Density Classification on Infinite Lattices and Trees

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Abstract. Consider an infinite graph with nodes initially labeled by independent Bernoulli random variables of parameter $p$. We want to find a (probabilistic or deterministic) cellular automaton or a finite-range interacting particle system that decides if $p$ is smaller or larger than $1/2$. Precisely, the trajectories should converge to the uniform configuration with only 0’s if $p < 1/2$, and only 1’s if $p > 1/2$. We present solutions to that problem on $\mathbb{Z}^d$, for any $d \geq 2$, and on the regular infinite trees. For $\mathbb{Z}$, we propose some candidates that we back up with numerical simulations.

Keywords: Cellular automata, interacting particle systems, density classification, percolation.

1 Introduction

Consider the configuration space $\{0,1\}^G$, where $G$ is a finite or countably infinite set with a group structure, the set of cells. We are interested in the density classification problem: given a configuration, decide in a decentralized way if it contains more 0’s or more 1’s. More precisely, the goal is to design a deterministic or random dynamical system on $\{0,1\}^G$, with local and homogeneous updating rules, whose trajectories converge to $0^G$, resp. $1^G$, if the initial configuration contains more 0’s, resp. more 1’s. To attack the problem, two natural instantiations of dynamical systems are considered, one with synchronous updates of the cells, and one with asynchronous updates. In the first case, time is discrete, all cells are updated at any time step, and the model is known as a Probabilistic Cellular Automaton (PCA) \cite{2}. A Cellular Automaton (CA) is a PCA in which the updating rule is deterministic. In the second case, time is continuous, cells are updated at random instants, at most one cell is updated at any given time, and the model is known as a (finite range) Interacting Particle System (IPS) \cite{15}.

The general spirit of the problem is that of distributed computing: gathering a global information by exchanging only local information. The challenge is two-fold: first, it is impossible to centralize the information (cells are indistinguishable); second, it is impossible to use classical counting techniques (cells contain only a binary information).
The density classification problem was originally introduced for rings of finite size \((G = \mathbb{Z}/n\mathbb{Z})\) and for synchronous models [16]. After experimentally observing that finding good rules to perform this task was difficult, it was shown that perfect classification with CA is impossible, that is, there exists no given CA that solves the density classification problem for all values of \(n\) [13]. This result however did not stop the quest for the best – although imperfect – models as nothing was known about how well CA could perform. The use of PCA opened a new path [5] and it was shown that there exist PCA that can solve the problem with an arbitrary precision [3].

The challenge is now to switch to infinite groups. First, we need to precise the meaning of “having more 0’s or more 1’s” in this context. Consider a random configuration on \(\{0, 1\}^G\) obtained by assigning independently to each cell a value 1 with probability \(p\) and a value 0 with probability \(1 - p\). A model “classifies the density” if the trajectories converge weakly to \(1^G\) for \(p > 1/2\), and to \(0^G\) for \(p < 1/2\). A couple of conjectures and negative results exist in the literature.

Density classification on \(\mathbb{Z}^d\) is considered in [1] under the name of “bifurcation”. The authors study variants of the famous voter model IPS [15, Ch. V] and they propose two instances that are conjectured to bifurcate. The density classification question has also been addressed for the Glauber dynamic associated to the Ising model at temperature 0, both for lattices and for trees [4,10,11]. The Glauber dynamic defines an IPS or PCA having \(0^G\) and \(1^G\) as invariant measures. Depending on the cases, there is either a proof that the Glauber dynamic does not classify the density, or a conjecture that it does with a proof only for densities sufficiently close to 0 or 1.

The density classification problem has been approached with different perspectives on finite and infinite groups, as emphasized by the results collected above. For finite groups, the problem is studied per se, as a benchmark for understanding the power and limitations of PCA as a computational model. The community involved is rather on the computer science side. For infinite groups, the goal is to understand the dynamics of specific models that are relevant in statistical mechanics. The community involved is rather on the theoretical physics and probability theory side.

The aim of the present paper is to investigate how to generalize the finite group approach to the infinite group case. We want to build models of PCA and IPS, as simple as possible, that correct random noises, even if the density of errors is close to 1/2. We consider the groups \(\mathbb{Z}^d\), whose Cayley graphs are lattices (Section 3), and the free groups, whose Cayley graphs are infinite regular trees (Section 4). In all cases, except for \(\mathbb{Z}\), we obtain both PCA and IPS models that classify the density. To the best of our knowledge, they constitute the first known such examples. The case of \(\mathbb{Z}\) is more complicated and could be linked to the so-called positive-rate conjecture [7]. We provide some potential candidates for density classification together with simulation experiments (Section 5).

For missing proofs and additional results, see [http://arxiv.org/abs/1111.4582](http://arxiv.org/abs/1111.4582)