ON PRIMORDIAL COSMOLOGICAL DENSITY FLUCTUATIONS IN THE EINSTEIN-CARTAN GRAVITY AND COBE DATA

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We study cosmological density fluctuations within a covariant and gauge-invariant fluid-flow approach for a perfect fluid in the Einstein-Cartan gravity and derive the corresponding Raychaudhuri type of inhomogeneous coupled differential evolution equations of the second order. It appears that the quantum fluctuations of spin trigger primordial density inhomogeneities at the scale of weak interactions. These inhomogeneities are then evolved precisely to the value measured by COBE mission at the scale of decoupling.

1 Introduction

As well as cosmography and nucleosynthesis, structure formation nowadays represents the most important part of our theoretical comprehension of the Universe \(^1\). Cosmological models with cold and hot dark matter, with or without the cosmological constant, can fairly well describe the formation and evolution of cosmological structures at small and large scales in the Universe, mostly within the Friedmann-Lemaître-Robertson-Walker metric models of the Einstein gravity \(^2\). However, the milestone assumption for the structure formation is that certain small density inhomogeneities cause the growth into large inhomogeneities observed today. These small primordial density
inhomogeneities were observed in 1992 by COBE mission as the large-angle anisotropy of CMBR [2].

In this paper we want to show that it is possible to solve the problem of the primordial density inhomogeneity within the Einstein-Cartan (EC) gravity without referring to the dynamics of the cosmological scalar field.

In the next section we derive the evolution equations for the density contrast within the Hawking fluid-flow approach [3] and with the covariant and gauge-invariant variables of Ellis et al. [4] in the EC gravity with perfect fluid described by Obukhov and Korotky [5].

The last section is devoted to the solution of inhomogeneous evolution equations for the spacetime with expansion, acceleration and torsion, as well as to the estimates of the density contrast at various scales, including the scale of decoupling of CMBR.

2 Inhomogeneous coupled evolution equations for the density contrast

The theory of small fluctuations in general relativity started with the work of Lifshitz [6] within a coordinate approach and it was formulated using the gauge-invariant variables by Bardeen [7].

Our task to study the fluctuations in a more general spacetime with non-vanishing expansion, acceleration, vorticity, shear and torsion, requires a different, more elegant and powerful approach, such as the fluid-flow formalism [3] supplied with covariant and gauge-invariant variables [4].

We start with the formulation of perfect fluid in the EC gravity described by Obukhov and Korotky [5]. Definitions, field equations and conservation equations, as a consequence of the Bianchi identities, look as it follows [5]:

\[
\tilde{\Gamma}^\alpha_{\beta \mu} = \Gamma^\alpha_{\beta \mu} + Q^\alpha_{\beta \mu} + Q_{\beta \mu} + Q_{\mu \beta}, \\
\Gamma^\alpha_{\beta \mu} = \frac{1}{2} g^{\alpha \nu} (\partial_{\beta} g_{\mu \nu} + \partial_{\mu} g_{\beta \nu} - \partial_{\nu} g_{\beta \mu}),
\]

\[
\tilde{R}^\alpha_{\beta \mu \nu} = \partial_{\mu} \tilde{\Gamma}^\alpha_{\beta \nu} - \partial_{\nu} \tilde{\Gamma}^\alpha_{\beta \mu} + \tilde{\Gamma}^\alpha_{\gamma \mu} \tilde{\Gamma}^\gamma_{\beta \nu} - \tilde{\Gamma}^\alpha_{\gamma \nu} \tilde{\Gamma}^\gamma_{\beta \mu}.
\]
\[ \Gamma^\alpha_{\beta\gamma} \rightarrow \Gamma^\alpha_{\beta\gamma} \rightarrow \bar{R}^\alpha_{\beta\gamma\delta} \rightarrow R^\alpha_{\beta\gamma\delta}, \quad \bar{\nabla}_\mu \rightarrow \nabla_\mu, \]

\[ Q^\alpha_{\beta\mu} = \kappa u^\alpha S_{\beta\mu}, \quad \kappa = \frac{8\pi G_N}{c^4}, \quad u^\mu S_{\mu\nu} = 0, \quad S^2 \equiv \frac{1}{2} S_{\mu\nu} S_{\mu\nu}, \]

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T^\text{eff}_{\mu\nu}, \quad (1) \]

\[ T^\text{eff}_{\mu\nu} = -p^\text{eff}_{\mu\nu} g_{\mu\nu} + u_\mu u_\nu (p^\text{eff} + \rho^\text{eff}) - 2 (g^{\alpha\beta} + u^\alpha u^\beta) \nabla_\alpha [u_\mu S_{\nu\beta}], \]

\[ (\alpha\beta) = \frac{1}{2} (\alpha\beta + \beta\alpha), \quad p^\text{eff} = p - \kappa S^2 - \Lambda, \quad \rho^\text{eff} = \rho - \kappa S^2 + \Lambda, \]

\[ u^\mu \nabla_\mu \rho = - (\rho + p) \nabla_\mu u^\mu, \quad (2) \]

\[ (\rho + p) a_\mu + (-\delta^\nu_\mu + u^\nu u_\mu) \nabla_\nu p + 2 \nabla_\nu (u^\nu a^\alpha S_{\mu\alpha}) + S_{\alpha\beta} \bar{R}^\alpha_{\beta\mu} u^\nu = 0, \quad (3) \]

To derive the Raychaudhuri type of evolution equations for expansion
and vorticity and the corresponding constraint equations, we contract Ricci
identities by various tensor structures that contain the four-velocity and the
projector orthogonal to the four-velocity \[ \Theta \equiv \nabla_\mu u^\mu, \quad a_\mu \equiv \dot{u}_\mu, \]

\[ \sigma_{\mu\nu} \equiv \frac{1}{2} (\nabla_\beta u_\alpha + \nabla_\alpha u_\beta) - \frac{1}{3} \Theta h_{\alpha\beta} h^\alpha_{\mu} h^\beta_{\nu}, \]

\[ \omega_{\mu\nu} \equiv \frac{1}{2} (\bar{\nabla}_\beta u_\alpha - \bar{\nabla}_\alpha u_\beta) h^\alpha_{\mu} h^\beta_{\nu}, \]

\[ \bar{\nabla}_\mu \rightarrow \nabla_\mu \Rightarrow \bar{\omega}_{\mu\nu} \rightarrow \omega_{\mu\nu}, \]

evolution equations:

\[ \dot{\Theta} = 2 \omega^2 + 2 Q^2 - 2 \sigma^2 - \frac{1}{3} \Theta^2 + \bar{\nabla}^\mu a_\mu - \frac{1}{2} \kappa (\rho + 3p - 2\Lambda), \quad (5) \]

\[ \dot{\omega}_{\perp\mu\nu} = - \frac{2}{3} \Theta \omega_{\mu\nu} + 2 \sigma_{\gamma\mu} \omega_{\gamma\nu} - \bar{\nabla}_{[\lambda} \dot{u}_{\sigma]} h^\sigma_{\mu} h^\lambda_{\nu}, \quad (6) \]
\[ \dot{\sigma}_{\alpha\beta} = -\sigma^\kappa \sigma_{\beta \kappa} + \omega_\alpha \sigma_{\beta \kappa} - \frac{2}{3} \Theta \sigma_{\alpha \beta} - \frac{1}{3} h_{\alpha \beta} (2\omega^2 + 2Q^2 - 2\sigma^2 + \nabla^\mu u_\mu) \]
\[ + h_{\alpha \mu} h_{\beta \nu} \nabla_{(\mu} u_{\nu)} - \dot{u}_\alpha \dot{u}_\beta, \]
\[ \omega^2 \equiv \frac{1}{2} \omega_{\mu \nu} \omega^{\mu \nu}, \quad \sigma^2 \equiv \frac{1}{2} \sigma_{\mu \nu} \sigma^{\mu \nu}, \]
\[ \dot{\omega}_{\alpha \beta} \equiv h_\gamma h_\delta u^\epsilon \nabla_\epsilon \omega_{\gamma \delta}, \quad [\alpha \beta] = \frac{1}{2} (\alpha \beta - \beta \alpha), \]

constraint equations:

\[ h^{\mu}_\nu \tilde{\nabla}_\mu \Theta = \frac{3}{2} [ (\tilde{\nabla}_\beta \omega_{\gamma}^\beta + \tilde{\nabla}_\beta \sigma_{\gamma}^\beta) h_{\gamma \nu} + \dot{u}^\beta (\omega_{\nu \beta} + \sigma_{\nu \beta}) ], \]
\[ \tilde{\nabla}_\alpha \tilde{\omega}_\alpha = -2 \tilde{\omega}_\alpha \dot{u}^\alpha, \quad \tilde{\omega}_\alpha = \frac{1}{2} \epsilon_{\alpha \beta \gamma \delta} \tilde{\omega}^{\gamma \delta} u^\beta. \]

To derive the above equations, we also use EC field equations, where necessary. Notice that in the evolution equation for expansion, the term \( \omega_{\mu \nu} Q^{\mu \nu} \) is cancelled by the same term on the right-hand side of the Ricci identity, contrary to the results obtained from the incomplete treatments of previous authors [9].

Following Ellis et al., we introduce covariant and gauge-invariant variables that are in fact the orthogonal spatial gradients of scalar density and expansion, thus describing in a more natural way "a real spatial fluctuation, rather than a fictitious time fluctuation" [4]:

\[ D_\mu \equiv R(t) \frac{\nabla_\mu \rho}{\rho} \equiv R(t) \chi_\mu; \quad Z_\mu \equiv R(t) \nabla_\mu \Theta \equiv R(t) Z_\mu, \]
\[ R(t) = \text{cosmic scale factor}, \quad \nabla_\mu \equiv h_{\mu \nu} \nabla^\nu. \]

It is important to underline that these covariant variables within the perfect-fluid model in the EC gravity are also gauge-invariant for the metric with vorticity, acceleration and shear, because of the time-dependence of mass density and pressure. In the presence of vorticity and acceleration there are no more hypersurfaces orthogonal to the fluid flow, but this is not a deficiency because the variables are defined and interpreted locally, with possible further local decomposition [4].

Acting on the evolution equations by the covariant derivative, and acknowledging the identity:
\[
(3) \nabla_\mu \dot{f} - (3) \nabla_\mu f \cdot = \dot{f} a_\mu + \frac{1}{3} \Theta (3) \nabla_\mu f + (3) \nabla_\delta f (\sigma_\mu + \omega_\mu), \quad (11)
\]

\[
(3) \nabla_\mu f \cdot \equiv u^\sigma h^\lambda_\mu \nabla_\sigma (h^\epsilon_\lambda \nabla f).
\]

one can immediately obtain the evolution equations for the density and expansion contrast vectors:

\[
h_\mu^\nu \dot{\chi}_\nu = \Theta (\frac{p}{\rho} - \frac{3}{3}) \chi_\mu - (\sigma_\mu + \omega_\mu) \chi_\nu - (1 + \frac{p}{\rho}) Z_\mu - \frac{\Theta}{\rho} J_\mu, \quad (12)
\]

\[
h_\mu^\nu \dot{Z}_\nu = G \dot{u}_\mu - B_\mu^\nu Z_\nu - (3) \nabla_\mu C - \frac{1}{2} \kappa \rho \chi_\mu - \frac{3}{2} \kappa J_\mu, \quad (13)
\]

\[
G = -2 \omega^2 - 2 Q^2 + 2 \sigma^2 + \frac{1}{3} \Theta^2 - \nabla^\nu \dot{u}_\nu - \kappa \rho - \kappa \Lambda,
\]

\[
B_\mu^\nu = \Theta \delta_\mu^\nu + \sigma_\mu^\nu + \omega_\mu^\nu,
\]

\[
C = 2 \sigma^2 - 2 \omega^2 - 2 Q^2 - \nabla^\nu \dot{u}_\nu,
\]

\[
J_\mu = 2 S_\mu^\nu u^\nu \nabla^\alpha \dot{a}_\alpha + h_\mu^\nu S_\alpha_\beta \tilde{R}^{\alpha\beta}_{\epsilon} u^\nu.
\]

By direct insertion we can write coupled inhomogeneous differential equations of the second order for the density-contrast vector:

\[
- \ddot{D}_\mu + \alpha_\mu^\nu \dot{D}_\nu + \beta_\mu^\nu D_\nu + \gamma_\mu = 0, \quad (14)
\]

\[
D \equiv (-D_\mu D^\mu)^\frac{1}{2}, \quad \ddot{D}_\mu \equiv u^\nu \nabla_\nu D_\mu, \quad \ddot{D}_\mu \equiv (\dot{D}_\mu),
\]

\[
\alpha_\mu^\nu = \left[ \frac{\dot{w}}{1 + w} + \Theta (w - \frac{1}{3}) + 2 \frac{\dot{R}}{R} \delta_\mu^\nu - \sigma_\mu^\nu - \omega_\mu^\nu - 2 u_\mu \dot{u}^\nu - B_\mu^\nu, \right. \quad (15)
\]

\[
\beta_\mu^\nu = \left[ - \frac{\Theta \dot{w} (w - \frac{1}{3})}{1 + w} - \frac{\dot{w}}{1 + w} \frac{\dot{R}}{R} + \Theta (w - \frac{1}{3}) + \dot{w} \Theta - \Theta (w - \frac{1}{3}) \frac{\dot{R}}{R} \right.
\]

\[
+ \left( \frac{\dot{R}}{R} \right)^2 + \frac{1}{2} \kappa \rho (1 + w) \left[ \delta_\mu^\nu + \frac{\dot{R}}{R} \left( \sigma_\mu^\nu + \omega_\mu^\nu \right) + \frac{\dot{w}}{1 + w} u_\mu \dot{u}^\nu - (\sigma_\mu^\nu + \omega_\mu^\nu) \right.
\]

\[
+ \frac{\dot{R}}{R} u_\mu \dot{u}^\nu - \dot{u}_\mu \dot{u}^\nu - u_\mu \dot{u}^\nu + (B_\mu^\nu + u_\mu \dot{u}^\nu) (\Theta (w - \frac{1}{3}) + \frac{\dot{R}}{R})
\]

\[
- (B_\mu^\lambda + u_\mu \dot{u}^\lambda) (\sigma_\lambda^\nu + \omega_\lambda^\nu + u_\lambda \dot{u}^\nu).
\]
\[\gamma_{\mu} = R[(\frac{\dot{w}}{1+w} + \frac{\dot{\Theta}}{\rho} + \frac{\dot{\Theta}}{\rho^2})J_{\mu} - \frac{\Theta}{\rho}J_{\mu} - \frac{\Theta}{\rho}(B_{\mu}^{\nu} + u_{\mu}\dot{u}^{\nu})J_{\nu} + (1 + w)(-G\dot{u}_{\mu} + (3)\nabla_{\mu}C + \frac{3}{2}\kappa J_{\mu})],\]

\[w \equiv \frac{p}{\rho}, \ c_s^2 \equiv \frac{dp}{d\rho} = w + \rho \frac{dw}{d\rho} = w + \rho \frac{\dot{w}}{\rho},
\]

\[\dot{w} = -(1 + w)(c_s^2 - w)\Theta, \ \dot{\rho} = -(p + \rho)\Theta.\]

In the next section we study these equations and explore its observable cosmological consequences.

3 Results and discussion

The standard description of the Universe usually contains only the Hubble expansion and perfect fluid. However, it was shown that vorticity and acceleration play a very important role in EC cosmology [10]. Namely, because of the strong binding force, the baryonic spins act coherently at the scale of weak interactions and consequently a fraction of the baryon mass density produces a large baryon spin density that takes effect as a bounce force avoiding and preventing cosmological singularity in the EC gravity precisely at the scale of weak interactions [10]. On the other hand, the spins of cold and hot dark matter can coherently contribute to spin and torsion at spacelike infinity when all dynamical degrees of freedom are frozen \((T_\gamma = 0)\). At spacelike infinity it is possible to make a relationship between basic cosmological observables within the EC gravity [10,11]:

\[|\omega_{\infty}| = \frac{\sqrt{3}}{2}\Sigma H_{\infty}, \ \Sigma = \frac{l}{k + l},\]

\[\rho_{\infty} = \frac{3}{4\pi G_N}H_{\infty}^2, \ \Lambda = -\frac{1}{2}\rho_{\infty}.\]

However, the consistency with the EC field equations for the perfect-fluid model requires the vanishing of vorticity and shear for the metric with nonvanishing expansion and acceleration \(m=0, r=R\) [3]:

\[ds^2 = dt^2 - R(t)^2(dx^2 + ka(x)^2dy^2) - r(t)^2dz^2 - 2R(t)b(x)dydt,\]
\[ b(x) = \sqrt{a(x)}, \ a(x) = Ae^{mx}, \ k, l, A, m = \text{const}, \]

EC equations ⇒ \( r = R, \ m = 0, \ \frac{\ddot{R}}{R} = \frac{\dot{R}^2}{R^2} \Rightarrow \sigma = \omega = 0. \)

This is not a serious obstacle because vorticity is small in comparison with expansion. One should improve the matter part of the field equations adding imperfect fluid terms, cosmic magnetic field, etc., if one wishes to develop a more detailed picture of the Universe. Anyhow, the \( \Sigma \) parameter remains constrained by the Hubble expansion- vorticity relationship. The Boltzmann equation for nonrelativistic fluid ensures that the present mass density does not differ significantly from that at spacelike infinity:

\[ \rho_m = \rho_{\text{CDM}} + \rho_\nu + \rho_B + \rho_\gamma, \]
\[ \rho_\nu = n_\nu (m_\nu c + m_\nu \mu + m_\nu \tau) + \frac{9}{2} n_\nu k_B T_\nu, \ p_\nu = 3 n_\nu k_B T_\nu, \]

\[ k_B = \text{Boltzmann constant}, \ n_i << \left( \frac{2 \pi m_i k_B T_i}{h^3} \right)^{3/2}, \]
\[ \Rightarrow \frac{\rho_\nu (T_{\nu,0}) - \rho_\nu (0K)}{\rho_\nu (T_{\nu,0})} = \mathcal{O}(10^{-2}), \text{ similarly for } \rho_{\text{CDM}} \text{ and } \rho_B. \]

Let us now look at the form of the coefficients of inhomogeneous evolution equations for the vector density contrast in the EC gravity with expansion and acceleration:

\[ \alpha_{\mu}^{\nu} = \left( \frac{\dot{w}}{1+w} + 3H(w - \frac{2}{3}) \right) \delta_{\mu}^{\nu} - 2u_{\mu} a^{\nu}, \]
\[ \beta_{\mu}^{\nu} = \left( 3 \frac{H \dot{w}}{1+w} + \left( w - \frac{1}{3} \right) \dot{\Theta} + 2(1 + 3w)H^2 + \frac{1}{2} \kappa \rho (1+w) \right) \delta_{\mu}^{\nu} \]
\[ + \left( \frac{\dot{w}}{1+w} + 3H(w - \frac{2}{3}) \right) u_{\mu} a^{\nu} - a_{\mu} a^{\nu} - u_{\mu} \dot{a}^{\nu}, \]
\[ \gamma_{\mu} = -(1+w) G a_{\mu} R(t). \]

By direct inspection we see that, for \( R/R_0 << 1 \), the terms with covariant derivatives are suppressed in comparison with the others, thus one can write for an approximate solution:
\[ D_\mu = \frac{4Q^2}{\kappa \rho} \sqrt{\bar{\alpha} \dot{R} R} \delta_\mu, \text{ for } R \simeq 10^{-16} \text{cm}, \quad (21) \]
\[ D_\mu = 2\sqrt{\bar{\alpha} \dot{R} R} \delta_\mu, \text{ for } \frac{R}{R_0} < 10^{-4}, \ w = \frac{1}{3}, \quad (22) \]
\[ D_\mu = 3\sqrt{\bar{\alpha} \dot{R} R} \delta_\mu, \text{ for } \frac{R}{R_0} > 10^{-4}, \ w = 0, \quad (23) \]
\[ Q^2 = \kappa^2 S^2, \ S^2 \simeq \left( \frac{h \rho_B}{m_B} \right)^2. \]

Searching for a small correction to this solution one has to insert the perturbed solution \( \delta D_2 + \delta D_2 \) to the coupled equations for \( D_0 \) and \( D_2 \) components (components \( D_{1,3} \), decoupled from the source term and \( D_{0,2} \), should vanish). Evidently, the corrections are negligible:

\[ \delta D_2 \simeq \delta D_2 \left( \frac{R}{R_0} \right)^4, \quad w = \frac{1}{3}, \]
\[ D_0 \simeq H^2 R \left( \frac{R}{R_0} \right)^4, \quad w = \frac{1}{3}. \]

The density contrast is then:

\[ \frac{\delta \rho}{\rho} \equiv D, \]
\[ D = \frac{4Q^2}{\kappa \rho} \sqrt{\bar{\alpha} R_0 R(t)} , \ R \simeq 10^{-16} \text{cm}, \ w = \frac{1}{3}; \]
\[ D = \frac{3}{2} \sqrt{\bar{\alpha} R_0 R(t)} , \ R \left( \frac{R}{R_0} \right) > 10^{-4}, \ w = 0; \]
\[ \dot{H} = 0, \ H = H_0. \]

Thus, quantum fluctuations of spin trigger the mass density inhomogeneities at the scale of weak interactions, and later on the density contrast evolves linearly in the cosmological scale \( R \) (even in the radiation-dominated epoch, contrary to the usual solution of the homogeneous evolution equations where it evolves quadratically), receiving the following value at the decoupling of CMBR (\( \sqrt{\bar{\alpha}} = \mathcal{O}(10^{-1}) \), \( R_0 \simeq H_0^{-1} \)):
\[
\frac{\delta T}{T} (\text{large angle}; \text{CDM dominance}) = \mathcal{O}(10^{-1}) \mathcal{D} \left( \frac{R}{R_0} \simeq 10^{-3} \right)
\]
\[
\Rightarrow \frac{\delta T}{T} \sim 10^{-5}.
\]

This result is consistent with the measurements of the COBE-DMR [2].

To conclude, one can say that within the EC gravity it is possible to solve fundamental cosmological problems: (1) the present mass density of the Universe: \( \Omega_{m,0} \simeq 2 \) [10] (2) the cosmological constant problem: \( \Omega_{\Lambda} \simeq -1 \) [1] (3) the absence of cosmological singularity: \( R_{\text{min}} \simeq 10^{-16} \text{cm} \) [10, 12] (4) the source of density inhomogeneities: quantum fluctuations of spin (this paper), (5) strength of the primordial density contrast (this paper).

The assumption that the physical space is not contractible (finite scale is fixed by spin-torsion effects or weak interactions) makes a connection between the EC gravity and the SU(3) conformal unification scheme for gauge interactions in particle physics [12] with the observed phenomenological consequences: (1) the scale of neutrino masses measured by the LSND and the SuperKamiokande [12] (2) anomalous enhancement of the strong coupling (Tevatron and HERA) [13] (3) anomalous b-quark electroweak couplings (LEP and SLD) [13].

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