SYMBOLIC APPROXIMATE TIME-OPTIMAL CONTROL

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ABSTRACT. There is an increasing demand for controller design techniques capable of addressing the complex requirements of today’s embedded applications. This demand has sparked the interest in symbolic control where lower complexity models of control systems are used to cater for complex specifications given by temporal logics, regular languages, or automata. These specification mechanisms can be regarded as qualitative since they divide the trajectories of the plant into bad trajectories (those that need to be avoided) and good trajectories. However, many applications require also the optimization of quantitative measures of the trajectories retained by the controller, as specified by a cost or utility function. As a first step towards the synthesis of controllers reconciling both qualitative and quantitative specifications, we investigate in this paper the use of symbolic models for time-optimal controller synthesis. We consider systems related by approximate (alternating) simulation relations and show how such relations enable the transfer of time-optimality information between the systems. We then use this insight to synthesize approximately time-optimal controllers for a control system by working with a lower complexity symbolic model. The resulting approximately time-optimal controllers are equipped with upper and lower bounds for the time to reach a target, describing the quality of the controller. The results described in this paper were implemented in the Matlab Toolbox Pessoa [1] which we used to work out several illustrative examples reported in this paper.

1. INTRODUCTION

Symbolic abstractions are simpler descriptions of control systems, typically with finitely many states, in which each symbolic state represents a collection or aggregate of states in the control system. The power of abstractions has been exploited in the computer science community over the years, and only recently started to gather the attention of the control systems community. In the present paper we analyze the suitability of symbolic abstractions of control systems to synthesize controllers enforcing both qualitative and quantitative specifications.

Qualitative specifications require the controller to preclude certain undesired trajectories from the system to be controlled. The term qualitative refers to the fact that all the desired trajectories are treated as being equally good. Examples of qualitative specifications include requirements given by means of temporal-logics, \( \omega \)-regular languages, or automata on infinite strings. These specifications are hard (if not impossible) to address with classical control design theories. In practice,
most solutions to such problems are obtained through hierarchical designs with supervisory controllers on the top layers. Such designs are usually the result of an ad-hoc process for which correctness guarantees are hard to obtain. Moreover, these kinds of designs require a certain level of insight that just the most experienced system designers possess. Recent work in symbolic control [2, 3, 4] has emerged as an alternative to ad-hoc designs.

In many practical applications, while there are plant trajectories that must be eliminated, there is also a need to select the best of the remaining trajectories. Typically, the best trajectory is specified by means of a cost or utility associated to each trajectory. The control design problem then requires the removal of the undesirable trajectories and the selection of the minimum cost or maximum utility trajectory. As a first step towards our objective of synthesizing controllers enforcing qualitative and quantitative objectives, we consider in the present paper the synthesis of time-optimal controllers for reachability specifications. A problem of this kind, widely studied in the robotics literature, is that of optimal kinodynamic motion planning. Such problem is known to easily become computationally hard [5]. We discuss in Section 4.4 where the complexity of solving this kind of problems resides when following our methods.

Since the illustrious seminal contributions in the 50’s by Pontryagin [6] and Bellman [7], the design of optimal controllers has remained a standing quest of the controls community. Despite the several advances since then, solving optimal control problems with complex geometries on the state space, constraints in the input space, and/or complex dynamics is still a daunting task. This has motivated the development of numerical techniques to solve complex optimization problems. A common method in the literature is to discretize the dynamics and apply optimal search algorithms on graphs such as Dijkstra’s algorithm [8, 9]. The philosophy behind such work is to show that by using finer discretizations, one obtains controllers that are arbitrarily close to the optimal controller. In contrast, our objective is not to approach the optimal solution asymptotically, but rather to effectively compute an approximate solution and to establish how much it deviates from the optimal one. Other techniques to solve complex optimal control problems include Mixed (Linear or Quadratic) Integer Programming [10] and SAT-solvers [11].

The approach we follow in the present paper is complementary to the aforementioned techniques and our contribution is twofold:

- At the theoretical level, we show that time-optimality information can be transferred from a system $S_a$ to a system $S_b$ when system $S_a$ is related to system $S_b$ by an approximate (alternating) simulation relation. Hence, we decouple the analysis of optimality considerations from the design of algorithms extracting a discretization $S_a$ from the original system $S_b$. Using this result, we show how to construct an approximately time-optimal controller for system $S_b$ from a time-optimal controller for system $S_a$. Moreover, we also provide bounds on how much the cost or utility of the approximately time-optimal controller deviates from the true cost or utility. These bounds are often conservative due to the, in general, non-deterministic nature of the abstractions used. However, these bounds can still be useful in practice as performance guarantees for the obtained solutions.
- At the practical level, we illustrate the practicality of our results by implementing them in the freely available Matlab toolbox Pessoa [12, 1].
report on several examples conducted in Pessoa to illustrate the feasibility of the proposed approach.

The proposed results are independent of the specific techniques employed in the construction of symbolic abstractions provided that the existence of approximately (alternating) simulations relations is established. The specific constructions reported in [13, 14] show that our assumptions can be met for a large class of systems, thus making the use of the proposed methods widely applicable. Furthermore, effective algorithms and data structures from computer science can be used to implement the proposed techniques, see for example the recent work on optimal synthesis [15]. In particular, the examples presented in the current paper, performed in the Matlab toolbox Pessoa, were implemented using Binary Decision Diagrams (BDD’s) [16] to store systems modeling both plants and controllers. The fact that BDD’s can be used to automatically generate hardware [17] or software [18] implementations of the controllers makes them specially attractive.

The paper is organized as follows: in Section 2 we review the notions of systems and relationships between systems. Section 3 formalizes the optimal control problem studied in this paper, and establishes relationships between the attainable costs for two systems related by (alternating) simulation relationships. Section 4 provides an algorithm to solve time-optimal control problems approximately by relying on symbolic abstractions. For the convenience of the readers wishing to solve concrete time-optimal problems, we provide a concise description of all the necessary steps in Section 1.3. Some illustrative examples are presented in Section 5 and Section 6 concludes the paper with a brief discussion.

2. Preliminaries

2.1. Notation. Let us start by introducing some notation that will be used throughout the present paper. We denote by \( \mathbb{N} \) the natural numbers including zero and by \( \mathbb{N}^+ \) the strictly positive natural numbers. With \( \mathbb{R}^+ \) we denote the strictly positive real numbers, and with \( \mathbb{R}_+^\ast \) the positive real numbers including zero. The identity map on a set \( A \) is denoted by \( 1_A \). If \( A \) is a subset of \( B \) we denote by \( i_A : A \rightarrow B \) or simply by \( i \) the natural inclusion map taking any \( a \in A \) to \( i(a) = a \in B \). The closed ball centered at \( x \in \mathbb{R}^n \) with radius \( \varepsilon \) is defined by \( B_\varepsilon(x) = \{ y \in \mathbb{R}^n \mid \| x - y \| \leq \varepsilon \} \). We denote by \( \text{int}(A) \) the interior of a set \( A \).

A normed vector space \( V \) is a vector space equipped with a norm \( \| \cdot \| \), as is well-known this induces the metric \( d(x, y) = \| x - y \|, x, y \in V \). Given a vector \( x \in \mathbb{R}^n \) we denote by \( x_i \) the \( i \)-th element of \( x \) and by \( \| x \| \) the infinity norm of \( x \); we recall that \( \| x \| = \max\{ |x_1|, |x_2|, ..., |x_n| \} \), where \( |x_i| \) denotes the absolute value of \( x_i \). We identify a relation \( R \subseteq A \times B \) with the map \( R : A \rightarrow 2^B \) defined by \( b \in R(a) \) iff \( (a, b) \in R \). For a set \( S \subseteq A \) the set \( R(S) \) is defined as \( R(S) = \{ b \in B : \exists a \in S, (a, b) \in R \} \). Also, \( R^{-1} \) denotes the inverse relation defined by \( R^{-1} = \{ (b, a) \in B \times A : (a, b) \in R \} \). We also denote by \( d : X \times X \rightarrow \mathbb{R}_0^+ \) a metric in the space \( X \) and by \( \pi_X : X_a \times X_b \times U_a \times U_b \rightarrow X_a \times X_b \) the projection sending \( (x_a, x_b, u_a, u_b) \in X_a \times X_b \times U_a \times U_b \) to \( (x_a, x_b) \in X_a \times X_b \).

2.2. Systems. In the present paper we use the mathematical notion of systems to model dynamical phenomena. This notion is formalized in the following definition:
Definition 2.1 (System [12]). A system $S$ is a sextuple $(X, X_0, U, \xrightarrow{} , Y, H)$ consisting of:

- a set of states $X$;
- a set of initial states $X_0 \subseteq X$
- a set of inputs $U$;
- a transition relation $\xrightarrow{} \subseteq X \times U \times X$;
- a set of outputs $Y$;
- an output map $H : X \to Y$.

A system is said to be:

- metric, if the output set $Y$ is equipped with a metric $d : Y \times Y \to \mathbb{R}_0^+$;
- countable, if $X$ is a countable set;
- finite, if $X$ is a finite set.

We use the notation $x \xrightarrow{u} y$ to denote $(x, u, y) \in \xrightarrow{}$. For a transition $x \xrightarrow{u} y$, state $y$ is called a $u$-successor, or simply successor. We denote the set of $u$-successors of a state $x$ by $\text{Post}_u(x)$. If for all states $x$ and inputs $u$ the sets $\text{Post}_u(x)$ are singletons (or empty sets) we say the system $S$ is deterministic. If, on the other hand, for some state $x$ and input $u$ the set $\text{Post}_u(x)$ has cardinality greater than one, we say that system $S$ is non-deterministic. Furthermore, if there exists some pair $(x, u)$ such that $\text{Post}_u(x) = \emptyset$ we say the system is blocking, and otherwise non-blocking. We also use the notation $U(x)$ to denote the set $U(x) = \{ u \in U | \text{Post}_u(x) \neq \emptyset \}$.

Nondeterminism arises for a variety of reasons such as modeling simplicity. Nevertheless, to every nondeterministic system $S_a$ we can associate a deterministic system $S_{d(a)}$ by extending the set of inputs:

Definition 2.2 (Associated deterministic system). The deterministic system $S_{d(a)} = (X_a, X_{a0}, U_{d(a)}, \xrightarrow{d(a)} , Y_a, H_a)$ associated with a given system $S_a = (X_a, X_{a0}, U_a, \xrightarrow{a} , Y_a, H_a)$, is defined by:

- $U_{d(a)} = U_a \times X_a$;
- $x \xrightarrow{(u,x')}_{d(a)} x'$ if there exists $x \xrightarrow{u}_{a} x'$ in $S_a$.

Sometimes we need to refer to the possible sequences of outputs that a system can exhibit. We call these sequences of outputs behaviors. Formally, behaviors are defined as follows:

Definition 2.3 (Behaviors [12]). For a system $S$ and given any state $x \in X$, a finite behavior generated from $x$ is a finite sequence of transitions:

$$y_0 \xrightarrow{} y_1 \xrightarrow{} y_2 \xrightarrow{} \ldots \xrightarrow{} y_{n-1} \xrightarrow{} y_n$$

such that $y_0 = H(x)$ and there exists a sequence of states $\{x_i\}$, and a sequence of inputs $\{u_i\}$ satisfying: $H(x_i) = y_i$ and $x_{i-1} \xrightarrow{u_{i-1}} x_i$ for all $0 \leq i < n$.

An infinite behavior generated from $x$ is an infinite sequence of transitions:

$$y_0 \xrightarrow{} y_1 \xrightarrow{} y_2 \xrightarrow{} y_3 \xrightarrow{} \ldots$$

such that $y_0 = H(x)$ and there exists a sequence of states $\{x_i\}$, and a sequence of inputs $\{u_i\}$ satisfying: $H(x_i) = y_i$ and $x_{i-1} \xrightarrow{u_{i-1}} x_i$ for all $i \in \mathbb{N}$. 
By $B_x(S)$ and $B_x^\infty(S)$ we denote the set of finite and infinite external behaviors generated from $x$, respectively. Sometimes we use the notation $y = y_0y_1y_2 \ldots y_n$, to denote external behaviors, and $y(k)$ to denote the $k$-th output of the behavior, i.e., $y_k$. A behavior $y$ is said to be maximal if there is no other behavior containing $y$ as a prefix.

Our objective is to design time-optimal controllers for control systems, which are formalized in the following definition:

**Definition 2.4** (Continuous-time control system). A continuous-time control system is a triple $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ consisting of:

- the state set $\mathbb{R}^n$;
- a set of input curves $\mathcal{U}$ whose elements are essentially bounded piece-wise continuous functions of time from intervals of the form $[a, b] \subset \mathbb{R}$ to $\mathbb{U} \subset \mathbb{R}^m$ with $a < 0 < b$;
- a smooth map $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}^n$.

A piecewise continuously differentiable curve $\xi : [a, b[ \to \mathbb{R}^n$ is said to be a trajectory or solution of $\Sigma$ if there exists $v \in \mathcal{U}$ satisfying:

$$\xi(t) = f(\xi(t), v(t)),$$

for almost all $t \in [a, b[$.

Although we have defined trajectories over open domains, we shall refer to trajectories $\xi : [0, \tau] \to \mathbb{R}^n$ defined on closed domains $[0, \tau], \tau \in \mathbb{R}^+$ with the understanding of the existence of a trajectory $\xi' : [a, b[ \to \mathbb{R}^n$ such that $\xi = \xi'_{|[0, \tau]}$. We also write $\xi_{x,v}(t)$ to denote the point reached at time $t \in [0, \tau]$ under the input $v$ from initial condition $x$; this point is uniquely determined, since the assumptions on $f$ ensure existence and uniqueness of trajectories.

### 2.3. Systems relations

The results we prove build upon certain simulation relations that can be established between systems. The first relation explains how a system can simulate another system.

**Definition 2.5** (Approximate Simulation Relation [13]). Consider two metric systems $S_a$ and $S_b$ with $Y_a = Y_b$, and let $\varepsilon \in \mathbb{R}^*_+$. A relation $R \subseteq X_a \times X_b$ is an $\varepsilon$-approximate simulation relation from $S_a$ to $S_b$ if the following three conditions are satisfied:

1. for every $(x_{a0}, x_{b0}) \in R$, there exists $(x_{a0}, x_{b0}) \in X_a \times X_b$ with $(x_{a0}, x_{b0}) \in R$;
2. for every $(x_{a}, x_{b}) \in R$ we have $d(H_a(x_{a}), H_b(x_{b})) \leq \varepsilon$;
3. for every $(x_{a}, x_{b}) \in R$ we have that $x_{a} \xrightarrow{u_a} x_{a}'$ in $S_a$ implies the existence of $x_{b} \xrightarrow{u_b} x_{b}'$ in $S_b$ satisfying $(x_{a}', x_{b}') \in R$.

We say that $S_a$ is $\varepsilon$-approximately simulated by $S_b$, or that $S_b$ $\varepsilon$-approximately simulates $S_a$, denoted by $S_a \preceq^\varepsilon S_b$, if there exists an $\varepsilon$-approximate simulation relation from $S_a$ to $S_b$.

When $S_a \preceq^\varepsilon S_b$, system $S_b$ can replicate the behavior of system $S_a$ by starting at a state $x_{b0} \in X_{b0}$ related to any initial state $x_{a0} \in X_{a0}$ and by replicating every transition in $S_a$ with a transition in $S_b$ according to (3). It then follows from (2) that the resulting behaviors will be the same up to an error of $\varepsilon$. If $\varepsilon = 0$ the second condition implies that two states $x_{a}$ and $x_{b}$ are related whenever their outputs are equal, i.e., $(x_{a}, x_{b}) \in R$ implies $H(x_{a}) = H(x_{b})$, and we say that the relation
is an exact simulation relation. When nondeterminism is regarded as adversarial, the notion of approximate simulation can be modified by explicitly accounting for nondeterminism.

**Definition 2.6** (Approximate alternating simulation relation [13]). Let $S_a$ and $S_b$ be metric systems with $Y_a = Y_b$ and let $\varepsilon \in \mathbb{R}_0^+$. A relation $R \subseteq X_a \times X_b$ is an $\varepsilon$-approximate alternating simulation relation from $S_a$ to $S_b$ if the following three conditions are satisfied:

1. For every $x_{a0} \in X_{a0}$ there exists $x_{b0} \in X_{b0}$ with $(x_{a0}, x_{b0}) \in R$;
2. For every $(x_a, x_b) \in R$ we have $d(H_a(x_a), H_b(x_b)) \leq \varepsilon$;
3. For every $(x_a, x_b) \in R$ and for every $u_a \in U_a(x_a)$ there exists $u_b \in U_b(x_b)$ such that for every $x_b' \in \text{Post}_{u_b}(x_b)$ there exists $x_a' \in \text{Post}_{u_a}(x_a)$ satisfying $(x_a', x_b') \in R$.

We say that $S_a$ is $\varepsilon$-approximately alternatingly simulated by $S_b$ or that $S_b$ $\varepsilon$-approximately alternately simulates $S_a$, denoted by $S_a \preceq_{\text{AS}} \varepsilon S_b$, if there exists an $\varepsilon$-approximate alternating simulation relation from $S_a$ to $S_b$.

Note that for deterministic systems the notion of alternating simulation degenerates into that of simulation. In general, the notions of simulation and alternating simulation are incomparable as illustrated by Example 4.21 in [13]. Also note that for any system $S_a$, its deterministic counterpart $S_d(a)$ satisfies $S_a \preceq_{\text{AS}} \varepsilon S_d(a)$. As in the case of exact simulation relations, we say a 0-approximate alternating simulation relation is an exact alternating simulation relation.

2.4. Composition of systems. The feedback composition of a controller $S_c$ with a plant $S_a$ describes the concurrent evolution of these two systems subject to synchronization constraints. In this paper we use the notion of extended alternating simulation relation to describe these constraints. The following formal definition is only used in the proof of Lemma 3.4. The readers not interested in the proof can simply replace the symbol $S_c \times_{\mathcal{F}^c} S_a$, defined below, with “controller $S_c$ acting on the plant $S_a$.”

**Definition 2.7** (Extended alternating simulation relation [13]). Let $R$ be an alternating simulation relation from system $S_a$ to system $S_b$. The extended alternating simulation relation $R^c \subseteq X_a \times X_b \times U_a \times U_b$ associated with $R$ is defined by all the quadruples $(x_a, x_b, u_a, u_b) \in X_a \times X_b \times U_a \times U_b$ for which the following three conditions hold:

1. $(x_a, x_b) \in R$;
2. $u_a \in U_a(x_a)$;
3. $u_b \in U_b(x_b)$ and for every $x_b' \in \text{Post}_{u_b}(x_b)$ there exists $x_a' \in \text{Post}_{u_a}(x_a)$ satisfying $(x_a', x_b') \in R$.

The interested reader is referred to [13] for a detailed explanation on how the following notion of feedback composition guarantees that the behavior of the plant is restricted by controlling only its inputs.

**Definition 2.8** (Approximate feedback composition [13]). Let $S_c$ and $S_a$ be two metric systems with the same output sets $Y_c = Y_a$, normed vector spaces, and let $R$ be an $\varepsilon$-approximate alternating simulation relation from $S_c$ to $S_a$. The feedback composition of $S_c$ and $S_a$ with interconnection relation $\mathcal{F} = R^c$, denoted by $S_c \times_{\mathcal{F}} S_a$, is the system $(X_F, X_F, U_F, H_F)$ consisting of:
\begin{itemize}
  \item $X_F = \pi_X(F) = R$;
  \item $X_{F0} = X_F \cap (X_{a0} \times X_{a0})$;
  \item $U_F = U_c \times U_a$;
  \item $(x_c, x_a) \xrightarrow{(u_c, u_a)} (x_c', x_a')$ if the following three conditions hold:
    \begin{enumerate}
      \item $(x_c, u_c, x_c') \in c$;
      \item $(x_a, u_a, x_a') \in a$;
      \item $(x_c, x_a, u_c, u_a) \in F$;
    \end{enumerate}
  \item $Y_F = Y_c = Y_a$;
  \item $H_F(x_c, x_a) = \frac{1}{2}(H(x_c) + H(x_a))$.
\end{itemize}

We also denote by $S_c \times_F S_a$ exact feedback compositions of systems, i.e., whenever $F = R^e$ with $R$ an exact ($\varepsilon = 0$) alternating simulation relation.

3. Time-optimal control and simulation relations

In this section we provide the main theoretical contribution of this paper by explaining how approximate simulation relations can be used to relate time-optimality information.

3.1. Problem definition. To simplify the presentation, we consider only systems in which $X_a = Y_a$ and $H_a = 1_{X_a}$. However, all the results in this paper can be easily extended to systems with $X_a \neq Y_a$ and $H_a \neq 1_{X_a}$ as we explain at the end of Section 3.

**Problem 3.1 (Reachability).** Let $S_a$ be a system with $Y_a = X_a$ and $H_a = 1_{X_a}$, and let $W \subseteq X_a$ be a set of outputs. Let $S_c$ be a controller and $R$ an alternating simulation relation from $S_c$ to $S_a$. The pair $(S_c, F)$, with $F = R^e$, is said to solve the reachability problem if there exists $x_0 \in X_{F0}$ such that for every maximal behavior $y \in B_{x_0}(S_c \times_F S_a) \cup B_{x_0}^\omega(S_c \times_F S_a)$, there exists $k(x_0) \in \mathbb{N}$ for which $y(k(x_0)) = y_{k(x_0)} \in W$.

We denote by $\mathcal{R}(S_a, W)$ the set of controller-interconnection pairs $(S_c, F)$ that solve the reachability problem for system $S_a$ with the target set $W$ as specification. For brevity, in what follows we refer to the pairs $(S_c, F)$ simply as controller pairs.

**Definition 3.2 (Entry time).** Let $S$ be a system and let $W \subseteq X$ be a subset of outputs. The entry time of $S$ into $W$ from $x_0 \in X_0$, denoted by $J(S, W, x_0)$, is the minimum $k \in \mathbb{N}$ such that for all maximal behaviors $y \in B_{x_0}(S) \cup B_{x_0}^\omega(S)$, there exists some $k' \in [0, k]$ for which $y(k') = y_{k'} \in W$.

If the set $W$ is not reachable from state $x_0$ we define $J(S, W, x_0) = \infty$. Note that asking in Definition 3.2 for the minimum $k$ is needed because $S$ might be a non-deterministic system, and thus there might be more than one behavior contained in $B_{x_0}(S) \cup B_{x_0}^\omega(S)$ and entering $W$.

If system $S$ is the result of the feedback composition of a system $S_a$ and a controller $S_c$ with interconnection relation $F$, i.e., $S = S_c \times_F S_a$, we denote by $\tilde{J}(S_c, F, S_a, W, x_{a0})$ the minimum entry time over all possible initial states of the controller related to $x_{a0}$:

$$\tilde{J}(S_c, F, S_a, W, x_{a0}) = \min_{x_{c0} \in X_{c0}} \{J(S_c \times_F S_a, W, (x_{c0}, x_{a0})) \mid (x_{c0}, x_{a0}) \in X_{F0}\}$$

The time-optimal control problem asks for the selection of the minimal entry time behavior for every $x_0 \in X_0$ for which $J(S, W, x_0)$ is finite.
**Problem 3.3** (Time-optimal reachability). Let $S_a$ be a system with $Y_a = X_a$ and $H_a = 1_{X_a}$, and let $W \subseteq X_a$ be a subset of the set of outputs of $S_a$. The time-optimal reachability problem asks to find the controller pair $(S^*_c, F^*) \in R(S_a, W)$ such that for any other pair $(S_c, F) \in R(S_a, W)$ the following is satisfied:

$$\forall x_{a0} \in X_{a0}, \quad \bar{J}(S_c, F, S_a, W, x_{a0}) \geq \bar{J}(S^*_c, F^*, S_a, W, x_{a0}).$$

**3.2. Entry time bounds.** The entry time $J$ acts as the cost function we aim at minimizing by designing an appropriate controller. The following Lemma, which is quite insightful in itself, explains how the existence of an approximate alternating simulation relates the minimal entry times of each system.

**Lemma 3.4.** Let $S_a$ and $S_b$ be two systems with $Y_a = X_a$, $H_a = 1_{X_a}$, $Y_b = X_b$ and $H_b = 1_{X_b}$, and let $W_a \subseteq X_a$ and $W_b \subseteq X_b$ be subsets of states. If the following two conditions are satisfied:

- $S_a \sim_{AS} S_b$ with the relation $R_c \subseteq X_a \times X_b$;
- $R_c(W_a) \subseteq W_b$

then the following holds:

$$(x_{a0}, x_{b0}) \in R_c \implies \bar{J}(S^*_c, F^*_a, S_a, W_a, x_{a0}) \geq \bar{J}(S^*_c, F^*_a, S_a, W_b, x_{b0})$$

where $(S^*_c, F^*_a) \in R(S_a, W_a)$ and $(S^*_c, F^*_b) \in R(S_b, W_b)$ denote the time-optimal controller pairs for their respective time-optimal control problems, and $x_{a0} \in X_{a0}$, $x_{b0} \in X_{b0}$.

**Proof.** We prove the result by parts. In the case when $\bar{J}(S^*_c, F^*_a, S_a, W_a, x_{a0}) = \infty$, the result is trivially true. Thus, we analyze the case when $\bar{J}(S^*_c, F^*_a, S_a, W_a, x_{a0}) < \infty$. In this case, we show that there exists a controller $S_c$ for $S_b$ such that:

$$\bar{J}(S_c, G, S_b, W_b, x_{b0}) \leq \bar{J}(S^*_c, F^*_a, S_a, W_a, x_{a0}).$$

This is proved by showing that for every maximal behavior $y^b \in B_{(x_{a0}, x_{b0})}(S_c \times S_b) \cup B_{(x_{a0}, x_{b0})}(S_c \times S_b)$ there exists a maximal behavior $y^a \in B_{(x_{a0}, x_{a0})}(S^*_c \times F^*_a S_a) \cup B_{(x_{a0}, x_{a0})}(S^*_c \times F^*_a S_a) \in$-related to $y^b$. The proof is finalized by noting that to be optimal, the controller $(S^*_c, F^*_a)$ has to satisfy:

$$\bar{J}(S^*_c, F^*_a, S_a, W_b, x_{b0}) \leq \bar{J}(S_c, G, S_b, W_b, x_{b0}) \leq \bar{J}(S^*_c, F^*_a, S_a, W_b, x_{b0})$$

for all $x_{a0} \in X_{a0}$ and $x_{b0} \in X_{b0}$ such that $(x_{a0}, x_{b0}) \in R_c$, hence proving the result.

We start defining the controller $S_c$ for system $S_b$. Let $R_a$ be the alternating simulation relation defining the interconnection relation $F^*_a = R^*_G$. We define an interconnection relation $G = R^*_G$ that allows us to use the system $S_c = S^*_c \times F^*_a S_a$ as a controller for system $S_b$. The interconnection relation $G = R^*_G$ is determined by the relation:

$$R_G = \{(x_{ca}, x_{a0}) \in (X^*_c \times X_a) \times X_b \mid (x_{ca}, x_{a0}) \in R_a \land (x_{a0}, x_{b0}) \in R_c\}.$$

Furthermore, one can easily prove (for a detailed explanation see Proposition 11.8 in [13]) that

$$S_c \times S_b \sim^2 S_a \times F^*_a S_a,$$

with the relation $R_{cb} \subseteq X_G \times X_c$:

$$R_{cb} = \{(x_c, x_c') \in X_G \times X^*_G \mid x_c = x_c'\}.$$
In order to show that for every maximal behavior \( y^b \in B_{(x,a,x_0)}(S_c \times G S_b) \) there exists an \( \varepsilon \)-related maximal behavior \( y^a \in B_{(x,a,x_0)}(S^*_c \times F^* a S_a) \cup B^\omega_{(x,a,x_0)}(S^*_c \times F^* a S_a) \), we first make the following remark: for any pair \((x,a,x_0)\) of related states \((x,a,x_0)\), by the definition of alternating simulation relation, if \( U_a(x_0) \neq \emptyset \) then \( U_b(x_0) \neq \emptyset \). From the definition of \( G \) it follows that for all \((x_c,a,x_0)\), \( x_0 \in X_G \) the pair \((x,a,x_0)\) belongs to \( R_c \). Thus, for any pair of related states \((x,a)\), \( x \in X_G \), there exists \( x_\emptyset \in X_G \), namely \((x,a)\), with \( x_\emptyset = (x_c,a,x_0) \), so that \( U_c(x_\emptyset) \neq \emptyset \implies U_b(x_\emptyset) \neq \emptyset \). The existence of the simulation relation \( \ref{eq:simrel} \) implies that for every behavior \( y^b \) there exists an \( \varepsilon \)-related behavior \( y^a \). Any infinite behavior is a maximal behavior, and thus we already know that for every (maximal) infinite behavior \( y^b \) there exists an \( \varepsilon \)-related (maximal) infinite behavior \( y^a \). Moreover, if \( y^b \) is a maximal finite behavior of length \( l \), the set of inputs \( U_G(y_l^b) \) is empty. As shown before, this implies that \( U_c(y_l^b) = \emptyset \), and thus \( y^a \) is also maximal, where \( y^a \) is the corresponding behavior of \( S^*_c \times F^* a S_a \) \( \varepsilon \)-related to \( y^b \).

We now show that \ref{claim:main} holds. For any initial state \( x_{a0} \) there exists an initial controller state \( x_{c0} \in R^{-1}(x_{a0}) \) of \( S^*_c \), such that every maximal behavior \( y^a \in B_{(x_{c0},x_{a0})}((S^*_c \times F^* a S_a) \cup B^\omega_{(x_{c0},x_{a0})}(S^*_c \times F^* a S_a)) \) reaches a state \( x \in W_a \) in the worst case after \( J(S^*_c, F^* a, S_a, W_a, x_{a0}) \) steps. We assume in what follows that the controller is initialized at \( x_{c0} \). Thus, as maximal behaviors of \( S^*_c \times G S_b \) are related by \( R_b \) to maximal behaviors of \( S^*_c \times F^* a S_a \), for any \( x_{a0} \in R_c(x_{a0}) \) every maximal behavior \( y^b \in B_{(x_{c0},x_{a0})}(S_c \times G S_b) \cup B^\omega_{(x_{c0},x_{a0})}(S_c \times G S_b) \) reaches some state \( x_b \in R_c(W_a) \) in at most \( J(S^*_c, F^* a, S_a, W_a, x_{a0}) \) steps. But then, from the second assumption, \( x_b \in R_c(W_a) \) implies that \( x_{b0} \in W_b \) and we have that

\[
J(S_c, G, S_b, W_b, x_{b0}) \leq J(S^*_c, F^* a, S_a, W_a, x_{a0})
\]

for all \( x_{a0} \in X_{a0} \) and \( x_{b0} \in X_{b0} \) such that \((x_{a0}, x_{b0}) \in R_c\).

The second assumption in Lemma \ref{lem:assumptions} requires the sets \( W_a \) and \( W_b \) to be related by \( R \). This assumption can always be satisfied by suitably enlarging or shrinking the target sets.

**Definition 3.5.** For any relation \( R \subseteq X_a \times X_b \) and any set \( W \subseteq X_b \), the sets \([W]_R, \overline{[W]}_R\) are given by:

\[
[W]_R = \{ x_a \in X_a \mid R(x_a) \subseteq W \},
\]

\[
\overline{[W]}_R = \{ x_a \in X_a \mid R(x_a) \cap W \neq \emptyset \}.
\]

The main theoretical result in the paper explains how to obtain upper and lower bounds for the optimal entry times in a system \( S_a \) by working with a related system \( S_b \).

**Theorem 3.6.** Let \( S_a \) and \( S_b \) be two systems with \( Y_a = X_a, H_a = 1_{X_a}, Y_b = X_b \) and \( H_b = 1_{X_b} \). If \( S_b \) is deterministic and there exists an approximate alternating simulation relation \( R \) from \( S_a \) to \( S_b \) such that \( R^{-1} \) is an approximate simulation relation from \( S_b \) to \( S_a \), i.e.:

\[
S_a \preceq_{AS} S_b \preceq_S S_a,
\]

...
then the following holds for any $W \subseteq X_b$ and $(x_{a0}, x_{b0}) \in R$:

$$J(S^*_{cb(a)}, F_d, S_{d(a)}, [W]_R, x_{b0}) \leq J(S^*_{cb}, F_b, S_b, W, x_{b0}) \leq J(S^*_{ca}, F, S_a, [W]_R, x_{a0})$$

where the controller pairs $(S^*_{cb}, F_b) \in \mathcal{R}(S_b, W)$, $(S^*_{ca}, F) \in \mathcal{R}(S_a, [W]_R)$ and $(S^*_{cd(a)}, F_d) \in \mathcal{R}(S_{d(a)}, [W]_R)$ are optimal for their respective time-optimal control problems.

**Proof.** Note that $S_b \preceq_{AS} S_{d(a)}$, by the assumed relation and both systems being deterministic. Also note that, by definition, $R([W]_R) \subseteq W$ and $R^{-1}(W) \subseteq [W]_R$. Then the proof follows from Lemma 3.4. □

**Remark 3.7.** If $S_b$ is not deterministic the inequality

$$\bar{J}(S^*_{cb}, F_b, S_b, W, x_{b0}) \leq J(S^*_{ca}, F, S_a, [W]_R, x_{a0})$$

still holds.

Theorem 3.6 explains how upper and lower bounds for the entry times in $S_b$ can be computed on $S_a$, hence decoupling the optimality considerations from the specific algorithms used to compute the abstractions. This possibility is of great value when $S_a$ is a much simpler system than $S_b$. We exploit this observation in the next section where $S_b$ denotes a control system and $S_a$ a much simpler symbolic abstraction.

### 4. Approximate Time-Optimal Control

Our ultimate objective is to synthesize time-optimal controllers to be implemented on digital platforms. The appropriate model for this analysis consists of a time-discretization of a control system.

**Definition 4.1.** The system $S_\tau(\Sigma) = (X_\tau, X_{\tau 0}, U_\tau, \rightarrow_{\tau}, Y_\tau, H_\tau)$ associated with a control system $\Sigma = (\mathbb{R}^n, \mathcal{U}, f)$ and with $\tau \in \mathbb{R}^+$ consists of:

- $X_\tau = \mathbb{R}^n$;
- $X_{\tau 0} = X_\tau$;
- $U_\tau = \{ v \in \mathcal{U} \mid \text{dom } v = [0, \tau] \}$;
- $x \rightarrow_{\tau} x'$ if there exist $v \in U_\tau$, and a trajectory $\xi_{xv} : [0, \tau] \rightarrow \mathbb{R}^n$ of $\Sigma$ satisfying $\xi_{xv}(\tau) = x'$;
- $Y_\tau = \mathbb{R}^n$;
- $H_\tau = 1_{\mathbb{R}^n}$.

A symbolic abstraction of a control system is a system in which its states represent aggregates or collections of states of the original control system. It has been shown in [2, 3, 14] that one can construct, under mild assumptions, symbolic abstractions in the form of finite systems $S_{abs}$ satisfying $S_{abs} \preceq_{AS} S_\tau(\Sigma) \preceq S_{b0}$ with arbitrary precision $\varepsilon$. Since $S_{abs}$ is a finite system, entry times for $S_{abs}$ can be efficiently computed by using algorithms in the spirit of dynamic programming or Dijkstra’s algorithm [19, 20]. It then follows from Theorem 3.6 that these entry times immediately provide bounds for the optimal entry time in $S_\tau(\Sigma)$. Moreover, the process of computing the optimal entry times for $S_{abs}$ provides us with a time-optimal controller for $S_{abs}$ that can be refined to an approximately time-optimal controller for $S_\tau(\Sigma)$. The refined controller is guaranteed to enforce the bounds for the optimal entry times in $S_\tau(\Sigma)$, computed in $S_{abs}$. 
4.1. Controller design. We now present a fixed point algorithm solving the time-optimal reachability problem for finite symbolic abstractions $S_{abs}$. We start by introducing an operator that help us define the time-optimal controller in a more concise way.

**Definition 4.2.** For a given system $S_{abs}$ and target set $W \subseteq X_{abs}$, the operator $G_W : 2^{X_{abs}} \rightarrow 2^{X_{abs}}$ is defined by:

$$G_W(Z) = \{ x_{abs} \in X_{abs} \mid x_{abs} \in W \lor \exists u_{abs} \in U_{abs}(x_{abs}) \text{ s.t. } \emptyset \neq \text{Post}_{u_{abs}}(x_{abs}) \subseteq Z \}.$$ 

A set $Z$ is said to be a fixed point of $G_W$ if $G_W(Z) = Z$. It is shown in [13] that when $S_{abs}$ is finite, the smallest fixed point $Z$ of $G_W$ exists and can be computed in finitely many steps by iterating $G_W$, i.e., $Z = \lim_{i \to \infty} G_W^i(\emptyset)$. Moreover, the reachability problem admits a solution if the minimal fixed point $Z$ of $G_W$ satisfies $Z \cap X_{abs0} \neq \emptyset$. The time-optimal controller pair can then be constructed from $Z$ as follows:

**Definition 4.3** (Time-optimal controller pair). For any finite system $S_{abs} = (X_{abs}, X_{abs0}, U_{abs}, X_{abs}, U_{abs}, X_{abs}, 1_{X_{abs}})$ and for any set $W \subseteq X_a$, the time-optimal controller pair $(S_{abs}^*, F^*)$ is given by the system $S_{abs}^* = (X_{cabs}, X_{cabs0}, U_{cabs}, X_{cabs}, U_{cabs}, X_{cabs}, 1_{X_{cabs}})$ and by the interconnection relation $F^* = R_{cabs}$ defined by:

- $R_{cabs} = \{ (x_{cabs}, x_{cabs}) \in X_{cabs} \times X_{cabs} \mid x_{cabs} = x_{cabs} \}$
- $Z = \lim_{i \to \infty} G_W^i(\emptyset)$
- $X_{cabs} = Z$
- $X_{cabs0} = Z \cap X_{abs0}$
- $x_{cabs} \xrightarrow{u_{cabs}} x'_{cabs}$ if there exists a $k \in \mathbb{N}^+$ such that $x_{cabs} \notin G_W^k(\emptyset)$ and $\emptyset \neq \text{Post}_{u_{abs}}(x_{cabs}) \subseteq G_W^k(\emptyset)$

where $\text{Post}_{u_{abs}}(x_{cabs})$ refers to the $u_{abs}$-successors in $S_{abs}$.

For more details about this controller design we refer the reader to Chapter 6 of [13].

4.2. Controller refinement. The time-optimal controller pair $(S_{cabs}^*, F^*)$ obtained in the previous section can be easily refined into a controller pair $(S_{ct}^*(\Sigma), F_{ct}^*)$ for $S_{ct}(\Sigma)$. Let $R_{abs}$ be the $\varepsilon$-approximate alternating simulation relation from $S_{abs}$ to $S_{ct}(\Sigma)$, then the refined controller $(S_{cabs}^*(\Sigma), F_{ct}^*)$ is given by the system $S_{ct}^* = (X_{ct}, X_{ct0}, U_{ct}, X_{ct}, U_{ct}, X_{ct}, 1_{X_{ct}})$ and by the interconnection relation $F_{ct} = R_{ct}^*$ defined by:

- $R_{ct} = \{ (x_{ct}, x_{ct}) \in X_{ct} \times X_{ct} \mid x_{ct} = x_{ct} \}$
- $X_{ct} = X_{ct}$
- $X_{ct0} = X_{ct0}$
- $x_{ct} \xrightarrow{u_{ct}} x'_{ct}$ if there exists $u_{abs} = u_{ct}$, $x_{cabs} \in R_{abs}(x_{ct})$ and $x'_{cabs} \in R_{ct}(x'_{ct})$ such that $x_{cabs} \xrightarrow{u_{cabs}} x'_{cabs}$

where we assumed $U_{abs} \subseteq U_{ct}$.

Intuitively, the refined controller enables all the inputs in $U_{cabs}(x_{cabs})$ at every state $x_{ct} \in X_{ct}$ of the system $S_{ct}(\Sigma)$ that is related by $R_{ct}$ to the state $x_{cabs} \in X_{cabs}$ of the abstraction $S_{abs}$. It is important to notice that this controller is nondeterministic, i.e., at a state $x_{ct}$ all the inputs in
U_{ct}(x_r) = \bigcup_{x_{abs} \in R_{x_{abs}}^{-1}(x_r)} U_{cabs}(x_{abs}) are available and they all enforce the cost bounds.

4.3. Approximate time-optimal synthesis in practice. The following is a typical sequence of steps to be followed when applying the presented techniques in practice.

1. Select a desired precision \( \varepsilon \). This precision is problem dependent and given by practical margins of error.
2. Construct a symbolic model. Given \( \varepsilon \) construct, using your favorite method, a symbolic model \( S_{abs} \) satisfying: \( S_{abs} \preceq_{AS} S_r(\Sigma) \preceq_{S} S_{abs} \). Such abstractions can be computed using Pessoa \([1, 12]\).
3. Compute the cost’s lower bound. This bound is obtained as:
   \[
   \tilde{J}(\theta) = \min\{ k \in \mathbb{N}^+ \mid x_{abs} \in G^k_{\phi_{|W|, \rho}}(\theta) \} - 1 \]
   This bound is obtained as:
   \[
   \tilde{J}(\theta) = \min\{ k \in \mathbb{N}^+ \mid x_{abs} \in G^k_{\phi_{|W|, \rho}}(\theta) \} - 1 \]
   with \( G_W \) defined for system \( S_{abs} \). The controller obtained when computing this bound, i.e. \( S_{abs}^* \), is the time-optimal controller for \( S_{abs} \) and approximately time-optimal for \( S_r(\Sigma) \) after refinement.
4. Compute the cost’s upper bound. This bound is obtained as:
   \[
   \tilde{J}(\theta) = \min\{ k \in \mathbb{N}^+ \mid x_{abs} \in G^k_{\phi_{|W|, \rho}}(\theta) \} - 1 \]
   with \( G_W \) defined for system \( S_{abs} \). The controller obtained when computing this bound, i.e. \( S_{abs}^* \), is the time-optimal controller for \( S_{abs} \) and approximately time-optimal for \( S_r(\Sigma) \) after refinement.
5. Iterate. If the obtained upper bound is not acceptable, refine the symbolic model so that the new model \( S_{abs2} \) satisfies:
   \[
   S_{abs} \preceq_{S} S_{abs2} \preceq_{AS} S_r(\Sigma) \]
   with \( \varepsilon' < \varepsilon \) and \( \varepsilon'' < \varepsilon \). In virtue of Theorem \([3,4]\) (and Remark \([3,7]\)) the upper bound will not increase. Moreover, it is our experience that, in general, the upper bound will improve by using more accurate symbolic models, i.e., \( \varepsilon' < \varepsilon \).

The more general case where \( X_r \neq Y_r \), \( H_r \neq 1_{X_r} \), and one is given an output target set \( W_{Yr} \subseteq Y \) can be solved in the same manner by using the target set \( W \subseteq X \) defined by \( W = H^{-1}(W_{Yr}) \).

4.4. Generalizations and Complexity. We briefly discuss in this section some simple generalizations of the proposed methods and the corresponding complexity. We first note that time-optimal synthesis can be combined with safety (qualitative) objectives when the specification is given as the requirement to satisfy both a safety constraint and a reachability requirement. A controller for such specifications can be obtained by first synthesizing the least restrictive controller enforcing the safety constraint and then solving a time-optimal reachability problem. In particular, this approach can be used for specifications given as a Linear Time Logic (LTL) formula of the kind \( \phi \land \diamond p \), where \( p \) is an atomic proposition denoting a set of states and \( \phi \) is a formula in the safe-LTL fragment of LTL \([21]\).

The general solution of a problem including qualitative and quantitative (time-optimal) specifications consists of five steps: abstraction of the control system; translation of the safe-LTL formula into a deterministic automaton recognizing all the behaviors satisfying the formula; composition of this automaton with the finite abstraction; synthesis of a controller by solving a safety game in the finite system.

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1The constructions in \([14]\) satisfy this property with \( \varepsilon = \eta/2 \), \( \varepsilon' = \eta'/2 \) and \( \varepsilon'' = \frac{n-\eta'}{2} \) by selecting \( \eta' = \frac{\rho}{p} \) with \( \rho > 1 \) an odd number and \( \theta = \varepsilon \), \( \theta' = \varepsilon' \).
resulting from the composition; and finally, the synthesis of the final controller as a solution to a time-optimal reachability game in the abstraction composed with the intermediate (safety) controller.

According to the five steps solution, the (time) complexity of solving these general problems can be split in terms of those steps. The abstraction problem, following the techniques in [14] [15] can be easily shown to have exponential complexity on the dimension of the control system; the translation of a safe-LTL formula into a deterministic automaton has doubly exponential complexity on the length of the formula [22]; composition of finite automata is a polynomial problem on the number of states of the composed automata; and, finally, the solution of reachability or safety games on finite automata also takes polynomial time in the number of states. This last step can be shown to be polynomial by noting that both problems admit a solution as the fixed-point of an operator [23] [13] that needs to be iterated at most as many times as the number of states of the finite automaton. This brief analysis indicates that the bottleneck, in general, lies on the abstraction process, as the translation of safe-LTL formulas, even though theoretically more complex, tends to be an easier problem due to the short length of the formulas used in practice.

5. Examples

To illustrate the provided results and its practical relevance we implemented the time-optimal controller design algorithm in Section 4 in the publicly available Matlab toolbox named Pessoa [1, 12]. All the run-time values for the examples where obtained on a MacBook with 2.2 GHz Intel Core 2 Duo processor and 4GB of RAM. The abstractions generated by Pessoa and used in the following examples, are obtained by discretizing the dynamics with sample time \( \tau \) and the state and input sets with discretization steps \( \eta \) and \( \mu \) respectively. We refer the readers to [14] where these abstractions are studied in detail. The precision \( \varepsilon \) of such abstractions can be adjusted by reducing the discretization parameters \( \eta \) and \( \mu \).

5.1. Double integrator. We illustrate the proposed technique on the classical example of the double integrator, where \( \Sigma \) is the control system:

\[
\dot{\xi}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \xi(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v(t)
\]

and the target set \( W \) is the origin, i.e., \( W = \{(0,0)\} \).

Following the steps presented in Section 4, first we select a precision \( \varepsilon \) which in this example we take as \( \varepsilon = 0.15 \). Next, we relax the problem by enlarging the target set to \( W = B_1((0,0)) \). We select as parameters for the symbolic abstraction \( \tau = 1 \), \( \mu = 0.1 \) and \( \eta = 0.3 \). Restricting the state set to \( X = B_{30}((0,0)) \subset \mathbb{R}^2 \) the state set of \( S_{\tau}(\Sigma) \) becomes finite and the proposed algorithms can be applied.

Constructing the abstraction \( S_{\tau}(\Sigma) \) in Pessoa took less than 5 minutes and the resulting model required 7.9 MB to be stored. The lower bound required about 50 milliseconds while computing the time-optimal controller required only 3 seconds and the controller was stored in 1 MB.

The approximately time-optimal controller \( S^*_c \) is depicted in Figure 1(a). We remind the reader that the obtained controller is non-deterministic. Hence, Figure 1(a) shows one of the valid inputs of the time-optimal controller at different locations of the state-space. The optimal controller to the origin is also shown in Figure 1(a) represented by the switching curve (thick blue line) dividing the state
space into regions where the inputs $u = 1$ (below the switching curve) and $u = -1$ (above the switching curve) are to be used. As expected, the partition produced by this switching curve does not coincide with the one found by our toolbox, as the time-optimal controller reported in [24] is not time-optimal to reach the set $W$ (it is just optimal when the target set is the singleton \{(0,0)\}).

Although the computed bounds are conservative, the cost achieved with the symbolic controller is quite close to the true optimal cost as illustrated in Figure 1(b) and Table 1. This is a consequence of the bounds relying entirely on the worst case scenarios induced by the non-determinism of the computed abstractions. In practice, the symbolic controller determines the actual state of the system every time it acquires a state measurement thus resolving the nondeterminism present in the abstraction. In Figure 1(b) we present the ratio between the cost to reach $W$, obtained from the symbolic controller, and the time-optimal controller. The time-optimal controller to reach the origin operates in continuous time and thus for some regions of the state-space the cost obtained will be smaller than one unit of time. On the other hand, the approximate time-optimal controller obtained with our techniques cannot obtain costs smaller than one unit of time, as it operates in discrete time. Hence, to make the comparison fair, in Figure 1(b) the costs achieved by the time-optimal controller smaller than one unit of time were saturated to a cost of 1 time unit. In Table 1 specific values of the time to reach the target set $W$ using the constructed controller are compared to the cost of reaching $W$ with the true time-optimal controller to reach the origin.

![Figure 1](image_url)

**Figure 1.** (a) Symbolic controller $S_c^*$. (b) Time to reach the target set $W$ represented as the ratio between the times obtained from the symbolic controller and the times obtained from the continuous time-optimal controller to reach the origin.

5.2. **Unicycle example.** With this example we want to persuade the reader of the potential of the presented techniques to solve control problems with both qualitative and quantitative specifications. The problem we consider now is to drive a unicycle through a given environment with obstacles. In this example both qualitative and quantitative specifications are provided. The avoidance of obstacles prescribes
SYMBOLIC APPROXIMATE TIME-OPTIMAL CONTROL

| Initial State | Continuous | Symbolic | Upper Bound | Lower Bound |
|---------------|------------|----------|-------------|-------------|
| (−6.1, 6.1)  | 12.83 s    | 14 s     | 29 s        | 9 s         |
| (−6, 6)      | 12.66 s    | 14 s     | 29 s        | 9 s         |
| (−5.85, 5.85)| 11.60 s    | 13 s     | 29 s        | 9 s         |
| (3.1, 0.1)   | 2.66 s     | 3 s      | 7 s         | 2 s         |
| (3, 0)       | 2.53 s     | 3 s      | 7 s         | 2 s         |
| (2.85, −0.1)| 2.38 s     | 3 s      | 7 s         | 2 s         |

Table 1. Times achieved in simulations by a time-optimal controller to reach the origin and the symbolic controller.

Figure 2. Unicycle trajectory under the automatically generated approximately time-optimal feedback controller (left figure) and the inputs employed: \(v\) in yellow and \(\omega\) in pink (right figure).

conditions that the trajectories should respect, thus establishing qualitative requirements of the desired trajectories. Simultaneously, a time-optimal control problem is specified by requiring the target set to be reached in minimum time, thus defining the quantitative requirements. Hence, the complete specification requires the synthesis of a controller disabling trajectories that hit the obstacles, and selecting, among the remaining trajectories, those with the minimum time-cost associated to them.

We consider the following model for the unicycle control system:

\[
\dot{x} = v \cos(\theta), \quad \dot{y} = v \sin(\theta), \quad \dot{\theta} = \omega
\]

in which \((x, y)\) denotes the position coordinates of the vehicle, \(\theta\) denotes its orientation, and \((v, \omega)\) are the control inputs, linear velocity and angular velocity respectively. The parameters used in the construction of the symbolic model are: \(\eta = 0.2, \mu = 0.1, \tau = 0.5\) seconds, and \(v \in [0, 0.5]\) and \(\omega \in [-0.5, 0.5]\). The problem to be solved is to find a feedback controller optimally navigating the unicycle from any initial position to the target set \(W = [4.6, 5] \times [1, 1.6] \times [-\pi, \pi]\), indicated with a red box in Figure 2 (with any orientation \(\theta\)), while avoiding the obstacles in the environment, indicated as blue boxes in Figure 2. The symbolic model was constructed in 179 seconds and used 11.5 MB of storage, and the approximately time-optimal controller was obtained in 5 seconds and required 3.5 MB of storage. In Figure 2 we present the result of applying the approximately time-optimal controller with the prescribed qualitative requirements (obstacle avoidance). The (approximately) bang-bang nature of the obtained controller can be appreciated in the right plot of this figure. For the initial condition \((1.5, 1, 0)\) the solution obtained, presented in Figure 2, required 44 seconds to reach the target set.
6. Discussion

We have proposed a computational approach to solve time-optimal control problems by resorting to symbolic abstractions. The obtained solutions provide explicit lower and upper bounds on the achievable cost. The employed techniques allow us to solve complex time-optimal control problems, with target sets, state sets and dynamics of very general nature.

The main theoretical result shows that symbolic abstractions which approximately alternatingly simulate a control system provide bounds for the achievable cost of time-optimal control problems. An algorithm has been provided to obtain these cost bounds by solving corresponding optimal control problems over the symbolic abstraction. Furthermore, this algorithm produces an approximately time-optimal symbolic controller that can be easily refined into a controller for the original system, as shown in Section 4.2. On the practical side, we have implemented the presented algorithms in the Pessoa toolbox resorting to binary decision diagrams as the underlying data structures. We have also illustrated the techniques using Pessoa on two examples, the last of which illustrates how symbolic models can be used to solve problems with both qualitative and quantitative requirements.

Future work will concentrate in the development of synthesis algorithms for combinations of general qualitative and quantitative specifications for control systems.

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