On Positive Realness for Stochastic Hybrid Singular Systems

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ABSTRACT This paper deals with the problem of passivity analysis and passivity-based synthesis for continuous-time stochastic hybrid singular systems (SHSSs). First, a set of linear matrix inequalities for the stochastic admissibility and passivity, which is also known as the positive real lemma, is obtained for continuous-time SHSSs. The proposed condition successfully holds a necessary and sufficient condition for the positive realness of SHSSs, whereas the existing papers in the literature have been handled only sufficient conditions. Next, the passivity-based control synthesis problem is also considered based on the new positive real lemma. The passivity-based stabilization criterion for the closed-loop system with its mode-dependent state-feedback control is expressed in terms of matrix inequality. Thus, by introducing an additional slack matrix to the non-convex conditions, the feasible conditions for the control gain are obtained in terms of linear matrix inequalities. Finally, a numerical example is provided to show the effectiveness of the result.

INDEX TERMS Stochastic system, hybrid system, positive real lemma, passivity-based control, linear matrix inequalities.

I. INTRODUCTION

The state-space systems which consist of system states and their first order differential equations are widely used in the field of control theory. However, in the real world, there exist the systems that cannot be modeled by only the differential equations of system state. For example, the large-scale systems require the algebraic equations of states to describe the interconnections between their subsystems [1]. Since the singular system can represent both differential and algebraic equations, it has been considered as an approximate model for the large scale system, the rolling ring drive model, mechanical system, planar manipulator and electrical circuit system [2].

Otherwise, physical systems usually contain changing components which occur randomly. Stochastic hybrid systems can be adopted to express the changes. Among them, Markovian jump systems, in which the operation point follows a Markov process, are one of well known examples of the stochastic hybrid systems [3]. For the systems that have both properties of singular systems and stochastic hybrid systems, stochastic hybrid singular systems (SHSSs) are considered as interesting research topics [4]. On its theoretical aspect, the convex conditions for stochastic admissibility and stabilization have been studied since few years ago [5]–[7].

On the other hand, handling external disturbances is an important issue because the disturbance negatively affects system performance or even makes the system be unstable. In particular, focusing on the relation between the system output and external disturbance has been a popular research approach in the literature. For example, the concept of passivity has been widely studied. Passivity is used for analyzing the stability of interconnected dynamic systems, which is also known as a positive real lemma (PRL) [8].

Many papers regarding to passivity have been published for linear systems [9], singular systems [10]–[12], switching systems [13] and stochastic hybrid systems [14], [15]. In the case of SHSSs, reliable passivity-based control problems were considered in [16]. For SHSSs with time-delay, the delay-dependent passivity analysis or stabilization
problems were dealt with in [17]–[19]. However, the above papers were based on the sufficient condition of the passivity for SHSSs. Through the careful examination, the authors catch that there isn’t the necessary and sufficient condition for the PRL of SHSSs. This is the motivation to carry out this study.

This paper proposes a new PRL for the continuous-time SHSSs. To obtain the necessary and sufficient condition for the passivity analysis, a mode-dependent Lyapunov function for stochastic admissibility and passivity is used. By introducing two slack variables which are coupled with the matrix in the null-space of the singular matrix in SHSSs, the proposed linear matrix inequality (LMI) conditions successfully hold the equality condition of the PRL. Based on the PRL, the stabilization condition via state-feedback control is also obtained in terms of LMIs. A numerical example illustrates the effectiveness of the result.

The notations used in this paper are fairly standard.

- For vector $x$ (or matrix $X$), the superscript $T$ denotes its transpose.
- For the symmetric matrices $X$ and $Y$, $X > (\geq) Y$ means that $X - Y > (\geq) Y$ is positive (semi-) definite.
- The notation $\text{sym}\{\cdot\}$ means the sum of itself and its transpose, i.e., $\text{sym}\{Z\} = Z + Z^T$.
- The identity matrix with dimension $r$ is represented by $I_r$. The notation $I$ is also used for the identity matrix with appropriate dimension.
- The notation $(\ast)$ represents an ellipsis of the terms which can be induced by symmetry.
- The notation $\mathcal{E}\{\cdot\}$ denotes the expectation value.

II. PRELIMINARIES

A. PROBLEM FORMULATION

Let us consider the following stochastic hybrid singular systems (SHSSs):

\[
\begin{align*}
\dot{x}(t) &= A(\mu_t)x(t) + B(\mu_t)u(t) + B_w(\mu_t)w(t), \\
y(t) &= C(\mu_t)x(t) + D(\mu_t)u(t) + D_w(\mu_t)w(t),
\end{align*}
\]

where $x(t) \in \mathcal{R}^n$ is the state, $u(t) \in \mathcal{R}^l$ is the control input, $w(t) \in \mathcal{R}^k$ is disturbance and $y(t) \in \mathcal{R}^m$ is the output. In this paper, it is assumed that disturbance belongs to the Hilbert space i.e., $\mathbb{L}_2 = \{w(t) \mid \int_0^\infty \mathbb{E}[|w(t)|^2]dt < \infty\}$. The notation $\{|\mu_t| \mid \text{for all } t > 0\}$ denotes the continuous-time Markov process on the probability space which obtains values in the finite set $\mathcal{N}_+ = \{1, 2, \ldots, N\}$. The probability of transition from mode $i$ to $j$ is defined such that

\[
\Pr(\mu_{t+\delta t} = j | \mu_t = i) = \begin{cases} 
\lambda_{ij}\delta t + o(\delta t), & \text{if } j \neq i, \\
1 + \lambda_{ii}\delta t + o(\delta t), & \text{otherwise},
\end{cases}
\]

where $\delta t > 0$, $\lim_{\delta t \to 0} (o(\delta t)/\delta t) = 0$, and $\lambda_{ij}$ is the transition rate from mode $i$ at time $t$ to mode $j$ at time $t + \delta t$. The mode transition rate matrix $\Lambda = [\lambda_{ij}]_{i,j \in \mathcal{N}_+}$ belongs to

\[
S_\Lambda = \left\{[\lambda_{ij}]_{i,j \in \mathcal{N}_+} \mid \sum_{j=1}^{N} \lambda_{ij} = 0, \lambda_{ij} \geq 0 \text{ for } i \neq j \right\}.
\]

To simplify the notations, the mode-dependent matrices can be rewritten such that

\[
\begin{bmatrix}
A(\mu_t = i) & B(\mu_t = i) & B_w(\mu_t = i) \\
C(\mu_t = i) & D(\mu_t = i) & D_w(\mu_t = i)
\end{bmatrix} \triangleq \begin{bmatrix}
A(i) & B(i) & B_w(i) \\
C(i) & D(i) & D_w(i)
\end{bmatrix}.
\]

Also, the matrix $E \in \mathcal{R}^{n \times n}$ is supposed to be singular, i.e., $\text{rank}(E) = r < n$. To handle the matrix $E$, four matrices $R \in \mathcal{R}^{(n-r) \times n}$, $S \in \mathcal{R}^{n \times (n-r)}$, $E_L \in \mathcal{R}^{n \times r}$, $E_R \in \mathcal{R}^{r \times n}$ are introduced, where $R$ and $S$ are of full rank, $RE = 0, ES = 0$, and $E_L E_R^T = E$.

The main objectives of this paper are obtaining the equivalent condition of the PRL for SHSSs in terms of LMIs, and constructing the mode-dependent passivity-based control based on the proposed PRL. Based on the following stochastic admissibility criterion and passivity criterion, the PRL for SHSSs will be suggested in the next section.

Lemma 1: ([20]) The SSHS (1) with $w(t) = 0$ is stochastically admissible if and only if there exist symmetric matrices $P(i) \in \mathcal{R}^{n \times n}$ and matrices $X(i) \in \mathcal{R}^{(n-r) \times (n-r)}$ such that for all $i \in \mathcal{N}_+$

\[
0 < E_L^T P(i) E_L,
\]

\[
0 > \text{sym}\{A(i)^T (P(i)E + R^T X(i)S^T)\} + \sum_{j=1}^{N} \pi_{ij} E^T P(j) E.
\]

Definition 1: ([21]) The SSHS (1)-(2) is said to be stochastically passive, under zero initial condition, if for any zero initial condition the inequality

\[
\mathcal{E}\left[\int_0^T w^T(t)y(t)dt\right] \geq 0,
\]

for any $T > 0$.

Also, the following lemma will be useful in the proof of Theorem 1.

Lemma 2: For an invertible matrix $X$, the following equality always holds:

\[
X(X + X^T)^{-1}X^T - X^T(X + X^T)^{-1}X = 0.
\]

Proof: Let us start the following term:

\[
X(X + X^T)^{-1}X^T = (X^{-T}X + X^T)^{-1}X^{-1}(X + X^T)X^{-1} - 1
\]

\[
= (X^{-1} + X^{-1})^{-1}
\]

\[
= (X^{-1} + X^{-1})^{-1}
\]

\[
= X^{-1}(X + X^T)X^{-1}
\]

\[
= X^T(X + X^T)^{-1}X.
\]

Therefore, the equation (9) always holds. This completes the proof. □
B. A POSITIVE REAL LEMMA FOR STOCHASTIC HYBRID SYSTEMS

The first aim of this paper is obtaining the PRL for SHSSs. To show the necessary and sufficient condition for SHSSs, it is necessary to use the PRL for stochastic hybrid systems.

Lemma 3: The stochastic hybrid system

\[
\begin{align*}
\dot{x}(t) &= A(i)x(t) + B_w(i)w(t), \\
y(t) &= C(i)x(t) + D_w(i)w(t)
\end{align*}
\]

is stochastically stable and passive if and only if there exist matrices \( P(i) \in \mathbb{R}^{n \times n} \) such that for all \( i \in N_+^* \)

\[
0 < P(i),
\]

\[
0 > \Upsilon_1(i),
\]

\[
\Upsilon_1(i) = \begin{bmatrix} (1, 1) & (1) \\ B_w(i)P(i) - C(i) & -\text{sym}[D_w(i)] \end{bmatrix},
\]

\[
(1, 1) = \text{sym}[A(i)^T P(i)] + \sum_{j=1}^{N} \lambda_{ij}P(j).
\]

Proof: (Sufficient) Let us select mode-dependent Lyapunov function candidate \( V(x(t)) = x^T(t)P(i)x(t) \), where \( V(x(t)) > 0 \) holds by (16). The weak infinitesimal operator acting on \( V(i, t) \) is known as follows:

\[
\nabla V(i, t) = 2x^T(t)P(i)x(t) + B_w(i)w(t)
\]

\[
\quad + \sum_{j=1}^{N} \lambda_{ij}P(j).
\]

Then we can obtain the following relation:

\[
\nabla V(i, t) - y^T(t)w(t) - w^T(t)y(t) = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \Upsilon_1(i) \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}.
\]

From the condition (17),

\[
0 > \nabla V(i, t) - y^T(t)w(t) - w^T(t)y(t).
\]

The above condition concludes to the following two state:

a) At \( w(t) = 0 \), the condition (22) becomes

\[
0 > \text{sym}[A(i)^T P(i)] + \sum_{j=1}^{N} \lambda_{ij}P(j),
\]

which means the stochastic stability of the hybrid system from [3].

b) At \( w(t) \neq 0 \), the condition (23) guarantees the following relation:

\[
0 < V(i, T) - V(i, 0) < 2\mathcal{E} \int_0^T w^T(t)y(t)dt,
\]

which means passive of the hybrid system from Definition 1.

Thus, the proposed conditions (16)-(17) guarantee the stochastic stability and passivity of the hybrid system.

(Necessary) Since the hybrid system is stochastically stable and passive, i.e., \( D_w(i) + D_w^T(i) > 0 \), there exist positive definite solution \( \tilde{P}(i) \) for the following algebraic Riccati equation (ARE):

\[
\begin{align*}
\text{sym}[A(i)^T \tilde{P}(i)] + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j) + (\tilde{P}(i)B_w(i) - C(i)) \\
\quad \times(D_w(i) + D_w^T(i))^{-1}(B_w(i)\tilde{P}(i) - C(i)) &= 0.
\end{align*}
\]

It means that the system matrix

\[
\tilde{A}(i) = A(i) + B_w(i)(D_w(i) + D_w^T(i))^{-1}B_w^T(i)\tilde{P}(i) - C(i)
\]

is stochastically stable, i.e., there exist positive definite matrix \( \tilde{P}(i) \) such that

\[
0 > \text{sym}[A(i)^T \tilde{P}(i)] + \sum_{j=1}^{N} \lambda_{ij}\tilde{P}(j)
\]

\[
+ (\tilde{P}(i)B_w(i) - C(i)(D_w(i) + D_w^T(i))^{-1}B_w^T(i)\tilde{P}(i),
\]

where \( \epsilon \) is arbitrary small positive number. By adding the conditions (26) and (28), we can obtain the following relation:

\[
0 > \text{sym}[A(i)^T P(i)] + \sum_{j=1}^{N} \lambda_{ij}P(j)
\]

where \( P(i) = \tilde{P}(i) + \epsilon\tilde{P}(i) \). Applying Schur complement to (29) concludes to (17). This completes the proof.

Remark 1: The passivity condition for stochastic hybrid systems has been considered in many papers in the literature. However, to the best of the authors’ knowledge, these papers shown only the sufficient condition for the PRL of stochastic hybrid systems. Since the necessary proof is required in the main section, we prove the necessary part.

III. MAIN RESULTS

In the main section, a necessary and sufficient condition for the PRL of SHSSs will be introduced. Next, based on the proposed PRL, state-feedback stabilization criterion will be derived in the next subsection.

A. POSITIVE REAL LEMMA FOR SINGULAR MARKOVIAN JUMP SYSTEMS

Theorem 1: The singular hybrid system (1)-(2) is stochastically admissible and strictly passive if and only if there exist symmetric matrices \( P(i) \in \mathbb{R}^{n \times n} \), non-singular matrices \( X(i) \in \mathbb{R}^{(n-r) \times (n-r)} \), and matrices \( Y(i) \in \mathbb{R}^{(n-r) \times k} \) such that for all \( i \in N_+^* \)

\[
0 < E_i^T P(i)E_i,
\]

\[
0 > \Upsilon_2(i),
\]

\[
\Upsilon_2(i) = \begin{bmatrix} M_1(i) & M_2(i) \\ M_2^T(i) & M_3(i) \end{bmatrix}.
\]
\[ \Omega(i) = P(i)E + R^TX(i)S^T, \]  
\[ M_1(i) = \text{sym} [A^T(i) \Omega(i)] + \sum_{j=1}^{N} \lambda_{ij}E^TP(j)E, \]  
\[ M_2(i) = B^T_{w}(i)\Omega(i) + Y^T(i)RA(i) - C(i), \]  
\[ M_3(i) = \text{sym} [B^T_{w}(i)R^TY(i) - D_w(i)]. \]

Proof: (Sufficient) First, we can construct a Lyapunov function candidate

\[ V(i) = x^T(t)E^TP(i)Ex(t) \geq 0 \]

from the condition (30). Then the weak infinitesimal operator \( \mathcal{V} \) of the Markov process acting on \( V(i) \) can be written as

\[ \mathcal{V}V(i) = 2x^T(t)E^TP(i)x(t) + \sum_{j=1}^{N} \lambda_{ij}E^TP(j)E \]

Since the matrix \( R \) belongs to the null-space of \( E \), we can put the free-variables \( X(i) \) and \( Y(i) \) into (38):

\[ \mathcal{V}V(i) = 2x^T(t)E^TP(i)x(t) + \sum_{j=1}^{N} \lambda_{ij}E^TP(j)E \]

\[ + 2x^T(t)SXT(i) + w^T(t)Y^T(i)REx(t) \]

\[ = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} M_1(i) & M_2(i) \\ M_2^T(i) & M_3(i) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \]  
\[ M_4(i) = B^T_{w}(i)\Omega(i) + Y^T(i)RA(i), \]  
\[ M_5(i) = \text{sym} [B^T(i)R^TY(i)]. \]

Then, to consider the definition of passivity in Definition 1, we can get the following relation:

\[ \mathcal{V}V(i) - w^T(t)y(t) - y^T(t)w(t) \]

\[ = \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T \begin{bmatrix} M_4(i) \\ M_5(i) \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}. \]  

By using the given condition (31), above condition holds the following inequality:

\[ 0 < V_i(T) - V_i(0) < E \left[ \int_0^T w^T(t)y(t)dt \right], \]

which guarantees the strictly passivity of SHSSs from Lemma 3. Also, it is clear that SHSSs (1) with \( w(t) = 0 \) is stochastically admissible from Lemma 1.

(Necessary) Without loss of generality, we can use the following matrices:

\[ E = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad E_L = \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \]

\[ R = G \begin{bmatrix} 0 & I_{n-r} \\ I_{n-r} & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & I_{n-r} \\ I_{n-r} & 0 \end{bmatrix}F, \]

where \( G, F \) are arbitrary invertible matrices. Suppose that SHSSs (1) is stochastically admissible. Every admissible singular systems can be expressed as follows:

\[ A(i) = \begin{bmatrix} A_1(i) & A_2(i) \\ A_3(i) & A_4(i) \end{bmatrix}, \]

when there exists \( A_{41}^{-1}(i) \in \mathcal{R}^{(n-r)\times(n-r)} \) due to impulse-free property and regularity of SHSSs. Then we can construct the equivalent system \( \hat{E}, \hat{A}(i), \hat{B}(i), C(i), D_w(i) \) by using

\[ U(i) = \begin{bmatrix} A_2(i)A_{41}^{-1}(i) \\ I \end{bmatrix}, \]

such that

\[ \hat{E} = U(i)E = E, \]

\[ \hat{A}(i) = U(i)A(i) = \begin{bmatrix} A_3(i) & A_4(i) \\ 0 & A_2(i) \end{bmatrix}, \]

\[ \hat{B}_w(i) = U(i)B_w(i) = \begin{bmatrix} B_1(i) \\ B_2(i) \end{bmatrix}, \]

\[ C(i) = C_1(i) - C_2(i)A_{41}^{-1}(i)A_3(i), \]

\[ \hat{A}(i) = A_1(i) - A_2(i)A_{41}^{-1}(i)A_3(i). \]

The equivalent system \( \hat{E}, \hat{A}(i), \hat{B}(i), C(i), D_w(i) \) can be represented as the following stochastically stable stochastic hybrid system:

\[ \dot{x}_1(t) = \hat{A}(i)x_1(t) + B_1(i)w(t), \]

\[ y(t) = \hat{C}(i)x(t) + \hat{D}(i)w(t), \]

where

\[ \hat{C}(i) = C_1(i) - C_2(i)A_{41}^{-1}(i)A_3(i), \]

\[ \hat{D}(i) = D_w(i) - C_2(i)A_{41}^{-1}(i)B_2(i), \]

and \( x_1(t) \in \mathcal{R}^{r} \) is non-zero part of \( Ex(t) \). From Lemma 3, the stochastic stability and passivity of the hybrid system (53)-(54) means the existence of positive definite matrix \( \hat{P}_i \in \mathcal{R}^{r\times r} \) such that

\[ 0 > \Theta(i), \]

\[ \Theta(i) = \begin{bmatrix} \mathcal{M}_{6}(i) & \mathcal{M}_{7}(i) \\ \mathcal{M}_{7}^{T}(i) & -\hat{D}(i) \end{bmatrix}, \]

\[ \mathcal{M}_{6}(i) = \text{sym} [\hat{A}(i)^T(i)\hat{P}(i)], \quad \mathcal{M}_{7}(i) = \sum_{j=1}^{N} \lambda_{ij}\hat{P}(j). \]

To construct the solutions \( P(i), X(i) \) and \( Y(i) \) for the proposed condition (30)-(31), let us set the following block matrix by using \( \hat{P}_i \) in (57):

\[ P(i) = \begin{bmatrix} \hat{P}(i) & Q(i) \\ Q^T(i) & 0 \end{bmatrix}, \]

where \( Q(i) \in \mathcal{R}^{r\times(n-r)} \) is to be determined. Then the proposed conditions (30)-(31) for the equivalent system \( \hat{E}, \hat{A}(i), \hat{B}(i), C(i), D_w(i) \) can be rewritten as

\[ 0 < \hat{P}(i), \]

\[ \begin{bmatrix} \mathcal{L}_1(i) & \mathcal{L}_2(i) \\ \mathcal{L}_2^T(i) & \mathcal{L}_3(i) \end{bmatrix}, \]

where

\[ \mathcal{L}_1(i) = \text{sym} [\hat{A}(i)^T(i)\hat{P}(i) + A_{41}^T(i)Q(i)] + \sum_{j=1}^{N} \lambda_{ij}\hat{P}(j). \]
\[
\begin{align*}
L_2(i) &= A^T_i(i)Q^T(i) + \bar{X}^T(i)A_3(i), \\
L_3(i) &= B^T_1(i)\bar{Y}(i) + B^T_3(i)Q^T(i) + \bar{Y}^T(i)A_3(i) - C_1(i), \\
L_4(i) &= B^T_2(i)\bar{X}(i) + \bar{Y}^T(i)A_4(i) - C_2(i), \\
L_5(i) &= \text{sym}(B^T_3(i)\bar{Y}(i) - D_3(i)), \\
\bar{X}(i) &= G^T X(i) F^T, \quad \bar{Y}(i) = G^T Y(i).
\end{align*}
\]

First, let us put \( \bar{X}(i) \) which holds \( 0 > \text{sym}(A^T_3(i)\bar{X}(i)) \triangleq -Z(i) \). Then we can apply Schur complement to (62):
\[
0 > \begin{bmatrix} L_1(i) & (\ast) \\ L_3(i) & L_5(i) \end{bmatrix} + \begin{bmatrix} L^T_1(i) \\ L_4(i) \end{bmatrix} Z^{-1}(i) \begin{bmatrix} L^T_2(i) \\ L_4(i) \end{bmatrix}^T = \Theta(i) + \begin{bmatrix} T_1(i) & (\ast) \\ T_2(i) & T_3(i) \end{bmatrix},
\]

\[
T_1(i) = \text{sym}(A^T_3(i)Q^T(i) + L^T_2(i)Z^{-1}(i)L_2(i)), \\
T_2(i) = B^T_3(i)Q^T(i) + \bar{Y}^T(i)A_3(i) - C_2(i)A_4^{-1}(i)A_3(i) + L_4(i)Z^{-1}(i)L_2(i), \\
T_3(i) = \text{sym}(B^T_3(i)\bar{Y}(i) - C_2(i)A_4^{-1}(i)B_2(i)) + L_4(i)Z^{-1}(i)L_2(i).
\]

If there exist the solutions \( \bar{Y}(i), Q(i) \) which hold
\[
0 > \begin{bmatrix} T_1(i) \\ T_2(i) \\ T_3(i) \end{bmatrix},
\]

then the existence of the solutions of proposed conditions (30)-(31) can be guaranteed by positive realness of SHSSs. Note that
\[
A_4^{-1}(i) + Z^{-1}(i)\bar{X}^T(i) = -Z^{-1}(i)A_3^{-1}(i)\bar{X}(i)A_4^{-1}(i).
\]

Let us rewritten the blocks \( T_1(i), T_2(i), T_3(i) \) in (68) as follows:
\[
T_1(i) = \text{sym}(Q(i)A_4(i)(A_3^{-1}(i) + Z^{-1}(i)\bar{X}^T(i))) \\
+ A_4^{-1}(i)\bar{X}(i)A_3^{-1}(i) + Q(i)Z_4(i)A_4^{-1}(i)Q^T(i) \\
= W_1(i)Z^{-1}(i)W_2^T(i) + A_4^{-1}(i)A_4^{-1}(i)K(i)A^{-1}_4(i)A_3(i),
\]

\[
T_2(i) = B^T_3(i)(A_4^{-1}(i) + \bar{X}(i)Z^{-1}(i)A_3(i)) \\
+ (Y^T(i)A_4(i) - C_2(i)A_4^{-1}(i)A_3(i)) \\
+ B^T_3(i)\bar{X}(i)Z^{-1}(i)\bar{X}^T(i) + (Y^T(i)A_4(i) - C_2(i)Z^{-1}(i)A_3(i)) \\
= W_1(i)Z^{-1}(i)W_2^T(i) + A_4^{-1}(i)A_4^{-1}(i)K(i)A^{-1}_4(i)A_3(i),
\]

\[
T_3(i) = \text{sym}(B^T_3(i)(A_4^{-1}(i) + \bar{X}(i)Z^{-1}(i)) \\
\times (A_4^{-1}(i)\bar{X}(i) - C_2(i)^2) \\
+ B^T_3(i)\bar{X}(i)Z^{-1}(i)\bar{X}^T(i)B_2(i) \\
+ (Y^T(i)A_4(i) - C_2(i)Z^{-1}(i)) \\
= W_2(i)Z^{-1}(i)W_2^T(i) + B^T_3(i)A_4^{-1}(i)K(i)A_4^{-1}(i)B_2(i).
\]

Note that \( K(i) = 0 \) from Lemma 2. Therefore, we can obtain the following relation:
\[
\begin{bmatrix} T_1(i) \\ T_2(i) \\ T_3(i) \end{bmatrix} = \begin{bmatrix} W_1(i) \\ W_2(i) \end{bmatrix} Z^{-1}(i) \begin{bmatrix} W_1(i) \\ W_2(i) \end{bmatrix}^T.
\]

Let us set the variables \( Q(i) \) and \( \bar{Y}(i) \) such that
\[
Q(i) = A_4^{-1}(i)A_4^{-T}(i)\bar{X}^T(i),
\]

\[
\bar{Y}(i) = A_4^{-T}(i)C_2(i) + \bar{X}(i)A_4^{-1}(i)B_2(i),
\]

which hold \( W_1(i) = 0 \) and \( W_2(i) \). Therefore, it is clear that there exist the feasible solutions \( P(i), X(i) \) and \( Y(i) \) for the LMIs (30)-(31) if SHSSs (1)-(2) is positive real. This completes the proof, which holds by the given condition that \( \Theta(i) < 0 \). This complete the proof.

Remark 2: To consider the passivity problem in terms of LMIs, the researches in the literature have been dealt with the sufficient condition of the PRL for SHSSs [16]–[19]. To obtain necessary and sufficient condition in terms of LMIs, it is the key idea to introduce two slack variables \( X(i) \) and \( Y(i) \) as [39], whereas the previous studies have been considered only one slack variable.

- The weak infinitesimal operator on \( V(i) \) is expressed as the function of \( x(t) \) and \( w(t) \). Thus, at most two slack variables can be applied.
- The variable \( X(i) \) acts for the stochastic admissibility criterion of SHSSs. This variable also can be seen in the sufficient conditions of the PRL for SHSSs in the literature.
- The other variable \( Y(i) \) is newly invented. Without \( Y(i) \), we cannot hold the condition \( W_2(i) = 0 \) because the variable \( \bar{X}(i) \) in \( W_2(i) \) already plays a role in holding \( \text{sym}(A^T_3(i)\bar{X}(i)) < 0 \). The process to find the solutions for \( W_1(i) = 0 \) and \( W_2(i) = 0 \) is mandatory for the necessary proof. Therefore, introducing the new variable \( Y(i) \) is essential part to show the necessary and sufficient condition for the PRL of SHSSs.

### B. PASSIVITY-BASED CONTROL SYNTHESIS

In this section, the passivity-based state-feedback control for SHSSs (1)-(2) is considered. For the following mode-dependent state-feedback control:
\[
u(t) = K(i)x(t),
\]

the closed-loop system is obtained as follows:
\[
\left\{
\begin{aligned}
E\dot{x}(t) &= A(i)x(t) + B(i)w(t), \\
y(t) &= C(i)x(t) + D(i)w(t).
\end{aligned}
\right.
\]

where \( \hat{A}(i) = A(i) + B(i)K(i) \) and \( \hat{C}(i) = C(i) + D(i)K(i) \).
Theorem 2: The closed-loop system (83)-(84) is stochastically admissible and passive if there exist symmetric matrices $\tilde{P}(i) \in \mathbb{R}^{n \times n}$, $\tilde{M}(i) \in \mathbb{R}^{(n-r) \times (n-r)}$, non-singular matrices $\tilde{X}(i) \in \mathbb{R}^{(n-r) \times (n-r)}$ and matrices $\tilde{Y}(i) \in \mathbb{R}^{(n-r) \times k}$, $K_1(i) \in \mathbb{R}^{l_x \times n}$, $K_2(i) \in \mathbb{R}^{l_y \times (n-r)}$ such that for all $i \in N_+$

$$0 < \tilde{P}(i)E^T \tilde{P}(i), \quad 0 < M(i), \quad 0 > \begin{bmatrix} N_1(i) & \ast & \ast & \ast \\ N_2(i) & N_3(i) & \ast & \ast \\ N_4(i) & N_5(i) & -M(i) & \ast \\ \tilde{X}(i) & 0 & 0 & \tilde{D}(i) \end{bmatrix},$$

where

$$N_1(i) = \text{sym}[A(i)\tilde{Q}_i + B(i)K_1(i)E^T + B(i)K_2(i)R] + \lambda_4E\tilde{P}(i)E^T,$$

$$N_2(i) = B^T_w(i) - C(i)\tilde{Q}_i - D(i)K_1(i)E^T - D(i)K_2(i)R,$$

$$N_3(i) = -\text{sym}[D_w(i)],$$

$$N_4(i) = -\tilde{R}^T \Phi(i)\tilde{X}(i)^T A(i)^T - \tilde{R}^T K_2(i)B(i)^T,$$

$$N_5(i) = \tilde{R}^T \tilde{X}(i)^T S(i)^T C(i)^T + \tilde{R}^T K_2(i)d(i)^T - Y(i),$$

$$\tilde{Q}_i = \tilde{P}(i)E + S\tilde{X}(i)R,$$

$$\tilde{D}(i) = -\text{diag}[E_{ik}^T \tilde{P}(i)]_{i \in N_+},$$

and apply the congruence transformation to (98):

$$0 > \Phi(i),$$

$$\Phi(i) = \text{sym} \begin{bmatrix} \tilde{A}(i)\tilde{Q}(i)U_1(i) \\ 0 \\ 0 \\ -\tilde{B}(i)U_2(i) \end{bmatrix},$$

$$U_1(i) = -A(i)\tilde{Q}(i)R^TY(i) + B_w(i),$$

$$U_2(i) = \sum_{j=1}^N \lambda_4E\tilde{P}(i)E^TP(j)E\tilde{P}(i)E^T,$$

The above condition is a non-convex condition because the variables $\tilde{P}(i)$, $\tilde{Q}_i$ and $Y(i)$ are coupled in the terms:

$$\tilde{A}(i)\tilde{Q}(i)R^TY(i) = A(i) + B(i)K(i) + \tilde{C}(i)\tilde{Q}(i)R^TY(i).$$

To obtain the feasible form of the condition (106), the following positive definite term is constructed by using a slack matrix $M(i) > 0$:

$$0 < S(i), \quad S(i) = \begin{bmatrix} \tilde{A}(i)\tilde{Q}(i)R^TY(i) \\ \tilde{C}(i)\tilde{Q}(i)R^TY(i) \end{bmatrix}M^{-1}(i) + \begin{bmatrix} 0 \\ 0 \end{bmatrix} Y(i) \quad Y(i)^T.$$

Then it is clear that the condition (106) holds by the following inequality:

$$0 > \Phi(i) + S(i),$$

$$\Phi(i) + S(i) = \begin{bmatrix} \text{sym}[\tilde{A}(i)\tilde{Q}(i)] + U_2(i) \\ B^T_w(i) - \tilde{C}(i)\tilde{Q}(i) \end{bmatrix} - \text{sym}[D_w(i)]$$

$$+ \begin{bmatrix} 0 \\ 0 \end{bmatrix} Y(i) \quad M(i),$$

$$\times M^{-1}(i) \begin{bmatrix} \tilde{A}(i)\tilde{Q}(i)R^TY(i) \\ \tilde{C}(i)\tilde{Q}(i)R^TY(i) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} Y(i) \quad M(i) \quad Y(i)^T.$$

Let us put $\tilde{Y}(i) = Y(i)M(i), \quad K_1(i) = K(i)\tilde{P}(i)$ and $K_2(i) = K(i)\tilde{X}(i)$. Then, the inequality (116) in turn becomes the condition (87) by applying the Schur complement with respect to the terms $M(i)$ and $\sum_{j=1}^N \lambda_4E\tilde{P}(i)E^TP(j)E\tilde{P}(i)E^T$. This completes the proof.

IV. NUMERICAL EXAMPLE

Consider the stochastic hybrid singular system (1)-(2) with two operating modes such that:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} -0.3 & 0.5 & 1.0 \\ 0.5 & -0.4 & -1.0 \\ -0.1 & 0.3 & -0.7 \end{bmatrix}.$$
The transition rate matrix for the two operation modes is shown in FIGURE 2. For the simulation, we set an initial state. The state trajectories of the closed-loop dynamic response are given by:

\[ x(0) = [3 - 1.5 - 2.5] \] and external disturbance such that

\[ w(t) = \begin{cases} 
1 + 0.2\sin(10t^2), & \text{if } 0.5 < t < 1 \\
-0.5 - 0.1\sin(10t^2), & \text{if } 3 < t < 3.5 \\
0, & \text{otherwise}
\end{cases} \] (127)

The mode generation and external disturbance are described in the FIGURE 2. It can be seen that Theorem 2 successfully stabilizes the system because all states in FIGURE 2 converges to zero.

V. CONCLUSION
This paper proposed the necessary and sufficient condition for the stochastic admissibility and passivity of SHSSs. First, the sufficient condition was derived by using a mode-dependent Lyapunov function. Next, to show the necessary proof, the authors introduced two slack variables by using zero constraint. Based on the proposed positive real lemma, stabilization criterion via mode-dependent state-feedback control was obtained in terms of LMIs. The result was verified by a numerical example.

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