The star-shapedness of a generalized numerical range

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Abstract

Let \( \mathcal{H}_n \) be the set of all \( n \times n \) Hermitian matrices and \( \mathcal{H}_m^n \) be the set of all \( m \)-tuples of \( n \times n \) Hermitian matrices. For \( A = (A_1, ..., A_m) \in \mathcal{H}_m^n \) and for any linear map \( L : \mathcal{H}_n^m \to \mathbb{R}^\ell \), we define the \( L \)-numerical range of \( A \) by

\[
W_L(A) := \{ L(U^*A_1U, ..., U^*A_mU) : U \in \mathbb{C}^{n \times n}, U^*U = I_n \}.
\]

In this paper, we prove that if \( \ell \leq 3 \), \( n \geq \ell \) and \( A_1, ..., A_m \) are simultaneously unitarily diagonalizable, then \( W_L(A) \) is star-shaped with star center at \( L \left( \frac{1}{n} \text{tr} A_1 I_n, ..., \frac{1}{n} \text{tr} A_m I_n \right) \).

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1 Introduction

Let \( \mathbb{C}^{n \times n} \) denote the set of all \( n \times n \) complex matrices, and \( A \in \mathbb{C}^{n \times n} \). The (classical) numerical range of \( A \) is defined by

\[
W(A) := \{ x^*Ax : x \in \mathbb{C}^n, x^*x = 1 \}.
\]

The properties of \( W(A) \) were studied extensively in the last few decades and many nice results were obtained; see [10, 13]. The most beautiful result is probably the Toeplitz-Hausdorff Theorem which affirmed the convexity of \( W(A) \); see [12, 17]. The generalizations of \( W(A) \) remain an active research area in the field.

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For any $A \in \mathbb{C}^{n \times n}$, write $A = A_1 + iA_2$ where $A_1, A_2$ are Hermitian matrices. Then by regarding $\mathbb{C}$ as $\mathbb{R}^2$, one can rewrite $W(A)$ as

$$W(A) := \{(x^*A_1 x, x^*A_2 x) : x \in \mathbb{C}^n, x^*x = 1\}.$$ 

This expression motivates naturally the generalization of the numerical range to the joint numerical range, which is defined as follows. Let $\mathcal{H}_n$ be the set of all $n \times n$ Hermitian matrices and $\mathcal{H}^m_n$ be the set of all $m$-tuples of $n \times n$ Hermitian matrices. The joint numerical range of $A = (A_1, ..., A_m) \in \mathcal{H}^m_n$ is defined as

$$W(A) = W(A_1, ..., A_m) := \{(x^*A_1 x, ..., x^*A_m x) : x \in \mathbb{C}^n, x^*x = 1\}.$$ 

It has been shown that for $m \leq 3$ and $n \geq m$, the joint numerical range is always convex [1]. This result generalizes the Toeplitz-Hausdorff Theorem. However, the convexity of the joint numerical range fails to hold in general for $m > 3$, see [1, 11, 14].

When a new generalization of numerical range is introduced, people are always interested in its convexity. Unfortunately, this nice property fails to hold in some generalizations. However, another property, namely star-shapedness, holds in some generalizations; see [5, 18]. Therefore, the star-shapedness is the next consideration when the generalized numerical ranges fail to be convex. A set $M$ is called star-shaped with respect to a star-center $x_0 \in M$ if for any $0 \leq \alpha \leq 1$ and $x \in M$, we have $\alpha x + (1-\alpha)x_0 \in M$. In [15], Li and Poon showed that for a given $m$, the joint numerical range $W(A_1, ..., A_m)$ is star-shaped if $n$ is sufficiently large.

Let $U_n$ be the set of all $n \times n$ unitary matrices. For $C \in \mathcal{H}_n$ and $A = (A_1, ..., A_m) \in \mathcal{H}^m_n$, the joint C-numerical range of $A$ is defined by

$$W_C(A) := \{(\text{tr}(CU^*A_1 U), ..., \text{tr}(CU^*A_m U)) : U \in U_n\},$$

where $\text{tr}(\cdot)$ is the trace function. When $C$ is the diagonal matrix with diagonal elements $1, 0, ..., 0$, then $W_C(A)$ reduces to $W(A)$. Hence the joint C-numerical range is a generalization of the joint numerical range. In [3], Au-Yeung and Tsing generalized the convexity result of the joint numerical range to the joint C-numerical range by showing that $W_C(A)$ is always convex if $m \leq 3$ and $n \geq m$. However $W_C(A)$ fails to be convex in general if $m > 3$. One may consult [6] and [7] for the study of the convexity of $W_C(A)$. The star-shapedness of $W_C(A)$ remains unclear for $m > 3$.

For $A = (A_1, ..., A_m) \in \mathcal{H}^m_n$, we define the joint unitary orbit of $A$ by

$$U_n(A) := \{(U^*A_1 U, ..., U^*A_m U) : U \in U_n\}.$$ 

For $C \in \mathcal{H}_n$, we consider the linear map $L_C : \mathcal{H}^m_n \to \mathbb{R}^m$ defined by

$$L_C(X_1, ..., X_m) = (\text{tr}(CX_1), ..., \text{tr}(CX_m)).$$

Then the joint C-numerical range of $A$ is the linear image of $U_n(A)$ under $L_C$. Inspired by this alternative expression, we consider the following generalized
numerical range of $A \in \mathcal{H}_n^m$. For $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ and linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$, we define

$$W_L(A) = L(U_n(A)) := \{L(U^*A_1U, ..., U^*A_mU) : U \in U_n\},$$

and call it the $L$-numerical range of $A$, due to [4]. Because $L_C$ is a special case of general linear maps $L$, the $L$-numerical range generalizes the joint $C$-numerical range and hence the classical numerical range.

In this paper, we shall study in Section two an inclusion relation of the $L$-numerical range of $m$-tuples of simultaneously unitarily diagonalizable Hermitian matrices and linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell = 2, 3$. This inclusion relation will be applied in Section three to show that the $L$-numerical ranges of $A$ under our consideration are star-shaped.

### 2 An Inclusion Relation for $L$-Numerical Ranges

The following results follow easily from the definition of the $L$-numerical range.

**Lemma 2.1.** Let $(A_1, ..., A_m) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ be linear. Then the following holds:

(i) $W_L(\alpha(A_1, ..., A_m) + \beta(I_n, ..., I_n)) = \alpha W_L(A_1, ..., A_m) + \beta W_L(I_n, ..., I_n)$ if $\alpha, \beta \in \mathbb{R}$;

(ii) $W_L(U^*A_1U, ..., U^*A_mU) = W_L(A_1, ..., A_m)$ for all unitary $U$.

In the following we shall consider those $A_1, ..., A_m$ which are simultaneously unitarily diagonalizable, i.e., there exists $U \in U_n$ such that $U^*A_1U, ..., U^*A_mU$ are all diagonal. Hence by Lemma 2.1, we assume without loss of generality that $A_1, ..., A_m$ are (real) diagonal matrices. For $d = (d_1, ..., d_n)^T \in \mathbb{R}^n$, we denote by $\text{diag}(d)$ the $n \times n$ diagonal matrix with diagonal elements $d_1, ..., d_n$. We first introduce a special class of matrices which is useful in studying the generalized numerical range; see [9, 16, 18].

An $n \times n$ real matrix $P = (p_{ij})$ is called a pinching matrix if for some $1 \leq s < t \leq n$ and $0 \leq \alpha \leq 1$,

$$p_{ij} = \begin{cases} \alpha, & \text{if } (i, j) = (s, s) \text{ or } (t, t), \\ 1 - \alpha, & \text{if } (i, j) = (s, t) \text{ or } (t, s), \\ 1, & \text{if } i = j \neq s, t, \\ 0 & \text{otherwise}. \end{cases}$$

**Definition 2.2.** Assume $D = (\text{diag}(d^{(1)}), ..., \text{diag}(d^{(m)})), \hat{D} = (\text{diag}(\hat{d}^{(1)}), ..., \text{diag}(\hat{d}^{(m)}))$ where $d^{(1)}, ..., d^{(m)}, \hat{d}^{(1)}, ..., \hat{d}^{(m)} \in \mathbb{R}^n$. We say $\hat{D} \prec D$ if there exist a finite number of pinching matrices $P_1, ..., P_k$ such that $\hat{d}^{(i)} = P_1P_2 \cdots P_kd^{(i)}$ for all $i = 1, ..., m$.

The following inclusion relation is the main result in this section.
Theorem 2.3. Let $D, \hat{D} \in \mathcal{H}_n^m$ and $n > 2$. If $\hat{D} < D$, then for any linear map $L: \mathcal{H}_n^m \to \mathbb{R}^3$, we have $W_L(\hat{D}) \subset W_L(D)$.

To prove Theorem 2.3, we need some lemmas. For $\theta, \phi \in \mathbb{R}$, let $T_{\theta,\phi} \in \mathcal{U}_n$ be defined by

$$T_{\theta,\phi} = \begin{pmatrix}
\cos \theta & \sin \theta e^{-i \phi} & 0 \\
-\sin \theta & \cos \theta e^{-i \phi} & 0 \\
0 & 0 & I_{n-2}
\end{pmatrix}.$$

Lemma 2.4. Let $D = (D_1, ..., D_m) \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices. Then for any linear map $L: \mathcal{H}_n^m \to \mathbb{R}^3$ and $U \in \mathcal{U}_n$, the set of points

$$E_L(D, U) := \{ L(U^* T_{\theta,\phi} D_1 T_{\theta,\phi} U, ..., U^* T_{\theta,\phi} D_m T_{\theta,\phi} U) : \theta \in [0, \pi], \phi \in [0, 2\pi] \}$$

forms an ellipsoid in $\mathbb{R}^3$.

Proof. Note that for any $L: \mathcal{H}_n^m \to \mathbb{R}^3$, we can always express $L$ as

$$L(X_1, ..., X_m) = \left( \text{tr} \left( \sum_{i=1}^{m} P_i X_i \right), \text{tr} \left( \sum_{i=1}^{m} Q_i X_i \right), \text{tr} \left( \sum_{i=1}^{m} R_i X_i \right) \right)$$

for some suitable $P_i, Q_i, R_i \in \mathcal{H}_n$, $i = 1, ..., m$. For $U \in \mathcal{U}_n$, we write $U P_i U^* = (p_{ij}^{(i)})$, $U Q_i U^* = (q_{jk}^{(i)})$, $U R_i U^* = (r_{jk}^{(i)})$ and $D_i = \text{diag}(d_1^{(i)}, ..., d_{n-1}^{(i)})$, $i = 1, ..., m$.

By direct computations, the first coordinate of points in $E_L(D, U)$ is

$$\text{tr} \left( \sum_{i=1}^{m} P_i U^* T_{\theta,\phi} D_i T_{\theta,\phi} U \right)$$

$$= \text{tr} \left( \sum_{i=1}^{m} D_i T_{\theta,\phi} U P_i U^* T_{\theta,\phi} \right)$$

$$= \frac{1}{2} \sum_{i=1}^{m} \left( d_1^{(i)} + d_2^{(i)} \right) (p_{11}^{(i)} + p_{22}^{(i)}) + \sum_{i=1}^{m} \sum_{j=1}^{n} d_j^{(i)} p_{j}^{(i)}$$

$$+ \frac{1}{2} \sum_{i=1}^{m} \left( d_1^{(i)} - d_2^{(i)} \right) (p_{11}^{(i)} - p_{22}^{(i)}) \cos 2\theta$$

$$+ \sum_{i=1}^{m} \left( d_1^{(i)} - d_2^{(i)} \right) \text{Re}(p_{21}^{(i)} e^{-iT \phi}) \sin 2\theta.$$

Similarly for the second and the third coordinates of points in $E_L(D, U)$. Note that for $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and $a_3, b_3, c_3 \in \mathbb{C}$, the points $(a_1, b_1, c_1) + (a_2, b_2, c_2) \cos 2\theta + \text{Re}(a_3 e^{-iT \phi}, b_3 e^{-iT \phi}, c_3 e^{-iT \phi}) \sin 2\theta$ form an ellipsoid in $\mathbb{R}^3$ when $\theta, \phi$ run through $[0, \pi]$ and $[0, 2\pi]$ respectively. Hence $E_L(D, U)$ is an ellipsoid in $\mathbb{R}^3$. \hfill \square

Note that $E_L(D, U) \subset W_L(D)$ for any $U \in \mathcal{U}_n$.
Lemma 2.5. Let $D \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices with $n > 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$, there exists $V \in \mathcal{U}_n$ such that $E_L(D,V)$ defined in Lemma 2.4 degenerates (i.e., $E_L(D,V)$ is contained in a plane in $\mathbb{R}^3$).

Proof. Following the notations in Lemma 2.4 and its proof, we let $\alpha_i = d^{(i)}_1 - d^{(i)}_2$ for $i = 1, \ldots, m$ and $P' = \sum_{i=1}^{m} \alpha_i P_i \in \mathcal{H}_n$. Since $n > 2$, by generalized interlacing inequalities for eigenvalues of Hermitian matrices (see [8]), there exist $V \in \mathcal{U}_n$ and $\alpha \in \mathbb{R}$ such that $V P' V^*$ has $\alpha I_2$ as leading $2 \times 2$ principal submatrix. For any matrix $M$, let $M_{ij}$ denote its $(i,j)$ entry. Then by taking $U = V$ in the proof of Lemma 2.4, the first coordinate of points in $E_L(D,V)$ is $a + b \cos 2\theta + c \sin 2\theta$ where

$$a = \frac{1}{2} \sum_{i=1}^{m} (d^{(i)}_1 + d^{(i)}_2) (p_{11}^{(i)} + p_{22}^{(i)}) + \sum_{i=1}^{m} \sum_{j=1}^{m} d^{(j)}_2 p_{ij}$$

$$b = \frac{1}{2} \sum_{i=1}^{m} \alpha_i [(VP_1V^*)_{11} - (VP_1V^*)_{22}]$$

$$= \frac{1}{2} \left( V \left( \sum_{i=1}^{m} \alpha_i P_i \right) V^* \right)_{11} - \frac{1}{2} \left( V \left( \sum_{i=1}^{m} \alpha_i P_i \right) V^* \right)_{22}$$

$$= \frac{1}{2} (VP' V^*)_{11} - \frac{1}{2} (VP' V^*)_{22}$$

$$= \frac{1}{2} \alpha - \frac{1}{2} \alpha = 0,$$

$$c = \sum_{i=1}^{m} \alpha_i \text{Re} \left( (VP_1V^*)_{21} e^{\sqrt{-1}\phi} \right)$$

$$= \text{Re} \left[ \left( V \left( \sum_{i=1}^{m} \alpha_i P_i \right) V^* \right)_{21} e^{\sqrt{-1}\phi} \right]$$

$$= \text{Re}((VP' V^*)_{21} e^{\sqrt{-1}\phi}) = 0.$$
Let $O$ denote the closure of $M$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell \geq 2$, where $A$ is always star-shaped for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^3$ and simultaneously unitarily diagonalizable $A_1, \ldots, A_m \in \mathcal{H}_n$. The following result is the essential element in our proof.

**Proposition 3.1.** [18] Let $\mathbb{P}_n$ be the set of all finite products of $n \times n$ pinching matrices. Then for $0 \leq \alpha \leq 1$, $\alpha J_n + (1 - \alpha) J_n$ is in the closure of $\mathbb{P}_n$, where $J_n$ is the $n \times n$ matrix with all entries equal $1/n$.

Note that for any $A \in \mathcal{H}_n^m$, $\mathcal{U}_n(A)$ is compact. Hence $W_L(A)$ is compact for all linear maps $L$.

**Theorem 3.2.** Let $D = (D_1, \ldots, D_m) \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices with $n \geq 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^3$, $W_L(D)$ is star-shaped with respect to star-center $L(\frac{nD_1}{n}, \ldots, \frac{nD_m}{n} I_n)$.

**Proof.** By Lemma 2.1, we may assume without loss of generality that $\text{tr} D_i = 0$ for $i = 1, \ldots, m$; otherwise we replace $D_i$ by $D_i - \frac{\text{tr} D_i}{n} I_n$. Let $D_i = \text{diag}(d(i))$ where $d(i) \in \mathbb{R}^n$, $i = 1, \ldots, m$. For any $0 \leq \alpha \leq 1$, we have $\alpha d(i) + (1 - \alpha) J_n$. Then for any $U \in \mathcal{U}_n$, by Proposition 3.1, Theorem 2.3 and the compactness of $W_L(D)$, we have $\alpha L(U^* DU) \in W_L(\alpha D) \subset W_L(D) = W_L(D)$ where $\overline{M}$ denotes the closure of $M$.

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$f : [0, 1] \to \mathcal{U}_n$ such that $f(0) = U$ and $f(1) = V$ where $V$ is defined in Lemma 2.5 and hence $E(D, f(1))$ degenerates. By continuity, there exists $t \in [0, 1]$ such that $L(U^* DU) \in E(D, f(t)) \subset W_L(D)$.

Using similar techniques, one can prove that Theorem 2.3 stills holds for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^2$ with $n \geq 2$. However, the following example shows that the inclusion relation in Theorem 2.3 fails to hold if $L : \mathcal{H}_n^m \to \mathbb{R}^2$ is linear with $\ell > 3$.

**Example 2.6.** Let $n \geq 2$, $d = (1, \ldots, 0)^T$, $\tilde{d} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)^T \in \mathbb{R}^n$ and let $O_k$ be the $k \times k$ zero matrix. Consider $D = (\text{diag}(d), O_n, \ldots, O_n)$, $\tilde{D} = (\text{diag}(\tilde{d}), O_n, \ldots, O_n) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell \geq 4$ defined by

$$L(X_1, \ldots, X_m) = (\text{tr}(PX_1), \text{tr}(QX_1), \text{tr}(RX_1), \text{tr}(SX_1), 0, \ldots, 0)$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus O_{n-2}, \quad Q = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus O_{n-2},$$

$$R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus O_{n-2}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus O_{n-2}.$$

Then we have $\tilde{D} \prec D$ and $(1, 0, \ldots, 0) \not\in W_L(\tilde{D})$, but $(1, 0, \ldots, 0) \not\in W_L(D)$.
For a linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$, by regarding it as a projection of some linear map $\overline{L} : \mathcal{H}_n^m \to \mathbb{R}^3$, we deduce the following corollary easily.

**Corollary 3.3.** Let $D = (D_1, ..., D_m) \in \mathcal{H}_n^m$ be an $m$-tuple of diagonal matrices with $n \geq 2$. Then for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$, $W_L(D)$ is star-shaped with respect to star-center $L \left( \frac{\text{tr}D_1}{n} I_n, ..., \frac{\text{tr}D_m}{n} I_n \right)$.

**Proof.** We only need to consider the case $n = 2$. We may assume without loss of generality that $m = 1$ and $D = \text{diag}(1, -1)$. For any linear map $L : \mathcal{H}_2 \to \mathbb{R}^2$, we express it as $L(X) := (\text{tr}(PX), \text{tr}(QX))$ for some $P, Q \in \mathcal{H}_2$. Then we have

$$W_L(D) = \{2(x^*Px, x^*Qx) - (\text{tr}P, \text{tr}Q) : x \in \mathbb{C}^n, x^*x = 1\} = 2W(P, Q) - (\text{tr}P, \text{tr}Q),$$

which is convex and contains the origin. This implies that $W_L(D)$ is star-shaped with respect to star-center $L \left( \frac{W_1}{n} I_2 \right)$, which is the origin. \hfill $\square$

Note that the star-shapedness of the $L$-numerical range for linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^\ell$ with $\ell > 3$ remains open in the diagonal case. Moreover, for general cases of $A = (A_1, ..., A_m)$ where $A_1, ..., A_m$ are not necessarily simultaneously unitarily diagonalizable and $L : \mathcal{H}_n^m \to \mathbb{R}^2$ with $m \geq 3$, the star-shapedness of $W_L(A)$ is also unclear. However, by applying a result in [4], we can show that $L \left( \frac{\text{tr}A_1}{n} I_n, ..., \frac{\text{tr}A_m}{n} I_n \right) \in W_L(A_1, ..., A_m)$ for all linear maps $L : \mathcal{H}_n^m \to \mathbb{R}^2$.

**Proposition 3.4** ([4], P. 23). Let $A_k = (a^{(k)}_{ij}) \in \mathcal{H}_n$, $k = 1, ..., m$. For $0 \leq \epsilon \leq 1$, define $A_k(\epsilon)$ as

$$A_k(\epsilon) = \begin{pmatrix} a^{(k)}_{11} & \epsilon a^{(k)}_{12} & \cdots & \epsilon a^{(k)}_{1n} \\ \epsilon a^{(k)}_{21} & a^{(k)}_{22} & \cdots & \epsilon a^{(k)}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon a^{(k)}_{n1} & \epsilon a^{(k)}_{n2} & \cdots & a^{(k)}_{nn} \end{pmatrix}, \quad k = 1, ..., m.$$

Then $W_L(A_1(\epsilon), ..., A_m(\epsilon)) \subseteq W_L(A_1, ..., A_m)$ for any linear map $L : \mathcal{H}_n^m \to \mathbb{R}^2$.

**Theorem 3.5.** Let $A = (A_1, ..., A_m) \in \mathcal{H}_n^m$ and $L : \mathcal{H}_n^m \to \mathbb{R}^2$ be linear. Then $L \left( \frac{\text{tr}A_1}{n} I_n, ..., \frac{\text{tr}A_m}{n} I_n \right) \in W_L(A)$.

**Proof.** Define $A_i(\epsilon)$ as in Proposition 3.4 and note that $\text{tr}A_i(\epsilon) = \text{tr}A_i$ for $i = 1, ..., m$. Hence by Corollary 3.3 and Proposition 3.4, we have

$$L \left( \frac{\text{tr}A_1}{n} I_n, ..., \frac{\text{tr}A_m}{n} I_n \right) \in W_L(A_1(0), ..., A_m(0)) \subseteq W_L(A_1, ..., A_m).$$

\hfill $\square$
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