Toral posets and the binary spectrum property

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Received: 8 January 2020 / Accepted: 15 March 2021 / Published online: 10 May 2021
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Abstract
We introduce a family of posets which generate Lie poset subalgebras of \( A_{n-1} = \mathfrak{sl}(n) \) whose index can be realized topologically. In particular, if \( \mathcal{P} \) is such a toral poset, then it has a simplicial realization which is homotopic to a wedge sum of \( d \) one-spheres, where \( d \) is the index of the corresponding type-A Lie poset algebra \( g_A(\mathcal{P}) \). Moreover, when \( g_A(\mathcal{P}) \) is Frobenius, its spectrum is binary, that is, consists of an equal number of 0’s and 1’s. We also find that all Frobenius, type-A Lie poset algebras corresponding to a poset whose largest totally ordered subset is of cardinality at most three have a binary spectrum.

Keywords Frobenius Lie algebra · Poset algebra · Spectrum · Index

Mathematics Subject Classification 2010 17B99 · 05E15

1 Introduction

In [3], Coll and Gerstenhaber introduce the notion of a “Lie poset algebra.” These Lie algebras are subalgebras of \( A_{n-1} = \mathfrak{sl}(n) \) which lie between the subalgebras of upper-triangular and diagonal matrices. We refer to such Lie algebras as type-A Lie poset algebras and find that they are naturally associated with the incidence algebras of posets [13] (see Sect. 3). In [5], the current authors establish formulas for the index of type-A Lie poset algebras corresponding to posets whose largest totally ordered subset is of cardinality at most three. The authors further characterize such posets which correspond to type-A Lie poset algebras with index zero. In this article, we initiate an investigation into the form of the spectrum of such Lie algebras.

Formally, the index of a Lie algebra \( g \) is defined as

\[
\text{ind } g = \min_{F \in g^*} \dim(\ker(B_F)),
\]

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where $B_F$ is the skew-symmetric Kirillov form defined by $B_F(x, y) = F([x, y])$, for all $x, y \in \mathfrak{g}$. Of particular interest are those Lie algebras which have index zero, and are called Frobenius.\footnote{Frobenius algebras are of special interest in deformation and quantum group theory stemming from their connection with the classical Yang–Baxter equation (see \cite{7,8}).} A functional $F \in \mathfrak{g}^*$ for which $\dim \ker(B_F) = \text{ind } \mathfrak{g} = 0$ is likewise called Frobenius. Given a Frobenius Lie algebra $\mathfrak{g}$ and a Frobenius functional $F \in \mathfrak{g}^*$, the map $\mathfrak{g} \to \mathfrak{g}^*$ defined by $x \mapsto B_F(x, -)$ is an isomorphism. The inverse image of $F$ under this isomorphism, denoted $\hat{F}$, is called a principal element of $\mathfrak{g}$. In \cite{11}, Ooms shows that the eigenvalues (and multiplicities) of $\text{ad}(\hat{F}) = [\hat{F}, -] : \mathfrak{g} \to \mathfrak{g}$ do not depend on the choice of principal element $\hat{F}$ (see also \cite{9}). It follows that the spectrum of $\text{ad}(\hat{F})$ is an invariant of $\mathfrak{g}$, which we call the spectrum of $\mathfrak{g}$\footnote{Our investigation into the spectral theory of type-A Lie poset algebras is largely motivated by the corresponding theory for seaweed algebras. Seaweed (or biparabolic) subalgebras of a complex semi-simple Lie algebra $\mathfrak{g}$ are intersections of two parabolic subalgebras whose sum is $\mathfrak{g}$ (see \cite{10,12}). It has been shown that the spectrum of seaweed subalgebras of the classical families of Lie algebras consists of an unbroken sequence of integers where the multiplicities of the eigenvalues form a symmetric distribution about one half (see \cite{2,4}).}.

In this article, we introduce a family of posets which generate type-A Lie poset algebras whose index can be realized topologically. In particular, if $\mathcal{P}$ is such a “toral poset” (see Definition 2), then it has a simplicial realization which is homotopic to a wedge sum of ind $\mathfrak{g}_A(\mathcal{P})$ one-spheres (see Theorem 7). Moreover, when $\mathfrak{g}_A(\mathcal{P})$ is Frobenius, its spectrum is binary, that is, consists of an equal number of 0’s and 1’s (see Theorem 10). We also find that all Frobenius, type-A Lie poset algebras corresponding to a poset whose largest totally ordered subset is of cardinality at most three have a binary spectrum (see Theorem 11). Extensive calculations suggest that all Frobenius, type-A Lie poset algebras have a binary spectrum.

The structure of the paper is as follows. In Sect. 2, we set the combinatorial definitions and notation related to posets, and in Sect. 3, we formally introduce type-A Lie poset algebras. Section 4 deals with the determination of the form of a principal element for certain Frobenius, type-A Lie poset algebras. Sections 5, 6, and 7 deal with the main objects of interest: toral posets—and their associated index and spectral theories.

## 2 Posets

A finite poset $(\mathcal{P}, \preceq_{\mathcal{P}})$ consists of a finite set $\mathcal{P}$ together with a binary relation $\preceq_{\mathcal{P}}$ which is reflexive, antisymmetric, and transitive. When no confusion will arise, we simply denote a poset $(\mathcal{P}, \preceq_{\mathcal{P}})$ by $\mathcal{P}$, and $\preceq_{\mathcal{P}}$ by $\preceq$. Throughout, we let $\preceq$ denote the natural ordering on $\mathbb{Z}$. Two posets $\mathcal{P}$ and $\mathcal{Q}$ are isomorphic if there exists a bijection $\varphi : \mathcal{P} \to \mathcal{Q}$ such that $p \preceq_{\mathcal{P}} q$ if and only if $\varphi(p) \preceq_{\mathcal{Q}} \varphi(q)$, for all $p, q \in \mathcal{P}$.

Let $\mathcal{P}$ be a finite poset and $x, y \in \mathcal{P}$. If $x \preceq y$ and $x \neq y$, then we call $x \preceq y$ a strict relation and write $x \prec y$. Let $\text{Rel}(\mathcal{P})$, $\text{Ext}(\mathcal{P})$, and $\text{Rel}_E(\mathcal{P})$ denote, respectively, the set of strict relations between elements of $\mathcal{P}$, the set of minimal and maximal elements of $\mathcal{P}$, and the set of strict relations between elements of $\text{Ext}(\mathcal{P})$. If $x \prec y$ and there exists no $z \in \mathcal{P}$ satisfying $x \prec z \prec y$, then $x \prec y$ is a covering relation. Covering relations are used to define a visual representation of $\mathcal{P}$ called the Hasse diagram—a
graph whose vertices correspond to elements of $\mathcal{P}$ and whose edges correspond to covering relations (see, for example, Fig. 1(a)). Extending the Hasse diagram of $\mathcal{P}$ by allowing all elements of $\text{Rel}(\mathcal{P})$ to define edges results in the comparability graph of $\mathcal{P}$ (see, for example, Fig. 1(b)).

A totally ordered subset $S \subset \mathcal{P}$ is called a chain. The height of $\mathcal{P}$ is one less than the largest cardinality of a chain in $\mathcal{P}$. One can define a simplicial complex $\Sigma_1(\mathcal{P})$ by having chains of cardinality $n$ in $\mathcal{P}$ define the $(n-1)$-dimensional faces of $\Sigma(\mathcal{P})$ (see, for example, Fig. 1 (c)). A subset $I \subset \mathcal{P}$ is an order ideal if given $y \in \mathcal{P}$ such that there exists $x \in I$ satisfying $y \prec x$, then $y \in I$. Similarly, a subset $F \subset \mathcal{P}$ is a filter if given $y \in \mathcal{P}$ such that there exists $x \in F$ satisfying $x \prec y$, then $y \in F$.

**Example 1** Consider the poset $\mathcal{P} = \{1, 2, 3, 4\}$ with $1 \prec 2 \prec 3$, then we have

\[
\text{Rel}(\mathcal{P}) = \{1 \prec 2, 1 \prec 3, 1 \prec 4, 2 \prec 3, 2 \prec 4\}, \quad \text{Ext}(\mathcal{P}) = \{1, 3, 4\}, \quad \text{and} \quad \text{Rel}_E(\mathcal{P}) = \{1 \prec 3, 1 \prec 4\}.
\]

Note that $\{1, 2\} \subset \mathcal{P}$ is both an order ideal and a chain of $\mathcal{P}$, but not a filter, while $\{3, 4\} \subset \mathcal{P}$ is a filter, but neither an order ideal nor a chain. See Fig. 1.

![Fig. 1 Hasse diagram of $\mathcal{P}$, comparability graph of $\mathcal{P}$, and $\Sigma(\mathcal{P})$](image)

### 3 Lie poset algebras

Let $k$ be an algebraically closed field of characteristic zero, and let $\mathcal{P}$ be a finite poset. The *associative incidence* (or *poset*) algebra $A(\mathcal{P}) = A(\mathcal{P}, k)$ is the span over $k$ of elements $E_{p_i,p_j}$, for $p_i \preceq p_j$, with multiplication given by setting $E_{p_i,p_j}E_{p_k,p_l} = E_{p_i,p_l}$ if $p_j = p_k$ and $0$ otherwise. The trace of an element $\sum c_{p_i,p_j}E_{p_i,p_j}$ is $\sum c_{p_i,p_i}$. We can equip $A(\mathcal{P})$ with the commutator bracket $[a, b] = ab - ba$, where juxtaposition denotes the product in $A(\mathcal{P})$, to produce the *Lie poset algebra* $\mathfrak{g}(\mathcal{P}) = \mathfrak{g}(\mathcal{P}, k)$.

**Example 2** If $\mathcal{P}$ is the poset of Example 1, then $\mathfrak{g}(\mathcal{P})$ is the span over $k$ of the elements of

\[
\{E_{1,1}, E_{2,2}, E_{3,3}, E_{4,4}, E_{1,2}, E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4}\}.
\]

Restricting to the trace-zero elements of $\mathfrak{g}(\mathcal{P})$ results in the *type-A Lie poset algebra* $\mathfrak{g}_A(\mathcal{P})$. 

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Example 3 If \( g(P) \) is as in Example 2, then \( g_A(P) \) is the span over \( k \) of the elements of
\[ \{ E_{1,1} - E_{2,2}, E_{2,2} - E_{3,3}, E_{3,3} - E_{4,4}, E_{1,2}, E_{1,3}, E_{1,4}, E_{2,3}, E_{2,4} \} . \]

Remark 1 Isomorphic posets correspond to isomorphic (type-A) Lie poset algebras.

Remark 2 Evidently, the definition of type-A Lie poset algebra given in this section is equivalent to the matrix definition given in the Introduction (cf. [3]).

We refer to posets \( P \) for which \( g_A(P) \) is Frobenius as Frobenius posets.

4 Frobenius functionals and principal elements

In this section, we develop a framework for analyzing the spectrum of Frobenius, type-A Lie poset algebras by determining the form of a particular principal element.

Given a finite poset \( P \) and \( B = \sum b_{p,q} E_{p,q} \in g_A(P) \), define \( E^*_{p,q} \in (g_A(P))^* \), for \( p, q \in P \) satisfying \( p < q \), by \( E^*_{p,q}(B) = b_{p,q} \). From any set \( S \) consisting of ordered pairs \((p, q)\) of elements \( p, q \in P \) satisfying \( p < q \), i.e., \( S \subset Rel(P) \), one can construct both a functional \( F_S = \sum_{(p,q) \in S} E^*_{p,q} \in (g_A(P))^* \) and a directed subgraph \( \Gamma_{F_S}(P) \) of the comparability graph of \( P \). In [9], Gerstenhaber and Giaquinto refer to such a functional as small if \( \Gamma_{F_S}(P) \) is a spanning subtree of the comparability graph of \( P \). Note if \( F_S \in (g_A(P))^* \) is a small functional, then \( \Gamma_{F_S}(P) \) naturally partitions the elements of \( P \) into the following disjoint subsets:

- \( U_{F_S}(P) \) consisting of all sinks in \( \Gamma_{F_S}(P) \),
- \( D_{F_S}(P) \) consisting of all sources in \( \Gamma_{F_S}(P) \), and
- \( O_{F_S}(P) \) consisting of those vertices which are neither sinks nor sources in \( \Gamma_{F_S}(P) \).

Remark 3 In [9], the authors establish a method for calculating the principal element \( \hat{F}_S = \sum_{p \in P} c_{p,p} E_{p,p} \) corresponding to a small, Frobenius functional \( F_S \) on a Frobenius, type-A Lie poset algebra. Their algorithm is equivalent to solving the following system of equations:

\[ c_{p,p} - c_{q,q} = 1, \text{ for } (p, q) \in S, \text{ and} \]
\[ \sum_{p \in P} c_{p,p} = 0. \]

Ongoing, we assume that every functional \( F \in (g_A(P))^* \) is of the form \( F_S \) for some \( S \subset Rel(P) \).

Theorem 1 Let \( P \) be a Frobenius poset. If \( F \in (g_A(P))^* \) satisfies the following conditions:

- \( F \) is small and Frobenius,
- \( U_F(P) \) is a filter of \( P \),
- \( D_F(P) \) is an order ideal of \( P \), and
- \( O_F(P) = \emptyset, \)

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then $F = \sum_{p \in \mathcal{P}} c_{p,p} E_{p,p}$ satisfies

$$c_{p,p} = \begin{cases} \frac{|\text{UFS}(\mathcal{P})|}{|p|}, & p \in D_{F_S}(\mathcal{P}); \\ -\frac{|\text{DFS}(\mathcal{P})|}{|p|}, & p \in U_{F_S}(\mathcal{P}). \end{cases}$$

**Proof** Assume $F = F_S$ for $S \subset \text{Rel}(\mathcal{P})$. To determine the form of $F_S$, we use the system of equations given in Remark 3.

Let $p_1 \in D_{F_S}(\mathcal{P})$ and $p_n \in U_{F_S}(\mathcal{P})$ with $(p_1, p_n) \in S$, i.e., $c_{p_1,p_1} - 1 = c_{p_n,p_n}$. Since $\Gamma_{F_S}(\mathcal{P})$ is connected, given $p \in D_{F_S}(\mathcal{P})$, there exists a path from $p_1$ to $p$ in $\Gamma_{F_S}(\mathcal{P})$. Assume that such a path is defined by the following sequence of vertices of $\Gamma_{F_S}(\mathcal{P})$: $p_1 = p_{i_0}, p_{i_1}, p_{i_2}, \ldots, p_{i_{m-1}}, p_{i_m} = p$. By our assumption that $O_{F_S} = \emptyset$, we must have

$$p_1 = p_{i_0} < p_{i_1} < \cdots < p_{i_{m-1}} < p_{i_m} = p,$$

so that

$$c_{p_{i_0}, p_{i_0}} - c_{p_{i_1}, p_{i_1}} = 1$$
$$c_{p_{i_2}, p_{i_2}} - c_{p_{i_1}, p_{i_1}} = 1$$
$$c_{p_{i_3}, p_{i_3}} - c_{p_{i_2}, p_{i_2}} = 1$$
$$c_{p_{i_4}, p_{i_4}} - c_{p_{i_3}, p_{i_3}} = 1$$
$$\vdots$$
$$c_{p_{i_m}, p_{i_m}} - c_{p_{i_{m-1}}, p_{i_{m-1}}} = 1.$$

Solving the above equations, we find that

$$c_{p_1,p_1} = c_{p_{i_0}, p_{i_0}} = c_{p_{i_1}, p_{i_1}} + 1 = c_{p_{i_2}, p_{i_2}} = \ldots = c_{p_{i_{m-1}}, p_{i_{m-1}}} + 1 = c_{p,p};$$

that is, $c_{p_1,p_1} = c_{p,p}$, for all $p \in D_{F_S}(\mathcal{P})$. Similarly, we find that $c_{p_n,p_n} = c_{p,p}$, for all $p \in U_{F_S}(\mathcal{P})$. Thus, $c_{p,p} = c_{p_1,p_1} - 1$, for all $p \in U_{F_S}(\mathcal{P})$, and the condition $\sum_{p \in \mathcal{P}} c_{p,p} = 0$ becomes $|\mathcal{P}| c_{p_1,p_1} = |U_{F_S}(\mathcal{P})|$. Therefore, $c_{p_1,p_1} = \frac{|U_{F_S}(\mathcal{P})|}{|\mathcal{P}|}$ and

$$c_{p,p} = \begin{cases} \frac{|U_{F_S}(\mathcal{P})|}{|\mathcal{P}|}, & p \in D_{F_S}(\mathcal{P}); \\ -\frac{|D_{F_S}(\mathcal{P})|}{|\mathcal{P}|}, & p \in U_{F_S}(\mathcal{P}). \end{cases}$$

$\square$
Remark 4 If $F \in (g_A(P))^*$ satisfies the conditions of Theorem 1, then one obtains a canonical choice of basis for $g_A(P)$:

$$\mathcal{B}_{P,F} = \{E_{p,q} \mid p, q \in P, p < q\} \cup \{E_{p,p} - E_{q,q} \mid E_{p,q}^* \text{ is a summand of } F\}.$$ 

This basis will prove useful in the analysis of the spectrum of Frobenius, type-A Lie poset algebras.

The following result is an immediate corollary to Theorem 1.

**Theorem 2** If $P$ is a Frobenius poset and $F \in (g_A(P))^*$ satisfies the conditions of Theorem 1, then the spectrum of $g_A(P)$ consists of 0’s and 1’s.

**Proof** To determine the spectrum of $g_A(P)$, we calculate the values $[\hat{F}, x]$, for $x \in \mathcal{B}_{P,F}$. To start, for $x \in \{E_{p,p} - E_{q,q} \mid E_{p,q}^* \text{ is a summand of } F\} \subset \mathcal{B}_{P,F}$, we must have $[\hat{F}, x] = 0 \cdot x$. It remains to consider basis elements of the form $E_{p,q} \in \mathcal{B}_{P,F}$. The analysis of such basis elements breaks into three cases:

**Case 1:** if $p, q \in U_F(P)$, then

$$[\hat{F}, E_{p,q}] = \left(-\frac{|D_F(P)|}{|P|} - \left(-\frac{|D_F(P)|}{|P|}\right)\right) \cdot E_{p,q} = 0 \cdot E_{p,q}.$$ 

**Case 2:** if $p, q \in D_F(P)$, then

$$[\hat{F}, E_{p,q}] = \left(\frac{|U_F(P)|}{|P|} - \frac{|U_F(P)|}{|P|}\right) \cdot E_{p,q} = 0 \cdot E_{p,q}.$$ 

**Case 3:** if $p \in D_F(P)$ and $q \in U_F(P)$, then

$$[\hat{F}, E_{p,q}] = \left(\frac{|U_F(P)|}{|P|} - \left(-\frac{|D_F(P)|}{|P|}\right)\right) \cdot E_{p,q}$$

$$= \left(\frac{|U_F(P)| + |D_F(P)|}{|P|}\right) \cdot E_{p,q} = 1 \cdot E_{p,q}.$$ 

Thus, as $\mathcal{B}_{P,F}$ forms a basis for $g_A(P)$, the spectrum of $g_A(P)$ consists of 0’s and 1’s. \hfill \square

Remark 5 Note that Theorem 2 only provides information about the spectrum of $ad(\hat{F})$, but not the multiplicities of the eigenvalues. By a result of Ooms ([11], Theorem 3.3 (1)), we may, in fact, conclude that there are an equal number of 0’s and 1’s. Even so, we establish this directly in subsequent sections.

In the next section, numerous examples of Frobenius posets will be given for which there exists a corresponding Frobenius functional with the properties listed in Theorems 1 and 2.
5 Toral-pairs

In this section, we introduce the notion of a toral-pair, consisting of a Frobenius poset together with a certain Frobenius functional, and we give numerous examples of such pairs. The posets of such pairs will form the “building blocks” used to construct the main objects of interest in this paper, toral posets.

Remark 6 Recall that we are assuming that every functional $F \in (g_A(P))^*$ is of the form $F_S$ for some $S \subset \text{Rel}(P)$. As such functionals $F \in (g_A(P))^*$ can also be viewed as elements of $(g(P))^*$—and we have occasion to consider both circumstances—we set the following notational convention: We denote the kernel of $B_F$, for $F \in (g(P))^*$, by $\ker(B_F)$, and we denote the kernel of $B_F$, for $F \in (g_A(P))^*$, by $\ker_A(B_F)$.

Definition 1 Given a Frobenius poset $P$ and a corresponding Frobenius functional $F \in (g_A(P))^*$, we call $(P, F)$ a toral-pair if $P$ satisfies

\begin{align*}
(P1) \quad |\text{Ext}(P)| &= 2 \text{ or } 3, \\
(P2) \quad \Sigma(P) \text{ is contractible, and} \\
(P3) \quad g_A(P) \text{ has a binary spectrum,}
\end{align*}

and $F$ satisfies

\begin{align*}
(F1) \quad F \text{ is small,} \\
(F2) \quad U_F(P) \text{ is a filter of } P, \quad D_F(P) \text{ is an order ideal of } P, \quad \text{and } O_F(P) = \emptyset, \\
(F3) \quad \Gamma_F \text{ contains all edges between elements of } \text{Ext}(P), \quad \text{and} \\
(F4) \quad B \in \ker(B_F) \text{ satisfies } E^*_{p, p}(B) = E^*_{q, q}(B), \text{ for all } p, q \in P, \quad \text{and } E^*_{p, q}(B) = 0, \text{ for all } p, q \in P \text{ satisfying } p \preceq q.
\end{align*}

Example 4 The posets illustrated in Fig. 2 can be paired with an appropriate functional to form a toral-pair (see Theorems 3, 4, and 5).

![Fig. 2 Posets of toral-pairs](image-url)
Remark 7 Here, we establish how we show that a functional is Frobenius.

(i) The general case
For a Lie algebra $g$, we can show that a functional $F \in g^*$ is Frobenius in the following way. Let $\{x_1, \ldots, x_n\}$ be a vector space basis for $g$ and let $B = \sum_{i=1}^{n} b_j x_j \in \ker(B_F)$. Now, determine the restrictions $F([x_i, B]) = 0$ places on the $b_j$, for $i = 1, \ldots, n$ and $j = 1, \ldots, n$. Finally, show that these restrictions imply that $B = 0$; that is,

$$\dim \ker(B_F) = 0 \geq \ind g \geq 0.$$

Specializing to type-A Lie poset algebras, we have the following.

(ii) Type-A Lie poset algebras
Given a poset $P = \{p_1, \ldots, p_n\}$ and a functional $F \in (g_A(P))^*$, it is shown in Appendix A that $\ker_A(B_F) = g_A(P) \cap \ker(B_F)$; that is, $B \in \ker_A(B_F)$ if and only if $B \in g(P)$ and

- $F([E_{p_i}, B]) = 0$, for $p_i \in P$,
- $F([E_{p_i}, E_{p_j}, B]) = 0$, for $p_i, p_j \in P$ satisfying $p_i \leq p_j$, and
- $\sum_{p_i \in P} E^*_{p_i, p_1}(B) = 0$.

Thus, to show that $F$ is Frobenius, we must show that the above restrictions imply $B = 0$.

Theorem 3 If $P_1 = \{p_1, p_2\}$ with $p_1 < p_2$ and $F_{P_1} = E^*_{p_1, p_2}$, then $(P_1, F_{P_1})$ forms a toral-pair.

Proof In order to simplify the notations, let $P = P_1$ and $F = F_{P_1}$. It is clear that $|\Ext(P)| = 2$ and $\Sigma(P)$ is contractible so that (P1) and (P2) of Definition 1 are satisfied. To show that $F$ satisfies (F4) of Definition 1 and is Frobenius on $g_A(P)$, take $B \in \ker(B_F)$. We have that

- $F([E_{p_1, p_2}, B]) = E^*_{p_2, p_2}(B) - E^*_{p_1, p_1}(B) = 0$ and
- $F([E_{p_1, p_1}, B]) = E^*_{p_1, p_2}(B) = 0$.

Thus,

$$E^*_{p_1, p_1}(B) = E^*_{p_2, p_2}(B) \quad (1)$$

and

$$E^*_{p_1, p_2}(B) = 0 \quad (2)$$

so that $F$ satisfies (F4) of Definition 1. Now, considering Remark 7 (ii), adding the condition

$$\sum_{p \in P} E^*_{p, p}(B) = 0$$

to (1), (2), and (2), we find that $\ker_A(B_F) = g_A(P) \cap \ker(B_F) = \{0\}$. Therefore, $g_A(P)$ is Frobenius with Frobenius functional $F$.

Given the form of the Frobenius functional $F$, we have that $F$ satisfies (F1) through (F4) of Definition 1 as follows:
• $F$ is clearly small,
• $D_F(\mathcal{P}) = \{p_1\}$ forms an order ideal of $\mathcal{P}$, $U_F(\mathcal{P}) = \{p_2\}$ forms a filter, and $O_F(\mathcal{P}) = \emptyset$,
• $\Gamma_F$ contains the only edge $(p_1, p_2)$ between elements of $Ext(\mathcal{P}_1)$, and
• the fact that $F$ satisfies (F4) was established above.

It remains to show that (P3) of Definition 1 is satisfied; that is, $g_A(\mathcal{P})$ has a spectrum consisting of an equal number of 0’s and 1’s. To determine the spectrum of $g_A(\mathcal{P})$, it suffices to calculate

$$[\hat{F}, x] = \left[ \frac{1}{2}(E_{p_1, p_1} - E_{p_2, p_2}), x \right],$$

for $x \in \mathcal{B}_{\mathcal{P}, F}$; but $\mathcal{B}_{\mathcal{P}, F}$ has only two elements: $E_{p_1, p_1} - E_{p_2, p_2}$ and $E_{p_1, p_2}$. The former is an eigenvector of $ad(\hat{F})$ with eigenvalue 0, and the latter is an eigenvector with eigenvalue 1. Therefore, $g_A(\mathcal{P})$ has a binary spectrum and $(\mathcal{P}, F)$ forms a toral-pair.

**Theorem 4** Each of the following pairs, consisting of a poset $\mathcal{P}$ and a functional $F_{\mathcal{P}}$, forms a toral-pair $(\mathcal{P}, F_{\mathcal{P}})$.

(i) $\mathcal{P}_2 = \{p_1, p_2, p_3, p_4\}$ with $p_1 < p_2 < p_3, p_4$ and

$$F_{\mathcal{P}_2} = E_{p_1, p_3}^* + E_{p_1, p_4}^* + E_{p_2, p_4}^*,$$

(ii) $\mathcal{P}_2^* = \{p_1, p_2, p_3, p_4\}$ with $p_1, p_2 < p_3 < p_4$, and

$$F_{\mathcal{P}_2^*} = E_{p_1, p_4}^* + E_{p_2, p_4}^* + E_{p_2, p_3}^*.$$

**Proof** We prove (i), as (ii) follows via a symmetric argument. In order to simplify the notations, let $\mathcal{P} = \mathcal{P}_2$ and $F = F_{\mathcal{P}_2}$. It is clear that $|Ext(\mathcal{P})| = 3$ and $\Sigma(\mathcal{P})$ is contractible so that (P1) and (P2) of Definition 1 are satisfied. To show that $F$ satisfies (F4) of Definition 1 and is Frobenius on $g_A(\mathcal{P})$, take $B \in \ker(B_F)$. We break the restrictions $B$ must satisfy into 3 groups:

**Group 1:**

• $F([E_{p_2, p_2}, B]) = E_{p_2, p_4}^*(B) = 0$,
• $F([E_{p_4, p_4}, B]) = -E_{p_1, p_4}^*(B) - E_{p_2, p_4}^*(B) = 0$,
• $F([E_{p_2, p_3}, B]) = -E_{p_1, p_2}^*(B) = 0$.

**Group 2:**

• $F([E_{p_1, p_1}, B]) = E_{p_1, p_3}^*(B) + E_{p_1, p_4}^*(B) = 0$,
• $F([E_{p_3, p_3}, B]) = -E_{p_1, p_3}^*(B) = 0$,
• $F([E_{p_1, p_2}, B]) = E_{p_2, p_3}^*(B) + E_{p_2, p_4}^*(B) = 0$.

**Group 3:**

• $F([E_{p_1, p_3}, B]) = E_{p_3, p_3}^*(B) - E_{p_1, p_1}^*(B) = 0$,
• $F([E_{p_1, p_4}, B]) = E_{p_4, p_4}^*(B) - E_{p_1, p_1}^*(B) = 0$.
\[ F([E_{p_2,p_4}, B]) = E^*_{p_4,p_4}(B) - E^*_{p_2,p_2}(B) - E^*_{p_1,p_2}(B) = 0. \]

The restrictions of the equations in Group 1 immediately imply that

\[ E^*_{p_2,p_4}(B) = E^*_{p_1,p_4}(B) = E^*_{p_1,p_2}(B) = 0. \]  \hspace{1cm} (3)

Combining the Group 1 restrictions to those of Group 2, we may conclude that

\[ E^*_{p_1,p_3}(B) = E^*_{p_2,p_3}(B) = 0. \]  \hspace{1cm} (4)

Finally, combining the restrictions of Group 1 to those of Group 3, we find that

\[ E^*_{p_i,p_i}(B) = E^*_{p_j,p_j}(B), \text{ for all } p_i, p_j \in \mathcal{P}. \]  \hspace{1cm} (5)

Equations (3), (4), and (5) establish that \( F \) satisfies (F4) of Definition 1. Now, considering Remark 7 (ii), adding the condition

\[ \sum_{p \in \mathcal{P}} E^*_{p,p}(B) = 0 \]

to (3), (4), and (5), we find that \( \ker_A(B_F) = g_A(\mathcal{P}) \cap \ker(B_F) = \{0\} \). Therefore, \( g_A(\mathcal{P}) \) is Frobenius with Frobenius functional \( F \).

Given the form of the Frobenius functional \( F \), we have that \( F \) satisfies (F1) through (F4) of Definition 1 as follows:

- \( F \) is clearly small,
- \( D_F(\mathcal{P}) = \{p_1, p_2\} \) forms an order ideal of \( \mathcal{P} \), \( U_F(\mathcal{P}) = \{p_3, p_4\} \) forms a filter, and \( O_F(\mathcal{P}) = \emptyset \),
- \( \Gamma_F \) contains the only edges, \((p_1, p_3)\) and \((p_1, p_4)\), between elements of \( \text{Ext}(\mathcal{P}) \), and
- the fact that \( F \) satisfies (F4) was established above.

It remains to show that (P3) of Definition 1 is satisfied; that is, \( g_A(\mathcal{P}) \) has a spectrum consisting of an equal number of 0’s and 1’s. To determine the spectrum of \( g_A(\mathcal{P}) \), it suffices to calculate

\[ [\widehat{F}, x] = \left[ \frac{1}{2}(E_{p_1,p_1} + E_{p_2,p_2} - E_{p_3,p_3} - E_{p_4,p_4}), x \right], \]

for \( x \in \mathcal{B}_{\mathcal{P}, F} \). Note that \( \mathcal{B}_{\mathcal{P}, F} \) can be partitioned into two sets:

\[ G_0 = \{ E_{p_1,p_1} - E_{p_3,p_3}, E_{p_1,p_1} - E_{p_4,p_4}, E_{p_2,p_2} - E_{p_4,p_4}, E_{p_1,p_2} \}, \]

which consists of eigenvectors of \( ad(\widehat{F}) \) with eigenvalue 0, and

\[ G_1 = \{ E_{p_1,p_3}, E_{p_1,p_4}, E_{p_2,p_3}, E_{p_2,p_4} \}, \]

which consists of eigenvectors with eigenvalue 1. As \( |G_0| = |G_1| \), we conclude that \( g_A(\mathcal{P}) \) has a binary spectrum and \((\mathcal{P}, F)\) forms a toral-pair. \hspace{1cm} \( \square \)
Theorem 5  Each of the following pairs, consisting of a poset $\mathcal{P}$ and a functional $F_{\mathcal{P}}$, forms a toral-pair $(\mathcal{P}, F_{\mathcal{P}})$.

(i) $\mathcal{P}_3 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ with $p_1 < p_2 < p_3, p_4; p_3 < p_5; \text{ and } p_4 < p_6, \text{ and}$

$$F_{\mathcal{P}_3} = E^*_{p_1, p_5} + E^*_{p_1, p_6} + E^*_{p_2, p_3} + E^*_{p_2, p_4} + E^*_{p_2, p_6};$$

(ii) $\mathcal{P}_3^* = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ with $p_1 < p_3; p_2 < p_4; \text{ and } p_3, p_4 < p_5 < p_6, \text{ and}$

$$F_{\mathcal{P}_3^*} = E^*_{p_1, p_6} + E^*_{p_2, p_6} + E^*_{p_3, p_5} + E^*_{p_4, p_5} + E^*_{p_2, p_5};$$

(iii) $\mathcal{P}_{4,n} = \{p_1, \ldots, p_n\}$ with $p_1 < p_2 < \ldots < p_{n-1}$ as well as $p_1 < p_2 < \ldots < p_{\lceil \frac{n}{2} \rceil} < p_n$, and

$$F_{\mathcal{P}_{4,n}} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} E^*_{p_i, p_{n-i}} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} E^*_{p_i, p_n};$$

(iv) $\mathcal{P}_{4,n}^* = \{p_1, \ldots, p_n\}$ with $p_1 < p_2 < \ldots < p_{\lfloor \frac{n}{2} \rfloor} < p_{\lfloor \frac{n}{2} \rfloor} + 1 < \ldots < p_n$ as well as $p_1 < p_2 < \ldots < p_{\lceil \frac{n}{2} \rceil} < p_n$, and

$$F_{\mathcal{P}_{4,n}^*} = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor-1} E^*_{p_i, p_{n+1-i}} + \sum_{i=\lceil \frac{n}{2} \rceil+1}^{n} E^*_{p_i, p_{\lceil \frac{n}{2} \rceil+1}};$$

(v) $\mathcal{P}_{5,n} = \{p_1, \ldots, p_{2n+1}\}$ with $p_i < p_j$ for $1 \leq i < 2n$ odd and $i+1 \leq j \leq 2n+1$ as well as $p_i < p_j$ for $1 < i < 2n$ even and $i+2 \leq j \leq 2n+1$, and

$$F_{\mathcal{P}_{5,n}} = E^*_{p_1, p_{2n+1}} + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor+1} E^*_{p_i, p_{2n}} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} E^*_{p_k, p_{2n-2k}} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} E^*_{p_{2k+1}, p_{2n-2k+1}};$$

(vi) $\mathcal{P}_{5,n}^* = \{p_1, \ldots, p_{2n+1}\}$ with $p_i < p_j$ for $1 \leq i < 2n$ odd and $i+2 \leq j \leq 2n+1$, as well as $p_i < p_j$ for $1 < i < 2n$ even and $i+1 \leq j \leq 2n+1$, and

$$F_{\mathcal{P}_{5,n}^*} = E^*_{p_1, p_{2n+1}} + \sum_{i=2}^{2n+1} E^*_{p_{i+1}, p_i} + \sum_{k=2}^{\lfloor \frac{n+1}{4} \rfloor} E^*_{p_{2k}, p_{2n-2k+4}} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} E^*_{p_{2k+1}, p_{2n-2k+1}}.$$

**Proof** Appendix B. \qed
Thus, the posets $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_2^*, \mathcal{P}_3, \mathcal{P}_3^*, \mathcal{P}_{4,n}, \mathcal{P}_{4,n}^*, \mathcal{P}_{5,n},$ and $\mathcal{P}_{5,n}^*$ along with the corresponding Frobenius functionals found in Theorems 3–5 form toral-pairs. In the next section, posets of toral-pairs are combined to form toral posets.

### 6 Toral posets

In this section, we define toral posets, which are constructed inductively from the posets of toral-pairs. Furthermore, we show that if $\mathcal{P}$ is a toral poset, then $\Sigma(\mathcal{P})$ is homotopic to a wedge sum of ind $g_A(\mathcal{P})$ one-spheres.

Let $(\mathcal{S}, F)$ be a toral-pair and $\mathcal{Q}$ be a poset. We define twelve ways of “combining” the posets $\mathcal{S}$ and $\mathcal{Q}$ by identifying minimal (resp., maximal) elements of $\mathcal{S}$ with minimal (resp., maximal) elements of $\mathcal{Q}$. If $|\operatorname{Ext}\mathcal{S}| = 2$, then $\operatorname{Ext}\mathcal{S} = \{a, b\}$ with either $a \prec \mathcal{S} b$ or $b \prec \mathcal{S} a$, and if $|\operatorname{Ext}\mathcal{S}| = 3$, then $\operatorname{Ext}\mathcal{S} = \{a, b, c\}$ with either $a \prec \mathcal{S} b, c$ or $b, c \prec \mathcal{S} a$. Further, assume $x, y, z \in \operatorname{Ext}\mathcal{Q}$. Since the construction rules are defined by identifying minimal elements and maximal elements of $\mathcal{S}$ and $\mathcal{Q}$, assume that if $a, b, c$ are identified with elements of $\mathcal{Q}$, then those elements are $x, y, z$, respectively. In order to simplify the notations, let $\sim_\mathcal{P}$ denote that two elements of a poset $\mathcal{P}$ are related, and let $\sim_\mathcal{P}$ denote that two elements are not related; that is, for $x, y \in \mathcal{P}$, $x \sim_\mathcal{P} y$ denotes that $x \prec_\mathcal{P} y$ or $y \prec_\mathcal{P} x$, and $x \sim_\mathcal{P} y$ denotes that both $x \not\prec_\mathcal{P} y$ and $y \not\prec_\mathcal{P} x$. The construction rules are as follows: If $|\operatorname{Ext}(\mathcal{S})| = 2$ or 3, then

- A1 denotes identifying $b \in \operatorname{Ext}\mathcal{S}$ with $y \in \operatorname{Ext}\mathcal{Q}$,
- C denotes identifying $a \in \operatorname{Ext}\mathcal{S}$ with $x \in \operatorname{Ext}\mathcal{Q}$,
- D1 denotes identifying $a, b \in \operatorname{Ext}\mathcal{S}$ with $x, y \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} y$,
- E1 denotes identifying $a, b \in \operatorname{Ext}\mathcal{S}$ with $x, y \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} y$.

If $|\operatorname{Ext}(\mathcal{S})| = 3$, then

- A2 denotes identifying $c \in \operatorname{Ext}\mathcal{S}$ with $z \in \operatorname{Ext}\mathcal{Q}$,
- B denotes identifying $b, c \in \operatorname{Ext}\mathcal{S}$ with $y, z \in \operatorname{Ext}\mathcal{Q}$,
- D2 denotes identifying $a, c \in \operatorname{Ext}\mathcal{S}$ with $x, z \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} z$,
- E2 denotes identifying $a, c \in \operatorname{Ext}\mathcal{S}$ with $x, z \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} z$,
- F denotes identifying $a, b, c \in \operatorname{Ext}\mathcal{S}$ with $x, y, z \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} y$ and $x \sim_\mathcal{Q} z$,
- G1 denotes identifying $a, b, c \in \operatorname{Ext}\mathcal{S}$ with $x, y, z \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} y$ and $x \sim_\mathcal{Q} z$,
- G2 denotes identifying $a, b, c \in \operatorname{Ext}\mathcal{S}$ with $x, y, z \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} y$ and $x \sim_\mathcal{Q} z$,
- H denotes identifying $a, b, c \in \operatorname{Ext}\mathcal{S}$ with $x, y, z \in \operatorname{Ext}\mathcal{Q}$, where $x \sim_\mathcal{Q} y$ and $x \sim_\mathcal{Q} z$.

**Definition 2** A poset $\mathcal{P}$ is called **toral** if there exists a sequence of toral-pairs $\{(S_i, F_i)\}_{i=1}^n$ along with a sequence of posets $\mathcal{S}_1 = \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \ldots \subset \mathcal{Q}_n = \mathcal{P}$ such that $\mathcal{Q}_i$ is formed from $\mathcal{Q}_{i-1}$ and $S_i$ by applying a rule from the set $\{A_1, A_2, B, C, D_1, D_2, E_1, E_2, F, G_1, G_2, H\}$, for $i = 2, \ldots, n$. Such a sequence $\mathcal{S}_1 = \mathcal{Q}_1 \subset \mathcal{Q}_2 \subset \ldots \subset \mathcal{Q}_n = \mathcal{P}$ is called a construction sequence for $\mathcal{P}$.
**Example 5** Let \( P \) be the toral poset constructed from the toral-pairs \( \{(S_i, F_i)\}_{i=1}^{5} \), where \( S_i = P_2 \), for \( i = 1, \ldots, 5 \), with attendant construction sequence \( S_1 = Q_1 \subset Q_2 \subset Q_3 \subset Q_4 \subset Q_5 = P \), where \( Q_2 \) is formed from \( Q_1 \) and \( S_2 \) by applying rule \( A_1 \), \( Q_3 \) is formed from \( Q_2 \) and \( S_3 \) by applying rule \( C \), \( Q_4 \) is formed from \( Q_3 \) and \( S_4 \) by applying rule \( D_1 \), and \( Q_5 = P \) is formed from \( Q_4 \) and \( S_5 \) by applying rule \( F \). See Fig. 3.

![Fig. 3](image)

**Fig. 3** Construction sequence of \( P \)

Coupling Remark 6 of [5] together with Theorem 20 of [5] yields the following.

**Theorem 6** If \( P \) is a toral poset, then \( \text{ind} g_A(P) = |\text{Rel}_E(P)| - |\text{Ext}(P)| + 1 \).

**Theorem 7** If \( P \) is a toral poset, then \( \Sigma(P) \) is homotopic to a wedge sum of \( \text{ind} g_A(P) \) one-spheres.

**Proof** Let \( P \) be a toral poset constructed from the toral-pairs \( \{(S_i, F_S)\}_{i=1}^{n} \) with construction sequence \( S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = P \). Since each \( \Sigma(S_i) \) is contractible, for \( i = 1, \ldots, n \), \( \Sigma(S_i) \) is homotopic to the Hasse diagram of \( S_i^{\text{Rel}_E(S_i)} \). Performing this homotopy sequentially for each \( \Sigma(S_i) \subset \Sigma(P) \) from \( i = 1 \) to \( n \), we arrive at the simplicial complex \( \Sigma(P)\)'s. Note that \( \Sigma(P)\)'s just the Hasse diagram of \( P_{\text{Ext}(P)} \). As \( \Sigma(P)\)'s is a connected graph with \( |\text{Rel}_E(P)| \) edges and \( |\text{Ext}(P)| \) vertices, \( \Sigma(P)\)'s is homotopic to a wedge sum of \( |\text{Rel}_E(P)| - |\text{Ext}(P)| + 1 \) one-spheres. Thus, considering Theorem 6, the result follows. \( \square \)

**Theorem 8** Let \( P \) be a toral poset constructed from the toral-pairs \( \{(S_i, F_S)\}_{i=1}^{n} \) with construction sequence \( S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = P \). Then, \( P \) is Frobenius if and only if \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rules from the set \( \{A_1, A_2, C, D_1, D_2, F\} \), for \( i = 2, \ldots, n \).

**Proof** Using Theorem 6, for \( i = 2, \ldots, n \), if \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule

- \( A_1 \), \( A_2 \), or \( C \), then \( \text{ind} g_A(Q_i) = \text{ind} g_A(Q_{i-1}) + 2 - 2 = \text{ind} g_A(Q_{i-1}) \);
- \( B \), \( E_1 \), or \( E_2 \), then \( \text{ind} g_A(Q_i) = \text{ind} g_A(Q_{i-1}) + 2 - 1 = \text{ind} g_A(Q_{i-1}) + 1 \);
- \( D_1 \) or \( D_2 \), then \( \text{ind} g_A(Q_i) = \text{ind} g_A(Q_{i-1}) + 1 - 1 = \text{ind} g_A(Q_{i-1}) \);
- \( F \), then \( \text{ind} g_A(Q_i) = \text{ind} g_A(Q_{i-1}) \);
- \( G_1 \) or \( G_2 \), then \( \text{ind} g_A(Q_i) = \text{ind} g_A(Q_{i-1}) + 1 \);
- \( H \), then \( \text{ind} g_A(Q_i) = \text{ind} g_A(Q_{i-1}) + 2 \).

Thus, as \( \text{ind} g_A(Q_1) = \text{ind} g_A(S_1) = 0 \), the result follows. \( \square \)
7 Toral functionals and spectrum

In this section, given a Frobenius, toral poset $\mathcal{P}$ constructed from the toral-pairs $\{(S_i, F_{S_i})\}_{i=1}^n$, we provide an inductive procedure for constructing a Frobenius functional $F_{\mathcal{P}} \in (g_A(\mathcal{P}))^*$ from the functionals $F_{S_i}$, for $i = 1, \ldots, n$. Coincidentally, we obtain an alternative proof that toral posets formed by applying rules from the set $\{A_1, A_2, C, D_1, D_2, F\}$ are Frobenius (see Theorem 9). Furthermore, we characterize the spectrum of all Frobenius, type-A Lie poset algebras which correspond to toral posets (see Theorem 10).

Remark 8 Let $\mathcal{P}$ be a toral poset constructed from the toral-pairs $\{(S_i, F_{S_i})\}_{i=1}^n$ with construction sequence $S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P}$. Throughout this section, in the notation of Sect. 6, if $Q = Q_{i-1}$ and $S = S_i$, for $i = 2, \ldots, n$, then we denote the elements of $\{a, x\}$ by $x_i$, $\{b, y\}$ by $y_i$, and $\{c, z\}$ by $z_i$.

Definition 3 If $\mathcal{P}$ is a Frobenius, toral poset constructed from the toral-pairs $\{(S_i, F_{S_i})\}_{i=1}^n$ with construction sequence $S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P}$, then define the “toral” functional $F_{Q_i} \in (g_A(Q_i))^*$, for $i = 1, \ldots, n$, as follows:

- $F_{Q_1} = F_{S_1}$;
- if $Q_i$ is formed from $Q_{i-1}$ and $S_i$, for $1 < i \leq n$, by applying rule
  - $A_1$, $A_2$, or $C$, then
    
    $$F_{Q_i} = F_{Q_{i-1}} + F_{S_i},$$
  
  - $D_1$, then
    
    $$F_{Q_i} = \begin{cases} 
    F_{Q_{i-1}} + F_{S_i} - E_{x_i,y_i}^* & \text{if } S_i \text{ has one minimal element;} \\
    F_{Q_{i-1}} + F_{S_i} - E_{y_i,x_i}^* & \text{if } S_i \text{ has one maximal element.}
    \end{cases}$$
  
  - $D_2$, then
    
    $$F_{Q_i} = \begin{cases} 
    F_{Q_{i-1}} + F_{S_i} - E_{x_i,z_i}^* & \text{if } S_i \text{ has one minimal element;} \\
    F_{Q_{i-1}} + F_{S_i} - E_{z_i,x_i}^* & \text{if } S_i \text{ has one maximal element.}
    \end{cases}$$
  
  - $F$, then
    
    $$F_{Q_i} = \begin{cases} 
    F_{Q_{i-1}} + F_{S_i} - E_{x_i,y_i}^* - E_{x_i,z_i}^* & \text{if } S_i \text{ has one minimal element;} \\
    F_{Q_{i-1}} + F_{S_i} - E_{y_i,x_i}^* - E_{z_i,x_i}^* & \text{if } S_i \text{ has one maximal element.}
    \end{cases}$$

Remark 9 Note that $E_{p,q}^*$ is a summand of $F_{Q_i}$ if and only if $E_{p,q}^*$ is a summand of $F_{Q_{i-1}}$ or $F_{S_i}$.
The following lemma is an immediate consequence of Definition 3.

**Lemma 1** If $\mathcal{P}$ is a toral poset constructed from the toral-pairs $\{(S_i, F_{S_i})\}_{i=1}^n$ with the construction sequence $S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P}$, then, for $i = 1, \ldots, n$,

- $F_{Q_i}$ is small,
- $D_{F_{Q_i}}(Q_i)$ is an order ideal of $Q_i$, $U_{F_{Q_i}}(Q_i)$ is a filter, and $O_{F_{Q_i}}(Q_i) = \emptyset$.

**Remark 10** Recall that for a poset $\mathcal{P}$, elements of $g(\mathcal{P})$ are of the form

$$\sum_{(p_i, p_j) \in \text{Rel}(\mathcal{P})} c_{p_i, p_j} E_{p_i, p_j} + \sum_{p_i \in \mathcal{P}} c_{p_i} E_{p_i, p_i}.$$ 

Let $\mathcal{P}$ be a poset formed by combining the posets $\mathcal{S}$ and $\mathcal{Q}$ by identifying minimal elements or maximal elements. If $B \in g(\mathcal{P})$, then let $B|_{\mathcal{Q}}$ denote the restriction of $B$ to basis elements of $g(\mathcal{Q})$ and $B|_{\mathcal{S}}$ denote the restriction of $B$ to basis elements of $g(\mathcal{S})$.

**Lemma 2** If $\mathcal{P}$ is a Frobenius, toral poset and $B \in g(\mathcal{P})$ satisfies $F_{\mathcal{P}}([E_{p, p}, B]) = 0$, for all $p \in \mathcal{P}$, then $E_{p, q}^*(B) = 0$, for $E_{p, q}^*$ a summand of $F_{\mathcal{P}}$.

**Proof** Assume $\mathcal{P}$ is constructed from the toral-pairs $\{(S_i, F_{S_i})\}_{i=1}^n$ with construction sequence $S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P}$. The proof is by induction on $i$. Throughout, we assume that $S_i$, for $i = 1, \ldots, n$, satisfies $\text{Ext}(S_i) = 3$ and contains a single minimal element; the other cases follow via a similar argument. By property (F1) of toral-pairs, we know that $\Gamma_{F_{S_i}}(S_i)$ is a tree, for $i = 1, \ldots, n$. Thus, for $i = 1, \ldots, n$, we can apply the following inductive procedure, denoted $\text{Proc}(i)$:

**Step 1:** Consider all degree-one vertices $p_1 \in S_i \setminus \text{Ext}(S_i)$ of $\Gamma_{F_{S_i}}(S_i) = \Gamma_1$. Suppose $p_1$ is adjacent to $q_1$ in $\Gamma_1$. Then,

$$F_{Q_i}([E_{p_1, p_1}, B]) = F_{S_i}([E_{p_1, p_1}, B|_{\mathcal{S}}]) = E_{p_1, q_1}^*(B) = 0 \text{ or } -E_{q_1, p_1}^*(B) = 0,$$

where $E_{p_1, q_1}^*$ (or $E_{q_1, p_1}^*$) is a summand of $F_{\mathcal{P}}$. Remove such $p_1$ and edges $(p_1, q_1)$ from $\Gamma_1$ to form the directed graph $\Gamma_2$.

**Step j:** Consider all degree-one vertices $p_j \in S_i \setminus \text{Ext}(S_i)$ of $\Gamma_j$. Suppose $p_j$ is adjacent to $q_j$ in $\Gamma_j$. Using the results of Step 1 through Step $j - 1$,

$$F_{Q_i}([E_{p_j, p_j}, B]) = F_{S_i}([E_{p_j, p_j}, B|_{\mathcal{S}}]) = E_{p_j, q_j}^*(B) = 0 \text{ or } -E_{q_j, p_j}^*(B) = 0,$$

where $E_{p_j, q_j}^*$ (or $E_{q_j, p_j}^*$) is a summand of $F_{\mathcal{P}}$. Remove such $p_j$ and edges $(p_j, q_j)$ from $\Gamma_{j-1}$ to form the directed graph $\Gamma_j$.

By properties (F1) and (F3) of toral-pairs and the fact that $S_i$ is finite, there must exist a finite $m_i$ for which $\Gamma_{m_i}$ consists solely of the elements of $\text{Ext}(S_i)$ along with all the edges between them. Note that this implies $E_{p, q}^*(B) = 0$, for $E_{p, q}^*$ a summand of $F_{Q_i}$ with $p, q \in S_i$ and either $p \in S_i \setminus \text{Ext}(S_i)$ or $q \in S_i \setminus \text{Ext}(S_i)$. 

\[ Springer\]
For the base case, \( i = 1 \), \( Q_1 = S_1 \) has maximal elements \( y_1, z_1 \) and minimal element \( x_1 \). Applying \( P_{roc}(1) \) it remains to consider \( E_{x_1,y_1}(B) \) and \( E_{x_1,z_1}(B) \); but the implications of \( P_{roc}(1) \) allow us to conclude that

\[
F_{Q_1}( [E_{y_1}, y_1], B) = -E_{x_1,y_1}(B) = 0
\]

and

\[
F_{Q_1}( [E_{z_1}, z_1], B) = -E_{x_1,z_1}(B) = 0.
\]

The base of the induction is thus established.

Now, assume the result holds for \( B \in g(Q_{i-1}) \), for \( 1 < i \leq n \). There are four cases to consider, based on the rules used in the construction sequence of \( \mathcal{P} \).

**Case 1:** \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule \( A_1 \) or \( A_2 \). Without loss of generality, assume \( Q_i \) is formed by applying rule \( A_1 \). The implications of \( P_{roc}(i) \) allow us to conclude that

\[
F_{Q_i}( [E_{z_i}, z_i], B) = F_{S_i}( [E_{z_i}, z_i], B) = -E_{x_i,z_i}(B) = 0,
\]

so that

\[
F_{Q_i}( [E_{x_i}, x_i], B) = F_{S_i}( [E_{x_i}, x_i], B) = E_{x_i,y_i}(B) = 0.
\]

Thus, \( E_{p,q}^*(B) = 0 \), for \( E_{p,q}^* \) a summand of \( F_{Q_i} \) with \( p, q \in S_i \) and either \( p \in S_i \setminus Q_i \) or \( q \in S_i \setminus Q_i \). Considering Definition 3, \( F_{Q_i}( [E_{p,p}, B]) = 0 \) reduces to \( F_{Q_{i-1}}( [E_{p,p}, B]|_{Q_{i-1}}) = 0 \), for \( p \in Q_{i-1} \). By the inductive hypothesis, this implies that \( E_{p,q}^*(B) = 0 \), for \( E_{p,q}^* \) a summand of \( F_{Q_i} \).

**Case 2:** \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule \( C \). The implications of \( P_{roc}(i) \) allow us to conclude that

\[
F_{Q_i}( [E_{y_i}, y_i], B) = F_{S_i}( [E_{y_i}, y_i], B) = -E_{x_i,y_i}(B) = 0
\]

and

\[
F_{Q_i}( [E_{z_i}, z_i], B) = F_{S_i}( [E_{z_i}, z_i], B) = -E_{x_i,z_i}(B) = 0.
\]

Thus, \( E_{p,q}^*(B) = 0 \), for \( E_{p,q}^* \) a summand of \( F_{Q_i} \) with \( p, q \in S_i \) and either \( p \in S_i \setminus Q_i \) or \( q \in S_i \setminus Q_i \). Considering Definition 3, \( F_{Q_i}( [E_{p,p}, B]) = 0 \) reduces to \( F_{Q_{i-1}}( [E_{p,p}, B]|_{Q_{i-1}}) = 0 \), for \( p \in Q_{i-1} \). By the inductive hypothesis, this implies that \( E_{p,q}^*(B) = 0 \), for \( E_{p,q}^* \) a summand of \( F_{Q_i} \).

**Case 3:** \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule \( D_1 \) or \( D_2 \). Without loss of generality, assume \( Q_i \) is formed by applying rule \( D_1 \). The implications of \( P_{roc}(i) \) allow us to conclude that

\[
F_{Q_i}( [E_{z_i}, z_i], B) = F_{S_i}( [E_{z_i}, z_i], B) = -E_{x_i,z_i}(B) = 0.
\]
Thus, $E^*_{p,q}(B) = 0$, for $E^*_{p,q}$ a summand of $F_{Q_i}$ with $p, q \in S_i$ and either $p \in S_i \setminus Q_i$ or $q \in S_i \setminus Q_i$. Considering Definition 3, $F_{Q_i}([E_{p,p}, B]) = 0$ reduces to $F_{Q_{i-1}}([E_{p,p}, B|_{Q_{i-1}}]) = 0$, for $p \in Q_{i-1}$. By the inductive hypothesis, this implies that $E^*_{p,q}(B) = 0$, for $E^*_{p,q}$ a summand of $F_{Q_i}$.

**Case 4:** $Q_i$ is formed from $Q_{i-1}$ and $S_i$ by applying rule F. The implications of $Proc(i)$ allow us to conclude that $E^*_{p,q}(B) = 0$, for $E^*_{p,q}$ a summand of $F_{Q_i}$ with $p, q \in S_i$ and either $p \in S_i \setminus Q_i$ or $q \in S_i \setminus Q_i$. Considering Definition 3, $F_{Q_i}([E_{p,p}, B]) = 0$ reduces to $F_{Q_{i-1}}([E_{p,p}, B|_{Q_{i-1}}]) = 0$, for $p \in Q_{i-1}$. By the inductive hypothesis, this implies that $E^*_{p,q}(B) = 0$, for $E^*_{p,q}$ a summand of $F_{Q_i}$.

The induction establishes the result. \hfill $\square$

**Lemma 3** Let $\mathcal{P}$ be a Frobenius, toral poset constructed from the toral-pairs $\{(S_i, F_i)\}_{i=1}^n$ with construction sequence $S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P}$. If $B \in \ker(B_{F_{Q_i}})$, for $1 < i \leq n$, then $B|_{Q_{i-1}} \in \ker(B_{F_{Q_{i-1}}})$ and $B|_{S_i} \in \ker(B_{F_{S_i}})$.

**Proof** Take $B \in \ker(B_{F_{Q_i}})$, for $1 < i \leq n$. Considering Remark 9, the equation $F_{Q_{i-1}}([E_{p,p}, B|_{Q_{i-1}}]) = 0$ (resp., $F_{S_i}([E_{p,p}, B|_{S_i}]) = 0$) consists of terms of the form $E^*_{p,q}(B)$ and $E^*_{r,p}(B)$, for summands $E^*_{p,q}$ and $E^*_{r,p}$ of $F_{Q_i}$ such that $p, q, r \in Q_{i-1}$ (resp. $p, q, r \in S_i$). Thus, by Lemma 2, $B \in \ker(B_{F_{Q_i}})$ must satisfy

$$F_{Q_{i-1}}([E_{p,p}, B|_{Q_{i-1}}]) = 0, \text{ for } p \in Q_{i-1},$$

and

$$F_{S_i}([E_{p,q}, B|_{S_i}]) = 0, \text{ for } p \in S_i.$$

It remains to consider restrictions placed on $B$ by equations of the form $F_{Q_i}([E_{p,q}, B]) = 0$ for $p, q \in Q_{i-1}$ (resp., $p, q \in S_i$); but combining Remark 9 with the fact that $Q_{i-1} \cap S_i \subset \text{Ext}(Q_i)$, it is immediate that

$$F_{Q_{i-1}}([E_{p,q}, B|_{Q_{i-1}}]) = F_{Q_i}([E_{p,q}, B]) = 0, \text{ for } p, q \in Q_{i-1},$$

and

$$F_{S_i}([E_{p,q}, B|_{S_i}]) = F_{Q_i}([E_{p,q}, B]) = 0, \text{ for } p, q \in S_i.$$

Thus, the result follows. \hfill $\square$

**Theorem 9** If $\mathcal{P}$ is a Frobenius, toral poset constructed from the toral-pairs $\{(S_i, F_i)\}_{i=1}^n$ with construction sequence $S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P}$, then $F_{Q_i} \in (\mathcal{g}_A(Q_i))^*$ is Frobenius, for $i = 1, \ldots, n$.

**Proof** We will show by induction on $i$ that $B \in \ker(B_{F_{Q_i}})$ satisfies

- $E^*_{p,p}(B) = E^*_{q,q}(B)$, for all $p, q \in Q_i$, and
- $E^*_{p,q}(B) = 0$, for all $p, q \in Q_i$ satisfying $p < q$,
for \( i = 1, \ldots, n \). Considering Remark 7 (ii), adding the restriction \( \sum_{p \in Q_i} E_{p,p}^*(B) = 0 \), we may conclude that \( \ker_A(B_{F_{Q_i}}) = g_A(Q_i) \cap \ker(B_{F_{Q_i}}) = \{0\} \), establishing the result.

For \( i = 1 \), the result is clear since \( (Q_1, F_{Q_1}) = (S_1, F_{S_1}) \) is a toral-pair. Assume the result holds for \( B \in \ker(B_{F_{Q_i}}) \), for \( 1 < i \leq n \). Take \( B \in \ker(B_{F_{Q_i}}) \). By Lemma 3, we know that

(a) \( B|_{Q_{i-1}} \in \ker(B_{F_{Q_{i-1}}}) \), and
(b) \( B|_{S_i} \in \ker(B_{F_{S_i}}) \).

Combining (a) with the inductive hypothesis, we find that \( E_{p,p}^*(B) = E_{q,q}^*(B) \), for all \( p, q \in Q_{i-1} \), and \( E_{p,q}^*(B) = 0 \), for all \( p, q \in Q_{i-1} \) satisfying \( p < q \). Furthermore, combining (b) with property (F4) of toral-pairs, we must also have that \( E_{p,p}^*(B) = E_{q,q}^*(B) \), for all \( p, q \in S_i \), and \( E_{p,q}^*(B) = 0 \), for all \( p, q \in S_i \) satisfying \( p < q \). Thus, \( E_{p,q}^*(B) = 0 \), for all \( p, q \in Q_i \) satisfying \( p < q \), and, since \( Q_i \) is connected, \( E_{p,p}^*(B) = E_{q,q}^*(B) \), for all \( p, q \in Q_i \). Hence, as stated above, considering Remark 7 (ii) we may conclude that \( \ker_A(B_{F_{Q_i}}) = \{0\} \). The result follows.

\[ \square \]

**Theorem 10**  
If \( \mathcal{P} \) is a Frobenius, toral poset, then the spectrum of \( g_A(\mathcal{P}) \) is binary.

**Proof** Assume \( \mathcal{P} \) is constructed from the toral-pairs \( \{(S_i, F_{S_i})\}_{i=1}^n \) with construction sequence \( S_1 = Q_1 \subset Q_2 \subset \ldots \subset Q_n = \mathcal{P} \). We will show that \( g_A(Q_i) \), for \( 1 \leq i \leq n \), has binary spectrum by induction on \( i \). Throughout, we assume that \( S_i \) satisfies \( Ext(S_i) = 3 \) and contains a single minimal element; the other cases follow via a similar argument.

For the base case, \( g_A(Q_1) = g_A(S_1) \) has a binary spectrum by property (P3) of toral-pairs. So, assume that \( g_A(Q_{i-1}) \), for \( 1 < i \leq n \), has a binary spectrum. By Lemma 1, the form of \( \widehat{F}_{Q_i} \) is given by Theorem 1. Thus, considering the proof of Theorem 2, to determine the spectrum of \( g_A(\mathcal{P}) \) it suffices to calculate the values \([\widehat{F}_{Q_i}, x]\) for \( x \in \widehat{B}_{Q_i, F_{Q_i}} \). Recall that \( E_{p,p} - E_{q,q} \in \widehat{B}_{Q_i, F_{Q_i}} \) is an eigenvector of \( ad(\widehat{F}_{Q_i}) \) with eigenvalue \( 0 \), \( E_{p,p} \in \widehat{B}_{Q_i, F_{Q_i}} \) with \( p, q \in D_{F_{Q_i}}(Q_i) \) or \( p, q \in U_{F_{Q_i}}(Q_i) \) has eigenvalue \( 0 \), and \( E_{p,q} \in \widehat{B}_{Q_i, F_{Q_i}} \) with \( p \in D_{F_{Q_i}}(Q_i) \) and \( q \in U_{F_{Q_i}}(Q_i) \) has eigenvalue \( 1 \). Considering Definition 3, we must have that \( D_{F_{Q_{i-1}}}(Q_{i-1}) \subset D_{F_{Q_i}}(Q_i) \) and \( U_{F_{Q_{i-1}}}(Q_{i-1}) \subset U_{F_{Q_i}}(Q_i) \) so that, by the inductive hypothesis, the elements of \( \widehat{B}_{Q_{i-1}, F_{Q_{i-1}}} \subset \widehat{B}_{Q_i, F_{Q_i}} \) contribute an equal number of 0’s and 1’s to the spectrum of \( g_A(Q_i) \). It remains to consider the eigenvalues corresponding to the elements of \( \widehat{B}_{Q_i, F_{Q_i}} \setminus \widehat{B}_{Q_{i-1}, F_{Q_{i-1}}} \). The collection of such basis elements breaks into four cases:

**Case 1:** \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule A1, A2, or C. In this case,

\[ \widehat{B}_{Q_i, F_{Q_i}} \setminus \widehat{B}_{Q_{i-1}, F_{Q_{i-1}}} = \widehat{B}_{S_i, F_{S_i}}. \]

By Definition 3, we have \( D_{F_{S_i}}(S_i) \subset D_{F_{Q_i}}(Q_i) \) and \( U_{F_{S_i}}(S_i) \subset U_{F_{Q_i}}(Q_i) \). Thus, by property (P3) of toral-pairs, we must have that the elements of \( \widehat{B}_{S_i, F_{S_i}} \) contribute an equal number of 0’s and 1’s to the spectrum of \( g_A(Q_i) \). Thus, \( g_A(Q_i) \) has a binary spectrum.

**Case 2:** \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule D1 or D2. Assume, without loss of generality, that \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule D1. In this

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case,
\[ \mathcal{B}_{Q_i} F_{Q_i} \setminus \mathcal{B}_{Q_{i-1}} F_{Q_{i-1}} = \mathcal{B}_{S_i} F_{S_i} \setminus \{ E_{x_i, x_i} - E_{y_i, y_i}, E_{x_i, y_i} \}. \]

As in Case 1, the basis elements of \( \mathcal{B}_{S_i} F_{S_i} \) contribute an equal number of 0's and 1's to the spectrum of \( g_A(Q_i) \). Since \( E_{x_i, x_i} - E_{y_i, y_i} \) is an eigenvector of \( ad(\hat{F}_{Q_i}) \) with eigenvalue 0, while \( E_{x_i, y_i} \) is an eigenvector with eigenvalue 1, it follows that \( g_A(Q_i) \) has a binary spectrum.

**Case 3:** \( Q_i \) is formed from \( Q_{i-1} \) and \( S_i \) by applying rule F. In this case,
\[ \mathcal{B}_{Q_i} F_{Q_i} \setminus \mathcal{B}_{Q_{i-1}} F_{Q_{i-1}} = \mathcal{B}_{S_i} F_{S_i} \setminus \{ E_{x_i, x_i} - E_{y_i, y_i}, E_{x_i, x_i} - E_{z_i, z_i}, E_{x_i, y_i}, E_{x_i, z_i} \}. \]

As in Case 1, the basis elements of \( \mathcal{B}_{S_i} F_{S_i} \) contribute an equal number of 0's and 1's to the spectrum of \( g_A(Q_i) \). Since \( E_{x_i, x_i} - E_{y_i, y_i} \) and \( E_{x_i, x_i} - E_{z_i, z_i} \) are eigenvectors of \( ad(\hat{F}_{Q_i}) \) with eigenvalue 0, while \( E_{x_i, y_i} \) and \( E_{x_i, z_i} \) are eigenvectors with eigenvalue 1, it follows that \( g_A(Q_i) \) has a binary spectrum. \( \square \)

As a corollary to Theorem 10, in conjunction with Theorem 11 of [5], we get the following succinct result.

**Theorem 11** If \( P \) is a Frobenius poset of height at most two, then \( g_A(P) \) is toral—and so has a binary spectrum.

**Remark 11** Extensive calculations suggest that Theorem 11 is true for posets of arbitrary height.

**Acknowledgements** The authors are indebted to Nicholas Russoniello and two anonymous referees for their careful reading of the manuscript and for a number of helpful suggestions which enhanced the exposition.

**Appendix A**

In this appendix, we show that given a finite poset \( P \) and \( F \in (g_A(P))^* \), that \( \ker_A(B_F) = g_A(P) \cap \ker(B_F) \). To do this, it suffices to show that for an element \( B \in g(P) \), the collection of restrictions placed on \( B \) by the conditions \( F([E_{p_1, p_1} - E_{p_1, p_1}, B]) = 0 \), for \( p_1 \in P \) and all \( p_1 \neq p_i \in P \), is equivalent to the restrictions placed by the conditions \( F([E_{p_1, p_1}, B]) = 0 \), for all \( p_i \in P \).

The conditions
\[ F([E_{p_1, p_1} - E_{p_1, p_1}, B]) = F([E_{p_1, p_1}, B]) - F([E_{p_1, p_1}, B]) = 0, \]
for all \( p_i \in P \) such that \( p_1 \neq p_i \), imply that there exists a scalar \( C \in k \) such that, for all \( p_i \in P \),
\[ F([E_{p_i, p_i}, B]) = C. \]
Thus,
\[ \sum_{p_i \in \mathcal{P}} F([E_{p_i}, p_i, B]) = |\mathcal{P}|C; \]
but
\[ \sum_{p_i \in \mathcal{P}} F([E_{p_i}, p_i, B]) = 0. \]

To see this, note that if \( E^*_{p_j, p_k}(B) \) is a term of \( \sum_{p_i \in \mathcal{P}} F([E_{p_i}, p_i, B]) \) (corresponding to \( F([E_{p_j}, p_j, B]) \)), then \( -E^*_{p_j, p_k}(B) \) must also be a term (coming from \( F([E_{p_k}, p_k, B]) \)). Thus, \( C = 0 \) and we have shown that the conditions
\[ F([E_{p_1}, p_1 - E_{p_i}, p_i, B]) = 0, \]
for \( p_1 \in \mathcal{P} \) and all \( p_1 \neq p_i \in \mathcal{P} \), imply that
\[ F([E_{p_i}, p_i, B]) = 0, \]
for all \( p_i \in \mathcal{P} \). The other direction is clear.

**Appendix B**

In this Appendix, we prove Theorem 5 of Sect. 5. We break the proof into three lemmas. Lemma 4 addresses the proof of Theorem 5 (i) and (ii), Lemma 5 addresses the proof of Theorem 5 (iii) and (iv), and Lemma 6 addresses the proof of Theorem 5 (v) and (vi).

**Lemma 4** Each of the following pairs, consisting of a poset \( \mathcal{P} \) and a functional \( F_\mathcal{P} \), forms a toral-pair \( (\mathcal{P}, F_\mathcal{P}) \).

1. \( \mathcal{P}_3 = \{ p_1, p_2, p_3, p_4, p_5, p_6 \} \) with \( p_1 < p_2 < p_3 < p_4; p_3 < p_5; \) and \( p_4 < p_6 \), and
   \[ F_{\mathcal{P}_3} = E^*_{p_1, p_5} + E^*_{p_1, p_6} + E^*_{p_2, p_4} + E^*_{p_2, p_6} \]
2. \( \mathcal{P}_3^* = \{ p_1, p_2, p_3, p_4, p_5, p_6 \} \) with \( p_1 < p_3; p_2 < p_4; \) and \( p_3, p_4 < p_5 < p_6 \), and
   \[ F_{\mathcal{P}_3^*} = E^*_{p_1, p_6} + E^*_{p_2, p_6} + E^*_{p_3, p_5} + E^*_{p_4, p_5} + E^*_{p_2, p_5} \]

**Proof** We prove (i), as (ii) follows via a symmetric argument. In order to simplify the notations, let \( \mathcal{P} = \mathcal{P}_3 \) and \( F = F_{\mathcal{P}_3} \). By definition, it is clear that \( |Ext(\mathcal{P})| = 3 \) and \( \Sigma(\mathcal{P}) \) is contractible so that (P1) and (P2) of Definition 1 are satisfied. Now, to see
that $F$ satisfies (F4) of Definition 1 and is Frobenius on $g_A(\mathcal{P})$, take $B \in \ker(B_F)$.

We break the restrictions $B$ must satisfy into three groups:

**Group 1:**
- $F([E_{p_5, p_5}, B]) = -E^*_{p_1, p_5}(B) = 0$,
- $F([E_{p_1, p_5}, B]) = E^*_{p_3, p_5}(B) = 0$,
- $F([E_{p_1, p_4}, B]) = E^*_{p_4, p_6}(B) = 0$,
- $F([E_{p_3, p_5}, B]) = -E^*_{p_2, p_3}(B) = 0$,
- $F([E_{p_4, p_4}, B]) = -E^*_{p_2, p_4}(B) = 0$,
- $F([E_{p_2, p_5}, B]) = -E^*_{p_1, p_2}(B) = 0$,
- $F([E_{p_3, p_5}, B]) = -E^*_{p_1, p_3}(B) = 0$.

**Group 2:**
- $F([E_{p_1, p_1}, B]) = E^*_{p_1, p_5}(B) + E^*_{p_1, p_6}(B) = 0$,
- $F([E_{p_2, p_2}, B]) = E^*_{p_2, p_3}(B) + E^*_{p_2, p_4}(B) + E^*_{p_2, p_6}(B) = 0$,
- $F([E_{p_6, p_6}, B]) = -E^*_{p_1, p_6}(B) - E^*_{p_2, p_6}(B) = 0$,
- $F([E_{p_1, p_2}, B]) = E^*_{p_2, p_5}(B) + E^*_{p_2, p_6}(B) = 0$,
- $F([E_{p_4, p_6}, B]) = -E^*_{p_1, p_4}(B) - E^*_{p_2, p_4}(B) = 0$.

**Group 3:**
- $F([E_{p_1, p_5}, B]) = E^*_{p_5, p_5}(B) - E^*_{p_1, p_1}(B) = 0$,
- $F([E_{p_1, p_6}, B]) = E^*_{p_6, p_6}(B) - E^*_{p_1, p_1}(B) = 0$,
- $F([E_{p_2, p_3}, B]) = E^*_{p_3, p_3}(B) - E^*_{p_2, p_2}(B) = 0$,
- $F([E_{p_2, p_4}, B]) = E^*_{p_4, p_4}(B) - E^*_{p_2, p_2}(B) + E^*_{p_4, p_6}(B) = 0$,
- $F([E_{p_2, p_6}, B]) = E^*_{p_6, p_6}(B) - E^*_{p_2, p_2}(B) - E^*_{p_1, p_2}(B) = 0$.

The restrictions of Group 1 immediately imply that

$$E^*_{p_1, p_2}(B) = E^*_{p_1, p_3}(B) = E^*_{p_1, p_5}(B) = E^*_{p_3, p_5}(B) = E^*_{p_2, p_3}(B) = E^*_{p_2, p_4}(B) = E^*_{p_4, p_6}(B) = 0.$$  (6)

Combining the restrictions of Group 1 with those of Group 2, we find that

$$E^*_{p_1, p_4}(B) = E^*_{p_1, p_6}(B) = E^*_{p_2, p_5}(B) = E^*_{p_2, p_6}(B) = 0.$$  (7)

Finally, combining the restrictions of Group 1 with those of Group 3, we find that

$$E^*_{p_i, p_j}(B) = E^*_{p_j, p_i}(B), \text{ for all } p_i, p_j \in \mathcal{P}. $$  (8)

Equations (6), (7), and (8) establish that $F$ satisfies (F4) of Definition 1. Now, considering Remark 7 (ii), adding the condition

$$\sum_{p \in \mathcal{P}} E^*_{p, p}(B) = 0$$

to (6), (7), and (8), we find that $\ker_A(B_F) = g_A(\mathcal{P}) \cap \ker(B_F) = \{0\}$. Therefore, $g_A(\mathcal{P})$ is Frobenius with Frobenius functional $F$. 

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Given the form of the Frobenius functional $F$, we have that $F$ satisfies (F4) of Definition 1 as follows:

- $F$ is clearly small,
- $D_F(\mathcal{P}) = \{p_1, p_2\}$ forms an order ideal in $\mathcal{P}$, $U_F(\mathcal{P}) = \{p_3, p_4, p_5, p_6\}$ forms a filter, and $O_F(\mathcal{P}) = \emptyset$,
- $\Gamma_F$ contains the only edges, $(p_1, p_5)$ and $(p_1, p_6)$, between elements of $\text{Ext}(\mathcal{P})$, and
- the fact that $F$ satisfies (F4) was established above.

It remains to show that (P3) of Definition 1 is satisfied; that is, $\mathbb{g}_A(\mathcal{P})$ has a spectrum consisting of an equal number of 0’s and 1’s. To determine the spectrum of $\mathbb{g}_A(\mathcal{P})$, it suffices to calculate $[\widehat{F}, x]$, for $x \in \mathcal{B}_{\mathcal{P}, F}$. Note that $\mathcal{B}_{\mathcal{P}, F}$ can be partitioned into two sets:

$$G_0 = \{E_{p_1, p_1} - E_{p_5, p_5}, E_{p_1, p_1} - E_{p_6, p_6}, E_{p_2, p_2} - E_{p_3, p_3}, E_{p_2, p_2} - E_{p_4, p_4}, E_{p_2, p_2} - E_{p_6, p_6}, E_{p_1, p_2}, E_{p_3, p_5}, E_{p_4, p_6}\},$$

which consists of eigenvectors of $ad(\widehat{F})$ with eigenvalue 0, and

$$G_1 = \{E_{p_1, p_5}, E_{p_1, p_6}, E_{p_1, p_3}, E_{p_1, p_4}, E_{p_2, p_5}, E_{p_2, p_6}, E_{p_2, p_3}, E_{p_2, p_4}\},$$

which are eigenvectors with eigenvalue 1. As $|G_0| = |G_1|$, we conclude that $\mathbb{g}_A(\mathcal{P})$ has a binary spectrum and $(\mathcal{P}, F)$ forms a toral-pair. □

**Lemma 5** Each of the following pairs, consisting of a poset $\mathcal{P}$ and a functional $F_\mathcal{P}$, forms a toral-pair $(\mathcal{P}, F_\mathcal{P})$.

(iii) $\mathcal{P}_{4,n} = \{p_1, \ldots, p_n\}$ with $p_1 < p_2 < \ldots < p_{n-1}$ as well as $p_1 < p_2 < \ldots < p_{\lceil \frac{n}{2} \rceil} < p_n$, and

$$F_{\mathcal{P}_{4,n}} = \sum_{i=1}^{\lceil \frac{n}{2} \rceil} E^*_{p_i, p_{n-i}} + \sum_{i=1}^{\lceil \frac{n}{2} \rceil} E^*_{p_i, p_n}.$$

(iv) $\mathcal{P}_{4,n}^* = \{p_1, \ldots, p_n\}$ with $p_1 < p_2 < \ldots < p_{\lceil \frac{n}{2} \rceil} - 1 < p_{\lceil \frac{n}{2} \rceil} + 1 < \ldots < p_n$ as well as $p_{\lceil \frac{n}{2} \rceil} < p_{\lceil \frac{n}{2} \rceil} + 1 < \ldots < p_n$, and

$$F_{\mathcal{P}_{4,n}^*} = \sum_{i=1}^{\lceil \frac{n}{2} \rceil - 1} E^*_{p_i, p_{n+1-i}} + \sum_{i=\lceil \frac{n}{2} \rceil + 1}^{n} E^*_{p_{\lceil \frac{n}{2} \rceil}, p_i}.$$

**Proof** We prove (iii), as (iv) follows via a symmetric argument. In order to simplify the notations, let $\mathcal{P} = \mathcal{P}_{4,n}$ and $F = F_{\mathcal{P}_{4,n}}$. By definition, it is clear that $|\text{Ext}(\mathcal{P})| = 3$ so that (P1) of Definition 1 is satisfied. To see that (P2) of Definition 1 is satisfied, i.e., $\Sigma(\mathcal{P})$ is contractible, note that $\Sigma(\mathcal{P})$ is formed by adjoining a $\lceil \frac{n}{2} \rceil$-simplex along a face; such a space is star-convex and thus contractible. Now, to
see that $F$ satisfies (F4) of Definition 1 and is Frobenius on $g_A(P)$, take $B \in \ker(B_F)$.
We break the restrictions $B$ must satisfy into three groups:

**Group 1:**
- $F(E_{p_{i-1},p_{n-i}}) = -E_{p_{i-1},p_{n-i}}^*(B) = 0$, for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$,
- $F(E_{p_{i-1},p_{j}}) = E_{p_{j},p_{n-i}}^*(B) = 0$, for $1 \leq i < j \leq \lfloor \frac{n-1}{2} \rfloor$,
- $F(E_{p_{j},p_{i}}) = -E_{p_{j},p_{i}}^*(B) = 0$, for $1 \leq j < i \leq \lfloor \frac{n}{2} \rfloor$,
- $F(E_{p_{n-j},p_{n-i}}) = -E_{p_{j},p_{n-i}}^*(B) = 0$, for $1 \leq i < j \leq \lfloor \frac{n}{2} \rfloor$.

**Group 2:**
- $F(E_{p_{i},p_{n-i}}) = E_{p_{i},p_{n-i}}^*(B) = 0$, for $n$ even,
- $F(E_{p_{i},p_{j}}) = E_{p_{j},p_{n-i}}^*(B) + E_{p_{i},p_{n}} = 0$, for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$,
- $F(E_{p_{j},p_{i}}) = E_{p_{j},p_{n-i}}^*(B) + E_{p_{i},p_{n}} = 0$, for $1 \leq i < j \leq \lfloor \frac{n}{2} \rfloor$.

**Group 3:**
- $F(E_{p_{i-1},p_{n-i}}) = E_{p_{n-i},p_{n-i}}^*(B) - E_{p_{i-1},p_{n-i}}^*(B) = 0$, for $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$,
- $F(E_{p_{n-i},p_{n}}) = E_{p_{n-i},p_{n}}^*(B) - E_{p_{n-i},p_{n}}^*(B) - \sum_{1 \leq j < i} E_{p_{j},p_{i}}^*(B) = 0$,
  for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$.

The restrictions of Group 1 immediately imply that

$$E_{p_{i},p_{n-i}}^*(B) = 0, \text{ for } 1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor,$$

$$E_{p_{j},p_{n-i}}^*(B) = 0, \text{ for } 1 \leq i < j \leq \lfloor \frac{n-1}{2} \rfloor,$$

$$E_{p_{j},p_{i}}^*(B) = 0, \text{ for } 1 \leq j < i \leq \lfloor \frac{n}{2} \rfloor,$$

and

$$E_{p_{i},p_{j}}^*(B) = 0, \text{ for } 1 \leq i < j \leq \lfloor \frac{n}{2} \rfloor.$$

Combining the restrictions of Group 1 with those of Group 2, we can conclude that

$$E_{p_{i},p_{n}}^*(B) = 0, \text{ for } 1 \leq i \leq \lfloor \frac{n}{2} \rfloor,$$

and

$$E_{p_{j},p_{n-i}}^*(B) = 0, \text{ for } 1 \leq i < j \leq \lfloor \frac{n}{2} \rfloor.$$

Finally, combining the restrictions of Group 1 with those of Group 3, we find that

$$E_{p_{i},p_{j}}^*(B) = E_{p_{j},p_{i}}^*(B), \text{ for } p_i, p_j \in P.$$

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Equations (9), (10), (11), (12), (13), (14), and (15) establish that $F$ satisfies (F4) of Definition 1. Now, considering Remark 7 (ii), adding the condition
\[
\sum_{p \in \mathcal{P}} E^*_p, p(B) = 0
\]
to (9), (10), (11), (12), (13), (14), and (15), we find that $\ker_A(B_F) = g_A(\mathcal{P}) \cap \ker(B_F) = \{0\}$. Therefore, $g_A(\mathcal{P})$ is Frobenius with Frobenius functional $F$.

Given the form of the Frobenius functional $F$, we have that $F$ satisfies (F1) through (F4) of Definition 1 as follows:
- $F$ is clearly small,
- $D_F(\mathcal{P}) = \{ p_i \mid 1 \leq i \leq \lfloor \frac{n}{2} \rfloor \}$ forms an order ideal in $\mathcal{P}$, $U_F(\mathcal{P}) = \{ p_i \mid \lfloor \frac{n}{2} \rfloor < i \leq n \}$ forms a filter, and $O_F(\mathcal{P}) = \emptyset$,
- $\Gamma_F$ contains the only edges, $(p_1, p_{n-1})$ and $(p_1, p_n)$, between elements of $\text{Ext}(\mathcal{P})$, and
- the fact that $F$ satisfies (F4) was established above.

It remains to show that (P3) of Definition 1 is satisfied; that is, $g_A(\mathcal{P})$ has a spectrum consisting of an equal number of 0’s and 1’s. To determine the spectrum of $g_A(\mathcal{P})$, it suffices to calculate $[\tilde{F}, x]$, for $x \in \mathcal{B}_{\mathcal{P}, F}$. Note that the elements of $\mathcal{B}_{\mathcal{P}, F}$ can be partitioned as follows:

\[
S_1 = \left\{ E_{p_i, p_j} \mid 1 \leq i < j \leq \left\lfloor \frac{n}{2} \right\rfloor \right\},
\]
\[
S_2 = \left\{ E_{p_i, p_{n-j}} \mid 1 \leq i, j \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } i \neq n - j \right\},
\]
\[
S_3 = \left\{ E_{p_{n-j}, p_{n-i}} \mid 1 \leq i < j \leq \left\lfloor \frac{n-1}{2} \right\rfloor \right\},
\]
\[
S_4 = \left\{ E_{p_i, p_n} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\},
\]
\[
S_5 = \left\{ E_{p_i, p_i} - E_{p_{n-i}, p_{n-i}} \mid 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \right\} \cup \left\{ E_{p_i, p_i} - E_{p_n, p_n} \mid 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \right\},
\]

where

\[
|S_1| = \sum_{i=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \left\lfloor \frac{n}{2} \right\rfloor - i \right) = \begin{cases} \frac{n^2-4n+3}{8}, & n \text{ odd}; \\ \frac{n^2-2n+8}{8}, & n \text{ even}, \end{cases}
\]

\[
|S_2| = \begin{cases} \left\lfloor \frac{n-1}{2} \right\rfloor, & n \text{ odd}; \\ \left\lfloor \frac{n-1}{2} \right\rfloor \left( \left\lfloor \frac{n-1}{2} \right\rfloor - 1 \right), & n \text{ even}, \end{cases}
\]

\[
|S_3| = \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( \left\lfloor \frac{n-1}{2} \right\rfloor - i \right) = \begin{cases} \frac{n^2-4n+3}{8}, & n \text{ odd}; \\ \frac{n^2-2n+8}{8}, & n \text{ even}, \end{cases}
\]

\[
|S_4| = \begin{cases} \frac{n^2-2n+1}{4}, & n \text{ odd}; \\ \frac{n^2-2n+8}{4}, & n \text{ even}, \end{cases}
\]

\[
|S_5| = \sum_{i=1}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( \left\lfloor \frac{n-1}{2} \right\rfloor - i \right) = \begin{cases} \frac{n^2-4n+3}{8}, & n \text{ odd}; \\ \frac{n^2-6n+8}{8}, & n \text{ even}. \end{cases}
\]
\[ |S_4| = \left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n-1}{2}, & n \text{ odd}; \\ \frac{n}{2}, & n \text{ even}, \end{cases} \]

and
\[ |S_5| = n - 1. \]

Furthermore, note that the elements contained in \( G_0 = S_1 \cup S_3 \cup S_5 \) are eigenvalues of \( \text{ad}(\hat{F}) \) with eigenvalue 0, while the elements contained in \( G_1 = S_2 \cup S_4 \) are eigenvectors with eigenvalue 1. Thus, as \( \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor = \frac{n-1}{2} \), when \( n \) is odd, and \( \lfloor \frac{n}{2} \rfloor = \frac{n-2}{2}, \) when \( n \) is even,
\[ |G_0| = |G_1| = \begin{cases} \frac{n^2-1}{4}, & n \text{ odd}; \\ \frac{n^2}{4}, & n \text{ even}. \end{cases} \]

Therefore, \( g_A(\mathcal{P}) \) has a binary spectrum and \((\mathcal{P}, F)\) is a toral-pair.

Lemma 6 Each of the following pairs, consisting of a poset \( \mathcal{P} \) and a functional \( F_\mathcal{P} \), forms a toral-pair \((\mathcal{P}, F)\).

(v) \( \mathcal{P}_{5,n} = \{p_1, \ldots, p_{2n+1}\} \) with \( p_i < p_j \) for \( 1 \leq i < 2n \) odd and \( i + 1 \leq j \leq 2n + 1 \) as well as \( p_i < p_j \) for \( 1 < i < 2n \) even and \( i + 2 \leq j \leq 2n + 1 \), and
\[ F_{\mathcal{P}_{5,n}} = E^*_{p_1, p_{2n+1}} + \sum_{i=1}^{2\lfloor \frac{n}{2} \rfloor + 1} E^*_{p_i, p_{2n}} + \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} E^*_{p_{2k}, p_{2n-2k}} + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} E^*_{p_{2k+1}, p_{2n-2k+1}}. \]

(vi) \( \mathcal{P}_{5,n}^* = \{p_1, \ldots, p_{2n+1}\} \) with \( p_i < p_j \) for \( 1 \leq i < 2n \) odd and \( i + 2 \leq j \leq 2n + 1 \), as well as \( p_i < p_j \) for \( 1 < i < 2n \) even and \( i + 1 \leq j \leq 2n + 1 \), and
\[ F_{\mathcal{P}_{5,n}^*} = E^*_{p_1, p_{2n+1}} + \sum_{i=2\lfloor \frac{n+1}{4} \rfloor + 1}^{2n+1} E^*_{p_{2i}, p_{2i-1}} + \sum_{k=2}^{\lfloor \frac{n+1}{2} \rfloor} E^*_{p_{2k-2}, p_{2n-2k+4}} + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} E^*_{p_{2k+1}, p_{2n-2k+1}}. \]

Proof We prove (v), as (vi) follows via a symmetric argument. In order to simplify the notations, let \( \mathcal{P} = \mathcal{P}_{5,n} \) and \( F = F_{\mathcal{P}_{5,n}} \). By definition, it is clear that \( |\text{Ext}(\mathcal{P})| = 3 \) so that (P1) of Definition 1 is satisfied. To see that (P2) of Definition 1 is satisfied, i.e., \( \Sigma(\mathcal{P}) \) is contractible, note that \( \Sigma(\mathcal{P}) \) is formed by taking \( n \) suspensions of a point; since the suspension of a contractible simplicial complex is contractible, the result follows. Now, to see that \( F \) satisfies (F4) of Definition 1 and is Frobenius on \( g_A(\mathcal{P}) \), take \( B \in \ker(B_F) \). We break the restrictions \( B \) must satisfy into six groups:

Group 1:
- \( F([E_{p_i, p_{2n+1}} B]) = -E^*_{p_i, p_i}(B) = 0 \), for \( 1 < i < 2n \) and \( i = 2n + 1 \),
- \( F([E_{p_1, p_1} B]) = E^*_{p_1, p_{2n}}(B) + E^*_{p_1, p_{2n+1}}(B) = 0. \)
Group 2:
- \( F([E_{p_{2n-2k}}, p_{2n-2k}, B]) = -E_{p_{2k}, p_{2n-2k}}^*(B) = 0 \), for \( 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2n-2k+1}}, p_{2n-2k+1}, B]) = -E_{p_{2k+1}, p_{2n-2k+1}}^*(B) = 0 \), for \( 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

Group 3:
- \( F([E_{p_{2k}}, p_{2k}, B]) = E_{p_{2k}, p_{2n-2k}}^*(B) + E_{p_{2k}, p_{2n}}^*(B) = 0 \), for \( 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2k+1}}, p_{2n-2k+1}, B]) = E_{p_{2k+1}, p_{2n-2k+1}}^*(B) + E_{p_{2k+1}, p_{2n}}^*(B) = 0 \), for \( 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

Group 4:
- \( F([E_{p_{2n-2k-2k_1}}, p_{2n}(B) - E_{p_{2k_1}, p_{2k}}^*(B) = 0 \), for \( 1 \leq k < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2k-2k_1+1}}, p_{2n}(B) - E_{p_{2k_1+1}, p_{2k}}^*(B) = 0 \), for \( 1 \leq k < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2k-2k_1}}, p_{2n}(B) - E_{p_{2k_1}, p_{2k_1+1}}^*(B) = 0 \), for \( 1 \leq k < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2n-2k-2k_1+1}}, p_{2n}(B) - E_{p_{2k_1+1}, p_{2k_1+1}}^*(B) = 0 \), for \( 1 \leq k < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

Group 5:
- \( F([E_{p_{2n-2k}}, p_{2n-2k_1}, B]) = -E_{p_{2k_1}, p_{2n-2k}}^*(B) = 0 \), for \( 1 \leq k_1 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2n-2k}}, p_{2n-2k_1+1}, B]) = -E_{p_{2k_1+1}, p_{2n-2k}}^*(B) = 0 \), for \( 1 \leq k_1 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2n-2k+1}}, p_{2n-2k_1}, B]) = -E_{p_{2k_1}, p_{2n-2k_1+1}}^*(B) = 0 \), for \( 1 \leq k_1 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2n-2k+1}}, p_{2n-2k_1+1}, B]) = -E_{p_{2k_1+1}, p_{2n-2k_1+1}}^*(B) = 0 \), for \( 1 \leq k_1 < k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).

Group 6:
- \( F([E_{p_1}, p_i, B]) = E_{p_1, p_i}^*(B) - E_{p_1, p_1}^*(B) = 0 \), for \( i = 2n, 2n+1 \),
- \( F([E_{p_{2k}}, p_{2n-2k}, B]) = E_{p_{2n-2k}, p_{2n-2k}}^*(B) - E_{p_{2k}, p_{2n}}^*(B) + E_{p_{2n-2k}, p_{2n}}^*(B) = 0 \), for \( 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \),
- \( F([E_{p_{2k+1}}, p_{2n-2k+1}, B]) = E_{p_{2n-2k+1}, p_{2n-2k+1}}^*(B) - E_{p_{2k+1}, p_{2k+1}}^*(B) + E_{p_{2n-2k+1}, p_{2n}}^*(B) = 0 \), for \( 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \).
• \( F([E_{p_1}, p_{2n}, B]) = E^*_{p_{2n}, p_{2n}}(B) - E^*_{p_1, p_1}(B) - \sum_{1 \leq j < i} E^*_{p_j, p_i}(B) = 0, \)
  for \( 1 \leq i \leq 2 \lfloor \frac{n}{2} \rfloor + 1. \)

The restrictions of Group 1 immediately imply that

\[ E^*_{p_1, p_i}(B) = 0, \text{ for } 1 < i \leq 2n + 1. \] (16)

Next, the restrictions of Group 2 immediately imply that

\[ E^*_{p_{2k}, p_{2n-2k}}(B) = E^*_{p_{2k+1}, p_{2n-2k+1}}(B) = 0, \text{ for } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \] (17)

Combining (16) and (17) with the restrictions of Group 3, we may conclude that

\[ E^*_{p_1, p_{2n}}(B) = E^*_{p_1, p_{2n+1}}(B) = 0, \text{ for } 1 \leq i < 2n. \] (18)

Equations (16) and (18) together with the restrictions of Group 4 allow us to conclude that

\[ E^*_{p_i, p_{2k}}(B) = E^*_{p_i, p_{2k+1}}(B) = 0, \text{ for } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } 1 \leq i < 2k. \] (19)

Note that the first six restrictions of Group 5 together with (17) and (19) imply that

\[ E^*_{p_i, p_{2n-2k}}(B) = 0, \text{ for } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } 1 \leq i = 2k + 1 < 2n - 2k, \] (20)

and

\[ E^*_{p_i, p_{2n-2k+1}}(B) = 0, \text{ for } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } 1 \leq i = 2k < 2n - 2k. \] (21)

Combining (20) and (21) with the last restriction of Group 5, we find that

\[ E^*_{p_{2k+1}, p_{2n-2k}}(B) = E^*_{p_{2k}, p_{2n-2k+1}}(B) = 0, \text{ for } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \] (22)

Thus,

\[ E^*_{p_i, p_{2n-2k}}(B) = E^*_{p_i, p_{2n-2k+1}}(B) = 0, \text{ for } 1 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ and } 1 \leq i < 2n - 2k. \] (23)

Finally, combining (16), (17), (18), (19), (20), (21), (22), and (23) with the restrictions of Group 6 implies that

\[ E^*_{p_i, p_j}(B) = E^*_{p_j, p_j}(B), \text{ for } 1 \leq i, j \leq 2n + 1. \] (24)
Equations (16), (17), (18), (19), (20), (21), (22), (23), and (24) establish that $F$ satisfies (F4) of Definition 1. Now, considering Remark 7 (ii), adding the condition

$$\sum_{P \in \mathcal{P}} E_{p,p}(B) = 0$$

to (16), (17), (18), (19), (20), (21), (22), (23), and (24), we find that $\ker_A(B_F) = g_A(\mathcal{P}) \cap \ker(B_F) = \{0\}$. Therefore, $g_A(\mathcal{P})$ is Frobenius with Frobenius functional $F$.

Given the form of the Frobenius functional $F$, we have that $F$ satisfies (F1) through (F4) of Definition 1 as follows:

- $F$ is clearly small,
- $D_F(\mathcal{P}) = \{p_i \mid 1 \leq i \leq 2\left[\frac{n+1}{2}\right] + 1\}$ forms an order ideal in $\mathcal{P}$, $U_F(\mathcal{P}) = \{p_i \mid 2\left[\frac{n-1}{2}\right] + 1 < i \leq 2n + 1\}$ forms a filter, and $O_F(\mathcal{P}) = \emptyset$,
- $\Gamma_F$ contains the only edges, $(p_1, p_{2n})$ and $(p_1, p_{2n+1})$, between elements of $\text{Ext}(\mathcal{P})$, and
- the fact that $F$ satisfies (F4) was established above.

It remains to show that (P3) of Definition 1 is satisfied; that is, $g_A(\mathcal{P})$ has a spectrum consisting of an equal number of 0’s and 1’s. To determine the spectrum of $g_A(\mathcal{P})$, it suffices to calculate $[\hat{F}, x]$, for $x \in \mathcal{B}_{\mathcal{P}, F}$. Note that the elements of $\mathcal{B}_{\mathcal{P}, F}$ can be partitioned as follows:

$$S_1^1 = \{E_{p_1, p_1} - E_{p_{2n+1}, p_{2n+1}}\},$$

$$S_1^2 = \left\{E_{p_1, p_i} - E_{p_{2n}, p_{2n}} \mid 1 \leq i \leq 2\left[\frac{n}{2}\right] + 1\right\},$$

$$S_1^3 = \left\{E_{p_{2k}, p_{2k}} - E_{p_{2n - 2k}, p_{2n - 2k}}, E_{p_{2k+1}, p_{2k+1}} - E_{p_{2n - 2k+1}, p_{2n - 2k+1}} \mid 1 \leq k \leq \left\lfloor\frac{n-1}{2}\right\rfloor\right\},$$

$$S_1 = S_1^1 \cup S_1^2 \cup S_1^3,$$

$$S_2 = \left\{E_{p_i, p_j} \mid 1 \leq i \leq 2\left[\frac{n-1}{2}\right] + 1, 2\left[\frac{n-1}{2}\right] + 1 < j \leq 2n + 1\right\},$$

$$S_3 = \{E_{p_i, p_j} \mid 1 \leq i < j \leq 2\left[\frac{n-1}{2}\right] + 1, p_i < p_j\},$$

$$S_4 = \{E_{p_i, p_j} \mid 2\left[\frac{n-1}{2}\right] + 1 < i < j \leq 2n + 1, p_i < p_j\},$$

where

$$|S_1| = 2n,$$

$$|S_2| = \begin{cases} n^2 + n, & n \text{ odd;} \\ n^2 + n, & n \text{ even;} \end{cases}$$

$$|S_3| = \begin{cases} \frac{n^2 - 2n + 1}{2}, & n \text{ odd;} \\ \frac{n^2}{2}, & n \text{ even;} \end{cases}$$
and

\[|S_4| = \begin{cases} 
\frac{n^2 - 1}{2}, & n \text{ odd;} \\
\frac{n^2 - 2n}{2}, & n \text{ even.}
\end{cases}\]

Furthermore, note that the elements contained in \(G_0 = S_1 \cup S_3 \cup S_4\) are eigenvectors of \(\text{ad}(\hat{F})\) with eigenvalue 0, while the elements of \(G_1 = S_2\) are eigenvectors with eigenvalue 1. Therefore,

\[|G_0| = |G_1| = \begin{cases} 
n^2 + n, & n \text{ odd;} \\
n^2 + n, & n \text{ even.}
\end{cases}\]

Consequently, we conclude that \(g_A(\mathcal{P})\) has a binary spectrum and \((\mathcal{P}, F)\) is a toral-pair. □

References

1. Belavin, A., Drindel’d, V.: Solutions of the classical Yang-Baxter equations for simple Lie algebras. Funct. Anal. Appl. 16, 159–180 (1982)
2. Cameron, A., Coll, V., Hyatt, M., Magnant, C.: The unbroken spectra of Frobenius seaweeds. J Algebraic Comb. (to appear)
3. Coll, V., Gerstenhaber, M.: Cohomology of Lie semidirect products and poset algebras. J. Lie Theory 26, 79–95 (2016)
4. Coll, V., Hyatt, M., Magnant. C.: The unbroken spectrum of type-A Frobenius seaweeds. J. Algebraic Combin., 1-17, (2016)
5. Coll, V., Mayers, N.: The index of Lie poset algebras. J. Combin. Theory Ser. A 177, 105331 (2021). https://doi.org/10.1016/j.jcta.2020.105331
6. Dergachev, V., Kirillov, A.: Index of Lie algebras of seaweed type. J. Lie Theory 10, 331–343 (2000)
7. Gerstenhaber, M., Giaquinto, A.: Boundary solutions of the classical Yang-Baxter equation. Lett. Math. Phys. 40, 337–353 (1997)
8. Gerstenhaber, M., Giaquinto, A.: Graphs, Frobenius functionals, and the classical Yang-Baxter equation. arXiv:0808.2423v1, August 18, (2008)
9. Gerstenhaber, M., Giaquinto, A.: The principal element of a Frobenius Lie algebra. Lett. Math. Phys. 88, 333–341 (2009)
10. Joseph, A.: On semi-invariants and index for biparabolic (seaweed) algebras. I. J. Algebra 305(1), 487–515 (2006)
11. Ooms, A.I.: On Frobenius Lie algebras. Comm. Algebra 8, 13–52 (1980)
12. Panyushev, D.: Inductive formulas for the index of seaweed Lie algebras. Mosc. Math. J. 1(2), 221–241 (2001)
13. Rota, G.: On the foundations of combinatorial theory, I: Theory of Möbius functions. Zeitschrift für Wahrscheinlichkeitsrechnung und verwandte Gebiete 2(4), 340–368 (1964)

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