A NOTE ON BERNSTEIN–SATO VARIETIES FOR TAME DIVISORS AND ARRANGEMENTS

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ABSTRACT. For strongly Euler-homogeneous, Saito-holonomic, and tame analytic germs we consider general types of multivariate Bernstein–Sato ideals associated to arbitrary factorizations of our germ. We show that these ideals are principal and the zero loci associated to different factorizations are related by a diagonal property. If, additionally, the divisor is a hyperplane arrangement, we obtain nice estimates for the zero locus of its Bernstein–Sato ideal for arbitrary factorizations and show the Bernstein–Sato ideal attached to a factorization into linear forms is reduced. As an application, we independently verify and improve upon an estimate of Maisonobe’s regarding standard Bernstein–Sato ideals for reduced, generic arrangements: we compute the Bernstein–Sato ideal for a factorization into linear forms and we compute its zero locus for other factorizations.

1. INTRODUCTION

Let $X$ be a smooth analytic space or a $\mathbb{C}$-scheme with dimension $n$ and consider the collection $F = (f_1, \ldots, f_r)$ of regular, analytic functions $f_k \in \mathcal{O}_X$. Throughout, $x$ will denote a point in $X$, $f = f_1 \cdots f_r$, and $\mathcal{D}_X$ will signify the sheaf of $\mathbb{C}$-linear differential operators. For $a = (a_1, \ldots, a_r) \in \mathbb{N}^r$ there is a $\mathcal{D}_X[S] := \mathcal{D}_X[s_1, \ldots, s_r]$-module (this is a standard polynomial ring extension where the $s_k$ are central variables)

$$\mathcal{D}_X[S]F^{S+a} := \mathcal{D}_X[s_1, \ldots, s_r]f_1^{s_1+a_1} \cdots f_r^{s_r+a_r},$$

where the action of a differential operator on

$$F^{S+a} := f_1^{s_1+a_1} \cdots f_r^{s_r+a_r}$$

is given by a formal application of the chain and product rule. As in the classical setting, where $F$ is just one function, we have a functional equation

$$b(S)F^S = PF^{S+a}$$

with $b(S) \in \mathbb{C}[S] := \mathbb{C}[s_1, \ldots, s_r]$ and $P \in \mathcal{D}_X[S]$ as well as a local version where each $f_k$ is regarded as its germ at $x$. Generalizing the Bernstein–Sato polynomial,

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we have the multivariate Bernstein–Sato ideal $B^n_{F,F}$, which consists of all the polynomials $b(S) \in \mathbb{C}[S]$ satisfying said local functional equation. Explicitly,

$$B^n_{F,F} := \{ b(S) \in \mathbb{C}[S] \mid b(S)F^S \in \mathcal{D}_{X,S}[S]F^{S+n} \}.$$ 

Finally, let $Z(B^n_{F,F})$ be the zero locus cut out by $B^n_{F,F}$:

$$Z(B^n_{F,F}) := V(\text{rad}(B^n_{F,F})).$$

Recently a lot of work has been done on multivariate Bernstein–Sato ideals and their zero loci. Let $1 = (1, \ldots, 1)$. While it had been shown by Sabbah and Gyoja, cf. [20] [22], that the codimension one components of $Z(B^n_{F,F})$ are rational hyperplanes, Maisonobe proved in [14] that any component of larger codimension can be translated into a codimension one component by an element of $\mathbb{Z}$. He did this by proving many useful homological properties of

$$\mathcal{D}_{X,S}[S]F^S \mathcal{D}_{X,S}[S]F^{S+1}$$

and analyzing various attached associated graded objects. Using some of these results, in [22] Budur, Van der Veer, Wu, and Zhou completed a proof of a conjecture of Budur’s from [7]. Namely, they showed that if $F = (f_1, \ldots, f_r)$ corresponds to the factorization $f = f_1 \cdots f_r$, then exponentiating $Z(B^n_{F,F})$ computes the rank one local systems on $X \setminus V(f)$ with nontrivial cohomology. (See also section 3.29 of [7].) This can be viewed as a generalization of Kashiwara and Malgrange’s result in [17], [13] that exponentiating the roots of the Bernstein–Sato polynomial computes the eigenvalues of the algebraic monodromy on nearby Milnor fibers. Indeed, in the multivariate setting, the role of the nearby cycle functor is replaced by the Sabbah specialization complex, cf. [7] [8] [9]. Most of these results have been generalized to $Z(B^n_{F,F})$ for $a \in \mathbb{N}^r$ arbitrary in [10].

Our main objective is to identify geometric conditions on $\text{Div}(f)$ so that $Z(B^n_{F,F})$ and/or $B^n_{F,F}$ has a particularly nice structure. The hypotheses concern the logarithmic derivations $\text{Der}_X(–\log f)$ of $\text{Div}(f)$, that is, the derivations that preserve $f$, as well as a sort of dual object introduced by Saito in [21]: the logarithmic differential forms $\Omega^*_X(\log f)$. We will often assume the following for $\text{Div}(f)$: it is strongly Euler-homogeneous (locally everywhere it has a singular derivation that acts as the identity on $f$); it is Saito-holonomic (the stratification of $X$ by the logarithmic derivations is locally finite); it is tame ($\Omega^*_X(\log f)$ satisfies a scaling projective dimension bound). This set of conditions was first considered by Walther in [24] in the univariate setting ($F = (f)$) and we considered them in the multivariate setting ($F = f_1 \cdots f_r; r > 1$) in [2].

While in general $B^n_{F,F}$ may not be principal (cf. [11] [5]), using some of our results from [2] as well as a crucial idea of Maisonobe’s from [16], we can obtain the following in Corollary 3.7.

**Theorem 1.1.** Suppose that $f$ is strongly Euler-homogenous, Saito-holonomic, and tame, $F$ corresponds to any factorization $f = f_1 \cdots f_r$, and $f_1^{a_1} \cdots f_r^{a_r}$ is not a unit. Then $B^n_{F,F}$ is principal.

This follows from the much more general statement Theorem 3.3 about $(n + 1)$-pure (Definition 2.1), relative holonomic (Definition 2.13) $\mathcal{D}_{X,S}[S]$-modules, that we spotlight since it is of general interest:
Theorem 1.2. Suppose that $M$ is a finite $\mathcal{D}_{X,\mathfrak{x}}[S]$-module that is relative holonomic and $(n+1)$-pure. Then its Bernstein–Sato ideal $B_M$, i.e. its $\mathbb{C}[S]$-annihilator, is principal.

The hurdle becomes verifying that, under the hypotheses of Theorem 1.1, $\mathcal{D}_{X,\mathfrak{x}}[S]F_S/\mathcal{D}_{X,\mathfrak{x}}[S]F_S+a$ satisfies the subtle homological property of purity. In Theorem 3.4 we prove something much stronger: it has only one nonvanishing dual Ext-module, i.e. it is $(n+1)$-Cohen–Macaulay (Definition 2.4).

In Lemma 4.20 of [7], Budur shows that information about the multivariate Bernstein–Sato ideal of a finer factorization of $f$ gives (potentially incomplete) data about the multivariate Bernstein–Sato ideal of a coarser factorization of $f$. Let $B_{f,\mathfrak{x}} \subseteq \mathbb{C}[s]$ be the ideal generated by the usual Bernstein–Sato polynomial of $f$ at $\mathfrak{x}$. Under the hypotheses of Theorem 1.1 we obtain in Theorem 3.15 a strengthening of Lemma 4.20 in loc. cit. While this result works for general $a$ and various factorizations, here is a special case:

Theorem 1.3. Suppose that $f$ is strongly Euler-homogenous, Saito-holonomic, and tame and $F$ corresponds to any factorization $f = f_1 \cdots f_r$. Then

$$Z(B_{f,\mathfrak{x}}) = Z(B_{F,\mathfrak{x}}^f) \cap \{s_1 = \cdots = s_r\}.$$  

Here points on the diagonal of $\mathbb{C}^r$ are naturally identified with points in $\mathbb{C}$.

This has many applications for arrangements. In Corollary 3.17 we finish the computation of $Z(B_{f,0}^1)$ and $Z(B_{f,0})$ for $f$ a central, possibly non-reduced, free arrangement and $F$ an arbitrary factorization by covering the subset of non-reduced cases not included in Theorem 1.4 of [3]. This finishes our certification, started in loc. cit., that the roots of the Bernstein–Sato polynomial of a free hyperplane are always combinatorial, regardless of any reducedness assumption. In Example 3.18 we extend Walther’s result that the b-function of a hyperplane arrangement is not necessarily combinatorial to the multivariate case. In Corollary 3.21 we extend Saito’s bounds on $Z(B_{f,0})$ from Theorem 1 of [22] to the tame, multivariate setting. The later bounds are somewhat precise since the hyperplanes in $Z(B_{f,0}) \cap [-1,0)$ are combinatorially determined by Theorem 4.11, Theorem 1.3 of [3].

As this paper was being completed, Wu proved, using different methods, the same diagonal result from Theorem 1.3 under more general hypotheses in Theorem 1.1 of [25]. While ours hypotheses are stronger, they are more geometric and verifiable; moreover, that ours assumptions imply his is not at all obvious, cf. Theorem 3.4. Using this diagonalization, he considered central, reduced, free arrangements and recovered in Theorem 1.2 of [25] a special case of our formula for $Z(B_{f,0})$ that originally appeared in Theorem 1.4 of [3].

Additionally, essentially because hyperplane arrangements have a factorization $F$ into linear forms, the Bernstein–Sato ideal attached to this special factorization has a particularly good property:

Theorem 1.4. Suppose that $f$ is a tame, possibly non-reduced hyperplane arrangement, $F$ corresponds to a factorization of $f$ into linear forms, and $f_1^{a_1} \cdots f_r^{a_r}$ is not a unit. Then $B_{F}^a = \text{rad}(B_{f}^a)$. 

Finally we consider central, reduced generic arrangements $f$ of $d$ hyperplanes in $\mathbb{C}^n$. In [23], Walther obtained a formula for the Bernstein–Sato polynomial for $f$. For $F$ a factorization into linear forms, Maisonobe found in [15] an element of the global Bernstein–Sato ideal $B^1_F$ for $d > n + 1$ and computed this ideal for $d = n + 1$. However, completely computing this ideal for $d > n + 1$ remained open. Using Walther’s formula, in Theorem 3.23, we consider any $d > n$, independently verify Maisonobe’s analysis of the $d = n + 1$ case, and explicitly determine this ideal for any $d > n$:

**Theorem 1.5.** Let $f$ be the reduced defining equation of a central, generic hyperplane arrangement in $\mathbb{C}^n$ with $d = \deg f > n$. If $F$ corresponds to a factorization of $f$ into irreducibles, then

\begin{equation}
B^1_F = \mathbb{C}[S] \cdot \prod_{k=1}^{d} (s_k + 1) \prod_{i=0}^{2d-n-2} \left( \sum_{k=1}^{d} s_k + i + n \right).
\end{equation}

If $F = (f_1, \ldots, f_r)$ corresponds to some other factorization, then $B^1_F$ is principal and

\begin{equation}
Z(B^1_F) = \left( \bigcup_{k=1}^{r} \{ s_k + 1 = 0 \} \right) \bigcup \left( \bigcup_{i=0}^{r} \left( \sum_{k=1}^{d} d_k s_k + i + n = 0 \right) \right).
\end{equation}

In Appendix A we document some basic properties of the logarithmic differential forms for non-reduced divisors. This is our attempt to fill an apparent gap in the literature, as previous texts seem to focus only on the reduced case.

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2. Preliminaries

Here we catalogue the various facts needed to prove the paper’s first set of results. The three subsections are divided up as follows: first, we collect basic lemmas about Auslander regular rings from [4, 9]; second, we review two different filtrations on $D_X[S]$ as well as a fundamental structure theorem from [9, 10]; third, we introduce the geometric hypotheses on $\text{Div}(f)$ we assume later in the paper.

2.1. Lemmas about Auslander Regular Rings.

We consider (noncommutative) rings $A$ that are positively filtered such that $\text{gr} A$ (the associated graded ring induced by the positive filtration) is a commutative, Auslander regular, Noetherian ring. By A.III.1.26 and A.IV.4.15 of [11] this implies $A$ is Zariskian filtered and Auslander regular. Let $M$ be a nonzero finitely generated left $A$-module. In this subsection we are mostly interested in the $A$-module $\text{Ext}^k_A(M, A)$ as well as the interplay between $M$ and $\text{gr} M$ (for a choice of good filtration on $M$). Note that because $A$ is Zariskian filtered, if $M$ is nonzero then $\text{gr} M$ is also nonzero (for any good filtration on $M$).

Before embarking let us emphasize the utility: both the total order filtration or relative order filtration of the next subsection are positive filtrations on $D_X[S]$ whose associated graded objects are commutative, Auslander regular, and Noetherian. So the facts below are germane.
Definition 2.1. For $A$ an Auslander regular ring and $0 \neq M$ a finitely generated $A$-module, the grade of $M$ is the smallest integer $k$ such that $\text{Ext}_A^k(M,A) \neq 0$. We say $M$ is pure of grade $j(M)$ if every nonzero submodule of $M$ has grade $j(M)$.

The grade can be computed on the associated graded side:

Proposition 2.2. (A.IV.4.15 [4]) Let $A$ be a positively filtered ring such that $\text{gr} A$ is a commutative, Noetherian, and Auslander regular ring and further assume that $0 \neq M$ a finitely generated left $A$-module. Then, for any good filtration on $M$, $j(M) = j(\text{gr} M)$.

The following lemma gives criteria to check purity:

Lemma 2.3. (A.IV.2.6, A.IV.4.11 [4]) For $A$ an Auslander regular ring and $0 \neq M$ a finitely generated left $A$-module, then

(i) $\text{Ext}_A^{j(M)}(M,A)$ is a pure $A$-module of grade $j(M)$;

(ii) $M$ is a pure $A$-module of grade $j(M)$ if and only if $\text{Ext}_A^k(\text{Ext}_A^k(M,A) = 0$ for all $k \neq j(M)$.

(iii) Suppose further that $A$ is positively filtered such that $\text{gr} A$ is commutative, Noetherian, and Auslander regular. If $M$ is a pure $A$-module, then there exists a good filtration on $M$ such that $\text{gr} M$ is a pure $\text{gr} A$-module of grade $j(M)$.

In [9] the authors introduce a noncommutative generalization of Cohen–Macaulay modules which will prove very useful for us throughout this paper.

Definition 2.4. For $A$ an Auslander regular ring and $0 \neq M$ a finitely generated left $A$-module, we say $M$ is $j$-Cohen–Macaulay if

$$\text{Ext}_A^k(M,A) = 0$$

precisely when $k \neq j$.

Purity and being $j$-Cohen–Macaulay are related by the following:

Proposition 2.5. For $A$ an Auslander regular ring and $0 \neq M$ a finitely generated left $A$-module, if $M$ is $j$-Cohen–Macaulay then $M$ is pure of grade $j$.

Proof. This is immediate by (i) and (ii) of Lemma 2.3. 

A crucial technique of this paper will be to transfer Cohen–Macaulay properties of $\text{gr} M$ to Cohen–Macaulay properties of $M$. The next lemma will be helpful.

Lemma 2.6. (A.IV.4.5 [4]) Let $A$ be a positively filtered ring and $0 \neq M$ a finitely generated left $A$-module. Then for a good filtration on $M$, there is a subquotient of $\text{Ext}_A^k(\text{gr} A, \text{gr} M, \text{gr} A)$.

In certain commutative settings, grade of $M$ and the purity of $M$ have geometric interpretations:

Proposition 2.7. (Proposition 4.5.1 [9]) Suppose $A$ is a commutative, regular, Noetherian domain and $0 \neq M$ a finitely generated $A$-module. Then

1. If $\dim A_m$ is the same for all $m \in \text{mSpec} A$, then $j(M) + \dim M = \dim A$.
2. $M$ is a pure $A$-module if and only if every associated prime of $M$ is minimal and $\dim A_p = j(M)$ for all minimal primes $p$ of $M$.

Finally, in our setting there is a notion of characteristic variety (defined entirely similarly to the $D_X$-module version) that is integral to this paper.
Definition 2.8. Let $A$ be a positively filtered, Noetherian ring such that $\text{gr}A$ is a commutative regular Noetherian ring. Let $0 \neq M$ be a finitely generated left $A$-module and $\Gamma$ a good filtration on $M$. Define the characteristic variety $\text{Ch}(M)$ of $M$ by

$$\text{Ch}(M) := V(\text{rad}(\text{ann}_{\text{gr}A} \text{gr}M))$$

where $\text{gr}M$ is defined with respect to $\Gamma$. In fact, $\text{Ch}(M)$ does not depend on the choice of good filtration of $M$.

2.2. Introduction to $\mathcal{D}_X[S]F^S/\mathcal{D}_X[S]F^{S+a}$.

Here we introduce two filtrations on $\mathcal{D}_X[S]$ that both extend the standard order filtration on $\mathcal{D}_X[S]$. We also recall some of the fundamental results about $\mathcal{D}_X[S]F^S/\mathcal{D}_X[S]F^{S+a}$ proved in [10] (see also [13]).

Definition 2.9. The total order filtration on $\mathcal{D}_X[S]$ is defined by extending the order filtration on $\mathcal{D}_X$ to $\mathcal{D}_X[S]$ by giving each $s_k$ weight one. More precisely, in local coordinates $(x, \partial)$ near $\frak x$ we can write any element of $\mathcal{D}_X[S]$ as $\sum_{v,w} g_{v,w} \partial^v s^w$ where $g_{v,w} \in \mathcal{O}_{x,v}$, $v \in \mathbb{Z}_{\geq 0}^n$, $w \in \mathbb{Z}_{\geq 0}^r$, $\partial^v = \partial_1^{v_1} \cdots \partial_n^{v_n}$, and $s^w = s_1^{w_1} \cdots s_r^{w_r}$. The elements of order at most $k$ are those that admit an expression above satisfying $|v| + |w| = v_1 + \cdots + v_n + w_1 + \cdots + w_r \leq k$ for each summand. The associated graded ring is $\text{gr}^t(\mathcal{D}_X[S])$.

Definition 2.10. The relative order filtration on $\mathcal{D}_X[S]$ is defined by extending the order filtration on $\mathcal{D}_X$ to $\mathcal{D}_X[S]$ by giving each $s_k$ weight zero. Using the expression defined above, elements of order at most $k$ satisfy $|v| \leq k$. The associated graded ring is $\text{gr}^{rel}(\mathcal{D}_X[S])$.

Under either the total order filtration or the relative order filtration, the associated graded rings satisfy the following isomorphisms locally:

$$\text{gr}^t(\mathcal{D}_X[S]) \simeq \mathcal{O}_{x,v}[y_1, \ldots, y_n][s_1, \ldots, s_r] \simeq \text{gr}^{rel}(\mathcal{D}_X[S]).$$

Here, $y_i$ corresponds to the principal symbol of $\partial_i$ (after picking coordinates) under the appropriate grading. Since either the relative or total order filtration is a positive filtration, this identification means we may apply the techniques from the previous subsection to $\mathcal{D}_X[S]$. (See [3] A.IV.3.6.) Now let us solidify some notation related to $\mathcal{D}_X[S]F^S$:

Convention 2.11. Given a factorization $F = (f_1, \ldots, f_r)$ of $f$ and $a \in \mathbb{N}^r$, we denote by $f^a$ the product $f_1^{a_1} \cdots f_r^{a_r}$. The multivariate Bernstein–Sato ideal of $F$ associated to $a$ at $\frak x$ is denoted by $B^{a}_{F,\frak x}$; the ideal generated by the Bernstein–Sato polynomial (or b-function) of $f$ associated to $a \in \mathbb{N}$ at $\frak x$ is $B^a_f$. When $a = 1$ we just write $B_{f,\frak x}$. When $F$ (or $f$) is global algebraic, we reserve $B^a_F$ and $B^a_f$ for the global algebraic versions of these objects, see subsection 3.3. By “multivariate” we mean $r > 1$; by “univariate” we mean $r = 1$. The reader should note that in [2][3] we used $\text{gr}_{(0,1,1)}(-)$ and $\text{gr}_{(0,1,0)}(-)$ instead of $\text{gr}^t(-)$ and $\text{gr}^{rel}(-)$ for the associated graded objects attached to the total order and relative order filtrations. In those papers we use $\text{V}(-)$ with respect to Bernstein–Sato ideals in the same way as we use $\text{Z}(-)$ here: to denote the zero locus. In this paper we use $\text{V}(-)$ for elements of $\mathcal{O}_X$ or when the input is reduced.

Definition 2.12. Consider a finitely generated $\mathcal{D}_X[S]$-module $M$. We denote its characteristic variety, cf. Definition 2.8, with respect to the relative order filtration by $\text{Ch}^{rel}(M)$; with respect to the total order filtration by $\text{Ch}^t(M)$. 
In [14], Maisonobe introduce a criterion on a $\mathcal{D}_X[S]$-module he calls majoré par une lagrangienne. This guarantees the relative characteristic variety has a very nice product structure. We adopt the verbiage of [11] where this is generalized slightly:

**Definition 2.13.** (Definition 3.2.3, [9]) A finitely generated $\mathcal{D}_X[S]$-module $M$ is relative holonomic if its relative characteristic variety satisfies

$$\text{Ch}^{\text{rel}}(M) = \bigcup_{\alpha} \Lambda_{\alpha} \times S_\alpha \subseteq T^*X \times \mathbb{C}^r,$$

where the $\Lambda_{\alpha}$ are irreducible, conical Lagrangian varieties in the cotangent bundle $T^*X$ and the $S_\alpha$ are irreducible algebraic varieties in $\mathbb{C}^r$. We denote by $p_2(\text{Ch}^{\text{rel}}(M))$ the image of the relative characteristic variety under the canonical projection $p_2: T^*X \times \mathbb{C}^r \to \mathbb{C}^r$.

Maisonobe demonstrated in [14] that $\mathcal{D}_{X,f}[S]F^S/\mathcal{D}_{X,f}[S]F^{S+1}$ has similar properties as in classical $F = (f)$ setting, cf. Resultat 2, Resultat 3, and Proposition 9 of loc. cit. Most of these properties were generalized in [9, 10]. We summarize the ones we need below. Note that (iv) requires interpreting $B_{f,x}^\alpha$ as the $\mathbb{C}[S]$-annihilator of $\mathcal{D}_{X,f}[S]F^S/\mathcal{D}_{X,f}[S]F^{S+\alpha}$.

**Theorem 2.14.** (Theorem 3.2.1 [10]; Lemma 3.4.1 [9]) Suppose that $f^a$ is not a unit. Then the following are true for the $\mathcal{D}_{X,f}[S]$-module $\mathcal{D}_{X,f}[S]F^S/\mathcal{D}_{X,f}[S]F^{S+a}$:

(i) it is relative holonomic;
(ii) it has grade $n + 1$;
(iii) $\dim \text{Ch}^{\text{rel}}(\mathcal{D}_{X,f}[S]F^S/\mathcal{D}_{X,f}[S]F^{S+1}) = n + r - 1$;
(iv) $Z(B_{f,x}^\alpha) = p_2(\text{Ch}^{\text{rel}}(\mathcal{D}_{X,f}[S]F^S/\mathcal{D}_{X,f}[S]F^{S+a}))$.

### 2.3. Hypotheses on the Logarithmic Data of $\text{Div}(f)$.

Here we introduce geometric conditions on the $\text{Div}(f)$ that were first considered in [24] and later in [2]. All of them involve the logarithmic information of $f$ and do not depend on the choice of defining equation of $f$. We generally do not assume $\text{Div}(f)$ is reduced and, as such (and following [24]), differentiate between the logarithmic data of $\text{Div}(f)$ and of $\text{Div}(f_{\text{red}})$. Most of our hypotheses hold irrespective of a reduced assumption, cf. Proposition A.6 Corollary A.7 as well as Appendix A at large. Nevertheless one should take heed of the terminal item of Remark 2.17.

**Definition 2.15.** Let $\mathcal{J}_{\text{Div}(f)} \subseteq \mathcal{O}_X$ be the ideal sheaf of $\text{Div}(f)$ and $\text{Der}_X$ the sheaf of derivations on $X$. The logarithmic derivations are the subsheaf $\text{Der}_X(-\log f)$ of $\text{Der}_X$ locally generated by the derivations $\delta$ such that $\delta \bullet (\mathcal{J}_{\text{Div}(f)}) \subseteq \mathcal{J}_{\text{Div}(f)}$.

Straightforward computations with the product rule imply that $\text{Der}_X(-\log f) = \text{Der}_X(-\log f_{\text{red}})$ and that the logarithmic derivations do not depend on the choice of defining equation. However, direct sum decompositions of the logarithmic derivations ala Remark 2.10 of [24] may depend on reducedness and/or the choice of defining equation. The following hypotheses helps to alleviate some of the issues involving choice of equation, cf. Remark 3.2 of loc. cit. and Remark 2.15 of [2].

**Definition 2.16.** We say $f$ is strongly Euler-homogeneous at $\mathfrak{r}$ if there is a derivation $E_{\mathfrak{r}} \in \text{Der}_X(-\log f)$ such that (1) $E_{\mathfrak{r}} \bullet f = f$ and (2) $E_{\mathfrak{r}}$ vanishes at $\mathfrak{r}$. We say $f$ is strongly Euler-homogeneous if it is strongly Euler-homogeneous for all $\mathfrak{r} \in V(f)$.
Remark 2.17. (a) The condition “strongly Euler-homogeneous” does not depend on the choice of local defining equation of \( \text{Div}(f) \): if \( E \cdot f = f \), then \( uE \cdot uf = uf \). If you remove condition (2), the choice may matter.

(b) Hyperplane arrangements are strongly Euler-homogeneous. At the origin, the Euler derivation \( \sum x_i \partial_i \) is a strong Euler-homogeneity; an appropriate coordinate change deals with the other points. Divisors that locally everywhere admit a choice of coordinates so that they can be defined by a “homogeneous polynomial” (with respect to an appropriate weight system) are known as quasi-homogeneous divisors and are also strongly Euler-homogeneous.

(c) If \( m \in \mathbb{Z}_{\geq 1} \), then \( f \) is strongly Euler-homogeneous at \( x \) if and only if \( f^m \) is strongly Euler-homogeneous at \( x \).

(d) Whether or not “\( f^a \) is strongly Euler-homogeneous at \( x \)” implies “\( f^b \) is strongly Euler-homogeneous at \( x \)” for arbitrary \( a, b \in \mathbb{Z}_{\geq 1} \) seems quite subtle. It is not known to us, even for quasi-homogeneous divisors. When \( f \) is a hyperplane arrangement this is true: up to coordinate change and a scaling, the Euler derivation is the requisite homogeneity. Essentially, this is because all the linear factors are homogeneous with respect to the same weight system. However, it seems possible that \( f \) may be quasi-homogeneous without all of its factors \( f_k \) being quasi-homogeneous; or, each \( f_k \) may be quasi-homogeneous with respect to different weight systems that somehow assemble into a quasi-homogeneous \( f \) with respect to a new weight system. Geometrically one might expect the aforementioned implication to hold at least for quasi-homogeneous \( f \) since it can be visualized as a \( \mathbb{C}^* \)-equivariant action on \( V(f) \).

In [21], Saito used the logarithmic derivations to stratify \( X \). Indeed, define an equivalence relation on \( X \) as follows: two points \( x \) and \( y \) are related if there is an open set \( U \subseteq X \) and a logarithmic derivation \( \delta \in \text{Der}_U(- \log f) \) that (1) does not vanish on \( U \) and (2) the integral curve of \( \delta \) passes through \( x \) and \( y \). The resulting stratification is the logarithmic stratification and we call the strata the logarithmic strata.

Definition 2.18. We say \( f \) is Saito-holonomic if the logarithmic stratification of \( X \) by \( \text{Der}_X(- \log f) \) is locally finite.

Remark 2.19. (a) As \( \text{Der}_X(- \log f) = \text{Der}_X(- \log f_{\text{red}}) \), the logarithmic stratification depends only on the reduced structure of \( f \).

(b) The \( n \)-dimensional logarithmic strata are the connected components of \( X \setminus V(f) \) and the \( (n-1) \)-dimensional strata are the smooth connected components of \( V(f_{\text{red}}) \).

(c) Hyperplane arrangements are Saito-holonomic, cf. Example 3.14 of [21].

(d) Consider \( f = z(x^5 + y^3 + zx^2y^3) \) from Section 4.2 of [11]. This is not Saito-holonomic: every point on the \( z \)-axis is a zero dimensional logarithmic stratum.

In [21], Saito also introduced (originally under a reducedness hypothesis) a sort of dual object to the logarithmic derivations:

Definition 2.20. Consider the classical sheaf of differential forms on \( X \), \( \Omega_X^* \), whose differential \( d \) is exterior differentiation. The sub-sheaf of logarithmic \( k \)-forms \( \Omega_X^k(\log f) \) satisfy

\[
\Omega_X^k(\log f) := \{ \omega \in \frac{1}{f} \Omega_X^k \mid d(\omega) \in \frac{1}{f} \Omega_X^{k+1} \}.
\]
We say $f$ is tame at $\mathfrak{r}$ if the projective dimension of the $\mathcal{O}_{X,\mathfrak{r}}$-module $\Omega^1_{X,\mathfrak{r}}(\log f)$ is at most $k$ for all $0 \leq k \leq n$; moreover, $f$ is simply tame if it so at each $\mathfrak{r} \in X$.

We say $f$ is free at $\mathfrak{r}$ if $\Omega^1_{X,\mathfrak{r}}(\log f)$ is a free $\mathcal{O}_{X,\mathfrak{r}}$-module; $f$ is simply free if it is so at each $\mathfrak{r} \in X$.

While Saito (and others) originally considered the logarithmic $k$-forms, freeness, and tameness only for reduced divisors, we do not make this restriction. To our knowledge, [23] is one of the first places where these objects were defined in the non-reduced setting. In Appendix A we catalogue the basic properties of $\Omega^1_{X,\mathfrak{r}}(\log f)$ for non-reduced $f$ and show that, up to $\mathcal{O}_X$-module isomorphisms, the non-reduced logarithmic $k$-forms can be identified with the reduced logarithmic $k$-forms, cf. Proposition A.6.

**Remark 2.21.** (a) The condition of freeness is stronger than tameness. By 1.7, 1.8 of [21], if $f_{\text{red}}$ is free, then $\Omega^1_{X,\mathfrak{r}}(\log f_{\text{red}}) \simeq \bigwedge^k \Omega^1_{X,\mathfrak{r}}(\log f_{\text{red}})$. (Saito’s originally argument assumes reducedness, but his argument for this item does not depend on this point. Also see item (c).)

(b) If $\dim X \leq 3$ then any $f \in \mathcal{O}_X$ is automatically tame. If $\dim X = 4$ then tameness is equivalent to $\text{proj dim } \Omega^1_{X,\mathfrak{r}}(\log f) \leq 1$. These facts follow from reflexivity of the logarithmic differential forms, cf. 1.7 of [21] for one case. Tameness and freeness only depend on the reduced structure of the divisor cut out by $f$, cf. Corollary [A.7]

### 3. Results on Multivariate Bernstein–Sato Ideals

We will now assume $\text{Div}(f)$ is strongly Euler-homogeneous, Saito-holonomic and tame. In this section we prove the first three theorems from the Introduction.

The general strategy is to pass the properties proved for $\text{gr}^r(\mathcal{D}_{X,S}[S] F^S)$ in [2] to $\mathcal{D}_{X,S}[S] F^S / \mathcal{D}_{X,S}[S] F^{S+a}$ and then to use the definition of relative holonomic and Theorem [2.14] to pass these properties to $\text{gr}^\text{rel}(\mathcal{D}_{X,S}[S] F^S / \mathcal{D}_{X,S}[S] F^{S+a})$.

#### 3.1. Criteria for $\mathcal{D}_{X,S}[S] F^S / \mathcal{D}_{X,S}[S] F^{S+a}$ to be $(n+1)$-Cohen–Macaulay.

In general, elements of $\mathcal{D}_{X,S}[S]$ that annihilate $F^S$ can be very complicated. The only such elements that are easy to identify are essentially encoded by the logarithmic derivations:

**Definition 3.1.** There is an $\mathcal{O}_X[S]$-linear map $\psi_F : \text{Der}_X(- \log f) \to \text{ann}_{\mathcal{D}_X[S]} F^S$

$$\psi_F(\delta) = \delta - \sum s_k \frac{\delta \cdot f_k}{f_k}.$$ We say $\text{ann}_{\mathcal{D}_X[S]} F^S$ is generated by derivations if

$$\text{ann}_{\mathcal{D}_X[S]} F^S = \mathcal{D}_X[S] \cdot \psi_F(\text{Der}_X(- \log f)).$$

By Theorem 2.29 of [2], if $f$ is strongly Euler-homogeneous, Saito-holonomic, and tame, then $\text{ann}_{\mathcal{D}_{X,S}[S]} F^S$ is generated by derivations. Moreover, by loc. cit., the associated graded object attached to the total order filtration has good properties:

**Theorem 3.2.** (Theorem 2.23, Corollary 2.28, Theorem 2.29 [2]) Suppose that $f$ is strongly Euler-homogenous, Saito-holonomic, and tame and $F$ corresponds to any factorization $f = f_1 \cdots f_r$. Then $\text{ann}_{\mathcal{D}_X[S]} F^S$ is generated by derivations and $\text{gr}^r(\text{ann}_{\mathcal{D}_X[S]} F^S)$ is locally a Cohen–Macaulay, prime ideal of dimension $n + r$. 


The Cohen–Macaulay properties for \( \text{gr}^2(\mathcal{D}_X[S]F^S) \) descend to \( \mathcal{D}_X[S]F^S \):

**Proposition 3.3.** Suppose that \( f \) is strongly Euler-homogenous, Saito-holonomic, and tame and \( F \) corresponds to any factorization \( f = f_1 \cdots f_r \). Then \( \mathcal{D}_X[S]F^S \) is \( n \)-Cohen–Macaulay.

**Proof.** We first show that the hypotheses imply \( \text{gr}^2(\mathcal{D}_X[S]) / \text{gr}^2(\text{ann}_{\mathcal{D}_X[S]}F^S) \) is a \( n \)-Cohen–Macaulay \( \text{gr}^2(\mathcal{D}_X[S]) \)-module. Since this module is a graded local \( \text{gr}^2(\mathcal{D}_X[S]) \)-module (with respect to the grading induced by the total order filtration), the vanishing of Ext-modules is a graded local condition. (See Proposition 1.5.15(c) of [6] where faithful exactness of localizations at graded maximal ideals is proved.) Thus, it suffices to show \( \text{gr}^2(\mathcal{D}_X[S]) / \text{gr}^2(\text{ann}_{\mathcal{D}_X[S]}F^S) \) is a \( n \)-Cohen–Macaulay module after localizing at the graded maximal ideal. This follows from Theorem 3.2 and a routine commutative algebra argument we summarize below.

Suppose \( R \) is a commutative, Cohen–Macaulay, Noetherian, local ring and \( M \) a finitely generated Cohen–Macaulay \( R \)-module of dimension \( \dim M \). We claim \( M \) is \( (\text{codim } M) \)-Cohen–Macaulay. By Auslander–Buchsbaum and since \( M \) is Cohen–Macaulay, we know that \( \dim M = \text{depth } M = \text{depth } R - \text{proj dim } M \). So \( \text{proj dim } M = \text{codim } M \). Since \( R \) is local, Cohen–Macaulay and \( M \) is finitely generated, height \( \text{ann } M + \dim M = \dim R \). Thus \( \text{proj dim } M = \text{height } \text{ann } M = \text{codim } M \).

Let \( j(M) \) be the grade of \( M \). Then to show \( M \) is \( (\text{codim } M) \)-Cohen–Macaulay it suffices to show \( j(M) = \text{proj dim } M \). Certainly \( j(M) \leq \text{proj dim } M \). On the other hand, let \( \ell \) be the length of a maximal regular \( R \)-sequence in \( \text{ann } M \). Then \( 0 \to R \to R \to R/R \cdot x \to 0 \) induces the short exact sequence \( 0 \to \text{Ext}^{\ell - 1}_R(M, R) \to \text{Ext}^{\ell - 1}_R(M, R/R \cdot x) \to \text{Ext}^\ell_R(M, R) \to 0 \). So if \( \text{Ext}^\ell_R(M, R) \neq 0 \), then \( \text{Ext}^{\ell - 1}_R(M, R/R \cdot x) \neq 0 \). Iterating this procedure, we see that \( j(M) \geq \ell \). Since \( R \) is local and Cohen–Macaulay, \( \ell = \text{height } \text{ann } M \) and the claim is proved.

So the above argument and Theorem 3.2 imply \( \text{gr}^2(\mathcal{D}_X[S]) / \text{gr}^2(\text{ann}_{\mathcal{D}_X[S]}F^S) \) is \( n \)-Cohen–Macaulay. Since \( \mathcal{D}_X[S]F^S \) is cyclically generated by \( F^S \), we have the obvious isomorphism \( \text{gr}^2(\mathcal{D}_X[S]F^S) \simeq \text{gr}^2(\mathcal{D}_X[S]) / \text{gr}^2(\text{ann}_{\mathcal{D}_X[S]}F^S) \). Recall that we can apply the lemmas in subsection 2.1 to \( \mathcal{D}_X[S] \) and the total order filtration. By Lemma 2.6 we see that there, for each \( k \), there is a good filtration \( \Gamma \) on \( \text{Ext}^k_{\mathcal{D}_X[S]}(\mathcal{D}_X[S]F^S, \mathcal{D}_X[S]) \) such that

\[
\text{gr}^2(\text{Ext}^k_{\mathcal{D}_X[S]}(\mathcal{D}_X[S]F^S, \mathcal{D}_X[S]))
\]

is a subquotient of

\[
\text{Ext}^k_{\mathcal{D}_X[S]}(\text{gr}^2(\mathcal{D}_X[S]F^S), \text{gr}^2(\mathcal{D}_X[S])).
\]

Since \( \mathcal{D}_X[S] \) is positively filtered by the total order filtration, for any good filtration on a finitely generated, left \( \mathcal{D}_X[S] \)-module \( M \), we know that if \( M \) is nonzero then \( \text{gr} M \) is nonzero. Therefore

\[
\text{Ext}^k_{\mathcal{D}_X[S]}(\mathcal{D}_X[S]F^S, \mathcal{D}_X[S]) = 0 \text{ for } k \neq n.
\]

That

\[
\text{Ext}^n_{\mathcal{D}_X[S]}(\mathcal{D}_X[S]F^S, \mathcal{D}_X[S]) \neq 0
\]

follows from, for instance, Proposition 3.2 and our computation of the grade of \( \text{gr}^2(\mathcal{D}_X[S]F^S) \) above. \( \square \)

The Cohen–Macaulay property descends further to \( \mathcal{D}_X[S]F^S / \mathcal{D}_X[S]F^{S+n} \).
This module is

Theorem 2.14, the grade of $B$

Note that, as one would hope,

Definition 3.5.

Applications to the Bernstein–Sato Ideal.

3.2. Applications to the Bernstein–Sato Ideal.

It behooves us to study a larger class of $\mathbb{C}[S]$-annihilators of finite $\mathcal{O}_{X,F}$-modules.

Definition 3.5. For a finite $\mathcal{O}_{X,F}$-module $M$, its Bernstein–Sato ideal is

$$B_M = \text{ann}_{\mathbb{C}[S]} M.$$  

Note that, as one would hope, $B_{F_x}^a$ is shorthand for the Bernstein–Sato ideal of $\mathcal{O}_{X,F}[S]/\mathcal{O}_{X,F}[S]^{S+a}$.

While our primary interest is $\mathcal{O}_{X,F}[S]/\mathcal{O}_{X,F}[S]^{S+a}$, the crucial property Theorem 2.14(iv) about Bernstein–Sato ideals applies more generally: if $M$ is relative holonomic, and with $p_2 : T^*X \times \text{Spec } \mathbb{C}[S] \to \text{Spec } \mathbb{C}[S]$ the canonical projection, then $Z(B_M) = p_2(\text{Ch}_{\text{rel}} M)$, cf. Lemma 3.4.1 of [9]. Also note that if $N \subseteq M$ are finite $\mathcal{O}_{X,F}[S]$-modules such that $M$ is relative holonomic, then relative holonomicity descends to $N$, cf. Lemma 3.2.2 and Proposition 3.2.5 of [9]. (The essence of the latter proposition originally appears in [14] using Maisonobe’s terminology majoré par une lagrangienne).

One difficulty is that the first item of Proposition 2.7 does not apply for a polynomial ring over the stalk of the analytic structure sheaf $\mathcal{O}_{X,F}$; there are maximal ideals of different heights. Nevertheless, Maisonobe showed the desired form of
Proposition 2.7(1) still does hold. We state a useful version combining \( \mathcal{D}_{X,F}[S] \)-data and associated graded data—recall grade can be computed on either side, cf. Proposition 2.2. Namely: if \( M \) is a finite \( \mathcal{D}_{X,F}[S] \)-module then

\[
(3.1) \quad j(M) + \dim(\text{Ch}^{\text{rel}} M) = 2n + r.
\]

(Recall that \( \mathcal{D}_{X,F}[S] = \mathcal{D}_{X,F}[s_1, \ldots, s_r]. \)) Maisonobe’s reasoning is succinctly described in more generality, and in English, in section 3.6 of [3] where this formula, as well as the philosophy for turning algebraic results into local analytic ones, is detailed.

Armed with these facts we can give criterion for a Bernstein–Sato ideal to be principal. The proof non-trivially builds on (and reveals the versatility of) an idea of Maisonobe’s originating in Theorem 2 of [16], that we also used in Proposition 3.13 of [3].

**Theorem 3.6.** Suppose that \( M \) is a finite \( \mathcal{D}_{X,F}[S] \)-module that is relative holonomic and \((n+1)\)-pure. Then its Bernstein–Sato ideal \( B_M \) is principal.

**Proof.** First we prove that \( Z(B_M) \) is purely codimension one. For this argument we only consider the relative order filtration on \( \mathcal{D}_{X,F}[S] \). By Lemma 2.3 there is a good filtration on \( M \) such that the associated graded module is a pure \( \text{gr}^{\text{rel}}(\mathcal{D}_{X,F}[S]) \)-module of grade \( n+1 \). Since the relative characteristic variety of any finitely generated \( \mathcal{D}_{X,F}[S] \) module does not depend on the choice of good filtration, Proposition 2.7(2) and equation (3.1) imply \( \text{Ch}^{\text{rel}}(\mathcal{D}_{X,F}[S]F^S/\mathcal{D}_{X,F}[S]F^{S+1}) \) is equidimensional of dimension \( n + r - 1 \). By assumption \( M \) is relative holonomic, so in the product structure of Definition 2.17 all the irreducible varieties \( S_\alpha \subseteq C' \) have dimension \( r - 1 \). And by the properties of the projection \( p_2 : T^*X \times \text{Spec} \mathbb{C}[S] \rightarrow \text{Spec} \mathbb{C}[S] \) outlined above (compare to Theorem 2.14)

\[
Z(B_M) = p_2(\text{Ch}^{\text{rel}}(B_M)) = \bigcup_\alpha S_\alpha.
\]

So \( Z(B_M) \) is purely codimension one.

Now we show the principality of \( \text{rad}(B_M) \) implies the principality of \( B_M \). Pick a generating set \( (b_1, \ldots, b_t) \) of \( B_M^{\text{rel}} \). As \( \text{rad}(B_{F,F}) \) is principal, we may write each \( b_i = \beta_i \alpha_i \) so that (1) \( \text{rad}(\mathbb{C}[S] \cdot \beta_i) = \text{rad}(B_{F,F}^{\text{rel}}) \) and (2) \( Z(\alpha_i) \cap Z(B_{F,F}^{\text{rel}}) \) has codimension at least 2. (This is possible since \( \mathbb{C}[S] \) is a UFD: factor \( b_i \) into irreducibles and group all the irreducible factors (and their powers) into those whose reduced form divide \( \text{rad}(B_M) \) and those who do not: the former constitute \( \beta_i \); the latter \( \alpha_i \).)

Now consider

\[
N_i = \beta_i \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+\alpha}} \subseteq \frac{\mathcal{D}_{X,F}[S]F^S}{\mathcal{D}_{X,F}[S]F^{S+\alpha}}.
\]

As \( b_i \in B_{F,F}^{\text{rel}} \), certainly \( \alpha_i \) kills \( N_i \). On the other hand, since \( N_i \) is a submodule of \( \mathcal{D}_{X,F}[S]F^S/\mathcal{D}_{X,F}[S]F^{S+\alpha} \), we also have the containment \( Z(\text{ann}_{\mathbb{C}[S]} N_i) \subseteq Z(B_{F,F}) \). Together, \( Z(\text{ann}_{\mathbb{C}[S]} N_i) \subseteq Z(B_{F,F}) \cap Z(\alpha_i) \), which has codimension at least 2. But \( N_i \) inherits \((n+1)\)-purity from \( \mathcal{D}_{X,F}[S]F^S/\mathcal{D}_{X,F}[S]F^{S+\alpha} \), which, from the prelude, implies that: if \( N_i \) is nonzero, \( Z(\text{ann}_{\mathbb{C}[S]} N_i) \) is purely codimension one. So we deduce \( N_i = 0 \).

The upshot is that the generators \( (b_1, \ldots, b_t) \) of \( B_{F,F} \) satisfies \( \text{rad}(\mathbb{C}[S] \cdot b_i) = \text{rad}(B_{F,F}^{\alpha}) \) for all \( i \). Now write \( d = \gcd(b_1, \ldots, b_t) \) and pick the ideal \( I \subseteq \mathbb{C}[S] \) so
that \((\mathbb{C}[S] \cdot d) \cdot I = B_{F,X}^a\). That is, \(I = (B_{F,X}^a : \mathbb{C}[S] \cdot d)\). We will be done if we show \(d\) kills \(\mathcal{D}_{X,t}[S]F^S/\mathcal{D}_{X,t}[S]F^{S+a}\), so consider

\[
N = d \frac{\mathcal{D}_{X,t}[S]F^S}{\mathcal{D}_{X,t}[S]F^{S+a}} \subseteq \frac{\mathcal{D}_{X,t}[S]F^S}{\mathcal{D}_{X,t}[S]F^{S+a}}.
\]

Like before, \(N\) inherits \((n+1)\)-purity. By construction \(I\) kills \(N\), so \(Z(\text{ann}_{\mathbb{C}[S]} N) \subseteq Z(I)\). And by construction \(Z(I)\) is at least codimension 2 (a codimension one component implies the gcd is too small). Again by the prelude, the \((n+1)\)-purity of \(N\) implies that if \(N\) is nonzero, then \(Z(\text{ann}_{\mathbb{C}[S]} N)\) is purely codimension one. So we conclude \(N\) must be zero. Thus \(\mathbb{C}[S] \cdot d = B_{F,X}^a\) and we are done. \(\square\)

Under our main working hypotheses we quickly obtain:

**Corollary 3.7.** Suppose that \(f\) is strongly Euler-homogenous, Saito-holonomic, and tame, \(F\) corresponds to any factorization \(f = f_1 \cdots f_r\), and \(f^a\) is not a unit. Then \(B_{F,X}^a\) is principal.

**Proof.** Once we show \(\mathcal{D}_{X,t}[S]F^S/\mathcal{D}_{X,t}[S]F^{S+a}\) is \((n+1)\)-pure this follows from Theorem 3.4. But we know something stronger: Theorem 3.4 says it is \((n+1)\)-Cohen–Macaulay. \(\square\)

**Remark 3.8.** (a) Theorem 3.4 only requires that \(\mathbb{C}[S]\) is a UFD, so a similar, and more general, result holds for Bernstein–Sato ideals over \(\mathcal{D}_X \otimes_{\mathbb{C}} R\) instead of \(\mathcal{D}_X[S]\) provided \(R\) is assumed to be a UFD in addition to the standard assumptions of: \(R\) is a commutative, finitely generated \(\mathbb{C}\)-algebra that is an integral domain, cf. \[9\]. That is, under these assumptions, the \(R\)-annihilator of a finite \((n+1)\)-pure module over \(\mathcal{D}_X \otimes_{\mathbb{C}} R\) will be principal. Working with \(\mathcal{D}_X \otimes_{\mathbb{C}} R\), where \(R\) is a localization of \(\mathbb{C}[S]\) at an ideal cutting out a closed variety of Spec \(\mathbb{C}[S]\), was used very effectively in loc. cit.; the above results thus apply to that setting.

(b) One can also obtain similar results in the algebraic \(D_X\)-module setting as there Proposition 2.7. (1) applies directly.

The module \(\mathcal{D}_{X,t}[S]F^S/\mathcal{D}_{X,t}[S]F^{S+a}\) may not be \((n+1)\)-pure and its corresponding Bernstein–Sato ideal may not be principal, as the following example from Bahloul and Oaku demonstrates (see also \[3\]).

**Example 3.9.** Consider the non-Saito-holonomic example (Example 2.19) of \(f = z(x^5 + y^5 + zx^2y^3)\) and \(F = (z, (x^5 + y^5 + zx^2y^3))\) from Section 4.2 of of \[1\]. By loc. cit. \(B_{F,0}\) has the zero dimensional components \((-i, -\frac{2}{5})\}_{i=2} \cup \{-2, -\frac{2}{5} \} \) and \(B_{F,0}^1\) is non-principal with three generators.

Now we turn to the second result of the introduction, that is, to comparing \(Z(B_{1,F,X}^1)\) and \(Z(B_{H,F,X}^1)\) where \(F\) and \(H\) correspond to different factorizations of \(f\). We write elements of \(\mathbb{C}^r\) as \(A = (A_1, \ldots, A_r) \in \mathbb{C}^r\) and reserve \(a\) for elements of \(N^r\). We require the following linchpin result from \[9\]:

**Proposition 3.10.** (Proposition 3.4.3 \[9\]) Suppose that \(\mathcal{D}_{X,t}[S]F^S/\mathcal{D}_{X,t}[S]F^{S+a}\) is \((n+1)\)-Cohen–Macaulay. Then

\[
A \in Z(B_{F,X}^a)\text{ if and only if } \frac{\mathcal{D}_{X,t}[S]F^S}{\mathcal{D}_{X,t}[S]F^{S+a} \otimes_{\mathbb{C}[S]} \mathbb{C}[S]} \cdot (s_1 - A_1, \ldots, s_r - A_r) \neq 0.
\]
This proposition will let us equate membership in the zero locus of the Bernstein–Sato ideal with behavior of a certain $\mathcal{D}_X$-linear map.

**Definition 3.11.** (cf. Definition 3.1 [2]) Let $\nabla^a : \mathcal{D}_{X,F}[S]F^S \to \mathcal{D}_{X,F}[S]F^S$ be the $\mathcal{D}_{X,F}$-linear map induced by sending each $s_k$ to $s_k + a_k$ (recall $a = (a_1, \ldots, a_r)$). So $P(S)F^S \mapsto P(S + a)F^{s + a}$ where $P(s) \in \mathcal{D}_{X,F}[S]$. Let $A = (A_1, \ldots, A_r) \in \mathbb{C}^r$ and $(S - A)\mathcal{D}_{X,F}[S]F^S$ correspond to the submodule in $\mathcal{D}_{X,F}[S]F^S$ generated by $s_1 - A_1, \ldots, s_r - A_r$. Since $\nabla^a$ sends $s_k - A_k$ to $s_k - (A_k - a_k)$, it induces a $\mathcal{D}_{X,F}$-linear map

$$\nabla^a : (S - A)\mathcal{D}_{X,F}[S]F^S \to (S - (A - a))\mathcal{D}_{X,F}[S]F^S.$$

When $a = 1$, this map was considered extensively in Sections 3, 4 of [2]. There it was denoted simply by $\nabla_A$. The next corollary answers positively a question raised in Section 3 of loc. cit.:  

**Corollary 3.12.** Suppose that $f$ is strongly Euler-homogeneous, Saito-holonomic, and tame and $F$ corresponds to any factorization $f = f_1 \cdots f_r$. Then the following are equivalent:

- (i) $A - 1 \notin Z(B_{F,r});$
- (ii) $\nabla^1_A$ is injective at $r$;
- (iii) $\nabla^1_A$ is surjective at $r$.

**Proof.** That (i) implies (ii) and (iii) was shown in Proposition 3.2 of [2]; that (ii) implies (iii) is the content of Theorem 3.11 of [2]. We must show (iii) implies (i). But this is immediate from Theorem 3.4, Proposition 3.10 and the fact that the cokernel of $\nabla_A$ at $r$ is exactly

$$\mathcal{D}_{X,F}[S]F^S/\mathcal{D}_{X,F}[S]F^{S+1} \otimes_{\mathbb{C}[S]} \mathbb{C}[S]/\mathbb{C}[S] \cdot (s_1 - (a_1 - 1), \ldots, s_r - (a_r - 1)).$$

□

Now we can begin to compare different factorizations of $f$.

**Definition 3.13.** Suppose $F$ corresponds to the factorization $f = f_1 \cdots f_r$. For a decomposition of $[r]$ as a disjoint union $\sqcup_{1 \leq t \leq m} I_t$, write $h_t = \prod_{j \in I_t} f_j$. If $H$ corresponds to the factorization $f = h_1 \cdots h_m$ of $f$, then we say $H$ is a *coarser* factorization of $F$. Now define $S_H$ as the ideal in $\mathbb{C}[S] = \mathbb{C}[s_1, \ldots, s_r]$ generated by $s_u - s_v$ for $u, v \in I_t$ and all choices of $1 \leq t \leq m$. And, finally, let $\Delta_H : \mathbb{C}^m \to \mathbb{C}^r$ be the embedding determined by $\mathbb{C}[S] \to \mathbb{C}[S]/S_H$:

$$\Delta_H(a_1, \ldots, a_m) = (b_1, \ldots, b_r)$$

where $b_u = a_t$ for all $u \in I_t$ and for all $t$. If $H$ corresponds to the trivial factorization $f = f$, then $\Delta_H$ is the diagonal embedding $a \mapsto (a, \ldots, a)$.

The following results from the fact both $\text{ann}_{\mathcal{D}_{X,F}[S]} F^S$ and $\text{ann}_{\mathcal{D}_{X,F}[S]} H^S$ are generated by derivations, see Theorem 2.29 of [2]:

**Proposition 3.14.** (cf. Proposition 2.33, Remark 3.3 [2]) Suppose $f$ is strongly Euler-homogeneous, Saito-holonomic, and tame and $F$ corresponds to any factorization $f = f_1 \cdots f_r$. If $H$ corresponds to a coarser factorization $f = h_1 \cdots h_m$ and
A \in \mathbb{C}^m$ then we have a commutative diagram of $\mathcal{D}_{X_f}$-modules and maps:

$$
\begin{array}{ccc}
\mathcal{D}_{X_f}[S]^{F^S} & \xrightarrow{\nabla_1} & \mathcal{D}_{X_f}[S]^{H^S} \\
(S - \Delta_H(A)) \mathcal{D}_{X_f}[S]^{F^S} & \xrightarrow{\nabla_2} & (S - A) \mathcal{D}_{X_f}[S]^{H^S} \\
\end{array}
$$

(3.2)

Proof. First show the horizontal maps are isomorphisms. We will deal with the case $H$ is the trivial factorization $f = f$--the more general case follows similarly. So $A = (a_1)$. Note that $\mathcal{D}_{X_f}[S] \cdot (S - \Delta_H(A)) = \mathcal{D}_{X_f}[S] \cdot (s_1 - s_2, \ldots, s_r - s_1 + a_1)$, cf. Definition 3.13. By the product rule, for a logarithmic derivation $\delta$, the image of $\psi_{X_f}[\delta]$ modulo $\mathcal{D}_{X_f}[S] \cdot S_H$ is precisely $\psi_{H_f}[\delta]$. Since $\mathcal{D}_{X_f}[S]^{F^S}$ and $\mathcal{D}_{X_f}[S]^{H^S}$ are both generated by derivations by Theorem 3.2, the horizontal maps are isomorphisms.

That the diagram is commutative follows from the fact $\nabla_1^\Delta (\Delta_H(A))$ is induced by sending $H \mapsto H^S + a$ (resp. $F^S \mapsto F^S + \Delta_H(A)$).

In general we only know that $\Delta_H(Z(B^1_{H_f})) \subseteq Z(B^1_{F_f}) \cap \Delta_H(\mathbb{C}^n)$, cf. Lemma 4.20 of [7]. In our setting, the commutative diagram (3.2) gives us equality:

Theorem 3.15. Suppose that $f$ is strongly Euler-homogeneous, Saito-holonomic, and tame and $F$ corresponds to any factorization $f = f_1 \cdots f_r$. Further, assume that $H = h_1 \cdots h_m$ corresponds to a coarser factorization of $f$, $a \in \mathbb{N}^m$ such that $h_1^{a_1} \cdots h_m^{a_m}$ is not a unit, and $A \in \mathbb{C}^m$. Then

$$
A \in Z(B^a_{H_f}) \text{ if and only if } \Delta_H(A) \in Z(B^a_{F_f}).
$$

(3.3)

Setting $H = (f)$ and $a = 1 \in \mathbb{N}$ we obtain

$$
Z(B_{f,\text{f}}) = Z(B^1_{F_f}) \cap \{s_1 = \cdots = s_r\}.
$$

Here points on the diagonal of $\mathbb{C}^r$ are naturally identified with points of $\mathbb{C}$.

Proof. Just as in the proof of Corollary 3.12 in the kernel of $\nabla_1^\Delta$ is $\mathcal{D}_{X_f}[S]^{H^S} / \mathcal{D}_{X_f}[S]^{H^S + a} \otimes \mathbb{C}[S] / \mathbb{C}[S] \cdot (s_1 - (A_1 - a_1), \ldots, (s_r - (A_r - a_r))$. So by Proposition 3.10 the surjectivity of $\nabla_1^\Delta$ precisely characterizes membership in $Z(B^a_{H_f})$. Now use the commutative diagram (3.2).

Example 3.16. Returning to our non-Saito-holonomic example $f = z(x^3 + y_5 + z^2y^3)$ then we have $\mathcal{D}_{X_f}[S]^{H^S} / \mathcal{D}_{X_f}[S]^{H^S + a} \otimes \mathbb{C}[S] / \mathbb{C}[S] \cdot (s_1 - (A_1 - a_1), \ldots, (s_r - (A_r - a_r))$. Macaulay2 verifies this diagonal property still holds: $Z(B^1_{F_f}) \cap \{s_1 = s_2\} = Z(B^1_{F_f})$.

3.3. Hyperplane Arrangements.

In this subsection we document some applications of the previous result for hyperplane arrangements. First, using the diagonal property (3.3) of Theorem 3.15 we will give formulae for the Bernstein–Sato ideals of any free, central, possibly non-reduced hyperplane arrangement (equipped with any factorization) and corresponding estimates when “freeness” is replaced with “tameness.” Second, we will build up the necessary notation and facts for our formulae for the Bernstein–Sato ideals attached to generic arrangements appearing in the next subsection. Additionally this will let us prove the factorization $F$ into linear forms has a very special
property: under the assumption of tameness, the attached Bernstein–Sato ideal equals its radical.

To begin with the applications, suppose \( f \) is a free, central hyperplane arrangement in \( X = \mathbb{C}^n \), cf. Definition 2.20. In Theorem 1 of [16], Maisonobe showed \( Z(B_{F,0}^1) \) is combinatorially determined provided \( f \) is reduced and \( F \) is a factorization of \( f \) into linear forms; in Theorem 1.4 of [3] we generalized this to arbitrary factorizations \( F \) of a reduced \( f \), certain factorizations \( F \) of a non-reduced \( f \), and computed \( Z(B_{f,0}) \) if \( f \) is a power of a reduced, central, free arrangement. We also outlined in Remark 4.28 of loc. cit. how to extend this result to any factorization \( F \) of a possibly non-reduced \( f \). Now we can do this:

**Corollary 3.17.** (cf. Remark 4.28 [3]) Suppose that \( f \) is a free, central, and possibly non-reduced hyperplane arrangement and \( F \) corresponds to any factorization \( f = f_1 \cdots f_r \). Then \( Z(B_{F,0}^1) \) is combinatorially determined and explicitly given by Theorem 1.4, equation (1.2) of [3]. In particular this gives a combinatorial formula for the roots of the Bernstein–Sato polynomial of \( f \).

**Proof.** By Theorem 1.4 of [3], there is an explicit combinatorial formula for \( B_{H,1}^1 \) for a free, central, possibly non-reduced \( f \) provided \( H \) is corresponds to a factorization into linear forms. It is immediate (and part of the construction) that if \( F \) is another factorization of \( f \), then \( \Delta_F^{-1}(Z(B_{H,0}^1)) \) agrees with the formula (1.2) of Theorem 1.4 of loc. cit. for \( V(B_F) \) even without the assumption “unmixed up to units.” (To use the formula therein, set \( f' = 1 \).) And recall that, cf. Convention 2.11, Remark 4.9 of [3], \( V(B_F) \) in [3] agrees with \( Z(B_{F,0}^1) \) here since the global Bernstein–Sato ideal agrees with the one at the origin by centrality.) The claim follows by (3.3) of Theorem 3.15. \( \square \)

As this paper was being finished, Wu (independently) used in [25] the same diagonal procedure as in Corollary 3.17 to recover a special case of our formula for \( Z(B_{f,0}) \) from Theorem 1.4 of [3]. Specifically he considered the case of free, central, and reduced arrangements. He obtains the diagonal property (3.3) by different means, cf. Theorem 1.1 in [25]. To deal with the non-reduced case in full generality, i.e. to cover the cases not included in our [3], one needs the duality formula Theorem 3.9 of [3], and the consequent symmetry formulae Theorem 3.16 and Corollary 3.18 from loc. cit. This is because the duality computations differ slightly in the non-reduced setting.

In [24], Walther demonstrated that the roots of the b-function of a hyperplane arrangement many not be combinatorially determined. Theorem 3.14 lets us extend this demonstration to multivariate Bernstein–Sato ideals:

**Example 3.18.** Consider Walther’s hyperplane arrangements (Example 5.10 [24]):
\[
\begin{align*}
f &= xyz(x + 3z)(x + y + z)(x + 2y + 3z)(2x + y + z)(2x + 3y + z)(2x + y + 4z); \\
g &= xy(x + 5z)(x + y + z)(x + 3y + 5z)(2x + y + z)(2x + 3y + z)(2x + y + 4z).
\end{align*}
\]
These have the same intersection lattice but different b-functions: \( f \) has \( -\frac{16}{5} \) as a root; \( g \) does not. See also Remark 4.14.(iv) of [22]. Since \( n = 3 \), Remark 2.21 shows \( f \) and \( g \) are tame. Since a factorization \( F \) of \( f \) (resp. \( G \) of \( g \)) amounts to grouping hyperplanes of \( V(f) = V(g) \) into sets, we can regard \( F \) and \( G \) as equivalent if they correspond to the same choices of hyperplanes. Thus if \( Z(B_{f,0}^1) = Z(B_{G,0}^1) \) for equivalent \( F \) and \( G \), then \( Z(B_{f,0}) \) would equal \( Z(B_{g,0}) \) by (3.3) of Theorem 3.15. This is impossible by construction.
It will hereafter be easier to work in the algebraic context. So $X = \mathbb{C}^n$, $f$ is a tame, central, and possibly non-reduced hyperplane arrangement equipped with an arbitrary factorization $F$. In this algebraic situation we can define multivariate Bernstein–Sato ideals using the Weyl algebra $A_n(\mathbb{C})$ as well as the naturally defined $A_n(\mathbb{C})[S]$-module $A_n(\mathbb{C})[S]F_S$. We will write $B^a_F$ for the Bernstein–Sato ideal corresponding to the algebraic functional equation using $A_n(\mathbb{C})[S]$ and $a$. Because $f$ is central, one can check that the local, analytically defined $B^a_{F,0}$ is the same as the global, algebraically defined Bernstein–Sato ideal $B^a_F$, see, for example, Remark 4.9. of [3].

One goal is to provide estimates to $Z(B^a_{F})$ that (1) are key to the formula for Bernstein–Sato ideals of generic arrangements and (2) are mild generalizations of the estimates appearing in [3]. (There $a = 1$ was exclusively considered.) We require notation particular to arrangements.

**Definition 3.19.** For $f$ a central, possibly non-reduced hyperplane arrangement of degree $d$ with a factorization into homogeneous linear terms $f = l_1 \cdots l_d$, we will usually denote $V(f)$ by $\mathcal{A} \subseteq \mathbb{C}^n$. We can define the intersection lattice $L(\mathcal{A})$ of $f$:

$$L(\mathcal{A}) = \left\{ \bigcap_{i \in I} V(l_i) \mid I \subseteq [d] \right\}.$$  

We say $X$ is an edge if $X \in L(\mathcal{A})$; we denote the rank of $X$ by $r(X)$. For an edge $X$, let $J(X) \subseteq [d]$ signify the largest subset of the hyperplanes $\{V(l_j)\}$ that contain $X$. (If $f$ is non-reduced, the “largest” assumption matters.) Then $X = \cap_{j \in J(X)} V(l_j)$.

To each edge $X$, we associate a subarrangement $f_X$ given by

$$f_X = \prod_{j \in J(X)} l_j$$

We denote the degree of $f_X$, that is $|J(X)|$, by $d_X$. The edge $X$ is decomposable if there is some change of coordinates so that $f_X$ can be written as a product of two hyperplanes using disjoint, nonempty, sets of variables. Otherwise $X$ is indecomposable. If $r(X) < n$, then $X$ is automatically decomposable as an arrangement in $\mathbb{C}^n$; in this case we decide if $X$ is decomposable after naturally viewing it in $\mathbb{C}^{r(X)}$.

For a different factorization $f = f_1 \cdots f_r$ we can identify (up to a unit) each $f_k$ with a collection of the $l_i$. Let $S_k \subseteq [d]$ be linear forms defining $f_k$:

$$f_k = \prod_{i \in S_k} l_i.$$  

The factorization $f = f_1 \cdots f_r$ induces a factorization on $f_X = f_{X,1} \cdots f_{X,r}$ with

$$f_{X,k} = \prod_{i \in J(X) \cap S_k} l_i.$$  

Write $\deg f_k$ as $d_k$, $\deg f_{X,k}$ as $d_{X,k}$.

First we slightly adapt an argument from [16] and [3] to find a particular nice element of $B^a_{F}$. The proof is essentially the same as in [3].

**Proposition 3.20.** (cf. Proposition 10 [16], Theorem 4.18 [3]) Suppose that $f$ is a central, possibly non-reduced hyperplane arrangement and $F$ a factorization of $f$.
into homogeneous linear terms. Then there exists an $N \in \mathbb{N}$ such that

$$
\prod_{X \in L(\omega')} \prod_{\ell=0}^{N} \left( \sum_{j \in J(X)} s_j + r(X) + \ell \right) \in B_{F}^{a}.
$$

**Proof.** First, pick $p = (p, \ldots, p) \in \mathbb{N}^r$ such that $a \leq p$ and $1 \leq p$ termwise. Clearly $B_{F}^{p} \subseteq B_{F}^{a}$ so it suffices to prove the claim for $a = p$. In Theorem 4.18 of [3] this is shown for $a = 1$. The inductive argument on the rank of $f$ used there works here with one slightly non-cosmetic change. By the induction hypothesis, for each $X \in L(\omega')$ of rank $n - 1$, there exists an operator $P_X$ of total order at most $k_X$ and a polynomial $b_X \in \mathbb{C}[S]$ such that $b_X \prod_{j \in J(X)} f_j^{s_j} = P_X \prod_{j \in J(X)} f_j^{s_j + p}$. In loc. cit. we pick a natural number $m$ greater than the max of all the total order’s of the differential operators coming from the inductive hypothesis therein. Here, if we pick $m > \max\{k_X + p \mid X \in L(\omega'), r(X) = n - 1\}$ then the argument in loc. cit. applies in this case as well. \qed

If $f$ is tame, we can use this nice element of $B_{F,0}^{a}$ to obtain a multivariate generalization of Saito’s Theorem 1 of [22]: the roots of the $b$-function of a reduced arrangement $f$ lie in $(-2 + \frac{1}{\deg f}, 0)$. For the reader’s sake, we include the result of Theorem 4.11 from [3] about certain hyperplanes necessarily in $Z(B_{F,0}^{1})$ due to tameness.

**Corollary 3.21.** Suppose that $f$ is a tame, central, and reduced hyperplane arrangement and $F$ corresponds to any factorization $f = f_1 \cdots f_r$. Then $Z(B_{F,0}^{1})$ is a union of hyperplanes,

$$
\bigcup_{X \in L(\omega')} \bigcup_{X \text{ indecomposable}} \left\{ \sum_{k=1}^{d_X - 1} \frac{d_X}{r} d_{X,k} s_k + r(X) + \ell = 0 \right\} \subseteq Z(B_{F,0}^{1})
$$

and

$$
Z(B_{F,0}^{1}) \subseteq \bigcup_{X \in L(\omega')} \bigcup_{X \text{ indecomposable}} \left\{ \sum_{k=1}^{[T_X - 1]} \frac{r}{\ell} d_{X,k} s_k + r(X) + \ell = 0 \right\}
$$

where $d = \deg f$, $d_X = \deg f_X = |J(X)|$, $d_{X,k} = \deg f_{X,k} = |J(X) \cap S_k|$, cf. Definition [3.19] and $T_X = 2d_X - \frac{d}{p} - r(X)$.

**Proof.** Since $f$ is tame, (3.5) follows from Theorem 4.11 of [3]; that $Z(B_{F,0}^{1})$ is a union of hyperplanes follows from Corollary [3.7] and Proposition [3.20]. If we replace $T_X$ with some very large $N \in \mathbb{N}$, then (3.6) follows from [3.20] The choice of $T_X$ follows from Theorem [3.15] and Theorem 1 in [22]. \qed

When $F$ corresponds to a factorization into linear forms, the nice element of $B_{F}^{a}$ from Proposition [3.20] is reduced. The same argument strategy from Theorem [3.16] then yields:

**Theorem 3.22.** Suppose that $f$ is a tame, central possibly non-reduced hyperplane arrangement, $F$ corresponds to a factorization of $f$ into linear forms, and $f^a$ is not a unit. Then $B_{F}^{a} = \text{rad}(B_{F}^{a})$.
Proof. It is equivalent to prove the result on the analytic side, i.e. $B^n_{F,0} = \ker(B^n_{F,0})$. By Proposition 3.20 we may find $b(S) \in B^n_{F,0}$ such that $b(S)$ cuts out a reduced hyperplane arrangement. By, say, Theorem 3.6, $\ker(B^n_{F,0})$ is principal. Pick a generator $\gamma(s)$. Write $b(S) = \gamma(s)b(S)$. Since $V(b(S))$ and $V(\gamma(s))$ are both reduced hyperplane arrangements, every component of $V(\gamma(s)) \cap V(b(S))/\gamma(s)$ has dimension at most $n - 2$.

Now consider

$$Q := \gamma(s)\frac{\mathcal{D}_{X,0}[S]^F}{\mathcal{D}_{X,0}[S]^{F+[a]}} \subseteq \frac{\mathcal{D}_{X,0}[S]^F}{\mathcal{D}_{X,0}[S]^{F+[a]}}.$$ 

Just as in Theorem 3.6, $Q$ inherits $(n+1)$-purity from $\mathcal{D}_{X,0}[S]/\mathcal{D}_{X,0}[S]^{F+[a]}$, is annihilated by $b(S)/\gamma(s)$, and its Bernstein–Sato ideal satisfies $Z(B_Q) \subseteq Z(B^n_{F,0}) = Z(\gamma(s))$. On one hand, purity implies $Z(B_Q)$ is purely codimension one; on the other hand, $Z(B_Q) \subseteq Z(\gamma(s)) \cap Z(b(S)/\gamma(s))$, which is, by construction, codimension at least two. We deduce $Q = 0$, $\gamma(s) \in B^n_{F,0}$, and $B^n_{F,0} = \ker(B^n_{F,0})$. □

We have now built up enough machinery to deal with generic arrangements.

3.4. Generic Arrangements.

A central arrangement of $d$ hyperplanes $\mathcal{A} \subseteq \mathbb{C}^n$ with $d > n$ is generic if the intersection of any collection of $n$ hyperplanes is exactly the origin. By [19] every generic arrangement is tame. We let $f$ be a reduced defining equation of $\mathcal{A}$ and we continue to use the notation introduced in Definition 3.19.

In [23], Walther obtained the following formula (see [22] for the multiplicity of $1$) for the Bernstein–Sato polynomial of a generic arrangement:

$$B_f = \mathbb{C}[s] \cdot (s+1)^{n-1} \prod_{i=0}^{2d-n-2} (s+\frac{i+n}{d}).$$ (3.7)

In [15], Maisonobe proved that if $F = (f_1, \ldots, f_d)$ corresponds to a factorization of $f$ into linear forms, then

$$\prod_{k=1}^{d} (s_k+1) \prod_{i=0}^{2d-n-2} \left( \sum_{k=1}^{d} s_k + i + n \right) \in B^1_F.$$ (3.8)

Moreover, if $d = n + 1$ he showed that the polynomial in (3.8) generates $B^1_F$. However, computing $B^1_F$ when $d > n + 1$ remained unsolved. Using Walther’s formula (3.8), we independently verify (3.8) and improve upon it by computing this Bernstein–Sato ideal explicitly:

**Theorem 3.23.** Let $f$ be the reduced defining equation of a central, generic hyperplane arrangement in $\mathbb{C}^n$ with $d = \deg f > n$. If $F$ corresponds to a factorization of $f$ into irreducibles, then

$$B^1_F = \mathbb{C}[S] \cdot \prod_{k=1}^{d} (s_k+1) \prod_{i=0}^{2d-n-2} \left( \sum_{k=1}^{d} s_k + i + n \right).$$ (3.9)

If $F = (f_1, \ldots, f_r)$ corresponds to some other factorization, then $B^1_F$ is principal and

$$Z(B^1_F) = \left( \bigcup_{k=1}^{r} \{s_k + 1 = 0\} \right) \bigcup \left( \bigcup_{i=0}^{2d-n-2} \bigcup_{k=1}^{r} \{d_k s_k + i + n = 0\} \right).$$ (3.10)
Proposition A.2. Suppose \( \log f \) defining equation 

\[
\Omega(3.12) \quad Z
\]

ential forms fit into the logarithmic de Rham complex 
\( \Omega \) which is well-defined since 
\( C \) diagonal of 
\( I \) where 
\( \gamma \) (3.11) a normal crossing divisor (and hence decomposable), Corollary A.1 implies 

\[
(3.11) \quad \gamma(S) = \prod_{k=1}^{d} (s_k + 1) \prod_{\ell=0}^{d-1} \left( \sum_{k=1}^{d} s_k + \ell + n \right) \prod_{i \in I} \left( \sum_{k=1}^{d} s_k + i + n \right),
\]

where \( I \subseteq \{d, d+1, \ldots, 2d-n-2\} \).

By Theorem A.1 and (3.11) we have, after naturally identifying points on the diagonal of \( \mathbb{C}^r \) with points in \( \mathbb{C} \),

\[
(3.12) \quad Z(B^1_{f,0}) = Z(B^1_{F,0}) \cap \{s_1 = \cdots = s_r\} = \left\{-\frac{i+n}{d} \big| 0 \leq i \leq d-1 \text{ or } i \in I \right\}.
\]

It is well known that the Bernstein–Sato polynomial of a central arrangement at the origin agrees with its global b-function. So (3.12) should agree with the roots computed by Walther’s formula (3.7). Thus \( I = \{d, d+1, \ldots, 2d-n-2\} \) and the explicit description of \( \gamma(S) \) verifies the first claim (3.9).

The statement about principality for other factorizations follows from Corollary 3.7. As for (3.10), use (3.9) and Theorem 3.15. \( \square \)

Appendix A. Logarithmic Differential Forms for Non-Reduced Divisors

In K. Saito’s original [21] logarithmic differential forms are considered only for reduced divisors. As the referee points out, there is no systematic taxonomy of the non-reduced analogue. We record some basic observations here in the setting of \( X \) smooth analytic (or \( \mathbb{C} \)-scheme) of dimension \( n \), though similar results hold in the algebraic setting. Throughout \( D = \sum n_i Z_i \), \( Z_i \) irreducible codimension one components, will be an effective divisor, i.e. \( n_i > 0 \), where we have made the sum finite for notational reasons. We allow \( D \) to be possibly non-reduced, i.e. some \( n_i \) may not equal one. When working with the reduced structure we use \( D_{\text{red}} \). Recall \( \mathcal{O}_X(D) = \{g \in \mathcal{O}_X(\pm D) \mid \text{div}(g) + D \geq 0\} \), that is, the sheaf of meromorphic functions with at worst poles of order \( n_i \) along each \( Z_i \). And \( \Omega^k_X(D) = \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(D) \) is the sheaf of meromorphic \( k \)-forms with poles of order at most \( n_i \) along each \( Z_i \). We denote by \( \Omega^k_X(\pm D) \) the meromorphic \( k \)-forms with poles of arbitrary order on (and only on) the \( Z_i \).

Definition A.1. The logarithmic differential \( k \)-forms \( \Omega^k_X(\log D) \) is the subsheaf of \( \Omega^k_X(\pm D) \) characterized by

\[
\Omega^k_X(\log D) = \{g \in \Omega^k_X(D) \mid d(g) \in \Omega^{k+1}(D)\},
\]

where \( d \) is the exterior derivative inherited from \( \Omega^k_X(\pm D) \). The logarithmic differential forms fit into the logarithmic de Rham complex

\[
\Omega^\bullet_X(\log D) = 0 \rightarrow \Omega^0_X(\log D) \xrightarrow{d} \Omega^1_X(\log D) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^r_X(\log D) \xrightarrow{d} 0,
\]

which is well-defined since \( \Omega^\bullet_X(\pm D) \) is itself a complex. When using an explicit defining equation \( f \) for \( D \) on a domain \( U \subseteq X \) we often replace “\( \log D \)” with “\( \log f \).”

Proposition A.2. Suppose \( f \) defines \( D \) on an open domain \( U \subseteq X \). For \( \eta \in \Omega^k_X(\pm D) \), the following are equivalent:
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(1) \( \eta \in \Omega^k_U(\log D) \);
(2) \( f\eta \in \Omega^k_U \) and \( fd(\eta) \in \Omega^{k+1}_U \);
(3) \( f\eta \in \Omega^k_U \) and \( df \wedge \eta \in \Omega_U^k \).

Proof. This is immediate from the formula \( d(f\eta) = df \wedge \eta + fd(\eta) \).

Proposition A.2 tracks with the first half of (1.1) of [21], but the rest of (1.1) seems to require \( D = D_{\text{red}} \). The explicit construction of \( g \) in (iii) of loc. cit. amounts to choosing \( g = \partial f / \partial x_j \) for some \( j \) such that \( \{ f = 0 \} \cap \{ \partial f / \partial x_j = 0 \} \) has codimension at least 2. When \( D \neq D_{\text{red}} \) this may not be possible: \( \text{codim} \text{Sing}(D) \geq 2 \) may fail. For instance, if \( f = x^2 \in \mathcal{O}_{C^2} \) we have

\[ \{ f = 0 \} \cap \{ \partial f / \partial x = 0 \} = \{ f = 0 \} \]

and \( \text{codim} \{ f = 0 \} = 1 \).

An unfortunate reality of the inability to generalize the proof of the equivalence of (iii) and (iv) in (1.1) of [21] to (2) and (3) of Proposition A.2 in the non-reduced setting, is that when \( D \) is non-reduced, the logarithmic differential forms along \( D \) may not be (and almost never are) closed under exterior product.

Example A.3. (Not necessarily an exterior algebra) Let \( f = x^2y^2 \) define our divisor \( D \subseteq X = C^2 \). Then \( (1/f)(adx + bdy) \) is in \( \Omega^1_X(\log f) \) precisely when

\[ (−2x^2ydy)a + (2xy^2)b dx dy \in f \cdot \Omega^2_X, \]

that is, when

\[ 2xy(by − ax) \in x^2y^2 \cdot \mathcal{O}_X. \]

This happens if and only if \( b \in x \cdot \mathcal{O}_X \) and \( a \in y \cdot \mathcal{O}_X \) meaning

\[ \Omega^1_X(\log f) = \mathcal{O}_X \cdot \frac{dx}{x^2y} \oplus \mathcal{O}_X \cdot \frac{dy}{xy^2}. \]

Therefore

\[ \wedge^2 \Omega^1_X(\log f) = \frac{1}{x^2y^2} \Omega^2_X = \frac{1}{f_{\text{red}}} \Omega^2_X \supseteq \frac{1}{f} \Omega^2_X = \Omega^2_X(\log f), \]

demonstrating that the logarithmic differential forms along a non-reduced divisor may not be an exterior algebra.

Recall the definition of the logarithmic derivations along \( D \):

\[ \text{Der}_X(-log D) = \{ \delta \in \text{Der}_X \mid \delta \cdot \mathcal{O}_X(-D) \subseteq \mathcal{O}_X(-D) \}, \]

i.e. these are vector fields that are tangent to \( D \). It is well known (and follows from the product rule) that \( \text{Der}_X(-log D) = \cap Z_i \text{ Der}_X(-log Z_i) \) and hence

\[ \text{Der}_X(-log D) = \text{Der}_X(-log D_{\text{red}}). \]

In the reduced case, contraction of a logarithmic \( k \)-form along a logarithmic derivation gives a logarithmic \( (k−1) \)-form. The same is true in the non-reduced situation:

Proposition A.4. Contracting along \( \chi \in \text{Der}_X(-log D) = \text{Der}_X(-log D_{\text{red}}) \) induces an \( \mathcal{O}_X \)-map

\[ \iota_\chi : \Omega^k_X(\log D) \to \Omega^{k-1}_X(\log D). \]
Proof. Let $\eta \in \Omega^k_U(\log D)$ and let $f$ be a defining equation for $D$ along a domain $U$. We use Proposition A.2. Certainly $f \iota_\chi(\eta) \in \Omega^{k-1}_U$ since contraction is $\scr{O}_X$-linear. We must show $df \wedge \iota_\chi(\eta) \in \Omega^k_U$. Observe:

$$\iota_\chi(df \wedge \eta) = \iota_\chi(df) \wedge \eta - df \wedge \iota_\chi(\eta).$$

By Proposition A.2 $\iota_\chi(df \wedge \eta) \in \Omega^k_U$; since $\chi \in \Der\langle - \log D \rangle$, $\iota_\chi(df) \wedge \eta \in \Omega^k_U$. □

In the non-reduced setting, the logarithmic derivations need not be (and almost never are) the $\scr{O}_X$-dual of $\Omega^1_X(\log D)$:

Example A.5. (Duality differs for non-reduced divisors) Take $f = x^2y^2$ and $D = \Var(f) \subseteq X = \mathbb{C}^2$ as in Example A.3. We have $\Der\langle - \log D \rangle = \Der\langle - \log D_{\text{red}} \rangle$ is freely generated by $x \partial_x$ and $y \partial_y$. Using the explicit description of $\Omega^1_X(\log D)$ from (A.1) one can check contracting along $\Der\langle - \log D \rangle$ gives a bilinear $\scr{O}_X$-map

$$\Omega^1_X(\log D) \times \Der\langle - \log D \rangle \rightarrow \scr{O}_X(D_{\text{red}}).$$

So $\Der\langle - \log D \rangle \neq \Hom_{\scr{O}_X}(\Omega^1_X(\log D), \scr{O}_X)$ in this non-reduced case.

Despite Example A.3 and Example A.5 at least as $\scr{O}_X$-modules, $\Omega^k_X(\log D)$ and $\Omega^k_X(\log D_{\text{red}})$ are closely linked and essentially the same up to isomorphism, thanks to:

Proposition A.6. We have the $\scr{O}_X$-module equality

$$\Omega^k_X(\log D) = \Omega^k_X(\log D_{\text{red}}) \otimes_{\scr{O}_X} \scr{O}_X(D - D_{\text{red}}),$$

under the canonical identification $\scr{O}_X(D) \otimes_{\scr{O}_X} \scr{O}_X(-D) = \scr{O}_X$.

Proof. It suffices to show that

(A.2) $\Omega^k_X(\log D) \otimes_{\scr{O}_X} \scr{O}_X(-D) = \Omega^k_X(\log D_{\text{red}}) \otimes_{\scr{O}_X} \scr{O}_X(-D_{\text{red}}).$

Write $D = \sum n_i Z_i$ as before. Proposition A.2 implies that $\mu \in \Omega^k_X(\log D)$ if and only if (1) $\mu \in \Omega^k_X(D)$ and (2) $dg \wedge \mu \in \Omega^{k+1}_X(D - n_i Z_i)$ for all $g \in \Omega^0_X(-D)$. But (2) is equivalent, by the product rule, to $dg \wedge \mu \in \Omega^{k+1}_X(D - n_i Z_i)$ for all $g \in \Omega^0_X(-n_i Z_i)$ and all $Z_i$. Therefore

(A.3) $\omega \in \Omega^k_X(\log D) \otimes_{\scr{O}_X} \scr{O}_X(-D) \iff \omega \in \Omega^k_X$ and

$$dg \wedge \omega \in \Omega^{k+1}_X(-Z_i) \forall g \in \Omega^0_X(-Z_i), \forall Z_i.$$

Note that (A.3) also applies for $D_{\text{red}}$. In particular, the second condition (after “and”) is the same. Therefore $\omega \in \Omega^k_X$ satisfies $\omega \in \Omega^k_X(\log D) \otimes_{\scr{O}_X} \scr{O}_X(-D)$ if and only if $\omega \in \Omega^k_X(\log D_{\text{red}}) \otimes_{\scr{O}_X} \scr{O}_X(-D_{\text{red}})$, which verifies (A.2). □

Corollary A.7. Let $D$ be a possibly non-reduced divisor and $D_{\text{red}}$ the associated reduced divisor. The following are true:

(1) $\Omega^0_X(\log D) = \Omega^0_X \otimes_{\scr{O}_X} \scr{O}_X(D - D_{\text{red}}) = \scr{O}_X(D - D_{\text{red}})$;
(2) $\Omega^{n-1}_X(\log D) \simeq \Der\langle - \log D \rangle$;
(3) $\Omega^k_X(\log D)$ is reflexive;
(4) $\Omega^k_X(\log D)$ is free if and only if $\Omega^k_X(\log D_{\text{red}})$ is free;
(5) $\Omega^k_X(\log D)$ is tame if and only if $\Omega^k_X(\log D_{\text{red}})$ is tame.
Proof. Because $\Omega^1_X(\log D_{\text{red}}) = \Omega^0_X = \mathcal{O}_X$, Proposition A.6 implies (1). The justification for (2) is already known in the non-reduced and reduced cases, and is given by a straightforward extension of the explicit map in Remark 2.4 (5) of [24]. (4) and (5) follow from the $\mathcal{O}_X$-module isomorphism $\Omega^k_X(\log D) \simeq \Omega^k_X(\log D_{\text{red}})$ given by Proposition A.6. (3) follows similarly, as the reflexivity of $\Omega^k_X(\log D_{\text{red}})$ was proved for $k = 1$ in (1.6) of [21] and $k \geq 1$ by Proposition 2.2 of [11].

We finish with the appropriate generalization of the perfect pairing between $\Omega^1_X(\log D_{\text{red}})$ and $\text{Der}_X(-\log D_{\text{red}})$ to the non-reduced setting. We augment a treatment in [18].

**Proposition A.8.** Contraction along vector fields induces the perfect pairing

$$\text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D) \times \Omega^1_X(\log D) \ni (\chi, \eta) \mapsto \iota_\chi(\eta) \in \mathcal{O}_X.$$ 

This naturally identifies $\text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D)$ and $\Omega^1_X(\log D)$ with each other’s $\mathcal{O}_X$-duals.

**Proof.** By Proposition A.4 and Corollary A.7, (1), contracting gives inclusions

$$\text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D) \subseteq \text{Hom}_{\mathcal{O}_X}(\Omega^1_X(\log D), \mathcal{O}_X).$$

and

$$\Omega^1_X(\log D) \subseteq \text{Hom}_{\mathcal{O}_X}(\text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D), \mathcal{O}_X).$$

We first show the reverse containment for (A.5). Note that

$$\text{Der}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D) \subseteq \text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D) \subseteq \text{Der}_X$$

and the two induced cokernels vanish outside of $D$. So the first Ext-module of the two cokernels vanish, meaning dualizing gives inclusions

$$\Omega^1_X \subseteq \text{Hom}_{\mathcal{O}_X}(\text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D), \mathcal{O}_X) \subseteq \Omega^1_X(D).$$

So $\eta \in \text{Hom}_{\mathcal{O}_X}(\text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D), \mathcal{O}_X)$ can be thought of as a one-form with poles of order at most $n_i$ along each $Z_i$ of $D$. To show the containment in (A.5) is an equality, it is enough to do this stalkwise; by Proposition A.2 it suffices to show $df \wedge \eta \in \Omega^1_{X,p}$ for $f$ defining $D$ at $p$. We may write $f = z_1^{n_1} \cdots z_m^{n_m}$ where $z_i$ defines $Z_i$ at $p$. Writing $\eta = (1/f) \sum_i a_i dx_i$, we must show that $df \wedge \eta \in \mathcal{O}_{X,p}$ or, explicitly, that

$$a_i \frac{\partial f}{\partial x_j} - \frac{\partial f}{\partial x_i} a_j \in \mathcal{O}_{X,p} \cdot f$$

for all $1 \leq i, j \leq n$. As $\mathcal{O}_{X,p}$ is a UFD, this reduces to showing

$$n_{k-1}z_k^{n_k-1} \left( a_i \frac{\partial z_k}{\partial x_j} - \frac{\partial z_k}{\partial x_i} a_j \right) \in \mathcal{O}_{X,p} \cdot z_k^{n_k}$$

for all $1 \leq k \leq m$, or, alternatively, that

$$a_i \frac{\partial f_{\text{red}}}{\partial x_j} - \frac{\partial f_{\text{red}}}{\partial x_i} a_j \in \mathcal{O}_{X,p} \cdot f_{\text{red}}.$$

As

$$\chi = \frac{\partial f_{\text{red}}}{\partial x_j} a_j - \frac{\partial f_{\text{red}}}{\partial x_i} a_j \in \text{Der}_X(-\log f_{\text{red}}),$$

we see that

$$\eta \text{ maps } f_{\text{red}} \chi \in \text{Der}_X(-\log D) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_{\text{red}} - D) \text{ into } \mathcal{O}_X.$$
This verifies (A.6), finishing the justification that (A.5) is actually an equality.

To certify (A.4) is an equality it is enough to dualize (into $O_X$) the (now verified) equality in (A.5): since $\text{Der}_X(- \log D) = \text{Der}_X(- \log D_{\text{red}})$ is reflexive (cf. (1.6) of [21]), the resultant equality of duals gives equality in (A.4).

□

Remark A.9. Note that whether or not $\Omega^k_X(\log D)$ is an exterior algebra is not actually used in any of the background material on which this paper relies. What are used are the $O_X$-module properties of $\Omega^k_X(\log D)$, in particular ones involving (2), (3), (4), (5) of Corollary A.7. (For example, while the Liouville complex of [24] is defined with an exterior product, the actual property invoked is that $\Omega^k_X(\log D) \wedge \Omega^1_X \subseteq \Omega^{k+1}_X(\log D)$ which is always true by Proposition A.2.)

REFERENCES

[1] R. Bahloul and T. Oaku. Local Bernstein-Sato ideals: algorithm and examples. J. Symbolic Comput., 45(1):46–59, 2010.
[2] D. Bath. Bernstein-sato varieties and annihilation of powers. Trans. Amer. Math. Soc., 373(12):8543–8582, 2020.
[3] D. Bath. Combinatorially determined zeroes of Bernstein–Sato Ideals for tame and free arrangements. J. Singul., 20:165–204, 2020.
[4] J.-E. Björk. Analytic D-modules and applications, volume 247 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1993.
[5] J. Briçon and H. Maynadier. Équations fonctionnelles généralisées: transversalité et principialité de l’idéal de Bernstein-Sato. J. Math. Kyoto Univ., 39(2):215–232, 1999.
[6] W. Bruns and J. Herzog. Cohen-Macaulay rings, volume 39 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1993.
[7] N. Budur. Bernstein-Sato ideals and local systems. Ann. Inst. Fourier (Grenoble), 65(2):549–603, 2015.
[8] N. Budur, Y. Liu, L. Saumell, and B. Wang. Cohomology support loci of local systems. Michigan Math. J., 66(2):295–307, 2017.
[9] N. Budur, R. van der Veer, L. Wu, and P. Zhou. Zero loci of Bernstein-Sato ideals. Invent. math., 225:45–72, 2021.
[10] N. Budur, R. van der Veer, L. Wu, and P. Zhou. Zero loci of Bernstein-Sato ideals – ii. Selecta Math. (N.S.), 27(3), 2021.
[11] G. Denham and M. Schulze. Complexes, duality and Chern classes of logarithmic forms along hyperplane arrangements. In Arrangements of hyperplanes—Sapporo 2009, volume 62 of Adv. Stud. Pure Math., pages 27–57. Math. Soc. Japan, Tokyo, 2012.
[12] A. Gyoja. Bernstein-Sato’s polynomial for several analytic functions. J. Math. Kyoto Univ., 33(2):399–411, 1993.
[13] M. Kashiwara. Vanishing cycle sheaves and holonomic systems of differential equations. In Algebraic geometry (Tokyo/Kyoto, 1982), volume 1016 of Lecture Notes in Math., pages 134–142. Springer, Berlin, 1983.
[14] P. Maisonobe. Filtration Relative, l’Idéal de Bernstein et ses pentes. Rend. Sem. Mat. Univ. Padova, to appear.
[15] P. Maisonobe. Idéal de Bernstein d’un arrangement central générique d’hyperplans. arXiv e-prints, page arXiv:1610.03357, Oct. 2016.
[16] P. Maisonobe. L’idéal de Bernstein d’un arrangement libre d’hyperplans linéaires. arXiv e-prints, page arXiv:1610.03356, Oct. 2016.
[17] B. Malgrange. Le polynôme de Bernstein d’une singularité isolée. In Fourier integral operators and partial differential equations (Colloq. Internat., Univ. Nice, Nice, 1974), pages 98–119. Lecture Notes in Math., Vol. 459. Springer, Berlin, 1975.
[18] D. Mond. Notes on logarithmic vector fields, logarithmic differential forms and free divisors (online lecture notes), 2012.
[19] L. L. Rose and H. Terao. A free resolution of the module of logarithmic forms of a generic arrangement. J. Algebra, 136(2):376–400, 1991.
[20] C. Sabbah. Proximité évanescente. I. La structure polaire d’un $\mathcal{D}$-module. *Compositio Math.*, 62(3):283–328, 1987.

[21] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):265–291, 1980.

[22] M. Saito. Bernstein-Sato polynomials of hyperplane arrangements. *Selecta Math. (N.S.)*, 22(4):2017–2057, 2016.

[23] U. Walther. Bernstein-Sato polynomial versus cohomology of the Milnor fiber for generic hyperplane arrangements. *Compos. Math.*, 141(1):121–145, 2005.

[24] U. Walther. The Jacobian module, the Milnor fiber, and the $D$-module generated by $f^n$. *Invent. Math.*, 207(3):1239–1287, 2017.

[25] L. Wu. Bernstein-Sato ideals and hyperplane arrangements. *arXiv e-prints*, page arXiv:2005.13502, May 2020.

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