Recursive Structure and Bandwidth of Hales-Numbered Hypercube

Xiaohan Wang, Xiaolin Wu,

*Department of Electrical and Computer Engineering*
*McMaster University*
*Hamilton, Ontario, Canada, L8S 4K1*

**Abstract**

The Hales numbered $n$-dimensional hypercube and the corresponding adjacency matrix exhibit interesting recursive structures in $n$. These structures lead to a very simple proof of the well-known bandwidth formula for hypercube, whose proof was thought to be surprisingly difficult. A related problem called hypercube antibandwidth, for which Harper proposed an algorithm, is also reexamined in the light of the above recursive structures, and a close form solution is found.

*Key words: Graph bandwidth, hypercube.*

1 Introduction

The problem of graph bandwidth has been extensively studied [1,2], and has found many applications such as parallel computations, VLSI circuit design, etc. In this paper we are particularly interested in the bandwidth of hypercubes. The study of hypercube bandwidth can guide the design of communication codes for error resilient transmission of signals over lossy networks such as the Internet [3].

First, we restate the definitions of vertex numbering and graph bandwidth, most of which are adopted from [4].

**Definition 1** A numbering of a vertex set $V$ is any function

$$\eta : V \to \{1, 2, \cdots, |V|\},$$

(1)

**Email addresses:** wangx28@mcmaster.ca (Xiaohan Wang), xwu@ece.mcmaster.ca (Xiaolin Wu).
which is one-to-one (and therefore onto).

A numbering \( \eta \) uniquely determines a total order, \( \leq_\eta \), on \( V \) as: \( u \leq_\eta v \) if \( \eta(u) < \eta(v) \). Conversely, a total order defined on \( V \) uniquely determines a numbering of the graph.

**Definition 2** The bandwidth of a numbering \( \eta \) of a graph \( G = (V, E) \) is

\[
bw(\eta) = \max_{\{u,v\} \in E} |\eta(u) - \eta(v)|.
\]  

(2)

**Definition 3** The bandwidth of a graph \( G \) is the minimum bandwidth over all numberings, \( \eta \), of \( G \), i.e.

\[
bw(G) = \min_{\eta} bw(\eta).
\]  

(3)

The graph of the \( n \)-dimensional cube, \( Q_n \), has vertex set \( \{0,1\}^n \), the \( n \)-fold Cartesian product of \( \{0,1\} \). Thus \( |V^{(n)}_Q| = 2^n \). \( Q_n \) has an edge between two vertices (\( n \)-tuples of 0s and 1s) if they differ in exactly one entry.

**Definition 4** The Hales order, \( \leq_H \), on \( V^{(n)}_Q \), is defined by \( u \leq_H v \) if

1. \( w(u) < w(v) \), or
2. \( w(u) = w(v) \) and \( u \) is greater than \( v \) in lexicographic order relative to the right-to-left order of the coordinates,

where \( w(\cdot) \) is the Hamming weight of a vertex of \( Q_n \). This total order determines a numbering, \( H^{(n)} : V^{(n)}_Q \to \{1,2,\cdots,2^n\} \), which is called Hales numbering.

**Theorem 5** (Harper, [4]) The Hales numbering minimizes the bandwidth of the \( n \)-cube, i.e.

\[
bw(H^{(n)}) = bw(Q^{(n)}).
\]  

(4)

**PROOF.** See Corollary 4.3 in [4]. \( \Box \)

**Theorem 6** (Harper, [4]) For the \( n \)-cube \( Q^{(n)} \), we have

\[
bw(Q^{(n)}) = \sum_{m=0}^{n-1} \left( m \left\lfloor \frac{m}{2} \right\rfloor \right).
\]  

(5)

Although the above result has been known for forty years, no proof seemed to appear in the literature. Harper posed the proof of Theorem 6 as an excise in his recent book [4], and noted “it is surprisingly difficult”. In the following section we present a rather simple proof. The proof also reveals some interesting effects of the Hales numbering on hypercubes.
2 Proof of the Bandwidth Formula for Hypercubes

To prove Theorem 6, we first need a lemma and some definitions.

Lemma 7 We define a $2^n \times n$ $(0, 1)$-matrix $S^{(n)}$ as

$$S^{(n)} = \begin{bmatrix} A_0^{(n)} \\ A_1^{(n)} \\ \vdots \\ A_n^{(n)} \end{bmatrix},$$

(6)

where $A_k^{(n)}$, $k = 0, 1, \cdots, n$, is an $\binom{n}{k} \times n$ $(0, 1)$-matrix satisfying the following recursive formula

$$A_k^{(n)} = \begin{bmatrix} A_{k-1}^{(n-1)} & 1 \\ A_k^{(n-1)} & 0 \end{bmatrix}, \quad k = 1, 2, \cdots, n - 1,$$

(7)

where $0$ and $1$ are column vectors containing only 0s and 1s respectively. As the base case, we have $A_0^{(n)} = 0^T$ and $A_n^{(n)} = 1^T$. Then the row vectors of $S^{(n)}$, from top to bottom, are all vertices of $Q^{(n)}$ in the increasing Hales order.

PROOF. From Definition 4, it is sufficient to show that the row vectors of $A_k^{(n)}$, $k = 0, 1, \cdots, n$, are all distinct vectors with Hamming weight $k$, which are sorted, from top to bottom, in the decreasing lexicographic order.

We prove by induction on $n$. The above assertion is trivially true for $n = 1$. Assume the assertion holds for $n - 1 \geq 1$. Now for $n$, $A_0^{(n)}$ is a vector of Hamming weight 0 and $A_n^{(n)}$ a vector of Hamming weight $n$, so the assertion trivially holds. For $1 \leq k \leq n - 1$, the first $\binom{n-1}{k-1}$ vectors of $A_k^{(n)}$ are all distinct and have Hamming weight $k$ by the induction assumption that all row vectors in $A_{k-1}^{(n-1)}$ are distinct and have Hamming weight $k - 1$. Further, these vectors are in the decreasing lexicographic order because they share the same rightmost bit and all vectors in $A_{k-1}^{(n-1)}$ are sorted. By the same argument the next $\binom{n-1}{k}$ vectors of $A_k^{(n)}$ are distinct, of Hamming weight $k$, and sorted in the decreasing lexicographic order as well. Combining the above facts and (7) concludes that the row vectors of $A_k^{(n)}$ are distinct, of Hamming weight $k$, and in the decreasing lexicographic order. \(\square\)

Definition 8 Given a graph $G = (V, E)$, for two vertex subsets $V_1 \subseteq V$ and $V_2 \subseteq V$ numbered by numberings $\eta_1$ and $\eta_2$ respectively, the adjacency matrix
of $V_1$ and $V_2$ is a $|V_1| \times |V_2|$ matrix $M$ such that for any $u \in V_1$ and $v \in V_2$

$$M(\eta_1(u), \eta_2(v)) = \begin{cases} 
1 & \text{if } \{u, v\} \in E; \\
0 & \text{otherwise.} 
\end{cases} \quad (8)$$

**Definition 9** The bandwidth of an $s \times t$ matrix $M$ is the maximum absolute value of the difference between the row and column indices of a nonzero element of that matrix, i.e.

$$bw(M) = \max_{1 \leq i \leq s, 1 \leq j \leq t} \{|i - j| \mid M(i, j) \neq 0\}. \quad (9)$$

**Remark 10** The bandwidth of a numbering $\eta$ of a graph $G$ is equal to the bandwidth of the adjacency matrix of $G$ numbered by $\eta$.

The bandwidth of a square matrix is obviously the maximum Manhattan distance from a nonzero element to the main diagonal of the matrix.

**Definition 11** For an $s \times t$ matrix $M$, its Manhattan radius $r(M)$ is defined by

$$r(M) = \max_{1 \leq i \leq s, 1 \leq j \leq t} \{|s - i + j| \mid M(i, j) \neq 0\}, \quad (10)$$

which is the maximum Manhattan distance from a nonzero element of $M$ to the position immediately to the left of the bottom-left corner of matrix $M$ (an imaginary matrix element $M(s, 0)$), as shown in Fig. 1. This imaginary matrix element $M(s, 0)$ is called the anchor of $M$.

![Fig. 1. Manhattan radius and anchor of a matrix.](image)

Let $M^{(n)}$ be the $2^n \times 2^n$ adjacency matrix of $Q^{(n)}$ numbered by $H^{(n)}$. Recall from Lemma 7 that matrix $S^{(n)}$ has as rows all vertices of $Q^{(n)}$ sorted by $H^{(n)}$. Consider the submatrices $A^{(n)}_k$ and $A^{(n)}_{k'}$ in $S^{(n)}$, and let the $\binom{n}{k} \times \binom{n}{k'}$ matrix $M^{(n)}_{k,k'}$ be the adjacency matrix between $A^{(n)}_k$ and $A^{(n)}_{k'}$. Then $M^{(n)}_{k,k'}$, $0 \leq k, k' \leq n$, form the $2^n \times 2^n$ adjacency matrix of the Hales numbered hypercube: $M^{(n)} = [M^{(n)}_{k,k'}]$. Obviously, $M^{(n)}_{k,k'}$ is an all-zero matrix if $|k - k'| \neq 1.$
Therefore, we have
\[
M^{(n)} = \begin{bmatrix}
0 & M_{0,1}^{(n)} & 0 & \cdots & 0 & 0 \\
M_{1,0}^{(n)} & 0 & M_{1,2}^{(n)} & \cdots & 0 & 0 \\
0 & M_{2,1}^{(n)} & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & M_{n-1,n}^{(n)} \\
0 & 0 & 0 & \cdots & M_{n,n-1}^{(n)} & 0
\end{bmatrix}.
\] (11)

The bandwidth of \(M^{(n)}\) equals to the maximum Manhattan distance from a nonzero element of \(M^{(n)}\) to the main diagonal of \(M^{(n)}\). Because of the symmetry of \(M^{(n)}\), the bandwidth of \(M^{(n)}\) is equal to the maximum Manhattan distance from a nonzero element of \(M_{k,k+1}^{(n)}\), \(k = 0, 1, \ldots, n - 1\), to the main diagonal of \(M^{(n)}\). Note that the anchors of \(M_{k,k+1}^{(n)}\) are all on the main diagonal. Therefore, by Definition 11 the bandwidth of \(M^{(n)}\) can be expressed in terms of Manhattan radii of \(M_{k,k+1}^{(n)}\):
\[
bw(M^{(n)}) = \max_{k=0,\ldots,n-1} r(M_{k,k+1}^{(n)}).
\] (12)

A pleasing recurrence structure of the Manhattan radius \(r(M_{k,k+1}^{(n)})\) affords us the following proof of Theorem 6.

**PROOF.** [Proof of Theorem 6] Because of Theorem 5 and Remark 10, we only need to show that the bandwidth of the adjacency matrix of \(Q^{(n)}\) with the Hales numbering \(H^{(n)}\) satisfies (5).

Rewrite (7) as,
\[
A_k^{(n)} = \begin{bmatrix}
A_{k-1}^{(n-1)} & 1 \\
A_{k-1}^{(n-1)} & 0
\end{bmatrix} \quad \text{and} \quad A_{k+1}^{(n)} = \begin{bmatrix}
A_{k}^{(n-1)} & 1 \\
A_{k}^{(n-1)} & 0
\end{bmatrix}, \quad k = 1, 2, \ldots, n - 1.
\] (13)

Then \(M_{k,k+1}^{(n)}\), the adjacency matrix between \(A_k^{(n)}\) and \(A_{k+1}^{(n)}\), can be divided into four submatrices. The top-left one is the adjacency matrix between \([A_{k-1}^{(n-1)} 1]\) and \([A_{k}^{(n-1)} 1]\), which equals to the adjacency matrix between \(A_{k-1}^{(n-1)}\) and \(A_{k}^{(n-1)}\), i.e. \(M_{k-1,k}^{(n-1)}\). Similarly, the bottom-right one is \(M_{k,k+1}^{(n-1)}\). Because there is no pair of Hamming distance one between \(A_{k-1}^{(n-1)}\) and \(A_{k}^{(n-1)}\), the top-right submatrix is an all-zero matrix. The bottom-left submatrix is the adjacency matrix between \([A_{k-1}^{(n-1)} 0]\) and \([A_{k}^{(n-1)} 1]\), which is an identity matrix \(I_{n-1}^{(n-1)}\) of
dimension \( \binom{n-1}{k} \). Namely,

\[
M^{(n)}_{k,k+1} = \begin{bmatrix}
M^{(n-1)}_{k-1,k} & 0 \\
I^{(n-1)}_k & M^{(n-1)}_{k,k+1}
\end{bmatrix}, \quad k = 1, 2, \ldots, n-1.
\] (14)

Because \( A^{(n)}_0 \) is the all zero vector and \( A^{(n)}_1 \) contains \( n \) vectors of Hamming weight 1, we have \( M^{(n)}_{0,1} = 1^T \).

\[
\begin{array}{cccccc|cccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

Fig. 2. The recursive structure of \( r(M^{(n)}_{k,k+1}) \), where \( n = 5 \) and \( k = 2 \).

It follows from (14) that the Manhattan radius \( r(M^{(n)}_{k,k+1}) \) equals to the maximum Manhattan distance from a nonzero element in submatrices \( M^{(n-1)}_{k-1,k} \), \( M^{(n-1)}_{k,k+1} \) or \( I^{(n-1)}_k \) to the anchor of \( M^{(n)}_{k,k+1} \), as illustrated in Figure 2. From the property of Manhattan distance and the fact that \( r(\cdot) > 0 \), we have for \( k = 1, 2, \ldots, n-1 \),

\[
r(M^{(n)}_{k,k+1}) = \max \left\{ \binom{n-1}{k} + r(M^{(n-1)}_{k-1,k}), \binom{n-1}{k} + r(M^{(n-1)}_{k,k+1}), r(M^{(n-1)}_{k,0}) \right\}
\]

\[
= \binom{n-1}{k} + \max \left\{ r(M^{(n-1)}_{k-1,k}), r(M^{(n-1)}_{k,k+1}) \right\},
\] (15)

and \( r(M^{(n)}_{0,1}) = 1 \) because \( M^{(n)}_{0,1} = 1^T \).

Now we prove Theorem 6 by induction on \( n \). It is trivial that when \( n = 1 \), \( r(M^{(1)}_{0,1}) = 1 = \binom{0}{0} \), and 0 = \( \left\lfloor \frac{m}{2} \right\rfloor \). Assume that

\[
r(M^{(n-1)}_{k,k+1}) \leq \sum_{m=0}^{n-2} \left( \frac{m}{2} \right), \quad k = 0, 1, \ldots, n-2,
\] (16)
where equality holds if \( k = \lfloor \frac{n-1}{2} \rfloor \). Then we have

\[
\begin{align*}
r(M_{k,k+1}) &= \left( \frac{n-1}{k} \right) + \max \left\{ r(M_{k-1,k}), r(M_{k,k+1}) \right\} \\
&\leq \left( \frac{n-1}{\lfloor \frac{n}{2} \rfloor} \right) + \sum_{m=0}^{n-2} \left( \frac{m}{\lceil \frac{m}{2} \rceil} \right) \\
&= \sum_{m=0}^{n-1} \left( \frac{m}{\lceil \frac{m}{2} \rceil} \right),
\end{align*}
\]

(17)
in which the equality holds if \( k = \lfloor \frac{n}{2} \rfloor \), because when \( n \) is even, \( \lfloor \frac{n}{2} \rfloor - 1 = \lfloor \frac{n-1}{2} \rfloor \), so \( r(M_{k-1,k}) \) achieves equality in (16); when \( n \) is odd, \( \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n-1}{2} \rfloor \), \( r(M_{k,k+1}) \) also achieves equality in (16). From (4), (12) and (17), Theorem 6 follows. \( \square \)

3 Antibandwidth problem

Another vertex numbering problem related to graph bandwidth is what we call antibandwidth problem. It is posed by reversing the objective of vertex numbering in that we now want to maximize the minimum distance between any adjacent pair of vertices.

**Definition 12** The antibandwidth problem of a graph \( G = (V, E) \) is defined as

\[
f(G) = \max_{\eta} \min_{\{v,w\} \in E} |\eta(v) - \eta(w)|,
\]

(18)
where \( \eta \) is a numbering of \( G \).

The antibandwidth problem has applications in code design for communications [3]. On hypercubes the antibandwidth problem has a very simple solution due to Harper [5].

**Corollary 13 (Harper, [5])** For the \( n \)-cube, first number the vertices with even Hamming weights and then number the vertices with odd Hamming weights, in the Hales order. The resulting numbering achieves \( f(G) \).

**PROOF.** See [5]. \( \square \)

In this section, we provide a close form formula for the solution of the antibandwidth problem on \( n \)-cubes, which is a new result.

**Theorem 14** For the \( n \)-cube \( Q^{(n)} \), we have

\[
f(Q^{(n)}) = 2^{n-1} - \sum_{m=0}^{n-2} \left( \frac{m}{\lceil \frac{m}{2} \rceil} \right).
\]

(19)
PROOF. The numbering described in Corollary 13 determines a new ordering of vertices

$$\tilde{S}^{(n)} = \begin{bmatrix}
A_0^{(n)} \\
A_2^{(n)} \\
\vdots \\
A_2^{(n)\frac{n}{2}} \\
A_4^{(n)} \\
\vdots \\
A_2^{(n)\frac{n}{2}+1}
\end{bmatrix}. \tag{20}$$

Similar to the proof of Theorem 6, we have the adjacency matrix of $Q^{(n)}$ with the vertices numbered in the order of (20)

$$\tilde{M}^{(n)} = \begin{bmatrix}
0 & 0 & \cdots & 0 & M_{0,1}^{(n)} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & M_{2,1}^{(n)} & M_{2,3}^{(n)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & M_{2[\frac{n}{2}], 2[\frac{n}{2}]+1}^{(n)} \\
0 & M_{3,2}^{(n)} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_{2[\frac{n}{2}]+1, 2[\frac{n}{2}]}^{(n)} & 0 & 0 & \cdots & 0
\end{bmatrix}. \tag{21}$$

From the symmetric structure of $\tilde{M}^{(n)}$, we only take into account the lower part of the matrix $\tilde{M}^{(n)}$. Then we have

$$f(\tilde{H}^{(n)}) = \min \left\{ \min_{k=1,3,\ldots,2[\frac{n}{2}]+1} \delta_{k,k-1}, \min_{k=1,3,\ldots,2[\frac{n}{2}]-1} \delta_{k,k+1} \right\}, \tag{22}$$

where $\delta_{k,k'}$ is the minimum Manhattan distance from a nonzero element in the submatrix $\tilde{M}_{k,k'}^{(n)}$ to the main diagonal of $\tilde{M}^{(n)}$. Take into account a row of submatrices $M_{k,k}^{(n)}$, $k = k + 1, k + 3, \cdots, 2[\frac{n}{2}], 1, 3, \cdots, k$ and apply the property of Manhattan distance, we have

$$\delta_{k,k+1} = \sum_{\kappa=k+1, k+3, \cdots, 2[\frac{n}{2}], 1, 3, \cdots, k} W(M_{\kappa,k}^{(n)}) - r(M_{k,k+1}^{(n)}), k = 1, 3, \cdots, 2[\frac{n-1}{2}]+1 \tag{23}$$
where $W(\cdot)$ is the width of a matrix and hence $W(M_{k,k}^{(n)}) = \binom{n}{k}$. Therefore

$$
\delta_{k,k+1} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{k} + \binom{n}{k+1} + \binom{n}{k+3} + \cdots + \binom{n}{\frac{n}{2}} - r(M_{k,k+1}^{(n)}).
$$

(24)

Similarly, we have for $k = 1, 3, \cdots, 2\left\lfloor \frac{n}{2} \right\rfloor - 1$,

$$
\delta_{k,k-1} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k+1} + \cdots + \binom{n}{\frac{n}{2}} - r(M_{k,k-1}^{(n)}).
$$

(25)

From (7) and similar to the analysis of (14), we derive the recursion form

$$
M_{k,k-1}^{(n)} = \begin{bmatrix}
M_{k-1,k-2}^{(n-1)} & I_{(n-1)}^{(k-1)} \\
0 & M_{k,k-1}^{(n-1)}
\end{bmatrix}, \quad k = 2, \cdots, n,
$$

(26)

where as the base case $M_{1,0}^{(n)} = 1$. The zero matrix at the bottom-left corner has dimension $\binom{n-1}{k-2} \times \binom{n-1}{k-2}$. Therefore,

$$
r(M_{k,k-1}^{(n)}) = \binom{n-1}{k} + \binom{n-1}{k-2} + r(I_{(n-1)}^{(k-1)})
= \binom{n-1}{k} + \binom{n-1}{k-2} + \binom{n-1}{k-1}.
$$

(27)

Substituting $r(M_{k,k-1}^{(n)})$ of (27) into (25), we have

$$
\delta_{k,k-1} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{k} + \binom{n}{k-1} + \binom{n}{k+1} + \cdots + \binom{n}{\frac{n}{2}} - \binom{n-1}{k} - \binom{n-1}{k-2} - \binom{n-1}{k-1}
= 2^{n-1},
$$

(28)

which can be easily established by considering the parity of $n$ and using the binomial coefficients relations $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ and $\sum_{k=0,\ldots,n-1} \binom{n-1}{k} = 2^{n-1}$.
Using the same justification and substituting $r(M_{k,k+1}^{(n)})$ of (17) into (24), we have

$$\delta_{k,k+1} = \binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{k} + \binom{n}{k+1} + \binom{n}{k+3} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor}$$

$$- \binom{n-1}{k} - \max\left\{ r(M_{k-1,k}^{(n-1)}), r(M_{k,k+1}^{(n-1)}) \right\}$$

$$= 2^{n-1} - \max\left\{ r(M_{k-1,k}^{(n-1)}), r(M_{k,k+1}^{(n-1)}) \right\}$$

$$\geq 2^{n-1} - \sum_{m=0}^{n-2} \binom{m}{\lfloor \frac{m}{2} \rfloor},$$

(29)

where equality holds when $k = \lfloor \frac{n-1}{2} \rfloor + 1$ or $\lfloor \frac{n-1}{2} \rfloor$, whichever being odd.

Combining (22), (28) and (29) completes the proof. \qed

References

[1] P. Z. Chinn, J. Chvátalová, A. K. Dewdney, N. E. Gibbs, The bandwidth problem for graphs and matrices - a survey, Journal of Graph Theory 6 (1982) 223–254.

[2] Y.-L. Lai, K. Williams, A survey of solved problems and applications on bandwidth, edgesum, and profile of graphs, Journal of Graph Theory 31 (2) (1999) 75–94.

[3] X. Wang, X. Wu, S. Dumitrescu, On optimal index assignment for MAP decoding of Markov sequences, IEEE International Symposium on Information Theory, Seattle, Washington, 2006, to be published.

[4] L. H. Harper, Global Methods for Combinatorial Isoperimetric Problems, Cambridge University Press, 2004.

[5] L. H. Harper, Optimal numbering and isoperimetric problems on graphs, Journal of Combinatorial Theory 1 (1966) 385–393.