Abstract. We describe a class of multivariate series rings generalizing the usual Robba ring over a $p$-adic field, and develop a partial slope theory for Frobenius modules over such rings. This involves passage to certain perfect closures, invocation of results from papers of the first author and R. Liu, and a descent argument. We also show how this construction gives rise to locally analytic vectors inside certain analogues of $(\phi, \Gamma)$-modules and to a conjectural generalization of the Cherbonnier-Colmez theorem on the overconvergence of $p$-adic Galois representations.

In $p$-adic Hodge theory, a prominent role is played by certain $p$-adic power series rings known as Robba rings. To be precise, the Robba ring with coefficients in a field $K$ (typically a finite extension of $\mathbb{Q}_p$) is the ring of formal sums $\sum_{n \in \mathbb{Z}} c_n \pi^n$ with coefficients in $K$ which converge on some annulus of the form $* < |\pi| < 1$. In practice, such rings typically come equipped with extra structure, especially a Frobenius endomorphism which on the subring of series with integral coefficients is a lift of some power of the Frobenius endomorphism modulo $p$.

A key structural result about Frobenius endomorphisms over Robba rings is the classification of semilinear Frobenius actions on finite projective modules over Robba rings in terms of certain numerical invariants called slopes; this is the focus of several prior papers by the author [12, 13, 14]. There is a strong analogy with the classification of vector bundles on curves, as most vividly demonstrated by the reinterpretation of slope theory in the work of Fargues and Fontaine [9] (see also [16 §6]). In particular, the case when all slopes are equal to zero (i.e., that of an étale Frobenius action) is closely related to Galois representations in much the same way that polystable vector bundles of degree 0 on a Riemann surface are related to unitary fundamental group representations via the theorem of Narasimhan–Seshadri [20].

In recent work of the author with Liu [16], some constructions have begun to appear which constitute relative versions of the theory of Frobenius actions over Robba ring. One important difference is that these constructions generalize not Robba rings themselves, on which the Frobenius action is not surjective, but their so-called perfect closures for the action of Frobenius. Another important difference is that the analogues of slopes are no longer single numbers, but rather functions on certain topological spaces which may be viewed as perfectoid spaces in the sense of Scholze [21].

In this paper, we expand upon these constructions by introducing a class of multivariate series rings, which we will refer to as multivariate Robba rings, which themselves have perfect closures fitting into the framework of [16]. One pressing reason to work with such a ring instead of its perfect closure is that one often considers actions of $p$-adic Lie groups on

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such rings, as in Fontaine’s theory of \((\varphi, \Gamma)\)-modules, and on the series ring this action is typically \textit{locally analytic}. This suggests the possibility of using this construction to develop a version of the \(p\)-adic Langlands correspondence generalizing the case for the group \(GL_2(\mathbb{Q}_p)\) exhibited by Colmez \([8]\); this may end up making contact with the group-theoretic approach of Schneider–Vignéras–Zábrádi \([23, 25]\). Some progress in this direction in the case \(F\) a finite unramified extension of \(\mathbb{Q}_p\), can be seen in the work of Berger \([3]\).

We conclude this introduction by describing the contents of this paper in more detail. We begin with the ring-theoretic setup of Robba rings and extended Robba rings corresponding to smooth affinoid algebras (and regular mixed affinoid algebras). We next give a descent result (Theorem 3.9) between multivariate Robba rings and their perfect closures, analogous to the descent step of the proofs of the slope filtration theorem in \([12, 13, 14]\); the key point is to split the embedding of a multivariate Robba ring into its perfect closure. We next show (Theorem 4.3) that the \(\acute{e}tale\) condition is in a certain sense local at the level of Berkovich analytic spaces; this is an easy consequence of results of \([16]\). Finally, we show how our constructions give rise to locally analytic representations of the Galois groups of \(p\)-adic Lie extensions of \(p\)-adic fields, culminating in a conjectural generalization (Conjecture 7.12) of the Cherbonnier-Colmez theorem on the overconvergence of Fontaine’s \((\varphi, \Gamma)\)-modules \([7]\).

Using Theorem 3.9 and Theorem 4.3 we establish Conjecture 7.12 in the setting considered by Berger in \([3]\): we consider crystalline representations of \(G_F\), for \(F\) a finite unramified extension of \(\mathbb{Q}_p\), and look at the action of the Galois group of the division field of a Lubin-Tate extension. We expect that additional cases of Conjecture 7.12 can be established by emulating the simplified proof of Cherbonnier-Colmez given by the first author in \([15]\); we will pursue this point elsewhere.

1. \textsc{Dagger lifts of some noetherian Banach algebras}

We start with a generalization of the observation that for \(k\) a perfect field of characteristic \(p\), the integral Robba ring over \(W(k)\) is an incomplete but henselian discrete valuation ring with residue field \(k((\pi))\). In this paper, we form similar lifts of some noetherian Banach algebras.

\textbf{Hypothesis 1.1.} Throughout this paper, let \(\mathfrak{o}\) be a complete discrete valuation ring with maximal ideal \(m\) and perfect residue field \(k\), and let \(\varpi\) be a generator of \(m\). Also, fix a constant \(\omega \in (0, 1)\).

\textbf{Definition 1.2.} For \(r > 0\), for \(m\) and \(n\) nonnegative integers, let \(\mathbf{A}\langle T_1, \ldots, T_m\rangle \langle\langle U_1, \ldots, U_n\rangle\rangle^{+r}\) be the completion of \(\mathfrak{o}[\pi, T_1, \ldots, T_m][[U_1, \ldots, U_n]][\pi^{-1}]\) for the \(r\)-Gauss norm

\[
\left| \sum_{i,j,\ldots,k} a_{i,j,\ldots,k} \pi^i T_1^{j_1} \cdots T_m^{j_m} U_1^{k_1} \cdots U_n^{k_n} \right|^r = \max_{i,j,\ldots,k} \{|a_{i,j,\ldots,k}| \omega^{ri}\}.
\]

Let \(\mathbf{A}\langle T_1, \ldots, T_m\rangle \langle\langle U_1, \ldots, U_n\rangle\rangle^{+}\) be the union of these rings over all \(r > 0\).

\textbf{Lemma 1.3.} The rings \(\mathbf{A}\langle T_1, \ldots, T_m\rangle \langle\langle U_1, \ldots, U_n\rangle\rangle^{+}\) are noetherian for all \(m, n \geq 0\).

\textit{Proof.} Let \(R\) be the \((m, \pi)\)-adic completion of \(\mathfrak{o}[\pi, T_1, \ldots, T_m][[U_1, \ldots, U_n]]\); since it is an adic completion of a polynomial ring over a discrete valuation ring, it is noetherian. The ring named in the lemma is a weakly complete finitely generated algebra over \(R\) for the ideal \(mR\) and the generator \(\pi^{-1}\); it is thus noetherian by a theorem of Fulton \([11]\). \(\square\)
**Definition 1.4.** Equip the field \( k((\pi)) \) with the \( \pi \)-adic norm for the normalization \( |\pi| = \omega \). A mixed affinoid algebra over \( k((\pi)) \) is a Banach algebra \( A \) over \( k((\pi)) \) obtained as a quotient of some ring of the form
\[
k[\pi] \langle T_1, \ldots, T_m \rangle \langle U_1, \ldots, U_n \rangle \langle \pi^{-1} \rangle
\]
for some \( m, n \) (again equipped with the \( \pi \)-adic norm). Such a ring is evidently noetherian.

A *dagger lift* of \( A \) is a flat \( \sigma \)-algebra \( S \) with \( S/mS \cong A \) admitting a surjection
\[
f : A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle \overset{\dagger}{\rightarrow} S
\]
for some \( m, n \); any such surjection is called a presentation of \( S \). By Lemma 1.3, \( S \) is noetherian.

Given a presentation \( f \) of \( S \), for \( r > 0 \) define the subrings
\[
S_r = f(A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle \langle \dagger \rangle^r)
\]
and the quotient norm \( |\cdot|_r \) on \( S_r \) induced via \( f \). While these constructions depend on \( f \), any two choices of \( f \) give the same rings \( S_r \) and equivalent norms \( |\cdot|_r \) for \( r \) sufficiently small, by Lemma 1.9 below.

**Remark 1.5.** Let \( S \) be a dagger lift of a mixed affinoid algebra over \( k((\pi)) \) equipped with a presentation. We will use frequently the fact that for \( 0 < s \leq r \) and \( x \in S_r \),
\[
|x|_s \leq |x|^{s/r}_r.
\]
A slightly stronger statement is that for \( t \in [0, 1] \),
\[
|x|_{s \downarrow t} \leq |x|^{t}_r |x|^{1-t}_s.
\]
Both theorems are proved by reducing to the case \( S_r = A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle \langle \dagger \rangle^r \), then reducing to the case \( x = \pi^i T_1^{j_1} \cdots T_m^{j_m} U_1^{k_1} \cdots U_n^{k_n} \), for which both inequalities become equalities.

**Remark 1.6.** Let \( S \) be a dagger lift of a mixed affinoid algebra over \( k((\pi)) \). For any \( x \in 1 + mS \), by Remark 1.5 we have \( |1 - x|_r < 1 \) for \( r > 0 \) sufficiently small, so \( x \) is a unit in \( S \). That is, \( mS \) is contained in the Jacobson radical of \( S \).

**Definition 1.7.** Let \( A \) be a mixed affinoid algebra over \( k((\pi)) \) and let \( S \) be a dagger lift of \( A \). For each positive integer \( n \), the quotient topology on \( S/m^nS \) induced by \( |\cdot|_n \) is independent of \( r \) provided that \( r \) is sufficiently small. By taking the inverse limit of these topologies, we obtain the weak topology on \( S \). It will follow from Corollary 1.10 that the definition of the weak topology depends only on \( S \) and not on the choice of a presentation.

**Lemma 1.8.** For \( i = 1, 2 \), let \( A_i \) be a mixed affinoid algebra over \( k((\pi)) \), let
\[
\overline{f}_i : k[\pi] \langle T_1, \ldots, T_m \rangle \langle U_1, \ldots, U_n \rangle \langle \pi^{-1} \rangle \rightarrow A_i
\]
be a surjective map, and let \( |\cdot|_i \) be the quotient norm on \( A_i \) induced via \( \overline{f}_i \). Let \( \overline{f} : A_1 \rightarrow A_2 \) be a continuous homomorphism of topological rings (but not necessarily a homomorphism of \( k((\pi)) \)-algebras), and suppose that the restriction of \( |\cdot|_2 \) to \( \overline{f}(k((\pi))) \) is multiplicative. Then there exist \( t, c > 0 \) such that for all \( \pi \in A_1 \),
\[
|\overline{f}(\pi)|_2 \leq c |\pi|^{t}_1.
\]
Proof. Put $u = \overline{g}(\pi)$; then $u$ is a unit in $A_2$ with inverse $u^{-1} = \overline{g}(\pi^{-1})$. Since $\bullet_2$ is multiplicative on $\overline{g}(k((\pi)))$ and $u$ is topologically nilpotent, we must have $0 < |u|_2 < 1$. Therefore there exists $t > 0$ for which $|u|_2 = \omega^t$. If we equip $A_2$ with the norm $|\bullet|_2^{1/t}$, we may then view it as a Banach algebra over $k((\pi))$ via $\overline{g}$, in which case $\overline{g}$ becomes a continuous and hence bounded morphism of Banach algebras over $k((\pi))$. This proves the claim. \[\square\]

Lemma 1.9. For $i = 1, 2$, let $A_i$ be a mixed affinoid algebra over $k((\pi))$, let $S_i$ be a dagger lift of $A_i$, let

$$f_i : A(T_1, \ldots, T_{m_i}) \langle \{U_1, \ldots, U_{n_i}\} \rangle \to S_i$$

be a presentation, put

$$S_{i,r} = f_i(A(T_1, \ldots, T_{m_i}) \langle \{U_1, \ldots, U_{n_i}\} \rangle^r),$$

let $|\bullet|_{i,r}$ be the quotient norm on $S_{i,r}$ induced from $|\bullet|_r$ via $f_i$, let

$$f_i : k[\pi][T_1, \ldots, T_{m_i}]\langle U_1, \ldots, U_{n_i} \rangle[\pi^{-1}] \to A_i$$

be the reduction of $f_i$ modulo $m$, and let $|\bullet|_i$ be the quotient norm on $A_i$ induced via $f_i$. Let $g : S_1 \to S_2$ be any ring homomorphism continuous for the weak topologies induced by the given presentations, let $\overline{g} : A_1 \to A_2$ be the reduction of $g$, and assume that the restriction of $|\bullet|_2$ to $\overline{g}(k((\pi)))$ is multiplicative. Then for $t$ as in Lemma 1.8 there exist $r_0, c > 0$ such that for $0 < r \leq r_0$, $g$ maps $S_{1,tr}$ to $S_{2,r}$ and

$$|g(x)|_2 \leq c^r |x|_{1,tr} \quad (x \in S_{1,tr}).$$

Proof. By assumption, there exists $c_1 > 0$ such that for all $\overline{x} \in A_1$,

$$|\overline{g}(\overline{x})|_2 \leq c_1 |\overline{x}|_1.$$  

Choose some $\epsilon > 0$. Choose $r_1 > 0$ such that $S_{1,tr_1}$ surjects onto $A_1$ and $S_{2,r_1}$ surjects onto $A_2$. Set

$$u_i = \begin{cases} \pi & i = 0 \\ T_i & i = 1, \ldots, m_1 \\ U_{i-m_1} & i = m_1 + 1, \ldots, m_1 + n_1. \end{cases}$$

For $i = 0, \ldots, m_1 + n_1$, if $g(u_i) = 0$, set $v_i = 0$; otherwise, define $v_i$ as follows. Let $h_i$ be the largest nonnegative integer for which $\varpi^{h_i}$ divides $g(u_i)$ and put $w_i = \varpi^{-h_i}g(u_i) \in S_{2,r_1}$. Lift $\overline{w}_i \in A_2$ to some

$$\overline{v}_i \in k[\pi][T_1, \ldots, T_{m_2}]\langle U_1, \ldots, U_{n_2} \rangle[\pi^{-1}]$$

with $|\overline{v}_i| \leq (1 + \epsilon)|\overline{w}_i|_2$. Lift $\overline{v}_i$ to

$$v_i' \in A(T_1, \ldots, T_{m_2})\langle \{U_1, \ldots, U_{n_2}\} \rangle^r_1$$

in such a way that every monomial $\varpi^{h_i}T_{1}^{i_{1}} \cdots T_{m_2}^{i_{m_2}}U_{1}^{k_{1}} \cdots U_{n_2}^{k_{n_2}}$ with a zero coefficient in $\overline{v}_i$ is lifted to zero. Then project $v_i'$ to $w_i' \in S_{2,r_1}$. The image of $w_i'$ in $A_2$ equals $\overline{w}_i$, so $w_i' \equiv w_i \pmod{\varpi}$. We may thus put $w_i'' = \varpi^{-1}(w_i' - w_i) \in S_{2,r_1}$ and lift $w_i''$ to

$$v_i'' \in A(T_1, \ldots, T_{m_2})\langle \{U_1, \ldots, U_{n_2}\} \rangle^r_1.$$  

Put

$$v_i = \varpi^{m_i}(v_i' + \varpi v_i'');$$

then $\varpi^{-m_i}v_i$ projects to $\overline{v}_i$ in $k[\pi][T_1, \ldots, T_{m_2}]\langle U_1, \ldots, U_{n_2} \rangle[\pi^{-1}]$ and to $w_i$ in $S_2$.  

Note that the special form of $v_i'$ ensures that for all $r > 0$,
\[ |v_i'|_r = |v_i|_r^r. \]
On the other hand, by Remark 1.5 for $0 < r \leq r_1$,
\[ |v_i''|_r \leq |v_i''|_{r_1}^{r/r_1}, \]
and if $v_i' \neq 0$ then there exists $r_0 > 0$ such that for $0 < r \leq r_0$,
\[ \omega |v_i''|_{r_1}^{r/r_1} < |v_i|_r^r. \]
Consequently, for $0 < r \leq r_0$ we have
\[ |v_i|_r = |\omega|^{m_1} |v_i|_r^r \leq |\omega|^{m_1} (1 + \epsilon)^r |v_i|_{r_1}^r. \]
In case $m_i = 0$, then $\overline{w_i} = \overline{g(x)}$ and so
\[ |v_i|_r \leq (1 + \epsilon)^r c_{i,1} |\overline{w_i}|_{r_1}^r = (1 + \epsilon)^r c_{i,1} |u_i|_{r_1}. \]
In case $m_i > 0$, by choosing $r_0$ small enough we can still ensure that for $0 < r \leq r_0$,
\[ |v_i|_r \leq (1 + \epsilon)^r c_{i,1} |u_i|_{r_1}. \]
Of course this also holds if $v_i = 0$.

Now for general $x \in S_{1,1}$ with $0 < r \leq r_0$, choose $\tilde{x} \in A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle^\dagger$ lifting $x$ with $|\tilde{x}|_{r_1} \leq (1 + \epsilon) |x|_{1,1}$. Write
\[ \tilde{x} = \sum_{h,j,s,k, s} \tilde{x}_{h,j_1,\ldots,j_m,k_1,\ldots,k_n} \pi^h T_1^{j_1} \cdots T_m^{j_m} U_1^{k_1} \cdots U_n^{k_n}, \]
then $g(x)$ is the image in $S_{2,2}$ of
\[ \sum_{h,j,s,k, s} \tilde{x}_{h,j_1,\ldots,j_m,k_1,\ldots,k_n} u_0^{h_1} u_1^{j_1} \cdots u_{m_1}^{k_1} u_{m_1+1}^{k_1} \cdots u_{m_1+n_2}^{k_1} \]
and for $0 < s \leq r$,
\[ \tilde{x}_{h,j_1,\ldots,j_m,k_1,\ldots,k_n} u_0^{h_1} v_1^{j_1} \cdots v_{m_1}^{k_1} v_{m_1+1}^{k_1} \cdots v_{m_1+n_2}^{k_1} \leq (1 + \epsilon)^s c_{s,1} |\tilde{x}|_{r_1} \leq (1 + \epsilon)^{2s} c_{s,1} |x|_{1,1}. \]
This yields the desired inequality with $c = (1 + \epsilon)^2 c_{1,1}$. \hfill \Box

**Corollary 1.10.** Let $A$ be a mixed affinoid algebra over $k((\pi))$ and let $S$ be a dagger lift of $A$. For $i = 1, 2$, let $f_i : A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle^\dagger \to S_i$ be a presentation, put
\[ S_{i,r} = f_i(A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle^\dagger, r), \]
and let $|\cdot|_{i,r}$ be the quotient norm on $S_{i,r}$ induced from $|\cdot|_r$ via $f_i$. Then there exist $r_0, c > 0$ such that for $0 < r \leq r_0$, $S_{1,r} = S_{2,r}$ and
\[ |x|_{1,r} \leq c^r |x|_{2,r}, \quad |x|_{2,r} \leq c^r |x|_{1,r} \quad (x \in S_{1,r}). \]

**Lemma 1.11.** For $i = 1, 2$, let $A_i$ be a mixed affinoid algebra over $k((\pi))$ and let $S_i$ be a dagger lift of $A_i$. Let $g : S_1 \to S_2$ be a ring homomorphism continuous for the weak topologies (which do not depend on the presentations by Corollary 1.10) whose reduction $\overline{g} : A_1 \to A_2$ satisfies the condition of Lemma 1.8. If $\overline{g}$ has one of the following properties:

(a) injective;
(b) surjective;
(c) finite;
(d) flat;
(e) faithfully flat;

then so does $g$.

**Proof.** Suppose that $\overline{g}$ is injective. Then $\ker(g) \subseteq \bigcap_{n=1}^{\infty} m^n S_1$, but the latter equals 0 because $mS_1$ is contained in the Jacobson radical of $S_1$ (Remark 1.15). This proves (a).

Suppose that $\overline{g}$ is finite; we prove that $g$ is finite by a variant of the proof of Lemma 1.9. Fix presentations
\[ f_i : A \langle T_1, \ldots, T_m \rangle \langle \langle U_1, \ldots, U_n \rangle \rangle \to S_i \]
and use these to define subrings $S_{i,r}$ of $S_i$, quotient norms $|\cdot|_{i,r}$ on $S_{i,r}$, and quotient norms $|\cdot|_i$ on $A_i$. Choose $r_0 > 0$ for which $S_{2,r_0}$ surjects onto $A_2$ and the conclusion of Lemma 1.9 holds for some $c > 0$. We can then find finitely many elements $w_1, \ldots, w_m \in S_{2,r_0}$ whose images in $A_2$ generate $A_2$ as an $A_1$-module. By the open mapping theorem [11, §I.3.3, Théorème 1], the map $A_1^n \to A_2$ given by $(y_1, \ldots, y_m) \mapsto (y_1)(y_1) w_1 + \cdots + (y_m)(y_m) w_m$ is strict; that is, for $t$ as in Lemma 1.8 there exists $c_1 > 0$ such that for any $\overline{\pi} \in A_2$, there exist $\overline{y}_1, \ldots, \overline{y}_m \in A_1$ with $\overline{g}(\overline{y}_1) w_1 + \cdots + \overline{g}(\overline{y}_m) w_m = \overline{\pi}$ and $|\overline{y}_1|^t, \ldots, |\overline{y}_m|^t \leq c_1 |\overline{\pi}|^2$.

Fix $\epsilon > 0$. Given $x \in S_{2,r_0}$, we construct elements $y_{n,j} \in S_{1,r_0}$ for $n = 0, 1, \ldots$ and $j = 1, \ldots, m$ as follows. Given $y_{0,j}, \ldots, y_{n-1,j}$ for which
\[ x \equiv \sum_{i=0}^{n-1} \sum_{j=1}^m g(\overline{w}^iy_{i,j}) w_m \quad (\mod \overline{w}^n), \]
put
\[ z = \overline{w}^{-n} \left( x - \sum_{i=0}^{n-1} \sum_{j=1}^m g(\overline{w}^iy_{i,j}) w_m \right) \in S_{2,r_0}, \]
and choose $y_{n,j} \in S_{1,r_0}$ with $\overline{g}(\overline{y}_1) w_1 + \cdots + \overline{g}(\overline{y}_m) w_m = \overline{z}$ and $|y_{n,j}|^t \leq c_1 \overline{r} (1 + \epsilon)^{n} |z|_{2,r_0}$. By induction, we have
\[ |\overline{w}^n y_{n,j}|^t_{1,r_0} \leq c_1^{n+1} \overline{r} (1 + \epsilon)^{n} \max_j |w_j|^r_{2,r_0}. \]

Put $y_j = \sum_{n=0}^{\infty} y_{n,j}$. As in the proof of Lemma 1.9 there exist $r_1 \in (0, r_0]$ and $c_2 > 0$ such that for $r \in (0, r_1]$, we have $y_j \in S_{1,r}$ and $|y_j|^t_{1,r} \leq c_2 |x|_{2,r}$. Since $S_{2,r_0}$ is dense in $S_{2,r_1}$, it follows that $S_{2,r_1}$ is finite over $S_{1,r_1}$. This proves (c); the same argument with $m = 1$ proves (b) (or see Remark 1.15).
Suppose that $\overline{g}$ is flat. To check that $g$ is flat, it suffices to check after localizing at a maximal ideal $I$ of $S_1$, which by Remark 1.6 must contain $mS_1$. Since the rings $S_i$ are noetherian by Lemma 1.3, it is enough to check flatness of the maps of completed local rings, for which we easily deduce flatness from flatness modulo $m$ plus the fact that $m$ is a principal ideal. This yields (d) and similarly (e).

**Corollary 1.12.** For $i = 1, 2$, let $A_i$ be a mixed affinoid algebra over $k((\pi))$ and let $S_i$ be a dagger lift of $A_i$. Let $g : S_1 \to S_2$ be a ring homomorphism whose reduction $\overline{g} : A_1 \to A_2$ is an isomorphism. Then $g$ is itself an isomorphism.

**Corollary 1.13.** Assume that $k$ is perfect of characteristic $p > 0$. Let $A$ be a regular mixed affinoid algebra over $k((\pi))$ (e.g., a smooth affinoid algebra over $k((\pi))$ thanks to [5, §7.3.2]) and let $S$ be a dagger lift of $A$. Let $g : S \to S$ be a ring homomorphism whose reduction $\overline{g} : A \to A$ is the $q$-power Frobenius map for some power $q$ of $p$. Then $g$ is finite flat.

**Proof.** The map $\overline{g}$ is easily seen to be finite. Since $A$ is regular, by working at the level of completed local rings we see that $\overline{g}$ is also flat. Then $g$ is finite flat by Lemma 1.11.

**Remark 1.14.** One could get a stronger version of Corollary 1.13 by adding logarithmic structures (thus allowing for some nonregular $A$), but we will not attempt to state such a result.

**Remark 1.15.** The proof of Lemma 1.11(c) gives a slightly stronger result: any finite collection of elements of $S_2$ whose images in $A_2$ generate $A_2$ as an $A_1$-module themselves generate $S_2$ as an $S_1$-module. This also follows directly from the statement of Lemma 1.11(c) using Remark 1.6 and Nakayama’s lemma.

**Remark 1.16.** Let $A$ be an affinoid algebra over $k((\pi))$ and let $S$ be a dagger lift of $A$. Then each affinoid localization $A \to B$ gives rise to a dagger lift $T$ of $B$ equipped with a morphism $S \to T$; since $A \to B$ is a flat morphism [5, Corollary 7.3.2/6], so then is $S \to T$ by Lemma 1.11(d).

### 2. Multivariate Robba rings

**Hypothesis 2.1.** For the remainder of the paper, assume that $k$ is perfect of characteristic $p > 0$ and that $\mathfrak{o}$ is of mixed characteristics. Let $A$ be a nonzero regular mixed affinoid algebra over $k((\pi))$. Starting from Definition 2.3, we write $R$ for the completion of the perfect closure of $A$ and fix an enhanced $q$-power Frobenius lift on $\mathcal{R}_A$ (see Definition 2.4).

**Definition 2.2.** Let $S$ be a dagger lift of $A$, and fix a presentation of $S$. For $r > 0$, let $\mathcal{R}_A^s$ be the Fréchet completion of $S_r[p^{-1}]$ for the norms $|\cdot|_s$ for $s \in (0, r]$. Put $\mathcal{R}_A = \bigcup_{r > 0} \mathcal{R}_A^r$; note that this definition does not depend on the choice of the presentation of $S$. Let $\mathcal{R}_A^{\text{int}}$ be the subring of $\mathcal{R}_A$ consisting of those elements $x$ for which $\limsup_{r \to 0^+} |x|_r \leq 1$; it carries a natural LF (limit of Fréchet) topology. Put $\mathcal{R}_A^{\text{bd}} = \mathcal{R}_A^{\text{int}}[p^{-1}]$.

**Lemma 2.3.** In Definition 2.2, the natural map $S \to \mathcal{R}_A^{\text{int}}$ is an isomorphism.

**Proof.** It is clear that the map is injective. To prove that it is surjective, pick any $x \in \mathcal{R}_A^{\text{int}}$, and let $S'$ be a dagger lift of $A$ inside $\mathcal{R}_A^{\text{int}}$ containing $x$ and the image of $S$ (this step is necessary because we do not yet have any finiteness property for $\mathcal{R}_A^{\text{int}}$). The map $S \to S'$ is then an isomorphism by Corollary 1.12 so $x$ belongs to the image of $S$. This proves the claim.
Definition 2.4. Let $q$ be a power of $p$. By an enhanced $q$-power Frobenius lift on $\mathcal{R}_A$, we will mean a locally compact topological (not necessarily commutative) monoid $\Gamma$ with a distinguished central element $\varphi$ generating a discrete submonoid, together with an action of $\Gamma$ on $\mathcal{R}_A^\text{int}$ which is continuous for the LF topology (that is, the action map $\Gamma \times \mathcal{R}_A^\text{int} \to \mathcal{R}_A^\text{int}$ is continuous) such that the induced action on $\mathcal{R}_A^\text{int}/m\mathcal{R}_A^\text{int}$ is faithful and identifies $\varphi$ with the $q$-power absolute Frobenius endomorphism. In the case where $\Gamma$ is generated by $\varphi$, we call the resulting action a plain $q$-power Frobenius lift.

Remark 2.5. In Fontaine’s theory of $(\varphi, \Gamma)$-modules, the relevant enhanced Frobenius lift consists of the product of a discrete free monoid generated by $\varphi$ with a compact group, and the label $\Gamma$ applies only to the compact factor. We prefer not to force a splitting of the monoid $\Gamma$ for flexibility in applications; see for instance Example 2.6.

The following situation is considered in [3].

Example 2.6. Let $K$ be a finite unramified extension of $\mathbb{Q}_p$ of degree $h$. Put $\omega = p^{-1}$ and $q = p^h$. Let $A$ be the completion of $\mathbb{F}_q[\overline{Y}_{0}^{\pm}, \ldots, \overline{Y}_{h-1}^{\pm}]$ for the Gauss norm of the form

$$|\overline{Y}_0| = p^{-1}, \ldots, |\overline{Y}_{h-1}| = p^{-p^{h-1}}.$$ 

We may view this as an affinoid algebra over $\mathbb{F}_q((\pi))$ by identifying $\overline{Y}_0$ with $\pi$. We construct a dagger lift of $A$ by taking the union of the completions of $\mathcal{O}_F[\overline{Y}_0^{\pm}, \ldots, \overline{Y}_{h-1}^{\pm}]$ for the Gauss norms of the form

$$|\overline{Y}_0| = p^{-r}, \ldots, |\overline{Y}_h| = p^{-p^{h-1}r}$$

for all $r > 0$.

Fix a Lubin-Tate formal $\mathcal{O}_K$-module of height $h$ and write $[\bullet] : \mathcal{O}_K \to T\mathcal{O}_K[[T]]$ for the formal $\mathcal{O}_K$-action. (For concreteness, one may take for example $[[p]](T) = pT + T^q$.) We define an enhanced $q$-power Frobenius lift $\Gamma$ on $\mathcal{R}_A$ by taking the semidirect product of the group $\Gamma_K = (\mathcal{O}_K^\times)^h$ with the free monoid generated by $\varphi_p$ acting on $(\mathcal{O}_K^\times)^h$ via a cyclic permutation of the factors. For this action, an element $\gamma$ of the $i$-th factor of $\mathcal{O}_K^\times$ within $\Gamma_K$ sends $Y_i$ to $[\gamma](Y_i)$ and fixing $Y_j$ for $j \neq i$. The element $\varphi_p$ sends $Y_i$ to $Y_{i+1}$ for $i = 0, \ldots, h - 2$ and sending $Y_{h-1}$ to $[p](Y_0)$. We take $\varphi$ to be $\varphi_p^h$, which is a $q$-power Frobenius lift even though $\varphi_p$ is not a $p$-power Frobenius lift.

The following situation is considered in [1]. The case $n = 1$ is also closely related to the setup of [6].

Example 2.7. Put $A = k((\pi))(T_1^\pm, \ldots, T_n^\pm)$. We construct a dagger lift of $A$ by taking the union of the completions of $\mathbb{F}_p[\pi^\pm, T_1^\pm, \ldots, T_n^\pm]$ for the Gauss norms of the form

$$|\pi| = p^{-r}, |T_1| = \cdots = |T_n| = 1$$

for all $r > 0$. We define an enhanced $p$-power Frobenius lift $\Gamma$ on $\mathcal{R}_A$ by taking the product of the group $\mathbb{Z}_p^\times \times \mathbb{Z}_p^n$ with the discrete monoid generated by $\varphi$. For this action, an element $\gamma$ of $\mathbb{Z}_p^\times$ sends $\pi$ to $(1 + \pi)^\gamma - 1$ and fixes $T_1, \ldots, T_n$; an element $\gamma$ of the $i$-th factor of $\mathbb{Z}_p^\times$ sends $T_i$ to $(1 + \pi)^\gamma T_i$ and fixes $\pi$ and $T_j$ for $j \neq i$; and $\varphi$ sends $\pi$ to $(1 + \pi)^p - 1$ and $T_i$ to $T_i^p$ for $i = 1, \ldots, n$. 

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To make contact with [16] [17], we form an analogue of Robba rings in which the mixed affinoid algebra $A$ is replaced by a perfect ring.

**Definition 2.8.** Let $R$ be a perfect Banach algebra over $\mathbb{F}_p$. As in [16] Definition 5.1.1, we define the *relative extended Robba ring* $\tilde{\mathcal{R}}_R$ as follows. For $r > 0$, let $\tilde{\mathcal{R}}_{int,r}$ be the subring of the ring $W(R)$ of $p$-typical Witt vectors consisting of those elements $x = \sum_{n=0}^{\infty} p^n [\overline{x}_n]$ for which $p^{-n} |\overline{x}_n|_r \to 0$ as $n \to \infty$, and define the norm $|\bullet|_r$ on $\tilde{\mathcal{R}}_{int,r}$ by the formula

$$\sum_{n=0}^{\infty} p^n |\overline{x}_n|_r = \max_{r} \{ p^{-n} |\overline{x}_n|_r \}.$$  

Then put $\tilde{\mathcal{R}}_{R}^{bd,r} = \tilde{\mathcal{R}}_{R}^{int,r}[p^{-1}]$, let $\tilde{\mathcal{R}}_R$ be the Fréchet closure of $\tilde{\mathcal{R}}_{R}^{bd,r}$ for the norms $|\bullet|_s$ for $s \in (0, r]$, and put $\mathcal{R}_R = \bigcup_{r > 0} \tilde{\mathcal{R}}_r$. For more on the basic properties of these rings, see [16] §5.

For the remainder of the paper, we take $R$ to be completion of the perfect closure $A^{perf}$ of $A$. In this case, the rings $\mathcal{R}_A$ and $\tilde{\mathcal{R}}_R$ are closely related.

**Definition 2.9.** Given an enhanced $q$-power Frobenius lift on $\mathcal{R}_A$, form the direct perfection of $\mathcal{R}_A^{int}$, i.e., the limit of the direct system

$$\mathcal{R}_A^{int} \xrightarrow{\varphi} \mathcal{R}_A^{int} \xrightarrow{\varphi} \cdots .$$

The $p$-adic completion of this ring may then be canonically identified with $W(A^{perf}) \otimes W(k) \mathfrak{o}$; this identification gives rise to a $\Gamma$-equivariant map $\mathcal{R}_A^{int} \to W(R) \otimes W(k) \mathfrak{o}$. By Lemma 2.10 below, this map factors through $\tilde{\mathcal{R}}_R^{int}$ and thus gives rise also to a map $\mathcal{R}_A \to \tilde{\mathcal{R}}_R$.

**Lemma 2.10.** Set notation as in Definition 2.9 and fix a presentation of $\mathcal{R}_A^{int}$.

(a) For any $c \in (0, 1)$, there exists $r_0 > 0$ such that for $r \in (0, r_0]$, for $x \in \mathcal{R}_A^{rq}$, we have $\varphi(x) \in \mathcal{R}_A^{r}$ and

$$|\varphi(x) - x^q|_r \leq c|x^q|_r .$$

(b) For any $r_0$ as in (a) for some $c$, the map $\mathcal{R}_A^{int,r} \to W(R) \otimes W(k) \mathfrak{o}$ factors through an inclusion $\mathcal{R}_A^{int,r} \to \tilde{\mathcal{R}}_R$ which is isometric for the norms $|\bullet|_r$.

(c) Within $\tilde{\mathcal{R}}_R^{int}$ we have

$$\mathcal{R}_A^{int,r} = \mathcal{R}_A^{int} \cap \tilde{\mathcal{R}}_R^{int,r} .$$

(d) The ring $\bigcup_{n=0}^{\infty} \varphi^{-n}(\mathcal{R}_A^{int,r}q^{-n})$ is dense in $\tilde{\mathcal{R}}_R^{r}$ with respect to $|\bullet|_r$.

*Proof.* Since the construction of the map $\mathcal{R}_A \to \tilde{\mathcal{R}}_R$ depends only on $\varphi$, we may assume that $\Gamma$ is generated by $\varphi$. We may then lift $\varphi$ up some presentation to reduce to the case $A = k[[\pi]](T_1, \ldots, T_m)[[U_1, \ldots, U_n]][\pi^{-1}]$. It then suffices to check (a) for $x = \pi, T_1, \ldots, T_m, U_1, \ldots, U_n$. However, for any single nonzero $x$ the desired inequality is clear because

$$\lim_{r \to 0^+} |x^q|_r = 1, \quad \limsup_{r \to 0^+} |\varphi(x) - x^q|_r \leq |\varphi| < 1 .$$

Thus (a) follows; we may then deduce (b) from (a) as in [12] Lemma 3.7. Parts (c) and (d) follow at once. \qed
Definition 2.11. Define the following subrings of $\tilde{R}_R$:

$$\tilde{R}_A^{\text{int}} = \bigcup_{n=0}^{\infty} \varphi^{-n}(\tilde{R}_A^{\text{int},r}q^{-n})$$

$$\tilde{R}_A^{\text{bd}} = \bigcup_{n=0}^{\infty} \varphi^{-n}(\tilde{R}_A^{\text{bd},r}q^{-n})$$

$$\tilde{R}_A^{\text{int}} = \bigcup_{n=0}^{\infty} \varphi^{-n}(\tilde{R}_A^{\text{int}}), \quad \tilde{R}_A^{\text{bd}} = \bigcup_{n=0}^{\infty} \varphi^{-n}(\tilde{R}_A^{\text{bd}}), \quad \tilde{R}_A = \bigcup_{n=0}^{\infty} \varphi^{-n}(\tilde{R}_A).$$

Note that none of these definitions depends on the choice of a presentation of $\tilde{R}_A^{\text{int}}$.

Remark 2.12. Recall that the functor of $p$-typical Witt vectors can be applied to arbitrary rings, not just perfect rings of characteristic $p$. In particular, the ring $W(A)$ may be identified with the subring of $W(R)$ consisting of those $x = \sum_{n=0}^{\infty} p^n [x_n] \in W(R)$ with $[x_n]_n \in A$ for all $n$. One then checks easily that the map $\tilde{R}_A^{\text{int}} \to W(R) \otimes_{W(k)} \phi$ factors through the ring $W(A) \otimes_{W(k)} \phi$. It is thus tempting to amend Definition 2.11 by replacing $\tilde{R}_A^{\text{int},r}(A)$ with $(W(A) \otimes_{W(k)} \phi) \cap \tilde{R}_A^{\text{int},r}$; however, note that this ring with $p$ inverted is identical to our ring $\tilde{R}_A^{\text{bd},r}$.

Lemma 2.13. Any finite projective module over $\tilde{R}_R^{\text{int}}$ is the base extension of some finite projective module over $\tilde{R}_A^{\text{int}}$.

Proof. Let $P$ be a finite projective module over $\tilde{R}_R^{\text{int}}$; we may assume $P \neq 0$. We can then find another finite projective module $Q$ over $\tilde{R}_R^{\text{int}}$ such that $F = P \oplus Q$ is free. The composition $F \to P \to F$ is a projector defined by some matrix $U$. Choose $r > 0$ for which $U$ is defined over $\tilde{R}_R^{\text{int},r}$; then $|U|_r \geq 1$. By Lemma 2.10(d), we can choose a matrix $V$ over $\tilde{R}_A^{\text{int},r}$ of the same size as $U$ such that $|V - U|_r < |U|_r^{-3}$. In particular, since $V^2 - V = V^2 - V - U^2 + U = (V - U)(V + U - 1)$, we have $|V^2 - V|_r < |U|_r^{-2}$.

We now define a sequence of matrices $W_0, W_1, \ldots$ by taking $W_0 = V$ and forming $W_{l+1}$ from $W_l$ via a modified Newton iteration:

$$W_{l+1} = 3W_l^2 - 2W_l^3.$$  

Since

$$W_{l+1} - W_l = (W_l^2 - W_l)(1 - 2W_l)$$

$$W_{l+1}^2 - W_l = (W_l^2 - W_l)^2(4W_l^2 - 4W_l - 3),$$

by induction on $l$ we have

$$|W_l - U|_r < |U|_r^{-2}$$

$$|W_l^2 - W_l|_r \leq |U|_r^{-2} (|V^2 - V|_r |U|_r^2)^{2^l}.$$  

Since each ring $\varphi^{-n}(\tilde{R}_A^{\text{int},r}q^{-n})$ is complete with respect to $|\cdot|_r$, the $W_l$ converge to a matrix $W$ over $\tilde{R}_A^{\text{int},r}$ with $|W - U|_r < |U|_r^{-2}$ and $W^2 = W$. Let $F_0$ be a free module over $\tilde{R}_A^{\text{int},r}$ equipped with an isomorphism $F_0 \otimes_{\tilde{R}_A^{\text{int},r}} \tilde{R}_R^{\text{int}} \cong F$. Let $P_0$ and $Q_0$ be the images of the projector on $F_0$ defined by $W$ and $1 - W$. We then obtain a map $P_0 \otimes_{\tilde{R}_A^{\text{int},r}} \tilde{R}_R^{\text{int}} \to P$ by embedding $P_0$ into $F_0$ and then projecting $F$ onto $P$; we similarly obtain a map $Q_0 \otimes_{\tilde{R}_A^{\text{int},r}} \tilde{R}_R^{\text{int}} \to Q$. Taking
the direct sum, we obtain the map $F \to F$ defined by the matrix
\[
UW + (1-U)(1-W) = UW - U^2 - W^2 + UW + 1
= U(W-U) - (W-U)W + 1,
\]
for which we see that $|UW + (1-U)(1-W) - 1|_r < 1$. The map is therefore an isomorphism, so in particular there exists an isomorphism $P_0 \otimes_R \tilde{\mathcal{R}}^\text{int}_R \cong P$.

**Definition 2.14.** By Corollary 1.13, $\varphi^{-1}(\mathcal{R}_A)$ is a finite projective module over $\mathcal{R}_A$. We can thus construct a $\mathcal{R}_A$-linear splitting $\varphi^{-1}(\mathcal{R}_A) = \mathcal{R}_A \oplus T$ and some $\mathcal{R}_A$-linear maps $U : T \to \mathcal{R}_A^m$, $V : \mathcal{R}_A^m \to T$ whose composition is the identity. Let $\Pi_0 : \varphi^{-1}(\mathcal{R}_A) \to \mathcal{R}_A$ be the first projection for the decomposition $\varphi^{-1}(\mathcal{R}_A) = \mathcal{R}_A \oplus T$ and let $\Pi_1, \ldots, \Pi_m : \varphi^{-1}(\mathcal{R}_A) \to \mathcal{R}_A$ be the composition of the second projection with the factors of $U$.

Applying powers of $\varphi^{-1}$ and assembling things together, we get $\mathcal{R}_A$-linear maps
\[
(2.14.1) \quad \tilde{\mathcal{R}}_A \to \bigoplus_S \mathcal{R}_A \to \tilde{\mathcal{R}}_A
\]
whose composition is the identity, with $S$ running over finite strings in the alphabet $\{0, \ldots, m\}$ not ending in 0. More precisely, for $S = s_1 \cdots s_l$, formally put $s_i = 0$ for $i > l$. Then the projection $\Pi_S : \varphi^{-n}(\mathcal{R}_A) \to \mathcal{R}_A$ indexed by $S$ is zero if $n < l$, and otherwise is the composition of the projections $\varphi^{-i}(\Pi_{s_i}) : \varphi^{-i-1}(\mathcal{R}_A) \to \varphi^{-i}(\mathcal{R}_A)$ for $i = n, \ldots, 1$.

**Lemma 2.15.** With notation as in Definition 2.14, we may complete in (2.14.1) to obtain $\mathcal{R}_A$-linear maps
\[
(2.15.1) \quad \tilde{\mathcal{R}}_R \to \bigoplus_S \mathcal{R}_A \to \tilde{\mathcal{R}}_R.
\]
More precisely, there exist $r_0 > 0, c \geq 1$ such that for $r \in (0, r_0]$ and $x \in \tilde{\mathcal{R}}_R$,
\[
(2.15.2) \quad c^{-r} |x|_r \leq \sup_S \{|\Pi_S(x)|_r\} \leq c^r |x|_r.
\]

**Proof.** Since the maps
\[
\varphi^{-1}(\mathcal{R}_A) \to \bigoplus_{i=0}^m \mathcal{R}_A \to \varphi^{-1}(\mathcal{R}_A)
\]
are $\mathcal{R}_A$-linear maps between finite projective modules over $\mathcal{R}_A$, by writing everything in terms of generators we may find $r_0 > 0$ and $c_0 \geq 1$ such that for $r \in (0, r_0]$ and $x \in \varphi^{-1}(\mathcal{R}_A)$,
\[
c_0^{-r} |x|_r \leq \sup_S \{|\Pi_S(x)|_r\} \leq c_0^r |x|_r.
\]
For each positive integer $n$, if we put $c_n = c_0^{1+q^{-1}+\cdots+q^{1-n}}$, then for $r \in (0, r_0]$ and $x \in \varphi^{-n}(\mathcal{R}_A)$ we have
\[
c_n^{-r} |x|_r \leq \sup_S \{|\Pi_S(x)|_r\} \leq c_n^r |x|_r.
\]
We thus have (2.15.2) for $c = c_0^{q/(q-1)}$. \qed

**Lemma 2.16.** Let $M$ be a finite projective module over $\mathcal{R}_A$, put $\tilde{M} = M \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_R$, and let $\tilde{N}$ be a direct summand of $\tilde{M}$. Then the intersection $N = \tilde{N} \cap M$ within $\tilde{M}$ is a direct summand of $M$. In particular, $N$ is a finite projective module over $\mathcal{R}_A$ and the induced map $N \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_R \to \tilde{N}$ is an isomorphism.
Proof. From Lemma 2.15, we obtain a $\mathcal{R}_A$-linear splitting of the inclusion $\mathcal{R}_A \hookrightarrow \tilde{\mathcal{R}}_R$. By tensoring with $M$, we obtain a splitting $\tilde{M} \to M$ of the inclusion $M \hookrightarrow \tilde{M}$; this splitting in turn restricts to a splitting $\tilde{N} \to N$ of the inclusion $N \hookrightarrow \tilde{N}$.

Since $\tilde{N}$ is a direct summand of $\tilde{M}$, there exists a $\tilde{\mathcal{R}}_R$-linear splitting $\tilde{M} \to \tilde{N}$ of the inclusion $\tilde{N} \hookrightarrow \tilde{M}$. The composition $M \to \tilde{M} \to \tilde{N} \to N$ is then a $\mathcal{R}_A$-linear splitting of the inclusion $N \hookrightarrow M$. $\square$

3. $(\varphi, \Gamma)$-modules

Definition 3.1. Put $S = \mathcal{R}_A$ or $S = \tilde{\mathcal{R}}_R$. A $(\varphi, \Gamma)$-module over $S$ is a finite projective $S$-module $M$ equipped with a semilinear action of $\Gamma$ which is continuous for the LF topology on $M$, such that for each $\gamma \in \Gamma$ the induced map $\gamma^* M \to M$ is an isomorphism. A $\varphi$-module is the same thing but with $\Gamma$ reduced to the monoid generated by $\varphi$.

Remark 3.2. In what follows, we will freely use the fact that any finite projective module over a normed ring carries a unique norm topology, obtained by choosing a presentation of the module as a direct summand of a finite free module and imposing the supremum norm induced by some basis of the latter. For a detailed derivation, see [16, Lemma 2.2.12].

Definition 3.3. Fix a presentation of $S$. For $0 < s \leq r$, let $\mathcal{R}_A^{[s,r]}$ be the Fréchet completion of $\mathcal{R}_A^{\text{int}}[p^{-1}]$ for the seminorms $|\cdot|_t$ for $t \in [s,r]$; when $s,r \in \mathbb{Q}$, this is an affinoid algebra over Frac $\mathfrak{o}$ under the supremum of $|\cdot|_r$ and $|\cdot|_s$.

Lemma 3.4. Choose $r_0 > 0$ as in Lemma 2.10, and choose $0 < s \leq r \leq r_0$. Let $N$ be a finite projective module over $\mathcal{R}_A^{[s/q,r]}$ equipped with an isomorphism

$$\varphi^* N \otimes_{\mathcal{R}_A^{[s/q,r]}} \mathcal{R}_A^{[s/q,r]} \cong N \otimes_{\mathcal{R}_A^{[s/q,r]}} \mathcal{R}_A^{[s/q,r]}.$$ 

Then $N$ lifts uniquely to a finite projective module $M_r$ over $\mathcal{R}_A$ equipped with an isomorphism

$$\varphi^* M_r \cong M_r \otimes_{\mathcal{R}_A^{[r/q]}} \mathcal{R}_A^{[r/q]}.$$ 

In particular, $N$ gives rise to a $\varphi$-module over $\mathcal{R}_A$ by forming $M_r \otimes_{\mathcal{R}_A^{[r/q]}} \mathcal{R}_A$.

Proof. By pulling back along $\varphi$ and glueing, we may lift $N$ to a finite projective module over $\mathcal{R}_A^{[r]}$ for each $t \in (0,r]$. These define a coherent sheaf $\mathcal{F}$ on a quasi-Stein space $X$, and we take $M_r$ to be the space of global sections of $\mathcal{F}$. One may show that $M_r$ is finite projective by following the proof of [18, Proposition 2.2.7]. Alternatively, it has been shown by Bellovin (see [2, §2.2]) that any coherent sheaf on a quasi-Stein space of uniformly bounded rank is generated by finitely many global sections. By applying this twice, we see that $M_r$ is finitely presented; it then formally follows that it is finite projective [18, Lemma 2.1.8]. $\square$

Definition 3.5. A $(\varphi, \Gamma)$-module $M$ over $\mathcal{R}_A$ (resp. $\tilde{\mathcal{R}}_R$) is étale if there exist a finite projective $\mathcal{R}_A^{\text{int}}$-module (resp. $\tilde{\mathcal{R}}_R^{\text{int}}$-module) $M_0$ equipped with a semilinear action of $\Gamma$ and a $\Gamma$-equivariant isomorphism $M_0 \otimes_{\mathcal{R}_A^{\text{int}}} \mathcal{R}_A \cong M$ (resp. $M_0 \otimes_{\tilde{\mathcal{R}}_R^{\text{int}}} \tilde{\mathcal{R}}_R \cong M$) such that for each $\gamma \in \Gamma$, the induced map $\gamma^* M_0 \to M_0$ is an isomorphism. Such a module $M_0$ is then called an étale model of $M$.

Lemma 3.6. Let $M, N$ be étale $(\varphi, \Gamma)$-modules over $\mathcal{R}_A$ (resp. $\tilde{\mathcal{R}}_R$) and let $M_0, N_0$ be étale models of $M, N$. Then any morphism $M \to N$ of $(\varphi, \Gamma)$-modules carries $M_0[p^{-1}]$ into $N_0[p^{-1}]$. In particular, $M_0[p^{-1}]$ depends only on $M$, not on the choice of $M_0$. 

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Proof. By [16] Proposition 7.3.6, there exists a perfect Banach algebra $S$ over $R$ such that $M_0 \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_S$ and $N_0 \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_S$ admit $\varphi$-invariant bases. The given morphism $M \to N$ then maps each element of the invariant basis of $M_0 \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_S$ into the submodule of $N_0 \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_S$ generated by the invariant basis over the subring of $\tilde{\mathcal{R}}_S$ fixed by $\varphi$. By [16] Corollary 5.2.4, the latter is contained in $\tilde{\mathcal{R}}_S^{\text{int}}$, so $M_0$ is carried into $N_0 \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_S^{\text{int}}$. This proves the claim. \hfill \Box

Lemma 3.7. Put $S = \tilde{\mathcal{R}}_R \hat{\otimes}_{\mathcal{R}_A} \tilde{\mathcal{R}}_R$ and let $S^{\text{int}}$ be the image of $\tilde{\mathcal{R}}_R \hat{\otimes}_{\mathcal{R}_A} \tilde{\mathcal{R}}_R^{\text{int}}$ in $S$. Let $F$ be a $d \times d$ matrix over $S^{\text{int}}$ and let $v$ be a column vector of length $d$ over $S$ such that $v = F \varphi(v)$. Then $v$ has entries in $S^{\text{int}}[p^{-1}]$.

Proof. We use an argument modeled on the proof of [14] Proposition 3.3.4. Set notation as in Lemma 3.7. Let $\mathcal{S}$ be the homomorphisms $t_1(x) = x \otimes 1, t_2(x) = 1 \otimes x$. Choose a basis $e_1, \ldots, e_d$ of $M_0$. We then have $\varphi(e_j) = \sum_i F_{ij} e_i$ for some invertible matrix $F$ over $\tilde{\mathcal{R}}_R^{\text{int}}$, and $t_1(e_j) = \sum_i U_{ij} t_2(e_i)$ for some invertible matrix $U$ over $S$. Applying $\varphi$, we find that $U_{ij}(F) = t_2(F)U$. By Lemma 3.7, $U$ has entries in $S^{\text{int}}[p^{-1}]$.

Lemma 3.8. Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A$ such that $\tilde{M} = M \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_R$ admits a free étale model $\tilde{M}_0$. Put $M_0 = M \cap \tilde{M}_0$ within $\tilde{M}$. Then $\bigcup_{n=0}^{\infty} \varphi^{-n}(M_0)$ is dense in $\tilde{M}_0$ for the LF topology of $\tilde{\mathcal{R}}_R$.

Proof. We use an argument modeled on the proof of [14] Theorem 3.1.3. Define the rings $S^{\text{int}}$, $\mathcal{S}$ as in Lemma 3.7. Let $t_1, t_2 : \tilde{\mathcal{R}}_R \to S$ be the homomorphisms $t_1(x) = x \otimes 1, t_2(x) = 1 \otimes x$. Choose a basis $e_1, \ldots, e_d$ of $M_0$. Then these maps define $\varphi(e_j) = \sum_i F_{ij} e_i$ for some invertible matrix $F$ over $\tilde{\mathcal{R}}_R^{\text{int}}$, and $t_1(e_j) = \sum_i U_{ij} t_2(e_i)$ for some invertible matrix $U$ over $S$. Applying $\varphi$, we find that $U_{ij}(F) = t_2(F)U$. By Lemma 3.7, $U$ has entries in $S^{\text{int}}[p^{-1}]$.

Theorem 3.9. Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A$ and put $\tilde{M} = M \otimes_{\mathcal{R}_A} \tilde{\mathcal{R}}_R$. Let $\tilde{M}_0$ be an étale model of $\tilde{M}$. Then the intersection $M_0 = M \cap \tilde{M}_0$ within $\tilde{M}$ is an étale model of $M$. Moreover, if $M_0$ is free as a module over $\tilde{\mathcal{R}}_R^{\text{int}}$, then $M_0$ is free as a module over $\mathcal{R}_A^{\text{int}}$.\hfill \Box
Proof. We first treat the case where \( \tilde{M}_0 \) is free. Choose a basis \( e_1, \ldots, e_d \) of \( \tilde{M}_0 \) and pick any \( r > 0 \). By Lemma 3.3 we can find \( e'_1, \ldots, e'_d \in \bigcup_{n=0}^{\infty} \varphi^{-n}(M_0) \) so that the matrix \( U \) over \( \tilde{R}_R \) for which \( e'_j = \sum_i U_{ij} e_i \) has entries in \( \tilde{R}_A \) and satisfies \( |U - 1|_s < 1 \) for \( s \in [r, qr] \). On the other hand, \( U \) also has entries with nonnegative \( p \)-adic valuation, so \( U \) is forced to be invertible. It follows that \( e'_1, \ldots, e'_d \) also form a basis of \( M_0 \). Choose \( n \) so that \( e'^n_1, \ldots, e'^n_d \in \varphi^{-n}(M_0) \); then \( \varphi^n(e'_1), \ldots, \varphi^n(e'_d) \) form a basis of \( M_0 \), so \( M_0 \) is an étale model of \( M \).

We next reduce the general case to the case of a free étale model. Since the claim depends only on \( \varphi \), we may assume that \( \varphi \) generates \( \Gamma \). By Lemma 2.13 plus the isomorphism \( \varphi^* \tilde{M}_0 \cong \tilde{M}_0 \), we can find a finite projective module \( M'_0 \) over \( \tilde{R}_A \) and an isomorphism \( M'_0 \otimes_{\tilde{R}_A} \tilde{R}_R \cong \tilde{M}_0 \) of modules over \( \tilde{R}_R \). We can then choose a finite projective module \( N_0 \) over \( \tilde{R}_A \) such that \( F_0 = M'_0 \oplus N_0 \) is free over \( \tilde{R}_A \). Put \( N = N_0 \otimes_{\tilde{R}_A} \tilde{R}_A \) and \( F = F_0 \otimes_{\tilde{R}_A} \tilde{R}_A \).

We may equip \( F \) with the structure of an étale \((\varphi, \Gamma)\)-module over \( \tilde{R}_A \) admitting \( F_0 \) as an étale model by decreeing that \( \varphi \) fixes some basis of \( F_0 \). We may then equip \( N \) with the structure of an étale \((\varphi, \Gamma)\)-module over \( \tilde{R}_A \) admitting \( N_0 \) as an étale model: define the action of \( \varphi \) on \( N_0 \) by mapping \( N_0 \) into \( F_0 \), applying \( \varphi \) on \( F_0 \) as described, then projecting \( F_0 \) to \( N_0 \).

Put \( \tilde{N} = N \otimes_{\tilde{R}_A} \tilde{R}_R \) and \( \tilde{N}_0 = N_0 \otimes_{\tilde{R}_A} \tilde{R}_R \); by applying the previous paragraph, we deduce that the intersection \((M \oplus N) \cap (\tilde{M}_0 \oplus \tilde{N}_0) = M_0 \oplus N_0 \) within \( \tilde{M} \oplus \tilde{N} \) is an étale model of \( M \oplus N \). Consequently, \( M_0 \) is an étale model of \( M \).

\( \Box \)

Corollary 3.10. Let \( M \) be a \((\varphi, \Gamma)\)-module over \( \tilde{R}_A \). If \( M \otimes_{\tilde{R}_A} \tilde{R}_R \) is étale, then so is \( M \).

4. Spreading the étale condition

We now link up with the study of \( \varphi \)-modules over relative extended Robba rings made in [16]. The basic strategy therein is to consider maps from \( R \) to complete nonarchimedean fields \( L \) and use the slope theory for \( \varphi \)-modules over \( \tilde{R}_L \) introduced previously by the author in [13]. Maps \( R \to L \) are parametrized up to a natural equivalence by the Gel’fand spectrum of \( R \) in the sense of Berkovich. (If one also keeps track of higher-rank valuations, one obtains a perfectoid space in the sense of Scholze [21], but this observation is not immediately relevant here.)

Hypothesis 4.1. Throughout §4 fix an element \( \alpha \) of the Gel’fand spectrum \( \mathcal{M}(A) \), the space of bounded multiplicative seminorms on \( A \) equipped with the evaluation topology. Note that there is a natural homeomorphism \( \mathcal{M}(A) \cong \mathcal{M}(R) \). There is also a homeomorphism \( R \to \mathcal{H}(\alpha) \) to a complete nonarchimedean field \( \mathcal{H}(\alpha) \) such that the restriction of the norm on \( \mathcal{H}(\alpha) \) computes \( \alpha \) on \( R \).

Definition 4.2. A rational localization of \( A \) is a homomorphism \( A \to B \) representing a rational subdomain of \( \mathcal{M}(A) \) (e.g., in the sense of [16] Definition 2.4.7); for such a homomorphism, the map \( \mathcal{M}(B) \to \mathcal{M}(A) \) identifies \( \mathcal{M}(B) \) with a closed subset of \( \mathcal{M}(A) \). We say that a rational localization \( A \to B \) encircles \( \alpha \) if \( \mathcal{M}(B) \) is a neighborhood of \( \alpha \) in \( \mathcal{M}(A) \); note that such neighborhoods form a neighborhood basis of \( \alpha \) in \( \mathcal{M}(A) \).

Theorem 4.3. Assume that \( \Gamma \) is generated by \( \varphi \). Let \( M \) be a \((\varphi, \Gamma)\)-module over \( \tilde{R}_A \) such that \( M \otimes_{\tilde{R}_A} \tilde{R}_{\mathcal{H}(\alpha)} \) is étale. Then there exists a rational localization \( A \to A' \) encircling \( \alpha \) such that \( M \otimes_{\tilde{R}_A} \tilde{R}_{A'} \) admits a free étale model, and in particular is étale.
Proof. By [16, Corollary 7.3.8], there exists a rational localization $R \to R'$ encircling $\alpha$ such that $M \otimes_{R_A} \bar{\mathcal{R}}_R'$ admits a free étale model. By approximating the parameters defining the rational localization, we can write it as the base extension of a rational localization of $\varphi^{-n}(A)$ for some nonnegative integer $n$; by raising the parameters to the $q^n$-th power, we can then write $R'$ as the completed perfect closure of $A'$ for some rational localization $A \to A'$ encircling $\alpha$. By Theorem 3.9 $M \otimes_{R_A} \mathcal{R}_{A'}$ admits a free étale model. \[\square\]

There are two problems with extending Theorem 4.3 to the case of general $\Gamma$: finding a rational localization stable under $\Gamma$ and finding an étale model stable under $\Gamma$. One way to circumvent both of these issues is to make the following construction.

**Definition 4.4.** Let $\mathcal{R}_{\alpha}^{bd}$ (resp. $\mathcal{R}_{\alpha}$) be the direct limit of $\mathcal{R}_{A'}^{bd}$ (resp. $\mathcal{R}_{A'}$) over all rational localizations $A \to A'$ encircling $\alpha$. Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_{\alpha}$ such that $M \otimes_{\mathcal{R}_{\alpha}} \bar{\mathcal{R}}_{\mathcal{H}(\alpha)}$ is étale. By Theorem 4.3, for any sufficiently small rational localization $A \to A'$ encircling $\alpha$, $M \otimes_{R_A} \mathcal{R}_{A'}$ as a $\varphi$-module admits an étale model $M_{0,A'}$. Let $M_0$ be the direct limit of the spaces $M_{0,A'}[p^{-1}]$ over all such localizations; it is a finite projective module over $\mathcal{R}_{\alpha}^{bd}$ on which $\Gamma$ acts (by Lemma 3.6), and the natural map $M_0 \otimes_{\mathcal{R}_{\alpha}^{bd}} \mathcal{R}_{\alpha} \to M \otimes_{R_A} \mathcal{R}_{\alpha}$ is an isomorphism.

**Remark 4.5.** If the identity in $\Gamma$ admits a neighborhood basis consisting of open submonoids, then every point of $\mathcal{M}(A)$ admits a neighborhood basis consisting of rational subdomains stable under open submonoids of $\Gamma$. Namely, every set of the form $\{\alpha \in \mathcal{M}(A) : \alpha(\overline{J}) \leq c\}$ or $\{\alpha \in \mathcal{M}(A) : \alpha(\overline{J}) \geq c\}$ is stable under some open submonoid (because the action of $\Gamma$ on $A$ is supposed to be continuous), and the finite intersections of such sets provide the desired neighborhoods.

An important special case of the previous paragraph is the case where $\Gamma$ is the product of the discrete monoid generated by $\varphi$ with a compact $p$-adic Lie group $G$. In this case, every point of $\mathcal{M}(A)$ admits a neighborhood basis consisting of rational subdomains stable under open subgroups of $G$, and hence also a neighborhood basis consisting of rational subgroups stable under all of $G$.

**Remark 4.6.** In general, $M \otimes_{R_A} \bar{\mathcal{R}}_{\mathcal{H}(\alpha)}$ admits a slope polygon according to a variant of the Dieudonné-Manin classification; see [16 §4] for a summary. As a function of $\alpha$, the slope polygon is lower semicontinuous [16 Theorem 7.4.5], which implies that étaleness at a point implies étaleness in an open neighborhood (as used in the proof of Theorem 4.3). In general, the slope polygon need not be constant in a neighborhood of $\alpha$. If it is, then by [16, Theorem 7.4.8] there exists a rational localization $A \to A'$ encircling $\alpha$ such that for $R'$ the completion of the perfect closure of $A'$, $M \otimes_{R_A} \bar{\mathcal{R}}_{R'}$ admits a slope filtration interpolating the Harder-Narasimhan filtration of $M \otimes_{R_A} \bar{\mathcal{R}}_{\mathcal{H}(\beta)}$ for each $\beta \in \mathcal{M}(A')$. This filtration has the property that each of its subquotients is a finite projective module over $\bar{\mathcal{R}}_{R'}$; it therefore descends to a filtration of $M \otimes_{R_A} \mathcal{R}_{A'}$ with the analogous property thanks to Lemma 2.16. Each subquotient of the resulting filtration is pure; that is, for some $c, d \in \mathbb{Z}$ with $c > 0$, the action of $p^c \varphi^d$ admits an étale model.
5. Local analyticity of group actions

In applications to $p$-adic Hodge theory, the monoid $\Gamma$ will typically contain an interesting $p$-adic Lie group; for instance, in Fontaine’s original theory of $(\varphi, \Gamma)$-modules, our monoid $\Gamma$ contains an open subgroup of $\mathbb{Z}_p^\times$. We now check that such group actions are locally analytic.

**Hypothesis 5.1.** Throughout §5, let $G$ be a $p$-adic Lie group contained in $\Gamma$.

**Lemma 5.2.** Equip $A$ with the quotient norm induced by some presentation. Then for any $c \in (0, 1)$, there exists a pro-$p$ compact open subgroup $H$ of $G$ such that for all $\gamma \in H$ and all $\pi \in A$,

$$|(\gamma - 1)(\pi)| < c |\pi|.$$  \hfill (5.2.1)

**Proof.** Let $\mathfrak{f} : k[\pi][T_1, \ldots, T_m][U_1, \ldots, U_n][\pi^{-1}] \to A$ be the chosen presentation. Since $G$ is a $p$-adic Lie group, its identity element admits a neighborhood basis consisting of pro-$p$ compact open subgroups. By hypothesis, the action of $G$ on $A$ is continuous, so for any fixed $\pi \in A$ and any $c > 0$ we can find a pro-$p$ compact open subgroup $H$ of $G$ such that (5.2.1) holds for all $\gamma \in H$. In particular, for any $c, \epsilon > 0$ with $c(1 + \epsilon)^{-1} < 1$, we can find $H$ so that for $u = \pi, \pi^{-1}, T_1, \ldots, T_m, U_1, \ldots, U_n$, for all $\gamma \in H$,

$$|(\gamma - 1)(\mathfrak{f}(u))| < c(1 + \epsilon)^{-1} |u|.$$  \hfill (5.2.2)

Using the twisted Leibniz rule

$$((\gamma - 1)(yz)) = ((\gamma - 1)(y)z + \gamma(y)(\gamma - 1)(z),$$

we get (5.2.2) also for $u = \pi T_1^{j_1} \cdots T_m^{j_m} U_1^{k_1} \cdots U_n^{k_n}$ for $i \in \mathbb{Z}$ and $j_1, \ldots, j_m, k_1, \ldots, k_n \geq 0$.

For arbitrary $\pi \in A$, we can lift $\pi$ to $x \in k[\pi][T_1, \ldots, T_m][U_1, \ldots, U_n][\pi^{-1}]$ with $|x| \leq (1 + \epsilon) |\pi|$ for any fixed $\epsilon > 0$. By writing $x$ as a sum $\sum_i x_i T_1^{j_1} \cdots T_m^{j_m} U_1^{k_1} \cdots U_n^{k_n}$ and applying (5.2.1) term-by-term, we get (5.2.2). \hspace{1cm} \Box

**Definition 5.3.** Let $M$ be a finite projective module over $\mathfrak{r}^r_R$. Choose a presentation of $M$ as a direct summand of a finite free module, then use this presentation to define norms $|v|_s$ for $s \in (0, r]$ as in Remark 4.2. We say that a subset $S$ of $M$ is uniformly $r$-analytic if for any $c \in (0, 1)$, there exists a pro-$p$ compact open subgroup $H$ of $G$ such that for all $\gamma \in H$, all $v \in S$, all $s \in (0, r]$, and all nonnegative integers $m$,

$$|(\gamma^m - 1)(v)|_s < \max\{p^{m-i}p^{-is} : i = 0, \ldots, m\} |v|_s.$$  \hfill (5.3.1)

Note that this condition does not depend on the choice of the presentation of $M$.

**Lemma 5.4.** For some $r_0 > 0$, for all $r \in (0, r_0]$ the subset $\mathcal{R}^r_A$ of $\mathcal{R}^r_R$ is uniformly $r$-analytic.

**Proof.** Thanks to the identity

$$\gamma^m - 1 = (\gamma - 1)^m + \sum_{i=1}^{m-1} \binom{m}{i} (\gamma - 1)^i,$$  \hfill (5.4.1)

and the fact that $\binom{m}{i}$ is divisible by $m - \lfloor \log_p i \rfloor$ for $1 < i < p^m$, we may reduce the verification of (5.3.1) at once to the case $m = 0$. This follows by imitating the proof of Lemma 5.2 using Lemma 1.9. \hspace{1cm} \Box
Theorem 5.5. Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A$ and put $\tilde{M} = M \otimes_{\mathcal{R}_A} \mathcal{R}_R$. Choose $r > 0$ and let $\tilde{M}_r$ be the unique descent of $\tilde{M}$ to $\mathcal{R}_R$ (obtained by applying Lemma 3.4). Then for any nonnegative integer $n$, the subset $\varphi^{-n}(M) \cap \tilde{M}_r$ of $\tilde{M}_r$ is uniformly $r$-analytic.

Proof. We may again reduce the verification of (5.3.1) to the case $m = 0$. By continuity of the action of $G$, we can choose $H$ so that (5.3.1) holds for $v$ running over the generators of $\varphi^{-n}(M) \cap \tilde{M}_r$ (at least for $n$ sufficiently large that this module is finitely generated over $\varphi^{-n}(\mathcal{R}_A^{q^n})$). The claim then follows using (5.2.3) and Lemma 5.4. □

Our choice of terminology is motivated by the following observation.

Lemma 5.6. Retain notation as in Definition 5.3. If $S$ is uniformly $r$-analytic, then for $H$ as in Definition 5.3, for any $s \in (0, r]$ the function $S \times \log(H) \to M$ given by $(v, t) \mapsto \exp(t)v$ is analytic in $\log(H)$, i.e., it is computed by a power series which uniformly over $S$ is convergent with respect to the norm $\|\cdot\|_s$.

Proof. When $\dim_{\mathbb{Q}_p} G = 1$, we may deduce the claim from the binomial expansion

$$\gamma^n = \sum_{i=0}^{\infty} \binom{n}{i} (\gamma - 1)^i;$$

the general case then follows using the Campbell-Hausdorff formula. □

6. APF extensions and perfectoid fields

We next recall the relationship between the theory of arithmetically profinite extensions of local fields of Fontaine–Wintenberger [10] and the theory of perfectoid fields introduced in [16] and [21] (and further discussed in [15]).

Definition 6.1. Suppose that $M/L/K$ are extensions of algebraic extensions of $\mathbb{Q}_p$ with $M/K$ Galois, and put $G = \text{Gal}(M/K)$ and $H = \text{Gal}(M/L)$. For $i \geq -1$, let $G^i$ be the $i$-th ramification subgroup of $G$ in the upper numbering. We say that $i$ is a ramification break of $L/K$ if $G^iH \neq G^{i+\epsilon}H$ for all $\epsilon > 0$. We denote by $i(M/K)$ the upper bound of those $i \geq -1$ for which $G^iH = G$. Note that none of these definitions depends on the choice of $M$.

Hypothesis 6.2. For the remainder of §6, let $K$ be a finite extension of $\mathbb{Q}_p$, let $K_{\infty}$ be an infinite algebraic extension of $K$, and let $L_{\infty}$ be an infinite Galois extension of $K$ containing $K_{\infty}$. Put $G = \text{Gal}(L_{\infty}/K)$ and $H = \text{Gal}(L_{\infty}/K_{\infty})$.

Definition 6.3. We say that $K_{\infty}/K$ is arithmetically profinite (APF) if $G^iH$ is open in $G$ for all $i$. Equivalently, the ramification breaks of $K_{\infty}/K$ form an increasing sequence $i_0 < i_1 < \cdots$ of rational numbers and the fields

$$K_j = K_{\infty} \cap L_{\infty}^{G_j} \quad (j = 0, 1, \ldots)$$

(whose union is evidently equal to $K_{\infty}$) are finite extensions of $K$. Note that any APF extension is almost totally ramified and almost totally wildly ramified.

We say that $K_{\infty}$ is strictly APF if $K_{\infty}$ is APF and

$$\liminf_{j \to \infty} \frac{i_j}{[K_j : K]} > 0,$$
where

\[ i'_j = \psi_{K_\infty/K}(i_j) = \int_0^{i_j} [G^0 : G^0 H^0] \, dv \]

is the index in the lower numbering corresponding to \( i_j \) in the upper numbering. Equivalently, we must have \( c(K_\infty/K) > 0 \) for

\[ c(K_\infty/K) = \liminf_{j \to \infty} \frac{p - 1}{p} \frac{i(K_{j+1}/K_j)}{[K_{j+1} : K_0]} . \]

**Lemma 6.4.** Suppose that \( K_\infty \) is strictly APF. Let \( v \) be the valuation on \( K_\infty \) normalized so that \( v(K^*) = \mathbb{Z} \). For \( j > 0 \), we have the following.

(a) For \( y \in K_{j+1} \), we have

\[ v(\text{Norm}_{K_{j+1}/K_j}(y) - y_{[K_{j+1}:K_j]}) \geq \frac{p}{p - 1} c(K_\infty/K) . \]

(b) For \( x \in K_j \), there exists \( y \in K_{j+1} \) with

\[ v(\text{Norm}_{K_{j+1}/K_j}(y) - x) \geq c(K_\infty/K) . \]

**Proof.** Part (a) is straightforward. Part (b) is [10, Lemme 3.1]. \( \square \)

We can now make contact with a more modern point of view.

**Proposition 6.5.** If \( K_\infty \) is strictly APF, then its completion is a perfectoid field in the sense of [16, Definition 3.5.1] or [21, Definition 3.1].

**Proof.** By definition, \( K_\infty \) is nondiscrete; by Lemma 6.4, there exists \( t \in \mathfrak{m}_{K_\infty} \) such that the Frobenius endomorphism of \( \mathfrak{o}_{K_\infty}/(t) \) is surjective. This completes the proof. \( \square \)

**Definition 6.6.** Suppose that \( K_\infty \) is strictly APF. By Proposition 6.5, we may apply the perfectoid correspondence [16, Theorem 3.5.3] (called tilting in [21, §3]) to the completion of \( K_\infty \) to produce a perfect field \( \tilde{E}_{K_\infty} \) of characteristic \( p \) which is complete with respect to a distinguished norm. The valuation subring of \( \tilde{E}_{K_\infty} \) is naturally identified with the inverse limit of \( \mathfrak{o}_{K_\infty}/(p) \) under Frobenius. We call \( \tilde{E}_{K_\infty} \) the perfect norm field associated to \( K_\infty \).

Let \( E_{K_\infty} \) be the inverse limit of the \( K_j \) under the norm maps. This set inherits an obvious multiplication map; it also admits a natural addition map under which

\[ (x_j)_{j=0}^{\infty} + (y_j)_{j=0}^{\infty} = \left( \lim_{j' \to \infty} \text{Norm}_{K_{j'/K_j}}(x_{j'} + y_{j'}) \right)_{j=0}^{\infty} . \]

It is shown in [10] that \( E_{K_\infty} \) is a local field of characteristic \( p \) whose valuation subring is the inverse limit of the \( \mathfrak{o}_{K_j} \) under the norm maps. Using Lemma 6.4, we may identify that valuation subring with the inverse limit of the rings \( \mathfrak{o}_{K_j}/(t) \) for some \( t \in \mathfrak{m}_{K_\infty} \) under the map induced by both \( \text{Norm}_{K_{j+1}/K_j} \) and \( x \mapsto x_{[K_{j+1}:K_j]} \). In particular, we may identify \( E_{K_\infty} \) with the completed perfect closure of \( E_{K_\infty} \). We call \( E_{K_\infty} \) the imperfect norm field associated to \( K_\infty \).
7. A CONJECTURE ON LOCALLY ANALYTIC VECTORS

We conclude with a conjecture closely related to the overconvergence theorem of Cherbonnier and Colmez.

**Hypothesis 7.1.** Throughout §7 assume Hypothesis 6.2, but assume further that $G$ is a $p$-adic Lie extension and that $L_\infty$ is almost totally ramified. Also fix some $r > 0$.

**Proposition 7.2.** The fields $K_\infty$ and $L_\infty$ are strictly APF, so (by Proposition 6.3) their completions are perfectoid.

*Proof.* This is a theorem of Sen [24].

**Definition 7.3.** Let $C_p$ be a completed algebraic closure of $K$ containing $L_\infty$. By Proposition 7.2 and the compatibility of the perfectoid correspondence with algebraic field extensions [16, Theorem 3.5.6], [21, Theorem 3.7], $C_p$ corresponds to a perfect field $\tilde{E}$ of characteristic $p$ on which $G_{K_\infty}$ acts.

**Lemma 7.4.** We have $\tilde{E}^{K_\infty} = \tilde{E}_{K_\infty}$, $\tilde{E}^{L_\infty} = \tilde{E}_{L_\infty}$.

*Proof.* For each finite extension $M$ of $K_\infty$ within $C_p$, the perfectoid correspondence assigns to $M$ a finite extension $\tilde{E}_M$ of $\tilde{E}$ within $\tilde{E}$. We then have an exact sequence

$$0 \to \tilde{E}_{K_\infty} \to \tilde{E}_{M} \to \tilde{E}_{M} \otimes_{\tilde{E}_{K_\infty}} \tilde{E}_M$$

where the last map is $x \mapsto x \otimes 1 - 1 \otimes x$. This sequence is moreover strict exact for the tensor product norm on the last factor, but since Frobenius acts bijectively on all terms of the sequence that coincides with the spectral seminorm. The latter in turn equals the supremum over the factors of $\tilde{E}_M \otimes_{\tilde{E}_{K_\infty}} \tilde{E}_M$ [16, Theorem 2.3.10]. We may then take the direct limit over $M$ and complete to obtain a strict exact sequence

$$0 \to \tilde{E}_{K_\infty} \to \tilde{E} \to \tilde{E} \otimes_{\tilde{E}_{K_\infty}} \tilde{E}$$

where again the tensor product seminorm coincides with the supremum over the factors of $\tilde{E} \otimes_{\tilde{E}_{K_\infty}} \tilde{E}$ by [16, Theorem 2.3.10]. Since those factors correspond to elements of $G_{K_\infty}$, any element of $\tilde{E}^{G_{K_\infty}}$ must map to zero in $\tilde{E} \otimes_{\tilde{E}_{K_\infty}} \tilde{E}$ and hence must lie in $\tilde{E}_{K_\infty}$. This proves the first inequality; taking $K_\infty = L_\infty$ yields the second inequality. □

**Definition 7.5.** Put $\tilde{A}^* = W(C_p)$, $\tilde{A}_{K_\infty} = W(\tilde{E}_{K_\infty})$, $\tilde{A}_{L_\infty} = W(\tilde{E}_{L_\infty})$.

Let $\tilde{A}^*_K$ be the subring of $\tilde{A}$ consisting of $r$-overconvergent Witt vectors as in [15, Lemma 1.7.2]; this ring carries the norm $|\bullet|_r$ defined by

$$\left|\sum_{n=0}^{\infty} p^n [\tau_n] \right|_r = \max_n \{p^{-n} |\tau_n|^r\}.$$ 

Put $\tilde{A}^*_K = \tilde{A}^* \cap \tilde{A}_K$.

**Definition 7.6.** Let $\tilde{A}^*_{K_\infty}$ be the subset of $\tilde{A}^*_{K_\infty}$ consisting of elements whose singleton sets are uniformly $r$-analytic for the action of $G$ on $\tilde{A}^*_L$. Note that this set is a union of uniformly $r$-analytic subrings of $\tilde{A}^*_{K_\infty}$. 19
**Example 7.7.** For any nonnegative integer $n$ and any $\mathcal{P} \in \varphi^{-n}(E_{K_{\infty}})$, for $\gamma \in G^D$ we have $|(\gamma - 1)(\mathcal{P})| \leq c^j |\mathcal{P}|$ for some $c \in (0, 1)$ independent of $j$. It follows that the Teichmüller lift $[\mathcal{P}]$ is uniformly $r$-analytic for all $r > 0$.

**Definition 7.8.** For $x \in \tilde{A}_{K_{\infty}}^{\dagger,r}$, an *affinoid model* of $x$ is a triple $(A, \alpha, y)$ where $A$ is a smooth affinoid algebra over $\mathbb{F}_p((\mathcal{P}))$ admitting a continuous $G$-action, $\alpha : A \to \tilde{E}_{K_{\infty}}$ is a bounded $G$-equivariant homomorphism, and $y \in \mathcal{R}_{K_{\infty}}^{\text{int}}$ is an element mapping to $x$ under the map $\mathcal{R}_{A}^{\dagger} \to \tilde{A}_{K_{\infty}}^{\dagger,r}$ induced by $\alpha$. We say that $x$ is an *affinoid element* of $\tilde{A}_{K_{\infty}}^{\dagger,r}$ if it admits an affinoid model. Let $\tilde{A}_{K_{\infty}}^{\dagger,r,\text{aff}}$ be the subring of affinoid elements of $\tilde{A}_{K_{\infty}}^{\dagger,r}$; by Lemma 5.4 we have $\tilde{A}_{K_{\infty}}^{\dagger,r,\text{aff}} \subseteq \tilde{A}_{K_{\infty}}^{\dagger,r,\text{an}}$.

**Conjecture 7.9.** We have $\tilde{A}_{K_{\infty}}^{\dagger,r,\text{aff}} = \tilde{A}_{K_{\infty}}^{\dagger,r,\text{an}}$.

**Remark 7.10.** One significant difficulty of Conjecture 7.9 is the lack of an obvious construction of $G$-stable affinoid models. One case where this does not arise is when $K_{\infty}$ is the division field of a formal group, in which case the formal group law can be used to construct such models (as in [3]). It may be possible to extend this construction to some additional cases in which $K_{\infty}$ arises by adjoining iterated inverse images of a finite rational map (e.g., the preperiodic points of an arithmetic dynamical system); the one-dimensional case of this has been treated by Cais and Davis (in preparation).

**Definition 7.11.** For $T$ a finite free $\mathbb{Z}_p$-module equipped with a continuous $G_K$-action, for $r > 0$ put

$$D_{K_{\infty}}^r(T) = (T \otimes_{\mathbb{Z}_p} \tilde{A}_{K_{\infty}}^{\dagger,r})^{G_{K_{\infty}}}.$$ 

This is a module over $\tilde{A}_{K_{\infty}}^{\dagger,r}$, and by the proof of [15] Theorem 2.4.5 (see also [16] §8), the induced map

$$D_{K_{\infty}}^r(T) \otimes_{\tilde{A}_{K_{\infty}}^{\dagger,r}} \tilde{A}_{K_{\infty}}^{\dagger,r} \to T \otimes_{\mathbb{Z}_p} \tilde{A}_{K_{\infty}}^{\dagger,r}$$

is an isomorphism. Let $D_{K_{\infty}}^r(T)_{\text{an}}$ be the set of uniformly $r$-analytic elements of $D_{K_{\infty}}^r(T)$ for the action of $G$ on $D_{L_{\infty}}^r(T)$; it is a module over $\tilde{A}_{K_{\infty}}^{\dagger,r,\text{an}}$.

**Conjecture 7.12.** Let $T$ be a finite free $\mathbb{Z}_p$-module equipped with a continuous $G_K$-action. Then for $r > 0$, the natural map

$$D_{K_{\infty}}^r(T)_{\text{an}} \otimes_{\tilde{A}_{K_{\infty}}^{\dagger,r,\text{an}}} \tilde{A}_{K_{\infty}}^{\dagger,r} \to D_{K_{\infty}}^r(T)$$

is an isomorphism.

**Remark 7.13.** For $K_{\infty} = K'_{\infty} = K(\mu_{p^\infty})$, Conjecture 7.12 follows immediately from the theorem of Cherbonnier and Colmez [7]. An alternate proof of that theorem has been described in [15], which we expect to lead to additional cases of Conjecture 7.12.

By combining our preceding results with work of Berger [3], we deduce the following additional case of Conjecture 7.12.

**Theorem 7.14.** Conjecture 7.12 holds in case $K$ is a finite unramified extension of $\mathbb{Q}_p$, $K_{\infty} = L_{\infty}$ is the division field of a Lubin-Tate formal $\mathfrak{o}_K$-module of height $h = [K : \mathbb{Q}_p]$, and $T$ is a crystalline representation.
Proof. Set notation as in Example 2.6. In [3 §3], Berger exhibits a \((\varphi, \Gamma_K)\)-equivariant map \(\mathcal{R}^\text{int}_A \to \tilde{A}_{K_\infty}^!\). This in turn induces a homomorphism \(R \to \tilde{E}\) of Banach algebras, which then gives rise to a map \(\mathcal{M}(\tilde{E}) \to \mathcal{M}(R)\) of Gel’fand spectra. Since \(\mathcal{M}(\tilde{E})\) is a point, the image of this map is a point \(\alpha\) of \(\mathcal{M}(R)\).

Berger also associates to \(T\) (see [3 Theorem 5.3]) a coadmissible reflexive module \(M\) over \(\mathcal{R}_A\) equipped with a semilinear continuous action of \(\Gamma\) with the property that \(M \otimes_{\mathcal{R}_A} \tilde{B}^!_{\text{rig}}\) is \(\alpha\)-étale. In particular, the rank of \(M\) does not jump up in a neighborhood of \(\alpha\), so for some \(r > 0\) and some rational localization \(A \to A'\) encircling \(\alpha\), \(M\) gives rise to a finite projective module over \(\mathcal{R}^r_{\mathcal{A}'}[q^r]\). By Lemma 3.4 \(M \otimes_{\mathcal{R}_A} \mathcal{R}_{A'}\) is itself finite projective and hence a \((\varphi, \Gamma)\)-module over \(\mathcal{R}_{A'}\).

Using Theorem 4.3, we may perform the construction of Definition 4.4 to choose a rational localization \(A \to A'\) such that as a \(\varphi\)-module, \(M \otimes_{\mathcal{R}_A} \mathcal{R}_{A'}\) admits an \(\alpha\)-étale model; using Remark 4.5 we may further ensure that \(\Gamma\) acts on \(A'\). This yields the desired result.

Remark 7.15. In Theorem 7.14 it is crucial to use power series in \(h\) variables. If one tries to make a similar construction using only a single variable, one is forced to limit consideration to representations whose Hodge-Tate weights vanish for all but one embedding of \(F\) into \(\mathbb{C}_p\) (see 3 [19]).

Remark 7.16. The motivation for Conjecture 7.12 is to approach a potential construction of the \(p\)-adic Langlands correspondence suggested by the work of Scholze–Weinstein [22] on moduli of \(p\)-divisible groups. Roughly speaking, given an \(n\)-dimensional representation \(V\) of \(G_K\), one expects the corresponding representation of \(\text{GL}_n(K)\) to be a sort of “universal \((\varphi, \Gamma)\)-module” of \(V\). That is, if we take \(K_\infty\) to be the division field of a formal \(\sigma_K\)-module of rank \(n\), then its Galois group \(G\) sits inside \(\text{GL}_n(K)\), and restricting the Langlands representation to \(G\) should produce something related to the locally analytic representation coming from Conjecture 7.12. Moreover, this construction should in some sense be compatible with families; that is, it should work uniformly over the Scholze–Weinstein moduli space.

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