Gaugings and other aspects in supergravity

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ABSTRACT

We discuss various topics in supergravity: gaugings, double field theory and $\mathcal{N}=2 \, D=4$ BPS multicenter black holes.

We introduce the main features of supergravity, focusing on the aspects of gauged supergravities. We study the embedding-tensor formalism as a tool that facilitates the construction of gauged supergravities due to its covariant formulation as well as its relevance in the description of the magnetic higher-rank field potentials.

In particular, we present a full study of the general gaugings of maximal $d=9$ as an example in which this formalism is applied. We obtain all the possible gaugings of the theory and its extended field content.

We also classify the orbits of gaugings of maximal and half-maximal $d=9,8,7$ supergravity and study their (non-)geometric origins by means of double field theory. By performing a generalized Scherk-Schwarz dimensional reduction of this T-duality-invariant formalism, we reproduce the orbits found by means of the embedding-tensor formalism.

Finally, we study a formalism to describe BPS multicenter solutions for $\mathcal{N}=2, \, D=4$ theories with quadratic prepotentials. Based on the charge vector space, this approach allows for the treatment of these solutions in a more general way.

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Gaugings and other aspects in supergravity

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Introduction

In this thesis we treat three well defined aspects of supergravity (SUGRA), being the possible gaugings of supergravity theories the central subject of study. Other relevant topics in the framework of supergravity, as specific topics of Double Field Theory formalism or multicenter black hole solutions in different scenarios, are also addressed throughout this thesis.

In order to contextualize the significance of these topics, we present a broad introduction with the main purpose of explaining the role that supergravity plays from the theoretical viewpoint of the current research in Physics. More specific and in-depth introductions of each of the topics treated in this work are presented at the beginning of the corresponding chapters.

Born in the late 1970s [1–3], supergravity is a quantum field theory that accounts gravity as a dynamical field. Namely, it is a theory that describes the gravitational interaction by means of a massless spin-2 particle. In addition, it is able to host additional fundamental interactions as internal symmetries.

However, the historical development and the reasons why supergravity is studied nowadays are completely different. In its origin, this theory was studied as a UV completion of the General Relativity theory that could host the internal symmetries of all the known interactions in the Universe. It was supposed to be divergenceless at high energies and that motivated its interest during the late 1970s and the 1980s decade. For $N = 1$ $d = 4$, it was proven that its divergences could not be avoided [4]. However, a recent result [5] strongly points towards the ultraviolet finiteness of $N = 8$ $D = 4$ supergravity.

Supergravity as an effective theory

The appearance of string theory supposed an important leap in the resurrection of supergravity. Supergravity strongly reappeared when it was discovered that the behavior of the superstring theories at a certain regime were equivalent to some specific supergravities (which, moreover, preserved the same amount of supersymmetries). In particular, supergravity describes the massless sector of the superstring theories. Let us see this explicitly.

The most general non-linear $\sigma$-model that describes a string coupled to different
non-trivial backgrounds is
\[
S = -\frac{T}{2} \int d^2\sigma \sqrt{|\gamma| \left[ (\gamma^{ij} g_{\mu\nu}(X) + e^{ij} B_{\mu\nu}(X)) \partial_{\mu} X^i \partial_{\nu} X^j - \alpha' \phi(X) R(\gamma) \right]}. \tag{1}
\]

Here, \(X^\mu\) are the spacetime coordinates of the string. The background fields are the spacetime metric \(g_{\mu\nu}\), a 2-form gauge potential \(B_{\mu\nu}\) and a scalar field \(\phi\). The worldsheet is parametrized by coordinates \(\sigma^i = (\sigma, \tau)\), \(\gamma_{ij}\) is the induced metric on the worldsheet and \(R(\gamma)\) is the scalar curvature of the worldsheet.

The action (1) is not conformally invariant. This is a dilemma, since scale invariance is a necessary requirement for the consistency of the theory when this \(\sigma\)-model is quantized. Thus, we can wonder what constraints have to be imposed on the fields such that Weyl invariance remains unbroken. Namely, we want to know what field configurations are the ones that guarantee this scale invariance. For this purpose, inspired by the problem of the dimensionful running coupling constants in other theories, a renormalization procedure can be performed\(^2\) in which the \(\beta\) functions calculated for each one of the fields have to vanish \(^6\),
\[
\beta_{\mu\nu}(g) = \alpha' \left[ R_{\mu\nu} - 2 \nabla_\mu \nabla_\nu \phi + \frac{1}{4} H_\mu^{\alpha\beta} H_{\nu\alpha\beta} \right] + O(\alpha'^2), \tag{2}
\]
\[
\beta_{\mu\nu}(B) = \alpha' e^{2\phi} \nabla^\rho (e^{-2\phi} H_{\mu\nu\rho}) + O(\alpha'^2), \tag{3}
\]
\[
\beta(\phi) = \frac{d - 26}{6} - \frac{\alpha'}{2} \left[ \nabla^2 \phi - (\partial \phi)^2 - \frac{1}{4} R - \frac{1}{48} H^2 \right] + O(\alpha'^2). \tag{4}
\]

For the critical dimension \(d = 26\), \(\beta(\phi)\) vanishes. Recasting properly these equations on the fields, we can construct an action that contains the fields \(\{g_{\mu\nu}, B_{\mu\nu}, \phi\}\) in such a way that the equations of motion arising from it are equivalent to these constraints. Namely, the minimization of the action
\[
S = \frac{g^2}{16\pi G_N^{(d)}} \int d^4x \sqrt{|g|} e^{-2\phi} \left[ R - 4 (\partial \phi)^2 + \frac{1}{2 \cdot 3!} H_{\mu\rho\sigma} H^{\mu\rho\sigma} - \frac{4}{3\alpha'} (d - 26) \right], \tag{5}
\]
with respect to the fields \(\{g_{\mu\nu}, B_{\mu\nu}, \phi\}\) is equivalent to the vanishing of the three \(\beta\) functions (2)-(4), \(\beta_{\mu\nu}(g) = \beta_{\mu\nu}(B) = \beta(\phi) = 0\). In particular, we obtain
\[
\frac{16\pi G_N^{(d)} e^{2(\phi - \phi_0)}}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} \sim \frac{1}{\alpha'} [\beta_{\mu\nu}(g) - 4 g_{\mu\nu} \beta(\phi)] + O(\alpha'^2), \tag{6}
\]
\[
\frac{16\pi G_N^{(d)} e^{2(\phi - \phi_0)}}{\sqrt{|g|}} \frac{\delta S}{\delta \phi} \sim - \frac{16}{\alpha'} \beta(\phi) + O(\alpha'^2), \tag{7}
\]
\[
\frac{16\pi G_N^{(d)} e^{2(\phi - \phi_0)}}{\sqrt{|g|}} \frac{\delta S}{\delta B^{\mu\nu}} \sim - \frac{1}{\alpha'} \beta_{\mu\nu}(B) + O(\alpha'^2). \tag{8}
\]
\(^2\)In this case, the fields play the role of running coupling constants.
By means of a conformal scaling on the metric, we define the so-called Einstein-frame metric $g_{E\mu\nu}$ as

$$ g_{\mu\nu} = e^{4\phi} g_{E\mu\nu}. \tag{9} $$

This new metric allows us to get rid of the scalar field factor $e^{-2\phi}$. Hence, in this Einstein frame, the action is rewritten as

$$ S = \frac{1}{16\pi G_N^{(d)}} \int d^d x \sqrt{|g_E|} \left[ R_E + \frac{4}{d-2} (\partial\phi)^2 + \frac{1}{2 \cdot 3!} e^{-\frac{4}{d-2} \phi} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{2(d-26)}{3\alpha'} e^{\frac{4}{d-2} \phi} \right]. \tag{10} $$

Then, we obtain that the low-energy limit effective action of the string common sector \[5\], coincides with the action of the NSNS sector of supergravities.

The possibility of projecting the low-energy behavior of string theory on a quantum field theory provides a useful scenario to investigate and understand different issues of string theory. For instance, the search of non-perturbative solutions or vacua of string theories is one of the main aspects that are exploited in the context of supergravity and that can be extrapolated to the string sector.

In the cases of interest for us, the low-energy effective action of the string theory is a supergravity theory. For example, the effective actions of type IIA and IIB string theories coincide with the $N = 2A$ and $N = 2B$ supergravity theories. In addition, type I and the two versions of the heterotic string (the ones with $SO(32)$ and $E_8 \times E_8$ gauge symmetries) coincide with different versions of $N = 1 D = 10$ supergravity.

Another intriguing aspect that supergravity possesses is its connection with the so-called $M$ theory. $N = 2A/2B$ SUGRAs have a UV completion, the type IIA/IIB string theories, respectively. These full theories not only include the massless modes of SUGRA, but also extended objects as strings (or branes).

This UV completion is a basis to conjecture the existence of a theory which, analogous to the string theories, entails $D = 11$ SUGRA as its low-energy effective theory. This is the hypothetical M theory. In addition, a relationship between M theory and string theory (and hence, SUGRA) strongly supports its existence. It is proven that the action of $N = 2A$ SUGRA theory is obtained by performing a dimensional reduction of $D = 11$ SUGRA on a circle \[7\],\[9\]. We can compare the factors of $N = 2A$ SUGRA and $D = 11$ SUGRA compactified on a circle. Hence, due to their different origins, we see obtain the following relation:

$$ R_{11} = \ell_s g_A, \tag{11} $$

where $R_{11}$ is the radius of compactification of the 11th dimension, $\ell_s$ is the characteristic string length and $g_A$ is the coupling constant of type IIA string theory. We see that at small radius, that is, when taking $N = 2A$ SUGRA description, the coupling is weak. However, at the strong-coupling regime, the radius grows and a new dimension becomes macroscopic. Therefore, since the UV completion of $N = 2A$ SUGRA is type
IIA theory, this suggests that the UV completion of $D = 11$ SUGRA corresponds to
the strong coupling limit of type IIA theory, the conjectured M theory.

In addition, since $D = 11$ SUGRA has no scalar fields nor dimensionful coupling
constants, M theory must also exhibit these features. This means that, unlike string
theory, M theory does not have a perturbative expansion and therefore, its treatment
is more difficult.

Once we have justified the importance and the rôle of supergravity from a stringy
viewpoint, we are going to dissect the main features of supergravity theories.

Supergravity is a quantum field theory in which local supersymmetry and General
Relativity coexist. The transformation parameter of supersymmetry is a spacetime
dependent spinor $\epsilon(x)$. The local character of this symmetry necessarily requires the
introduction of a corresponding gauge field, which in this case must be a spinor. Then,
it is the gravitino, $\psi_\mu(x)$, a spin-$\tfrac{3}{2}$ particle, the fermionic field that carries out this
action. However, this is not all what we need. The supergravity algebra, the so-called
superPoincaré algebra, implies the following anticommutating relation,

$$\{Q^i_\alpha, Q^j_\beta\} = (\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij}. \tag{12}$$

Here, $Q^i_\alpha$ are the supercharges, where $i$ is an index that denotes the number of su-
percharges and $\alpha$ is a spinorial index, $C$ is a charge conjugation matrix and $P_\mu$
are the generators of the translations. We expect that gauging of supersymmetry leads to
gauging of translations. Then, since local translations are part of the general coordinate
transformations, we also expect that the gravitational field $g_{\mu\nu}(x)$ (or alternatively, the
vielbein $e_{\mu}^{\ a}$) behaves as a gauge field.

Hence, we see how superPoincaré algebra is the tailor that sews local supersymmetry
and general coordinate transformations together so that supergravity results properly
outfitted.

From the phenomenological viewpoint, one wonders whether it is worth considering
such a scenario like supergravity when one wants to obtain results that could be tested
by particle experiments in laboratories. Despite of the relative recent result on the
renormalizability of $N = 8 \ D = 4$ supergravity and the absence of results for $N < 8$,
the answer to this question is positive provided that supergravity is considered an
effective phenomenological theory arising from a UV completing theory.

This scenario is similar to the one of the old Fermi theory [11], in which the weak
interaction is described by means of a dimensionful coupling constant $[G_F] = -2$. It is
know that the 4-fermion interaction is valid for carrying out a description of the weak
interaction at the scale energy $E \approx M_W$, where $M_W$ is the mass of the $W^-$
boson, one of the three force carriers of the weak force. However, for $E \gg M_W$, this theory breaks
down and the genuine Glashow-Weinberg-Salam theory [12] [11] is required for a suitable
description of Nature. That is, although the Fermi theory is non-renormalizable, its
results are correct at a certain regime.

\footnote{For a very pedagogical explanation, we recommend [10].}
In a similar fashion, for \( E \approx M_P \), being \( M_P \) the Planck mass, one must use the UV completion of supergravity: the superstring or M theories. However, for the regime \( E \ll M_P \) is a good approximation to work with supergravity. Consequently, from these arguments we conclude that the study of supergravity becomes crucial. It is the link between the possible final theory of elementary particles, strings or any other extended object and the low-energy effective theory which has to reproduce, at least, the \( SU(3) \times SU(2) \times U(1) \) Standard Model that describes the electromagnetic, weak and strong interactions that exist in our Universe [15].

**Dimensional reduction and hidden symmetries**

Aside from the existing parallelism between string/M theories and supergravity, we can formulate SUGRA theories for dimensions \( d \leq 11 \). Starting out from higher-dimensional supergravities, one obtains new supergravity theories in lower dimensions by means of a dimensional reduction mechanism\(^4\). The symmetry structure of the resulting theory depends very much on the geometric properties of the internal manifold on which we compactify.

**Maximal supergravities** (namely, the supergravity theories that host the maximum number of supercharges) in several dimensions are related by dimensional reduction. When compactifying a \( D \)-dimensional supergravity theory on a \( T^n \) \( n \)-torus, we obtain a \( d \)-dimensional supergravity, with \( d = D - n \). In order to consistently construct the lower-dimensional theory, one has to decompose the higher-dimensional fields into fields that transform covariantly under gauge symmetries and diffeomorphisms of the lower-dimensional theory. This rearrangement of the degrees of freedom is necessary for building and classifying the supermultiplets in a covariant way.

Let us briefly discuss the dimensional reduction of a \( D \)-dimensional toy-model based on gravity coupled to an antisymmetric 2-form gauge field on a \( T^n \) \( n \)-torus. The model is given by

\[
\mathcal{L} \propto e^{\left( \frac{1}{2} \hat{R} + \frac{1}{4} \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} \hat{H}^{\hat{\mu}\hat{\nu}\hat{\rho}} \right)},
\]

where \( \hat{e} \) is the determinant of the vielbein, \( \hat{R}(\hat{g}) \) is the scalar curvature \( \hat{H}_{\hat{\mu}\hat{\nu}\hat{\rho}} = 3 \partial_{[\hat{\mu}} \hat{B}_{\hat{\nu}\hat{\rho}]} \) and \( \hat{\mu}, \hat{\nu}, \hat{\rho} = 1, \ldots, D \). The dimensional reduction scheme consists of the following redefinitions: the \( \frac{(d+n)(d+n+1)}{2} \) degrees of freedom of \( \hat{g}_{\hat{\mu}\hat{\nu}} \) are decomposed into

\[
\frac{D(D+1)}{2} \quad \frac{d(d+1)}{2} \quad d \times n \quad \frac{n(n+1)}{2}
\]

\[
\hat{g}_{\hat{\mu}\hat{\nu}} \rightarrow g_{\mu\nu}, \quad A^m_{\mu}, \quad g_{mn}, \quad \left( \frac{n(n+1)}{2} \right)
\]

and similarly, the \( \frac{(d+n)(d+n-1)}{2} \) degrees of freedom of \( \hat{B}_{\hat{\mu}\hat{\nu}} \) are rearranged into

\[
\frac{D(D-1)}{2} \quad \frac{d(d-1)}{2} \quad d \times n \quad \frac{n(n-1)}{2}
\]

\[
\hat{B}_{\hat{\mu}\hat{\nu}} \rightarrow B_{\mu\nu}, \quad B^m_{\mu}, \quad B_{mn}, \quad \left( \frac{n(n-1)}{2} \right)
\]

\(^4\)Very complete lectures on this topic are [16] [17].
Here, $g_{\mu\nu}$ and $B_{\mu\nu}$ are the $d$-dimensional metric and gauge potential, respectively. $A^m_\mu$ and $B^m_\mu$ are $d \times n$ vector fields whereas $g_{mn}$ and $b_{mn}$ are two symmetric and antisymmetric $n \times n$ scalar matrices, respectively.

The diffeomorphisms acting on the torus coordinates \( \{x^m\}_{m=1,n} \),

$$x^m \rightarrow U^m_n x^n,$$

act on the scalar matrices as follows:

$$g \rightarrow U^T g U, \quad B \rightarrow U^T B U.$$  \hspace{1cm} (17)

The $U$ matrices generate the $GL(n)$ group, which contains the rotation group $SO(n)$ as a subgroup. In addition, special gauge transformations whose transformation parameter is proportional to $\Lambda_{mn}x^n$ induce a shift on $B_{mn}$,

$$B_{mn} \rightarrow B_{mn} + \Lambda_{[mn]}.$$  \hspace{1cm} (18)

Hence, we are able to identify $n^2 + \frac{1}{2}n(n-1) = \frac{1}{2}n(3n-1)$ transformations. However, it turns out that there exist additional transformations which do not have a higher-dimensional origin \[18\]. These $\frac{1}{2}n(n-1)$ extra transformations, combined with the previous ones, imply an enhancement on the global symmetry of the lower-dimensional theory, so that the action becomes $SO(n,n)$ invariant \[19\]. In addition, the coset space parametrized by the scalar fields also results improved to $SO(n,n)/SO(n) \times SO(n)$.

These so-called hidden symmetries occur in many other scenarios and some attempts have been done to try to justify them. The most successful ones are methods that have to do with decompositions of Kac-Moody algebras. These approaches are the $E_{10}$ and $E_{11}$ formalisms. By means of the decomposition of $E_{10}$ or $E_{11}$ (at the level of the algebras) into the global symmetry group $G_D$ of the $D$-dimensional SUGRA times a residual factor $A_{D-1}$,

$$E_{11} = G_D \times A_{D-1},$$

they provide the full field content of a given supergravity theory.

In particular, the case of the $SO(n,n)$ symmetry is understood as a realization of the T-duality symmetry that takes place at the level of string theory, which interchanges string momenta and winding modes.

### Gauged supergravities

As it was seen, we can formulate diverse supergravity theories for dimensions $d \leq 11$. By means of a dimensional reduction mechanism, we can construct new supergravities with different features and these features depend on the geometry of the compactified manifold.

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\[5\] $A_{D-1}$ represents the diffeomorphisms of the $D$-dimensional spacetime. Further work on this aspect can be found in \[20-22\].
For instance, if we compactify a $D$-dimensional supergravity (namely, 11- or 10-dimensional theories) over a $T^n$ torus, we obtain a $d$-dimensional supergravity with a larger global symmetry group and an Abelian gauge symmetry. The Abelian character of the local symmetry has to do with the commutativity of the generators of the group manifold $T^n$. In this way, we obtain the so-called *ungauged theories* for every dimension. The local symmetry of the ungauged supergravities corresponds to the $U(1)^n$ Abelian gauge symmetry of the vector fields

$$\delta_{\Lambda} A_{\mu}^M = d\Lambda^{(0)M}, \quad (20)$$

where $\Lambda^{(0)M}$ is a 0-form gauge parameter. The number of generators of the gauge group corresponds to the number of vector fields. A *gauging* or *gauged deformation* turns this Abelian local symmetry into a non-Abelian local one.

If we consider more geometrically complicated compactification manifolds, the new lower-dimensional theories that emerge will enjoy a non-Abelian gauge symmetry. These are the so-called *gauged supergravities*, which entail the main part of this manuscript. Gauged supergravities are the only supersymmetric deformations of maximal supergravity that preserve supersymmetry.\(^6\) Whatever the dimensional reduction scheme it is, the gauge parameters of the theory must depend on the compactification parameters. Namely, if the gauging arises from a compactification with non-trivial fluxes (i.e., background values for the higher-dimensional gauge fields), a certain brane configuration or any kind of torsion of the compactification manifold, the gauge parameters must exhibit a dependence on the variables that govern these phenomena. This scenario is schematically illustrated in Figure 1.

Up to now, the general statement is that given a certain compactification scheme, a certain gauged supergravity arises. However, despite of flux compactification is a confident and straightforward mechanism to generate gauged supergravities, deformations can also be done without following this path. Indeed, the first deformed theories\(^{25–29}\) were constructed by adding the ingredients that the theory required step by step and assembling them properly.

From the decade of the 1980s to late 2000s, a wide variety of gaugings for different supergravities were found. However, since their search was mainly inspired in group theoretical arguments, the quest became harder as long as the global symmetry group grew. Taking into account that the only restriction is that the gauged symmetry group has to be a subgroup of the global symmetry group, the set of all the possible gaugings (including non-semisimple algebras) still result too broad to perform a complete analysis. Nevertheless, there appeared a new tool that, applied on a certain ungauged theory, systematically scanned all the possible gaugings of that supergravity. This is the so-called *embedding tensor mechanism*\(^{30–34}\), which basically promotes a certain subgroup $G$ of the global symmetry $G_0$ to be gauged in a covariant way. If we denote by $t_\alpha$ the generators of $g = \text{Lie } G$ and let $M = 1, \ldots, n_V$ label the $n_V$ vector fields of

\(^6\)Up to now, the only known exceptions are the massive IIA Romans’ supergravity\(^{23}\) and a massive deformation for the $N = 4$ $D = 6$ supergravity\(^{24}\).
the ungauged theory, then the embedding tensor, \( \vartheta \), describes the embedding of \( G \) into \( G_0 \) by means of the gauge generators

\[
X_M = \vartheta_M^\alpha t_\alpha .
\]  

(21)

Then, the deformation parameters are identified with the non-zero components of the embedding tensor. \( X_M \), as the gauge generators of the theory, appear in the covariant derivative,

\[
D_\mu = \partial_\mu + gA_\mu^M X_M .
\]  

(22)

That is, the embedding tensor formalism, acts as a caretaker of the covariance, deciding (by means of the constraints that act on it) what linear combinations of the global symmetry generators are the ones that preserve the covariance and the supersymmetry of the theory.

On the other hand, the fact of deforming a supergravity theory implies certain collateral adjustments that are essential for keeping covariance and supersymmetry unbroken. The main ones are the following:

- The standard derivatives have to be replaced by the covariant derivatives (22) to provide the local character of the promoted subgroup. Therefore, some quantities, as the field strengths, result modified.

- The modification of the field strengths by means of the covariant derivatives and the condition that they have to transform covariantly imply the introduction of new couplings in their definition and in the gauge transformation of the fields, the so-called \textit{Stückelberg couplings}.

- The SUSY transformations of the fermion fields are modified by the addition of the so-called \textit{fermion shifts}, which are linear in the deformation parameters.

- A scalar potential is generated and can be expressed as a sum of the squares of the fermion shifts. Thus, it is quadratic in the deformation parameters.

Let us focus on the modified field strengths. As we said in the second item, it is necessary the introduction of Stückelberg couplings to guarantee the covariance of the field strengths. Schematically, the ‘deformed’ field strength and gauge transformation of an arbitrary \( p \)-form gauge field \( C^{(p)\alpha} \), where \( \alpha \) is an index of the representation of \( G \) under which \( C^{(p)} \) transforms, result

\[
\mathcal{F}^{(p+1)\alpha} = \mathcal{D}C^{(p)\alpha} + \cdots + Z^\alpha_1 C^{(p+1)_1} ,
\]

\[
\delta_\Lambda C^{(p)\alpha} = \mathcal{D}\Lambda^{(p-1)\alpha} + \cdots + Z^\alpha_1 \Lambda^{(p)_1} .
\]  

(23)

The Stückelberg couplings are the tensors \( Z^\alpha_I \), which are supposed to be linear in the embedding tensor, \( C^{(p+1)_I} \) is the \((p + 1)\)-form gauge field realized on a certain representation of \( G \), denoted by the index \( I \). The tensors \( \Lambda^{(p-1)\alpha} \) and \( \Lambda^{(p)_I} \) are \((p - 1)\)-
and $p$-form gauge parameters living in their respective representations and can be understood as the generalizations of the 0-form gauge parameter that appears in \[20\]. Hence, the St"uckelberg couplings connect the leading-order $(p+1)$-form with the field strength of a $p$-form. The completion of this structure from the vector fields to the top-forms is the so-called tensor hierarchy and, by means of it, we have access to the full field content of the theory, including the magnetic dual gauge fields. In particular, we obtain information about the $(d - 1)$- and $d$-form gauge fields, which are related to the parameter deformations and the constraints that filter the valid gaugings, respectively.

In summary, following the many subtleties of the embedding tensor formalism, we are able to obtain all the possible gaugings (as well as all their possible combinations) of a given ungauged supergravity.

**Non-geometric fluxes and T-duality constructions**

Thus, once we possess a mechanism that provides all the possible non-Abelian deformations of an ungauged supergravity, it seems natural to compare these results with the ones obtained by dimensional reduction of the higher-dimensional supergravities. When this analysis is done, the situation is the following: not all the gaugings arising from the embedding tensor formalism can be obtained by means of a dimensional reduction procedure. Namely, despite of using a wide variety of compactification
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schemes \[^{35–37}\], there is no higher-dimensional geometric explanation for some of the non-Abelian deformations.

However, inspired by duality covariance arguments, it was proven \[^{38}\] that by performing T-duality transformations on the gauge and geometric fluxes (the ones associated to non-trivial $H_{\mu\nu\rho}$ and $g_{\mu\nu}$ backgrounds, respectively), there appeared new fluxes that could not be reached by dimensional reduction but which, nevertheless, were found by means of the embedding tensor formalism. These are the so-called non-geometric fluxes. The transformations that were applied to the gauge and geometric fluxes are given by the so-called Buscher’s rules, which relate two different string backgrounds. That is, by applying T-duality transformations on fluxes that have a higher-dimensional origin, we obtain new fluxes that cannot be justified by compactification.

Therefore, the situation is the following: roused by T-duality arguments, there are some new fluxes that seem to be associated to the gaugings that are not reached by the standard dimensional reduction procedure. Are we missing any ingredient in the compactification procedure that has to do with T-duality? The appearance of non-geometric fluxes in \[^{38}\] suggests the realization of T-duality as a fundamental symmetry of the theory, rather than just being a symmetry of the compactified versions of string theory. Thus, in order to provide a scenario in which non-geometric fluxes naturally emerge (and, hence the missed supergravity deformations associated to them), new constructions have been investigated in the literature whose main feature is the inclusion of the T-duality symmetry group as a true symmetry of the theory.

The origin of T-duality could give us a hint on the importance of the winding modes in the compactification procedure. Let us assume that we have two different backgrounds in a $(d + 1)$-dimensional theory, $\{g, B, \phi\}$ and $\{g', B', \phi'\}$. Then, if we perform a dimensional reduction on a circle of radius $R$ of the theory turning on the unprimed background and a dimensional reduction on a circle of radius $R'$ of the theory with the primed background turned on, we obtain an equivalent theory. Hence, what T-duality does, by means of the Buscher’s rules, is to establish a relation between $\{g, B, \phi\}$ and $\{g', B', \phi'\}$. At the string theory level, this can be understood as an interchange of winding modes and momentum modes.

At this point, we wonder whether the fact of not considering the winding modes in a compactification scheme could lead to the loss of any flux generated by the wrapping of the closed string around a certain geometry or around nontrivial 1-cycles in spacetime. This could qualitatively justify the necessity of taking the winding modes of the string in our compactification scheme into account.

There exist two main approaches to promote T-duality from a hidden symmetry to a truly global symmetry. One direction is the so-called generalized complex geometry \[^{39–41}\], in which the internal manifold enjoys a particular bundle structure such that the corresponding gauge fields span the whole T-duality symmetry group. Another construction consists of doubling the internal coordinates by adding winding modes as the dual spacetime coordinates. Naïvely, this winding would contribute to the generation of the non-geometric fluxes in the same way as a compactification on a
twisted double torus does \cite{42}.

The last procedure has been recently improved into the so-called Double Field Theory (DFT). This construction is a T-duality invariant reformulation of supergravity in 10+10 dimensions, where the new set of coordinates are related to winding modes. The first formulation of DFT involved the \((10 + 10)\)-dimensional metric \(g_{ij}\), a 2-form field \(B_{ij}\) and a scalar dilaton field \(\phi\), which correspond to the field content of the bosonic common sector of strings. Later on, motivated by the search of an \(O(D,D)\) invariant theory, these fields were encoded into the so-called generalized metric \(\mathcal{H}_{MN}\),

\[
\mathcal{H} = \begin{pmatrix}
g^{ij}
& -g^{ik}b_{kj}

b_{ik}g^{kj}
& g_{ij} - b_{ik}g^{kl}b_{lj}
\end{pmatrix},
\]

which is \(O(D,D)\) invariant by construction. The scalar dilaton, multiplied by the determinant of the metric, becomes T-duality invariant,

\[
e^{-2\phi} = \sqrt{|g|e^{-2\phi}}.
\]

Once the metric and the 2-form are unified into this generalized metric, one wonders whether it is possible to define certain ‘generalized diffeomorphisms’ such that the diffeomorphisms and the gauge symmetry that act on the metric and the 2-form, respectively, result unified in a similar way. This leads to the definition of a generalized gauge parameter \(\xi^M\),

\[
\xi^M = (\tilde{\Lambda}_i, \Lambda^i),
\]

made out of the parameters of both symmetries. Hence, we can think of a generalized Lie derivative,

\[
\mathcal{L}_\xi V^M = \xi^P \partial_P V^M + (\partial^M \xi_P - \partial_P \xi^M) V^P,
\]

which will bring up the definition of a suitable generalized bracket such that the generalized Lie derivatives, which are associated to the gauge transformations of the fields, close properly.

Once that DFT is properly defined, one can perform Scherk-Schwarz dimensional reductions on it, so that one expects to catch the gaugings associated to the non-geometric fluxes. First attempts resulted successful to formally reproduce \(N = 4 \equiv D\) supergravities \cite{43,44}. However, it was shown \cite{45} that a relaxation of some of the constraints of DFT formalism enabled a full description of the gaugings of a theory. The predictability power of DFT for describing gaugings of maximal and half-maximal supergravities in dimensions \(d = 9, 8, 7\) was confirmed in \cite{46}

**Supergravity extremal black holes**

Once we have presented some key features of supergravity theories and their understanding from a string theory perspective, we can study the existence of solutions,
how solutions in supergravity are modified to those in pure gravity and how symmetric
they are with respect to the underlying theory: what is the influence of preserving a
fraction of the supersymmetric charges in our solution.

In particular, due to its simplicity and its conceptual richness, extremal black hole
solutions in $D = 4$ will be our guinea pigs to carry out this brief primer. They are also
the subject of the third part of this work. Extremal black holes are particular solutions
of (super)gravity and possess the minimal amount of mass allowed by their charge,
$M = \sqrt{Q^2 + P^2}$ (see Chapter 5 for details). Their temperature vanishes, making them
stable solutions under Hawking radiation. Solutions as the Bertotti-Robinson metric
or, for multicenter configurations, the Majumdar-Papapetrou metric,

$$ds^2 = -H^{-2}(x)dt^2 + H^2(x)dx^i dx^i;$$  \(28\)

represent extremal black holes. In the last expression, $H(x)$ is a harmonic function,
$\Delta_3 H = 0$.

Let us see some consequences of having local supersymmetry in our theory. For
every extended supersymmetry, the algebra contains central charges, \textit{i.e.} operators
that commute with all the generators of the algebra. For the case of $N = 2$ $D = 4$,
which will be the case we will treat, there is one complex central charge $Z$. From the
supersymmetry algebra, we can infer that all massive representations satisfy a mass
bound, which is given by this central charge,

$$M \geq |Z|.$$  \(29\)

States that saturate this bound, $M = |Z|$, are the \textit{BPS states}.

Minimal $N = 2$ $D = 4$ SUGRA is a suitable scenario in which Einstein-Maxwell
theory is naturally embedded. The pure supergravity multiplet is spanned by the
graviton (vielbein) $e_{\mu}^{\alpha}$, the gravitini $\psi_{\mu i}$, $i = 1, 2$ and gauge field $A_{\mu}$, which in this
context is usually called \textit{graviphoton}.

The central charge transformations are $U(1)$ symmetries and the graviphoton plays
the rô of the gauged field to guarantee its local nature. This central charge is related
to the electric and magnetic charges by the relation \[47\]

$$Z = Q + iP.$$  \(30\)

Hence, the classical bound $M = \sqrt{Q^2 + P^2}$ translates into the supersymmetric one,
$M = |Z|$.

Typically, a non-trivial field configuration has less symmetry than the vacuum.
However, there exist certain solutions that preserve different portions of the symmetries.
For instance, axisymmetric black holes are solutions of Einstein-Maxwell theory that
keep rotations as isometries of the metric. These isometries are generated by the Killing
vectors. The situation in the context of supersymmetry transformations is comparable
to this one. If we are able to find supersymmetric transformation parameters $\epsilon(x)$ (in
the previous case, the transformation parameters are the Killing vectors) such that
a particular field configuration is invariant under these transformations, we have the fermionic analogue of an isometry. Due to their fermionic nature, these parameters are called *Killing spinors*.

For the above case, proving the invariance of the fields means to put to zero the supersymmetric transformations of all the fields of the theory,

$$\delta_\epsilon \{e_\mu^a, A_\mu, \psi^i_\mu\} = 0.$$  \hfill (31)

Since we are interested in purely bosonic solutions, we truncate the fermion fields to zero,

$$\psi^i_\mu = 0.$$  \hfill (32)

On the other hand, the fact of having a spinorial transformation parameter implies that all the supersymmetric transformations of the bosonic fields are odd in fermions. Hence, we automatically have

$$\delta_\epsilon \text{(boson)} = 0,$$  \hfill (33)

so that the remaining non-trivial condition is

$$\delta_\epsilon \psi^i_\mu = 0,$$  \hfill (34)

which is an equation for $\epsilon(x)$ (the so-called *Killing spinor equation*). In order to make this problem more tractable, some assumptions on $e_\mu^a$ and $A_\mu$ can be done. For instance, we can assume static ansatze for the gauge field and the metric. One can check that the aforementioned Bertotti-Robinson and Majumdar-Papapetrou metrics allows the existence of some spinor $\epsilon(x)$ that satisfies (34). In the former case, the full set of supercharges remains unbroken, whereas the latter only preserves half of it.

From a general point of view, we can consider black holes as solitonic solutions. Solitons are broadly defined as time-independent, non-perturbative, non-singular\(^7\) localized solutions of classical equations of motion with finite energy in a field theory \[48\].

The method presented above can be used to construct different supersymmetric solitonic solutions of supergravity theories in various dimensions. Such solutions in $d$ space-time dimensions are alternatively called $p$-branes \[49\] if they are localized in $d - 1 - p$ spatial coordinates and independent of the other $p$ spatial coordinates, where $p < d - 1$. The $p = 0$ case (0-brane) corresponds to a point particle; $p = 1$ case is called a string; $p \geq 2$ cases are known as membranes.

The discovery of string dualities in the second string revolution led to a new picture in the knowledge of solutions of the theories. The knowledge of non-perturbative solitons in these scenarios is essential for the understanding of different regimes of string/M theories by means of dualities applied on them. An example of this is the work done in \[50\]-\[53\]. In particular, S-duality provided a bridge between the strong

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\(^7\)In gravity contexts, solutions whose singularities are covered by event horizons are also admitted.
coupling limit of a given string theory and its dual theory that turned out to be weakly coupled. The result of applying dualities to black hole is a dual description of string excitations. The so-called string-black hole correspondence \[54,55\] predicts the black hole entropy in terms of string states and gives an explanation for the final state of a Schwarzschild BH.

In general, Supergravity reproduces the Einstein-Hilbert action coupled to a certain number of matter fields, whose specific content depends on the theory under analysis. Generically, these fields are a bunch of scalar fields (moduli), spin-1/2 fermions, spin-1 gauge fields and \( N \) gravitini, where the latter behave as the gauge fields of the local supersymmetry. At this point, we can wonder what are the simplest BH solutions in this scenario where additional fields are present and their relation to the pure gravity ones.

The so-called attractor mechanism may be described as follows in simple terms: the scalar fields approach fixed values at the BH horizon, that are only determined by the charge configuration. The asymptotic values of the moduli are forgotten even though the dynamics is completely valid and the fixed point represents the equilibrium of the system. In fact, the flow of the scalars towards the horizon behaves as a gradient flow towards a fixed point, which is the minimum of a function called black hole potential. The entropy is shown to be proportional to the black hole potential evaluated in the horizon \[56\]. On the other hand, non-supersymmetric extremal solutions (non-BPS states) also exhibit this attractor behavior. However, in this case not all the scalar fields of the vector multiplets become stabilized in terms of the BH conserved charges at the event horizon. Some of them generate flat directions at the minimum of the potential \[57\]. Even though, the entropy of non-BPS BHs also depends on the dyonic charges, as in the supersymmetric case \[57,58\].

Once we have reviewed from a general perspective some topics of supergravity that are going to be treated in this manuscript, we are going to describe how this dissertation is organized as well as the content of each of its sections.

**Outline of this work**

The work of this thesis is arranged by following a comprehensible progression. In the following paragraphs, we will enumerate each chapter, with a brief description of its content.

Chapter 1 is an introductory presentation of the main features of ungauged supergravities, emphasizing in supersymmetry as one of its pillars. We give a digest of the maximal higher-dimensional supergravities due to their importance in the following chapters.

In Chapter 2 gauged supergravities and the embedding tensor formalism are studied. We will dissect the structure of these deformed theories as well as the consequences of inserting the embedding tensor inside an ungauged supergravity.

After that, we give a full example of how the embedding tensor scans all the possible
gaugings of $D = 9$ maximal supergravity. This is done in Chapter 3. We construct the tensor hierarchy of the gauged theory and compare our results with the ones based in the $E_{11}$ formalism.

Chapter 4 deals about flux compactifications and how do they motivate one of the clashes between string theory and supergravity. As we said, the gauged supergravities that the embedding tensor allows us to construct and the ones generated by flux compactification of higher-dimensional theories do not coincide. In this chapter, we use Double Field Theory to solve this problem, at least, for all maximal and half-maximal $D = 9, 8, 7$ supergravities.

In Chapter 5 we present a study of some aspects of extremal multicenter black hole solutions in certain $N = 2$ $D = 4$ supergravity models. In the context of special geometry, we provide a formalism to obtain explicit composite black hole solutions with an arbitrary number of centers for any arbitrary quadratic prepotential.

Finally, Chapter 6 summarizes and synthesizes the main results and conclusions of the work done in this dissertation. Various prospects and further projects are shown as possible candidate ideas to address in a near future.

Several appendices are included. Appendix A treats general aspects of T duality. Appendix B includes general notation and definitions and more results obtained in Chapter 3. Appendix C shows some technical material used in the development of the calculus of Chapter 4.

In page 175 we provide the list of publications on which the thesis is based, as well as other works that have been done during the PhD period.
Chapter 1

Supergravity: a primer

In this chapter, we will introduce some basic aspects of supersymmetry and supergravity theories. We will show how supersymmetry restricts and casts the field content of the theory depending on the dimension in which we formulate our theory. Furthermore, we will show a catalog of the maximal higher-dimensional theories in \( D = 11, 10, 9 \).

1.1 Supersymmetry essentials

It is generally assumed that the exact or approximate symmetry groups of the known fundamental laws of nature are (at least locally) isomorphic to direct products of the spacetime Poincaré group and compact Lie groups \([15]\) representing internal symmetries. The internal symmetry concept (as isospin) was initially introduced in Physics by Heisenberg in 1932 \([59]\) and quickly expanded by the \( SU(4) \supset SU(2) \times SU(2)_{iso-s} \) Wigner model \([60]\). However, prior to the establishment of QCD and the current Standard Model (SM) of particles and interactions, symmetries that extended Poincaré symmetry (or its non-relativistic limit including spin) in a non-trivial way were suggested in the early 1960s as a way of formulating a viable theory of hadronic physics (see for example \([61–67]\), also \([60]\)).

In one of these extensions the older, non-relativistic, \( SU(4) \) Wigner ‘supermultiplet’ model was extended to \( SU(6) \) (see \([61]\) and references therein)\(^1\). This group has a subgroup \( SU(2) \times SU(3) \) identified with the direct product of the non-relativistic spin group \( SU(2) \) and a \( SU(3) \) internal-symmetry group. The full \( SU(6) \) theory proposes to treat the ordinary spin on the same footing as the isotopic spin and hypercharge. It mixes the spin and \( SU(3) \) coordinates so that particles with different spin as well as with different isospin and strangeness can lie in the same supermultiplet: quarks \( u, d, s \), with spin up and down belong to the fundamental representation \( 6 \). Mesons and baryons belong to the \( 35 \) and \( 56 \) representations obtained from the product \( q\bar{q} \) and

\(^1\)The term ‘supermultiplet’, as in ‘the \( SU(4) \) Wigner supermultiplet’, apparently appears for the first time in 1964 \([61]\), having nothing to do with the concept of ‘supersymmetry’.
Later on, the (partial) success of the $SU(6)$ theories in explaining some aspects of the classification and properties of hadrons raised the possibility of a relativistic symmetry group which was not simply a direct product of Poincaré and internal symmetry. The extension of the theories to include special relativity was however very problematic. A way of extension was by searching a larger group which included the $SU(6)$ and the Lorentz groups as subgroups. One fitting candidate of this kind was found to be the $SL(6, \mathbb{C})$ group [61], which contains $SL(2) \times SL(3)$ as a subgroup. However, this extension seemed to be impossible without considering a higher 36-dimensional spacetime. Moreover, it would have to admit either an infinite number of one-particle states or a continuous mass distribution for a given particle state [67].

Other group structures were explored, as for example the $\tilde{U}(8)$ and $\tilde{U}(12)$ theories [64,67]. They were based on a covariant merging of isospin and spacetime symmetries including higher-dimensional gamma matrices generators. However, these models became very problematic; since the free Lagrangian was not invariant with respect to the symmetry group (only the interaction part resulted invariant), the physical states did not form a unitary representation of it.

All attempts to find such a group were clearly unsuccessful. At the same time, there appeared a set of no-go theorems [68,70], the Coleman-Mandula theorem [68] the strongest among them, which showed that the symmetry group of a consistent 4-dimensional relativistic quantum field theory with a finite number of massive particles is necessarily the direct product of the internal symmetry group and the Poincaré group.

Typically, these theorems showed that a physical field theory with a finite number of definite mass particles and with an analytical $S$ matrix without any of these groups as symmetries, did not allow anything but trivial scattering, in the forward and backward directions.

Superalgebras, as a way of avoiding the no-go theorems and extending the concept of symmetry, were introduced in particle physics for the first time in 1966 by Miyazawa [71,72]. They were used to introduce spinor currents, in addition to the algebra of vector currents. These currents are bilinear combinations of both bosonic and fermionic fields. As a result mesons and baryons of different spins appear in an unified way in the same multiplet.

The four-dimensional Poincaré superalgebra was developed in 1971 by Gol’fand & Likthman developed [75]. Ramond [76] and Neveu & Schwarz [77] developed superstrings and the supersymmetric extensions of a non Lie algebra, the Virasoro algebra.

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2 In a relativistic wave equation, the spin indices are so tightly related to the coordinates (as we may see in the Dirac equation) that it is impossible to decouple the spin variables from the coordinates even in the free Hamiltonian.

3 These $\tilde{U}$ groups are generated by Lie algebras including gamma matrices and $su(2)$ or $su(3)$ algebras.

4 Lie superalgebras appeared, though not in a central role, in some mathematical contexts in the 1960s [73,74].
Volkov & Akulov [78] and Wess & Zumino [79] wrote different realizations of supersymmetric field theories, even without being aware of the earlier work done by Gol’fand & Likthman. In particular, the Wess-Zumino model [79–82] was the first widely known example of an interacting 4-dimensional quantum field theory with supersymmetry.\footnote{At this time, QCD and the full SM with their symmetry product of the Poincaré group and local Lie groups were well established and apparently there was not need for further developments.}

Superspace formalism was introduced in 1974 [83].

In 1975, Haag, Lopuszański, and Sohnius published [84] a general proof that weakened the assumptions of the Coleman-Mandula theorem by allowing both commuting and anticommuting symmetry generators. There is a nontrivial extension of the Poincaré algebra, the supersymmetry algebra, which is the most general symmetry of the $S$ matrix of a quantum field theory. More in detail, the theorem may be summarized as follows: the most general Lie algebra of generators of supersymmetries and ordinary symmetries of the $S$ matrix in a massive theory involves the following Bose type operators: the energy-momentum operators $P_\mu$; the generators of the homogeneous Lorentz group $M_{\mu\nu}$; and a finite number of scalar charges. It will involve, in addition, Fermi-type operators, all of which commute with the translations and transform like spinors under the homogeneous Lorentz group.

### 1.1.1 Clifford algebras and spinors

The transformation properties of Bose and Fermi generators under the Lorentz group imply restrictions on the number of each of these types of generators and, indirectly, on the number and signature of spacetime dimensions. These restrictions are trivial for the case of Bose generators: a vector representation in any $D$-dimensional spacetime has always $D$ components. The situation is less trivial for the Fermi generators. They carry a spinorial representation of the Lorentz group which makes convenient the detailed study of the representation theory of Clifford algebras.

Clifford algebras are relevant in Physics due to the fact that their representations can be used to construct specific representations of symmetry groups, the spinorial representations. In particular, a representation of the $D$-dimensional Clifford algebra can be used to construct a representation of the $D$-dimensional Lorentz algebra $\mathfrak{so}(1, D-1)$. More in detail, if we define a set of gamma matrices $\{\gamma_\mu\}_{\mu=0,\ldots,D-1}$ which satisfy a Clifford algebra with associated metric $\eta_{\mu\nu} = \text{diag} (-, +, \ldots, +)$,

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \mathbb{1},$$

then the matrices

$$\Sigma^{S}_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$$

The model consists of a single chiral superfield (composed of a complex scalar and a spinor fermion) whose cubic superpotential leads to a renormalizable theory. The action of the free massless Wess-Zumino model is invariant under the transformations generated by a superalgebra allowing both commuting and anticommuting symmetry generators.
are generators for a spinorial representation $S$ of the Lorentz group. The exponentiation of these generators gives a Lorentz transformation

$$\Gamma^S(\Lambda) = \exp \left( \frac{1}{2} \omega^{\mu\nu} \Sigma_{\mu\nu}^S \right).$$  \hfill (1.3)

It can be shown that for a given dimension $D$, there is only one inequivalent irreducible representation of the Clifford algebra whose dimension is $2^{[D/2]}$. The elements of this $2^{[D/2]}$-dimensional vector representation space, where the algebra acts are the Dirac spinors.

Irreducible representations of Clifford algebras may lead to reducible Lorentz representations depending on the dimension of the spacetime. For instance, even dimensions allow the existence of $2^{[D/2]-1}$-dimensional irreducible representations. This can be easily seen by defining a matrix $\gamma^{D+1}$,

$$\gamma^{D+1} = i(-1)^{\frac{D+2}{4}-1}\gamma^0 \cdots \gamma^{D-1}.$$  \hfill (1.4)

This chirality matrix is traceless, squares to unity, half of its eigenvalues are +1s and the other half are -1s. It is natural then to split Dirac spinors into the direct sum of the subspaces of spinors with different eigenvalues. The elements of each of these subspaces are called Weyl spinors and satisfy, by definition, the Weyl or chirality condition,

$$\frac{1}{2} (1 \pm \gamma^{D+1}) \chi = \chi,$$  \hfill (1.5)

where $\chi$ is an arbitrary spinor. The so-called left- and right-handed spinors correspond to the eigenvectors with eigenvalues +1 and -1, respectively.

We can also reduce Dirac spinors using the fact that, since $\gamma_\mu$ satisfies (1.1), then $\gamma^*_\mu$ and $\gamma^T_\mu$ do as well. This implies the existence of some isomorphisms relating these representations. One isomorphism, represented by $C$, relates

$$C\gamma_{\mu\nu}C^{-1} = -\gamma^T_{\mu\nu}.$$  \hfill (1.6)

The matrix $C$ is called a charge conjugate matrix and allows to define a charge-conjugate spinor,

$$\hat{\lambda} = \lambda^TC.$$  \hfill (1.7)

We can look for spinors whose charge-conjugate spinors are proportional to their Dirac conjugate $\bar{\lambda}$ defined by

$$\bar{\lambda} \equiv i\lambda^1\gamma^0.$$  \hfill (1.8)

That is, spinors satisfying

$$\hat{\lambda} = a\bar{\lambda} = \lambda^TC = ai\lambda^1\gamma^0.$$  \hfill (1.9)

This is a ‘reality’ condition for the spinors. The ones that fulfill it are called Majorana spinors. Sometimes chirality and Majorana conditions may be simultaneously satisfied.

We schematically show in Table 1.1 some characteristics of the spinorial irreducible representations for any dimension.
### 1.1. Supersymmetry essentials

| $D \mod 8$ | spinor irreps | real components | $R$-symmetry |
|------------|---------------|-----------------|--------------|
| 1, 3       | M             | $2^{(D-1)/2}$   | $SO(N)$      |
| 2          | MW            | $2^{D/2-1}$     | $SO(N_L) \times SO(N_R)$ |
| 4, 8       | M             | $2^{D/2}$       | $U(N)$       |
| 5, 7       | D             | $2^{(D+1)/2}$   | $USp(2N)$    |
| 6          | W             | $2^{D/2}$       | $USp(2N_L) \times USp(2N_R)$ |

Table 1.1: We show the different irreducible spinorial representations for every dimension $D$ and the number of real components. Depending on the dimension $D \mod 8$, we can have Dirac ($D$), Weyl ($W$), Majorana ($M$) or Majorana-Weyl ($MW$) representations. In addition, we show the $R$-symmetry group for every dimension, where $N$ and $(N_L, N_F)$ denote the number of supersymmetric charges preserved.

### 1.1.2 SUSY algebras and their representations

From a mathematical point of view, a Lie superalgebra is an algebra based on a $\mathbb{Z}_2$ graded vector space. The physical Bose and Fermi elements will be, respectively, the grade 0 and grade 1 algebra vectors.

A Lie superalgebra $\mathfrak{s}$ satisfies the following properties:

- **$\mathfrak{s}$ is a mod 2 graded vector space over $\mathbb{C}$.** I.e., it admits a map
  \[
  \text{gr} : \mathfrak{s} \rightarrow \mathbb{Z}_2 ,
  \]
  which decomposes $\mathfrak{s}$ into $\mathfrak{s}^{(0)}$ and $\mathfrak{s}^{(1)}$ in such a way that
  \[
  \text{gr}(B) = 0 \mod 2 \quad \forall B \in \mathfrak{s}^{(0)} ,
  \]
  \[
  \text{gr}(F) = 1 \mod 2 \quad \forall F \in \mathfrak{s}^{(1)} .
  \]

- **$\mathfrak{s}$ is endowed with a binary operation, the bracket $[,]$, which is bilinear, superanticommutative and mod 2 grade additive,**

  This means that, given $A, B \in \mathfrak{s}$, we have $[A, B] = -[B, A]$ in all cases but one, where both $A$ and $B$ are Fermi in which case $[A, B] = +[B, A]$. The mod 2 grade additivity means that denoting the grades $a, b, c$ of $A, B, C \in \mathfrak{s}$ respectively, if we have $[A, B] = C$ then $a + b = c$ (mod 2) has to be satisfied.

  \[
  \{A, B\} = (-1)^{1+\text{gr}(A)\text{gr}(B)} \{B, A\} ,
  \]
  \[
  \text{gr}(\{A, B\}) = \text{gr}(A) + \text{gr}(B) .
  \]

- **The bracket operation obeys the superJacobi identity**

  \[
  (-1)^{1+\text{gr}(C)\text{gr}(A)} \{\{A, B\}, C\}
  
  +(-1)^{1+\text{gr}(A)\text{gr}(B)} \{\{B, C\}, A\}
  
  +(-1)^{1+\text{gr}(B)\text{gr}(C)} \{\{C, A\}, B\} = 0 .
  \]
This reduces to the ordinary Jacobi identity in all cases but one: when any two of the elements \( A, B, C \) are Fermi and the third one is Bose, in which case one of the three usual Jacobi terms has its sign flipped.

The simple finite-dimensional Lie superalgebras over \( \mathbb{C} \) are fully classified \[85, 86\]. There are eight infinite families, a continuum \( D(2|1; \alpha) \) of 17-dimensional exceptional superalgebras, and one exceptional superalgebra each in dimensions 31 and 40. The special linear \( \mathfrak{sl}(m|n) \) and the orthosymplectic \( \mathfrak{osp}(m|n) \) superalgebras are the most relevant ones from the physical point of view. The superalgebra \( \mathfrak{osp}(4|N) \), which has as bosonic Lie algebra \( \mathfrak{so}(3,2) \times \mathfrak{so}(N) \), corresponds to the AdS superalgebra. The superconformal one is \( \mathfrak{su}(2,2|N) \), which has as Lie algebra \( \mathfrak{so}(4,2) \times \mathfrak{su}(N) \times \mathfrak{u}(1) \).

It is of interest to us superalgebras which include the Poincaré group. The Poincaré superalgebra (the superalgebra whose bosonic sector is strictly the Poincaré algebra) is spanned by the generators \( \{ \mathcal{P}_\mu, \mathcal{M}_{\mu\nu}, \mathcal{Q}_i^\alpha \} \). These generators satisfy the following relations:

\[
[M_{\mu\nu}, M^{\rho\sigma}] = -2\delta_{\rho}^{(\sigma} M_{\nu)\mu}, \quad [P_{\mu}, M_{\nu\rho}] = \eta_{\mu[\nu} P_{\rho]}, \quad [P_{\mu}, P_{\nu}] = 0,

[M_{\mu\nu}, Q^i_\alpha] = -\frac{1}{4}(\gamma_{\mu\nu})_\alpha^\beta Q^i_\beta, \quad [P_{\mu}, Q^i_\alpha] = 0,

\{Q^i_\alpha, Q^j_\beta\} = (\gamma^\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij}. \tag{1.13}
\]

The last term implies that two internal fermionic transformations lead to a spacetime translation. Here it is realized the basic feature of SUSY, the interplay between spacetime and some other internal symmetry\[7\].

For superalgebras including the Poincaré group, the number of supercharges (or grade 1 generators) turns out to be a multiple of the number of real components of an irreducible spinor. This is required by Lorentz invariance itself, since in this case the components of an irreducible spinor transform into each other. Thus, the supercharges \( Q^i_\alpha \) carry two indices: \( i = 1, \ldots, N \), where \( N \) is, in principle, an arbitrary integer, and \( \alpha \) is an irreducible spinor index.

Some elementary properties

Some well-known important properties can be straightforwardly inferred from the Fermi sector of the Poincaré superalgebra. Let us take as an example the simplest \( (N = 1) \) supersymmetric extension of the Poincaré algebra, which can be written in terms of two complex Weyl spinors and their conjugates with the following anticommutation relations:

\[
\{Q_\alpha, Q_\beta\} = \{Q^\dagger_\alpha, Q^\dagger_\beta\} = 0, \quad \{Q_\alpha, \bar{Q}_\beta\} = 2(\sigma^\mu)_{\alpha\beta} P_\mu.
\tag{1.14}
\]

\[7\]Incidentally we observe here the spin-statistics connection at work: Fermi half-integer spin generators have to be anticommuting.
Contracting the first relation with $(\bar{\sigma}^\nu)_{\beta\alpha}$, we have
\[ 4P_\nu = (\bar{\sigma}^\nu)_{\beta\alpha} \{Q_\alpha, \bar{Q}_\beta\}. \] (1.15)

In a quantum theory, the superalgebra generators are operators in the Hilbert space of the system which includes bosonic and fermionic states. Single particle states fall into irreducible representations of the algebra, the supermultiplets. Since the fermionic generators commute with $P^\mu P_\mu$, all particles in a supermultiplet have the same mass.

The time component $P_0$ corresponds to the Hamiltonian operator, which can be written as
\[ 4P_0 = 4H = \sum_\alpha \{Q_\alpha, \bar{Q}_\alpha\} = \sum_\alpha \{Q_\alpha, Q^\dagger_\alpha\} = \sum_\alpha (Q_\alpha Q^\dagger_\alpha + Q^\dagger_\alpha Q_\alpha). \] (1.16)

The expected value of the Hamiltonian in an arbitrary state $|s\rangle$ is given by
\[
\langle s|H|s\rangle = \frac{1}{4} \sum_\alpha \langle s|(Q_\alpha Q^\dagger_\alpha + Q^\dagger_\alpha Q_\alpha)|s\rangle \\
= \frac{1}{4} \sum_\alpha \sum_{s'} \langle s|Q_\alpha|s'\rangle \langle s'|Q^\dagger_\alpha|s\rangle + \langle s|Q^\dagger_\alpha|s\rangle \langle s'|Q_\alpha|s\rangle \\
= \frac{1}{2} \sum_\alpha \sum_{s'} |\langle s'|Q_\alpha|s\rangle|^2 \geq 0, \] (1.17)

where we have introduced the closure relation $1 = \sum_{s'} |s'\rangle \langle s'|$. Thus, we conclude that in a supersymmetric quantum theory, any physical state $|s\rangle$ must have non-negative energy. The inequality saturates if the ground (or vacuum) state denoted by $|0\rangle$ is annihilated by a SUSY generator $Q_\alpha|0\rangle = 0$. In this case, one talks of absence of spontaneous SUSY symmetry breaking.

Since $Q_\alpha$ has spinorial indices, when it acts on a bosonic state of the Hilbert space it produces a spinor, fermionic state. Hence, any supermultiplet has both bosonic and fermionic states. Moreover, one can show that the number of bosonic states is equal to the number of fermionic ones for each supermultiplet with non-zero energy.

Using the SUSY algebra properties, one can construct the corresponding algebra representations, that is, the detailed particle supermultiplet content [87]. Since all the particles in the supermultiplet have the same mass, one can independently study the massive and massless cases. In both cases, the SUSY algebra reduces to a Clifford algebra of raising and lowering anticommuting operators. By combining the Clifford algebra representation theory and maximal weight techniques, one can construct the entire massive or massless multiplets repeatedly applying “raising” $Q^\dagger$ operators to a given maximal spin state. As an example, the so-called massive (massless) ‘chiral’ multiplet is formed by starting with a spin-0 state: it contains a Majorana (Weyl)

They have, in addition, the same charge corresponding to any possible gauge symmetry.
fermion and a complex scalar. The massive vector multiplet is formed from a spin-$\frac{1}{2}$ initial state and contains two Majorana fermions, a massive spin-1 vector and a real scalar. The massless vector multiplet turns out to be composed of a Weyl fermion and a massless spin-1 boson.

There exists a physical upper bound for $N$, the number of spinorial charges. If $N \geq 9$, massless representations necessarily contain some undesirable particles of higher spin $s \geq 5/2$.

If we restrict ourselves to theories with particles of spin $s \leq 2$ (and not more than one time-like coordinate), the maximum number of supercharges that we can have is 32 and the theory may live in dimensions no higher than 11. We will refer to these SUSY theories with the maximal number of supercharges as maximal. In $D = 1, 3, 4, 5, 7, 8, 9, 11$, the supersymmetric algebra is classified by a positive integer $N$, whereas for $D = 2, 6, 10$, the SUSY algebra is classified by two integers $(N_L, N_R)$ which, at least, one of them has to be non-zero. $N_L$ and $N_R$ represent the number of left-handed and right-handed supersymmetries, respectively.

For $N = 1$, the super-Poincaré algebra is invariant under a multiplication of the fermionic charges $Q_\alpha$ by a phase. The corresponding symmetry group, called $U(1)_R$, is the simplest example of an additional symmetry at the level of the supercharges. The so-called $R$-symmetry is an automorphism of the fermionic sector, which transforms different supercharges into each other. For extended SUSY ($N \geq 2$), it becomes a non-Abelian group. Formally, it is defined as the largest subgroup of the automorphism group of the SUSY algebra that commutes with Lorentz transformations. In Table 1.1, it is shown the $R$-symmetry group for any dimension.

Central charges

SUSY algebras with $N \geq 2$ can be extended by adding ‘central charge’ operators. These can be Lorentz scalar $Z^i$ [84] or ‘tensorial’ $Z^{ij}_{\mu_1 \mu_2 \ldots}$ central charges [88,89]. They appear in the anticommutator of two SUSY generators as

$$\{Q_i^\alpha, Q_j^\beta\} = (\gamma_\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij} + \sum_k (\Gamma^{\mu_1 \ldots \mu_k} C)_{\alpha\beta} Z^{ij}_{\mu_1 \ldots \mu_k}.$$  \hfill (1.18)

The possible combinations of central extensions will depend on the dimension and characteristics of the theory. For example, for $D = 11$, we have [90]

$$\{Q_\alpha, Q_\beta\} = (\gamma_\mu C^{-1})_{\alpha\beta} P_\mu \delta^{ij} + \sum_n (\Gamma^{\mu_1 \mu_2} C)_{\alpha\beta} Z_{\mu_1 \mu_2} + \sum_n (\Gamma^{\mu_1 \ldots \mu_5} C)_{\alpha\beta} Z_{\mu_1 \ldots \mu_5}.$$  \hfill (1.19)

For a pure scalar central charge we have [84] ($\epsilon = i\sigma^2$):

$$\{Q^i_\alpha, Q^j_\beta\} = 2\epsilon_{\alpha\beta} Z^{ij},$$

$$\{Q^{\dagger}_\alpha, Q^{\dagger}_\beta\} = 2\epsilon_{\dot{\alpha}\dot{\beta}} Z^{ij\dagger},$$

$$\{Q_\alpha, Q^\beta_\dot{\beta}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu.$$  \hfill (1.20)
1.2. Supergravity

It is possible to choose a set of states \( \{|M,Z^{ij}\}_{i<j} \) which are simultaneously eigenstates of \( P_\mu P_\mu \) and \( Z^{ij} \). The corresponding \( N \times N \) matrix \( (Z^{ij}) \) is antisymmetric in its indices. This matrix can be skew-diagonalized to \( N/2 \) real eigenvalues. Thus, for example, for \( N = 2 \) one can write

\[
(Z^{ij}) = Z(\epsilon^{ij}), \tag{1.21}
\]

where \( Z \) is a real parameter which can be assigned, in addition to the mass, to any corresponding multiplet. By a redefinition of the supercharges, and ensuring that all the states of the supermultiplet have non-negative norm, one arrives to the inequality \( M \geq |Z| \). (1.22)

This is an example of a Bogmol’nyi-Prasad-Sommerfeld (BPS) bound [92,93]. In particular, for massless states, we have \( Z = 0 \). The states that saturate the inequality, \( M = |Z| \), have zero norm. Namely, they are annihilated by a fraction (a half, a quarter or an eighth) of the supercharges.

The structure of the unitary representations of the algebra is different for the cases \( M > |Z| \) and \( M = |Z| \): the supermultiplets with \( M = |Z| \) are smaller (called short multiplets) than those corresponding to \( M > |Z| \) (long multiplets). The short multiplets are also called BPS multiplets because they are related to BPS monopoles [88,89,92,93].

1.2 Supergravity

Supergravity theories are field theories that remain invariant under local supersymmetry [1,2,94–99], i.e. under super-Poincaré transformations with spacetime dependent commuting and anticommuting parameters. Because of the underlying supersymmetry algebra, the invariance under local supersymmetry implies the invariance under spacetime diffeomorphisms. Therefore these theories are necessarily theories of gravity. Supergravity [9] was quickly generalized for several dimensions and for additional \( N \) supersymmetric charges. The number of supercharges in a spinor depends on the dimension and the signature of spacetime. Supergravity theories do not contain any fields that transform as symmetric tensors of rank higher than two under Lorentz transformations. Thus, the limit on the number of supercharges cannot be satisfied in a spacetime of arbitrary dimension. Supergravity can be formulated, in spacetimes with Lorentz signatures, in any number of dimensions up to 11 [100].

About supermultiplets, the most common ones that appear in supergravity are the following:

---

9It can be considered that supergravity was initially proposed in 1973 by Volkov in [78] where there appears an action invariant under local spacetime transformations with 10 commuting and 4 anticommuting parameters.
Gravity supermultiplet. The field content satisfies \( s_{\text{max}} = 2 \). It hosts the graviton plus \( N \) gravitini at least.

Vector/gauge supermultiplet. Here \( s_{\text{max}} = 1 \). They exist for \( N \leq 4 \) theories. The gauge fields of those multiplets can gauge an extra Yang-Mills-like group that commutes with supercharges and it is not part of the superalgebra.

Chiral supermultiplet. \( s_{\text{max}} = 1/2 \). In \( D = 4 \) theories, they only exist for \( N = 1 \). Supersymmetry requires the scalars to span a Kähler-Hodge manifold. They must transform under the gauge group defined by the vector multiplet.

Hypermultiplets. They are the equivalent chiral multiplets for \( N = 2 \). They also must transform under the gauge group. In this case, the scalars must parametrize a quaternionic Kähler manifold.

Tensor multiplets. They include antisymmetric tensors \( T_{\mu\nu\ldots} \). In some cases, they can be dualized to scalar or vector fields and thus, be included in the other multiplets.

1.3 Maximal higher-dimensional supergravities

In the next sections we are going to inspect the \( D = 9, D = 10 \) and \( D = 11 \) maximal supergravities.

1.3.1 \( D = 11 \) supergravity

In 1978, Cremmer, Julia and Scherk (CJS) [101] found the classical action for an 11-dimensional supergravity theory. Up to now, this is the only known classical 11-dimensional theory with local supersymmetry and no fields of spin higher than two.

Other 11-dimensional theories are known that are quantum-mechanically inequivalent to the CJS theory, but classically equivalent. That is, they reduce to the CJS theory when one imposes the classical equations of motion. For example, in [102], it is found a \( D = 11 \) supergravity with local \( SU(8) \) invariance.

The field content of CJS \( D = 11 \) supergravity is

\[
\{ e_\mu^a, C_{\mu\nu\rho}, \psi_\mu \}.
\]

That is, there is an ‘elfbein’ \( e_\mu^a \), a Majorana gravitino field \( \psi_\mu \) and a 3-rank antisymmetric gauge field \( C_{\mu\nu\rho} \). Together with chiral \( (2,0) \) supergravity in \( D = 6 \), it is the
1.3. Maximal higher-dimensional supergravities

only \( Q \geq 16 \) theory without a scalar field. Its full action reads

\[
S = \frac{1}{2\kappa^2} \int d^{11}x e^{\alpha\mu} e^{\beta\nu} R_{\mu\nuab} - \bar{\psi}_\mu \gamma^{\mu\rho} D_\nu \psi_\rho - \frac{1}{24} F^{\mu
u\rho\sigma} F_{\mu
u\rho\sigma} \\
- \frac{\sqrt{2}}{192} \bar{\psi}_\mu \left( \gamma^{\alpha\beta\delta\nu} + 12 \gamma^{\alpha\beta} g^{\gamma\delta} g^{\delta\nu} \right) \psi_\rho \left( F_{\alpha\beta\gamma\delta} + \tilde{F}_{\alpha\beta\gamma\delta} \right) \\
- \frac{2v^2}{(144)^2} e^{-1} e^{\alpha\beta\gamma\delta\nu} e^{\alpha\beta\gamma\delta\nu} F_{\alpha\beta\gamma\delta} C_{\mu\nu\rho} ,
\]

where the Ricci scalar and the covariant derivative, respectively

\[ R = R(\omega) \quad \text{and} \quad D_\mu = D_\mu \left( \frac{1}{2}(\omega + \tilde{\omega}) \right), \]

depend on the spinorial connection \( \omega \) and its supercovariant version \( \tilde{\omega} \). In components, we have for these and other quantities,

\[
\omega_{\muab} = \omega_{\muab}(e) + K_{\muab} , \\
\tilde{\omega}_{\muab} = \omega_{\muab}(e) + K_{\muab} - \frac{1}{8} \bar{\psi}_\nu \gamma^{\nu} \muab \psi_\rho , \\
K_{\muab} = - \frac{1}{4} \left( \tilde{\psi}_\mu \gamma_\rho \psi_\nu - \bar{\psi}_\nu \gamma_\mu \psi_\rho + \bar{\psi}_\nu \gamma_\mu \psi_\rho \right) + \frac{1}{8} \bar{\psi}_\nu \gamma^{\nu} \muab \psi_\rho , \\
F_{\mu\nu\rho\sigma} = 4 \partial_{[\mu} C_{\nu\rho\sigma]} + \frac{3v^2}{2} \bar{\psi}_{[\mu} \gamma_{\nu]} \psi_{\rho\sigma]} ,
\]

where \( \psi_\rho = e_\rho^a \psi_\mu \), and the covariant derivative \( D_\mu \) acts on the spinors as follows:

\[
D_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{8} (\omega + \tilde{\omega})_{\muab} \gamma^{ab} \psi_\nu .
\]

Taking into account that \( F = dC \) is the field strength of the 3-form \( C_{\mu\nu\rho} \), the bosonic field equations and the Bianchi identity are

\[
R_{\mu\nu} = \frac{1}{72} g_{\mu\nu} F_{\rho\sigma\lambda\tau} F^{\rho\sigma\lambda\tau} - \frac{1}{6} F_{\mu\rho\sigma\lambda} F^{\mu\rho\sigma\lambda} , \\
\partial_{[\mu} e F^{\nu\rho\sigma]} = \frac{1}{1152} \sqrt{2} e^{\nu\rho\sigma\lambda\tau\alpha\beta\delta\kappa\pi} F_{\lambda\tau\alpha\beta} F_{\gamma\delta\kappa\pi} , \\
\partial_{[\mu} F_{\nu\rho\sigma\lambda]} = 0 .
\]

An alternative form for the second equation is

\[
\partial_{[\mu} H_{\nu\rho\sigma\lambda\tau\alpha\beta]} = 0 ,
\]

where \( H_{\mu\nu\rho\sigma\lambda\tau\alpha\beta} \) is the dual field strength,

\[
H_{\mu\nu\rho\sigma\lambda\tau\alpha\beta} = \frac{1}{72} e e_{\mu\nu\rho\sigma\lambda\tau\alpha\beta\delta\kappa\pi} F^{\beta\gamma\delta\kappa} - \frac{1}{\sqrt{2}} F_{[\mu\nu\rho\sigma\lambda\tau\alpha\beta]} C_{\lambda\tau\alpha\beta} .
\]

Let us analyze the constant \( \kappa^{-2} \) that multiplies the Lagrangian and carries dimension [mass]\(^9\). We can see that, in principle, it is undetermined and depends on fixing some length scale. If we apply the following shift on the fields (an \( \mathbb{R}^+ \) symmetry),

\[
e_\mu^a \rightarrow e^{-\alpha} e_\mu^a , \quad \psi_\mu \rightarrow e^{-\alpha/2} \psi_\mu , \quad C_{\mu\nu\rho} \rightarrow e^{-3\alpha} C_{\mu\nu\rho} ,
\]
the Lagrangian rescales as
\[ \mathcal{L}_{11} \rightarrow e^{-9\alpha} \mathcal{L}_{11}. \] (1.31)

This is the so-called trombone symmetry \[103\] and it is manifest only at the level of the equations of motion. This scaling could be reabsorbed into a redefinition of \( \kappa_{11}^{-2} \),
\[ \kappa_{11}^2 \rightarrow e^{-9\alpha} \kappa_{11}^2. \] (1.32)

For other supergravities in arbitrary \( D \) dimensions, we observe a similar behavior. In general, we could make the following redefinitions:
\[ g_{\mu\nu} \rightarrow e^{-2\alpha} g_{\mu\nu}, \quad \mathcal{L}_D \rightarrow e^{(2-D)\alpha} \mathcal{L}_D, \quad \kappa_D^2 \rightarrow e^{(2-D)\alpha} \kappa_D^2. \] (1.33)

### 1.3.2 \( D = 10 \) supergravities

In \( D = 10 \) we have Majorana-Weyl (MW) irreducible spinors. The maximal supersymmetry is \( N = 2 \), which gives rise to two discrete and inequivalent possibilities, \( N = (1,1) \) with opposite chiralities and \( N = (2,0) \), with same chirality. They correspond to the \( N = 2A \) and \( N = 2B \) theories, respectively.

**\( N = 2A \) supergravity**

The \( N = 2A \) 10-dimensional theory can be obtained by dimensional reduction of \( D = 11 \) supergravity on a circle. Its field content is given by
\[ \{ g_{\mu\nu}, \phi, B_{\mu\nu}, C^{(3)}_{\mu\rho}, C^{(1)}_{\mu}, \psi_{\mu}^{\pm}, \chi_{\mu}^{\pm} \}. \] (1.34)

The bosonic fields are split into the NSNS sector (the graviton \( g_{\mu\nu} \), the dilaton \( \phi \) and the 2-form \( B \)) and the RR sector (the 3-form \( C^{(3)} \) and the graviphoton \( C^{(1)} \)), whereas the fermionic content consists of 2 MW gravitini \( \psi_{\mu}^{\pm} \) and 2 MW dilatinos \( \chi_{\mu}^{\pm} \). The gravitini and the two dilatinos have opposite chiralities.

The bosonic part of the Lagrangian is
\[ \mathcal{L}_{2A} = e \left\{ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{-\phi} |H|^2 - \frac{1}{2} \sum_{d=1,3} e^{(4-d)\phi/2} |G^{(d+1)}|^2 - \frac{1}{2} \star (dC^{(3)} \wedge dC^{(3)} \wedge B) \right\}, \] (1.35)

where \( H = dB \) is the field strength associated to the NSNS 2-form \( B \) and \( G^{(d+1)} \) is the modified field strength of \( C^{(d)} \),
\[ G^{(d+1)} = dC^{(d)} - dB \wedge C^{(d-2)}, \] (1.36)

\[10\] Lower-dimensional theories inherit the trombone symmetry. This can be justified by dimensional reduction arguments \[17\].
for $d = 1, 3$. In this case we have two different $\mathbb{R}^+$ symmetries: one is the trombone symmetry, a symmetry of the field equations analogous to the existing one in $D = 11$ supergravity and the other one is a symmetry of the Lagrangian, which acts on the field as follows:

$$e^\phi \rightarrow \lambda e^\phi, \quad B \rightarrow \lambda^{1/2} B, \quad C^{(1)} \rightarrow \lambda^{1/2} C^{(1)}, \quad C^{(3)} \rightarrow \lambda^{-1/4} C^{(3)}.$$  \hfill (1.37)

$N = 2B$ supergravity

The field content of $N = 2B$ $D = 10$ supergravity is given by

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi, C^{(0)}, C^{(2)}, C^{(4)}_{\mu\nu\rho\sigma}, \psi^I_{\mu}, \lambda^I\},$$  \hfill (1.38)

where $I = 1, 2$. The bosonic fields are contained in the NSNS common sector (the graviton $g_{\mu\nu}$, the dilaton $\phi$ and the 2-form $B$) and the RR sector (the axion $C^{(0)}$, the 2-form $C^{(2)}$ and the 4-form $C^{(4)}$), whereas the fermionic sector consists of 2 MW gravitini $\psi^\pm_\mu$ and 2 MW dilatinos $\chi^\pm$. The rank-4 antisymmetric tensor is supposed to have a self-dual field strength. Since this is a $N = (2, 0)$ theory, both gravitini have the same chirality. Both dilatinos also have the same chirality but opposite to that of the gravitini.

The Lagrangian of the bosonic sector is given by

$$\mathcal{L}_{2B} = \epsilon \left\{ R - \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} e^{-\phi} |H|^2 - \frac{1}{2} \sum_{d=0,2,4} |G^{(d+1)}|^2 - \frac{1}{2} \star (C^{(4)} \wedge dC^{(2)} \wedge B) \right\},$$  \hfill (1.39)

where $H = dB$ is, again, the field strength of $B$ and $G^{(d+1)}$ is given by

$$G^{(d+1)} = dC^{(d)} - dB \wedge C^{(d-2)}.$$  \hfill (1.40)

for $d = 0, 2, 4$.

The 5-form field strength $G^{(5)}$ satisfies a self-duality condition,

$$G^{(5)} = \star G^{(5)}.$$  \hfill (1.41)

This condition does not follow from the equations of motion the $N = 2B$ action, but has to be imposed as an extra constraint \[104\].

The $N = 2B$ theory enjoys two symmetries: a trombone scaling symmetry and a $SL(2, \mathbb{R})$ symmetry. The former, as in the 11-dimensional case, is only realized on-shell whereas the latter is realized at the level of the Lagrangian and acts on the fields as follows. Considering an $SL(2, \mathbb{R})$ element

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),$$  \hfill (1.42)
the fields transform as
\[\tau \rightarrow a\tau + b \frac{c\tau + d}{c\tau + d}, \quad B^\alpha \rightarrow (\Lambda^{-1})^\alpha_\beta B^\beta, \quad C^{(4)} \rightarrow C^{(4)},\]
\[\psi_\mu \rightarrow \left(\frac{c\tau + d}{c\tau + d}\right)^{1/4} \psi_\mu, \quad \chi \rightarrow \left(\frac{c\tau^* + d}{c\tau + d}\right)^{3/4} \chi,\]
where the complex scalar \(\tau\) encodes the two real ones
\[\tau \equiv C^{(0)} + ie^{-\phi},\]
and the pair of 2-form fields are arranged into a doublet
\[B^\alpha \equiv (-B, C^{(2)}).\]

1.3.3 \(N = 2\) \(D = 9\) supergravity

Next, we are going to study the maximal supergravity in \(D = 9\). There is only one undeformed (i.e. ungauged, massless) maximal (i.e. \(N = 2\), containing no dimensionful parameters in their action, apart from the overall Newton constant) 9-dimensional supergravity [105].

The theory has as (classical) global symmetry group \(SL(2, \mathbb{R}) \times (\mathbb{R}^+)^2\). The \((\mathbb{R}^+)^2\) symmetries correspond to scalings of the fields, the first of which, that we will denote by \(\alpha\), acts on the metric and only leaves the equations of motion invariant while the second of them, which we will denote by \(\beta\), leaves invariant both the metric and the action. The \(\alpha\) rescaling corresponds to a trombone symmetry.

Both the dimensional reduction of the massless \(N = 2A, d = 10\) theory and that of the \(N = 2B, d = 10\) theory on a circle give the same \(N = 2, d = 9\) theory [13].

The fundamental (electric) fields of this theory are,
\[\{e_\mu^a, \varphi, \tau \equiv \chi + ie^{-\phi}, A_I^\mu, B_{\mu\nu}^i, C_{\mu\nu\rho}, \psi_\mu, \tilde{\lambda}, \lambda, \}\.\]
where \(I = 0, i\), with \(i, j, k = 1, 2\) and \(i, j, k = 1, 2, 3\). The complex scalar \(\tau\) parametrizes an \(SL(2, \mathbb{R})/U(1)\) coset that can also be described through the symmetric \(SL(2, \mathbb{R})\) matrix

\[\text{Type IIB string theory breaks } SL(2, \mathbb{R}) \text{ into its discrete subgroup } SL(2, \mathbb{Z}). \text{ This group contains the so-called S-duality transformation that flips the sign of the dilaton } \phi \text{ in a background with vanishing axion } C^{(0)}. \text{ Explicitly, this is done by choosing } a = d = 0 \text{ and } b = -c = 1 \text{ in the } SL(2, \mathbb{R}) \text{ transformation. Because of its very definition, S-duality turns out to be a non-perturbative duality relating the strong- and weak-coupling regimes.}

\[\text{This discussion follows closely that of Ref. [106] in which the higher-dimensional origin of each symmetry is also studied. In particular, we use the same names and definitions for the scaling symmetries and we reproduce the table of scaling weights for the electric fields.}

\[\text{This is a property related to the T duality between type IIA and IIB string theories compactified on circles [107,108] and from which the type II Buscher’s rules can be derived [104]}.\]

\[\text{Sometimes we need to distinguish the indices 1, 2 of the 1-forms (and their dual 6-forms) from those of the 2-forms (and their dual 5-forms). We will use boldface indices for the former and their associated gauge parameters.}\]
\[ \mathcal{M} \equiv e^{\phi} \begin{pmatrix} |\tau|^2 & \chi \\ \chi & 1 \end{pmatrix}, \quad \mathcal{M}^{-1} \equiv e^{\phi} \begin{pmatrix} 1 & -\chi \\ -\chi & |\tau|^2 \end{pmatrix}. \] (1.47)

The field strengths of the electric 4-forms are, in our conventions, given by

\[ F^I = dA^I, \] (1.48)
\[ H^i = dB^i + \frac{1}{2} \delta_i(A^0 \wedge F^i + A^i \wedge F^0), \] (1.49)
\[ G = d[C - \frac{1}{6} \varepsilon_{ij} A^{0ij}] - \varepsilon_{ij} F^i \wedge (B^j + \frac{1}{2} \delta^j_i A^0j) , \] (1.50)

and are invariant under the gauge transformations

\[ \delta_{\Lambda} A^I = -d\Lambda^I, \] (1.51)
\[ \delta_{\Lambda} B^i = -d\Lambda^i + \delta_i \left[ \Lambda^I F^0 + A^0 F^i + \frac{1}{2} \left( A^0 \Lambda A^I + A^I \Lambda A^0 \right) \right] , \] (1.52)
\[ \delta_{\Lambda} \left[ C - \frac{1}{6} \varepsilon_{ij} A^{0ij} \right] = -d\Lambda - \varepsilon_{ij} \left( F^i \Lambda A^j + \Lambda^i \wedge H^j - \delta_{\Lambda} A^i \wedge B^j + \frac{1}{2} \delta^j_i A^0j \wedge \delta_{\Lambda} A^j \right) . \] (1.53)

The bosonic action is, in these conventions, given by

\[ S = \int \left\{ -\star R + \frac{1}{2} d\phi \wedge \star d\phi + \frac{1}{2} \left[ d\phi \wedge \star d\phi + e^{2\phi} d\chi \wedge \star d\chi \right] + \frac{1}{2} e^{\sqrt{2}\phi} F^0 \wedge \star F^0 \right. \\
\left. + \frac{1}{2} e^{3/2} (\mathcal{M}^{-1})_{ij} F^i \wedge \star F^j + \frac{1}{2} e^{-3/2} (\mathcal{M}^{-1})_{ij} H^i \wedge \star H^j + \frac{1}{2} e^{\sqrt{2}\phi} G \wedge \star G \right. \\
\left. - \frac{1}{2} \left[ G + \varepsilon_{ij} A^i \wedge (H^j - \frac{1}{2} \delta^j_i A^0j \wedge F^0) \right] \wedge \left\{ \left[ G + \varepsilon_{ij} A^i \wedge (H^j - \frac{1}{2} \delta^j_i A^0j \wedge F^0) \right] \wedge A^0 \right. \\
\left. \left. - \varepsilon_{ij} \left( H^i - \delta^i_j A^0j \wedge F^0 \right) \wedge (B^j - \frac{1}{2} \delta^j_i A^0j) \right) \right\} \right\} . \] (1.54)

The kinetic term for the \( SL(2, \mathbb{R}) \) scalars \( \phi \) and \( \chi \) can be written in the alternative forms

\[ \frac{1}{2} \left[ d\phi \wedge \star d\phi + e^{2\phi} d\chi \wedge \star d\chi \right] = \frac{d\tau \wedge \star d\tau}{2(3\pi m_r)^2} = \frac{1}{4} \text{Tr} \left[ d\mathcal{M} \mathcal{M}^{-1} \wedge \star d\mathcal{M} \mathcal{M}^{-1} \right] , \] (1.55)

the last of which is manifestly \( SL(2, \mathbb{R}) \)-invariant. The Chern-Simons term of the action (the last two lines of Eq. (1.54)) can also be written in the alternative form

\[ -\frac{1}{2} d \left[ C - \frac{1}{6} \varepsilon_{ij} A^{0ij} - \varepsilon_{ij} A^i \wedge B^j \right] \wedge \{ d \left[ C - \frac{1}{6} \varepsilon_{ij} A^{0ij} - \varepsilon_{ij} A^i \wedge B^j \right] \wedge A^0 \\
- \varepsilon_{ij} d \left( B^i - \frac{1}{2} \delta^i_j A^0j \right) \wedge \left( B^j - \frac{1}{2} \delta^j_i A^0i \right) \} , \] (1.56)
that has an evident 11-dimensional origin.

The equations of motion of the scalars, derived from the action above, are

\[ d \star d \varphi - \frac{2}{\sqrt{7}} e^{\frac{4}{\sqrt{7}} \varphi} F^0 \wedge \star F^0 - \frac{3}{2\sqrt{7}} e^{\frac{3}{\sqrt{7}} \varphi} (\mathcal{M}^{-1})_{ij} F^i \wedge \star F^j \]

\[ + \frac{1}{2\sqrt{7}} e^{-\frac{1}{\sqrt{7}} \varphi} (\mathcal{M}^{-1})_{ij} H^i \wedge \star H^j - \frac{1}{\sqrt{7}} e^{\frac{2}{\sqrt{7}} \varphi} G \wedge \star G = 0, \quad (1.57) \]

\[ d \left[ \star \frac{d\bar{\tau}}{(3m\tau)^2} \right] - i \frac{d\tau \wedge \star d\bar{\tau}}{(3m\tau)^3} - \partial_\tau (\mathcal{M}^{-1})_{ij} [F^i \wedge \star F^j + H^i \wedge \star H^j] = 0, \quad (1.58) \]

and those of the fundamental $p$-forms ($p \geq 1$), after some algebraic manipulations, take the form

\[ d \left( e^{\frac{4}{\sqrt{7}} \varphi} \star F^0 \right) = -e^{-\frac{1}{\sqrt{7}} \varphi} \mathcal{M}^{-1}_{ij} F^i \wedge \star H^j + \frac{1}{2} G \wedge G, \quad (1.59) \]

\[ d \left( e^{\frac{3}{\sqrt{7}} \varphi} \mathcal{M}^{-1}_{ij} \star F^j \right) = -e^{\frac{3}{\sqrt{7}} \varphi} \mathcal{M}^{-1}_{ij} F^0 \wedge \star F^j + \varepsilon_{ij} e^{\frac{3}{\sqrt{7}} \varphi} H^j \wedge \star G, \quad (1.60) \]

\[ d \left( e^{-\frac{1}{\sqrt{7}} \varphi} \mathcal{M}^{-1}_{ij} \star H^j \right) = \varepsilon_{ij} e^{\frac{3}{\sqrt{7}} \varphi} F^j \wedge \star G - \varepsilon_{ij} H^j \wedge G, \quad (1.61) \]

\[ d \left( e^{\frac{2}{\sqrt{7}} \varphi} \star G \right) = F^0 \wedge G + \frac{1}{2} \varepsilon_{ij} H^i \wedge H^j. \quad (1.62) \]

The study of this theory and its possible deformations will be extensively addressed in Chapter 3.
Chapter 2

Gauged supergravities and the embedding tensor

At any dimension, with the remarkable exception of \( D = 11 \), there are some deformations of the known basic supergravity theories. These deformations may consist of the introduction of a superpotential and modifications of kinetic terms, a gauging of the R-symmetry group in extended supergravities, or of the global symmetries of the theory, etc.

We restrict ourselves to gauged supergravities, i.e. supergravities in which vector fields gauge a Yang-Mills group. In this case the number of generators of the gauge group (including Abelian components) equals the number of 110.

Important examples of known basic supergravities are those constructed by dimensional reduction. For example, the dimensional reduction of the common sector of the 10-dimensional supergravities on a \( T^n \) \( n \)-torus, provides a theory with Abelian gauge symmetry and a manifest global symmetry, \( O(n,n) \) in this case. The Kaluza-Klein vector fields and the genuine vector fields combine into \( 2n \) vector fields which transform as the fundamental representation of \( G \equiv O(n,n) \). The scalar fields take values in the coset \( O(n,n)/O(n) \times O(n) \).

One is typically interested in gauged supergravities that arise as deformations of this theory. One way of deforming the theory is to perform the dimensional reduction on twisted tori with fluxes. Another possibility consists of promoting a subgroup \( G_0 \subset G \) to a local symmetry gauged by the existing vector fields in the theory.

A possible systematic approach to the problem of gauging a theory consistently is provided by the embedding tensor formalism\(^1\). This formalism, introduced in refs. \[30\textendash}34\] allows the study of the most general deformations of field theories and, in particular, of supergravity theories \[114\textendash}122\]. In this formalism, if the generators of \( G \) are denoted by \( t_\alpha \), then the generators \( X_M \) of the subgroup \( G_0 \subset G \) to be gauged are conveniently specified by the embedding tensor \( \vartheta_M^\alpha \), so that

\[ X_M = \vartheta_M^\alpha t_\alpha. \tag{2.1} \]

\(^1\)For recent reviews see refs. \[111\textendash}113\].
All the terms in the (purely bosonic) deformed action, except a possible scalar potential, are completely determined by gauge invariance (that is, by $G_0$ throughout the embedding tensor) and by the requirement of recovering the undeformed action in the ungauged limit ($\vartheta_M^a \to 0$). The scalar potential could be in principle any $G$-invariant function of the scalar fields. In the supersymmetric case, this is the bosonic sector of a gauged supergravity action. In addition, the fermionic sector results modified by the addition of fermionic mass terms to keep supersymmetry preserved.

Supersymmetry leads to further restrictions. In general, it rules out some of the possible gauge groups (by imposing certain linear constraints on the components of the embedding tensor) and determines the form of the scalar potential. The ungauged action is manifestly invariant under $G$ global transformations, while the couplings of the gauged action would break this to a subgroup. However, it becomes invariant if the embedding tensor behaves as a spurionic object and is allowed to transform under the gauge group according to its index structure.

**Extended objects and gauged SUGRAS**

The construction of gauged supergravities is not only important from the viewpoint of enlarging the catalogue of known supergravities, but also as a way of completing our knowledge of extended objects in supergravity and string theory. Gauged supergravities become relevant in the study of string configurations and flux compactifications. They are a reliable scenario under which string theory results can be tested.

The discovery of the relation between RR $(p+1)$-form potentials in 10-dimensional type II supergravity theories and D-branes [123] made it possible to associate most of the fields of the string low-energy effective field theories (supergravity theories in general) to extended objects (branes) of diverse kinds: fundamental, Dirichlet, solitonic, Kaluza-Klein, etc. This association has been fruitfully used in two directions: to infer the existence of new supergravity fields from the known existence in the String Theory of a given brane or string state and *vice versa*. Thus, the knowledge of the existence of D$p$-branes with large values of $p$ made it necessary to learn how to deal consistently with the magnetic duals of the RR fields that were present in the standard formulations of the supergravity theories constructed decades before, because in general it is impossible to dualize and rewrite the theory in terms of the dual magnetic fields. The existence of NSNS $(p+1)$-forms in the supergravity theories that could also be dualized made it necessary to include solitonic branes dual to the fundamental ones (strings, basically).

The search for all the extended states of string theory has motivated the search for all the fields that can be consistently introduced in the corresponding supergravity theories, a problem that has no simple answer for the $d$-, $(d-1)$ and $(d-2)$-form fields, which are not the duals of electric fields already present in the standard formulation, at least in any obvious way. The branes that would couple to them can play important rôles in String Theory models, which makes this search more interesting.
U-duality arguments, systematic studies of the possible consistent supersymmetry transformation rules for $p$-forms in the 10-dimensional maximal supergravities \cite{124-129} or the use of the conjectured infinite dimensional $E_{11}$ symmetry \cite{130-132} have been used to determine the bosonic extended field content of maximal supergravity in different dimensions.

Another possible systematic approach to this problem is provided by the embedding-tensor gauging formalism presented before. One of the main features of this formalism is that it requires the systematic introduction of new higher-rank potentials which are related by Stückelberg gauge transformations. This structure is known as the tensor hierarchy \cite{33,34,119,133-135} and can be taken as the (bosonic) extended field content of the theory. In Supergravity Theories one may need to take into account additional constraints on the possible gaugings, but, if the gauging is allowed by supersymmetry, gauge invariance will require the introduction of all the fields in the associated tensor hierarchy.

This formalism cannot be used in the most interesting cases, $N = 1, d = 11$ and $N = 2A, B, d = 10$ Supergravity, because these theories cannot be gauged because they do not have 1-forms ($N = 1, d = 11$ and $N = 2B, d = 10$) or the 1-form transforms under the only (Abelian) global symmetry ($N = 2A, d = 10$). Only $N = 2A, d = 10$ can be deformed through the introduction of Romans’ mass parameter, but the consistency of this deformation does not seem to require the introduction of any higher-rank potentials. The dimensional reduction to $d = 9$ of these theories, the unique $d = 9$ maximal supergravity, though, has 3 vector fields, and their embedding tensor formalism can be used to study all its possible gaugings and find its extended field content.

This chapter is dedicated to the introduction of the basic aspects of the embedding tensor formalism. We will study how to gauge a given supergravity theory, i.e. we will choose a subgroup $G_0 \subset G$ and promote it to a local symmetry. This is a covariant formalism that preserves and guarantees the covariance of the final theory. In the next chapter this formalism will be employed to find all the possible gaugings of the $d = 9$ maximal supergravity.

## 2.1 The embedding tensor

The rôle of the vector fields already existing in a theory are crucial in any gauging procedure. The vectors $A_{\mu}^{M}$ of a typical ungauged theory transform under a group $G$ of global transformations and under an Abelian gauge symmetry $U(1)^{n_V}$, where $n_V$ is the number of vector fields in the theory. If $\xi^\alpha$ is the transformation parameter of the global symmetry $G$ and $\Lambda^M$ is the transformation parameter of $U(1)^{n_V}$ symmetry, the fields transform as

$$
\delta_{\xi}A_{\mu}^{M} = -\xi^{\alpha}(t_{\alpha})_{N}^{M}A_{\mu}^{N}, \quad \delta_{\Lambda}A_{\mu}^{M} = \partial_{\mu}\Lambda^{M},
$$

(2.2)
2. Gauged supergravities and the embedding tensor

where \( t_\alpha \) are the generators of \( G \), \( M = 1, \ldots, n_V \) is an index of the fundamental representation and \( \alpha = 1, \ldots, \dim G \) is an index of the adjoint representation.

In general, any other generic field \( V^{(r)} \) of the theory transforming under a certain representation (symbolically denoted by the superindex \( (r) \)) of the global symmetry group \( G \), will transform as

\[
\delta V^{(r)} = \Lambda^\alpha t^{(r)}_\alpha V ,
\]

where \( t^{(r)}_\alpha \) are the group generators in the corresponding representation.

The aim of the gauging procedure is to promote an undetermined subgroup (or subgroups) \( G_0 \subset G \) to a local symmetry. Let us assume a subset of generators \( X_M \subset g = \text{Lie } G \) to be the candidates to be gauged. The explicit embedding of \( G_0 \) into \( G \) is given by a \((n_V \times \dim G)\) matrix \( \vartheta_{M\alpha} \). This is the so-called embedding tensor, which describes the relation between the global and the gauge candidate generators

\[
X_M = \vartheta_{M\alpha}(t^{(r)}_\alpha) \in g .
\]

This is a relation at the level of the abstract Lie algebras. For any specific representation \( (r) \) of the algebra, of the algebra this is translated to a relation of the form

\[
X^{(r)}_M = \vartheta_{M\alpha}(t^{(r)}_\alpha) .
\]

In particular, for the fundamental representation, we have

\[
X_{MN}^P = \vartheta_{M\alpha}(t^{(r)}_\alpha)_N^P .
\]

In general, the closure of the algebra generated by the gauge generators \( X_M \) is not guaranteed. In principle, we have

\[
[X_M, X_N] = -Z_{MN}^P X_P - X_{MN}^P X_P ,
\]

where the gauge generators are split into their symmetric and antisymmetric parts, respectively;

\[
X_{MN}^P = Z_{(MN)}^P + X_{[MN]}^P .
\]

In the following paragraphs, we will see how this situation is solved.

On the other hand, the local symmetry that we want to establish is implemented by means of the following covariant derivative

\[
\partial_\mu \rightarrow D_\mu = \partial_\mu - A_\mu^M X_M ,
\]

where \( X_M \) is realized in the corresponding representation of the object on which the derivative is applied upon. The new covariant derivatives will guarantee the covariance of the theory under the local symmetry group. The global covariance of the theory is

\footnote{The representation label will be generally suppressed if there is no ambiguity.}
2.1. The embedding tensor

also preserved along the procedure. Only when we choose a particular gauge group $G_0$, i.e. a particular $\vartheta_M^\alpha$ configuration, $G$ gets broken. Indeed, the dimension of the final gauge group is the rank of $\vartheta_M^\alpha$, which fulfills $\text{rank}(\vartheta_M^\alpha) \leq \min(n_V, \text{dim } G)$. In order to respect the global $G$ covariance of the theory, the embedding tensor is considered as a spurionic field with global and local transformation properties. Thus, gauge transformations are supposed to act on it in the corresponding way:

$$\delta \Lambda \vartheta_M^\alpha = \Lambda^N X_{NM}^P \vartheta_P^\alpha - \Lambda^N X_{N\beta}^\alpha \vartheta_M^\beta$$

$$= \Lambda^N \vartheta_N^\beta \left( t_{\beta M}^P \vartheta_P^\alpha - f_{\beta \gamma}^\alpha \vartheta_M^\gamma \right), \quad (2.10)$$

where we have used that, in the adjoint representation, $X_{N\gamma}^\alpha = \vartheta_M^\beta f_{\beta \gamma}^\alpha$ and $f_{\alpha \beta \gamma}$ are the structure constants of $G$. The constraints \( (2.10) \), obtained by demanding the gauge invariance of the embedding tensor, is a set of second degree constraints in $\vartheta_M^\alpha$, the so-called quadratic constraints (QC). The QC guarantee the closure of the algebra of the gauge generators for any representation. After its imposition, we have

$$[X_M, X_N] = -X_{MN}^P X_P. \quad (2.11)$$

so we can check that

$$Z^P_{MN} X_P = 0. \quad (2.12)$$

However, if we define $X_{[MN]}^P$ to be the structure constants, we realize that

$$X_{[MN]}^P X_{[QP]}^R + X_{[QM]}^P X_{[NP]}^R + X_{[NQ]}^P X_{[MP]}^R = -Z^R_{P[Q} X_{MN]}^P. \quad (2.13)$$

That is, Jacobi identity is satisfied upon contracting with $X_R$, due to the condition \( (2.12) \). This is enough for the QC \( (2.10) \) to be satisfied.

The embedding tensor components can be decomposed into irreducible representations. In general, we have

$$\vartheta_M^\alpha := V' \otimes g_0 = \theta_1 \oplus \theta_2 \oplus \cdots \oplus \vartheta_k, \quad (2.14)$$

where $V'$ is the conjugate representation of the fundamental $V$, $g_0$ is the adjoint representation and $\theta_i$ are several irreps. In a theory with a given number of bosonic $p$-form fields transforming in different representations, gauge consistency of the tensor hierarchy usually implies the existence of additional linear constraints on the embedding tensor. In addition, there may also exists a linear constraint (LC) arising from supersymmetry (this constraint is not necessarily independent of the linear constraints arising from the bosonic sector). That is, SUSY kills some of the representations of the embedding tensor. In Chapter 3, we will see how this restriction explicitly appears when we study the closure of the supersymmetric transformations of the fields in $D = 9$ maximal supergravity.

Then, the linear constraints restrict the r.h.s. of \( (2.14) \). In Table 2.1, we have the resulting representations of the embedding tensor in maximal theories. For half-maximal supergravities, the structure is similar. Thus, the classification of all the
Table 2.1: In this table we show some aspects of maximal supergravities in various dimensions. $G$ is the global symmetry group and $H$ is its maximal compact subgroup. We show the representation of the vector fields and the embedding tensor, where subindices refer to the weights of the corresponding representation with respect to the $\mathbb{R}^+$ scaling.

| $D$ | $G$         | $H$         | $\tilde{\varphi}$ scalars | vectors | $\vartheta$ |
|-----|-------------|-------------|-----------------------------|---------|-------------|
| 9   | $\mathbb{R}^+ \times SL(2)$ | $SO(2)$     | $1_{+4} + 2_{-3}$           | $2_{+3} + 3_{-4}$ |
| 8   | $SL(2) \times SL(3)$          | $SO(2) \times SO(3)$ | $7$ (2, 3')                   | (2, 3) + (2, 6') |
| 7   | $SL(5)$                   | $SO(5)$     | $14$ 10'                     | $15 + 40'$ |
| 6   | $SO(5, 5)$                | $SO(5) \times SO(5)$ | $25$ 16                       | $144$ |
| 5   | $E_{6(6)}$                | $USp(8)$    | $42$ 27'                     | $351$ |
| 4   | $E_{7(7)}$                | $SU(8)$     | $70$ 56                      | $912$ |

possible gaugings of a given theory reduces to the search and analysis of solutions of the quadratic and linear constraints. Moreover, the counting of inequivalent gaugings or identification of the different orbits is also a non-trivial problem to be solved.

### 2.2 Deformed tensor gauge algebra

Once we have introduced a covariant derivative, it might be a natural ansatz to define a generalized field strength by the expression

$$ F_{\mu \nu}^M = 2\partial_{[\mu}A_{\nu]}^M + X_{[NP]}^M A_{\mu}^N A_{\nu}^P. \quad (2.15) $$

However this is a too na"ıve hypothesis, since it does not transform covariantly,

$$ \delta_\Lambda F_{\mu \nu}^M = -\Lambda^P X_{PN}^M F_{\mu \nu}^N + 2 Z_{MQ}^M \left( \Lambda^P F_{\mu \nu}^Q - A_{[\mu}^P \delta A_{\nu]}^Q \right). \quad (2.16) $$

Only when $Z_{MQ}^M$ vanishes, the field strength transforms covariantly.

The condition of keeping $G$ covariance is the responsible of this situation. We are performing a redundant description of the gauging in terms of the $n_V$ generators $X_M$. In general, since the dimension of the gauge group is smaller than that of the global symmetry group, $n_V$, not all of the $X_M$ generators are linearly independent. For some cases, we can split the vector fields into two groups

- $A_\mu^m$, which transform in the adjoint of $G_0$,
- $A_\mu^i$, which transform in some representation of $G_0$,

so that $Z_{MQ}^M = 0$, and $Z_{iMQ}^M \neq 0$. For some particular examples, this can be done and the problem can be circumvented. However, a general procedure is required.

\[ \text{\footnote{For some explicit examples, see \cite{113}.}} \]
Let us now define the generalized field strength by the expression

\[ symbolicallyfuF^{(1)}_{\mu\nu} = F_{\mu\nu}^M + Z^M_{PQ}B_{\mu\nu}^{PQ}, \]

where \( B_{\mu\nu}^{PQ} \) are 2-forms which maybe belong to the field content of the corresponding theory. Then, we can balance the contribution of the non-covariant terms of (2.16) if the gauge transformations of the 1- and 2-form fields are

\[ \delta A_\mu^M = D_\mu \Lambda^M - Z^M_{PQ} \Xi^{PQ}_\mu, \]

\[ \delta B^{MN}_{\mu\nu} = 2D_{[\mu} \Xi^{MN}_{\nu]} - 2\Lambda^M F^{(1)}_{\mu\nu} + 2A_\mu^{(M} \delta A^{N)}_{\nu}], \]

where \( \Xi^{MN}_\mu \) is a 1-form gauge parameter. Thus, we have a Stückelberg-type coupling between the vector fields and the antisymmetric 2-forms (see [136] for the original introduction of the Stückelberg mechanism). This is a typical situation in massive deformations of supergravities [23].

Moreover, the quantity \( Z^M_{PQ} \) is restricted to live in the representation in which the \( B^{MN}_{\mu\nu} \) do. Then, since \( Z^M_{PQ} \) depends on the embedding tensor by construction, this condition entails a linear restriction on the embedding tensor and its allowed representations.

This procedure can be extended to the existing higher-order rank \( p \)-forms of the theory. As a consequence, a new set of 3-forms have to be properly added to the field strength of the 2-form \( B_{\mu\nu} \) and its gauge transformation. This mechanism necessarily brings to light all the \( p \)-form fields of a given theory. Schematically, we have a tower of relations as

\[ F^{(2)}_M = DA^M + \cdots + Z^{MI} C^{(2)}_I, \]

\[ F^{(3)}_I = DC^{(2)}_I + \cdots + Z^{IA} C^{(3)}_A, \]

\[ \ldots \]

\[ F^{(n)}_P = DC^{(n-1)}_P + \cdots + Z^{PW} C^{(D)}_W, \]

where the indices \( \{ M, I, A, P, W \} \) denote the different representations of \( G \) under which the 1-, the 2-, the 3-, the \( (n-1)\)- and the \( n \)-form fields transform, respectively. The gauge variations of the previous field strengths would be

\[ \delta F^{(2)}_M = D(\delta A^M) + \cdots + Z^{MI} \delta C^{(2)}_I, \]

\[ \delta F^{(3)}_I = D(\delta C^{(2)}_I) + \cdots + Z^{IA} \delta C^{(3)}_A, \]

\[ \ldots \]

\[ \delta F^{(n)}_P = D(\delta C^{(n-1)}_P) + \cdots + Z^{PW} \delta C^{(n)}_W, \]

where the gauge transformation of the fields are given by

\[ \delta A^M = D\Lambda^M + \cdots - Z^{MI} \Xi^{(1)}_I, \]

\[ \delta C^{(2)}_I = D\Xi^{(1)}_I + \cdots - Z^{IA} \Sigma^{(2)}_A, \]

\[ \ldots \]

\[ \delta C^{(n-1)}_P = D\Delta^{(n-2)}_P + \cdots - Z^{PW} \Delta^{(n-1)}_W. \]
The elements $\Lambda^M, \Xi^I, \Sigma^A, \Delta^P, \Delta^W$ are 0-, 1-, 2-, $(n-2)$- and $(n-1)$-form gauge parameters, respectively.

We thus realize that not only covariant derivatives are necessary as new ingredients to gauge a theory, but also St"uckelberg-like couplings between $p$-forms and $(p+1)$-forms become crucial. In particular, they are essential to construct suitable field transformations and guarantee the covariance of the field strengths.

Another consequence of the new gaugings is that the new field strengths do not satisfy the standard Bianchi identities. It can be seen that they satisfy a hierarchy of coupled deformed Bianchi identities, which schematically has the following structure:

\[
D F^{(2)M} = Z^{MI} F^{(3)}_I, \\
D F^{(3)I} = \cdots + Z^{IA} F^{(4)}_A, \\
\ldots \\
D F^{(n-1)P} = \cdots + Z^{PQ} F^{(n)}_Q. \tag{2.22}
\]

A detailed analysis of the higher rank tensor gauge transformations allows us to determine the full field content of the theory, including the $D$- and $(D-1)$-forms, which are non-propagating fields.

### 2.2.1 The deformed Lagrangian

Once we have studied the impact of the gaugings in the group structure of supergravities, let us focus on the Lagrangian of the deformed gauged theory. This study is valid for theories that admit a Lagrangian description, otherwise this treatment is performed in a similar fashion at the level of the equations of motion. The first, straightforward, modifications to be introduced are the covariantization of the derivatives and the replacement of the Abelian field strengths by the fully covariant ones.

Next, it is necessary the modification of the topological Chern-Simons terms of the ungauged version and the addition of a potential.

Concerning the fermionic sector, we require the addition of new mass terms for the spinorial fields in order to keep SUSY invariance. On the other hand, it is indispensable the modification of the supersymmetric variations of the fermions by means of the so-called fermion shifts. These two subtle enhancements ensure the supersymmetric invariance of the action (or, alternatively, the equations of motion).

We have seen how the St"uckelberg couplings connect the $p$- and $(p+1)$-forms throughout the field strengths of the former. This could be, in principle, problematic, since they could imply new equations of motion. However, these contributions combine into first order equations of motions, which show nothing but the fact that they are the on-shell dual fields of the ungauged theory. They enter as Lagrange multipliers-like equations in the Lagrangian.

It is important to point out the conceptual difference between this situation and the so-called democratic formulations of supergravities [125], in which all the dual fields...
2.2. Deformed tensor gauge algebra

are introduced in the action in an egalitarian way and the duality relations must be added by hand.

Once the gaugings are properly implemented in the theory, local supersymmetry invariance of the Lagrangian has to be imposed. The SUSY variations of the new St"uckelberg couplings of the field strengths have to be canceled by new terms of the lagrangian. Let us consider a truly covariant field strength of the form $F^{(p)} \sim F^{(p)} + Z C^{(p)}$ (c.f. \eqref{2.17}). Let us focus on a generic kinetic term $F^{(p)} \wedge \star F^{(p)}$, which is schematically given by

$$F^{(p)} \wedge \star F^{(p)} \sim F^{(p)} \wedge \star F^{(p)} + 2 Z C^{(p)} \wedge \star F^{(p)} + Z Z C^{(p)} \wedge \star C^{(p)} .$$

In general, the SUSY variations of the field strengths and $p$-forms, at second-order in fermions, have the following structures:

$$
\begin{align*}
\delta_\epsilon F^{(p)} & \sim A d (\bar{\epsilon} \gamma^{(p-1)} \cdots \gamma \lambda) + B d (\bar{\epsilon} \gamma^{(p-2)} \cdots \gamma \psi_\mu) , \\
\delta_\epsilon C^{(p)} & \sim D \bar{\epsilon} \gamma^{(p)} \cdots \gamma \lambda + E \bar{\epsilon} \gamma^{(p-1)} \cdots \gamma \psi_\mu ,
\end{align*}
$$

where $\lambda$ is an arbitrary spin-1/2 field, $\psi_\mu$ is a gravitino and $A, B, C, D$ are functions that may depend on the scalar fields. Then, if we focus on the SUSY variation of the Lagrangian kinetic term for $F$, we obtain that

$$
\delta_\epsilon (F^{(p)} \wedge \star F^{(p)}) = 2 \left[ F^{(p)} \wedge \star \delta_\epsilon F^{(p)} + F^{(p)} \wedge \star Z \delta_\epsilon C^{(p)} \right. \\
\left. + \delta_\epsilon F^{(p)} \wedge \star Z C^{(p)} + Z Z C^{(p)} \wedge \star \delta_\epsilon C^{(p)} \right] .
$$

The first term also appears in the ungauged theory and does not imply any problem (as we can check by demanding $\theta \to 0 \Rightarrow Z \to 0$). The rest of the terms depend on the embedding tensor, so one possibility to cancel them is the addition of some fermionic mass terms that explicitly depend linearly on the embedding tensor. Generically, these terms have the following structure

$$
\mathcal{L}_{\text{fm}} = \bar{\psi}_\mu a A^{\mu \nu} ab \psi_\nu b + \bar{\chi}^m B^a_{\mu a} b \psi_\mu a + \bar{\chi}^m C_{ab} \chi^n + \text{h.c.} ,
$$

where $\psi_\mu a$ and $\chi^m$ are generic gravitini and spin $\frac{1}{2}$ fermions, respectively. The indices $a, b$ and $m, n$ belong to some representations of the maximal compact subgroup $H$ of $G$. Hence, the tensors $A_{\mu a b}$, $B_{\mu a}$ and $C_{\mu n}$ which, by construction, depend on the embedding tensor and may depend on the scalar fields, transform under $H \subset G$. In addition, the presence of these new terms requires the modification of the supersymmetric transformation rules of the fermion fields\footnote{For instance, if we study the supersymmetric transformation of the gravitini mass term $\bar{\psi}_\mu a A^{\mu \nu} ab \psi_\nu b$, we obtain the following pattern:

$$
\delta_\epsilon (\bar{\psi}_\mu a A^{\mu \nu} ab \psi_\nu b) \sim 2 (\bar{\psi}_\mu a A^{\mu \nu} ab \delta_\epsilon \psi_\nu b) \sim 2 (\bar{\psi}_\mu a A^{\mu \nu} ab D_\nu \psi_\mu b) + \cdots .
$$

Then, considering the structure of the fermionic transformation rules, the only cancellation of this term arises from $\delta_\epsilon (\bar{\psi}_\mu a A^{\mu \nu} ab D_\nu \psi_\mu b)$). However, there is not any contribution in $\delta_\epsilon \psi_\mu$ proportional to $\theta_M$. Thus, we need to modify the supersymmetric rules of the fermion fields to include these terms.} The new terms are required...
to depend linearly on the embedding tensor. The appropriate modifications result to depend on $A_{\mu}{}^{ab}$ and $B_{\mu}{}^{ma}$:

\[
\delta_\epsilon \psi_\mu^a = \delta_0 \psi_\mu^a + A_\mu{}^{ab} \epsilon_b,
\]
\[
\delta_\epsilon \chi^m = \delta_0 \chi^m + B^{ma} \epsilon_a,
\]

(2.28)

where $\delta_0$ denotes the supersymmetric transformation of the ungauged theory. These extra terms are known as fermion shifts. As a consequence, new terms proportional to $\vartheta^2$ are generated by the action of the fermion shifts on (2.26). This requires the inclusion of one more term, a scalar potential, which schematically has the form

\[
\mathcal{L}_{pot} = -e V = -e \left( B_{\mu}{}^{ma} B^\mu{}_{ma} - A_\mu{}^{ab} A^\mu{}_{ab} \right).
\]

(2.29)

$V$ can be rewritten in terms of the embedding tensor. In general, it can be expressed as

\[
V = V^{MN}{}_{\alpha\beta} \vartheta_M{}^\alpha \vartheta_M{}^\beta,
\]

(2.30)

where $V^{MN}{}_{\alpha\beta}$ is a scalar dependent matrix.

Summarizing, in this chapter we have introduced some basic aspects of gauged supergravities and the embedding tensor formalism. In the next chapter, we will show an exhaustive study of all of the gauged supergravities that the maximal $D = 9$ supergravity can host, by using the embedding tensor to scan all the valid gaugings.
Chapter 3

Gaugings in $N = 2 \ d = 9$ supergravity

After having studied the embedding tensor formalism, it is illustrative to apply it to a non-trivial theory. In this case, we have chosen $N = 2 \ d = 9$ supergravity, since the size of its global symmetry group, $\mathbb{R}^+ \times \text{SL}(2, \mathbb{R})$, allows to carry out the full implementation of the formalism.

3.1 Introduction

We use the embedding tensor method to construct the most general maximal gauged (massive) supergravity in $d = 9$ dimensions and to determine its extended field content. Some gaugings of the maximal $d = 9$ supergravity have been obtained in the past by generalized dimensional reduction \cite{137} of the 10-dimensional theories with respect to the $\text{SL}(2, \mathbb{R})$ global symmetry of the $N = 2B$ theory \cite{109, 138, 139} or other rescaling symmetries \cite{140}. All these possibilities were systematically and separately studied in Ref. \cite{106}, taking into account the dualities that relate the possible deformation parameters introduced with the generalized dimensional reductions. However, the possible combinations of deformations were not studied, and, as we will explain, some of the higher-rank fields are associated to the constraints on the combinations of deformations. Furthermore, we do not know if other deformations, with no higher-dimensional origin (such as Romans’ massive deformation of the $N = 2A, d = 10$ supergravity) are possible.

Our goal in this chapter will be to make a systematic study of all these possibilities using the embedding-tensor formalism plus supersymmetry to identify the extended-field content of the theory, finding the rôle played by the possible 7-, 8- and 9-form potentials, and compare the results with the prediction of the $E_{11}$ approach. We expect to get at least compatible results, as in the $N = 2, d = 4, 5, 6$ cases studied in \cite{122} and \cite{142}.

\footnote{An $\text{SO}(2)$-gauged version of the theory was directly constructed in Ref. \cite{141}.}
This chapter is organized as follows: in Section 3.2 we review the undeformed maximal 9-dimensional supergravity and its global symmetries. In Section 3.3 we study the possible deformations of the theory using the embedding-tensor formalism and checking the closure of the local supersymmetry algebra for each electric $p$-form of the theory. In Section 3.4 we summarize the results of the previous section describing the possible deformations and the constraints they must satisfy. We discuss the relations between those results and the possible 7-, 8- and 9-form potentials of the theory and how these results compare with those obtained in the literature using the $E_{11}$ approach. Section 3.5 contains our conclusions. Our conventions are briefly discussed in Appendix B.1. The Noether currents of the undeformed theory are given in Appendix B.3. A summary of our results for the deformed theory (deformed field strengths, gauge transformations and covariant derivatives, supersymmetry transformations etc.) is contained in Appendix B.4.

3.2 More on the maximal ungauged $d = 9$ supergravity

We have seen some aspects of the only undeformed maximal $N = 2$ 9-dimensional supergravity in Section 1.3.3. Now, we are going to perform a detailed analysis of its symmetries and its magnetic field content.

3.2.1 Global symmetries

The undeformed theory has as (classical) global symmetry group $SL(2, \mathbb{R}) \times (\mathbb{R}^+)^2$. The $(\mathbb{R}^+)^2$ symmetries correspond to scalings of the fields, the first of which, that we will denote by $\alpha$, acts on the metric and only leaves the equations of motion invariant while the second of them, which we will denote by $\beta$, leaves invariant both the metric and the action. The $\beta$ rescaling corresponds to the so-called trombone symmetry which may not survive to higher-derivative string corrections.

One can also discuss two more scaling symmetries $\gamma$ and $\delta$, but $\gamma$ is just a subgroup of $SL(2, \mathbb{R})$ and $\delta$ is related to the other scaling symmetries by

$$\frac{4}{9} \alpha - \frac{8}{3} \beta - \gamma - \frac{1}{2} \delta = 0 \, .$$

We will take $\alpha$ and $\beta$ as the independent symmetries. The weights of the electric fields under all the scaling symmetries are given in Table 3.1. We can see that each of the three gauge fields $A_I^\mu$ has zero weight under two (linear combinations) of these three symmetries: one is a symmetry of the action, the other is a symmetry of the equations of motion only. The 1-form that has zero weight under a given rescaling is

\footnote{This discussion follows closely that of Ref. [106] in which the higher-dimensional origin of each symmetry is also studied. In particular, we use the same names and definitions for the scaling symmetries and we reproduce the table of scaling weights for the electric fields.}
3.2. More on the maximal ungauged $d = 9$ supergravity

Table 3.1: The scaling weights of the electric fields of maximal $d = 9$ supergravity.

| $\alpha$ | $9/7$ | $6/\sqrt{7}$ | $0$ | $0$ | $3$ | $0$ | $0$ | $3$ | $3$ | $3$ | $9/14$ | $-9/14$ | $-9/14$ | $9$ |
|-------|------|------|-----|-----|-----|-----|-----|-----|-----|-----|-------|------|------|-----|
| $\beta$ | $0$ | $\sqrt{7}/4$ | $3/4$ | $-3/4$ | $1/2$ | $-3/4$ | $0$ | $-1/4$ | $1/2$ | $-1/4$ | $0$ | $0$ | $0$ | $0$ |
| $\gamma$ | $0$ | $0$ | $-2$ | $2$ | $0$ | $1$ | $-1$ | $1$ | $-1$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $\delta$ | $8/7$ | $-4/\sqrt{7}$ | $0$ | $0$ | $0$ | $2$ | $2$ | $2$ | $4$ | $4/7$ | $-4/7$ | $-4/7$ | $4/7$ | $8$ |

precisely the one that can be used to gauge that rescaling, but this kind of conditions are automatically taken into account by the embedding-tensor formalism and we will not have to discuss them in detail.

The action of the element of $SL(2, \mathbb{R})$ given by the matrix

$$(\Omega^i_j) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1,$$  

(3.2)
on the fields of the theory is

$$\tau' = \frac{a \tau + b}{c \tau + d}, \quad M'_{ij} = \Omega^k_i \Omega^l_j M_{kl},$$  

$$A^i' = \Omega^i_j A^j, \quad B^{i'} = \Omega^i_j B^j,$$  

$$\psi'_\mu = e^{2il} \psi_\mu, \quad \lambda = e^{2il} \lambda,$$

$$\tilde{\lambda}' = e^{-2il} \tilde{\lambda}, \quad \epsilon' = e^{2il} \epsilon.$$  

(3.3)

where

$$e^{2il} = \frac{c \tau^* + d}{c \tau + d}.$$  

(3.4)

The rest of the fields ($e^a_\mu, \varphi, A^0_\mu, C_{\mu\nu\rho}$) are invariant under $SL(2, \mathbb{R})$.

We are going to label the 5 generators of these global symmetries by $T_A$, $A = 1, \cdots, 5$. $\{T_1, T_2, T_3\}$ will be the 3 generators of $SL(2, \mathbb{R})$ (collectively denoted by $\{T_m\}$, $m = 1, 2, 3$), and $T_4$ and $T_5$ will be, respectively, the generators of the rescalings $\alpha$ and $\beta$. Our choice for the generators of $SL(2, \mathbb{R})$ acting on the doublets of 1-forms $A^i$ and 2-forms $B^a$ is

$$T_1 = \frac{1}{2} \sigma^3, \quad T_2 = \frac{1}{2} \sigma^1, \quad T_3 = \frac{i}{2} \sigma^2,$$  

(3.5)

where the $\sigma^m$ are the standard Pauli matrices, so

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = -T_1, \quad [T_3, T_1] = -T_2.$$  

(3.6)

Then, the $3 \times 3$ matrices corresponding to generators acting (contravariantly) on the 3 1-forms $A^i$ (and covariantly on their dual 6-forms $\tilde{A}_I$ to be introduced later) are
\[(T_1)_{j'} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \quad (T_2)_{j'} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \quad (T_3)_{j'} = \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 0 & i\sigma^2 \end{pmatrix},
\]
\[(T_4)_{j'} = \text{diag}(3,0,0), \quad (T_5)_{j'} = \text{diag}(1/2,-3/4,0). \quad (3.7)\]

We will sometimes denote this representation by $T^{(3)}_A$. The $2 \times 2$ matrices corresponding to generators acting (contravariantly) on the doublet of 2-forms $B^i$ (and covariantly on their dual 5-forms $\tilde{B}_i$ to be introduced later) are

\[(T_1)_{ji} = \frac{1}{2} \sigma^3, \quad (T_2)_{ji} = \frac{1}{2} \sigma^1, \quad (T_3)_{ji} = \frac{1}{2} \sigma^2, \quad (T_4)_{ji} = \text{diag}(3,3), \quad (T_5)_{ji} = \text{diag}(-1/4,1/2). \quad (3.8)\]

We will denote this representation by $T^{(2)}_A$. The generators that act on the 3-form $C$ (sometimes denoted by $T^{(1)}_A$) are

\[T_1 = T_2 = T_3 = 0, \quad T_4 = 3, \quad T_5 = -1/4. \quad (3.9)\]

We will also need the generators that act on the magnetic 4-form $\tilde{C}$, also denoted by $T^{(1)}_A$,

\[\tilde{T}_1 = \tilde{T}_2 = \tilde{T}_3 = 0, \quad \tilde{T}_4 = 6, \quad \tilde{T}_5 = 1/4. \quad (3.10)\]

We define the structure constants $f^{ABC}$ by

\[[T_A, T_B] = f^{ABC} T_C. \quad (3.11)\]

The symmetries of the theory are isometries of the scalar manifold ($\mathbb{R} \times SL(2, \mathbb{R})/U(1)$). The Killing vector associated to the generator $T_A$ will be denoted by $k_A$ and will be normalized so that their Lie brackets are given by

\[[k_A, k_B] = -f^{ABC} k_C. \quad (3.12)\]

The $SL(2, \mathbb{R})/U(1)$ factor of the scalar manifold is a Kähler space with Kähler potential, Kähler metric and Kähler 1-form, respectively given by

\[K = -\log \Im \tau = \phi, \quad g_{\tau \tau^*} = \partial_\tau \partial_{\tau^*} K = \frac{1}{4} e^{2\phi}, \quad Q = \frac{1}{2} (\partial_\tau K d\tau - \text{c.c.}) = \frac{1}{2} e^\phi d\chi. \quad (3.13)\]

In general, the isometries of the Kähler metric only leave invariant the Kähler potential up to Kähler transformations:

\[\mathcal{L}_{k_m} K = k_m^\tau \partial_\tau K + \text{c.c.} = \lambda_m(\tau) + \text{c.c.}, \quad \mathcal{L}_{k_m} Q = -\frac{i}{2} d\lambda_m, \quad (3.14)\]
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where the $\lambda_m$ are holomorphic functions of the coordinates that satisfy the equivariance property

$$\mathcal{L}_{k_m} \lambda_n - \mathcal{L}_{k_n} \lambda_m = -f_{mn}^p \lambda_p.$$  \hspace{1cm} (3.15)

Then, for each of the $SL(2, \mathbb{R})$ Killing vectors $k_m$, $m = 1, 2, 3$, it is possible to find a real Killing prepotential or momentum map $\mathcal{P}_m$ such that

$$k_m \tau^* = \mathcal{G}_{\tau^*} k_m \tau = i \partial_{\tau^*} \mathcal{P}_m,$$

$$k_m \tau \partial_\tau \mathcal{K} = i \mathcal{P}_m + \lambda_m,$$

$$\mathcal{L}_{k_m} \mathcal{P}_n = -f_{mn}^p \mathcal{P}_p.$$  \hspace{1cm} (3.16)

The non-vanishing components of all the Killing vectors are

$$k_1 \tau = \tau, \quad k_2 \tau = \frac{1}{2}(1 - \tau^2), \quad k_3 \tau = \frac{1}{2}(1 + \tau^2), \quad k_4 \tau = 0, \quad k_5 \tau = -\frac{3}{4} \tau.$$  \hspace{1cm} (3.17)

and

$$k_4 \varphi = 6/\sqrt{7}, \quad k_5 \varphi = \sqrt{7}/4.$$  \hspace{1cm} (3.18)

The holomorphic functions $\lambda_m(\tau)$ take the values

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{2} \tau, \quad \lambda_3 = -\frac{1}{2} \tau,$$

and the momentum maps are given by:

$$\mathcal{P}_1 = \frac{1}{2} e^\phi \chi, \quad \mathcal{P}_2 = \frac{1}{4} e^\phi (1 - |\tau|^2), \quad \mathcal{P}_3 = \frac{1}{4} e^\phi (1 + |\tau|^2).$$  \hspace{1cm} (3.20)

These objects will be used in the construction of $SL(2, \mathbb{R})$-covariant derivatives for the fermions.

3.2.2 Magnetic fields

As it is well known, for each $p$-form potential with $p > 0$ one can define a magnetic dual which in $d - 9$ dimensions will be a $(7 - p)$-form potential. Then, we will have magnetic 4-, 5- and 6-form potentials in the theory.

A possible way to define those potentials and identify their $(8 - p)$-form field strengths consists in writing the equations of motion of the $p$-forms as total derivatives. Let us take, for instance, the equation of motion of the 3-form $C$ Eq. (1.62). It can be written as

$$3$$

The holomorphic and anti-holomorphic components are defined by $k = k^\tau \partial_\tau + \text{c.c.} = k^\chi \partial_\chi + k^\phi \partial_\phi$. 

The holomorphic functions $\lambda_m(\tau)$ take the values

$$\lambda_1 = -\frac{1}{2}, \quad \lambda_2 = \frac{1}{2} \tau, \quad \lambda_3 = -\frac{1}{2} \tau,$$

and the momentum maps are given by:

$$\mathcal{P}_1 = \frac{1}{2} e^\phi \chi, \quad \mathcal{P}_2 = \frac{1}{4} e^\phi (1 - |\tau|^2), \quad \mathcal{P}_3 = \frac{1}{4} e^\phi (1 + |\tau|^2).$$  \hspace{1cm} (3.20)

These objects will be used in the construction of $SL(2, \mathbb{R})$-covariant derivatives for the fermions.
\[ d \frac{\partial \mathcal{L}}{\partial G} = d \left\{ e^{\frac{2}{\sqrt{7}} \phi} \star G - \left[ G + \epsilon_{ij} A^i \wedge \left( H^j - \frac{1}{2} \delta^j_i A^j \wedge F^0 \right) \right] \wedge A^0 ight\} = 0. \] (3.21)

We can transform this equation of motion into a Bianchi identity by replacing the combination of fields on which the total derivative acts by the total derivative of a 4-form which we choose for the sake of convenience:

\[ d \left[ \tilde{C} - C \wedge A^0 - \frac{3}{4} \epsilon_{ij} A^{0i} \wedge B^j \right] = e^{\frac{2}{\sqrt{7}} \phi} \star G - \left[ G + \epsilon_{ij} A^i \wedge \left( H^j - \frac{1}{2} \delta^j_i A^j \wedge F^0 \right) \right] \wedge A^0 + \frac{1}{2} \epsilon_{ij} \left( H^i - \delta^i_j A^j \wedge F^0 \right) \wedge \left( B^j - \frac{1}{2} \delta^j_i A^0 \right), \] (3.22)

where \( \tilde{C} \) will be the magnetic 4-form. This relation can be put in the form of a duality relation

\[ e^{\frac{2}{\sqrt{7}} \phi} \star G = \tilde{G}, \] (3.23)

where we have defined the magnetic 5-form field strength

\[ \tilde{G} \equiv d \tilde{C} + C \wedge F^0 - \frac{1}{21} \epsilon_{ij} A^{0ij} \wedge F^0 - \epsilon_{ij} \left( H^i - \frac{1}{2} dB^i \right) \wedge B^j. \] (3.24)

The equation of motion for \( \tilde{C} \) is just the Bianchi identity of \( G \) rewritten in terms of \( \tilde{G} \).

In a similar fashion we can define a doublet of 5-forms \( \tilde{B}_i \) with field strengths denoted by \( \tilde{H}_i \), and a singlet and a doublet of 6-forms \( \tilde{A}_0, \tilde{A}_i \) with field strengths denoted, respectively, by \( \tilde{F}_0 \) and \( \tilde{F}_i \). The field strengths can be chosen to have the form.

\[ \text{With this definition } \tilde{G} \text{ will have exactly the same form that we will obtain from the embedding tensor formalism.} \]
# 3.2. More on the maximal ungauged $d = 9$ supergravity

| 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  |
|----|----|----|----|----|----|----|----|----|----|
| $J_A$ | $A_I^I$ | $B^i$ | $C$ | $C$ | $B_i$ | $A_I^A$ | $A_{(8)}^A$ | $A_{(9)}$ | $\tilde{A}_{(8)}$ | $\tilde{A}_{(9)}$ |

| $F^I$ | $H^i$ | $G$ | $G$ | $H_i$ | $\tilde{F}_I^I$ | $\tilde{F}_{(8)}^A$ | $\tilde{F}_{(9)}^A$ |

Table 3.2: Electric and magnetic forms and their field strengths.

\[
\tilde{H}_i = d\tilde{B}_i - \delta_{ij}B^j \wedge G + \delta_{ij}\tilde{C} \wedge F^j + \frac{1}{2}\delta_{ij} \left( A^0 \wedge F^j + A^j \wedge F^0 \right) \wedge C + \frac{1}{2}\delta_{ij}\varepsilon_{kl}B^{jk} \wedge F^l, \tag{3.25}
\]

\[
\tilde{F}_0 = d\tilde{A}_0 + \frac{1}{2}C \wedge G - \varepsilon_{ij}F^i \wedge \left( \delta^{jk}\tilde{B}_k - \frac{2}{3}B^j \wedge C \right)
\]

\[
- \frac{1}{18}\varepsilon_{ij}A^i \wedge \left( \tilde{G} - F^0 \wedge C - \frac{1}{2}\varepsilon_{kl}B^k \wedge H^l \right)
\]

\[
- \frac{1}{6}\varepsilon_{ij}A^i \wedge \left( B^j \wedge G - C \wedge H^j - \frac{2}{3}\delta^j_l\tilde{C} \wedge F^l - \varepsilon_{kl}B^{jk} \wedge F^l \right), \tag{3.26}
\]

\[
\tilde{F}_1 = d\tilde{A}_1 + \delta_{ij} \left( B^j + \frac{7}{18}\delta^j_k A^{0k} \right) \wedge \tilde{G} - \delta_{ij}F^0 \wedge \tilde{B}_j - \frac{1}{9}\delta_{ij} \left( 8A^0 \wedge F^j + A^j \wedge F^0 \right) \wedge \tilde{C}
\]

\[
- \frac{1}{3}\delta_{ij}\varepsilon_{lm} \left( B^j + \frac{1}{3}\delta^j_k A^{0k} \right) \wedge B^l \wedge H^m - \frac{1}{6}\delta_{ij}\varepsilon_{kl} \left( A^0 \wedge H^j - B^j \wedge F^0 \right) \wedge A^k \wedge B^l
\]

\[
- \frac{1}{6}A^0 \wedge F^0 \wedge \delta_{ij} \left( \frac{7}{2}A^j \wedge C + \delta^j_k\varepsilon_{lm}A^{lm} \wedge B^k \right), \tag{3.27}
\]

and the duality relations are

\[
\tilde{H}_i = e^{-\frac{i}{\sqrt{7}}\varphi} \mathcal{M}^{-1}_{ij} \star H^j, \tag{3.28}
\]

\[
\tilde{F}_0 = e^{\frac{4}{\sqrt{7}}\varphi} \star F^0, \tag{3.29}
\]

\[
\tilde{F}_1 = e^{\frac{3}{\sqrt{7}}\varphi} \mathcal{M}^{-1}_{ij} \star F^j. \tag{3.30}
\]

The situation is summarized in Table 3.2. The scaling weights of the magnetic fields are given in Table 3.3.

This dualization procedure is made possible by the gauge symmetries associated to all the $p$-form potentials for $p > 0$ (actually, by the existence of gauge transformations with constant parameters) and, therefore, it always works for massless $p$-forms with
### Table 3.3: The scaling weights of the magnetic fields of maximal \( d = 9 \) supergravity can be determined by requiring that the sum of the weights of the electric and magnetic potentials equals that of the Lagrangian. The scaling weights of the 7-, 8- and 9-forms can be determined in the same way after we find the entities they are dual to (Noether currents, embedding-tensor components and constraints, see Section 3.4).

| \( \mathbb{R}^+ \) | \( C \) | \( B_2 \) | \( B_1 \) | \( A_2 \) | \( A_1 \) | \( A_0 \) |
|---|---|---|---|---|---|---|
| \( \alpha \) | 6 | 6 | 6 | 9 | 9 | 6 |
| \( \beta \) | 1/4 | -1/2 | +1/4 | 0 | +3/4 | -1/2 |
| \( \gamma \) | 0 | 1 | -1 | 1 | -1 | 0 |
| \( \delta \) | 4 | 6 | 6 | 6 | 6 | 8 |

\( p > 0 \) and generically fails for 0-form fields. However, in maximal supergravity theories at least, there is a global symmetry group that acts on the scalar manifold and whose dimension is larger than that of the scalar manifold. Therefore, there is one Noether 1-form current \( j_A \) associated to each of the generators of the global symmetries of the theory \( T_A \). These currents are conserved on-shell, i.e. they satisfy

\[
d \star j_A = 0,
\]
on-shell, and we can define a \((d-2)\)-form potential \( \tilde{A}^A_{(d-2)} \) by

\[
d \tilde{A}^A_{(d-2)} = G^{AB} \star j_B,
\]

where \( G^{AB} \) is the inverse Killing metric of the global symmetry group, so that the conservation law (dynamical) becomes a Bianchi identity.

Thus, while the dualization procedure indicates that for each electric \( p \)-form with \( p > 0 \) there is a dual magnetic \((7-p)\)-form transforming in the conjugate representation, it tells us that there are as many magnetic \((d-2)\)-form duals of the scalars as the dimension of the global group (and not of as the dimension of the scalar manifold) and that they transform in the co-adjoint representation. Actually, since there is no need to have scalar fields in order to have global symmetries, it is possible to define magnetic \((d-2)\)-form potentials even in the total absence of scalars.\(^5\)

According to these general arguments, which are in agreement with the general results of the embedding-tensor formalism \[121\,122\,133\,135\], we expect a triplet of 7-form potentials \( \tilde{A}^n_{(7)} \) associated to the \( SL(2, \mathbb{R}) \) factor of the global symmetry group \[109\] and two singlets \( \tilde{A}^4_{(7)}, \tilde{A}^5_{(7)} \) associated to the rescalings \( \alpha, \beta \) (see Table 3.2).

Finding or just determining the possible magnetic \((d-1)\)- and \( d \)-form potentials in a given theory is more complicated. In the embedding-tensor formalism it is natural to expect as many \((d-1)\)-form potentials as deformation parameters (embedding-tensor components, mass parameters etc.) can be introduced in the theory since the rôle of the \((d-1)\)-forms in the action is that of being Lagrange multipliers enforcing their

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\(^5\)See Refs. \[121\,122\] for examples.
constancy. The number of deformation parameters that can be introduced in this theory is, as we are going to see, very large, but there are many constraints that they have to satisfy to preserve gauge and supersymmetry invariance. Furthermore, there are many Stückelberg shift symmetries acting on the possible \((d-1)\)-form potentials. Solving the constraints leaves us with the independent deformation parameters that we can denote by \(m^\sharp\) and, correspondingly, with a reduced number of \((d-1)\)-form potentials \(\tilde{A}^\sharp_{(d-1)}\) on which only a few Stückelberg symmetries (or none at all) act.

The \(d\)-form field strengths \(\tilde{F}^\sharp_{(d)}\) are related to the scalar potential of the theory through the expression \[ \tilde{F}^\sharp_{(d)} = \frac{1}{2} \ast \frac{\partial V}{\partial m^\sharp}. \]

Thus, in order to find the possible 8-form potentials of this theory we need to study its independent consistent deformations \(m^\sharp\). We will consider this problem in the following section.

In the embedding-tensor formalism, the \(d\)-form potentials are associated to constraints of the deformation parameters since they would be the Lagrange multipliers enforcing them in the action \([118]\). If we do not solve any of the constraints there will be many \(d\)-form potentials but there will be many Stückelberg symmetries acting on them as well. Thus, only a small number of \textit{irreducible} constraints that cannot be solved and of associated \(d\)-forms may be expected in the end, but we have to go through the whole procedure to identify them. This identification will be one of the main results of the following section.

However, this is not the end of the story for the possible 9-forms. As it was shown in Ref. [122] in 4-, 5- and 6-dimensional cases, in the ungauged case one can find more \(d\)-forms with consistent supersymmetric transformation rules than predicted by the embedding-tensor formalism. Those additional fields are predicted by the Kač-Moody approach \([142]\). However, after gauging, the new fields do not have consistent, independent, supersymmetry transformation rules to all orders in fermions, and have to be combined with other \(d\)-forms, so that, in the end, only the number of \(d\)-forms predicted by the embedding-tensor formalism survive.

This means that the results obtained via the embedding-tensor formalism for the 9-forms have to be interpreted with special care and have to be compared with the results obtained with other approaches.

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6The embedding-tensor formalism gives us a reason to introduce the \((d-1)\)-form potentials based on the deformation parameters but the \((d-1)\)-form potentials do not disappear when the deformation parameters are set equal to zero.

7The \((d-1)\)-form potentials that “disappear” when we solve the constraints are evidently associated to the gauge-fixing of the missing Stückelberg symmetries.

8In general, the quadratic constraints cannot be used to solve some deformation parameters in terms of the rest. For instance, in this sense, if \(a\) and \(b\) are two of them, a constraint of the form \(ab = 0\) cannot be solved and we can call it \textit{irreducible}.

9The insufficiency of first-order in fermions checks was first noticed in Ref. [128].
The closure of the local supersymmetry algebra needs to be checked on all the fields in the tensor hierarchy predicted by the embedding-tensor formalism and, in particular, on the 9-forms to all orders in fermions. However, given that gauge invariance is requirement for local supersymmetry invariance, we expect consistency in essentially all cases with the possible exception of the 9-forms, according to the above discussion. In the following section we will do this for the electric fields of the theory.

3.3 Deforming the maximal \( d = 9 \) supergravity

In this section we are going to study the possible deformations of \( d = 9 \) supergravity, starting from its possible gaugings using the embedding-tensor formalism and constructing the corresponding tensor hierarchy \[30\] \[34\] \[133\] \[135\] up to the 4-form potentials.

If we denote by \( \Lambda^I(x) \) the scalar parameters of the gauge transformations of the 1-forms \( A^I \) and by \( \alpha^A \) the constant parameters of the global symmetries, we want to promote

\[
\alpha^A \longrightarrow \Lambda^I(x) \vartheta_I^A,
\]

where \( \vartheta_I^A \) is the embedding tensor, in the transformation rules of all the fields, and we are going to require the theory to be covariant under the new local transformations using the 1-forms as gauge fields.

To achieve this goal, starting with the transformations of the scalars, the successive introduction of higher-rank \( p \)-form potentials is required, which results in the construction of a tensor hierarchy. Most of these fields are already present in the supergravity theory or can be identified with their magnetic duals but this procedure allows us to introduce consistently the highest-rank fields (the \( d-\), \( d-1 \)- and \( d-2 \)-form potentials), which are not dual to any of the original electric fields. Actually, as explained in Section \[3.2.2\], the highest-rank potentials are related to the symmetries (Noether currents), the independent deformation parameters and the constraints that they satisfy, but we need to determine these, which requires going through this procedure checking the consistency with gauge and supersymmetry invariance at each step.

Thus, we are going to require invariance under the new gauge transformations for the scalar fields and we are going to find that we need new couplings to the gauge 1-form fields (as usual). Then we will study the modifications of the supersymmetry transformation rules of the scalars and fermion fields which are needed to ensure the closure of the local supersymmetry algebra on the scalars. Usually we do not expect modifications in the bosons’ supersymmetry transformations, but the fermions’ transformations need to be modified by replacing derivatives and field strengths by covariant derivatives and covariant field strengths and, furthermore, by adding fermion shifts. The local supersymmetry algebra will close provided that we impose certain constraints on the embedding tensor components and on the fermion shifts.
3.3. Deforming the maximal $d = 9$ supergravity

Repeating this procedure on the 1-forms (which requires the coupling to the 2-forms) etc.
we will find a set of constraints that we can solve, determining the independent components
of the deformation tensors[^10] and the fermions shifts. Some constraints (typically quadratic in
deforation parameters) have to be left unsolved and we will have to take them into account
in the end of this procedure.

As a result we will identify the independent deformations of the theory and the
constraints that they satisfy. From this we will be able to extract information about
the highest-rank potentials in the tensor hierarchy.

3.3.1 The 0-forms $\varphi$, $\tau$

Under the global symmetry group, the scalars transform according to

$$
\delta_\alpha \varphi = \alpha^A k_A \varphi, \quad \delta_\alpha \tau = \alpha^A k_A \tau,
$$

where the $\alpha^A$ are the constant parameters of the transformations, labeled by $A = 1, \cdots, 5$, and
where $k_A \varphi$ and $k_A \tau$ are the corresponding components of the Killing
vectors of the scalar manifold, given in Eq. (3.18) (Eq. (3.17)).

According to the general prescription Eq. (3.32), we want to gauge these symmetries
making the theory invariant under the local transformations

$$
\delta_\Lambda \varphi = \Lambda^I \varphi I A k_A \varphi, \quad \delta_\Lambda \tau = \Lambda^I \varphi I A k_A \tau,
$$

where $\Lambda^I(x)$, $I = 0, 1, 2$, are the 0-form gauge parameters of the 1-form gauge fields
$A^I$ and $\varphi I A$ is the embedding tensor.

To construct gauge-covariant field strengths for the scalars it is enough to replace
their derivatives by covariant derivatives.

Covariant derivatives

The covariant derivatives of the scalars have the standard form

$$
\mathcal{D} \varphi = d \varphi + A^I \varphi I A k_A \varphi, \quad \mathcal{D} \tau = d \tau + A^I \varphi I A k_A \tau,
$$

and they transform covariantly provided that the 1-form gauge fields transform as

$$
\delta_\Lambda A^I = -\mathcal{D} \Lambda^I + Z^I, \Lambda^i,
$$

where the $A^i$, $i = 1, 2$, are two possible 1-form gauge parameters and $Z^I$ is a possible
new deformation parameter that must satisfy the orthogonality constraint

$$
\varphi I A Z^I = 0.
$$

[^10]: As we are going to see, besides the embedding tensor, one can introduce many other deformation
tensors.
Furthermore, it is necessary that the embedding tensor satisfies the standard quadratic constraint

\[ \partial_I^A T_{AJ}^K \partial_K^C - \partial_I^A \partial_J^B f_{AB}^C = 0, \]  

(3.38)

that expresses the gauge-invariance of the embedding tensor.

As a general rule, all the deformation tensors have to be gauge-invariant and we can anticipate that we will have to impose the constraint that expresses the gauge-invariance of \( Z_{Ii} \), namely

\[ X_{JK}^I Z_{K_i}^i - X_{Ji}^j Z_{I j}^j = 0, \]  

(3.39)

where

\[ X_{IJ}^K \equiv \partial_I^A T_{AJ}^K, \quad X_{Ji}^j \equiv \partial_J^A T_{Ai}^j. \]  

(3.40)

Supersymmetry transformations of the fermion fields

We will assume for simplicity that the supersymmetry transformations of the fermion fields in the deformed theory have essentially the same form as in the undeformed theory but covariantized (derivatives and field strengths) and, possibly, with the addition of fermion shifts which we add in the most general form:

\[ \delta \epsilon \psi_\mu = \mathcal{D}_\mu \epsilon + f \gamma_\mu \epsilon + k \gamma_\mu \epsilon^* + \frac{i}{2} \epsilon e^{-\frac{2}{\sqrt{7}} \sqrt{\varphi} + \frac{1}{2} \phi} \left( \frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu \right) F^0 \epsilon 
\]

\[ - \frac{1}{8 \sqrt{2}} e^{\sqrt{\varphi} + \frac{1}{2} \phi} \left( \frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu \right) (F^1 - \tau F^2) \epsilon^* 
\]

\[ - \frac{i}{8} e^{-\frac{1}{3} \sqrt{\varphi}} f_{AB} \gamma_\mu \left( \frac{3}{7} \gamma_\mu \gamma^{(3)} + \gamma^{(3)} \gamma_\mu \right) (H^1 - \tau H^2) \epsilon^* 
\]

\[ - \frac{1}{4} e^{\frac{1}{2} \sqrt{\varphi}} \left( \frac{1}{7} \gamma_\mu \gamma^{(4)} - \gamma^{(4)} \gamma_\mu \right) G \epsilon, \]  

(3.41)

\[ \delta \epsilon \bar{\lambda} = i \mathcal{D}_\mu \varphi \epsilon^* + \bar{g} \epsilon + \bar{h} \epsilon^* - \frac{1}{\sqrt{7}} e^{-\frac{2}{\sqrt{7}} \sqrt{\varphi}} F^0 \epsilon^* - \frac{3i}{2 \sqrt{2} \sqrt{7}} e^{\frac{3}{2} \sqrt{\varphi} \frac{1}{2} \phi} (F^1 - \tau F^2) \epsilon 
\]

\[ - \frac{1}{2 \sqrt{3} \sqrt{7}} e^{-\frac{1}{2 \sqrt{3}} \sqrt{\varphi} + \frac{1}{2} \phi} (H^1 - \tau^* H^2) \epsilon - \frac{i}{4 \sqrt{7}} e^{\frac{1}{2} \sqrt{\varphi}} G \epsilon^*, \]  

(3.42)

\[ \delta \epsilon \lambda = - e^\phi \mathcal{D}_\mu \epsilon^* + e \epsilon + h \epsilon^* - \frac{1}{2 \sqrt{2}} e^{\frac{3}{2} \sqrt{\varphi} + \frac{1}{2} \phi} (F^1 - \tau F^2) \epsilon 
\]

\[ + \frac{1}{2 \sqrt{3}} e^{-\frac{1}{2 \sqrt{3}} \sqrt{\varphi} + \frac{1}{2} \phi} (H^1 - \tau H^2) \epsilon. \]  

(3.43)
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In these expressions, $f, k, g, h, \tilde{g}, \tilde{h}$ are six functions of the scalars and deformation parameters to be determined, the covariant field strengths have the general form predicted by the tensor hierarchy (to be determined) and the covariant derivatives of the scalars have the forms given above. Furthermore, in $\delta \psi_\mu$, $\mathcal{D}_\mu \epsilon$ stands for the Lorentz- and gauge-covariant derivative of the supersymmetry parameter, which turns out to be given by

$$\mathcal{D}_\mu \epsilon \equiv \{ \nabla_\mu + \frac{i}{2} \left[ \frac{1}{2} e^\phi \mathcal{D}_\mu^5 \chi + A^I_\mu \partial_I m P_m \right] + \frac{9}{14} \gamma_\mu A^I_\mu \partial_I \gamma^4 \} \epsilon \tag{3.44}$$

where $P_m$ are the momentum maps of the holomorphic Killing vectors of $SL(2, \mathbb{R})$, defined in Eq. (3.16) and given in Eq. (3.20), $\nabla_\mu$ is the Lorentz-covariant derivative and

$$\mathcal{D}_\mu^5 \chi \equiv \partial_\mu \chi - \frac{3}{4} A^I_\mu \partial_I \gamma^5 \chi \tag{3.45}$$

is the derivative of $\chi$ covariant only with respect to the $\beta$ rescalings. It can be checked that $\mathcal{D}_\mu \epsilon$ transforms covariantly under gauge transformations if and only if the embedding tensor satisfies the standard quadratic constraint Eq. (3.38).

An equivalent expression for it is

$$\mathcal{D}_\mu \epsilon = \{ \nabla_\mu + \frac{i}{2} \left[ \frac{1}{2} e^\phi \mathcal{D}_\mu^5 \chi - A^I_\mu \partial_I m \gamma_m \right] + \frac{9}{14} \gamma_\mu A^I_\mu \partial_I \gamma^4 \} \epsilon \tag{3.46}$$

where $\gamma_m$, $m = 1, 2, 3$, of $SL(2, \mathbb{R})$ and defined in Eq. (3.16) and given in Eq. (3.19) and where now

$$\mathcal{D}_\mu \chi \equiv \partial_\mu \chi + A^I_\mu \partial_I A^k k_A \chi \tag{3.47}$$

is the total covariant derivative of $\chi$ (which is invariant under both the $\alpha$ and $\beta$ scaling symmetries as well as under $SL(2, \mathbb{R})$).

The actual form of the $(p + 1)$-form field strengths will not be needed until the moment in which study the closure of the supersymmetry algebra on the corresponding $p$-form potential.

**Closure of the supersymmetry algebra on the 0-forms $\varphi, \tau$**

We assume that the supersymmetry transformations of the scalars are the same as in the undeformed theory

$$\delta_\epsilon \varphi = -\frac{i}{4} \bar{\epsilon} \gamma^* \lambda + \text{h.c.} \tag{3.48}$$

$$\delta_\epsilon \tau = -\frac{1}{2} e^{-\phi} \bar{\epsilon} \gamma^* \lambda \tag{3.49}$$

To lowest order in fermions, the commutator of two supersymmetry transformations gives
\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \varphi = \xi^\mu \mathcal{D}_\mu \varphi + \text{Re}(\tilde{h}) b - \text{Im}(\tilde{g}) c + \text{Re}(\tilde{g}) d, \tag{3.50} \]

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \tau = \xi^\mu \mathcal{D}_\mu \tau + e^{-\phi} [g(c - id) - i hb], \tag{3.51} \]

where \( \xi^\mu \) is one of the spinor bilinears defined in Appendix B.1.1 that clearly plays the role of parameter of the general coordinate transformations and \( a, b, c, d \) are the scalar bilinears defined in the same appendix.

In the right hand side of these commutators, to lowest order in fermions, we expect a general coordinate transformation (the Lie derivative \( \mathcal{L}_\xi \) of the scalars with respect to \( \xi^\mu \)) and a gauge transformation which has the form of Eq. (3.34) for the scalars. Therefore, the above expressions should be compared with

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \varphi = \mathcal{L}_\xi \varphi + \Lambda^I \partial_I A^A \varphi, \tag{3.52} \]

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] \tau = \mathcal{L}_\xi \tau + \Lambda^I \partial_I A^A \tau, \tag{3.53} \]

from which we get the relations

\[ \text{Re}(\tilde{h}) b - \text{Im}(\tilde{g}) c + \text{Re}(\tilde{g}) d = (\Lambda^I - a^I) \partial_I A^A \varphi, \tag{3.54} \]

\[ g(c - id) - i hb = e^\phi (\Lambda^I - a^I) \partial_I A^A \tau, \tag{3.55} \]

which would allow us to determine the fermion shift functions if we knew the gauge parameters \( \Lambda^I \). In order to determine the \( \Lambda^I \)'s we have to close the supersymmetry algebra on the 1-forms. In these expressions and in those that will follow, we use the shorthand notation

\[ a^I \equiv \xi^\mu A^I \mu, \quad b^i_\mu \equiv \xi^\nu B^i_\nu \mu, \quad c_{\mu\nu} \equiv \xi^\rho C_{\rho\mu\nu}, \quad \text{etc.} \tag{3.56} \]

### 3.3.2 The 1-forms \( A^I \)

The next step in this procedure is to consider the 1-forms that we just introduced to construct covariant derivatives for the scalars.

#### The 2-form field strengths \( F^I \)

The gauge transformations of the 1-forms are given in Eq. (3.36) and we first need to determine their covariant field strengths. A general result of the embedding-tensor formalism tells us that we need to introduce 2-form potentials in the covariant field
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strengths. In this case only have the $SL(2, \mathbb{R})$ doublet $B^i$ at our disposal and, therefore, the 2-form field strengths have the form

$$F^I = dA^I + \frac{1}{2} X_{JK}^I A^J \wedge A^K + Z^I_i B^i,$$

(3.57)

where $X_{JK}^I$ has been defined in Eq. (3.40) and $Z^I_i$ is precisely the deformation tensor we introduced in Eq. (3.36). $F^I$ will transform covariantly under Eq. (3.36) if simultaneously the 2-forms $B^i$ transform according to

$$\delta \Lambda B^i = -D \Lambda i - 2h_{IJ}^i \left[ \Lambda^I F^J + \frac{1}{2} A^I \wedge \delta \Lambda A^J \right] + Z^I_i \Lambda,$$

(3.58)

where $h_{IJ}^i$ and $Z^i$ are two possible new deformation tensors the first of which must satisfy the constraint

$$X_{(JK)}^I + Z^I_i h_{JK} = 0,$$

(3.59)

while $Z^i$ must satisfy the orthogonality constraint

$$Z^I_i Z^i = 0.$$

(3.60)

Both of them must satisfy the constraints that express their gauge invariance:

$$X_{Ij}^i h_{JK}^j - 2X_{I(L} h_{K)j}^i = 0,$$

(3.61)

$$X_I Z^i - X_{Ij}^i Z^j = 0,$$

(3.62)

where

$$X_I \equiv \partial I A^{(1)}_A.$$

(3.63)

Closure of the supersymmetry algebra on the 1-forms $A^I$

We assume, as we are doing with all the bosons, that the supersymmetry transformations of the 1-forms of the theory are not deformed by the gauging, so they take the form

$$\delta_{\epsilon} A^0_\mu = \frac{i}{2} \epsilon^{\mu} \gamma^\phi \epsilon \left( \psi_\mu - \frac{i}{\sqrt{7}} \gamma_\mu \tilde{\lambda}^* \right) + \text{h.c.},$$

(3.64)

$$\delta_{\epsilon} A^1_\mu = \frac{i}{2} \epsilon^* \gamma^\phi \epsilon \left( \bar{\psi}_\mu - \frac{i}{\sqrt{7}} \gamma_\mu \lambda^* \right) + \text{h.c.},$$

(3.65)

$$\delta_{\epsilon} A^2_\mu = \frac{i}{2} \epsilon^{\mu} \gamma^\phi \epsilon \left( \bar{\psi}_\mu - \frac{i}{\sqrt{7}} \gamma_\mu \lambda^* \right) + \text{h.c.}$$

(3.66)

The commutator of two of them gives, to lowest order in fermions,
\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^0_{\mu} = \xi^\nu F^0_{\nu \mu} - \mathcal{D}_\mu \left( e^{\frac{2}{\sqrt{7}} \varphi} b \right) + \frac{2}{\sqrt{7}} e^{\frac{2}{\sqrt{7}} \varphi} \left\{ \left[ \Re(\tilde{h}) - \sqrt{7} \Im(f) \right] \xi_\mu \right. \\
+ \left. \left[ \Re(\tilde{g}) - \sqrt{7} \Im(k) \right] \sigma_\mu + \left[ \Im(\tilde{g}) - \sqrt{7} \Re(k) \right] \rho_\mu \right\}, \tag{3.67} \]

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^1_{\mu} = \xi^\nu F^1_{\nu \mu} - \partial_\mu \left( e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} (\chi d + e^{-\phi} c) \right) \]
\[ -A^1_\mu \left[ \frac{1}{2}(\partial_1^2 - \frac{2}{3} \partial_1^3) e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} (\chi d + e^{-\phi} c) + \frac{1}{2}(\partial_1^2 + \partial_1^3) e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} d \right] \]
\[ -2e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} \left\{ e^{\phi} \left[ \Im(m(k)) + \frac{3}{4\sqrt{7}} \Re(g) - \frac{1}{4} \Re(h) \right] + e^{-\phi} \left[ -\Re(f) - \frac{3}{4\sqrt{7}} \Im(m(g)) - \frac{1}{4} \Im(m(g)) \right] \right\} \xi_\mu \]
\[ -2e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} \left\{ e^{\phi} \left[ -\Re(f) - \frac{3}{4\sqrt{7}} \Im(m(g)) - \frac{1}{4} \Im(m(g)) \right] + e^{-\phi} \left[ \Im(m(f)) + \frac{3}{4\sqrt{7}} \Re(h) + \frac{1}{4} \Re(h) \right] \right\} \sigma_\mu, \tag{3.68} \]

and

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^2_{\mu} = \xi^\nu F^2_{\nu \mu} - \partial_\mu \left( e^{\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} d \right) \]
\[ -A^2_\mu \left[ \frac{1}{2}(\partial_1^2 - \partial_1^3) e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} (\chi d + e^{-\phi} c) - \frac{1}{2} \partial_1^4 e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} d \right] \]
\[ -2e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} \left\{ \Im(m(k)) + \frac{3}{4\sqrt{7}} \Re(g) - \frac{1}{4} \Re(h) \right\} \xi_\mu \]
\[ -2e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} \left\{ -\Re(f) - \frac{3}{4\sqrt{7}} \Im(m(g)) + \frac{1}{4} \Im(m(g)) \right\} \rho_\mu \]
\[ -2e^{-\frac{3}{2\sqrt{7}} \varphi + \frac{1}{2} \delta} \left\{ \Im(m(f)) + \frac{3}{4\sqrt{7}} \Re(h) - \frac{1}{4} \Re(h) \right\} \sigma_\mu, \tag{3.69} \]

where \( \sigma_\mu \) and \( \rho_\mu \) are spinor bilinears defined in Appendix B.1.1.

The closure of the local supersymmetry algebra requires the commutators to take the form

\[ [\delta_{\epsilon_1}, \delta_{\epsilon_2}] A'_\mu = \mathcal{L}_\xi A'_\mu - \mathcal{D}_\mu A'_\mu + Z'_I A'_\mu, \tag{3.70} \]

which will only happen if gauge parameters \( A'_\mu \) are given by
\[ \Lambda^0 = a^0 + e^{\frac{2}{3} \varphi} b, \]
\[ \Lambda^1 = a^1 + e^{-\frac{3}{4} \varphi + \frac{1}{2} \phi} (\chi d + e^{-\phi} c), \] (3.71)
\[ \Lambda^2 = a^2 + e^{-\frac{3}{4} \varphi + \frac{1}{2} \phi} d, \]

and the 1-form gauge parameters \( \Lambda^i_{\mu} \) satisfy the relations

\[
\left[ \text{Re}(\tilde{h}) - \sqrt{7} \text{Im}(f) \right] \xi_\mu + \left[ \text{Re}(\tilde{g}) - \sqrt{7} \text{Im}(k) \right] \sigma_\mu + \left[ \text{Im}(\tilde{g}) - \sqrt{7} \text{Re}(k) \right] \rho_\mu \\
= \frac{\sqrt{7}}{2} e^{-\frac{2}{3} \varphi} Z^0_i \left[ \Lambda^i_{\mu} - (b_{i}^\mu - h_{IJ} d^I A^J_{\mu}) \right] (3.72)
\]

\[
\left\{ \chi \left[ \text{Im}(k) + \frac{3}{4} \sqrt{7} \text{Re}(\tilde{g}) - \frac{1}{4} \text{Re}(\tilde{g}) \right] + e^{-\phi} \left[ -\text{Re}(k) - \frac{3}{4} \sqrt{7} \text{Im}(\tilde{g}) - \frac{1}{4} \text{Im}(\tilde{g}) \right] \right\} \xi_\mu \]
\[ + \left\{ \chi \left[ -\text{Re}(f) - \frac{3}{4} \sqrt{7} \text{Im}(\tilde{h}) + \frac{1}{4} \text{Im}(\tilde{h}) \right] + e^{-\phi} \left[ -\text{Im}(f) - \frac{3}{4} \sqrt{7} \text{Re}(\tilde{h}) - \frac{1}{4} \text{Re}(\tilde{h}) \right] \right\} \rho_\mu \]
\[ + \left\{ \chi \left[ \text{Im}(f) + \frac{3}{4} \sqrt{7} \text{Re}(\tilde{h}) - \frac{1}{4} \text{Re}(\tilde{h}) \right] + e^{-\phi} \left[ -\text{Re}(f) - \frac{3}{4} \sqrt{7} \text{Im}(\tilde{h}) - \frac{1}{4} \text{Im}(\tilde{h}) \right] \right\} \sigma_\mu , \]
\[ = -\frac{1}{2} e^{\frac{3}{4} \varphi^2 - \frac{1}{2} \phi} Z^1_i \left[ \Lambda^i_{\mu} - (b_{i}^\mu - h_{IJ} d^I A^J_{\mu}) \right] (3.73) \]

\[
\left[ \text{Im}(k) + \frac{3}{4} \sqrt{7} \text{Re}(\tilde{g}) - \frac{1}{4} \text{Re}(\tilde{g}) \right] \xi_\mu + \left[ -\text{Re}(f) - \frac{3}{4} \sqrt{7} \text{Im}(\tilde{h}) + \frac{1}{4} \text{Im}(\tilde{h}) \right] \rho_\mu \\
+ \left[ \text{Im}(f) + \frac{3}{4} \sqrt{7} \text{Re}(\tilde{h}) - \frac{1}{4} \text{Re}(\tilde{h}) \right] \sigma_\mu , \]
\[ = -\frac{1}{2} e^{\frac{3}{4} \varphi^2 - \frac{1}{2} \phi} Z^2_i \left[ \Lambda^i_{\mu} - (b_{i}^\mu - h_{IJ} d^I A^J_{\mu}) \right] (3.74) \]

Using the values of the parameters \( \Lambda^i \) that we just have determined in the relations Eqs. (3.54) and (3.55) we can determine some of the fermions shifts:

\[ \Re(\tilde{h}) = \varphi_0 A k_A \varphi e^{\frac{2}{3} \varphi} , \] (3.75)
\[ \tilde{g} = (\varphi_1 A \tau^* + \varphi_2 A) k_A \varphi e^{-\frac{3}{4} \varphi + \frac{1}{2} \phi} , \] (3.76)
\[ h = i \varphi_0 A k_A \varphi e^{\frac{2}{3} \varphi + \phi} , \] (3.77)
\[ g = \varphi_1 A k_A \varphi e^{-\frac{3}{4} \varphi + \frac{1}{2} \phi} . \] (3.78)
As a matter of fact, \( g \) is overdetermined: we get two different expression for it that give the same value if and only if

\[
(\vartheta_1^A \tau + \vartheta_2^A)k_A^\tau = 0,
\]

which, upon use of the explicit expressions of the holomorphic Killing vectors \( k_A^\tau \) in Section 3.2.1, leads to the following linear constraints on the components of the embedding tensor:

\[
\begin{align*}
\vartheta_2^2 + \vartheta_2^3 &= 0, \\
\vartheta_1^2 + \vartheta_1^3 + 2\vartheta_2^1 - \frac{3}{2}\vartheta_2^5 &= 0, \\
\vartheta_2^2 - \vartheta_2^3 - 2\vartheta_1^1 + \frac{3}{2}\vartheta_2^5 &= 0, \\
\vartheta_1^2 - \vartheta_1^3 &= 0.
\end{align*}
\]

These constraints allow us to express 4 of the 15 components of the embedding tensor in terms of the remaining 11, but we are only going to do this after we take into account the constraints that we are going to find in the closure of the local supersymmetry algebra on the doublet of 2-forms \( B^i \).

The values of \( g, h, \tilde{g}, \tilde{h} \) and the above constraints are compatible with those of the primary deformations found in Ref. [106].

### 3.3.3 The 2-forms \( B^i \)

In the previous subsection we have introduced a doublet of 2-forms \( B^i \) with given gauge transformations to construct the 2-form field strengths \( F^I \). We now have to construct their covariant field strengths and check the closure of the local supersymmetry algebra on them.

#### The 3-form field strengths \( H^i \)

In general we need to introduce 3-form potentials to construct the covariant 3-form field strengths and, since in maximal 9-dimensional supergravity, we only have \( C \) at our disposal, the 3-form field strengths will be given by

\[
H^i = \mathcal{D}B^i - h_{ij} A^i \wedge dA^j - \frac{1}{6} X_{[ij}^k h_{k]} A^i A^j A^k + Z C,
\]

and they transform covariantly under the gauge transformations of the 1- and 2-forms that we have previously determined provided if the 3-form \( C \) transforms as

\[
\delta_\Lambda C = -\mathcal{D}C + g_{Ii} \left[ -\Lambda^I H^i - F^I \wedge \Lambda^i + \delta_\Lambda A^I \wedge B^i - \frac{1}{3} h_{JK} A^{IJK} \wedge \delta_\Lambda A^K \right] + Z \Lambda. \tag{3.82}
\]
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where $g_{Ii}$ and $Z$ are two possible new deformation parameters. $g_{Ii}$ must satisfy the constraint

$$2h_{IJ} J^J + X_{Ij}^i + Z^i g_{IJ} = 0 ,$$

(3.83)

while $Z$ must satisfy the orthogonality constraint

$$Z^i Z = 0 .$$

(3.84)

Both must by gauge-invariant, which implies the constraints

$$X_{IJ} g_{L} + X_{IJ} g_{JL} - X_{IJ} = 0 ,$$

(3.85)

$$Z^i g_{IJ} = 0 ,$$

(3.86)

where

$$\tilde{X}_I \equiv \vartheta I^A T^{(1)}_A .$$

(3.87)

Using the constraints obeyed by the deformation parameters and the explicit form of the 2-form field strengths $F^I$ we can rewrite the 3-form field strengths in the useful form

$$H^i = \mathcal{D} B^i - h_{IJ} A^I \wedge F^J + \frac{1}{6} X_{[IJ} g_{KL] A^{IJ}} - \frac{1}{2} X_{Ij} A^j \wedge B^i + Z^i (C - \frac{1}{2} g_{IJ} A^I \wedge B^J) .$$

(3.88)

Closure of the supersymmetry algebra on the 2-forms $B^i$

In the undeformed theory, the supersymmetry transformation rules for the 2-forms are

$$\delta \epsilon B^1 = \tau^* \epsilon^{\frac{1}{2}\gamma} + \frac{1}{2} \phi \left[ \epsilon^{\gamma} \gamma_{[\mu} [\sigma] - \frac{1}{8} \epsilon^{\gamma} \gamma_\mu \bar{\lambda}, \right]$$

$$- \delta^1_1 \left( A^0_{[\mu] \delta \epsilon A^1_{[\nu]} + A^1_{[\mu] \delta \epsilon A^0_{[\nu]}} \right) + \text{h.c. ,}$$

(3.89)

$$\delta \epsilon B^2 = \epsilon^{\frac{1}{2}\gamma} + \frac{1}{2} \phi \left[ \epsilon^{\gamma} \gamma_{[\mu} [\sigma] - \frac{1}{8} \epsilon^{\gamma} \gamma_\mu \bar{\lambda}, \right]$$

$$- \delta^2_1 \left( A^0_{[\mu] \delta \epsilon A^1_{[\nu]} + A^1_{[\mu] \delta \epsilon A^0_{[\nu]}} \right) + \text{h.c. .}$$

(3.90)

The last terms in both transformations are associated to the presence of derivatives of $A^1$ and $A^2$ in the field strengths of $B^1$ and $B^2$ in the undeformed theory (see Eq. (1.49)).
In the deformed theory, the terms \(- (A^0 \wedge dA^1 + A^1 \wedge dA^0)\) are replaced by more general couplings \(- h_{IJ}^i A^I \wedge dA^J\) and, therefore, it would be natural to replace the last terms in \(\delta \epsilon_i B_{\mu \nu}\) by

\[- 2h_{IJ}^i A^I_{[\mu} \delta \epsilon A^J_{\nu]} \cdot (3.91)\]

In the commutator of two supersymmetry transformations on the 2-forms, these terms give the right contributions to the terms \(-2h_{IJ}^i A^I F^J\) of the gauge transformations (see Eq. (3.58)). However, these terms must receive other contributions in order to be complete and it turns out that the only terms of the form \(-2h_{IJ}^i A^I F^J\) that can be completed are precisely those of the undeformed theory, which correspond to

\[h_{0j}^j = - \frac{1}{2} \delta_{i}^j \cdot (3.92)\]

In order to get more general \(h_{IJ}^i\)'s it would be necessary to deform the fermions' supersymmetry rules, something we will not do here. Furthermore, the structure of the Chern-Simons terms of the field strengths is usually determined by the closure of the supersymmetry algebra at higher orders in fermions and it is highly unlikely that a more general structure of the Chern-Simons terms will be allowed by supersymmetry. Therefore, from now on, we will set \(h_{IJ}^i\) to the above value and we will set the values of the deformation tensors in the Chern-Simons terms of the higher-rank field strengths, to the values of the undeformed theory. Using the above value of \(h_{IJ}^i\) in the constraints in which it occurs will help us to solve them, sometimes completely, as we will see. Nevertheless, we will keep using the notation \(h_{IJ}^i\) for convenience.

Using the identity

\[\xi^p H^i_{\rho \mu \nu} - 2h_{IJ}^i A^I_{\mu} \mathcal{L}_\xi A^J_{\nu} = \mathcal{L}_\xi B^i_{\mu \nu} - 2\Omega_{[\mu}(b^i_{\nu]} - h_{IJ}^i a^I A^J_{[\nu]}))\]

\[- 2h_{IJ}^i a^I F^J_{\mu \nu}\]

\[+ Z^i \left( c_{\mu \nu} - g_{1j} a^I B^j_{\mu \nu} + \frac{2}{3} g_{1j} h_{1K}^j a^I A^K_{\mu \nu} \right), (3.93)\]

we find that the local supersymmetry algebra closes on the \(B^i\)'s in the expected form (to lowest order in fermions)

\[[\delta_{c1}, \delta_{c2}] B^i_{\mu \nu} = \mathcal{L}_\xi B^i_{\mu \nu} + \delta_\Lambda B^i_{\mu \nu}, (3.94)\]

where \(\delta_\Lambda B^i_{\mu \nu}\) is the gauge transformation given in Eq. (3.58) in which the 0-form gauge parameters \(\Lambda^I\) are as in Eqs. (3.71), the 1-form gauge parameters \(\Lambda^i_{\mu}\) are given by

\[\Lambda^i_{\mu} = \lambda^i_{\mu} + b^i_{\mu} - h_{IJ}^i a^I A^J_{\mu}, (3.95)\]

where
3.3. Deforming the maximal $d = 9$ supergravity

\[
\lambda^1_\mu \equiv e^{\frac{1}{\sqrt{2}} \varphi + \frac{1}{2} \phi} (\chi \sigma_\mu - e^{-\phi} \rho_\mu),
\]

(3.96)

and the shift term is given by

\[
Z^1 [\Lambda_{\mu\nu} - (c_{\mu\nu} - g_{IJi} B^i_{\mu\nu} + \frac{2}{3} g_{JJ} h_{IK} a^I A^{JK}_{\mu\nu})] = e^{\frac{1}{\sqrt{2}} \varphi + \frac{1}{2} \phi} \left( \left( \frac{1}{2} \Im(g) - 4 \Re(k) + \frac{1}{2 \sqrt{7}} \Im(\tilde{g}) \right) \chi \right.
\]

\[- \left( \frac{1}{2} \Re(g) + 4 \Im(m) - \frac{1}{2 \sqrt{7}} \Re(\tilde{g}) \right) e^{-\phi} \right] \xi_{\mu\nu},
\]

(3.97)

\[
Z^2 [\Lambda_{\mu\nu} - (c_{\mu\nu} - g_{IJi} B^i_{\mu\nu} - \frac{2}{3} g_{JJ} h_{IK} a^I A^{JK}_{\mu\nu})] = e^{\frac{1}{\sqrt{2}} \varphi + \frac{1}{2} \phi} \left( \left( \frac{1}{2} \Im(g) - 4 \Re(k) + \frac{1}{2 \sqrt{7}} \Im(\tilde{g}) \right) \xi_{\mu\nu} \right).
\]

(3.98)

Now, let us analyze the constraints that involve $h_{IJi}$. From those that only involve the embedding tensor we find seven linear constraints that imply those in Eqs. (3.80) and that can be used to eliminate seven components of the embedding tensor:

\[
\vartheta_2^1 = 0, \quad \vartheta_1^2 = \frac{3}{4} \vartheta_2^5, \quad \vartheta_1^3 = \frac{3}{4} \vartheta_2^5,
\]

\[
\vartheta_0^4 = -\frac{1}{6} \vartheta_0^5,
\]

(3.99)

leaving the eight components (a triplet of $SL(2, \mathbb{R})$ in the upper component, a singlet and two doublets of $SL(2, \mathbb{R})$ in the lower components)

\[
\vartheta_0^m, \quad m = 1, 2, 3, \quad \vartheta_0^5, \quad \vartheta_1^4, \quad \vartheta_1^5, \quad \mathbf{i} = 1, 2,
\]

(3.100)

as the only independent ones. These components correspond to the eight deformation parameters of the primary deformations studied in Ref. [106]. More precisely, the relation between them are

\[
\vartheta_0^m = m_m, \quad (m = 1, 2, 3) \quad \vartheta_1^4 = -m_1, \quad \vartheta_1^5 = \tilde{m}_4,
\]

\[
\vartheta_0^5 = -\frac{16}{3} m_{\text{IB}}, \quad \vartheta_2^4 = m_{\text{IA}}, \quad \vartheta_2^5 = m_4.
\]

(3.101)

From the constraints that relate $h_{IJi}$ to $Z^i$, $Z^i_{\mu}$ and $g_{IJ}$ we can determine all these tensors, up to a constant $\zeta$, in terms of the independent components of the embedding tensor:
\[Z^I_j = \partial_0^m (T_m)^I_j - \frac{3}{4} \partial_0^5 \delta^I_j \delta^I_1, \quad Z^0_i = 3 \partial_1^i + \frac{1}{2} \partial_i^5,\]  
(3.102)

\[g_{0i} = 0, \quad g_{ij} = \varepsilon_{ij}.\]

The constant \(\zeta\) is the coefficient of a Chern-Simons term in the 4-form field strength and, therefore, will be completely determined by supersymmetry.

Finally, using all these results in Eqs. (3.72-3.74) we find

\[k = -\frac{9}{11} e^{-\frac{3}{2\sqrt{7}} \phi + \frac{1}{2} \phi}(\vartheta^1 4 \tau + \vartheta^2 4),\]  
(3.103)

\[\Im m(f) = \frac{3}{28} \vartheta^0 5 e^{\frac{2}{\sqrt{7}} \varphi},\]  
(3.104)

\[\Re e(f) + \frac{3}{4\sqrt{7}} \Im m(\tilde{h}) = \frac{1}{4} e^{\frac{2}{\sqrt{7}} \varphi + \phi} \left\{ \left( \frac{1}{2} (\vartheta_0^2 + \vartheta_0^3) + (\vartheta_0^1 - \frac{3}{4} \vartheta_0^5) \chi \right) \right\},\]  
(3.105)

which determines almost completely all the fermion shifts. We find that, in order to determine completely \(\Re e(f)\) and \(\Im m(\tilde{h})\), separately, one must study the closure of the supersymmetry algebra on the fermions of the theory or on the bosons at higher order in fermions. The result is

\[\Re e(f) = \frac{1}{4} e^{\frac{2}{\sqrt{7}} \varphi} \vartheta_0^m P_m,\]  
(3.106)

\[\Im m(\tilde{h}) = \frac{4}{7} e^{\frac{2}{\sqrt{7}} \varphi} \vartheta_0^m P_m.\]  
(3.107)

All these results are collected in Appendix B.4.

### 3.3.4 The 3-form \(C\)

In the next step we are going to consider the last of the fundamental, electric \(p\)-forms of the theory, the 3-form \(C\), whose gauge transformation is given in Eq. (3.82).

#### The 4-form field strength \(G\)

The 4-form field strength \(G\) is given by

\[G = \mathcal{D} C - g_{Ij} \left( F^I - \frac{1}{2} Z^I_j B_j \right) \wedge B^i - \frac{1}{3} h_{IK} g_{Ij} A^{Ij} \wedge dA^K + Z \tilde{C},\]  
(3.108)

and it is covariant under general gauge transformations provided that the 4-form \(\tilde{C}\) transforms as
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\[
\delta \tilde{\Lambda} \tilde{C} = -\mathfrak{D} \tilde{\Lambda} - \tilde{g}_I [\Lambda^I G + C \wedge \delta \Lambda A^I + F^I \wedge \Lambda + \frac{1}{12} g_{ij} h_{KL}^J A^{IJK} \wedge \delta \Lambda A^J] \\
- \tilde{g}_{ij} [2H^i \wedge \Lambda^j - B^i \wedge \delta \Lambda B^j + 2h_{Ij} A^j \wedge \Lambda + \delta \Lambda A^j] \\
- \tilde{g}_{IJK} [3\Lambda^I F^{JK} + 2(F^I - Z^I B^i) \wedge A^j \wedge \delta \Lambda A^K - \frac{1}{4} X_{LM}^J A^{ILM} \wedge \delta \Lambda A^K] \\
+ Z^i \tilde{\Lambda}_i, \tag{3.109}
\]

where the new deformation tensors that we have introduced, \( \tilde{g}_I, \tilde{g}_{ij} = -\tilde{g}_{ji} \) and \( \tilde{g}_{IJK} = \tilde{g}_{(IJK)} \), are subject to the constraints

\[
g_{I[i} Z^I_{j]} + Z \tilde{g}_{ij} = 0, \tag{3.110}
\]

\[
X^I + g_{Ii} Z^i + Z \tilde{g}_I = 0, \tag{3.111}
\]

\[
h_{(IJ}^i g_{K)i} - Z \tilde{g}_{IJK} = 0, \tag{3.112}
\]

plus the constraints that express the gauge invariance of the new deformation parameters

\[
\tilde{X}^I \tilde{g}_{IJ} - X^I_{IJ} \tilde{g}_{K} = 0, \tag{3.113}
\]

\[
\tilde{X}^I \tilde{g}_{ij} - 2X^I_{[i} \tilde{g}^i_{j]} = 0, \tag{3.114}
\]

\[
\tilde{X}^I \tilde{g}_{JK} - 3X^I_{(J} \tilde{g}^i_{KL)} = 0 . \tag{3.115}
\]

**Closure of the supersymmetry algebra on the 3-form \( C \)**

Taking into account the form of \( \delta \epsilon C_{\mu \nu \rho} \) in the undeformed case and the form of the field strength \( G \), we arrive at the following Ansatz for the supersymmetry transformation of the 3-form \( C \):

\[
\delta \epsilon C_{\mu \nu \rho} = -\frac{3}{2} e^{-\frac{1}{\sqrt{7}} \phi} \epsilon_{\mu \nu} \left( \psi_\rho + \frac{1}{\sqrt{7}} \tilde{\lambda} \right) + \text{h.c.} + 3 \delta \epsilon A_{[\mu}^I (g_{Ii} B^i_{\nu \rho} + \frac{2}{3} h_{IJ}^i g_{K} A^{JK} |_{\nu \rho} ) . \tag{3.116}
\]

The last two terms are written in terms of the tensors \( g_{Ii} \) and \( h_{IJ}^i \). In the undeformed theory these tensors have values which are determined by supersymmetry (at orders in fermions higher than we are considering here) and that cannot be changed in the
deformed theory, as we already discussed when we considered the 2-forms for $h_{IJ}$. Thus, $h_{IJ}$ is given by Eq. (3.92) and $g_{Ii}$ is given by Eqs. (3.102) with $\zeta = +1$.

Using the identity

$$
\xi^\nu G_{\nu\rho} + 3\mathcal{L}A^{I}_{[\mu} \left[ g_{Ii}B^{i}_{|\nu]} + \frac{2}{3} h_{IJ} g_{KI} A^{JK}_{|\nu]} \right] =
$$

$$
= \mathcal{L}_{\xi} C_{\mu\nu} - 3\mathfrak{D}_{[\mu} \left[ (c_{|\nu]} - g_{IJ} a^{I} B^{J}_{|\nu]} + \frac{2}{3} g_{IJ} h_{IK} A^{JK}_{|\nu]} \right] 
+ g_{Ii} [-a^{I} H^{i}_{|\mu\rho} - 3F^{I}_{|\mu\nu}(b^{i}_{|\nu]} - h_{JK} a^{J} A^{K}_{|\rho}] 
+ Z \left\{ \bar{\epsilon}_{\mu\nu} - \bar{g}_{i} a^{I} C_{|\nu]} + 3\bar{g}_{ij} B^{i}_{|\nu]} (b^{i}_{|\nu] - h_{JK} a^{J} A^{K}_{|\rho]} - 12\bar{g}_{IKL} a^{I} A^{J[|\mu\nu]} 
- 3h_{IJ} \bar{g}_{ij} a^{I} A^{J[|\mu\nu]} + \frac{1}{4} \left( \bar{g}_{IL} g_{JK} h_{IJ} + 3X_{JK}^{M} \bar{g}_{ILM} a^{I} A^{JKL}_{|\mu\nu]} \right) \right\} ,
$$

(3.117)

one can see that the local supersymmetry algebra closes into a general coordinate transformation plus a gauge transformation of $C$ of the form Eq. (3.82) with

$$
\Lambda_{\mu\nu} = e^{\sqrt{\phi}} \xi_{\mu\nu} + \left( c_{\mu\nu} - g_{IJ} a^{I} B^{J}_{|\mu\nu} - \frac{2}{3} g_{IJ} h_{IK} A^{JK}_{|\mu\nu} \right) ,
$$

(3.118)

and with the identification

$$
Z \left\{ \tilde{\Lambda}_{\mu\nu} - \bar{\epsilon}_{\mu\nu} + \tilde{g}_{i} a^{I} C_{|\nu]} + 3\tilde{g}_{ij} B^{i}_{|\nu]} (b^{i}_{|\nu] - h_{JK} a^{J} A^{K}_{|\rho]} - 12\tilde{g}_{IKL} a^{I} A^{J[|\mu\nu]} 
- 3\tilde{g}_{ij} h_{IJ} a^{I} A^{J[|\mu\nu]} + \frac{1}{4} \left( \tilde{g}_{IL} g_{JK} h_{IJ} + 3\tilde{g}_{ILN} X_{JK}^{N} a^{I} A^{JKL}_{|\mu\nu]} \right) \right\} = 6e^{-\sqrt{\phi}} \left[ \Im(f) + \frac{1}{6\sqrt{\phi}} \Re(h) \right] \xi_{\mu\nu} ,
$$

(3.119)

Comparing Eq. (3.118) with Eqs. (3.97) and (3.98) we find that

$$
Z^{1} = X_{2} = 3\vartheta_{2}^{4} - \vartheta_{2}^{5} , \quad Z^{2} = -X_{1} = -3\vartheta_{1}^{4} + \vartheta_{1}^{5}.
$$

(3.120)

To make further progress it is convenient to compute the 5-form $\tilde{G}$ since it will contain the tensors $\tilde{g}_{ij}, \tilde{g}_{ij}, \tilde{g}_{IJ}$ that appear in the above expression. These tensors cannot be deformed (just as it happens with $h_{IJ}$) and their values can be found by comparing the general form of $\tilde{G}$ with the value found by duality, Eq. (3.24).

The generic form of the magnetic 5-form field strength $\tilde{G}$ is

$$
\tilde{G} = \mathcal{D} \tilde{C} - \tilde{g}_{i} \left( (F^{J} - Z^{J} B^{J}) \wedge C + \frac{1}{12} g_{Kij} h_{MN} A^{JMK} \wedge dA^{N} \right) + 2\tilde{g}_{ij} \left( H^{i} - \frac{1}{2} \mathcal{D} B^{i} \right) \wedge B^{j} - \tilde{g}_{JKL} \left( A^{J} \wedge dA^{KL} + \frac{3}{4} X_{MN} A^{JMN} \wedge dA^{K} \right) + Z^{j} \tilde{B}_{i} ,
$$

(3.121)
and comparing this generic expression with Eq. (3.24) we find that
\[
\tilde{g}_I = -\delta_I^0, \quad \tilde{g}_{ij} = -\frac{1}{2} \varepsilon_{ij}, \quad \tilde{g}_{IJK} = 0.
\] (3.122)

Plugging these values into the constraints that involve \( Z \) Eqs. (3.84),(3.86), and (3.110-3.112) we find that it must be related to \( \vartheta_0^5 \) by
\[
Z = -\frac{3}{4} \vartheta_0^5, \quad (3.123)
\]
and that \( \vartheta_0^5 \) must satisfy the two doublets of quadratic constraints
\[
\vartheta_1^4 \vartheta_0^5 = 0, \quad (3.124)
\]
\[
\vartheta_1^5 \vartheta_0^5 = 0. \quad (3.125)
\]

Plugging our results into all the other constraints between deformation tensors, we find that all of them are satisfied provided that the quadratic constraints
\[
\varepsilon^{ij} \vartheta_1^4 \vartheta_1^5 = 0, \quad (3.126)
\]
\[
\vartheta_0^m \left( 12 \vartheta_1^4 + 5 \vartheta_1^5 \right) = 0, \quad (3.127)
\]
\[
\vartheta_1^4 \left( \vartheta_0^m T_m \right)_i^j = 0, \quad (3.128)
\]
are also satisfied. This set of irreducible quadratic constraints that cannot be used to solve some deformation parameters in terms of the rest in an analytic form, and to which the 9-form potentials of the theory may be associated as explained in Section 3.2.2 is one of our main results.

### 3.4 Summary of results and discussion

In the previous section we have constructed order by order in the rank of the \( p \)-forms the supersymmetric tensor hierarchy of maximal 9-dimensional supergravity, up to \( p = 3 \), which covers all the fundamental fields of the theory.

As it usually happens in all maximal supergravity theories, all the deformation parameters can be expressed in terms of components of the embedding tensor. Furthermore, we have shown that gauge invariance and local supersymmetry allow for one triplet, two doublets and one singlet of independent components of the embedding tensor
\[
\vartheta_0^m, \quad m = 1, 2, 3, \quad \vartheta_0^5, \quad \vartheta_1^4, \quad \vartheta_1^5, \quad i = 1, 2. \quad (3.129)
\]
They can be identified with the deformation parameters studied in Ref. \[106\]:

\[
\begin{align*}
\vartheta_0^m &= m_m, \quad (m = 1, 2, 3) \\
\vartheta_1^4 &= -m_{11}, \\
\vartheta_1^5 &= \tilde{m}_4, \\
\vartheta_0^5 &= -\frac{16}{3}m_{\text{IIB}}, \\
\vartheta_2^4 &= m_{\text{IIB}}, \\
\vartheta_2^5 &= m_4.
\end{align*}
\] (3.130)

This proves, on the one hand, that no more deformations are possible and, on the other hand, that all the deformations of maximal 9-dimensional supergravity have a higher-dimensional origin, as shown in Ref. \[106\].

Furthermore, we have also shown that it is not possible to give non-zero values to all the deformation parameters at the same time, since they must satisfy the quadratic constraints

\[
\begin{align*}
\vartheta_0^m \left( 12\vartheta_1^4 + 5\vartheta_1^5 \right) &\equiv Q^m_i = 0, \\
\vartheta_1^4\vartheta_0^5 &\equiv Q_1^4 = 0, \\
\vartheta_1^5\vartheta_0^5 &\equiv Q_5^5 = 0, \\
\vartheta_j^4 \left( \vartheta_0^m T_m \right)_i^j &\equiv Q_i = 0, \\
\varepsilon^{ij}\vartheta_1^4\vartheta_1^5 &\equiv Q = 0,
\end{align*}
\] (3.131-3.135)

all of which are related to gauge invariance.

Using these results, we can now apply the arguments developed in Section 3.2.2 to relate the number of symmetries (Noether currents), deformation parameters, and quadratic constraints to the numbers (and symmetry properties) of 7-, 8- and 9-forms of the theory. Our results can be compared with those presented in Ref. \[143\] (Table 6) and Ref. \[144\] (Table 3) and found from $E_{11}$ level decomposition.

Associated to the symmetry group of the equations of motion of the theory, $SL(2, \mathbb{R}) \times \mathbb{R}^2$ there are 5 Noether currents $j_A$ that fit into one triplet and two singlets of $SL(2, \mathbb{R})$ and are explicitly given in Appendix B.3. Their weights are given in Table 3.4. They
3.4 Summary of results and discussion

\[ R^+ \quad \vartheta_0^1 \quad \vartheta_0^2 - \vartheta_0^3 \quad \vartheta_0^2 + \vartheta_0^3 \quad \vartheta_1^4, \vartheta_1^5 \quad \vartheta_1^4, \vartheta_2^5 \quad \vartheta_0^5 \]

| \(\alpha\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 | 0 |
|\(\beta\) | -1/2 | -5/4 | 1/4 | 3/4 | 0 | 0 | -1/2 |
|\(\gamma\) | 0 | 2 | -2 | -1 | 1 | 1 | 0 |
|\(\delta\) | 0 | 0 | 0 | -2 | -2 | 0 | 0 |

Table 3.5: Weights of the embedding tensor components

\[ R^+ \quad Q_1^1 \quad Q_2^1 \quad Q_1^{2-3} \quad Q_2^{2-3} \quad Q_1^{2+3} \quad Q_2^{2+3} \quad Q_1^4, Q_1^5 \quad Q_3^4, Q_5^5 \quad Q_1 \quad Q_2 \quad Q \]

| \(\alpha\) | -3 | -3 | -3 | -3 | -3 | -3 | -3 | -3 | -3 | -3 | 0 |
|\(\beta\) | 1/4 | -1/2 | -1/2 | -5/4 | 1 | 1/4 | 1/4 | -1/2 | 1/4 | -1/2 | 3/4 |
|\(\gamma\) | -1 | 1 | 1 | 3 | -3 | -1 | -1 | 1 | -1 | 1 | 0 |
|\(\delta\) | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -4 | |

Table 3.6: Weights of quadratic constraints components.

can be dualized as explained in Section 3.2.2 into a triplet and two singlets of 7-forms \(\tilde{A}_{(7)}\) whose weights are given in Table 3.7. In Refs. [143, 144] the \(\beta\) rescaling has not been considered. As mentioned before, it corresponds to the so-called trombone symmetry which may not survive to higher-derivative string corrections. The associated 7-form singlet \(\tilde{A}_5^{(7)}\) does not appear in their analysis. The weights assigned in those references to the fields correspond to one third of the weight of the \(\alpha\) rescaling in our conventions.

Associated to each of the \(SL(2, \mathbb{R})\) multiplets of independent embedding-tensor components there is a dual multiplet of 8-forms \(\tilde{A}_{(8)}\) (i.e. one triplet, two doublets and one singlet) whose weights are given in Table 3.7. The doublet and singlet associated to the gauging of the trombone symmetry using the doublet and singlet of 1-forms are missing in Refs. [143, 144], but the rest of the 8-forms and their weights are in perfect agreement with those obtained from \(E_{11}\). Given the amount of work that it takes to determine which are the independent components of the embedding tensor allowed by supersymmetry, this is a quite non-trivial test of the consistency of the \(E_{11}\) and the embedding-tensor approaches.

Finally, associated to each of the quadratic constraints that the components of the embedding tensor must satisfy \(Q_i^m, Q_1^4, Q_5^5, Q_i, Q\) there is a 9-form potential \(\tilde{A}_{(9)}\). The weights of these potentials are given in Table 3.7. If we set to zero the embedding-tensor components associated to the trombone symmetry \(\vartheta_A^5\), the only constraints which are not automatically solved are

\[ Q_i^m = 12 \vartheta_0^m \vartheta_1^4 = 0, \quad Q_1 = \vartheta_1^4 (\vartheta_0^m T_m)_j^1 \varepsilon_{kl}. \] (3.136)

The first of these constraints can be decomposed into a quadruplet and a doublet: rewriting \(Q_i^m\) in the equivalent form

\[ Q_{(jk)} = \vartheta_1^4 (\vartheta_0^m T_m)_j^1 \varepsilon_{kl}. \] (3.137)
the quadruplet corresponds to the completely symmetric part $Q_{ijk}$ and the doublet to

$$\varepsilon^{jk} Q_{j(kl)} = -Q_1,$$

which is precisely the other doublet. Therefore, we get the quadruplet and one doublet of 9-forms with weight 4 under $\alpha/3$, while one more doublet is found in Refs. [143,144].

This situation is similar to the one encountered in the $N = 2$ theories in $d = 4, 5, 6$ dimensions [122]. In those cases, the Kač-Moody (here $E_{11}$) approach predicts one doublet of $d$-form potentials more than the embedding-tensor formalism [142]. However, it can be seen that taking the undeformed limit of the results obtained in the embedding-tensor formalism, one additional doublet of $d$-forms arises because some Stückenber shifts proportional to deformation tensors that could be used to eliminate them, now vanish. Furthermore, the local supersymmetry algebra closes on them as independent fields.

By analogy with what happens in the $N = 2$ theories in $d = 4, 5, 6$ dimensions, the same mechanism can make our results compatible with those of the $E_{11}$ approach (up to the trombone symmetry): we expect the existence of two independent doublets of 9-forms in the undeformed theory but we also expect new Stückenber transformations in the deformed theory such that one a combination of them is independent and the supersymmetry algebra closes.

This possibility (and the exclusion of any further 9-forms) can only be proven by the direct exploration of all the possible candidates to 9-form supersymmetry transformation rules, to all orders in fermions, something that lies outside the boundaries of this work.

## 3.5 Conclusion remarks

In this chapter we have applied the embedding-tensor formalism to the study of the most general deformations (i.e. gaugings and massive deformations) of maximal 9-dimensional supergravity. We have used the complete global $SL(2, \mathbb{R}) \times \mathbb{R}^2$ symmetry of its equations of motion, which includes the so-called trombone symmetry. We have found the constraints that the deformation parameters must satisfy in order to preserve both gauge and supersymmetry invariance (the latter imposed through the closure of the local supersymmetry algebra to lowest order in fermions). We have used most of
the constraints to express some components of the deformation tensors in terms of a few components of the embedding tensor which we take to be independent and which are given in Eq. (3.129). At that point we have started making contact with the results of Ref. [106], since those independent components are precisely the 8 possible deformations identified there. All of them have a higher-dimensional origin discussed in detail in Ref. [106]. The field strengths, gauge transformations and supersymmetry transformations of the deformed theory, written in terms of the independent deformation tensors, are collected in Appendix B.4.

The 8 independent deformation tensors are still subject to quadratic constraints, given in Eq. (3.131), but those constraints cannot be used to express analytically some of them in terms of the rest, and, therefore, we must keep the 8 deformation parameters and we must enforce these irreducible quadratic constraints.

In Section 3.4 we have used our knowledge of the global symmetries (and corresponding Noether 1-forms), the independent deformation tensors and the irreducible quadratic constraints of the theory, together with the general arguments of Section 3.2.2 to determine the possible 7-, 8- and 9-forms of the theory (Table 3.7), which are dual to the Noether currents, independent deformation tensors and irreducible quadratic constraints. We have compared this spectrum of higher-rank forms with the results of Refs. [143, 144], based on $E_{11}$ level decomposition. We have found that, in the sector unrelated to the trombone symmetry, which was excluded from that analysis, the embedding-tensor formalism predicts one doublet of 9-forms less than the $E_{11}$ approach. However, both predictions are not contradictory: the extra doublet of 9-forms may not survive the deformations on which the embedding-tensor formalism is built: new 9-form Stückelberg shifts proportional to the deformation parameters may occur that can be used to eliminate it so only one combination of the two 9-form doublets survives. This mechanism is present in the $N = 2 \ d = 4, 5, 6$ theories [122], although the physics behind it is a bit mysterious.

We can conclude that we have satisfactorily identified the extended field content (the tensor hierarchy) of maximal 9-dimensional supergravity and, furthermore, that all the higher-rank fields have an interpretation in terms of symmetries and gaugings. This situation is in contrast with our understanding of the extended field content of the maximal 10-dimensional supergravities ($N = 2A, B$) for which the $E_{11}$ approach can be used to get a prediction of the higher-rank forms (which turns out to be correct [126, 128] but the embedding-tensor approach apparently cannot be used [11] for this end. This seems to preclude an interpretation for the 9- and 10-form fields in terms of symmetries and gaugings [12] at least if we insist in the standard construction of the tensor hierarchy that starts with the gauging of global symmetries. Perhaps a more general point of view is necessary.

\[\text{11} \] In the $N = 2B$ case there are no 1-forms to be used as gauge fields and in the $N = 2A$ case the only 1-form available is not invariant under the only rescaling symmetry available.

\[\text{12} \] The 8-form fields are dual to the Noether currents of the global symmetries.
Chapter 4

DFT and Duality orbits of non-geometric fluxes

After having studied gauged supergravities as deformations of the ungauged theories, we will study the gaugings that arise from dimensional reductions of higher-dimensional supergravities. The existence of a mismatch between the catalog of gaugings (probably obtained with the help of the embedding tensor formalism) and the ones that arise from compactification has motivated the formulation of theories that include T-duality as a true symmetry. We will study of double field theory (DFT), one of these T-duality proposals, is able to reproduce the whole set of gaugings that the embedding tensor formalism supplies.

4.1 Introduction

In the context of half-maximal and maximal supergravities, not only does supersymmetry tightly organize the ungauged theory, but also it strictly determines the set of possible deformations (i.e. gaugings).

When compactifying heterotic, type II or eleven-dimensional supergravity on a given background, one obtains lower-dimensional effective theories whose features depend on the fluxes included in the compactification procedure and, in particular, on the amount of supersymmetry preserved by the chosen background. When some supersymmetry is preserved during the compactification, the effective theories under consideration are then gauged supergravities. Compactification can be considered then a way of “deforming” supergravities.

As we have seen in Chapter the embedding tensor formalism enable us to formally describe all the possible deformations in a single universal formulation, which therefore completely restores duality covariance. Not all the deformations obtained in this way have a clear higher-dimensional origin, in the sense that they can be obtained by means of a certain compactification of ten or eleven dimensional supergravity.

One of the most interesting open problems concerning flux compactifications is
4. DFT and Duality orbits of non-geometric fluxes

to reproduce, by means of a suitable flux configuration, a given lower-dimensional gauged supergravity theory. Although this was done in particular cases (see for example [145][146]), an exhaustive analysis remains to be done. This is due to fact that, on the one hand we lack a classification of the possible gauging configurations allowed in gauged supergravities and, on the other hand, only a limited set of compactification scenarios are known. Typically, to go beyond the simplest setups one appeals to dualities. The paradigmatic example [38] starts by applying T-dualities to a simple toroidal background with a non-trivial two-form generating a single $H_{abc}$ flux. By T-dualizing this setup, one can construct a chain of T-dualities leading to new backgrounds (like twisted-tori or T-folds) and generating new (dual) fluxes, like the so-called $Q_a^{bc}$ and $R^{abc}$. It is precisely by following duality covariance arguments in the lower-dimensional effective description that non-geometric fluxes [38] were first introduced in order to explain the mismatch between particular flux compactifications and generic gauged supergravities.

Here we would like to emphasize that all these (a priori) different T-duality connected flux configurations by definition lie in the same orbit of gaugings, and therefore give rise to the same lower-dimensional physics. In order to obtain a different gauged supergravity, one should consider more general configurations of fluxes, involving for example combinations of geometric and non-geometric fluxes, that can never be T-dualized to a frame in which the non-geometric fluxes vanish. For the sake of clarity, we depict this concept in Figure 4.1.

\begin{center}
\includegraphics[width=\textwidth]{duality_orbits.png}
\end{center}

\textit{Figure 4.1:} The space of flux configurations sliced into duality orbits (vertical lines). Moving along a given orbit corresponds to applying dualities to a certain flux configuration and hence it does not imply any physical changes in the lower-dimensional effective description. Geometric fluxes only constitute a subset of the full configuration space. Given an orbit, the physically relevant question is whether (orbit 2 between A and B) or not (orbit 1) this intersects the geometric subspace. We refer to a given point in an orbit as a representative.
4.1. Introduction

Non-geometric fluxes are the inevitable consequence of string dualities, and only a theory which promotes such dualities to symmetries could have a chance to describe them together with geometric fluxes and to understand their origin in a unified way. From the viewpoint of the lower-dimensional effective theory, it turns out that half-maximal and maximal gauged supergravities give descriptions which are explicitly covariant with respect to T- and U-duality respectively. This is schematically depicted in Table 4.1, even though only restricted to the cases we will address in this work.

In recent years, a new proposal (DFT) aiming to promote T-duality to a fundamental symmetry in field theory has received increasing interest. It is named Double Field Theory (DFT) since T-duality invariance requires a doubling of the spacetime coordinates, by supplementing them with dual coordinates associated to the stringy winding modes, whose dynamics can become important in the compactified theory. Recently it has been pointed out how to obtain gaugings of $\mathcal{N} = D = 4$ supergravity by means of twisted double torus reductions of DFT, even though at that stage, the so-called weak and strong constraints imposed for consistency of DFT represented a further restriction that prevented one from describing the most general gaugings that solve the Quadratic Constraints (QC) of gauged supergravity.

Subsequently, an indication has been given that gauge consistency of DFT does not need the weak and strong constraints. Following this direction, we could wonder whether relaxing these constraints can provide a higher-dimensional origin for all gaugings of extended supergravity in DFT.

The aim of our work will be to assess to what extent DFT can improve our description of non-geometric fluxes by giving a higher-dimensional origin to orbits which do not follow from standard supergravity compactifications. We will call such orbits of gaugings non-geometric (in Figure 4.1 they are represented by orbit 1).

As a starting point for this investigation, we will address the problem in the context of maximal and half-maximal gauged supergravities in seven dimensions and higher, where the global symmetry groups are small enough to allow for a general classification of orbits, without needing to consider truncated sectors. We will show that in the half-maximal supergravities in seven and higher-dimensions, where the classifications of orbits can be done exhaustively, all the orbits (including geometric and non-geometric) admit an uplift to DFT, through Scherk-Schwarz (SS) compactifications on appro-

| $D$ | T-duality | U-duality |
|-----|-----------|-----------|
| 9   | O(1,1)    | $\mathbb{R}^+ \times \text{SL}(2)$ |
| 8   | $\text{O}(2,2) = \text{SL}(2) \times \text{SL}(2)$ | $\text{SL}(2) \times \text{SL}(3)$ |
| 7   | $\text{O}(3,3) = \text{SL}(4)$ | $\text{SL}(5)$ |

Table 4.1: The various T- and U-duality groups in $D > 6$. These turn out to coincide with the global symmetry groups of half-maximal and maximal supergravities respectively.
appropriate backgrounds. We provide explicit backgrounds for every orbit, and discuss their (un)doubled nature. The result is that truly doubled DFT provides the appropriate framework to deal with orbits that can not be obtained from supergravity. In contrast, in maximal supergravities in eight and higher-dimensions, all orbits are geometric and hence can be obtained without resorting to DFT.

The chapter is organized as follows. In Sections 4.2 and 4.3 we present a general introduction to flux compactifications and different methods used in T-duality covariant constructions.

In Section 4.4, we introduce and motive DFT emphasizing in the aspect of its SS compactification and its connection with gauged supergravities. We will explicitly show how the gaugings in the effective theory are related to the compactification ansatz, in order to make a link with the results of the following sections. In Section 4.4.2 we present the classification of consistent gaugings in maximal supergravity in terms of U-duality orbits. In particular, we work out the $D = 9$ and $D = 8$ orbits. In both cases we are able to show that all the duality orbits have a geometric origin in compactifications of ten dimensional supergravity. In Section 4.4.3 we classify the consistent gaugings in half-maximal supergravity in terms of T-duality orbits. In particular, we work out the $D = 8$ and $D = 7$ orbits. Here we encounter the first orbits lacking a geometric higher-dimensional origin. We show that such orbits do follow from dimensional reductions of DFT. Finally, our conclusions are presented in Section 4.5. We defer a number of technical details on gauge algebras and 't Hooft symbols to the Appendix C.

4.2 Flux compactification: a primer

4.2.1 Geometric fluxes

Let us briefly introduce the geometric fluxes origin from Scherk-Schwarz (SS) compactifications of supergravities. We will closely follow the references [17,35,148].

Let us consider the common “NSNS” bosonic sector of supergravity, spanned by a $D$-dimensional metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, a 2-form field $\hat{B}_{\hat{\mu}\hat{\nu}}$ and a dilaton $\hat{\phi}$. This sector is shared by all the superstring-derived theories. Its effective action in the “string frame” is given by (cf. (5))

$$S = \frac{g_s^2}{16\pi G_{(d)}^N} \int d^D x \sqrt{|\hat{g}|} e^{-2\phi} \left[ R - 4(\partial\phi)^2 + \frac{1}{2\cdot3!} H_{\mu\nu\rho}H^{\mu\nu\rho} \right]. \quad (4.1)$$

This action can arise as part of the low-energy effective action of the bosonic string with $D = 26$ or from the “common sector” of the heterotic or type II superstrings in $D = 10$ dimensions. All of the fields will depend on $D$ space-time coordinates. The standard Kaluza-Klein dimensional reduction of the previous action on a $n$-torus $T^n$ would give a theory in $d = D - n$ dimensions with a manifest $O(n,n)$ global invariance and a $U(1)^{2n}$ gauge symmetry. The scalar fields would take values in the coset $O(n,n)/O(n) \times O(n)$. If one is interested in gauged supergravities that arise
4.2. Flux compactification: a primer

as deformations of this theory, one possibility of deformations consists of promoting a 2n-dimensional subgroup $G_0 \subset O(n,n)$ to a local symmetry gauged by the vector fields already existing on it using the embedding tensor formalism (see previous chapters).

Another possibility of deformation is to perform a more general dimensional reduction. Let us for this purpose split the coordinates as follows:

$$x^\hat{\mu} = (x^\mu, y^m),$$  \hspace{1cm} (4.2)

where $y^m, m = 1, \ldots, n$ are compact space directions and $x^\mu, \mu = 1, \ldots, d$ are spacetime directions. The fields must be decomposed into representations of the symmetry group of the lower-dimensional theory,

$$\hat{g}_{\mu\nu} = \left( \begin{array}{c|c} \hat{g}_{\mu\nu} & \hat{A}_\mu^p \hat{A}_\nu^q \\ \hline \hat{g}_{mp} \hat{A}_p^\nu & \hat{g}_{\mu\nu} \end{array} \right),$$  \hspace{1cm} (4.3)

$$\hat{b}_{\mu\nu} = \left( \begin{array}{c|c} \hat{b}_{\mu\nu} - \frac{1}{2} \left( \hat{A}_\mu^p \hat{V}_\nu^p - \hat{A}_\nu^p \hat{V}_\mu^p \right) + \hat{A}_\mu^p \hat{A}_\nu^q \hat{b}_{pq} & \hat{V}_\mu - \hat{b}_{mp} \hat{A}_\mu^p \\ \hline -\hat{V}_\nu + \hat{b}_{mp} \hat{A}_\nu^p & \hat{b}_{\mu\nu} \end{array} \right),$$  \hspace{1cm} (4.4)

where $\hat{A}_\mu^m$ and $\hat{V}_mn$ are vector fields and $\hat{g}_{mn}$ and $\hat{b}_{mn}$ are symmetric and antisymmetric scalar matrices, respectively. In principle, all the fields in the matrices above depend on both $(x^\mu, y^m)$ coordinates.

A reduction ansatz, expressing the dependence of these D-dimensional fields on the effective fields that will live in d dimensions (unhatted) is necessary. We can assume an ansatz in which these fields do not depend on the compact coordinates, as for example:

$$\hat{g}_{\mu\nu} = g_{\mu\nu}(x), \quad \hat{g}_{mn} = u^a_{\ m}(y)u^b_{\ n}(y)g_{\ ab}(x),$$

$$\hat{b}_{\mu\nu} = b_{\mu\nu}(x), \quad \hat{b}_{mn} = u^a_{\ m}(y)u^b_{\ n}(y)b_{ab}(x) + v_{mn}(y),$$

$$\hat{A}_\mu^m = u^a_{\ m}(y)A^a_\mu(x), \quad \hat{V}_mn = u^a_{\ m}(y)V^a_{\ mn}(x),$$

$$\hat{\phi} = \phi(x).$$  \hspace{1cm} (4.5)

Thus, we are left with a d-dimensional metric and a 2-form plus 2n vector fields, $A^a_\mu$, and $V_{\ mn}$, and $n^2 + 1$ scalar fields $(g_{\ ab}, b_{ab}, \phi)$. The y-dependent elements $u^a_{\ m}(y)$ and $v_{mn}(y)$ carry the deformation of the compactified manifold, and they have to combine in such a way that there is not y-dependence in the effective action. \[^1\]

The SS reduction of the gauge transformation parameters implies new contributions to the gauge transformations of the effective fields. For a detailed discussion, we refer to \[^{17,35}\]. Schematically, if we have a D-dimensional gauge parameter

$$\hat{\lambda}^\hat{\mu} = (e^\mu, \Lambda^m),$$  \hspace{1cm} (4.6)

and an arbitrary vector field of the type

$$\hat{V}^\hat{\mu} = (V^\mu(x), u^a_{\ m}(y)V^a_\mu(x)), $$  \hspace{1cm} (4.7)

\[^1\]We note the formal similarity of the $u^a_{\ m}$ quantities with a ‘vielbein’.
DFT and Duality orbits of non-geometric fluxes

The effective Lie derivative gets modified. Namely, if
\[ \hat{\mathcal{L}} \hat{V}^\mu = \hat{\lambda}^\nu \partial_\nu \hat{V}^\mu - \hat{\phi} \partial_\mu \hat{\lambda}^\mu \]  
(4.8)
is the \( D \)-dimensional Lie derivative, the (unhatted) effective Lie derivative results
\[ \mathcal{L}_c V^a = \hat{\mathcal{L}}_c V^a + f_{bc}{}^a \Lambda^b V^c , \]  
(4.9)
where
\[ f_{ab}{}^c = u_a{}^m \partial_m u_b{}^n u_c^\epsilon_n - u_b{}^m \partial_m u_a{}^n u_c^\epsilon_n . \]  
(4.10)
These structure constants are known as metric fluxes, due to the role that \( u \) plays on the definition of the lower-dimensional metric.

Inspired by \( O(n,n) \), we can rearrange the fields and gauge parameters into \( O(n,n) \) multiplets,
\[ \xi = (\epsilon_\mu, \epsilon^\mu, \Lambda^A), \]  
\[ \Lambda^A = (\lambda_a, \lambda^a), \]  
\[ A^A{}_\mu = (V^a{}_\mu, A^a {}_\mu), \]  
\[ M_{AB} = \left( \begin{array}{cc} g_{ab} & -g_{ac} b^d b_d \\ b_{ac} g^c b \end{array} \right), \]  
(4.11)
where indices \( A, B = 1, \ldots, 2n \) are raised and lowered by means of the metric
\[ \eta_{AB} = \left( \begin{array}{cc} 0 & \delta^a b \\ \delta_a b & 0 \end{array} \right). \]  
(4.12)
The gauge transformations of the effective fields result modified and their dependence on the compact manifold is reflected in the structure constants \( f_{ABC} \),
\[ \delta_\xi g_{\mu\nu} = \mathcal{L}_c g_{\mu\nu} , \]  
\[ \delta_\xi b_{\mu\nu} = \mathcal{L}_c b_{\mu\nu} + (\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu) , \]  
\[ \delta_\xi A^A {}_\mu = \mathcal{L}_c A^A {}_\mu - \partial_\mu \Lambda^A + f_{BC}{}^A \Lambda^B C^C {}_\mu , \]  
\[ \delta_\xi M_{AB} = \mathcal{L}_c M_{AB} + f_{AC}{}^D \Lambda^C M_{DB} + f_{BC}{}^D \Lambda^C M_{AD} . \]  
(4.13)
The structure constants \( f_{ABC} \) have non-vanishing components
\[ f_{ab}{}^c = u_a{}^m \partial_m u_b{}^n u_c^\epsilon_n - u_b{}^m \partial_m u_a{}^n u_c^\epsilon_n , \]  
\[ f_{abc} = 3(\partial_\mu v_{bc} + f_{[ab}{}^d v_{cd]}), \]  
(4.14)
so that \( f_{a}{}^{bc} = f^{abc} = 0 \).

Substituting the ansatz (4.5) into the \( D \)-dimensional SUGRA bosonic action (4.1), we have the effective d-dimensional action
\[ S = \int d^d x \sqrt{|g|} e^{-2\phi} \left( R + 4(\partial \phi)^2 - \frac{1}{4} M_{AB} F^{A\mu\nu} F_{B \mu\nu} - \frac{1}{12} G_{\mu\nu\rho} G^{\mu\nu\rho} + \frac{1}{8} D_\mu M_{AB} D^\mu M^{AB} - V \right) , \]  
(4.15)
4.2. Flux compactification: a primer

\[ F^A_{\mu \nu} = 2 \partial_{[\mu} A^A_{\nu]} - f_{BC}{}^A A^B_{\mu} A^C_{\nu}, \]
\[ G_{\mu \nu \rho} = 3 \partial_{[\mu} b_{\nu \rho]} - f_{ABC} A^A_{\mu} A^B_{\nu} A^C_{\rho} + 3 \partial_{[\mu} A^A_{\nu} A^A_{\rho]}, \]

and the covariant derivative is
\[ D_{\mu} M_{AB} = \partial_{\mu} M_{AB} - f_{AD}{}^C A^D_{\mu} M_{CB} - f_{BD}{}^C A^D_{\mu} M_{AC}. \]

In addition, a scalar potential naturally arises. This is given by the expression
\[ V = \frac{1}{4} f_{DA}{}^C f_{CB}{}^D M^{AB} + \frac{1}{12} f_{AC}{}^E f_{BD}{}^F M^{AB} M^{CD} M_{EF} + \frac{1}{6} f_{ABC} f^{ABC}. \]

The structure constants that have appeared as a consequence of the dimensional reduction of the 2-form and metric fields are called geometric fluxes due to their geometrical reduction origin. In the literature, they are also denoted as
\[ H_{abc} \equiv f_{abc}, \quad \omega_{ab}{}^c \equiv f_{ab}{}^c. \]

This is a notation which we will frequently use in the following sections.

**Beyond geometric fluxes**

If in the ansatz (4.5), we choose
\[ g_{ab}(x) = \delta_{ab}, \quad b_{ab}(x) = 0, \]
the twist matrices \( u \) and \( v \) can be understood as the background fields associated to the vielbein and the 2-form that live in the compact space. Since T duality exchanges metric and 2-form components by means of the Buscher’s rules (A.13), these geometric fluxes can be transformed into each other as well. Let us study a simple setting of these fluxes to see explicitly how this applies [38].

Let us consider a compactification on a 3-torus with a non-trivial 2-form, e.g.
\[ \hat{g}_{mn} = \delta_{mn}, \quad b_{23} = C y^1, \]
whose associated twist matrices are
\[ u_m{}^a = \delta_m{}^a, \quad v_{23} = C y^1. \]

The corresponding fluxes are
\[ H_{123} = C, \quad \omega_{12}{}^3 = \omega_{23}{}^1 = \omega_{31}{}^2 = 0. \]

Since these backgrounds enjoy isometries in the \( y^2 \) and \( y^3 \) directions, we can perform T duality transformations on these directions. So, applying (A.13), we get certain \( g_{mn} \) and \( b_{mn} \),
\[ ds^2 = g_{mn} dy^m dy^n = (dy^1)^2 + (dy^2)^2 + (dy^3 + C y^1 dy^2)^2, \quad b_{mn} = 0, \]
which imply the following fluxes:

\[ H_{123} = \omega_{23}^1 = \omega_{31}^2 = 0, \quad \omega_{12}^3 = C. \]  

(4.26)

By simple inspection, we notice that these fluxes still can be T-dualized in the direction \( y^2 \). Again, using the Buscher’s rules, they transform into

\[ ds^2 = g_{mn} dy^m dy^n = (dy^1)^2 + \frac{1}{1 + (Cy^1)^2} \left[ (dy^2)^2 + (dy^3)^2 \right], \quad b_{23} = - \frac{C y^1}{1 + (Cy^1)^2}. \]  

(4.27)

The non-vanishing component of the 2-form is associated to a new flux, which in the literature is called \( Q_{123} \).

Symbolically we have built T-duality transformations, such that

\[ H_{abc} \overset{T_c}{\longleftrightarrow} \omega_{ab}^c, \quad \overset{T_b}{\longleftrightarrow} Q_a^{bc}, \quad \overset{T_a}{\longleftrightarrow} R_{abc}. \]  

(4.28)

The first T-duality transformation \( T_c \) relates the metric and gauge fluxes. The second one, \( T_b \), produces the so-called \( Q \) fluxes, which describe locally geometric backgrounds despite of not being globally well-defined.

A last T-duality transformation, \( T_a \) in the diagram below, would generate the \( R \) fluxes. and since there are no isometries in the \( y^1 \) direction, there does not exist even a local description for these background fluxes.

\[ H_{abc} \overset{T_c}{\leftrightarrow} \omega_{ab}^c \overset{T_b}{\leftrightarrow} Q_a^{bc} \overset{T_a}{\leftrightarrow} R_{abc} \]  

(4.29)

Thus, T-duality would allow to transform a single non-geometric flux into a geometric one. However, a configuration of both geometric and non-geometric fluxes simultaneously turned on such that T-duality is not capable of converting all the non-geometric fluxes into geometric ones is a special situation. This kind of setting is called duality orbit of a non-geometric flux and is treated in Section 4, where we explicitly show that a standard SS reduction is not able to reproduce it. This fact turns out to wonder whether we need extra ingredients in our compactification procedure to get these additional fluxes.

We realize that T duality is going to be crucial in the development and inclusion of these non-geometric backgrounds. Indeed, the way in that these fluxes have emerged suggests a new framework in which T-duality becomes a true symmetry of the genuine theory, instead of appearing after the compactification.

### 4.3 T-duality covariant constructions

Several approaches have been developed to solve the problem of getting non-geometric fluxes in a natural and covariant formalism. We can distinguish three different trends. The first one is the doubled geometry, in which the local charts or patches that define
the background geometry are slightly modified. Another possibility is the so-called
*generalized complex geometry*, which is defined on a manifold whose bundle structure
is extended to include new elements. Finally, there exists the *double field theory* formalism, which suggests the doubling of spatial coordinates, associating the new ones
to their corresponding dual winding modes.

Despite of the different approaches under which these theories are built, their aim
is the same: to be able to host T-duality as a global symmetry by construction.

### 4.3.1 Doubled Geometry

The distinctive characteristic of doubled geometry is that given a manifold the group
of transition functions between overlapping coordinate charts is generalized to include,
in addition to diffeomorphisms and gauge transformations, duality transformations. When these duality transformations are T-duality transition functions the manifold
equipped with the extra structure is named a *T-fold* \[149,150\].

In \[151\], \(O(n,n)\) duality twist reductions have been performed by making use of
this T-fold structure. Later on, dimensional reduction over twisted doubled tori were
performed to include non-geometric fluxes configurations in \[12,152,153\].

### 4.3.2 Generalized Complex Geometry

The starting point of this approach consists of a modification of the tangent bundle
structure associated to the manifold. The main idea is the treatment of the tangent
and cotangent space at the same level, without distinguishing them. In its original
formulation \[154\], a new generalized tangent bundle is constructed by the direct sum
of both spaces,

\[
X + \xi \in TM_n \oplus T^* M_n. \tag{4.30}
\]

The elements of such a space are formal sums of a vector field and a one-form.

This generalized bundle induces a natural metric \(\mathcal{I}\),

\[
\mathcal{I}(X + \xi, Y + \eta) \equiv \frac{1}{2} (\iota_Y \xi + \iota_X \eta), \tag{4.31}
\]

where \(\iota_Y \xi \equiv Y^m \xi_m\). In the coordinate basis \((\partial_m, dx^m)\), the metric is realized by the matrix

\[
\mathcal{I} = \frac{1}{2} \begin{pmatrix}
0 & 1_n \\
1_n & 0
\end{pmatrix}. \tag{4.32}
\]

\[2\] Similarly, U-folds with U-duality transition functions, or mirror-folds with mirror symmetry transition functions can be defined. Locally, T-folds or U-folds require each coordinate chart or patch to be the product of a torus with some open set, while a mirror-folds have a Calabi-Yau fibration.
Thus, a *generalized almost-complex structure* on this bundle is defined as an endomorphism \( J \),

\[
J : T M_n \oplus T^* M_n \to T M_n \oplus T^* M_n
\]  

(4.33)
such that

\[
J^2 = -I_{2n}
\]

and

\[
J^T \mathcal{I} J = \mathcal{I}.
\]

Following the parallelism of an almost-complex structure, a generalized Lie bracket can be defined. This is the so-called *Courant bracket*, which is defined as

\[
[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\iota_X \eta - \iota_Y \xi).
\]

(4.34)

A *generalized complex structure* is a generalized almost complex structure such that the space of smooth bundle sections is closed under the Courant bracket. This bracket is also defined in double field theory, as we will see in the next section. Interesting monographs dealing with generalized complex geometry are for example [155–160].

### 4.3.3 Double field theory

Double field theory (DFT) can be understood as a T duality invariant formulation of string theory and supergravity. That is, it contains T duality as a symmetry of the theory by construction. From the stringy point of view this is achieved by doubling the spacetime coordinates and associating the winding modes of the strings to the new dual coordinates that are required to be introduced to have T-duality as a symmetry. Its original version was developed to describe the dynamics of closed strings on tori [147]. However, due to the successful development of a background independent version [161], it was quickly used to perform SS reductions over different manifolds. These dimensional compactifications done in a DFT scenario allowed to obtain the gaugings associated to the electric sector of \( N = 4 \ D = 4 \) supergravity, thus establishing a relation between DFT and non-geometric fluxes.

We will show here a brief introduction to the main features of DFT and its relation with gauged supergravities. Let us introduce the necessary ingredients of DFT and some notation. For a \( D \)-dimensional spacetime with \( d \) non-compact spacetime coordinates and \( n \) compact dimensions \((D = d + n)\), the fields depend on coordinates

\[
X^M = (\tilde{x}_i, x^i) = (\tilde{x}_\mu, \tilde{y}_m, x^\mu, y^m),
\]

(4.35)

where \( M = 1, \ldots, 2D \) is an \( O(D, D) \) index. The \( 2D \) coordinates can be split into the genuine \( D \) spacetime coordinates \( x^i \) and their dual coordinates, \( \tilde{x}_\mu \). In addition, the \( i \) index can be split into extended and compact coordinates, \( i = \{ \mu, m \} \), where \( \mu = \{ \mu, m \} \)
1, \ldots, d represents extended coordinates and \( m = 1, \ldots, n \) runs over the compactified coordinates.

Any fields and gauge parameters of DFT is supposed to be annihilated by the differential operator

\[
\partial_i \tilde{\partial}^i \Phi = 0. \tag{4.36}
\]

Where \( \Phi \) denotes any field or gauge parameter of the theory. This is the DFT weak constraint (WC). A background independent action is constructed \[161\] under a stronger restriction: \( (4.36) \) must hold not only for any field or gauge parameter, but for any product of them. This is the so-called strong constraint (SC). If we define a generalized field \( \mathcal{E}_{ij} \) in terms of the metric and the 2-form,

\[
\mathcal{E}_{ij} \equiv g_{ij} + b_{ij}, \tag{4.38}
\]

and a T-duality invariant scalar field \( d \)[4]

\[
e^{-2d} \equiv \sqrt{|g|} e^{-2\phi}, \tag{4.39}
\]

the background independent action is given by

\[
S = \int d^d x d^d \tilde{x} e^{-2d} \left[ -\frac{1}{4} g^{ik} g^{jl} \mathcal{D}_k \mathcal{D}_l \mathcal{E}_{ij} + \frac{1}{4} g^{kl} \left( \mathcal{D}^j \mathcal{E}_{ik} \mathcal{D}^i \mathcal{E}_{jl} + \tilde{\mathcal{D}}^j \mathcal{E}_{ik} \tilde{\mathcal{D}}^i \mathcal{E}_{jl} \right) \right]. \tag{4.40}
\]

The derivative operators \( \mathcal{D}^i, \tilde{\mathcal{D}}^i \) are defined as

\[
\mathcal{D}_i \equiv \frac{\partial}{\partial x^i} - \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}^k}, \quad \tilde{\mathcal{D}}_i \equiv \frac{\partial}{\partial x^i} + \mathcal{E}_{ik} \frac{\partial}{\partial \tilde{x}^k}. \tag{4.41}
\]

This action is invariant under the \( O(D, D) \) T-duality group, which acts on the fields as follows:

\[
\mathcal{E}'(X') = \frac{a \mathcal{E}(X) + b}{c \mathcal{E}(X) + d}, \quad d'(X') = d(X), \quad X' = hX, \tag{4.42}
\]

where \( h \) is

\[
h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(D, D), \quad h^T \eta h = \eta \text{ with } \eta = \begin{pmatrix} 0 & \mathbb{1}_D \\ \mathbb{1}_D & 0 \end{pmatrix}. \tag{4.43}
\]

In its original stringy formulation, DFT was restricted to satisfy the level matching condition

\[
L_0 - \bar{L}_0 = 0, \tag{4.37}
\]

arising for closed string theory. This condition translates to the WC.

The context should be enough to clarify the difference between the dimension ‘\( d \)’ and the scalar field ‘\( d \)’.
4. DFT and Duality orbits of non-geometric fluxes

This action can be rewritten in terms of the so-called generalized metric, $\mathcal{H}_{MN}$. This is a $2D \times 2D$ symmetric matrix constructed from the $D \times D$ matrices $g_{ij}$ and $b_{ij}$, with the remarkable property that it transforms as an $O(D, D)$ tensor,

$$\mathcal{H} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}.$$  \hfill (4.44)

Under $h \in O(D, D)$ transformations, the fields transform as

$$\mathcal{H}_{MN}(X) \rightarrow h_M^P h_N^Q \mathcal{H}_{PQ}(hX), \quad d(X) \rightarrow d(hX),$$  \hfill (4.45)

For cases in which $h$ corresponds to a T-duality transformation, it reproduces the corresponding Buscher’s rules [A.13] for $\{g_{ij}, b_{ij}, \phi\}$. In fact, it has been shown that these transformation rules allow the possibility of performing a T-duality transformation in non-isometric directions [150–152, 162]. Then, in terms of this generalized metric formulation, the original action is rewritten as

$$S = \int d^d x d^d \tilde{x} e^{-2d} \left( \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{PQ} \partial_N \mathcal{H}_{PQ} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{PQ} \partial_Q \mathcal{H}_{MP} \right.$$  

$$\left. - 2 \partial_M d \partial_N \mathcal{H}^{MN} + 4 \mathcal{H}^{MN} \partial_M d \partial_N d \right).$$  \hfill (4.46)

Gauge invariance of the action and the closure of the algebra of DFT happens upon the weak (WC) and strong (SC) versions of (4.36), which in $O(D, D)$ indices are rewritten, respectively, as

$$\partial_M \partial^M A = 0, \quad \partial_M A \partial^M B = 0,$$  \hfill (4.47)

where $A, B$, again refers to any field and/or gauge parameter. Gauge transformations of the fields $\{\mathcal{H}, d\}$ are driven by the transformation rules of $\mathcal{E}_{ij}$,

$$\delta \xi \mathcal{H}^{MN} = \xi^P \partial_P \mathcal{H}^{MN} + (\partial^M \xi_P - \partial_P \xi^M)\mathcal{H}^{PN} + (\partial^N \xi_P - \partial_P \xi^N)\mathcal{H}^{MP},$$

$$\delta \xi d = \xi^M \partial_M d - \frac{1}{2} \partial_M \xi^M.$$  \hfill (4.48)

This motivated the definition of a generalized Lie derivative $\hat{\mathcal{L}}_\xi$ such that, for an arbitrary $O(D, D)$ tensor $V^M_N$,

$$\hat{\mathcal{L}}_\xi V^M_N = \xi^P \partial_P V^M_N + (\partial^M \xi_P - \partial_P \xi^M)V^P_N - (\partial^N \xi_P - \partial_P \xi^N)V^M_P.$$  \hfill (4.49)

Then, the field transformations are rewritten as

$$\delta \xi \mathcal{H}^{MN} = \hat{\mathcal{L}}_\xi \mathcal{H}^{MN},$$

$$\delta \xi d = \hat{\mathcal{L}}_\xi d.$$  \hfill (4.50)
Upon the SC constrain, this generalized Lie derivative (and thus the gauge transformations) close under the Courant or C-bracket (equivalent to the one defined in the previous section),

\[ [\xi_1, \xi_2]_C^M = 2\xi_1^M \partial_N \xi_2^N - \xi_1^N \partial_M \xi_2^N . \] (4.52)

The SC results essential in this DFT development in this way. However, some deficiencies to this formulation arose \cite{43, 44}. In these works, some, but not all of the gaugings of \( N = 4 \) \( D = 4 \) SUGRA were obtained by SS reductions of DFT. Indeed, the gaugings associated to non-geometric fluxes could be geometrized by performing suitable T-duality transformations as the ones shown before. This, together with the presence of the constraint (4.47), led to think about a new reformulation of DFT in which the SC (4.47) would be somehow relaxed and genuine non-geometric fluxes would be captured as consequence.

In ref. \cite{45}, DFT was formulated without imposing any constraint at the very beginning. Without this constraint the requirements of gauge invariance of the action, the closure of the generalized Lie derivatives and the generalized Jacobi identities are not automatically satisfied. When the SS compactification is performed on the theory, it is shown that (4.47) is indeed a sufficient but not a necessary condition for the consistency of the theory. In particular, they find a less restrictive condition under which the 3 previous requirements are fulfilled. These relaxed constraints are

\[ \partial_M \partial^M \hat{A} = 0, \quad \partial_M \hat{A} \partial^M \hat{B} = 0, \] (4.53)

where \( \hat{A}, \hat{B} \) denote any effective (that is, living in the lower-dimensional theory) field and/or gauge parameter. That is, while (4.47) is required not only for the lower-dimensional fields but also for the fields of the higher-dimensional theory, the new constraints (4.53) are only imposed on fields living in the lower-dimensional theory. Moreover, not only the 3 consistency constraints (gauge invariant action, closure of the gauge transformations, Courant-like Jacobi identities) are satisfied, but an additional term, which is killed by (4.47), can be this time added to the action,

\[ \int d^d x d^d \tilde{x} e^{-2d} \frac{1}{2} \partial_M \mathcal{E}^a_P \partial^M \mathcal{E}^b_Q S_{ab} \eta^{PQ} . \] (4.54)

Actually, this term becomes crucial for matching fluxes and gaugings, as we will verify in the following sections.

## 4.4 Duality orbits of non-geometric fluxes

As we have mentioned in the last section, compactifications in duality covariant constructions such as generalized geometry and double field theory have proven to be suitable frameworks to reproduce gauged supergravities containing non-geometric fluxes. However, it is a priori unclear whether these approaches only provide a reformulation of old results, or also contain new physics. To address this question, we classify
the T- and U-duality orbits of gaugings of (half-)maximal supergravities in dimensions seven and higher. It turns out that all orbits have a geometric supergravity origin in the maximal case, while there are non-geometric orbits in the half-maximal case. We show how the latter are obtained from compactifications of double field theory. Some technical material used in the development of this chapter can be found in Appendix C.1. The results of this chapter were first obtained in refs. [46,163,164].

4.4.1 Orbits from double field theory

While toroidal compactifications of DFT lead to half-maximal ungauged supergravities, SS compactifications on more general double spaces are effectively described by gauged supergravities like the ones we will analyze in the next sections. If the internal space is restricted in such a way that there always exists a frame without dual coordinate dependence, the only orbits allowed in the effective theory are those admitting representatives that can be obtained from compactifications of ten dimensional supergravity. This is not the most general case, and we will show that some orbits require the compact space to be truly doubled, capturing information of both momentum and winding modes.

Recently in ref. [45], a new set of solutions to the constraints for DFT has been found. For these solutions the internal dependence of the fields is not dynamical, but fixed. The constraints of DFT restrict the dynamical external space to be undoubled, but allows for a doubling of the internal coordinates as long as the QC for the gaugings are satisfied. Interestingly, these are exactly the constraints needed for consistency of gauged supergravity, so there is a priori no impediment to uplift any orbit to DFT in this situation. In fact, in the following sections we show that all the orbits in half-maximal $D = 7, 8$ gauged supergravities can be reached from twisted double tori compactifications of DFT.

DFT and (half-)maximal gauged supergravities

In the SS procedure, the coordinates $X^M$ are split into external directions $X = (\tilde{x}_i, x^i)$ and compact internal $Y = (\tilde{y}_i, y^i)$ coordinates. The former set contains pairs of $O(D, D)$ dual coordinates, while the latter one contains pairs of $O(n, n)$ dual coordinates, with $d = D + n$. This means that if a given coordinate is external (internal), its dual must also be external (internal), so the effective theory is formally a (gauged) DFT. The SS procedure is then defined in terms of a reduction ansatz, that specifies the dependence of the fields in $(X, Y)$

$$\mathcal{H}_{MN}(X, Y) = U(Y)^A_M \hat{H}(X)_{AB} U(Y)^B_N , \quad d(X, Y) = \hat{d}(X) + \lambda(Y) .$$

(4.55)

Here the hatted fields $\hat{H}$ and $\hat{d}$ are the dynamical fields in the effective theory, parametrizing perturbations around the background, which is defined by $U(Y)$ and $\lambda(Y)$. The matrix $U$ is referred to as the twist matrix, and must be an element of $O(n, n)$. It contains a DFT T-duality index $M$, and another index $A$ corresponding to the T-duality
4.4. Duality orbits of non-geometric fluxes

Duality orbits of non-geometric fluxes

When DFT is evaluated on the reduction ansatz, the twists generate the gaugings of the effective theory

$$f_{ABC} = 3\eta_{[A}(U^{-1})^B(U^{-1})^C] \partial_M U^D \partial_M ,$$

(4.56)

$$\xi_A = \partial_M (U^{-1})^A - 2(U^{-1})^A \partial_M \lambda ,$$

(4.57)

where $f_{ABC}$ and $\xi_A$ build the generalized structure constants of the gauge group in the lower-dimensional theory.

Although $U$ and $\lambda$ are $Y$ dependent quantities, the gaugings are forced to be constants in order to eliminate the $Y$ dependence from the lower dimensional theory. When the external-internal splitting is performed, namely $d = D + n$, the dynamical fields are written in terms of their components which are a $D$-dimensional metric, a $D$-dimensional 2-form, $2n$ $D$-dimensional vectors and $n^2$ scalars. These are the degrees of freedom of half-maximal supergravities. Since these fields are contracted with the gaugings, one must make sure that after the splitting the gaugings have vanishing Lorentzian indices, and this is achieved by stating that the twist matrix is only non-trivial in the internal directions. Therefore, although formally everything is covariantly written in terms of $O(d, d)$ indices $A, B, C, ...$, the global symmetry group is actually broken to $O(n, n)$. We will not explicitly show how this splitting takes place, and refer to \cite{43} for more details. In this work, for the sake of simplicity, we will restrict to $O(n, n)$ global symmetry groups, without additional vector fields.

There are two possible known ways to restrict the fields and gauge parameters in DFT, such that the action is gauge invariant and the gauge algebra closes. On the one hand, the weak and strong constraints can be imposed, which in this context they read as

$$\partial_M \partial^M A = 0 , \quad \partial_M A \partial^M B = 0 ,$$

(4.58)

where $A$ and $B$ generically denote products of (derivatives of) fields and gauge parameters. When this is the case, one can argue \cite{161} that there is always a frame in which the fields do not depend on the dual coordinates. On the other hand, in the SS compactification scenario, it is enough to impose the weak and strong constraints only on the external space (i.e., on hatted quantities)

$$\partial_M \hat{A} = 0 , \quad \partial_M \hat{A} \partial^M \hat{B} = 0 ,$$

(4.59)

and impose QC for the gaugings

$$f_{E[AB} f^{E}{}_{C]D} = 0 .$$

(4.60)

This second option is more natural for our purposes, since these constraints exactly coincide with those of half-maximal gauged supergravities (which are undoubled theories in the external space, and contain gaugings satisfying the QC).

\footnote{We are working under the assumption that the structure constants not only specify the gauging, but all couplings of the theory. Reproducing the correct structure constants therefore implies reproducing the full theory correctly, as has been proven in $D = 4$ and $D = 10$ \cite{43,44,165}.}
Notice that if a given $U$ produces a solution to the QC, any T-dual $U$ will also. Therefore, it is natural to define the notion of twist orbits as the sets of twist matrices connected through T-duality transformations. If a representative of a twist orbit generates a representative of an orbit of gaugings, one can claim that the twist orbit will generate the entire orbit of gaugings. Also, notice that if a twist matrix satisfies the weak and strong constraints, any representative of its orbit will, so one can define the notions of undoubled and truly doubled twist orbits.

Non-geometry VS weak and strong constraint violation

Any half-maximal supergravity can be uplifted to the maximal theory whenever the following constraint holds:

$$\epsilon_{ABC} f^{ABC} = 0 \ .$$  

(4.61)

This constraint plays the role of an orthogonality condition between geometric and non-geometric fluxes. Interestingly, the constraint (4.61) evaluated in terms of the twist matrix $U$ and $\lambda$ can be rewritten as follows (by taking relations (4.56) and (4.57) into account)

$$\epsilon_{ABC} f^{ABC} = -3 \partial_D U^A_P \partial_D (U^{-1})^P_A - 24 \partial_D \lambda \partial_D \lambda + 24 \partial_D \partial_D \lambda .$$  

(4.62)

The RHS of this equation is zero whenever the background defined by $U$ and $\lambda$ satisfies the weak and strong constraints. This immediately implies that any background satisfying weak and strong constraints defines a gauging which is upliftable to the maximal theory. Conversely, if an orbit of gaugings in half-maximal supergravity does not satisfy the extra constraint (4.61), the RHS of this equation must be non-vanishing, and then the strong and weak constraint must be relaxed. In conclusion, the orbits of half-maximal supergravity that do not obey the QC of the maximal theory require truly doubled twist orbits, and are therefore genuinely non-geometric. This point provides a concrete criterion to label these orbits as non-geometric. Also, notice that these orbits will never be captured by non-geometric flux configurations obtained by T-dualizing a geometric background.

For the sake of clarity, let us briefly review the definitions that we use. A twist orbit is non-geometric if it doesn’t satisfy the weak/strong constraint, and geometric if it does. Therefore, the notion of geometry that we consider is local, and we will not worry about global issues (given that the twist matrix is taken to be an element of the global symmetry group, the transition functions between coordinate patches are automatically elements of $O(n,n)$). On the other hand an orbit of gaugings is geometric

---

6 $D = 4$ half-maximal supergravity is slightly different because its global symmetry group features an extra SL(2) factor; for full details, see [166,167].

7 However, we would like to stress that, in general, it is not true that an orbit satisfying the QC constraints of maximal supergravity (4.61) is necessarily generated by an undoubled twist orbit. An example can be found at the end of Section 4.4.3.
4.4. Duality orbits of non-geometric fluxes

if it contains a representative that can be obtained from 10 dimensional supergravity (or equivalently from a geometric twist orbit), and it is non-geometric if it does not satisfy the constraints of maximal supergravity.

We have now described all the necessary ingredients to formally relate dimensional reductions of DFT and the orbits of half-maximal gauged supergravities. In particular, in what follows we will:

1. Provide a classification of all the orbits of gaugings in maximal and half-maximal supergravities in $D \geq 7$.

2. Explore mechanisms to generate orbits of gaugings from twists, satisfying
   - $U(\mathbb{Y}) \in O(n,n)$
   - Constant $f_{ABC}$
   - $f_{E[AB}f^{EC]}D = 0$

3. Show that in the half-maximal theories all the orbits of gaugings can be obtained from twist orbits in DFT.

4. Show that in the half-maximal theories the orbits that satisfy the QC of maximal supergravity admit a representative with a higher-dimensional supergravity origin. For these we provide concrete realizations in terms of undoubled backgrounds in DFT. Instead, the orbits that fail to satisfy (4.61) require, as we argued, truly doubled twist orbits for which we also provide concrete examples.

5. Show that there is a degeneracy in the space of twist orbits giving rise to the same orbit of gaugings. Interestingly, in some cases a given orbit can be obtained either from undoubled or truly doubled twist orbits.

In the next sections we will classify all the orbits in (half-)maximal $D \geq 7$ supergravities, and provide the half-maximal ones with concrete uplifts to DFT, explicitly proving the above points.

Parametrizations of the duality twists

Here we would like to introduce some notation that will turn out to be useful in the uplift of orbits to DFT. We start by noting the double internal coordinates as $Y^A = (\tilde{y}_a, y^a)$ with $a = 1, \ldots, n$. As we saw, the SS compactification of DFT is defined by the twists $U(\mathbb{Y})$ and $\lambda(\mathbb{Y})$. The duality twist $U(\mathbb{Y})$ is not generic, but forced to be an element of $O(n,n)$, so we should provide suitable parametrizations. One option is the light-cone parametrization, where the metric of the (internal) global symmetry group is taken to be of the form

$$\eta_{AB} = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \quad (4.63)$$
The most general form of the twist matrix is then given by
\[ U(Y) = \begin{pmatrix} e & 0 \\ 0 & e^{-T} \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ -B & 1_n \end{pmatrix} \begin{pmatrix} 1_n & \beta \\ 0 & 1_n \end{pmatrix}, \] (4.64)
with \( e \in \text{GL}(n) \) and \( B \) and \( \beta \) are generic \( n \times n \) antisymmetric matrices. When \( \beta = 0 \), \( e = e(y^a) \) and \( B = B(y^a) \), the matrix \( e \) can be interpreted as a \( n \)-dimensional internal vielbein and \( B \) as a background 2-form for the \( n \)-dimensional internal Kalb-Ramond field \( b \). Whenever the background is of this form, we will refer to it as geometric (notice that this still does not determine completely the background, which receives deformations from scalar fluctuations). In this case the gaugings take the simple form
\[ f_{abc} = 3(e^{-1})^a_{[\alpha}(e^{-1})^\beta_{\beta}(e^{-1})^\gamma_{c]\partial_{[\alpha}B_{\beta\gamma]}, \]
\[ f^a_{bc} = 2(e^{-1})^\beta_{[\alpha}(e^{-1})^\gamma_{\beta}\partial_{\beta}e^a_{\gamma}], \]
\[ f^{ab}_{\ c} = f^{abc} = 0. \] (4.65)

If we also turn on a \( \beta(y^a) \), the relation of \( e, B \) and \( \beta \) with the internal \( g \) and \( b \) is less trivial, and typically the background will be globally well defined up to \( \text{O}(n,n) \) transformations mixing the metric and the two-form (this is typically called a T-fold). In this case, we refer to the background as locally geometric but globally non-geometric, and this situation formally allows for non-vanishing \( f_{abc} \) and \( f^{abc} \). Finally, if the twist matrix is a function of \( \tilde{y}_a \), we refer to the background as locally non-geometric. Notice however, that if it satisfies the weak and strong constraints, one would always be able to rotate it to a frame in which it is locally geometric, and would therefore belong to an undoubled orbit.

Alternatively, one could also define the *Cartesian* parametrization of the twist matrix, by taking the metric of the (internal) global symmetry group to be of the form
\[ \eta_{AB} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}. \] (4.66)
This formulation is related to the light-cone parametrization through a \( \text{SO}(2n) \) transformation, that must also rotate the coordinates. In this case the relation between the components of the twist matrix and the internal \( g \) and \( b \) is non-trivial. We will consider the \( \text{O}(n,n) \) twist matrix to contain a smaller \( \text{O}(n-1,n-1) \) matrix in the directions \( (y^2, ..., y^n, \tilde{y}_2, ..., \tilde{y}_n) \) fibred over the flat directions \( (y^1, \tilde{y}_1) \). We have seen that this typically leads to constant gaugings.

Of course these are not the most general parametrizations and ansatz, but they will serve our purposes of uplifting all the orbits of half-maximal supergravity to DFT. Interesting works on how to generate gaugings from twists are [42].

### 4.4.2 U-duality orbits of maximal supergravities

Following the previous discussion of DFT and its relevance for generating duality orbits, we turn to the actual classification of these. In particular, we start with orbits
under U-duality of gaugings of maximal supergravity. Moreover, we will demonstrate that all such orbits do have a higher-dimensional supergravity origin.

Starting with the highest dimension for maximal supergravity, \( D = 11 \), no known deformation is possible here. Moreover, in \( D = 10 \) maximal supergravities, the only possible deformation occurs in what is known as massive IIA supergravity\(^8\)\([23]\). It consists of a Stückelberg-like way of giving a mass to the 2-form \( B_2 \). Therefore, such a deformation cannot be interpreted as a gauging. The string theory origin of this so-called Romans’ mass parameter is nowadays well understood as arising from D8-branes \([123]\). Furthermore, its DFT uplift has been constructed in ref. \([170]\). Naturally, the structure of possible orbits becomes richer when going to lower dimensions. In what follows we will perform the explicit classification in dimensions nine and eight.

**Orbits and origin of the \( D = 9 \) maximal case**

**Maximal \( D = 9 \) gauged supergravity**

The maximal (ungauged) supergravity in \( D = 9 \)\([105]\) can be obtained by reducing either massless type IIA or type IIB supergravity in ten dimensions on a circle. The global symmetry group of this theory is taken here to be

\[
G_0 = \mathbb{R}^+ \times \text{SL}(2)
\]

Note that \( G_0 \) is the global symmetry of the action and hence it is realized off-shell, whereas the on-shell symmetry has an extra \( \mathbb{R}^+ \) with respect to which the Lagrangian has a non-trivial scaling weight. This is normally referred to as the trombone symmetry. As a consequence, the on-shell symmetry contains three independent rescalings \([17,106]\), which we summarize in Table 4.2. The full field content consists of the following

| ID | \( e_\mu \) | \( A_\mu \) | \( A_\mu^1 \) | \( A_\mu^2 \) | \( B_{\mu\nu}^1 \) | \( B_{\mu\nu}^2 \) | \( C_{\mu\nu\rho} \) | \( e^\nu \) | \( \chi \) | \( e^\phi \) | \( \psi_\mu \) | \( \lambda \) | \( \bar{\lambda} \) | \( \mathcal{L} \) |
|----|-------------|------------|-------------|-------------|----------------|----------------|----------------|-------------|-----------|-------------|-------------|-----------|-----------|---------|
| \( \alpha \) | \( \frac{9}{7} \) | 3 | 0 | 0 | 3 | 3 | 3 | \( \frac{6}{\sqrt{7}} \) | 0 | 0 | \( \frac{9}{11} \) | \( -\frac{9}{11} \) | 9 |
| \( \beta \) | 0 | \( \frac{1}{2} \) | \( -\frac{3}{4} \) | 0 | \( -\frac{1}{4} \) | \( \frac{1}{4} \) | \( \frac{\sqrt{7}}{4} \) | \( -\frac{3}{4} \) | \( \frac{3}{4} \) | 0 | 0 | 0 |
| \( \gamma \) | 0 | 0 | 1 | \( -1 \) | 1 | \( -1 \) | 0 | 2 | \( -2 \) | 0 | 0 | 0 |
| \( \delta \) | \( \frac{8}{7} \) | 0 | 2 | 2 | 2 | 2 | 4 | \( -\frac{4}{\sqrt{7}} \) | 0 | 0 | \( \frac{4}{7} \) | \( -\frac{4}{7} \) | 8 |

Table 4.2: The scaling weights of the nine-dimensional fields. As already anticipated, only three rescalings are independent since they are subject to the following constraint: \( 8\alpha - 48\beta - 18\gamma - 9\delta = 0 \). As the scaling weight of the Lagrangian \( \mathcal{L} \) shows, \( \beta \) and \( \gamma \) belong to the off-shell symmetries, whereas \( \alpha \) and \( \delta \) can be combined into a trombone symmetry and an off-shell symmetry.

\(^8\)Throughout this section we will not consider the trombone gaugings giving rise to theories without an action principle, as discussed in e.g. \([106,140,168,169]\).
objects (see also Chapter 3 for more details) which arrange themselves into irreducible representations of $\mathbb{R}^+ \times \text{SL}(2)$:

\[
9D: \quad \varepsilon^a_{\mu}, A_{\mu}, A^{i}_{\mu}, B_{\mu\nu}^i, C_{\mu
u\rho}, \varphi, \tau = \chi + i e^{-\phi}; \quad \psi_{\mu}, \lambda, \tilde{\lambda},
\]

where $\mu, \nu, \ldots$ denote 9-dimensional curved spacetime, $a, b, \ldots$ 9-dimensional flat spacetime and $i, j, \cdots$ fundamental $\text{SL}(2)$ indices respectively.

The general deformations of this theory have been studied in detail in Chapter 3 (see also ref. [164]), where both embedding tensor deformations and gaugings of the trombone symmetry have been considered. For the present scope we shall restrict ourselves to the first ones. The latter ones would correspond to the additional mass parameters $m_{\text{IIB}}$ and $(m_{11}, m_{\text{IIA}})$ in refs [106, 164], which give rise to theories without an action principle.

The vectors of the theory $\{A_{\mu}, A^{i}_{\mu}\}$ transform in the $V' = 1_{(+4)} \oplus 2_{(-3)}$ of $\mathbb{R}^+ \times \text{SL}(2)$, where the $\mathbb{R}^+$ scaling weights are included as well. The resulting embedding tensor deformations live in the following tensor product

\[
g_0 \otimes V = 1_{(-4)} \oplus 2_{(+3)} \oplus 3_{(-4)} \oplus 4_{(+3)}.
\]

The Linear Constraint (LC) projects out the $4_{(+3)}$, the $1_{(-4)}$ and one copy of the $2_{(+3)}$ since they would give rise to inconsistent deformations. As a consequence, the consistent gaugings are parametrized by embedding tensor components in the $2_{(+3)} \oplus 3_{(-4)}$. We will denote these allowed deformations by $\theta^i$ and $\kappa^{ij}$.

The closure of the gauge algebra and the antisymmetry of the brackets impose the following Quadratic Constraints (QC)

\[
\epsilon_{ij} \theta^i \kappa^{jk} = 0, \quad 2_{(-1)} \quad (4.69)
\]

\[
\theta^{(i} \kappa^{jk)} = 0, \quad 4_{(-1)} \quad (4.70)
\]

**The $\mathbb{R}^+ \times \text{SL}(2)$ orbits of solutions to the QC**

The QC (4.69) and (4.70) turns out to be very simple to solve; after finding all the solutions, we studied the duality orbits, i.e. classes of those solutions which are connected via a duality transformation. The resulting orbits of consistent gaugings in this case are presented in Table 4.3.

**Higher-dimensional geometric origin**

The four different orbits of maximal $D = 9$ theory have the following higher-dimensional origin in terms of geometric compactifications [171]:

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9The $\mathbb{R}^+$ factor in the global symmetry is precisely the combination $\left(\frac{4}{3} \alpha - \frac{4}{3} \delta\right)$ of the different rescalings introduced in ref. [106].
4.4. Duality orbits of non-geometric fluxes

| ID | $\theta^i$ | $\kappa^{ij}$ | gauging |
|----|-------------|----------------|---------|
| 1  | diag(1,1)   |                | SO(2)   |
| 2  | (0,0)       | diag(1,-1)     | SO(1,1) |
| 3  | diag(1,0)   |                | $\mathbb{R}_\gamma^+$ |
| 4  | (1,0)       | diag(0,0)      | $\mathbb{R}_\beta^+$ |

Table 4.3: All the U-duality orbits of consistent gaugings in maximal supergravity in $D = 9$. For each of them, the simplest representative is given. The subscripts $\beta$ and $\gamma$ refer to the rescalings summarized in Table 4.2.

- **Orbits 1 – 3**: These come from reductions of type IIB supergravity on a circle with an SL(2) twist.

- **Orbit 4**: This can be obtained from a reduction of type IIA supergravity on a circle with the inclusion of an $\mathbb{R}_\beta^+$ twist.

**Orbits and origin of the $D = 8$ maximal case**

**Maximal $D = 8$ gauged supergravity**

The maximal (ungauged) supergravity in $D = 8$ \(^{172}\) can be obtained by reducing eleven-dimensional supergravity on a $T^3$. The global symmetry group of this theory is

$$G_0 = \text{SL}(2) \times \text{SL}(3).$$

The full field content consists of the following objects which arrange themselves into irreps of $\text{SL}(2) \times \text{SL}(3)$:

$$8D : \underbrace{e_{\mu}^a, A_{\mu}^{am}, B_{\mu\nu m}, C_{\mu\nu p}, L_{I}^{m}}_{\text{bosonic dof's}}, \underbrace{\phi, \chi}_{\text{bosonic dof's}}, \underbrace{\psi_{\mu}, \chi_{I}}_{\text{fermionic dof's}},$$

where $\mu, \nu, \cdots$ denote eight-dimensional curved spacetime, $a, b, \cdots$ eight-dimensional flat spacetime, $m, n, \cdots$ fundamental $\text{SL}(3)$, $I, J, \cdots$ fundamental $\text{SO}(3)$ and $\alpha, \beta, \cdots$ fundamental $\text{SL}(2)$ indices respectively. The six vector fields $A_{\mu}^{am}$ in (4.71) transform in the $V' = (2, 3')$. There are eleven group generators, which can be expressed in the adjoint representation $\mathfrak{g}_0$.

The embedding tensor $\Theta$ then lives in the representation $\mathfrak{g}_0 \otimes V$, which can be decomposed into irreducible representations as

$$\mathfrak{g}_0 \otimes V = 2 \cdot (2, 3) \oplus (2, 6') \oplus (2, 15) \oplus (4, 3).$$

The LC restricts the embedding tensor to the $(2, 3) \oplus (2, 6') ^{112}$. It is worth noticing that there are two copies of the $(2, 3)$ irrep in the above composition; the LC imposes a relation between them \([113]\). This shows that, for consistency, gauging some $\text{SL}(2)$
generators implies the necessity of gauging some $\text{SL}(3)$ generators as well. Let us denote the allowed embedding tensor irrep’s by $\xi_{\alpha m}$ and $f^{(mn)}_{\alpha}$ respectively.

The quadratic constraints (QC) then read \[ 163, 173 \]

\[ \epsilon^{\alpha \beta} \xi_{\alpha p} \xi_{\beta q} = 0 \quad (1, 3') \quad (4.73) \]

\[ f_{(\alpha np} \xi_{\beta m)} = 0 \quad (3, 3') \quad (4.74) \]

\[ \epsilon^{\alpha \beta} (\epsilon_{mqr} f^{\alpha qn} f^{\beta rp} + f_{\alpha np} \xi_{\beta m}) = 0 \quad (1, 3') \oplus (1, 15) \quad (4.75) \]

Any solution to the QC (4.73), (4.74) and (4.75) specifies a consistent gauging of a subgroup of $\text{SL}(2) \times \text{SL}(3)$ where the corresponding generators are given by

\[ (X_{\alpha m})_{\beta}^\gamma = \delta_{\alpha}^\gamma \xi_{\beta m} - \frac{1}{2} \delta_{\beta}^\gamma \xi_{\alpha m} \quad (4.76) \]

\[ (X_{\alpha m})_{n}^p = \epsilon_{mnp} f^{\alpha qn} - \frac{3}{4} \left( \delta_{m}^p \xi_{\alpha n} - \frac{1}{3} \delta_{m}^p \xi_{\alpha m} \right) \quad (4.77) \]

The $\text{SL}(2) \times \text{SL}(3)$ orbits of solutions to the QC

We exploited an algebraic geometry tool called the Gianni-Trager-Zacharias (GTZ) algorithm \[ 174 \]. This algorithm has been computationally implemented by the \textsc{Singular} project \[ 175 \] and it consists in the primary decomposition of ideals of polynomials. After finding all the solutions to the QC by means of the algorithm mentioned above, one has to group together all the solutions which are connected through a duality transformation, thus obtaining a classification of such solutions in terms of duality orbits. The resulting orbits of consistent gaugings\[ 163 \] in this case are presented in Table \[ 4.4 \].

Higher-dimensional geometric origin

- **Orbits 1 – 5:** These stem from reductions of eleven-dimensional supergravity on a three-dimensional group manifold of type A in the Bianchi classification \[ 176 \]. The special case in orbit 1 corresponds to a reduction over an $\text{SO}(3)$ group manifold and it was already studied in ref. \[ 172 \].

- **Orbit 6:** This can be obtained from a reduction of maximal nine-dimensional supergravity on a circle with the inclusion of an $\mathbb{R}^+$ twist inside the global symmetry group.

- **Orbits 7 – 9:** These can come from the same reduction from $D = 9$ but upon inclusion of a more general $\mathbb{R}^+ \times \text{SL}(2)$ twist.

- **Orbit 10:** This orbit seems at first sight more complicated to be obtained from a dimensional reduction owing to its non-trivial $\text{SL}(2)$ angles. Nevertheless, it turns out that one can land on this orbit by compactifying type IIB supergravity on a

\[ \text{orbit 3} \]

Similarly, the possible vacua of the different theories have been analyzed \[ 163 \]. It was found that only orbit 3 has maximally symmetric vacua.
4.4. Duality orbits of non-geometric fluxes

| ID | $f_+^{mn}$       | $f_-^{mn}$       | $\xi_+$  | $\xi_-$  | gauging                                    |
|----|-----------------|-----------------|----------|----------|--------------------------------------------|
| 1  | diag(1, 1, 1)   |                 |          |          | SO(3)                                      |
| 2  | diag(1, 1, -1)  |                 |          |          | SO(2, 1)                                   |
| 3  | diag(1, 1, 0)   | diag(0, 0, 0)   | (0, 0, 0)| (0, 0, 0)| ISO(2)                                     |
| 4  | diag(1, -1, 0)  |                 |          |          | ISO(1, 1)                                  |
| 5  | diag(1, 0, 0)   |                 |          |          | CSO(1, 0, 2)                               |
| 6  | diag(0, 0, 0)   | diag(0, 0, 0)   | (1, 0, 0)| (0, 0, 0)| Solv$_2 \times$ Solv$_3$                  |
| 7  | diag(1, 1, 0)   |                 |          |          |                                            |
| 8  | diag(1, -1, 0)  | diag(0, 0, 0)   | (0, 0, 1)| (0, 0, 0)| Solv$_2 \times$ Solv$_3$                  |
| 9  | diag(1, 0, 0)   |                 |          |          |                                            |
| 10 | diag(1, -1, 0)  | $\begin{pmatrix}1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | $\frac{2}{9}(0, 0, 1)$ | (0, 0, 0) | Solv$_2 \times$ SO(2) $\ltimes$ Nil$_3(2)$ |

Table 4.4: All the U-duality orbits of consistent gaugings in maximal supergravity in $D = 8$. For each of them, the simplest representative is given. We denote by Solv$_2 \subset$ SL(2) and Solv$_3 \subset$ SL(3) a solvable algebra of dimension 2 and 3 respectively. To be more precise, Solv$_2$ identifies the Borel subgroup of SL(2) consisting of $2 \times 2$ upper-triangular matrices. Solv$_3$, instead, is a Bianchi type V algebra.

circle with an SL(2) twist and then further reducing on another circle with $\mathbb{R}^+ \times$ SL(2) twist given by the residual little group leaving invariant the intermediate nine-dimensional deformation.

Remarks on the $D = 7$ maximal case

The general deformations of the maximal theory in $D = 7$ are constructed and presented in full detail in ref. [115]. For the present aim we only summarize here a few relevant facts.

The global symmetry group of the theory is SL(5). The vector fields $A_\mu^{MN} = A_\mu^{[MN]}$ transform in the $10'$ of SL(5), where we denote by $M$ a fundamental SL(5) index. The embedding tensor $\Theta$ takes values in the following irreducible components

$$10 \otimes 24 = 10 \oplus 15 \oplus 40' \oplus 175.$$  \hspace{1cm} (4.78)

The LC restricts the embedding tensor to the $15 \oplus 40'$, which can be parametrized by the following objects

$$Y_{(MN)}, \quad \text{and} \quad Z^{[MN],P} \quad \text{with} \quad Z^{[MN,P]} = 0 .$$  \hspace{1cm} (4.79)

The generators of the gauge algebra can be written as follows

$$(X_{MN})_P^Q = \delta^Q_M Y_{NP} - 2 \epsilon_{MNPQR} Z^{RS,Q} .$$  \hspace{1cm} (4.80)
or, identically, if one wants to express them in the $10$,

$$(X_{MN})^P_{RS} = 2 (X_{MN})^P_{[P} R^S]_{Q]}.$$  \hspace{1cm} (4.81)

The closure of the gauge algebra and the antisymmetry of the brackets imply the following QC

$$Y_{MQ} Z^{Q,N,P} + 2 \epsilon_{MRSTU} Z^{RS,N} Z^{TU,P} = 0,$$  \hspace{1cm} (4.82)

which have different irreducible pieces in the $5' \oplus 45' \oplus 70'$. Unfortunately, in this case, both the embedding tensor deformations and the quadratic constraints reach a level of complexity that makes an exhaustive and general analysis difficult. Such analysis lies beyond the scope of our work.

### 4.4.3 T-duality orbits of half-maximal supergravities

After the previous section on maximal supergravities, we turn our attention to theories with half-maximal supersymmetry. In particular, in this section we will classify the orbits under T-duality of all gaugings of half-maximal supergravity. We will only consider the theories with duality groups $\mathbb{R}^+ \times \text{SO}(d,d)$ in $D = 10 - d$, which places a restriction on the number of vector multiplets. For these theories we will classify all duality orbits, and find a number of non-geometric orbits. Furthermore, we demonstrate that double field theory does yield a higher-dimensional origin for all of them.

Starting from $D = 10$ half-maximal supergravity without vector multiplets, it can be seen that there is no freedom to deform this theory, rendering this case trivial. In $D = 9$, instead, we have the possibility of performing an Abelian gauging inside $\mathbb{R}^+ \times \text{SO}(1,1)$, which will depend on one deformation parameter. However, this is precisely the parameter that one expects to generate by means of a twisted reduction from $D = 10$. This immediately tells us that non-geometric fluxes do not yet appear in this theory. In order to find the first non-trivial case, we will have to consider the $D = 8$ case.

**Orbits and origin of the $D = 8$ half-maximal case**

**Half-maximal $D = 8$ gauged supergravity**

Half-maximal supergravity in $D = 8$ is related to the maximal theory analyzed in the previous section by means of a $\mathbb{Z}_2$ truncation. The action of such a $\mathbb{Z}_2$ breaks $\text{SL}(2) \times \text{SL}(3)$ into $\mathbb{R}^+ \times \text{SL}(2) \times \text{SL}(2)$, where $\text{SL}(2) \times \text{SL}(2) = O(2,2)$ can be interpreted as the T-duality group in $D = 8$ as shown in Table 4.1. The embedding of $\mathbb{R}^+ \times \text{SL}(2)$ inside $\text{SL}(3)$ is unique and it determines the following branching of the fundamental representation

$$3 \rightarrow 1_{(+2)} \oplus 2_{(-1)},$$

$$m \rightarrow (\bullet, i),$$
where the $\mathbb{R}^+$ direction labeled by $\bullet$ is parity even, whereas $i$ is parity odd, such as the other $\text{SL}(2)$ index $\alpha$. In the following we will omit all the $\mathbb{R}^+$ weights since they do not play any role in the truncation.

The embedding tensor of the maximal theory splits in the following way

$$(2,3) \rightarrow [2,1] \oplus (2,2),$$

$$(2,6') \rightarrow [2,1] \oplus (2,2) \oplus [2,3],$$

where all the crossed irrep’s are projected out because of $\mathbb{Z}_2$ parity. This implies that the consistent embedding tensor deformations of the half-maximal theory can be described by two objects which are doublets with respect to both $\text{SL}(2)$’s. Let us denote them by $a_{\alpha i}$ and $b_{\alpha i}$. This statement is in perfect agreement with the Kač-Moody analysis performed in ref. [118]. The explicit way of embedding $a_{\alpha i}$ and $b_{\alpha i}$ inside $\xi_{\alpha m}$ and $f_{\alpha mn}$ is given by

$$f_{\alpha i} \epsilon = f_{\alpha i} = \epsilon^{ij} a_{\alpha j}, \quad (4.83)$$

$$\xi_{\alpha i} = 4 b_{\alpha i}. \quad (4.84)$$

The QC given in (4.73), (4.74) and (4.75) are decomposed according to the following branching

$$(1,3') \rightarrow (1,1) \oplus [1,2],$$

$$(3,3') \rightarrow (3,1) \oplus [3,2],$$

$$(1,15) \rightarrow (1,1) \oplus 2 \cdot (1,2) \oplus 2 \cdot (1,3) \oplus [1,4].$$

As a consequence, one expects the set of $\mathbb{Z}_2$ even QC to consist of 3 singlets, a $(3,1)$ and 2 copies of the $(1,3)$. By plugging (4.83) and (4.84) into (4.73), (4.74) and (4.75), one finds

$$\epsilon^{\alpha \beta} \epsilon^{ij} b_{\alpha i} b_{\beta j} = 0, \quad (1,1) \quad (4.85)$$

$$\epsilon^{\alpha \beta} \epsilon^{ij} a_{\alpha i} b_{\beta j} = 0, \quad (1,1) \quad (4.86)$$

$$\epsilon^{\alpha \beta} \epsilon^{ij} a_{\alpha i} a_{\beta j} = 0, \quad (1,1) \quad (4.87)$$

$$\epsilon^{ij} a_{(\alpha i} b_{\beta j)} = 0, \quad (3,1) \quad (4.88)$$

$$\epsilon^{\alpha \beta} a_{(\alpha i} b_{\beta j)} = 0. \quad (1,3) \quad (4.89)$$

With respect to what we expected from group theory, we seem to be finding a $(1,3)$ less amongst the even QC. This could be due to the fact that $\mathbb{Z}_2$ even QC can be sourced by quadratic expressions in the odd embedding tensor components that we truncated away. After the procedure of turning off all of them, the two $(1,3)$’s probably collapse to the same constraint or one of them vanishes directly.

The above set of QC characterizes the consistent gaugings of the half-maximal theory which are liftable to the maximal theory, and hence they are more restrictive than the pure consistency requirements of the half-maximal theory. In order to single
4. DFT and Duality orbits of non-geometric fluxes

Table 4.5: All the T-duality orbits of consistent gaugings in half-maximal supergravity in $D=8$. For each of them, the simplest representative is given. Solv$_2$ refers again to the solvable subgroup of SL(2) as already explained in the caption of Table 4.4.

| ID | $a_{ai}$ | $b_{ai}$ | gauging         |
|----|----------|----------|----------------|
| 1  | $\text{diag}(\cos \alpha, 0)$ | $\text{diag}(\sin \alpha, 0)$ | Solv$_2 \times \text{SO}(1,1)$ |
| 2  | $\text{diag}(1,1)$ | $\text{diag}(-1,-1)$ | SL(2) $\times$ SO(1,1) |
| 3  | $\text{diag}(1,-1)$ | $\text{diag}(-1,1)$ | SL(2) $\times$ SO(1,1) |

Out only these we need to write down the expression of the gauge generators and impose the closure of the algebra. The gauge generators in the $(2, 2)$ read

\[
(X_{ai})_{\beta j} \gamma^k = \frac{1}{2} \delta^k_\beta \epsilon_{ij} \epsilon^{kl} a_{ai} a_{bj} + \delta^{\gamma}_{\beta} \delta^k_i b_{ai} + \frac{1}{2} \delta^\gamma_\beta \delta^k_j b_{ai} + \epsilon_{\alpha\beta} \epsilon^{\gamma\delta} \delta^k_i b_{ai} .
\] (4.90)

The closure of the algebra generated by (4.90) implies the following QC

\[
\begin{align*}
\epsilon^{\alpha\beta} \epsilon^{ij} (a_{ai} a_{bj} - b_{ai} b_{bj}) & = 0 , & (1, 1) \\
\epsilon^{\alpha\beta} \epsilon^{ij} (a_{ai} b_{bj} + b_{ai} b_{bj}) & = 0 , & (1, 1) \\
\epsilon^{ij} a_{ai} a_{bj} & = 0 , & (3, 1) \\
\epsilon_{\alpha\beta} a_{ai} b_{bj} & = 0 . & (1, 3)
\end{align*}
\] (4.91) (4.92) (4.93) (4.94)

To facilitate the mapping of gaugings $a_{ai}$ and $b_{ai}$ with the more familiar $f_{ABC}$ and $\xi_A$ in the DFT language, we have written a special section in the appendix C.2. The mapping is explicitly given in (C.16).

The O(2, 2) orbits of solutions to the QC

After solving the QC given in (4.91), (4.92), (4.93) and (4.94) again with the aid of SINGULAR, we find a 1-parameter family of T-duality orbits plus two discrete ones. The results are all collected in Table 4.5.

Higher-dimensional geometric origin

The possible higher-dimensional origin of the three different orbits is as follows:

- **Orbit 1**: This orbit can be obtained by performing a two-step reduction of type I supergravity. In the first step, by reducing a circle, we can generate an $\mathbb{R}^+ \times \text{SO}(1,1)$ gauging of half-maximal $D=9$ supergravity. Subsequently, we reduce such a theory again on a circle with the inclusion of a new twist commuting with the previous deformation. Also, these orbits include a non-trivial $\xi_A$ gauging, so we will not address it from a DFT perspective.
• **Orbits 2 – 3:** These do not seem to have any obvious geometric higher-dimensional origin in supergravity. In fact, they do not satisfy the extra constraints (4.61), so one can only hope to reproduce them from truly doubled twist orbits in DFT. Therefore we find that, while the half-maximal orbits in $D = 9$ all have a known geometric higher-dimensional origin, this is not the case for the latter two orbits in $D = 8$. We have finally detected the first signals of non-geometric orbits.

**Higher-dimensional DFT origin**

As mentioned, the orbits 2 and 3 lack of a clear higher-dimensional origin. Here we would like to provide a particular twist matrix giving rise to these gaugings. We chose to start in the Cartesian framework, and propose the following form for the SO(2, 2) twist matrix

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh(m y^1 + n \tilde{y}_1) & 0 & \sinh(m y^1 + n \tilde{y}_1) \\ 0 & 0 & 1 & 0 \\ 0 & \sinh(m y^1 + n \tilde{y}_1) & 0 & \cosh(m y^1 + n \tilde{y}_1) \end{pmatrix}. \tag{4.95}$$

This is in fact an element of SO(1, 1) lying in the directions $(\tilde{y}_2, y^2)$, fibred over the double torus $(\tilde{y}_1, y^1)$. Here, the coordinates are written in the Cartesian formulation, so we must rotate this in order to make contact with the light-cone case.

For this twist matrix, the weak and strong constraints in the light-cone formulation read $(m + n)(m - n) = 0$, while the QC are always satisfied. The gaugings are constant, and when written in terms of $a_{\alpha i}$ and $b_{\alpha i}$ we find

$$a_{\alpha i} = -b_{\alpha i} = \text{diag} \left( -\frac{m + n}{2 \sqrt{2}}, \frac{m - n}{2 \sqrt{2}} \right), \tag{4.96}$$

so orbit 2 is obtained by choosing $m = 0, n = -2 \sqrt{2}$, and orbit 3 by choosing $m = -2 \sqrt{2}, n = 0$. Notice that in both cases the twist orbit is truly doubled, so we find the first example of an orbit of gaugings without a clear supergravity origin, that finds an uplift to DFT in a truly doubled background.

**Orbits and origin of the $D = 7$ half-maximal case**

**Half-maximal $D = 7$ gauged supergravity**

A subset of half-maximal gauged supergravities is obtained from the maximal theory introduced in Section 4.4.2 by means of a $\mathbb{Z}_2$ truncation. Thus, we will in this section perform this truncation and carry out the orbit analysis in the half-maximal theory. As we already argued before, this case is not only simpler, but also much more insightful from the point of view of understanding T-duality in gauged supergravities and its relation to DFT.
The action of our $\mathbb{Z}_2$ breaks $\text{SL}(5)$ into $\mathbb{R}^+ \times \text{SL}(4)$. Its embedding inside $\text{SL}(5)$ is unique and it is such that the fundamental representation splits as follows

$$5 \rightarrow 1_{(+4)} \oplus 4_{(-1)}.$$  \hfill (4.97)

After introducing the following notation for the indices in the $\mathbb{R}^+$ and in the $\text{SL}(4)$ directions

$$M \rightarrow (\triangledown, m),$$  \hfill (4.98)

we assign an even parity to the $\triangledown$ direction and odd parity to $m$ directions.

The embedding tensor of the maximal theory splits according to

$$15 \rightarrow 1 \oplus 4 \oplus 10,$$  \hfill (4.99)

$$40' \rightarrow \wedge^3 \oplus 6 \oplus 10' \oplus 20,$$  \hfill (4.100)

where again, as in Section 4.4.3, all the crossed irreps are projected out because of $\mathbb{Z}_2$ parity. This implies that the embedding tensor of the half-maximal theory lives in the $1 \oplus 6 \oplus 10 \oplus 10'$ and hence it is described by the following objects

$$\theta, \xi_{[mn]}, M_{(mn)} , \tilde{M}^{(mn)}.$$  \hfill (4.101)

This set of deformations agrees with the decomposition $D_8^{+++} \rightarrow A_3 \times A_6$ given in ref. [118]. The objects in (4.101) are embedded in $Y$ and $Z$ in the following way

$$Y_{\triangledown} = \theta,$$  \hfill (4.102)

$$Y_{mn} = \frac{1}{2} M_{mn},$$  \hfill (4.103)

$$Z^{mn,\triangledown} = \frac{1}{8} \xi^{mn},$$  \hfill (4.104)

$$Z^{m\triangledown,n} = -Z^{\triangledown m,n} = \frac{1}{16} \tilde{M}^{mn} + \frac{1}{16} \xi^{mn},$$  \hfill (4.105)

where for convenience we defined $\xi^{mn} = \frac{1}{2} \epsilon^{mnpq} \xi_{pq}$.

Now we will obtain the expression of the gauge generators of the half-maximal theory by plugging the expressions (4.102) – (4.105) into (4.80). We find

$$(X_{mn})_p^q = \frac{1}{2} \delta^q_m M_{np} - \frac{1}{4} \epsilon_{m n p r} \left( \tilde{M} + \xi \right)^r_q,$$  \hfill (4.106)

which extends the expression given in ref. [177] by adding an antisymmetric part to $\tilde{M}$ proportional to $\xi$. Note that the $\xi$ term is also the only one responsible for the

\footnote{The $\mathbb{Z}_2$ element with respect to which we are truncating is the following $\text{USp}(4) = \text{SO}(5)$ element

$$\alpha = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$$

projecting out half of the supercharges.}
trace of the gauge generators which has to be non-vanishing in order to account for $\mathbb{R}^+$ gaugings.

The presence of such a term in the expression (4.106) has another consequence: the associated structure constants that one writes by expressing the generators in the $\mathbf{6}$ $(X_{mn})_{pq}^{rs}$ will not be automatically antisymmetric in the exchange between $mn$ and $pq$. This implies the necessity of imposing the antisymmetry by means of some extra QC\textsuperscript{12}

The QC of the maximal theory are branched into

\begin{align}
5' & \rightarrow 1 \oplus \mathfrak{X}, \\
45' & \rightarrow \mathfrak{X} \oplus 6 \oplus 15 \oplus 26, \\
70' & \rightarrow 1 \oplus \mathfrak{X} \oplus \mathfrak{X} \oplus 10' \oplus 15 \oplus 36'.
\end{align}

By substituting the expressions (4.102) – (4.105) into the QC (4.82), one finds

\begin{align}
\theta \xi_{mn} &= 0, \quad (6) \\
(\tilde{M}^{mp} + \xi^{mp}) M_{pq} &= 0, \quad (1 \oplus 15) \\
M_{mp} \xi^{pn} - \xi_{mp} (\tilde{M}^{pn} + \xi^{pm}) &= 0, \quad (1 \oplus 15) \\
\theta \tilde{M}^{mn} &= 0. \quad (10')
\end{align}

Based on the Kač-Moody analysis performed in ref. \[118\], the QC constraints of the half-maximal theory should only impose conditions living in the $1 \oplus 6 \oplus 15 \oplus 15$. The problem is then determining which constraint in the $1$ is already required by the half-maximal theory and which is not.

By looking more carefully at the constraints (4.110) – (4.113), we realize that the traceless part of (4.111) exactly corresponds to the Jacobi identities that one gets from the closure of the algebra spanned by the generators (4.106), whereas the full (4.112) has to be imposed to ensure antisymmetry of the gauge brackets. Since there is only one constraint in the $6$, we do not have ambiguities there\textsuperscript{13}.

We are now able to write down the set of QC of the half-maximal theory:

\begin{align}
\theta \xi_{mn} &= 0, \quad (6) \\
(\tilde{M}^{mp} + \xi^{mp}) M_{pq} - \frac{1}{4} (\tilde{M}^{mp} M_{np}) \delta_{q}^{m} &= 0, \quad (15) \\
M_{mp} \xi^{pn} + \xi_{mp} \tilde{M}^{pn} &= 0, \quad (15) \\
\epsilon^{mnpq} \xi_{mn} \xi_{pq} &= 0. \quad (1)
\end{align}

\textsuperscript{12}The QC which ensure the antisymmetry of the gauge brackets are given by $(X_{mn})_{pq}^{rs} X_{rs} + (mn \leftrightarrow pq) = 0$, where $X$ is given in an arbitrary representation.

\textsuperscript{13}We would like to stress that the parameter $\theta$ within the half-maximal theory is a consistent deformation, but it does not correspond to any gauging and hence QC involving it cannot be derived as Jacobi identities or other consistency constraints coming from the gauge algebra.
4. DFT and Duality orbits of non-geometric fluxes

We are not really able to confirm whether (4.114) is part of the QC of the half-maximal theory, in the sense that there appears a top-form in the $6$ from the $D_8^{+++}$ decomposition but it could either be a tadpole or a QC. This will however not affect our further discussion, in that we only consider orbits of gaugings in which $\theta = 0$. The extra QC required in order for the gauging to admit an uplift to maximal supergravity are

\begin{align}
\tilde{M}^{mn} M_{mn} &= 0, \quad (1) \\
\theta \tilde{M}^{mn} &= 0. \quad (10')
\end{align}

The $O(3,3)$ orbits of solutions to the QC in the $10 \oplus 10'$

The aim of this section is to solve the constraints summarized in (4.114), (4.115), (4.116) and (4.117). We will start by considering the case of gaugings only involving the $10 \oplus 10'$. This restriction is motivated by flux compactification, as we will try to argue later on.

The only non-trivial QC are the following

\[ \tilde{M}^{mp} M_{pm} - \frac{1}{4} \left( \tilde{M}^{pq} M_{pq} \right) \delta^m_n = 0, \quad (4.120) \]

which basically implies that the matrix product between $M$ and $\tilde{M}$, which in principle lives in the $1 \oplus 15$, has to be pure trace. We made use of a GL(4) transformation in order to reduce $M$ to pure signature; as a consequence, the QC (4.120) imply that $\tilde{M}$ is diagonal as well $[178]$. This results in a set of eleven 1-parameter orbits of solutions to the QC which are given in Table 4.6.

As we will see later, some of these consistent gaugings in general include non-zero non-geometric fluxes, but at least in some of these cases one will be able to dualize the given configuration to a perfectly geometric background.

Higher-dimensional geometric origin

Ten-dimensional heterotic string theory compactified on a $T^3$ gives rise to a half-maximal supergravity in $D = 7$ where the $\text{SL}(4) = \text{SO}(3, 3)$ factor in the global symmetry of this theory can be interpreted as the T-duality group. The set of generalized fluxes which can be turned on here is given by

\[ \{ f_{abc}, f_{ab}^c, f_a^{bc}, f^{abc} \} \equiv \{ H_{abc}, \omega_{ab}^c, Q_a^{bc}, R^{abc} \}, \quad (4.121) \]

where $a, b, c = 1, 2, 3$.

These are exactly the objects that one obtains by decomposing a three-form of $\text{SO}(3, 3)$ with respect to its $\text{GL}(3)$ subgroup. The number of independent components

\footnote{We would like to point out that the extra discrete generator $\eta$ of $O(3,3)$ makes sure that, given a certain gauging with $M$ and $\tilde{M}$, it lies in the same orbit as its partner with the role of $M$ and $-\tilde{M}$ interchanged.}
4.4. Duality orbits of non-geometric fluxes

This gives rise to the following dictionary between the non-semisimple gauge algebras $f_M$ such that the matching between the embedding tensor deformations ($M_{mn}$, $\tilde{M}^{mn}$) and the generalized fluxes given in (4.121) now perfectly works. The explicit mapping between vectors of $SO(3,3)$ expressed in light-cone coordinates and two-forms of $SL(4)$ can be worked out by means of the $SO(3,3)$ 't Hooft symbols ($G_A$)mn (see Appendix C.2). This gives rise to the following dictionary between the $M$ and $\tilde{M}$-components and the fluxes given in (4.121)

$$M = \text{diag} \left( H_{123}, Q_1^{23}, Q_2^{31}, Q_3^{12} \right), \quad \tilde{M} = \text{diag} \left( R^{123}, \omega_1^{23}, \omega_2^{31}, \omega_3^{12} \right). \quad (4.123)$$

Table 4.6: All the T-duality orbits of consistent gaugings in half-maximal supergravity in $D = 7$. Any value of $\alpha$ parametrizes inequivalent orbits. More details about the non-semisimple gauge algebras $f_1$, $f_2$, $h_1$, $h_2$, $g_0$ and $l$ are given in appendix C.1.

| ID | $M_{mn} / \cos \alpha$ | $M^{mn} / \sin \alpha$ | range of $\alpha$ | gauging |
|----|-------------------|------------------|----------------|---------|
| 1  | diag(1,1,1,1)     | diag(1,1,1,1)    | $-\pi < \alpha \leq \pi$ | $\{ \text{SO}(4), \alpha \neq \pi, \text{SO}(3), \alpha = \pi \}$ |
| 2  | diag(1,1,1,−1)    | diag(1,1,1,−1)   | $-\pi < \alpha < \frac{\pi}{2}$ | $\text{SO}(3,1)$ |
| 3  | diag(1,1,−1,−1)   | diag(1,1,−1,−1)  | $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$ | $\{ \text{SO}(2,2), \alpha \neq \frac{\pi}{4}, \text{SO}(2,1), \alpha = \frac{\pi}{4} \}$ |
| 4  | diag(1,1,1,0)     | diag(0,0,1,0)    | $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ | ISO(3) |
| 5  | diag(1,1,−1,0)    | diag(0,0,1,0)    | $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ | ISO(2,1) |
| 6  | diag(1,1,0,0)     | diag(0,1,1,0)    | $-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$ | $\{ \text{CSO}(2,0,2), \alpha \neq \frac{\pi}{4}, f_1 (\text{Solv}_6), \alpha = \frac{\pi}{4} \}$ |
| 7  | diag(1,1,0,0)     | diag(0,0,1,−1)   | $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ | $\{ \text{CSO}(2,0,2), |\alpha| < \frac{\pi}{4}, \text{CSO}(1,1,2), |\alpha| > \frac{\pi}{4}, g_0 (\text{Solv}_6), |\alpha| = \frac{\pi}{4} \}$ |
| 8  | diag(1,1,0,0)     | diag(0,0,0,1)    | $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ | $h_1 (\text{Solv}_6)$ |
| 9  | diag(1,−1,0,0)    | diag(0,0,1,−1)   | $-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$ | $\{ \text{CSO}(1,1,2), \alpha \neq \frac{\pi}{4}, f_2 (\text{Solv}_6), \alpha = \frac{\pi}{4} \}$ |
| 10 | diag(1,−1,0,0)    | diag(0,0,0,1)    | $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ | $h_2 (\text{Solv}_6)$ |
| 11 | diag(1,0,0,0)     | diag(0,0,0,1)    | $-\frac{\pi}{4} < \alpha \leq \frac{\pi}{4}$ | $\{ l (\text{Nil}_6(3)), \alpha \neq 0, \text{SO}(1,0,3), \alpha = 0 \}$ |

of the above fluxes (including traces of $\omega$ and $Q$) amounts to $1 + 9 + 9 + 1 = 20$, which is the number of independent components of a three-form of $SO(3,3)$. Nevertheless, the three-form representation is not irreducible since the Hodge duality operator in 3+3 dimensions squares to 1. This implies that one can always decompose it in a self-dual (SD) and anti-self-dual (ASD) part

$$10 \oplus 10' \text{ of SL(4)} \longleftrightarrow 10_{\text{SD}} \oplus 10_{\text{ASD}} \text{ of SO(3,3)}, \quad (4.122)$$

such that the matching between the embedding tensor deformations ($M_{mn}$, $\tilde{M}^{mn}$) and the generalized fluxes given in (4.121) now perfectly works. The explicit mapping between vectors of $SO(3,3)$ expressed in light-cone coordinates and two-forms of $SL(4)$ can be worked out by means of the $SO(3,3)$ 't Hooft symbols ($G_A$)mn (see Appendix C.2). This gives rise to the following dictionary between the $M$ and $\tilde{M}$-components and the fluxes given in (4.121)
The QC given in equations (4.114)-(4.117) enjoy a symmetry in the exchange
\[(M, \xi) \leftrightarrow (-\tilde{M}, -\xi)\].  
(4.124)

The discrete $\mathbb{Z}_2$ transformation $\eta$ corresponds to the following $O(3,3)$ element with determinant $-1$
\[
\eta = \begin{pmatrix} 0 & 1_3 \\ 1_3 & 0 \end{pmatrix},
\]
(4.125)
which can be interpreted as a triple T-duality exchanging the three compact coordinates $y^a$ with the corresponding winding coordinates $\tilde{y}_a$ in the language of DFT.

Now we have all the elements to analyze the higher dimensional origin of the orbits classified in Table 4.6.

- **Orbits 1 – 3:** These gaugings are non-geometric for every $\alpha \neq 0$; for $\alpha = 0$, they correspond to coset reductions of heterotic string theory. See e.g. the $S^3$ compactification in ref. [179] giving rise to the SO(4) gauging. This theory was previously obtained in ref. [180] as $\mathcal{N} = 2$ truncation of a maximal supergravity in $D = 7$.

- **Orbits 4 – 5:** For any value of $\alpha$ we can always dualize these representatives to the one obtained by means of a twisted $T^3$ reduction with $H$ and $\omega$ fluxes.

- **Orbits 6 – 7:** For any $\alpha \neq 0$ these orbits could be obtained from supergravity compactifications on locally-geometric T-folds, whereas for $\alpha = 0$ it falls again in a special case of the reductions described for orbits 4 and 5.

- **Orbits 8 – 11:** For any value of $\alpha$, these orbits always contain a geometric representative involving less general $H$ and $\omega$ fluxes.

To summarize, in the half-maximal $D = 7$ case, we encounter a number of orbits which do not have an obvious higher-dimensional origin. To be more precise, these are orbits 1, 2 and 3 for $\alpha \neq 0$. The challenge in the next subsection will be to establish what DFT can do for us in order to give these orbits a higher-dimensional origin. Again, before reading the following subsections we refer to the Section 4.4.1 for a discussion of what we mean by light-cone and Cartesian formulations.

**Higher-dimensional DFT origin**

First of all we would like to show here how to capture the gaugings that only involve (up to duality rotations) fluxes $H_{abc}$ and $\omega_{ab^c}$. For this, we start from the light-cone formulation, and propose the following Ansatz for a *globally geometric twist* (involving
4.4. Duality orbits of non-geometric fluxes

\[ e = \begin{pmatrix} 1 & 0 & \omega_3 \sin(\omega_1 \omega_3 y^2) \\ 0 & \cos(\omega_2 \omega_3 y^1) & -\frac{\omega_2}{\omega_3} \cos(\omega_1 \omega_3 y^2) \sin(\omega_2 \omega_3 y^1) \\ 0 & \frac{\omega_2}{\omega_3} \sin(\omega_2 \omega_3 y^1) & \cos(\omega_1 \omega_3 y^2) \cos(\omega_2 \omega_3 y^1) \end{pmatrix}, \quad (4.126) \]

\[ B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & H y^1 \cos(\omega_1 \omega_3 y^2) & 0 \\ 0 & -H y^1 \cos(\omega_1 \omega_3 y^2) & 0 & 0 \end{pmatrix}, \quad (4.127) \]

\[ \lambda = -\frac{1}{2} \log(\cos(\omega_1 \omega_3 y^2)) . \quad (4.128) \]

This is far from being the most general ansatz, but it serves our purposes of reaching a large family of geometric orbits. The parameters \( \omega_i \) can be real, vanishing or imaginary, since \( U \) is real and well-behaved in these cases. The QC, weak and strong constraints are all automatically satisfied, and the gaugings read

\[ M = \text{diag}(H, 0, 0, 0), \quad \tilde{M} = \text{diag}(0, \omega_1^2, \omega_2^2, \omega_3^2) . \quad (4.129) \]

From here, by choosing appropriate values of the parameters the orbits 4, 5, 8, 10 and 11 can be obtained. Indeed these are geometric as they only involve gauge and (geo)metric fluxes.

Secondly, in order to address the remaining orbits, we consider an SO(2, 2) twist \( U_4 \) embedded in O(3, 3) in the following way

\[ U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix}, \quad U_4 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \lambda = 0 . \quad (4.130) \]

This situation is analog to the SO(1, 1) twist considered in the \( D = 8 \) case, but with a more general twist. Working in the Cartesian formulation, one can define the generators and elements of SO(2, 2) as

\[ [t_{IJ}]_K^L = \delta^K_J \eta_{IJ} \eta_{KL}, \quad U_4 = \exp(t_{IJ} \phi^{IJ}) , \quad (4.131) \]

where the rotations are generated by \( t_{12} \) and \( t_{34} \), and the boosts by the other generators. Also, we take \( \phi^{IJ} = \alpha^{IJ} y^1 + \beta^{IJ} \tilde{y}_1 \) to be linear.

From the above SO(2, 2) duality element one can reproduce the following orbits employing a locally geometric twist (including \( e, B \) and \( \beta \) but only depending on \( y \), usually referred to as a T-fold):

- **Orbit 6** can be obtained by taking

\[ (6) \quad \alpha^{12} = -\beta^{12} = -\frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) , \quad \alpha^{34} = -\beta^{34} = -\frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) . \]

and all other vanishing.
• **Orbits 7** and **9** can be obtained by the following particular identifications

\[
\begin{align*}
\phi^{14} &= \phi^{23} , \quad \phi^{12} = \phi^{34} \quad \text{and} \quad \phi^{13} = \phi^{24} .
\end{align*}
\]

(7) \[ \alpha^{14} = -\beta^{14} = \frac{1}{\sqrt{2}} \sin \alpha , \quad \alpha^{12} = -\beta^{12} = \frac{1}{\sqrt{2}} \cos \alpha , \quad \alpha^{13} = \beta^{13} = 0 , \]

(9) \[ \alpha^{14} = -\beta^{14} = \frac{1}{\sqrt{2}} \sin \alpha , \quad \alpha^{12} = \beta^{12} = 0 , \quad \alpha^{13} = \beta^{13} = \frac{1}{\sqrt{2}} \cos \alpha . \]

All these backgrounds satisfy both the weak and the strong constraints and hence they admit a locally geometric description. This is in agreement with the fact that the simplest representative of **orbits 6, 7** and **9** given in Table 4.6 contains \( H, \omega \) and \( Q \) fluxes but no \( R \) flux.

Finally, one can employ the same SO(2,2) duality elements with different identifications to generate the remaining orbits with a **non-geometric twist** (involving both \( y \) and \( \tilde{y} \) coordinates):

• **Orbits 1, 3** can be again obtained by considering an SO(2) \( \times \) SO(2) twist with arbitrary \( \phi^{12} \) and \( \phi^{34} \):

(1) \[ \alpha^{12} = -2 \sqrt{2} (\cos \alpha + \sin \alpha) , \quad \beta^{34} = 2 \sqrt{2} (\cos \alpha - \sin \alpha) , \quad \alpha^{34} = \beta^{12} = 0 , \]

(3) \[ \alpha^{34} = -2 \sqrt{2} (\cos \alpha + \sin \alpha) , \quad \beta^{12} = 2 \sqrt{2} (\cos \alpha - \sin \alpha) , \quad \alpha^{12} = \beta^{34} = 0 . \]

• **Orbit 2** can be obtained by means of a different SO(2,2) twist built out of the two rotations and two boosts subject to the following identification

\[
\phi^{14} = \phi^{23} , \quad \phi^{12} = \phi^{34} . \tag{4.132}
\]

(2) \[ \alpha^{14} = \beta^{12} = \frac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha) , \quad \alpha^{12} = -\beta^{14} = -\frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) . \]

These backgrounds violate both the weak and the strong constraints for \( \alpha \neq 0 \). This implies that these backgrounds are truly doubled and they do not even admit a locally geometric description.

Finally, let us also give an example of degeneracy in twist orbits-space reproducing the same orbit of gaugings. The following twist

\[
\phi^{12} = \phi^{13} , \quad \phi^{34} = \phi^{24} , \quad \phi^{23} = \phi^{14} = 0 \quad \tag{4.133}
\]

(6) \[ \alpha^{13} = -\frac{1}{\sqrt{2}} (\cos \alpha + \sin \alpha) , \quad \beta^{24} = \frac{1}{\sqrt{2}} (\cos \alpha - \sin \alpha) , \quad \alpha^{24} = \beta^{13} = 0 , \]

also reproduces the **orbit 6**, but in this case through a non-geometric twist. What happens in this case is that although the twist matrix does not satisfy the weak/strong constraints, the contractions in (4.62) cancel.
4.5 Concluding remarks

In the research summarized in this chapter we have addressed the notion of non-geometry, by classifying the explicit orbits of consistent gaugings of different supergravity theories, and considering the possible higher-dimensional origins of these. The results turn out to be fundamentally different for the cases of U-duality orbits of maximal supergravities, and T-duality orbits of half-maximal theories.

In the former case we have managed to explicitly classify all U-duality orbits in dimensions $8 \leq D \leq 11$. This led to zero, one, four and ten discrete orbits in dimensions $D = 11, 10, 9$ and 8, respectively, with different associated gauge groups. Remarkably, we have found that all of these orbits have a higher-dimensional origin via some geometric compactification, be it twisted reductions or compactifications on group manifolds or coset spaces. In our parlance, we have therefore found that all U-duality orbits are geometric. The structure of U-duality orbits is therefore dramatically different from the sketch of Figure 4.1 in the introduction. Although a full classification of all orbits in lower-dimensional cases becomes increasingly cumbersome, we are not aware of any examples that are known to be non-geometric. It could therefore hold in full generality that all U-duality orbits are necessarily geometric.

This is certainly not the case for T-duality orbits of gaugings of half-maximal supergravities. In this case, we have provided the explicit classification in dimensions $7 \leq D \leq 10$ (where in $D = 7$ we have only included three-form fluxes). The numbers of distinct families of orbits in this case are zero, one, three and eleven in dimensions $D = 10, 9, 8$ and 7, respectively, which includes both discrete and one-parameter orbits. A number of these orbits do not have a higher-dimensional origin in terms of a geometric compactification. Such cases are orbits 2 and 3 in $D = 8$ and orbits 1, 2 and 3 in $D = 7$ for $\alpha \neq 0$. Indeed, these are exactly the orbits that do not admit an uplift to the maximal theory. As proven in Section 4.4.1, all such orbits necessarily violate the weak and/or strong constraints, and therefore need truly doubled backgrounds. Thus, the structure of T-duality orbits is very reminiscent of Figure 4.1 in the introduction. Given the complications that already arise in these simpler higher-dimensional variants, one can anticipate that the situation will be similar in four-dimensional half-maximal supergravity.

Fortunately, the formalism of double field theory seems tailor-made to generate additional T-duality orbits of half-maximal supergravity. Building on the recent generalization of the definition of double field theory [45], we have demonstrated that all T-duality orbits, including the non-geometric ones in $D = 7, 8$, can be generated by a twisted reduction of double field theory. We have explicitly provided duality twists for all orbits. For locally-geometric orbits the twists only depend on the physical coordinates $y$, while for the non-geometric orbits these necessarily also include $\tilde{y}$. Again, based on our exhaustive analysis in higher-dimensions, one could conjecture that also in lower-dimensional theories, all T-duality orbits follow from this generalized notion of double field theory.
At this point we would like to stress once more that a given orbit of gaugings can be generated from different twist orbits. Therefore, there is a degeneracy in the space of twist orbits giving rise to a particular orbit of gaugings. Interestingly, as it is the case of orbit 6 in $D = 7$ for instance, one might find two different twist orbits reproducing the same orbit of gaugings, one violating weak and strong constraints, the other one satisfying both. Our notion of a locally geometric orbit of gaugings is related to the existence of at least one undoubled background giving rise to it. However, this ambiguity seems to be peculiar of gaugings containing $Q$ flux. These can, in principle, be independently obtained by either adding a $\beta$ but no $\tilde{y}$ dependence (locally geometric choice, usually called T-fold), or by including non-trivial $\tilde{y}$ dependence but no $\beta$ (non-geometric choice) [43].

Another remarkable degeneracy occurs for the case of semi-simple gaugings, corresponding to orbits 1 – 3 in $D = 7$. For the special case of $\alpha = 0$, we have two possible ways of generating such orbits from higher-dimensions: either a coset reduction over a sphere or analytic continuations thereof, or a duality twist involving non-geometric coordinate dependence. Therefore $d$-dimensional coset reductions seem to be equivalent to $2d$-dimensional twisted torus reductions (with the latter in fact being more general, as it leads to all values of $\alpha$). Considering the complications that generally arise in proving the consistency of coset reductions, this is a remarkable reformulation that would be interesting to understand in more detail. Furthermore, when extending the notion of double field theory to type II and M-theory, this relation could also shed new light on the consistency of the notoriously difficult four-, five- and seven-sphere reductions of these theories.

Our results mainly focus on Scherk-Schwarz compactifications leading to gauged supergravities with vanishing $\xi_M$ fluxes. In addition, we have restricted to the NSNS sector and ignored $\alpha'$-effects. Also, we stress once again that relaxing the strong and weak constraints is crucial in part of our analysis. If we kept the weak constraint, typically the Jacobi identities would lead to backgrounds satisfying also the strong constraint [45]. However, from a purely (double) field theoretical analysis the weak constraint is not necessary. A sigma model analysis beyond tori would help us to clarify the relation between DFT without the weak and strong constraints and string field theory on more general backgrounds.
Chapter 5

Studies on $N = 2$ extremal multicenter black holes

In this chapter we present a systematic study of extremal, stationary, multicenter black hole solutions in ungauged four dimensional Einstein-Maxwell $N = 2$ supergravity theories minimally coupled to scalars, i.e. theories with quadratic prepotentials.

We show how it is possible to derive in a systematic and straightforward way a fully analytic, explicit description of the multicenter black holes, the attractor mechanism and their properties making an intensive use of the matrices $(\mathcal{S}_N, \mathcal{S}_F)$ and their adjoints with respect to the symplectic product (to be defined here). The compatibility of these matrices with respect to the symplectic product makes possible the definition of an associated inner product for which these matrices are unitary. This unitarity suggests the decomposition of the $2n_v + 2$ dimensional ($n_v$ the number of vector multiplets) symplectic space into a subspace generated by the center charge vectors $q_a$, and their associated vectors $\mathcal{S}^\dagger q_a$ and its orthogonal complement subspace.

In particular, this decomposition results useful for understanding some questions related to multicenter black holes, as the entropy increasing effects in the fragmentation of a single center black holes into two or more centers, or the extremality of the solutions by simple considerations of the dimensions of each subspace. The results presented here are easily extendable to general prepotentials or even theories without them.

The study we are going to developed in this chapter is based on the work done in [181].

5.1 Introduction

In this chapter, we are interested in general, stationary, multicenter black hole solutions in ungauged four dimensional $N = 2$ supergravity theories coupled to an arbitrary number of $N = 2$ vector multiplets. The action of the theory can be determined, in the framework of special geometry, in terms of a holomorphic section $\Omega$ of the scalar manifold. The set of field equations and Bianchi identities associated to the action is
Studies on \( N = 2 \) extremal multicenter black holes invariant under the group of symplectic transformations \( Sp(2n_v + 2, \mathbb{R}) \). This group acts linearly on the section \( \Omega \), becoming this a symplectic vector which can be written as \( \Omega = (X^I, F_I) \), with \( I = (0, n_v) \).

Black hole solutions in \( N = 2 \) \( D = 4 \) supergravity have been extensively studied for a long term by now.

The values of the \( n_v \) scalar fields constitute the moduli space of the theory. A distinctive feature of many of these theories is that the, possibly disconnected, black hole horizon acts as an attractor for the scalar fields present in the spectrum. The values of the moduli at any of the horizon components does not depend on their asymptotic values, but only on the symplectic vector of charge assigned to that horizon component \[183,189\]. The embedding of the duality group of the moduli space into the symplectic group \( Sp(2n_v + 2, \mathbb{R}) \) establishes, in general, a relation between the upper and lower components of \( \Omega \), \( F_I = F_I(X^J) \). In some cases, \( F_I \) is the derivative of a single function, the prepotential \( F = F(X^I) \). The choice of a particular embedding determines the full Lagrangian of the theory and whether a prepotential exists \[200,201\].

In this chapter, we focus in general quadratic prepotentials. These theories include the simplest examples of special Kähler homogeneous manifolds, the \( \mathbb{C}P^n \equiv SU(1,n)/U(1) \times SU(n) \) case.

These models correspond to Maxwell-Einstein \( N = 2 \) supergravities minimally coupled to \( n_v \) vector multiplets. They lead to phenomenologically interesting \( N = 1 \) minimally coupled supergravities \[202\]. Theories derived from particular examples of these quadratic prepotentials have been studied in detail. The case \( n_v = 1 \) corresponds to the \( SU(1,1)/U(1) \) axion-dilaton black hole (see for example \[203,204\] or \[205\]) with prepotential \( F = -iX^0X^1 \).

The aim of this study is the explicit, detailed study of stationary multicenter black hole solutions with any number of scalar fields, the study of the properties of the bosonic field solutions and their global and local properties. For this purpose, we make a systematic use of, some previously well-known objects of the theory, the stabilization matrices, and some new ones, their symplectic “adjoints”. These stabilization matrices, named \( S_F, S_N \) along this work, are related to the vector kinetic matrix and the matrix of second derivatives of the prepotential. They are real \( Sp(2n_v + 2, \mathbb{R}) \) matrices, isometries of the symplectic quadratic form, connecting the real and imaginary parts of the special geometry sections. Their adjoints with respect to the symplectic product \( S_N^\dagger, S_F^\dagger \) are defined and shown to lie inside the Lie algebra of the isometry group. They are such that (for any \( S = S_F, S_N \)) \( S + S^\dagger = 0 \). This property, together with \( S^2 = -1 \), make these matrices unitary with respect to the symplectic product, \( SS^\dagger = 1 \). We show how it is possible to derive or rederive again in a systematic and straightforward way a fully analytic, explicit description of the multicenter black holes and their properties.

\[ ^1 \text{See, for example, refs. [56,182–192]. Multicenter black holes have been treated in refs. [193–199].} \]
(attractor mechanism, central charge, horizon areas, masses, . . .) making an intensive use of these stabilization matrices, their adjoints and the algebraic properties of both.

As we will show, the properties of these matrices, specially their symplectic unitarity property, suggest the convenience of the separation of the $(2n_a + 2)$-dimensional symplectic space into a $2n_a$ dimensional subspace generated by the $n_a$ center charges $q_a$ and their associated vectors $S^aq_a$ (or $S\hat{q}_a$) and its orthogonal complement subspace (possibly of dimension zero depending on the number and on the linear dependency of center charge vectors). For quadratic prepotentials, this separation into “charge-longitudinal” and “transversal” subspaces can be made global by choosing $S = S_F$. A similar, but local, scalar dependent separation can be advantageously considered also for generic prepotentials, or even theories without them. The projection of any symplectic vector appearing in the theory (for example a subset of the charge vectors themselves or vectors characterizing the black hole ansatz at infinity) in terms of these new bases appears as a promising technique. The use of this projection allows, in particular, the understanding of questions as entropy increasing effects in the fragmentation of a single center black holes into two o more centers, or the extremality of the solutions, in terms, for example, of simple considerations of the dimensions of each of the charge-longitudinal and transversal subspaces.

Although we have focused in the study of minimally coupled theories with quadratic prepotentials, the main techniques, properties and expressions presented are extendable to theories governed by general prepotentials or even theories without them.

This study is organized as follows. In Section 5.2, we present a brief introduction of the Reissner-Nordström black hole and the concept of extremality. Section 5.3 treats the attractor mechanism. In Section 5.4, we present some well-known basic aspects of $N = 2$ $D = 4$ supergravity theories and their formulation in terms of special and symplectic geometry. In Section 5.5, we first introduce the matrices $S_{N,F}$, stressing some of their known properties and deriving new ones. We also construct projective operators (as well as their corresponding symplectic adjoints) based on these matrices. After the consideration of the attractor mechanism in terms of these projectors, we enter in a full explicit description of multicenter black hole solutions, their horizons and their asymptotic properties. This is done in Sections 5.6 and 5.7. We finally present Section 5.8, which contains a summary and discussion of our work, as well as an outlook on further proposals.

5.2 Reissner-Nordström: a window to extremality

In this section, we review some elementary properties of the Reissner-Nordström black hole, laying stress on its extremal case.

Our starting point is the Einstein-Maxwell action in 4 dimensions,

$$\mathcal{L} = \int d^4x \sqrt{g} \left( R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right), \quad (5.2)$$
which allows charged black holes as solutions. For the sake of simplicity, we consider a static and spherically symmetric metric ansatz. The most general solution of the field equations satisfying these requirements is

\[ ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} \left[ dr^2 + r^2 d\Omega^2 \right], \quad (5.3) \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and \( U(r) \) is the warp factor. Imposing the same symmetry conditions on the Maxwell field, the field strength 2-form is restricted to be

\[ F = P \sin \theta d\theta \wedge d\phi + Q dt \wedge \frac{1}{r^2} dr, \quad (5.4) \]

where the constants \( P \) and \( Q \) can be interpreted as the magnetic and electric charges, respectively. Solving the field equations derived from (5.2), we get that the Reissner-Nordström metric is given by (5.3) with warp factor

\[ e^{2U(r)} = 1 - \frac{2M}{r} + \frac{P^2 + Q^2}{r^2}. \quad (5.5) \]

This solution enjoys a singularity at \( r = 0 \) hidden by the horizons at \( r = r_\pm \), which appear when the metric element vanishes \( (e^{2U(r_\pm)} = 0) \),

\[ r_\pm = M \pm \sqrt{M^2 - (P^2 + Q^2)}. \quad (5.6) \]

Both \( r_\pm \) values are real when \( M^2 \geq P^2 + Q^2 \). We define so-called extremality parameter \( c \) as the

\[ c = r_+ - r_- = \sqrt{M^2 - (P^2 + Q^2)}, \quad (5.7) \]

The properties of the solution depend on the values of the mass and the electromagnetic charges. For \( c > 0 \) then the solution describes a non-extremal black hole with two, interior and exterior, horizons. Its surface gravity \( \kappa_S \) and exterior horizon area are, respectively, given by

\[ A = 4\pi(c + M)^2, \quad (5.8) \]
\[ \kappa_S = \frac{4\pi c}{A}. \quad (5.9) \]

The two horizons coincide when \( c = 0 \) or, equivalently, when

\[ M^2 = P^2 + Q^2. \quad (5.10) \]

In this case, the surface gravity vanishes and the horizon area is given exclusively in terms of the charge,

\[ A = 4\pi(P^2 + Q^2)^2. \quad (5.11) \]

The kind of black hole that results for \( c = 0 \) is called extremal black hole. In the case \( c < 0 \), the event horizons disappear and the singularity at \( r = 0 \) becomes a naked
singularity. The Schwarzschild black hole and the Minkowski spacetime are special cases for, respectively, $M > 0, P = Q = 0$ and $M = P = Q = 0$ values.

Let us focus on the extremal case. By introducing a radial coordinate $v = r - M$, the metric can be expressed as

$$ds^2 = -\left(1 + \frac{M}{v}\right)^{-2} dt^2 + \left(1 + \frac{M}{v}\right)^2 \left[dv^2 + v^2 d\Omega^2\right]. \quad (5.12)$$

The horizon is now at $v = 0$ and the near-horizon metric for $v \to 0$ is

$$ds_{NH}^2 = -\frac{v^2}{M^2} dt^2 + \frac{M^2}{v^2} dv^2 + M^2 d\Omega^2. \quad (5.13)$$

Defining a new coordinate $z = M^2/v$, this metric is rewritten as

$$ds^2 = M^2 \frac{z^2}{v^2} (-dt^2 + dz^2) + M^2 d\Omega^2. \quad (5.14)$$

This line element describes the direct product of two manifolds, $AdS_2 \times S^2$, where the $AdS$ scale $L$ and the radius of the sphere $r_S$ coincide, $L = r_S = M$. This metric is the so-called Robinson-Bertotti metric [206][207].

The Reissner-Nordström solution can be expressed in isotropic coordinates. Under the transformation

$$r = \rho \left[\left(1 + \frac{M}{2\rho}\right)^2 - \left(\frac{e}{2\rho}\right)^2\right], \quad (5.15)$$

where $e^2 \equiv P^2 + Q^2$, the metric element becomes

$$ds^2 = -H_1(\rho)^2 dt^2 + H_2(\rho)^2 \left(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2 \theta d\phi^2\right). \quad (5.16)$$

The functions $H_1(\rho)$ and $H_2(\rho)$ are

$$H_1(\rho) = \frac{1 - \left(\frac{M}{2\rho}\right)^2 + \left(\frac{e}{2\rho}\right)^2}{1 + \frac{M}{\rho} + \left(\frac{M}{2\rho}\right)^2 - \left(\frac{e}{2\rho}\right)^2}, \quad (5.17)$$

$$H_2(\rho) = 1 + \frac{M}{\rho} + \left(\frac{M}{2\rho}\right)^2 - \left(\frac{e}{2\rho}\right)^2. \quad (5.18)$$

In the extremal limit $c = 0$, or $M^2 = P^2 + Q^2$, the coordinate transformation becomes linear, $r = \rho + M$, and the metric results

$$ds^2 = -H^{-2} dt^2 + H^2 dx^2, \quad (5.19)$$

where the factor

$$H(\rho) = 1 + \frac{M}{\rho} \quad (5.20)$$
is an elementary solution of a Laplace equation (in the 3-space \((\rho, \theta, \phi)\), \(\Delta_3 H = 0\)). This is the extremal Reissner-Nordstrom metric in isotropic form. It is not casual that the factor \(H\) is a harmonic function, since it can been understood as a special case of a more general family of solutions of Einstein-Maxwell action without spherical symmetry that we review on continuation.

The Majumdar-Papapetrou solution \(^{[208],[209]}\) can be considered a non-spherical generalization of the Reissner-Nordström solution. Let us take the metric element

\[
ds^2 = -H^{-2}(x)dt^2 + H^2(x)dx^2. \tag{5.21}
\]

The Einstein-Maxwell equations of motion reduce to the following equation of motion for the warp factor \(H(x)\)

\[
\Delta_3 H = 0, \tag{5.22}
\]

with \(\Delta_3\) the 3-dimensional Laplacian. The electromagnetic field strength, solution to the Einstein-Maxwell equations, results

\[
F_{ti} = -\cos \theta \partial_i H^{-1}, \\
F_{ij} = \sin \theta \epsilon_{ijk} \partial^k H. \tag{5.23}
\]

The Laplace equation (5.22) is satisfied, in particular, by harmonic functions with an (arbitrary) number of point singularities,

\[
H \equiv e^{-U} = 1 + \sum_n \frac{m_n}{|x - x_n|}. \tag{5.24}
\]

The charges \(e_i\) inside a closed surface surrounding each point singularity may be identified by computing the flux of the electromagnetic field through the aforementioned surface. They result equal to the residues \(m_n\) at any of the singularities of the function (5.24),

\[
e_i^2 = m_i^2. \tag{5.25}
\]

These types of geometries, with \(m_n \geq 0, \forall n\), have event horizons with spherical topology and represent genuine black hole solutions (in fact, they are the only Majumdar-Papapetrou solutions\(^2\) with this property \(^{[210]}\)). In the case of a single point singularity solution, one recovers the extremal Reissner-Nordström black hole. In the general case, they can be seen as an arbitrary, static, configuration of single Reissner-Nordström black holes. These multicenter, static, black hole solutions are in static equilibrium with the gravitational and the electrostatic forces cancelling each other.\(^3\) The solution can be seen as a simple example of BPS configuration. In fact, this type of solutions emerge as a BPS solutions of \(N = 2\) supergravity \(^{[211],[212]}\).

\(^2\)Not only that: also the only IWP solutions with that property.

\(^3\)Newtonian point charged particles can remain in static equilibrium if all the charges share the same sign and satisfy \(|e_i| = m_i\), no matter how arranged they are \(^{[210]}\).
A further generalization to stationary solutions is possible. A class of stationary solutions of the pure Einstein-Maxwell equations are given by \[ ds^2 = -(H\bar{H})^{-1}(dt + \omega)^2 + H\bar{H}dx^2 , \] where \( \omega \) is a purely spatial 1-form \( \omega = \omega_i dx^i \) and \( H = H(x) \) is any complex solution to the 3-dimensional Laplace equation \[ \Delta_3 H = 0 . \] (5.27)

The term \( \omega \) is given by the equation \[ \star_3 d\omega = 2\text{Im} (\bar{H}dH) . \] (5.28)

A integrability condition for this equation is given by the complex Laplace equation above. In particular, if \( H \) is real (or purely imaginary) then \( d\omega = 0 \) and we can write, by the Poincaré lemma, \( \omega = d\lambda \). If we do a translation on the time coordinate, \( t \to t + \lambda \), we recover the Majumdar-Papapetrou class of solutions. These are the so-called Israel-Wilson-Perjes (IWP) solutions. Similarly, as we have seen before, particular solutions of the Laplace equation with a finite number of point singularities are given by \[ H \equiv e^{-U} = 1 + \sum_n \frac{m_n}{|x - x_n|} , \] but in this case the parameters \( m_n \) and \( x_n \) are allowed to be complex.

For example \[213], the Kerr-Newman solution with \( M^2 = Q^2 \) corresponds to a simple case of \((5.29)\) for the values \( n = 1, m_1 = M \) and \( x_1 = (0, 0, ia) \), where \( a, M \in \mathbb{R} \), the former related to the form \( \omega \). One can consider a generalized solution with similarly defined real \( m_n \) and arbitrary complex parameters \( x_n \). The resulting metric will represent the field of a set of arbitrarily spinning, charged Kerr-like particles in neutral equilibrium. The single point source solution has a naked singularity and no horizons. Superposition of a number of solutions also generally results in naked singularities except in some special cases for some concrete configuration of the parameters. However, it can be shown that in this case, the solution becomes static and reduces to the Majumdar-Papapetrou class \[210].

Another simple case of the IWP solution includes a single point singularity at \( x_1 = 0 \) and \( m_1 = M + iN \), it is given by \((r = |x|)\) \[210] \[ H = 1 + \frac{M + iN}{r} . \] (5.30)

The 1-form \( \omega \) is given by (up to an additive constant) \[ \omega = \frac{2N \cos \theta - 1}{r \sin \theta} d\phi . \] (5.31)
This is proportional to $N$, so for $N = 0$ we recover the static case. Let us take the generic case, with $n$ complex quantities $m_i = M_i + iN_i$ and arbitrary real parameters $x_i$. At large distances the corresponding function $H$ is given by

$$H \sim 1 + \sum_n \frac{M_n + iN_n}{r}.$$  

(5.32)

We note that for $r \to \infty$, the behavior becomes that of a Majumdar-Papapetrou solution if the imaginary part of (5.32) is zero, namely, if the following condition is satisfied

$$\sum_n N_n = 0.$$  

(5.33)

The only IWP solutions with point singularities which represent black hole solutions are those for which all the imaginary parts, $N_i$, are null. That is, they are nothing but the Majumdar-Papapetrou static solutions \[215\] (see also \[210\]).

### 5.3 Black holes in SUGRA and the attractor mechanism

We will study now black hole solutions in gravity theories that contain gauge and scalar fields (as for example it happens in supergravity). An important mechanism appears in these theories, the *attractor mechanism*. This phenomenon was originally discovered for BPS extremal black holes in $N = 2$ supergravity theories \[56, 182, 216\]. The flow of the scalar fields towards the horizon exhibits the feature of a gradient flow towards a fixed point, which, in the supergravity case, is the minimum of a function related to the central charge of the SUSY algebra. Among other properties, a basic feature of the attractor mechanism is that the ADM mass is minimized, for fixed values of the conserved charges carried by the black hole, when the scalar fields are constant (they take their attractor values through the spacetime).

In the last years, the attractor mechanism has been investigated for extremal black holes in non-supersymmetric theories, in theories beyond GR as $D = 5$ Gauss-Bonnet gravity \[217\], as well as for non-BPS extremal solutions in $N \geq 1$ supersymmetric theories.

A generic Lagrangian describing the bosonic sector of $D = 4$ supergravity coupled to scalars and $n_v$ vector multiplets is of the form

$$S = \int d^4x \left( R \ast 1 - \frac{1}{2} g_{ij}(\phi) d\phi^i \wedge \ast d\phi^j + \frac{1}{4} \mathcal{I}_{\Lambda\Sigma}(\phi) F^\Lambda \wedge F^\Sigma + \frac{1}{4} \mathcal{R}_{\Lambda\Sigma}(\phi) F^\Lambda \wedge \ast F^\Sigma \right),$$

(5.34)

where $g_{ij}(\phi)$ is the metric of the scalar $\sigma$-model, and $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ describe the couplings terms of the vector fields. In particular, $\mathcal{R}_{\Lambda\Sigma}$ is the generalization of the $\theta$-angle
terms in presence of scalar and vector fields. We assume that there is no-scalar potential.

We are interested here in finding single center, static, extremal, spherically symmetric and charged black hole solutions. In addition, we assume asymptotical flatness. A suitable ansatz for such requirements is of the form

$$ds^2 = -e^{2U} dt^2 + e^{-2U} \left( \frac{c^4}{\sinh^4(cz)} dz^2 + \frac{c^2}{\sinh^2(cz)} d\Omega^2 \right),$$

(5.35)

where the constant $c$ is an extremality parameter. The $z$ coordinate runs from $z = -\infty$ (horizon) to $z = 0$ (spatial infinity). The unknown function $U = U(z)$ is such that $\exp(-2U(z \to 0)) = \exp(-2Mz) \to 1$ (asymptotic flatness). At $z \to -\infty$ we require, in order to ensure a finite horizon area ($c \neq 0$),

$$U(z \to -\infty) = cz,$$
$$U'(z \to -\infty) = c.$$  

(5.36)

In the extremal limit $c \to 0$, we recover the metric

$$ds^2 = -e^{2U} dt^2 + e^{-2U} \left( \frac{1}{z^2} dz^2 + \frac{1}{z^2} d\Omega^2 \right).$$

(5.37)

In this case, the condition of having a finite horizon area implies the boundary condition for the regime $z \to -\infty$,

$$\exp(-2U) \to \frac{A}{4\pi} z^2.$$  

(5.38)

A similar static, spherically symmetric ansatz can be introduced for the gauge fields. Due to the structure of the couplings in (5.34) and the dependence of $\mathcal{R}_{\Sigma\Lambda}$ and $\mathcal{I}_{\Sigma\Lambda}$ on the moduli, the Bianchi identities are

$$dF^\Lambda = 0,$$
$$dG_\Lambda = d \left( \mathcal{R}_{\Lambda\Sigma} F^\Sigma - \mathcal{I}_{\Lambda\Sigma} \star F^\Sigma \right) = 0,$$  

(5.39)

where the second equation defines the quantity $G$. This set of equations remains invariant when performing a symplectic rotation of the field strengths.

Electric and magnetic conserved charges can be defined in terms of the field strengths and their duals,

$$\frac{1}{4\pi} \int F^\Lambda = p^\Lambda,$$
$$\frac{1}{4\pi} \int G_\Lambda = q_\Lambda.$$  

(5.40)

We can introduce the pair of potentials $(A^\Sigma, A_\Sigma)$, corresponding to the symplectic vector of 2-forms $(F^\Sigma, G_\Sigma)$ with the required symmetry:

$$A^\Lambda = \chi^\Lambda(r) dt - p^\Lambda \cos \theta d\phi,$$
$$A_\Lambda = \psi_\Lambda(r) dt - q_\Lambda \cos \theta d\phi.$$  

(5.41)
The electric-magnetic duality relation imposes the constraint
\[ \chi' = e^{2U} \mathcal{I}'(q_i - R_i p_i) . \] (5.42)

At this point, we can write the Einstein field equations for the metric and the
gauge field ansätze, (5.35) and (5.41) respectively. The equations of motion for the
gauge fields may then be directly solved. The equations of motion for metric and
scalar fields simplify to the equations \[ U'' - e^{2U} V_{BH} = 0, \] (5.43)
\[ (U')^2 + \frac{1}{2} g_{ij} \phi_i' \phi_j' - e^{2U} V_{BH} - c^2 = 0, \] (5.44)
\[ \phi'' + \Gamma_{jk} i \phi_i' \phi_k' - e^{2U} g^{ij} \partial_j V_{BH} = 0, \] (5.45)
where \( V_{BH} \) and \( \Gamma_{jk} \) are scalar functions.

The non-linear system of second order differential equations (5.43)-(5.45) is com-
plemented by the asymptotic boundary conditions for the metric at infinity (flatness)
and at the horizon (finite area condition for \( c \to 0 \)) and, in principle, by two initial
or boundary conditions for each of the scalar fields. However, only one of these two
theoretically possible conditions for each of the scalars survives. This is due to the
properties of this non-linear system, the existence of the first order constraint equation
(5.44) and the requirement of everywhere regularity of the solutions. In fact, in the
extremal case \( c \to 0 \) the value of the scalars and their first derivatives will be fixed at
the horizon. Only the values of the scalars at infinity will remain as free parameters of
the theory. We will see more details in what follows.

The quantity \( V_{BH} \) is the black hole potential, which encodes the terms of the energy
momentum tensor corresponding to the vector fields that appear in the Lagrangian \[56\].
It can be written as
\[ V_{BH} = -\frac{1}{2} Q^T M Q , \] (5.46)
where \( M \) is a scalar-dependent matrix (see Sections \[5,4 and 5.5\]) and \( Q \) is a symplectic
charge vector,
\[ Q = \left( \begin{array}{c} p^A \\ q_A \end{array} \right) . \] (5.47)
By making use of the scalar matrix \( M \), we can rewrite the gauge field strengths in a
covariant way as
\[ \left( \begin{array}{c} F \\ G \end{array} \right) = e^{2U} \hat{\Omega} M \left( \begin{array}{c} p^A \\ q_A \end{array} \right) dt \wedge dz - \left( \begin{array}{c} p^A \\ q_A \end{array} \right) \sin \theta d\theta \wedge d\phi , \] (5.48)
where \( \hat{\Omega} \) is the symplectic metric
\[ \hat{\Omega} = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) . \] (5.49)
At this point, let us consider the possibility of deriving the field equations above from an effective Lagrangian. Let us take the following Lagrangian

$$\mathcal{L} = (U')^2 + \frac{1}{2} g_{ij} \phi^i \phi^j + e^{2U} V_{BH}(\phi) + c^2 ,$$

(5.50)

which depends on the “fields” $U(z)$, $\phi(z)$. The Lagrangian does not explicitly depend on $z$, Noether’s theorem implies that the effective ‘energy’

$$\mathcal{E} \equiv (U')^2 + \frac{1}{2} g_{ij} \phi^i \phi^j - e^{2U} V_{BH} - c^2$$

(5.51)

is a constant. The Euler-Lagrange equations corresponding to the fields $U(z), \phi^i(z)$ agree with the equations (5.43) and (5.45), respectively. However, the condition (5.44) has to be implemented by hand, as an additional constraint,

$$\mathcal{E} = 0 .$$

(5.52)

Thus, this effective action plus the constraint $\mathcal{E} = 0$ is equivalent to the system of differential equations.

From the behavior of the constraint equation (5.52) at spatial infinity (considering that $\exp(-2U(z \to 0)) = \exp(-2Mz) \to 1$, we get the following constraint between the black hole mass $M$, the scalar charges$^4$ and the potential at infinity:

$$M^2 + \frac{1}{2} g_{ij} \Sigma^i \Sigma^j - V_{BH}(\phi^i_{\infty}) = c^2 .$$

(5.53)

The equations (5.43)-(5.45) can be solved by constant values $\phi^i(z) = \tilde{\phi}^i = \phi^i_{\infty}$ for the scalar fields. This is possible if these values represent a critical point of the effective potential, i.e. if

$$\partial_i V_{BH}(\tilde{\phi}^i, q, p) = 0 .$$

(5.54)

The black hole charges are the only parameters that appear in (5.54). Thus the extremal points will be solved in terms of them,

$$\tilde{\phi}^i = \tilde{\phi}^i(p, q) .$$

(5.55)

The value of BH potential at the minimum is a constant given by

$$\hat{V}_{BH} = V_{BH}(\tilde{\phi}^i(p, q), p, q) .$$

(5.56)

$^4$The scalar charges $\Sigma^i$ of the black hole are defined by (at spatial infinity, $z \to 0$)

$$\phi^i = \phi^i_{\infty} + z \frac{\Sigma^i}{r} + O(z^2) .$$
The equations (5.43) and (5.44) for the warp factor at the horizon can be directly solved giving

\[ U''(z) = (U'(z))^2, \]
\[ U(z) = -\log(r_H z), \]

where \( r_H = \sqrt{V_{BH}} \). This implies that the black hole entropy is

\[ S_{BH} = \frac{A}{4} = \pi \tilde{V}_{BH}(q,p). \]

If we take the extremal case \( c = 0 \), from equation (5.53), we get

\[ M^2 = V_{BH}(\phi_\infty) = \tilde{V}_{BH}(q,p). \]

Next, we will study general, non-constant scalar solutions focusing in the extremal \( c = 0 \) case. The critical values of the effective black hole potential represent possible attractor values for the moduli scalars (provided positivity of the Hessian). The moduli and their derivatives will have the same value at the horizon, whilst their asymptotic values may be varied freely. We will see how, in this general case, the same universal properties (as the attractor mechanism and the area of extremal black holes) can be deduced only by demanding a regular behaviour of the geometry and the moduli near the horizon.

If the scalar fields and their derivatives do not blow up near the horizon, the following asymptotic expression is valid for \( z \to \infty \)

\[ \phi^i = \tilde{\phi}_h^i + \frac{a^i}{z} + \mathcal{O}(z^2). \]

By making use of this series and of the relation (for the extremal \( c \to 0 \) case) \( \exp(-2U)_h = \frac{z^2 A}{4\pi} \), the differential equations (5.43)-(5.45) near the horizon become, at leading orders,

\[ \frac{1}{z^2} - \frac{4\pi}{A z^2} V_{BH,h} = 0, \]
\[ \frac{1}{z^2} + \frac{a_1}{z^4} - \frac{4\pi}{A z^2} V_{BH,h} + \mathcal{O} \left( \frac{1}{z^6} \right) = 0, \]
\[ \frac{a_2}{z^3} + \frac{a_3}{z^4} - \frac{4\pi}{A z^2} (g^{ij} \partial_j V_{BH})_h + \mathcal{O} \left( \frac{1}{z^6} \right) = 0, \]

or, equivalently,

\[ 1 - \frac{4\pi}{A} V_{BH,h} = 0, \]
\[ a_1 + z^2 \left( 1 - \frac{4\pi z^2}{A} V_{BH,h} \right) + \mathcal{O} \left( \frac{1}{z} \right) = 0, \]
\[ a_2 z + a_3 - z^2 \frac{4\pi}{A} (g^{ij} \partial_j V_{BH})_h + \mathcal{O} \left( \frac{1}{z} \right) = 0. \]

5Extremal black hole solutions in which the scalar fields take constant values are usually called double-extreme black holes.

6Here, we assume the vanishing of the scalar charges, \( \Sigma^i = 0 \).
with $a_1, a_2, a_3$ are arbitrary constants. If we compare the coefficients order by order in $1/z$, we obtain that, in order to the solutions not blow up at the horizon, the following conditions should be imposed:

\[ A = 4\pi V_{BH,h}, \quad (5.68) \]
\[ \phi^i(z \to -\infty) = 0, \quad (5.69) \]
\[ \left( \frac{\partial V_{BH}}{\partial \phi^i} \right)_h = 0. \quad (5.70) \]

These conditions show that the area of the horizon of extremal black holes coincides with the area of the horizon of double-extremal black holes with the same values of charges and is given by the value of $V_{BH}$ (cf. (5.65)). Moreover, the entropy of the black hole, related to the area of the horizon, will be determined by the charges. In addition, we see that the values of the moduli at the horizon can be considered as free initial conditions, since they are given by the minimization of the effective black hole potential (5.67). That is, the horizon is an attractor point [182,183,216,219].

**Supergravity central charge and flow equations**

Let us consider in more detail the special case of $N = 2$ supergravity, for which the scalar manifold is a special Kähler manifold. For $N = 2$ theories, special geometry can be used and the expressions are somehow simplified. The black hole potential is given by

\[ V_{BH} = |Z|^2 + 4g^{ij}\partial_i|Z|\bar{\partial}_j|Z|, \quad (5.71) \]

where $Z$ is the central charge of the $N = 2$ SUSY algebra. This central charge is defined by the special Kähler geometry, as it is discussed in the next section. Thus, the Lagrangian (5.50) and the constraint (5.51) become

\[ \mathcal{L} = (U')^2 + g_{ij}\phi^{i'}\phi^{j'} + e^{2U}(|Z|^2 + 4g^{ij}\partial_i|Z|\bar{\partial}_j|Z|), \quad (5.72) \]
\[ (U')^2 + g_{ij}\phi^{i'}\phi^{j'} = e^{2U}(|Z|^2 + 4g^{ij}\partial_i|Z|\bar{\partial}_j|Z|). \quad (5.73) \]

The energy constraint is an equality between two different sums of squares with the same weight $e^{2U}$. So, a possible ansatz for the solution would be

\[ U' = \pm e^U|Z|, \]
\[ \phi^{i'} = \pm 2e^U g^{ij}\bar{\partial}_j|Z|. \quad (5.74) \]

It can be checked that this ansatz is also a solution for the equations of motion (5.43)-(5.45) when the same signs are chosen. Thus, this is a reduction of the original second
order system to a first order system governed by $|Z|$. Due to asymptotic flatness arguments, the physical sign is fixed and the equations become

\[
U' = -e^U |Z|, 
\]

\[
\phi' = -2e^U g^{ij} \bar{\partial}_j |Z|. 
\]

These same first order equations can be obtained by analyzing the Killing spinor equations for the theory. The conditions for the gravitino and gaugino supersymmetry transformations

\[
\delta \psi_A^\mu = 0, 
\]

\[
\delta \lambda_A^i = 0. 
\]

are equivalent to (5.75) and (5.76), respectively.\(^{10}\)

By evaluating the equations (5.75)-(5.76) at infinity and at the horizon, similarly as in the previous section, we infer that the central charge fully determines the solution. The fixed values of the scalars at horizon are given by the minimization condition

\[
\partial_i |Z|_h = 0, 
\]

whose critical points are also a critical point for the black hole potential. Solutions corresponding to a critical point describes a supersymmetric extremal black hole. The central charge at the horizon is fixed in terms of the discrete charges

\[
|Z|_h = |Z(p, q, \phi^h(p, q))|_h. 
\]

The special Kähler nature of the scalar manifold guarantees that the second derivative of the central charge is such that

\[
\partial_i \partial_j |Z| = g_{ij} |Z| > 0, 
\]

i.e. all the critical points are minima of the central charge. No matter what the values of the scalars are at infinity, they will be driven towards the minimum of the central charge. This constitutes an attractor behaviour.

The extremality condition for the central charge was brought to a purely algebraic and equivalent form in [221–225] under the condition that the special geometry is not singular.

\(^{10}\)Actually, the Killing spinor equation for gauginos implies a new first order equation for a phase factor. However, it is also related to the Kähler connection and once the flow equations (5.75) - (5.76) are fulfilled, this additional equation is automatically satisfied, showing that the phase factor is not an independent quantity.
5.4. \( N = 2 \) \( D = 4 \) SUGRA and Special Kähler geometry

The field content of the \( N = 2 \) supergravity theory coupled to vector multiplets consists of

\[
\{ e_\mu^a, A_\mu^I, z^\alpha, \psi_\mu^r, \lambda_\nu^\alpha \}, \tag{5.82}
\]

with \( \alpha = 1, \ldots, n_v \), and \( I = 0, \ldots, n_v \). The theory also contains some hypermultiplets, which can be safely taken as constant or neglected (further details can be found in \([194]\), whose notation and concepts we generally adopt). The bosonic \( N = 2 \) action can be written as

\[
S = \int_{M(4d)} R * 1 + G_{\alpha \bar{\beta}} dz^\alpha \wedge \star d\bar{z}^\beta + F^I \wedge G_I. \tag{5.83}
\]

The fields \( F^I, G_I \) are not independent. Whilst \( F^I \) is given by \( F^I = dA^I, G_I \) is a set of combinations of the \( F^I \) and their Hodge duals,

\[
G_I = a_{IJ} F^J + b_{IJ} \star F^J, \tag{5.84}
\]

with scalar-dependent coefficients \( a_{IJ} \) and \( b_{IJ} \).

Abelian charges with respect the \( U(1)^{n_v+1} \) local symmetry of the theory are defined by means of the integrals of the gauge field strengths. The total charges of the geometry are

\[
q \equiv (p^I, q_I) \equiv \frac{1}{2\pi^2} \int_{S_n} (F^I, G_I). \tag{5.85}
\]

Similar charges can be defined for specific finite regions.

The theory is defined, in the special geometry formalism, by the introduction of some projective scalar coordinates \( X^I \), as for example, ‘special’ projective coordinates \( z^\alpha \equiv X^I/X^0 \). By introducing a covariantly holomorphic section of a symplectic bundle, \( V \), we are able to arrange \( 2n_v \) quantities that transform as a vector under symplectic transformations at any point of the manifold. \( V \) has the following structure

\[
V = V(z, \bar{z}) \equiv \langle V^I | V_I \rangle, \tag{5.86}
\]

and satisfies the following identities\(^{11}\)

\[
\langle V | \bar{V} \rangle \equiv V^I \omega V \equiv \bar{V}^I V_I - V^I \bar{V}_I = -i. \tag{5.87}
\]

The scalar kinetic term metric is given by

\[
G_{\alpha \bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} \mathcal{K}, \tag{5.88}
\]

\(^{11}\)We choose a basis such that \( \omega = \begin{pmatrix} 0 & \mathbb{I}_{n_v} \\ \mathbb{I}_{n_v} & 0 \end{pmatrix} \).
where the Kähler potential $\mathcal{K}$ is defined by the relations $V = \exp(-\mathcal{K}/2)\Omega$ being $\Omega \equiv (X^I, F_I)$ a holomorphic section and

$$e^{-\mathcal{K}} = i (X^I F_I - X^J F_J) = i \langle \Omega | \overline{\Omega} \rangle.$$  \hspace{1cm} (5.89)

In $N = 2$ theories, the central charge $Z$ can be expressed as a linear function on the charge space:

$$Z(z^\alpha, q) \equiv \langle V | q \rangle = e^{\mathcal{K}/2} (p^I F_I - q_I X^I) \hspace{1cm} (5.90)$$

The embedding of the isometry group of the scalar manifold metric $G_{\alpha\beta}$, into the symplectic group fixes, through the Kähler potential $\mathcal{K}$, a functional relation between the lower and upper parts of $V$ and $\Omega \ [226][227]$.

$$F_I = F_I(X^I), \hspace{1cm} V_I = V_I(V^I).$$  \hspace{1cm} (5.91)

There always exists a symplectic frame under which the theory can be described in terms of a single holomorphic function, the prepotential $F(X)$. It is a second degree homogeneous function on the projective scalar coordinates $X^I$, such that $F_I(X) = \partial_I F(X)$. For simplicity, we will assume the existence of such prepotential along this study although the results will not depend on such existence. Using the notation $F_{IJ} = \partial_I \partial_J F$, the lower and upper components of $\Omega$ are related by

$$F_I = F_{IJ}X^J.$$  \hspace{1cm} (5.92)

The lower and upper components of $V$ are related by a field dependent matrix $N_{IJ}$, which is determined by the special geometry relations [183]

$$V_I = N_{IJ}V^J, \quad D_i \bar{V}_I = N_{IJ}D_i \bar{V}^J.$$  \hspace{1cm} (5.93)

The matrix $N$, which also fixes the vector couplings $(a_{IJ}, b_{IJ})$ in the action, can be related to $F_{IJ}$ [228] by

$$N_{IJ} = \bar{F}_{IJ} + T_I T_J,$$  \hspace{1cm} (5.95)

where the quantities $T_i$ are proportional to the projector of the graviphoton, whose flux defines the $N = 2$ central charge [228]. For our purposes, it is convenient to write this relation between the $N_{IJ}$ and $F_{IJ}$ quantities as

$$N_{IJ} \equiv F_{IJ} + N^\perp_{IJ}$$

$$= F_{IJ} - 2i \text{Im} \left( F_{I,J} \right) + 2i \frac{\text{Im} \left( F_{I,K} \right) L^K \text{Im} \left( F_{J,Q} \right) L^Q}{L^J \text{Im} \left( F_{P,Q} \right) L^P},$$  \hspace{1cm} (5.96)
where we have decomposed the matrix $N_{IJ}$ into "longitudinal" (the $F_{IJ}$ themselves) and "transversal" parts ($N_{IJ}^\perp$). The perpendicular term (defined by the expression above) annihilates $L^I$, or any multiple of it,

$$N_{IJ}^\perp(\alpha L^J) = 0.$$  

(5.97)

From this, (5.93) can be written as

$$V_I = N_{IJ} L^J = (F_{IJ} + N_{IJ}^\perp) L^J = F_{IJ} L^J.$$  

(5.98)

Thus, the upper and lower components of $V$ and $\Omega$ are connected by the same matrix $F_{IJ}$.

The existence of functional dependencies among the upper and lower components of the vectors $V$ or $\Omega$ imply further relations between their respective real and imaginary parts. They are related by symplectic matrices $S(N), S(F) \in Sp(2n_v + 2, \mathbb{R})$ which are respectively associated to the quantities $N_{IJ}, F_{IJ}$ as follows:

$$\text{Re} (\Omega) = S(F) \text{Im} (\Omega),$$  

(5.99)

$$\text{Re} (V) = S(N) \text{Im} (V) = S(F) \text{Im} (V).$$  

(5.100)

The last expression is obtained by means of the relation (5.98). These same relations (5.99)-(5.100) are valid for any complex multiple of $\Omega$ or $V$. It is straightforward to show, for example, that for any $\lambda \in \mathbb{C}$, we have

$$\text{Re} (\lambda V) = S(N) \text{Im} (\lambda V) = S(F) \text{Im} (\lambda V).$$  

(5.101)

The matrix $S(F)$ is of the form

$$S(F) = \begin{pmatrix} 1 & -\text{Re} (F_{IJ})^t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Im} (F_{IJ}) & 0 \\ 0 & \text{Im} (F_{IJ})^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\text{Re} (F_{IJ}) & 1 \end{pmatrix}. $$  

(5.102)

Similarly, the same result applies for $S(N)$ with $F_{IJ} \rightarrow N_{IJ}$.

In $N = 2$ theories, the matrix $S(N)$ always exhibits a moduli dependence $[202]$. However, this is not the case for $S(F)$. We will focus in this work on the particular case of theories with quadratic prepotentials $[13]$

$$F(X) = \frac{1}{2} F_{IJ} X^I X^J,$$  

(5.103)

where $F_{IJ}$ is a complex, constant, symmetric matrix. Then, the corresponding matrix $S(F)$ is a field-independent, constant matrix. We can assume that $\text{Re} (F_{IJ}) = 0$ and $\text{Im} (F_{IJ})$ is negative definite. In what follows, we will use the notation $S \equiv S(F)$. The condition $e^{-K} > 0$ and the expression (5.89) imply a restriction on the prepotential. We will write this restriction in a convenient form in section (5.5) in terms of the positivity of a quadratic form.

---

12The matrix $S_N$ is related to $\mathcal{M}$, the matrix that appears in the black hole effective potential $V_{BH} = -\frac{1}{2} q^t \mathcal{M} q$, by $S(N) \omega = \mathcal{M}$.

13Or, equivalently, $D = 4$ theories with U-duality groups of “degenerate type $E_7$” $[202]$. 

General supersymmetric stationary solutions

The most general stationary (time independent) 4-dimensional metric compatible with supersymmetry can be written in the IWP form \[213,214,229,\]
\[ds^2 = e^{2U} (dt + \omega)^2 - e^{-2U} d\mathbf{x}^2.\] (5.104)

Supersymmetric \( N = 2 \) supergravity solutions can be constructed systematically following well-established methods \[194\]. The 1-form \( \omega \) and the function \( e^{-2U} \) are related in these theories to the Kähler potential and connection, \( \mathcal{K}, Q \, \[229\]. \) We demand asymptotic flatness, \( e^{-2U} \to 1 \) together with \( \omega \to 0 \) for \( \| \mathbf{x} \| \to \infty \) BPS field equation solutions for the action above (for example, quantities that appear in the metric, as \( e^{-2U} \) or \( \omega \)) can be written in terms of the following real symplectic vectors \( \mathcal{R} \) and \( \mathcal{I} \)

\[
\mathcal{R} = \frac{1}{\sqrt{2}} \text{Re} \left( \frac{V}{X} \right), \quad \mathcal{I} = \frac{1}{\sqrt{2}} \text{Im} \left( \frac{V}{X} \right). \] (5.105)

\( X \) is an arbitrary complex function of space coordinates such that \( 1/X \) is harmonic. The \( 2n_v + 2 \) components of \( \mathcal{I} \) and \( \mathcal{R} \) are real harmonic functions in \( \mathbb{R}^3 \). There is an algebraic relation between \( \mathcal{R} \) and \( \mathcal{I} \) and the solutions can be written in terms only of the vector \( \mathcal{I} \). Due to the relations \[5.99,5.100\] and \[5.101\], we can write the following stabilization equation

\[ \mathcal{R} = \mathcal{S} \mathcal{I}. \] (5.107)

In practice, specific solutions are determined by giving a particular, suitable, ansatz for the symplectic vector \( \mathcal{I} \) as a function of the spacetime coordinates.

Using these symplectic vectors we rewrite the only independent metric component as

\[
e^{-2U} = e^{-\mathcal{K}} = \frac{1}{2|X|^2} = \langle \mathcal{R} | \mathcal{I} \rangle = \langle \mathcal{S} \mathcal{I} | \mathcal{I} \rangle. \] (5.108)

Similarly, the time independent 3-dimensional 1-form \( \omega = \omega_i dx^i \) satisfies the equation

\[ d\omega = 2 \langle \mathcal{I} | \ast_3 d \mathcal{I} \rangle, \] (5.109)

where \( \ast_3 \) is the Hodge dual on flat \( \mathbb{R}^3 \), together with the integrability condition

\[ \langle \mathcal{I} | \Delta \mathcal{I} \rangle = 0. \] (5.110)

The asymptotic flatness condition implies

\[ \langle \mathcal{R}_\infty | \mathcal{I}_\infty \rangle = \langle \mathcal{S} \mathcal{I}_\infty | \mathcal{I}_\infty \rangle = 1. \] (5.111)
The gauge field equations of motion and Bianchi identities can be directly solved in terms of spatially dependent harmonic functions \( [194] \). The modulus of the central charge function defined in (5.90) can be written, taking into account (5.108), as

\[
|Z(q)|^2 e^{-2U} = |\langle R|q \rangle|^2 + |\langle I|q \rangle|^2.
\]

At spatial infinity, assuming asymptotic flatness (5.111), we arrive to

\[
|Z_\infty(q)|^2 = |\langle R_\infty|q \rangle|^2 + |\langle I_\infty|q \rangle|^2.
\]

The, assumed time independent, \( n_\omega \) complex scalar fields \( z^\alpha \) solutions to the field equations, are given in this formalism by

\[
z^\alpha = \Omega^\alpha \Omega^0 = V^\alpha V^0 = \frac{R^\alpha + iI^\alpha}{R^0 + iI^0}.
\]

This is, in general, a formal expression as the \( I \) or \( R \) quantities may be scalar dependent \( [13] \).

These scalar fields can, in principle, take any values \( (z_\infty) \) at infinity. These values will appear as free parameters in the ansatz that we give for \( I \). Nevertheless, according to the attractor mechanism, the moduli adjust themselves at some fixed points.

We are interested in this work in extremal, single- or multi-center black hole-type solutions determined by an \( I \) ansatz with point-like singularities of the form

\[
I = I_\infty + \sum_a \frac{q_a}{|x - x_a|},
\]

where \( a = 1, \ldots, n_a \) being the number \( n_a \) arbitrary and \( q_a = (p_a I, q_a I) \) and \( I_\infty \) real, constant, symplectic vectors.

For this kind of solutions, the quantities \( I_\infty \) are related to the values at infinity of the moduli while the “charge” vectors \( q_a \) are related to their values at the fixed points. The fixed values of the scalars, \( z(x) \rightarrow z(x_a) = z_\infty^a \), are the solutions of the following attractor equations \( [182,183,185] \):

\[
q^a = \text{Re} \left( 2i \bar{Z}(z_\infty^a) V(z_\infty^a) \right).
\]

The prepotential performs its influence throughout \( V \) and \( Z \) (cf. (5.90)). The scalar attractor values are independent of their asymptotic values and only depend on the discrete charges \( z_\infty^a = z_\infty^a(q_I) \).

Single center black hole solutions are known to exist for all regions of the moduli scalars at infinity, under very mild conditions on the charge vector. In the multicenter case, for fixed charge vectors, not all the positions \( x_a \) in the ansatz (5.115) are allowed. The integrability condition (5.110) imposes necessary conditions on the relative positions and on the moduli at spatial infinity (through \( I_\infty \)) for the existence of a solution. In this framework, a particular black hole solution is completely determined by a triplet of charge vectors, distances and values of the moduli at infinity \( (q_a, x_a, z_\infty^a) \).

\[14\] Even for a non scalar dependent ansatz \( I \), the matrix \( S \) is, in general, scalar dependent.
5.5 The stabilization matrix and the attractor equations

Let us consider now the attractor equations (5.116), in more detail. We will use the properties of the stabilization matrix $S$ to solve them in a purely algebraic way to obtain some properties and give some explicit expressions for the scalars at the fixed points.

For this purpose, we first establish some well-known properties of $S_N, S \equiv S_F$ and define new matrices: some projector operators associated to them and their respective symplectic adjoints.

It can be shown by explicit computation that the real symplectic matrices $S_N, S \equiv S_F \in Sp(2n_v + 2, \mathbb{R})$ defined by (5.99)-(5.100) and whose explicit expressions are given by (5.102), satisfy the relations

$$S_N^2 = S_F^2 = -1.$$  \hspace{1cm} (5.117)

From this, it is possible as well as convenient to define the projector operators for the matrix $S$ (similarly for $S_N$) as

$$P_{\pm} = \frac{1 \pm iS}{2}.$$  \hspace{1cm} (5.118)

They satisfy the following straightforward properties

$$P_{\pm}^2 = P_{\pm},$$
$$SP_{\pm} = \mp iP_{\pm},$$
$$\left(P_{\pm}\right)^* = P_{\mp},$$  \hspace{1cm} (5.119)

and, for $X, Y$ arbitrary real vectors,

$$P_{\pm}X = P_{\pm}Y \Rightarrow X = Y.$$  \hspace{1cm} (5.120)

According to (5.119), the $P_{\pm}$ are the projectors into the eigenspaces of the matrix $S$. The symplectic space $W$ can be decomposed into eigenspaces of the matrix $S$:

$$W = W^+ \oplus W^-,$$  \hspace{1cm} (5.121)

where $W^\pm = P_{\pm}W$. Complex conjugation interchanges the $W^+$ and $W^-$ subspaces, $(P_{\pm})^* = P_{\mp}$, so that both subspaces are isomorphic to each other.

We note that we can rewrite a stabilization relation similar to (5.101) in a different way with the help of these projection operators $P_{\pm}$. For any vector $V \in W$ for which there is a relation between its real and imaginary parts of the form $Re\,(V) = SIm\,(V)$, we have the relations (for an arbitrary $\lambda \in \mathbb{C}$)

$$\lambda V = Re\,(\lambda V) + iIm\,(\lambda V)$$
$$= 2iP_-Im\,(\lambda V)$$
$$= 2P_-Re\,(\lambda V).$$  \hspace{1cm} (5.122)
Thus, the full vector $V$ can be reconstructed by applying one of the projectors either from its real or imaginary part. We see that such vectors are fully contained in the subspace $W^-$ or, equivalently, they are eigenvectors of $S$

$$SV = 2iSP_\text{-Im}(V) = 2P_\text{-Im}(V) = iV.$$  \hspace{1cm} (5.123)

We will find convenient to define the adjoint operator of the matrix $S$, $S^\dagger$, with respect to the symplectic bilinear product such that, for any vectors $A, B$,

$$\langle SA|B \rangle = \langle A|S^\dagger B \rangle.$$  \hspace{1cm} (5.124)

A straightforward computation shows that $S^\dagger$ is given by

$$S^\dagger = -\Omega S^t \Omega.$$  \hspace{1cm} (5.125)

Under the assumption of a symmetric $F_{IJ}$ matrix, it is given by

$$S^\dagger = -S.$$  \hspace{1cm} (5.126)

In summary, the matrix $S$ is skew-adjoint with respect to $\omega$ and its square is $S^2 = -I$. It fulfills an “unitarity” condition $S^\dagger S = I$.

In mathematical terms, $S$ defines an (almost) complex structure on the symplectic space. This complex structure preserves the symplectic bilinear form, the matrix $S$ is an isometry of the symplectic space,

$$\langle SA|SB \rangle = \langle A|B \rangle.$$  \hspace{1cm} (5.127)

From (5.126), we see that $S$ is an element of the symplectic Lie algebra $\mathfrak{sp}(2n_v + 2)$.

Moreover, the bilinear form defined by

$$g(X, Y) \equiv \langle SX|Y \rangle$$  \hspace{1cm} (5.128)

is symmetric. This can be easily seen:

$$g(X, Y) = \langle SX|Y \rangle = \langle Y|S^\dagger X \rangle = \langle SY|X \rangle = g(Y, X).$$  \hspace{1cm} (5.129)

We will apply these properties to the study of the attractor equations. In general, the matrices $S_N, S_F$ are scalar dependent. Only one of them, $S_F$, is constant, in the case of quadratic prepotentials. Let us write $S_N^f = S_N(z = z_f) \quad S_F^f = S_F(z = z_f)$ for the matrices evaluated at (anyone of) the fixed points. Let us use the sub/superindex $f$ to denote any quantity at the fixed points. For instance, $Z_f \equiv Z(z^f)$ or $V_f \equiv V(z^f)$.

If we multiply both sides of (5.116) by $S_N^f = S_N(z = z_f)$, we arrive to

$$S_N^f q^a = S_N^f \text{Re}(2i\bar{Z}^f V^f) = S_F^f \text{Re}(2i\bar{Z}^f V^f) = S q^a.$$  \hspace{1cm} (5.130)
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where we have used the properties (5.100) and (5.101).\footnote{Following [228], we note that $V_{BH} = |Z_i|^2 + |Z|^2 = -\frac{1}{2}q^I\mathcal{S}(N)\omega q$ and $|Z_i|^2 - |Z|^2 = \frac{1}{2}q^I\mathcal{S}(F)\omega q$. At the fixed points, we have $Z_i = 0$, so that $|Z|^2 = -\frac{1}{2}q^I\mathcal{S}(N)\omega q = -\frac{1}{2}q^I\mathcal{S}(F)\omega q$. This last equation is satisfied by a solution of (5.116).}

The attractor equations can be written yet in another alternative way. By using (5.122) and (5.116), we can write

$$i\bar{Z}_fV_f = 2P_i\bar{Z}_fV_f$$

$$= P_-, q, \quad (5.131)$$

or its conjugate equation

$$-iZ_f\bar{V}_f = P_+ q. \quad (5.132)$$

That is, the attractor equations simply equal (a multiple of) the vector $V$ (which, as we have seen above lies on the subspace $W^-$) with the part of the charge vector which lies on such a subspace.

From (5.131)-(5.132), by taking symplectic products, we obtain

$$|Z_f|^2 \langle V_f | \bar{V}_f \rangle = \langle P_- | P_+ \rangle = \langle q | P_+ q \rangle$$

$$= -\frac{i}{2} \langle \mathcal{S}q | q \rangle. \quad (5.133)$$

If we insert the constraint $\langle V | \bar{V} \rangle = -i$, we arrive in a straightforward and purely algebraic way to the well known formula

$$|Z_i|^2 = \frac{1}{2} \langle \mathcal{S}q | q \rangle, \quad (5.134)$$

which relates the absolute value of the central charge at any fixed point to a quadratic expression on the charge. It is obvious from (5.134) that the positivity of the quadratic form $g(q, q) = \langle \mathcal{S}q | q \rangle$ (at least locally at all the fixed points) is a necessary consistency condition for the existence of solutions to the attractor mechanism.

Moreover the mathematical consistency condition $e^{-\mathcal{K}} > 0$ can be written as (cf. (5.89))

$$e^{-\mathcal{K}} = i \langle \Omega | \bar{\Omega} \rangle = 2 \langle \text{Re} (\Omega) | \text{Im} (\Omega) \rangle$$

$$= 2 \langle \mathcal{S} \text{Im} (\Omega) | \text{Im} (\Omega) \rangle > 0. \quad (5.135)$$

This last equation is automatically satisfied for a definite positive quadratic form $g$ at any point.

Positivity (which is physically imposed by (5.134) and (5.135)) and symmetry (demanded by (5.129)), implies that the bilinear form $g(X, Y)$ is an inner product. In addition, a hermitian form $h$ can be defined from it and from the symplectic form. We define

$$h(X, Y) = \langle \mathcal{S}X | Y \rangle + i \langle X | Y \rangle, \quad (5.136)$$
which can be written in terms of the projection operators $P_\pm$ as

\[
h(X,Y) = 2i \langle P_- X | Y \rangle = 2i \langle P_- X | P_+ Y \rangle.
\] (5.137)

The three defined structures $\{g, \omega, S\}$ form a compatible triple, each structure can be specified by the two others.\(^\text{16}\)

Let us address now to the problem of obtaining the values of the moduli at the fixed points and at infinity. The values of the scalar fields at the fixed points can be computed by an explicit expression, which involves only the matrix $S_F$. The fixed values of the $n_v$ complex scalars $z^\alpha_f(q)$ (at a generic fixed point with charge $q$) are given, using the expressions (5.114) and (5.131), by

\[
z^\alpha_f(q) = \frac{(SI)_\alpha^\alpha + iI^\alpha}{(SI)^0_\alpha + iI^0} = \frac{(S + iI)^\alpha}{((S + iI)^0)} = \frac{(P_- q)^\alpha}{(P_- q)^0}.
\] (5.138)

That is, the fixed values of the scalars are given in terms of the projection of the charges into the eigenspaces of the matrix $S$. For quadratic prepotentials, where this matrix is a constant, this is a complete, explicit solution of the attractor equations.

The values of the $n_v$ complex scalars at spatial infinity, $|x| \to \infty$ are given by (using again (5.114) and defining $I_\infty = \lim_{|x| \to \infty} I$, we are not assuming any particular ansatz for $I$ at this moment)

\[
z^\alpha_\infty = \lim_{|x| \to \infty} \frac{(P_+ I)^\alpha}{(P_+ I)^0} = \frac{(P_+ I_\infty)^\alpha}{(P_+ I_\infty)^0}.
\] (5.139)

According to this formula, the ‘moduli’ $z^\alpha_\infty$ are simple rational functions of the $2n_v + 2$ real constant components of $I_\infty$. They are thus independent of the fixed attractor values (5.138) (at least for a vector $I$ with only point like singularities, as (5.115)).

We note that the expression (5.139) is formally identical to the expression (5.138), since both give the moduli values at a fixed point in terms of the charges, where the roles of $I_\infty$ and $q$ are exchanged. It is suggesting then to write an “effective attractor equation” at infinity, where the rôle of the center charge is played by the vector $I_\infty$.

That is, the scalar solutions of the equation

\[
I_\infty = \text{Re} \left(2i\bar{Z}V\right)_{|\infty},
\] (5.140)

are those precisely given by (5.139).

One can extract some algebraic relations for the vectors $I_\infty$ and $q^a$ and the equations (5.138)-(5.139) in specific cases, for example for solutions with constant scalars. Let us...
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assume a one-scalar theory and $z_f = z_\infty(\neq 0)$. In this case, equations (5.138)-(5.139) imply the projective equality ($\lambda \in \mathbb{R}$ an arbitrary non-zero, constant)

$$ P_{-I_\infty} = \lambda P_{-q}. $$

(5.141)

which, due to (5.120), implies

$$ I_\infty = \lambda q. $$

(5.142)

In addition, the asymptotic flatness condition (5.111) implies

$$ \lambda^2 = \frac{1}{\langle Sq|q \rangle} = \frac{1}{2|Z_f|^2}. $$

(5.143)

The consistency of the last equation is assured by the positivity of the quadratic form $\langle Sq|q \rangle$. Thus, we can finally arrive to a characterization of the $I_\infty$ parameters in the case of constant scalar solutions

$$ I_\infty = \pm \frac{q}{\sqrt{\langle Sq|q \rangle}}. $$

(5.144)

Similar arguments can be stated in the multicenter case.

Let us finish this section with some qualitative remarks. We have arrived to the expressions (5.138)-(5.139) which can be written, in terms of the projective complex, vector $\Omega = (X^I, F_I)$, as

$$ \Omega_{fix} = P_{-q}, $$

$$ \Omega_\infty = P_{-I_\infty}. $$

(5.145)

We could have predicted these expressions a priori if SUSY solutions are uniquely determined by the symplectic real vectors $q_a$, then the also symplectic but complex vector $\Omega = (X^I, F_I)$ must be related to these vectors in a linear way, respecting symplectic covariance at the same time. Moreover, the symplectic sections $\Omega$ (or $V$) lie on the subspace $W^-$, one eigenspace of the stabilization matrix $S$. The only possibility for such a relation would be the expressions in (5.145), where precisely appear the projections of $q$ or $I_\infty$ into such subspace. These expressions, evaluated at the points of maximal symmetry (the horizon and infinity), are equivalent forms of the standard horizon attractor equations and the generalized attractor equation at infinity presented here.

5.6 Complete solutions for quadratic prepotentials

We have got some general results in the previous section without using a concrete form for the solutions, for $I$. In this section we will make use of the ansatz (5.115) for theories with quadratic prepotentials to obtain a full characterization of the solutions.

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$^{17}$Extending arguments presented in [230] (and references therein).
Let us insert the ansatz (5.115) into the general expression for the complex scalars, (5.114). The values for the time independent \( n \) complex scalar solutions to the field equations are explicitly given by

\[
z^\alpha(x) = \left( \frac{P_\infty}{P_0} \right)^\alpha = \left( \frac{P_\infty}{P_0} \right)^\alpha + \sum_a \frac{(P_\infty q_a)^\alpha}{|x-x_a|}.
\]

(5.146)

This equation is a simple rational expression for the value of the scalar fields in the whole space. The fields and their derivatives are regular everywhere, including the fixed points (there could be singularities for special charge configurations which make zero the denominator of (5.146)).

The expression (5.146) interpolates between the values at the fixed points and at infinity. After some simple manipulations, it can be written as

\[
z^\alpha(x) = c_\infty^\alpha(x) z_\infty^\alpha + c_a^\alpha(x) z_a^\alpha,
\]

(5.147)

where \( c_\infty^\alpha(x) \) and \( c_a^\alpha(x) \) are spatial dependent complex functions such that

\[
\begin{align*}
c_\infty^\alpha(x) + c_a^\alpha(x) &= 1, \\
c_\infty^\alpha(\infty) &= 1, \\
c_\infty^\alpha(x_a) &= 0, \\
\lim_{x \to x_a} c_a^\alpha(x) &= \delta_{ab}.
\end{align*}
\]

(5.148)

For a single center solution, we note that if \( z_\infty^\alpha = z_f^\alpha \) then the scalar fields are constant in all the space.

It is straightforward to see that the attractor mechanism is automatically fulfilled by the ansatz (5.115). The value of \( z^\alpha \) at any center \( x_a \) is given, by taking the corresponding limit in (5.146), by

\[
z^\alpha(x_a) = \frac{(P_\infty q_a)^\alpha}{(P_0 q_a)^\alpha} = z_f^\alpha(q_a),
\]

(5.149)

where, after the second equality, we have used the fixed point expression (5.138), which is a direct consequence of the attractor equations.

On the other hand, the solution at the spatial infinity recovers spherical symmetry. Again, taking limits, we have (with \( |x| \equiv r \))

\[
z^\alpha(r \to \infty) = \frac{r P_\infty^\alpha + \sum_a (P_\infty q_a)^\alpha}{r P_0^\alpha + \sum_a (P_0 q_a)^\alpha}
\]

\[
= (1 - c^\alpha(r)) z_\infty^\alpha + c^\alpha(r) z_f^\alpha(Q),
\]

(5.150)

where \( z_f(Q) \) is the fixed point scalar value which would correspond, according to the attractor equations, to a total charge \( Q = \sum_a q_a \). The asymptotically interpolating functions appearing above are

\[
c^\alpha(r) = \frac{1}{1 + \frac{r}{r_0}},
\]

(5.151)
with the (non-zero) scale parameter
\[ r_0^\alpha = \frac{\sum_a (P - q_a)^0}{\sum_a (P - \mathcal{I}_\infty)^0}. \] (5.152)

They are such that
\[ c^\alpha(0) = 1, \]
\[ c^\alpha(\infty) = 0. \] (5.153)

The scalar charges \( \Sigma^\alpha \) associated to the scalar fields can be simply defined by the asymptotical series
\[ z^\alpha(r \to \infty) = z^\alpha_\infty + \frac{\Sigma^\alpha}{r} + \mathcal{O}\left(\frac{1}{r^2}\right). \] (5.154)

Expanding (5.150), we have
\[ z^\alpha(r \to \infty) = z^\alpha_\infty + \frac{r_0^\alpha (z^\alpha_f(Q) - z^\alpha_\infty)}{r} + \mathcal{O}\left(\frac{1}{r^2}\right), \] (5.155)
and thus the scalar charges are given by
\[ \Sigma^\alpha = r_0^\alpha (z^\alpha_f(Q) - z^\alpha_\infty). \] (5.156)

In the special case of a single center solution, the expression (5.156) is in agreement with the well known fact that the scalar charges vanish for double extremal black holes. In the multicenter case, from this formula we infer a similar result: the scalar charges vanish if
\[ z^\alpha_\infty = z^\alpha_f(Q), \] (5.157)
where \( Q \) is the sum of the individual charges. Obviously, in this case this does not mean that the scalars are constant in all the space. Thus the conditions (5.157) could be considered a convenient generalization of double extremal solutions in the multicenter case. By taking into account the considerations of the previous section, (5.144), a candidate vector \( \mathcal{I}_\infty \) corresponding to such a solution would be of the form
\[ \mathcal{I}_\infty = \pm \frac{Q}{\langle S Q | Q \rangle}, \] (5.158)
whereas the scalar fields would be given at any point by
\[ z^\alpha(x) = c^\alpha_\infty(x) z^\alpha_f(Q) + c^\alpha_a(x) z^\alpha_f(q_a). \] (5.159)

The integrability condition for the equation determining \( \omega \) are, for any charge center \( q_b \),
\[ \langle \mathcal{I}_\infty | q_b \rangle + \sum_a \langle q_a | q_b \rangle \frac{r_{ab}}{r_{bb}} = 0, \] (5.160)
where \( r_{ab} = |\mathbf{x}_a - \mathbf{x}_b| \). The solutions for these equations give the possible intercenter positions.

Let us see the consequence of the integrability equations for a double extremal two center configuration. In this case, if \( \mathcal{I}_\infty = \lambda Q \), we have

\[
0 = \lambda \langle Q | q_1 \rangle + \frac{\langle q_1 | q_2 \rangle}{r_{12}} \\
= \lambda \langle q_2 | q_1 \rangle + \frac{\langle q_1 | q_2 \rangle}{r_{12}} \\
= \langle q_2 | q_1 \rangle \left( \lambda - \frac{1}{r_{12}} \right). \tag{5.161}
\]

Comparing this last equation with (5.158) we conclude that the double extremal inter-center distance is given by

\[
r_{12}|_{\text{doub. extm}} = \langle S \mathcal{Q} | Q \rangle. \tag{5.162}
\]

### 5.6.1 Near horizon and infinity geometry

Let us now study the gravitational field. The metric has the form given by (5.104), with the asymptotic flatness conditions

\[-g_{rr} = \langle R_\infty | \mathcal{I}_\infty \rangle = 1 \quad \text{and} \quad \omega(x \to \infty) \to 0.\]

For point-like sources, as those represented by the ansatz (5.115), the compatibility equation (5.110) takes the form (see, for example [194])

\[
N \equiv \sum_a \langle \mathcal{I}_\infty | q_a \rangle = \langle \mathcal{I}_\infty | Q \rangle = 0. \tag{5.163}
\]

An explicit computation of the total field strength shows that (5.163) is equivalent to the requirement of absence of NUT charges: only after imposing the condition \( N = 0 \), the overall integral of the \( (F^I, G_I) \) field strengths at infinity is equal to \( Q = \sum q_a \).

Another consequence of the condition \( N = 0 \), which can be checked by direct computation from (5.28), is that the 1-form \( \omega \) evaluated at each horizon of the multicenter solution is the same and is equal to its value at spatial infinity, which can be taken to be zero.

Let us write a more explicit expression for the \( g_{rr} \) component at any space point. We can write, using the ‘stabilization equation’ (5.107) and the ansatz (5.115), the expression

\[
\langle R | \mathcal{I} \rangle = \left\langle S \mathcal{I}_\infty + \sum_a \frac{S q_a}{|\mathbf{x} - \mathbf{x}_a|} | \mathcal{I}_\infty + \sum_b \frac{q_b}{|\mathbf{x} - \mathbf{x}_b|} \right\rangle \\
= 1 + \sum_b \frac{1}{|\mathbf{x} - \mathbf{x}_b|} \left( \langle S \mathcal{I}_\infty | q_b \rangle + \langle S q_b | \mathcal{I}_\infty \rangle \right) + \sum_{a,b} \frac{\langle S q_a | q_b \rangle}{|\mathbf{x} - \mathbf{x}_a| |\mathbf{x} - \mathbf{x}_b|} \\
= 1 + 2 \sum_b \frac{\langle S \mathcal{I}_\infty | q_b \rangle}{|\mathbf{x} - \mathbf{x}_b|} + \sum_{a,b} \frac{\langle S q_a | q_b \rangle}{|\mathbf{x} - \mathbf{x}_a| |\mathbf{x} - \mathbf{x}_b|}, \tag{5.164}
\]

where we have used the property $S^\dagger = -S$ and the asymptotic flatness condition $\langle SI_\infty | I_\infty \rangle = 1$. We introduce now the quantities

$$M_a \equiv \langle SI_\infty | q_a \rangle ,$$

(5.165)

$$A_{ab} \equiv \langle S q_a | q_b \rangle ,$$

(5.166)

where $A_{ab}$ is symmetric in its indices due to the property (5.129).

With these definitions, we can finally write the expression for the metric element as

$$-g_{rr} = \langle R | I \rangle = 1 + 2 \sum_b M_b | x - x_b | + \sum_{a,b} A_{ab} | x - x_a | | x - x_b |.$$  

(5.167)

If the metric element (5.167) describes a black hole, then the right part should be kept always positive and finite for any finite $|x|$. Its positivity is ensured as long as the “mass” $M_a$ and “area” $A_{ab}$ parameters are positive. But less strict conditions can be imposed, for the positivity it is sufficient that the matrix $(A_{ab})$ is (semi-) definite positive. This is guaranteed by the fact that this matrix is the Gram matrix of a set of (linearly independent or not) vectors $q_a$ with the inner product $g$ (see discussion in Section 5.5).

**Behaviour at fixed points and at infinity**

We will define new quantities the mass $M_{ADM}$ and $A_\infty$ from the behaviour of the metric at infinity. At spatial infinity $|x| \rightarrow \infty$, $\frac{1}{|x - x_a|} \rightarrow 1/r$, the metric element (5.167) becomes spherically symmetric:

$$-g_{rr} = \langle R | I \rangle = 1 + 2 \sum_a M_a r + \sum_{a,b} A_{ab} \frac{1}{r^2} + O \left( \frac{1}{r^3} \right).$$

(5.168)

The second equation defines $M_{ADM}$ and $A_\infty$. Comparing both expressions and using (5.165), (5.166) and $Q = \sum_a q_a$, we have

$$M_{ADM} = \sum_a M_a = \langle SI_\infty | Q \rangle ,$$

(5.169)

$$A_\infty = \sum_{a,b} A_{ab} = \langle S Q | Q \rangle .$$

(5.170)

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18 Also consider that $-g_{rr} \sim e^{-K}$.  
19 If the vectors $q_a$ are not linearly independent, then $(A_{ab})$ is only semi definite positive, positive definite otherwise.  
20 From the mathematical point of view it is not necessary that the $M_b$ are all positive. The quadratic form $-g_{rr}(y) = 1 + 2\sum M_a y_a + \sum A_{ab} y_a y_b$ is strictly convex, and hence has a global minimum as long as $A_{ab}$ is positive definite. Positivity of this minimum ($-g_{rr} > 0$) is guaranteed in this case at least if $M_a (A^{-1})_{ab} M_b < 1$. 
The expression for the central charge at infinity, \( (5.113) \), becomes then

\[
|Z_\infty|^2 = M^2_{\text{ADM}} + N^2, \tag{5.171}
\]

where \( N \) is defined by \( (5.163) \). The compatibility condition \( N = 0 \) is equivalent to the saturation of a BPS condition

\[
|Z_\infty|^2 = M^2_{\text{ADM}} = |\langle SI_\infty|Q \rangle|^2. \tag{5.172}
\]

The \( M_{\text{ADM}} \) quantity, opposed to \( A_\infty \), depends on the scalar values at infinity through the implicit dependence of \( I_\infty \) on them. These can take arbitrary values there, or at least can be chosen in a continuous range. In the single center case, for any given charge vector, one can obtain a certain particular solution by setting the scalar fields to constant values \( (z_\alpha^f = z_\alpha^\infty) \), giving this the minimal possible \( M_{\text{ADM}} \) mass \( [184] \).

For multicenter solutions and generic non-trivial charge vectors, it is not possible to have constant scalar fields. Nevertheless, we can still proceed to the minimization of \( M_{\text{ADM}}(z_\infty^\alpha) \),

\[
\frac{\partial M_{\text{ADM}}}{\partial z_\alpha^\infty} \bigg|_{z_\infty,\text{min}} = 0, \tag{5.173}
\]

with respect to the scalar fields at infinity for a given configuration. On view of the relation \( (5.156) \), if this minimum coincides with \( z_\alpha^\infty = z_f(Q) \), we would have full analogy with the one center case.

That is indeed the case. We can show that, for a given configuration of charges,

\[
|Z_\infty|^2 = M^2_{\text{ADM}} \leq \langle SQ|Q \rangle = A_\infty, \tag{5.174}
\]

the equality appears at the minimum \( (z_\alpha^\infty)_{\text{min}} = z_f(Q) \). The proof is a simple application of the fact that the bilinear form \( \langle SX|Y \rangle \) is an inner product. If we apply the Cauchy-Schwartz inequality to \( (5.172) \) and then the asymptotical flatness condition, we have

\[
M^2_{\text{ADM}} = |\langle SI_\infty|Q \rangle|^2 \leq \langle SI_\infty|I_\infty \rangle \langle SQ|Q \rangle = \langle SQ|Q \rangle. \tag{5.175}
\]

The saturation of the inequality \( (5.175) \) happens when \( I_\infty \) is of the form

\[
I_\infty = \lambda Q, \tag{5.176}
\]

where \( \lambda \in \mathbb{R} \). Such \( I_\infty \) trivially satisfies the \( N = 0 \) condition, \( \langle I_\infty|Q \rangle = \langle Q|Q \rangle = 0 \) and therefore the values of the scalar fields at infinity are given by \( z_\alpha^\infty = z_f(Q) \). Thus, this configuration is a multicenter generalization of the double extremal solutions.

Let us proceed now to the study of the geometry near the centers. For \( x \to x_a \), the metric element given by \( (5.167) \) becomes spherically symmetric. Moreover, it can be shown that, by fixing additive integration constants, we can take \( \omega_a = \omega(x \to x_a) = 0 \)
at the same time that $\omega_\infty = \omega(x \to \infty) = 0$. As a consequence, the metric at any of
the horizon components with charge $q_a$ approaches an $AdS_2 \times S^2$ metric of the form
\[ ds^2 = \frac{r^2}{\langle S q_a | q_a \rangle} dt^2 - \frac{\langle S q_a | q_a \rangle}{r^2} d\mathbf{x}^2. \]  
(5.177)
This is a Robinson-Bertotti metric, of the form (5.13). Positivity of $-g_{rr}$ in any of
the fixed point limits is ensured if we request that, for all the center charges $q_a$
\[ \langle S q_a | q_b \rangle > 0. \]  
(5.178)
The parameter $M_{RB}$ appearing there satisfies a charge extremal condition of the form
\[ M_{RB} = \langle S q_a | q_a \rangle. \]  
(5.179)
The near horizon geometry is thus, completely determined in terms of the individual
horizon areas $S_{h,a} = \langle S q_a | q_a \rangle$. The horizon area $S_h$ is the sum of the areas of its
disconnected parts
\[ S_h = \sum_a S_{h,a} = \sum_a \langle S q_a | q_a \rangle \]
\[ = 2 \sum_a |Z_{f,a}|^2. \]  
(5.180)
This expression can be compared with the area corresponding to a single center black
hole with the same total charge $Q = \sum_a q_a$, which is given by $S_h(q = Q) = \langle SQ|Q \rangle$.

5.7 Other properties: charge vector expansions

Given generic real charge vectors $(q_1, q_2, \ldots, q_n)$ one can define a subspace of $W$
generated by eigenvectors of the matrix $S$ associated to the center charges, directly of
the form
\[ B(q_n) \equiv \text{Span}(P_{\pm q_1}, \ldots, P_{\pm q_n}), \]  
(5.181)
or, equivalently, in the slightly modified basis
\[ B(q_n) \equiv \text{Span}(q_1, \ldots, q_n, S q_1, \ldots, S q_n). \]  
(5.182)
In particular, we can consider the subspace $B(q_{na})$ generated by the $n_a$ pairs $(q_a, S q_a)$ of center charges, whose dimension is, in general, $\dim B(q_{na}) \leq 2n_a$. The dimension
of the orthogonal complement to this space, $B(q_{na})^\perp$, i.e. those vectors $s$ such that
$\langle q|s \rangle = \langle S q|s \rangle = 0$ is, generically, $\dim B(q_{na})^\perp = 2(n_v - n_a) + 2 \quad \text{[21]}$. This dimension is
zero for one scalar, one center black holes ($n_v = 0, n_a = 1$). The set of vectors $(q_a,
\quad \text{[21]}$Or, $B(q_{na})^\perp$ is defined as the set of vectors $s$ such that $h(s, q) = 0$ for all $q \in (q_{na})$, where $h$ is
the hermitian inner product defined in Section 5.3.}
$S_{q_a}$ may form themselves a (maybe overcomplete) basis for the $(2n_v + 2)$ symplectic space. Otherwise, they can be extended with as many other vectors as necessary to complete such a basis. Naturally, other bases are possible or convenient, for example bases including linear combinations of the charge vectors, the total charge vector $Q$, $I_\infty$, etc.

We will use several expansions of different quantities in such a basis formed by charge and extra vectors, to get different results. In a first illustrative case, we will get a bound on the black hole areas $S_h, A_\infty$. In the second place, by decomposition of the $I_\infty$ vector, we will study different properties. In particular, we will see how the extremality of the solutions imposes strong conditions on such extra vectors.

### 5.7.1 A bound on $S_h$

The relation between the asymptotic “area” $A_\infty$ and the multicenter horizon area, or horizon entropy $S_h$, is simply

$$A_\infty = \langle SQ|Q \rangle = \sum_{a,b} \langle S_{qa}|qb \rangle = S_h + 2 \sum_{a<b} \langle S_{qa}|qb \rangle. \quad (5.183)$$

Taking into account the positivity of the quantities $\langle S_{qa}|qb \rangle$, cf. (5.178), we arrive to

$$A_\infty - S_h \geq 0. \quad (5.184)$$

For one center solution we always have $A_\infty = S_h$. For the case of two centers, for example, with charges $q_{1,2}$ the difference is

$$A_\infty - S_h = 2 \langle Sq_1|q_2 \rangle > 0. \quad (5.187)$$

We can use a combination of Cauchy-Schwartz and Jensen inequalities applied to the scalar product which appear in the last equation to write the expression

$$2|\langle Sq_1|q_2 \rangle| \leq 2\sqrt{\langle Sq_1|q_1 \rangle \langle Sq_2|q_2 \rangle} \leq \langle Sq_1|q_1 \rangle + \langle Sq_2|q_2 \rangle = S_h. \quad (5.188)$$

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22 This is in agreement with [205] where it has been shown that for quadratic prepotentials, the single center BPS extremal black hole area with charge $Q = q_1 + q_2$ is always larger than the corresponding two-center area

$$S_h(Q = q_1 + q_2) \geq S_{h,1} + S_{h,2}, \quad (5.185)$$

or, into account that $A_\infty$ is also the area of the equivalent single black hole with the same total charge $A_{\infty q_1,q_2} = S_h(Q = q_1 + q_2)$

$$A_\infty - S_h = 2 \langle Sq_1|q_2 \rangle \geq 0. \quad (5.186)$$
Then, we arrive to the bound

\[ 0 \leq A_\infty - S_h = S_h, \quad (5.189) \]

or, equivalently\(^{23}\)

\[ S_h \leq A_\infty \leq 2S_h. \quad (5.190) \]

### 5.7.2 Decomposition of \( I_\infty \) and double extremality

We will decompose now the vector \( I \) into a basis of charge and extra vectors. For the sake of simplicity we will discuss here the case of a single center solution and an arbitrary number of scalars. We will see, in particular, how the extremality of the solutions imposes strong conditions on such extra vectors. In addition, we will show, using this decomposition, the double extremality of the black hole solutions for quadratic prepotentials.

Let us decompose the vector \( I_\infty \) in the following way (with \( \langle S q | q \rangle \neq 0 \))

\[ I_\infty = \alpha q + \beta S q + \gamma s + \epsilon S s, \quad (5.191) \]

where \( \alpha, \beta, \gamma, \epsilon \in \mathbb{R} \) and \( s \) is an arbitrary but fixed, normalized vector such that \( s \in B(q_{na}, S q_{na})^\perp \), i.e.

\[ \langle s | q \rangle = \langle s | S q \rangle = 0, \]

\[ \langle S s | s \rangle = 1. \quad (5.192) \]

Such vector \( s \) can be always determined by a modified Gram-Schmidt procedure for a given pair of vectors \((q, S^\dagger q)\). By projecting the relation \((5.191)\) over any of the individual vectors \(q, S q\), we get

\[ \langle I_\infty | q \rangle = \beta \langle S q | q \rangle, \]

\[ \langle I_\infty | S q \rangle = -\alpha \langle S q | q \rangle. \quad (5.193) \]

Using the expressions \((5.163), (5.169)\) and \((5.170)\), we can rewrite these last two expressions respectively as

\[ N = \beta A_\infty, \]

\[ M_{ADM} = -\alpha A_\infty. \quad (5.194) \]

\(^{23}\)For the multicenter case, using only \((5.183)\) and applying Cauchy-Schwartz and the triangle inequalities, we get the slightly less restrictive bound

\[ |A_\infty - S_h| \leq S_h. \]
from where we read the values of the $\alpha, \beta$ coefficients in terms of some other, more physical, parameters. The condition $N = 0$ implies that $\beta = 0$, hence the $I_\infty$ vector does not contain any component in the “$S_q$” direction.

Let us consider now the asymptotic flatness condition and apply the ansatz (5.191) for $I_\infty$, but without imposing at this moment the $N = 0$ condition. We have, using the $\alpha, \beta$ values, the definition $\Delta^2 \equiv (\gamma^2 + \epsilon^2)$ and (5.171), the expression

$$1 = \langle S I_\infty | I_\infty \rangle = (\alpha^2 + \beta^2) \langle S q | q \rangle + (\gamma^2 + \epsilon^2) \langle S s | s \rangle = \frac{M^2_{ADM} + N^2}{A_\infty^2} \langle S q | q \rangle + \Delta^2, \quad (5.195)$$

or, equivalently,

$$|Z_\infty|^2 = M^2_{ADM} + N^2 = \langle S q | q \rangle (1 - \Delta^2). \quad (5.196)$$

The BPS condition $|Z_\infty| = M_{ADM} = \langle S q | q \rangle$ is only fulfilled if $N = 0$ (in concordance with (5.172)) and the additional condition $\Delta = 0$. The parameter $\Delta$ is an “extremality” parameter.

The vanishing of these quantities can be directly seen by imposing extremality in the metric elements, by requesting extremal RN black hole type metric or, $-g_{rr} \sim f^2$ with $f$ an spatially harmonic function. The metric component $g_{rr}$ is

$$-g_{rr} = 1 + 2 \frac{M_{ADM}}{r} + \frac{\langle S q | q \rangle}{r^2} = 1 + 2 \frac{M_{ADM}}{r} + \frac{(M^2_{ADM} + N^2)/(1 - \Delta^2)}{r^2} = \left(1 + \frac{M_{ADM}}{r}\right)^2 + \frac{1}{r^2} \frac{1}{1 - \Delta^2} \left(M^2_{ADM}\Delta^2 + N^2\right). \quad (5.197)$$

The metric element is of the form $-g_{rr} \sim f^2$ with $f$ an spatially harmonic function if and only if the second part of the previous expression is zero, that is, if and only if

$$M^2_{ADM}\Delta^2 + N^2 = 0. \quad (5.198)$$

Thus, the conditions $N = 0$ and $\Delta = 0$ (which is equivalent to $\gamma = \epsilon = 0$ in (5.191)) are necessary conditions to recover an extremal RN black hole type metric. In this case, the central charge at infinity is

$$|Z_\infty|^2 = M^2_{ADM} = \langle S q | q \rangle. \quad (5.199)$$

We see that the vanishing of the non-extremality parameter $\Delta$ is equivalent to require that $I_\infty$ is fully contained in the subspace $\text{Span}(q, S q)$, whereas the condition $N = 0$ further restricts it to be proportional to the vector charge $I_\infty = q/M_{ADM}$. In this case, after imposing the conditions $N = \Delta = 0$, we can finally write

$$I = \frac{q}{M_{ADM}} \left(1 + \frac{M_{ADM}}{r}\right). \quad (5.200)$$
As a consequence of having $I_\infty = q/M_{\text{ADM}}$ the scalar fields $z^\alpha$ are constant everywhere and equal to their values at the fixed point (see (5.139) and the discussion in Section 5.6). It might be interesting to remark that in this expression the “unphysical” vector $\mathcal{I}$ appears written in terms of the physical quantities $q$ and $M_{\text{ADM}}$ which can be input by hand from the beginning.

### 5.8 Summary and concluding remarks

We have presented a systematic study of general, stationary, multicenter black hole solutions in ungauged four dimensional Einstein-Maxwell $N=2$ supergravity theories minimally coupled to scalars, i.e. theories with quadratic prepotentials. An important part of our analysis has been based on the matrices $S_F, S_N$ and their symplectic adjoints. These matrices are isometries of the symplectic bilinear form. Their adjoints with respect to the symplectic product $S_N^\dagger, S_F^\dagger$, which fulfills the property $S^2 = -\mathbb{I}$, are shown to lay inside the Lie algebra of the isometry group, they are such that $S + S^\dagger = 0$. They are “unitary”, $SS^\dagger = \mathbb{I}$, with respect to the symplectic product. Inner products, $g, h$, are defined. The three defined structures $(g, \omega, S)$ form a compatible triple, each structure can be specified by the two others. The symplectic $2n_a + 2$ dimensional space $W$ is decomposed into eigenspaces of the matrix $S$. Projection operators over these subspaces are considered.

Using the properties of these matrices, it is shown in particular that symplectic vectors (for which a stabilization equation relating their imaginary and real parts, $\text{Re} (X) = S \text{Im} (X)$ is valid) are inside the subspace $W^-$, one eigenspace of the matrix $S$.

We derive using pure algebraic properties, some alternative expressions for the attractor equations, (5.131) or (5.132). In this form, the attractor equations simply equal (a multiple of) the vector $V$, which lies on the subspace $W^-$, with the part of the charge vector which lies on such a subspace. We show some properties of the central charge modulus which can be expressed as a norm of a charge vector induced by the inner product $g$.

Similarly, the values of the scalars at the fixed points and at infinity are given by explicit expressions, (5.138) and (5.139), respectively. By these formulas, the values of the scalar fields at the fixed points and at infinity are given in terms of the projection of the charges into the eigenspaces of the matrix $S$.

Supposing a generic multicenter ansatz, (5.115), (which depends on the center charges $q_a$ and the value at infinity $I_\infty$) and a new form of the attractor equations, we have derived, or rederived in a simple way, different relations. The scalar field solutions are explicitly given by (5.146)-(5.147). In particular, we study some properties of configurations for which $z^\alpha_\infty = z^\alpha_f(Q)$. For these configurations, the scalar charges
vanish, cf. (5.156), and the vector $\mathcal{I}_\infty$ is of the form

$$\mathcal{I}_\infty = \frac{Q}{\sqrt{|SQ|Q}}.$$  

(5.201)

In fact, the vanishing of the scalar charges is shown to be equivalent to the vanishing of the quantities $(z_\infty^\alpha - z_f^\alpha(Q))$. This is in close analogy with the single center case, in which the vanishing of the scalar charges is a necessary and sufficient condition for the double extremality of the black hole [184].

The study of the near horizon and infinity geometry of the black hole lead us to the consideration of the area-like quantities

$$A_{ab} = \langle S_q^a | q_b \rangle$$

and

$$A_\infty = \sum_{a,b} A_{ab},$$

in addition to the horizon areas $S_{h,a} = \langle S_q^a | q_a \rangle$. The metric element is written as (cf. (5.167))

$$-g_{rr} = \langle R | \mathcal{I} \rangle = 1 + 2 \sum_b M_b \frac{1}{|x - x_b|} + \sum_{a,b} \frac{A_{ab}}{|x - x_a| |x - x_b|},$$

(5.202)

which is positive and finite for any finite $|x|$ if the matrix $(A_{ab})$ is definite positive. This is guaranteed by the fact that this matrix is the Gram matrix of a set of (linearly independent) vectors $q_a$ with the inner product $g$.

We proceed to the minimization of $M_{\text{ADM}}(z_\infty^\alpha) (M_{\text{ADM}}^2 = \langle SQ\rangle)$,

$$\frac{\partial M_{\text{ADM}}}{\partial z_\infty^\alpha} \bigg|_{z_{\text{min}}} = 0,$$

(5.203)

with respect to the scalars at infinity for a given charge configuration. We can show that, for a given charge configuration, we have

$$|Z_\infty|^2 = M_{\text{ADM}}^2 \leq \langle SQ \rangle = A_\infty.$$

(5.204)

where the equality appears at the minimum $(z_\infty^\alpha)_{\text{min}} = z_f(Q)$. The proof of this relation is a simple application of the Cauchy-Schwartz inequality to the inner product $\langle SX | Y \rangle$ and the asymptotic flatness condition.

The near horizon geometry is completely determined in terms of the individual horizon areas $S_{h,a} = \langle S_q^a | q_a \rangle$. The total horizon area $S_h$ is the sum of the areas of its disconnected parts

$$S_h = \sum_a S_{h,a} = \sum_a \langle S_q^a | q_a \rangle$$

$$= 2 \sum_a |Z_{f,a}|^2.$$

(5.205)

We use expansions of different quantities in terms of symplectic charge and extra vectors to get a series of different results. We get, for example different bounds of the quantities $S_h, A_\infty$. For the case of two centers with charges $q_{1,2}$ the relation between
both quantities is given by \( A_\infty - S_h = 2 \langle Sq_1|q_2 \rangle \), cf. (5.187). Using some simple general arguments, we arrive to the bound

\[
S_h \leq A_\infty \leq 2S_h .
\]

Finally, we have studied diverse properties and given some explicit expression of the quantity \( I_\infty \) by expanding this vector in a certain symplectic basis of the form (5.191),

\[
I_\infty = \alpha q + \beta S q + \gamma s + \epsilon S s .
\]

We arrive to the expression for the central charge

\[
|Z_\infty|^2 = M_{ADM}^2 + N^2 = \langle S q |q \rangle (1 - \Delta^2) ,
\]

where \( \Delta^2 = \gamma^2 + \epsilon^2 \). The condition \( |Z_\infty| = M_{ADM} = \langle S q |q \rangle \) is fulfilled if \( N = 0 \) and \( \Delta = 0 \). The vanishing of parameter \( \Delta \) is equivalent to demanding \( I_\infty \) to be fully contained in the subspace \( \text{Span}(q, S q) \). We finally arrive to an explicit expression for the solution ansatz \( I \), which for this case results

\[
I = \frac{q}{M_{ADM}} \left( 1 + \frac{M_{ADM}}{r} \right) .
\]

As a consequence of having \( I_\infty = q/M_{ADM} \), the scalar fields \( z^\alpha \) are constant everywhere and equal to their values at the fixed point. In this expression the ‘unphysical’ vector \( I \) is written in terms of the physical quantities \( q \) and \( M_{ADM} \), which can be input by hand from the beginning.

The projection of any symplectic vector that appears in the theory (for example, a subset of the charge vectors themselves or vectors characterizing the black hole ansatz at infinity) in terms of these new bases might be of general interest. The use of this projection, as it has been shown here, allows the understanding of questions as the entropy effects in the fragmentation of a single center black hole into a multicenter one. It also simplifies the study of the extremality of the solutions in terms, for example, of simple dimensional considerations of each of the charge-longitudinal and transversal subspaces.

In this study, we have focused on minimal coupling theories with quadratic prepotentials. It is of interest to study to which extent, and which modifications are needed, to apply the main techniques, properties and expressions presented here to the study of extremal and non extremal solutions in theories with general prepotentials (where the matrix \( S \) is not constant) or even theories without them.
Chapter 6

Conclusions and prospects

This work comprises an analysis of diverse theoretical topics of supergravity with three well differentiated parts: first, the study of gauged supergravities in higher (9D) dimensions within the embedding tensor formalism. The second part addresses the study of maximal and half-maximal gauged supergravities in $D = 9, 8, 7$. By using the double field theory formalism, we classify which ones have a higher-dimensional geometric origin or, otherwise, are obtained by means of a generalized Scherk-Scharzw reduction of DFT, in which the dual coordinates have a crucial importance. Finally, extremal multicenter black hole solutions have been considered in the context of some specific $N = 2$ supergravity theories, emphasizing on those coming from special geometry and quadratic prepotentials. We present full conclusions at the end of any of the three parts and we refer to them. Here we collect a summary of these conclusions.

The first part treats the study and classification of maximal gauged supergravities in $d = 9$ by means of the embedding tensor formalism. This formalism is a covariant tool to generate all possible gauged supergravities from a basic given theory. It scans along all the possible combinations of the global symmetry generators catching all the gaugings allowed by the global symmetry that the ungauged theory enjoys. Maximal $D = 9$ supergravity is a feasible example on which perform this analysis due to its relatively simple field content and group structure.

We have applied the embedding-tensor formalism to the study of the most general deformations ($i.e.$ gaugings and massive deformations) of maximal 9-dimensional supergravity. We have used the complete global $SL(2, \mathbb{R}) \times \mathbb{R}^2$ symmetry of its equations of motion, which includes the so-called trombone symmetry. We have found the constraints that the deformation parameters must satisfy in order to preserve both gauge and supersymmetry invariance (the latter imposed through the closure of the local supersymmetry algebra to lowest order in fermions). We have used most of the constraints to express some components of the deformation tensors in terms of a few components of the embedding tensor which we take to be independent and which are given in Eq. (3.129). At that point we have started making contact with the results of Ref. [106], since those independent components are precisely the 8 possible deformations identified there. All of them have a higher-dimensional origin discussed in detail.
6. Conclusions and prospects

in Ref. [106]. The field strengths, gauge transformations and supersymmetry transformations of the deformed theory, written in terms of the independent deformation tensors, are collected in Appendix B.4.

The 8 independent deformation tensors are still subject to quadratic constraints, given in Eq. (3.131), but those constraints cannot be used to express analytically some of them in terms of the rest, and, therefore, we must keep the 8 deformation parameters and we must enforce these irreducible quadratic constraints.

In Section 3.4 we have used our knowledge of the global symmetries (and corresponding Noether 1-forms), the independent deformation tensors and the irreducible quadratic constraints of the theory, together with the general arguments of Section 3.2.2 to determine the possible 7-, 8- and 9-forms of the theory (Table 3.7), which are dual to the Noether currents, independent deformation tensors and irreducible quadratic constraints. We have compared this spectrum of higher-rank forms with the results of Refs. [143, 144], based on $E_{11}$ level decomposition. We have found that, in the sector unrelated to the trombone symmetry, which was excluded from that analysis, the embedding-tensor formalism predicts one doublet of 9-forms less than the $E_{11}$ approach. However, both predictions are not contradictory: the extra doublet of 9-forms may not survive the deformations on which the embedding-tensor formalism is built: new 9-form Stückelberg shifts proportional to the deformation parameters may occur that can be used to eliminate it so only one combination of the two 9-form doublets survives. This mechanism is present in the $N = 2 \ d = 4, 5, 6$ theories [122], although the physics behind it is a bit mysterious.

Such a powerful mechanism as the embedding tensor seems to be a suitable tool in the search of a complete catalog of gaugings for every supergravity theory in different dimensions. Depending on the aim of our research and how witty we use it, we can face different problems. The completion of this catalog of deformations is still a intriguing task that suggests to be addressed by using this technique, as recent results show [231].

Another problem that the embedding tensor simplifies is the search of vacua for these gauged supergravities. The fact of having a scalar potential conveniently expressed in terms of $\vartheta$, together with techniques that translate our search from the moduli space to the flux background spaces [232], simplifies very much the exploration of vacua of a determined theory, as can be checked in [233–237].

The second part treats gauged supergravities and their origin from SS compactifications of higher-dimensional supergravities. Once we have a tool that provides all the possible deformations of a given supergravity, we decided to use it to extend this classification to lower dimensional theories. We performed the orbit classification of maximal and half-maximal $D = 9, 8, 7$ theories. The aim of this work is not only interesting by itself, but also results a reference to understand what orbits have a geometric origin, in the sense of arising from a SS compactification of a higher-dimensional theory. Since there is a mismatch between the existence of some gauged supergravities and the gaugings that arise from flux compactifications, several T duality constructions emerged to justify the information leak that occurs when a dimensional reduction pro-
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The procedure is done. Once again, we want to remark that the embedding tensor formalism is essential because it provides all the possible gaugings and guarantees the existence of no more than the ones found. In other case, we would work with a set of gaugings without being sure that of the existence of more gaugings and hence, the comparison with the flux compactification gaugings could not be performed in a systematic way.

In Chapter 4 we have provided a litmus test to the notion of non-geometry, by classifying the explicit orbits of consistent gaugings of different supergravity theories, and considering the possible higher-dimensional origins of these. The results turn out to be fundamentally different for the cases of U-duality orbits of maximal supergravities, and T-duality orbits of half-maximal theories.

In the former case we have managed to explicitly classify all U-duality orbits in dimensions $8 \leq D \leq 11$. This led to zero, one, four and ten discrete orbits in dimensions $D = 11, 10, 9$ and $8$, respectively, with different associated gauge groups. Remarkably, we have found that all of these orbits have a higher-dimensional origin via some geometric compactification, be it twisted reductions or compactifications on group manifolds or coset spaces. In our parlance, we have therefore found that all U-duality orbits are geometric. The structure of U-duality orbits is therefore dramatically different from the sketch of figure 1 in the introduction. Although a full classification of all orbits in lower-dimensional cases becomes increasingly cumbersome, we are not aware of any examples that are known to be non-geometric. It could therefore hold in full generality that all U-duality orbits are necessarily geometric.

This is certainly not the case for T-duality orbits of gaugings of half-maximal supergravities. In this case, we have provided the explicit classification in dimensions $7 \leq D \leq 10$ (where in $D = 7$ we have only included three-form fluxes). The numbers of distinct families of orbits in this case are zero, one, three and eleven in dimensions $D = 10, 9, 8$ and $7$, respectively, which includes both discrete and one-parameter orbits. A number of these orbits do not have a higher-dimensional origin in terms of a geometric compactification. Such cases are orbits $2$ and $3$ in $D = 8$ and orbits $1, 2$ and $3$ in $D = 7$ for $\alpha \neq 0$. Indeed, these are exactly the orbits that do not admit an uplift to the maximal theory. As proven in section 4.4.1 all such orbits necessarily violate the weak and/or strong constraints, and therefore need truly doubled backgrounds. Thus, the structure of T-duality orbits is very reminiscent of figure 1 in the introduction. Given the complications that already arise in these simpler higher-dimensional variants, one can anticipate that the situation will be similar in four-dimensional half-maximal supergravity.

Fortunately, the formalism of double field theory seems tailor-made to generate additional T-duality orbits of half-maximal supergravity. Building on the recent generalization of the definition of double field theory \[45\], we have demonstrated that all T-duality orbits, including the non-geometric ones in $D = 7, 8$, can be generated by a twisted reduction of double field theory. We have explicitly provided duality twists for all orbits. For locally-geometric orbits the twists only depend on the physical coordinates $y$, while for the non-geometric orbits these necessarily also include $\tilde{y}$. Again,
6. Conclusions and prospects

based on our exhaustive analysis in higher-dimensions, one could conjecture that also in lower-dimensional theories, all T-duality orbits follow from this generalized notion of double field theory.

At this point we would like to stress once more that a given orbit of gaugings can be generated from different twist orbits. Therefore, there is a degeneracy in the space of twist orbits giving rise to a particular orbit of gaugings. Interestingly, as it is the case of orbit 6 in $D = 7$ for instance, one might find two different twist orbits reproducing the same orbit of gaugings, one violating weak and strong constraints, the other one satisfying both. Our notion of a locally geometric orbit of gaugings is related to the existence of at least one undoubled background giving rise to it. However, this ambiguity seems to be peculiar of gaugings containing $Q$ flux. These can, in principle, be independently obtained by either adding a $\beta$ but no $\tilde{y}$ dependence (locally geometric choice, usually called T-fold), or by including non-trivial $\tilde{y}$ dependence but no $\beta$ (non-geometric choice) [43].

Another remarkable degeneracy occurs for the case of semi-simple gaugings, corresponding to orbits 1 – 3 in $D = 7$. For the special case of $\alpha = 0$, we have two possible ways of generating such orbits from higher-dimensions: either a coset reduction over a sphere or analytic continuations thereof, or a duality twist involving non-geometric coordinate dependence. Therefore $d$-dimensional coset reductions seem to be equivalent to $2d$-dimensional twisted torus reductions (with the latter in fact being more general, as it leads to all values of $\alpha$). Considering the complications that generally arise in proving the consistency of coset reductions, this is a remarkable reformulation that would be interesting to understand in more detail. Furthermore, when extending the notion of double field theory to type II and M-theory, this relation could also shed new light on the consistency of the notoriously difficult four-, five- and seven-sphere reductions of these theories.

Our results mainly focus on Scherk-Schwartz compactifications leading to gauged supergravities with vanishing $\xi_M$ fluxes. In addition, we have restricted to the NSNS sector and ignored $\alpha'$-effects. Also, we stress once again that relaxing the strong and weak constraints is crucial in part of our analysis. If we kept the weak constraint, typically the Jacobi identities would lead to backgrounds satisfying also the strong constraint [45]. However, from a purely (double) field theoretical analysis the weak constraint is not necessary. A sigma model analysis beyond tori would help us to clarify the relation between DFT without the weak and strong constraints and string field theory on more general backgrounds. We hope to come back to this point in the future.

At this point, we wonder whether we could generalize this study to lower dimensions. Unfortunately, this is a considerable more complicated goal, due to how the global symmetry groups quickly grow. This means that the classification of the orbits is extraordinarily difficult. However, some questions based on some insights of our results could be set out. What is the relation between geometric orbits and maximal supergravities? That is, is there any underlying reason why the maximal theories
analyzed only host geometric orbits? On the other hand, we wonder whether all the
gaugings of half-maximal theories have a description in terms of DFT. What about 1/4-
BPS states? Is DFT powerful enough to reproduce those solutions? Do these states
violate even the relaxed version of the strong constraint? What about the supersym-
metric completion of DFT? This is an issue that has already been addressed 238.
Finally, a sizzling problem is the generalization of DFT towards the M theory goal.
Some recent constructions have recently been proposed 239–241.

The third part of the manuscript treats the multicenter black hole solutions in
$N=2$ theories. Despite different solutions have been working out since long time ago,
it is not trivial to find a set of parameters that satisfy the physical constraints of these
solutions.

In Chapter 5 we have presented a systematic study of general, stationary, mul-
ticenter black hole solutions in ungauged four dimensional Einstein-Maxwell $N=2$
supergravity theories minimally coupled to scalars, i.e. theories with quadratic pre-
 potentials. An important part of our analysis has been based on the matrices $S_F, S_N$
and their symplectic adjoints. These matrices are isometries of the symplectic bilinear
form. Their adjoints with respect to the symplectic product $S_F^\dagger, S_N^\dagger$, which fulfills the
property $S^2 = -1$, are shown to lay inside the Lie algebra of the isometry group,
they are such that $S + S^\dagger = 0$. They are “unitary”, $SS^\dagger = 1$, with respect to the
symplectic product. Inner products, $g, h$, are defined. The three defined structures
$(g, \omega, S)$ form a compatible triple, each structure can be specified by the two others.
The symplectic $2n_r + 2$ dimensional space $W$ is decomposed into eigenspaces of the
matrix $S$. Projection operators over these subspaces are considered.

Using the properties of these matrices, it is shown in particular that symplectic
vectors (for which a stabilization equation relating their imaginary and real parts,
$\text{Re}(X) = S \text{Im}(X)$ is valid) are inside the subspace $W^-$, one eigenspace of the matrix
$S$.

We derive using pure algebraic properties, some alternative expressions for the
attractor equations, (5.131) or (5.132). In this form, the attractor equations simply
equal (a multiple of) the vector $V$, which lies on the subspace $W^-$, with the part of the
charge vector which lies on such a subspace. We show some properties of the central
charge modulus which can be expressed as a norm of a charge vector induced by the
inner product $g$.

Similarly, the values of the scalars at the fixed points and at infinity are given by
explicit expressions, (5.138) and (5.139), respectively. By these formulas, the values of
the scalar fields at the fixed points and at infinity are given in terms of the projection
of the charges into the eigenspaces of the matrix $S$.

Supposing a generic multicenter ansatz, (5.115), (which depends on the center
charges $q_a$ and the value at infinity $I_\infty$) and a new form of the attractor equations,
we have derived, or rederived in a simple way, different relations. The scalar field sol-
utions are explicitly given by (5.146)-(5.147). In particular, we study some properties
of configurations for which $z_\infty^\alpha = z_f^\alpha(Q)$. For these configurations, the scalar charges
vanish, cf. (5.156), and the vector $I_{\infty}$ is of the form

$$
I_{\infty} = \frac{Q}{\sqrt{\langle S | Q \rangle}}.
$$

(6.1)

In fact, the vanishing of the scalar charges is shown to be equivalent to the vanishing of the quantities $(z_{\infty}^a - z_f^a(Q))$. This is in close analogy with the single center case, in which the vanishing of the scalar charges is a necessary and sufficient condition for the double extremality of the black hole [184].

The study of the near horizon and infinity geometry of the black hole lead us to the consideration of the area-like quantities $A_{ab} = \langle S q_a | q_b \rangle$ and $A_{\infty} = \sum_{ab} A_{ab}$, in addition to the horizon areas $S_{h,a} = \langle S q_a | q_a \rangle$. The metric element is written as (cf. (5.167))

$$
-g_{rr} = \langle R | I \rangle = 1 + 2 \sum_b M_b |x - x_b| + \sum_{a,b} A_{ab} |x - x_a||x - x_b|,
$$

(6.2)

which is positive and finite for any finite $|x|$ if the matrix $(A_{ab})$ is definite positive. This is guaranteed by the fact that this matrix is the Gram matrix of a set of (linearly independent) vectors $q_a$ with the inner product $g$.

We proceed to the minimization of $M_{ADM}(z_{\infty}^a) (M_{ADM}^2 = \langle S I_{\infty} | Q \rangle)$,

$$
\frac{\partial M_{ADM}}{\partial z_{\infty}^a} \bigg|_{z_{\infty}^a = z_{\min}} = 0,
$$

(6.3)

with respect to the scalars at infinity for a given charge configuration. We can show that, for a given charge configuration, we have

$$
|Z_{\infty}|^2 = M_{ADM}^2 \leq \langle S Q | Q \rangle = A_{\infty},
$$

(6.4)

where the equality appears at the minimum $(z_{\infty}^a)_{min} = z_f(Q)$. The proof of this relation is a simple application of the Cauchy-Schwartz inequality to the inner product $\langle SX | Y \rangle$ and the asymptotic flatness condition.

The near horizon geometry is completely determined in terms of the individual horizon areas $S_{h,a} = \langle S q_a | q_a \rangle$. The total horizon area $S_h$ is the sum of the areas of its disconnected parts

$$
S_h = \sum_a S_{h,a} = \sum_a \langle S q_a | q_a \rangle
= 2 \sum_a |Z_{f,a}|^2.
$$

(6.5)

We use expansions of different quantities in terms of symplectic charge and extra vectors to get a series of different results. We get, for example different bounds of the quantities $S_h, A_{\infty}$. For the case of two centers with charges $q_{1,2}$ the relation between
6. Conclusions and prospects

Both quantities is given by \( A_\infty - S_h = 2 \langle Sq_1|q_2 \rangle \), cf. (5.187). Using some simple general arguments, we arrive to the bound

\[
S_h \leq A_\infty \leq 2S_h .
\]  

(6.6)

Finally, we have studied diverse properties and given some explicit expression of the quantity \( I_\infty \) by expanding this vector in a certain symplectic basis of the form (5.191),

\[
I_\infty = \alpha q + \beta Sq + \gamma s + \epsilon Ss .
\]  

(6.7)

We arrive to the expression for the central charge

\[
|Z_\infty|^2 = M^2_{ADM} + N^2 = \langle Sq|q \rangle (1 - \Delta^2) ,
\]  

(6.8)

where \( \Delta^2 = \gamma^2 + \epsilon^2 \). The condition \( |Z_\infty| = M_{ADM} = \langle Sq|q \rangle \) is fulfilled if \( N = 0 \) and \( \Delta = 0 \). The vanishing of parameter \( \Delta \) is equivalent to demanding \( I_\infty \) to be fully contained in the subspace \( \text{Span}(q, Sq) \). We finally arrive to an explicit expression for the solution ansatz \( I \), which for this case results

\[
I = \frac{q}{M_{ADM}} \left( 1 + \frac{M_{ADM}}{r} \right) .
\]  

(6.9)

As a consequence of having \( I_\infty = q/M_{ADM} \), the scalar fields \( \varepsilon^\alpha \) are constant everywhere and equal to their values at the fixed point. In this expression the ‘unphysical’ vector \( I \) is written in terms of the physical quantities \( q \) and \( M_{ADM} \), which can be input by hand from the beginning.

The projection of any symplectic vector that appears in the theory (for example, a subset of the charge vectors themselves or vectors characterizing the black hole ansatz at infinity) in terms of these new bases might be of general interest. The use of this projection, as it has been shown here, allows the understanding of questions as the entropy effects in the fragmentation of a single center black hole into a multicenter one. It also simplifies the study of the extremality of the solutions in terms, for example, of simple dimensional considerations of each of the charge-longitudinal and transversal subspaces.

In this study, we have focused on minimal coupling theories with quadratic prepotentials. It is of interest to study to which extent, and which modifications are needed, to apply the main techniques, properties and expressions presented here to the study of extremal and non extremal solutions in theories with general prepotentials (where the matrix \( S \) is not constant) or even theories without them.
Appendix A

Nuts and bolts: T-duality

In the framework of supergravities considered as low energy effective field theories of string theories, the global symmetries of the SUGRAs are seen to correspond to dualities of the string theories \[242\].

Some of these string dualities are essentially perturbative and the worldsheet approach is valid to be studied. For instance, T-duality \[243\], that relates string theories compactified on circles of radius $R$ and dual radius $R' = 1/R$, is an exact symmetry at all orders in string perturbation theory \[244\]. However, the so-called $S$-duality, is non-perturbative in the string coupling constant and cannot be studied using the standard worldsheet approach. Finally, $U$-duality is another duality that includes $S$- and T-duality and is considered directly related to the existence of the so-called M theory.

We will show some basic ideas of T-duality in the next paragraphs.

The bosonic string

We will restrict to the string common sector. We will follow \[245, 246\]. Since T-duality relates different theories compactified on a circle, we will choose the effective action (5) as the one on which to perform the dimensional reduction. We will get a $D = \hat{D} - 1$ dimensional theory that will enjoy this duality. Let us assume the following standard KK reduction ansatz,

\[
\hat{e}_{\hat{\mu}} = \begin{pmatrix} e_\mu^a & kA_\mu \\ 0 & k \end{pmatrix}, \quad \phi = \hat{\phi} - \frac{1}{2} \ln k,
\]

\[
\hat{B}_{\mu\nu} = B_{\mu\nu} - A_{[\mu}B_{\nu]} , \quad \hat{B}_{\mu z} = B_{\mu} ,
\]

where $\hat{\mu} = \{\mu, z\}$, i.e. hatted indices and fields are defined on $\hat{D}$ dimensions and the unhatted ones correspond to $D$ dimensions. We will refer as $z$ the compactified coordinate. After integrating over the compact coordinate, the reduced effective action is

\[
S \sim \int d^Dx \sqrt{|g|} e^{-2\phi} \left( R - 4(\partial\phi)^2 + \frac{1}{2 \cdot 3!} H^2 + (\partial \log k)^2 - \frac{1}{4} k^2 F^2(A) - \frac{1}{4} k^{-2} F^2(B) \right),
\]
where $F(A)$ and $F(B)$ are the field strengths of the vector fields $A_\mu$ and $B_\mu$, respectively. We can check the invariance of this action under the transformation rules

$$A_\mu \rightarrow B_\mu, \quad B_\mu \rightarrow A_\mu, \quad k \rightarrow k^{-1}, \quad (A.3)$$

so that the KK scalar gets inverted and the KK vector and the winding vector are interchanged. Two interpretations can be done: first, we compactify a string background, T dualize it, and decompactify it into a different background. Second, we have two different compactifications of a given background; these compactifications give the same $D$-dimensional background and thus, are dual.

The way in that these two backgrounds are related is described by an isometry. These field relations are known as Buscher’s rules [247] [249].

\[
\hat{e}^a_z = \mp \hat{e}^a_z \frac{1}{g_{zz}}, \quad \hat{e}^\mu_z = \hat{e}^\mu_z \mp \frac{\hat{g}_{\mu z}}{\hat{g}_{zz}}, \\
\hat{B}'_{\mu z} = \hat{g}_{\mu z} \frac{1}{\hat{g}_{zz}}, \quad \hat{B}'_{\mu\nu} = \hat{B}_{\mu\nu} + 2 \hat{g}_{[\mu|z} \hat{B}_{|\nu]z} \frac{1}{\hat{g}_{zz}} \hat{g}_{zz}, \quad (A.4)
\]

Now, at the string level, let us study T-duality applied to the $\sigma$-model of the bosonic string introduced in [1], without considering the dilaton term, since it does not play any relevant role in this classic approach. Let us assume [1] with hatted fields running over hatted indices. Then, decomposing the $\hat{D}$-dimensional fields into $D$-dimensional fields using (A.1), we have

\[
S = -\frac{T}{2} \int d^2 \sigma \sqrt{\gamma} \left[ \gamma^{ij} g_{ij} - k^2 F^2 \right] + \frac{T}{2} \int d^2 \sigma \sqrt{\gamma} \epsilon^{ij} \left[ B_{ij} + A_i B_j - 2 F_i B_j \right], \quad (A.5)
\]

where $g_{ij}$, $B_{ij}$, $A_i$, $B_i$ are the pullbacks of the $D$-dimensional metric, KR 2-form, KK and winding vectors respectively. $F_i$ is the field strength of the $Z$ coordinate,

$$F_i = \partial_i Z + A_i, \quad (A.6)$$

which reflects the following shift invariance:

$$\delta \Lambda Z = -\Lambda(x), \quad \delta \Lambda A_\mu = \partial_\mu \Lambda. \quad (A.7)$$

This invariance implies the following conserved current

$$P^i_z = T (k^2 F^i - \star B^i), \quad (A.8)$$

whose associated magnetic-like conserved current is

$$W^i_z = T \star F^i - \star A^i. \quad (A.9)$$
Their associated charges are the momentum of the string in the compact dimension and the winding number, respectively. Then, if we perform a Poincaré duality transformation on the $Z$ coordinate, $Z \rightarrow Z'$, by using the Bianchi identity of $F_i$ and its equation of motion, we have

\[ S' = -\frac{T}{2} \int d^2 \sigma \sqrt{|\gamma|} \left[ \gamma^{ij} g_{ij} - k^{-2} F'^2 \right] + \frac{T}{2} \int d^2 \sigma \sqrt{|\gamma|} \epsilon^{ij} [B_{ij} + B_i A_j - 2F'_i A_j] , \]

(A.10)

where

\[ F'_i = \partial_i Z' + B_i . \]

(A.11)

This action coincides with the original one when we make the field replacements \[ A.3 \]. We find that its conserved currents, $P^{i}_{Z'}$, and $W^{i}_{Z'}$, are closely related to those of the original theory,

\[ P^{i}_{Z'} = W^{i}_{Z'}, \quad W^{i}_{Z'} = P^{i}_{Z'} . \]

(A.12)

Thus, we summarize that T-duality inverts the compactification radius and interchanges momentum modes with winding modes, leaving invariant the mass spectrum and performing a parity transformation on the right-moving modes.

For type II superstrings, this parity transformation changes the chirality of the spinors and the overall result is that the $N = (1, 1)$ type IIA theory can be mapped into the $N = (2, 0)$ type IIB version. This relation holds for any value of the radius, in particular it relates the limits $R \rightarrow 0$ and $R \rightarrow \infty$. For the case of $N = 2A$ and $N = 2B$ supergravity theories, there is a discrete symmetry relating the two supergravity theories when both of them are reduced to 9 dimensions \[ 250 \]. A generalization of the Buscher’s rules can be established \[ 104, 109 \] when one performs dimensional reductions from $N = 2A$ and $N = 2B$ to $D = 9$ and identifies the same fields from the two different reduction schemes \[ 246 \],

\[ \hat{J}_{\mu \nu} = \hat{g}_{\mu \nu} - \frac{\hat{g}_{\mu z} \hat{g}_{\nu z} - B_{\mu z} \hat{B}_{\nu z}}{g_{zz}}, \quad \hat{J}_{\mu y} = \frac{\hat{B}_{\mu y}}{g_{zz}}, \]

\[ \hat{B}_{\mu \nu} = \hat{B}_{\mu \nu} + \frac{\hat{g}_{\mu z} \hat{B}_{\nu z} - \hat{B}_{\mu z} \hat{g}_{\nu z}}{g_{zz}}, \quad \hat{B}_{\mu y} = \frac{\hat{g}_{\mu z}}{g_{zz}}, \]

\[ \hat{\phi} = \phi - \frac{1}{2} \ln |\hat{g}_{zz}|, \quad \hat{\phi} = -\frac{1}{2} \ln |\hat{g}_{zz}|, \]

\[ \hat{C}^{(2n)}_{\mu_1 \cdots \mu_{2n}} = \hat{C}^{(2n+1)}_{\mu_1 \cdots \mu_{2n+1}} + 2n \hat{B}_{[\mu_1] z} \hat{C}^{(2n-1)}_{\mu_{2n+1} \cdots \mu_{2n+1}}, \quad \hat{C}^{(2n)}_{\mu_1 \cdots \mu_{2n-1} y} = -\hat{C}^{(2n-1)}_{\mu_1 \cdots \mu_{2n-1}} + (2n - 1) \frac{\hat{g}_{[\mu_1] z} \hat{C}^{(2n-1)}_{\mu_{2n-1} \cdots \mu_{2n-1}}}{g_{zz}}. \]

On the other hand, T-duality effects on the heterotic superstrings result in the transformation laws of the heterotic whose gauge group is $E_8 \times E_8$ into the heterotic theory with $SO(32)$ as a gauge group, and vice versa \[ 251 \].
T-duality in type I string theory is even more subtle. We can obtain the effective action of type I by considering type IIB and truncating it using one of its $\mathbb{Z}_2$ symmetries plus the inclusion of an O9-plane and 32 D9-branes.\footnote{Reference \cite{246} pedagogically shows how to do it.} The T-duality between type IIB and type IIA theories implies the existence of the so-called type I' [252], which can be interpreted as a rotation of the space where we compactify. This implies the interchange of Neumann and Dirichlet boundary conditions for certain coordinates.

The examples of T-duality that we have discussed are only the tip of a mathematical iceberg: there exist additional dualities known as mirror symmetries, in which different 10-dimensional string theories compactified on Calabi-Yau manifolds are related to each other [253].
Appendix B

Gaugings in $N = 2$ $D = 9$ supergravity

B.1 Conventions

We follow the conventions of Ref. [106]. In particular, we use mostly plus signature $(-, +, \cdots, +)$ and the gamma matrices satisfy

$$\gamma^*_a = -\gamma_a, \quad \gamma_a = \eta_{aa} \gamma^+_a.$$  \hfill (B.1)

The Dirac conjugate of a spinor $\epsilon$ is defined by

$$\bar{\epsilon} \equiv \epsilon^\dagger \gamma_0.$$ \hfill (B.2)

Then, we have

$$ (\bar{\epsilon} \gamma^{(n)} \lambda)^* = a_n \epsilon^* \gamma^{(n)} \lambda^*, $$ \hfill (B.3)

$$ (\bar{\epsilon} \gamma^{(n)} \lambda)^* = b_n \bar{\lambda} \gamma^{(n)} \epsilon, $$

where the signs $a_n$ and $b_n$ are given in Table B.1

B.1.1 Spinor bilinears

We define the following real bilinears of the supersymmetry parameters $\epsilon_1$ and $\epsilon_2$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|---|---|
| $a_n$ | - | + | - | + | - | + | - | + | - |
| $b_n$ | + | - | - | + | - | - | + | + | - |

Table B.1: Values of the coefficients $a_n$ and $b_n$ defined in Eqs. (B.3).
\[ \bar{\epsilon}_2 \epsilon_1 \equiv a + i b, \quad \text{(B.4)} \]
\[ \bar{\epsilon}_2 \epsilon_1^* \equiv c + i d, \quad \text{(B.5)} \]
\[ \bar{\epsilon}_2 \gamma_{\mu_1 \cdots \mu_n} \epsilon_1 \equiv \xi_{\mu_1 \cdots \mu_n} + i \zeta_{\mu_1 \cdots \mu_n}, \quad \text{(B.6)} \]
\[ \bar{\epsilon}_2 \gamma_{\mu_1 \cdots \mu_n} \epsilon_1^* \equiv \sigma_{\mu_1 \cdots \mu_n} + i \rho_{\mu_1 \cdots \mu_n}, \quad \text{(B.7)} \]

### B.2 Relation with other conventions

The electric fields used in this paper are related to those used in Ref. [109] (which uses a mostly minus signature) as follows:

\[ K = e^{\frac{\chi}{\sqrt{7}}} \phi, \quad \text{(B.8)} \]
\[ \lambda \equiv C^{(0)} + i e^{-\phi} = \tau \equiv \chi + i e^{-\phi}, \quad \text{(B.9)} \]
\[ A_{(1)} = A^0, \quad \text{(B.10)} \]
\[ A_{(1)} = A^i, \quad \text{(B.11)} \]
\[ A_{(2)} = B^i + \frac{i}{2} A^{0i}, \quad \text{(B.12)} \]
\[ A_{(3)} = -C + \frac{1}{2} \varepsilon_{ij} A^i \wedge B^j - \frac{1}{12} \varepsilon_{ijkl} A^{0ij}, \quad \text{(B.13)} \]
\[ A_{(4)} = -\tilde{C} + C \wedge A^0 - \frac{1}{4} \varepsilon_{ij} B^i \wedge A^{0j}. \quad \text{(B.14)} \]

The field strengths are related by
B.3. Noether currents

\[ F_{(2)} = F^0, \quad (B.15) \]
\[ F_{(2)} = F^i, \quad (B.16) \]
\[ F_{(3)} = H^i, \quad (B.17) \]
\[ F_{(4)} = -G, \quad (B.18) \]
\[ F_{(5)} = -\tilde{G}. \quad (B.19) \]

The relation with the fields used in Ref. [106] (which also uses mostly plus signature) is given by (our fields are in the r.h.s. of these equations)

\[ B^i = -(B^i + \frac{1}{2} A^{0i}), \quad (B.20) \]
\[ C = -(C - \frac{1}{6} \varepsilon_{ij} A^{0ij}), \quad (B.21) \]

while the field strengths are related by

\[ H^i = -H^i, \quad (B.22) \]
\[ G = -G. \quad (B.23) \]

The rest of the fields are identical.

B.3 Noether currents

The Noether 1-form currents of the undeformed theory \( j_A \) are given by

\[ \star j_m = \star d M_{ij} (M^{-1})_{jk} T_{mi}^k + e^{\frac{1}{\sqrt{7}} \varphi} (M^{-1})_{ij} T_{mk}^i A^k \wedge \star F^j \]
\[ + T_{mk}^i \left[ e^{-\frac{1}{\sqrt{7}} \varphi} M_{ij}^{-1} (B^k - \frac{1}{2} A^{0k}) \wedge \star H^j + \frac{1}{2} \varepsilon_{ij} \left( -2 e^{\frac{2}{\sqrt{7}} \varphi} A^j \wedge B^k \wedge \star G \right. \right. \]
\[ + (B^j - A^{0j}) \wedge B^k \wedge G + \varepsilon_{ln} A^l \wedge B^{jk} \wedge (H^n - \frac{1}{2} A^n \wedge F^0) \]
\[ \left. + \frac{1}{4} \varepsilon_{ln} A^{0ln} \wedge B^k \wedge H^j \right), \quad (B.24) \]
\[ \ast j_4 = \frac{6}{\sqrt{7}} \ast d\varphi + 3 \left[ e^{-\frac{1}{\sqrt{7}} \varphi} A^0 \ast F^0 + e^{-\frac{1}{\sqrt{7}} \varphi} M_{ij}^{-1} (B^i + \frac{1}{2} A^{0i}) \ast H^j + e^{\frac{2}{\sqrt{7}} \varphi} (C - \frac{1}{6} \varepsilon_{ij} A^{0[ij]} \ast G \right. \\
\left. + A^0 \ast (C + \varepsilon_{ij} A^i \ast B^j) \ast G \right] + \frac{3}{2} \varepsilon_{ij} \left[ (-C + \varepsilon_{kl} A^k \ast B^l - \frac{7}{12} \varepsilon_{kl} A^{0kl}) \ast B^i \ast H^j \right. \\
\left. - \frac{3}{2} A^{0i} \ast C \ast H^j + (A^i \ast B^j - \frac{1}{2} A^{0[ji]} \ast F^0 \ast C) \right] , \quad (B.25) \]

\[ \ast j_5 = \frac{\sqrt{7}}{4} \ast d\varphi - \frac{3}{8} \ast \tau d\tau + \text{c.c.} \left( \frac{3\tau}{4} \sqrt{7} T_{50}^0 A^0 \ast \ast F^0 + \frac{2}{\sqrt{7}} \varepsilon_{ijk} - \frac{1}{\sqrt{7}} T_{5k}^i \ast F^j \right. \\
\left. + e^{-\frac{1}{\sqrt{7}} \varphi} M_{ij}^{-1} \left[ T_{5k}^i (B^k - \frac{1}{2} A^{0k}) + \frac{1}{4} A^{0i} \right] \ast H^j \right. \\
\left. + e^{\frac{2}{\sqrt{7}} \varphi} (T_5 C - \frac{1}{12} \varepsilon_{ij} A^{0[ij]} - T_{5k}^i \varepsilon_{ij} (A^k \ast B^j - \frac{1}{6} A^{0kj})) \right) \ast F^j \right. \\
\left. + \frac{1}{12} \varepsilon_{ij} \left[ T_{5k}^i (-2 B^{jk} + 3 A^{0[j} \ast B^{k]} - 5 A^{0k} \ast B^j) - \frac{1}{2} A^{0i} \ast B^j \right] \ast G \right. \\
\left. + \frac{1}{12} \varepsilon_{ij} \left[ T_{5k}^i (2 \varepsilon_{ln} A^l \ast B^{nk} - \varepsilon_{ln} A^{0ln} \ast B^k) - T_5 (6 A^{0i} + B^i) \ast C - \frac{1}{12} \varepsilon_{kl} A^{0kl} \right. \right. \\
\left. \ast H^j \right) \right. \\
\left. + \varepsilon_{ij} \varepsilon_{ln} T_{5k}^i \left[ \frac{5}{6} A^{0[jk} \ast B^{k]} - A^{0[ji} \ast B^k + \frac{1}{2} A^k \ast B^{jl]} \right] \ast H^n \right. \\
\left. + T_5 \left[ A^0 \ast C \ast G + \frac{1}{2} \varepsilon_{ij} (B^j + \frac{1}{2} A^{0[j}) \ast A^i \ast F^0 \ast C \right] \right] \quad (B.26) \]

### B.4 Final results

In this Appendix we give the final form of the deformed covariant field strengths, covariant derivatives, gauge and supersymmetry transformations in terms of the independent deformation parameters given in Eq. \( (B.29) \). We must bear in mind that they are assumed to satisfy the irreducible quadratic constraints given in Eq. \( (B.31) \) and only then the field strengths etc. have the right transformation properties.

The covariant derivatives of the scalar fields are given by

\[ \mathcal{D}\varphi = -\frac{137}{24\sqrt{7} \varphi} \varphi A_0^5 A^0 + \left( -\frac{\sqrt{7}}{4} \partial_1^4 + \frac{6}{\sqrt{7}} \partial_1^5 \right) A^i , \quad (B.27) \]

\[ \mathcal{D}\tau = \partial_0^m k_m \tau A^1 - \frac{3}{4} \partial_0^5 \tau A^0 + \frac{3}{4} \left( \partial_1^5 \tau + \partial_2^5 \right) A^1 - \tau A^2 , \quad (B.28) \]
and their gauge transformations are explicitly given by

\[ \delta \Lambda \varphi = -\frac{137}{24\sqrt{7}} \vartheta_0^5 \Lambda^0 + \left( -\frac{\sqrt{7}}{4} \vartheta_1^4 + \frac{6}{\sqrt{7}} \vartheta_1^5 \right) \Lambda^1, \]

(B.29)

\[ \delta \Lambda \tau = \vartheta_0^m k_m \tau \Lambda^0 - \frac{3}{4} \vartheta_0^5 \Lambda^0 + \frac{3}{4} \left( \vartheta_1^5 \tau + \vartheta_2^5 \right) \left( \Lambda^1 - \tau \Lambda^2 \right). \]

(B.30)

The deformed \( p \)-form field strengths are given by

\[ F^0 = dA^0 - \frac{1}{2} \left( 3 \vartheta_1^4 + \frac{1}{2} \vartheta_1^5 \right) A^0 + \left( 3 \vartheta_1^4 + \frac{1}{2} \vartheta_1^5 \right) B^1, \]

(B.31)

\[ F^i = dA^i + \frac{1}{2} \left( \vartheta_0^m (T_m^{(3)})_j A^0 - \frac{3}{4} \delta_t^i \vartheta_0^5 A^0 + \frac{3}{2} \varepsilon_t^i \vartheta_0^5 A^1 \right) B^j - \frac{3}{4} \delta_t^i \vartheta_0^5 B^1, \]

(B.32)

\[ H^i = \mathcal{D} B^i + \frac{1}{2} \left( A^0 \wedge dA^i + A^i \wedge dA^0 \right) + \frac{1}{6} \varepsilon^{ij} \left( 3 \vartheta_j^4 + \frac{1}{2} \vartheta_j^5 \right) A^{012} \]

\[ + \varepsilon^{ij} \left( 3 \vartheta_j^4 - \frac{1}{4} \vartheta_j^5 \right) C, \]

(B.33)

\[ G = \mathcal{D} C - \varepsilon_{ij} \left[ F^i \wedge B^j - \frac{1}{2} \delta_t^i \left( A^i \wedge dA^j - \frac{1}{2} d(A^{0j}) \right) \right] + \frac{1}{2} \left( \varepsilon_{ij} \vartheta_0^m (T_m^{(2)})_k B^{jk} - \frac{3}{2} \vartheta_0^5 B^{12} \right) + Z \tilde{C}, \]

(B.34)

where the covariant derivatives acting on the different fields are given by

\[ \mathcal{D} B^i = dB^i + \vartheta_0^m (T_m^{(2)})_j A^0 \wedge B^j - \frac{3}{4} \delta_t^i \vartheta_0^5 A^0 \wedge B^1 \]

\[ + \left( 3 \vartheta_t^4 - \frac{1}{2} \vartheta_t^5 \right) A^t \wedge B^i + \frac{3}{4} \delta_t^i \vartheta_t^5 A^t \wedge B^k, \]

(B.35)

\[ \mathcal{D} C = dC - \frac{3}{4} \vartheta_0^5 A^0 \wedge C - (3 \vartheta_1^4 - \frac{1}{2} \vartheta_1^5) A^1 \wedge C. \]

(B.36)

The field strengths transform covariantly under the gauge transformations.
\[ \delta_\Lambda A^0 = -\mathcal{D}\Lambda^0 + (3\partial_4^4 + \frac{1}{2}\partial_5^5) A^0, \]  
(B.37)

\[ \delta_\Lambda A^i = -\mathcal{D}\Lambda^i + \partial_0^m(T_m^{(3)})_j^i A^j - \frac{3}{4}\delta_1^1\partial_0^5 A^1, \]  
(B.38)

\[ \delta_\Lambda B^i = -\mathcal{D}\Lambda^i + F^0 \wedge A^i + F^1 A^0 + \frac{1}{2} (A^0 \wedge \delta_\Lambda A^i + A^i \wedge \delta_\Lambda A^0) + \epsilon^{ij} \left(3\partial_4^4 - \frac{1}{4}\partial_5^5\right) \Lambda, \]  
(B.39)

\[ \delta_\Lambda \left(C - \frac{1}{6} \varepsilon_{ij} A^{0i} \right) = -\mathcal{D}\Lambda - \varepsilon_{ij} \left(\Lambda^i H^j + F^1 \wedge \Lambda^j - \delta_\Lambda A^i \wedge B^j\right) - \frac{1}{2} \varepsilon_{ij} A^{0i} \delta_\Lambda A^j + Z\Lambda, \]  
(B.40)

where the covariant derivatives of the different gauge parameters are given by

\[ \mathcal{D}\Lambda^0 = d\Lambda^0 + (3\partial_4^4 + \frac{1}{2}\partial_5^5) A^i \Lambda^0, \]  
(B.41)

\[ \mathcal{D}\Lambda^i = d\Lambda^i + \partial_0^m(T_m^{(3)})_j^i A^j + \frac{3}{4}\delta_1^1\partial_0^5 A^0 A^1 + \frac{3}{4}\varepsilon_{ij} \partial_0^5 A^k A^1, \]  
(B.42)

\[ \mathcal{D}\Lambda^i = d\Lambda^i + \partial_0^m(T_m^{(2)})_j^i A^j \wedge A^i + (3\partial_4^4 - \frac{1}{4}\partial_5^5) A^k \wedge A^i \]  

\[ + \frac{3}{4}\delta_1^1 \partial_0^5 A^i \wedge A^k, \]  
(B.43)

\[ \mathcal{D}\Lambda = d\Lambda - \frac{3}{4}\partial_0^5 A^0 \wedge A + (3\partial_4^4 - \frac{1}{4}\partial_5^5) A^i \wedge A. \]  
(B.44)

The supersymmetry transformation rules of the fermion fields are given by

\[ \delta_\epsilon \psi_\mu = \mathcal{D}_\mu \epsilon + f^{\gamma\mu\nu} \epsilon + k^{\gamma\mu\nu} \epsilon^* + \frac{i}{8\sqrt{7}} e^{-\sqrt{7} \phi} \left(\frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu\right) F^0 \epsilon \]  

\[ - \frac{1}{8\sqrt{7}} e^{\frac{1}{2} \sqrt{7} \phi} + \frac{i}{2} \phi \left(\frac{5}{7} \gamma_\mu \gamma^{(2)} - \gamma^{(2)} \gamma_\mu\right) (F^1 - \tau F^2) \epsilon^* \]  

\[ - \frac{1}{8\sqrt{7}} e^{-\frac{1}{2} \sqrt{7} \phi} \left(\frac{3}{7} \gamma_\mu \gamma^{(3)} + \gamma^{(3)} \gamma_\mu\right) (H^1 - \tau H^2) \epsilon^* \]  

\[ - \frac{1}{8\sqrt{7}} e^{-\sqrt{7} \phi} \left(\frac{1}{7} \gamma_\mu \gamma^{(4)} - \gamma^{(4)} \gamma_\mu\right) G \epsilon, \]  
(B.45)
\[ \delta \dot{\lambda} = i \mathcal{D} \varphi^* + \bar{g} \epsilon + \tilde{h} \epsilon^* - \frac{1}{\sqrt{\pi}} e^{-\frac{2}{3} \sqrt{7} \Phi} F^0 \epsilon^* - \frac{3i}{2 \sqrt{7}} e^{\frac{1}{2} \sqrt{7} \Phi} \frac{1}{3}(F^1 - \tau^* F^2) \epsilon \]

\[ - \frac{1}{2 \cdot 3 \sqrt{7}} e^{-\frac{2}{3} \sqrt{7} \Phi} \frac{1}{3}(F^1 - \tau^* F^2) \epsilon \]

\[ (B.46) \]

\[ \delta \lambda = -e^{\Phi} \mathcal{D} \tau^* + \bar{g} \epsilon + \tilde{h} \epsilon^* - \frac{i}{\sqrt{\pi}} e^{-\frac{2}{3} \sqrt{7} \Phi} \frac{1}{3}(F^1 - \tau^* F^2) \epsilon \]

\[ + \frac{1}{2 \cdot 3 \sqrt{7}} e^{-\frac{2}{3} \sqrt{7} \Phi} \frac{1}{3}(F^1 - \tau^* F^2) \epsilon , \]

\[ (B.47) \]

where

\[ \mathcal{D}_\mu \epsilon = \{ \nabla_\mu + \frac{i}{2} \left[ \frac{1}{2} e^{\phi} \mathcal{D}_\mu \chi + A'_\mu \vartheta_l m P_m \right] + \frac{9}{16} \gamma_\mu A'_l \vartheta_l \} \epsilon , \]

\[ (B.48) \]

\[ \mathcal{D}_\mu \chi = \partial_\mu \chi - \frac{3}{4} A'_\mu \vartheta_l \chi , \]

\[ (B.49) \]

and where the fermion shifts are given by

\[ f = \frac{1}{14} e^{\frac{2}{3} \sqrt{7} \Phi} \left( \vartheta_0^m P_m + \frac{3i}{2} \vartheta_0^5 \right) , \]

\[ (B.50) \]

\[ k = -\frac{9}{14} e^{-\frac{2}{3} \sqrt{7} \Phi} \left( \vartheta_1^4 \tau + \vartheta_2^4 \right) , \]

\[ (B.51) \]

\[ \tilde{g} = e^{-\frac{2}{3} \sqrt{7} \Phi} \left[ \frac{6}{7} \left( \vartheta_1^4 \tau^* + \vartheta_2^4 \right) + \frac{\sqrt{7} \Phi}{2} \left( \vartheta_1^5 \tau^* + \vartheta_2^5 \right) \right] , \]

\[ (B.52) \]

\[ \tilde{h} = \frac{4}{\sqrt{7}} e^{\frac{2}{3} \sqrt{7} \Phi} \left( \frac{3}{16} \vartheta_0^5 + \vartheta_0^m P_m \right) , \]

\[ (B.53) \]

\[ g = \frac{3}{4} e^{-\frac{2}{3} \sqrt{7} \Phi} \left( \vartheta_1^5 \tau + \vartheta_2^5 \right) , \]

\[ (B.54) \]

\[ h = i e^{\frac{2}{3} \sqrt{7} \Phi} \left( \vartheta_0^m k_m \tau - \frac{3}{4} \vartheta_0^5 \tau \right) . \]

\[ (B.55) \]

The supersymmetry transformations of the bosonic fields are

\[ \delta \varphi = -i \frac{1}{4} e^{-\Phi} \lambda^* + h.c. , \]

\[ (B.56) \]

\[ \delta \tau = -\frac{1}{2} e^{-\Phi} \epsilon \lambda , \]

\[ (B.57) \]
\[\delta_\epsilon A_{\mu}^0 = \frac{i}{2} e^{\frac{1}{\sqrt{7}}} \bar{\epsilon} \left( \psi_\mu - \frac{i}{\sqrt{7}} \gamma_\mu \lambda^* \right) + \text{h.c.}, \quad (B.58)\]

\[\delta_\epsilon A_{\mu}^1 = \frac{i}{2} \tau^* e^{-\frac{1}{2\sqrt{7}}} \tilde{\psi} + \frac{i}{2} \phi \left( \bar{\epsilon}^* \psi_\mu - \frac{i}{4} \bar{\epsilon} \gamma_\mu \lambda + \frac{3i}{4\sqrt{7}} \bar{\epsilon}^* \gamma_\mu \lambda^* \right) + \text{h.c.}, \quad (B.59)\]

\[\delta_\epsilon A_{\mu}^2 = \frac{i}{2} e^{-\frac{1}{2\sqrt{7}}} \bar{\psi} + \frac{i}{2} \phi \left( \bar{\epsilon}^* \psi_\mu - \frac{i}{4} \bar{\epsilon} \gamma_\mu \lambda + \frac{3i}{4\sqrt{7}} \bar{\epsilon}^* \gamma_\mu \lambda^* \right) + \text{h.c.} \quad (B.60)\]

\[\delta_\epsilon B^1 = \tau^* e^{\frac{1}{2\sqrt{7}}} \bar{\psi} + \frac{i}{2} \phi \left[ \bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} - \frac{i}{8} \bar{\epsilon} \gamma_{\mu\nu} \lambda - \frac{i}{8\sqrt{7}} \bar{\epsilon}^* \gamma_{\mu\nu} \lambda^* \right] + \text{h.c.} \quad (B.61)\]

\[\delta_\epsilon B^2 = \frac{1}{2} \tau^* e^{\frac{1}{2\sqrt{7}}} \bar{\psi} + \frac{i}{2} \phi \left[ \bar{\epsilon}^* \gamma_{[\mu} \psi_{\nu]} - \frac{i}{8} \bar{\epsilon} \gamma_{\mu\nu} \lambda - \frac{i}{8\sqrt{7}} \bar{\epsilon}^* \gamma_{\mu\nu} \lambda^* \right] + \text{h.c.} \quad (B.62)\]

\[\delta_\epsilon C_{\mu\nu\rho} = -\frac{3}{2} e^{-\frac{1}{\sqrt{7}}} \bar{\epsilon} \gamma_{\mu\nu} \left( \psi_\rho + \frac{i}{\sqrt{7}} \lambda^* \right) + \text{h.c.} + 3 \delta_\epsilon A_{[\mu}^I \left( g_{IJ} B_{[\nu\rho]}^I + \frac{3}{2} h I J K g_{K [\nu\rho]} A_{J [\mu]} \right) . \quad (B.63)\]
Appendix C

Duality orbits of non-geometric fluxes

C.1 Different solvable and nilpotent gaugings

In section 4.4.3 we have studied the T-duality orbits of gaugings in half-maximal $D = 7$ supergravity and for each of them, we identified the gauge algebra and presented the results in table 4.6. Since there is no exhaustive classification of non-semisimple algebras of dimension 6, we would like to explicitly give the form of the algebras appearing in table 4.6.

Solvable algebras

The CSO(2, 0, 2) and CSO(1, 1, 2) algebras

The details about these algebras can be found in ref. 254; we summarise here some relevant facts.

The six generators are labelled as $\{t_0, t_i, s_i, z\}_{i=1,2}$, where $t_0$ generates SO(2) (SO(1,1)), under which $\{t_i\}$ and $\{s_i\}$ transform as doublets

$[t_0, t_i] = \epsilon^j_i t_j$ , $[t_0, s_i] = \epsilon^j_i s_j$ ,

(C.1)

where the Levi-Civita symbol $\epsilon^j_i$ has one index lowered with the metric $\eta_{ij} = \text{diag}(\pm 1, 1)$ depending on the two different signatures. $z$ is a central charge appearing in the following commutators

$[t_i, s_j] = \delta_{ij} z$ .

(C.2)

The Cartan-Killing metric is diag$(\mp 1, 0, \ldots, 0)$, where the $\mp$ is again related to the two different signatures.
The $\mathfrak{f}_1$ and $\mathfrak{f}_2$ algebras

These are of the form $\text{Solv}_4 \times \text{U}(1)^2$. The 4 generators of $\text{Solv}_4$ are labeled by $\{t_0, t_i, z\}_{i=1,2}$, where $t_0$ generates $\text{SO}(2)$ ($\text{SO}(1,1)$), under which $\{t_i\}$ transform as a doublet

$$[t_0, t_i] = \epsilon^j_i t_j \ ,$$
$$[t_i, t_j] = \epsilon_{ij} z \ .$$

(C.3)

(C.4)

The Cartan-Killing metric is $\text{diag}(\mp 1, 0, \cdots , 0).$

The $\mathfrak{h}_1$ and $\mathfrak{h}_2$ algebras

The 6 generators are $\{t_0, t_i, s_i, z\}_{i=1,2}$ and they satisfy the following commutation relations

$$[t_0, t_i] = \epsilon^j_i t_j \ , \quad [t_0, s_i] = \epsilon^j_i s_j + t_i \ ,$$
$$[t_i, s_j] = \delta_{ij} z \ , \quad [s_i, s_j] = \epsilon_{ij} z \ .$$

(C.5)

The Cartan-Killing metric is $\text{diag}(\mp 1, 0, \cdots , 0).$

The $\mathfrak{g}_0$ algebra

The 6 generators are $\{t_0, t_i, z\}_{i=1,\cdots,4}$, where $t_0$ transforms cyclically the $\{t_I\}$ amongst themselves such that

$$[[[t_I, t_0] , t_0] , t_0] = t_I \ ,$$

(C.6)

and

$$[t_1, t_3] = [t_2, t_4] = z \ .$$

(C.7)

Note that this algebra is solvable and not nilpotent even though its Cartan-Killing metric is completely zero.

Nilpotent algebras

The $\text{CSO}(1,0,3)$ algebra

The details about this algebra can be again found in ref. 254; briefly summarizing, the 6 generators are given by $\{t_m, z^m\}_{m=1,2,3}$ and they satisfy the following commutation relations

$$[t_m, t_n] = \epsilon_{mnp} z^p \ ,$$

(C.8)

with all the other brackets being vanishing. The order of nilpotency of this algebra is 2.
C.2. SO(2, 2) and SO(3, 3) ’t Hooft symbols

The algebra

The 6 generators \( \{t_1, \cdots, t_6\} \) satisfy the following commutation relations

\[
[t_1, t_2] = t_4 , \quad [t_1, t_4] = t_5 , \quad [t_2, t_4] = t_6 .
\]  

(C.9)

The corresponding central series reads

\[
\{ t_1, t_2, t_3, t_4, t_5, t_6 \} \supset \{ t_4, t_5, t_6 \} \supset \{ t_5, t_6 \} \supset \{ 0 \} ,
\]  

(C.10)

from which we can immediately conclude that its nilpotency order is 3.

C.2 SO(2, 2) and SO(3, 3) ’t Hooft symbols

In section 4.4.1 we discuss the origin of a given flux configuration from DFT backgrounds specified by twist matrices \( U \). The deformations of half-maximal supergravity in \( D = 10 - d \) which can be interpreted as the gauging of a subgroup of the T-duality group \( O(d, d) \) can be described by a 3-form of \( O(d, d) f_{ABC} \) which represents a certain (non-)geometric flux configuration.

In \( D = 8 \) and \( D = 7 \), the T-duality group happens to be isomorphic to \( SL(2) \times SL(2) \) and \( SL(4) \) respectively. As a consequence, in order to explicitly relate flux configurations and embedding tensor orbits, we need to construct the mapping between T-duality irrep’s and irrep’s of \( SL(2) \times SL(2) \) and \( SL(4) \) respectively.

From the \( (2, 2) \) of \( SL(2) \times SL(2) \) to the 4 of SO(2, 2)

The ’t Hooft symbols \( (G_A)^{\alpha i} \) are invariant tensors which map the fundamental representation of SO(2, 2) (here denoted by \( A \)), into the \( (2, 2) \) of \( SL(2) \times SL(2) \)

\[
v^{\alpha i} = (G_A)^{\alpha i} v^A ,
\]  

(C.11)

where \( v^A \) denotes a vector of SO(2, 2) and the indices \( \alpha \) and \( i \) are raised and lowered by means of \( \epsilon_{\alpha \beta} \) and \( \epsilon_{ij} \) respectively. \( (G_A)^{\alpha i} \) and \( (G_A)^{\alpha i}_{\beta j} \) satisfy the following identities

\[
(G_A)^{\alpha i}_{\beta j} (G_B)^{\beta j} = \eta_{AB} ,
\]  

(C.12)

\[
(G_A)^{\alpha i} (G_A)^{\beta j} = \epsilon^{\alpha \beta} \epsilon^{ij} ,
\]  

(C.13)

where \( \eta_{AB} \) is the SO(2, 2) metric.

After choosing light-cone coordinates for SO(2, 2), our choice for the tensors \( (G_A)^{\alpha i} \) is the following

\[
(G_1)^{\alpha i} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} ,
\]  

\[
(G_2)^{\alpha i} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ,
\]  

\[
(G_1)^{\alpha i} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,
\]  

\[
(G_2)^{\alpha i} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} .
\]  

(C.15)
By making use of the mapping \[ (C.11) \], we can rewrite the structure constants \((X_\alpha)_{\beta^\gamma}^k\) as a 3-form of \(SO(2,2)\) as follows:

\[
f_{ABC} = (X_\alpha)_{\beta^\gamma}^k (G_A)^\alpha_i (G_B)^{\beta^\gamma} (G_C)_{\gamma^k} .
\] (C.16)

**From the 6 of SL(4) to the 6 of SO(3,3)**

The ’t Hooft symbols \((G_A)^{mn}\) are invariant tensors which map the fundamental representation of \(SO(3,3)\), i.e. the 6 into the anti-symmetric two-form of SL(4)

\[
v^{mn} = (G_A)^{mn} v^A,
\] (C.17)

where \(v^A\) denotes a vector of \(SO(3,3)\). The two-form irrep of SL(4) is real due to the role of the Levi-Civita tensor relating \(v^{mn}\) to \(v_{mn}\)

\[
v_{mn} = \frac{1}{2} \epsilon_{mnpq} v^{pq}.
\] (C.18)

The ’t Hooft symbols with lower SL(4) indices \((G_A)_{mn}\) carry out the inverse mapping of the one given in \( (C.17) \). The tensors \((G_A)^{mn}\) and \((G_A)_{mn} = \frac{1}{2} \epsilon_{mnpq} (G_A)^{pq}\) satisfy the following identities

\[
(G_A)^{mn} (G_B)^{mn} = 2 \eta_{AB},
\] (C.19)

\[
(G_A)_{mp} (G_B)^{pm} + (G_B)_{mp} (G_A)^{pm} = -\delta_m^n \eta_{AB},
\] (C.20)

\[
(G_A)_{mp} (G_B)^{pq} (G_C)_{qr} (G_D)^{rs} (G_E)^{st} (G_F)^{ln} = \delta_m^n \epsilon_{ABCDEF},
\] (C.21)

where \(\eta_{AB}\) and \(\epsilon_{ABCDEF}\) are the \(SO(3,3)\) metric and Levi-Civita tensor respectively.

After choosing light-cone coordinates for \(SO(3,3)\) vectors, our choice of the ’t Hooft symbols is

\[
(G_1)^{mn} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (G_2)^{mn} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\] (C.22)

\[
(G_3)^{mn} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad (G_1)^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},
\] (C.23)

\[
(G_2)^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (G_3)^{mn} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\] (C.24)

Thus, we can rewrite the structure constants in the 6, \((X_m)_{pq}^{rs}\), arising from \((4.106)\) as a 3-form of \(SO(3,3)\) as follows:

\[
f_{ABC} = (X_m)_{pq}^{rs} (G_A)^{mn} (G_B)^{pq} (G_C)^{rs} .
\] (C.25)
Publications

List of publications arising from the research done during the PhD period.

1. J. Fernandez-Melgarejo and E. Torrente-Lujan, “N=2 SUGRA BPS Multi-center solutions, quadratic prepotentials and Freudenthal transformations,” to appear in JHEP. [arXiv:1310.4182 [hep-th]]

2. J. Fernandez-Melgarejo, T. Ortin, and E. Torrente-Lujan, “Maximal Nine Dimensional Supergravity, General gaugings and the Embedding Tensor,” [Fortsch.Phys. 60 (2012) 1012–1018, arXiv:1209.3774 [hep-th]]

3. G. Dibitetto, J. Fernandez-Melgarejo, D. Marques, and D. Roest, “Duality orbits of non-geometric fluxes,” [Fortsch.Phys. 60 (2012) 1123–1149, arXiv:1203.6562 [hep-th]]

4. E. A. Bergshoeff, J. Fernandez-Melgarejo, J. Rosseel, and P. K. Townsend, “On 'New Massive' 4D Gravity,” [JHEP 1204 (2012) 070, arXiv:1202.1501 [hep-th]]

5. L. Granda, E. Torrente-Lujan, and J. Fernandez-Melgarejo, “Non-minimal kinetic coupling and Chaplygin gas cosmology,” [Eur.Phys.J. C71 (2011) 1704, arXiv:1106.5482 [hep-th]]

6. J. Fernandez-Melgarejo, T. Ortin, and E. Torrente-Lujan, “The general gaugings of maximal d=9 supergravity,” [JHEP 1110 (2011) 068, arXiv:1106.1760 [hep-th]]

7. M. Picariello, B. Chauhan, C. Das, Fernandez-Melgarejo, D. Montanino, et al., “Neutrino Dipole Moments and Solar Experiments,” [arXiv:0907.0637 [hep-ph]]
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