GEOMETRIC CRYSTALS ON UNIPOTENT GROUPS AND GENERALIZED YOUNG TABLEAUX

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Abstract
We define geometric/unipotent crystal structure on unipotent subgroups of semi-simple algebraic groups. We shall show that in $A_n$-case, their ultra-discretizations coincide with crystals obtained by generalizing Young tableaux.

Key words: Geometric crystal, Unipotent groups, Generalized Young tableaux

1 Introduction

The notion of crystals is initiated by Kashiwara ([3],[4],[5]), which influences over many areas in mathematics, in particular, combinatorics and representation theory, e.g., combinatorics of Young tableaux (= semi-standard tableaux), piece-wise linear combinatorics, etc. Indeed, in [7], we succeed in describing the crystal bases for classical quantum algebras by using Young tableaux. One feature of crystal theory is that it produces many piece-wise linear formulae ([5],[11],[12],[13]).

Theory of geometric crystals is introduced by Berenstein and Kazhdan [1] in semi-simple setting and is extended to Kac-Moody setting in [10], which is a kind of geometric analogue of Kashiwara’s crystal theory. More precisely, let $G$ be a Kac-Moody group over $\mathbb{C}$, $T$ be its maximal torus and $I$ be a finite index set of its simple roots. For an ind-(algebraic)variety $X$, morphisms $e_i: \mathbb{C}^\times \times X \to X$ ($i \in I$) and $\gamma: X \to T$, the triplet $(X, \gamma, \{e_i\}_{i \in I})$ is called a geometric crystal if they satisfy the conditions as in Definition 2.2. Geometric crystals are not only analogy of crystals, but also has certain categorical correspondence to crystals, which is called a tropicalization/ultra-discretization. It is so remarkable that this correspondence reproduces several piece-wise linear formulae in the theory of crystals from subtraction free(=positive) rational formulae in geometric crystals ([10]) as follows:

\[
\begin{array}{cccc}
\text{(Geometric Crystals)} & \overset{\text{ultra-discretization}}{\longrightarrow} & \text{(Crystals)} \\
\times \times y, \ x/y, \ x + y & \overset{\text{tropicalization}}{\longrightarrow} & x + y, \ x - y, \ \max(x, y)
\end{array}
\]

Furthermore, this correspondence reproduces the tensor product structure of crystals from the product structure of geometric crystals ([11]).

Let $B$ be a Borel subgroup of $G$ and $W$ be the Weyl group associated with $G$. Any finite Schubert variety $X_w \subset X := G/B$ has a natural geometric crystal structure([11],[10]). Then, in semi-simple setting we know that the whole flag variety $X := G/B$ holds a geometric crystal structure. But, in general Kac-Moody setting, we do not have any natural geometric crystal structure on the flag variety $X$. The opposite unipotent subgroup $U^-$ can be seen as

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an open dense subset of $X$. In this paper, we present some sufficient condition for existence of geometric(unipotent) crystal structure on $U^-$ and then on $X$, which is described as follows: if there exists a morphism $T : U^- \to T$ satisfying the condition as in Lemma then we obtain $U$-morphism $F : U^- \to B^-$ and then the associated unipotent crystal structure, which means the existence of a geometric crystal structure on $U^-$. In semi-simple cases, there exists such morphism which is given by matrix coefficients. In particular, for $G = SL_{n+1}(\mathbb{C})$ case, we present its geometric crystal structure explicitly and reveal that it corresponds to the crystals called generalized Young tableaux, which is a sort of “limit” of usual Young tableaux and forms a free $\mathbb{Z}$-lattice of rank $\frac{n(n+1)}{2}$. In more general cases, e.g., affine cases, the existence of such morphisms is not yet known, which is our further problem.

The article is organized as follows: in Sect.2, we review the notion of geometric crystals, unipotent crystals and the tropicalization/ultra-discretization correspondence. In Sect.3, we consider geometric crystal on a unipotent subgroup $U^- \subset G$ and in Sect.4, the explicit geometric crystal structure on $U^- \subset SL_{n+1}(\mathbb{C})$ is described. In the final section, we give a tropicalization/ultra-discretization correspondence between geometric crystals on $U^-$ and generalized Young tableaux.

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2 Geometric Crystals and Unipotent Crystals

2.1 Kac-Moody algebras and Kac-Moody groups

Fix a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$, where $I$ be a finite index set. Let $(t, \{a_i\}_{i \in I}, \{h_i\}_{i \in I})$ be the associated root data, where $t$ be the vector space over $\mathbb{C}$ with dimension $|I| + \text{corank}(A)$, and the set of simple roots $\{a_i\}_{i \in I} \subset t^*$ and the set of simple co-roots $\{h_i\}_{i \in I} \subset t$ are linearly independent indexed sets satisfying $a_i(h_j) = a_{ij}$.

The Kac-Moody Lie algebra $g = g(A)$ associated with $A$ is the Lie algebra over $\mathbb{C}$ generated by $t$, the Chevalley generators $e_i$ and $f_i$ ($i \in I$) with the usual defining relations $(\mathbb{R}, \mathbb{R})$. There is the root space decomposition $g = \bigoplus_{\alpha \in \Delta} g_\alpha$. Denote the set of roots by $\Delta := \{\alpha \in t^* | \alpha \neq 0, g_\alpha \neq (0)\}$. Set $Q := \sum_i \mathbb{Z}a_i$, $Q^+ := \sum_i \mathbb{Z}_{\geq 0}a_i$ and $\Delta_+ := \Delta \cap Q^+$. An element of $\Delta_+$ is called a positive root. Let $\omega$ be the Chevalley involution of $g$ defined by $\omega(e_i) = -f_i, \omega(f_i) = -e_i$ and $\omega(h) = -h$ for $h \in t$. Let $L(\Lambda) (\Lambda \in P_+: \text{set of dominant weights})$ be an irreducible integrable highest weight module with highest weight $\Lambda$ and $\pi_\Lambda : g \to \text{End}(L(\Lambda))$ be the $g$-action. The action $\pi_\Lambda^* : = \pi_\Lambda \circ \omega$ defines a $g$-module structure on $L(\Lambda)$, which is called the contragredient module of $L(\Lambda)$ and denoted $L^*(\Lambda)$. Let us fix a highest weight vector $u_\Lambda \in L(\Lambda)$ and denote it by $u_\Lambda^*$ in $L^*(\Lambda)$. We obtain a unique $g$-invariant bilinear form $\langle \ , \ \rangle$ on $L(\Lambda) \times L^*(\Lambda)$ such that $\langle u_\Lambda, u_\Lambda^* \rangle = 1$.

Define simple reflections $s_i \in \text{Aut}(t)$ ($i \in I$) by $s_i(h) := h - a_i(h)h_i$, which generate the Weyl group $W$. We also define the action of $W$ on $t$ by $s_i(\lambda) := \lambda - a_i(h_i)\alpha_i$. Set $\Delta^\text{re} := \{|w(\alpha)| \in W, i \in I\}$, whose element is called a real root.

Let $G$ be the Kac-Moody group associated with the derived Lie algebra $g'$ defined in $[\mathbb{R}]$. Set $U_\alpha := \exp g_\alpha (\alpha \in \Delta^\text{re})$, which is an one-parameter subgroup of $G$ and $G$ is generated by $U_\alpha (\alpha \in \Delta^\text{re})$. Let $U_{\pm \alpha}$ be the subgroups generated by $U_{\pm \alpha} (\alpha \in \Delta^\text{re} = \Delta^\text{re} \cap Q^+)$, i.e., $U_{\pm} := \langle U_{\pm \alpha} (\alpha \in \Delta^\text{re}) \rangle$, which is called the unipotent subgroup of $G$. Here note that if $g$ is a semi-simple Lie algebra, then $G$ is a usual semi-simple algebraic group over $\mathbb{C}$.

For any $i \in I$, there exists a unique homomorphism; $\phi_i : SL_2(\mathbb{C}) \to G$ such that

$$x_i(t) := \phi_i \left( \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp t e_i, \ y_i(t) := \phi_i \left( \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp t f_i \ (t \in \mathbb{C}).$$

Set $G_i := \phi_i(SL_2(\mathbb{C})), T_i := \phi_i(\{\text{diag}(t, t^{-1}) | t \in \mathbb{C}\})$ and $N_i := N_{G_i}(T_i)$. Let $T$ (resp. $N$) be the subgroup of $G$ generated by $T_i$ (resp. $N_i$), which is called a maximal torus in $G$. 

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and $B^\pm := U^\pm T$ be the Borel subgroup of $G$. We have the isomorphism $\phi : W \to N/T$ defined by $\phi(s_i) = N_iT/T$. An element $s_i := x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$. Define $R(w)$ for $w \in W$ by

$$R(w) := \{(i_1, i_2, \ldots, i_l) \in I| w = s_{i_1}s_{i_2}\cdots s_{i_l}\},$$

where $l$ is the length of $w$. We associate to each $w \in W$ its standard representative $\bar{w} \in N_G(T)$ by $\bar{w} = s_{i_1}s_{i_2}\cdots s_{i_l}$ for any $(i_1, i_2, \ldots, i_l) \in R(w)$.

We have the following (as for ind-variety and ind-group, see [3]):

**Proposition 2.1** ([3]). (i) Let $G$ be a Kac-Moody group and $U^\pm$, $B^\pm$ be its subgroups as above. Then $G$ is an ind-group and $U^\pm$, $B^\pm$ are its closed ind-subgroups.

(ii) The multiplication maps

$$T \times U \to B \quad U^- \times T \to B^-$$

are isomorphisms of ind-varieties.

### 2.2 Geometric Crystals

In this subsection, we review the notion of geometric crystals ([1], [10]).

Let $(a_{ij})_{i, j \in I}$ be a symmetrizable generalized Cartan matrix and $G$ be the associated Kac-Moody group with the maximal torus $T$. An element in $\text{Hom}(T, C^\times)$ (resp. $\text{Hom}(C^\times, T)$) is called a character (resp. co-character) of $T$. We define a simple co-root $\alpha_i^\vee \in \text{Hom}(C^\times, T)$ ($i \in I$) by $\alpha_i^\vee(t) := T_i$. We have a pairing $(\alpha_i^\vee, \alpha_j) = a_{ij}$.

Let $X$ be an ind-variety over $C$, $\gamma : X \to T$ be a rational morphism and a family of rational morphisms $e_i : C^\times \times X \to X$ ($i \in I$);

$$e_i^\gamma : C^\times \times X \to X$$

$$(c, x) \mapsto e_i^\gamma(x).$$

For a word $i = (i_1, i_2, \ldots, i_l) \in R(w)$ ($w \in W$), set $\alpha^{(l)} := \alpha_{i_l}, \alpha^{(l-1)} := s_{i_l}(\alpha_{i_{l-1}}), \ldots, \alpha^{(1)} := s_{i_1}\cdots s_{i_l}(\alpha_{i_1})$. Now for a word $i = (i_1, i_2, \ldots, i_l) \in R(w)$ we define a rational morphism $\epsilon_i : T \times X \to X$ by

$$(t, x) \mapsto e_i^\gamma(x) := e_{i_1}^{\alpha^{(1)}(t)}e_{i_2}^{\alpha^{(2)}(t)}\cdots e_{i_l}^{\alpha^{(l)}(t)}(x).$$

**Definition 2.2.** (i) The triplet $\chi = (X, \gamma, \{e_i\}_{i \in I})$ is a geometric crystal if it satisfies

$$e_1^\gamma(x) = x$$

and

$$\gamma(e_i^\gamma(x)) = \alpha_i^\gamma(c)\gamma(x),$$

for any $x \in W$, and any $i, i' \in R(w).$ (2.1)

(ii) Let $(X, \gamma_X, \{e_i^X\}_{i \in I})$ and $(Y, \gamma_Y, \{e_i^Y\}_{i \in I})$ be geometric crystals. A rational morphism $f : X \to Y$ is a morphism of geometric crystals if $f$ satisfies that

$$f \circ e_i^X = e_i^Y \circ f, \quad \gamma_X = \gamma_Y \circ f.$$

In particular, if a morphism $f$ is a birational isomorphism of ind-varieties, it is called an isomorphism of geometric crystals.

The following lemma is a direct result from [1] [Lemma 2.1] and the fact that the Weyl group of any Kac-Moody Lie algebra is a Coxeter group [2] [Proposition 3.13].
Lemma 2.3. The relations (2.2) are equivalent to the following relations:

\[
\begin{align*}
\epsilon_i^1 e_j^1 &= \epsilon_j^1 e_i^1 & \text{if } \langle \alpha_j^\vee, \alpha_i \rangle = 0, \\
\epsilon_i^1 e_j^2 e_i^1 &= \epsilon_j^2 e_i^1 e_j^1 & \text{if } \langle \alpha_j^\vee, \alpha_i \rangle = \langle \alpha_j, \alpha_i \rangle = -1, \\
\epsilon_i^1 e_j^1 e_i^1 &= e_j^1 e_i^1 e_j^1 & \text{if } \langle \alpha_j, \alpha_i \rangle = -2, \langle \alpha_j^\vee, \alpha_i \rangle = -1, \\
\epsilon_i^1 e_j^1 e_i^1 e_j^1 &= e_j^1 e_i^1 e_j^1 e_i^1 & \text{if } \langle \alpha_j^\vee, \alpha_j \rangle = -3, \langle \alpha_j^\vee, \alpha_i \rangle = -1,
\end{align*}
\]

Remark. If \( \langle \alpha_j^\vee, \alpha_j \rangle, \langle \alpha_j^\vee, \alpha_i \rangle \geq 4 \), there is no relation between \( e_i \) and \( e_j \).

2.3 Unipotent Crystals

In the sequel, we denote the unipotent subgroup \( U^+ \) by \( U \). We define unipotent crystals (see [1]) associated to Kac-Moody groups. The definitions below follow [1], [10].

Definition 2.4. Let \( X \) be an ind-variety over \( \mathbb{C} \) and \( \alpha : U \times X \rightarrow X \) be a rational \( U \)-action such that \( \alpha \) is defined on \( \{ e \} \times X \). Then, the pair \( X = (X, \alpha) \) is called a \( U \)-variety. For \( U \)-varieties \( X = (X, \alpha_X) \) and \( Y = (Y, \alpha_Y) \), a rational morphism \( f : X \rightarrow Y \) is called a \( U \)-morphism if it commutes with the action of \( U \).

Now, we define the \( U \)-variety structure on \( B^- = U^{-1}T \). By Proposition 2.1, \( B^- \) is an ind-subgroup of \( G \) and then is an ind-variety over \( \mathbb{C} \). The multiplication map in \( G \) induces the open embedding: \( B^- \times U \hookrightarrow G \); this is a birational isomorphism. Let us denote the inverse birational isomorphism by \( g \):

\[
g : G \rightarrow B^- \times U.
\]

Then we define the rational morphisms \( \pi^- : G \rightarrow B^- \) and \( \pi : G \rightarrow U \) by \( \pi^- : = \text{proj}_{B^-} \circ g \) and \( \pi : = \text{proj}_U \circ g \). Now we define the rational \( U \)-action \( \alpha_{B^-} \) on \( B^- \) by

\[
\alpha_{B^-} := \pi^- \circ m : U \times B^- \rightarrow B^-,
\]

where \( m \) is the multiplication map in \( G \). Then we obtain \( U \)-variety \( B^- = (B^-; \alpha_{B^-}) \).

Definition 2.5. (i) Let \( X = (X, \alpha) \) be a \( U \)-variety and \( f : X \rightarrow B^- \) be a \( U \)-morphism. The pair \( (X, f) \) is called a unipotent \( G \)-crystal or, for short, unipotent crystal.

(ii) Let \( (X, f_X) \) and \( (Y, f_Y) \) be unipotent crystals. A \( U \)-morphism \( g : X \rightarrow Y \) is called a morphism of unipotent crystals if \( f_X = f_Y \circ g \). In particular, if \( g \) is a birational isomorphism of ind-varieties, it is called an isomorphism of unipotent crystals.

We define a product of unipotent crystals following [1]. For unipotent crystals \( (X, f_X) \), \( (Y, f_Y) \), define a morphism \( \alpha_{X \times Y} : U \times X \times Y \rightarrow X \times Y \) by

\[
\alpha_{X \times Y}(u, x, y) := (\alpha_X(u, x), \alpha_Y(\pi(u \cdot f_X(x)), y)).
\]

(2.3)

If there is no confusion, we use abbreviated notation \( u(x, y) \) for \( \alpha_{X \times Y}(u, x, y) \).

Theorem 2.6 ([1]). (i) The morphism \( \alpha_{X \times Y} \) defined above is a rational \( U \)-morphism on \( X \times Y \).

(ii) Let \( m : B^- \times B^- \rightarrow B^- \) be a multiplication morphism and \( f = f_{X \times Y} : X \times Y \rightarrow B^- \) be the rational morphism defined by

\[
f_{X \times Y} := m \circ (f_X \times f_Y).
\]

Then \( f_{X \times Y} \) is a \( U \)-morphism and then, \( (X \times Y, f_{X \times Y}) \) is a unipotent crystal, which we call a product of unipotent crystals \( (X, f_X) \) and \( (Y, f_Y) \).

(iii) Product of unipotent crystals is associative.
2.4 From unipotent crystals to geometric crystals

We have the canonical projection \( \xi_i : U^- \to U_{-\alpha_i} \) \( (i \in I) \) (see [11]). Now, we define the function on \( U^- \) by

\[
\chi_i := y_i^{-1} \circ \xi_i : U^- \to U_{-\alpha_i} \to \mathbb{C},
\]

and extend this to the function on \( B^- \) by \( \chi_i(u \cdot t) := \chi_i(u) \) for \( u \in U^- \) and \( t \in T \). For a unipotent \( G \)-crystal \((X, f_X)\), we define a function \( \varphi_i := \varphi_{iX} : X \to \mathbb{C} \) by

\[
\varphi_i := \chi_i \circ f_X,
\]

and a rational morphism \( \gamma_X : X \to T \) by

\[
\gamma_X := \text{proj}_T \circ f_X : X \to B^- \to T,
\]

where \( \text{proj}_T \) is the canonical projection. Suppose that the function \( \varphi_i \) is not identically zero on \( X \). We define a rational morphism \( e_i : \mathbb{C}^x \times X \to X \)

\[
e_i(x) := x_i \left( \frac{e-1}{\varphi_i(x)} \right)(x).
\]

Theorem 2.7 ([1]). For a unipotent \( G \)-crystal \((X, f_X)\), suppose that the function \( \varphi_i \) is not identically zero for any \( i \in I \). Then the rational morphisms \( \gamma_X : X \to T \) and \( e_i : \mathbb{C}^x \times X \to X \) as above define a geometric \( G \)-crystal \((X, \gamma_X, \{e_i\}_{i \in I})\), which is called the induced geometric \( G \)-crystals by unipotent \( G \)-crystal \((X, f_X)\).

Due to the product structure of unipotent crystals, we can deduce a product structure of geometric crystals derived from unipotent crystals, which is a counterpart of tensor product structure of Kashiwara’s crystals. We omit the explicit statement here (see [1],[10]).

2.5 Crystals

The notion “crystal” is introduced as a combinatorial object by abstracting the properties of “crystal bases”, which has, in general, no corresponding \( U_q(\mathfrak{g}) \)-module.

Definition 2.8. A crystal \( B \) is a set endowed with the following maps:

\[
\begin{align*}
wt : B & \to P, \\
\varepsilon_i : B & \to \mathbb{Z} \sqcup \{ -\infty \}, \\
\varphi_i : B & \to \mathbb{Z} \sqcup \{ -\infty \} \quad \text{for} \quad i \in I, \\
\hat{e}_i : B \sqcup \{ 0 \} & \to B \sqcup \{ 0 \}, \\
\hat{f}_i : B \sqcup \{ 0 \} & \to B \sqcup \{ 0 \} \quad \text{for} \quad i \in I, \\
\hat{e}_i(0) & = \hat{f}_i(0) = 0,
\end{align*}
\]

those maps satisfy the following axioms: for all \( b, b_1, b_2 \in B \), we have

\[
\begin{align*}
\varphi_i(b) & = \varepsilon_i(b) + \langle h_i, wt(b) \rangle, \\
wt(\hat{e}_i b) & = wt(b) + \alpha_i \quad \text{if} \quad \hat{e}_i b \in B, \\
wt(\hat{f}_i b) & = wt(b) - \alpha_i \quad \text{if} \quad \hat{f}_i b \in B, \\
\hat{e}_i b_2 = b_1 & \iff \hat{f}_i b_1 = b_2 \quad (b_1, b_2 \in B), \\
\varepsilon_i(b) & = -\infty \implies \hat{e}_i b = \hat{f}_i b = 0.
\end{align*}
\]

The operators \( \hat{e}_i \) and \( \hat{f}_i \) are called the Kashiwara operators. Indeed, if \((L, B)\) is a crystal base, then \( B \) is a crystal.

Remark. A pre-crystal is an object satisfying the conditions (2.6)–(2.8).
Let us define \( \hat{s}_i : B \to B \) \((i \in I)\) by

\[
\hat{s}_i(b) = \begin{cases} 
    e^{-\langle wt(b), h_i \rangle}(b) & \text{if } \langle wt(b), h_i \rangle < 0, \\
    e^{\langle wt(b), h_i \rangle}(b) & \text{if } \langle wt(b), h_i \rangle \geq 0.
\end{cases}
\]

Here note that we have \( \hat{s}_i^2 = id_B \).

**Definition 2.9.** Let \( B \) be a crystal.

(i) If the actions by \( \{s_i\}_{i \in I} \) define the action of the Weyl group \( W \) on \( B \), we call \( B \) a \textit{W-crystal}.

(ii) If \( \hat{e}_i \) or \( \hat{f}_i \) is bijective, then we call \( B \) a \textit{free crystal}.

Note that if \( B \) is a free crystal, then \( \hat{f}_i = \hat{e}_i^{-1} \). We frequently denote a free crystal \( B \) by \((B, \text{wt}, \{\hat{e}_i\}_{i \in I})\).

### 2.6 Positive structure and Ultra-discretizations/Tropicalizations

Let us recall the notions of positive structure and ultra-discretization/tropicalization.

The setting below is simpler than the ones in \([11,10]\), since it is sufficient for our purpose. Let \( T = (\mathbb{C}^\times)^l \) be an algebraic torus over \( \mathbb{C} \) and \( X^*(T) \cong \mathbb{Z}^l \) (resp. \( X_*(T) \cong \mathbb{Z}^l \)) be the lattice of characters (resp. co-characters) of \( T \). Set \( R := \mathbb{C}(c) \) and define

\[
v : R \setminus \{0\} \to \mathbb{Z} \quad f(c) \mapsto \deg(f(c)).
\]

Here note that for \( f_1, f_2 \in R \setminus \{0\} \), we have

\[
v(f_1 f_2) = v(f_1) + v(f_2), \quad v\left(\frac{f_1}{f_2}\right) = v(f_1) - v(f_2)
\]

(2.11)

Let \( f = (f_1, \cdots, f_n) : T \to T' \) be a rational morphism between two algebraic tori \( T = (\mathbb{C}^\times)^m \) and \( T' = (\mathbb{C}^\times)^n \). We define a map \( \hat{f} : X_*(T) \to X_*(T') \) by

\[
(\hat{f}(\xi))(c) := (e^{v(f_1(\xi(c)))}, \cdots, e^{v(f_n(\xi(c)))}),
\]

where \( \xi \in X_*(T) \). Since \( v \) satisfies (2.11), the map \( \hat{f} \) is an additive group homomorphism. If we identify \( X_*(T) \) (resp. \( X_*(T') \)) with \( \mathbb{Z}^m \) (resp. \( \mathbb{Z}^n \)) by \( \xi(c) = (c^1, \cdots, c^m) \leftrightarrow (l_1, \cdots, l_m) \in \mathbb{Z}^m \), we write

\[
\hat{f}(l_1, \cdots, l_m) := (v(f_1(\xi(c))), \cdots, v(f_n(\xi(c)))).
\]

A rational function \( f(c) \in \mathbb{C}(c) \) \((f \neq 0)\) is \textit{positive} if \( f \) can be expressed as a ratio of polynomials with positive coefficients.

**Remark.** A rational function \( f(c) \in \mathbb{C}(c) \) is positive if and only if \( f(a) > 0 \) for any \( a > 0 \) (pointed out by M. Kashiwara).

If \( f_1, f_2 \in R \) are positive, then we have (2.11) and

\[
v(f_1 + f_2) = \max(v(f_1), v(f_2)).
\]

(2.12)

**Definition 2.10** \([11]\). Let \( f = (f_1, \cdots, f_n) : T \to T' \) between two algebraic tori \( T, T' \) be a rational morphism as above. It is called \textit{positive} if the following two conditions are satisfied:

(i) For any co-character \( \xi : \mathbb{C}^\times \to T \), the image of \( \xi \) is contained in \( \text{dom}(f) \).
(ii) For any co-character $\xi : \mathbb{C}^* \to T$, any $f_i(\xi(c)) \ (i \in I)$ is a positive rational function.

Denote by $\text{Mor}^+(T, T')$ the set of positive rational morphisms from $T$ to $T'$.

**Lemma 2.11** ([1]). For any positive rational morphisms $f \in \text{Mor}^+(T_1, T_2)$ and $g \in \text{Mor}^+(T_2, T_3)$, the composition $g \circ f$ is in $\text{Mor}^+(T_1, T_3)$.

By Lemma 2.11 we can define a category $\mathcal{T}_+$ whose objects are algebraic tori over $\mathbb{C}$ and arrows are positive rational morphisms.

**Lemma 2.12** ([1]). For any algebraic tori $T_1$, $T_2$, $T_3$, and positive rational morphisms $f \in \text{Mor}^+(T_1, T_2)$, $g \in \text{Mor}^+(T_2, T_3)$, we have $g \circ f = \hat{g} \circ \hat{f}$.

By this lemma, we obtain a functor $\mathcal{U}D : \mathcal{T}_+ \to \text{Set}$ defined by $f : T \to T'$, $$f \mapsto (f : X_*(T) \to X_*(T')).$$

**Definition 2.13** ([1]). Let $\chi = (X, \gamma, \{e_i\}_{i \in I})$ be a geometric crystal, $T'$ be an algebraic torus and $\theta : T' \to X$ be a birational isomorphism. The isomorphism $\theta$ is called positive structure on $\chi$ if it satisfies

(i) the rational morphism $\gamma \circ \theta : T' \to T$ is positive.

(ii) For any $i \in I$, the rational morphism $e_i,\theta : \mathbb{C}^* \times T' \to T'$ defined by $e_i,\theta(c, t) := \theta^{-1} \circ e_i \circ \theta(t)$ is positive.

Let $\theta : T \to X$ be a positive structure on a geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$. Applying the functor $\mathcal{U}D$ to positive rational morphisms $e_i,\theta : \mathbb{C}^* \times T' \to T'$ and $\gamma \circ \theta : T' \to T$ (the notations are as above), we obtain

$\hat{e}_i := \mathcal{U}D(e_i,\theta) : \mathbb{Z} \times X_*(T) \to X_*(T)$
$\hat{\gamma} := \mathcal{U}D(\gamma \circ \theta) : X_*(T') \to X_*(T)$.

Now, for given positive structure $\theta : T' \to X$ on a geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$, we associate the triplet $(X_*(T'), \hat{\gamma}, \{\hat{e}_i\}_{i \in I})$ with a free pre-crystal structure (see [1 2.2]) and denote it by $\mathcal{U}D_{\theta, T}(\chi)$. By Lemma 2.12, we have the following theorem:

**Theorem 2.14.** For any geometric crystal $\chi = (X, \gamma, \{e_i\}_{i \in I})$ and positive structure $\theta : T' \to X$, the associated pre-crystal $\mathcal{U}D_{\theta, T}(\chi) = (X_*(T'), \hat{\gamma}, \{\hat{e}_i\}_{i \in I})$ is a free $W$-crystal (see [1 2.2]).

We call the functor $\mathcal{U}D$ “ultra-discretization” instead of “tropicalization” unlike in [1]. And for a crystal $B$, if there exists a geometric crystal $\chi$, an algebraic torus $T$ in $\mathcal{T}_+$, and a positive structure $\theta$ on $\chi$ such that $\mathcal{U}D_{\theta, T}(\chi) \cong B$ as crystals, we call $\chi$ a tropicalization of $B$.

### 3 Geometric crystals on unipotent groups

In this section, we associate a geometric/unipotent crystal structure with unipotent subgroup $U^-$ of semi-simple algebraic group $G$. In particular, for $G = SL_n(\mathbb{C})$ we describe it explicitly.
3.1 $U$-variety structure on $U^-$

In this subsection, suppose that $G$ is a Kac-Moody group as in Sect.2. As mentioned in, the Borel subgroup $B^-$ has a $U^-$-variety structure. By the similar manner, we define $U^-$-variety structure on $U^-$. As in 2.3, the multiplication map $m$ in $G$ induces an open embedding: $m : U^- \times B \hookrightarrow G$, then this is a birational isomorphism. Let us denote the inverse birational isomorphism by $h$:

$$h : G \rightarrow U^- \times B.$$ 

Then we define the rational morphisms $\pi^- : G \rightarrow B^-$ and $\pi^+ : G \rightarrow B$ by $\pi^- := \text{proj}_{U^-} \circ h$ and $\pi^+ := \text{proj}_B \circ h$. Now we define the rational $U^-$-action $\alpha_{U^-}$ on $U^-$ by

$$\alpha_{U^-} := \pi^- \circ m : U \times U^- \rightarrow U^-,$$

Then we obtain

**Lemma 3.1.** A pair $U^- = (U^-, \alpha_{U^-})$ is a $U^-$-variety on a unipotent subgroup $U^- \subset G$.

3.2 Unipotent/Geometric crystal structure on $U^-$

In order to define a unipotent crystal structure on $U^-$, let us construct a $U^-$-morphism $F : U^- \rightarrow B^-$. The multiplication map $m$ in $G$ induces an open embedding: $m : U^- \times T \times U \hookrightarrow G$, which is a birational isomorphism. Thus, by the similar way as above, we obtain the rational morphism $\pi^0 : G \rightarrow T$. Here note that we have

$$\pi^-(x) = \pi^-(x)\pi^0(x) \quad (x \in G).$$ (3.1)

Now, we give a sufficient condition for existence of $U^-$-morphism $F$.

**Lemma 3.2.** Let $T : U^- \rightarrow T$ be a rational morphism satisfying:

$$T(\pi^-(xu)) = \pi^0(xu)T(u), \quad \text{for } x \in U \text{ and } u \in U^-.$$ (3.2)

Defining a morphism $F : U^- \rightarrow B^-$ by

$$F : U^- \rightarrow B^- \quad \begin{array}{c}
\text{u} \\
\mapsto
\end{array} uT(u),$$ (3.3)

then the morphism $F$ is a $U^-$-morphism $U^- \rightarrow B^-$. 

**Proof.** We may show

$$F(\alpha_{U^-}(x, u)) = \alpha_B^-(x, F(u)), \quad \text{for } x \in U \text{ and } u \in U^-.$$ (3.4)

As for the left-hand side of (3.4), we have

$$F(\alpha_{U^-}(x, u)) = \pi^-((xu)T(u)) = \pi^-(xu)\pi^0(xu)T(u),$$

where the last equality is due to (3.2). On the other hand, the right-hand side of (3.4) is written by:

$$\alpha_B^-(x, F(u)) = \pi^-(xuT(u)) = \pi^-(xuT(u))\pi^0(xuT(u)) = \pi^-(xu)\pi^0(xu)T(u)$$

where the second equality is due to (3.1) and the third equality is obtained by the fact that $T(u) \in T \subset B$. Now we get (3.4).

Let us verify that there exists such $U^-$-morphism $F$ or rational morphism $T$ for semisimple cases. Suppose that $G$ is semisimple in the rest of this section.
Let \( \Lambda_i \in P_+ \) (\( i = 1, \cdots, n \)) be a fundamental weight and \( L(\Lambda_i) \) be a corresponding irreducible highest weight \( g \)-module, where \( g \) is a complex semi-simple Lie algebra associated with \( G \). Let \( L^*(\Lambda_i) \) be a contragredient module of \( L(\Lambda_i) \) as in Sect.2 and fix a highest (resp. lowest) weight vector \( u_{\Lambda_i} \) (resp. \( v^{\ast}_{\Lambda_i} \)) and \( v^*_i \in L^*(\Lambda_i) \) be the same vector as \( v_{\Lambda_i} \) such that \( \langle v_{\Lambda_i}, v^*_i \rangle = 1 \). Now, let us define a function \( f_i : U^- \to \mathbb{C} \) (\( i \in I \)) as a matrix coefficient:

\[
f_i(g) = \langle g \cdot u_{\Lambda_i}, v^*_i \rangle.
\]

We define a rational morphism \( \mathcal{T} : U^- \to T \) by

\[
\mathcal{T}(u) := \prod_{i \in I} \alpha^\vee_i (f_i(u)^{-1}).
\]

and define a morphism \( F : U^- \to B^- \) by

\[
F(u) := u \cdot \prod_{i \in I} \alpha^\vee_i (f_i(u)^{-1}).
\]

**Lemma 3.3.** The morphism \( F : U^- \to B^- \) is a \( U \)-morphism.

**Proof.** Let us verify that \( \mathcal{T} \) satisfies \( \mathcal{T} \). For \( x \in U \) and \( u \in U^- \) such that \( xu \in \text{Im}(U^- \times T \times U \hookrightarrow G) \), let \( u^- \in U^- \), \( u^0 \in T \) and \( u^+ \in U \) be the unique elements satisfying \( u^- u^+ = xu \), i.e., \( \pi^- (xu) = u^- \), \( \pi^0 (xu) = u^0 \) and \( \pi (xu) = u^+ \). Since \( \langle \ , \ \rangle \) is a contragredient bilinear form and the fact that \( g \cdot v^*_i = v^*_i \) for any \( g \in U^- \), we have

\[
\langle xu \cdot u_{\Lambda_i}, v^*_i \rangle = \langle u \cdot u_{\Lambda_i}, \omega(x) \cdot v^*_i \rangle = \langle u \cdot u_{\Lambda_i}, v^*_i \rangle.
\]

On the other hand, since \( g \cdot u_{\Lambda_i} = u_{\Lambda_i} \) for \( g \in U \), we have

\[
\langle xu \cdot u_{\Lambda_i}, v^*_i \rangle = \langle \pi^0 (xu) \cdot u_{\Lambda_i}, v^*_i \rangle = \Lambda_i (\pi^0 (xu)) \langle \pi^- (xu) \cdot u_{\Lambda_i}, v^*_i \rangle,
\]

where \( \Lambda_i \in X^*(T) \) such that \( \Lambda_i (\alpha^\vee_j (c)) = e^{h_j} \cdot j \). Hence, by (3.8), (3.9), we have

\[
f_i(\pi^- (xu)) = \Lambda_i (\pi^0 (xu))^{-1} \langle uu_{\Lambda_i}, v^*_i \rangle = \Lambda_i (\pi^0 (xu))^{-1} f_i(u).
\]

By the formula

\[
\prod_{i} \alpha^\vee_i (\Lambda_i (t)) = t, \quad (t \in T),
\]

and the definitions of \( \mathcal{T} \) and \( F \), we obtained (3.2).

**Corollary 3.4.** Suppose that \( G \) is semi-simple. Then \( (U^-, F) \) is a unipotent crystal.

As we have seen in (2.1) we can associate geometric crystal structure with the unipotent subgroup \( U^- \) since it has a unipotent crystal structure.

It is trivial that the function \( \varphi_i : U^- \to \mathbb{C} \) is not identically zero. Thus, defining the morphisms \( e_i : \mathbb{C}^\times \times U^- \to U^- \) and \( \gamma_{U^-} : U^- \to T \) by

\[
e_i(c, u) := e_i(u) := x_i(\frac{c - 1}{\varphi_i(u)})(u), \quad \gamma_{U^-}(u) := \mathcal{T}(u), \quad (u \in U^- \text{ and } c \in \mathbb{C}^\times),
\]

It follows from Theorem (2.7)

**Theorem 3.5.** If \( G \) is semi-simple, then the triplet \( \chi_{U^-} := (U^-, \gamma_{U^-}, \{e_i\}_{i \in I}) \) is a geometric crystal.
4 \textit{SL}_{n+1}(\mathbb{C})\text{-case}

We see the result of the previous section in the $\text{SL}_{n+1}(\mathbb{C})$-case more explicitly.

We identify unipotent subgroup $U^-$ with the set of lower triangular matrices whose diagonal part is an identity matrix.

First, let us describe the morphism $\mathcal{T} : U^- \to \mathcal{T}$. For $i \in I := \{1, \cdots, n\}$ and $u = (a_{ij})_{1 \leq i, j \leq n+1} \in U^-$, let $u^{(i)}$ be the submatrix with size $i$ as:

$$u^{(i)} := (a_{ij})_{1 \leq i+2 \leq n+1, 1 \leq j \leq i},$$

i.e.,

$$u = \begin{pmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 \\ u^{(i)} \\ 1 \end{pmatrix} \in U^-$$

and set $m_i(u) := \det(u^{(i)})$.

Let $V = \mathbb{C}^{n+1}$ be the $n + 1$-dimensional vector space with the basis $\{u_1, u_2, \cdots, u_{n+1}\}$. We can identify $V$ with the vector representation $L(\Lambda_1)$ of $\mathfrak{sl}_{n+1}$ by the standard way. Indeed, the explicit actions are given by:

$$e_i(u) = \delta_{i+1, j} u_{i-1} \quad f_i(u) = \delta_{i, j} u_{i+1},$$

where $e_i = E_{i, i+1}$ and $f_i = E_{i+1, i}$ (matrix unit), and we set $e_i(u_1) = 0$ and $f_i(u_{n+1}) = 0$ for all $i \in I$, which implies that $u_1$ is the highest weight vector and $u_{n+1}$ is the lowest weight vector. Then we have the isomorphism between the fundamental representation $L(\Lambda_k) \ (1 \leq k \leq n)$ and the $k$-th anti-symmetric tensor module $\Lambda^k(V)$. Let us fix

$$u_{\Lambda_k} := u_1 \wedge u_2 \wedge \cdots \wedge u_k \quad (\text{resp. } v_{\Lambda_k} := u_{n-k+2} \wedge u_{n-k+3} \wedge \cdots \wedge u_{n+1}) \quad (4.1)$$

the highest (resp. lowest) weight vector in $L(\Lambda_k) \cong \Lambda^k(V)$. In this setting, we have

\begin{lemma}
$f_i \equiv m_i$ on $U^-$ for all $i = 1, \cdots, n$.
\end{lemma}

\textbf{Proof.} For $g = (g_{ij}) \in U^-$, we have

$$g \cdot u_i = u_i + \sum_{i < j} g_{ij} u_j.$$ 

Let us see the coefficient of the vector $v_{\Lambda_k} := u_{n-k+2} \wedge u_{n-k+3} \wedge \cdots \wedge u_{n+1}$ in $g \cdot u_{\Lambda_k}$. We have

$$g \cdot u_{\Lambda_k} = g \cdot u_1 \wedge g \cdot u_2 \wedge \cdots \wedge g \cdot u_k = (u_1 + \sum_{1 < j} g_{j1} u_j) \wedge \cdots \wedge (u_k + \sum_{k < j} g_{jk} u_j).$$

Thus, the coefficient of the vector $u_{j_1} \wedge \cdots \wedge u_{j_k}$ is $g_{j_1} \cdots g_{j_k}$. Hence, we obtain the coefficient of $v_{\Lambda_k}$ as

$$\sum_{\sigma \in S_k} \text{sgn}(\sigma) g_{n-\sigma(1)+2} \cdots g_{n-\sigma(k)+2} = m_i(u),$$

by using $v_{\sigma(1)} \wedge v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} = \text{sgn}(\sigma) v_1 \wedge v_2 \wedge \cdots \wedge v_k$. On the other hand, the coefficient of the lowest weight vector gives the function $f_i(u)$. Then we get the desired result. \qed
Next, let us see the action of $e_i^\alpha$ on $U^-$. Indeed, the action of $e_i^\alpha$ is described simply by:

$$e_i^\alpha(u) = x_i\left(\frac{\phi_i(u)}{\alpha - 1}\right) \cdot u \cdot x_i\left(\frac{1 - \alpha}{\alpha \phi_i(u)}\right) \cdot \alpha_i^{-1}(\alpha).$$

Here, for the later purpose, we consider the following subset $B^u$ of $U^-$ and describe the action of $e_i^\alpha$ on it:

$$B^u := \left\{ Y(a) = \begin{array}{c}
\times y_n(a_{1,n})y_{n-1}(a_{1,n-1})\cdots y_1(a_{1,1}) \\
\times y_n(a_{2,n})\cdots y_2(a_{2,2}) \\
\times y_n(a_{n,n})
\end{array} : a_{i,j} \in C^\times \right\} \subset U^-. \quad (4.2)$$

It is easy to see that $B^u$ is an open dense subset in $U^-$ and isomorphic to the algebraic torus $T_0 := (C^\times)^{n(n+1)/2}$ by:

$$T_0 = (C^\times)^{n(n+1)/2} \xrightarrow{\sim} B^u, \quad a = (a_{i,j})_{1 \leq i \leq j \leq n} \mapsto Y(a),$$

which gives a birational isomorphism $\theta : T_0 = (C^\times)^{n(n+1)/2} \to B^u \hookrightarrow U^-$. Furthermore, we have

**Lemma 4.2.** $\varphi_i(Y(a)) = \sum_{i=1}^k a_{k,i}$, $f_i(Y(a)) = \prod_{k=1}^n a_{n+k,i}.$

**Proof.** Since for $u \in U^-$, $\phi_i(u)$ is given as a $(i+1,i)$-entry and the $(i+1,i)$-entry of $Y(a)$ is $\sum_{k=1}^k a_{k,i}$, we obtained the first result.

For a word $\omega = i_1, \cdots, i_m$, we set $f_\omega := f_1 \cdots f_{i_m}$. For a fixed reduced longest word $\omega_0 = n, \cdots, 1, n-1, n-2, \cdots, 2, 1$, there exists the unique subword

$$\omega_i := n-i+1, \cdots, 1, n-i+2, \cdots, 2, \cdots, n-1, \cdots, i, \cdots, i,$$

such that

$$f_{\omega_i}u_{\Lambda_i} = v_{\Lambda_i}, \quad (4.3)$$

where $u_{\Lambda_i}$ (resp. $v_{\Lambda_i}$) is the highest (resp. lowest) weight vector of $L(\Lambda_i)$ as in [4.1]. Since $f_{\omega_i}^2(L(\Lambda_k)) = \{0\}$, we have $\varphi_i(a) = 1 + a_{i+1}$ on $L(\Lambda_k)$ and then

$$Y(a) = (1 + a_{1,n}f_n) \cdots (1 + a_{1,1}f_1) \cdots (1 + a_{n,n}f_n)$$

$$= \sum_{\text{subword } \omega_0} a_{\omega} f_\omega,$$

where $a_{\omega}$ is a coefficient of $f_\omega$ and $a_{i+1} = 1$ if $\omega$ is empty. Hence by (4.3), we have

$$f_{\omega_i}(Y(a)) = \langle Y(a)u_{\Lambda_i}, v_{\Lambda_i}^\alpha \rangle = a_{\omega_i} = \prod_{k=1}^n a_{k,i}.$$

Let us see the rational action $e_i^\alpha : B^u \to B^u$.

**Proposition 4.3.** We have $e_i^\alpha Y((a_{k,j})_{1 \leq k \leq j \leq n}) = Y((a_{k,j}^\prime)_{1 \leq k \leq j \leq n}).$

$$a_{k,j}^\prime := \begin{cases}
C_k^{(i)} a_{k,i-1} & \text{if } j = i - 1, \\
C_k^{(i)} a_{k,i} & \text{if } j = i, \\
C_k^{(i)} C_k^{(i)} & \text{if } j = i + 1, \\
a_{k,j} & \text{otherwise},
\end{cases} \quad (4.4)$$

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where
\[ C_k^{(i)} := \frac{\alpha(a_{1,i} + \cdots + a_{k,i}) + a_{k+1,i} + \cdots + a_{i,i}}{a_{1,i} + \cdots + a_{i,i}} \quad (1 \leq k \leq i \leq n). \]

**Proof.** We recall the formula:
\[ x_i(a)y_j(b) = \begin{cases} y_j(b)x_i(a) & \text{if } i = j, \\ \frac{b}{x_i(a)} \alpha_i^\gamma(1 + ab)x_i(\frac{a}{1 + ab}) & \text{if } i \neq j, \end{cases} \]
(4.5)
\[ \alpha_i^\gamma(a)x_j(b) = x_j(a^\gamma b)\alpha_i^\gamma(a), \quad \alpha_i^\gamma(a)y_j(b) = y_j(a^\gamma b)\alpha_i^\gamma(a). \] (4.6)

Using these formula repeatedly, we have
\[ x_i(c) \cdot Y(a) = Y(a') \cdot \alpha_i^\gamma(1 + c\varphi_i(Y(a))) \cdot x_i(\frac{c}{1 + c\varphi_i(Y(a))}), \]
where \( c = (\alpha - 1)/(a_{1,i} + \cdots + a_{i,i}) \) and \( \varphi_i(Y(a)) = a_{1,i} + \cdots + a_{i,i} \).

Now, we consider the following birational isomorphism:
\[ \xi : T_0 = (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \rightarrow T_0 = (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}, \]
\[ (a_{i,j})_{1 \leq i \leq j \leq n} \rightarrow (A_{i,j})_{1 \leq i \leq j \leq n}, \]
where
\[ A_{i,j} := \frac{a_{i,j}a_{i-1,j-1} \cdots a_{1,j-i+1}}{a_{i-1,j}a_{i-2,j-1} \cdots a_{1,j-i+2}} \quad (1 \leq i \leq j \leq n). \]
The inverse morphism is given by
\[ a_{i,j} := \frac{A_{i,j}A_{i-1,j} \cdots A_{1,j}}{A_{i-1,j-1}A_{i-2,j-1} \cdots A_{1,j-1}} \quad (1 \leq i \leq j \leq n). \] (4.7)

We can describe explicitly
\[ \xi \circ e_i^\alpha \circ \xi^{-1} : (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \rightarrow (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}, \]
\[ (A_{k,j})_{1 \leq k \leq j \leq n} \mapsto (A'_{k,j})_{1 \leq k \leq j \leq n}, \]
where
\[ A'_{k,j} = \begin{cases} A_{k,j} & \text{if } j \neq i, i - 1, \\ \alpha_{k,j}^{(i)} \cdot A_{k,i-1} & \text{if } j = i - 1, \\ (\alpha_{k,j}^{(i)})^{-1} \cdot A_{k,i} & \text{if } j = i, \end{cases} \]
(4.8)
\[ \alpha_{k,j}^{(i)} = \alpha \sum_{1 \leq j \leq k} \frac{\prod_{l=1}^{j} A_{l,i}}{\prod_{l=1}^{j} A_{l,i-1}} + \sum_{k < j \leq i} \frac{\prod_{l=1}^{j} A_{l,i}}{\prod_{l=1}^{j} A_{l,i-1}}. \]
Set \( \hat{\theta} := \theta \circ \xi^{-1} : (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \rightarrow (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \rightarrow U^-. \)

**Theorem 4.4.** The morphism \( \hat{\theta} \) gives a positive structure on the geometric crystal \( \chi_{U^-} \).

**Proof.** The explicit form of
\[ \hat{\theta}^{-1} \circ e_i^\alpha \circ \hat{\theta} : (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \times (\mathbb{C}^\times)^{\frac{n(n+1)}{2}} \rightarrow (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}, \]
\[ (A_{k,j})_{1 \leq k \leq j \leq n} \mapsto (A'_{k,j})_{1 \leq k \leq j \leq n}. \]
is given as (4.8), which is trivially positive. Then let us show the positivity of $\gamma_{U^-} \circ \hat{\theta}$. For $Y(a) \in B^n$, we have $\gamma_{U^-}(Y(a)) = \prod_i \alpha_i^y(f_i(Y(a))^{-1})$ and by Lemma 4.2, the explicit form of $f_i(Y(a))$ is given. Substituting (4.7) in it, we obtain

$$\gamma_{U^-} \circ \hat{\theta}((A_{k,j})_{1 \leq k \leq j \leq n}) = \gamma_{U^-} \circ \hat{\theta} \circ \xi^{-1}((A_{k,j})_{1 \leq k \leq j \leq n}) = \prod_{i=1}^{n} \alpha_i^y(\prod_{1 \leq k \leq i \leq j \leq n} A_{k,j})^{-1},$$

which implies that $\gamma \circ \hat{\theta}$ is positive.

5 Tropicalization of Geometric Crystals on $U^-$ and generalized Young Tableaux

5.1 Crystal structure on Young tableaux

Let us recall the crystal structure on Young tableaux where the terminology “Young tableaux” means “semi-standard tableaux” in [7]. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$, set

$$B(\lambda) = \{ \text{Young tableau of shape } \lambda \text{ with contents } 1, 2, \cdots, n+1 \},$$

which gives an $A_n$-crystal of irreducible highest weight $U_q(sl_{n+1})$-module $V(\lambda)$[7].

In order to describe the action of $\tilde{e}_i(b)$ ($\beta \geq 0$) explicitly, let us recall how to construct $B(\lambda)$ following [7],[5].

Let $V := V(\Lambda_1)$ be the vector representation of $U_q(sl_{n+1})$, which is the irreducible highest weight module with the highest weight $\Lambda_1$ and let

$$B := \{ 1, 2, \cdots, n+1 \}$$

be the crystal of $V$. The explicit actions of $\tilde{e}_i$ and $\tilde{f}_i$ are given as follows(7):

$$\tilde{e}_i(j) = \delta_{i+1,j} - 1, \quad \tilde{f}_i(j) = \delta_{i,j+1} + 1.$$  

We realize $B(\lambda)$ by embedding into $B^N (N = |\lambda|)$, which follows the way of embedding $V(\lambda) \hookrightarrow V^N$. In [7], the “Japanese reading” is introduced, which gives the embedding by reading entries in a Young tableau column by column. But here we take so-called “arabic reading”[5], which gives the embedding by reading entries in a Young tableau row by row from right to left since it matches what we do below.

Example 5.1. (i) Japanese reading

$$[a \ b \ c \ d] \otimes [e] \otimes [f] \otimes [g] \in B^\otimes 7$$

(ii) Arabic reading

$$[a \ b \ c \ d] \otimes [e] \otimes [f] \otimes [g] \in B^\otimes 7$$
The description of the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(\lambda)$ in [7] is as follows: Let $\{(+),(0)\}$ (resp. $\{(0)\}$) be the crystal of the irreducible $U_q(sl_2)$-module $\mathcal{V}$ (resp. $\mathcal{V}(0)$). If we consider the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on a tensor product $B_N$, we can identify (7.2.1.),

$$i = (+), \quad i+1 = (-), \quad j = (0) \quad (j \neq i, i+1). \quad (5.1)$$

Let $b \in B(\lambda)$ be in the following form:

$$b = v_1 \otimes \cdots \otimes v_{i+1}, \quad (5.2)$$

where $B_i, j := \mathcal{V}(j$ in the $i$-th row $).$ If we consider the actions of $\tilde{e}_i$ and $\tilde{f}_i$, by the “arabic reading” and (5.1) we can identify:

$$b = v_1 \otimes \cdots \otimes v_{i+1}, \quad \text{where} \quad \beta = \sum_{1 \leq k \leq i+1} \beta_k^{(i)}.$$ 

For any $i \in I$ and $\beta \in \mathbb{Z}_{\geq 0}$ there exist unique $\beta_k^{(i)} \in \mathbb{Z}_{\geq 0} (1 \leq k \leq i+1)$ such that

$$\tilde{e}_i^\beta(v_1 \otimes \cdots \otimes v_{i+1}) = \tilde{e}_i^{\beta_1^{(i)}}(v_1) \otimes \cdots \otimes \tilde{e}_i^{\beta_{i+1}^{(i)}}(v_{i+1}), \quad (5.4)$$

and $\beta = \sum_{1 \leq k \leq i+1} \beta_k^{(i)}$. Note that on each component, we have

$$\tilde{e}_i^{\beta_k^{(i)}}(v_k) := (-)^{\otimes (B_{k,i+1} - \beta_k^{(i)})} \otimes (+)^{\otimes (B_{k,i} + \beta_k^{(i)})}. \quad (5.5)$$

Let us see the explicit form of $\beta_k^{(i)}$, in order to describe the action of $\tilde{e}_i^\beta$ on $b$. For the purpose, we prepare the following formula:

**Lemma 5.2 (5).** Let $B_1, B_2, \cdots, B_l$ be crystals. For $v_k \in B_k$ and $i \in I$, set $b_k := \varepsilon_i(v_k) - \sum_{1 \leq j < k} \langle h_i, \text{wt}(v_j) \rangle$. Then, we have

$$\tilde{e}_i^\beta (v_1 \otimes \cdots \otimes v_l) = \tilde{e}_i^{c_1}(v_1) \otimes \cdots \otimes \tilde{e}_i^{c_l}(v_l),$$

where

$$c_k = \max(c + \max_{1 \leq j \leq k} (b_j), \max_{k < j \leq l} (b_j)) - \max(c + \max_{1 \leq j \leq k} (b_j), \max_{k \leq j \leq l} (b_j)). \quad (5.5)$$
Applying this lemma to (5.4), we have

**Proposition 5.3.** Under the setting (5.3) and (5.4),

\[
\beta_k^{(i)} = \max \left( \beta + \max_{1 \leq j \leq k} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right), \max_{k < j \leq i} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right) \right) - \max \left( \beta + \max_{1 \leq j < k} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right), \max_{k \leq j \leq i} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right) \right) \quad (1 \leq k \leq i),
\]

\[
\beta_{i+1}^{(i+1)} = 0
\]

Proof. By Lemma 5.2, we have \( \varepsilon_i(v_k) = B_{k,i+1}, \langle h_i, wt(v_j) \rangle = B_{j,i} - B_{j,i+1} \). Applying this to Lemma 5.2, we obtain

\[
\beta_k^{(i)} = \max \left( \beta + \max_{1 \leq j \leq k} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right), \max_{k < j \leq i} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right) \right) - \max \left( \beta + \max_{1 \leq j < k} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right), \max_{k \leq j \leq i} \left( \sum_{l=1}^{j} B_{l,i+1} - \sum_{l=1}^{j-1} B_{l,i} \right) \right). \quad (5.6)
\]

Since \( B_{i+1,i+1} \leq B_{i,i} \), we have

\[
\sum_{l=1}^{i} B_{l,i+1} - \sum_{l=1}^{i-1} B_{l,i} \geq \sum_{l=1}^{i+1} B_{l,i+1} - \sum_{l=1}^{i} B_{l,i}.
\]

Hence, we can neglect \( j = i + 1 \) in the formula above. \( \square \)

**Remark.** The formula \( \beta_k^{(i)} \) does not depend on \( B_{i,i} \) or \( B_{i+1,i+1} \).

### 5.2 Generalized Young tableaux and its crystal structure

Let \( B \in B(\lambda) \) be a Young tableau as in (5.2). The \( B_{i,j} \)'s have several constraints, e.g.,

\[
B_{i,j} \geq 0, \quad \sum_{i \leq j \leq k} B_{i,j} \geq \sum_{i+1 \leq j \leq k+1} B_{i+1,j},
\]

which come from the conditions for being Young tableaux.

Now, forgetting such constraints on \( B_{i,j} \)'s, we obtain a free \( \mathbb{Z} \)-lattice \( B^\dagger \):

\[
B^\dagger := \{(B_{i,j})_{1 \leq i \leq j \leq n+1} \mid B_{i,j} \in \mathbb{Z}\} = \mathbb{Z}^n(n+1),
\]

Now, we define the action of \( \varepsilon_i^\beta \ (\beta \geq 0) \) on \( B^\dagger \) by

\[
\varepsilon_i^\beta ((B_{k,j})_{1 \leq k < j \leq n+1}) = ((B_{k,j} + \beta_{k,j})_{1 \leq k < j \leq n+1}), \quad \beta_{k,j} := \begin{cases} \beta_k^{(i)} & \text{if } j = i, \\ -\beta_k^{(i+1)} & \text{if } j = i + 1, \\ 0 & \text{otherwise} \end{cases}
\]

(5.7)

Here note that in the definition of \( B^\dagger \), \( B_{i,i} \)'s do not appear since the formula \( \beta_k^{(i)} \) does not depend on \( B_{i,i} \)'s as mentioned in the remark of the last subsection.

The explicit action of the Kashiwara operator \( \tilde{e}_i \) (resp. \( \tilde{f}_i \)) on \( B^\dagger \) is given by (5.7) taking \( \beta = 1 \) (resp. \( \beta = -1 \)). Indeed, the crystal structure of \( B^\dagger \) is described as follows:
Theorem 5.4. By the setting (5.8), (5.9) and (5.10), we obtain a free crystal $B^\sharp$.

Proof. It suffices to check the axioms (2.6)-(2.10) in Definition 2.8 and the bijectivity of $\tilde{e}_i$ or $\tilde{f}_i$. Indeed, (2.6)-(2.8) are trivial from (5.8), (5.9) and (5.10). The assumption of (2.10) never occurs. Thus, we may show that $\tilde{e}_i\tilde{f}_i = \text{id} = \tilde{f}_i\tilde{e}_i$. For $v = (B_{i,j})$, set $p := M_i(v)$, which implies

$$b_k^{(i)}(v) := \sum_{1 \leq l \leq k} B_{i,l+1} - \sum_{1 \leq l \leq k} B_{i,l},$$

$$\varepsilon_i(v) := \max_{1 \leq k \leq n} \{ b_k^{(i)}(v) \},$$

$$w_t(v) := - \sum_{i=1}^{k} \left( \sum_{i+1 \leq l \leq k+1} B_{k,j} \right) \alpha_i,$$

$$\varphi_i(v) := \langle h_i, wt(v) \rangle + \varepsilon_i(v),$$

(5.8)

$$m_i = m_i(v) := \min \{ k | 1 \leq k \leq i, b_k^{(i)}(v) = \varepsilon_i(v) \}$$

$$M_i = M_i(v) := \max \{ k | 1 \leq k \leq i, b_k^{(i)}(v) = \varepsilon_i(v) \}.$$  

The actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $v = (B_{i,j})$ are given by

$$\tilde{f}_i : \begin{cases} B_{k,j} \to B_{k,i} & \text{if } (k,j) \neq (M_i,i), (M_i,i+1) \\ B_{M_{i,i}} \to B_{M_{i,i}} - 1 & \text{if } (k,j) = (M_i,i) \\ B_{M_{i,i+1}} \to B_{M_{i,i+1}} + 1 & \text{if } (k,j) = (M_i,i+1) \end{cases}$$

(5.9)

$$\tilde{e}_i : \begin{cases} B_{k,j} \to B_{k,i} & \text{if } (k,j) \neq (m_i,i), (m_i,i+1), \\ B_{m_{i,i}} \to B_{m_{i,i}} + 1 & \text{if } (k,j) = (m_i,i) \\ B_{m_{i,i+1}} \to B_{m_{i,i+1}} - 1 & \text{if } (k,j) = (m_i,i+1) \end{cases}$$

(5.10)

Theorem 5.4. By the setting (5.8), (5.9) and (5.10), we obtain a free crystal $B^\sharp$.

Remark. It is unknown whether the crystal graph of $B^\sharp$ is connected or not.
Applying the following correspondence to \( x \cdot y \leftrightarrow x + y \)
\( x/y \leftrightarrow x - y \)
\( x + y \leftrightarrow \max(x, y) \)
\( i \leftrightarrow i + 1 \)
we obtain \( \alpha^{(i)}_k \leftrightarrow \beta^{(i)}_k \) and then

\[
\mathcal{U}D_{\theta,T_0}(e_i^c) = \tilde{e}_i^c, \quad \mathcal{U}D_{\theta,T_0}(\gamma) = wt,
\]

(5.11)

(\( T_0 := (\mathbb{C}^\times)^{n(n+1)/2} \)), which implies the following theorem:

**Theorem 5.5.** We have \( \mathcal{U}D_{\theta,T_0}(\chi_U^-) = (B^t, wt, \{\tilde{e}_i\}_{i \in I}) \), i.e., the geometric crystal \( \chi_U^- \)
on \( U^- \subset SL_{n+1}(\mathbb{C}) \) defined in Sect 4 is a tropicalization of the crystal \( B^t = (B^t, wt, \{\tilde{e}_i\}_{i \in I}) \).

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