**A $\nu = 2/5$ Paired Wavefunction**

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We construct a wavefunction, generalizing the well known Moore-Read Pfaffian, that describes spinless electrons at filling fraction $\nu = 2/5$ (or bosons at filling fraction $\nu = 2/3$) as the ground state of a very simple three body potential. We find, analogous to the Pfaffian, that when quasiholes are added there is a ground state degeneracy which can be identified as zero-modes of the quasiholes. The zero-modes are identified as having semionic statistics. We write this wavefunction as a correlator of the Virasoro minimal model conformal field theory $\mathcal{M}(5,3)$. Since this model is non-unitary, we conclude that this wavefunction is a quantum critical state. Nonetheless, we find that the overlaps of this wavefunction with exact diagonalizations in the lowest and first excited Landau level are very high, suggesting that this wavefunction may have experimental relevance for some transition that may occur in that regime.

**I. INTRODUCTION**

The vast majority of quantum Hall states observed experimentally in the lowest Landau level (LLL) are very accurately described in terms of composite fermions (or equivalently the hierarchy wavefunctions). Despite these successes, there are a number of quantum Hall states that remain much less well understood and may require more exotic explanations. For example, there exist a few experimentally observed quantum Hall plateaus in the LLL that do not fit into the usual hierarchy-composite-fermion framework. Also, in the first excited Landau level (ILL), even the “simple” observed filling fractions (such as $\nu = 2 + 1/3$ and $2 + 2/5$) appear from numerics to have strong differences from the corresponding states in the LLL. Of the non-hierarchy exotic states that have been proposed, perhaps the best understood is the Moore-Read Pfaffian, which is thought to describe the plateau observed at $\nu = 2 + 1/2$, and whose quasiparticle excitations have exotic nonabelian statistics. However, for the neighboring experimentally observed plateau at $\nu = 2 + 2/5$ there are at least two competing trial states which have been proposed: the hierarchy (composite fermion) state, and the (particle-hole conjugate of the) $Z_3$ Read-Rezayi parafermion state—a generalization of the Pfaffian which has an even richer nonabelian structure. Another case where more exotic states may occur is in the quantum Hall effect of rotating bosons.

In this paper we will study another type of generalization of the Pfaffian that gives a different trial state at $\nu = 2/5$ (or $2 + 2/5$) for spinless electrons. We call this new wavefunction the “Gaffnian”, for reasons described below. The Hamiltonian that generates this state as its highest density zero energy state of a simple Hamiltonian. We then consider what happens when additional flux is added to the system. As mentioned above, in the presence of quasiholes, there is a ground state degeneracy stemming from semionic zero-modes. Since some of the analytic manipulations are messy, we relegate some of the details to rather lengthy appendices. In particular, the demonstration that the Gaffnian is a unique ground state of this Hamiltonian, and the explicit counting of quasihole states is put in appendix. However, for the interested reader, this appendix shows explicitly the mechanism by which the semionic occur. In section we construct the Gaffnian as a correlator of the $\mathcal{M}(5,3)$ Virasoro minimal model conformal field theory. In section we examine results of exact diagonalizations. We look at low energy excitations to find evidence of criticality, and we also discover that the overlap of the Gaffnian with the hierarchy wavefunction is remarkably high (we also find high overlap with exact diagonalizations of systems with interactions close to Coulomb). Finally, in section

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we give a brief discussion of some of our results.

II. THE GAFFNIAN WAVEFUNCTION

Before constructing our new trial wavefunction, for motivation we review construction of Laughlin wavefunctions. Constrained to a single Landau level (LLL or 1LL), the relative angular momentum of two fermions $L_2$ must be odd and positive. Thus the minimum relative angular momentum is $L_2^{\text{min}} = 1$ (For bosons, $L_2^{\text{min}} = 0$ and $L_2$ must be even). We define a projection operator $P_2^p$ to project out (i.e., to keep only) states where any three particles have relative angular momentum less than $L_2^{\text{min}} + p$ (with $p$ even). This projection operator can serve as a Hamiltonian. The Laughlin $\nu = 1/(L_2^{\text{min}} + p)$ state is the unique highest density (zero energy) ground state of $P_2^p$ (with $L_2^{\text{min}} + p$ odd for fermions and even for bosons).

We now generalize this construction. In a single Landau level the relative angular momentum of three fermions $L_3$ has a minimum value $L_3^{\text{min}} = 3$ (For bosons, $L_3^{\text{min}} = 0$). It is not hard to show that $L_3 \neq L_3^{\text{min}} + 1$ is dictated by symmetry, but any other $L_3 \geq L_3^{\text{min}}$ is allowed. We analogously define a projection operator $P_3^q$ to project out (i.e., to keep only) states where any three particles have relative angular momentum less than $L_3^{\text{min}} + p$ which will serve as our Hamiltonian. It is well known that the Pfaffian (at $\nu = 1/2$ for fermions and $\nu = 1$ for bosons) is the unique highest density (zero energy) ground state of the Hamiltonian $P_3^q$. In Ref. 15, another state, known as the “Haffnian” is shown to be the unique highest density (zero energy) ground state of $P_3^4$ (which is a $\nu = 1/3$ state for fermions and $\nu = 1/2$ state for bosons). Using the method of Ref. 14, 15, we can show that the Hamiltonian $P_3^q$ also has a unique highest density (zero energy) ground state. The argument is straightforward and is given in the appendix of this paper. This unique state occurs at $\nu = 2/5$ for fermions ($\nu = 2/3$ for bosons), and is the focus of this paper. Since this new state lies between the $p = 3$ Pfaffian and the $p = 4$ Haffnian, we alpha-phonetically interpolate and dub this new state the “Gaffnian”.

Before commencing our study of the Gaffnian, we note that several other states can be constructed analogously. By considering general $k$-particle angular momenta $L_k$, we can construct a general $P_k^p$. In Ref. 3 it was shown that the Hamiltonian $P_k^2$ generates the $Z_{k-1}$ Read-Rezayi state (the $Z_2$ state being the Pfaffian). Study of several other values of $p$ and $k$ is given in Ref. 10 by three of the current authors.

Knowing that a unique quantum Hall ground state exists for the Hamiltonian $P_3^3$, we set about describing the properties of the Gaffnian. We begin by writing down the wavefunction explicitly.

We will represent a particle’s coordinate as an analytic variable $z = x + iy$ which is simply the complex representation of the particle position $r$. On the spherical geometry, $z$ is the stereographic projection of the position on the sphere of radius $R$ to the plane. All distances are measured in units of the magnetic length. We can write any single particle wavefunction as an analytic function times a measure $\mu(r)$. On the disk the measure is the usual gaussian factor $\mu(r) = e^{-|z|^2/4}$ whereas on the sphere we can construct a general $P_k^p$ (with stereographic projection to the plane) the measure is $\mu(r) = [1 + |z|^2/(4R^2)]^{-(1+N_\phi/2)}$ with $N_\phi$ ($=-2R^2$ when the magnetic length is unity) being the total number of flux penetrating the sphere. On the sphere the degree of the polynomial $\psi(z)$ ranges from $z^0$ to $z^{N_\phi}$ giving a complete basis of the $N_\phi + 1$ states of the LLL. We will not write the measure explicitly, instead writing all wavefunctions simply as analytic functions (which must be fully symmetric for bosons and fully antisymmetric for fermions).

It is convenient to think, for a moment about bosons at $\nu = 2/3$. Since the Hamiltonian $P_3^3$ puts no restriction on the two particle angular momentum, there is no restriction against two bosons being at the same point $z_0$. However, when a third particle approaches, it must approach the other two such that the overall angular momentum of the three particles is $p \geq 3$, i.e., the wavefunction vanishes as $(z_3 - z_0)^p$. (In this sense, the Gaffnian, like the Pfaffian and Haffnian, is a paired state in the spirit of that originally proposed in Ref. 17.) The Gaffnian wavefunction can be written explicitly as:

$$\Psi = \tilde{S} \prod_{a < b \leq N/2} (z_a - z_b)^{2+q} \prod_{N/2 < c < d} (z_c - z_d)^{2+q} \prod_{e \leq N/2 < f} (z_e - z_f)^{1+q} \prod_{g \leq N/2} \frac{1}{(z_g - z_{g+N/2})}$$

(1)

where $q = 0$ corresponds to a bosonic ($\nu = 2/3$) wavefunction and $q = 1$ is a fermionic ($\nu = 2/5$) wavefunction. We have assumed the number $N$ of particles is even and $\tilde{S}$ means symmetrize or antisymmetrize over all particle coordinates for bosons or fermions respectively. One can confirm directly that the above wavefunction for $q = 0$ correctly has the property that it does not vanish as two particles come to the same position but vanishes as three powers as the third particle arrives. Further, we show in appendix A that this is the unique densest wavefunc-
This value of flux is the same as that for the standard hierarchy $\nu = 2/5$ states$^{12}$. This should not be a surprise, since some of the first trial wavefunctions for $\nu = 2/5$ (for fermions) in the hierarchy were based on pairing$^{12}$. In the appendix we analytically establish that this state is the unique zero energy state of the Hamiltonian $P_3^1$ at this flux. We have numerically confirmed this fact by explicitly diagonalizing the Hamiltonian $P_3^1$ with up to $N = 12$ particles on the spherical geometry and up to $N = 10$ particles on the torus. (We note that on the torus the Gaffnian occurs at $N_\phi = (3/2 + q)N$ meaning there is no "shift", which is always the case on the torus).

In appendix $\text{C}$ we consider possible generalizations of the form of the Gaffnian wavefunction (Eq. 1). In particular, we find trial states for wavefunctions of the Jain hierarchy $\nu = p/(mp+1)$ (with $m$ odd for bosons and even for fermions) with the same value of the flux as the usual Jain sequence. Since (as we will discuss below) the Gaffnian is distinct from the hierarchy (or Jain) states, we suspect that these trial states are similarly distinct from the usual Jain states. However, we leave detailed study of these wavefunctions for further work.

Since the Gaffnian is a paired state$^{8,17}$, we expect that each additional flux added will correspond to two quasiholes, each with charge $e^* = e\nu/2$ with $-e$ the charge on the elementary underlying "electron" (or underlying boson for $q = 0$). Generally, we define the number of extra flux added to the Gaffnian ground state to be

$$n = N_\phi - (3N/2 - 3 + q(N - 1)) \tag{3}$$

(compare Eq. 2). Note that $n$ here is defined so that it is half integer if $N$ is odd. To construct wavefunctions in the presence of $n$ (integral) additional flux, we can insert a factor of

$$\prod_{a \leq N/2; j \leq n} (z_a - w_j) \prod_{N/2 < b \leq N; n < k \leq 2n} (z_b - w_k) \tag{4}$$

into the above wavefunction inside the symmetrization where the $w$'s indicate the quasihoole positions. However, for $2n$ fixed quasihole positions, there are apparently $\binom{2n}{n}$ inequivalent ways to choose which of the positions $w_j$ are labelled with an index $j \leq n$ and which with an index $j > n$. One might expect that the different groupings of the positions into these two groups generate equally many inequivalent quasihole wavefunctions. The fact that we find more than one independent quasihole wavefunction means that there are zero-modes associated with these quasiholes. Analogous to the Pfaffian$^{14}$, however, it turns out that there are many linear dependencies between these many different wavefunctions.

In appendix $\text{A}$ we show explicitly how to count the zero energy ground state degeneracy of the system with Hamiltonian $P_3^1$ at any flux (Strictly speaking the appendix only addresses the case of $N$ and $2n$ even, although the odd case proceeds similarly). We find that the degeneracy of zero energy states is given by

$$\sum_{F_{\text{max}}}^{F_{\text{max}}} \sum_{F_{\text{min}}}^{F_{\text{max}}} \left(\frac{(N - F)/2 + 2n}{2n}\right) \left(\frac{n + F/2 - 1}{F}\right) \tag{5}$$

where the maximum value of $F$ is given by $F_{\text{max}} = \min(N, 2n - 2)$. To verify this result, we have numerically performed exact diagonalizations. For every case we have examined, we find perfect agreement between this analytic rule and the results of our exact diagonalization of the Hamiltonian $P_3^1$. (We have examined $N = 4, 6$ with $n \leq 6$, $N = 8$ with $n \leq 4$, $N = 10$ with $n \leq 2$, $N = 5$ with $n \leq 7/2$ and $N = 7$ with $n \leq 5/2$).

The first term in Eq. 5 corresponds to the positional degeneracy of the quasiholes and can be thought of as $2n$ bosons in $(N - F)/2 + 1$ orbitals. The second term is the degeneracy of the zero-modes and can be thought of as $F$ fermions in $n + F^2 - 1$ orbitals. Since the number of orbitals changes half as fast as number of particles, these zero-modes are a realization of semionic exclusion statistics$^{19}$.

The form of Eq. 5 is quite analogous to the zero-mode counting expressions found for the Pfaffian$^{14}$, Haffnian$^{15}$, and Read-Rezayi States$^{2,20}$. However, in those cases the zero-modes have fermionic, bosonic, and parafermionic statistics respectively. For the fermionic (Pfaffian) case we put $F$ fermions in a fixed number $n$ orbitals$^{14}$. For the bosonic (Haffnian) case$^{15}$, we put $F$ fermions in $n + F - 2$ orbitals (which is equivalent to putting $F$ bosons in $n - 1$ orbitals). The Gaffnian case is quite naturally an interpolation between these two cases. (The Read-Rezayi parafermion case cannot be phrased in this language so easily$^{20}$).

As with the Pfaffian$^{14}$, Haffnian$^{15}$, and Read-Rezayi$^{2,20}$ cases, the structure of Eq. 5 also tells us how to decompose these degenerate states into angular momentum multiplets. We simply calculate the multiplets of the $2n$ bosons in $(N - F)/2 + 1$ orbitals and also the multiplets of the $F$ fermions in $n + F^2 - 1$ orbitals and then add these together using standard angular momentum addition rules. An explicit example of this angular momentum addition is given in appendix $\text{B}$.

As discussed above, we can also look at wavefunctions with fixed quasiparticle positions. The number of linearly independent states should just be given by the zero-mode contribution to the above equation

$$D_n = \sum_{F_{\text{min}}}^{F_{\text{max}}} \left(\frac{n + F/2 - 1}{F}\right) \tag{6}$$

Indeed by generating wavefunctions (described by Eq. 1) inserted into Eq. 1 numerically and checking for linear
We now write this Gaffnian wavefunction as a correlator of a conformal field theory (CFT)\textsuperscript{21}. Making the connection to CFT has, in the past, been extremely powerful in understanding states with nonabelian statistics (See for example Refs. 6, 7, 8, 9). For example, the structure of a CFT can tell us about behavior of the degenerate space under adiabatic braiding of quasiholes.\textsuperscript{22} A CFT describing a paired state should contain a field $\psi$ with fusion relation $\psi \times \psi \sim 1$ such that it has operator product expansion

$$\psi(z)\psi(w) \sim (z-w)^{-2\Delta_\psi}[1 + \ldots]$$ \hspace{1cm} (7)

with 1 the identity, $\Delta_\psi$ the conformal weight (or dimension) of $\psi$, and dots representing less singular terms. We can then construct a paired wavefunction

$$\Psi = \left\langle \prod_{i=1}^{N} \psi(z_i) \prod_{i<j} (z_i - z_j)^{2\Delta_\psi + q} \right\rangle$$ \hspace{1cm} (8)

Repeating the arguments which are presented in Ref. 6 it is clear that (for $q = 0$, i.e., for bosons) this wavefunction will not vanish as 2 particles come to the same position since the (fractional) Jastrow factor precisely cancels the singularity of the operator product expansion. However, the wavefunction vanishes as $z^{4\Delta_\psi}$ powers when the third particle approaches the other two (since there are three (fractional) Jastrow factors and only one singularity). The Moore-Read Pfaffian\textsuperscript{6} is described in this way by the Ising CFT, also known as the $\mathcal{M}(4,3)$ minimal model\textsuperscript{23}, which contains such a field $\psi$ with weight $\Delta_\psi = 1/2$ so the wavefunction vanishes as $z^3$ as three particles come to the same point. The Gaffnian is correspondingly described by one of the simplest generalizations of the Ising CFT, known as the minimal model $\mathcal{M}(5,3)$. This CFT has a field $\psi$ with $\Delta_\psi = 3/4$ so that the wavefunction vanishes as $z^3$ as three particles coalesce (for $q = 0$). The dimensions and fusion rules for the three independent fields ($\psi, \sigma$, and $\varphi$) in this model are given in Fig. 1. The fusion of the field $\sigma$ with the field $\psi$ gives us the operator product expansion\textsuperscript{21}

$$\psi(z)\sigma(w) \sim (z-w)^{-1/2}\varphi(w) + \ldots$$ \hspace{1cm} (9)

where here the exponent $-1/2$ is determined by the conformal weights in Fig. 2 as $\Delta_\sigma - \Delta_\psi = \Delta_\varphi$. As described in Ref. 6 this power of 1/2 means that the quasihole created by the field $\sigma$ must have charge $e^* = e\nu/2$ consistent with our expectation for a paired state. To see how this happens we write a general wavefunction in the presence of 2n quasiholes as

$$\Psi = \left\langle \prod_{j=1}^{2n} \sigma(w_j) \prod_{i=1}^{N} \psi(z_i) \right\rangle$$

$$\prod_{i<j}(z_i - z_j)^{3/2+q} \prod_{i=1}^{N} \prod_{j=1}^{2n} (z_i - w_j)^{1/2}$$ \hspace{1cm} (10)

Given the operator product expansion Eq. 9 the final exponent in Eq. 10 must have power $1/2$ so that the wavefunction is single valued in the $z$’s. This Jastrow factor then pushes precisely a charge $e\nu/2$ away from each quasihole.

We can also use the fusion rules to count the degeneracy of the 2n quasihole state. The degeneracy is given by the number of ways the $\sigma$ fields in Eq. 10 can fuse together to form the identity. This is illustrated graphically as the number of paths through the Bratteli diagram\textsuperscript{22} shown in Fig. 2. The number of paths is $\text{Fib}(2n-1)$, which is consistent with the result of our above counting formula. If the number of particles $N$ is even, then we pair the $\psi$ fields to form identities, and the $\sigma$ fields must also pair to form the identity. However, if $N$ is odd, we can only form the identity if the $\sigma$ fields fuse to form one more $\psi$ that can then fuse to form the identity with the remaining $\psi$ field.

One may ask how we know that we have the correct conformal field theory (particularly in light of the fact that classification of all conformal field theories is an ongoing research field). The fact that we have a paired state at filling fraction $\nu = 2/3$ for bosons (i.e., the fact

| $\Delta$ | $\psi$ | $\varphi$ | $\sigma$ |
|---------|--------|----------|---------|
| $3/4$   | $\psi$ | $\varphi$ | $\sigma$ |
| $1/5$   | $\psi$ | $\sigma$ | $\varphi$ |
| $-1/20$ | $\psi$ | $\varphi$ | $\sigma$ |

FIG. 1: In the Virasoro minimal model conformal field theory $\mathcal{M}(3,5)$, there are three nontrivial fields, $\psi$, $\varphi$, and $\sigma$ with dimensions $\Delta$ given in the left table and fusion algebra given in right table.
that the wavefunction does not vanish when two particles come together] means we must have a field $\psi$ which fuses with itself to form the identity. The fact that the Hamiltonian forces the wavefunction to vanish as three powers when three particles come together further fixes the dimension $\Delta_\psi$. It is easy enough to show that the only Virasoro minimal model conformal field theory with such a field is $\mathcal{M}(5,3)$. If we further insist that the charge of the quasihole should be $e\nu/2$, as is expected for a paired state, this fixes the exponent of the final factor in Eq. 114 and this in turn fixes $\Delta_\sigma - \Delta_\psi = 1/4$. We must also insist that the fusion relations for fusing many quasiparticles with each other has the form of the Bratteli diagram in Fig. 2. Finally, one can look at the subleading behavior of the wavefunction as particles approach each other to extract the central charge of the theory, which again is consistent with $\mathcal{M}(5,3)$ (we do not perform this calculation here). These restrictions place serious constraint on any possible conformal field theory we would like to use to represent the gaffnian state. Certainly there is no “simple” (i.e., minimal model) theory other than $\mathcal{M}(5,3)$ with the required properties. However, we have not proven that no other theory exists.

The conformal field theory $\mathcal{M}(5,3)$ is nonunitary. This highly suggests that the Gaffnian wavefunction does not represent a true phase, but rather represents a quantum critical point. The argument for this goes as follows: The edge state theory in $1+1$ dimensions of a quantum Hall state should be described by the same conformal field theory as the bulk 2 dimensional theory. However, since the edge state theory is a dynamical theory, it must be unitary. If we have a trial wavefunction that is generated by a nonunitary theory, apparently the only way out of this conundrum is that the edge state theory does not exist; i.e., edge excitations do not stay on the edge, but leak into the bulk. This could indeed be the case if the ground state has arbitrarily low energy excitations in the thermodynamic limit. This could in turn occur if the wavefunction represents a quantum critical point. Indeed, there have been past examples of critical quantum Hall states which are described by nonunitary CFTs. While there is no strict proof that a nonunitary conformal field theory necessarily implies a critical state, there is also no understanding of how anything else could occur.

### IV. EXACT DIAGONALIZATIONS

We now turn to exact diagonalizations. Strictly speaking, the Hamiltonian $P^0_{ijk}$ has been defined to be a projection operator that acts on the full wavefunction (to keep any states where any three particles have relative angular momentum less than three). As such, this Hamiltonian has eigenvalues that are either zero (for the zero energy space) or unity. A more physical version of this Hamiltonian can be written as

$$H = \sum_{i<j<k} (V_{5,0} P^0_{ijk} + V_{5,2} P^2_{ijk})$$

(11)

where we have defined a general three body operator $P^p_{ijk}$ which projects out (i.e., keeps) any component of the wavefunction where the three particles $i$, $j$ and $k$ have relative angular momentum $L_{\text{min}} + p$. (On the sphere, one defines $P^p_{ijk}$ to project out (i.e., keep) any cluster of three particles with total angular momentum $3N_0/2 - p$).

![FIG. 3: Lowest neutral gap excitation of the Gaffnian as a function of system size (using the Hamiltonian in Eq. 11) in units of $V_{5,0} = V_{5,2}$. Data is shown for $N = 8, 10, 12$ particles. The solid is a linear fit of all three data points. The dashed line is a fit of the two larger systems only (suggesting that if we could access even larger systems, the extrapolations might be even closer to zero). This data suggests the possibility that the gap may extrapolate to zero in the thermodynamic limit, as would be expected for a critical state. However, from the available numerical data, we cannot exclude the possibility that it extrapolates to a finite value.]

Note that three particles cannot have relative angular momentum of $L_{\text{min}} + 1$, so this Hamiltonian gives energy to any case where the relative angular momentum of any cluster of three particles is less than three. (}
Some readers may have assumed that the form of Eq. 1 is what we meant all along when we have been writing $P_3^1$, as we were not very explicit about what we meant.) Since the Hamiltonian Eq. 1 gives energy to any cluster of three particles with relative angular momentum less than 3, it has precisely the same zero energy space as $P_3^3$. However, the excitation spectrum here is different, and is dependent on the values of $V_{3,0}$ and $V_{3,2}$.

Let us first examine the issue of criticality. In Fig. 3, we show the lowest energy neutral excitation of $H$ (from Eq. 1) as a function of system size for $N = 8, 10, 12$ on a spherical geometry with $V_{3,0} = V_{3,2}$ (We have chosen to look at bosons on a sphere because we can go to larger systems). As can be seen in the figure it appears that the gap extrapolates to a positive value, but it is not possible to rule out extrapolation to a zero value which would be a sign of criticality. Furthermore, changing the ratio of $V_{3,0}/V_{3,2}$ (data not shown) does not appear to substantially affect the ratio of the extrapolated energy to the reference energy of the gap for $N = 10$.

![Overlap Squared](image)

**FIG. 4:** Squared overlaps of trial states with the exact ground state at $\nu = 2/5$ on a sphere with 10 electrons, as we vary the Coulomb interaction. Solid line is the overlap of the Gaffnian wavefunction with the exact ground state. The dashed line is the hierarchy $2/5$ state with the exact ground state. The top is results for the lowest Landau level, the bottom is the first excited Landau level. In the horizontal direction the interaction is varied around the Coulomb interaction by adding an additional $\delta V_1$ Haldane pseudopotential.

We now turn to the question of whether the Gaffnian is physically relevant to the physics of 2D electron systems. We have performed exact diagonalization on a spherical geometry for 10 electrons in the lowest Landau level (LLL) and first excited Landau Level (1LL), and we have varied the electron-electron interaction in the neighborhood of the Coulomb interaction by varying the Haldane pseudopotential $V_1$. In Fig. 2a, we show the overlap of the exact ground state with our trial wavefunction. Results are shown for the Gaffnian (solid) and the hierarchy $2/5$ state (dashed). Over a range of $V_1$ both trial states have quite good overlaps with the ground state considering that the zero angular momentum Hilbert space has 52 dimensions (Note that for many of the well known numerical cases where extremely large overlaps have been reported, the dimension of the available Hilbert space is much smaller than this). Near the regime of $V_1$ where the overlaps drop, we believe the system is in the Read-Rezayi phase (although at a different value of flux on the sphere). Since both the Gaffnian and hierarchy states have such large overlaps with the ground state, they necessarily have large overlaps with each other, although in the thermodynamic limit they become orthogonal.

We have also performed exact diagonalization on the torus geometry. Here, the Gaffnian ground state is found to be doubly degenerate (in addition to the usual center of mass degeneracy). The two zero energy ground states are distinguished by a parity quantum number. The state with positive parity again has extremely high overlap with the hierarchy state, similar to the overlaps on the sphere. As on the sphere, both of these have a high overlap with the exact ground state for a wide range of interactions. However, we do not find that the exact ground state has an even approximate double degeneracy in the regimes where the overlaps of the Gaffnian and the hierarchy are large. Approximate double degeneracy of the ground state is found where we believe the Read-Rezayi state is the proper ground state.

**V. DISCUSSION**

If the Gaffnian does turn out to be a critical state, as suggested here, this then raises the question as to what the neighboring phases are. It is reasonable to think of one being a "strong pairing" phase (albeit one that cannot be easily described within BCS theory which may correspond to the hierarchy wavefunction itself). This would be quite natural considering the high mutual overlaps of the hierarchy and Gaffnian.

The nature of the state on the opposite side of the transition is a bit harder to guess at. One possibility is that is the Read-Rezayi state. This would make some sense because of the similar ground state degeneracy. Here, we imagine that as we approach the transition from the hierarchy side, the putative zero energy state would drop continuously and hit zero energy at the Gaffnian critical point. It would then stay at zero energy through the Read-Rezayi phase. On the other hand, we should note that there is a notable topological difference between the Read-Rezayi and Gaffnian state, which is more evident on the sphere as they occur at different values of flux.

Yet another possible candidate for a state that might occur nearby is a charge density wave state. We leave the project of sorting out the details of this transition for future work.

If the Gaffnian is in fact a critical state, this means that the concept of "nonabelian" statistics may not be
well defined. Indeed, the idea of statistics describes what happens to a system when particles are adiabatically exchanged. Since definition of adiabatic usually requires any perturbation to the system to be on a timescale slower than $\hbar/\Delta$ with $\Delta$ the minimum gap in the system, if the system has gapless excitations, there is generally no way to have adiabaticity. One might ask whether any remnant of the idea of nonabelian statistics still remain. This is a question that is hard to answer without knowing the details of what these “critical” low energy excitations are.

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APPENDIX A: ANALYTIC COUNTING OF ZERO ENERGY STATES

In this appendix we will enumerate all possible zero energy states of the Hamiltonian $P^3$ on the sphere with any number of particles and at any given flux. Our approach will be in two parts. In section A.1 we will write down a linearly independent set of zero energy states (and we will count them). It is this section that shows most clearly how the semionic zero-modes arise. Then in section A.2 we will show that these wavefunctions are indeed zero energy states of $P^3$. Finally, we will show in A.3 that these states indeed form a complete set of the zero energy states. The arguments here are quite similar to those given in Refs. 14, 15. However, here the situation is more complicated as our zero-modes are not simple fermions and bosons as in those two references. Note that throughout this appendix we will focus on the case where $N$ (and therefore $2n$) is even. The $N$ odd case is a relatively simple generalization.

1. Counting States

We start with the requirement that $\psi$ vanishes as three powers as any three particles approach each other. The wavefunction Eq. 1 clearly provides one such solution (at a given value of flux). We will call this the Gaffnian ground state. In this section we propose a more general form for wavefunctions when there are some arbitrary number $2n$ quasiholes (or $n$ additional flux) added to the ground state, and we will count the number of such states that are linearly independent.

Inspired by the work of Refs. 14, 15 and analogous to the Pfaffian, Haffnian and Haldane-Rezayi states, we write our proposed wavefunction in a form with broken and unbroken pairs. Let us declare that $F$ of the $N$ total particles are unpaired. Restrictions on $F$ will be determined later. We then propose the following form for our wavefunctions:

$$
\psi_G = \hat{S} \left[ \prod_{1 \leq a < b \leq N} (z_a - z_b)^{2+q} \prod_{1 \leq a < c, d \leq N} (z_a - z_d)^{2+q} \prod_{1 \leq c < d \leq N} (z_c - z_d)^{1+q} \prod_{1 \leq e < f \leq N} (z_e - z_f)^{1+q} \prod_{1 \leq g \leq N-2} \frac{\Phi(z_g + \frac{e}{2}, z_g + \frac{N+e}{2}; w_1, \ldots, w_{2n})}{z_g + \frac{e}{2} - z_g + \frac{N+e}{2}} \left( \prod_{i=1}^{\frac{N}{2}+\tau} \prod_{j=\frac{N}{2}+1}^{N} (z_i - z_j)^{1+q} \Omega(z_1, \ldots, z_{N/2}; z_{N/2+1}, \ldots, z_{N}) \right) \right] (A1)
$$

where as above $\hat{S}$ either symmetrizes (for bosons, even $q$) or antisymmetrizes (for fermions, odd $q$) over all of the $z$ coordinates. Here we have defined $\Phi$ to be the Read-Rezayi quasihole insertion:

$$
\Phi(z_1, z_2; w_1, \ldots, w_{2n}) = (A2)
$$

$$
\frac{1}{(n!)^2} \sum_{\tau \in S_{2n}} \prod_{i=1}^{n} (z_{\tau(2i-1)} - w_{\tau(2i-1)})(z_{2i} - w_{\tau(2i)})
$$

and $\Omega$ is a wavefunction for the zero-modes to be determined later. (The sum over $\tau \in S_{2n}$ is the sum over permutations of the $2n$ variables $w$). We now specialize to the case of $q = 0$ (bosons) for simplicity. Since the particles must be at the flux $N_\psi = 3(N/2 - 1) + n$ we will deduce that the highest degree of the unpaired particle coordinates is $n - 1$. To see how this is deduced we start by defining $A \equiv \{z_1, \ldots, z_{N/2}\}$, and $B \equiv \{z_{N/2+1}, \ldots, z_N\}$. We then simply count up powers of $z_k$ appearing in $\psi_G$...
\[ \psi_G = S \left[ \prod_{i<j} (z_{A_i} - z_{A_j})^2 \prod_{i<j} (z_{B_i} - z_{B_j})^2 \prod_{i,j} \Phi(z_{A_i}, z_{B_j}) \prod_{i,j} \Phi(z_{A_i}, z_{B_j}) \right] \right] \Omega(\ldots) \] (A3)

Here \( S \) symmetrizes over all coordinates \( z \). In this equation, we have written beneath each term the number of powers of \( z_k \) occurring. Thus adding up the powers, we conclude that \( \Omega \) is some polynomial in unpaired coordinates of degree \( m_i : 0 \leq m_i \leq n-1 - \frac{N}{2} \) for each unpaired coordinate \( z_i \). Notice also, that this puts a restriction on \( F \): \( F \leq 2n - 2 \), and since obviously \( F \leq N \), we obtain

\[ F \leq \min(2n - 2, N) \] (A4)

as written above in the main text. The maximum degree of \( \Omega \) occurs when \( F = 0 \) and is given by \( n - 1 \).

To see how many linearly independent wavefunctions we have for given \( \{N, n, F\} \) we proceed as follows. We choose \( N-F \) (necessarily even here) of the \( N \) coordinates and group them together in pairs \( \{(z_{a_i}, z_{b_i})\} \) for \( i = 1, \ldots (N-F)/2 \) with \( a_i, b_i \in [1, \ldots N] \) and \( a_i \neq a_j, b_i \neq b_j \) for \( i \neq j \) and \( a_i \neq b_j \) for all \( i, j \). We then bring together the position of the paired particles to coordinates \( \bar{z}_i \). In other words we set \( z_{a_i} = z_{b_i} = \bar{z}_i \) for \( i = 1, \ldots (N-F)/2 \). Taking this limit selects out a particular group of terms from the original full symmetrization that do not vanish. In particular the nonvanishing terms are the terms in which a factor of \( \Phi(\bar{z}_a, z_{b_1}, \ldots z_{b_{N-F}})^{-1} \) appears for each pair \( (z_{a_i}, z_{b_i}) \). The other terms will have a factor of \( (z_{a_i} - z_{b_i}) \) in the numerator, and will vanish in these limits. After taking these limits we are left with

\[ \tilde{\psi} = S' \prod_{i<j} (\bar{z}_i - \bar{z}_j)^6 \prod_k \Phi(\bar{z}_k; w_1, \ldots, w_{2n}) \prod_{1 \leq a < b \leq \frac{N}{2}} (z_a - z_b)^2 \prod_{\frac{N}{2}+1 \leq c < d \leq \frac{N}{2}+\frac{F}{2}} (z_c - z_d)^2 \] (A5)

where \( S' \) symmetrizes over \( \{\bar{z}_i\} \) and \( \{z_1, \ldots, z_{\frac{N}{2}}, z_{\frac{N+1}{2}}, \ldots, z_{\frac{N+F}{2}}\} \) separately. In other words, \( S' \) is what remains of the original full symmetrization over \( N \) particles. The underlined factors contain the dependence of \( \tilde{\psi} \) on \( \bar{z} \), and are symmetric in \( \{\bar{z}_i\} \), while the doubly underlined factor is symmetric in \( \{z_1, \ldots, z_{\frac{N}{2}}, z_{\frac{N+1}{2}}, \ldots, z_{\frac{N+F}{2}}\} \) as well. Thus, the symmetrization \( S' \) reduces to \( S'' \) which symmetrizes over unpaired particles only (because the expression is already symmetric in \( \bar{z}_i \)), and we can rewrite the wavefunction as

\[ \tilde{\psi} = (\tilde{\psi}_{LL})^2 S'' \left\{ \left[ \Omega(\ldots) \prod_{l=1}^{\frac{N}{2}} \prod_{c=1}^{\frac{F}{2}} (z_c - \bar{z}_l)^3 \prod_{m=1}^{\frac{N}{2}+\frac{F}{2}} (z_m - z_l)^3 \prod_{k=1}^{\frac{N}{2}+\frac{F}{2}} \Phi(\bar{z}_k) \right] \times \prod_{i<j} (\bar{z}_i - \bar{z}_j)^6 \prod_k \Phi(\bar{z}_k) \right\} \] (A6)

where \( \tilde{\psi}_{LL} \) is a Laughlin-Jastrow factor in the unpaired particle coordinates. We thus discover that \( \Omega(\ldots) \) can always be taken to be fully symmetric in its arguments (any nonsymmetric parts vanish when symmetrized). We can
thus think of this as a bosonic wavefunction for the zero
modes. However, we’ve already determined the maximal
degree of $\Omega(\cdots)$ to be $n - 1 - \frac{F}{2}$. The minimal degree is
obviously 0, so we have a total of $n - \frac{F}{2}$ orbitals in which
to put $F$ bosons. There are $(n - \frac{F}{2})_{F}^{+} = (n + \frac{F}{2} - 1)$
such linear independent wavefunctions. This is equivalent
to placing $F$ fermions in $n + F/2 - 1$ orbitals. Since
the number of orbitals changes half as fast as the number
of particles we put in them, these particles have semionic
exclusion statistics\(^{15}\).

2. Zero Energy

We will continue on to demonstrate that the linearly
independent set of wavefunctions we have just written
down is in fact a complete set of zero energy states of
the Hamiltonian $P^{3}$. First, however, we show that these
wavefunctions are indeed zero energy states. The wave-
function for any zero energy state must vanish as three
or more powers when three particle positions come to the
same point. On the sphere, this is equivalent to re-
stricting the total angular momentum of the cluster of
three particles to be no greater than $3N_{\phi}/2 - 3$.

First we’ll show that the proposed wavefunctions $\psi_{G}$
are zero energy eigenstates of this Hamiltonian. For the
ground state, i.e. no additional flux ($n = 0$) we have
$N_{\phi} = 3(N/2 - 1)$. Let us look at the $(ijk) = (z_{i}, z_{j}, z_{k})$
triplet. We want to know what the highest total angular
momentum is for this triplet in our wavefunction $\psi_{G}$.
The wavefunction can be rewritten (à la Haldane\(^{12,16}\)) as a
sum of terms proportional to $L_{ijk} = L_{ij} + L_{jk} + L_{ki}$
where $F_{ij}=\psi_{ij}$ is an eigenstate of $L_{ij}$, the 3 particle relative
angular momentum operator, and $F_{total}(z_{i}, z_{j}, z_{k})$ is an eigenstate
of $L_{ij}$, the 3 particle total angular momentum operator.
Note that, as above, we will always focus on $q = 0$ for
simplicity (The $q \neq 0$ case is a relatively minor general-
ization). To find the total angular momentum, we look at
the maximal degree of $z_{i}^{\alpha} z_{j}^{\beta} z_{k}^{\gamma}$ in $F_{total}$, and find the total
angular momentum $L = \max(\frac{1}{2}(\alpha + \beta + \gamma))$. To find the
maximum total angular momentum we must consider all
possible ways to have chosen the triplet $(z_{i}, z_{j}, z_{k})$ from the
many terms in the wavefunction. In particular, we must look at all cases of which coordinate is one of the
paired variables, and which is unpaired, as well as looking
at which variable is an A-coordinate, and which is a
B-coordinate. Here we are looking at the relative angular
momentum of a given triplet in each of the many
terms of the symmetrization sum. All possibilities are
enumerated next.

- Case 1: $i, j, k \in A, i < j < k$.
Here we have

\[
\alpha = \deg_{z_{i}} \psi_{G} = 2\left[\frac{N}{2} - 1\right] + \frac{N}{2} - 1
\]

\[
\beta = \deg_{z_{j}} \psi_{G} = 2\left[\frac{N}{2} - 2\right] + \frac{N}{2} - 1
\]

\[
\gamma = \deg_{z_{k}} \psi_{G} = 2\left[\frac{N}{2} - 3\right] + \frac{N}{2} - 1
\]

Using $L = \frac{1}{2}(\alpha + \beta + \gamma)$ and with $N_{\phi} = 3(N/2 - 1)$ we obtain in this case $L = \frac{3}{2}N_{\phi} - 6$.

- Case 2a: $i, j \in A, i < j; k \in B$ with pairing of the form $(ia)(jb)(ck)$, i.e. terms of the form

\[
\Phi(z_{i}, z_{a}) \Phi(z_{j}, z_{b}) \Phi(z_{c}, z_{k})
\]

(A8)

Here we have

\[
\alpha = \deg_{z_{i}} \psi_{G} = 2\left[\frac{N}{2} - 1\right] + \frac{N}{2} - 1
\]

\[
\beta = \deg_{z_{j}} \psi_{G} = 2\left[\frac{N}{2} - 2\right] + \frac{N}{2} - 1
\]

\[
\gamma = \deg_{z_{k}} \psi_{G} = 2\left[\frac{N}{2} - 1\right] + \frac{N}{2} - 2
\]

Similarly, adding up these powers we obtain an angular momentum $L = \frac{3}{2}N_{\phi} - 4$.

- Case 2b: $i, j \in A, i < j; k \in B$ with pairing of the form $(ia)(jk)$, i.e. terms of the form

\[
\Phi(z_{i}, z_{a}) \Phi(z_{j}, z_{k})
\]

(A10)

Here we have

\[
\alpha = \deg_{z_{i}} \psi_{G} = 2\left[\frac{N}{2} - 1\right] + \frac{N}{2} - 1
\]

\[
\beta = \deg_{z_{j}} \psi_{G} = 2\left[\frac{N}{2} - 2\right] + \frac{N}{2} - 1
\]

\[
\gamma = \deg_{z_{k}} \psi_{G} = 2\left[\frac{N}{2} - 1\right] + \frac{N}{2} - 2
\]

which results in an angular momentum $L = \frac{3}{2}N_{\phi} - 3$.

These cases are the only possibilities. Thus the highest
total angular momentum for any triplet is $\frac{3}{2}N_{\phi} - 3$ and so
the proposed wavefunction $\psi_{G}$ is a zero energy eigenstate
of the Gaffnian Hamiltonian $P^{3}$ as claimed.

3. Completeness

Now we show that the proposed wavefunctions span the
complete set of zero energy states of the Gaffnian
Hamiltonian. To do this we will construct the most gen-
eral zero energy eigenstate and show that it takes the
form of our proposed wavefunction. Take the following
zero energy wavefunction $\psi_{G} = \psi_{LJ}^{2} \phi_{G}$ where here
$\psi_{LJ}$ is the Laughlin-Jastrow factor for all of the
particles. Consider the behavior of $\psi_{G}$ as particles in an
arbitrary triplet $(ijk)$ approach each other, while the other
particles remain far away from the three.
The wavefunction vanishes as $6 + Q$ powers as these three particles come together. This is equivalent to saying that the total angular momentum of three particles is $L = \frac{N_\alpha}{2} - (6 + Q)$ (Since we’re on the sphere, the maximum angular momentum of each particle is $\frac{N_\alpha}{2}$). Any relative angular momentum reduces the total by a corresponding amount. Furthermore, by analyticity of $\psi_G$, we must have $q_{nn} \geq -2$.

Now, in order for $\psi_C$ to be a zero energy state of the Gaffnian Hamiltonian, we must have $Q \geq -3$ (so that the relative angular momentum of the cluster is greater than or equal to $3 = 6 + Q$). From here on we’ll concentrate on the $\phi_G$ factor of the eigenstates, restoring the ubiquitous $\psi_{L,j}^i$ at the end. Allowed forms in the Laurent expansion of $\phi_G$ as $(ijk)$ approach each other are

$$\psi_G \propto \frac{1}{(z_i - z_j)^2 (z_j - z_k)^2 (z_k - z_i)}$$ (A13)

$$\frac{1}{(z_i - z_j)^2 (z_j - z_k)^2}$$ (A14)

$$\frac{1}{(z_i - z_j)(z_j - z_k)(z_k - z_i)}$$ (A15)

as well as the same terms with $(ijk)$ permuted. However, it is easy to see that the second two forms reduce to the the first since expression A13 is equivalent to

$$\frac{-1}{(z_i - z_j)^2 (z_j - z_k) + (z_i - z_j)(z_j - z_k)^2}$$ (A16)

and expression A15 is equivalent to

$$\frac{-1}{(z_k - z_i)^2 (z_j - z_k) + (z_k - z_i)(z_j - z_k)^2}$$ (A17)

It follows then, that it’s enough to consider forms of type of Eq. A13 for triplets $(ijk)$ (as well as the same form with permutations of $(ijk)$). When $(ijk) \rightarrow (i)$, the most general zero energy eigenstate should have the form

$$\phi_G \propto \frac{F(z_i, z_j, z_k)}{(z_i - z_j)^2 (z_j - z_k)}$$ (A18)

(or a form like this with any permutation of $(ijk)$) where $F(\ldots)$ must be analytic (i.e., with no poles).

Now arbitrarily pair up and relabel the particles, e.g. $(z_{A_1}, z_{B_1}), (z_{A_2}, z_{B_2}), \ldots, (z_{A_{N/2}}, z_{B_{N/2}})$. Look at the most singular part of $\phi_G$ as particles within these pairs approach each other, while pairs are kept separated.

$$\phi_G \propto \frac{1}{(z_{A_1} - z_{B_1})^2 (z_{A_2} - z_{B_2})^2} \cdots$$

$$\frac{1}{(z_{A_{N/2}} - z_{B_{N/2}})^2} \phi_{\phi}$$ (A19)

Since we’ve isolated the most singular part of $\phi_G$, it’s clear that $\phi_{\phi}$ cannot contain factors $(z_{A_i} - z_{B_i})^{-1}$. If we now consider triplets $(A_1, B_1, k) \forall k \notin \{A_1, B_1\}$, and bring particle $k$ close to $(A_1, B_1)$ it’s clear that $\phi_{\phi}$ must contain a factor of $(z_{A_1} - z_k)^{-1}$ or $(z_{B_1} - z_k)^{-1}$, but not both, in order to satisfy the requirements on the pole structure deduced above. We might be led to naively define

$$\phi_{\phi} = \sum_j \left( \prod_{m,n} \frac{\phi_{1,j}}{(z_{A_1} - z_{D_{mn}}^i)(z_{C_{n}^i} - z_{B_1})} \right)$$ (A20)

where $C_j \cup D_j = \{z_i\}; C_j \cap D_j = \emptyset$, (i.e. a partition of the set of particle coordinates), and $j$ indexes all possible partitions. However, the pole structure places further restrictions on the sets $C$ and $D$. In particular, $A_i$ and $B_i$ cannot both be in $C$ or in $D$. Otherwise, supposing $A_i, B_i \in C$, we have (schematically) the following:

$$\phi_G \propto \prod_{i \neq j} \frac{1}{(z_{A_i} - z_{B_j})^2} \phi_{\phi}$$ (A21)

and we immediately recognize, that the triplet $(A_1, A_1, B_1)$ has too many poles ($Q < -3$). We conclude, that for $i$-th pair only one factor is allowed in $\phi_{\phi}$, either $(z_{A_i} - z_{A_i})^{-1}$, or $(z_{A_i} - z_{B_i})^{-1}$. That is, the partitions are such that $C$ includes only one representative of any pair, and $D$ includes the complementary member of this pair, i.e. necessarily $A_i \in C, B_i \in D$ or $B_i \in C, A_i \in D$. At this point we can recognize that our notation of $C$ and $D$ is redundant, and that we can rewrite

$$\phi_{\phi} \propto \sum_{k \in \text{Partitions}} \left( \prod_{i \neq j} \frac{\tilde{\phi}_k}{(z_{A_i}^k - z_{B_j}^k)} \right)$$ (A22)

where $\tilde{\phi}_k$ cannot contain any more poles, and the conventions are that for all $k$ the same coordinates are paired up, i.e. $\{A_i^k, B_i^k\} = \{A_i^j, B_i^j\}$ are equal as sets. The difference between partition $k$ and partition $k'$ is the order of coordinates within a pair, i.e. we could have $z_{A_1^k} = z_{B_{k'}}^k, z_{B_1^k} = z_{A_{k'}}^k$, or $z_{A_1^k} = z_{A_{k'}}^k, z_{B_1^k} = z_{B_{k'}}^k$. Clearly the sum over $k$ is a subset of symmetrization over all particles.

Finally, we should also include the exchange of pairs $(z_{A_1^k}, z_{B_1^k}) \leftrightarrow (z_{A_{k'}}, z_{B_{k'}}^k)$, since the most singular part is symmetric under this exchange, and arrive at
By analyzing the symmetry of denominators we find

\[
\phi_{\frac{N}{2}} = \sum_{\text{pairings}} \sum_{k \in \text{Partitions}} \left( \prod_{i \neq j} \frac{\tilde{\phi}_k}{(z_{A_i} - z_{B_j})} \right)
\]  

(A23)

Then for the particular choice of pairs we’ll have

\[
\phi_G \propto \prod_i \frac{1}{(z_{A_i} - z_{B_i})^2} \left[ \sum_{\text{pairings}} \sum_{k \in \text{Partitions}} \left( \prod_{i \neq j} \frac{\tilde{\phi}_k}{(z_{A_i} - z_{B_j})} \right) \right]
\]  

(A24)

and recover the whole eigenfunction by symmetrization over all particles and multiplication by the Jastrow factor squared:

\[
\psi_G = \psi_L^2 S \left\{ \prod_i \frac{1}{(z_{A_i} - z_{B_i})^2} \left[ \sum_{\text{pairings}} \sum_{k \in \text{Partitions}} \left( \prod_{i \neq j} \frac{\tilde{\phi}_k}{(z_{A_i} - z_{B_j})} \right) \right] \right\}
\]  

(A25)

The wavefunction of the densest state has as few zeros as possible, and to find it we may choose \( \tilde{\phi}_k \equiv 1 \), then

\[
\psi_G = \psi_L^2 S \left\{ \sum_{\text{pairings}} \sum_{k \in \text{Partitions}} \left( \prod_i \frac{1}{(z_{A_i} - z_{B_i})^2} \prod_{i \neq j} \frac{1}{(z_{A_i} - z_{B_j})} \right) \right\} \equiv
\]  

(A26)

\[
\psi_L^2 S \left[ \prod_i \frac{1}{(z_{A_i} - z_{B_i})^2} \prod_{i \neq j} \frac{1}{(z_{A_i} - z_{B_j})} \right]
\]  

(A27)

This is can be recognized as the proposed Gaffnian wavefunction with no broken pairs and no added flux. To obtain the states of lower density we need to consider the case of non-constant \( \tilde{\phi}_k(z_{A_1}, z_{B_1}; \ldots; z_{A_{N/2}}, z_{B_{N/2}}) \). By analyzing the symmetry of denominators we find that \( \tilde{\phi}_k \) must be symmetric under the exchange of pairs \((z_{A_i}, z_{B_i}) \leftrightarrow (z_{A_j}, z_{B_j})\). We now claim that a complete basis for functions that satisfy this symmetry condition is given by functions of the form

\[
\sum_{\tau \in \mathcal{S}} \prod_{i=1}^{N/2} f_i(z_{A_{\tau(i)}}, z_{B_{\tau(i)}})
\]  

(A28)

where the \( f_i \)'s are chosen from a basis for arbitrary polynomials of their two arguments. While this may seem to be a strange way to write a basis for the polynomial \( \tilde{\phi}_k(z_{A_1}, z_{B_1}; \ldots; z_{A_{N/2}}, z_{B_{N/2}}) \) this is actually a form well known to physicists. To see this, imagine a system of \( N/2 \) bosons where the “position” of each boson is specified by two coordinates \((z_1, z_2)\). The functions \( f_i \) are basis functions for the single “particle” positions. All multiparticle states can be written as symmetrized (bosonic) linear combinations of the occupations of these basis states.

Consider now the case, when we’ve added \( n \) quanta of flux to the Gaffnian ground state. The highest degree of \( f_i() \) is \( n \), and we could choose basis polynomials \( f_i(z_1, z_2) \) of the form \( z_1^{n_1} z_2^{n_2} \) with \( 0 \leq n_1, n_2 \leq n \). The dimension of this space is \((n + 1)^2\). However, a different basis set turns out to be more useful. Specifically, it is useful to separate functions \( f_i \) that vanish in the limit \( z_1 \to z_2 \), from ones that do not.

We choose a basis for our space of \( f_i \) which decomposes into two disjoint sub-bases: the symmetric \( z_1^{n_1}, z_2^{n_2}, z_1^{n_1} z_2^{n_2} \) with \( 0 \leq n_1 \leq n_2 \leq n \) and the antisymmetric \( z_1^{n_1} z_2^{n_2} - z_1^{n_2} z_2^{n_1} \) with \( 0 \leq n_1 \leq n_2 \leq n \). The dimensions of subspaces spanned by them are \( \frac{1}{2} n(n + 1) \) and \( \frac{1}{2} (n + 1)n \) respectively. Clearly the span of the antisymmetric sub-basis vanishes as \( z_1 \to z_2 \). The quotient of the full space by the span of the antisymmetric sub-basis is just the span of the symmetric sub-basis \( \mathcal{S} \), i.e., symmetric polynomials. Of these, polynomials which vanish as \( z_1 \to z_2 \) are spanned by \( (z_1 - z_2)^2(z_1^{n_1}, z_2^{n_2}, z_1^{n_2} z_2^{n_1}) \), with \( 0 \leq n_1 \leq n_2 \leq n - 2 \). The dimension of this subspace is \( \frac{1}{2} n(n - 1) \).

The quotient of \( \mathcal{S} \) by the subspace of the vanishing symmetric polynomials has dimension \( 2n + 1 \) and contains symmetric polynomials in \( 2 \) variables that do not vanish in the limit \( z_1 \to z_2 \), we’ll call this quotient \( \mathcal{Q} \). However, by considering the Taylor expansion of \( \text{Read-Rezayi pairing form} \) \( \Phi(z_1, z_2) \) given in Eq. A2 we’ve already found a set of \( 2n + 1 \) linearly independent symmetric polynomials in \( 2 \) variables, thus we may choose them as the basis of this quotient space.

Now given a choice of \( \tilde{\phi}_k \) we obtain a zero energy state of the Hamiltonian. Further, all possible zero energy states can be written in this way. We can now decompose
any \( \phi_k \) into basis polynomials \( f_i \) of the above described form. Let our choice be such that \( f_i() \) for \( 1 \leq i \leq \frac{N}{2} \) belong to the subspace of polynomials that vanish as \( z_1 \rightarrow z_2 \) and \( f_i() \) for \( \frac{N}{2} + 1 \leq i \leq \frac{N}{2} \) belong to the complementary subspace, i.e. \( Q \). Then each vanishing \( f_i() \) simplifies with the appropriate factor in the denominator of \( \phi_G \) producing a “broken pair”, and the remaining factors form what we above called \( \Omega() \), whereas the product of non-vanishing \( f_i() \) can be reexpressed as a linear combination of Read-Rezayi pairing forms \( \Phi() \). Thus we conclude that the most general zero energy eigenstate of \( H_G \) is of the conjectured form, and therefore we counted the complete degeneracy of eigenstates for a given value of additional flux \( n \).

**APPENDIX B: AN EXAMPLE OF ANGULAR MOMENTUM ADDITION**

We would like to determine the full angular momentum spectrum of the zero energy states of the Hamiltonian \( P_{3/2}^3 \) using Eq. 19. Here we will consider the example of \( N = 4 \) particles and \( n = 3 \) (6 quasiholes). Eq. 19 tells us that we should have a total number of zero energy states given by the sum of three terms corresponding to \( F = 0, 2, 4 \). For \( F = 4 \) we have (6 bosons in 1 orbitals) \( \otimes \) (4 fermions in 4 orbitals). Both 6 bosons in 1 orbital on 4 fermions in 4 orbitals have \( L = 0 \), so overall this is an \( L = 0 \) state. The \( F = 2 \) case is more tricky. Here we have (6 bosons in 2 orbitals) \( \otimes \) (2 fermions in 3 orbitals).

First we take 6 bosons in 2 orbitals. When there are two orbitals with two fermions in three orbitals with two fermions in 3 ways, which have \( L_z = 1, 0, -1 \) so we recognize this as \( L = 1 \). Now we must add together the angular momentum of (6 bosons in 2 orbitals) \( \otimes \) (2 fermions in 3 orbitals). This means we need to add \( L = 3 \) with \( L = 1 \). By the usual angular momentum addition rules we obtain \( L = 2, 3, 4 \). Finally, we turn to the \( F = 0 \) case. Here we have (6 bosons in 3 orbitals) \( \otimes \) (0 fermions in 2 orbitals). The 0 fermions in 2 orbitals clearly has \( L = 0 \). It is a simple exercise to count up the possibilities for 6 bosons in 3 orbitals. We discover that this has \( L = 0, 2, 4, 6 \). Putting together all of the results we find that the zero energy states of the Hamiltonian \( P_{3/2}^3 \) for \( N \) particles with \( n = 3 \) occur at angular momentum \( L = 0, 0, 2, 2, 3, 4, 4, 6 \) which agrees with the results of exact diagonalizations.

**APPENDIX C: FURTHER GENERALIZED WAVEFUNCTIONS**

Although there may be many possible ways to generalize Gaffnian wavefunctions, the form written in Eq. 18 suggests a generalization from paired to clustered wavefunctions where instead of dividing the particles into two groups, we divide the particles into \( g \)-groups. Let us assume the number of particles \( N \) in the system is divisible by \( g \) and write \( N = gn \). We then write the wavefunction

\[
\Psi = \tilde{S} \left\{ \prod_{a=1}^{g} \left( \prod_{(a-1)n+1 \leq i \leq an} (z_i - z_j) \right) \right\} \prod_{1 \leq a < b \leq g} \prod_{i=1}^{\frac{n}{(a-1)n+i}} \frac{1}{z_i - z_{(b-1)n+i}} \prod_{1 \leq i \leq n} (z_i - z_j)^m \right\}
\]

(C1)

with \( m \geq 1 \) where again \( \tilde{S} \) symmetrizes or antisymmetrizes for bosons (odd \( m \)) or fermions (even \( m \)) respectively. Counting powers of \( z \) we discover that this wavefunction occurs at flux

\[
N_\phi = (N/g - 1) - (g - 1) + m(N - 1) \quad \text{(C2)}
\]

\[
= (1 + g/m)N - (g + m) \quad \text{(C3)}
\]

corresponding to a filling fraction \( \nu = g/(gm + 1) \) which is just the Jain sequence. Furthermore the precise value of the flux (the shift) is also in agreement with the Jain series. This construction clearly reproduces the Gaffnian for \( g = 2 \). For the bosonic case \( (m = 1) \) for general \( g \) this construction produces a wavefunction that does not vanish when \( g \) particles come to the same point, but vanishes as \( g + 1 \) powers as the \( g + 1 \)st particle arrives at that point. However, for \( g > 2 \) this trial wavefunction is not the densest possible wavefunction with this particular property. Nonetheless, we believe that this, and other related wavefunctions can generally be constructed with simple projection rules. For example, for the \( g = 3, m = 1 \) case of Eq. (C1) this wavefunction is the unique densest wavefunction that does not vanish as 3 particles come together, that always vanishes as at least 4 powers when 4 particles come together, and vanishes faster than 4 powers if particles are brought together in groups of 2 and then two groups of 2 are brought together.
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