On groups whose subnormal subgroups are inert

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Abstract. A subgroup $H$ of a group $G$ is called inert if for each $g \in G$ the index of $H \cap H^g$ in $H$ is finite. We show that for a subnormal subgroup $H$ this is equivalent to being strongly inert, that is for each $g \in G$ the index of $H$ in the join $\langle H, H^g \rangle$ is finite for all $g \in G$. Then we give a classification of soluble-by-finite groups $G$ in which subnormal subgroups are inert in the cases $G$ has no nontrivial periodic normal subgroups or $G$ is finitely generated.

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1 Introduction

The class $\mathcal{T}$ of groups in which subnormal subgroups are normal and its generalizations received much attention in the literature. In particular, classes $\mathcal{T}^*$ (resp. $\mathcal{T}_*$) of groups $G$ in which for each subnormal subgroup $H$ it holds $|H^G : H| < \infty$ (resp. $|H : H_G| < \infty$) were studied in [2] (resp. [8]). Here, as usual, by $H^G$ (resp. $H_G$) we denote the smallest (resp. largest) normal subgroup of $G$ containing $H$ (resp. contained in $H$). In both cases, such an $H$ is inert (in the terminology of [1] and [12]), that is commensurable to its conjugates. Recall that two subgroups $H$ and $K$ of a group are told commensurable iff the index of $H \cap K$ in both $H$ and $K$ is finite. Commensurability is an equivalence relation, clearly.

In this paper, we regard classes $\mathcal{T}^*$ and $\mathcal{T}_*$ in the same framework by considering the class $\mathcal{T}$ of groups whose subnormal subgroups are inert. Actually, in Proposition [1] we show that an inert subnormal subgroup $H$ of a group $G$ is strongly inert indeed (in the terminology of [7]), that is $H$ has

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the property that $|\langle H, H^g \rangle : H|\) is finite for all $g \in G$. Clearly, strongly inert subgroups are inert.

A group whose all subgroups are inert is said inertial in [12] where, in the context of generalized soluble groups, a characterization of inertial groups with some finiteness conditions was given. Then groups whose subgroups are strongly inert have been studied in [7]. Inertial groups also received attention in the context of locally finite groups (see [1] for example).

For terminology, notation and basic facts we refer to [10] and [11]. In particular, by dihedral group on an abelian group $A$ we mean a group $\langle x \rangle \ltimes A$ where $x$ acts faithfully on $A$ as the inversion map. Moreover, an automorphism $\gamma$ of a group $A$ is said a itl power automorphism iff it maps each subgroup into itself. Thus $a^\gamma = a^n$ for all $a \in A$ where $n = n(a) \in \mathbb{Z}$. If $A$ is bounded, the group of power automorphism of its is well-known to be finite. If $A$ is abelian non-periodic, then the only power automorphisms of $A$ are the identity and the inversion map (see [11]).

Recall that Theorem A of [12] states that if $G$ is a hyper-(abelian or finite) inertial group, then it is abelian or dihedral, provided it has no non-trivial periodic normal subgroups. In Sect. 2, by Theorem ̃A, we give a corresponding characterization of ̃T-groups by substituting the class of dihedral groups by that of semidihedral groups, which are defined below.

In Theorem B of [12] it is shown that a finitely generated hyper-(abelian or finite) group is inertial iff it has a finite index torsion-free abelian normal subgroup on which elements of $G$ induce power automorphisms. In our Theorem ̃B in Sect. 3 we show that a corresponding statement holds for soluble-by-finite ̃T-groups (see also Proposition 3). This answers Question C in [9].

2 Preliminary results

Our first result seems to be missing in the literature.

**Proposition 1** Inert subnormal subgroups are strongly inert.

**Proof.** This follows from next lemma.

**Lemma 1** Let $H$ be an inert subnormal subgroup of a group $G$ and $K \leq G$. If $|K : (H \cap K)|$ is finite, then $|\langle H, K \rangle : H|$ is finite as well.
Proof. We may assume \( G = \langle H, K \rangle = H^G K \) and proceed by induction on the subnormality defect \( d \) of \( H \) in \( G \), the statement being trivial if \( d \leq 1 \). Moreover, since \( |G : H| = |G : H^G| \cdot |H^G : H| \) where \( |G : H^G| \leq |K : (H \cap K)| =: n < \infty \), we only have to show \( |H^G : H| \) is finite. To this aim note that \( H^G \) is the join of at most \( n \) conjugates of \( H \). Thus the statement follows from the following claim: for any positive integer \( r \), any subgroup \( H^+ = \langle H_1, \ldots, H_{r+1} \rangle \) where each \( H_i \) is conjugate to \( H \). Thus the statement follows from the following claim: for any positive integer \( r \), any subgroup \( H^+ = \langle H_1, \ldots, H_{r+1} \rangle \) where each \( H_i \) is conjugate to \( H \). Then, denoting commensurability by \( \sim \), by induction on \( r \), we have that \( L := \langle H_1, \ldots, H_r \rangle \sim H \sim H_{r+1} \). Further, since \( H_{r+1} \) has subnormality defect at most \( d - 1 \) in \( H^G \) we may apply induction hypothesis (on \( d \)) to the group \( H^G \) and its subgroups \( H_{r+1} \) and \( L \). We have that \( |\langle H_1, \ldots, H_{r+1} \rangle : H_{r+1}| \) is finite. Thus \( H^+ \sim H_{r+1} \sim H \). □

The class of \( \tilde{T} \)-groups is closed with respect to forming normal subgroups and homomorphic images, clearly. Let us give further instances of \( \tilde{T} \)-groups.

Lemma 2 Let \( G_1 \leq G_0 \) be normal subgroups of a group \( G \) with \( G_1 \) and \( G/G_0 \) finite. If any subnormal subgroup of \( G_0/G_1 \) is inert in \( G/G_1 \), then \( G \) is \( \tilde{T} \).

Proof. Let \( H \) be a subnormal subgroup of \( G \). On the one hand, \( HG_1 \cap G_0 \) is subnormal in \( G \) hence inert. On the other hand, \( H \) and \( HG_1 \cap G_0 \) are commensurable. Thus \( H \) is inert as it is commensurable to an inert subgroup. □

Recall that in [3] an automorphism of an abelian group \( A \) is said to be an inertial automorphism iff it maps each subgroup of \( A \) to a commensurable one. Clearly, a \( \tilde{T} \)-group acts on its normal abelian sections by means of inertial automorphisms. In [3] we noticed that inertial automorphisms of \( A \) form a group \( IAut(A) \) and that, if \( A \) is torsion-free, then \( IAut(A) \) consists of maps \( \gamma \) defined by \( (a\gamma)^n = a^m \ \forall a \in A \), where coprime integers \( m, n \) are uniquely defined (with \( n > 0 \)). We call such a map rational power automorphisms and write \( \gamma = m/n \). Clearly, we are just considering \( A \) as a \( \mathbb{Q}^\pi \)-module, where \( \mathbb{Q}^\pi \) is the ring of rationals whose denominator is a \( \pi \)-number and \( \pi \) is the set of primes \( p \) such that \( A^p = A \). For a complete description of inertial automorphisms (when \( A \) is any abelian group) see [4] and [5], where we call rational power automorphisms just "multiplications".

3
because of the additive notation for $A$. As usual, we denote by $\mathbb{Q}_\pi$ the additive group of the ring $\mathbb{Q}_\pi$.

By next statement we recall some facts which follow from Theorem 1 of [3] and Proposition 2.2 and Theorem A of [4]. Here we deal with finitely generated subgroups of $IAut(A)$, while for the structure of the whole group $IAut(A)$ we refer to [6].

Recall Let $\Gamma$ be a finitely generated group of inertial automorphisms of an abelian group $A$. Then,
1) if $A$ is torsion-free and $\pm 1 \neq \gamma \in Aut(A)$, then $\gamma$ is inertial iff $\gamma = m/n \in \mathbb{Q}$ and $A^n = A^m = A$ has finite rank, thus $IAut(A)$ is abelian in this case;
2) if $A$ is bounded, then there is a finite $\Gamma$-invariant subgroup $F \leq A$ such that $\Gamma$ acts on $A/F$ by means of power automorphisms;
3) if $A$ is periodic, then for each $X \leq A$ there is a $\Gamma$-invariant subgroup $X^\Gamma \geq X$ such that $X^\Gamma/X$ is finite;
4) if $A$ is any, then there is a $\Gamma$-invariant torsion-free subgroup $V \leq A$ such that $A/V$ is periodic.

Let us next introduce a class of groups whose all subnormal subgroups are (strongly) inert.

Definition A group $G$ is said to be semidihedral on a torsion-free abelian subgroup $A$ if $G = K \rtimes A$ and $K$ acts faithfully on $A$ by means of inertial automorphisms.

We warn that the word semidihedral has been used with a different meaning in other areas of group theory. It is clear that above $K$ is abelian and if its elements induce power automorphism (that is power by the rationals $\pm 1$), then $G$ is abelian or dihedral. Note that in a dihedral group $G$ all subgroups $H$ are inert, as $|H : H_G| \leq 2$ (while subgroups with order 2 are not strongly inert). By next statement we give details about semidihedral groups and in Theorem A we see that any semidihedral group is $\tilde{T}$.

Proposition 2 Let $A$ be a torsion-free abelian normal subgroup of a group $G$. Then
1) $G$ is semidihedral on $A$ iff $A = C_G(A)$ and $G$ acts on $A$ by means of inertial automorphisms; in this case $A$ is uniquely determined as $A = \text{Fit}(G)$, moreover $G/A$ is isomorphic to a multiplicative group of rationals;
2) if $G$ is semidihedral on $A$ and $G_0$ is a non-abelian subgroup of finite index in $G$, then $G_0$ is semidihedral on $A \cap G_0$. 

4
Proof. (1) Assume $G$ semidihedral on $A$. Since non-trivial inertial automorphisms of $A$ are rational power, they are fixed-point-free, so that $A = C_G(A) = \text{Fit}(G)$. Conversely, if $A = C_G(A)$, we claim that $G$ splits on $A$. In fact, for any $x \in G \setminus A$, the subgroup $N := \langle x, A \rangle = \langle x \rangle \rtimes A$ has trivial centre. By 11.4.21 of [11], up to equivalence there exists a unique extension of $N$ by $Q = G/N$ with coupling the natural homomorphism $Q \to \text{Out}N$. Thus $G$ is isomorphic to the subgroup $G/A \rtimes A$ of the holomorph of $A$. Last part of the statement follows from Recall 1.

(2) Let $A_0 := A \cap G_0$. Every element of $G_0 \setminus A_0$ acts fixed-point-free on $A$ and so $C_{G_0}(A_0) = A_0$. Hence $G_0$ is semidihedral on $A_0$ by (1).  

We have now a statement corresponding to Theorem A in [12]. Notice that a semidihedral group has no non-trivial periodic normal subgroups.

Theorem $\tilde{A}$ A hyper-(abelian or finite) group $G$ without non-trivial periodic normal subgroups is a $\tilde{T}$-group iff it is semidihedral on a torsion-free abelian subgroup.

Proof. Suppose $G$ is a $\tilde{T}$-group. By Theorem A of [12], any torsion-free nilpotent normal subgroup of $G$ is abelian. Thus $A := \text{Fit}(G)$ is abelian and by Recall 1 it follows that $G/C_G(A)$ is abelian, too. Suppose, by contradiction, that $A \neq C := C_G(A)$. Since $G$ is hyper-(abelian or finite), there exists a $G$-invariant subgroup $U$ of $C$ properly containing $A$ such that $U/A$ is finite or abelian. In the latter case $U$ is nilpotent and so $U = A$, a contradiction. Then $U/A$ is finite, so $U$ is centre-by-finite and $U'$ is finite. Then $U' = 1$, a contradiction again. Hence $A = C$ and $G$ is semidihedral on $A$ by Proposition 2(1).

Conversely, let $G = K \rtimes A$ be a semidihedral group. If $A$ has infinite rank, then $G$ is dihedral and every subgroup is inert. Then assume $A$ has finite rank. Let $H$ be a subnormal subgroup of $G$. If $H \leq A$, then $H$ is inert as $K$ acts on $A$ by means of inertial automorphisms. Otherwise, by Proposition 2(1), there is an element $h \in H \setminus A$ acting on $A$ as a rational $m/n \neq 1$. If $H$ has subnormality defect $i$, we have $H \geq [H, A] \geq A^{(m-n)i}$. Since $A$ has finite rank, then $A/A^{(m-n)i}$ and $|HA : H|$ are finite, so $H$ is strongly inertial, hence inertial, as $|H^G : H|$ is finite. Thus $G$ is a $\tilde{T}$-group.

By next proposition we can apply Theorem $\tilde{A}$ to groups $G$ whose maximum normal torsion subgroup $\tau(G)$ is finite.
Proposition 3 Consider following properties for a group $G$:

i) $G$ has a semidihedral normal subgroup with finite index $G_0$ such that $G$ acts by means of rational power automorphisms on $A_0 := \text{Fit}(G_0)$ (therefore $G$ acts trivially on $G_0/A_0$);

ii) $G$ has a finite normal subgroup $F$ such that $G/F$ is semidihedral.

Then (i) implies (ii). Moreover (i) and (ii) are equivalent, provided $G$ has finite rank.

Proof. Let (i) hold. By Recall 1, $G/C_{G}(A_0)$ is abelian. Thus for any $g \in G$ and $g_0 \in G_0$, we have $[g,g_0] \subseteq C_{G_0}(A_0) = A_0$ by Proposition 2(1). Hence $g$ acts trivially on $G_0/A_0$. Then, let $C := C_{G}(A_0)$. Since $C \cap G_0 = A_0$, we have that $C/A_0$ is finite. It follows that $C'$ and $F/C' := \tau(C/C')$ are finite as well. Thus $F$ is finite.

We claim that $\bar{G} := G/F$ is semidihedral on $\bar{C}$ (use bar notation). To show this, note that $G$ acts by means of inertial automorphisms on $A_0$ where $C/A_0$ is finite. Then $G$ does the same on the whole of $C$, by an argument as in Lemma 2. On the other side $C_{\bar{G}}(\bar{C}) = \bar{C}$ as if $\bar{x} \in C_{\bar{G}}(\bar{C})$, then $[\bar{x},A_0] \subseteq A_0 \cap F = 1$. Therefore the claim follows by Proposition 2(1).

Assume now (ii) holds and $G$ has finite rank. Let $n := |F|$, $A_1/F := \text{Fit}(G/F)$ and $C := C_{A_1}(F)$. Then $F \cap C \leq Z(C)$ and $C/(F \cap C)$ is abelian. Thus $[C^n,C^n] \leq (F \cap C)^n = 1$. Therefore $C^n$ is abelian and $A_0 := C^{m^2}$ is torsion-free abelian and has finite index, say $s$, in $A_1$. By using bar notation in $\bar{G} = G/A_0$, let $\bar{G}_1 := C_{\bar{G}}(\bar{A}_1)$. Then $[\bar{G}_1^n,\bar{G}_1^n] \leq \bar{A}_1^n = 1$ and so $\bar{G}_1^n$ is abelian. Moreover $\bar{G}_0 := \bar{G}_1^{2^n}$ is torsion-free abelian and has finite index in $\bar{G}$, as $\bar{G}$ as finite rank. Finally $G_0 \cap F \leq A_0 \cap F = 1$ and so $G_0$ is semidihedral on $A_0$ by Proposition 2(2), as it is $G$-isomorphic to a finite index normal subgroup of $G/F$.

3 Main result

We consider now finitely generated groups.

Lemma 3 Let $G = \langle g_1, ..., g_r \rangle$ be a finitely generated group with a torsion-free abelian normal subgroup $A$ such that $G/A$ is abelian. If each $g_i$ acts on $A$ by means of rational power automorphism $m_i/n_i \in \mathbb{Q}$, then $A$ is a free $\mathbb{Q}^\pi$-module with finite rank where $\pi = \pi(m_1...m_r,n_1...n_r)$ is a finite set of primes.
Proof. In the natural embedding of $\bar{G} := G/C_G(A)$ in $IAut(A)$, each generator $\bar{g}_i$ corresponds to the rational power by $m_i/n_i \in \mathbb{Q}$, (see Recall 1). Thus the ring $\mathbb{Z}[\bar{G}]$ corresponds to a subring of $End(A)$ isomorphic to $\mathbb{Q}^\pi$.

Since $G/A$ is finitely presented, we have that $A$ is finitely $G$-generated and $A$ is a finitely generated $\mathbb{Q}^\pi$-module. Then $A$ is isomorphic to a direct sum of finitely many quotient of the additive group $\mathbb{Q}_\pi$. Moreover $A$ is a free $\mathbb{Q}^\pi$-module, as it is torsion-free as an abelian group. □

Proposition 4 A finitely generated semidihedral group $G$ is of type $G = K \rtimes A$ where:
- $A$ is isomorphic to the sum of finitely many copies of $\mathbb{Q}_\pi$
- $K = \langle k_1, ..., k_s \rangle$ and each $k_i$ acts faithfully on $A$ by means of rational power automorphism $m_i/n_i \in \mathbb{Q}$,
- $\pi = \pi(m_1...m_sn_1...n_s)$ is a finite set of primes. □

Notice that the above $G = K \rtimes A$ may be obtained by a sequence of finitely many HNN-extensions starting with a free abelian group with finite rank as a base group. Generarily, such extensions are not ascending and $A$ is not finitely presented, an easy example being the extension of $\mathbb{Q}_{\{2,3\}}$ by the (inertial) rational power automorphism $\gamma = 2/3$, see Proposition 11.4.3 of [10]. On the other hand, since finitely generated semidihedral groups have finite rank, for such groups conditions (i) and (ii) of Proposition 3 are equivalent.

We have a statement corresponding to Theorem B in [12].

Theorem ˜B For a finitely generated soluble-by-finite group $G$, the following are equivalent:
i) $G$ is a $\tilde{T}$-group;
ii) $G$ has a finite normal subgroup $F$ such that $G/F$ is a semidihedral group.

Proof. Clearly, (ii) implies (i) by Lemma 2 and Theorem ˜A. On the other hand, by Theorem ˜A, (ii) is equivalent to saying that the maximum normal periodic subgroup $\tau(G)$ is finite. To prove that this follows from (i), we treat first some particular cases as lemmas.

Lemma 4 Let $G$ be a finitely generated group with a normal subgroup $N$ such that $G/N$ is abelian and $G$ acts on $N/N'$ by means of inertial automorphisms. If $N'$ is finite, then $\tau(G)$ is finite.
Proof. By arguing mod $N'$ we may assume $N$ is abelian.

If $N$ is periodic, then it is bounded, as it is $G$-finitely generated. Thus by Recall 2 there is a finite $G$-subgroup $F \leq N$ such that $G$ acts by means of power automorphisms on $N/F$. We may assume $F = 1$. Then $G/C_G(N)$ is finite, as a group of power automorphisms of a bounded abelian group. Therefore the nilpotent group $C_G(N)$ is polycyclic and $\tau(G)$ is finite.

If $N$ is any abelian group, then, by Recall 4, there is a torsion-free $G$-subgroup $V \leq N$ such that $N/V$ is periodic. By the above $\tau(G/V)$ is finite, whence $\tau(G)$ is finite.

□

Lemma 5 Let $G$ be a finitely generated $\tilde{T}$-group. If $G'$ nilpotent of class 2 and $G''$ is a $p$-group, then $\tau(G)$ is finite.

Proof. By Lemma 4, we have that $\tau(G/G'')$ is finite. Then by Theorem A and Proposition 3 there is a subgroup $G_0$ of finite index in $G$ such that $G_0 \geq G''$ and $G_0/G''$ is torsion-free semidihedral. Then $T := \tau(G_0)$ is abelian. Applying again Lemma 4 to $G_0/G''$, we have that $T/G''$ is finite. Then we have $T = AF$, where $F$ is a finite subgroup, which may be assumed $G$-invariant, by Recall 3. Thus we may factor out $F$. Denoting $N := G'$, we reduced to the following picture:

- $G = \langle g_1, ..., g_r \rangle$ is finitely generated with a nilpotent subgroup $N$ with class 2 such that $G/N$ is abelian, $N/N'$ is torsion-free and $N' = \tau(G)$ is a $p$-group;
- each $g_i$ acts by means of a rational power automorphism, say $m_i/n_i \in \mathbb{Q}$, on $N/N'$.

By Lemma 3 $N/N'$ is a free $\mathbb{Q}^\pi$-module of finite rank where $\pi := \pi(m_1...m_r n_1...n_r)$. Further, we have that for each $a, b \in N$ and $g \in G$, if $a^{ng} = a^m z_1$ and $b^{ng} = b^m z_2$ with $z_1, z_2 \in N' \leq Z(N)$, hence $[a, b]^{ng} = [a^m z_1, b^m z_2] = [a^m, b^m] = [a, b]^m$. Since $N' = \{[a, b] | a, b \in N\}$, we have $p \not\in \pi$.

Now, by a standard argument (see 5.2.5 in [12]), we have that the $p$-group $\tau(G) = N'$ is finite as it is isomorphic to an epimorphic image of $N/N' \otimes N/N'$ which is a direct sum of finitely many copies of $\mathbb{Q}_\pi$ where $p \not\in \pi$. □
Proof of Theorem ˜B (concluded). It remains to show that if \( G \) is a soluble finitely generated \( \mathcal{F}_T \)-group, then \( \tau(G) \) is finite. By induction on the derived length of \( G \) we may assume \( G \) has a normal abelian subgroup \( A \) such that \( \tau(G/A) \) is finite. By Theorem ˜ and Proposition 8 there is a subgroup \( G_0 \) of finite index in \( G \) such that \( G_0/A \) is torsion-free semidihedral of finite rank. We may assume \( G := G_0 \).

Consider first the case \( A \) is a \( p \)-group. If \( A \) is unbounded and \( B \leq A \) is such that \( A/B \) is a Prüfer group, then, by Recall 3, such is \( A/B^G \). Consider \( \tilde{G} := G/B^G \). Then \( \tilde{G}/C_G(A) \) is abelian and \( \tilde{G}' \) is nilpotent of class 2, as \( G'' \leq A \). Then we may apply Lemma 5 and we have that \( \tau(\tilde{G}) \) is finite. Thus \( A = \tau(G) \) is finite, a contradiction. Thus \( A \) is bounded and, by Recall 2, there is a finite \( G \)-invariant subgroup \( F \leq A \) such that \( G \) acts on \( A/F \) by means of power automorphisms. Then \( G/C_G(A/F) \) is abelian and \( A = \tau(G) \) is finite, again by Lemma 5.

Let \( A \) be any periodic abelian group. By the above, its primary components are finite. If by contradiction \( A \) is infinite, there is \( B \leq A \) such that \( A/B \) is infinite with cyclic primary components. By Recall 3, we may assume \( B := B^G \) to be \( G \)-invariant. Then \( A/B \) has finite Prüfer rank. Hence \( \tilde{G} := G/B \) has finite Prüfer rank (and is finitely generated). Thus, by Corollary 10.5.3 of [10], \( \tilde{G} \) is a minimax group indeed. Therefore its periodic normal subgroups have Min (the minimal condition on subgroups) while \( \tilde{A} \) has not, a contradiction.

In the general case, since \( G \) acts by means of inertial automorphisms on \( A \), by Recall 4 there is a torsion-free \( G \)-invariant subgroup \( V \) of \( A \) such that \( A/V \) is periodic. By the above, \( \tau(G/V) \) is finite, whence \( \tau(G) \) is finite. \( \Box \)

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