_{\text{reg}}$. This defines a real analytic distribution $p \mapsto L_p$ of complex hyperplanes in $TH_{\text{reg}}$. When this distribution is integrable in the sense of Frobenius, we say that $H$ is a Levi-flat hypersurface. The resulting foliation in $H$, denoted by $\mathcal{L}$, is known as Levi foliation. A normal form for such an object was given by E. Cartan [6, Th. IV]: for each $p \in H_{\text{reg}}$, there are holomorphic coordinates $(z_1, \ldots, z_N)$ in a neighborhood $U$ of $p$ such that

$$H_{\text{reg}} \cap U = \{ \text{Im}(z_N) = 0 \}. $$

As a consequence, the leaves of $\mathcal{L}$ have local equations $z_N = c$, for $c \in \mathbb{R}$.

Cartan’s local trivialization allows the extension the Levi foliation to a non-singular holomorphic foliation in a neighborhood of $H_{\text{reg}}$ in $M$, which is unique as a germ around $H_{\text{reg}}$. In general, it is not possible to extend $\mathcal{L}$ to a singular holomorphic foliation in a neighborhood of $H_{\text{reg}}$. There are examples of Levi-flat hypersurfaces whose Levi foliations extend to k-webs in the ambient space $\mathbb{R}^N$. However there is an extension in some “holomorphic lifting” of $M$. If a foliation $\mathcal{F}$ in the ambient space $M$ coincides with the Levi foliation on $H_{\text{reg}}$, we say either that $H$ is invariant by $\mathcal{F}$ or that $\mathcal{F}$ is tangent to $H$.

Locally, germs of codimension one foliations at $(\mathbb{C}^N, 0)$ tangent to real analytic Levi-flat hypersurfaces are given by the levels of meromorphic functions — possibly holomorphic — according to a theorem by D. Cerveau and A. Lins Neto. Questions involving the global integrability of codimension one foliations in $\mathbb{P}^N$ tangent to Levi-flat hypersurfaces where addressed by J. Lebl in [5]. For instance, if $H$ is a real algebraic Levi-flat hypersurface tangent to a codimension one foliation $\mathcal{F}$ in $\mathbb{P}^N$, then $\mathcal{F}$ admits a rational first integral $R$ and there is a real algebraic curve $S \subset \mathbb{C}$ such that $H \subset R^{-1}(S)$.

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Our goal in this paper is to establish local and global integrability results for foliations tangent to real analytic Levi-flat subsets. A real analytic subset $H \subset M$, where $M$ is an $N$-dimensional complex manifold, is called Levi-flat if it has real dimension $2n + 1$ and its regular part $H_{\text{reg}}$ is foliated by complex varieties of dimension $n$ (Section 3, Definition 3.1). This is called Levi foliation and $n$ is the Levi dimension of $H$. When $N = n + 1$, we recover the definition of Levi-flat hypersurface. This object appears in M. Brunella’s study on Levi-flat hypersurfaces \cite{Brunella}, as the result of the lifting of a Levi-flat hypersurface to the projectivized cotangent bundle of the ambient space by means of the Levi foliation (see Section 8).

Key ingredients in the study of integrability properties of Levi-flat hypersurfaces are Segre varieties. Their structure is used in Brunella’s geometric proof for the local integrability of foliations tangent to Levi-flat hypersurfaces \cite{Brunella} as well as in Lebl’s global integrability results \cite{Lebl}. Segre varieties for Levi-flat subsets are the cornerstone of our work. Their definition, along with main properties, are presented in Section 4. Recently, a research paper on Levi flat subsets, also founded on the study of Segre varieties, has been released \cite{Recent}. It has an approach to Segre varieties slightly different from ours, although leading to equivalent constructions.

Given a Levi-flat subset $H$ of Levi dimension $n$, there is a unique complex variety $H^i$ of dimension $n + 1$, called intrinsic complexification or $i$-complexification, defined in a neighborhood of $H_{\text{reg}}$ containing $H_{\text{reg}}$ \cite[Th. 2.5]{Brunella}. If $H$ is tangent to a foliation $\mathcal{F}$ of dimension $n$ in the ambient space, then $H^i$ is invariant by $\mathcal{F}$. Our integration results are stated in terms of the $i$-complexification $H^i$ and the foliation $\mathcal{F}^i$, the restriction of $\mathcal{F}$ to $H^i$. For real analytic Levi-flat subsets in projective spaces, we can state the following theorem, to be proved along Sections 5 and 6:

**Theorem A.** Let $H \subset \mathbb{P}^N$, $N > 3$, be a real analytic Levi-flat subset of Levi dimension $n$ invariant by a $n$-dimensional holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^N$. Suppose that $n > (N - 1)/2$. If the Levi foliation $\mathcal{L}$ has infinitely many algebraic leaves, then:

1. the $i$-complexification $H^i$ of $H$ extends to an algebraic variety in $\mathbb{P}^N$;
2. the foliation $\mathcal{F}^i = \mathcal{F}|_{H^i}$ has a rational first integral $R$ in $H^i$;
3. there exists a real algebraic curve $S \subset \mathbb{C}$ such that $H \subset R^{-1}(S)$. In particular $H$ is semialgebraic.

For a real algebraic Levi-flat subset $H \subset \mathbb{P}^N$, the $i$-complexification $H^i$ is algebraic. If further $H$ is invariant by a global $n$ dimensional holomorphic foliation, then the same elements of the proof of Theorem A give that assertions (2) and (3) are also true in this case.

In the local point of view, we have the following integrability result:

**Theorem B.** Let $\mathcal{F}$ be a germ of holomorphic foliation of dimension $n$ at $(\mathbb{C}^N, 0)$ tangent to a germ of real analytic Levi-flat subset $H$ of Levi dimension $n$. Suppose that $\text{Sing}(H^i)$, the singular set of the $i$-complexification of $H$, has codimension at least two. Then $\mathcal{F}^i$ admits a meromorphic first integral.

The proof of this theorem, in Section 7, relies on the integration techniques of Brunella’s geometric proof for Cerveau-Lins Neto’s local integrability theorem \cite{Brunella}. Lastly, we illustrate our main results with some examples in Section 8.
This article is a partial compilation of the results of the Ph.D. thesis of the first author [3], written under the supervision of the second and third authors. They all express their gratitude to R. Rosas and B. Scárdua for suggestions in the development of this work.

2. Mirroring and complexification

Consider coordinates $z = (z_1, \ldots, z_N)$ in $\mathbb{C}^N$, where $z_j = x_j + iy_j$, and the complex conjugation $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_N)$, where $\bar{z}_j = x_j - iy_j$. We will employ the standard multi-index notation. For instance, if $\mu = (\mu_1, \ldots, \mu_N) \in \mathbb{Z}_{\geq 0}^N$ then $z^\mu = (z_1^{\mu_1}, \ldots, z_N^{\mu_N})$. We also fix the following notation for rings of germs at $(\mathbb{C}^N, 0)$:

- $\mathcal{O}_N = \mathbb{C}\{z_1, \ldots, z_N\}$ is the ring of germ of complex analytic functions;
- $\mathcal{A}_N = \mathbb{C}\{z_1, \ldots, z_N, \bar{z}_1, \ldots, \bar{z}_N\} = \mathbb{C}\{x_1, y_1, \ldots, x_N, y_N\}$ is the ring of germs of real analytic functions with complex values;
- $\mathcal{A}_{NR} \subset \mathcal{A}_N$ is the ring of germs of real analytic functions with real values.

A germ of function $\phi(z) = \sum_{\mu, \nu} a_{\mu \nu} z^\mu \bar{z}^\nu$ in $\mathcal{A}_N$ is in $\mathcal{A}_{NR}$ if and only if $\phi(z) = \bar{\phi}(\bar{z})$ for all $z$, which is equivalent to $a_{\mu \nu} = \bar{a}_{\nu \mu}$ for all $\mu, \nu$.

Let $\mathbb{C}^{N\ast}$ be the space with the opposite complex structure of $\mathbb{C}^N$, having complex coordinates $w = (w_1, \ldots, w_N) = \bar{z}$. The conjugation map $\Gamma : z = x + iy \mapsto x - iy = w$ defines a biholomorphism between $\mathbb{C}^N$ and $\mathbb{C}^{N\ast}$. This correspondence is referred to as mirroring. In general, given a subset $A \subset \mathbb{C}^N$, its mirror is the set

$$A^\ast = \Gamma(A) = \{\bar{z}; z \in A\} \subset \mathbb{C}^{N\ast}.$$ 

Given a complex function $\phi$ in $A \subset \mathbb{C}^N$, its mirror $\phi^\ast$ is the function in $A^\ast \subset \mathbb{C}^{N\ast}$ given by

$$\phi^\ast(w) = \bar{\phi}(\bar{w}).$$

For instance, if $\phi(z) = \sum_{\mu} a_{\mu} z^\mu$ is complex analytic, then its mirror

$$\phi^\ast(w) = \sum_{\mu} a_{\mu} \bar{w}^\mu = \sum_{\mu} \bar{a}_{\mu} w^\mu$$

is complex analytic. In the same way, if $\phi \in \mathcal{A}_{NR}$ has a development in power series $\phi(z) = \sum_{\mu, \nu} a_{\mu \nu} z^\mu \bar{z}^\nu$, where $z \in \mathbb{C}^N$, then its mirror function $\phi^\ast \in \mathcal{A}_{NR}$ has a power series expansion

$$\phi^\ast(w) = \sum_{\mu, \nu} a_{\mu \nu} \bar{w}^\mu \bar{w}^\nu = \sum_{\mu, \nu} \bar{a}_{\mu \nu} w^\mu \bar{w}^\nu = \sum_{\mu, \nu} a_{\mu \nu} w^\mu \bar{w}^\nu,$$

where $w \in \mathbb{C}^{N\ast}$. It follows from this discussion that, if $A \subset \mathbb{C}^N$ is a (real or complex) analytic subset, so is its mirror $A^\ast \subset \mathbb{C}^{N\ast}$.

This mirroring procedure can be applied to other geometric objects. For example, to an analytic $p$–form $\eta = \sum_I \alpha_I(z) dz_I$, where $I = (i_1, \ldots, i_p)$ and $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$, we associate the $p$–form $\eta^\ast = \sum_I \alpha_I^\ast(w) dw_I$. A germ of holomorphic foliation $\mathcal{F}$ of codimension $p$ at $(\mathbb{C}^N, 0)$, defined by a $p$–form $\eta$ — that is integrable and locally decomposable outside the singular set — engenders its mirror $\mathcal{F}^\ast$, which is the foliation of codimension $p$ defined by $\eta^\ast$ whose leaves are the mirroring of those of $\mathcal{F}$ (see the Appendix for the definition of holomorphic foliation).
We consider $\mathbb{C}^N \times \mathbb{C}^{N*} \simeq \mathbb{C}^{2N}$ with coordinates $(z, w)$, the embedding
\[ i : \mathbb{C}^N \to \mathbb{C}^N \times \mathbb{C}^{N*} \]
\[ z \mapsto (z, \bar{z}) \]
and the diagonal subset
\[ \Delta := i(\mathbb{C}^N) = \{(z, w) \in \mathbb{C}^N \times \mathbb{C}^{N*}; w = \bar{z}\}. \]

Given a germ of analytic function $\phi \in \mathcal{A}_{\text{NR}}$ we say that a connected neighborhood $U$ of $0 \in \mathbb{C}^N$ is reflexive for $\phi$ or $\phi$-reflexive if $\phi(z, w)$ converges in $U \times U^* \subset \mathbb{C}^N \times \mathbb{C}^{N*}$.

For a germ of map $\phi = (\phi_1, ..., \phi_k) \in (\mathcal{A}_{\text{NR}})^k$, a $\phi$-reflexive neighborhood is one that is $\phi_j$-reflexive for every $j = 1, ..., k$.

Let $\phi \in \mathcal{A}_{\text{NR}}$ be a real function with development in power series $\phi(z) = \sum_{\mu, \nu} a_{\mu\nu} z^\mu \bar{z}^\nu$. The \textit{complexification} of $\phi$ is the germ of complex function $\phi^C \in \mathcal{O}_{2N}$ defined at the origin $0 \in \mathbb{C}^N \times \mathbb{C}^{N*}$ by the series
\[ \psi^C(z, w) = \sum_{\mu, \nu} a_{\mu\nu} z^\mu w^\nu. \]

If $U \subset \mathbb{C}^N$ is a $\phi$-reflexive neighborhood, then this series converges in $U \times U^*$. The complexification of a germ of map $\phi = (\phi_1, ..., \phi_k) \in (\mathcal{A}_{\text{NR}})^k$ is the germ of complex map $\phi^C \in (\mathcal{O}_{2N})^k$ defined by $\phi^C = (\phi^C_1, ..., \phi^C_k)$.

Let $H$ be a germ of real analytic variety at $(\mathbb{C}^N, 0)$. As before, we denote by $H_{\text{reg}}$ its regular part. The singular part of $H$, denoted by $H_{\text{sing}}$, consists of the points in $H \setminus \overline{\mathcal{H}}_{\text{reg}}$. Let $\mathcal{I}(H)$ denote the ideal of $H$ in $\mathcal{A}_{\text{NR}}$. Since $\mathcal{A}_{\text{NR}}$ is Noetherian, we can take a system of generators $\phi_1, ..., \phi_k$ of $\mathcal{I}(H)$ and associate a map $\phi = (\phi_1, ..., \phi_k) \in (\mathcal{A}_{\text{NR}})^k$ that is called generating map of $H$. We have the definition:

\textbf{Definition 2.1.} The \textit{extrinsic complexification} or simply complexification $H^C$ of $H$ is the germ of complex analytic variety at $(\mathbb{C}^N \times \mathbb{C}^{N*}, 0)$ defined by the equation $\phi^C(z, w) = 0$.

If $U$ is $\phi$-reflexive neighborhood, then $H^C$ is realized as
\[ H^C = \{(z, w) \in U \times U^*; \phi(z, w) = 0\}. \]

The set $H^C$ is the smallest germ of complex analytic subset at $(\mathbb{C}^N \times \mathbb{C}^{N*}, 0)$ containing $H_{\Delta} := i(H) = H^C \cap \Delta$. It is evident from the definition that the complexification respects inclusions: if $H_1 \subset H_2$ are germs of real analytic varieties then $H_1^C \subset H_2^C$. This notion of complexification, introduced by H. Cartan in [7], has the following properties:

(i) $H^C \supset H_{\Delta}$;
(ii) every germ of holomorphic function vanishing over $H_{\Delta}$ also vanishes over $H^C$;
(iii) the irreducible components of the real analytic variety $H$ are in correspondence, by complexification, to the irreducible components of the complex analytic variety $H^C$. In particular, $H$ is irreducible if and only if $H^C$ is irreducible.

Let us examine the effect of the complexification procedure on complex varieties. Take $X \subset (\mathbb{C}^N, 0)$ a germ of complex analytic variety whose ideal in $\mathcal{O}_N$ is generated by $f_1, \ldots, f_k$. Seen as a real analytic variety, the corresponding generators of the ideal of $X$ in $\mathcal{A}_{\text{NR}}$ are $\phi_j = \text{Re}(f_j) = (f_j + \bar{f}_j)/2$ and $\psi_j = \text{Im}(f_j) = (f_j - \bar{f}_j)/2i$, for $j = 1, \ldots, k$. 

We consider $\mathbb{C}^{2N}$ with coordinates $(z, w)$, the embedding
\[ i : \mathbb{C}^N \to \mathbb{C}^N \times \mathbb{C}^{N*} \]
\[ z \mapsto (z, \bar{z}) \]
Thus, the complexification $X^C$ in $(\mathbb{C}^N \times \mathbb{C}^{N^*}, 0)$ is the complex analytic variety defined by the system of equations
\[
\phi_j(z, w) = \frac{f_j(z) + \bar{f}_j(w)}{2} = \frac{f_j(z) + \bar{f}_j^*(w)}{2} = 0
\]
and
\[
\psi_j(z, w) = \frac{f_j(z) - \bar{f}_j(w)}{2i} = \frac{f_j(z) - \bar{f}_j^*(w)}{2i} = 0,
\]
for $j = 1, \ldots, k$, which is equivalent to
\[
f_j(z) = 0 \quad \text{and} \quad f_j^*(w) = 0 \quad \text{for} \quad j = 1, \ldots, k.
\]
We therefore conclude that $X^C = X \times X^*$. In particular, we have that $\dim \mathbb{C} X^C = 2 \dim \mathbb{C} X$.

### 3. Levi-flat subsets, local aspects

Essentially, real analytic Levi-flat subsets are real analytic subsets of odd real dimension $2n + 1$ foliated by complex varieties of complex dimension $n$. When the real codimension is one, we are in the case of Levi-flat hypersurfaces. We give the precise definition:

**Definition 3.1.** Let $H \subset M$ be a real analytic subset of real dimension $2n + 1$, where $M$ is an $N$-dimensional complex manifold, $N \geq 2$ and $1 \leq n \leq N - 1$. We say that $H$ is a **Levi-flat** subset if the distribution of tangent spaces
\[
\mathcal{L} : H_{\text{reg}} \subset \mathbb{C}^N \rightarrow T\mathbb{C}^N \cong \mathbb{C}^N
\]
has dimension $n$ and is integrable in the sense of Frobenius.

The regular part of $H$ is a CR-variety, of CR-dimension $n+1$, carrying an $n-$dimensional foliation with complex leaves. We use the qualifier “Levi” for the foliation, its leaves and its dimension. The foliation itself is also denoted by $\mathcal{L}$, its dimension is called $\mathcal{L}$-dimension and denoted by $\dim \mathcal{L}$ or $\dim \mathcal{C} H$. The leaf through by $p \in H_{\text{reg}}$ is denoted by $L_p$. Also, we say that $N$ a the **ambient dimension** of $H$. Most of the time we are concerned with local properties of Levi-flat subsets. In this case, an open set $U \subset \mathbb{C}^N$ plays the role of $M$ in the definition. The notion of Levi-flat subset germifies and, in general, we do not distinguish a germ at $(\mathbb{C}^N, 0)$ from its realization in some neighborhood $U$ of $0 \in \mathbb{C}^N$.

A trivial model for a Levi-flat subset of $\mathcal{L}$-dimension $n$ in $\mathbb{C}^N$ is provided by
\[
H = \{ z = (z', z'') \in \mathbb{C}^{n+1} \times \mathbb{C}^{N-n-1}; \ \text{Im}(z_{n+1}) = 0, z'' = 0 \},
\]
where $z' = (z_1, \ldots, z_{n+1})$ and $z'' = (z_{n+2}, \ldots, z_N)$. The Levi foliation is given by
\[
\{ z = (z', z'') \in \mathbb{C}^{n+1} \times \mathbb{C}^{N-n-1}; \ z_{n+1} = c, z'' = 0, \ \text{with} \ c \in \mathbb{R} \}.
\]
This trivial model is in fact a local form for Levi-flat subsets. This was mentioned in [3] without an explicit proof, which we give for the sake of completeness:

**Proposition 3.2.** Let $H$ be a Levi-flat subset of $\mathcal{L}$-dimension $n$ and ambient dimension $N$. Then, at each $p \in H_{\text{reg}}$, there are local holomorphic coordinates $(z', z'') \in \mathbb{C}^{n+1} \times \mathbb{C}^{N-n-1}$ such that $H$ has the local form (4).
Proof. Since $H_{\text{reg}}$ is a CR-subvariety, for some $k$ with $2 \leq k \leq N$, there are local holomorphic coordinates $t = (t', t'') \in \mathbb{C}^k \times \mathbb{C}^{N-k}$ at $p$ such that $H_{\text{reg}} \subset \{t'' = 0\} \cong \mathbb{C}^k$ is a generic subvariety, that is, $H_{\text{reg}}$ is defined by $d$ real functions in $\mathbb{C}^k$ whose complex differentials are $\mathbb{C}$-linearly independent [1, Cor. 1.8.10]. This gives

$$\dim_{\mathbb{R}}H_{\text{reg}} + d = 2k \quad \text{and} \quad \dim_{\mathbb{C}}T^{(1,0)}H_{\text{reg}} + d = k.$$ 

Combining these equations, we obtain

$$k = \dim_{\mathbb{R}}H_{\text{reg}} - \dim_{\mathbb{C}}T^{(1,0)}H_{\text{reg}} = (2n + 1) - n = n + 1.$$ 

We found that $H_{\text{reg}}$ is as a real analytic Levi-flat hypersurface in the complex variety $\{t'' = 0\}$. It then suffices to apply E. Cartan’s normal form $[1]$ to the coordinates $t'$ in order to get the coordinates $z'$ and take $z'' = t''$.

In the local form $[1]$, $\{z'' = 0\}$ corresponds to the unique local $(n + 1)$–dimensional complex subvariety of the ambient space containing the germ of $H_{\text{reg}}$ at $p$. These local subvarieties glue together forming a complex variety defined in a whole neighborhood of $H_{\text{reg}}$. It is analytically extendable to a neighborhood of $\overline{H_{\text{reg}}}$ by the following theorem:

**Theorem 3.3** (Brunella [1]). Let $M$ be an $N$–dimensional complex manifold and $H \subset M$ be a real analytic Levi-flat subset of $\mathcal{L}$-dimension $n$. Then, there exists a neighborhood $V \subset M$ of $\overline{H_{\text{reg}}}$ and a unique complex variety $X \subset V$ of dimension $n + 1$ containing $H$.

The variety $X$ is the realization in the neighborhood $V$ of a germ of complex analytic variety around $H$. We denote it — or its germ — by $H'$ and call it *intrinsic complexification* or $i$–complexification of $H$. It plays a central role in the theory of Levi-flat subsets we develop. The notion of intrinsic complexification also appears in [19] with the name of Segre envelope.

In this article we are mostly interested in real analytic Levi-flat subsets which are invariant by holomorphic foliations in the ambient space. A real analytic Levi-flat subset $H \subset M$ is *invariant* by an $n$–dimensional singular holomorphic foliation $\mathcal{F}$ on $M$ if the Levi leaves are leaves of $\mathcal{F}$. We also say that $\mathcal{F}$ is *tangent* to $H$. If $H$ is invariant by a foliation, the same holds for its $i$–complexification:

**Proposition 3.4.** Let $H \subset M$ be a real analytic Levi-flat subset of $\mathcal{L}$-dimension $n$, where $M$ is a complex manifold of dimension $N$. If $H$ is invariant by an $n$-dimensional holomorphic foliation $\mathcal{F}$ on $M$, then its $i$-complexification $H'$ is also invariant by $\mathcal{F}$.

**Proof.** We have $\mathcal{F}|_{H_{\text{reg}}} = \mathcal{L}$, where $\mathcal{L}$ is the Levi foliation. The problem is local, so we can work in a local trivialization $[1]$, in which the $i$-complexification is defined by $z'' = 0$ and the Levi leaves are given by $\{z_{n+1} = c, z'' = 0\}$, where $c \in \mathbb{R}$. Let $\vec{v} = (v_1, \ldots, v_{n+2}, \ldots, v_N)$ be a local vector field tangent to $\mathcal{F}$. For each $i = n + 2, \ldots, N$, every $\zeta = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z_{n+1} \in \mathbb{C}$ sufficiently small, it holds

$$v_i(\zeta, \text{Re}(z_{n+1}), 0, \ldots, 0) \equiv 0,$$

and thus

$$v_i(\zeta, z_{n+1}, 0, \ldots, 0) \equiv 0.$$ 

This says that $H'$ is invariant by $\vec{v}$.

When $H$ is invariant by the foliation $\mathcal{F}$, we denote by $\mathcal{F}' = \mathcal{F}|_H$, the restriction of $\mathcal{F}$ to $H'$. Note that $\mathcal{F}'$ has codimension one in $H'$. 


Proposition 3.5. Let $H$ be a germ of real analytic Levi-flat subset. Then $H^C$ is a subset of $H^1 \times H^{*1}$ of complex codimension one.

Proof. Since $H \subset H^1$, it is a consequence of the comments in Section 3 that

$$H^C \subset (H^1)^C = H^1 \times H^{*1}.$$  

Now, this inclusion must be proper since, otherwise, given a defining map $\phi$ for $H$, the complexification $\phi^C$ would vanish over $H^1 \times H^{*1}$, which would imply that $\phi$ itself would vanish over $H^1$. Finally, if $L$ is the closure of a Levi leaf of $H$, which is an analytic set of dimension $\dim_C H$ (see Proposition 4.7 below), then $L \times L^* = L^C \subset H^C$. That is, $H^C$ contains infinitely many complex varieties of codimension two in $H^1 \times H^{*1}$. This implies that the codimension of $H^C$ in $H^1 \times H^{*1}$ is strictly lower than two, which gives the result. \hfill \Box

Denote by $\pi_1 : H^C \to H^1$ and $\pi_2 : H^C \to H^{*1}$ the restrictions of the two canonical projections to $H^C \subset \mathbb{C}^N \times \mathbb{C}^{N^*} \simeq \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$. The following fact appeared in the proof of Theorem 3.3. Its usefulness motivates an explicit statement:

Proposition 3.6. Let $H$ be a germ of real analytic Levi-flat subset. Then given $p \in \overline{H}_{reg}$, we have

$$\pi_1(H^C_{(p, \bar{p})}) = H^1_p \quad \text{and} \quad \pi_2(H^C_{(p, \bar{p})}) = H^{*1}_p,$$

where the sets involved are germs of $H^C$, $H^1$ and $H^{*1}$ at $(p, \bar{p})$, $p$ and $\bar{p}$, respectively.

4. Segre varieties of Levi-flat subsets

Let $H$ be a germ of real analytic Levi-flat subset at $(\mathbb{C}^N, 0)$, $\phi = (\phi_1, \ldots, \phi_k) \in (\mathcal{A}_{NR})^k$ be a generating map and $U$ be a $\phi$-reflexive neighborhood.

Definition 4.1. For each $p \in H^1 \cap U$, the set

$$\Sigma_p(U, \phi) := \{ z \in U; \phi(z, \bar{p}) = 0 \} \cap H^1 \subset U \cap H^1$$

is called Segre variety at $p$ associated to the generating map $\phi$ and to the $\phi$-reflexive neighborhood $U$.

The Segre variety $\Sigma_p(U, \phi) \subset U$ is a closed analytic set that contains $p$ if and only if $p \in H$. It does not depend on the generating map and on the neighborhood $U$ of $0 \in \mathbb{C}^N$ in the following sense: if $\psi$ is another generating map of $H$ and $V$ is a $\psi$-reflexive neighborhood of $0 \in \mathbb{C}^N$, then there exists a neighborhood of the origin $W \subset V \cap U$ such that whenever $p \in W \cap H^1$ it holds $\Sigma_p(U, \phi) \cap W = \Sigma_p(V, \psi) \cap W$. In particular, the germ at $p$ of the Segre variety is well defined. It will be denote by $\Sigma_p$. It contains $p$ if and only if $p \in H$.

Recall that, by Proposition 3.3, we have $H^C \subset H^1 \times H^{*1}$. Let $\pi_1 : H^C \to H^1$ and $\pi_2 : H^C \to H^{*1}$ be the canonical projections. For $p \in H^1$, if we identify $H^1 \times \{ p \} \simeq H^1$, we have, by (3) and (2),

$$\pi_2^{-1}(\bar{p}) = \{ z \in H^1; \phi^C(z, \bar{p}) = 0 \} = \Sigma_p. \quad (5)$$

Similarly, under the identification $\{ p \} \times H^{*1} \simeq H^{*1}$, we have that

$$\pi_1^{-1}(p) = \{ w \in H^{*1}; \phi^C(p, w) = 0 \} = \Sigma^*_p. \quad (6)$$

We have the following result:
Proposition 4.2. Let $W \subset U \subset \mathbb{C}^N$ be an open set and $\phi(z, \bar{z})$ be a real analytic map in $U$. Suppose that $L \subset W$ is a complex variety such that $\phi(z, \bar{z}) = 0$ for all $z \in L$. Then, for each fixed $p \in L$, we have $\phi(z, \bar{p}) = 0$ for all $z \in L$.

Proof. Without loss of generality, we can suppose that $W = U$ and that $U$ is $\phi$-reflexive. Let $H = \{\phi(z, \bar{z}) = 0\} \subset U$. Our hypothesis is that $L \subset H$. Taking complexifications, we find $L \times L^* \subset \mathbb{C}^C \subset \mathbb{C}^N \times \mathbb{C}^{N^*}$. Given $p \in L$, we have

$$L \times \{\bar{p}\} \subset H^C \cap (\mathbb{C}^N \times \{\bar{p}\}) = \{\phi^C(z, \bar{p}) = 0\}.$$  

This is equivalent to $L \subset \{\phi(z, \bar{p}) = 0\}$, which is the desired result. $\square$

As a consequence, if $H$ is a Levi-flat subset and $L_p$ is the Levi leaf at $p \in \overline{L_{reg}}$, then $L_p \subset \Sigma_p$, which gives codim $\mathbb{C}_{H_i}(\Sigma_p) \leq$ codim $\mathbb{C}_{H_i}(L_p) = 1$. This remark motivates the following definition:

Definition 4.3. Let $H$ be a germ of real analytic Levi-flat subset. The point $p \in H$ is said to be Segre degenerate or simply $S$-degenerate if

$$\text{codim}_{\mathbb{C}_{H_i}}(\Sigma_p) = 0.$$  

When codim $\mathbb{C}_{H_i}(\Sigma_p) = 1$, the point $p \in H$ is called Segre ordinary or $S$-ordinary. We denote by $S_d$ the set of $S$-degenerate points of $H$.

For a germ $\phi \in \mathcal{A}_{NR}$ and for a $\phi$-reflexive neighborhood $U$, equation (3) gives that, whenever $(p, \bar{q}) \in U \times U^*$,

$$\phi^C(q, \bar{p}) = 0 \iff \phi^C(q, \bar{q}) = 0 \iff \phi^C(p, \bar{q}) = 0.$$  

This applied to the components of a generating map $\phi$ of a Levi-flat subset $H$ and to a $\phi$-reflexive neighborhood $U$ gives the following:

Proposition 4.4. We have $q \in \Sigma_p(U, \phi)$ if and only if $p \in \Sigma_q(U, \phi)$. In particular, if $p \in S_d$, then $p \in \Sigma_q$ for every $q \in U \cap H_i$.

We have the following proposition:

Proposition 4.5. $S_d$ is a complex analytic variety.

Proof. Following the above notation, we have

$$(7) \quad S_d = \{p \in U \cap H^i; \phi^C(z, \bar{p}) = 0 \forall z \in U \cap H^i\} = \{p \in U \cap H^i; \phi^C(p, \bar{z}) = 0 \forall z \in U \cap H^i\},$$

and then

$$S_d = \left(\bigcap_{z \in U \cap H^i} \{\phi^C(p, \bar{z}) = 0\}\right) \cap H^i.$$  

This defines $S_d$ as a complex analytic set. $\square$

It is worth commenting that $S_d$ is a proper subset of $H^i$. Indeed, otherwise, by (4), $\phi^C$ would vanish over $H^i \times H^{i*}$. This would happen if and only if $\phi^C \equiv 0$, which is impossible.

It is a known fact that the set of $S$-degenerated points of a Levi-flat hypersurface form a complex subvariety of codimension at least two contained in $H_{sing}$ [16]. For Levi-flat subsets we can state the following:
**Proposition 4.6.** $S_d$ has codimension at least two in $H^i$.

**Proof.** We first suppose that $n = \dim_{\mathbb{C}} H = 1$, so that $\dim_{\mathbb{R}} H = 3$ and $\dim_{\mathbb{C}} H^i = 2$. By contradiction, suppose that there exists a one-dimensional irreducible component $\Gamma \subset S_d$. We have $\Sigma_p = H^i$ for every $p \in \Gamma$. As before, let $\pi_2 : H^C \subset H^i \times H^{i*} \rightarrow H^{i*}$ be the projection in the second coordinate. Then, by (8), we have $\pi_2^{-1}(\bar{p}) \simeq \Sigma_p = H^i$ for every $p \in \Gamma$. Therefore $\pi_2^{-1}(\Gamma^*) = H^i \times \Gamma^*$ is a three-dimensional variety. On the other hand, $H^C$ is irreducible and thus $\pi_2^{-1}(\Gamma^*) = H^C$, which gives $\Gamma^* = \pi_2(H^C) = H^{i*}$. This is a contradiction, since $\Gamma^*$ is properly contained in $H^{i*}$.

The general case $n = \dim_{\mathbb{C}} H > 1$ follows from the particular one by taking planar sections. Consider a complex plane $\alpha$ of codimension $n - 1$ simultaneously transversal to $H$, $H^i$ and $H_{\text{sing}}$. The sets $H_{\alpha} = H \cap \alpha$ and $H^i_{\alpha} = H^i \cap \alpha$ have dimensions $\dim_{\mathbb{R}} H_{\alpha} = 3$ and $\dim_{\mathbb{C}} H^i_{\alpha} = 2$. By the minimality property, we have that $H^i_{\alpha} = (H_{\alpha})^i$ is the $\nu$-complexification of $H_{\alpha}$. Let $\phi$ be a defining map for $H$. If $(S_{\alpha})_d$ denotes the set of $S$-degenerated points of $H_{\alpha}$, we have

$$(S_{\alpha})_d = \{ p \in H^i_{\alpha}; \phi|_{\alpha}(z, \bar{p}) \equiv 0 \text{ on } H^i_{\alpha} \} \supseteq \{ p \in H^i; \phi(z, \bar{p}) \equiv 0 \text{ on } H^i \} \cap \alpha = S_d \cap \alpha.$$

The particular case gives that $(S_{\alpha})_d$ is formed by isolated points, which is enough to conclude that $\dim_{\mathbb{C}, H} S_d \geq 2$. □

Levi leaves of a real analytic Levi-flat hypersurface are closed analytic varieties. The same holds for Levi-flat subsets:

**Proposition 4.7.** The Levi leaves of a germ of Levi-flat subset are closed analytic sets.

**Proof.** Indeed, by Proposition [19, Proposition 2.1], every Levi leaf contains $S$-ordinary points. Thus, if $p \in H_{\text{reg}}$ is $S$-ordinary and $L_p$ is the corresponding Levi leaf, we have $\dim_{\mathbb{C}} L_p = \dim_{\mathbb{C}} \Sigma_p$. Since $L_p \subset \Sigma_p$, we conclude that $L_p$ is a component of the analytic set $\Sigma_p$. □

**Remark.** For a germ of real analytic Levi-flat subset $H$ at $(\mathbb{C}^N, 0)$, a point $p \in H_{\text{sing}}$ is said to be dicritical if it belongs to (the closure of) infinitely many leaves of $\mathcal{L}$. The main result in [19] states that the notions of dicriticalness and Segre degeneracy coincide for real analytic Levi-flat subsets.

### 5. Levi flat subsets in projective spaces

In this section we present some results on real analytic Levi-flat subsets in the complex projective space $\mathbb{P}^N = \mathbb{P}^N_{\mathbb{C}}$. If $H \subset \mathbb{P}^N$ is a real analytic variety, then the natural projection

$$\sigma : \mathbb{C}^{N+1} \setminus \{0\} \rightarrow \mathbb{P}^N$$

identifies $H$ with the complex cone

$$(8) \quad H_{\kappa} := \{ z \in \mathbb{C}^{N+1} \setminus \{0\}; \sigma(z) \in H \} \cup \{0\},$$

which is a real analytic subvariety in $\mathbb{C}^{N+1} \setminus \{0\}$. When $H$ is Levi-flat, $H_{\kappa}$ naturally inherits the Levi structure of $H$ and $\dim_{\mathbb{L}} H_{\kappa} = \dim_{\mathbb{L}} H + 1$. We have that $H$ is algebraic if and only if $H_{\kappa}$ is analytic at $0 \in \mathbb{C}^{N+1}$ [19, Proposition 2.1]. Thus, in the real algebraic case, some of the local constructions done so far can be repeated for the germ of $H_{\kappa}$ at $(\mathbb{C}^{N+1}, 0)$.

For instance, we can extend the construction of the (extrinsic) complexification for a real projective algebraic variety $H \subset \mathbb{P}^N$. Consider the ideal $\mathcal{I}(H_{\kappa})$ in $\mathbb{C}[z, \bar{z}]$, where
$z = (z_1, \ldots, z_{N+1})$ are coordinates of $\mathbb{C}^{N+1}$, and take a system of generators $\phi_1, \ldots, \phi_k$, where, for $j = 1, \ldots, k$, each $\phi_j$ is a bihomogeneous polynomial of bidegree $(d_j, d_j)$ in the variables $(z, \bar{z})$. Their complexifications define a complex variety $H_\kappa^C$ in $\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}$, which goes down to an algebraic subvariety $H^C \subset \mathbb{P}^N \times \mathbb{P}^N$ called (extrinsic) projective complexification of $H$. Note that $H^C$ inherits the properties of the local complexification $H_\kappa^C$. We summarize this in the following:

**Proposition 5.1.** Let $H \subset \mathbb{P}^N$ be a real algebraic variety. Then $H^C \subset \mathbb{P}^N \times \mathbb{P}^N$ is a complex algebraic variety, which is irreducible if and only if $H$ is.

We now examine the intrinsic complexification $H^\iota$ of a real analytic Levi-flat subset $H \subset \mathbb{P}^N$. In principle, by pasting local $\iota$-complexifications, we build $H^\iota$ as a complex analytic variety of dimension $\dim_L H + 1$ defined in an open neighborhood of $\mathbf{II}_{\text{reg}}$. When $H$ is algebraic, $H^\iota$ extends to an algebraic subset of $\mathbb{P}^N$, as shown in:

**Proposition 5.2.** Let $H \subset \mathbb{P}^N$ be an irreducible real algebraic Levi-flat subset of $L$-dimension $n$. Then its $\iota$-complexification $H^\iota$ extends to an $(n + 1)$-dimensional algebraic variety in $\mathbb{P}^N$.

**Proof.** We associate to $H$ its projective cone $H_\kappa$, which is analytic and irreducible as a germ at $(\mathbb{C}^{N+1}, 0)$. Let $H_\kappa^C$ denote its complexification at $(\mathbb{C}^{N+1} \times \mathbb{C}^{N+1}, 0)$. By Proposition 4.0, we have $\pi_1(H_\kappa^C) = H_\kappa^C$, where $H_\kappa^C$ is the $\iota$-complexification of $H_\kappa$. By Proposition 5.1, $H^C \subset \mathbb{P}^N \times \mathbb{P}^N$ is complex algebraic and so is its image $\pi^C_1(H^C) \subset \mathbb{P}^N$ by the projection $\pi^C_1: \mathbb{P}^N \times \mathbb{P}^N \to \mathbb{P}^N$ in the first coordinate. Note that the cone associated with $\pi^C_1(H^C)$ is $(\pi^C_1(H^C))_\kappa = \pi_1(H_\kappa^C) = H_\kappa^C$. Finally, $H_\kappa^C$ is the cone of an irreducible algebraic variety in $\mathbb{P}^N$ of dimension $n + 1$ which contains $H$. The result follows from the uniqueness of the intrinsic complexification as a germ around $H$. □

Next we look at Segre varieties of a Levi-flat algebraic subset $H$. We identify $H$ with its algebraic cone $H_\kappa$ at $(\mathbb{C}^{N+1}, 0)$ and take a system of bihomogeneous generators $\phi_1, \ldots, \phi_k \in \mathbb{C}[z, \bar{z}]$ for the ideal $\mathcal{I}(H_\kappa)$. By Proposition 5.3, the $\iota$-complexification $H_\kappa^\iota$ is algebraic. It then follows from Definition 3.1 that the Segre varieties of $H_\kappa$ are algebraic. An arbitrary Levi leaf of $H_\kappa$ contains $S$-ordinary points and, at each of these points, it is a component of the corresponding Segre variety. This gives the following:

**Proposition 5.3.** The Levi leaves of a real algebraic Levi-flat subset in $\mathbb{P}^N$ are algebraic.

As we observed, when a Levi-flat subset $H \subset \mathbb{P}^N$ is real analytic, its $\iota$-complexification in principle is defined in a neighborhood of $\mathbf{II}_{\text{reg}}$. However, in certain cases, we can apply extension results of analytic varieties in order to prove that $H^\iota$ extends to an algebraic variety in $\mathbb{P}^N$. For instance, we can use of the following theorem:

**Theorem 5.4.** (Chow, [4]) Let $Z \subset \mathbb{P}^N$ be an algebraic set of dimension $n$ and $V$ be a connected neighborhood of $Z$ in $\mathbb{P}^N$. Then any analytic subvariety of dimension higher than $N - n$ in $V$ that intersects $Z$ extends algebraically to $\mathbb{P}^N$.

This allows us to state the following extension result for the $\iota$-complexification $H^\iota$:

**Proposition 5.5.** Let $H \subset \mathbb{P}^N$ be a real analytic Levi-flat subset of $\dim_L H = n$ such that $N > 3$ and $n > \frac{N - 1}{2}$. If the Levi foliation has an algebraic leaf, then $H^\iota$ extends to an algebraic variety in $\mathbb{P}^N$. 

Proof. We have $\dim_{\mathbb{C}} L = n$, where $L$ is the Levi leaf which supposed to be algebraic, and $\dim_{\mathbb{C}} H^i = n + 1$. Since $n > (N - 1)/2$, we find $\dim_{\mathbb{C}} H^i = n + 1 > N - n$. The result then follows from Chow’s Theorem. \[\square\]

A foliation of codimension one in $\mathbb{P}^N$ tangent to an algebraic Levi-flat hypersurface has a rational first integral [13, Theorem 6.6]. We can state a version of this result in the context of Levi-flat subsets. We consider a real analytic Levi flat subset of dimension $q$, which is a codimension one foliation on $H^i$, which in principle is a singular variety. We make use of the following result on the integrability of foliations in projective manifolds:

**Theorem 5.6.** (X. Gómez-Mont, [13]) Let $\mathcal{F}$ be a singular holomorphic foliation of codimension $q$ on an irreducible projective manifold $M$. Assume that every leaf $L$ of $\mathcal{F}$ is a quasi-projective subvariety of $M$. Then there exist a projective manifold $X$ of dimension $q$ and a rational map $f : M \to X$ such that the leaves of $\mathcal{F}$ are contained in the fibers of $f$.

We also need the following generalization of Darboux-Jouanolou Theorem [14]:

**Theorem 5.7.** (E. Ghys, [11]) Let $\mathcal{F}$ be a singular holomorphic foliation of codimension one on a smooth, compact and connected analytic complex manifold. If $\mathcal{F}$ has infinitely many closed leaves, then $\mathcal{F}$ has a meromorphic first integral and, therefore, all its leaves are closed.

In order to apply the above theorems, we have to desingularize the $\iota$-complexification $H^i$ using Hironaka’s Dessingularization’s Theorem [13]; there exists a manifold $\tilde{H}^i$ and a proper bimeromorphic morphism $\pi : \tilde{H}^i \to H^i$ such that:

(i) $\pi : \tilde{H}^i \setminus (\pi^{-1}(\text{Sing}(H^i))) \to H^i \setminus \text{Sing}(H^i)$ is an isomorphism.

(ii) $\pi^{-1}(\text{Sing}(H^i))$ is a simple normal crossing divisor.

Note that if the real analytic Levi-flat subset $H \subset \mathbb{P}^N$ is tangent to an abient foliation $\mathcal{F}$ on $\mathbb{P}^N$, then $\mathcal{F}^i$, being the restriction of $\mathcal{F}$ to $H^i$, lifts by the desingularization map to a foliation $\tilde{\mathcal{F}}^i$ on $\tilde{H}^i$.

We then have the main result of this section:

**Proposition 5.8.** Let $H \subset \mathbb{P}^N$ be a real analytic Levi-flat subset of dimension $n$ invariant by a holomorphic foliation $\mathcal{F}$ in $\mathbb{P}^N$. Suppose that the $\iota$-complexification $H^i$ extends to an algebraic variety in $\mathbb{P}^N$ — which happens, for instance, if $N > 3$ and $n > (N - 1)/2$. If the Levi foliation $\mathcal{L}$ has infinitely many algebraic leaves, then $\mathcal{F}^i = \mathcal{F}|_{H^i}$ has a rational first integral.

Proof. Let $\pi : \tilde{H}^i \to H^i$ be a desingularization map. $H^i$ is compact and so is $\tilde{H}^i$. We lift $\mathcal{F}^i$ to an $n$-dimensional foliation $\tilde{\mathcal{F}}^i$ on $\tilde{H}^i$. Our hypothesis gives that $\tilde{\mathcal{F}}^i$ has infinitely many closed leaves and thus the same holds for $\tilde{\mathcal{F}}^i$. By Theorem 5.7, $\tilde{\mathcal{F}}^i$ admits a meromorphic first integral in $\tilde{H}^i$. So, all leaves of $\tilde{\mathcal{F}}^i$ are compact. Besides, their $\pi$-images are compact leaves of $\mathcal{F}^i$ in $H^i$. Finally, by Theorem 5.6, there exists a one-dimensional projective manifold $X$ and a rational map $f : H^i \to X$ whose fibers contain the leaves of $\mathcal{F}^i$. The rational first integral is obtained by composing $f$ with a non-constant rational map $r : X \to \mathbb{P}^1$. \[\square\]
When a Levi-flat subset $H \subset \mathbb{P}^N$ is algebraic, assembling the conclusions of Propositions 5.2 and 5.3, the same argument of the proof of Proposition 5.8 gives the following integrability result:

**Corollary 5.9.** Let $H \subset \mathbb{P}^N$ be an algebraic Levi-flat subset invariant by a holomorphic foliation $F$. Then $H^1$ is algebraic and $F^1$ has a rational first integral.

6. **Rational functions and Levi-flat subsets**

Let $R$ be a rational function on $\mathbb{P}^N$ and $S \subset \mathbb{C}$ be a real algebraic curve. Then $\overline{R^{-1}(S)}$ is a Levi-flat hypersurface [15, Prop. 5.1]. An equivalent result — with a similar proof — can be stated in the context of this paper:

**Proposition 6.1.** Let $X \subset \mathbb{P}^N$ be an irreducible $(n + 1)$-dimensional algebraic variety, $R$ be a rational function on $X$ and $S \subset \mathbb{C}$ be a real algebraic curve. Then the set $\overline{R^{-1}(S)}$ is an algebraic Levi-flat subset of $L$-dimension $n$ whose $\nu$-complexification is $X$.

Our goal in this section is to prove that, with the additional hypothesis that the Levi-flat subset is tangent to a foliation in the ambient space, a reciprocal of this result can be proved by adapting the techniques of [15, Theorem 6.1].

**Proposition 6.2.** Let $H$ be a germ of irreducible real analytic Levi-flat subset of $\dim_C H = n$ at $(\mathbb{C}^N, 0)$. Suppose that $F$ is a non-constant meromorphic function in $H^1$, such that $\text{codim}_{\mathbb{C}} \text{Ind}(F) \geq 2$, which is constant along the Levi leaves. If $0 \in H \cap \text{Ind}(F)$, then there exists an algebraic one-dimensional subset $S \subset \mathbb{C}$ such that $H \subset \overline{F^{-1}(S)}$.

**Proof.** Since the proof goes as that of [15, Lemma 5.2], we just review its main steps and verify that they adapt to our context. It is sufficient to consider the case $n = 1$, for which $\dim_{\mathbb{C}} H^1 = 2$ and $0 \in H^1$ is an isolated point of indeterminacy of $F$ — the general case $n > 1$ reduces to this particular one by cutting $H$ by an $(n - 1)$-plane $\alpha$ in general position, as we did in Proposition 5.4. Write $F = f/g$, where $f$ and $g$ are holomorphic functions in $H^1$, without common factors, and consider the map

$$\Phi : z \in H^1 \mapsto (f(z), g(z)) \in \mathbb{C}^2$$

The crucial fact is that $\Phi(H)$ is semianalytic, an open subset of an analytic variety $K$ of the same dimension. In fact, the map

$$\Psi^C : (z, w) \in H^1 \times (H^1)^* \mapsto (f(z), g(z), f^*(w), g^*(w)) \in \mathbb{C}^4$$

is finite and thus, by the Finite Map Theorem, $\Psi^C(H^1)$ is an analytic variety. Therefore, considering

$$\tilde{\Phi} : z \in H^1 \mapsto (f(z), g(z), \overline{f(z)}, \overline{g(z)}) \in \mathbb{C}^4,$$

we have that $\tilde{\Phi}(H) \subset \Psi^C(H^1) \cap \Delta$ is open and thus it is semianalytic. Note that $\Psi^C(H^1) \cap \Delta \subset \mathbb{C}^4$ can be defined by functions that depend only on the two first coordinates. Thus, taking the projection $\pi : \mathbb{C}^4 \to \mathbb{C}^2$, $\pi(z_1, z_2, z_3, z_4) = (z_1, z_2)$, we have that $\Phi(H) = \pi(\tilde{\Phi}(H)) \subset \pi_1(\Psi^C(H^1) \cap \Delta) = K$ is also semianalytic.
Note that $\Phi(H)$ contains infinitely many complex lines through the origin and thus, if $r(z, \bar{z}) = \sum_{j,k} r_{j,k}(z, \bar{z})$ is a defining function for $K$, written in bihomogeneous terms of bidegree $(j, k)$, then $r_{j,k}(z, \bar{z}) \equiv 0$ for all $(j, k)$, meaning that $K$ is real algebraic. Next, project the algebraic set
\[
\{(z, \xi) \in \mathbb{C}^2 \times \mathbb{C}; z \in K \text{ and } \xi z_2 = z_1\}
\]
in the $\xi$-variable. By Tarski-Seidenberg Theorem \[27\], this projection is semialgebraic, so it lies in a one-dimensional algebraic set $S \subset \mathbb{C}$. Thus $K \subset G^{-1}(S)$. Since $\phi(H) \subset K$ and $F = G \circ \phi$, we conclude that $H \subset \overline{F^{-1}(S)}$. \[\square\]

Remark that if $X \subset \mathbb{P}^N$ is an algebraic complex variety of dim$_\mathbb{C}X \geq 2$, then any rational function in $X$ admits points of indeterminacy. This gives us the following:

**Proposition 6.3.** Let $F$ be a holomorphic foliation in $\mathbb{P}^N$ tangent to a real analytic Levi-flat subset $H$ of dim$_\mathbb{C}H = n$. Suppose that $F^i$ has a rational first integral $R$. Then there exists a real algebraic curve $S \subset \mathbb{C}$ such that $H \subset \overline{R^{-1}(S)}$.

**Proof.** Write $\overline{\Pi_{reg}} = \cup_{\ell} L_{\ell}$, where $L_{\ell}$ are irreducible complex analytic subvarieties given by the closures of Levi leaves of $F$, which are levels of the rational function $R$. Taking $p \in \text{Ind}(R)$, then $p \in H$, since $p \in \cap L_{\ell}$. Applying Proposition 5.2 at $p$, we find a one-dimensional algebraic subset $S \subset \mathbb{C}$ such that, locally, $H \subset \overline{R^{-1}(S)}$. Since $\overline{\Pi_{reg}} = \cup \ell L_{\ell}$ and $p \in L_{\ell}$ for every $\ell$, then $H \subset \overline{R^{-1}(S)}$. \[\square\]

With this proposition, we accomplish the proof of Theorem 4.

**Proof of Theorem 4.** By Proposition 5.8, $F^i = F|_{H^i}$ has a rational first integral in $H^i$, say $R$. The result then follows from Proposition 6.3. \[\square\]

In a similar way, the combination of Proposition 5.3 and the Corollary 5.9 gives:

**Corollary 6.4.** Let $H \subset \mathbb{P}^N$ an algebraic Levi-flat subset invariant by a foliation $F$ in $\mathbb{P}^N$. Then there exist a rational function $R$ in $H^i$ and a real algebraic curve $S \subset \mathbb{C}$ such that $H \subset \overline{R^{-1}(S)}$.

7. **A comment on Brunella’s integration techniques**

In this section we explain how the techniques of \[1\] can be adapted in order to prove Theorem 3. Recall the conditions of its statement: we have a germ of real analytic Levi-flat subset $H$ at $(\mathbb{C}^N, 0)$, of dim$_\mathbb{C}H = n$ and codim$_\mathbb{C}\text{Sing}(H^i) \geq 2$, invariant by a germ of holomorphic foliation $F$ of dimension $n$. We start by remarking that, by applying the Transversality Lemma (stated a proved in the Appendix) and taking transverse plane sections, we can suppose that dim$_\mathbb{C}H = 1$ and that $H^i$ has an isolated singularity at $0 \in \mathbb{C}^N$. We have the following Lemma:

**Lemma 7.1.** Let $H$ be a real analytic Levi-flat subset of $\mathcal{L}$-dimension 1 at $(\mathbb{C}^N, 0)$ invariant by a germ of holomorphic foliation $F$. Then, for each $p \in H^i \setminus \{0\}$, the mirror of Segre variety $\Sigma_p \subset (H^i)^*$ is a non-empty curve invariant by the mirror foliation $F^*$. Besides, if $p$ and $q$ are on the same leaf of $F^*$, then $\Sigma_p = \Sigma_q$. 
Proof: The fact that $\Sigma_p^* \subset H^*$ is non-empty for every $p \in H^1$ sufficiently near $0 \in C^N$ follows from Proposition 3.4. Since $\text{codim}_C S_d \geq 2$ and $\dim H^1 = 2$, we can suppose that $0 \in C^N$ is the only Segre degenerate point, implying that $\Sigma_p^*$ is a curve in $H^*$ for each $p \in H^1 \setminus \{0\}$. Take the two-dimensional foliation $\mathcal{F} \times \mathcal{F}^*$ in $H^1 \times H^*$ whose leaf through $(p, q, r) \in (H^1 \setminus \{0\}) \times (H^* \setminus \{0\})$ is $L_{p, q} = L_{p, q} \times L_{q, r}$, where $L_p$ denotes the leaf of $\mathcal{F}$ through $p$. Consider the analytic complex set of tangencies between $\mathcal{F} \times \mathcal{F}^*$ and $H^C \subset H^1 \times H^*$, denoted by $\text{Tang}(\mathcal{F} \times \mathcal{F}^*, H^C) \subset H^C$. Since $H^\Delta \subset \text{Tang}(\mathcal{F} \times \mathcal{F}^*, H^C)$, the minimality of the complexification implies that $$(H^\Delta)^C = H^C = \text{Tang}(\mathcal{F} \times \mathcal{F}^*, H^C).$$

Denote, as before $\pi_1 : H^C \subset H^1 \times H^* \rightarrow H^1$ the projection in the first coordinate. Then, for each $p \in H^1 \setminus \{0\}$, the fiber $\pi_1^{-1}(p) = \Sigma_p^*$ is a one-dimensional analytic set tangent to $\mathcal{F} \times \mathcal{F}^*$. Thus $\Sigma_p^*$ is invariant by $\mathcal{F}^*$ and is composed by a finite union of leaves of $\mathcal{F}^*$. It follows that, for a fixed leaf $L$ of $\mathcal{F}$, the inverse image $\pi_1^{-1}(L) \subset H^C$ is invariant by $\mathcal{F} \times \mathcal{F}^*$ and has the form $\pi_1^{-1}(L) = L \times \bigcup_{\lambda \in \Lambda} L^*_\lambda$, where the $L^*_\lambda$’s are leaves of $\mathcal{F}^*$ and $\Lambda$ is a finite set. In particular, if $p$ and $q \in L$, we have $\Sigma_p^* = \{p\} \times \bigcup_{\lambda \in \Lambda} L^*_\lambda$ and $\Sigma_q^* = \{q\} \times \bigcup_{\lambda \in \Lambda} L^*_\lambda$. Identifying these with $\bigcup_{\lambda \in \Lambda} L^*_\lambda$, we obtain $\Sigma_p^* = \Sigma_q^*$. 

Theorem 3 is a straight consequence of the proposition below, for which the above lemma is a key ingredient. It restates Propositions 2 and 4 of [4] and its proof follows the very same steps as those in Brunella’s paper. The only difference is that here we should also take into account the desingularization divisor of the $\nu$-complexification $H^1$. The hypothesis on the codimension of $\text{Sing}(H^1)$ is needed in order to apply Levi’s extension theorem for meromorphic functions.

Proposition 7.2. Let $\mathcal{F}$ be a germ of one-dimensional holomorphic foliation at $(C^N, 0)$ tangent to a germ of analytic real Levi-flat subset $H$ of $\text{dim}_C H = 1$. Suppose that the $\nu$-complexification $H^1$ has an isolated singularity at origin and that one of the two following conditions is satisfied:

1. For every $p \in H^1 \setminus \{0\}$, the mirror of Segre variety $\Sigma_p^*$ is a proper analytic curve in $H^*$ passing through the origin;
2. For every $p \in H^*$, the mirror of Segre variety $\Sigma_p^*$ is a proper analytic curve in $H^*$ passing through the origin when $p = 0$.

Then $\mathcal{F}$ has a first integral that is purely meromorphic in case (1) and holomorphic in case (2).

8. Examples

Let $Z$ be a real analytic Levi-flat hypersurface in a complex manifold $X$ of $\text{dim}_C X = n + 1$. Let $\mathbb{P}T^*X$ be the cotangent bundle projectivization, which is a $\mathbb{P}^n$-bundle over $X$ whose dimension is $N = 2n + 1$. Denote by $\rho$ the projection $\mathbb{P}T^*X \rightarrow X$. The regular part $Z_{\text{reg}}$ of $Z$ can be lifted to $\mathbb{P}T^*X$, since, for any $z \in Z_{\text{reg}},$

$$T^*_z Z_{\text{reg}} = T_z Z_{\text{reg}} \cap J(T_z Z_{\text{reg}}) \subset T_z X$$

is a complex hyperplane. Let $H_{\text{reg}}$ be the lifting of $Z_{\text{reg}}$ in $\mathbb{P}T^*X$. Fix $y \in H_{\text{reg}}$ such that $\rho(y) = z \in Z_{\text{reg}}$. It follows from 3 that there exists a neighborhood $V \subset \mathbb{P}T^*X$ of $y$ and a germ of complex variety $Y_y$ at $y$ of dimension $n + 1$ containing $H_{\text{reg}}$ on $\mathbb{P}T^*X$. 
We have that $H = \overline{H}_{reg}$ is a germ at $y$ of Levi-flat subset of $\text{dim}_C H = n$ on $M = \mathbb{P}T^* X$. The gluing of the local varieties $Y_y$ produces its $\iota$-complexification $H^\iota$. By this procedure, any real analytic Levi-flat hypersurface in a complex manifold $X$ induces a real analytic Levi-flat subset in $\mathbb{P}T^* X$.

When $X = \mathbb{P}^{n+1}$, its projectivized cotangent bundle is isomorphic to the incidence variety

$$\Upsilon = \{(p, \alpha) \in \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}; p \in \alpha\},$$

where $\mathbb{P}^{n+1}$ denotes the parameter space of all hyperplanes in $\mathbb{P}^{n+1}$ (see [18, p. 27]). Therefore, when considering a real analytic Levi-flat hypersurface $Z$ in $\mathbb{P}^{n+1}$, what we get is a real analytic Levi-flat subset $H$ in $\Upsilon$. However $\Upsilon$ is not a complex projective space and our main results on global integrability cannot be applied in this situation.

A canonical way to generate Levi-flat subsets is by intersecting Levi-flat hypersurfaces with complex analytic subvarieties. The examples of real analytic Levi subsets we present below are based on this principle.

**Example 8.1.** Let $H = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4; \bar{z}_3 z_2 - \bar{z}_2 z_3 = 0, \ z_4 = 0\}$. Then $H$ is a real analytic Levi-flat subset in $\mathbb{C}^4$, with degenerate singularities along the $z_1$-axis. The leaves of the Levi foliation are $L_c = \{z_3 = z_2 c, \ z_4 = 0\}$ for $c \in \mathbb{R}$. Note that the $\iota$-complexification of $H$ is the hyperplane $H^\iota = \{z_4 = 0\}$. On the other hand, since $H$ is a complex cone in $\mathbb{C}^4 \setminus \{0\}$, we get that $H$ induces a Levi-flat subset in $\mathbb{P}^3$ that satisfies the hypothesis of Theorem [A]. The foliation $\mathcal{F}$ given by the polynomial 1-form $\omega = z_2 d\bar{z}_3 - z_3 d\bar{z}_2$ defines a holomorphic foliation on $\mathbb{P}^3$ tangent to $H$. Moreover, $\mathcal{F}$ has a rational first integral $R = z_3/z_2$, which clearly defines a rational first integral on $H^\iota$.

**Example 8.2.** In $\mathbb{C}^4$ with coordinates $(z_1, z_2, z_3, z_4)$, take

$$H = \{z_1^2 \bar{z}_3^2 - z_1 \bar{z}_3 |z_2|^2 + z_1 z_3 \bar{z}_2^2 - 2 |z_1|^2 |z_3|^2 + \bar{z}_1 \bar{z}_3 z_2^2 - z_3 \bar{z}_1 |z_2|^2 + z_3^2 \bar{z}_1^2 = 0, \ z_4 = 0\}.$$  

Then $H$ is a real analytic Levi-flat subset foliated by the 2-planes

$$L_c = \{z_1 + cz_2 + c^2 z_3 = 0, \ z_4 = 0\}$$

for $c \in \mathbb{R}$. Again, the $\iota$-complexification is $H^\iota = \{z_4 = 0\}$. Naturally, $H$ defines a real analytic Levi-flat subset in $\mathbb{P}^3$ but, in this case, $H$ is not invariant by an ambient holomorphic foliation. Note that, by elimination of $c$ in the system of equations

$$\begin{cases} z_1 + cz_2 + c^2 z_3 = 0 \\ dz_1 + cdz_2 + c^2 dz_3 = 0 \end{cases}$$

we obtain a holomorphic 2-web tangent to $H$ in $\mathbb{P}^3$.

**Example 8.3.** We present next a real analytic non-algebraic Levi-flat subset of $\mathbb{P}^3$ of $\mathcal{L}$-dimension 1, having an algebraic $\iota$-complexification and containing infinitely many algebraic leaves in its Levi foliation. However, it is not invariant by a global holomorphic foliation on $\mathbb{P}^3$. This shows that, in Theorem [A], the assumption of the existence of a global foliation is essential in order to get semi-algebraicity. We adapt an example in [17], whose construction is summarized in the following lemma:

**Lemma 8.4 ([17]).** Let $S \subset \mathbb{R}^2$ be a connected compact real analytic curve without singularities. Let $H$ be the complex cone defined by

$$\tilde{H} = \{(z_0, z_1, z_2) \in \mathbb{C}^3; z_0 = z_1 x + z_2 y \text{ for } (x, y) \in S\} \cup \{z \in \mathbb{C}^3; z_1 \bar{z}_2 = \bar{z}_1 z_2\}. $$
Then $\tilde{H}$ is a real analytic Levi-flat hypersurface in $\mathbb{C}^3 \setminus \{0\}$ whose canonical projection $\sigma(\tilde{H})$ is a real analytic Levi-flat hypersurface in $\mathbb{P}^2$. Besides, if $S$ is not contained in any proper real algebraic curve in $\mathbb{R}^2$, then $\sigma(\tilde{H})$ is not algebraic.

Let us now take the projection $\nu : \mathbb{C}^4 \setminus \{0\} \to \mathbb{C}^3 \setminus \{0\}$ defined by $\nu(z_0, z_1, z_2, z_3) = (z_0, z_1, z_2)$ and the real analytic complex cone defined by $H' = \nu^{-1}(\tilde{H}) \cap \{(z_0, z_1, z_2, z_3) \in \mathbb{C}^4 \setminus \{0\}; z_0z_3 - z_1z_2 = 0\}$. Hence $H = \sigma(H')$ is a real analytic subvariety in $\mathbb{P}^3$. We have that $H \subset \mathbb{P}^3$ is Levi-flat with $\dim \mathcal{L}H = 1$ and its intrinsic complexification is the quadric $Q \subset \mathbb{P}^3$ defined by $z_0z_3 - z_1z_2 = 0$. Moreover, if we pick $S \subset \mathbb{R}^2$ real analytic but non-algebraic, we obtain a real analytic non-algebraic Levi-flat subset $H \subset \mathbb{P}^3$.

Finally, we assert that $H$ is not tangent to a one-dimensional holomorphic foliation in $\mathbb{P}^3$. In fact, without loss of generality and possibly translating $S$, we assume that for all $x \in \mathbb{R}$ small enough, there exists at least two distinct points $y_1, y_2 \in \mathbb{R}$ such that $(x, y_1), (x, y_2) \in S$. Given such a $x \neq 0$, there are at least two distinct leaves of the Levi-flat subset $H$ passing through $[x : 1 : 0 : 0] \in H$, corresponding the hyperplanes of equations $z_0 = z_1x + z_2y_1$ and $z_0 = z_1x + z_2y_2$. Then, around these points, the Levi foliation cannot be tangent to an ambient holomorphic foliation.

\section*{Appendix}

Let $M$ be an $N$-dimensional complex manifold whose cotangent sheaf is $\Omega_M = \mathcal{O}(T^*M)$. An $n$-dimensional holomorphic foliation $\mathcal{F}$ on $M$, where $1 \leq n < N$, is the object defined by an analytic coherent subsheaf $\mathcal{C}$ of $\Omega_M$ of rank $N-n$ satisfying the following properties (see \cite{20} for details):

(i) $d\mathcal{C}_p \subset (\Omega_M \wedge \mathcal{C})_p$ for every $p \in M \setminus \operatorname{Sing}(\mathcal{C})$ (integrability condition);

(ii) $\operatorname{Sing}(\Omega_M/\mathcal{C})$ is a set of codimension two. This is the singular set of $\mathcal{F}$ and denoted by $\operatorname{Sing}(\mathcal{F})$.

We call $\mathcal{C}$ the conormal sheaf of $\mathcal{F}$. Recall that the singular set of a coherent sheaf is the set of points where its stalks fail to be free modules over the structural sheaf. Outside $\operatorname{Sing}(\mathcal{F})$, the conormal sheaf is the sheaf of sections of a rank $N-n$ vector subbundle of $T^*M$, defining an integrable holomorphic distribution of subspaces of dimension $N-n$. 

on $T^*M$ and, thus, a regular holomorphic foliation of dimension $n$ on $M$. Then, since $\text{codim}_C \text{Sing}(\mathcal{F}) \geq 2$, the foliation $\mathcal{F}$ is locally induced by holomorphic $(N - n)$-forms which are locally decomposable outside $\text{Sing}(\mathcal{F})$ and satisfy the integrability condition. We emphasize that our definition does not ask $\mathcal{F}$ to be a reduced foliation. By definition, this happens when $\mathcal{C}$ is a full sheaf, that is, whenever $U \subset M$ is open and $\omega$ is a holomorphic section of $\Omega_M$ over $U$ such that is also a section of $\mathcal{C}$ over $U \setminus \text{Sing}(\mathcal{F})$, then it is a section of $\mathcal{C}$ over $U$.

We finish by proving a transversality lemma that has been used in Theorem 3. First a definition. Let $\mathcal{F}$ be a germ of singular holomorphic foliation of dimension $n$ at the origin of $M = \mathbb{C}^N$ with conormal sheaf $\mathcal{C}$, where $1 < n < N$. Let $\alpha$ be a germ of hyperplane through $0 \in \mathbb{C}^N$ and denote by $\Omega_\alpha$ its cotangent sheaf. We say that $\alpha$ is in general position with or transverse to $\mathcal{F}$ if the singular set of $(\Omega_M/\mathcal{C})_\alpha \cong \Omega_\alpha/(\mathcal{C}_\alpha)$ has codimension at least two. Thus, $\mathcal{C}_\alpha$ is the conormal sheaf of a foliation of dimension $n - 1$ in $(\alpha, 0) \cong (\mathbb{C}^{n-1}, 0)$ that will be denoted by $\mathcal{F}_\alpha$.

**Lemma** (Transversality). Let $\mathcal{F}$ be a germ of singular holomorphic foliation of dimension $n$ at $(\mathbb{C}^N, 0)$. Then the set of hyperplanes through $0 \in \mathbb{C}^N$ transverse to $\mathcal{F}$ form a generic subset in the Grassmannian $\text{Gr}_0(N - 1, N) \cong \mathbb{P}^{N-1}$. 

**Proof.** We have the following fact: if $\omega$ is a germ of holomorphic 1–form at $0 \in \mathbb{C}^N$ (not necessarily integrable) with singular set of codimension at least two, then the set of hyperplanes through $0 \in \mathbb{C}^N$ transverse to $\omega$ is generic in $\text{Gr}_0(N - 1, N) \cong \mathbb{P}^{N-1}$. This is actually a consequence of the proof of [3, Lemma 10]. The conormal sheaf $\mathcal{C}$ of $\mathcal{F}$ is coherent and thus, generated by finitely many sections at $0 \in \mathbb{C}^N$, say $k$ holomorphic 1–forms $\omega_1, \ldots, \omega_k$. For each $i = 1, \ldots, k$, we can cancel one-codimensional singular components of $\omega_i$, obtaining holomorphic 1–forms $\tilde{\omega}_i$ such that $\text{Sing}(\tilde{\omega}_i) \geq 2$. Note that, since we are not assuming that $\mathcal{F}$ is reduced, each $\tilde{\omega}_i$ does not necessarily define a section of $\mathcal{C}$, yielding however a section outside $\text{Sing}(\mathcal{F})$. The set of hyperplanes transverse to each $\tilde{\omega}_i$ is a generic set $\Gamma_i \subset \text{Gr}_0(N - 1, N)$. Let $\Gamma_0$ denote the generic set of hyperplanes transverse to $\text{Sing}(\mathcal{F})$ and consider the set $\Gamma = \Gamma_0 \cap \Gamma_i$. Then $\Gamma \subset \text{Gr}_0(N - 1, N)$ is a generic set formed by hyperplanes transverse to $\mathcal{F}$. In fact, fix $\alpha \in \Gamma$. Let $S_\alpha = \text{Sing}(\mathcal{F}) \cap \alpha$ and $S_i = \text{Sing}(\tilde{\omega}_i|_{\alpha})$ for $i = 1, \ldots, k$. Then $S = \bigcup_{i=0}^k S_i$ is a germ of analytic subset in $(\alpha, 0) \cong (\mathbb{C}^{N-1}, 0)$ of codimension at least two. We assert that $\text{Sing}(\mathcal{F}_|\alpha) \subset S$. Indeed, if $p \in \alpha \setminus S$, then $p \not\in \text{Sing}(\mathcal{F})$ and thus there are 1–forms $\omega_{i1}, \ldots, \omega_{iN-n}$, all of them non singular at $p$, such that

$$T_p\mathcal{F} = \{\omega_{i1}(p) = \cdots = \omega_{iN-n}(p) = 0\}.$$ 

But $H$ is transverse to each $\tilde{\omega}_{i\ell}$ — and also to $\omega_{i\ell}$ — at $p$, giving that $p$ is not a singular point for $\mathcal{F}_|\alpha$. \qed

REFERENCES

[1] M. S. Baouendi, P. Ebenfelt, and L. P. Rothschild. Real submanifolds in complex space and their mappings, volume 47 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1999.

[2] J. Bretas. Folheações holomorfas tangentes a subconjuntos Levi-flat (Portuguese). PhD Thesis, Universidade Federal de Minas Gerais, Brazil, 2016.

[3] M. Brunella. Singular Levi-flat hypersurfaces and codimension one foliations. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 6(4):661–672, 2007.
[4] M. Brunella. Some remarks on meromorphic first integrals. *Enseign. Math. (2)*, 58(3-4):315–324, 2012.

[5] C. Camacho, A. Lins Neto, and P. Sad. Foliations with algebraic limit sets. *Ann. of Math. (2)*, 136(2):429–446, 1992.

[6] E. Cartan. Sur la géométrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes. *Ann. Mat. Pura Appl.*, 11(1):17–90, 1933.

[7] H. Cartan. Variétés analytiques réelles et variétés analytiques complexes. *Bull. Soc. Math. France*, 85:77–99, 1957.

[8] D. Cerveau and A. Lins Neto. Local Levi-flat hypersurfaces invariants by a codimension one holomorphic foliation. *Amer. J. Math.*, 133(3):677–716, 2011.

[9] W. L. Chow. On meromorphic maps of algebraic varieties. *Ann. of Math. (2)*, 89:391–403, 1969.

[10] A. Fernández-Pérez. On Levi-flat hypersurfaces with generic real singular set. *J. Geom. Anal.*, 23(4):2020–2033, 2013.

[11] É. Ghys. À propos d’un théorème de J.-P. Jouanolou concernant les feuilles fermées des feuilletages holomorphes. *Rend. Circ. Mat. Palermo (2)*, 49(1):175–180, 2000.

[12] X. Gómez-Mont. Integrals for holomorphic foliations with singularities having all leaves compact. *Ann. Inst. Fourier (Grenoble)*, 39(2):451–458, 1989.

[13] H. Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79* (1964), 109–203; ibid. (2), 79:205–326, 1964.

[14] J-P. Jouanolou. *Équations de Pfaff algébriques*, volume 708 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[15] J. Lebl. Algebraic Levi-flat hypervarieties in complex projective space. *J. Geom. Anal.*, 22(2):410–432, 2012.

[16] J. Lebl. Singular set of a Levi-flat hypersurface is Levi-flat. *Math. Ann.*, 355(3):1177–1199, 2013.

[17] J. Lebl. Singular Levi-flat hypersurfaces in complex projective space induced by curves in the Grassmannian. *Internat. J. Math.*, 26(5):1550036, 17, 2015.

[18] J. V. Pereira and L. Pirio. An invitation to web geometry, volume 2 of *IMPA Monographs*. Springer, Cham, 2015.

[19] S. Pinchuk, R. Shafikov, and A. Sukhov. Segre envelopes of singular Levi-flat sets. *Pre-print*, arXiv:1606.09294, 2016.

[20] T. Suwa. *Indices of vector fields and residues of singular holomorphic foliations*, Actualités Mathématiques. [Current Mathematical Topics]. Hermann, Paris, 1998.

[21] L. van den Dries. *Tame topology and o-minimal structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

Jane Bretas
Departamento de Física e Matemática
Centro Federal de Educação Tecnológica de Minas Gerais
Av. Amazonas, 7675 – Belo Horizonte, BRAZIL
janebretas@des.cefetmg.br

Arturo Fernández-Pérez
Departamento de Matemática
Universidade Federal de Minas Gerais
Av. Antônio Carlos, 6627  C.P. 702
30123-970 – Belo Horizonte – MG, BRAZIL
arturofp@mat.ufmg.br

Rogério Mol
Departamento de Matemática
Universidade Federal de Minas Gerais
