THE GEOMETRY OF TWISTED CONJUGACY CLASSES IN WREATH PRODUCTS

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To Robert Zimmer on the occasion of his 60th birthday.

ABSTRACT. We give a geometric proof based on recent work of Eskin, Fisher and Whyte that the lamplighter group $\mathbb{Z}_n$ has infinitely many twisted conjugacy classes for any automorphism $\varphi$ only when $n$ is divisible by 2 or 3, originally proved by Gonçalves and Wong. We determine when the wreath product $G \wr \mathbb{Z}$ has this same property for several classes of finite groups $G$, including symmetric groups and some nilpotent groups.

1. Introduction

We say that a group $G$ has property $R_\infty$ if any automorphism $\varphi$ of $G$ has an infinite number of $\varphi$-twisted conjugacy classes. Two elements $g_1, g_2 \in G$ are $\varphi$-twisted conjugate if there is an $h \in G$ so that $hg_1\varphi(h)^{-1} = g_2$. The study of this property is motivated by topological fixed point theory, and is discussed below. Groups with property $R_\infty$ include

(1) Baumslag-Solitar groups $BS(m,n) = \langle a, b | ba^m b^{-1} = a^n \rangle$ except for $BS(1,1)$. [FG2]
(2) Generalized Baumslag-Solitar (GBS) groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic as well as any group quasi-isometric to a GBS group. [L, TWo2]
(3) Lamplighter groups $\mathbb{Z}_n \wr \mathbb{Z}$ iff $2|n$ or $3|n$. [GW1]
(4) Some groups of the form $\mathbb{Z}^2 \times \mathbb{Z}$. [GW2]
(5) Non-elementary Gromov hyperbolic groups. [LL, F]
(6) The solvable generalization $\Gamma$ of $BS(1,n)$ which is defined by the short exact sequence $1 \rightarrow \mathbb{Z}[\frac{1}{n}] \rightarrow \Gamma \rightarrow \mathbb{Z}^k \rightarrow 1$ as well as any group quasi-isometric to $\Gamma$. [TWo]
(7) The mapping class group. [FG1]
(8) Relatively hyperbolic groups. [FG1]
(9) Any group with a characteristic and non-elementary action on a Gromov hyperbolic space. [TWh, FG1]

These results fall into two categories: those which show the property holds using a group presentation, and those which show that property $R_\infty$ is geometric in some way, whether invariant under quasi-isometry, or dependent on an action of a group on a particular space. The final statement on the list is the most general, relying only on the existence of a certain group action. This was proven independently in [TWh] and in the Appendix of [FG1] using similar methods, and provides a wealth of examples of groups with this property.

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The study of the finiteness of the number of twisted conjugacy classes arises in Nielsen fixed point theory. Given a selfmap \( f : X \to X \) of a compact connected manifold \( X \), the nonvanishing of the classical Lefschetz number \( L(f) \) guarantees the existence of fixed points of \( f \). Unfortunately, \( L(f) \) yields no information about the size of the set of fixed points of \( f \). However, the Nielsen number \( N(f) \), a more subtle homotopy invariant, provides a lower bound on the size of this set. For \( \dim X \geq 3 \), a classical theorem of Wecken [We] asserts that \( N(f) \) is a sharp lower bound on the size of this set, that is, \( N(f) \) is the minimal number of fixed points among all maps homotopic to \( f \). Thus the computation of \( N(f) \) is a central issue in fixed point theory.

Given an endomorphism \( \varphi : \pi \to \pi \) of a group \( \pi \), the \( \varphi \)-twisted conjugacy classes are the orbits of the (left) action of \( \pi \) on \( \pi \) via \( \sigma \cdot \alpha \mapsto \sigma \alpha \varphi(\sigma)^{-1} \). Given a selfmap \( f : X \to X \) of a compact connected polyhedron, the fixed point set \( \text{Fix}_f = \{ x \in X | f(x) = x \} \) is partitioned into fixed point classes, which are identical to the \( \varphi \)-twisted conjugacy classes where \( \varphi = f^k \) is the homomorphism induced by \( f \) on the fundamental group \( \pi_1(X) \). The Reidemeiser number \( R(f) \) is the number of \( \varphi \)-twisted conjugacy classes, and is an upper bound for \( N(f) \). When \( R(f) \) is finite, this provides additional information about the cardinality of the set of fixed points of \( f \).

For a certain class of spaces, called Jiang spaces, the vanishing of the Lefshetz number implies that \( N(f) = 0 \) as well, and a nonzero Lefshetz number, combined with a finite Reidemeister number, implies that \( N(f) = R(f) \). As the Reidemeister number is much easier to calculate than the Lefshetz number, this provides a valuable tool for computing the cardinality of the set of fixed points of the map \( f \). Jiang’s results have been extended to selfmaps of simply connected spaces, generalized lens spaces, topological groups, orientable coset spaces of compact connected Lie groups, nilmanifolds, certain nilpotent spaces where \( C \) denotes the class of finite groups, certain solvmanifolds and infra-homogeneous spaces (see, e.g., [Wo2], [Wo3]). Groups \( G \) which satisfy property \( R_{\infty} \), that is, every automorphism \( \varphi \) has \( R(\varphi) = \infty \), will never be the fundamental group of a manifold which satisfies the conditions above.

In this paper, we study the (in)finiteness of the number of \( \varphi \)-twisted conjugacy classes for automorphisms \( \varphi : G \wr \mathbb{Z} \to G \wr \mathbb{Z} \) where \( G \wr \mathbb{Z} \) is the restricted wreath product of a finite group \( G \) with \( \mathbb{Z} \). When \( G = \mathbb{Z}/n \), so that \( G \wr \mathbb{Z} \) is a “lamplighter” group, we prove that \( G \wr \mathbb{Z} \) has property \( R_{\infty} \) iff 2 or 3 divides \( n \) using the geometry of the Cayley graph of these groups. Our proofs rely on recent work of A. Eskin, D. Fisher and K. Whyte identifying all quasi-isometries of \( L_n \), and classifying these groups up to quasi-isometry. [EFW] Our theorem was originally proven in [GW] using algebraic methods. The geometric techniques used below extend to certain other groups of this form, but combining geometric and algebraic methods we determine several larger classes of finite groups \( G \) for which \( G \wr \mathbb{Z} \) has property \( R_{\infty} \). In particular, \( S_n \wr \mathbb{Z} \) has this property, for \( n \geq 5 \), yielding the corollary that every group of the form \( G \wr \mathbb{Z} \) can be embedded into a group which has property \( R_{\infty} \).

We note that property \( R_{\infty} \) is not geometric in the sense that it is preserved under quasi-isometry. Namely, \( \mathbb{Z}^d \wr \mathbb{Z} \) is quasi-isometric to \( (\mathbb{Z}/2) \wr \mathbb{Z} \) by [EFW] but according to [GW] the former has property \( R_{\infty} \) while the latter does not. It also fails to be geometric when one considers cocompact lattices in \( \text{Sol} \). Let \( A, B \in GL(2, \mathbb{Z}) \) be matrices whose traces have absolute value greater than two. We know that \( \mathbb{Z}^2 \rtimes_A \mathbb{Z} \) and \( \mathbb{Z}^2 \rtimes_B \mathbb{Z} \) are always quasi-isometric, as they are both cocompact lattices in \( \text{Sol} \), but they may not both have property \( R_{\infty} \). However, there are classes of groups for which this property is invariant under quasi-isometry: these include the solvable Baumslag-Solitar groups, their solvable generalization \( \Gamma \) given above and the generalized Baumslag-Solitar groups.
2. Background on twisted conjugacy

2.1. Twisted conjugacy. We say that a group $G$ has property $R_\infty$ if, for any $\varphi \in \text{Aut}(G)$, we have $R(\varphi) = \infty$. The main technique we use for computing $R(\varphi)$ is as follows. We consider groups which can be expressed as group extensions, for example $1 \to A \to B \to C \to 1$. Suppose that an automorphism $\varphi \in \text{Aut}(B)$ induces the following commutative diagram, where the vertical arrows are group homomorphisms, that is, $\varphi|_A = \varphi'$ and $\overline{\varphi}$ is the quotient map induced by $\varphi$ on $C$:

$$
\begin{array}{ccc}
1 & \longrightarrow & A \\
\downarrow \varphi' & & \downarrow \varphi \\
1 & \longrightarrow & A
\end{array}
\quad
\begin{array}{ccc}
& i & \longrightarrow & B \\
& \downarrow & & \downarrow \overline{\varphi} \\
& p & \longrightarrow & C \\
\end{array}
\quad
\begin{array}{ccc}
1 & \longrightarrow & 1
\end{array}
$$

(1)

Then we obtain a short exact sequence of sets and corresponding functions $\hat{i}$ and $\hat{p}$:

$$
R(\varphi') \xrightarrow{i} R(\varphi) \xrightarrow{p} R(\overline{\varphi})
$$

(2)

where if $\overline{1}$ is the identity element in $C$, we have $\hat{i}(R(\varphi')) = \hat{p}^{-1}(\{1\})$, and $\hat{p}$ is onto. To ensure that both $\varphi'$ and $\overline{\varphi}$ are both automorphisms, we need the following lemma. Recall that a group is Hopfian if every epimorphism is an automorphism.

**Lemma 2.1.** If $C$ is Hopfian, then $\varphi' \in \text{Aut}(A)$ and $\overline{\varphi} \in \text{Aut}(C)$.

**Proof.** Since $\varphi$ is an automorphism, the commutativity of $\{1\}$ implies that $\varphi'$ is injective and $\overline{\varphi}$ is surjective. Since $C$ is Hopfian, $\overline{\varphi} \in \text{Aut}(C)$. Suppose $a \in A - \varphi'(A)$. It follows that $\varphi^{-1}(i(a))$ is an element of $B - i(A)$ so $p(\varphi^{-1}(i(a)))$ is non-trivial in $C$. This means that

$$
\overline{\varphi}(p(\varphi^{-1}(i(a)))) = p\varphi(\varphi^{-1}(i(a))) = p(i(a))
$$

is non-trivial, a contradiction. Thus, such an element $a$ cannot exist and so $\varphi'$ is also surjective. \qed

The following result is straightforward and follows from more general results discussed in [Wo].

**Lemma 2.2.** Given the commutative diagram labeled $\{1\}$ above,

1. if $R(\overline{\varphi}) = \infty$ then $R(\varphi) = \infty$,
2. if $C$ is finite or $\text{Fix}(\overline{\varphi}) = 1$, and $R(\varphi') = \infty$ then $R(\varphi) = \infty$.

When $C \cong \mathbb{Z}$, we have one of two situations:

3. the map $\overline{\varphi}$ is the identity, in which case $R(\overline{\varphi}) = \infty$ and hence $R(\varphi) = \infty$, or
4. $\overline{\varphi}(t) = t^{-1}$ and $R(\overline{\varphi}) = 2$. In this case, $R(\varphi) < \infty$ iff $R(\varphi') < \infty$ and $R(t \cdot \varphi') < \infty$, in which case $R(\varphi) = R(\varphi') + R(t \cdot \varphi')$.

To show that a group $G$ does not have property $R_\infty$, it suffices to produce a single example of $\varphi \in \text{Aut}(G)$ with $R(\varphi) < \infty$. When $G$ additionally has a semidirect product structure, such an automorphism is often constructed as follows. Write $G = A \rtimes B$ as $1 \to A \to G \to B \to 1$, with $\Theta : B \to \text{Aut}(A)$. We can use automorphisms $\varphi' : A \to A$ and $\overline{\varphi} : B \to B$ to construct the following commutative diagram defining $\varphi \in \text{Aut}(G)$, provided that

$$
\varphi'(\Theta(b)(a)) = \Theta(\overline{\varphi}(b))(\varphi'(a))
$$
for $b \in B$ and $a \in A$:

$$1 \rightarrow A \xrightarrow{i} G \xrightarrow{p} B \rightarrow 1$$

Through careful selection of the maps $\varphi'$ and $\overline{\varphi}$, we can sometimes create an example of an automorphism of $G$ with finite Reidemeister number using the above diagram and Lemma 2.2. We should point however that the map $\varphi$ so constructed using $\varphi'$ and $\overline{\varphi}$ is not unique.

3. Lamplighter groups and Diestel-Leader graphs

We begin with a discussion of the geometry of the lamplighter groups $L_n = \mathbb{Z}_n \wr \mathbb{Z}$ and later address more general groups $G \wr \mathbb{Z}$ where $G$ is any finite group.

The standard presentation of $L_n$ is $(a, t | a^n = 1, [a^i, a^j] = 1)$ where $x^y$ denotes the conjugate $yxy^{-1}$. Equivalently, we recall that a wreath product is simply a certain semidirect product, and write $L_n$ as $(\bigoplus_{i=-\infty}^{\infty} A_i) \rtimes \mathbb{Z}$, with each $A_i \cong \mathbb{Z}_n$, fitting into the split short exact sequence

$$0 \rightarrow \bigoplus_{i=-\infty}^{\infty} A_i \rightarrow L_n \rightarrow \mathbb{Z} \rightarrow 0$$

where the generator of $\mathbb{Z}$ is taken to be $t$ from the presentation given above.

The “lamplighter” picture of elements of this group is the following. Take a bi-infinite string of light bulbs placed at integer points along the real line, each of which has the integer part of $a$ single bulb in position $j$ with $i \in \mathbb{Z}$ corresponding to bulbs indexed by negative and non-negative integers: Understanding group elements via this picture, the generator $a$ bulbs to some allowable states, leaving the lamplighter at a fixed integer.

$L_e$ element of position to the right, and $a$ changes the state of the current bulb under consideration. Thus each element of $L_n$ can be interpreted as a series of instructions to illuminate a finite collection of light bulbs to some allowable states, leaving the lamplighter at a fixed integer.

Understanding group elements via this picture, the generator $a_j = t^j a t^{-j}$ of $A_j$ in $\bigoplus_{i=-\infty}^{\infty} A_i$ has a single bulb in position $j$ illuminated to state $a$, and the cursor at the origin of $\mathbb{Z}$. This leads to two possible normal forms for $g \in L_n$, as described in [CT], separating the word into segments corresponding to bulbs indexed by negative and non-negative integers:

$$rf(g) = a_{i_1}^{e_1} a_{i_2}^{e_2} \cdots a_{i_k}^{e_k} a_{-j_1}^{f_1} a_{-j_2}^{f_2} \cdots a_{-j_l}^{f_l} t^m$$

or

$$lf(g) = a_{-j_1}^{f_1} a_{-j_2}^{f_2} \cdots a_{-j_l}^{f_l} a_{i_1}^{e_1} a_{i_2}^{e_2} \cdots a_{i_k}^{e_k} t^m$$

with $i_k > \ldots i_2 > i_1 \geq 0$ and $j_1 > \ldots j_2 > j_1 > 0$ and $e_i$, $f_j$ in the range $\{-h, \ldots, h\}$, where $h$ is the integer part of $\frac{n}{2}$. When $n$ is even, we omit $a^{-h}$ to ensure uniqueness, since $a^h = a^{-h}$ in $\mathbb{Z}_{2h}$.

If we consider the normal form as a series of instructions for acting on the lamplighter picture to create these group elements, then the normal form $rf(g)$ first illuminates bulbs at or to the right of the origin, and $lf(g)$ first illuminates the bulbs to the left of the origin. It is proven in [CT] that the word length of $g \in L_n$ with respect to the finite generating set $\{a, t\}$, using the notation of either normal form given above, is

$$\sum e_i + \sum f_j + \min\{2j_l + i_k + |m - i_k|, 2i_k + j_l + |m + j_l|\}.$$
However, to use a Diestel-Leader graph as the Cayley graph of $L_n$, we must consider the generating set \{t, ta, ta^2, \ldots, ta^{n-1}\} for $L_n$.

There is another generating set for $L_2$ worth noting. R. Grigorchuk and A. Zuk in [GZ] show that this group is an example of an automata group. The natural generating set arising from this interpretation is \{a, ta\}. Grigorchuk and Zuk compute the spectral radius with respect to this generating set and find that it is a discrete measure.

3.1. Diestel-Leader graphs. We now describe explicitly the Cayley graph for the lamplighter group $L_m$ with respect to one particular generating set. This Cayley graph is an example of a Diestel-Leader graph, which we define in full generality as follows.

For positive integers $m \leq n$, the Diestel-Leader graph $DL(m, n)$ is a subset of the product of the regular trees of valence $m + 1$ and $n + 1$, which we denote respectively as $T_1$ and $T_2$. We orient these trees so that each vertex has $m$ (resp. $n$) outgoing edges. By fixing a basepoint, we can use this orientation to define a height function $h_i : T_i \rightarrow \mathbb{Z}$. The Diestel-Leader graph $DL(m, n)$ is defined to be the subset of $T_1 \times T_2$ for which $h_1 + h_2 = 0$. This definition lends itself to the following pictorial representation. Each tree has a distinguished point at infinity which determines the height function. If we place these points for the two trees at opposite ends of a page, the Diestel-Leader graph can be seen as those pairs of points, one from each tree, lying on the same horizontal line. See Figure 1 for a portion of the graph $DL(3, 3)$.

![Figure 1](image)

**Figure 1.** Part of the Cayley graph $DL(3, 3)$ of $L_3$ with the point $Id = (x_0, y_0)$ labeled. The integers on the edges of the diagram represent the height in each tree. The point $(x_0, y_0)$ represents the identity element of the group.

When $n = m$, we will see that $DL(m, m)$ is the Cayley graph of the lamplighter group $L_m$ with respect to a particular generating set. This fact was first noted by Moeller and Neumann [MN]. Both Woess [Woe] and Wortman [Wor] describe slightly different methods of understanding this model of the lamplighter groups; our explanation is concrete in a different way than either of theirs.

The group $L_m$ is often presented by $L_m \cong \langle a, t | a^n = 1, [a^t, a^u] = 1 \rangle$. However, the Cayley graph resulting from this presentation is rather untractable. When we take the generating set \{t, ta, ta^2, \ldots, ta^{m-1}\} we obtain the Cayley graph $DL(m, m)$ for $L_m$. Thus a group element $g \in L_m$ corresponds to a pair of points $(c, B)$ where $c \in T_1$ and $B \in T_2$ so that $h_1(c) + h_2(B) = 0$. Our notational convention will be to use lower case letters for vertices in $T_1$ and capital letters for vertices in $T_2$, with the exception of the identity, which we always denote $(x_0, y_0)$.

The action of the group $L_m$ on $DL(m, m)$ is as follows:

1. $t$ translates up in height in $T_1$ and down in height in $T_2$, so that the condition $h_1 + h_2 = 0$ is preserved, and
2. if $(v, W)$ is a point in $DL(m, m)$, let $v_0, v_1, \ldots, v_{m-1}$ be the vertices in $T_1$ adjacent to $v$ with $h_1(v_i) = h_1(v) + 1$. Then $a$ performs a cyclic rotation among these vertices, so that
\begin{align*}
a \cdot (v_i, W) &= (v_{(i+1) \pmod{m}}, W). \end{align*}

Conjugates of \(a\) by powers of \(t\) perform analogous rotations on the vertices with a common parent in \(T_2\) while fixing the coordinate in \(T_1\).

We now describe how to identify group elements with vertices of \(DL(m, m)\). Express \(g \in L_m\) using the generating set \(\{t, ta, ta^2, \ldots, ta^{m-1}\}\) and inverses of these elements. For example, let \(m = 3\) and take \(a = t^{-1}(ta)\). We describe a path in \(DL(m, m)\) from the identity to \(a\), using the expression \(a = t^{-1}(ta)\) and the action of the group on the graph. We introduce coordinates on the trees as needed, but always designate the identity as \((x_0, y_0)\). From a given vertex \((x, Y) \in DL(m, m)\), there are edges emanating from this vertex labeled by all these generators and their inverses. The edge labeled by a generator of the form \(ta^i\) for \(i = 0, 1, \ldots, m - 1\) leads to a point \((x', Y')\) with \(h_1(x') = h_1(x) + 1\) and \(h_2(Y') = h_2(Y) - 1\), and the edge labeled by a generator of the form \((ta^i)^{-1}\) leads to a point \((x', Y')\) with \(h_1(x') = h_1(x) - 1\) and \(h_2(Y') = h_2(Y) + 1\).

Notice that when leaving \((x, Y)\) via an edge labeled \((ta^i)^{-1}\), there is only one possible vertex in \(T_1\) adjacent to the vertex \(x\) at lower height. When tracing out a path from the identity to a particular element, we must remember which path was taken when it seems like several generators correspond to traversing the same edge in one of the two trees. This will be illustrated in an example below.

We begin by finding the group element \(a = t^{-1}(ta)\) as a vertex in \(DL(3, 3)\). We read the generators from left to right as a series of instructions for forming a path in \(DL(3, 3)\) from the identity to \(a\). Using the group action on \(DL(3, 3)\) described above, the generator \(t\) always increases the height function in \(T_1\) while decreasing it in \(T_2\), and the action of \(a\) (resp. \(a^{-1}\)) is to cyclically rotate the edge we traverse in \(T_1\) (resp. \(T_2\)).

For example, in \(DL(3, 3)\) we trace a path to \(a = t^{-1}(ta)\) in each tree as follows. The edge emanating from \(Id = (x_0, y_0)\) labeled \(t^{-1}\) must decrease the height in \(T_1\) and increase it in \(T_2\). This determines a vertex \(x'\) in \(T_1\) and \(Y'\) in \(T_2\). The edge emanating from \((x', Y')\) labeled by \(ta\) must increase the height function in \(T_1\) and decrease it in \(T_2\). There is a unique vertex in \(T_2\) satisfying the conditions necessary for a coordinate in the terminus of this edge, namely \(y_0\). In \(T_1\), since the initial edge was labeled \(t^{-1}\), we do not traverse up that edge, but rather the cyclically adjacent edge, ending at a point whose coordinates we label \((x_1, y_0)\). Using the coordinates in Figure 2 below, we see that \(a^2 = t^{-1}(ta^2) = (x_2, y_0)\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{cayley_graph.png}
\caption{The path in the Cayley graph \(DL(3, 3)\) of \(L_3\) from \(Id = (x_0, y_0)\) to the group element \(a = t^{-1}(ta)\).}
\end{figure}

For a more involved example, consider the group elements \(t^{-1}(ta^2)^{-1}t(ta)\) and \((ta)^{-1}(ta^2)^{-1}t(ta)\). In Figure 3 we trace out the paths in \(T_1\) leading from \(x_0\) to the first coordinate of these two elements in \(DL(3, 3)\). This example illustrates the cyclic labeling of edges determined by the downward path of edges and the action of \(a\) on the graph. It is not hard to see that the \(T_2\) coordinate of both of these group elements is \(y_0\).
The $T_1$ coordinates of the paths in $DL(3,3)$ leading from the identity $(x_0, y_0)$ to the group elements $t^{-1}(ta^2)^{-1}t(ta)$ and $(ta)^{-1}(ta^2)^{-1}t(ta)$, respectively. The $T_2$ coordinates of the final point in each path is $y_0$.

Writing $L_m$ as a group extension:

$$0 \rightarrow \bigoplus_{i=-\infty}^{\infty} A_i \rightarrow L_m \rightarrow \mathbb{Z} \rightarrow 0 \quad (5)$$

where $A_i \cong \mathbb{Z}_m$, we see that the map onto $\mathbb{Z}$ is determined by the exponent sum on all instances of the generator $t$ in any word representing $g \in L_m$, using the generating set $\{t, a\}$. Thus the kernel of this map is simply those group elements in which there are equal numbers of $t$ and $t^{-1}$. It is easy to see that when we change to the generating set $\{t, ta, \ldots, ta^{m-1}\}$ these elements can be characterized in the same manner. Thus in $DL(m,m)$ the points corresponding to the group elements in the kernel of this map again have coordinates with height 0 in both trees.

To understand this picture more thoroughly, we identify in $DL(m,m)$ the points corresponding to the different factors of $A_i \cong \mathbb{Z}_m$. First suppose that $i \leq 0$. The points in $A_i$ all have the form $t^{-i}a^kt^i$ with respect to the $\{t, a\}$ generating set for $L_m$, for $0 \leq k \leq m-1$, and $t^{-(i+1)}(ta^k)t^i$ with respect to the $\{t, ta, ta^2, \ldots, ta^{m-1}\}$ generating set. Thus they are easy to find in $DL(m,m)$: the coordinate in $T_2$ will be $y_0$, and in $T_1$, follow the unique path which takes $i + 1$ edges decreasing in height, cyclically rotate over $k$ edges, proceed up in height one edge, then regardless of $k$, proceed up the identical path in each subtree labeled by $t$ at each step until you reach height zero. The factors of $A_i$ for $i > 0$ are reversed in that points in such a factor all have the first coordinate in $T_1$ equal to $x_0$ and the second coordinate is found in $T_2$ in an analogous manner.

Using the lamplighter picture to understand the group, a natural subset to identify is the collection of elements with bulbs illuminated (to any state) within a given set of positions, say between $-i$ and zero, for some $i \geq 0$. The situation we describe below is completely analogous for $i < 0$, with
the two trees interchanged. Let $l$ be the line in $DL(m, m)$ which is the orbit of the identity under the group generator $t$. Viewing $T_1$ and $T_2$ simply as trees, and not as part of $DL(m, m)$, the line $l$ in $DL(m, m)$ determines a line $l_i \subset T_i$ for each $i$, for $i = 1, 2$.

When $-i \leq 0$, to determine this set of points, we recall that in the generating set corresponding to $DL(m, m)$, any instance of a generator $(ta^k)^{\pm 1}$ for $k$ strictly greater than zero will illuminate a bulb in the lamplighter picture for the group. The position of this bulb is determined by the exponent sum on all instances of the generator $t$, whether as part of $(ta^k)$ or not, within the word up until the string $a^k$ is read. Thus all group elements of this form can be expressed as follows:

$$t^{-i(i+1)} \Pi_{j=1}^{i+1} \alpha_j$$

where $\alpha_i$ or $\alpha_i^{-1}$ lies in $\{t, ta, ta^2, \ldots, ta^{m-1}\}$. It is not hard to see that all of these group elements have $y_0$ for their $T_2$ coordinate. When $i > 0$, the points will share $x_0$ as their $T_1$ coordinate. Returning to $i \leq 0$, we identify the $T_1$ coordinates of these points as follows. Let $x_{i+1}$ be the point on $l_1$ at height $-(i + 1)$. Then the $3^{i+1}$ points which can be reached in $T_1$ by a path of length $i + 1$ which increases in height at each step are precisely all the points of the form given in (6). The points at height zero in the tree $T_1$ which are shown in Figure 1, when paired with $y_0 \in T_2$, give the complete collection of elements of $L_3$ with bulbs illuminated in all combinations of positions between $-2$ and zero, inclusive.

3.2. Results of A. Eskin, D. Fisher and K. Whyte. In our proofs below, we rely on results of A. Eskin, D. Fisher and K. Whyte concerning the classification of the Diestel-Leader graphs up to quasi-isometry. [EFW] These results are relevant since any automorphism of $\mathbb{Z}_n \wr \mathbb{Z}$ can be viewed as a quasi-isometry of the Cayley graph $DL(n, n)$.

Introduce the following coordinates on $DL(m, n)$: a point in $DL(m, n)$ can be uniquely identified by the triple $(x, y, z)$ where $x \in \mathbb{Q}_m$, $y \in \mathbb{Q}_n$ and $z \in \mathbb{Z}$. Here, $\mathbb{Q}_m$ denotes the $m$-adic rationals, which we view as the ends of the tree $T_1$ once the height function has been fixed. The $n$-adic rationals correspond in an analogous way to the ends of $T_2$. Restricting to $T_1$, any $x \in \mathbb{Q}_m$ determines a unique line in $T_1$ on which the height function is strictly increasing. This line has a unique point at height $z$. Similarly, the line in $T_2$ determined by $y \in \mathbb{Q}_n$ contains a unique point at height $-z$. This pair of points together forms a single point in $DL(m, n)$ that is uniquely identified in this way.

Following [EFW], we define a product map $\hat{\phi} : DL(m, n) \to DL(m', n')$ as follows. Namely, $\hat{\phi}$ is a product map if it is within a bounded distance of a map of the form $(x, y, z) \mapsto (f(x), g(y), q(z))$ or $(x, y, z) \mapsto (g(y), f(x), q(z))$ where $f : \mathbb{Q}_m \to \mathbb{Q}_{m'}$ (or $\mathbb{Q}_n$), $g : \mathbb{Q}_n \to \mathbb{Q}_{n'}$ (or $\mathbb{Q}_m$) and $q : \mathbb{R} \to \mathbb{R}$.

A product map is called standard if it is the composition an isometry and a product map in which $q$ is the identity, and $f$ and $g$ are bilipschitz.

They determine the form of any quasi-isometry between Diestel-Leader graphs in the following theorem, which is stated in [EFW] and proven there for $m \neq n$, and proven in the remaining case in [EFW2].

**Theorem 3.1** ([EFW], Theorem 2.3). *For any $m \leq n$, any $(K, C)$-quasi-isometry $\varphi$ from $DL(m, n)$ to $DL(m', n')$ is within bounded distance of a height respecting quasi-isometry $\varphi'$. Furthermore, the bound is uniform in $K$ and $C$.*

Below, we use the fact that any height-respecting quasi-isometry is at a bounded distance from a standard map. Eskin, Fisher and Whyte describe the group of standard maps of $DL(m, n)$ to itself; this group is isomorphic to $(Bilip(\mathbb{Q}_m) \times Bilip(\mathbb{Q}_n)) \rtimes \mathbb{Z}/2\mathbb{Z}$ when $m = n$ and $(Bilip(\mathbb{Q}_m) \times Bilip(\mathbb{Q}_n))$ otherwise. They deduce that this is the quasi-isometry group of $DL(m, n)$. The extra factor of $\mathbb{Z}_2$ in the case $m = n$ reflects the fact that the trees in this case may be interchanged.
This result has the following implications for our work below. The product map structure implies that a quasi-isometry takes a line in \( T_1 \) and maps it to within a uniformly bounded distance of a line in either \( T_1 \) or \( T_2 \), depending on whether the quasi-isometry interchanges the two tree factors or not. The same is true when we begin with a line in \( T_2 \). This is a geometric fact not dependent on the group structure, and we use it repeatedly below.

4. Counting twisted conjugacy classes in \( L_n = \mathbb{Z}_n \wr \mathbb{Z} \)

In this section, we present a geometric proof of the following theorem, first proven in \([GW1]\). We follow the outline of the lemmas in \([GW1]\), providing proofs based on the geometry of the group and the results of A. Eskin, D. Fisher and K. Whyte in \([EFW]\).

**Theorem 4.1** \([GW1], \text{Theorem 2.3}\). Let \( m \geq 2 \) be a positive integer with its prime decomposition

\[
m = 2^{e_1} 3^{e_2} \Pi_i p_i^{e_i}
\]

where each \( p_i \) is a prime greater than or equal to 5. Then there exists an automorphism \( \varphi \) of \( \mathbb{Z}_m \wr \mathbb{Z} \) with \( R(\varphi) < \infty \) if and only if \( e_1 = e_2 = 0 \).

Let \( \varphi \in \text{Aut}(L_n) \). We first must show that the infinite direct sum \( \bigoplus A_i \) where each \( A_i \cong \mathbb{Z}_n \) is characteristic in \( L_n \), regardless of whether \( (n,6) = 1 \), allowing us to obtain “vertical” maps on the short exact sequence defining \( L_n \) given in Equation \([5]\). Let \( DL(n,n) \) be the Cayley graph of \( L_n \) with respect to the generating set \( \{ t, ta, ta^2, \ldots, ta^{n-1} \} \). We label the identity using the coordinates \((x_0, y_0)\) and label other points as needed. In general, we use lower case letters for coordinates in the first tree \( T_1 \), and capital letters to denote coordinates in the second tree \( T_2 \).

**Lemma 4.2.** Let \( \varphi \in \text{Aut}(L_n) \). Then \( \varphi(\bigoplus A_i) \subset \bigoplus A_i \).

**Proof.** Let \( a = t^{-1}(ta) = (b, y_0) \). Since \( \varphi(Id) = \varphi(x_0, y_0) = (x_0, y_0) \), we have one of two situations: a quasi-isometry of \( DL(n,n) \) which interchanges the tree factors will take any line in \( T_1 \) through \( x_0 \) to within a uniformly bounded distance of a line in \( T_2 \) passing through \( y_0 \), and a quasi-isometry which does not interchange the tree factors will coarsely preserve the set of lines in \( T_1 \) through \( x_0 \).

Respectively, we have that \( \varphi(a) = \varphi(b, y_0) = (x_0, C) \) or \( \varphi(b, y_0) = (c, y_0) \). In either case it is clear that \( \varphi(a) \) must lie at height zero in \( DL(n,n) \), that is, \( \varphi(a) \in \bigoplus A_i \). If \( a_i \) is the generator of the \( i \)-th copy of \( A \) in \( \bigoplus A_i \), then \( a_i = t a_i t^{-1} \), and it is easy to see that \( a_i \) either has coordinates \((x_0, D)\) or \((d, x_0)\) in \( DL(n,n) \). Thus the same argument shows that \( \varphi(a_i) \in \bigoplus A_i \) and the lemma follows. \( \square \)

Lemma 4.2 guarantees the following commutative diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & \bigoplus A_i & \longrightarrow & L_n & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
\varphi' \downarrow & & \varphi \downarrow & & \varphi \downarrow & & \\
1 & \longrightarrow & \bigoplus A_i & \longrightarrow & L_n & \longrightarrow & \mathbb{Z} & \longrightarrow & 1
\end{array}
\]

(7)

In particular, if \( \varphi(t) = t \), it follows immediately from Lemma [2.2] that \( R(\varphi) = \infty \). In all that follows, we assume that \( \varphi(t) = -t \), which necessarily means that \( \varphi \) interchanges the tree factors in \( DL(n,n) \). Since the fixed point set of \( \varphi \) is trivial, if we can show that \( R(\varphi') = \infty \) it will follow from Lemma 2.2 that \( R(\varphi) = \infty \) as well.

When \( n \) is prime, we are able to determine the image of the generator \( a_i \) of the \( i \)-th copy of \( A_i \) in \( \bigoplus A_i \) under any automorphism \( \varphi \in \text{Aut}(L_n) \) using only the geometry of the group. The simplest
Lemma 4.3. Let \( G = \mathbb{Z}_p \wr \mathbb{Z} = (\bigoplus \mathbb{Z}_p) \times \mathbb{Z} \) where \( p \) is prime, and let \( \varphi \) be any automorphism of \( G \). Denote by \( a_i \) the generator of the \( i \)-th copy of \( \mathbb{Z}_p \) in \( \bigoplus (\mathbb{Z}_p) \). For each \( j \in \mathbb{Z}, \exists i \in \mathbb{Z} \) and \( k \in \{1, 2, \ldots, n-1\} \) so that \( \varphi^k(a_i) = a_j \).

Proof. Let \( \varphi'(a_0) = a_1^c_1 a_2^c_2 \cdots a_k^c_k a_{-j}^d_1 a_j^d_2 \cdots a_{-j}^d_j \). This expression yields a particular collection of bulbs that are illuminated in the lamplighter picture corresponding to this element, with \( a_{\min} \) and \( a_{\max} \), respectively, the left- and right-most of these illuminated bulbs. Then \( \varphi'(a_i) = \varphi'(t^i \cdot a_0) = \varphi(t^i) \cdot \varphi(a_0) \). Since \( \varphi(t^i) \) subtracts \( i \) from the indices of all terms in \( \varphi'(a_0) \), we see that \( \varphi'(a_i) \) is just the same series of illuminated bulbs as in \( \varphi'(a_0) \) shifted \( i \) units to the left.

Suppose that \( \varphi'(x) = a_j \), where \( x = a_1^{c_1} a_2^{c_2} \cdots a_k^{c_k} a_{-m_1}^{d_1} a_{-m_2}^{d_2} \cdots a_{-m_j}^{d_j} \). Since \( p \) is prime, no exponents in \( \varphi'(a_i^k) \) or \( \varphi'(a_i^{d_i}) \) become congruent to 0 mod \( p \), and thus each element of the form \( \varphi'(a_i^k) \) or \( \varphi'(a_i^{d_i}) \) has the same number of illuminated bulbs in its lamplighter picture as \( \varphi(a_0) \).

If we assume that \( a_{\min} \neq a_{\max} \), it is clear that \( \varphi'(x) \) has illuminated bulbs in positions \( \min - m_i \) and \( \max + n_k \) and thus cannot be equal to \( a_j \). We conclude that both \( a_{\min} = a_{\max} \) and \( x = a_i^k \) for some \( i \in \mathbb{Z} \).

Since \( p \) is prime, by taking powers of \( a_i^k \) it follows from Lemma 4.3 that \( \varphi'(a_i) = a_j^k \). Thus, if \( \varphi'(a_0) = a_j^k \), then \( \varphi'(a_i) = a_j^{k-i} \) and \( \varphi'(a_{j-i}) = a_i^k \) for the same value of \( k \), and we obtain by restriction a map \( \varphi' : A_i \bigoplus A_{j-i} \rightarrow A_i \bigoplus A_{j-i} \).

We now recall the definition of \( \varphi' \)-twisted conjugacy classes in \( \bigoplus A_i \). If \( g_1, g_2 \in \bigoplus A_i \) are \( \varphi' \)-twisted conjugate, then there exists \( h \in \bigoplus A_i \) so that \( hg_1 \varphi'(h)^{-1} = g_2 \). Since \( \bigoplus A_i \) is abelian, we rewrite this equation as \( h \varphi'(h)^{-1} = g_1^{-1} g_2 \). Switching to additive notation, we see that if \( Id - \varphi' : A_i \bigoplus A_{j-i} \rightarrow A_i \bigoplus A_{j-i} \) is invertible, then \( R(\varphi') < \infty \). In particular, if we can find a fixed point of \( \varphi' \), we guarantee that \( Id - \varphi' \) is not surjective and hence not invertible. In the proof of Theorem 4.1, we prove that \( \varphi' \) must have a fixed point if \( n \) is 2 or 3, and if \( (n, 2) = (n, 3) = 1 \) then it is possible to construct automorphisms \( \varphi' \) so that \( Id - \varphi' \) is surjective. This geometric method will allow us to prove that \( L_2 \) and \( L_3 \) have property \( R_\infty \), as well as prove that when \( (n, 6) = 1, L_n \) does not have this property. When \( (n, 6) \neq 1 \), it will follow from the following lemma that \( L_n \) does have property \( R_\infty \).

Lemma 4.4. If \( (n, 6) \neq 1 \) then there is a characteristic subgroup \( H_i \triangleleft L_n \) with quotient \( \mathbb{Z}_n \) where \( i | n \), for \( i = 2 \) or 3, and \( H_i \cong L_i \).

Proof. Suppose that \( 2 | n \). Let \( i = 2 \), and if \( \tau \) is the generator for \( \mathbb{Z}_n \) then the element \( \tau^n \) is the unique element of order 2 in \( \mathbb{Z}_n \). Consider the subgroup \( H_2 = \langle \tau^2 \rangle \triangleleft \mathbb{Z} \) of \( L_n \), where \( \mathbb{Z} = \langle t \rangle \). It is clear that \( H_2 \) is isomorphic to \( L_2 \) and that \( H_2 \) contains \( \bigoplus \langle \langle \tau^2 \rangle \rangle_j \), the subgroup of \( L_n \) consisting of elements of order 2. It is easy to see that every automorphism \( \varphi \in Aut(L_n) \) sends \( \bigoplus \langle \langle \tau^2 \rangle \rangle_j \) to itself, and the subgroup generated by \( (1, t) \) to itself, where \( 1 \in \bigoplus \langle \langle \tau^2 \rangle \rangle_j \) and \( t \in \mathbb{Z} \). Since \( H_2 \) is generated by \( \langle \tau^2, 1 \rangle \) and \( (1, t) \), it follows that \( H_2 \) is characteristic in \( L_n \). Now \( L_n/H_2 \cong \mathbb{Z}_{n/2} \). The case when \( 3 | n \) and we let \( i = 3 \) is similar.

One can also show that if \( (n, 6) \neq 1 \) then there is a characteristic subgroup \( K_i \) of \( L_n \) such that \( L_n/K_i \cong L_i \) for \( i = 2 \) or 3.
Proof of Theorem 4.1. Consider \( \varphi \in \text{Aut}(L_n) \), and suppose that \( \varphi'(a_0) = a_k^i \). Since \( \varphi \) is an automorphism, it is also a quasi-isometry. By [EFW] it is a uniformly bounded distance from a product map. Let \( K \) and \( C \) be the quasi-isometry constants of this product map. We note that since we are assuming that \( \varphi(t) = -t \), this product map interchanges the two tree factors, that is, it maps lines in the first tree factor in \( DL(n, n) \) to within a uniformly bounded distance \( D \) of lines in the second tree factor, where the constant \( D \) depends on \( K \) and \( C \).

We know that \( \varphi' : \bigoplus A_i \to \bigoplus A_i \), and our assumption that \( \varphi'(a_0) = a_k^i \) yields the restriction \( \varphi' : A_i \bigoplus A_{j-i} \to A_i \bigoplus A_j \). We call this subset of \( \bigoplus A_i \) a block. We now explain the effect on these blocks of the geometric fact that \( \varphi \) is a uniformly bounded distance from product map. Recall that the value of \( j \) is determined by the image of \( a_0 \), that is, \( \varphi'(a_0) = a_k^i \). We always consider \( i > j \), so that one factor in the block has a positive index, and the other a negative index. This ensures that coordinates can be introduced on the \( n^2 \) points in this block so that if \( a_i^{m_1} = (x_0, B) \) and \( a_j^{m_2} = (c, y_0) \), then \( a_i^{m_1} a_j^{m_2} \) will have coordinates \((c, B)\).

Since \( \varphi \) is within a uniformly bounded distance \( D \) of a product map, we additionally choose \( i > KD \), where \( K \) is the maximum of the bilipshitz constants of the maps on the tree factors. (Note that this lower bound is perhaps much larger than required.) This ensures that when we consider the coordinates in \( T_2 \) of the points \( a_k^i \) and in \( T_1 \) of the points \( a_k^j \) for \( 0 \leq k \leq n \), these coordinates are at least distance \( 2KD \) apart, using the standard metric on the tree which assigns each edge length one. This is easy to see by writing \( a_i^k = t^{(i-1)l} t^k t^{-i} \); when \( i < 0 \), the second coordinates of these points, as \( k \) varies, is always \( y_0 \). The first coordinates, pairwise, are distance \( 2i > 2KD \) apart in \( T_1 \). More importantly, if \( l \) is any line in \( T_2 \) through the second coordinate of \( a_k^i \) on which the height function is strictly decreasing, this product map takes \( l \) to within \( D \) of a line in \( T_1 \); the \( D \)-neighborhood of this line, by construction, can contain a unique first coordinate of one of the points in \( A_{j-i} \).

Thus the fact that lines in \( T_1 \) are mapped to within a uniformly bounded distance of lines in \( T_2 \) tells us that there is a surjective function \( f_1 \) from the set of \( T_1 \) coordinates of the points in \( A_{j-i} \) to the set of \( T_2 \) coordinates of the points in \( A_i \). In addition, there is a surjective function \( f_2 \) from the set of \( T_2 \) coordinates of the points in \( A_i \) to the set of \( T_1 \) coordinates of the points in \( A_{j-i} \). Moreover, these maps combine in the following way, as follows from the definition of a product map given in [EFW]. If \((x, y) \in A_i \bigoplus A_{j-i}\), then \( \varphi'(x, y) = (f_2(y), f_1(x)) \).

We begin with the case \( n = 2 \). Consider the restriction \( \varphi' : A_i \bigoplus A_{j-i} \to A_i \bigoplus A_{j-i} \), where \( j \) is defined by the image of \( a_0 \) under \( \varphi' \). Then the set \( A_i \bigoplus A_{j-i} \) consists of four points. Assume that \( i > j \) and that \( i \) was chosen to be sufficiently large, as described above. The first choice ensures that the index of one factor is negative, and the other is positive; the second choice guarantees that the points within the block \( A_i \bigoplus A_{j-i} \) have coordinates in the appropriate trees which are at least \( D \) units apart from each other. We describe the points in \( A_i \bigoplus A_{j-i} \) using both the lamplighter picture of \( L_2 \) as well as giving coordinates in the graph \( DL(2, 2) \), as follows:

1. the identity \((x_0, y_0)\)
2. a single bulb illuminated in position \( i \), where we introduce the coordinates \( a_i = (x_0, B) \)
3. a single bulb illuminated in position \( j - i \), where we introduce the coordinates \( a_{j-i} = (c, y_0) \)
4. bulbs illuminated in positions \( i \) and \( j - i \), which necessarily has coordinates \((c, B)\).

Note that it is exactly the fact that we ensured that \( i \) and \( j - i \) have opposite signs which determines the coordinates of \( a_i a_{j-i} \) to be \((c, B)\).

We use the geometric result of [EFW] that \( \varphi \) is a uniformly bounded distance from a product map, which is bilipshitz on each tree and then interchanges the tree factors, to show that \((c, B)\) is a fixed
point for \( \varphi' \). Let \( f_1 \) and \( f_2 \) be the functions on the coordinates of the points of \( A_i \oplus A_{j-i} \) defined above. We must have \( f_1(x_0) = y_0 \) and \( f_2(y_0) = x_0 \) since the identity \((x_0, y_0)\) is preserved under any automorphism. Lemma 4.3 implies that \( f_1(B) = c \) and \( f_2(c) = B \). This forces \( \varphi'(c, B) = (c, B) \); the existence of a fixed point for \( \varphi' \) says that \( Id - \varphi' \mid _{A_i \oplus A_{j-i}} \) is not invertible when \( n = 2 \). Thus within each block of the form \( A_i \oplus A_{j-i} \), for \( i > j \), there are at least two distinct \( \varphi' \)-twisted conjugacy classes. By choosing group elements in \( A_i \) with entries in the different \( \varphi' \)-twisted conjugacy classes within each block, we can create infinitely many different \( \varphi' \)-twisted conjugacy classes in \( \bigoplus A_i \). We conclude that \( R(\varphi') \) and hence \( R(\varphi) \) are infinite.

In the case \( n = 3 \), the blocks \( A_i \oplus A_{j-i} \) have nine elements, and we have the additional algebraic fact that \( a_i^2 = a_i^{-1} \). We have two choices for the maps \( f_1 \) and \( f_2 \), as follows.

1. If \( \varphi'(a_i) = a_{j-i} \) and \( \varphi'(a_{j-i}) = a_i \) then it is clear that \( f_1(c_1) = B_1 \) and \( f_2(B_1) = c_1 \), resulting in \((c_1, B_1)\) being fixed under \( \varphi' \).
2. If \( \varphi'(a_i) = a_{j-i}^2 \) and \( \varphi'(a_{j-i}) = a_i^2 \) then we know that \( f_1(c_1) = B_2 \) and \( f_2(B_2) = c_2 \). This forces \( f_1(c_2) = B_1 \) and \( f_2(B_2) = c_1 \). We see immediately that \((c_1, B_2)\) is fixed under \( \varphi' \).

In either case, the existence of a non-trivial fixed point ensures that \( L_3 \) has property \( R_{\infty} \).

When \( n > 3 \) is odd and \((n, 3) = 1\), we use the blocks \( A_{j-i} \oplus A_i \) to construct \( \varphi' : A_i \oplus A_{j-i} \rightarrow A_i \oplus A_{j-i} \) with no nontrivial fixed points, under the assumption that \( \varphi'(a_0) = a_0^j \). Thus the map \( Id - \varphi' \) will be surjective, hence invertible, on each block, and we see that there must be a single \( \varphi' \)-twisted conjugacy class in \( \bigoplus A_i \). This then implies that \( L_n \) does not have property \( R_{\infty} \). We begin by assuming that \( \varphi' : \bigoplus A_i \rightarrow \bigoplus A_i \) is an automorphism with \( \varphi'(a_0) = a_0^j \), and that \( \overline{\varphi}(t) : \mathbb{Z} \rightarrow \mathbb{Z} \) is given by \( \overline{\varphi}(t) = t^{-1} \). The compatibility condition given in Equation 3 ensures that these automorphisms together induce an automorphism of \( L_n = \mathbb{Z}_n \wr \mathbb{Z} \).

Since \( \varphi'(a_0) = a_0^j \), we again obtain a map on blocks, where the block \( A_i \oplus A_{j-i} \oplus A_{j-i} \) consists of \( n^2 \) points; we introduce coordinates on these points as follows.

| \( a_i^0 \) | \( a_i^1 \) | \( a_i^2 \) | \( \cdots \) | \( a_i^{n-1} \) |
|---|---|---|---|---|
| \( a_{j-i} \) | \( (x_0, y_0) \) | \( (x_0, B_1) \) | \( (x_0, B_2) \) | \( \cdots \) | \( (x_0, B_{n-1}) \) |
| \( a_{j-i} \) | \( (c_1, y_0) \) | \( (c_1, B_1) \) | \( (c_1, B_2) \) | \( \cdots \) | \( (c_1, B_{n-1}) \) |
| \( a_{j-i} \) | \( (c_2, y_0) \) | \( (c_2, B_1) \) | \( (c_2, B_2) \) | \( \cdots \) | \( (c_2, B_{n-1}) \) |
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( a_{j-i}^{n-1} \) | \( (c_{n-1}, y_0) \) | \( (c_{n-1}, B_1) \) | \( (c_{n-1}, B_2) \) | \( \cdots \) | \( (c_{n-1}, B_{n-1}) \) |

These points determine \( n \) points in each tree, all at height zero, namely \( \{x_0, c_1, \ldots, c_{n-1}\} \) in \( T_1 \) and \( \{y_0, B_1, \ldots, B_{n-1}\} \) in \( T_2 \). Choose \( i > \max\{j, KD\} \) as above.

We now show that when \((n, 2) = (n, 3) = 1\), we can construct maps \( f_1 \) and \( f_2 \) so that the induced automorphism \( \varphi' : A_i \oplus A_{j-i} \rightarrow A_i \oplus A_{j-i} \) has no non-trivial fixed points, and thus \( Id - \varphi' : A_i \oplus A_{j-i} \rightarrow A_i \oplus A_{j-i} \) is invertible, which implies that \( R(\varphi') < \infty \). Suppose that \( \varphi'(a_i) = a_i^2 \) and \( \varphi'(a_{j-i}) = a_{j-i}^2 \). To obtain a fixed point, we must solve the equations \( \varphi'(a_i^k) = a_{j-i}^{2k(mod n)} \) and \( \varphi'(a_{j-i}^{2k(mod n)}) = a_i^k \) for \( k \), that is, \( 3k \equiv 0(mod n) \) which has a solution unless \( 3|n \). Thus when \( n = 3 \), the map \( 1 - \varphi' : A_i \rightarrow A_i \) is invertible on each block, hence invertible on \( \bigoplus A_i \).

Since we are trying to show that \( R(\varphi) < \infty \), we refer to part (4) of Lemma 2.2. We know that \( R(\overline{\varphi}) = \#\text{Coker}(1 - \overline{\varphi}) = 2 \), so \( \overline{\varphi} \) has two twisted conjugacy classes: \([1]\) and \([t]\). Recalling Equation 2, we must count the twisted conjugacy classes lying over the classes \([1]\) and \([t]\). Over \([1]\) we have \( R(\varphi') \) twisted conjugacy classes, and over \([t]\) we have \( R(t \cdot \varphi') \). Since \( \text{Fix}\overline{\varphi} = \{1\} \), it follows that
$R(\varphi) = R(\varphi') + R(t \cdot \varphi').$ The action of $t$ shifts the block $A_i \bigoplus A_{j-i}$ by increasing all the indices by one, and thus the geometry of the lamplighter picture for $L_n$ easily implies that $R(\varphi') = R(t \cdot \varphi')$ and hence $R(\varphi) = 2R(\varphi') < \infty$.

When $2|n$, we construct the following short exact sequence of groups:

$$1 \rightarrow \mathbb{Z}_2 \wr \mathbb{Z} \rightarrow \mathbb{Z}_n \wr \mathbb{Z} \rightarrow \mathbb{Z}_2 \rightarrow 0$$

It is proven in Lemma 4.2 that $\mathbb{Z}_2 \wr \mathbb{Z}$ is characteristic in $\mathbb{Z}_n \wr \mathbb{Z}$, allowing us to obtain a commutative diagram based on the above short exact sequence and any $\varphi \in \text{Aut}(\mathbb{Z}_n \wr \mathbb{Z})$. We proved above that $\mathbb{Z}_2 \wr \mathbb{Z}$ has property $R_\infty$ and it follows from Lemma 2.2 (2) that $\mathbb{Z}_n \wr \mathbb{Z}$ has property $R_\infty$ as well. The argument is analogous when $3|n$.

When $G$ is a finite abelian group, $G \wr \mathbb{Z}$ has as its Cayley graph $DL(|G|, |G|)$, with respect to the generating set \{t, tg_1, tg_2, \ldots, tg_{n-1}\} where \{g_i\} represent all elements of $G$. Gonçalves and Wong [GW1] prove the following general result concerning which groups of this form have property $R_\infty$.

**Theorem 4.5 ([GW1], Theorem 3.7).** Let

$$G = \bigoplus_j (\mathbb{Z}_{p_j})^{r_j}$$

be a finite abelian group, where the $p_j$ are distinct primes. Then for all automorphisms $\varphi \in \text{Aut}(G)$, $R(\varphi) = \infty$ if and only if $p_j = 2$ or $3$ for some $j$ and $r_j = 1$.

5. Counting twisted conjugacy classes in $G \wr \mathbb{Z}$

We end this paper by studying property $R_\infty$ for general lamplighter groups of the form $G \wr \mathbb{Z}$, where $G$ is an arbitrary finite group. The Cayley graph of $G \wr \mathbb{Z}$ is again the Diestel-Leader graph $DL(n, n)$, where $n = |G|$, with respect to the generating set \{tg | g \in G\}.

We prove several results about when these general lamplighter groups have property $R_\infty$, but do not give a complete classification of which groups of this form have or do not have the property. The first two results are algebraic, using various expressions of $G$ as a group extension, and we end with geometric results relying on $DL(n, n)$. We first note that as for $\mathbb{Z}_n \wr \mathbb{Z}$, we have the following short exact sequence in which $\bigoplus G_i$ is characteristic:

$$1 \rightarrow \bigoplus G_i \rightarrow G \wr \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 1$$

**Lemma 5.1.** Let $\varphi \in \text{Aut}(G \wr \mathbb{Z})$. Then $\varphi(\bigoplus G_i) \subset \bigoplus G_i$.

**Proof.** Let $|G| = n$ and put coordinates on $DL(n, n)$ as we did in the case of the lamplighter group $L_n = \mathbb{Z}_n \wr \mathbb{Z}$: denote the identity by $(x_0, y_0)$ and the generators of $G_0$, the copy of $G$ in $\bigoplus G$ indexed by $0$, by $(x_0, B_i)$.

Let $\varphi \in \text{Aut}(G \wr \mathbb{Z})$. The argument given in Lemma 4.3 quoting the result on [FPW] again applies, and we conclude that $\varphi(x_0, B_i) = (c_i, y_0)$, for some coordinates $c_i$. Regardless of the actual vertices of the trees represented by the $c_i$, we conclude that $\varphi(x_0, B_i)$ lies at height zero in $DL(n, n)$, that is, $\varphi(x_0, B_i) \in \bigoplus G_i$. Since $\varphi(G_0) \subset \bigoplus G_i$ and the action of $t$ is to translate between the factors of $G$ in the sum $\bigoplus G_i$, we see that $\bigoplus G_i$ is characteristic in $G \wr \mathbb{Z}$. □
Applying Lemma 5.1 we see that the following diagram is commutative, for \( \phi \in Aut(G \wr Z) \).

\[
\begin{array}{cccccc}
1 & \to & \bigoplus G_i & \to & G \wr Z & \to & Z & \to & 1 \\
\varphi' & \downarrow & \varphi & \downarrow & \varphi & \downarrow & Id, g & \to & \bigoplus G_i & \to & G \wr Z & \to & Z & \to & 1
\end{array}
\]

(8)

Let \( G = \{Id, g_1, g_2, \ldots, g_n\} \). To denote the elements of the \( i \)-th copy of \( G \) in \( \bigoplus G_i \), we use the notation \( g_{i,j} \) where \( i \) is the index of \( G_i \) in \( \bigoplus G_i \) and \( j \in \{1,2,\ldots,n\} \). The symbol \( Id_i \) will denote the identity in \( G_i \).

If \( G \) itself can be written as a group extension \( 1 \to A \to G \to C \to 1 \) then one always obtains the following short exact sequences, although in both cases the kernel is not necessarily a characteristic subgroup of \( G \wr Z \):

\[
1 \to \bigoplus A_i \to G \wr Z \to C \wr Z \to 1 \quad \text{and} \quad 1 \to A \wr Z \to G \wr Z \to C \to 1
\]

(9)

Given an arbitrary finite group \( G \), we have two natural short exact sequences:

\[
1 \to [G,G] \to G \to G^{Ab} \to 1 \quad \text{and} \quad 1 \to Z(G) \to G \to G/Z(G) \to 1
\]

where \([G,G]\) and \( Z(G)\) denote the commutator subgroup and the center of \( G \), respectively. Both of these subgroups are characteristic in \( G \). To obtain the related commutative diagrams, we must first prove the following lemma.

**Lemma 5.2.** The subgroups \( \bigoplus ([G,G])_i \) and \( Z(G) \wr Z \) of \( G \wr Z \) are both characteristic.

**Proof.** We show that \( \bigoplus ([G,G])_i \) is characteristic in \( \bigoplus G_i \) and thus in \( G \wr Z \). The commutator subgroup \( [\bigoplus G_i, \bigoplus G_i] \) has the following form. Let \( a, b \in \bigoplus G_i \). Then

\[
a = \sum_s g_{\sigma(s),j_s} \quad \text{and} \quad b = \sum_t g_{\tau(t),j_t}
\]

where \( \sigma \) and \( \tau \) are injective functions from \( \{1,2,3,\ldots,k\} \) and \( \{1,2,\ldots,l\} \), respectively, to \( Z \) so that \( \sigma(1) < \sigma(2) < \cdots < \sigma(k) \) and \( \tau(1) < \tau(2) < \cdots < \tau(l) \). When we form \([a,b]\), one of two things must occur:

1. If \( \sigma(s) = \tau(t) \) for some \( t \), then the coordinate of \([a,b]\) in \( G_{\sigma(s)} \) is \( g_{\sigma(s),j_s} g_{\tau(t),j_t} \).
2. If \( \sigma(s) \neq \tau(t) \) for any \( t \), then the coordinate of \([a,b]\) in \( G_{\sigma(s)} \) is \( g_{\sigma(s),j_s} g_{\sigma(s),j_s}^{-1} = Id_{\sigma(s)} \). The same is true for indices \( \tau(t) \) which are not equal to \( \sigma(s) \) for any \( s \).

Since each non-identity coordinate of \([a,b]\) is a commutator of \( G \), we see that \( [\bigoplus G_i, \bigoplus G_i] \subset \bigoplus ([G,G])_i \), and the other inclusion is clear. Thus \( \bigoplus ([G,G])_i \) must be a characteristic subgroup.

To show that \( Z(G) \wr Z \) is characteristic in \( G \wr Z \), we first show that \( \bigoplus Z(G)_i \) is characteristic in \( \bigoplus G_i \). Since the group operation between elements of \( \bigoplus G_i \) involves component-wise multiplication, it follows immediately that \( Z(\bigoplus G_i) = \bigoplus Z(G)_i \), and thus this subgroup is characteristic.

Let \( \varphi \in Aut(G \wr Z) \). We obtain automorphisms \( \varphi' : \bigoplus G_i \to \bigoplus G_i \) and \( \varphi : Z \to Z \); we assume the latter is given by \( \varphi(t) = t^{-1} \). Since \( \bigoplus Z(G)_i \) is characteristic in \( \bigoplus G_i \), we can restrict to obtain \( \varphi' : \bigoplus Z(G)_i \to \bigoplus Z(G)_i \). Since \( \varphi' \) and \( \varphi \) satisfy the compatibility condition given in equation 6 we induce a map on \( Z(G) \wr Z \) which makes the following diagram commute and must be the
restriction of ϕ to Z(G) ⊙ Z:

\[ 1 \longrightarrow \bigoplus Z(G)_i \longrightarrow Z(G) \bowtie Z \longrightarrow Z \longrightarrow 1 \]

(10)

\[ 1 \longrightarrow \bigoplus Z(G)_i \longrightarrow Z(G) \bowtie Z \longrightarrow Z \longrightarrow 1 \]

From this we conclude that Z(G) \bowtie Z is a characteristic subgroup of G \bowtie Z.

It follows from Lemma 5.2 that given any ϕ ∈ Aut(G \bowtie Z), we obtain the following commutative diagrams:

\[ 1 \longrightarrow \bigoplus ([G,G])_i \longrightarrow G \bowtie Z \longrightarrow G^{Ab} \bowtie Z \longrightarrow 1 \]

(11)

\[ 1 \longrightarrow \bigoplus ([G,G])_i \longrightarrow G \bowtie Z \longrightarrow G^{Ab} \bowtie Z \longrightarrow 1 \]

which is a special case of (9) given above, and

\[ 1 \longrightarrow Z(G) \bowtie Z \longrightarrow G \bowtie Z \longrightarrow G/Z(G) \longrightarrow 1 \]

(12)

\[ 1 \longrightarrow Z(G) \bowtie Z \longrightarrow G \bowtie Z \longrightarrow G/Z(G) \longrightarrow 1 \]

Here, the projection G \bowtie Z → G/Z(G) is given by

\[
\left( \sum_{k} a_{ik}, t^j \right) \mapsto \prod_{k} [a_{ik}]
\]

where [a] is the image of a ∈ G in G/Z(G) and \( i_1 < ... < i_m \).

We will always use ϕ' and ϕ for the induced homomorphisms on the kernel and on the quotient, respectively. We note that the two maps in the diagrams above labeled ϕ' (resp. ϕ) denote different maps in the two diagrams.

Denote by \( \mathfrak{A} \) the family of finite abelian groups A such that A\bowtie Z has property \( R_\infty \). This family was completely determined in [GW1], and is listed above as Theorem 4.5. Similarly, let \( \mathcal{L} \) be the family of finite groups G such that G \bowtie Z has property \( R_\infty \). We use both the algebra and the geometry of these lamplighter groups to determine some conditions under which G ∈ \( \mathcal{L} \).

**Theorem 5.3.** Let G be a finite group. If G^{Ab} ∈ \( \mathfrak{A} \) or Z(G) ∈ \( \mathfrak{A} \) then G ∈ \( \mathcal{L} \).

**Proof.** First suppose that G^{Ab} ∈ \( \mathfrak{A} \). It follows from (11) and Lemma 2.2 that G \bowtie Z has property \( R_\infty \), provided that ϕ is an automorphism of G^{Ab} \bowtie Z. By Lemma 2.1, it suffices to show that G^{Ab} \bowtie Z is Hopfian. Since G^{Ab} is finite and abelian, Theorem 3.2 of [G] shows that G^{Ab} \bowtie Z is residually finite. Now, G^{Ab} \bowtie Z is finitely generated so we conclude that it is Hopfian.

Next, suppose that Z(G) ∈ \( \mathfrak{A} \), which yields \( R(\varphi') = \infty \). The inclusion i : Z(G) \bowtie Z → G \bowtie Z induces a function \( \hat{i} : R(\varphi') \rightarrow R(\varphi) \) where R(ψ) denotes the set of ψ-twisted conjugacy classes. Since G/Z(G) is finite, so is the number of fixed points of \( \overline{\varphi} \). The reasoning in part (4) of Lemma 2.2 shows that R(ϕ) is in 1-1 correspondence with R(ϕ') modulo the action of Fix\( \overline{\varphi} \). Thus R(ϕ) = \( R(\varphi') \rightarrow \infty \) as well.

\[ \square \]
Using Theorem 4.5, one can easily determine when \( Z(G) \in \mathfrak{A} \). We now list some examples of groups with property \( R_\infty \) which follow immediately from Theorem 5.3.

**Example 5.4** (Dihedral groups \( D_{2n} \) for odd \( n \)). Let \( G = D_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2 \) be the dihedral group of order 6. In this case, \( |G| \cong \mathbb{Z}_3 \) while \( Z(G) = \{1\} \). Since \( G^{ab} \cong \mathbb{Z}_2 \in \mathfrak{A} \), it follows from Theorem 5.3 that \( D_6 \in \mathfrak{L} \), that is, \( D_6 \not\cong \mathbb{Z} \) has property \( R_\infty \). Moreover, if \( n \) is odd and \( D_{2n} \) is the dihedral group of order \( 2n \) then \( \mathbb{Z}_n \) is a characteristic subgroup of \( D_{2n} \cong \mathbb{Z}_n \times \mathbb{Z}_2 \). Using the commutative diagram (8) and the fact that \( \mathbb{Z}_2 \in \mathfrak{A} \), it follows from part (1) of Lemma 2.2 and Theorem 5.3 that \( D_{2n} \in \mathfrak{L} \) when \( n \) is odd.

**Example 5.5** (Dihedral groups \( D_{2n} \) for even \( n \)). When \( n \) is even, \( D_{2n} \cong \mathbb{Z}_n \times \mathbb{Z}_2 \), then \( D_{2n} \not\cong \mathbb{Z} \) also has property \( R_\infty \). If we take \( \mathbb{Z}_n \cong \langle t \rangle \), then the center of \( D_{2n} \) is isomorphic to \( \mathbb{Z}_2 \cong \langle t^{\frac{1}{2}} \rangle \). Thus \( Z(D_{2n}) \in \mathfrak{A} \) and it follows from Theorem 5.3 that \( D_{2n} \not\cong \mathbb{Z} \) has property \( R_\infty \) for all \( n > 0 \).

**Example 5.6** (Quaternion group of order 8). Let \( G = \mathbb{Q}_8 \) be the quaternion group of order 8. In this case, \( |G| = Z(G) \cong \mathbb{Z}_2 \). Since \( \mathbb{Z}_2 \in \mathfrak{A} \), it follows from Theorem 5.3 that \( Q_8 \in \mathfrak{L} \).

Combining Theorem 4.1 with Examples 5.4 and 5.5, we have proven the following.

**Proposition 5.7.** If \( G \) is any finite group with order \( 2p \), where \( p \) is an odd prime, then \( G \not\cong \mathbb{Z} \) has property \( R_\infty \).

**Proof.** It is an elementary theorem in algebra that any group of order \( 2p \), where \( p \) is an odd prime, must be either cyclic or dihedral. \( \square \)

If \( Z(G) \) or \( G^{ab} \) is unknown, the following special condition may be applicable, which allows us to extend our results to some nilpotent groups, dependent on order.

**Theorem 5.8.** Let \( G \) be a finite group with a unique Sylow 2-group \( S_2 \). If \( Z(S_2) \in \mathfrak{A} \) then \( G \in \mathfrak{L} \).

Similarly, if \( G \) has a unique Sylow 3-group \( S_3 \) with \( Z(S_3) \in \mathfrak{A} \), then \( G \in \mathfrak{L} \).

Before proving Theorem 5.8, we state two corollaries.

**Corollary 5.9.** Let \( G \) be a finite nilpotent group whose order is divisible by either 2 or 3, and let \( S_2 \) and/or \( S_3 \) denote, respectively, the unique Sylow 2- and/or 3-subgroups. If \( Z(S_i) \in \mathfrak{A} \), for \( i = 2 \) or \( i = 3 \), then \( G \not\cong \mathbb{Z} \) has property \( R_\infty \).

Some elementary group theory leads to an additional corollary.

**Corollary 5.10.** Let \( p \) and \( q \) be prime, and let \( G \) be a finite nonabelian group whose order is one of:

1. \( 2p^n \), where \( 2 < p \),
2. \( 3q^m \), where \( 3 < q \),
3. \( 2p^n q^m \), where \( 2 < p < q \), or
4. \( 3p^n q^m \), where \( 3 < p < q \).

Then \( G \not\cong \mathbb{Z} \) has property \( R_\infty \).

**Proof.** It is an exercise in group theory to check that groups with the above orders have either a unique Sylow 2 subgroup of order 2 or a unique Sylow 3 subgroup of order 3. The conclusion then follows directly from Theorem 5.8. \( \square \)

We now return to the proof of Theorem 5.8.
Proof of Theorem 5.8. We prove the theorem in the case that $G$ has a unique Sylow 2-subgroup $S_2$; the case for the unique Sylow 3-subgroup is analogous. It follows immediately from Theorem 5.3 that $S_2 \lhd Z$ has property $R_\infty$. We will show that $S_2 \lhd Z$ is a characteristic subgroup of $G \wr Z$ and the theorem then follows from Lemma 2.2, the short exact sequence $1 \to S_2 \lhd Z \to G \wr Z \to G/S_2 \to 1$, and the fact that a unique Sylow subgroup is normal.

Let $\varphi \in \text{Aut}(G \wr Z)$, $\varphi'$ its restriction to $\bigoplus G_i$ and $\overline{\varphi}$ its projection to $Z$. Since $S_2$ contains all elements of $G$ whose order is a power of 2, the same is true for $\bigoplus (S_2)_i \subset \bigoplus G_i$. Since order is preserved under automorphism, $\varphi'(\bigoplus (S_2)_i) \subset \bigoplus (S_2)_i$ and this subgroup is characteristic. We now have vertical maps from the short exact sequence $1 \to \bigoplus (S_2)_i \to S_2 \lhd Z \to Z \to 1$ where the map on the kernel is the restriction of $\varphi'$ and the map on the quotient is $\overline{\varphi}$. These maps induce a map on $S_2 \lhd Z$ which must be the restriction of $\varphi$ to $S_2 \lhd Z$. Thus $S_2 \lhd Z$ is a characteristic subgroup of $G \wr Z$.

We extend these results further for simple groups, and as a consequence of Theorem 5.12 below, determine that $A_n \wr Z$ and $S_n \wr Z$ have property $R_\infty$ for all $n \geq 5$.

Lemma 5.11. Let $G$ be a finite simple group, and $\varphi \in \text{Aut}(G \wr Z)$. For each $j \in Z$, there exists $i \in Z$ so that $\varphi'(G_i) = G_j$.

Proof. Since $G$ is finite, there is an $r \in Z^+$ so that $\varphi'(G_0) \subset \bigcup_{k=1}^r G_{i_k}$, where the $i_k$ are distinct indices. Consider the homomorphism defined by composing $\varphi'(G_0)$ with projection onto the first factor of $\bigcup_{k=1}^r G_{i_k}$. Since $G$ is simple, there can be no kernel, so this is an automorphism of $G$. This argument holds for any factor in $\bigcup_{k=1}^r G_{i_k}$. We obtain a set $\{\xi_k\}$ of automorphisms of $G$ so that $\xi_k : G \to G_{i_k}$. We observe that since each $\xi_k$ is an automorphism, for all nontrivial $g \in G_0$ and for all values of $k$, the element $\xi_k(g)$ is nontrivial.

As the action of $t$ on $\bigoplus G_i$ is by translation, the same set of automorphisms can be used to determine the image of any element of $G_i$ in $\bigoplus G_i$.

Since we began with a group automorphism, and $\bigoplus G_i$ is a characteristic subgroup of $G \wr Z$, there is some $x \in \bigoplus G_i$ so that $\varphi'(x) = g_j$. Now the proof of Lemma 4.3 allows us to conclude that $x \in G_i$ for some $i$. Since the action of $t$ on $\bigoplus G_i$ is by translation, there is a single automorphism $\xi : G \to G$ which is used to determine the image of any element in $\bigoplus G_i$ under $\varphi \in \text{Aut}(G \wr Z)$.

Using Lemma 5.11, we can now prove the following theorem.

Theorem 5.12. Let $G$ be a finite simple group whose outer automorphism group is trivial. Then $G \in \mathcal{L}$, that is, $G \wr Z$ has property $R_\infty$.

Proof. It follows from Lemma 5.11 that $\varphi' : \bigoplus G_i \to \bigoplus G_i$ preserves blocks of the form $G_i \bigoplus G_{j-i}$. We mimic the proof of Theorem 4.1 and consider the restriction of $\varphi'$ to each block. If $G$ has no outer automorphisms, then $\varphi'|_{G_i}$ must be conjugation by the same group element on each block. Since conjugation by $g \in G$ always has $g$ as a nontrivial fixed point, the element $(g, g) \in G_i \bigoplus G_{j-i}$ will be a nontrivial fixed point for $\varphi' : G_i \bigoplus G_{j-i} \to G_i \bigoplus G_{j-i}$. Note that the number of $\varphi$-twisted conjugacy classes need not be the index of the subgroup $(\text{Id} - \varphi)$ when the group is non-abelian. Instead, the number of such classes is given by the number of ordinary conjugacy classes $[x]$ for which $[x] = [\varphi(x)]$ (see e.g. [FH, Theorem 5]). In particular, a nontrivial fixed point of $\varphi$ yields a class other than that of the identity element. Thus there are at least two $\varphi$-twisted conjugacy classes on each block, and the theorem follows.
Since both $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are finite simple groups with trivial outer automorphism groups, Theorem 5.12 yields the immediate corollary that $L_2$ and $L_3$ have property $R_\infty$.\cite{GW1}

It follows immediately that many of the finite simple groups lie in $\mathcal{L}$, including:

1. the Matthieu groups $M_{11}$, $M_{23}$ and $M_{24}$,
2. the Conway groups $C_1$, $C_2$ and $C_3$,
3. the Janko groups $J_1$ and $J_4$,
4. the baby monster $B$ and the Fischer-Griess monster $M$,
5. other sporadic groups: the Fischer group $Fi_{23}$, the Held group $He$, the Harada-Norton group $HN$, the Lyons group $Ly$, and the Thompson group $Th$.

In the case of $A_n$ for $n = 5$ and $n \geq 7$, which has outer automorphism group isomorphic to $\mathbb{Z}_2$, we obtain the following corollary.

**Corollary 5.13.** The alternating group $A_n$ for $n \geq 5$, $n \neq 6$, is in $\mathcal{L}$, that is, $A_n \wr \mathbb{Z}$ has property $R_\infty$.

**Proof.** For $n \neq 6$, we have $Out(A_n) = \mathbb{Z}_2$. This single outer automorphism is conjugation by an odd permutation, and thus preserves a conjugacy class within $A_n$. Since this outer automorphism is defined up to inner automorphisms, we can choose it so that there is a fixed point in $A_n$. Then the argument in the proof of Theorem 5.12 works verbatim, that is, the automorphism $\varphi'$ when restricted to the blocks $G_j \oplus G_{i-j}$ again must have a non-trivial fixed point. \hfill $\square$

Since $A_n$ is a subgroup of index 2 in $S_n$, we can apply reasoning similar to the proof of Lemma 5.11 to obtain the following proposition.

**Proposition 5.14.** For $n \geq 5$, the symmetric group $S_n \in \mathcal{L}$, that is, $S_n \wr \mathbb{Z}$ has property $R_\infty$.

**Proof.** Since $A_n$ is a subgroup of index 2 in $S_n$, we obtain the following short exact sequence:

$$1 \rightarrow \bigoplus (A_n)_i \rightarrow S_n \wr \mathbb{Z} \rightarrow \mathbb{Z}_2 \wr \mathbb{Z} \rightarrow 1$$

Since $\mathbb{Z}_2 \wr \mathbb{Z}$ is finitely generated and residually finite by \cite[Theorem 3.2]{G}, it is Hopfian so that Lemma 2.1 applies. Therefore, if we can show that $\bigoplus (A_n)_i$ is a characteristic subgroup of $\bigoplus (S_n)_i$, then the proposition follows from Lemma 2.2 and Theorem 4.1.

Following the proof of Lemma 5.11, suppose that $\varphi'(G_0) \subset \cup_{k=1}^{\infty} G_{ik}$. When we compose with the projection onto any of these $G_{ik}$ factors, we obtain one of two possible images for $G \cong S_n$: either $S_n$ or $\mathbb{Z}_2$, where the latter arises when the kernel of the homomorphism is $A_n$. In either case, the image of $A_n$ under the composition of $\varphi'$ and this projection map must be $A_n$. Now restrict the map $\varphi'$ to $\bigoplus (A_n)_i$ initially and it follows that $\bigoplus (A_n)_i$ is a characteristic subgroup of $\bigoplus (S_n)_i$. \hfill $\square$

The following corollary follows immediately from Proposition 5.14 since any finite group can be embedded into $S_n$ for some value of $n$.

**Corollary 5.15.** Every group of the form $G \wr \mathbb{Z}$ where $G$ is a nontrivial finite group can be embedded into a group which has property $R_\infty$.

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