The Pentagonal Fuzzy Number: Its Different Representations, Properties, Ranking, Defuzzification and Application in Game Problems

Avishek Chakraborty 1,3, Sankar Prasad Mondal 2, Shariful Alam 3, Ali Ahmadian 4,*
Norazak Senu 4, Debasish De 5 and Soheil Salahshour 6

1 Department of Basic Science, Narula Institute of Technology, Agarpara, Kolkata-700109, India; avishek.chakraborty@nit.ac.in
2 Department of Natural Science, Maulana Abul Kalam Azad University of Technology, West Bengal, Haringhata, Nadia, West Bengal 741249, India; sankar.res07@gmail.com
3 Department of Mathematics, Indian Institute of Engineering Science and Technology, Shibpur, Howrah-711103, India; salam50in@yahoo.co.in
4 Laboratory of Computational Sciences and Mathematical Physics, Institute for Mathematical Research, Universiti Putra Malaysia, Serdang, Selangor 43400 UPM, Malaysia; norazak@upm.edu.my
5 Department of Computer Science and Engineering, Maulana Abul Kalam Azad University of Technology, West Bengal, Haringhata, Nadia, West Bengal 741249, India; dr.debashis.de@gmail.com
6 Young Researchers and Elite Club, Mobarakeh Branch, Islamic Azad University, Mobarakeh, Iran; soheilsalahshour@yahoo.com
* Correspondence: ahmadian.hosseini@gmail.com

Received: 30 November 2018; Accepted: 15 January 2019; Published: 16 February 2019

Abstract: In this paper, different measures of interval-valued pentagonal fuzzy numbers (IVPFN) associated with assorted membership functions (MF) were explored, considering significant exposure of multifarious interval-valued fuzzy numbers in neoteric studies. Also, the idea of MF is generalized somewhat to nonlinear membership functions for viewing the symmetries and asymmetries of the pentagonal fuzzy structures. Accordingly, the construction of level sets, for each case of linear and nonlinear MF was also carried out. Besides, defuzzification was undertaken using three methods and a ranking method, which were also the main features of this framework. The developed intellects were implemented in a game problem by taking the parameters as PFNs, ultimately resulting in a new direction for modeling real world problems and to comprehend the uncertainty of the parameters more precisely in the evaluation process.

Keywords: pentagonal fuzzy number (PFN); non-linear interval-valued fuzzy number; ranking; defuzzification

1. Introduction

1.1. Uncertainty and Uncertainty Measure

Uncertainty theory plays important role in modeling for engineering and science problems. Recently, huge developments have taken place in this area. Analysis based on uncertainty theory is regularly required in the following fields, although it is not limited to these areas; mathematical modeling, physics, chemistry, economics, artificial intelligence, legal fact-finding, medical science, business administration, psychology, decision sciences, etc. Various concepts have been formulated to measure uncertainty. Fuzzy logic involves the construction of an approximate acumen mechanism to tackle the uncertainty related with human behavior. Many activities and approaches have been
designed to counteract or curtail the uncertainty that is generated by decision making, however, there are few comprehensive theories available in the literature. Researchers are currently striving to construct a common approach.

There are some important differences between different types of imprecise or uncertain parameters.

If we consider an Interval number, the following observations can be made:

(i) The fact belongs to some certain interval
(ii) The concept of the membership function is missing.

If we consider a Fuzzy number [1–3], the following observation can be made:

(i) The conception of belongingness of the component has to be considered
(ii) The concept of membership function is derived.

If we consider an Intuitionistic fuzzy number [4], the following observations can be made:

(i) The conception of belongingness, or non-belongingness of the component has to be considered
(ii) The concepts of the membership, or non-membership function are derived

If we take a Neutrosophic fuzzy number [5], the following observations can be made:

(i) The conception of truthfulness, falsity and indeterminacy of the component have to be considered
(ii) The concept of membership function for truthfulness, falsity and indeterminacy are derived

We can easily see the idea by using a Figure 1 that shows how different type of PFN can be formed as below:

![Figure 1. Different uncertain parameter.](image)

1.2. Fuzzy Sets and Number
Zadeh [1] invented a new idea, which is known as fuzzy sets theory (FST). The basic theory of uncertainty has been used with immense success in different fields. Chang and Zadeh [2] discovered the main idea of fuzzy sets and numbers. Mathematicians have further studied the different result on that theory [3,6] and the impressive and considerable development of ideas and different application of FST have resulted in the topic gaining a great deal of attention.

1.3. Interval-Valued Fuzzy Numbers (IVFNs)

Several studies have been done on the topic of IVFNs. We reviewed some of these papers and then chose the topics we would cover. Our investigation is summarized below in Table 1.

| Authors Information | Membership Function’s Type | Main Contribution in Theatrical Improvement | Application Area |
|---------------------|---------------------------|-------------------------------------------|------------------|
| Guijun and Xiaoping [7] | General fuzzy cases | Define IVFN and interval distribution number | Generalized pseudo-probability metric spaces and pseudo metric spaces |
| Anitha and Parvathi [8] | Correlation coefficient of IVFN | Properties of IVFN | Information energy |
| Lin [9] | Triangular Linear | Ranking using signed distance methods | Job-shop scheduling problem |
| Wei and Chen [10] | Trapezoidal Linear | Found similarity measures between two IVFN | Fuzzy risk analysis |
| Kalaichelvi et. al. [11] | Not take the membership function concept | Extended the concept of IVFN to Soft IVFN | Fuzzy matrix theory |
| Kumar and Singh [12] | Triangular linear | Found the signed distance | Fuzzy fault tree |
| Abirami and Dinagar [13] | LR type | Found distance function | Project network |
| Su [14] | Triangular linear | Arithmetic operation and comparison | Linear programming problem |
| Bhatia and Kumar [15] | Triangular linear | Ranking for solving LPP | Linear programming problem |
| Mondal [16] | Triangular linear and non-linear | Arithmetic operation | Fuzzy differential equation |
| Ebrahimnejad [17] | Generalized trapezoidal linear | Arithmetic operation on LPP | Linear programming problem |
| Dahooie et. al. [18] | Trapezoidal and triangular linear | Defuzzification and fuzzy additive ratio assessment method | Oil and gas well drilling projects |

1.4. Pentagonal Fuzzy Numbers

Many researchers have investigated PFN with various types of membership functions. In this part we study the available papers that are related to PFN (see below).

From our survey, we can say that membership functions are taken as linear with symmetry at both ends for the majority of cases. However, what is the scenario if the non-linear membership function or interval-valued membership functions concept is taken and there is asymmetry on both
ends and the concept of generalized fuzzy numbers is involved? Obviously, the formations are very different. Also, the results are different from previous cases. In this paper, we try to take all the possibilities for the formation of PFNs into account. For details we have to see Table 2.

Table 2: Studies done on PFNs.

| Authors Information       | Membership Function’s Type | Main Contribution in Theatrical Improvement | Application Area                        |
|---------------------------|----------------------------|--------------------------------------------|----------------------------------------|
| Panda and Pal [19]        | Linear and symmetry on both end | exponent operation and arithmetic operation | Fuzzy matrix theory                    |
| Anitha and Parvathi [20]  | Linear                     | Find expected crisp value                  | Inventory management                   |
| Helen and Uma [21]        | Linear                     | Defining the parametric representation     | Proofing the arithmetic operation and find the ranking |
| Siji and Kumari [22]      | Linear membership, non-membership functions | Arithmetic operation and find the ranking | Networking problem                     |
| Raj and Karthik [23]      | Linear                     | Arithmetic operation                       | Neural network problem                 |
| Dhanamandand Parimaldevi [24] | Linear                  | Find the ranking using circumcenter of centroids method | Multi item and multi-objective inventory management problem |
| Pathinathan and Ponnivalavan [25] | Reverse order linear | Arithmetic operation                       | Defining reverse order fuzzy number     |
| Ponnivalavan and Pathinathan [26] | Linear membership, non-membership function | Arithmetic operation                       | Find score and accuracy function        |
| Annie Christi and Kasthuri [27] | Linear membership, non-membership function | Arithmetic operation and ranking            | Transportation problem                  |
| Mondal and Mandal [28]    | Linear and non-linear      | Define symmetric and asymmetric PFN         | Fuzzy equation                          |

1.5. Verbal Phrases in Uncertainty Theory

An important consideration is how can we relate the concept of uncertainty theory to practical examples and what is the verbal phrase for a particular type of uncertainty.

Example 1. Suppose some mountaineers want to calculate the approximate height of a mountain range. They have different points of view after looking the mountain from different angles and in different situations so they use different kinds of uncertain parameters. The parameters might be anything from an interval number, a triangular fuzzy number, a trapezoidal fuzzy number, etc.

Example 2. Suppose some traffic sergeants want to compute the traffic intensity in a congested crossing in a certain time domain. They have different points of view of the congestion in their mind so they use different kinds of uncertain parameters. As before, we can take any one of the uncertain parameters.

The verbal phrase for different types of uncertain numbers for the above problems are shown in Table 3.
### Table 3. Verbal phrase of different uncertain parameters.

| Type of Uncertain Parameter | Verbal Phrase | Quantity Information |
|-----------------------------|---------------|----------------------|
| Interval Number             | [Low, High]   | Example 1: The height lies in the range [2000,3000] ft. |
|                             |               | Example 2: The car number is between the interval [1000,2000] cars |
| Triangular Fuzzy Number     | [Low, Medium, High] | Example 1: The height lies in the fuzzy number set [2010,2050,2150] ft. |
|                             |               | Example 2: The car number is in the fuzzy number set [1000,1050,1200] cars |
| Trapezoidal Fuzzy Number    | [Low, Low Mean, High Mean, High] | Example 1: The height lies in the fuzzy number set [2000,2205,2705,2300] ft. |
|                             |               | Example 2: The car number is in the fuzzy number set [1000,1025,1075,1200] cars |

### 1.6. Ranking and Defuzzification

The concept of ranking and the defuzzification methods for any fuzzy numbers are not novel among decision makers. But what is the basic concept of the ranking and the defuzzification methods and what are their inter-relations?

Ranking a fuzzy number involves measuring up to two fuzzy numbers, and defuzzification is a technique whereby the fuzzy number is renewed to an approximated crisp number. Just as the decision maker takes two concepts that are the same, similarly, for this problem we have to convert the fuzzy number to a corresponding crisp number and compare the number on the basis of crisp values.

### 1.7. Motivation

Fuzzy sets theory plays several significant roles in the theory of uncertainty for modeling. An important issue is that if anybody wants to take a PFN, then what should its pictorial representations (uncertainty quantification area) look like? How should we define the membership functions? From this viewpoint, we formulated different types of PFN that may be a good choice for a decision maker in a practical scenario.

### 1.8. Novelties

There are several published works where pentagonal fuzzy sets have been formulated and decision makers have applied these to various fields. However, there are still many important scopes to be worked on for PFN. A summary of the work we have done on PFN is as follows:

(i) The development of different types of interval-valued PFN, i.e., symmetric linear PFN, asymmetric linear PFN, symmetric nonlinear PFN, and asymmetric nonlinear PFN were defined.

(ii) The representation of the said PFNs in parametric form was defined.
(iii) The ranking and defuzzification of PFN were done.
(iv) The number was applied to a fuzzy game theory problem.

We can easily see by the following Figure 2 for the difference and relation between defuzzification and ranking.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure2.png}
\caption{Defuzzification and Ranking concept.}
\end{figure}

1.9. Structure of the Paper

The article is structured as follows as shown in the Figure 3:

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Figure3.png}
\caption{Structure of the paper.}
\end{figure}
2. Preliminaries

**Definition 1.** Fuzzy set: Let us take a set \( A \), which is defined by \( A = \{ (x, \mu_A(x)) : x \in A, \mu_A(x) \in [0,1] \} \). If in the pair \( (x, \mu_A(x)) \), the first one, \( x \) belongs to the classical set \( A \) and the second one, \( \mu_A(x) \) belongs to the interval \([0,1]\), then set \( A \) is called a fuzzy set. Here \( \mu_A(x) \) is called a Membership function.

**Definition 2.** Interval-valued fuzzy set (IVFS): [29] An IVFS \( A \) on \( \mathbb{R} \) is defined by
\[
A = \{ (x, (\mu_{\text{up}}(x), \mu_{\text{low}}(x)) : x \in \mathbb{R} \}
\]
where \( x \in \mathbb{R} \) and \( \mu_{\text{up}}(x), \mu_{\text{low}}(x) \) maps \( \mathbb{R} \) into \([0, \lambda]\), \( \mu_{\text{up}}(x) \) maps \( \mathbb{R} \) into \([0, \omega]\). \( \forall x \in \mathbb{R}, \mu_{\text{up}}(x) \leq \mu_{\text{low}}(x) \).

\( \lambda \) and \( \omega \) are the maximum value of upper and lower membership function, respectively.

**Definition 3.** Non-linear interval-valued fuzzy number (IVFN):[29] An IVFN is denoted by \( A = \{(a_1, b, c; \lambda), (a, b, c; \omega) ; n_1, n_2, n_3, n_4\} \)

3. PFN and Its Different Representations

In this section we extend special types of PFNs in different viewpoints.

**Definition 4.** Pentagonal fuzzy number (PFN):[28] A PFN \( A = (a_1, a_2, a_3, a_4, a_5) \) should satisfy the following condition:

1. \( \mu_A(x) \) is a continuous function in the interval \([0,1]\)
2. \( \mu_A(x) \) is strictly non-decreasing continuous function on the intervals \([a_1, a_2]\) and \([a_2, a_3]\)
3. \( \mu_A(x) \) is strictly non-increasing continuous function on the intervals \([a_3, a_4]\) and \([a_4, a_5]\)

**3.1. Linear PFN with Symmetry**

**Definition 5.** Linear PFN with symmetry: [28] A linear PFN is written as \( A_{LS} = (a_1, a_2, a_3, a_4, a_5; k) \) whose corresponding membership function is written as
\[
\mu_{A_{LS}}(x) = \begin{cases} 
  \frac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x \leq a_2 \\
  1 - (1 - k) \frac{x - a_2}{a_3 - a_2} & \text{for } a_2 \leq x \leq a_3 \\
  1 & \text{for } x = a_3 \\
  1 - (1 - k) \frac{a_4 - x}{a_4 - a_3} & \text{for } a_3 \leq x \leq a_4 \\
  \frac{k a_5 - x}{a_5 - a_4} & \text{for } a_4 \leq x \leq a_5 \\
  0 & \text{for } x > a_5 
\end{cases}
\]
Definition 6. \( \alpha \)-cut or the parametric form of linear PFN with symmetry: \cite{28} \( \alpha \)-cut or parametric form of LPFNS is written by the formulae

\[
A_\alpha = \{ x \in X | \mu_{A LS}(x) \geq \alpha \}
\]

\[
A_{1L}(\alpha) = a_1 + \frac{\alpha}{k} (a_2 - a_1) \text{ for } \alpha \in [0, k]
\]

\[
A_{2L}(\alpha) = a_2 + \frac{\alpha}{1 - k} (a_3 - a_2) \text{ for } \alpha \in [k, 1]
\]

\[
A_{2R}(\alpha) = a_4 - \frac{\alpha}{1 - k} (a_4 - a_3) \text{ for } \alpha \in [k, 1]
\]

\[
A_{1R}(\alpha) = a_5 - \frac{\alpha}{k} (a_5 - a_4) \text{ for } \alpha \in [0, k]
\]

where \( A_{1L}(\alpha) \), \( A_{2L}(\alpha) \) is the nondecreasing function with respect to \( \alpha \) and \( A_{2R}(\alpha) \), \( A_{1R}(\alpha) \) is the decreasing function with respect to \( \alpha \).

Remark 1. The main idea of the symmetric PFN is that the left picked point is equal to the right picked point (see in Figure 4 the same picked value is \( k \)).

![Figure 4. Linear PFN with symmetry.](image)

3.2. Non-Linear Pentagonal Fuzzy Number with Symmetry

Definition 7. Non-Linear PFN with symmetry: A linear PFN is written as \( \tilde{A}_{LNS} = (a_1, a_2, a_3, a_4, a_5; k)_{(n_1, n_2; m_1, m_2)} \) where the membership function is written as

\[
\mu_{\tilde{A}_{LNS}}(x) = \begin{cases} 
  k \left( \frac{x - a_1}{a_2 - a_1} \right)^{n_1} & \text{for } a_1 \leq x \leq a_2 \\
  1 - (1 - k) \left( \frac{x - a_2}{a_3 - a_2} \right)^{n_2} & \text{for } a_2 \leq x \leq a_3 \\
  1 & \text{for } x = a_3 \\
  1 - (1 - k) \left( \frac{a_4 - x}{a_4 - a_3} \right)^{m_1} & \text{for } a_3 \leq x \leq a_4 \\
  k \left( \frac{a_5 - x}{a_5 - a_4} \right)^{m_2} & \text{for } a_4 \leq x \leq a_5 \\
  0 & \text{for } x > a_5
\end{cases}
\]

Definition 8. \( \alpha \)-cut or parametric form of non-linear PFN with symmetry: \( \alpha \)-cut or parametric form of LPFNS is written by the formulae
\[
A_{\alpha} = \{ x \in X | \mu_{A_{\alpha}}(x) \geq \alpha \} 
\]

\[
= \begin{cases} 
A_{1L}(\alpha) = a_1 + \frac{\alpha}{k} (a_2 - a_1) & \text{for } \alpha \in [0, k] \\
A_{2L}(\alpha) = a_2 + \left(1 - \frac{\alpha}{k} \right) (a_3 - a_2) & \text{for } \alpha \in [k, 1] \\
A_{2R}(\alpha) = a_4 - \left(1 - \frac{\alpha}{k} \right) (a_4 - a_3) & \text{for } \alpha \in [k, 1] \\
A_{1R}(\alpha) = a_5 - \frac{\alpha}{k} (a_5 - a_4) & \text{for } \alpha \in [0, k] 
\end{cases}
\]

where \( A_{1L}(\alpha), A_{2L}(\alpha) \) are increasing functions with respect to \( \alpha \) and \( A_{2R}(\alpha), A_{1R}(\alpha) \) are decreasing functions with respect to \( \alpha \).

Remark 2. The basic idea of the mentioned number in Figure 5 is that the left picked value and right picked value are same but the boundary of the fuzzy area be supposed to not linear. The membership function can be formed as a non-linear function. So, we have to present the non-linearity on the membership function as an addition.

4. Interval-Valued PFN
4.1. Linear Interval-Valued PFN
4.1.1. Linear Pentagonal Interval-Valued Fuzzy Number with Symmetry

A linear pentagonal interval-valued fuzzy number is written as \( \tilde{A}_{LPIS} = \{(a_1, a_2, c, a_4, a_5; k, p), (b_1, b_2, c, b_4, b_5; w, q)\} \) whose upper membership function and lower membership function are defined as follows

\[
\mu_{\tilde{A}_{LPIS}}^{\text{upper}}(x) = \begin{cases} 
p \frac{x-a_1}{a_2-a_1} & \text{for } a_1 \leq x \leq a_2 \\
k - (k-p) \frac{x-a_2}{c-a_2} & \text{for } a_2 \leq x \leq c \\
k & \text{for } x = c \\
k - (k-p) \frac{a_4-x}{a_4-c} & \text{for } c \leq x \leq a_4 \\
p \frac{a_5-x}{a_5-a_4} & \text{for } a_4 \leq x \leq a_5 \\
0 & \text{for } x > a_5 
\end{cases}
\]
\[
\mu_{\Delta_{\text{LPIS}}} (x) = \begin{cases} 
q \frac{x - b_1}{b_2 - b_1} & \text{for } b_1 \leq x \leq b_2 \\
w - (w - q) \frac{x - b_2}{b_3 - b_2} & \text{for } b_2 \leq x \leq c \\
w & \text{for } x = c \\
w - (w - q) \frac{b_4 - x}{b_4 - c} & \text{for } c \leq x \leq b_4 \\
q \frac{b_5 - x}{b_5 - b_4} & \text{for } b_4 \leq x \leq b_5 \\
0 & \text{for } x > b_5 
\end{cases}
\]

4.1.2. The $\alpha$-Cut of a Linear Pentagonal Interval-Valued Fuzzy Number with Symmetry

Corresponding Alpha-cut ($\alpha$-cut) are defined for the upper and lower membership functions as follows.

The $\alpha$-cut or parametric representation of LPIS is represented by the formulae

\[
A_{\text{LPIS}} = \bigcup_{a_2} [A_{1L}^u(\alpha_2), A_{2L}^u(\alpha_2), A_{1R}^u(\alpha_2), A_{2R}^u(\alpha_2)] \bigcup \bigcup_{a_1} [A_{1L}^u(\alpha_1), A_{2L}^u(\alpha_1), A_{1R}^u(\alpha_1), A_{2R}^u(\alpha_1)]
\]

where,

\[
A_{1L}^u(\alpha_2) = a_1 + \frac{\alpha_2}{p}(a_2 - a_1) \text{ for } \alpha_2 \in [0, p] \\
A_{2L}^u(\alpha_2) = a_2 + \frac{k - \alpha_2}{k - p}(c - a_2) \text{ for } \alpha_2 \in [p, k] \\
A_{1R}^u(\alpha_2) = a_5 - \frac{\alpha_2}{p}(a_5 - a_4) \text{ for } \alpha_2 \in [0, p] \\
A_{2R}^u(\alpha_2) = a_4 - \frac{k - \alpha_2}{k - p}(a_4 - c) \text{ for } \alpha_2 \in [p, k] \\
A_{1L}^l(\alpha_1) = b_1 + \frac{\alpha_1}{q}(b_2 - b_1) \text{ for } \alpha_1 \in [0, q] \\
A_{2L}^l(\alpha_1) = b_2 + \frac{w - \alpha_2}{w - q}(c - b_2) \text{ for } \alpha_1 \in [q, w] \\
A_{1R}^l(\alpha_1) = b_5 - \frac{\alpha_1}{q}(b_5 - b_4) \text{ for } \alpha_1 \in [0, q] \\
A_{2R}^l(\alpha_1) = b_4 - \frac{w - \alpha_1}{w - q}(b_4 - c) \text{ for } \alpha_1 \in [q, w]
\]

where $A_{1L}^u(\alpha_2), A_{2L}^u(\alpha_2), A_{1L}^u(\alpha_1), A_{2L}^u(\alpha_1)$ are increasing functions with respect to $\alpha_2, \alpha_1$, respectively, and $A_{1R}^u(\alpha_2), A_{2R}^u(\alpha_2), A_{1R}^u(\alpha_1), A_{2R}^u(\alpha_1)$ are decreasing functions with respect to $\alpha_2, \alpha_1$, respectively.
Remark 3. If in Figure 6 \( a_1 = b_1, a_2 = b_2, a_4 = b_4, a_5 = b_5, k = w, p = q \) then, the interval-valued PFN becomes simply PFN.

4.1.3. Linear Pentagonal Interval-Valued Fuzzy Number with Asymmetry

A linear pentagonal interval-valued fuzzy number is written as \( \mathcal{A}_{\text{LPIAS}} = \{(a_1, a_2, c, a_4, a_5; k, p, r), (b_1, b_2, c, b_4, b_5; w, q, s)\} \) whose upper membership function and lower membership function are defined as follows

\[
\mu^{\text{LU}}_{\mathcal{A}_{\text{LPIAS}}}(x) = \begin{cases} 
\frac{x - a_1}{a_2 - a_1} & \text{for } a_1 \leq x \leq a_2 \\
\frac{k - (k - p)(x - a_2)}{c - a_2} & \text{for } a_2 \leq x \leq c \\
k & \text{if } x = c \\
\frac{(k - r)(a_4 - x)}{a_4 - c} & \text{for } c \leq x \leq a_4 \\
\frac{r}{a_5 - a_4} & \text{for } a_4 \leq x \leq a_5 \\
0 & \text{for } x > a_5 
\end{cases}
\]

and

\[
\mu^{\text{LL}}_{\mathcal{A}_{\text{LPIAS}}}(x) = \begin{cases} 
\frac{x - b_1}{b_2 - b_1} & \text{for } b_1 \leq x \leq b_2 \\
w - (w - q) \frac{x - b_2}{b_3 - b_2} & \text{for } b_2 \leq x \leq c \\
w & \text{for } x = c \\
w - (w - s) \frac{b_4 - x}{b_4 - c} & \text{for } c \leq x \leq b_4 \\
s & \text{for } b_4 \leq x \leq b_5 \\
0 & \text{for } x > b_5 
\end{cases}
\]

4.1.4. The \( \alpha \)-Cut of a Linear Pentagonal Interval-Valued Fuzzy Number with Asymmetry

Corresponding Alpha-cut (\( \alpha \)-cut) are defined for the upper and lower membership functions as follows.

The \( \alpha \)-cut or parametric form of LPIS is represented by the formulae
\[ A_{LPIASa} = \bigcup \left[ A_{1L}^u(\alpha_2), A_{1L}^l(\alpha_2), A_{1R}^u(\alpha_2), A_{1R}^l(\alpha_2) \right] \]
\[ \bigcap \bigcup \left[ A_{1L}^l(\alpha_1), A_{1L}^r(\alpha_1), A_{1R}^l(\alpha_1), A_{1R}^r(\alpha_1) \right] \]

where,

\[ A_{1L}^u(\alpha_2) = a_1 + \frac{\alpha_2}{p} (a_2 - a_1) \quad \text{for} \quad \alpha_2 \in [0, p] \]
\[ A_{1L}^l(\alpha_2) = a_2 + \frac{k - \alpha_2}{k - p} (c - a_2) \quad \text{for} \quad \alpha_2 \in [p, k] \]
\[ A_{1R}^u(\alpha_2) = a_5 - \frac{\alpha_2}{r} (a_5 - a_4) \quad \text{for} \quad \alpha_2 \in [0, r] \]
\[ A_{1R}^l(\alpha_2) = a_4 - \frac{k - \alpha_2}{k - r} (a_4 - c) \quad \text{for} \quad \alpha_2 \in [r, k] \]
\[ A_{2L}^l(\alpha_1) = b_1 + \frac{\alpha_1}{q} (b_2 - b_1) \quad \text{for} \quad \alpha_1 \in [0, q] \]
\[ A_{2L}^r(\alpha_1) = b_2 + \frac{w - \alpha_2}{w - q} (c - b_2) \quad \text{for} \quad \alpha \in [q, w] \]
\[ A_{2R}^l(\alpha_1) = b_5 - \frac{\alpha_1}{s} (b_5 - b_4) \quad \text{for} \quad \alpha_1 \in [0, s] \]
\[ A_{2R}^r(\alpha_1) = b_4 - \frac{w - \alpha_2}{w - s} (b_4 - c) \quad \text{for} \quad \alpha_1 \in [s, w] \]

where \( A_{1L}^u(\alpha_2), A_{1L}^l(\alpha_2), A_{1R}^u(\alpha_2), A_{1R}^l(\alpha_2) \) are increasing functions with respect to \( \alpha_2, \alpha_1 \), respectively, and \( A_{1R}^u(\alpha_2), A_{1R}^l(\alpha_2), A_{1L}^r(\alpha_1), A_{1L}^l(\alpha_1) \) are decreasing functions with respect to \( \alpha_2, \alpha_1 \), respectively.

**Remark 4.** If in Figure 7 \( p = r, q = s \) then, the above number becomes symmetric PFN.

4.2. Non-Linear Interval-Valued Fuzzy Number

4.2.1. Non-Linear Pentagonal Interval-Valued Fuzzy Number with Symmetry
A non-linear interval-valued pentagonal fuzzy number can be written as \( \tilde{A}_{NPIFNS} = \{(a_1, a_2, c, a_4, a_5; k, p)_{(n_1, n_2; m_1, m_2)}, (b_1, b_2, c, b_4, b_5; w, q)_{(n_1, n_2; m_1, m_2)}\} \), whose upper and lower membership function can be written as follows

\[
\mu^{U}_{\tilde{A}_{NPIFNS}}(x) = \begin{cases} 
 p \left( \frac{x - a_1}{a_2 - a_1} \right)^{n_1} & \text{if } a_1 \leq x \leq a_2 \\
 k - (k - p) \left( \frac{x - a_2}{c - a_2} \right)^{n_2} & \text{for } a_2 \leq x \leq c \\
k f x = c & \\
k - (k - p) \left( \frac{a_4 - x}{a_4 - c} \right)^{m_1} & \text{for } c \leq x \leq a_4 \\
p \left( \frac{a_5 - x}{a_5 - a_4} \right)^{m_2} & \text{for } a_4 \leq x \leq a_5 \\
0 & \text{for } x > a_5 
\end{cases}
\]

and

\[
\mu^{L}_{\tilde{A}_{NPIFNS}}(x) = \begin{cases} 
 q \left( \frac{x - b_1}{b_2 - b_1} \right)^{n_1} & \text{for } b_1 \leq x \leq b_2 \\
w - (w - q) \left( \frac{x - b_2}{c - b_2} \right)^{n_2} & \text{for } b_2 \leq x \leq c \\
w f x = c & \\
w - (w - q) \left( \frac{b_4 - x}{b_4 - c} \right)^{m_1} & \text{for } c \leq x \leq b_4 \\
q \left( \frac{b_5 - x}{b_5 - b_4} \right)^{m_2} & \text{for } b_4 \leq x \leq b_5 \\
0 & \text{for } x > b_5 
\end{cases}
\]

4.2.2. \( \alpha \)-Cut of Non-Linear Pentagonal Interval-Valued Fuzzy Number with Symmetry

Corresponding Alpha-cut (\( \alpha \)-cut) are defined for the upper and lower membership functions as follows.

The \( \alpha \)-cut or parametric form of NPIFNS is represented by the formulae

\[
A_{NLPFNS\alpha} = \bigcup_{\alpha_2} A_{1U}(\alpha_2), A_{2U}(\alpha_2), A_{1L}(\alpha_2), A_{2L}(\alpha_2)\bigcup_{\alpha_1} A_{1R}(\alpha_1), A_{2R}(\alpha_1), A_{1l}(\alpha_1), A_{2L}(\alpha_1)
\]

where,

\[
A_{1U}(\alpha_2) = a_1 + \left( \frac{\alpha_2}{p} \right)^{n_1} (a_2 - a_1) \text{ for } \alpha_2 \in [0, p]
\]

\[
A_{2U}(\alpha_2) = a_2 + \left( \frac{k - \alpha_2}{k - p} \right)^{n_2} (c - a_2) \text{ for } \alpha_2 \in [p, k]
\]

\[
A_{1L}(\alpha_2) = a_5 - \left( \frac{\alpha_2}{p} \right)^{m_2} (a_5 - a_4) \text{ for } \alpha_2 \in [0, p]
\]

\[
A_{2L}(\alpha_2) = a_4 - \left( \frac{k - \alpha_2}{k - p} \right)^{m_1} (a_4 - c) \text{ for } \alpha_2 \in [p, k]
\]

\[
A_{1R}(\alpha_1) = b_1 + \left( \frac{\alpha_1}{q} \right)^{n_1} (b_2 - b_1) \text{ for } \alpha_1 \in [0, q]
\]

\[
A_{2R}(\alpha_1) = b_2 + \left( \frac{w - \alpha_1}{w - q} \right)^{n_2} (c - b_2) \text{ for } \alpha_1 \in [q, w]
\]
\[ A_{1R}^*(\alpha) = b_5 - \left(\frac{\alpha_1}{q}\right)^{m_2} (b_5 - b_4) \text{ for } \alpha \in [0, q] \]

\[ A_{2R}^*(\alpha) = b_4 - \left(\frac{w - \alpha_2}{w - q}\right)^{m_1} (b_4 - c) \text{ for } \alpha \in [q, w] \]

where \( A_{1L}^*(\alpha_2), A_{2L}^*(\alpha_2), A_{1L}^*(\alpha_1), A_{2L}^*(\alpha_1) \) are increasing functions with respect to \( \alpha_2, \alpha_1 \), respectively, and \( A_{1R}^*(\alpha_2), A_{2R}^*(\alpha_2), A_{1L}^*(\alpha_1), A_{2L}^*(\alpha_1) \) are decreasing functions with respect to \( \alpha_2, \alpha_1 \), respectively.

Several different types of figures are given below for different values of \( n_1, n_2; m_1, m_2 \).

**Figure 8.** Non-linear interval-valued PFN with symmetry for \( n_1, n_2, m_1, m_2 > 1 \).

**Figure 9.** Non-linear interval-valued PFN with symmetry for \( n_1, n_2 > 1, m_1, m_2 < 1 \).
The above Figures 8–11 represents different verity of Non-linear interval-valued PFN with symmetry.

4.2.3. Non-Linear Pentagonal Interval-Valued Fuzzy Number with Asymmetry

A non-linear pentagonal fuzzy number can be written as $\tilde{A}_{NPIFNAS} = \{ (a_1, a_2, c, a_4, a_5; k, p, s)_{(n_1, n_2, m_1, m_2)} , (b_1, b_2, c, b_4, b_5; w, q, t)_{(n_1, n_2, m_1, m_2)} \}$, whose upper and lower membership function can be written as follows.
\( \mu_{\text{NPIFNAS}}(x) = \begin{cases} 
p \left( \frac{x - a_1}{a_2 - a_1} \right)^{n_1} & \text{if } a_1 \leq x \leq a_2 \\
k - (k-p) \left( \frac{x - a_2}{c - a_2} \right)^{n_2} & \text{if } a_2 \leq x \leq c \\
k - (k-s) \left( \frac{a_4 - x}{a_4 - c} \right)^{m_1} & \text{if } c \leq x \leq a_4 \\
s \left( \frac{a_5 - x}{a_5 - a_4} \right)^{m_2} & \text{if } a_4 \leq x \leq a_5 \\
0 & \text{if } x > a_5 \end{cases} \)

and

\( \mu_{\text{NPIFNAS}}(x) = \begin{cases} 
q \left( \frac{x - b_1}{b_2 - b_1} \right)^{n_1} & \text{if } b_1 \leq x \leq b_2 \\
w - (w-q) \left( \frac{x - b_2}{c - b_2} \right)^{n_2} & \text{if } b_2 \leq x \leq c \\
w - (w-t) \left( \frac{b_4 - x}{b_4 - c} \right)^{m_2} & \text{if } c \leq x \leq b_4 \\
t \left( \frac{b_5 - x}{b_5 - b_4} \right)^{m_2} & \text{if } b_4 \leq x \leq b_5 \\
0 & \text{if } x > b_5 \end{cases} \)

4.2.4. \( \alpha \)-Cut of Non-Linear Pentagonal Interval-Valued Fuzzy Number with Asymmetry

Corresponding Alpha-cut (\( \alpha - \text{cut} \)) are defined for the upper and lower membership functions as follows.

The \( \alpha \)-cut or parametric form of NPIFNAS is represented by the formulae

\[ A_{\text{NPIFNAS}a} = \bigcup_{a_2} \left[ A_{1L}^U(a_2), A_{1L}^U(a_2), A_{1R}^U(a_2), A_{1R}^U(a_2), A_{1L}^U(a_2) \right] \]

\[ \bigoplus \bigcup_{a_1} \left[ A_{1L}^L(a_1), A_{1L}^L(a_1), A_{1R}^L(a_1), A_{1R}^L(a_1), A_{1R}^L(a_1) \right] \]

where,

\[ A_{1L}^U(a_2) = a_1 + \left( \frac{a_2}{p} \right)^{n_1} (a_2 - a_1) \text{ for } a_2 \in [0, p] \]

\[ A_{1L}^U(a_2) = a_2 + \left( \frac{k - a_2}{k - p} \right)^{n_2} (c - a_2) \text{ for } a_2 \in [p, k] \]

\[ A_{1R}^U(a_2) = a_5 - \left( \frac{a_2}{s} \right)^{m_2} (a_5 - a_4) \text{ for } a_2 \in [0, s] \]

\[ A_{1R}^U(a_2) = a_4 - \left( \frac{k - a_2}{k - s} \right)^{m_1} (a_4 - c) \text{ for } a_2 \in [s, k] \]

\[ A_{1L}^L(a_1) = b_1 + \left( \frac{a_1}{q} \right)^{n_1} (b_2 - b_1) \text{ for } a_1 \in [0, q] \]

\[ A_{1L}^L(a_1) = b_2 + \left( \frac{w - a_1}{w - q} \right)^{n_2} (c - b_2) \text{ for } a_1 \in [q, w] \]

\[ A_{1R}^L(a_1) = b_5 - \left( \frac{a_1}{t} \right)^{m_2} (b_5 - b_4) \text{ for } a_1 \in [0, t] \]

\[ A_{1R}^L(a_1) = b_4 - \left( \frac{w - a_1}{w - t} \right)^{m_1} (b_4 - c) \text{ for } a_1 \in [t, w] \]
where $A_{1L}^u(a_2), A_{2L}^u(a_2), A_{1R}^l(a_1), A_{2R}^l(a_1)$ are increasing functions with respect to $a_2, a_1$, respectively, and $A_{1L}^d(a_2), A_{2L}^d(a_2), A_{1R}^r(a_1), A_{2R}^r(a_1)$ are decreasing functions with respect to $a_2, a_1$, respectively.

Several different types of figures are given below for different values of $n_1, n_2; m_1, m_2$.

![Figure 12](image1.png) Nonlinear interval-valued PFN with asymmetry for $n_1, n_2, m_1, m_2 > 1$.

![Figure 13](image2.png) Non-linear interval-valued PFN with asymmetry for $n_1, n_2 > 1, m_1, m_2 < 1$. 

The above Figures 12–15 represents different verity of Non-linear interval-valued PFN with symmetry.

4.3. Verbal Phrase for the PFN and Interval-Valued PFN

In the case of real-life problems such as the salary of laborers in a factory, everything depends on lots of parameters. For example, there are some laborers whose salaries are very low, some are moderately low, and some have low and moderate salaries while others have high and very high salaries owing to years of experience and their designations. To identify such distinct cases, we need the pentagonal fuzzy number in different forms. Thus, we use verbal phrases to identify its nature.

**VL-Very Low, L-Low, M-Medium, H-High, M-Medium, VH-Very High**
In this particular game problem, we use different kind of strategies for player A and B. Thus, the members of the payoff matrix are of the pentagonal fuzzy type, and we also use corresponding verbal phrases for different kinds of members. For the details about the problem see Tables 4, 5.

| Strategy | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------|-------|-------|-------|-------|
| $A_1$    | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) |
| $A_2$    | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) |
| $A_3$    | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) |
| $A_4$    | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) | (VL,M,H,M,L) |

Verbal Phrase for Interval-Valued Pentagonal Fuzzy Number:

In the case of interval-valued problems, there is a finite range for the interval whose income is low, medium or high. It is specified as very low, low, low mean, large mean, high, very high, etc. Within this finite range the membership function actually varies. Generally, it is denoted by

**VL- Very Low, L-Low, LM-Low Mean, HM-High Mean, H-High, VH-Very High**

| Strategy | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------|-------|-------|-------|-------|
| $A_1$    | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) |
| $A_2$    | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) |
| $A_3$    | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) |
| $A_4$    | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) | (VL,HM,VH,HM,L; L,L,M,VH,H,VL) |

5. Defuzzification of Linear Symmetric PFN

5.1. The Defuzzification Method

The defuzzification process is very important for a problem for two important reasons:

(i) Those who are not familiar with the fuzzy concept can relate to the result or solution.

(ii) The crispified value of the fuzzy solutions is identified.

Defuzzification is a method for constructing an irrefutable outcome in fuzzy logic, given the fuzzy sets and the equivalent degree of membership functions. There are many defuzzification skills and some of the useful tools are as follows:

(1) Centre of area (COA) [30]
(2) Largest of maxima (LOM)[31]
(3) Smallest of maxima (SOM)[32]
(4) Bisector of area (BOA) [33]
(5) Mean of maxima (MOM)
(6) Regular weighted point (RWP)
(7) Graded mean integration value (GMIV)
(8) Centre of approximated interval (COAI).

5.2. Defuzzification of Non-Linear Symmetric PFN Based on Centroid Method

We proposed a method to compute the defuzzification of non-linear symmetric PFN as,
\[ R = \frac{\int_{a_1}^{a_2} x \mu(x) \, dx + \int_{a_2}^{a_3} x \mu(x) \, dx + \int_{a_3}^{a_4} x \mu(x) \, dx + \int_{a_4}^{a_5} x \mu(x) \, dx}{\int_{a_1}^{a_2} \mu(x) \, dx + \int_{a_2}^{a_3} \mu(x) \, dx + \int_{a_3}^{a_4} \mu(x) \, dx + \int_{a_4}^{a_5} \mu(x) \, dx} \]

\[ = \frac{\int_{a_1}^{a_2} \left[ k \left( \frac{x-a_1}{a_2-a_1} \right)^{n_1} \right] \, dx + \int_{a_2}^{a_3} \left[ 1 - (1-k) \left( \frac{x-a_2}{a_3-a_2} \right)^{n_2} \right] \, dx + \int_{a_3}^{a_4} \left[ 1 - (1-k) \left( \frac{x-a_3}{a_4-a_3} \right)^{n_3} \right] \, dx + \int_{a_4}^{a_5} \left[ k \left( \frac{x-a_4}{a_5-a_4} \right)^{m_1} \right] \, dx}{\int_{a_1}^{a_2} \left[ k \left( \frac{x-a_1}{a_2-a_1} \right)^{n_1} \right] \, dx + \int_{a_2}^{a_3} \left[ 1 - (1-k) \left( \frac{x-a_2}{a_3-a_2} \right)^{n_2} \right] \, dx + \int_{a_3}^{a_4} \left[ 1 - (1-k) \left( \frac{x-a_3}{a_4-a_3} \right)^{n_3} \right] \, dx + \int_{a_4}^{a_5} \left[ k \left( \frac{x-a_4}{a_5-a_4} \right)^{m_1} \right] \, dx} \]

where,

\[ P = \int_{a_1}^{a_2} \left\{ k \left( \frac{x-a_1}{a_2-a_1} \right)^{n_1} \right\} \, dx \]

\[ + \int_{a_2}^{a_3} \left\{ 1 - (1-k) \left( \frac{x-a_2}{a_3-a_2} \right)^{n_2} \right\} \, dx \]

\[ + \int_{a_3}^{a_4} \left\{ 1 - (1-k) \left( \frac{a_4-x}{a_4-a_3} \right)^{n_3} \right\} \, dx \]

\[ + \int_{a_4}^{a_5} \left\{ k \left( \frac{a_5-x}{a_5-a_4} \right)^{m_1} \right\} \, dx \]

\[ Q = \int_{a_1}^{a_2} \left\{ k \left( \frac{x-a_1}{a_2-a_1} \right)^{n_1} \right\} \, dx \]

\[ + \int_{a_2}^{a_3} \left\{ 1 - (1-k) \left( \frac{x-a_2}{a_3-a_2} \right)^{n_2} \right\} \, dx \]

\[ + \int_{a_3}^{a_4} \left\{ 1 - (1-k) \left( \frac{a_4-x}{a_4-a_3} \right)^{n_3} \right\} \, dx \]

\[ + \int_{a_4}^{a_5} \left\{ k \left( \frac{a_5-x}{a_5-a_4} \right)^{m_1} \right\} \, dx \]

After computing we have

\[ P = k(a_2-a_1)^2 + ka_2(a_2-a_1) + \frac{(a_2^2 - a_1^2)}{2} - \frac{(1-k)(a_3-a_2)^2}{2} \frac{n_1+1}{n_1+2} - \frac{a_2(1-k)(a_3-a_2)}{m_2+2} + \frac{k(a_5-a_4)^2}{m_2+2} \frac{m_1+2}{m_1+1} \]

\[ Q = \frac{k(a_2-a_1)}{n_1+1} + k(a_3-a_2) - \frac{(1-k)(a_3-a_2)^2}{n_2+1} + k(a_4-a_3) - \frac{(1-k)(a_4-a_3)^2}{m_1+1} + k(a_5-a_4) - \frac{(1-k)(a_5-a_4)^2}{m_2+1} \]

Hence,

\[ R = \frac{L + M}{N} \]

where,

\[ L = \frac{k(a_2-a_1)^2}{n_1+1} + ka_2(a_2-a_1) + \frac{(a_2^2 - a_1^2)}{2} - \frac{(1-k)(a_3-a_2)^2}{n_1+1} - \frac{a_2(1-k)(a_3-a_2)}{n_1+1} \]

\[ M = \frac{(a_2^2 - a_1^2)}{2} + \frac{(1-k)(a_4-a_3)^2}{m_1+1} - \frac{a_4(1-k)(a_4-a_3)}{m_1+1} + \frac{k(a_5-a_4)^2}{m_2+2} + \frac{ka_5(a_5-a_4)}{m_2+2} \]
And

\[ N = \frac{k(a_2 - a_1)}{n_1 + 1} + \frac{k(a_3 - a_2)}{n_2 + 1} - \frac{(1 - k)(a_3 - a_2)}{n_2 + 1} + \frac{k(a_4 - a_3)}{m_1 + 1} \]

For the linear pentagonal fuzzy number, we have \( n_1 = 1, n_2 = 1, m_1 = 1, m_2 = 1 \)

\[ R_{\text{New}} = \frac{k(a_2 - a_1)^2}{2} + \frac{k(a_3 - a_2)^2 + (1 - k)(a_3 - a_2)}{2} + \frac{k(a_4 - a_3)^2}{2} \]

For \( k = 1, n_1 = 1, n_2 = 1, m_1 = 1, m_2 = 1 \) we have

\[ P' = \frac{a_2 + a_3 + a_5 a_4 - a_1 a_2 - a_3^2}{6} \]

\[ Q' = \frac{a_5 + a_4 - a_2 - a_1}{2} \]

So, \( R' = \frac{(a_2 + a_3 + a_5 a_4 - a_1 a_2 - a_3^2 - a_1^2)}{3(a_5 + a_4 - a_2 - a_1)} \)

5.3. Defuzzification of Linear Symmetric PFN Based on Mean of Alpha (\( \alpha \))-Cut Method

The left and right \( \alpha \)-cut of a non-linear heptagonal fuzzy number are

\[ L^{-1}(\alpha) = a_1 + \left( \frac{\alpha}{k} \right) (a_2 - a_1) \text{ for } \alpha \in [0, k] \]

\[ R^{-1}(\alpha) = a_5 - \left( \frac{\alpha}{k} \right) (a_5 - a_4) \text{ for } \alpha \in [0, k] \]

\[ L^{-1}(\alpha) = a_2 + \left( \frac{1 - \alpha}{k} \right) (a_3 - a_2) \text{ for } \alpha \in [k, 1] \]

\[ R^{-1}(\alpha) = a_4 - \left( \frac{1 - \alpha}{k} \right) (a_4 - a_3) \text{ for } \alpha \in [k, 1] \]

We proposed the mean of interval method for defuzzification as

\[ \bar{A} = \frac{1}{2} \left( \int_{a=0}^{1} (L^{-1}(\alpha) + R^{-1}(\alpha)) d\alpha \right) \]

\[ = \int_{a=0}^{k} \frac{(L^{-1}(\alpha)+R^{-1}(\alpha))d\alpha}{2} + \frac{1}{a=k} \left( L^{-1}(\alpha)+R^{-1}(\alpha) \right)d\alpha \]

\[ = \int_{a=0}^{k} \left( a_1 + \left( \frac{\alpha}{k} \right) (a_2 - a_1) \right) \left( a_5 - \left( \frac{\alpha}{k} \right) (a_5 - a_4) \right) d\alpha + \frac{1}{a=k} \left( a_2 + \left( \frac{1 - \alpha}{k} \right) (a_3 - a_2) \right) \left( a_4 - \left( \frac{1 - \alpha}{k} \right) (a_4 - a_3) \right) d\alpha \]

\[ = \frac{a_1 k^2 (a_2 - a_1) + a_5 k^2 (a_5 - a_4)}{n_1 + 1} + \frac{a_2 (1 - k)(a_3 - a_2) + a_4 (1 - k)(a_4 - a_3)}{m_1 + 1} \]

For the linear pentagonal fuzzy number, we have \( n_1 = 1, n_2 = 1, m_1 = 1, m_2 = 1 \)

Thus,
\[
\bar{A} = a_1 k + \frac{k(a_2 - a_1)}{2} + a_5 k - \frac{k(a_5 - a_4)}{2} + a_2 (1 - k) - \frac{(1-k)(a_4 - a_3)}{2} + a_4 (1 - k) + \frac{(1-k)(a_3 - a_2)}{2}
\]

For \(k = 1, n_1 = 1, n_2 = 1, m_1 = 1, m_2 = 1\), we have

\[
\bar{A} = \frac{(a_5 + a_4 + a_2 + a_1)}{4}
\]

5.4. Defuzzification of Linear Symmetric PFN Based on Removal of Area Method for Linear Pentagonal Fuzzy Number

We consider different types of areas of the corresponding linear PFN as shown below.

Figure 16. First step for the Removal of Area method

Figure 17. Second step for the Removal of Area method
Then, we find the following.

$$R_{1}(\bar{A},0) = \text{Area of shaded region for Figure 16} = \frac{(a_1 + a_2)}{2} \cdot k$$

$$R_{2}(\bar{A},0) = \text{Area of shaded region for Figure 17} = \frac{(a_2 + a_3)}{2} \cdot (1 - k)$$

$$R_{3}(\bar{A},0) = \text{Area of shaded region for Figure 18} = a_3 \cdot 1 = a_3$$

$$R_{4}(\bar{A},0) = \text{Area of shaded region for Figure 19} = \{a_4 \cdot 1 - \frac{(a_4 - a_3)}{2} \cdot (1 - k)\}$$

$$R_{5}(\bar{A},0) = \text{Area of shaded region for Figure 20} = \frac{(a_5 + a_4)}{2} \cdot k$$

Hence,

$$R(\bar{D},0) = \frac{R_{1}(\bar{A},0) + R_{2}(\bar{A},0) + R_{3}(\bar{A},0) + R_{4}(\bar{A},0) + R_{5}(\bar{A},0)}{5}$$

$$= \frac{(a_1 + a_2)}{2} \cdot k + \frac{(a_2 + a_3)}{2} \cdot (1 - k) + a_3 + \frac{(a_4 - a_3)}{2} \cdot (1 - k) + \frac{(a_5 + a_4)}{2} \cdot k$$

For $k = 1$, $R(\bar{D},0) = \frac{a_1 + a_2 + 2a_3 + 3a_4 + a_5}{10}$

5.5. Comparison of the Above Three Defuzzification Methods

We already found the analytical result for the defuzzification value for the pentagonal fuzzy number. Then, we compared the two methods numerically as follows in the Table 6:
### Remark 5
In the numerical study above, we show that calculating the defuzzification value by the centroid method and mean of alpha-cut method gives almost the same result.

### 6. Ranking for the Pentagonal Fuzzy Number

#### 6.1. Basic Concept of Ranking Fuzzy Numbers

The concept of ranking fuzzy numbers is very important for decision making problems. Researchers have presented many different reasons for finding the ranking of fuzzy numbers [34–57]. In this section we find a new concept for finding the ranking of PFNs.

#### 6.2. Ranking of Pentagonal Fuzzy Numbers

We proposed a new way to find the ranking of PFNs. In Figure 21 the pentagon is divided into three triangles and one rectangle. We can find the ranking by using the centroid formula of triangles and rectangles.

![Figure 21. Figure for PFN.](image)

Then, considering the above figure where \((a_1, 0), (a_2, k), (a_3, 1), (a_4, k), (a_5, 0)\) are the vertices of the pentagonal fuzzy numbers and \(0 < k < 1\):

- \(T_1\) represents the centroid of the corresponding triangle whose vertices are \((a_1, 0), (a_2, 0), (a_2, k)\).
- \(T_2\) represents the centroid of the corresponding triangle whose vertices are \((a_4, 0), (a_5, 0), (a_4, k)\).
- \(T_3\) represents the centroid of the corresponding triangle whose vertices are \((a_1, 1), (a_2, k), (a_4, k)\).
- \(T_4\) represents the centroid of the corresponding triangle whose vertices are \((a_2, k), (a_2, 0), (a_4, 0), (a_4, k)\).

Now, the centroids are in the form:

\[
T_1 = \left( \frac{a_1 + 2a_2 - k}{3} \right)
\]
\[ T_3 = \left( \frac{a_2 + a_3 + a_4}{3}, \frac{1 + 2k}{3} \right) \]
\[ T_2 = \left( \frac{a_5 + 2a_4}{3}, \frac{k}{3} \right) \]
\[ T_4 = \left( \frac{a_2 + a_4}{2}, \frac{k}{2} \right) \]

If we consider the average of these, we can obtain the new ranking as
\[ R(F_H) = \left( \frac{2a_1 + 9a_2 + 2a_3 + 9a_4 + 2a_5}{24}, \frac{11k + 2}{24} \right) \]

6.3. Working Rule to Find the Ranking of a Pentagonal Fuzzy Number

\( F(R) \) is a set of PFNs defined on the set of real numbers, and the ranking of PFNs is actually a function \( R: F(R) \rightarrow R \) which maps each fuzzy number into a real line.

Suppose we consider two different PFNs \( A \) and \( B \), which are to be ranked. Then using the above method and after finding the \( R(A) \) and \( R(B) \) by using the previous formula we can easily say that if

1. \( R(A) > R(B) \) then \( A > B \)
2. \( R(A) < R(B) \) then \( A < B \)
3. \( R(A) = R(B) \) then \( A \approx B \)

The flowchart of the proposed method is given below in Figure 22.

![Flowchart](image)

Figure 22. Ranking between two interval valued PFN.

6.4. Numerical Computation

| No. of Example | Components | \( R(F_H) \) | Conclusion |
|---------------|------------|--------------|------------|
| Example 1     | \( A(1,2,3,4,5;0.8) \) | (3.0,0.45) | \( B < C < B < A \) |
| Set-1         | \( B(-1,0,2,4,5;0.8) \) | (2.0,0.45) |           |
| Set-2         | \( C(0,1,3,4,5;0.8) \) | (1.0,0.45) |           |
Remark 6. By using the above concept of ranking PF’s, we can easily compare them (see Table 7).

7. Application of Pentagonal Fuzzy Number in a Game Problem in a Fuzzy Environment Using the Dominance Method

Let us consider two different players, A and B playing a game of carom, with A₁,A₂,…,Aₙ strategies and B₁,B₂,…,Bₘ strategies, respectively. We also consider that each player can choose pure strategies. Let, aₘₙ be the elements of the pay-off matrix. The strategy of player A is denoted by Aₚ and the strategy of player B is denoted by Bᵠ.

Saddle point: In a game problem, the max-min for A and min-max for B should be equal, therefore, the corresponding game is assumed to have a saddle point or an equilibrium point. The gain at the saddle point, which is the arrangement in the associated matrix where the maximum of the row minima correspond with the minimum of the column maxima, is said to be the value of the game.

We represent the max-min value of the game by gamma(γ), and the min-max value of the game by γ̅. The game is said to be fair. If γ = 0 = γ̅.

The game is supposed to be strictly ascertainable. If γ = γ̅ = γ̅.

7.1. Operation for Solving a Game Problem Using Fuzzy

Step 1: First, we have to examine whether a saddle point will exist or not. If it exists in the given problem, then we need to find it in a direct way. However, if it does not exist, then we need to follow the second step.

Step 2: Compare column strategies.
(1) In the given pay-off matrix, if the elements of Column A ≤ elements of Column B, that is, Column A strategy will fully dominate over column B strategy, then, according to the rule we need to delete column B strategy from the given pay-off matrix.

(2) In the given pay-off matrix, we tally each column strategy with all other column strategies and omit high strategies as far as possible.

Step 3: Compare row strategies.
(i) In the given pay-off matrix, if the elements of Row A ≥ elements of Row B, Row A strategy will dominate over Row B strategy. So, omit Row B strategy from the given payoff matrix.

(ii) In the given pay-off matrix, we tally each row strategy with all possible row strategies and omit low strategies as far as possible.

(iii) The game may decrease to a single cell giving an order about the value of the game and optimal master plan of the players. If not, then moves to step 4.

Step 4: The dominance rule should not be based on the dominance of pure strategies only. A given approach can be dominated if it gives us a poor result to an usual of two or more other pure master plan.

7.2. Numerical Problems

Example 1. Let us consider a fuzzy game pay-off matrix where two different players are A and B (Table 8):

Example 2

| Set-1   | Ā(−1,0,0.2,0.3,0.04;0.6) | (0.079,0.36) |
|---------|-------------------------|------------|
| Set-2   | ā(−1,−0.5,0.04,0.05;0.6) | (−0.79,0.36) |
| Set-3   | Ĉ(−1−0.6,−0.3,2,0.5;0.6) | (−0.22,0.36) |
| Set-4   | ė(−1,−0.2,−0.1,0.2,0.3;0.6) | (−0.067,0.36) |

Example 2 gives B < Ĉ < ė < Ā.
We apply the rule of defuzzification using Alpha (α)-cut method,
\[
\bar{A} = a_1k + \frac{\alpha(a_2-a_1)}{2} + a_5k - \frac{\alpha(a_5-a_4)}{2} + a_2(1-k) - \frac{(1-k)(a_2-a_3)}{2} + a_4(1-k) + \frac{(1-k)(a_5-a_2)}{2}
\]
to convert the pentagonal fuzzy number into a crisp number, then we have the modified pay-off matrix as Table 9:

| Strategy | B1 | B2 | B3 | B4 |
|----------|----|----|----|----|
| A1       | 2  | 1  | 4  | 0  |
| A2       | 3  | 4  | 2  | 4  |
| A3       | 4  | 2  | 4  | 0  |
| A4       | 0  | 4  | 0  | 8  |

Now all elements of A1 are less than or equal to A3, here, A3 dominates A1. So, according to the dominance properties, our table becomes Table 10:

| Strategy | B2 | B3 | B4 |
|----------|----|----|----|
| A2       | 3  | 4  | 2  |
| A3       | 4  | 2  | 4  |
| A4       | 0  | 4  | 0  |

Now all elements of B3 are less than or equal to elements of B1, here B1 dominates B3. So, according to the dominance properties our table becomes Table 11:

| Strategy | B2 | B3 | B4 |
|----------|----|----|----|
| A2       | 4  | 2  | 4  |
| A3       | 2  | 4  | 0  |
| A4       | 4  | 0  | 8  |

Now, the convex combination of B3, B4 (we take the average), that is, 3, 2, 4 is dominated by B2. Hence, we have Table 12,

| Strategy | B3 | B4 |
|----------|----|----|
| A2       | 2  | 4  |
| A3       | 4  | 0  |
| A4       | 0  | 8  |

Now, the convex combination of A3, A4 (we take the average), that is, 2, 4 is dominated by A2. Hence, we have Table 13,
Hence, the optimal solution to the main problem is for $A(0,0,\frac{2}{3},\frac{1}{3})$ for $B(0,0,\frac{2}{3},\frac{1}{3})$ and the corresponding game value is $\frac{6}{3}$.

**Example 2.** Let us consider another fuzzy game pay-off matrix where two different players are $A$ and $B$ for different pentagonal fuzzy numbers as a pay-off matrix member Table 14.

| Strategy | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------|-------|-------|-------|-------|
| $A_1$    | (1,2,3,4,5;0.5) | (0,1,2,3,4;0.5) | (3,4,5,6,7;0.5) | (-2,-1,0,1,2;0.5) |
| $A_2$    | (2,3,4,5,6;0.5) | (3,4,5,6,7;0.5) | (1,2,3,4,5;0.5) | (3,4,5,6,7;0.5) |
| $A_3$    | (3,4,5,6,7;0.5) | (1,2,3,4,5;0.5) | (3,4,5,6,7;0.5) | (-2,-1,0,1,2;0.5) |
| $A_4$    | (-2,-1,0,1,2;0.5) | (2,3,4,5,6;0.5) | (-2,-1,0,1,2;0.5) | (7,8,9,10,11;0.5) |

After crispification we have a pay-off matrix as Table 15.

| Strategy | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------|-------|-------|-------|-------|
| $A_1$    | 3     | 2     | 5     | 0     |
| $A_2$    | 4     | 5     | 3     | 5     |
| $A_3$    | 5     | 3     | 5     | 0     |
| $A_4$    | 0     | 4     | 0     | 9     |

Hence, the optimal solution to the main problem is for $A(0,0,\frac{9}{14},\frac{5}{14})$ for $B(0,0,\frac{9}{14},\frac{5}{14})$ and the corresponding game value is $\frac{45}{14}$.

**Example 3.** Let us consider another fuzzy game pay-off matrix where two different players are $A$ and $B$ for different pentagonal fuzzy numbers as a pay-off matrix member Table 16.

| Strategy | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------|-------|-------|-------|-------|
| $A_1$    | (2,3,4,5,6;0.5) | (2,3,4,5,6;0.5) | (4,5,6,7,8;0.5) | (-2,-1,0,1,2;0.5) |
| $A_2$    | (3,4,5,6,7;0.5) | (4,5,6,7,8;0.5) | (3,4,5,6,7;0.5) | (4,5,6,7,8;0.5) |
| $A_3$    | (4,5,6,7,8;0.5) | (2,3,4,5,6;0.5) | (4,5,6,7,8;0.5) | (-2,-1,0,1,2;0.5) |
| $A_4$    | (-2,-1,0,1,2;0.5) | (4,5,6,7,8;0.5) | (-2,-1,0,1,2;0.5) | (8,9,10,11,12;0.5) |

After crispification we have a pay-off matrix as Table 17.

| Strategy | $B_1$ | $B_2$ | $B_3$ | $B_4$ |
|----------|-------|-------|-------|-------|
| $A_1$    | 4     | 3     | 6     | 0     |
| $A_2$    | 5     | 6     | 4     | 6     |
| $A_3$    | 6     | 4     | 6     | 0     |
| $A_4$    | 0     | 5     | 0     | 10    |

Hence, the optimal solution to the main problem is for $A(0,0,\frac{5}{8},\frac{3}{8})$ for $B(0,0,\frac{5}{8},\frac{3}{8})$ and the corresponding game value is $\frac{15}{4}$. 
Remark 7. In the above problem we take different three sets of the pay-off matrix with symmetric PFN and after utilizing the concept of defuzzification we solve the game problem. We can see that corresponding optimal strategies are changed, and also the game values are changed. Thus, we can conclude that for different types of pentagonal fuzzy numbers, there will be different kinds of optimal solutions with different types of game value result.

8. Conclusions and Scope of Future Research

In this paper, we extended the characteristics of pentagonal fuzzy numbers to interval-valued fuzzy numbers and pentagonal fuzzy number. Bearing in mind the symmetric and asymmetric structures of a pentagonal shape, the membership functions were comprehensively investigated for both cases. The linear and non-linear for symmetric and asymmetric membership functions along with three cuts were also added to the discussion. The defuzzification of the pentagonal fuzzy number was carried out by three methods, namely: the centroid method, mean of Alpha \((\alpha)\)-cut method, and the removal of area method. The ranking between two IVFNswas found by the centroid formula method. Efficaciously, the produced solutions exhibited IVPFN structures that contain all the possible outcomes of the governing model. Thus, the following conclusion were made:

- The expansion of the interval-valued pentagonal fuzzy numbers and functions, adds a new tool for modeling different aspects of science, engineering and other environmental studies.
- The classifications of linear and nonlinear membership functions provided an appropriate strategy for decision makers.
- Detailed illustrations of membership functions, \(\alpha\)-cuts and ranking, and defuzzification provide all the required information in one platform to model any real-world problem.

In our forthcoming work, we will apply these definitions to other types of fuzzy numbers such as hexagonal fuzzy numbers, and diamond shape fuzzy numbers and use them to model different dynamics of applied sciences, such as multi-criterion decision making, optimization, networking problem, etc.

Author Contributions: All authors have contributed equally to this paper.

Funding: This research was financially supported by the Ministry of Education, Malaysia under FRGS Grant (Project No.: 01-01-18-2031FR).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh, L.A. Fuzzy sets. *Inf. Control* **1965**, *8*, 338–353.
2. Chang, S.S.L.; Zadeh, L.A. On fuzzy mappings and control.*IEEE Trans. Syst. Man Cybern.* **1972**, *2*, 30–34.
3. Dubois, D.; Prade, H. Operations on fuzzy numbers. *Int. J. Syst. Sci.* **1978**, *9*, 613–626.
4. Atanassov, K.T. *Intuitionistic Fuzzy Sets*; VII ITKR’s Session: Sofia, Bulgarian, 1983.
5. Smarandache, F. A Unifying Field in Logics Neutrosophy: NeutrosophicProbability;American Research Press: Rehoboth, DE, USA, 1998.
6. Dubois, D.; Prade, H. Fundamental of Fuzzy Sets. *Volume 7, The Handbooks of Fuzzy Sets*, Springer, 2000.
7. Guijun, W.; Xiaoping, L. The applications of interval-valued fuzzy numbers and interval distribution numbers.*Fuzzy Sets Syst.* **1998**, *98*, 331–335.
8. Wang, G.; Li, X. Correlation and information energy of interval-valued fuzzy number.*Fuzzy Sets Syst.* **2001**, *103*, 169–175.
9. Lin, F.T. Fuzzy job-shop scheduling based on ranking level \((\alpha, \beta)\) interval-valued fuzzy numbers.*IEEE Trans. Fuzzy Syst.* **2002**, *10*, 510–522.
10. Wei, S.H.; Chen, S.M. A new approach for fuzzy risk analysis based on similarity measures of generalized fuzzy numbers.*Expert Syst. Appl.* **2009**, *36*, 589–598.
11. Kalaichelvi, A.; Malini, P.H.; Janofer, K. Application of interval fuzzy matrices and interval valued fuzzy soft sets in the analysis of the factors influencing high scores in higher secondary examinations. *Int. J. Math. Sci. Appl.* 2012, 2, 777–780.

12. Kumar, P.; Singh, S.B. Fuzzy fault tree analysis using level (?,?,?) interval-valued fuzzy numbers. *Ind. Eng. Lett.* 2015, 5, 36–42.

13. Abirami, D.; Dinagar, D.S. On L-R type interval valued fuzzy numbers in critical path analysis. *Int. J. Fuzzy Math. Arch.* 2015, 6, 777–780.

14. Su, J.-S. Fuzzy programming based on interval-valued fuzzy numbers and ranking. *Ind. Eng. Lett.* 2015, 5, 36–42.

15. Abirami, D.; Dinagar, D.S. On L-R type interval valued fuzzy numbers in critical path analysis. *Int. J. Fuzzy Math. Arch.* 2015, 6, 77–83.

16. Kumar, P.; Singh, S.B. Fuzzy fault tree analysis using level (?,?,?) interval-valued fuzzy numbers. *Ind. Eng. Lett.* 2015, 5, 777–780.

17. Bhatia, N.; Kumar, A. Sensitivity analysis for interval valued fully fuzzy linear programming problems. *J. Appl. Res. Technol.* 2015, 10, 871–884.

18. Mondal, S.P. Differential equation with interval valued fuzzy number and its applications. *Int. J. Syst. Assur. Eng. Manag.* 2015, 7, 131–139.

19. Panda, A.; Pal, M. A study on pentagonal fuzzy number and its corresponding matrices. *Pac. Sci. Rev. B Hum. Soc. Sci.* 2015, 1, 131–139.

20. Anitha, P.; Parvathi, P. An Inventory Model with Stock Dependent Demand, two parameter Weibull Distribution Deterioration in a fuzzy environment. In Proceedings of the 2016 Online International Conference on Green Engineering and Technologies (IC-GET), 16–21 May 2016; pp. 1-8.

21. Helen, R.; Uma, G. A new operation and ranking on pentagonal fuzzy numbers. *Int. J. Math. Sci. Appl.* 2015, 5, 341–346.

22. Siji, S.; Kumari, K.S. An Approach for Solving Network Problem with Pentagonal Intuitionistic Fuzzy Numbers Using Ranking Technique. *Middle-East J. Sci. Res.* 2016, 24, 2977–2980.

23. Raj, A.V.; Karthik, S. Application of Pentagonal Fuzzy Number in Neural Network. *Int. J. Math. Appl.* 2016, 4, 149–154.

24. Dhanamandand, K.; Parimaldevi, M. Cost analysis on a probabilistic multi objective-multi item inventory model using pentagonal fuzzy number. *Glob. J. Appl. Math. Math. Sci.* 2016, 9, 151–163.

25. Pathinathan, T.; Ponnivalavan, K. Reverse order Triangular, Trapezoidal and Pentagonal Fuzzy Numbers. *Ann. Pure Appl. Math.* 2015, 9, 107–117.

26. Ponnivalavan, K.; Pathinathan, T. Intuitionistic pentagonal fuzzy number. *ARPN J. Eng. Appl. Sci.* 2015, 10, 5446–5450.

27. Christi, M.S.A.; Kasthuri, B. Transportation Problem with Pentagonal Intuitionistic Fuzzy Numbers Solved Using Ranking Technique and Russell’s Method. *Int. J. Eng. Res. Appl.* 2016, 6, 82–86.

28. Mondal, S.P.; Mandal, M. Pentagonal fuzzy number, its properties and application in fuzzy equation. *Future Comput. Inform. J.* 2017, 2, 110–117.

29. Mondal, S.P.; Mandal, M.; Bhattacharya, D. Non-linear interval-valued fuzzy numbers and their application in difference equations. *Granul. Comput.* 2018, 3, 177–189.

30. Chu, T.C.; Huang, K.S.; Chang, T.M. COA defuzzification method for evaluating Cpk under fuzzy environments. *J. Discret. Math. Sci. Cryptogr.* 2004, 7, 271–280.

31. Perumal, L.; Nagi, F.H. Largest of maximum (LOM) method for switching fuzzy control system. *Aust. J. Electr. Electron. Eng.* 2008, 4, 167–178.

32. Tóth-Laufer, E.; Takács, M. The Effect of Aggregation and Defuzzification Method Selection on the Risk Level Calculation. In Proceedings of the 10th IEEE Jubilee International Symposium on Applied Machine Intelligence and Informatics, Herlány, Slovakia, 26–28 January 2012.

33. Mondal, S.P.; Khan, N.A.; Razzaq, O.A.; Tudu, S.; Roy, T.K. Adaptive strategies for system of fuzzy differential equation: Application of arms race model. *J. Math. Comput. Sci.* 2018, 18, 192–205.

34. Abbasbandy, S.; Asady, B. Ranking of fuzzy numbers by sign distance. *Inform. Sci.* 2006, 176, 2405–2416.

35. Abbasbandy, S.; Hajjari, T. A new approach for ranking of trapezoidal fuzzy numbers. *Comput. Math. Appl.* 2009, 57, 413–419.
36. Abbasbandy, S.; Hajjari, T. An improvement on centroid point method for ranking of fuzzy numbers. *J. Sci. IAU* **2011**, *7*, 109–119.
37. Asady, B. The revised method of ranking LR fuzzy number based on deviation degree. *Expert Syst. Appl.* **2010**, *37*, 5056–5060.
38. Chen, S.J.; Chen, S.M. A new method for handling multicriteria fuzzy decision making problems using FN-IOWA operators. *Cybern. Syst.* **2003**, *34*, 109–137.
39. Chen, S.J.; Chen, S.M. Fuzzy risk analysis based on the ranking of generalized trapezoidal fuzzy numbers, *Appl. Intell.* **2007**, *26*, 1–11.
40. Chen, S.-M.; Chen, J.-H. Fuzzy risk analysis based on the ranking of generalized fuzzy numbers with different heights and different spreads. *Expert Syst. Appl.* **2009**, *36*, 6833–6842.
41. Deng, Y.; Liu, Q. A TOPSIS-based centroid index ranking method of fuzzy numbers and its application in decision-making. *Cybern. Syst.* **2005**, *36*, 581–595.
42. Deng, Y.; Zhu, Z.F.; Liu, Q. Ranking fuzzy numbers with an area method using ofgyration. *Comput. Math. Appl.* **2006**, *51*, 1127–1136.
43. Hajjari, T. Ranking of fuzzy numbers based on ambiguity degree. *Aust. J. Basic Appl. Sci.* **2011**, *5*, 62–69.
44. Hajjari, T. On deviation degree methods for ranking fuzzy numbers. *Aust. J. Basic Appl. Sci.* **2011**, *5*, 750–758.
45. Wang, Z.-X.; Liu, Y.-J.; Fan, Z.-P.; Feng, B. Ranking L-R fuzzy numbers based on deviation degree. *Inform. Sci.* **2009**, *176*, 2070–2077.
46. Chen, L.H.; Lu, H.W. An approximate approach for ranking fuzzy numbers based on left and right dominance. *Comput. Math. Appl.* **2001**, *41*, 1589–1602.
47. Chen, L.H.; Lu, H.W. The preference order of fuzzy numbers. *Comput. Math. Appl.* **2002**, *44*, 1455–1465.
48. Liu, X.W.; Han, S.L. Ranking fuzzy numbers with preference weighting function expectation. *Comput. Math. Appl.* **2005**, *49*, 1455–1465.
49. Cheng, C.H. A new approach for ranking fuzzy numbers by distance method. *Fuzzy Sets Syst.* **1998**, *95*, 307–317.
50. Chu, T.; Tsao, C. Ranking fuzzy numbers with an area between the centroid point and orginal point. *Comput. Math. Appl.* **2002**, *43*, 111–117.
51. Wang, Y.J.; Lee, H.S. The revised method of ranking fuzzy numbers with an area between the centroid and original points. *Comput. Math. Appl.* **2008**, *55*, 2033–2042.
52. Halgamuge, S.; Runkler, T.; Glesner, M. On the neural defuzzification methods. In Proceeding of the 5th IEEE International Conference on Fuzzy Systems, New Orleans, LA, USA, 8–11 September 1996; pp. 463–469.
53. Song, Q.; Leland, R.P. Adaptive learning defuzzification techniques and applications. *Comput. Math. Appl.* **1996**, *31*, 321–329.
54. Yager, R.R. Knowledge-based defuzzification. *Fuzzy Sets Syst.* **1996**, *80*, 177–185.
55. Filev, D.P.; Yager, R.R. A generalized defuzzification method via BADD distributions. *Int. J. Intell. Syst.* **1991**, *6*, 687–697.
56. Jiang, T.; Li, Y. Generalized defuzzification strategies and their parameter learning procedure. *IEEE Trans. Fuzzy Syst.* **1996**, *4*, 64–71.
57. Ramon, E. *Moore, Methods and Applications of Interval Analysis*; Studies in Applied and Numerical Mathematics; SIAM: Philadelphia, PA, USA, 1979.

© 2019 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (http://creativecommons.org/licenses/by/4.0/).