On unitary representations of $GL_{2n}$ distinguished by the symplectic group

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Abstract

We provide a family of representations of $GL_n$ over a $p$-adic field that admit a non-vanishing linear functional invariant under the symplectic group (i.e. representations that are $Sp(n)$-distinguished). This is a generalization of a result of Heumos–Rallis. Our proof uses global methods. The results of [Omer Offen, Eitan Sayag, Global mixed periods and local Klyachko models for the general linear group, submitted for publication] imply that the family at hand contains all irreducible, unitary representations that are distinguished by the symplectic group.

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Let $F$ be a $p$-adic field and let $G = GL_{2n}(F)$. We denote by

$$H_x = \{ g \in G \mid g'xg = x \}$$

the symplectic group associated with the skew symmetric matrix $x \in G$. We further denote by $H$ the group $H_{\epsilon_{2n}}$ where

$$\epsilon_{2n} = \begin{pmatrix} -w_n & w_n \\ w_n & -w_n \end{pmatrix}$$

and $w_n$ is the $n \times n$ permutation matrix with unit anti-diagonal.

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Definition 1. A representation $\pi$ of $G$ is called $H$-distinguished if

$$\text{Hom}_H(\pi, \mathbb{C}) \neq 0.$$ 

In this work we show that a certain family of irreducible, unitary representations of $G$ are distinguished by the symplectic group $H$. In an upcoming work [OS] we show, in particular, that this family exhausts the irreducible, unitary, $H$-distinguished representations of $G$.

Our interest in local symplectic periods is motivated by the work of Klyachko over finite fields [Kl84]. In [HR90], Heumos and Rallis began the study of an analogue for $p$-adic fields. A survey of their work, with some motivation and the relation to periods of automorphic forms, can also be found in [Heu93]. Let us briefly describe the problem at hand.

Let $\psi$ be an additive character of $F$ and let $U_q$ denote the group of upper triangular unipotent matrices in $GL_q(F)$. We denote by

$$\psi_q(u) = \psi(u_{1,2} + \cdots + u_{q-1,q})$$

the associated character of $U_q$. We will also denote by $H_{2q}$ the symplectic group $H_{\epsilon_{2q}}$. For $0 \leq k \leq \lfloor \frac{q}{2} \rfloor$, let $H_{q,k}$ be the subgroup of $GL_q(F)$ of matrices of the form

$$\begin{pmatrix} u & X \\ 0 & h \end{pmatrix}$$

where $u \in U_{q-2k}$, $h \in H_{2k}$ and $X \in M(q-2k) \times 2k(F)$. We denote by $\psi_{q,k}$ the character

$$\psi_{q,k} \begin{pmatrix} u & X \\ 0 & h \end{pmatrix} = \psi_{q-2k}(u).$$

We refer to the spaces

$$\mathcal{M}_{q,k} = \text{Ind}_{H_{q,k}}^{GL_q(F)}(\psi_{q,k})$$

as Klyachko models. The model $\mathcal{M}_{2n,n}$ is referred to as a symplectic model and the Klyachko models interpolate between a Whittaker model and (if $q$ is even) a symplectic model. An irreducible representation $\pi$ of $GL_q(F)$ is said to have the Klyachko model $\mathcal{M}_{q,k}$ if

$$\text{Hom}_{GL_q(F)}(\pi, \mathcal{M}_{q,k}) \neq 0.$$ 

Note that a representation is $H$-distinguished if and only if it has a symplectic model. In [Kl84], Klyachko showed that each irreducible representation of $GL_q$ over a finite field has a unique Klyachko model. In [HR90], Heumos and Rallis provide evidence that every irreducible, unitary representation of $GL_q(F)$ has a Klyachko model. In fact, they prove this fact for $q \leq 4$. They also show that irreducible, unitary representations can imbed in at most one of the different Klyachko models $\mathcal{M}_{q,k}$. We refer to [Heu93, p. 143] for the local conjecture and its global analogue. The present work is a step towards proving the conjectures of [Heu93, p. 143]. In [OS], we will show that any irreducible unitary representation has a Klyachko model. Moreover, we specify the model it has in terms of the Tadic parameter of the representation. The exact description of a Klyachko model for any unitary representation, together with the main result of the present
work and the unitary disjointness of models [HR90, Theorem 3.1], imply that the representations that we consider in Theorem 1 are precisely all irreducible, unitary representations that are distinguished by $H$.

To state our main theorem we briefly review Tadic’s classification of the unitary dual of $G$ [Tad86]. Denote by $\nu$ the character $g \mapsto |\det g|$ on $GL_q(F)$ for any $q$. For representations $\pi_i$ of $GL_{q_i}(F)$, $i = 1, \ldots, t$, and for $q = q_1 + \cdots + q_t$, we denote by $\pi_1 \times \cdots \times \pi_t$ the representation of $GL_q(F)$ obtained from $\pi_1 \otimes \cdots \otimes \pi_t$ by normalized parabolic induction. For a representation $\tau$ of $GL_q(F)$ and $\alpha \in \mathbb{R}$, we denote $\pi(\tau, \alpha) = \nu^{\alpha \tau} \times \nu^{-\alpha \tau}$. For representations $\pi_i$ of $GL_{q_i}(F)$ set $\pi = \pi_1 \otimes \cdots \otimes \pi_t$ and let $\lambda = (\lambda_1, \ldots, \lambda_t) \in \mathbb{C}^t$. We denote $\pi[\lambda] = \nu^{\lambda_1} \pi_1 \otimes \cdots \otimes \nu^{\lambda_t} \pi_t$ and

$$I(\pi, \lambda) = \nu^{\lambda_1} \pi_1 \times \cdots \times \nu^{\lambda_t} \pi_t.$$ 

Let

$$\Lambda_m = \left( \frac{m - 1}{2}, \frac{m - 3}{2}, \ldots, \frac{1 - m}{2} \right) \in \mathbb{R}^m.$$ 

A representation of $GL_r(F)$ is called square integrable if its matrix coefficients are square integrable modulo the center. Square integrable representations are in particular unitary. For a square integrable representation $\delta$ of $GL_r(F)$, the representation $I(\delta^{\otimes m}, \Lambda_m)$ has a unique irreducible quotient which we denote by $U(\delta, m)$. Let

$$B_u = \left\{ U(\delta, m), \pi(U(\delta, m), \alpha) : \delta\text{-square integrable, } m \in \mathbb{N}, |\alpha| < \frac{1}{2} \right\}.$$ 

A representation of the form $\sigma_1 \times \cdots \times \sigma_t$ where $\sigma_i \in B_u$, is irreducible and unitary. Any irreducible, unitary representation of $GL_q(F)$ for some $q$ has this form and is uniquely determined by the multi-set of $\sigma_i$’s. This is the classification of Tadic. Our main result is the following.

**Theorem 1.** Let $\pi = \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_s$ be a unitary representation of $G$, such that $\sigma_i = U(\delta_i, 2m_i) \in B_u$ and $\tau_i = \pi(U(\delta_i, 2m_i), \alpha_i) \in B_u$. Then $\pi$ is $H$-distinguished.

In fact, we prove in Proposition 2 that $\pi$ is $H$-distinguished for a wider family of—not necessarily unitary—representations. Theorem 1 is a generalization of a result of Heumos–Rallis. They showed in [HR90] that the representations $U(\delta, 2)$ are $H$-distinguished. Their argument is the following. First, they construct a non-vanishing $H$-invariant functional on $I(\delta \otimes \delta, \Lambda_2)$. This representation has length 2 and its unique irreducible subrepresentation has a Whittaker model. The existence of an $H$-invariant functional on $U(\delta, 2)$ is therefore a consequence of the fact that irreducible generic representations are not $H$-distinguished. This is a special case of [HR90, Theorem 3.1]. The method of proof of Heumos–Rallis does not generalize directly to the case $m > 1$. In Remark 1 we explain where the difficulties lie. Our proof of Theorem 1 is in two steps. We first use global methods to show that the building blocks $U(\delta, 2m)$ are $H$-distinguished. We introduce in (1) a non-zero $H$-invariant functional $j_H$ on $I(\delta^{\otimes 2m}, \Lambda_{2m})$ and imbed $I(\delta^{\otimes 2m}, \Lambda_{2m})$ as the local component of a certain global representation $I(\sigma^{\otimes 2m}, \Lambda_{2m})$ induced from cuspidal.
The functional \( j_H \) is then the corresponding local component of a certain factorizable, period \( j_H \) on \( I(\sigma \otimes 2m, \Lambda_{2m}) \) and \( U(\delta, 2m) \) is the local component of the unique irreducible quotient \( L(\sigma, 2m) \) of \( I(\sigma \otimes 2m, \Lambda_{2m}) \). We then use the results of [Off06b] to show that \( j_H \) is not identically zero and factors through \( L(\sigma, 2m) \). The second step consists of showing that symplectic periods on the building blocks can be induced. Our proof of this fact is rather technical. The idea, due to Heumos–Rallis, is to apply Bernstein’s principle of meromorphic continuation. This requires convergence of a certain complicated integral dependent on a complex parameter in some right half-plane. We accomplish this in Lemma 2 using an integration formula of Jacquet–Rallis [JR92]. In fact, we now know that the hereditary property of symplectic periods follows from a recent work of Delorme–Blanc [DB].

1. Symplectic period on the building blocks

Let \( \delta \) be a square integrable representation of \( GL_r(F) \) and let \( n = mr \). We construct an explicit non-zero and \( H \)-invariant linear form \( l_H \) on the representation \( U(\delta, 2m) \) of \( G = GL_{2n}(F) \). For a permutation \( w \in S_{2m} \) in \( 2m \) variables, let \( M(w) \) be the standard intertwining operator

\[
M(w) : I(\delta \otimes 2m, \Lambda_{2m}) \to I(\delta \otimes 2m, w \Lambda_{2m}).
\]

Let \( w' = w'_{2m} \) be the permutation defined by

\[
w'(2i - 1) = i, \quad w'(2i) = 2m + 1 - i, \quad i = 1, \ldots, m.
\]

Let \( M \) be the standard Levi of type \( (r, \ldots, r) \) of \( G \) and let \( M_H = M \cap H \). Up to a scalar, there is a unique \( M_H \)-invariant form on \( \delta \otimes 2m \) which we denote by \( l_{M_H} \). Indeed, \( l_{M_H} \) is given by the pairing of \( \delta \otimes m \) with its contragradient. Let \( K \) be the standard maximal compact subgroup of \( G \) and set \( K_H = K \cap H \). The linear form

\[
j_H(\varphi) = \int_{K_H} l_{M_H}(M(w')\varphi(k)) \, dk \quad (1)
\]

is a non-zero \( H \)-invariant form on \( I(\delta \otimes 2m, \Lambda_{2m}) \). Indeed, this is shown in [Off06b, §3], when \( \delta \) is any irreducible, generic, unitary representation. To obtain a symplectic period on \( U(\delta, 2m) \) it is therefore enough to show that the form \( j_H \) factors through the unique irreducible quotient \( U(\delta, 2m) \) of \( I(\delta \otimes 2m, \Lambda_{2m}) \).

**Remark 1.** If \( \delta \) is supercuspidal, we can show that the representation \( I(\delta \otimes 2m, \Lambda_{2m}) \) has a decomposition series for which no factor (except \( U(\delta, 2m) \)) is \( H \)-distinguished. When \( m = 1 \), the same is true for any square integrable \( \delta \). This was the key point in the proof of [HR90, Theorem 11.1]. For \( m > 1 \) and \( \delta \) square integrable this is in general no longer true. The representation \( I(\delta \otimes 2m, \Lambda_{2m}) \) may have more then one decomposition factor which is \( H \)-distinguished. For this reason the method of proof of Heumos–Rallis does not generalize directly to \( U(\delta, 2m) \). To overcome this problem we use a global approach.

We imbed our local problem in a global setting. In order to construct locally a non-vanishing symplectic period, we construct a global, decomposable symplectic period and apply [Off06b] to show that it factors through the unique irreducible quotient.
Proposition 1. The form \( j_H \) on \( I(\delta \otimes 2m, A_{2m}) \) factors through \( U(\delta, 2m) \), i.e. it defines a non-zero \( H \)-invariant form on \( U(\delta, 2m) \).

Proof. We start with the following lemma.

Lemma 1. Let \( \delta \) be a square integrable representation of \( GL_r(F) \). There is a number field \( k \), a place \( v \) of \( k \) so that \( F = k_v \) and a cuspidal automorphic representation \( \sigma \) of \( GL_r(\mathbb{A}_k) \) so that \( \delta = \sigma_v \).

Proof. The lemma follows from the proof of Proposition 5.15 in [Rog83].

Let \( k, v \) and \( \sigma \) be as in Lemma 1. Let \( P \) be the standard parabolic subgroup of \( G \) with Levi \( M \). For \( \lambda = (\lambda_1, \ldots, \lambda_{2m}) \in \mathbb{C}^{2m} \) denote

\[
I(\sigma \otimes 2m, \lambda) = \text{Ind}_{P(A_k)}^{G(A_k)} (|\det|_{\mathbb{A}_k}^{\lambda_1} \sigma \otimes |\det|_{\mathbb{A}_k}^{\lambda_2} \sigma \otimes \cdots \otimes |\det|_{\mathbb{A}_k}^{\lambda_{2m}} \sigma).
\]

Let \( L(\sigma, 2m) \) be the unique irreducible quotient of \( I(\sigma \otimes 2m, A_{2m}) \). Then \( I(\delta \otimes 2m, A_{2m}) \) is the local component of \( I(\sigma \otimes 2m, A_{2m}) \) and \( U(\delta, 2m) \) is the local component of \( L(\sigma, 2m) \) at \( v \). Let

\[
j_H(\varphi) = \int_{K_H} \int_{M_H \backslash M_H(\mathbb{A}_k)} (M_{-1}(w')\varphi)(mk) \, dm \, dk
\]

where \( M_{-1}(w') \) is the multi-residue at \( A_{2m} \) of the standard intertwining operator

\[
M(w', \lambda) : I(\sigma \otimes 2m, \lambda) \rightarrow I(\sigma \otimes 2m, w'\lambda).
\]

It is shown in [Off06b], that the form \( j_H \) is a non-zero \( H(A_{k,f}) \)-invariant form on \( I(\sigma \otimes 2m, A_{2m}) \), where \( \mathbb{A}_{k,f} \) is the ring of finite adèles of \( k \). It is decomposable into local factors \( j_H = \bigotimes_w j_{H, w} \) and \( j_{H, v} \) is proportional to \( j_H \) given by (1). Let \( E_{-1} \) denote the intertwining operator that projects \( I(\sigma \otimes 2m, A_{2m}) \) → \( L(\sigma, 2m) \). It is also decomposable. In [Off06a] it is shown that \( j_H \) factors through \( L(\sigma, 2m) \), i.e. there is a linear form \( l_H \) on \( L(\sigma, 2m) \) that makes the following diagram commute:

\[
\begin{array}{ccc}
I(\sigma \otimes 2m, A_{2m}) & \xrightarrow{E_{-1}} & L(\sigma, 2m) \\
\downarrow j_H & & \downarrow l_H \\
\mathbb{C} & \xrightarrow{j_{H,v}} & L(\sigma, 2m)
\end{array}
\]

Fix a decomposable element \( \varphi_0 = \bigotimes_w \varphi_{0,w} \in I(\sigma \otimes 2m, A_{2m}) \) such that \( j_H(\varphi_0) \neq 0 \). For each \( \varphi_v \in I(\delta \otimes 2m, A_{2m}) \) denote \( \varphi = \bigotimes_{w \neq v} \varphi_{0,w} \otimes \varphi_v \). If \( \varphi_v \) is in the kernel of the projection \( I(\delta \otimes 2m, A_{2m}) \rightarrow U(\delta, 2m) \) then \( \varphi \) is in the kernel of \( E_{-1} \) and therefore \( j_H(\varphi) = 0 = j_{H,v}(\varphi_v) \). Thus \( j_{H,v}(\varphi_v) = 0 \). This shows that \( j_{H,v} \) factors through \( U(\delta, 2m) \). The proposition follows.

We only needed to introduce global notation for the proof of Proposition 1. For the remainder of this work we remain strictly in a local setting. Recall that \( j_H \) is the \( H \)-invariant form on
I(δ⊗2, A2m) defined by (1). It follows from Proposition 1 that there exists an $H$-invariant form $l_H$ on $U(\delta, 2m)$ such that $j_H = l_H \circ M(w_{2n})$.

2. Induction of the symplectic period

In this section we fix irreducible, square integrable representations $\delta_i$ of $GL_{r_i}(F)$, $i = 1, \ldots, t$. We also fix $\alpha_1, \ldots, \alpha_t \in \mathbb{R}$ and positive integers $m_1, \ldots, m_t$.

**Proposition 2.** The representation

$$J = v^{\alpha_1} U(\delta_1, 2m_1) \times \cdots \times v^{\alpha_t} U(\delta_t, 2m_t)$$

is distinguished by $H$.

The rest of this work is devoted to the proof of Proposition 2. Let $k_i = m_i r_i$ and let $k = (2k_1, \ldots, 2k_t)$ be a partition of $2n$. Let $Q = LV$ be the standard parabolic subgroup of $G$ of type $k$, and let $x$ be the skew symmetric matrix

$$x = \text{diag}(e_{2k_1}, \ldots, e_{2k_t}).$$

We denote $Q_x = Q \cap H_x$ and let $P = MU$ be the standard parabolic of $G$ of type

$$\left(\prod_{i=1}^t r_i, \ldots, \prod_{i=1}^t r_i\right).$$

Its Levi component is $M = M_1 \times \cdots \times M_t$ where $M_i$ is the standard Levi of $GL_{2k_i}$ of type $(r_i, \ldots, r_i)$. We denote by $M_{i,H}$ the intersection of $M_i$ with the symplectic group $H_{2k_i}$ and by $K_{i,H}$ the intersection of $H_{2k_i}$ with the standard maximal compact subgroup of $GL_{2k_i}(F)$. In [Off06b] we provided an $H$-filtration of induced representations and a useful description of their composition factors, using the geometric lemma of Bernstein–Zelevinsky. The filtration of $J$ is parameterized by $Q \setminus G/H$. Let $l_i$ be the symplectic period on $U(\delta_i, 2m_i)$ introduced in Section 1. It gives rise to a period on the first composition factor coming from the open double coset. Let $\eta \in G$ be such that $x = \eta e_{2n}^t \eta$. Then $\eta H \eta^{-1} = H_x$ and $Q \eta H$ is the open double coset. It is a consequence of Frobenious reciprocity that on the subspace of $J$, of functions supported on $Q \eta H$ we obtain a non-zero $H$-invariant functional defined by the formula

$$l_H(\varphi) = \int_{(H \cap \eta^{-1} Q \eta) \setminus H} (l_1 \otimes \cdots \otimes l_t)(\varphi(\eta h)) \, dh$$

$$= \int_{Q_x \setminus H_x} (l_1 \otimes \cdots \otimes l_t)(\varphi(h \eta)) \, dh. \quad (4)$$

However, this integral needs not converge on the fully induced space $J$. We follow the ideas of [HR90] to bypass this obstacle. We let

$$J_s = \text{Ind}_Q^G(\delta_Q^s \otimes (v^{\alpha_1} U(\delta_1, 2m_1) \otimes \cdots \otimes v^{\alpha_t} U(\delta_t, 2m_t)))$$
where \( \delta_Q \) is the modulus function of \( Q \). Denote by \( l_{s,H} \) the linear form on \( J_s \) defined by the right-hand side of (4). We show that for \( \Re s \) large enough and for \( \varphi \in J_s \), the integral defining \( l_{s,H}(\varphi) \) is absolutely convergent. It will then follow from the uniqueness of symplectic periods [HR90, Theorem 2.4.2], and from Bernstein’s principle of meromorphic continuation as used in [HR90, pp. 277–278], that \( J_s \) has a non-zero symplectic period, which is a rational function of \( q^s \), where \( q \) is the cardinality of the residual field of \( F \). This will provide a non-zero symplectic period on \( J = J_0 \). Indeed, there will be an integer \( m \) so that \( sml_{s,H} \) is holomorphic and non-zero at \( s = 0 \). We therefore only need to show that for \( \Re s \gg 0 \) and for \( \varphi \in J_s \), the integral on the right-hand side of (4) is absolutely convergent. Let

\[
I'_s = \text{Ind}_P^G \left( \delta_Q^s |_P \otimes \left( \nu_{\alpha_1} \delta_1^{2m_1} [A_{2m_1}] \otimes \cdots \otimes \nu_{\alpha_t} \delta_t^{2m_t} [A_{2m_t}] \right) \right).
\]

Let \( E_i \) denote the projection from \( I(\delta \otimes \nu_{\alpha_i} \delta_i^{2m_i}, \Lambda_{2m_i}) \) to \( U(\delta_i, \Lambda_{2m_i}) \). The projection \( E = E_1 \otimes \cdots \otimes E_t \) gives rise to a projection \( \tilde{E}_s : I'_s \to J_s \) given by

\[
(\tilde{E}_s(f))(g) = E(f(g)).
\]

It is easy to see that if \( \varphi = \tilde{E}_s(f), f \in I'_s \), then

\[
l_{s,H}(\varphi) = \int_{Q \setminus H} \left( j'_1 \otimes \cdots \otimes j'_t \right)(f(h\eta)) \, dh \tag{5}
\]

where \( j'_i = l_i \circ E_i \) is the non-zero symplectic period on \( I(\delta_i^{2m_i}, A_{2m_i}) \) introduced in (1). We let \( j'_{s,H} \) be the linear form on \( I'_s \) defined by the right-hand side of (5). Let

\[
I_s = \text{Ind}_P^G \left( \delta_Q^s |_P \otimes \left( \nu_{\alpha_1} \delta_1^{2m_1} [w'_{2m_1}, A_{2m_1}] \otimes \cdots \otimes \nu_{\alpha_t} \delta_t^{2m_t} [w'_{2m_t}, A_{2m_t}] \right) \right)
\]

and let \( w' = \text{diag}(w'_{2m_1}, \ldots, w'_{2m_t}) \). Then \( M(w') \) is the standard intertwining operator from \( I'_s \) to \( I_s \). Making the \( j'_i \)'s explicit we observe that

\[
j'_{s,H} = j_{s,H} \circ M(w'),
\]

where \( j_{s,H} \) is the linear form on \( I_s \) given by

\[
j_{s,H}(\varphi) = \int_{Q \setminus H} \int_{K_{1,H} \times \cdots \times K_{t,H}} (l_{M_1,H} \otimes \cdots \otimes l_{M_t,H})(f(\text{diag}(k_1, \ldots, k_t)h\eta)) \, d(k_1, \ldots, k_t) \, dh \tag{6}
\]

and \( l_{M_i,H} \) is the \( M_i,H \)-invariant form on \( \delta_i^{2m_i} \). It is left to prove the following.

**Lemma 2.** For \( \Re s \gg 0 \) and \( f \in I_s \), the integral (6) is absolutely convergent.
Proof. It will be convenient to use the integration formula of Jacquet–Rallis [JR92] or rather its generalization to \( Q \setminus H \) given in [Off06a]. We will need to introduce some new notation. We will try to minimize the notation and details and focus only on the information we need for our proof of convergence. More details regarding the integration formula can be found in [Off06a, §5]. For \( Y = (y_1, \ldots, y_m) \in F^m \) let
\[
\|Y\| = \max_{i=1}^m |y_i|
\]
and let
\[
\lambda(Y) = \max(\|Y\|, 1).
\]
For \( X = (X_1, \ldots, X_{t-1}) \) where \( X_i \in M_{2(k_1+\cdots+k_{i-1})+k_i}(F) \), we define a unipotent matrix \( \sigma_k(X) \in G \) by recursion on \( t \) as follows. Let \( k' \) be the partition of \( 2n \) defined by \( k' = (2k_1, \ldots, 2k_{t-2}, 2k_{t-1}+2k_t) \). Define
\[
\sigma_k(X) = \begin{pmatrix}
1_{2(k_1+\cdots+k_{t-2})} & 1_{k_{t-1}} & 1_{k_{t-1}} & \cdots & 1_{k_{t-1}} & 1_{k_t} \\
1_{k_{t-1}} & 1_{k_{t-1}} & 1_{k_{t-1}} & \cdots & 1_{k_{t-1}} & 1_{k_t} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1_{k_1} & 1_{k_1} & 1_{k_1} & \cdots & 1_{k_1} & 1_{k_t} \\
1_{k_1} & 1_{k_1} & 1_{k_1} & \cdots & 1_{k_1} & 1_{k_t} \\
1_{k_1} & 1_{k_1} & 1_{k_1} & \cdots & 1_{k_1} & 1_{k_t}
\end{pmatrix} \sigma_{k'}(X_1, \ldots, X_{t-2}).
\]
For our purpose, it is enough to give the integration formula for the \( H \)-invariant measure on \( Q \setminus H \), for functions \( \phi \) on \( G \) which are left \( U \)-invariant. There is a function \( \gamma(X) \) such that for functions \( \phi \) as above we have
\[
\int_{Q \setminus H} \phi(h) dh = \int_{K \cap H} \gamma(X) \phi(\sigma_k(X)k) dX dk
\]
where \( K_{H_k} = K \cap H_k \). On the factor \( \gamma(X) \) all we need to know is that there are constant \( c \) and \( m \) such that
\[
\gamma(X) \leq c \left( \prod_{i=1}^{t-1} \lambda(X_i) \right)^m.
\]
For \( f \in I_s \) we therefore have
\[
j_{s,H}(I_s(\eta^{-1})f) = \int_{K \cap H} \gamma(X) \int_{K_1,H \times \cdots \times K_t,H} (l_{M_1,H} \otimes \cdots \otimes l_{M_t,H}) \times (f(\diag(k_1, \ldots, k_t)\sigma_k(X)k)) d(k_1, \ldots, k_t) dX dk.
\]
Since \( f \) is \( K \)-finite, fixing a basis \( \{f_j\} \) of \( I_s(K)f \), there are smooth functions \( a_j \) on \( K \) such that \( I_s(k)(f) = \sum_j a_j(k)f_j \). It follows that \( j_{s,H}(I_s(\eta^{-1})f) \) is the finite sum over \( j \) of \( \int_{K \cap H} a_j(k) dk \) times
\[
\int_{K_1,H \times \cdots \times K_t,H} \gamma(X) (l_{M_1,H} \otimes \cdots \otimes l_{M_t,H})(f_j(\diag(k_1, \ldots, k_t)\sigma_k(X))) d(k_1, \ldots, k_t) dX.
\]
To prove the proposition it is therefore enough to show that the integral

\[
\int \left( \prod_{i=1}^{t-1} \lambda(X_i) \right)^m \int_{K_1, H \times \cdots \times K_t, H} \left| (l_{M_1, H} \otimes \cdots \otimes l_{M_t, H}) \right| \times \left( f \left( \text{diag}(k_1, \ldots, k_t) \sigma_k(X) \right) \right) d(k_1, \ldots, k_t) dX
\]

is convergent. For any matrix \( g \) we will denote by \( \| \epsilon_i g \| \) the maximum of the absolute values of the \( i \times i \) minors in the lower \( i \) rows of \( g \). For each \( j \in \{1, t\} \), \( i \in \{1, 2m_j\} \), let \( R_{i,j} = ir_j + 2 \sum_{q=j+1}^{t} k_q \). We write the coordinates of each \( A_{2m_j} \) as \( A_{2m_j} = (\mu_1^j, \ldots, \mu_{2m_j}^j) \) (in fact the convergence is proved for \( \mu_i^j \) arbitrary). Let \( \mu = (\Lambda_{2m_1}, \ldots, \Lambda_{2m_t}) \in \mathbb{R}^{2(m_1 + \cdots + m_t)} \). For \( p \in P \) with diagonal blocks \( p_i^j \in GL_{r_j}(F) \), \( j \in \{1, t\} \), \( i \in \{1, 2m_j\} \), we denote \( p^\mu = \prod_{i,j} |\det p_i^j|^{\mu_i^j} \). If we write an Iwasawa decomposition of \( g \in G \) with respect to \( P \) as \( g = p(g) \kappa(g) \) then we have

\[
f(g) = \delta_Q^\mu \left( p(g) \right) p(g)^{\mu + \rho_P} \left( \bigotimes_{j=1}^{t} \delta_j^{\otimes 2m_j} \right) (p(g)) f(\kappa(g))
\]

where \( \rho_P \) is half the sum of positive roots with respect to the parabolic \( P \) of \( G \). If \( g = pk \) where \( p \in P \) has diagonal blocks denoted as before we may write

\[
|\det p_i^j| = \frac{\| \epsilon_{R_{2m_j+1-i,j}g} \|}{\| \epsilon_{Rg} \|}
\]

where \( R = R_{2m_j-i,j} \) if \( i < 2m_j \) and \( R = R_{1,j+1} \) otherwise. In other words we may find \( \lambda \in \mathbb{R}^{2(m_1 + \cdots + m_t)} \) dependent only on \( \mu \) so that

\[
f(g) = \delta_Q^\mu \left( p(g) \right) \prod_{i,j} |\epsilon_{R_{i,j}g}|^{\lambda_i^j} \left( \bigotimes_{j=1}^{t} \delta_j^{\otimes 2m_j} \right) (p(g)) f(\kappa(g))
\]

The integral (7) then becomes

\[
\int \left( \prod_{i=1}^{t-1} \lambda(X_i) \right)^m \int_{K_1, H \times \cdots \times K_t, H} \delta_Q^\mu \left( p(\sigma_k(X)) \right) \prod_{i,j} |\epsilon_{R_{i,j}} \text{diag}(k_1, \ldots, k_t) \sigma_k(X)|^{\lambda_i^j} \times \left| (l_{M_1, H} \otimes \cdots \otimes l_{M_t, H}) \right| \left( \bigotimes_{j=1}^{t} \delta_j^{\otimes 2m_j} \right) (p(\text{diag}(k_1, \ldots, k_t) \sigma_k(X))) \\
\times f(\kappa(\text{diag}(k_1, \ldots, k_t) \sigma_k(X))) \right) d(k_1, \ldots, k_t) dX.
\]
We first claim that the expression in the absolute value is now bounded, independently of \(k_1, \ldots, k_t\) and \(X\). Indeed, \(f\) being smooth, obtains only finitely many values on \(K\) and therefore it is enough to bound

\[
(l_{M_1, H} \otimes \cdots \otimes l_{M_t, H})\left(\left( \bigotimes_{j=1}^t \delta_j^{2m_j} \right) \left( p(\text{diag}(k_1, \ldots, k_t) \sigma_k(X)) \right) v \right)
\]

for any \(v\) in the space of \(\bigotimes_{j=1}^t \delta_j^{2m_j}\). We may further assume that \(v\) decomposes as \(v = v_{1,1} \otimes v_{1,2} \otimes \cdots \otimes v_{t,1} \otimes v_{t,2}\) where \(v_{i,j}\) lies in the space of \(\delta_j^{m_i}\) for \(j = 1, 2\). For \(p \in M\) we denote by \(p_i^j \in GL_{r_i}(F)\) its diagonal blocks as before. The map

\[
p \mapsto (l_{M_1, H} \otimes \cdots \otimes l_{M_t, H})\left(\left( \bigotimes_{j=1}^t \delta_j^{2m_j} \right) (p) v \right)
\]

is a matrix coefficient of the unitary representation \(\bigotimes_{j=1}^t \delta_j^{m_j}\) evaluated at

\[
\text{diag}\left( (\tilde{p}_1^{1})^{-1} p_1^1, \ldots, (\tilde{p}_{m_1+1}^{1})^{-1} p_{m_1}^1, (\tilde{p}_2^{2})^{-1} p_1^2, \ldots, (\tilde{p}_{m_2+1}^{2})^{-1} p_{m_2}^2, \ldots, (\tilde{p}_{m_t+1}^t)^{-1} p_{m_t}^t \right).
\]

Here \(\tilde{g} = w_q^t g^{-1} w_q\) for \(g \in GL_q(F)\). Matrix coefficients of unitary representations are bounded. It is therefore enough to show that for \(\text{Re } s\) large enough the expression

\[
\int \left( \prod_{i=1}^{t-1} \lambda(X_i) \right)^m \int_{K_{1,H} \times \cdots \times K_{t,H}} \delta_q^s(p(\sigma_k(X))) \times \prod_{i,j} \| \epsilon_{R_{i,j}} \text{diag}(k_1, \ldots, k_t) \sigma_k(X) \|^{\lambda_i^{j}} d(k_1, \ldots, k_t) dX
\]

converges. In order to bound the integrand in (9), we will use the following two claims.

**Claim 1.** There exists an \(N\) such that

\[
1 \leq \| \epsilon_{R_{i,j}} \text{diag}(k_1, \ldots, k_t) \sigma_k(X) \| \leq \left( \prod_{i=1}^{t-1} \lambda(X_i) \right)^N.
\]

**Claim 2.**

\[
\delta_q^s(p(\sigma_k(X))) \leq \left( \prod_{i=1}^{t-1} \lambda(X_i) \right)^{-s}.
\]
The upper bound in Claim 1 is obvious. We show the lower bound. To avoid ambiguity of notation let us denote by $k_{i,H}$ the elements of $K_{i,H}$. Note that the lower $R_{1,j}$ rows of $\text{diag}(k_1,H,\ldots,k_t,H)\sigma_k(X)$ have the form
\[
\begin{pmatrix}
\ast & k_{j,H} \\
\ast & \text{diag}(k_{j+1,H},\ldots,k_t,H)\sigma_{2(k_{j+1},\ldots,2k_t)}(X_{j+1},\ldots,X_{t-1})
\end{pmatrix}
\]
where we put $\ast$ in each block that will play no role for us. For each $i \in [1,2m_j]$ there is an $ir_j \times ir_j$ minor $A$ in the lower $ir_j$ rows of $k_{i,H}$ of absolute value 1. Together with the lower right $2(k_{j+1}+\cdots+k_t) \times 2(k_{j+1}+\cdots+k_t)$-block of $\text{diag}(k_1,H,\ldots,k_t,H)\sigma_k(X)$ we get that
\[
\begin{pmatrix}
A \\
\ast & \text{diag}(k_{j+1,H},\ldots,k_t,H)\sigma_{2(k_{j+1},\ldots,2k_t)}(X_{j+1},\ldots,X_{t-1})
\end{pmatrix}
\]
is an $R_{i,j} \times R_{i,j}$ minor of absolute value 1 in the lower $R_{i,j}$ rows of the matrix $\text{diag}(k_1,H,\ldots,k_t,H)\sigma_k(X)$. This shows Claim 1. To show Claim 2, we note that if $|\det g| = 1$ then
\[
\delta_Q(p(g)) = \prod_{j=1}^{t-1} \|\epsilon_{R_{1,j}} g\|^{-2k_{t-1-j}-2k_{t-j}}.
\]
It can be proved as in [Off06a, Lemma 5.5], that
\[
\|\epsilon_{R_{1,j}} p(\sigma_k(X))\| \geq \lambda(X_{t-j}).
\]
Claim 2 readily follows. Using the two claims, we bound the integral (9) by replacing each term $\|\epsilon_{R_{1,j}} \text{diag}(k_1,\ldots,k_t)\sigma_k(X)\|^{\lambda_j}$ by 1 if $\lambda_j \leq 0$ and by a certain fixed and large enough power of $(\prod_{i=1}^{t-1} \lambda(X_i))$ otherwise. It is shown in [JR92] that for any $q$ the integral
\[
\int_{Fq} \lambda(Y)^{-s} dY
\]
is convergent for $s \gg 0$. The lemma therefore follows from the two claims. □

This concludes the proof of Proposition 2 and in particular also of Theorem 1.

References

[DB] Philippe Blanc, Patrick Delorme, Vecteurs distributions $H$-invariants de représentations induites, pour un espace symétrique réductif $p$-adique $G/H$, arXiv: math.RT/0412435.

[Heu93] Michael J. Heumos, Models and periods for automorphic forms on $GL_n$, in: Representation Theory of Groups and Algebras, in: Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 135–144.

[HR90] Michael J. Heumos, Stephen Rallis, Symplectic-Whittaker models for $GL_n$, Pacific J. Math. 146 (2) (1990) 247–279.

[JR92] Hervé Jacquet, Stephen Rallis, Symplectic periods, J. Reine Angew. Math. 423 (1992) 175–197.

[KIJ84] A.A. Klyachko, Models for complex representations of the groups $GL(n,q)$, Mat. Sb. (N.S.) 48 (2) (1984) 365–378.

[Off06a] Omer Offen, On symplectic periods of discrete spectrum of $GL_{2n}$, Israel J. Math. 154 (2006) 253–298.

[Off06b] Omer Offen, Distinguished residual spectrum, Duke Math. J. 134 (2) (2006) 313–357.
[OS] Omer Offen, Eitan Sayag, Global mixed periods and local Klyachko models for the general linear group, submitted for publication.

[Rog83] Jonathan D. Rogawski, Representations of GL(n) and division algebras over a $p$-adic field, Duke Math. J. 50 (1) (1983) 161–196.

[Tad86] Marko Tadić, Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case), Ann. Sci. École Norm. Sup. (4) 19 (3) (1986) 335–382.