Multivariate Splines and Polytopes

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Abstract

In this paper, we use multivariate splines to investigate the volume of polytopes. We first present an explicit formula for the multivariate truncated power, which can be considered as a dual version of the famous Brion’s formula for the volume of polytopes. We also prove that the integration of polynomials over polytopes can be dealt with by the multivariate truncated power. Moreover, we show that the volume of the cube slicing can be considered as the maximum value of the box spline. Based on this connection, we give a simple proof for Good’s conjecture, which has been settled before by probability methods.

Keywords: Box Splines; Multivariate Truncated Powers; Polytopes; Unit Cubes

1 Introduction

Box splines and multivariate truncated powers were first introduced in [5] and [8], respectively. They have a wide range of important and varied applications in numerical analysis and approximation theory. From the point of view of discrete geometry, box splines and multivariate truncated powers are closely related to the volume of cube slicing and the volume of polytopes, respectively. However, people working in the discrete geometry do not seem to be fully aware of the means of multivariate splines. The aim of this paper is to recast some problems related to the computation of volumes of polytopes and solve them by multivariate splines. We believe that the means of multivariate splines shed some light on problems concerning polytopes.
The main results in this paper are as follows. The exact computation of the volume of a polytope $P$ is an important and difficult problem which has close ties to various mathematical areas. Brion’s formula in continuous form (see [2, 7]), which is well known in discrete geometry, gives an explicit formula for the volume of polytopes. However, the formula requires the vertex representation of polytopes and the generators for each vertex cone. Based on the multivariate exponential truncated power [19], we give an explicit formula for the multivariate truncated power, which can be regarded as a dual version of Brion’s formula. In [13], Lasserre gave a recursive formula for computing the volume of polytopes, which has become a popular method today. We re-prove the formula by an iterative formula for the multivariate truncated power [16], which was presented by Micchelli.

Integration of continuous functions over polytopes has important applications. For example, in most finite element integration methods, the domain of integration is decomposed into polytopes. Hence the integration of real functions over polytopes is always required. In [12] an exact formula for the integration of polynomials over simplices is presented. An iterative formula for computing the integration over polytopes is also given in [14]. We shall show that integration of polynomials over polytopes can be dealt with by the multivariate truncated power and consequently we present an explicit formula for the integral of polynomials over polytopes. As continuous functions on a compact set can be uniformly approximated by polynomials, this result provides an approximate formula for integrating continuous functions over polytopes. Moreover, this result also shows that we can compute the integrals of polynomials over polytopes by calculating the volumes of polytopes.

The volume of cube slicing is another active research topic in discrete geometry (see [22]). Suppose that $Q_n := [0, 1]^n$ is the unit cube in $\mathbb{R}^n$ and $H$ is an $n - 1$ dimensional hyperplane of $\mathbb{R}^n$ through of its center. According to [11], Good conjectured that $\text{vol}(H \cap Q_n) \geq 1$. Hensley unexpectedly introduced a
probability method into the study of this conjecture and finally solved it\[11\].

In fact, the conjecture can be reformulated as the following box spline problem:

\[
\max_x B(x|(a_1, \ldots, a_n)) \geq \frac{1}{\sqrt{\sum_{i=1}^n a_i^2}},
\]

(1.1)

where \(B(x|(a_1, \cdots, a_n))\) is a univariate box spline (cf. Section 3), and \(a_i\) are positive real numbers for \(1 \leq i \leq n\). Based on the Fourier transform of the box spline, we give a simple proof of (1.1). Hence we present a spline method for proving the conjecture.

The problem of computing the volume of the intersection of \(Q_n\) and an \((n-1)\)-hyperplane is also interesting and it can be traced back to Pólya’s thesis. In \[18\], the authors derived a formula for the volume of such domains using combinatorial methods. Using box splines, we give an explicit formula for the volume of convex bodies obtained by intersecting \(Q_n\) and a \(j\)-hyperplane, where \(j\) is a positive integer < \(n\). The formula in \[18\] may be considered as a special case of ours.

The paper is organized as follows. After recalling some definitions and notations in Section 2, we show (Section 3) the connection between the multivariate truncated power and the volume of polytopes. In Section 4, we transform the integration of polynomials over polytopes to a problem concerning the multivariate truncated power. In Section 5, we investigate the volume of cube slicing using box splines. Finally, Section 6 illustrates the application of the formulas given in this paper with some examples.

## 2 Definitions and Notations

A convex polytope \(P\) is the convex hull of a finite set of points in \(\mathbb{R}^d\). Throughout this paper, we shall omit the qualifier “convex” since we confine our discussion to such polytopes. Moreover, we use \(d\)-polytope to mean a \(d\)-dimensional polytope. When the polytope \(P\) is defined as the convex hull of a finite set of points in \(\mathbb{R}^d\), the finite set is called as a vertex representation or simply \(V\)-representation of \(P\), while, if \(P\) is defined as \(\{x \in \mathbb{R}_+^n \mid Mx = b\}\) for some \(s \times n\) matrix \(M\) and
s-vector \(b\), then the pair \((M, b)\) is called a half space representation or simply \(\mathcal{H}\)-representation. For a vertex \(v\) of \(P\), we define the vertex cone of \(v\) as the smallest cone with vertex \(v\) that contains \(P\). If \(P \subset \mathbb{R}^n\) is a \(d\)-polytope, then let \(\text{vol}_n(P)\) denote the \(d\)-dimensional volume of \(P\) in \(\mathbb{R}^n\). For a rational polytope \(P\), i.e., a polytope whose vertices have rational coordinates, let \(\mathbb{R}P\) denote the space that is spanned by the vertex vectors of \(P\). The lattice points in \(\mathbb{R}P\) form an Abelian group of rank \(d\), i.e., \(\mathbb{R}P \cap \mathbb{Z}^d\) is isomorphic to \(\mathbb{Z}^d\). Hence there exists an invertible affine linear transformation \(T : \mathbb{R}P \to \mathbb{R}^d\) satisfying \(T(\mathbb{R}P \cap \mathbb{Z}^n) = \mathbb{Z}^d\). The relative volume of \(P\), denoted as \(\text{vol}(P)\), is just the \(d\)-dimensional volume of the image \(T(P) \subset \mathbb{R}^d\). For more detailed information about the relative volume, the reader is referred to [20].

Throughout this paper, \(\mathbb{Z}_+\) and \(\mathbb{R}_+\) denote the non-negative integer and non-negative real sets, respectively. Given a set \(D\), let \(\chi_D(x) = 1\) if \(x \in D\), otherwise let \(\chi_D(x) = 0\). Elements of \(\mathbb{R}^n\) can be regarded as row or column \(s\)-vectors according to circumstances. Let \(M\) be an \(s \times n\) matrix. Then \(M\) can be considered as a multiset of its columns. The cone spanned by \(M\), denoted by \(\text{cone}(M)\), is the set \(\{ \sum_{m \in M} a_m m | a_m \geq 0 \text{ for all } m \}\). Moreover, we set \([[M]] := \{ \sum_{m \in M} a_m m | 0 \leq a_m < 1, \text{ for all } m \in M \}\). Furthermore, we use \(#A\) to denote the cardinality of the finite set \(A\). “·” stands for scalar product and \(\| \cdot \|\) for the Euclidean norm. \(E_{s \times s}\) denotes the \(s \times s\) identity matrix. As a final piece of notation, \(A^{-T} := (A^{-1})^T\) when \(A\) is an invertible matrix.

### 3 Multivariate truncated powers and the volume of polytopes

Let \(M\) be an \(s \times n\) real matrix with \(\text{rank}(M) = s\). Recall that \(M\) is also viewed as the multiset of its column vectors. Throughout this section we always assume that the convex hull of \(M\) does not contain the origin. The multivariate truncated power \(T(\cdot | M)\) associated with \(M\), first introduced by Dahmen [8], is
where $\mathcal{D}(\mathbb{R}^s)$ is the space of test functions on $\mathbb{R}^s$. If we define $P := \{y \in \mathbb{R}^n_+ \mid My = x\}$, then (see [6])

$$T(x|M) = \frac{\text{vol}_n(P)}{\sqrt{\det(MM^T)}}.$$  \hspace{1cm} (3.2)

Note that

$$\text{vol}_n(P) = \sqrt{\det(MM^T)} \#\{\lfloor MT \rfloor \cap \mathbb{Z}^s\}$$

provided $M$ is an integer matrix (see [1]). Hence we have

$$T(x|M) = \frac{\text{vol}(P)}{\#\{\lfloor MT \rfloor \cap \mathbb{Z}^s\}}$$  \hspace{1cm} (3.3)

provided $M$ is an integer matrix. In particular, if $E_{s \times s} \subset M$ then $T(x|M) = \text{vol}(P)$, since $\#\{\lfloor MT \rfloor \cap \mathbb{Z}^s\} = 1$. So, the relative volume of polytopes $P$ can be obtained by computing $T(x|M)$.

In the following, we shall give an explicit formula for $T(x|M)$. We first introduce the multivariate exponential truncated power $E_c(x|M)$ associated with a complex vector $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$ and a matrix $M$. $E_c(x|M)$ is the distribution given by the rule (see [19]):

$$\int_{\mathbb{R}^s} E_c(x|M) \phi(x) dx = \int_{\mathbb{R}^n_+} \exp(-c \cdot u) \phi(Mu) du, \quad \phi(x) \in \mathcal{D}(\mathbb{R}^s).$$  \hspace{1cm} (3.4)

It is convenient for us to index the constants $c_1, \ldots, c_n$ by an element $m_i \in M$, that is, we set $c_{m_i} := c_i$ for $i = 1, \ldots, n$. For the submatrix $M' = (m_{i_1}, \ldots, m_{i_k})$ in $M$ we set $c_{M'} := (c_{i_1}, \ldots, c_{i_k})$ and $M'/c_{M'} := (m_{i_1}/c_{i_1}, \ldots, m_{i_k}/c_{i_k})$.

We recall an explicit formula for $E_c(\cdot|M)$. In this formula, we denote, given a square invertible $Y \subset M$, $\theta_Y := Y^{-T}c_Y$ and $\alpha_Y := \prod_{y \in M \setminus Y} (\theta_Y \cdot y - c_Y)^{-1}$.

Lemma 3.1. (19)

$$E_c(x|M) = \sum_{Y \subset M} \alpha_Y E_{c_Y} (x|Y),$$  \hspace{1cm} (3.5)

for all $c \in \mathbb{C}^n$ such that the denominators in $\alpha_Y$ do not vanish.
Using Lemma 3.1, we can now give an explicit formula for $T(x|M)$.

**Theorem 3.1.**

$$T(x|M) = \frac{1}{(n-s)!} \sum_{Y \subseteq M \atop \# Y = \text{rank}(Y) = s} \alpha_Y \det Y^{-1} (-\theta_Y \cdot x)^{n-s} \chi_{\text{cone}(Y)}(x),$$

for all $c \in \mathbb{C}^n$ such that the denominators on the right-hand side do not vanish, where both $\alpha_Y$ and $\theta_Y$ are defined in Lemma 3.1.

**Proof.** For an invertible $Y \subseteq M$, one has

$$E_{c_Y}(x|Y) = \frac{1}{\det Y} \exp (-\theta_Y \cdot x) \chi_{\text{cone}(Y)}(x).$$

According to Lemma 3.1, for $\rho \in \mathbb{R} \setminus 0$, we have

$$E_{\rho c_Y}(x|M) = \rho^{-n+s} \sum_{Y \subseteq M \atop \# Y = \text{rank}(Y) = s} \alpha_Y E_{\rho c_Y}(x|Y).$$

Then the Taylor expansion of $E_{\rho c_Y}(x|M)$ about 0 in the variable $\rho$ is

$$E_{\rho c_Y}(x|M) = \sum_{l=0}^{\infty} \rho^{l-n+s} p_l(x), \quad (3.6)$$

where

$$p_l(x) = \frac{1}{l!} \sum_{Y \subseteq M \atop \# Y = \text{rank}(Y) = s} \alpha_Y \det Y^{-1} (-\theta_Y \cdot x)^l \chi_{\text{cone}(Y)}(x).$$

The definition of $T(x|M)$ implies that it is the constant term in (3.6). Hence, we have $T(x|M) = p_{n-s}(x)$. The theorem follows. \qed

Brion’s formula, which is obtained by Brion’s Theorem, is useful for computing the relative volumes of polytopes. We state it here.

**Theorem 3.2.** (11) Suppose that $\mathcal{P}$ is a simple rational convex $d$-polytope. For a vertex cone $K_v$ of $\mathcal{P}$, fix a set of generators $w_1(v), w_2(v), \ldots, w_d(v) \in \mathbb{Z}^d$. Then

$$\text{vol}(\mathcal{P}) = \frac{(-1)^d}{d!} \sum_{v \text{ a vertex of } \mathcal{P}} \frac{(v \cdot c)^d \det(w_1(v), \ldots, w_d(v))}{\prod_{k=1}^d (w_k(v) \cdot c)}$$

for all $c \in \mathbb{C}^d$ such that the denominators on the right-hand side do not vanish.
Remark 3.1. Brion’s formula requires the $V$-representation and the generators for each vertex cone, while the formula presented in Theorem 3.1 requires the $H$-representation. Hence, the formula in Theorem 3.1 can be considered as a dual version of Brion’s.

We next turn to another formula for computing the relative volume of polytopes. We first introduce an iterative formula for calculating the multivariate truncated power.

Theorem 3.3. (13) Let $M$ be an $s \times n$ matrix with columns $m_1, \ldots, m_n \in \mathbb{R}^s \setminus \{0\}$ such that the origin is not contained in $\text{conv}(M)$. Suppose that $n > s + 1$. For any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and $x = \sum_{j=1}^{n} \lambda_j m_j$, we have

$$T(x|M) = \frac{1}{n-s} \sum_{j=1}^{n} \lambda_j T(x|M \setminus m_j).$$

(3.7)

In [13], Lasserre found a formula which expressed the volume of polytopes as the combination of the volumes of the faces. We describe it as follows. Consider the convex polytope defined by

$$D(b) := \{x \in \mathbb{R}^n \mid Ax \leq b\},$$

where $A$ is an $s \times d$ matrix and $b$ an $s$-vector. The $i$th face of $D(b)$ is defined by

$$D_i(b) := \{x \in \mathbb{R}^d \mid a_i \cdot x = b_i, Ax \leq b\},$$

where $a_i$ is the $i$th row of $A$. We set $V(d, A, b) = \text{vol}_d(D(b))$ and set $V_i(d - 1, A, b) = \text{vol}_d(D_i(b))$. Now we can describe Lasserre’s formula and present a proof by (3.7).

Theorem 3.4. (13) If $V(d, A, b)$ is differentiable at $b$, then

$$V(d, A, b) = \frac{1}{d} \sum_{i=1}^{s} \frac{b_i}{\|a_i\|} V_i(d - 1, A, b).$$

(3.8)

Proof. Without loss of generality, we can suppose that all points in $D(b)$ are non-negative, i.e., $D(b) := \{x \in \mathbb{R}_+^d \mid Ax \leq b\}$. We first consider the case where each entry in $A$ is an integer. By (3.2) and (3.3), when $A$ is an integer matrix,

$$T(b|M) = \text{vol}(P) = V(d, A, b),$$
where $P := \{x \in \mathbb{R}_{+}^{d+s} \mid Mx = b\}$ and $M := (A, E_{s \times s})$. Let $e_i$ be the $s$-vector with 1 at the $i$th position and 0 for $j \neq i$. Using (3.7), we obtain

$$T(b|M) = \frac{1}{d} \sum_{i=1}^{s} b_i T(b|M \setminus e_i).$$

(3.9)

Note that the $(d-1)$-polytope $D_i(b)$ lies in a hyperplane $\{x \in \mathbb{R}^d \mid a_i \cdot x = b_i\}$ and that the $(n-1)$-dimensional volume of the unit parallelogram in the hyperplane is

$$\frac{\|a_i\|}{\gcd(a_{i1}, \ldots, a_id)} \quad \text{(cf. [1])},$$

where $a_{ij}$ is the $j$th entry in the vector $a_i$. So, one has

$$\frac{\text{vol}(D_i(b))}{\text{vol}(D_i(b))} = \frac{\|a_i\|}{\gcd(a_{i1}, \ldots, a_id)}.$$

(3.10)

By (3.3) and (3.10), we have

$$T(b|M \setminus e_i) = \frac{\text{vol}(D_i(b))}{\text{vol}(D_i(b))} = \frac{\text{vol}(D_i(b))}{\|a_i\|} = V_i(d-1, A, b)$$

Substituting $T(b|M) = V(d, A, b)$ and $T(b|M \setminus e_i) = \frac{V_i(d-1, A, b)}{\|a_i\|}$ into (3.9), we get (3.8). By taking limit, (3.8) holds for any matrix $A$.

4 Integration of polynomials over polytopes

In this section, we consider the problem of integrating of polynomials over polytopes. Since each polynomial can be written as the sum of monomials, we only consider the monomial case. For every $k = (k_1, \ldots, k_n) \in \mathbb{Z}_{+}^n$ and $M = (m_1, \ldots, m_n)$ we set

$$M^k := \left(\overbrace{m_1, \ldots, m_1}^{k_1+1}, \overbrace{m_2, \ldots, m_2}^{k_2+1}, \ldots, \overbrace{m_n, \ldots, m_n}^{k_n+1}\right).$$

The following theorem shows that the integration of monomials can be handled by the multivariate truncated power.

**Theorem 4.1.** Suppose $k = (k_1, \ldots, k_n) \in \mathbb{Z}_{+}^n$ and $f(u) = \prod_{j=1}^{n} u_j^{k_j}$. Set $P := \{u \in \mathbb{R}_{+}^n \mid Mu = x\}$. Then

$$T(x|M^k) = \frac{1}{k! \cdot \sqrt{\det(MM^T)}} \int_P f(u) du,$$

where $k! := k_1! \cdots k_n!$.
Proof. We set
\[ T_k(x|M) := \frac{1}{\sqrt{\det(M M^T)}} \int_P f(u) du \]
and consider its Laplace transform, i.e.,
\[ \hat{T}_k(\omega|M) := \int_{\mathbb{R}^n} \exp(-\omega \cdot x) T_k(x|M) dx. \]

For every \( u \) in \( P := \{ u \in \mathbb{R}_+^n | Mu = x \} \), we can write \( u \) in a unique way as \( u = z + y \) where \( z \in \ker(M) \) and \( y \perp \ker(M) \). Note that \( x = My \) and hence
\[ \sqrt{\det(M M^T)} dy = dx \] (see [6]). Then we have
\[
\int_{\mathbb{R}^n} \exp(-\omega \cdot x) T_k(x|M) dx \\
= \frac{1}{\sqrt{\det(M M^T)}} \int_{\mathbb{R}^n} \exp(-\omega \cdot x) \int_{\{ u \in \mathbb{R}_+^n \mid Mu = x \}} f(u) du dx \\
= \int_{y \perp \ker M} \exp(-\omega \cdot My) \int_{z \in \ker M} \chi_{\mathbb{R}_+^n}(z + y) f(z + y) dz dy \\
= \int_{y \perp \ker M} \int_{z \in \ker M} \chi_{\mathbb{R}_+^n}(z + y) f(z + y) \exp(-\omega \cdot M(z + y)) dz dy \\
= \int_{\mathbb{R}_+^n} f(u) \exp(-\omega \cdot Mu) du = k! \cdot \prod_{j=1}^n \frac{1}{(\omega \cdot m_j)^{k_j+1}}.
\]

Also, note that
\[ \hat{T}(\omega|M^k) = \prod_{j=1}^n \frac{1}{(\omega \cdot m_j)^{k_j+1}}. \]

Hence the theorem follows from the inverse theorem for Laplace transform. \( \square \)

5 Multivariate box splines and the volume of cube slicing

The multivariate box spline \( B(\cdot|M) \) associated with \( M \) is the distribution given by the rule (see [4, 5])
\[
\int_{\mathbb{R}^n} B(x|M) \phi(x) dx = \int_{[0,1]^n} \phi(Mu) du, \ \phi \in \mathcal{D}(\mathbb{R}^n).
\] (5.1)

According to [6], one has
\[
B(x|M) = \frac{\text{vol}_n(P \cap [0,1]^n)}{\sqrt{\det(M M^T)}},
\] (5.2)
where $P := \{ y \in \mathbb{R}^n_+ \mid My = x \}$. The formula (5.2) shows the connection between the box spline and the volume of cube slicing. Based on this connection, we can study some interesting problems concerning the unit cube.

Recall that $Q_n$ is the unit cube in $\mathbb{R}^n$. We define $H$ is an $n−1$ dimensional hyperplane of $\mathbb{R}^n$ through the center of $Q_n$, i.e.,

$$H := \{ y \in \mathbb{R}^n \mid a_1y_1 + \cdots + a_my_m = \sum_{i=1}^m a_i/2 \},$$

where $1 \leq m \leq n$, $a_i$ is a real number for $1 \leq i \leq m$. Based on the symmetry of $Q_n$, we can suppose $a_i > 0$ for $1 \leq i \leq m$. We set $A := (a_1, \ldots, a_m)$. Then from (5.2) we have

$$\text{vol}_n(H \cap Q_n) = \sqrt{\text{det}(AA^T)} B ((a_1 + \cdots + a_m)/2|A).$$

By the symmetry of the box spline, $B(x|A)$ achieves its maximum value at $(a_1 + \cdots + a_m)/2$. So, as stated before, Good’s conjecture is equivalent to

$$\max_x B(x|A) \geq \frac{1}{\sqrt{\sum_{i=1}^m a_i^2}}, \quad (5.3)$$

where $1 \leq m \leq n$. We next present a spline method for proving (5.3).

**Theorem 5.1.**

$$\max_x B(x|(a_1, \ldots, a_m)) \geq \frac{1}{\sqrt{\sum_{i=1}^m a_i^2}},$$

where $a_i$ are positive real numbers. The equality holds if and only if $m = 1$.

**Proof.** Set

$$C(x|A) := B(x + \sum_{i=1}^m a_i/2|A).$$

The Fourier transform of $C(x|A)$ is

$$\hat{C}(\omega|A) = \prod_{i=1}^m \frac{\sin(\omega a_i/2)}{\omega a_i/2}.$$ 

We can see that $\hat{C}(0|A) = 1$. According to the definition of Fourier transform, we conclude that

$$\int_0^\infty t^2 C(t|A)dt = -\frac{1}{2} \hat{C}''(0|A) = \frac{\sum_{i=1}^m a_i^2}{24}. \quad (5.4)$$
Put

\[ S(t) := \int_0^t C(x|A)dx. \]

By

\[ \left( \max_x t\ C(x|A) \right)^2 \geq S(t)^2 \]

we have

\[
\left( \max_x C(x|A) \right)^2 \int_0^\infty t^2 C(t|A) dt \geq \int_0^\infty S(t)^2 C(t|A) dt \\
= \frac{1}{3} \int_0^\infty dS(t)^3 dt = \frac{1}{24},
\]

where the last equality follows from

\[ \int_0^\infty C(x|A)dx = \frac{\hat{C}(0|A)}{2} = \frac{1}{2}. \]

We combine (5.4) and (5.5) to obtain

\[ \max_x B(x|A) = \max_x C(x|A) \geq \frac{1}{\sqrt{\sum_{i=1}^m a_i^2}}. \]

From (5.5), we see that the equality holds if and only if \( m = 1. \)

In the following theorem, we present an explicit formula for the volume of the \( j \)-slice of \( Q_n \). The formula given in [13] is the \((j = n - 1)\)-case in the following theorem.

**Theorem 5.2.** Suppose that \( M \) is an \((n - j) \times n\) matrix with \( \text{rank}(M) = n - j \) and let \( P := \{ y \in \mathbb{R}_+^n \mid My = x \} \). Denote \( \Xi = \{0, 1\}^n \) and \( |\varepsilon| = \sum_{i=1}^n \varepsilon_i \). Then we have

\[
\text{vol}_n(P \cap Q_n) = \frac{\sqrt{\det(\text{det}(M^TM))}}{j!} \sum_{Y \subseteq M \atop \#Y = \text{rank}(Y) = s} \alpha_Y |\det(Y)|^{-1} \sum_{\varepsilon \in \Xi} (-1)^{|\varepsilon|} (-\theta_Y \cdot (x - M\varepsilon))^j \chi_{\text{cone}(Y)}(x - M\varepsilon),
\]

for all \( c \) such that the denominators on the right-hand side do not vanish, where both \( \alpha_Y \) and \( \theta_Y \) are defined in Lemma 3.1.
Proof. By (5.2), one has
\[ \text{vol}_n(P \cap Q_n) = \sqrt{\det(\mathbf{M}_n^T)B(x|M)}. \]

Now we present an explicit formula for \( B(x|M) \). Recall that \( B(x|M) = \nabla M \nabla^T (x|M) \) (see [6]), where \( \nabla M := \prod_{i=1}^{n} \nabla_m \) and \( \nabla_m, T(x|M) = T(x|M) - T(x - m_i|M) \).

By the formula and the property of the difference, the following formula can be obtained (see [17]):
\[ B(\cdot|M) = \sum_{\varepsilon \in \Xi} (-1)^|\varepsilon|T(\cdot - M\varepsilon|M). \] (5.6)

The theorem is proved by putting (3.6) into (5.6). \( \square \)

6 Examples

Example 6.1. Set
\[ D(z) := \{ y \in \mathbb{R}^2_+ | y_1 + y_2 \leq z; -2y_1 + 2y_2 \leq z; 2y_1 - y_2 \leq z \}. \]

The volume of \( D(z) \) has been calculated in [13] using Cauchy’s Residue theorem. Here we can obtain it directly. Based on (5.6), we have \( \text{vol}(D(z)) = T(z|M) \) where
\[ M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \end{pmatrix}, \quad z = (z, z, z)^T. \]

We use \( m_i \) to denote the \( i \)th column in \( M \). A brief calculation shows that the cones spanned by the square matrices \( (m_1, m_2, m_4), (m_1, m_2, m_5), (m_1, m_3, m_4), (m_2, m_3, m_5) \) and \( (m_3, m_4, m_5) \) contain \( z \). We select \( c = (1, 1, 1, 1, 1/2) \) in Theorem 5.1 and obtain that \( T(z|M) = \frac{17}{48}z^2 \) which agrees with the result presented in [13].

Example 6.2. Set \( \Omega_d := \{ y \in \mathbb{R}^d_+ | \sum_{i=1}^{d} y_i \leq 1 \} \). We consider the problem of integrating of monomials over \( \Omega_d \), i.e.,
\[ J_d := \int_{\Omega_d} y_1^{k_1} \cdots y_d^{k_d} \, dy_1 \cdots dy_d. \]
The value of \( J_d \) is also calculated in [10, 21]. Based on Theorem 4.4, we can compute it easily. We set \( e_d := (1, \ldots, 1) \in \mathbb{Z}^d \). Using Theorem 4.4 we have

\[
J_d = k_1! \cdots k_d! \ T(1|e_d \subseteq \sum_{i=1}^d k_i +d+1).
\]

It is well known that \( T(x|e_d) = \frac{x^{d-1}}{(d-1)!} \). Hence, we have

\[
J_d = k_1! \cdots k_d! \ T(1|e_d \subseteq \sum_{i=1}^d k_i +d+1) = \frac{k_1! \cdots k_d!}{(\sum_{i=1}^d k_i + d)!}.
\]

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