Dynamic Fuzzy $(\lambda, \eta)$–Function Rough Sets

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Abstract. By learning function $S$-rough sets, we find that transfer functions play the roles in practice are that define which elements into the set $Q$ and which elements left the set $Q$. But the changed elements have been mapped into what kinds of elements that are not very important, so it is not necessary to intensive study the transfer functions. Thus we can defines a dynamic fuzzy sets in the set $D$ to reconsider the fates of the elements of $Q$. By using function $S$-rough sets and dynamic fuzzy sets, the new definition of dynamic fuzzy $(\lambda, \eta)$–function rough sets are given, then the mathematical structure and characteristics of dynamic fuzzy $(\lambda, \eta)$–function rough sets are discussed. The structural relations of dynamic fuzzy $(\lambda, \eta)$–function rough sets and function $S$-rough sets are deeply deliberated. Dynamic fuzzy $(\lambda, \eta)$–function rough sets are the general form of function $S$-rough sets, then function $S$-rough sets are the special cases of dynamic fuzzy $(\lambda, \eta)$–function rough sets. Dynamic fuzzy $(\lambda, \eta)$–function rough sets are the new direction in the study of rough sets. In terms of practical application, dynamic fuzzy $(\lambda, \eta)$–function rough sets can achieve some better results than the other methods.

Keywords. Dynamic fuzzy sets; function $S$-rough sets; dynamic fuzzy $(\lambda, \eta)$–function rough sets; the relations.

1. Introduction
In 1982, rough set is firstly proposed by Pawlak [1], and has aroused the great interest of many scholars. A lot of research works on theory and application of rough sets have been done. Over the last decade or so, rough sets have become the new academic hotspot field of artificial intelligence, and have been widely used in many other fields, for example. Based on rough sets, Professor Kai-quan Shi also firstly gave the conception of function $S$-rough sets [2] and studied their relative theory. A series of theoretical researches and practical applications on function $S$-rough sets have been obtained that are very good results. The scholars who study on the function of $S$-rough sets have gradually increased with considerable scale. By learning function $S$-rough sets, we find that transfer functions play the roles in practice are that define which elements into the set $Q$ and which elements left the set $Q$. But the changed elements have been mapped into what kinds of elements that are not very important, so it is not necessary to intensive study the transfer functions. Thus we can defines a dynamic fuzzy sets [3-6] in the set $D$ to reconsider the fates of the elements of $Q$, so as to meet the definitions of function $S$-rough sets, and put forward the study on the theory of rough sets. At the same
time, this new method makes the definition of function S-rough sets simple and convenient, and easy to combine the new theory with the existing theoretical knowledge, and more conducive to in-depth research on rough sets.

In order to simplify the notation, $[\mu(x)]$ is denoted by $\mu$; $D(x)$ is denoted by $D$; $Q(x)$ is denoted by $Q$.

2. Preliminaries

2.1. Dynamic Fuzzy Sets [5]

Definition 2.1: Let $U$ is a domain of functions and $T$ is a nonempty set, a mapping that from $U \times T$ to $[0, 1]$ is called dynamic fuzzy set on $U$.

2.2. Rough Sets

Definition 2.2: Given the knowledge base $K = (U, R)$, for each subset $X \in U$ and an equivalence relation $R \in \text{ind}(K)$, the set $X$ can be divided by the description for the basic set of $R$. Considering two subsets:

$$R_-(X) = \{Y \in U / R : Y \subseteq X\}$$

$$R^+(X) = \{Y \in U / R : Y \cap X \neq \phi\}$$

which are called the lower and upper approximation of $X$ respectively.

The upper approximation and lower approximation can also be expressed by the following equation:

$$R_-(X) = \{x \in U : [x]_R \subseteq X\}$$

$$R^+(X) = \{x \in U : [x]_R \cap X \neq \phi\}$$

2.3. Function S-rough Sets [2]

Definition 2.3: Given $Q \subseteq D$, $Q^o$, $Q'$ are two direction $S$-function sets of $Q$, if $Q^o = Q \cup \{v \mid v \in D, v \in Q, f(v) = \mu \in Q\}$ and $Q' = Q - \{\mu | \mu \in Q, \tilde{f}(\mu) = v \notin Q\}$. $Q'$ is the $f$-extension and $\tilde{f}$-wither of $Q$, if $Q'^f = \{v \mid v \in D, v \in Q, f(v) = \mu \in Q\}$ and $Q'^\tilde{f} = \{\mu | \mu \in Q, \tilde{f}(\mu) = v \in Q\}$.

Definition 2.4: Given $Q \subseteq D$, $Q^*, Q'$ are two directions $S$-function sets(two direction singular function sets)of $Q$, $Q^* = Q^o \cup Q'$, then $(R,F)(Q^*)$ are respectively the lower approximation and upper approximation of $Q^*$, if $(R,F)(Q^*) = \cup[\mu] = \{\mu | \mu \in D, [\mu] \subseteq Q^*\}$ and $(R,F)(Q^*) = \cup[\mu] = \{\mu | \mu \in D, [\mu] \cap Q^* \neq \phi\}$.

3. Dynamic Fuzzy $(\lambda, \eta)$-Function Rough Sets

Function S-rough sets have two functions, one is function of moving in, and another is function of moving out. In fact, function S-rough sets are only rough sets on the dynamic fuzzy sets. By using function S-rough sets and dynamic fuzzy sets, the new definition of dynamic fuzzy $(\lambda, \eta)$-function rough sets are given, then the mathematical structure and characteristics of dynamic fuzzy $(\lambda, \eta)$-function rough sets are discussed. Dynamic fuzzy $(\lambda, \eta)$-function rough sets are more simple and easier to understand and accept.

Definition 3.1: Suppose that $D$ is a domain of functions, $\tilde{D} = \{\mu, \alpha_\beta(\mu, t)\}$, $\mu \in D$, $0 \leq \alpha_\beta(\mu) \leq 1$, $t$
is a dynamic fuzzy sets on $D$, and $Q = Q_{(\lambda, \eta)}(0) = \{\mu \in D | \alpha_{D}(\mu, 0) \geq \lambda \} = \{\mu_1, \mu_2, \ldots, \mu_m\} \subset D$ is the set of initial function when $t = 0$; then the set $Q_{(\lambda, \eta)}(t) = \{\mu \in D | \alpha_{D}(\mu, t) \geq \lambda \}$ and $\alpha_{D}(\mu, 0) \leq \eta$ and $t > 0$. $Q_{(\lambda, \eta)}^{-1}(t) = \{\mu \in Q | \alpha_{D}(\mu, t) \leq \eta \}$ and $t > 0$, where $0 \leq \eta \leq \lambda \leq 1$, are respectively called the dynamic fuzzy $(\lambda, \eta)$-moving in set, the dynamic fuzzy $(\lambda, \eta)$-moving out set and the dynamic fuzzy $(\lambda, \eta)$-swing set of $Q$ on the dynamic fuzzy sets $\tilde{D}$, $(Q_{(\lambda, \eta)}^0(t), Q_{(\lambda, \eta)}^{-1}(t), Q_{(\lambda, \eta)}^{0.5}(t))$ is called the dynamic fuzzy $(\lambda, \eta)$-migration group of $Q$ on the dynamic fuzzy set $\tilde{D}$.

Definition 3.2: Suppose that $D$ is a domain of functions, $Q = \{\mu_1, \mu_2, \ldots, \mu_m\} \subset D$ is the set of initial function when $t = 0$, $\tilde{D}$ is an dynamic fuzzy sets on $D$. $(Q_{(\lambda, \eta)}^0(t), Q_{(\lambda, \eta)}^{-1}(t), Q_{(\lambda, \eta)}^{0.5}(t))$ is called the dynamic fuzzy $(\lambda, \eta)$-migration group of $Q$ on the dynamic fuzzy sets $\tilde{D}$, then $Q^* = (Q - Q_{(\lambda, \eta)}^{-1}(t)) \cup Q_{(\lambda, \eta)}^0(t) \cup Q_{(\lambda, \eta)}^{0.5}(t)$ is called the dynamic fuzzy $(\lambda, \eta)$-migration set of $Q$ on the dynamic fuzzy set $\tilde{D}$.

Definition 3.3: Let $D$ is a domain of functions, $Q = \{\mu_1, \mu_2, \ldots, \mu_m\} \subset D$ is the set of initial function when $t = 0$, $\tilde{D}$ is an dynamic fuzzy sets on $D$. $Q^*$ is the dynamic fuzzy $(\lambda, \eta)$-migration set of $Q$ on the dynamic fuzzy sets, $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_L\}$ is the attribute set of $[\mu]$; $R$ is the equivalence relation of function which is defined on $D$, $[\mu]$ is the $R$-equivalence class of function, then $(R,D)(Q^*) = \{\mu, \mu \in D, [\mu] \subseteq D\}$ and $(R,D)(Q^*) = \{\mu, \mu \in D, [\mu] \cap Q^* \neq \phi\}$ are respectively called the dynamic fuzzy $(\lambda, \eta)$-lower approximation and the dynamic fuzzy $(\lambda, \eta)$-upper approximation.

Definition 3.4: The pair of $(R,D)(Q^*)$ and $(R,D)(Q^*)$ is called the dynamic fuzzy $(\lambda, \eta)$-function rough set of $Q \subseteq D$, which is denoted by $((R,D)(Q^*),(R,D)(Q^*))$; $BN_R$ $(Q^*)$, $(R,D)(Q^*)-(R,D)(Q^*)$ is called the boundary of the dynamic fuzzy $(\lambda, \eta)$-function rough set $((R,D)(Q^*),(R,D)(Q^*))$.

Example 3.1: Let six judges to assess the quality of the three papers, the results of the review of the judges are six functions, denoted as $g_1, g_2, g_3, g_4, g_5, g_6$. In order to facilitate, three papers are recorded as 1, 2, 3, and which is the domain. “Good” is recorded as 1 and “bad” is recorded as -1, which is the range. Then six functions defined as follows:

$g_1: x \to y$ 1→0 2→1 3→1;  $g_2: x \to y$ 1→1 2→0 3→0

$g_3: x \to y$ 1→0 2→0 3→0;  $g_4: x \to y$ 1→1 2→1 3→1

$g_5: x \to y$ 1→0 2→1 3→0;  $g_6: x \to y$ 1→1 2→0 3→1

And which is the domain of the functions $D = \{g_1, g_2, g_3, g_4, g_5, g_6\}$. $R$ is that the results have the same number of 0 and 1 are put into the same class, then the equivalence class about $R$ on $D$ are respectively $\{g_1, g_6\}, \{g_2, g_5\}, \{g_3\}, \{g_4\}$. Suppose $\Delta = \{\Delta_1 = \text{selecting results only have one “0”,} \Delta_2 = \text{selecting results only have one “1”,} \Delta_3 = \text{selecting results only have two “0”,} \Delta_4 = \text{selecting results only have two “1”,} \Delta_5 = \text{selecting results all are “0”,} \Delta_6 = \text{selecting results all are}$. 


“1”), then \{g_1, g_6\} have attributes\{ \alpha_1, \alpha_4\}, \{g_2, g_3\} have attributes\{ \alpha_2, \alpha_3\}, \{g_3\} has attributes\{ \alpha_5\}, \{g_4\} has attributes\{ \alpha_6\}, three of the six experts are the chairmen, so their opinions must be pay attention to, and which is \(Q = \{g_1, g_2, g_4\}\). But because there are the objections to the results of the selection, after a period of time, it is need to review the results. Due to practical problems, there are no time and energy to focused review three papers, so only assume that the original review comments are unchanged. But now the status of six experts has changed, and the value of their opinion has changed. \(Q\) will be naturally affected. At this time, the degree of attention of the six expert are need to reconsider, and need to build a dynamic fuzzy sets \(D\), i.e. \(\tilde{D}(0) = (g_1,1),(g_2,1),(g_3,0),(g_4,1),(g_5,0),(g_6,0)\), \(\tilde{D}(t) = (g_1,0.1),(g_2,1),(g_3,0),(g_4,0.7),(g_5,0.5)\), \(t > 0\). Given \(\lambda = 0.7, \eta = 0.2\), then by definition, 2.1, definition, 2.2, definitions 2.3 and definitions 2.4, it is easy to get \(Q_{0.7,0.2,0.5}(t) = \{g_6\}\), so \(Q^\alpha = \{g_2, g_4, g_5, g_6\}\) \(t > 0\) is the group of new chairman, then \(R(D)\) \(t\) \(Q^\alpha = \{g_2, g_4, g_5\}\), \(R(D)\) \(t\) \(Q^\alpha = \{g_1, g_2, g_4, g_5, g_6\}\), \(Bnr(Q^\alpha) = \{g_1, g_6\}\). Therefore, when papers are reviewed, the importance of \(g_2, g_4, g_5\) is the first to be considered, the selection results \(g_1\) and \(g_6\) are appropriate considered.

4. The Basic Properties of Dynamic Fuzzy \((\lambda, \eta)\)-Function Rough Sets

Theorem 4.1: Suppose that \(D\) is a domain of functions, \(Q = \{\mu_1, \mu_2, \ldots, \mu_n\} \subseteq D\) is the set of initial function when \(t = 0\), \(\tilde{D}\) is an dynamic fuzzy sets on \(D\), \(Q^\alpha\) is an dynamic fuzzy \((\lambda, \eta)\)-migration set of \(Q\) on the dynamic fuzzy sets \(\tilde{D}\), \((R(D))(Q^\alpha), (\bar{R}(D))(Q^\alpha))\) is called the dynamic fuzzy \((\lambda, \eta)\)-function rough set, then

1. \((R(D))(Q^\alpha) \subseteq (\bar{R}(D))(Q^\alpha)\);
2. \((R(D))\phi = \overline{\phi} = (\bar{R}(D))\phi, (R(D))(D) = D = (\bar{R}(D))(D)\);
3. \((R(D))\neg Q = \neg (\bar{R}(D))(Q), (R(D))\neg Q = \neg (\bar{R}(D))(Q)\);
4. \((R(D))((R(D))(Q^\alpha)) = (\bar{R}(D))((R(D))(Q^\alpha)) = (R(D))(Q^\alpha)\);
5. \((R(D))(\bar{R}(D))(Q^\alpha)) = (R(D))(\bar{R}(D))(Q^\alpha) = (\bar{R}(D))(Q^\alpha)\).

The above theorem 4.1 is easy to be proved, so is omitted.

5. The Relationship between Dynamic Fuzzy \((\lambda, \eta)\)-Function Rough Sets and Function S-Rough Sets [2]

Theorem 5.1: Let \(Q \subseteq D, g = \{g_1, g_2, \ldots, g_n\}\) is the family migration of the functions on \(D\), \(Q^g = \{\nu \in D|\nu \notin Q, g(\nu) = \mu \in Q\}\) is \(g\)-expand on \(Q\), then build a dynamic fuzzy set \(\tilde{D} = \{\mu, \alpha_B(\mu, t)\}, \mu \in D, 0 \leq \alpha_B(\mu, t) \leq 1, t \geq 0\} \subseteq D\), where

\[
\alpha_B(\mu, 0) = \begin{cases} 1 & \mu \in Q \\ 0 & \mu \notin Q \end{cases}, \quad \alpha_B(\mu, t) = \begin{cases} 1 & \mu \in Q \cup Q^g \\ 0 & \mu \notin Q \cup Q^g \end{cases}, \quad t > 0,
\]

\[\alpha_B(\mu, t) = \frac{1}{\eta} \int_0^t \alpha_B(\mu, s) ds, \quad 0 < \eta \leq 1. \]
are two characteristic functions, then \( \lambda, \eta \) \-characteristic functions, is the dynamic fuzzy
\( \mathbf{\mu} \) \= \( \mathbf{\alpha}_D(\mu, t) \geq \lambda \) and \( \mathbf{\alpha}_D(\mu, 0) \leq \eta \) and \( t > 0 \) \} is the dynamic fuzzy 
\( (\lambda, \eta) \)-moving in set, and when \( \lambda = 1, \eta = 0 \), \( Q_{(\lambda, \eta)}(t) = Q^\# \) and \( Q_{(\lambda, \eta)}(0) = Q \).

Theorem 5.2: Suppose that \( Q \subset D, \mathcal{G} = \{ g_1, g_2, ..., g_m \} \) is the family migration of the functions on \( D, Q^\# = \{ \mu \in D | g(\mu) = \mathbf{\nu} \notin Q \} \), then build a dynamic fuzzy set
\( \mathcal{D} = \{ (\mu, \alpha_D(\mu, t)) | \mu \in D, t \geq 0 \} \) on \( D \), where
\[
\alpha_D(\mu, 0) = \begin{cases} 0 & \mu \in Q \\ 1 & \mu \in D - Q \end{cases}, \quad \alpha_D(\mu, t) = \begin{cases} 0 & \mu \notin (D - Q) \cup Q^\# \\ 1 & \mu \in (D - Q) \cup Q^\# \end{cases}, \quad t > 0,
\]
then
\( Q_{(\lambda, \eta)}(t) = \{ \mu \in Q | \alpha_D(\mu, t) \leq \eta \) and \( t > 0 \} \) is the dynamic fuzzy \( (\lambda, \eta) \)-moving out set, and when \( \lambda = 1, \eta = 0 \), \( Q_{(\lambda, \eta)}(t) = Q^\# \).

Theorem 5.3: Suppose that \( Q \subset D, G = \{ g_1, g_2, ..., g_m \} \) and \( \mathcal{F} = \{ g_1, g_2, ..., g_m \} \) are the family migration of the functions on \( D, Q^\# = (Q - Q^\#) \cup Q^\# \) is bidirectional \( S \)-set on \( Q \), then build a dynamic fuzzy set
\( \mathcal{D} = \{ (\mu, \alpha_D(\mu, t)) | \mu \in D, t \geq 0 \} \) on \( D \), where
\[
\alpha_D(\mu, 0) = \begin{cases} 0 & \mu \in Q \\ 1 & \mu \in D - Q \end{cases}, \quad \alpha_D(\mu, t) = \begin{cases} 1 & \mu \in Q^\# \\ 0 & \mu \notin Q^\# \end{cases}, \quad t > 0,
\]
then
\( Q_{(\lambda, \eta)}(t) = \{ \mu \in D | \alpha_D(\mu, t) = 1 \) and \( \alpha_D(\mu, 0) = 0 \) and \( t > 0 \} \) is the dynamic fuzzy \( (\lambda, \eta) \)-swing set, and when \( \lambda = 1, \eta = 0 \), \( Q_{(\lambda, \eta)}(t) = Q^\# \).

Proof: By definition 2.1, when \( \lambda = 1, \eta = 0 \), it is easy to get \( Q_{(\lambda, \eta)}(0) = \{ \mu \in D | \alpha_D(\mu, t) \geq 1 \)
and \( \alpha_D(\mu, 0) \leq 0 \) and \( t > 0 \} = \{ \mu \in D | \alpha_D(\mu, t) = 1 \) and \( \alpha_D(\mu, 0) = 0 \) and \( t > 0 \} = Q^\# ; \)
\( Q_{(\lambda, \eta)}^{-1}(t) = \{ \mu \in Q | \alpha_D(\mu, t) \leq \eta \) and \( t > 0 \} = \{ \mu \in D | \alpha_D(\mu, t) = 0, \alpha_D(\mu, 0) = 1 \) and \( t > 0 \} = Q^\# . \)
Thus \( Q - Q_{(\lambda, \eta)}(0) \cup Q_{(\lambda, \eta)}^{-1}(t) = (Q - Q^\#) \cup Q^\# = Q^\# \) can be got, i.e. the original proposition is proved.

Theorem 5.4: Suppose that \( AS(Q^\#) = \{ \mu \in D | \mathbf{v} \in Q, g(\mu) = \mu \notin Q \} \cup \{ \mu \in D | \mu \notin Q, \mathbf{v} \notin Q \} \)
is the generated assistant set of the function of two direction \( S \)-rough sets \( (\mathcal{R}, \mathcal{G})(Q^\#), (\mathcal{R}, \mathcal{G})(Q^\#) \), which are recorded as
\( \mathcal{Q}_1 = \{ \mu \in D | \mathbf{v} \in Q, g(\mu) = \mu \notin Q \} \),
\( \mathcal{Q}_2 = \{ \mu \in D | \mu \notin Q \), \( g(\mu) = \mathbf{v} \notin Q \), build a dynamic fuzzy set
\( \mathcal{D} = \{ (\mu, \alpha_D(\mu, t)) | \mu \in D, t \geq 0 \} \) on \( D \), where
\[
\alpha_D(\mu, 0) = \begin{cases} 0 & \mu \in Q \\ 1 & \mu \notin Q \end{cases}, \quad \alpha_D(\mu, t) = \begin{cases} 1 & \mu \in Q^\# \\ 0 & \mu \notin Q^\# \end{cases}, \quad t > 0.
\]
when \( \lambda = 1, \eta = 0 \), \( Q_{(\lambda, \eta)}(0.5) = \{ \mu \in D | 0 < \alpha_D(\mu, t) < 1, t > 0 \} = Q^3 = \mathcal{Q}_1 \cup \mathcal{Q}_2 = AS(Q^\#) . \)

By theorem 5.1, 5.2 and 5.3, the function \( S \)-rough set \( ((\mathcal{R}, \mathcal{G})(Q^\#), (\mathcal{R}, \mathcal{G})(Q^\#)) \) of \( Q \subset D \) can
be converted into the dynamic fuzzy $(1,0)$–function rough set $(\bar{R},\bar{D})(Q^*)$, $(\bar{R},\bar{D})(Q^*)$ by constructing the dynamic fuzzy sets $\tilde{D}$ on $D$. Thus the function $S$–rough sets are the special case of the dynamic fuzzy $(\lambda, \eta)$–function rough sets. At the same time, the boundary of the function $S$–rough set $Bnr (Q^*) = (\bar{R},G)(Q^*) - (\bar{R},G)(Q^*)$ on $Q^* \subset D$ can be converted into the boundary of the dynamic fuzzy $(\lambda, \eta)$–function rough set $((\bar{R},D)(Q^*), (\bar{R},D)(Q^*))$.

By theorem 5.4, assistant sets of function $S$-rough sets is easy to be transformed to dynamic fuzzy $(\lambda, \eta)$–swing set, which will be transformed into the category of dynamic fuzzy $(\lambda, \eta)$–function rough sets to be discussed.

In conclusion, the mainly concepts of function $S$-rough sets can almost completely be equivalent of the ones of dynamic fuzzy $(\lambda, \eta)$–function rough sets. In this way, the function $S$-rough sets can be forward pushed a step, but also the theory and practice can be closer.

6. Summary
The dynamic fuzzy $(\lambda, \eta)$–function rough sets are the promotion of the function $S$-rough sets, and which are the needs of theoretical research should be in accordance with the facts, and which are easier to be understood and mastered. In this paper, only the basic concepts on the dynamic fuzzy $(\lambda, \eta)$–function of rough sets and the function $S$-rough sets are discussed, on the relationship between them in other areas haven’t been deeply studied due to the length of the paper. Next, the practical application value of the dynamic fuzzy $(\lambda, \eta)$–function rough sets should be studied in the further.

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