Noncommutative Martin-Löf randomness: on the concept of a random sequence of qubits

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Martin-Löf’s definition of random sequences of cbits as those not belonging to any set of constructive zero Lebesgue measure is reformulated in the language of Algebraic Probability Theory.

The adoption of the Pour-El Richards theory of computability structures on Banach spaces allows us to give a natural noncommutative extension of Martin-Löf’s definition, characterizing the random elements of a chain Von Neumann algebra.

In the particular case of the minimally informative noncommutative alphabet our definition reduces to the definition of a random sequence of qubits.

I. INTRODUCTION

Probability Theory may be defined as that branch of Mathematics studying random phenomena. The mathematical foundation of Classical Probability Theory was given by A.N. Kolmogorov in terms of Measure Theory: a classical probability space is nothing but a measure space in which the measure is normalized to one [1].

As far as physical phenomena are concerned, Quantum Physics, differing in this by Classical Physics, is not deterministic: according to the Modern Statistical Interpretation (that we adhere) quantum measurement is a random phenomenon [2].

In spite of being formalizable in the language of Classical Probability Theory, quantum randomness presents some peculiarity that led Feynman to implicitly suggest that its deep comprehension requires the introduction of a Quantum Probability Theory different from the classical one:

1. the composition property is satisfied by classical-probability amplitudes and not by their square-moduli [3];

2. a generic quantum system can’t be simulated efficiently by a classical universal computer [3], implying that the complexity class QP of the problems soluble with certainty in the worst case in polynomial time by a (quantum-probabilistically - non-deterministic) - computer is different both from the class P of the problems soluble with certainty in the worst case in polynomial time by a deterministic classical computer and from the class NP of the problems soluble with certainty in the worst case in polynomial time by a non-deterministic classical computer [3], [4] (i.e. quantum non-determinism is different both from classical determinism and from classical non-determinism, a thing unfortunately never considered in all the discussions about the possibility of a deterministic completion of quantum mechanics).

The adoption of the W*-algebraic language, in particular, allows a unified formulation of both Classical and Quantum Physics, the classical case being characterized by the abelianity of the observables’ algebra [5], [6], [7], leading through Noncommutative Measure Theory, to the delineation of Quantum Probability Theory as the noncommutative generalization of Classical Probability Theory.

Now it is well known that, curiously, Classical Probability Theory is not self-contained as far as the definition of a random sequence on a finite alphabet is concerned: its appropriate characterization, given by Martin-Löf [8], [9], [10], requires notions from a field of Mathematics very far from Measure Theory, i.e. Classical Recursion Theory [11].

In this paper we analyze the same issue for Quantum Probability Theory: a brief review of Martin-Löf’s randomness in sect. I is reformulated, in sect. II in the language of W*-algebras.

The adoption of the Pour-El Richards theory of computability structures on Banach spaces allows us to give a natural noncommutative extension of Martin-Löf definition, characterizing the random elements of a chain Von Neumann algebra.

In the particular case of the minimally informative noncommutative alphabet our definition reduces to the definition of a random sequence of qubits.

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II. MARTIN-LOF RANDOM SEQUENCES

Given a finite alphabet $\Sigma$ let us denote with $\Sigma^*$ the set of the strings on $\Sigma$ and with $\Sigma^\infty$ the set of the sequences on $\Sigma$. Since:

$$\text{card}(\Sigma^*) = \aleph_0$$ (2.1)

$$\text{card}(\Sigma^\infty) = \aleph_1$$ (2.2)

we can make, modulo recursive codings, the following identifications:

$$\Sigma \cong \{0, 1\}$$ (2.3)

$$\Sigma^* \cong \mathbb{N}$$ (2.4)

$$\Sigma^\infty \cong (0, 1]$$ (2.5)

Denoted by $\mathcal{F}_{\text{cylinder}}$ the $\sigma$-algebra of the cylinder sets on $\Sigma$ let us consider a probability measure $P$ on the measurable space $(\Sigma^\infty, \mathcal{F}_{\text{cylinder}})$.

The more natural way to characterize the random sequences on $\Sigma$ would seem to be the following: introduced the notion of:

**DEF II.1 (NOT-CONSTRUCTIVE P-TYPICAL SUBSETS OF $\Sigma^\infty$)**

$$P - \text{Typ}(\Sigma)_{\text{NC}} \equiv \{ S \subseteq \Sigma^\infty : \neg(\forall \epsilon > 0 \exists \{A_n \subset \Sigma^\infty\}_{n \in \mathbb{N}} : S \subset \bigcup_{n \in \mathbb{N}} A_n \land \sum_{n=0}^{\infty} P(A_n) < \epsilon) \}$$ (2.6)

it should be natural to call random those sequences on $\Sigma$ not belonging to any not-constructive $P$-typical set.

Such an approach is, anyway, invalidated by the following:

**Theorem II.1 (on the not self-sufficiency of Measure Theory to define randomness)**

$$P - \text{Typ}(\Sigma)_{\text{NC}} = \emptyset \forall \text{ probability measure } P \text{ on } (\Sigma^\infty, \mathcal{F}_{\text{cylinder}})$$ (2.7)

The above approach was saved by P. Martin-Lof at the prize of requiring the introduction of ingredients not belonging to Measure Theory but to Classical Recursion Theory: introduced, in fact, the following notion:

**DEF II.2 (CONSTRUCTIVE P-TYPICAL SUBSETS OF $\Sigma^\infty$)**

$$P - \text{Typ}(\Sigma)_C \equiv \{ S \subseteq \Sigma^\infty : \neg(\forall \epsilon > 0 \exists \{A_n \subset \Sigma^\infty\}_{n \in \mathbb{N}} : \{A_n\} \text{ is r.e. } \land S \subset \bigcup_{n \in \mathbb{N}} A_n \land \sum_{n=0}^{\infty} P(A_n) < \epsilon) \}$$ (2.8)

depending on the recursion-theoretic notion of recursive-enumerability (r.e.-bility) and given a sequence $\{\omega_n\}_{n \in \mathbb{N}}$ on $\Sigma$ we may characterize its randomness in the following way:

**DEF II.3 ( $\{\omega_n\}_{n \in \mathbb{N}}$ is Martin-Lof random )**

$$\{\omega_n\}_{n \in \mathbb{N}} \notin S \forall S \in P_{\text{Lebesgue}} - \text{Typ}(\Sigma)_C$$ (2.9)

where $P_{\text{Lebesgue}} - \text{Typ}(\Sigma)_C$ is the Lebesgue probability measure on $\Sigma^\infty$. 

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III. MARTIN-LOF RANDOMNESS IN THE LANGUAGE OF VON NEUMANN ALGEBRAS

Demanding to the monograph of W. Thirring [7], [8] for any notion about Von Neumann algebras and their link with Quantum Mechanics, let us introduce the following notion:

**DEF III.1 (ALGEBRAIC PROBABILITY SPACE)**

\((A, \omega)\):

- A is a W*-algebra
- \(\omega \in S(A)\) is a state on A

The notion of algebraic probability space is a noncommutative generalization of the notion of classical probability space, i.e. the concepts of classical probability space and abelian algebraic probability space are conceptually equivalent as is shown by the following facts:

1. given a classical probability space \((S, \mathcal{F}, P)\) let us observe that it can also be described in a different, conceptually equivalent, way as the couple \((A, \omega)\) where:
   - A is the abelian W*-algebra \(L^\infty(S, P)\)
   - \(\omega \in S(A)\) is the state on A defined by:
     \[
     \omega(a) = \int_S a(x) dP(x) \quad a \in A
     \]  
     (3.1)

2. given an abelian W*-algebra \(A\) and a state on it \(\omega \in S(A)\), there exist a classical probability space \((S, \mathcal{F}, P)\) such that \(A = L^\infty(S, P)\) and \(\omega = \omega_P\).

Let us, then, reformulate the notion of Martin-Lof randomness in the algebraic language substituting to the classical probability space \((S, \mathcal{F}, P)\) the corresponding abelian algebraic probability space \((L^\infty(S, P), \omega_P)\); then to a subset \(A \subset \Sigma^\infty\) corresponds its characteristic function \(\chi_A\):
Given a function $f \in L^\infty(\Sigma^\infty, P)$ we have, then, clearly, that:

**Theorem III.2**

$$f \text{ is Martin–Löf random } \iff f \notin \forall S \in \omega_{\text{Lebesgue}} - TY P[L^\infty(\Sigma^\infty, P)]_C$$

As far as the definitions DEF:II.1 and DEF:II.2 are concerned we have already remarked that they involve, through the constructivity assumption, a field of Mathematics very far from Measure Theory, i.e. Recursion Theory.

More precisely they involve that standard part of Recursion Theory that, through robust theorems of equivalence between different approaches, is univoquely determined, i.e. Classical Recursion Theory [14].

Such a lucky situation doesn’t occur, as we will see in the next paragraph, when their noncommutative extensions are considered.

### IV. NONCOMMUTATIVE CONSTRUCTIVE MEASURE THEORY AND RANDOMNESS

Condensed in a slogan the leit-motif of Noncommutative Geometry, corroborated by the observation presented in the previous paragraph, consists in looking at an algebraic probability space $(A, \omega)$ as ” a kind of ” $(L^\infty(S, P), \omega_P)$ with $S$ a ” noncommutative space ” and $P$ a noncommutative probability measure on $S$ [9].

Looking at the definition DEF:III.3 the issue of looking for an its natural noncommutative extension would appear as a typical research program of Noncommutative Geometry.

Such a definition, and the consequent characterization of Martin-Löf randomness given by Theorem III.2 anyway, involving Recursion Theory, belongs to a peculiar, surprisingly unexplored, field of research: Noncommutative Constructive Measure theory.

Given an algebraic probability space $(A, \omega)$ a naive extension of the definition DEF:III.3 would sound as follow:

**Trial definition IV.1 (CONSTRUCTIVE $\omega$-TYPICAL SETS OF $A$)**

$$\omega - TY P[A]_C \equiv \{ B \subseteq A : \neg(\forall \epsilon > 0 \exists \{ a_n \in A \}_{n \in \mathbb{N}} : \text{eq.4.2} \land \text{eq.4.3} \land \text{eq.4.4} \}$$

where:

- $\{ a_n \}$ is recursively enumerable
- $B \subseteq \text{span}(\{ a_n \}_{n \in \mathbb{N}})$
- $\sum_{n=0}^{\infty} \omega(a_n^* a_n) < \epsilon$

Granted, for the moment, that such a definition is correct, it leads immediately to the following definition:

**Trial definition IV.2 ( $a \in A$ is random )**

$$a \notin B \forall B \in \omega_{\text{Lebesgue}} - TY P[A]_C$$

The definitions DEF:IV.1 and DEF:IV.2 are vague and imprecise owed to the following:

**Remark IV.1** it doesn’t exist, in Recursion Theory, a standard definition of a recursively-enumerable sequence on a Von Neumann algebra

**Remark IV.2** it doesn’t exist any notion of a Lebesgue-state on a $W^\ast$-algebra

As far as REMARK:IV.1 is concerned a particularly promising, though not-univoquely determined, extension of Computable Analysis to the theory of Banach spaces has been realized by M. Pour-El and J.I. Richards [15].

Given a Banach space $B$ on the real/complex field Pour-El and Richards introduce the following notion:
DEF IV.1 (COMPUTABILITY STRUCTURE on B) a specification of a subset $S$ of the set of all the sequences in $B$ identified as the set of the computable sequences on $B$ :

AXIOM IV.1 (linear forms)

HP:

$\{x_n\}$ and $\{y_n\}$ computable sequences in B
$\{\alpha_{n,k}\}, \{\beta_{n,k}\}$ two recursive double sequence of real/complex numbers
'd recursive function

$s_n \equiv \sum_{k=0}^{d(n)} \alpha_{n,k}x_k + \beta_{n,k}y_k$

TH:

$\{s_n\} \in S$

AXIOM IV.2 (limits)

HP:

$x_{n,k}$ computable double sequence in B

$r - \lim_{k \to \infty} x_{n,k} = x_n$

TH:

$\{x_n\} \in S$

AXIOM IV.3 (norms)

HP:

$\{x_n\} \in S$

TH:

\{$\|x_n\|$\} is a recursive sequence of real numbers.

These axioms contains some notion we have to specify.
First of all let us observe that, given a sequence $\{x_n\}$ of real or complex numbers, the fact that each element of the sequence is recursive, and can, consequently, be effectively approximated to any desired degree of precision by a computer program $P_n$ given in advance doesn’t imply the recursivity of the whole sequence since there might not exist a way of combining the sequence of programs $\{P_n\}$ in a unique program $P$ computing the whole sequence $\{x_n\}$.

Given a double sequence $\{x_{n,k} \in \mathbb{R}\}$ and an other sequence $\{x_n\}$ of real numbers:

$$\lim_{k \to \infty} x_{n,k} = x_n \quad \forall n \in \mathbb{N}$$  \hfill (4.6)

DEF IV.2 ( $\{x_{n,k}\}$ CONVERGES RECURSIVELY TO $\{x_n\} (r - \lim_{k \to \infty} x_{n,k} = x_n)$ )

$$\exists e : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \in \Delta_0^0 : (k > e(n,N) \Rightarrow |r_k - x| \leq \frac{1}{2^N}) \forall n \in \mathbb{N}, \forall N \in \mathbb{N}$$  \hfill (4.7)

DEF IV.3 ( $\{x_n\}_{n \in \mathbb{N}}$ IS RECURSIVE )

$$\exists \{r_{n,k} \in \mathbb{Q}\}_{n,k \in \mathbb{N}} : |r_{n,k} - x_n| \leq \frac{1}{2^k}$$  \hfill (4.8)

DEF IV.4 ( $\{z_n \in \mathbb{C}\}_{n \in \mathbb{N}}$ )
\{\Re(z_n)\}_{n\in\mathbb{N}} and \{\Im(z_n)\}_{n\in\mathbb{N}} are recursive \hspace{1cm} (4.9)

The above argument should clarify why the definition of a computability structure on a Banach space \(B\) is made through a proper specification of the computable sequences in \(B\) and not, simply, by the specification of a proper set of the computables vectors.

The notion of a computable vector, instead, is immediately induced by the assignment on \(B\) of a computability structure \(S\).

**DEF IV.5 (COMPUTABLE VECTORS OF \(B\))**

\[ \Delta^0(B) \equiv \{x \in B : \{x, x, x, \ldots\} \in S\} \hspace{1cm} (4.10) \]

The Axioms Axiom [IV.1], Axiom [IV.2] and Axiom [IV.3] have a transparent intuitive meaning: since a Banach space is made up of a vector space \(V\), a norm on \(V\) and the completeness-condition for such a norm, it is natural to require analogous effective conditions for the set of computable sequences.

Unfortunately such axioms do not provide the axiomatic definition of a unique structure for a Banach space \(B\) since \(B\) admits, generally, more computability-structures.

This, anyway, doesn’t relativize the whole approach thanks to the existence of a suppletive condition whose satisfiability results in the invoked univocity.

Given a computability structure \(S\) on a Banach space \(B\):

**DEF IV.6 (EFFECTIVE GENERATING SET for \(B\))**

\( \{e_n\} \in S : \text{linear} - \text{span}(\{e_n\}) \text{is dense in } B \) \hspace{1cm} (4.11)

**DEF IV.7 (B is EFFECTIVELY SEPARABLE)**

\[ \exists \{e_n\} \text{ effective generating set for } B \hspace{1cm} (4.12) \]

Fortunately Pour-El and Richards proved the following:

**Theorem IV.1 (of stability)**

HP:

- \(B\) Banach space
- \(S_1, S_2\) effectively separable computability structures on \(B\)
- \(\{e_n\} \in S_1 \cap S_2\) effective generating set for \(B\)

TH:

\[ S_1 = S_2 \]

Let us now return to our algebraic probability space \((A, \omega)\) and let us suppose that \(A\) is endowed with a computability structure \(S\) eventually associated to some effective generating set of observables physically known to be effectively measurable.

Then it would appear meaningful to give meaning to the vague locution “\(\{a_n \in A\}_{n\in\mathbb{N}}\) is recursively-enumerable ” in eq [IV.1] substituting it with the precise condition “\(\{a_n \in A\}_{n\in\mathbb{N}} \in S\)”.

As far as Remark [IV.2] is concerned, let us observe that \(P_{\text{Lebesgue}}\), being the ” uniform probability measure ” on \(\Sigma^\infty\), is that of maximum entropy.

Then it appears natural to think that the role of the unprecised state \(\omega_{\text{Lebesgue}}\) in [IV.2] must be played by a state on \(A\) of maximum entropy according to the following definition:

**DEF IV.8 (ENTROPY of a state \(\omega\) on \(A\))**

\[ S(\omega) \equiv \{ \sum_i \lambda_i S(\omega_i, \omega) : \sum_i \lambda_i \omega_i = \omega \} \hspace{1cm} (4.13) \]

where the supremum is taken over all the decompositions of \(\omega\) in countable convex combinations of other states and where \(S(\omega_i, \omega)\) is the Umegaki-Araki relative entropy of \(\omega_i\) w.r.t. \(\omega\) [16].

Collecting our considerations about Remark [IV.3] and Remark [IV.2] we are, then, lead to introduce the following definitions:
DEF IV.9 (CONSTRUCTIVE $\omega$-TYPICAL SETS OF $(A,S)$)

$$\omega - TYP[A]_C \equiv \{ B \subseteq A : \neg(\forall \varepsilon > 0 \exists \{ a_n \in A \}_{n \in \mathbb{N}} : eq.4.15 \land eq.4.3 \land eq.4.4) \} \tag{4.14}$$

where:

$$\{ a_n \}_{n \in \mathbb{N}} \in S \tag{4.15}$$

DEF IV.10 ($a \in S$ is random)

$$a \notin B \forall B \in \omega_{Maximum\ Entropy} - TYP[A]_C \tag{4.16}$$

where $\omega_{Maximum\ Entropy}$ is the state on $A$ of maximum entropy provided such a state exists.

Though the definition DEF IV.10 was given for an arbitrary W*-algebra endowed with a computability structure, its physical relevance arises when the particular case of chain-algebras is considered.

Let us observe, in fact, that, in the previous paragraph, we didn’t consider arbitrary probability spaces but the particular probability spaces $(\Sigma^\infty, F_{cylinder}, P)$ on the sequences on a finite alphabet $\Sigma$; the physically relevant applications of the definition IV.10 correspond, then, to the case of algebraic probability spaces on the set of the sequences on a ”noncommutative alphabet”.

Such particular algebraic probability spaces are nothing but the well known chain - quasi-local - algebras of the one-dimensional quantum lattice spins systems usual in Quantum Statistical Mechanics \[16\].

As far as usual Martin-Lof randomness is concerned the relations 2.1 and 2.2 explain why all finite alphabets are, as far as the definition of randomness is concerned, absolutely equivalent and justifies, consequently, the reduction of the analysis to the alphabet with minimal classical information, i.e to the one bit alphabet $\{0, 1\}$.

In the same way, we can, as far as the definition of a random noncommutative sequence is concerned, restrict the analysis to the alphabet of minimal ”noncommutative information”, i.e. to the 1-qubit alphabet $\{0, 1\}_{NONCOMMUTATIVE}$ with observable algebra \[1\]:

$$M_2(\mathbb{C}) \equiv \{ \bar{\alpha} \cdot \bar{\sigma} + \beta I, \bar{\alpha} \in \mathbb{C}^3, \beta \in \mathbb{C} \} \tag{4.17}$$

where $\bar{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ is the 3-vector of Pauli matrices and $I$ is the bidimensional identity matrix.

Since both the measurements of spin-components and the preservation of the state are effective operations, it appears physically reasonable to consider $A$ endowed with the computability structure $S$ individuated by the condition that the basis of $A$:

$$S_{spin} \equiv \{ \sigma_1, \sigma_2, \sigma_3, I \} \tag{4.18}$$

is effective.

The computable matrices in $M_2(\mathbb{C})$ are, informally speaking, those obtainable by elements of $S_{spin}$ with ”effective linear combinations”.

As an example let us consider the following matrices:

$$m_{Collatz}(\bar{n}) \equiv \frac{1}{2} (1 + (-1)^{p_{Collatz}(|\bar{n}|+1)} \bar{n} \cdot \bar{\sigma}) \bar{n} \in \mathbb{N}^3 \tag{4.19}$$

where $p_{Collatz} : \mathbb{N} \to \{0, 1\}$ is the Collatz predicate:

$$p_{Collatz}(n) = (\text{cond } \exists m \in \mathbb{N} : T^m(n) = 1, 1, 0) \tag{4.20}$$

while $T^m(n)$ is the sequence:

$$T^m(n) \equiv (\text{cond } m = 1, n, (\text{ cond } T^{m-1}(n) \text{ is even }, \frac{T^{m-1}(n)}{2}, 3T^{m-1}(n))) \tag{4.21}$$

where I have adopted Mc Carthy’s LISP notation for the conditional definitions \[17\]. The well-known recursive-undecidability \[18\] of the:

**Conjecture IV.1 (of Collatz)**
\[ p_{\text{Collatz}}(n) = 1 \quad \forall n \in \mathbb{N} \quad (4.22) \]

implies the recursive-undecidability of the associated statement in term of the matrices \( m_{\text{Collatz}}(\vec{n}) \):

**Conjecture IV.2 ( of Collatz in terms of the W*-algebra \( M_2(\mathbb{C}) \))** \( m_{\text{Collatz}}(\vec{n}) \) is the projector associated to spin \( +1/2 \) along the component \( \vec{n} \), \( \forall \vec{n} \in \mathbb{N}^3 \)

The matrices \( m_{\text{Collatz}}(\vec{n}) \) are given by a non-effective linear combination of the elements of \( S_{\text{spin}} \) and are, then, non-computable.

Let us, then, consider the 1-qubit spin chain:

\[ A \equiv \text{norm} - \text{completion}(A_\mathbb{Z}) \quad , \quad A_\mathbb{Z} \equiv \bigotimes_{\mathbb{Z}} M_2(\mathbb{C}) \quad (4.23) \]

A state of maximum entropy on \( A \) is, clearly, that represented by the density matrix:

\[ \rho_{\text{uniform}} \equiv \times_{\mathbb{Z}} \text{diagonal}(\frac{1}{2}, \frac{1}{2}) \quad (4.24) \]

describing the thermodinamical equilibrium of a one-dimensional lattice of spins-1/2 at infinite temperature.

Furthermore, the computability structure \( S_{\text{spin}} \) induces, naturally, the computability structure on \( A \):

\[ S \equiv \times_{\mathbb{Z}} S_{\text{spin}} \quad (4.25) \]

Given, then, a sequence of qubits \( a \in A \), i.e. an infinitely long word on the alphabet \( \{0,1\} \), if minimal noncommutative information, we have, as a particular case of the general definition DEF IV.10, that:

\[ a \text{ is random } \iff a \notin B \forall B \in \omega_{\text{uniform}} - TYP[A]_C \quad (4.26) \]

where \( \omega_{\text{uniform}} - TYP[A]_C \) is the set of all the constructive \( \omega_{\text{uniform}} \) - typical sets of \( (A, S_{\text{chain}}) \), i.e.:

\[ \omega - TYP[A]_C \equiv \{ B \subseteq A : \neg(\forall \epsilon > 0 \exists \{ a_n \in A \}_{n \in \mathbb{N}} : \text{eq.4.28} \land \text{eq.4.29} \} \quad (4.27) \]

with:

\[ \{ a_n \}_{n \in \mathbb{N}} \in S_{\text{chain}} \quad (4.28) \]

\[ \sum_{n=0}^{\infty} \omega_{\text{uniform}}(a_n^*a_n) < \epsilon \quad (4.29) \]

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