MARKET EFFECTS OF LOYALTY AND COST FACTORS IN A
PRICE DISCRIMINATION ENVIRONMENT

Theja Tulabandhula, Aris Ouksel
Information and Decision Sciences, University of Illinois at Chicago

February 22, 2021

ABSTRACT

Product cost heterogeneity across firms and loyalty models of customers are two topics that have garnered limited attention in prior studies on competitive price discrimination. Costs are generally assumed negligible or equal for all firms, and loyalty is modeled as an additive bias in customer valuations. We extend these previous treatments by considering cost asymmetry and a richer class of loyalty models in a game-theoretic model involving two asymmetric firms. Here firms may incur different non-negligible product costs, and customers can have firm-specific loyalty levels. We characterize the effects of loyalty levels and product cost difference on market outcomes such as prices, market share and profits. Our analysis and numerical simulations shed new light on market equilibrium structures arising from the interplay between product cost difference and loyalty levels.

Keywords Customer loyalty, price discrimination, product cost, Markov equilibrium.
1 Introduction

Firms may incur different product costs due to various reasons including but not limited to differences in logistics, production, marketing, sales, distribution, service, technology, financial administration, information resources and general administration [Cooper and Kaplan, 1988]. The relationship between product cost and prices, and therefore profits, is well-understood in practice. However, an established approach in competitive price discrimination studies is to assume that product cost is either negligible or equal for all firms [Shaffer and Zhang, 2000; Chen et al., 2001; Ouksel and Eruysal, 2011]. While this assumption may simplify mathematical derivations and allows one to focus on other aspects of competition, it obscures a fundamental question as to the impact of product cost asymmetry on price discrimination, and therefore competition in a segmented market. Well-established price discrimination definitions in the literature clearly display importance of product cost. For example, Stugler [1987] argues that price discrimination exists when two similar products are sold at prices that are in different ratios to their marginal costs. Similarly, Philips [1983] states that price discrimination occurs when the same provider sells two varieties of the same commodity at different net prices, where net price is simply the price of a commodity adjusted to take into account the product cost. Even though definitions of Stugler [1987] and Philips [1983] are widely accepted and emphasize the importance of product cost, the models used to study competitive price discrimination ignore product cost as either negligible or equal for all firms. This in fact rules out the possibility that product cost has any impact on market equilibrium conditions.

Loyalty of customers is a key driver that determines much of the pricing and marketing decisions of a large variety of firms in today’s world. In industries such as internet services, telephone services and subscription services (such as Spotify, Netflix or Overleaf), customers regularly switch between different providers. This is also evident in markets such as cloud services (e.g., AWS or Google Cloud). Impact of such customer loyalty behavior, which manifests in terms of switching costs when customers make purchases, has received some attention in the literature [Somaiini and Einav, 2013; Rhodes, 2014; Villas-Boas, 2015; Cabral, 2016]. As evidenced by these studies, loyalty characteristics of consumers play a pivotal role in how firms make their decisions, for instance, how they price their goods in the market. Much of prior literature which takes loyalty into account works specifically with simple additive models of such behavior. These works also omit the time varying characteristics of loyalty levels. Both these simplistic assumptions artificially limit the nature of market outcomes that are possible. For instance, as we show in this work, loyalty models can interact with product cost asymmetry to reveal new market characteristics that hitherto have been under-explored or completely missed in the literature.

Our work posits that taking both cost asymmetry and loyalty models into account has clear implications on competition. Thus, our goal is to study the market effects of a combination of loyalty and cost factors in a competitive price discrimination environment. We consider a game theoretic model with two asymmetric firms where firms incur different product costs and customers exhibit varying degrees of loyalty levels, where the latter is parametrized by a general parametric model that subsumes prior works such as [Somaiini and Einav, 2013; Rhodes, 2014; Villas-Boas, 2015]. Firms view the market as composed of two homogeneous submarkets: a strong submarket where loyal customers are ex-ante willing to pay a premium; and a weak submarket where customers ex-ante prefer to purchase from a rival firm. One firm’s strong submarket is its rival’s weak submarket. The loyalty levels are stochastic and not necessarily similar in the two submarkets. We show that product cost difference and the loyalty levels in the two submarkets are both important factors in driving competition, and thus determine firm prices, market shares and profits.

Our key contributions are as follows:

- We provide a comprehensive study of the impact of product cost asymmetry and loyalty levels on competition in single and multi-period (specifically, infinite horizon) settings. The case where product cost is considered to be negligible or equal across firms represents only a special case. Further, we also show that assuming marginal prices (or net prices) misses capturing several of these impacts on market outcomes. This is primarily due to the way the costs and loyalty levels interact with each other to determine equilibria, as well as the fact that we can explicitly capture temporal cost trends in the infinite horizon setting (e.g., variations in recurring expenses, increase or decrease in costs due to changes in the underlying technologies etc.).

- In the single stage setting, customers are a priori loyal to one of the two firms according to a parameterized stochastic loyalty model. We show the value of considering costs explicitly in addition to pricing, and identify multiple market structures that depend on the relationship between costs and loyalty levels. These results are derived for both the newly introduced multiplicative loyalty model (Section 4) and the previously known additive loyalty model (Section 5). For instance, for the multiplicative setting, we identify four regimes, which are characterized by how the costs relate to the loyalty model parameters, and derive explicit market share and profit values at equilibrium in each regime. Three of these regimes have not been investigated before.

- In the infinite horizon setting, firms are forward looking, and customers purchase in each period. We evaluate market outcomes such as profits and market shares when both loyalty characteristics and product costs
remain fixed as well as when they evolve over time. These results are based on the notion of a Markov equilibrium [Maskin and Tirole, 2001], and extend and complement the results in [Somaini and Einav, 2013], Rhodes [2014] and Villas-Boas [2015]. For instance, customers are short-lived (e.g., two periods) in these prior works unlike our setting.

Our analytical results are complemented by numerical simulations to cover cases where closed form expressions for market outcomes are hard to derive. Note that these results capturing the impact of cost asymmetry and rich loyalty modeling can be easily extended to multiple firms, and can also take into account product differentiation by situating the products on a Hotelling line [Hotelling, 1990]. To summarize, our main contribution in this paper is the introduction of product cost in a game-theoretic price discrimination model that captures a fairly general consumer loyalty behavior, and the analysis of the impact of product cost differences and loyalty characteristics on competition, prices and market share.

2 Literature Review

Like [Stigler, 1987] and [Phelps, 1983], we argue that having different prices in the market does not necessarily imply price discrimination. We use Phelps [1983]'s definition of price discrimination in this study and argue that price discrimination exists due to different net prices: price of a product minus its cost. Previous competitive price discrimination studies assumed that product cost is either considered negligible or equal for all firms. Therefore, price and net price are considered to be equivalent in those studies and product cost is implicitly assumed to have no impact on competition. This however, leads to discarding some important market equilibrium structures parameterized by product cost differences and loyalty models of consumers, as shown in this paper. Works such as Shaffer and Zhang [2000], Chen et al. [2001], Oukssel and Erulyal [2011] studied price discrimination for firms that incurred the same product cost, and thus avoided the issue of evaluating the effect of product cost differences on prices, market share and firm profitability. Product costs are not necessarily the same in our setting, and thus competition takes place on both product costs and profits, raising the question of the effect of a marginal product cost change on prices and market share.

Dewan et al. [2003] and Liu and Serfes [2005] use Salop's Model [Salop, 1979] to study price discrimination. In this model, points distributed around a circle represent both products/firms and customers. A customer incurs a transportation cost to acquire a product, which is proportional to the distance between the customer and the product/firm. Differences between transportation costs in Salop's model can be considered as capturing loyalty, as the farther away firm has to match the difference in transportation costs to make a sale. Other works such as Somaini and Einav [2013], Rhodes [2014], Villas-Boas [2015], Cabral [2016] define loyalty as switching costs which are additive in nature. Building on these previous foundational works, we work with a fairly general model of loyalty and describe various market structures that arise due to cost differences interacting with the loyalty model. Analogous to our work, the interaction between loyalty levels and market entry difference between two firms was the focus of [Demirhan et al., 2007], although there the loyalty model considered is still a simplistic additive variant.

Unlike prior works such as Farrell and Klemperer [2007], Villas-Boas [2015], we consider an infinite horizon for both the firms and customers to ensure that there no end-of-game effects. Somaini and Einav [2013] consider an infinite horizon oligopolistic dynamic price competition with switching costs (loyalty) and study customer retention and acquisition strategies, proving the existence and uniqueness of a specific Markov equilibrium. Their loyalty model is additive and subsumed by our more general analysis. While they do explicitly take product costs into account, the focus of the paper is not on exhibiting how these costs influence the equilibrium prices. Moreover, the customers are short-lived, living for two time periods. Following this line of work, Rhodes [2014] also considers a two period setting with a very similar analysis for a duopoly. Unlike both these, we remove explicit product differentiation (Hotelling line) in order to better isolate the nature of loyalty effects on market outcomes. While Rhodes [2014] wants to answer the question of whether switching costs increase prices, our goals are focused on the impact of costs and loyalty on absolute prices and other market outcomes. Finally, we focus on customer segmentation, through the notions of strong and weak sub-markets, which are markedly absent from these two prior works.

Cabral [2016] considers a setting similar to ours under an additive switching cost structure and focused on the impact of switching costs on market outcomes ignoring product cost differences. We expand their analysis to multiplicative as well as a more general loyalty model, while at the same time characterizing additional equilibrium outcomes due to the interplay between product cost differences and loyalty, a phenomenon ignored in their analysis. In Choe et al. [2018], the authors consider a two-period duopoly setting where firms price in the first period without discrimination (i.e., there are no strong or weak sub-markets a priori unlike our setting) as a result of which market shares are established. In the second period, the firms post both a poaching price for their weak sub-markets as well as a personalized price for their strong sub-markets. Similar to our setting, Kehoe et al. [2018] consider an infinite horizon duopoly game with multiple products, where two firms interact with a single buyer. Firms price their varieties taking into account the
ex-ante probability that the product is desired by the customer and update it using Bayes rule. Firms also choose which variety to offer in each period. Equilibrium conditions that dictate the prices and their effects on the efficiency of the market are considered, and while the decision space is richer, effects of cost asymmetry and loyalty are completely sidelined.

3 Competition under Product Cost Asymmetry and Loyalty

In this section, we introduce a general model that captures loyalty behavior and product cost asymmetry. We first discuss the single stage setting, where firms and customers interact in a single period, followed by the infinite horizon setting, where the firms and customers interact with each other over multiple periods. In both settings, we explicitly capture: (a) loyalty, which is a customer’s preference towards a firm/product that measures the extent to which she is impervious to pricing enticements by a rival firm, and (b) product costs, which are the costs incurred by firms to produce a unit of the (single) product that they are offering in the market. Under these two effects, we characterize the resultant market structure and price competition. In Sections 4 and 5 we specialize the general loyalty model considered here, and obtain crisp results in two important and practically motivated scenarios.

3.1 Single Stage Setting

Firms with Costs: There are two firms, A and B that produce a similar product. Their product costs are \( c_A \geq 0 \) and \( c_B \geq 0 \) respectively. We denote the prices offered by A to two sets of customers as \((p^A_\alpha, p^B_\alpha)\): customers in set \( \alpha \) ex-ante prefer A (and thus belong to its strong sub-market) and are offered \( p^A_\alpha \), and customers in set \( \beta \) ex-ante prefer B (so they belong to A’s weak sub-market) and are offered \( p^B_\alpha \). Analogously, firm B offers prices \((p^B_\beta, p^A_\beta)\).

Customers with Loyalty: Each customer purchases exactly one unit (the market is assumed to be covered) and belongs to exactly one of the two sets: \( \alpha \) and \( \beta \), which expresses their ex-ante preference for one firm over the other. If she belongs to set \( \alpha \), she can buy from the non-preferred firm depending on her idiosyncratic loyalty level towards firm A (e.g., if firm A charges a large premium). In other words, a customer may purchase from her non-preferred firm if the preferred firm decides to charge a premium higher than her loyalty level can tolerate. Because of such loyalty effects, her inclination to purchase from the non-preferred firm depends on how much discount she can obtain by switching. Consider a generic customer belonging to set \( \alpha \). Her idiosyncratic utilities (which are influenced by her loyalty to each of the firms) from purchasing the product from firms A and B are \( U^A_\alpha - p^A_\alpha \) and \( U^B_\alpha - p^B_\alpha \) respectively. We will model the net loyalty-aware valuation by this customer for the product by firm A (i.e., the difference \( U^A_\alpha - U^B_\alpha \)) using a random variable \( \xi \) supported on the real line \( \mathbb{R} \). That is, let \( U^A_\alpha - U^B_\alpha = g_\alpha(\xi) \), where \( g_\alpha() \) is a scalar (potentially non-linear) invertible function that parameterizes the loyalty level. This customer will purchase from firm A if the realization of random variable \( \xi \) is such that \( g_\alpha(\xi) \geq p^A_\alpha - p^B_\alpha \). That is, if the price premium charged by her preferred firm is less than the her loyalty level, then she tolerates the premium and buys from her preferred firm. Otherwise, she switches to her non-preferred firm.

Thus, her ex-ante probability of purchasing from A is given by \( 1 - F(h_\alpha(p^A_\alpha - p^B_\alpha)) \), where \( F \) is the distribution function associated with \( \xi \) (we do not assume a finite or bounded support for random variable \( \xi \) a priori) and the scalar function \( h_\alpha() \) is the inverse of \( g_\alpha() \). For notational convenience, let \( \xi^\alpha = h_\alpha(p^A_\alpha - p^B_\alpha) \) be a fixed scalar threshold. Thus, the probability of purchase can be re-expressed as \( 1 - F(\xi^\alpha) \) (we hide the dependence of \( \xi^\alpha \) on \( p^A_\alpha \) on \( p^B_\alpha \) when the context is clear). Analogously, the ex-ante probability that a customer from set \( \beta \) purchases the product from firm B is given by \( 1 - F(\xi^\beta) \), where the fixed scalar threshold \( \xi^\beta = h_\beta(p^B_\beta - p^A_\beta) \).

Assumption 1. We make the following assumptions: (i) We assume that the firms have prior access to, or are able to learn/estimate the loyalty model functions \( g_\alpha \) and \( g_\beta \). (ii) In addition to the customers knowing their membership in sets \( \alpha \) and \( \beta \), we assume that the firms have the ability to classify all customers in the market as being in set \( \alpha \) or \( \beta \) perfectly. (iii) Firms have the ability to offer different prices to different customers (e.g., \( p^A_\alpha \) and \( p^\beta_\alpha \) to their strong and weak sub-markets respectively). (iv) The product costs are such that \( c_A \geq c_B \geq 0 \). By symmetry, the case where \( c_A < c_B \) will not be examined. (v) The functions \( h_\alpha() \) and \( h_\beta() \) are differentiable.

Even though a firm may have less information about the loyal customers of its rival, in many cases it can still obtain enough information from internal and external data sources to predict loyalty function parameters (i.e., parameters of \( g_\alpha \) and \( g_\beta \)). This does not assume that the firms know the individual customer’s idiosyncratic loyalty level, as is the case in several related studies, including [Shaffer and Zhang][2000]. The question of how firms build customer loyalty itself (e.g., though branding and marketing exercises) and estimate the parameters of the loyalty model is not a primary focus of this paper, and we assume that this capability exists and allows firms to price-discriminate among the customers present in the market. The knowledge of which customers belong to which sub-market can be considered as a relatively
Firm’s Objective: The objective of the firms is to post prices that maximize their profits in the presence of competition. Each firm has a single product (with infinite inventory) to sell and offers two prices, one to their loyal following (namely, their strong sub-market), another one to their rival’s loyal customers (namely, their weak sub-market). Given the above demand functions, firm A’s profit maximization problem can be written as:

\[
\max_{p_A^\alpha \geq c_A, p_B^\alpha \geq c_A} (p_A^\alpha - c_A)\theta(1 - F(\xi^\alpha)) + (p_B^\beta - c_A)(1 - \theta)F(\xi^\beta).
\]

(1)

It is easy to see that the problem above is separable across the two price variables, yielding two 1-dimensional problems. An analogous optimization problem can be written for firm B.

Intuitively, if a firm charges a premium to its loyal following, some of its least loyal customers (the ones with relatively low idiosyncratic loyalty levels) end up making purchases from its competitor. A higher premium improves profit margin; however, the firm’s market share for its loyal following also shrinks. Therefore, a firm should be mindful of the trade off between market share and profit margin. Further, the firm cannot charge a premium for its weak sub-market (its rival’s loyal customers) as those customers already do not prefer its product. Therefore, the firm has to undercut its rival for its weak sub-market. If the firm offers a substantial lower price compared to its rival, its market share for its weak sub-market improves. However, its profit margin declines. Again, the firm should be mindful of the trade off between market share and profit margin. In our setting, firms independently and simultaneously determine their pricing strategies in a non-cooperative game. In this game, an equilibrium strategy profile (here, the four prices above constitute a strategy profile) is such that neither firm can improve its profits by unilaterally changing its own set of prices. Our problem is essentially a general sum game with an infinite strategy space for each player. The following proposition states that if an equilibrium strategy exists, it satisfies the following first order conditions.

**Proposition 1.** The following implicit equations must be satisfied by any unconstrained equilibrium solution of prices offered by the firms A and B:

\[
\begin{align*}
    p_A^\alpha &= c_A + \frac{1 - F(\xi^\alpha)}{f(\xi^\alpha)h_\alpha(p_A^\alpha - p_B^\alpha)}, \\
    p_B^\alpha &= c_A + \frac{F(\xi^\beta)}{f(\xi^\beta)h_\beta(p_B^\beta - p_A^\alpha)}, \\
    p_A^\beta &= c_B + \frac{1 - F(\xi^\beta)}{f(\xi^\beta)h_\beta(p_B^\beta - p_A^\beta)}, \text{ and} \\
    p_B^\beta &= c_B + \frac{F(\xi^\alpha)}{f(\xi^\alpha)h_\alpha(p_A^\alpha - p_B^\beta)},
\end{align*}
\]

where \( \xi^\alpha = h_\alpha(p_A^\alpha - p_B^\alpha) \) and \( \xi^\beta = h_\beta(p_B^\beta - p_A^\beta) \) respectively.

The proposition above can be used to prove the existence of an unique equilibrium for specific choices of the loyalty function (for example, see Sections 4.1 and 5.1 and the distribution of the underlying random variable \( \xi \), as long as the
prices are unconstrained (i.e., they are not limited by the costs). In reality, the prices are constrained by costs when the probabilities of purchase approach 0 or 1. Additionally, cost lower bounds can also become binding constraints if the first derivatives of inverses of loyalty functions, viz., \( h'_\alpha(p^\alpha_A - p^\alpha_B) \) and \( h'_\beta(p^\beta_B - p^\beta_A) \) become negative over their domains.

Constraints on prices can also manifest from customer behavior. For instance, for any realistic loyalty model, we should ensure that if \( p^\beta_B > p^\alpha_A \), then the probability of a customer from set \( \alpha \) purchasing from firm \( B \) is zero, which leads to a constraint on the price space. This happens for instance, when \( F \) is supported between \([0, \infty)\) and \( g_\alpha, g_\beta \) are the identity functions. In this sense, the value \( \max(0, p^\alpha_A - p^\beta_B) \) is the discount being offered by firm \( B \) to its weak sub-market and similarly, \( \max(0, p^\beta_B - p^\alpha_A) \) is the discount being offered by firm \( A \) to its weak sub-market.

It is precisely due to these binding constraints, which can manifest due to the loyalty functions and their relationship with the product costs, that we obtain new market structures, with potentially different equilibria. This is explored in detail in Sections 4 and 5 for two important loyalty models. These new market structures would not be easily discernible if one directly uses net prices (e.g., \( p^\alpha_A - c_A \)) in their analysis. As we will see later, the relationship between the optimal prices and the product cost differences is indeed nonlinear, and depends on their interactions with the loyalty model parameters.

Given an equilibrium strategy profile \( (p^\alpha_A, p^\beta_A, p^\alpha_B, p^\beta_B) \), the ex post market shares and profits in the single stage setting can be easily recovered from Equation 1. For example, if \( F \) is uniform between \([0, 1]\) and \( g_\alpha, g_\beta \) are the identity functions, the size of the ex post strong sub-market of firm \( A \) decreases proportionally to the difference in prices \( p^\alpha_A \) and \( p^\beta_B \) that customers loyal to \( A \) observe. Assuming \( p^\alpha_A > p^\beta_B \), the higher the difference, the larger is the premium being charged by firm \( A \), and smaller is the ex post market share.

### 3.2 Infinite Horizon Setting

**Forward Looking Firms:** In this setting, firms \( A \) and \( B \) price their products and customers respond by deciding their purchases, in a repeated fashion over an infinite horizon. In each period, when the firms make their current pricing decisions, they can consider future profits. In this sense, the firms can be forward looking. In particular, firms \( A \) and \( B \) discount their future profits using scalar valued time invariant discount factors \( \delta_A, \delta_B \in [0, 1] \) respectively. Since we assume that both firms know whether a given customer is in their strong sub-market or weak sub-market (see Assumption 1), they can focus on competing for business with each given customer, independent of other customers. Their profit from every customer in the market can then be aggregated to get the overall trends in profits, market share, etc. The need for a firm to be forward looking is rooted in the idea that they can initially sway the customers in their weak sub-market by posting low enough prices and then charge premium prices once they have become part of their strong sub-market. In this sense, the firms are willing to price lower today to have the option of pricing higher tomorrow, thus being able to reap larger aggregated profits overall. Further, it is natural to assume that the firms follow the principle of *time value of money*, where they value the profit obtained today higher than the same level of profit in the future, which results in the discount factors defined above.

While in each period \( t \), a customer can either belong to set \( \alpha \) or to set \( \beta \), their loyalty level for that period, captured using the underlying random variable \( \xi \), is assumed to be i.i.d across time. This is not a limitation because the loyalty functions \( g_\alpha \) and \( g_\beta \) (which take \( \xi \) as an input) depend on the ex-ante membership of the customer in sets \( \alpha \) and \( \beta \) respectively, thus exhibiting a *Markov property*. For a generic customer, let her loyalty random variables across time be denoted by the sequence \( \{\xi_i\} \), where \( i \in \{\alpha, \beta\} \) depending on which set she was part of immediately before time period \( t \). If she belongs to the set \( \alpha \) initially, then \( i \) at time \( t = 0 \) is equal to \( \alpha \).

We start with the setting where the firms are myopic, followed by the setting where the firms are forward looking. In each of these settings, we aim to derive characterizations of the prices that firms \( A \) and \( B \) offer in steady state in each period, along with their resulting profits and market shares.

#### 3.2.1 Myopic Firms

Let \( \delta_A = \delta_B = 0 \), i.e., let both the firms be myopic. Assuming that initially a customer was part of firm \( A \)'s strong sub-market, her probability of purchasing from \( A \) at time period \( t = 1 \) is given by \( 1 - F(\xi^\alpha) \) (see Section 3.1). This probability depends on the prices chosen by the two firms at \( t = 1 \). Depending on the outcome (which is driven by the realization \( \xi_A(1) \)), she can either remain in the set \( \alpha \) or move to set \( \beta \). If she does move to set \( \beta \) at the end of \( t = 1 \) by virtue of purchasing from firm \( B \), then her probability of purchasing from set \( A \) at \( t = 2 \) now changes to \( F(\xi^\beta) \), which depends on the prices set by the two firms at \( t = 2 \).
In the single stage setting (Section 3.1), the optimal prices of both firms did not depend on the initial market shares in their strong sub-markets ($\theta$ and $1 - \theta$ respectively). This property is carried over to the infinite horizon setting with myopic firms, and hence the prices will remain the same in every period as long as other quantities (such as $c_A, c_B, g_\alpha$ and $g_\beta$) remain the same. On the other hand, the market shares of both firms change over time, as shown below. We will use $\theta_t$ to denote the time-varying size/market share of firm A’s strong sub-market. Let $p_A^\alpha$, $p_A^\beta$, $p_B^\alpha$ and $p_B^\beta$ denote the optimal steady state prices in any given period (they are invariant to time index $t$). Then the expected profit function of time $t$ are as follows.

**Lemma 1.** Given $p_A^\alpha, p_A^\beta, p_B^\alpha$ and $p_B^\beta$, if $F(\xi^\alpha) + F(\xi^\beta) \in (0, 2)$, then the market share of firm A at the end of time period $t$ is given as:

$$\theta_t = \theta(1 - F(\xi^\alpha) - F(\xi^\beta))^t + F(\xi^\beta) \frac{1 - (1 - F(\xi^\alpha) - F(\xi^\beta))^t}{F(\xi^\alpha) + F(\xi^\beta)},$$

where $\theta$ is the initial market share at $t = 0$, $\xi^\alpha = h_\alpha(p_A^{\alpha *} - p_B^{\alpha *})$ and $\xi^\beta = h_\alpha(p_B^{\beta *} - p_A^{\beta *})$. The market share of firm B at the end of time period $t$ is simply $1 - \theta_t$. Further,

$$\theta_\infty = \frac{F(\xi^\beta)}{F(\xi^\alpha) + F(\xi^\beta)}.$$

As shown above, the eventual market shares of firms A and B do not depend on the initial market shares. The size of the set of customers loyal to firm A is proportional to the ratio $F(\xi^\beta)/(F(\xi^\alpha) + F(\xi^\beta))$. This ratio is closer to 1 if $F(\xi^\beta)$ is closer to 1 and $F(\xi^\alpha)$ is closer to 0. In other words, if the prices $p_A^\alpha, p_A^\beta, p_B^\alpha$ and $p_B^\beta$ are such that the price $p_B^\beta$ charged by firm B to its strong sub-market is much larger (i.e., a large premium) compared to the price offered by its rival, it leads to a large value for $F(\xi^\beta)$. At the same time, if the price $p_A^\beta$ charged by firm A to its strong sub-market is comparable to the that offered by its competitor, it leads to a small value for $F(\xi^\alpha)$. If the firms were symmetric (i.e., $c_A = c_B$) then $p_A^\beta = p_B^\beta$ and $p_A^\alpha = p_B^\alpha$ and $\theta_\infty = 1/2$, giving equal market shares to both firms. The expected profit of each firm at time $t$ is simply the previous period market share multiplied by the current period’s marginal price. Thus, the profit for firm A at time $t$ is:

$$(p_A^{\alpha *} - c_A)\theta_{t-1}(1 - F(h_\alpha(p_A^{\alpha *} - p_B^{\alpha *})) + (p_A^{\beta *} - c_A)(1 - \theta_{t-1})F(h_\beta(p_B^{\beta *} - p_A^{\beta *})),

where $\theta_t$ is given in Lemma 1.

**3.2.2 Forward Looking Firms**

Next, we consider the case when $\delta_A, \delta_B \in (0, 1)$. In this case, the firms need to compute prices in each period that take into account the expected value of future profits. We start with focusing on a single myopic customer, which can be aggregated later to get the market shares and overall profits. The expected long-term value that firm A obtains by making a potential sale to a customer in its strong and weak sub-markets is as follows:

1. Firm A can make a sale to a customer currently in its strong sub-market at a price $p_A^\alpha$ to obtain the following expected long-term value:

$$V_A^\alpha(p_A^\alpha) = (1 - F(\xi^\alpha))(p_A^\alpha - c_A + \delta_A V_A^{\alpha *}) + F(\xi^\alpha)\delta_A V_A^{\beta *},$$

where $V_A^{\alpha *}$ is the optimal value obtained by firm A in the next time-step if this customer remains in its strong sub-market and $V_A^{\beta *}$ is the optimal value obtained by firm A from the next time-step if the customer moves to its weak sub-market.

2. Firm A can make a sale to a customer currently in its weak sub-market at a price $p_A^\beta$ to obtain the following expected long-term value:

$$V_A^\beta(p_A^\beta) = F(\xi^\beta)(p_A^\beta - c_A + \delta_A V_A^{\alpha *}) + (1 - F(\xi^\beta))\delta_A V_A^{\beta *}.$$

Similarly, we can write the expected long-term value functions of firm B, one each for a customer in its strong and weak sub-markets as follows:

$$V_B^\alpha(p_B^\alpha) = (1 - F(\xi^\alpha))(p_B^\alpha - c_B + \delta_B V_B^{\alpha *}) + F(\xi^\alpha)\delta_B V_B^{\beta *},$$

and

$$V_B^\beta(p_B^\beta) = F(\xi^\beta)(p_B^\beta - c_B + \delta_B V_B^{\beta *}) + (1 - F(\xi^\beta))\delta_B V_B^{\alpha *}.$$

For the rest of this section, we make the following assumption:
Assumption 2. We assume that the cost constraints \((p_A^a \geq p_A^\alpha \geq c_A\) and \(p_B^\beta \geq p_B^\alpha \geq c_B\)) and order constraints \((p_A^a \geq p_B^\alpha\) and \(p_B^\beta \geq p_A^\alpha\)) are non-binding.

While this assumption does not hold in general when firms solve for value maximizing steady-state prices, especially for arbitrary product costs and loyalty functions, it allows for partial understanding of the mechanics of the infinite horizon game between the two firms, which we tackle first. We will relax the above assumption for specific loyalty models in Sections 4.2 and 5.2. For now, we characterize the necessary conditions that any equilibrium strategy profile should satisfy under this assumption, as shown below.

Proposition 2. Under Assumption 2, the optimal prices should satisfy the following implicit equations:

\[
\begin{align*}
p_A^a &= c_A + \frac{1 - F(\xi^\alpha)}{f(\xi^\alpha)h_\alpha(p_A^\alpha - p_B^\alpha)} - \delta_A(V_A^\alpha - V_A^{\alpha*}), \\
p_A^\alpha &= c_A + \frac{F(\xi^\beta)}{f(\xi^\beta)h_\alpha(p_A^\beta - p_B^\beta)} - \delta_A(V_A^\alpha - V_A^{\alpha*}), \\
p_B^\beta &= c_B + \frac{1 - F(\xi^\beta)}{f(\xi^\beta)h_\beta(p_B^\beta - p_A^\beta)} - \delta_B(V_B^{\beta*} - V_B^\beta), \quad \text{and} \\
p_B^\alpha &= c_B + \frac{F(\xi^\alpha)}{f(\xi^\alpha)h_\beta(p_B^\alpha - p_A^\alpha)} - \delta_B(V_B^{\beta*} - V_B^\alpha).
\end{align*}
\]

Again, note that the candidate equilibrium prices that we obtained in Proposition 2 may become invalid when cost and order constraints (see Assumption 2) are imposed. When we compare the above result to the unconstrained prices in Proposition 1, we observe that each candidate optimal price has an additional additive term that is the discounted difference of expected long term value of customers in a firm’s strong sub-market and its weak sub-market.

Substituting these candidate prices back into the value function expressions in Equations 5-8, we can get the following equations for the difference of optimal values (i.e., the difference between the optimal value from the strong sub-market and the weak sub-market), one for each firm:

\[
\begin{align*}
V_A^{\alpha*} - V_A^\alpha &= \frac{1}{1 - \delta_A + \delta_A(F(\xi^\alpha) + F(\xi^\beta))} \left( (1 - F(\xi^\alpha))(p_A^\alpha - c_A) - F(\xi^\beta)(p_A^\beta - c_A) \right), \\
V_B^{\beta*} - V_B^\alpha &= \frac{1}{1 - \delta_B + \delta_B(F(\xi^\alpha) + F(\xi^\beta))} \left( (1 - F(\xi^\beta))(p_B^\beta - c_B) - F(\xi^\alpha)(p_B^\alpha - c_B) \right).
\end{align*}
\]

Further, the optimal values can also be obtained by solving the following system of equations:

\[
\begin{align*}
(1 - \delta_A(1 - F(\xi^\alpha)))V_A^{\alpha*} - \delta_A F(\xi^\alpha)V_A^\alpha &= (1 - F(\xi^\alpha))(p_A^\alpha - c_A), \\
-\delta_A F(\xi^\beta)V_A^\alpha + (1 - \delta_A(1 - F(\xi^\beta)))V_A^{\beta*} &= F(\xi^\beta)(p_A^\beta - c_A), \\
(1 - \delta_B(1 - F(\xi^\beta)))V_B^{\beta*} - \delta_B F(\xi^\beta)V_B^\beta &= (1 - F(\xi^\beta))(p_B^\beta - c_B), \quad \text{and} \\
-\delta_B F(\xi^\alpha)V_B^\beta + (1 - \delta_B(1 - F(\xi^\alpha)))V_B^{\alpha*} &= F(\xi^\alpha)(p_B^\alpha - c_B).
\end{align*}
\]

Similarly, substituting the expressions of the candidate prices in the definition of \(\xi^\alpha\) and \(\xi^\beta\), we get:

\[
\begin{align*}
\xi^\alpha &= h_\alpha \left( c_A - c_B + \frac{1 - 2F(\xi^\alpha)}{f(\xi^\alpha)h_\alpha(p_A^\alpha - p_B^\alpha)} - \delta_A(V_A^\alpha - V_A^{\alpha*}) + \delta_B(V_B^{\beta*} - V_B^\beta) \right), \\
\xi^\beta &= h_\beta \left( c_B - c_A + \frac{1 - 2F(\xi^\beta)}{f(\xi^\beta)h_\beta(p_B^\beta - p_A^\beta)} - \delta_B(V_B^{\beta*} - V_B^\beta) + \delta_A(V_A^{\alpha*} - V_A^{\alpha*}) \right).
\end{align*}
\]

It is important to note that these thresholds depend not only on the product costs but also on the parameterization of the loyalty functions. Depending on the primitives \(g_a(), g_\beta()\), \(F, c_A\) and \(c_B\), prices and market shares when viewed as functions of costs and loyalty parameters can be highly non-linear and/or discontinuous.

So far, we have shown how firms can maximize long term value in the infinite horizon setting, and obtained necessary conditions that a candidate strategy profile should satisfy in order to maximize profits when pricing constraints are non-binding. In the rest of the paper, we make another key regularity assumption about one of the primitives, namely the distribution function \(F\), as shown below.
Assumption 3. We assume that the random variable $\xi$ is such that $F(\xi)$ and $F(\xi)^{-1}$ are strictly increasing functions of $\xi$ in its domain.

Under Assumptions 2 and 3, Equations (13) and (16) can be used to show that the candidate optimal prices obtained by maximizing the value functions in Equations (5) and (8) constitute a unique Markov equilibrium [Maskin and Tirole, 2001] of the infinite horizon non-cooperative game between the two firms. We will establish this key result and derive specific function forms for the optimal prices in two specific loyalty settings (see Sections 4.2 and 5.2), which we also complement with numerical results. These results will together demonstrate comprehensively, the non-trivial effects of loyalties and costs on market outcomes.

4 Cost Asymmetry and Multiplicative Loyalty

In the multiplicative loyalty model (ML), we assume that the loyalty function has the following form: $g_\alpha(\xi) = l_\alpha \xi$ (its inverse is given by $h_\alpha(y) = y/l_\alpha$) where parameter $l_\alpha \in \mathbb{R}_+$. Thus, given prices $p_A^\alpha$ and $p_B^\alpha$, the probability of a customer belonging to the set $\alpha$ purchasing from firm $A$ is $1 - F(\xi^{\alpha})$, where $\xi^{\alpha} = (p_B^\alpha - p_A^\alpha)/l_\alpha$. When the support of random variable $\xi$ is restricted to $[0,1]$, the loyalty model provides two insights. First, the parameter $l_\alpha$ (as well as $l_\beta$) can be interpreted as the maximum loyalty level a customer can have. Second, it also suggests the following constraint on pricing: if a customer is offered a higher price by non-preferred firm, then she only purchases from its loyal firm. On the other hand, if she is offered a very high price by her preferred firm relative to the non-preferred firm (normalized by her maximum loyalty level), then she will definitely not purchase from her loyal firm. Since customers have varying degrees of loyalty levels, very loyal customers will tolerate higher premiums. For the rest of this section, we will assume that $F$ is uniform on $[0,1]$ (extension to other invertible distribution functions is straightforward).

4.1 Single Stage Setting

Recall that in this setting, the firms compete only once. The demand functions for the strong and weak sub-markets of firm $A$ under the ML model are as follows:

$$D_A^{ss}(p_A^\alpha, p_B^\alpha) = \theta \left( 1 - F\left( \frac{p_A^\alpha - p_B^\alpha}{l_\alpha} \right) \right),$$

and

$$D_A^{ws}(p_A^\beta, p_B^\beta) = (1 - \theta) F\left( \frac{p_B^\beta - p_A^\beta}{l_\beta} \right).$$

Under the choices made for $F$, $g_\alpha$ and $g_\beta$ above, our analysis reveals four distinct price discrimination classes based on the interplay of maximum loyalty levels ($l_\alpha$ and $l_\beta$) and the magnitude of product cost difference ($c_A - c_B$). Equilibrium conditions are determined for each of the following sub-cases in Propositions 3-6 below, which are mutually exclusive and exhaustive (see Figure 1):

- Region I: $l_\beta \leq c_A - c_B \leq 2l_\alpha$ (see Proposition 3).
- Region II: $c_A - c_B \geq \min\{2l_\alpha, l_\beta\}$ (see Proposition 4).
- Region III: $c_A - c_B \leq \min\{2l_\alpha, l_\beta\}$ (see Proposition 5).
- Region IV: $2l_\alpha \leq c_A - c_B \leq l_\beta$ (see Proposition 6).

**Proposition 3.** Under Region I, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms $A$ and $B$ for the ML model with $F \sim U[0,1]$ are as follows:

$$p_A^\alpha = \frac{1}{3}(2c_A + c_B + 2l_\alpha), \quad p_A^\beta = c_A, \quad p_B^\beta = c_A, \quad \text{and} \quad p_B^\beta = \frac{1}{3}(2c_B + c_A + l_\alpha).$$

**Proposition 4.** Under Region II, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms $A$ and $B$ for the ML model with $F \sim U[0,1]$ are as follows:

$$p_A^\alpha = c_A, \quad p_A^\beta = c_A, \quad p_B^\beta = c_A, \quad \text{and} \quad p_B^\beta = c_A - l_\beta.$$  

**Proposition 5.** Under Region III, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms $A$ and $B$ for the ML model with $F \sim U[0,1]$ are as follows:

$$p_A^\alpha = \frac{1}{3}(2c_A + c_B + 2l_\alpha), \quad p_A^\beta = \frac{1}{3}(2c_A + c_B + l_\beta),$$

and

$$p_B^\beta = \frac{1}{3}(2c_B + c_A + 2l_\beta), \quad \text{and} \quad p_B^\beta = \frac{1}{3}(2c_B + c_A + l_\alpha).$$
Proposition 6. Under Region IV, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms A and B for the ML model with $F \sim U[0,1]$ are as follows:

$$
\begin{align*}
    p^\alpha_A &= c_A, \\
    p_B^\alpha &= \frac{1}{3} (2c_A + c_B + l_\beta), \\
    p_B^\beta &= \frac{1}{3} (2c_B + c_A + 2l_\beta), \\
    p_A^\alpha &= c_A - l_\alpha.
\end{align*}
$$

What is the impact of product costs and loyalty levels on market equilibrium? The results above show that the game is in equilibrium regardless of the product cost difference and the loyalty model parameters. Figure 1 shows how the four regions related to each other. Both x- and y-axes represent $c_A - c_B$, the horizontal dashed line corresponds to cases where $l_\beta = c_A - c_B$, and the vertical dashed line is for the case $2l_\alpha = c_A - c_B$. The figure captures the entire space of possible combinations of product cost differences divided into regions based on their relationship with the loyalty model parameters. The interplay of these two aspects (cost asymmetry and loyalty) together determines market equilibrium conditions, and we elaborate on them below.

4.1.1 Discussion

Our analysis above clearly illustrates that ignoring product cost in competitive price discrimination studies leads to disregarding a large number of realistic competitive price discrimination market equilibria. Previous studies dealt only with the case represented by the origin in this graph, i.e., the case where product cost difference $(c_A - c_B)$ is zero (see Region III). Region III represents the cases where product cost difference is small compared to the maximum loyalty levels (recall that this interpretation is true when $F \sim U[0,1]$) to have any significant impact on the market structure. The two firms are able to sell to each other’s strong and weak sub-markets. To the best of our knowledge, many of the previous competitive duopolistic price discrimination studies – where product cost is either ignored or the difference in product costs is negligible – can be grouped within this class.

We cannot expect the same outcome for the other regions. In Region I, the high-cost firm is able to sell its product to some of its loyal following in its strong sub-market, but it is unable to make any inroads into its weak sub-market. The low-cost firm is able to prevent penetration of its rival into its strong sub-market. In Region IV, the high-cost firm is able to sell into its weak sub-market, but, it cannot do the same for its own loyal customers in its strong sub-market. Region II represents the scenario where the low-cost firm drives its rival out of business.

As mentioned earlier, the well-established approaches in competitive price discrimination literature, i.e., those assuming product cost as either negligible or equal for all firms, rule out the cases in Regions I, II or IV, and yet their market equilibrium conditions are significantly different in price, market share and profitability than in Region III, as seen above. Thus, Regions I, II and IV, which together represent a large class of competitive price discrimination market equilibrium cases, allow for a clearer understanding of the impact of costs and the assumed loyalty model on the market structure.
(a) Prices seen by firm A’s strong sub-market
(b) Prices seen by firm A’s weak sub-market
(c) Market share
(d) Profit
(e) Probability of purchase

Figure 2: Single stage market outcomes under multiplicative loyalty: Optimal prices, market shares, profits of firms, and probability of purchase of customers of types $\alpha$ and $\beta$ as a function of $c_A - c_B$. Here, $c_B = 0, l_\alpha = 4, l_\beta = 3$ and $\theta = 0.8$.

**High-Cost Firm’s Strong Sub-market:** We now discuss how firm prices and market shares change with product costs, focusing on the high-cost firm’s strong sub-market. As this is the strong market for the high-cost firm (A), it constitutes a loyal following that can pay a premium for its product. However, the high-cost firm is disadvantaged due to its high product cost at the same time. Even though the high-cost firm is the preferred firm for the customers in this set, it allows its rival to make inroads into its loyal customer base. This is not solely due to its disadvantage in product costs. Even for the case where both firms have exactly the same costs ($c_A = c_B = 0$), firm A cannot undercut firm B to retain all of its loyal customers. Firm A can maximize its profit by charging a higher premium to its loyal customers. Therefore, some of its least loyal customers cannot tolerate the premium and switch to the rival firm. This scenario shows a natural trade-off between market share and prices: the premium charged by a firm may be increased only to the point where loss in market share starts having a negative impact on overall profitability.

Figure 2 illustrates firm prices with varying degrees of product cost difference. While x-axis represent $c_A - c_B$, y-axis is prices. The origin corresponds to studies assuming product cost as either negligible or equal for all firms. The solid lines show firm prices. The dashed vertical line represents the point where the cost difference is equal to twice the loyalty parameter $l_\alpha$. When the cost difference is less than $2l_\alpha$, firms A and B charge $\frac{1}{3}(2c_A + c_B + 2l_\alpha)$ and $\frac{1}{3}(2c_B + c_A + l_\alpha)$ respectively. Firm A, which is the high-cost firm, is also a high-priced firm in this sub-market. Quadrant I and III in Figure 1 correspond to the case where $c_A - c_B < 2l_\alpha$. Here, high-cost firm is able to charge a premium to some of its loyal customers. An increase in high-cost firm’s product cost will cause both firms to raise their prices. For a $\delta$ increase in firm A’s product cost, high-cost firm (firm A) increases its prices by $\frac{2}{3}\delta$. On the other hand, low-cost firm (firm B) increments its price by $\frac{1}{3}\delta$. As a change in product cost affects the prices differently, high-cost firm increases its price, lowers its profit margin and reduces market share. As firm A’s product cost keeps going up, its price will increase; however, the change in price does not match the increases in its costs. Therefore, the high-cost firm’s profit margin slowly diminishes. At the same time, its market share decreases as its rival doesn’t increase its prices as much as the high-cost firm does. Therefore, more customers will switch to the low-cost firm. Thus, the low-cost firm improves its price, profit margin and market share at the same time. When product cost difference reaches $2l_\alpha$, the high-cost firm is driven out of business in its strong sub-market. Even though, customers prefer its products, the high-cost firm cannot offer affordable prices anymore. When the cost difference is equal or greater than $2l_\alpha$, firms A and B charge $c_A$ and $c_A - l_\alpha$ respectively. Quadrant II and IV in Figure 1 correspond to the case where $c_A - c_B \geq 2l_\alpha$. Low-cost firm undercuts its rival and captures all of the customers in the market. As the product cost difference is considerably high, firm A cannot protect its loyal customer base.

There are three important implications for high-cost firm’s strong sub-market. First, low-cost firm does not capture all of its rival’s loyal customers when product cost difference reaches $l_\alpha$. Why would the low-cost firm not undercut
its rival and capture all of its rival’s loyal customer when it has the ability to do it? After all, high-cost firm can no longer block its rival when product cost difference reaches \( l_A \). And the answer is that both firms are mindful of the trade-off between price and market share. Low-cost firm can undercut its rival to increase its market share; however, the discounts offered are only to the point where the decrease in product price along with increase in market share starts having a negative impact on overall profitability. Second, a change in product cost does not translate into a perfectly correlated adjustment of the prices. Recall that the high-cost firm increases its prices by \( \frac{2}{3} \delta \) for a \( \delta \) increase in its product cost. In other words, firm \( A \) absorbs some of the increases in product cost. Third, and most importantly, customer loyalty, which is captured by parameter \( l_A \), is extremely vital for the survival of high-cost firm. As the loyalty level goes up, the vertical line where high-cost firm is driven out of business starts to move to the right. High-cost firm is able to tolerate greater difference in product costs.

**High-Cost Firm’s Weak Sub-market:** We next look at the high-cost firm’s weak sub-market. As this is the weak market for high-cost firm, customers prefer its rival’s products. Moreover, high-cost firm is also disadvantaged due to its high product cost. Figure 2b illustrates firm prices with varying degrees of product cost difference. While x-axis represents \( c_A - c_B \), y-axis is prices. Similar to Figure 2a, the origin corresponds to previous studies where product cost as either negligible or equal for all firms. The solid lines again show firms’ prices. The dashed vertical line is \( c_A - c_B = l_B \). The dashed vertical line in Figure 2b is the same dashed horizontal line in Figure 1. Firm \( B \), which is the low-cost firm, is the high-price firm in this sub-market. When the cost difference is less than \( l_B \), firms \( A \) and \( B \) charge \( \frac{1}{3} (2c_A + c_B + l_B) \) and \( \frac{1}{3} (2c_B + c_A + 2l_B) \) respectively.

Quadrants III and IV in Figure 1 correspond to the case where \( c_A - c_B < l_B \). The low-cost firm is able to charge a premium to some of its loyal customers. An increase in high-cost firm’s product cost will cause both firms to raise their prices. Similar to the previous case, high-cost firm increases its prices by \( \frac{2}{3} \delta \) for a \( \delta \) increase in its product cost. On the other hand, low-cost firm increases its price by \( \frac{1}{3} \delta \). The change in high-cost firm’s product cost also plays a similar role as it did in high-cost firm’s strong sub-market. High-cost firm’s price will go up; however, the change in price does not match the increase in its costs. Its profit margin slowly diminishes. At the same time, its market share decreases as its rival doesn’t increase its prices as much as the high-cost firm does. Therefore, more customers will switch to the low-cost firm, which improves its price, profit margin and market share at the same time. When product cost difference reaches \( l_B \), high-cost firm is driven out of business in its weak sub-market. As low-cost firm is customers’ preferred brand in this segment and has a cost advantage, it can retain all of its loyal following by matching the high-cost firm’s price.

When the cost difference is equal or greater than \( l_B \), firms \( A \) and \( B \) charge the same price \( c_A \). Quadrants I and II in Figure 1 correspond to the case where \( c_A - c_B \geq l_B \). Low-cost firm does not have to undercut its rival to retain its loyal customers in this case, as all of the customers in this segment already prefer the low-cost firm. As the product cost difference is considerably high, low-cost firm is able to protect its loyal customer base. There is one important implication for the high-cost firm’s weak sub-market. First, even though the low-cost firm is the preferred firm for the customers in this segment, it allows its rival to make inroads into its loyal customer base. Why would the low-cost firm allow its rival to make inroads into its loyal customer base? After all, as the preferred brand for the customers in this segment, low-cost firm has flexibility to offer the same price to retain all of its loyal customers. Yet, our results show that low-cost firm is willing to lose some of its loyal customers. In other words, low-cost firm can increase its profit simply by charging a higher premium to its loyal customers. As a consequence, some of its least loyal customers may find the premium so high that they switch to the high-cost firm. This is again due to the trade-off between market share and prices.

### 4.2 Infinite Horizon Setting

As in Section 3.2, we first give results for the setting where \( \delta_A = \delta_B = 0 \), and then characterize equilibrium prices in the unconstrained setting when \( \delta_A = \delta_B (= \delta_F \), a common discount value) \( > 0 \).

#### 4.2.1 Myopic Firms

As before, let \( \theta \) be the initial market share at \( t = 0 \). Recall that the optimal prices as derived in Propositions 3-6 remain valid in this setting. Thus, following Lemma 1, we can derive market shares and profits under the ML model as shown below.

**Lemma 2.** The market share of firm \( A \) at any time index \( t \), namely \( \theta_t \), under the multiplicative loyalty model with \( F \sim U[0, 1] \) is given as \( \theta_t = \theta_1 \eta_1^t + \eta_2 (1 - \eta_1^t) \), where \( \eta_1 \) and \( \eta_2 \) are defined as follows.
• Region I: \( \eta_1 = \frac{2}{3} - \frac{c_A - c_B}{3\alpha} \), and \( \eta_2 = 0 \). Further, \( \theta_\infty = 0 \). In the special case when \( c_A = c_B \), \( \theta_t = \theta(\frac{2}{3})^t \) and \( \theta_\infty = 0 \).

• Region II: \( \eta_1 = 0 \), and \( \eta_2 = 0 \). Further, \( \theta_\infty = 0 \).

• Region III:

\[
\eta_1 = \frac{1}{3} + \left( \frac{1}{3\beta} - \frac{1}{3\alpha} \right)(c_A - c_B), \quad \text{and} \quad \eta_2 = \frac{(l_\beta - (c_A - c_B))}{3\beta(1 - \eta_1)}.
\]

Further, \( \theta_\infty = \eta_2 \). There are two special cases: (a) when \( c_A = c_B \), \( \theta_t = \frac{\theta}{3} + \frac{1}{3}(1 - \frac{1}{3^t}) \) and \( \theta_\infty = \frac{1}{2} \); and (b) when \( l_\alpha = l_\beta, \theta_t = \frac{\theta}{3} + \frac{1}{2}(1 - \frac{c_A - c_B}{l_\alpha})(1 - \frac{1}{3^t}) \) and \( \theta_\infty = \frac{1}{2}(1 - \frac{c_A - c_B}{l_\beta}) \).

• Region IV:

\[
\eta_1 = \frac{c_A - c_B - l_\beta}{3l_\beta}, \quad \text{and} \quad \eta_2 = \frac{l_\beta - (c_A - c_B)}{4l_\beta - (c_A - c_B)}.
\]

Further, \( \theta_\infty = \eta_2 \).

In the special case when \( c_A = c_B \), \( \theta_t = \theta(\frac{1}{3})^t + \frac{1}{2}(1 - (\frac{1}{3})^t) \) and \( \theta_\infty = \frac{1}{3} \).

The market share of firm B at the end of time period \( t \) is simply \( 1 - \theta_t \) (similarly at steady state it is \( 1 - \theta_\infty \)).

From the above, it is immediately clear that the steady state market shares of the high-cost firm are zero in Regions I and II. In Regions III and IV, the market share depends on the loyalty parameters and the cost asymmetry. For instance, in Region IV, which corresponds to \( 2l_\alpha \leq c_A - c_B \leq l_\beta \), the steady state market share of firm A depends on how much smaller the product cost difference is when compared to the loyalty level of its weak sub-market. If the loyalty level \( l_\beta \) is large, then its market share is small. Further, if its own product cost is large, then this also decreases its market share. Noticeably, its market share does not depend on the maximum loyalty level of its loyal customers.

4.2.2 Forward Looking Firms

It is harder to characterize succinctly the equilibrium conditions for the infinite horizon in general. Below, we ignore constraints on prices (as discussed in Section 3.2) and show that in this case, it is indeed possible to achieve a unique Markov equilibrium. Further, the result is obtained for any distribution function \( F \) that satisfies Assumption 3.

**Proposition 7.** For the multiplicative loyalty model, under Assumption 3 there exists a unique Markov equilibrium when \( \delta_A = \delta_B = \delta_F > 0 \), where firms price based on whether the customer bought their product in the immediate preceding period. This equilibrium is characterized by the following fixed point equations for thresholds \( \xi^\alpha \) and \( \xi^\beta \):

\[
\left( \xi^\alpha - \frac{c_A - c_B}{l_\alpha} \right) \left( \frac{1 - \delta_F}{\delta_F} + F(\xi^\beta) + 1 \right) + \frac{2F(\xi^\alpha) - 1}{f(\xi^\alpha)} \left( \frac{1 - \delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta) \right) + \frac{F(\xi^\alpha)}{f(\xi^\alpha)} = \frac{(1 - F(\xi^\beta))l_\beta}{f(\xi^\beta)l_\alpha} - F(\xi^\beta) \left( \frac{l_\beta}{l_\alpha} \xi^\alpha + \frac{c_A - c_B}{l_\alpha} \right),
\]

and

\[
\left( \xi^\beta - \frac{c_B - c_A}{l_\beta} \right) \left( \frac{1 - \delta_F}{\delta_F} + F(\xi^\alpha) + 1 \right) + \frac{2F(\xi^\beta) - 1}{f(\xi^\beta)} \left( \frac{1 - \delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta) \right) + \frac{F(\xi^\beta)}{f(\xi^\beta)} = \frac{(1 - F(\xi^\alpha))l_\alpha}{f(\xi^\alpha)l_\beta} - F(\xi^\alpha) \left( \frac{l_\alpha}{l_\beta} \xi^\alpha + \frac{c_B - c_A}{l_\beta} \right).
\]

The above thresholds can be used in conjunction with Equations 9 to 14 to obtain the optimal prices, profits and resulting market shares. Because the thresholds are implicitly defined, in the following, we numerically solve for them in order to obtain the dependence of key market metrics on cost asymmetry and loyalty parameters.
Figure 3: Infinite horizon setting market outcomes under multiplicative loyalty where the constraints are non-binding: Optimal prices, market shares, profits of firms, and probability of purchase of customers of types $\alpha$ and $\beta$ as a function of $c_A - c_B$. Here, $c_B = .2, l_\alpha = 3, l_\beta = 4, \delta_F = .4$.

4.2.3 Discussion

In Figure 3, we plot the prices, market shares and profits of firms when $\delta_A = \delta_B = \delta_F = 0.4$ and prices are not constrained. That is, the values of costs and the loyalty parameters are chosen such that the constraints on prices are non-binding. From Figures 3a and 3b, we can infer that the prices vary linearly with different slopes as the cost asymmetry increases. Further, as the cost difference between firm $A$ and firm $B$ increases, firm $A$ loses significant market share and profit, as its myopic loyal consumers increasingly prefer to purchase from its rival instead.

When the prices are constrained, the situation changes quite a bit (see Figure 4). For example, in Figure 4b, we can observe that $p_\beta^B$ converges to $p_\alpha^A$ in a nonlinear way. Similarly, the rate of change of market share and profit as a function of cost asymmetry is also non-linear (Figures 3c and 3d). For the plots in Figure 4, numerical computation of equilibria is performed using a dynamic stochastic game solver that uses the Homotopy method [Eibelshäuser and Poensgen, 2019].

The gist of the computational strategy is as follows. Instead of solving for the Markov equilibrium using, say, Kuhn-Tucker conditions, we simplify the underlying non-linear optimality equations using logit choice. In particular, we discretize the action space and assume that the action probabilities take a logit/softmax form. It turns out that the corresponding logit Markov quantal response equilibrium (QRE) is easier to solve computationally. We solve for QREs at various temperatures (similar to the simulation annealing procedure used for global optimization) that control the steepness of the logit functions. As the temperature parameter approaches infinity, the solution concept approximates the Markov equilibrium. Thus, the QRE solutions are linked together by the Homotopy method from the numerical analysis literature to get the desired Markov equilibrium.

For both the constrained and the unconstrained instances, we omit characterization of distinct regions (due to the non-availability of closed-form expressions describing the boundaries as seen in Section 4.1 Figure 1). Nonetheless, the nonlinear trends of various market outcomes seen in Figure 4 provides convincing evidence of the non-trivial impact that cost asymmetry and the assumed loyalty model can have.
Figure 4: Infinite horizon setting market outcomes using numerical simulations under multiplicative loyalty where the constraints are binding: Optimal prices, market shares, profits of firms, and probability of purchase of customers of types $\alpha$ and $\beta$ as a function of $c_A - c_B$. Here, $c_B = 0.2$, $l_\alpha = 3$, $l_\beta = 4$, $\delta_F = 0.4$. In contrast to Figure 3, the equilibrium is computed for cost differences up to 4 units.

5 Cost Asymmetry and Additive Loyalty

In the additive loyalty model (AL), the loyalty function is given by $g_\alpha(\xi) = \xi + s_\alpha$ (its inverse is given by $h_\alpha(y) = y - s_\alpha$). Thus, given prices $p_\alpha^A$ and $p_\beta^B$, the probability of a customer belonging to the set $\alpha$ purchasing from firm $A$ is $1 - F(\xi^\alpha - p_\alpha^A - p_\beta^B - s_\alpha)$. The parameter $s_\alpha \geq 0$ can be interpreted as the bias in the loyalty level (which is driven by $s_\alpha$ and additionally by the random variable $\xi$ as well). For instance, if $E[\xi] = 0$, then $s_\alpha$ represents the overall non-random loyalty or inclination of a customer from set $\alpha$ to purchase from firm $A$. From a different point of view, if $\xi$ is supported on the interval $[-B, B]$ for some positive scalar $B$, then $B + s_\alpha$ can be interpreted as the maximum loyalty level exhibited by any customer. The parametric model $g_\alpha(\xi) = \xi + s_\alpha$ has been used in prior work such as Somaini and Einav [2013], Rhodes [2014], Villas-Boas [2015] and Cabral [2016], where the symbol $s$ (without market segment subscript) is used and is referred to as the sub-market agnostic switching cost. We will comment on the differences between our approach and these prior works whenever relevant below, although note that our unified general treatment of the loyalty in this paper (with multiple parametric models of which the additive version is but one) and their impact on market outcomes in conjunction with non-zero product costs significantly extends these prior works. Similar to Section 4, we will assume that $F$ is uniform on $[0, 1]$ for the rest of this section unless otherwise noted (extension to other invertible distribution functions is straightforward).

5.1 Single Stage Setting

Just as before, in this setting, there is a one-shot competition between the firms. Specializing the demand functions to the AL model gives us the following expressions (the demand function related to the firm $B$ is analogous):

\[
D_\alpha^A(p_\alpha^A, p_B^\beta) = \theta(1 - F(p_\alpha^A - p_B^\beta - s_\alpha)), \quad \text{and} \quad D_\alpha^A(p_A^\beta, p_B^\beta) = (1 - \theta)F(p_B^\beta - p_A^\beta - s_\beta). \tag{21}
\]

Under the choices made for $F$, $g_\alpha$ and $g_\beta$ above, our analysis reveals five distinct price discrimination regimes based on the interplay of maximum loyalty levels ($s_\alpha$ and $s_\beta$) and the magnitude of product cost difference ($c_A - c_B$). Equilibrium conditions are determined for each of the following sub-cases in Propositions 8-12 below, which are mutually exclusive and exhaustive (see Figure 5):

- **Region I:** $1 - s_\beta \leq c_A - c_B \leq s_\alpha - 1$ (see Proposition 8).
• Region II: \( \max(1 - s_\beta, s_\alpha - 1) \leq c_A - c_B \leq s_\alpha + 2 \) (see Proposition 9).
• Region III: \( \max(1 - s_\beta, s_\alpha + 2) \leq c_A - c_B \) (see Proposition 10).
• Region IV: \( c_A - c_B \leq \min(s_\alpha - 1, 1 - s_\beta) \) (see Proposition 11).
• Region V: \( s_\alpha - 1 \leq c_A - c_B \leq \min(s_\alpha + 2, 1 - s_\beta) \) (see Proposition 12).

As can be inferred from Propositions 8-12, the general sum game between the two firms stays in equilibrium throughout. The possible combinations of product cost differences and loyalty model parameters lead to a variety of market outcomes, which is visually captured in Figure 5.

**Proposition 8.** Under Region I, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms A and B for the AL model with \( F \sim U[0, 1] \) are as follows:

\[
p_A^A = c_B + s_\alpha, \quad p_A^B = c_A, \quad p_B^B = c_A + s_\beta, \quad \text{and} \quad p_B^A = c_B.
\]  

**Proposition 9.** Under Region II, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms A and B for the AL model with \( F \sim U[0, 1] \) are as follows:

\[
p_A^A = \frac{1}{3}(2c_A + c_B + s_\alpha + 2), \quad p_A^B = c_A, \quad p_B^B = c_A + s_\beta, \quad \text{and} \quad p_B^A = \frac{1}{3}(c_A + 2c_B - s_\alpha + 1).
\]  

**Proposition 10.** Under Region III, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms A and B for the AL model with \( F \sim U[0, 1] \) are as follows:

\[
p_A^A = c_A, \quad p_A^B = c_A, \quad p_B^B = c_A + s_\beta, \quad \text{and} \quad p_B^A = c_A - s_\alpha - 1.
\]  

**Proposition 11.** Under Region IV, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms A and B for the AL model with \( F \sim U[0, 1] \) are as follows:

\[
p_A^A = c_B + s_\alpha, \quad p_A^B = \frac{1}{3}(c_B + 2c_A - s_\beta + 1), \quad p_B^B = \frac{1}{3}(2c_B + c_A + s_\beta + 2), \quad \text{and} \quad p_B^A = c_B.
\]  

**Proposition 12.** Under Region V, the unique pure Nash equilibrium prices for the strong and weak sub-markets for firms A and B for the AL model with \( F \sim U[0, 1] \) are as follows:

\[
p_A^A = \frac{1}{3}(2c_A + c_B + s_\alpha + 2), \quad p_A^B = \frac{1}{3}(c_B + 2c_A - s_\beta + 1), \quad p_B^B = \frac{1}{3}(2c_B + c_A + s_\beta + 2), \quad \text{and} \quad p_B^A = \frac{1}{3}(c_A + 2c_B - s_\alpha + 1).
\]  

As can be inferred from Propositions 8-12, the general sum game between the two firms stays in equilibrium throughout. The possible combinations of product cost differences and loyalty model parameters lead to a variety of market outcomes, which is visually captured in Figure 5.
5.1 Discussion

Similar to the single stage setting for the ML model, nonlinear trends of market outcomes with respect to cost asymmetry are also observed in this setting. From Propositions 8-12 and Figure 6, we can again see that a variety of market outcomes that were not previously considered in the literature can transpire. In fact, the regions of parameter space where they happen are significant. The previously studied regime where there was no cost asymmetry would fall under Region IV (and to be more specific, is represented by the origin). In some of the regions, both firm $A$ and $B$ are able to sell to their weak sub-markets in addition to their strong sub-markets, carefully trading off profitability and market shares. At the same time, the high cost firm ($A$) is unable to sell to its loyal following in its strong sub-market while also not being able to make inroads into its weak sub-market in other regions. Thus, Regions I, II, III, and V together represent a class of competitive price discrimination market equilibrium cases that have hitherto gone unnoticed.

5.2 Infinite Horizon Setting

We start with the setting when both $\delta_A = \delta_B = 0$, and then discuss the setting where $\delta_A = \delta_B (= \delta_F$, a common discount value) $> 0$.

5.2.1 Myopic Firms

As we have seen before, the equilibrium prices computed in the single stage setting remain valid here. Using Lemma 1, we can obtain the following expressions for the market shares (and profits can be derived analogously). With $\theta$ as the initial market share of firm $A$ at time $t = 0$, we get the following results.

**Lemma 3.** The market share of firm $A$ at any time index $t$, namely $\theta_t$, under the Additive Loyalty model (ML) with $F \sim U[0, 1]$ is given as:

- **Region I:** $\theta_t = \theta, \theta_{\infty} = \theta$.
- **Region II:** $\theta_t = \theta \left(\frac{2-c_A+c_B+s_\alpha}{3}\right)^t, \theta_{\infty} = 0$.
- **Region III:** $\theta_t = 0, \theta_{\infty} = 0$. 

Figure 6: Single stage market outcomes under additive loyalty: Optimal prices, market shares, profits of firms, and probability of purchase of customers of types $\alpha$ and $\beta$ as a function of $c_A - c_B$. Here, $c_B = 0.6, s_\alpha = 1.1, s_\beta = 0.5$ and $\theta = 0.8$. 
• Region IV:

\[ \theta_t = 1 - (1 - \theta) \left( \frac{c_A - c_B + s_\beta + 2}{3} \right)^t . \]

Further, \( \theta_\infty = 1 \).

• Region V:

\[ \theta_t = \theta(s_\alpha + s_\beta - 1)^t + \frac{c_A - c_B + s_\beta - 1}{s_\alpha + s_\beta - 2} \left( 1 + (s_\alpha + s_\beta - 1)^t \right) . \]

Further, \( \theta_\infty = \frac{c_A - c_B + s_\beta - 1}{s_\alpha + s_\beta - 2} . \)

The market share of firm B at the end of time period t is simply \( 1 - \theta_t \) (similarly at steady state it is \( 1 - \theta_\infty \)).

From the above lemma, it is clear that the steady state market shares of the high-cost firm (firm A) are 0 in Regions I, II and III. In Region IV, the market share is 1. Finally, in Region V, the market share depends on the loyalty parameters \((s_\alpha, s_\beta)\) and the cost asymmetry. For instance, in Region V, which corresponds to \( s_\alpha - 1 \leq c_A - c_B \leq \min(1 - s_\beta, s_\alpha + 2) \), the steady state market share of firm A depends on how large the product cost difference and the loyalty level of its weak sub-market \((s_\beta)\) are. If either of them are large, then its market share is large in most cases. Noticeably, its market share does depend on the loyalty level of its loyal customers (i.e., on \( s_\alpha \)), unlike Region IV of the multiplicative loyalty setting (see Lemma 2).

5.2.2 Forward Looking Firms

Similar to the multiplicative setting in Section 4.2.2, we will first characterize the equilibrium condition for the general case where \( F \) satisfies Assumption 3 and when there are no restrictions on the prices (also see Sections 3.2).

Proposition 13. For the Additive loyalty model (AL), under Assumption 2 there exists a unique Markov equilibrium when \( \delta_A = \delta_B = \delta_F > 0 \), where firms price based on whether the customer bought their product in the immediate preceding time period. This equilibrium is characterized by the following fixed point equations for thresholds \( \xi^\alpha \) and \( \xi^\beta \):

\[
\begin{align*}
(\xi^\alpha - (c_A - c_B - s_\alpha)) \left( 1 - \frac{\delta_F}{\delta_F} + F(\xi^\beta) + 1 \right) + \frac{2F(\xi^\alpha) - 1}{f(\xi^\alpha)} \left( 1 - \frac{\delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta) \right) + \frac{F(\xi^\alpha)}{f(\xi^\alpha)} &= 1 - F(\xi^\beta) \left( \xi^\beta + c_A - c_B + s_\beta \right), \\
(\xi^\beta - (c_B - c_A - s_\beta)) \left( 1 - \frac{\delta_F}{\delta_F} + F(\xi^\alpha) + 1 \right) + \frac{2F(\xi^\beta) - 1}{f(\xi^\beta)} \left( 1 - \frac{\delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta) \right) + \frac{F(\xi^\beta)}{f(\xi^\beta)} &= 1 - F(\xi^\alpha) \left( \xi^\alpha + c_B - c_A + s_\alpha \right).
\end{align*}
\]

Similar to the Proposition 7, the above thresholds can be used in conjunction with Equations 9-14 to obtain the optimal prices, profits and resulting market shares. Because the thresholds are implicitly defined, in the following, we numerically solve for their values and characterize the dependence of key market metrics on cost asymmetry and loyalty parameters.

5.2.3 Discussion

Firstly, recall that customer who bought firm A’s product in the immediate preceding time period belongs to their strong sub-market. Similarly, a customer who bought firm B’s product in the immediate preceding time period belongs to firm A’s weak sub-market. Also note that, similar to the analysis in Section 4.2, the above pricing strategy is at the level of an individual customer. In order to get market level metrics, we need to aggregate over the total number of customers (or take into account their proportion).

Next, we also see the following intuitive relationship between firm A’s pricing across its strong and weak markets when compared to firm B’s. In particular, Equations 9-12 yield the following for the AL model:

\[
p^\alpha_A - p^\beta_A - \frac{1}{f(\xi^\alpha)} = p^\beta_B - p^\alpha_B - \frac{1}{f(\xi^\beta)}.
\]

(28)
This indicates that when $\xi$ is uniform, the difference in the optimal prices charged by firm $A$ is the same as the difference in optimal prices charged by firm $B$. Further, this equivalence is invariant to the costs $c_A, c_B$ as well as the loyalty parameters $s_\alpha$ and $s_\beta$. Further, this holds even when the long term discounting done by each of the firm is different (i.e., when $\delta_A \neq \delta_B$).

Figure 7 shows how the prices, market shares and profits of firms evolve when $\delta_A = \delta_B = \delta_F = 0.6$ and prices are not constrained. This is enabled by choosing a suitable range of cost asymmetry and the loyalty parameters. From Figures 7a and 7b we can infer that the prices vary linearly with different slopes as the cost asymmetry increases (similar to the multiplicative loyalty model). In the depicted (narrow) regime of cost asymmetry, firm $A$ starts losing significant market share and profit, as its myopic loyal consumers increasingly prefer to purchase from its rival.

When the prices are constrained, the market outcomes start evolving non-linearly with cost asymmetry (see Figure 8). For example, in Figure 8a, we can observe that $p^A_\alpha$ has to initially increase faster than $p^B_\alpha$ for firm $A$ to maximize its profit, and in Figure 8b, we can observe that $p^B_\beta$ converges to $p^A_\beta$ in a nonlinear way signifying that the lower cost firm $B$ just needs to match rival firm’s prices to maximize profit and market share. Similarly, the rate of change of market share and profit as a function of cost asymmetry is also non-linear (Figures 7c and 7d) with saturating trends for the former and runaway trends for the latter (i.e., with firm $B$ making much more profit). Similar to the multiplicative loyalty model setting, numerical computation of equilibria driving the plots in Figure 8 is performed using the dynamic stochastic game solver.

Although we once again omit the characterization of distinct regions (due to the non availability of closed-form expressions describing the boundaries as seen in Section 5.1 Figure 5), it is clearly evident that cost asymmetry and additive loyalty have significant impact on market outcomes, an aspect that was under-explored in prior literature.
Future Directions

Our paper has investigated cost asymmetry and loyalty for a fairly limited collection of market settings. First, we have assumed that both firms can identify customers with perfect accuracy. But in practice different firms have different insights about individual customer preferences due to the varying degree of customer data that is available to them. Despite continuous improvement in data collection and advances in information technology, firms do routinely mis-classify customers. A firm with less customer information is more likely to classify customers erroneously. An incorrectly classified customer may not purchase the product as anticipated. One can incorporate classification errors as well as learning customer preferences in future models and study their implications. A significant direction of improvement along similar lines would be to consider an evolving learning process implemented by each firm and interleave it with firm decision making each period [Mansour et al., 2018].

Second, the analytical results presented here are restricted to affine loyalty functions. This can potentially be extended to include a non-loyal customer base (these always purchase the lowest priced product) and a collection of customers who can be segmented into weakly loyal, strongly loyal and moderately loyal customers. This refined segmentation can capture markets that tend to be more volatile (e.g., certain groceries), markets which are brand driven (e.g., airlines and product companies such as Apple or Microsoft), as well as markets where there is almost no loyalty. It is interesting to analyze how firms harvest their strongly loyal customers, pay to stay the moderately loyal customers, and pay to switch the weakly loyal customers of the rival firm, while still being able to profit from the non-loyal customer base.

Lastly, a few further extensions of our analysis that are promising are as follows: (1) While we did not consider forward looking customers in this work, one can rely on prior works to build suitable extended games and study the interplay of costs and loyalty. (2) While we considered a parametric impact of loyalty on prices, there are multiple other behavioral factors that have been considered in competitive pricing settings recently [Amaldoss and He, 2018; Choi et al., 2018]. Combining these effects in a multidimensional parametric setting where prices depend on the entire customer’s profile would improve the fidelity of the inferences drawn. (3) Relaxing the assumption that the market is covered, and accounting for more than one firm (with possibility of collusion) that can have varied state dependent risk tolerances/discount factors can make the modeling more realistic and the conclusions more actionable.
7 Conclusion

Competitive price discrimination models often assume product cost as either negligible or equal for all firms, and therefore it has no impact on profitability and on market share. We argue that this is a significant oversimplification. To the best of our knowledge, no price discrimination study takes into account the likelihood that firms could also compete on product cost difference. To investigate the effect of product cost difference between firms on market outcomes in the presence of customer loyalty, we built a game-theoretic model with two asymmetric firms. This enabled us to analyze the impact of varying degrees of product cost differences and loyalty levels of consumers on competition, profits, and other market outcomes in both single period and infinite horizon settings. Our results show that the interplay of product cost difference and loyalty characteristics together can determine a variety of market outcomes at equilibrium.

References

Wilfred Amaldoss and Chuan He. Reference-dependent utility, product variety, and price competition. Management Science, 64(9):4302–4316, 2018.

Luis Cabral. Dynamic pricing in customer markets with switching costs. Review of Economic Dynamics, 20:43–62, 2016.

Yuxin Chen, Chakravarthi Narasimhan, and Z John Zhang. Individual marketing with imperfect targetability. Marketing Science, 20(1):23–41, 2001.

Chongwoo Choe, Stephen King, and Noriaki Matsushima. Pricing with cookies: Behavior-based price discrimination and spatial competition. Management Science, 64(12):5669–5687, 2018.

Michael Choi, Anovia Yifan Dai, and Kyungmin Kim. Consumer search and price competition. Econometrica, 86(4):1257–1281, 2018.

Robin Cooper and Robert S Kaplan. Measure costs right: make the right decisions. Harvard Business Review, 66(5):96–103, 1988.

Didem Demirhan, Varghese S Jacob, and Srinivasan Raghunathan. Strategic IT investments: The impact of switching cost and declining IT cost. Management Science, 53(2):208–226, 2007.

Rajiv Dewan, Bing Jing, and Abraham Seidmann. Product customization and price competition on the internet. Management Science, 49(8):1055–1070, 2003.

Steffen Eibelshäuser and David Poensgen. dsgamesolver: A python program for computing markov perfect equilibria of dynamic stochastic games. Available at SSRN 3316631, 2019.

Joseph Farrell and Paul Klemperer. Coordination and lock-in: Competition with switching costs and network effects. Handbook of Industrial Organization, 3:1967–2072, 2007.

Harold Hotelling. Stability in competition. In The Collected Economics Articles of Harold Hotelling, pages 50–63. Springer, 1990.

Patrick J Kehoe, Bradley J Larsen, and Elena Pastorino. Dynamic competition in the era of big data. Technical report, Working paper, Stanford University and Federal Reserve Bank of Minneapolis, 2018.

Qihong Liu and Konstantinos Serfes. Imperfect price discrimination, market structure, and efficiency. Canadian Journal of Economics/Revue canadienne d’économique, 38(4):1191–1203, 2005.

Yishay Mansour, Aleksandrs Slivkins, and Zhiwei Steven Wu. Competing bandits: Learning under competition. Conference on Innovations in Theoretical Computer Science, 2018.

Eric Maskin and Jean Tirole. Markov perfect equilibrium: I. observable actions. Journal of Economic Theory, 100(2):191–219, 2001.

Aris M Ouksel and Ferdi Eruysal. Loyalty intelligence and price discrimination in a duopoly. Electronic Commerce Research and Applications, 10(5):520–533, 2011.

Louis Phlips. The economics of price discrimination. Cambridge University Press, 1983.

Andrew Rhodes. Re-examining the effects of switching costs. Economic Theory, 57(1):161–194, 2014.

Steven C Salop. Monopolistic competition with outside goods. The Bell Journal of Economics, pages 141–156, 1979.

Greg Shaffer and Z John Zhang. Competitive coupon targeting. Marketing Science, 14(4):395–416, 1995.

Greg Shaffer and Z John Zhang. Pay to switch or pay to stay: preference-based price discrimination in markets with switching costs. Journal of Economics & Management Strategy, 9(3):397–424, 2000.
A Proofs

A.1 Proof of Proposition [1]

Proof. The result follows by noting that the profit optimization problem for each firm separates across variables. We are interested in the unconstrained variant of this optimization. Thus, firm $A$ can maximize with respect to $p_A^\alpha$ and $p_A^\beta$ independent of each other. Taking the derivatives with respect to $p_A^\alpha$ and $p_A^\beta$ and setting them to 0 gives:

$$-(p_A^\alpha - c_A)f(\xi^\alpha)h'_\alpha(p_A^\alpha - p_B^\alpha) + (1 - F(\xi^\alpha)) = 0,$$

$$-(p_A^\beta - c_A)f(\xi^\beta)h'_\beta(p_A^\beta - p_B^\beta) + (1 - F(\xi^\beta)) = 0.$$  

A similar set of equations can also be obtained for firm $B$. Rearranging the terms yields the desired first order conditions (FOC).

A.2 Proof of Lemma [1]

Proof. The proof follows by a straightforward substitution of the definition of market shares for the single stage setting from Section 3.1. Given the initial market share $\theta$ and the optimal prices, the market share at $t = 1$, i.e., $\theta_1$ is given as:

$$\theta_1 = \theta(1 - F(\xi^\alpha)) + (1 - \theta)F(\xi^\beta),$$

$$\theta_2 = \theta_1(1 - F(\xi^\alpha)) + (1 - \theta_1)F(\xi^\beta),$$

and so on. As a result,

$$\theta_t = \theta(1 - F(\xi^\alpha) - F(\xi^\beta))^t + F(\xi^\beta)\sum_{j=0}^{t-1}(1 - F(\xi^\alpha) - F(\xi^\beta))^j.$$  

Finally using the fact that $F(\xi^\alpha) + F(\xi^\beta) \in (0, 2)$, we get the desired result. The result when $t \to \infty$ also follows naturally.

A.3 Proof of Proposition [2]

Proof. Because we are assuming that there are no price constraints, we can differentiate Equations [5][8] with respect to price variables to obtain the following implicit equations:

$$-(p_A^\alpha - c_A)f(\xi^\alpha)h'_\alpha(p_A^\alpha - p_B^\alpha) + (1 - F(\xi^\alpha)) - \delta_A f(\xi^\alpha)h'_\alpha(p_A^\alpha - p_B^\alpha)(V_A^\alpha - V_A^\alpha) = 0,$$

$$-(p_A^\beta - c_A)f(\xi^\beta)h'_\beta(p_A^\beta - p_B^\beta) + (1 - F(\xi^\beta)) - \delta_B f(\xi^\beta)h'_\beta(p_A^\beta - p_B^\beta)(V_B^\beta - V_B^\beta) = 0,$$

$$-(p_B^\beta - c_B)f(\xi^\beta)h'_\beta(p_B^\beta - p_A^\beta) + (1 - F(\xi^\beta)) - \delta_B f(\xi^\beta)h'_\beta(p_B^\beta - p_A^\beta)(V_B^\beta - V_B^\beta) = 0,$$

Rearranging the terms gives us the desired implicit equations.

A.4 Proof of Lemma [2]

Proof. The four market share expressions are obtained by using Lemma [1] with Propositions [3][6].
A.5 Proof of Proposition 7

Proof. To show that we have a Markov equilibrium, we first show that \( \xi^\alpha \) is unique for any fixed \( \xi^\beta \) and vice versa. And then we assert that \( V_A^\alpha, V_B^\alpha, V_A^\beta, V_B^\beta \) and \( V_A^{\beta*}, V_B^{\beta*} \) are also unique, implying that the corresponding prices (shown in Equations 9 and 12) constitute a unique Markov equilibrium.

Let \( \gamma = (c_A - c_B)/l_\alpha \). Noting that \( \delta_A = \delta_B = \delta_F, h_\alpha(p^\alpha_A - p^\alpha_B) = (p^\alpha_A - p^\alpha_B)/l_\alpha \) and \( h'_\alpha(p^\alpha_A - p^\alpha_B) = 1/l_\alpha \), we can simplify the expression for \( \xi^\alpha \) in Equation 13 as follows:

\[
\xi^\alpha = h_\alpha \left( c_A - c_B + \frac{1 - 2F(\xi^\alpha)}{f(\xi^\alpha)h'_\alpha(p^\alpha_A - p^\alpha_B)} - \delta_A (V^{\alpha*}_A - V^\alpha_A) + \delta_B (V^{\beta*}_B - V^\alpha_B) \right) \\
= \frac{1}{l_\alpha} \left( c_A - c_B + \frac{1 - 2F(\xi^\alpha)}{f(\xi^\alpha)} - \delta_A (V^{\alpha*}_A - V^\alpha_A) + \delta_B (V^{\beta*}_B - V^\alpha_B) \right) \\
= \gamma + \frac{1 - 2F(\xi^\alpha)}{f(\xi^\alpha)} - \frac{\delta_F}{l_\alpha} (V^{\alpha*}_A - V_A^\alpha - (V^{\beta*}_B - V_B^{\beta*})).
\]

From Equations 13 and 14 we know that:

\[
V^{\alpha*}_A - V_A^{\beta*} = \frac{1}{1 - \delta_F + \delta_F(F(\xi^\alpha) + F(\xi^\beta))} \left( (1 - F(\xi^\alpha))(p^\alpha_A - c_A) - F(\xi^\beta)(p^\beta_A - c_A) \right), \quad \text{and}
\]

\[
V^{\beta*}_B - V_B^{\alpha*} = \frac{1}{1 - \delta_F + \delta_F(F(\xi^\alpha) + F(\xi^\beta))} \left( (1 - F(\xi^\beta))(p^\beta_B - c_B) - F(\xi^\alpha)(p^\alpha_B - c_B) \right).
\]

Substituting these in the above expression for \( \xi^\alpha \), we get:

\[
\xi^\alpha = \gamma + \frac{1 - 2F(\xi^\alpha)}{f(\xi^\alpha)} - \frac{\delta_F}{l_\alpha(1 - \delta_F + \delta_F(F(\xi^\alpha) + F(\xi^\beta)))} \left( (1 - F(\xi^\alpha))(p^\alpha_A - c_A) - F(\xi^\beta)(p^\beta_A - c_A) - (1 - F(\xi^\beta))(p^\beta_B - c_B) - F(\xi^\alpha)(p^\alpha_B - c_B) \right).
\]

(29)

In the above expression, we intend to replace prices with \( \xi^\alpha \) and \( \xi^\beta \) and then segregate all terms involving \( \xi^\alpha \) to the left. This allows us to inspect if the left hand side is monotonic for every value of \( \xi^\beta \). As we show below, this is the case. We start with replacing terms involving prices with terms involving \( \xi^\alpha \) and \( \xi^\beta \). Let

\[
T = \left( (1 - F(\xi^\alpha))(p^\alpha_A - c_A) - F(\xi^\beta)(p^\beta_A - c_A) \right) \\
- \left( (1 - F(\xi^\beta))(p^\beta_B - c_B) - F(\xi^\alpha)(p^\alpha_B - c_B) \right),
\]

\[
= p^\alpha_A - c_A - F(\xi^\alpha)p^\alpha_A + F(\xi^\alpha)c_A - F(\xi^\beta)p^\beta_A + F(\xi^\beta)c_A \\
- (p^\beta_B - c_B - F(\xi^\beta)p^\beta_B + F(\xi^\beta)c_B - F(\xi^\alpha)p^\alpha_B + F(\xi^\alpha)c_B)
\]

\[
= p^\alpha_A - c_A - F(\xi^\alpha)p^\alpha_A + F(\xi^\alpha)c_A - F(\xi^\beta)p^\beta_A + F(\xi^\beta)c_A \\
- (p^\beta_B - c_B + F(\xi^\beta)p^\beta_B - F(\xi^\beta)c_B + F(\xi^\alpha)p^\alpha_B - F(\xi^\alpha)c_B)
\]

\[
= p^\alpha_A - p^\beta_B - l_\alpha \gamma - F(\xi^\alpha)(p^\alpha_A - p^\alpha_B) + F(\xi^\alpha)c_A \gamma + F(\xi^\beta)(p^\beta_B - p^\beta_A) + F(\xi^\beta)c_B \gamma,
\]

where in the last step we substituted the definition of \( \gamma \) and grouped a few terms together. Further, from the definition of \( \xi^\alpha \) and \( \xi^\beta \), we have \( p^\alpha_B - p^\beta_B = l_\alpha \xi^\alpha \) and \( p^\beta_B - p^\alpha_B = l_\beta \xi^\beta \). Thus,

\[
T = p^\alpha_A - p^\beta_B - l_\alpha \gamma - F(\xi^\alpha)c_A \gamma + F(\xi^\alpha)c_B \gamma + F(\xi^\beta)(p^\beta_B - p^\beta_A) + F(\xi^\beta)c_A \gamma.
\]

From Equations 11 and 12 of optimal prices, we also know that:

\[
p^\beta_B = \frac{1 - F(\xi^\beta)}{f(\xi^\beta)h'_\beta(p^\beta_B - p^\beta_A)} + p^\beta_B - \frac{F(\xi^\alpha)}{f(\xi^\alpha)h'_\alpha(p^\alpha_A - p^\alpha_B)} \\
= \frac{1 - F(\xi^\beta)}{f(\xi^\beta)l_\beta + p^\alpha_B} - \frac{F(\xi^\alpha)}{f(\xi^\alpha)l_\alpha}.
\]

23
Again using the identity \( p_B^\alpha = p_A^\alpha - s_\alpha \xi^\alpha \) we get,

\[
p_B^\beta = \frac{1 - F(\xi^\beta)}{f(\xi^\beta)} l_\beta + p_A^\alpha - l_\alpha \xi^\alpha - \frac{F(\xi^\alpha)}{f(\xi^\alpha)} l_\alpha,
\]

\[
\Rightarrow p_A^\alpha - p_B^\beta = \frac{1 - F(\xi^\beta)}{f(\xi^\beta)} l_\beta + l_\alpha \xi^\alpha + \frac{F(\xi^\alpha)}{f(\xi^\alpha)} l_\alpha.
\]

Thus, the term \( T \) can be updated as:

\[
T = -\frac{1 - F(\xi^\beta)}{f(\xi^\beta)} l_\beta + l_\alpha \xi^\alpha + \frac{F(\xi^\alpha)}{f(\xi^\alpha)} l_\alpha
- l_\alpha \gamma - F(\xi^\alpha) l_\alpha \xi^\alpha + F(\xi^\alpha) l_\alpha \gamma + F(\xi^\beta) l_\beta \xi^\beta + F(\xi^\beta) l_\alpha \gamma,
\]

\[
= -\frac{1 - F(\xi^\beta)}{f(\xi^\beta)} l_\beta + l_\alpha (\xi^\alpha - \gamma)
+ \frac{F(\xi^\alpha)}{f(\xi^\alpha)} l_\alpha - F(\xi^\alpha) l_\alpha (\xi^\alpha - \gamma) + F(\xi^\beta) l_\beta \xi^\beta + F(\xi^\beta) l_\alpha \gamma.
\]

Rearranging terms in Equation 29 and using the definition of \( T \) above, we get:

\[
(\xi^\alpha - \gamma) \left( \frac{1 - \delta_F}{\delta_F} F(\xi^\alpha) + F(\xi^\beta) \right)
+ \frac{2 F(\xi^\alpha)}{f(\xi^\alpha)} \left( \frac{1 - \delta_F}{\delta_F} F(\xi^\alpha) + F(\xi^\beta) \right)
= -\frac{1}{l_\alpha} T.
\]

Bringing the terms involving \( \xi^\alpha \) in the expression \( T \) to the left hand side, we get:

\[
(\xi^\alpha - \gamma) \left( \frac{1 - \delta_F}{\delta_F} F(\xi^\beta) + 1 \right)
+ \frac{2 F(\xi^\alpha)}{f(\xi^\alpha)} \left( \frac{1 - \delta_F}{\delta_F} F(\xi^\beta) + F(\xi^\beta) \right)
+ \frac{F(\xi^\alpha)}{f(\xi^\alpha)} \left( \frac{1 - \delta_F}{\delta_F} F(\xi^\beta) + F(\xi^\beta) l_\alpha \gamma \right),
\]

\[
= \frac{1}{l_\alpha} \left( -\frac{1 - F(\xi^\beta)}{f(\xi^\beta)} l_\beta + F(\xi^\beta) l_\beta \xi^\beta + F(\xi^\beta) l_\alpha \gamma \right),
\]

\[
= \frac{(1 - F(\xi^\beta)) l_\beta}{f(\xi^\beta) l_\alpha} - F(\xi^\beta) \left( \frac{l_\beta}{l_\alpha} \xi^\beta + \gamma \right).
\]

Under Assumption 3 we know that \( \frac{2 F(\xi^\alpha)}{f(\xi^\alpha)} \) and \( \frac{F(\xi^\alpha)}{f(\xi^\alpha)} \) are increasing functions of \( \xi^\alpha \). Thus, each term on the left hand side of the above expression is monotonically increasing with \( \xi^\alpha \). Equating the left hand side to a constant on the right hand side (for any fixed \( \xi^\beta \)) implies that there is a unique solution for \( \xi^\alpha \). An analogous claim can be made for \( \xi^\beta \) as well. These two results imply that there are unique solutions for \( V_A^{\alpha *}, V_A^{\beta *}, V_B^{\beta *}, V_B^{\alpha *}, \) and further imply the unique Markov prices \( (p_A^\alpha, p_A^\beta, p_B^\beta, p_B^\alpha) \) given in Equations 9-12.

\[
\square
\]

A.6 Proof of Proposition 13

**Proof.** Similar to the proof of Proposition 13 above, to show that we have a Markov equilibrium, we first show that \( \xi^\alpha \) is unique for any fixed \( \xi^\beta \) and vice versa. And then we assert that \( V_A^{\alpha *}, V_A^{\beta *}, V_B^{\beta *}, \) and \( V_B^{\alpha *}, \) are also unique, implying that the corresponding prices (shown in Equations 9-12) constitute a unique Markov equilibrium.

Let \( \gamma = c_A - c_B - s_\alpha \). Noting that \( \delta_A = \delta_B = \delta_F, h_\alpha(p_A^\alpha - p_B^\alpha) = p_A^\alpha - p_B^\alpha - s_\alpha \) and \( h_\alpha'(p_A^\alpha - p_B^\alpha) = 1 \), we can simplify the expression for \( \xi^\alpha \) in Equation 15 as follows:

\[
\xi^\alpha = \gamma + \frac{1 - 2 F(\xi^\alpha)}{f(\xi^\alpha)} - \delta_F (V_A^{\alpha *} - V_B^{\alpha *}) - (V_B^{\beta *} - V_A^{\beta *}).
\]
We start with replacing terms involving prices with terms involving
where we substituted the definition of
Again using the identity
This allows us to inspect if the left hand side is monotonic for every value of \( T \)
In the above expression, we intend to replace prices with \( \xi \)
From Equations 13 and 14, we know that:
\[ V_A^\alpha - V_A^\beta = \frac{1}{1 - \delta_F + \delta_F(F(\xi^\alpha) + F(\xi^\beta))} \left((1 - F(\xi^\alpha))(p_A^\alpha - c_A) - F(\xi^\beta)(p_A^\beta - c_A)\right), \]
\[ V_B^\beta - V_B^\alpha = \frac{1}{1 - \delta_F + \delta_F(F(\xi^\alpha) + F(\xi^\beta))} \left((1 - F(\xi^\beta))(p_B^\beta - c_B) - F(\xi^\alpha)(p_B^\alpha - c_B)\right). \]
Substituting these in the above expression for \( \xi^\alpha \), we get:
\[
\xi^\alpha = \gamma + \frac{1 - 2F(\xi^\alpha)}{f(\xi^\alpha)} - \frac{\delta_F}{1 - \delta_F + \delta_F(F(\xi^\alpha) + F(\xi^\beta))} \left((1 - F(\xi^\alpha))(p_A^\alpha - c_A) - F(\xi^\beta)(p_B^\alpha - c_B)\right) - F(\xi^\beta)(p_B^\beta - c_B) - (1 - F(\xi^\beta))(p_B^\alpha - c_B).
\] (30)
In the above expression, we intend to replace prices with \( \xi^\alpha \) and \( \xi^\beta \) and then segregate all terms involving \( \xi^\alpha \) to the left. This allows us to inspect if the left hand side is monotonic for every value of \( \xi^\beta \). As we show below, this is the case. We start with replacing terms involving prices with terms involving \( \xi^\alpha \) and \( \xi^\beta \). Let
\[
T = \left((1 - F(\xi^\alpha))(p_A^\alpha - c_A) - F(\xi^\beta)(p_A^\beta - c_A)\right) - (1 - F(\xi^\beta))(p_B^\beta - c_B) - F(\xi^\beta)(p_B^\alpha - c_B) + F(\xi^\alpha)\gamma + F(\xi^\alpha)s_\alpha + F(\xi^\beta)\gamma + F(\xi^\beta)s_\alpha,
\]
where we substituted the definition of \( \gamma \) and grouped a few terms together. Further, from the definition of \( \xi^\alpha \) and \( \xi^\beta \), we have \( p_A^\alpha - p_B^\beta - s_\alpha = \xi^\alpha \) and \( p_B^\beta - p_A^\alpha = \xi^\beta \). Thus,
\[
T = p_A^\alpha - p_B^\beta - \gamma - s_\alpha - F(\xi^\alpha)\xi^\alpha + F(\xi^\alpha)\gamma + F(\xi^\beta)\xi^\beta + F(\xi^\beta)(\gamma + s_\alpha + s_\beta).
\]
From Equations 11 and 12 of optimal prices, we also know that:
\[
p_B^\beta = \frac{1 - F(\xi^\beta)}{f(\xi^\beta)} + p_B^\alpha - \frac{F(\xi^\alpha)}{f(\xi^\alpha)}.
\]
Again using the identity \( p_B^\alpha = p_A^\alpha - s_\alpha - \xi^\alpha \) we get,
\[
\Rightarrow p_A^\alpha - p_B^\beta - s_\alpha = \xi^\alpha - \frac{1 - F(\xi^\beta)}{f(\xi^\beta)} + \frac{F(\xi^\alpha)}{f(\xi^\alpha)}.
\]
Thus, the term \( T \) can be updated as:
\[
T = \xi^\alpha - \frac{1 - F(\xi^\beta)}{f(\xi^\beta)} + \frac{F(\xi^\alpha)}{f(\xi^\alpha)} - \gamma - F(\xi^\alpha)\xi^\alpha + F(\xi^\alpha)\gamma + F(\xi^\beta)\xi^\beta + F(\xi^\beta)(\gamma + s_\alpha + s_\beta).
\]
Rearranging terms in Equation 30 and using the definition of \( T \) above, we get:
\[
(\xi^\alpha - \gamma) \left(\frac{1 - \delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta)\right) + \frac{2F(\xi^\alpha) - 1}{f(\xi^\alpha)} \left(\frac{1 - \delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta)\right) = -T.
\]
Bringing the terms involving $\xi^\alpha$ in the expression $T$ to the left hand side, we get:

$$
(\xi^\alpha - \gamma) \left( \frac{1 - \delta_F}{\delta_F} + F(\xi^\beta) + 1 \right) + \frac{2F(\xi^\alpha) - 1}{f(\xi^\alpha)} \left( \frac{1 - \delta_F}{\delta_F} + F(\xi^\alpha) + F(\xi^\beta) \right) + \frac{F(\xi^\alpha)}{f(\xi^\alpha)}
$$

$$
= \frac{(1 - F(\xi^\beta))}{f(\xi^\beta)} - F(\xi^\beta) (\xi^\beta + \gamma + s_\alpha + s_\beta).
$$

Under Assumption 3, we know that $\frac{2F(\xi^\alpha) - 1}{f(\xi^\alpha)}$ and $\frac{F(\xi^\alpha)}{f(\xi^\alpha)}$ are increasing functions of $\xi^\alpha$. Thus, each term on the left hand side of the above expression is monotonically increasing with $\xi^\alpha$. Equating the left hand side to a constant on the right hand side (for any fixed $\xi^\beta$) implies that there is a unique solution for $\xi^\alpha$. An analogous claim can be made for $\xi^\beta$ as well. These two results imply that there are unique solutions for $V_A^{\alpha*}$, $V_A^{\beta*}$, $V_B^{\beta*}$ and $V_B^{\alpha*}$, and further imply the unique Markov prices $(p_A^\alpha, p_A^\beta, p_B^\beta, p_B^\alpha)$ given in Equations 9-12.