First Order Linear Homogeneous Fuzzy Ordinary Differential Equation Based on Lagrange Multiplier Method

Sankar Prasad Mondal*, Tapan Kumar Roy1

(1) Department of Mathematics, Bengal Engineering and Science University, Shibpur, Howrah-711103, West Bengal, India

Abstract
In this paper the First Order Linear Fuzzy Ordinary Differential Equations are described. Here coefficients and /or initial condition of said differential equation are taken as the Generalized Triangular Fuzzy Numbers (GTFNs). The solution procedure of this Fuzzy Differential Equation is developed by Lagrange Multiplier Method. An imprecise barometric pressure problem is described.

Keywords: Fuzzy Differential Equation, Generalized Triangular fuzzy number, First Order differential equation.

1 Introduction

It is seen that in recent years the topic of Fuzzy Differential Equations (FDEs) has been rapidly grown. In the year 1987, the term “fuzzy differential equation” was introduced by Kandel and Byatt [6]. To study FDE there have been many conceptions for the definition of fuzzy derivative. Chang and Zadeh [7] was someone who first introduced the concept of fuzzy derivative, later on it was followed up by Dobois and Prade [8] who used the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [9], Goetschel and Voxman [10], Seikkala [11] and Friedman et al. [12, 13]. Buckley and Feuring [14] have been comparing the different solutions through which one may obtain to the fuzzy differential equations using various derivatives of fuzzy function that have been presented in the various literature. Bede introduced a strongly generalized differentiability of fuzzy functions in [15] and studied in [16, 17]. The initial value problem for fuzzy differential equation (FIVP) has been studied by Kaleva in [18, 19] and by Seikkala in [20]. Buckley and Feuring [21,22] and Buckley et al. [23] gave a very basic formulation of a fuzzy first-order initial value problem. At first they found the crisp solution, then fuzzified it and then checked to see if it satisfies the Fuzzy System of differential Equation. Initial condition of a FDE is taken as different type of fuzzy number. It was supposed that the initial conditions are triangular by Buckley, Feuring, Hayashi [1]. Trapezoidal fuzzy number to solve the FDE
was used by C. Duraisamy, B. Usha [2]. Initial condition as LR type fuzzy number was used in B. Bede, S.G.Gal, L. Stefanini [5].

There have been many suggestions for study of fuzzy differential equations. One of the suggestions is S. Salahshour et al [37], Tofigh Allahviranloo, Soheil Salahshour [38]. It is seen in several paper like Chen and Chen [3], Mahapatra and Roy [4] that generalized fuzzy number are used for solving real life problem and still no one has used generalized fuzzy number for solving the FDE problem with fuzzy parameters.

Fuzzy differential equations play a significant role in the field of biology, engineering, physics and other sciences. For example, in population models [23], civil engineering [25], bioinformatics and computational biology [26], quantum optics and gravity [27], modeling hydraulic [28], HIV model [29], decay model [30], predator-prey model [31], population dynamics model [32], Friction model [33], Growth model [34], Bacteria culture model [35], bank account and drug concentration problem [36], application in Laplace transform [39], Integro-differential equation [40]. First order linear fuzzy differential equations are among all the Fuzzy Differential Equation which has many applications.

2 Preliminary concepts

Definition 2.1. Fuzzy Set: A fuzzy set \( \hat{A} \) in a universe of discourse \( X \) is defined as the following set of pairs \( \hat{A} = \{(x, \mu_{\hat{A}}(x)) : x \in X\} \). Here \( \mu_{\hat{A}} : X \rightarrow [0,1] \) is a mapping called the membership value of \( x \in X \) in a fuzzy set \( \hat{A} \).

Definition 2.2. Height: The height \( h(\hat{A}) \), of a fuzzy set \( \hat{A} = (x, \mu_{\hat{A}}(x)) : x \in X\) is the largest membership grade obtained by any element in that set i.e. \( h(\hat{A})=\sup \mu_{\hat{A}}(x) \).

Definition 2.3. Convex Fuzzy sets: A fuzzy set \( \hat{A} = \{(x, \mu_{\hat{A}}(x)) \} \subseteq X \) is called convex fuzzy set if all \( A_{\alpha} \) for every \( \alpha \in [0,1] \) are convex sets i.e. for every element \( x_1 \in A_{\alpha} \) and \( x_2 \in A_{\alpha} \) and \( \lambda x_1 + (1 - \lambda)x_2 \in A_{\alpha} \) \( \forall \lambda \in [0,1] \). Otherwise the fuzzy set is called non-convex fuzzy set.

Definition 2.4. \( \alpha \)-Level or \( \alpha \)-cut of a fuzzy set: The \( \alpha \)-level set (or interval of confidence at level \( \alpha \) or \( \alpha \)-cut) of the fuzzy set \( \hat{A} \) of \( X \) is a crisp set \( A_{\alpha} \) that contains all the elements of \( X \) that have membership values in \( A \) greater than or equal to \( \alpha \) i.e. \( A_{\alpha} = \{x : \mu_{\hat{A}}(x) \geq \alpha, x \in X, \alpha \in [0,1]\} \).

Definition 2.5. Fuzzy Number: A fuzzy number is an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function. A fuzzy number is a convex and normal fuzzy set.

Definition 2.6. Generalized Fuzzy number (GFN): Generalized Fuzzy number \( \hat{A} \) as
\[
\hat{A} = (a_1, a_2, a_3, a_4; \omega),
\]
where \( 0 < \omega \leq 1 \), and \( a_1, a_2, a_3, a_4 \) \( a_1 < a_2 < a_3 < a_4 \) are real numbers. The generalized fuzzy number \( \hat{A} \) is a fuzzy subset of real line \( R \), whose membership function \( \mu_{\hat{A}}(x) \) satisfies the following conditions:
1) $\mu_{\tilde{A}}(x) : R \rightarrow [0, 1]$
2) $\mu_{\tilde{A}}(x) = 0$ for $x \leq a_1$
3) $\mu_{\tilde{A}}(x)$ is strictly increasing function for $a_1 \leq x \leq a_2$
4) $\mu_{\tilde{A}}(x) = w$ for $a_2 \leq x \leq a_3$
5) $\mu_{\tilde{A}}(x)$ is strictly decreasing function for $a_3 \leq x \leq a_4$
6) $\mu_{\tilde{A}}(x) = 0$ for $a_4 \leq x$

Definition 2.7.
Generalized TFN: If $a_2 = a_3$ then $\tilde{A}$ is called a GTFN as $\tilde{A} = (a_1, a_2, a_4; \omega)$ or $(a_1, a_3, a_4; \omega)$ with membership function $\mu_{\tilde{A}}(x) = \begin{cases} 
\omega \frac{x-a_1}{a_2-a_1} & \text{if } a_1 \leq x \leq a_2 \\
\omega \frac{a_4-x}{a_4-a_2} & \text{if } a_2 \leq x \leq a_4 \\
0 & \text{otherwise}
\end{cases}$

Definition 2.8.
TFN: If $a_2 = a_3$, $\omega = 1$ then $\tilde{A}$ is called a TFN as $\tilde{A} = (a_1, a_2, a_3)$ or $\tilde{A} = (a_1, a_3, a_4)$

Definition 2.9.
Type of GTFN: If $\tilde{A} = (a_1, a_2, a_3; \omega)$ is a GTFN then it is called
1) symmetric GTFN if $a_2 - a_1 = a_3 - a_2$
2) Non symmetric type 1 \((l_s > r_s)\) GTFN if \(a_2 - a_1 > a_3 - a_2\)

3) Non symmetric type 2 \((l_s < r_s)\) GTFN if \(a_2 - a_1 < a_3 - a_2\)

4) Left GTFN if \(\tilde{A}\) is written as \(\tilde{A}_{GTFN}^L = (a_1, a_2, a_3; \omega)\)
5) Right GTFN if $\tilde{A}$ is written as $\tilde{A}_{GTFN}^R = (a_2, a_3; \omega_2)$

![Diagram](attachment:image.png)

Figure 7

**Definition 2.10.**

**Equality of two GTFN:** Two fuzzy number $\tilde{A} = (a_1, a_2, a_3; \omega_1)$ and $\tilde{B} = (b_1, b_2, b_3; \omega_2)$ are equal when $a_1 = b_1, a_2 = b_2, a_3 = b_3$ and $\omega_1 = \omega_2.$

**3 Solution of System of Differential Equation by De Alemberts Method:**

Consider the system of first order differential equation

\[
\begin{align*}
\frac{dx}{dt} &= a_1 x + b_1 y \\
\frac{dy}{dt} &= a_2 x + b_2 y
\end{align*}
\]

(1)

Multiplying the second equation by some constant $\lambda$ and add termwise to the first equation we get

\[
\frac{d(x + \lambda y)}{dt} = (a_1 + \lambda a_2) x + (b_1 + \lambda b_2) y = (a_1 + \lambda a_2) (x + \frac{b_1 + \lambda b_2}{a_1 + \lambda a_2} y)
\]

(2)

Chose the number $\lambda$ so that

\[
\frac{b_1 + \lambda b_2}{a_1 + \lambda a_2} = \lambda
\]

(3)

Then (2) reduces to an equation linear in $x + \lambda y$

\[
\frac{d(x + \lambda y)}{dt} = (a_1 + \lambda a_2) (x + \lambda y)
\]

Which on integrating gives

\[
x + \lambda y = C e^{(a_1 + \lambda a_2)}
\]

(4)

If equation (3) has distinct real roots $\lambda_1$ and $\lambda_2$, then we obtain two first integrals of system (1) from (4) and so the integration of the system will be completed.

**4 Solution Procedure of 1st Order Linear Homogeneous FODE**

The solution procedures of 1st order linear homogeneous FODE of Type-I, Type-II and Type-III are described. Here fuzzy numbers are taken as GTFNs.

**4.1. Solution Procedure of 1st Order Linear Homogeneous FODE of Type-I**

Consider the initial value problem

\[
\frac{dx}{dt} = k x
\]

(5)
with fuzzy Initial Condition (IC) \( \tilde{x}(t_0) = \tilde{y}_0 = (y_1, y_2, y_3; \omega) \).
Let \( \tilde{x}(t) \) be a solution of FODE (5) with \( \alpha \)-cut \( x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)] \)
and \( (\tilde{y}_0)_\alpha = [y_1 + \frac{a_1y_0}{\omega}, y_3 - \frac{ar_2}{\omega}] \forall \alpha \in [0, \omega], 0 < \omega \leq 1 \)
Here we solve the given problem for \( k > 0 \) and \( k < 0 \) respectively.

**Case 4.1.1.**

When \( k > 0 \)
The FODE (5) becomes \( \frac{dx_i(t, \alpha)}{dt} = kx_i(t, \alpha) \quad (i = 1, 2) \) \( \quad \) (6)
\( \therefore \) The solution is
\[ x_1(t, \alpha) = \left( y_1 + \frac{a_1y_0}{\omega} \right) e^{k(t-t_0)}, \quad x_2(t, \alpha) = \left( y_3 - \frac{ar_2}{\omega} \right) e^{k(t-t_0)} \] \( \quad \) (7)
Here \( \frac{\partial}{\partial \alpha}[x_1(t, \alpha)] = \frac{\gamma_0}{\omega} e^{k(t-t_0)} > 0, \quad \frac{\partial}{\partial \alpha}[x_2(t, \alpha)] = -\frac{r_2}{\omega} e^{k(t-t_0)} < 0 \)
and \( x_1(t, \omega) = y_2 e^{k(t-t_0)} = x_2(t, \omega) \).
So the solution is a generalized fuzzy number. The \( \alpha \)-cut of the strong solution is
\[ x(t, \alpha) = \left[ \left( y_1 + \frac{a_1y_0}{\omega} \right), \left( y_3 - \frac{ar_2}{\omega} \right) \right] e^{k(t-t_0)}. \]
So the solution of (5) is \( \tilde{x}(t) = \tilde{y}_0 e^{k(t-t_0)}. \)

**Case 4.1.2.**

When \( k < 0 \), let \( k = -m \) where \( m \) is a positive real number.
The FODE (5) becomes
\[ \frac{dx_1(t, \alpha)}{dt} = -mx_2(t, \alpha) \] \( \quad \) (8)
\[ \frac{dx_2(t, \alpha)}{dt} = -mx_1(t, \alpha) \] \( \quad \) (9)
Multiplying Equation (9) by \( \lambda \) and then adding with equation (8) we get
\[ \frac{d}{dt}[x_1(t, \alpha) + \lambda x_2(t, \alpha)] = -\lambda m[x_1(t, \alpha) + \frac{1}{\lambda} x_2(t, \alpha)] \] \( \quad \) (10)
Let \( \frac{1}{\lambda} = \lambda \) and \( x_1(t, \alpha) + \lambda x_2(t, \alpha) = z \)
Therefore (10) becomes \( \frac{dz}{dt} = -\lambda mz \) \( \quad \) (11)
Therefore the solution of is, \( z = Ce^{-\lambda mt} \) and \( \lambda = \pm 1 \)
Therefore,
\[ x_1(t, \alpha) + x_2(t, \alpha) = C_1 e^{-mt} \] \( \quad \) (12)
and
\[ x_1(t, \alpha) - x_2(t, \alpha) = C_2 e^{mt} \] \( \quad \) (13)
Using initial condition from we get (12) and (13) we get
\[ C_1 = \left( y_1 + y_3 + \frac{a_1y_0 - r_2}{\omega} \right) e^{mt_0} \quad \text{and} \quad C_2 = \left( y_1 - y_3 + \frac{a_1y_0 + r_2}{\omega} \right) e^{-mt_0} \]
Solving (12) and (13) we get
\[ x_1(t, \alpha) = \frac{1}{2} (C_1 e^{-\alpha t} + C_2 e^{\alpha t}) \]
\[ = \frac{1}{2} \left( \gamma_1 + \gamma_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} \]
\[ x_2(t, \alpha) = \frac{1}{2} (C_1 e^{-\alpha t} - C_2 e^{\alpha t}) \]
\[ = \frac{1}{2} \left( \gamma_1 + \gamma_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{\gamma_0} + r_{\gamma_0}) e^{m(t-t_0)} \]

Now
\[ \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] = \frac{1}{2 \omega} \left( l_{\gamma_0} - r_{\gamma_0} \right) e^{-m(t-t_0)} + \frac{1}{2 \omega} \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \]
\[ \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] = \frac{1}{2 \omega} \left( l_{\gamma_0} - r_{\gamma_0} \right) e^{-m(t-t_0)} - \frac{1}{2 \omega} \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \]

and \( x_1(t, \omega) = y_2 e^{-m(t-t_0)} = x_2(t, \omega) \)

Here three cases arise.

Case 4.1.2.1.

When \( l_{\gamma_0} = r_{\gamma_0} \)

Then \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0, \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \) and \( x_1(t, \omega) = x_2(t, \omega) \)

Hence
\[ \left[ \frac{1}{2} \left( \gamma_1 + \gamma_3 \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right. \]
\[ \left. \frac{1}{2} \left( \gamma_1 + \gamma_3 \right) e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right] \]

is the \( \alpha \)-cut of the strong solution of the FODE (5).

So, \( \bar{x}(t) = \frac{y_2 + \tilde{y}_2}{2} e^{-m(t-t_0)} + \bar{0} \left( y_2 - \gamma_1 \right) e^{m(t-t_0)} \) where \( \bar{0} = (-1,0,1) \) be a symmetric TFN is the solution of (5)

Case 4.1.2.2.

When \( l_{\gamma_0} < r_{\gamma_0} \) then \( \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \).

So in this case we get the strong solution if \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0 \) i.e. \( t > t_0 + \frac{1}{2m} \log \left[ \frac{r_{\gamma_0} - \gamma_0}{l_{\gamma_0} + r_{\gamma_0}} \right] \)

Hence
\[ \left[ \frac{1}{2} \left( \gamma_1 + \gamma_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right. \]
\[ \left. \frac{1}{2} \left( \gamma_1 + \gamma_3 + \frac{\alpha}{\omega} (l_{\gamma_0} - r_{\gamma_0}) \right) e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) \left( l_{\gamma_0} + r_{\gamma_0} \right) e^{m(t-t_0)} \right] \]
is the $\alpha$-cut of the strong solution of the FODE (5) if $t > t_0 + \frac{1}{2m} \log \frac{r_{y_0} - r_{y_0}}{l_{y_0} + r_{y_0}}$.

Case 4.1.2.3.

When $l_{y_0} > r_{y_0}$ then $\frac{\partial}{\partial \alpha}[x_1(t, \alpha)] > 0$

So in this case we get the strong solution if $\frac{\partial}{\partial \alpha}[x_2(t, \alpha)] < 0$ i.e. $t > t_0 + \frac{1}{2m} \log \frac{r_{y_0} - r_{y_0}}{l_{y_0} + r_{y_0}}$

Hence

$$\left\{ \frac{1}{2} \left[ y_1 + y_3 + \frac{\alpha}{\omega} (l_{y_0} - r_{y_0}) \right] e^{-m(t-t_0)} + \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{y_0} + r_{y_0}) e^{m(t-t_0)}, \right.$$ 

$$\left. \frac{1}{2} \left[ y_1 + y_3 + \frac{\alpha}{\omega} (l_{y_0} - r_{y_0}) \right] e^{-m(t-t_0)} - \frac{1}{2} \left( \frac{\alpha}{\omega} - 1 \right) (l_{y_0} + r_{y_0}) e^{m(t-t_0)} \right\}$$

is the $\alpha$-cut of the strong solution of the FODE (5) if $t > t_0 + \frac{1}{2m} \log \frac{r_{y_0} - r_{y_0}}{l_{y_0} + r_{y_0}}$.

For case Case 4.1.2.2, Case 4.1.2.3 the solution is

$$\tilde{x}(t) = \frac{1}{2} \tilde{f} e^{-m(t-t_0)} + \frac{y_3 - y_1}{2} \tilde{0} e^{m(t-t_0)} \quad \text{where} \quad \tilde{f} = (y_1 + y_3, 2y_2, y_1 + y_3), \quad \tilde{0} = (-1, 0, 1) \quad \text{are two symmetric TFN.}$$

4.2. Solution Procedure of 1st Order Linear Homogeneous FODE of Type-II

Consider the initial value problem

$$\frac{dx}{dt} = \tilde{k} x$$

with IC $x(t_0) = \gamma$

where $\tilde{k} = (\beta_1, \beta_2, \beta_3; \lambda)$

Let $\tilde{x}(t)$ be the solution of FODE (14)

Let $x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)]$ be the $\alpha$-cut of the solution and the $\alpha$-cut of $\tilde{k}$ be

$$\tilde{k}_\alpha = \left[ \beta_1 + \frac{\alpha \tilde{k}}{\lambda}, \beta_3 - \frac{\alpha \tilde{k}}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1$$

Here we solve the given problem for $\tilde{k} > 0$ and $\tilde{k} < 0$ respectively.

Case 4.2.1.

when $\tilde{k} > 0$

The FODE (14) becomes

$$\frac{dx_i(t, \alpha)}{dt} = k_i(\alpha)x_i(t, \alpha) \quad \text{for} \quad i = 1, 2$$

$\therefore$ From (14) we get the solution as

$$x_1(t, \alpha) = \gamma e^{\left( \beta_1 + \frac{\alpha \tilde{k}}{\lambda} \right)(t-t_0)}$$

and $x_2(t, \alpha) = \gamma e^{\left( \beta_3 - \frac{\alpha \tilde{k}}{\lambda} \right)(t-t_0)}$

Here $\frac{\partial}{\partial \alpha}[x_1(t, \alpha)] = \frac{y_1 \tilde{k}}{\lambda} (t-t_0) e^{\left( \beta_1 + \frac{\alpha \tilde{k}}{\lambda} \right)(t-t_0)} > 0$

and $\frac{\partial}{\partial \alpha}[x_2(t, \alpha)] = -\frac{y_3 \tilde{k}}{\lambda} (t-t_0) e^{\left( \beta_3 - \frac{\alpha \tilde{k}}{\lambda} \right)(t-t_0)} < 0$

and $x_1(t, \beta) = \gamma e^{\beta_2(t-t_0)} = x_2(t, \beta)$
Hence the $\alpha$-cut of the strong solution of FODE (14) is

$$x(t, \alpha) = \gamma \left[ e^{\left( \beta_1 \frac{\alpha}{\lambda} \right) (t-t_0)} e^{\left( \beta_3 - \frac{\alpha}{\lambda} \right) (t-t_0)} \right]$$

Case 4.2.2.

When $\bar{\kappa} < 0$, let $\bar{\kappa} = -\bar{m}$, where $\bar{m} = (\beta_2, \beta_3, \beta_3; \lambda)$ is a positive GTFN.

So $(\bar{m})_\alpha = [m_1(\alpha), m_2(\alpha)] = \left[ \beta_1 + \frac{a m}{\lambda}, \beta_3 - \frac{a m}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1$

The FODE (14) becomes

$$\frac{dx_1(t,\alpha)}{dt} = -m_2(\alpha)x_2(t, \alpha) \quad (16)$$

and

$$\frac{dx_2(t,\alpha)}{dt} = -m_1(\alpha)x_1(t, \alpha) \quad (17)$$

Multiplying equation (17) by $\lambda$ and adding with equation (16) we get

$$\frac{d}{dt} \left[ x_1(t, \alpha) + \lambda x_2(t, \alpha) \right] = -\lambda m_1 \left[ x_1(t, \alpha) + \frac{m_2(\alpha)}{\lambda m_1(\alpha)} x_2(t, \alpha) \right] \quad (18)$$

Now chose $\lambda = \frac{m_2(\alpha)}{\lambda m_1(\alpha)}$ and let $x_1(t, \alpha) + \lambda x_2(t, \alpha) = z$ then from (18) we get

$$\frac{dz}{dt} = -\lambda m_1 z \quad (19)$$

Therefore the solution is $z = C e^{-\lambda m_1 t}$, and $\lambda = \pm \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}}$

i.e.,

$$x_1(t, \alpha) + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} x_2(t, \alpha) = C_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}} t \quad (20)$$

and

$$x_1(t, \alpha) - \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} x_2(t, \alpha) = C_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}} t \quad (21)$$

Solving (20) and (21) we get

$$x_1(t, \alpha) = \frac{1}{2} \left[ C_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}} t + C_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}} t \right] \quad (22)$$

and

$$x_2(t, \alpha) = \frac{1}{2} \sqrt{\frac{m_1(\alpha)}{m_2(\alpha)}} \left[ C_1 e^{-\sqrt{m_1(\alpha)m_2(\alpha)}} t + C_2 e^{\sqrt{m_1(\alpha)m_2(\alpha)}} t \right] \quad (23)$$

Using initial condition from (20) and (21) we get

$$C_1 = \gamma \left( 1 + \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} \right) e^{\sqrt{m_1(\alpha)m_2(\alpha)}} t_0 \quad \text{and} \quad C_2 = \gamma \left( 1 - \sqrt{\frac{m_2(\alpha)}{m_1(\alpha)}} \right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}} t_0$$

Using this value from (22) and (23) we get

$$x_1(t, \alpha) = \frac{\gamma}{2} \left( \left( 1 - \frac{m_2(\alpha)}{m_1(\alpha)} \right) e^{\sqrt{m_1(\alpha)m_2(\alpha)}} (t-t_0) + \left( 1 + \frac{m_2(\alpha)}{m_1(\alpha)} \right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}} (t-t_0) \right)$$

$$x_2(t, \alpha) = \frac{\gamma}{2} \left( \left( 1 - \frac{m_2(\alpha)}{m_1(\alpha)} \right) e^{\sqrt{m_1(\alpha)m_2(\alpha)}} (t-t_0) + \left( 1 + \frac{m_2(\alpha)}{m_1(\alpha)} \right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}} (t-t_0) \right)$$

(24)
and
\[
x_2(t, \alpha) = \frac{1}{\sqrt{2}} \left\{ \frac{m_1(\alpha)}{m_2(\alpha)} - \left( 1 - \frac{m_2(\alpha)}{m_1(\alpha)} \right) e^{\sqrt{m_1(\alpha)m_2(\alpha)}(t-t_0)} + \left( 1 + \frac{m_2(\alpha)}{m_1(\alpha)} \right) e^{-\sqrt{m_1(\alpha)m_2(\alpha)}(t-t_0)} \right\} \\
= \frac{1}{\sqrt{2}} \left\{ -\sqrt{\frac{\beta_1 + \frac{\alpha k}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}} \left( 1 - \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha k}{\lambda}} \right) e^{\sqrt{\frac{\beta_1 + \frac{\alpha k}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}(t-t_0)}} + \left( 1 + \frac{\beta_3 - \frac{\alpha r_m}{\lambda}}{\beta_1 + \frac{\alpha k}{\lambda}} \right) e^{-\sqrt{\frac{\beta_1 + \frac{\alpha k}{\lambda}}{\beta_3 - \frac{\alpha r_m}{\lambda}}(t-t_0)}} \right\} \quad (25)
\]

Now if \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0, \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \) and \( x_1(t, \lambda) \leq x_2(t, \lambda) \) then we get the strong solution.

### 4.3. Solution Procedure of 1st Order Linear Homogeneous FODE of Type-III

Consider the initial value problem
\[
\frac{dx}{dt} = \bar{k} x
\]
with fuzzy IC \( \bar{x}(t_0) = \bar{y}_0 = (\gamma_1, \gamma_2, \gamma_3; \omega) \), where \( \bar{k} = (\beta_1, \beta_2, \beta_3; \lambda) \)

Let \( \bar{x}(t) \) be the solution of FODE (26).

Let \( x(t, \alpha) = [x_1(t, \alpha), x_2(t, \alpha)] \) be the \( \alpha \)-cut of the solution.

Also \( \bar{x}(t_0) = [\gamma_1 + \frac{\alpha k}{\eta}, \gamma_2 - \frac{\alpha r_k}{\eta}] \) \( \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1 \)

and \( \gamma_0 \alpha = [\gamma_1 + \frac{\alpha y_0}{\eta}, \gamma_2 - \frac{\alpha r_y}{\eta}] \) \( \forall \alpha \in [0, \omega], 0 < \omega \leq 1 \)

Let \( \eta = \min(\lambda, \omega) \)

Here we solve the given problem for \( \bar{k} > 0 \) and \( \bar{k} < 0 \) respectively.

#### Case 4.3.1.

when \( \bar{k} > 0 \)

The FODE (26) becomes
\[
\frac{dx(t, \alpha)}{dt} = \left[ \beta_1 + \frac{\alpha k}{\eta}, \beta_3 - \frac{\alpha r_k}{\eta} \right] x_i(t, \alpha) \quad \text{for} \quad i = 1, 2
\]

∴ The solution is
\[
x_1(t, \alpha) = \left( \gamma_1 + \frac{\alpha y_0}{\eta} \right) e^{\left( \beta_1 + \frac{\alpha k}{\eta} \right)(t-t_0)}
\]

and
\[
x_2(t, \alpha) = \left( \gamma_2 - \frac{\alpha r_y}{\eta} \right) e^{\left( \beta_3 - \frac{\alpha r_k}{\eta} \right)(t-t_0)}
\]

Now if \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0, \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \) and \( x_1(t, \eta) \leq x_2(t, \eta) \) then we get the strong solution.

#### Case 4.3.2.

when \( \bar{k} < 0 \) then \( \bar{k} = -\bar{m} \) where \( \bar{m} = (\beta_1, \beta_2, \beta_3; \lambda) \) is a positive GTFN.

Then \( \bar{m}(\alpha) = \left[ \beta_1 + \frac{\alpha l_m}{\lambda}, \beta_3 - \frac{\alpha r_m}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1 \)

Let \( \eta = \min(\lambda, \omega) \)

when \( \bar{k} < 0 \) then \( \bar{k} = -\bar{m} \) where \( \bar{m} = (\beta_1, \beta_2, \beta_3; \lambda) \) is a positive GTFN.

Then \( \bar{m}(\alpha) = \left[ \beta_1 + \frac{\alpha l_m}{\lambda}, \beta_3 - \frac{\alpha r_m}{\lambda} \right] \forall \alpha \in [0, \lambda], 0 < \lambda \leq 1 \)

Let \( \eta = \min(\lambda, \omega) \)
The FODE (26) becomes
\[
\frac{dx_1(t, \alpha)}{dt} = -\left(\beta_3 - \frac{ar_m}{\eta}\right)x_2(t, \alpha)
\]  
(28)

and
\[
\frac{dx_2(t, \alpha)}{dt} = -\left(\beta_1 + \frac{al_m}{\eta}\right)x_1(t, \alpha)
\]  
(29)

Multiplying equation (29) by \(\lambda\) and adding with equation (28) we get
\[
\frac{d}{dt}[x_1(t, \alpha) + \lambda x_2(t, \alpha)] = -\lambda\left(\beta_1 + \frac{al_m}{\eta}\right)[x_1(t, \alpha) + \frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}x_2(t, \alpha)]
\]  
(30)

Taking \(\lambda = \frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}\) and let \(z = x_1(t, \alpha) + \lambda x_2(t, \alpha)\)

We get \(\lambda = \pm \frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}\) and \(\frac{dz}{dt} = -\lambda\left(\beta_1 + \frac{al_m}{\eta}\right)z\)

Therefore the solution gives \(z = Ce^{-\lambda\left(\beta_1 + \frac{al_m}{\eta}\right)t}\) i.e.,
\[
x_1(t, \alpha) + \frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}x_2(t, \alpha) = C_1e^{-\lambda\left(\beta_1 + \frac{al_m}{\eta}\right)\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}t}
\]  
(31)

and
\[
x_1(t, \alpha) - \frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}x_2(t, \alpha) = C_2e^{-\lambda\left(\beta_1 + \frac{al_m}{\eta}\right)\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}t}
\]  
(32)

Using initial condition from (31) and (32) we get
\[
C_1 = \left\{y_1 + \frac{al_{y_0}}{\omega} + \sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}}y_3 - \frac{ar_{y_0}}{\omega}\right\} e^{-\sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}t_0}
\]

and
\[
C_2 = \left\{y_1 + \frac{al_{y_0}}{\omega} - \sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}}y_3 - \frac{ar_{y_0}}{\omega}\right\} e^{-\sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}t_0}
\]

Now from equation (31) and (32) we get
\[
x_1(t, \alpha) = \frac{1}{2}\left\{y_1 + \frac{al_{y_0}}{\omega} + \sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}}y_3 - \frac{ar_{y_0}}{\omega}\right\} e^{-\sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}t} - \frac{1}{2}\left\{y_1 + \frac{al_{y_0}}{\omega} - \sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}}y_3 - \frac{ar_{y_0}}{\omega}\right\} e^{-\sqrt{\frac{\lambda(\beta_1 + \frac{al_m}{\eta})}{\lambda(\beta_1 + \frac{al_m}{\eta})}(t-t_0)}
\]  
(33)
and

\[ x_2(t, \alpha) = \frac{1}{2} \left( \frac{\beta_3 - \alpha \eta}{\eta} \right)^2 \left( y_1 + \alpha t y_0 \right) + \left( \frac{\beta_3 - \alpha \eta}{\eta} \right) \left( y_3 - \frac{\alpha \eta}{\omega} \right) e^{-\left( \frac{\beta_1 + \alpha \eta}{\eta} \right) \left( \beta_3 - \frac{\alpha \eta}{\eta} \right) (t-t_0)} - \left( y_1 + \frac{\alpha t y_0}{\omega} - \left( \frac{\beta_3 - \alpha \eta}{\eta} \right) \left( y_3 - \frac{\alpha \eta}{\omega} \right) e^{-\left( \frac{\beta_1 + \alpha \eta}{\eta} \right) \left( \beta_3 - \frac{\alpha \eta}{\eta} \right) (t-t_0)} \right) \]  

(34)

Here also if \( \frac{\partial}{\partial \alpha} [x_1(t, \alpha)] > 0, \frac{\partial}{\partial \alpha} [x_2(t, \alpha)] < 0 \) and \( x_1(t, \eta) \leq x_2(t, \eta) \) then we get the strong solution.

5 Applications

The barometric pressure \( y \) (in inches of mercury) at an altitude of \( x \) miles above the sea level decreases at a rate proportional to the current pressure according to the model \( \frac{dy}{dx} = -0.2y \). where \( y = 30 \) inches when \( x = 0 \). Find the barometric pressure (a) at the top of Mt.St.Helens (8364 feet) and (b) at the top of Mt.McKinley (20,320 feet).  

The fuzzy environment the problem is

(i) \( \frac{dy}{dx} = -ky \) when \( k = 0.2 \) and \( y(0) = (28,30,33; 0.8) \)

(ii) \( \frac{dy}{dx} = -ky \) when \( k = (0.17,0.2,0.22; 0.7) \) and \( y(0) = 30 \)

(iii) \( \frac{dy}{dx} = -ky \) when \( k = (0.19,0.2,0.22; 0.8) \) and \( y(0) = (28,30,31; 0.7) \)

Solution:

(i) The solution is

\[ y_1(t, \alpha) = \frac{1}{2} (61 - 1.25\alpha)e^{-0.2t} + 2.5(1.25\alpha - 1)e^{0.2t} \]

\[ y_2(t, \alpha) = \frac{1}{2} (61 - 1.25\alpha)e^{-0.2t} - 2.5(1.25\alpha - 1)e^{0.2t} \]

Table 1

| \( \alpha \) | \( t = 8364 \text{ feet} = 1.5840 \text{ miles} \) | \( t = 20,320 \text{ feet} = 3.8484 \text{ miles} \) |
|---|---|---|
| | \( y_1(t, \alpha) \) | \( y_2(t, \alpha) \) | \( y_1(t, \alpha) \) | \( y_2(t, \alpha) \) |
| 0 | 18.7867 | 25.6504 | 8.7287 | 19.5241 |
| 0.1 | 19.1702 | 25.1758 | 9.3745 | 18.8204 |
| 0.2 | 19.5536 | 24.7013 | 10.0203 | 18.1168 |
| 0.3 | 19.9371 | 24.2268 | 10.6660 | 17.4131 |
| 0.4 | 20.3205 | 23.7523 | 11.3118 | 16.7095 |
| 0.5 | 20.7039 | 23.2778 | 11.9575 | 16.0058 |
| 0.6 | 21.0874 | 22.8033 | 12.6033 | 15.3022 |
| 0.7 | 21.4708 | 22.3288 | 13.2491 | 14.5985 |
| 0.8 | 21.8543 | 21.8543 | 13.8948 | 13.8948 |

From the above Table 1 we see that for this particular value of \( t \), \( y_1(t, \alpha) \) is an increasing function, \( y_2(t, \alpha) \) is a decreasing function and \( y_1(t, 0.8) = y_2(t, 0.8) \) for \( t = 1.5840 \) miles and \( t = 3.8484 \) miles. Hence this is strong solution.
(ii) The solution is given by

\[ y_1(t, \alpha) = 15 \left[ \left( 1 - \sqrt{\frac{0.22 - 0.0285\alpha}{0.17 + 0.0428\alpha}} \right) e^{\sqrt{0.17 + 0.0428\alpha}(0.22 - 0.0285\alpha)t} + \left( 1 + \sqrt{\frac{0.22 - 0.0285\alpha}{0.17 + 0.0428\alpha}} \right) e^{-\sqrt{0.17 + 0.0428\alpha}(0.22 - 0.0285\alpha)t} \right] \]

\[ y_2(t, \alpha) = 15 \left[ \frac{0.17 + 0.0428\alpha}{0.22 - 0.0285\alpha} \left[ - \left( 1 - \sqrt{\frac{0.22 - 0.0285\alpha}{0.17 + 0.0428\alpha}} \right) e^{\sqrt{0.17 + 0.0428\alpha}(0.22 - 0.0285\alpha)t} + \left( 1 + \sqrt{\frac{0.22 - 0.0285\alpha}{0.17 + 0.0428\alpha}} \right) e^{-\sqrt{0.17 + 0.0428\alpha}(0.22 - 0.0285\alpha)t} \right] \]

| \alpha | \text{t = 8364 feet = 1.5840 miles} | \text{t = 20,320 feet = 3.8484 miles} |
|--------|--------------------------------|--------------------------------|
|        | \( y_1(t, \alpha) \) | \( y_2(t, \alpha) \) | \( y_1(t, \alpha) \) | \( y_2(t, \alpha) \) |
| 0      | 20.7999 | 23.2133 | 10.8891 | 17.2096 |
| 0.1    | 20.9526 | 23.2222 | 11.3286 | 16.7535 |
| 0.2    | 21.1044 | 22.8301 | 11.7643 | 16.2917 |
| 0.3    | 21.2555 | 22.6371 | 12.1969 | 15.8243 |
| 0.4    | 21.4058 | 22.4433 | 12.6255 | 15.3514 |
| 0.5    | 21.5553 | 22.2485 | 13.0505 | 14.8730 |
| 0.6    | 21.7040 | 22.0528 | 13.4716 | 14.3893 |
| 0.7    | 21.8519 | 21.8563 | 13.8888 | 13.9003 |

From the above Table 2 we see that for this particular value of \( t \), \( y_1(t, \alpha) \) is an increasing function, \( y_2(t, \alpha) \) is a decreasing function and \( y_1(t, 0.7) < y_2(t, 0.7) \) for \( t = 1.5840 \) miles and \( t = 3.8484 \) miles. Hence this is strong solution.

(iii) The solution is given by

\[ y_1(t, \alpha) = \frac{1}{2} \left[ \left( 28 + 2.85\alpha \right) - \left( 31 - 1.42\alpha \right) \sqrt{\frac{0.22 - 0.0285\alpha}{0.19 + 0.0142\alpha}} \right] e^{\sqrt{0.19 + 0.0142\alpha}(0.22 - 0.0285\alpha)t} + \left( 28 + 2.85\alpha \right) + \left( 31 - 1.42\alpha \right) \sqrt{\frac{0.22 - 0.0285\alpha}{0.19 + 0.0142\alpha}} e^{-\sqrt{0.19 + 0.0142\alpha}(0.22 - 0.0285\alpha)t} \]

\[ y_2(t, \alpha) = \frac{1}{2} \sqrt{\frac{0.19 + 0.0142\alpha}{0.22 - 0.0285\alpha}} \left[ - \left( 28 + 2.85\alpha \right) - \left( 31 - 1.42\alpha \right) \sqrt{\frac{0.22 - 0.0285\alpha}{0.19 + 0.0142\alpha}} \right] e^{\sqrt{0.19 + 0.0142\alpha}(0.22 - 0.0285\alpha)t} + \left( 28 + 2.85\alpha \right) + \left( 31 - 1.42\alpha \right) \sqrt{\frac{0.22 - 0.0285\alpha}{0.19 + 0.0142\alpha}} e^{-\sqrt{0.19 + 0.0142\alpha}(0.22 - 0.0285\alpha)t} \]
| $\alpha$ | $t = 8364$ feet = 1.5840 miles | $t = 20,320$ feet = 3.8484 miles |
|---------|-----------------------------|-----------------------------|
| 0       | $y_1(t, \alpha)$ | $y_2(t, \alpha)$ | $y_1(t, \alpha)$ | $y_2(t, \alpha)$ |
| 0.1     | 18.6693 | 25.1222 | 8.1980 | 19.2353 |
| 0.2     | 19.2080 | 24.7629 | 9.1278 | 18.5881 |
| 0.3     | 19.7441 | 24.4053 | 10.0504 | 17.9429 |
| 0.4     | 20.2777 | 24.0495 | 10.9657 | 17.2995 |
| 0.5     | 20.8086 | 23.6954 | 11.8738 | 16.6580 |
| 0.6     | 21.3369 | 23.3431 | 12.7744 | 16.0185 |
| 0.7     | 21.8627 | 22.9925 | 13.6677 | 15.3810 |
|         | 22.3858 | 22.6435 | 14.5534 | 14.7456 |

From the above Table 3 we see that for this particular value of $t$, $y_1(t, \alpha)$ is an increasing function, $y_2(t, \alpha)$ is a decreasing function and $y_1(t, 0.7) < y_2(t, 0.7)$ for $t = 1.5840$ miles and $t = 3.8484$ miles. Hence this is strong solution.

6 Conclusion and future work

In this paper we have solved first order linear homogeneous fuzzy ordinary differential equation. The fuzzy number is taken as GTFN. We have also discussed three cases: (i) Initial value as fuzzy number, (ii) initial value and coefficients as fuzzy number, (iii) coefficients are fuzzy number. The solution procedure is developed by Lagrange Multiplier Method. Imprecise barometric pressure problem is considered. Same problem can be solved by using Generalized Trapezoidal Fuzzy Number and Generalized L-R type Fuzzy Number. One can follow the same method for first or higher order linear non-homogeneous and homogeneous fuzzy ordinary differential equation. This process can be followed for any economical or bio-mathematical model and in engineering sciences.

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