Newforms of Half-integral Weight: The Minus Space Counterpart

Ehud Moshe Baruch and Soma Purkait

Abstract. We study genuine local Hecke algebras of the Iwahori type of the double cover of \(\text{SL}_2(\mathbb{Q}_p)\) and translate the generators and relations to classical operators on the space \(S_{k+1/2}(\Gamma_0(4M))\), \(M\) odd and square-free. In [9] Manickam, Ramakrishnan, and Vasudevan defined the new space of \(S_{k+1/2}(\Gamma_0(4M))\) that maps Hecke isomorphically onto the space of newforms of \(S_{2k}(\Gamma_0(2M))\). We characterize this new space as a common \(-1\)-eigenspace of a certain pair of conjugate operators that come from local Hecke algebras. We use the classical Hecke operators and relations that we obtain to give a new proof of the results in [9] and to prove our characterization result.

1 Introduction

Let \(M\) be odd and square-free and let \(k\) be a positive integer. In a remarkable work, Niwa [10], comparing the traces of Hecke operators, proved the existence of Hecke isomorphism between \(S_{k+1/2}(\Gamma_0(4M))\), the space of holomorphic cusp forms of weight \(k + 1/2\) on the congruence subgroup \(\Gamma_0(4M)\) and \(S_{2k}(\Gamma_0(2M))\), the space of weight \(2k\) cusp forms on \(\Gamma_0(2M)\). In [5, 6] Kohnen considers a certain Hecke operator on \(S_{k+1/2}(\Gamma_0(4M))\), which is an analogue of Niwa’s operator at level 4. This operator has two eigenvalues, one positive and one negative, and the Kohnen plus space is the eigenspace of the positive eigenvalue. Kohnen considers a new space, \(S_{k+1/2}^n(\Gamma_0(4M))\), inside his plus space and proves that this new subspace is Hecke isomorphic to \(S_{2k}^n(\Gamma_0(M))\), the space of newforms of weight \(2k\) and level \(M\). From Kohnen’s results, it is clear that the Niwa map sends the Kohnen plus space to a subspace of old forms inside \(S_{2k}(\Gamma_0(2M))\). In a subsequent work, Manickam, Ramakrishnan, and Vasudevan [9] define the new space of \(S_{k+1/2}^n(\Gamma_0(4M))\) that maps Hecke isomorphically onto \(S_{2k}^n(\Gamma_0(2M))\), the space of newforms of weight \(2k\) and level \(2M\). Our main objective in this paper is to give a common eigenspace characterization for this new space of \(S_{k+1/2}(\Gamma_0(4M))\) in terms of certain finitely many pairs of conjugate operators.

This is a continuation of our earlier work in [2], where we use local Hecke algebras to give an eigenspace characterization of the space of integral weight newforms. The local Hecke algebra method allows us to obtain the newspace of Manickam et al. in a different way, and we show that it is the common \(-1\)-eigenspace of Kohnen’s operator, a conjugate of Kohnen’s operator, and pairs of \(p\)-adic analogues of Kohnen’s operator and their conjugates for each prime dividing \(M\). We call this new space the minus space at level \(4M\).
Our results are motivated by the results of Loke and Savin [8], who interpreted
the Kohnen plus space in representation theory language. For the case \( M = 1 \), Loke
and Savin defined another space of half-integer weight forms that they showed is
“conjugate” to the Kohnen plus space. This means that it is an image of the Kohnen
plus space by an invertible Hecke operator and is isomorphic to the Kohnen plus space
as a Hecke module. We show that the Kohnen plus space and the space considered
by Loke and Savin do not intersect and that their sum maps isomorphically to the
space of old forms \( S_{2k}^{\text{old}}(\Gamma_0(2)) \) under the Niwa map. We define the minus space
at level 4 to be the orthogonal complement of the direct sum under the Petersson
inner product and show that it is mapped isomorphically under the Niwa map to
\( S_{2k}^{\text{new}}(\Gamma_0(2)) \), the space of newforms on \( \Gamma_0(2) \). We characterize this space as a com-
mon eigenspace of two Hecke operators: the Niwa operator used by Kohnen to define
the Kohnen plus space and a conjugate of the Niwa operator that was considered by
Loke and Savin. The minus space is the intersection of the negative eigenspaces of
both operators. We normalize the negative eigenvalue to be \(-1\) as in [2]. Our de-
scription of the minus space at level 4 is completely analogous to our description of
the new space \( S_{2k}^{\text{new}}(\Gamma_0(2)) \) in [2], where we showed that \( S_{2k}^{\text{new}}(\Gamma_0(2)) \) is the common
\(-1\)-eigenspace of two Hecke operators. To summarize the case of \( M = 1 \), we show that
the space \( S_{k+1/2}(\Gamma_0(4)) \) decomposes into a direct sum of three spaces: the Kohnen
plus space, a “conjugate” of the Kohnen plus space given by Loke and Savin, and the
minus space. The Kohnen plus space and its conjugate are indistinguishable as Hecke
modules, which is the same as saying that they are mapped under the Niwa map to
“conjugate” spaces of old forms. The minus space is different as a Hecke module from
both spaces.

In order to generalize this result for \( M \) odd and square-free, we consider certain
\( p \)-adic Hecke algebras for every prime \( p \) dividing \( M \). Our work follows that of Loke
and Savin, who studied a certain 2-adic Hecke algebra that allowed them to give a
representation theoretic interpretation of the Kohnen plus space and to introduce the
operator that is a conjugate of Niwa’s operator and the space that is a “conjugate” to
Kohnen’s plus space.

We compute genuine local Hecke algebras, of the Iwahori type with genuine qua-
dratic central character, for \( \widehat{\text{SL}}_2(\mathbb{Q}_p) \), the double cover of \( \text{SL}_2(\mathbb{Q}_p) \), and prove that this
is isomorphic to the Iwahori Hecke algebra of \( \text{PGL}_2(\mathbb{Q}_p) \). In [13], Savin obtained de-
scription of Iwahori-type Hecke algebras for coverings of simply connected Chevally
group \( G \neq \text{SL}_2 \). We are not aware of any such results for \( \text{SL}_2 \), apart from the work of
Loke and Savin [8] for the 2-adic case that we generalize for any odd prime \( p \).

In our \( p \)-adic Hecke algebra, we consider two \( p \)-adic operators that give rise to
conjugate classical Hecke operators which, when used along with Niwa’s operator and
its conjugate, allow us to define our minus space at level \( 4M \). We note that these two
\( p \)-adic operators are \( p \)-adic analogues of Niwa’s operator and its conjugate. We give
two descriptions of the minus space: one description as an orthogonal complement of
a certain sum of subspaces and another description as a common \(-1\)-eigenspace of the
Niwa operator, its conjugate, and a pair of conjugate operators for each prime dividing
\( M \). This again is completely analogous to our description of the space of newforms
of weight \( 2k \) for \( \Gamma_0(2M) \) given in [2, Theorem 1]. We show that the minus space of
weight $k + 1/2$ at level $4M$ is isomorphic as a Hecke module to the space of newforms of weight $2k$ at level $2M$.

Due to the Hecke isomorphism and multiplicity, it is clear that the minus space we define is identical to the newspace of [9]. In particular, we obtain a new proof of the Hecke isomorphism in [9]. We note that our description of the minus space as an orthogonal complement differs from the description of the newspace in [9]. We elaborate this point in Remark 6.27.

Our paper is divided up as follows. In Section 2, we set up notation following Shimura’s work on half-integral weight forms and recall Gelbart’s theory of the double cover of $\text{SL}_2(\mathbb{Q}_p)$. In Section 3, we define a genuine Hecke algebra of the double cover of $\text{SL}_2(\mathbb{Q}_p)$ modulo certain subgroups and a genuine central character and give its presentation using generators and relations. In particular, we recall the work of Loke and Savin when $p = 2$. In Section 4, we translate certain elements in our $p$-adic Hecke algebra to classical Hecke operators on $S_{k+1/2}(\Gamma_0(4M))$. We obtain two classical operators: $\widetilde{Q}_p$ with eigenvalues $p$ and $-1$ and an involution $\widetilde{W}_p$. We further consider $\widetilde{Q}_p'$, which is a conjugate of $\widetilde{Q}_p$ by $\widetilde{W}_p$. We check that these operators are self-adjoint with respect to the Petersson inner product. We recall Kohnen’s classical operator $Q$ on $S_{k+1/2}(\Gamma_0(4M))$, which he uses to describe his plus space. We show that his operator $Q$ comes from the 2-adic Hecke algebra considered by Loke and Savin. Let $\widetilde{Q}_2 := \left( \frac{2}{2k+1} \right) Q/\sqrt{2}$ and let $\widetilde{Q}_2$ be conjugate of $\widetilde{Q}_2'$ by an involution $\widetilde{W}_4$. The operators $\widetilde{Q}_p'$ and $\widetilde{Q}_p$ are $p$-adic analogues of Kohnen’s operator $Q$ and its conjugate. In Section 5, we define our minus space $S_{k+1/2}^-(\Gamma_0(4M))$ and prove our main result.

**Theorem** Let $S_{k+1/2}^-(\Gamma_0(4M)) \subseteq S_{k+1/2}(\Gamma_0(4M))$ be the common $-1$-eigenspace of operators $\widetilde{Q}_p$ and $\widetilde{Q}_p'$ for all primes $p$ dividing $2M$. Then $S_{k+1/2}^-(\Gamma_0(4M))$ has a basis of eigenforms for all the operators $T_q^2$ where $q$ is a prime coprime to $2M$ and all the operators $U_{p^2}$ where $p$ is a prime dividing $2M$, and maps isomorphically under the Niwa map onto the space $S_{2k}^{\text{new}}(\Gamma_0(2M))$.

We are certain that the Hecke algebra approach can be employed to give a newform theory for the space of half-integral weight forms of a general level. Indeed, in [3] we use the methods developed in this paper to define the minus space at level $8M$, $M$ odd and square-free, and show that the minus space at level $8M$ is Hecke isomorphic to $S_{2k}^{\text{new}}(\Gamma_0(4M))$. This generalizes Ueda and Yamana’s work in [17]. Please refer to Remark 6.34 for more details. We plan to use the results in this paper to study Whittaker functions associated with automorphic forms coming from Hecke eigenforms in the minus space. As an application, we plan to generalize the Kohnen–Zagier formula for the twisted central L-values of an integer weight modular form of level $2M$.

# 2 Preliminaries and Notation

Let $k, N$ denote positive integers. Let $\Gamma_0(N)$ be the subgroup of $\text{SL}_2(\mathbb{Z})$ consisting of matrices of the form $\left( \begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix} \right)$ (mod $N$). We denote by $S_k(\Gamma_0(N))$ the space of holomorphic cusp forms of weight $k$ on the group $\Gamma_0(N)$. For each prime $p$ not dividing $N$,
we have the Hecke operator $T_p$ on $S_k(\Gamma_0(N))$ whose action on $q$-expansion can be given as follows: if $f = \sum_{n=1}^{\infty} a_n q^n \in S_k(\Gamma_0(N))$ then $T_p(f) = \sum_{n=1}^{\infty} (a_{pn} + p^k a_{n/p}) q^n$.

For $m \in \mathbb{N}$, let $U_m, V(m)$ be given by the following action on any formal $q$-series:

$$U_m \left( \sum_{n=1}^{\infty} a_n q^n \right) = \sum_{n=1}^{\infty} a_{mn} q^n, \quad V(m) \left( \sum_{n=1}^{\infty} a_n q^n \right) = \sum_{n=1}^{\infty} a_n q^{mn}.$$

It is well known that $V(m)$ maps $S_k(\Gamma_0(N))$ to $S_k(\Gamma_0(mN))$ and if $m | N$, then $U_m$ is an operator on $S_k(\Gamma_0(N))$.

We briefly recall the theory of half-integral weight modular forms [14]. Let $\mathcal{S}$ be the set of all ordered pairs $(\alpha, \phi(z))$ where $\alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \text{GL}_2^+ (\mathbb{R})$ and $\phi(z)$ is a holomorphic function on the upper half plane $\mathbb{H}$ such that $\phi(z)^2 = t \det(\alpha)^{-1/2} (cz + d)$ with $t$ in the unit circle $S^1 := \{ z \in \mathbb{C} : |z| = 1 \}$. Then $\mathcal{S}$ is a group under the following operation:

$$\left( \alpha, \phi(z) \right) \left( \beta, \psi(z) \right) = \left( \alpha \beta, \phi(\beta z) \psi(z) \right).$$

Let $P : \mathcal{S} \to \text{GL}_2^+ (\mathbb{R})$ be the homomorphism given by the projection map onto the first coordinate.

Let $\zeta = (\alpha, \phi(z)) \in \mathcal{S}$. Define the slash operator $\left[ \zeta \right]_{k+1/2}$ on functions $f$ on $\mathbb{H}$ by

$$f \left[ \zeta \right]_{k+1/2} = f(az)(\phi(z))^{-2k+1}.$$

Let $N$ be divisible by 4 and let $\alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N)$. Define the automorphy factor

$$j(\alpha, z) = \varepsilon_d^{-1} \left( \frac{\zeta}{d} \right) (cz + d)^{1/2},$$

where $\varepsilon_d = 1$ or $i$ according to whether $d \equiv 1$ or $3$ (mod 4) and $\left( \frac{\zeta}{d} \right)$ is as in Shimura’s notation. Let

$$\Delta_0(N) := \left\{ \alpha \ast = (\alpha, j(\alpha, z)) \in \mathcal{S} \mid \alpha \in \Gamma_0(N) \right\} \leq \mathcal{S}.$$

The map $L : \Gamma_0(N) \to \mathcal{S}$ given by $\alpha \mapsto \alpha \ast$ defines an isomorphism onto $\Delta_0(N)$. Thus, $P|_{\Delta_0(N)}$ and $L$ are inverse of each other. Denote by $\Delta_1(N)$ the image of $\Gamma_1(N)$.

Let $\chi$ be an even Dirichlet character modulo $N$. Let $S_{k+1/2}(\Gamma_0(N), \chi)$ be the space of cusp forms of weight $k+1/2$, level $N$, and character $\chi$ consisting of $f \in S_{k+1/2}(\Delta_1(N))$ such that $f \left[ \alpha \ast \right]_{k+1/2} = \chi(d) f(z)$ for all $\alpha \in \Gamma_0(N)$. In particular, when $\chi$ is trivial, $S_{k+1/2}(\Gamma_0(N), \chi) = S_{k+1/2}(\Delta_0(N))$. In this case we will simply denote the space by $S_{k+1/2}(\Gamma_0(N))$.

Let $\xi$ be an element of $\mathcal{S}$ such that $\Delta_0(N)$ and $\xi^{-1} \Delta_0(N) \xi$ are commensurable. Then we have an operator $\left[ \Delta_0(N) \xi \Delta_0(N) \right]_{k+1/2}$ on $S_{k+1/2}(\Gamma_0(N))$ defined by

$$f \left[ \Delta_0(N) \xi \Delta_0(N) \right]_{k+1/2} = \det(\xi)^{(2k-3)/4} \sum_{\nu} f \left[ \xi \nu \right]_{k+1/2},$$

where $\Delta_0(N) \xi \Delta_0(N) = \bigcup_{\nu} \Delta_0(N) \xi_{\nu}$.

Let $\xi = \left( \begin{smallmatrix} 1 & s \\ 0 & p^{-1} \end{smallmatrix} \right), \ p^{1/2}$. If $p$ is a prime dividing $N$, then by [14, Proposition 1.5],

$$f \left[ \Delta_0(N) \xi \Delta_0(N) \right]_{k+1/2} = p^{(2k-3)/2} \sum_{s=0}^{p-1} f \left[ \left( \begin{smallmatrix} 1 & s \\ 0 & p \end{smallmatrix} \right), p^{1/2} \right]_{k+1/2}(z),$$
thus if \( f = \sum_{n=1}^{\infty} a_n q^n \), then \( f \left[ [\Delta_0(N) \xi \Delta_0(N)]_{k+1/2} = \sum_{n=1}^{\infty} a_p p^n q^n = U_p^\xi(f) \). If \( p \) is a prime such that \( (p, N) = 1 \), then the Hecke operator \( T_{p^2} \) is defined by

\[
T_{p^2}(f) = f \left[ [\Delta_0(N) \xi \Delta_0(N)]_{k+1/2}.
\]

We will be studying local Hecke algebra of the double cover of \( SL_2 \). We next recall Gelbart’s \([4]\) description of the double cover. Let \( p \) be any prime (including the infinite prime). The group \( SL_2(\mathbb{Q}_p) \) has a non-trivial central extension by \( \mu_2 = \{ \pm 1 \} : \)

\[
1 \rightarrow \mu_2 \rightarrow \tilde{SL}_2(\mathbb{Q}_p) \rightarrow SL_2(\mathbb{Q}_p) \rightarrow 1
\]

\[
\{ (1, \pm 1) \} \quad (g, \pm 1) \quad \rightarrow \quad g
\]

We use the 2-cocycle defined below to determine the double cover \( \tilde{SL}_2(\mathbb{Q}_p) \). Let \((\cdot, \cdot)_p \) be the Hilbert symbol over \( \mathbb{Q}_p \). For \( g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Q}_p) \), define

\[
\tau(g) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0. \end{cases}
\]

If \( p = \infty \), set \( s_p(g) = 1 \), while for a finite prime \( p \)

\[
s_p(g) = \begin{cases} (c, d)_p & \text{if } cd \neq 0 \text{ and } \text{ord}_p(c) \text{ is odd}, \\ 1 & \text{otherwise}. \end{cases}
\]

Define the 2-cocycle \( \sigma_p \) on \( SL_2(\mathbb{Q}_p) \) as follows:

\[
\sigma_p(g, h) = (\tau(gh) \tau(g), \tau(gh) \tau(h))_p s_p(g) s_p(h) s_p(gh).
\]

Then the double cover \( \tilde{SL}_2(\mathbb{Q}_p) \) is the set \( SL_2(\mathbb{Q}_p) \times \mu_2 \) with the group law:

\[
(g, \epsilon_1)(h, \epsilon_2) = (gh, \epsilon_1 \epsilon_2 \sigma_p(g, h)).
\]

For any subgroup \( H \) of \( SL_2(\mathbb{Q}_p) \), we will denote by \( \overline{H} \) the complete inverse image of \( H \) in \( \tilde{SL}_2(\mathbb{Q}_p) \).

We consider the following subgroups of \( SL_2(\mathbb{Z}_p) \):

\[
K^0_0(p^n) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}_p) : c \in p^n\mathbb{Z}_p \right\},
\]

\[
K^1_0(p^n) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}_p) : c \in p^n\mathbb{Z}_p, a \equiv 1 \pmod{p^n \mathbb{Z}_p} \right\}.
\]

By [4, Proposition 2.8] for odd primes \( p \), \( \tilde{SL}_2(\mathbb{Q}_p) \) splits over \( SL_2(\mathbb{Z}_p) \). Thus, \( SL_2(\mathbb{Z}_p) \) is isomorphic to the direct product \( SL_2(\mathbb{Z}_p) \times \mu_2 \) and \( K^0_0(p) \) is isomorphic to \( K^0_0(p) \times \mu_2 \). It follows from [4, Corollary 2.13] that the center \( M_p \) of \( \tilde{SL}_2(\mathbb{Q}_p) \) is simply the direct product \( \{ \pm 1 \} \times \mu_2 \). Thus, any genuine character is given by a non-trivial character of \( \mu_2 \times \mu_2 \).

However \( \tilde{SL}_2(\mathbb{Q}_2) \) does not split over \( SL_2(\mathbb{Z}_2) \) but instead splits over the subgroup \( K^2_0(4) \). In this case, the center \( M_2 \) of \( \tilde{SL}_2(\mathbb{Q}_2) \) is a cyclic group of order 4 generated by \( (-1, 1) \) and so a genuine central character is given by sending \( (-1, 1) \) to a primitive fourth root of unity.
We set up more notation. For \( s \in \mathbb{Q}_p, \ t \in \mathbb{Q}_p^\times \), let us define the following elements of \( \text{SL}_2(\mathbb{Q}_p) \):

\[
x(s) = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad y(s) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \quad w(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix}, \quad h(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.
\]

Let \( N = \{(x(s), e) : s \in \mathbb{Q}_p, \ e = \pm 1\} \), \( \overline{N} = \{(y(s), e) : s \in \mathbb{Q}_p, \ e = \pm 1\} \) and \( T = \{(h(t), e) : t \in \mathbb{Q}_p^\times, \ e = \pm 1\} \) be the subgroups of \( \overline{\text{SL}_2(\mathbb{Q}_p)} \). Then the normalizer \( N_{\text{SL}_2(\mathbb{Q}_p)}(T) \) of \( T \) in \( \overline{\text{SL}_2(\mathbb{Q}_p)} \) consists of elements \( (h(t), e), (w(t), e) \) for \( t \in \mathbb{Q}_p^\times \). We note the following useful relations: for \( s, t \in \mathbb{Q}_p^\times \) and \( u, \ v \in \mathbb{Q}_p \), we have

\[(2.1) \quad (h(s), 1)(h(t), 1) = (h(st), (s, t)_p), \]
\[(w(s), 1)(w(t), 1) = (h(-st^{-1}), (s, t)_p), \]
\[(h(s), 1)(w(t), 1) = (w(st), (s, -t)_p), \]
\[(w(s), 1)(h(t), 1) = (w(st^{-1}), (-s, t)_p), \]
\[(h(s), 1)(x(u), 1) = \begin{pmatrix} s & su \\ 0 & s^{-1} \end{pmatrix}, 1 \), \]
\[(x(u), 1)(h(s), 1) = \begin{pmatrix} s & s^{-1}u \\ 0 & s^{-1} \end{pmatrix}, 1 \), \]
\[(h(s), 1)(y(u), 1) = \begin{pmatrix} s & 0 \\ s^{-1}u & s^{-1} \end{pmatrix}, \sigma_p(h(s), y(u))) \), \]
\[(y(u), 1)(h(s), 1) = \begin{pmatrix} s & 0 \\ su & s^{-1} \end{pmatrix}, \sigma_p(y(u), h(s))) \), \]

where

\[
\sigma_p(h(s), y(u)) = \sigma_p(y(u), h(s)) = \begin{cases} 1 & \text{if } u = 0, \\ (s, u)_p & \text{if } u \neq 0, \text{ord}_p(su) \text{ even}, \\ (s, s)_p & \text{if } u \neq 0, \text{ord}_p(su) \text{ odd}, \end{cases}
\]

\[
(w(t), 1)(x(u), 1) = \begin{pmatrix} 0 & t \\ -t^{-1} & -t^{-1}u \end{pmatrix}, \sigma_p(w(t), x(u))) \), \]
\[(x(u), 1)(w(t), 1) = \begin{pmatrix} -ut^{-1} & t \\ -t^{-1} & 0 \end{pmatrix}, 1 \), \]
\[(w(t), 1)(y(v), 1) = \begin{pmatrix} tv & t \\ -t^{-1} & 0 \end{pmatrix}, 1 \), \]
\[(y(v), 1)(w(t), 1) = \begin{pmatrix} 0 & t \\ -t^{-1} & tv \end{pmatrix}, \sigma_p(y(v), w(t))) \), \]

where

\[
\sigma_p(w(t), x(u)) = \begin{cases} (-t, -u)_p & \text{if } u \neq 0, \text{ord}_p(t) \text{ odd}, \\ 1 & \text{otherwise}, \end{cases}
\]
and
\[
\sigma_p(y(v), w(t)) = \begin{cases} 
(-t, v)_p & \text{if } u \neq 0, \text{ord}_p(t) \text{ odd,} \\
1 & \text{otherwise,}
\end{cases}
\]
\[
(x(u), 1)(y(v), 1) = \left( \begin{array}{cc}
1 + uv & 0 \\
\frac{u}{v} & 1 
\end{array} \right),
\]
\[
(y(v), 1)(x(u), 1) = \left( \begin{array}{cc}
1 & u + v \\
\frac{u}{1} & 1 
\end{array} \right),
\]
\[
(x(u), 1)(x(v), 1) = \left( \begin{array}{cc}
1 & u \\
\frac{0}{1} & 1 
\end{array} \right),
\]
\[
(y(v), 1)(y(u), 1) = \left( \begin{array}{cc}
1 & 0 \\
\frac{u + v}{1} & 1 
\end{array} \right),
\]

where
\[
\sigma_p(y(v), x(u)) = \begin{cases} 
(v, uv + 1)_p & \text{if } v(uv + 1) \neq 0, \text{ord}_p(v) \text{ odd,} \\
1 & \text{otherwise.}
\end{cases}
\]

For any subgroup \(S\) of \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\), we further let \(N^S = N \cap S\), \(T^S = T \cap S\), and \(\overline{N}^S = \overline{N} \cap S\).

3 A Local Hecke Algebra of \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\)

Loke and Savin [8] studied a genuine local Hecke algebra of \(\tilde{\text{SL}}_2(\mathbb{Q}_2)\) corresponding to \(K^2_0(4)\) and a genuine central character, and gave an interpretation of Kohnen’s plus space at level 4 in terms of certain elements in this 2-adic Hecke algebra. In this section we recall their work on the 2-adic Hecke algebra. We then study genuine Iwahori Hecke algebra for \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\) corresponding to \(K^p_0(p)\) and a genuine character of \(M_p\) for a general odd prime \(p\).

Let \(p\) be any finite prime and let \(C^\infty_c(\tilde{\text{SL}}_2(\mathbb{Q}_p))\) be the space of locally constant, compactly supported complex-valued functions on \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\). For an open compact subgroup \(S\) of \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\) and a genuine character \(\gamma\) of \(S\) (that is, a character of \(S\) that acts nontrivially on \(\mu_2\)), let \(H(S, \gamma)\) be the subalgebra of \(C^\infty_c(\tilde{\text{SL}}_2(\mathbb{Q}_p))\) defined as follows:

\[
\{ f \in C^\infty_c(\tilde{\text{SL}}_2(\mathbb{Q}_p)) : f(\tilde{k}g\tilde{k}') = \bar{\gamma}(\tilde{k})\overline{\gamma}(\tilde{k}')f(\tilde{g}) \text{ for } \tilde{g} \in \tilde{\text{SL}}_2(\mathbb{Q}_p), \tilde{k}, \tilde{k}' \in S \}.
\]

Then \(H(S, \gamma)\) is a \(C\)-algebra under the convolution, which, for any \(f_1, f_2 \in H(S, \gamma)\), is defined by
\[
f_1 * f_2(\tilde{h}) = \int_{\tilde{\text{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{g}) f_2(\tilde{g}^{-1}\tilde{h})d\tilde{g} = \int_{\tilde{\text{SL}}_2(\mathbb{Q}_p)} f_1(\tilde{h}\tilde{g}) f_2(\tilde{g}^{-1})d\tilde{g},
\]
where \(d\tilde{g}\) is the Haar measure on \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\) such that the measure of \(S\) is one. We call \(H(S, \gamma)\) the genuine Hecke algebra of \(\tilde{\text{SL}}_2(\mathbb{Q}_p)\) with respect to \(S\) and \(\gamma\). We can sometimes denote \(f_1 * f_2\) simply by \(f_1 f_2\).
For certain $S$ and $\gamma$, we would like to describe the algebra $H(S, \gamma)$ using generators and relations. In order to do so, we need to first compute the support of $H(S, \gamma)$. We say that $H(S, \gamma)$ is supported on $\mathcal{g} \in \mathcal{S}_{\mathcal{L}}(\mathbb{Q}_p)$ if there exists $f \in H(S, \gamma)$ such that $f(\mathcal{g}) \neq 0$. We use the following lemmas to compute the support.

**Lemma 3.1** Let $S_\mathcal{g} = S \cap \mathcal{g}S\mathcal{g}^{-1}$. Then $H(S, \gamma)$ is supported on $\mathcal{g}$ if and only if for every $\mathcal{k} \in S_\mathcal{g}$ we have $\gamma([\mathcal{k}^{-1}, \mathcal{g}^{-1}]) = 1$, where $[\mathcal{\ldots}, \mathcal{\ldots}]$ is the usual commutator bracket.

**Lemma 3.2** The function $\alpha_\mathcal{g}: S_\mathcal{g} \to \mathbb{C}$ defined by $\alpha_\mathcal{g}(\mathcal{k}) = \gamma([\mathcal{k}^{-1}, \mathcal{g}^{-1}])$ is a character of $S_\mathcal{g}$.

In order to compute the support using above lemmas, we need certain results on cocycle multiplication. We note them in the appendix.

We also note that the following well-known lemmas will be useful in computing convolutions.

**Lemma 3.3** Let $f_1, f_2 \in H(S, \gamma)$ such that $f_1$ is supported on $S\mathcal{g}S = \bigcup_{i=1}^m \mathcal{a}_iS$ and $f_2$ is supported on $S\mathcal{g}S = \bigcup_{j=1}^n \mathcal{b}_jS$. Then

$$f_1 \ast f_2(\mathcal{h}) = \sum_{i=1}^m f_1(\mathcal{a}_i) f_2(\mathcal{a}_i^{-1}\mathcal{h}),$$

where the nonzero summands are precisely for those $i$ for which there exist a $j$ such that $\mathcal{h} \in \mathcal{a}_i\mathcal{b}_jS$.

For $\mathcal{g} \in \mathcal{S}_{\mathcal{L}}(\mathbb{Q}_p)$ let $\mu(\mathcal{g})$ denote the number of disjoint left (right) $S$ cosets in the decomposition of the double coset $\mathcal{g}S\mathcal{g}$.

**Lemma 3.4** Let $\mathcal{g}, \mathcal{h} \in \mathcal{S}_{\mathcal{L}}(\mathbb{Q}_p)$ be such that $\mu(\mathcal{g})\mu(\mathcal{h}) = \mu(\mathcal{gh})$. Let $f_1, f_2 \in H(S, \gamma)$ be supported on $S\mathcal{g}S$ and $S\mathcal{h}S$, respectively. Then $f_1 \ast f_2$ is precisely supported on $S\mathcal{g}\mathcal{h}S$ and $f_1 \ast f_2(\mathcal{g}\mathcal{h}) = f_1(\mathcal{g})f_2(\mathcal{h})$.

### 3.1 Local Hecke Algebra of $\mathcal{S}_{\mathcal{L}}(\mathbb{Q}_2)$ Modulo $\mathcal{K}_0(4)$

Let $S = \mathcal{K}_0(4)$ and let $\gamma$ be a genuine character of $M_2$ determined by its value on $(-I, 1)$. Since $\mathcal{K}_0(4)$ is the direct product $\mathcal{K}_0(4) \times M$, we can extend $\gamma$ to a genuine character of $\mathcal{K}_0(4)$ by setting it trivial on $\mathcal{K}_0(4)$. Loke and Savin described $H(S, \gamma)$ for the above choice of $S$ and $\gamma$ as follows.

Using relations in (2.1), extend $\gamma$ to the normalizer $N_{\mathcal{S}_{\mathcal{L}}(\mathbb{Q}_2)}(T)$ by defining $\gamma((h(2^n), 1)) = 1$ for all integers $n$ and $\gamma((w(1), 1)) = 1 + \gamma((-I, 1))/\sqrt{2}$, a primitive 8th root of unity. For $n \in \mathbb{Z}$, define the elements $\mathcal{J}_n$ and $\mathcal{U}_n$ of $H(\mathcal{K}_0(4), \gamma)$ supported respectively on the $\mathcal{K}_0^2(4)$ double cosets of $(h(2^n), 1)$ and $(w(2^{-n}), 1)$ such that

$$\mathcal{J}_n(\mathcal{k}(h(2^n), 1)\mathcal{k}') = \bar{\gamma}(\mathcal{k})\bar{\gamma}((h(2^n), 1))\bar{\gamma}(\mathcal{k}'),$$

$$\mathcal{U}_n(\mathcal{k}(w(2^{-n}), 1)\mathcal{k}') = \bar{\gamma}(\mathcal{k})\bar{\gamma}((w(2^{-n}), 1))\bar{\gamma}(\mathcal{k}'),$$

for $\mathcal{k}, \mathcal{k}' \in \mathcal{K}_0^2(4)$. 
**Theorem 3.5** (Loke–Savin [8]) For \( m, n \in \mathbb{Z} \),

(i) if \( mn \geq 0 \), then \( T_m \ast T_n = T_{m+n} \);
(ii) \( U_1 \ast T_n = U_{n+1} \) and \( T_n \ast U_1 = U_{1-n} \);
(iii) \( U_1 \ast U_n = T_{n-1} \) and \( U_n \ast U_1 = T_{1-n} \).

The Hecke algebra \( H(K_0^2(A, \gamma)) \) is generated by \( U_0 \) and \( U_1 \) modulo relations \( (U_0 - 2\sqrt{2})(U_0 + \sqrt{2}) = 0 \) and \( U_1^2 = 1 \).

### 3.2 Iwahori Hecke Algebra of \( \widetilde{SL}_2(\mathbb{Q}_p) \) Modulo \( K_0^p(p) \), \( p \) Odd

Fix an odd prime \( p \). Let \( S = \overline{K_0^p(p)} \). Let \( y \) be a character of \( K_0^p(p) \) such that it is trivial on \( K_1^p(p) \). Since \( \frac{K_1^p(p)}{K_1^p(p)} \cong (\mathbb{Z}/p\mathbb{Z})^\times \), we can define \( y \) by a character of \( (\mathbb{Z}/p\mathbb{Z})^\times \). We use the same symbol \( y \) to denote a genuine character of \( S \) by defining \( y(A, e) = ey(A) \) for \( A \in K_0^p(p) \). We call \( H(S, y) \) with the above choice of \( S \) and \( y \) to be the genuine Iwahori Hecke algebra of \( \widetilde{SL}_2(\mathbb{Q}_p) \) with central character \( y \). Our main result in this subsection is to describe this Iwahori Hecke algebra using generators and relations when \( y \) is quadratic.

In the rest of this subsection, we denote \( K_0^p(p) \) simply by \( K_0 \). We first note the following lemma.

**Lemma 3.6** A complete set of representatives for the double cosets of \( \widetilde{SL}_2(\mathbb{Q}_p) \) mod \( \overline{K_0} \) is given by \( (h(p^n), 1), (w(p^{-n}), 1) \), where \( n \) varies over integers.

We need to compute the support of \( H(\overline{K_0}, y) \). Fix an integer \( n \). Let \( A = h(p^n) \) and \( \widetilde{A} = (A, e_1) \). We shall show that \( H(\overline{K_0}, y) \) is supported on \( \widetilde{A} \). We have

\[
S_{\widetilde{A}} = \left\{ \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm 1 \right) \in \widetilde{SL}_2(\mathbb{Z}_p) : \text{ord}_p(c) \geq \text{max}\{-2n+1, 1\}, \text{ord}_p(b) \geq \text{max}\{2n, 0\} \right\}.
\]

We check that \( S_{\widetilde{A}} \) has a triangular decomposition \( S_{\widetilde{A}} = N^{S_{\widetilde{A}}} T^{S_{\widetilde{A}}} \overline{N}^{S_{\widetilde{A}}} \), where \( T^{S_{\widetilde{A}}} = T^{\overline{K_0}} \), \( N^{S_{\widetilde{A}}} = \{(x(s), \pm 1) : \text{ord}_p(s) \geq \text{max}\{2n, 0\}\} \), and \( \overline{N}^{S_{\widetilde{A}}} = \{(y(t), \pm 1) : \text{ord}_p(t) \geq \text{max}\{-2n+1, 1\}\} \).

By Lemma 3.1 and 3.2, it is enough to check that the value of \( y \) on the commutator \([B, e_2]^{-1}, (A, e_1)^{-1}\) is 1 for any \((B, e_2) \in N^{S_{\widetilde{A}}}\), \( T^{S_{\widetilde{A}}} \), \( \overline{N}^{S_{\widetilde{A}}} \), respectively.

By Lemma A.3, for \( B = (x(s), e_2) \in N^{S_{\widetilde{A}}} \), we get

\[
[(B, e_2)^{-1}, (A, e_1)^{-1}] = \left( \begin{pmatrix} 1 & sp^{-2n}-s \\ 0 & 1 \end{pmatrix}, 1 \right);
\]

for \( B = (h(u), e_2) \in T^{S_{\widetilde{A}}} \), we get \([(B, e_2)^{-1}, (A, e_1)^{-1}] = (I, 1) \); and for \( B = (y(t), e_2) \in N^{S_{\widetilde{A}}} \), we get that

\[
[(B, e_2)^{-1}, (A, e_1)^{-1}] = \left( \begin{pmatrix} 1 & 0 \\ (p^{2n}-1)t & 1 \end{pmatrix}, 1 \right).
\]

Since each of them belongs to \( K_1^p(p) \times \{1\} \), we are done.
Next let $A = w(p^{-n})$. We show that $H(\overline{K_0}, \gamma)$ is supported on $\overline{A} = (A, e_1)$ provided $\gamma(u^2) = 1$ for all units $u$ in $\mathbb{Z}_p$. In this case, we have

$$S_{\overline{A}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \pm 1 \right\} \in \text{SL}_2(\mathbb{Z}_p) : \text{ord}_p(c) \geq \max\{2n, 1\}, \text{ord}_p(b) \geq \max\{-2n + 1, 0\} \right\},$$

and $S_{\overline{A}}$ has a triangular decomposition $S_{\overline{A}} = N^{S_{\overline{A}}} T^{S_{\overline{A}}} N^{S_{\overline{A}}}$, where $T^{S_{\overline{A}}} = T^{K_0}$, $N^{S_{\overline{A}}} = \{(x(s), \pm 1) : \text{ord}_p(s) \geq \max\{-2n + 1, 0\}\}$, $N^{S_{\overline{A}}} = \{(y(t), \pm 1) : \text{ord}_p(t) \geq \max\{2n, 1\}\}$. By Lemma A.3, for $B = (x(s), e_2) \in N^{S_{\overline{A}}}$, we get

$$[(B, e_2)^{-1}, (A, e_1)^{-1}] = \left( \begin{pmatrix} 1 + s^2 p^{2n} & -s \\ -sp^{2n} & 1 \end{pmatrix}, 1 \right),$$

so $\gamma$ takes value 1 on this commutator. In the case $B = (y(t), e_2) \in N^{S_{\overline{A}}}$, we have

$$B^{-1} A^{-1} BA = \begin{pmatrix} 1 & -p^{-2n}t \\ -t & 1 + p^{-2n}t^2 \end{pmatrix},$$

so $s_p(B^{-1} A^{-1} BA) = 1$ if either $-t(1 + p^{-2n}t^2) = 0$ or $\text{ord}_p(t)$ is even. Assume that $-t(1 + p^{-2n}t^2) \neq 0$ and $\text{ord}_p(t)$ is odd. Then $s_p(B^{-1} A^{-1} BA) = (-t, 1 + p^{-2n}t^2)_p = (-p, 1 + p^{-2n}t^2)_p$. Let $u = 1 + p^{-2n}t^2$. Since $\text{ord}_p(t) \geq \max\{2n, 1\}$, we have $u \equiv 1 \pmod{p\mathbb{Z}_p}$. Hence, $s_p(B^{-1} A^{-1} BA) = (-p, u)_p = \left( \frac{u}{p} \right) = 1$. So in this case also $\gamma$ takes value 1.

For $B = (h(u), e_2) \in T^{S_{\overline{A}}}$,

$$[(B, e_2)^{-1}, (A, e_1)^{-1}] = \left( \begin{pmatrix} 1/u^2 & 0 \\ 0 & u^2 \end{pmatrix}, 1 \right),$$

so $\gamma([(B, e_2)^{-1}, (A, e_1)^{-1}]) = \gamma(u^2)$.

Thus, if $\gamma(u^2) = 1$ for all units $u$ in $\mathbb{Z}_p$, then $H(\overline{K_0}, \gamma)$ is supported on $(w(p^{-n}), e)$. In particular, this holds if our choice of $\gamma$ is quadratic. Thus, we have the following proposition.

**Proposition 3.7** If $\gamma$ is a quadratic character then $H(\overline{K_0}, \gamma)$ is supported on the double cosets of $\overline{K_0}$ represented by $(h(p^n), 1)$ and $(w(p^{-n}), 1)$, as $n$ varies over integers.

We now obtain the generators and relations in $H(\overline{K_0}, \gamma)$ when $\gamma$ is quadratic.

We consider the character $\gamma$ of $\overline{K_0}$ to be the genuine character of the center $M_p$ and extend it to the normalizer group $N_{\text{SL}_2(\mathbb{Q}_p)}(T)$ as follows.

Let $\varepsilon_p = 1$ or $i$ depending on whether $p \equiv 1$ or 3 (mod 4), thus $\varepsilon_p^2 = \left( \frac{-1}{p} \right)$. Let $t = p^n u \in \mathbb{Q}_p^\times$, where $n \in \mathbb{Z}$ and $u$ is a unit in $\mathbb{Z}_p$. Define

$$\gamma((h(t), 1)) = \begin{cases} \gamma((h(u), 1)) & \text{if } n \text{ is even}, \\ \varepsilon_p \left( \frac{u}{p} \right) \gamma((h(u), 1)) & \text{if } n \text{ is odd}. \end{cases}$$

It is easy to see that $\gamma$ thus defined is a character of $T$. 
We now extend $\gamma$ to the normalizer $N_{\text{GL}_2(\mathbb{Q}_p)}(T)$ by defining $\gamma((w(1), 1)) := 1$ and using the relation

$$(w(t), 1) = (h(t), 1)(w(1), 1)(I, (-1, t^{-1})_p).$$

Thus, for $t = p^n u$ as above,

$$\gamma((w(t), 1)) = \begin{cases} 
\gamma((h(u), 1)) & \text{if } n \text{ is even}, \\
\varepsilon_p(-u/p) \gamma((h(u), 1)) & \text{if } n \text{ is odd}.
\end{cases}$$

We define the elements $T_n$ and $U_n$ of $H(K_0, \gamma)$ supported respectively on the double cosets of $(h(p^n), 1)$ and $(w(p^{-n}), 1)$ such that

$$T_n(\kappa(h(p^n), 1)\kappa') = \overline{\gamma(\kappa)}\overline{\gamma((h(p^n), 1))\gamma(\kappa')}),$$

$$U_n(\kappa(w(p^{-n}), 1)\kappa') = \overline{\gamma(\kappa)}\overline{\gamma((w(p^{-n}), 1))\gamma(\kappa')}$$

for $\kappa, \kappa' \in K_0$. Thus, Proposition 3.7 implies that $T_n, U_n$, as $n$ varies over integers form a $\mathbb{C}$-basis for $H(K_0, \gamma)$ when $\gamma$ is quadratic.

In order to obtain relations amongst $T_n$ and $U_n$, we note the following lemma, which can be obtained by using the triangular decomposition of $K_0$.

**Lemma 3.8**

(i) For $n \geq 0$,

$$K_0 h(p^n)K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} x(s) h(p^n)K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} K_0 h(p^n) y(ps).$$

(ii) For $n \geq 1$,

$$K_0 h(p^{-n})K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} y(ps) h(p^{-n})K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} K_0 h(p^{-n}) x(s).$$

(iii) For $n \geq 1$,

$$K_0 w(p^{-n})K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} y(ps) w(p^{-n})K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} K_0 w(p^{-n}) y(ps).$$

(iv) For $n \geq 0$,

$$K_0 w(p^n)K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} x(s) w(p^n)K_0 = \bigcup_{s \in \mathbb{Z}_p/p^n\mathbb{Z}_p} K_0 w(p^n) x(s).$$

**Proposition 3.9**

We have the following relations:

(i) If $mn \geq 0$, then $T_m * T_n = T_{m+n}$.

(ii) For $n \geq 0$, $U_1 * T_n = U_{n+1}$ and $T_n * U_1 = U_{n+1}$.

(iii) For $n \geq 0$, $U_0 * T_n = U_{n}$ and $T_n * U_0 = U_{n-1}$.

(iv) For $n \geq 1$, $U_0 * U_n = \overline{\gamma((-1, 1))} T_n$ and $U_n * U_0 = \overline{\gamma((-1, 1))} T_{n-1}$.

**Proof**

We prove (i) and the second part of (iv). The rest are similar.

For (i), let $mn \geq 0$. We can assume both $m, n \geq 0$. It follows from Lemma 3.8 and 3.4 that $T_m * T_n$ is precisely supported on the double coset $K_0(h(p^{a+m}), 1)K_0$ and that

$$T_m * T_n((h(p^m), 1)(h(p^n), 1)) = T_m((h(p^m), 1)) T_n((h(p^n), 1)).$$
Let \( m \) and \( n \) both be even. Then \((h(p^m),1)(h(p^n),1) = (h(p^{m+n}),1)\) and so
\[
\mathcal{T}_m \ast \mathcal{T}_n((h(p^{m+n}),1)) = \mathcal{T}_m((h(p^m),1)) \mathcal{T}_n((h(p^n),1))
\[
= \overline{\gamma}(h(p^m),1) \overline{\gamma}(h(p^n),1)
\[
= 1 = \mathcal{T}_{m+n}((h(p^{m+n}),1)),
\]
hence \( \mathcal{T}_m \ast \mathcal{T}_n = \mathcal{T}_{m+n} \). Next suppose both \( m \) and \( n \) are odd, so \( m + n \) is even. Then
\((h(p^m),1)(h(p^n),1) = (h(p^{m+n}),1)(I, (p, p)_p)\) and so
\[
\mathcal{T}_m \ast \mathcal{T}_n((h(p^{m+n}),1)) = \overline{\gamma}((I, (p, p)_p)) \mathcal{T}_m((h(p^m),1)) \mathcal{T}_n((h(p^n),1))
\[
= \left( -\frac{1}{p} \right) \overline{\gamma}(h(p^m),1) \overline{\gamma}(h(p^n),1) = \left( -\frac{1}{p} \right) \overline{\varepsilon_p}
\[
= 1 = \mathcal{T}_{m+n}((h(p^{m+n}),1)).
\]
Now suppose \( m \) is odd and \( n \) is even (or vice versa), so \( m + n \) is odd. In this case,
\((h(p^m),1)(h(p^n),1) = (h(p^{m+n}),1)\) and so
\[
\mathcal{T}_m \ast \mathcal{T}_n((h(p^{m+n}),1)) = \overline{\varepsilon_p} = \mathcal{T}_{m+n}((h(p^{m+n}),1)),
\]
and we are done.

For (iv), let \( n \geq 1 \). As before, using Lemma 3.8 and 3.4, we know that \( \mathcal{U}_n \ast \mathcal{U}_0 \) is supported on the double coset \( K_0(h(p^{-n}),1)K_0 \) and that
\[
\mathcal{U}_n \ast \mathcal{U}_0((w(p^{-n}),1)(w(1),1)) = \mathcal{U}_n((w(p^{-n}),1)) \mathcal{U}_0((w(1),1)).
\]
We have \((w(p^{-n}),1)(w(1),1) = (h(p^{-n}),1)(-I, (p^{-n}, -1)_p)\), and so
\[
\overline{\gamma}((-I,1)) \mathcal{U}_n \ast \mathcal{U}_0((h(p^{-n}),1))
\[
= (p^{-n}, -1)_p \mathcal{U}_n((w(p^{-n}),1)) \mathcal{U}_0((w(1),1))
\[
= \begin{cases} 
\left( -\frac{1}{p} \right) \overline{\varepsilon_p}(\frac{1}{p}) = \overline{\varepsilon_p} & \text{if } n \text{ is odd,} \\
1 & \text{if } n \text{ is even}
\end{cases}
\[
= \mathcal{T}_{-n}((h(p^{-n}),1)),
\]
and thus \( \mathcal{U}_n \ast \mathcal{U}_0 = \overline{\gamma}((-I,1))\mathcal{T}_{-n}. \)

We consider two choices for \( \gamma \) as a character of \((\mathbb{Z}/p\mathbb{Z})^*\): either \( \gamma \) is trivial or \( \gamma \) is given by the Kronecker symbol \( \gamma = \left( \frac{-1}{p} \right) \). Then we have the following proposition.

**Proposition 3.10**  
(i) \( \mathcal{U}_0^2 = \begin{cases} 
(p-1)\mathcal{U}_0 + p & \text{if } \gamma \text{ is trivial,} \\
\left( -\frac{1}{p} \right) p & \text{if } \gamma = \left( \frac{-1}{p} \right).
\end{cases} \)

(ii) \( \mathcal{U}_1^2 = \begin{cases} 
p & \text{if } \gamma \text{ is trivial,} \\
\varepsilon_p(p-1)\mathcal{U}_1 + \left( -\frac{1}{p} \right) p & \text{if } \gamma = \left( \frac{-1}{p} \right).
\end{cases} \)

(iii) If \( \gamma \) is trivial, then \( \mathcal{T}_1 \ast \mathcal{U}_1 = p \mathcal{U}_0 \) and \( \mathcal{T}_{-1} = \left( \frac{1}{p} \right) \mathcal{U}_1 \ast \mathcal{T}_1 \ast \mathcal{U}_1. \)

**Proof**  
For (i), we use Lemma 3.3 to check that \( \mathcal{U}_0^2 \) is at most supported on the double cosets \( K_0 \) and \( K_0(w(1),1)K_0 \). Thus, we need to only compute the values of \( \mathcal{U}_0^2 \) at \((I,1)\).
and \((w(1), 1)\). Using Lemma 3.8 and 3.3, we have

\[
\begin{align*}
\mathcal{U}_0^2((I, 1)) &= \sum_{s=0}^{p-1} \mathcal{U}_0((x(s), 1)(w(1), 1)) \mathcal{U}_0((w(1), 1)^{-1}(x(s), 1)^{-1}) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_0((w(1), 1)) \mathcal{U}_0((w(-1), 1)(x(-s), 1)) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_0((h(-1), 1)(w(1), 1)(x(-s), 1)) \\
&= \sum_{s=0}^{p-1} \bar{y}(-1) = \begin{cases} 
p & \text{if } y \text{ is trivial,} \\
\left(\frac{-1}{p}\right) p & \text{if } y = \left(\frac{-1}{p}\right),
\end{cases}
\end{align*}
\]

where the third equality follows from the relation \((h(-1), 1)(w(1), 1) = (w(-1), (-1, 1)_p)\) by equation (2.1).

Similarly, we get that

\[
\begin{align*}
\mathcal{U}_0^2((w(1), 1)) &= \sum_{s=0}^{p-1} \mathcal{U}_0((x(s), 1)(w(1), 1)) \mathcal{U}_0((w(1), 1)^{-1}(x(s), 1)^{-1}(w(1), 1)) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_0\left(\begin{pmatrix} 0 & -1 \\ 1 & -s \end{pmatrix}, 1\right)(w(1), 1) \\
&= \sum_{s=0}^{p-1} \mathcal{U}_0((y(s), 1)) = \sum_{s=0}^{p-1} \mathcal{U}_0((y(s), 1)),
\end{align*}
\]

since \(\mathcal{U}_0((I, 1)) = 0\) (as \((I, 1)\) is not in the support of \(\mathcal{U}_0\)). It is easy to check that for \(1 \leq s \leq p - 1\),

\[
(y(s), 1) = \begin{pmatrix} 1 & 1/s \\ 0 & 1 \end{pmatrix}(w(1), 1) \begin{pmatrix} -s & -1 \\ 0 & -1/s \end{pmatrix}, 1) \in \overline{K_0}(w(1), 1)\overline{K_0},
\]

and hence

\[
\begin{align*}
\mathcal{U}_0^2((w(1), 1)) &= \sum_{s=1}^{p-1} \bar{y}(-1/s) = \sum_{s=1}^{p-1} y(s) = \begin{cases} 
p - 1 & \text{if } y \text{ is trivial,} \\
\sum_{s=1}^{p-1} \left(\frac{s}{p}\right) = 0 & \text{if } y = \left(\frac{-1}{p}\right).
\end{cases}
\end{align*}
\]

Thus, if we write \(\mathcal{U}_0^2 = c_1 \mathcal{U}_0 + c_2\), we get that

\[
c_1 = \begin{cases} 
p - 1 & \text{if } y \text{ is trivial,} \\
0 & \text{if } y = \left(\frac{-1}{p}\right),
\end{cases} \quad c_2 = \begin{cases} 
p & \text{if } y \text{ is trivial,} \\
\left(\frac{-1}{p}\right) p & \text{if } y = \left(\frac{-1}{p}\right).
\end{cases}
\]

Now we prove (ii). Again using Lemma 3.3, we see that \(\mathcal{U}_1^2\) is at most supported on the double cosets \(\overline{K_0}\) and \(\overline{K_0}(w(p^{-1}), 1)\overline{K_0}\). So we need to find the values of \(\mathcal{U}_1^2\)
at \((I, 1)\) and \((w(p^{-1}), 1)\). Using Lemma 3.8 and 3.3,

\[
\mathcal{U}_1^2((I, 1)) = \sum_{s=0}^{p-1} \mathcal{U}_1((y(ps), 1)(w(p^{-1}), 1)) \mathcal{U}_1((w(p^{-1}), 1)^{-1}(y(ps), 1)^{-1})
\]

\[
= \sum_{s=0}^{p-1} \mathcal{U}_1((w(p^{-1}), 1)) \mathcal{U}_1((w(-p^{-1}), 1)(y(-ps), 1))
\]

\[
= \sum_{s=0}^{p-1} \varepsilon_p\left(\frac{-1}{p}\right) \mathcal{U}_1((h(-1), (-p, -1)_{p})(w(p^{-1}), 1))
\]

\[
= \sum_{s=0}^{p-1} \varepsilon_p\left(\frac{-1}{p}\right) y(-1)\left(\frac{-1}{p}\right) \varepsilon_p\left(\frac{-1}{p}\right)
\]

\[
= y(-1)p = \begin{cases} p & \text{if } y \text{ is trivial,} \\ \left(\frac{-1}{p}\right)p & \text{if } y = \left(\frac{-1}{p}\right). \end{cases}
\]

Finally, we have

\[
\mathcal{U}_1^2((w(p^{-1}), 1))
\]

\[
= \sum_{s=0}^{p-1} \mathcal{U}_1((y(ps), 1)(w(p^{-1}), 1)) \mathcal{U}_1((w(-p^{-1}), 1)(y(-ps), 1)(w(p^{-1}), 1))
\]

\[
= \sum_{s=0}^{p-1} \varepsilon_p\left(\frac{-1}{p}\right) \mathcal{U}_1(\left(\left(\frac{s}{p} \quad -p^{-1} \right), (p^2, -p^2s)_p\right)(w(p^{-1}), 1))
\]

\[
= \sum_{s=0}^{p-1} \varepsilon_p\left(\frac{-1}{p}\right) \mathcal{U}_1((x(s/p), (p, -p)_p)) = \sum_{s=1}^{p-1} \varepsilon_p\left(\frac{-1}{p}\right) \mathcal{U}_1((x(s/p), 1))
\]

Now we check that for \(1 \leq s \leq p - 1\),

\[
(x(s/p), 1)(I, \left(\frac{s}{p}\right)) = \left(\begin{array}{c} s \\ \frac{s}{1/s} \\ 1 \end{array}\right) (w(p^{-1}), 1) \left(\begin{array}{c} 1 \\ \frac{p}{s} \\ 0 \end{array}\right),
\]

and so

\[
\mathcal{U}_1^2((w(p^{-1}), 1)) = \sum_{s=1}^{p-1} \varepsilon_p\left(\frac{-1}{p}\right) \left(\frac{s}{p}\right) \varepsilon_p(1/s) \varepsilon_p\left(\frac{-1}{p}\right)
\]

\[
= \sum_{s=1}^{p-1} \left(\frac{-s}{p}\right) \varepsilon(1/s)
\]

\[
= \begin{cases} \sum_{s=1}^{p-1} \left(\frac{-s}{p}\right) = 0 & \text{if } y \text{ is trivial,} \\ \sum_{s=1}^{p-1} \left(\frac{-s}{p}\right) \left(\frac{-1}{p}\right) = \left(\frac{-1}{p}\right)(p - 1) & \text{if } y = \left(\frac{-1}{p}\right). \end{cases}
\]

Thus, if we write \(\mathcal{U}_1^2 = c_1 \mathcal{U}_1 + c_2\), we get that

\[
c_1 = \begin{cases} 0 & \text{if } y \text{ is trivial,} \\ \varepsilon_p(p - 1) & \text{if } y = \left(\frac{-1}{p}\right), \end{cases} \quad c_2 = \begin{cases} p & \text{if } y \text{ is trivial,} \\ \left(\frac{-1}{p}\right)p & \text{if } y = \left(\frac{-1}{p}\right). \end{cases}
\]
For (iii) let $\gamma$ be a trivial character. From Proposition 3.9(iv), we have $\mathcal{U}_0 \ast \mathcal{U}_1 = \mathcal{T}_1$. Right multiplication by $\mathcal{U}_1$ on both sides and using (ii) above give $\mathcal{T}_1 \ast \mathcal{U}_1 = p \mathcal{U}_0$. Further, using the same proposition, we get that $\mathcal{T}_{-1} = \mathcal{U}_1 \ast \mathcal{U}_0 = (1/p) \mathcal{U}_1 \ast \mathcal{T}_1 \ast \mathcal{U}_1$. ■

**Remark 3.11** We compare the $p$-adic operator $\mathcal{U}_1$ with Ueda’s classical operator $Y_p$ [16, Proposition 1.27], which satisfies a similar relation. In particular, if we consider operator $\mathcal{U}'_1 = \overline{e}_p \mathcal{U}_1$, then in the case $\gamma$ is trivial, we have

$$(\mathcal{U}'_1)^2 = (\overline{e}_p \mathcal{U}_1)^2 = \varepsilon_p^2 p = \left( \frac{-1}{p} \right) p,$$

while in the case $\gamma = \left( \frac{\omega}{p} \right)$, we have

$$(\mathcal{U}'_1)^2 = (\overline{e}_p \mathcal{U}_1)^2 = \overline{e}_p^2 \left( \varepsilon_p (p - 1) \mathcal{U}_1 + p \left( \frac{-1}{p} \right) \right) = (p - 1) \mathcal{U}'_1 + p.$$ 

Thus, $\mathcal{U}'_1$ satisfies exactly the same relations as the operator $Y_p$.

**Theorem 3.12** The “genuine” Iwahori Hecke algebra $H(K^p_0(p), \gamma)$ for $\gamma$ trivial or $\left( \frac{\omega}{p} \right)$ is generated as a $\mathbb{C}$-algebra by $\mathcal{U}_0$ and $\mathcal{U}_1$ with the defining relations given by the above proposition.

**Proof** We let $A$ be an abstract algebra generated by $\tilde{\mathcal{U}}_0$ and $\tilde{\mathcal{U}}_1$ with defining relations as (i) and (ii) of Proposition 3.10. We have a homomorphism from $A$ to $H(\gamma)$ mapping $\tilde{\mathcal{U}}_0$ to $\mathcal{U}_0$ and $\tilde{\mathcal{U}}_1$ to $\mathcal{U}_1$. It follows from Proposition 3.9 that this homomorphism is onto. We let $M$ be the kernel of this homomorphism. Using relations (i) and (ii), it follows that $M$ is a linear combination of words of the form $\tilde{\mathcal{U}}_0 \tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_0 \cdots$ and $\tilde{\mathcal{U}}_1 \tilde{\mathcal{U}}_1 \cdot \tilde{\mathcal{U}}_1 \cdots$. There are four possibilities for the beginning and ending of such a word and each one is mapped by the homomorphism to a different basis element (again using Proposition 3.9). It follows that $M = 0$. ■

**Remark 3.13** We note that the Hecke algebras $H(K^p_0(p), \gamma)$ for $\gamma$ trivial or $\left( \frac{\omega}{p} \right)$ are isomorphic (with roles of $\tilde{\mathcal{U}}_0$, $\tilde{\mathcal{U}}_1$ switched after suitable normalization). Further, these are isomorphic to the Iwahori Hecke algebra of $\text{PGL}_2(\mathbb{Q}_p)$ giving what Loke and Savin called, local Shimura correspondence at odd primes.

The Hecke algebra generators and relations described above allow a study of the representation theory of the maximal compact with $(K^p_0(p), \gamma)$ equivariant vectors and also the infinite dimensional genuine representations of $\tilde{\mathbb{SL}}(2)$ with such vectors. We will pursue this study in a subsequent work.

4 Translation of Adelic to Classical

In this section, following Gelbart [4] and Waldspurger [18], we review the connection between automorphic forms on $\tilde{\mathbb{SL}}_2(\mathbb{A})$ and classical modular forms of half-integral
weight. We use this connection to translate certain elements in the $p$-adic Hecke algebra described in the previous section into classical operators and thus obtain relations satisfied by these classical operators.

Let $\mathbb{A} = \mathbb{A}_\mathbb{Q}$ be the adele ring of $\mathbb{Q}$ and $\widetilde{SL}_2(\mathbb{A}) = SL_2(\mathbb{A}) \times \{ \pm 1 \}$ with the group law: for $g = (g_v), h = (h_v) \in SL_2(\mathbb{A})$ and $e_1, e_2 \in \{ \pm 1 \}$,

$$(g, e_1)(h, e_2) = (gh, e_1 e_2 \sigma(g, h)), \text{ where } \sigma(g, h) = \prod_v \sigma_v(g_v, h_v).$$

The group $\widetilde{SL}_2(\mathbb{A})$ splits over $SL_2(\mathbb{Q})$, and the splitting is given by

$$s_\mathbb{Q} : SL_2(\mathbb{Q}) \to \widetilde{SL}_2(\mathbb{A}), g \mapsto (g, s_\mathbb{A}(g)), \text{ where } s_\mathbb{A}(g) = \prod_v s_v(g).$$

By [4, Proposition 2.16], for $\alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_1(N), s_\mathbb{A}(\alpha) = \left( \frac{\zeta}{\overline{\zeta}} \right)_s$ unless $c = 0$, in which case $s_\mathbb{A}(\alpha) = 1$. Here, $\left( \frac{\zeta}{\overline{\zeta}} \right)_s = \left( \frac{\zeta}{\overline{\zeta}} \right)(c, d)_\infty$.

**Lemma 4.1** Let $4 \mid N$. For $\alpha = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma_0(N), we have

$$s_\mathbb{A}(\alpha) = \begin{cases} \left( \frac{\zeta}{\overline{\zeta}} \right)_s (c, d)_2 & \text{if } c \neq 0 \text{ and } \text{ord}_2(c) \text{ is even}, \\ \left( \frac{\zeta}{\overline{\zeta}} \right)_s (c, d) & \text{if } c \neq 0 \text{ and } \text{ord}_2(c) \text{ is odd}, \\ 1 & \text{if } c = 0. \end{cases}$$

**Proof** If $c = 0$, then $s_v(\alpha) = 1$ for all places $v$, and so $s_\mathbb{A}(\alpha) = 1$.

Suppose $c \neq 0$. Since $\alpha \in \Gamma_0(N)$ and $4 \mid N$, $d$ is odd and coprime to $c$. By Definition, for any finite prime $q$, we have $s_q(\alpha) = (c, d)_q$ if $\text{ord}_q(c)$ is odd and is 1 otherwise. Hence

$$s_\mathbb{A}(\alpha) = \prod_{q \text{ finite}} s_q(\alpha) = \prod_{\text{ord}_q(c) \text{ odd}} (c, d)_q \prod_{\text{ord}_q(c) \text{ even} > 0} (c, d)_q.$$ 

It follows from the proof of [4, Proposition 2.16] (the proof only uses that $d$ is odd and coprime to $c$), that $\left( \frac{\zeta}{\overline{\zeta}} \right)_s (c, d)_q$. Now

$$\prod_{\text{ord}_q(c) \text{ odd}} (c, d)_q = \prod_{q | c} (c, d)_q \prod_{\text{ord}_q(c) \text{ even} > 0} (c, d)_q = \left( \frac{\zeta}{\overline{\zeta}} \right)_s \prod_{\text{ord}_q(c) \text{ even} > 0} (c, d)_q.$$ 

So we just need to show that $\prod_{\text{ord}_q(c) \text{ even} > 0} (c, d)_q (c, d)_2$ if $\text{ord}_2(c)$ is even and is 1 if $\text{ord}_2(c)$ is odd (note that $\text{ord}_2(c) \geq 2$). Let $p$ be any odd prime such that $\text{ord}_p(c)$ is even and $> 0$. Let $c = p^{2n}u$, where $u$ is unit in $\mathbb{Z}_p$. Then $(c, d)_p = (u, d)_p = 1$ as both $d, u$ are units in $\mathbb{Z}_p$. Hence we are done.

For $\tilde{g} = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right), z \in \mathbb{H}$, define

$$\tilde{g}(z) = \frac{az + b}{cz + d} \quad \text{and} \quad J(\tilde{g}, z) = \epsilon(cz + d)^{1/2}.$$ 

By [4, Lemma 3.3], $J(\tilde{g}, z)$ satisfies the automorphy condition i.e.,

$$J(\tilde{g}\tilde{h}, z) = J(\tilde{g}, \tilde{h}z)J(\tilde{h}, z).$$
Let \( f \in S_{k+1/2}(\Gamma_0(N)) \), where \( 4 \mid N \) and \( \alpha \in \Gamma_0(N) \). Then considering \( \overline{\alpha} = (\alpha, s_\alpha(\alpha)) \) \( \in \widetilde{\text{SL}}_2(\mathbb{R}) \), using the above lemma, we have
\[
f(\overline{\alpha}z) = \left( \frac{c}{d} \right) (e_\alpha^{-1})^{2k+1}(cz + d)^{k+1/2} f(z)
\]
\[
= \left( \frac{c}{d} \right) (e_\alpha^{-1})^{2k+1}s_\alpha(\alpha)J(\overline{\alpha}, z)^{2k+1} f(z)
\]
\[
= \begin{cases}
(k(\theta), 1) & \text{if } c = 0, \\
(k(\theta), -1) & \text{if } c \neq 0 \text{ and } \text{ord}_2(c) \text{ is odd}, \\
(c, d)_{\infty} (e_\alpha^{-1}J(\overline{\alpha}, z))^{2k+1} f(z) & \text{if } c \neq 0 \text{ and } \text{ord}_2(c) \text{ is even}.
\end{cases}
\]

For \( \theta \in \mathbb{R} \), let
\[
k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

Define \( \widetilde{K}_\infty := \{ \widetilde{k}(\theta) : \theta \in (-2\pi, 2\pi) \} \), where
\[
\widetilde{k}(\theta) = \begin{cases}
(k(\theta), 1) & \text{if } -\pi < \theta \leq \pi, \\
(k(\theta), -1) & \text{if } -2\pi < \theta \leq -\pi \text{ or } \pi < \theta \leq 2\pi.
\end{cases}
\]

Then \( \widetilde{K}_\infty \) is a maximal compact subgroup of \( \widetilde{\text{SL}}_2(\mathbb{R}) \) and \( \widetilde{k}(\theta) \mapsto e^{i\frac{2\pi}{\theta}} \) is a genuine character of \( \widetilde{K}_\infty \). Let
\[
K_1(N) := \prod_{q < \infty} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}_q) : c \equiv 0, \text{ and } a, d \equiv 1 \pmod{N\mathbb{Z}_q} \right\}.
\]

Recall the strong approximation theorem for \( \widetilde{\text{SL}}_2(\mathbb{A}) \): every element \( \widetilde{g} \in \widetilde{\text{SL}}_2(\mathbb{A}) \) can be written as
\[
\widetilde{g} = (\alpha, s_\alpha(\alpha))\widetilde{g}_\infty(k_1, 1),
\]
where \( (\alpha, s_\alpha(\alpha)) \in s_\mathcal{O}(\text{SL}_2(\mathbb{Q})) \), \( k_1 \in K_1(N) \) and \( \widetilde{g}_\infty \in \widetilde{\text{SL}}_2(\mathbb{R}) \) determined up to left multiplication by elements in \( s_\mathcal{O}(\Gamma_0(N)) \).

We follow the notation of Waldspurger [18]. Let \( \chi \) be an even Dirichlet character modulo \( N \). Write \( \chi_0 = \chi\left(\frac{-1}{\cdot}\right)^k \). Define \( \overline{\gamma}_2 \) on \( \mathbb{Z}_2^* \) as
\[
\overline{\gamma}_2(t) = \begin{cases}
1 & \text{if } t \equiv 1 \pmod{4\mathbb{Z}_2}, \\
-i & \text{if } t \equiv 3 \pmod{4\mathbb{Z}_2},
\end{cases}
\]
and for \( k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K^*_0(4) \), define
\[
\overline{\epsilon}_2(k_0) = \begin{cases}
\overline{\gamma}_2(d)^{-1}(c, d)_2 s_2(k_0) & \text{if } c \neq 0, \\
\overline{\gamma}_2(d) & \text{if } c = 0.
\end{cases}
\]

Let \( \chi_0 \) also denote the idelic character (of \( \mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times \)) corresponding to the Dirichlet character \( \chi_0 \) (it will be clear from the context when we consider \( \chi_0 \) to be idelic or Dirichlet character) and \( \chi_0, p \) be the \( p \)-component of \( \chi_0 \). Let \( A_{k+1/2}(N, \chi_0) \) denote the set of functions \( \Phi : \widetilde{\text{SL}}_2(\mathbb{A}) \rightarrow \mathbb{C} \) satisfying the following properties:
(i) \( \Phi(s_Q(\alpha) \bar{g}(k_1, 1)) = \Phi(\bar{g}) \) for all \( k_1 \in \Pi_{q \neq 1 N} \) \( SL_2(\mathbb{Z}_q) \), \( \alpha \in SL_2(\mathbb{Q}) \), \( \bar{g} \in \mathcal{S}L_2(\mathbb{A}) \).

(ii) \( \Phi \) is genuine, that is, \( \Phi((I, \zeta) \bar{g}) = \zeta \Phi(\bar{g}) \) for \( \zeta \in \mu_2 \).

(iii) For odd primes \( p \) such that \( p^n \mid N \), \( \Phi(\bar{g}(k_0, 1)) = \chi_{0,p}(d) \Phi(\bar{g}) \) for all \( k_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_0^p(p^n) \).

(iv) If \( 2^n \mid N \) (n \geq 2), \( \Phi(\bar{g}(k_0, 1)) = \bar{e}_2(k_0) \chi_{0,2}(d) \Phi(\bar{g}) \) for all \( k_0 \in K_0^2(2^n) \).

(v) \( \Phi(\bar{g}(k(\theta))) = e^{i \frac{2\pi}{d} \theta} \Phi(\bar{g}) \) for all \( k(\theta) \in \widehat{K} \).

(vi) \( \Phi \) is smooth as a function of \( \mathcal{S}L_2(\mathbb{R}) \) and satisfies the differential equation \( \Delta \Phi = -[((2k + 1)(2k - 3)/16] \Phi \), where \( \Delta \) is the Casimir operator.

(vii) \( \Phi \) is square integrable, that is, \( \int_{\mathcal{S}L_2(\mathbb{Q}))} |\Phi(\bar{g})|^2 d\bar{g} < \infty \).

(viii) \( \Phi \) is cuspidal, that is, \( \int_{\mathcal{N}_0 \setminus \mathcal{N}_\mathbb{A}} \Phi((1 \chi, \gamma) \bar{g}) d\alpha = 0 \) for all \( \bar{g} \in \mathcal{S}L_2(\mathbb{A}) \).

By [18, Proposition 3], there exists an isomorphism between
\[
A_{k+1/2}(N, \chi_0) \to S_{k+1/2}(\Gamma_0(N), \chi)
\]
given by \( \Phi \mapsto f_\Phi \), where for \( z \in \mathbb{H} \),
\[
f_\Phi(z) = \Phi(\bar{g}_\infty) f(\bar{g}_\infty, i)^{2k+1},
\]
where \( \bar{g}_\infty \in \mathcal{S}L_2(\mathbb{R}) \) is such that \( \bar{g}_\infty(i) = z \). The inverse map is given by \( f \mapsto \Phi_f \), where, for \( g \in \mathcal{S}L_2(\mathbb{A}) \), if \( \bar{g} = (\alpha, s_\mathfrak{b}(\alpha)) \bar{g}_\infty(k_1, 1) \),
\[
\Phi_f(\bar{g}) = f(\bar{g}_\infty(i)) f(\bar{g}_\infty, i)^{-2k-1}.
\]
This isomorphism induces a ring isomorphism of spaces of linear operators,
\[
q: \text{End}_\mathbb{C}(A_{k+1/2}(N, \chi_0)) \to \text{End}_\mathbb{C}(S_{k+1/2}(\Gamma_0(N), \chi))
\]
given by \( q(T)(f) = f(T(\Phi_f)) \).

4.1 \( N = 4M \), \( M \) **Odd and** \( p \mid M \)

Let \( p \) be an odd prime and let \( N = 4M \) with \( M \) odd such that \( p \) strictly divides \( M \).
In this subsection, we translate the elements \( \mathcal{J}_1, \mathcal{U}_1, \mathcal{U}_0, \) and \( \mathcal{F}_1 \) in the \( p \)-adic Hecke algebra to certain classical operators on \( S_{k+1/2}(\Gamma_0(4M), \chi) \). We restrict ourselves to \( \chi \) being the trivial character modulo \( 4M \). In this case, \( \chi_0 = (\frac{-1}{\cdot})^k \) has conductor either 1 or 4, and so \( \chi_{0,p} \) is trivial on \( \mathbb{Z}_p^\times \), while \( \chi_{0,2} \) acts by \( \chi_{0,1} = \chi_0 \) on \( \mathbb{Z}_2^\times \).

Let \( \gamma \) be the character on \( (\mathbb{Z}_p/\mathbb{p}\mathbb{Z}_p)^\times \) induced by \( \chi_{0,p}|\mathbb{p}\mathbb{Z}_p^\times \) (so in the current case, \( \gamma \) is trivial). Then Iwahori Hecke algebra \( H(K_0^p(p), \gamma) \) is a subalgebra of \( \text{End}_\mathbb{C}(A_{k+1/2}(N, \chi_0)) \) via the following action: for \( T \in H(K_0^p(p), \gamma) \) and \( \Phi \in A_{k+1/2}(N, \chi_0) \),
\[
T(\Phi)(\bar{g}) = \int_{\mathcal{S}L_2(\mathbb{Q}_p)} T(\bar{x}) \Phi(\bar{g} \bar{x}) d\bar{x}.
\]

**Proposition 4.2** Let \( M \) be positive integer such that \( p \) strictly divides \( M \). Let \( \chi \) be the trivial character modulo \( 4M \) and let \( \gamma \) be induced by \( \chi_{0,p} \). Let \( \mathcal{J}_1, \mathcal{U}_1, \mathcal{U}_0, \mathcal{F}_1 \in H(K_0^p(p), \gamma) \) and \( f \in S_{k+1/2}(\Gamma_0(4M), \chi) \). Then
(i) \( \left( \frac{-1}{p} \right)^k q(\mathcal{T}_1)(f)(z) = p^{-k-1/2} \sum_{s=0}^{p^2-1} f \left( \frac{z+s}{p^2} \right) = p^{(3-2k)/2} U_p^k(f) \).

(ii) \( q(\mathcal{U}_1)(f)(z) = \overline{\xi}_p \left( \frac{-1}{p} \right) \left( \frac{M/p}{p} \right) \sum_{s=0}^{p^2-1} f \left( [\alpha_s, \phi_{a_s}] \right)_{k+1/2}(z) \), where

\[
\alpha_s = \left( \frac{p^2 n - 4 M m s}{4 p M(1-s)} \right) \in M_2(\mathbb{Z})
\]

has determinant \( p^2 \) and \( m, n \in \mathbb{Z} \) such that \( p n - (4M/p)m = 1 \) and \( \phi_{a_s}(z) = (4M(1-s)z + 1)^{1/2} \).

(iii) \( q(\mathcal{U}_0)(f)(z) = \sum_{s=0}^{p^2-1} f \left( [\beta_s, \phi_{\beta_s}] \right)_{k+1/2}(z) \), where \( \beta_s = \left( \frac{M_{1} n p - 4 M s}{4M_{1} z + (np - 4 M_s)} \right)^{1/2} \).

(iv) \( q(\mathcal{T}_{-1})(f)(z) = \left( \frac{-1}{p} \right)^k \sum_{s=0}^{p^2-1} f \left( [\gamma_s, \phi_{\gamma_s}(z)] \right)_{k+1/2}(z) \), where \( \gamma_s = \left( \frac{p^2}{4M_1 s} \right) \) and \( \phi_{\gamma_s}(z) = (-4(M/p)s z + p^{-1})^{1/2} \).

**Proof** For (i), let \( \tilde{g}_\infty = (g_\infty, 1) \in \tilde{\text{SL}}_2(\mathbb{R}) \) such that \( \tilde{g}_\infty i = z \). Then using decomposition in Lemma 3.8, we have

\[
\mathcal{T}_1(\Phi_f)(\tilde{g}_\infty) = \sum_{s=0}^{p^2-1} \overline{\gamma}(h(p), 1) \Phi_f(\tilde{g}_\infty(x(s), 1)(h(p), 1))
\]

\[
= \overline{\xi}_p \sum_{s=0}^{p^2-1} \Phi_f(\tilde{g}_\infty(x(s), 1)(h(p), 1))
\]

\[
= \overline{\xi}_p \sum_{s=0}^{p^2-1} \Phi_f(s_{\mathcal{Q}}(A_s)\tilde{g}_\infty(x(s), 1)(h(p), 1)),
\]

where for each \( 0 \leq s \leq p^2 - 1 \), we take \( A_s = h(p^{-1})x(-s) = \left( \frac{p^{s_1} - p^{s_1}}{p} \right) \in \text{SL}_2(\mathbb{Q}) \)

(note that \( \Phi_f(s_{\mathcal{Q}}(\alpha)\tilde{g}) = \Phi_f(\tilde{g}) \) for any \( \alpha \in \text{SL}_2(\mathbb{Q}), \tilde{g} \in \tilde{\text{SL}}_2(\mathbb{A}) \)). Clearly \( s_{\mathcal{Q}}(A_s) = 1 \) for all primes \( \nu \), so \( s_{\mathcal{Q}}(A_s) = (A_s, 1) \). The infinite-component of

\[
\begin{pmatrix} (A_s, 1) \\ \tilde{g}_\infty \\ (x(s), 1)(h(p), 1) \\ p \end{pmatrix}
\]

is \( (A_s, 1)\tilde{g}_\infty \), for a prime \( q \) such that \( (q, 2M) = 1 \) the \( q \)-component is \( (A_s, 1) \in \text{SL}_2(\mathbb{Z}_q) \)

\times \{1\}, for a prime \( r \) such that \( (r, 2p) = 1 \) and \( r^b \parallel M \), the \( r \)-component is \( (A_s, 1) \in K^r_{\nu}(r) \times \{1\} \), the 2-component is \( (A_s, 1) \in K^2_{\nu}(4) \times \{1\} \), and the \( p \)-component is \( (A_s, 1)(x(s), 1)(h(p), 1) = (I, (p, p), p) = (I, \left( \frac{-1}{p} \right)) \).

Since \( \chi \) is trivial, \( \chi_{0,2} = \left( \frac{-1}{p} \right)^k \), while \( \chi_{0,p} \) and \( \chi_{0,r} \) are trivial. So the 2-component acts by

\[
\tilde{\bar{\varepsilon}}_2(A_s) \chi_{0,2}(p) = \tilde{\bar{\gamma}}_2(p) \chi_{0,2}(p) = \overline{\xi}_p \left( \frac{-1}{p} \right)^k,
\]

and the \( p \)-component acts by \( \left( \frac{-1}{p} \right) \).
Thus,

\[ T_1(\Phi_f)(\tilde{g}_\infty) = \varepsilon_p \sum_{s=0}^{p^2-1} \Phi_f(s_{\mathbb{Q}}(A_s)\tilde{g}_\infty(x(s),1)(h(p),1)) = \left(\frac{-1}{p}\right)^k \left(\frac{-1}{p}\right)^{p^2-1} \Phi_f(A_s\tilde{g}_\infty,1) \]

\[ = \left(\frac{-1}{p}\right)^k \sum_{s=0}^{p^2-1} f(A_s\tilde{g}_\infty(i))J((A_s\tilde{g}_\infty,1),i)^{-2k-1}. \]

Consequently,

\[ q(T_1)(f)(z) = T_1(\Phi_f)(\tilde{g}_\infty)J((g_\infty,1),i)^{2k+1} = \left(\frac{-1}{p}\right)^k p^{-k-1/2} \sum_{s=0}^{p^2-1} f(z+s)\left(\frac{z}{p^2}\right). \]

For (ii) we need the following decomposition (we use \((4, M) = 1\):

\[ K_0 w(p^{-1})K_0 = \bigcup_{s \in \mathbb{Z}_p/p\mathbb{Z}_p} y(4Ms)w(p^{-1})K_0. \]

Taking \(\tilde{g}_\infty\) such that \(\tilde{g}_\infty i = z\), we have

\[ \mathcal{U}_i(\Phi_f)(\tilde{g}_\infty) = \varepsilon_p \left(\frac{-1}{p}\right) \sum_{s=0}^{p^2-1} \Phi_f(\tilde{g}_\infty(y(4Ms),1)(w(p^{-1}),1)). \]

Since \(p\) is coprime to \(4M/p\), we fix \(m, n \in \mathbb{Z}\) such that \(pn - (4M/p)m = 1\). For \(0 \leq s \leq p - 1\), take

\[ A_s = \begin{pmatrix} pn & m \\ 4M & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4Ms & 1 \end{pmatrix} = \begin{pmatrix} pn - 4Ms & m \\ 4M(1-s) & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{Q}). \]

Since \(s_n(A_s) = 1\) for all primes, \(v\) we have \(s_{\mathbb{Q}}(A_s) = (A_s,1)\). As before, the \(\infty\)-component of

\[ \Phi_{\mathbb{Q}}(A_s)\tilde{g}_\infty \begin{pmatrix} y(4Ms),1 \\ w(p^{-1}),1 \end{pmatrix} \]

is \((A_s,1)\tilde{g}_\infty\), for a prime \(q\) such that \((q, 2M) = 1\) the \(q\)-component is \((A_s,1) \in \text{SL}_2(\mathbb{Z}_q)\times\{1\}\), for a prime \(r\) such that \((r, 2p) = 1\) and \(r^b\parallel M\); the \(r\)-component is \((A_s,1) \in K_{\Phi}^\prime(r^b)\times\{1\}\) and the 2-component is \((A_s,1) \in K_{\Phi}^\prime(4)\times\{1\}\) (as \((2, 2)\)-th entry of \(A_s\) is 1). For the \(p\)-component we check that

\[ (A_s,1) = \begin{pmatrix} pn & m/p \\ 4M & 1 \end{pmatrix}, \]

and so

\[ (A_s,1)(y(4Ms),1)(w(p^{-1}),1) = \begin{pmatrix} -m & n \\ -p & 4M/p \end{pmatrix} \begin{pmatrix} M/p \end{pmatrix}. \]

Since \(\chi\) is trivial, as before, the \(r\)-component acts trivially, the \(p\)-component acts by \(\left(\frac{M/p}{p}\right)\) (as \(\chi_0, p(4M/p) = 1\), and the 2-component by \(\varepsilon_2(A_s)\chi_0,2(1) = 1\). Thus,
we have
\[
\mathcal{U}_1(\Phi_f)(\tilde{g}_\infty) = \mathcal{E}_p\left(\frac{-1}{p}\right) \sum_{j=0}^{p-1} \Phi_f(s_{Q_j}(A_s)\tilde{g}_\infty(y(4M_s), 1)(w(p^{-1}), 1))
\]
\[
= \mathcal{E}_p\left(\frac{-1}{p}\right) \left(\frac{M/p}{p}\right) \sum_{j=0}^{p-1} \Phi_f((A_s, 1)(g_\infty, 1))
\]
\[
= \mathcal{E}_p\left(\frac{-1}{p}\right) \left(\frac{M/p}{p}\right) \sum_{j=0}^{p-1} f((A_s, 1)z)J((A_s, 1), z)^{-2k-1}J((g_\infty, 1), i)^{-2k-1}.
\]
So we have
\[
q(\mathcal{U}_1(f)(z)) = \mathcal{E}_p\left(\frac{-1}{p}\right) \left(\frac{M/p}{p}\right) \sum_{j=0}^{p-1} f((A_s, 1)z)J((A_s, 1), z)^{-2k-1}.
\]
Let \(\alpha_s = A_s(p_0^0)\) and \(\phi_{\alpha_s}(z) = (4M(1-s)z + 1)^{1/2}\). Then
\[
q(\mathcal{U}_1(f)(z)) = \mathcal{E}_p\left(\frac{-1}{p}\right) \left(\frac{M/p}{p}\right) \sum_{j=0}^{p-1} f\left(\left(\frac{p^2n - 4mMs}{4pM(1-s)z + p}\right)\right) (4M(1-s)z + 1)^{-k-1/2}
\]
\[
= \mathcal{E}_p\left(\frac{-1}{p}\right) \left(\frac{M/p}{p}\right) \sum_{j=0}^{p-1} f\left(\left[\left(\alpha_s, \phi_{\alpha_s}\right)\right]_{k+1/2}(z)\right).
\]
For (iii), using Lemma 3.8, we have
\[
\mathcal{U}_0(\Phi_f)(\tilde{g}_\infty) = \sum_{j=0}^{p-1} \Phi_f(\tilde{g}_\infty(x(s), 1)(w(1), 1)).
\]
Let \(m, n \in \mathbb{Z}\) such that \(pn - (4M/p)m = 1\) and let \(M_1 = M/p\). For \(0 \leq s \leq p - 1\), take
\[
A_s = \begin{pmatrix} 1 & -s + m \\ -4M_1s + np & -4M_1 \\ \end{pmatrix} \in \Gamma_1(4M_1).
\]
By Lemma 4.1, we have \(s_{\Lambda_1}(A_s) = \begin{pmatrix} -4M_1 \\ -4M_1s + np \end{pmatrix} = 1\). Thus, the \(\infty\)-component of \(s_{Q_j}(A_s)\tilde{g}_\infty(x(s), 1)(w(1), 1)\) is \((A_s, 1)(g_\infty, 1)\); for a prime \(q\) such that \((q, 2M) = 1\) the \(q\)-component is \((A_s, 1) \in \text{SL}_2(\mathbb{Z}_q) \times \{1\}\); if \(r\) is an odd prime such that \(r^b \parallel M\) then the \(r\)-component is \((A_s, 1) \in K_0^r(r^b) \times \{1\}\); and the 2-component is \((A_s, 1) \in K_0^r(4) \times \{1\}\). For the \(p\)-component, since \(\text{ord}_p(4M_1) = 0\), we have
\[
\begin{pmatrix} 1 & m \\ 4M_1 & np \end{pmatrix} 1(x(-s), 1) = (A_s, 1),
\]
\[
\begin{pmatrix} 1 & m \\ 4M_1 & np \end{pmatrix} 1(w(1), 1) = \begin{pmatrix} -m \\ -np \end{pmatrix} 1(4M_1, \beta).
\]
where \(\beta\) is either \((4M_1, -1)_p\) or \((4M_1, np)_p\) depending on whether \(\text{ord}_p(np)\) is odd or even. In either case it is clear that \(\beta\) is 1. Thus, the \(p\)-component is \((\begin{pmatrix} -m \\ -np \end{pmatrix} 1, 4M_1, \beta) \in K_0 \times \{1\}\).
Since $\chi$ is trivial, the $p$-component and $r$-component act trivially, and the 2-component acts by \( \tilde{\varepsilon}(A_s) = r_{0,2}(-4M_1s + np) = (4M_1, -4M_1s + np)_{2s_2}(A_s) \), which clearly equals 1. Thus,

$$\mathcal{U}_0(\Phi_f)(\tilde{g}_\infty) = \sum_{s=0}^{p-1} \Phi_f((A_s, 1)\tilde{g}_\infty)$$

$$= \sum_{s=0}^{p-1} f(A_s z) F((A_s, 1), z)^{-2k-1} F((\tilde{g}_\infty, 1), i)^{-2k-1}$$

and consequently

$$q(\mathcal{U}_0(f))(z) = \sum_{s=0}^{p-1} f\left( \frac{z + (m - s)}{4M_1 z + (np - 4M_1s)} \right) (4M_1z + (np - 4M_1s))^{-k-1/2}.$$ 

For (iv), using $K_0 h(p^{-1}) K_0 = \bigcup_{s \in \mathbb{Z}/p^2\mathbb{Z}} y(4Ms) h(p^{-1})$, we have

$$\mathcal{T}_{-1}(\Phi_f)(\tilde{g}_\infty) = \overline{e_p} \sum_{s=0}^{p^2-1} \Phi_f(\tilde{g}_\infty(y(4Ms), 1)(h(p^{-1}), 1)).$$

For $0 \leq s \leq p^2 - 1$, take $A_s = h(p) y(-4Ms) = \left( \begin{array}{cc} p & 0 \\ -4(M/p)s & p^{-1} \end{array} \right) \), then $s_q(A_s) = (A_s, \xi_s)$, where

$$\xi_s := \begin{cases} 1 & \text{if } s = 0, \\ 1 & \text{if ord}_p(s) = 1 \text{ and ord}_2(s) \text{ odd}, \\ \left( \frac{-1}{p} \right) \left( \frac{Ms}{p}, p \right) & \text{if ord}_p(s) = 1 \text{ and ord}_2(s) \text{ even}, \\ \left( \frac{-1}{p} \right) \left( \frac{Ms}{p}, p \right) & \text{if } (s, p) = 1 \text{ and ord}_2(s) \text{ odd}, \\ \left( \frac{Ms}{p}, p \right) & \text{if } (s, p) = 1 \text{ and ord}_2(s) \text{ even}. \end{cases}$$

We verify the above formula for $\xi_s$ in the case when ord$_p(s) = 1$ and ord$_2(s)$ is even; the other cases follow similarly. Clearly, ord$_p(s) = 1$ and ord$_2(s)$ even imply that ord$_p(-4M/p)s) = 1$ and ord$_2(-4M/p)s)$ is even. So we have $s_2(A_s) = 1, s_p(A_s) = (-4Ms/p, p^{-1})_p$, and by definition, $s_\infty(A_s) = 1$. For any prime $q$, note that $(-4Ms/p, p^{-1})q = (-Ms, p)_q = (Ms, p)_q$. So

$$\xi_s = \prod_v s_v(A_s) = (Ms, p)_p \prod_{q: (4Ms/p)_q = 1, \text{ord}_q(4Ms) \text{ odd}} (Ms, p)_q.$$ 

If ord$_q(4Ms)$ is even, then $(Ms, p)_q = (u, p)_q$ for some unit $u$ in $\mathbb{Z}_q$, so $(Ms, p)_q = 1$. Thus, using the product formula $\prod_v (Ms, p)_v = 1$, we have

$$\xi_s = \prod_v s_v(A_s) = (Ms, p)_p \prod_{q: (4Ms/p)_q = 1} (Ms, p)_q = (Ms, p)_2.$$ 

Since $(p, p)_2 = \left( \frac{-1}{p} \right)$, we get that $\left( \frac{-1}{p} \right) \left( \frac{Ms}{p}, p \right)_2 = (Ms, p)_2$, and we are done. Thus, we have

$$\mathcal{T}_{-1}(\Phi_f)(\tilde{g}_\infty) = \overline{e_p} \sum_{s=0}^{p^2-1} \xi_s \Phi_f((A_s, 1)\tilde{g}_\infty(y(4Ms), 1)(h(p^{-1}), 1)).$$
Now the \( \infty \)-component of \( (A_s, 1)\bar{g}_\infty, y(4Ms), 1)(h(p^{-1}), 1) = (A_s, 1)\bar{g}_\infty \), for a prime \( q \) such that \( (q, 2M) = 1 \) the \( q \)-component is \( (A_s, 1) \in \text{SL}_2(\mathbb{Z}_q) \times \{1\} \), if \( r \) is an odd prime coprime to \( p \) such that \( r^b \| M \), then the \( r \)-component belongs to \( K^*_0(r^b) \times \{1\} \), the 2-component is

\[
\begin{pmatrix}
   p & 0 \\
-4(M/p)s & p^{-1}
\end{pmatrix} \in K^*_0(4) \times \{1\},
\]

and the \( p \)-component is \((A_s, 1)(y(4Ms), 1)(h(p^{-1}), 1), \) which is precisely equal to \((I, \eta_s)\), where

\[
\eta_s := \begin{cases}
   \left( \frac{-1}{p} \right) & \text{if } s = 0, \\
   1 & \text{if ord}_p(s) = 1, \\
   \left( \frac{-1}{p} \right)(M_s/p)_p & \text{if } (s, p) = 1.
\end{cases}
\]

Since \( \chi \) is trivial, \( \chi_{0, p} \) is trivial, and so the \( p \)-component acts on \( \Phi_f \) simply by multiplication by \( \eta_s \). Next we look at how the 2-component acts on \( \Phi_f \). Since \( \chi_{0, 2} = \left( \frac{-1}{A} \right)^k \), we get that

\[
\bar{\epsilon}_2(A_s)\chi_{0, 2}(p^{-1}) = \begin{cases}
   \bar{\epsilon}_2(p^{-1})\chi_{0, 2}(p^{-1}) & \text{if } s = 0, \\
   \bar{\epsilon}_2(p^{-1}) = -4(M/p)s, p^{-1})_2^s(A_s)\chi_{0, 2}(p^{-1}) & \text{if } s \neq 0,
\end{cases}
\]

\[
= \vartheta_s := \begin{cases}
   \vartheta_s(p^{-1})_k & \text{if } s = 0, \\
   \vartheta_s(p^{-1})^{k+1}(M_s/p)_p & \text{if } s \neq 0 \text{ and ord}_2(s) \text{ odd}, \\
   \vartheta_s(p^{-1})^{k+1}(&M_s/p)_p & \text{if } s \neq 0 \text{ and ord}_2(s) \text{ even}.
\end{cases}
\]

One can check that

\[
\vartheta_s\eta_s = \varrho_p(-\frac{-1}{p})_k \xi_s,
\]

and so

\[
\mathcal{T}_1(\Phi_f)(\mathbb{g}_\infty) = \bar{\epsilon}_p \sum_{s=0}^{p^2-1} \xi_s\vartheta_s\eta_s\Phi_f((A_s, 1)\mathbb{g}_\infty) = \left( \frac{-1}{p} \right)^k \sum_{s=0}^{p^2-1} \Phi_f((A_s, 1)\mathbb{g}_\infty).
\]

Thus,

\[
q(\mathcal{T}_1)(f)(z) = \left( \frac{-1}{p} \right)^k \sum_{s=0}^{p^2-1} f\left( \frac{p^2z}{-4Msz+1} \right) \left( \frac{-4Msz+1}{p} \right)^{-k-1/2}
\]

\[
= \left( \frac{-1}{p} \right)^k \sum_{s=0}^{p^2-1} f\left( [\gamma_s, \phi_y(z) \right]_k^{-1/2}(z),
\]

where \( y_s = \left( \frac{-1}{-4Ms} \right)^k \) and \( \phi_y(z) = \left( -4(M/p)s + p^{-1} \right)^{1/2} \).

Let \( \bar{Q}_p := q((\mathbb{U}_0) \) and \( \bar{W}_{p^2} := q(p^{-1/2}\mathbb{U}_1) \). Then we have the following corollary.

**Corollary 4.3** On \( S_{k+1/2}(\Gamma_o(4M)) \), we have the following:

(i) \( \bar{W}_{p^2} \) is an involution;

(ii) \( (\bar{Q}_p - p)(\bar{Q}_p + 1) = 0 \);
(iii) \( \bar{Q}_p = \left( \frac{-1}{p} \right)^k p^{1-k} U_p \bar{W}_p \).

(iv) if \( f \in S_{k+1/2}(\Gamma_0(4M/p)) \), then \( \bar{Q}_p(f) = pf \).

**Proof** The proof of (i) to (iii) follows by using Propositions 3.10 and 4.2. For (iv) we use Proposition 4.2(ii).

We further define an operator \( \bar{Q}_p' \) on \( S_{k+1/2}(\Gamma_0(4M)) \) to be the conjugate of \( \bar{Q}_p \) by \( \bar{W}_p \), i.e., \( \bar{Q}_p' = \bar{W}_p \bar{Q}_p \bar{W}_p \). Thus, \( \bar{Q}_p' \) satisfies the same quadratic relation as \( \bar{Q}_p \), and we have \( \bar{Q}_p' = \left( \frac{-1}{p} \right)^k p^{1-k} \bar{W}_p U_p \).

**Remark 4.4** We note that for a prime \( q \) such that \( (q, 2M) = 1 \), one can similarly obtain the usual Hecke operator \( \mathcal{T}_q^2 \) on \( S_{k+1/2}(\Gamma_0(4M)) \). In particular, if we take \( \mathcal{T}_1 := X_{(\bar{h}(q), 1)} \in \text{H}(\text{SL}_2(\mathbb{Z}_q), \gamma_q) \), then \( q(\mathcal{T}_1) = \left( \frac{-1}{p} \right)^k p^{3-2k/2} \mathcal{T}_q^2 \).

Moreover, if \( p \) and \( q \) are distinct primes such that \( p^n, q^m \) strictly divide \( N \), then the operators \( S \in \text{H}(K_0^q(p^n), \gamma_p) \) and \( \mathcal{T} \in \text{H}(K_0^q(q^m), \gamma_q) \) in \( \text{End}_C(S_{k+1/2}(\Gamma_0(N))) \) commute.

In particular, the operators \( \bar{Q}_p, \bar{W}_p \) on \( S_{k+1/2}(\Gamma_0(4M)) \) that we defined above commute with \( \mathcal{T}_q^2 \) for primes \( q \) coprime to \( 2M \).

**Remark 4.5** Let \( f \in S_{k+1/2}(\Gamma_0(2^\nu M)) \), where \( \nu \geq 2 \). Then we have exactly the same statement as Proposition 4.2 for the action on \( f \) with \( M \) replaced by \( 2^\nu M \). In particular, if \( f \in S_{k+1/2}(\Gamma_0(2^\nu M/p)) \), then \( \bar{Q}_p(f) = pf \). The results of the next section on self-adjointness also hold similarly.

### 4.2 Self-adjointness

Let \( M \) be odd such that \( p \mid M \). In this subsection, we check that the operators \( \bar{W}_p, \bar{Q}_p, \) and \( \bar{Q}_p' \) are self-adjoint operators on \( S_{k+1/2}(\Gamma_0(4M)) \). The property of self-adjointness will be used to give a description of our minus space in terms of common eigenspaces.

**Proposition 4.6** The operator \( \bar{W}_p \) is self-adjoint with respect to the Petersson inner product.

**Proof** We write
\[
\bar{W}_p(f) = \frac{\bar{w}_p}{\sqrt{p}} \left( \frac{-1}{p} \right)^k \left( \frac{M/p}{p} \right) S_p(f), \quad S_p(f) := \sum_{j \in \mathbb{Z}/p\mathbb{Z}} f_j \left[ (\alpha_s, \phi_{\alpha_s}(z)) \right]_{k+1/2},
\]
where
\[
(\alpha_s, \phi_{\alpha_s}(z)) = \left( \begin{pmatrix} p^2n - 4ms & m \\ 4pM(1-s) & p \end{pmatrix}, (4M(1-s)z + 1)^{1/2} \right) \in \mathcal{S},
\]
and \( n, m \) are integers such that \( pn - (4M/p)m = 1 \).

We will show that \( \langle S_p(f), g \rangle = \left( \frac{-1}{p} \right) \langle f, S_p(g) \rangle \). We write \( S_p = S_{1,p} + S_{2,p} \), where \( S_{1,p} \) consists of the \( s = 0 \) term and \( S_{2,p} \) consists of rest of the terms. Also, let \( M_1 = M/p \).
We first consider $S_{2,p}$. For $s \neq 0$, as $pn - 4M_1ms = 1 + 4M_1m(1 - s)$ it is clear that $pn - 4M_1ms$ and $4M(1 - s)$ are relatively coprime, hence there exists integers $u, \nu$ such that $u(pn - 4M_1ms) + 4\nu M(1 - s) = 1$. In particular, this implies that $-4M_1msu \equiv 1 \pmod{p}$. Since $-4M_1m \equiv 1 \pmod{p}$, we get that $su \equiv 1 \pmod{p}$.

We take

$$X = \begin{pmatrix} u & v \\ -4M(1 - s) & pn - 4M_1ms \end{pmatrix} \in \Gamma_0(4M);$$

then $X^* = (X, j(X, z))$, where

$$j(X, z) = \left( -\frac{4M(1 - s)}{pn - 4M_1ms} \right) \left(\frac{-4M(1 - s)z + (pn - 4M_1ms)}{pn - 4M_1ms} \right)^{1/2},$$

as $pn - 4M_1ms \equiv 1 \pmod{4}$. Since $f$ has level $4M$, we have

$$f \left[ \left( \alpha_s, \phi_a(z) \right) \right]_{k+1/2} = f \left[ X^* \right]_{k+1/2} \left[ \left( \alpha_s, \phi_a(z) \right) \right]_{k+1/2}.$$

We claim that in $\mathfrak{G}$,

$$X^*(\alpha_s, \phi_a(z)) = \left( \begin{pmatrix} p & um + \nu p \\ 0 & p \end{pmatrix} \right)^{1/2} \left( \begin{pmatrix} u \\ p \end{pmatrix} \right).$$

It is easy to see equality in the matrix component, also $j(X, \alpha_s, z)\phi_a(z)$ simplifies to just $\left( \frac{-4M(1 - s)}{pn - 4M_1ms} \right)$. So we only need to check equality of the Kronecker symbols $\left( \frac{-4M(1 - s)}{pn - 4M_1ms} \right) = \left( \frac{u}{p} \right)$. While making a choice of $m$ and $n$ so that $pn - 4M_1m = 1$, we can choose $m$ to be a negative integer so that for $1 \leq s \leq p - 1$, $pn - 4M_1ms = 1 + 4M_1m(1 - s) > 0$. So we have

$$\left( \frac{-4M(1 - s)}{pn - 4M_1ms} \right) = \left( \frac{-4M_1m(1 - s)}{1 + 4M_1m(1 - s)} \right) \left( \frac{p}{1 + 4M_1m(1 - s)} \right) \left( \frac{m}{1 + 4M_1m(1 - s)} \right) \left( \frac{1}{1 + 4M_1m(1 - s)} \right) \left( \frac{p}{1 + 4M_1m(1 - s)} \right).$$

Note that

$$\left( \frac{p}{1 + 4M_1m(1 - s)} \right) = \left( \frac{1 + 4M_1m(1 - s)}{p} \right) = \left( \frac{pn - 4M_1ms}{p} \right) = \left( \frac{u}{p} \right).$$

If $m$ is odd, clearly $\left( \frac{m}{1 + 4M_1m(1 - s)} \right) = 1$. Also, if $m = 2^v m'$, where $v \geq 1$ and $m'$ is odd, then

$$\left( \frac{m}{1 + 4M_1m(1 - s)} \right) = \left( \frac{2^v}{1 + 4M_1m(1 - s)} \right)^v \left( \frac{m'}{1 + 4M_1m(1 - s)} \right)^v = 1,$$

since in this case we have $1 + 4M_1m(1 - s) \equiv 1 \pmod{8}$. Thus our claim is proved. Consequently, we have

$$S_{2,p}(f) = \sum_{u \in (\mathbb{Z}/p)\times} f \left[ \left( \begin{pmatrix} p & um \\ 0 & p \end{pmatrix} \right)^{1/2} \right]_{k+1/2}.$$

Since the adjoint of $\left[ \left( \begin{pmatrix} p & um \\ 0 & p \end{pmatrix} \right)^{1/2} \right]_{k+1/2}$ is $\left[ \left( \begin{pmatrix} p & -um \\ 0 & p \end{pmatrix} \right)^{1/2} \right]_{k+1/2}$, the adjoint of $S_{2,p}$ is $\left( \frac{-1}{p} \right) S_{2,p}$, i.e., $\langle S_{2,p}(f), g \rangle = \left( \frac{-1}{p} \right) \langle f, S_{2,p}(g) \rangle$. 

Next we consider the term
\[ S_{1,p}(f) = \left[ \left( \left( \frac{p^2 n}{4pM} \right)^m, (4MZ + 1)^{1/2} \right) \right]_{k+1/2}. \]

For this case, we can choose \( m \) to be a positive integer. Let \( \gamma_p := (a \ b) \in \text{SL}_2(\mathbb{Z}) \) such that \( \gamma_p \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mod p \) and \( \gamma_p \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod 8M_1m \) (this is possible since \( (p, 8M_1m) = 1 \)). We may also choose \( c, d \) above so that \( c < 0 \) and \( d > 0 \). We claim that
\[ S_{1,p}(f) = \left[ \left( \left( \frac{p a}{p^2 c}, \frac{b}{p d} \right), \left( \frac{M_1}{p} \right)^{\left( \frac{c}{d} \right)} \right) (cpz + d)^{1/2} \right]_{k+1/2}. \]

Let
\[ Y = \begin{pmatrix} a - 4bM_1 \\ pc - 4Md \\ \frac{-ma + bpn}{p} \end{pmatrix}. \]

Then \( Y \in \text{SL}_2(\mathbb{Z}) \) and \( pc - 4Md \equiv 0 \mod 4M \), so \( Y \in \Gamma_0(4M) \). We further note that \( -mc + dpn \equiv 1 \mod 4 \), \( dpn - mc = d(1 + 4M_1m) - mc > 0 \). To prove the claim, we need to check that
\[ Y \left( \left( \frac{p^2 n}{4pM} \right)^m \right), (4MZ + 1)^{1/2} \right] = \left( \left( \frac{p a}{p^2 c}, \frac{b}{p d} \right), \left( \frac{M_1}{p} \right)^{\left( \frac{c}{d} \right)} \right) (cpz + d)^{1/2}. \]

As before, matrix equality is easy to check and the automorphy factor of the left-hand side equals kronecker symbol \( \left( \frac{pc - 4Md}{-cm + dpn} \right) \) times \( (pcz + d)^{1/2} \). So we need to show that
\[ \left( \frac{pc - 4Md}{-cm + dpn} \right) = \left( \frac{M_1}{p} \right) \left( \frac{c}{d} \right). \]

Now
\[ \left( \frac{pc - 4Md}{-cm + dpn} \right) = \left( \frac{pc}{pc} \right) \left( \frac{c - 4M_1d}{-cm + dpn} \right) = \left( \frac{-cm + dpn}{p} \right) \left( \frac{c - 4M_1d}{-cm + dpn} \right) = \left( \frac{M_1}{p} \right) \left( \frac{c - 4M_1d}{-cm + dpn} \right). \]

Since \( (m, -cm + dpn) = 1 \) we can write \( \left( \frac{c - 4M_1d}{-cm + dpn} \right) = \left( \frac{d + cm - dpn}{-cm + dpn} \right) \left( \frac{-cm + dpn}{p} \right) \left( \frac{-cm + dpn}{-cm + dpn} \right). \) We have
\[ \left( \frac{d + cm - dpn}{-cm + dpn} \right) \left( \frac{-cm + dpn}{p} \right) = \left( \frac{d + cm - dpn}{-cm + dpn} \right) \left( \frac{-cm + dpn}{p} \right) = \left( \frac{c}{d} \right) \left( \frac{m}{d} \right). \]

We finally check that \( \left( \frac{m}{d} \right) \left( \frac{-cm + dpn}{-cm + dpn} \right) = 1. \) If \( m \) is odd,
\[ \left( \frac{m}{-cm + dpn} \right) = \left( \frac{dpn}{m} \right) \left( \frac{m}{m} \right) = 1 = \left( \frac{m}{d} \right). \]

If \( m = 2^v m', v \geq 1 \) then \( dpn - cm \equiv 1 \mod 8 \) and so
\[ \left( \frac{m}{-cm + dpn} \right) = \left( \frac{2}{-cm + dpn} \right)^v \left( \frac{m'}{-cm + dpn} \right) = \left( \frac{dpn}{m'} \right) = 1 = \left( \frac{m}{d} \right). \]

Thus, our claim is proved.
Next we note that
\[
\left( \begin{array}{cc}
p a & b \\
p^2 c & p d
\end{array} \right) \left( \begin{array}{c}
c/pz + d \\
d
\end{array} \right)^{1/2} = \left( \begin{array}{c}
1 \\
p
\end{array} \right) \left( \begin{array}{c}
0 \\
p
\end{array} \right) \left( \begin{array}{c}
p^{1/4} \\
p^{1/4}
\end{array} \right) y^*_p \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4} =: \zeta_p,
\]
and so \( S_{1,p}(f) = \left( \frac{M_b}{p} \right) f \left[ \zeta_p \right]_{k+1/2} \).

We check similarly that
\[
\left( \begin{array}{cc}
1 & 0 \\
p
\end{array} \right) \left( \begin{array}{c}
p^{1/4} \\
p
\end{array} \right) (y^*_p)^2 \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4} = \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
-1 \\
p
\end{array} \right) Z^*,
\]
where
\[
Z = \left( \begin{array}{cc}
a^2 + bc & \frac{a b + b d}{p} \\
p c (a + d) & bc + d^2
\end{array} \right) \in \Gamma_0(4M)
\]
and so
\[
f \left[ \left( \begin{array}{cc}
1 & 0 \\
p
\end{array} \right) \left( \begin{array}{c}
p^{1/4} \\
p
\end{array} \right) (y^*_p)^2 \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4} \right]_{k+1/2} = \left( \frac{-1}{p} \right) f.
\]

Note that
\[
\left( \zeta_p \right)^2 = \left( \begin{array}{cc}
1 & 0 \\
p
\end{array} \right) \left( \begin{array}{c}
p^{1/4} \\
p
\end{array} \right) y^*_p \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4} = \left( \begin{array}{cc}
1 & 0 \\
p
\end{array} \right) y^*_p \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4} = \left( \begin{array}{cc}
1 & 0 \\
p
\end{array} \right) \left( \begin{array}{c}
p \\
p 0
\end{array} \right)^2 \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4} = \left( \begin{array}{cc}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p 0
\end{array} \right) \left( \begin{array}{c}
p \\
p
\end{array} \right) p^{1/4}.
\]

Thus,
\[
f \left[ \left( \zeta_p \right)^2 \right]_{k+1/2} = \left( \frac{-1}{p} \right) f, \quad \text{i.e., } f \left[ \zeta_p \right]_{k+1/2} = \left( \frac{-1}{p} \right) f \left[ \zeta_p \right]_{k+1/2}.
\]

Since the adjoint of \( \zeta_p \) is \( \zeta_p^{-1} \), we get \( \langle S_{1,p}(f), g \rangle = \left( \frac{-1}{p} \right) \langle f, S_{1,p}(g) \rangle \).

Thus, \( \langle S_p(f), g \rangle = \left( \frac{-1}{p} \right) \langle f, S_p(g) \rangle \). So
\[
\langle \tilde{W}_p^2(f), g \rangle = \frac{\varepsilon_p}{\sqrt{p}} \left( \frac{-M_1}{p} \right) \langle S_p(f), g \rangle = \frac{\varepsilon_p}{\sqrt{p}} \left( \frac{M_1}{p} \right) \langle f, S_p(g) \rangle = \langle f, \tilde{W}_p^2(g) \rangle.
\]

Hence, we are done.
Next we want to show that \( \tilde{\mathcal{Q}}_p = q(\mathcal{U}_0) \) is self-adjoint. We use the relations \( \mathcal{U}_1 \mathcal{T}_1 \mathcal{U}_1 = p \mathcal{T}_{-1} \) and \( \mathcal{T}_1 \mathcal{U}_1 = p \mathcal{U}_0 \) (Proposition 3.10(iii)). Thus, we have

\[
\{ q(\mathcal{U}_0) f, g \} = \frac{1}{p} \{ q(\mathcal{T}_1) q(\mathcal{U}_1) f, g \}.
\]

Since by the above theorem \( q(\mathcal{U}_1) \) is self-adjoint, we get that

\[
\{ f, q(\mathcal{U}_0) g \} = \frac{1}{p} \{ f, p q(\mathcal{U}_0) g \} = \frac{1}{p} \{ f, q(\mathcal{T}_1) q(\mathcal{U}_1) g \}
= \frac{1}{p} \left( \frac{1}{p} q(\mathcal{U}_1) q(\mathcal{T}_1) q(\mathcal{U}_1) q(\mathcal{U}_1) g \right)
= \frac{1}{p} \{ q(\mathcal{U}_1) f, q(\mathcal{T}_{-1}) g \}.
\]

Since \( q(\mathcal{U}_1) \) is surjective, it follows that \( q(\mathcal{U}_0) \) is self-adjoint if and only if the adjoint of \( q(\mathcal{T}_{-1}) \) is \( q(\mathcal{T}_1) \). We now show that the adjoint of \( q(\mathcal{T}_{-1}) \) is \( q(\mathcal{T}_1) \).

Consider elements \( \xi = \left( \begin{smallmatrix} 1 & 0 \\ 0 & p^j \end{smallmatrix} \right) \) and \( \eta = \left( \begin{smallmatrix} 0 & 1 \\ p^j & 0 \end{smallmatrix} \right) \) in \( \mathcal{G} \). We can choose \( \beta_s \) such that \( \Gamma_0(4M) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p^j \end{smallmatrix} \right) \Gamma_0(4M) = \bigcup \Gamma_0(4M) \beta_s = \bigcup \beta_s \Gamma_0(4M) \). So by [14, Propositions 1.1, 1.2], we have \( \Delta_0(4M) \xi \Delta_0(4M) = \bigcup \Delta_0(4M) \xi_s = \bigcup \xi_s \Delta_0(4M) \), where \( P(\xi_s) = \beta_s \).

Since \( \Delta_0(4M) \eta \Delta_0(4M) = \Delta_0(4M) \xi^{-1} \Delta_0(4M) \left( \begin{smallmatrix} p^j & 0 \\ 0 & p^j \end{smallmatrix} \right) \), it follows that

\( \Delta_0(4M) \eta \Delta_0(4M) = \bigcup \Delta_0(4M) \xi_{-1}^{-1} \left( \begin{smallmatrix} p^j & 0 \\ 0 & p^j \end{smallmatrix} \right) \).

Thus, for \( f, g \in S_{k+1/2}(\Gamma_0(4M)) \), we have

\[
\langle f \mid [\Delta_0(4M) \xi \Delta_0(4M)]_{k+1/2}, g \rangle = \langle p^{2k-3/2} \sum_s f \mid [\xi_s]_{k+1/2}, g \rangle
= \langle f, p^{2k-3/2} \sum_s g \mid [\xi_{-1}]_{k+1/2} \rangle
= \langle f, g \mid [\Delta_0(4M) \eta \Delta_0(4M)]_{k+1/2} \rangle,
\]

as elements of the type \( (aI, 1) \) belong to the center of \( \mathcal{G} \) and act trivially via the slash operator.

Using the triangular decomposition we check that

\[
\Gamma_0(4M) \left( \begin{smallmatrix} p^j & 0 \\ 0 & 1 \end{smallmatrix} \right) \Gamma_0(4M) = \bigcup_{s=0}^{p^j-1} \Gamma_0(4M) \left( \begin{smallmatrix} p^j & 0 \\ 0 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ -4Ms & 1 \end{smallmatrix} \right),
\]

and so

\[
\Delta_0(4M) \eta \Delta_0(4M) = \bigcup_{s=0}^{p^j-1} \Delta_0(4M) \eta \left( \begin{smallmatrix} 1 & 0 \\ -4Ms & 1 \end{smallmatrix} \right), (-4Msz + 1)^{1/2}
= \bigcup_{s=0}^{p^j-1} \Delta_0(4M) \left( \begin{smallmatrix} p^j & 0 \\ -4Ms & 1 \end{smallmatrix} \right), (-4(M/p)sz + p^{-1})^{1/2}
\]

Thus it follows from parts (i) and (iv) of Proposition 4.2 that

\[
g \mid [\Delta_0(4M) \eta \Delta_0(4M)]_{k+1/2} = \left( \frac{-1}{p} \right)^k p^{(2k-3)/2} q(\mathcal{T}_{-1})(g),
\]
and \( f \left[ \Delta_0(4M) \xi \Delta_0(4M) \right]_{k+1/2} = \left( \frac{-1}{2} \right)^k p^{(2k-3)/2} q(\mathcal{J}_1)(f) \). Thus by equation (4.1), we obtain the following proposition.

**Proposition 4.7** The operator \( q(\mathcal{J}_{-1}) \) is adjoint of \( q(\mathcal{J}_1) \), and consequently \( \widetilde{Q}_p \) is self-adjoint with respect to the Petersson inner product.

### 4.3 Translating Elements of 2-adic Hecke Algebra and Kohnen’s Plus Space

Following Niwa and Kohnen’s work, Loke and Savin gave an interpretation of Kohnen’s plus space at level 4 in terms of certain elements in the 2-adic Hecke algebra described previously. In this subsection, we describe Kohnen’s plus space at level 4M for M odd in a similar way.

Let \( \chi \) be the trivial character modulo 4; thus, \( \chi_0 = \left( \frac{-1}{2} \right)^k \). Let \( \gamma \) be a character of \( M_2 \) such that \( \gamma((I,1)) = -i^{2k+1} \) and let \( \varphi_\gamma := \gamma((w(1), 1)) \). Then, for any \( k_0 = \left( \binom{b}{d} \right) \in K_2^0(4) \) we have

\[
\overline{\varepsilon}_k(k_0) \chi_{0,2}(d) = \gamma((k_0, 1)).
\]

**Proposition 4.8** (Loke–Savin [8]) For \( \mathcal{J}_1, \mathcal{U}_1 \in H(K_2^0(4), \chi) \) and \( f \in S_{k+1/2}(\Gamma_0(4), \chi) \), the following hold.

(i) \( q(\mathcal{J}_1)(f)(z) = 2^{(3-2k)/2} U_4(f)(z) \);  
(ii) \( q(\mathcal{U}_1)(f)(z) = \left( \frac{2}{2k+1} \right) W_4(f)(z) \), where the operator \( W_4 \) is given by \( W_4(f)(z) = (-2iz)^{-k-1/2} f(-1/4z) \) and \( \left( \frac{2}{2k+1} \right) \) is the usual Kronecker symbol.

Niwa [10] considered operator \( R = W_4 U_4 \) on \( S_{k+1/2}(\Gamma_0(4), \chi) \), proved that it is self-adjoint and that \( (R - \alpha_1)(R - \alpha_2) = 0 \), where \( \alpha_1 = \left( \frac{2}{2k+1} \right) 2^k, \alpha_2 = -\frac{\alpha_1}{2} \). Kohnen [5] defined his plus space \( S_{k+1/2}^+(\Gamma_0(4)) \) at level 4 to be the \( \alpha_1 \)-eigenspace of \( R \) in \( S_{k+1/2}(\Gamma_0(4)) \). It follows from the above proposition that \( S_{k+1/2}^+(\Gamma_0(4)) \) is the 2-eigenspace of \( q(\mathcal{U}_1) q(\mathcal{J}_1)/\sqrt{2} \) and hence that of \( q(\mathcal{U}_2)/\sqrt{2} \).

In the case of level \( 4M \) with \( M \) odd and \( \chi \) a trivial character modulo \( 4M \), Kohnen [6] defines a classical operator \( Q \) on \( S_{k+1/2}(\Gamma_0(4M)) \) in order to obtain his plus space. The operator \( Q \) is defined by

\[
Q := [\Delta_0(4M, \chi) \rho \Delta_0(4M, \chi)], \quad \rho = \left( \begin{array}{cc} 4 & 1 \\ 0 & 4 \end{array} \right), \ e^{\pi i/4}.
\]

By [6, Proposition 1], \( Q \) is self-adjoint and satisfies \((Q - \alpha)(Q - \beta) = 0\), where \( \alpha = (-1)^{(k+1)/2} 2^{\sqrt{2}}, \beta = -\alpha/2 \), and the plus space \( S_{k+1/2}^+(\Gamma_0(4M)) \) is precisely the \( \alpha \)-eigenspace of \( Q \).

**Proposition 4.9** Let \( f \in S_{k+1/2}(\Gamma_0(4M)) \) with \( M \) odd. Then we have

\[
Q(f) = \left( \frac{2}{2k+1} \right) q(\mathcal{U}_2)(f) = \left( \frac{2}{2k+1} \right) q(\mathcal{U}_1) q(\mathcal{J}_1)(f).
\]

Consequently, \( S_{k+1/2}^+(\Gamma_0(4M)) \) is the 2-eigenspace of \( q(\mathcal{U}_1) q(\mathcal{J}_1)/\sqrt{2} \).
Proof  Following [6, Proposition 1], we can write
\[ Q(f) = \sum_{j=0}^{\frac{4}{3}} f \left( \left[ [p] \right]_{k+1/2} \left[ \left[ \frac{1}{4Ms} \ 0 \ 
abla \left[ \frac{4 + 4Ms}{16Ms} \ 1 \right]_{k+1/2} \left( 4Msz + 1 \right)^{1/2} \right]_{k+1/2} \right. \]
\[ = e^{-2k+1/4} \sum_{j=0}^{\frac{4}{3}} f \left( \left[ \frac{4}{4Ms} \ (-1) \ e^{-\pi i/4} \left[ \frac{1}{4Ms} \ 0 \right]_{k+1/2} \left( 4Msz + 1 \right)^{1/2} \right]_{k+1/2} \right). \]
and its adjoint
\[ \tilde{Q}(f) = \sum_{j=0}^{\frac{4}{3}} f \left( \left[ \frac{4}{4Ms} \ (-1) \ e^{-\pi i/4} \left[ \frac{1}{4Ms} \ 0 \right]_{k+1/2} \left( 4Msz + 1 \right)^{1/2} \right]_{k+1/2} \right). \]
Since \( Q \) is self-adjoint, \( Q = \tilde{Q} \).

We now compute \( q(U_2)(f) \). Let \( \tilde{g}_\infty \in \tilde{SL}_2(\mathbb{R}) \) be such that \( \tilde{g}_\infty i = z \). Using \( K_\mathfrak{p}(4)w(2^{-2})K_\mathfrak{p}(4) = \cup_{s \in \mathbb{Z}/4 \mathbb{Z}} y(4M(1-s))w(2^{-2})K_\mathfrak{p}(4) \) (from [8, Proposition 3]), we get
\[ U_2(\Phi_f)(\tilde{g}_\infty) = \sum_{s=0}^{3} \Phi_f(\tilde{g}_\infty(y(4M(1-s)),1)(w(2^{-2}),1)). \]
Take
\[ A_s = \left( 1 - \left( \frac{-1}{M} \right) \frac{1}{M} / 4 \right) \in SL_2(\mathbb{Q}), \]
so \( s_Q(A_s) = (A_s, 1) \). The \( \infty \)-component of
\[ s_Q(A_s) \tilde{g}_\infty \left( y(4M(1-s)),1 \right)(w(2^{-2}),1) \]
is \( (A_s, 1)\tilde{g}_\infty \), for a prime \( q \) such that \( (q, 2M) = 1 \) the \( q \)-component is \( (A_s, 1) \in SL_2(\mathbb{Z}_q) \times \{1\}, \) for an odd prime \( p \) such that \( p^b \parallel M \), the \( p \)-component is \( (A_s, 1) \in K_\mathfrak{p}(p^b) \times \{1\} \) while the 2-component is
\[ (A_s, 1) \left( y(4M(1-s)),1 \right)(w(2^{-2}),1) = \left( \left( \frac{-1}{M} \right) \frac{1-M(\frac{q}{4})}{M} \right), \]
Since \( M \) is odd, it is clear that \( \frac{1-M(\frac{q}{4})}{M} \in \mathbb{Z}_2 \) and so the 2-component is in \( K_\mathfrak{p}(4) \times \{1\} \).
The \( p \)-component acts trivially, while the 2-component acts by \( (\tilde{g}_\infty(M))^{-1} \)
\[ (-1, M)_{2, \chi_{0,2}}(M) := \omega_M. \]
Hence,
\[ q(U_2)(f)(z) := \tilde{q}_8 \omega_M \sum_{s=0}^{3} f(A_s z) j(A_s, z)^{-2k-1} \]
\[ = \tilde{q}_8 \omega_M \sum_{s=0}^{3} f \left( \frac{4 - 4M(\frac{-1}{M})sz - (\frac{-1}{M})}{16MsZ + 4} (4MsZ + 1)^{-k/2} \right. \]
We note that
\[ e^{-2k+1/4} = \left( \frac{2}{2k+1} \right)^{1+i^{2k+1}} = \left( \frac{2}{2k+1} \right)^{\frac{1}{2}} = \left( \frac{2}{2k+1} \right)^{\frac{1}{2}}. \]
Thus, when \( M \equiv 1 \) (mod 4), since \( \omega_M = 1 \), comparing the expression of \( \tilde{Q} \) and \( q(U_2) \), we see that \( \tilde{Q}(f) = (\frac{2}{2k+1}) q(U_2)(f) \). In the case \( M \equiv 3 \) (mod 4), we get that \( \omega_M = -i(-1)^k \)
so \( \left( \frac{2}{M} \right) \overline{\phi_M} \omega_M = e^{-2(2k+1)\pi i/4} \), and consequently \( Q(f) = \left( \frac{2}{M} \right) q(\mathcal{U}_2)(f) \). Since by Theorem 3.5, \( \mathcal{U}_2 = \mathcal{U}_1 \ast \mathcal{I}_1 \), we get that \( \mathcal{U}_2 = \left( \frac{2}{M} \right) q(\mathcal{I}_1)(f) \). Hence, we are done.

The last statement follows, since \((-1)^{[(k+1)/2]} = \left( \frac{2}{M} \right) \).

As before, we can translate \( \mathcal{I}_1 \), \( \mathcal{U}_1 \), \( \mathcal{U}_0 \in H(K_0(4), \gamma) \) to classical operators on \( S_{k+1/2}(\Gamma_0(4M)) \).

**Proposition 4.10** For \( f \in S_{k+1/2}(\Gamma_0(4M)) \),

(i) \( q(\mathcal{I}_1)(f)(z) = 2^{3-2k/2}U_4(f)(z) = \sum_{s=0}^3 \overline{f}(W, \phi_W(z))k_{k+1/2}(z) \),

(ii) \( q(\mathcal{U}_1)(f)(z) = \overline{\phi_M}(\frac{1}{M})^{k+3/2} \sum_{s=0}^3 \overline{f}[A_s, \phi_A(z)]k_{k+1/2}(z) \), where \( W = \left( \frac{4n}{M} \right) \) with \( m, n \in \mathbb{Z} \) such that \( 4n - mM = 1 \) and \( \phi_W(z) = (2Mz + 2)^{1/2} \).

(iii) \( q(\mathcal{U}_0)(f)(z) = \overline{\phi_M}(\frac{1}{M})^{k+3/2} \sum_{s=0}^3 \overline{f}[A_s, \phi_A(z)]k_{k+1/2}(z) \), where \( A_s = \left( \frac{n}{M} \right) \) with \( m, n \in \mathbb{Z} \) such that \( 4n - mM = 1 \) and \( \phi_W(z) = (Mz + 4 - Ms)^{1/2} \).

Define \( \widetilde{Q}_2 := q(\mathcal{U}_0)/\sqrt{2} \). It follows from the relation \( \mathcal{U}_0 = \mathcal{I}_1 \mathcal{U}_1 \) that \( \widetilde{Q}_2 = q(\mathcal{I}_1)q(\mathcal{I}_2)/\sqrt{2} \). One can also observe it directly from the above proposition. Let \( \overline{\mathcal{W}}_4 := q(\mathcal{U}_1) \). Thus, \( \overline{\mathcal{W}}_4 \) is an involution. Let \( \overline{\mathcal{Q}_2} \) be the conjugate of \( \overline{Q}_2 \) by \( \overline{\mathcal{W}}_4 \). Thus, \( \overline{\mathcal{Q}_2} = 2^{1-k}U_4 \overline{\mathcal{W}}_4 \) and \( \overline{\mathcal{Q}_2} = 2^{1-k} \overline{\mathcal{W}}_4 U_4 \). The Kohnen's plus space at level 4M is the 2-eigenspace of \( \overline{Q}_2 \). Note that \( \overline{Q}_2 \) and \( \overline{Q}_2 \) are self-adjoint with respect to the Petersson inner product. The operators \( \overline{Q}_2' \) and \( \overline{Q}_2' \) are \( p \)-adic analogues of Kohnen's operator \( \overline{Q}_2 \) and its conjugate \( \overline{Q}_2 \).

**Remark 4.11** We note that \( q(\mathcal{U}_1) \) in the above proposition can also be given by the following expression:

\[
q(\mathcal{U}_1)(f)(z) = \overline{\phi_M}(\frac{2}{M})(\frac{1}{M})^{k+3/2} \sum_{s=0}^3 \overline{f}[W, \phi_W(z)]k_{k+1/2}(z),
\]

where \( W = \left( \frac{4n}{M} \right) \) with \( m, n \in \mathbb{Z} \) such that \( 8n - mM = 1 \) and \( \phi_W(z) = (2Mz + 4)^{1/2} \).

We shall use this expression of \( q(\mathcal{U}_1) \) in [3].

5 **Eigenvalues of \( U_p \)**

For every positive integer \( n \) and a modular form \( F \), let \( F_n(z) := V(n)F(z) = F(nz) \).

Let \( M \) be a positive integer such that \( p \) divides \( M \). If \( F \in S_{2k}(\Gamma_0(M)) \), then by well-known action of \( T_p \) and \( U_p \), we have

\[
U_p(F)(z) = T_p(F)(z) - p^{2k-1}F_p(z).
\]

Assume that \( F \in S_{2k}(\Gamma_0(M)) \) is a primitive Hecke eigenform and \( a_p \) is the \( p \)-th Fourier coefficient of \( F \). Then \( T_p(F) = a_pF \). It is known that \( |a_p| \) is real and by the Ramanujan conjecture proved by Deligne we have that \( |a_p| \leq 2p^{(2k-1)/2} \).

**Lemma 5.1**

(i) If \( (p, n) = 1 \), then \( U_p(F_n) = a_pF_n - p^{2k+1}F_{np} \).

(ii) If \( p \mid n \), then \( U_p(F_n) = F_{n/p} \).
Minus Space of Half-integral Weight

\textbf{Proof} It is well known that if \((p, n) = 1\) then \(V(n) T_p(F) = T_p V(n) F\). Hence, using (5.1) and that \(F\) is a primitive Hecke eigenform, we get that
\[
U_p(F_n) = T_p(F_n) - p^{2k-1}F_{np} = V(n) T_p(F) - p^{2k-1}F_{np}
\]
\[
= V(n) a_p F - p^{2k-1}F_{np} = a_p F_n - p^{2k-1}F_{np}.
\]
For (i) write \(n = mp\). Then
\[
U_p(F_n)(z) = \frac{1}{p} \sum_{k=0}^{p-1} F_{mp} \left( \frac{z + k}{p} \right) = \frac{1}{p} \sum_{k=0}^{p-1} F_m(z + k) = F_{n/p}(z).
\]

Thus, \(U_p\) stabilizes the two dimensional subspace spanned by \(F_n\) and \(F_{np}\) for \((p, n) = 1\). We will compute the eigenvalues of \(U_p\) on this space. If \(G = \lambda F_n + \beta F_{np}\) is an eigenfunction of \(U_p\) then it follows from part (ii) of the above lemma that \(\lambda \neq 0\). Hence, we can assume that \(\lambda = 1\). We have
\[
U_p(F_n + \beta F_{np}) = (a_p + \beta)F_n - p^{2k-1}F_{np}.
\]
It is clear from above that \(\beta\) cannot be zero and that \(G\) is an eigenfunction if and only if \(a_p + \beta = -p^{2k-1}/\beta\) with eigenvalue \(a_p + \beta\). Hence, \(\beta^2 + a_p \beta + p^{2k-1} = 0\), and we have
\[
\beta = \frac{-a_p \pm \sqrt{a_p^2 - 4p^{2k-1}}}{2}.
\]
The eigenvalues of \(U_p\) on the subspace \(\langle F_n, F_{np} \rangle\) are
\[
a_p + \beta = \frac{a_p \pm \sqrt{a_p^2 - 4p^{2k-1}}}{2}.
\]

\textbf{Proposition 5.2} If an eigenvalue \(\lambda\) of \((U_p)^2\) on the two dimensional subspace spanned by \(F_n\) and \(F_{np}\) is real, then \(\lambda = \pm p^{2k-1}\).

\textbf{Proof} Using the Ramanujan conjecture, we can see that the eigenvalues of \(U_p\) are real or purely imaginary if and only if \(a_p = \pm 2p^{k-1/2}\) or \(a_p = 0\). In those cases, the eigenvalue of \((U_p)^2\) are precisely \(\pm p^{2k-1}\).

\section{The Minus Space of Half-integral Weight Forms}

Let \(M\) be odd and square-free. In this section, we use the operators and relations that we obtained in Section 4 to define the minus space \(S_{k+1/2}^-(\Gamma_0(4M))\) of weight \(k + 1/2\) and level \(4M\). We show that there is an Hecke algebra isomorphism between \(S_{k+1/2}^-(\Gamma_0(4M))\) and \(S_{2k}^\text{new}(\Gamma_0(2M))\), and we give a common eigenspace characterization of \(S_{k+1/2}^-(\Gamma_0(4M))\). It follows that this minus space is identical to the newspace in [9].

For the sake of clarity, we start by defining the minus space at level 4 and at level \(4p\) for \(p\) an odd prime. After that we treat the general case of level \(4M\).
6.1 Minus Space for $\Gamma_0(4)$

We recall the following theorem of Niwa, which was obtained by proving equality of traces of Hecke operators.

**Theorem 6.1** (Niwa [10]) Let $M$ be odd and square-free. There exists an isomorphism of vector spaces $\psi : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ satisfying

$$T_p(\psi(f)) = \psi(T_{p^2}(f)) \text{ for all primes } p \text{ coprime to } 2M.$$ 

Moreover, if $f \in S_{k+1/2}(\Gamma_0(4))$, then we further have $U_2(\psi(f)) = \psi(U_4(f))$.

We also recall the Shimura lift [14]: For $t$ a positive square-free integer, there is a linear map $\text{Sh}_t : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ given by

$$\text{Sh}_t \left( \sum_{n=1}^{\infty} a_n q^n \right) = \sum_{n=1}^{\infty} \left( \sum_{d|n} \left( \frac{-1}{d} \right)^k \left( \frac{t}{d} \right)^{k-1} a \left( t \frac{n^2}{d^2} \right) \right) q^n.$$ 

We note the following observations [11]:

(a) $\text{Sh}_t$ need not be injective, but if $\text{Sh}_t(f) = 0$ for all square-free $t$, then $f = 0$.
(b) $\text{Sh}_t$ commutes with all Hecke operators, i.e., $T_p(\text{Sh}_t(f)) = \text{Sh}_t(T_{p^2}(f))$ for all primes $p$ coprime to $2M$ and $U_p(\text{Sh}_t(f)) = \text{Sh}_t(U_p(f))$ for all primes $p$ dividing $2M$.

We denote $S_{k+1/2}^+(\Gamma_0(4))$ simply by $S^+(4)$. We note the following theorem of Kohnen.

**Theorem 6.2** (Kohnen [5])

(i) $\dim(S^+(4)) = \dim(S_{2k}(\Gamma_0(1)))$.
(ii) $S^+(4)$ has a basis of eigenforms for all the operators $T_{p^2}$, $p$ odd.
(iii) If $f$ is such an eigenform, then $\psi(f)$ is an old form and $\psi(f) = \lambda F + \beta F_2$, where $F \in S_{2k}(\Gamma_0(1))$ is a primitive eigenform determined by the eigenvalues of $f$.

Define $A_{k+1/2}^+(\Gamma_0(4)) := \widetilde{W}_4 S_{k+1/2}^+(\Gamma_0(4))$, which we shall simply denote by $A^+(4)$. We know that $S^+(4)$ is the 2-eigenspace of $\widetilde{Q}_2$, hence $A^+(4)$ is the 2-eigenspace of $\widetilde{Q}_2$. Since $\widetilde{W}_4$ is invertible, we can use the above theorem of Kohnen to get that $\dim(A^+(4)) = \dim(S_{2k}(\Gamma_0(1)))$ and the following corollary.

**Corollary 6.3**

(i) $A^+(4)$ has a basis of eigenforms under $T_{p^2}$ for all $p$ odd.
(ii) $\psi$ maps $A^+(4)$ into the space of old forms in $S_{2k}(\Gamma_0(2))$.

**Proof** Let $f \in S^+(4)$ be an eigenform under $T_{p^2}$ for all $p$ odd satisfying $T_{p^2}(f) = \lambda_p f$. Since $\widetilde{W}_4$ commutes with all such $T_{p^2}$, we get that $g = \widetilde{W}_4 f \in A^+(4)$ is also an eigenform under all $T_{p^2}$ with eigenvalues $\lambda_p$. By Theorem 6.1, $\psi(f)$ and $\psi(g)$ are eigenforms in $S_{2k}(\Gamma_0(2))$ under all $T_p$ with the same set of eigenvalues $\lambda_p$. Since $\psi(f)$ is an old form, it follows from Atkin–Lehner [1] that $\psi(g)$ is also an old form (belonging to the same two dimensional subspace spanned by $F$ and $F_2$).
We note the following key proposition, showing that the sum $S^+(4) + A^+(4)$ is a direct sum. We see analogues of this result in Subsections 6.2 and 6.3.

**Proposition 6.4** $S^+(4) \cap A^+(4) = \{0\}$.

**Proof** Suppose there is a nonzero $f \in S^+(4) \cap A^+(4)$. We can assume that $f$ is an eigenform under $T_{p^2}$ for all $p \text{ odd}$ (since $T_{p^2}$ stabilizes the intersection $S^+(4) \cap A^+(4)$). Since $A^+(4)$ and $S^+(4)$ are respectively the 2-eigenspaces of $\widetilde{Q}_2$ and $\widetilde{Q}_2'$, we have $\widetilde{Q}_2(f) = 2f = \widetilde{Q}_2'(f)$. Using the relations $\widetilde{Q}_2 = 2^{1-k}U_4W_4$, $\widetilde{Q}_2' = 2^{1-k}\widetilde{W}_4U_4$ and $\widetilde{W}_4^2 = 1$, we get that $U_4^2 = 2^{2k-2}\widetilde{Q}_2\widetilde{Q}_2'$ and thus

$$(U_4^2(f) = 2^{2k}f.$$ Applying $\psi$ to the above equation, we get that $(U_4^2(\psi(f)) = 2^{2k}\psi(f)$. Now $\psi(f)$ belongs to the subspace spanned by $F$ and $F_2$ for some primitive form $F \in S_{2k}(\Gamma_0(1))$, and by Proposition 5.2, the eigenvalues of $(U_4^2)$ on this subspace are either non-real or $\pm 2^{2k-1}$. This is a contradiction.

Define $S^{-}_{k+1/2}(\Gamma_0(4))$ to be the orthogonal complement of $S^+(4) \oplus A^+(4)$. Since $\widetilde{Q}_2$ and $\widetilde{Q}_2'$ are Hermitian it follows that $S^{-}_{k+1/2}(\Gamma_0(4))$ is the common eigenspace with the eigenvalue $-1$ of the operators $\widetilde{Q}_2$ and $\widetilde{Q}_2'$. We write $S^{-}_{k+1/2}(\Gamma_0(4))$ simply by $S^-(4)$. So we have

$$S^{-}_{k+1/2}(\Gamma_0(4)) = S^+(4) \oplus A^+(4) \oplus S^-(4).$$

**Theorem 6.5** $S^-(4)$ has a basis of eigenforms for all the operators $T_{p^2}$, $p$ odd; these eigenforms are also eigenfunctions under $U_4$. If two eigenforms in $S^-(4)$ share the same eigenvalues for all $T_{p^2}$, then they are scalar multiples of each other. $\psi$ induces a Hecke algebra isomorphism:

$$S^-(4) \cong S_{2k}^{\text{new}}(\Gamma_0(2)).$$

**Proof** Since $\psi$ maps $S^+(4) \oplus A^+(4)$ into $S_{2k}^{\text{old}}(\Gamma_0(2))$ and dim($S^+(4) \oplus A^+(4)$) = $2 \text{dim}(S_{2k}^{\text{old}}(\Gamma_0(1))) = \text{dim}(S_{2k}^{\text{old}}(\Gamma_0(2)))$, we get that $\psi$ maps this direct sum onto $S_{2k}^{\text{old}}(\Gamma_0(2))$.

Now $T_{p^2}$ commutes with $\widetilde{Q}_2$ and $\widetilde{Q}_2'$ for every odd prime $p$, so we get that $T_{p^2}$ stabilizes $S^-(4)$, hence it has a basis of eigenforms for all $T_{p^2}$ with $p$ odd.

If $f$ is such an eigenform, then $F := \psi(f)$ is an eigenform in $S_{2k}(\Gamma_0(2))$ under all $T_p$, $p$ odd. By Atkin-Lehner [1], $F$ is either an old form or a newform. Since $\psi$ is injective, it follows that $F$ must be a newform. So $\psi$ maps the space $S^-(4)$ into the space $S_{2k}^{\text{new}}(\Gamma_0(2))$. By equality of dimensions, we get that $\psi$ is an isomorphism of $S^-(4)$ onto $S_{2k}^{\text{new}}(\Gamma_0(2))$. Consequently, by [1] an eigenform in $S^-(4)$ under all $T_{p^2}$ for $p$ odd is uniquely determined up to scalar multiplication.

Further for such an eigenform $f$, by [1, Theorem 3], $U_2(F) = -2^{k-1}\lambda(2)F$, where $\lambda(2) = \pm 1$. Thus, $\psi(U_4(f)) = U_2(F) \in S_{2k}^{\text{new}}(\Gamma_0(2))$, so $U_4(f)$ belongs to $S^-(4)$. Since $U_4$ commutes with $T_{p^2}$ for all $p$ odd, we get that $U_4(f)$ is an eigenform under all $T_{p^2}$ with the same eigenvalues as $f$ and hence is a scalar multiple of $f$. ■
6.2 Minus Space for $\Gamma_0(4p)$ for $p$ an Odd Prime

In this subsection, we need the involution $\tilde{W}_{p^2}$ and the operators $U_{p^2}$, $\tilde{Q}_p$ and $\tilde{Q}_p' = \tilde{W}_{p^2} \tilde{Q}_p \tilde{W}_{p^2}$ on $S_{k+1/2}(\Gamma_0(4p))$ that we defined in Section 4.

Consider the subspace $\mathcal{V}(1)$ of $S_{2k}(\Gamma_0(2p))$ coming from the old forms at level 1, that is,

$$\mathcal{V}(1) = S_{2k}(\Gamma_0(1)) \oplus V(2) S_{2k}(\Gamma_0(1)) \oplus V(p) S_{2k}(\Gamma_0(1)) \oplus V(2p) S_{2k}(\Gamma_0(1)).$$

We consider the eigenvalues of $(U_p)^2$ on $\mathcal{V}(1)$.

**Lemma 6.6** The operator $U_p$ stabilizes $\mathcal{V}(1)$. If an eigenvalue $\lambda$ of $(U_p)^2$ on this space is real, then $\lambda = \pm p^{2k-1}$.

**Proof** For a primitive Hecke eigenform $F$ in $S_{2k}(\Gamma_0(1))$, consider the four dimensional subspace spanned by $F, F_2, F_p, F_{2p}$. Then $\mathcal{V}(1)$ is a direct sum of such four dimensional subspaces. By Lemma 5.1, $U_p$ preserves the two dimensional subspace spanned by $F$ and $F_p$ and the two dimensional subspace spanned by $F_2$ and $F_{2p}$. It follows by Proposition 5.2 that the eigenvalues of $(U_p)^2$ on these two dimensional subspaces are either non-real or $\pm p^{2k-1}$.

Let $R := S_{k+1/2}^r(\Gamma_0(4)) \oplus A_{k+1/2}^+(\Gamma_0(4))$. Then we have the following proposition.

**Proposition 6.7** $R \cap \tilde{W}_{p^2} R = \{0\}$.

**Proof** Let $f \neq 0$ belong to the intersection. We can again assume that $f$ is an eigenform under $T_{q^2}$ for all primes $q$ coprime to $2p$. Since, by Corollary 4.3(iv) $S_{k+1/2}(\Gamma_0(4))$ is contained in the $p$-eigenspace of $\tilde{Q}_p$ and so $\tilde{W}_{p^2} S_{k+1/2}(\Gamma_0(4))$ is contained in the $p$-eigenspace of $\tilde{Q}_p'$, we have $\tilde{Q}_p(f) = pf = \tilde{Q}_p'(f)$. Using $\tilde{Q}_p = (\frac{-1}{p})^k p^{1-k} U_{p^2} \tilde{W}_{p^2}$, we get that $(U_{p^2})^2 = p^{2k-2} \tilde{Q}_p \tilde{Q}_p'$, and thus

$$(U_{p^2})^2(f) = p^{2k} f.$$

Since $f \neq 0$, there exists a square-free integer $t$ such that the Shimura lift $\text{Sh}_t(f) \neq 0$. Applying this $\text{Sh}_t$ to the above equation, we get that $(U_p)^2(\text{Sh}_t(f)) = p^{2k} \text{Sh}_t(f)$. Since $\text{Sh}_t$ commutes with all the Hecke operators we get that $\text{Sh}_t(f) \in \mathcal{V}(1)$. But by Lemma 6.6, the eigenvalues of $(U_p)^2$ on $\mathcal{V}(1)$ are either non-real or $\pm p^{2k-1}$ leading to a contradiction.

**Corollary 6.8** Niwa’s map $\psi$ maps $R \oplus \tilde{W}_{p^2} R$ isomorphically onto $\mathcal{V}(1)$.

**Proof** As before (see Corollary 6.3(ii)) $\psi$ maps $R \oplus \tilde{W}_{p^2} R$ into $\mathcal{V}(1)$. It follows from the equality of dimensions that the map is onto.
This space is a direct sum of two dimensional subspaces spanned by \( F \) and \( F_p \), where \( F \) is a primitive Hecke eigenform in \( S_{2k}^{\text{new}}(\Gamma_0(2)) \). Using Proposition 5.2, we have the following lemma.

**Lemma 6.9**  If an eigenvalue \( \lambda \) of \((U_p)^2\) on \( \mathcal{V}(2) \) is real, then \( \lambda = \pm p^{2k-1} \).

Since (by Theorem 6.5) \( \psi \) maps \( S_{k+1/2}^{-}(\Gamma_0(4)) \) isomorphically onto \( S_{2k}^{\text{new}}(\Gamma_0(2)) \), it follows that \( \psi \) maps \( \tilde{W}_p \cdot S_{k+1/2}^{-}(\Gamma_0(4)) \) into the space \( \mathcal{V}(2) \). The proof of the following is identical to that of Proposition 6.7.

**Proposition 6.10**  \( S_{k+1/2}^{-}(\Gamma_0(4)) \cap \tilde{W}_p \cdot S_{k+1/2}^{-}(\Gamma_0(4)) = \{0\} \).

**Corollary 6.11**  \( \psi \) maps \( S_{k+1/2}^{-}(\Gamma_0(4)) \) isomorphically onto \( \mathcal{V}(2) \).

Finally, we consider the following subspace of \( S_{2k}(\Gamma_0(2p)) \) coming from the old forms at level \( p \):

\[
\mathcal{V}(p) = S_{2k}^{\text{new}}(\Gamma_0(p)) \oplus \mathcal{V}(2) \cdot S_{2k}^{\text{new}}(\Gamma_0(p)).
\]

This space is a direct sum of two dimensional subspaces spanned by \( F \) and \( F_2 \), where \( F \) is a primitive Hecke eigenform in \( S_{2k}^{\text{new}}(\Gamma_0(p)) \). We have the following lemma.

**Lemma 6.12**  If an eigenvalue \( \lambda \) of \((U_2)^2\) on \( \mathcal{V}(p) \) is real, then \( \lambda = \pm 2^{2k-1} \).

Let \( S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \) be the new space inside the plus space in \( S_{k+1/2}(\Gamma_0(4p)) \). Kohnen [6, Theorem 2] proved that \( \psi \) maps \( S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \) into \( \mathcal{V}(p) \) and the dimension of \( S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \) equals the dimension of \( S_{2k}^{\text{new}}(\Gamma_0(p)) \). Then as before, \( \psi \) maps \( \tilde{W}_4 \cdot S_{k+1/2}^{+,\text{new}}(\Gamma_0(4)) \) into \( \mathcal{V}(p) \), and we have the following proposition and corollary.

**Proposition 6.13**  \( S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \cap \tilde{W}_4 \cdot S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) = \{0\} \).

**Corollary 6.14**  \( \psi \) maps \( S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \) isomorphically onto \( \mathcal{V}(p) \).

We define the following subspace of \( S_{k+1/2}(\Gamma_0(4p)) \),

\[
E := R \oplus \tilde{W}_p \cdot R \oplus S_{k+1/2}^{-}(\Gamma_0(4)) \oplus \tilde{W}_p \cdot S_{k+1/2}^{-}(\Gamma_0(4)) \\
\quad \oplus S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \oplus \tilde{W}_4 \cdot S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)).
\]

By Corollary 6.8, 6.11, and 6.14, we get that \( \psi \) maps the space \( E \) isomorphically onto the old space \( S_{2k}^{\text{old}}(\Gamma_0(2p)) \). We define the minus space to be the orthogonal complement of \( E \) under the Petersson inner product; that is,

\[
S_{k+1/2}^{-}(\Gamma_0(4p)) := E^\perp.
\]

**Theorem 6.15**  \( S_{k+1/2}^{-}(\Gamma_0(4p)) \) has a basis of eigenforms for all the operators \( T_q \), where \( q \) is a prime coprime to \( 2p \), uniquely determined up to scalar multiplication. \( \psi \) maps the space \( S_{k+1/2}^{-}(\Gamma_0(4p)) \) isomorphically onto the space \( S_{2k}^{\text{new}}(\Gamma_0(2p)) \).
Proof Since the operators $T_{q^2}$, with $(q, 2p) = 1$ stabilize the space $E$ and since they are self-adjoint with respect to the Petersson inner product, it follows that they stabilize the space $S_{k+1/2}(\Gamma_0(4p))$; hence, $S_{k+1/2}(\Gamma_0(4p))$ has a basis of eigenforms for all such operators $T_{q^2}$. If $f$ is such an eigenform, then $\psi(f) \in S_{2k}(\Gamma_0(2p))$ is also an eigenform for all the operators $T_q, (q, 2p) = 1$, and thus (by [1]) $\psi(f)$ is either an old form or a newform. Since $\psi$ is injective and maps $E$ onto $S_{2k}^{\text{old}}(\Gamma_0(2p))$, it follows that $\psi(f)$ is a newform. Thus, $\psi$ maps the space $S_{k+1/2}^{-}(\Gamma_0(4p))$ into the space $S_{2k}^{\text{new}}(\Gamma_0(2p))$. By equality of dimensions, we get that $\psi$ maps the space $S_{k+1/2}^{-}(\Gamma_0(4p))$ isomorphically onto $S_{2k}^{\text{new}}(\Gamma_0(2p))$. Consequently, an eigenform in $S_{k+1/2}^{-}(\Gamma_0(4p))$ is uniquely determined up to multiplication by a scalar.

Corollary 6.16 Let $f \in S_{k+1/2}^{-}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators $T_{q^2}$, $q$ prime, and $(q, 2p) = 1$. Then $\widetilde{W}_p f = \beta(p) f$, $\widetilde{W}_4 f = \beta(2) f$, where $\beta(p) = \pm1$, $\beta(2) = \pm1$.

Proof Let $g = \widetilde{W}_p f$. Since $\widetilde{W}_p$ commutes with all the operators $T_{q^2}$ for $(q, 2p) = 1$, we get that $g$ is an eigenform for all the operators $T_{q^2}$ with the same eigenvalues as $f$. Since $\psi(f)$ is a newform, it follows by [1] that $\psi(g)$ is a scalar multiple of $\psi(f)$. Since $\psi$ is an isomorphism we get that $g$ is a scalar multiple of $f$. Since $\widetilde{W}_p$ is an involution, we get that the scalar is $\pm1$. The same proof applies to $\widetilde{W}_4$.

Let $f \in S_{k+1/2}^{-}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators $T_{q^2}$ as above. It follows that $F = \psi(f)$ is a Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(2p))$ for all the operators $T_q$, $(q, 2p) = 1$. Since the Shimura lift $\text{Sh}_t(f)$ is also an eigenform for all the operators $T_q$ with the same eigenvalues as $F$, it follows from [1] that $\text{Sh}_t(f)$ is a scalar multiple of $F$ (which could be zero). Also, $U_p(F) = -p^{k-1}\lambda(p)F$, where $\lambda(p) = \pm1$ and $U_2(F) = -2k-1\lambda(2)F$, where $\lambda(2) = \pm1$.

Proposition 6.17 Let $f \in S_{k+1/2}^{-}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators $T_{q^2}$, $q$ prime and $(q, 2p) = 1$. Then

$$U_p f = -p^{k-1}\lambda(p)f, \quad U_4 f = -2^{k-1}\lambda(2)f,$$

where $\lambda(p) = \pm1$ and $\lambda(2) = \pm1$ are defined as above.

Proof Let $g = U_p f$. Then $\text{Sh}_t(g) = U_p \text{Sh}_t(f) = -p^{k-1}\lambda(p)\text{Sh}_t(f)$ for every positive square-free integer $t$. It follows that $\text{Sh}_t(g - p^{k-1}\lambda(p)f) = 0$ for all such $t$ implying $g - p^{k-1}\lambda(p)f = 0$, which is what we need. For the prime 2, the proof is the same.

Proposition 6.18 Let $f \in S_{k+1/2}^{-}(\Gamma_0(4p))$. Then $\widetilde{Q}_p f = -f = \widetilde{Q}_p' f$ and $\widetilde{Q}_2 f = -f = \widetilde{Q}_2' f$.

Proof Let $f \in S_{k+1/2}^{-}(\Gamma_0(4p))$ be a Hecke eigenform for all the operators $T_{q^2}$, $(q, 2p) = 1$. Since $\widetilde{Q}_p = (1/p)^k p^{1-k} U_p \widetilde{W}_p$ and $\widetilde{Q}_2 = 2^{1-k} U_4 \widetilde{W}_4$, it follows from Corollary 6.16 and Proposition 6.17 that $f$ is an eigenform for the operators $\widetilde{Q}_p, \widetilde{Q}_p'$, $\widetilde{Q}_2$, and $\widetilde{Q}_2'$ with eigenvalues $\pm1$. However, the eigenvalues of $\widetilde{Q}_p, \widetilde{Q}_p'$ are $p$ and $-1$, respectively.
and the eigenvalues of $\tilde{Q}$ and $\tilde{Q}'$ are 2 and −1, hence the eigenvalues have to be −1. Since $S_{k+1/2}(\Gamma_0(4p))$ has a basis of such eigenforms, we get the result.

**Theorem 6.19** Let $f \in S_{k+1/2}(\Gamma_0(4p))$. Then $f \in S_{k+1/2}^-(\Gamma_0(4p))$ if and only if 
$$\tilde{Q}_p(f) = -f = \tilde{Q}'_p(f) \text{ and } \tilde{Q}_2(f) = -f = \tilde{Q}'_2(f).$$

**Proof** If $f \in S_{k+1/2}^-(\Gamma_0(4p))$, then by Proposition 6.18 the conditions hold. Now assume that $f \in S_{k+1/2}(\Gamma_0(4p))$ is in the intersection of −1-eigenspaces of $\tilde{Q}_p$, $\tilde{Q}'_p$, $\tilde{Q}_2$, and $\tilde{Q}'_2$. For every $g \in S_{k+1/2}(\Gamma_0(4))$, we have $\tilde{Q}_p(g) = pg$. Since $\tilde{Q}_p$ is self-adjoint,
$$-(f, g) = (\tilde{Q}_p f, g) = (f, \tilde{Q}_p g) = p(f, g),$$

implying $(f, g) = 0$. Thus, $f$ is orthogonal to $R \oplus S_{k+1/2}^-(\Gamma_0(4))$. For every $g \in \tilde{W}'_p; S_{k+1/2}(\Gamma_0(4))$, we have $\tilde{Q}'_p(g) = pg$, and the same argument shows that $(f, g) = 0$ implying $f$ is orthogonal to $\tilde{W}'_p; (R \oplus S_{k+1/2}^-(\Gamma_0(4)))$. Since Kohnen's plus space is the 2-eigenspace of $\tilde{Q}'_2$, for $g \in S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p))$ we have $\tilde{Q}'_2(g) = 2g$; consequently, for $g \in \tilde{W}_4 S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p))$, we have $Q_2(g) = 2g$. Hence, $(f, g) = 0$ for such $g$; that is, $f$ is orthogonal to $S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p)) \oplus \tilde{W}_4 S_{k+1/2}^{+,\text{new}}(\Gamma_0(4p))$. It follows that $f \in S_{k+1/2}(\Gamma_0(4p))$.

**6.3 Minus Space for $\Gamma_0(4M)$ for $M$ Odd and Square-free**

Let $M \neq 1$ be an odd and square-free natural number. Write $M = p_1 p_2 \cdots p_k$. For each $i = 1, \ldots, k$ let $M_i = M/p_i$. Since $S_{k+1/2}(\Gamma_0(4M_i))$ is contained in the $p_i$-eigenspace of $\tilde{Q}_p$, (Corollary 4.3(4)), following the proof of Proposition 6.7 we obtain the following proposition.

**Proposition 6.20** $S_{k+1/2}(\Gamma_0(4M_i)) \cap \tilde{W}'_p; S_{k+1/2}(\Gamma_0(4M)) = \{0\}$.

**Corollary 6.21** The Niwa map $\psi : S_{k+1/2}(\Gamma_0(4M)) \to S_{2k}(\Gamma_0(2M))$ maps $S_{k+1/2}^-(\Gamma_0(4M)) \oplus \tilde{W}'_p; S_{k+1/2}(\Gamma_0(4M))$ isomorphically onto $S_{2k}(\Gamma_0(2M)) \oplus V(p_1) S_{2k}(\Gamma_0(2M))$.

Let $S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M))$ be the new space inside the Kohnen plus subspace of $S_{k+1/2}(\Gamma_0(4M))$. Then similarly we have the following proposition.

**Proposition 6.22** $S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M)) \cap \tilde{W}_4 S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M)) = \{0\}$.

**Corollary 6.23** $\psi$ maps $S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M)) \oplus \tilde{W}_4 S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M))$ isomorphically onto $S_{2k}^{\text{new}}(\Gamma_0(M)) \oplus V(2) S_{2k}^{\text{new}}(\Gamma_0(M))$.

We let $B_i = S_{k+1/2}(\Gamma_0(4M_i)) \oplus \tilde{W}'_p; S_{k+1/2}(\Gamma_0(4M_i))$, $i = 1, \ldots, k$. Define
$$E := \sum_{i=1}^k B_i \oplus S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M)) \oplus \tilde{W}_4 S_{k+1/2}^{+,\text{new}}(\Gamma_0(4M)).$$
Proposition 6.24  Under $\psi$, the space $E$ maps isomorphically onto the old space $S_{2k}^{\text{old}}(\Gamma_0(2M))$.

Proof  This follows from Corollaries 6.21 and 6.23 and from the decomposition

$$S_{2k}^{\text{old}}(\Gamma_0(2M)) = \left( \sum_{i=1}^{k} S_{2k}(\Gamma_0(2M_i)) \oplus V(p_i)S_{2k}(\Gamma_0(2M_i)) \right) \oplus \left( S_{2k}^{\text{new}}(\Gamma_0(M)) \oplus V(2)S_{2k}^{\text{new}}(\Gamma_0(M)) \right).$$

We now define the minus space to be the orthogonal complement of $E$, under the Petersson inner product, that is,

$$S_{k+1/2}^{-}(\Gamma_0(4M)) := E^\perp.$$

Let $f \in S_{k+1/2}^{-}(\Gamma_0(4M))$ be a Hecke eigenform for all the operators $T_{q^2}$ where $q$ is an odd prime satisfying $(q, M) = 1$. Let $\psi(f) = F$. The proofs of the following results are identical to the proofs in the previous subsections.

Proposition 6.25  $F$ is up to a scalar a primitive Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(2M))$.

Theorem 6.26  The space $S_{k+1/2}^{-}(\Gamma_0(4M))$ has a basis of eigenforms for all the operators $T_{q^2}$ where $q$ is an odd prime satisfying $(q, M) = 1$. Under $\psi$, the space $S_{k+1/2}^{-}(\Gamma_0(4M))$ maps isomorphically onto the space $S_{2k}^{\text{new}}(\Gamma_0(2M))$. If two forms in $S_{k+1/2}^{-}(\Gamma_0(4M))$ have the same eigenvalues for all the operators $T_{q^2}$, $(q, 2M) = 1$, then they are same up to a scalar factor.

In particular, the minus space $S_{k+1/2}^{-}(\Gamma_0(4M))$ has strong multiplicity one property in the full space; that is, if $f_1$ and $f_2$ are Hecke eigenforms in $S_{k+1/2}^{-}(\Gamma_0(4M))$ with the same eigenvalues for all $T_{q^2}$, $(q, 2M) = 1$ and if $f_1$ is a nonzero element of the minus space $S_{k+1/2}^{-}(\Gamma_0(4M))$, then $f_2$ is a scalar multiple of $f_1$.

Remark 6.27  Our results in Theorems 6.5, 6.15, and 6.26 give another proof of [9, Theorem 5]. We note that in [9] the old space is defined using the operators $U_{p^2}$ for $p | 2M$, while our definition uses Atkin–Lehner type operators $\tilde{W}_{p^2}$. The operators $U_{p^2}$, $\tilde{W}_{p^2}$ and $\tilde{Q}_p$ come from the local Hecke algebra element corresponding to the double cosets of $(h(p), 1)$, $(w(p^{-1}), 1)$ and $(w(1), 1)$, respectively, and our proofs essentially depend on relations among these operators that we derive from the local Hecke algebra. Since $S^+(4)$ is the 2-eigenspace of $\tilde{Q}_p$, we indeed have $S^+(4) = \tilde{Q}_p^2S^+(4) = \tilde{W}_4U_4S^+(4)$, which implies equality of spaces, $U_4S^+(4) = \tilde{W}_4S^+(4) = A^+(4)$. Thus, $U_4\tilde{W}_4A^+(4) = A^+(4)$. However, $U_4A^+(4)$ need not equal $S^+(4)$ as noted in Example 6.33 in the next subsection. In the case of odd primes $p$, dividing $M$, the space $S_{k+1/2}^{-}(\Gamma_0(4M))$ is contained in the $p$-eigenspace of $\tilde{Q}_p$, which in particular implies that $U_{p_1}W_{p_1}S_{k+1/2}^{-}(\Gamma_0(4M_i)) = S_{k+1/2}^{-}(\Gamma_0(4M_i))$, but as before we do not expect the spaces $U_{p_1}W_{p_1}S_{k+1/2}^{-}(\Gamma_0(4M_i))$ and $\tilde{W}_{p_1}S_{k+1/2}^{-}(\Gamma_0(4M_i))$ to be equal inside $S_{k+1/2}^{-}(\Gamma_0(4M_i))$. We illustrate this using the following reasoning, which needs to be proved. Consider the simple case $M = 4p$, $p$ an odd prime. In this case, if $U_{p^2}S^-(4) = \tilde{W}_{p^2}S^-(4)$, then the corresponding picture in the integral
weight should be $U_p S_{2k}^{\text{new}}(\Gamma_0(2)) = W_p S_{2k}^{\text{new}}(\Gamma_0(2)) = V_p S_{2k}^{\text{new}}(\Gamma_0(2))$ (where the last equality was shown in [2]). If $S_{2k}^{\text{new}}(\Gamma_0(2))$ is non-zero, then the action of $U_p$ (see Lemma 5.1) and the fact that $S_{2k}^{\text{new}}(\Gamma_0(2)) \cap V_p S_{2k}^{\text{new}}(\Gamma_0(2)) = \{0\}$ leads to a contradiction. Since representation theoretically $A^+(4)$ corresponds to $S_{2k}(\Gamma_0(1))$, using the same reasoning, we do not expect the spaces $U_4 A^+(4)$ and $S^+(4)$ to be equal.

Let $f \in S_{k+1/2}^-(\Gamma_0(4M))$ be a Hecke eigenform for all the operators $T_{q^2}, (q, 2M) = 1$. Then $\psi(f) = F$ is a Hecke eigenform in $S_{2k}^{\text{new}}(\Gamma_0(2M))$ for all operators $T_q, (q, 2M) = 1$. By [1], for all primes $p$ such that $p|M$, $U_p(F) = -p^{-k-1}\lambda(p)F$ where $\lambda(p) = \pm 1$ and $U_2(F) = -2^{k-1}\lambda(2)F$ where $\lambda(2) = \pm 1$.

**Proposition 6.28** Let $f \in S_{k+1/2}^-(\Gamma_0(4M))$ be a Hecke eigenform for all the operators $T_{q^2}, q$ prime, $(q, 2M) = 1$. Then for all primes $p$ such that $p|M$,

$$U_p(f) = -p^{-k-1}\lambda(p)f \quad \text{and} \quad U_2(f) = -2^{k-1}\lambda(2)f,$$

where $\lambda(p) = \pm 1$ and $\lambda(2) = \pm 1$ are defined as above.

Following [14, Theorem 1.9] we have the following corollary.

**Corollary 6.29** Let $f = \sum_{n=1}^{\infty} a_n q^n \in S_{k+1/2}^-(\Gamma_0(4M))$ be a Hecke eigenform for all Hecke operators, i.e., $T_{q^2}(f) = \omega_q f$ for all primes $(q, 2M) = 1$ and $U_p(f) = \omega_p f$ for all primes $p|2M$. Let $F = \sum_{n=0}^{\infty} A_n q^n \in S_{2k}^{\text{new}}(\Gamma_0(2M))$ be the unique normalized primitive form determined by $f$, i.e., $A_p = \omega_p$ for all primes $p$. Then for a fundamental discriminant $D$ such that $(-1)^k D > 0$,

$$L(s - k + 1, \left(\frac{D}{n}\right)) \sum_{n=1}^{\infty} a_n |n^2| n^{-s} = a(|D|) \sum_{n=1}^{\infty} A_n n^{-s}.$$

We finally give the characterization of our minus space. The proofs of the following proposition and theorem are as before.

**Proposition 6.30** Let $f \in S_{k+1/2}^-(\Gamma_0(4M))$. Then for every prime $p$ dividing $M$ we have $\tilde{Q}_p(f) = -f = \tilde{Q}_p'(f)$ and $\tilde{Q}_2(f) = -f = \tilde{Q}_2'(f)$.

**Theorem 6.31** Let $f \in S_{k+1/2}^-(\Gamma_0(4M))$. Then $f \in S_{k+1/2}^-(\Gamma_0(4M))$ if and only if $\tilde{Q}_p(f) = -f = \tilde{Q}_p'(f)$ for every prime $p$ dividing $M$ and $\tilde{Q}_2(f) = -f = \tilde{Q}_2'(f)$.

### 6.4 Some Examples

We complete this section by giving two examples. For simplicity we shall denote plus and minus spaces $S_{k+1/2}^+(\Gamma_0(4M))$ and $S_{k+1/2}^-(\Gamma_0(4M))$ by $S_{k+1/2}^+(4M)$ and $S_{k+1/2}^-(4M)$.

We use Shimura decomposition [15] and recall the following notation: for a primitive Hecke eigenform $F$ of weight $2k$ and level dividing $2M$, $S_{k+1/2}(4M, F)$ denotes the subspace of $S_{k+1/2}(\Gamma_0(4M))$ consisting of forms that are Shimura-equivalent to $F$. 


(i.e., forms \( f \) that are eigenforms under \( T_p^2 \) with the same eigenvalues as \( F \) under \( T_p \) for almost all odd primes \( p \) coprime to \( M \)).

**Example 6.32**  The space \( S_{3/2}(\Gamma_0(28)) \) is one dimensional and is spanned by
\[
f = q - q^2 - q^4 + q^7 + q^8 - q^9 + q^{14} - 2q^{15} + q^{16} + 3q^{18} - 2q^{21} + \cdots.\]
Then by Shimura decomposition,
\[
S_{3/2}(\Gamma_0(28)) = \bigoplus_{\Gamma \in S^\text{new}_2(\Gamma_0(M))} S_{3/2}(28, F) = S_{3/2}(28, F_{14}),
\]
as there are no primitive Hecke eigenforms of weight 2 at level 1, 2, 7, and \( F_{14} \in S^\text{new}_2(\Gamma_0(14)) \) is the only primitive Hecke eigenform at level 14. In particular, we have \( S^+_{3/2}(28) = \{0\} \) and \( S^-_{3/2}(28) = S_{3/2}(\Gamma_0(28)) = \{f\} \).

**Example 6.33**  The space \( S_{17/2}(\Gamma_0(12)) \) is 13-dimensional. We first give the Shimura decomposition of \( S_{17/2}(\Gamma_0(12)) \). We note that there are seven primitive Hecke eigenforms of weight 16 and level dividing 6, namely, \( F_1 \) of level 1, \( G_2 \) of level 2, \( H_3 \), \( K_3 \) of level 3 each and \( L_6, M_6, N_6 \) each of level 6. Using Shimura decomposition algorithm in [12] we have

\[
(6.1) \quad S_{17/2}(\Gamma_0(12)) = S_{17/2}(12, F_1) \oplus S_{17/2}(12, G_2) \oplus S_{17/2}(12, H_3) \oplus S_{17/2}(12, K_3) \oplus S_{17/2}(12, L_6) \oplus S_{17/2}(12, M_6) \oplus S_{17/2}(12, N_6),
\]
where \( S_{17/2}(12, F_1) \) is the four-dimensional space spanned by
\[
\begin{align*}
f_1 &= q + 88q^4 + 513q^9 + 3024q^{12} - 4368q^{13} - 13760q^{16} + 33264q^{21} + \cdots, \\
f_2 &= 11q^2 + 64q^4 + 232q^7 - 1408q^8 + 4608q^9 + 190q^{10} - 6578q^{11} + \cdots, \\
f_3 &= 9q^3 - 64q^4 + 189q^6 - 232q^7 - 190q^{10} + 1152q^{12} - 3328q^{13} + \cdots, \\
f_4 &= q^5 - 11q^8 + 18q^9 - 9q^{12} - 116q^{17} + 344q^{20} - 99q^{21} - 189q^{24} + \cdots;
\end{align*}
\]
the space \( S_{17/2}(12, G_2) \) is two-dimensional and is spanned by
\[
\begin{align*}
g_1 &= q + 21q^3 - 128q^4 - 609q^6 + 3192q^7 + 5313q^9 - 12810q^{10} + \cdots, \\
g_2 &= 3q^2 + 7q^3 - 203q^6 - 384q^8 - 416q^9 + 2706q^{11} - 896q^{12} + \cdots;
\end{align*}
\]
the space \( S_{17/2}(12, H_3) \) is two-dimensional and is spanned by
\[
\begin{align*}
h_1 &= q^5 + 7q^8 - 27q^{12} - 80q^{17} + 56q^{20} + 189q^{21} + 81q^{24} + 231q^{29} + \cdots, \\
h_2 &= 7q^2 - 27q^3 + 81q^6 - 896q^8 + 854q^{11} + 3456q^{12} - 1876q^{14} + \cdots;
\end{align*}
\]
the space \( S_{17/2}(12, K_3) \) is two-dimensional and is spanned by
\[
\begin{align*}
k_1 &= q - 362q^4 - 2187q^9 - 11826q^{12} + 19032q^{13} + 51940q^{16} + \cdots, \\
k_2 &= 1971q^3 + 13184q^4 + 31266q^6 - 20158q^7 + 271340q^{10} + \cdots;
\end{align*}
\]
the last three summands in (6.1) are one-dimensional, each with \( S_{17/2}(12, L_6) \) spanned by
\[
\begin{align*}
l_1 &= 13q^2 + 129q^3 + 736q^5 + 1323q^6 + 1664q^8 + 5918q^{11} + 16512q^{12} + \cdots;
\end{align*}
\]
the space \( S_{17/2}(12, M_6) \) spanned by
\[
m_1 = q^3 - 18q^6 - 42q^7 - 12q^{10} + 128q^{12} + 384q^{13} - 126q^{15} - 1074q^{19} + 896q^{21} + \cdots;
\]
and the space \( S_{17/2}(12, N_6) \) spanned by
\[
n_1 = 16q - 1539q^3 - 2048q^4 - 5994q^6 - 50178q^7 - 34992q^9 - 2460q^{10} + \cdots.
\]
We can also check (using bound in [7]) that the Kohnen's plus space \( S^+_{17/2}(12) \) is four-dimensional. Indeed,
\[
S^+_{17/2}(12) = \langle f_1, f_4, h_1, k_1 \rangle = S^+_{17/2}(4) \oplus \tilde{W}_9 S^+_{17/2}(4) \oplus S^+_{17/2}(12),
\]
with \( S^+_{17/2}(4) = \langle f_1 - 336f_4 \rangle \) and \( S^+_{17/2}(12) = \langle h_1, k_1 \rangle \). Note that from Remark 6.27, \( A^+_{17/2}(4) = U_4(S^+_{17/2}(4)) \), so \( A^+_{17/2}(4) = \langle U_4(f_1 - 336f_4) \rangle = \langle 88f_1 + 336f_2 + 672f_3 - 115584f_4 \rangle \) and \( S^+_{17/2}(4) = \langle g_1 + 3g_2 \rangle \) (again we use Shimura decomposition algorithm to get the explicit forms in \( S^+_{17/2}(4) \) and \( S^+_{17/2}(4) \)). One can further check that \( U_4(A^+_{17/2}(4)) \) does not equal \( S^+_{17/2}(4) \); indeed, \( A^+_{17/2}(4) \) is spanned by a form with \( q \)-expansion given by
\[
88q + 3696q^2 + 6048q^3 - 13760q^4 - 115584q^5 + 127008q^6 - 77952q^7 + 798336q^8 + \cdots,
\]
and so
\[
U_4(A^+_{17/2}(4)) = \langle -13760q + 798336q^2 + 1306368q^3 - 5855744q^4 + \cdots \rangle,
\]
which is clearly not equal to \( S^+_{17/2}(4) \).

Thus, we have
\[
S_{17/2}(12, F_1) = R \oplus \tilde{W}_9 R, \quad \text{where} \quad R = S^+_{17/2}(4) \oplus A^+_{17/2}(4),
\]
\[
S_{17/2}(12, G_2) = S^+_{17/2}(4) \oplus \tilde{W}_9 S^+_{17/2}(4),
\]
\[
S_{17/2}(12, H_3) \oplus S^+_{17/2}(12, K_3) = S^+_{17/2}(12) \oplus \tilde{W}_4 S^+_{17/2}(12),
\]
\[
S_{17/2}(12, L_6) \oplus S^+_{17/2}(12, M_6) \oplus S^+_{17/2}(12, N_6) = \langle l_1, m_1, n_1 \rangle = S^+_{17/2}(12).
\]

**Remark 6.34**

(i) In general, \( S^+_{k+1/2}(\Gamma_0(4M)) = \bigoplus_F S^+_{k+1/2}(4M, F) \), where \( F \) runs through all primitive Hecke eigenforms of weight \( 2k \) and level \( 2M \).

(ii) The Kohnen plus space is given by a well-known Fourier coefficient condition. But we do not expect any such Fourier coefficient condition for forms in our minus space, as is also evident from the above examples. We note that in [17], Ueda and Yamana define generalized Kohnen plus space of level \( 8M \) and show that the newspaces inside this plus space is Hecke isomorphic to \( S^2_{2k}(\Gamma_0(2M)) \). In [3], we obtain a self-adjoint involution on \( S^+_{k+1/2}(\Gamma_0(8M)) \) coming from an element in a certain 2-adic Hecke algebra of \( \tilde{S}_L \) of level 8 that is not inside the corresponding 2-adic Hecke algebra of \( \tilde{S}_L \) of level 4. We observe that the plus space defined by Ueda-Yamana is precisely the \( +1 \)-eigenspace of this involution and that their plus newspaces is a "conjugate" of \( S^+_{k+1/2}(\Gamma_0(4M)) \). We define the minus space at level \( 8M \) and show that this space is contained inside the \( -1 \)-eigenspace of the involution and hence satisfy a Fourier coefficient condition that is exactly opposite to the Kohnen's plus space.
Fourier coefficient condition. Since this involution on $S_{k+1/2}(\Gamma_0(8M))$ does not preserve the space $S_{k+1/2}(\Gamma_0(4M))$, we do not expect Fourier coefficient condition for $S_{k+1/2}(\Gamma_0(4M))$. For more details, please refer to [3].

A Some Observations on Cocycle Multiplication

Let $p$ denote any prime. In this appendix we note down some useful observations on the multiplication in $\tilde{\text{SL}}_2(\mathbb{Q}_p)$ by cocycle $\sigma_p$.

Recall the Hilbert symbol $(\cdot, \cdot)_p$ defined on $\mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$. For an odd prime $p$ it can be given by the following formula: For $a, b$ coprime to $p$,

$$(p^sa, p^tb)_p = \left(\frac{-1}{p}\right)^s \left(\frac{a}{p}\right)^t \left(\frac{b}{p}\right)^s.$$

Thus, $(p, p)_p = (\frac{-1}{p})$ and $(-p, u)_p = (p, u)_p = (\frac{u}{p})$, where $u$ is a unit in $\mathbb{Z}_p$. For the prime 2, if $a, b$ are odd, then

$$(2^sa, 2^tb)_2 = (-1)^{(a-1)(b-1)/2} \left(\frac{2}{|a|}\right)^t \left(\frac{2}{|b|}\right)^s.$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Q}_p)$. For $(A, \varepsilon_1) \in \tilde{\text{SL}}_2(\mathbb{Q}_p)$, $(A, \varepsilon_1)^{-1} = (A^{-1}, \varepsilon_1 \sigma_p(A, A^{-1}))$, where

(i) if $c = 0$, then $\sigma_p(A, A^{-1}) = (a, a)_p = (d, d)_p$.
(ii) if $c \neq 0$ and $\text{ord}_p(c)$ is even, then $\sigma_p(A, A^{-1}) = 1$.
(iii) if $c \neq 0$ and $\text{ord}_p(c)$ is odd, then

$$\sigma_p(A, A^{-1}) = \begin{cases} 
(c, d)_p & \text{if } d \neq 0, \ a \neq 0, \\
(c, d)_p & \text{if } d \neq 0, \ a = 0, \\
(-c, a)_p & \text{if } d = 0, \ a \neq 0, \\
1 & \text{if } d = 0, \ a = 0.
\end{cases}$$

In particular, if $A \in \{ x(p^n), y(p^n), w(p^n) \}_{n \in \mathbb{Z}}$, then $\sigma_p(A, A^{-1}) = 1$. For $A = h(p^n)$ with $n \in \mathbb{Z}$, if $p = 2$, then $\sigma_p(A, A^{-1}) = 1$; however, if $p$ is an odd prime, then

$$\sigma_p(A, A^{-1}) = \begin{cases} 
1 & \text{if } n \text{ even}, \\
\left(\frac{-1}{p}\right) & \text{otherwise}.
\end{cases}$$

Let $(A, \varepsilon_1), (B, \varepsilon_2) \in \tilde{\text{SL}}_2(\mathbb{Q}_p)$. The following lemmas can be easily obtained using the cocycle formula.

**Lemma A.1** We have $[(B, \varepsilon_2)^{-1}, (A, \varepsilon_1)^{-1}] = (B^{-1}A^{-1}BA, \xi)$, where $\xi = \sigma_p(A, A^{-1}) \sigma_p(B, B^{-1}) \sigma_p(B, A) \sigma_p(A^{-1}, BA) \sigma_p(B^{-1}, A^{-1}BA)$. 
Lemma A.2  The $\sigma_p$-factor (\(\xi\) factor above) of \([(B, e_2)^{-1}, (A, e_1)^{-1}]\) equals the product
\[
(\tau(B), \tau(B^{-1}))_p \cdot (\tau(A), \tau(A^{-1}))_p \cdot (\tau(BA)\tau(B), \tau(BA)\tau(A))_p \\
\cdot \left(\tau(A^{-1}BA)\tau(A^{-1}), \tau(A^{-1}BA)\tau(BA)\right)_p \\
\cdot \left(\tau(B^{-1}A^{-1}BA)\tau(B^{-1}), \tau(B^{-1}A^{-1}BA)\tau(A^{-1}BA)\right)_p \cdot s_p(B^{-1}A^{-1}BA).
\]

In the proofs for checking the support of our local Hecke algebra (Section 3) we need the following lemma.

Lemma A.3  Let \(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Q}_p)\).

(i) If \(B = x(s)\), where \(s \neq 0\), then \(\sigma_p\)-factor is
\[
\begin{cases} 
(-sc^2, 1 - cds)_p & \text{if } sc^2(1 - cds) \neq 0 \text{ and } \text{ord}_p(s) \text{ is odd,} \\
1 & \text{otherwise.}
\end{cases}
\]

(ii) If \(B = h(u)\), where \(u \neq \pm 1\), then \(\sigma_p\)-factor is
\[
\begin{cases} 
(ac(1 - u^2), 1 + (1 - u^2)bc)_p & \text{if } ac(1 - u^2)(1 + (1 - u^2)bc) \neq 0 \text{ and } \text{ord}_p(ac(1 - u^2)) \text{ is odd,} \\
1 & \text{otherwise.}
\end{cases}
\]

(iii) If \(B = y(t)\), where \(t \neq 0\), then \(\sigma_p\)-factor is
\[
\begin{cases} 
((a^2 - 1)t + abt^2, 1 + abt + b^2t^2)_p & \text{if } ((a^2 - 1)t + abt^2)(1 + abt + b^2t^2) \neq 0 \text{ and } \text{ord}_p((a^2 - 1)t + abt^2) \text{ is odd,} \\
1 & \text{otherwise.}
\end{cases}
\]

In each of the above cases, the \(\sigma_p\)-factor is simply \(s_p(B^{-1}A^{-1}BA)\).

Proof  For (i) let \(B = x(s)\), where \(s \neq 0\). Then we have
\[
BA = \begin{pmatrix} a + sc & b + sd \\ c & d \end{pmatrix}, \quad A^{-1}BA = \begin{pmatrix} 1 + cds & sd^2 \\ -sc^2 & 1 - cds \end{pmatrix},
\]
\[
B^{-1}A^{-1}BA = \begin{pmatrix} 1 + cds + s^2c^2 & sd^2 - s + cds^2 \\ -sc^2 & 1 - cds \end{pmatrix}.
\]

It is easy to see that \((\tau(B), \tau(B^{-1})) = 1\) and that
\[
\left(\tau(A), \tau(A^{-1})\right) = \left(\tau(A^{-1}BA)\tau(A^{-1}), \tau(A^{-1}BA)\tau(BA)\right)_p \\
= \begin{cases} 
1 & \text{if } c \neq 0, \\
(d, a)_p & \text{otherwise.}
\end{cases}
\]

Further, one can check that \((\tau(BA)\tau(B), \tau(BA)\tau(A)) = 1\) and also
\[
\left(\tau(B^{-1}A^{-1}BA)\tau(B^{-1}), \tau(B^{-1}A^{-1}BA)\tau(A^{-1}BA)\right)_p = 1.
\]
Finally, we have
\[ s_p(B^{-1}A^{-1}BA) = \begin{cases} (-sc^2, 1 - cds) & \text{if } sc^2(1 - cds) \neq 0 \text{ and } \text{ord}_p(s) \text{ is odd}, \\ 1 & \text{otherwise.} \end{cases} \]

By using Lemma A.2, multiplying all the above terms, we get the required \( \sigma_p \)-factor.

For (ii) we proceed similarly. Let \( B = h(u) \), where \( u \neq \pm 1 \). Then
\[
BA = \begin{pmatrix} ua & ub \\ u^{-1}c & u^{-1}d \end{pmatrix}, \quad A^{-1}BA = \begin{pmatrix} uad - u^{-1}bc & bd(u - u^{-1}) \\ ac(u^{-1} - u) & u^{-1}ad - ubc \end{pmatrix},
\]
\[
B^{-1}A^{-1}BA = \begin{pmatrix} 1 + (1 - u^{-2})bc & bd(1 - u^{-2}) \\ ac(1 - u^2) & 1 + (1 - u^2)bc \end{pmatrix}.
\]

We have \( \left( \tau(B), \tau(B^{-1}) \right)_p = (u, u^{-1})_p \). Also, \( (\tau(A), \tau(A^{-1})) = 1 \text{ if } c \neq 0 \text{ and } (d, a)_p \text{ otherwise.} \)

We check that
\[
\left( \tau(BA) \tau(B), \tau(BA) \tau(A) \right)_p = \begin{cases} (c, u^{-1}) & \text{if } c \neq 0, \\ (d, u^{-1}) & \text{otherwise,} \end{cases}
\]
\[
\left( \tau(A^{-1}BA) \tau(A^{-1}), \tau(A^{-1}BA) \tau(BA) \right)_p
\]
\[
= \begin{cases} (-a(u^{-1} - u), u^{-1}) & \text{if } ac \neq 0 \\ (bu, -b)_p & \text{if } a = 0 \text{ and } c \neq 0 \\ (du^{-1}, a)_p & \text{if } a \neq 0 \text{ and } c = 0, \end{cases}
\]
\[
\left( \tau(B^{-1}A^{-1}BA) \tau(B^{-1}), \tau(B^{-1}A^{-1}BA) \tau(A^{-1}BA) \right)_p
\]
\[
= \begin{cases} (ac(u^{-1} - u), u^{-1}) & \text{if } ac \neq 0 \\ (bc, u)_p = (-1, u)_p & \text{if } a = 0 \text{ and } c \neq 0 \\ (-ad, u)_p & \text{if } a \neq 0 \text{ and } c = 0, \end{cases}
\]
and
\[
s_p(B^{-1}A^{-1}BA) = \begin{cases} \left( ac(1 - u^2), 1 + (1 - u^2)bc \right)_p & \text{if } ac(1 - u^2)(1 + (1 - u^2)bc) \neq 0 \text{ and } \text{ord}_p(ac(1 - u^2)) \text{ is odd}, \\ 1 & \text{otherwise.} \end{cases}
\]

Again, by multiplying all the above terms we get the required \( \sigma_p \)-factor.

For (iii), let \( B = y(t) \), where \( t \neq 0 \). Then
\[
BA = \begin{pmatrix} a & b \\ at + c & bt + d \end{pmatrix}, \quad A^{-1}BA = \begin{pmatrix} 1 - abt & -b^2t \\ a^2t & 1 + abt \end{pmatrix},
\]
\[
B^{-1}A^{-1}BA = \begin{pmatrix} 1 - abt & -b^2t \\ (a^2 - 1)t + abt^2 & 1 + abt + b^2t^2 \end{pmatrix}.
\]
As before, $(\tau(B), \tau(B^{-1})) = (t, -t) \neq (1, -1)$, and $(\tau(A), \tau(A^{-1})) = 1$ if $c \neq 0$ and $(d, a) \neq (1, -1)$ otherwise. One can compute (using $ad - bc = 1$ in the Hilbert symbol calculations) that

$$\left( \tau(BA) \tau(B), \tau(BA) \tau(A) \right) = \begin{cases} 
(t(at + c), -ct) & \text{if } a \neq -c/t \text{ and } c \neq 0, \\
(-c, a) & \text{if } a = -c/t \text{ and } c \neq 0, \\
(a, -dt) & \text{if } c = 0,
\end{cases}$$

and

$$\left( \tau(A^{-1}BA) \tau(A^{-1}), \tau(A^{-1}BA) \tau(BA) \right) = \begin{cases} 
(t(at + c), -ct) & \text{if } a \neq -c/t \text{ and } c \neq 0 \text{ and } a \neq 0, \\
1 & \text{if } a \neq -c/t \text{ and } c \neq 0 \text{ and } a = 0, \\
(-c, a) & \text{if } a = -c/t \text{ and } c \neq 0, \\
(a, at) & \text{if } c = 0.
\end{cases}$$

All the above factors clearly multiply to 1. Also it turns out that

$$\left( \tau(B^{-1}A^{-1}BA), \tau(B^{-1}A^{-1}BA) \tau(A^{-1}BA) \right) = 1,$$

so we get the required $\sigma_p$-factor.

We also note the triangular decomposition of $K_0^p(p^n)$.

**Lemma A.4** We have a triangular decomposition

$$\overline{K_0^p(p^n)} = N_K^0(p^n) \cap K_0^p(p^n) \cap K_0^p(p^n).$$

More precisely, for $(A, e) = \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, e \right) \in \overline{K_0^p(p^n)}$, $$(A, e) = (x(s), 1)(h(u), 1)(y(t), 1)(I, e\delta)$$

where

$$u = d^{-1}, \quad s = d^{-1}b, \quad t = d^{-1}c,$$

and

$$\delta = \begin{cases} 
1 & c = 0, \\
(d, -1) & c \neq 0, \text{ord}_p(c) \text{ is odd}, \\
(-c, d) & c \neq 0, \text{ord}_p(c) \text{ is even}.
\end{cases}$$

**Proof** Clearly,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & bd^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & cd^{-1} \end{pmatrix}.$$ 

Let $u = d^{-1}$, $s = bd^{-1}$, $t = cd^{-1}$. Since

$$x(s)h(u)y(t) = \begin{pmatrix} u & su^{-1} \\ 0 & u^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} u + su^{-1}t & su^{-1} \\ tu^{-1} & u^{-1} \end{pmatrix},$$

we get that

$$(x(s), 1)(h(u), 1)(y(t), 1) = (x(s)h(u)y(t), \delta) = (A, \delta),$$
where

\[ \delta = \sigma(x(s), h(u)) \sigma(x(s)h(u), y(t)) = \begin{cases} 1 & t = 0, \\ (u, -1)_p & t \neq 0, \text{ord}_p(t) \text{ is odd}, \\ (t, u)_p & t \neq 0, \text{ord}_p(t) \text{ is even}. \end{cases} \]

Substituting \( u, s, t \) in terms of \( b, c, d \), we get \( \delta \) as in the statement. ■

References

[1] A. O. L. Atkin and J. Lehner, *Hecke operators on \( \Gamma_0(m) \).* Math. Ann. 185(1970), 134–160. https://doi.org/10.1007/BF01359701

[2] E. M. Baruch and S. Purkait, *Hecke algebras, new vectors and newforms on \( \Gamma_0(m) \).* Math. Zeit. 287(2017), 705–733. https://doi.org/10.1007/s00209-017-1842-y

[3] E. M. Baruch and S. Purkait, *Newforms of half-integral weight: the minus space of \( S_{k+1/2}(\Gamma_0(8M)) \).* Israel J. Math. 232(2019), 41–73. https://doi.org/10.1007/s11856-019-1873-7

[4] S. Gelbart, *Weil's representation and the spectrum of the metaplectic group.* Lecture Notes in Mathematics, 530, Springer-Verlag, Berlin, 1976.

[5] W. Kohnen, *Modular forms of half-integral weight on \( \Gamma_0(4) \).* Math. Ann. 248(1980), 249–266. https://doi.org/10.1007/BF01420529

[6] W. Kohnen, *Newforms of half-integral weight.* J. Reine Angew. Math. 333(1982), 32–72. https://doi.org/10.1515/crll.1982.333.32

[7] N. Kumar and S. Purkait, *A note on the Fourier coefficients of half-integral weight modular forms.* Arch. Math. (Basel) 102(2014), no. 4, 369–378. https://doi.org/10.1007/s00013-014-0622-8

[8] H. Y. Loke and G. Savin, *Representations of the two-fold central extension of \( \text{SL}_2(\mathbb{Q}_2) \).* Pacific J. Math. 247(2010), 435–454. https://doi.org/10.2140/pjm.2010.247.435

[9] M. Manickam, B. Ramakrishnan, and T. Vasudevan, *On the theory of newforms of half-integral weight.* J. Number Theory 34(1990), 210–224. https://doi.org/10.1016/0022-001X(90)90151-G

[10] S. Niwa, *On Shimura's trace formula.* Nagoya Math. J. 66(1977), 183–202.

[11] S. Purkait, *On Shimura's decomposition.* Int. J. Number Theory 9(2013), 1431–1445. https://doi.org/10.1142/S179304211350036X

[12] S. Purkait, *Hecke operators in half-integral weight.* J. Théor. Nombres Bordeaux 26(2014), 233–251.

[13] G. Savin, *On unramified representations of covering groups.* J. Reine Angew. Math. 566(2004), 111–134. https://doi.org/10.1515/crll.2004.001

[14] G. Shimura, *On modular forms of half integral weight.* Ann. of Math. 97(1973), 440–481. https://doi.org/10.2307/1970831

[15] G. Shimura, *The critical values of certain zeta functions associated with modular forms of half-integral weight.* J. Math. Soc. Japan 33(1981), 649–672. https://doi.org/10.2969/jmsj/03340649

[16] M. Ueda, *On twisting operators and newforms of half-integral weight.* Nagoya Math. J. 131(1993), 135–205. https://doi.org/10.1017/S002776300000045X

[17] M. Ueda and S. Yamana, *On newforms for Kohnen plus spaces.* Math. Z. 264(2010), 1–13. https://doi.org/10.1007/s00209-009-0449-9

[18] J.-L. Waldspurger, *Sur les coefficients de Fourier des formes modulaires de poids demi-entier.* J. Math. Pures Appl. (9) 60(1981), 375–484.

Department of Mathematics, Technion, Haifa, 32000, Israel
e-mail: embaruch@math.technion.ac.il

Department of Mathematics, Tokyo Institute of Technology, Japan
e-mail: somapurkait@gmail.com