I. INTRODUCTION

Quantum fluctuations of a scalar field can enter a tachyonic regime where their frequency becomes imaginary. Such regime can occur if the scalar field is either coupled to a strong stationary external potential or strongly self-interacting with a potential exhibiting spontaneous symmetry breaking. In the tachyonic regime the system is essentially restructured. Its fluctuations Hamiltonian becomes unbounded from below, while the Hilbert space of states acquires an indefinite metric. All these changes are indicative of a new, metastable phase.

In the present work, we aim to derive a quantum kinetic equation describing the production of the tachyonic modes for a self-interacting neutral massive scalar field. Particle production in the tachyonic regime has been extensively studied so far in various models of spontaneous symmetry breaking. In these studies, the occupation number of produced particles has been estimated at the end of the metastable phase. Herein we suggest to study the full time evolution of the momentum distribution of the tachyonic modes using a kinetic description.

The decay of the metastable vacuum state has been discussed in different ways, including the semiclassical approach, the classical lattice field theory, the two-particle irreducible effective action formalism. Our approach is based on the canonical quantization of the tachyonic modes.

The plan of the paper is as follows. In Sec. II we introduce the model and identify the tachyonic regime. In Sec. III we perform the quantization of the tachyonic modes. A quantum kinetic equation is derived in Sec. IV. We conclude with summary in Sec. V.

II. TACHYONIC REGIME

We consider a general scalar field model with the Lagrangian density

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} m^2 \varphi^2 - V(\varphi),$$  \hspace{1cm} (1)

where $V(\varphi)$ is a self-interaction potential which contains orders $\varphi^3$ and higher without derivative terms and $m$ is the mass of the scalar field. The model is defined in a finite volume $L^3$, $-L/2 \leq x_i \leq L/2$, $i = 1, 2, 3$. The continuum limit is $\int \frac{d^3k}{(2\pi)^3}$.

From (1) we obtain the Klein-Gordon type equation of motion for the field $\varphi(\vec{x}, t)$:

$$(\Box + m^2) \varphi = J \equiv - \frac{\delta V}{\delta \varphi},$$  \hspace{1cm} (2)

where the non-linear current $J$ is also determined by the self-interaction.

Following the mean-field approximation, we decompose $\varphi(\vec{x}, t)$ into its space-homogeneous vacuum mean value $\phi(t) = \langle \varphi(\vec{x}, t) \rangle$ and fluctuations $\chi$

$$\varphi(\vec{x}, t) = \phi(t) + \chi(\vec{x}, t)$$  \hspace{1cm} (3)

with $\langle \chi(\vec{x}, t) \rangle = 0$. The mean field is treated as a classical background field defined with respect to the in-vacuum $|0\rangle$ as

$$\phi(t) \equiv \langle \varphi(\vec{x}, t) \rangle = \frac{1}{L^3} \int d^3x \langle 0 | \varphi(\vec{x}, t) | 0 \rangle,$$  \hspace{1cm} (4)

so that in the limit $t \to -\infty \phi(t) \to 0$, while the fluctuations are quantized and take place at all times.

Using Eq.(3) provides the following decomposition for the current

$$J(\phi + \chi) = J(\phi) + \frac{\delta J(\phi)}{\delta \phi} \chi + \mathcal{J}(\phi, \chi),$$  \hspace{1cm} (5)

where $\mathcal{J}(\phi, \chi)$ includes terms of second and higher orders in $\chi$. 

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\[ \mathcal{J}(\phi, \chi) = \frac{1}{2} \frac{\delta^2 J(\phi)}{\delta \phi^2} \chi^2 + \ldots \] 

Substituting Eq. (3) also into Eq. (2) and taking the mean value \( \langle \ldots \rangle \) yields the vacuum mean field equation

\[ \ddot{\phi} + m^2 \phi - J(\phi) = \langle \mathcal{J} \rangle, \]

where the overdot indicates the derivative with respect to time, while the equation of motion for the quantum fluctuations reads

\[ (\Box + m_{eff}^2)\chi = \mathcal{J} - \langle \mathcal{J} \rangle \]

with

\[ m_{eff}^2 \equiv m^2 + \frac{\delta^2 V(\phi)}{\delta \phi^2}. \] 

For \( \frac{\delta^2 V}{\delta \phi^2} > 0 \), the effective mass squared is positive at all times. However, if \( \frac{\delta^2 V}{\delta \phi^2} < 0 \), \( m_{eff}^2 \) becomes negative for \( |\frac{\delta^2 V}{\delta \phi^2}| > m^2 \) indicating a tachyonic regime.

In terms of the Fourier components \( \chi(k, t) \), Eq. (8) takes the form

\[ \ddot{\chi}(k, t) + \omega^2(k)\chi(k, t) = F_{\chi}(k, t), \]

where

\[ F_{\chi}(k, t) \equiv \mathcal{J}(k, t) - \sqrt{V(k)}\delta_{k,0} \]

and \( \mathcal{J}(k, t) \) is the Fourier transform of the current \( \mathcal{J}(\vec{x}, t) \),

\[ \mathcal{J}(\vec{k}, t) \equiv \frac{1}{L^{3/2}} \int d^3x e^{-i\vec{k}\cdot\vec{x}}\mathcal{J}(\vec{x}, t), \]

while

\[ \omega^2(k) \equiv k^2 + m_{eff}^2(t) \]

is the time-dependent frequency squared of the fluctuations. In the tachyonic regime, \( \omega^2(k) \) can be negative.

Whether the system evolves in the tachyonic or non-tachyonic regime is dynamically fixed by the time-dependent critical momentum:

\[ \tilde{k}_c^2 = \begin{cases} \left| \frac{\delta^2 V}{\delta \phi^2} \right| - m^2, & \text{otherwise} \\ 0, & \text{otherwise} \end{cases} \]

All momentum modes below \( \tilde{k}_c^2 \) are tachyonic. For \( \frac{\delta^2 V}{\delta \phi^2} > -m^2 \), the critical momentum is zero, since the frequency is always positive and no tachyonic modes can appear.

The system can enter the tachyonic regime in different ways: gradually when the critical momentum becomes nonzero very slowly, or discontinuously when tachyonic modes appear suddenly on a short time scale. In any case, the critical momentum changes in tune with the time dependence of the vacuum mean field \( \phi \). If \( \phi \) oscillates, then the same momentum mode can change its nature during the time evolution.

Eqs. (2) and (10) are exact, self-consistently coupled and include back-reactions. The vacuum mean field modifies the equation for fluctuations via a time dependent frequency, while the fluctuations react back on the vacuum mean field via the source term \( \langle \mathcal{J} \rangle \) in Eq. (2) and on the fluctuations themselves via the “external force” term \( F_{\chi}(k, t) \) in Eq. (10).

### III. QUANTIZATION

The Hamiltonian density corresponding to (1) is

\[ \mathcal{H} = \frac{1}{2} \pi^2 + \frac{1}{2}(\nabla \varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi), \]

where \( \pi \) is the momentum canonically conjugate to \( \varphi \). With the decomposition for the potential

\[ V(\phi + \chi) = V(\phi) - J(\phi)\chi + \frac{1}{2}(m_{eff}^2 - m^2)\chi^2 + \nabla(\phi, \chi), \]

orders \( \chi^3 \) and higher being included into \( \nabla(\phi, \chi) \), we deduce from (13) the Hamiltonian density governing the dynamics of the fluctuations

\[ \mathcal{H}_\chi \equiv \frac{1}{2} \pi_\chi^2 + \frac{1}{2}(\nabla \chi)^2 + \frac{1}{2}m_{eff}^2\chi^2 + \nabla(\phi, \chi). \]

In terms of the Fourier components \( \chi(k, t) \) and \( \pi_\chi(k, t) \), the fluctuations Hamiltonian reads

\[ H_\chi = \int d^3x \mathcal{H}_\chi \]

\[ = \frac{1}{2} \sum_{\vec{k} \geq \vec{k}_c} \left( \pi_\chi^1(\vec{k}, t)\pi_\chi^1(\vec{k}, t) + \omega^2(\vec{k})\chi^1(\vec{k}, t)\chi^1(\vec{k}, t) \right) \]

\[ + \frac{1}{2} \sum_{\vec{k} < \vec{k}_c} \left( \pi_\chi^1(\vec{k}, t)\pi_\chi^1(\vec{k}, t) - \nu^2(\vec{k})\chi^1(\vec{k}, t)\chi^1(\vec{k}, t) \right) \]

\[ + L^3/2\nabla(\vec{k} = 0, t), \]

where \( \nu^2(\vec{k}) = -\omega^2(\vec{k}) > 0 \) for \( \vec{k}^2 < \vec{k}_c^2 \), and

\[ \chi^1(\vec{k}, t) = \chi(-\vec{k}, t), \]

\[ \pi^1_\chi(\vec{k}, t) = \pi_\chi(-\vec{k}, t) \]

for all momentum modes.

The non-tachyonic and tachyonic modes contribution to the Hamiltonian (13) represents a collection of positive and inverted (repulsive) oscillators, respectively. Both types of modes are coupled. Their interaction is described by the last term in Eq. (13), \( \nabla(\vec{k}, t) \) being the Fourier transform of the potential \( V(\vec{x}, t) \).
For the standard, non-tachyonic modes, we introduce the annihilation and creation operators by
\[ \chi(\vec{k}, t) = \Gamma_{k}(t)a(\vec{k}, t) + \Gamma_{k}^{\dagger}(t)a^{\dagger}(-\vec{k}, t), \] (21)
and
\[ \pi_{\chi}(\vec{k}, t) = -i\omega_{k}(t)\left[\Gamma_{k}(t)a(-\vec{k}, t) - \Gamma_{k}^{\dagger}(t)a^{\dagger}(\vec{k}, t)\right], \] (22)
where
\[ \Gamma_{k}(t) = \frac{1}{\sqrt{2\omega_{k}(t)}}\exp\{-i\Theta_{k}(\omega_{k}, t)\}, \] (23)
and \( \Theta_{k}(\omega_{k}, t) \) is a phase which in the in-limit takes the form \( \omega_{k}t = \sqrt{k^2 + m^2}. \)

Eqs. (21) and (22) are well-known expressions for the real frequency oscillations. The first term in the Hamiltonian \( \chi^{\dagger} \) – we denote it by \( H_{n}^{\dagger} \) – becomes up to a c-number:
\[ H_{n}^{\dagger} = \sum_{\vec{k}}^{2}>\vec{k}_{c}^{2}\omega_{k}(t)N_{n}^{\dagger}(\vec{k}, t), \] (24)
where \( N_{n}^{\dagger}(\vec{k}, t) \equiv a^{\dagger}(\vec{k}, t)a(\vec{k}, t) \) is the non-tachyonic modes number density operator.

For the modes with \( \vec{k}^2 < \vec{k}_{c}^{2} \), \( \omega_{k} = \pm i\nu_{k} = \pm i\sqrt{k^2 - \vec{k}^2} \) and one of the phase factors in the ansatz \( \Gamma(31) \Gamma_{k}(t) \) or \( \Gamma^{\dagger}_{k}(t) \), grows exponentially in time. Instead of oscillations we have an exponential growth of long wavelength quantum fluctuations with momenta \( \vec{k}^2 < \vec{k}_{c}^{2} \). This is the so-called tachyonic instability [15].

Making the transition \( \omega_{k} \rightarrow \nu_{k} \) in Eq.(22) yields the following ansatz for the negative frequency squared fluctuations
\[ \chi(\vec{k}, t) \rightarrow \chi_{t}(\vec{k}, t) = \frac{1}{\sqrt{2\nu_{k}}} \left( e^{\nu_{k}t}a(\vec{k}, t) + e^{-\nu_{k}t}a^{\dagger}(-\vec{k}, t) \right), \] (25)
where \( \nu_{k}(\nu_{k}, t) = -i\Theta_{k}(\omega_{k}, t) \). Introducing
\[ \sigma_{1}(\vec{k}, t) \equiv \frac{1}{\sqrt{2\nu_{k}}}\cosh\nu_{k}\cdot(a(\vec{k}, t) + a^{\dagger}(-\vec{k}, t)), \] (26)
\[ \sigma_{2}(\vec{k}, t) \equiv -\frac{1}{\sqrt{2\nu_{k}}}\sinh\nu_{k}\cdot(a(\vec{k}, t) - a^{\dagger}(-\vec{k}, t)), \] (27)
which obey the hermiticity condition \( \sigma_{1}^{\dagger}(\vec{k}, t) = \sigma_{1}(\vec{k}, t) \), we rewrite Eq.(23) as
\[ \chi_{t}(\vec{k}, t) = \sigma_{1}(\vec{k}, t) + i\sigma_{2}(\vec{k}, t) \] (28)
with \( \chi_{t}^{\dagger}(\vec{k}, t) \neq \chi_{t}(-\vec{k}, t) \), i.e. the ansatz (27) is non-Hermitian.

The canonically conjugate momentum is transformed as
\[ \pi_{\chi}(\vec{k}, t) \rightarrow \pi_{\chi,t}(\vec{k}, t) = \pi_{\sigma_{1}}(\vec{k}, t) + i\pi_{\sigma_{2}}(\vec{k}, t), \] (29)
where
\[ \pi_{\sigma_{1}}(\vec{k}, t) \equiv \nu_{k}\cosh\nu_{k}\cdot\sigma_{2}(\vec{k}, t), \] (30)
\[ \pi_{\sigma_{2}}(\vec{k}, t) \equiv -\nu_{k}\tanh\nu_{k}\cdot\sigma_{1}(\vec{k}, t). \] (31)
The commutation relations for \( \sigma_{1}, \sigma_{2} \)-fields are
\[ \left[ \sigma_{1}(\vec{k}, t), \sigma_{2}(\vec{\nu}, t) \right] = -\frac{i}{2\nu_{k}}\sinh2\nu_{k}\cdot\delta_{\vec{k},\vec{\nu}} \] (32)
all other commutators vanishing.

Analytically continuing the ansatz (21) in the frequency to imaginary values leads therefore to a non-Hermitian field. This is not acceptable because we require the hermiticity conditions, Eqs.(19)-(20), to be valid for all momentum modes and at all steps of our consideration. In addition, such non-Hermitian field is known to violate causality [17].

To define the Hermitian tachyonic fluctuations we can use either \( \sigma_{1}(\vec{k}, t) \) or \( \sigma_{2}(\vec{k}, t) \) instead of \( \chi_{t}(\vec{k}, t) \). Without loss of generality, we choose \( \sigma_{1}(\vec{k}, t) \) and introduce the field
\[ \sigma_{t}(\vec{k}, t) \equiv \frac{1}{\cosh\nu_{k}}\sigma_{1}(\vec{k}, t). \] (33)
Its canonically conjugate momentum is
\[ \pi_{\sigma,t}(\vec{k}, t) \equiv \frac{1}{\cosh\nu_{k}}\pi_{\sigma,1}(\vec{k}, t). \] (34)

With Eqs.(33) and (34), the second term in the Hamiltonian (18) takes the form
\[ H_{n}^{\dagger} = \sum_{\vec{k}}^{2}<\vec{k}_{c}^{2}\nu_{k}(t)N^{\dagger}(\vec{k}, t), \] (35)
where
\[ N^{\dagger}(\vec{k}, t) \equiv -\frac{1}{2}\left(a^{\dagger}(\vec{k}, t)a^{\dagger}(-\vec{k}, t) + a(-\vec{k}, t)a(\vec{k}, t)\right). \] (36)

Since the spectrum of an inverted oscillator is purely continuous, the tachyonic modes are not really “particle” ones [14]. In contrast with the case of the standard, non-tachyonic modes where the eigenfunctions of \( H_{n}^{\dagger} \) coincide with those of the number operator, the tachyonic modes are not eigenoperators of
\[ N^{\dagger} \equiv \sum_{\vec{k}}^{2}<\vec{k}_{c}^{2}\frac{1}{2}\sigma_{1}^{\dagger}(\vec{k}, t), \] (37)
\[ N^{\dagger}, a(\vec{k}, t) = a^{\dagger}(-\vec{k}, t), \] (38)
\[ N^{\dagger}, a^{\dagger}(\vec{k}, t) = -a(-\vec{k}, t), \] (39)
so that \( a(\vec{k},t),a^\dagger(\vec{k},t) \) in Eq.(35) can not be viewed as creation and annihilation operators.

However, once complex values are allowed for energy, the particle interpretation can be kept for the tachyonic modes as well. Let us introduce

\[
\alpha_{1(2)}(\vec{k},t) = \frac{1 \pm i}{2} a(\vec{k},t) + \frac{1 \pm i}{2} a(\vec{k},t),
\]

(40)

where the upper signs correspond to the subscript 1 and the lower one to 2. These new mode operators are Hermitian and fulfill the algebra

\[
[\alpha_{1}(\vec{k},t),\alpha_{1}^\dagger(\vec{p},t)] = [\alpha_{2}(\vec{k},t),\alpha_{2}^\dagger(\vec{k},t)] = 0,
\]

(41)

\[
[\alpha_{1}(\vec{k},t),\alpha_{2}^\dagger(\vec{p},t)] = i\delta_{\vec{k},\vec{p}}.
\]

(42)

The Fock representation for the algebra (41)-(42) is constructed by using an indefinite metric. Indeed, if \(|0; t\rangle \) is an instantaneous vacuum state defined by

\[
\alpha_{1(2)}(\vec{k},t)|0; t\rangle = \alpha_{2(1)}(\vec{k},t)|0; t\rangle = 0 \quad \text{for} \quad (k_i) > 0,
\]

(43)

where \((k_i) = (k_1, k_2, k_3)\), then for the excited states

\[
|\alpha_{1(2)}; \vec{k}, t\rangle \equiv \alpha_{1(2)}^\dagger(\vec{k},t)|0; t\rangle, \quad (k_i) > 0,
\]

(44)

the inner product is vanishing or imaginary,

\[
\langle \alpha_{1(2)}; \vec{k}, t|\alpha_{1(2)}(\vec{p},t) = 0,
\]

\[
\langle \alpha_{1}; \vec{k}, t|\alpha_{1}(\vec{p},t) = i\delta_{\vec{k},\vec{p}}.
\]

(46)

The indefinite inner product is related to the existence of associated eigenvectors of \( H_\chi \).

The density of \( N^t \) becomes

\[
N^t(\vec{k},t) = -iN^t_\alpha(\vec{k},t)
\]

\[
\equiv -\frac{1}{2}
\left(\alpha_{1}^\dagger(\vec{k},t)\alpha_{2}(\vec{k},t) + \alpha_{2}^\dagger(\vec{k},t)\alpha_{1}(\vec{k},t)\right),
\]

\[
\alpha_{1(2)}(\vec{k},t)
\]

being eigenoperators of

\[
N^t_\alpha = \sum_{\vec{k} \in \mathbb{R}^2 \atop (k_i) > 0} 2N^t_\alpha(\vec{k},t)
\]

(48)

with real eigenvalues,

\[
\left[N^t_\alpha, \alpha_{1(2)}(\vec{k},t)\right] = \pm \alpha_{1(2)}(\vec{k},t).
\]

(49)

For the instantaneous vacuum, \( N^t_\alpha|0; t\rangle = 0 \), while for the excited states \( N^t_\alpha \) counts excitations. For the state

\[
|n\alpha_1; \vec{k}_1, \vec{k}_2, ..., \vec{k}_n; t\rangle
\]

\[
\equiv \alpha_{1}^\dagger(\vec{k}_1,t)\alpha_{2}^\dagger(\vec{k}_2,t) \cdot \cdot \cdot \alpha_{2}^\dagger(\vec{k}_n,t)|0; t\rangle,
\]

all \((k_{1,i}, ..., k_{n,i}) > 0,\)

for instance,

\[
N^t_\alpha|n\alpha_2; \vec{k}_1, \vec{k}_2, ..., \vec{k}_n; t\rangle = n|n\alpha_2; \vec{k}_1, \vec{k}_2, ..., \vec{k}_n; t\rangle,
\]

(51)

i.e. \( N^t_\alpha \) plays the role of the “number operator”.

In the space with indefinite metric, the Hamiltonian \( H^t_\chi \) is pseudoadjoint \( \dagger \) and its eigenvalues are imaginary. If \(|\varepsilon; t\rangle \) is an eigenstate of \( H^t_\chi \) with eigenvalue \( \varepsilon \), then for the state \( \alpha_{1}^\dagger(\vec{k},t)|\varepsilon; t\rangle \) we obtain

\[
H^t_\chi \alpha_{1}^\dagger(\vec{k},t)|\varepsilon; t\rangle = (\varepsilon + iv_k)\alpha_{2}^\dagger(\vec{k},t)|\varepsilon; t\rangle,
\]

(52)

i.e. \( \alpha_{2}^\dagger(\vec{k},t)|\varepsilon; t\rangle \) is also an eigenstate of \( H^t_\chi \) with the eigenvalue shifted by \( iv_k \). Neglecting for a moment the third term in the right-hand side of Eq.(42), we see that the eigenvalues of the total Hamiltonian \( H^t_\chi + H^t_\chi \) are complex, the corresponding eigenfunctions representing unstable states.

**IV. KINETIC EQUATION**

In the mean-field approximation, the quantum fluctuations are treated perturbatively. This is necessary, in particular, for the derivation of the kinetic equation. One of basic points of the kinetic formulation is the postulate of asymptotic completeness \[19\]. The postulate specifies the set of possible states of the system in the infinite past as a complete set of states of freely moving non-interacting particles. For the system with a self-interaction, this postulate can be applied only in a few lower orders of perturbations when the interaction potential vanishes in the in-limit due to the vanishing of the vacuum mean field.

In higher orders, the quantum fluctuations dominate, the corresponding terms in the interaction potential surviving in the infinite past.

We limit our consideration to the third order term in \( \nabla \langle \phi, \chi \rangle \) neglecting all higher orders. In addition, we use a Hartree-type approximation that in the second and third orders consists of the factorization

\[
\chi^2 \rightarrow \langle \chi^2 \rangle, \quad \chi^3 \rightarrow 3\langle \chi^2 \rangle \chi.
\]

(53)

For the non-tachyonic modes, the form of the kinetic equation is well-known and was given in different models \[20,21\]. It determines the time evolution of the occupation number density

\[
N^{nt}(\vec{k},t) \equiv \langle 0|N^{nt}(\vec{k},t)|0\rangle
\]

(54)

which defines the number of particles of a given state characterized by the momentum \( \vec{k}^2 > \vec{k}_0^2 \) at time \( t \). An increase in the occupation number density is interpreted as particle production.

Herein we focus on the time evolution of

\[
N^t(\vec{k},t) \equiv \langle 0|N^t_\alpha(\vec{k},t)|0\rangle
\]

(55)

which defines the momentum distribution of the tachyonic modes. We start with the tachyonic Hamiltonian equations of motion.
\[ \dot{\sigma}_t(\vec{k}, t) = \pi^+_\sigma(\vec{k}, t), \]  
\[ \dot{\pi}_{\sigma,t}(\vec{k}, t) = \nu_k^2 \sigma^*_\sigma(\vec{k}, t) + \mathcal{J}(\vec{k}, t), \]  

where the current \( \mathcal{J}(\vec{k}, t) \) represents the self-interaction potential contribution. With the factorization (53), the self-interaction potential and current take the form

\[ \nabla(\vec{k} = 0, t) = -\frac{1}{2} \frac{\delta^2 J(\phi)}{\delta \phi^2} \langle \chi^2 \rangle \sigma_t(0, t) \]  

and

\[ \mathcal{J}(\vec{k}, t) = \frac{1}{2} L^{3/2} \frac{\delta^2 J(\phi)}{\delta \phi^2} \langle \chi^2 \rangle \delta_{k,0}. \]  

respectively. Using the relations

\[ \alpha_{1(2)}(\vec{k}, t) = \sqrt{\frac{\nu_k}{2}} (\sigma_t(\vec{k}, t) \mp \frac{1}{\nu_k} \pi^+_{\sigma,t}(\vec{k}, t)) \]  

yields then the equations for \( \alpha_{1(2)}(\vec{k}, t) \):

\[ \dot{\alpha}_{1(2)}(\vec{k}, t) \pm \nu_k \alpha_{1(2)}(\vec{k}, t) - \frac{\dot{\nu}_k}{2 \nu_k} \alpha_{1(2)}(\vec{k}, t) = \mp \frac{1}{\sqrt{2 \nu_k}} \mathcal{J}(\vec{k}, t). \]  

Taking next the time derivative of \( \mathcal{N}_t(\vec{k}, t) \) we find:

\[ \dot{\mathcal{N}}_t(\vec{k}, t) = \frac{\nu_k}{2 \nu_k} \left( \mathcal{C}_1(\vec{k}, t) + \mathcal{C}_2(\vec{k}, t) \right), \]  

where we have defined the time-dependent one-particle correlation functions

\[ \mathcal{C}_{1(2)}(\vec{k}, t) = \langle 0 | \alpha_{1(2)}(-\vec{k}, t) \alpha_{1(2)}(\vec{k}, t) | 0 \rangle. \]  

Since \( \langle \chi(\vec{x}, t) \rangle = 0 \), the vacuum expectation values for the zero momentum mode operators \( \sigma_t(0, t) \) and \( \pi_{\sigma,t}(0, t) \) are equal to zero, and , as a result, the current \( \mathcal{J}(\vec{k}, t) \) drops out of Eq.(62).

The functions \( \mathcal{C}_{1(2)}(\vec{k}, t) \) obey the equations

\[ \dot{\mathcal{C}}_{1(2)}(\vec{k}, t) = \frac{\nu_k}{\nu_k} \mathcal{N}_t(\vec{k}, t) \mp 2 \nu_k \mathcal{C}_{1(2)}(\vec{k}, t). \]  

Their formal solution is

\[ \mathcal{C}_{1(2)}(\vec{k}, t) = \int_{t_0}^t dt' \frac{\nu_k(t')}{\nu_k(t')} \mathcal{N}_t(\vec{k}, t') \times \exp \{ \pm 2 (\vartheta_k^d(t') - \vartheta_k^d(t)) \}, \]  

where

\[ \vartheta_k^d(t) = \int_{t_0}^t dt' \nu_k(t'). \]  

and \( t_0 \) is a moment of time at which the tachyonic regime starts. If \( t_0 = -\infty \), then the in-vacuum can be chosen as an initial state of the system.

Although we have not assumed that the frequency \( \nu_k \) varies adiabatically slowly in time and the phase \( \vartheta_k \) in the ansatz (23) is a general function of \( \omega_k \) and \( t \), it is just the “adiabatic” phase (66), i.e. the phase which looks exactly like the one in the adiabatic case, that enters this solution. Substituting it into Eq.(62), we obtain a closed equation for \( \mathcal{N}_t(\vec{k}, t) \):

\[ \dot{\mathcal{N}}_t(\vec{k}, t) = \frac{\dot{\nu}_k}{\nu_k} \int_{t_0}^t dt' \frac{\nu_k(t')}{\nu_k(t')} \mathcal{N}_t(\vec{k}, t') \times \cosh \left[ 2 \vartheta_k^d(t') - 2 \vartheta_k^d(t) \right] \]  

This kinetic equation determines the time evolution of the momentum distribution of the tachyonic modes produced in the fluctuations of the scalar field. As seen from the definition (47), the tachyonic modes production is symmetric in the momentum space, \( \mathcal{N}_t(\vec{k}, t) = \mathcal{N}_t(-\vec{k}, t) \) for all times \( t \).

Eq.(67) has non-Markovian character due to the explicit dependence of its right-hand side - the source term – on the time evolution of \( \mathcal{N}_t(\vec{k}, t) \) and therefore involves memory effects starting from \( t_0 \). For the real particle modes, the source term is known to contain a time integration over the statistical factor \( (1 \pm 2 \mathcal{N}(\vec{k}, t)) \), where the plus sign corresponds to bosons and the minus one to fermions. For the tachyonic modes , this factor reduces to \( 2 \mathcal{N}(\vec{k}, t) \) reflecting once more the fact that tachyons are not real particles .

In our approximation, the vacuum mean field equation becomes

\[ \ddot{\phi} + m^2 \phi = J(\phi) + \frac{1}{2} \frac{\delta^2 J}{\delta \phi^2} \langle \chi^2 \rangle, \]  

where the \( \langle \chi^2 \rangle \)-term represents the back-reaction of the fluctuations on the vacuum mean field and provides damping of the oscillations of \( \phi \). The initial conditions for both Eqs.(67) and (68) are specified by the model under study.

The vacuum mean value of \( \chi^2 \) is given by

\[ \langle \chi^2 \rangle = \frac{1}{L^2} \sum_{\vec{k} \gg \vec{k}_c} \langle 0 | \chi(\vec{k}, t) \chi(-\vec{k}, t) | 0 \rangle \]

\[ + \frac{1}{L^2} \sum_{\vec{k} \gg \vec{k}_c} \langle 0 | \sigma_t(\vec{k}, t) \sigma_t(-\vec{k}, t) | 0 \rangle, \]  

the bi-linear operator expressions here being assumed to be normal-ordered with respect to the instantaneous vacuum state. Both types of modes, tachyonic and non-tachyonic, contribute to \( \langle \chi^2 \rangle \), so a proper inclusion of the back-reactions effects can be achieved only by the complete treatment of all momentum modes.

Taking the vacuum expectation value of the fluctuations Hamiltonian yields

\[ \langle 0 | H_\chi | 0 \rangle = E_\chi - \frac{1}{2} \Gamma_\chi, \]  

(70)
where the non-tachyonic modes contribute to the energy of the metastable vacuum state

\[ E_\chi \equiv \sum_{\vec{k} \, 2 > \vec{k}_c^2} \omega_k(t)N^m(\vec{k}, t), \tag{71} \]

while the tachyonic ones to its decay rate,

\[ \Gamma_\chi \equiv 2 \sum_{\vec{k} \, 2 < \vec{k}_c^2} \nu_k(t)N^t(\vec{k}, t). \tag{72} \]

\section*{V. SUMMARY}

For the model of a self-interacting scalar field, we have derived a non-Markovian quantum kinetic equation determining the momentum distribution of the tachyonic modes. These modes are produced in quantum fluctuations of the scalar field around its vacuum mean value when the system is in a metastable phase. The kinetic and vacuum mean field equations are coupled, the latter including the back-reaction term, while the collisions effects are neglected.

Despite the fact that the fluctuations Hamiltonian is not bounded from below in the tachyonic regime, the conservation of energy prevents any catastrophic production of tachyons. If the system starts in a false, metastable vacuum state and then undergoes the transition to a lower energy density, stable one, the tachyonic regime stops as soon as all the potential energy of the false vacuum state is transferred into the quantum fluctuations. We have shown that the tachyonic modes contribute to the decay rate of this state, so their intensive production results in its rapid decay.

The kinetic equation obtained is hoped to be useful for the numerical study of the tachyonic modes production in various problems, in particular, in cosmology \cite{9} and heavy-ion collisions \cite{13,14,15}. The complete study requires the inclusion of higher orders effects when the quantum fluctuations interact with each other. Its realization within the kinetic formulation would provide further insight into the dynamics of the tachyonic regime.

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