Involutions of real intervals

Gaetano Zampieri
Università di Verona
Dipartimento di Informatica
strada Le Grazie 15, 37134 Verona, Italy
E-mail: gaetano.zampieri@univr.it

Dedicated to Jorge Sotomayor for his 70th birthday

Abstract

This paper shows a simple construction of the continuous involutions of real intervals in terms of the continuous even functions. We also study the smooth involutions defined by symmetric equations. Finally, we review some applications, in particular the characterization of the isochronous potentials by means of smooth involutions.

1 Introduction

An involution is a function that is its own inverse. This is an important object in all mathematical fields. We are going to consider continuous involutions on real intervals only.

Proposition 1.1. Let \( h : J \to J \) be continuous function on the interval \( J \subseteq \mathbb{R} \) which is the inverse of itself and does not coincide with the identity function \( \text{id}_J \). Then \( h \) is strictly decreasing and has a unique fixed point \( \bar{x} = h(\bar{x}) \).

To appear in Annales Polonici Mathematici.

2010 Mathematics Subject Classification: Primary 39B22; Secondary 37J45.

Key words and phrases: Involutions of real intervals; even functions; symmetric equations; isochronous potentials.
Proof. The function $h$ is strictly monotonic being continuous and injective on an interval. Let us prove that $h$ strictly decreases. Suppose it does not, then it is increasing. Since $h \neq id_J$ then $h(x_0) \neq x_0$ for some $x_0 \in J$. If $x_0 < h(x_0)$ then $h(x_0) < h(h(x_0)) = x_0$ a contradiction; similarly, $x_0 > h(x_0)$ implies $h(x_0) > x_0$. Thus $h$ decreases and the function $k(x) = x - h(x)$ strictly increases. Fixed points of $h$ coincide with zeros of $k$ and there is one zero at most since $k$ strictly increases. Consider a point $x_1 \in J$. If $k(x_1) = 0$ then $x_1$ is the unique fixed point of $h$. If $k(x_1) > 0$, then $x_1 > h(x_1)$ and $k(h(x_1)) = h(x_1) - h(h(x_1)) = h(x_1) - x_1 = -k(x_1) < 0$, so, by the continuity of $k$, there exists $\bar{x} \in (h(x_1), x_1)$ such that $k(\bar{x}) = 0$, namely $\bar{x}$ is the fixed point of $h$. If $k(x_1) < 0$ we can argue similarly.

These and other general properties are well known. Involutions are solutions to the celebrated Babbage functional equation $\phi^n(x) = x$, in the case $n = 2$, see the book [K] Chap. XV, by Kuczma, in particular Thms. 15.3 and 15.2, and Lemma 15.1. See also Kuczma, Choczewski and Ger [KCG Chap. 11], and Section 1 of [PR Chap. VIII] by Przeworska-Rolewicz where $h$ as above is called a Carleman function.

For $\bar{x}$ as above, the function $x \mapsto h(x + \bar{x}) - \bar{x}$ is also an involution and has 0 as fixed point, conversely $x \mapsto h(x - \bar{x}) + \bar{x}$ has fixed point $\bar{x}$ if $h(0) = 0$.

In the sequel we shall consider non-trivial involutions $h \neq id_J$, with $J$ open interval and $h(0) = 0$, moreover we are going to study smooth $h \in C^1$ involutions. By the chain rule we have $h'(h(x))h'(x) = 1$ so $h'(x) \neq 0$ at all $x$, more precisely $h'(x) < 0$ since we excluded the identity, and $h'(0) = -1$ necessarily. So, the present paper uses the following terminology:

**Definition 1.2.** A continuous function $h$ of an open interval $J \subseteq \mathbb{R}$ onto itself is called an *involution* if

\begin{equation}
(1.1) \quad h^{-1} = h, \quad 0 \in J, \quad h(0) = 0, \quad h \neq id_J.
\end{equation}

In particular it is called a smooth involution if $h \in C^1$, so it is a $C^1$ diffeomorphism with $h'(0) = -1$.

Of course $h(x) = -x$ is an involution on the whole $\mathbb{R}$. The following piecewise-linear example is taken from [PR] p. 177

\begin{equation}
(1.2) \quad h(x) = \begin{cases} 
-x/\lambda, & x \leq 0 \\
-\lambda x, & x > 0 
\end{cases} \quad x \in \mathbb{R}, \quad \lambda > 0.
\end{equation}
A very simple smooth involution which seems to be “new” is
\begin{equation}
(1.3) \quad h(x) = \ln\left(2 - e^x\right), \quad x < \ln 2.
\end{equation}

2 Constructing continuous involutions

Our main result is the following:

**Theorem 2.1.** Let \( h : J \to J \) be a (continuous) involution as in Definition 1.3, then \( k(x) := x - h(x) \) is a homeomorphism \( J \to I \) with \( I \) symmetric open interval, and the function \( P : I \to \mathbb{R} \) defined by \( P(y) = 2k^{-1}(y) - y \) satisfies \( P(0) = 0 \) and is even. Vice versa, if \( P : I \to \mathbb{R}, \) with \( P(0) = 0, \) is a continuous even function on a symmetric open interval such that the function \( K : I \to J, \)
\begin{equation}
(2.1) \quad K(y) = \frac{1}{2}(y + P(y)),
\end{equation}
is a homeomorphism onto some \( J, \) then \( h(x) := x - k(x), k = K^{-1}, \) is an involution on \( J. \)

**Proof.** If \( h : J \to J \) is an involution then \( k(x) := x - h(x) \) is strictly increasing as we already saw in the proof of Proposition 1.1, so a homeomorphism onto some open interval \( I \) as well known. The interval \( I \) is symmetric since
\[ y = k(x) = x - h(x) \in I \implies -y = -k(x) = k(h(x)) \in I. \]
Next, \( h(x) = k^{-1}(k(h(x))) = k^{-1}(-k(x)) \) and
\[ P(y) := 2k^{-1}(y) - y = -2y + 2k^{-1}(y) + y = -2k(k^{-1}(y)) + 2k^{-1}(y) + y = -2k^{-1}(y) + 2h(k^{-1}(y)) + 2k^{-1}(y) + y = 2k^{-1}(-y) + y = P(-y). \]
This fact and \( P(0) = 0 \) prove the first sentence. To prove the second sentence let us plug \( k(x) = K^{-1}(x) \) into (2.1), and then plug \(-k(x)\)
\[ x = \frac{1}{2}(k(x) + P(k(x))), \quad k^{-1}(-k(x)) = \frac{1}{2}(-k(x) + P(-k(x))) = \]
\[ = \frac{1}{2}(-k(x) + P(k(x))) \implies x - k^{-1}(-k(x)) = k(x) = x - h(x). \]
Thus \( h(x) = k^{-1}(-k(x)) \) which shows that \( h \) is a homeomorphism, \( h \neq id_J \) (otherwise \( k = -k \) so \( k \equiv 0 \), and \( h(h(x)) = x \) for all \( x \in J \) namely \( h^{-1} = h. \)
Finally \( P(0) = 0 \) implies \( k(0) = 0 \) and \( h(0) = 0. \) \( \square \)
Since $I = k(J)$ we have that $I = (\inf J - \sup J, \sup J - \inf J)$ when both \(\inf J, \sup J \in \mathbb{R}\), and $I = \mathbb{R}$ otherwise.

Of course, if we consider an arbitrary $C^1$ even function $P$, with $P(0) = 0$, then $P'(0) = 0$, and formula (2.1) restricted to the maximal symmetric open interval $I$ where $K'(y) > 0$, defines a strictly increasing diffeomorphism onto an interval $J$, and $h(x) := x - k(x)$, with $k = K^{-1}$, is a smooth involution on $J$.

For instance, starting from $P(y) = y^2/8$, $y \in \mathbb{R}$, formula (2.1) defines $K(y) = y/2 + y^2/16$ and $K'(y) > 0$ if and only if $y > -4$. So $K$ is injective on the symmetric open interval $I = (-4, 4)$ and a homeomorphism $I \rightarrow J = K(I) = (-1, 3)$. The function $k(x) = K^{-1}(x) = -4 + 4\sqrt{1 + x}$ and finally we get the involution $h(x) = x - k(x)$ namely

\begin{equation}
(2.2) \quad h : (-1, 3) \rightarrow (-1, 3), \ x \mapsto x + 4 - 4\sqrt{1 + x}.
\end{equation}

If we start from $P(y) = y^6$, we have 

\[ k : J = \left(\frac{-5}{12\sqrt[3]{7/5}}, \frac{7}{12\sqrt[3]{7/5}}\right) \rightarrow I = \left(\frac{-1}{6\sqrt[3]{7/5}}, \frac{1}{6\sqrt[3]{7/5}}\right) \]

and $h : J \rightarrow J$ is the non elementary algebraic function in Figure 2.

To illustrate the first part of the theorem, consider the piecewise linear involution (1.2). It gives

\[ k(x) = \begin{cases} 
(1 + \lambda)x/\lambda, & x \leq 0 \\
(1 + \lambda)x, & x > 0
\end{cases} \]

and

\[ k^{-1}(y) = \begin{cases} 
\lambda y/(1 + \lambda), & y \leq 0 \\
y/(1 + \lambda), & y > 0
\end{cases} \]
and the even function $P(y) = 2k^{-1}(y) - y$ is

\begin{equation}
(2.3) \quad P(y) = \frac{1 - \lambda}{1 + \lambda} |y|, \quad y \in \mathbb{R}, \quad \lambda > 0.
\end{equation}

Finally, the smooth involution \(1.3\) gives

\[ k(x) = x - \ln(2 - e^x), \quad k(-\infty, \ln 2) = \mathbb{R}, \quad k^{-1}(y) = \ln \frac{2}{1 + e^{-y}}, \]

and the following even function

\begin{equation}
(2.4) \quad P(y) = -y + 2 \ln \frac{2}{1 + e^{-y}} = -2 \ln \cosh \frac{y}{2}, \quad y \in \mathbb{R}.
\end{equation}

Functional equations relating involutory and even functions are studied in Schwerdtfeger [S], however our simple Theorem 2.1 seems to be new.

## 3 Involution given by symmetric equations

The condition $h^{-1} = h$ is equivalent to the symmetry of the graph of $h$ with respect to the diagonal; indeed $(x, h(x))$ has $(h(x), x)$ as symmetric point and this coincides with the point $(h(x), h(h(x)))$ of the graph. For example, consider the hyperbola $yx = 1$, which is symmetric with respect to the diagonal. In order to fulfill the further condition $h(0) = 0$, we translate its...
point \((1,1)\) to the origin. In this way we get \((y+1)(x+1) = 1\), which can be solved for \(y\) as \(y = -x/(1+x)\). If we finally take the branch that goes through the origin we arrive at the following involution

\[(3.1) \quad h(x) = -\frac{x}{1+x}, \quad x \in J = (-1, +\infty).\]

Involutions are preserved by homothety:

**Remark 3.1.** Let \(a \in \mathbb{R} \setminus \{0\}\) and \(h\) be an involution on \((b,c)\), then \(\tilde{h}(x) = h(ax)/a\) is an involution on \((b/a, c/a)\) if \(a > 0\), on \((c/a, b/a)\) otherwise.

In this way \((3.1)\) gives the following 1-parameter family of involutions

\[(3.2) \quad h(x) = -\frac{x}{1+ax}, \quad x \in J = \begin{cases} (-1/a, +\infty), & a > 0 \\ (-\infty, +\infty), & a = 0 \\ (-\infty, -1/a), & a < 0 \end{cases}\]

These are the only involutions that are rational functions of \(x\) as shown in Cima, Mañosas, and Villadelprat [CMV].

Aczél in [A] and Shisha and Mehr in [SM] obtain injective functions \(\hat{h} : \mathbb{R} \to \mathbb{R}\) such that \(\hat{h}^{-1} = \hat{h}\) from symmetric functions \(f : \mathbb{R}^2 \to \mathbb{R}\), \(f(x,y) = f(y,x)\). The paper [SM] suppose that for every \(x \in \mathbb{R}\) there exists a unique \(y\) to be denoted by \(\hat{h}(x)\) such that \(f(x,y) = 0\), then \(\hat{h}\) satisfies \(\hat{h}^{-1} = \hat{h}\). In particular, \(f(x,y) = x^3 + y^3 - a\) gives \(\hat{h}(x) = \sqrt[3]{a - x^3}\) which has the fixed point \(\bar{x} = \sqrt[3]{a}/2\). The function \(x \mapsto \hat{h}(x + \bar{x}) - \bar{x}\), i.e.

\[(3.3) \quad \mathbb{R} \to \mathbb{R}, \quad x \mapsto \sqrt[3]{a - \left( x + \sqrt[3]{a}/2 \right)^3} - \sqrt[3]{a}/2,\]

is an involution in the sense of Definition 1.2. For \(a \neq 0\) it is non-differentiable at \(x = \sqrt[3]{a} - \sqrt[3]{a}/2\).

We consider smooth symmetric equations in order to use the implicit function theorem:

**Proposition 3.2.** Let \(f : \Omega \to \mathbb{R}\) be a \(C^1\) function on the open set \(\Omega \subseteq \mathbb{R}^2\) such that: \((0,0) \in \Omega\), \(f(0,0) = 0\), and

\[(3.4) \quad (x,y) \in \Omega \implies (y,x) \in \Omega, \quad f(y,x) = f(x,y).\]

Let \(\Gamma\) be the connected component of \(f^{-1}(0)\) that contains the origin. Suppose that \(\partial_2 f(x,y) \neq 0\) for all \((x,y) \in \Gamma\). Then \(\Gamma\) is the graph of a smooth involution \(h\). All smooth involutions can be obtained this way.
Proof. By the implicit function theorem, $\Gamma$ is the graph of a $C^1$ function $h$ with $h(0) = 0$. Let $J$ be the projection of $\Gamma$ on the $x$-axis. It is an open interval since $\Gamma$ is open and connected. From (3.4) we have $\partial_1 f(x, y) = \partial_2 f(y, x)$ for $(x, y) \in \Omega$ so $\partial_1 f$ never vanishes on $\Gamma$ and has the same sign as $\partial_2 f$. We deduce that

$$h'(x) = \frac{-\partial_1 f(x, h(x))}{\partial_2 f(x, h(x))} < 0, \quad x \in J,$$

and in particular $h'(0) = -1$. Finally, $f(h(y), y) = f(y, h(y)) = 0$ shows that $h^{-1} = h$.

Let us prove the last sentence. Let $h : J \to J$ be a smooth involution and define $f(x, y) := x + y - h(x) - h(y)$. This is a $C^1$ function on $J \times J$, $f(0, 0) = 0$, and $f(y, x) = f(x, y)$. We have $\partial_2 f(x, y) = 1 - h'(y) > 0$ for all $(x, y) \in J \times J$. The graph of $h$ coincides with $f^{-1}(0)$. Indeed, if $y = h(x)$ then $x = h(y)$ and $f(x, y) = 0$; conversely, if $f(x, y) = 0$ then $k(y) = y - h(y) = -(x - h(x)) = -k(x)$, so $y = k^{-1}(k(y)) = k^{-1}(-k(x)) = k^{-1}(k(h(x))) = h(x)$.

For instance, let us consider the following function on the whole $\mathbb{R}^2$

$$\tilde{f}(x, y) = \left((x + 1)^3 + (y + 1)^3 - 2\right)(x + y + 2),$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{involution.png}
\caption{Global involution (3.3) for $a = 2$.}
\end{figure}
let $\tilde{\Gamma}$ be the cubic plane curve $(x + 1)^3 + (y + 1)^3 - 2 = 0$ and let $L$ be the straight line $x + y + 2 = 0$. The connected sets $\tilde{\Gamma}$ and $L$ are disjoint, moreover $(0, 0) \in \tilde{\Gamma}$. We have $\partial_2 f(x, y) = 0$ for $(x, y) \in \tilde{\Gamma}$ when $y = -1$ namely at the point $(c, -1) \in \tilde{\Gamma}$ with $c = \sqrt{2} - 1$. So we define $\Omega = (-1, c) \times (-1, c)$, an open square with $(0, 0) \in \Omega$, and we have that the restriction $f = \tilde{f}|\Omega$, satisfies (3.4). The connected component of $f^{-1}(0)$ that contains the origin is $\Gamma = \tilde{\Gamma} \cap \Omega$ and $\partial_2 f(x, y) \neq 0$ for all $(x, y) \in \Gamma$. Therefore $\Gamma$ is the graph of a smooth involution $h$. In this case we can even write the explicit formula which is the following restriction of the involution (3.3) for $a = 2$

\begin{equation}
(3.5) \quad h : (-1, \sqrt{2} - 1) \to (-1, \sqrt{2} - 1), x \mapsto \sqrt{2} - (x + 1)^3 - 1.
\end{equation}

The thick curve in Figure 4 is the graph of this smooth involution which is a piece of the non-smooth graph $\tilde{\Gamma}$, compare with Figure 3. The straight line below $\tilde{\Gamma}$ is $L$.

4 Isochronous potentials by involutions

An equilibrium point of a planar vector field is called a (local) center if all orbits in a neighborhood are periodic and enclose it. The center is isochronous
if all periodic orbits have the same period. The smooth involutions can be used to construct the isochronous centers for the scalar equation \( \ddot{x} = -g(x) \) as proved in the 1989 paper [Z2], by the present author. There are other different approaches to such isochronous centers, which do not involve involutions, in particular the 1961 Urabe’s paper [U1], see also [U2].

**Theorem 4.1.** Let \( h : J \to J \) be a smooth involution, \( \omega > 0 \), and define

\[
V(x) = \frac{\omega^2}{8} (x - h(x))^2, \quad x \in J.
\]

Then the origin is an isochronous center for \( \ddot{x} = -g(x) \), where \( g(x) = V'(x) \), namely all orbits which intersect the \( J \) interval of the \( x \)-axis in the \( x, \dot{x} \)-plane, are periodic and have the same period \( 2\pi/\omega \). Vice versa, let \( g \) be continuous on a neighborhood of \( 0 \in \mathbb{R} \), \( g(0) = 0 \), suppose there exists \( g'(0) > 0 \), and the origin is an isochronous center for \( \ddot{x} = -g(x) \), then there exist an open interval \( J, 0 \in J \), which is a subset of the domain of \( g \), and an involution \( h : J \to J \) such that (4.1) holds with \( V(x) = \int_0^x g(s)ds \) and \( \omega = \sqrt{g'(0)} \).

The potential \( V \) of an isochronous center is called an isochronous potential. The proof is included in the proof of Proposition 1 in [Z2] as a particular case. Formula (4.1) corresponds to formula (6.2) in the paper [Z2]. A detailed proof can be also found in the recent [Z3]; see Theorem 2.1 and Corollary 2.2 in [Z3]. This last paper also contains the following necessary conditions for a smooth enough potential \( V \) to be isochronous:

\[
V^{(4)}(0) = \frac{5V''(0)^2}{3V''(0)}, \quad V^{(6)}(0) = \frac{7V'''(0)V^{(5)}(0)}{V''(0)} - \frac{140V'''(0)^4}{9V''(0)^3},
\]

which can be deduced by taking successive derivatives of the involution relation \( h(h(x)) \equiv x \) at \( x = 0 \). We can consider the necessary condition at any even order derivative, provided that \( V \) admits that derivative.

Inserting the involution (3.2) into formula (4.1) we obtain the following isochronous potential

\[
V(x) = \frac{\omega^2}{8} x^2 \left( \frac{2 + ax}{1 + ax} \right)^2, \quad x \in J = \begin{cases} (-1/a, +\infty), & a > 0 \\ (-\infty, +\infty), & a = 0 \\ (-\infty, -1/a), & a < 0. \end{cases}
\]

This is the only isochronous rational potential as proved in [CV].
The paper [GZ], by Gorni and the present author, studies the global isochronous potentials $V : \mathbb{R} \to \mathbb{R}$ in terms of smooth involutions. In particular it gives implicit examples and new explicit ones. Also, the paper [GZ] revisits Stillinger and Dorignac global isochronous potentials in terms of involutions which are given by hyperbolas in Stillinger’s case.

5 Instability under some attractive central forces

The paper [Z1] considers the differential system

\begin{equation}
\begin{aligned}
\dot{x} &= -xf(x), \\
\dot{y} &= -yf(x), \\
f(0) &= 1,
\end{aligned}
\end{equation}

where $f$ is continuous near 0. It represents the motion under a particular attractive central force which is not a gradient. The origin of $\mathbb{R}^2$ is a (local) center for the first equation $\ddot{x} = -xf(x)$. Let us introduce the potential $V(x) = \int_0^x sf(s)ds$. For a suitable open interval $J \ni 0$, the potential $V$ is strictly increasing on $J \cap \mathbb{R}^+$, strictly decreasing on $J \cap \mathbb{R}^-$, and for each point $x \in J$ there is a unique point $h(x) \in J$ with $V(h(x)) = V(x)$, and $xh(x) < 0$ for $x \neq 0$. We easily see that the function $h$ is a smooth involution. The origin in $\mathbb{R}^4$ is Lyapunov stable for (5.1) if and only if for $x \neq 0$ in a neighborhood of 0 we have

\begin{equation}
\frac{1}{V(x)} = \frac{1}{2} \left( \frac{1}{x} - \frac{1}{h(x)} \right)^2,
\end{equation}

see formula (4.3) in [Z1]. In particular, if $f$ is even then so is $V(x)$ and $h(x) = -x$, formula (5.2) is equivalent to $V(x) = x^2/2$ and we have stability if and only if $f$ is constant in a neighborhood of 0. This particular case was studied in [ZB] with a different approach.

In Figure 5 you can see the projection on the $x,y$-plane of a solution to (5.1) for $f(x) = 1 + x^2$. In this case, the origin is an unstable equilibrium for (5.1). The initial condition for the solution in Figure 5 is $(x(0), \dot{x}(0), y(0), \dot{y}(0)) = (0.4, 0, 0, 0.5)$, on the left $t \in [0, 8]$, in the central picture $t \in [0, 14]$, and on the right $t \in [0, 38]$. It is an unbounded motion (see [Z1] for details).
6 Functional-differential equations with involutions

Consider the following problem which involves the involution (3.1) on the interval $(-1, +\infty)$, the parameter $a \in \mathbb{R}$, and the initial datum $y_0 \in \mathbb{R}$ at $t = 0$

\[
\begin{align*}
    y'(t) &= a y(h(t)), \\
    y(0) &= y_0,
\end{align*}
\]

If $y(t)$ is a $C^1$ solution then it is $C^2$. By differentiation we get

\[
y''(t) = a h'(t) y'(h(t)) = -\frac{a^2}{(1+t)^2} y(h(h(t))) = -\frac{a^2}{(1+t)^2} y(t).
\]

So (6.1) is equivalent to the ordinary Cauchy problem

\[
\begin{align*}
    y''(t) &= -\frac{a^2}{(1+t)^2} y(t), \\
    y(0) &= y_0, \\
    y'(0) &= a y_0,
\end{align*}
\]

Figure 5: Projection on the $x, y$-plane of a solution to (5.1).
The solution is defined on the whole $(-1, \infty)$. For $|a| > 1/2$:

$$y(t) = y_0 \sqrt{1 + t} \left( \cos (c \ln(1 + t)) + \frac{2a-1}{2c} \sin (c \ln(1 + t)) \right),$$

where $c := \sqrt{4a^2 - 1}/2$. While for $a = 1/2$:

$$y(t) = y_0 \sqrt{1 + t}.$$

For $a = -1/2$:

$$y(t) = y_0 \sqrt{1 + t} (1 - \ln(1 + t)).$$

For $|a| < 1/2$:

$$y(t) = \frac{y_0}{2b} (1 + t)^{1-b} \left( b + 1 - 2a + (b - 1 + 2a) (1 + t)^b \right),$$

where $b := \sqrt{1 - 4a^2}$. This is just an example of functional-differential equations of Carleman type, a general theory is treated in Chapter VIII of Przeworska-Rolewicz [PR] where references by other authors are quoted. Equations with involutions are also studied in [BT], [CI], [DI], [SW], [W], [W1], [W2], [WW].

Acknowledgements

This research was partly supported by the PRIN “Equazioni differenziali ordinarie e applicazioni”.

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