Derivation of Probability Distribution Function for Noisy Signal

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Abstract: The work in this paper produces mathematically derivation for Probability density function (PDF) of proposed noisy signal model. Two types of noise are considered: zero mean Gaussian multiplicative noise and phase noise. Cramer-Rao bound (CRB) expression of frequency estimator for single tone sinusoidal signal under zero mean Gaussian multiplicative noise and additive Gaussian noise (AWGN) are derived. The obtained simulation results support investigating mathematical and comparative study.

Keywords: multiplicative noise; phase noise; frequency estimator; CRB; AWGN.

1. Introduction

Frequency estimation of signal under Gaussian noise is one of the most popular problem that concern with many applications [1]. This applications include radar, radio frequency identification and resonance sensor systems [1]. In practice, these applications destroyed by multiplicative noise which caused random amplitude modulation effect [2]. For an estimation problem the measurement used to determine the variance of the best possible estimator, is Cramer-Rao bound (CRB) [3]. The easiest theoretical bound used to determined physical limitation of estimator was CRB, which is large than the variance of unbiased estimator [3]. Computable CRB expressions on phase and frequency estimator of single and multiple tone signal in presence of zero and nonzero Gaussian multiplicative noise, was derived in [2]. Ananthram Swami in [4] derived CRB for parameters of signal under two kind of additive and multiplicative noise: non-Gaussian and colored model. In [5] CRB of frequency estimator based on correlation method, was investigated for signals in fading channels. Finite-sample CRB and the large sample CRB (asymptotic Cramer Rao bound) of estimator for signal under complex-valued and noncircular multiplicative noise was derived [6].
However, none of the literature derive the probability function of proposed signal model. Therefore, this work presents deriving pdf of proposed noisy signal model under multiplicative noise and additive white Gaussian noise. Then the CRB of noisy signal distribution produced with respect to estimated frequency.

The paper is organized as follows: The signal and observation models as well as its distribution are defined in section 2. In section 3, simulation result and performance comparison.

2. Proposed model and distribution of noisy single-tone sinusoid signal under multiplicative noise

Let the signal to be noisy single-tone sinusoid as follows:

\[ y(t) = m(t) \cos(\omega_o t + \varnothing_o) + \epsilon(t) \]  

(1)

Where \( m(t) \) is multiplicative white Gaussian noise with zero mean and variance \( \sigma_m^2 \). \( \omega_o \) is the frequency of the signal, \( \varnothing_o \) is the initial phase and \( \epsilon(t) \) is an additive white Gaussian noise with zero mean and variance \( \sigma_a^2 \). Where \( m(t) \) and \( \epsilon(t) \) are independent process.

Without loss of generality, consider \( \varnothing_o = 0 \). It can be shown that the product \( m(t) \cos(\omega_o t) \) is Gaussian with normal distribution.

**Theorem (2.1)**

*if \( m \sim N(\mu, \sigma_m^2) \) then \( y(t) \sim N(\mu, \sigma^2) \)*

**Proof:**

*if \( x \sim N(\mu, \sigma^2) \) then* Gaussian probability density function (pdf) \( f(x) \) is [7]:

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty \]  

(2)

In addition, Moment Generated Function is:

\[ M_X(t) = e^{\mu t} + \frac{\sigma^2 t^2}{2} \]  

(3)

Where \( \mu \) is mean and is variance \( \sigma^2 \) of \( x \)

*if \( \mu = 0 \), then* \( f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \)

\[ M_X(t) = \frac{\sigma^2 t^2}{2} \]

*if \( y(t) = m(t) \sin \theta(t) + \epsilon(t) \) where \( m \sim N(0, \sigma_m^2) \), and \( \epsilon \sim N(0, \sigma_a^2) \) then* 

\[ M_y(r) = E(e^{yr}) = E(e^{im(t) \sin \theta(t) + \epsilon(t)r}) \]

\[ = E(e^{\epsilon(t)r}).E(e^{im(t) \sin \theta(t)r}) \]

\[ = M_{\epsilon(t)}(r).M_{m(t)}(r \sin \theta(t)) \]

\[ = (e^{\frac{1}{2} \sigma_a^2 r^2}).e^{\frac{1}{2} \sigma_m^2 r^2 (\sin \theta(t))^2} \]

\[ = e^{\frac{1}{2} \sigma_m^2 (\sin \theta(t))^2 + \sigma_a^2 r^2} \]

\[ \Rightarrow y(t) \sim N(0, \sigma_m^2 (\sin \theta(t))^2 + \sigma_a^2)) \]
\[ g(y) = \frac{1}{\sqrt{2\pi} (\sigma_m^2 + \sigma_n^2)} e^{-(y(t))^2/(2(\sigma_m^2 + \sigma_n^2))} \]

(4)

since \( y(t) = m(t) \sin \theta(t) + \epsilon(t) \)

\[ g(y) = \frac{1}{\sqrt{2\pi}(\sigma_m^2 + \sigma_n^2)} e^{-(m(t)\sin \theta(t) + \epsilon(t))^2/(2(\sigma_m^2 + \sigma_n^2))} \]

(5)

For \( N \) samples of \( y(t) \), the above Gaussian pdf can be written as:

\[ g(y) = \frac{1}{(2\pi)^{N/2} \prod_{n=0}^{N-1} (\sigma_m^2 + \sigma_n^2)^{1/2}} \sum_{n=0}^{N-1} \frac{-(m(n)\sin (w_0 n) + \epsilon(n))^2}{2(\sigma_m^2 + \sigma_n^2)} \]

(6)

After deriving probability distribution function in section 2, the procedure of deriving CRB mathematically in the presence of zero mean Gaussian additive and multiplicative noise, as a number of well-defined steps below[7]:

**Input**: probability density function of signal \( g(y) \)

**Output**: variance of \( w_0 \)

**Step1**: find log likelihood function of pdf in equation (6) as below:

\[ \ln(g(y)) = -\frac{N}{2} \ln(2\pi) - \frac{1}{2} \sum_{n=0}^{N-1} \ln((\sigma_m^2 + \sigma_n^2)) - \frac{1}{2} \sum_{n=0}^{N-1} \frac{(m(n)\sin (w_0 n) + \epsilon(n))^2}{(\sigma_m^2 + \sigma_n^2)} \]

(7)

**Step2**: derive first and second derivative of above log function in equation (7) as:

\[ \frac{\partial^2 \ln(g(y))}{\partial w_0^2} = \frac{\partial^2 \left( -\frac{N}{2} \ln(2\pi) \right)}{\partial w_0^2} - \frac{\partial^2 \left( \frac{1}{2} \sum_{n=0}^{N-1} \ln((\sigma_m^2 + \sigma_n^2)) \right)}{\partial w_0^2} - \frac{\partial^2 \left( \frac{1}{2} \sum_{n=0}^{N-1} \frac{(m(n)\sin (w_0 n) + \epsilon(n))^2}{(\sigma_m^2 + \sigma_n^2)} \right)}{\partial w_0^2} \]
Where

\[ \frac{\partial^2}{\partial w_0^2} \left( \frac{-N}{2} \ln(2\pi) \right) = 0 \]

\[ \frac{\partial^2}{\partial w_0^2} \left( \frac{-1}{2} \sum_{n=0}^{N-1} \ln(\sigma_m^2 n^2 + \sigma_a^2) \right) = \sum_{n=0}^{N-1} \frac{\sigma_m^2 n^2 (\sigma_m^2 \sin(\omega_0 n)^2 + 2\sigma_a^2 \sin(\omega_0 n)^2 - \sigma_a^2)}{\left( (\sigma_m^2 \sin(\omega_0 n)^2 + \sigma_a^2) \right)^2} \]

\[ \frac{\partial^2}{\partial w_0^2} \left( \frac{-1}{2} \sum_{n=0}^{N-1} \frac{(m(t) \sin(\omega_0 n) + \epsilon(t))^2}{(\sigma_m^2 \sin(\omega_0 n))^2 + \sigma_a^2} \right) \]

\[ \frac{\partial^2}{\partial w_0^2} \left( \frac{-1}{2} \sum_{n=0}^{N-1} \frac{(m(t) \sin(\omega_0 n) + \epsilon(t))^2}{(\sigma_m^2 \sin(\omega_0 n))^2 + \sigma_a^2} \right) = -\frac{1}{2} \sum_{n=0}^{N-1} \left( \sigma_m^2 \sin(\omega_0 n)^6 + 3 \sigma_m^4 \sigma_a^2 \sin(\omega_0 n)^4 + 3 \sigma_m^2 \sigma_a^4 \sin(\omega_0 n)^2 + \sigma_a^6 \right) \]

\[ \times \left( -2 \sigma_m^2 \sin(\omega_0 n)^2 \sin(\omega_0 n)^4 + 4 m(n) \epsilon(n) n^2 \sigma_m^2 + 4m(n) \epsilon(n) n^2 (\sigma_m^4 + 3 \sigma_m^2 \sigma_a^2) \sin(\omega_0 n)^3 \right) \]

\[ -2m(n) \epsilon(n) n^2 (6 \sigma_m^2 \sigma_a^2 + \sigma_a^4) \sin(\omega_0 n)^3 + 4 n^2 (-\sigma_m^4 \epsilon(n)^2 + \sigma_m^2 \sigma_a^2 m(n)^2) \sin(\omega_0 n)^2 \]

\[ -2 n^2 (m(n)^2 (3 \sigma_m^2 \sigma_a^2 + 2 \sigma_a^4) - n^2 (3 \sigma_m^2 + 2 \sigma_m \sigma_a^2)) \sin(\omega_0 n)^2 + 2n^2 (-\sigma_m^2 \sigma_a^2 \epsilon(n)^2 + \sigma_a^4 m(n)^2) \]

Step3: find expected value of above derivative with:

\[ E(m(n)) = 0 \text{ and } E(\epsilon(n)) = 0 \]

\[ E(m(n)^2) = (\text{var}(m(n)) + E(m(n))) = \sigma_m^2 \]

\[ E(\epsilon(n)^2) = (\text{var}(\epsilon(n)) + E(\epsilon(n))) = \sigma_a^2 \]

Step4: with Neglecting to the sum of sin function as N grows as in below lemma[2]:

\[ \lim_{N \to \infty} \sum_{n=0}^{N-1} \left( \frac{n}{N} \right) e^{j\omega_0 n} = 0 \]

The expected value of second derivative of log function becomes:

\[ E \left( \frac{\partial^2 \ln(g(y))}{\partial w_0^2} \right) = \sum_{n=0}^{N-1} \frac{-\sigma_m^2 n^2 \sigma_a^2}{\sigma_a^2} = \sum_{n=0}^{N-1} \frac{-\sigma_m^2 n^2}{\sigma_a^2} \]
Where:

\[ \sum_{n=0}^{N-1} n^2 = \frac{N(N - 1)(2N - 1)}{6} \]

And

\[ SNR = \frac{\sigma_m^2}{\sigma_a^2} \]

Then:

\[ E\left(\frac{\partial^2 \ln(g(y))}{\partial w_0^2}\right) = -\frac{\sigma_m^2 N(N - 1)(2N - 1)}{6\sigma_a^2} \approx -\frac{\sigma_m^2 N^3}{6\sigma_a^2} \approx -\frac{SNR^3}{6} \]

Finally, CRB can be formulated as:

\[ CRB(w_0) \equiv \text{var}(w_0) \geq \frac{1}{E\left(\frac{\partial^2 \ln(g(y))}{\partial w_0^2}\right)} \approx \frac{6}{N(N - 1)(2N - 1)SNR} \]

\[ \approx \frac{6}{N^3SNR} \] (8)

3. Proposed model and distribution of noisy single-tone sinusoid signal under phase noise

Let the signal to be noisy single-tone sinusoid as follows:

\[ y(t) = A \sin(\omega_o t + \phi_o(t)) + \epsilon(t) \] (9)

Where \( A \) is constant, \( \omega_o \) is the frequency of the signal, \( \phi_o(t) \) is an phase noise with zero mean and variance \( \sigma_p^2 \) and \( \epsilon(t) \) is an additive white Gaussian noise with zero mean and variance \( \sigma_a^2 \). Where \( \phi_o(t) \) and \( \epsilon(t) \) are independent process.

**Theorem (3.1)**

Let \( y(t) = \sin(\omega_o t + \phi_o(t)) \)  
\[ y(t) \sim N(\mu, \sigma^2) \]

if \( \phi_o(t) \sim N(\mu, \sigma^2) \) then

**Proof:**

Let \( \phi_o(t) = x \) then \( x = \sin^{-1}(y) - \omega_o \)
\[
\frac{dy}{dx} = \cos(\omega_0 + x) \Rightarrow dx = \frac{dy}{\cos(\sin^{-1} y)}
\]

Probability density function of \(y(t)\) is [7]:

\[
g(y) = \int g(x) \ast dx \quad \text{(10)}
\]

\[
g(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\sin^{-1}(y) - \omega_0)^2}{2\sigma^2}} \quad \text{(11)}
\]

\[
\Rightarrow g(y) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(\sin^{-1}(y) - \omega_0)^2}{2\sigma^2}} \ast \frac{dy}{\cos(\sin^{-1} y)}
\]

\[
\Rightarrow g(y) = \frac{1}{2} \text{erf} \left( \frac{\sin^{-1}(x) - \omega_0}{\sigma \sqrt{2}} \right) = \frac{1}{2} \cdot 2 \cdot \frac{2}{\sqrt{\pi}} \int_0^y e^{-\frac{(\sin^{-1}(x) - \omega_0)^2}{2\sigma^2}} \quad \text{(12)}
\]

Where \(\text{erf}\) is error function which equal \(\frac{2}{\sqrt{\pi}} \int_0^y e^{-\frac{(\sin^{-1}(x) - \omega_0)^2}{2\sigma^2}}\)

4. Simulation Result

Maximum likelihood (ML) estimator using Fast Fourier Transform (FFT) with interpolated peak estimation based frequency estimation is used to estimate frequency of noisy signal under Gaussian multiplicative noise. The above algorithms with signal model under multiplicative noise (MN) as per equation (1) are simulated using MATLAB.

The signal-to-noise ratio (SNR) of noisy signal under effect of both AWGN and MN is defined as follows [8, 2]:

\[
\text{SNR} = \frac{p_x}{p_m} = \frac{p_{xm}}{p_n}
\]

where \(p_x\) being the signal power, \(p_m\) being the MN power, and \(p_n\) is the additive noise power.

The simulated signal has total time length \(L = 10\) s, the sampling interval is \(T_s = 0.001\) s, and the number of samples is given by \(N = \lfloor L/T_s \rfloor\). The signal amplitude is \(A = 1\) volt, \(\omega_0\) is angle frequency \(\omega_0 = 2\pi f_0\), where \(f_0 = 23\) Hz. MN has been with different MN power \((-50\) dB, \(0\) dB, \(30\) dB) respectively.

Figures (1), (2), (3) and (4) show the mean-square error (MSE) of frequency estimation versus SNR using FT methods with different MN power \((-50\) dB, \(0\) dB, \(30\) dB) for different value of \(N\).

The curve of mean square error (MSE) of estimated frequency in figure (1-4) provides information about lower band of error, which represent the asymptote representing of CRB as in [4]. Meanwhile these figures show the decrease of MSE as SNR and \(N\) increases.

Figures (5), (6), (7) and (8) shows the curve of CRB versus SNR as in equation (8), for MN power \((-50\) dB, \(0\) dB, \(30\) dB) for different value of \(N\).

Clearly, the curve of CRB in figures (5-8) breakdown at high SNR especially with high value of \(N\).
It can be seen that our result is an agreement with the result obtained by [8] and [2].

In comparison between figures (1-4) and figures (5-8) the MSE which represent the variance of estimated frequency, is more than CRB as in equation (8).

As a result, these simulation figures support the obtained Mathematical equation of CRB in section (8) and vice versa.

5. Conclusion

In this paper, a probability function distribution of noisy signal in the presence of additive Gaussian noise and zero mean Gaussian multiplicative noise is derived. Meanwhile, the asymptotic CRB is derived. Simulations results show that the variance of estimated frequency represented by CRB curve inversely proportional with high SNR and large value of data length $n$.

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Figure (1): The Mean-Squared Error (MSE) of frequency estimation versus SNR using FT estimator with different value of Multiplicative noise power and $N = 10001$

Figure (2): The Mean-Squared Error (MSE) of frequency estimation versus SNR using FT estimator with different value of Multiplicative noise power and $N = 20001$
Figure (3): The Mean-Squared Error (MSE) of frequency estimation versus SNR using FT estimator with different value of Multiplicative noise power and $N = 50001$

Figure (4): The Mean-Squared Error (MSE) of frequency estimation versus SNR using FT estimator with different value of Multiplicative noise power and $N = 70001$
Figure (5): Cramer Rao Bound (CRB) of frequency estimator versus SNR with different value of Multiplicative noise power and $N = 10001$

Figure (6): Cramer Rao Bound (CRB) of frequency estimator versus SNR with different value of Multiplicative noise power and $N = 20001$
Figure (7): Cramer Rao Bound (CRB) of frequency estimator versus SNR with different value of Multiplicative noise power and $N = 50001$

Figure (8): Cramer Rao Bound (CRB) of frequency estimator versus SNR with different value of Multiplicative noise power and $N = 70001$