Commutative Associative Binary Operations on a Set with Five Elements

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Abstract. The main goal of this paper is to count commutative as well as associative binary operation on five element set, by using partition and composition of mapping. This is achieved using algorithm based on partition

Introduction
There is easy calculation to counting binary operation on a set with n element which is \(n^{n^2}\). If set has only five elements, then number of binary operations is as very large \(298,023,223,876,953,125\). Similarly, there is easy calculation to counting commutative binary operation on a set with n element which is \(n^{n(n+1)/2}\). If set has only five elements, then number of commutative binary operations is also large \(30,517,578,125\). In [9] authors count associative binary operations on five element set which comes as 183,732. But no easy calculation and no educated guess, seems to give the answer to following question:-

“How many binary operations on a five elements are associative as well as commutative?”

The objective of this paper is to answer this question. In other words, how many of the 30,517,578,125 different commutative binary operations on five-element set are associative or how many of the 183,732 different associative binary operations on five-element set are commutative.

Operations on a set with two, three and four elements set
Form [2, 3, 7, 10], we have following results
\begin{itemize}
\item Associative binary operation on a set of two elements is 8
\item Associative binary operation on a set of three elements is 113
\item Associative binary operation on a set of four elements is 3492
\item Commutative Associative binary operation on a set of two elements is 6
\item Commutative Associative binary operation on a set of three elements is 63
\item Commutative Associative binary operation on a set of four elements is 1140
\end{itemize}
Operations on a set with five elements set

As mentioned in the introduction, the number of possible binary operations on a set of four elements is $4,294,957,296$. Of these $1,048,576$ are commutative. We now proceed to answer the question: How many binary operations on a set of four elements are associative as well as commutative?

Algorithm for finding number of commutative as well as associative binary operation on n-element

Algorithm given below is taken from [3, 4, 10]. The analysis of the commutative associative binary operations on n-element set $S$ will now divide into 3 steps:-

1). Partition the set of $n^2$ mappings in such a way that element of same partition can be obtained by using one-one and onto mapping from $S$ onto $S$.
2). Rearrange the partition according to their order of any element of the partition. (Say order as $k$).
3). Calculate the contribution towards number of associative as well as commutative binary operations when one row is fill by the first element of $i^{th}$ partition which can also fill $k-1$ more rows and remaining row $n-k$ can be filled by $i^{th}$ and onwards partitions (if any) with two conditions given below.

Conditions: - Before starting calculation, firstly we insured that no commutative associative table counted twice. For this, we make some rules:-

(i) If we fixed $r^{th}$ row from any element $i^{th}$ partition (which has order $k$) then we can fill at least $k-1$ more rows. Remaining unfilled rows can be filled with element of $i^{th}$ and onwards partition (if any) however selected element of $i^{th}$ partition cannot fill the unfilled rows before selected row.

(ii) If table contains $n$ different entry of $i^{th}$ partition then contribution towards number of commutative associative operations counted is $1/n$.

4.1 Step 1st

If set $S$ has $n$ elements, then total number of mapping possible from set $S$ to $S = n^n$

In our problem $n=5$, then total number of mapping possible from set $S$ to $S = 3125$

Here we consider $S=\{0,1,2,3,4\}$ and first digit of each mapping comes from 0, second comes from 1, third comes from 2, fourth comes from 3 and fifth comes from 4.

| Partition No | First element of Partition | Total number of element in partition |
|--------------|----------------------------|------------------------------------|
| 1            | 00000                      | 5                                  |
| 2            | 00001                      | 60                                 |
| 3            | 00004                      | 20                                 |
| 4            | 00011                      | 60                                 |
| 5            | 00012                      | 60                                 |
| 6            | 00013                      | 120                                |
| 7            | 00014                      | 120                                |
| 8            | 00033                      | 60                                 |
| 9            | 00034                      | 30                                 |
| 10           | 00043                      | 30                                 |
| 11           | 00111                      | 20                                 |
| 12           | 00112                      | 120                                |
| 13           | 00114                      | 60                                 |
| 14           | 00122                      | 60                                 |
| Partition No | First element of Partition | Order of an element of partition | Generated elements belongs to partition |
|-------------|---------------------------|----------------------------------|----------------------------------------|
| 1           | 10342                     | 6                                | 1,2,1,40,21,1,47                       |
| 2           | 12340                     | 5                                | 2,2,2,2,47                             |
| 3           | 00123                     | 4                                | 3,27,25,46                             |
| 4           | 10023                     | 4                                | 4,23,34,41                             |
| 5           | 12003                     | 4                                | 5,15,42,15                             |
| 6           | 12300                     | 4                                | 6,35,6,45                              |
| 7           | 02341                     | 4                                | 7,39,7,47                              |
| 8           | 00013                     | 3                                | 8,25,46                                |
| 9           | 00112                     | 3                                | 9,25,46                                |
| 10          | 00124                     | 3                                | 10,22,44                               |
| 11          | 02113                     | 3                                | 11,42,33                               |
| 12          | 02311                     | 3                                | 12,12,45                               |
| 13          | 10002                     | 3                                | 13,41,34                               |

4.2 Step 2nd
| 14 | 10012 | 3 | 14, 41, 34 |
|---|---|---|---|
| 15 | 12001 | 3 | 15, 15, 42 |
| 16 | 00122 | 3 | 16, 26, 46 |
| 17 | 00143 | 3 | 17, 43, 36 |
| 18 | 10022 | 3 | 18, 41, 34 |
| 19 | 12000 | 3 | 19, 19, 43 |
| 20 | 00342 | 3 | 20, 20, 45 |
| 21 | 01342 | 3 | 21, 21, 47 |
| 22 | 00014 | 2 | 22, 44 |
| 23 | 00133 | 2 | 23, 41 |
| 24 | 00322 | 2 | 24, 42 |
| 25 | 00001 | 2 | 25, 46 |
| 26 | 00011 | 2 | 26, 46 |
| 27 | 00012 | 2 | 27, 46 |
| 28 | 00114 | 2 | 28, 44 |
| 29 | 00134 | 2 | 29, 43 |
| 30 | 00243 | 2 | 30, 45 |
| 31 | 01322 | 2 | 31, 45 |
| 32 | 02211 | 2 | 32, 43 |
| 33 | 02112 | 2 | 33, 42 |
| 34 | 10001 | 2 | 34, 41 |
| 35 | 10043 | 2 | 35, 45 |
| 36 | 00043 | 2 | 36, 43 |
| 37 | 00111 | 2 | 37, 46 |
| 38 | 10000 | 2 | 38, 44 |
| 39 | 02143 | 2 | 39, 47 |
| 40 | 01243 | 2 | 40, 47 |
| 41 | 00033 | 1 | 41 |
| 42 | 00224 | 1 | 42 |
| 43 | 00034 | 1 | 43 |
| 44 | 00004 | 1 | 44 |
| 45 | 00234 | 1 | 45 |
| 46 | 00000 | 1 | 46 |
| 47 | 01234 | 1 | 47 |

### 4.3 3rd Step

| Partition-No | First element of Partition | Assumed row | Contribution towards number of commutative as well as associative |
|--------------|-----------------------------|-------------|---------------------------------------------------------------|
| 1            | 10342                       | 1           | 0                                                             |
|              | 10342                       | 2           | 0                                                             |
|              | 10342                       | 3           | 0                                                             |
|              | 10342                       | 4           | 0                                                             |
|              | 10342                       | 5           | 0                                                             |
| 2            | 12340                       | 1           | 6                                                             |
|              | 12340                       | 2           | 6                                                             |
|              | 12340                       | 3           | 6                                                             |
|              | 12340                       | 4           | 6                                                             |
|              | 12340                       | 5           | 6                                                             |
|   |     | 1   | 0   |
|---|-----|-----|-----|
| 3 | 00123 | 2   | 0   |
|   | 00123 | 3   | 0   |
|   | 00123 | 4   | 120 |
|   | 00123 | 5   | 120 |
| 4 | 10023 | 1   | 0   |
|   | 10023 | 2   | 0   |
|   | 10023 | 3   | 0   |
|   | 10023 | 4   | 120 |
|   | 10023 | 5   | 120 |
| 5 | 12003 | 1   | 0   |
|   | 12003 | 2   | 0   |
|   | 12003 | 3   | 0   |
|   | 12003 | 4   | 120 |
|   | 12003 | 5   | 120 |
| 6 | 12300 | 1   | 120 |
|   | 12300 | 2   | 60  |
|   | 12300 | 3   | 60  |
|   | 12300 | 4   | 60  |
|   | 12300 | 5   | 0   |
| 7 | 02341 | 1   | 0   |
|   | 02341 | 2   | 15  |
|   | 02341 | 3   | 15  |
|   | 02341 | 4   | 15  |
|   | 02341 | 5   | 15  |
| 8 | 00013 | 1   | 0   |
|   | 00013 | 2   | 0   |
|   | 00013 | 3   | 0   |
|   | 00013 | 4   | 660 |
|   | 00013 | 5   | 360 |
| 9 | 00112 | 1   | 0   |
|   | 00112 | 2   | 0   |
|   | 00112 | 3   | 300 |
|   | 00112 | 4   | 0   |
|   | 00112 | 5   | 240 |
| 10| 00124 | 1   | 0   |
|   | 00124 | 2   | 0   |
|   | 00124 | 3   | 240 |
|   | 00124 | 4   | 240 |
|   | 00124 | 5   | 0   |
| 11| 02113 | 1   | 0   |
|   | 02113 | 2   | 0   |
|   | 02113 | 3   | 0   |
|   | 02113 | 4   | 120 |
|   |   |   |   |
|---|---|---|---|
|   | 02113 | 5 | 120 |
| 12 | 02311 | 1 | 0  |
|    | 02311 | 2 | 120 |
|    | 02311 | 3 | 60  |
|    | 02311 | 4 | 60  |
|    | 02311 | 5 | 0  |
| 13 | 10002 | 1 | 0  |
|    | 10002 | 2 | 0  |
|    | 10002 | 3 | 300 |
|    | 10002 | 4 | 120 |
|    | 10002 | 5 | 360 |
| 14 | 10012 | 1 | 0  |
|    | 10012 | 2 | 0  |
|    | 10012 | 3 | 480 |
|    | 10012 | 4 | 180 |
|    | 10012 | 5 | 180 |
| 15 | 12001 | 1 | 120 |
|    | 12001 | 2 | 120 |
|    | 12001 | 3 | 60  |
|    | 12001 | 4 | 0  |
|    | 12001 | 5 | 0  |
| 16 | 00122 | 1 | 0  |
|    | 00122 | 2 | 0  |
|    | 00122 | 3 | 360 |
|    | 00122 | 4 | 60  |
|    | 00122 | 5 | 0  |
| 17 | 00143 | 1 | 0  |
|    | 00143 | 2 | 0  |
|    | 00143 | 3 | 0  |
|    | 00143 | 4 | 0  |
|    | 00143 | 5 | 0  |
| 18 | 10022 | 1 | 0  |
|    | 10022 | 2 | 0  |
|    | 10022 | 3 | 360 |
|    | 10022 | 4 | 60  |
|    | 10022 | 5 | 0  |
| 19 | 12000 | 1 | 420 |
|    | 12000 | 2 | 30  |
|    | 12000 | 3 | 30  |
|    | 12000 | 4 | 0  |
|    | 12000 | 5 | 0  |
| 20 | 00342 | 1 | 0  |
|    | 00342 | 2 | 0  |
|    | 00342 | 3 | 40  |
|   |   |   |
|---|---|---|
| 21 | 01342 | 1  
|    | 01342 | 2  
|    | 01342 | 3  
|    | 01342 | 4  
|    | 01342 | 5  |
| 22 | 00014 | 1  
|    | 00014 | 2  
|    | 00014 | 3  
|    | 00014 | 4  
|    | 00014 | 5  |
| 23 | 00133 | 1  
|    | 00133 | 2  
|    | 00133 | 3  
|    | 00133 | 4  
|    | 00133 | 5  |
| 24 | 00322 | 1  
|    | 00322 | 2  
|    | 00322 | 3  
|    | 00322 | 4  
|    | 00322 | 5  |
| 25 | 00001 | 1  
|    | 00001 | 2  
|    | 00001 | 3  
|    | 00001 | 4  
|    | 00001 | 5  |
| 26 | 00011 | 1  
|    | 00011 | 2  
|    | 00011 | 3  
|    | 00011 | 4  
|    | 00011 | 5  |
| 27 | 00012 | 1  
|    | 00012 | 2  
|    | 00012 | 3  
|    | 00012 | 4  
|    | 00012 | 5  |
| 28 | 00114 | 1  
|    | 00114 | 2  
|    | 00114 | 3  
|    | 00114 | 4  
|    | 00114 | 5  |
| 29 | 00134 | 1  
|    | 00134 | 2  |
|   |   |   |   |
|---|---|---|---|
| 00134 | 3 | 480 |
| 00134 | 4 | 0 |
| 00134 | 5 | 0 |
| 30 | 00243 | 1 | 0 |
| 00243 | 2 | 0 |
| 00243 | 3 | 0 |
| 00243 | 4 | 420 |
| 00243 | 5 | 420 |
| 31 | 01322 | 1 | 0 |
| 01322 | 2 | 0 |
| 01322 | 3 | 720 |
| 01322 | 4 | 360 |
| 01322 | 5 | 0 |
| 32 | 02111 | 1 | 0 |
| 02111 | 2 | 600 |
| 02111 | 3 | 60 |
| 02111 | 4 | 0 |
| 02111 | 5 | 0 |
| 33 | 02112 | 1 | 0 |
| 02112 | 2 | 180 |
| 02112 | 3 | 180 |
| 02112 | 4 | 0 |
| 02112 | 5 | 0 |
| 34 | 10001 | 1 | 960 |
| 10001 | 2 | 360 |
| 10001 | 3 | 0 |
| 10001 | 4 | 0 |
| 10001 | 5 | 0 |
| 35 | 10043 | 1 | 40 |
| 10043 | 2 | 20 |
| 10043 | 3 | 0 |
| 10043 | 4 | 20 |
| 10043 | 5 | 20 |
| 36 | 00043 | 1 | 0 |
| 00043 | 2 | 0 |
| 00043 | 3 | 0 |
| 00043 | 4 | 270 |
| 00043 | 5 | 270 |
| 37 | 00111 | 1 | 0 |
| 00111 | 2 | 600 |
| 00111 | 3 | 20 |
| 00111 | 4 | 0 |
| 00111 | 5 | 0 |
| 38 | 10000 | 1 | 1300 |
|   |   |   |
|---|---|---|
| 10000 | 2 | 20 |
| 10000 | 3 | 0 |
| 10000 | 4 | 0 |
| 10000 | 5 | 0 |
| 39 | 02143 | 1 |
|    |     |   |
| 02143 | 2 | 5 |
| 02143 | 3 | 5 |
| 02143 | 4 | 5 |
| 02143 | 5 | 5 |
| 40 | 01243 | 1 |
|    |     |   |
| 01243 | 2 | 0 |
| 01243 | 3 | 0 |
| 01243 | 4 | 90 |
| 01243 | 5 | 90 |
| 41 | 00033 | 1 |
|    |     |   |
| 00033 | 2 | 0 |
| 00033 | 3 | 0 |
| 00033 | 4 | 1040 |
| 00033 | 5 | 0 |
| 42 | 00224 | 1 |
|    |     |   |
| 00224 | 2 | 0 |
| 00224 | 3 | 270 |
| 00224 | 4 | 0 |
| 00224 | 5 | 0 |
| 43 | 00034 | 1 |
|    |     |   |
| 00034 | 2 | 0 |
| 00034 | 3 | 0 |
| 00034 | 4 | 70 |
| 00034 | 5 | 70 |
| 44 | 00004 | 1 |
|    |     |   |
| 00004 | 2 | 0 |
| 00004 | 3 | 0 |
| 00004 | 4 | 0 |
| 00004 | 5 | 75 |
| 45 | 00234 | 1 |
|    |     |   |
| 00234 | 2 | 0 |
| 00234 | 3 | 0 |
| 00234 | 4 | 0 |
| 00234 | 5 | 0 |
| 46 | 00000 | 1 |
|    |     |   |
| 00000 | 2 | 0 |
| 00000 | 3 | 0 |
| 00000 | 4 | 0 |
| 00000 | 5 | 0 |
Conclusion
The conclusion of this paper is that among the 30,517,578,125 different commutative binary operations on a Five-element set, $S=\{0,1,2,3,4\}$, there are exactly 30730 operations which are associative. In other words, there exist exactly 30730 five-element commutative semi groups.

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