Coherent Polariton Radiation and Light Localization

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Abstract

A system of resonance atoms is placed in a medium with developed polariton effect. In the spectrum of polariton states there can exist a band gap. If an atom with a resonance frequency inside the polariton gap is incorporated into the medium, the atomic spontaneous emission can be strongly suppressed. This is what is called the localization of light. Nevertheless, an ensemble of resonance atoms inside the polariton gap can radiate if their coherent interaction is sufficiently strong. Conditions when coherent polariton radiation can appear and the properties of this coherent polariton emission are studied.
1 Light Localization

The phenomenon of light localization appears in three–dimensional periodic dielectric structures, in which, due to periodicity, an electromagnetic band gap develops. Then spontaneous emission with a frequency inside the band gap can be rigorously forbidden [1–3], because of a severe depression in the photon density of states for those frequencies which remain in the spectral gap between the upper and lower branches. This kind of samples, in which photon band gap develops because of the structure periodicity in real space, has been called photonic band–gap materials. The appearance of gaps in the spectrum of photon states, due to real–space periodicity, is similar to the formation of gaps in the spectrum of electrons in a periodic lattice potential [4].

If resonance atoms are incorporated in a band–gap material, so that their transition frequency is inside the gap, then the effect of light localization [2,3] can arize. To formulate explicitly what this effect means, let us consider the average

\[ s(t) \equiv \langle \sigma^z(t) \rangle \]

of the population–difference operator \( \sigma^z \). The average here implies the statistical or, at zero temperature, quantum–mechanical average. Under the localization of light one understands [5,6] that

\[ \lim_{t \to \infty} s(t) > -1. \]

The light localization becomes possible due to the formation of photon–atom bound states [5–7]. If a collective of identical impurity atoms is incorporated into a medium with a photon band gap, so that their transition frequency is inside the gap, and their spacing is much less than the transition wavelength, then a photonic impurity band is formed within the photonic band gap [7]. Electromagnetic coupling of neighboring atoms takes place by means of an effective resonance dipole–dipole interaction. If this interaction is sufficiently strong, then electromagnetic radiation can propagate inside the impurity band [7].
The formation of photon band gaps in photonic band–gap materials is similar to the well known polariton effect of the appearance of photon bands due to the interaction of light with collective excitations of dense medium [8,9]. Physical processes are, actually, the same in both types of materials. The difference is only in the nature of scatterers which light interacts with. In artificial photonic band–gap materials, a suppression of the photon density of states over a narrow frequency range results from multiple photon scattering by spatially correlated scatterers. In natural dense media, such as dielectrics or semiconductors, a frequency gap for propagating electromagnetic modes develops as a result of the photon interaction with optical collective excitations, such as optical phonons, magnons, excitons, and so on. Photons in a medium, coupled with collective excitations, are called polaritons.

When a single resonance atom is placed in a frequency dispersive medium whose polariton spectrum has a gap, and the atomic transition frequency lies inside this gap, then a polariton–atom bound state appears causing a significant suppression of spontaneous emission [10,11]. The physical picture explaining this suppression is as follows. Let us imagine an atom in a medium, with the atomic transition frequency within the polariton gap. If this atom is initially excited, then it tends to become deexcited emitting a photon. However, since the propagation of photons inside the polariton gap is prohibited, the emitted photon is scattered, by collective excitations, back and is again absorbed by the atom. Thus, the atom cannot get rid of a photon and is doomed to stay excited. Conversely, if the atom is initially in the ground state, it continues to be in that state, since there are no photons around to excite it. The suppression of spontaneous emission of an atom is termed localization of light. As is explained above, the effect of light localization can be expressed as the inequality $s(t) > -1$ for the average population difference, valid for all times.

The formation of polariton–atom bound states has been studied for a stationary case [10,11]. In dynamical picture, the population difference of an atom
satisfies the equation
\[ \frac{ds}{dt} = -\gamma_1(s - \zeta), \]
in which \( \gamma_1 \) is a level width and \( \zeta \), a stationary value of the population difference defined by a solution to the stationary problem. From the dynamical equation one has
\[ s(t) = (s_0 - \zeta)e^{-\gamma_1 t} + \zeta, \]
where \( s_0 \equiv s(0) \). If the stationary value \( \zeta > -1 \), then \( \lim_{t \to \infty} s(t) = \zeta > -1 \), which implies the localization of light. The complete suppression of emission corresponds to \( \zeta = s_0 \); then \( s(t) = \zeta \). Note that the linewidth \( \gamma_1 \) is caused by vacuum quantum fluctuations and is always nonzero, irrespectively what medium the atom is placed into.

If a collection of resonance impurity atoms is doped into a medium with a polariton band gap, then, in the same way as for photonic band-gap materials [7], an impurity band can be formed within the polariton gap [12,13]. Then electromagnetic radiation can propagate in such an impurity band. In order that such an impurity band be formed, the spacing of resonance impurity atoms in the medium should be much smaller than the radiation wavelength. If it is so, then for a group of atoms a sufficiently strong effective interaction, caused by photon exchange, can develop. This interaction collectivizes the atoms that start radiating coherently [14]. In this way, the suppression of spontaneous emission for a single atom can be overcome by a group of atoms radiating coherently.

The situation when a single atom cannot radiate inside the polariton gap but a collective of strongly interacting atoms can radiate reminds the following related case. If a sample with a polariton band gap is irradiated by a monochromatic electromagnetic wave with a frequency within the polariton gap, then the incident light cannot propagate through this medium because of total reflection. However, if the incident intensity is large enough, the light can penetrate into the dense media even when propagating inside the polariton gap [15,16]. In such
a case, analogously to that of coherent radiation, the possibility of the radiation propagation inside the band gap is due to nonlinear effects.

Here and in what follows we use the term "atom" in the general sense, implying under a resonance atom any two-level object. Depending on radiation frequencies, this could be atoms as such, molecules, nuclei, or quantum dots and wells. The latter case is of special importance for semiconductors. Really, the polariton effect is well developed in many semiconductors, for instance, in CuCl, CuBr, CdSe, ZnSe, GaAs, GaSb, InAs, AlAs, SiC. The characteristic frequencies, where the polariton band gap arises, are as follows [4] (see also [12–14]). For example, in GaAs the polariton gap having the width $\Delta \equiv \Omega_2 - \Omega_1 = 4 \times 10^{12} \text{s}^{-1}$ lies between $\Omega_1 = 5.1 \times 10^{13} \text{s}^{-1}$ and $\Omega_2 = 5.5 \times 10^{13} \text{s}^{-1}$; the linewidth being $\gamma_1/\Omega_1 = 1.2 \times 10^{-5}$. In SiC the polariton gap $\Delta = 3 \times 10^{13} \text{s}^{-1}$ is between $\Omega_1 = 1.5 \times 10^{14} \text{s}^{-1}$ and $\Omega_2 = 1.8 \times 10^{14} \text{s}^{-1}$; with the linewidth $\gamma_1/\Omega_1 = 3 \times 10^{-6}$. In all cases, for the relaxation parameter $\gamma_2$ one has $\gamma_2/\Omega_1 \sim 10^{-2}$. As is seen, the polariton band gap in such semiconductors is located in the infrared region. Therefore, resonance radiation for this region of frequencies could be presented by quantum dots and wells. Keeping in mind the feasibility of different radiating objects, we continue, for the sake of simplicity, to use the term "resonance atoms".

2 Basic Equations

The total Hamiltonian is given by the sum

$$\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_m + \hat{H}_{af} + \hat{H}_{mf},$$  \hspace{1cm} (1)

consisting of atomic, $\hat{H}_a$, field, $\hat{H}_f$, matter, $\hat{H}_m$, atom–field, $\hat{H}_{af}$, and matter–field, $\hat{H}_{mf}$, Hamiltonians. In the atomic Hamiltonian

$$\hat{H}_a = \frac{1}{2} \sum_{i=1}^{n} \omega_0 (1 + \sigma_i^z)$$  \hspace{1cm} (2)
the index \( i \) enumerates the atoms, \( \omega_0 \) is a transition frequency, and \( \sigma_i^z \) is a population difference operator. Here and in what follows we set \( \hbar \equiv 1 \). The field Hamiltonian

\[
\hat{H}_f = \frac{1}{8\pi} \int \left[ \vec{E}(\vec{r})^2 + \vec{H}(\vec{r})^2 \right] d\vec{r}
\]  

(3)

has the standard form in which \( \vec{E} \) is electric field and \( \vec{H} = \nabla \times \vec{A} \) is magnetic field, with a vector potential \( \vec{A} \) satisfying the Coulomb gauge condition \( \nabla \cdot \vec{A} = 0 \). The Hamiltonian of matter represents optic collective excitations and can be modelled by an ensemble of oscillators,

\[
\hat{H}_m = \sum_{i=1}^{N'} \frac{\vec{p}_i^2}{2m} + \frac{1}{2} \sum_{ij} \sum_{\alpha\beta} D_{ij}^{\alpha\beta} u_i^\alpha u_j^\beta ,
\]  

(4)

where the index \( i = 1, 2, \ldots, N' \) enumerate lattice sites, \( \vec{p}_i \) and \( \vec{u}_i \) are momentum and displacement operators, respectively, and \( D_{ij}^{\alpha\beta} \) is a dynamical matrix. The atom–field interaction is described by the Hamiltonian

\[
\hat{H}_{af} = -\frac{1}{c} \sum_{i=1}^{N} J_a(\vec{r}_i) \vec{A}_i(\vec{r}_i),
\]  

(5)

which corresponds to the dipole approximation with the transition current

\[
\vec{J}_a(\vec{r}_i) = i\omega_0 \left( \sigma_i^+ \vec{d}^* - \sigma_i^- \vec{d} \right),
\]  

(6)

where \( \sigma_i^+ \) and \( \sigma_i^- \) are the rising and lowering operators, respectively, and \( \vec{d} \) is a transition dipole. The matter–field interaction can be presented as

\[
\hat{H}_{mf} = -\frac{1}{c} \sum_{j=1}^{N'} \vec{J}_m(\vec{r}_j) \vec{A}(\vec{r}_j),
\]  

(7)

with the matter current

\[
\vec{J}_m(\vec{r}_j) = \frac{e}{m} \vec{p}_j,
\]  

(8)

in which \( e \) and \( m \) are charge and mass, respectively.

The commutation relations for the operators introduced above are

\[
\left[ \sigma_i^+, \sigma_j^- \right] = \delta_{ij} \sigma_i^z , \quad \left[ \sigma_i^z , \sigma_j^\pm \right] = \pm 2\delta_{ij} \sigma_i^\pm ,
\]  

6
\[ [E^\alpha(\vec{r}), A^\beta(\vec{r}')] = i4\pi c\delta_{\alpha\beta}\delta(\vec{r} - \vec{r}'). \]

Using these relations and the Heisenberg equations of motion, we get the Maxwell operator equations

\[
\frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\vec{E}, \quad \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{H} - \frac{4\pi}{c} \vec{J},
\]

with the total density of current

\[
\vec{J}(\vec{r}) = \sum_{i=1}^{N} \vec{J}_a(\vec{r}_i) \delta(\vec{r} - \vec{r}_i) + \sum_{j=1}^{N'} J_m(\vec{r}_j) \delta(\vec{r} - \vec{r}_j).
\]

For the atomic variables, we find

\[
\frac{d\sigma^-}{dt} = -i\omega_0\sigma^- + k_0\sigma^+ \vec{d}^\ast \cdot \vec{A}_i,
\]

and

\[
\frac{d\sigma^+}{dt} = -2k_0(\sigma^+_i \vec{d}^\ast + \sigma^-_i \vec{d}) \cdot \vec{A}_i,
\]

where the notation

\[
\vec{A}_i \equiv \vec{A}(\vec{r}_i, t), \quad k_0 \equiv \frac{\omega_0}{c}
\]

is used. From (9), with the Coulomb gauge condition, we have the wave equation

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{A} = -\frac{4\pi}{c} \vec{J},
\]

whose solution reads

\[
\vec{A}(\vec{r}, t) = \vec{A}_v(\vec{r}, t) + \frac{1}{c} \int \frac{\vec{J}(\vec{r}', t - |\vec{r} - \vec{r}'|/c) d\vec{r}'}{|\vec{r} - \vec{r}'|},
\]

where \( \vec{A}_v \), being a solution of the related homogeneous equation, corresponds to vacuum fluctuations. Substituting the density of current (10) into (14) yields for the vector potential at the point \( \vec{r}_i \) the expression

\[
\vec{A}_i(t) = \vec{A}_v(\vec{r}_i, t) + \vec{A}_a(\vec{r}_i, t) + \vec{A}_m(\vec{r}_i, t),
\]

in which the first term is caused by vacuum fluctuations, the second term,

\[
\vec{A}_a(\vec{r}_i, t) = ik_o \sum_{j(\neq i)}^N \frac{1}{r_{ij}} \left[ \sigma^+_j \left( t - \frac{r_{ij}}{c} \right) \vec{d}^\ast - \sigma^-_j \left( t - \frac{r_{ij}}{c} \right) \vec{d} \right],
\]

7
\[ r_{ij} \equiv |\vec{r}_{ij}|, \quad \vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j, \]

is a vector potential generated by radiating atoms, and the last term,

\[ \vec{A}_m (\vec{r}, t) = \frac{1}{c} \sum_{j \neq i} \frac{1}{r_{ij}} \vec{J}_m \left( \vec{r}_j, t - \frac{r_{ij}}{c} \right), \tag{17} \]

is due to local electric currents in the medium. In the vector potentials (16) and (17) the self–action parts are excluded. Instead, we shall add to Eqs. (11) and (12) the terms describing the level width and the line width,

\[ \gamma_1 = \frac{2}{3} k_0^3 d_0^2 = \frac{1}{T_1}, \quad \gamma_2 = \frac{1}{T_2}, \]

where \( d_0 \equiv |\vec{d}| \). In this way, introducing the effective electric induction

\[ \vec{D}_i (t) \equiv k_0 \left[ \vec{A}_v (\vec{r}, t) + \vec{A}_m (\vec{r}, t) \right], \tag{18} \]

we come to the equations

\[ \frac{d\sigma^-}{dt} = -(i\omega_0 + \gamma_2)\sigma^- + \sigma^+ \vec{d} \cdot \vec{D}_i + \]

\[ + ik_0^2 \sigma^+ \vec{d} \cdot \sum_{j \neq i} \frac{1}{r_{ij}} \left[ \sigma^+ \left( t - \frac{r_{ij}}{c} \right) \vec{d}^* - \sigma^- \left( t - \frac{r_{ij}}{c} \right) \vec{d} \right] \tag{19} \]

and

\[ \frac{d\sigma^+}{dt} = -\gamma_1 (\sigma^+ - \zeta) + 2(\sigma^+ \vec{d} + \sigma^- \vec{d}) \cdot \vec{D}_i - \]

\[ - i2k_0^2 (\sigma^+ \vec{d} + \sigma^- \vec{d}) \sum_{j \neq i} \frac{1}{r_{ij}} \left[ \sigma^+ \left( t - \frac{r_{ij}}{c} \right) \vec{d}^* - \sigma^- \left( t - \frac{r_{ij}}{c} \right) \vec{d} \right]. \tag{20} \]

The retardation effects in these equations can be treated in the quasirelativistic approximation. This means the following. In the nonrelativistic limit, when \( c \to \infty \) and \( k_0 \to 0 \), from (19) would follow \( \sigma^- \sim \exp(-i\omega_0 t) \). In the quasirelativistic approximation, we set

\[ \sigma^- \left( t - \frac{r_{ij}}{c} \right) \simeq \sigma^- (t) \exp(ik_0 r_{ij}) \tag{21} \]
Define the statistical averages

\[ u_i \equiv < \sigma_i^- >, \quad s_i \equiv < \sigma_i^z > \]  \tag{22}

over atomic degrees of freedom. Then from (19) and (20), in the semiclassical approximation, we obtain

\[ \frac{du_i}{dt} = -(i\omega_0 + \gamma_2)u_i + s_i (\vec{d}_i \cdot \vec{D}_i) + ik_0^3 s_i \cdot \vec{d} \cdot \sum_{j(\neq i)}^N (\varphi_{ij}^* u_j^* \vec{d} - \varphi_{ij} u_j \vec{d}) \]  \tag{23}

and

\[ \frac{ds_i}{dt} = -\gamma_1 (s_i - \zeta) - 2(u_i^* \vec{d}^* + u_i \vec{d}) \cdot \vec{D}_i -
\]

\[ -i2k_0^3(u_i^* \vec{d}^* + u_i \vec{d}) \cdot \sum_{j(\neq i)}^N (\varphi_{ij}^* u_j^* \vec{d} - \varphi_{ij} u_j \vec{d}), \]  \tag{24}

where

\[ \varphi_{ij} \equiv \frac{\exp(i k_0 r_{ij})}{k_0 r_{ij}}. \]

The semiclassical approximation is a kind of the mean-field approximation. In the spirit of these, we may make the following mean-field approximation

\[ \sum_{j(\neq i)}^N \varphi_{ij} u_j \approx u_i \sum_{j(\neq i)}^N \varphi_{ij} \equiv u_i \varphi_i, \]

where

\[ \varphi_i \equiv \sum_{j(\neq i)}^N \varphi_{ij} = \sum_{j(\neq i)}^N \frac{\exp(i k_0 r_{ij})}{k_0 r_{ij}}. \]

The factors \( \varphi_{ij} \) and \( \varphi_i \) describe local fields.

Introduce the local-field shift

\[ \Delta L \equiv \gamma_2 g' s, \quad g' \equiv k_0^3 d_0^2 \sum_{j(\neq i)}^N \cos(k_0 r_{ij}) \]  \tag{25}

also called the cooperative Lamb shift [17], and the effective atom–atom coupling parameter

\[ g \equiv \frac{k_0^3 d_0^2}{\gamma_2} \sum_{j(\neq i)}^N \frac{\sin(k_0 r_{ij})}{k_0 r_{ij}}. \]  \tag{26}
These quantities enter into the definitions of the effective radiation frequency and radiation width,
\[ \Omega \equiv \omega_0 + \Delta_L \, , \quad \Gamma \equiv \gamma_2(1 - gs) \, , \] (27)
respectively.

Involving these notations and keeping in mind that
\[ u_i = u(\vec{r}_i, t) \, , \quad s_i = s(\vec{r}_i, t) \, , \quad \vec{D}_i = \vec{D}(\vec{r}_i, t) \, , \quad \varphi_i = \varphi(\vec{r}_i) \, , \]
we transform Eqs. (23) and (24) to the form
\[ \frac{du}{dt} = -(i\Omega + \Gamma)u + s \, \vec{d}^\ast \cdot \vec{D} + ik_0^3 s \varphi^\ast u^\ast (\vec{d}^\ast)^2 \] (28)
and
\[ \frac{ds}{dt} = -4\gamma_2 g |u|^2 - \gamma_1 (s - \zeta) - 2(u^\ast \vec{d}^\ast + u \vec{d}) \cdot \vec{D} - \]
\[ -i2k_0^3 \left[ \varphi^\ast (u^\ast \vec{d}^\ast)^2 - \varphi(u \vec{d})^2 \right] . \] (29)
Since $u$ is a complex variable, we have to supplement Eqs. (28) and (29) by an equation for either $u^\ast$ or $|u|^2$. For instance, for $|u|^2$ we get
\[ \frac{d|u|^2}{dt} = -2\Gamma |u|^2 + s(u^\ast \vec{d}^\ast + u \vec{d}) \cdot \vec{D} + \]
\[ + ik_0^3 s \left[ \varphi^\ast (u^\ast \vec{d}^\ast)^2 - \varphi(u \vec{d})^2 \right] . \] (30)
Equations (29) and (30) give for the Bloch vector the equation
\[ \frac{d}{dt} \left( s^2 + 4|u|^2 \right) = -8\gamma_2 |u|^2 - 2\gamma_1 (s - \zeta) s . \]
The derived equations (28), (29), and (30) are the basic equations describing nonequilibrium processes in the system of resonance atoms interacting with polariton field.
3 Scale Separation

Equations (28), (29), and (30) can be solved using the scale separation approach [18–20]. To start with, we need to define what small parameters we have.

The standard small parameters are related to the relaxation parameters \( \gamma_1 \) and \( \gamma_2 \), for which
\[
\frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1. \tag{31}
\]
It is reasonable to suppose that
\[
\left| \frac{\Delta_L}{\Omega} \right| < 1, \quad \left| \frac{\Gamma}{\Omega} \right| < 1, \tag{32}
\]
although \( \Gamma \) can become much larger than \( \gamma_2 \). Assume also that
\[
\left| \frac{\vec{d} \cdot \vec{D}}{\Omega} \right| < 1, \tag{33}
\]
which means that the interaction of atoms with matter does not change drastically the properties of the atoms. Under the validity of small parameters (31) to (33), the variable \( u \) has to be considered as fast, compared to \( s \) and \( |u|^2 \) that are to be treated as slow. Accepting the variables \( s \) and \( |u|^2 \) as quasi–integrals of motion, we keep them fixed when solving Eq. (28). Then the solution for the fast variable is
\[
u(t) = u_0 G_1(t) + u^*_0 G_2(t) + \\
+ s \vec{d} \int_0^t \left[ G_1(t - \tau) + G_2(t - \tau) \right] \vec{D}(\tau) d\tau, \tag{34}
\]
where \( u_0 \equiv u(0) \) and the Green functions are
\[
G_1(t) = \left( \frac{\lambda_1 - a^*}{\lambda_1 - \lambda_2} \right) e^{\lambda_1 t} - \left( \frac{\lambda_2 - a^*}{\lambda_1 - \lambda_2} \right) e^{\lambda_2 t}, \\
G_2(t) = \frac{b}{\lambda_1 - \lambda_2} \left( e^{\lambda_1 t} - e^{\lambda_2 t} \right), \\
a = -(i\Omega + \Gamma), \quad b = -ik_0^3 s \varphi^*(\vec{d}^*)^2, \\
\lambda_{1,2} = \frac{1}{2} \left[ a + a^* \pm \sqrt{(a-a^*)^2 + 4|b|^2} \right].
\]
Taking into account the existence of small parameters, we may write

$$\lambda_{1,2} = \pm i\Omega - \Gamma,$$

and (34) can be simplified to

$$u(t) = e^{-(i\Omega+\Gamma)t} \left[ u_0 + s d^* \int_0^t e^{(i\Omega+\Gamma)\tau} \hat{D}(\tau)d\tau \right]. \quad (35)$$

The found fast variable (35) is to be substituted into the equations (29) and (30) for the slow variables and the right-hand sides of these equations are to be averaged over time and over the degrees of freedom corresponding to collective excitations of matter [18–20]. Recall that the quantities in (22) were defined as the averages over atomic degrees of freedom. The double averaging, over time and over the matter degrees of freedom, for a function $F(t)$, depending on time and on the operators of collective excitations, is defined as

$$\langle\langle F \rangle\rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \langle F(t) \rangle dt, \quad (36)$$

where the angle brackets imply the statistical averaging over the matter degrees of freedom. The usage of the same angle brackets for denoting the statistical averaging over the atomic and over matter degrees of freedom should not yield confusion, since at the present stage the atomic degrees of freedom do not arise being averaged out earlier. Therefore, in the definition (36) and in what follows the statistical averaging always concerns only the matter degrees of freedom.

Let us introduce the parameter

$$\alpha \equiv \langle\langle e^{-\Gamma t} \int_0^t e^{(i\Omega+\Gamma)\tau} \hat{d}^* \cdot \hat{D}(\tau)d\tau \rangle \rangle^2 \rangle \quad (37)$$

characterizing the strength of interaction between the atoms and matter. Thus, the quantity (37) can be called the atom–medium coupling parameter.

When substituting the fast variable (35) into the equations (29) and (30) for the slow variables and averaging, according to (36), the right-hand sides of the
latter equations, we may notice the following useful property. The fast variable (35) can be written as the sum
\[ u = u_1 + u_2 \]
of the term
\[ u_1 = u_0 \exp\left\{-(i\Omega + \Gamma)t\right\}, \]
not depending on the field \( \vec{D} \) of matter, and of the term
\[ u_2 = \exp\left\{-(i\Omega + \Gamma)t\right\} \int_0^t \exp\left\{(i\Omega + \Gamma)\tau\right\} \vec{D}(\tau) d\tau, \]
depending on the matter field. Define the function
\[ w \equiv |u_1|^2 = |u_0|^2 \exp\left\{-2\Gamma t\right\}. \]
By this definition, the function \( w \) must satisfy an equation that follows from the equation for \( |u|^2 \) where the matter field \( \vec{D} \) is set zero. For \( |u|^2 \) we have
\[ |u|^2 = w + u_1^* u_2 + u_2^* u_1 + |u_2|^2. \]
When averaging, according to (36), we take into account that
\[ \langle\langle u_1^* u_2 + u_2^* u_1 \rangle\rangle = 0. \]
This is because of two reasons. First, the terms \( u_1 \) and \( u_2 \) oscillate, in general, with different frequencies. Second, the term \( u_2 \) is a linear combination of operators of collective excitations in matter. In this way, averaging \( |u|^2 \) over fast variables, we get
\[ |u|^2 = w + \langle\langle |u_2|^2 \rangle\rangle, \quad \langle\langle |u_2|^2 \rangle\rangle = \alpha s^2. \]
This consideration suggests that it is convenient to introduce the slow variable
\[ w \equiv |u|^2 - \alpha s^2, \quad (38) \]
for which the evolution equation should have a form simpler than for \( |u|^2 \). Really, averaging the equations (29) and (30) for the slow variables, we obtain
\[ \frac{ds}{dt} = -4g\gamma_2 w - \gamma_1(s - \zeta), \quad (39) \]
where the transformation (38) is used, and
\[
\frac{dw}{dt} = -2\gamma_2(1 - gs)w .
\]  
(40)

From these two equations one can derive one equation
\[
\frac{d^2 s}{dt^2} + (2 + \gamma - 2gs)\frac{ds}{dt} - 2\gamma gs^2 + 2\gamma(1 + g\zeta)s - 2\gamma\zeta = 0 ,
\]
in which \(\gamma = \gamma_1/\gamma_2\) and time is measured in units of \(\gamma_2^{-1}\).

**4 Models of Matter**

Before analysing equations (39) and (40), let us consider some examples defining concretely the matter field \(\vec{D}\). Suppose, first, that the matter consists of a set of random scatterers, such that
\[
\vec{d} \cdot \vec{D} = \xi ,
\]  
(41)
where \(\xi\) is a stochastic field defined by the averages
\[
< \xi >= 0 , \quad < |\xi|^2 > = \gamma^2 .
\]  
(42)
Then the coupling parameter (37) is
\[
\alpha = \frac{\gamma^2}{\Omega^2} .
\]  
(43)

As another example, consider the matter modelled by the white noise, when
\[
\vec{d} \cdot \vec{D} = \xi(t) ,
\]  
(44)
where the white–noise stochastic variable \(\xi(t)\) is defined by the averages
\[
< \xi(t) >= 0 , \quad < \xi^*(t)\xi(t') >= 2\gamma\delta(t-t') ,
\]  
(45)
where the angle brackets mean a stochastic averaging. Then for the coupling parameter, we get
\[
\alpha = \frac{\gamma}{\Gamma} .
\]  
(46)
In the third example, we model the matter by an oscillator, so that

$$\vec{d} \cdot \vec{D} = \gamma \left( b_\omega e^{-i\omega t} + b_\omega^\dagger e^{i\omega t} \right),$$

(47)

where $b_\omega$ and $b_\omega^\dagger$ are the annihilation and creation operators satisfying the Bose statistics, and for which the statistical averaging gives

$$< b_\omega^\dagger b_\omega > = n_\omega, \quad < b_\omega b_\omega^\dagger > = 1 + n_\omega,$$

(48)

with $n_\omega$ being an effective weight of excitations of a frequency $\omega$. Then the coupling parameter (37) is

$$\alpha = |\gamma|^2 \left[ \frac{n_\omega}{(\Omega - \omega)^2 + \Gamma^2} + \frac{1 + n_\omega}{(\Omega + \omega)^2 + \Gamma^2} \right].$$

(49)

The strongest coupling between the impurity atoms and matter happens at the resonance, when $\omega = \Omega$, and $\alpha \approx n_\omega |\gamma/\Gamma|^2$.

Finally, we consider a more realistic situation when the effective electric induction of matter is defined by the relations (18), (17), and (8), so that

$$\vec{D}_i = \frac{e k_0}{mc} \sum_{j \neq i}^N \frac{1}{r_{ij}} \vec{p}_j \left( t - \frac{r_{ij}}{c} \right),$$

(50)

with the momentum operator

$$\vec{p}_j \left( t \right) = -i \sum_{ks} \left( \frac{m \omega_{ks}}{2N'} \right)^{1/2} \vec{e}_{ks} \times$$

$$\times \left[ b_{ks} \exp \{ i(\vec{k} \cdot \vec{r}_j - \omega_{ks} t) \} - b_{ks}^\dagger \exp \{ -i(\vec{k} \cdot \vec{r}_j - \omega_{ks} t) \} \right],$$

(51)

in which $\omega_{ks} = \omega_{-ks}$ is a spectrum of collective excitations; $\vec{k}$ being a wave vector; $s = 1, 2, 3$, a polarization index; $\vec{e}_{ks}$ is a polarization vector; $N'$ is the number of lattice sites. The annihilation and creation operators of collective excitations satisfy the Bose statistics and have the following statistical averages

$$< b_{ks}^\dagger b_{k's'} > = n_{ks} \delta_{kk'} \delta_{ss'}, \quad < b_{ks} b_{k's'} > = 0.$$

(52)

In this case, for the coupling parameter (37), we obtain

$$\alpha = \frac{k_0 r_c}{2N'} \sum_{ks} f_{ks} \gamma_{ks} \omega_{ks} \left[ \frac{n_{ks}}{(\Omega - \omega_{ks})^2 + \Gamma^2} + \frac{1 + n_{ks}}{(\Omega + \omega_{ks})^2 + \Gamma^2} \right].$$

(53)
where
\[ \gamma_{ks} \equiv k_0^3 | \vec{d} \cdot \vec{e}_{ks} |^2, \quad r_e \equiv \frac{e^2}{m \sqrt{\epsilon}} , \]
and
\[ f_{ks} \equiv \left| \sum_{j \neq i}^{N'} \exp \left\{ i \left( \vec{k} \cdot \vec{r}_{ij} + \frac{\omega_{ks}}{c} \right) \right\} \right|^2. \]

Again, it is clear that the coupling parameter (53) is the most strongly influenced by resonance collective excitations with \( \omega_{ks} \approx \Omega. \)

5 Transient Regime

Consider the times shorter than \( \gamma^{-1}. \) Then the term containing \( \gamma_1 \) in (39) can be omitted. In this case, using the second relation from (27), we have
\[ \frac{d\Gamma}{dt} = 4g^2 \gamma_2 w , \quad \frac{dw}{dt} = -2\Gamma w . \]

These two equations can be reduced to one,
\[ \frac{d^2\Gamma}{dt^2} + 2\Gamma \frac{d\Gamma}{dt} = 0 , \]
integrating which we get
\[ \frac{d\Gamma}{dt} + \Gamma^2 = \gamma_0^2 , \]
\( \gamma_0 \) being an integration constant. The last equation is a Riccati equation whose solution is
\[ \Gamma = \gamma_0 \tanh \left( \frac{t - t_0}{\tau_0} \right) , \quad \gamma_0 \equiv \frac{1}{\tau_0} , \]
where \( t_0 \) is another integration constant. Using relation (27) gives
\[ s = -\frac{\gamma_0}{g \gamma_2} \tanh \left( \frac{t - t_0}{\tau_0} \right) + \frac{1}{g} . \]

The first equation in (56), together with (57), yields
\[ w = \frac{\gamma_0^2}{4g^2 \gamma_2^2} \sech^2 \left( \frac{t - t_0}{\tau_0} \right) . \]
And from (38) we find

$$|u|^2 = \frac{\gamma_0^2}{4g^2\gamma_2^2} \operatorname{sech}^2\left(\frac{t - t_0}{\tau_0}\right) + \alpha s^2.$$  

(60)

The integration constants $\gamma_0$ and $t_0$ are to be defined from the initial conditions

$$u(0) = u_0, \quad s(0) = s_0.$$  

(61)

From the latter we obtain for the effective radiation width $\gamma_0$ the expression

$$\gamma_0^2 = \Gamma_0^2 + 4g^2\gamma_2^2 \left(|u_0|^2 - \alpha_0 s_0^2\right),$$  

(62)

in which $\alpha_0$ is $\alpha$ at $t = 0$, when $s = s_0$,

$$\Gamma_0 \equiv \gamma_2(1 - gs_0),$$

and the delay time

$$t_0 = \frac{\tau_0}{2} \ln \left|\frac{\gamma_0 - \Gamma_0}{\gamma_0 + \Gamma_0}\right|.$$  

(63)

The radiation width can be written in the form

$$\gamma_0 = 2|gs_0|\gamma_2(\alpha_c - \alpha_0)^{1/2},$$  

(64)

where the critical atom–matter coupling parameter

$$\alpha_c \equiv \frac{(1 - gs_0)^2 + 4g^2|u_0|^2}{4gs_0^2}$$  

(65)

is introduced. Expression (64) is a direct consequence of (62) for all $g$ and $\alpha_0$.

It is necessary to stress that the atom–matter coupling parameter (37) cannot surpass the critical value (65). If this would happen, then the radiation width (64) would become imaginary and, instead of (57), we would have

$$\Gamma = -|\gamma_0|\tan\left(\frac{t - t_0}{\tau_0}\right), \quad |\tau_0| \equiv \frac{1}{|\gamma_0|},$$

$$|\gamma_0| = 2\gamma_2|gs_0|\sqrt{\alpha_0 - \alpha_c} \quad (\alpha_0 > \alpha_c).$$

The delay time (63) would be

$$t_0 = |\tau_0|\arctan\left(\frac{1 - gs_0}{2|gs_0|\sqrt{\alpha_0 - \alpha_c}}\right).$$
And for the solutions (58) and (59) we would get
\[ s = s_0 \text{sgn}(g s_0) \sqrt{\alpha_0 - \alpha_c} \tan \left( \frac{t - t_0}{\tau_0} \right) + \frac{1}{g}, \]
\[ w = -s_0^2 (\alpha_0 - \alpha_c) \sec^2 \left( \frac{t - t_0}{\tau_0} \right). \]

The effective width \( \Gamma \), as well as the solutions \( s \) and \( w \), become divergent at \( t = t_n \),
\[ t_n = t_0 + \frac{\pi}{2} (1 + 2n)|\tau_0| \quad (n = 0, 1, 2\ldots). \]

Certainly, this behaviour is unphysical and it means that some conditions, under which the method of scale separation has been used, are, probably, not valid any more. This is really the case since, when \( \Gamma \) and \( s \) diverge, conditions (32) do not hold true. Consequently, when \( \alpha_0 > \alpha_c \), we cannot separate solutions onto fast and slow, all of them oscillating equally fast. At the same time, the existence of slow solutions is a characteristic feature of developed coherence. Thus, the absence of slow solutions suggests that coherence cannot emerge in the system. From the physical point of view, all this sounds quite understandable. There should be a threshold for the strength of interactions of atoms with matter after which such strong interactions destroy the correlation between atoms, thus, destroying their coherence. In this way, the inequality
\[ \alpha_0 < \alpha_c \quad (66) \]
is a necessary condition for the applicability of the scale separation approach and, at the same time, a condition for the possibility of coherent radiation of doped atoms.

The maximal level of coherence develops at the time \( t = t_0 \) when
\[ s(t_0) = \frac{1}{g}, \quad w(t_0) = s_0^2(\alpha_c - \alpha_0), \quad |u(t_0)|^2 = (\alpha_c - \alpha_0)s_0^2 + \frac{\alpha}{g^2}. \quad (67) \]

For the times much longer than \( t_0 \), Eqs. (58) to (60) give
\[ s \simeq \frac{1}{g} \left( 1 - \frac{\gamma_0}{\gamma_2} \right) \quad (t \gg t_0), \]
\[ w \simeq \frac{\gamma_0^2}{g^2\gamma_2} \exp(-2\gamma_0 t), \quad (68) \]

\[ |u|^2 \simeq \frac{\gamma_0^2}{g^2\gamma_2} \exp(-2\gamma_0 t) + \frac{\alpha}{g^2} \left(1 - \frac{\gamma_0}{\gamma_2}\right)^2. \]

However, the asymptotic behaviour given by (68) is valid only for \( t \ll T_1 \).

Consider the case when both the atom–matter and atom–atom coupling parameters are small, i.e.

\[ \alpha_0 \ll \alpha_c, \quad |g| \ll 1. \quad (69) \]

Using the first of inequalities in (69), we have from (64)

\[ \gamma_0 \simeq \gamma_2 \left[(1 - gs_0)^2 + 4g^2|u_0|^2\right]^{1/2} \left(1 - \frac{\alpha_0}{2\alpha_c}\right). \]

The latter expression, with the second inequality in (69), reduces to

\[ \gamma_0 \simeq \gamma_2(1 - gs_0) \left(1 - \frac{\alpha_0}{2\alpha_c}\right). \]

The critical parameter (65) becomes

\[ \alpha_c \simeq (4g^2s_0^2)^{-1} \quad (g \ll 1). \quad (70) \]

Employing this, we find

\[ \gamma_0 \simeq \gamma_2(1 - gs_0 - 2\alpha_0 g^2 s_0^2), \quad (71) \]

valid for small coupling parameters as in (69). For the delay time (63), we get

\[ t_0 \simeq \frac{1 + gs_0}{2\gamma_2} \ln \left|\alpha_0 g^2 s_0^2\right|, \quad (72) \]

which tends to \(-\infty\) if either \(\alpha_0\) or \(g\) tends to zero. This implies that, under conditions (69), an essential level of coherence does not evolve.

Let us analyse the case when

\[ \alpha_0 \ll \alpha_c, \quad |g| \gg 1. \quad (73) \]

Then the critical parameter (65) is

\[ \alpha_c \simeq \frac{1}{4s_0^2} \left(s_0^2 + 4|u_0|^2 - 2\frac{s_0}{g}\right). \quad (74) \]
The radiation width (64) becomes
\[
\gamma_0 \simeq \gamma_2 \frac{|g|}{\sqrt{s_0^2 + 4|u_0|^2}} \left( s_0^2 + 4|u_0|^2 - 2\alpha_0 s_0^2 - \frac{s_0}{g} \right),
\] (75)
with the corresponding radiation time
\[
\tau_0 \simeq \frac{T_2}{|g|\sqrt{s_0^2 + 4|u_0|^2}}.
\] (76)

For the delay time (63), we find
\[
t_0 \simeq \frac{\tau_0}{2} \ln \frac{|g|(s_0^2 + 4|u_0|^2 - 2\alpha_0 s_0^2) + gs_0\sqrt{s_0^2 + 4|u_0|^2}}{|g|(s_0^2 + 4|u_0|^2 - 2\alpha_0 s_0^2) - gs_0\sqrt{s_0^2 + 4|u_0|^2}}.
\] (77)

If the process develops from an initially incoherent state, when \(u_0 = 0\), then the radiation width (75) is
\[
\gamma_0 \simeq \gamma_2 |g s_0| \left( 1 - 2\alpha_0 - \frac{1}{gs_0} \right) \quad (u_0 = 0).
\] (78)

Thence, the delay time (77) becomes
\[
t_0 \simeq \frac{T_2}{2|g s_0|} \ln \left| \frac{1 - 2\alpha_0 + \varepsilon}{1 - 2\alpha_0 - \varepsilon} \right|,
\] (79)
where
\[\varepsilon \equiv \text{sgn}(gs_0) = \pm 1.\]

As far as for \(u_0 = 0\) and \(|g| \gg 1\), the critical parameter (74) is
\[
\alpha_c \simeq \frac{1}{4} \quad (|g| \gg 1, \ u_0 = 0),
\]
then the inequality \(\alpha_0 \ll \alpha_c\) implies \(\alpha_0 \ll 1\). Hence, we may simplify (79) as
\[
t_0 \simeq \frac{T_2}{2gs_0} |\ln \alpha_0|.
\] (80)

After the time (80), the population difference tends to
\[
s \simeq -\varepsilon s_0 (1 - 2\alpha_0) + \frac{1 + \varepsilon}{g} \quad (t \gg t_0).
\] (81)
If $g > 0$, then $\varepsilon s_0 = |s_0|$, while for $g < 0$, one has $\varepsilon s_0 = -|s_0|$. Combining both these cases, we get $\varepsilon s_0 = \text{sgn}(g)|s_0|$. Assume now that the atom–matter coupling parameter is close to its critical value (65), but the atom–atom coupling is arbitrary, 

$$\frac{\left|\alpha_0 - \alpha_c\right|}{\alpha_c} \ll 1 \quad (\forall g) \quad (82)$$

Then the radiation width (64) tends to zero, as $\alpha_0 \to \alpha_c$, and respectively, the radiation time $\tau_0 \equiv \gamma_0^{-1}$ tends to infinity. For the delay time (63), we find

$$t_0 \simeq \frac{2|gs_0|T_2}{(1 - gs_0)^2} (\alpha_c - \alpha_0)^{1/2} \quad , (83)$$

while the radiation time is

$$\tau_0 = \frac{T_2}{2|gs_0|} (\alpha_c - \alpha_0)^{-1/2} \quad . (84)$$

When $\alpha_0 \to \alpha_c$, then $t_0 \to 0$, and for the functions (58) and (59) we have

$$s \simeq \frac{1}{g} - 2|s_0|\text{sgn}(g)\sqrt{\alpha_c - \alpha_0} \left(1 - 2e^{-2\gamma_0 t}\right) \quad ,$$

$$w \simeq 4s_0^2 (\alpha_c - \alpha_0)e^{-2\gamma_0 t} \quad (t > t_0) \quad . (85)$$

There is a suppression of self–organized coherence in the system of atoms, their radiation being almost completely due to the pumping by matter excitations

$$s \approx \frac{1}{g} \quad , \quad w \approx 0 \quad , \quad |u|^2 \approx \frac{\alpha}{g^2} \quad .$$

Although the coherent relaxation may happen provided that $\tau_0 \ll T_2$, that is,

$$|gs_0|(\alpha_c - \alpha_0)^{1/2} \gg 1 \quad ,$$

which corresponds to superradiant emission.

Note that analysing the properties of the solutions to equations (39) and (40), we talk about radiation processes keeping in mind the following. The total radiation intensity of atoms can be approximately defined in the usual way as

$$I(t) = -N\hbar\omega_0 \frac{ds}{dt} \quad . (86)$$
For a more accurate definition of radiation intensity see e.g. Refs. [21, 22]. From (86), using equation (39), we find

\[ I(t) = I_{coh}(t) + I_{inc}(t), \]  

(87)

where the first term

\[ I_{coh}(t) = 4Ng\hbar\omega_0\gamma_2 w \]  

(88)

had the meaning of the coherent radiation intensity, and the second,

\[ I_{inc}(t) = N\hbar\omega_0\gamma_1(s - \zeta), \]  

(89)

corresponds to the intensity of incoherent radiation. The latter is always proportional to the number of atoms \( N \), while the radiation intensity (88) is proportional to \( Ng \). For a concentrated sample, whose linear size is much smaller that the radiation wavelength, we have \( g \approx N \), and the radiation intensity (88) becomes proportional to \( N^2 \), which is typical of superradiance. In this way, the solutions \( s \) and \( w \) define the temporal behaviour of the incoherent radiation intensity (89) and of the coherent radiation intensity (88), respectively. For instance, using the solution \( w \) given by (59), with the radiation width (64), we obtain the intensity of coherent radiation

\[ I_{coh}(t) = 4Ng\hbar\omega_0\gamma_2 s_0^2(\alpha_c - \alpha_0)\text{sech}^2 \left( \frac{t - t_0}{\tau_0} \right). \]  

(90)

The latter shows that, if \( \alpha_0 \to \alpha_c \), then \( I_{coh} \to 0 \).

6 Close–to–Stationary Regime

In the previous section the transitient regime is considered corresponding to times \( t \ll T_1 \). For the times comparable or larger that \( T_1 \), we cannot neglect any more the term with \( \gamma_1 \) in equation (39). In the intermediate stage, when \( t \sim T_1 \), an exact solution of Eq. (39) and (40) is not available. Here we have to resort to
numerical calculations, which will be the subject of a separate paper. But it is possible to give an analysis for asymptotically large times, when \( t \gg T_1 \).

The following analysis assumes that \( g \neq 0 \). Since, if \( g = 0 \), the solutions to Eqs. (39) and (40) are

\[
\begin{align*}
    s &= \zeta + (s_0 - \zeta)e^{-\gamma_1 t}, \\
    w &= \left(|u_0|^2 - \alpha s_0^2\right)e^{-2\gamma_2 t} \quad (g = 0),
\end{align*}
\]

which describes the relaxation process of a single atom. In such a case, if there is the localization of light, then \( \zeta = s_0 \), and (91) gives \( s = s_0 \).

If \( N \) impurity atoms are doped into the matter, then \( g \neq 0 \). The resonance dipole–dipole interactions of a pair of atoms with a transition frequency inside the photon gap have been studied in several works [7,23,24]. The conclusion of these studies is that two closely spaced atoms, with transition frequencies in the gap, interact with each other by means of the virtual photon exchanges much in the same way as the atoms in vacuum. That is, if the atoms are separated from each other by a spacing much larger than the radiation wavelength, then each of them can be considered as a single atom. If the transition frequency of such an atom is inside the gap, then the phenomenon of light localization occurs. However, if the atoms are close to each other, with a spacing much smaller than the radiation wavelength, then they practically do not experience the existence of the gap [7,23,24].

Consider the close–to–stationary regime, when \( t \gg T_1 \). Equations (39) and (40) can be written as

\[
\begin{align*}
    \frac{ds}{dt} &= V_1 , \\
    \frac{dw}{dt} &= V_2 ,
\end{align*}
\]

with the right–hand sides

\[
\begin{align*}
    V_1 &= -4g\gamma_2 w - \gamma_1 (s - \zeta) , \\
    V_2 &= -2\gamma_2 (1 - gs)w .
\end{align*}
\]
Stationary points, or fixed points for Eqs. (92), (93), are given by the condition
\( V_1 = V_2 = 0 \). This yields two stationary points:

\[
S_1^* = \zeta , \quad W_1^* = 0 \tag{94}
\]

and

\[
S_2^* = \frac{1}{g} , \quad W_2^* = -\frac{\gamma_1(1 - g\zeta)}{4\gamma_2 g^2} \tag{95}
\]

The stability of these fixed points can be defined by the Lyapunov analysis. To this end, we need to find the eigenvalues of the Jacobian matrix

\[
\hat{J} = \begin{bmatrix}
\frac{\partial V_1}{\partial s} & \frac{\partial V_1}{\partial w} \\
\frac{\partial V_2}{\partial s} & \frac{\partial V_2}{\partial w}
\end{bmatrix} \tag{96}
\]

These eigenvalues are given by the expression

\[
\lambda^\pm = -\frac{1}{2} \left\{ \gamma_1 + 2\gamma_2 (1 - gs) \pm \left[ (\gamma_1 - 2\gamma_2 (1 - gs))^2 - 32\gamma_2^2 g^2 w \right]^{1/2} \right\} . \tag{97}
\]

Substituting here the values corresponding to the fixed points yields the Lyapunov exponents. For the stationary point (94), we have

\[
\lambda_1^+ = -\gamma_1 , \quad \lambda_1^- = -2\gamma_2 (1 - g\zeta) , \tag{98}
\]

and for the stationary point (95), we find

\[
\lambda_2^\pm = -\frac{\gamma_1}{2} \left\{ 1 \pm \left[ 1 + 8\frac{\gamma_2}{\gamma_1} (1 - g\zeta) \right]^{1/2} \right\} . \tag{99}
\]

The analysis of the Lyapunov exponents (98) and (99) shows that if

\[
g\zeta < 1 , \tag{100}
\]

then the fixed point (94) is a stable node, and the fixed point (95) is a saddle point. When

\[
g\zeta = 1 , \tag{101}
\]
both fixed points merge together becoming neutral, since $\lambda_1^- = \lambda_2^- = 0$. In this case, the system of equations (39) and (40) is structurally unstable. Equality (101) defines a bifurcation point. For the interval

$$1 < g\zeta \leq 1 + \frac{\gamma_1}{8\gamma_2},$$  \hfill (102)

the fixed point (94) is a saddle point, while that (95) is a stable node. For all $g\zeta > 1$, the point (94) is a saddle point. If

$$g\zeta > 1 + \frac{\gamma_1}{8\gamma_2},$$  \hfill (103)

the stationary point (95) becomes a stable focus, since the Lyapunov exponents (99) take the form

$$\lambda_2^\pm = -\frac{\gamma_1}{2} \pm i\omega_\infty,$$  \hfill (104)

where

$$\omega_\infty \equiv \frac{\gamma_1}{2} \left[ \frac{8\gamma_2 (g\zeta - 1)}{\gamma_1} - 1 \right]^{1/2}.$$  

Suppose that for a single atom there occurs the localization of light, so that $\zeta = s_0$. If many resonant atoms are doped into matter, but their interactions through the polariton exchange are not strong enough, so that $g_0s_0 < 1$, then the light localization prevails. This means that the light remains confined in the vicinity of the atoms. The confinement of light is demonstrated by the fact that the stationary point (94) is a stable node with $s_1^* = s_0$. But if the resonant interaction between the atoms is sufficiently strong, so that

$$g_0s_0 > 1,$$  \hfill (105)

the deconfinement of light happens. Then the fixed point (95) becomes stable, while the point (94) looses its stability. The deconfinement of light is not complete, since $s_2^* = 1/g < s_0$, but it is partial. The portion of light that remains confined decreases with increasing $g$. The qualitative change of the asymptotic behaviour of solutions to a system of differential equations is called, in dynamical
theory, the dynamical phase transition. In our case, this happens if $g\zeta = 1$. The equality $g_{s_0} = 1$ separates the regions where light is localized ($g_{s_0} < 1$) and where it is deconfined ($g_{s_0} > 1$). Therefore, the dynamical phase transition occurring at $g_{s_0} = 1$ corresponds to a transition that may be called the *deconfinement of light* or *photon deconfinement*.

When the resonant interaction between atoms is so strong that inequality (103) holds true, then the stable stationary point (95) is a focus. This means that the solutions to the equations (39) and (40) display an oscillatory regime of motion when approaching the stationary point (95). Such an oscillatory motion is similar to that found for a concentrated system with the resonant frequency near the edge of a photonic band gap [5] and to that for two atoms with transition frequencies inside or slightly outside a photonic band gap [24].

### 7 Coupling Parameters

There are several characteristic quantities defining the behaviour of the system. These are the initial conditions $s_0 \equiv s(0)$ and $u_0 \equiv u(0)$ and the coupling parameters $g$, $g'$, and $\alpha$. Below we study the typical values of the latter.

Recall that the coupling parameters $g'$, defined in (25), and $g$, given in (26), have appeared in the evolution equations when treating the retardation effects in the quasirelativistic approximation (21). Without the latter approximation, we should deal with the integral–type equations [25]. Thus, the atom–atom coupling parameters $g'$ and $g$ describe the retardation or local–field effects. The values of these parameters essentially depend on the shape of the sample and on the spacing between atoms [17].

Accepting the equality $\gamma_1 = 2k_0^3d_0^2/3$, we have from (25) and (26)

$$g = \frac{3\gamma_1}{2\gamma_2} \sum_{j \neq i}^N \frac{\sin(k_0r_{ij})}{k_0r_{ij}}, \quad g' = \frac{3\gamma_1}{2\gamma_2} \sum_{j \neq i}^N \frac{\cos(k_0r_{ij})}{k_0r_{ij}}.$$

If the radiation wavelength $\lambda = 2\pi/k_0$ is much smaller than the mean spacing,
a, between the atoms, then the sums in \( g \) and \( g' \) can be of any sign but with absolute values less than unity. As far as usually \( \gamma_1 \ll \gamma_2 \), the absolute values \(|g|\) and \(|g'|\) are small. If \(|g| \ll 1\) and \(|g'| \ll 1\), the impurity atoms almost do not interact with each other and their behaviour is practically the same as that of a collection of single atoms.

In the opposite case, when \( \lambda \gg a \), the sums in \( g \) and \( g' \) can be estimated with a good approximation \[17\] as

\[
\sum_{j \neq i}^{N} \frac{\sin(k_0r_{ij})}{k_0r_{ij}} \approx \sum_{j \neq i}^{N} \frac{\cos(k_0r_{ij})}{k_0r_{ij}} \approx \rho \lambda^3 ,
\]

where \( \rho \equiv N/V \) is the density of the doped atoms and it is assumed that \( \lambda \) is less than the linear sizes of the sample in all directions. Then we have

\[
g \approx g' \approx \frac{3\gamma_1}{2\gamma_2 \lambda} \rho \lambda^3 .
\]

The value of the atom–matter coupling parameter (37) essentially depends on the peculiarity of the atom–matter interaction. As the models of Section 4 show, one should expect that \( \alpha \ll 1 \). If the transition frequency of the doped atoms lies outside the polariton band gap, then the atom–matter resonance is possible, when \( \omega_{ks} \sim \omega_0 \). Moving the atomic frequency into the gap makes such a resonance more and more difficult. Far inside the gap, where there are no elementary excitations of matter, this resonance becomes impossible. It follows from (53) that the relation between the atom–matter coupling parameter, \( \alpha_{out} \), corresponding to the case when the atomic frequency is outside the gap, and the parameter \( \alpha_{ins} \), when the frequency is far inside the gap, is roughly speaking, as

\[
\frac{\alpha_{out}}{\alpha_{ins}} \sim \frac{\Omega^2}{\Gamma^2} ,
\]

for a sufficiently large polariton band gap. The decrease of \( \alpha \) leads, as is clear from (80), to the increase of the delay time \( t_0 \).

One can also notice that the coupling parameters \( g \) and \( \alpha \) are not independent, but \( \alpha \) depends on \( g \). This dependence, for \( g \gg 1 \), is approximately as \( \alpha \sim g^{-2} \).
Therefore, strong atomic interactions diminish $\alpha$, thus, increasing the delay time (80). During the interval $0 \leq t < t_0$, there is a temporal localization of light even for rather large parameters $g$, such that $g\zeta > 1$, but then the process of photon deconfinement starts. If $t_0$ becomes comparable with $T_1$, one cannot omit in Eqs. (39), (40) the term containing $\gamma_1$. Then one should resort to numerical solution of these equations, which will be considered in a separate paper.

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