On the Distribution of Prime Multiplets

Doron Gepner

Department of Physics
Weizmann Institute
Rehovot 76100, Israel

ABSTRACT

The probability of finding a prime multiplet, i.e., a sequence of primes $p$ and $p + a_i, i = 1 \ldots m$, being all primes where $p$ is some prime less than the integer $n$ is naively $1/\log(n)^{m+1}$. It is shown that, in reality, it is proportional to this probability by a constant factor which depends on $a_i$ and $m$ but not on $n$, for large $n$. These constants are appellated as PDF (prime distribution factors). Moreover, it is argued that the PDF depend on the $a_i$ in a "week" way, only on the prime factors of the differences $a_i - a_j$ and not on their exponents. For example $p$ and $p+2^s$ will have the exact same probability for all integer $s > 0$. The exact formulae for the PDF ratios are given. Moreover, the actual 'basic' PDF’s are calculated exactly and are shown to be less than 1, which indicates that primes ‘attract’ each other. An exact asymptotic formula for the number of basic multiplets is given.
Prime numbers have arisen curiosity for a long time. Their distribution received attention by the conjecture by Gauss that between 2 and \( n \) there are on the average \( \frac{n}{\log(n)} \) primes. This was proved in 1896 by Hadamard and de La Vallée Poussin. Therefore, the probability of finding a prime less than \( n \) is \( \frac{1}{\log(n)} \), for large enough \( n \). Recently attention was given to multiplets of primes, by the work of Goldstone and Yildrim [1]. For example, twin primes, such as \( p \) and \( p + 2 \) both being primes. Or, sequences such as \( p, p + 2, p + 6 \) and \( p + 8 \), all being prime.

In this letter I wish to make several observations on prime multiplets, described below. Suppose that \( a_i, i = 1, \ldots, m \) is a list of \( m \) positive integers, in ascending order. Then we may ask what is the probability that \( p \) and \( p + a_i \) will all be primes, where \( p \) is a prime less than \( n \). Let us denote the number of such primes by \( N(n, a_i) \). Statistically, if this would be independent events, the average number should be

\[
\frac{n}{\log(n)^{m+1}}. \tag{1}
\]

The probability is, actually, different. Moreover the ratio between the measured probability and the naive one is a constant, typical for the list of numbers \( a_i \). This constant, \( f(a_i) \), is defined by

\[
f(a_i) = \frac{n}{N(n, a_i) / \log(n)^{m+1}}, \tag{2}
\]

where \( n \) is large enough. If \( N(n, a_i) \) is zero, we define \( f(a_i) \) to be infinite. For example, for \( p \) and \( p + 2 \) we find that the constant is 0.66 approximately, with \( n \) about \( 10^6 \). Another example is \( p, p + 2, p + 6 \). Here we find for \( 10^6 \) that the factor is approximately 0.27. We call \( f(a_i) \) prime distribution factors or PDF. As can be seen later, these numbers are off the limiting values, which are calculated later in this note, eq. (14). For example, the actual limit for \( p \) and \( p + 2 \) is 0.757392..., and not 0.66. To calculate the exact numbers, directly, one needs to go to extremely large numbers. However, the ratios of PDF’s for the same \( m \) can be reliably estimated for a few million primes.
Indeed, for many different \( a_i \) that we tried, the ratio eq. (2) exists and can be measured easily. This leads us to conjecture number (1):

**Conjecture (1):** For all sequence \( a_i \) the limit on the r.h.s. of eq. (2) exists and enables us to define the PDF, \( f(a_i) \).

One may wander how does the PDF \( f(a_i) \) depends on \( a_i \). For example, \( p \) and \( p + a \) being both primes where \( a \) is any even number. For this case a remarkable phenomena happens. The PDF is dependent only on the list of prime factors in the decomposition of \( a \). I.e., if

\[
a = \prod p_i^{s_i},
\]

where \( p_i \) are some primes, then the PDF depends only on \( p_i \) and not the exponents \( s_i \). For example, for \( a = 2, 4, 8, \) etc, it is the same PDF, or for all \( a = 2^n \). Similarly, for \( a = 2^b3^c \) it is the same for any integers \( b, c > 0 \).

One may wander then how this phenomena extends to larger chains, \( m > 1 \). It turns out to obey a simple rule too. The PDF \( f(a_i) \) depends only on the primes composing the number

\[
x = \gcd_{i,j}\{a_i - a_j\},
\]

where \( i, j \) range on all possible differences \( i, j = 0 \ldots m \) where \( a_0 = 0 \) is set by convention, and on \( m \). For example, \( p, p + 2 \) and \( p + 6 \) will have the same PDF as \( p, p + 4 \) and \( p + 96 \) since the gcd eq. (4) is composed only from the primes 2 and 3, in both cases. Another example is \( p, p + 8 \) and \( p + 12 \), being all primes. It is easy to enumerate many examples of this kind. This leads us to the following second conjecture,

**Conjecture (2):** The PDF eq. (2), \( f(a_i) \), depend only on the primes composing the gcd, eq. (4), where \( a_0 \) is set to zero, and \( i, j = 0 \ldots m \).

To summarize, this is an interesting observation on the distribution of prime multiplets. One could expect to see the naive probability when the pairs are widely apart, i.e., say, \( p \) and \( p + a \) where \( a \) is large, and thus to become independent. This
is not at all the case. For $a = 2^n$, according to conjecture (2) it is always the same probability. $f$ depends only on the prime factors. This is definitely an indication that the distribution of primes, rather than being random, is "non-locally" strongly correlated. More on the distribution later in this note.

Let us turn now to some examples. We start by considering twins, $m = 1$, of the form $p$ and $p + a$. These are some sample calculations.

We checked all the primes up to the 400000th prime which is 5800079. For the case $a = 2$ we find that there are exactly 36826 pairs, starting with 3, 5 and 5, 7, etc. The ratio eq. (2) is then $f \approx 0.6494$.

For $a = 4$ we find almost the same number, according to conjecture (2), i.e., 36707 pairs. For $a = 6$ we find 73187 pairs with $f \approx 0.32676$. For $a = 12$, again it is almost the same number 73449, again verifying conjecture (2). For $a = 14$ there are 43993 pairs and $f = 0.543606$. For $a = 30$ we find 97825 pairs and $f = 0.244466$.

Actually, if one considers the ratios of these numbers they are very close to simple rational numbers, e.g. $f(6)/f(2) \approx 0.5$ this leads us to an exact formula for the binary pairs,

$$f(a) = f(2) \prod_{i} \frac{p_i - 2}{p_i - 1}, \quad (5)$$

where $i$ ranges over all prime factors of the number $a$, except 2. We verified this formula for many cases, e.g. $a = 70, 210, 30, 14, \text{ etc.}$, and it is exact to few tenth of a percent, for the first million primes.

This leads us to the third conjecture.

**Conjecture (3):** The ratios of the numbers $f(a_i)$ for a fixed $m$ are given by simple rational numbers. For $m = 1$ the ratio is given by eq. (5).

Let us turn now to examples of triplets. The simplest one is $p, p + 2$ and $p + 6$. The number $x$ is always divisible by two and three. We find for the triplets from one to 5800079 the factor $f_0 = 0.278193$. For the triplet $p, p + 2$ and $p + 12$ we find
\( f = 0.182965 \). For \( p, p + 2 \) and \( p + 14 \) we find \( f = 0.222554 \). For \( p, p + 6, p + 70 \) we find 0.14695. Again we note that the ratios of these numbers are rational and that \( f \) is given by

\[
f(a_i) = f_0 \prod_{p_i} \frac{p_i - 3}{p_i - 2},
\]

where \( p_i \) ranges over all prime divisors of \( x \), eq. (4), excluding 2 and 3.

From these two cases we can guess the general formula for any \( m \)-plet. It is given by

\[
f(a_i) = C(m) \prod_{p_i} \frac{p_i - m - 1}{p_i - m - 1 + g_i},
\]

where \( p_i \) ranges over all prime factors of \( x \), eq. (4), which are greater than \( m + 1 \), and \( g_i \) is the number of independent differences \( a_i - a_j \) divisible by \( p_i \).

This formula is the conjecture:

**Conjecture (4):** The PDF are given by eq. (7).

Now, let us turn our attention to the constants \( C(m) \). These are the PDF for a multiplet divisible only by primes less or equal to \( m + 1 \), where \( m + 1 \) is the length of the multiplets. These we term, basic multiplets. For example for \( m = 1 \), \( N(x, m) \) is the number of pairs \( p \) and \( p + 2 \), which are both prime, \( p \leq x \).

Interestingly, there is a conjecture for this number [2], and ref. therein; for a review see [3], which is

\[
N(x, 1) \approx k \int_2^x \frac{du}{\log(u)^2},
\]

where

\[
k = 2 \prod_{p>2} \{1 - \frac{1}{(p-1)^2}\} = 1.32032...
\]

This conjecture was verified in calculations, e.g. for \( x = 10^9 \) there are 3424506 pairs, agreeing very well with this formula which gives 3425230.
As we have found, this formula enjoys a generalization to all basic \( m \)plets, for any \( m \). The number \( N(x, m) \) is conjectured to be approximately,

\[
k(m) = z(m) \prod_{\substack{p > m + 1 \\ p \text{ prime}}} \left(1 - \frac{1}{(p - q + 1)^{m+1}}\right),
\]

(10)

where \( q \) is the highest prime less or equal to \( m + 1 \), and

\[
N(x, m) = k(m) \int_{m+1}^{x} \frac{du}{\log(u)^{m+1}},
\]

(11)

and where \( z(m) \) is an integer conjectured to be

\[
z(m) = (m + 1)(m - 2) \ldots (m + 1 - 3t).
\]

(12)

where \( t \) is the highest integer such that \( m + 1 - 3t > 0 \).

Let us give several examples to this formula. For the basic triplet we take \( p \), \( p + 2 \) and \( p + 6 \). Up to the 400000 prime there are 5520 such triplets. From the formula it comes to 5580, up to \( x = 5800079 \). Up to the \( 10^6 \) primes there are 12092 basic triplets, and from the formula, eq. (11), 12170. Up to the \( 2 \times 10^6 \) there are 21953 triplets, and from the formula 22099, up to the prime 32452843. This concludes the evidence for triplets, giving credence to the formula eq. (11), for the case of triplets.

Eq. (11) can be checked also for quadruplets, \( m = 3 \). Since there is less statistics, we expect the convergence to be worse. For the basic quadruplet we take \( p \), \( p + 2 \), \( p + 6 \) and \( p + 8 \). Up to the 400,000th prime, we find 591 such quadruplets, up to the prime 5800079. The formula gives 551.54. For the first \( 10^6 \) primes we find 1229 primes up to 15485863. Eq. (11) gives 1115.5. We checked also \( 2 \times 10^6 \). Here we find 2052 primes up to 32452843. The formula gives 1923 primes, or it is about 5% off. We believe that bigger primes will indeed converge to the asymptotic equation (11).
Next we check quintuplets. Here we take, \( p, p+2, p+6, p+8 \) and \( p+12 \). We get good agreement with the asymptotic formula, eq. (11). For the first \( 4 \times 10^5 \) primes we have 109 such quintuplet, where the formula gives 103, up to the prime 5800079. For the \( 10^6 \) numbers we have 205 multiplets, where the formula gives 191.36, up to 15485863. We checked also the first \( 10^7 \) primes. We find 336 quintuplets, where the formula gives 311.6, up to the prime 179424673.

Lastly, we checked the asymptotic formula for sextuplets. Here we take for the basic multiplet \( p, p+2, p+6, p+8, p+12 \) and \( p+18 \). Up to 5800079 there are 15 such sextuplets, whereas the formula gives 16.09. Up to 15485863 there are 20 multiplets, whereas the formula gives 25.99. For 86028121 we get 57 multiplets, whereas the formula gives 68.61.

These results encourage us to believe that with further calculations a rather exact correspondence could be seen, and that the formula eq. (11) is asymptotically exact.

Now, we come to the question of determining the basic PDF, \( C(m) \), eq. (7). The function eq. (11) has the limit

\[
\lim_{x \to \infty} \frac{\log(x)^{m+1}/x}{\int_{m+1}^{x} \frac{du}{\log(u)^{m+1}}} = 1,
\]

as is easy to see by performing the integral by parts, giving this up to negligible pieces. This implies that the PDF \( C(m) \) is

\[
C(m) = 1/k(m),
\]

Interestingly, \( C(m) \) is less than one, e.g., \( C(1) = 0.757392... \), \( C(2) = 0.34997... \), implying that these multiplets are more frequent than what may be naively expected. This shows that, in fact, the primes ”attract” each other.

This forms our last conjecture:
Conjecture (5): The number of basic $m$-plets up to the number $x$ is given by eq. (11). $1/k(m)$ is the basic PDF, equal to $C(m)$, eq. (10), and it is always less than one.

There is actually a probabilistic way to understand eq. (11). Consider the pairs $p$ and $p + 2$. The probability of either being prime is $2/p$. By the sieve method then the number of prime pairs less or equal to $x$ is approximated by

$$M = \frac{x}{2} \prod_{\substack{p > 2 \ 2 \text{prime}}} \sqrt{x} \left\{ 1 - \frac{2}{p} \right\}.$$

(15)

This generalizes trivially to the higher multiplets, where the probability is $(m + 1)/p$, thus replacing 2 with $m + 1$.

$$M(m) = xZ(m) \prod_{\substack{p > m + 1 \ 2 \text{prime}}} \sqrt{x} \left\{ 1 - \frac{m + 1}{p} \right\},$$

(16)

where

$$Z(m) = \prod_{\substack{u \leq m + 1 \ 2 \text{prime}}} \frac{1}{u}.$$  

(17)

In passing, we note another way of expressing $M(m)$,

$$b(m) = x/M(m) = Z(m)^{-1} \sum_{t} \frac{(m + 1)^{l(t)}}{t},$$

(18)

where $t$ is any number whose prime factors are all primes less than $\sqrt{x}$, including of course all numbers up to $x$, which is not divisible by the primes less or equal to $m + 1$, and

$$l(\prod p_{r}^{s_{r}}) = \sum s_{r},$$

(19)

where $p_{r}$ are primes bigger than $m + 1$. 

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Now, take a non-basic multiplet, e.g., \( p \) and \( p + 2q \), \( q \) prime. Then for the prime \( q \) the probability changes to

\[
\frac{(1 - 1/q)}{(1 - 2/q)} = \frac{(q - 1)}{(q - 2)},
\]

since it is enough to check only one prime. This inverse ratio becomes

\[
\frac{q - 2}{q - 1},
\]

which is precisely the PDF we found, eq. (5). For several primes indeed the probability is a product of all such factors.

For \( m \) greater than one, the probabilistic argument indeed gives eq. (7). To see this, suppose that a given prime \( p \) divides several independent \( a_{ij} = a_i - a_j \), in the notation of eq. (4). Denote the maximal number of such divisible differences by \( g \). From the probabilistic argument it follows that the probability ratio is

\[
K = \frac{(1 - (m + 1 - g)/p)}{(1 - (m + 1)/p)} = \frac{(p - m - 1 + g)/(p - m - 1)},
\]

since it is enough to check only \( m + 1 - g \) numbers divisible by \( p \), instead of the basic \( m + 1 \) numbers, and the PDF is

\[
C(m)/K,
\]

exactly verifying eq. (7). We checked this in examples, and indeed it works, e.g., for \( p \), \( p + 10 \) and \( p + 30 \), we find that the ratio of PDF is 0.5 in accordance with eq. (22).

Thus, it appears that a probabilistic argument indeed explains the values of the PDF’s. This is an indication that these values are probabilistic, and that this simple sieve argument gives the exact values.
Actually, it is not difficult to compute the sieve product $M(m)$, eq. (16). We need two identities proven by Martens (1874) [3]:

$$\sum_{\substack{p \leq x \\ p \text{ prime}}} \frac{1}{p} = \log \log(x) + B + O\left(\frac{1}{\log(x)}\right),$$

(24)

where $B$ is a constant equal to $B = 0.2616...$ and

$$\prod_{\substack{p \leq x \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \frac{e^{-c}}{\log(x)} \left(1 + O\left(\frac{1}{\log(x)}\right)\right),$$

(25)

where $c$ is Euler’s constant $c = 0.577215...$.

Now, consider the product, eq. (25). It is the sieve for single prime $s$, i.e., the number of primes up to $x$ is given by

$$r_0 x \prod_{\substack{p \leq \sqrt{x} \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \frac{x}{\log(x)} \left[1 + O\left(\frac{1}{\log(x)}\right)\right],$$

(26)

where we used eq. (25) and set

$$r_0 = \exp(c)/2,$$

(27)

to get the correct result. $r_0$ is less, and close to one, $r_0 = 0.890536$. We conclude that the sieve sum needs to be corrected by a factor. So we redefine the sieve product,

$$M(m) = Z(m) x r_m \prod_{\substack{q > m+1 \\ q \text{ prime}}} \left(1 - \frac{m+1}{q}\right),$$

(28)

where $r_m$ is a factor close to one, as yet to be determined.
We can now take the log of eq. (28). We find that

\[ H(m) = \log \left( \frac{M(m)}{xZ(m)} \right) = \sum_{q > m+1 \atop q \text{ prime}} \log \left( 1 - \frac{m+1}{q} \right). \]  

(29)

Expanding the log in series we obtain,

\[ H(m) = -(m+1) \sum_q \frac{1}{q} - (m+1)^2 \sum_q \frac{1}{2q^2} - \ldots, \]  

(30)

where the sum over \( q \) is as above. Now, the second terms and above are convergent, so they can be replaced by a constant. For the first term, we use eq. (24), implying that

\[ H(m) = -(m+1) \log \log(x) + y(m) \]  

(31)

where \( y(m) \) is some constant. Exponentiating we find an expression for the sieve product \( M(m) \),

\[ M(m) = k(m) \frac{x}{\log(x)^{m+1}} \left( 1 + O(\frac{1}{\log(x)}) \right), \]  

(32)

where we set the constant \( r_m \) to give the asymptotic formula eq. (14). For example, we have \( r_1 = 0.7931 \ldots \) and \( r_2 = 0.7060 \ldots \), etc.

Eq. (32) is an exact result, and it shows that indeed the PDF’s are as conjectured.

The two conjectures eq. (7) and eq. (11) allow us to give a good estimate for the occurrence of any multiplet, which is exact, it appears, for large enough numbers.

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