Semi-Invariant Terms for Gauged Non-Linear $\sigma$-Models

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Abstract

We determine all the terms that are gauge-invariant up to a total spacetime derivative (“semi-invariant terms”) for gauged non-linear sigma models. Assuming that the isotropy subgroup $H$ of the gauge group is compact or semi-simple, we show that (non-trivial) such terms exist only in odd dimensions and are equivalent to the familiar Chern-Simons terms for the subgroup $H$. Various applications are mentioned, including one to the gauging of the Wess-Zumino-Witten terms in even spacetime dimensions. Our approach is based on the analysis of the descent equation associated with semi-invariant terms.
1 Introduction

The Lagrangian of a dynamical system invariant under a given symmetry is invariant itself up to a total spacetime derivative, \( \delta \mathcal{L} = \partial_{\mu} k^{\mu} \). In some cases, one can redefine the Lagrangian without modifying its Euler-Lagrange derivatives, \( \mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \partial_{\mu} l^{\mu} \), so that the new Lagrangian is strictly invariant, \( \delta \mathcal{L}' = 0 \). In other cases, however, this is impossible. A well-known example is the Chern-Simons term for Yang-Mills theory in odd spacetime dimensions [1], which is invariant only up to an unremovable total derivative.

Terms for which \( \delta \mathcal{L} = \partial_{\mu} k^{\mu} \) but \( m \neq m' - \partial_{\mu} m^{\mu} \) with \( \delta \mathcal{L}' = 0 \) are sometimes called (non trivial) “semi-invariant terms” and we shall adopt here this convenient terminology.

Given a theory, it is important to know all the terms that can be added to the action without destroying the assumed symmetries. While the identification of the manifestly invariant terms is usually rather straightforward, it is in general more difficult to determine systematically the semi-invariant ones. The purpose of this letter is to solve this question for gauged nonlinear sigma models. As it is well known, these models occur repeatedly in supergravity [2, 3, 4].

Our work is motivated by the beautiful paper [5], where a method for constructing a new class of semi-invariant terms in even spacetime dimensions, with topological connotation, is described. As pointed out there, a necessary condition for these terms to exist is that the gauge group be non semi-simple. The question was further studied in [6], where it was proved that if the isotropy subgroup \( H \) of the gauge group reduces to the identity, then, no matter what the gauge group is, the terms of [5] differ from strictly gauge-invariant terms by a total derivative and thus are not truly semi-invariant. We show here that this conclusion remains valid whenever \( H \) is compact or semi-simple. In fact, the only available non-trivial semi-invariant terms are then the Chern-Simons terms based on the subgroup \( H \), but these exist only in odd dimensions. More precisely, any semi-invariant term is proportional to a Chern-Simons term up to terms that are strictly gauge-invariant and up to a total derivative. Thus, under the assumption that \( H \) is compact or semi-simple, non-trivial semi-invariant terms are exhausted by the familiar Chern-Simons family.

Our approach is based on the analysis of the BRST descent equation associated with semi-invariant terms.
2 Description of $\sigma$-Models

We denote the gauge group by $G$, its Lie algebra by $\mathcal{G}$ and the Yang-Mills connection by $A^A_\mu$. The Lie group $G$ may be non-compact since non-compact groups are typically the rule in supergravity. In addition to the sigma-model variables, we allow for the presence of “matter” fields $y^i$ transforming in some linear representation of $G$.

The gauge group $G$ may not act transitively on the manifold $M$ of the non-linear sigma model. For instance, $M$ could be a homogeneous space $G'/H'$ invariant under a bigger group $G'$ of which $G$ is only a proper subgroup. In that case, we assume the existence of a slice through every point of $M$, so that $M$ splits into $G$-orbits, which are smooth submanifolds. The $G$-orbits are by definition - homogeneous spaces for $G$ and thus of the form $G/H$ where the isotropy subgroup $H$ depends in general on the orbits. In a stratum where the isometry subgroup is constant, one may locally introduce coordinates that parametrize the points in the coset manifold $G/H$ and coordinates $\psi^\Delta$ that are constant along the orbits. These latter transform in the trivial linear representation of $G$ and can simply be included among the “matter fields” $y^i$’s. They are gauge-invariant while the coordinates along the orbits could be gauged away by fixing the gauge (as we shall see, the $\psi^\Delta$’s contribute to the BRST cohomology but not the coordinates along the orbits).

Thus, provided one allows for extra matter fields and includes the coordinates $\psi^\Delta$ among them, one can assume that $G$ (the gauge group) acts transitively on the (smooth) manifold of the non-linear model, which has the form $G/H$. This will be done from now on. The subgroup $H$ is assumed to be compact or semi-simple.

First, we recall a few properties of coset spaces. For that matter, it is convenient to split the generators $T_A$ of $\mathcal{G}$ into two sets, the generators of the Lie algebra of $H$ ($\mathcal{H}$) and the remaining ones,

$$T_A = \{T_\alpha, T_a\}, \quad (2.1)$$

where capital latin indices refer to $\mathcal{G}$, small greek indices to $\mathcal{H}$ and small latin indices to the chosen supplementary subspace $\mathcal{K}$. The commutators

$$[T_A, T_B] = f^C_{AB} T_C, \quad [T_\alpha, T_\beta] = f^\gamma_{\alpha\beta} T_\gamma \quad (2.2)$$

How to patch strata together may presumably be achieved along the lines of [8] combined with the results concerning linear representations, but we have not investigated this question here.
define the structure constants of $G$ and $H$ respectively. Because $H$ is semi-simple or compact, the coset space is reductive. This means that one can choose the supplementary generators $T_a$ so that the commutators $[T_a, T_\alpha]$ involve only the $T_b$'s,

$$[T_a, T_\beta] = f^c_{a\beta} T_c. \quad (2.3)$$

Thus, the structure constants $f^c_{a\beta}$ vanish and the generators $T_a$ define a representation of $H$.

Coset representatives may be taken to be of the form

$$U(\xi) = \exp\{\xi^a T_a\}, \quad (2.4)$$

with some real fields $\xi^a$, the number of which equals the dimension of $G/H$. The $\xi^a$ provide local coordinates in the vicinity of $\xi^a = 0$. We shall in a first approach restrict the sigma-model fields to be in a star-shaped neighbourhood of $\xi^a = 0$. How to deal with global features (which are easily incorporated) is discussed in the conclusions.

The group $G$ acts on the left as

$$gU(\xi) = U(\xi')h(\xi, g) \quad (2.5)$$

with $g \in G$ and $h \in H$ (note that right action conventions were adopted in [6]). The infinitesimal transformation property of the local parameters $\xi^a$ is derived through the parametrization

$$g = \exp\{-\epsilon^A T_A\}, \quad h(\xi, g) = \exp\{-u^\alpha(\xi, \epsilon) T_\alpha\} \quad (2.6)$$

leading to

$$(1 - \epsilon^A T_A)U(\xi) = (U(\xi) + \partial_a U(\xi) \delta \xi^a)(1 - u^\alpha(\xi, \epsilon) T_\alpha). \quad (2.7)$$

Setting

$$U^{-1}\partial_a U = \mu^A_a(\xi) T_A \quad (2.8)$$

and

$$U^{-1}T_A U = k^B_A(\xi) T_B \quad (2.9)$$

one gets

$$\delta \xi^a = \Omega^a_A(\xi) \epsilon^A, \quad (2.10)$$

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with $\Omega_A^\alpha(\xi)$ defined through

$$\Omega_A^\alpha \mu_a^b = -k_A^b \tag{2.11}$$

[The matrix $\mu_a^b(\xi)$ is invertible in a vicinity of $\xi^a = 0$ since one has $\mu_a^b(\xi) = \delta^b_a + O(\xi)$ from (2.4) and (2.8). Note also $\mu_a^a = O(\xi)$ and $k_A^B(\xi) = \delta_A^B + O(\xi)$. The matrices $k_A^B(\xi)$ are in fact matrices of the adjoint representation of $G$ and thus clearly invertible]. The $\Omega_A^a \partial_a$ are the Killing vectors of the coset manifold. They obey the relation $[\Omega_A^a, \Omega_B^b] = f_{ABC}^D \Omega_D^d$, or in coordinates

$$\Omega_A^b \partial_b \Omega^a_B - \Omega_B^b \partial_b \Omega^a_A = \Omega^a_A f^D_{AB} \tag{2.12}$$

as well as $\Omega_a^b = -\delta_a^b + O(\xi)$ and $\Omega_a^a = O(\xi)$. Similarly, one gets

$$u^\alpha = \Omega_A^\alpha \epsilon^A \tag{2.13}$$

with

$$\Omega_A^\alpha(\xi) = k_A^\alpha + \mu_a^a \Omega_A^a. \tag{2.14}$$

For later purposes, we observe that $\Omega_B^a = \delta_B^a + O(\xi)$ and $\Omega_a^a = O(\xi)$. It follows that the matrix $\Omega_B^A$ is invertible (in the vicinity of the identity), so that one can express the gauge parameters $\epsilon^A$ in terms of the field variations $\delta\xi^a$ and the $u^\alpha$’s.

The gauge transformations of the gauged non-linear $\sigma$ model $G/H$ (with $G$ the gauge group) read

$$\delta_c A^A_\mu = D^{(A)}_\mu \epsilon^A, \quad \delta_c \xi^a = \Omega_A^a \epsilon^A \tag{2.15}$$

where $D^{(A)}_\mu$ is the covariant derivative with connection $A^A_\mu$, $D^{(A)}_\mu \epsilon^A = \partial_\mu \epsilon^A + f_{BC}^A A^B_\mu \epsilon^C$. The question is to find the most general gauge invariant function of the fields and their derivatives up to a total divergence.

### 3 BRST Differential

If $\mathcal{L}$ is invariant under gauge transformations up to a total derivative,

$$\delta_c \mathcal{L} = \partial_\mu k^\mu \tag{3.1}$$

then $k^\mu$ itself is not arbitrary but subject to some definite conditions obtained by evaluating the commutators of the second variations of $\mathcal{L}$. 


It is convenient to analyse this question in terms of the BRST differential
taken here to act from the left and explicitly defined through

\[ \gamma A^A_\mu = D^{(A)}_\mu C^A, \]  
\[ \gamma \xi^a = \Omega^a_C C^A, \]  
\[ \gamma y^i = -Y^i_A y^j C^A, \]  
\[ \gamma C^A = -\frac{1}{2} f^{A}_{BC} C^B C^C. \]  

(3.2) (3.3) (3.4) (3.5)

where the matrices \( Y_A \) are the generators of the representation of the \( y \)'s,
\( [Y_A, Y_B] = f^A_{BC} Y_C \), and where \( C^A \) are the ghosts. The BRST differential acts
also on the so-called antifields. Since we are interested only in terms that are
off-shell gauge invariant (up to a total divergence), we shall, however, not
include them. Comments on more general deformations are briefly given at
the end.

Local forms are exterior forms on spacetime which depend on the fields,
the ghosts and their derivatives. A density \( \mathcal{L} \) that is gauge-invariant up to
a total divergence defines, in dual terms, a local \( n \)-form with ghost number
zero that is \( \gamma \)-closed modulo \( d \),

\[ \gamma a + db = 0. \]  

(3.6)

A redefinition of \( \mathcal{L} \) by a total divergence is clearly equivalent to

\[ a \to a + \gamma c + de \]  

(3.7)

since there is no form in negative ghost number in the absence of antifields
\((c \equiv 0)\). The question under study is thus equivalent to determining whether
elements in the cohomology \( H^0(\gamma|d) \) have a representative that is strictly
\( \gamma \)-exact.

The advantage of reformulating the question in these cohomological terms
is that there are well-developed techniques for handling the BRST cohomol-
ogy \( H^0(\gamma|d) \). The idea is that the equation (3.7) implies \( \gamma b + de = 0 \) for
some \((n-2)\)-form \( e \) of ghost number two. By continuing in the same way,
one gets a chain of equations \( \gamma a + db = 0, \gamma b + de = 0, \gamma e + df = 0 \) (the
"descent"), the last two of which read \( \gamma m + dn = 0 \) and \( \gamma n = 0 \). If \( n \) is trivial
in \( H(\gamma) \), \( n = \gamma u \), then one can absorb it in redefinitions \( m \to m' = m - du \)
and $n \to n - \gamma u = 0$) and shorten the descent by one step, the last equation being now $\gamma m' = 0$.

In a first stage, one thus determines all non trivial solutions of $\gamma n = 0$, without restriction on the ghost number. In a second stage, one determines which among these solutions can be lifted all the way back up to form-degree $n$. Both problems have been studied in the literature.

4 The $\gamma$-cohomology

Our strategy for computing the $\gamma$-cohomology is to perform an appropriate change of variables in the “jet-space” coordinatized by the fields, the ghosts and their derivatives. This change of variables is adapted to the symmetry and combines the analysis of [8] for coset models with ungauged symmetry with the change of variables of [6] for the gauged principal models ($H$ reduced to the identity).

The starting point is the observation that if one decomposes the Lie algebra-valued form $U(\xi)^{-1}(d + A)U(\xi)$ in the basis $(T_a, T_\alpha)$,

$$ U(\xi)^{-1}(d + A)U(\xi) = p^a(\xi, A)T_a + v^\alpha(\xi, A)T_\alpha, \quad (4.1) $$

one generates a quantity $p^a_\mu$ that is covariant under the action of $H$, and a field $v^{\alpha}_\mu$ that behaves as an $H$-gauge connection,

$$ \gamma p^a_\mu = -f^a_{\gamma b}p^b_\mu D^\gamma \quad (4.2) $$
$$ \gamma v^{\alpha}_\mu = f^{\alpha}_{\beta \gamma}v^{\beta}_\mu D^\gamma + \partial_\mu D^\alpha \quad (4.3) $$

The ghosts $D^\alpha$ are identical to the $u^\alpha(\xi, \epsilon)$ that occurred in the parametrization of $h(\xi, g)$, with the gauge parameters $\epsilon^A$ being replaced by the ghosts $C^A$,

$$ D^\alpha(\xi) = u^\alpha(\xi, C) = \Omega^\alpha_B(\xi)C^B. \quad (4.4) $$

The nilpotence of $\gamma$ requires the $D^\alpha$ to transform as

$$ \gamma D^\alpha = -\frac{1}{2}f^\alpha_{\beta \gamma}D^\beta D^\gamma \quad (4.5) $$

We complete the redefinition of the ghosts by introducing the abelian ghosts $D^a$, which are the BRST variations of the coset space coordinates $\xi^a$,

$$ D^a = \gamma \xi^a = \Omega^a_B(\xi)C^B, \quad \gamma D^a = 0 \quad (4.6) $$
This redefinition of the ghosts is clearly invertible in the vicinity of the identity, since the matrix $\Omega^A_B(\xi)$ is invertible. Thus, one can trade the $C^A$ for the $D^A$,

$$D^A = \Omega^A_B(\xi)C^B, \quad C^A = \omega^A_B(\xi)D^B. \quad (4.7)$$

The $H$-connection $v$ can be used to construct a $v$-covariant derivative of $p$,

$$D^{(v)}_{\mu} p^\nu = \partial^\mu p^\nu + \left[v^\mu, p^\nu\right], \quad \gamma D^{(v)}_{\mu} p^\nu = [D^{(v)}_{\mu} p^\nu, D] \quad (4.8)$$

The suffix $v$ has been introduced to distinguish $D^{(v)}_{\mu}$ from $D^{(A)}_{\mu}$.

The crucial observation now is that one can use (4.1) to express the Yang-Mills connection $A^A_{\mu}$ in terms of the variables $\xi^a$, $\partial^\mu \xi^a$, $p^a_{\mu}$ and $v^a_{\mu}$ (in the vicinity of the identity). Indeed, the matrix multiplying $A^A_{\mu}$ in (4.1) is the matrix $k^B_A$ of (2.9), which is invertible. The relation $A^A_{\mu} = A^A_{\mu}(\xi^a, \partial^\mu \xi^a, p^a_{\mu}, v^a_{\mu})$ may clearly be prolonged to the derivatives of $A^A_{\mu}$, which thus may be expressed in an invertible way through the ordinary derivatives of $\xi^a$ and $v^a_{\mu}$ as well as the $v$-covariant derivatives of $p^a_{\mu}$. Furthermore, the relation between the original ghosts $C^A$ and the redefined ones, $C^A = \omega^A_B D^B$, may also be prolonged to the subsequent derivatives.

In the same way, all the linearly transforming fields $y^i$ may be combined with the coset representative $U(\xi)$ written in the adequate representation, to form $H$-covariant fields together with their $v$-covariant derivatives. If for instance $y^i$ transforms in the representation $D$ of $G$ generated by $(Y_A)^i_j$, $y^i = D(g)y$, then the quantity $\tilde{y} = D(U^{-1}(\xi))y$ transforms only under $H$, $\tilde{y}^i = D(h)\tilde{y}^i$, with $h$ given by (2.5). In terms of the BRST differential, $\gamma \tilde{y}^i = -Y^i_{\alpha j} \tilde{y}^j D^\alpha$. The $v$-covariant derivative $D^{(v)} \tilde{y}$ shares the same transformation property.

Thus, one may take as redefined set of jet-space coordinates the local coset coordinates $\xi^a$ and the new ghosts $D^a$, together with their ordinary derivatives; the $H$-gauge connection $v^a_{\mu}$ with its ordinary derivatives; and the $H$-covariant quantities $p^a_{\mu}$ and $\tilde{y}^i$ with their $v$-covariant derivatives.

The variables $\xi^a$ and $D^a$ form BRST contractible pairs since $\gamma \xi^a = D^a$ and $\gamma D^a = 0$. The same is true for their derivatives. Thus the BRST cohomology involves only the $H$-connection $v^a_{\mu}$ and the the ghosts $D^a$, as well as the $H$-covariant objects $p^a_{\mu}$ and $\tilde{y}^i$, together with their $v$-covariant derivatives. The problem of computing $H(\gamma)$ for the Yang-Mills + gauged non-linear sigma model $G/H$ with gauge group $G$ is thus reduced to the problem of computing $H(\gamma)$ for a Yang-Mills model with gauge group $H$.
and fields transforming in some definite linear representation of $H$. This is a well-known problem whose solution may be found in [9, 10, 11]. Let $[\tilde{\chi}]_c$ denote the $H$-covariant fields $p^a_\mu$, $\tilde{y}^i$ and $F^{(v)\alpha}_{\mu\nu}$ and their successive $v$-covariant derivatives. Then, any $\gamma$-cocycle takes the form

$$\gamma a = 0 \Rightarrow a = P^L_{\text{inv}}([\tilde{\chi}]_c)\omega^L_{\text{Lie}}(D) + \gamma b$$ (4.9)

where $P^L_{\text{inv}}$ is an invariant polynomial and $\omega^L_{\text{Lie}}(D)$ stands for a basis of the Lie algebra cohomology of $H$. Since $H$ is compact or semi-simple, this cohomology is generated by the "primitive elements" which are the ghosts of the abelian factors as well as $trD^3$ and traces of appropriate higher powers of $D$ for the non-abelian factors (see [12] for more information).

Two extreme cases should be mentioned. In the principal case ($H = \{e\}$), the variables $v^a_\mu$ are absent and the $p^a_\mu$ reduce to the $\tilde{I}^a_\mu$ of [3]. At the opposite end, when $H = G$ ($G/G$-model), $U(\xi) = I$ (so that $\xi^a = \partial_\xi^a = \cdots = 0$), the variables $p^a_\mu$ are zero and $v^a_\mu \equiv v^A_\mu = A^A_\mu$: one recovers the standard results for the Yang-Mills theory with gauge group $G$.

5 $\gamma$-cocycles modulo $d$

The previous analysis provides the $\gamma$-cohomology for all values of the ghost numbers. In particular, at ghost number zero (no $\omega^L_{\text{Lie}}(D)$ in (4.3)), one sees that the most general gauge-invariant polynomial is an invariant polynomial in the $H$-covariant fields $p^a_\mu$, $F^{(v)\alpha}_{\mu\nu}$, $\tilde{y}^i$ and their covariant derivatives. There are no other strictly invariant terms.

To determine the terms that are gauge-invariant only up to a total divergence, one must, through the descent equation, determine the possible bottoms (solutions of $\gamma a = 0$) that can be lifted. Again, this problem has been systematically studied [3, 10, 11] with the result that the most general solution of the equation $\gamma a + db = 0$ is given, at ghost number zero, by

$$a = c + CS + dm$$ (5.1)

where $c$ is strictly invariant ($\gamma c = 0$) and thus of the above form while $CS$ stands for a linear combination with constant coefficients of the Chern-Simons terms for the effective gauge group $H$, which are available only in odd dimensions. The Chern-Simons terms do not involve the “matter” fields.
\[ \tilde{y}^i \text{ or } p_\mu^a \text{ and their derivatives, but only the vector potential } v^\alpha \text{ and its field strength. In particular, one way rewrite the Chern-Simons terms for } G \text{ as Chern-Simons terms for } H \text{ modulo strictly gauge-invariant terms and total derivatives.} \]

Similarly, the terms constructed in [3], which are defined in even dimensions, differ from strictly gauge-invariant terms by total divergences. This was explicitly verified in [6] in the principal case with abelian gauge group. A non-Abelian example follows from the analysis of [13]. The gauge group is three-dimensional and its Lie algebra reads

\[ [T_0, T_1] = -2T_1, \quad [T_0, T_2] = T_2, \quad [T_1, T_2] = 0. \]  

(5.2)

The subgroup generated by \( T_0 \) is non-compact.

The construction of [5] relies on a non-trivial cocycle \( c_{A,BC} \) of the Lie-algebra cohomology with value in the symmetric tensor product of the adjoint representation, which we take to be as in [13]

\[ c_{1,22} = 2, \quad c_{2,12} = c_{2,21} = -1, \]  

(5.3)
others vanish. Besides the Yang-Mills field, we consider the field \( \xi^A \) taking value in the group \( G \) (principal case). The corresponding semi-invariant terms take the form

\[ S_{AB}(\xi^A)F^A \wedge F^B + \frac{2}{3}C_{A,BC}A^A \wedge A^B \wedge (dA^C - \frac{3}{8}f_{DE}A^D \wedge A^E) \]  

(5.4)

One easily verifies that (5.4) is gauge-invariant up to a non-vanishing surface term provided \( S_{AB} = S_{BA} \) transforms as

\[ \delta_\epsilon S_{AB} = (2S_{C(AB)D} + C_{D,AB})\epsilon^D \]  

(5.5)

One may take for \( S_{AB}(\xi) \)

\[ S_{00} = 2(\xi^2)^2\xi^1, \quad S_{01} = \frac{1}{2}(\xi^2)^2, \quad S_{02} = -2\xi^1\xi^2, \]  

(5.6)

\[ S_{11} = 0, \quad S_{12} = -\xi^2, \quad S_{22} = 2\xi^1 \]  

(5.7)

up to terms that transform homogeneously and clearly lead to strictly gauge-invariant contributions to the action. In this case (5.4) reads explicitly

\[ 2(\xi^2)^2\xi^0 \wedge F^0 + (\xi^2)^2F^0 \wedge F^1 - 4\xi^1\xi^2F^0 \wedge F^2 - 2\xi^2F^1 \wedge F^2 + 2\xi^1F^2 \wedge F^2 + 2A^1 \wedge A^2 \wedge dA^2 \]  

(5.8)
By a direct calculation, one easily checks that (5.8) differs from the strictly
gauge-invariant term
\[ 2p^1 \wedge p^2 \wedge dp^2 \]  
(5.9)
by a total derivative. Here, the \( p^A \)'s are the invariant one-forms
\[ p^0 = I^0, \quad p^1 = (\exp(-2\xi^0))[I^1 + 2\xi^1 I^0], \quad p^2 = (\exp\xi^0)[I^2 - \xi^2 I^0] \]  
(5.10)
with
\[ I^0 = d\xi^0 - A^0, \quad I^1 = d\xi^1 - 2\xi^1 d\xi^0 - A^1, \quad I^2 = d\xi^2 + \xi^2 d\xi^0 - A^2. \]  
(5.11)

Even though \( c_{A,BC} \) defines a non-trivial Lie-algebra cohomological class (for \( G \)), there is no obstruction in rewriting (5.8) in a manifestly gauge-invariant way by adding a total derivative. The algebraic obstructions for doing so are given by the Lie-algebra cohomology of \( H \), and are absent here. [In addition, as mentioned in the conclusions, the De Rham cohomology of \( G \) controls whether the construction done in a star-shaped neighbourhood of the identity in field space can be extended to the whole of the field manifold. There is no problem here since the De Rham cohomology of \( G \) is trivial; and indeed, (5.9) is globally defined.]

6 Gauged Wess-Zumino-Witten Terms

The problem investigated in this letter is the problem of finding the most general term that can be added to the Lagrangian while preserving \( G \)-gauge-invariance (up to a total derivative). This problem has an indirect bearing on the problem of gauging the Wess-Zumino-Witten term, which we first briefly review.

The Wess-Zumino-term [14, 15] for the ungauged non-linear sigma model in \( n \) spacetime dimensions reads
\[ W = \int_X \sigma^* h \]  
(6.1)
where \( X \) is a \((n + 1)\)-dimensional manifold with spacetime as boundary and where \( h \) is a closed \((n + 1)\)-form on the manifold \( M \) of the scalar fields that
is invariant under $G$,

$$\mathcal{L}_{\Omega_A} h = 0, \quad dh = 0. \quad (6.2)$$

In (6.1), $\sigma^* h$ is the pull-back of $h$ to $X$. One may write $h = db$ locally in field space. Thus, one can transform (6.1) as a $n$-dimensional integral over spacetime

$$W = \int d^n x b_{a_1...a_n} (\xi(x)) \partial_{\alpha_1} \xi^{a_1} \partial_{\alpha_2} \xi^{a_2} \cdots \partial_{\alpha_n} \xi^{a_n} \epsilon^{\alpha_1...\alpha_n} \quad (6.3)$$

for configurations of the fields in a star-shaped neighbourhood of $\xi = 0$.

Because $h$ is invariant under $G$, the $n$-form $b$ is invariant under $G$ up to a total derivative

$$\mathcal{L}_{\Omega_A} b = dk_A \quad (6.4)$$

This guarantees that the Wess-Zumino-Witten term is (semi-)invariant under the rigid $G$-transformations generated by the Killing vectors $\Omega^a_A$. Again, it may not be possible to add a total derivative to the integrand of (6.3) to make it strictly $G$-invariant. There may be algebraic restrictions for doing this, independently of whether $b$ can be globally defined in the whole of $M$. For instance, a non-zero translation-invariant (= constant) $(n+1)$-form in $\mathbb{R}^k$ is closed and cannot be written as the exterior derivative of a translation-invariant $n$-form since the exterior derivative of a constant $n$-form vanishes. The obstruction exists even though the De Rham cohomology of $\mathbb{R}^k$ is trivial; it can be detected by a local analysis.

One says that the WZW term (6.1) can be gauged if one can add to it terms involving the $G$-connection $A^A_\mu$ so as to make it invariant under local $G$-transformations up to a total derivative. We claim that in even spacetime dimensions, the WZW term can be gauged if and only if one can make the integrand of (6.3) strictly invariant under $G$-transformations by adding to it a total derivative. In that case, the gauging is of course direct. If one cannot (locally) replace $b$ in (6.3) by a term that is strictly invariant under rigid $G$-transformations, the gauging is impossible.

This is an immediate consequence of our analysis. Indeed, if the WZW term can be gauged, then the gauged WZW term must be in the list given above of the terms that can be added to the Lagrangian without destroying the gauge symmetry. Thus, the gauged WZW term (if it can be gauged) must be one of the possible semi-invariant terms for (2.15).
In even dimensions, there is no Chern-Simons term and all semi-invariant terms differ from strictly invariant ones by total derivatives. Thus, in particular, the gauged WZW term (if it exists) takes the form \( WZW_{\text{gauged}} = \int (\text{Invariant} + dk) \). Setting the connection \( A_\mu^A \) equal to zero in this relationship yields the desired result. Note that the argument does not work in odd spacetime dimensions. There are gaugeable WZW terms that are not described by invariant forms \( b \). Upon gauging, these lead to the Chern-Simons terms for \( H \), an information that one can use to explicitly construct them.

Because the conditions for gauging [16, 17] are not usually formulated in those terms, we find it instructive to verify explicitly the (local) equivalence of the gauged WZW term to a strictly \( G \)-invariant term in the simplest case, namely, two dimensions. As stated above, the obstructions to that equivalence can be detected by a local analysis, so we work in a stratum of the manifold \( M \) of the scalar fields where the isotropy group is constant. We consider in fact a region of the form \( B \times G/H \) where \( B \) is a star-shaped open set with \( G \)-invariant coordinates \( \psi^\Delta \). We keep explicitly the \( B \)-factor to check that it is indeed irrelevant for the present purposes.

Let \( N \) be the counting operator for the \( d\psi \)'s,

\[
N = d\psi^\Delta \frac{\partial}{\partial (d\psi^\Delta)}. \tag{6.5}
\]

The 3-form \( h \) splits into a sum

\[
h = h^{(0)} + h^{(1)} + h^{(2)} + h^{(3)} \tag{6.6}
\]

where \( Nh^{(k)} = kh^{(k)} \), i.e., \( h^{(k)} \) contains \( k \) \( d\psi^\Delta \) and \( (3-k) \) \( d\xi^a \). Since the Killing vectors \( \Omega^a_A \) depend only on the \( \xi \)'s, one has \([N, \mathcal{L}_{\Omega_A}] = 0\). Furthermore, \( d = d_0 + d_1 \), \([N, d_0] = 0\), \([N, d_1] = d_1 \) with \( d_0 = (\partial/\partial \xi^a) d\xi^a \) and \( d_1 = (\partial/\partial \psi^\Delta) d\psi^\Delta \).

Because \( dh = 0 \) and \( \mathcal{L}_{\Omega_A} h = 0 \), one has \( d_1 h^{(3)} = 0 \) and \( \mathcal{L}_{\Omega_A} h^{(3)} = 0 \). The Poincaré lemma in \( B \) implies \( h^{(3)} = d_1 u^{(2)} \) where \( u^{(2)} \) is a 2-form that may be taken to be invariant, \( \mathcal{L}_{\Omega_A} u^{(2)} = 0 \). Indeed, the \( \xi^a \) are external parameters for \( d_1 \) so the standard homotopy for \( d_1 \) (not \( d \!\!) \), obtained by integrating along rays in \( B \), is easily verified to commute with \( \mathcal{L}_{\Omega_A} \). If one substracts \( du^{(2)} \) from \( h \), one gets a 3-form \( h' \) without components containing three \( d\psi \)'s, \( h = h' + du^{(2)} \), \( h' = h^{(0)} + h^{(1)} + h^{(2)} \).
By repeating the analysis for \( h''(2) \) and then \( h''(1) \), one easily concludes that \( h \) takes the form
\[
h = \tilde{h} + du
\]
where both \( u \) and \( \tilde{h} \) are invariant
\[
\mathcal{L}_{\Omega_A} u = 0, \quad \mathcal{L}_{\Omega_A} \tilde{h} = 0
\]
and where \( \tilde{h} \) reduces to its 0-th component,
\[
\tilde{h} = \tilde{h}^{(0)}.
\]
One has also \( d\tilde{h} = 0 \), which implies both \( d_1\tilde{h} = 0 \) and \( d_0\tilde{h} = 0 \). The first condition means that \( \tilde{h} \) has no \( \psi^\Delta \)-dependence and is thus a 3-form defined entirely on \( G/H \). Since \( h \) is gaugeable if and only if \( \tilde{h} \) is (recall that \( u \) in (6.7) is \( G \)-invariant), we have completely eliminated the \( B \)-factor. We can thus assume without loss of generality that \( M \) reduces to a single orbit \( G/H \). This will be done in the sequel; we shall also drop the tilde on \( \tilde{h} \).

As shown in [16, 17], the 3-form \( h \) on \( G/H \) leads to a \( G \)-gaugeable WZW term if and only if it fulfills
\[
i_{\Omega_A} h = dv_A
\]
for some 1-forms \( v_A \) on \( G/H \) that are required to obey
\[
i_{\Omega_A} v_B + i_{\Omega_B} v_A = 0
\]
as well as
\[
\mathcal{L}_{\Omega_A} v_B = f_{AB}^C v_C.
\]
Now, the algebraic condition (6.11) implies that \( v_A \) takes the form
\[
v_A = i_{\Omega_A} b
\]
for some 2-form \( b \) on \( G/H \) that is uniquely determined from \( v_A \). Indeed, given \( v_A \) and \( \Omega_A \), the equations (6.13), which read explicitly \( \Omega_A^b b_{ab} = v_A^a \), form a system of linear, inhomogeneous equations for \( b_{ab} \). The system possesses at most a solution because the Killing vectors span the tangent space as \( G \) is transitive, so \( \Omega_A^b c_{ab} = 0 \) implies \( c_{ab} = 0 \). The solution exists because the consistency conditions of (6.13) are precisely (6.11).
Substituting (6.13) into (6.12) yields then

\[ i_{\Omega_B} (\mathcal{L}_{\Omega_A} b) = 0. \]  

(6.14)

Again, since the Killing vectors span the whole tangent space to \( G/H \), this implies

\[ \mathcal{L}_{\Omega_A} b = 0. \]  

(6.15)

The 2-form \( b \) is thus invariant. Inserting (6.13) into (6.10) and using (6.13) shows then that \( H \) is the exterior derivative of an invariant 2-form. A different way to arrive at the same conclusion is to consider the descent associated with (6.3) and to observe that there is no element in \( H(\gamma) \) with ghost number two and form-degree zero, or ghost number one and form-degree one, which could serve as non trivial bottom. Thus, the descent is effectively trivial and one may redefine the integrand of (6.3) to make it strictly invariant. [The strictly-invariant expression equivalent to the gauged WZW term may however not be easy to work out explicitly in practice, or may not be convenient].

7 Comments and Conclusions

Our results are somewhat disappointing because they show that the familiar \( H \)-Chern-Simons terms exhaust all semi-invariant terms when \( H \) is compact or semi-simple. There are no others. In particular, in even dimensions, the construction of [5] provides semi-invariant terms that turn out to be equivalent to strictly invariant ones. In retrospect, this is perhaps not too surprising since one knows that \( G \)-invariance boils down to \( H \)-invariance in the coset space case \( G/H \) with reductive embedding of \( H \). To get new terms, one needs thus to consider more general situations.

The above analysis assumed that the field variables of the sigma-model were restricted to a (star-shaped) neighbourhood of the identity. One may easily take into account the non-trivial topology of field space along the lines of [15, 6]. The most convenient way to doing so is to work in the lifted formulation, in which the sigma-model variables are \( G \)-elements and the right action of \( H \) is gauged by means of a gauge field without kinetic term. One just finds, besides the Chern-Simons terms, additional semi-invariant terms, namely the \( G \)-"winding number" terms. These well-known terms are
the pull-back to spacetime of representatives of the De Rham cohomological classes of $G$. Like $\theta$-terms, they can be written (locally in field space but not globally) as total divergences and do not contribute therefore to the equations of motion. Chern-Simons like terms are not of this type since they do modify the equations of motion and thus can be detected even in a local approach in field space. The details are given in [19], where the inclusion of the antifields is also treated and shown not to alter the main conclusions. One can, also in the presence of antifields, reduce the problem to the much studied standard Yang-Mills problem with gauge group $H$.

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References

[1] S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (N. Y.) 140 (1982) 372.

[2] K. Cahill, Phys. Rev. D 18 (1978) 2930;
B. Julia and J. F. Luciani, Phys. Lett. B 90 (1980) 270;
B. de Wit, P. G. Lauwers, R. Philippe and A. Van Proeyen, Phys. Lett. B 135 (1984) 295;
C. M. Hull, Phys. Lett. B 142 (1984) 39.

[3] C. M. Hull, A. Karlhede, U. Lindström and M. Roček, Nucl. Phys. B 266 (1986) 1.

[4] R. Percacci and E. Sezgin, Properties of Gauged Sigma Models, contribution to R. Arnowitt Fest, hep-th/9810183.

[5] B. de Wit, C. M. Hull and M. Roček, Phys. Lett. B 184 (1987) 233.
[6] M. Henneaux and A. Wilch, *Phys. Rev.* D **58** (1998) 025017.

[7] R. S. Palais, *Ann. Math.* **73** (1961) 295;  
J. Isenberg and J. E. Marsden, *Phys. Rep.* **89** (1982) 862.

[8] S. Weinberg, *Phys. Rev.* **166** (1968) 1568;  
S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** (1969) 2239;  
C. G. Callan, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** (1969) 2247;  
A. Salam and Strathdee, *Ann. Phys. (N.Y.)* **141** (1982) 316;  
P. van Nieuwenhuizen, in Proceedings of *Supersymmetry and supergravity ’84*, B. de Wit, P. Fayet and P. van Nieuwenhuizen eds., World Scientific (1984), p. 469.

[9] M. Dubois-Violette, M. Talon, C. Viallet *Commun. Math. Phys.* **102** (1985) 105.

[10] F. Brandt, N. Dragon, M. Kreuzer, *Nucl. Phys.* B **332** (1990) 224.

[11] M. Dubois-Violette, M. Henneaux, M. Talon and C. M. Viallet, *Phys. Lett.* B**289** (1992) 361.

[12] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature and cohomology*, vol. III, Academic Press (New York: 1976).

[13] B. de Wit, P. G. Lauwers and A. Van Proeyen, *Nucl. Phys.* B **255** (1985) 569.

[14] J. Wess and B. Zumino, *Phys. Lett.* B **37** (1971) 95.

[15] E. Witten, *Commun. Math. Phys.* **92** (1984) 455.

[16] C. M. Hull and B. Spence, *Phys. Lett.* B **232** (1989) 204; *Nucl. Phys.* B **353** (1991) 379.

[17] I. Jack, D.R.T. Jones, N. Mohammedi and H. Osborn, *Nucl. Phys.* B **332** (1990) 359.

[18] G. Barnich, F. Brandt and M. Henneaux, *Nucl. Phys.* B **455** (1995) 357.

[19] A. Wilch, Ph. D Thesis (Brussels) (1999).