Abstract. We extend a quantized skew Howe duality result for Type A algebras to orthogonal types via a seesaw. We develop an operator commutant version of the First Fundamental Theorem of invariant theory for $U_q(\mathfrak{so}_n)$ using a double centralizer property inside a quantized Clifford algebra. We obtain a multiplicity-free decomposition of tensor powers of the $U_q(\mathfrak{so}_{2n})$ spin representation by explicitly computing joint highest weights with respect to an action of $U_q(\mathfrak{so}_{2n}) \otimes U'_q(\mathfrak{so}_m)$. Clifford algebras are an essential feature of our work: they provide a unifying framework for classical and quantized skew Howe duality results that can be extended to include orthogonal algebras of types BD.

1. Introduction

In this article we develop skew Howe duality for orthogonal types, both classical and quantum. The results in this article extend those developed in [Abo22]. Thus we recommend the reader to review [Abo22].

The classical situation is as follows. Let $G$ and $H$ be reductive subgroups of a complex orthogonal group, such that $G$ is the centralizer of $H$ and vice versa. We will refer to $G$ and $H$ as a dual reductive pair. We will restrict the projective spin representation of $O(2nm)$ to $G \times H$ and decompose it into irreducibles. More precisely, we will construct a seesaw [Bum04, Proposition 38.4], as depicted in the following diagram.

\[
\begin{array}{c}
S^\otimes m \cong \Lambda(C^{nm}) \cong \Lambda(C^n)^\otimes m \\
\downarrow \boxed{\Lambda(C^{nm})} \\
O(2nm) \\
\end{array}
\]

\[
\begin{array}{c}
O(2n) \\
\downarrow \\
GL(n) \\
\end{array} \quad \quad \quad \\
\begin{array}{c}
O(2m) \\
\downarrow \\
GL(m) \\
\end{array}
\]

\[
\begin{array}{c}
SO(n) \\
\end{array} \quad \quad \quad \\
\begin{array}{c}
SO(m) \\
\end{array}
\]

(1.1)

Here $\Lambda(C^{nm})$ is the $O(2nm)$, or rather the $Pin(2nm)$, spin module. The group $Pin(2nm) \subset Cl(C^{nm} \otimes (C^{nm})^*)$ is the double (spin) cover of $O(2nm)$. Dashed lines indicate commuting embeddings. In the third row, we need the full orthogonal group, instead of the special orthogonal group, in order to ensure a multiplicity-free decomposition of $\Lambda(C^{nm})$.

We are mainly interested in the quantum case. Quantum groups are more closely related to enveloping algebras than Lie groups, so we work at the Lie algebra level throughout. We use the commuting embeddings of [Abo22, Section 2.2] to construct explicit homomorphisms as described by the following

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We may think of the algebra \( so_{2p} \), defined in 3.4, as the Lie algebra for the full pin group \( Pin(\mathbb{C}^{2p}) \cong Spin(\mathbb{C}^{2p}) \times \mathbb{Z}_2 \), which is a double-cover of \( O(\mathbb{C}^{2p}) \). We assume any space of the form \( V \oplus V^* \) is equipped with the canonical symmetric bilinear form arising from the duality pairing between \( V \) and \( V^* \). The exterior algebras are isomorphic as modules of the Clifford algebra \( Cl(\mathbb{C}^n \oplus \mathbb{C}^m) \cong Cl(\mathbb{C}^{nm}) \).

The group homomorphisms \( GL(p) \rightarrow SO(2p) \), for any \( p \), which induce the Lie algebra embeddings \( gl_p \rightarrow so_{2p} \), are key. They allow us to identify certain \( so_{2p} \)-generators with \( gl_p \)-generators, so we can use our work from [Abo22] to prove analogous duality results for orthogonal groups. The induced map \( gl_p \rightarrow so_{2p} \) has a quantum analogue embedding \( U_q(gl_p) \rightarrow U_q(so_{2p}) \), which helps transfer our skew \( U_q(gl_n) \otimes U_q(gl_m) \)-duality [Abo22, Theorem 3.17] to a result for Types BD.

The inclusion maps \( O(p) \hookrightarrow GL(p) \) are also important. They result from the restriction of \( GL(p) \) to a subgroup of invertible isometries and they induce Lie algebra embeddings \( so_p \hookrightarrow gl_p \). These maps allow us to identify \( so_p \)-generators with elements of \( gl_p \) and base our constructions on the results of [Abo22]. However, the embeddings \( so_p \hookrightarrow gl_p \) have no quantum analogue \( U_q(so_p) \hookrightarrow U_q(gl_p)! \)

Since there is no algebra map \( U_q(so_p) \rightarrow U_q(gl_n) \), in the quantum case we must instead work with the non-standard deformation \( U'_q(so_m) \) of \( so_m \). The non-standard deformation \( U'_q(so_m) \) may be understood as a quantization of the compact form of \( so_m \) that is compatible with the embeddings \( U'_q(so_m) \supset U'_q(so_{m-1}) \supset \cdots \supset U'_q(so_3) \) and whose modules can be constructed explicitly using bases indexed by Gelfand-Tsetlin patterns as in the classical case [Gel97]. The algebra \( U'_q(so_m) \) was first introduced in [GK91] and its representation theory has been studied in e.g. [IK05, IK00] and most recently in [Wen20b]. See also the references therein. Letzter studied the co-ideal algebra structure of \( U'_q(so_m) \) in [Let00,Let19].

It is a remarkable feature of the quantum case in the orthogonal setting that the commutant of the \( U_q(so_m) \) spin action is not generated by a Drinfeld-Jimbo quantum group. Instead, the centralizer is described in terms of \( U'_q(so_m) \), which we realize as a co-ideal subalgebra of \( U_q(gl_m) \). While [KL08] obtains some triangular decomposition for \( U'_q(so_m) \), this non-standard deformation does not support an analogue of an “upper triangular” Borel subalgebra. This means that the method of identifying commuting actions of simple root vectors in terms of Clifford algebra operators, which succeeded in deriving a \( U_q(gl_n) \otimes U_q(gl_m) \)-duality result via embeddings into a quantized Clifford algebra in [Abo22], does not directly carry over to the orthogonal setting. However, since we realize \( U'_q(so_m) \) as a bona fide subalgebra of \( U_q(gl_m) \) using explicit formulas in terms of the standard \( U_q(gl_m) \) generators, we can still use the results of [Abo22] to obtain a quantized duality theorem for Types BD.

In Section 4 we construct embeddings of \( U_q(so_{2p}) \) and \( U'_q(so_m) \) into the quantum Clifford algebra \( Cl_q(\mathbb{C}^{nm}) \) to develop the seesaw depicted in Diagram (1.3). The quantized Clifford algebra was first defined by Hayashi in [Hay90]. In this article we use a similar version due to Kwon [Kwo14]. In [AS22] we study a more general version in depth and obtain new results on the algebraic structure and representation theory of quantized Clifford algebras, including a description of the center calculation, a factorization as a tensor product, and a complete list of irreducible representations.
Much like in the classical case, our embeddings rely crucially on our skew Howe duality result for Type A [Abo22, Theorem 3.17].

\[
\Lambda_q(V^{(n)}) \otimes \Lambda_q(V^{(m)}) = S_q^{\otimes m} \cong \Lambda_q(V^{(nm)}) \\
\Lambda_q(V^{(n)}) \otimes \Lambda_q(V^{(nm)}) \\
U_q(\mathfrak{so}_{2n}) \quad U_q(\mathfrak{so}_{2m}) \\
U_q(\mathfrak{gl}_n) \quad U_q(\mathfrak{gl}_m) \\
U_q'(\mathfrak{so}_n) \quad U_q'(\mathfrak{so}_m)
\]

(1.3)

In the top row we have isomorphisms of $U_q(\mathfrak{so}_{2n})$-modules. To obtain multiplicity-free decompositions, we extend the $U_q(\mathfrak{so}_{2n})$-action to the full orthogonal quantum group $U_q(\mathfrak{so}_{2n})$, recalled in Definition 4.7.

The algebra $U_q(\mathfrak{so}_n)$ is a semidirect product $U_q(\mathfrak{so}_n) \rtimes \mathbb{Z}_2$ that serves as the quantum analogue of $O(n)$.

Braided exterior algebras were first defined in [BZ05] as $U_q(\mathfrak{g})$-module analogues of the classical exterior algebras. In [Abo22, Section 3.1] we recall the construction of $\Lambda_q(V^{(n)})$, including $Cl_q(n)$-module structure, in detail.

Contrary to the classical situation, no representation of the quantized Clifford algebra $Cl_q(p) \to \text{End} \left( \Lambda_q(V^{(p)}) \right)$ is faithful. This phenomenon explains how we may obtain non-commuting homomorphic images of $U_q(\mathfrak{so}_{2n})$ and $U_q'(\mathfrak{so}_n)$ inside $Cl_q(nm)$ that nevertheless induce commuting actions on the $Cl_q(nm)$-module $\Lambda_q(V^{(nm)})$.

There are related results in the literature. The dual reductive pair $(U_q'(\mathfrak{so}_n), U_q(\mathfrak{so}_{2m}))$ appears in [ST18]. The authors prove their duality result by developing webs and diagrammatical categories. The argument used in [ST18] is quite different from ours, which is based on $q$-Clifford algebras. In addition, the results in [ST18] do not include a theory for the $U_q(\mathfrak{so}_n)$-spin module.

However, the recent pre-print [Wen20a] indeed discusses a duality result for tensor powers of the quantum spinor module. The author discovered this pre-print while this work was in preparation. While Wenzl does use the quantized Clifford algebra in [Wen20a], his work does not include an explicit joint highest weight vector calculation and it does not comment on the nature of the duality in relation to $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-duality. In [Wen20a], Wenzl obtains an action of $U_q'(\mathfrak{so}_n)$ on $S_q^{\otimes m}$ by quantizing the canonical $O(n)$-invariant element in $\mathbb{C}^n \otimes \mathbb{C}^n$. In contrast, in this work we realize $U_q'(\mathfrak{so}_m)$ as a co-ideal subalgebra of $U_q(\mathfrak{gl}_m)$ and we use our results from [Abo22]. In addition, this work benefits from a new understanding of the representation theory of quantized Clifford algebras, including a calculation of their center, which is developed in [AS22].

Various papers deal with the classical situation. For instance, [HK22, GGL22] discusses skew duality results in the classical symplectic case. In addition, [NPS21] studies skew Howe duality for various classical reductive dual pairs.

We note an important application of our results. With the skew quantum Howe duality results of this article in hand, we may describe generators of a braid group representation on $\text{End}_{U_q'(\mathfrak{so}_{2n})}(S_q^{\otimes m})$ arising from the braiding on the category of finite-dimensional modules over $U_q(\mathfrak{so}_{2n})$ explicitly in terms of quantum Clifford algebra operators. In other words, we use our skew quantum Howe duality results to construct solutions of the Yang-Baxter equation that centralize the $U_q(\mathfrak{so}_{2n})$-action on $\Lambda_q(V^{(nm)})$. These solutions and associated braid group representations have been considered by Rowell and Wang, and by Rowell and Wenzl, in [RW11] and [RW17]. They describe the quantum computation model based on metaplectic anyons [HNW13,HNW14]. We hope the explicit description made possible by our
duality Theorem 4.19 paves the way for a detailed study of the associated braid representations that can be used to prove the conjectures in [CW15] regarding the universality of the quantum computation model based on metaplectic anyons.

To sum up, this article develops \textit{operator commutant versions} of the First Fundamental Theorem of invariant theory, in the spirit of [How95, Section 4.3.4], for the quantum group $U_q(\mathfrak{so}_n)$ and its spin module. The Section 3 recalls relevant details of the classical case and Section 4 develops new results in the quantum setting in three steps. First, Section 4.1 explains how the spin action factors through a quantized Clifford algebra. Then Section 4.2 obtains an action of a coideal subalgebra $U'_q(\mathfrak{so}_m)$ in $U_q(\mathfrak{gl}_m)$ that commutes with the $U_q(\mathfrak{so}_n)$-action on tensor powers $S^\otimes m$ of its spin module. Finally, Section 4.3 achieves a multiplicity-free decomposition of $S^\otimes m$ by constructing joint highest weights with respect to the action of $U_q(\mathfrak{so}_n) \otimes U'_q(\mathfrak{so}_m)$.

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\section{Notation and conventions}
In this article we use the notation fixed in [Abo22, Section 2]. For convenience and concreteness, we recall that if $\mathfrak{g}$ is a complex semisimple Lie algebra there exists a unique non-degenerate symmetric invariant bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ such that

\begin{equation}
\langle H_i, H_j \rangle = d_j^{-1} a_{ij}, \quad \langle H_i, E_j \rangle = \delta_{ij} = 0, \quad \langle E_i, F_j \rangle = 0, \quad \text{and} \quad \langle E_i, F_j \rangle = d_i^{-1} \delta_{ij},
\end{equation}

for all $i, j$ [Kac83, Theorem 2.2]. When $\mathfrak{g} = \mathfrak{gl}_n$ we take $\langle \cdot, \cdot \rangle$ to be the non-degenerate trace bilinear form of the natural representation. The form is normalized so that

\begin{equation}
\langle \alpha, \alpha \rangle = 2
\end{equation}

for short roots. In the same spirit we record some relevant Cartan matrices:

\begin{equation}
A_n = \begin{bmatrix}
2 & -1 & & \\
-1 & 2 & -1 & \\
& \ddots & \ddots & \\
& & -1 & 2 & -1 \\
& & & 2
\end{bmatrix},
\end{equation}

\begin{equation}
B_n = \begin{bmatrix}
A_{n-1} & 0 & & \\
& \ddots & \ddots & \\
& & 0 & 0 \\
0 & \cdots & 0 & -2 & 0
\end{bmatrix}, \quad \text{and} \quad D_n = \begin{bmatrix}
A_{n-1} & 0 & & \\
& \ddots & \ddots & \\
& & 0 & 0 & -1 \\
& & & 0 & 0
\end{bmatrix}.
\end{equation}

We label the matrices by the Lie type of the root system to which they are associated. The corresponding diagonal root lengths matrices are described by $d = (1, \ldots, 1)$ for the root systems of types $A_n, D_n$, and by $d = (2, \ldots, 2, 1)$ for type $B_n$. 

\begin{equation}
\end{equation}

\begin{equation}
\begin{bmatrix}
A_{n-1} & 0 & & \\
& \ddots & \ddots & \\
& & 0 & 0 \\
& & & 0 & 0
\end{bmatrix}.
\end{equation}
3. The classical case

In this section we develop orthogonal duality theory in the classical case. In particular, we (re)-prove the classical $O(n) \times SO(m)$-duality Theorem 3.6, this time using a double centralizer property inside a Clifford algebra. This theorem is well-known to experts but our method lays the foundations for our treatment of the more difficult quantum case in Section 4.

We prove Theorem 3.6 in three steps. First, in Section 3.1 we show that for any complex vector space $V$ there are actions of $\mathfrak{so}(V \oplus V^*)$ and of $\mathfrak{so}(V)$ on the exterior algebra $\Lambda(V)$ that factor through the Clifford algebra $Cl(V \oplus V^*)$. Then we construct commuting embeddings of $\mathfrak{so}_{2n}$ and $\mathfrak{so}_m$ into $Cl(C^{nm} \oplus (C^{nm})^*)$ in Section 3.2. Finally, we compute a multiplicity-free decomposition of $\Lambda(C^{nm})$ as an $\mathfrak{o}_{2n} \otimes \mathfrak{so}_m$-module in Section 3.3.

As in [Abo22], we work at the Lie algebra level throughout, since we are most interested in the quantum case and it is the enveloping algebra $U(\mathfrak{g})$, rather than the Lie group $G = \exp(\mathfrak{g})$, that more closely resembles $U_q(\mathfrak{g})$.

3.1. Two orthogonal actions on the spin module $S$. Consider any complex vector space $V$ and let $\langle \cdot, \cdot \rangle$ denote the symmetric bilinear form on $V \oplus V^*$ arising from the dual pairing between $V$ and $V^*$; explicitly,

$$\beta((v, f), (w, h)) = f(w) + h(v), \quad v, w \in V, \ f, h \in V^*.$$ 

In [AS22, Section 1] we review the Clifford algebra $Cl(V \oplus V^*)$ on $V \oplus V^*$ and its spin action on the exterior algebra $\Lambda(V)$ via inner and exterior multiplication operators $t_f$ and $e_v$. This treatment is fairly standard and may be found in various sources, e.g. [Bum04, Chapter 31] or [GW09, Chapter 6]. When necessary or convenient, we view $Cl(V \oplus V^*)$ as a Lie algebra with bracket given by the usual algebra commutator: $[A, B] = AB - BA$.

There are two actions of orthogonal Lie algebras on $\Lambda(V)$. On one hand, there is a spin action of $\mathfrak{so}(V \oplus V^*)$ on $\Lambda(V)$ that makes the following diagram commute. On the other, there is an $\mathfrak{so}(V)$-action obtained by restricting the $\mathfrak{gl}(V)$-action on $\Lambda(V)$, described in [Abo22, Section 2.1], to a subalgebra of skew-symmetric operators.

$$\mathfrak{so}(V \oplus V^*) \longrightarrow \text{End}(\Lambda(V))$$

$$\mathfrak{gl}(V)$$

In this article we are interested in these actions in so far as they motivate results in the quantized setting. Therefore since $U_q(\mathfrak{so}_m)$ does not embed into $U_q(\mathfrak{gl}_m)$, we would not gain much by studying the action on $\Lambda(V)$ of a positive Borel subalgebra in $\mathfrak{so}(V)$ with respect to a chosen weight basis.

Thus in this subsection we focus on the spin action of $\mathfrak{so}(V \oplus V^*)$, which we quantize in Section 4.1 using a map $U_q(\mathfrak{so}_{2n}) \rightarrow Cl_q(n)$. This action factors through $Cl(V \oplus V^*)$ as follows. Let $\{A, B\} = AB + BA$ and consider the following simple identity, valid in any associative algebra containing $V \oplus V^*$:

$$[[v, w], u] = v\{w, u\} + \{w, u\}v - w\{v, u\} - \{v, u\}w \quad u, v, w \in V \oplus V^*.$$  

(3.2)

As an identity in $Cl(V \oplus V^*)$, Relation (3.2) reads

$$[[v, w], u] = 2((w, u)v - (v, u)w).$$

A straightforward calculation shows the operator $X_{v,w}(u) = (w, u)v - (v, u)w$ is skew-symmetric with respect to the form $\langle \cdot, \cdot \rangle$, so we obtain an isomorphism of Lie algebras $\mathfrak{so}(V \oplus V^*) \rightarrow Cl(V \oplus V^*)$.

Lemma 3.1. [GW09, Lemma 6.2.2] Let $\gamma: V \oplus V^* \rightarrow Cl(V \oplus V^*)$ denote the natural inclusion map. There is an injective homomorphism of Lie algebras $\varphi: \mathfrak{so}(V \oplus V^*) \rightarrow Cl(V \oplus V^*)$ satisfying

$$\varphi(X_{v,w}(u)) = \frac{1}{2} [\gamma(v), \gamma(w)] \quad \text{and} \quad [\varphi(X_{v,w}), u] = X_{v,w}(u)$$

for every $u, v, w \in V \oplus V^* \subset Cl(V \oplus V^*)$. 
From now on we let \( S = \bigwedge (V) \) denote the \emph{spin module} over \( \mathfrak{so}(V \oplus V^*) \) equipped with the action defined by Lemma 3.1.

The subspaces \( V \) and \( V^* \) are Lagrangian in \( V \oplus V^* \), so \( \mathfrak{so}(V \oplus V^*) \) has a three-step grading of the form

\[
\mathfrak{so}(V \oplus V^*) \cong \mathfrak{so}^{(2,0)} \oplus \mathfrak{so}^{(1,1)} \oplus \mathfrak{so}^{(0,2)}.
\]

Here

\[
\mathfrak{so}^{(2,0)} = \text{span} (\varepsilon_v \varepsilon_w \mid v, w \in V),
\]

\[
\mathfrak{so}^{(1,1)} = \text{span} \left( \frac{1}{2} [\varepsilon_v, \iota f] \mid v \in V, f \in V^* \right),
\]

\[
\mathfrak{so}^{(0,2)} = \text{span} (\iota f \iota g \mid f, g \in V^*).
\]

The subspaces \( \mathfrak{so}^{(2,0)} \) and \( \mathfrak{so}^{(0,2)} \) generate abelian subalgebras, both normalized by \( \mathfrak{so}^{(1,1)} \). If we choose an \( \mathfrak{so}(V \oplus V^*) \)-weight basis of the \( 2n \)-dimensional \( V \oplus V^* \) that is isotropic with respect to \( (\cdot, \cdot) \) then \( \mathfrak{so}^{(1,1)} \) corresponds to the set of \( (2n) \times (2n) \) block diagonal matrices with blocks of size \( n \times n \).

Remark 3.2. The notation \( \mathfrak{so}^{(0,2)} \), \( \mathfrak{so}^{(1,1)} \), and \( \mathfrak{so}^{(2,0)} \) is motivated by the following observations. Each element in \( \mathfrak{so}^{(0,2)} \) is a product of two \emph{lowering operators} mapping \( \bigwedge^p (W) \) into \( \bigwedge^{p-2} (W) \). Similarly, each element of \( \mathfrak{so}^{(2,0)} \) is a product of two \emph{raising operators} taking \( \bigwedge^p (W) \) into \( \bigwedge^{p+2} (W) \). The elements of \( \mathfrak{so}^{(1,1)} \) are products of a raising and a lowering operator, thus preserving components of homogeneous degree.

The subalgebra \( \mathfrak{so}^{(1,1)} \) is isomorphic to \( \mathfrak{gl}(V) \). Recall that \( \mathfrak{gl}(V) \cong V \otimes V^* \) and \( \{ \varepsilon_v, \iota f \} = f(v) \) by [AS22, Relation (6)]. Therefore

\[
\varphi(X_{v,f}) = \frac{1}{2} [\varepsilon_v, \iota f] = \varepsilon_v \iota f - \frac{1}{2} f(v),
\]

showing \( \mathfrak{so}^{(1,1)} \cong \mathfrak{gl}(V) \). In particular, the element \( X = v \otimes f \) in \( V \otimes V^* \cong \mathfrak{gl}(V) \subset \mathfrak{so}(V \oplus V^*) \) acts on \( \bigwedge (V) \) by the rightmost operator in Relation (3.4).

Note that the \( \mathfrak{so}^{(1,1)} \cong \mathfrak{gl}(V) \)-action on \( \bigwedge (V) \) differs from the \( \mathfrak{gl}(V) \)-action defined by [Abo22, Relation (7)] by constants. In particular, \( f(v) \) is the trace of the endomorphism \( X = v \otimes f \in \mathfrak{gl}(V) \) because \( X \) is rank 1 and its only non-trivial eigenvalue is \( f(v) \). These constants do not alter commutation relations, but they do affect the action of both \( \mathfrak{gl}(V) \) and \( GL(V) \) on \( \bigwedge (V) \). At the Lie group level, subtracting the trace normalizes the \( GL(V) \)-action by a factor of \( \det^{-1/2} \). One consequence of this normalization is that the action of \( GL(V) \), or rather its two-fold cover, on \( \bigwedge (V) \), is now self-contragradient [How95].

Lemma 3.1 describes an \( \mathfrak{so}(V \oplus V^*) \)-action on \( \bigwedge (V) \) that factors through \( Cl(V \oplus V^*) \). We are interested in a quantum version of this action. In the quantum setting operators are described by their action on a weight basis, so we describe the map defined by Lemma 3.1 explicitly in terms of an \( \mathfrak{so}(V \oplus V^*) \)-weight basis in preparation of the quantum case.

To begin, let \( v_1, \ldots, v_n \) denote a basis of \( V \) and let \( v_{-n}, \ldots, v_{-1} \) denote the corresponding dual basis of \( V^* \), chosen so that \( v_{-j}(v_i) = \delta_{ij} \). Then \( v_1, \ldots, v_n, v_{-n}, \ldots, v_{-1} \) is an isotropic basis of \( V \oplus V^* \) and there is an injective map \( \mathfrak{so}(V \oplus V^*) \rightarrow \text{Mat}_{2n}(\mathbb{C}) \) satisfying

\[
\begin{align*}
E_i & \rightarrow M_{i,i+1} - M_{i-1,i}, & F_i & \rightarrow M_{i+1,i} - M_{i,i-1}, & i < n \\
E_n & \rightarrow M_{n-1,n} - M_{n,n-1}, & F_n & \rightarrow M_{n,n-1} - M_{n-1,n} \\
H_i & \rightarrow M_{ii} + M_{i-1,i-1} - (M_{i+1,i+1} + M_{i,i-1}), & i < n \\
H_n & \rightarrow M_{n-1,n-1} + M_{nn} - (M_{n+1,n+1} + M_{n,n-1}).
\end{align*}
\]

The \( M_{ij} \) denote matrix units with respect to the \( v_i \) basis defined by \( M_{ij}v_k = \delta_{jk}v_i \). The action of the \( H_i \) is diagonal, so \( v_1, \ldots, v_n, v_{-n}, \ldots, v_{-1} \) is in fact simultaneously a \( \mathfrak{gl}(V) \)- and an \( \mathfrak{so}(V \oplus V^*) \)-weight basis. Our choice of weight basis defines an isomorphism \( V \oplus V^* \cong \mathbb{C}^n \oplus (\mathbb{C}^n)^* \) and from now we denote
$\mathfrak{so}(V \oplus V^*)$ by $\mathfrak{so}_{2n}$. Under the isomorphism defined by Relation (3.5), the image of $\mathfrak{so}_{2n}$ is the set of traceless matrices $X$ satisfying $XJ + JXT = 0$, with $J := \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}$.

We obtain a basis for $\bigwedge(V)$ using the vectors $\bar{v}(\ell)$ defined by [Abo22, Relation (8)]. On the Clifford algebra side, we consider the generators $\psi_i = \iota_{v_i}$ and $\psi_i^\dagger = \bar{v}_{\ell_i}$, for $i = 1, \ldots, n$, much like in [Abo22, Section 2.1]. For convenience, we recall that the $\psi_i$ and $\psi_j^\dagger$ satisfy the canonical anticommutation relations

$$\psi_i \psi_j + \psi_j \psi_i = \psi_i^\dagger \psi_j^\dagger + \psi_j^\dagger \psi_i^\dagger = 0 \quad \text{and}$$

$$\psi_i^\dagger \psi_j + \psi_j^\dagger \psi_i = \delta_{ij},$$

and that they act by lowering and raising operators: for any $\bar{v}(\ell)$ in $\bigwedge(V)$,

$$\psi_i \bar{v}(\ell) = (-1)^{\ell_i + \cdots + \ell_{i-1}} \bar{v}(\ell - e_i),$$

$$\psi_i^\dagger \bar{v}(\ell) = (-1)^{\ell_i + \cdots + \ell_{i-1}} \bar{v}(\ell + e_i).$$

The following proposition defines the map $\mathfrak{so}_{2n} \rightarrow Cl(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ explicitly with respect to the $v_i$ basis.

**Proposition 3.3.** Recall the map $\Phi_n : \mathfrak{gl}_n \rightarrow Cl(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ defined in [Abo22, Proposition 2.2]. There is a Lie algebra homomorphism $\Phi_n^D : \mathfrak{so}_{2n} \rightarrow Cl(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ satisfying $\Phi_n^D(X) = \Phi_n(X)$ whenever $X \in \mathfrak{gl}_n \subset \mathfrak{so}_{2n}$ and

$$E_n \rightarrow \psi_{n-1}^\dagger \psi_n^\dagger,$$

$$F_n \rightarrow \psi_{-n} \psi_{-n+1},$$

$$H_n \rightarrow \psi_{n-1} \psi_{-n+1} + \psi_n^\dagger \psi_{-n} - 1.$$

**Proof.** This is simply the map defined by Lemma 3.1 in terms of the $Cl(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ generators described by Relations (11) and (12). \qed

We note that $S = \bigwedge(V)$ is not an irreducible $\mathfrak{so}_{2n}$-module. Rather, it is the sum of two irreducible components: as an $\mathfrak{so}_{2n}$-module,

$$\bigwedge(V) \cong S_+ \oplus S_-.$$  

Here $S_{\pm}$ denotes the irreducible $\mathfrak{so}_{2n}$ module with highest weight $(\frac{1}{2}, \ldots, \frac{1}{2})$. Regardless, $S$ is an irreducible module of the full orthogonal Lie algebra $U(\mathfrak{o}_n)$.

**Definition 3.4.** The Hopf algebra $U(\mathfrak{o}_n) = U(\mathfrak{so}_n) \rtimes \mathbb{Z}_2$ is generated by the enveloping algebra $U(\mathfrak{so}_n)$ and the additional generator $t$, subject to the following relations. If $n$ is odd, then $t$ commutes with every generator. If $n = 2r$ is even, then

$$tE_{r-1}t^{-1} = E_r, \quad tE_rt^{-1} = E_{r-1},$$

$$tF_{r-1}t^{-1} = F_r, \quad tF_rt^{-1} = F_{r-1},$$

$$tH_{r-1}t^{-1} = H_r, \quad tH_rt^{-1} = H_{r-1},$$

and $t$ commutes with all other generators. The element $t$ is group-like, which means we extend the comultiplication $\Delta$ and the antipode $S$ of $U(\mathfrak{so}_r)$ to $U(\mathfrak{o}_n)$ by specifying that $\Delta(t) = t \otimes t$ and $S(t) = t^{-1}$.

Note that when $n = 2r$ is even, the conjugation action of $t$ on the $\mathfrak{so}_n$ simple positive root vectors induces the $D_r$ Dynkin diagram automorphism swapping the two leaf nodes attached at the trivalent vertex.

In general, we may extend any $\mathfrak{so}_n$-module to a $U(\mathfrak{o}_n)$-module by specifying the action of $t$. The defining relations (4.2), together with $t^2 = 1$, imply that when $n = 2r + 1$ is odd, $t$ must act by $\pm 1$ on
each $\mathfrak{so}_n$-weight space. In fact since $t$ commutes with every $\mathfrak{so}_n$ generator, Schur’s lemma implies that $t$ must act by a scalar on any irreducible $\mathfrak{so}_n$-module.

Conversely, when $n = 2r$ is even, $t$ induces the map $\mu \to \bar{\mu}$ on the weight lattice of $V$, with $\mu = (\mu_1, \ldots, \mu_r)$ and $\bar{\mu} = (\mu_1, \ldots, \mu_{r-1}, -\mu_r)$: if $v_\mu$ is a weight vector of weight $\mu$, then $tv_\mu$ is a weight vector of weight $\nu$ determined by the relations

$$\langle \nu, \alpha_i \rangle = \langle \mu, \alpha_i \rangle \quad \text{for } i < r - 1,$$
$$\langle \nu, \alpha_{r-1} \rangle = \langle \mu, \alpha_r \rangle,$$
$$\langle \nu, \alpha_r \rangle = \langle \mu, \alpha_{r-1} \rangle,$$

which imply $\nu = \bar{\mu}$. Note that if $v_\mu$ is a highest weight vector of the $\mathfrak{so}_n$ action, then $tv_\mu$ is also a highest weight vector with respect to $\mathfrak{so}_n$, since it is annihilated by every $E_i$ generator. Therefore, if $\mu \neq \bar{\mu}$, or equivalently if $\mu_r \neq 0$, any irreducible $\mathfrak{so}_n$-module containing an $\mathfrak{so}_n$-highest weight vector of weight $\mu$ splits into two irreducible $\mathfrak{so}_n$-modules upon restriction. Alternatively, if $\mu = \bar{\mu}$, or equivalently if $\mu_r = 0$, then $t$ preserves the highest weight space and there is an irreducible module for each possible action of $t$. Since $t^2 = 1$, there are exactly two inequivalent irreducible $\mathfrak{so}_n$-modules in this case.

To summarize, the $\mathfrak{so}_n$ and $\mathfrak{so}_n$ representations are related as follows. The irreducible $\mathfrak{so}_n$-modules are parametrized by dominant highest weights $\mu$ satisfying

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{r-1} \geq |\mu_r|.$$

Given a partition $\mu$, let $\mu'$ denote its conjugate and suppose $V_\mu$ is an irreducible $\mathfrak{so}_n$-module with highest weight $\mu$. There are two possibilities.

(i) If $\mu_r = 0$, then there are exactly two non-isomorphic $\mathfrak{so}_n$-modules whose restriction to $\mathfrak{so}_n$ is isomorphic to $V_\mu$: one is labeled by $\mu$ and the other by $\mu^1$. The Young diagram corresponding to $\mu^1$ is identical to the one corresponding to $\mu$ except for its first column, which has $n - \mu_1'$ boxes.

(ii) Alternatively, if $\mu_r \neq 0$, then there is exactly one $\mathfrak{so}_n$-module corresponding to $\mu$, and its restriction to $\mathfrak{so}_n$ decomposes as $V_\mu \oplus V_{\bar{\mu}}$. The corresponding $\mathfrak{so}_n$-module is parametrized by the partition $\mu$ with $\mu_r > 0$.

In any case, we see that when $n = 2r$ is even, the irreducible $U(\mathfrak{so}_n)$-modules are parametrized by partitions with at most $n$ parts satisfying

$$\mu_1' + \mu_2' \leq n. \quad (3.7)$$

### 3.2. Commuting embeddings into the Clifford algebra.

As in [Abo22, Section 2.2], now suppose $V = U \otimes W$ with $\dim U = n$ and $\dim W = m$. In this subsection, we construct commuting embeddings of $\mathfrak{so}(U \oplus U^*)$ and $\mathfrak{so}(W)$, and of $\mathfrak{so}(U)$ and $\mathfrak{so}(W \oplus W^*)$, into the Clifford algebra

$$\text{CL}((U \otimes W) \oplus (U \otimes W)^*) \cong \text{End} \left( \bigwedge (U \otimes W) \right)$$

as in Diagram (1.2). These embeddings rely on the maps $\lambda: \mathfrak{gl}_n \to \text{CL}(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*)$ and $\rho: \mathfrak{gl}_m \to \text{CL}(\mathbb{C}^{nm} \oplus (\mathbb{C}^{nm})^*)$ defined in [Abo22, Proposition 2.6] and [Abo22, Proposition 2.7]. In fact, the constructions in this section are analogous to those in [Abo22, Section 2.2].

Much like in [Abo22, Relation (15)], we map $\mathfrak{so}(U \oplus U^*)$ into $\mathfrak{so}(U \oplus U^*) \otimes \mathfrak{gl}(W)$, which can be seen as a subalgebra of $\mathfrak{so}(U \oplus U^*) \otimes \mathfrak{so}(W \oplus W^*) \subseteq \mathfrak{so}(V \oplus V^*)$, by tensoring with the identity. Then we use the map of Proposition 3.3 to embed $\mathfrak{so}(V \oplus V^*)$ into $\text{CL}(V \oplus V^*)$. The resulting $\mathfrak{so}(U \oplus U^*)$-action on $S_{\otimes m}$ coincides with the action obtained by composing the spin action described in Section 3.1 with the comultiplication. We obtain a commuting action of $\mathfrak{so}(W)$ by restricting the $\mathfrak{gl}(W)$-action on $\bigwedge (V)$ described in [Abo22, Proposition 2.7].

Dually, we tensor with the identity to embed $\mathfrak{so}(W \oplus W^*)$ into $\mathfrak{gl}(U) \otimes \mathfrak{so}(W \oplus W^*)$, which is a subalgebra of $\mathfrak{so}(U \oplus U^*) \otimes \mathfrak{so}(W \oplus W^*) \subseteq \mathfrak{so}(V \oplus V^*)$, and then we compose with the map defined in Proposition 3.3. Again, we obtain a commuting copy of $\mathfrak{so}(U) \subseteq \mathfrak{gl}(U)$ by restricting the action defined in [Abo22, Proposition 2.6]. Alternatively, we may obtain these embeddings by reversing the roles of $U$.
and $W$. In this case, the preferred factorization of $\Lambda((C^n)^m)$ is into $n$ tensor factors of the $\mathfrak{so}(W \oplus W^*)$-spin module $\Lambda((C^n)^m)$, instead of $m$ factors of the $\mathfrak{so}(U \oplus U^*)$-spin module $\Lambda((C^n)^m)$.

Ultimately we are interested in quantum versions of these embeddings, so we define them explicitly with respect to the $\mathfrak{gl}(V)$-weight basis of $V = U \otimes W$ defined in [Abo22, Section 2.2]. This basis may be extended to the $\mathfrak{so}(V \oplus V^*)$-weight basis $v_1, \ldots, v_n, v_1^*, \ldots, v_n^*$ of $V \oplus V^*$ by appending the corresponding dual basis of $V^*$.

The next proposition describes an explicit embedding $\mathcal{L}$ that takes $\mathfrak{so}_m \twoheadrightarrow \mathfrak{so}(U \oplus U^*)$ into $\mathcal{O}(C^{nm} + C^{nm})^*$ and makes the following diagram commute. This proposition directly motivates Proposition 4.9 in the quantum case.

\[
\begin{array}{cccccc}
\mathfrak{sl}_n & \xrightarrow{\Delta^{(m-1)}} & \mathfrak{sl} \otimes \mathfrak{sl} & & \xrightarrow{\Phi_n} & CI((C^n \oplus (C^n)^*) \otimes (C^n \oplus (C^n)^*)) \\
& & \downarrow & \downarrow & \downarrow & \text{id} \\
\mathfrak{so}_{2n} & \xrightarrow{\Delta^{(m-1)}} & \mathfrak{so}_{2n} \otimes \mathfrak{so}_{2n} & \xrightarrow{(\Phi_n^D)^{\otimes m}} & CI((C^n \oplus (C^n)^*) \otimes (C^n \oplus (C^n)^*)) & \xrightarrow{\text{id}} \text{End}(\Lambda((C^n)^{\otimes m})) \\
& & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathfrak{sl}_n & \xrightarrow{\Delta^{(m-1)}} & \mathfrak{sl} \otimes \mathfrak{sl} & \xrightarrow{\Phi_n} & CI((C^n \oplus (C^n)^*) \otimes (C^n \oplus (C^n)^*)) & \xrightarrow{\text{id}} \text{End}(\Lambda((C^n)^{\otimes m})) \\
\end{array}
\]  

Here $\Delta: U(g) \to U(g)^{\otimes 2}$ denotes the comultiplication in the enveloping algebra. Recall Proposition 3.3 defines $\Phi_n^D$. We take $\Gamma_n$ as in [AS22, Proposition 1.7] and $\lambda$ and $\Phi_n$ as in [Abo22, Propositions 2.2 and 2.6].

**Proposition 3.5.** There is a Lie algebra homomorphism $\mathcal{L}: \mathfrak{so}_{2n} \to CI((C^{nm} + (C^{nm})^*)$ satisfying $\mathcal{L}(X) = \lambda(X)$ for every $X$ belonging to the subalgebra $\mathfrak{gl}_n \subset \mathfrak{so}_{2n}$ and

\[
\begin{align*}
E_n & \rightarrow \sum_{j=1}^m \psi^\dagger_{n-1+(j-1)n}\psi^\dagger_{n+(j-1)n}, \\
F_n & \rightarrow \sum_{j=1}^m \psi_{n+(j-1)n}\psi_{n-1+(j-1)n}, \\
H_n & \rightarrow -m + \sum_{j=1}^m \left(\psi^\dagger_{n-1+(j-1)n}\psi_{n-1+(j-1)n} + \psi^\dagger_{n+(j-1)n}\psi_{n+(j-1)n}\right).
\end{align*}
\]

**Proof.** This map is the composition $\mathcal{L} = \Gamma_n \circ (\Phi_n^{D \otimes m} \circ \Delta^{(m-1)})$ of known Lie algebra maps illustrated in Diagram (3.8). \hfill \square

An immediate corollary of Proposition 3.5 is that $S^{\otimes m} \cong \Lambda((C^{nm})$ as an $\mathfrak{so}_{2n}$-module.

As explained in Section 4.1, we will not gain much in the quantum case by describing the commuting embedding of $\mathfrak{so}(W)$ into $CI(V \oplus V^*)$ explicitly in terms of root vectors because the analogous commuting factor in the quantum case is the *non-standard deformation* $U_q(\mathfrak{so}_n)$, which does not have an analogue of a positive Borel subalgebra. Thus we avoid discussing the embedding of $\mathfrak{so}(W)$ here.

### 3.3. Multiplicity-free decomposition of $S^{\otimes m}$.

In this section we compute a multiplicity-free decomposition of $\Lambda((C^{nm})$ as a $U(\mathfrak{o}_n) \otimes U(\mathfrak{so}_n)$-module. Recall that $S \cong \Lambda((C^n)$ and $\Lambda((C^{nm}) \cong \Lambda((C^n)^{\otimes m}$ as a $U(\mathfrak{o}_n)$-module.

Section 3.1 explains that the irreducible representations of $U(\mathfrak{o}_n)$ are parametrized by partitions $\mu$ such that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_r \leq n$, with $\mu'$ denoting the conjugate of $\mu$. When $m = 2r$ is even, the irreducible representations of $\mathfrak{so}_n$ are labeled by dominant weights $\nu$ such that $\nu_1 \geq \nu_2 \geq \cdots \geq |\nu_r|$. Conversely, if $m = 2r + 1$ is odd, the irreducible representations of $\mathfrak{so}_n$ are labeled by dominant weights $\nu$ such that $\nu_1 \geq \nu_2 \geq \cdots \geq |\nu_r| \geq 0$. In any case, either all $\nu_j$ are integers or all $\nu_j \equiv 1/2 \mod \mathbb{Z}$. 


The map
\[ \mu \to \bar{\mu}, \quad \text{with} \quad \bar{\mu}_i = \frac{n}{2} - \mu_{r+1-i} \quad (3.9) \]
defines a bijection between the set of irreducible representations \( V_\mu \) of \( U(\mathfrak{so}_n) \) for which \( \mu_1 \leq r \) and the set of irreducible \( \mathfrak{so}_m \)-representations \( V_\bar{\mu} \) for which \( \bar{\mu}_1 \leq n/2 \) and \( n/2 - \bar{\mu}_i \) is an integer for \( 1 \leq i \leq r \).

**Theorem 3.6.** Let \( S \cong S_+ \oplus S_- \) denote the \( \mathfrak{so}_{2n} \)-spin module. As a \( U(\mathfrak{so}_n) \otimes U(\mathfrak{so}_m) \)-module, \( S^\otimes m \cong \wedge(\mathbb{C}^{nm}) \) is multiplicity-free. In particular, we have
\[ S^\otimes m \cong \bigoplus_{\mu} V^{(n)}_\mu \otimes V^{(m)}_\bar{\mu} \]
as a \( U(\mathfrak{so}_n) \otimes U(\mathfrak{so}_m) \)-module. The sum ranges over all partitions \( \mu \) that fit in a \((2n) \times r\) rectangle and satisfy \( \mu_1' + \mu_2' \leq 2n \). In the decomposition \( V^{(n)}_\mu \) denotes the irreducible \( U(\mathfrak{so}_n) \)-module with highest weight and \( \mu \), \( V^{(m)}_\bar{\mu} \) denotes an irreducible \( \mathfrak{so}_m \)-module indexed by \( \bar{\mu} \). Consequently, \( U(\mathfrak{so}_n) \) and \( U(\mathfrak{so}_m) \) generate mutual commutants in \( \text{End} (\wedge(\mathbb{C}^{nm})) \).

**Proof.** In [Abo22, Theorem 2.10] we compute the decomposition of \( \wedge(\mathbb{C}^{nm}) \) into isotypic components of \( \mathfrak{gl}_n \otimes \mathfrak{gl}_m \). Notice the subspace \( \mathfrak{so}^{(1,1)} \subset \mathfrak{so}_{2n} \) described in Relation (3.3) is a normalization of \( \mathfrak{gl}_n \). In addition, it is the Levi component of the parabolic subalgebra \( \mathfrak{so}^{(1,1)} \oplus \mathfrak{so}^{(0,2)} \) of \( \mathfrak{so}_{2n} \), whose nilradical is \( \mathfrak{so}^{(0,2)} \). Thus in any irreducible \( \mathfrak{so}_{2n} \)-module \( M \) the space
\[ \ker \mathfrak{so}^{(0,2)} = \{ v \in M \mid Xv = 0, \text{ for all } X \in \mathfrak{so}^{(0,2)} \} \]
is an irreducible \( \mathfrak{so}^{(1,1)} \)-module. Moreover, the irreducible representation of the Levi component \( \mathfrak{so}^{(1,1)} \) characterizes the \( \mathfrak{so}_{2n} \)-module containing it. This means the irreducible \( \mathfrak{so}_{2n} \)-modules appearing in the decomposition of \( \wedge(\mathbb{C}^{nm}) \) are the \( \mathfrak{gl}_n \)-isotypic components that appear in the skew \( \mathfrak{gl}_n \otimes \mathfrak{gl}_m \)-duality [Abo22, Theorem 2.10] and are annihilated by \( \mathfrak{so}^{(0,2)} \). These modules are parametrized by a subset of dominant weights fitting in an \( n \times m \) rectangle.

The condition \( \mu_1' + \mu_2' \leq 2n \) arises when we consider the relationship between \( \mathfrak{so}_n \) and \( \mathfrak{so}_m \)-modules, explained in Section 3.1. The restriction \( \mu_1 \leq r \) arises when we consider the commuting \( \mathfrak{so}_m \)-action. Corollary 3.3.2 in [How95] proves that an irreducible \( \mathfrak{gl}_m \)-module parametrized by \( \mu \) contains a highest weight with respect to the subalgebra \( \mathfrak{so}_m \subset \mathfrak{gl}_m \) only if every row of \( \mu \) is even. Thus the irreducible \( \mathfrak{gl}_m \)-modules appearing in the decomposition of the skew \( \mathfrak{gl}_n \otimes \mathfrak{gl}_m \)-duality result that are isotypic with respect to the restricted \( \mathfrak{so}_m \)-action correspond to \( \mathfrak{so}_{2n} \)-modules with highest weight \( \mu \) satisfying \( \mu_1 \leq r \).

The exact correspondence can be computed using an explicit Borel subalgebra for \( \mathfrak{so}_m \). This calculation does not have a quantum analogue, so we omit it here. \( \square \)

Of course we obtain an analogous decomposition of \( \mathfrak{so}_n \otimes \mathfrak{so}_{2m} \)-modules by considering the isomorphism \( S^\otimes n \cong \wedge(\mathbb{C}^m)^\otimes n \cong \wedge(\mathbb{C}^n) \) of \( \mathfrak{so}_{2m} \)-modules. Here we let \( S_m \) denote the \( \mathfrak{so}_{2m} \)-spin module.

### 4. The Quantum Case

In this section we prove quantized skew duality results for Types BD using our constructions for Type A from [Abo22, Section 3]. Theorem 4.19 is our main result. In particular, we identify \( U_q(\mathfrak{gl}_n) \) as a subalgebra of \( U_q(\mathfrak{so}_{2n}) \) and we realize \( U_q(\mathfrak{so}_m) \) as a subalgebra of \( U_q(\mathfrak{gl}_m) \) to extend our \( U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m) \)-duality [Abo22, Theorem 3.17] to the orthogonal setting via the seesaw depicted in Diagram (1.3). Alternatively, one could realize \( U_q(\mathfrak{so}_n) \) as a subalgebra of \( U_q(\mathfrak{gl}_n) \) and extend the \( U_q(\mathfrak{gl}_n) \)-action to a \( U_q(\mathfrak{so}_{2n}) \)-action in order to prove a

In [Abo22, Section 3] we learned that the actions of \( \mathfrak{gl}_n \) and \( \mathfrak{gl}_m \) on \( \wedge(\mathbb{C}^{nm}) \) can be generalized to the quantum setting to obtain actions of \( U_q(\mathfrak{gl}_n) \) and \( U_q(\mathfrak{gl}_m) \) on \( \wedge_q(V^{nm}) \) by understanding the action of generating root vectors as products of Clifford algebra operators. In the quantum case we have an analogue \( U_q(\mathfrak{gl}_n) \to U_q(\mathfrak{so}_{2n}) \) of the diagonal embedding \( \mathfrak{gl}(V) \to \mathfrak{so}(V \oplus V^*) \), so we emulate the strategy of [Abo22, Section 3]: in Section 4.2 we construct a map \( \mathcal{L}_q : U_q(\mathfrak{so}_{2n}) \to Cl_q(nm) \) that restricts
to the map $\lambda: U_q(\mathfrak{gl}_n) \to Cl_q(nm)$ defined in [Abo22, Section 3.2] on the subalgebra $U_q(\mathfrak{gl}_n) \subset U_q(\mathfrak{so}_{2n})$. However, $U_q(\mathfrak{so}_n)$ does not embed into $U_q(\mathfrak{gl}_n)$, so we cannot directly quantize the action of simple root vectors in $\mathfrak{so}(V)$.

This is a remarkable feature of the quantum case. The non-standard deformation of $\mathfrak{so}_m$ does not support an analogue of a Borel subalgebra. However, there is an analogue of a Cartan subalgebra in $U_q(\mathfrak{so}_m)$ and every irreducible $U_q'(\mathfrak{so}_m)$-module has a basis indexed by Gelfand-Tsetlin patterns as in the classical case. Thus in the quantum case, the main obstacle in finding joint highest weight vectors in order to decompose $S^{\otimes m}$ is diagonalizing the Cartan subalgebra. This is achieved in Section 4.3.

4.1. The spin module $S$ as a braided exterior algebra. Recall the notation of [Abo22, Section 3.1]. In particular, let $V^{(n)}$ denote the natural $U_q(\mathfrak{gl}_p)$-module and consider the braided exterior algebra $\Lambda_q(V^{(n)})$ defined by [Abo22, Relation (3)] using the $R$-matrix of $U_q(\mathfrak{gl}_n)$.

In this subsection, we define actions of the orthogonal quantum groups $U_q(\mathfrak{so}_{2n})$ and $U_q^+(\mathfrak{so}_{2n+1})$ on $\Lambda_q(V^{(n)})$. These actions factor through the $Cl_q(n)$-action on $\Lambda_q(V^{(n)})$ defined in [Abo22, Relation (3)] and they are compatible with the $U_q(\mathfrak{gl}_n)$-module algebra structure defined in [Abo22, Proposition 3.5], in the sense that the following diagrams commute:

$$
\begin{align*}
U_q(\mathfrak{gl}_n) \otimes \mathbb{D}_n &\longrightarrow U_q(\mathfrak{so}_{2n}) \\
\Phi_{q,n} &\downarrow \\
Cl_q(n) &\quad \Phi_{q,n} \\
&\downarrow \\
U_q^+(\mathfrak{so}_{2n+1}) &\quad \Phi_{q,n}^B
\end{align*}
$$

(4.1)

Propositions 4.1 and 4.3 define the maps $\mathbb{D}_n$ and $\Phi_{q,n}$. In addition, the $U_q(\mathfrak{so}_{2n})$-action on $\Lambda_q(V^{(n)})$ motivates the embedding of $U_q(\mathfrak{so}_{2n})$ into the quantum Clifford algebra $Cl_q(nm)$ presented in Section 4.2.

Much like the quantum group $U_q(\mathfrak{gl}_n)$, the orthogonal quantum groups $U_q(\mathfrak{so}_{2n})$ and $U_q^+(\mathfrak{so}_{2n+1})$ also map into quantum Clifford algebra $Cl_q(n)$. These embeddings thereby define $spin$ actions on $\Lambda_q(V^{(n)})$.

Note that in this section we still deal with the braided exterior algebra $\Lambda_q(V^{(n)})$ as in [Abo22, Relation (3)] using the $U_q(\mathfrak{gl}_n)$, and not the $U_q(\mathfrak{so}_{2n})$, $R$-matrix. Regardless, we will no longer consider the underlying algebra structure of $\Lambda_q(V^{(n)})$: we merely extend the $U_q(\mathfrak{gl}_n)$-module structure.

Proposition 4.1. Recall the map $\Phi_{q,n}$ of [Abo22, Proposition 3.5]. There is an algebra map $\Phi_{q,n}^\mathbb{D}: U_q(\mathfrak{so}_{2n}) \to Cl_q(n)$ satisfying $\Phi_{q,n}^\mathbb{D}(X) = \Phi_{q,n}(X)$ for $X$ belonging to the subalgebra $U_q(\mathfrak{sl}_n) \subset U_q(\mathfrak{so}_{2n})$ and

$$
\begin{align*}
\Phi_{q,n}^\mathbb{D}(E_n) &= \psi_{n-1}^\dagger \psi_n^\dagger \\
\Phi_{q,n}^\mathbb{D}(F_n) &= \psi_n \psi_{n-1} \\
\Phi_{q,n}^\mathbb{D}(K_n) &= (q \omega_{n-1} \omega_n)^{-1}.
\end{align*}
$$

Remark 4.2. As in the classical case, we use the normalized general linear action. In particular, recall Relation (3.4) and the comments surrounding it. The normalized action differs from the $\mathfrak{gl}_n$-action defined by [Abo22, Relation (7)] only on matrices with non-trivial trace. Thus the two in fact coincide on the subalgebra $\mathfrak{sl}_n \subset \mathfrak{so}_{2n}$. In addition, we note there is an algebra map $U_q(\mathfrak{so}_{2n}) \to Cl_q(n)$ satisfying

$$
\begin{align*}
E_n &\to \varepsilon_n^q \varepsilon_n^{q^{-1}} = q^{-1} \prod_{p=1}^{n-2} \omega_p^{-2} \omega_{n-1}^{-1} \psi_n^\dagger \psi_{n-1}^\dagger \\
F_n &\to \ell_{n-1}^q \ell_n^q = \prod_{p=1}^{n-2} \omega_p^2 \omega_{n-1} \psi_n \psi_{n-1} \\
K_n &\to (q \omega_{n-1} \omega_n)^{-1}.
\end{align*}
$$
This map makes $\bigwedge_q(V^{(n)})$ into a $U_q(\mathfrak{so}_{2n})$-module algebra in the sense of [Mon93, Definition 4.1.1]. In this section we do not consider the underlying algebra structure of the $U_q(\mathfrak{so}_{2n})$-module $\bigwedge_q(V^{(n)})$, so we focus on the action defined in Proposition 4.1.

**Proof.** The claim follows from a calculation. Since $\Phi_{q,n}$ is an algebra map, it suffices to check the relations involving the images $\bar{E}_n, \bar{F}_n, \bar{K}_n$ of $E_n, F_n, K_n$ under $\mathcal{D}_n$. Let $A = [a_{ij}]$ denote the $n \times n$ Cartan matrix of $\mathfrak{so}_{2n}$, as in Relation (2.2).

First notice that $\bar{K}_i \bar{E}_n \bar{K}_i^{-1} = \bar{E}_n$ and similarly $\bar{K}_n \bar{E}_i \bar{K}_n^{-1} = \bar{E}_i$ whenever $i < n - 2$. When $i \geq n - 2$, $K_i$ and $E_n$ contain non-commuting factors, and we calculate that

$$\bar{K}_i \bar{E}_n \bar{K}_i^{-1} = q^{-1} (\omega_i^{-1}, \omega_i), \psi^\dagger_n \psi_n (\omega_i^{-1}, \omega_i)^{-1} = \begin{cases} q^{-1} \bar{E}_n, & \text{if } i = n - 2 \\ \bar{E}_i, & \text{if } i = n - 1. \end{cases}$$

Similarly,

$$\bar{K}_n \bar{E}_i \bar{K}_n^{-1} = q^{-1} (\omega_n^{-1}, \omega_n) (\psi^\dagger_n, \psi_n) (\omega_n^{-1}, \omega_n)^{-1} = \begin{cases} q^{-1} \bar{E}_i, & \text{if } i = n - 2 \\ \bar{E}_n, & \text{if } i = n - 1. \end{cases}$$

Finally, we calculate that

$$\bar{K}_n \bar{E}_n \bar{K}_n^{-1} = q^{-1} (\omega_n^{-1}, \omega_n) (\psi^\dagger_n, \psi_n) (\omega_n^{-1}, \omega_n)^{-1} = q^2 \bar{E}_n.$$ Combining results, we conclude that $\bar{K}_i \bar{E}_n \bar{K}_i^{-1} = q^{a_{ii}} \bar{E}_j$, as desired. Applying the $*$-operation defined by [AS22, Relation (10)] shows that $\bar{K}_i \bar{F}_j \bar{K}_i^{-1} = q^{-a_{ii}} \bar{F}_j$ as well.

Using [AS22, Relation (6)], we find that

$$[\bar{E}_n, \bar{F}_n] = [\psi_n^\dagger \psi_n^\dagger, \psi_n \psi_n] = - \frac{(q \omega_n^{-1}, \omega_n) - (q \omega_n^{-1}, \omega_n)^{-1}}{q - q^{-1}} = \frac{\bar{K}_n - \bar{K}_n^{-1}}{q - q^{-1}}.$$ To conclude, we verify the quantum Serre relations. If $i \neq n - 2$, then $[\bar{E}_n, \bar{E}_i] = 0$ because $\bar{E}_n$ and $\bar{E}_i$ share no common $\phi_i$ factors, with $\phi_i$ denoting either $\psi_i$ or $\psi_i^\dagger$ as usual. In addition, $[\bar{E}_n, \bar{E}_n] = 0$ because $\bar{E}_n$ and $\bar{E}_n^\dagger$ share a common $\psi_n^\dagger$ factor. Finally, we consider the alternative form of the quantum Serre relation (9) in [Abo22] with $(i, j) = (n, n - 2)$. Observe that

$$[\bar{E}_n, \bar{E}_n] = q^{-1} (\omega_n^{-1}, \omega_n) (\psi_n^\dagger, \psi_n) (\omega_n^{-1}, \omega_n)^{-1} = \omega_n^{-1} \psi_n^\dagger \psi_n^\dagger.$$ Each term in the $q^{-1}$-commutator $[\bar{E}_n, [\bar{E}_n, \bar{E}_n]] = q^{-1}$ contains a factor of $(\psi_n^\dagger)^2 = 0$, so it vanishes. The relations for the $\bar{F}_i$ follow from these by an application of the $*$-structure defined in [AS22, Relation (10)].

We can also embed the odd orthogonal quantum group $U_q^\pm(\mathfrak{so}_{2n+1})$ into $\text{Cl}_q(n)$.}

**Proposition 4.3.** Recall the map $\Phi_{q,n}$ of [Abo22, Proposition 3.5]. There is an algebra map $\Phi^B_{q,n} : U_q^\pm(\mathfrak{so}_{2n+1}) \to \text{Cl}_q(n)$ satisfying $\Phi^B_{q,n}(X) = \Phi_{q,n}(X)$ for $X$ belonging to the subalgebra $U_q(\mathfrak{sl}_n) \subset U_q^\pm(\mathfrak{so}_{2n+1})$ and

$$\Phi^B_{q,n}(E_n) = \psi_n^\dagger,$$$$
$$\Phi^B_{q,n}(F_n) = \psi_n,$$$$
$$\Phi^B_{q,n}(K_n) = q^{\frac{1}{2}} \omega_n^{-1}.$$ **Remark 4.4.** The quantum Clifford algebra parameter is still $q$, but we extend the base field to include $q^2$. 

Proof. The claim again follows from a calculation. It suffices to check the relations involving the images \( E_i, \bar{F}_i, K_i \) of \( E_n, F_n, K_n \) under \( \Phi_{q,n}^3 \). In this case let \( A = [a_{ij}] \) denote the \( n \times n \) Cartan matrix of \( \mathfrak{so}_{2n+1} \), as in Relation (2.2).

Note that \( \bar{K}_i \bar{E}_i \bar{K}_i^{-1} = \bar{E}_i \) and similarly \( \bar{K}_i \bar{F}_i \bar{K}_i^{-1} = \bar{F}_i \) whenever \( i < n - 1 \). Conversely if \( i \geq n - 1 \), \( K_i \) and \( E_n \) contain non-commuting factors and

\[
\bar{K}_{n-1} \bar{E}_n \bar{K}_{n-1}^{-1} = (\omega_n \gamma_1 \omega_n \gamma_1 \omega_n) = q^{-1} \bar{E}_n = q^\frac{1}{2} d_{a_n - 1, n} \bar{E}_n.
\]

Similarly,

\[
\bar{K}_n \bar{E}_{n-1} \bar{K}_n^{-1} = q^{-1} \omega_n (\omega_n \gamma_1 \omega_n \gamma_1 \omega_n) = q^{-1} \bar{E}_i = q^\frac{1}{2} d_{a_n, n-1} \bar{E}_n.
\]

Finally, we calculate that

\[
\bar{K}_n \bar{E}_n \bar{K}_n^{-1} = \omega_n (\omega_n \gamma_1 \omega_n \gamma_1 \omega_n) = q \bar{E}_n = q^\frac{1}{2} d_{a_n n} \bar{E}_n,
\]

to conclude that \( \bar{K}_i \bar{E}_j \bar{K}_i^{-1} = (q^\frac{1}{2})^{a_{ij}} \bar{E}_j \), with \( q_i = q^{d_i} \).

Next, notice that

\[
[\bar{E}_n, \bar{F}_n] = [\psi_n, \bar{E}_n] = \frac{(q^\frac{1}{2} \gamma_1)}{(q^\frac{1}{2} - q^{-\frac{1}{2}})} = \frac{K_n - K_n^{-1}}{q^\frac{1}{2} - q^{-\frac{1}{2}}},
\]

To finish the proof, we verify the Serre relations. Recall the identity (9) in [Abo22]. First notice that \( [\bar{E}_n, \bar{E}_j] = 0 \) when \( j < n - 1 \) because \( \bar{E}_n \) and \( \bar{E}_j \) do not share common factors. Next we find that

\[
[\bar{E}_{n-1}, \bar{E}_n] = q^{-1} \omega_{n-1} (\psi_{n-1} \gamma_1 \psi_{n-1} - q \psi_{n} \psi_{n-1}) = q^{-1} \omega_{n-1}^2 \psi_{n-1},
\]

so \( [\bar{E}_{n-1}, [\bar{E}_{n-1}, \bar{E}_n] = q^{-1} \omega_{n-1}^3 \psi_{n-1} \) indeed vanishes because each summand contains a factor of \( (\psi_{n-1})^2 = 0 \).

Finally, since \( a_{n, n-1} = -2 \), we see that

\[
\sum_{p=0}^3 \left[ \frac{3}{p} \right] q^{p/2} [\bar{E}_n^{p} \bar{E}_{n-1} \bar{E}_n^{3-p}] = 0
\]

because each summand contains a factor of \( \bar{E}_n^3 = 0 \).

The unverified relations involving \( \bar{F}_n \) follow from the corresponding relations involving \( \bar{E}_n \) by an application of the \( * \)-structure defined by [AS22, Relation (10)]. \( \square \)

The homomorphisms of Propositions 4.1 and 4.3 immediately yield the decomposition of \( \bigwedge_q (V(n)) \) as a \( U_q(\mathfrak{so}_{2n}) \)- and as a \( U_q(\mathfrak{so}_{2n+1}) \)-module. Since the maps identify \( U_q(\mathfrak{sl}_n) \) as a subalgebra of \( U_q(\mathfrak{so}_{2n}) \) and \( U_q(\mathfrak{so}_{2n+1}) \), respectively, we see that the highest weight vectors with respect to the orthogonal quantum group action are the \( U_q(\mathfrak{sl}_n) \)-highest weight vectors that are also annihilated by \( E_n \).

**Proposition 4.5.** As a \( U_q(\mathfrak{so}_{2n}) \)-module, the braided exterior algebra \( \bigwedge_q (V(n)) \) defined by [Abo22, Relation (3)] decomposes as

\[
\bigwedge_q (V(n)) \cong S_+ \oplus S_-.
\]

Here \( S_\pm \) denote the irreducible \( U_q(\mathfrak{so}_{2n}) \)-modules of highest weight \((\frac{1}{2}, \ldots, \frac{1}{2}, \pm \frac{1}{2})\).

**Proof.** The \( U_q(\mathfrak{so}_{2n}) \) generators \( E_i, F_i, K_i \), for \( i = 1, \ldots, n-1 \) generate the subalgebra \( U_q(\mathfrak{sl}_n) \subset U_q(\mathfrak{so}_{2n}) \), so the highest weight vectors of the \( U_q(\mathfrak{so}_{2n}) \)-action are \( U_q(\mathfrak{gl}_n) \)-highest weight vectors that are also annihilated by \( E_n \). Recall the basis \( v(\ell) \) of \( \bigwedge_q (V(n)) \) defined by [Abo22, Relation (6)]. The highest weight vectors with respect to the \( U_q(\mathfrak{sl}_n) \)-action are \( v(\gamma_j) \), for \( j = 0, 1, \ldots, n \), with \( \gamma_j = \sum_{k=0}^j e_j \). Clearly, \( v(\ell) \in \ker D(E_n) \) only if \( \ell_{n-1} + \ell_n \geq 1 \), so there are exactly two highest weight vectors:

\[
v(1, \ldots, 1, 1) \quad \text{and} \quad v(1, \ldots, 1, 0)
\]

of weights \((1/2, \ldots, 1/2, 1/2)\) and \((1/2, \ldots, 1/2, -1/2)\), respectively. \( \square \)
We will henceforth refer to the $U_q(\mathfrak{so}_{2n})$-module $S = \bigwedge_q V(n)$ as the quantum spin module. We note that $S$ may also be equipped with a $U_{q^{1/2}}(\mathfrak{so}_{2n+1})$-module structure. A similar kernel calculation shows that $\bigwedge_q V(n)$ is irreducible as a $U_{q^{1/2}}(\mathfrak{so}_{2n+1})$-module. This is analogous to the classical case: the irreducible $SO(C^{2n+1})$-module $\bigwedge(C^n)$ splits as $S_+ \oplus S_-$ when viewed as an $SO(C^{2n})$-module.

**Proposition 4.6.** As a $U_{q^{1/2}}(\mathfrak{so}_{2n+1})$-module, $\bigwedge_q V(n)$ is irreducible. It has a highest weight vector of weight $(\frac{1}{2}, \ldots, \frac{1}{2})$.

**Proof.** The proof again relies on [Abo22, Proposition 3.8], which decomposes $\bigwedge_q V(n)$ as a $U_q(\mathfrak{gl}_n)$-module. In this case $v(\ell) \in \ker B(E_n)$ only if $\ell_n = 1$ so $E_n$ only annihilates a single $U_q(\mathfrak{gl}_n)$ highest weight vector, namely $v_\mu = v(1, \ldots, 1)$. Its weight with respect to the $U_{q^{1/2}}(\mathfrak{so}_{2n+1})$-action is determined by

\[
\langle \mu, \alpha_i \rangle = 0, \quad \text{for } i = 1, \ldots n - 1, \quad \text{and} \quad \langle \mu, \alpha_n \rangle = 1,
\]

since $K_i \triangleright v_\mu = q^{\beta_i} v_\mu$. Here $\langle \cdot, \cdot \rangle$ is the bilinear form on the root space normalized so that $\langle \alpha, \alpha \rangle = 2$ for short roots, as in Relation (4). Solving using $\alpha_i = e_i - e_{i+1}$ for $i = 1, \ldots n - 1$ and $\alpha_n = e_n$ yields $\mu = (\frac{1}{2}, \ldots, \frac{1}{2})$, as claimed. \hfill \square

Although $S$ is not irreducible as a $U_q(\mathfrak{so}_{2n})$-module, it is an irreducible representation of the full orthogonal quantum group $U_q(\mathfrak{so}_{2n})$. This Hopf algebra is the quantum analogue of the Hopf algebra $U(\mathfrak{o}_n)$ defined in 3.4.

**Definition 4.7.** The quantum group $U_q(\mathfrak{o}_n) = U_q(\mathfrak{so}_n) \times \mathbb{Z}_2$ is generated by the enveloping algebra $U_q(\mathfrak{so}_n)$ and the additional generator $t$, subject to the following relations. If $n$ is odd, then $t$ commutes with every generator. If $n = 2r$ is even, then

\[
t E_{r-1} t^{-1} = E_r, \quad t E_r t^{-1} = E_{r-1}, \\
t F_{r-1} t^{-1} = F_r, \quad t F_r t^{-1} = F_{r-1}, \\
t K_{r-1} t^{-1} = K_r, \quad t K_r t^{-1} = K_{r-1},
\]

and $t$ commutes with all other generators. The element $t$ is group-like, which means we extend the comultiplication $\Delta$ and the antipode $S$ of $U_q(\mathfrak{so}_n)$ to $U_q(\mathfrak{o}_n)$ by specifying that $\Delta(t) = t \otimes t$ and $S(t) = t^{-1}$.

We may extend any $U_q(\mathfrak{so}_n)$-module to a $U_q(\mathfrak{o}_n)$-module; the $U_q(\mathfrak{o}_n)$-modules are related to the $U_q(\mathfrak{so}_n)$-modules exactly as in the classical case. Section 3.2 explains the relationship.

In particular, $S$ is an irreducible $U_q(\mathfrak{so}_{2n})$-module because $t$ must permute the highest weight spaces of the $U_q(\mathfrak{so}_{2n})$-action. In the odd case, $S$ is already irreducible as a $U_{q^{1/2}}(\mathfrak{so}_{2n+1})$-module, and it remains so when we extend the action to $U_q(\mathfrak{so}_{2n+1})$.

To conclude, we record the decomposition of $S \otimes S$ as a $U_q(\mathfrak{so}_{2n})$-module, for convenience. The decomposition is analogous to the classical situation.

**Proposition 4.8.** As a $U_q(\mathfrak{so}_{2n})$-module,

\[S \otimes S \cong \bigoplus_{j=0}^{2n} Y_j.\]

Here, $Y_j$ denotes the irreducible $U_q(\mathfrak{so}_{2n})$-module of highest weight $\gamma_j = \sum_{k=0}^j e_k$.

**Proof.** The decomposition can be computed directly using the Brauer-Klimyk formula. Alternatively, it is Relation (4.8) specialized with $m = 2$. \hfill \square
4.2. Commuting actions on $S^\otimes m$. Recall the construction of the irreducible $U_q(\mathfrak{so}_{2n})$ spin module $S = \bigwedge_q(V^{(n)})$ described in Section 3.2. In this section we study the commutant of the quantum group action on the tensor product $S^\otimes m$. We begin by showing that $S^\otimes m \cong \bigwedge_q(V^{(nm)})$ as a $U_q(\mathfrak{so}_{2n})$-module, and we show that the action of $U_q(\mathfrak{so}_{2n})$ on $S^\otimes m$ factors through the quantum Clifford algebra $Cl_q(nm)$. We then use the construction of \cite[Section 3]{Abo22} to obtain an action of the co-ideal subalgebra $U'_q(\mathfrak{so}_m) \subset U_q(\mathfrak{gl}_{nm})$ on $\bigwedge_q(V^{(nm)})$ that also factors through $Cl_q(nm)$. This action commutes with that of the subalgebra $U_q(\mathfrak{sl}_n) \subset U_q(\mathfrak{so}_{2n})$ automatically by \cite[Proposition 3.16]{Abo22}. We conclude by showing that the $U'_q(\mathfrak{so}_m)$ on $S^\otimes m \cong \bigwedge_q(V^{(nm)})$ indeed commutes with whole the $U_q(\mathfrak{so}_{2n})$ action.

Motivated by the classical embedding of Proposition 3.3, we begin by constructing an algebra map $\mathcal{L}_q: U_q(\mathfrak{so}_{2n}) \to Cl_q(nm)$ that extends the homomorphism $\lambda_q: U_q(\mathfrak{gl}_n) \to Cl_q(nm)$ defined in \cite[Proposition 3.10]{Abo22}. This map equips the braided exterior algebra $\bigwedge_q(V^{(nm)})$ with a $U_q(\mathfrak{so}_{2n})$-module structure making the following diagram commute.

\[
\begin{array}{ccccccccc}
U_q(\mathfrak{sl}_n) & \xrightarrow{\tilde{\Delta}^{(m-1)}} & U_q(\mathfrak{sl}_n)^{\otimes m} & \xrightarrow{\Phi^{\otimes m}} & Cl_q(n)^{\otimes m} & \xrightarrow{\pi_0^{\otimes m}} & \text{End}\left(\bigwedge_q(V^{(n)})^{\otimes m}\right) \\
\downarrow & & \downarrow & & \downarrow & & \\
U_q(\mathfrak{so}_{2n}) & \xrightarrow{\tilde{\Delta}^{(m-1)}} & U_q(\mathfrak{so}_{2n})^{\otimes m} & \xrightarrow{(\Phi^{\otimes m})_n} & Cl_q(n)^{\otimes m} & \xrightarrow{\pi_0^{\otimes m}} & \text{End}(S^{\otimes m}) \\
\downarrow & & \downarrow & & \downarrow & & \\
U_q(\mathfrak{sl}_n) & \xrightarrow{\lambda_q} & Cl_q(nm) & \xrightarrow{\pi_0} & \text{End}\left(\bigwedge_q(V^{(nm)})\right) \\
\end{array}
\]

Here $\tilde{\Delta}$ denotes the comultiplication map satisfying

\[
\tilde{\Delta}(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \tilde{\Delta}(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \text{and}
\]

\[
\tilde{\Delta}(L_i) = L_i \otimes L_i,
\]

as in \cite[Relation (13)]{Abo22}. Recall Proposition 4.1 defines $\Phi^{\mathcal{D}}_{q,n}$. We take $\Gamma_q$ and $\pi_0$ as in \cite[Theorem 2.11 and proposition 2.16]{AS22} and $\lambda_q$ and $\Phi_{q,n}$ as in \cite[Propositions 3.5 and 3.10]{Abo22}.

**Proposition 4.9.** Recall the notation of Proposition 3.10. In addition, define

\[
\kappa_{n,<j} = \prod_{p=0}^{j-1} (q\omega_{n-1+p}\omega_{n+p})^{-1} \quad \text{and} \quad \kappa_{n,>j} = \prod_{p=j}^{n-1} (q\omega_{n-1+p}\omega_{n+p})^{-1},
\]

by taking an appropriate product of $\omega$ generators in the $(n-1)st$ and $n$th rows to the left, and respectively to the right, of the $j$th column.

There is an algebra homomorphism $\mathcal{L}_q: U_q(\mathfrak{so}_{2n}) \to Cl_q(nm)$ satisfying

\[
E_n \mapsto \sum_{j=0}^{m-1} \psi_{n-1+j,n}^{\dagger}\psi_{n+j,n} \kappa_{n,<j}, \quad F_n \mapsto \sum_{j=0}^{m-1} \kappa_{n,>j} \psi_{n+j,n} \psi_{n-1+j,n}, \quad \text{and} \quad K_n \mapsto \kappa_{n,<m}
\]

and $\mathcal{L}_q(X) = \lambda_q(X)$ for every $X$ belonging to the subalgebra $U_q(\mathfrak{sl}_n) \subset U_q(\mathfrak{so}_{2n})$.

**Proof.** Notice that $\mathcal{L}_q = \Gamma_q \circ \mathcal{D}^{\otimes m} \circ \tilde{\Delta}^{(m-1)}$ is a composition of algebra maps. \qed
The commutativity of Diagram (4.3) immediately yields an isomorphism of $U_q(\mathfrak{so}_{2n})$-modules between the $m$-fold tensor product of the quantum spin module $S$ and the braided exterior algebra $\Lambda_q(V^{(nm)})$.

**Corollary 4.10.** As $U_q(\mathfrak{so}_{2n})$-module, $S^{\otimes m} \cong \Lambda_q(V^{(nm)})$.

**Remark 4.11.** Notice that $S^{\otimes m}$ is no longer a $U_q(\mathfrak{so}_{2n})$-module algebra in the sense of [Mon93, Definition 4.1.1], unless we deform the underlying algebra structure of $\Lambda_q(V^{(nm)})$ using a twisted multiplication as in [LZZ10, Theorem 2.3].

Notice $L_q$ maps each $K_i$ to a diagonal operator with respect to the $v(\ell)$ basis of $\Lambda_q(V^{(nm)})$ defined by [Abo22, Relation (6)]. Later we will need the weight of each $v(\ell)$ with respect to the action of $U_q(\mathfrak{so}_{2n})$, so we compute it in the next lemma.

Recall that each $v(\ell)$ determines a state of occupied and vacant positions in an $n \times m$ grid corresponding to the following arrangement of weight vectors in $V^{(nm)}$.

$$
\begin{array}{cccc}
v_{11} & v_{12} & v_{13} & \cdots & v_{1m} \\
v_{21} & v_{22} & v_{23} & \cdots & v_{2m} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nm}
\end{array}
$$

Here we use the shorthand $v_{ij} = v_{i+(j-1)n}$. The $i$th component of $v(\ell)$'s weight is determined by the state of the $i$th row in the rectangular array: each occupied position adds $+1/2$ while each vacant position contributes $-1/2$.

**Lemma 4.12.** For each $\ell \in \{0,1\}^{nm}$, let $v(\ell)$ denote the basis vector of $\bigwedge_q(V^{(nm)})$ defined by [Abo22, Relation (6)]. Its weight with respect to the action of $U_q(\mathfrak{so}_{2n})$ via $L_q$ is

$$\ell 1_m - \frac{m}{2} 1_n.$$

Here $\ell$ is viewed as an $n \times m$ matrix and $1_p$ denotes the $p$-vector of all ones.

**Proof.** The weight $\mu$ is uniquely determined by the equations

$$q^{(\mu,\alpha)} v(\ell) = L_q(K_i) v(\ell) = \prod_{j=0}^{m-1} \omega_{i+jn}^{-1} \omega_{i+1+jn} v(\ell) = q^{\sum_{j=1}^{m} \ell_{ij} - \sum_{j=1}^{m} \ell_{i+1,j}} v(\ell)$$

for $i = 1, \ldots, n - 1$, and

$$q^{(\mu,\alpha)} v(\ell) = L_q(K_i) v(\ell) = \prod_{j=0}^{m-1} (q^{\omega_{n-1+jn} \omega_{n+jn}})^{-1} v(\ell) = q^{-m+\sum_{j=1}^{m} \ell_{n-1,j} + \sum_{j=1}^{m} \ell_{nj}} v(\ell).$$

Combining equations implies that

$$\mu_n = \frac{1}{2} (\mu, \alpha_n - \alpha_{n-1}) = -\frac{m}{2} + \sum_{j=1}^{m} \ell_{nj}.$$

Backward substitution into the remaining equations yields the desired result.

We now turn to studying the commutant of the $U_q(\mathfrak{so}_{2n})$-action on $S^{\otimes m} \cong \Lambda_q(V^{(nm)})$. The classical duality result for orthogonal groups relies on the embedding $O(\mathbb{C}^m) \subset GL(\mathbb{C}^m)$ at the Lie group level, which induces a map $\mathfrak{so}_m \to \mathfrak{gl}_m$ at the level of Lie algebras. The last map has no quantum analogue. In other words, there is no algebra homomorphism $U_q(\mathfrak{so}_m) \to U_q(\mathfrak{gl}_m)$.

Notwithstanding, the commutant of the $U_q(\mathfrak{so}_{2n})$ action on $S^{\otimes m}$ is generated by the non-standard deformation $U'_q(\mathfrak{so}_m)$ of the Lie algebra $\mathfrak{so}_m$, which can be realized as a co-ideal subalgebra of $U_q(\mathfrak{gl}_m)$. Realizing $U'_q(\mathfrak{so}_m)$ as a subalgebra of $U_q(\mathfrak{gl}_m)$ allows us to use our skew $U_q(\mathfrak{gl}_m) \otimes U_q(\mathfrak{gl}_m)$-duality result [Abo22, Theorem 3.17] for Type A in order to compute the centralizer of $U_q(\mathfrak{so}_{2n})$ in $\text{End}(S^{\otimes m})$. 
Definition 4.13. The non-standard deformation \( U'_q(\mathfrak{so}_m) \) of \( \mathfrak{so}_m \) is the unital associative algebra generated by \( B_1, \ldots, B_{m-1} \) subject to the relations \( B_i B_j = B_j B_i \), whenever \( i \neq j \), and
\[
B_j^2 B_{j \pm 1} - (q + q^{-1}) B_j B_{j \pm 1} B_j + B_{j \pm 1} B_j^2 = B_{j \pm 1}.
\] (4.6)

Remark 4.14. Relation (4.6) is sometimes referred to as the \( q \)-Serre relation. In particular, see [RW17]. Note that it differs from the Serre relation defining a Drinfeld-Jimbo quantum group in that the right hand side is not identically zero.

Proposition 4.15. Let \( E_j, F_j, K_j \), for \( j = 1, \ldots, m-1 \) denote generators for \( U_q(\mathfrak{sl}_m) \subset U_q(\mathfrak{gl}_m) \). The algebra \( U'_q(\mathfrak{so}_m) \) can be realized as a left co-ideal subalgebra of \( U_q(\mathfrak{gl}_m) \) by setting
\[
B_j = \sqrt{-1}(F_j - q^{-1} K_j E_j).
\]

Proof. First notice that the algebra generated by the \( B_j \) is a co-ideal subalgebra of \( U_q(\mathfrak{gl}_m) \): for any \( j = 1, \ldots, m-1 \),
\[
-\sqrt{-1} \Delta(B_j) = F_j \otimes 1 + K_j^{-1} \otimes F_j - q(K_j^{-1} \otimes K_j^{-1})(E_j \otimes K_j + 1 \otimes E_j)
= B_j \otimes 1 + K_j^{-1} \otimes B_j \in U_q(\mathfrak{sl}_m) \otimes U'_q(\mathfrak{so}_m).
\]

Next observe that if \( |i - j| > 1 \) then \([ E_i, F_j ] = [ F_i, E_j ] = 0 \) by the quantum Serre relations in \( U_q(\mathfrak{gl}_m) \), so \([ B_i, B_j ] = 0 \) as well.

It remains to show that the \( B_j \) satisfy Relation (4.6). This follows from a straightforward yet somewhat tedious and unenlightening calculation that is left to the reader as an exercise.

Next we embed \( U'_q(\mathfrak{so}_m) \) into \( Cl_q(nm) \). We have done all the heavy lifting already: we need only combine Proposition 4.15, which expresses each generator of \( U'_q(\mathfrak{so}_m) \) in terms of \( U_q(\mathfrak{gl}_m) \) generators, with [Abo22, Proposition 3.15], which maps \( U_q(\mathfrak{gl}_m) \) into \( Cl_q(nm) \). The resulting map equips the \( Cl_q(nm) \)-module \( S^\otimes m \cong \bigwedge_q(V^{(nm)}) \) with a \( U'_q(\mathfrak{so}_m) \)-action.

Proposition 4.16. Recall the map \( \rho_q: U_q(\mathfrak{gl}_m) \to Cl_q(nm) \) defined in [Abo22, Proposition 3.15]. There is an algebra map \( \mathcal{R}_q: U'_q(\mathfrak{so}_m) \to Cl_q(nm) \) resulting from the composition
\[
U'_q(\mathfrak{so}_m) \longrightarrow U_q(\mathfrak{gl}_m) \overset{\rho_q}{\longrightarrow} Cl_q(nm)
\]

We conclude this section by showing that \( \mathcal{L}_q: U_q(\mathfrak{so}_2n) \to Cl_q(nm) \) and \( \mathcal{R}_q: U'_q(\mathfrak{so}_m) \to Cl_q(nm) \) induce commuting actions on the \( Cl_q(nm) \)-module \( S^\otimes m \).

Proposition 4.17. The embeddings \( \mathcal{L}_q \) and \( \mathcal{R}_q \) defined in Propositions 4.9 and 4.16 induce commuting subalgebras of \( \text{End} \left( \bigwedge_q(V^{(nm)}) \right) \).

Proof. The proof follows from a calculation. In [Abo22, Proposition 3.16] we show \( U_q(\mathfrak{gl}_n) \subset U_q(\mathfrak{so}_2n) \) and \( U_q(\mathfrak{gl}_m) \) induce commuting actions on \( \bigwedge_q(V^{(nm)}) \). Proposition 4.15 realizes \( U'_q(\mathfrak{so}_m) \) as a subalgebra of \( U_q(\mathfrak{gl}_m) \), so it suffices to show
\[
[\mathcal{L}_q(K^{(2n)}_n), \mathcal{R}_q(B_i)] = [\mathcal{L}_q(E^{(2n)}_n), \mathcal{R}_q(B_i)] = [\mathcal{L}_q(F^{(2n)}_n), \mathcal{R}_q(B_i)] = 0.
\]

We use superscripts on quantum group generators to indicate the algebra to which they belong; for instance \( E^{(2n)}_n \) belongs to \( U_q(\mathfrak{so}_2n) \) while \( F^{(m)}_i \) lives in \( U_q(\mathfrak{gl}_m) \).

First notice that
\[
\mathcal{L}_q(K^{(2n)}_n) \mathcal{R}_q(F^{(m)}_j) \mathcal{L}_q(K^{(2n)}_n)^{-1} = \sum_{b=1}^{n} \left( \prod_{a=0}^{m-1} \omega_{n-1+bn} \omega_{n+an} \right) \psi_{b+(j-1)n} \psi_{b+jn} \left( \prod_{a=0}^{m-1} \omega_{n-1+an} \omega_{n+an} \right) = \mathcal{R}_q(F^{(m)}_j).
\]
We also have \( \mathcal{L}_q(K_n^{(2n)}) \mathcal{R}_q(F_j^{(m)}) \mathcal{L}_q(K_n^{(2n)})^{-1} = \mathcal{R}_q(F_j^{(m)}) \) for each \( j = 1, \ldots, m-1 \), so \( \mathcal{L}_q(K_n^{(2n)}) \equiv \mathcal{R}_q(F_j^{(m)}) = 0 \).

Next we compute \( \mathcal{L}_q(E_n^{(2n)}) \mathcal{R}_q(F_j^{(m)}) \) in two steps. For each \( j = 1, \ldots, m-1 \), let \( A_j \) denote the central anticommutator \( \{ \psi_{n-1+(j-1)n} \psi_{n-1+(j-1)n} \} \). By sliding all \( \omega_a \) generators to the left of a product and rearranging the \( \phi_a \) generators when possible, we see all terms in the commutator \( \mathcal{L}_q(E_n^{(2n)}) \mathcal{R}_q(F_j^{(m)}) \) vanish except for two:

\[
\mathcal{L}_q(E_n^{(2n)}) \mathcal{R}_q(F_j^{(m)}) = \sum_{n=1}^{m-1} \sum_{b=1}^{n-1} \kappa_{n,b} \psi_{n-1+(a-1)n} \psi_{n+b+jn} \psi_{b+(j-1)n}^\dagger
\]

Similarly, all summands in \( \mathcal{L}_q(E_n^{(2n)}) \mathcal{R}_q((K_j^{(m)})^{-1}E_j^{(m)}) \) vanish except for two:

\[
\mathcal{L}_q(E_n^{(2n)}) \mathcal{R}_q((K_j^{(m)})^{-1}E_j^{(m)}) = \mathcal{R}_q(K_j^{(m)})^{-1} \sum_{n=1}^{m-1} \sum_{b=1}^{n-2} \kappa_{n,b} \psi_{n-1+(a-1)n} \psi_{n+b+jn} \psi_{b+(j-1)n}^\dagger
\]
like terms to obtain and the commuting endomorphism algebras generated by subalgebras of weight $U$ roots and weights of commutants. This is Theorem 4.19. It may be verified using a calculation similar to the one above and it is left as an exercise for the reader. All of these computations follows from

\[
\mathcal{L}_q(E_n^{(2n)}), \mathcal{R}_q(B_j) = [\mathcal{L}_q(E_n^{(2n)}), \mathcal{R}_q(F_j^{(m)})] - q [\mathcal{L}_q(E_n^{(2n)}), \mathcal{R}_q((K_j^{(m)})^{-1}E_j^{(m)})]
\]

\[
= \kappa_{n,j} \left( \mathcal{L}_q(F_n^{(2n)}) \right) = q^{-1} \kappa_{n,j+1} \omega_{n+1}^{j+1} \left( \psi_{n+1}^{j+1} \psi_{n+1}^{j+1} - \omega_{n+1}^{j+1} \right).
\]

Every anticommutator $\{ \psi_a, \psi_a \}$ is central, so it acts as the identity on $\wedge_q(V)^{(nm)}$ [AS22, Lemma 2.8]. This means $(1 - q^{-1} \omega_{n+1}^{j+1} A_{j+1}) \psi_{n+1}^{j+1} = (A_j - q^{-1} \omega_{n+1}^{j+1}) \psi_{n+1}^{j+1}$ act as zero. Hence Relation (4.7) implies $\mathcal{L}_q(E_n^{(2n)})$ and $\mathcal{R}_q(B_j)$ induce commuting module endomorphisms.

It remains to check that $\mathcal{L}_q(E_n^{(2n)})$ and $\mathcal{R}_q(B_j)$ induce commuting endomorphisms. This follows from a calculation similar to the one above and it is left as an exercise for the reader. All of these computations may be verified using SAGEMath.

4.3. Multiplicity-free decomposition of $S^\otimes m$. In the previous section we found homomorphic images of $U_q(\mathfrak{so}_{2n})$ and $U'_q(\mathfrak{so}_m)$ in the quantum Clifford algebra $\mathcal{C}_q(nm)$. These maps generate commuting subalgebras of $\text{End} \left( \wedge_q(V)^{(nm)} \right)$. Recall Section 4.2 extends the $U_q(\mathfrak{so}_{2n})$-module $S$ into an irreducible $U_q(\mathfrak{so}_{2n})$-module. It turns out that $S^\otimes m \cong \wedge_q(V)^{(nm)}$ is irreducible as a $U_q(\mathfrak{so}_{2n}) \otimes U'_q(\mathfrak{so}_m)$-module and the commuting endomorphism algebras generated by $U_q(\mathfrak{so}_{2n})$ and $U'_q(\mathfrak{so}_m)$ are in fact each others commutants. This is Theorem 4.19.

Before we state our skew $U_q(\mathfrak{so}_{2n}) \otimes U'_q(\mathfrak{so}_m)$ duality theorem, we briefly discuss the representation theory of $U'_q(\mathfrak{so}_m)$. For the rest of this section, we assume $m = 2r$ or $m = 2r + 1$. There is a notion of roots and weights of $U'_q(\mathfrak{so}_m)$ and we may identify them with vectors in $\mathbb{R}^r$, as usual. Here the analogue of the Cartan subalgebra is the algebra $\mathfrak{b}'$ generated by $B_1, B_3, \ldots, B_{2r-1}$. A vector $v$ in a $U'_q(\mathfrak{so}_m)$-module is said to have weight $\mu$ if

\[
B_{2j-1} v = [\mu_j] v
\]

for all $j = 1, \ldots, r$. As usual, $[n] = (q^n - q^{-n})/(q - q^{-1})$ denotes a $q$-integer.
Theorem 4.18. [GK91] For each dominant $\mathfrak{so}_m$ weight $\mu$, there exists a finite dimensional simple $U_q'(\mathfrak{so}_m)$-module $W_{\mu}$ with highest weight $\mu$ and the same weight multiplicities as the corresponding $U(\mathfrak{so}_m)$-module.

The finite-dimensional $U_q'(\mathfrak{so}_m)$-module $W_{\mu}$ may be realized explicitly using a basis indexed by Gelfand-Tsetlin patterns as in the classical case [GK91]. This means $v_{\mu}$ is a highest weight vector in $V_{\mu}$ if it is a weight vector of dominant weight $\mu$.

Theorem 4.19. Let $S \cong S_+ \oplus S_-$ denote the so-called $U_q'(\mathfrak{so}_{2n})$ “spin” module. As a $U_q(\mathfrak{so}_{2n}) \otimes U_q'(\mathfrak{so}_m)$-module, $S^\otimes m \cong \bigwedge_q (V^{(nm)})$ is multiplicity-free. In particular,

$$S^\otimes m \cong \bigoplus_{\mu} V_{\mu} \otimes W_{\mu}$$

(4.8)

as a $U_q(\mathfrak{so}_{2n}) \otimes U_q'(\mathfrak{so}_m)$-module. The sum ranges over all partitions $\mu$ fitting in a $(2n) \times r$ rectangle. $V_{\mu}$ denotes the irreducible $U_q'(\mathfrak{so}_m)$-module indexed by $\mu$, $W_{\mu}$ denotes the irreducible $U_q(\mathfrak{so}_{2n})$-module indexed by $\nu$, and we take $\mu$ as in Relation (3.9).

Consequently, $U_q(\mathfrak{so}_{2n})$ and $U_q'(\mathfrak{so}_m)$ generate mutual commutants in $\text{End} (S^\otimes m)$.

Much like [Abo22, Theorem 3.17] proves that the irreducibles appearing in the decomposition of the braided exterior algebra $\bigwedge_q (V^{(nm)})$ as a $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-module are parametrized by the same weights as the irreducible components in the decomposition of $\bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m)$ as a $\mathfrak{gl}_n \otimes \mathfrak{gl}_m$-module, Theorem 4.19 shows that the $U_q(\mathfrak{so}_{2n}) \otimes U_q'(\mathfrak{so}_m)$-isotypic components in the decomposition of $S^\otimes m \cong \bigwedge_q (V^{(nm)})$ are labeled by the same weights as the irreducibles in the decomposition of $\bigwedge (\mathbb{C}^n \otimes \mathbb{C}^m)$ as a $U(\mathfrak{so}_{2n}) \otimes U(\mathfrak{so}_m)$-module.

To prove Theorem 4.19, we argue as in our proof of the skew duality result for Type A [Abo22, Theorem 3.17]. The first step is to identify, for each component in the decomposition of Relation (4.8), an element of $S^\otimes m \cong \bigwedge_q (V^{(nm)})$ that is a highest weight vector with respect to the joint $U_q(\mathfrak{so}_{2n}) \otimes U_q'(\mathfrak{so}_m)$-action. Keeping in mind the relationship between $U_q(\mathfrak{so}_{2n})$ and $U_q'(\mathfrak{so}_m)$-modules, as described in Section 4, we then use a dimension count and appeal to the classical result to conclude.

Our construction of joint $U_q(\mathfrak{so}_{2n}) \otimes U_q'(\mathfrak{so}_m)$-highest weight vectors begins with the highest weight vectors with respect to the $U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_m)$-action constructed in [Abo22, Lemma 3.19]: for each partition $\nu$ with at most $n$ parts satisfying $\nu_1 \leq m$, the corresponding highest weight vector $v_{\nu}$ is the product of all basis vectors in Relation (4.5) that fit in the Young diagram defined by $\nu$.

Now consider a partition $\mu$ with at most $n$ parts satisfying $\mu \leq r$. Notice $\nu$ can have twice as many columns as $\mu$. For each $\nu = (\nu_1, \ldots, \nu_n)$ with $0 \leq \nu_i \leq m$, let

$$v_{\nu}^* = (v_{1+(m-1)n} \cdots v_{1+(m-\nu_1-1)n}) \cdots (v_{1+(m-1)n} \cdots v_{1+(m-\nu_n-1)n}).$$

As illustrated in Figure 1, $v_{\nu}^*$ determines the state obtained by reflecting the state determined by $v(\sum_{i=1}^{n} \sum_{j=1}^{\nu_i} e_{i+(j-1)n})$ across the vertical mid-segment.

![Figure 1](image-url)
For each $p = (p_1, \ldots, p_r)$, with $0 \leq p_j \leq n$, define

$$w^p = \prod_{j=1}^r \left( \prod_{a=1}^{p_j} v_{n-a+2(j-1)n} \right)$$

The blue boxes in Figure 2 illustrate the vectors $w^p$ when $(n, m) = (3, 4)$ and $p = (2, 1)$. Let $\#_k(p)$ denote the number of entries in $p$ that are at least $k$, and set $p^c = (n - \#_1(p), \ldots, n - \#_n(p))$.

Notice $0 \leq (p^c)_i \leq m$ for each $i = 1, \ldots, n$. For any tuple $z$ in $\mathbb{Z}_+^n$, let $2z$ denote $(2z_1, \ldots, 2z_n)$. Now define

$$\xi_p = v^*_{2p^c} w^p. \quad (4.9)$$

**Figure 2.** The element $\xi_\mu = v^*_{2\mu^c} w^\mu$ of $\wedge_q(V^{(nm)})$ represented by states of occupied and vacant positions in an $n \times m$ grid, with $(n, m) = (3, 4)$, $\mu = (2, 1)$, and $p = (2, 1)$.

$$\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{33} \\
\end{array}
\end{matrix}
\xrightarrow{\mathcal{R}_q(F_1^{(m)})}
\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{23} & \rho_{24} \\
\rho_{33} \\
\end{array}
\end{matrix}
+ \begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{33} \\
\end{array}
\end{matrix}
$$

$$\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{23} & \rho_{24} \\
\rho_{32} & \rho_{33} \\
\end{array}
\end{matrix}
\xrightarrow{\mathcal{R}_q(F_1^{(m)})}
\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{23} & \rho_{24} \\
\rho_{32} & \rho_{33} \\
\end{array}
\end{matrix}
$$

$$\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{23} & \rho_{24} \\
\rho_{33} & \\
\end{array}
\end{matrix}
\xrightarrow{\mathcal{R}_q(F_1^{(m)})}
\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{23} & \rho_{24} \\
\rho_{33} & \\
\end{array}
\end{matrix}
$$

**Figure 3.** Shifting occupied positions in the 1st pair of columns to the right by applying $F_2^{(m)} \in U_q(\mathfrak{gl}_m)$ to the $U_q(\mathfrak{so}_{2n})$ weight vector $\xi_p$, with $p = (2, 1)$ and $(n, m) = (3, 4)$. Here $\alpha_q, \alpha'_q$, and $\beta_q$ are some coefficients.

The next lemma records useful properties of $\xi^p$. For any partition $\mu$, we use $\mu'$ to denote the conjugate of $\mu$. 

$$\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{23} & \rho_{24} \\
\rho_{33} & \\
\end{array}
\end{matrix}
\xrightarrow{\mathcal{R}_q(F_1^{(m)})}
\begin{matrix}
\begin{array}{cccc}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{23} & \rho_{24} \\
\rho_{33} & \\
\end{array}
\end{matrix}
$$
Lemma 4.20. For any $p = (p_1, \ldots, p_r)$ with $p_j \leq n$, let $\xi^p$ denote the vector defined by Relation (4.9). Then $\xi^p$ is

1. always nonzero.
2. a weight vector with respect to the action of $U_q(\mathfrak{sl}_{2n})$ with weight $\mu$ if $p_j = n - \mu'_j$ for some partition $\mu$ with at most $n$ parts satisfying $\mu'_1 \leq r$.
3. a highest weight vector in the $(p_j + 1)$-dimensional module $U_q(\mathfrak{sl}_2) \otimes \xi^p$ of the copy of $U_q(\mathfrak{sl}_2) \subset U_q(\mathfrak{gl}_m)$ generated by $E_{2j-1}^{(m)}$, $E_{2j}^{(m)}$, $F_{2j-1}^{(m)}$.
4. a highest weight vector with respect to the action of $U_q(\mathfrak{gl}_n) \subset U_q(\mathfrak{so}_{2n})$. In particular, it is annihilated by $L_q(E_i^{(n)}),$ for $i = 1, \ldots, n - 1.$

Proof. To see that $\xi^p$ is nonzero, notice that $v_{-\mu_j}^* \xi^p$ and $w^p$ do not share common factors. Lemma 4.12 determines the weight of $\xi^p$. When $p_j = n - \mu'_j$, each position in the $n \times m$ grid is occupied by $v_{2\mu}$, occupied by $w^p$, or it is paired bijectively to a box occupied by $w^p$. Since each occupied (vacant) position in the $i$th row contributes $+1/2$ ($-1/2$) to the $i$th component of the weight, the positions occupied by $v_{2\mu}^*$ contribute $\mu$ while boxes occupied by $w^p$ have no overall contribution to the weight.

Figures 3 and 4 sketch the proof of (3). The general case follows from observing that the formulas defining $R_{\gamma}$ in Proposition 4.9 imply that, e.g., when $p_j \geq 0$, each application of $R_{\gamma}(E_{2j-1}^{(m)})$ on $\xi^p$ moves one occupied position in the $(2j - 1)$st column below the $\mu'_j$th row to the right, and a further application of $R_{\gamma}(F_{2j-1}^{(m)})$ when all the occupied positions below the $\mu'_j$th row in the $(2j)$th column kills the vector.

An analogous story holds for $R_{\gamma}(E_{2j-1}^{(m)})$ when $p_j \leq 0$.

Statement (4) follows immediately from the definition of $L_q : U_q(\mathfrak{gl}_n) \to Cl_q(nm)$ in Proposition 3.10, since every box above an occupied box in $\xi^p$ is occupied.

We are now ready to construct the $U_q'(\mathfrak{so}_m)$ weight vectors in $S^{\otimes m}$.

Lemma 4.21. Let $m = 2r$ or $m = 2r + 1$. Consider any $p = (p_1, \ldots, p_r)$, with $-n \leq p_j \leq n$, for each $j = 1, \ldots, r$. Define $\text{abs}(p) = (|p_1|, \ldots, |p_r|)$ and let $s = (s_1, \ldots, s_r)$, with $s_j = \text{sgn}(p_j) \in \{\pm 1\}$, denote the sign of $p_j$. Set $I_{p_j} = \{0, 1, 2, \ldots, |p_j|\}$, and let $I^p = I_{p_1} \times \cdots \times I_{p_r}$. For each $a \in I$, let $\theta(a) = -\frac{1}{2} \sum_j (|p_j| - a_j)(|p_j| - a_j - 1)$, and set $|a|! = |a_1|! \cdots |a_r|!$ and $F^a = \prod_{j=1}^r R_q(F_{2j-1}^{(a)}).$ Now define

$$
\Xi_p = \frac{i^{\text{abs}(p)} F^a}{|a|!} \xi_{\text{abs}(p)}.
$$

(4.10)

Here $i = \sqrt{-1}$ and $\cdot \cdot$ denotes the usual dot product of $s$ and $a$.

Then $\Xi_p$ is a $U_q'(\mathfrak{so}_m)$ weight vector of weight $p$ when $m = 2r$ is even. When $m = 2r + 1$ is odd, multiply $\Xi_p$ by $v(\sum_{i=1}^n c_{i+j-1})$ to fill its last column and obtain a $U_q'(\mathfrak{so}_m)$ weight vector of weight $p + (1/2, \ldots, 1/2, +1/2).$ Similarly, multiply $\Xi_p$ by $v(\sum_{i=1}^{n-1} c_{i+j-1})$ to obtain a weight vector of weight $p + (1/2, \ldots, 1/2, -1/2).$
Proof. We show that $\Xi_p$ is a $[p_j]$-eigenvector of each $B_{2j-1}$. This boils down to a calculation in the copy of $U_q(\mathfrak{sl}_2) \subset U_q(\mathfrak{g}_m)$ generated by $E_{2j-1}, F_{2j-1}, K_{2j-1}$, since $[E_{2j-1}, F_{2k-1}] = [F_{2j-1}, F_{2k-1}] = 0$ whenever $j \neq k$. In what follows we only consider the $j$th inner sum defining $\Xi_p$.

Applying $F_{2j-1}$ merely shifts the index of summation:

$$F_{2j-1} \triangleright \sum_{b=0}^{[p_j]} i^{s_j b} \frac{q^{\theta(b)} R_q(F_{2j-1})}{[b]!} \xi_{abs(p)}(F_{2j-1})^b \xi_{abs(p)} = i^{s_j b} \left\{ i^{s_j(b+1)} \frac{q^{\theta(b+1) - ([p_j] - (b+1))} R_q(F_{2j-1})}{[b]!} \xi_{abs(p)}(F_{2j-1})^b \xi_{abs(p)} \right\}.$$

In the second equality we used the identity $\theta(b) = -\frac{1}{2}([p_j] - b - 1) ([p_j] - b - 2) - [p_j] + b - 1 = \theta(b + 1) - ([p_j] - (b + 1))$. The term with $F_{2j-1}^{[p_j] + 1}$ vanishes because $\xi_{abs(p)}$ is a highest weight vector in a $(|p_j| + 1)$-dimensional $U_q(\mathfrak{sl}_2)$-module.

Next we calculate

$$qK_{2j-1}^{-1} E_{2j-1} \triangleright \sum_{b=0}^{[p_j]} i^{s_j b} \frac{q^{\theta(b)} R_q(F_{2j-1})}{[b]!} \xi_{abs(p)}(F_{2j-1})^b \xi_{abs(p)}.$$

The second equality follows from the $U_q(\mathfrak{sl}_2)$ identity

$$[E_{2j-1}, F_{2j-1}^b] = [b] F_{2j-1}^{b-1} q^{-(b-1)} K_{2j-1} - q^{-(b-1)} K_{2j-1}^{-1} q - q^{-1}$$

that is proved in, e.g., Lemma VI.1.3 of [Kas95], and the easy identity $\theta(b) = -\frac{1}{2} (|p_j| - (b - 1)) (|p_j| - b) + (|p_j| - (b - 1)) - 1 = \theta(b - 1) + (|p_j| - (b - 1)) - 1$.

Let $\mathcal{I}_p = \mathcal{I}_{p_1} \times \cdots \times \mathcal{I}_{p_{j-1}} \times \mathcal{I}_{p_j} \times \cdots \times \mathcal{I}_{p_{m-1}}$ denote the set obtained by omitting $\mathcal{I}_{p_j}$ from the product. Taking the $b = 0$ term from the second calculation, the $b = |p_j|$ term from the first, and adding the rest, we find that $\Xi_p$ is indeed a $[p_j]$-eigenvector of $B_{2j-1}$:

$$B_{2j-1} \triangleright \Xi_p = iR_q(F_{2j-1} - q^{-1} K_{2j-1}) E_{2j-1} \triangleright \Xi_p.$$

$$= i^{1 - s_j} \sum_{a \in \mathcal{I}_p} i^{s_aj} \frac{q^{\theta(a)}}{[a]!} \left( q^{\frac{1}{2} |p_j| - (|p_j| - 1) [p_j]} \right)$$

$$+ \sum_{b=1}^{[p_j]} i^{s_j b} \frac{R_q(F_{2j-1})^{p_j}}{[b]!} q^{\theta(b)} \left( q^{b [p_j] - b} + q^{-((|p_j| - b) [b]}) \right) \xi_{abs(p)}(F_{2j-1})^b \xi_{abs(p)}.$$
shows that

\[
\eta q^m [n] = \eta q^{-m} [n] = [n + m].
\]

Finally, we obtain joint highest weight vectors in \( S^{\otimes m} \cong \wedge_q (V^{(nm)}) \) with respect to the \( U_q (\mathfrak{so}_{2n}) \otimes U'_q (\mathfrak{so}_m) \) action.

**Lemma 4.22.** Let \( \mu \) denote a partition with at most \( n \) parts satisfying \( \mu_1 \leq r \) and take \( \bar{\mu} \) as in Relation (3.9) and \( \Xi_{p} \) as in Relation (4.10). Then \( \Xi_{\bar{\mu}} \) is a \( U_q (\mathfrak{so}_{2n}) \) highest weight vector of weight \( \mu \).

**Proof.** Lemma 4.20 shows that \( \xi_{\bar{\mu}} \) has weight \( \mu \) with respect to the \( U_q (\mathfrak{so}_{2n}) \) action and that \( \mathcal{L}_q (E_{i}^{(2n)}) \xi_{\bar{\mu}} = 0 \) for each \( i < n \). Since \( \mathcal{L}_q \) and \( \mathcal{R}_q \) induce commuting module endomorphisms by Proposition 4.17, \( \Xi_{\bar{\mu}} \) also has weight \( \mu \) with respect to the \( U_q (\mathfrak{so}_{2n}) \) action and \( \mathcal{L}_q (E_{i}^{(m)}) \Xi_{\bar{\mu}} = 0 \) for each \( i < n \).

It remains to show \( \Xi_{\bar{\mu}} \) is also annihilated by \( E_n \in U_q (\mathfrak{so}_{2n}) \). This is clear when \( \bar{\mu}_1 \leq 1 \), since every position in the \((n - 1)\)st row of \( \xi_{\bar{\mu}} \) is already occupied. Let \( \eta = \mathcal{L}_q (E_{i}^{(2n)}) \xi_{\bar{\mu}} \) and suppose \( \bar{\mu}_1 \geq 2 \). Then \( \mathcal{R}_q (F_{2j-1})^{j-1} \eta \) is also annihilated by \( E_n \), so \( \eta \) generates \( U_q (\mathfrak{so}_{n}) \)-module of dimension \( \bar{\mu}_1 - 1 \) by the same \( \mathcal{L}_q \) and \( \mathcal{R}_q \) induce commuting module endomorphisms. \( \eta \) must also be \( U'_q (\mathfrak{so}_m) \) weight vector of weight \( \bar{\mu} \). But this contradicts the eigenvalue calculation of Lemma 4.21 unless \( \eta \) is zero. \( \square \)

(Proof of Theorem 4.19). Combined with Relation (4.10), Lemma 4.22 proves that \( \Xi_{\bar{\mu}} \) is a joint highest weight vector for the \( U_q (\mathfrak{so}_{2n}) \otimes U'_q (\mathfrak{so}_m) \) action on \( S^{\otimes m} \) for each partition \( \mu \) with at most \( n \) parts satisfying \( \mu_1 \leq r \).

Now consider partitions \( \mu \) with at most \( 2n \) parts satisfying \( \mu_1 + \mu_2 \leq 2n \). If \( \mu \neq \mu \), then we obtain two linearly independent joint highest weight vectors \( \Xi_{\bar{\mu}} \) and \( \Xi_{\bar{\mu}'} \) with the same \( U_q (\mathfrak{so}_{2n}) \) weight, but distinct \( U'_q (\mathfrak{so}_m) \) weight. However, these generate inequivalent \( U_q (\mathfrak{so}_{n}) \)-modules. Conversely if \( \mu = \mu \), then there is a unique irreducible \( U_q (\mathfrak{so}_{n}) \)-module that extends to an irreducible \( U'_q (\mathfrak{so}_m) \)-module.

Hence, we have exhibited a joint highest weight vector for each irreducible module appearing in the decomposition Relation (4.8). Since the dimension of each \( U_q (\mathfrak{so}_{2n}) \)- and each \( U'_q (\mathfrak{so}_m) \)-module equals that of its classical counterpart, the theorem follows: a dimension count together with the classical duality result guarantee that we have exhausted every possible irreducible component. \( \square \)

As in the classical case, we obtain a decomposition of \( \bigwedge_q (V^{(nm)}) \) as a \( U_q (\mathfrak{so}_n) \otimes U_q (\mathfrak{so}_{2m}) \)-modules by considering the isomorphism of \( U_q (\mathfrak{so}_n) \)-modules \( S_m^{\otimes n} \cong \bigwedge_q (V^{(m)})^{\otimes n} \cong \bigwedge_q (V^{(nm)}) \). This decomposition completes the seesaw of Diagram (1.3).

**REFERENCES**

[Abo22] Willie Aboumrad. Skew Howe duality for \( U_q (\mathfrak{gl}_n) \) via quantized Clifford algebras. 2022.

[AS22] Willie Aboumrad and Travis Scrimshaw. Algebraic structure and representation theory of quantized Clifford algebras. (In preparation), 2022.

[Bun04] Daniel Bump. *Lie Groups*. Springer, second edition, 2004. doi:10.1007/978-1-4757-4094-3.

[BZ05] Arkady Berenstein and Sebastian Zwicknagl. Braided symmetric and exterior algebras. 2005. URL: https://arxiv.org/abs/math/0504155, doi:10.48550/ARXIV.MATH/0504155.

[CW15] Shawn X. Cui and Zhenghan Wang. Universal quantum computation with metaplectic anyons. *Journal of Mathematical Physics*, 56(3):032202, 2015. doi:10.1063/1.4914941.

[GGL22] Thomas Gerber, Jérémie Guilhot, and Cédric Lecouvey. Generalised Howe duality and injectivity of induction: The symplectic case. *Combinatorial Theory*, 2(2), 2022. doi:10.5070/c62257878.

[GJ97] A. M. Gavriliuk and N. Z. Iorgov. q-deformed algebras \( U_q (\mathfrak{so}_n) \) and their representations. 1997. URL: https://arxiv.org/abs/q-alg/9709036, doi:10.48550/ARXIV.Q-ALG/9709036.

[GK91] A. M. Gavriliuk and A. U. Klimyk. q-deformed orthogonal and pseudo-orthogonal algebras and their representations. *Letters in Mathematical Physics*, 21(3):215-220, 1991. doi:10.1007/bf00420371.
[GW09] Roe Goodman and Nolan Wallach. *Symmetry, representations, and invariants*. Springer, first edition, 2009. doi:10.1007/978-0-387-79852-3.

[Hay90] Takahiro Hayashi. Q-analogues of Clifford and Weyl algebras-spinor and oscillator representations of quantum enveloping algebras. *Communications in Mathematical Physics*, 127(1):129–144, 1990. doi:10.1007/bf02096497.

[HK22] Taehyeok Heo and Jae-Hoon Kwon. Combinatorial howe duality of symplectic type. *Journal of Algebra*, 600:1–44, Jun 2022. doi:10.1016/j.jalgebra.2022.01.037.

[HNW13] Matthew B. Hastings, Chetan Nayak, and Zhenghan Wang. Metaplectic anyons, majorana zero modes, and their computational power. *Physical Review B*, 87(16), 2013. doi:10.1103/physrevb.87.165421.

[HNW14] Matthew B. Hastings, Chetan Nayak, and Zhenghan Wang. On metaplectic modular categories and their applications. *Communications in Mathematical Physics*, 330(1):45–68, 2014. doi:10.1007/s00220-014-2044-7.

[IK00] N.Z. Iorgov and A.U. Klimyk. Representations of the nonstandard (twisted) deformation $U_q(s^o(n))$ for $q$ a root of unity. *Czechoslovak Journal of Physics*, 50(11):1257–1263, 2000. doi:10.1023/a:1022813108279.

[IK05] N.Z. Iorgov and A.U. Klimyk. Classification theorem on irreducible representations of the $q$-deformed algebra $U_q(s^o(n))$. *International Journal of Mathematics and Mathematical Sciences*, (2):225–262, 2005. doi:10.48550/ARXIV.MATH/0702482.

[Kac83] V.G. Kac. *Infinite Dimensional Lie Algebras*. Cambridge University Press, 1983. doi:10.1007/978-1-4757-1382-4.

[Kas95] Christian Kassel. *Quantum Groups*. Springer-Science, first edition, 1995. doi:10.1007/978-1-4612-0783-2.

[KL08] Stefan Kolb and Gail Letzter. The center of quantum symmetric pair coideal subalgebras. *Representation Theory of the American Mathematical Society*, 12(12):294–326, 2008. doi:10.1090/s1088-4165-08-00332-4.

[Kwo14] Jae-Hoon Kwon. Q-deformed Clifford algebra and level zero fundamental representations of quantum affine algebras. *Journal of Algebra*, 399:927–947, 2014. doi:10.1016/j.jalgebra.2013.10.026.

[Let00] Gail Letzter. Harish-chandra modules for quantum symmetric pairs. *Representation Theory of the American Mathematical Society*, 4(5):64–96, 2000. doi:10.1090/s1088-4165-00-00087-x.

[Let19] Gail Letzter. Cartan subalgebras for quantum symmetric pair coideals. *Representation Theory of the American Mathematical Society*, 23(3):88–153, 2019. doi:10.1090/ert/523.

[LZZ10] G. I. Lehrer, Hechun Zhang, and R. B. Zhang. A quantum analogue of the First Fundamental Theorem of classical invariant theory. *Communications in Mathematical Physics*, 301(1):131–174, 2010. doi:10.1007/s00220-010-1143-3.

[Mon93] Susan Montgomery. Hopf algebras and their actions on rings. *CBMS Regional Conference Series in Mathematics*, 1993. doi:10.1090/cbms/092.

[NPS21] Anton Nazarov, Olga Postnova, and Travis Scrimshaw. Skew howe duality and limit shapes of Young diagrams, 2021. URL: https://arxiv.org/abs/2111.12426, doi:10.48550/ARXIV.2111.12426.

[RW11] Eric C. Rowell and Zhenghan Wang. Localization of unitary braid group representations. *Communications in Mathematical Physics*, 311(3):595–615, 2011. doi:10.1007/s00220-011-1386-7.

[RW17] Eric Rowell and Hans Wenzl. SO(N)_2 braid group representations are Gaussian. *Quantum Topology*, 8(1):1–33, 2017. doi:10.4171/qt/85.

[ST18] Antonio Sartori and Daniel Tubbenhauer. Webs and $q$-howe dualities in types BCD. *Transactions of the American Mathematical Society*, 371(10):7387–7431, 2018. doi:10.1090/tran/7583.

[The22] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 9.6.0)*, 2022. URL: https://www.sagemath.org.

[Wen20a] Hans Wenzl. Dualities for spin representations, 2020. URL: https://arxiv.org/abs/2005.11299, doi:10.48550/ARXIV.2005.11299.

[Wen20b] Hans Wenzl. On representations of $U_q(s^o(n))$. *Transactions of the American Mathematical Society*, 373(5):3295–3322, 2020. doi:10.1090/tran/7983.

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