On the Hamilton-Waterloo Problem with cycle lengths of distinct parities

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Abstract

Let $K^*_v$ denote the complete graph $K_v$ if $v$ is odd and $K_v - I$, the complete graph with the edges of a 1-factor removed, if $v$ is even. Given non-negative integers $v, M, N, \alpha, \beta$, the Hamilton-Waterloo problem asks for a 2-factorization of $K^*_v$ into $\alpha C_M$-factors and $\beta C_N$-factors. Clearly, $M, N \geq 3$, $M \mid v$, $N \mid v$ and $\alpha + \beta = \left\lfloor \frac{v-1}{2} \right\rfloor$ are necessary conditions.

Very little is known on the case where $M$ and $N$ have different parities. In this paper, we make some progress on this case by showing, among other things, that the above necessary conditions are sufficient whenever $M \mid N$, $v > 6N > 36M$, and $\beta \geq 3$.

Keywords: 2-Factorizations, Resolvable Cycle Decompositions, Cycle Systems, Generalized Oberwolfach Problem, Hamilton-Waterloo Problem.

1 Introduction

As usual, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of a simple graph $G$, respectively. Also, we denote by $tG$ the vertex-disjoint union of $t > 0$ copies of $G$.

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A factor of $G$ is a spanning subgraph of $G$; in particular, a 1-factor is a factor which is 1-regular and a 2-factor is a factor which is 2-regular and hence consists of a collection of cycles. A 2-factor of $G$ containing only one cycle is a Hamiltonian cycle. We denote by $C_\ell$ a cycle of length $\ell$ (briefly, an $\ell$-cycle), by $(x_0, x_1, \ldots, x_{\ell-1})$ the $\ell$-cycle with edges $x_0x_1, x_1x_2, \ldots, x_{\ell-1}x_0$, and by $K_v$ the complete graph on $v$ vertices. By $K^*_v$ we mean the graph $K_v$ when $v$ is odd and $K_v - I$, where $I$ is a single 1-factor, when $v$ is even.

A 2-factorization of a simple graph $G$ is a set of 2-factors of $G$ whose edge sets partition $E(G)$. It is well known that a regular graph has a 2-factorization if and only if every vertex has even degree. However, if we specify a particular 2-factor, $F$ say, and ask for all the factors to be isomorphic to $F$ the problem becomes much harder. Indeed, if $G \cong K^*_v$, we have the Oberwolfach Problem, which is well known to be hard. A survey of the well-known results on this problem, updated to 2006, can be found in [13, Section VI.12]. For more recent results we refer the reader to [10, 8, 9, 23, 24].

Given a simple graph $G$ and a collection of graphs $\mathcal{H}$, an $\mathcal{H}$-factor of $G$ is a set of vertex-disjoint subgraphs of $G$, each isomorphic to a member of $\mathcal{H}$, which between them cover every point in $G$. An $\mathcal{H}$-factorization of $G$ is a set of edge-disjoint $\mathcal{H}$-factors of $G$ whose edges partition the edge set of $G$. When $\mathcal{H}$ consists of a single graph $H$, we speak of $H$-factors and $H$-factorizations of $G$ respectively. If $H$ is a Hamiltonian cycle of $G$ and there exists an $H$-factorization of $G$ (briefly, a Hamiltonian factorization), then $G$ is called Hamiltonian factorable.

We call a factor whose components are pairwise isomorphic a uniform factor. The problem of factoring $K^*_v$ into pairwise isomorphic uniform 2-factors has been solved [1, 2, 16].

**Theorem 1.1** ([1, 2, 16]). Let $v, \ell \geq 3$ be integers. There is a $C_\ell$-factorization of $K^*_v$ if and only if $\ell \mid v$, except that there is no $C_3$-factorization of $K^*_6$ or $K^*_12$.

Given a graph $G$, we denote by $G[n]$ the lexicographic product of $G$ with the empty graph on $n$ points. Specifically, the vertex set of $G[n]$ is $V(G) \times \mathbb{Z}_n$ (where $\mathbb{Z}_n$ denotes the cyclic group of order $n$) and $(x, i)(y, j) \in E(G[n])$ if and only if $xy \in E(G)$, $i, j \in \mathbb{Z}_n$. Note that $G[n_1][n_2] \cong G[n_1n_2]$.

The existence problem for a $C_\ell$-factorization of the complete equipartite graph has been completely solved by Liu [20, 21].
Theorem 1.2 (20, 21). Let \( \ell, t \) and \( z \) be positive integers with \( \ell \geq 3 \). There exists a \( C_\ell \)-factorization of \( K_t[z] \) if and only if \( \ell \mid tz, (t-1)z \) is even, further \( \ell \) is even when \( t = 2 \), and \((\ell, t, z) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}\).

We provide a straightforward generalization of Theorem 1.2 to \( C_\ell[n] \)-factorizations of \( K_t[z^n] \).

Corollary 1.3. Given four positive integers \( \ell, n, t \) and \( z \) with \( \ell \geq 3 \), there exists a \( C_\ell[n] \)-factorization of \( K_t[z^n] \) whenever \( \ell \mid tz, (t-1)z \) is even, \( \ell \) is even when \( t = 2 \), and \((\ell, t, z) \notin \{(3, 3, 2), (3, 6, 2), (3, 3, 6), (6, 2, 6)\}\).

Proof. Theorem 1.2 guarantees the existence of a \( C_\ell \)-factorization of \( K_t[z] \). By expanding each point of this factorization by \( N \), we obtain a \( C_\ell[n] \)-factorization of \( K_t[z^n] \).

A well-known variant of the Oberwolfach Problem is the Hamilton-Waterloo Problem \( \text{HWP}(G; F, F'; \alpha, \beta) \), which asks for a factorization of a specified graph \( G \) into \( \alpha \) copies of \( F \) and \( \beta \) copies of \( F' \), where \( F \) and \( F' \) are distinct 2-factors of \( G \). We denote by \( \text{HWP}(G; F, F') \) the set of \((\alpha, \beta)\) for which there is a solution to \( \text{HWP}(G; F, F'; \alpha, \beta) \). In the case where \( F \) and \( F' \) are uniform with cycle lengths \( M \) and \( N \), respectively, we refer to \( \text{HWP}(G; M, N; \alpha, \beta) \) and \( \text{HWP}(G; M, N) \) as appropriate. Further, if \( G = K_v^* \), we refer to \( \text{HWP}(v; M, N; \alpha, \beta) \) and \( \text{HWP}(v; M, N) \) respectively. We note the following necessary conditions for the case of uniform factors.

Theorem 1.4. Let \( G \) be a graph of order \( v \), and let \( M, N, \alpha \) and \( \beta \) be non-negative integers. In order for a solution of \( \text{HWP}(G; M, N; \alpha, \beta) \) to exist, \( M \) and \( N \) must be divisors of \( v \) greater than 2, and \( G \) must be regular of degree \( 2(\alpha + \beta) \).

This problem has received much interest recently. For more details and some history on the problem, we refer the reader to [11, 12]. These two papers deal with the case where both \( M \) and \( N \) are odd positive integers and provide an almost complete solution to the Hamilton-Waterloo Problem \( \text{HWP}(v; M, N; \alpha, \beta) \) for odd \( v \). If \( M \) and \( N \) are both even, then \( \text{HWP}(v; M, N; \alpha, \beta) \) has a solution except possibly when \( \alpha = 1 \) or \( \beta = 1 \) [7], whereas this problem is completely solved when \( M \) and \( N \) are even and \( M \) is a divisor of \( N \) [8].

In this paper, we deal with the challenging case where \( M \) and \( N \) have different parities. In fact, the only known results on \( \text{HWP}(v; M, N; \alpha, \beta) \)
when \( M \neq N \pmod{2} \) concern the case \((M, N) = (3, 4)\) which is completely solved in \([6, 14, 22, 25]\), and the cases where \((M, N) = (3, v)\) \([3]\), \((M, N) = (3, 6n)\) \([3]\) or \((M, N) = (4, N)\) \([17, 22]\) which are all still open.

In this paper, we make further progress by showing the following.

**Theorem 1.5.** Let \( M, N, v, \alpha, \beta \) be positive integers such that \( N > M \geq 3 \) and \( M \) is an odd divisor of \( N \). Then, \((\alpha, \beta) \in \text{HWP}(v; M, N)\) if and only if \( N \mid v \) and \( \alpha + \beta = \left\lfloor \frac{v - 1}{2} \right\rfloor \) except possibly when at least one of the following conditions holds:

1. \( \beta = 1 \);
2. \( \beta = 2, N \equiv 2M \pmod{4M} \);
3. \( N \in \{2M, 6M\} \);
4. \( v \in \{N, 2N, 4N\} \);
5. \((M, v) = (3, 6N)\).

In the next section we introduce some tools and provide some powerful methods which we use in Section 3 where we prove a result (Theorem 3.5) on factorizations of \( C_M[n] \), the lexicographic product of an \( M \)-cycle and the empty graph on \( n \) vertices. In Section 4 we prove the main result of this paper, Theorem 1.5.

## 2 Preliminaries

In this section we state some known results and develop the tools we will need for the 2-factorizations. We use \([a, b]\) to denote the set of integers from \( a \) to \( b \) inclusive; clearly, \([a, b]\) is empty when \( a > b \).

### 2.1 Cayley graphs

We will make use of the notion of a Cayley graph on an additive group \( \Gamma \). Given \( S \subseteq \Gamma \setminus \{0\} \), the *Cayley Graph* \( \text{cay}(\Gamma, S) \) is a graph with vertex set \( \Gamma \) and edge set \( \{a(d + a) \mid a \in \Gamma, d \in S\} \). When \( \Gamma = \mathbb{Z}_N \) this graph is known as a *circulant graph*. We note that the edges generated by \( d \in S \) are the same as those generated by \( -d \in -S \), so that \( \text{cay}(\Gamma, S) = \text{cay}(\Gamma, \pm S) \), and that the degree of each point is \( |S \cup (-S)| \).
Given a set \( S \subseteq \Gamma \), we denote by \( C_m[S] \ (m \geq 3) \) the graph with point set \( \mathbb{Z}_m \times \Gamma \) and edges \((i, x)(i+1, d+x), i \in \mathbb{Z}_m, x \in \Gamma \) and \( d \in S \). In other words, \( C_m[S] = \text{cay}(\mathbb{Z}_m \times \Gamma, \{1\} \times S) \); hence, it is \( 2|S| \)-regular. It is straightforward to see that if \( \Gamma \) has order \( n \), then \( C_m[n] \cong C_m[\Gamma] \); hence, \( C_m[S] \) is a subgraph of \( C_m[n] \). We will sometimes denote the vertex \((i, x)\) of \( C_m[S] \) by \( i_x \).

We will make use of the following two results due to Bermond, Favaron and Mahéo \([5]\) and Westlund \([26]\), which provide sufficient conditions for the existence of a Hamiltonian factorization of a connected Cayley graph of degree 4 and 6.

**Theorem 2.1** \([5]\). Any connected 4-regular Cayley graph on a finite Abelian group has a Hamiltonian factorization.

**Theorem 2.2** \([26]\). If \( X = \text{cay}(A, \{e_1, e_2, e_3\}) \) is a 6-regular Cayley graph, \( A \) is an abelian group of even order generated by both \( \{e_1, e_2\} \) and \( \{e_2, e_3\} \), and \( e_2 \) has index at least four in \( A \), then \( X \) has a Hamiltonian factorization.

We use these two results to show the existence of a hamiltonian factorization of a special connected 6-regular subgraph of \( C_M[n] \).

**Lemma 2.3.** Let \( n \geq 4 \) be even and let \( M \geq 3 \) be such that \( Mn \equiv 0 \ (\text{mod} \ 4) \). Then, \( C_M[\{\frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 1\}] \) factorizes into three \( C_M[n] \)-factors.

**Proof.** We recall that \( C_M[\{\frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 1\}] = \text{cay}(\mathbb{Z}_M \times \mathbb{Z}_n, \{e_1, e_2, e_3\}) \) where \( (e_1, e_2, e_3) = ((1, \frac{n}{2} - 1), (1, \frac{n}{2}), (1, \frac{n}{2} + 1)) \).

We first note that for any \( x \in \mathbb{Z}_n \) the set \{(1, x), (1, x+1)\} is a system of generators of \( \mathbb{Z}_M \times \mathbb{Z}_n \). In fact, \((0, 1) = (1, x +1) - (1, x)\) and \((1, 0) = (x +1)(1, x) - x(1, x +1)\); therefore, any element of \( \mathbb{Z}_M \times \mathbb{Z}_n \) is a linear combination of \{(1, x), (1, x +1)\}. It then follows that both \( \{e_1, e_2\} \) and \( \{e_2, e_3\} \) generate \( \mathbb{Z}_M \times \mathbb{Z}_n \), hence \( C_M[\{\frac{n}{2} - 1, \frac{n}{2}, \frac{n}{2} + 1\}] \) is a connected 6-regular graph.

We denote by \( \langle e_2 \rangle \) the subgroup of \( \mathbb{Z}_M \times \mathbb{Z}_n \) generated by \( e_2 \), and by \( |\mathbb{Z}_M \times \mathbb{Z}_n : \langle e_2 \rangle| \) the index of \( \langle e_2 \rangle \) in \( \mathbb{Z}_M \times \mathbb{Z}_n \). It is not difficult to check that \( |\mathbb{Z}_M \times \mathbb{Z}_n : \langle e_2 \rangle| = n \) or \( \frac{n}{2} \) according to whether \( M \) is even or odd. Since by assumption \( Mn \equiv 0 \ (\text{mod} \ 4) \) and \( n \geq 4 \), we have that \( |\mathbb{Z}_M \times \mathbb{Z}_n : \langle e_2 \rangle| \geq 4 \) when either \( M \) is even or \( M \) is odd and \( n \neq 4 \); in these cases, the assertion follows from Theorem 2.2. If \( M \) is odd and \( n = 4 \), then \( C_M[\{1, 2, 3\}] \) can be decomposed into \( C_M[\{1\}] \), which is a Hamiltonian cycle, and \( C_M[\{2, 3\}] \) which is a connected 4-regular Cayley graph and, by Theorem 2.1, it has a Hamiltonian factorization, and this completes the proof.\[\Box\]
2.2 Constructing factors of $C_M[n]$

In Section 3 we will make use of the following result which provide sufficient conditions for the existence of a solution to $HWP(C_\ell[T]; g\ell, h\ell; \alpha, |T| - \alpha)$, where $T$ is a subset of $\Gamma = \mathbb{Z}_n$ and $g, h$ are positive divisors of $n$. This result is proven in [12] for an arbitrary group $\Gamma$.

**Theorem 2.4** (Theorem 2.9, [12]). Let $n$ be a positive integer, and let $g$ and $g'$ be positive divisors of $n$. Also, let $T$ be a subset of $\mathbb{Z}_n$ and $\ell \geq 3$. Suppose there exists a $|T| \times \ell$ matrix $A = [a_{ij}]$ with entries from $T$ satisfying the following properties:

1. $\alpha$ rows of $A$ have sum an element of order $g$ in $\mathbb{Z}_n$, and the remaining $|T| - \alpha$ rows have sum an element of order $g'$ in $\mathbb{Z}_n$;
2. each column of $A$ is a permutation of $T$.

Then $(\alpha, |T| - \alpha) \in HWP(C_\ell[T]; g\ell, g'\ell)$. Moreover, if we also have that:

3. $T$ is closed under taking negatives,

then $(\alpha, |T| - \alpha) \in HWP(C_m[T]; gm, g'm)$ for any $m \geq \ell$ with $m \equiv \ell \pmod{2}$.

Note that Theorem 2.4 gives a $C_g\ell$-factorization of $C_\ell[T]$ when $\alpha = |T|$.

We finally state the following well-known result which has been proven in [15] in a much more general form.

**Lemma 2.5.** $C_M[2]$ has a Hamiltonian factorization for every $M \geq 3$.

2.3 Skolem sequences

In some of our constructions in Section 3 we will make use of Skolem sequences, which we now define in a slightly more general form.

**Definition 2.6** (Skolem sequences). A Skolem sequence of order $\nu \geq 0$ is a sequence of $\nu + 1$ pairs $(a_0, b_0), (a_1, b_1), \ldots, (a_\nu, b_\nu)$ such that

1. $b_i - a_i = i$ for every $i \in [0, \nu]$;
2. $\bigcup_{i=1}^{\nu} \{a_i, b_i\} = [x, x + 2\nu]$ for some integer $x$. 
In this case, we say that the Skolem sequence covers the interval $[x, x + 2\nu]$.

We point out that in the literature, the term Skolem sequence is only used when $(x, a_0) = (1, 2\nu + 1)$. When $(x, a_0) = (1, 2\nu)$, such a sequence is usually referred to as a hooked Skolem sequence. In all other cases in which $x = 1$, one speaks of an $a_0$-extended Skolem sequence.

We recall the following existence results for Skolem sequences.

**Theorem 2.7** ([1]). There exists a Skolem sequence of order $\nu$ for every $\nu \geq 0$

Note that given a Skolem sequence $(a_0, b_0), (a_1, b_1), \ldots, (a_\nu, b_\nu)$ covering the interval $[x, x + 2\nu]$ and an integer $t$, it is clear that $(a_0 + t, b_0 + t), (a_1 + t, b_1 + t), \ldots, (a_\nu + t, b_\nu + t)$ is still a Skolem sequence which covers the interval $[x + t, x + 2\nu + t]$. Therefore, the above theorem implies what follows.

**Corollary 2.8.** Every interval of length $2\nu + 1$ can be covered by a Skolem sequence.

### 3 Determining HWP($C_M[n]; M, Mn$)

In this section, we provide sufficient conditions for a solution of HWP($C_M[n]; M, Mn$) to exists. We will make use of Theorem 2.4 to factorize large subgraphs of $C_M[n]$ by constructing suitable matrices with entries in $\mathbb{Z}_n$, and use Theorems 2.1 and 2.2 to factorize what is possibly left over. For this reason, given any integers $x$ and $y$ such that $0 < \ell = y - x < n$, we define two $(\ell + 1) \times 2$ matrices below, denoted by $A(x, y)$ and $B(x, y)$, with entries in $\mathbb{Z}_n$:

| $A(x, y)$ | $B(x, y)$ if $\ell$ is odd | $B(x, y)$ if $\ell$ is even |
|-----------|---------------------------|---------------------------|
| $\begin{bmatrix} x & -x \\ x + 1 & -(x + 1) \\ \vdots & \vdots \\ x + \ell & -(x + \ell) \end{bmatrix}$ | $\begin{bmatrix} x & -(x + 1) \\ x + 1 & -x \\ \vdots & \vdots \\ x + \ell & -(x + \ell + 1) \\ x + \ell - 1 & -(x + \ell) \\ x + \ell & -(x + \ell - 1) \end{bmatrix}$ | $\begin{bmatrix} x & -(x + 1) \\ x + 1 & -x \\ \vdots & \vdots \\ x + 2i & -(x + 2i + 1) \\ x + 2i + 1 & -(x + 2i) \\ \vdots & \vdots \\ x + \ell - 4 & -(x + \ell - 3) \\ x + \ell - 3 & -(x + \ell - 4) \\ x + \ell - 2 & -(x + \ell - 1) \\ x + \ell - 1 & -(x + \ell) \\ x + \ell & -(x + \ell - 2) \end{bmatrix}$ |
Further, if \( y < x \), we set \( A(x, y) = \emptyset = B(x, y) \). Finally, \( A(x, x) = [x - x] \). Note that \( B(x, y) \) is not defined when \( y = x \).

We note that when \( x \leq y \) each of the rows in \( A(x, y) \) sums to 0. Similarly, when \( x < y \) each of the rows in \( B(x, y) \) sums to \( \pm 1 \), unless \( y - x \) is even, in which case the last row of \( B(x, y) \) sums to 2.

We first consider the problem in which \( n \) is odd.

**Lemma 3.1.** Let \( M, n \geq 3 \) with \( n \) odd, and let \( 0 \leq \beta \leq n \). Then \((\alpha, \beta) \in \text{HWP}(C_M[n]; M, Mn)\) except possibly when \( \beta = 1 \).

**Proof.** Let \( T \) be the \( n \times 2 \) matrix defined as 
\[
T = \begin{bmatrix}
A(1, \alpha) \\
B(\alpha + 1, n)
\end{bmatrix}.
\]
Also, let \( T' \) be the \( n \times 3 \) matrix obtained from \( T \) by replacing each row \([m_1, m_2]\) with \([m_1, m_2, m_2]\). Here \( m_2 \) is well defined as an element of \( \mathbb{Z}_n \), since \( n \) is odd.

Clearly, each of the first \( \alpha \) rows of \( T \) sums to 0, whereas each of the remaining \( \beta \) rows sums to \( \pm 1 \) or \( \pm 2 \) (which are elements of order \( n \) in \( \mathbb{Z}_n \) since \( n \) is odd). Further, each column of \( T \) and \( T' \) is a permutation of \( \mathbb{Z}_n \). Therefore, by applying Theorem 2.4 to \( T \) and \( T' \), it follows that \((\alpha, \beta) \in \text{HWP}(C_M[n]; M, Mn)\) for any \( M \geq 3 \). \( \Box \)

Note that the above Lemma has been independently proven in [18] with different techniques. An alternative proof in the case where \( M \) is odd can be found in [12].

The following three lemmas deal with the case where \( n \) is even.

**Lemma 3.2.** If \( n \geq 2 \) is even and \( M \geq 3 \), then \((n, 0) \in \text{HWP}(C_M[n]; M, Mn)\) except when \( M \) is odd and \( n = 2 \) and possibly when \( M \) is odd and \( n = 6 \).

**Proof.** We first consider the case where \( M \geq 3 \) is odd. It is not difficult to check that there is no \( C_M \)-factorization of \( C_M[2] \). Therefore, let \( n \geq 4 \) be even with \( n \neq 6 \). By Theorem 1.2 there exists a \( C_3 \)-factorization \( \mathcal{F} = \{F_1, F_2, \ldots, F_n\} \) of \( C_3[n] \), where \( F_i = \{C_{ij} \mid j \in [1, n]\} \) and \( C_{ij} = (c_{ij}^0, c_{ij}^1, c_{ij}^2) \). Without loss of generality we can assume \( c_{ij}^2 = (2, j) \) for any \( j \in [1, n] \).

Now, for each \( i, j \in [1, n] \) we define the \( M \)-cycle \( \overline{C_{ij}} = (\overline{c}_{ij}^0, \overline{c}_{ij}^1, \ldots, \overline{c}_{ij}^{M-1}) \) as follows:
\[
\overline{c}_{ij}^h = \begin{cases}
    c_{ij}^h & \text{if } h = 0, 1, 2, \\
    (h, j + i) & \text{if } h \text{ is odd and } 3 \leq h < M, \\
    (h, j) & \text{if } h \text{ is even and } 4 \leq h < M.
\end{cases}
\]
Finally, set $F_i = \{C_{ij} | j \in [1, n]\}$ and $\overline{F} = \{F_i | i \in [1, n]\}$. It is not difficult to check that each $F_i$ is a $C_M$-factor of $C_M[n]$ and $\overline{F}$ is a $C_M$-factorization of $C_M[n]$.

If $M \geq 4$ is even, it is enough to apply Theorem 2.4 to the $n \times M$ block matrix $T = [A(1, n) \ A(1, n) \ \cdots \ A(1, n)]$.

Note that a result similar to Lemma 3.2 has been proven in [18] in the case where $M \geq 3$ is odd and $n > 1$.

Lemma 3.3. Let $n \geq 2$ be even, $M \geq 3$, and $0 < \beta \leq n$. Then $(n - \beta, \beta) \in \text{HWP}(C_M[n]; M, Mn)$ whenever the following conditions simultaneously hold:

1. $\beta \equiv \frac{Mn}{2} \pmod{2}$;

2. if $Mn \equiv 2 \pmod{4}$ and $n > 2$, then $\beta \neq 1$.

Proof. We consider four cases depending on whether $n \equiv 0, 2 \pmod{4}$ and $M \equiv 0, 1 \pmod{4}$. In each of these cases, we will construct an $(n \times c)$ matrix $T$, where $\{2, 3\} \ni c \equiv M \pmod{2}$, satisfying the following conditions:

1. each column of $T$ is a permutation of $\mathbb{Z}_n$;

2. $T$ has $\alpha = n - \beta$ rows each of which sums to 0;

3. $T$ has $\beta$ rows each of which sums to $\pm 1$, or $\left\{ \frac{n}{2} \pm 1 \text{ if } n \equiv 0 \pmod{4}, \frac{n}{2} \pm 2 \text{ if } n \equiv 2 \pmod{4} \right\}$.

Note that $\frac{n}{2} \pm 1$ is coprime to $n$ if and only if $n \equiv 0 \pmod{4}$; therefore, $\frac{n}{2} \pm 1$ has order $n$ in $\mathbb{Z}_n$. Similarly, $\frac{n}{2} \pm 2$ has order $n$ in $\mathbb{Z}_n$ if and only if $n \equiv 2 \pmod{4}$. The assertion then follows by applying Theorem 2.4 to $T$.

We first consider the case where $n \equiv 2 \pmod{4}$ and $M$ is even; thus, by assumption, we have that $\beta$ is even. If $n = 2$, then $\beta = 2$ (since, by assumption, $\beta > 0$) and we set $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We now assume that $n \geq 6$.

For $i \in \{2, 4, 6\}$ we first define the $6 \times 2$ matrix $C_i$ as follows:

$$C_2 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ 1 & -1 \\ \frac{n}{2} & 2 \\ 2 & -2 \\ -2 & \frac{n}{2} \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 0 \\ \frac{n}{2} & 2 \\ 2 & -2 \\ -2 & \frac{n}{2} \end{bmatrix}, \quad C_6 = \begin{bmatrix} 0 & 1 \\ 2 & -1 \\ -1 & 0 \\ \frac{n}{2} & 2 \\ 1 & -2 \\ -2 & \frac{n}{2} \end{bmatrix}.$$
Clearly, each column of $C_i$ uses up all integers in $[-2, 2] \cup \{\frac{n}{2}\}$. Also, $i$ rows of $C_i$ sum to $\pm 1$ or $\frac{n}{2} \pm 2$, which are all elements of order $n$ in $\mathbb{Z}_n$. Each of the remaining $6 - i$ rows sums to 0. Now, for each value of $\beta$, we define an $n \times 2$ matrix $T$ satisfying conditions 1-3 as follows:

| $\beta = 2$ | $4 \leq \beta \equiv i \pmod{4}$ with $i \in \{4, 6\}$ |
|-----------------|---------------------|
| $T = \begin{bmatrix} A(3, \frac{n}{2} - 1) \\ -A(3, \frac{n}{2} - 1) \\ C_\beta \end{bmatrix}$ | $T = \begin{bmatrix} A(\frac{\beta - i}{2} + 3, \frac{n}{2} - 1) \\ -A(\frac{\beta - i}{2} + 3, \frac{n}{2} - 1) \\ B(3, \frac{\beta - i}{2} + 2) \\ -B(3, \frac{\beta - i}{2} + 2) \\ C_i \end{bmatrix}$ |

We now let $n \equiv 2 \pmod{4}$ and $M$ be odd. Note that, by assumption, we have that $\beta > 0$ is odd. If $n = 2$, we set $T = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. We now assume that $n \geq 6$ and we note that by condition 2 we have that $\beta \geq 3$. For $i \in \{3, 5\}$ we define the $6 \times 2$ matrix $C_i$ as follows:

$C_3 = \begin{bmatrix} -1 & 2 \\ -2 & \frac{n}{2} \\ 0 & 0 \\ 1 & -1 \\ 2 & -2 \\ \frac{n}{2} & 1 \end{bmatrix}$

$C_5 = \begin{bmatrix} -1 & 2 \\ 0 & -1 \\ 1 & 0 \\ -2 & \frac{n}{2} \\ 2 & -2 \\ \frac{n}{2} & 1 \end{bmatrix}$

Clearly, both columns of $C_i$ use up all integers in $[-2, 2] \cup \{\frac{n}{2}\}$. Also, each of the first $i - 1$ rows of $C_i$ sums to $\pm 1$ or $\frac{n}{2} - 2$, the last row of $C_i$ sums to $\frac{n}{2} + 1$, and the remaining $6 - i$ rows sum to 0. We now define an $n \times 2$ matrix $R$ according to the possible values of $\beta$:

$$R = \begin{bmatrix} A(\frac{\beta - i}{2} + 3, \frac{n}{2} - 1) \\ -A(\frac{\beta - i}{2} + 3, \frac{n}{2} - 1) \\ B(3, \frac{\beta - i}{2} + 2) \\ -B(3, \frac{\beta - i}{2} + 2) \\ C_i \end{bmatrix}$$ where $3 \leq \beta \equiv i \pmod{4}$ with $i \in \{3, 5\}$.

Clearly, each column of $R$ is a permutation of $\mathbb{Z}_n$. Further, $R$ has $\alpha$ rows whose sum is 0, and $\beta - 1$ rows each of which sums to $\pm 1$ or $\frac{n}{2} \pm 2$, whereas the
last row sums to $\frac{n}{2} + 1$. To construct the requisite $(n \times 3)$ matrix $T$ satisfying conditions $1 - 3$, we consider a Skolem sequence $\{(a_i, b_i) \mid i \in [0, n/2 - 1]\}$ covering $[1, n-1]$ (which exists by Corollary 2.8) and replace each element $i \in \left[-\frac{n}{2} + 1, \frac{n}{2}\right]$ in the first column of $R$ with the pair $(x_i, y_i)$ defined below:

$$
(x_i, y_i) = \begin{cases} 
(b_i, -a_i) & \text{if } i \in [0, \frac{n}{2} - 1], \\
(a_{i-n}, b_{i-n}) & \text{if } i \in [-1, \frac{n}{2} - 1], \\
(0, 0) & \text{if } i = \frac{n}{2}.
\end{cases}
$$

(1)

It is not difficult to check that the new matrix $T$ satisfies conditions $1 - 3$. In fact, the first column (resp., second column) of $T$ uses up all integers in $[1, n]$ (resp., $[-1, n]$), therefore they are both permutations of $\mathbb{Z}_n$. We also point out that the above substitution preserves the sum of each row, except for the last row of $T$, which is $[0 \ 0 \ 1]$, and thus sums to $1$, and therefore yields a $C_{Mn}$-factor.

Now, let $n \equiv 0 \pmod{4}$; thus, by assumption, $\beta > 0$ is even. For $i \in \{0, 2, 4\}$ we define the $4 \times 2$ matrix $C_i$ as follows:

$$
C_0 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ -1 & 1 \\ \frac{n}{2} & \frac{n}{2} \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ \frac{n}{2} & 1 \\ -1 & \frac{n}{2} \end{bmatrix}, \quad C_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \frac{n}{2} & -1 \\ -1 & \frac{n}{2} \end{bmatrix}.
$$

Clearly, both columns of $C_i$ use up all integers in $[-1, 1] \cup \{\frac{n}{2}\}$. Also, $i$ rows of $C_i$ sum to $1$ or $\frac{n}{2} \pm 1$, whereas the remaining $4 - i$ row sums to $0$. If $M$ is even, we define an $n \times 2$ matrix $T$ satisfying conditions $1 - 3$ as follows:

$$
2 \leq \beta \equiv i \pmod{4} \text{ with } i \in \{2, 4\}
$$

$$
T = \begin{bmatrix} A\left(\frac{\beta - i}{2} + 2, \frac{n}{2} - 1\right) \\
-A\left(\frac{\beta - i}{2} + 2, \frac{n}{2} - 1\right) \\
B(2, \frac{\beta - i}{2} + 1) \\
-B(2, \frac{\beta - i}{2} + 1) \\
C_i
\end{bmatrix}
$$

If $M$ is odd, to construct the required $(n \times 3)$ matrix satisfying conditions $1 - 3$, we consider a Skolem sequence $\{(a_i, b_i) \mid i \in [0, n/2 - 1]\}$ of $[1, n-1]$ (which exists by Corollary 2.8) and replace each element $i$ in the first column of $T$ with the pair $(x_i, y_i)$ defined in equation (1). It is not difficult to check that the new matrix satisfies conditions $1 - 3$ and this completes the proof. □
Lemma 3.4. Let \( n \geq 2 \) be even, \( M \geq 3 \), and \( 0 < \beta \leq n \). Then \((n - \beta, \beta)\) is in HWP\( (C_M[n]; M, Mn)\) whenever the following conditions simultaneously hold:

1. \( \beta \equiv \frac{Mn}{2} + 1 \pmod{2}; \)
2. if \( Mn \equiv 0 \pmod{4} \), then \( \beta \neq 1; \)
3. if \( Mn \equiv 2 \pmod{4} \) and \( n > 2 \), then \( \beta \neq 2. \)

Proof. We first consider the case where \( Mn \equiv 0 \pmod{4} \); hence, by assumption, we have that \( \beta \) is odd and \( \beta \geq 3 \), thus \( n \geq 4 \). Let \( T \) be the \((n - 3) \times 2\) matrix with entries in \( \mathbb{Z} \) defined as follows:

\[
T = \begin{bmatrix}
A(-\frac{n}{2} + 2, -\frac{n}{2} + 1 + \alpha) \\
B(-\frac{n}{2} + 2 + \alpha, \frac{n}{2} - 2)
\end{bmatrix}.
\]

Clearly, each column of \( T \) is a permutation of \( \mathbb{Z} \setminus \{\frac{n}{2} \pm 1, \frac{n}{2}\} \), each of the first \( \alpha \) rows of \( T \) sums to 0, whereas each of the remaining \( \beta - 3 \) sums to \( \pm 1 \).

We now construct an \((n - 3) \times 3\) matrix \( T' \) by modifying \( T \) as follows. By Corollary 2.8, there is a Skolem sequence \( \{(a_i, b_i) \mid i \in [0, \frac{n}{2} - 1]\} \) covering \([-\frac{n}{2} + 2, \frac{n}{2} - 2]\). To construct \( T \) we replace each element \( i \) in the first column of \( T' \) with the pair \((x_i, y_i)\) defined below:

\[
(x_i, y_i) = \begin{cases}
(b_i, -a_i) & \text{if } i \in [0, \frac{n}{2} - 2], \\
(a_{-i}, -b_{-i}) & \text{if } i \in [-1, \frac{n}{2} - 2].
\end{cases}
\]

(2)

It is not difficult to check that each of the first two columns of \( T' \) uses up all integers in \([-\frac{n}{2} + 2, \frac{n}{2} - 2]\), therefore they are both permutations of \( \mathbb{Z} \setminus \{\frac{n}{2} \pm 1, \frac{n}{2}\} \). We also point out to the reader that the above substitution preserves the sum of each row. Therefore, by applying Theorem 2.4 to \( T \) and \( T' \), it follows that \((n - \beta, \beta - 3)\) is in HWP\( (C_M[\mathbb{Z_n} \setminus \{\frac{n}{2} \pm 1, \frac{n}{2}\}], M, Mn)\).

In view of Lemma 2.3, \((0, 3)\) is in HWP\( (C_M[\{\frac{n}{2} \pm 1, \frac{n}{2}\}], M, Mn)\), therefore \((n - \beta, \beta)\) is in HWP\( (C_M[n]; M, Mn)\).

We finally assume that \( Mn \equiv 2 \pmod{4} \); hence, by assumption, \( M \) is odd, \( n \equiv 2 \pmod{4} \), and \( \beta > 0 \) is even. If \( n = 2 \) then \( (0, 2) \) is in HWP\( (C_M[2]; M, 2M)\) by Lemma 2.5. Therefore, we can assume that \( n > 2 \), hence \( \beta \geq 4 \) (condition 3). First, let \( T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \) be an \((n - 2) \times 2\) matrix with entries in \( \mathbb{Z}_n \setminus \{\frac{n}{2} - 1, \frac{n}{2}\} \) where:

\[
T_1 = \begin{bmatrix}
A(-\frac{n}{2} + 3, -\frac{n}{2} + \alpha + 2) \\
B(-\frac{n}{2} + \alpha + 3, \frac{n}{2} - 2)
\end{bmatrix}
\text{ and } \ T_2 = \begin{bmatrix}
-\frac{n}{2} + 1 & -\frac{n}{2} - 2 \\
-\frac{n}{2} + 2 & -\frac{n}{2} + 1
\end{bmatrix}.
\]
Note that each column of $T$ is a permutation of $\mathbb{Z}_n \setminus \{\frac{n}{2} - 1, \frac{n}{2}\}$; also, each of the first $\alpha$ rows of $T_1$ sums to 0, whereas each of the following $\beta - 4$ rows sums to $\pm 1$.

We now construct an $(n-2) \times 3$ matrix $T'$ as follows. By Corollary 2.8 there is a Skolem sequence $\{(a_i, b_i) \mid i \in [0, \frac{n}{2} - 2]\}$ covering $[-\frac{n}{2} + 2, \frac{n}{2} - 2]$. As before, to construct $T'$ we replace each element of the second column of $T$, say $i \in [-\frac{n}{2} + 1, \frac{n}{2} - 2]$, with the pair $(x_i, y_i)$ defined below:

$$
(x_i, y_i) = \begin{cases} 
(b_i, -a_i) & \text{if } i \in [0, \frac{n}{2} - 2], \\
(a_i, -b_i) & \text{if } i \in [-1, \frac{n}{2} - 2], \\
(-\frac{n}{2} + 1, -\frac{n}{2} + 1) & \text{if } i = -\frac{n}{2} + 1.
\end{cases}
$$

(3)

It is not difficult to check that each of the columns of $T'$ uses up all integers in $[-\frac{n}{2} + 1, \frac{n}{2} - 2]$, that is, each of them is a permutation of $\mathbb{Z}_n \setminus \{\frac{n}{2} - 1, \frac{n}{2}\}$. We also point out that the substitution $i \mapsto (x_i, y_i)$ preserves the sum of each row, except that the last row of $T'$ sums to $\frac{n}{2} + 4$ which is coprime to $n$. Therefore, by applying Theorem 2.4 to $T$, it follows that $(n-\beta, \beta - 2) \in \text{HWP}(C_M[\mathbb{Z}_n \setminus \{\frac{n}{2} - 1, \frac{n}{2}\}; M, Mn])$. By Lemma 2.1, $(0, 2) \in \text{HWP}(C_M[\{\frac{n}{2} - 1, \frac{n}{2}\}; M, Mn])$, therefore $(n-\beta, \beta) \in \text{HWP}(C_M[n]; M, Mn)$. \hfill $\Box$

Lemmas 3.1 – 3.4 clearly yield the following result.

**Theorem 3.5.** Let $n \geq 2$, $M \geq 3$, and $0 \leq y \leq n$. Then $(n-y, y) \in \text{HWP}(C_M[n]; M, Mn)$ except possibly when at least one of the following conditions holds:

1. $y = 1$ and $(n, (-1)^M) \neq (2, -1)$;
2. $y = 2 < n \equiv 2 \pmod{4}$ and $M$ is odd;
3. $(y, n) \in \{(0, 2), (0, 6)\}$ and $M$ is odd.

### 4 Determining $\text{HWP}(v; M, Mn)$

In this section we prove the main result of this paper which concerns the existence of a solution to $\text{HWP}(K^n_v; M, N; \alpha, \beta)$ when $M$ is a divisor of $N$. Note that when $\alpha = 0$ or $\beta = 0$, this problem is equivalent to determining a $C_\ell$-factorization of $K^n_v$ and in this case a complete solution is provided by Theorem 1.1.
We denote by $\text{HW}(G; M, N; \alpha, \beta)$ any solution to $\text{HWP}(G; M, N; \alpha, \beta)$, that is, any factorization of $G$ into $\alpha$ $C_M$-factors and $\beta$ $C_N$-factors. We first prove the following lemma which provides sufficient conditions for the existence of an $\text{HW}(G; M, N; \alpha, \beta)$ for a given graph $G$.

**Lemma 4.1.** Let $M, N, \alpha, \beta$ be positive integers with $M$ being a divisor of $N$ and $N > M \geq 3$. Also, assume that $G$ has a factorization into $r \geq 2$ $C_M[n]$-factors where $n = N/M$. Then, $(\alpha, \beta) \in \text{HWP}(G; M, N)$ if and only if $\alpha + \beta = rn$, except possibly when at least one of the following conditions holds:

(i) $\beta = 1$;
(ii) $\beta = 2 < n \equiv 2 \pmod{4}$ and $M$ is odd;
(iii) $n = 2$, $M$ is even, and $\beta$ is odd;
(iv) $n = 2$, $M$ is odd, and $\beta < r$;
(v) $n = 6$, $M$ is odd, and $\beta < 3r$.

**Proof.** Set $n = N/M$ and note that $n \geq 2$ since $N > M$. By assumption, $G$ has a $C_M[n]$-factorization $G = \{G_1, G_2, \ldots, G_r\}$ with $r \geq 2$. It follows that $G$ is a regular graph of degree $2rn$. Now note that if $(\alpha, \beta) \in \text{HWP}(G; M, N)$, then $G$ has degree $2(\alpha + \beta)$, therefore $\alpha + \beta = rn$.

We now show sufficiency; hence, we assume that $\alpha + \beta = rn$. We will proceed by applying Theorem 3.5 to factorize each of the $r C_M[n]$-factors $G_i$ into an $\text{HW}(C_M[n]; M, Mn; \alpha_i, \beta_i)$ where $\alpha = \sum_i \alpha_i$ and $\beta = \sum_i \beta_i$ for $i \in [1, r]$. Clearly, this will result in an $\text{HW}(G; M, N; \alpha, \beta)$.

Set $\beta = xn + y$, with $0 \leq x < r$ and $0 \leq y < n$; note that by assumption $\beta > 0$, and by exception (i) we have that $\beta \neq 1$, hence $(x, y) \not\in \{(0, 0), (0, 1)\}$. We first assume that $n \not\in \{2, 6\}$. By taking into account exceptions (ii), the following condition holds:

(a) if $(x, y) = (0, 2)$ (i.e., $\beta = 2$) and $M$ is odd, then $n \not\equiv 2 \pmod{4}$.

We start with the case where $y \not\in \{1, 2\}$ and apply Theorem 3.5 to fill $x C_M[n]$-factors with an $\text{HW}(C_M[n]; M, Mn; 0, n)$, one $C_M[n]$-factor with an $\text{HW}(C_M[n]; M, Mn; n - y, y)$, and the rest with an $\text{HW}(C_M[n]; M, Mn; n, 0)$. If $(x, y) = (0, 2)$, then in view of condition (a) we can apply Theorem 3.5 to fill 1 $C_M[n]$-factor with an $\text{HW}(C_M[n]; M, Mn; n - y, y)$ and the rest with
an HW($C_M[n]; M, Mn; n, 0$). We finally consider the case where $y \in \{1, 2\}$ and $x > 0$. We again apply Theorem 3.5 to fill $x - 1$ $C_M[n]$-factors with an HW($C_M[n]; M, Mn; 0, n$). If $n \geq 4$, we proceed by filling one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; 2, n - 2$) and one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; n - y - 2, y + 2$). If $n = 3$ and $y = 1$, then we proceed by filling two $C_M[n]$-factors with an HW($C_M[n]; M, Mn; 1, 2$). If $n = 3$ and $y = 2$, then we fill one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; 0, 3$) and one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; 3 - y, y$). We fill the remaining $r - x - 1$ $C_M[n]$-factors with an HW($C_M[n]; M, Mn; n, 0$).

Now, we consider the case where $n \in \{2, 6\}$ and $M$ is even. Note that when $n = 2$, then $\beta$ is even (exception (iii)), that is, $y = 0$. If $y \neq 1$, then we apply Theorem 3.5 to fill $x$ $C_M[n]$-factors with an HW($C_M[n]; M, Mn; 0, n$), one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; n - y, y$), and the rest with an HW($C_M[n]; M, Mn; n, 0$). If $y = 1$, then $n = 6$ and $x > 0$ (since $(x, y) \neq (0, 1)$). We apply again Theorem 3.5 to fill $x - 1$ $C_M[n]$-factors with an HW($C_M[n]; M, Mn; 0, n$), one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; 1, n - 1$), one $C_M[n]$-factor with an HW($C_M[n]; M, Mn; n - 2, 2$), and the rest with an HW($C_M[n]; M, Mn; n, 0$).

We finally assume that $n \in \{2, 6\}$ and $M$ is odd, and set $\beta = x'r + y'$, with $0 \leq x' < n$ and $0 \leq y' < r$. In view of exceptions (iv)–(v) we have that $x' \geq 1$ when $n = 2$, and $x' \geq 3$ when $n = 6$. We can then apply Theorem 3.5 to fill $y'$ $C_M[n]$-factors with an HW($C_M[n]; M, Mn; n - x' - 1, x' + 1$) and the remaining $(r - y')$ $C_M[n]$-factors with an HW($C_M[n]; M, Mn; n - x', x'$), and this completes the proof.

We are now ready to prove the main result of this paper.

**Theorem 1.5.** Let $M, N, v, \alpha, \beta$ be positive integers such that $N > M \geq 3$ and $M$ is an odd divisor of $N$. Then, $(\alpha, \beta) \in \text{HWP}(v; M, N)$ if and only if $N \mid v$ and $\alpha + \beta = \left\lfloor \frac{v - 1}{2} \right\rfloor$ except possibly when at least one of the following conditions holds:

(i) $\beta = 1$;

(ii) $\beta = 2$, $N \equiv 2M \pmod{4M}$;

(iii) $N \in \{2M, 6M\}$;

(iv) $v \in \{N, 2N, 4N\}$;
(v) $(M, v) = (3, 6N)$.

**Proof.** We first note that by Theorem 1.4 if $(\alpha, \beta) \in HWP(v; M, N)$, then necessarily $\alpha + \beta = \lfloor \frac{v-1}{2} \rfloor$, and both $M$ and $N$ are divisors of $v$.

We now show sufficiency; therefore, let $(v, M, N, \alpha, \beta)$ a quintuple which satisfies the assumptions of this theorem. Therefore, $v = Mns$ where $n = N/M$ and $s$ is a suitable positive integer. Also, in view of the possible exceptions $(i) - (v)$, we can assume that the following conditions simultaneously hold:

$$
\begin{align*}
\beta &\neq 1, \beta \neq 2 \text{ when } n \equiv 2 \pmod{4}, \\
&s \notin \{1, 2, 4\}, \text{ and } (M, s) \neq (3, 6). 
\end{align*}
$$

We now set $w = Mn^t$ where $t = s$ if $s$ is odd, otherwise $t = s/2$. Note that in both cases we have $t \geq 3$, since $s \notin \{1, 2, 4\}$.

We factorize $K_v^*$ into $G_0 = tK_w^*$ and $G_1 = K_t[w]$. We start by applying Theorem 1.1 which guarantees the existence of either a $C_M$- or a $C_N$-factorization of $G_0$ as we choose. Therefore, this step will yield either $\gamma C_M$-factors or $\gamma C_N$-factors decomposing $G_0$, where $\gamma = \lfloor \frac{w-1}{2} \rfloor$. More precisely, let $(\alpha_0, \beta_0)$ be the pair defined as follows:

$$
(\alpha_0, \beta_0) = \begin{cases} 
(\gamma, 0) & \text{if } \beta < \gamma + 3, \\
(0, \gamma) & \text{if } \beta \geq \gamma + 3,
\end{cases}
$$

and apply Theorem 1.1 to fill $G_0$ with an $HW(G_0; M, N; \alpha_0, \beta_0)$. Since $(M, s) \neq (3, 6)$, by applying Corollary 1.3 with $z = Mn^t$ we obtain a $C_M[n]$-factorization of $K_t[w]$ containing at least three factors. By taking into account Lemma 1.1 and conditions (4), it follows that there exists an $HW(G_1; M, N; \alpha - \alpha_0, \beta - \beta_0)$ which we use to fill $G_1$ and this completes the proof. \qed

We point out that the above result has been proven in [12] in the case in which both $M$ and $N$ are odd, but gives new results when $M$ is odd and $N$ is even.

The following corollary easily follows.

**Corollary 4.2.** Let $M \geq 3$ be an odd divisor of $N$. The necessary conditions for the solvability of $HWP(v; M, N; \alpha, \beta)$ are sufficient whenever $v > 6N > 36M$ and $\beta \geq 3$. 

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