GROUPS WITH FRAMES OF TRANSLATES

HARTMUT FÜRHR, VIGNON OUSSA

ABSTRACT. Let $G$ be a locally compact group with left regular representation $\lambda_G$. We say that $G$ admits a frame of translates if there exist a countable set $\Gamma \subset G$ and $\varphi \in L^2(G)$ such that $(\lambda_G(x)\varphi)_{x \in \Gamma}$ is a frame for $L^2(G)$. The present work aims to characterize locally compact groups having frames of translates, and to this end, we derive necessary and/or sufficient conditions for the existence of such frames. Additionally, we exhibit surprisingly large classes of Lie groups admitting frames of translates.

1. Introduction

Throughout this paper, $G$ denotes a second countable locally compact group $G$ with Haar measure $\mu_G$. We denote the associated $L^2$-space by $L^2(G)$. $G$ acts on this space unitarily via the left regular representation, which we denote by $\lambda_G$.

We briefly recall the definitions of frames and Bessel sequence: A system $(\eta_i)_{i \in I}$ of vectors in a Hilbert space is called a frame if there exist constants $0 < A \leq B < \infty$ such that, for all $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, \eta_i \rangle|^2 \leq B\|f\|^2.$$ 

The constants $A, B$ are called frame bounds. We refer to $A$ as a lower frame bound and to $B$ as an upper frame bound. If $(\eta_i)_{i \in I}$ only admits an upper frame bound then we say that $(\eta_i)_{i \in I}$ is a Bessel family.

We are interested in groups of the following type:

**Definition 1.1.** $G$ admits a frame of translates, or is an FT group if there exists a family $\Gamma \subset G$ and $\varphi \in L^2(G)$ such that the family $(\lambda_G(x)\varphi)_{x \in \Gamma} \subset L^2(G)$ is a frame; i.e., there exist constants $0 < A \leq B < \infty$ such that, for all $g \in L^2(G)$,

$$A\|g\|^2 \leq \sum_{x \in \Gamma} |\langle g, \lambda_G(x)\varphi \rangle|^2 \leq B\|g\|^2.$$ 

The next remark lists some known or expected results. Part (b) is [3, Theorem 1.2], and the proof given in [3] relies on fairly technical concepts such as Beurling density.

**Remark 1.2.**

(a) If $G$ is discrete, then $(\lambda_G(x)\delta_e)_{x \in G}$ is an orthonormal basis of $L^2(G) = \ell^2(G)$, where $\delta_e$ denotes the Kronecker delta at the neutral element. Hence $G$ is an FT-group.

(b) $G = \mathbb{R}^d$ is not an FT-group.

The main objective of the present work is to investigate solutions to the following question

**Question 1.3.** Which non-discrete locally compact groups possess the FT property?
Intuitively, one may not expect many positive answers to Question 1.3 outside the discrete case. However, our results will show that FT groups are surprisingly not rare.

Remark 1.4. Instead of using subsets $\Gamma \subset G$, one may pose the central question of our paper with reference to families $(x_i)_{i \in I} \subset G$ and associated families of vectors $(\lambda_G(x_i)\varphi)_{i \in I}$. The difference between the two is that the latter allows repetitions of elements. However, it is easy to see that $(\lambda_G(x_i)\varphi)_{i \in I}$ is a frame if and only if $(\lambda_G(x)\varphi)_{x \in \Gamma}$ is a frame, where $\Gamma = \{x_i : i \in I\}$, and additionally, $\sup_{\varphi \in \Phi} \|i \in I : y = x_i\| < \infty$. Hence exchanging subsets for families does not affect the FT property, and we will freely switch between the two notions.

In the course of this paper, we will freely use notions from frame theory, representation theory of locally compact groups and Lie theory; our primary references for these topics are [2], [9], and [13] respectively.

2. Necessary criteria

In this section, we will consider various necessary conditions for frames of the type $(\lambda_G(x)\varphi)_{x \in \Gamma}$. These conditions will either concern the family $\Gamma$ of shifts, the function $\varphi$, or the group. In our analysis, we proceed precisely in this order.

Definition 2.1. Let $\Gamma$ be a subset of $G$. We say that $\Gamma$ is \textbf{(left) relatively separated} if $\sup_{x \in G} \|\Gamma \cap xU\| < \infty$ for some and hence all relatively compact neighborhoods $U$ of the identity. Next, we say that $\Gamma$ is $\textbf{V-separated}$ if for some relatively compact neighborhood $V$ of the identity, the family $(xV)_{x \in \Gamma}$ consists of pairwise disjoint sets. $\Gamma$ is called \textbf{separated} if it is $V$-separated for some suitable $V$.

The following result has been rediscovered several times in frame theory. We rephrase it for our setting.

Lemma 2.2. If $(\lambda(\gamma)\varphi)_{\gamma \in \Gamma}$ is a Bessel family then $\Gamma$ must be relatively separated.

Proof. Suppose that $\Gamma$ is not relatively separated. We consider the function $x \mapsto \langle \varphi, \lambda(x)\varphi \rangle$ defined over $G$. Since $x \mapsto \langle \varphi, \lambda(x)\varphi \rangle$ is continuous, there exists an open set $V$ around the identity element such that $\inf \{\|\langle \varphi, \lambda(x)\varphi \rangle : x \in V\| = \mu > 0$. Next, for an arbitrary natural number $N$, there exists $y \in G$ such that $yV$ contains at least $N$ elements from $\Gamma$. 
Next, let $\Gamma_N = yV \cap \Gamma$. As a result,

$$\sum_{\gamma \in \Gamma} |\langle (y) \varphi, \lambda(\gamma) \varphi \rangle|^2 \geq \sum_{\gamma \in \Gamma_N} |\langle (y) \varphi, \lambda(\gamma) \varphi \rangle|^2$$

$$= \sum_{\gamma \in \Gamma_N} |\langle \varphi, \lambda(y^{-1} \gamma) \varphi \rangle|^2$$

$$= \sum_{y\alpha \in \Gamma_N} |\langle \varphi, \lambda(y^{-1}y\alpha) \varphi \rangle|^2$$

$$= \sum_{\alpha \in y^{-1}\Gamma_N} |\langle \varphi, \lambda(\alpha) \varphi \rangle|^2$$

$$\geq \#(y^{-1}\Gamma_N) \cdot \mu^2$$

$$\geq N \cdot \mu^2 = \left(\frac{N \cdot \mu^2}{\|\varphi\|^2}\right) \|\lambda(y) \varphi\|^2.$$

Now since $N$ is arbitrary, it follows that $\langle (\gamma) \varphi \rangle_{\gamma \in \Gamma}$ is not a Bessel sequence. \hfill \Box

The following result is formulated in [8, Lemma 3.3], with proof attributed to [7]. Since we need it in the following, and the argument in [7] is given for the slightly different context of admissible coverings, we include a short proof.

**Lemma 2.3.** Let $\Gamma \subset G$ denote a relatively separated set of a locally compact group $G$. Then $\Gamma$ is the finite union of separated sets.

**Proof.** Fix a relatively compact and symmetric neighborhood $V \subset G$ of the identity. Then relative separatedness of $\Gamma$ yields that

$$\sup_{\gamma \in \Gamma} \sharp \{\gamma' \in \Gamma : \gamma' \cap \gamma V \neq \emptyset\} \leq \sup_{x \in G} \sharp \{\gamma' \in \Gamma : \gamma' \subset xV^2\} = m < \infty.$$

Zorn’s Lemma allows to choose a subset $\Gamma_1 \subset \Gamma$ that is $V$-discrete and maximal with respect to inclusion. If $\Gamma_1 = \Gamma$, then $\Gamma$ itself is $V$-discrete. We continue this procedure of choosing a maximal $V$-discrete $\Gamma_{s+1} \subset \Gamma \setminus \bigcup_{j \leq s} \Gamma_j$, as long as the complement is nonempty. We claim that this procedure breaks off after at most $m+1$ steps. Assuming that $\gamma_0 \in \Gamma \setminus \bigcup_{j \leq m+1} \Gamma_j$, we find for every fixed $1 \leq j \leq m+1$ that $\gamma_0 \in \Gamma \setminus \bigcup_{i<j} \Gamma_i$, and by maximality of $\Gamma_j$, $\Gamma_j \cup \{\gamma_0\}$ is not $V$-discrete. Hence there exists $\gamma_j \in \Gamma_j$ such that $\gamma_0 V \cap \gamma_j V \neq \emptyset$. Since the $\Gamma_j$ are pairwise disjoint, this entails

$$\sharp \{\gamma' \in \Gamma : \gamma' \cap \gamma_0 V \neq \emptyset\} \geq m + 1,$$

contrary to our choice of $m$. \hfill \Box

We next derive necessary conditions on the function $\varphi$ giving rise to frames of translates. Our aim is to show that for non-discrete groups such functions must necessarily be somewhat pathological. For instance, bounded functions with compact support will not do. In fact, we will be able to exclude a substantially larger space of functions, namely a particular **Wiener amalgam space**.

We define a local maximum function, as follows: Fix a compact neighborhood $U$ of the identity. Given a measurable function $f$ on $G$, we define

$$f_U^\sharp(x) = \text{ess sup}_{y \in xU} |f(y)|.$$
Next, given $1 \leq p \leq \infty$, we define the Wiener amalgam spaces $W(L^\infty, L^p)$ as the space of Borel functions $f$ for which the respective norm

$$\|f\|_{W(L^\infty, L^p)} = \|f_U^\sharp\|_p$$

is finite. It is well-known that, up to equivalence, the Wiener amalgam norm does not depend on the choice of $U$.

Note that compactly supported, bounded functions $\varphi$ are contained in $W(L^\infty, L^p)$, for all $1 \leq p \leq \infty$. For the following result, we also need the following. The convolution of two functions $f, \varphi$ on $G$ is defined as the integral

$$(f * \varphi)(x) = \int_G f(y) \varphi(y^{-1}x) \, dy$$

and $\varphi^*(x) = \overline{\varphi(x^{-1})}$. Then a straightforward calculation gives $(f * \varphi^*)(x) = \langle f, \lambda_G(x) \varphi \rangle$, for all $f, \varphi \in L^2(G)$.

Now the next proposition excludes compactly supported bounded functions from frame generation.

**Proposition 2.4.** Let $G$ be non-discrete. Let $\varphi \in L^2(G)$ be such that $\varphi^* \in W(L^\infty, L^2)$. Then there does not exist a discrete set $\Gamma \subset G$ such that $(\lambda_G(x)\varphi)_{x \in \Gamma}$ is a frame of $L^2(G)$.

**Proof.** Suppose by contradiction that $(\lambda_G(x)\varphi)_{x \in \Gamma}$ is a frame with lower frame bound $A$. Then, for all $f \in L^2(G)$, we have

$$\|f\|_2^2 \leq A^{-1} \sum_{x \in \Gamma} |f * \varphi^*(x)|^2. \tag{1}$$

By Lemmas 2.2 and 2.3, we can write $\Gamma = \bigcup_{i=1}^n \Gamma_i$ disjointly, and each $\Gamma_i$ is $U$-discrete, for a suitable symmetric neighborhood $U$ of the identity. Hence we get

$$\sum_{x \in \Gamma} |f * \varphi^*(x)|^2 = \sum_{i=1}^n \sum_{x \in \Gamma_i} |f * \varphi^*(x)|^2$$

$$= \sum_{i=1}^n \sum_{x \in \Gamma_i} \frac{1}{|U|} \int_{xU} |f * \varphi^*(x)|^2 \, dy$$

$$\leq \sum_{i=1}^n \sum_{x \in \Gamma_i} \frac{1}{|U|} \int_{U} \left| (f * \varphi^*)^\sharp_U(y) \right|^2 \, dy$$

$$\leq \frac{n}{|U|} \left\| (f * \varphi^*)^\sharp_U \right\|_2^2.$$

Applying the well-known pointwise estimate

$$\left\| (f * g)^\sharp_U \right\|_2 \leq \left\| (f * (\varphi^*)^\sharp_U) \right\|_2,$$

valid for arbitrary measurable functions $f, g$ yields

$$\left\| (f * \varphi^*)^\sharp_U \right\|_2 \leq \left\| (f * (\varphi^*)^\sharp_U) \right\|_2.$$
Coming back to (1), we thus obtain
\[ \|f\|_2 \leq \left( \frac{n}{A|U|} \right)^{1/2} \| |f| \ast (\varphi^*)_U^2 \|_2. \]

On the other hand, Young’s inequality yields for all \( f \in L^1(G) \cap L^2(G) \), that
\[ \left\| |f| \ast (\varphi^*)_U^2 \right\|_2 \leq \|f\|_1 \| (\varphi^*)_U^2 \|_2 = \|f\|_1 \| \varphi^* \|_{W(L^\infty, L^2)}, \]
and the Wiener amalgam norm is finite by assumption.

In summary, we have shown
\[ (2) \quad \|f\|_2 \leq \left( \frac{n}{A|U|} \right)^{1/2} \| \varphi^* \|_{W(L^\infty, L^2)} \|f\|_1, \]
for all \( f \in L^1(G) \cap L^2(G) \). Now replacing \( f \) with the indicator function of \( U \) with \( \lambda_G(U) \rightarrow 0 \), yields the desired contradiction. \( \square \)

We next derive various classes of groups that are not FT.

**Corollary 2.5.** If \( G \) is compact, then it is FT if and only if it is finite.

**Proof.** By Lemma 2.2 and the fact that relatively separated subsets of compact groups are finite, \( L^2(G) \) is finite-dimensional whenever \( G \) is a compact FT group. \( \square \)

**Theorem 2.6.** Let \( G \) be non-discrete, satisfying the following property: The inverse of any subset of \( G \) which is relatively separated is also relatively separated. Then \( G \) is not an FT group.

**Proof.** Suppose by ways of contradiction that the stated assumptions hold and that there exists \( \varphi \in L^2(G) \) and \( \Gamma \subset G \) such that \( (\lambda(\gamma) \varphi)_{\gamma \in \Gamma} \) is a frame for \( L^2(G) \). Thus, \( (\lambda(\gamma) \varphi)_{\gamma \in \Gamma} \) is a Bessel sequence. By Lemma 2.2, \( \Gamma \) must be relatively separated. By assumption, \( \Gamma^{-1} \) is also relatively separated. Furthermore, according to Lemma 2.3, \( \Gamma^{-1} \) can be written as a disjoint union of separated sets. Let \( \Gamma^{-1} = \bigcup_{k=1}^s \Psi_k \) such that each collection \( \{xV : x \in \Psi_k\} \) consists of essentially disjoint subsets of \( G \). Then
\[ \sum_{\gamma \in \Gamma} |(\chi_V, \lambda(\gamma) \varphi)|^2 = \sum_{\gamma \in \Gamma} |(\chi_V, \chi_V \cdot \lambda(\gamma) \varphi)|^2 \]
\[ \leq \|\chi_V\|^2 \left( \sum_{k=1}^s \sum_{\gamma \in \Psi_k} \|\chi_V \cdot \lambda(\gamma) \varphi\|^2 \right) \]
\[ = \|\chi_V\|^2 \left( \sum_{k=1}^s \sum_{\gamma \in \Psi_k} \int_V |\varphi(\gamma^{-1}x)|^2 dx \right) \]
\[ = \|\chi_V\|^2 \sum_{k=1}^s \left( \int_{\bigcup_{\gamma \in \Psi_k} \gamma^{-1}V} |\varphi(x)|^2 dx \right). \]

Note that since \( \varphi \) is square-integrable,
\[ \int_{\bigcup_{\gamma \in \Psi_k} \gamma^{-1}V} |\varphi(x)|^2 dx \leq \int_G |\varphi(x)|^2 dx = \|\varphi\|^2 < \infty. \]
By Lebesgue’s Dominated Convergence Theorem, taking a nested family of relatively compact and open sets converging to the singleton containing the identity in $G$, we obtain:
\[
\lim_{V \to \{e\}} \int_{\bigcup_{\gamma \in \Phi} \gamma^{-1}V} |\varphi(x)|^2 \, dx = 0.
\]
Thus for any $\epsilon > 0$, there exists a sufficiently small open set $V$ around the identity such that
\[
\sum_{\gamma \in \Gamma} |\langle \chi_V, \lambda(\gamma) \varphi \rangle|^2 \leq \epsilon \|\chi_V\|^2.
\]
This violates the lower frame bounds condition and gives us the desired contradiction. □

This observation allows to generalize [3, Theorem 1.2] to a larger class of groups called [IN]-groups: groups having a compact neighborhood of the identity which is invariant under all inner automorphisms.

**Lemma 2.7.** If $G$ is an [IN]-group then every relatively separated subset of $G$ has a relatively separated inverse. In particular, nondiscrete [IN]-groups are not FT.

**Proof.** Let $G$ be an [IN]-group. Then by definition, there exists a compact neighborhood $W \subset G$ of the identity which is conjugation-invariant. Next, let $\Gamma$ be a subset of $G$ which is relatively separated. By assumption, $\sup_{x \in G} \sharp (\Gamma \cap xW) < \infty$. On the other hand for any $x \in G$, note that
\[
\Gamma \cap xW = \Gamma \cap (xWx^{-1}) x = \Gamma \cap Wx,
\]
and thus
\[
(\Gamma \cap xW)^{-1} = \Gamma^{-1} \cap x^{-1}W^{-1}.
\]
Consequently, since inversion on $G$ is bijective,
\[
\sup_{y \in G} \sharp (\Gamma^{-1} \cap yW^{-1}) = \sup_{x \in G} \sharp (\Gamma^{-1} \cap x^{-1}W^{-1}) = \sup_{x \in G} \sharp (\Gamma \cap xW) < \infty
\]
which proves that $\Gamma^{-1}$ is relatively separated. □

**Remark 2.8.** Clearly, Lemma 2.7 applies to abelian groups, thus it directly generalizes [3, Theorem 1.2].

A result due to Iwasawa [15, Theorem 2] yields that a connected topological group $G$ is an [IN]-group if and only if the topological commutator of $G$ is compact.

A nonabelian group to which this applies is the reduced Weyl-Heisenberg group, which is the quotient of the simply connected, connected Heisenberg group by a central discrete subgroup. More generally, Lemma 2.7 also implies that no step-two nilpotent Lie group with compact center is FT.

It is currently open which groups have the property that inverses of relatively separated sets are relatively separated again. As the previous remark shows, some nonabelian nilpotent Lie groups do. However, simply connected nilpotent Lie groups generally do not:

**Lemma 2.9.** If $G$ is a nonabelian, simply connected connected nilpotent Lie group, then there exists a separated set $\Gamma \subset G$ such that $\Gamma^{-1}$ is not relatively separated.
Proof. We start out by considering the special case that \( G \) is the simply connected and connected three-dimensional Heisenberg Lie group, with Lie algebra spanned by

\[
X_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

with non-trivial Lie brackets \([X_1, X_2] = X_3\). It is easy to verify that

\[
\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X,Y]) = \exp \begin{bmatrix} 0 & x_1 + y_1 & x_3 + y_3 + \frac{x_1 y_2 - x_2 y_1}{2} \\ 0 & 0 & x_2 + y_2 \\ 0 & 0 & 0 \end{bmatrix}.
\]

Next, we endow the Heisenberg Lie group with the following quasi-norm

\[
\|\exp \begin{bmatrix} 0 & x_1 & x_3 \\ 0 & 0 & x_2 \\ 0 & 0 & 0 \end{bmatrix}\| = \left(\left(x_1^2 + x_2^2\right)^2 + x_3^4\right)^{1/4}.
\]

This induces a left-invariant quasi-metric between two arbitrary elements by

\[
d(\exp(x_1 X_1 + x_2 X_2 + x_3 X_3), \exp(y_1 X_1 + y_2 X_2 + y_3 X_3)) = \|\exp(-x_1 X_1 - x_2 X_2 - x_3 X_3) \exp(y_1 X_1 + y_2 X_2 + y_3 X_3)\|
\]

\[
= \left(\left((y_1 - x_1)^2 + (y_2 - x_2)^2\right)^2 + \left(y_3 - x_3 + \frac{x_2 y_1 - x_1 y_2}{2}\right)^4\right)^{1/4}.
\]

For any natural number \( N \in \mathbb{N} \), we define

\[
U_N = \begin{bmatrix} 0 & N^2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad V_{N, \ell} = \begin{bmatrix} 0 & N^2 & \ell \\ 0 & 0 & 2N^2 \\ 0 & 0 & 0 \end{bmatrix}, \quad \ell = 1, \ldots, N.
\]

Let

\[
\Gamma = \{\exp(-V_{M, \ell}) : M \in \mathbb{N}, \ell = 1, \ldots, M\}.
\]

Note that, for all \( N \in \mathbb{N} \),

\[
d(\exp U_N, \exp V_{N, \ell}) = \frac{\ell}{N^2} \leq \frac{1}{N}.
\]

In other words, the ball with center \( U_N \) and radius \( N^{-1} \) contains at least \( N \) elements of \( \Gamma^{-1} \), which shows that \( \Gamma^{-1} \) is not relatively separated. On the other hand, it is easy to verify for distinct \( \exp(V_{N, \kappa}), \exp(V_{M, \ell}) \in \Gamma \), that

\[
d(\exp(-V_{N, \kappa}), \exp(-V_{M, \ell})) \geq 1.
\]

Thus, \( \Gamma \) is left-uniformly discrete.

Now, if \( G \) is a simply connected, connected nonabelian nilpotent Lie group of dimension larger than three, then by Kirillov’s lemma [4], \( G \) contains a closed subgroup \( H \) that is isomorphic to the simply connected Heisenberg group. By the case of the three-dimensional Heisenberg Lie group, there exists a separated (in \( H \)) \( \Gamma \subset H \) with the property that \( \Gamma^{-1} \) is not separated (in \( H \)). However, for subsets of \( H \), being separated in \( H \) is the same as being separated in \( G \). \qed
A further interesting property of FT groups is a kind of universal sampling theorem. First some terminology:

**Definition 2.10.** Given a unitary representation \( \pi \) of a locally compact group \( G \) acting on a Hilbert space \( \mathcal{H}_\pi \) and \( \eta \in \mathcal{H}_\pi \), we define the associated wavelet transform \( V_\eta : \mathcal{H}_\pi \to C_b(G) \) as

\[
V_\eta u(x) = \langle u, \pi(x) \eta \rangle.
\]

\( \eta \) is called **admissible** if \( V_\eta : \mathcal{H}_\pi \to L^2(G) \) is well-defined and isometric. We call \( \pi \) **strongly square-integrable** if there exists an admissible vector \( \eta \in \mathcal{H}_\pi \).

It is well-known that admissible vectors associated to a strongly square-integrable representations give rise to weak-sense inversion formulae

\[
u = \int_G V_\eta u(x) \pi(x) \eta \, dx.
\]

We can now derive a discretization result for generalized wavelet transforms over FT groups. Observe here that the sampling set is **universal**, i.e., it is picked independently of the representation whose inversion formula it discretizes.

**Theorem 2.11.** Assume that \( G \) has a frame \( (\lambda_G(x) \varphi)_{x \in \Gamma} \) of translates for \( L^2(G) \). Then for every strongly square-integrable representation \( \pi \), there exists \( \eta \in \mathcal{H}_\pi \) such that \( (\pi(x) \eta)_{x \in \Gamma} \) is a frame.

**Proof.** Let \( \psi \in \mathcal{H}_\pi \) be an admissible vector for \( \pi \). Then \( V_\psi \) is a unitary equivalence between \( \mathcal{H}_\pi \) and \( \mathcal{H}_{\pi, \psi} = V_\psi(\mathcal{H}_\pi) \), and the latter is a closed, left-invariant subspace of \( L^2(G) \). The orthogonal projection onto \( \mathcal{H}_{\pi, \psi} \) is \( P = V_\psi V_\psi^* \). It follows that \( (P \lambda_G(x) \varphi)_{x \in \Gamma} \) is a frame of \( \mathcal{H}_{\pi, \psi} \). Then, since \( V_\psi^* : \mathcal{H}_{\pi, \psi} \to \mathcal{H}_\pi \) is a unitary equivalence intertwining the group actions, we finally see that \( \eta = V_\psi^* \varphi \) is as required. \( \square \)

**Remark 2.12.** There is an alternative proof available of the fact that the reduced Weyl-Heisenberg group is not FT, which makes use of the Theorem [2.11] in combination with well-known necessary density conditions for Gabor frames, as derived in [3].

For the remainder of this section we concentrate on discrete groups. For the following results, recall that a Riesz basis of a Hilbert space is a frame \( (\eta_i)_{i \in I} \) satisfying the additional inequality

\[
\sum_{i \in I} |c_i|^2 \leq C \left\| \sum_{i \in I} c_i \eta_i \right\|^2.
\]

**Theorem 2.13.** Let \( G \) denote a discrete group, \( \Gamma \subset G \) and \( \varphi \in \ell^2(G) \). Then the following are equivalent:

(a) \( (\lambda_G(x))_{x \in \Gamma} \) is a frame of \( \ell^2(G) \).

(b) \( \Gamma = G \), and \( (\lambda_G(x))_{x \in G} \) is Riesz basis of \( \ell^2(G) \).

**Proof.** We only need to prove (a) \( \Rightarrow \) (b). We first show that \( \Gamma \subset G \) is uniformly dense. We prove this by contradiction. Let \( A \) denote the lower frame bound, and pick \( U \subset G \) finite such that

\[
\sum_{x \notin U} |\varphi(x)|^2 < A.
\]
The assumption that $\Gamma$ is not uniformly dense implies the existence of $y \in G$ such that $yU \cap \Gamma$ is empty. We then get

$$A \leq \sum_{x \in \Gamma} |\langle 1_y, \lambda_G(x) \varphi \rangle|^2$$

$$= \sum_{x \in y^{-1} \Gamma} |\varphi(x)|^2$$

$$\leq \sum_{x \notin U} |\varphi(x)|^2 < A$$

by choice of $y$ and $A$, respectively. This is the desired contradiction.

Hence $\Gamma$ is uniformly dense, which means that $(\lambda_G(x) \varphi)_{x \in G}$ is a frame as well. Thus the associated frame operator $S_\varphi = V_\varphi^* V_\varphi : f \mapsto f * \varphi * \varphi$ is a bounded, self-adjoint operator with bounded inverse, commuting with left translations on $G$. Hence, letting $\eta = S_\varphi^{-1/2} \varphi$ yields a tight frame generator, i.e., we have that $f \mapsto f * \eta^*$ is an isometry, or equivalently, that

$$f = V_\eta^* V_\eta f = f * \eta^* * \eta,$$

holds for all $f \in \ell^2(G)$. In particular

$$\delta_e = \delta_e * \eta^* * \eta = \eta^* * \eta,$$

and thus $\|\eta\|^2 = (\eta^* * \eta)(e) = 1$. We thus obtain that $(\lambda_G(x) \eta)_{x \in G} \subset \ell^2(G)$ is a Parseval frame consisting of unit vectors; and it is well known that such frames are actually orthonormal bases.

But then $(\lambda_G(x) \varphi)_{x \in G} = (S_\varphi^{1/2} \lambda_G(x) \eta)_{x \in G}$ is a Riesz basis, as the image of an orthonormal basis under an invertible operator. In particular, the system $(\lambda_G(x) \varphi)_{x \in \Lambda}$ is incomplete in $\ell^2(G)$, for every proper subset $\Lambda$ of $G$. Thus $\Gamma = G$ follows.

We conjecture that a group $G$ has a Riesz basis of translates if and only if it is discrete.

### 3. Sufficient criteria

The results established so far do not seem to point towards the existence of nondiscrete FT groups. As Proposition 2.4 shows, the functions $\varphi$ occurring in frames of translates are necessarily somewhat pathological, an observation that seems to raise the bar somewhat further. Nonetheless, the remainder of this paper will show that FT groups exist in abundance. The strategy for proving such a result rests on two observations. The first one is the following remarkably general discretization result, recently established by Freeman and Speegle [10, Theorem 1.3]:

**Theorem 3.1.** Let $(\eta_x)_{x \in X} \subset \mathcal{H}$ denote a family of bounded vectors in a separable Hilbert space, measurably indexed by $x \in X$, where $(X, \mu)$ is a $\sigma$-finite measure space. Assume that $(\eta_x)_{x \in X}$ is a continuous frame with respect to $\mu$, i.e., there exist constants $0 < A \leq B < \infty$ such that

$$\forall g \in \mathcal{H} : A \|g\|^2 \leq \int_X |\langle g, \eta_x \rangle|^2 d\mu(x) \leq B \|g\|^2.$$

Then there exists a countable family $(x_i)_{i \in I}$ such that $(\eta_{x_i})_{i \in I}$ is a frame.
The main consequence of this result is the following theorem which reveals a first, large class of FT groups.

**Theorem 3.2.** Let $G$ be type I and non-unimodular. Then $G$ is an FT group.

**Proof.** By [11, 12], $\lambda_G$ has an admissible vector $\eta$. I.e., the family $(\lambda_G(x)\eta)_{x \in G}$ is a continuous frame with respect to $\mu_G$. Theorem 3.1 yields a family $(x_i)_{i \in I} \subset G$ such that $(\lambda_G(x_i)\eta)_{i \in I}$ is a frame. \qed

The second observation enlarges the class of FT further, by considering the restriction of the regular representation to a suitable closed subgroup.

**Definition 3.3.** Given a representation $\pi$ of $G$, we say that $\pi$ has infinite multiplicity if $\pi \simeq \infty \cdot \pi$.

**Remark 3.4.** Let $G$ denote a type I group. Then the Plancherel transform of $G$ gives rise to a unique direct integral decomposition

$$\lambda_G \simeq \int_{\hat{G}} m_\sigma \cdot \sigma d\nu_G(\sigma),$$

where $\nu_G$ is the Plancherel measure of $G$, and the multiplicity $m_\sigma$ with which $\sigma \in \hat{G}$ enters the Plancherel decomposition is equal to the Hilbert space dimension of $\mathcal{H}_\sigma$. In particular, $\lambda_G$ has infinite multiplicity if and only if $m_\sigma = \infty$, for $\nu_G$-almost every $\sigma$.

Important classes of groups for which this holds are the nonunimodular type I groups, and the nonabelian connected nilpotent Lie groups. In both cases, $\nu_G$-almost all irreducible representations are induced from subgroups of infinite index: For the nonunimodular case, this was established in [6]; for the nilpotent case, it follows by Kirillov’s orbit method, see [4]. But the representation spaces associated to these representations are then infinite-dimensional.

It is worthwhile noting that if a representation $\lambda_G$ has infinite multiplicity, then

$$\lambda_G \simeq m \cdot \lambda_G$$

holds as well, for all natural numbers $m$.

The following result exploits an idea due to Iverson [14, Theorem 3.6] for our purposes.

**Theorem 3.5.** Let $H < G$ denote a closed subgroup that has FT, and such that $\lambda_H$ has infinite multiplicity. Then $G$ is FT. In fact, there exists a vector $\varphi \in L^2(G)$ and $\Gamma \subset H$ such that $(\lambda_G(x)f)_{x \in \Gamma} \subset L^2(G)$ is a frame.

**Proof.** Using a measurable set of coset representatives $C$ mod $H$, we have a Borel isomorphism $H \times C \ni (h, x) \mapsto hx \in G$. Furthermore, this isomorphism intertwines the left action $H \times (H \times C) \ni (h_1, (h_2, x)) \mapsto (h_1 h_2, x) \in H \times C$ with the left action of $H$ on $G$. By Weil’s formula, we can then identify the Haar measure on $G$ with the product of Haar measure on $H$, and a suitable choice of measure on the quotient. Since the latter is Borel isomorphic to $C$, this identification induces a unitary equivalence $L^2(G) \to L^2(H) \otimes L^2(C)$.

Since the Borel isomorphism intertwines the respective actions on $H \times C$ and $G$, the unitary map intertwines $\lambda_G|_H$ with $\lambda_H \otimes 1$, where $1$ denotes the trivial representation acting on $L^2(C)$.
Hence, denoting by $\kappa \in \mathbb{N} \cup \{\infty\}$ the Hilbert space dimension of $L^2(C)$, we find
\[ \lambda_G|_H \simeq \kappa \cdot \lambda_H \simeq \lambda_H. \]
By assumption on $H$, there exists a frame of translates in $L^2(H)$, and the image of this frame under the intertwining operator $\lambda_H \simeq \lambda_G|_H$ has the desired properties. \(\Box\)

The simplest example of a type I nonunimodular group is the $ax+b$-group, the semidirect product $\mathbb{R} \rtimes \mathbb{R}^+$. Note that $SL(2, \mathbb{R})$ contains the closed subgroup
\[ \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} : a > 0, b \in \mathbb{R} \right\} \]
which is isomorphic to the $ax+b$ group. Hence Theorem 3.5 implies the following:

**Corollary 3.6.** $SL(2, \mathbb{R})$ is an FT group.

Note that $SL(2, \mathbb{R})$ is unimodular. Hence unimodular FT groups do exist. Moreover, let $p, q$ be natural numbers satisfying $p+q > 2$. Next, let
\[ SO(p, q) = \left\{ M \in GL(p+q, \mathbb{R}) : M^T J(p, q) M = J(p, q) \right\} \]
where $J(p, q) = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ such that the trace of $J(p, q)$ is equal to $p- q$. Then $SO(p, q)$ is a closed subgroup of $GL(p+q, \mathbb{R})$ and the following holds true.

**Corollary 3.7.** Given natural numbers $p, q$ such that $p+q > 2$, $SO(p, q)$ is an FT group

**Proof.** We first observe that every element $X$ of the Lie algebra of $SO(p, q)$ can be written in block-form as
\[ X = \begin{bmatrix} Z & S \\ S^t & Y \end{bmatrix} \quad \text{for some } (Z, Y) \in \mathfrak{so}(p) \times \mathfrak{so}(q). \]
As such, if $q = 1$ then
\[ A = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \in \mathfrak{so}(p, 1) \]
and $[A, X] = X$. Thus, $\exp(\mathbb{R}X) \exp(\mathbb{R}A)$ is a closed nonunimodular subgroup of $SO(p, 1)$ which is isomorphic to the $ax+b$-group and its follows that $SO(p, 1)$ is FT. More generally, letting
\[ I = \{(1, p+1), (p+1, 1)\} \quad \text{and} \quad J = \{(1, p), (p, p+1), (p+1, p)\} \]
and defining matrices $B, Y \in \mathfrak{gl}(p+q, \mathbb{R})$ with entries satisfying
\[ B_{jk} = \begin{cases} 1 & \text{if } (j, k) \in I \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad Y_{jk} = \begin{cases} 1 & \text{if } (j, k) \in J \\ -1 & \text{if } (j, k) = (p, 1) \\ 0 & \text{otherwise} \end{cases} \]
it is easy to verify that $[B, Y] = Y$. Thus, the $ax+b$-group $\exp(\mathbb{R}Y) \exp(\mathbb{R}B)$ is a closed subgroup of $SO(p, q)$ and it follows immediately that $SO(p, q)$ is FT as well. \(\Box\)
Remark 3.8. Since $SL(n, \mathbb{R})$ and $SL(n, \mathbb{C})$, for $n \geq 2$, contain closed isomorphic copies of $SL(2, \mathbb{R})$, all of these groups are FT groups again. The same reasoning applies to the symplectic groups $Sp(2n, \mathbb{R})$ and $Sp(2n, \mathbb{C})$: One has $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$, and the higher-dimensional groups contain closed isomorphic copies of $Sp(2, \mathbb{R})$. Identical reasoning entails the FT property for the metaplectic groups in arbitrary dimensions.

Remark 3.9. We can use previous results to construct a rather unusual shearlet frame. Consider the matrix group 
\[ H = \left\{ \pm \begin{bmatrix} a & b \\ 0 & a^{1/2} \end{bmatrix} : a > 0, b \in \mathbb{R} \right\}. \]
Let $G = \mathbb{R}^2 \rtimes H$. The natural affine action of $G$ on $\mathbb{R}^2$ gives rise to the quasi-regular representation on $L^2(\mathbb{R}^2)$, which is known to be strongly square-integrable \[5\]. The generalized wavelet transform associated with this group is the so-called shearlet transform. Shearlet frames are typically constructed by choosing a lattice $\Gamma \subset \mathbb{R}^2$ and a discrete subset $H_d \subset H$ and considering families of the kind 
\[ (\pi(hx, h)\psi)_{x \in \Gamma, h \in H_d}, \]
for suitable well-chosen functions $\psi$; see \[1\] for an early source using this type of construction. Note that $H$ is a closed subgroup of $G$ that is isomorphic to the $ax + b$-group. Hence combining Theorems \[2.11\] and \[3.5\], we can now show that there exists a frame of the type 
\[ (\pi(0, h)\varphi)_{h \in H_d}, \]
i.e., using only dilations! This example generalizes easily to shearlet groups in arbitrary dimension.

4. Lie groups with FT

Using the tools developed in the previous sections, we will now explore the FT property on a class of Lie groups known as exponential Lie groups. To present the findings of this section, we need the following. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. We say that $\mathfrak{g}$ is of type $\mathbf{R}$ if for every $X \in \mathfrak{g}$, the eigenvalues of the endomorphism $ad_{\mathfrak{g}}(X)$ are purely imaginary.

The lower central series of $\mathfrak{g}$ is inductively defined as follows $C^1\mathfrak{g} = \mathfrak{g}$ and $C^{j+1}\mathfrak{g} = [\mathfrak{g}, C^j\mathfrak{g}]$ for $j > 0$. Moreover, a Lie algebra is nilpotent if there exists a natural number $k$ such that $C^k\mathfrak{g}$ is trivial. According to Engel’s characterization, a Lie algebra $\mathfrak{g}$ is nilpotent if and only if for every $X \in \mathfrak{g}$, the endomorphism $ad_{\mathfrak{g}}(X) : Y \mapsto [X, Y]$ is nilpotent (see \[13\], Theorem 5.2.8.) The derived series of $\mathfrak{g}$ is a decreasing collection $D^j\mathfrak{g}$ of ideals in $\mathfrak{g}$ defined as $D^0\mathfrak{g} = \mathfrak{g}$ and $D^{j+1}\mathfrak{g} = [D^j\mathfrak{g}, D^j\mathfrak{g}]$ for $j \in \mathbb{N}$.

A Lie algebra $\mathfrak{g}$ is solvable if its derived series reaches the trivial algebra in finitely many steps. Suppose for now that $\mathfrak{g}$ is a real solvable Lie algebra of dimension $n$. Then $ad_{\mathfrak{g}}(\mathfrak{g})$ is a solvable algebra of endomorphisms acting on $\mathfrak{g}$. Next, let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$. As a $\mathfrak{g}_\mathbb{C}$-module, the Lie algebra $\mathfrak{g}_\mathbb{C}$ has a Jordan-Hölder sequence 
\[ \{0\} = \mathfrak{g}_\mathbb{C}^{(0)} \subset \mathfrak{g}_\mathbb{C}^{(1)} \cdots \subset \mathfrak{g}_\mathbb{C}^{(n-1)} \subset \mathfrak{g}_\mathbb{C}^{(n)} = \mathfrak{g}_\mathbb{C}. \]
That is, each $g^{(k)}_C$ is an ideal and $\dim g^{(k)}_C = k$. Moreover, the action of $g_C$ on the vector space $g^{(k)}_C / g^{(k-1)}_C$ where $1 \leq k \leq n$ defines a linear form on $g_C$ called a root of $g$. We say that $g$ is an **exponential solvable Lie algebra** if it does not possess any root with nonzero purely imaginary value. In other words, each root of $g$ is given by $X \mapsto \lambda(X) (1 + i\alpha)$ where $\lambda \in g^*$ and $\alpha \in \mathbb{R}$.

Next, let $G = \exp g$ be a simply connected and connected Lie group with solvable Lie algebra $g$. Then it is known that $g$ is exponential if and only if the exponential map determines an analytic diffeomorphism between $g$ and $G$. In this case, $G$ is called an **exponential solvable Lie group**.

The following theorem summarizes our results

**Theorem 4.1.** Exponential solvable Lie groups which are not nilpotent are FT.

It is worthwhile noting that for all FT groups mentioned in Theorem 1.1 the set $\Gamma \subset G$ of shifts generating the frame can be chosen as a subset of a closed subgroup of dimension at most 3, regardless of the dimension of $G$ itself.

The case of nonabelian simply connected, connected nilpotent Lie groups is currently open. There are several reasons why this class is interesting: A comprehensive answer would ultimately settle the exponential solvable case. Moreover, it would help clarify the relationship between the FT property and the property that inverses of relatively separated sets are relatively separated again. Recall that the latter implies the negation of the former, and we currently have no example that the converse does not hold in general.

If such a general converse holds, it implies that all nonabelian simply connected, connected nilpotent Lie groups are not FT, via Lemma 2.9. However, with current knowledge, all we can do is to reduce the discussion to certain test cases. For the following proposition, we introduce $T(n, \mathbb{R})$ for the group of upper triangular matrices in $GL(n, \mathbb{R})$ with ones on the diagonal.

**Proposition 4.2.**

(a) Every nonabelian simply connected, connected nilpotent Lie group is FT if and only if the Heisenberg group is FT.

(b) No nonabelian simply connected, connected nilpotent Lie group is FT if and only if for all $n \geq 3$, $T(n, \mathbb{R})$ is not FT.

**Proof.** Both statements are consequences of Theorem 3.5 via the observation that if $G$ is nonabelian and simply connected, nilpotent, there exist embeddings

$$H \subset G \subset T(n, \mathbb{R})$$

as closed subgroups, where $H$ is isomorphic to a Heisenberg group, and $n$ is sufficiently large. Here the first embedding is due to Kirillov’s lemma [4], the second is a well-known consequence of Engel’s Theorem.

There are two solvable Lie algebras which are of crucial importance in this work. The first one is the $ax+b$ Lie algebra which is a two-dimensional solvable Lie algebra spanned by $A, X$ such that $[A, X] = X$. The second one: the Grélaud’s algebra is a three-dimensional solvable Lie algebra spanned by $A, Y_1, Y_2$ with non-trivial Lie brackets

$$[A, Y_1] = Y_1 + \beta Y_2 \quad \text{and} \quad [A, Y_2] = -\beta Y_1 + Y_2$$
for some nonzero real number $\beta$.

**Lemma 4.3.** If $\mathfrak{g}$ is not of type $R$ then $\mathfrak{g}$ admits a Lie subalgebra $\mathfrak{h} < \mathfrak{g}$ which is either isomorphic to the $ax+b$ algebra or the Grélaud’s algebra.

**Proof.** By assumption, one of the following must hold.

Case 1: There exists $X \in \mathfrak{g}$ with a nonzero real eigenvalue $\lambda$. As such, there exists an eigenvector $Y$ for $\text{ad}(X)$ with corresponding nonzero eigenvalue $\lambda$ satisfying $[\lambda^{-1}X,Y] = Y$ and $\mathfrak{h} = \mathbb{R}Y + \mathbb{R}X$ as desired.

Case 2: Suppose that Case 1 does not hold. Then there exists $X \in \mathfrak{g}$ such that $\alpha + i\beta$ is an eigenvalue for the endomorphism $\text{ad}(X)$ and $\alpha \neq 0$. Thus, there exist $Y_1, Y_2 \in \mathfrak{g}$ such that $[X,Y_1] = \alpha Y_1 + \beta Y_2$ and $[X,Y_2] = -\beta Y_1 + \alpha Y_2$. Next, we claim that $Y_1, Y_2$ must commute. Otherwise, a straightforward application of Jacobi’s identity yields $[X,[Y_1,Y_2]] = 2\alpha [Y_1,Y_2]$. However, this contradicts the fact that $X$ does not have a nonzero real eigenvalue. Finally, setting $\mathfrak{h} = \mathbb{R}\text{-span}\{X,Y_1,Y_2\}$ gives the desired result. \qed

**Lemma 4.4.** Let $\mathfrak{g}$ be an $n$-dimensional exponential solvable Lie algebra. Then the following statements are equivalent

1. $\mathfrak{g}$ is not a nilpotent Lie algebra
2. There exists a Lie subalgebra of $\mathfrak{h}$ of $\mathfrak{g}$ such that $\exp \mathfrak{h}$ is a closed type I non-unimodular Lie group

**Proof.** To prove that (2) $\implies$ (1), suppose that there exists a Lie subalgebra of $\mathfrak{h}$ of $\mathfrak{g}$ such that $\exp \mathfrak{h}$ is a type I non-unimodular Lie group. Since nilpotent Lie groups are unimodular, this subalgebra cannot be nilpotent. Then $\mathfrak{g}$ is not a nilpotent Lie algebra since every subalgebra of a nilpotent Lie algebra is nilpotent (see [4], Proposition 1.1.6). Conversely, let us suppose that $\mathfrak{g}$ is an exponential solvable Lie group which is not nilpotent. By Engel’s Theorem, there exists $X \in \mathfrak{g}$ such that $\text{ad}_g(X)$ is not nilpotent. The fact that $\mathfrak{g}$ has no purely imaginary roots together with Lemma 4.3 imply that $G$ admits a subalgebra isomorphic to the $ax+b$ Lie algebra or Grélaud’s algebra. Since $G$ is exponential, the associated Lie subgroup is simply connected and closed, and since it is exponential solvable, it is of type I. Therefore (1) $\implies$ (2) holds. \qed

**Proof of Theorem 4.1.** The proof of Theorem 4.1 is a direct consequence of Lemma 4.4 and Theorem 3.5

**References**

[1] David Bernier and Keith F. Taylor. Wavelets from square-integrable representations. *SIAM J. Math. Anal.*, 27(2):594–608, 1996.
[2] Ole Christensen. *An introduction to frames and Riesz bases*. Applied and Numerical Harmonic Analysis. Birkhäuser Boston, Inc., Boston, MA, 2003.
[3] Ole Christensen, Baiqiao Deng, and Christopher Heil. Density of Gabor frames. *Appl. Comput. Harmon. Anal.*, 7(3):292–304, 1999.
[4] Laurence Corwin and Frederick P Greenleaf. *Representations of Nilpotent Lie Groups and Their Applications: Volume 1, Part 1, Basic Theory and Examples*, volume 18. Cambridge university press, 1990.
[5] Stephan Dahlke, Gitta Kutyniok, Peter Maass, Chen Sagiv, Hans-Georg Stark, and Gerd Teschke. The uncertainty principle associated with the continuous shearlet transform. *Int. J. Wavelets Multiresolut. Inf. Process.*, 6(2):157–181, 2008.

[6] M. Duflo and Calvin C. Moore. On the regular representation of a nonunimodular locally compact group. *J. Functional Analysis*, 21(2):209–243, 1976.

[7] Hans G. Feichtinger and Peter Gröbner. Banach spaces of distributions defined by decomposition methods. *I. Math. Nachr.*, 123:97–120, 1985.

[8] Hans G. Feichtinger and K. H. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions. I. *J. Funct. Anal.*, 86(2):307–340, 1989.

[9] Gerald B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[10] Daniel Freeman and Darren Speegle. The discretization problem for continuous frames. *arXiv preprint arXiv:1611.06469*, 2016.

[11] Hartmut Führ. Admissible vectors for the regular representation. *Proc. Amer. Math. Soc.*, 130(10):2959–2970 (electronic), 2002.

[12] Hartmut Führ. *Abstract harmonic analysis of continuous wavelet transforms*, volume 1863 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005.

[13] Joachim Hilgert and Karl-Hermann Neeb. *Structure and geometry of Lie groups*. Springer Science and Business Media, 2011.

[14] Joseph W. Iverson. Subspaces of $L^2(G)$ invariant under translation by an abelian subgroup. *J. Funct. Anal.*, 269(3):865–913, 2015.

[15] Kenkichi Iwasawa. Topological groups with invariant compact neighborhoods of the identity. *Annals of Mathematics*, pages 345–348, 1951.