Local solution to the $G_2$–monopole equation with prescribed tangent cone and $G_2$–structure

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Given a $G_2$–structure $(\phi, \psi)$, on the $G_2$–monopole equation

$$F_A \wedge \psi + \ast \phi (d_A u) = 0$$

the following theorem is true.

**Theorem 1.** Let $B_O(R) \subset \mathbb{R}^7$ be an arbitrary ball centred at the origin. For any smooth $G_2$–structure $(\phi, \psi)$ defined over $B_O(R)$, and any smooth $SO(m)$–bundle $\eta \to \mathbb{S}^6$ equipped with a Hermitian-Yang-Mills connection $A_0$, there exists a $G_2$–monopole which is defined in a smaller ball and asymptotic to $A_0$ exponentially at $O$.

**Remark 2.** Not every singular elliptic equation admits a local solution. For example, Brezis-Cabrè [1] showed that the equation $\Delta u = -\frac{u^2 + 1}{|x|^2}$ does not admit any solution defined near the origin. In contrast, our theorem says that the singular $G_2$–monopole equation is always locally solvable. In particular, for any smooth $G_2$–structure defined near $O$, it yields a $G_2$–monopole tangent to the canonical connection on $\mathbb{S}^6$ (see [2] and [14]). We hope this could help to construct $G_2$–instantons with point singularities on a closed 7–manifold.

**Remark 3.** We expect the local solution to be highly non-unique. There exists a solution whose exponential rate is arbitrarily close to 1 [see (11) and the discussion below (4)].

The monopole equation in $G_2$–setting first appeared in [4] by Donaldson-Segal. For highly-related later development on $G_2$ or other kinds of monopoles (instantons), we refer the interested readers to the work done by Sa Earp-Walpuski [10], Walpuski [12], Oliveira ([6], [7], [9]), Foscolo [5], Charbonneau-Harland [2], Xu [14], and the references therein.

**Proof of Theorem 7:** Near $O$, $(\phi, \psi)$ is a small perturbation of $[\phi(O), \psi(O)]$. Using a sophisticated version of the rescaling in page 6-9 of [3], we show in the following that Theorem 1 is a direct corollary of Theorem 1.13 in [13].

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Let the coordinate vector of \( B_O(R) \subset \mathbb{R}^7 \) be \( v = \begin{bmatrix} v_1 \\ \vdots \\ v_7 \end{bmatrix} \). All the balls below are centred at \( O \). By Lemma 3.7 in [11], there exists a linear transformation \( L \) such that under the new coordinate \( y = Lv, \phi(O) \) is the Euclidean \( G_2 \)-form i.e.

\[
\phi(O) = dy^{123} - dy^{145} - dy^{167} - dy^{246} + dy^{257} - dy^{347} - dy^{356}.
\] (2)

It suffices to work under the \( y \)-coordinate, under which \( \phi \) is defined in \( B(R_0) \) for some \( R_0 > 0 \) depending on \( R \) and \( L \). We bring in the bundle \( \eta \) as defined over \( S^6(1) \) (the unit sphere), and then view it as a bundle over \( \mathbb{R}^7 \setminus O \) pulled back from the natural spherical projection (Remark 2.3 in [13]). The connection \( A_0 \) is pulled-back to be a \( G_2 \)-instanton on \( \mathbb{R}^7 \setminus O \) with respect to \( \phi(O) \).

We write \( \phi = \Sigma_{ijk}\phi_{ijk}dy^{ijk} \). Let \( \Gamma \) denote the map \( x = \Gamma(y) = \lambda y \) from \( B(\frac{1}{\lambda}) \) to \( B(\frac{1}{4}) \). To the \( x \)-coordinate, exactly as in the previous paragraph, we can also pull back the bundles \( \eta, \text{ad}\eta \), and \( A_0 \) (denoted the same as in \( y \)-coordinate). Since they are objects on \( S^6 \), they are invariant under \( \Gamma \). Let

\[
\tilde{\phi} = \Sigma_{ijk}\Gamma^{-1,\ast}(\phi_{ijk})dx^{ijk} = \lambda^3 \Gamma^{-1,\ast}\phi, \quad \text{where} \ \phi_{ijk} \ \text{is the same as above.} \ (3)
\]

Let \( c_{\phi} \) denote \( C\Sigma_{ijk}|\phi_{ijk} - \phi_{ijk}(O)|c_{\phi}^{5|[B(\frac{1}{\lambda})]|} \), where \( C \) is a proper universal constant (which could be different in various context), and \( c_{\phi} \) means the \( C^5 \)-norm in \( y \)-coordinate. By chain-rule we have for any \( x \) that

\[
|\nabla_x^k(\Gamma^{-1,\ast}\phi_{ijk} - \phi_{ijk}(O))|(x) \leq \frac{c_{\phi}}{\lambda^k}, \quad \text{for all integer } k \in [1, 5] \text{ and } x \in B(\frac{1}{4}),
\]

\( \nabla_x \) is as below (5). Moreover,

\[
|(|\Gamma^{-1,\ast}\phi_{ijk} - \phi_{ijk}(O))|(x) = |\phi_{ijk} - \phi_{ijk}(O))|(y) \leq \frac{c_{\phi}}{\lambda} \quad \text{when } x \in B(\frac{1}{4}) \ (y \in B(\frac{1}{4\lambda})).
\]

Therefore

\[
|\tilde{\phi} - \tilde{\phi}(O)|c_{\phi}^{5|[B(\frac{1}{\lambda})]|} \leq \frac{c_{\phi}}{\lambda}. \quad C^5 \text{ means the } C^5 - \text{norm in } x\text{-coordinate.} \ (4)
\]

We actually have a \((\tilde{\phi}, \tilde{\psi})\)-monopole on \( B(\frac{1}{4}) \) of exponential rate arbitrarily close to 1. To see this, for any \( 1 > \theta > 0 \), choose \( \rho \in (-\frac{5}{2}, \theta - \frac{5}{2}) \) such that the condition in Definition 2.21 of [13] is satisfied. Let \( \delta_0 \) be small enough with respect to \( A_0 \) and \( \rho \), Theorem 1.13 in [13] (and the rate of convergence given by the proof of it) is directly applicable. Therefore, let \( \lambda \) be large enough such that \( \frac{c_{\phi}}{\lambda} < \frac{\delta_0}{5} \) and \( \frac{1}{\lambda} < \frac{\rho}{2} \), there exists a \((\tilde{\phi}, \tilde{\psi})\)-monopole \((A, \sigma)\) over \( B(\frac{1}{4}) \) i.e.

\[
F_A \wedge \tilde{\psi} + \ast_{\tilde{g}} d_A \sigma = 0 \text{ over } B(\frac{1}{4}), \ \tilde{g} \text{ is the metric of } \tilde{\phi}. \ (5)
\]
Moreover, let $\delta_0$ be even smaller if necessary, by the proof of Theorem 1.13 in section 5 of [13] (also see Definition 2.9 and Theorem 5.1 therein), $A$ is of exponential rate $1 - \theta$ and order 3 i.e.

$$|x|^{l+1}|\nabla^l_x (A - A_0)|(x) \leq |x|^{1-\theta},$$

where $|x|$ is just the usual norm of $x$, and $\nabla_x$ is the ordinary derivative (of the components of $A - A_0$) under the natural charts as in Definition 2.10 of [13] (of course here $\eta \to S^6$ and $A_0$ might be trivialized by more than 2 coordinate neighbourhoods, but this does not make any difference).

Pulling back both sides of (5) via $\Gamma$, we obtain

$$F_{A^*} \wedge (\lambda^4 \psi) + \Gamma^*[\star_{\tilde{g}} (d_A \sigma)] = 0 \text{ over } B\left(\frac{1}{4\lambda}\right), \quad A^* = \Gamma^* A. \quad (7)$$

Using

$$\Gamma^*[\star_{\tilde{g}} (d_A \sigma)] = \star_{\Gamma^* \tilde{g}} \Gamma^* (d_A \sigma), \quad \Gamma^* \tilde{g} = \lambda^2 g, \quad \text{and } \star_{\lambda^2 g} = \lambda^5 \star g \text{ (see Remark 4)}, \quad (8)$$

where $g$ is the metric of $\phi$, we obtain

$$F_{A^*} \wedge \psi + \star_{g} d_{A^*} (\lambda \sigma^*) = 0, \quad \sigma^* = \Gamma^* \sigma. \quad (9)$$

The pair $(A^*, \lambda \sigma^*)$ is the monopole we desire. Since $\Gamma^* A_0 = A_0$ (as a connection, see the paragraph above (3)), the estimate (6) means

$$|y|^{l+1}|\nabla^l_y (A^* - A_0)|(y) \leq \lambda^{l-\theta} |y|^{1-\theta},$$

where $l$ is as in (6), and $\nabla_y$ is as under (6) but in $y$-coordinate. Since $A_0$ is smooth on $S^6(1)$, we directly verify by (11) that

$$|y|^{l+1}|\nabla^l_{A_0} (A^* - A_0)|(y) \leq C \lambda^{l-\theta} |y|^{1-\theta}, \quad (11)$$

where $C$ is a constant depending only on $A_0$.

The proof of Theorem 1 is complete.

**Remark 4.** Under a fixed coordinate basis, for any $G_2$-structure $\phi$, the components of the co-associative form $\psi$ and the associated metric $g$ only depend on the components of $\phi$. Moreover, the dependence is via a composition only of power functions, fractions, and polynomials in terms of the components of $\phi$. Thus we directly verify (7) and (8).

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