NOTES ON NON-ARCHIMEDEAN TOPOLOGICAL GROUPS

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Dedicated to Professor Dikran Dikranjan on his 60th birthday

Abstract. We show that the Heisenberg type group $H_X = (\mathbb{Z}_2 \oplus V) \rtimes V^*$, with the discrete Boolean group $V := C(X, \mathbb{Z}_2)$, canonically defined by any Stone space $X$, is always minimal. That is, $H_X$ does not admit any strictly coarser Hausdorff group topology. This leads us to the following result: for every (locally compact) non-archimedean $G$ there exists a (resp., locally compact) non-archimedean minimal group $M$ such that $G$ is a group retract of $M$. For discrete groups $G$ the latter was proved by S. Dierolf and U. Schwanengel [8]. We unify some old and new characterization results for non-archimedean groups. We show also that any epimorphism into a non-archimedean group must be dense.

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1. Introduction and preliminaries

A topological group is non-archimedean if it has a local base at the identity consisting of open subgroups. This class of groups coincides with the class of topological subgroups of the homeomorphism groups $\text{Homeo}(X)$, where $X$ runs over Stone spaces (compact zero-dimensional spaces) and $\text{Homeo}(X)$ carries the usual compact open topology. Recall that by Stone’s representation theorem, there is a duality between the category of Stone spaces and the category of Boolean algebras. The class $\mathcal{NA}$ of non-archimedean groups and their actions on ultra-metric and Stone spaces have many applications. For instance, in non-archimedean functional analysis, in descriptive set theory, computer science, etc. See, e.g., [44, 3, 25, 24] and references therein.

In the present paper we provide some applications of generalized Heisenberg groups, with emphasis on minimality properties, in the theory of $\mathcal{NA}$ groups and actions on Stone spaces.

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Recall that a Hausdorff topological group $G$ is minimal (Stephenson [47] and Doichinov [13]) if it does not admit a strictly coarser Hausdorff group topology, or equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological group embedding.

If otherwise is not stated all topological groups and spaces in this paper are assumed to be Hausdorff. We say that an additive topological group $(G, +)$ is a Boolean group if $x + x = 0$ for every $x \in G$. As usual, a $G$-space $X$ is a topological space $X$ with a continuous group action $\pi : G \times X \to X$ of a topological group $G$. We say that $X$ is a $G$-group if, in addition, $X$ is a topological group and all $g$-translations, $\pi^g : X \to X$, $x \mapsto gx := \pi(g, x)$, are automorphisms of $X$. For every $G$-group $X$ we denote by $X \rtimes G$ the corresponding topological semidirect product.

To every Stone space $X$ we associate a (locally compact, 2-step nilpotent) Heisenberg type group

$$H_X = (\mathbb{Z}_2 \oplus V) \rtimes V^*,$$

where $V := C(X, \mathbb{Z}_2)$ is a discrete Boolean group which can be identified with the group of all clopen subsets of $X$ (symmetric difference is the group operation). $V^* := \text{Hom}(V, \mathbb{Z}_2)$ is the compact group of all group homomorphisms into the two element cyclic group $\mathbb{Z}_2$. $V^*$ acts on $\mathbb{Z}_2 \oplus V$ in the following way: every $(f, (a, x)) \in V^* \times (\mathbb{Z}_2 \oplus V)$ is mapped to $(a + f(x), x) \in \mathbb{Z}_2 \oplus V$. The group operation on $H_X$ is defined as follows: for

$$u_1 = (a_1, x_1, f_1), \ u_2 = (a_2, x_2, f_2) \in H_X$$

we define

$$u_1u_2 = (a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2).$$

In Section 4 we study some properties of $H_X$ and show in particular (Theorem 4.1) that the (locally compact) Heisenberg group $H_X = (\mathbb{Z}_2 \times V) \rtimes V^*$ is minimal and non-archimedean for every Stone space $X$.

Every Stone space $X$ is naturally embedded into $V^* := \text{Hom}(V, \mathbb{Z}_2)$ by the natural map $\delta : X \to V^*$, $x \mapsto \delta_x$ where $\delta_x(f) := f(x)$. Every $\delta_x$ can be treated as a 2-valued measure on $X$. Identifying $X$ with $\delta(X) \subset V^*$ we get a restricted evaluation map $V \times X \to \mathbb{Z}_2$ which in fact is the evaluation map of the Stone duality. Note that the role of $\delta : X \to V^*$ for a compact space $X$ is similar to the role of the Gelfand map $X \to C(X)^*$, representing $X$ via the point measures.

For every action of a group $G \subset \text{Homeo}(X)$ on a Stone space $X$ we can deal with a $G$-space version of the classical Stone duality. The map $\delta : X \to V^*$ is a $G$-map of $G$-spaces. Every continuous group action of $G$ on a Stone space $X$ is automorphizable in the sense of [29], meaning that $X$ is a $G$-subspace of a $G$-group $K$. This contrasts the case of general compact spaces (see [29]). More generally, we study (Theorem 6.3) also metric and uniform versions of automorphizable actions.

Furthermore, a deeper analysis shows (Theorem 4.4) that every topological subgroup $G \subset \text{Homeo}(X)$ induces a continuous action of $G$ on $H_X$ by automorphisms such that the corresponding semidirect product $H_X \rtimes G$ is a minimal group.

We then conclude (Corollary 4.5) that every (locally compact) non-archimedean group is a group retract of a (resp., locally compact) minimal non-archimedean group. It covers a result of Dierolf and Schwanengel [8] (see also Example 3.5 below).
which asserts that every discrete group is a group retract of a locally compact non-archimedean minimal group.

Section 2 contains additional motivating results and questions. Several interesting applications of generalized Heisenberg groups can be found in the papers [25, 31, 32, 13, 84, 10, 11, 46].

Studying the properties of the Heisenberg group $H_X$, we get a unified approach to several (mostly known) equivalent characterizations of the class $\mathcal{N}A$ of non-archimedean groups (Lemma 3.2 and Theorem 5.1). In particular, we show that the class of all topological subgroups of $\text{Aut}(K)$, for compact abelian groups $K$, is precisely $\mathcal{N}A$.

A morphism $f : M \to G$ in a category $C$ is an epimorphism if there are no different morphisms $g, h : G \to F$ in $C$ such that $gf = hf$. In the category of Hausdorff topological groups a morphism with dense range is obviously an epimorphism. K.H. Hofmann asked in late 1960’s whether the converse is true. This epimorphism problem was answered by Uspenskij [50] in the negative. Nevertheless, in many natural cases indeed the epimorphism $M \to G$ must be dense. For example, Nummela [39] has shown it in the case that the co-domain $G$ is either locally compact or having the coinciding left and right uniformities. Using a criterion of Pestov [40] and the uniform automorphizability of certain actions by non-archimedean groups (see Theorem 6.5) we prove in Theorem 6.7 that any epimorphism $f : M \to G$ into a non-archimedean group $G$ must be dense.

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2. Minimality and Group Representations

Clearly, every compact topological group is minimal. Trivial examples of non-minimal groups are: the group $\mathbb{Z}$ of all integers (or any discrete infinite abelian group) and $\mathbb{R}$, the topological group of all reals. By a fundamental theorem of Prodanov and Stoyanov [38] every abelian minimal group is precompact. For more information about minimal groups see review papers of Dikranjan [9] and Comfort-Hofmann-Remus [6], a book of Dikranjan-Prodanov-Stoyanov [12] and a recent book of Lukacs [26].

Unexpectedly enough many non-compact naturally defined topological groups are minimal.

**Remark 2.1.** Recall some nontrivial examples of minimal groups.

1. Prodanov [37] showed that the $p$-adic topologies are the only precompact minimal group topologies on $\mathbb{Z}$.
2. Symmetric topological groups $S_X$ (Gaughan [18]).
3. Homeo($\{0,1\}^\mathbb{N}$) (see Gamarnik [17]) and also Uspenskij [52] for a more general case).
4. Homeo(0,1] (Gamarnik [17]).
5. The semidirect product $\mathbb{R} \ltimes \mathbb{R}_+$ (Dierolf-Schwanengel [8]). More general cases of minimal (so-called admissible) semidirect products were studied by Remus and Stoyanov [43]. By [31], $\mathbb{R}^n \ltimes \mathbb{R}_+$ is minimal for every $n \in \mathbb{N}$. 
(6) Every connected semisimple Lie group with finite center, e.g., $SL_n(R)$, $n \geq 2$ (Remus and Stoyanov [43]).

(7) The full unitary group $U(H)$ (Stoyanov [48]).

One of the immediate difficulties is the fact that minimality is not preserved by quotients and (closed) subgroups. See for example item (5) with minimal $\mathbb{R} \times \mathbb{R}_+$ where its canonical quotient $\mathbb{R}_+$ (the positive reals) and the closed normal subgroup $\mathbb{R}$ are nonminimal. As a contrast note that in a minimal abelian group every closed subgroup is minimal [12].

In 1983 Pestov raised the conjecture that every topological group is a group retract of a minimal group. Note that if $f : M \to G$ is a group retraction then necessarily $G$ is a quotient of $M$ and also a closed subgroup in $M$. Arhangel’skii asked the following closely related questions:

**Question 2.2.** ([3], [36]) Is every topological group a quotient of a minimal group? Is every topological group a closed subgroup of a minimal group?

By a result of Uspenskij [51] every topological group is a subgroup of a minimal group $M$ which is Raikov-complete, topologically simple and Roelcke-precompact.

Recently a positive answer to Pestov’s conjecture (and hence to Question 2.2 of Arhangel’skii) was obtained in [34]. The proof is based on methods (from [28]) of constructing minimal groups using group representations on Banach spaces and involving generalized Heisenberg groups.

According to [28] every locally compact abelian group is a group retract of a minimal locally compact group. It is an open question whether the same is true in the nonabelian case.

**Question 2.3.** ([28], [34] and [6]) Is it true that every locally compact group $G$ is a group retract (at least a subgroup or a quotient) of a locally compact minimal group?

A more general natural question is the following:

**Question 2.4.** [28] Let $\mathcal{K}$ be a certain class of topological groups and $\text{min}$ denotes the class of all minimal groups. Is it true that every $G \in \mathcal{K}$ is a group retract of a group $M \in \mathcal{K} \cap \text{min}$ ?

So Corollary 4.5 gives a partial answer to Questions 2.3 and 2.4 in the class $\mathcal{K} := N\mathcal{A}$ of non-archimedean groups.

**Remark 2.5.** Note that by [34] Theorem 7.2] we can present any topological group $G$ as a group retraction $M \to G$, where $M$ is a minimal group having the same weight and character as $G$. Furthermore, if $G$ is Raikov-complete then $M$ also has the same property. These results provide in particular a positive answer to Question 2.4 in the following basic classes: second countable groups, metrizable groups, Polish groups.

2.1. Minimality properties of actions.

**Definition 2.6.** Let $\alpha : G \times X \to X$, $\alpha(g, x) = gx$ be a continuous action of a Hausdorff topological group $(G, \sigma)$ on a Hausdorff topological space $(X, \tau)$. The action $\alpha$ is said to be:
(1) algebraically exact if $\ker \alpha := \{ g \in G : gx = x \ \forall x \in X \}$ is the trivial subgroup $\{ e \}$.

(2) topologically exact (t-exact, in short) if there is no strictly coarser, not necessarily Hausdorff, group topology $\sigma' \subsetneq \sigma$ on $G$ such that $\alpha$ is $(\sigma', \tau, \tau)$-continuous.

Remark 2.7. (1) Every topologically exact action is algebraically exact. Indeed, otherwise $\ker \alpha$ is a non-trivial subgroup in $G$. Then the preimage group topology $\sigma' \subset \sigma$ on $G$ induced by the onto homomorphism $G \to G/\ker \alpha$ is not Hausdorff (in particular, it differs $\sigma$) and the action remains $(\sigma', \tau, \tau)$-continuous.

(2) On the other hand, if $\alpha$ is algebraically exact then it is topologically exact if and only if for every strictly coarser Hausdorff group topology $\sigma' \subset \sigma$ on $G$ the action $\alpha$ is not $(\sigma', \tau, \tau)$-continuous. Indeed, since $\alpha$ is algebraically exact and $(X, \tau)$ is Hausdorff then every coarser group topology $\sigma'$ on $G$ which makes the action $(\sigma', \tau, \tau)$-continuous must be Hausdorff.

Let $X$ be a locally compact group and $\text{Aut}(X)$ be the group of all automorphisms endowed with the Birkhoff topology (see [19, §26] and [12, p. 260]). Some authors use the name Braconnier topology (see [5]).

The latter is a group topology on $\text{Aut}(X)$ and has a local base formed by the sets

$$\mathcal{B}(K, O) := \{ f \in \text{Aut}(X) : f(x) \in O x \text{ and } f^{-1}(x) \in O x \ \forall x \in K \}$$

where $K$ runs over compact subsets and $O$ runs over neighborhoods of the identity in $X$. In the sequel $\text{Aut}(X)$ is always equipped with the Birkhoff topology. It equals to the Arens $g$-topology [8, 10]. If $X$ is compact then the Birkhoff topology coincides with the usual compact-open topology. If $X$ is discrete then the Birkhoff topology on $\text{Aut}(X) \subset X^X$ coincides with the pointwise topology.

Lemma 2.8. In each of the following cases the action of $G$ on $X$ is t-exact:

(1) [28] Let $X$ be a locally compact group and $G$ be a subgroup of $\text{Aut}(X)$.

(2) Let $G$ be a topological subgroup of $\text{Homeo}(X)$, the group of all automorphisms of a compact space $X$ with the compact open topology.

(3) Let $G$ be a subgroup of $\text{Is}(X, d)$ the group of all isometries of a metric space $(X, d)$ with the pointwise topology.

Proof. Straightforward. □

2.2. From minimal dualities to minimal groups. In this subsection we recall some definitions and results from [28, 34].

Let $E, F, A$ be abelian additive topological groups. A map $w : E \times F \to A$ is said to be biadditive if the induced mappings

$$w_x : F \to A, w_f : E \to A, w_x(f) := w(x, f) =: w_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$.

A biadditive mapping $w : E \times F \to A$ is separated if for every pair $(x_0, f_0)$ of nonzero elements there exists a pair $(x, f)$ such that $w(x_0) \neq 0_A$ and $f_0(x) \neq 0_A$. 

A continuous separated biadditive mapping \( w : (E, \sigma) \times (F, \tau) \to A \) is \textit{minimal} if for every coarser pair \((\sigma_1, \tau_1)\) of Hausdorff group topologies \(\sigma_1 \subseteq \sigma, \tau_1 \subseteq \tau\) such that \( w : (E, \sigma_1) \times (F, \tau_1) \to A \) is continuous, it follows that \(\sigma_1 = \sigma\) and \(\tau_1 = \tau\).

Let \( w : E \times F \to A \) be a continuous biadditive mapping. Consider the action: \( w^{\tau} : F \times (A \oplus E) \to A \oplus E, \quad w^{\tau}(f, (a, x)) = (a + w(x, f), x) \). Denote by \( H(w) = (A \oplus E) \times F \) the topological semidirect product of \( F \) and the direct sum \( A \oplus E \). The group operation on \( H(w) \) is defined as follows: for a pair
\[
\begin{align*}
  u_1 &= (a_1, x_1, f_1), \\
  u_2 &= (a_2, x_2, f_2)
\end{align*}
\]
we define
\[
u_1u_2 = (a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2)\]
where, \( f_1(x_2) = w(x_2, f_1) \). Then \( H(w) \) becomes a Hausdorff topological group which is said to be a \textit{generalized Heisenberg group} (induced by \( w \)).

Let \( G \) be a topological group and let \( w : E \times F \to A \) be a continuous biadditive mapping. A continuous \textit{birepresentation} of \( G \) in \( w \) is a pair \((\alpha_1, \alpha_2)\) of continuous actions by group automorphisms \( \alpha_1 : G \times E \to E \) and \( \alpha_2 : G \times F \to F \) such that \( w \) is \( G \)-invariant, i.e., \( w(gx, gf) = w(x, f) \).

The birepresentation \( \psi \) is said to be \( t \)-exact if \( \ker(\alpha_1) \cap \ker(\alpha_2) = \{e\} \) and for every strictly coarser \textit{Hausdorff} group topology on \( G \) the birepresentation does not remain continuous. For instance, if one of the actions \( \alpha_1 \) or \( \alpha_2 \) is \( t \)-exact then clearly \( \psi \) is \( t \)-exact.

Let \( \psi \) be a continuous \( G \)-birepresentation
\[
\psi = (w : E \times F \to A, \alpha_1 : G \times E \to E, \alpha_2 : G \times F \to F).
\]
The topological semidirect product \( M(\psi) := H(w) \ltimes_\pi G \) is said to be the \textit{induced group}, where the action \( \pi : G \times H(w) \to H(w) \) is defined by
\[
\pi(g, (a, x, f)) = (a, gx, gf).
\]

**Fact 2.9.** Let \( w : E \times F \to A \) be a minimal biadditive mapping and \( A \) is a minimal group. Then

1. [11] Corollary 5.2] The Heisenberg group \( H(w) \) is minimal.
2. (See [28] Theorem 4.3 and [34]) If \( \psi \) is a \( t \)-exact \( G \)-birepresentation in \( w \) then the induced group \( M(\psi) \) is minimal.

**Fact 2.10.** [28] Let \( G \) be a locally compact abelian group and \( G^* := \text{Hom}(G, \mathbb{T}) \) be the dual (locally compact) group. Then the canonical evaluation mapping
\[
G \times G^* \to \mathbb{T}
\]
is minimal and the corresponding Heisenberg group \( H = (\mathbb{T} \oplus G) \ltimes G^* \) is minimal.

3. Some facts about non-archimedean groups and uniformities

3.1. **Non-archimedean uniformities.** For information on uniform spaces, we refer the reader to [16] (in terms of entourages) and to [22] (via coverings). If \( \mu \) is a uniformity for \( X \) in terms of coverings, then the collection of elements of \( \mu \) which are \textit{finite} coverings of \( X \) forms a base for a topologically compatible uniformity for \( X \) which we denote by \( \mu_{\text{fin}} \) (the precompact replica of \( \mu \)).
A partition of a set $X$ is a covering of $X$ consisting of pairwise disjoint subsets of $X$. Due to Monna (see [44, p.38] for more details), a uniform space $(X, \mu)$ is non-archimedean if it has a base consisting of partitions of $X$. In terms of entourages, it is equivalent to saying that there exists a base $\mathcal{B}$ of the uniform structure such that every entourage $P \in \mathcal{B}$ is an equivalence relation. Equivalently, iff its large uniform dimension (in the sense of Isbell [22, p. 78]) is zero.

A metric space $(X, d)$ is said to be an ultra-metric space (or, isosceles [24]) if $d$ is an ultra-metric, i.e., it satisfies the strong triangle inequality

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}.$$}

The definition of ultra-semimetric is the same as ultra-metric apart from the fact that the condition $d(x, y) = 0$ need not imply $x = y$. For every ultra-semimetric $d$ on $X$ every $\varepsilon$-covering $\{B(x, \varepsilon) : x \in X\}$ by the open balls is a clopen partition of $X$.

Furthermore, a uniformity is non-archimedean iff it is generated by a system $\{d_i\}_{i \in I}$ of ultra-semimetrics. The following result (up to obvious reformulations) is well known. See, for example, [22] and [21].

**Lemma 3.1.** Let $(X, \mu)$ be a non-archimedean uniform space. Then both $(X, \mu_{fin})$ and the uniform completion $(\hat{X}, \hat{\mu})$ of $(X, \mu)$ are non-archimedean uniform spaces.

### 3.2. Non-archimedean groups.

The class $\mathcal{NA}$ of all non-archimedean groups is quite large. Besides the results of this section see Theorem 5.1 below. The prodiscrete (in particular, the profinite) groups are in $\mathcal{NA}$. All $\mathcal{NA}$ groups are totally disconnected and for every locally compact totally disconnected group $G$ both $G$ and $Aut(G)$ are $\mathcal{NA}$ (see Theorems 7.7 and 26.8 in [19]). Every abelian $\mathcal{NA}$ group is embedded into a product of discrete groups.

The minimal groups $(\mathbb{Z}, \tau_p), S_X, \text{Homeo} (\{0, 1\}^{\mathbb{N}})$ (in items (1), (2) and (3) of Remark 2.1) are non-archimedean. By Theorem 4.1 the Heisenberg group $H_X = (\mathbb{Z}^2 \oplus V) \ltimes V^*$ is $\mathcal{NA}$ for every Stone space $X$. It is well known that there exist $2^{\mathbb{N}}$-many nonhomeomorphic metrizable Stone spaces.

Recall that every topological group can be identified with a subgroup of $\text{Homeo}(X)$ for some compact $X$ and also with a subgroup of $Is(M, d)$, topological group of isometries of some metric space $(M, d)$ endowed with the pointwise topology, [49]. Similar characterizations are true for $\mathcal{NA}$ with compact zero-dimensional spaces $X$ and ultra-metric spaces $(M, d)$. See Lemma 3.2 and Theorem 5.1 below.

We will use later the following simple observations. Let $X$ be a Stone space (compact zero-dimensional space) and $G$ be a topological subgroup of $\text{Homeo}(X)$. For every finite clopen partition $P = \{A_1, \ldots, A_n\}$ of $X$ define the subgroup

$$M(P) := \{g \in G : gA_k = A_k \forall 1 \leq k \leq n\}.$$}

Then all subgroups of this form defines a local base (subbase, if we consider only two-element partitions $P$) of the original compact-open topology on $G \subset \text{Homeo}(X)$. So for every Stone space $X$ the topological group $\text{Homeo}(X)$ is non-archimedean. More generally, for every non-archimedean uniform space $(X, \mu)$ consider the group $\text{Unif}(X, \mu)$ of all uniform automorphisms of $X$ (that is, the bijective functions $f : X \to X$ such that both $f$ and $f^{-1}$ are $\mu$-uniform). Then
**Lemma 3.2.** The following assertions are equivalent:

1. $G$ is a non-archimedean topological group.
2. The right (left) uniformity on $G$ is non-archimedean.
3. $\dim \beta G = 0$, where $\beta G$ is the maximal $G$-compactification \[35\] of $G$.
4. $G$ is a topological subgroup of $\text{Homeo}(X)$ for some compact zero-dimensional space $X$ (where $w(X) = w(G)$).
5. $G$ is a topological subgroup of $\text{Unif}(Y, \mu)$ for some non-archimedean uniformity $\mu$ on a set $Y$.

**Proof.** For the sake of completeness we give here a sketch of the proof. The equivalence of (1) and (3) was established by Pestov \[41, \text{Prop. 3.4}\]. The equivalence of (1), (2) and (3) is \[35, \text{Theorem 3.3}\].

(1) $\Rightarrow$ (2) Let $\{H_i\}_{i \in I}$ be a local base at $e$ (the neutral element of $G$), where each $H_i$ is an open (hence, clopen) subgroup of $G$. Then the corresponding decomposition of $G = \bigcup_{g \in G} H_i g$ by right $H_i$-cosets defines an equivalence relation $\Omega_i$ and the set $\{\Omega_i\}_{i \in I}$ is a base of the right uniform structure $\mu_r$ on $G$.

(2) $\Rightarrow$ (3) If the right uniformity $\mu$ is non-archimedean then by Lemma 3.1 the completion $(\hat{X}, \hat{\mu}_{\text{fin}})$ of its precompact replica (Samuel compactification of $(X, \mu)$) is again non-archimedean. Now recall (see for example \[35\]) that this completion is just the greatest $G$-compactification $\beta G$ (the $G$-space analog of the Stone-Čech compactification) of $G$.

(3) $\Rightarrow$ (4) A result in \[27\] implies that there exists a zero-dimensional proper $G$-compactification $X$ of the $G$-space $G$ (the left action of $G$ on itself) with $w(X) = w(G)$. Then the natural homomorphism $\varphi: G \to \text{Homeo}(X)$ is a topological group embedding.

(4) $\Rightarrow$ (5) Trivial because $\text{Homeo}(X) = \text{Unif}(X, \mu)$ for compact $X$ and its unique compatible uniformity $\mu$.

(5) $\Rightarrow$ (1) The non-archimedean uniformity $\mu$ has a base $\mathfrak{B}$ where each $P \in \mathfrak{B}$ is an equivalence relation. Then the subsets

$$M(P) := \{g \in G : \ (gx, x) \in P \ \forall x \in X\},$$

form a local base of $G$. Observe that $M(P)$ is a subgroup of $G$. \qed

$\mathcal{N}A$-ness of a dense subgroup implies that of the whole group. Hence the Raikov-completion of $\mathcal{N}A$ groups are again $\mathcal{N}A$. Subgroups, quotient groups and (arbitrary) products of $\mathcal{N}A$ groups are also $\mathcal{N}A$. Moreover the class $\mathcal{N}A$ is closed under group extensions.

**Fact 3.3.** \[20, \text{Theorem 2.7}\] If both $N$ and $G/N$ are $\mathcal{N}A$, then so is $G$.

For the readers convenience we reproduce here the proof from \[20\].

**Proof.** Let $U$ be a neighborhood of $e$ in $G$. We shall find an open subgroup $H$ contained in $U$. We choose neighborhoods $U_0$, $V$ and $W$ of $e$ in $G$ as follows. First let $U_0$ be such that $U_0^2 \subseteq U$. By the assumption, there is an open subgroup $M$ of $N$ contained in $N \cap U_0$. Let $V \subseteq U_0$ be open with $V = V^{-1}$ and $V^3 \cap N \subseteq M$. We denote by $\pi$ the natural homomorphism $G \to G/N$. Since $\pi(V)$ is open in $G/N$, it...
contains an open subgroup \( K \). We set \( W = V \cap \pi^{-1}(K) \). We show that \( W^2 \subseteq WM \).

Suppose that \( w_0, w_1 \in W \). Since \( \pi(w_0), \pi(w_1) \in K \), we have \( \pi(w_0w_1) \in K \). So there is \( w_2 \in W \) with \( \pi(w_2) = \pi(w_0w_1) \). Then \( w_2^{-1}w_0w_1 \in N \cap W^3 \subseteq M \), and hence \( w_0w_1 \in w_2M \). Using this result and also the fact that \( M \) is a subgroup of \( N \) we obtain by induction that \( W^k \subseteq WM \) for all \( k \in \mathbb{N} \). Now let \( H \) be the subgroup of \( G \) generated by \( W \). Clearly, \( H = \bigcup_{k=1}^{\infty} W^k \). Then \( H \) is open and

\[
H \subseteq WM \subseteq U_0^2 \subseteq U
\]
as desired. \( \Box \)

**Corollary 3.4.** Suppose that \( G \) and \( H \) are non-archimedean groups and that \( H \) is a \( G \)-group. Then the semidirect product \( H \ltimes G \) is non-archimedean.

**Example 3.5.** (Dierolf and Schwanengel [8]) Every discrete group \( H \) is a group retract of a locally compact non-archimedean minimal group.

More precisely, let \( \mathbb{Z}_2 \) be the discrete cyclic group of order 2 and let \( H \) be a discrete topological group. Let \( G := \mathbb{Z}_2^H \) be endowed with the product topology. Then

\[
\sigma : H \to \text{Aut}(G), \quad \sigma(k)((x_h)_{h \in H}) := (x_{hk})_{h \in H} \quad \forall k \in H, \quad (x_h)_{h \in H} \in G
\]
is a homomorphism. The topological semidirect (wreath) product \( G \ltimes \sigma H \) is a locally compact non-archimedean minimal group having \( H \) as a retraction.

Corollary 3.5 below provides a generalization.

4. **The Heisenberg group associated to a Stone space**

Let \( X \) be a Stone space. Let \( V = (V(X), \Delta) \) be the discrete group of all clopen subsets in \( X \) with respect to the symmetric difference. As usual one may identify \( V \) with the group \( V := C(X, \mathbb{Z}_2) \) of all continuous functions \( f : X \to \mathbb{Z}_2 \).

Denote by \( V^* := \text{hom}(V, \mathbb{T}) \) the Pontryagin dual of \( V \). Since \( V \) is a Boolean group every character \( V \to \mathbb{T} \) can be identified with a homomorphism into the unique 2-element subgroup \( \Omega_2 = \{1, -1\} \), a copy of \( \mathbb{Z}_2 \). The same is true for the characters on \( V^* \), hence the natural evaluation map \( w : V \times V^* \to \mathbb{T} \) (\( w(x, f) = f(x) \)) can be restricted naturally to \( V \times V^* \to \mathbb{Z}_2 \). Under this identification \( V^* := \text{hom}(V, \mathbb{Z}_2) \) is a closed (hence compact) subgroup of the compact group \( \mathbb{Z}_2^V \). Clearly, the groups \( V \) and \( \mathbb{Z}_2 \), being discrete, are non-archimedean. The group \( V^* = \text{hom}(V, \mathbb{Z}_2) \) is also non-archimedean since it is a subgroup of \( \mathbb{Z}_2^V \).

In the sequel \( G \) is an arbitrary non-archimedean group. \( X \) is its associated Stone space, that is, \( G \) is a topological subgroup of \( \text{Homeo}(X) \) (see Lemma 3.2). \( V \) and \( V^* \) are the non-archimedean groups associated to the Stone space \( X \) we have mentioned at the beginning of this subsection. We intend to show using the technique introduced in Subsection 2.2, among others, that \( G \) is a topological group retract of a non-archimedean minimal group.

**Theorem 4.1.** For every Stone space \( X \) the (locally compact 2-step nilpotent) Heisenberg group \( H = (\mathbb{Z}_2 \oplus V) \ltimes V^* \) is minimal and non-archimedean.

**Proof.** Using Fact 2.10 (or, by direct arguments) it is easy to see that the continuous separated biadditive mapping

\[
w : V \times V^* \to \mathbb{Z}_2
\]
is minimal. Then by Fact 2.9.1 the corresponding Heisenberg group $H$ is minimal. $H$ is non-archimedean by Corollary 3.4.

Lemma 4.2. Let $G$ be a topological subgroup of Homeo $(X)$ for some Stone space $X$ (see Lemma 3.2). Then $w(G) \leq w(X) = w(V) = |V| = w(V^*)$.

Proof. Use the facts that in our setting $V$ is discrete and $V^*$ is compact. Recall also that (see e.g., \[16\, Thm. 3.4.16\])

$$w(C(A, B)) \leq w(A) \cdot w(B)$$

for every locally compact Hausdorff space $A$ (where the space $C(A, B)$ is endowed with the compact-open topology).

The action of $G \subset \text{Homeo}(X)$ on $X$ and the functoriality of the Stone duality induce the actions on $V$ and $V^*$. More precisely, we have

$$\alpha : G \times V \to V, \quad \alpha(g, A) = g(A)$$

and

$$\beta : G \times V^* \to V^*, \quad \beta(g, f) := gf, \quad (gf)(A) = f(g^{-1}(A)).$$

Every translation under these actions is a continuous group automorphism. Therefore we have the associated group homomorphisms:

$$i_\alpha : G \to \text{Aut}(V)$$

$$i_\beta : G \to \text{Aut}(V^*)$$

The pair $(\alpha, \beta)$ is a birepresentation of $G$ on $w : V \times V^* \to \mathbb{Z}_2$. Indeed,

$$w(gf, g(A)) = (gf)(g(A)) = f(g^{-1}(g(A))) = f(A) = w(f, A).$$

Lemma 4.3. (1) Let $G$ be a topological subgroup of Homeo $(X)$ for some Stone space $X$. The action $\alpha : G \times V \to V$ induces a topological group embedding $i_\alpha : G \hookrightarrow \text{Aut}(V)$.

(2) The natural evaluation map

$$\delta : X \to V^*, \quad x \mapsto \delta_x, \quad \delta_x(f) = f(x)$$

is a topological $G$-embedding.

(3) The action $\beta : G \times V^* \to V^*$ induces a topological group embedding $i_\beta : G \hookrightarrow \text{Aut}(V^*)$.

(4) The pair $\psi := (\alpha, \beta)$ is a $t$-exact birepresentation of $G$ on $w : V \times V^* \to \mathbb{Z}_2$.

Proof. (1) Since $V$ is discrete, the Birkhoff topology on $\text{Aut}(V)$ coincides with the pointwise topology. Recall that the topology on $G$ inherited from Homeo $(X)$ is defined by the local subbase

$$H_A := \{g \in G : gA = A\}$$

where $A$ runs over nonempty clopen subsets in $X$. Each $H_A$ is a clopen subgroup of $G$. On the other hand the pointwise topology on $i_\alpha(G) \subset \text{Aut}(V)$ is generated by the local subbase of the form

$$\{i_\alpha(g) \in i_\alpha(G) : gA = A\},$$

So, $i_\alpha$ is a topological group embedding.

(2) Straightforward.
(3) Since $V^*$ is compact, the Birkhoff topology on $\text{Aut}(V^*)$ coincides with the compact open topology.

The action of $G$ on $X$ is t-exact. Hence, by (2) it follows that the action $\beta$ cannot be continuous under any weaker group topology on $G$. Now it suffices to show that the action $\beta$ is continuous.

The topology on $V^* \subset \mathbb{Z}_2^V$ is a pointwise topology inherited from $\mathbb{Z}_2^V$. So it is enough to show that for every finite family $A_1, A_2, \cdots, A_m$ of nonempty clopen subsets in $X$ there exists a neighborhood $O$ of $e \in G$ such that $(g\psi)(A_k) = \psi(g^{-1}(A_k))$ for every $k$. Since $(g\psi)(A_k) = \psi(g^{-1}(A_k))$ we may define $O := \cap_{k=1}^m H_{A_k}$

(Another way to prove (3) is to combine (1) and \textbf{[19, Theorem 26.9]}).

(4) $\psi = (\alpha, \beta)$ is a birepresentation as we already noticed before this lemma. The t-exactness is a direct consequence of (1) or (3) together with Fact \textbf{[2.8.1].} \hfill \Box

\textbf{Theorem 4.4.} The topological group 

$$M := M(\psi) = H(w) \times G = ((\mathbb{Z}_2 \oplus V) \times V^*) \times G$$

is a non-archimedean minimal group.

\textit{Proof.} By Corollary \textbf{3.4} $M$ is non-archimedean. Use Theorem \textbf{4.1}, Lemma \textbf{4.3} and Fact \textbf{2.9} to conclude that $M$ is a minimal group. \hfill \Box

\textbf{Corollary 4.5.} Every (locally compact) non-archimedean group $G$ is a group retract of a (resp., locally compact) minimal non-archimedean group $M$ where $w(G) = w(M)$.

\textit{Proof.} Apply Theorem \textbf{4.4} taking into account Fact \textbf{2.8.1} and the local compactness of the groups $\mathbb{Z}_2, V, V^*$ (resp., $G$). \hfill \Box

\textbf{Remark 4.6.} Another proof of Corollary \textbf{4.5} can be obtained by the following way. By Lemma \textbf{4.3} a non-archimedean group $G$ can be treated as a subgroup of the group of all automorphisms $\text{Aut}(V^*)$ of the compact abelian group $V^*$. In particular, the action of $G$ on $V^*$ is t-exact. The group $V^*$ being compact is minimal. Since $V^*$ is abelian one may apply \textbf{[28, Cor. 2.8]} which implies that $V^* \times G$ is a minimal topological group. By Lemmas \textbf{3.2} and \textbf{4.2} we may assume that $w(G) = w(V^* \times G)$.

5. More characterizations of non-archimedean groups

The results and discussions above lead to the following list of characterizations (compare Lemma \textbf{3.2}).

\textbf{Theorem 5.1.} The following assertions are equivalent:

(1) $G$ is a non-archimedean topological group.
(2) $G$ is a topological subgroup of the automorphisms group (with the pointwise topology) $\text{Aut}(V)$ for some discrete Boolean ring $V$ (where $|V| = w(G)$).
(3) $G$ is embedded into the symmetric topological group $S_\kappa$ (where $\kappa = w(G)$).
(4) $G$ is a topological subgroup of the group $\text{Is}(X, d)$ of all isometries of an ultra-metric space $(X, d)$, with the topology of pointwise convergence.
The right (left) uniformity on $G$ can be generated by a system of right (left) invariant ultra-semimetrics.

$G$ is a topological subgroup of the automorphism group $\text{Aut}(K)$ for some compact abelian group $K$ (with $w(K) = w(G)$).

Proof. (1) ⇒ (2) As in Lemma 4.3.1.

(2) ⇒ (3) Simply take the embedding of $G$ into $S_V \cong S_\kappa$, with $\kappa = |V| = w(G)$.

(3) ⇒ (4) Consider the two-valued ultra-metric on the discrete space $X$ with $|X| = \kappa$.

(4) ⇒ (5) For every $z \in X$ consider the left invariant ultra-semimetric $\rho_z(s, t) := d(sz, tz)$.

Then the collection $\{\rho_z\}_{z \in X}$ generates the left uniformity of $G$.

(5) ⇒ (1) Observe that for every right invariant ultra-semimetric $\rho$ on $G$ and $n \in \mathbb{N}$ the set

$$H := \{g \in G : \rho(g, e) < 1/n\}$$

is an open subgroup of $G$.

(3) ⇒ (6) Consider the natural (permutation of coordinates) action of $S_\kappa$ on the usual Cantor additive group $\mathbb{Z}_2^\kappa$. It is easy to see that this action implies the natural embedding of $S_\kappa$ (and hence, of its subgroup $G$) into the group $\text{Aut}(\mathbb{Z}_2^\kappa)$.

(6) ⇒ (1) Let $K$ be a compact abelian group and $K^*$ be its (discrete) dual. By [19, Theorem 26.9] the natural map $\nu : g \mapsto \tilde{g}$ defines a topological anti-isomorphism of $\text{Aut}(K)$ onto $\text{Aut}(K^*)$. Now, $K^*$ is discrete, hence, $\text{Aut}(K^*)$ is non-archimedean as a subgroup of the symmetric group $S_{K^*}$. Since $G$ is a topological subgroup of $\text{Aut}(K)$ we conclude that $G$ is also non-archimedean (because its opposite group $\nu(G)$ being a subgroup of $\text{Aut}(K^*)$ is non-archimedean). □

Remark 5.2. (1) Note that the universality of $S_N$ among Polish groups was proved by Becker and Kechris (see [4, Theorem 1.5.1]). The universality of $S_\kappa$ for N.A. groups with weight $\leq \kappa$ can be proved similarly. It appears in the work of Higasikawa, [20, Theorem 3.1].

(2) Isometry groups of ultra-metric spaces studied among others by Lemin and Smirnov [25]. Note for instance that [25, Theorem 3] implies the equivalence (1) ⇔ (4). Lemin [23] established that a metrizable group is non-archimedean iff it has a left invariant compatible ultra-metric.

(3) In item (6) of Theorem 5.1 it is essential that the compact group $K$ is abelian. For every connected non-abelian compact group $K$ the group $\text{Aut}(K)$ is not N.A. containing a nontrivial continuous image of $K$.

(4) Every non-archimedean group admits a topologically faithful unitary representation on a Hilbert space. It is straightforward for $S_X$ (hence, also for its subgroups) via permutation of coordinates linear action.

6. AUTOMORPHIZABLE ACTIONS AND EPIMORPHISMS IN TOPOLOGICAL GROUPS

Resolving a longstanding principal problem by K. Hofmann, Uspenskij [50] has shown that in the category of Hausdorff topological groups epimorphisms need not have a dense range. Dikranjan and Tholen present in [14] a rather direct proof of this important result of Uspenskij. Pestov gave later a criterion [40, 42] (Fact 6.1) which we will use below in Theorem 6.7. This criterion is closely related to the
natural concept of the free topological $G$-group $F_G(X)$ of a $G$-space $X$ introduced by the first author [29]. It is a natural $G$-space version of the usual free topological group. A topological (uniform) $G$-space $X$ is said to be automorphizable if $X$ is a topological (uniform) $G$-subspace of a $G$-group $Y$ (with its right uniform structure). Equivalently, if the universal morphism $X \to F_G(X)$ of $X$ into the free topological (uniform) $G$-group $F_G(X)$ of the (uniform) $G$-space $X$ is an embedding.

**Fact 6.1.** (Pestov [40, 42]) Let $f : M \to G$ be a continuous homomorphism between Hausdorff topological groups. Denote by $X := G/H$ the left coset $G$-space, where $H$ is the closure of the subgroup $f(M)$ in $G$. The following are equivalent:

1. $f : M \to G$ is an epimorphism.
2. The free topological $G$-group $F_G(X)$ of the $G$-space $X$ is trivial.

Triviality in (2) means, ‘as trivial as possible’, isomorphic to the cyclic discrete group.

Let $X$ be the $n$-dimensional cube $[0,1]^n$ or the $n$-dimensional sphere $S_n$. Then by [29] the free topological $G$-group $F_G(X)$ of the $G$-space $X$ is trivial for every $n \in \mathbb{N}$, where $G = \text{Homeo} (X)$ is the corresponding homeomorphism group. So, one of the possible examples of an epimorphism which is not dense can be constructed as the natural embedding $H \hookrightarrow G$ where $G = \text{Homeo} (S_1)$ and $H = G_z$ is the stabilizer of a point $z \in S_1$. The same example serve as an original counterexample in the paper of Uspenskij [50].

In contrast, for Stone spaces, we have:

**Proposition 6.2.** Every continuous action of a topological group $G$ on a Stone space $X$ is automorphizable (in $\mathcal{NA}$). Hence the canonical $G$-map $X \to F_G(X)$ is an embedding.

**Proof.** Use item (2) of Lemma 4.3. □

Roughly speaking this result says that the action by conjugations of a subgroup $H$ of a non-archimedean group $G$ on $G$ reflects all possible difficulties of the Stone actions. Below in Theorem 6.5 we extend Proposition 6.2 to a much larger class of actions on non-archimedean uniform spaces, where $X$ need not be compact. This will be used in Theorem 6.7 about epimorphisms into $\mathcal{NA}$-groups.

**Definition 6.3.** [30] Let $\pi : G \times X \to X$ be an action and $\mu$ be a uniformity on $X$. We say that the action is $\pi$-uniform if for every $\varepsilon \in \mu$ and $g_0 \in G$ there exist: $\delta \in \mu$ and a neighborhood $O$ of $g_0$ in $G$ such that

$$(gx, gy) \in \varepsilon \quad \forall (x, y) \in \delta, \ g \in O.$$  

It is an easy observation that if the action $\pi : G \times X \to X$ is $\pi$-uniform and all orbit maps $\tilde{x} : G \to X$ are continuous then $\pi$ is continuous.

**Lemma 6.4.** [30] Let $\mu$ be a uniformity on a $G$-space $X$ which generates its topology. Then the action $\pi : G \times X \to X$ is $\pi$-uniform in each of the following cases:

1. $X$ is a $G$-group and $\mu$ is the right or left uniformity on $X$.
2. $X$ is the coset $G$-space $G/H$ with respect to the standard right uniformity (which is always compatible with the topology).
3. $\mu$ is the uniformity of a $G$-invariant metric.
(4) $X$ is a compact $G$-space and $\mu$ is the unique compatible uniformity on $X$.

A function $|| \cdot || : G \to [0, \infty)$ on an abelian group $(G, +)$ is an ultra-norm if

$||u|| = 0 \Leftrightarrow u = 0$, $||u|| = || - u||$ and

$||u + v|| \leq \max\{||u||, ||v||\}$ $\forall u, v \in G$.

A group $(G, +, || \cdot ||)$ with an ultra-norm $|| \cdot ||$ is an ultra-normed space. The definition of an ultra-seminorm is understood. It is easy to see that if the topology on $(G, +)$ can be generated by a system of ultra-seminorms then $G$ is a non-archimedean group (cf. Theorem 5.1, the equivalence $(1) \Leftrightarrow (5)$) and its right (=left) uniformity is just the uniform structure induced on $G$ by the given system of ultra-seminorms. Every abelian non-archimedean metrizable group admits an ultra-norm (see Theorems 6.4 and 6.6 in [53]).

6.1. Arens-Eells linearization theorem for actions. Recall that the well known Arens-Eells linearization theorem (cf. [2]) asserts that every uniform (metric) space can be (isometrically) embedded into a locally convex vector space (resp., normed space). For a metric space $(X, d)$ one can define a real normed space $(A(X), || \cdot ||)$ as the set of all formal linear combinations

$$\sum_{i=1}^{n} c_i(x_i - y_i)$$

where $x_i, y_i \in X$ and $c_i \in \mathbb{R}$. For every $u \in A(X)$ one may define the norm by

$$||u|| := \inf \left\{ \sum_{i=1}^{n} |c_i|d(x_i, y_i) : u = \sum_{i=1}^{n} c_i(x_i - y_i) \right\}.$$

Now if $(X, z)$ is a pointed space with some $z \in X$ then $x \mapsto x - z$ defines an isometric embedding of $(X, d)$ into $A(X)$ (as a closed subset).

This theorem on isometric linearization of metric spaces can be naturally extended to the case of non-expansive semigroup actions provided that the metric is bounded [33], or, assuming only that the orbits are bounded [45]. Furthermore, suppose that an action of a group $G$ on a metric space $(X, d)$ with bounded orbits is only uniform in the sense of Definition 6.3 (and not necessarily non-expansive). Then again such an action admits an isometric $G$-linearization on a normed space.

Here we give a non-archimedean version of Arens-Eells type theorem for uniform group actions.

**Theorem 6.5.** Let $\pi : G \times X \to X$ be a continuous $\pi$-uniform action of a topological group $G$ on a non-archimedean Hausdorff uniform space $(X, \mu)$.

1. Then there exist a $\mathcal{N}A$ Hausdorff Boolean $G$-group $E$ and a uniform $G$-embedding

$$\alpha : X \hookrightarrow E$$

such that $\alpha(X)$ is closed. Hence, $(X, \mu)$ is uniformly $G$-automorphizable (in $\mathcal{N}A$).

2. Let $(X, d)$ be an ultra-metric space and suppose there exists a $d$-bounded orbit $Gx_0$ for some $x_0 \in X$. Then there exists an ultra-normed Boolean $G$-group $E$ and an isometric $G$-embedding $\alpha : X \hookrightarrow E$ such that $\alpha(X)$ is closed.
(3) Every ultra-metric space is isometric to a closed subset of an ultra-normed Boolean group.

Proof. (1) Every non-archimedean uniformity $\mu$ on $X$ can be generated by a system $\{d_j\}_{j \in J}$ of ultra-semimetrics. Furthermore one may assume that $d_j \leq 1$. Indeed, every uniform partition of $X$ leads to the naturally defined $0, 1$ ultra-semimetric. We can suppose in addition that $X$ contains a $G$-fixed point $\theta$. Indeed, adjoining if necessary a fixed point $\theta$ and defining $d_j(x, \theta) = d_j(\theta, x) = 1$ for every $x \in X$, we get again an ultra-semimetric.

Furthermore one may assume that for any finite collection $d_{j_1}, d_{j_2}, \ldots, d_{j_m}$ from the system $\{d_j\}_{j \in J}$ the ultra-semimetric $\max\{d_{j_1}, d_{j_2}, \ldots, d_{j_m}\}$ also belongs to our system.

Consider the free Boolean group $(P_\infty(X), +)$ over the set $X$. The elements of $P_\infty(X)$ are finite subsets of $X$ and the group operation $+$ is the symmetric difference of subsets. The zero element (represented by the empty subset) we denote by $0$. Clearly, $u = -u$ for every $u \in P_\infty(X)$.

For every nonzero $u = \{x_1, x_2, x_3, \ldots, x_m\} \in P_\infty(X)$, define the support $\text{supp}(u)$ as $u$ treating it as a subset of $X$. So $x \in X$ is a support element of $u$ iff $x \in \{x_1, x_2, x_3, \ldots, x_m\}$. Let us say that $u$ is even (odd) if the number of support elements $m$ is even (resp., odd). Define the natural homomorphism $\text{sgn} : P_\infty(X) \to \mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$, where, $\text{sgn}(u) = \overline{0}$ iff $u$ is even. We denote by $E$ the subgroup $\text{sgn}^{-1}(\overline{0})$ of all elements in $P_\infty(X)$.

Consider the natural set embedding

$$\iota : X \hookrightarrow P_\infty(X), \quad \iota(x) = \{x\}.$$  

Sometimes we will identify $x \in X$ and $\iota(x) = \{x\} \in P_\infty(X)$.

Define also another embedding of sets

$$\alpha : X \to E, \quad \alpha(x) = x - \theta.$$  

Observe that $\alpha(x) - \alpha(y) = \iota(x) - \iota(y) = x - y$ for every $x, y \in X$.

By a configuration we mean a finite subset of $X \times X$ (finite relations). Denote by $\text{Conf}$ the set of all configurations. We can think of any $\omega \in \text{Conf}$ as a finite set of some pairs

$$\omega = \{(x_1, x_2), (x_3, x_4), \ldots, (x_{2n-1}, x_{2n})\},$$  

where all $\{x_i\}_{i=1}^{2n}$ are (not necessarily distinct) elements of $X$. If $x_i \neq x_k$ for all distinct $1 \leq i, k \leq 2n$ then $\omega$ is said to be normal. For every $\omega \in \text{Conf}$ the sum

$$u := \sum_{i=1}^{2n} x_i = \sum_{i=1}^{n} (x_{2i-1} - x_{2i}).$$  

necessarily belongs to $E$ and we say that $\omega$ represents $u$ or, that $\omega$ is an $u$-configuration. Notation $\omega \in \text{Conf}(u)$. We denote by $\text{Norm}(u)$ the set of all normal configurations of $u$. If $\omega \in \text{Norm}(u)$ then necessarily $\omega \subseteq \text{supp}(u) \times \text{supp}(u)$ and $\text{supp}(u) = \{x_1, x_2, \ldots, x_{2n}\}$. So, $\text{Norm}(u)$ is a finite set for any given $u \in E$.

Let $j \in J$. Our aim is to define an ultra-seminorm $\| \cdot \|_j$ on the Boolean group $(E, +)$ such that $d_j(x, y) = \|x - y\|_j$. For every configuration $\omega$ we define its
For every even nonzero element \( u \in E \) and every \( u \)-configuration
\[
\omega = \{(x_1, x_2), (x_3, x_4), \ldots, (x_{2n-1}, x_{2n})\}
\] define the following elementary reductions:

1. Deleting a trivial pair \((t, t)\). That is, deleting the pair \((x_{2i-1}, x_{2i})\) whenever \( x_{2i-1} = x_{2i} \).
2. Define the trivial inversion at \( i \) of \( \omega \) as the replacement of \((x_{2i-1}, x_{2i})\) by the pair in the reverse order \((x_{2i}, x_{2i-1})\).
3. Define the basic chain reduction rule as follows. Assume that there exist distinct \( i \) and \( k \) such that \( x_{2i} = x_{2k-1} \). We delete in the configuration \( \omega \) two pairs \((x_{2i-1}, x_{2i}), (x_{2k-1}, x_{2k})\) and add the following new pair \((x_{2i-1}, x_{2k})\).

Then in all three cases we get again an \( u \)-configuration. The reductions \(1\) and \(2\) do not change the \( d_j\)-length of the configuration. Reduction \(3\) cannot exceed the \( d_j\)-length.

**Proof.** Comes directly from the axioms of ultra-semimetric. In the proof of \(3\) observe that
\[
x_{2i-1} + x_{2i} + x_{2k-1} + x_{2k} = x_{2i-1} + x_{2k}
\] in \( E \). This ensures that the new configuration is again an \( u \)-configuration. \( \square \)

**Claim 2:** For every even nonzero element \( u \in E \) and every \( u \)-configuration \( \omega \) there exists a normal \( u \)-configuration \( \nu \) such that \( \varphi_j(\nu) \leq \varphi_j(\omega) \).

**Proof.** Using Claim 1 after finitely many reductions of \( \omega \) we get a normal \( u \)-configuration \( \nu \) such that \( \varphi_j(\nu) \leq \varphi_j(\omega) \). \( \square \)

Now we define the desired ultra-seminorm \( || \cdot ||_j \). For every \( u \in E \) define
\[
||u||_j = \inf_{\omega \in \text{Conf}(u)} \varphi_j(\omega).
\]

**Claim 3:** For every nonzero \( u \in E \) we have
\[
||u||_j = \min_{\omega \in \text{Normal}(u)} \varphi_j(\omega).
\]

**Proof.** By Claim 2 it is enough to compute \( ||u||_j \) via normal \( u \)-configurations. So, since \( \text{Norm}(u) \) is finite, we may replace \( \inf \) by \( \min \). \( \square \)

**Claim 4:** \( || \cdot ||_j \) is an ultra-seminorm on \( E \).

**Proof.** Clearly, \( ||u||_j \geq 0 \) and \( ||u||_j = || - u ||_j \) (even \( u = -u \)) for every \( u \in E \). For the \( 0 \)-configuration \( \{(\theta, \theta)\} \) we obtain that \( ||0||_j \leq d_j(\theta, \theta) = 0 \). So \( ||0||_j = 0 \). We have to show that
\[
||u + v||_j \leq \max\{||u||_j, ||v||_j\} \quad \forall u, v \in E.
\]
Assuming the contrary, there exist configurations
\[
\{(x_i, y_i)\}_{i=1}^n, \quad \{(t_i, s_i)\}_{i=1}^m
\]
with
\[ u = \sum_{i=1}^{n} (x_i - y_i), \quad v = \sum_{i=1}^{m} (t_i - s_i) \]
such that
\[ ||u + v||_{j} > c := \max \{ \max_{1 \leq i \leq n} d_j(x_i, y_i), \max_{1 \leq i \leq m} d_j(t_i, s_i) \} \]
but this contradicts the definition of \( ||u + v||_{j} \) because
\[ u + v = \sum_{i=1}^{n} (x_i - y_i) + \sum_{i=1}^{m} (t_i - s_i) \]
and hence
\[ \omega := \{(x_1, y_1), \ldots, (x_n, y_n), (t_1, s_1), \ldots, (t_m, s_m)\} \]
is a configuration of \( u + v \) with \( ||u + v||_{j} > \varphi_j(\omega) = c \), a contradiction to the definition of \( || \cdot ||_{j} \).

Claim 5: \( \alpha : (X, d_j) \hookrightarrow (E, || \cdot ||_{j}), \alpha(x) = x - \theta \) is an isometric embedding, that is,
\[ ||x - y||_{j} = d_j(x, y) \quad \forall \; x, y \in X. \]

Proof. By Claim 3 we may compute via normal configurations. For the element \( u = x - y \neq 0 \) only possible normal configurations are \{\( (x, y) \)\} or \{\( (y, x) \)\}. So \( ||x - y||_{j} = d_j(x, y) \).

Claim 6: For any given \( u \in E \) with \( u \neq 0 \) we have
\[ ||u||_{j} \geq \min \{ d_j(x_i, x_k) : x_i, x_k \in \text{supp}(u), x_i \neq x_k \}. \]

Proof. Easily comes from Claims 2 and 3.

Claim 7: For any given \( u \in E \) with \( u \neq 0 \) there exists \( j_0 \in J \) such that \( ||u||_{j_0} > 0 \).

Proof. Since \( u \neq 0 \) we have at least two elements in \( \text{supp}(u) \). Since \( (X, \mu) \) is Hausdorff the system \( \{d_j\}_{j \in J} \) of ultra-semimetrics separates points of \( X \). So some finite subsystem \( d_{j_1}, d_{j_2}, \ldots, d_{j_m} \) separates points of \( \text{supp}(u) \). By our assumption the ultra-semimetric \( d_{j_0} := \max \{d_j, d_{j_2}, \ldots, d_{j_m} \} \) belongs to our system \( \{d_j\}_{j \in J} \). Then
\[ \min \{ d_{j_0}(x_i, x_k) : x_i, x_k \in \text{supp}(u), x_i \neq x_k \} > 0. \]
Claim 6 implies that \( ||u||_{j_0} > 0 \).

It is easy to see that the family \( \{|| \cdot ||_{j}\}_{j \in J} \) of ultra-seminorms induces a non-archimedean group topology on the Boolean group \( E \) and a non-archimedean uniformity \( \mu_\ast \) which is the right (=left) uniformity on \( E \). By Claim 7 the topology on \( E \) is Hausdorff.

We have the natural group action
\[ \pi : G \times E \to E, (g, u) \mapsto gu \]
induced by the given action $G \times X \to X$. Clearly, $g(u + v) = gu + gv$ for every $(g, u, v) \in G \times E \times E$. So this action is by automorphisms. Since $g\theta = \theta$ for every $g \in G$ it follows that $\alpha : X \to E$ is a $G$-embedding.

Now we show that the action $\pi$ of $G$ on $E$ is uniform and continuous. Indeed, the original action on $(X, \mu)$ is $\pi$-uniform. Hence, for every $j \in J$, $\varepsilon > 0$ and $g_0 \in G$, there exist: a finite subset $\{j_1, \ldots, j_n\}$ of $J$, $\delta > 0$ and a neighborhood $O(g_0)$ of $g_0$ in $G$ such that

$$
d_j(gx, gy) \leq \varepsilon \quad \forall \max_{1 \leq i \leq n} d_j(x, y) \leq \delta, \ g \in O.
$$

Then by Claim 3 it is easy to see that

$$
\|gu\|_j \leq \varepsilon \quad \forall \max_{1 \leq i \leq n} \|u\|_j, \ g \in O.
$$

This implies that the action $\pi$ of $G$ on $(E, \mu_\ast)$ is uniform. Claim 5 implies that $\alpha : X \hookrightarrow E$ is a topological $G$-embedding. Since $\alpha(X)$ algebraically spans $E$ it easily follows that every orbit mapping $G \to E$, $g \mapsto gu$ is continuous for every $u \in E$. So we can conclude that $\pi$ is continuous (see the remark after Definition 6.3) and $E$ is a $G$-group.

Finally we check that $\alpha(X)$ is closed in $E$. Let $u \in E$ and $u \notin \alpha(X)$. Since $u - x + \theta \neq 0$ for every $x \in X$, we can suppose that there are at least two elements in $\text{supp}(u) \cap (X \setminus \{\theta\})$. Similarly to the proof of Claim 7 we may choose $j_0 \in J$ and $\varepsilon_1 > 0$ such that

$$
\varepsilon_1 := \min\{d_{j_0}(x_i, x_k) : x_i, x_k \in \text{supp}(u), x_i \neq x_k\} > 0.
$$

Furthermore, one may assume in addition that

$$
\varepsilon_2 := \min\{d_{j_0}(x_i, \theta) : x_i \in \text{supp}(u), x_i \neq \theta\} > 0.
$$

Define $\varepsilon_0 := \min\{\varepsilon_1, \varepsilon_2\}$.

For every $x \in X$, every normal configuration $\omega$ of $u - x + \theta \neq 0$ contains an element $(s, t)$ such that $\{s, t\} \subset \text{supp}(u) \cup \{\theta\}$. Therefore,

$$
\varphi_{j_0}(\omega) \geq d_{j_0}(s, t) \geq \varepsilon_0.
$$

So by Claim 3 we obtain $\|u - x + \theta\|_j \geq \varepsilon_0$ for every $x \in X$.

Summing up we finish the proof of (1).

(2) The proof in the second case is similar. We only explain why we may suppose that $X$ contains a $G$-fixed point. Indeed, as in the paper of Schröder [45, Remark 5] we can look at $(X, d)$ as embedded into the space $\exp(X)$ of all bounded closed subsets endowed with the standard Hausdorff metric $d_H$ defined by

$$
d_H(A, B) := \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\}.
$$

The closure $cl(Gx_0)$ of the orbit $Gx_0$ in $X$ is bounded and defines an element $\theta \in \exp(X)$. Consider the metric subspace $X' := X \cup \{\theta\} \subset \exp(X)$. It is easy to see that the induced action of $G$ on $X'$ is well defined and remains uniform (Definition 6.3) with respect to the metric $d_H|_{X'}$. Clearly, $\theta$ is a $G$-fixed point in $X'$. This implies that all orbit maps $G \to X'$ are continuous. It follows that the action of $G$ on $X'$ is continuous (see the remark after Definition 6.3).
Finally observe that since $d$ is an ultra-metric the Hausdorff metric $d_H$ on $\exp(X)$ is also an ultra-metric. Hence, $d_H|_{X'}$ is an ultra-metric on $X'$. To prove the strong triangle inequality for $d_H$ we will use the following lemma.

**Lemma 6.6.** Let $(X, d)$ be an ultra-metric space and $A, B, C$ subsets of $X$. Then
\[
\sup_{a \in A} d(a, C) \leq \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, C) \}
\]

*Proof.* Let $M := \sup_{a \in A} d(a, C)$. Assuming the contrary,
\[
M > d(a, B) \quad \forall a \in A
\]
and also
\[
M > d(b, C) \quad \forall b \in B.
\]
Set $a_0 \in A$. Since $M > d(a, B) \quad \forall a \in A$, we have in particular $M > d(a_0, B)$. So there exists $b_0 \in B$ such that $M > d(a_0, b_0)$. Now, $M > d(b, C) \quad \forall b \in B$, hence, there exists $c_0 \in C$ such that $M > d(b_0, c_0)$. Since $d$ is an ultra-metric we obtain that $M > d(a_0, c_0) \geq d(a_0, C)$. Since $a_0$ is an arbitrary element of $A$ we get that $M > \sup_{a \in A} d(a, C) = M$. This clearly contradicts our assumption. \hfill \Box

We can now prove the strong triangle inequality for $d_H$. Using Lemma 6.6 twice we obtain that
\[
\sup_{a \in A} d(a, C) \leq \max \{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, C) \}
\]
and also (by switching $A \leftrightarrow C$)
\[
\sup_{c \in C} d(A, c) \leq \max \{ \sup_{b \in B} d(A, b), \sup_{c \in C} d(B, c) \}.
\]
This implies that
\[
d_H(A, C) \leq \max \{ d_H(A, B), d_H(B, C) \}.
\]

(3) Directly follows from (2). \hfill \Box

**Theorem 6.7.** Let $G$ be a non-archimedean group. If a continuous homomorphism $f : M \to G$ is an epimorphism in the category of Hausdorff topological groups then $f(M)$ is dense in $G$.

*Proof.* Denote by $H$ the closure of the subgroup $f(M)$ in $G$. We have to show that $H = G$. Assuming the contrary consider the *nontrivial* Hausdorff coset $G$-space $G/H$. Recall that the sets
\[
\tilde{U} := \{(aH, bH) : bH \subseteq UaH\},
\]
where $U$ runs over the neighborhoods of $e$ in $G$, form a uniformity base on $G/H$. This uniformity (called the right uniformity) is compatible with the quotient topology (see for instance [2]).

The fact that $G$ is $\mathcal{NA}$ implies that the right uniformity on $G/H$ is non-archimedean. Indeed, if $\mathcal{B}$ is a local base at $e$ consisting of clopen subgroups then $\tilde{\mathcal{B}} := \{ \tilde{U} : U \in \mathcal{B} \}$ is a base for the right uniformity of $G/H$ and its elements are equivalence relations. To see this just use the fact that $H$ as well as
the elements of $\mathcal{B}$ are all subgroups of $G$. By Lemma 6.4.2 the natural left action $\pi : G \times G/H \to G/H$ is $\pi$-uniform. Obviously this action is also continuous. Hence, we can apply Theorem 6.5.1 to conclude that the nontrivial $G$-space $X := G/H$ is $G$-automorphizable in $\mathcal{N} \mathcal{A}$. In particular, we obtain that there exists a non-trivial equivariant morphism of the $G$-space $X$ to a Hausdorff $G$-group $E$. This implies that the free topological $G$-group $F_G(X)$ of the $G$-space $X$ is not trivial. Now by the criterion of Pestov (Fact 6.1) we conclude that $f : M \to G$ is not an epimorphism.

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