DAVEY-STEWARTSON EQUATION FROM A ZERO CURVATURE AND A SELF-DUALITY CONDITION

J. C. Brunelli
and
Ashok Das
Department of Physics and Astronomy
University of Rochester
Rochester, NY 14627

Abstract

We derive the two equations of Davey-Stewartson type from a zero curvature condition associated with SL(2,R) in 2 + 1 dimensions. We show in general how a 2 + 1 dimensional zero curvature condition can be obtained from the self-duality condition in 3 + 3 dimensions and show in particular how the Davey-Stewartson equations can be obtained from the self-duality condition associated with SL(2,R) in 3 + 3 dimensions.
I. Introduction:

Integrable field theories are quite interesting and a lot of their properties are well studied in 1 + 1 dimensions [1,2]. In 2 + 1 dimensions, however, only a few are known and very little is known about their structure. While the KP equation (Kadomtsev-Petviashvilli equation) [3] has been the subject of intense studies in recent years, the Davey-Stewartson equations [4] – which are also interesting in their own right – have not received as much attention. In particular, even though one knows the matrix Lax pair for this equation [5], a scalar Lax formulation or even a zero curvature formulation [2,6] for the system does not exist. Similarly, the Gelfand-Dikii bracket structures [7] for such systems are also not understood. In fact, it is not known how most of the interesting properties of 1 + 1 dimensional theories extend to higher space-time dimensions. In this letter, we make a modest contribution to such studies. In particular, we show, in sec. II how the Davey-Stewartson equation can be obtained from a zero curvature condition [2,6] associated with SL(2,R) in 2 + 1 dimensions. In sec. III, we further derive these equations from the self-duality conditions [8-10] associated with SL(2,R) in 3 + 3 dimensions. We end with a brief conclusion in sec. IV.

II. Zero Curvature Formulation:

There are two Davey-Stewartson equations [4,5] in 2 + 1 dimensions commonly known as DSI and DSII equations and both are known to be integrable. DSI has the form

\[ i\dot{q} = -\frac{1}{2} (\partial_1^2 + \partial_2^2)q - (qr + \partial_2\phi)q \]
\[ (\partial_1^2 - \partial_2^2)\phi - 2\partial_2(qr) = 0 \]

where \( q \) and \( \phi \) are the basic dynamical variables. \( q \) is complex while \( \phi \) is real and \( r = \pm q^* \).

In our notation \( \partial_0, \partial_1 \) and \( \partial_2 \) represent derivatives with respect to \( t, x, y \) respectively. The other equation, DSII, on the other hand, has the structure

\[ i\dot{q} = -\frac{1}{2} (\partial_1^2 - \partial_2^2)q + (qq^* + \partial_2\phi)q \]
\[ (\partial_1^2 + \partial_2^2)\phi + 2\partial_2(qq^*) = 0 \]
We note here that Eq. (2) can be obtained from Eq. (1) by letting $y \rightarrow -iy$, $\phi \rightarrow i\phi$ and identifying $r = -q^*$. Similarly, we can obtain Eq. (1) from Eq. (2) by letting $y \rightarrow -iy$, $\phi \rightarrow i\phi$ and identifying $q^* = -r$. Since DSI and DSII can be obtained from each other through simple redefinitions, we will concentrate, for simplicity, on only one of them, namely, DSII given in Eq. (2).

The nonlinear equations (2) can also be written in terms of linear equations of the form \[ i\partial_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \partial_1 + i\partial_2 & -q \\ -q^* & \partial_1 - i\partial_2 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0 \] (3)

\[ i\partial_0 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \partial_2^2 + \frac{i}{2}((\partial_1 - i\partial_2)\phi) + \frac{1}{2}q^*q & iq\partial_2 - \frac{1}{2}((\partial_1 - i\partial_2)q) \\ iq^*\partial_2 + \frac{1}{2}((\partial_1 + i\partial_2)q^*) & -\partial_2^2 + \frac{i}{2}((\partial_1 + i\partial_2)\phi) - \frac{1}{2}q^*q \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \] (4)

The consistency of Eqs. (3) and (4), then give rise to the Davey-Stewartson equations (2). Eqs. (3) and (4), therefore, define the Lax pair for the system, but note that these Lax operators

\[ L = \begin{pmatrix} \partial_1 + i\partial_2 & -q \\ -q^* & \partial_1 - i\partial_2 \end{pmatrix} \] (5)

\[ B = \begin{pmatrix} \partial_2^2 + \frac{i}{2}((\partial_1 - i\partial_2)\phi) + \frac{1}{2}q^*q & iq\partial_2 - \frac{1}{2}((\partial_1 - i\partial_2)q) \\ iq^*\partial_2 + \frac{1}{2}((\partial_1 + i\partial_2)q^*) & -\partial_2^2 + \frac{i}{2}((\partial_1 + i\partial_2)\phi) - \frac{1}{2}q^*q \end{pmatrix} \] (6)

unlike the 1 + 1 dimensional case are matrix Lax operators. Furthermore, even though these operators are matrices, the consistency condition is not equivalent to a zero curvature condition.

To obtain a zero curvature formulation \[2, 6\] of the Davey-Stewartson equations, let us note that these equations can be thought of as a higher dimensional generalization of the nonlinear Schrödinger equation in 1+1 dimension. The nonlinear Schrödinger equation, on the other hand, is known to arise from a zero curvature condition associated with $\text{SL}(2, \mathbb{R})$. 

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in $1 + 1$ dimensions [2]. Thus, let us choose $\text{SL}(2, \mathbb{R})$ as the appropriate group for our potentials. With the explicit representations of the generators

$$
\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
$$

(7)

let us choose

$$
A_0 = -\frac{i}{2} (qq^* + \partial_2 \phi + 3\lambda^2)\sigma_3 - 3\lambda q^* \sigma_+ - 3\lambda q \sigma_-
$$

$$
A_1 = \sqrt{2} i\lambda \sigma_3 + \frac{1}{\sqrt{2}} q^* \sigma_+ + \frac{1}{\sqrt{2}} q \sigma_-
$$

$$
A_2 = -\frac{i\lambda}{\sqrt{2}} \sigma_3 + \frac{1}{\sqrt{2}} q^* \sigma_+ + \frac{1}{\sqrt{2}} q \sigma_-
$$

(8)

Here $\lambda$ is a constant parameter. The curvatures (field strengths) associated with these potentials can be easily constructed from

$$
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]
$$

(9)

Requiring the curvatures to vanish, we expect nine equations. $F_{01} = 0$ gives

$$
\partial_1 (qq^* + \partial_2 \phi) = 0
$$

(10)

$$
\dot{q} + 3\sqrt{2} \lambda \partial_1 q + i(qq^* + \partial_2 \phi - 9\lambda^2)q = 0
$$

(11)

$$
\dot{q}^* + 3\sqrt{2} \lambda \partial_1 q^* - i(qq^* + \partial_2 \phi - 9\lambda^2)q^* = 0
$$

(12)

Similarly, $F_{02} = 0$ gives

$$
\partial_2 (qq^* + \partial_2 \phi) = 0
$$

(13)

$$
\dot{q} + 3\sqrt{2} \lambda \partial_2 q + i(qq^* + \partial_2 \phi + 9\lambda^2)q = 0
$$

(14)

$$
\dot{q}^* + 3\sqrt{2} \lambda \partial_2 q^* - i(qq^* + \partial_2 \phi + 9\lambda^2)q^* = 0
$$

(15)

Finally, $F_{12} = 0$ leads to only two equations

$$
(\partial_1 - \partial_2)q - 3\sqrt{2} i\lambda q = 0
$$

(16)

$$
(\partial_1 - \partial_2)q^* + 3\sqrt{2} i\lambda q^* = 0
$$

(17)
We note from Eq. (16) that

$$3\sqrt{2} i\lambda (\partial_1 + \partial_2) q = (\partial_1^2 - \partial_2^2) q$$

Adding Eqs. (11) and (14) and using Eq. (18) we obtain

$$\dot{q} + \frac{3}{\sqrt{2}} \lambda (\partial_1 + \partial_2) q + i(qq^* + \partial_2\phi) q = 0$$

or,

$$i\dot{q} + \frac{1}{2} (\partial_1^2 - \partial_2^2) q - (qq^* + \partial_2\phi) q = 0$$

which is, of course, the same as the first of Eq. (2). Similarly, from Eqs. (10), (13), and (17) we obtain

$$\partial_2 \left[ (\partial_1^2 + \partial_2^2) \phi + 2\partial_2(qq^*) \right] = 0$$

which with a suitable field redefinition of \(\phi\) can be written as

$$(\partial_1^2 + \partial_2^2) \phi + 2\partial_2(qq^*) = 0$$

This is the second part of the Davey-Stewartson equations (Eq. (2)). Thus, we see that both the Davey-Stewartson equations can be derived from a zero curvature condition associated with SL(2,R) in 2 + 1 dimensions.

### III. Self-Dual Formulation:

It is well known that the 1 + 1 dimensional integrable models can be obtained from a self-duality condition associated with Yang-Mills field strengths belonging to SL(2,R) in 2 + 2 dimensions [8-10]. The KP equation is also known to result from a self-duality condition on Yang-Mills field strengths belonging to SL(2,R) in 3 + 3 dimensions [11]. Since Davey-Stewartson equation is a 2 + 1 dimensional equation like the KP equation, we examine the self-duality conditions in 3 + 3 dimensions. The self-duality conditions in dimensions higher than four are quite tricky and we refer the interested readers to ref. 11 for details. Here we only note that the self-duality conditions on the field strengths in 3 + 3
dimensions lead to
\[ F'_{02} = 0 \]
\[ F'_{05} = 0 \]
\[ F'_{25} = 0 \]
\[ F'_{13} = 0 \]
\[ F'_{14} = 0 \]
\[ F'_{34} = 0 \]
\[ F'_{01} + F'_{23} + F'_{54} = 0 \] (22)

Here
\[ F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu + [A'_\mu, A'_\nu] \quad \mu, \nu = 0, 1, \ldots, 5 \] (23)

are the 3+3 dimensional field strength tensors (We use primes to emphasize that these are 3+3 dimensional objects.). Let us next choose \( x^0 = t, \ x^2 = x, \ x^5 = y \) and if we identify
\[ A'_0 = A_0 = -\frac{i}{2} (qq^* + \partial_2 \phi + 3\lambda^2) \sigma_3 - 3\lambda q^* \sigma_+ - 3\lambda q \sigma_- \]
\[ A'_1 = A_1 = \sqrt{2} i\lambda \sigma_3 + \frac{1}{\sqrt{2}} q^* \sigma_+ + \frac{1}{\sqrt{2}} q \sigma_- \]
\[ A'_2 = A_2 = \frac{i\lambda}{\sqrt{2}} \sigma_3 + \frac{1}{\sqrt{2}} q^* \sigma_+ + \frac{1}{\sqrt{2}} q \sigma_- \] (24)

then the first three equations of (22) would merely give the Davey-Stewartson equation of Eq. (2) as we have seen in the previous section. The rest of the equations can be solved by noting that with the reduction conditions (This is different from ref. 11.)
\[ \partial_0 - \partial_1 = 0 = \partial_0 - \partial_4 \]
\[ \partial_2 - \partial_3 = 0 \] (25)

and with the identification
\[ A'_1 = A'_4 = A'_0 \quad \text{and} \quad A'_3 = A'_2 \] (26)
we have

\[ F'_{13} = \partial_1 A'_3 - \partial_3 A'_1 + [A'_1, A'_3] = \partial_0 A'_2 - \partial_2 A'_0 + [A'_0, A'_2] = F'_02 = 0 \]
\[ F'_{14} = \partial_1 A'_4 - \partial_4 A'_1 + [A'_1, A'_4] = \partial_0 A'_0 - \partial_0 A'_0 + [A'_0, A'_0] = 0 \]
\[ F'_{34} = \partial_3 A'_4 - \partial_4 A'_3 + [A'_3, A'_4] = \partial_2 A'_2 - \partial_2 A'_2 + [A'_2, A'_2] = -F'_02 = 0 \]
\[ F'_{01} = \partial_0 A'_1 - \partial_1 A'_0 + [A'_0, A'_1] = \partial_0 A'_0 - \partial_0 A'_0 + [A'_0, A'_0] = 0 \]
\[ F'_{23} = \partial_2 A'_3 - \partial_3 A'_2 + [A'_2, A'_3] = \partial_2 A'_2 - \partial_2 A'_2 + [A'_2, A'_2] = 0 \]
\[ F'_{54} = \partial_5 A'_4 - \partial_4 A'_5 + [A'_5, A'_4] = \partial_5 A'_0 - \partial_0 A'_5 + [A'_5, A'_0] = -F'_05 = 0 \]  

(27)

Thus, we see that the rest of the equations in (22) are automatically satisfied with our choice of reduction by virtue of the first three equations which with the ansatz of Eq. (24) lead to the Davey-Stewartson equation. This shows how the Davey-Stewartson equations can be obtained from the self-duality condition on Yang-Mills field strengths belonging to SL(2, \( \mathbb{R} \)) in 3 + 3 dimensions. This result, in fact, is quite general and shows that any 2 + 1 dimensional equation which can be formulated as a zero curvature condition, can also be obtained from a self-duality condition on Yang-Mills fields in 3 + 3 dimensions with our choice of reduction and gauge potential identifications given in Eqs. (25) and (26). This result is, in fact, quite similar to the general procedure outlined for self-duality in 2 + 2 dimensions [12]. This, therefore, gives an alternate to the derivation of the KP equation from a self-duality condition proposed in ref. 11 also.

**IV. Conclusion:**

We have shown how the two Davey-Stewartson equations can be obtained from a zero curvature condition associated with SL(2, \( \mathbb{R} \)) in 2 + 1 dimensions. We have also shown how a general equation in 2 + 1 dimensions resulting from a zero curvature condition can be obtained from a self-duality condition on Yang-Mills field strengths in 3 + 3 dimension with appropriate reduction. In particular, we have shown that the Davey-Stewartson equations can be obtained from a self-duality condition associated with field strengths belonging to SL(2, \( \mathbb{R} \)).
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