Tight Performance Bounds for Compressed Sensing With Group Sparsity

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Abstract

Compressed sensing refers to the recovery of a high-dimensional but sparse vector using a small number of linear measurements. Minimizing the $\ell_1$-norm is among the more popular approaches for compressed sensing. A recent paper by Cai and Zhang has provided the “best possible” bounds for $\ell_1$-norm minimization to achieve robust sparse recovery (a formal statement of compressed sensing). In some applications, “group sparsity” is more natural than conventional sparsity. In this paper we present sufficient conditions for $\ell_1$-norm minimization to achieve robust group sparse recovery. When specialized to conventional sparsity, these conditions reduce to the known “best possible” bounds proved earlier by Cai and Zhang. This is achieved by stating and proving a group robust null space property, which is a new result even for conventional sparsity. We also derive bounds for the $\ell_p$-norm of the residual error between the true vector and its approximation, for all $p \in [1, 2]$. These bounds are new even for conventional sparsity and of course also for group sparsity, because previously error bounds were available only for the $\ell_2$-norm.

1 Introduction

Compressed sensing refers to the recovery of sparse entities such as vectors, images and matrices from a small number of measurements. The simplest version of the compressed sensing problem can be stated as follows: Suppose $n$ and $k \ll n$ are given integers, and that $x \in \mathbb{R}^n$ is a “$k$-sparse vector.” This means that $x$ has $k$ or fewer nonzero components, but their locations are unknown. The objective is to design an integer $m$ and a matrix $A \in \mathbb{R}^{m \times n}$ such that, from $y = Ax$, it is possible to recover $x$ exactly. In this set-up, $m$ is called the number of measurements, $A$ is called the measurement matrix, and $y$ is called the vector of measurements.

One of the most popular approaches to compressed sensing is $\ell_1$-norm minimization. Specifically, an approximation $\hat{x}$ to the unknown vector $x$ is constructed as

$$\hat{x} := \arg\min_z \|z\|_1 \text{ s.t. } Az = y.$$  

(1)

This approach is introduced in [1, 2], and is referred to as “basis pursuit.” In these papers, empirical evidence was presented to show that basis pursuit works; but there were no theoretical

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results. The $\ell_1$-norm minimization approach is sometimes referred to as the LASSO formulation, due its similarity to the LASSO formulation for sparse regression [3]. The LASSO formulation was popularized in a series of papers [4, 5, 6, 7] that provided a theoretical justification. A paper by Cai and Zhang [8] provides the best possible bounds, or to put it another way, tight bounds for sparse recovery using $\ell_1$-norm minimization. The results are stated precisely in the next section. But roughly speaking, if the columns of the matrix $A$ are chosen so that they are nearly orthogonal, known as the restricted isometry property (RIP), then $\hat{x}$ will equal $x$ if $x$ is sparse to begin with. More precise statements are given below.

In some applications, the number of nonzero elements of $x$ is of less interest than the number of “groups” in which $x$ has nonzero components. A typical application arises in biology, where the integer $n$ (dimension of $x$) represents the number of genes under study, which are then grouped into various pathways. In this setting, a collection of genes that act in concert is known as a “biological pathway.” The objective is to find a small number of genes that can accurately predict the measured vector $y$. If there are two explanations of the measured vector $y$, one of which involves $k_1$ genes but all lying in distinct pathways, while another involves $k_2 > k_1$ genes, but belonging to very few distinct pathways, then biologists would prefer the latter explanation, as it is “less group sparse.” In order to formalize this notion mathematically, over the years some variants of LASSO have been proposed, such as the Group LASSO (GL) [9] and the Sparse Group LASSO (SGL) [10].

In the GL formulation, the index set $\{1, \ldots, n\}$ is partitioned into $g$ disjoint sets $G_1, \ldots, G_g$, and the objective function $\|z\|_1$ in (1) is replaced by

$$
\|z\|_{\text{GL}} := \sum_{i=1}^{g} \|z_{G_i}\|_2, 
$$

where $z_{G_i}$ denotes the projection of the vector $z$ onto the components in $G_i$. In some formulations the quantity

$$
\|z\|_{\text{GL}} := \sum_{i=1}^{g} \frac{\|z_{G_i}\|_2}{|G_i|^{1/2}}
$$

is used, so as to normalize for the size of the group $G_i$; however, we do not use this convention in this paper. A further refinement of GL is the sparse group LASSO (SGL), in which the group structure is as before, but the objective function is now defined as

$$
\|z\|_{\text{SGL},\mu} := \sum_{i=1}^{g} [(1 - \mu)\|z_{G_i}\|_1 + \mu\|z_{G_i}\|_2],
$$

where before $\mu \in [0, 1]$ is some adjustable parameter.

In the literature, some evidence is presented to show that the GL and SGL formulations sometimes work. However, until the publication of [11], there was no theoretical justification to show that these algorithms to achieve group sparse recovery. The paper [11] provides a unified theory that embraces several algorithms that have been proposed in the literature for both “conventional” as well as group sparsity. Specifically, the $\ell_1$-norm is replaced by a general “group-decomposable” norm, and it is shown that both the GL and SGL norms are group-decomposable. Then sufficient conditions for recovery of group sparse vectors are presented. The main shortcoming of the results in [11] is that, when specialized to the case of conventional sparsity and $\ell_1$-norm minimization, the bounds are not the best possible bounds established in [8]. The objective of the present paper is therefore to state and prove bounds for $\ell_1$-norm minimization to achieve group sparse recovery, which include the bounds of [8] as a special case in the case of conventional sparsity. Because
the bounds in [8] are tight, so are the bounds proved here, in the sense that conventional sparsity is a special case of group sparsity. In addition to providing sufficient conditions for group sparse recovery, we also state and prove a useful result on “group robust null space property.” It is worth mentioning that the property is new even for conventional sparsity. Using this property, we are able to establish bounds on the $\ell_p$-norm of the residual error $\hat{x} - x$ for all $p \in [1, 2]$, whereas in [8], only bounds on $\|\hat{x} - x\|_2$ are given. Moreover, in [8] the case of noiseless and noisy measurements are treated separately, whereas in the present approach both are treated in a common setting. Therefore, aside from proving sufficient conditions for group sparse recovery, the present paper also improves upon existing theory even for conventional sparsity.

The paper is organized as follows. Section 2 contains a review of the definitions of robust sparse recovery for both conventional sparsity, and the results in [8] giving tight bounds for robust sparse recovery. Our new results on group sparsity are given in Section 3. The specialization of our results to conventional sparsity, which incidentally results in improved error estimates in comparison with [8], are presented in Section 4.

## 2 Review of Definitions and Known Results

Throughout this paper, the symbol $[n]$ denotes the set $\{1, \ldots, n\}$ whenever $n$ is an integer. If $x \in \mathbb{R}^n$, then $\text{supp}(x)$ denotes the support of $x$; that is

$$\text{supp}(x) := \{i \in [n] : x_i \neq 0\}.$$  

A vector $x$ is said to be $k$-sparse if $|\text{supp}(x)| \leq k$. The set of all $k$-sparse vectors in $\mathbb{R}^n$ is denoted by $\Sigma_k$. If $\| \cdot \|$ is a norm on $\mathbb{R}^n$, then

$$\sigma_k(x, \| \cdot \|) := \min_{z \in \Sigma_k} \|x - z\|$$

is known as the $k$-sparsity index of $x$. It is obvious that $x \in \Sigma_k$ if and only if $\sigma_k(x, \| \cdot \|) = 0$. Also, for fixed $x$ and $k$, the $k$-sparsity index $\sigma_k(x, \| \cdot \|)$ depends on the norm $\| \cdot \|$. Note that if $p \in [1, \infty]$, then in principle the sparsity index $\sigma_k(x, \| \cdot \|_p)$ is easy to determine. Let $\Lambda_0$ denote the index set corresponding to the $k$ largest components of $x$ by magnitude. Then

$$\sigma_k(x, \| \cdot \|_p)\|x_{\Lambda_0}\|_p, \forall p \in [1, \infty].$$

Next we come to notions of group sparsity, group sparsity index, and related concepts.

**Definition 1** Let $\mathcal{G} = \{G_1, \ldots, G_g\}$ be a partition of $[n]$ such that $|G_i| \leq k$ for all $i$. If $S \subseteq \{1, \ldots, g\}$, define $G_S := \cup_{i \in S} G_i$. A subset $\Lambda \subseteq [n]$ is said to be group $k$-sparse if there exists a subset $S \subseteq \{1, \ldots, g\}$ such that $\Lambda = G_S$, and in addition, $|\Lambda| \leq k$. The collection of all group $k$-sparse subsets of $[n]$ is denoted by $\text{GkS}$. A vector $u \in \mathbb{R}^n$ is said to be group $k$-sparse if its support set $\text{supp}(u)$ is contained in a group $k$-sparse set.

See [11] for more details about this definition. In particular, every group $k$-sparse set is $k$-sparse (has cardinality no larger than $k$), but the converse is not true in general.

**Definition 2** Given an integer $k$, let $\text{GkS}$ denote the collection of all group $k$-sparse subsets of $[n]$, and define

$$\sigma_{k, \mathcal{G}}(x, \| \cdot \|) := \min_{\Lambda \in \text{GkS}} \|x - x_\Lambda\| = \min_{\Lambda \in \text{GkS}} \|x_{\Lambda_0}\|$$

(5)

to be the group $k$-sparsity index of the vector $x$ with respect to the norm $\| \cdot \|$ and the group structure $\mathcal{G}$.

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It is obvious that if \( g = n \), and each group \( G_i \) is the singleton set \( \{ i \} \), then group sparsity and group sparsity index reduce respectively to \( k \)-sparsity and \( k \)-sparsity index. Note that, because \( \text{GkS} \) is in general a strict subset of the set of all \( k \)-sparse sets, it follows that
\[
\sigma_k(x, \| \cdot \|) \leq \sigma_{k,G}(x, \| \cdot \|).
\]

Next we introduce the restricted isometry property, which plays a major role in compressed sensing. This property was introduced in [4].

**Definition 3** A matrix \( A \in \mathbb{R}^{m \times n} \) is said to satisfy the **restricted isometry property (RIP)** of order \( k \) with constant \( \delta_k \) if
\[
(1 - \delta_k)\|u\|_2^2 \leq \|Au\|_2^2 \leq (1 + \delta_k)\|u\|_2^2, \quad \forall u \in \Sigma_k.
\]

To analyze group sparsity, we introduce an extension of RIP; this concept is introduced in [12, 11].

**Definition 4** A matrix \( A \in \mathbb{R}^{m \times n} \) is said to satisfy the **group restricted isometry property (GRIP)** of order \( k \) with constant \( \delta_k \) if
\[
1 - \delta_k \leq \min_{\Lambda \in \text{GkS}} \min_{\text{supp}(z) \subseteq \Lambda} \frac{\|Az\|_2^2}{\|z\|_2^2} \leq \max_{\Lambda \in \text{GkS}} \max_{\text{supp}(z) \subseteq \Lambda} \frac{\|Az\|_2^2}{\|z\|_2^2} \leq 1 + \delta_k.
\]

As mentioned above, the set of group \( k \)-sparse vectors can be strictly smaller than the set of \( k \)-sparse vectors. Consequently, in general, the GRIP constant of order \( k \) can be smaller than the RIP constant of order \( k \). When probabilistic methods are used to construct the measurement matrix \( A \), often we require fewer measurements to achieve group sparse recovery than sparse recovery. See for example [11, Section 6]. This is why we study group sparsity.

Next we define precisely what is meant recovery. Suppose \( A \in \mathbb{R}^{m \times n} \), known as the measurement map, and \( \Delta : \mathbb{R}^m \rightarrow \mathbb{R}^n \), known as the decoder map. Then, for an unknown vector \( x \), the measured vector \( y \) equals \( Ax + \eta \), where \( \eta \) is a measurement noise. Now \( \hat{x} = \Delta(y) = \Delta(Ax + \eta) \) is the “recovered” vector which we hope is a good approximation to the true but unknown \( x \).

**Definition 5** Suppose \( A \in \mathbb{R}^{m \times n} \), known as the measurement map, and \( \Delta : \mathbb{R}^m \rightarrow \mathbb{R}^n \), known as the decoder map. Then the pair \((A, \Delta)\) is said to achieve **robust sparse recovery of order** \( k \) if there exist constants \( C_1, C_2 \) such that, for all \( \eta \in \mathbb{R}^m \) with \( \|\eta\|_2 \leq \epsilon \), we have that
\[
\|\Delta(Ax + \eta) - x\|_2 \leq C_1\sigma_k(x, \| \cdot \|_1) + C_2\epsilon.
\]

**Definition 6** Suppose \( A \in \mathbb{R}^{m \times n} \), known as the measurement map, and \( \Delta : \mathbb{R}^m \rightarrow \mathbb{R}^n \), known as the decoder map. Then the pair \((A, \Delta)\) is said to achieve **robust group sparse recovery of order** \( k \) if there exist constants \( D_1, D_2 \) such that, for all \( \eta \in \mathbb{R}^m \) with \( \|\eta\|_2 \leq \epsilon \), we have that
\[
\|\Delta(Ax + \eta) - x\|_2 \leq D_1\sigma_{k,G}(x, \| \cdot \|_1) + D_2\epsilon.
\]

Note that (10) guarantees perfect recovery of group \( k \)-sparse vectors, which is the main aim of group sparsity studies. Because the set of group sparse vectors is smaller than the set of sparse vectors, in principle group sparse recovery should be possible even if sparse recovery is not. In other words, the aim of group sparsity studies is to identify sufficient conditions for group sparse recovery that are strictly weaker than known sufficient conditions for sparse recovery.
Now we present some important results from [8] on when $\ell_1$-norm minimization can achieve robust sparse recovery. In this set-up, first proposed in [5], the decoder is defined via

$$\hat{x} = \arg\min_{z} \|z\|_1 \text{ s.t. } \|y - Az\|_2 \leq \epsilon.$$  \hspace{1cm} (11)

**Theorem 1** (See [8, Theorem 2.1]) Suppose $A$ satisfies the RIP of order $tk$ for some number $t \geq 4/3$ such that $tk$ is an integer, with $\delta_{tk} < \sqrt{(t-1)/t}$. Then the recovery procedure in (11) achieves robust sparse recovery of order $k$.

**Theorem 2** (See [8, Theorem 2.2]) Let $t \geq 4/3$. For all $\gamma > 0$ and all $k \geq 5/\gamma$, there exists a matrix $A$ satisfying the RIP of order $tk$ with constant $\delta_{tk} \leq \sqrt{(t-1)/t} + \gamma$ such that the recovery procedure in (11) fails for some $k$-sparse vector.

Taken together, these two theorems show that the bound $\delta_{tk} < \sqrt{(t-1)/t}$ is tight – every matrix $A$ that satisfies this condition can achieve robust $k$-sparse recovery, whereas at least one matrix that violates this condition fails to achieve robust $k$-sparse recovery. Therefore an important criterion for assessing a sufficient condition for group sparse recovery is to see whether that sufficient condition reduces to Theorem 1 in the case of conventional sparsity. This is achieved in this paper.

In [11], a general framework is put forward, in which the $\ell_1$-norm is replaced by an arbitrary “group-decomposable” norm. The theory proposed in [11] includes LASSO, GL, and SGL all within a common framework. In that sense it is a major step forward. When specialized to LASSO ($\ell_1$-norm minimization), the condition for robust sparse recovery to order $k$ becomes $\delta_{2k} < \sqrt{2} - 1 \approx 0.414$, a condition first derived in [6]. However, as shown in Theorem 1, the “best possible bound” on $\delta_{2k}$ is $1/\sqrt{2} \approx 0.707$. This suggests that the method of proof adopted in [11] can be improved. That is precisely the purpose of the present paper. To keep things simple, we forego the generality of the problem formulation in [11], and restrict the discussion just to $\ell_1$-norm minimization (LASSO). By doing so, we are able to prove theorems for group sparse recovery that reduce to the “best possible” bound in Theorem 1 for conventional sparsity.

The method of proof in this paper is based on the notion of group robust null space property, which is a generalization of the robust null space property as in [13, Definition 4.21]. A consequence of this is that, even for conventional sparsity, we are able to study both noiseless and noisy measurements in a common setting, unlike in [8] where the two cases are treated separately. Moreover, whereas the paper [8] gives bounds for only $\|\hat{x} - x\|_2$, we are able to give bounds for $\|\hat{x} - x\|_p$ for $p \in [1, 2]$. These bounds also improve on those in [13, Theorem 4.25].

### 3 Bounds for Robust Group Sparse Recovery

In this section we state and prove two results, namely Theorem 3 and Theorem 4. Theorem 3 establishes a property called the group robust null space property. Not only is this result new in the case of group sparsity, but its special case to conventional sparsity is also new, as stated in Section 4. The second result gives bounds on the residual error $\|\hat{x} - x\|_p$ for $p \in [1, 2]$. These bounds are also new for both group as well as conventional sparsity. In order to establish all of these results, we begin with a lemma.

#### 3.1 Polytope Decomposition Lemma

The key to the results in [8] is Lemma 1.1 of that paper, which the authors call the “polytope decomposition lemma.” In this subsection we generalize this lemma to the case of group sparsity.
Before presenting the lemma, we introduce a couple of terms. Given a vector \( v \in \mathbb{R}^n \), we define the \textbf{group support set of} \( v \), denoted by Gsupp\((v)\), as

\[
\text{Gsupp}(v) := \{ j \in [g] : v_{G_j} \neq 0 \}.
\]

Thus Gsupp\((v)\) denotes the subset of the groups on which \( v \) has a nonzero support. Obviously \(|\text{Gsupp}(v)|\) is the number of distinct groups on which \( v \) is supported. Next, we define

\[
m_{\text{max}} := \max_{j \in [g]} |G_j|, m_{\text{min}} := \min_{j \in [g]} |G_j|.
\]

Recall that, by assumption, \( m_{\text{max}} \leq k \). Also, for conventional sparsity, each \( G_j \) is a singleton, whence \( m_{\text{max}} = m_{\text{min}} = 1 \). By substituting these values in the various expressions below, the bounds for group sparsity can be readily specialized to conventional sparsity.

\textbf{Lemma 1} \textit{Given a vector} \( v \in \mathbb{R}^n \) \textit{such that,}

\[
\|v_{G_j}\|_1 \leq \alpha, \forall j \in [g], \text{ and } \|v\|_1 \leq s\alpha
\]

\textit{for some integer} \( s \), \textit{there exist an integer} \( N \) \textit{and vectors} \( u_i, i \in [N] \) \textit{such that}

\begin{itemize}
  \item \( \text{supp}(u_i) \subseteq \text{supp}(v), \forall i \in [N] \).
  \item \( \|u_i\|_1 = \|v\|_1, \forall i \in [N] \).
  \item \( u_i \) \textit{is} \( sm_{\text{max}} \)-\textit{sparse} \textit{for each} \( i \), \textit{and finally}
  \item \( v \) \textit{is a convex combination of} \( u_i, i \in [N] \).
\end{itemize}

\textbf{Remarks:} In the case of conventional sparsity, \( m_{\text{max}} = 1 \), in which case all vectors \( u_i \) are \( s \)-sparse, which is precisely [8, Lemma 1.1].

\textbf{Proof:} The proof is by induction. Define a subset of \( \mathbb{R}^n \) as follows:

\[
X := \{ v \in \mathbb{R}^n : \|v_{G_j}\|_1 \leq \alpha \forall j \in [g], \|v\|_1 \leq s\alpha \}.
\]

Note that if each \( G_j \) is a singleton, then

\[
\|v_{G_j}\|_1 \leq \alpha \forall j \in [g] \iff \|v\|_\infty \leq \alpha,
\]

which is the hypothesis in [8, Lemma 1.1]. To begin the inductive process, suppose \(|\text{Gsupp}(v)| \leq s \). Then \( v \) is itself \( sm_{\text{max}} \)-sparse. So we can take \( N = 1 \) and \( u_1 = v \). Now suppose that the lemma is true for all \( v \in X \) such that \(|\text{Gsupp}(v)| = r - 1 \) where \( r - 1 \geq s \). It is shown that the lemma is also true for all \( v \in X \) satisfying \(|\text{Gsupp}(v)| = r \).

Let \( Q \subseteq [g] \) denote the index set \( \{ j \in [g] : v_{G_j} \neq 0 \} \), and observe that \(|Q| = |\text{Gsupp}(v)| = r \) by assumption. Then \( v \) can be expressed as \( v = \sum_{j \in Q} v_{G_j} \). Now arrange the vectors \( v_{G_j} \) in decreasing order of their \( \ell_1 \)-norm. Denote the permuted vectors as \( p_1 \) through \( p_r \). Define \( a_i := \|p_i\|_1 \), and \( \hat{p}_i = (1/a_i)p_i \). Then each \( \hat{p}_i \) has unit \( \ell_1 \)-norm. Moreover \( a_i \geq a_{i+1} \) for all \( i \), and

\[
v = \sum_{i=1}^{r} p_i = \sum_{i=1}^{r} a_i \hat{p}_i.
\]
Also, because the $\ell_1$-norm is decomposable and the $p_i$ have nonoverlapping support sets, it follows that

$$
\|v\|_1 = \sum_{i=1}^r p_i \|p_i\|_1 = \sum_{i=1}^r a_i.
$$

Now define a set

$$
D := \{\beta \in [r-1] : \sum_{i=1}^r a_i \beta \leq (r - \beta)\alpha\}.
$$

Then $1 \in D$ because $r \sum_{i=1}^r a_i = \|v\|_1 \leq s\alpha \leq (r - 1)\alpha$.

Therefore $D$ is nonempty. Now, by a slight abuse of notation, let $\beta$ again denote the largest element of the set $D$. This implies that

$$
\sum_{i=1}^r a_i \leq (r - \beta)\alpha, \quad \sum_{i=\beta+1}^r a_i > (r - \beta - 1)\alpha.
$$

Define the constants

$$
b_t := \frac{1}{r - \beta} \sum_{i=\beta}^r a_i - a_t, \beta \leq t \leq r.
$$

Since the first term on the right side is independent of $t$, and $a_{t+1} \leq a_t$, it follows that $b_{t+1} \geq b_t$. Also

$$
b_\beta = \frac{1}{r - \beta} \sum_{i=\beta}^r a_i - a_\beta = \frac{1}{r - \beta} \sum_{i=\beta+1}^r a_i - \frac{r - \beta - 1}{r - \beta} a_\beta \geq \frac{1}{r - \beta} \left[ \sum_{i=\beta+1}^r a_i - (r - \beta - 1)\alpha \right] > 0,
$$

where the last two steps follow from $a_i \leq \alpha$ for all $i$, and from the second inequality in (15). Also, it is easy to verify that

$$
\sum_{i=\beta}^r a_i = (r - \beta) \sum_{i=\beta}^r b_i
$$

Next, for $t = \beta, \ldots, r$, define

$$
w_t := \sum_{i=1}^{\beta-1} a_i \hat{p}_i + \left( \sum_{i=\beta}^r b_i \right) \sum_{i=\beta, i \neq t}^r \hat{p}_i, \lambda_t := \frac{b_t}{\sum_{i=\beta}^r b_i}.
$$

Now observe that

$$
0 < \lambda_t < 1, \sum_{t=\beta}^r \lambda_t = 1, \text{ and } v = \sum_{t=\beta}^r \lambda_t w_t.
$$
Next, \( \text{supp}(w_t) \subseteq \text{supp}(v) \) for all \( t \). Moreover, \( |G\text{supp}(w_t)| \leq r - 1 \) for all \( t \), because the corresponding term \( \hat{p}_t \) is missing from the summation in (17). Also, note that each \( \hat{p}_i \) has unit \( \ell_1 \)-norm. Therefore, for each \( t \) between \( \beta \) and \( r \), we have that
\[
\|w_t\|_1 = \sum_{i=1}^{\beta-1} a_i + (r - \beta) \sum_{i=\beta}^{r} b_i
\]
\[
= \sum_{i=1}^{\beta-1} a_i + \sum_{i=\beta}^{r} a_i = \sum_{i=1}^{r} a_i = \|v\|_1.
\]
Therefore each \( w_t \in X \). By the inductive assumption, each \( w_t \) has a convex decomposition as in the statement of the lemma. It follows that \( v \) is also a convex combination as in the statement of the lemma. This completes the inductive step. \( \square \)

**Lemma 2** Let \( u_i, i \in [N] \) be the vectors in the convex combination of Lemma 1. Then
\[
\|u_i\|_2^2 \leq \frac{sm_{\text{max}}}{m_{\text{min}}} \alpha^2, \forall i \in [N]. \tag{18}
\]

**Proof:** Fix the index \( i \in [N] \). Define the index set
\[
B_i := \{ j \in [g] : (u_i)_{G_j} \neq 0 \}.
\]
Let \( c_i = |B_i| \). Because \( u_i \) is \( sm_{\text{max}} \)-sparse, it follows that \( c_i \leq \frac{sm_{\text{max}}}{m_{\text{min}}} \). Moreover, for each index \( j \in B_i \), we have that
\[
\|(u_i)_{G_j}\|_2 \leq \|(u_i)_{G_j}\|_1 \leq \|u_i\|_1 = \alpha.
\]
Now observe that
\[
u_i = \sum_{j \in B_i} (u_i)_{G_j}.
\]
Since there are at most \( c_i \) terms in the above summation, and each term has Euclidean norm no larger than \( \alpha \), the desired conclusion (18) follows. \( \square \)

### 3.2 Group Robust Null Space Property

In this subsection we present the first of our new results for group sparsity. In order to do this, we introduce a new concept.

**Definition 7** A matrix \( A \in \mathbb{R}^{m \times n} \) is said to satisfy the \( \ell_2 \) **group robust null space property (GRNSP)** with constants \( \rho \in (0, 1), \tau \in \mathbb{R}_+ \), if, for all \( h \in \mathbb{R}^n \) and all sets \( S \in GkS \), it is true that
\[
\|h_S\|_2 \leq \frac{\rho}{\sqrt{k}} \|h_{S^c}\|_1 + \frac{\tau}{\sqrt{k}} \|Ah\|_2, \tag{19}
\]

This definition extends the so-called robust null space property (see e.g. [13, Definition 4.17]) to group sparsity. Note that the definition in [13] is slightly different, in the presence of the constant \( \sqrt{k} \). It is a ready consequence of Schwarz’ inequality that, if \( A \) satisfies the \( \ell_2 \) GRNSP, then the following is also true:
\[
\|h_S\|_1 \leq \rho \|h_{S^c}\|_1 + \tau \|Ah\|_2. \tag{20}
\]
The main result of this subsection states that if the matrix $A$ satisfies the GRIP then it also satisfies the GRNSP. To facilitate the statement and proof of the main theorem, we define a few constants. Suppose that $A$ satisfies the GRIP of order $t_k$ with constant $\delta_{tk} = \delta$. Then define

$$\mu := \sqrt{(t-1)t - (t-1)},$$

$$a := [\mu(1-\mu) - \delta(0.5 - \mu + \mu^2)]^{1/2},$$

$$b := \mu(1-\mu)\sqrt{1 + \delta}, c := \left[ \frac{\delta^2 m_{\text{max}}^2}{2(t-1)m_{\text{min}}} \right]^{1/2},$$

$$\rho := c/a, \tau := b\sqrt{k/a^2}. \quad (24)$$

Now we come to the key result that allows us to establish robust group $k$-sparse recovery. Note that, even in the case of conventional sparsity, the following result is new.

**Theorem 3** Suppose that the matrix $A$ satisfies the GRIP of order $t_k$ with constant $\delta_{t_j} = \delta$. If $\delta$ satisfies

$$\delta < \mu(1-\mu) \left( \frac{\mu^2 m_{\text{max}}^2}{2(t-1)m_{\text{min}}} + 0.5 - \mu + \mu^2 \right)^{-1}, \quad (25)$$

then $A$ satisfies the $\ell_2$ GRNSP with constants $\rho, \tau$ defined in (24).

**Remark:** In the case of conventional sparsity, $m_{\text{max}} = m_{\text{min}} = 1$, in which case the bound in (25) becomes

$$\delta < \mu(1-\mu) \left( \frac{\mu^2}{2(t-1)} + 0.5 - \mu + \mu^2 \right)^{-1}. \quad (25)$$

It is tedious but straightforward to substitute for $\mu$ from (21) and to verify that the right side of this inequality equals $\sqrt{(t-1)/t}$; this is precisely the bound given by [8] and stated here as Theorem 1.

**Proof:** Choose an index set $\Lambda_0 \in \text{GkS}$ such that

$$\|h_{\Lambda_0}\| = \min_{S \in \text{GkS}} \|h_{S}\|_1.$$ 

Denote $h - h_{\Lambda_0} = h_{\Lambda_0}$ by $h^*$. Define subsets $S_1$ and $S_2$ of $[g]$ as follows:

$$S_1 = \left\{ j \in [g] : \|h_{G,j}^*\|_1 > m_{\text{max}} \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}, \forall j \in [g] \right\},$$

$$S_2 = \left\{ j \in [g] : \|h_{G,j}^*\|_1 \leq m_{\text{max}} \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}, \forall j \in [g] \right\}.$$ 

Let $GS_1 = \cup_{j \in S_1} G_j$ and $GS_2 = \cup_{j \in S_2} G_j$. Now define

$$h^{(0)} = h_{\Lambda_0}, h^{(1)} = h_{GS_1}^*, h^{(2)} = h_{GS_2}^*.$$ 

Then we have

$$h_{\Lambda_0} = h^* = h_{GS_1}^* + h_{GS_2}^* = h^{(1)} + h^{(2)}.$$ 

Let $r = |S_1|$, and note that

$$r \leq \frac{k(t-1)}{m_{\text{max}}}.$$ 

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This is because, by the manner in which we defined the set $S_1$, it follows that

$$\|h_{\Lambda_0}\|_1 \geq \|h^{(1)}\|_1 > r\max \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}.$$  

Now we establish upper bounds on $\|h^{(2)}\|_1$ and $\|h^{(2)}_{G_j}\|_1$. Because of the definition of set $S_1$, it follows that

$$\|h^{(1)}\|_1 \geq r\max \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}. \quad (26)$$

Moreover, by the decomposability of $\| \cdot \|_1$,

$$\|h^{(2)}\|_1 = \|h_{\Lambda_0}\|_1 - \|h^{(1)}\|_1.$$

Using (26) we get

$$\|h^{(2)}\|_1 \leq \|h_{\Lambda_0}\|_1 - r\max \frac{\|h_{\Lambda_0}\|_1}{k(t-1)} = \left[k(t-1) \frac{m_{\max}}{m_{\max} - r}\right] \max \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}. \quad (27)$$

By the definition of set $S_2$

$$\|h^{(2)}_{G_j}\|_1 \leq m_{\max} \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}, \forall j \in [g]. \quad (28)$$

From (27) and (28), we see that the vector $h^{(2)}$ satisfies the hypotheses of Lemma 1 with

$$\alpha = m_{\max} \frac{\|h_{\Lambda_0}\|_1}{k(t-1)}, s = \left[k(t-1) \frac{m_{\max}}{m_{\max} - r}\right].$$

Therefore we can apply Lemma 1 to $h^{(2)}$. So $h^{(2)}$ can be represented as a convex combination

$$h^{(2)} = \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_N u_N,$$

where each $u_i$ is group $(k(t-1) - r\max)$-sparse, $h^{(1)}$ is group $(r\max)$-sparse, and $h^{(0)}$ is group $k$-sparse. Therefore, for each $i \in [N]$, the vector $u_i + h^{(1)} + h^{(0)}$ is group $tk$-sparse, because

$$k(t-1) - r\max + r\max + k = tk.$$

Now let, for all $i \in [N]$,

$$x_i = \frac{1}{2} \left(h^{(0)} + h^{(1)}\right) + \frac{\mu}{2} u_i,$$

$$z_i = \frac{1-2\mu}{2} \left(h^{(0)} + h^{(1)}\right) - \frac{\mu}{2} u_i,$$

$$\gamma = x_i + z_i = (1-\mu) \left(h^{(0)} + h^{(1)}\right),$$

$$\beta_i = x_i - z_i = \mu \left(h^{(0)} + h^{(1)} + u_i\right).$$

Then

$$\sum_{i=1}^N \lambda_i \langle A\gamma, A\beta_i \rangle = \left\langle A\gamma, A \sum_{i=1}^N \lambda_i \beta_i \right\rangle = \mu (1-\mu) \langle A(h^{(0)} + h^{(1)}) , Ah \rangle. \quad (29)$$
However, for each index set $i$, we have that

$$\langle A\gamma, A\beta_i \rangle = \langle Ax_i + Az_i, Ax_i - Az_i \rangle = \|Ax_i\|^2 - \|Az_i\|^2.$$ 

Therefore it follows that

$$\sum_{i=1}^{N} \lambda_i \left(\|Ax_i\|^2 - \|Az_i\|^2\right) = \mu(1 - \mu) \left\langle A(h^{(0)}) + h^{(1)}, Ah \right\rangle,$$

$$\sum_{i=1}^{N} \lambda_i \|Ax_i\|^2 = \sum_{i=1}^{N} \lambda_i \|Az_i\|^2$$

$$+ \mu(1 - \mu) \left\langle A(h^{(0)}) + h^{(1)}, Ah \right\rangle.$$

Since $x_i$, $z_i$, $(h^{(0)} + h^{(1)})$ are all group $tk$-sparse, it follows from the GRIP and Schwarz’ inequality that

$$(1 - \delta) \sum_{i=1}^{N} \lambda_i \|x_i\|^2 \leq (1 + \delta) \sum_{i=1}^{N} \lambda_i \|z_i\|^2$$

$$+ \mu(1 - \mu)\|A(h^{(0)}) + h^{(1)}\|_2 \cdot \|Ah\|_2.$$

Since $h^{(0)}$, $h^{(1)}$ and $u_i$ have disjoint support sets, it follows that, for all $i \in [N]$, we have

$$\|x_i\|^2 = 0.25 \left(\|(h^{(0)} + h^{(1)})\|^2 + \mu^2 \|u_i\|^2\right),$$

$$\|z_i\|^2 = 0.25 \left[(1 - 2\mu)^2\|(h^{(0)} + h^{(1)})\|^2 + \mu^2 \|u_i\|^2\right].$$

Substituting these relationships, multiplying both sides by 4, and noting that $\sum_{i=1}^{N} \lambda_i = 1$, leads to

$$(1 - \delta) \cdot \left[\|(h^{(0)} + h^{(1)})\|^2 + \mu^2 \sum_{i=1}^{N} \lambda_i \|u_i\|^2\right]$$

$$\leq (1 + \delta) \left[(1 - 2\mu)^2\|(h^{(0)} + h^{(1)})\|^2 \right.$$  

$$\left. + \mu^2 \sum_{i=1}^{N} \lambda_i \|u_i\|^2\right]$$

$$+ 4\mu(1 - \mu)\|A(h^{(0)}) + h^{(1)}\|_2 \cdot \|Ah\|_2,$$

or upon rearranging,

$$\|(h^{(0)} + h^{(1)})\|^2 \cdot \left[(1 - \delta) - (1 + \delta)(1 - 2\mu)^2\right]$$

$$\leq 2\delta \mu^2 \sum_{i=1}^{N} \lambda_i \|u_i\|^2$$

$$+ 4\mu(1 - \mu)\|A(h^{(0)} + h^{(1)})\|_2 \cdot \|Ah\|_2.$$
Recall that
\[ \alpha = m_{\text{max}} \frac{\|h_{\Lambda_0}^i\|_1}{k(t-1)}, \quad s = \left[ \frac{k(t-1)}{m_{\text{max}}} - r \right], \]
Substituting these values into (18), we get that
\[ \|u_i\|_2^2 \leq \frac{k(t-1) - rm_{\text{max}}}{m_{\text{min}}} m_{\text{max}}^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k^2(t-1)^2} \]
\[ \leq \frac{k(t-1)}{m_{\text{min}}} m_{\text{max}}^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k^2(t-1)^2} \]
\[ = \frac{m_{\text{max}}^2}{m_{\text{min}}} \frac{\|h_{\Lambda_0}^i\|_1^2}{k(t-1)} \]
Substituting this bound, which is independent of \( i \), into (32), we get
\[ \|(h^{(0)} + h^{(1)})\|_2 \leq 2\delta \mu^2 m_{\text{max}}^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k(t-1)} \]
\[ \leq 4\mu(1 - \mu) \sqrt{1 + \delta} \|(h^{(0)} + h^{(1)})\|_2 \cdot \|Ah\|_2. \]
Denote \( \|(h^{(0)} + h^{(1)})\|_2 \) by \( f \) and invoke the definition of the constants \( a, b, c \) from (22) and (23). This gives
\[ 4f^2 a^2 \leq 4c^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k} + 4bf \|Ah\|_2, \]
or after dividing both the sides by 4 and rearranging,
\[ f^2 a^2 - bf \|Ah\|_2 \leq c^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k}. \]
The next step is to complete the square on left side of the above inequality.
\[ f^2 a^2 - bf \|Ah\|_2 + \frac{b^2}{4a^2} \|Ah\|_2^2 \leq \frac{b^2}{4a^2} \|Ah\|_2^2 + c^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k}, \]
or equivalently,
\[ \left[ af - \frac{b}{2a} \|Ah\|_2 \right]^2 \leq \frac{b^2}{4a^2} \|Ah\|_2^2 + c^2 \frac{\|h_{\Lambda_0}^i\|_1^2}{k}. \]
Taking the square root on both sides, and using the obvious inequality that \( \sqrt{x^2 + y^2} \leq x + y \) whenever \( x, y \geq 0 \), leads to
\[ af - (b/2a)\|Ah\|_2 \leq (b/2a)\|Ah\|_2 + c \frac{\|h_{\Lambda_0}^i\|_1}{\sqrt{k}}, \]
or upon rearranging and replacing \( f \) by \( \|(h^{(0)} + h^{(1)})\|_2 \),
\[ a\|(h^{(0)} + h^{(1)})\|_2 \leq (b/a)\|Ah\|_2 + c \frac{\|h_{\Lambda_0}^i\|_1}{\sqrt{k}}. \]
Dividing both the sides by $a$ and observing that $h_{\Lambda_0} = h^{(0)}$ and
\[ \|h^{(0)}\|_2 \leq \|(h^{(0)} + h^{(1)})\|_2, \]
we get
\[ \|h_{\Lambda_0}\|_2 \leq \|(h^{(0)} + h^{(1)})\|_2 \leq \frac{b}{a^2} \|Ah\|_2 + \frac{c}{a} \|h_{\Lambda_0}\|_1 \sqrt{k}, \]
This inequality is of the form (19) with $\rho, \tau$ given as in (24). The only thing left to prove is that, if $\delta$ satisfies the bound (25), then $\rho < 1$. This is equivalent to $c/a < 1$, or $c^2 < a^2$. This part is routine algebra and is omitted. □

3.3 Bounds on the Residual Error

In this subsection we present explicit bounds on the residual error when LASSO is used to reconstruct the unknown vector from a possibly noisy measurement. Thus the measurement vector $y$ equals $Ax + \eta$ where $\|\eta\|_2 \leq \epsilon$ and the recovery procedure is to define
\[ \hat{x} = \arg \min_z \|z\|_1 \text{ s.t. } \|Az - y\|_2 \leq \epsilon. \] (32)

**Theorem 4** With the estimate $\hat{x}$ defined as in (11), let $h = \hat{x} - x$ denote the residual error vector. Then
\[ \|h\|_1 \leq \frac{2}{1 - \rho} \left[ (1 + \rho) \sigma_{k,G} + 2\tau \epsilon \right], \] (33)
\[ \|h\|_p \leq \frac{2}{1 - \rho} \left\{ \left[ \frac{\rho}{k^{1/p}} + (1 + \rho) \right] \sigma_{k,G} + \left( \frac{1}{k^{1/p}} + 2 \right) \tau \epsilon \right\}, \] (34)
where both $\rho$ and $\tau$ are defined in (24).

**Proof:** Let $x_{S_0}, x_{S_1}, \ldots, x_{S_b}$ be an optimal group $k$-sparse decomposition of $x$. Then
\[ \|x_{S_0} + h_{S_0}\|_1 + \|x_{S_0} + h_{S_0}\|_1 \leq \|x_{S_0}\|_1 + \|x_{S_0}\|_1. \]
Applying triangle inequality twice to the left hand side of the above inequality, we get
\[ \|x_{S_0}\|_1 - \|h_{S_0}\|_1 - \|x_{S_0}\|_1 + \|h_{S_0}\|_1 \leq \|x_{S_0}\|_1 + \|x_{S_0}\|_1. \]
Cancelling the common term $nmn x_{S_0} e_1$ and denoting $\|x_{S_0}\|$ by $\sigma_{k,G}(x, \| \cdot \|_1) = \sigma_{k,G}$, we get
\[ \|h_{S_0}\|_1 - \|h_{S_0}\|_1 \leq 2 \sigma_{k,G} \] (35)
Now let $h_{\Lambda_0}, h_{\Lambda_1}, \ldots, h_{\Lambda_s}$ be an optimal group $k$-sparse decomposition of $h$. Then
\[ \|h_{\Lambda_0}\|_1 \geq \|h_{\Lambda_0}\|_1, \text{ and } \|h_{\Lambda_0}\|_1 \leq \|h_{S_0}\|_1. \]
Using the above facts in (35), we get
\[ \|h_{\Lambda_0}\|_1 - \|h_{\Lambda_0}\|_1 \leq 2 \sigma_{k,G}. \] (36)
Next, because both $x$ and $\hat{x}$ are feasible for the optimization problem in (11), we get

$$\|Ah\|_2 = \|(A\hat{x} - y) - (Ax - y)\|_2 \leq 2\epsilon.$$ 

Using the inequality (20) and the above fact, we have that

$$\|h_{\Lambda_0}\|_1 \leq \rho\|h_{\Lambda_0^c}\|_1 + 2\tau\epsilon. \quad (37)$$

Now the two inequalities (36) and (37) can be neatly expressed in the form

$$\begin{bmatrix} 1 & -1 \\ -\rho & 1 \end{bmatrix} \begin{bmatrix} \|h_{\Lambda_0^c}\|_1 \\ \|h_{\Lambda_0}\|_1 \end{bmatrix} \leq \begin{bmatrix} 2\sigma_{k,G} \\ 2\tau\epsilon \end{bmatrix}. \quad (38)$$

Let the $M$ denote the coefficient matrix on the left hand side. Then, because $\rho < 1$, it follows that all elements of

$$M^{-1} = \frac{1}{1 - \rho} \begin{bmatrix} 1 & 1 \\ \rho & 1 \end{bmatrix}$$

are positive. Therefore we can multiply both the sides of (38) by $M^{-1}$, which gives

$$\begin{bmatrix} \|h_{\Lambda_0^c}\|_1 \\ \|h_{\Lambda_0}\|_1 \end{bmatrix} \leq \frac{1}{1 - \rho} \begin{bmatrix} 1 & 1 \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 2\sigma_{k,G} \\ 2\tau\epsilon \end{bmatrix} = \frac{1}{1 - \rho} \begin{bmatrix} 2(\sigma_{k,G} + \tau\epsilon) \\ 2(\rho\sigma_{k,G} + \tau\epsilon) \end{bmatrix}. \quad (39)$$

Finally using the triangle inequality, we get

$$\|h\|_1 \leq \|h_{\Lambda_0^c}\|_1 + \|h_{\Lambda_0}\|_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \|h_{\Lambda_0^c}\|_1 \\ \|h_{\Lambda_0}\|_1 \end{bmatrix} \leq \frac{2(1 + \rho)\sigma_{k,G} + 2\tau\epsilon}{1 - \rho}.$$ 

This is the same as (33).

Next we derive bounds on $\|h\|_p$ for $p \in [1,2]$. From the triangle inequality,

$$\|h\|_p \leq \|h_{\Lambda_0}\|_p + \|h_{\Lambda_0^c}\|_p. \quad (40)$$

Now we will obtain the upper bound for both of the terms in right hand side of (40). It is easy to show that

$$\|h_{\Lambda_0^c}\|_p \leq \|h_{\Lambda_0^c}\|_1.$$ 

Using the above fact and (39), we get

$$\|h_{\Lambda_0^c}\|_p \leq \frac{2}{1 - \rho}[(1 + \rho)\sigma_{k,G} + 2\tau\epsilon]. \quad (41)$$

It is a ready consequence of Hölder’s inequality that

$$\|h_{\Lambda_0}\|_p \leq k^{1/p - 1/2}\|h_{\Lambda_0}\|_2.$$ 

Using the above fact and the $\ell_2$-GRNS property, we get
\[
\|h_{A_0}\|_p \leq k^{1/p-1} (\rho \|h_{A_0}\|_1 + 2\tau \epsilon)
\]
(42)

Now we have from (39) that
\[
\|h_{A_0}\|_1 \leq \frac{2}{1-\rho} (\sigma_{k,G} + \tau \epsilon)
\]
Substituting from the above into (42) gives
\[
\|h_{A_0}\|_p \leq \frac{1}{k^{1-1/p}} \left[ \frac{2\rho}{1-\rho} (\sigma_{k,G} + \tau \epsilon) + \frac{2(1-\rho)\tau \epsilon}{1-\rho} \right],
\]
or after rearranging
\[
\|h_{A_0}\|_p \leq \frac{1}{k^{1-1/p}} \left[ \frac{2}{1-\rho} \sigma_{k,G} + \tau \epsilon \right].
\]
(43)

Adding (41) and (43) gives the final estimate, namely
\[
\|h\|_p \leq \frac{2}{1-\rho} \left\{ \left[ \frac{\rho}{k^{1-1/p}} + (1+\rho) \right] \sigma_{k,G} + \left( \frac{1}{k^{1-1/p}} + 2 \right) \tau \epsilon \right\},
\]
which is the same as (34).

\[\square\]

4 New Results for Conventional Sparsity

5 Conclusions

In this paper we have presented sufficient conditions for $\ell_1$-norm minimization to achieve robust group sparse recovery. When specialized to conventional sparsity, these conditions reduce to the known “best possible” bounds proved earlier. We have also derived bounds for the $\ell_p$-norm of the residual error between the true vector $x$ and its approximation $\hat{x}$, for all $p \in [1, 2]$. These bounds are new even for conventional sparsity and of course also for group sparsity.

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