Integrable Model of Topological SO(5) Superfluidity

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Assisted by general symmetry arguments and a many-body invariant, we introduce a phase of matter that constitutes a topological SO(5) superfluid. Key to this finding is the realization of an exactly solvable model that displays some similarities with a minimal model of superfluid 3He. We study its quantum phase diagram and correlations, and find exotic superfluid as well as metallic phases in the repulsive sector. At the critical point separating trivial and non-trivial superfluid phases, our Hamiltonian reduces to the globally SO(5)-symmetric Gaudin model with a degenerate ground manifold that includes quartet states. Most importantly, the exact solution permits uncovering of an interesting non-pair-breaking mechanism for superfluids subject to external magnetic fields. Non-integrable modifications of our model lead to a strong-coupling limit of our metallic phase with a ground state manifold that shows an extensive entropy.

Introduction — Exactly-solvable models of quantum many-body systems are theoretical constructions key to uncover physical mechanisms or effects resulting from competing interactions [1–3]. The case of spin-1/2 particles with SO(5)-symmetric p-wave interactions is particularly compelling because it can give rise to non-trivial spin-triplet Cooper-pair topological phases with no equivalent in SU(2)-symmetric couplings. For instance, it is well-known that the emergent SO(3) L symmetry in liquid 3He, contained in SO(5), is responsible for topological classification of the defects of its exotic superfluid phases [4]. Similar mechanisms could be at play in unconventional uranium-based metallic ferromagnetic superconductors, where strong external magnetic fields can even revive superconductivity [5]. A theoretical understanding of these mechanisms is therefore a prerequisite to engineering materials or synthetic matter with exotic magnetic superfluid behavior [6].

SO(5)-symmetric models have a long history in nuclear physics as a description of isovector (isospin 1) pairing between protons and neutrons. The earliest version of an integrable model consisting of a unique SO(5) algebra, describing a proton-neutron system, was presented in Refs. [7, 8]. The generalization of the exact solution to many SO(5) copies or, equivalently, to many non-degenerate single particle orbitals arose as an extension of the Richardson-Gaudin (RG) models [9, 10] to rank 2 algebras [11, 12]. In condensed matter physics, the systems closest to admitting an SO(5)-symmetric representation are arguably superfluid 3He [13–16] and non-p-wave systems [17–19], but, as far as we know, there are no corresponding integrable interacting models.

In this work, we study the quantum phase diagram of the fermionic (c_{kσ}, c_{kσ}^\dagger) SO(5) Hamiltonian expressed in momentum (k) space as

\[ H = \sum_{k,\sigma=\uparrow,\downarrow} \epsilon_k c_{k\sigma}^\dagger c_{k\sigma} - \sum_{k,k'} \Delta_{kk'} \left( \frac{T_k^+ \cdot T_{k'}^- + T_k^- \cdot T_{k'}^+}{2} \right) - \sum_{k,k'} W_{kk'} S_k^z S_{k'}^z - \sum_{k,k'} V_{kk'} N_k N_{k'} - \hbar \sum_k S_k^z \]  

The operator \( T_{\mu k}^\pm \) creates a spin-triplet fermion pair \( (k, -k) \) with spin-projection \( \mu = \pm 1, 0 \). Magnetic Heisenberg \( \langle S_k \cdot S_{k'} \rangle \) and density-density \( \langle N_k N_{k'} \rangle \) interactions complete the minimal set required to close an SO(5) algebra, with a fermionic representation [20]

\[ T_{\mu k}^- = \frac{(-1)^{\mu+1}}{(2 \delta_{\mu,\pm 1} + \sqrt{2} \delta_{\mu,0})} \sum_{\sigma,\sigma'} c_{-k\sigma'} (i \sigma^\mu \sigma') c_{k\sigma}, \]

\[ S_k^\mu = \frac{1}{2} \sum_{\sigma,\sigma'} (c_{k\sigma'}^\dagger \sigma^\mu \sigma c_{k\sigma} + c_{-k\sigma}^\dagger \sigma^\mu \sigma' c_{-k\sigma'}), \]

\[ N_k = \sum_\sigma (c_{k\sigma}^\dagger c_{k\sigma} + c_{-k\sigma}^\dagger c_{-k\sigma}), \]

where \( \sigma^\pm = \sigma \pm i \sigma_y, \sigma^0 = \sigma_z \) are Pauli matrices and \( T_{\mu k}^- = (T_{\mu k}^-)^\dagger \). For an appropriate choice of (separable) interactions \( \Delta_{kk'} = W_{kk'} = 4V_{kk'} \), Hamiltonian (1) is exactly-solvable independently of spatial dimensionality.

In the attractive pairing sector, the model displays trivial and non-trivial topological superfluid phases, separated by a critical point that is globally SO(5)-symmetric. At this point the ground state manifold is macroscopically degenerate with pair and quartet correlations. The application of a magnetic field \( h \) [21] leads to a remarkable magnetized superfluid, where spin-triplet pairs transition, without pair breaking, between different spin projections as in the B to A first-order transition in superfluid 3He. This mechanism is absent in SU(2) pairing models. Finally, the repulsive sector shows a metallic phase whose strong-coupling limit is adiabatically connected to a flat band model with exponentially-degenerate ground states, giving an extensive entropy similar to holographic models of strange metals [22].

Exactly-solvable SO(5) model — RG integrable models are defined by a set of integrals of motion \( R_k \) fulfilling the integrability condition \( \{ R_k, R_{k'} \} = 0 \), such that their linear combination realizes a Hamiltonian as (10). The exact eigenpectrum of the integrals of motion, and co-
responding Hamiltonian, may be found with algebraic complexity by solving the RG ansatz equations.

These models may be formulated in terms of a generalized Gaudin algebra \([3, 23–25]\). Starting from the rational SO(5) RG integrals of motion \([12]\), we form the set

\[
R_k = \left(1 + \frac{\Delta}{2}\right)N_k - \frac{\Delta}{2} \sum_{k' \neq k} Z_{kk'} \vec{T}_k \cdot \vec{T}_{k'},
\]

where \(N_k = N_k/2 - 1\), and \(\vec{T}_k \cdot \vec{T}_{k'}\) is the SO(5) Gaudin interaction. The function \(Z_{kk'} = \frac{2}{J_{kk'} - J_{kk'}}\) is a particular case of a more general function interpolating between hyperbolic and trigonometric SU(2) RG models \([3, 26, 27]\). The parameters \(\Delta\), \(q\), and \(\eta_k\) are arbitrary real numbers with the restriction that \(\eta_k \neq \eta_{k'}\) for \(k \neq k'\) to avoid singularities.

In Eq. (3) and for the remainder of this paper, sums are taken over momenta with \(k_x > 0\) to avoid double counting. Each pair of momenta \(\pm k\) labels a level with a corresponding irreducible representation (irrep) of SO(5) characterized by seniority \(\nu_k\) and reduced spin \(s_k\) quantum numbers. The \(l\) levels correspond to a lattice with \(L = 2l\) sites, since each level incorporates two modes in \(k\)-space.

Eigenvalues and eigenvectors of the integrals of motion are determined by two sets of spectral parameters: pairs \(e_{\alpha}, \alpha = 1, \ldots, N_c\), and wavefunction parameters \(\omega_{\beta}, \beta = 1, \ldots, N_c\), that are roots of the two sets of RG (Bethe) equations

\[
\frac{1}{q} = 2 \sum_{\alpha' \neq \alpha} Z_{\alpha' \alpha} - \sum_{\beta} Z_{\beta \alpha} + \sum_k \left(\frac{\nu_k}{2} - 1 + s_k\right) Z_{k \alpha},
\]

and

\[
-\frac{\Delta}{q} = -\sum_{\beta' \neq \beta} Z_{\beta' \beta} + \sum_k Z_{k \beta} Z_{k \beta},
\]

with \(Z_{\alpha' \alpha} = Z(e_{\alpha'}, e_\alpha), Z_{\beta \alpha} = Z(\omega_{\beta}, e_\alpha),\) and \(Z_{k \alpha} = Z(\eta_k, e_\alpha)\). The number of pairs \(N_c\) is equal to the number of spin-1 fermion pairs and relates to the total fermion number as \(N = 2N_c + \sum_k \nu_k\). The number of wavefunction parameters is \(N_c = N_c + \sum_k (S^x_k + s_k)\). We emphasize that while the dimension of the Hilbert space grows exponentially with the number of levels \(L\) and \(N_c\), the complexity of the exact solution grows only linearly, allowing exact treatment of very large systems.

Numerical solution of the RG equations must navigate the singularities that arise whenever the spectral parameters approach each other or the level parameters \(\eta_k\). To avoid these singularities, we add modulated imaginary parts to \(\eta_k\) while iteratively solving to a desired coupling \(q\) and then incrementally remove these to achieve physical results \([27]\).

In terms of the variables \(e_\alpha\) and \(\omega_\beta\), the integrals of motion \(R_k\) have eigenvalues

\[
r_k = \frac{\nu_k}{2} - 1 + q s_k \sum_{\beta} Z_{k \beta} + q \left(1 - \frac{\nu_k}{2} - s_k\right) \sum_{\alpha} Z_{k \alpha}
- q \sum_{k' \neq k} \left[\left(\frac{\nu_k}{2} - 1\right) \left(\frac{\nu_{k'}}{2} - 1\right) + s_k s_{k'}\right] Z_{k k'}.\]

To obtain the corresponding eigenstates, we need operators

\[
S^+_{\beta} = \sum_k Z_{k \beta} S^x_k, \quad T^{+\alpha}_{\mu} = \sum_k Z_{k \alpha} T_{\mu k},
\]

and \(\hat{T}^+_{\alpha}\), defined by its action on \(T^{+\alpha}_{\mu}\):

\[
T^{+\alpha}_{\mu+1} \hat{T}^+_{\alpha} = \delta_{\alpha \alpha'} \left(\frac{T^{+\alpha}_{\mu+1}}{\mu} + 1\right)\]

Then, the eigenstates can be written as \([20]\)

\[
|\Psi\rangle = \prod_{\alpha=1}^{N_c} T^+_{-1} \prod_{\beta=1}^{N_c} \left(S^+_{\beta} - \sum_{\alpha'=1}^{N_c} \hat{T}^+_{\alpha'} Z_{\alpha' \beta}\right) |\Lambda\rangle,
\]

where \(|\Lambda\rangle\) is a vacuum state characterized by \(\nu_k\) and \(s_k\) with \(S^x_{\beta} |\Lambda\rangle = T^{+\alpha}_{\mu} |\Lambda\rangle = 0\). In most cases, the ground state is built from the empty \((\nu_k = s_k = 0\) for all \(k\)) vacuum \(|0\rangle\). The exception occurs when sufficiently strong repulsive pairing couplings break pairs.

When \(e_k = \eta_k\), \(\Delta_{kk'} = W_{kk'} = 4V_{kk'} = (g/L)\eta_k\eta_{k'}\), and \(h = 0\), Hamiltonian (1) can be written as a linear combination of the integrals of motion \([20]\)

\[
H = \frac{2}{1 - q \sum_k \eta_k} \sum_k \eta_k R_k + \text{constant} = \left(1 - \frac{g}{g_c}\right) \sum_k \eta_k N_k - \frac{g}{L} \sum_{k, k'} \eta_k \eta_{k'} \vec{T}_k \cdot \vec{T}_{k'} + \frac{gL}{g_c}.
\]

Here, we define \(qL = -g/(1 - g/g_c)\) with \(g_c^{-1} L = \sum_k \eta_k\) and set \(\Delta = 0\) (letting \(\Delta \neq 0\) has the effect of assigning a different kinetic energy to spin-up versus spin-down fermions). At \(g = g_c\), \(q\) becomes singular and \(H\) reduces to the (globally) SO(5)-symmetric Gaudin model, as it is evident from the second line in (10). Adding a uniform magnetic field \(h\) does not break integrability and is discussed below.

Using the eigenvalues \(r_k\) from Eq. (6), the total energy for a system of density \(\rho = N/L\) is

\[
\mathcal{E}(N) = \frac{2 \sum_k \eta_k r_k - 1 - q \sum_k \eta_k}{1 - q \sum_k \eta_k},
\]

up to a constant dependent on the vacuum state \(|\nu\rangle\) \([20]\) while total energy per site (energy density) will be indicated by \(e = \mathcal{E}(N)/L\). Other observables may also be computed from the integrals of motion using the Hellmann-Feynman theorem, i.e., momentum distribution \((N_k) = 2(r_k - q \delta r_k/\delta q + 1)\).
Hamiltonian (10) displays a particle-hole symmetry $\mathcal{P}$. Under the map $\mathcal{P} |c_{k\beta}^\dagger = |c_{-k\bar{\beta}}^\dagger$, $\mathcal{P} |c_k = -|c_{-k}^\dagger$, $H$ transforms as $\mathcal{P}^2 H(\rho, g) |\mathcal{P} = \alpha H (2 - \rho, \alpha g)$ up to an additive constant, where $\alpha^{-1} = 2g/g_c - 1$. At $g = g_c$, $\alpha = 1$ and $H$ is particle-hole symmetric.

Quantum phase diagram: One-dimensional case

To illustrate the physics arising from our SO(5) model we now work in one spatial dimension. The momenta for periodic boundary conditions are $k_j = \frac{2\pi j}{L}$, $j = -L/2, -L/2 + 1, \ldots, L/2 - 1$, which leaves isolated modes at $k = 0$ and $k = -\pi$ that cannot participate in pairing interactions and therefore do not correspond to RG levels. When using these boundary conditions, we ignore all interactions on the $k = -\pi$ mode to preserve integrability. The effect of the ignored interactions diminishes in the thermodynamic limit ($L \to \infty$ with $\rho$ and $g$ fixed). To avoid this finite-size effect, the majority of our calculations utilize antiperiodic boundary conditions, under which all momenta come in pairs ($|k|, -|k|$) corresponding to RG levels: $k_j = \frac{\pi}{L}(2j + 1)$, $j = -L/2, -L/2 + 1, \ldots, L/2 - 1$.

We linearize the dispersion close to the Fermi points by choosing $\eta_k = k$ and $\epsilon_k = |k|$ (in units of $\hbar v_F$ where $\hbar$ is Planck’s constant and $v_F$ the Fermi velocity). Because $\eta_k = -\eta_{-k}$, interaction coefficients $\Delta_{kk'} = \eta_k \eta_{k'}$ have the antisymmetry necessary for $p$-wave pairing: $\Delta_k = \Delta_{-k-k'} = -\Delta_{kk'}$. In coordinate space, the Fourier-transformed coefficients $\Delta_{ij}$ linking sites at $r_i$ and $r_j$ decay as $(r_i - r_j)^{-1}$ [28, 29], and show alternating sign $(-1)^{i-j}$ [20].

We next examine the various phases that emerge in the phase diagram of the SO(5) RG Hamiltonian (10).

Topological superfluid phase — For attractive couplings $g > 0$, the ground state of Hamiltonian (10) is a superfluid of spin-triplet pairs. At the density-independent critical coupling $g = g_c$, the system undergoes a topological phase transition with an accompanying change in occupation number ($N_k$) around zero momentum ($k = 0$), as seen in the inset of Fig. 1. The system transitions from a weak-pairing topologically non-trivial SO(5) superfluid into a strong-pairing trivial superfluid gapped phase. The transition is signaled by a divergence in $\partial^2E_0/\partial g^2$, the second-order derivative of the ground state energy density. (Figure 2 illustrates this along with distribution of spectral parameters in the complex plane).

To understand the topological nature of these superfluid phases, we need a many-body (bulk) topological invariant distinguishing them. Ref. [28] introduces a fermion parity switch for spinless fermions that distinguishes $p$-wave topological phases of an SU(2) model. Our SO(5) model consists of spinful fermions and therefore requires a generalization of the fermion parity to $P_N(\phi) = \text{sign} (c_0^{\text{even}}(\phi) - c_0^{\text{odd}}(\phi))$, (12)

where the ground-state energies are defined as $E_0^{\text{even}}(\phi) = E_0^0(N)$ and $E_0^{\text{odd}}(\phi) = \frac{1}{2}(c_0^0(N + 2) + c_0^0(N - 2))$ for fermion number $N$ divisible by four, such that the $N\pm 2$-particle states have $N_l = N_i$ odd. This differs from the SU(2) case (where $c_0^0$ is the average of $N\pm 1$-particle energies) due to the spin-degeneracy of the $k = 0$ mode. The quantity $\phi = 0(2\pi)$ represents periodic (antiperiodic) boundary conditions and corresponds to enclosing a flux $\Phi = \phi\rho|\pi g_0/2$ in a ring geometry with anomalous flux quantum $\Phi_0$ [28, 29].

In the topologically trivial phase ($g > g_c$), $P_N(\phi) = 1$ for both periodic and antiperiodic boundary conditions (Fig. 2). For $g < g_c$, a parity switch is observed, with $P_N(0) = -1$ and $P_N(2\pi) = 1$. This can be linked back to occupation of the zero-momentum state that exists for $\phi = 0$: in the topologically non-trivial phase, it is energetically advantageous to occupy both $|\sigma = \uparrow, \downarrow\rangle k = 0$ single-particle states instead of forming a pair. In the pairing-dominated trivial phase, the $k = 0$ states are vacated in favor of an additional pair at finite momentum.

Unlike what is seen in other SU(2) RG models, there is a macroscopic degeneracy at the critical point involving multiple states from each sector with fixed $N$ and $M = \sum_k S_k^z$, with an accompanying global SO(5) symmetry generated by operators $I^\alpha = \sum_k I^\alpha_k$ where $I^\alpha_k$, $\alpha = 1, \ldots, 10$, is any generator of the SO(5) algebra in Eq. (2). At $g = g_c$, Hamiltonian (10) becomes the SO(5) Gaudin model $\sum_{k,k'} \eta_k \eta_{k'} \tilde{T}_k \tilde{T}_{k'}$, and the ground-state solutions to the RG equations have all pairings $\epsilon_{\alpha} = 0$. Those equations then simplify to a single set for variables $\omega_{\beta}, \sum_{\beta' \neq \beta} Z_{\beta' \beta} = \sum_k s_k Z_{\beta k}$, $\beta = 1, \ldots, N_{\omega}$. Each independent solution corresponds to a degenerate eigenstate. The entire energy spectrum at this point can be classified according to the degenerate SO(5) global irreps constructed from the coupling of the $l$ SO(5)$k$ irreps $\{n_k, s_k\}$ of each level. The chain decomposition $SO(5) \supset U(2) \supset U_\sigma(1)$ [30] classifies the complete set of eigenstates in terms of the fermion number $N$ and spin content $S$ in each global irrep. The
FIG. 2. Second derivative of the ground-state energy density $e_0$ with respect to coupling for a mean-field solution [20] with $L = 2000$ (solid line), and the exact solution at $L = 128$, both at quarter filling ($\rho = 1/2$). Lower insets: the variables $e_0$ (diamonds) and $\omega_0$ (dots) are plotted with their imaginary parts on the $y$-axis and their real parts on the $x$-axis at $g/g_c = 0.9$ (left) and 1.1 (right). An animation of these spectral parameters as a function of $g$ is included in the supplemental material [20]. Upper insets: energy difference between odd and even sectors is plotted as a function of boundary condition $\phi$ for a quarter-filled system with 32 fermions at $g/g_c = 0.7$ (left) and 1.75 (right).

Wavefunctions constituting the ground state irrep are defined in terms of $S = 0$ quartet creation operator, $Q^+ = \sum_{k,k'} (T^{+}_{1k}T^{-}_{1k'} + T^{+}_{1k'}T^{-}_{1k} - T^{+}_{0k}T^{-}_{0k})$, $S = 1$ global pair operators $T^+ = \sum_k T^+_{0k}$, and spin lowering operator $S^- = \sum_k S^-_{0k}$. For an even number of particles $N \leq L$ ($N > L$ states can be determined by particle-hole transformation), these states are

$$|N_Q, S, M\rangle = (S^-)^S-M (Q^+)^{N_Q} (T^+)^{S} |0\rangle. \quad (13)$$

Since $N = 4N_Q + 2S$, the possible values of spin are $S = N/2, N/2 - 1, \ldots, 1$ or 0, with $S = 0$ representing the pure quartet state. From this, we find the degeneracy of the even $N$ ground-state manifold: $d_{N,M}^{even} = \frac{\min(N,2L-N)-2|M|}{4} + 1$, where $[x]$ is the largest integer less than or equal to $x$. The energy of these states is $\mathcal{E}^{even} = -3\hbar^2 \sum_k k^2$.

The $N+1$ ($N$ even) particle ground-state irrep has $N$ particles in a wavefunction of the form (13) with spin $S_e$ and one unpaired particle in the lowest momentum level $k_m$, giving total spin $S = S_e \pm 1/2$ with possible values $S = N/2, N/2 - 1, \ldots, 1/2$. The number of particles is then $N = 4N_Q + 2S_e + 1$. From the available spins and the additional two-fold degeneracy arising from the two momenta ($\pm k_m$) of the unpaired particle, the degeneracy of the odd-sector ground-state subspace is: $d_{N,M}^{odd} = \frac{\min(N,2L-N)-2|M|}{2} + 1$. The energy of these states $\mathcal{E}^{odd} = \mathcal{E}^{even} + \hbar \omega_{k_m} (1 + \frac{\hbar^2}{2m^2} k_{m}^2)$ simplifies to the even-$N$ energy plus the kinetic energy of the unpaired fermion in the thermodynamic limit.

The presence of quartets in a Hamiltonian such as (10) deserves mention. We are only aware of the significance of quartet correlations in atomic nuclei [31, 32] and in exotic phases of cold spin-3/2 fermionic atoms [17, 19]. It is important, then, to establish the interactions that take our system away from its $g = g_c$ critical point and stabilize a quartet, as opposed to a paired, ground state. To this end, we compare the quartet, $\Delta_4(N) = \Delta_0(N + 2) + \Delta_0(N - 2) - 2\Delta_0(N)/2$ (see for example [33]), and paired, $\Delta_2(N) = \Delta_0(N + 1) + \Delta_0(N - 1) - 2\Delta_0(N)/2$, gaps in our Hamiltonian (10) as a function of $g$. If $\Delta_4(N) \sim \Delta_2(N)$, we say that there are significant quartet correlations in the ground state for that value of $g$. Our analysis indicates that quartet correlations become more relevant in the repulsive sector and, in the attractive sector, for pairing-only (non-integrable) interactions [20].

SO(5) magnetic superfluid – An interesting physical mechanism emergers when our SO(5) system (10) is subject to an external magnetic field $\hbar$ as in Eq. (1). At low temperatures and pressures superfluid $^4$He, known to have both $p$-wave pairing and ferromagnetic interactions [13], displays transitions between non-magnetic (B) and magnetic (A) superfluid phases as function of an applied magnetic field. The A and B phases of superfluid $^4$He are associated with the mean-field wavefunctions proposed by Anderson, Brinkman, and Morel (ABM) and Balian and Werthamer (BW), respectively [13]. The BW state is a simple generalization of BCS principles to spin-triplet (rather than spin-singlet) pairs, and in the SO(5) language is a superposition of $T^0_{-1}$, $T^0_{0}$ and $T^+_{1}$ operators acting on the vacuum. The ABM state is structured similarly, but allows only like-spin fermion pairs, ruling out the channel generated by $T^0_{0}$. Experimentally, it is known that in absence of a magnetic field, the B phase is the only possible superfluid at zero temperature. With the addition of a magnetic field, both phases become accessible at zero temperature along with the spin-polarized superfluid A1 phase [34].

Interestingly, our model demonstrates a series of first-order magnetically-driven transitions between different spin-triplet superfluids with no pair-breaking, which may provide insight into magnetic superfluidity. To determine the ground state energy of the Hamiltonian (10), it suffices to find $\mathcal{E}_0(N, M)$, the lowest energy of the $\hbar = 0$ Hamiltonian for each possible value of $S^z$ (a conserved quantity) and determine which value of $M$ gives the lowest total energy $\mathcal{E}_0(h) = \min_M (\mathcal{E}_0(N, M) - hM)$. This process simplifies for $2g > g_c$, since above this coupling we find the ground state has no unpaired fermions for
any value of $h$. From this formula, it is clear that the magnetization $M_z = \partial E_0/\partial h$ is equal to $M$. Rather than breaking pairs, the magnetic field changes the balance of $-1$, 0, and 1 pairs. This manifests as a series of first-order phase transitions as $M_z$ jumps between integers with the same parity as $N_c$, as illustrated in Fig. 3. The minimum value of $h$ for nonzero magnetization goes to zero in the thermodynamic limit, while the final transition occurs at a value, $h = h_c$, that remains finite in that limit. A similar type of mechanism may be at play in superfluid $^3$He leading to the emergence of the A1 phase. Crucially, the $h \neq 0$ ground states of our model are merely the lowest-energy solutions to the $h = 0$ problem at the same coupling $g$ in a sector with $N_ \uparrow \neq N_ \downarrow$, and so shares topological and superfluid properties with the $h = 0$ ground state at the same coupling $g$.

This non pair-breaking mechanism, already encoded in the exact solution, can be modeled at the mean-field level by introducing the SO(5) generalized coherent state [35]

$$\psi = e^{\sum_k z_k (x_1 T_{0k} + x_2 T_{1k}^+ + x_3 T_{2k}^-)} |0\rangle,$$

where $x_1^2 + x_2^2 + x_3^2 = 1$ and $\{z_k\}$ are variational parameters [20]. As $\{x_{1,2,3}\}$ goes from 0 to 1, this state goes from having only $M = 0$ pairs through a state with a mixture of all three pairing channels, similar to the BW state, to a state with only like-spin pairs, similar to the ABM wavefunction.

Metallic phases — For repulsive couplings ($g < 0$), the ground state has a momentum distribution with a discontinuity at the Fermi momentum $k_F$ (Fig. 1), suggesting a ground state almost identical to a noninteracting Fermi gas $|\psi_{\text{nonint}}\rangle = \frac{1}{\sqrt{N}} \prod_{k=-k_F}^{k_F} |c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger |0\rangle$ with energy $E_{\text{nonint}}(N) = \langle \psi_{\text{nonint}} | H | \psi_{\text{nonint}} \rangle$. This discontinuity persists even for strongly repulsive couplings, unlike what is usually observed in an interacting Fermi liquid [3, 36]. In the thermodynamic limit, the ground state energy density $e_0$ converges to $e_{\text{nonint}} = \frac{3}{4} \rho^2 - \frac{3}{4} \rho^4 + O(1/L)$ [20].

One may wonder whether (10) has a flat-band limit in the strong coupling ($g \rightarrow -\infty$) limit. The SU(2) RG model shares similar metallic properties with the SO(5) model for low couplings, but the flat-band Hamiltonian $H_{\text{plane}}$ has an exponentially degenerate ground-state manifold [20]. This limiting case has been studied, for instance, in fractional quantum Hall liquids [37], and its importance lies in the non-Fermi liquid behavior that manifests due to a high density of states near the ground state. Due to the presence of effective single-particle terms in the interaction, the SO(5) model in (10) does not exhibit high degeneracy in this limit. Instead, a level crossing occurs at a nonuniversal coupling where the ground state gains a nonzero seniority independent of system size [20]. By removing all single-particle terms, one arrives at a special case of the Hamiltonian (1) with an exponentially degenerate ground state in the flat band (pure interaction) limit. A detailed discussion of this behavior is beyond the scope of this paper [20].

**Concluding Remarks** — We have presented an exactly-solvable model displaying SO(5) topological superfluidity. Its relevance lies in providing a new non-pair-breaking mechanism for magnetic superfluids, of relevance for liquid $^3$He or other exotic spin-triplet $p$-wave superfluids. At a critical coupling separating trivial and non-trivial topological superfluids, the model reduces to an (global) SO(5) Gaudin Hamiltonian. These phases show quartet correlations that become more prevalent as magnetic and density interactions are quenched. The repulsive phases of the model are also of interest, in particular, in relation to non-Fermi liquid behavior; they deserves further study. Finally, we would like to make connection to a seemingly unrelated phenomenon. The positive semi-definite (frustration-free) Haldane-Rezayi Hamiltonian [38, 39] $H = \sum_{j<L} J_j$, $J_j = \frac{1}{2} \sum_{k,k'} \eta_k \eta_{k'} \langle T_{j,k}^+ \cdot \hat{T}_{j,k'}^\dagger \rangle$ with $T_{0,k} = (c_{j,k,k}^\dagger c_{j,-k,k}^\dagger + c_{j,k,k}^\dagger c_{j,-k,k}^\dagger)/\sqrt{2}$, $T_{1,k} = c_{j+k,k}^\dagger c_{j-k,k}^\dagger$, $T_{-1,k} = c_{j+k,k}^\dagger c_{j-k,k}^\dagger$, defined in a cylinder $(k,k' \in [-j,j] \text{ are angular momenta indexes})$, stabilizes a gapless zero mode at filling fraction $\nu = 1/2$ representing a non-Abelian fractional quantum Hall trial state [40]. We have shown that (positive semi-definite) Hamiltonian $H_j$ is an element of SO(5) but, as a corollary of this work, it is not integrable à la RG. As it is a (repulsive) pairing-only Hamiltonian, it is expected that quartet correlations become relevant. Interestingly, each $H_j$ has a macroscopically degenerate zero-energy subspace and the intersection of their kernels result into the Haldane-Rezayi state.
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I. The SO(5) algebra: commutation relations and fermionic representation

The generators of SO(5) satisfy the commutation relations

\[ [S^z_k, \vec{T}^+_k] = \delta_{k,k'} \begin{pmatrix} -T^+_{-1k} \\ 0 \\ T^+_{1k} \end{pmatrix} \] (1)

\[ [S^z_k, S^\pm_{k'}] = \pm \delta_{k,k'} S^z_k \] (2)

\[ [S^+_k, \vec{T}^+_k] = \sqrt{2} \delta_{k,k'} \begin{pmatrix} T^+_{0k} \\ T^+_{1k} \end{pmatrix} \] (3)

\[ [S^-_k, \vec{T}^-_k] = 2 \delta_{k,k'} S^z_k \] (4)

\[ [N_k, \vec{T}^-_k] = 2 \delta_{k,k'} \vec{T}^+_k \] (5)

\[ [T^-_{-1k}, \vec{T}^+_k] = \frac{1}{\sqrt{2}} \delta_{k,k'} \begin{pmatrix} \sqrt{2}(1 + S^z_k - \frac{1}{2} N_k) \\ -S^z_k \\ 0 \end{pmatrix} \] (6)

\[ [T^-_{0k}, \vec{T}^+_k] = \frac{1}{\sqrt{2}} \delta_{k,k'} \begin{pmatrix} -S^-_k \\ \sqrt{2}(1 - \frac{1}{2} N_k) \\ -S^z_k \end{pmatrix} \] (7)

\[ [T^-_{1k}, \vec{T}^+_k] = \frac{1}{\sqrt{2}} \delta_{k,k'} \begin{pmatrix} 0 \\ -S^-_k \\ \sqrt{2}(1 - S^z_k - \frac{1}{2} N_k) \end{pmatrix} \] (8)
with all other commutators between these operators equal to zero. Here, we use notation $\vec{T}^\pm_k = (T^\pm_{\pm1k}, T^\pm_{\pm0k}, T^\pm_{1k})$ for the spherical components of the vector operator.

These generators form four (non-commuting) SU(2) subalgebras on each level: one from the spin operators $\{S^+_k, S^-_k, S^z_k\}$, and one from each pair creation operator:

$$\left\{ T^\pm_{\mu k}, T^\mp_{\mu k}, \frac{1}{2} N_k - 1 + \mu S^z_k \right\}, \quad \mu \in \{-1, 0, 1\}.$$  \hspace{1cm} (10)

The SO(5) algebra has Casimir operators

$$T^2_k = T^+_k \cdot T^-_k + T^-_k \cdot T^+_k + S^2_k + \left( \frac{N_k}{2} - 1 \right)^2.$$  \hspace{1cm} (11)

where $S^2_k = \frac{1}{2} (S^+_k S^-_k + S^-_k S^+_k) + S^z_k S^z_k$. These commute with all elements of the algebra, and have eigenvalues for each level determined by seniority and reduced spin quantum numbers $\nu_k$ and $s_k$:

$$\langle T^2_k \rangle = s_k (s_k + 1) + \left( \frac{\nu_k}{2} - 1 \right)^2 - 3 \left( \frac{\nu_k}{2} - 1 \right) \equiv t^2_k.$$  \hspace{1cm} (12)

The generators of a local (acting on modes at $\{+k, -k\}$) SO(5) algebra can be written in a fermionic ($c^\dagger_{k\sigma}$) representation as

$$T^\pm_{\pm1k} = c^\dagger_{k\sigma \pm} c^\dagger_{-k\sigma \mp} = (T^\pm_{\pm1k})^\dagger,$$

$$T^\pm_{0k} = \frac{1}{\sqrt{2}} \left( c^\dagger_{k\sigma} c^\dagger_{-k\bar{\sigma}} + c^\dagger_{-k\sigma} c^\dagger_{k\bar{\sigma}} \right) = (T^\pm_{0k})^\dagger,$$

$$N_k = n_{k\uparrow} + n_{-k\uparrow} + n_{k\downarrow} + n_{-k\downarrow},$$

$$S^z_k = \frac{1}{2} \left( n_{k\uparrow} + n_{-k\uparrow} - n_{k\downarrow} - n_{-k\downarrow} \right),$$

$$S^+ = c^\dagger_{k\sigma} c^\dagger_{k\bar{\sigma}} c^\dagger_{-k\sigma} c^\dagger_{-k\bar{\sigma}} = (S^-)^\dagger$$

with $\sigma_+ = \uparrow, \sigma_- = \downarrow$.

II. THE SO(5) RICHARDSON-GAUDIN HAMILTONIAN

The exactly-solvable Hamiltonian

$$H = \sum_k \eta_k N_k - \frac{g}{\mathcal{L}} \sum_{kk'} \eta_k \eta_{k'} \left( \vec{T}^+_k \cdot \vec{T}^-_{k'} + \vec{T}^-_k \cdot \vec{T}^+_{k'} + \vec{S}_k \cdot \vec{S}_{k'} + \frac{1}{4} N_k N_{k'} \right)$$  \hspace{1cm} (14)

can be expressed as a linear combination of the RG integrals of motion

$$R_k = \frac{1}{2} N_k - 1 + q \sum_{k' \neq k} Z_{kk'} \vec{T}_k \cdot \vec{T}_{k'}$$  \hspace{1cm} (15)

with $\vec{T}_k \cdot \vec{T}_{k'} = \vec{T}^+_k \cdot \vec{T}^-_{k'} + \vec{T}^-_k \cdot \vec{T}^+_{k'} + \vec{S}_k \cdot \vec{S}_{k'} + (N_k/2 - 1) (N_{k'}/2 - 1)$.

Summing the integrals of motion with coefficients $\eta_k$ and using the antisymmetry of the function $Z_{kk'} = \frac{\eta_k \eta_{k'}}{\eta_k - \eta_{k'}}$, we arrive at

$$\sum_k \eta_k R_k = \frac{1}{2} \sum_k \eta_k (N_k - 2) + \frac{q}{2} \sum_{k, k' \neq k} (\eta_k - \eta_{k'}) Z_{kk'} \vec{T}_k \cdot \vec{T}_{k'} = \frac{1}{2} \sum_k \eta_k (N_k - 2) + \frac{q}{2} \sum_{k, k' \neq k} \eta_k \eta_{k'} \vec{T}_k \cdot \vec{T}_{k'}.$$

Now, we add in the diagonal elements of the double sum, which are the SO(5) Casimirs (11) whose eigenvalues are the quantum numbers $t^2_k$ (12), and arrive at

$$\sum_k \eta_k R_k = \frac{1}{2} \left( 1 - q \sum_{k'} \eta_{k'} \right) \sum_k \eta_k N_k + \frac{q}{2} \sum_{k, k'} \eta_k \eta_{k'} \left( \vec{T}^+_k \cdot \vec{T}^-_{k'} + \vec{T}^-_k \cdot \vec{T}^+_{k'} + \vec{S}_k \cdot \vec{S}_{k'} + \frac{1}{4} N_k N_{k'} \right) + \Gamma(q).$$  \hspace{1cm} (16)
\[ \Delta_{ij} \text{ (Eq. (17) as a function of the displacement } r_i - r_j \text{ with } r_i = i \text{ (lattice constant set to unity) for periodic (crosses) and antiperiodic (dots) boundary conditions in a system of size } L = 40. \text{ The gray line is } L/(r_i - r_j), \text{ showing the } 1/r \text{ decay of } |\Delta_{ij}|. \]

where \( \Gamma(q) = -\sum_k \eta_k \left( 1 - \frac{q}{2} \sum_{k'} \eta_{k'} + \frac{q}{2} \eta_k \eta_{k'} \right) \). Subtracting this constant, rescaling by a factor of \( 2/(1 - q \sum_{k'} \eta_{k'}) \), and defining the physical coupling \( g/L \equiv -q/(1 - q \sum_{k'} \eta_{k'}) \) gives the Hamiltonian (14).

In one spatial dimension, we can take the Fourier transform of the interaction coefficients \( \delta_{kk'} = \eta_k \eta_{k'} \) to understand their behavior in coordinate space:

\[ \Delta_{ij} = \frac{1}{L} \sum_{k,k'} \eta_k \eta_{k'} e^{ikr_i} e^{ik' r_j}. \]  

In Fig. 1, we plot the behavior of these coefficients for \( \eta_k = k \) with periodic and antiperiodic boundary conditions. As is seen in the plot, these behave as \( \Delta_{ij} \propto (-1)^{i-j}/(r_i - r_j) \).

III. SO(5) RICHARDSON-GAUDIN WAVEFUNCTIONS

It is difficult to intuitively understand the Richardson-Gaudin (RG) wavefunctions from their formal description. To illustrate the form of these wavefunctions, it is useful to consider small systems. As a simplification, we will restrict ourselves to the zero-seniority case, which removes the \( S^+_0 \) operator.

First, consider cases in which \( N_e = N_\omega \), corresponding to the ground state sector when there is no external magnetic field.

For \( N_e = N_\omega = 1 \), the eigenstates are simply

\[ |\Psi\rangle = Z(e_1, \omega_1) \sum_k Z(\eta_k, e_k) T^+_0 |0\rangle. \]

For \( N_e = N_\omega = 2 \), the wavefunction is already much more complicated:

\[ |\Psi\rangle = \sum_{k_1, k_2} \left( C^{1-1}_{k_1, k_2} T^+_{1k_1} T^+_{-1k_2} + C^{00}_{k_1, k_2} T^+_{0k_1} T^+_{0k_2} \right) |0\rangle \]

where the coefficients are

\[ C^{1-1}_{k_1, k_2} = Z(e_1, \omega_1) Z(e_1, \omega_2) Z(\eta_{k_1}, e_1) Z(\eta_{k_2}, e_2) + Z(e_2, \omega_1) Z(e_2, \omega_2) Z(\eta_{k_1}, e_2) Z(\eta_{k_2}, e_1), \]

\[ C^{00}_{k_1, k_2} = Z(e_1, \omega_1) Z(e_2, \omega_2) Z(\eta_{k_1}, e_1) Z(\eta_{k_2}, e_2) + Z(e_2, \omega_1) Z(e_1, \omega_2) Z(\eta_{k_1}, e_2) Z(\eta_{k_2}, e_1). \]
It can be shown that these coefficients obey the relation
$$
\sum_{k_1, k_2} C^{1-1}_{k_1, k_2} = -2 \sum_{k_1, k_2} C^{00}_{k_1, k_2}.
$$
Moving one more size up, the \( N_e = N_\omega = 3 \) wavefunction can be written as
$$
|\Psi\rangle = \sum_{k_1, k_2} \left( C^{10-1}_{k_1, k_2} T^+_{1k_1} T^+_{-1k_2} + C^{00}_{k_1, k_2} T^+_{0k_1} T^+_{0k_2} T^+_{0k_3} \right)
$$
(20)
To express these coefficients concisely, we write them as permanents and sum over the symmetric group \( S_3 \) to consider permutations of \( \{\omega_1, \omega_2, \omega_3\} \):
$$
C^{00}_{k_1, k_2, k_3} = \text{perm} \left( \begin{array}{ccc}
Z(e_1, \omega_1)Z(\eta_{k_1}, e_1) & Z(e_2, \omega_1)Z(\eta_{k_1}, e_2) & Z(e_3, \omega_1)Z(\eta_{k_1}, e_3) \\
Z(e_1, \omega_2)Z(\eta_{k_2}, e_1) & Z(e_2, \omega_2)Z(\eta_{k_2}, e_2) & Z(e_3, \omega_2)Z(\eta_{k_2}, e_3) \\
Z(e_1, \omega_3)Z(\eta_{k_3}, e_1) & Z(e_2, \omega_3)Z(\eta_{k_3}, e_2) & Z(e_3, \omega_3)Z(\eta_{k_3}, e_3)
\end{array} \right)
$$
(21)
$$
C^{10-1}_{k_1, k_2, k_3} = \sum_{\{a, b, c\} \in S_3} \text{perm} \left( \begin{array}{ccc}
Z(e_1, \omega_a)Z(e_1, \omega_b)Z(\eta_{k_1}, e_1) & Z(e_2, \omega_a)Z(e_2, \omega_b)Z(\eta_{k_1}, e_2) & Z(e_3, \omega_a)Z(e_3, \omega_b)Z(\eta_{k_1}, e_3) \\
Z(e_1, \omega_c)Z(\eta_{k_2}, e_1) & Z(e_2, \omega_c)Z(\eta_{k_2}, e_2) & Z(e_3, \omega_c)Z(\eta_{k_2}, e_3) \\
Z(\eta_{k_3}, e_1) & Z(\eta_{k_3}, e_2) & Z(\eta_{k_3}, e_3)
\end{array} \right)
$$
(22)
This permanent structure is also present in the previously considered eigenstates, and in fact the creation operators for these states may also be expressed as permanents. Unlike singlet-pairing states, eigenstates of our Hamiltonian can be fully or partially spin-polarized without breaking pairs due to the different \( S^z \) pairing channels. The \( N_e = 2 \) states are fairly simple: Starting with the trivial case \( N_e = 2, N_\omega = 0 \),
$$
|\Psi\rangle = \sum_{k_1, k_2} Z(\eta_{k_1}, e_1)Z(\eta_{k_2}, e_2)T^+_{1k_1} T^+_{-1k_2}.
$$
(23)
Now with \( N_e = 2 \) and \( N_\omega = 1 \), we get
$$
|\Psi\rangle = \sum_{k_1, k_2} C^{0-1}_{k_1, k_2} T^+_{0k_1} T^+_{-1k_2}
$$
(24)
The spin-up polarized wavefunctions will have equivalent forms with \(-1\) and \(1\) subscripts interchanged. As a final example, consider the case \( N_e = 3, N_\omega = 2 \). Here, the wavefunction is
$$
|\Psi\rangle = \sum_{k_1, k_2, k_3} \left( C^{1-1-1}_{k_1, k_2, k_3} T^+_{1k_1} T^+_{-1k_2} T^+_{-1k_3} + C^{00-1}_{k_1, k_2, k_3} T^+_{0k_1} T^+_{0k_2} T^+_{0k_3} \right)
$$
(25)
with coefficients
$$
C^{1-1-1}_{k_1, k_2, k_3} = \frac{1}{2} \text{perm} \left( \begin{array}{ccc}
Z(e_1, \omega_1)Z(e_1, \omega_2)Z(\eta_{k_1}, e_1) & Z(e_2, \omega_1)Z(e_2, \omega_2)Z(\eta_{k_1}, e_1) & Z(e_3, \omega_1)Z(e_2, \omega_1)Z(\eta_{k_1}, e_3) \\
Z(\eta_{k_2}, e_1) & Z(\eta_{k_2}, e_1) & Z(\eta_{k_2}, e_3) \\
Z(\eta_{k_3}, e_1) & Z(\eta_{k_3}, e_2) & Z(\eta_{k_3}, e_3)
\end{array} \right)
$$
(26)
$$
C^{00-1}_{k_1, k_2, k_3} = \text{perm} \left( \begin{array}{ccc}
Z(e_1, \omega_1)Z(\eta_{k_1}, e_1) & Z(e_1, \omega_2)Z(\eta_{k_1}, e_2) & Z(e_1, \omega_3)Z(\eta_{k_1}, e_3) \\
Z(e_2, \omega_1)Z(\eta_{k_2}, e_1) & Z(e_2, \omega_2)Z(\eta_{k_2}, e_2) & Z(e_2, \omega_3)Z(\eta_{k_2}, e_3) \\
Z(\eta_{k_3}, e_1) & Z(\eta_{k_3}, e_2) & Z(\eta_{k_3}, e_3)
\end{array} \right).
$$
(27)
Similarly to the \( N_e = N_\omega = 2 \) case, these coefficients satisfy a relationship
$$
\sum_{k_1, k_2, k_3} C^{1-1-1}_{k_1, k_2, k_3} = -2 \sum_{k_1, k_2, k_3} C^{00-1}_{k_1, k_2, k_3}.
$$

### IV. SO(5) Mean-Field Solutions

We start with the SO(5) coherent state
$$
|\Psi\rangle = e^{\sqrt{2} \sum_k z_k \Gamma^+_k} |0\rangle
$$
(28)
where \( \Gamma^+_k = \sqrt{(1-x)T^+_{0k} + \sqrt{x/2}} (T^+_{1k} - T^+_{-1k}) \). Variational parameters \( \{z_k\} \) and \( x \) can be chosen to minimize the energy of this wavefunction with the Hamiltonian (14).

The \( \{z_k\} \) variables related to the \( u_k \) and \( v_k \) factors used in BCS superconductivity as \( z_k = v_k/u_k \) with restriction \( |u_k|^2 + |v_k|^2 = 1 \). As \( x \) goes from 0 to 1, the wavefunction goes from having only \( S^z = 0 \) pairs through a state with...
a mixture of all three pairing channels similar to the BW state to a state with only \( S^z = \pm 1 \) pairs, similar to the ABM wavefunction. For \( x \neq 0 \), the state (28) is no longer an eigenstate of \( S^z \) but preserves the zero magnetization \( \langle S^z \rangle = 0 \).

To approximate the ground state, we minimize the expectation value of \( H' = H - \lambda \sum_k N_k \), where \( H \) is the integrable Hamiltonian of Eq. (9) of the main paper and \( \lambda \) is a Lagrange multiplier fixing the particle number \( \langle \Psi | \sum_k N_k | \Psi \rangle = N \). Simple calculations show

\[
\langle \Psi | H' | \Psi \rangle = 4 \sum_k (\eta_k - \lambda) \frac{z_k^2}{1 + z_k^2} - 4G \sum_{kk'} \eta_k \eta_{k'} \frac{1 + z_k z_{k'}}{1 + z_k^2 (1 + z_{k'}^2)}.
\]

Note that the energy is independent of \( x \), meaning that the mean-field state is a degenerate mixture of the three spin components with \( \langle S^z \rangle = 0 \).

By taking the derivatives with respect to the variational parameters \( z_k \) and setting them to zero, we find the extremum conditions that give the set of mean-field equations (for all \( k > 0 \)) plus constraint equation

\[\begin{align*}
0 &= (\eta_k - \lambda) z_k - G \eta_k \sum_{k'} \eta_{k'} z_{k'} \frac{1 - z_k^2 + 2z_k z_{k'}}{1 + z_k^2 (1 + z_{k'}^2)} = 0 \quad (29a) \\
N &= \sum_k \frac{z_k^2}{1 + z_k^2} \quad (29b).
\end{align*}\]

These are the set of mean-field equations that define the parameters \( \{ z_k \} \) and \( \lambda \). However, it is more clear and convenient to express them in terms of the more familiar \( \{ u_k, v_k \} \) BCS factors. Equations (29a) and (29b) transform to

\[\begin{align*}
0 &= (\eta_k - \lambda) u_k v_k - G \eta_k (u_k^2 - v_k^2) \sum_{k'} \eta_{k'} u_{k'} v_{k'} - 2G \eta_k u_k v_k \sum_{k'} \eta_{k'} v_{k'}^2, \quad (30a) \\
N &= 4 \sum_k v_k^2. \quad (30b)
\end{align*}\]

We now define the gap \( \Delta \), the Fock term \( \Gamma \), and the single-particle Hartree-Fock (HF) energies \( \varepsilon_k \) as

\[\Delta = 2G \sum_k \eta_k u_k v_k, \quad \Gamma = -G \sum_k \eta_k v_k^2, \quad \varepsilon_k = \eta_k (1 + 2\Gamma) - \lambda.\]

Inserting in (30a) and solving for \( \{ u_k, v_k \} \),

\[\begin{align*}
u_k^2 &= \frac{1}{2} \left[ 1 + \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + \eta_k^2 \Delta^2}} \right], \\
u_k^2 &= \frac{1}{2} \left[ 1 - \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + \eta_k^2 \Delta^2}} \right] \quad (31).
\end{align*}\]

The mean-field equations for the three unknowns \( \Delta, \Gamma \) and \( \lambda \) are then

\[\begin{align*}
1 &= G \sum_k \frac{\eta_k^2}{\sqrt{[\eta_k (1 + 2\Gamma) - \lambda]^2 + \eta_k^2 \Delta^2}}, \quad (32a) \\
N &= L - 2 \sum_k \frac{\eta_k (1 + 2\Gamma) - \lambda}{\sqrt{[\eta_k (1 + 2\Gamma) - \lambda]^2 + \eta_k^2 \Delta^2}}, \quad (32b) \\
\Gamma &= -\frac{G}{2} \sum_k \eta_k \left[ 1 - \frac{\eta_k (1 + 2\Gamma) - \lambda}{\sqrt{[\eta_k (1 + 2\Gamma) - \lambda]^2 + \eta_k^2 \Delta^2}} \right]. \quad (32c)
\end{align*}\]

Once the system of equations is solved, the ground state energy can be evaluated in terms of these parameters as

\[\mathcal{E} = -\frac{1}{G} (\Delta^2 + 4\Gamma(1 + \Gamma))\]
In the thermodynamic limit, this becomes a set of integral equations

\[ 1 = g \frac{1}{2\pi} \int_0^\pi \eta_k^2 \frac{dk}{\sqrt{\eta_k (1 + 2\Gamma) - \lambda}^2 + \eta_k^2 \Delta^2} \]  
\[ \rho = 1 - \frac{1}{2\pi} \int_0^\pi \eta_k (1 + 2\Gamma) - \lambda \frac{dk}{\sqrt{\eta_k (1 + 2\Gamma) - \lambda}^2 + \eta_k^2 \Delta^2} \]  
\[ \Gamma = - \frac{g}{4\pi} \int_0^\pi \eta_k \left[ 1 - \frac{\eta_k (1 + 2\Gamma) - \lambda}{\sqrt{\eta_k (1 + 2\Gamma) - \lambda}^2 + \eta_k^2 \Delta^2} \right] dk \]

(33a)\hspace{1cm} (33b)\hspace{1cm} (33c)

with corresponding energy density \( e = -\frac{1}{g} \left( \Delta^2 + 4\Gamma(1 + \Gamma) \right) \).

For repulsive couplings (\( G < 0 \)), there is no mean-field solution (\( \Delta = 0 \)). The lowest-energy Slater determinant is the ground state of the noninteracting Fermi gas in one dimension (i.e. our Hamiltonian with \( G = 0 \)): 

\[ |\Psi_{\text{nonint}}\rangle = \prod_{k} c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger |0\rangle \]

for any coupling \( G < 0 \) with energy

\[ \langle \Psi_{\text{nonint}} | H | \Psi_{\text{nonint}} \rangle = 4 \sum_{k=0}^{k_F} \epsilon_k - G \left( 3 \sum_{k>0} \epsilon_k^2 + 4 \left( \sum_{k=0}^{k_F} \epsilon_k \right)^2 \right) \]

(34)

where \( k_F \) is the momentum of the highest-energy occupied single-particle state.

A. Ground state energy scaling

We would like to understand which unique properties of these models persist into the thermodynamic limit (i.e. are present in macroscopic systems). Here, we focus on the ground state energy density \( e_0 = \mathcal{E}_0 / L \). In previous work on the SU(2) RG models, it was possible to directly compute the exact energy density in the thermodynamic limit [1], but for our purposes, it suffices to plot energy density as a function of \( 1/L \) and predict the intercept at \( 1/L = 0 \). To aid in this prediction, we perform a least-squares fit of this data. These results are plotted in Fig. 2 for \( \rho = 1/2 \) in the metallic, trivial superfluid, and nontrivial superfluid phases. Due to the particle-hole symmetry of the model, we expect qualitatively similar results for other densities.

In all cases, the dominant scaling of both the mean-field and exact energy densities is \( 1/L \) with corrections of higher order in \( 1/L \), which we fit using the function

\[ e(L) = e_\infty + a/L + b/L^2 + c/L^3 \]

(35)

where \( e_\infty \) is the energy density \( e \) as \( L \rightarrow \infty \).

For the results in Fig. 2, we find \( e_\infty \) consistent with the infinite-size mean-field results up to a relative difference of at most \( 10^{-4} \). \( e_\infty \) and other fit coefficients for the cases plotted in Fig. 2 are presented in Table I.

| \( g/g_c \) | \( \rho \) | \( e_\infty \) | \( a \) | \( b \) | \( c \) |
|---|---|---|---|---|---|
| -2 | 1/2 | 0.22089 | 12.3 | -0.3 | -6.8 |
| -2 | 3/2 | 3.755 | 8.4 | -14.8 | -23.6 |
| 0.8 | 1/2 | 0.1178 | -5.1 | 1.0 | 3.7 |
| 0.8 | 3/2 | 0.4498 | -5.1 | 0.3 | 4.2 |
| 1.2 | 1/2 | 0.1737 | -7.6 | 0.5 | 5.8 |
| 1.2 | 3/2 | 0.5014 | -7.6 | 1.3 | 7.5 |

TABLE I. Parameters of the least-squares fit for exact and mean-field energies plotted in Fig. 2. The values for \( e_\infty \) extrapolated from the RG results agree with the directly-computed mean-field results for all digits included in the table.

V. QUARTETS IN THE SO(5) MODEL

Quartet states are collective states of four-fermion. As mentioned in the main work, at \( g = g_c \) and for \( N \) divisible by four, the Hamiltonian (14) has a degenerate ground state with eigenstates made of singlet (S=0) quartets. Among
FIG. 2. Least-squares fit of energy density $e = E/L$ as a function of $1/L$ and couplings $g/g_c = -2$ (solid), 0.8 (dashes) and 1.6 (dots) for mean-field (red) and exact (black) results. Crosses and circles mark exact and mean-field solutions, respectively, while for repulsive couplings the energy density was computed continuously from Eq. (34). The circles on the $y$-axis mark the directly-computed thermodynamic values of the mean-field energy densities.

them the pure quartet ground state wavefunction

$$|\Psi_Q\rangle = \left( \sum_{kk'} \left( T_{1k}^+ T_{-1k'}^+ + T_{-1k}^+ T_{1k'}^+ - T_{0k} T_{0k'} \right) \right)^{N/4} |0\rangle$$

with total spin $S = 0$. Quartets show up elsewhere in the exact results. For instance, as we can see from (18), the general, unpolarized 4 fermion ground state also has a quartet form.

What is not as easily determined from exact results is whether or not the ground state for any $N$ away from $g_c$ supports ground state correlations. Rather than try to measure these correlations directly, we calculate the quartet gap:

$$\Delta_4(N) = \frac{\varepsilon_0(N + 2) + \varepsilon_0(N - 2) - 2\varepsilon_0(N)}{2}.$$  

Analogously to the pair (or charge) gap $\Delta_2(N) = (\varepsilon_0(N + 1) + \varepsilon_0(N - 1) - 2\varepsilon_0(N))/2$, this quantity for $N$ divisible
by 4 represents the energy added to the $N$ particle ground state when a quartet is broken into pairs. The advantage of this compared to other correlation measures is that it can be calculated for large systems using our exact solutions.

It can be easily shown that at $g = 0$ (where the ground state energy for $\eta_k = k$ is simply $4 \sum_{k=0}^{k_F} k$), the quartet gap has the value

$$\Delta_{4}^{\text{nonint}}(N) = \begin{cases} \frac{\pi}{2L} & N \text{ divisible by } 4 \\ 0 & \text{else.} \end{cases}$$

This value clearly goes to zero in the limit $L \to \infty$ and is due to purely kinematic effects. As such, it is important to compare the quartet gaps away from $g = 0$ to understand to what extent they are due to legitimate quartet correlations. Additionally, at $g = g_c$ the quartet gap is zero because the ground state energy for all even values of $N$ is equal. In this case, the zero quartet gap is a consequence of the existence of purely pair-based state that are degenerate with the quartet state.

Quartet gaps, along with pair gaps, are plotted in Fig. 3 as a function of $g/g_c$. As can be seen, quartet correlations are most prevalent when the spin-spin interactions are removed from the Hamiltonian, becoming comparable to the pairing correlations for certain densities and couplings. As these modifications break the integrability of the model, these calculations were undertaken using exact-diagonalization in small systems. As such, the thermodynamic behavior of these quantities has yet to be determined.
VI. HIGHLY DEGENERATE FLAT BAND LIMITS IN PAIRING HAMILTONIANS

The hyperbolic SU(2) Richardson-Gaudin Hamiltonian

\[
H = \sum_k \eta_k c_k^\dagger c_k - \frac{g}{L} \sum_{k,k'} \eta_k \eta_{k'} c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}
\]  

(39)

depicts \(p\)-wave pairing for spinless (or more correctly, fully spin-polarized) fermions. This model exhibits topological superfluidity for \(g > 0\) [1]. As with the SO(5) RG Hamiltonian, the ground state for \(g < 0\) is a metal with atypical excitations. What distinguishes the SU(2) case is a flat-band limit of the exactly-solvable model, initially introduced to study fractional Quantum Hall physics [2]:

\[
H_\infty = \lim_{g \to -\infty} \frac{1}{g} H = \sum_{k,k'} \eta_k \eta_{k'} c_k^\dagger c_{-k}^\dagger c_{-k'} c_{k'}.
\]  

(40)

Here, solutions to the SU(2) RG equations may be classified by the number of singular pairons. All solutions with all pairons finite give zero energy in the limit due to the factor of \(1/G\). The only other possible solutions have finite energy with precisely one pairon diverging. By counting the ways of arranging the divergent pairons and subtracting this number from the total number of available states, the degeneracy of the ground state is found to be

\[
d_d^{0} = \max \left( 1, \frac{L/2}{N/2} - \left( \frac{L/2}{N/2} - 1 \right) \right)
\]  

(41)

for a system with \(N/2\) fermion pairs on \(L\) sites [2]. Extrapolating to all seniorities gives total ground state degeneracy

\[
d_d^{\text{total}} = \max \left( 1, \frac{L}{N} - \left( \frac{L}{N} - 2 \right) \right).
\]  

(42)

As a fraction of the total Hilbert space dimension \(|\mathcal{H}| = (L/2)^{N/2}\), this is

\[
\frac{d_d^{\text{total}}}{|\mathcal{H}|} = \max \left( 0, \frac{1 - 2\rho}{(1 - \rho)^2} \right)
\]  

(43)

in the thermodynamic limit.

A relative of our exactly-solvable Hamiltonian (14),

\[
H' = \sum_k \epsilon_k N_k - G_T \sum_{k,k'} \eta_k \eta_{k'} \tilde{T}_k^+ \cdot \tilde{T}_{k'}^- - G_S \sum_{k,k'} \eta_k \eta_{k'} S_k^+ S_{k'}^- - \frac{3G}{2} \sum_k \eta_k^2 N_k
\]  

(44)

also exhibits an exponentially degenerate, zero energy ground state in the limit \(\epsilon_k \to 0\) with \(\eta_k\) finite, although the precise formula for this degeneracy cannot be derived from the RG equations as was done for the SU(2) model [2].

The number of ground states is plotted as a function of system size in Fig. 4. In this zero kinetic-energy limit with \(\epsilon = 0\), this is the Haldane-Rezayi Hamiltonian [3]. Near this limit, this Hamiltonian has an extremely high density of states close to the ground state, creating a nontrivial excitation spectrum and non-Fermi liquid behavior. For lower couplings, both the exactly-solvable SO(5) Hamiltonian (14) and Hamiltonian (44) have Fermi-liquid-like behavior.

We can contrast this behavior with two limits of the exactly-solvable SO(5) Hamiltonian. First, we copy the \(p\)-wave pairing for spinless (or more correctly, fully spin-polarized) fermions. This model exhibits topological superfluidity for \(g > 0\) [1]. As with the SO(5) RG Hamiltonian, the ground state for \(g < 0\) is a metal with atypical excitations. What distinguishes the SU(2) case is a flat-band limit of the exactly-solvable model, initially introduced to study fractional Quantum Hall physics [2]:

\[
H = \sum_k \left( \eta_k - \frac{3G}{2} \eta_k^2 \right) N_k - G \sum_{k,k'} \eta_k \eta_{k'} \left( 2\tilde{T}_k^+ \cdot \tilde{T}_{k'}^- + S_k^+ S_{k'}^- + \frac{1}{4} N_k N_{k'} \right)
\]  

(45)

with additional single-particle terms from the commutators of \(S_k^+ S_{k'}^-\) and diagonal entries of the \(S_k^+ S_{k'}^-\) and \(N_k N_{k'}\) terms. As these scale with \(G\), it is impossible to have the many-body interactions have more than a \(O(1)\) strength relative to the single-particle terms.

Unlike the SU(2) case [1, 2], these single-particle terms cannot be absorbed into the kinetic energy by rescaling and defining a new coupling, since they have coefficients \(\eta_k^2\) rather than \(\eta_k\). The rational SU(2) Hamiltonian with
density-density interactions studied in Ref. [4] behaves similarly to the SO(5) case in the infinite-coupling limit, for the same reasons. The Hamiltonian (44) is arrived at by simply removing all these terms.

At the critical point $G_c$, the integrable SO(5) Hamiltonian is proportional to $\sum_{kk'} \eta_k \eta_{k'} \vec{T}_k \cdot \vec{T}_{k'}$, again superficially similar to Eq. (40). As with the $G \to -\infty$ limit just examined, single-particle terms appearing in the interaction spoil the exponential degeneracy by creating an effective kinetic energy. Although there is a ground state degeneracy at this point, it scales only linearly with the system size, and thus represents a vanishing fraction of the full Hilbert space in the thermodynamic limit.

To understand the nature of excitations in these models, we make use of the spectral function $A(k, \sigma, \omega) = A^+(k, \sigma, \omega) + A^-(k, \sigma, \omega)$, here defined as

$$A^\pm(k, \sigma, \omega) = \frac{1}{\pi} \text{Im} \sum_{\alpha} \frac{\langle \Psi_\alpha | c_{k\sigma}^\pm | \Psi_0 \rangle}{E_\alpha - E_{\alpha \pm} - i\epsilon}, \quad c_{k\sigma}^+ = c_{k\sigma}^\dagger = (c_{-k\sigma})^\dagger,$$

where $\alpha$ indexes all eigenstates $|\Psi_\alpha(N \pm 1)\rangle$ of the Hamiltonian with energies $E_\alpha(N \pm 1)$ and $\epsilon$ is a positive, infinitesimal number. Due to symmetries of the systems we are concerned with, we will take $\sigma = \uparrow$ for all of the following $(A(k, \omega) = A(k, \uparrow, \omega)$. We plot these functions for various modifications of the SO(5) Hamiltonian in Fig. 5.

A single peak in $A(k, \omega)$ for $k$ fixed indicates single-particle excitations of the $N$-particle ground state are close to eigenstates of the Hamiltonian, allowing for accurate quasiparticle descriptions of these eigenstates. This peak is located at a value of $\omega$ equal to the energy difference between the $N$ and $N \pm 1$-particle states, which when $k = k_F$ (the Fermi momentum), corresponds to the Fermi energy $\mu$. This is emblematic of Fermi liquid states, where the infinitesimally thin peak of a noninteracting Fermi gas acquires a finite width and therefore gives the quasiparticles a finite lifetime. Such behavior is seen in SO(5) Hamiltonians with kinetic energy and only Heisenberg or (not pictured) density-density interactions. As (repulsive) pairing interactions are added, a number of additional peaks are formed away from $\omega = \mu$. As plotted, we choose $k$ slightly above the Fermi momentum so the main peak is in $A^+(k, \omega)$ (as the additional fermion is created/annihilated above the Fermi level), while the additional behavior is seen in $A^-(k, \omega)$.
FIG. 5. Spectral function for various SO(5) Hamiltonians in a system with $L = 8$ at three-quarter filling ($\rho = 3/2$) and $g/g_c = -1$. The spectral function (46) is calculated for the lowest-energy unoccupied mode, and with $\epsilon = 0.2$ to smooth out the delta peaks. The modified Hamiltonian is Eq. (44). In all cases, $A^+(k, \omega)$ has a large peak at or near the Fermi energy $\mu = \frac{1}{2}(E_0(N+1) - E_0(N-1))$. In the cases of kinetic energy plus Heisenberg interactions and (not pictured) purely kinetic or kinetic plus density-density, this is the only peak in the spectral function, indicating single-particle excitations. In the other cases, a number of peaks appear in $A^-(k, \omega)$, as discussed in the text. The lower plot shows just $A^-(k, \omega)$ for these same systems over a smaller range in $\omega - \mu$ to emphasize features not visible in the upper plots.
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