Elementary Divisors of the Shapovalov Form on the Basic Representation of Kac-Moody Algebras

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1. Introduction/Prospectus

Let \( g \) be the simply-laced Kac-Moody algebra of type \( X^{(1)}_i \) (i.e. \( X = A, D, \) or \( E \)) and let \( V = V(\Lambda_0) \) be the basic representation of \( g \). Let \( |0\rangle \) be a vacuum vector and define the lattice \( V_Z = U_Z|0\rangle \) in \( V \), where \( U_Z \) is the \( Z \)-subalgebra of the universal enveloping algebra of \( g \) generated by the divided powers

\[
\frac{e_i^n}{n!} \quad \text{and} \quad \frac{f_i^n}{n!}, \quad (i = 0, \ldots, l, n \geq 1)
\]

in the Chevalley generators.

Let \( \tau : g \to g \) be the antilinear Chevalley antiautomorphism defined on the Chevalley generators by

\[
\tau(e_i) = f_i, \quad \tau(f_i) = e_i.
\]

This involution extends to an involution of the universal enveloping algebra \( U(g) \). The Shapovalov form, \( \langle \cdot, \cdot \rangle_S \), is the unique Hermitian form on \( V \) satisfying

\[
\langle |0\rangle, |0\rangle_S = 1 \quad \text{and} \quad \langle xv, v' \rangle_S = \langle v, \tau(x)v' \rangle_S
\]

for \( x \in U(g) \) and all \( v, v' \in V \). The restriction of the Shapovalov form to \( V_Z \) gives a symmetric bilinear form

\[
\langle \cdot, \cdot \rangle_S : V_Z \times V_Z \to \mathbb{Z}.
\]

The main result of this paper is an algorithm for calculating the invariant factors of this form. We give formulas for the invariant factors of the Gram matrix of this form on each weight space of \( V_Z \), provided the powers of the primes occurring in \( l + 1 \) are not too large.
Indeed, we begin by proving a relationship between the Shapovalov form on $V$ and a family of bilinear forms $\langle \cdot, \cdot \rangle_s, (s \in \mathbb{Z}_{\geq 0})$, on the ring $\Lambda_\mathbb{C} = \mathbb{C}[p_1, p_2, \ldots]$ of symmetric functions defined by

$$\langle p_\lambda, p_\mu \rangle_s = \delta_{\lambda\mu} s^{l(\lambda)} z_\lambda$$

where $p_\lambda$ is a power sum symmetric function, and for $\lambda = (1^{m_1}2^{m_2} \cdots)$,

$$z_\lambda = \prod_{r \geq 1} (r^{m_r} \cdot m_r!).$$

This form restricts to a symmetric bilinear form on $\Lambda := \Lambda_\mathbb{Z} = \mathbb{Z}[h_1, h_2, \ldots]$, where $h_n$ is the $n$th complete homogeneous symmetric function. There is a simple relationship between $V_\mathbb{Z}$ and $\Lambda$. Indeed, it turns out that the weight spaces of $V$ are of the form $w\Lambda_0 - d\delta$, where $w$ is an element of the Weyl group of type $X$, $\delta$ is the null root, and $d$ is a positive integer (see [3], section 12.6). We have

**Theorem 1.1.** Let $a_1, \ldots, a_l$ be the invariant factors of the Cartan matrix for a simple finite dimensional Lie algebra of $ADE$ type. Let $a_{k_1}^{(r)}, \ldots, a_{k_h}^{(r)}$ be the invariant factors of the form $\langle \cdot, \cdot \rangle_{a_k}$ on the degree $r$ component of $\Lambda$ (here $h = |\text{Par}(r)|$ is the number of partitions of $r$). Then, the invariant factors of the Shapovalov form on the $(w\Lambda_0 - d\delta)$-weight space of $V_\mathbb{Z}$ are

$$\left\{ \prod_{k=1}^l a_{k_i}^{(d_k)} : d_1 + \cdots + d_l = d, 1 \leq i_k \leq |\text{Par}(d_k)| \right\}.$$

This reduces the problem to calculating the invariant factors of the form $\langle \cdot, \cdot \rangle_s$ on $\Lambda$ for positive integers $s$.

We further reduce the problem as follows. Let $X_s$ denote the matrix $(\langle m_\lambda, h_\mu \rangle_s)_{\lambda, \mu}$ ($m_\lambda$ is a monomial symmetric function). The $m_\lambda$ also form a basis for $\Lambda$, so we may compute the invariant factors of $X_s$. This calculation is equivalent to calculating the Smith normal form $S(X_s)$ of $X_s$. We have:

**Theorem 1.2.** Let $s, t \in \mathbb{Z}_{\geq 0}$. Then, $X_{st} = X_sX_t$. In particular, if $(s, t) = 1$, then $S(X_{st}) = S(X_s)S(X_t)$.

Therefore, to calculate invariant factors on $V_\mathbb{Z}$, it is enough to know the invariant factors of the form $\langle \cdot, \cdot \rangle_{p^r}$ on $\Lambda$ for every prime $p$ and $r \geq 1$. We calculate these numbers in the case when $r \leq p$. 
For an integer $a$, define the number

$$d_p(a) = \sum_{j \geq 1} \left\lfloor \frac{a}{p^j} \right\rfloor$$

and for a partition $\lambda = (1^{m_1(\lambda)} 2^{m_2(\lambda)} \ldots)$, define

$$D_r(\lambda) = \prod_{(n,p) = 1} \prod_{i=0}^{r-1} p^{[(r-i)m_{p \cdot n}(\lambda) + d_p(m_{p \cdot n}(\lambda))]_i}$$

Then,

**Theorem 1.3.** Let $r \leq p$. Then, the elementary divisors of the form $\langle \cdot, \cdot \rangle_{p^r}$ on the $d$th graded component of $\Lambda$ are

$$\{D_r(\lambda) | \lambda \vdash d\}.$$  

We conjecture that this result holds for all $r$ based on computational evidence, but have yet to understand the proof.

This result has a number of consequences. First, assume $g$ is of type $A_{l-1}^{(1)}$. Let $H_n$ be the Iwahori-Hecke algebra associated to the symmetric group $S_n$ at a primitive $l$th root of unity. In [1] and [5], it was shown that there is an isomorphism between the basic representation $V_{\mathbb{Z}}$ of $g$ and the direct sum $K = \bigoplus K_n$ of Grothendieck groups $K_n$ of finitely generated projective $H_n$-modules. Under this isomorphism there is a correspondence between blocks of $K$ and weight spaces of $V_{\mathbb{Z}}$. The Shapovalov form on $V_{\mathbb{Z}}$ corresponds to the Cartan pairing

$$([P], [Q]) = \dim \text{Hom}(P, Q)$$

between projective modules $P$ and $Q$. As a consequence, this paper gives the invariant factors of the blocks of the Hecke algebra $H_n$ at an $l$th root of unity when $l = \prod_i p_i^{r_i}$ satisfies $r_i \leq p_i$ for all $i$.

In [4], Külshammer, Olsson and Robinson develop an $l$-modular representation theory of symmetric groups for $l$ not necessarily prime. They conjecture what the invariant factors of the blocks should be. We expect that the answer given in this paper will confirm their conjecture.

Finally, we expect that these results will extend to include twisted affine Kac-Moody algebras.
2. REDUCTION TO THE HEISENBERG SUBALGEBRA OF \( \mathfrak{g} \)

Let \( \mathfrak{g}' \) be the finite dimensional simple Lie algebra of type \( X_l \) with root system \( \Phi \), simple roots \( \alpha'_1, \ldots, \alpha'_l \), and root lattice \( Q = \bigoplus_{i=1}^l \mathbb{Z} \alpha'_i \). Let \( \mathfrak{h}' = \mathbb{C} \otimes Q \subset \mathfrak{g}' \) be its Cartan subalgebra. Recall that

\[
\mathfrak{g} = \mathfrak{g}' \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d
\]

where \( c \) is the one dimensional central extension of \( \mathfrak{g}' \otimes \mathbb{C}[t, t^{-1}] \) and \( d \) is the scaling element. Let \( (\cdot | \cdot) \) be a standard invariant form on \( \mathfrak{g} \), normalized so that \( (\alpha | \alpha) = 2 \) for each root \( \alpha \in Q \).

Notice that \( \mathfrak{g} \) contains a copy of the Heisenberg Lie algebra \( \mathfrak{t} = \mathfrak{t}^+ \oplus \mathbb{C}c \oplus \mathfrak{t}^- \), where \( \mathfrak{t}^\pm = \bigoplus_{n>0} \mathfrak{h}' \otimes t^n \).

In particular, \( \mathfrak{t} \) is generated by elements \( \alpha'_i(n) \) \((i = 0, \ldots, l, n \in \mathbb{Z})\), where \( \alpha'_0 \) is the longest root of \( \Phi \) and

\[
\alpha'_i(n) = \alpha_i \otimes t^n.
\]

The commutation relations for \( \mathfrak{t} \) are given by

\[
[\alpha'_i(n), \alpha'_j(m)] = n(\alpha_i | \alpha_j) \delta_{n, -m} c.
\]

We may view the symmetric algebra \( S(\mathfrak{t}^-) \) as a \( \mathfrak{t} \)-module so that \( c \) acts as 1, elements of \( \mathfrak{t}^- \) act by multiplication and elements of \( \mathfrak{t}^+ \) annihilate 1. It is \( \mathbb{Z} \)-graded by setting

\[
\deg(h \otimes \mathfrak{t}^-^n) = n
\]

for \( h \in \mathfrak{h}' \) and \( n \geq 1 \).

Define generating series

\[
H_{\alpha'_i}(z) = \exp \left( \sum_{n \geq 1} \frac{\alpha'_i(-n)z^n}{n} \right) \quad \text{and} \quad K_{\alpha'_i}(z) = \left( -\sum_{n \geq 1} \frac{\alpha_i(n')z^n}{n} \right)
\]

viewed as elements of \( \text{End}(S(\mathfrak{t}^-))[[z^{\pm1}]] \).

Next, for \( n \geq 1 \) and \( i = 0, \ldots, l \), define

\[
y_n^{(i)} = \alpha'_i(-n)/n \quad \text{and} \quad x_n^{(i)} = \sum_{k_1+2k_2+\cdots=n} \frac{(y_1^{(i)})^{k_1}}{k_1!} \frac{(y_2^{(i)})^{k_2}}{k_2!} \frac{(y_3^{(i)})^{k_3}}{k_3!} \cdots.
\]
Note that
\[ H^{(i)}(t) := H_{\alpha'_i}(t) = \exp \left( \sum_{n \geq 1} \frac{y_{n}^{(i)}}{n} t^n \right) = 1 + \sum_{n \geq 1} x_{n}^{(i)} t^n. \]

The \( y_{n}^{(i)}, i = 1, \ldots, n, n \geq 1 \), form a basis for \( t^- \). Hence \( S(t^-) \) is the free polynomial algebra
\[ B_C := \bigotimes_{i=1}^{l} \mathbb{C}[y_{n}^{(i)}|n \geq 1]. \]

Since the transition matrix from the \( x_{n}^{(i)} \) to the \( y_{n}^{(i)}/n \) is unitriangular, we obtain a \( \mathbb{Z} \)-form
\[ B_Z = \bigotimes_{i=1}^{l} \mathbb{Z}[x_{n}^{(i)}|n \geq 1] \subset B_C \]
for \( B_C \). Moreover, since \( \alpha'_0 \) is a \( \mathbb{Z} \)-linear combination of simple roots, the elements \( x_{n}^{(0)} \) belong to the lattice \( B_Z \). Finally, \( B_Z \) inherits a \( \mathbb{Z} \)-grading from the grading on \( S(t^-) \) given by
\[ \deg(y_{n}^{(i)}) = \deg(x_{n}^{(i)}) = n \]
for \( i = 1, \ldots, l \).

Fix \( d \geq 0 \). It was shown in ([2], lemma 4.1) that the Gram matrix of the Shapovalov form on the \( (w\Lambda_0 - d\delta) \) weight space of \( V_Z \) is related to the Gram matrix of the Shapovalov form on the degree \( d \) component of \( B_Z \) in a unimodular way.

Inspiration for this paper comes from ([2], 3.5). The relevant fact for our purposes is that
\[ K_{\alpha'_i}(z)H_{\alpha'_j}(w) = H_{\alpha'_j}(w)K_{\alpha'_i}(z)(1 - zw)^{-a_{ij}} \]
where \( a_{ij} = (\alpha'_i|\alpha'_j) \) is the \((i,j)\)-entry in the Cartan matrix for \( g' \). In the next section, we will show how to extract the entries of the Gram matrix of the Shapovalov form from certain coefficients in the generating series
\[ \prod_{i,j=1}^{l} \prod_{s,t \geq 1} (1 - z_{s}^{(i)}w_{t}^{(j)})^{-a_{ij}}. \]
3. The Shapovalov Form on the Ring of Symmetric Functions

Throughout the paper, we will use notation from MacDonald, [6]. In particular, for a partition \( \lambda \), \( m_\lambda \), \( h_\lambda \), and \( p_\lambda \) will denote the monomial, homogeneous, and power sum symmetric functions, respectively.

Now, since \( B_\mathbb{C} \) is a free polynomial algebra, we may identify it with the \( l \)-fold tensor product
\[
\Lambda := \bigotimes_{i=1}^{l} \Lambda^{(i)}
\]
via \( x_n^{(i)} = h_n^{(i)} \). Here, \( \Lambda^{(i)} \) is the ring of symmetric functions in the variables \( z_n^{(i)} \) (or any other letter) and \( h_n^{(i)}(z) := h_n(z^{(i)}) \) is the \( n \)th homogeneous symmetric function in the variables \( z_n^{(i)} \), \( n \geq 1 \).

This induces an identification of \( B_\mathbb{C} \) with \( \Lambda_\mathbb{C} = \bigotimes_{i=1}^{l} \Lambda^{(i)}_\mathbb{C} \) so that \( y_n^{(i)} = p_n^{(i)} / n \).

We may index bases of \( \Lambda \) (resp. \( \Lambda_\mathbb{C} \)) as follows:

**Notation 3.1.** Let \( I = \{1, 2, \ldots, l\} \), and for a partition \( \lambda = (\lambda_1 \geq \lambda_2, \ldots \geq \lambda_k > 0) \), let \( I(\lambda) = I^k \). If \( \underline{i} = (i_1, \ldots, i_k) \in I(\lambda) \), set
\[
\begin{align*}
\lambda^{(i)}_1 = & \lambda^{(i)}_{i_1}, \\
\lambda^{(i)}_k = & \lambda^{(i)}_{i_k}, \\
p^{(i)}_\lambda = & p^{(i)}_{\lambda_{i_1}} \cdots p^{(i)}_{\lambda_{i_k}}, \\
m^{(i)}_\lambda = & m^{(i)}_{\lambda^{(1)}_{i_1}} \cdots m^{(i)}_{\lambda^{(l)}_{i_k}}
\end{align*}
\]
where \( \lambda^{(j)} \) is the partition consisting of the parts \( \lambda_r \) satisfy \( i_r = j \). (Note the difference in the formula describing the monomial symmetric functions. This is due to the fact that the \( m_\lambda \) are not multiplicative.)

Let \( \Omega(\lambda) = \{ \underline{i} \in I(\lambda) | i_j \leq i_{j+1} \text{ whenever } \lambda_j = \lambda_{j+1} \} \). Then the set \( \{ (\lambda, \underline{i}) | \lambda \in \text{Par}(d), \underline{i} \in \Omega(\lambda) \} \) is in one-to-one correspondence with the dimension of the \( d \)th graded component \( \Lambda_d \) of \( \Lambda \).

It is worth noting another indexing system for bases of \( \Lambda \).

**Notation 3.2.** Given \( \lambda \in \text{Par}(d) \) and \( \underline{i} \in \Omega(\lambda) \), let \( \lambda^{(i)}_r \) be the partition consisting of the parts \( \lambda_r \) satisfy \( i_r = j \). Define the multipartition \( \underline{\lambda} := (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(l)}) \), set \( |\underline{\lambda}| = \sum |\lambda^{(i)}| \). Define
\[
\begin{align*}
\lambda^{(i)}_1 = & \lambda^{(1)}_{i_1}, \\
\lambda^{(i)}_k = & \lambda^{(l)}_{i_k}, \\
p^{(i)}_\lambda = & p^{(i)}_{\lambda^{(1)}_{i_1}} \cdots p^{(i)}_{\lambda^{(l)}_{i_k}}, \\
m^{(i)}_\lambda = & m^{(i)}_{\lambda^{(1)}_{i_1}} \cdots m^{(i)}_{\lambda^{(l)}_{i_k}}
\end{align*}
\]
Then, the set of all multipartitions \( \underline{\lambda} \) with \( |\underline{\lambda}| = d \) is in one-to-one correspondence with the dimension of \( \Lambda_d \).
When working with multipartition \( \Delta \), it will sometimes be useful to refer to the associated partition \( \Sigma \) defined by \( \Sigma = \sum_{i=1}^{l} \lambda^{(i)} \). Here, given two partitions \( \sigma = (i_{m_{i}(\sigma)}) \) and \( \rho = (i_{m_{i}(\rho)}) \), \( \sigma + \rho = (i_{m_{i}(\sigma)+m_{i}(\rho)}) \).

To go back to the notation of 3.1, note that \( h_{\lambda} = h_{\Sigma}^{(j)} \), where \( j = (i_{k}, \ldots, i_{1}) \) is obtained by writing \( \Sigma = (\lambda^{(1)}, \ldots, \lambda^{(k)}) > 0 \) according to the rule that \( i_{j} \leq i_{j+1} \) if \( \lambda_{j} = \lambda_{j+1} \).

Now, we transport the Shapovalov form to \( \Lambda \) using the identification above. It follows from (2), 5.3 that

\[
\langle p_{\lambda}, p_{\mu} \rangle_{S} = \delta_{\lambda\mu}a_{i_{j}j_{1}} \cdots a_{i_{k}j_{k}}z_{\lambda}
\]

where, for \( \lambda = (1^{m_{1}}2^{m_{2}} \cdots) \),

\[
z_{\lambda} = \prod_{r \geq 1} (r^{m_{r}} \cdot m_{r}!).
\]

We will now record some facts about transition matrices from [6]. Let \( L := M(p, m) \) denote the transition matrix from the power sums to the monomial symmetric functions. Using Notation 3.1, \( L = (L_{\lambda\mu}) \) where

\[
p_{\lambda} = \sum_{\mu} L_{\lambda\mu}m_{\mu}.
\]

The transition matrix \( A = M(h, p) \) is related to \( L \) by

\[
A = L^{t}z^{-1}
\]

where \( z = \text{diag}(z_{\Sigma}) \) is diagonal and \( \Sigma \) is defined in (2).

We also have the unimodular transition matrix \( N = M(h, m) \). Using standard properties of transition matrices ([6], Ch.I, (6.3)) we deduce that

\[
A = M(h, p)
= M(h, m)M(m, p)
= NL^{-1}.
\]

Let \( C = (h_{\lambda}, h_{\mu})_{S} \Sigma_{\lambda}^{\mu} \) and let \( B \) be the matrix

\[
B = (p_{\lambda}, z_{\Sigma}^{-1}p_{\mu})_{S}^{\lambda\mu}.
\]
This matrix is block diagonal in the sense that \( B_{\lambda\mu} = 0 \) unless \( \Sigma_{\lambda} = \Sigma_{\mu} \) (see (2) above). Then, by the relevant definitions,

\[
A^{-1}C(A^{-1})^t = Bz
\]

whence

\[
C = ABzA^t = NL^{-1}Bzz^{-1}L
\]

so that

\[
N^{-1}C = L^{-1}BL.
\]

Notice that \( N \) is a unimodular matrix, so the elementary divisors of \( N^{-1}C \) are the same as those for \( C \). Notice too that

\[
(N^{-1}C)_{\alpha\beta} = \sum_{\lambda} N^{-1}_{\alpha\lambda}C_{\lambda\beta} = \sum_{\lambda} N^{-1}_{\alpha\lambda}\langle h_{\lambda}, h_{\beta} \rangle_S = \langle \sum_{\lambda} N^{-1}_{\alpha\lambda} h_{\lambda}, h_{\beta} \rangle_S = \langle m_{\alpha}, h_{\beta} \rangle_S.
\]

We are now in a position to calculate a generating series for the entries of \( N^{-1}C = (\langle m_{\lambda}, h_{\mu} \rangle_S)_{\lambda\mu} \). In fact, we prove a more general proposition that will be useful in the next section. For this Proposition, we use notation \[\text{(3.1)}\] In this notation, \( L = (L_{\lambda\mu}) \) where

\[
p^{(i)}_{\lambda} = \sum_{\mu, j \in \Omega(\mu)} L^{i,j}_{\lambda\mu}m_{j}.
\]

**Proposition 3.3.** Let \( \tilde{P} = (p_{ij}) \) be an \( l \times l \) matrix and define

\[
P = \bigoplus_{\lambda \in \text{Par}} S^{m_{\lambda}(\lambda)}(\tilde{P}) \otimes S^{m_{2}(\lambda)}(\tilde{P}) \otimes \cdots.
\]

Then \( (L^{-1}PL)_{\alpha\beta}^{i,j} \) is the coefficient of \( h^{(i)}_{\alpha}m^{(j)}_{\beta}(x) \) in the product

\[
\prod_{i,j=1}^{l} \prod_{s,t} \left(1 - x^{(i)}_{s} y^{(j)}_{t}\right)^{-p_{ij}}.
\]
Proof: First, we describe what we mean by $S^m(\tilde{P})$. To this end, label the rows (resp. columns) of $\tilde{P}$ with integers $1, 2, \ldots, l$ from left to right (resp. top to bottom). Then, the rows (resp. columns) of $S^m(\tilde{P})$ are labelled by $m$-tuples $(i_1 \leq i_2 \leq \cdots \leq i_m) =: \mathbf{i}$ and the $(i, j)$ entry of $S^m(\tilde{P})$ is

$$p_{i_1 j_1} p_{i_2 j_2} \cdots p_{i_m j_m}.$$  

Notice here that the rows and columns of the $\lambda$ block of $P$,

$$S^{m_1(\lambda)}(\tilde{P}) \otimes S^{m_2(\lambda)}(\tilde{P}) \otimes \cdots,$$

are naturally labelled by $\Omega(\lambda)$.

We deduce the following from [3], Proposition 7.7.4

$$\prod_{i,j=1}^{l} \prod_{s,t \geq 1} (1 - x_s^{(i)} y_t^{(j)})^{-p_{ij}} = \prod_{i,j=1}^{l} \exp \left( \sum_{n \geq 0} \frac{1}{n} p_{n}^{(i)}(x) p_{n}^{(j)}(y) \right)^{p_{ij}}$$

$$= \prod_{i,j=1}^{l} \prod_{n \geq 0} \exp \left( \frac{p_{n}^{(i)}(x) p_{n}^{(j)}(y)}{n} \right)$$

$$= \sum_{\lambda \vdash l} \sum_{\alpha} P_{\lambda}^{(i)} p_{\alpha}^{(j)} (x) p_{\lambda}^{(i)} (y).$$

Moreover, we have that

$$(L^{-1} P L)_{\alpha \beta}^{\lambda} = \sum_{\lambda \vdash l} (L^{-1})_{\alpha \lambda}^{\lambda} p_{\lambda}^{(i)} \chi_{\lambda}^{(j)} L_{\lambda \beta}^{(i)}.$$  

Thus,

$$\prod_{i,j=1}^{l} \prod_{s,t \geq 1} (1 - x_s^{(i)} y_t^{(j)})^{-p_{ij}} = \sum_{\lambda \vdash l} \sum_{\alpha} P_{\lambda}^{(i)} p_{\alpha}^{(j)} (x) p_{\lambda}^{(i)} (y)$$

$$= \sum_{\lambda \vdash l} \sum_{\alpha} P_{\lambda}^{(i)} p_{\alpha}^{(j)} \chi_{\lambda}^{(j)} \left( \sum_{n \vdash \lambda} (L^{-1})_{\alpha \lambda}^{\lambda} z_{\lambda} p_{\lambda}^{(i)} (x) \right)$$

$$\times \left( \sum_{\beta \vdash \lambda} L_{\beta \lambda}^{(i)} m_{\beta}^{(j)} (y) \right)$$

$$= \sum_{\lambda \vdash l} \sum_{\alpha} \sum_{\beta \vdash \lambda} (L^{-1})_{\alpha \lambda}^{\lambda} p_{\lambda}^{(i)} \chi_{\lambda}^{(j)} L_{\beta \lambda}^{(i)} h_{\alpha}^{(i)} (x) m_{\beta}^{(j)} (y).$$
Corollary 3.4. The coefficient of $h_\alpha(x)m_\beta(y)$ in the generating series

$$\prod_{i,j=1}^l \prod_{s,t} \left(1 - x_s^{(i)} y_t^{(j)}\right)^{-a_{ij}}$$

is $\langle m_\alpha, h_\beta \rangle_S$.

Proof: This follows from the fact that

$$B = \bigoplus_{\lambda = (1^{m_1}2^{m_2}\ldots)} S^{m_1}(X) \otimes S^{m_2}(X) \otimes \cdots$$

where $X = (a_{ij})_{i,j=1}^l$ is the Cartan matrix of the simple Lie algebra of type $X_l$ ($X = ADE$). See (2, 5.3) for details.

4. SOME REDUCTIONS

4.1. Block Diagonalization of the Gram Matrix. Let $X = (a_{ij})$ be the Cartan matrix for the simple finite dimensional Lie algebra of type $X_l$ ($X = A, D, E$). We will show that the elementary divisors of the Shapovalov form depend only on the elementary divisors of $X$. To this end, let $\tilde{Q} = (q_{ij})$ and $\tilde{T} = (t_{ij})$ be unimodular matrices such that $\tilde{Q}X\tilde{T} = \text{diag}(a_1, a_2, \ldots, a_l)$. Construct matrices $Q$ and $T$ by the formulae:

$$Q = \bigoplus_{\lambda \in \text{Par}} S^{m_1}(\tilde{Q}) \otimes S^{m_2}(\tilde{Q}) \otimes \cdots$$

and

$$T = \bigoplus_{\lambda \in \text{Par}} S^{m_1}(\tilde{T}) \otimes S^{m_2}(\tilde{T}) \otimes \cdots$$

where $\lambda = (1^{m_1}2^{m_2}\ldots)$. Then, there exist bases $(q_\lambda)$ and $(t_\lambda)$ for $\Lambda_C$ defined by $Q = M(q, p)$ and $T = M(t, z^{-1}p)^t$. Note that

$$QBT = \bigoplus_{\lambda} S^{m_1}(\tilde{Q}X\tilde{T}) \otimes S^{m_2}(\tilde{Q}X\tilde{T}) \otimes \cdots$$

so

$$\langle q_\lambda, t_\mu \rangle_S = \delta_{\lambda, \mu} a_1^{(\lambda(1))} \cdots a_l^{(\lambda(l))}$$

where $\lambda = (\lambda(1), \ldots, \lambda(l))$ is a multipartition.

Define bases $(a_\lambda)$ and $(b_\lambda)$ for $\Lambda_C$ by the formulae:

$$Y := M(a, h)^t = L^{-1}QL$$

and

$$Z := M(b, m) = L^{-1}TL.$$
Proposition 4.1. 

\[ \prod_{i,j \; k=1}^{l} (1-x^{(k)}_i y^{(k)}_j)^{-a_k} = \sum_{\alpha, \beta} \langle m_\alpha, h_\beta \rangle s a_\alpha b_\beta. \]

Proof: Let 

\[ R := M(z^{-1}, p, a) = L^{-1}Q^{-1} \quad \text{and} \quad S := M(p, b) = T^{-1}L. \]

Then, 

\[ N^{-1}C = L^{-1}BL = (L^{-1}Q^{-1})(QBT)(T^{-1}L) = R(QBT)S. \]

Therefore, 

\[ \langle m_\alpha, h_\beta \rangle S = (N^{-1}C)_{\alpha \beta} = \sum_{\lambda} a_1^{l(\lambda)} \cdots a_l^{l(\lambda)} R_{\alpha \lambda} S_{\lambda \beta}. \]

Now, by a calculation similar to the one given in Proposition 3.3 one has

\[ \prod_{i,j \; k=1}^{l} (1-x^{(k)}_i y^{(k)}_j)^{-a_k} = \prod_{k=1}^{l} \prod_{n \geq 0} \exp \left( \frac{a_k}{n} p_n^{(k)}(x)p_n^{(k)}(y) \right) \]

\[ = \sum_{\lambda} a_1^{l(\lambda)} \cdots a_l^{l(\lambda)} R_{\alpha \lambda} a_\alpha(x) \]

\[ \times \left( \sum_\beta S_{\lambda \beta} b_\beta(y) \right) \]

\[ = \sum_{\alpha \beta} \left( \sum_{\lambda} a_1^{l(\lambda)} \cdots a_l^{l(\lambda)} R_{\alpha \lambda} S_{\lambda \beta} \right) a_\alpha(x) b_\beta(y) \]

\[ = \sum_{\alpha \beta} \langle m_\alpha, h_\beta \rangle S a_\alpha(x) b_\beta(y). \]

Now, if we define bases \((c_\lambda)\) and \((d_\lambda)\) for \(\Lambda\) by the formulae \(M(c, m) = Y\) and \(M(d, h)^t = Z\), we obtain

\[ \sum_{\alpha, \beta} \langle m_\alpha, h_\beta \rangle S a_\alpha(x) b_\beta(y) = \sum_{\alpha, \beta} \langle c_\alpha, d_\beta \rangle S h_\alpha(x) m_\beta(y). \]
Since the \( c_\lambda \) and \( d_\lambda \) are obtained from \( h_\lambda \) and \( m_\lambda \) by unimodular change, the matrix

\[
Y N^{-1} C Z = (\langle c_\mu, d_\nu \rangle s)_{\mu, \nu}
\]

has the same elementary divisors as \( N^{-1} C \).

4.2. From the MacDonald Pairing to the Shapovalov Form. From now on, let \( \Lambda \) (resp. \( \Lambda_C \)) be the usual ring of symmetric functions over \( \mathbb{Z} \) (resp. \( \mathbb{C} \)). Given any real number \( s > 0 \), consider the scalar product \( \langle \cdot, \cdot \rangle_s \) on \( \Lambda_C \) defined on the power sum symmetric functions by the formula

\[
\langle p_\lambda, p_\mu \rangle_s = \delta_{\lambda\mu} s^{l(\lambda)} z_\lambda
\]
as in (\cite{6}, Ch.VI, section 10). We call this form the \( s \)-form on \( \Lambda_C \).

Since the following term occurs frequently, we make the abbreviation:

\[
\Pi(x, y) := \prod_{i,j} (1 - x_i y_j)^{-1}.
\]

**Proposition 4.2.** If \((u_\lambda)\) and \((v_\lambda)\) are two bases for \( \Lambda \), then

\[
\Pi(x, y)^s = \sum_{\lambda, \mu} \langle u_\lambda^*, v_\mu^* \rangle_s u_\lambda(x)v_\mu(y)
\]

where \((u_\lambda^*)\) (resp. \((v_\lambda^*)\)) is the dual basis to \((u_\lambda)\) (resp. \((v_\lambda)\)) with respect to the form \( \langle \cdot, \cdot \rangle_1 \) on \( \Lambda \).

**Proof:** Write \( M(u, h) = (a_{\lambda\mu})_{\lambda, \mu} \) and \( M(v, m) = (b_{\lambda\mu})_{\lambda, \mu} \). Then, \( M(u^*, m) = (M(u, h)^{-1})^t \) and \( M(v^*, h) = (M(v, m)^{-1})^t \) (see \cite{6}, Ch.I, 6.3). Thus,

\[
\sum_{\lambda, \mu} \langle u_\lambda^*, v_\mu^* \rangle_s u_\lambda(x)v_\mu(y) = \sum_{\lambda, \mu} \left( \sum_{\alpha} a_{\alpha \lambda}^{(-1)} m_\alpha \sum_{\beta} b_{\beta \mu}^{(-1)} h_\beta \right)_s \times \left( \sum_{\sigma} a_{\lambda \sigma} h_\sigma(x) \right) \left( \sum_{\rho} b_{\mu \rho} m_\rho(y) \right)
\]

\[
= \sum_{\alpha, \beta, \sigma, \rho} a_{\alpha \lambda}^{(-1)} a_{\lambda \sigma} \left( \sum_{\mu} b_{\beta \mu}^{(-1)} b_{\mu \rho} \right) \times \langle m_\alpha, h_{\beta} \rangle_s h_\sigma(x)m_\rho(y)
\]

\[
= \sum_{\alpha, \beta} \langle m_\alpha, h_{\beta} \rangle_s h_\alpha(x)m_{\beta}(y)
\]

\[
= \Pi(x, y)^s.
\]
The last equality follows from Proposition 3.3 by taking the matrix $\tilde{P}$ to be the $1 \times 1$ matrix $(s)$.

**Theorem 4.3.** Let $a_1, \ldots, a_l$ be the invariant factors of the Cartan matrix for a simple finite dimensional Lie algebra of ADE type. Let $a_{k1}^{(r)}, \ldots, a_{kh}^{(r)}$ be the invariant factors of the form $\langle \cdot, \cdot \rangle_{a_k}$ on the degree $r$ component of $\Lambda$ (here $h = |\text{Par}(r)|$ is the number of partitions of $r$). Then, the invariant factors of the Shapovalov form on the $(w\Lambda_0 - d\delta)$-weight space of $V^Z$ are

$$\left\{ \prod_{k=1}^l a_{ki_k}^{(d_k)} : d_1 + \cdots + d_l = d, 1 \leq i_k \leq |\text{Par}(d_k)| \right\}.$$

**Proof:** The coefficient of $h_\lambda(x)m_\mu(y)$ in the product

$$\prod_{i,j} \prod_{k=1}^l (1 - x_i^{(k)} y_j^{(k)})^{-a_k}$$

is $\langle c_\lambda, d_\mu \rangle_S$, where $c_\lambda$ and $d_\lambda$ are dual to $a_\lambda$ and $b_\mu$ with respect to the 1-form on $\Lambda$ (see Proposition 4.1 and subsequent remarks). Since $c_\lambda$ and $d_\mu$ are obtained from $m_\lambda$ and $h_\mu$ by unimodular change, the matrix $(\langle c_\lambda, d_\mu \rangle_S)$ has the same invariant factors as $C$.

On the other hand, by Proposition 4.2 the coefficient of $h_\lambda(x)m_\mu(y)$ is

$$\prod_{k=1}^l \langle m_{\lambda(k)}, h_{\mu(k)} \rangle_{a_k}.$$ 

In particular,

$$\langle (c_\Delta, d_\mu) \rangle_{\Delta, \mu} = \left( \prod_{k=1}^l \langle m_{\lambda(k)}, h_{\mu(k)} \rangle_{a_k} \right)_{\Delta, \mu}.$$ 

It is just left to observe that the invariant factors of the matrix on the right hand side of the equality above are

$$\left\{ \prod_{k=1}^l a_{ki_k}^{(d_k)} : d_1 + \cdots + d_l = d, 1 \leq i_k \leq |\text{Par}(d_k)| \right\}.$$
4.3. Splitting the Gram Matrix across Primes. For each positive integer $s$, let $X_s = (m_{\lambda \mu} s_{\lambda \mu})_{\lambda \mu}$ denote the Gram matrix of the $s$-form on $\Lambda$, and

$$B_s = \text{diag}\{s^{(\lambda)}\}.$$  

Then, it follows from the calculation done in Proposition 3.3 (with the matrix $(a_{ij})$ taken to be the $1 \times 1$ matrix $(s)$) that

$$X_s = L^{-1} B_s L.$$  

Notice that

$$X_{st} = L^{-1} B_{st} L = L^{-1} B_s B_t L = (L^{-1} B_s L)(L^{-1} B_t L) = X_s X_t.$$  

In particular, if $s$ and $t$ are positive integers satisfying $(s, t) = 1$, then

$$(\det X_s, \det X_t) = 1.$$  

Indeed, this follows since $\det(X_s)$ is a power of $s$. By [7, Theorem II.15],

$$S(X_{st}) = S(X_s) S(X_t),$$  

where $S(X)$ is the Smith normal form of a matrix $X$. This reduces the problem to calculating the elementary divisors of $X_{p^r} = (X_p)^{p^r}$.

5. The Invariant Factors

5.1. Higher Homogeneous Functions. In the previous sections, we have reduced the problem of calculating the invariant factors of the Shapovalov form to calculating the invariant factors of the Gram matrix of the $p^r$-form on the ring of symmetric functions. Thanks to Proposition 1.2 we have identified the entries in this matrix as coefficients of the generating series $\Pi(x, y)^{p^r}$. We have the following:

Lemma 5.1.

$$\Pi(x, y)^s = \sum_{\mu} \prod_{i=1}^{l(\mu)} \left( \sum_{\lambda \vdash \mu_i} \left( \sum_{\mu} \left( \frac{s}{l(\lambda)} \prod_{i=1}^{l(\lambda)} \left( \frac{1}{m_i(\lambda)} m_{i}(\lambda), m_2(\lambda), \ldots \right) h_{\lambda}(x) \right) m_{\mu}(y) \right) \right).$$
Proof: This is a straightforward calculation.

\[
\prod_i (1 - x_i y_j)^{-s} = \left( 1 + \sum_{n \geq 1} h_n(x) y_j^n \right)^s \\
= \sum_{k \geq 0} \binom{s}{k} \left( \sum_{n \geq 1} h_n(x) y_j^n \right)^k \\
= \sum_{n \geq 0} \sum_{k \geq 0} \binom{s}{k} \left( \sum_{\alpha} h_{\alpha}(x) \right) y_j^n
\]

where the last sum is over all \( k \)-tuples \( \alpha = (\alpha_1, \ldots, \alpha_k) \) such that \( \alpha_i \geq 1 \) for all \( i \), and \( \sum \alpha_i = n \).

Now, \( S_k \) acts on \( \mathbb{Z}_{\geq 0}^k \) by permuting the coordinates, and the size of the orbit of a partition \( \lambda \in \mathbb{Z}_{\geq 0}^k \) is

\[
\binom{k}{m_1(\lambda), m_2(\lambda), \ldots}.
\]

Hence,

\[
\sum_{\alpha} h_{\alpha}(x) = \sum_{\lambda \vdash n} \left( \binom{k}{m_1(\lambda), m_2(\lambda), \ldots} \right) h_{\lambda}(x).
\]

If we collect the coefficients of \( y_j^n \) in the expansion above, we obtain

\[
\prod_i (1 - x_i y_j)^{-s} = \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} \binom{s}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} \right) h_{\lambda}(x) y_j^n.
\]

Now, let \( \mu = (\mu_1, \mu_2, \ldots) \). Then, by the calculation above, the coefficient of \( y_{j_1}^{\mu_1} y_{j_2}^{\mu_2} \cdots \) in the generating series \( \Pi(x,y)^s \) is

\[
\prod_{i=1}^{l(\mu)} \left( \sum_{\lambda \vdash \mu_i} \binom{s}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} \right) h_{\lambda}(x).
\]

Hence, the Lemma.

Let \( h_{\alpha}^{(r)}(x) \) be the coefficient of \( t^n \) in the generating series

\[
H(t)^{p^r} = \prod_{i \geq 1} (1 - x_i t)^{-p^r}
\]

and \( h_{\lambda}^{(r)} = h_{\lambda_1}^{(r)} h_{\lambda_2}^{(r)} \cdots \). Then,

\[
\Pi(x,y)^{p^r} = \sum_{\lambda} h_{\lambda}^{(r)}(x) m_\lambda(y).
\]
Notice that
\[
\prod_{i \geq 1} (1 - x_it)^{-p^r} = \left( \prod_{i \geq 1} (1 - x_it)^{-p^{r-1}} \right)^p.
\]

Thus, applying an argument similar to the proof of Lemma 5.1, we obtain that
\[
\text{(3)} \quad h_n^{(r)} = \sum_{\lambda \vdash n} \left( p \binom{p}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), \ldots, m_n(\lambda)} \right) h_{\lambda}^{(r-1)}.
\]

Let \( \Lambda^{(r)} = \bigoplus_{\lambda} \mathbb{Z} h_{\lambda}^{(r)} \). Then, we have a sequence of sublattices
\[
\Lambda = \Lambda^{(0)} \supset \Lambda^{(1)} \supset \cdots \supset \Lambda^{(r)} \supset \cdots.
\]

5.2. Divisibility Properties of Higher Homogeneous Functions. First, we will examine some divisibility properties of the coefficient of \( h_{\lambda}^{(r-1)}(x)m_\mu(y) \) in the generating series
\[
\Pi(x, y)^{p^r}.
\]

To this end, we have the following Proposition:

**Proposition 5.2.** If \( \lambda \) is a partition of \( n \), then
\[
\binom{p}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), \ldots, m_n(\lambda)}
\]
is divisible by \( p \) unless \( \lambda = \left( \left( \frac{n}{p} \right)^p \right) \).

**Proof:** First, if \( l(\lambda) < p \), then \( p \) divides \( \binom{p}{l(\lambda)} \). On the other hand, if \( l(\lambda) = p \) and \( \lambda \neq \left( \left( \frac{n}{p} \right)^p \right) \), then \( m_i(\lambda) < p \) for all \( i \), so \( p \) divides
\[
\binom{l(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots} = \binom{p}{m_1(\lambda), m_2(\lambda), \ldots}.
\]

This Proposition entitles us to define the following integers:

**Definition 5.3.** If \( \lambda \vdash n \) and \( \lambda \neq ((n/p)) \), set
\[
(4) \quad c_n(\lambda) = \frac{1}{p} \binom{p}{l(\lambda)} \binom{l(\lambda)}{m_1(\lambda), \ldots, m_n(\lambda)}
\]
and \( c_{pn}((n^p)) = 0 \).

**Lemma 5.4.** \( c_{pn}(p\lambda) = c_n(\lambda) \).
Proof: We have \( l(\lambda) = l(p\lambda) \). Also \( m_{pk}(p\lambda) = m_k(\lambda) \), and if \((k, p) = 1\), then \( m_k(p\lambda) = 0 \). Thus,

\[
\begin{align*}
\frac{c_{pn}(p\lambda)}{p} &= \frac{1}{p} \left( \frac{p}{l(p\lambda)} \right) \left( \frac{l(p\lambda)}{m_p(p\lambda), m_{2p}(p\lambda), \ldots, m_{pn}(p\lambda)} \right) \\
&= \frac{1}{p} \left( \frac{p}{l(\lambda)} \right) \left( \frac{l(\lambda)}{m_1(\lambda), \ldots, m_n(\lambda)} \right) \\
&= c_n(\lambda).
\end{align*}
\]

\[\blacksquare\]

Corollary 5.5. Let \( \lambda \) be a partition. Then,

1. \( h_{p\lambda}^{(r)} \equiv (h_\lambda^{(r-1)})^p \mod p\Lambda^{(r-1)} \);
2. If \( \lambda \) contains a part prime to \( p \), then \( h_\lambda^{(r)} \equiv 0 \mod p\Lambda^{(r-1)} \).

5.3. A More Suitable Pair of Bases. Our goal is to find the invariant factors of the matrix whose coefficients are the coefficients of \( h_\lambda(x)m_\mu(y) \) in the generating series

\[
\Pi(x, y)^{p^r} = \sum_\lambda h_\lambda^{(r)}(x)m_\lambda(y).
\]

Unfortunately, the transition matrix \( M(h^{(r)}, h) \) is complicated. In this subsection, we construct a new family of bases \( (g_{\lambda}^{(i, r)}) \) and \( (M_{\lambda}) \) for \( \Lambda \) so that \((g_{\lambda}^{(0, r)})\) and \( (M_{\lambda}) \) are bases for \( \Lambda \),

\[
\Pi(x, y)^{p^r} = \sum_\lambda g_{\lambda}^{(r, r)}(x)M_{\lambda}(y)
\]

and the transition matrix \( M(g^{(r, r)}, g^{(0, r)}) \) is relatively simple. In view of Proposition 4.2, the coefficient of \( g_{\lambda}^{(0, r)}(x)M_{\mu}(y) \) is \(((g_{\lambda}^{(0, r)})^*, M_{\mu}^*)_{p^r} \). The matrix with these entries has the same invariant factors as our original matrix.

Lemma 5.6. For each \( r \geq 1 \) and \( 0 \leq i \leq r \), there exist multiplicative bases \( (g_{\lambda}^{(i, r)}) \) for \( \Lambda^{(i)} \) satisfying

\[
\begin{align*}
\text{i:} & \quad \text{For } (\lambda, p) = 1, \quad g_{\lambda}^{(r, r)} = p^{r-i}g_{\lambda}^{(i, r)}; \\
\text{ii:} & \quad \text{For any } l \geq 1, i \geq 1, \quad g_{pl}^{(i, r)} = pg_{pl}^{(i-1, r)} + (g_{pl}^{(i-1, r)})^p; \\
\text{iii:} & \quad g_{l}^{(i, r)} = \sum_{\lambda \vdash l} c_{\lambda}^{(r-i)}(\lambda)h_{\lambda} \text{ with } c_{\lambda}^{(i)}((\lambda)) = 1 \text{ and } c_{pl}^{(r-i)}(p\lambda) = c_{l}^{(r-i)}(\lambda) \text{ for all admissible } i \text{'s.}
\end{align*}
\]

In particular, \( (g_{\lambda}^{(i, r)}) \) is a multiplicative basis for \( \Lambda^{(i)} \) and the transition matrix \( M(g^{(i, r)}, h^{(i)}) \) is upper unitriangular.
Proof: The case \( r = 1 \). Apply induction on the \( p \)-adic valuation of \( l \). If \( l = n \) is an integer prime to \( p \), set \( g_n^{(1,1)} = h_n^{(1)} \) and

\[
g_n^{(0,1)} = \frac{1}{p} g_n^{(1,1)} = \sum_{\lambda \vdash n} c_n(\lambda) h_\lambda^{} \]

(see equations 3 and 4).

For the inductive step, assume that for some \( l \geq 1 \) we have found \( g_l^{(0,1)} \) satisfying

\[
g_l = \sum_{\lambda \vdash l} c_l^{(1)}(\lambda) h_\lambda^{} \]

for integer coefficients \( c_l^{(1)}(\lambda) \). Set

\[
g_l^{(1,1)} = \sum_{\lambda \vdash l} c_l^{(1)}(\lambda) h_\lambda^{(1)}.
\]

Then, by Corollary 5.5,

\[
g_{pl}^{(1,1)} \equiv \sum_{\lambda \vdash l} c_l^{(1)}(\lambda) (h_\lambda^{} p) \equiv (g_l^{(0,1)})^p \]

modulo \( pA \). Set

\[
g_{pl}^{(0,1)} = \frac{1}{p} [g_{pl}^{(1,1)} - (g_l^{(0,1)})^p],
\]

and note that equation (5) guarantees that \( \text{iii} \) holds.

The case \( r \geq 2 \). Assume by induction that the lemma holds for all smaller \( r \). In particular, we have integers \( c_l^{(s)} \) for all \( s < r \). Hence, for \( i > 0 \), set

\[
g_l^{(i,r)} = \sum_{\lambda \vdash l} c_l^{(r-i)}(\lambda) h_\lambda^{(i)}.
\]

Then, \( \text{iii} \) holds for \( i > 0 \). Moreover, \( \text{ii} \) holds for \( i > 1 \). Indeed, \( \text{ii} \) is equivalent to

\[
\sum_{\lambda \vdash pl} c_{pl}^{(r-i)}(\lambda) h_\lambda^{(i)} = p \left( \sum_{\lambda \vdash pl} c_{pl}^{(r-i+1)}(\lambda) h_\lambda^{(i-1)} \right) + \left( \sum_{\lambda \vdash l} c_l^{(r-i+1)}(\lambda) h_\lambda^{(i-1)} \right)^p.
\]

By induction, we have

\[
g_{pl}^{(i-1,r-1)} = pg_{pl}^{(i-2,r-1)} + (g_l^{(i-2,r-1)})^p
\]

or

\[
\sum_{\lambda \vdash pl} c_{pl}^{(r-i)}(\lambda) h_\lambda^{(i-1)} = p \left( \sum_{\lambda \vdash pl} c_{pl}^{(r-i+1)}(\lambda) h_\lambda^{(i-2)} \right) + \left( \sum_{\lambda \vdash l} c_l^{(r-i+1)}(\lambda) h_\lambda^{(i-2)} \right)^p.
\]

Since \( M(h^{(i)}, h^{(i-1)}) = M(h^{(i-1)}, h^{(i-2)}) \), this implies equation (6). Similarly, one checks that \( \text{i} \) holds for all \( i > 0 \).
Now, we define $g_i^{(0,r)}$ inductively by the formulas
\[
g_i^{(0,r)} = \begin{cases} 
1/p g_i^{(1,r)} & \text{if } (n,p) = 1, \\
1/p [g_p^{(1,r)} - g_i^{(0,r)}] & \text{else.}
\end{cases}
\]

To finish the proof, we check by induction on the $p$-adic valuation of $l$ that $g_i^{(0,r)} \in \Lambda$, the coefficient of $h_l$ in $g_i^{(0,r)}$ is 1, and $c^{(r-1)}_{pl}(p\lambda) = c^{(r)}_i(\lambda)$. For the induction base, observe that when $(n,p) = 1$, $g_i^{(1,r)}$ is a linear combination of $h^{(1)}_{\lambda}$ satisfying the conditions of Corollary 5.5(2), so $g_i^{(0,r)} \in \Lambda$. It is also clear from equation (8) that the coefficient of $h_n$ in $g_i^{(0,r)}$ is 1. Next, we prove that
\[
c^{(r-1)}_{pl}(p\lambda) = c^{(r)}_i(\lambda).
\]

Indeed, the number $c^{(r)}_i(\lambda)$ is the coefficient of $h_\lambda$ in
\[
g_i^{(0,r)} = \frac{1}{p} g_i^{(1,r)} = \frac{1}{p} \left[ \sum_{\mu \vdash n} c^{(r-1)}_{n}(\mu)h^{(1)}_{\mu} \right].
\]

On the other hand, $c^{(r-1)}_{pl}(p\lambda)$ is the coefficient of $h_{p\lambda}$ in
\[
g_{pl}^{(0,r)} = \frac{1}{p} \left[ g_p^{(1,r)} - g_i^{(0,r)} \right]
= \frac{1}{p} \left[ \sum_{\nu \vdash pn} c^{(r-1)}_{pn}(\nu)h_{\nu} - \left( \sum_{\mu \vdash n} c^{(r-1)}_{n}(\mu)h^{(1)}_{\mu} \right) \right].
\]

No term of the form $h_{p\lambda}$ comes from
\[
\left( \sum_{\mu \vdash n} c^{(r-1)}_{n}(\mu)h^{(1)}_{\mu} \right)^p,
\]
as every partition occurring there contains a part prime to $p$. Hence, $c^{(r-1)}_{pl}(p\lambda)$ is the coefficient of $h_{p\lambda}$ in
\[
\frac{1}{p} \sum_{\nu \vdash pn} c^{(r-1)}_{pn}(\nu)h_{\nu}.
\]

It follows from equation (8) that $h_{p\lambda}$ appears in $h^{(1)}_{\nu}$ only if $\nu$ is of the form $p\mu$ for $\mu \vdash n$. So (7) follows by comparing (8) and (9), since by induction we know that $c^{(r-2)}_{pn}(p\mu) = c^{(r-1)}_{n}(\mu)$. 

For the inductive step, we have
\[ g_{pl}^{(1,r)} = \sum_{\lambda \vdash pl} c_{pl}^{(r-1)}(\lambda) h_{\lambda}^{(1)} = \sum_{\lambda \vdash r-1} c_{\lambda}^{(r)}(h_{\lambda})^p = (g_{l}^{(0,r)})^p \]
modulo \(p\Lambda\). This shows that \(g_{pl}^{(0,r)} \in \Lambda\). Moreover, equation (3) together with the definition of \(g_{pl}^{(0,r)}\) imply that the coefficient of \(h_{pl}\) in \(g_{pl}^{(0,r)}\) is 1.

Finally, we show that \(c_{pl}^{(r-1)}(p\lambda) = c_{pl}^{(r)}(\lambda)\). The number \(c_{pl}^{(r-1)}(p\lambda)\) is the coefficient of \(h_{p\lambda}\) in
\[ g_{p^2l}^{(0,r-1)} = \frac{1}{p}[g_{p^2l}^{(1,r-1)} - (g_{pl}^{(0,r-1)})^p] \]
and the number \(c_{pl}^{(r)}(\lambda)\) is the coefficient of \(h_{\lambda}\) in
\[ g_{pl}^{(0,r)} = \frac{1}{p}[g_{pl}^{(1,r)} - (g_{l}^{(0,r)})^p] \]
By induction, we may assume that the coefficient of \(h_{p\lambda}\) in \((g_{pl}^{(0,r-1)})^p\) is the same as the coefficient of \(h_{\lambda}\) in \((g_{l}^{(0,r)})^p\). Now, argue as above that the coefficient of \(h_{p\lambda}\) in \(g_{p^2l}^{(1,r-1)}\) is the same as the coefficient of \(h_{\lambda}\) in \(g_{pl}^{(1,r)}\).

**Corollary 5.7.** There exists a basis \((M_{\lambda})\) for \(\Lambda\) such that
\[ \Pi(x, y)^{pr} = \sum_{\lambda} g_{\lambda}^{(r,r)}(x) M_{\lambda}(y) \]

**Proof:** Let \(Z = M(h^{(r)}, g^{(r,r)})\). By Lemma 5.6 iii, it follows that \(Z\) is the same for all \(r\). Now, define \((M_{\lambda})\) by \(M(M, m) = Z^t\). Then,
\[ \Pi(x, y)^{pr} = \sum_{\lambda} h_{\lambda}^{(r)}(x) m_{\lambda}(y) = \sum_{\lambda, \mu} Z_{\lambda \mu} g_{\mu}^{(r,r)}(x) m_{\lambda}(y) = \sum_{\mu} g_{\mu}^{(r,r)}(x) \left( \sum_{\lambda} Z_{\lambda \mu} m_{\lambda}(y) \right) = \sum_{\mu} g_{\mu}^{(r,r)}(x) M_{\mu}(y) \]
5.4. Invariant factors of the $p$-form on $\Lambda$. For ease of notation, we omit superscripts in this section. That is, set $g_\lambda := g_\lambda^{(0,1)}$. Then, by Corollary 5.7 the entries of the Gram matrix of the $p$-form on $\Lambda$ are the coefficients of $g_\lambda(x)M_\mu(y)$ in

$$
\Pi(x, y)^p = \sum_{\mu} \prod_{(n,p)=1} (pg_n)^{m_{n}(\mu)} \prod_{i \geq 1} (pg_{p^i n} + (g_{p^{i-1} n})^p)^{m_{p^i n}(\mu)} M_\mu(y).
$$

Let $n$ be an integer prime to $p$, and $i \geq 1$. Define elements $G_{(n)} = g_n$, $G_{(p^i n)} = g_{p^i n}$, and

$$
G_{(p^i n)} = pG_{(p^{i-1} n)} + (G_{(p^{i-1} n)})^p = pg_{p^i n} + (g_{p^{i-1} n})^p.
$$

If $m$ is any integer, let $m = \sum a_i p^i$ be its $p$-adic decomposition. We set

$$
G_{(n)^m} = \prod_i (G_{(p^i n)})^{a_i} \quad \text{and} \quad G_{((p^i n)^m)} = (G_{(p^i n)})^m.
$$

For $\lambda = (1^{m_1} 2^{m_2} \cdots)$, set

$$
G_\lambda = G_{(1^{m_1})} G_{(2^{m_2})} \cdots.
$$

We will demonstrate below that $(G_\lambda)$ forms a basis for $\Lambda$.

**Lemma 5.8.** There exist integers $C_{ij}(\lambda)$ such that

$$
(G_{(p^i n^{i-j})})^{p^j} = \sum_{\lambda=p^i n} C_{ij}(\lambda) G_\lambda
$$

and

(1) $C_{ij}((n^{i-j})) = \pm p^{\frac{j}{p-1}}$, and

(2) $C_{ij}(\lambda) = 0$ unless $\lambda = (p^i n \geq p^j n \geq \cdots)$.

**Proof:** Proceed by induction on $i \geq 0$ and $0 \leq j \leq i$. For $i = 0$, or $i > 0$ and $j = 0$ the lemma holds trivially. Suppose that $i, j > 0$ and write

$$
\varphi_{i-1,j-1}(G) = \sum_{\lambda \neq (n^{i-j-1})} C_{i-1,j-1}(\lambda) G_\lambda.
$$
Now,

\[
(G_{(n^s)^{-j}})^{p^j} = ((G_{(n^s)^{-j}})^{p^{j-1}})^p \\
= \left( \varphi_{i-1,j-1}(G) \pm p^{(p^{j-1}-1)/(p-1)} G_{(n^s)^{-1}} \right)^p
\]

(12)

\[
\sum_{k=0}^{p-1} \binom{p}{k} \left( \pm p^{(p^{j-1}-1)/(p-1)} \varphi_{i-1,j-1}^{-1}(G) \right)^{p-k} G_{(n^s)^{-1}}^k
\]

\[
\pm p^{(p^j-p)/(p-1)} G_{(n^s)^{-1}}^p.
\]

Induction applies to the first summand in (12). By (10), we have that

\[
(G_{(n^s)^{-1}})^p = G_{(p' n)} - p G_{(n^s)^{-1}}.
\]

Therefore,

\[
p^{(p^j-p)/(p-1)} G_{(n^s)^{-1}}^p = p^{(p^j-p)/(p-1)} (G_{(p' n)} - p G_{(n^s)^{-1}})
\]

\[
= p^{(p^j-p)/(p-1)} G_{(p' n)} - p^{(p^j-p)/(p-1)+1} G_{(n^s)^{-1}}.
\]

Hence, the lemma. \(\blacksquare\)

**Corollary 5.9.** \((G_\lambda)\) forms a basis for \(\Lambda\).

**Proof:** It is easy to write \(G_\lambda\) as an integral linear combination of \(g_\mu\)'s. Conversely, write

\[
g_\lambda = \prod_{(n,p)=1} \prod_{i \geq 0} (g_{p^i n})^{m_{p^i n}(\lambda)}
\]

(13)

\[
= \prod_{(n,p)=1} \prod_{i \geq 0} (G_{(p^i n)})^{m_{p^i n}(\lambda)}.
\]

and expand using (10) and (11) above. By Lemma 5.8(2), it is enough to expand a product of the form

\[
\prod_{i \geq 0} (G_{(p^i)})^{m_i},
\]

which can be obtained by induction on \(\sum_i m_i\). Indeed, if all \(m_i < p\), there is nothing to do. Otherwise, find the smallest \(i\) such that \(m_i > p\), and let \(m_i = \sum_{j \geq 0} a_j^i p^j\) be the \(p\)-adic expansion of \(m_i\). We have

\[
(G_{(p^i)})^{m_i} = \prod_{j \geq 0} (G_{(p^i)})^{p^j a_j^i}
\]

\[
= \prod_{j \geq 0} \left( (\text{const.}) G_{(p^{i+j})} \right)^{a_j^i} + (*).
\]
It follows from Lemma 5.8(2) that (⋯) is a linear combination of terms having fewer than \(m_i\) components of the form \(G_{(n^p)}\). Observing that \(\sum_{j \geq 0} a_j^i < m_i\), completes the induction.

In this basis, we have that

\[
\Pi(x, y)^p = \sum_{\mu} \prod_{(n, p) = 1} p^{m_n(\mu)} (G_{(n)}(x))^{m_n(\mu)} \prod_{i \geq 1} (G_{((p')^n m_{p^a(n)})}(x)) M_{\mu}(y).
\]

We need to analyze the term \((G_{(n)}(m_n(\mu)))\) in the generating series above. To this end, let \(m_n(\mu) = m_n = \sum_i a_i^n p^i\) be its p-adic decomposition. Then,

\[
(G_{(n)})^{m_n} = \prod_i (G_{(n)})^{a_i^n p^i}.
\]

Recall the definition of \(d_p(a)\) given by equation (1) on page 4.

**Corollary 5.10.** The coefficient of \(G_{(n^m n)}\) in the product \((G_{(n)}(m_n))\) is \(p^{d_p(m_n(\lambda))}\).

**Proof:** By Lemma 5.8 the coefficient of \(G_{(n^m n)}\) in the product \((G_{(n)}(m_n))\) is

\[
\pm \prod_{j \geq 1} p^{a_j^n \left( \frac{p^j - 1}{p - 1} \right)}.
\]

Observe that

\[
\sum_{j \geq 1} a_j^n \left( \frac{p^j - 1}{p - 1} \right) = \sum_{j \geq 1} a_j^n \left( 1 + p + \cdots + p^{j-1} \right)
\]

\[
= \sum_{j \geq 1} \left( \frac{a_j^n p^j}{p^k} \right)\frac{p^j}{p^k}
\]

\[
= \sum_{k \geq 1} \left( \frac{\sum_{j \geq 1} a_j^n p^j}{p^k} \right)
\]

\[
= d_p(m_n).
\]

**Corollary 5.11.** The matrix whose entries are the coefficients of \(G_{\lambda}(x) M_{\mu}(y)\) in the product \(\Pi(x, y)^p\) is upper triangular. Moreover, if \(\lambda = (1^{m_1} 2^{m_2} \cdots)\) and \(m_n = \sum_i a_i^n p^i\) is the p-adic decomposition of \(m_n\), then the coefficient of \(G_{\lambda}(x) M_{\lambda}(y)\) is

\[
\pm \prod_{(n, p) = 1} p^{m_n + d_p(m_n)}.
\]

The following proposition will complete the proof of Theorem 1.3 for the special case \(r = 1\).
Proposition 5.12. The coefficient of $G_\lambda(x)M_\mu(y)$ in the product $\Pi(x, y)^p$ is divisible by that of $G_\lambda(x)M_\lambda(y)$.

Proof: First, observe that by Corollary 5.10, the coefficient of $G_\lambda(x)M_\mu(y)$ is

$$\prod_{(n, p) = 1} p^{m_n(\lambda) + d_p(m_n(\lambda))}.$$  

Let $m_n(\mu) = \sum_{j \geq 0} a_n^\mu(p^j)$ be the $p$-adic expansion of $m_n(\mu)$. Then, Lemma 5.8 gives us that the coefficient of $M_\mu(y)$ in $\Pi(x, y)^p$ is

$$\prod_{(n, p) = 1} p^{m_n(\mu)} \prod_{j \geq 0} \left( \sum_{\sigma \in p^n} C_j(\sigma) G_\sigma(x) \right) \prod_{i \geq 1} \left( G_{\left( (p^j) m_{p^n(\mu)} \right)}(x) \right).$$

Using property (2) of the lemma, we see that if $C_j(\sigma) \neq 0$ and $\sigma \neq (n^p^r)$, then $\sigma$ must have fewer than $p^j$ parts prime to $p$. Hence, the $\lambda$ that occur when expanding the expression above have fewer parts prime to $p$ than $\mu$ and, therefore, the coefficient of $G_\lambda(x)M_\mu(y)$ is divisible by $\prod_{(n, p) = 1} p^{m_n(\lambda)}$. Finally, property (1) of the lemma implies that each time $(n^p^r)$ appears in $\lambda$, the coefficient of $G_\lambda(x)M_\mu(y)$ is divisible by $p^{d_p(p^r)}$.

The following corollary is immediate.

Corollary 5.13. There exists a basis $(N_\lambda)$ for $\Lambda$ such that

$$\Pi(x, y)^p = \sum_\lambda \prod_{(n, p) = 1} p^{m_n(\lambda) + d_p(m_n(\lambda))} G_\lambda(x) N_\lambda(y).$$

5.5. Outline of the Approach to the $p^r$-form. The basis $(G_\lambda)$ for $\Lambda$ constructed in the previous section (together with computational evidence) suggests an approach to finding the invariant factors of the $p^r$-form on $\Lambda$. Indeed, suppose that we can construct a basis $(G_\lambda^{(r)})$ with the following properties:

(P1) For $i \geq 0$,

$$G_\lambda^{(r)}_{(p^i n)} = \frac{1}{[p^r - i]} g_{p^i n}^{(r, r)}$$

where $[x]$ is the ceiling function.

(P2) For $0 \leq i < r$ and $j \geq 0$,

$$G_\lambda^{(r)}_{((p^i)n)^{p^j}} = \sum_{\lambda \in p^i + j n} X_{ij}(\lambda) g_\lambda^{(0, r)}$$

for integer coefficients $X_{ij}(\lambda)$ satisfying $X_{ij}(\lambda (p^i + j n)) = p^i$ and $X_{ij}(\lambda) = 0$ unless $\lambda = (p^k n \geq p^{k+1} n \geq \cdots)$. 
(P3) For 1 \leq i < r and j \geq 0,
\[ \Pi((p^i n)^{p^j}) = (\Pi((p^{i-1} n)^{p^j}))^p + p \Pi((p^{i-1} n)^{p^j} + 1). \]

(P4) For i \geq r,
\[ \Pi((p^i n)) = (\Pi((p^{i-1} n)^{p^{i-r}}))^p + p \Pi((p^{i-1} n)^{p^{i-r} + 1}). \]

(P5) For a partition \( \lambda = (1^{m_1} 2^{m_2} \ldots) \),
\[ G^{(r)}_{\lambda} = \prod_i \Pi^{(r)}_{(i^{m_i})} \]
where, if \( m = \sum_{j \geq 0} a_j p^j \) is the p-adic expansion of \( m \), then
\[ G^{(r)}_{(p^i n)^m} = \begin{cases} \prod_{j \geq 0} (\Pi_{(p^{i-1} n)^{p^j}})^{a_j} & \text{if } 0 \leq i < r; \\ (\Pi_{(p^i n)^m}) & \text{otherwise.} \end{cases} \]

By taking \( i = 0 \) in property (P2) it follows that we indeed have a basis for \( \Lambda \).
Indeed, it is easy to write the \( G^{(r)}_{\lambda} \) in terms of \( g^{(0,r)}_{\lambda} \). To write the \( g^{(0,r)}_{\lambda} \) in terms of \( G^{(r)}_{\lambda} \), we show how to write \( g^{(0,r)}_{p^i n} \) in terms of \( G^{(r)}_{\lambda} \) by induction on \( i \), starting from \( g^{(0,r)}_{n} = G^{(r)}_{(n)} \). Now, assume that we can write \( g^{(0,r)}_{p^i n} \) in terms of \( G^{(r)}_{\lambda} \) for all \( j < i \).

Then,
\[ G^{(r)}_{(n^m)} = \sum_{\lambda \vdash p^i n} X_{0i}(\lambda) g^{(0,r)}_{\lambda} \]
with \( X_{0i}(p^i n) = 1 \) and \( X_{0i}(\lambda) = 0 \) unless \( \lambda = (p^{k_1} n \geq p^{k_2} n \geq \ldots) \). In particular, induction applies to the \( g^{(0,r)}_{\lambda} \) occurring with nonzero coefficients. Since both sets \( (G^{(r)}_{\lambda}) \) and \( (g^{(0,r)}_{\lambda}) \) are labelled by partitions, we conclude that we have a basis.

In this basis, we have
\[ \Pi(x, y)^{p^r} = \sum_{\lambda} \prod_{(n, p) = 1} \prod_{i=0}^{r-1} \left( \Pi^{(r)}_{(p^{i-1} n)}(x) \right)^{m_{\lambda}(\lambda)} \prod_{l \geq 1} \left( \Pi^{(r)}_{(p^l n)}(x) \right)^{M_{\lambda}(y)}. \]

It remains to multiply out the terms \( (\Pi^{(r)}_{(p^i n)^{p^j}})^{m_{\lambda}(\lambda)} \) using (P3) and (P4).

Indeed, we have

**Lemma 5.14.** For 0 \( \leq i < r \), there exist integers \( C_{ijk}(\lambda) \) such that
\[ (G^{(r)}_{(p^i n)^{p^j}})_{p^{k}} = \sum_{\lambda \vdash p^{i+j} n} C_{ijk}(\lambda) G^{(r)}_{\lambda} \]
and
\[ C_{ijk}((p^i n)^{p^j}) = \pm p^{\frac{k-1}{p-1}}, \]
and
(2) \( C_{ijk}(\lambda) = 0 \) unless \( \lambda = (p^{i_1}n \geq p^{i_2}n \geq \cdots) \geq ((p^i n)^{p^j}). \)

The proof of this Lemma is similar to the proof of Lemma 5.8 using (P3) and (P4) above. The analogous corollaries to Lemma 5.8 are also true, and their proofs require no new techniques. In particular, one has

**Corollary 5.15.** For \( 0 \leq i < r \), the coefficient of \( G_{(p^i n)^m}^{(r)} \) in the product \((G_{(p^i n)^m})^m\) is \( p^{d_p(m)}. \)

**Corollary 5.16.** The matrix whose entries are the coefficients of \( G_{\lambda}^{(r)}(x)M_{\mu}(y) \) in the product \( \Pi(x, y)^{p^r} \) is upper triangular. Moreover, if \( \lambda = (1^{m_1}2^{m_2} \cdots) \), then the coefficient of \( G_{\lambda}^{(r)}(x)M_{\lambda}(y) \) is

\[
D_r(\lambda) = \prod_{(n,p)=1} (r-1) \prod_{i=0}^{r-1} p^{(r-i)m_{\mu^n} + d_p(m_{\mu^n})}.
\]

(recall \( D_r(\lambda) \) from (1) on page 3.)

**Proposition 5.17.** The coefficient of \( G_{\lambda}^{(r)}(x)M_{\mu}(y) \) in the product \( \Pi(x, y)^{p^r} \) is divisible by the coefficient of \( G_{\lambda}^{(r)}(x)M_{\lambda}(y) \).

Hence, we arrive at the conjecture

**Conjecture 5.18.** The invariant factors of the Gram matrix of the \( p^r \)-form on the degree \( d \) component of \( \Lambda \) are

\[
\{D_r(\lambda)\mid \lambda \vdash d\}.
\]

In the following sections, we show that the conjecture is true under the assumption that \( r \leq p. \)

**Remark 5.19.** For the sake of the lemmas and propositions above, conditions (P3) and (P4) are overly strict. Instead, we could replace these with the following:

(P3') For \( 1 \leq i < r \) and \( j \geq 0, \)

\[
\left(G_{(p^{i-1}n)^{p^j}}^{(r)}\right)^p = \sum_{\lambda\vdash p^{i+j}n} Y_{ij}(\lambda)G_{\lambda}^{(r)}
\]

where

- \( Y_{ij}(((p^{i-1}n)^{p^{j+1}})) = \pm p; \)
- \( Y_{ij}(\lambda) = 0 \) unless \( \lambda = (p^{i_1}n \geq p^{i_2}n \geq \cdots) \geq ((p^{i-1}n)^{p^{j+1}}). \)
(P4') For $i \geq r$,
\[
\left( G^{(r)}_{((p^{r-1} - n)p^{r-i})} \right)^p = \sum_{\lambda \vdash p^n} Y_{i,i-r}(\lambda) G^{(0,r)}_{\lambda}
\]
where
- $Y_{i,i-r} \left( ((p^{r-1} - n)p^{r-i+1}) \right) = \pm p$;
- $Y_{i,i-r}(\lambda) = 0$ unless $\lambda = (p^{i_1}n \geq p^{i_2}n \geq \cdots \geq (p^{r-1} - n)p^{r-i+1})$.

In fact, for the case $r = p$, we will only obtain conditions (P3') and (P4'). The problem with these conditions is that they do not make satisfactory inductive hypotheses for calculations below.

### 5.6. Invariant factors of the $p^r$-form on $\Lambda$, for $r < p$.

Fix $r \geq 2$, and assume that for $s < r$ we have constructed a basis $(G^{(s)}_{\lambda})$ for $\Lambda$ satisfying properties (P1)-(P5) of §5.5. We take as our base case the basis $(G^{(1)}_{\lambda}) := (G_{\lambda})$ constructed in §5.4.

Define a basis $(\hat{G}_{\lambda})$ for $\Lambda$ inductively by the formula
\[
M(g^{(0,r)}_{(0,pn)}, \hat{G}) = M(g^{(0,r-1)}_{(0,pn)}, G^{(r-1)}_{(pn)}).
\]

Then, since $M(g^{(r-1,1)}_{(0,pn)}, g^{(0,r)}_{(0,pn)}) = M(g^{(r-1,1)}_{(0,pn)}, g^{(0,r-1)}_{(0,pn)})$, property (P1) implies that for $1 \leq i < r$,
\[
g^{(r,r)}_{(p^n)} = pg^{(r-1,r)}_{(p^n)} + (g^{(r-1,1)}_{(p^n)})^p = p(p^{r-i+1} \hat{G}_{(p^n)}) + (p^{r-i} \hat{G}_{(p^n)})^p = 0
\]
modulo $p^{r-i} \Lambda$. We may therefore define
\[
G^{(r)}_{(p^n)} = \frac{1}{[p^{r-i}]} g^{(r,r)}_{(p^n)}
\]
for all $i \geq 0$. (Note that $[p^{r-i}] = 1$ for $i \geq r$.)

Next, we construct $G^{(r)}_{((p^{r-i} - n)p^j)}$ by induction on $0 \leq i < r$ and $0 \leq j \leq i$ so that we obtain properties (P2) and (P3). Notice here that $G^{(r)}_{(n)} = g^{(0,r)}_{(n)}$, so (P2) holds, and (P3) is vacuous.

Now, let $1 < i \leq r-1$. Assume that for $k < i$ and $1 \leq l \leq k$ we have constructed $G^{(r)}_{((p^{k-1} - n)p^l)}$ such that
\[
G^{(r)}_{((p^{k-1} - n)p^l)} = \hat{G}_{((p^{k-1} - n)p^l)} + p^{k-1} \varphi_{k,l}(g)
\]
(15)
where, here and throughout the paper, \( \varphi_{k,i}(g) \) is a polynomial in the \( g_{p^{m}n}^{(0,r)} \) with \( m < k \). (Notice that equation (15) and induction on \( r \) imply that properties (P2) and (P3) hold for \( k < i \.)

By property (P3),
\[
G^{(r)}_{(p^{i}n)} = \hat{G}_{(p^{i}n)} + p^{(p-1)(r-i)} \left( \hat{G}_{(p^{i-1}n)} \right)^{p}
\]
\[
= \left[ p\hat{G}_{((p^{i-1}n)p)} + (\hat{G}_{(p^{i-1}n)})^{p} + p^{(p-1)(r-i)} \left( \hat{G}_{(p^{i-1}n)} \right)^{p} \right],
\]
and, by equation (15),
\[
\hat{G}_{(p^{i-1}n)} = G^{(r)}_{(p^{i-1}n)} - p^{i} \varphi_{i-1,0}(g).
\]
Thus,
\[
(16) \quad G^{(r)}_{(p^{i}n)} = p\hat{G}_{((p^{i-1}n)p)} + (G^{(r)}_{(p^{i-1}n)} - p^{i} \varphi_{i-1,0}(g))^{p}
\]
\[
+ p^{(p-1)(r-i)} \left( \hat{G}_{(p^{i-1}n)} \right)^{p}.
\]
Since \( p > r \) and \( i \leq r - 1 \), it follows that
\[
(p-1)(r-i) > (r-1)(r - (r-1)) = r - 1 \geq i.
\]
Thus,
\[
G^{(r)}_{(p^{i}n)} \equiv p\hat{G}_{((p^{i-1}n)p)} + (G^{(r)}_{(p^{i-1}n)})^{p}
\]
modulo \( p^{i+1} \). Set
\[
G^{(r)}_{((p^{i-1}n)p)} = \frac{1}{p} \left[ G^{(r)}_{(p^{i}n)} - (G^{(r)}_{(p^{i-1}n)})^{p} \right].
\]
By induction on \( i \), property (P2) holds for \( G^{(r)}_{(p^{i-1}n)} \). Together with equation (16), this implies that
\[
G^{(r)}_{((p^{i-1}n)p)} = \hat{G}_{((p^{i-1}n)p)} + p^{i} \varphi_{i,1}(g).
\]
Next, let \( 1 \leq j < i \) and assume by induction that we have constructed
\[
G^{(r)}_{((p^{i-j}n)n^{j})} = \hat{G}_{((p^{i-j}n)n^{j})} + p^{i-j+1} \varphi_{i,j}(g).
\]
Then, by property (P3) and equation (15),
\[
(17) \quad G^{(r)}_{((p^{i-j}n)n^{j})} = \left[ p\hat{G}_{((p^{i-j}n)n^{j+1})} + (G^{(r)}_{((p^{i-j}n)n^{j})} - p^{i-j} \varphi_{i-1,j}(g))^{p} \right]
\]
\[
+ p^{i-j+1} \varphi_{i,j}(g).
\]
Thus,

\[ G^{(r)}_{(p^{i-j-1}n^j)} = p\hat{G}^{(r)}_{(p^{i-j-1}n^j)\rho^{i+1}} + (G^{(r)}_{(p^{i-j-1}n^j)})^p \]

modulo \( p^{i-j+1} \Lambda \).

Set

\[ G^{(r)}_{((p^{i-j-1}n^j)\rho^{i+1})} = \frac{1}{p} \left[ G^{(r)}_{((p^{i-j-1}n^j)^\rho)} - (G^{(r)}_{((p^{i-j-1}n^j)^\rho)})^p \right]. \]

By induction on \( i \), property (P2) holds for \( G^{(r)}_{((p^{i-j-1}n^j)^\rho)} \). Therefore, equation (17) implies that

\[ G^{(r)}_{((p^{i-j-1}n^j)^{\rho^{i+1}})} = \hat{G}^{(r)}_{((p^{i-j-1}n^j)^{\rho^{i+1}})} + p^{i+j} \varphi_{i,j+1}(g). \]

This completes the construction of the \( G^{(r)}_{((p^{i-j-1}n^j)^\rho)} \) for \( i < r \).

The last step in the construction is to obtain the elements \( G^{(r)}_{(p^{i-1}n)} \) for \( i \geq r \) satisfying properties (P3) and (P4) of \[5.5 \text{ or } 1.3 \]. To this end, assume that \( i \geq r \) and that

\[ G^{(r)}_{(p^{i-1}n)} = \hat{G}^{(r)}_{(p^{i-1}n)} + p^{r-1}\varphi_{i-1,0}(g). \]

(Note that we have the right to make this assumption because when \( i = r \), we have

\[ G^{(r)}_{(p^{r-1}n)} = \hat{G}^{(r)}_{(p^{r-1}n)} + p^{r-1}(\hat{G}^{(r)}_{(p^{r-2}n)}))^p \]

and \( p > r \).)

Then,

\[ G^{(r)}_{(p^{i-1}n)} = \frac{1}{p} \left[ G^{(r)}_{(p^{i-1}n)} - (G^{(r)}_{(p^{i-1}n)})^p \right] \]

modulo \( p^{r+1} \Lambda \). Set

\[ G^{(r)}_{((p^{r-1}n)^{\rho^{i-r}})} = \frac{1}{p} \left[ G^{(r)}_{(p^{r-1}n)} - (G^{(r)}_{(p^{r-1}n)})^p \right] = \hat{G}^{(r)}_{(p^{r-1}n)} + p^{r}\varphi_{i,1}(g). \]

By induction on \( r \),

\[ \hat{G}^{(r)}_{(p^{r-1}n)} = p\hat{G}^{(r)}_{((p^{r-2}n)^{\rho^{i-r}})} + (\hat{G}^{(r)}_{((p^{r-2}n)^{\rho^{i-r}})})^p \]
and, by induction on \(i\),

\[
G^{(r)}_{((p-2)\cdot n^{i-2})} = \hat{G}_{((p-2)\cdot n^{i-1})} + p^{r-1}\varphi_{i-1,1}(g).
\]

Thus,

\[
G^{(r)}_{((p-1)\cdot n^{i-r+1})} = p\hat{G}_{((p-2)\cdot n^{i-r+1})} + \left(G^{(r)}_{((p-2)\cdot n^{i-r})} - p^{r-1}\varphi_{i-1,1}(g)\right)^p + p^r\varphi_{i,1}(g)
\]

\[
= \left(G^{(r)}_{((p-2)\cdot n^{i-r-1})}\right)^p
\]

modulo \(p^r\Lambda\). Set

\[
G^{(r)}_{((p-2)\cdot n^{i-r+2})} = \frac{1}{p} \left[G^{(r)}_{((p-1)\cdot n^{i-r+1})} - \left(G^{(r)}_{((p-2)\cdot n^{i-r})}\right)^p\right]
\]

\[
= \hat{G}_{((p-2)\cdot n^{i-r+1})} + p^{r-1}\varphi_{r,2}(g).
\]

Now, assume that \(2 \leq j < r\) and that

\[
G^{(r)}_{((p-j)\cdot n^{i-r+j})} = \hat{G}_{((p-j-1)\cdot n^{i-r+j-1})} + p^{r-j+1}\varphi_{i,j}(g).
\]

By induction on \(r\)

\[
\hat{G}_{((p-j)\cdot n^{i-r+j-1})} = p\hat{G}_{((p-j-1)\cdot n^{i-r+j})} + (\hat{G}_{((p-j-1)\cdot n^{i-r+j-1})})^p
\]

and, by induction on \(i\),

\[
G^{(r)}_{((p-j-1)\cdot n^{i-r+j})} = \hat{G}_{((p-j-1)\cdot n^{i-r+j-1})} + p^{r-j}\varphi_{i-1,j}(g).
\]

Hence,

\[
G^{(r)}_{((p-j)\cdot n^{i-r+j})} = p\hat{G}_{((p-j-1)\cdot n^{i-r+j})} + \left(G^{(r)}_{((p-j-1)\cdot n^{i-r+j})} - p^{r-j}\varphi_{i-1,j}(g)\right)^p + p^{r-j+1}\varphi_{i,j}(g)
\]

\[
= \left(G^{(r)}_{((p-j-1)\cdot n^{i-r+j})}\right)^p
\]

modulo \(p^{r-j+1}\Lambda\). Set

\[
G^{(r)}_{((p-j-1)\cdot n^{i-r+j+1})} = \frac{1}{p} \left(G^{(r)}_{((p-j)\cdot n^{i-r+j})} - \left(G^{(r)}_{((p-j-1)\cdot n^{i-r+j})}\right)^p\right)
\]

\[
= \hat{G}_{((p-j-1)\cdot n^{i-r+j})} + p^{r-j}\varphi_{i,j+1}(g).
\]

This completes the construction of \(G^{(r)}_{p\cdot n}\) satisfying properties (P3) and (P4).
5.7. The $p^2$-form on $\Lambda$. Here we will extend the result one step further. There are two reasons for considering this case. First, the nontrivial invariant factor of the Cartan matrix of type $D_{2l+1}$ is 4. Therefore, by Corollary 1.8, we need to know the invariant factors of the $2^2$-form for this case. Also, the approach to this case lends insight into why the general proof is so difficult.

We begin the construction the way we did in the previous section. As before, we deduce that for $i < p$

$$G_{(p,n)}^{(p)} = \hat{G}_{(p,n)} + p^{(p-1)(p-1)}(\hat{G}_{(p-1,n)})^p$$

$$= p\hat{G}_{((p-1)n,p)} + (1 + p^{(p-1)(p-1)})(G_{(p-1,n)})^p - p^{p-1}(p-1)(\hat{G}_{(p-3,n)})^p.$$

Note that when $i < p - 1$, $(p-1)(p-i) > p-1 > i$, hence

$$G_{(p,n)}^{(p)} = p\hat{G}_{((p-1)n,p)} + (G_{(p-1,n)})^p$$

modulo $p^{i+1}\Lambda$. Therefore, in this case we can construct $G_{(p,n)}^{(p)}$ satisfying property (P3) exactly as we did in the previous section.

When $i = p - 1$, we make the following adjustment. We have that

$$G_{(p-1,n)}^{(p)} = \hat{G}_{(p-1,n)} + p^{p-1}(\hat{G}_{(p-2,n)})^p$$

$$= p\hat{G}_{((p-2)n,p)} + (1 + p^{p-1})(G_{(p-2,n)})^p - p^{2(p-1)}(\hat{G}_{(p-3,n)})^p$$

$$= p\hat{G}_{((p-2)n,p)} + (1 + p^{p-1})(G_{(p-2,n)})^p$$

modulo $p^2\Lambda$. Therefore, set

$$G_{((p-2)n,p)}^{(p)} = \frac{1}{p}[G_{(p-1,n)}^{(p)} - (G_{(p-2,n)})^p]$$

$$= \hat{G}_{((p-2)n,p)} + p^{p-2}(G_{(p-2,n)})^p + p^{p-1}\varphi_{p-1,1}(g).$$

Next, assume by induction that $1 \leq j < p - 1$ and that we have constructed

$$G_{((p^2-j-1)n,p^2)}^{(p)} = \hat{G}_{((p^2-j-1)n,p^2)} + p^{p-j-1}(G_{(p-2,n)})^p + p^{p-j}\varphi_{p-1,j}(g).$$

By induction on $r (= p)$

$$\hat{G}_{((p^2-j-1)n,p^2)} = p\hat{G}_{((p^2-j-2)n,p^2+1)} + (\hat{G}_{((p^2-j-2)n,p^2)})^p,$$

and (by induction on $i = p - 1$)

$$G_{((p^2-j-2)n,p^2)}^{(p)} = p\hat{G}_{((p^2-j-2)n,p^2+1)} + (G_{(p-2,n)})^p + p^{p-j-1}\varphi_{p-2,j}(g).$$
Assume by induction that for $1 \leq p < q$, we have
\[
G_{(p^{q-1}n)}^{(p)} = p\hat{G}_{(p^{q-2}n)}^{(p)} + p^{q-1}\left(\frac{G_{(p^{q-2}n)}^{(p)}}{p^{q-2}n}\right)^p
\]
modulo $p^{q-1}A$. Set
\[
G_{(p^{q-2}n)}^{(p)} = \frac{1}{p}\left[G_{(p^{q-1}n)}^{(p)} - (G_{(p^{q-2}n)}^{(p)})^p\right]
\]
This completes the construction of $G_{(p^{q-1}n)}^{(p)}$ for $i < p$ satisfying property (P3).

Assume that $i = p^k \geq p$. We will construct elements $G_{(p^{q-1}n)}^{(p)}$ by induction on $k$ and $j$. To this end, when $k = 0$, we have
\[
G_{(p^{q-1}n)}^{(p)} = p\hat{G}_{(p^{q-2}n)}^{(p)} + (G_{(p^{q-2}n)}^{(p)})^p
\]
modulo $p^{q-1}A$. Set
\[
G_{(p^{q-2}n)}^{(p)} = \frac{1}{p}\left[G_{(p^{q-1}n)}^{(p)} - (G_{(p^{q-2}n)}^{(p)})^p\right]
\]
Assume by induction that for $1 \leq j < p - 1$
\[
G_{(p^{q-2}n)}^{(p)} = \hat{G}_{(p^{q-3}n)}^{(p)} - p^{q-1}\sum_{l=0}^{j-1}\left(\frac{G_{(p^{q-3}n)}^{(p)}}{p^{q-3}n}\right)^p
\]
By induction on $r (= p)$,
\[
\hat{G}_{(p^{q-j}n)}^{(p)} = \hat{G}_{(p^{q-j-1}n)}^{(p)} + (\hat{G}_{(p^{q-j-1}n)}^{(p)})^p,
\]
and, by induction on $i (= p)$,
\[
G_{(p^{q-j-1}n)}^{(p)} = \hat{G}_{(p^{q-j-1}n)}^{(p)} + p^{q-j-1}(G_{(p^{q-2}n)}^{(p)})^p + p^{q-j}\varphi_{p-1,j}(g).
\]
Hence

\[
G_{(p^{n-1})}^{(p)} = p \hat{G}_{(p^{n-1})}^{(p)} + (\hat{G}_{(p^{n-1})}^{(p)})^p
\]

\[
- p^{p-j} \sum_{l=0}^{j-1} \left( G_{(p^{l-1})}^{(p)} \right)^{p-1} \left( G_{(p^{n-2})}^{(p)} \right)^p + p^{p-j+1} \varphi_{p,j}(g)
\]

\[
= p \hat{G}_{(p^{n-1})}^{(p)}
\]

\[
+ (G_{(p^{n-1})}^{(p)})^p - p^{p-j-1} (G_{(p^{n-1})}^{(p)})^{p-1} G_{(p^{n-2})}^{(p)}
\]

\[
- p^{p-j} \sum_{l=0}^{j-1} \left( G_{(p^{l-1})}^{(p)} \right)^{p-1} \left( G_{(p^{n-2})}^{(p)} \right)^p + p^{p-j+1} \varphi_{p,j}(g)
\]

\[
\equiv p \hat{G}_{(p^{n-1})}^{(p)} + (G_{(p^{n-1})}^{(p)})^p
\]

\[
- p^{p-j} \sum_{l=0}^{j} \left( G_{(p^{l-1})}^{(p)} \right)^{p-1} \left( G_{(p^{n-2})}^{(p)} \right)^p
\]

modulo \( p^{p-j+1} \Lambda \). Set

\[
G_{(p^{n-1})}^{(p)} = \frac{1}{p} \left[ G_{(p^{n-1})}^{(p)} - (G_{(p^{n-1})}^{(p)})^p \right]
\]

\[
= \hat{G}_{(p^{n-1})}^{(p)} + p^{p-j} \varphi_{p,j+1}(g)
\]

\[
- p^{p-j-1} \left[ \sum_{l=0}^{j} \left( G_{(p^{l-1})}^{(p)} \right)^{p-1} \left( G_{(p^{n-2})}^{(p)} \right)^p \right].
\]

When \( j = p - 2 \), we have

\[
G_{(p^{p-1})}^{(p)} = \hat{G}_{(p^{p-1})}^{(p)} - p \left[ \sum_{l=0}^{p-2} \left( G_{(p^{l-1})}^{(p)} \right)^{p-1} \left( G_{(p^{n-2})}^{(p)} \right)^p \right]
\]

\[
+ p^2 \varphi_{p,p-1}(g).
\]

As before,

\[
\hat{G}_{(p^{p-1})}^{(p)} = p \hat{G}_{(p^{p-1})}^{(p)} + (\hat{G}_{(p^{p-1})}^{(p)})^p,
\]

but this time

\[
G_{(p^{p-1})}^{(p)} = \hat{G}_{(p^{p-1})}^{(p)} + (G_{(p^{p-1})}^{(p)})^p + p \varphi_{p-1,p-1}(g).
\]
Therefore,

\[
G_{((pn)^{p^p-1})}^{(p)} = p\widehat{G}_{(np^p)} + (\widehat{G}_{np^{p-1}})^p
- \sum_{l=0}^{p-2} p^{\left(\frac{G^{(p)}}{(p^p-1-1)n^{p^p-1}}\right)}^{p-1} \left(\frac{G^{(p)}}{(p^p-2n)}\right)^p + p^2 \varphi_{p,p-1}(g)
\]

\[
= p\widehat{G}_{(np^p)} + (\widehat{G}_{np^{p-1}})^p - (G^{(p)}_{(p^p-1-n)} - G^{(p)}_{(p^p-2n)})^p - p\varphi_{p-1,p-1}(g)
- \sum_{l=0}^{p-2} p^{\left(\frac{G^{(p)}}{(p^p-1-1)n^{p^p-1}}\right)}^{p-1} \left(\frac{G^{(p)}}{(p^p-2n)}\right)^p + p^2 \varphi_{p,p-1}(g)
\]

\[
= p\widehat{G}_{(np^p)} + (\widehat{G}_{np^{p-1}})^p - (G^{(p)}_{(p^p-1-n)} - G^{(p)}_{(p^p-2n)})^p - p\varphi_{p-1,p-1}(g)
- \sum_{l=0}^{p-2} p^{\left(\frac{G^{(p)}}{(p^p-1-1)n^{p^p-1}}\right)}^{p-1} \left(\frac{G^{(p)}}{(p^p-2n)}\right)^p + p^2 \varphi_{p,p-1}(g)
\]

modulo \(p^2\Lambda\). Set

\[
G^{(p)}_{(np^p)} = \frac{1}{p} \left[ G^{(p)}_{(np^p)^{p^p-1}} - (G^{(p)}_{np^{p-1}} - G^{(p)}_{np^{p-2}})^p \right]
\]

\[
= \widehat{G}_{np^p} - \left[ \sum_{l=0}^{p-2} p^{\left(\frac{G^{(p)}}{(p^p-1-1)n^{p^p-1}}\right)}^{p-1} \left(\frac{G^{(p)}}{(p^p-2n)}\right)^p \right] + p\varphi_{p,p}(g).
\]

We are now ready to complete the result. To this end, assume that \(k > 0\) and that

(11)

\[
G^{(p)}_{((p^p)^{p^p-1})^{p^k}} = \widehat{G}_{(p^p)^{p^k-1}} + p^p \varphi_{p^k,p-1}(g)
\]

\[
+ (-1)^k p^{p-1} \prod_{m=0}^{k-1} \left(\frac{G^{(p)}}{(p^p-1-n)^{p^m}}\right)^{p-1} \left(\frac{G^{(p)}}{(p^p-2n)}\right)^p.
\]

(12) For \(j \geq 1\),

\[
G^{(p)}_{((p^p-1-n)^{p^j})^{p^k}} = \widehat{G}_{((p^p-1-n)^{p^j})^{p^k}} + p^{p-1} \varphi_{p^{k-1},p^j}(g)
\]

\[
+ (-1)^k p^{p-1} \sum_{0 \leq i_0 \leq \cdots \leq i_{k-1} \leq j} \prod_{m=0}^{k-1} \left(\frac{G^{(p)}}{(p^p-i_{m-1}-1)^{p^m+m^m}}\right)^{p-1} \left(\frac{G^{(p)}}{(p^p-2n)}\right)^p.
\]
Notice that when \( k = 1 \) this agrees with the previous case.

Now, observe that

\[
G_{(p^k)}^{(p)} = g_{p^{k+1}}^{(p)}
\]

\[
= pg_{p^{k+1}}^{(p-1)} + (g_{p^{k+1}}^{(p-1, p)})^p
\]

\[
= pG_{(p^k)}^{(p)} + (G_{(p^k)})^p
\]

\[
= pG_{(p^k)}^{(p)} + \left(G_{(p^k)}^{(p)} \right) - (-1)^k p^{p-1}
\]

\[
\times \prod_{m=0}^{k-1} \left( G_{(p^k)}^{(p)} \right) p^{p-1} \left( G_{(p^k)}^{(p)} \right) + p^p \varphi_{p^k}
\]

\[
\equiv pG_{(p^k)}^{(p)} + (G_{(p^k)})^p
\]

\[
+ (-1)^{k+1} p^p \left( G_{(p^k)}^{(p)} \right) \prod_{m=0}^{k-1} \left( G_{(p^k)}^{(p)} \right) p^{p-1} \left( G_{(p^k)}^{(p)} \right)
\]

\[
\equiv pG_{(p^k)}^{(p)} + (G_{(p^k)})^p
\]

\[
+ (-1)^{k+1} p^p \prod_{m=0}^{k} \left( G_{(p^k)}^{(p)} \right) p^{p-1} \left( G_{(p^k)}^{(p)} \right)
\]

modulo \( p^{p+1} \). Set

\[
G_{(p^k)}^{(p)} = \frac{1}{p} \left[ G_{(p^k)}^{(p)} - (G_{(p^k)}^{(p)})^p \right]
\]

\[
= \widehat{G}_{(p^k)}^{(p)} + (-1)^{k+1} p^{p-1} \prod_{m=0}^{k} \left( G_{(p^k)}^{(p)} \right) p^{p-1} \left( G_{(p^k)}^{(p)} \right)
\]

\[
+ p^p \varphi_{p^k}
\]

Next assume that \( 1 \leq j < p - 1 \) and

\[
G_{(p^j)}^{(p)} = \widehat{G}_{(p^j)}^{(p)} + p^{p-j+1} \varphi_{p^j}
\]

\[
+ (-1)^{k+1} p^p \sum_{0 \leq i \leq p-j \leq k \leq j} \left( G_{(p^j)}^{(p)} \right) p^{p-1} \left( G_{(p^j)}^{(p)} \right)
\]

By induction on \( r \),

\[
\widehat{G}_{(p^j)}^{(p)} = p\widehat{G}_{(p^{j-1})}^{(p^{j+1})} + (\widehat{G}_{(p^{j-1})}^{(p^{j+1})})^p,
\]
and, by induction on $k$,
\[
G^{(p)}_{(p^p-j-1_n)n+j+k} = \hat{G}_{(p^p-j-1_n)n+j+k} + p^{p-j} \varphi_{p+k-1,j+k}(g)
\]
\[+(-1)^k(p^{p-j}) \sum_{0 \leq l_0 \leq \ldots \leq l_k \leq j} \left[ \prod_{m=0}^{k-1} \left( G^{(p)}_{(p^p-lm-1_n)n+lm+m} \right)^{p-1} \left( G^{(p)}_{(p^p-2_n)} \right)^{p} \right].
\]

Hence, modulo $p^{p-j+1} \Lambda$,
\[
G^{(p)}_{(p^p-j-1_n)n+j+k} = p \hat{G}_{(p^p-j-1_n)n+j+k+1} + (\hat{G}_{(p^p-j-1_n)n+j+k})^p + p^{p-j+1} \varphi_{p+k,j+k}(g)
\]
\[+(-1)^k(p^{p-j}) \sum_{0 \leq l_0 \leq \ldots \leq l_k \leq j} \left[ \prod_{m=0}^{k-1} \left( G^{(p)}_{(p^p-lm-1_n)n+lm+m} \right)^{p-1} \left( G^{(p)}_{(p^p-2_n)} \right)^{p} \right].
\]

Set
\[
G_{(p^p-j-1_n)n+j+k+1} = \frac{1}{p} \left[ G_{(p^p-j-1_n)n+j+k} - (G^{(p)}_{(p^p-j-1_n)n+j+k})^p \right]
\]
\[+(-1)^k(p^{p-j}) \sum_{0 \leq l_0 \leq \ldots \leq l_k \leq j} \left[ \prod_{m=0}^{k-1} \left( G^{(p)}_{(p^p-lm-1_n)n+lm+m} \right)^{p-1} \left( G^{(p)}_{(p^p-2_n)} \right)^{p} \right].
\]

We have verified the inductive assumptions above!

When $j = p - 2$, we have
\[
G^{(p)}_{(pn)p^{k-1}} = \hat{G}_{(pn)p^{k-1}} + p^2 \varphi_{p+k,p+k-1}(g)
\]
\[+(-1)^{k+1}p \sum_{0 \leq l_0 \leq \ldots \leq l_k \leq p-2} \left[ \prod_{m=0}^{k} \left( G^{(p)}_{(p^p-lm-1_n)n+lm+m} \right)^{p-1} \left( G^{(p)}_{(p^p-2_n)} \right)^{p} \right].
\]

As usual
\[
\hat{G}_{(pn)p^{k-1}} = p \hat{G}_{(np^{k-1})} + (\hat{G}_{(np^{k-1})})^p
\]

But this time
\[
G^{(p)}_{(np^{k-1})} = \hat{G}_{(np^{k-1})} + p^2 \varphi_{p+k-1,p+k-1}(g)
\]
\[+(-1)^k \sum_{0 \leq l_0 \leq \ldots \leq l_k \leq p-1} \left[ \prod_{m=0}^{k-1} \left( G^{(p)}_{(p^p-lm-1_n)n+lm+m} \right)^{p-1} \left( G^{(p)}_{(p^p-2_n)} \right)^{p} \right].
\]
Hence
\[
G_{(\lambda, (p^k + k))} = p \hat{G}_{(\lambda, (p^k + k))} + p \sum_{0 \leq k \leq p-2} \left[ \prod_{m=0}^{k} G_{(\lambda, (p^k + k + m))} \right]^{p-1} \left( G_{(p^{p-2} \lambda)} \right)^p
\]
\[
= \left( G_{(\lambda, (p^k + k))} + p \sum_{0 \leq k \leq p-2} \left[ \prod_{m=0}^{k} G_{(\lambda, (p^k + k + m))} \right]^{p-1} \left( G_{(p^{p-2} \lambda)} \right)^p \right)^p
\]
modulo $p^2 \lambda$. (Note that, in the last line of the computation above, we have used $G_{(\lambda, (p^k + k))} = G_{(\lambda, (p^k + k + m))} \equiv G_{(\lambda, (p^k + k + m + 1))} \equiv x \mod p^2$.)

Set
\[
G_{(\lambda, (p^k + k))} = \frac{1}{p} \left( G_{(\lambda, (p^k + k))} + p \sum_{0 \leq k \leq p-1} \prod_{m=0}^{k} G_{(\lambda, (p^k + k + m))} \right)^{p-1} \left( G_{(p^{p-2} \lambda)} \right)^p
\]
\[
= \hat{G}_{(\lambda, (p^k + k))} + \sum_{0 \leq k \leq p-2} \prod_{m=0}^{k} G_{(\lambda, (p^k + k + m))} \left( G_{(p^{p-2} \lambda)} \right)^p
\]
This completes the construction of $G_{(\lambda, (p^k + k))}$.

5.8. Statement of the Main Result.

Theorem 5.20. Let $r \leq p$. Then, there exists a bases $(G_{(\lambda, (p^k + k))})$ and $(N_{(\lambda, (p^k + k))})$ for $\Lambda$ such that
\[
\Pi(x, y)^{p^r} = \sum_{\lambda} D_r(\lambda) G_{(\lambda, (p^k + k))} (x) N_{(\lambda, (p^k + k))} (y).
\]
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