A DECOMPOSITION FOR THE SCHRÖDINGER EQUATION WITH APPLICATIONS TO BILINEAR AND MULTILINEAR ESTIMATES

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Abstract. A new decomposition for frequency-localized solutions to the Schrödinger equation is given which describes the evolution of the wavefunction using a weighted sum of Lipschitz tubes. As an application of this decomposition, we provide a new proof of the bilinear Strichartz estimate as well as the multilinear restriction theorem for the paraboloid.

1. Introduction. This paper introduces a way of describing solutions to the Schrödinger equation, which bounds the evolution of the mass distribution of a solution using a weighted sum of Lipschitz tubes. There are several ways in which this description involving tubes differs from the existing wavepacket decomposition used in restriction theory. First, whereas the wavepacket decomposition breaks the solution up into tubes of width $R^{1/2}$ and length $R$, the tubes appearing in this ‘decomposition’ have unit width and can be made arbitrarily long. Second, the tubes involved here are curved instead of straight. Finally, the decomposition involves no cancellation. Whereas in the usage of the wavepacket decomposition it is often necessary to induct on scales in order to resolve possible cancellations between tubes, all pieces of the decomposition introduced here are positive so there is no opportunity for cancellation. The lack of cancellation to exploit and the relatively loose control of the shape of the tubes means that this decomposition seems unable to prove many dispersive inequalities used in the literature on the Schrödinger equation. However, the decomposition is useful enough to prove the bilinear Strichartz estimate and a special case of the multilinear restriction estimate.

Now let us set up the general situation for our results. Let $u: \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$ be a solution to the Schrödinger equation

$$i\partial_t u + \Delta u = 0$$

with initial condition $u(x, 0) = u_0(x) \in L^2(\mathbb{R}^d)$. For convenience we will write $u_t$ for the function $u_t(x) = u(t, x)$. If $u_t$ is localized in frequency space, so that the Fourier transform $\text{supp} \hat{u}_t \subset B_1$ is contained in the unit ball, then physical intuition about the Schrödinger equation suggests that $u_t$ has two important properties. First, because of the uncertainty principle, one expects that $u_t$ (and thus $|u_t|^2$) does not vary much (is “locally constant”) on unit scales. Moreover, because $u_t$ has bounded momentum, one expects that $u_t$ enjoys something akin to finite speed of

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propagation. Together these suggest that it is possible to describe the evolution of the mass $|u_t|^2$ in terms of discrete “packets” that travel along finite-speed paths. The point of this paper is to make this intuition precise.

Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a differentiable path, which might represent the motion of one such packet. We say that $\gamma$ has speed at most $V$ if $|\nabla \gamma| \leq V$. We define the tube of width $r$ centered at $\gamma$, written $T_{\gamma,r}$, by

$$T_{\gamma,r} := \{(x,t) \in \mathbb{R}^d \times \mathbb{R} : |x - \gamma(t)| \leq r\}.$$ 

We will use $T_{\gamma,r}$ and its indicator function interchangeably. Then our main result is the following decomposition.

**Theorem 1.1 (Skinny Lipschitz Tube decomposition).** For a given dimension $d$, there exists a radius $r$, a speed limit $V$, and a constant $C$ with the following property:

If $u_t$ solves the Schrödinger equation (1) with frequency localization $\text{supp} \hat{u}_0 \subset B_1$, then for any $R > 0$, there exists a countable collection of paths $\{\gamma_i\}$ with speed at most $V$ and weights $\{w_i\}$ such that

$$|u_t(x)|^2 \leq \sum_{i} w_i T_{\gamma_i,r}(x,t)$$

for any $(x,t) \in \mathbb{R}^d \times [-R,R]$, and moreover this bound is not too lossy on average, the sense that

$$\sum_{i} r^d w_i \leq C \int_{\mathbb{R}^d} |u_t|^2 \, dx.$$  

**Remark 1.** By rescaling the weights $w_i$ so that $\sum_{i} w_i = 1$, it is possible to think of them as probabilities. In this interpretation, $w_i$ is the probability that the particle represented by the wavefunction $u_t$ takes the path described by $\gamma$. However, since the paths are far from unique, this is not a very accurate description.

The proof of this theorem has two main ingredients. The first ingredient is a pair of inequalities which demonstrate two key properties of $u_t$. The first inequality says that $u_t$ is locally constant, in the sense that $|u_t(x)|^2$ can be bounded pointwise in terms of a weighted average centered at $x$. The second says that $|u_t|^2$ has finite speed. The statement of the finite speed property is slightly more complex, but suffice it to say that this is where we encode the “bounded momentum” intuition. The second ingredient of the proof is combinatorial. We state and prove an analogue of Theorem 1.1 in a discrete situation. The main idea here is to construct discrete paths one step at a time. The existence of these one-step paths is guaranteed by the Max-Flow Min-Cut theorem.

By itself, Theorem 1.1 is unsuitable for applications which require a variety of frequency localizations. This is easily fixed with an application of scaling and Galilean symmetries.

**Corollary 1.** Let $u_t$ be a solution to the Schrödinger equation with frequency localization $\text{supp} \hat{u}_0 \subset B_{\rho}(\xi)$, where $\xi \in \mathbb{R}^d$ is some average momentum, and $\rho$ is the frequency uncertainty. Let $r,V,C > 0$ be the same constants given in Theorem 1.1, and let $T > 0$ be arbitrary. Then there exists a set of paths $\{\gamma_i\}_{i=1}^{\infty}$ with $|\nabla \gamma_i - \xi| < V \rho$, and weights $\{w_i\}_{i=1}^{\infty}$ such that

$$|u_t(x)|^2 \leq \sum_{i} w_i T_{\gamma_i,r,T\rho}(x,t)$$
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and

\[ \sum_i (r^d w_i) \leq C \int_{\mathbb{R}^d} |u_t(x)|^2 dx. \]  (4)

Proof of Corollary 1 using Theorem 1.1. Consider the function

\[ u'(x,t) = e^{i(|\xi|^2/\rho^2 - \xi \cdot x/\rho)} u(x\rho^2 + t\xi - \xi^2/\rho), \]

which is still a solution to the Schrödinger equation (by Galilean and scale invariance), and which moreover satisfies the frequency localization \( \text{supp} \hat{u}' \subset B_1 \). Apply Theorem 1.1 to \( u' \) and then apply the scaling and Galilean symmetries again in reverse.

As mentioned above, Theorem 1.1 and Corollary 1 seem unsuitable for some applications. In particular, one apparent obstacle is that Theorem 1.1 provides no guarantee about the dispersive behavior of the tubes. It may be that there is a single tube which remains coherent for all time. Thus, Theorem 1.1 cannot by itself prove something such as the Strichartz inequality. However, there are examples of useful estimates in which dispersion does not play as large a role. The two we illustrate here are the bilinear Strichartz estimate and the multilinear restriction theorem.

1.1. Bilinear Strichartz. The bilinear Strichartz estimate was introduced by Bourgain in [4]. It handles the interaction between high-frequency and low-frequency solutions to the Schrödinger equation. To state it, let

\[ \mathcal{A}_N := \{\xi; N/2 \leq |\xi| \leq 2N\} \]

denote the annulus at scale \( N \).

Theorem 1.2 (Bilinear Strichartz [4]). Let \( M \ll N \), and let \( u_0, v_0 \in L^2(\mathbb{R}^d) \) have the frequency localizations \( \text{supp} \hat{u} \subset \mathcal{A}_N \), \( \text{supp} \hat{v} \subset \mathcal{A}_M \). Then

\[ \|u_t v_t\|_{L^2_x L^\infty_t(\mathbb{R}^d \times \mathbb{R})} \leq C \frac{M^{(d-1)/2}}{N^{1/2}} \|u_0\|_{L^2_x(\mathbb{R}^d)} \|v_0\|_{L^2_x(\mathbb{R}^d)}. \]  (5)

This estimate is one of the ingredients in the I-method, which has been used to prove global well-posedness for the nonlinear Schrödinger equation for rough data [5]. The first proof of the bilinear Strichartz estimate relied heavily on properties of the Fourier transform which do not easily generalize to non-Euclidean settings. Since the first proof by Bourgain, Tao has given a more physical-space proof in [13]. Moreover it seems that the wavepacket approach to the bilinear Strichartz estimate for the wave equation described in [11] might work for the Schrödinger equation as well. But to my knowledge, the only proof of the bilinear Strichartz estimate on a manifold is due to Hani [8]. Hani used this estimate and the I-method to prove a global well-posedness estimate for the cubic nonlinear Schrödinger equation in [9]. The approach in this paper to the bilinear Strichartz estimate is quite different from any mentioned above, and I hope that it may be extended to non-Euclidean settings such as hyperbolic space.
1.2. Multilinear restriction. The multilinear restriction estimate was first stated by Bennett, Carbery, and Tao [1]. It is a version of the restriction problem which is simplified to handle only quantitatively transverse interactions. Here we work with a very special case, since we only want to show how the decomposition of Theorem 1.1 can be used.

**Theorem 1.3** (Special case of multilinear restriction [1]). Let \( u_i \) be solutions to the Schrödinger equation for \( 0 \leq i \leq d \). Let \( r = (10^d)^{-1} \) and suppose that \( u_0 \) has frequency support \( \text{supp} \hat{u}_0 \subset B_r(0) \) and the \( u_i \) have frequency support \( \text{supp} \hat{u}_i \subset B_r(e_i) \), where \( e_i \) is any orthonormal basis for \( \mathbb{R}^d \). Then for every \( \varepsilon > 0 \) there is a constant \( C_\varepsilon > 0 \) such that

\[
\int_{|x|<R} \int_{|t|<R} \prod_{i=0}^{d} |u_i(t, x)|^{2/d} \leq C_\varepsilon R \prod_{i=0}^{d} \|u_i\|_{L_x^2}^{2/d}
\]

for any \( R > 0 \).

This kind of estimate has been used to make progress on the restriction problem [3] and more recently to prove the \( \ell^2 \) decoupling conjecture [2]. We remark that the main difficulty in proving the multilinear restriction is an incidence geometry problem about tubes (see [7] for a short discussion of the problem), and our only contribution is to provide a new derivation of the restriction estimate from the geometric problem. In particular this derivation avoids the use of induction on scales.

1.3. Plan of the paper. The paper is organized as follows. Section 2 contains a derivation of the locally constant and finite speed properties to frequency localized solutions. Section 3 proves a discrete analogue of the decomposition. In Section 4, the full proof of Theorem 1.1 is given. Section 5 proves the bilinear Strichartz estimate, Theorem 1.2, and the multilinear restriction estimate, Theorem 1.3, using the tube decomposition in Corollary 1.

2. Properties of frequency-localized solutions. In this section we prove variants of the locally constant (LC) and finite speed (FS) properties for solutions \( u_t \) to the Schrödinger equation satisfying

\[
\text{supp} \hat{u}_0 \subset B_1.
\]

2.1. Locally constant. We begin with the locally constant property. We say that a bounded function \( f \in L^\infty(\mathbb{R}^d) \) is localized to the unit ball if there exists some constant \( C \) such that \( |f(x)| \leq C \) and

\[
|f(x)| \leq C|x|^{-10d}, \quad |\nabla f(x)| \leq C|x|^{-10d-1}.
\]

**Proposition 1.** There exists some positive function \( \mu \) which is localized to the unit ball and such that for any function \( u_0 \) satisfying the frequency localization (7), and for any \( x \in \mathbb{R}^d \),

\[
\sup_{y \in B_1(x)} |u_0(y)|^2 \leq C \int |u_0(y)|^2 \mu(x-y) \, dy.
\]

Moreover, \( \mu \) can be chosen such that the translates of \( \mu \) form a partition of unity,

\[
\sum_{a \in \mathbb{Z}^d} \mu(x-a) = 1,
\]

\(^1\)In the statement of this theorem we drop the notation that \( u_t(x) = u(t, x) \).
and such that
\[ c|x|^{-10d} \leq \mu. \]  
\[ (9) \]

**Remark 2.** The extra conditions \((8)\) and \((9)\) can be tacked on by starting with some \(\mu''\) localized to the unit ball and then making it larger. At first glance this seems quite lossy; the point however is that in the proof we will use the bound \((\text{LC})\) only once, and the extra mass in \(\mu\) will allow us to arrive at a finite speed property (in the next subsection) which has no loss of constants, and which will be iterated many times.

**Remark 3.** The bound in \((9)\) could be tweaked in several ways. For the purposes of the next section, it would suffice to know that \(\mu\) is smooth (approximately locally constant on unit scales) and that
\[ \int_E |\partial_i \mu| \leq C \int_{\partial E} \mu \]
whenever \(E \subset \mathbb{R}^d\) satisfies \(E \cap B_1 = \emptyset\). This is a more general condition which may be useful in non-Euclidean settings, but the easiest way I know to verify it is to enforce that \(\mu\) has polynomial decay, hence the bound \((9)\).

**Proof.** Let \(\chi \in L^2(\mathbb{R}^d)\) satisfy \(\hat{\chi}(\xi) = 1\) for \(|\xi| \leq 1\) and \(\hat{\chi}(\xi) = 0\) outside \(|\xi| \geq 2\), with a smooth cutoff in between. By choosing the function appropriately, we can enforce that \(\chi\) localized to the unit ball. The Fourier localization of \(u_0\) yields \(u_0 = \chi \ast u_0\). This allows us to bound, using Cauchy-Schwartz,
\[ |u(x)|^2 = |\chi \ast u(x)|^2 = \left| \int u(y) \chi(x-y) \, dy \right|^2 \]
\[ \leq \left( \int |\chi(x-y)| \, dy \right) \int |u(y)|^2 |\chi(x-y)| \, dy \]
\[ = c_d \int |u(y)|^2 |\chi(x-y)| \, dy. \]

Now define the function \(\mu''\) by
\[ \mu''(y) = \sup_{|z-y| \leq 1} |\chi(z)|. \]
The fact that \(\chi\) is localized to the unit ball implies that \(\mu''\) is also, with a larger constant. We now modify \(\mu''\) to satisfy \((8)\) and \((9)\). As long as these modifications only increase \(\mu'\), we can ensure that \((\text{LC})\) is still satisfied.

First, since \(\mu''\) is localized to the unit ball, we can find some function \(\mu'\) which is smooth, is localized to the unit ball, has polynomial decay \(\mu' \geq c|x|^{-10d}\), and satisfies \(\mu'' \leq \mu'\). Moreover, each dilation \(\mu_{R'}(x) = \mu'(x/R)\) is still localized to the unit ball and satisfies \((9)\) with different constants. By making \(R\) large enough the periodic function
\[ p(x) = \sum_{a \in \mathbb{Z}^d} \mu(x - a) \]
becomes very smooth, so that \(\mu = \mu'(x)/p(x)\) which satisfies \((8)\) still obeys all previous bounds. \(\square\)
2.2. Finite speed. In this section we prove a version of the finite speed property for frequency localized solutions to the Schrodinger equation. We first establish some notation for the section. Define the set
\[ H := \{ h \in \mathbb{Z}^d; \max_{1 \leq i \leq d} |h_i| \leq 1 \}, \]
which has cardinality \(3^d\) and is the cube centered at the origin. Given any set \( A \subset \mathbb{Z}^d\), we let \( A + H \) denote the sumset
\[ A + H := \{ a + h \in \mathbb{Z}^d; a \in A, h \in H \}. \]
In other words, \( A + H \) consists of all lattice points in \( \mathbb{Z}^d\) which are within a distance 1 of \( A \) in the \( \ell^\infty\) norm. We also define \( \partial A = (A + H) \setminus A \), the lattice points that are within 1 of \( A \) but are not in \( A \).

We will also be using \( \mu \) and its translates from Proposition 1. For convenience, let \( \mu_a(x) = \mu(x - a) \) be the function \( \mu \) centered at the lattice point \( a \in \mathbb{Z}^d\). Moreover, for any set \( B \subset \mathbb{Z}^d\), we write \( \mu_B = \sum_{b \in B} \mu_b \).

Proposition 2. Let \( u_0 \) have frequency localization as in (7) and \( u_t = e^{it\Delta} u_0 \) be the free Schrodinger evolution of \( u_0 \). Moreover let \( \mu \) satisfy the conclusions of Proposition 1.

Then there exists some time \( \tau > 0 \) and such that for any \( A \subset \mathbb{Z}^d\) and \( 0 < t < \tau \),
\[ \int |u_t|^2 \mu_A \leq \int |u_0|^2 \mu_{A+H}, \quad \int |u_0|^2 \mu_A \leq \int |u_t|^2 \mu_{A+H}. \] (FS)

Remark 4. By time reversal symmetry, the equations in (FS) are equivalent, so we only focus on proving the first.

We will need a few lemmas in order to complete the proof of this proposition.

Lemma 2.1. There exists a constant \( C > 0 \) such that for any \( A \subset \mathbb{Z}^d \) and any \( 1 \leq j \leq d \),
\[ |\partial_j \mu_A| \leq C |\mu_{\partial A}|. \] (11)

Proof. Let \( x \in \mathbb{R}^d \), and let \( \text{dist}(x, \partial A) \) denote the minimum distance between \( x \) and a point in \( \partial A \). We split the analysis into cases. The first case occurs when \( x \) is near the boundary of \( A \), so that \( \text{dist}(x, \partial A) < 100d \). Then \( \mu_{\partial A}(x) = \Omega_d(1) \), and \( |\partial_j \mu_A| = O_d(1) \), so the bound holds for some constant.

The second case is when \( x \) is in the “exterior” of \( A \), so \( \text{dist}(x, A) > 100d \). Then we write, expanding the definition of \( \mu_A \), applying the triangle inequality, and using the fact that \( \mu \) is localized to the unit ball
\[ |\partial_j \mu_A| \leq C \sum_{a \in A} |x - a|^{-10d-1}. \]

We would like to turn this sum into an integral. Let \( Q(A) \subset \mathbb{R}^d \) denote the union of unit cubes centered at the points of \( A \). Since we are well away from the origin, the function \( |y|^{-10d-1} \) doesn’t change much on unit scales, so
\[ \sum_{a \in A} |x - a|^{-10d-1} \leq C \int_{Q(A)} |x - a|^{-10d-1} da. \]
Now observe that the integrand can be written as $-9d \partial_i (x_i |x|^{-10d-1})$, so by integrating by parts we obtain
\[
\int_{Q(A)} |x-a|^{-10d-1} \, da \leq c \int_{\partial Q(A)} |x-a|^{-10d} \, d\sigma(a),
\]
where $d\sigma$ denotes the surface measure on the exposed faces of the cubes. We can assign each face of an exposed cube to a point in $\partial A$, with each point in $\partial A$ being chosen at most $2^d$ times. Again because we are away from the origin, the function $|y|^{-10d}$ doesn’t change much on unit scales,
\[
\int_{\partial Q(A)} |x-a|^{-10d} \, d\sigma(a) \leq C \sum_{a \in \partial A} |x-a|^{-10d}.
\]
Applying (9) and chaining the inequalities we can conclude that
\[
|\partial_1 \mu_A| \leq C|\mu_{\partial A}|.
\]
Finally we must handle the case that $x$ is in the “interior” of $A$, so that $\text{dist}(x, A) < 100d$. This time the bound
\[
|\partial_1 \mu_A| \leq C \sum_{a \in A} |x-a|^{-10d-1}
\]
is much too lossy because of the cancellation involved. To exploit this cancellation, we observe that because $\mu$ forms a partition of unity,
\[
|\partial_1 \mu_A| = |\partial_j \mu_{A^c}|,
\]
where $A^c = \mathbb{Z}^d \setminus A$ is the complement of $A$. Now $\text{dist}(x, A^c) > 100d$, so apply the same argument as above.

**Lemma 2.2.** Let $K \in S(\mathbb{R}^d)$ be a Schwartz function (smooth and rapidly decaying). Then there exists $C > 0$, independent of $K$, such that for any $B \subset \mathbb{Z}^d$,
\[
|K * \mu_B(x)| \leq C \left( \int |K(y)|(1 + |y|^{10d}) \, dy \right) \mu_B(x) \tag{12}
\]

**Proof.** By the triangle inequality and translation invariance, it suffices to show that
\[
|K * \mu(x)| \leq C \mu(x).
\]
Now let $y \in \mathbb{R}^d$. We claim that
\[
\mu(x-y) \leq C(1 + |y|^{10d}) \mu(x). \tag{13}
\]
Again we split the analysis into cases. The first case occurs when $|x| \leq 50|y|$ and $|y| > 10$. Then $\mu(x) \geq c|50y|^{-10d}$, so
\[
\mu(x-y) \leq C \leq C|y|^{10d} \mu(x).
\]
The second case is $|x| \leq 50|y|$ and $|y| \leq 10$. Then both $\mu(x)$ and $\mu(x-y)$ are $\Theta(1)$. Finally, if $|x| > 50|y|$ then
\[
\mu(x-y) \sim \mu(x)
\]
because both are $\Theta(|x|^{-10d})$. This concludes the proof of the claim (13).

We use this claim to bound the convolution:
\[
|K * \mu(x)| = \left| \int K(y) \mu(x-y) \, dy \right| \leq \int |K(y)||\mu(x-y)| \, dy
\leq C \mu(x) \int |K(y)|(1 + |y|^{10d}) \, dy.
\]
Upon taking $\tau < 1$ and applying Lemma 2.1 followed by Lemma 2.2, we can bound the right hand side $V$ degree. The nodes represent a discretization of the space (in our case $\mathbb{R}^2$) come up with the following set-up. We have a digraph $G$ around according to Lipschitz paths. This can be discretized in space and time to $\mathbb{R}^3$. The discrete situation.

Proof of Proposition 2. Subtract $|u_0|^2 \mu_A$ from both sides of (FS), apply the fundamental theorem of calculus, and use the local conservation of mass

$$\partial_t |u_t|^2 = \partial_j \operatorname{Im}(u_t \partial_j u_t)$$

to note that it suffices to show

$$\int (|u_t|^2 - |u_0|^2) \mu_A = \int \left( \int_0^t \partial_s |u_s|^2 \, ds \right) \mu_A$$

$$= -\int \left( \int_0^t \partial_j \operatorname{Im}(\pi_j u_s) \, ds \right) \mu_A \leq \int |u_0|^2 \mu_{\partial A}.$$  

We may integrate the left hand side by parts. Moreover we use $u_s = e^{is\Delta} u = e^{is\Delta} (\chi * u) = (e^{is\Delta} \chi) * u$, and define $K_s = e^{is\Delta} \chi$. We can rewrite this inequality as

$$\operatorname{Im}\left( \int_0^t \int (K_s * \pi_0)(\partial_j K_s * u_0) \partial_j \mu_A \right) \leq \int |u_0|^2 \mu_{\partial A}.$$  

We now need a few facts about $K_s$ and $\partial K_s$. They are both concentrated in the ball $|x| \leq s$ and have rapidly decaying tails (the estimate on the tails is a repeated application of integration by parts). In particular, the quantities

$$\int |K_s(y)|(1 + |y|^{10d}) \, dy, \quad \int |\partial_j K_s(y)|(1 + |y|^{10d}) \, dy$$

are uniformly bounded near $s = 0$. We can therefore apply Cauchy-Schwartz several times to bound the left hand side by

$$\operatorname{Im}\left( \int_0^t \int (K_s * \pi_0)(\partial_j K_s * u_0) \partial_j \mu \right)$$

$$\leq \int_0^t \int (|K_s| * |u_0|)(|\partial_j K_s| * |u_0|)|\partial_j \mu_A|$$

$$\leq C \int_0^t \left( \int (|K_s| * |u_0|)^2 |\partial_j \mu_A| \right)^{1/2} \left( \int (|\partial_j K_s| * |u_0|)^2 |\partial_j \mu_A| \right)^{1/2}$$

$$\leq C \tau \sup_{0 < s < \tau, 1 \leq j \leq d} \left\{ \int |u_0|^2 |K_s| * |\partial_j \mu_A|, \int |u_0|^2 |\partial_j K_s| * |\partial_j \mu_A| \right\}.$$  

By applying Lemma 2.1 followed by Lemma 2.2, we can bound the right hand side by

$$C \tau \int |u_0|^2 |\mu_{\partial A}|.$$  

Upon taking $\tau < 1/C$ we are done. \hfill $\Box$

3. The discrete situation. In the continuous situation, we have an evolving mass distribution $|u_t|^2$ and we would like to understand it in terms of packets moving around according to Lipschitz paths. This can be discretized in space and time to come up with the following set-up. We have a digraph $G = (V, E)$ with bounded degree. The nodes represent a discretization of the space (in our case $V = \mathbb{Z}^d$), and edges represent possible movements of mass within a timestep (in our case $(a, b) \in \mathbb{Z}^{2d}$ is an edge if $|a - b|_{\ell^\infty} \leq 1$). Instead of an evolving mass distribution

\footnote{In our notation, a digraph has directed edges but it is allowed for both the edges $(u, v)$ and $(v, u)$ to exist, and there is no requirement that $u \neq v.$}
|u_{i}|^2$ there is a sequence of mass distributions $w_i : V \to \mathbb{R}^+$ for $i = 1, 2, \cdots, N$. We say a sequence of vertices $p = (p(1), \cdots, p(N))$ is a path of length $N$ if $(p(i), p(i+1))$ is an edge for each $1 \leq i < N$. The set of all such paths is denoted $\mathcal{P}_N$. We want to describe this evolution of mass as a sum of packets that each move according to different paths.

**Question 1.** Is it possible to find a weighting $\alpha : \mathcal{P}_N \to \mathbb{R}^+$ on the set of paths $\mathcal{P}_N$ of length $N$ such that for all $1 \leq i \leq N$, and all vertices $v \in V$,

$$w_i(v) = \sum_{p \in \mathcal{P} : p(i) = v} \alpha(p)?$$

Sometimes it is impossible to come up with such a weighting. For a trivial example, there may be no edges in $E$ at all, and so there can be no paths. The only allowable mass distribution is the trivial one, $w_i = 0$. A less trivial constraint is that $w_i$ must satisfy a conservation of mass. Indeed, observe that

$$\sum_{v \in V} w_i(v) = \sum_{v \in V} \sum_{p \in \mathcal{P} : p(i) = v} \alpha(p) = \sum_{p \in \mathcal{P}} \alpha(p)$$

is independent of $i$. In fact, $w_i$ must satisfy a *local* conservation law. To state it, we make a few definitions. For $A \subset V$ define $N^+(A)$, the outgoing neighborhood of $A$, to be the set

$$N^+(A) := \{v \in V ; (a, v) \in E \text{ for some } a \in A\}.$$

Similarly define the incoming neighborhood,

$$N^-(A) := \{v \in V ; (v, a) \in E \text{ for some } a \in A\}.$$

By an abuse of notation we will write $N^+(v)$ instead of $N^+(\{v\})$ when $A$ consists of a single vertex $v$.

The key property here is that if $p$ is a path and $p(i) \in A$, then $p(i+1) \in N^+(A)$ (and $p(i-1) \in N^-(A)$). Thus,

$$\sum_{a \in A} w_i(a) = \sum_{p(i) \in A} \sum_{p(i+1) \in N^+(A)} \alpha(p) = \sum_{b \in N^+(A)} \sum_{b \in N^-} \alpha(p) = \sum_{b \in N^+(A)} w_i(b).$$

Likewise

$$\sum_{a \in A} w_i(a) \leq \sum_{b \in N^-} w_{i-1}(b).$$

The answer to our question above is that these local conservation laws are not only necessary but also sufficient.

**Proposition 3.** Let $G = (V, E)$ as above, and let $w_i : V \to \mathbb{R}^+$ be a sequence of positive weights on $V$. Suppose that for every $A \subset V$, and $1 \leq i < N$ the local conservation law holds,

$$\sum_{a \in A} w_i(a) \leq \sum_{b \in N^+(A)} w_{i+1}(b), \quad (14)$$

and in addition the global conservation law holds,

$$\sum_{v \in V} w_1(v) = \sum_{v \in V} w_N(v). \quad (15)$$
Then there exists a weighting \( \alpha : \mathcal{P}_N \to \mathbb{R}^+ \) on the set of paths of length \( N \) such that for every \( 1 \leq i \leq N \) and every \( v \in V \),
\[
w_i(v) = \sum_{p(i)=v} \alpha(p). \tag{16}
\]

First we prove the Proposition in the case \( N = 2 \). Then we deduce the case of general \( N \) from the \( N = 2 \) case.

3.1. **The case** \( N = 2 \). When \( N = 2 \), Proposition 3 is a consequence of the Max-Flow Min-Cut (MFMC) theorem. An introduction to the MFMC theorem can be found online, for example see [10, 12]. The theorem is originally due to Ford and Fulkerson [6]. Usually this theorem is stated in the case of a finite network, but here our vertex set \( V \) is infinite. In general this can be problematic, but the weights we consider are absolutely summable so there is no real difficulty.

Now we set up notation so that we may state the MFMC theorem. Let \( U \) be a set of nodes with two distinguished elements, \( s \) and \( t \), which represent the source and sink respectively. Let \( c : U \times U \to \mathbb{R}^+ \). We say \( c \) is a capacity function if \( c(u, v) > 0 \) implies \( c(v, u) = 0 \). That is, if we think of \( c \) as being the capacity of a network of pipes connecting the nodes of \( U \), the pipes must only go in one direction. A cut is a subset \( S \subset U \) such that \( s \in S \) and \( t \notin S \). Given a capacity function \( c \), we define the capacity of the cut \( S \), written \( \text{cap} S \), by
\[
\text{cap} S = \sum_{u \in S, v \notin S} c(u, v).
\]

The dual object of a cut is a flow. A flow is a function \( f : U \times U \to \mathbb{R}^+ \) satisfying the Kirchoff’s laws
\[
f_{\text{in}}(v) := \sum_{u \in U} f(u, v) = \sum_{u \in U} f(v, u) =: f_{\text{out}}(v)
\]
for all \( v \in U \setminus \{s,t\} \). The value of a flow is given by
\[
\text{val} f = \sum_{u \in U} f(s, u).
\]

Because \( f \) satisfies Kirchoff’s laws except on \( s \) and \( t \), we may also write
\[
\text{val} f = \sum_{u \in U} f(u, t).
\]

The MFMC theorem relates the maximum value of a flow to the minimum capacity of a cut.

**Theorem 3.1** (Max-Flow Min-Cut [6]). Let \( U \) be a finite vertex set with source \( s \in U \) and sink \( t \in U \), and let \( c : U \times U \to \mathbb{R}^+ \) be a capacity on \( U \). Then there exists a flow \( f : U \times U \to \mathbb{R}^+ \) with \( f \leq c \) such that
\[
\text{val} f = \min_{S \subset U} \text{cap} S,
\]
where the minimum ranges over cuts \( S \subset U \) with \( s \in S \) and \( t \notin S \).

Notice that flows, which are weights on edges, can also be thought of weights on the set of paths of length 2. This provides the main connection between MFMC and the \( N = 2 \) case of Proposition 3. To make this connection more explicit, we consider a graph with a single source vertex \( s \), two ‘copies’ \( V_1 = V \times \{1\} \) and \( V_2 = V \times \{2\} \).
of the original vertex set $V$, and a sink vertex $V$. The source vertex $s$ flows into $V_1$ with capacities determined by the weights $w_1$, and these in turn flow into $V_2$ according to the connectivity of the graph $G$, and finally $V_2$ flows into the sink $t$ with capacities determined by $w_2$. Precisely, let $U = \{s, t\} \cup V_1 \cup V_2$ be the vertex set of our flow network, with capacity $c : U \times U \to \mathbb{R}^+$ defined by

$$c(u, u') = \begin{cases} w_1(u) & \text{if } u = s, u' \in V_1 \\ +\infty & \text{if } u \in V_1, u' \in N^+(u) \\ w_2(u) & \text{if } u \in V_2, u' = t \\ 0 & \text{else.} \end{cases} \quad (17)$$

If we could apply MFMC directly to $U$ with capacity function $c$, then the local conservation laws (14) and (15) would ensure that there is a suitable flow.

Unfortunately $U$ as described may be infinite. We therefore need to apply MFMC to finite subnetworks of $U$. Let $A \subset V$ be a finite subset of $V$, and let $U_A = \{s, t\} \cup A_1 \cup N^+(A_1) \subset U$, where we think of $A_1 \subset V_1$ and $N^+(A_1) \subset V_2$. We can also define the capacity $c_A = c|_{U_A}$, which is simply the restriction of $c$ to $U_A$. We can apply MFMC to the network described by the nodes $U_A$ and capacity $c_A$ to obtain the following result.

**Lemma 3.2.** Let $A \subset V$, and let $U_A$ and $c_A$ be as described above. Then there exists a flow $f_A : U_A \times U_A \to \mathbb{R}^+$ such that $f_A \leq c_A$ and

$$f_A(s, u) = w_1(u)$$

for every $u \in A$.

**Proof.** Consider the flow network on $U_A$ with capacity $c_A$. Let $S \subset U_A$ be a cut. We can write $S = \{s\} \cup S_1 \cup S_2$, where $S_1 \subset A_1$ and $S_2 \subset N^+(A)$. Now we write $\text{cap} S$ using the definition (17) of $c$

$$\text{cap} S = \sum_{u \in A_1 \setminus S_1} c(s, u) + \sum_{u \in S_1} \sum_{v \in N^+(S_1) \setminus S_2} c(u, v) + \sum_{v \in S_2} c(v, t)$$

$$= \sum_{u \in A_1 \setminus S_1} w_1(u) + \sum_{u \in S_1} \sum_{v \in N^+(S_1) \setminus S_2} w_1(v) + \sum_{v \in S_2} w_2(v).$$

Define a subset $S'_1 \subset S_1$ by

$$S'_1 := \{u \in S_1; N^+(u) \subset S_2\}.$$ 

If $u \in S_1 \setminus S'_1$, then there is some $v \in N^+(u)$ such that $v \notin S_2$, so that $c(u, v) = w_1(u)$ contributes $w_1(u)$ to the sum in $\cap S$. Moreover we have $S_2 \subset N^+(S'_1)$, so

$$\text{cap} S \geq \sum_{u \in A_1 \setminus S'_1} w_1(u) + \sum_{v \in N^+(S'_1)} w_2(v) \geq \sum_{u \in A_1 \setminus S'_1} w_1(u) + \sum_{u \in S'_1} w_1(u) + \sum_{u \in A_1} w_1(u)$$

where we have used the local conservation law (14) with $i = 1$. The expression on the RHS is exactly $\text{cap}(\{s\})$, so by MFMC there exists a flow $f_A$ on $U_A$ with $f_A \leq c_A$ such that

$$\text{val} f_A = \sum_{u \in A} w_1(u).$$
On the other hand, since \( f_A(s, u) \leq c(s, u) = w_1(u) \),

\[
\text{val } f_A = \sum_{u \in A} f_A(s, u) \leq \sum_{u \in A} w_1(u).
\]

Thus \( f_A(s, u) = w_1(u) \) for all \( u \in A \), as desired. \( \square \)

A compactness and limiting argument allows us to make a conclusion about the entire network \( U \). Formulated in slightly different notation, we obtain the case of Proposition 3.

**Lemma 3.3.** Let \( G = (V, E) \) be as in Proposition 3, and let \( w_1 \) and \( w_2 \) be two weights on \( V \) satisfying (14) with \( i = 1 \) and (15) for \( N = 2 \). Then there exists a function \( f : E \to \mathbb{R}^+ \) on the edges such that

\[
w_1(v) = \sum_{u \in N^+(v)} f(v, u)
\]

and

\[
w_2(v) = \sum_{u \in N^-(v)} f(u, v).
\]

**Proof.** Let \( A_k \subset V \) be an increasing sequence of sets, \( A_1 \subset A_2 \subset \cdots \), such that for every \( v \in V \) there is some \( k \) such that \( v \in A_k \). Let \( U_A \) be the network described above. For each \( k \), Lemma 3.2 provides some flow \( f_k \) such that \( f_k \leq c \) and \( f_k(s, u) = w_1(u) \) when \( u \in V \). We show that, up to subsequence, \( f_k \) converges to a flow \( f \) on \( U_A \) in \( \ell^1 \). This is done by a compactness argument.

Indeed, let \( \varepsilon > 0 \). We will prove the existence of a subsequence \( k(1), k(2), \cdots \) such that

\[
\sum_{(u, u') \in U \times U} |f_{k(i)}(u, u') - f_{k(j)}(u, u')| < \varepsilon
\]

for all \( i, j \geq 0 \). First choose \( K \) so large that

\[
\sum_{u \in V \setminus A_K} w_1(u) \leq \varepsilon,
\]

which is possible since \( w_1 \) is absolutely summable and \( A_k \) is an increasing sequence which converges to \( V \).

There are finitely many pairs \((u, u') \in U_{A_K} \times U_{A_K} \), so by compactness we can find a sequence \( k(1), k(2), \cdots \) such that

\[
\sum_{(u, u') \in U_{A_K} \times U_{A_K}} |f_{k(i)}(u, u') - f_{k(j)}(u, u')| \leq \varepsilon
\]

for all \( i, j \). Now we write

\[
\sum_{(u, u') \in U \times U} |f_{k(i)}(u, u') - f_{k(j)}(u, u')|
\]

\[
= \sum_{u \in A_K} |f_{k(i)}(s, u) - f_{k(j)}(s, u)| + \sum_{u \in A_K \cap V_2} |f_{k(i)}(u, v) - f_{k(j)}(u, v)|
\]

\[
+ \sum_{v \in N^+(A_K)} |f_{k(i)}(v, t) - f_{k(j)}(v, t)| + \sum_{v \in A_K \cap V_2} |f_{k(i)}(u, v) - f_{k(j)}(u, v)|.
\]
We bound each term separately. Because of (19) the last term is bounded by \( \varepsilon \).

The first term is bounded by \( 2\varepsilon \) due to the triangle inequality, the fact that \( f \leq c \), and the definition of \( A_K \). The next two terms are actually identical to the first, once you apply the triangle inequality and the Kirchoff’s laws for \( f_{k(i)} \) and \( f_{k(j)} \). We have therefore shown (18).

A standard diagonalization argument now shows the existence of a subsequence, which we will not relabel explicitly, which is Cauchy in \( \ell^1 \). More precisely, this means for every \( \varepsilon > 0 \) there exists \( M \) such that when \( n, m > M \),

\[
\sum_{(u,u') \in U \times U} |f_n(u, u') - f_m(u, u')| < \varepsilon.
\]

By the completeness of \( \ell^1 \), this converges in \( \ell^1 \) to some function \( f : U \times U \to \mathbb{R}^+ \).

This convergence is strong enough to ensure that \( f \) is still a flow, and moreover \( f(s,u) = w_1(u) \) for all \( u \in V \). This, combined with the Kirchoff’s laws, implies that

\[
w_1(u) = \sum_{v \in N^+(u)} f(u, v)
\]

for all \( u \in V \). Moreover, Kirchoff’s laws and the global conservation law (15) ensure

\[
\sum_{u \in V} w_1(u) = \text{val } f = \sum_{v \in V_2} f(v, t) \leq \sum_{v \in V} w_2(v) = \sum_{u \in V} w_1(v).
\]

Since \( f(v, t) \leq w_2(v) \) pointwise, this equality enforces \( f(v, t) = w_2(v) \) for all \( v \in V \). We have therefore shown that \( f \) has the desired properties.

\[\square\]

3.2. The case \( N > 2 \). This short section just shows how to iterate the \( N = 2 \) case to complete the proof of Proposition 3.

\textbf{Proof of Proposition 3.} We start with a simple observation. By (14), since \( N^+(V) \subset V \),

\[
\sum_{v \in V} w_1(u) \leq \sum_{v \in V} w_2(v) \leq \cdots \leq \sum_{v \in V} w_N(v).
\]

But by (15) all of these inequalities are actually equalities.

Now we consider the graph on the vertices \( V_{[N]} = V \times [N] \), where \((u,i) \in V_{[N]} \) is connected to \((v, i+1) \in V_{[N]} \) if \( v \in N^+(u) \). Each edge can be uniquely assigned a triple \((i, u, v)\) where \( 1 \leq i < N \) and \( v \in N^+(u) \). Denote by \( E_{[N-1]} \) the set of such triples. We write \( V_{[N]} = V_1 \cup V_2 \cup \cdots \cup V_N \) where \( V_i = V \times \{i\} \). Similarly we write \( E_{[N-1]} = E_1 \cup E_2 \cup \cdots \cup E_{N-1} \), where each \( E_i \) has the first coordinate of its triple equal to \( i \). We would like to assign a weighting function \( f : E_{[N-1]} \to \mathbb{R}^+ \) to each of these edges such that for each \( 1 \leq i < N \) and \( v \in V \),

\[
w_i(u) = \sum_{v \in N^+(u)} f(i, u, v), \tag{20}\]

and for each \( 1 < i \leq N \),

\[
w_i(v) = \sum_{u \in N^-(v)} f(i - 1, u, v). \tag{21}\]

This can be done by finding, for each \( 1 \leq j < N \), a flow \( f_j : E_i \to \mathbb{R}^+ \) such that (20) is satisfied for \( i = j \) and (21) is satisfied for \( i = j + 1 \). But the existence of \( f_j \) is exactly what is guaranteed by Lemma 3.3. Now simply define \( f \) on \( E_{[N-1]} \) by \( f|_{E_j} = f_j \).
Finally we use $f$ to define a weight on each path. We approach this with a probabilistic interpretation. Divide all weights $w_i$ and the flow $f$ by the normalizing factor $\sum_{v \in V} w_1(v)$ to obtain new weights $\tilde{w}_i$ and a new flow $\tilde{f}$ which still obeys (20) and (21). Now consider the following random process: let $\nu_1 \in V$ be a random variable with probability distribution given by $\tilde{w}_1$. Given $\nu_k \in V$, independently choose a $\nu_{k+1} \in N^+(\nu_k)$ with probability distribution given by

$$P(\nu_{k+1} = v|\nu_k) = f(k, \nu_k, v)/w_k(\nu_k).$$

Notice that for this probability to make sense we would need

$$1 = P(\nu_{k+1} \in N(\nu_k)|\nu_k) = \frac{1}{w_k(\nu_k)} \sum_{v \in N(\nu_k)} f(k, \nu_k, v),$$

which is given by (20). Once $\nu_i$ is chosen for all $i \in [N]$, we have a randomly constructed path $\rho \in \mathcal{P}_N$. Let $\tilde{\alpha}(p) := P(\rho = p)$ be the probability distribution of $p$.

Now observe that for any $v \in V$, by the construction of $\rho$ and $\tilde{\alpha}$,

$$\tilde{w}_1(v) = P(\nu_1 = v) = P(\rho(1) = v) = \sum_{p(1) = v} \tilde{\alpha}(p),$$

which up to the normalizing factor of $\sum_{v \in V} w_1(v)$ establishes (16) for $i = 1$. By way of induction, suppose (16) is true for some $1 \leq i < N$. Then for any $v \in V$,

$$P(\nu_{i+1} = v) = \sum_{u \in N^{-}(v)} P(\nu_i = u \text{ and } \nu_{i+1} = v)$$

$$= \sum_{u \in N^{-}(v)} P(\nu_i = u)P(\nu_{i+1} = v|\nu_i = u)$$

$$= \sum_{u \in N^{-}(v)} \tilde{w}_i(u)(\tilde{f}(i, u, v)/\tilde{w}_i(u)) = \sum_{u \in N(v)} \tilde{f}(i, u, v) = \tilde{w}_{i+1}(v)$$

by (21). But of course we can also write

$$P(\nu_{i+1} = v) = P(\rho(i + 1) = v) = \sum_{p(i+1) = v} \tilde{\alpha}(p).$$

Thus by induction, and undoing the normalization, we have constructed our desired $\alpha$. \hfill \Box

4. The skinny Lipschitz tube decomposition. In this section we demonstrate how to combine the bounds from Section 2 and the discrete problem from Section 3 to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u_t$ be a solution to the Schrodinger equation with frequency localization (7). Let $\mu$ be given by Proposition 1, $H$ defined as in (10), and $\tau > 0$ given by Proposition 2. We will discretize to the lattice $\mathbb{Z}^d \times (\tau \mathbb{Z}) \subset \mathbb{R}^d \times \mathbb{R}$; for convenience we will reserve the letters $a, b$ for general elements of $\mathbb{Z}^d$, $h$ will always denote an element of $H$, and $x, y$ will always be used for generic elements of $\mathbb{R}^d$.

We define for $n \in \mathbb{Z}$ the weight function $m : \mathbb{Z}^d \times (\tau \mathbb{Z}) \to \mathbb{R}^+$ by

$$m(a, nt) = \int |u_{nt}|^2 \mu_{a + H}.$$
First observe that $m$ satisfies a conservation law since the translates of $\mu$ form a partition of unity:

$$\sum_{a \in \mathbb{Z}^d} m(a, n\tau) = \int |u_{n\tau}|^2 \mu_{a+H} = |H| \int |u_{n\tau}|^2 \mu_{H}$$

$$= |H| \int |u_{n\tau}|^2 = |H| \|u_{0}\|_{L^2}^2. \quad (22)$$

Now we would like to show that $m$ also inherits a finite speed property from (FS). Let $A \subset \mathbb{Z}^d$ and let $n \in \mathbb{Z}$. We would like to show that

$$\sum_{a \in A} m(a, n\tau) \leq \sum_{b \in A + H} m(b, (n+1)\tau). \quad (23)$$

Expand the expression on the left using the definition of $m$:

$$\sum_{a \in A} m(a, n\tau) = \int |u_{n\tau}|^2 \mu_{a+H} = \int |u_{n\tau}|^2 \sum_{a \in A} \sum_{h \in H} \mu_{a+h}.$$  

A particular $a' \in A + H$ may contribute several times in the above integral. Let $E_{\geq i} = \{a' \in A + H; |A \cap (a' - H)| \geq i\}$, so that in other words $a' \in E_{\geq i}$ means there exists at least $i$ distinct choices of $h \in H$ such that $a' - h \in A$. Then we can write the above sum a different way, obtaining

$$\sum_{a \in A} m(a, n\tau) = \sum_{i=1}^{|H|} \int |u_{n\tau}|^2 \mu_{E_{\geq i}}.$$  

Now apply the finite speed bound (FS),

$$\sum_{a \in A} m(a, n\tau) \leq \sum_{i=1}^{|H|} \int |u_{(n+1)\tau}|^2 \mu_{E_{\geq i} + H}.$$  

Similarly, if we define $F_{\geq i} = \{b' \in A + 2H; |(A + H) \cap (b' - H)| \geq i\}$, then we can write

$$\sum_{a \in A} m(a, (n+1)\tau) = \sum_{i=1}^{|H|} \int |u_{(n+1)\tau}|^2 \mu_{F_{\geq i}}.$$  

Thus, to prove (23) it would suffice to show $E_{\geq i} + H \subset F_{\geq i}$. Indeed, if $b \in E_{\geq i} + H$ then $b = a' + h$ for some $a' \in E_{\geq i}$ and $h \in H$, so by the definition of $E_{\geq i}$,

$$i \leq |\{(a, h') \in A \times H; a' = a + h'\}|$$

$$= |\{(a, h') \in A \times H; b = (a + h')\}|$$

$$\leq |\{(c, h') \in (A + H) \times H; b = c + h'\}|$$

This last inequality implies $b \in F_{\geq i}$. We have therefore shown (23).

We are nearing a situation where we can apply Proposition 3. Let $R > 0$, and let $N = \lceil 2R/\tau \rceil$ (rounded to the next even number for convenience). We consider the graph on $G = (\mathbb{Z}^d, E)$ on $\mathbb{Z}^d$ where $(a, b) \in E$ if and only if $a \in b + H$. Define the sequence of weights $w_i : \mathbb{Z}^d \to \mathbb{R}^+\text{ for } 1 \leq i \leq N$ by

$$w_i(a) = m(a, (i - N/2)\tau).$$
Now (23) implies (14), and the conservation of mass from (22) implies (15). Thus by Proposition 3 there is a weight \( \alpha: P_N \to \mathbb{R}^+ \) on the set of paths of length \( N \) on \( G \) such that

\[
m(a, (i - N/2)\tau) = \sum_{p(i)=a} \alpha(p).
\]

Let \( p \) be one such path. Choose \( r = 10d \), and let \( \gamma_p \) be the piecewise linear path with \( \gamma_p((i - N/2)\tau) = p(i) \). By taking \( V = C_d/\tau \), we obtain the velocity bound \( |\nabla \gamma_p| \leq V \). Consider the tube decomposition function

\[
f(x, t) = \sum_{p \in P_N} \alpha(p) T_{\gamma_p, r}(x, t).
\]

Now we verify that \( f(x, t) \) is a pointwise upper bound for \( |u_t(x)|^2 \). Let \( a \in \mathbb{Z}^d \) and \( 1 \leq i \leq N \). Suppose that \( (x, t) \in \mathbb{R}^d \times \mathbb{R} \) is contained in the prism with bottom-left corner \((a, (i - N/2)\tau)\), meaning that \( a_j \leq x_j < a_j + 1 \), \((i - N/2)\tau \leq t < (i - N/2 + 1)\tau\) for \( 1 \leq j \leq d \). Now let \( p \in P_N \) be some path with \( p(i) = a \). Then because we chose \( r \) large enough, and because \( \gamma_p((i - N/2)) = p(i) = a \), it follows that \((x, t) \in T_{\gamma_p, r}\).

Using this observation we can make the bound

\[
f(x, t) = \sum_{p \in P_N} \alpha(p) T_{\gamma_p, r}(x, t) \geq \sum_{p(i)=a} \alpha(p) = m(a, (i - N/2)\tau).
\]

Now by (LC) and (FS), there exists some absolute constant \( C \) such that

\[
|u_t(x)|^2 \leq C \int |u_t|^2 \mu_a \leq C \int |u_{(i-N/2)\tau}|^2 \mu_{a+H} = C m(a, n\tau).
\]

Thus for every \((x, t)\) with \(|t| < R\),

\[
|u_t(x)|^2 \leq C \sum_{p \in P_N} \alpha(p) T_{\gamma_p, r}(x, t),
\]

which is exactly (with slightly different letters) the bound (2).

Now we verify (3). Indeed, by (22),

\[
\sum_{p \in P_N} (10d)^d \alpha(p) \leq C \sum_{a \in \mathbb{Z}^d} w_1(a) = C |H| \|u_0\|_{L^2}^2.
\]

5. Applications to bilinear and multilinear estimates. In this section we prove the bilinear Strichartz estimate, Theorem 1.2 and the multilinear restriction theorem, Theorem 1.3 using the tube decomposition. Neither result is new, but I think that these proofs illustrate a few interesting points. Perhaps one of the main lessons is that neither the bilinear Strichartz nor the multilinear restriction estimates are truly dispersive. Indeed observe that the decomposition described in Theorem 1.1 does not rule out the possibility that there is only one tube. Thus by itself it cannot imply any dispersive estimates.
5.1. **Proof of bilinear Strichartz.** In this part we prove Theorem 1.2 using the decomposition given by Theorem 1.1. I believe this proof illustrates something intuitive about the estimate — roughly speaking the intuition is that a fast and a slow particle cannot be in the same place for very long. This intuition is expressed in the proof by observing that a “fast” and “slow” tube must have a small region of intersection.

Recall the definition of the annulus,

\[ A_N := \{ \xi; N/2 \leq |\xi| \leq 2N \}. \]

We define in addition the annulus

\[ A^*_N := \{ \xi; N/4 \leq |\xi| \leq 4N \}. \]

**Proof of Theorem 1.2.** Let \( u_0, v_0 \in L^2(\mathbb{R}^d) \) have frequency localizations \( \text{supp } \hat{u}_0 \subset A_N \) and \( \text{supp } \hat{v}_0 \subset A_M \) with \( M \ll N \).

Let \( V \) be the constant appearing in Theorem 1.1 (which is the same constant appearing in Corollary 1), and let \( \{ \xi_i \}_{i=1}^{100V^d} \subset A_1 \) be a set of points such that the balls with radius \( 1/(10V) \) centered at \( \{ \xi_i \} \) cover \( A_1 \) and are contained in \( A^*_1 \):

\[ A_1 \subset \bigcup_{i=1}^{100V^d} B_{1/10V}(\xi_i) \subset A^*_1. \]

Using a partition of unity on (a scaled version of) this covering, write the decompositions

\[ u_0 = \sum_{i=1}^{100V^d} u_i, \quad v_0 = \sum_{i=1}^{100V^d} v_i \]

such that \( \text{supp } \hat{u}_i \subset B_{N/(10V)}(N\xi_i) \) and \( \text{supp } \hat{v}_i \subset B_{M/(10V)}(M\xi_i) \). We can also demand that these decompositions are nearly orthogonal in the sense that

\[ \sum_{i=1}^{100V^d} \|u_i\|_{L^2_{t,x}}^2 \leq 2\|u\|_{L^2_{t,x}}^2 \]

and likewise with \( v_i \). Using this decomposition and the Cauchy-Schwartz inequality, we obtain

\[ \|uv\|_{L^2_{t,x}(\mathbb{R}^d \times \mathbb{R})} \leq (100V)^d \sum_{i=1}^{100V^d} \sum_{j=1}^{100V^d} \|u_i v_j\|_{L^2_{t,x}(\mathbb{R}^d \times \mathbb{R})}. \]

Now we bound these cross-terms \( \|u_i v_j\|_{L^2_{t,x}(\mathbb{R}^d \times \mathbb{R})} \). We will show that, for any time limit \( R \), the inequality

\[ \|u_i v_j\|_{L^2_{t,x}(\mathbb{R}^d \times [-R,R])} \leq CN^{-1}M^{-1-1} \|u_i\|_{L^2_{t,x}} \|v_j\|_{L^2_{t,x}} \]

holds with a constant independent of \( R \). Since this bound is uniform in \( R \) we can conclude a global in time estimate.

From Corollary 1 we obtain paths \( \{ \gamma_n \}_{n=1}^{\infty} \) with weights \( \{ w_n \}_{n=1}^{\infty} \) such that for all \( |t| < R \),

\[ |u_i(x,t)|^2 \leq \sum_{n=1}^{\infty} w_n T_{\gamma_n,rN^{-1}}(x,t). \]
For convenience we will write $T_n$ instead of $T_{\gamma_n \times N^{-1}}$. We also use Corollary 1 on $v_j$ to obtain paths $\{\gamma'_m\}_{m=1}^\infty$ with weights $\{w'_m\}_{m=1}^\infty$ such that

$$|v_j(x,t)|^2 \leq \sum_{m=1}^\infty w'_m T'_m(x,t)$$

where $T'_m(x,t) = T_{\gamma'_m \times M^{-1}}$, and $|t| < R$. Applying these pointwise bounds, we estimate

$$\int_{-R}^R \int_{\mathbb{R}^d} |u_i(x,t)|^2 |v_j(x,t)|^2 \, dx \, dt \leq \sum_{m=1}^\infty \sum_{n=1}^\infty w_n w'_m \int_{-R}^R \int_{\mathbb{R}^d} T_n(x,t) T'_m(x,t) \, dx \, dt.$$  

(25)

Now the paths $\gamma_n$ and $\gamma'_m$ defining $T_n$ and $T'_m$ have velocities satisfying

$$|\nabla \gamma_n - N \xi_i| \leq N/10$$

and

$$|\nabla \gamma'_m - M \xi_j| \leq M/10.$$

Since $\xi_i, \xi_j \in A_1$, we can conclude that $\nabla \gamma_n \subset A^*_N$ and $\nabla \gamma'_m \subset A^*_M$. Thus the tubes $T_n$ and $T'_m$ can meet at most once, in a region of spatial width $N^{-1}$ and only for a duration of $(MN)^{-1}$. That is,

$$\int_{\mathbb{R}^d} T_n(x,t) T'_m(x,t) \, dx \, dt \leq CN^{-1-d} M^{-1}.$$  

Plugging this back into (25) and then applying (4),

$$\|u_i v_j\|_{L^2_{t,x}(\mathbb{R}^d \times \mathbb{R})}^2 \leq C \sum_{m=1}^\infty \sum_{n=1}^\infty N^{-1-d} M^{-1} w_n w'_m$$

$$\leq C N^{-1} M^{d-1} \left(\sum_{m=1}^\infty M^{-d} w'_m\right) \left(\sum_{n=1}^\infty N^{-d} w_n\right)$$

$$\leq C N^{-1} M^{d-1} \|u_i\|_{L^2_t}^2 \|v_j\|_{L^2_t}^2.$$  

Notice that the constants here are independent of $R$ as guaranteed by the statement of Corollary 1. The choice of the time limit $R$ does significantly affect the number of tubes used in the decomposition, but it does not affect the quality of the bounds.

Finally applying the almost orthogonality of the decomposition allows us to conclude

$$\sum_{i=1}^{(100V)^d} \sum_{j=1}^{(100V)^d} \|u_i v_j\|_{L^2_{t,x}(\mathbb{R}^d \times \mathbb{R})}^2 \leq C N^{-1} M^{d-1} \left(\sum_{i=1}^{(100V)^d} \|u_i\|_{L^2_t}^2\right) \left(\sum_{j=1}^{(100V)^d} \|v_j\|_{L^2_t}^2\right)$$

$$\leq 2CN^{-1} M^{d-1} \|u\|_{L^2_t}^2 \|v\|_{L^2_t}^2.$$  

The theorem is finished by taking square roots. \(\square\)

5.2. Multilinear restriction. We can also prove the multilinear restriction estimate using the multilinear Kakeya problem for Lipschitz tubes, proved by Guth in [7]. First let us set up some notation. Let $\{v_i\}_{i=1}^n \subset \mathbb{R}^n$ be a collection of unit vectors. We say that $\{v_i\}$ is $\nu$-transverse if $|v_1 \wedge v_2 \wedge \cdots \wedge v_n| \geq \nu$. Let $\Gamma_i = \{\gamma_{i,j}\}_{j=1}^{N_i}$ be a collection of Lipschitz curves. Given $\delta > 0$, we say that $\Gamma_i$ is $\delta$-close to parallel with $v_i$ if for any tangent vector $\tilde{t}$ of a curve $\gamma_{i,j}$, $|\tilde{t} - v_j| < \delta$. Let $T_{i,j}$ denote the tube of unit radius centered along the curve $\gamma_{i,j}$.
Lemma 5.2. For every Schrödinger equation which are highly localized in frequency.

\[ \text{Theorem 5.1 (Multilinear Kakeya for Lipschitz tubes [7])} \]

\[ \text{For every } \varepsilon > 0 \text{ and } \nu > 0, \text{ there is a } C_{\varepsilon, \nu} > 0 \text{ and a } \delta \text{ such that the following holds: If } \{v_i\} \subset \mathbb{R}^n \text{ is a set of unit vectors that is } \nu\text{-transverse, and each } \Gamma_i = \{\gamma_{i,j}\}_{j=1}^{N_i} \text{ is a set of Lipschitz paths which are } \delta\text{-close to } v_i, \text{ and } w_{i,j} \text{ is any positive collection of weights, then} \]

\[ \int_{B_R} \prod_{i=1}^{n} \left( \sum_{j=1}^{N_i} w_{i,j} T_{i,j} \right)^{\frac{1}{n-1}} \leq C_{\varepsilon} R^\nu \prod_{i=1}^{n} \left( \sum_{j=1}^{N_i} w_{i,j} \right)^{\frac{1}{n-1}}. \]  \hspace{1cm} (26)

This theorem implies a version of multilinear restriction for solutions to the Schrödinger equation which are highly localized in frequency.

Lemma 5.2. For every \( \varepsilon > 0 \) there exists a \( C_{\varepsilon} \) and a \( \delta > 0 \) such that the following holds: Let \( r = (10d)^{-1} \) and suppose \( \{\xi_i\}_{i=0}^{\infty} \) is a set of vectors with \( \xi_0 \in B_r(0) \) and \( \xi_i \in B_r(v_i) \) for \( 1 \leq i \leq d \). If also \( u_i \) are solutions to the Schrodinger equation with \( \sup \dot{u}_i \in B_\delta(\xi_i) \), then

\[ \int |x| < R \int |t| < R \sum_{i=0}^{d} |u_i(t,x)|^{2/d} dt \leq C_{\varepsilon} R^\nu \sum_{i=0}^{d} \|u_i\|_{L_x^2}^{2/d} \]

for all \( R > 0 \).

\textbf{Proof.} Consider the vectors \( v'_i = (1, -2\xi'_i) \in \mathbb{R}^{d+1} \), and let \( v_i = v'_i/|v'_i| \). The vectors \( \{v_i\} \) correspond to the unit normals to the paraboloid \( P = \{\tau = |\xi|^2\} \subset \mathbb{R}^{d+1} \) at the points \((|\xi|^2, \xi)\). Because of the constraint on the positions of \( \xi_i \), we can guarantee that the set \( \{v_i\} \) is \( \nu\)-transverse for some absolute constant \( \nu > 0 \).

Let \( \varepsilon > 0 \) and let \( C_{\varepsilon} \) and \( \delta \) be given according to Theorem 5.1 with this constant \( \nu \) and with \( n = d + 1 \). Let \( V \) be the same constant that appears in Theorem 1.1, and define \( \delta = \delta/V \).

Now if \( \sup \dot{u}_i \subset B_\delta(\xi_i) \), then according to Corollary 1, it is possible to find paths \( \{\gamma_{i,j}\}_{j=1}^{\infty} \) such that \( |\nabla \gamma_{i,j} - \xi_i| < \delta \) and weights \( \{w_{i,j}\} \) such that

\[ |u_i(t,x)|^2 \leq \sum_j w_{i,j} T_{\gamma_{i,j},r \delta^{-1}}(t,x) \]

and

\[ \sum_j (r \delta^{-1})^d w_{i,j} \leq C \|u_i\|_{L_x^2}. \]

From now on we will simply write \( T_{i,j} \) for the tube \( T_{\gamma_{i,j},r \delta^{-1}} \) centered at \( \gamma_{i,j} \) with width \( r \delta^{-1} \). Thinking of \( \gamma_{i,j} \) as a curve in \( \mathbb{R}^{d+1} \), where the first coordinate is time and the remaining \( d \) coordinates are spatial, the condition \( |\nabla \gamma_{i,j} - \xi_i| < \delta \) ensures that the curves \( \{\gamma_{i,j}\} \) are \( \delta\)-close to \( v_i \). Since the radius of the tubes \( T_{i,j} \) is \( r \delta^{-1} \), which is much larger than 1, we may by splitting the tubes into thinner pieces think of them as having radius 1 (buying simplicity at the cost of an ignorable factor of \( \delta^\nu \)). Thus we may apply Theorem 5.1 to obtain

\[ \int |x| < R/2 \int |t| < R/2 \prod_{i=0}^{d} |u_i(t,x)|^{2/d} dt \leq \int_{B_R} \prod_{i=0}^{d} \left( \sum_{j=1}^{\infty} w_{i,j} T_{i,j} \right)^{1/d} \]

\[ \leq C_{\varepsilon} R^\nu \prod_{i=0}^{d} \left( \sum_{j=1}^{\infty} w_{i,j} \right)^{1/d} \leq C_{\varepsilon} R^\nu \prod_{i=0}^{d} \|u_i\|_{L_x^2}^{2/d}. \]
The multilinear restriction theorem, Theorem 1.3, follows upon decomposing each of the \( d + 1 \) solutions into finitely many pieces with frequency localizations given by \( \delta \), and then summing the finitely many contributions.

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