Two different approaches for consistency of intuitionistic multiplicative preference relation using directed graph

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Abstract
Consistency is an important issue that causes wide public concern of decision makers in the decision-making process. The lack of consistency in preference relations results in a vague solution. The main goal of this paper is to achieve the consistent intuitionistic multiplicative preference relation using a graphical approach. We have proposed two different approaches of the consistency for intuitionistic multiplicative preference relation (IMPR). In the first approach, we propose an algorithm to achieve the consistency of IMPR by using the cycles of various lengths in a directed graph. The second approach proves isomorphism between IMPRs and asymmetric multiplicative preference relations. The result explores the methodologies developed for asymmetric multiplicative preference relations in the case of IMPRs and achieves the consistency of asymmetric multiplicative preference relations using a directed graph. Sometimes the decision maker may not be able to provide the complete reference. So the above-said method is applied for an incomplete IMPR; consistency plays an important role here. The examples are provided to illustrate both the ways in all cases.

Keywords
Isomorphism · Consistency · Directed graph · Cycle · Intuitionistic multiplicative preference relation · Asymmetric multiplicative preference relation

1 Introduction

In decision-making problems, preference relation is one of the most valuable tools decision makers (DMs) use, as it can express their views over the alternatives/criteria. Plenty of research has been done in the past decades about preference relations. Among these efforts, the fuzzy preference relations (Orlovsky 1978) and the multiplicative preference relations (Saaty 1980) are two types of preference relations that many researchers widely study. The multiplicative preference relations have been intensely studied and successfully applied in the analytic hierarchy process (AHP)(Saaty 1980). In fuzzy preference relation (FPR), the expert provides the preference information using the 0-1 scale, and a multiplicative preference relation (MPR) (Saaty 1980), in which the (1/9)-9 ratio scale is used to measure the intensity of the pairwise comparison of alternatives.

The essential elements indicate that one alternative is before the other in these preference relations. However, sometimes DMs may also need to give the degree that an option is not before the other in practical problems. In these cases, DM is unsure about the preference information, namely hesitation degree (uncertainty degree). Intuitionistic fuzzy preference relations (IFPRs) (Xu 2007) and intuitionistic multiplicative preference relations (IMPRs) (Xia et al. 2013) are proposed to indicate the positive information \(x_i\) is preferred to \(x_j\), and simultaneously the degree that cannot be determined by the DM.

Consistency, a fundamental property of preference relations, in the decision-making process, has drawn much attention in many decision-making fields. The level of agreement among the preference data given by the individual DMs measures consistency (Xia et al. 2013). Based on the multiplicative consistency property, Xia et al. 2013 discussed the consistency of FPRs and developed an algorithm to improve the consistency level of FPRs. Herrera-Viedma et al. (2004)
developed a method for constructing consistent FPRs and using the additive transitivity property of the FPRs, and they proposed a characterization of the consistency property.

To improve the consistency of an MPR, Xu and Wei (1999) proposed an algorithm to derive a definite reciprocal matrix with acceptable consistency. Ergu et al. (2011) developed a method to measure the consistency for MPRs.

Xu et al. (2013) first proposed the ordinal consistency of an FPR, the authors proposed how to find the cycles in the direct graphs. In addition, they also proposed the process of calculating the ordinal consistency index and positioning each cycle, as well as the process of finding inconsistent judgments in the preference relationship. To repair the inconsistency in FPR, they developed an algorithm to find and eliminate the 3 cycles on the graph. Xu et al. (2019) first proposed the ordinal consistency of incomplete fuzzy linguistic preference relations. Also, they developed one novel algorithm to judge whether an incomplete FLPR is ordinally consistent. To rectify the inconsistency of an incomplete FLPR, they developed another algorithm and improved it for ordinally consistent. Using the proposed algorithms, they identify and rectify the ordinal inconsistency of the complete FLPRs. Xu et al. (2018) studied both the ordinal and multiplicative consistencies for FPRs. To measure the degree of ordinal consistency for FPRs, they introduced a new ordinal consistency index and developed a new multiplicative inconsistency identification and modification method for FPR. Xu et al. (2021) developed an algorithm to rectify the multiplicative and ordinal inconsistencies for FPRs. Xu et al. (2019) proposed a graphical method to visualize and rectify different inconsistencies for FPRs. They draw Gower plots to visualize the ordinal and cardinal inconsistencies on a two-dimensional plane using ordinal and cardinal consistency. They formulated some optimization models to rectify different types of inconsistencies, respectively. Xu et al. (2016) proposed a distance-based methodology is proposed to deal with ordinal and additive inconsistencies for FPR.

Many researchers are committed to using IFPR to make decisions in uncertain environments (Xu and Liao 2015). The definitions of feasible-region-based consistency and priority derivation methods were proposed for IFPRs (Behret 2014; Gong et al. 2011, 2009).

Jiang et al. (2015) discussed the consistency and acceptably consistent of an IMPR. Based on it, two approaches were developed to complement all missing elements of incomplete IMPRs. Ren et al. (2016) defined the IMWGA operator. They proposed an iterative process to adjust the inconsistent IMPRs into an acceptably consistent one and gradually repair the maximum in conformity, making fewer changes to the DMs’ original opinions. Also, they provide an adjustment process to restore and improve the consistency of inconsistent IMPR. Zhang and Guo (2017) developed a linear programming-based algorithm to check and adjust the flexibility of an IMPR. As well, they discuss the relationships between an IMPR and a normalized intuitionistic multiplicative weight vector and developed two group decision methods based on complete and incomplete IMPRs, respectively. Zhang and Pedrycz (2018) manage the unity and consensus of IMPR and establish a consistent and consensus-based method through various proposed goal programming models to address group decision making (GDM) with IMPR. In this paper, our work focuses on only IMPRs.

Nishizawa (1995) proposed an algorithm checking for the consistency of MPRs by using the cycle of a directed graph. After that, Nishizawa (1996) proposed two algorithms to find the various odd and even length cycles in the incomplete directed graph. Taking inspiration from Nishizawa (1996), in this paper, we have developed two approaches to improve the consistency of IMPR. In the first approach, we have formed an ordered-pair binary matrix from the IMPR, which is split into two binary arrays and using Nishizawa (1996) algorithm, we have checked the consistency of IMPR.

The subsequent methodology of this paper is to demonstrate the isomorphism between the set of IMPR and the set of asymmetric multiplicative preference relations. This outcome can accordingly be abused to utilize techniques created for MPRs to the instance of IMPR and, eventually, to expand the utilization of IMPR in decision making and to overcome the computation of the complexity mentioned above. In other words, this result will allow taking advantage of mature and well-defined methodologies developed for MPRs while controlling the flexibility of IMPR to model vagueness/uncertainty.

As shown in work mentioned above, the preference relations are generally complete information. In the decision-making process, DM required \( \frac{n(n-1)}{2} \) judgment at each level to present an entire preference relation, and when \( n \) is large, it becomes a tedious task. Here and there, DM may not yet have a good understanding of a particular question, and thus he/she cannot make a direct comparison between every two objects. Therefore it is sometimes necessary to allow the DM to skip some dubious comparisons flexibly. Thus, due to the DMs lack of time and busy schedule, incomplete preference relations are sometimes obtained. In this case, the whole process may slow down. Therefore, both approaches are applied in an incomplete IMPR scenario in this paper.

Indeed, an issue that can be addressed using the mentioned equivalence is the presence of incomplete IMPR in the decision-making process. Using a directed graph (Nishizawa 1996), we check the consistency of asymmetric multiplicative preference relation. A new section has been added that compares the two approaches and improves the consistency of IMPR.

The rest of the paper is arranged as follows. Section 2 represents some basic concepts of MPRs and IMPRs. We have developed two approaches to check the consistency of
IMPR given in Sects. 3 and 4. Section 3 demonstrates the first approach in which we have developed an algorithm that improves the consistency of IMPRs and is discussed with some examples. This section includes incomplete IMPRs also. The second approach in Sect. 4, the set of IMPR and the set of asymmetric multiplicative preference relations, has proved mathematically isomorphic. This section also gives numerical examples for both complete and incomplete intuitionistic scenarios. In this section, we have compared the two approaches and improved the consistency of IMPR. Section 5 gives a comparative analysis of the proposed methodology relevant to the context. Concluding remarks are given in the last section.

2 Preliminaries

Several basic concepts related to MPR and IMPR are introduced, and accordingly, the idea of incomplete IMPR is defined.

Definition 1 (Saaty 1980) MPR $R = (r_{ij})_{n \times n}$ is called multiplicative consistent if it satisfied the multiplicative transitivity property

$$r_{ij} = r_{ik}r_{kj}, \ \forall \ i, j, k = 1, 2, \ldots, n.$$ \hspace{1cm} (1)

In this case, there exists a positive vector $w = (w_1, w_2, \ldots, w_n)^T$ such that $r_{ij} = \frac{w_i}{w_j}$ satisfying the conditions $\sum_{i=1}^n w_i = 1$, $w_i \geq 0$, $i, j \in N$, where $w_i$ is the priority weight.

In general cases, equation 1 does not always hold; therefore, the MPRs are inconsistent. Saaty (1980) gave a method to measure the consistency degree of an MPR for an inconsistent MPR $R$, by the concept of consistency index (CI) and consistency ratio (CR).

$$CI = \frac{\lambda_{max} - n}{n - 1}, \quad CR = \frac{CI}{RI_n}$$ \hspace{1cm} (2)

where $\lambda_{max}$ and $n$ are the largest eigenvalue and the order of the MPR $R$, respectively. $RI_n$ is the random index that depends on the orders of the preference relation matrices given in Table 1.

Definition 2 An MPR $R$ is said to be acceptably consistent if $CR \leq 0.1$, else $R$ is announced not acceptably consistent (or simply inconsistent) and returned to an expert to have a re-look at the preferences evaluations.

Expressly, if the value of the consistency ratio $(CR) < 0.1$, the comparison matrix is acceptable. There is no standard for matrix consistency. According to some experiments and experiences, the degree of inconsistency is 10% of the degree of inconsistency still acceptable.

Nishizawa, in 1995, proposed an algorithm for the consistency of MPRs by using the cycle of a directed graph. Later, Nishizawa (1996) proposed two algorithms to find the cycles of various odd and even lengths using an incomplete directed graph. For finding cycles, vertex matrix that is denoted by $V$ of order $n \times n$ is needed, among them, $n$ is the number of vertices, corresponding to the element $(i, j)$, which is a directed graph determined by $V(i, j)$. If one points $i$ is connected to another point $j$ by an arrow, say “$i \rightarrow j$,” then $V(i, j) = 1$ otherwise 0. Pairwise comparison data are represented by $\theta$, where $\theta$ is the parameter whose binary AHP value is greater than 1 (Saaty 1977). Nishizawa (1996) proposed two algorithms for even and odd-length cycles in the incomplete directed graph. From the result of Nishizawa (1996), one can easily judge the consistency of the comparison matrix. For the sake of convenience, the following two algorithms are given by Nishizawa (1996) for finding the cycle of odd and even length.

Algorithm 1 (Finding even length cycle) Nishizawa (1996) discussed the cycle of even length $2m$ $(m = 2, 3, \ldots)$, of $n$ elements be the form of $(i_0, i_1, \ldots, i_{m-1}, j_0, j_1, \ldots, j_{m-1})$.

1. Find the vertex matrix $V$ and $V^m$.
2. Find pair of numbers $i_0$ and $j_0$ with $V^m(i_0, j_0) > 0$ and $V^m(j_0, i_0) > 0$, where $i_0 < j_0$. Without these pairs, there are no cycles that are $2m$ in length.
3. Find a pair of numbers $i_\alpha(\alpha > i_0)$ and $j_\alpha(\alpha > i_0)$ under the conditions $V(i_{\alpha-1}, i_\alpha) = 1$, $V^m(i_\alpha, j_\alpha) > 0$, $V^m(j_\alpha, i_\alpha) > 0$, and $V(j_{\alpha-1}, j_\alpha) = 1$ for each $\alpha = m - 1$.
4. Confirm $V(i_{m-1}, j_0) = 1$ and $V(j_{m-1}, i_0) = 1$.
5. Repeat step 2 to 4 for all pairs of $i_0$, $j_0$ where $i_0 < j_0$.

Algorithm 2 (Finding odd length cycle) Nishizawa (1996) discussed the cycle of odd length $2m - 1$ $(m = 2, 3, \ldots)$, of $n$ elements be the form of $(i_0 i_1 \ldots i_{m-2} j_1 i_1 i_3 \ldots i_{m-2} j_2 i_0 j_1 \ldots j_{m-1} j_{m-2} \ldots j_2 j_0)$ if $m$ is odd, and $(i_0 i_2 \ldots i_{m-2} j_3 \ldots j_1 i_3 i_1 i_3 \ldots i_{m-3} j_{m-2} \ldots j_2 j_0)$ if $m$ is even.

1. Find the vertex matrix $V$, $V^{m-1}$, and $V^m$.
2. Find pair of numbers $i_0$, $k$ and $j_0$ with $V^{m-1}(i_0, k) > 0$ and $V^m(k, i_0) > 0$, $V^{m-1}(j_0, k) > 0$, $V^m(j_0, k) > 0$, and $V(j_0, i_0) = 1$, where $i_0 < k$, $i_0 < j_0$. Without these pairs, there are no cycles of length $2m - 1$.
3. Find a pair of numbers $(i_1(\alpha > i_0)$ and $j_1(\alpha > i_0)$ with $V^{m-1}(i_1, i_0) > 0$, $V^{m-1}(i_1, j_0) > 0$, $V(k, i_1) = 1$, $V^{m-1}(j_0, j_1) > 0$, $V^m(j_1, j_0) > 0$, and $V(j_1, k) = 1$.

| Table 1 | Random Index |
|---------|-------------|
| n      | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| RI     | 0  | 0  | 0.52 | 0.89 | 1.11 | 1.25 | 1.35 | 1.40 | 1.45 | 1.49 |
4. Find a pairs of numbers \( i_a > i_0 \) and \( j_a > i_0 \) under the conditions \( V^m(i_a-1, i_a) > 0, V^{m-1}(i_a, i_a-1) > 0, V(i_a-2, i_a) = 1, V^{m-1}(j_a-1, j_a) > 0, V^m(j_a, j_a-1) > 0, \) and \( V(j_a, j_a-2) = 1 \) for each \( \alpha = 2 \sim m - 2. \)

5. Confirm \( V(i_m-2, j_m-3) = 1 \) and \( V(i_m-3, j_m-2) = 1. \)

6. Repeat step 2 to 4 for all pairs of \( i_o, k, j_0 \) where \( i_0 < k, i_0 < j_0. \)

If no cycles have been found in the directed graph, then the comparison matrix is consistent. The comparison matrix is inconsistent if at least one cycle is found in the incomplete directed graph. In case of inconsistency, they find the minimum covering sets (Nishizawa 1995) among the cycles, and then they eliminate the path of cycles such as the comparison matrix is consistent.

In this paper, we have discussed two approaches for checking the consistency of IMPRs. Here we are extending the above method of Nishizawa (1996) in the IMPR scenario. For this, we recall some basic concepts of IMPRs, incomplete IMPRs, and the consistency property. Xia et al. (2013) extended MPR into IMPR.

**Definition 3** (Xia et al. 2013) Let \( \tilde{R} = [\tilde{r}_{ij}(x_i, x_j)]_{n \times n} \) be an intuitionistic multiplicative preference relation (IMPR), where \( \tilde{r}_{ij}(x_i, x_j) = (u(x_i, x_j), v(x_i, x_j)), i, j \in N, \) is an intuitionistic multiplicative number (IMN), and \( u(x_i, x_j) \) and \( v(x_i, x_j) \) indicates certainty degree to which \( x_i \) is preferred to \( x_j \) and \( v(x_i, x_j) \) is the certainty degree to which \( x_j \) is not preferred to \( x_i \), and it satisfies the following characteristics:

\[
1/9 \leq u(x_i, x_j), v(x_i, x_j) \leq 9, u(x_i, x_j) = v(x_i, x_j), \quad u(x_i, x_j) = u(x_j, x_i),
\]

\[
u(x_i, x_i) = v(x_i, x_i) = 1, \quad 0 < u(x_i, x_j) v(x_j, x_j) \leq 1, \quad \forall i, j \in N.
\]

If \( u(x_i, x_j), v(x_i, x_j) = 1, \forall i, j \in N, \) the IMPR is equivalent to an MPR. The value of \( u(x_i, x_j), v(x_i, x_j) \) is not arbitrary. For easy understanding, \( u(x_i, x_j) \) and \( v(x_i, x_j) \) are denoted by \( u_{ij} \) and \( v_{ij} \), respectively.

Xu (2013) has given the consistent property of IMPR \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} = (u_{ij}, v_{ij})_{n \times n} \) based on the transitive property:

\[
(u_{ij}, v_{ij}) = (u_{ik}u_{kj}, v_{ik}v_{kj}), \quad \text{for all } i, j, k \in N \quad \text{and} \quad i \leq k \leq j.
\]  

It is to note that equation (3) is restricted for the condition \( i \leq k \leq j \), while the transitive property of an MPR is unconstrained, which satisfies for all \( i, j, k \in N \). If the equation (3) is utilized to check the consistency of an IMPR for all \( i, j, k \in N \), the transitivity and consistency properties sometimes do not hold. This is because when ‘\( k \)’ comes from the row of the lower triangular matrix, the equation does not hold. For example,

\[
\begin{pmatrix}
(1, 1) & (1/2, 1) & (1, 1/2) \\
(1/2, 1) & (1, 1) & (2, 1/2) \\
(1/2, 1/2, 1) & (2, 1, 1/2) & (1, 1)
\end{pmatrix}
\]

is a consistent IMPR given by Xu et al. (2013). Jiang et al. (2015) relax the condition \( i \leq k \leq j \); it follows that \( a_{23} = (u_{21}u_{12}, v_{21}v_{12}) = (1, 1/4) \). But \( a_{23} = (2, 1/2) \neq (1, 1/4) \). To overcome this type of transitivity limitation, Jiang et al. (2015) proposed a more general consistency property of an IMPR split into two MPRs by using the formula

\[
c_{ij} = \begin{cases}
u_{ij} & i < j, \\1/\nu_{ij} & i > j, \\
1 & i = j.
\end{cases} \quad \text{(4)}
\]

where the MPRs \( C = (c_{ij})_{n \times n} \) and \( D = (d_{ij})_{n \times n} \) are preferred and non-preferred information matrix given by the DM with respect to the alternative \( x_i \) over \( x_j \). Based on the above concept, Jiang et al. (2015) defined the consistent IMPR.

Jiang et al. (2015) relax the condition \( i \leq k \leq j \) and proposed a more general consistency property of IMPR to overcome this type of transitivity limitation.

**Definition 4** (Jiang et al. 2015) An IMPR \( \tilde{R} = (\tilde{r}_{ij})_{n \times n} \) is said to be consistent if both MPRs \( C \) and \( D \) obtained by splitting the IMPR \( \tilde{R} \) are consistent such that

\[
c_{ij} = c_{ik}c_{kj}, \quad d_{ij} = d_{ik}d_{kj} \quad \forall i, j, k \in N \quad \text{(5)}
\]

In this paper, we have developed two different approaches to check the consistency of IMPR and incomplete IMPRs, which are discussed in Sects. 3 and 4, respectively.

### 3 Consistency for IMPR and incomplete IMPR

The absence of consistency in decision making with preference relations is a big challenge to bring about conflicting conclusions. Numerous strategies on consistency measure and improvement of preference relations with various structures have been exhibited progressively. This section checks the consistency of both IMPR and incomplete IMPR.

Let \( \tilde{R} \) be the IMPR

\[
\tilde{R} = \begin{pmatrix}
(u_{11}, v_{11}) & (u_{12}, v_{12}) & \cdots & (u_{1j}, v_{1j}) & \cdots & (u_{1n}, v_{1n}) \\
(u_{21}, v_{21}) & (u_{22}, v_{22}) & \cdots & (u_{2j}, v_{2j}) & \cdots & (u_{2n}, v_{2n}) \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
(u_{ni}, v_{ni}) & (u_{nj}, v_{nj}) & \cdots & (u_{ni}, v_{nj}) & \cdots & (u_{nn}, v_{nn})
\end{pmatrix}
\]
Two different approaches for consistency...

We define a ordered pair vertices matrix $V = \{v_{ij}\} = \{(a, b)\}$ as follows, where $v_{ij}$ is an ordered pair $(a, b)$ such that

$$
\begin{align*}
\text{For } i < j & \quad \text{if } u_{ij} > 1 \text{ then } a = 1, \text{ otherwise } a = 0 \\
\text{ if } v_{ij} > 1 \text{ then } b = 1, \text{ otherwise } b = 0 \\
\text{For } i > j & \quad \text{if } \frac{1}{u_{ij}} > 1 \text{ then } a = 1, \text{ otherwise } a = 0 \\
\text{ if } \frac{1}{v_{ij}} > 1 \text{ then } b = 1, \text{ otherwise } b = 0 \\
\text{For } i = j & \quad \text{both } a = 0 \text{ and } b = 0.
\end{align*}
$$

(6)

To illustrate this, consider the following IMPR

$$(1, 1) \quad (1/2, 2/3) \quad (1/5, 5) \quad (2/3, 1/2) \quad (1, 1) \quad (4/5, 3/4) \quad (5, 1/5) \quad (3/4, 4/5) \quad (1, 1)$$

Using the binary order paired vertex matrix, $V$ is given below.

$$V = \begin{pmatrix}
(0, 0) & (0, 0) & (0, 1) \\
(1, 0) & (1, 0) & (0, 0)
\end{pmatrix}
$$

Here, the binary order paired vertex matrix is split into two vertex matrices, i.e., lower vertex matrix $V_L$ containing the lower element of order pair and upper vertex matrix $V_U$ containing the upper element of order pair.

$$V_L = \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
1 & 1 & 0
\end{pmatrix}, \quad V_U = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
$$

Using the concept of order pair, we have defined a new definition of consistent IMPR.

**Definition 5** The IMPR is consistent if both the corresponding lower and upper vertex matrices are consistent. If there is no cycle found from the lower and upper vertex matrices, then both the lower and upper vertex matrices are consistent using algorithm 1 and 2. Otherwise, it is inconsistent.

In this section, we have applied the graphical approach of Nishizawa (1996) to check the consistency of both complete IMPRs and incomplete IMPRs scenario. We have developed an algorithm to illustrate the above method.

**Algorithm 3** Let us consider an IMPR.

1. Order pair vertex matrix is obtained using equation 6. In the case of incomplete IMPRs, missing elements are treated as zero in the vertex matrix.
2. The ordered pair vertex matrix is split into the lower vertex matrix and upper vertex matrix.
3. Then apply algorithm 1 and 2 of Sect. 2 for finding the cycle of even and odd length on both the vertex matrix.
4. If any cycle is found, then the vertex matrix is inconsistent, otherwise consistent.
5. IMPR is consistent if both the vertex lower matrix and the upper matrix are consistent; otherwise, the matrix is inconsistent.
6. If the IMPR matrix is inconsistent, then remove the minimum number of the path that covers the cycles.

An example illustrates the above-said method.

**Example 1** Let us consider a decision-making problem with four sets of alternatives $x_i, (i = 1, 2, 3, 4)$. The decision maker judge these four alternatives by pairwise comparison and provides his/her judgement as follows: $\tilde{r}_{12} = (1/2, 1/4)$, $\tilde{r}_{13} = (2, 1/8)$, $\tilde{r}_{14} = (2/3, 1/4)$, $\tilde{r}_{23} = (5, 1/7)$, $\tilde{r}_{24} = (7/5, 2/3)$, $\tilde{r}_{34} = (6, 1/7)$. The matrix representation is given by

$$\tilde{R}_1 = \begin{pmatrix}
(1, 1) & (1/2, 1/4) & (2, 5/7) & (3/4, 1) \\
(1/2, 1/4) & (1, 1) & (5, 1/7) & (7/5, 23/4) \\
(2, 5/7) & (5, 1/7) & (1, 1) & (6, 1/7) \\
(3/4, 1) & (7/5, 23/4) & (1, 1) & (6, 1/7)
\end{pmatrix}
$$

By using equation 6, the binary order pair vertex matrix is

$$V = \begin{pmatrix}
(0, 0) & (0, 0) & (1, 0) & (0, 0) \\
(1, 1) & (0, 0) & (1, 0) & (1, 0) \\
(0, 1) & (0, 1) & (0, 0) & (1, 0) \\
(1, 1) & (0, 1) & (0, 1) & (0, 0)
\end{pmatrix}
$$

The lower and upper vertex matrices are

$$V_L = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \quad V_U = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{pmatrix}
$$

There is only one cycle of length 3 present in the lower vertex matrix $V_L$. Directly, we can find the cycle from Fig. 1. To find cycles of length 3, we use step (2) to step (5) of odd-length algorithm (see algorithm 2), where $m = 2$. The format of a cycle of length 3 is $(i_0 - k - j_0)$.

![Fig. 1 Directed graph of $V_L$ of example 1](Image 354x96 to 496x169)
At the starting we have

\[ V_2 \]

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Then, IMPR

judges these seven alternatives by pairwise comparison and finding the step (2) of algorithm 2 of odd length cycle, i.e., 

\[ V_L(1, 3) = 1, V_L^2(3, 1) = 1, V_L(3, 4) = 1, V_L^2(4, 3) = 1, \]

and 

\[ V_L(4, 1) = 1. \]

Then, the cycle of length 3 is \((1 – 3 – 4)\). There is no other cycle available in \(V_L\). Similarly, there is no cycle of any length in Fig. 2. The lower vertex matrix 

\[ V_L \]

is inconsistent, and the upper vertex matrix \(V_U\) is consistent. Therefore, the complete IMPR is inconsistent. Apply an extinguishing cycles algorithm based on minimum covering sets to provide the cause of inconsistency (see Nishizawa 1995). The cycle-arc incidence matrix is given in Table 2.

From Table 2, we get the several pairs of edges that cover the cycle. We have to choose the pair to eliminate the cycle. In this example, if we will change the pair of the vertex from any one of them from \((3, 4)\) to \((4, 3)\) in the original IMPR \(\vec{R}_1\), then both the lower and upper vertex matrix is consistent. Then, IMPR \(\vec{R}_1\) is also consistent.

**Example 2** Let us consider a decision-making problem with seven sets of alternatives \(x_i\), \((i = 1, 2, \ldots, 7)\). The DM judges these seven alternatives by pairwise comparison and provides his/her judgment. The matrix representation of the decision-maker judgment is given by

\[ R_2 = \begin{pmatrix}
(1, 1) & (\frac{5}{7}, \frac{1}{7}) & (\frac{5}{7}, \frac{1}{7}) & (\frac{5}{7}, \frac{1}{7}) & (\frac{1}{4}, \frac{1}{4}) & (\frac{1}{4}, \frac{1}{4}) \\
(\frac{1}{4}, \frac{1}{4}) & (1.1) & (\frac{5}{7}, \frac{1}{7}) & \ldots & \frac{1}{4}, \frac{1}{4} \\
(\frac{1}{4}, \frac{1}{4}) & \ldots & \ldots & \ldots & \frac{1}{4}, \frac{1}{4} \\
(\frac{1}{4}, \frac{1}{4}) & \ldots & \ldots & \ldots & \frac{1}{4}, \frac{1}{4} \\
(\frac{1}{4}, \frac{1}{4}) & \ldots & \ldots & \ldots & \frac{1}{4}, \frac{1}{4} \\
\end{pmatrix}
\]

Using equation 6, the binary order pair vertex matrix is given by

\[ V = \begin{pmatrix}
(0, 0) & (1, 0) & (1, 0) & (1, 0) & (1, 0) & (0, 0) & (0, 1) \\
(1, 0) & (0, 0) & (1, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 1) \\
(0, 0) & (0, 0) & (1, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 1) \\
(0, 1) & (1, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 1) \\
(1, 0) & (1, 0) & (1, 0) & (1, 0) & (1, 0) & (0, 0) & (0, 0) \\
(1, 0) & (1, 0) & (1, 0) & (1, 0) & (1, 0) & (0, 0) & (0, 0) \\
\end{pmatrix}
\]

The above vertex matrix is split into two binary arrays.

\[ V_L = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}, \quad V_U = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

By applying algorithm 1 and 2, there is no cycle found in 

\[ V_L (\text{Fig. 3}), \]

and one cycle of length three, i.e., \((3 \rightarrow 5 \rightarrow 4)\) is located in \(V_U\). In order to find the cycle of length 3 in \(V_U\) we use step (2) to step (5) of algorithm 2, where \(m = 2\). The form of the cycle of length 3 is \((i_0 \rightarrow k \rightarrow j_0)\). To locate these elements, we need \(V_U\) and \(V_U^2\). For this \(V_U^2\) is as follows:
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\[ V_U^2 = \begin{pmatrix} 
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
3 & 0 & 0 & 1 & 0 & 1 \\
3 & 1 & 0 & 0 & 1 & 1 \\
2 & 1 & 1 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \]

At the starting we have \( i_0 = 3, \ k = 5, \ j_0 = 4, \) satisfying \( V_U(3, 5) = 1, \ V_U^2(5, 3) = 1, \ V_U(5, 4) = 1, \ V_U^2(4, 5) = 1, \) and \( V_U(4, 3) = 1. \) Then, the cycle of length 3 is \((3 - 5 - 4).\) Similarly there is no cycle of any length in Fig. 4. The lower vertex matrix \( V_L \) is consistent, and the upper vertex matrix \( V_U \) is inconsistent. Therefore, the complete IMPR is inconsistent.

We apply an extinguishing cycles algorithm based on minimum covering sets to provide the cause of inconsistency (see Nishizawa 1995). The cycle-arc incidence matrix is given in Table 3. From Table 3, we get the several pairs of edges that cover the cycle. We have to choose the pair to eliminate the cycle. In this example, if we will change the pair \((3, 4)\) to \((4, 3)\) in the original IMPR \(\tilde{R}_2,\) then both the lower and upper vertex matrix are consistent. Then, IMPR \(\tilde{R}_2\) is consistent.

The above-said method is also applied in the incomplete IMPRs scenario, which is given in the following subsection.

### 3.1 Consistency of incomplete IMPR

In a decision-making problem, the reality of the situation may prove that the decision maker might not have a decent comprehension of a specific inquiry, and along these lines, he/she cannot make an immediate correlation between two choices or criteria. Therefore, it is more fitting and adaptable to avoid a few similarities, and in those cases, the decision maker may like to express their judgments with incomplete preference relation. As indicated by the already talked about, the decision makers may not give exactly \(\frac{n(n-1)}{2}\) judgments in practical decision making, and the incomplete IMPR will be introduced. So, it is essential to investigate incomplete preference relations as a useful tool in the decision-making problem, and many research results have been developed. Herrera-Viedma et al. (2007) proposed the meaning of incomplete preference relation. The idea of IMPRs is to reach out to the circumstances where the preference data given by DM is incomplete.

**Definition 6** (Jiang et al. 2015) An IMPR \(\tilde{r}_{ij} = (u_{ij}, v_{ij})_{n \times n}\) is called an incomplete IMPR if some elements in it are missing, and all available elements satisfy the characteristics of IMPR stated in Definition 3.

For an incomplete preference relation, it is very much essential for known elements to satisfy consistency. We are applying the same graphical approach in incomplete IMPRs scenario. For an incomplete comparison case, which includes unknown pairwise comparisons, we use the measure of inconsistency by the number of the cycle in the graph corresponding to known elements of the IMPR matrix. For the incomplete preference relation, if one wants to find the inconsistency, one condition is that the incomplete preference relation is acceptable; that is, in its directed graph, there is no isolated point.

**Example 3** Let us consider a decision-making problem with six sets of alternatives \(x_i, (i = 1, 2, 3, 4, 5, 6).\) The decision maker judge these six alternatives by pairwise comparison and provides his/her judgement as follows: \(\tilde{r}_{12} = (1/5, 2), \tilde{r}_{13} = (5, 1/6), \tilde{r}_{15} = (1/9, 8), \tilde{r}_{16} = (2, 1/6),\)
The lower and upper vertex matrices are given below

\[
V_L = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}, \quad V_U = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

To find the cycle of length 3 in the lower vertex matrix \(V_L\), we need \(V_L^2\) and \(V_L^3\). For this \(V_L^2\) is as follows:

\[
V_L^2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

At the starting we have \(i_0 = 1, k = 3, j_0 = 5\), satisfying the step (2) of algorithm 2, i.e., \(V_L(1, 3) = 1, V_L^2(3, 1) = 1, V_L(3, 5) = 1, V_L^2(5, 3) = 1\), and \(V_L(5, 1) = 1\). Then the cycle of length 3 is \((1 - 3 - 5)\).

Similarly, using algorithm 2, there is also one cycle of length 3 in \(V_U\). Here also, for finding the cycle of length 3 in \(V_U\), we need \(V_U\) and \(V_U^2\). For this \(V_U^2\) is as follows:

\[
V_U^2 = \begin{pmatrix}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0
\end{pmatrix}
\]

Also we have \(i_0 = 1, k = 5, j_0 = 3\), satisfying \(V_U(1, 5) = 1, V_U^2(5, 1) = 1, V_U(5, 3) = 1, V_U^2(3, 5) = 1\), and \(V_U(3, 1) = 1\). Then, the cycle of length 3 is \((1 - 5 - 3)\).

Both the lower and upper vertex matrices \(V_L\) and \(V_U\) are inconsistent (See Figs. 5 and 6). Therefore, the incomplete IMPR is inconsistent. To suggest the cause of inconsistency, we apply an algorithm for extinguishing cycles based on minimum covering sets. The cycle-arc incidence matrix is given in Table 4.

| Cycle   | (1,5) | (1,3) | (3,5) |
|---------|-------|-------|-------|
| (1 - 3 - 5) | 1     | 1     | 1     |
| (1 - 5 - 3) | 1     | 1     | 1     |
From Table 4, we get several pairs of the edges that cover the cycle. We have to choose the couple to eliminate the cycle. In this example, if we will change the pair (3, 5) to (5, 3) in the original IMPR $\tilde{R}_4$, then both the lower and upper vertex matrices are consistent. Then, incomplete IMPR $\tilde{R}_4$ is also consistent.

**Example 4** Let us take an example of incomplete IMPRs of order $10 \times 10$ given as below.

$$\tilde{R}_4 = \begin{bmatrix}
(1, 1) & * & (\frac{1}{2}, \frac{1}{3}) & (\frac{1}{3}, 2) & (3, \frac{1}{6}) & * & (\frac{8}{9}, \frac{1}{8}) & (\frac{7}{9}, 4) & (5, \frac{1}{5}) \\
* & (1, 1) & (\frac{2}{3}, \frac{6}{19}) & (\frac{1}{2}, \frac{1}{4}) & (\frac{1}{2}, 1) & * & (\frac{3}{5}, 5) & * & (\frac{1}{3}, 3) \\
(\frac{1}{5}, \frac{2}{3}) & (\frac{1}{6}, \frac{3}{5}) & (1, 1) & * & (\frac{2}{3}, \frac{4}{9}) & (4, \frac{7}{8}) & (7, \frac{1}{8}) & (\frac{1}{5}, 5) & * \\
(2, \frac{1}{3}) & (\frac{1}{2}, \frac{2}{3}) & (\frac{3}{5}, \frac{5}{7}) & (1, 1) & * & (3, \frac{7}{9}) & (\frac{5}{7}, 2) & * & (2, \frac{1}{3}) \\
(\frac{1}{6}, 3) & (1, \frac{1}{3}) & * & * & (1, 1) & (\frac{5}{7}, 7) & (8, \frac{1}{5}) & (\frac{1}{6}, 3) & * & (4, \frac{5}{7}) \\
* & * & (\frac{1}{2}, \frac{1}{2}) & (\frac{2}{3}, \frac{3}{7}) & (7, \frac{5}{8}) & (1, 1) & * & (\frac{1}{3}, 4) & (2, \frac{1}{3}) \\
* & (\frac{5}{7}, \frac{3}{4}) & (\frac{1}{5}, \frac{4}{7}) & (\frac{3}{5}, \frac{5}{8}) & * & (1, 1) & (\frac{5}{7}, 6) & (\frac{1}{5}, 7) & (2, \frac{4}{7}) \\
(\frac{1}{4}, \frac{7}{8}) & * & (\frac{3}{5}, \frac{7}{8}) & * & (6, \frac{4}{7}) & (1, 1) & * & (6, \frac{1}{5}) \\
(4, \frac{1}{7}) & (3, \frac{1}{5}) & (5, \frac{1}{7}) & (\frac{2}{3}, 2) & * & (4, \frac{1}{7}) & (2, \frac{1}{3}) & * & (1, 1) & (\frac{3}{2}, 2) \\
(\frac{1}{5}, \frac{5}{4}) & * & * & (\frac{1}{3}, 4) & (\frac{1}{2}, \frac{2}{3}) & (\frac{2}{3}, 4) & (\frac{1}{5}, 6) & (2, \frac{1}{3}) & (1, 1)
\end{bmatrix}$$

The binary order pair vertex matrix is given by

$$V = \begin{bmatrix}
(0, 0) & (0, 0) & (0, 0) & (0, 1) & (1, 0) & (0, 0) & (0, 0) & (1, 0) & (0, 0) & (0, 0) \\
(0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) & (0, 0) \\
(1, 1) & (1, 1) & (0, 0) & (0, 0) & (0, 0) & (1, 0) & (1, 0) & (1, 0) & (1, 0) & (1, 0) \\
(1, 0) & (1, 1) & (1, 1) & (0, 0) & (0, 0) & (1, 0) & (1, 0) & (0, 0) & (1, 0) & (0, 0) \\
(1, 0) & (1, 1) & (1, 0) & (0, 0) & (0, 0) & (1, 0) & (1, 0) & (0, 0) & (1, 0) & (0, 0) \\
(1, 0) & (1, 0) & (0, 1) & (0, 1) & (1, 0) & (0, 0) & (0, 0) & (1, 0) & (1, 0) & (1, 0) \\
(1, 0) & (1, 0) & (0, 1) & (0, 1) & (1, 0) & (0, 0) & (0, 0) & (1, 0) & (1, 0) & (1, 0) \\
(1, 0) & (1, 0) & (0, 1) & (0, 1) & (1, 0) & (0, 0) & (0, 0) & (1, 0) & (1, 0) & (1, 0) \\
(0, 1) & (0, 0) & (0, 0) & (0, 0) & (0, 1) & (0, 1) & (0, 1) & (0, 1) & (0, 1) & (0, 1) \\
(0, 1) & (0, 0) & (0, 0) & (0, 0) & (0, 1) & (0, 1) & (0, 1) & (0, 1) & (0, 1) & (0, 1)
\end{bmatrix}$$

The lower and upper vertex matrices are given below:

$$V_L = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 2 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix}, \quad V_U = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}$$

There are three cycles of length 3 found in the lower vertex matrix, i.e., $(1 - 10 - 9), (6 - 10 - 9)$ and $(7 - 10 - 9)$. In order to find the cycle of length 3 in $V_L$, we use step (2) to step (5) of algorithm 2, where $m = 2$. The form of the cycle of length 3 is $(i_0 - k - j_0)$. To find the element of the cycle, we need $V_L^2$ as follows:

For the cycle $(1 - 10 - 9)$, we have $i_0 = 1, k = 10, j_0 = 9$, satisfying $V_L(1, 10) = 1, V_L^2(10, 1) > 0, V_L(10, 9) > 0, V_L^2(9, 10) > 0, V_L(9, 1) = 1$. Then, the cycle of length 3 is $(1 - 10 - 9)$. Similarly for the cycle $(6 - 10 - 9)$, we have $i_0 = 6, k = 10, j_0 = 9$, such that $V_L(6, 10) > 0, V_L^2(10, 6) > 0, V_L(10, 9) > 0, V_L^2(9, 10) > 0, V_L(9, 6) = 1$. For the cycle $(7 - 10 - 9)$, we have $i_0 = 7, k = 10, j_0 = 9$, such that $V_L(7, 10) > 0, V_L^2(10, 7) > 0, V_L(10, 9) > 0, V_L^2(9, 10) > 0, V_L(9, 7) = 1$.

![Fig. 8 Directed graph of $V_U$ of example 4](image-url)
Next we try to find out the cycle of length 4. The form of the cycle of length 4 is \((i_0 - i_1 - j_0 - j_1)\). By using algorithm 1, there are five cycle of length 4 that is \((5 - 10 - 9 - 6), (1 - 10 - 9 - 3), (3 - 6 - 10 - 9), (3 - 7 - 10 - 9)\), and \((3 - 8 - 10 - 9)\). To find cycle of length 4, we need \(V_L^3\). For the cycle \((5 - 10 - 9 - 6)\), we have \(i_0 = 5, j_0 = 9\), satisfying step (2) of algorithm 2, since \(V_L^3(5, 9) > 0\) and \(V_L^3(5, 9) > 0\), and we have \(i_1 = 10\) and \(j_1 = 6\) satisfying step (3) of even length cycle algorithm 2 for \(\alpha = 1\). Since \(V_L(5, 10) = 1, V_L(10, 6) > 0, V_L(6, 10) > 0\) and \(V_L(9, 6) = 1\). After that, it is confirm \(V_L(10, 9) = 1\) and \(V_L(6, 5) = 1\). Then, we have cycle of length 4 is \((5 - 10 - 9 - 6)\). Similarly cycle of length 4 is \((1 - 10 - 9 - 3), (3 - 6 - 10 - 9), (3 - 7 - 10 - 9)\), and \((3 - 8 - 10 - 9)\). To find the cycle of length 5, using the algorithm for odd cycles, we need \(V_L, V_L^2\) and \(V_L^3\). For this \(V_L^3\) is as follows:

\[
V_L^3 = \begin{bmatrix}
1 & 4 & 1 & 0 & 0 & 1 & 2 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 4 & 0 & 4 & 5 \\
1 & 5 & 0 & 6 & 1 & 5 & 3 & 3 & 10 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 3 & 0 & 0 & 4 & 0 & 4 & 1 & 3 & 7 \\
1 & 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 3
\end{bmatrix}
\]

The form of the cycle of length 5 is \((i_0 - j_1 - k - i_1 - j_0)\). We have \(i_0 = 3, k = 7, j_0 = 9\), satisfying \(V_L^3(3, 7) > 0, V_L^3(7, 9) > 0, V_L^3(9, 7) > 0\) and \(V_L(9, 3) = 1\). For \(\alpha = 1\), we have to find out \(i_1\) and \(j_1\) where \(i_1 = 10\) and \(j_1 = 8\) with \(V_L^3(3, 10) > 0, V_L^3(10, 3) > 0, V_L(7, 10) = 1, V_L(9, 8) > 0, V_L(8, 9) > 0\) and \(V_L(8, 7) = 1\). After that, it is confirm \(V_L(10, 9) = 1\) and \(V_L(3, 8) = 1\). Then, we have cycle of length 5 which is \((3 - 8 - 7 - 10 - 9)\). Similarly cycle of length 5 are obtained, i.e., \((3 - 6 - 5 - 10 - 9), (1 - 8 - 10 - 9 - 3), (1 - 5 - 10 - 9 - 3), and (5 - 7 - 10 - 9 - 6)\).

Next we will try to find the cycle of length 6. The form of the cycle of length 6 is \((i_0 - i_1 - i_2 - j_0 - j_1 - j_2)\). We have \(i_0 = 1, j_0 = 10\) with \(V_L^3(1, 10) > 0, V_L^3(10, 1) > 0\). For \(\alpha = 1\), we have to find the vertices \(i_1 = 5\) and \(j_1 = 9\) satisfying step (3) of algorithm 1 satisfying \(V_L(1, 5) = 1, V_L^3(5, 9) > 0, V_L^3(9, 5) > 0, V_L(10, 9) = 1\). Similarly, for \(\alpha = 2\) we have \(i_2 = 7\) and \(j_2 = 3\) satisfying step (4) of algorithm 2, i.e., \(V_L(5, 7) = 1, V_L^3(7, 3) > 0, V_L^3(3, 7) > 0, V_L(9, 3) = 1\). After that, it’s confirm \(V_L(7, 10) = 1\) and \(V_L(3, 1) = 1\). Then, we have cycle of length 6 is \((1 - 5 - 7 - 10 - 9 - 3)\). Other cycle of length 6 is given by \((1 - 8 - 7 - 10 - 9 - 3), (3 - 6 - 5 - 7 - 10 - 9), (3 - 8 - 5 - 7 - 10 - 9)\), and \((1 - 8 - 5 - 10 - 9 - 3)\). Similarly, one cycle of length 7 is found that is \((1 - 8 - 5 - 7 - 10 - 9 - 3)\). There is no cycle of lengths 8, 9, and 10 detected. For easy understanding, all the cycles of \(V_U\) are presented in Table 5 (Fig. 7).

Similarly, the possible cycles of the upper vertex matrix \(V_U\) are given in Table 6 (Fig. 8).

Since both the lower and upper matrices \(V_L\) and \(V_U\) are inconsistent; therefore, IMPR \(R_S\) is also inconsistent.

To suggest the cause of inconsistency, we apply an algorithm for extinguishing cycles based on minimum covering sets (Nishizawa 1995). In this example, from Tables 5 and 6, the paths \(2 - 7, 3 - 4, 9 - 10\) cover all the cycle. If we will change the pair \((3, 4)\) to \((4, 3), (2, 7)\) to \((7, 2), (9, 10)\) to \((10, 9)\), in the original IMPR \(R_S\), then the lower vertex matrix \(V_L\) and the upper vertex matrix \(V_U\) are both consistent. Then, IMPR \(R_S\) is also consistent.

### 4 Isomorphism between intuitionistic multiplicative preference relations (IMPR) and asymmetric multiplicative preference relation (AMPR)

Based on the asymmetrical relation concept, we have define asymmetric multiplicative preference relation as follows.

**Definition 7** Let \(X = \{x_1, x_2, \ldots, x_n\}\) be a finite set of alternatives and \(N = \{1, 2, \ldots, n\}, M = \{1, 2, \ldots, m\}\). An asymmetric MPR (AMPR) on \(X\) is characterized by a matrix \(A_{x_{xy}} = (a_{ij}) \subset X \times X\) with

\[
a_{ij} \in [1/9, 9], a_{ii} = 1, a_{ij} = a_{ji}, a_{ij} \leq 1, i, j \in N
\]

where \(a_{ij}\) represents the preferred degree of the alternative \(x_i\) over \(x_j\).

If \(a_{ij} = a_{ji} = 1\), then AMPR becomes an MPR.
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This section discusses the equivalence between the set of IMPR and the AMPR that leads to an AMPR from a given IMPR.

We obtained the vertex matrix from the AMPR and checked for consistency by using a directed graph in the IMPR scenario.

Consider an IMPR

\[ \tilde{R} = \begin{pmatrix} (u_{11}, v_{11}) & (u_{12}, v_{12}) & \cdots & (u_{1i}, v_{1i}) & \cdots & (u_{1n}, v_{1n}) \\ (u_{21}, v_{21}) & (u_{22}, v_{22}) & \cdots & (u_{2i}, v_{2i}) & \cdots & (u_{2n}, v_{2n}) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (u_{ni}, v_{ni}) & (u_{n2}, v_{n2}) & \cdots & (u_{ni}, v_{ni}) & \cdots & (u_{nn}, v_{nn}) \end{pmatrix} \]

The above relation can completely be characterized using just its upper triangular part, because the intuitionistic multiplicative element \((u_{ij}, v_{ij})\) is the mirror image of \((u_{ji}, v_{ji})\).

\[ U \tilde{R} = \begin{pmatrix} (u_{11}, v_{11}) & (u_{12}, v_{12}) & \cdots & (u_{1i}, v_{1i}) & \cdots & (u_{1n}, v_{1n}) \\ (u_{22}, v_{22}) & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ (u_{ii}, v_{ii}) & \cdots & \cdots & (u_{ii}, v_{ii}) & \cdots & (u_{nn}, v_{nn}) \end{pmatrix} \]

and this can be represented equivalently as the following matrix

\[ R = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1i} & \cdots & u_{1n} \\ v_{12} & u_{22} & \cdots & u_{2i} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ v_{ni} & v_{n2} & \cdots & v_{ni} & \cdots & v_{nn} \end{pmatrix} \]

Since \(u_{ij} = v_{ji}\) and \(v_{ij} = u_{ij}\), the above matrix \(R\) becomes

\[ R = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1i} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2i} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ u_{ni} & u_{n2} & \cdots & u_{ni} & \cdots & u_{nn} \end{pmatrix} \]

is an asymmetric MPR.

Let \(\tilde{Q}\) denote the set of IMPRs, where \(\tilde{Q} = \{ \tilde{R} = (\tilde{r}_{ij})|\forall i, j : \tilde{r}_{ij} = (u_{ij}, v_{ij})\}, u_{ij}, v_{ij} \in [1/9, 9], u_{ii} = v_{ii} = 1, u_{ij} = v_{ji}, u_{ij} = v_{ij}, 0 \leq u_{ij}v_{ij} \leq 1\) and \(\mathfrak{M}\) be the set of MPRs \(\mathfrak{M} = \{ R = (r_{ij})|\forall i, j : r_{ij} \in [1/9, 9]\}. Define a mapping \(f : [1/9, 9] \times [1/9, 9] \rightarrow [1/9, 9]\) by the function \(f(x_1, x_2) = x_1\). We can define the following mapping, \(F : \tilde{Q} \rightarrow \mathfrak{M}\) between the set of IMPRs \(\tilde{Q}\) and the set of MPRs, \(\mathfrak{M}\)

\[ \{ f(\tilde{r}_{ij}) \} = \{ u_{ij} \} \ i.e., R = F(\tilde{R}) \]

The following properties can be proved.
Proposition 1 Function $F$ is well defined, i.e., for given $\tilde{R} \in \tilde{Q} \Rightarrow f(\tilde{R}) \in \triangledown$.

Proof Let $\tilde{R} = (\tilde{r}_{ij}) \in \tilde{Q}$. Here $\tilde{r}_{ij} = (u_{ij}, v_{ij}) \Rightarrow f(\tilde{r}_{ij}) = f(u_{ij}, v_{ij}) = u_{ij} \in R$. □

Proposition 2 Function $F$ is one-one.

Proof Let $\tilde{R}_1 = (\tilde{r}_{ij}^1)$ and $\tilde{R}_2 = (\tilde{r}_{ij}^2)$ are IMPR such that $F(\tilde{R}_1) = F(\tilde{R}_2)$. Then, we have that

$$f(\tilde{r}_{ij}^1) = f(\tilde{r}_{ij}^2) \forall i, j \iff u_{ij}^1 = u_{ij}^2 \forall i, j.$$

Because of the conditions of $u_{ij}^1 = v_{ij}^1$ and $u_{ij}^2 = v_{ij}^2$, then it is obvious that $v_{ij}^1 = v_{ij}^2$, $\forall i, j$. Therefore, we have that

$$(u_{ij}^1, v_{ij}^1) = (u_{ij}^2, v_{ij}^2) \iff \tilde{R}_1 = \tilde{R}_2 \forall i, j.$$ □

For the function to be onto, the following conditions to be verified:

$$\forall R \in \triangledown \exists \tilde{R} \in \tilde{Q} : F(\tilde{R}) = R.$$ By the definition of $F$ and $\tilde{Q}$, we have that $R = (r_{ij}) = (u_{ij})$ satisfies:

$$0 \leq r_{ij} r_{ji} = u_{ij} u_{ji} \leq 1.$$ Thus, $R$ is asymmetric multiplicative preference relation that proves the range of the function $F$ is the subset of MPRs which are asymmetric.

Remark 1 Asymmetric MPR is not an MPR. It is a subset of MPR.

Theorem 1 The set of intuitionistic multiplicative preference relations is isomorphic to a set of asymmetric multiplicative preference relations.

Proof We know that when $\tilde{R} \in \tilde{Q}$ has hesitancy degree always zero, we have that:

$$u_{ij} v_{ij} = 1, \forall i, j$$ (7)

In this case, $F(\tilde{R}) = R$ is also reciprocal, i.e., $r_{ij} r_{ji} = 1 \forall i, j$. The proof of this is quite simple as we have the following:

$$\forall i, j : r_{ij} = f(\tilde{r}_{ij}) = u_{ij} \land r_{ji} = f(\tilde{r}_{ji}) = u_{ji}.$$ Since $\tilde{R} \in \tilde{Q}$ then we have that $u_{ji} = v_{ij} \forall i, j$ and by using equation 7 it is $r_{ij} r_{ji} = u_{ij} u_{ji} = u_{ij} v_{ij} = 1 \forall i, j$. □

Example 5 Isomorphic asymmetric multiplicative preference relation of IMPR of example 2 is

$$I_{\tilde{R}_2} = \begin{pmatrix}
1 & 5 & 7 & 3 & 5 & 1 & 4 \\
4 & 1 & 5 & 3 & 1 & 1 & 4 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
7 & 4 & 5 & 1 & 5 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 3 & 3 & 1 & 7 \\
5 & 3 & 7 & 3 & 3 & 1 \\
\end{pmatrix}$$

The vertex matrix of the above asymmetric MPR is

$$V = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\end{pmatrix}$$

By using algorithm 1 and 2 of odd and even length cycle, there is no cycle present in the vertex matrix $V$. Hence $\tilde{R}_2$ is consistent. Also, we can directly see from Fig. 9.

Example 6 Isomorphic asymmetric MPR of example 3 is

$$I_{\tilde{R}_3} = \begin{pmatrix}
1 & 1 & \frac{1}{5} & \frac{1}{5} & 2 \\
2 & 1 & \frac{1}{4} & \frac{1}{4} & 3 \\
\frac{1}{6} & 1 & 1 & 6 & * \\
* & \frac{1}{5} & 1 & \frac{1}{5} & * \\
8 & \frac{1}{3} & 4 & 1 & * \\
\frac{1}{6} & \frac{1}{5} & * & * & 1 \\
\end{pmatrix}$$

The directed graph of example 5 is shown in Fig. 9.
Using algorithm 1 and 2, there is one cycle of length 3, i.e., (1 – 3 – 5) is present. Therefore, \( \bar{R}_3 \) is inconsistent (Fig. 10).

In this example, if we will change the pair (3, 5) to (5, 3) in the original incomplete IMPR \( \bar{R}_3 \), then it is consistent.

**Example 7** Isomorphic asymmetric MPR of example 4 is

\[
V = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Vertex matrix of the asymmetric MPR, \( I_{\bar{R}_3} \) is

\[
I_{\bar{R}_3} = \begin{pmatrix}
1 & * & \frac{1}{2} & \frac{1}{3} & 3 & * & \frac{8}{3} & \frac{1}{7} & 5 \\
* & 1 & \frac{2}{3} & \frac{1}{4} & * & \frac{1}{5} & * & \frac{1}{5} & * \\
\frac{1}{5} & \frac{6}{10} & 1 & \frac{3}{5} & * & 2 & 4 & 7 & \frac{1}{8} & * \\
2 & \frac{1}{2} & 1 & * & 3 & 5 & * & 2 & * \\
\frac{1}{6} & 1 & * & * & \frac{1}{8} & 8 & \frac{1}{5} & * & 4 \\
* & \frac{1}{2} & 7 & 1 & * & \frac{1}{5} & 2 \\
* & 5 & \frac{1}{6} & \frac{1}{9} & * & 1 & \frac{1}{9} & 7 & 4 \\
\frac{1}{5} & \frac{1}{8} & * & 3 & 6 & 1 & * & 6 \\
4 & 3 & 5 & \frac{1}{2} & * & 4 & 2 & * & 1 & \frac{1}{3} \\
\frac{1}{5} & * & * & \frac{1}{8} & \frac{1}{5} & 1 & \frac{1}{8} & \frac{2}{1} & 1 \\
\end{pmatrix}
\]

Using algorithm 1 and 2, there is one cycle of length 3, i.e., (1 – 3 – 5) is present. Therefore, \( \bar{R}_3 \) is inconsistent (Fig. 10).

In this example, if we will change the pair (3, 5) to (5, 3) in the original incomplete IMPR \( \bar{R}_3 \), then it is consistent.

**Example 7** Isomorphic asymmetric MPR of example 4 is

\[
V = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Vertex matrix of \( I_{\bar{R}_4} \) is

\[
I_{\bar{R}_4} = \begin{pmatrix}
1 & * & \frac{1}{2} & \frac{1}{3} & 3 & * & \frac{8}{3} & \frac{1}{7} & 5 \\
* & 1 & \frac{2}{3} & \frac{1}{4} & * & \frac{1}{5} & * & \frac{1}{5} & * \\
\frac{1}{5} & \frac{6}{10} & 1 & \frac{3}{5} & * & 2 & 4 & 7 & \frac{1}{8} & * \\
2 & \frac{1}{2} & 1 & * & 3 & 5 & * & 2 & * \\
\frac{1}{6} & 1 & * & * & \frac{1}{8} & 8 & \frac{1}{5} & * & 4 \\
* & \frac{1}{2} & 7 & 1 & * & \frac{1}{5} & 2 \\
* & 5 & \frac{1}{6} & \frac{1}{9} & * & 1 & \frac{1}{9} & 7 & 4 \\
\frac{1}{5} & \frac{1}{8} & * & 3 & 6 & 1 & * & 6 \\
4 & 3 & 5 & \frac{1}{2} & * & 4 & 2 & * & 1 & \frac{1}{3} \\
\frac{1}{5} & * & * & \frac{1}{8} & \frac{1}{5} & 1 & \frac{1}{8} & \frac{2}{1} & 1 \\
\end{pmatrix}
\]

Using algorithm 1 and 2, there is one cycle of length 3, i.e., (1 – 3 – 5) is present. Therefore, \( \bar{R}_3 \) is inconsistent (Fig. 10).

In this example, if we will change the pair (3, 5) to (5, 3) in the original incomplete IMPR \( \bar{R}_3 \), then \( \bar{R}_3 \) is consistent.

**4.1 Comparison between two different approaches:**

By using the algorithm of odd and even length cycle 1 and 2, the corresponding cycles are given in Table 7 (Fig. 11).

Therefore, \( \bar{R}_4 \) is inconsistent. In Table 7, it is conclude the path 9 – 10 that cover all the cycle. If we will change the pair (9, 10) to (10, 9), in the original IMPR \( \bar{R}_4 \), then \( \bar{R}_4 \) is consistent.

**Table 7** All possible cycles of \( V \)

| Length | Cycles |
|--------|--------|
| 3      | (1 10 9), (6 10 9), (7 10 9) |
| 4      | (5 10 9 6), (1 8 10 9), (1 5 10 9), (3 7 10 9), (3 8 10 9), (3 6 10 9) |
| 5      | (3 8 7 10 9), (3 6 5 10 9), (1 8 7 10 9), (1 8 5 10 9), (3 7 10 9 6), (3 8 5 10 9), (1 5 7 10 9) |
| 6      | (1 8 5 7 10 9), (5 7 10 9 3 6), (3 6 5 7 10 9), (3 8 5 7 10 9), (5 7 10 9 3 8) |

\( 7 \sim 10 \) Nothing

**Fig. 10** Directed graph of 6

**Fig. 11** Directed graph of Example 7
the path 3—5 covers the cycle. We have to choose the couple to eliminate the cycle. In this case, if we will change the pair (3, 5) to (5, 3) in the original IMPR $\tilde{R}_3$, then both the lower and upper vertex is consistent. Then, IMPR $\tilde{R}_3$ is also consistent. Therefore, approach 2 is better than approach 1 because less no of the cycle is found in approach 2.

In example 4, according to approach 1, 19 cycles are found in the lower vertex matrix $V_L$ and 78 cycles are found in $V_U$ that are given in Tables 5 and 6. Similarly, according to approach 2, 21 cycles are found that are given in Table 7. In approach 1, we conclude that if we change the path $9 \rightarrow 10$, $4 \rightarrow 3$, and $2 \rightarrow 7$, then the IMPR $\tilde{R}_4$ is consistent. But according to approach 2, if we will change the pair $(9, 10)$ to $(10, 9)$, in the original IMPR $R_4$, then $\tilde{R}_4$ is consistent. Therefore, approach 2 is better than approach 1.

5 Comparative analysis with the existing methods

A lot of research has been developed in the literature in MPR, incomplete MPR, IMPR, incomplete IMPRs, and their consistency. In this section, we have compared our methodology with these existing techniques in the literature.

Comparison with Xu et al. (2013)

Chiclana et al. (2001) studied the transformation function between reciprocal MPR with values in the interval scale (1/9, 9) and reciprocal FPRs with values in [0, 1].

Proposition 3 (Chiclana et al. 2001) Suppose that $X = \{x_1, x_2, \ldots, x_n\}$, and associated with it a reciprocal multiplicative preference relation $A = (a_{ij})_{n \times n}$ with $a_{ij} \in [1/9, 9]$, then the corresponding reciprocal fuzzy preference relation, $P = (p_{ij})_{n \times n}$ with $p_{ij} \in [0, 1]$, associated with $A$ is given as follows: $p_{ij} = \frac{1}{2}(1 + \log_9a_{ij})$

After splitting the IMPR into two MPRs, the two MPRs is converted into two FPRs using proposition 3. Xu et al. (2013) proposed the ordinal consistency index to measure the degree of ordinal consistency of an FPR, which counts the unreasonable 3-cycles in a directed graph representing the FPR. Using this concept, we check the consistency of IMPR. Let us take example 1

$$\tilde{R}_1 = \begin{pmatrix}
(1, 1) & (\frac{1}{3}, \frac{1}{2}) & (2, \frac{1}{3}) & (\frac{3}{2}, 1)
\end{pmatrix}
\begin{pmatrix}
(\frac{1}{3}, \frac{1}{2}) & (1, 1) & (5, \frac{1}{2}) & (\frac{3}{2}, \frac{3}{2})
\end{pmatrix}
\begin{pmatrix}
(\frac{3}{2}, 2) & (\frac{1}{3}, 5) & (1, 1) & (6, \frac{1}{2})
\end{pmatrix}
\begin{pmatrix}
(1, 1) & (\frac{3}{2}, \frac{3}{2}) & (\frac{3}{2}, 6) & (1, 1)
\end{pmatrix}
$$

This IMPR $\tilde{R}_1$ is split into two MPRs using equation 6 that is given below

$$A_1 = \begin{pmatrix}
1 & \frac{1}{2} & \frac{2}{3} \\
2 & 1 & \frac{5}{2} \\
\frac{1}{3} & \frac{1}{2} & 1 \\
\frac{2}{3} & \frac{1}{2} & 6 \\
\frac{3}{2} & \frac{1}{2} & 1
\end{pmatrix}
A_2 = \begin{pmatrix}
1 & \frac{1}{4} & \frac{1}{8} & \frac{1}{4} \\
4 & 1 & \frac{7}{5} & \frac{1}{2} \\
8 & 7 & 1 & 1 \\
4 & 3 & 2 & 7 \\
1
\end{pmatrix}$$

Using Proposition 3, the above two MPRs converted into two FPRs, i.e., $P_1$ and $P_2$, are given by

$$P_1 = \begin{pmatrix}
0.5 & 0.3423 & 0.6577 & 0.4077 \\
0.6577 & 0.5 & 0.8662 & 0.5766 \\
0.3423 & 0.1338 & 0.5 & 0.9077 \\
0.5923 & 0.4234 & 0.0923 & 0.5 \\
\end{pmatrix}
$$

$$P_2 = \begin{pmatrix}
0.5 & 0.1845 & 0.0268 & 0.1845 \\
0.8155 & 0.5 & 0.0572 & 0.4077 \\
0.9732 & 0.9428 & 0.5 & 0.0572 \\
0.8155 & 0.5923 & 0.9428 & 0.5 \\
\end{pmatrix}
$$

According to Xu et al. (2013), the adjacency matrix of $P_1$ is

$$E = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}
$$

$$E^3 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

The Hadamard product of $E^2$ and $E^T$ is denoted by $B$ that is given below.

$$B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
$$

Thus, $\frac{\text{tr}(E^3)}{3} = \frac{\sum_{r=1}^{4} \sum_{j=1}^{n} b_{rj}}{3} = 1$. Then, there is one 3-cycle in the FPR $P_1$. To identify the cycle, we will collect all the $3 \times 3$ principal sub-matrices of $B$ are

$$\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 4 \\
2 & 3 & 4 \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
2 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
2 & 0 & 0 \\
2 & 0 & 0 \\
3 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
3 & 1 & 0 \\
4 & 0 & 0 \\
4 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
1 & 3 & 4 \\
1 & 0 & 0 \\
3 & 1 & 0 \\
4 & 0 & 1 \\
\end{pmatrix}
$$

There is no zero row or column in the principal submatrix $B[1, 3, 4]$. Therefore, there is one 3-cycle in digraph of $P_1$, i.e., $V_1 \rightarrow V_4 \rightarrow V_3 \rightarrow V_1$ (see Fig. 12). The inconsistent entries are $r_{31}^{(0)}, r_{14}^{(0)}, r_{43}^{(0)}$. In our method also we got same inconsistent entries. Thus, $P_1$ is order
inconsistent and each entry appears once time in the cycle. So, \( r_{31}^{(0)} = 0.3423, r_{41}^{(0)} = 0.4077, r_{43}^{(0)} = 0.0923 \). According to Xu et al. (2013), we choose the entry that is closer to 0.5. From these above three values \( r_{14}^{(0)} = 0.4077 \) is near to 0.5. Therefore, take \( r_{14}^{(1)} = 1 - 0.4077 = 0.5923 \), and \( r_{41}^{(1)} = 0.4077 \) in \( P_1 \) matrix and then apply the same procedure; it becomes consistent. Similarly, the adjacency matrix corresponding to \( P_2 \) is given by

\[
E = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 
\end{bmatrix}
\]

The Hadamard product of \( E^2 \) and \( E^T \) is denoted by \( B \) that is given below.

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}
\]

Here, \( \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} = 0 \) then there are no 3-cycles found in \( P_2 \) matrix (Fig. 13). In our method also, the MPR \( A_2 \) has no cycles. Therefore, \( P_2 \) is consistent. Since both the FPRs \( P_1, P_2 \) are consistent, the IMPR is consistent.

Let us take another example 2

\[
\tilde{R}_2 = \begin{pmatrix}
(1, 1) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \\
(\frac{1}{2}, \frac{1}{2}) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \\
(\frac{1}{2}, \frac{1}{2}) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \\
(\frac{1}{2}, \frac{1}{2}) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \\
(\frac{1}{2}, \frac{1}{2}) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) \\
(\frac{1}{2}, \frac{1}{2}) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right) & \left(\frac{1}{2}, \frac{1}{2}\right)
\end{pmatrix}
\]

This IMPR \( R_2 \) is split into two MPRs using equation 6 that is given below

\[
A_1 = \begin{pmatrix}
1 & 5 & 7 & 3 & 5 & 1 & 1 \\
3 & 1 & 5 & 5 & 3 & 1 & 1 \\
5 & 1 & 3 & 3 & 5 & 3 & 4 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}, \quad A_2 = \begin{pmatrix}
1 & 1 & 1 & 1 & 3 & 5 & 5 \\
4 & 1 & 1 & 1 & 1 & 3 \\
9 & 4 & 1 & 1 & 5 & 5 & 5 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 3 & 3 & 1 & 1 & 1 \\
7 & 4 & 1 & 1 & 3 & 3 & 7 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

The MPRs converted into FPRs, that is, \( P_1 \) and \( P_2 \), are given by

\[
P_1 = \begin{pmatrix}
0.5 & 0.6162 & 0.9428 & 0.75 & 0.6162 & 0.5 & 0.1845 \\
0.3838 & 0.5 & 0.6162 & 0.6162 & 0.3838 & 0.25 & 0.1845 \\
0.0572 & 0.3838 & 0.5 & 0.5 & 0.3838 & 0.25 & 0 \\
0.3838 & 0.6162 & 0.8155 & 0.6162 & 0 & 0.25 & 0.0572 \\
0.5 & 0.75 & 0.75 & 0.75 & 0.5 & 0.5072 \\
0.8155 & 0.8155 & 1 & 1 & 0.9428 & 0.9428 & 0.5
\end{pmatrix}
\]

\[
P_2 = \begin{pmatrix}
0.5 & 0.1845 & 0 & 0.0572 & 0.0572 & 0.3838 & 0.6162 \\
0.8155 & 0.5 & 0.1845 & 0.1845 & 0.5 & 0.5 & 0.75 \\
1 & 0.8155 & 0.5 & 0.3838 & 0.6162 & 0.75 & 0.9428 \\
0.9428 & 0.8155 & 0.6162 & 0.5 & 0.3838 & 0.75 & 0.9428 \\
0.9428 & 0.5 & 0.3838 & 0.6162 & 0.5 & 0.75 & 0.75 \\
0.6162 & 0.5 & 0.25 & 0.25 & 0.25 & 0.5 & 0.75 \\
0.3838 & 0.25 & 0.0572 & 0.0572 & 0.25 & 0.25 & 0.5
\end{pmatrix}
\]

The adjacency matrix of the FPR \( P_1 \) is

\[
E = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}, \quad E^3 = \begin{pmatrix}
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 5 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Thus, \( \frac{\text{trac}(E^3)}{3} = 0 \). There are no 3-cycles. Similarly, the adjacency matrix of \( P_2 \) is

\[
E = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad E^3 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 0 & 1 & 0 & 1 & 6 \\
3 & 1 & 0 & 0 & 1 & 1 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The Hadamard product of \( E^2 \) and \( E^T \) is denoted by \( B \) that is given below

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Thus, \( \frac{\text{trac}(E^3)}{3} = \sum_{i=1}^{3} \sum_{j=1}^{3} b_{ij} = 1 \). Then, there is one 3-cycle. To identify the cycle we have 35, \( 3 \times 3 \) principal sub-matrices of \( B \). Out of these, the principal sub-matrix \( B[3, 4, 5] \) has no zero row and column. Therefore, there are one 3-cycles in the digraph of adjacency matrix of \( P_2 \), i.e., \( V_1 \rightarrow V_4 \rightarrow V_3 \rightarrow V_1 \). The inconsistent entries are \( r_{34}^{(0)}, r_{45}^{(0)}, r_{53}^{(0)} \). In our method, also we got same inconsistent entries. Thus, \( P_1 \) is order inconsistent and each entry appears once in the cycle. So, \( r_{34}^{(0)} = 0.3838 \), \( r_{45}^{(0)} = 0.3838 \), \( r_{53}^{(0)} = 0.0.3838 \). According to Xu et al. (2013), we choose the entry that is closer to 0.5. From these above three values, take any one of them. Take \( r_{34}^{(1)} = 1 - 0.3838 = 0.6162 \) and \( r_{43}^{(1)} = 0.3838 \) in \( P_1 \) matrix and then apply the same procedure; it becomes consistent.

Both the methods are equivalent. We are getting the same cycles in both methods. In our method, we can take any matrix order and check the consistency. But Xu et al. (2013) methods are complicated to find the inconsistency entries from the principal sub-matrix as for \( n \)-order matrix, the number of principal sub-matrices of order 3 is \( n_{C_3} \).

**Comparison with Jiang et al. (2015)**

Jiang et al. (2015) proposed the consistency property of an IMPR. They proposed two approaches to find the missing elements of incomplete IMPRs, i.e., “estimating step” and “adjusting step.” To calculate the initial values of the missing element, a geometric mean method is used in the estimating step. To improve the initial values, two different approaches are developed: the local optimizations model and the iterative method. Jiang et al. (2015) method, once getting the missing value, the consistency of IMPR will be checked by splitting the IMPRs into two MPRs. In this paper consistency of arbitrary IMPRs is checked using the cyclic length of a directed graph. We have taken one example of Jiang et al. (2015) paper. Let us consider the example 2 of Jiang et al. (2015)

\[
A = \begin{bmatrix}
(1, 1) & (3/2, *) & (1/3, 1) & (1/2, 1) \\
(3/2, *) & (1, 1) & (1/4, 1/2) & (1, 1) \\
(1, 1/3) & (2, 1/4) & (1, 1) & (1/3, 1) \\
(1, 1/2) & (2, *) & (1, 1/3) & (1, 1)
\end{bmatrix}
\]

The order pair vertex matrix of IMPR \( A \) is given

\[
V_A = \begin{bmatrix}
0 & 0 & (1, 0) & (0, 0) & (0, 0) \\
0 & 0 & (0, 0) & (0, 0) & (0, 1) \\
(1, 0) & (1, 0) & (0, 0) & (0, 0) \\
(1, 0) & (0, 0) & (1, 0) & (0, 0)
\end{bmatrix}
\]

The lower and upper vertex matrices are

\[
V_L = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}, \quad V_U = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

From Figs. 14 and 15, we conclude that no cycle is found in both the lower and upper vertex matrices. Therefore, the incomplete IMPR \( A \) is consistent with the result of Jiang et al. (2015). The methods, i.e., our graphical methods and the Jiang et al. (2015) method, are equivalent. But in the graphical approach we can directly check the consistency of IMPR without finding the missing elements from the incomplete IMPR.
Comparison with Zhang and Pedrycz (2017)

Zhang and Pedrycz (2017) defined a new definition of consistency IMPR and proved that Xu’s (Xu 2013) definition of consistency and (Jiang et al. 2015) definition of consistency, are the special case of consistency defined by Zhang and Pedrycz (2017). According to Zhang and Pedrycz (2017), consistency is independent of alternative labels. For a multi-criteria decision-making (MCDM) problem with $n$ decision alternatives, the DMs pairwise comparison information can be structured by differently labeling the $n$ alternatives and yields $n!$ IMPRs.

We take example 1 from Zhang and Pedrycz (2017).

Consider three alternatives $A$, $B$, $C$ in an MCDM problem. A DM employs the pairwise comparison method to elicit his/her judgments. There are six possible cases for labeling the three alternatives. If the three alternatives $A$, $B$, $C$ are labeled by $x_1$, $x_2$ and $x_3$, then the DMs pairwise judgments are given by the following IMPR.

$$
\begin{align*}
&x_1 : A & x_2 : B & x_3 : C \\
&A & \begin{pmatrix}
1, 1 & \left(\frac{3}{5}, \frac{3}{5}\right) \\
\left(\frac{3}{5}, \frac{3}{5}\right) & 1, 1 \\
\left(\frac{3}{5}, \frac{3}{5}\right) & \left(\frac{3}{5}, \frac{3}{5}\right)
\end{pmatrix}
\end{align*}
$$

It is consistent with Jiang’s (Jiang et al. 2015) method. According to our method, the order pair vertex matrix.

$$
V = \begin{bmatrix}
(0, 0) & (0, 1) & (0, 0) \\
(0, 1) & (0, 0) & (0, 0) \\
(1, 1) & (0, 1) & (0, 0)
\end{bmatrix}
$$

The lower and upper vertex matrices are

$$
V_L = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{bmatrix}, \quad V_U = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

From the directed graph (Fig. 16), both $V_L$ and $V_U$ are consistent because there is no circle was found. So, IMPR is consistent.

If the alternative $A$, $B$, $C$ are labeled by $x_2$, $x_3$, $x_1$, then the DMs pairwise judgments should be re-structured by the following IMPR, i.e.,

$$
\begin{align*}
&x_1 : C & x_2 : A & x_3 : B \\
&C & \begin{pmatrix}
(1, 1) & \left(\frac{3}{5}, \frac{3}{5}\right) \\
\left(\frac{3}{5}, \frac{3}{5}\right) & 1, 1 \\
\left(\frac{3}{5}, \frac{3}{5}\right) & \left(\frac{3}{5}, \frac{3}{5}\right)
\end{pmatrix}
\end{align*}
$$

This re-structured IMPR is inconsistent, according to Jiang et al. (2015). But it is consistent, according to Zhang and Pedrycz (2017). Now the same re-structured IMPR is consistent according to our method. The ordered pair vertex matrix is

$$
V = \begin{bmatrix}
(0, 0) & (0, 0) & (0, 0) \\
(1, 1) & (0, 0) & (0, 1) \\
(1, 0) & (1, 0) & (0, 0)
\end{bmatrix}
$$

The lower and upper vertex matrix is

$$
V_L = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 1 & 0
\end{bmatrix}, \quad V_U = \begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

From the directed graph 17, both $V_L$ and $V_U$ are consistent because there is no circle was found. So, re-structured IMPR is consistent. Therefore, the re-structured IMPR is consistent according to Zhang and Pedrycz (2017) and our method.

Comparison with Jin et al. (2018)

Jin et al. (2018) proposed a method to derive normalized intuitionistic multiplicative weights based on order consistent IMPR for decision making. Also, Jin et al. (2018) constructed an optimization model to generate the normalized intuitionistic multiplicative weights of IMPR and compute the optimal deviation values to improve the consistency of the given IMPR, such that the repaired IMPR is constructed. To perform the comparison, we have taken example 2 of (Jin et al. 2018) the IMPR $P = (p_{ij})$, i.e.,
According to Jin et al. (2018), the IMPR $P$ is non-order consistent. Then, they obtained the optimal normalized intuitionistic multiplicative weights using the goal programming model (M-2) of Jin et al. (2018). To construct the repaired consist IMPR, they utilize the optimal deviation values. Therefore, they get the repaired IMPR is consistent. Here, we solve the same IMPR by using our method. The binary order pair vertex matrix of IMPR $P$ is

$$P = \begin{pmatrix}
(1, 1) & \left(\frac{5}{7}, \frac{1}{2}\right) & (7, \frac{1}{3}) & (3, \frac{1}{4}) \\
(\frac{1}{2}, \frac{5}{7}) & (1, 1) & \left(\frac{5}{7}, \frac{1}{4}\right) & (\frac{5}{7}, \frac{1}{6}) \\
(\frac{1}{2}, 7) & \left(\frac{5}{7}, \frac{1}{2}\right) & (1, 1) & (1, \frac{1}{2}) \\
(\frac{1}{2}, 3) & \left(\frac{5}{7}, \frac{1}{6}\right) & (\frac{1}{2}, 1) & (1, 1)
\end{pmatrix}$$

The lower and upper vertex matrices are,

$$V = \begin{pmatrix}
(0, 0) & (1, 0) & (1, 0) & (1, 0) \\
(0, 1) & (0, 0) & (1, 0) & (1, 0) \\
(0, 1) & (0, 1) & (0, 0) & (0, 0) \\
(0, 1) & (0, 1) & (0, 1) & (0, 0)
\end{pmatrix}$$

$$V_L = \begin{pmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}; \quad V_U = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}$$

From Fig. 18, it is clear that there is no cycle is found from the lower and upper vertex matrices. The $V_L$ and $V_U$ are consistent, and then the IMPR is also consistent. The IMPR $P$ are consistent with our method. Therefore, both the methods gave the same result that proves the validity of our result, and also, in our method, there is no need to solve an optimization problem.

### 6 Conclusion

In this paper, we have proposed two approaches to characterize the consistency for intuitionistic multiplicative preference relation (IMPR). In the first approach, we propose an algorithm to check the consistency of IMPR by using the cycles of various lengths in a directed graph, and the same procedure applies for incomplete IMPRs also. The second approach proved isomorphic between the set of IMPRs and asymmetric multiplicative preference relations. Also, consistency property is then checked of asymmetric preference relation using a directed graph that is used to get the consistency of IMPR.

Our future research will study the consistency of hesitant fuzzy preference relation HFPR using the cyclic length of a directed graph with the best additive consistency index, the worst additive consistency index, and average additive consistency index of an HFPR. Also, we can check the consistency of incomplete HFPR using the cyclic approach that will extend to propose a new procedure for the group analytic hierarchy process to deal with multi-criteria group decision-making problems.

Nowadays, linguistic large-scale group decision-making problems are a handy tool in the decision-making process. Our future research will study the consistency of IMPR with multi-granular linguistic information based on the developed model. It will be more interesting to analyze the consistency of IMPR with multi-granular linguistic information using the cyclic length of the directed graph of IMPR. Also, our studies
can be extended to the social network group decision-making framework.

Also, Our future research will study the consensus reaching process with IMPRs with graphical approach; Consensus reaching for social network group decision making by considering leadership and bounded confidence; Consensus reaching for group decision making with multi-granular unbalanced linguistic information: A bounded confidence and minimum adjustment based approach.

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Declarations

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References

Behret H (2014) Group decision making with intuitionistic fuzzy preference relations. Knowl-Based Syst 70:33–43
Chiclana F, Herrera F, Herrera-Viedma E (2001) Integrating multiplicative preference relations in a multipurpose decision-making model based on fuzzy preference relations. Fuzzy Sets Syst 122:277–291
Erguv D, Kou G, Peng Y, Shi Y (2011) A simple method to improve the consistency ratio of the pair-wise comparison matrix in ANP. Eur J Oper Res 213:246–259
Gong ZW, Li FJ, Forrest J, Zhao Y (2011) The optimal priority models of the intuitionistic fuzzy preference relation and application in selecting industries with higher meteorological sensitivity. Expert Syst Appl 38:4394–4402
Gong ZW, Li LS, Zhou FX, Yao TX (2009) Goal programming approaches to obtain the priority vectors from the intuitionistic fuzzy preference relations. Comput Ind Eng 57:1187–1193
Herrera-Viedma E, Chiclana F, Herrera F, Alonso S (2007) Group decision-making model with incomplete fuzzy preference relations based on additive consistency. IEEE Trans Syst Man Cybern - Part B 37:176–189
Herrera-Viedma E, Herrera F, Chiclana F, Luque M (2004) Some issues on consistency of reciprocal relations. Eur J Oper Res 154:98–109
Jiang Y, Xu Z, Yu X (2015) Group decision making based on incomplete intuitionistic multiplicative preference relations. Inf Sci 295:33–52
Jin F, Ni Z, Pei L, Chen H, Li Y (2018) Goal programming approach to derive intuitionistic multiplicative weights based on intuitionistic multiplicative preference relations. Int J Mach Learn Cyber 9:641–650
Nishizawa K (1995) A consistency improving method in binary AHP. J Op Res Soc Japan 38(1):21–33
Nishizawa K (1996) A method to find elements of cycles in an incomplete directed graph and its Applications- Binary AHP and Petri Nets. Comput Math Appl 33:33–46
Orlovsky SA (1978) Decision making with a fuzzy preference relation. Fuzzy Sets Syst 1:155–167
Ren P, Xu Z, Liao H (2016) Intuitionistic multiplicative analytic hierarchy process in group decision making. Comput Ind Eng 101:513–524
Saaty TL (1977) A scaling method for priorities in hierarchy structures. J Math Psychol 15:234–281
Saaty TL (1980) The analytic hierarchy process. McGraw-Hill, New York
Xia MM, Xu ZS, Chen J (2013) Algorithms for improving consistency or consensus of reciprocal [0, 1]-valued preference relations. Fuzzy Sets Syst 216:108–133
Xia MM, Xu ZS, Liao HC (2013) Preference relations based on intuitionistic multiplicative information. IEEE Trans Fuzzy Syst 21:113–135
Xu Y, Herrera F (2019) Visualizing and rectifying different inconsistencies for fuzzy reciprocal preference relations. Fuzzy Sets Syst 362:85–109
Xu Y, Herrera F, Wang H (2016) A distance-based framework to deal with ordinal and additive inconsistencies for fuzzy reciprocal preference relations. Inf Sci 328:189–205
Xu Y, Li M, Cabrerizo FJ, Chiclana F, Herrera-Viedma E (2021) Algorithms to detect and rectify multiplicative and ordinal inconsistencies of fuzzy preference relations. IEEE Trans Syst, Man, Cybern: Syst 51(6):3498–3511
Xu Y, Ma F, Herrera F (2019) Revisiting inconsistent judgments for incomplete fuzzy linguistic preference relations: algorithms to identify and rectify ordinal inconsistencies. Knowl-Based Syst 163:305–319
Xu Y, Patnayakuni R, Wang H (2013) The ordinal consistency of a fuzzy preference relation. Inf Sci 224:152–164
Xu Y, Wang Q, Cabrerizo FJ, Herrera-Viedma E (2018) Methods to improve the ordinal and multiplicative consistency for reciprocal preference relation. Appl Soft Comput 67:479–493
Xu ZS (2007) Intuitionistic preference relations and their application in group decision making. Inf Sci 177(11):2363–2379
Xu ZS (2013) Priority weight intervals derived from intuitionistic multiplicative preference relations. IEEE Trans Fuzzy Syst 21:642–654
Xu Z, Liao H (2015) A survey of approaches to decision making with intuitionistic fuzzy preference relations. Knowl-Based Syst 80:131–142
Xu Z, Wei CP (1999) A consistency improving method in the analytic hierarchy process. Eur J Oper Res 116:443–449
Zhang Z, Guo C (2017) Deriving priority weights from intuitionistic multiplicative preference relations under group decision-making settings. J Op Res Soc 68(12):1582–1599
Zhang Z, Pedrycz W (2018) Goal programming approaches to managing consistency and consensus for intuitionistic multiplicative preference relations in group decision-making. IEEE Trans Fuzzy Syst. https://doi.org/10.1109/TFUZZ.2018.2818074
Zhang Z, Pedrycz W (2017) Models of mathematical programming for intuitionistic multiplicative preference relations. IEEE Trans Fuzzy Syst 25(4):945–957

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