Extremal results regarding $K_6$-minors in graphs of girth at least 5

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Abstract. We prove that every 6-connected graph of girth $\geq 6$ has a $K_6$-minor and thus settle Jorgensen’s conjecture for graphs of girth $\geq 6$. Relaxing the assumption on the girth, we prove that every 6-connected n-vertex graph of size $\geq 3\frac{1}{5}n - 8$ and of girth $\geq 5$ contains a $K_6$-minor.

Preamble. Whenever possible notation and terminology are that of [2]. Throughout, a graph is always simple, undirected, and finite. $G$ always denotes a graph. We write $\delta(G)$ and $d_G(v)$ to denote the minimum degree of $G$ and the degree of a vertex $v \in V(G)$, respectively. $\kappa(G)$ denotes the vertex connectivity of $G$. The girth of $G$ is the length of a shortest circuit in $G$. Finally, the cardinality $|E(G)|$ is called the size of $G$ and is denoted $|G|$. $|V(G)|$ is called the order of $G$ and is denoted $|G|$.

§1 Introduction. A conjecture of Jorgensen postulates that the 6-connected graphs not containing $K_6$ as a minor are the apex graphs, where a graph is apex if it contains a vertex removal of which results in a planar graph. The 6-connected apex graphs contain triangles. Consequently, if Jorgensen’s conjecture is true, then a 6-connected graph of girth $\geq 4$ contains a $K_6$-minor. Noting that the extremal function for $K_6$-minors is at most $4n - 10$ [4] (where $n$ is the order of the graph), our first result in this spirit is that

1.1. a graph of size $\geq 3n - 7$ and girth at least 6 contains a $K_6$-minor.

So that,

1.2. every 6-connected graph of girth $\geq 6$ contains a $K_6$-minor;

This settles Jorgensen’s conjecture for graphs of girth $\geq 6$. Relaxing the assumption on the girth in 1.1 we prove the following.

1.3. A 6-connected graph of size $\geq 3\frac{1}{5}n - 8$ and girth at least 5 contains a $K_6$-minor.

Remark. In our proofs of 1.1 and 1.3 the proofs of claims (1.1)A-B) and (1.3)A-D) follow the approach of [3].

§2 Preliminaries. Let $H$ be a subgraph of $G$, denoted $H \subseteq G$. The boundary of $H$, denoted by $\text{bnd}_G H$ (or simply $\text{bnd} H$), is the set of vertices of $H$ incident with $E(G) \setminus E(H)$. By $\text{int}_G H$ (or simply $\text{int} H$) we denote the subgraph induced by $V(H) \setminus \text{bnd} H$. If $v \in V(G)$, then $N_H(v)$ denotes $N_G(v) \cap V(H)$.

Let $k \geq 1$ be an integer. By $k$-hammock of $G$ we mean a connected subgraph $H \subseteq G$ satisfying $|\text{bnd} H| = k$. A hammock $H$ coinciding with its boundary is called trivial, degenerate
if $|H| = |bndH| + 1$, and fat if $|H| \geq |bndH| + 2$. A proper subgraph of $H$ that is a $k$-hammock is called a proper $k$-hammock of $H$. A fat $k$-hammock is called minimal if all its proper $k$-hammocks, if any, are trivial or degenerate. Clearly, every fat $k$-hammock contains a minimal fat $k$-hammock. \hfill (2.1)

Let $H$ be a fat $2$-hammock with $bndH = \{u,v\}$. By capping $H$ we mean $H + uv$ if $uv \notin E(H)$ and $H$ if $uv \in E(H)$. In the former case, $uv$ is called a virtual edge of the capped of $H$. The set $bndH$ is called the window of the capping.

Let now $\kappa(G) = 2$ and $\delta(G) \geq 3$. By the standard decomposition of 2-connected graphs into their 3-connected components \cite[Section 9.4]{I}, such a graph has at least two minimal fat 2-hammocks whose interiors are disjoint and that capping of each is 3-connected. Such a capping is called an extreme 3-connected component.

A $k$-(vertex)-disconnector, $k \geq 1$, is called trivial if removal of which isolates a vertex. Otherwise, it is called nontrivial. A graph is called essentially $k$-connected if all its $(k-1)$-disconnectors are trivial. If each $(k-1)$-disconnector $D$ isolates a vertex and $G - D$ consists of precisely 2 components (one of which is a singleton) then $G$ is called internally $k$-connected.

Suppose $\kappa(G) \geq 1$ and that $D \subseteq V(G)$ is a $\kappa(G)$-disconnector of $G$. Then, $G[C \cup D]$ is a fat $\kappa(G)$-hammock for every non-singleton component $C$ of $G - D$. In particular, we have that

2.2. if $\kappa(G) \geq 1$, $\delta(G) \geq 3$, and $D \subseteq V(G)$ is a nontrivial $\kappa(G)$-disconnector of $G$, then $G$ has at least two fat minimal $\kappa(G)$-hammocks whose interiors are disjoint.

2.3. If $\kappa(G) \geq 1$, $\delta(G) \geq 3$, $e \in E(G)$, and $G$ has a nontrivial $\kappa(G)$-disconnector, then $G$ has a minimal fat $\kappa(G)$-hammock $H$ such that if $e \in E(H)$, then $e$ is spanned by $bndH$.

Let $H$ be a $k$-hammock. By augmentation of $H$ we mean the graph obtained from $H$ by adding a new vertex and linking it with edges to each vertex in $bndH$.

2.4. Suppose $\kappa(G) = 3$ and that $H$ is a minimal fat 3-hammock of $G$. Then, an augmentation of $H$ is 3-connected.

Proof. Let $H'$ denote the augmentation and let $\{x\} = V(H') \setminus V(H)$. Assume, to the contrary, that $H'$ has a minimum disconnector $D$, $|D| \leq 2$. If $H' - D$ has a component containing $x$, then $H$ has a nontrivial $|D|$-hammock; contradicting the assumption that $\kappa(G) = 3$. Hence, $x \in D$. As $x$ is 3-valent, $H' - D$ has a component $C$ containing a single member of $bndH' (= N_{H'}(x))$, say $u$. Since $\delta(G) \geq 3$, $|N_C(u) \setminus D| \geq 1$ so that $(D \setminus \{x\}) \cup \{u\}$ is a disconnector of $H$ of size $\leq 2$ not containing $x$ and hence also a disconnector of $G$; contradiction. \hfill \qed

2.5. Suppose $\kappa(G) = 3$ and that $H$ is a triangle free minimal fat 3-hammock of $G$ such that $e \in E(G[bndH])$. Then, an augmentation of $H - e$ is 3-connected.

Proof. Let $H'$ be the augmentation of $H - e$, let $\{x\} = V(H') \setminus V(H)$, and let $e = tw$ such that $t, w \in N_{H'}(x)$. By $2.4$, $\kappa(H' + e) \geq 3$. Suppose that $\kappa(H') < 3$, then $H'$ contains
a 2-disconnector, say \( \{u, v\} \), so that \( H' = H_1 \cup H_2 \), \( H'\{u, v\} = H_1 \cap H_2 \) and such that \( x \in V(H_i) \) for some \( i \in \{1, 2\} \). Unless \( x \in \{u, v\} \), then \( t, w \in V(H_i) \). Thus, if \( x \notin \{u, v\} \), then \( \{u, v\} \) is a 2-disconnector of \( H' + e \); contradiction.

Suppose then that, without loss of generality, \( x = u \). Thus, since \( x \) is 3-valent, there exists an \( i \in \{1, 2\} \) such that \( |N_{H_i}(x) \setminus \{v\}| = 1 \). As \( \{x, v\} \) is a minimum disconnector of \( H' \), it follows that \( H_i \setminus \{x, v\} \) is connected so that \( N_{H_i}(x) \cup \{v\} \) is the boundary of a 2-hammock of \( G \); such must be trivial as \( \kappa(G) = 3 \), implying that \( |V(H_i)| = \{x, v, z\} \), where \( z \in \{t, w\} \).

We may assume that \( x \) is not adjacent to \( v \); for otherwise, \( |N_{H_{3-i}}(x) \setminus \{v\}| = 1 \) so that the minimality of the disconnector \( \{x, v\} \) implies that \( H_{3-i} \setminus \{x, v\} \) is connected and consequently that \( N_{H_{3-i}}(x) \cup \{v\} \) is the boundary of a 2-hammock of \( G \); since such must be trivial we have that \( H \) is a triangle (consisting of \( \{t, v, w\} \)) contradicting the assumption that \( H \) is triangle-free.

Hence, since \( H \) is triangle free and since each member of \( \{v\} \cup N_{H_{3-i}}(x) \) has at least two neighbors in \( H_{3-i} \), \( \{v\} \cup N_{H_{3-i}}(x) \) is the boundary of a proper fat 3-hammock of \( H \); contradiction to \( H \) being minimal.■

The maximal 2-connected components of a connected graph are called its blocks. Such define a tree structure for \( G \) whose leaves are blocks and are called the leaf blocks of \( G \) [2].

We conclude this section with the following notation. Let \( H \subseteq G \) be connected (possibly \( H \) is a single edge). By \( G/H \) we mean the contraction minor of \( G \) obtained by contracting \( H \) into a single vertex. We always assume that after the contractions the graph is kept simple; i.e., any multiple edges resulting from a contraction are removed.

§3 Truncations. Let \( \mathcal{F} \) be a family of graphs (possibly infinite). A graph is \( \mathcal{F} \)-free if it contains no member of \( \mathcal{F} \) as a subgraph. A graph \( G \) is nearly \( \mathcal{F} \)-free if it is either \( \mathcal{F} \)-free or has a breaker \( x \in V(G) \cup E(G) \) such that \( G - x \) is \( \mathcal{F} \)-free. A breaker that is a vertex is called a vertex-breaker and an edge-breaker if it is an edge.

An \( \mathcal{F} \)-truncation of an \( \mathcal{F} \)-free graph \( G \) is a minor \( H \) of \( G \) that is nearly \( \mathcal{F} \)-free such that either \( H \subseteq G \) (and then it has no breaker) or \( H \) contains a breaker \( x \) such that \( H - x \subseteq G \). In the former case, the truncation is called proper; in the latter case, the truncation is improper with \( x \) as its breaker and \( H - x \) as its body. An improper truncation is called an edge-truncation if its breaker is an edge and a vertex-truncation if its breaker is a vertex. A vertex-truncation is called a 3-truncation if its breaker is 3-valent.

3.1. Let \( \mathcal{F} \) be a graph family such that \( K_3 \in \mathcal{F} \) and let \( G \) be \( \mathcal{F} \)-free with \( \delta(G) \geq 3 \). Then \( G \) has an essentially 4-connected \( \mathcal{F} \)-truncation \( H \) such that:

\[ \begin{align*}
& (\texttt{3.1.1}) \ |H| \geq 4; \text{ and} \\
& (\texttt{3.1.2}) \text{ if } H \text{ is a vertex-truncation then it is a 3-truncation and } |H| \geq 5.
\end{align*} \]

Proof. Let \( \mathcal{H} \) denote the 3-connected truncations of \( G \).

\[ \begin{align*}
& (\texttt{3.1.1}) \mathcal{H} \text{ is nonempty. In particular, } \mathcal{H} \text{ contains a truncation } H \text{ with } |H| \geq 4 \text{ so that if improper then it is an edge-truncation with edge-breaker } e \text{ such that } \kappa(H - e) = 2.
\end{align*} \]

Subproof. We may assume that \( G \) is connected. Let \( B \) be a leaf block of \( G \) (possibly \( B = G \)). If \( \kappa(B) \geq 3 \), then \( \texttt{3.1.1} \) follows (by setting \( H = B \)) as \( B \) is a proper truncation of \( G \). As-
sume then that \( \kappa(B) = 2 \) and let \( H \) be an extreme 3-connected component of \( B \) with window \( \{x, y\} \). Now, \( H \in \mathcal{H} \) with possibly \( xy \) an edge-breaker. If \( H \) is improper, then \( \kappa(H - xy) = 2 \). Note that \( \delta(G) \geq 3 \) implies that \( |H| \geq 4 \) in both cases. \( \square \)

If \( \mathcal{H} \) contains a proper or an edge-truncation that is essentially 4-connected, then (3.1) follows. Suppose then that

\[
\mathcal{H} \text{ has no proper or edge-truncations that are essentially 4-connected.} \quad (3.2)
\]

Subproof. Let \( H \in \mathcal{H} \) such that if improper then \( H \) and \( e \) are as in (3.1). By (3.2) and 2.3, \( H \) has a minimal fat 3-hammock \( H' \) such that if \( e \in E(H') \), then \( e \) is spanned by the boundary of \( H' \). Let \( H'' \) be the graph obtained from an augmentation of \( H' \) by removing \( e \) if it is spanned by \( bndH' \). Let \( \{x\} = V(H'') \setminus V(H') \).

By 2.4 and 2.5, \( \kappa(H'') \geq 3 \) so that \( H'' \in \mathcal{H} \) with \( x \) as a potential 3-valent vertex-breaker and (3.1) follows.

Finally, note that \( |intH'| \geq 2 \) so that \( |H''| \geq 5 \). \( \square \)

Next, we show the following.

(3.1) If \( \mathcal{H} \) contains a 3-truncation \( X \) of order \( \geq 5 \), then \( \mathcal{H} \) contains essentially 4-connected 3-truncations \( Y \) such that \( 5 \leq |Y| \leq |X| \).

Subproof. Let \( H^* \in \mathcal{H} \) be a 3-truncation of order \( \geq 5 \) with the order of its body minimized. We show that \( H^* \) is essentially 4-connected. Let \( x \) denote the vertex-breaker of \( H^* \). By the minimality of \( H^* \),

any minimal fat 3-h hammock \( T \) of \( H^* \) with \( x \notin V(T) \) satisfies \( T = H^* - x \) (3.3)

(so that \( bndT = N_{H^*}(x) \)).

Assume now, towards contradiction, that \( H^* \) is not essentially 4-connected so that it contains nontrivial 3-disconnectors and at least two minimal fat 3-hammocks that may meet only at their boundary, by 2.2. By (3.3), existence of at least two such hammocks implies that \( x \) belongs to every nontrivial 3-disconnector and thus to the boundary of every minimal fat 3-hammock. As \( x \) is 3-valent, there is a minimal fat 3-hammock \( T \) of \( H^* \) with \( x \) on its boundary such that \( N_T(x) = \{y\} \). As \( T \) is a minimal fat 3-hammock, \( V(T) \) consists of \( x, y \), the two members of \( bndT \setminus \{x\} \), and an additional vertex \( u \). As \( \delta(G) \geq 3 \), \( uy \in E(T) \), \( u \) is adjacent to both members of \( bndT \setminus \{x\} \) and \( y \) is adjacent to at least one member of \( bndT \setminus \{x\} \). Hence, \( K_3 \subseteq T - x \subseteq H^* - x \) so that \( x \) is not a breaker; contradiction. \( \square \)

Assuming (3.2), then, by (3.1), there are 3-connected 3-truncations of \( G \) of order \( \geq 5 \) so that an essentially 4-connected 3-truncation of \( G \) exists by (3.1). \( \blacksquare \)

3.4. Let \( \mathcal{F} \) be a graph family such that \( \{K_3, K_{2,3}\} \subseteq \mathcal{F} \), then \( G \) has an internally 4-connected
\( F \text{-truncation satisfying (3.1 I-2) and if such is a vertex-truncation then it is a 3-truncation.} \)

**Proof.** Let \( T \) denote the essentially 4-connected truncations of \( G \) that are either proper, or edge-truncations, or 3-truncations; \( T \) is nonempty by 3.1. Let \( \alpha(T) \) denote the least \( k \) such that \( T \) contains a proper truncation of order \( k \) or an improper edge-truncation of order \( k \). Let \( \beta(T) \) denote the least \( k \) such that \( T \) contains an improper 3-truncation with its body of order \( k \). Let \( H \in T \) such that \( |H| = \min\{\alpha(T), \beta(T) + 1\} \) and let \( x \) denote its breaker if improper.

We show that \( H \) is internally 4-connected. To see this, assume, to the contrary, that \( H \) is not internally 4-connected and let \( D \) be a 3-disconnector of \( H \) such that \( H - D \) consists of \( \geq 3 \) components at least one of which is a singleton (since \( H \) is essentially 4-connected). Let \( C \) denote the non-singleton components of \( H - D \). Since \( K_{2,3} \in F \), \( |C| \geq 1 \)

Suppose \( J = H[C \cup D] \) is a 3-hammock of \( H \), for some \( C \in \mathcal{C} \), that does not meet \( x \) in its interior (if \( x \) exists). By the choice of \( H \),

\[
\text{for each fat 3-hammock } X \text{ of } J \text{ either } x \in bndX \text{ or } x \in E(H[bndX]).
\] (3.5)

Indeed, for otherwise, an augmentation of a minimal fat 3-hammock of \( X \) is a 3-truncation of order \( \geq 5 \) of \( G \) that belongs to \( \mathcal{H} \) and has order \( < |H| \), where \( \mathcal{H} \) is as in the proof of 3.1 existence of such a 3-truncation of \( G \) implies that \( G \) has an essentially 4-connected 3-truncation of order \( \geq 5 \), by (3.1 C), and such has order \( < |H| \) contradicting the choice of \( H \). Consequently, the assumption that the interior of \( J \) does not meet \( x \) implies that

\[
\text{if } J \text{ exists, then } x \in D \cup E(H[D]).
\] (3.6)

Suppose now that \( J \) has a minimal fat 3-hammock \( J' \) (possibly \( J' = J \)) with \( x \in bndJ' \) so that \( x \in D \), by (3.6). \( |D| = \kappa(H) \) imply that \( x \) is incident with each component of \( H - D \) so that \( |N_{intJ'}(x)| = 1 \), as \( x \) is 3-valent. The minimality of \( J' \) then implies that \( |intJ'| = 2 \) so that \( J' - x \) contains a \( K_3 \) (see proof of (3.1 C) for the argument) and thus \( x \) is not a breaker of \( H \); contradiction.

Suppose next that \( J' \) is a minimal fat 3-hammock of \( J \) whose boundary vertices span \( x \) (as an edge). Then, an augmentation of \( J' - x \) belongs to \( \mathcal{H} \), by 2.5 and such contains an essentially 4-connected 3-truncation of \( G \), by (3.1 C), of order \( < |H| \). Hence,

\[
J \text{ (if exists) has no minimal fat 3-hammock } J' \text{ with } x \in bndJ' \cup E[H[bndJ']].
\] (3.7)

If \( J \) exists, then (3.5) and (3.7) are contradictory. Thus, to obtain a contradiction and hence conclude the proof of 3.4 we show that a 3-hammock such as \( J \) exists. This is clear if \( |C| \geq 2 \) as then at least one member of \( C \) does not meet \( x \). Suppose then that \( |C| = 1 \) so that \( H - D \) consists of two singleton components, say \( \{u, v\} \), and the single member \( C \) of \( C \). \( D \cup \{u, v\} \) induce a \( K_{2,3} \), say \( K \). Since \( K_{2,3} \in F \) and \( x \) is a breaker, \( K \) contains \( x \) so that \( C \) does not; hence, \( H[C \cup D] \) is the required 3-hammock.

For \( k \geq 4 \), a graph that is nearly \( \{K_3, C_4, \ldots, C_{k-1}\} \)-free is called *nearly k-long*. That is, \( G \) is nearly \( k \)-long if either it has girth \( \geq k \) or it has a breaker \( x \in V(G) \cup E(G) \) such that \( G - x \) has girth \( \geq k \).
A nearly 5-long graph is nearly \( \{K_3, C_4\}\)-free; such is also nearly \( \{K_3, K_{2,3}\}\)-free. In addition, a 3-connected nearly 5-long truncation has order \( \geq 5 \). Consequently, we have the following consequence of 3.4.

3.8. A graph with girth \( \geq k \geq 5 \) and \( \delta \geq 3 \) has an internally 4-connected nearly \( k \)-long truncation of order \( \geq 5 \) and if such is a vertex-truncation then it is a 3-truncation.

§4 Nearly long planar graphs. For a plane graph \( G \), we denote its set of faces by \( F(G) \) and by \( X_G \) its infinite face.

4.1. Let \( G \) be a 2-connected plane graph of girth \( \geq 6 \), and let \( S \subseteq V(G) \) be the 2-valent vertices of \( G \). Then, \( |S| \geq 6 \).

Proof. By Euler’s formula:

\[ |E(G)| = |V(G)| + |F(G)| - 2. \] (4.2)

Since \( G \) is 2-connected, every vertex in \( V(G) \setminus S \) is at least 3-valent so that

\[ 2|E(G)| \geq 3(|V(G)| - |S|) + 2|S|. \] (4.3)

As \( G \) is of girth \( \geq 6 \) and 2-connected (and hence every edge is contained in exactly two distinct faces) then:

\[ 2|E(G)| \geq 6|F(G)|. \] (4.4)

Substituting (4.2) in (4.3),

\[ 2(|V(G)| + |F(G)| - 2) \geq 3(|V(G)| - |S|) + 2|S| \Rightarrow |V(G)| \leq 2|F(G)| + |S| - 4 \] (4.5)

Substituting (4.2) in (4.4),

\[ 2(|V(G)| + |F(G)| - 2) \geq 6|F(G)| \Rightarrow |V(G)| \geq 2|F(G)| + 2 \] (4.6)

From (4.5) and (4.6),

\[ 2|F(G)| + 2 \leq 2|F(G)| + |S| - 4 \Rightarrow |S| \geq 6 \] (4.7)

Hence, the proof follows. ■

From 4.1 we have that:

4.8. A nearly 6-long internally 4-connected graph is nonplanar.

4.9. Let \( G \) be a nearly 5-long internally 4-connected planar graph and suppose that if \( G \) has a vertex-breaker, then it also has a vertex-breaker which is a 3-valent vertex. Then, \( |G| \geq 11 \).
Proof. Define $S \subseteq V(G) \cup E(G)$ as follows. If $G$ is of girth $\geq 5$ set $S := \emptyset$; otherwise set $S := \{x\}$, where $x \in V(G) \cup E(G)$ is a breaker of $G$ so that if $x \in V(G)$ then $x$ is 3-valent. Then, $G - S$ is 2-connected, and has at most three 2-valent vertices. Hence,

$$2|E(G)| \geq 3(|V(G)| - 3) + 6. \quad (4.10)$$

As $G - S$ is of girth $\geq 5$ and $G$ is 2-connected then:

$$2|E(G)| \geq 5|F(G)|. \quad (4.11)$$

Substituting (4.2) in (4.10),

$$2(|V(G)| + |F(G)| - 2) \geq 3(|V(G)| - 3) + 6 \Rightarrow |F(G)| \leq (|V(G)| + 1)/2 \quad (4.12)$$

Substituting (4.2) in (4.11),

$$2(|V(G)| + |F(G)| - 2) \geq 5|F(G)| \Rightarrow |F(G)| \geq (2|V(G)| - 2)/3 \quad (4.13)$$

From (4.12) and (4.13),

$$(|V(G)| + 1)/2 \leq (2|V(G)| - 2)/3 \Rightarrow |V(G)| \geq 11 \quad (4.14)$$

Hence, the proof follows.

4.15. A 2-connected plane graphs $G$ satisfying the following does not exist.

1) $G$ has girth $\geq 5$;
2) each member of $V(G) - V(X_G)$ is at least 4-valent; and
3) $G$ has a set $S \subseteq V(X_G)$, $|S| \leq 3$ (possibly $S = \emptyset$) with each of its members 2-valent and each member of $V(X_G) - S$ at least 3-valent.

Proof. Assume towards contraction that the claim is false. We will use the Discharging Method to obtain a contradiction to Euler’s formula. The discharging method starts by assigning numerical values (known as charges) to the elements of the graph. For $x \in V(H) \cup F(H)$, define $ch(x)$ as follows.

(CH.1) $ch(v) = 6 - d_H(v)$, for any $v \in V(H)$.
(CH.2) $ch(f) = 6 - 2|f|$, for any $f \in F(H) - \{X_H\}$.
(CH.3) $ch(X_H) = -5\frac{2}{3} - 2|X_H|$.

Next, we show that

$$\sum_{x \in V(H) \cup F(H)} ch(x) = \frac{1}{3} \quad (4.16)$$
Proof.  

\[ \sum_{x \in V(H) \cup F(H)} ch(x) = -\frac{2}{3} - 2|x_H| + \sum_{f \in F(H) - X_H} (6 - 2|f|) + \sum_{v \in V(H)} (6 - d(v)) \]


= \[ -\frac{2}{3} - 2|x_H| + 6(|f(H)| - 1) + \sum_{f \in F(H) - X_H} (-2|f|) + \sum_{v \in V(H)} (6 - d(v)) \]


= \[ -\frac{2}{3} + 6(|f(H)| - 1) - 2(2|E|) + 6|V(H)| - 2|E(H)| \]


= \[ 6(F(H) - E(H) + V(H)) - 11\frac{2}{3} = \frac{1}{3} \]

Next the charges are locally redistributed according to the following discharging rules:

(DIS.1) If \( v \) is 2-valent, then \( v \) sends \( 3\frac{1}{3} \) to \( X_G \) and \( \frac{4}{3} \) to the other face incident to it.

(DIS.2) If \( v \) is 3-valent, then \( v \) sends \( 1\frac{5}{8} \) to \( X_G \) and \( \frac{4}{5} \) to every other face incident to it.

(DIS.3) If \( v \) is at least 4-valent, then \( v \) sends \( \frac{4}{5} \) to each incident face.

For \( x \in V(G) \cup F(G) \), let \( ch^*(x) \) (denoted as the modified charge) be the resultant charge after modification of the initial charges according to (DIS.1-3). We obtain a contradiction to (4.16) by showing that \( ch^*(x) \leq 0 \) for every \( x \in V(H) \cup F(H) \). This is clearly implied by the following claims proved below.

(A) \( ch^*(v) \leq 0 \), for each \( v \in V(H) \).
(B) \( ch^*(f) \leq 0 \), for each \( f \in F(H) - \{X_H\} \).
(C) \( ch^*(X_H) \leq 0 \).

Observe that according to DIS.(1)-(3), faces do not send charge and vertices do not receive charge.

Proof of (A). It is sufficient to consider vertices \( v \) satisfying \( d_G(v) \geq 5 \). Indeed, if \( d_H(v) \geq 6 \), then \( ch(v) = ch^*(v) \leq 0 \) by (CH.1). If \( 2 \leq d_G(v) \leq 3 \), then it is easily seen by (CH.1) and (DIS.1-2) that \( ch^*(v) = 0 \). If \( 4 \leq d_G(v) \leq 5 \), then, by (CH.1) and (DIS.3), \( ch^*(v) = 6 - d_H(v) - \frac{4}{5}d_G(v) \leq 0 \). □

Proof of (B). Let \( f \in F(H) - \{X_H\} \). By (DIS.1-3), \( f \) receives a charge of \( \frac{4}{5} \) from every vertex incident to it. Hence, together with (CH.2), \( ch^*(f) = 6 - 2|f| + \frac{4}{5}|f| \leq 0 \). (The last inequality follows as \( |f| \geq 5 \).) □

Proof of (C). Let \( S_1 \subseteq V(X_G) \) be the set of 3-valent vertices of \( X_G \), and let \( S_2 = V(X_G) - (S \cup S_1) \). By (CH.3), (DIS.1-3) and as \( |S| \leq 3 \), we see that \( ch^*(f) = -5\frac{2}{3} - 2|X_G| + 3\frac{1}{5}|S| + 1\frac{5}{8}|S_1| + \frac{5}{8}|S_2| \leq -5\frac{2}{3} - 2|X_G| + 3 \times 3\frac{1}{5} + 1\frac{5}{8}(|X_G| - 3) = -\frac{5}{8}|X_G| - \frac{11}{12} \leq 0 \). □
§5. \textit{K}_5\text{-}minors in internally 4-connected graphs. By \(V_8\) we mean \(C_8\) together with 4 pairwise overlapping chords. By \(TG\) we mean a subdivided \(G\).

The following is due to Wanger.

5.1. \cite{6} \textbf{Theorem 4.6} If \(G\) is 3-connected and \(TV_8 \subseteq G\) then either \(G \cong V_8\) or \(G\) has a \(K_5\)-minor.

The following structure theorem was proved independently by Kelmans \cite{7} and Robertson \cite{8}.

5.2. \cite{7} Let \(G\) be internally 4-connected with no minor isomorphic to \(V_8\). Then \(G\) satisfies one of the following conditions:

\begin{enumerate}
\item[(5.2.1)] \(G\) is planar;
\item[(5.2.2)] \(G\) is isomorphic to the line graph of \(K_{3,3}\);
\item[(5.2.3)] there exist a \(uv \in E(G)\) such that \(G - \{u, v\}\) is a circuit;
\item[(5.2.4)] \(\left|G\right| \leq 7\);
\item[(5.2.5)] there is an \(X \subseteq V(G), \left|X\right| \leq 4\) such that \(\|G - X\| = 0\).
\end{enumerate}

From 5.1 and 5.2 we deduce that

5.3. \textit{A nearly 5-long internally 4-connected nonplanar \(G\) has a \(K_5\)-minor.}

\textbf{Proof.} We may assume that \(G \not\cong V_8\) and that \(G\) has no \(V_8\)-minor. The former since \(V_8\) is not nearly 5-long and the latter by 5.1. Hence, \(G\) satisfies one of (5.2.1-5). As \(G\) is nonplanar, by assumption, and the line graph of \(K_{3,3}\) has a \(K_5\)-minor (and is not nearly 5-long) it follows that \(G\) satisfies one of (5.2.3-5).

If \(G\) is of girth \(\leq 4\), let \(a \in V(G) \cup E(G)\) be a breaker of \(G\); otherwise (if \(G\) has girth \(\geq 5\)) let \(a\) be an arbitrary vertex of \(G\). If \(a \in V(G)\), put \(b := a\); otherwise let \(b\) be some end of \(a\). By definition, \(G - b\) has girth \(\geq 5\).

\begin{enumerate}
\item[(5.3.1)] \(G - \{u, v\}\) is not a circuit for any \(u, v \in V(G)\) so that \(G\) does not satisfy (5.2.3).
\end{enumerate}

\textit{Subproof.} For suppose not; and let \(C := G - \{u, v\} = \{x_0, \ldots, x_{k-1}\}\), where \(k \geq 3\) is an integer.

Suppose first that \(b \in \{u, v\}\) and assume, without loss of generality, that \(u = b\). Then, \(k \geq 5\). As \(v\) is at least 3-valent, there exists \(0 \leq i \leq k - 1\) so that \(vx_i \in E(G)\). Since \(G - b\) has girth \(\geq 5\), \(vx_{i+1}, vx_{i+2} \notin E(G)\) (subscript are read modulo \(k\)). Since \(x_{i+1}\) and \(x_{i+2}\) are at least 3-valent in \(G\), each is adjacent to \(u\). But then \(\{u, x_i, x_{i+3}\}\) is a 3-disconnector of \(G\) separating \(\{x_{i+1}, x_{i+2}\}\) from \(\{v, x_{i+4}\}\) (note that since \(k \geq 5\), \(x_{i+1}, x_{i+2} \neq x_{i+4}\)); a contradiction to \(G\) being internally 4-connected.

Suppose then that \(x_i = b\), for some \(0 \leq i \leq k - 1\). Hence, exactly one of \(v\) and \(u\) is adjacent to \(x_{i+1}\) and exactly one to \(x_{i+2}\) (this is true since every vertex of \(C\) is adjacent to \(v\) or \(u\), and if say, \(v\), is adjacent to both \(x_{i+1}\) and \(x_{i+2}\) then \(G - b\) contains a triangle). If \(x_{i+3} \neq x_i\), then \(x_{i+3}\) is adjacent to one of \(u\) and \(v\). If \(x_i = x_{i+3}\), then \(C\) is a circuit of length three, and \(V(G) = 5\). Both cases contradict the fact that \(G\) is nearly 5-long.\(\square\)
\(5.3\) If \(|G| \geq 8\) so that \(G\) does not satisfy \((5.2)\).

Subproof. For suppose \(|G| \leq 7\). As \(G\) is internally 4-connected, \(G-b\) is 2-connected. Since \(G-b\) is of girth \(\geq 5\), then \(G-b\) contains an induced circuit \(C\) of length \(\geq 5\). Hence \(|G| \geq 6\). If \(|G| = 6\), then \(G = C \cup b\) and then \(G\) is planar; a contraction. If \(|G| = 7\) then \(G\) is a circuit plus two vertices and we get a contradiction to \((5.3)\). Hence, \(V(G) \geq 8.\)

To reach a contradiction we show that \((5.2)\) is not satisfied by \(G\). For suppose it is satisfied and let \(X\) be as in \((5.2)\) and let \(Y = V(G) - X\). As \(V(G) \geq 8\), then \(|Y| \geq 4\) and every vertex of \(Y\) is adjacent to at least three vertices in \(X\). But then it is easily seen that \(G\) is of girth \(\leq 4\) but contains no edge- or vertex-breaker; a contradiction. ■

Let \(G\) be a plane graph. By jump over \(G\) we mean a path \(P\) internally-disjoint of \(G\) whose ends are not cofacial in \(G\).

\(5.4\). Let \(G\) be an internally 4-connected nearly 5-long plane graph and let \(P\) be a jump over \(G\). Then, \(G\) has a \(K_5\)-minor with every branch set meeting \(V(G)\).

Proof. Put \(G' := G \cup P\). (By possibly contracting \(P\)) we may assume that \(P\) is an edge \(e\) with both ends in \(G\). Suffices now to show that \(G'\) has a \(K_5\)-minor. Suppose \(G'\) has no such minor. We may assume that \(G' \neq V_8\), since \(V_8\) with any edge removed is not internally 4-connected, and that \(G'\) has no \(V_8\)-minor, by \(5.1\). Since \(G'\) is nonplanar, \(|G'| \geq |G| \geq 11\), by \(4.9\) and and the line graph of \(K_3,3\) has a \(K_5\)-minor, we have that \(G'\) satisfies \((5.2)\) or \((5.2.5)\). We show that both options lead to a contradiction to the definition of \(G\).

Suppose \((5.2.3)\) is satisfied. Set \(C := G' - \{u, v\} = \{x_0, \ldots, x_{k-1}\}\), where \(k \geq 9\) is an integer. If \(e \notin E(C)\), then a contraction is obtained by showing that \(G-e-\{v, u\}\) cannot be a circuit. The proof is exactly the same as the proof of \((5.3)\) with \(G-e\) instead of \(G\).

Hence we may assume that \(e \in E(C)\); so let \(e = x_ix_{i+1}\), for some \(0 \leq i \leq k-1\) (subscript are read modulo \(k\)). Observe that \(d_{G'}(x_i), d_{G'}(x_{i+1}) \geq 4\). Hence, in \(G\), each of \(x_i\) and \(x_{i+1}\) is adjacent to both \(u\) and \(v\).

By assumption that \((5.2.3)\) is satisfied, \(uw, v \in E(G)\), and we see that one of \(u\) or \(v\) is a breaker, say \(u\). Hence, \(ux_{i+2}, vx_{i+3} \notin E(G)\). But then, since and \(d_G(x_{i+1}), d_G(x_{i+2}) = 3\), the set \(\{u, x_{i+1}, x_{i+4}\}\) is a 3-disconnector of \(G\) (note that since \(k \geq 9\), \(x_{i+1}, x_{i+4}\) are distinct) separating \(\{x_{i+2}, x_{i+3}\}\) from \(\{x_{i+5}, x_{i+6}\}\); a contradiction. Hence \((5.2.3)\) is not satisfied.

Suppose \((5.2.5)\) is satisfied. As \(V(G) \geq 11\), it is easily seen that \(G(= G' - e)\) is of girth \(\leq 4\) but has no edge- or vertex-breaker; a contradiction. This concludes the proof. ■

By society we mean a pair \((G, \Omega)\) consisting of a graph \(G\) and a cyclic permutation \(\Omega\) over a finite set \(\Omega \subseteq V(G)\). Let \(\Omega = \{v_1, \ldots, v_k\}\), \(k \geq 4\). Two pairs of vertices \(\{s_1, t_1\} \subseteq \Omega\) and \(\{s_2, t_2\} \subseteq \Omega\) are said to overlap along \((G, \Omega)\) if \(\{s_1, s_2, t_1, t_2\}\) occur in \(\Omega\) in this order along \(\Omega\).

Two vertex disjoint paths \(P\) and \(P'\) of \(G\) that are both internally-disjoint of \(\Omega\) are said to form a cross on \((G, \Omega)\) if their ends are in \(\Omega\) and these overlap along \((G, \Omega)\).

\(5.5\). [9, Lemma (2.4)] Let \((G, \Omega)\) be a society. Then either

1. \((G, \Omega)\) admits a cross in \(G\), or
Let $C$ be a circuit in a plane graph $G$. Then the clockwise ordering of $V(C)$ induced by the embedding of $G$ defines a cyclic permutation on $V(C)$ denoted $\Omega_C$ and we do not distinguish between the cyclic shifts of this order. Then, $(G, \Omega_C)$ is a society with $\Omega_C = V(C)$. Throughout, we omit this notation when dealing with such societies of circuits of plane graphs and instead say that $C$ is a society of $G$.

5.6. Let $G$ be a 3-connected plane graph of order $\geq 5$ and let $P$ and $P'$ be vertex disjoint paths that are internally-disjoint of $G$ and whose ends are contained in a facial circuit $f$ of $G$. If $P \cup P'$ form a cross on $f$, then $G \cup P \cup P'$ contains a $K_5$-minor with every branch set meeting $V(G)$.

Proof. Clearly, $V(G) \neq V(f)$. Since the facial circuits of a 3-connected plane graph are it induced nonseparating circuits \cite{5}, we have that $G - V(f)$ is connected so that $f \cup P \cup P'$ have a $K_4$-minor which is completed into a $K_5$-minor by adding a fifth branch set that is $G - V(f)$ (as $f$ is an induced circuit). \hfill \Box

§6 Proof of 1.1. Let $\mathcal{H} = \{H \subseteq G : H$ is connected, $|G/H| \geq 5$, and $\|G/H\| \geq 3|G/H| - 7\}$. $\mathcal{H}$ contains every member of $V(G)$ as a singleton and thus nonempty. Let $H_0 \in \mathcal{H}$ be maximal in $(\mathcal{H}, \subseteq)$, $H_1 = G[N_G(H_0)]$, and let $G_0 = G/H_0$, where $z_0 \in V(G_0)$ represents $H_0$. Let $G_1 = G_0 - z_0$ and note that $G_1 \subseteq G$.

$|G_0| = 5$ implies that $\|G_0\| \geq 8$ so that $\|G_1\| \geq 4$ and contains a $k$-circuit with $k < 5$; contradiction to the assumption that $G$ has girth at least 6. Thus, we may assume that

(1.1A) $|G_0| \geq 6$.

Let $x \in V(H_1)$ and put $G'_0 = G_0/z_0x$. $|G'_0| \geq 5$, by (1.1A). Thus, the maximality of $H_0$ in $(\mathcal{H}, \subseteq)$ implies that $\|G'_0\| \leq 3|G'_0| - 8$. Thus, $\|G_0\| - \|G'_0\| \geq 3|G_0| - 7 - 3(|G_0| - 1) + 8 \geq 4$; implying that $z_0x$ is common to at least three triangles so that $d_{H_1}(x) \geq 3$. It follows then that

(1.1B) $\delta(H_1) \geq 3$.

Let $H$ be an internally 4-connected nearly 6-long truncation of $H_1$, by \cite{3.8} such is nonplanar by \cite{4.8} and has a $K_5$-minor by \cite{5.3} Consequently, $G_0$ has a $K_6$-minor. \hfill \Box

§7 Proof of 1.3. In a manner similar to that presented in the proof of 1.1 let $\mathcal{H} = \{H \subseteq G : H$ is connected, $|G/H| \geq 5$, and $\|G/H\| \geq 3|G/H| - 8\}$ (such is nonempty) and let $H_0, H_1, G_0, z_0, G_1$ be as in the proof of 1.1

$|G_0| = 5$ implies that $\|G_0\| \geq 8$ so that $\|G_1\| \geq 4$ and contains a $k$-circuit with $k < 5$; contradiction to the assumption that $G$ has girth at least 5. Thus, we may assume that

(1.3A) $|G_0| \geq 6$. 

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Let \( x \in V(H_1) \) and put \( G_0' = G_0/z_0x. \) \(|G_0'| \geq 5, \) by \((1.3)A). \) Thus, the maximality of \( H_0 \) in \((\mathcal{H}, \subseteq) \) implies that \( \|G_0'\| \leq 3\frac{1}{5}|G_0| - 9. \) Thus, \( \|G_0\| - \|G_0'\| \geq 3\frac{1}{5}|G_0| - 8 - 3\frac{1}{5}(|G_0| - 1) + 9 \geq 4; \) implying that \( z_0x \) is common to at least three triangles so that \( d_{H_1}(x) \geq 3. \) It follows then that

\[(1.3)B \] \( \delta(H_1) \geq 3; \)

implying that

\[(1.3)C \] \( \delta(G_0) \geq 4. \)

Next, we prove that

\[(1.3)D \] \( \kappa(G_0) \geq 5. \)

To see \((1.3)D\), let \( T \subseteq V(G) \) be a minimum disconnector of \( G_0 \) and assume, towards contradiction, that \( |T| \leq 4. \) As \( \kappa(G) \geq 6, z_0 \in T. \) Let then \( y = |N_{G_0}(z_0) \cap T| \) and let \( \mathcal{C} \) denote the components of \( G_0 - T. \) Choose \( C \in \mathcal{C} \) and put \( H_1 = G_0[C \cup T] \) and \( H_2 = G_0 - C. \)

Let \( H_i' \) be the graph obtained from \( G_0 \) by contracting \( H_{3-i} \) into \( z_0 \) (note that minimality of \( T \) implies that each of its members is incident with each member of \( \mathcal{C} \), for \( i = 1, 2. \) As \( |H_i| \geq 5, \) by \((1.3)C), then \( |H_i'| \geq 5, \) for \( i = 1, 2. \) The maximality of \( H_0 \) in \((\mathcal{H}, \subseteq) \) then implies that \( \|H_i'\| \leq 3\frac{1}{5}|H_i'| - 9. \)

As \( z_0x \in E(H_i') \) for each \( x \in T' = T \setminus \{z_0\}, \) for \( i = 1, 2, \) it follows that

\[ \|G_0\| + y + 2(|T'| - y) + \|G_0[T']\| \leq \|H_1'\| + \|H_2'\| \leq 3\frac{1}{5}(|G_0| + |T|) - 18. \] (7.1)

As \( \|G_0\| \geq 3\frac{1}{5}|G_0| - 8, \) we have that

\[ 8 + \|G_0[T']\| \leq \frac{1}{5}|T| + y. \] (7.2)

Now, \( |T| \leq 4 (\) by assumption\( ), so that \( y \leq 3, \) and \( \|G_0[T']\| \geq 0. \) Consequently, the right hand size of \((7.2)\) does not exceed \( 7.8. \) This contradiction establishes \((1.3)D\).

Let \( \mathcal{B} \) denote the bridges of \( H_1 \) in \( G_1. \) We may assume that \( \mathcal{B} \) is nonempty. Otherwise, \( G_1 \) coincides with \( H_1 \) so that \( H_1 \) is a nonplanar 4-connected graph of girth \( \geq 5 \) and thus containing a \( K_5 \)-minor by \((5.3). \) Consequently, \( G_0 \) has a \( K_5 \)-minor and \((1.3) \) follows.

Let \( H \) be an internally 4-connected nearly 5-long truncation of \( H_1, \) by \((3.8). \) We may assume that \( H \) is planar for otherwise \( H \) has a \( K_5 \)-minor, by \((5.3) \) so that \( G_0 \) has a \( K_5 \)-minor and \((1.3) \) follows. Let \( x \) denote the breaker of \( H, \) if such exists in \( H. \) Let \( B_1 = \emptyset \) if \( x \) does not exist (so that \( H \subseteq G \) or is an edge-breaker. Otherwise (i.e., if \( x \) is a vertex-breaker), \( B_1 \) denotes the members of \( \mathcal{B} \) with attachment vertices in the subgraph of \( H_1 \) contracted into \( x. \) Put \( B_2 = B \setminus B_1, \)

Fix an embedding of \( H \) in the plane. No member of \( \mathcal{B} \) defines a jump over \( H \) for otherwise the union of \( H \) and such a jump has has a \( K_5 \)-minor with every branch set meeting \( V(H), \) by \((5.4). \) Hence, every member of \( \mathcal{B} \) has all of its attachment vertices confined to a single face of \( H. \)

By \textit{patch} we mean a face \( f \) of \( H \) together with all members of \( \mathcal{B} \) attaching to \( V(f). \) Patches not meeting \( x \) in case it is a vertex-breaker are called \textit{clean} (so that if \( x \) does not exist or is
an edge-breaker, then every patch is clean). $f$ is called the *rim* of the patch. If $P$ is a patch with rim $f$, then by $(P, \Omega_f)$ we mean a society with $\Omega_f = V(f)$ and $\Omega_f$ is the clockwise order on $V(f)$ defined by the embedding of $f$ in the plane.

(1.3) Let $H'$ denote the union of $H$ and all members of $B_2$. Then, $H'$ is planar.

To see (1.3) it is sufficient to show that every clean patch is planar. Indeed, since any two faces of $H$ meet either at a single vertex or at a single edge, the union of any number of planar patches results in a planar graph.

Let $P$ be a clean patch with rim $f$. If $(P, \Omega_f)$ contains a cross, then the union of $H$ and such a cross has a $K_5$-minor, by [5,6] with every branch set meeting $V(H)$; so that $G_0$ has a $K_6$-minor and (1.3) follows. Assume then that $(P, \Omega_f)$ has no cross and is nonplanar. Then, $P = P_1 \cup P_2, P_1 \cap P_2 = P[D]$ and $|D| \leq 3$ such that $V(f) \subseteq V(P_1)$ and $|V(P_2) \setminus V(P_1)| \geq 2$, by [5,5]. Hence, $\{z_0\} \cup D$ is a $k$-disconnecter of $G_0$ with $k \leq 4$; contradicting (1.3). It follows that $P$ is planar so that (1.3) follows.

If $x$ is a vertex-breaker, then let $C$ be the vertices of $H$ cofacial with $x$. 4-connectivity of $G_1$ implies that every vertex in $H' - \{x\} - C$ is at least 4-valent in $H' - x$. As $x$ is 3-valent in this case, by (3.1), we have that $H' - x$ is a 2-connected planar graph of girth $\geq 5$ has an embedding in the plane with each vertex not in $X_{H' - x}$ at least 4-valent, and each vertex in $X_{H' - x}$ at least 3-valent except for at most 3 vertices which are at least 2-valent. By [4.15], $H' - x$ is does not exist; contradiction.$\blacksquare$

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