A note on eigenvalues and Hamiltonian properties of $k$-connected graphs

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Abstract

Let $\lambda_1(G)$ and $\mu_1(G)$ denote the spectral radius and the Laplacian spectral radius of a graph $G$, respectively. Li in [Electronic J. Linear Algebra 34 (2018) 389-392] proved sharp upper bounds of $\lambda_1(G)$ based on the connectivity to assure a connected graph to be Hamiltonian and traceable, respectively. In this paper, we present best possible upper bounds of $\lambda_1(G)$ for $k$-connected graphs to be Hamiltonian-connected and homogeneously traceable, respectively. Furthermore, best possible upper bounds of $\mu_1(G)$ to predict $k$-connected graphs to be Hamiltonian-connected, Hamiltonian and traceable are originally proved, respectively.

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1 Introduction

We consider simple, undirected and connected graphs with undefined terms and notation reference to [3]. As in [3], $\overline{G}$, $\alpha(G)$, $\kappa(G)$, $\delta(G)$ and $d(v)$ denote the complement, the stability number (also call the independence number), the connectivity, the minimum degree of a graph $G$ and the degree of $v$ in $G$, respectively. The join of $G$ and $H$, denoted by $G \vee H$, is the graph obtained from a disjoint union of $G$ and $H$ by adding all possible edges between them. Let $K_{a,b}$ denote complete bipartite graphs on $n$ vertices, where $a + b = n$.

A well-known result of Whitney [14] states that $\kappa(G) \leq \delta(G)$ for any graph $G$. A graph $G$ is $k$-connected if $\kappa(G) \geq k$. A cycle passing through all the vertices of a graph is called a Hamiltonian cycle. A path passing through all the vertices of a graph is called a Hamiltonian path. The graph $G$ is called Hamiltonian-connected if every two vertices of $G$ are connected

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by a Hamiltonian path. A graph containing a Hamiltonian cycle is called a Hamiltonian graph.

Surely all Hamilton-connected graphs are Hamiltonian. A graph $G$ is called homogeneously traceable if for each $v \in V(G)$, there is a Hamiltonian path in $G$ with initial vertex $v$. Clearly every Hamiltonian graph is homogeneously traceable. A graph containing a Hamiltonian path is said to be traceable. Hence these four Hamiltonian properties weaken in turn.

The adjacency matrix of $G$ is the $n \times n$ matrix $A(G) = (a_{ij})$, where $a_{ij} = 1$ if $v_i$ and $v_j$ are adjacent and otherwise $a_{ij} = 0$. Let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the adjacency spectrum of $G$. Let $D(G)$ be the diagonal matrix of the vertex degrees of $G$. The matrix $L(G) = D(G) - A(G)$ is known as the Laplacian matrix of $G$. Let $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ be the Laplacian eigenvalues of $G$. We call $\lambda_1(G)$ and $\mu_1(G)$ the spectral radius and the Laplacian spectral radius of $G$, respectively.

There have been many studies on spectral conditions which warrant Hamiltonian properties of a graph, as can be seen in [6, 9, 10, 11, 12, 15, 16, 17], among others. However, hardly any of these results involve Laplacian eigenvalues. Very recently, R. Li [8] proved sufficient conditions of $\lambda_1(G)$ based on the connectivity to assure a connected graph to be Hamiltonian and traceable, respectively.

**Theorem 1.1** (R. Li [8]) Let $G$ be a graph of order $n \geq 3$ with connectivity $\kappa$ and minimum degree $\delta(G)$. If $\lambda_1(G) \leq \delta \sqrt{\frac{\kappa - s + 1}{n - \kappa + s - 1}}$, then each of the following holds.

(i) $G$ is Hamiltonian if and only if $G \not\cong K_{\kappa, s - 1}$ for $s = 0$.
(ii) $G$ is traceable if and only if $G \not\cong K_{\kappa, s - 1}$ for $s = -1$, where $n \geq 12$.

One of our goals is to show best possible upper bounds of $\lambda_1(G)$ for $k$-connected graphs to be Hamiltonian-connected and homogeneously traceable, respectively. Another goal of this research is to initiate studies to find best possible upper bounds of $\mu_1(G)$ to predict $k$-connected graphs to be Hamiltonian-connected, Hamiltonian and traceable, respectively. The main results are as follows.

**Theorem 1.2** Let $G$ be a $k$-connected graph of order $n \geq 3$ and minimum degree $\delta(G)$. If $\lambda_1(G) \leq \delta \sqrt{\frac{k - s + 1}{n - k + s - 1}}$, then $G$ is Hamiltonian-connected if and only if $G \not\cong K_{k, s - 1}$ for $s = 1$.

As a consequence of Theorem 1.2, an upper bound on $\lambda_1(K_1 \nabla G)$ to assure a $k$-connected graph to be homogeneously traceable is obtained.

**Corollary 1.3** Let $G$ be a $k$-connected graph of order $n \geq 3$ and minimum degree $\delta(G)$. If $\lambda_1(K_1 \nabla G) \leq (\delta + 1) \sqrt{\frac{k + 1}{n - k}}$, then $G$ is homogeneously traceable.

**Theorem 1.4** Let $G$ be a $k$-connected graph of order $n \geq 3$ and minimum degree $\delta(G)$. If $\mu_1(G) < \frac{n - k + s - 1}{n \delta}$, then each of the following holds.

(i) $G$ is Hamiltonian-connected for $s = 1$.
(ii) $G$ is Hamiltonian for $s = 0$.
(iii) $G$ is traceable for $s = -1$. 


Noting that the results of Theorem 1.4 are also best possible in the sense that the condition of the theorem can not be weakened. Let us consider $G \cong K_{k,k-s+1}$. From Lemma 2.4, we have $\mu_1(K_{k,k-s+1}) = 2k - s + 1$. Clearly the graph $G \cong K_{k,k-s+1}$ satisfies $\mu_1(G) = \frac{n\delta}{n-k+s-1}$ for $s = 1, 0, -1$. However, $G \cong K_{k,k}$ is not Hamiltonian-connected for $s = 1$, $G \cong K_{k,k+1}$ is not Hamiltonian for $s = 0$. $G \cong K_{k,k+2}$ is not traceable for $s = -1$.

In the next section, we display some tools to be deployed in our arguments. The proofs of the main results are in the subsequent section.

2 Preliminaries

We in this section will present some lemmas that will be useful in our arguments.

\textbf{Lemma 2.1} (Dirac [5], Ore [13]) Let $G$ be a graph of order $n \geq 3$ and minimum degree $\delta(G)$. If $\delta(G) \geq \frac{n + s}{2}$, then each of the following holds.

(i) $G$ is Hamiltonian-connected for $s = 1$.

(ii) $G$ is Hamiltonian for $s = 0$.

(iii) $G$ is traceable for $s = -1$.

\textbf{Lemma 2.2} (Chvátal and Erdős [4]) Let $G$ be a $k$-connected graph of order $n \geq 3$. If $\alpha(G) \leq k - s$, then each of the following holds.

(i) $G$ is Hamiltonian-connected for $s = 1$.

(ii) $G$ is Hamiltonian for $s = 0$.

(iii) $G$ is traceable for $s = -1$.

\textbf{Lemma 2.3} (Exercise 18.1.6, Bondy and Murty [3]) Let $G$ be a graph. Then $G$ is homogeneously traceable if and only if $K_1 \nabla G$ is Hamiltonian-connected.

\textbf{Lemma 2.4} (Anderson and Morely [1]) Let $G$ be a graph of order $n \geq 2$. Then $\mu_1(G) \leq n$ with equality if and only if $\overline{G}$ is disconnected.

The main tool in our paper is the eigenvalue interlacing technique described below. Given two non-increasing real sequences $\theta_1 \geq \theta_2 \geq \cdots \geq \theta_n$ and $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_m$ with $n > m$, the second sequence is said to \textit{interlace} the first one if $\theta_i \geq \eta_i \geq \theta_{n-m+i}$ for $i = 1, 2, \ldots, m$. The interlacing is \textit{tight} if exists an integer $k \in [0, m]$ such that $\theta_i = \eta_i$ for $1 \leq i \leq k$ and $\theta_{n-m+i} = \eta_i$ for $k + 1 \leq i \leq m$.

Consider an $n \times n$ real symmetric matrix

$$
M = \begin{pmatrix}
M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\
M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
M_{m,1} & M_{m,2} & \cdots & M_{m,m}
\end{pmatrix},
$$

whose rows and columns are partitioned according to a partitioning $X_1, X_2, \ldots, X_m$ of $\{1, 2, \ldots, n\}$. The \textit{quotient matrix} $R(M)$ of the matrix $M$ is the $m \times m$ matrix whose entries are the average
row sums of the blocks \( M_{i,j} \) of \( M \). The partition is \textbf{equitable} if each block \( M_{i,j} \) of \( M \) has constant row (and column) sum.

**Lemma 2.5** \((\text{Brouwer and Haemers} [2, 7])\) Let \( M \) be a real symmetric matrix. Then the eigenvalues of every quotient matrix of \( M \) interlace the ones of \( M \). Furthermore, if the interlacing is tight, then the partition is equitable.

### 3 Proofs

**Proof of Theorem 1.2.** By observation, \( K_{k,k} \) is not Hamiltonian-connected since there does not exist a Hamiltonian path between any pair of vertices belonging to the same part of the bipartition of \( G \). Therefore, it suffices to prove the sufficiency.

By contradiction, we assume that \( G \not\cong K_{k,k} \) and \( G \) is not Hamiltonian-connected. First, we prove the following Claim.

**Claim.** \( n \geq 2k \).

In fact, if \( n \leq 2k - 1 \), then \( \delta(G) \geq \kappa(G) \geq k \geq \frac{n+1}{2} \). By Lemma 2.1(i), \( G \) is Hamiltonian-connected, a contradiction. Claim is completed.

By Lemma 2.2(i), then \( \alpha(G) \geq k \), and thus there exists an independent set \( X \) in \( G \) such that \(|X| = k\). Let \( X = \{u_1, u_2, \ldots, u_k\} \) and \( Y = V(G) - X = \{v_1, v_2, \ldots, v_{n-k}\} \). Let \( d_1(v_i) = |N(v_i) \cap X| \) and \( d_2(v_i) = |N(v_i) \cap Y| \) for \( 1 \leq i \leq n-k \). Clearly \( \sum_{i=1}^{n-k} d_1(v_i) = \sum_{i=1}^{n-k} d_2(v_i) \).

Accordingly, the quotient matrix \( R(A) \) of \( A(G) \) on the partition \((X, Y)\) is as follows.

\[
R(A) = \left( \begin{array}{cc}
\sum_{i=1}^{n-k} d_1(v_i) & \frac{k}{n-k} \\
\frac{k}{n-k} & \sum_{i=1}^{n-k} d_2(v_i)
\end{array} \right).
\]

Let \( \lambda_1(R(A)) \geq \lambda_2(R(A)) \) be the eigenvalues of \( R(A) \). By Lemma 2.5,

\[
\lambda_1(G) \geq \lambda_1(R(A)) \geq \lambda_{n-1}(G), \quad \lambda_2(G) \geq \lambda_2(R(A)) \geq \lambda_n(G).
\]

From Perron-Frobenius Theorem, we have \( \lambda_1(G) \geq |\lambda_n(G)| \). Hence

\[
\lambda_1^2(G) \geq -\lambda_1(G)\lambda_n(G) \geq -\lambda_1(R(A))\lambda_2(R(A)) = -\det(R(A)) \tag{1}
\]

\[
= \frac{\sum_{i=1}^{k} d(u_i)}{k} \cdot \frac{\sum_{i=1}^{k} d(u_i)}{n-k} - \frac{\sum_{i=1}^{k} d(u_i)}{k} \cdot \frac{\sum_{i=1}^{k} d(u_i)}{n-k} \cdot \frac{k}{n-k} \geq \frac{k\delta(G) \cdot k\delta(G) \cdot k}{n-k} \geq \lambda_1^2(G).
\]

It follows that all the inequalities in (1) must be equalities. Hence we must have \( \lambda_1(G) = -\lambda_n(G) \), \( \lambda_1(G) = \lambda_1(R(A)) \), \( \lambda_n(G) = \lambda_2(R(A)) \) and \( d(u_i) = \delta(G) \) for \( 1 \leq i \leq k \). So 0 = \( \lambda_1(G) + \lambda_n(G) \) = \( \lambda_1(R(A)) + \lambda_2(R(A)) \) = \( \frac{\sum_{i=1}^{n-k} d_2(v_i)}{n-k} \), and thus \( d_2(v_i) = 0 \) for \( 1 \leq i \leq n-k \). Therefore, \( G \) is a bipartite graph with the partition \((X, Y)\). Since

\[
\delta(G) = \frac{\sum_{i=1}^{k} d(u_i)}{k} = \frac{\sum_{i=1}^{n-k} d_1(v_i)}{n-k} \cdot \frac{n-k}{k} = \frac{\sum_{i=1}^{n-k} d_2(v_i)}{n-k} \cdot \frac{n-k}{k} \geq \delta(G) \cdot \frac{n-k}{k},
\]
we have \( n \leq 2k \). By Claim, then \( n = 2k \). Hence \( n - k = k \) and \( d(v_i) = \delta(G) \). Note that \( d(u_i) = d(v_i) = \delta(G) \geq s(G) \geq k \). Then \( G \cong K_{k,k} \), contrary to our assumption. This completes the proof of the theorem. \( \square \)

**Proof of Corollary 1.3.** By assumption, \( K_1 \nabla G \) is \( k+1 \)-connected of order \( n+1 \) and minimum degree \( \delta(G) + 1 \). Note that \( K_1 \nabla G \not\cong K_{k,k} \). By Theorem 1.2, \( K_1 \nabla G \) is Hamiltonian-connected. By Lemma 2.3, then \( G \) is homogeneously traceable. \( \square \)

In the proof of Theorem 1.4, we say that a graph possesses Hamiltonian properties, which means that the graph is Hamiltonian-connected, Hamiltonian or traceable.

**Proof of Theorem 1.4.** We divide our proof into two cases.

**Case 1.** \( \delta > n - k + s - 1 \).

As \( \delta \geq k \) and \( \delta > n - k + s - 1 \), we have \( \delta > \frac{n+k-1}{2} \).

**Case 1.1.** \( n = 2t+1 \) is odd, where \( t \geq 1 \).

If \( s = 1 \), then \( \delta > \frac{2t+1}{2} \), and we have \( \delta \geq t+1 = \frac{n+k}{2} \). By Lemma 2.1(i), \( G \) is Hamiltonian-connected. If \( s = 0 \), then \( \delta > t \), and we have \( \delta \geq t+1 = \frac{n+k}{2} \). By Lemma 2.1(ii), \( G \) is Hamiltonian-connected.

**Case 1.2.** \( n = 2t+2 \) is even, where \( t \geq 1 \).

If \( \delta > t+1 \), and we have \( \delta \geq t+2 > \frac{n+k}{2} \). By Lemma 2.1(i), \( G \) is Hamiltonian-connected. If \( s = 0 \), then \( \delta > t+\frac{1}{2} \), and we have \( \delta \geq t+1 = \frac{n+k}{2} \). By Lemma 2.1(ii), \( G \) is Hamiltonian-connected.

**Case 2.** \( \delta \leq n - k + s - 1 \).

Now, \( \frac{n+k-1}{n-k+s-1} \leq n \). Suppose, to the contrary, that \( G \) does not possess Hamiltonian properties. By Lemma 2.2, \( \alpha(G) \geq k-s+1 \), and then there exists an independent set \( X \) in \( G \) such that \( |X| = k-s+1 \). Let \( X = \{ u_1, u_2, \ldots, u_{k-s+1} \} \) and \( Y = V(G) - X = \{ v_1, v_2, \ldots, v_{n-k+s-1} \} \).

Let \( d_1(v_i) = |N(v_i) \cap X| \) and \( d_2(v_i) = |N(v_i) \cap Y| \) for \( 1 \leq i \leq n-k+s-1 \). Clearly \( \sum_{i=1}^{k-s+1} d(u_i) = \sum_{i=1}^{n-k+s-1} d_1(v_i) \). Accordingly, the quotient matrix \( R(L) \) of \( L(G) \) on the partition \((X,Y)\) becomes:

\[
R(L) = \begin{pmatrix}
\sum_{i=1}^{k-s+1} \frac{d(u_i)}{k-s+1} & \cdots & \sum_{i=1}^{k-s+1} \frac{d(u_i)}{n-k+s-1} \\
\sum_{i=1}^{n-k+s-1} \frac{d_1(v_i)}{k-s+1} & \cdots & \sum_{i=1}^{n-k+s-1} \frac{d_1(v_i)}{n-k+s-1}
\end{pmatrix} = \begin{pmatrix}
\sum_{i=1}^{k-s+1} \frac{d(u_i)}{k-s+1} & \cdots & \sum_{i=1}^{k-s+1} \frac{d(u_i)}{n-k+s-1} \\
\sum_{i=1}^{n-k+s-1} \frac{d_1(v_i)}{k-s+1} & \cdots & \sum_{i=1}^{n-k+s-1} \frac{d_1(v_i)}{n-k+s-1}
\end{pmatrix}.
\]

Let \( \mu_1(R(L)) \geq \mu_2(R(L)) = 0 \) be the eigenvalues of \( R(L) \). By algebraic manipulation, we have

\[
\mu_1(R(L)) = \left( \frac{1}{k-s+1} + \frac{1}{n-k+s-1} \right) \sum_{i=1}^{k-s+1} d(u_i).
\]

By Lemma 2.5, then

\[
\mu_1(G) \geq \mu_1(R(L)) = \left( \frac{1}{k-s+1} + \frac{1}{n-k+s-1} \right) \sum_{i=1}^{k-s+1} d(u_i) \\
\geq \left( \frac{1}{k-s+1} + \frac{1}{n-k+s-1} \right) (k-s+1) \delta = \frac{n\delta}{n-k+s-1}.
\]
This contradicts the assumption of this theorem.

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