Tensor Lagrangians, Lagrangians equivalent to the Hamilton-Jacobi equation and relativistic dynamics

Alexander Gersten
Department of Physics
Ben-Gurion University of the Negev
84105 Beer-Sheva, Israel
alex.gersten@gmail.com

January 27, 2016

Abstract

We deal with Lagrangians which are not the standard scalar ones. We present a short review of tensor Lagrangians, which generate massless free fields and the Dirac field, as well as vector and pseudovector Lagrangians for the electric and magnetic fields of Maxwell’s equations with sources. We introduce and analyse Lagrangians which are equivalent to the Hamilton-Jacobi equation and recast them to relativistic equations.

Key Words, scalar Lagrangians, tensor Lagrangians, Hamilton-Jacobi equation, relativistic dynamics

(In Foundations of Physics: Found Phys (2011) 41: 88–98)

1 Introduction

Maxwell’s equations are a peculiar case for which there does not exist a scalar Lagrangian generating the equations of the electric and magnetic fields. There exist scalar Lagrangians generating equations for the electromagnetic potentials. Maxwell’s equations played a very important role in the development of theoretical physics. They played a crucial role in the development of special relativity as they were Lorentz invariant. They even encompass the one photon quantum equation\(^1\) \(^2\) \(^3\) \(^4\). One would expect to see the Planck constant in a quantum equation, but we have observed that the Planck constant cancels out in massless free field (homogeneous) equations\(^4\) \(^5\); that explains its absence in Maxwell’s equations.
Maxwell’s equations have a large amount of important symmetries. Most of them were described in the book of Fushchych and Nikitin. We have also developed methods for finding symmetries of Maxwell’s equations, but now we realize that they were a drop in the infinite sea of symmetries.

The Lagrangian formulation of the interaction of charged particles with the electromagnetic potentials was crucial in the development of classical and quantum electrodynamics. The Lagrangian formalism is extremely important in theoretical physics, yet it is not uniquely defined and not always applicable. Our aim is to bring forward new approaches and new perspectives. In the present paper we deal with Lagrangians which are not the standard scalar Lagrangians.

Tensor Lagrangians were derived in the past for all massless free fields and the Dirac field. Vector Lagrangians were constructed for the electric and magnetic fields of the Maxwell equations with currents. These Lagrangians gave rise to a large number of conserved currents. As these Lagrangians are not well known, a short review is given in Sec. 2.

In Sec. 3 Lagrangians equivalent to the Hamilton-Jacobi equation (HJE) will be presented. In addition to our previous work, the relativistic dynamics will be exposed in more details. Similarly to the HJE’s, these Lagrangians generate all possible trajectories. The merit of this approach is that it generates equations for the first integrals of the HJE namely the conjugate momenta and energy, thus it can be thought as a shortcut to the HJE. The main difference between standard classical mechanics and our approach is that in the standard classical mechanics the coordinates and momenta are functions of time only. In our approach the momenta are functions of the coordinates and the coordinates are functions of time. Thus the coordinate-dependent momenta can be considered as the fields of classical mechanics.

Summary and conclusions will be given in Sec. 4.

## 2 Tensor Lagrangians

In this section a short review of tensor Lagrangians for free fields and vector Lagrangians for Maxwell equations with sources will be given. Repeated indices will induce the summation convention.

### 2.1 Tensor Lagrangians for free massless fields and for the Dirac field

Morgan and Joseph considered the Pauli-Fiertz equation for a massless free field of spin $s$, represented by a completely symmetric spinor $\varphi_{A_1...A_{2s}}$ with 2s indices

$$\partial^{A_1\hat{B}_1} \varphi_{A_1...A_{2s}} = 0, \quad \text{or} \quad \partial^{A_1\hat{B}_1} \varphi_{\hat{B}_1...\hat{B}_{2s}} = 0$$

(1)
where $\partial^{A\dot{B}}$ is the derivative operator in spinor form. They found that the Pauli-Fierz equations can be obtained from the tensor Lagrangian

$$L_{A_2...A_2,\dot{B}_2...\dot{B}_2} = \frac{i}{2} \left[ \varphi_{A_1...A_2} \partial^{A_1\dot{B}_1} \varphi_{\dot{B}_1...\dot{B}_2} - \left( \partial^{A_1\dot{B}_1} \varphi_{A_1...A_2} \right) \varphi_{\dot{B}_1...\dot{B}_2} \right].$$

The Dirac equation is obtained from the Lagrangian

$$L = \frac{i}{2} \left[ \varphi_{A} \partial^{A} \varphi_{B} - \left( \partial^{A} \varphi_{A} \right) \varphi_{B} \right] + \frac{i}{2} m \left( \chi_{A} \partial^{A} \chi_{B} + \varphi_{B} \partial^{A} \chi_{A} \right) + \text{h.c.}.$$  \hspace{1cm} (3)

### 2.2 Vector and pseudovector Lagrangians for the electromagnetic field with sources

For the full set of Maxwell’s equations

$$\partial_{\mu} F^{\mu \nu} = j^{\nu}, \quad \partial_{\mu} \bar{F}^{\mu \nu} = 0,$$  \hspace{1cm} (4)

where $F_{\mu \nu}$ is the electromagnetic field tensor and $\bar{F}^{\mu \nu}$ its dual, Sudbery \cite{9} found the following 4-vector Lagrangian

$$L_{\alpha} = \bar{F}^{\mu \nu} \partial_{\nu} F_{\mu \alpha} - F^{\mu \nu} \partial_{\nu} \bar{F}_{\mu \alpha} - 2 \bar{F}_{\alpha \mu} j^{\mu}$$  \hspace{1cm} (5)

Fushchych, Krivskiy and Simulik \cite{10} found an infinite number of Lagrangians which generate Eqs.\textsuperscript{(4)}. Using their notation

$$Q_{\mu} = \partial_{\nu} F, \quad R_{\mu} = \partial_{\nu} \bar{F}_{\mu \nu},$$  \hspace{1cm} (6)

Eqs.\textsuperscript{(5)} become

$$Q_{\mu} = j_{\mu}, \quad R_{\mu} = 0, \quad \mu = 0, 1, 2, 3,$$  \hspace{1cm} (7)

They consider the tensor $T_{\mu \rho \sigma}$ and pseudotensor $T'_{\mu \rho \sigma}$ with respect to the total Poincaré group $\tilde{P}(1,3)$

$$T_{\mu \rho \sigma} = a \left[ g_{\mu \rho} (Q_{\sigma} - j_{\sigma}) - g_{\mu \sigma} (Q_{\rho} - j_{\rho}) \right] + b \epsilon_{\mu \rho \sigma \nu} R^{\nu},$$  \hspace{1cm} (8)

$$T'_{\mu \rho \sigma} = a' \left[ g_{\mu \rho} R_{\sigma} - g_{\mu \sigma} R_{\rho} \right] + b' \epsilon_{\mu \rho \sigma \nu} (Q_{\nu} - j_{\nu}),$$  \hspace{1cm} (9)

where $a, b, a', b'$ are constant coefficients and $\epsilon^{\mu \nu \rho \sigma}$ is a completely antisymmetric unit tensor, $\epsilon^{0123} = 1$. They proved that for any $ab = 0 = a'b'$ each of the set of equations

$$T_{\mu \rho \sigma} = 0,$$  \hspace{1cm} (10)

$$T'_{\mu \rho \sigma} = 0,$$  \hspace{1cm} (11)
is equivalent to the Maxwell equations Eqs. (4). They considered the following Lagrangian:

\[ L_{\mu} = a_1 F_{\mu\nu} Q^\nu + a_2 F_{\mu\nu} \bar{R}^\nu + a_3 \epsilon F_{\mu\nu} R^\nu + a_4 \epsilon F_{\mu\nu} \bar{Q}^\nu + a_5 \bar{F}_{\mu\nu} Q^\nu + a_6 \bar{F}_{\mu\nu} R^\nu + a_7 \epsilon \bar{F}_{\mu\nu} \bar{Q}^\nu + a_8 \epsilon \bar{F}_{\mu\nu} Q^\nu + (q_1 F_{\mu\nu} + q_2 \epsilon \bar{F}_{\mu\nu}) j^\nu, \] (12)

where

\[ \bar{Q}_{\mu} \equiv \partial_{\nu} \bar{F}_{\mu\nu}, \quad \bar{R}_{\mu} \equiv \partial_{\nu} \epsilon \bar{F}_{\mu\nu}, \quad \epsilon \bar{F}_{\mu\nu} \equiv \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \bar{F}_{\rho\sigma} \]

They proved that Maxwell equations are generated from the Lagrangian Eqs. (12) if and only if the following conditions on the coefficients are

\[ a_8 - a_2 = a = -b' = -q_1 \equiv -q = 0, \]
\[ a_6 - a_4 = a' = -b \neq 0, \]
\[ a_1 - a_3 - a_6 - a_8 = a_2 + a_4 + a_5 - a_7 = 0. \] (13)

3 Lagrangians equivalent to the Hamilton-Jacobi equation

We will start with nonrelativistic dynamics and consider n-dimensional Lagrangians

\[ L(x_1, ..., x_n, \dot{x}_1, ..., \dot{x}_n, t) \equiv L(x, \dot{x}, t), \quad \dot{x} = \frac{dx}{dt}. \] (14)

Hamilton’s principle of least action

\[ S(x) = \int_{t_1}^{t_2} L(x, \dot{x}, t) dt = \min, \] (15)

where \( S \) is the action, leads to the Euler-Lagrange equations

\[ \frac{\partial L}{\partial x_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i}, \quad i = 1, ..., n \] (16)

or to the Hamilton’s equations

\[ \frac{\partial H(x, p, t)}{\partial x_k} = -\dot{p}_k, \quad \frac{\partial H(x, p, t)}{\partial p_k} = \dot{x}_k, \quad k = 1, ..., n, \] (17)

where \( H \) is the Hamiltonian and \( p_i \) the momenta. The solutions are trajectories \( x_i(t) \).

Let us adopt another approach. Let us change Eq. (15) to

\[ S(x, t) = \int_{t_0}^{t} L(x, \dot{x}, t') dt' = \min, \] (18)
and go to the Hamiltonian formalism replacing \( L \) with a Lagrangian \( L_{HJ} \) which we will show later on, will generate the HJE.

\[
L_{HJ}(x, \dot{x}, t) = p_i(x, t)\dot{x}_i - H(x, p(x, t), t), \quad i = 1, ..., n, \tag{19}
\]

From Eq. (18) and Eq. (19) we obtain

\[
dS(x, t) = L_{HJ} dt = p_i(x, t) dx_i - H(x, p(x, t), t)dt, \quad i = 1, ..., n, \tag{20}
\]

from which the HJE immediately results

\[
\frac{\partial S(x, t)}{\partial x_j} = p_j(x, t), \quad \frac{\partial S(x, t)}{\partial t} = -H(x, p, t). \tag{21}
\]

The Euler-Lagrange equations for the Lagrangian \( L_{HJ} \) of Eq. (19) are

\[
\frac{\partial L_{HJ}}{\partial x_j} - \frac{d}{dt} \frac{\partial L_{HJ}}{\partial \dot{x}_j} = \left( \frac{\partial p_i}{\partial x_j} \dot{x}_i - \frac{\partial H}{\partial p_i} \right) \frac{\partial p_i}{\partial x_j} - \frac{\partial H}{\partial x_j} - \frac{d}{dt} p_j. \tag{22}
\]

Later on we will also use the relations

\[
\frac{\partial L_{HJ}}{\partial x_j} = \frac{d}{dt} \frac{\partial L_{HJ}}{\partial \dot{x}_j} = \frac{d}{dt} p_j \equiv \dot{p}_j, \quad \frac{\partial L_{HJ}}{\partial x_i} = p_j, \quad \frac{d}{dt} p_j = \frac{\partial p_j}{\partial x_k} \dot{x}_k + \frac{\partial p_j}{\partial t}, \tag{23}
\]

and

\[
dL_{HJ}(x, \dot{x}, t) = \frac{\partial L_{HJ}}{\partial x_i} dx_i + \frac{\partial L_{HJ}}{\partial \dot{x}_i} d\dot{x}_i + \frac{\partial L_{HJ}}{\partial t} dt = \dot{p}_i dx_i + p_i d\dot{x}_i + \frac{\partial L_{HJ}}{\partial t} dt, \tag{24}
\]

in order to find a non-linear equation for the momenta.

3.1 The general case

Using Eq. (19) and Eq. (24) we have

\[
dH(x, p(x, t), t) = \dot{x}_i dp_i + p_i d\dot{x}_i - dL_{HJ} = \dot{x}_i dp_i - \frac{\partial L_{HJ}}{\partial x_i} dx_i - \frac{\partial L_{HJ}}{\partial t} dt, \tag{25}
\]

from which we deduce that

\[
\frac{\partial H}{\partial x_i} = -\frac{\partial L_{HJ}}{\partial x_i}, \quad \frac{\partial H}{\partial p_i} = \dot{x}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L_{HJ}}{\partial t}, \tag{26}
\]

but

\[
\frac{\partial H}{\partial x_i} = -\frac{\partial L_{HJ}}{\partial x_i} = -\frac{d}{dt} p_i = -\dot{p}_i,
\]

and Hamilton-like equations are obtained

\[
\frac{\partial H(x, p(x, t), t)}{\partial x_i} = -\dot{p}_i(x, t), \quad \frac{\partial H(x, p(x, t), t)}{\partial p_i} = \dot{x}_i. \tag{27}
\]
Noting that
\[ \dot{p}_i(x, t) = \frac{d}{dt} p_i(x, t) = \frac{\partial p_i(x, t)}{\partial x_j} \dot{x}_j + \frac{\partial p_i(x, t)}{\partial t}, \]  
and substituting the expression for \( \dot{x}_i \) in Eq. (27) we obtain a non-linear equation for the momenta
\[ \frac{\partial H(x, p(x, t), t)}{\partial x_i} + \frac{\partial p_i(x, t)}{\partial x_j} \frac{\partial H(x, p(x, t), t)}{\partial p_j} + \frac{\partial p_i(x, t)}{\partial t} = 0. \]  
(29)

Let us look for first integrals of Eq. (29). Let us consider
\[ dH(x, p(x, t), t) = \dot{x}_i dp_i + \frac{\partial H}{\partial x} dx_i + \frac{\partial H}{\partial t} dt = \dot{x}_i dp_i - \dot{p}_i dx_i + \frac{\partial H}{\partial t} dt, \]  
and divide Eq. (30) by \( dt \), we obtain
\[ \frac{dH(x, p(x, t), t)}{dt} = \dot{x}_i \dot{p}_i - \dot{p}_i \dot{x}_i + \frac{\partial H}{\partial t} \]  
i.e.
\[ \frac{dH(x, p(x, t), t)}{dt} = \frac{\partial H(x, p(x, t), t)}{\partial t}. \]  
(31)

3.2 The case of time independent Hamiltonian

If the Hamiltonian is time independent, the solutions of the HJE have the form
\[ S(x, t) = W(x) - Et + C \]  
(32)
where \( E \) and \( C \) are constants. The HJE (Eq. (21)) becomes
\[ \frac{\partial S(x, t)}{\partial x_j} = \frac{\partial W(x)}{\partial x_j} = p_j(x), \quad H(x, \nabla S) = E, \]  
(33)
i.e. the momenta are functions of the coordinates only, describing all possible trajectories with the same total energy \( E \). The Hamiltonian, using Eq. (31) becomes a constant of motion
\[ \frac{dH(x, p(x))}{dt} = 0. \]  
(34)

From Eq. (228b) we have
\[ \frac{\partial p_i}{\partial x_k} - \frac{\partial p_k}{\partial x_i} = \frac{\partial^2 S}{\partial x_k \partial x_i} - \frac{\partial^2 S}{\partial x_i \partial x_k} = 0 \]  
(35)

Let us show that
\[ \frac{dH(x, p(x))}{dx_k} = 0. \]  
(36)
Indeed, using Eq. (30), Eq. (35) and Eq. (33) we have

\[ \frac{dH}{dx_k} = \dot{x}_i \frac{dp_i}{dx_k} - \dot{p}_k \frac{dx_k}{dx_i} = \dot{x}_i \left( \frac{dp_i}{dx_k} - \frac{dp_k}{dx_i} \right) = 0 \]

Hence we can deduce that the Hamiltonian is the constant of motion for all trajectories

\[ H(x, p(x)) = E. \tag{37} \]

The non linear equation for the momenta Eq. (29) becomes now

\[ \frac{\partial H(x, p(x))}{\partial x_i} + \frac{\partial p_i(x)}{\partial x_j} \frac{\partial H(x, p(x))}{\partial p_j} = 0 \tag{38} \]

From equations Eq. (38), Eq. (35) and Eq. (37) one can find all the trajectories \( p_i(x) \) having the same energy \( E \).

3.2.1 Example 1

Let us consider, as a special case the Hamiltonian in one dimension

\[ H(x, p(x)) = \frac{p^2(x)}{2m} + V(x) \tag{39} \]

We can start from Eq. (37), but for better illustration we use Eq. (29), which becomes

\[ \frac{p}{m} \frac{dp}{dx} = - \frac{dV}{dx} \tag{40} \]

Eq. (40) can be recast into

\[ \frac{d}{dx} \left( \frac{p^2}{2m} + V \right) = 0, \tag{41} \]

hence \( H \) is a constant in \( x \) and may depend only on time

\[ H(x, p(x)) = \frac{p^2(x)}{2m} + V(x) = E(t). \tag{42} \]

Moreover if \( H \) does not depend explicitly on time

\[ \frac{\partial H}{\partial t} = 0 \implies E = \text{const.}, \tag{43} \]

and the total energy is conserved (of course we could have started from this point using Eq. (37))

\[ \frac{p^2(x)}{2m} + V(x) = E. \tag{44} \]
The momentum as a (double valued) field is given by

\[ p(x) = \pm \sqrt{2mE - V(x)}. \]  

(45)

Usually the sign ambiguity can be resolved using physical considerations. The trajectories may be obtained from Eq.(27), which in our case is

\[ \dot{x} = \frac{p(x)}{m}, \]  

(46)

and by integrating

\[ dt = \frac{m}{p(x)} dx = \frac{\pm mdx}{\sqrt{2mE - V(x)}}. \]  

(47)

The action can be evaluated from Eq.(21) and Eq.(32)

\[ S(x, t; x_0, t_0) = \pm \int_{x_0}^x \sqrt{2mE - V(x')} dx' - E(t - t_0). \]  

(48)

### 3.3 Relativistic dynamics of one particle

We will show that Eq.(20), although derived from nonrelativistic mechanics, can be considered as a relativistic one if the momenta and energy (the Hamiltonian) belong to the same 4-vector.

If we consider Eq.(20) as a relation between two Lorentz scalars, then the relativistic Lagrangian can be defined as (using the metric (+,-,-,-))

\[ L_{\text{rel}} = -p_\mu \frac{dx^\mu}{d\tau} = -p_i(x) \frac{dx_i}{d\tau} - H(x, p) \frac{dt}{d\tau}; \]  

(49)

\[ \mu = 0, 1, 2, 3; \quad i = 1, 2, 3; \quad x_0 = ct; \quad p_0 = H/c; \quad \dot{x}^i = \frac{dx^i}{d\tau}, \]

where \( \tau \) is the proper time.

The action to be minimized is

\[ S(x, \tau) = \int_{\tau_0}^\tau L_{\text{rel}} \left( x, \frac{dx}{d\tau}, \tau' \right) d\tau' = \int_t^t L_{\text{rel}} \frac{d\tau'}{dt} dt' = \int_t^t \frac{dS}{dt'} dt' = \int_{t_0}^t L_{\mu, \dot{\tau}} dt' = S(x, t), \]

(50)

i.e. the same as in the nonrelativistic case, provided \( p_i(x) \) and \( H(x, p(x))/c \) are components of the same 4-vector. Therefore all equations previously derived in this section can be used for relativistic dynamics with the Lagrangian

\[ L_{\mu, \dot{\tau}} = -p_i(x) \dot{x}^i - H(x, p). \]  

(51)

The equations are not covariant but relativistically invariant.
3.3.1 Relativistic charged particle in a static electromagnetic field

Barut\(^{[53]}\) has shown that a proper Hamiltonian for this case (which is the energy) is

\[
H(x,p) = e\varphi(x) + c \left[ p_i(x) - \frac{e}{c} A_i(x) \right] \left[ p_i(x) - \frac{e}{c} A_i(x) + m^2 c^2 \right]^{\frac{1}{2}},
\]  

(52)

where \(\varphi(x)\) and \(A_i(x)\) are the scalar and vector electromagnetic potentials respectively. In order to find all the trajectories of the momenta with a constant energy \(E\) one has to solve Eq. (38) and Eq. (37).

3.3.2 Example 2, free particle with mass \(m\)

The (time independent) Hamiltonian \(H\) is

\[
H/c = \sqrt{m^2 c^2 + p_i(x) p_i(x)},
\]  

(53)

Using Eq. (57) we find that the trajectories with constant energy \(E\) satisfy

\[
p_i(x) p_i(x) = \frac{E^2}{c^2} - m^2 c^2.
\]  

(54)

From Eq. (56) we have

\[
\frac{dH}{dx} = \frac{cp_i(x) \frac{dp_i(x)}{dx}}{\sqrt{m^2 c^2 + p_i(x) p_i(x)}} = 0; \quad p_i(x) \frac{dp_i(x)}{dx} = 0,
\]  

(55)

and from Eq. (55) one can deduce that

\[
\frac{dp_i(x)}{dx} = 0,
\]  

(56)

i.e. all the trajectories are straight lines with the constraint given by Eq. (54).

In order to find a particular trajectory one has to give initial conditions and solve the Hamilton equation (Eq. (27))

\[
\frac{\partial H(x,p(x))}{\partial p_i} = \dot{x}_i,
\]

and use the constraint as given by Eq. (54).

3.3.3 Example 3, relativistic charged particle in a static electric field

\[
\frac{\partial H(x,p(x))}{\partial x_i} + \frac{\partial p_i(x)}{\partial x_j} \frac{\partial H(x,p(x))}{\partial p_j} = e \frac{\partial \varphi(x)}{\partial x_i} + c \frac{\partial p_i(x)}{\partial x_j} \left[ (p_i(x)p_i(x) + m^2 c^2)^{\frac{1}{2}} \right] = 0
\]  

and

\[
e\varphi(x) + c \left[ p_i(x)(p_i(x) + m_0^2 c^2)^{\frac{1}{2}} \right] = E,
\]  

(58)
combining the two equations we find
\[ c^2 \frac{\partial p_i(x)}{\partial x_j} p^j(x) = \frac{\partial \varphi(x)}{\partial x_i} [e \varphi(x) - E], \quad i = 1, 2, 3 \] (59)
The absolute value of the momentum can be found from Eq. (58), namely
\[ c^2 p_i(x) p^i(x) = (E - e \varphi(x))^2 - m_0^2 c^4. \] (60)
The components of the momentum can be found by solving Eq. (59) with initial conditions. An exact solution for an arbitrary potential can be given if it depends on one coordinate only, for instance
\[ \varphi(x) = \varphi(x_1). \]
In this case from Eq. (59) we find
\[ \frac{\partial p_2(x)}{\partial x_j} = \frac{\partial p_3(x)}{\partial x_j} = 0; \quad j = 1, 2, 3; \quad p_2(x) = P_2 = \text{const}; \quad p_3(x) = P_3 = \text{const}. \]
and from Eq. (60)
\[ cp_1(x) = \pm [(E - e \varphi(x_1))^2 - m_0^2 c^4 - c^2 P_2^2 - c^2 P_3^2]^{1/2}. \]

4 Summary and conclusions

In the present paper we have dealt with Lagrangians which are not the standard scalar Lagrangians. It is well known that the Maxwell equations for the electric and magnetic fields cannot be derived from a scalar Lagrangian. Only the equations for the vector potentials can be derived from a scalar one.

We gave a short review of tensor Lagrangians applied for free fields and the Dirac field \[8\], and vector and pseudovector Lagrangians generating the Maxwell’ equations \[9, 10\]. In the cited papers the symmetries of the Lagrangians were used to derive new and even an infinite amount of conserved currents.

In Sec. 3 Lagrangians equivalent to the Hamilton-Jacobi equation (HJE) were presented in addition to our previous work \[11\], emphasizing the relativistic dynamics. Similarly to the HJE’s, these Lagrangians generate all possible trajectories. We have shown that a Lagrangian of the form as given in Eq. (19), generates the HJE. We have found the equations of the momentum field, they are given in Eq. (29). The advantage of this approach is that it generates equations for the first integrals of the HJE namely the conjugate momenta and energy, without solving the HJE. The case of time independent Hamiltonian is dealt in subsection 3.2. We have shown that the solutions (Eq. (34) and Eq. (36)) satisfy
\[ \frac{dH(x, p(x))}{dt} = \frac{dH(x, p(x))}{dx_1} = 0, \] and the energy \( E \) is the constant of motion for all trajectories of the family \( H(x, p(x)) = E \), which is the first integral of the momentum field equations Eq. (29) for time independent Hamiltonians.
The main difference between standard classical mechanics\textsuperscript{12} and our approach is that in the standard classical mechanics the coordinates and momenta are functions of time only. In our approach the momenta are functions of the coordinates for all possible trajectories.

In subsection 3.3 we have dealt with relativistic dynamics and we have shown that the non-relativistic formalism can be used provided the momenta and (the equal to energy) Hamiltonian belong to the same 4-vector.

References

[1] A. Gersten, ”Maxwell equations as the one-photon quantum equation”. \textit{Foundations of Physics Letters} \textbf{12}, 291-8 (1999),

[2] A. Gersten, ”Maxwell equations - the one-photon quantum equation”. \textit{Foundations of Physics}, \textbf{13}, 1211-1231, (2001).

[3] I. Bialynicki-Birula, ”Photon wave function”, \textit{Progress in Optics} \textbf{36}, 245 (1996)

[4] S. Esposito, \textit{Found. Phys.} \textbf{28}, 231-44 (1999)

[5] A. Gersten, ”Quantum equations for massless particles of any spin”. \textit{Foundations of Physics Letters}, \textbf{13}, 185-192, 2000.

[6] V.I. Fushchich and A.G. Nikitin, \textit{Symmetries of Maxwell’s Equations}, Reidel, Dordrecht 1987.

[7] A. Gersten, Conserved Currents of the Maxwell Equations with Electric and Magnetic Sources, \textit{Ann. Fond. Louis de Broglie} \textbf{21}, 67-79 (1996).

[8] T.A. Morgan and D.W. Joseph, ”Tensor Lagrangians and Generalized Conservation Laws for Free Fields”, \textit{Nuovo Cimento} \textbf{39}, 494-503 (1965).

[9] A. Sudbery, ”A vector Lagrangian for the electromagnetic field”, \textit{J. Phys. A: Math. Gen.} \textbf{19}, L33-L36 (1986)

[10] W.I. Fushchych , I.Yu Krivskiy and V.M. Simulik, ”On vector and pseudovector Lagrangians for electromagnetic field”, \textit{Nuovo Cimento B}, \textbf{103}, 423-429 (1989). Also in W.I. Fushchych, \textit{Scientific Works 3}, 506-510 (2001).

[11] A. Gersten, ”Field Approach to Classical Mechanics”, \textit{Found. Phys.} \textbf{35}, 1433-43 (2005).

[12] H. Goldstein, \textit{Classical Mechanics, 2nd Ed.}, Addison-Wesley, Reading, Massachusetts 1980.

[13] W. Pauli and M. Fierz, \textit{Proc. Roy. Soc.} (London) \textbf{A73}, 211 (1939)
[14] E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave-Equations*, Hafner, New York 1953.

[15] A.O. Barut, *Electrodynamics and Classical Theory of Fields and Particles*, Dover, New York 1980.