ON THE SPLITTING METHOD FOR THE NONLINEAR
SCHRÖDINGER EQUATION WITH INITIAL DATA IN $H^1$

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Abstract. In this paper, we establish a convergence result for the operator
splitting scheme $Z_\tau$ introduced by Ignat [12], with initial data in $H^1$, for the
nonlinear Schrödinger equation:

$$i \partial_t u = \Delta u + i\lambda |u|^p u, \quad u(x, 0) = \phi(x),$$

where $p > 0$, $\lambda \in \{-1, 1\}$ and $(x, t) \in \mathbb{R}^d \times [0, \infty)$. We prove the $L^2$ convergence
of order $O(\tau^{1/2})$ for the scheme with initial data in the space $H^1(\mathbb{R}^d)$ for the
energy-subcritical range of $p$.

1. Introduction. Consider the following Cauchy problem of the nonlinear
Schrödinger equation in $\mathbb{R}^{d+1}$:

$$\begin{cases}
\partial_t u = i\Delta u + i\lambda |u|^p u, & (x, t) \in \mathbb{R}^d \times \mathbb{R}, \\
u(x, 0) = \phi(x),
\end{cases}$$

(1.1)

where $\lambda \in \{-1, 1\}$. Nonlinear Schrödinger equations appear in various models of
quantum mechanics (see, e.g., [3, 23, 24]). In this paper, we are concerned with operator
splitting schemes, which are useful for the numerical computation of semilinear-
type equations (1.1). The idea of such schemes is to divide the problem (1.1) into
a linear flow and a nonlinear flow, as described below.

We define $N(t)\phi$ as the solution of the flow

$$\begin{cases}
\partial_t u = i\lambda |u|^p u, & x \in \mathbb{R}^d, \ t > 0, \\
u(x, 0) = \phi(x), & x \in \mathbb{R}^d,
\end{cases}$$

that is, $N(t)\phi = \exp(it\lambda |\phi|^p)\phi$. On the other hand, we set $S(t)\phi$ as the solution of the
linear Schrödinger propagation

$$\begin{cases}
\partial_t u = i\Delta u, & x \in \mathbb{R}^d, \ t > 0, \\
u(x, 0) = \phi(x), & x \in \mathbb{R}^d,
\end{cases}$$

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which admits the Fourier multiplier formula $S(t)\phi = e^{it\Delta} \phi$. Then we split the flow of (1.1) into the flows $N(t)$ and $S(t)$ with a small switching time. Namely, for a fixed time interval $[0, T]$ and a small value $\tau > 0$, we can consider the Lie approximation

$$Z(n\tau)\phi = \left(S(\tau)N(\tau)\right)^n \phi, \quad 0 \leq n \tau \leq T,$$

or the Strang approximation

$$Z(n\tau)\phi = (S(\tau/2)N(\tau)S(\tau/2))^n \phi, \quad 0 \leq n \tau \leq T.$$

The convergence of these two schemes has been studied by Besse et al. [2] for globally Lipschitz continuous nonlinearities and by Lubich [18] for Schrödinger-Poisson and cubic NLS equations with initial data in the space $H^1(\mathbb{R}^3)$. Recently, Eilinghoff et al. [5] established the convergence result for $H^{2+2\epsilon}(\mathbb{R}^d)$ with $\epsilon \in (0, 1)$ and $1 \leq d \leq 3$. On the other hand, Ignat and Zuazua [13, 14] and Ignat [11] developed various numerical schemes for which they proved Strichartz type estimates to obtain the convergence of the schemes with initial data of low regularity. Also, Ignat [12] introduced the following modified version of the splitting scheme:

$$Z_\tau(n\tau) = \left(S_\tau(\tau)N(\tau)\right)^n \Pi_\tau \phi. \quad (1.2)$$

Here, $S_\tau(t)$ denotes the frequency localized Schrödinger flow given by

$$S_\tau(t)\phi = S(t)\Pi_\tau \phi,$$

where

$$\Pi_\tau \phi(\xi) = \chi(\tau^{1/2} \xi) \hat{\phi}(\xi), \quad \xi \in \mathbb{R}^d, \quad (1.3)$$

and $\chi \in C^N(\mathbb{R}^d)$ is a cut-off function supported in $B^d(0, 2)$ such that $\chi = 1$ on $B^d(0, 1)$, where $N \in \mathbb{N}$ is some large number. In fact, it is sufficient to set $N = 2d$.

The aim of this paper is to determine an improved estimate for the splitting scheme $Z_\tau(n\tau)$. In particular, we prove the convergence result for $p$ in the energy-subcritical range when the initial data $\phi$ belongs to the space $H^1(\mathbb{R}^d)$. Before stating our result, we recall the previous results.

- (Lubich [18]) Let $d = 3$ and $p = 2$. Suppose that $\phi \in H^4(\mathbb{R}^3)$, and consider a time $T > 0$ such that $\sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{R}^3)} < \infty$. Then, the Strang approximation $Z$ satisfies

$$\max_{0 \leq n \tau \leq T} \|Z(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^3)} \leq \tau^2 C(T, \phi) \quad \text{and}$$

$$\max_{0 \leq n \tau \leq T} \|Z(n\tau) - u(n\tau)\|_{H^2(\mathbb{R}^3)} \leq \tau C(T, \phi).$$

- (Eilinghoff-Schaubelt-Schratz [5]) Let $1 \leq d \leq 3$ and $p = 2$. For the Strang approximation $Z$, we assume that $\phi \in H^{2+2\epsilon}(\mathbb{R}^d)$ with $\epsilon \in (0, 1)$ and that $T > 0$ is a positive such that $\sup_{0 \leq t \leq T} \|u(t)\|_{H^{2+2\epsilon}(\mathbb{R}^d)} < \infty$. Then we have

$$\max_{0 \leq n \tau \leq T} \|Z(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau^{1+\epsilon} C(T, \phi).$$

For the Lie approximation $Z$, we assume that $\phi \in H^2(\mathbb{R}^d)$ and that $T > 0$ is a positive such that $\sup_{0 \leq t \leq T} \|u(t)\|_{H^2(\mathbb{R}^d)} < \infty$. Then we have

$$\max_{0 \leq n \tau \leq T} \|Z(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau C(T, \phi).$$

- (Ignat [12]) Let $1 \leq d \leq 3$ and $1 \leq p < \frac{4}{3}$. For any $\phi \in H^2(\mathbb{R}^d)$ and any time $T > 0$, the approximation $Z_\tau$ in (1.2) satisfies

$$\max_{0 \leq n \tau \leq T} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau C(d, p, T, \|\phi\|_{H^2}).$$
The splitting method for the nonlinear Schrödinger equation

For the initial data in $H^s$ ($s > 0$), the order $s/2$ is understood as the natural order barrier for the convergence rate of the Strang and Lie splitting schemes (see [19]). In the following theorems, we provide the convergence results for initial data in $H^1(\mathbb{R}^d)$.

**Theorem 1.1.** Let $d \geq 1$ and $0 < p < \frac{4}{d}$. For any $\phi \in H^1(\mathbb{R}^d)$ and any time $T > 0$, the approximation $Z_\tau$ satisfies
\[
\max_{0 \leq n \tau < T} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau^{1/2}C(d, p, T, \|\phi\|_{H^1})
\]
for any $\tau \in (0, 1)$.

**Theorem 1.2.** Let $1 \leq d < 6$ and $1 \leq p < p_d$. Suppose that $\phi \in H^1(\mathbb{R}^d)$, and consider a time $T > 0$ such that $\sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{R}^d)} < \infty$. Then, the approximation $Z_\tau$ satisfies
\[
\max_{0 \leq n \tau < T} \|Z_\tau(n\tau) - u(n\tau)\|_{L^2(\mathbb{R}^d)} \leq \tau^{1/2}C(d, p, T, \phi)
\]
for any $\tau \in (0, 1)$. Here, $p_d = \infty$ if $d = 1, 2$ and $p_d = \frac{4}{d-2}$ if $d \geq 3$.

**Remark 1.3.** In fact, we can find an upper bound of $C(d, p, T, \|\phi\|_{H^1})$ in Theorem 1.1 such as
\[
C(d, p, T, \|\phi\|_{H^1}) \leq C_{d, p} \exp \left( \exp \left( \tilde{C}_{d, p} T^{c_2(d, p)} \|\phi\|_{H^1}^{c_2(d, p)} \right) \right)
\]
for some positive constants $C_{d, p}$, $\tilde{C}_{d, p}$, $c_1(d, p)$ and $c_2(d, p)$.

In order to obtain a convergence result with the low regularity assumption, Strichartz-type estimates are employed in [12] along with the Duhamel-type formula for $Z_\tau$, given by
\[
Z_\tau(n\tau) = S_\tau(n\tau)\phi + \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \frac{N(\tau)}{\tau} Z_\tau(k\tau), \quad n \geq 1, \quad (1.4)
\]
which is similar to the Duhamel formula of the solution $u$ to (1.1), expressed as
\[
u(t) = S(t)\phi + i\lambda \int_0^t S(t-s)|u|^p u(s)ds, \quad t \geq 0. \quad (1.5)
\]
In the above, the operator $I$ in (1.4) is the identity operator, and the formula (1.4) is obtained as follows (refer to (2.11) in [12]):
\[
Z_\tau(n\tau) = \left( S_\tau(\tau) + S_\tau(\tau)(N_\tau - I) \right) Z_\tau((n-1)\tau)
\]
\[
= S_\tau(\tau) Z_\tau((n-1)\tau) + S_\tau(\tau)(N_\tau - I) Z_\tau((n-1)\tau)
\]
\[
= \left( S_\tau(2\tau) Z_\tau((n-2)\tau) + S_\tau(2\tau)(N_\tau - I) Z_\tau((n-2)\tau) \right)
\]
\[
+ S_\tau(\tau)(N_\tau - I) Z((n-1)\tau)
\]
\[
= S_\tau(n\tau) Z_\tau(0) + \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau)(N_\tau(\tau) - I) Z_\tau(k\tau).
\]
A key ingredient of the convergence analysis in [12] is to obtain the stability (uniformly in $\tau \in (0, 1)$) of the scheme $Z_\tau$ in the discrete space $\ell^q(n\tau \in I; L^r(\mathbb{R}^d))$. 
Here, we introduce a few notations. For any interval \( I \subset [0, \infty) \), we define the space \( L^q(n \tau \in I; L^r(\mathbb{R}^d)) \) consisting of functions defined on \( \tau \mathbb{Z} \cap I \) with values in \( L^r(\mathbb{R}^d) \), the norm of which is given by
\[
\|u\|_{L^q(n \tau \in I; L^r(\mathbb{R}^d))} = \left\{ \begin{array}{ll}
\left( \frac{1}{q} \sum_{n \tau \in I} \|u(n \tau)\|_{L^r(\mathbb{R}^d)}^q \right)^{1/q} & \text{if } 1 < q < \infty, \\
\sup_{n \tau \in I} \|u(n \tau)\|_{L^r(\mathbb{R}^d)} & \text{if } q = \infty.
\end{array} \right.
\]

In the present work, we take into account the nonlinearity \(|u|^p u\) in the energy-subcritical range \( p \in (0, p_d) \), where \( p_d \) is defined by
\[
p_d = \begin{cases} 
\frac{4}{d-2} & \text{if } d \geq 3 \\
\frac{d}{\infty} & \text{if } d = 1, 2.
\end{cases}
\]

A pair \((q, r) \in [2, \infty] \times [2, \infty]\) is called an admissible pair if \( 2/q + d/r = d/2 \), \((q, r, d) \neq (2, \infty, 2)\).

Lastly, we always denote by \((q_0, r_0)\) the admissible pair \((q_0, r_0) = (\frac{4(p+2)}{dp}, p + 2)\).

The well-posedness theory for (1.1) with initial data \( \phi \in H^1(\mathbb{R}^d) \) is well understood as follows (See, e.g., [3]).

**Theorem A.** Let \( d \geq 1 \) and \( 0 < p < p_d \), and suppose that \( \phi \in H^1(\mathbb{R}^d) \). Then, there is a time \( T_{\text{max}} = T(d, p, \phi) \in (0, \infty] \) such that a solution \( u \in C \left( [0, T_{\text{max}}]; H^1(\mathbb{R}^d) \right) \) to (1.1) exists in the sense of the Duhamel formula (1.5). Moreover, for any \( T < T_{\text{max}} \) and any admissible pairs \((q, r)\), there is a positive constant \( M_1 = M_1(d, p, T, \phi) > 0 \) such that
\[
\|u\|_{L^q([0,T];H^1)} + \|u\|_{L^q([0,T];W^{1,r})} \leq M_1.
\]

In addition, one of the following is true:

- The solution \( u \) exists globally, i.e., \( T_{\text{max}} = \infty \) and \( \sup_{t \in [0, \infty)} \|u(t)\|_{H^1(\mathbb{R}^d)} < \infty \).
- The solution \( u \in C \left( [0, T_{\text{max}}]; H^1(\mathbb{R}^d) \right) \) exists for a maximal time interval \( T_{\text{max}} < \infty \), and
\[
\lim_{t \to T_{\text{max}}} \|u(t)\|_{H^1(\mathbb{R}^d)} = \infty.
\]

It is well known that if \( 0 < p < \frac{4}{d} \), then we always have \( T_{\text{max}} = \infty \), due to the mass conservation property. In this case, we may take \( M_1 = C_{d,p} \|\phi\|_{H^1} \) in (1.6) with a suitable constant \( C_{d,p} > 0 \). In addition, if the equation is defocusing, i.e., \( \lambda = -1 \) in (1.1), then the solution \( u \) exists globally for \( 0 < p < p_d \) by the energy conservation law. We refer the reader to Sections 4 and 5 of Cazenave [3] for the details.

The theorem below is one of the main contributions of this paper.

**Theorem 1.4.** Let \( d \geq 1 \) and \( 0 < p < p_d \). Suppose that \( \phi \in H^1(\mathbb{R}^d) \), and consider a time \( T > 0 \) such that \( \sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{R}^d)} < \infty \). Suppose that there is a constant \( M_2 = C(d, p, T, \phi) \) such that the stability of \( Z_\tau \) given by
\[
\|Z_\tau(n \tau)\|_{L^\infty([n \tau \in [0,T];H^1)} + \|Z_\tau(n \tau)\|_{L^p([n \tau \in [0,T];W^{1,r})} \leq M_2
\]
holds for all \( \tau \in (0, 1) \). Then we have the \( L^2 \) convergence of the difference between \( u \) and \( Z_\tau \)
\[
\max_{0 \leq n \tau \leq T} \|Z_\tau(n \tau) - u(n \tau)\|_{L^2(\mathbb{R}^d)} \leq \tau^{1/2} C(d, p, M_1, M_2) \exp \left( C_{d,p} (T - 1 + M_2)^{\frac{2(p+2)}{dp}} \right).
\]
The constant $M_1 > 0$ above is the one introduced in (1.6) and the constant $C(d,p, M_1, M_2) > 0$ is described in the proof.

By Theorem 1.4, it is enough to prove the stability (1.7) of $Z_\tau$ in order to establish the convergence of $Z_\tau$. Namely, to prove Theorem 1.1 and Theorem 1.2, it is enough to obtain the global (in-time) stability of $Z_\tau$ with $\phi \in H^1(\mathbb{R}^d)$ in the space $\ell^q(n\tau \in [0, T]; W^{1, r}(\mathbb{R}^d))$ for any $T < T_{\text{max}}$, which are the contents of Theorem 1.5 and Theorem 1.6 below. When $0 < p < \frac{4}{d}$, the stability result of $Z_\tau$ with $\phi \in L^2(\mathbb{R}^d)$ in the space $\ell^q(n\tau \in [0, T]; L'(\mathbb{R}^d))$ was obtained in [12, Theorem 1.1] for every admissible pair $(q, r)$. A crucial observation in the proof of Theorem 1.5 is that the $L^2(\mathbb{R}^d)$ norm of $Z_\tau$ does not increase. On the other hand, the $H^1(\mathbb{R}^d)$ norm of $Z_\tau$ may increase. This is one of the main difficulties to obtain the desired global (in-time) stability in Theorem 1.6.

**Theorem 1.5.** Let $d \geq 1$ and $0 < p < \frac{4}{d}$, and suppose that $\phi \in H^1(\mathbb{R}^d)$. Then, for any time $T > 0$ and any admissible pair $(q, r)$, the approximation $Z_\tau$ satisfies

$$\|Z_\tau(n\tau)\|_{\ell^q(n\tau \in [0, T]; W^{1, r})} \leq C_{d,p} \exp \left(\tilde{C}_{d,p} T \max \left\{\|\phi\|_{H^1}^{\frac{2p(p+1)}{4p-2d}}, \|\phi\|_{L^2}^{\frac{4p}{4p-2d}}\right\}\right)\|\phi\|_{H^1}$$

for any $\tau \in (0, 1)$, where $C_{d,p}$ and $\tilde{C}_{d,p}$ are two positive constants determined by $d$ and $p$.

**Theorem 1.6.** Let $1 \leq d < 6$ and $1 \leq p < p_d$. Suppose that $\phi \in H^1(\mathbb{R}^d)$, and consider a time $T > 0$ such that $\sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{R}^d)} < \infty$. Then, for any admissible pair $(q, r)$, there is a constant $C(d, p, T, \phi) > 0$ such that the approximation $Z_\tau$ satisfies

$$\|Z_\tau(n\tau)\|_{\ell^q(n\tau \in [0, T]; W^{1, r})} \leq C(d, p, T, \phi)$$

for any $\tau \in (0, 1)$.

Towards the above global $H^1$ stability results, we first prove the corresponding local (in-time) stability result (see Proposition 4.1). Next, for any $T > 0$ such that $\sup_{0 \leq t \leq T} \|u(t)\|_{H^1(\mathbb{R}^d)}$ is finite, we extend the local $H^1$ stability to the global $H^1$ stability on $[0, T]$ by an inductive argument after dividing $[0, T]$ into small subintervals. The inductive step is provided separately for the cases $0 < p < \frac{4}{d}$ and $1 \leq p < p_d$, as described below.

The case $0 < p < \frac{4}{d}$ is the simplest one. In this case we obtain an inductive estimate for proving Theorem 1.5 by combining the local $H^1$ stability (Proposition 4.1) with the $\ell^q(n\tau \in [0, T]; L'(\mathbb{R}^d))$ stability result on $Z_\tau$, obtained by Ignat [12]. More specifically, we will see that given an interval $[0, R]$ for which the $H^1$ stability of $Z_\tau$ is known, it is possible to extend the interval $[0, R]$ to $[0, R + T_1]$ for $T_1 > 0$ if $T_1^{\frac{1}{4-d}}\|Z_\tau(n\tau)\|_{\ell^q(n\tau \in [0, R + T_1]; L')} < \text{a specific constant}$.

The above procedure is difficult to apply in the mass critical and super-critical cases $\frac{4}{d} \leq p < p_d$, since we do not have a priori bound on $\|Z_\tau(n\tau)\|_{\ell^q(n\tau \in [0, T]; L')}$. Instead, we shall iterate the local $H^1$ stability result of Proposition 4.1 in a direct way to obtain the global $H^1$ stability. In this argument, the major challenge is to obtain a bound for the possible growth of the $\|Z_\tau\|_{H^1(\mathbb{R}^d)}$ when iterating the local $H^1$ stability result. To deal with this issue, we observe and prove that for initial $\phi \in H^1(\mathbb{R}^d)$, the scheme $Z_\tau$ converges to the solution $u$ in $H^1(\mathbb{R}^d)$ on an interval $[0, R]$ (see (5.6)). This enables to control the norm $\|Z_\tau(R)\|_{H^1}$ by $\|u(R)\|_{H^1} + \epsilon$ for some $\epsilon > 0$ when $\tau$ is small enough. Then we may apply the local $H^1$ stability result to $Z_\tau$ from initial data $Z_\tau(R)$ to find the
$H^1$ stability of $Z_\tau$ on an additional interval of suitable length. By developing this idea further, we shall prove Theorem 1.6.

We remark that there are also interesting variants of the splitting methods for rough initial data [16, 19, 20] and high-order method [10, 25, 26]. The long time behavior is also studied in the works [7, 8], and we recall a detail regarding the well-posedness result for (1.1). In Section 3, we present the proof of Theorem 1.1 for the energy-subcritical case $0 < p < p_d$. In Section 4, we prove the local $H^1$ stability result on the splitting scheme $Z_\tau$, and we prove Theorem 1.5 which is the mass-subcritical case $0 < p < \frac{4}{d}$. In Sections 5-6, we prove Theorem 1.6 which is the energy-subcritical case $1 \leq p < p_d$.

**Notations.**

- The $C$ is a generic constant depending only on dimension $d$, which may change from line to line.
- If a constant depends on some other values, we mark it as $C_T$ (depending on time $T$) or $C(d, p, T, \phi)$ (depending on dimension $d$, nonlinear exponent $p$, time $T$ and initial data $\phi$). Their values may change from line to line.
- For $0 < a < b < \infty$, we often write $\| \cdot \|_{\ell^p(a, b; B)}$ as $\| \cdot \|_{\ell^p(a, b; \mathcal{B})}$ for $B = W^{k,q}(\mathbb{R}^d)$ or $L^q(\mathbb{R}^d)$.
- We often simply denote $W^{k,q}(\mathbb{R}^d)$ as $W^{k,q}$, and similarly for $H^k(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$.
- The pair $(q_0, r_0)$ denotes the admissible pair $\left(\frac{4(p+2)}{p-2}, p + 2\right)$.
- The notation $p'$ is the conjugate of $p$, namely $p' = \frac{p}{p-1}$.
- We write ‘local’ to mean ‘local-in-time’ for the sake of simplicity.
- The notation ‘stable’ means ‘stable uniformly in $\tau \in (0, 1)$’.
- $\nabla f$ always means $\nabla_x f$ even if $f$ is a time-space function.
- In Section 5-6, we precisely write $Z_{\tau}^\phi$ and $u^\phi$ to denote the flow $Z_\tau$ and the solution $u$ corresponding to the initial data $\phi$.

### 2. Preliminary lemmas.

**Theorem 2.1.** Let $(q, r)$ and $(q, \tilde{r})$ be any admissible pairs. Then, there exist $C_{d,q}, C_{d,q, \tilde{q}} > 0$ such that

$$\| S_\tau(\cdot) \phi \|_{L^q(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C_{d,q} \| \phi \|_{L^2(\mathbb{R}^d)}, \quad (2.1)$$

$$\left\| \int_{\mathbb{R}} S_\tau(-s)f(s)ds \right\|_{L^2(\mathbb{R}^d)} \leq C_{d,q} \| f \|_{L^{q'}(\mathbb{R}, L^{\tilde{r}'}(\mathbb{R}^d))}.$$


These estimates for Theorem 2.2 (Theorem 2.1) versions of the Strichartz estimates, obtained by Ignat [12], hold for all $\phi$ (Lemma 4.5) and Tao [15] to all admissible pairs, including the endpoint case $(q, r) = (2, \frac{2d}{d-2})$. These estimates for $S(t)$ are extended easily to the frequency localized operator $S_{\tau}(t)$ using that $S_{\tau}(t)\phi = S(t)(\Pi_{\tau}\phi)$ and (1.3). Next we recall the time discrete versions of the Strichartz estimates, obtained by Ignat [12].

**Theorem 2.2** ([12, Theorem 2.1]). Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any admissible pairs. Then, there exist $C_{d,q}, C_{d,q, \tilde{q}} > 0$ such that

$$\|S_{\tau}(\cdot)\phi\|_{\ell^{q}(\tau Z; L^{r}(\mathbb{R}^{d}))} \leq C_{d,q}\|\phi\|_{L^{2}(\mathbb{R}^{d})},$$

(2.2)

and

$$\left\|\frac{\tau}{n} \sum_{n \in \mathbb{Z}} S_{\tau}(-n\tau)f(n\tau)\right\|_{\ell^{q}(\tau Z; L^{r}(\mathbb{R}^{d}))} \leq C_{d,q, \tilde{q}}\|f\|_{\ell^{q}(\tau Z; L^{\tilde{r}}(\mathbb{R}^{d}))},$$

(2.3)

hold for all $\phi \in L^{2}(\mathbb{R}^{d})$ and $f \in \ell^{q}(\tau Z; L^{\tilde{r}}(\mathbb{R}^{d}))$.

Since the operators $S_{\tau}$ and $\nabla$ commute, the above result immediately implies the following corollary.

**Corollary 2.3.** Let $(q, r)$ and $(\tilde{q}, \tilde{r})$ be any admissible pairs. Then, there exist $C_{d,q}, C_{d,q, \tilde{q}} > 0$ such that

$$\|S_{\tau}(\cdot)\phi\|_{\ell^{q}(\tau Z; W^{1, r}(\mathbb{R}^{d}))} \leq C_{d,q}\|\phi\|_{H^{1}(\mathbb{R}^{d})},$$

(2.4)

and

$$\left\|\frac{\tau}{n} \sum_{k=\infty}^{n-1} S_{\tau}((n - k)\tau)f(k\tau)\right\|_{\ell^{q}(\tau Z; W^{1, r}(\mathbb{R}^{d}))} \leq C_{d,q, \tilde{q}}\|f\|_{\ell^{q}(\tau Z; W^{1, \tilde{r}}(\mathbb{R}^{d}))},$$

(2.5)

hold for all $\phi \in H^{1}(\mathbb{R}^{d})$ and $f \in \ell^{q}(\tau Z; W^{1, \tilde{r}}(\mathbb{R}^{d}))$.

By combining the Strichartz estimates in Theorems 2.1 and 2.2 with the Christ-Kiselev lemma [4], the following result was derived.

**Corollary 2.4** ([12, Lemma 4.5]). For any admissible pairs $(q, r)$ and $(\tilde{q}, \tilde{r})$ with $(q, \tilde{q}) \neq (2, 2)$, we have

$$\left\|\int_{s < n\tau} S_{\tau}(n\tau - s)f(s)ds\right\|_{\ell^{q}(\tau Z; L^{r}(\mathbb{R}^{d}))} \leq C_{d,q, \tilde{q}}\|f\|_{L^{q}(\tau Z; L^{\tilde{r}}(\mathbb{R}^{d}))}.$$  

(2.6)

In the remaining part of this section, we prove some basic estimates that are used frequently in this paper.
Lemma 2.5. For any $p \in (0, \infty)$, there exists a positive constant $c_p > 0$ such that
\[
\left| \frac{N(\tau) - I}{\tau} v - \frac{N(\tau) - I}{\tau} w \right| \leq c_p |v - w|(|v|^p + |w|^p) \tag{2.7}
\]
and
\[
\left| \frac{\exp(i\tau|v|^p)}{\tau} - \frac{\exp(i\tau|w|^p)}{\tau} \right| \leq |v|^p |w| \tag{2.8}
\]
hold for all $v, w \in \mathbb{C}$. Furthermore, for weakly differentiable $f : \mathbb{R}^d \to \mathbb{C}$, we have a pointwise estimate
\[
\left| \nabla \left( \frac{N(\tau) - I}{\tau} f \right) \right| \leq (p + 1)|f|^p |\nabla f|. \tag{2.9}
\]
Proof. The estimates (2.7) and (2.8) follow from a direct calculation using the mean value theorem (see also Lemma 4.2 in [12]). For the last estimate, we notice that
\[
\left| \nabla \left( \frac{N(\tau) - I}{\tau} f \right) \right| = \left| \nabla \left( \frac{\exp(i\tau|f|^p)}{\tau} - 1 \right) \right| \\
\leq \left( \frac{\exp(i\tau|f|^p)}{\tau} - 1 \right) \nabla f \right| + \left| \exp(i\tau|f|^p) p |f|^{p-2} \Re \{f \nabla f \} \right| \\
\leq (p + 1)|f|^p |\nabla f|,
\]
where (2.8) is used for the last inequality. The lemma is proved. \hfill \Box

Lemma 2.6. For any $1 \leq q \leq r < \infty$ and $\phi : \mathbb{R}^d \to \mathbb{C}$, we have
\[
\| \Pi_\tau \phi - \phi \|_{L^r(\mathbb{R}^d)} \leq C \tau^{1/2} \|(-\Delta)^{1/2} \phi \|_{L^r(\mathbb{R}^d)}, \tag{2.10}
\]
\[
\| \Pi_\tau \phi \|_{L^r(\mathbb{R}^d)} \leq C \| \phi \|_{L^r(\mathbb{R}^d)}, \tag{2.11}
\]
\[
\| \nabla (\Pi_\tau \phi) \|_{L^q(\mathbb{R}^d)} \leq C \tau^{-\frac{d}{2}} \| \phi \|_{L^r(\mathbb{R}^d)}, \tag{2.12}
\]
and
\[
\| \Pi_\tau \phi \|_{L^q(\mathbb{R}^d)} \leq C \tau^{\frac{d}{2} \left( \frac{1}{q} - \frac{1}{r} \right)} \| \phi \|_{L^r(\mathbb{R}^d)}. \tag{2.13}
\]
Proof. The estimates (2.10) and (2.11) follow from the basic multiplier theory (see, e.g., Theorem 4.4 in [12] and Theorem 5.2.2 in [9]). In order to show that (2.12) and (2.13), we notice from the definition (1.3) that
\[
\Pi_\tau \phi(x) = (K_\tau * \phi)(x), \tag{2.14}
\]
where $K_\tau(x) = \tau^{-\frac{d}{2}} \hat{\chi}(\tau^{-1/2}x)$. Since $\chi \in C^N(B^d(0,2))$ with $N = 2d$, its inverse Fourier transform $\hat{\chi}$ admits the decay property $|\hat{\chi}(\xi)| \leq C_{d,N} (1 + |\xi|)^{-N}$. In addition, by the equality $\hat{\xi}_k \hat{\chi}(\xi) = i(x_k \chi)(\xi)$ we have $|\hat{\xi}_k \hat{\chi}(\xi)| \leq C_{d,N} (1 + |\xi|)^{-N}$, and
\[
\hat{\xi}_k (\Pi_\tau \phi)(x) = \tau^{-\frac{d+1}{2}} (\hat{\xi}_k \hat{\chi})(\tau^{-1/2}x) * \phi(x). \tag{2.15}
\]
By applying the Young’s inequality, we obtain that
\[
\| \nabla (\Pi_\tau \phi) \|_{L^q(\mathbb{R}^d)} \leq C \tau^{-\frac{d+1}{2}} \| \nabla \hat{\chi}(\tau^{-1/2} \cdot) \|_{L^1(\mathbb{R}^d)} \| \phi \|_{L^r(\mathbb{R}^d)} \\
\leq C \tau^{-\frac{d}{2}} \| \phi \|_{L^r(\mathbb{R}^d)},
\]
which gives estimate (2.12). By applying the Young’s inequality to (2.14) with $q < r$,
\[
\| \Pi_\tau \phi \|_{L^q(\mathbb{R}^d)} \leq C \| K_\tau \|_{L^r(\mathbb{R}^d)} \| \phi \|_{L^r(\mathbb{R}^d)},
\]
where $1/r + 1 = 1/\alpha + 1/q$. This verifies the estimate (2.13). The proof is finished. \hfill \Box
Lemma 2.7. For any admissible pairs \((q, r)\) and \((\tilde{q}, \tilde{r})\), there is a constant \(C_{d,q,\tilde{q}} > 0\) such that
\[
\left\| \int_{s<nT} S_\tau(n\tau - s)f(s)ds - \tau \sum_{k=-\infty}^{n-1} S_\tau(n\tau - k\tau)f(k\tau) \right\|_{L^q(T\mathbb{Z}, L^{r}(\mathbb{R}^d))} \leq C_{d,q,\tilde{q}} \tau^{1/2}\|f\|_{L^{q_\tau}(\mathbb{R}, L^{r_\tau}(\mathbb{R}^d))} + C_{d,q,\tilde{q}} \tau \|\partial_\tau f\|_{L^{q_\tau}(\mathbb{R}, L^{r_\tau}(\mathbb{R}^d))}
\]
hold for all test functions \(f \in S(\mathbb{R}^{d+1})\).

Proof. First, we recall the following estimate from Lemma 4.3 and Lemma 4.6 in [12]:
\[
\left\| \int_{s<nT} S_\tau(n\tau - s)f(s)ds - \tau \sum_{k=-\infty}^{n-1} S_\tau(n\tau - k\tau)f(k\tau) \right\|_{L^q(T\mathbb{Z}, L^{r}(\mathbb{R}^d))} \leq C_{d,q,\tilde{q}} \tau \|\nabla^2 f\|_{L^{q_\tau}(\mathbb{R}, L^{r_\tau}(\mathbb{R}^d))} + C_{d,q,\tilde{q}} \tau \|\partial_\tau f\|_{L^{q_\tau}(\mathbb{R}, L^{r_\tau}(\mathbb{R}^d))},
\]
where \((q, r)\) and \((\tilde{q}, \tilde{r})\) are any admissible pairs. We notice that \(S_\tau(t)f(x) = S_\tau(t)\Pi_{r}f(x)\), by definition of \(S_\tau\). Using this and (2.17) with Lemma 2.6, we obtain that
\[
\left\| \int_{s<nT} S_\tau(n\tau - s)f(s)ds - \tau \sum_{k=-\infty}^{n-1} S_\tau(n\tau - k\tau)f(k\tau) \right\|_{L^q(T\mathbb{Z}, L^{r}(\mathbb{R}^d))} \leq C_{d,q,\tilde{q}} \tau \|\nabla^2 f\|_{L^{q_\tau}(\mathbb{R}, L^{r_\tau}(\mathbb{R}^d))} + C_{d,q,\tilde{q}} \tau \|\partial_\tau f\|_{L^{q_\tau}(\mathbb{R}, L^{r_\tau}(\mathbb{R}^d))},
\]
This proves the estimate (2.16). \(\square\)

Lemma 2.8. Under the assumption (1.6), we have
\[
\|\Pi_r u\|_{L^{q_0}(0,T;L^{r_0})} + \|\Pi_r u\|_{L^{q_0}(0,T;L^{r_0})} \leq C_{d,p} \tau^{-1/2}(1 + T_{\rho_0}^{1/2})\mathcal{M}_{r_0}^{p+1}.
\]
(2.18)

Proof. The proof is a simple combination of the Hölder’s inequality and the Sobolev embedding. We consider the first term of the left hand side in (2.18). By Lemma 2.6, we have
\[
\|\Pi_r u\|_{L^{q_0}(0,T;L^{r_0})} \leq C_{d,p} \tau^{-\frac{d}{2}(\frac{p+1}{p+2} + \frac{1}{q_0})} \|u\|_{L^{2}(2p+1)q_0(0,T;L^{r_0})}
\]
for all \(r_1 \leq (2p+1)r_0\). We choose the value of \(r_1 > 0\) separately for the cases that \(\frac{3}{d} \leq p < p_d\) and \(0 < p < \frac{3}{d}\) such as
\[
\frac{1}{r_1} = \left\{ \begin{array}{ll}
\frac{1}{2p+1} \left(\frac{p+1}{p+2} + \frac{1}{q_0}\right) & \text{if } \frac{3}{d} \leq p < p_d, \\
\frac{1}{(2p+1)r_0} & \text{if } 0 < p < \frac{3}{d}.
\end{array} \right.
\]
First, we consider the case that \(\frac{3}{d} \leq p < p_d\). In this case, we take \(\frac{1}{r_1} = \frac{1}{2p+1} \left(\frac{p+1}{p+2} + \frac{1}{q_0}\right) < 1\) and choose \(q_2\) and \(r_2\) such that
\[
\frac{1}{q_2} := \frac{1}{(2p+1)q_0} = \frac{4(p+2) - dp}{4(p+2)(2p+1)} \quad \text{and} \quad \frac{1}{r_2} := \frac{1}{2} - \frac{2}{dq_2} = \frac{1}{2} - \frac{4(p+2) - dp}{2d(p+2)(2p+1)}
\]
so that \((q_2, r_2)\) is an admissible pair whenever \(\frac{3}{d} \leq p < p_d\). Then, by (2.19) and the Sobolev embedding \(W^{s, r_2} \subseteq L^{r_1}\) (with \(\frac{1}{r_1} + \frac{d}{2} = \frac{1}{r_2}\)), we obtain that

\[
\|\Pi_r u\|^{2p+1}_{L^0(0,T; L^0')} \leq C_{d,p} \tau^{-\frac{1}{2}} \|u\|^{2p+1}_{L^{r_2}(0,T; W^{s, r_2})} \\
\leq C_{d,p} \tau^{-\frac{1}{2}} \|u\|^{2p+1}_{L^{r_2}(0,T; W^{s, r_2})},
\]

where

\[
s = \frac{d}{2} - \frac{d + 6}{2(2p + 1)}.
\]

One may check that \(s \in [0, 1]\) whenever \(\frac{3}{d} \leq p \leq p_d\). Therefore, we have that

\[
\|\Pi_r u\|^{2p+1}_{L^0(0,T; L^0')} \leq C_{d,p} \tau^{-\frac{1}{2}} \|u\|^{2p+1}_{L^{r_2}(0,T; W^{s, r_2})} \\
\leq C_{d,p} \tau^{-\frac{1}{2}} \|u\|^{2p+1}_{L^{r_2}(0,T; W^{s, r_2})} \leq C_{d,p} \tau^{-\frac{1}{2}} M_1^{2p+1}
\]

from (1.6). For the case that \(0 < p < \frac{3}{d}\), we set \(r_1 = (2p + 1)r_0'\) in (2.19). Using the Hölder’s inequality in time variable and the Sobolev embedding, we have that

\[
\|\Pi_r u\|^{2p+1}_{L^0(0,T; L^0')} \leq T^\frac{1}{p} \|u\|^{2p+1}_{L^\infty(0,T; L^{2p+1}r_0')} \leq C_{d,p} T^\frac{1}{p} \|u\|^{2p+1}_{L^\infty(0,T; H^{s})},
\]

where

\[
s = \frac{d}{2} - \frac{d(p + 1)}{(p + 2)(2p + 1)}.
\]

Because \(s \in [0, 1]\) whenever \(0 \leq p \leq \frac{3}{d}\), this is bounded by \(C_{d,p} T^\frac{1}{p} M_1^{2p+1}\) thanks to (1.6). Thus, we have the upper bound

\[
\|\Pi_r u\|^{2p+1}_{L^0(0,T; L^0')} \leq C_{d,p} \left(\tau^{-\frac{1}{2}} + T^\frac{1}{p}\right) M_1^{2p+1}
\]

for all \(0 < p < p_d\). Next, for the second term of the left hand side in (2.18), we apply the Hölder’s inequality and Lemma 2.6 again to obtain

\[
\left\|\Pi_r u \Pi_r (|u|^p u)\right\|_{L^0(0,T; L^0')} \leq \left\|\Pi_r u\right\|_{L^{r_2}(0,T; L^{2p+1}r_0')} \left\|\Pi_r (|u|^p u)\right\|_{L^{r_2}(0,T; L^{2p+1}r_0')} \\
\leq C T^{-\frac{1}{2}} \left(\frac{2p+1}{r_1} - \frac{1}{r_0}\right) \left\|u\right\|_{L^{r_2}(0,T; L^{2p+1}r_0')} \left\|u\right\|_{L^{r_2}(0,T; L^{2p+1}r_0')} \\
= C T^{-\frac{1}{2}} \left(\frac{2p+1}{r_1} - \frac{1}{r_0}\right) \left\|u\right\|_{L^{r_2}(0,T; L^{2p+1}r_0')} \left\|u\right\|_{L^{r_2}(0,T; L^{2p+1}r_0')}
\]

for any \(r_1 \leq (2p + 1)r_0'\). The right hand side is same as the right hand side of (2.19). Hence the desired estimate follows in the same way.

We conclude the section by recalling the Hölder’s inequalities for \(L^\infty(I; L^{\infty}(\mathbb{R}^d))\) and \(L^{r_0}(I; L^{r_0}(\mathbb{R}^d))\) which we use frequently:

\[
\|f\|^p_{L^\infty(0,T; L^{r_0})} \leq |T|^\frac{1}{r_0} \|f\|^p_{L^\infty(0,T; L^{r_0})} \|g\|_{L^\infty(0,T; L^{r_0})}
\]

and

\[
\|f\|^p_{L^{r_0}(0,T; L^{r_0})} \leq |T|^\frac{1}{r_0} \|f\|^p_{L^{r_0}(0,T; L^{r_0})} \|g\|_{L^{r_0}(0,T; L^{r_0})}.
\]
3. The convergence result for the difference between \( Z_\tau \) and \( u \) in \( L^2 \). In this section, we prove Theorem 1.4. The key ingredient is to use the Duhamel formulas of \( Z_\tau \) and \( u \) given in (1.4) and (1.5) respectively.

Proof of Theorem 1.4. Let us fix a time \( T \in (0, \infty) \) such that \( \sup_{0 \leq t \leq T} \| u(t) \|_{H^1} < \infty \). Then, from Theorem A and the assumption of Theorem 1.4, we see that \( u \) and \( Z_\tau \) satisfy the following estimates

\[
\| u \|_{L^\infty(0,T;H^1)} + \| u \|_{L^0(0,T;W^{1,\infty})} \leq M_1,
\]

\[
\| Z_\tau(n\tau) \|_{L^\infty(0,T;H^1)} + \| Z_\tau(n\tau) \|_{L^0(0,T;W^{1,\infty})} \leq M_2.
\]

For our purpose, it is sufficient to estimate \( Z_\tau(n\tau) - \Pi_{\tau} u(n\tau) \) instead of \( Z_\tau(n\tau) - u(n\tau) \), because we have that

\[
\| u(n\tau) - \Pi_{\tau} u(n\tau) \|_{L^\infty(0,T;L^2)} \leq C \tau^{1/2} \| u(n\tau) \|_{L^\infty(0,T;H^1)} \leq C \tau^{1/2} M_1
\]

by (2.10) and (1.6). Now, we take \( T_* > 0 \) as

\[
T_* = \alpha_{d,p} \left( M_1 + M_2 \right)^{- \frac{2p(p+2)}{4(d-2)p}} \tag{3.1}
\]

where the small constant \( \alpha_{d,p} > 0 \) will be chosen later.

First, for the case of \( \tau \in \left[T_*/4, 1\right) \), we can obtain the desired estimate easily as follows:

\[
\max_{n \in \mathbb{N}} \| Z_\tau(n\tau) - u(n\tau) \|_{L^2} \leq \max_{n \in \mathbb{N}} \| Z_\tau(n\tau) \|_{L^2} + \max_{n \in \mathbb{N}} \| u(n\tau) \|_{L^2}
\]

\[
\leq 2 \| \phi \|_{L^2} \leq \tau^{1/2} \left( \frac{4}{\sqrt{T_*}} \right) \| \phi \|_{L^2}
\]

\[
\leq 4 \tau^{1/2} \alpha_{d,p}^{-1/2} \left( M_1 + M_2 \right)^{\frac{p(p+2)}{4(d-2)p}} \| \phi \|_{L^2}.
\]

Now, assume \( 0 < \tau < T_*/4 \). We take \( R \in \left(T_*/2, T_*\right) \) such that \( R/\tau \in \mathbb{N} \). To employ an induction argument, we split \([0, T] \) as

\[
[0, T] = \bigcup_{j=0}^{N-1} (jR, (j+1)R) \cup [NR, T]
\]

\[
=: \bigcup_{j=0}^{N-1} I_j \cup I_N, \tag{3.2}
\]

where \( N \in \mathbb{N} \) is chosen so that \( NR \leq T < (N+1)R \). For each \( j \in \{0, 1, \cdots, N\} \) we choose \( m_j \in \mathbb{N} \) such that \( m_j \tau = jR, \) i.e. \( m_j = j(R/\tau) \). Then

\[
\| Z_\tau - \Pi_{\tau} u \|_{L^2(I_j;L^\prime)} = \left\| Z_\tau(m_j \tau + n\tau) - \Pi_{\tau} u(m_j \tau + n\tau) \right\|_{L^2(0,R;L^\prime)}, \tag{3.3}
\]

where, if \( j = N \), we regard the interval \((0, R)\) as replaced by \((0, T - NR) \subset (0, T)\).

By considering \( Z_\tau(m_j \tau) \) as initial data for each \( j = 0, 1, \cdots, N \), the formula (1.4) can be written as

\[
Z_\tau(m_j \tau + n\tau) = S_\tau(n\tau)Z_\tau(m_j \tau) + \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \frac{N(\tau) - I}{\tau} Z_\tau(m_j \tau + k\tau), \quad n \geq 1.
\]

(3.4)

By combining this with (1.5), we obtain the following decomposition:

\[
Z_\tau(m_j \tau + n\tau) - \Pi_{\tau} u(m_j \tau + n\tau) = A_1(j) + A_2(j) + A_3(j) + A_4(j),
\]
where

\[ A_1(j) := S_\tau(nt\tau)\Big(Z_\tau(m_j\tau - \Pi_\tau u(m_j\tau)) \Big) \]

\[ A_2(j) := S_\tau(nt\tau)\Big(\Pi_\tau u(m_j\tau) - u(m_j\tau) \Big) \]

\[ A_3(j) := \tau \sum_{k=0}^{n-1} S_\tau(nt\tau - kr) \left( \frac{N(\tau) - I}{\tau} Z_\tau(m_j\tau + kr) - \frac{N(\tau) - I}{\tau} \Pi_\tau u(m_j\tau + kr) \right) \]

\[ A_4(j) := \tau \sum_{k=0}^{n-1} S_\tau(nt\tau - kr) \frac{N(\tau) - I}{\tau} \Pi_\tau u(m_j\tau + kr) - i\lambda \int_0^\tau S_\tau(nt\tau - s)|u|^p u(m_j\tau + s)ds. \]

For \((q, r) \in \{(q_0, r_0), (\infty, 2)\}\), we apply the triangle inequality to obtain that

\[ \left\| Z_\tau(m_j\tau + nt\tau) - \Pi_\tau u(m_j\tau + nt\tau) \right\|_{\ell^q(0,R;L^r)} \leq \| A_1(j) \|_{\ell^q(0,R;L^r)} + \| A_2(j) \|_{\ell^q(0,R;L^r)} + \| A_3(j) \|_{\ell^q(0,R;L^r)} + \| A_4(j) \|_{\ell^q(0,R;L^r)}. \]  

(3.5)

By using the Strichartz estimate (2.2), we get the estimate

\[ \| A_1(j) \|_{\ell^q(0,R;L^r)} = \left\| S_\tau(nt\tau)\Big(Z_\tau(m_j\tau - \Pi_\tau u(m_j\tau)) \Big) \right\|_{\ell^q(0,R;L^r)} \leq C_{d,q} \left\| Z_\tau(m_j\tau) - \Pi_\tau u(m_j\tau) \right\|_{L^2}. \]

(3.6)

Secondly, we use the inequality (2.2), (2.10) with (1.6) to deduce that

\[ \| A_2(j) \|_{\ell^q(0,R;L^r)} = \left\| S_\tau(nt\tau)\Big(\Pi_\tau u(m_j\tau) - u(m_j\tau) \Big) \right\|_{\ell^q(0,R;L^r)} \leq C_{d,q} \left\| \Pi_\tau u(m_j\tau) - u(m_j\tau) \right\|_{L^2} \leq C_{d,q} r^{1/2} \left\| (-\Delta)^{1/2} u(m_j\tau) \right\|_{L^2} \leq C_{d,q} r^{1/2} M_1. \]

(3.7)

Next, to estimate \( A_3(j) \), we use (2.7) to find that

\[ \left| \frac{N(\tau) - I}{\tau} Z_\tau - \frac{N(\tau) - I}{\tau} \Pi_\tau u \right| \leq C_p |Z_\tau - \Pi_\tau u| \left( |Z_\tau|^p + |\Pi_\tau u|^p \right). \]

(3.8)

After applying the Strichartz estimate (2.3) to \( A_3(j) \), we use (3.8) with the Hölder’s inequality (2.21) and the fact that \( R \leq T_\ast \) to find

\[ \| A_3(j) \|_{\ell^q(0,R;L^r)} \leq C_{d,p} \left\| Z_\tau - \Pi_\tau u \right\|_{\ell^p(0,R;L^r)} \left( |Z_\tau|^p + |\Pi_\tau u|^p \right)(nt\tau + m_j\tau) \leq C_{d,p} T_\ast^{-\frac{1}{p} - \frac{1}{p_0}} \left\| Z_\tau - \Pi_\tau u \right\|_{\ell^p(0,R;L^r)}^p \left( |Z_\tau|^p + |\Pi_\tau u|^p \right)^{p - p_0}(nt\tau + m_j\tau). \]

(3.9)

To proceed further, we use the Sobolev embedding \( H^1(\mathbb{R}^d) \rightarrow L^{r_0}(\mathbb{R}^d) \) (thanks to \( r_0 < p_d + 2 \), (1.6) and (1.7) to obtain

\[ \| \Pi_\tau u(n\tau + m_j\tau) \|_{\ell^p(0,R;L^{r_0})} \leq C_{d,p} \| u \|_{\ell^p(0,R;H^1)} \leq C_{d,p} M_1, \]

\[ \| Z_\tau(n\tau + m_j\tau) \|_{\ell^p(0,R;L^{r_0})} \leq C_{d,p} \| Z_\tau \|_{\ell^p(0,R;H^1)} \leq C_{d,p} M_2. \]

By inserting these estimates into (3.9), we get

\[ \| A_3(j) \|_{\ell^q(0,R;L^r)} \leq C_{d,p} (M_1^p + M_2^p) T_\ast^{-\frac{1}{p} + \frac{1}{p_0}} \left\| (Z_\tau - \Pi_\tau u)(nt\tau + m_j\tau) \right\|_{\ell^p(0,R;L^r)}^p. \]

(3.10)
where we used the identity
\[
\frac{1}{q_0} - \frac{1}{q_0} = \frac{4 - (d - 2)p}{2(p + 2)}.
\]
(3.11)

Now, by choosing \(\alpha_{d,p} > 0\) small enough in (3.1), we deduce from (3.10) the following estimate
\[
\|A_3(j)\|_{L^p(0, R; L^r)} \leq \frac{1}{2}\|Z_\tau - \Pi_r u(n_\tau + m_\tau)\|_{L^p(0, R; L^{\alpha_0})}.
\]
(3.12)

Lastly, for \(A_4(j)\), we claim the following estimate:
\[
\max_{j=0,1,\ldots,N}\left\{\|A_4(j)\|_{L^p(0, R; L^r)} + \|A_4(j)\|_{L^p(0, R; L^{\alpha_0})}\right\} \leq C_{d,p} \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(M_1^{p+1} + M_1^{2p+1}),
\]
(3.13)

whose verification is given in Lemma 3.1 below. Now we are ready to finish the proof. From (3.6), (3.7), (3.12) and (3.13), we have
\[
\begin{align*}
&\left\|Z_\tau(m_\tau n_\tau) - \Pi_u(m_\tau n_\tau)\right\|_{L^p(0, R; L^r)} \\
&\leq C_{d,p}\left\|Z_\tau(m_\tau) - \Pi_u(m_\tau)\right\|_{L^2} + \frac{1}{2}\left\|Z_\tau(m_\tau n_\tau) - \Pi_u(m_\tau n_\tau)\right\|_{L^p(0, R; L^{\alpha_0})} \\
&\quad + C_{d,p} \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(M_1^{p+1} + M_1^{2p+1}).
\end{align*}
\]
(3.14)

Since (3.14) holds for \((q, r) \in \{(q_0, r_0), (\infty, 2)\}\), we may first take \((q, r) = (q_0, r_0)\) to get
\[
\begin{align*}
&\left\|Z_\tau(m_\tau n_\tau) - \Pi_u(m_\tau n_\tau)\right\|_{L^p(0, R; L^r)} \\
&\leq 2C_{d,p}\left\|Z_\tau(m_\tau) - \Pi_u(m_\tau)\right\|_{L^2} + 2C_{d,p} \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(M_1^{p+1} + M_1^{2p+1})
\end{align*}
\]
and use this in (3.14) with \((q, r) = (\infty, 2)\) to obtain
\[
\begin{align*}
&\left\|Z_\tau(m_\tau n_\tau) - \Pi_u(m_\tau n_\tau)\right\|_{L^p(0, R; L^2)} \\
&\leq 2C_{d,p}\left\|Z_\tau(m_\tau) - \Pi_u(m_\tau)\right\|_{L^2} + 2C_{d,p} \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(M_1^{p+1} + M_1^{2p+1}).
\end{align*}
\]
Then, by utilizing (3.3), we arrive at the following estimates:
\[
\begin{align*}
&\left\|Z_\tau - \Pi_u\right\|_{L^p(0, R; L^2)} \leq C_{d,p} \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(1 + M_1^{2p+1}), \\
&\left\|Z_\tau - \Pi_u\right\|_{L^p(0, R; L^2)} \leq C_{d,p}\left\|Z_\tau - \Pi_u\right\|_{L^p(0, R; L^2)} + C_{d,p} \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(M_1^{p+1} + M_1^{2p+1}),
\end{align*}
\]
for \(j = 0, 1, \ldots, N - 1\). Inductively, this implies that
\[
\left\|Z_\tau - \Pi_u\right\|_{L^p(0, R; L^2)} \leq \tau^{1/2}(1 + T_{\alpha}^{\frac{1}{2}})(M_1^{p+1} + M_1^{2p+1})\left(\sum_{i=0}^{j} (C_{d,p})^{i+1}\right)
\]
for all \(j = 0, 1, \ldots, N\). Since \(N \leq 2T/	au_{\alpha}\) with \(\tau_{\alpha}\) given in (3.1), we get
\[
\left\|Z_\tau - \Pi_u\right\|_{L^p(0, R; L^2)} \leq \tau^{1/2}\left[1 + (M_1 + M_2)^{2(p+1)}\right](M_1^{p+1} + M_1^{2p+1})\sum_{j=0}^{[2T/T_{\alpha}]} (C_{d,p})^{j+1}.
\]
(3.15)
Choosing $C_{d,p} \geq 2$ if necessary, we have

$$\sum_{j=1}^{[2T/T_0]} (C_{d,p})^{j+1} = \frac{C_{d,p}^{[2T/T_0]+2}}{C_{d,p} - 1} - 1 \leq C_{d,p}^{[2T/T_0]+2} \leq C_{d,p}^2 \exp \left( \tilde{C}_{d,p} T (M_1 + M_2)^{(2\nu - 2p) / (4p - 2)} \right)$$

(3.16)

with a suitable constant $\tilde{C}_{d,p} > 0$. Inserting this estimate into the above, we obtain the desired estimate (1.8). This completes the proof. \(\square\)

In the remaining part of this section, we prove the estimate (3.13) used in the above proof.

**Lemma 3.1.** Under the assumption (1.6), we have

$$\left\| \tau \sum_{k=0}^{n-1} S_\tau (n\tau - k\tau) \frac{N(\tau) - I}{\tau} \Pi_\tau u(k\tau) - i\lambda \int_0^{n\tau} S_\tau (n\tau - s) |u|^p u(s) ds \right\|_{L^r(0,T;L^s)}$$

$$\leq C_{d,p} \tau^{1/2} (1 + T^{\nu_0}) (M_1^{p+1} + M_1^{2p+1}).$$

(3.17)

for all admissible pairs $(q, r)$ and $\tau \in (0, 1)$. Here, the constant $M_1 > 0$ above is the one introduced in (1.6).

**Proof.** Recall that $(q_0, r_0)$ denote the admissible pair $(\frac{4(p+2)}{p}, p+2)$. Also, we denote that

$$B_1(u) := \frac{N(\tau) - I}{\tau} \Pi_\tau u(s)$$

$$B_2(u) := \frac{N(\tau) - I}{\tau} \Pi_\tau u - i\lambda |\Pi_\tau u|^p \Pi_\tau u$$

$$B_3(u) := i\lambda \left( |\Pi_\tau u|^p \Pi_\tau u - |u|^p u \right)$$

Now, in order to show (3.17), we perform the decomposition

$$\tau \sum_{k=0}^{n-1} S_\tau (n\tau - k\tau) \frac{N(\tau) - I}{\tau} \Pi_\tau u(k\tau) - i\lambda \int_0^{n\tau} S_\tau (n\tau - s) |u|^p u(s) ds$$

$$= \left( \tau \sum_{k=0}^{n-1} S_\tau (n\tau - k\tau) \frac{N(\tau) - I}{\tau} \Pi_\tau u(k\tau) - \int_0^{n\tau} S_\tau (n\tau - s) \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) ds \right)$$

$$+ \left( \int_0^{n\tau} S_\tau (n\tau - s) \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) ds - i\lambda \int_0^{n\tau} S_\tau (n\tau - s) |u|^p u(s) ds \right).$$

(3.18)

By Lemma 2.7, we can find an upper bound of the first term of (3.18) as

$$\left\| \tau \sum_{k=0}^{n-1} S_\tau (n\tau - k\tau) \frac{N(\tau) - I}{\tau} \Pi_\tau u(k\tau) - \int_0^{n\tau} S_\tau (n\tau - s) \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) ds \right\|_{L^r(0,T;L^s)}$$

$$\leq C_{d,p} \tau^{1/2} \left\| \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) \right\|_{L^{q_0'}(0,T;W^{1,q_0'})} + C_{d,p} \left\| \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) \right\|_{W^{1,q_0'}(0,T;L^{r_0'})}$$

$$= C_{d,p} \tau^{1/2} \left\| B_1(u) \right\|_{L^{q_0'}(0,T;W^{1,q_0'})} + C_{d,p} \left\| B_1(u) \right\|_{W^{1,q_0'}(0,T;L^{r_0'})}$$
for all admissible pair \((q, r)\). On the other hand, for the second term of (3.18), we begin the estimate by splitting
\[
\frac{N(\tau) - I}{\tau} \Pi_\tau u - i \lambda |u|^p u = \left( \frac{N(\tau) - I}{\tau} \Pi_\tau u - i \lambda |\Pi_\tau u|^p \Pi_\tau u \right) + i \lambda \left( |\Pi_\tau u|^p |\Pi_\tau u - |u|^p u \right)
= B_2(u) + B_3(u).
\]
Then, we can apply (2.6) to obtain that
\[
\left\| \int_0^{n\tau} S_\tau (n\tau - s) \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) ds - i \lambda \int_0^{n\tau} S_\tau (n\tau - s) |u|^p u(s) ds \right\|_{L^q(0,T;L^r)} \\
\leq \left\| \int_0^{n\tau} S_\tau (n\tau - s) B_2(u) ds \right\|_{L^q(0,T;L^r)} + \left\| \int_0^{n\tau} S_\tau (n\tau - s) B_3(u) ds \right\|_{L^q(0,T;L^r)} \\
\leq C \| B_2(u) \|_{L^{q_0}(0,T;L^{r_0})} + C \| B_3(u) \|_{L^{q_0}(0,T;L^{r_0})}
\]
for all admissible pair \((q, r)\). Thus, it is enough to show that
\[
\frac{1}{T^{1/2}} \| B_1(u) \|_{L^{q_0}(0,T;W_{1,0}^{1, r_0})} + \frac{T}{r_0} \| B_1(u) \|_{W^{1, q_0}(0,T;L_{r_0})} + \| B_2(u) \|_{L^{q_0}(0,T;L^{r_0})} + \| B_3(u) \|_{L^{q_0}(0,T;L^{r_0})} \leq C_{d,p} T^{1/2} \left( 1 + T^{\frac{1}{q_0}} \right) \left( M_1^{p+1} + M_1^{2p+1} \right).
\]
(3.19)

For the first term on the right hand side of (3.19), we apply Lemma 2.5, the Hölder inequality (2.20), Lemma 2.6 and the Sobolev embedding \(H^1(\mathbb{R}^d) \rightarrow L^{\infty}(\mathbb{R}^d)\) to obtain that
\[
\left\| B_1(u) \|_{L^{q_0}(0,T;W_{1,0}^{1, r_0})} \\
= \left\| \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) \right\|_{L^{q_0}(0,T;W_{1,0}^{1, r_0})} \\
\leq \left\| \Pi_\tau u \|_{L^{q_0}(0,T;L^{r_0})} + C_p \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} \right\|_{L^{q_0}(0,T;L^{r_0})} \\
\leq C_p T^{\frac{1}{q_0} - \frac{1}{r_0}} \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} + \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} \right\|_{L^{q_0}(0,T;L^{r_0})} \\
\leq C_{d,p} T^{\frac{1}{q_0} - \frac{1}{r_0}} \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;H^1)} \left\| u \right\|_{L^{q_0}(0,T;W_{1,0}^{1, r_0})},
\]
which is then bounded by \(C_{d,p} T^{\frac{1}{q_0} - \frac{1}{r_0}} M_1^{p+1} \), owing to Theorem A. This bound is fit to (3.17) because we have \(T^{\frac{1}{q_0} - \frac{1}{r_0}} \leq 1 + T^{\frac{1}{q_0}}\) due to \(q_0 \leq q_0\) as \(q_0 \geq 2\).

Next, we estimate the second term on the right hand side of (3.19). Similarly to (2.9), we deduce that
\[
\left\| B_1(u) \right\|_{W_{1,0}^{1, r_0}(0,T;L_{r_0})} \\
= \left\| \frac{N(\tau) - I}{\tau} \Pi_\tau u(s) \right\|_{L^{q_0}(0,T;L^{r_0})} \\
\leq C_p \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} + C_p \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} \\
\leq C_p \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} + C_p \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})} \left\| \Pi_\tau u \right\|_{L^{q_0}(0,T;L^{r_0})},
\]
where the second inequality follows from the identities \(\partial_t \Pi_\tau u = \Pi_\tau \partial_t u\) and \(\Pi_\tau u = i\Delta u + i\lambda |u|^p u\). By the Hölder’s inequality (2.20) and Lemma 2.6, the first term on
the right hand side of (3.20) is bounded by
\[
\left\| \Pi_r u \left| \Pi_r \Delta u \right|_{L^6(0,T;L^{6})} \right\| \leq \left\| \Pi_r u \right\|_{L^6(0,T;L^{6})} \left\| \Pi_r \Delta u \right\|_{L^6(0,T;L^{6})} \\
\leq C_{d,p} \tau^{-\frac{1}{2}} T_{\tau}^{-\frac{1}{6}} \left\| \Pi_r u \right\|_{L^6(0,T;L^{6})} \left\| \Pi_r \nabla u \right\|_{L^6(0,T;L^{6})},
\]
which is bounded by \(C_{d,p} \tau^{-\frac{1}{2}} T_{\tau}^{-\frac{1}{6}} M_{\tau}^{p+1}\) owing to Theorem A. Also, for the second term of (3.20), we can apply Lemma 2.8 to bound it by \(C_{d,p} \tau^{-1/2} (1 + T_{\tau}^{\frac{1}{6}}) M_{\tau}^{p+1}\).

For the third term of (3.19), we recall that \(N(\tau) = e^{i\tau\lambda}|a|^p a\) for \(a \in \mathbb{C}\). Then by using the mean value theorem, one has
\[
|B_2(u)| = \left| \frac{N(\tau) - I}{\tau} \Pi_r u - i\lambda|\Pi_r u|^p \Pi_r u \right| \leq C\tau \left| \Pi_r u \right|^2, \]
which leads to
\[
|B_2(u)|_{L^6(0,T;L^6)} \leq C\tau \left| \Pi_r u \right|^2_{L^6(0,T;L^6)}.
\]
Then, by Lemma 2.8, the latter term is bounded by \(C_{d,p} \tau^{1/2} (1 + T_{\tau}^{\frac{1}{6}}) M_{\tau}^{2p+1}\).

For the last term of (3.19), we proceed as follows:
\[
\left| B_3(u) \right|_{L^6(0,T;L^6)} = \left| \left| \Pi_r u \right|_{L^6(0,T;L^6)}^p \Pi_r u - \left| u \right|_{L^6(0,T;L^6)}^p \right|_{L^6(0,T;L^6)} \\
\leq C_{d,p} \tau^{1/2} T_{\tau}^{-\frac{1}{6}} \left| \left| \Pi_r u \right|_{L^6(0,T;L^6)}^p \left( \left| \Pi_r u \right|_{L^6(0,T;L^6)} + \left| u \right|_{L^6(0,T;L^6)} \right) \right|_{L^6(0,T;L^6)} \\
\leq C_{d,p} \tau^{1/2} T_{\tau}^{-\frac{1}{6}} \left| \left| \left| \Pi_r u \right|_{L^6(0,T;L^6)}^p \left| \Pi_r u \right|_{L^6(0,T;L^6)} \right|_{L^6(0,T;L^6)} \right|_{L^6(0,T;L^6)},
\]
where we have used Lemma 2.6 and the Sobolev embedding. The latter term is bounded by \(C_{\tau}^{1/2} T_{\tau}^{-\frac{1}{6}} M_{\tau}^{p+1}\) from Theorem A.

Collecting the above estimates, we obtain (3.19). The proof is complete. \(\square\)

4. Global \(H^1\) stability of \(Z_{\tau}\) for \(0 < p < \frac{3}{4}\). In this section, we first prove the local \(H^1\) stability result on the scheme \(Z_{\tau}\) in the space \(\ell^q(n \tau \in I; W^{1,r}(\mathbb{R}^d))\) with initial data \(\phi\) in \(H^1(\mathbb{R}^d)\) and \(p \in (0, p_d)\). After that, we will prove Theorem 1.5 corresponding to the case that \(0 < p < \frac{1}{4}\). The proof for this case will be derived by combining the global \(L^2\) stability result (see Lemma 4.3.4) and the local \(H^1\) stability of \(Z_{\tau}\) (see Proposition 4.1).

By considering the discrete Strichartz estimate (2.4), we can take a constant \(C = C(d,p) \in [1, \infty)\) such that
\[
C = \max \left\{ \sup_{\tau \in (0,1)} \sup_{\phi \in H^1} \left\| S_{\tau} \phi \right\|_{\ell^\infty(n \tau \in I; W^{1,r}(\mathbb{R}^d))} + \left\| S_{\tau} \phi \right\|_{\ell^\infty(n \tau \in I; W^{1,r}(\mathbb{R}^d))} \right\|_{H^1}, 1 \right\}. \tag{4.1}
\]
Then, we have the following proposition whose proof is inspired by [12].

**Proposition 4.1** (Local \(H^1\) stability). Let \(d \geq 1\) and \(0 < p < p_d\), and suppose that \(\phi \in H^1(\mathbb{R}^d)\). Then there exists a constant \(\beta_{d,p} > 0\) such that the \(Z_{\tau}\) satisfies
\[
\left\| Z_{\tau}(n \tau) \right\|_{\ell^q(n \tau \in I; W^{1,r}(\mathbb{R}^d))} \leq 4C \left\| \phi \right\|_{H^1(\mathbb{R}^d)} \quad \text{for all} \quad \tau \in (0,1), \tag{4.2}
\]
where \((q,r) \in \{(q_0,r_0),(\infty,2)\}\) and \(T_0 > 0\) is defined by
\[
T_0 = \beta_{d,p} \|\phi\|_{H^1}^{\frac{2(q+r)}{q}}.
\] (4.3)

Here, the \(\beta_{d,p} > 0\) is a constant determined by \(d\) and \(p\).

Proof. To obtain the estimate (4.2), we consider the following set
\[
\Lambda = \{ N \in \mathbb{N} \cup \{0\} : \{\|Z_t(k\tau)\|_{\ell^q(\omega_0,N_r,\tau; W^{1,r_0})} + \|Z_t(k\tau)\|_{\ell^q(\omega_0,N_r,\tau; H^1)} \leq 4C\|\phi\|_{H^1}\}.
\] (4.4)

If \(\Lambda\) is an infinite set, then (4.2) follows trivially. Therefore, we suppose that \(\Lambda\) is a finite set, and let \(N_\ast\) be the largest element of \(\Lambda\). It is then sufficient to find a lower bound on \(N_\ast\), as the form of \(N_\ast \geq T_0/\tau\) for \(T_0 > 0\) defined in (4.3) with a suitable choice of \(\beta_{d,p} > 0\).

First we verify that the set \(\Lambda\) is non-empty. Indeed, by the definitions of \(Z_t\) and \(\mathbf{C}\) given in (1.2) and (4.1) respectively, we find that
\[
\tau^{\frac{1}{q}}\|Z_t(0)\|_{W^{1,r_0}} + \|Z_t(0)\|_{H^1} = \tau^{\frac{1}{q}}\|S_t(0)\|_{W^{1,r_0}} + \|S_t(0)\|_{H^1}
\]
\[
\leq \|S_t(\tau)\|_{\ell^q(\omega_0,N_r,\tau; W^{1,r_0})} + \|S_t(\tau)\|_{\ell^q(\omega_0,N_r,\tau; H^1)}
\]
\[
\leq \mathbf{C}\|\phi\|_{H^1},
\]
which means that \(0 \in \Lambda\).

Let \((q,r)\) denote either \((q_0,r_0)\) or \((\infty,2)\). Then, using the Duhamel formula (1.4) we have
\[
\left(\sum_{n=0}^{N_\ast+1} \|Z_t(n\tau)\|_{W^{1,r}}^{\frac{1}{q}}\right)^{1/q}
\]
\[
\leq \|S_t(n\tau)\|_{\ell^q(\omega_0,N_r,\tau; W^{1,r})}
\]
\[
+ \left\| \sum_{k=0}^{n-1} S_t(n\tau - k\tau) \frac{N(\tau) - I}{\tau} Z_t(k\tau) \right\|_{\ell^q(\omega_0,N_r,\tau; W^{1,r})}
\]
\[
\leq \mathbf{C}\|\phi\|_{H^1} + \left\| \sum_{k=0}^{n-1} S_t(n\tau - k\tau) \frac{N(\tau) - I}{\tau} Z_t(k\tau) \right\|_{\ell^q(\omega_0,N_r,\tau; W^{1,r})},
\] (4.5)

where (4.1) is used for the second inequality. We can bound the last term of (4.5) by applying the Strichartz estimate (2.5) as follows:
\[
\left\| \sum_{k=0}^{n-1} S_t(n\tau - k\tau) \frac{N(\tau) - I}{\tau} Z_t(k\tau) \right\|_{\ell^q(\omega_0,N_r,\tau; W^{1,r})}
\]
\[
\leq C_{d,p} \left\| \frac{N(\tau) - I}{\tau} Z_t(n\tau) \right\|_{\ell^q(0\leq n\tau \leq N_\ast; W^{1,r_0})}.\] (4.6)

To estimate the right hand side of (4.6), we apply Lemma 2.5 and the Hölder’s inequality (2.21). Then,
\[
\left\| \frac{N(\tau) - I}{\tau} Z_t(n\tau) \right\|_{\ell^q(0\leq n\tau \leq N_\ast; W^{1,r_0})}
\]
\[
\leq \left\| \frac{N(\tau) - I}{\tau} Z_t(n\tau) \right\|_{\ell^q(0\leq n\tau \leq N_\ast; L^{r_0})} + \left\| \nabla \left( \frac{N(\tau) - I}{\tau} Z_t(n\tau) \right) \right\|_{\ell^q(0\leq n\tau \leq N_\ast; L^{r_0})}.
\]
Indeed, estimate (4.5) holds for any admissible pair and its right hand side is $\beta_{N}$.

In the proof above, we note that the estimate (4.2) holds for any admissible $p$. Combining estimates (4.6) and (4.7) in (4.5), we obtain

To proceed further, we utilize the Sobolev embedding $H^1(\mathbb{R}^d) \to L^{q_r}(\mathbb{R}^d)$ and the fact that $N_\ast \in \Lambda$. Then we get

Combining estimates (4.6) and (4.7) in (4.5), we obtain

for all $(q, r) \in \{(q_0, r_0), (\infty, 2)\}$. Consequently,

This estimate yields that $N_\ast$ obeys the following estimate

Indeed, if (4.9) does not hold, then it follows directly from (4.8) that $N_\ast + 1 \in \Lambda$, in view of definition (4.4). However, it is impossible by the maximality of $N_\ast$. Thus (4.9) is true, and hence

where we used the identity (3.11). This shows that (4.2) is true with the choice of $\beta_{d, p} = (2C_{d, p}A^{p+1}C_p)^{-\frac{2(p+2)}{4(p+2)-d}}$ in (4.3). The proof is finished. \hfill \Box

Remark 4.2. In the proof above, we note that the estimate (4.2) holds for any admissible $(q, r)$, i.e.,

Indeed, estimate (4.5) holds for any admissible pair and its right hand side is bounded as in (4.6).

Before showing the global $H^1$ stability of $Z_\tau$ for the case of $0 < p < \frac{4}{d}$, we recall the $L^2$ stability result on $Z_\tau$ from [12].

Lemma 4.3 (Theorem 1.1 in [12], pages 3030–3032 in detail). For $d \geq 1$, $0 < p < \frac{4}{d}$ and $\phi \in L^2(\mathbb{R}^d)$, there exist a constant $\tilde{\beta}_{d, p} > 0$ and a time $\tilde{T}_0 = \tilde{\beta}_{d, p}\|\phi\|_{L^2}^{-\frac{4p}{8-4p}}$ such that

holds for all $k \in \mathbb{N} \cup \{0\}$. 

Now we are ready to prove the global $H^1$ stability of $Z_{\tau}$ for the mass-subcritical case.

**Proof of Theorem 1.5.** Consider $\phi \in H^1(\mathbb{R}^d)$ and the solution $u$ to (1.1) with initial data $\phi$. Let us set $T_1 > 0$ by

$$T_1 := \gamma_{d,p} \min \{ T_0, \tilde{T}_0 \} = \gamma_{d,p} \min \left\{ \beta_{d,p} \| \phi \|_{H^1}^{\frac{2(p+2)}{p+2d}} \gamma_{d,p} \| \phi \|_{L^2}^{\frac{4p}{d}} \right\}, \quad (4.10)$$

where a constant $\gamma_{d,p} \in (0, 1)$ will be chosen later. Here the constants $\beta_{d,p}$ and $\tilde{\beta}_{d,p}$ are given in (4.3) and Lemma 4.3.

In the case of $\tau \in [T_1/2, 1)$, we use the identity $Z_{\tau} = (\Pi_{T/2})^2Z_{\tau}$ from the definition of $Z_{\tau}$ in (1.2). By the Hölder’s inequality in time, (2.11), (2.12) and (2.13) in Lemma 2.6, we have

$$\| Z_{\tau}(n\tau) \|_{L^q(T_1, T_2; W^{1, r})} \leq \left( \frac{2T}{\tau} \right)^{\frac{4}{q}} \sup_{n \in [0, T]} \| (\Pi_{T/2})^2 Z_{\tau}(n\tau) \|_{W^{1, r}}$$

where we used also the definition of admissible pairs $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$ and the fact that $L^2$ norm of $Z_{\tau}$ does not increase, as proved by Ignat [12], that is,

$$\sup_{n \in [0, \infty)} \| Z_{\tau}(n\tau) \|_{L^2} \leq \| \phi \|_{L^2}.$$

Using the fact that $\tau \geq T_1/2$ and (4.10), we have

$$\| Z_{\tau}(n\tau) \|_{L^q(T_1, T_2; W^{1, r})} \leq C_{d,q} \left( \frac{T_1}{\tau} \right)^{\frac{4}{q}} \| \phi \|_{L^2}$$

for some constant $c_{d,p} > 0$, which proves the theorem for the case $\tau \in [T_1/2, 1)$.

Now, we consider $\tau \in (0, T_1/2)$ and we choose a value $R \in (T_1/2, T_1]$ such that $R/\tau \in \mathbb{N}$. We set $I_j = [jR, (j + 1)R)$ for $j \in \mathbb{N} \cup \{0\}$. For each $j \in \mathbb{N} \cup \{0\}$ we choose $m_j \in \mathbb{N}$ such that $m_j \tau = jR$, i.e., $m_j = j(R/\tau)$. Then

$$\| Z_{\tau}(n\tau) \|_{L^q(I_j; W^{1, r})} \leq \| Z_{\tau}(m_j \tau + n\tau) \|_{L^q(0, R; W^{1, r})} \quad (4.11)$$

for any admissible pair $(q, r)$. By applying the Strichartz estimates of Corollary 2.3 to the Duhamel formula of $Z_{\tau}(m_j \tau + n\tau)$, we obtain

$$\| Z_{\tau}(m_j \tau + n\tau) \|_{L^q(0, R; W^{1, r})} \leq \| S_{\tau}(n\tau) Z_{\tau}(m_j \tau) \|_{L^q(0, R; W^{1, r})}$$

where $S_{\tau}(n\tau)$ is the Duhamel formula of $Z_{\tau}(m_j \tau + n\tau)$. The proof is now complete.
By applying Lemma 4.3, we estimate the right hand side as follows:

\[ \left\| \frac{N(\tau) - I}{\tau} Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} \]

where we can estimate the last term using Lemma 2.5 and the Hölder’s inequality as

\[ \left\| \frac{N(\tau) - I}{\tau} Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} \leq C_d,q \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} + \left\| \nabla \left( \frac{N(\tau) - I}{\tau} Z_\tau(m_j \tau + n \tau) \right) \right\|_{\ell^0_0(0, R; L^{r_0}_0)} \]

\[ \leq R^{1 - \frac{dp}{4}} \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; L^{r_0}_0)} + (p + 1) \left\| \nabla Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; L^{r_0}_0)}. \]

Here we also used the equalities

\[ \frac{1}{r_0} = p + 1 \quad \text{and} \quad \frac{1}{q_0} - \frac{(p + 1)}{q_0} = 1 - \frac{dp}{4}. \]

By applying Lemma 4.3, we estimate the right hand side as follows:

\[ \left\| \frac{N(\tau) - I}{\tau} Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} \leq C_d,p R^{1 - \frac{dp}{4}} \left\| \phi \right\|_{L^2}^p \left( \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; L^{r_0}_0)} + \left\| \nabla Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; L^{r_0}_0)} \right) \]

\[ \leq C_d,p \left( 2^\gamma_d,p \tilde{\beta}_d,p \right)^{1 - \frac{dp}{4}} \left\| \phi \right\|_{L^2}^p \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)}. \]

We insert this estimate into (4.12). Then, choosing \( \gamma_d,p > 0 \) smaller in (4.10) if necessary, we arrive at the following estimate:

\[ \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} \leq C_d,q \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} + \frac{1}{2} \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} \]

for any \( j \in \mathbb{N} \). This estimate with \( (q, r) = (q_0, r_0) \) yields

\[ \left\| Z_\tau(m_j \tau + n \tau) \right\|_{\ell^0_0(0, R; W^{1, r}_0)} \leq 2C_d,p \left\| Z_\tau(n \tau) \right\|_{L^\infty(J_{n-1}; H^1)}. \]

By inserting this back into (4.13) and using (4.11), we obtain

\[ \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(J_{n}; W^{1, r})} \leq 2C_d,p \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(J_{n-1}; H^1)}, \]

for any admissible pair \( (q, r) \) and \( j \in \mathbb{N} \). Here we may regard that the constant \( C_d,p \) is larger than 1/2. By applying this with \( (q, r) = (\infty, 2) \), we finally deduce that for any \( T > 0, \)

\[ \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(0, T; H^1)} \leq \sup_{0 \leq j \leq \left\lfloor \frac{2T}{T_1} \right\rfloor} \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(J_j; H^1)} \leq (2C_d,p)^{\left\lfloor \frac{2T}{T_1} \right\rfloor} \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(J_0; H^1)} \leq \exp \left( \frac{2T}{T_1} \log(2C_d,p) \right) \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(J_0; H^1)} \]

where we have used Proposition 4.1 for the final inequality. Combining this with (4.14) we get an upper bound

\[ \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(0, T; W^{1, r})} \leq \exp \left( C_d,p T \max \left\{ \left\| \phi \right\|_{H^1}^{\frac{2p(\gamma_d,p)}{dp}}, \left\| \phi \right\|_{L^2}^{\frac{4p}{dp}} \right\} \right) \left\| Z_\tau(n \tau) \right\|_{\ell^0_0(J_0; H^1)}, \]
where the definition of $T_1$ of (4.10) is used. The proof is finished.

5. More lemmas for the case of $p \geq 1$. In Section 5-6, we explicitly write $Z_r^\phi$ and $u^\phi$ to denote the flow $Z_r$ and the solution $u$ corresponding to the initial data $\phi$. We introduce the well-posedness theory on (1.1) for $p \geq 1$ as follows.

**Theorem B** (Theorem 5.3.1 in [3]). Let $d \geq 1$, $1 \leq p < p_d$, and $(q, r)$ be any admissible pair. For any $M \geq 1$, there is a time $T_0 = c_{d,p} M^{\frac{2(p+2)}{d(p-2)}}$ with some absolute constant $c_{d,p} > 0$ such that the following statements hold:

- If $\phi_1, \phi_2 \in H^1(\mathbb{R}^d)$ with $\|\phi_1\|_{H^1}, \|\phi_2\|_{H^1} \leq M$, there is a constant $C_{d,p} > 0$ such that
  \[\|u^{\phi_1} - u^{\phi_2}\|_{L^\infty(0, T; W^{1,r})} \leq C_{d,p} \|\phi_1 - \phi_2\|_{H^1}.\]

- If $\psi \in H^2(\mathbb{R}^d)$ with $\|\psi\|_{H^1} \leq M$, there is a constant $M_3 = M_3(d, p, M, \psi) > 0$ such that
  \[\|u^{\psi}\|_{L^\infty(0, T; W^{2,r})} + \|u^{\psi}\|_{L^\infty(0, T; W^{2,r})} \leq M_3.\]

In the following result, we obtain a stability of $Z_r^\phi$ similarly to that of $u^\phi$ given in Theorem B. It will be essential for the global $H^1$ stability of $Z_r$ for the energy-subcritical case.

**Proposition 5.1.** Let $d \geq 1$, $1 \leq p < p_d$, and $(q, r)$ be any admissible pair. For any $M \geq 1$, there is a constant $\beta_{d,p} > 0$

\[T_2 := \beta_{d,p} M^{-\frac{4(p+2)}{p-2}} (\leq T_0)\]

such that the following statements hold:

- If $\phi_1, \phi_2 \in H^1(\mathbb{R}^d)$ with $\|\phi_1\|_{H^1}, \|\phi_2\|_{H^1} \leq M$, there is a constant $C_{d,p} \geq 1$
  such that
    \[\|Z_{\phi_1}^\tau - Z_{\phi_2}^\tau\|_{\ell^\infty(0, T; W^{1,r})} \leq C_{d,p} \|\phi_1 - \phi_2\|_{H^1(\mathbb{R}^d)}\]
  for any $\tau \in (0, 1)$.

- If $\psi \in H^2(\mathbb{R}^d)$ with $\|\psi\|_{H^1} \leq M$, there is a constant $C(d, p, M, \psi) > 0$
  such that
    \[\|Z_{\psi}^\tau - u^{\psi}\|_{\ell^\infty(0, T; W^{1,r})} \leq \tau^{1/2} C(d, p, M, \psi)\]
  for any $\tau \in (0, 1)$.

- If $\phi \in H^1(\mathbb{R}^d)$ with $\|\phi\|_{H^1} \leq M/2$, then
  \[\lim_{\tau \to 0} \|Z_{\phi}^\tau - u^{\phi}\|_{\ell^\infty(0, T; H^1)} = 0.\]

The proof of (5.4) is motivated by the proof of (5.1) while we apply the Duhamel-type formulas (1.4) for $Z_r^\phi$ instead of the Duhamel-type formulas (1.5) for $u^\phi$. The idea of the proof of (5.5) is similar to that of the proof of (1.8) in Theorem 1.4.

To establish Proposition 5.1, we introduce a technical lemma.

**Lemma 5.2.** Suppose that $d \geq 1$ and $1 \leq p < p_d$, and let $(q_0, r_0)$ denote the admissible pair $(\frac{4(p+2)}{d(p-2)}, p+2)$. Then, for any time interval $I$ and functions $v, w : I \times \mathbb{R}^d \to C$, we have

\[\|\frac{N(\tau)}{\tau} v - \frac{N(\tau)}{\tau} w\|_{\ell^\infty(0, I; W^{1,r_0})},\]

\[\leq C|I|^{\frac{1}{q_0} - \frac{1}{p}} \|v - w\|_{\ell^\infty(I, H^1)} \left(\|v\|_{\ell^{q_0}(I, W^{1,r_0})} + \|w\|_{\ell^{q_0}(I, W^{1,r_0})}\right) \left(\|v\|^{p-1}_{\ell^{q_0}(I, H^1)} + \|w\|^{p-1}_{\ell^{q_0}(I, H^1)}\right)\]

\[+ \tau \|\nabla w\|_{\ell^{q_0}(I, L^{r_0})} \|w\|^{2(p-1)}_{\ell^{q_0}(I, H^1)} + \tau \|\nabla w\|_{\ell^{q_0}(I, L^{r_0})} \|w\|^{2(p-1)}_{\ell^{q_0}(I, H^1)}\].
Proof. The proof basically follows from Lemma 2.5 and estimates in [3, Section 4.4]. Firstly, we apply (2.7) and the Hölder’s inequality, to deduce that
\[
\left\| \frac{N(\tau) - I}{\tau} v - \frac{N(\tau) - I}{\tau} w \right\|_{L^0(\Gamma; L'_{\theta})} \leq C \left| I - \frac{1}{\tau} \right| \left\| v - w \right\|_{L^\infty(\Gamma; L'_{\theta})} \left( \| v \|_{L^0(\Gamma; L'_{\theta})} \| v \|_{L^1(\Gamma; L'_{\theta})} + \| w \|_{L^0(\Gamma; L'_{\theta})} \| w \|_{L^1(\Gamma; L'_{\theta})} \right)
\]
\[
\leq C \left| I - \frac{1}{\tau} \right| \left\| v - w \right\|_{L^\infty(\Gamma; H^1)} \left( \| v \|_{L^0(\Gamma; L'_{\theta})} \| v \|_{L^1(\Gamma; H^1)} + \| w \|_{L^0(\Gamma; L'_{\theta})} \| w \|_{L^1(\Gamma; H^1)} \right),
\]
where the Sobolev embedding is used for the last inequality, with the fact that \( r_0 = p + 2 < p_d + 2 \).

Next, by differentiating and rearranging, we have that
\[
\nabla \left( \frac{N(\tau) - I}{\tau} v \right) - \nabla \left( \frac{N(\tau) - I}{\tau} w \right) = \nabla \left( \frac{e^{i\tau \lambda} |v|^p - 1}{\tau} v \right) - \nabla \left( \frac{e^{i\tau \lambda} |w|^p - 1}{\tau} w \right) = i\lambda p \left( \frac{e^{i\tau \lambda} |v|^p - 1}{\tau} |v|^p \nabla v - e^{i\tau \lambda} |w|^p |w|^p \nabla w \right) + \left( \frac{e^{i\tau \lambda} |v|^p - 1}{\tau} - \frac{e^{i\tau \lambda} |w|^p - 1}{\tau} \right) \nabla v + \left( \frac{e^{i\tau \lambda} |w|^p - 1}{\tau} \right) (\nabla v - \nabla w),
\]
which together with Lemma 2.5 yields that
\[
\left\| \nabla \left( \frac{N(\tau) - I}{\tau} v \right) - \nabla \left( \frac{N(\tau) - I}{\tau} w \right) \right\|_{L^0(\Gamma; L'_{\theta})} \leq p \left\| e^{i\tau \lambda} |v|^p |v|^p \nabla v - e^{i\tau \lambda} |w|^p |w|^p \nabla w \right\|_{L^0(\Gamma; L'_{\theta})} + \left\| (|v|^p - |w|^p) \nabla v \right\|_{L^0(\Gamma; L'_{\theta})} + \left\| |w|^p |\nabla w - \nabla v| \right\|_{L^0(\Gamma; L'_{\theta})}.
\]
Using the following identity
\[
e^{i\tau \lambda} |v|^p \nabla v - e^{i\tau \lambda} |w|^p |w|^p \nabla w = \left( |v|^p \nabla v - |w|^p \nabla w \right) e^{i\tau \lambda} |v|^p |w|^p \nabla w \left( e^{i\tau \lambda} |v|^p - e^{i\tau \lambda} |w|^p \right),
\]
we can decompose the first term on the right hand side of (5.7) into
\[
\left\| e^{i\tau \lambda} |v|^p |v|^p \nabla v - e^{i\tau \lambda} |w|^p |w|^p \nabla w \right\|_{L^0(\Gamma; L'_{\theta})} \leq \left\| |v|^p (\nabla v - \nabla w) \right\|_{L^0(\Gamma; L'_{\theta})} + \left\| (|v|^p - |w|^p) \nabla w \right\|_{L^0(\Gamma; L'_{\theta})},
\]
From the Hölder’s inequality and the Sobolev embedding, we estimate the first right term of (5.9) as
\[
\left\| |v|^p (\nabla v - \nabla w) \right\|_{L^0(\Gamma; L'_{\theta})} \leq |I| \frac{1}{\tau} \frac{1}{\tau} \left\| |v|^p \right\|_{L^0(\Gamma; L^{2\theta_0/p})} \left\| \nabla v - \nabla w \right\|_{L^\infty(\Gamma; L^2)} \leq |I| \frac{1}{\tau} \frac{1}{\tau} \left\| |v|^p \right\|_{L^0(\Gamma; L^{2\theta_0})} \left\| |v|^p \right\|_{L^1(\Gamma; L^1)} \left\| v - w \right\|_{L^\infty(\Gamma; L^1)} \left\| v - w \right\|_{L^\infty(\Gamma; H^1)},
\]
and by using $\frac{dp}{2r_0} \leq 1$ for $p < p_d$ and the H"older’s inequality, we get

$$\left\| v \right\|_{\dot{H}^k(\mathbb{R}^d)}^{p} \leq \left\| v \right\|_{\dot{H}^k(\mathbb{R}^d)}^{p}$$

$$\leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

$$\leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

For the second right term of (5.9), we use an inequality

$$\left\| v \right\|_{\dot{H}^k(\mathbb{R}^d)}^{p} \leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

$$\leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

For the second right term of (5.9), we use an inequality

$$\left\| v \right\|_{\dot{H}^k(\mathbb{R}^d)}^{p} \leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

$$\leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

For the third right term of (5.8), we notice that

$$\left\| v \right\|_{\dot{H}^k(\mathbb{R}^d)}^{p} \leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

Then, similarly to (5.10) we obtain

$$\left\| v \right\|_{\dot{H}^k(\mathbb{R}^d)}^{p} \leq C \left( \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} + \left\| \nabla v \right\|_{L^2(\mathbb{R}^d)} \right)^{p}$$

Combining the above estimates gives the desired bound for the first term of (5.7), with help of the Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{r_0}(\mathbb{R}^d)$. The second and third terms of the right hand side of (5.7) can be estimated as we did for the right hand sides of (5.9). The proof is done. $\square$

Now, we are ready to show Proposition 5.1.

**Proof of Proposition 5.1.** We divide the proof into three parts corresponding to (5.4), (5.5), and (5.6).

**Proof of (5.4).** Let $\phi_1$ and $\phi_2 \in H^1(\mathbb{R}^d)$ such that $\|\phi_1\|_{H^1} \leq M$ and $\|\phi_2\|_{H^1} \leq M$. We consider the difference between the Duhamel formulas of $Z_{\tau}^{\phi_1}$ and $Z_{\tau}^{\phi_2}$ provided by (1.4). Then by applying the Strichartz estimate (2.3), we have

$$\left\| Z_{\tau}^{\phi_1}(n\tau) - Z_{\tau}^{\phi_2}(n\tau) \right\|_{0(T_2;W^{1,\tau})}$$

$$= \left\| S_{\tau}(n\tau)(\phi_1 - \phi_2) + \sum_{k=0}^{n-1} S_{\tau}((n-k)\tau) \left( \frac{N(\tau) - I}{\tau} Z_{\tau}^{\phi_1} - \frac{N(\tau) - I}{\tau} Z_{\tau}^{\phi_2} \right) (k\tau) \right\|_{0(T_2;W^{1,\tau})}$$

$$\leq C \left\| \phi_1 - \phi_2 \right\|_{H^1(\mathbb{R}^d)} + C \left\| \frac{N(\tau) - I}{\tau} Z_{\tau}^{\phi_1} - \frac{N(\tau) - I}{\tau} Z_{\tau}^{\phi_2} \right\|_{0(T_2;W^{1,\tau})}$$

(5.11)
for any admissible pair \((q, r)\). By the local \(H^1\) stability of Proposition 4.1 with choosing \(\beta_{d, p} > 0\) in (5.3) small enough, for \((q, r) \in \{(q_0, r_0), (\infty, 2)\}\), we have

\[
\|Z_{\tau}^{\phi_1}(n\tau)\|_{\ell^q(0, T_2; W^{1, r})} \leq \|Z_{\tau}^{\phi_2}(n\tau)\|_{\ell^q(0, T_0; W^{1, r})} \leq 4CM, \quad j = 1, 2,
\]

(5.12)
since \(T_2 < T_0\). We proceed to use bound (5.12) to estimate the last term of (5.11). By applying Lemma 5.2 and the Sobolev embedding in (5.11) along with (5.12), we obtain that

\[
\|Z_{\tau}^{\phi_1}(n\tau) - Z_{\tau}^{\phi_2}(n\tau)\|_{\ell^q(0, T_2; W^{1, r})} \\
\leq C\|\phi_1 - \phi_2\|_{H^1(\mathbb{R}^d)} + C T_2^{\frac{1}{p} - \frac{1}{q}} \|
\frac{\partial \phi_1}{\partial \tau}\|_{\ell^q(0, T_2; L^p)} + \|
\frac{\partial \phi_2}{\partial \tau}\|_{\ell^q(0, T_2; L^p)}
\]

(5.13)

To estimate the last term of (5.13), we apply (2.13) in Lemma 2.6 to get

\[
\|Z_{\tau}^{\phi_1}(n\tau)\|_{\ell^q(0, T_2; W^{1, r})} \leq C T_2^{\frac{1}{p} - \frac{1}{q}} \|
\frac{\partial \phi_1}{\partial \tau}\|_{\ell^q(0, T_2; L^p)} + \|
\frac{\partial \phi_2}{\partial \tau}\|_{\ell^q(0, T_2; L^p)}
\]

This implies that

\[
\|Z_{\tau}^{\phi_1}(n\tau) - Z_{\tau}^{\phi_2}(n\tau)\|_{\ell^q(0, T_2; W^{1, r})} \leq C T_2^{\frac{1}{p} - \frac{1}{q}} \|
\frac{\partial \phi_1}{\partial \tau}\|_{\ell^q(0, T_2; L^p)} + \|
\frac{\partial \phi_2}{\partial \tau}\|_{\ell^q(0, T_2; L^p)}
\]

(5.14)

We remark that \(1 - \frac{dp}{2(p+2)} > 0\) and \(2p - 1 > 0\) since \(1 < p < p_d\), so the final bound of (5.14) is uniform in \(\tau\). Inserting the inequality (5.14) into (5.13) with the fact \(C, M \geq 1\), we have

\[
\|Z_{\tau}^{\phi_1}(n\tau) - Z_{\tau}^{\phi_2}(n\tau)\|_{\ell^q(0, T_2; W^{1, r})} \\
\leq C\|\phi_1 - \phi_2\|_{H^1(\mathbb{R}^d)} + C d M T_2^{\frac{1}{p} - \frac{1}{q}} (4CM)^{2p} \|
\frac{\partial \phi_1}{\partial \tau}\|_{\ell^q(0, T_2; L^p)} + \|
\frac{\partial \phi_2}{\partial \tau}\|_{\ell^q(0, T_2; L^p)}
\]

Now, we choose \(T_2 = \left(\frac{2C d M (4CM)^{2p}}{\|\phi_1 - \phi_2\|_{H^1(\mathbb{R}^d)}}\right)^{\frac{1}{0.99 - 0.99}}\). Then, the estimate above with \((q, r) = (\infty, 2)\) yields that

\[
\|Z_{\tau}^{\phi_1}(n\tau) - Z_{\tau}^{\phi_2}(n\tau)\|_{\ell^q(0, T_2; H^1)} \leq 2C\|\phi_1 - \phi_2\|_{H^1(\mathbb{R}^d)}
\]

By inserting this into (5.13) for general admissible pairs \((q, r)\), we obtain (5.4).

**Proof of (5.5).** Assume that \(\psi \in H^2(\mathbb{R}^d)\) with \(\|\psi\|_{H^1} \leq M\). For our aim, it is sufficient to estimate \(Z_{\tau}^{\psi}(n\tau) - \Pi_{\tau} u_{\psi}(n\tau)\) instead of \(Z_{\tau}^{\psi}(n\tau) - u_{\psi}(n\tau)\), because we have

\[
\|u_{\psi}(n\tau) - \Pi_{\tau} u_{\psi}(n\tau)\|_{\ell^q(0, T_2; W^{1, r})} \leq C T_2^{1/2} \|u_{\psi}(n\tau)\|_{\ell^q(0, T_2; W^{2, r})} \\
\leq C T_2^{1/2} M_3 = T_2^{1/2} C(d, p, M, \psi)
\]
thanks to (2.10) and (5.2). By utilizing the Duhamel formulas (1.5) and (1.4), we decompose \( Z^\psi_\tau (n\tau) - \Pi_\tau u^\psi(n\tau) \) as
\[
Z^\psi_\tau (n\tau) - \Pi_\tau u^\psi(n\tau) \\
= \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \left( \frac{N(\tau)}{\tau} Z^\psi_\tau(k\tau) - \frac{N(\tau)}{\tau} \Pi_\tau u^\psi(k\tau) \right) \\
+ \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \frac{N(\tau)}{\tau} \Pi_\tau u^\psi(k\tau) - i\lambda \int_0^{n\tau} S_\tau(n\tau - s)|u^\psi|^p u^\psi(s)ds.
\]
Let \((q,r)\) be an admissible pair. We proceed to find an estimate for the \( \ell^q(0,T_2; W^{1,r}) \) norm of \( Z^\psi_\tau - \Pi_\tau u^\psi \) using the decomposition (5.15). Firstly, we estimate the first term on the right hand side of (5.15). For this, by applying the Strichartz estimates (2.3) and Lemma 5.2 in order, we arrive at the following estimate
\[
\left\| \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \left( \frac{N(\tau)}{\tau} Z^\psi_\tau(k\tau) - \frac{N(\tau)}{\tau} \Pi_\tau u^\psi(k\tau) \right) \right\|_{\ell^q(0,T_2; W^{1,r})} \\
\leq C \left\| N(\tau) - I Z^\psi_\tau(k\tau) - \frac{N(\tau)}{\tau} \Pi_\tau u^\psi(k\tau) \right\|_{\ell^q(0,T_2; W^{1,r})}
\]
\[
\leq CT_2^{\frac{1}{2}} \left\| Z^\psi_\tau - \Pi_\tau u^\psi \right\|_{\ell^q(0,T_2; H^1)} \left\| \left( |Z^\psi_\tau|_{\ell^q(0,T_2; W^{1,r})} + \|\Pi_\tau u^\psi\|_{\ell^q(0,T_2; W^{1,r})} \right) \right\|
\]
\[
\times \left( |Z^\psi_\tau|_{\ell^p(0,T_2; H^1)} + \|\Pi_\tau u^\psi\|_{\ell^p(0,T_2; H^1)} \right) + \tau |\nabla Z^\psi_\tau|_{\ell^q(0,T_2; L^\infty)} \left\| \left| Z^\psi_\tau \right|_{2p-1} \right\|_{\ell^q(0,T_2; L^{2p-1})}.
\]
On the other hand, we recall from Proposition 4.1 with the \( T_2 \leq T_0 \) given in (5.3) that \( Z^\psi_\tau \) enjoys the following local stability
\[
\left\| Z^\psi_\tau \right\|_{\ell^q(0,T_2; H^1)} \leq 4CM \quad \text{and} \quad \left\| Z^\psi_\tau \right\|_{\ell^q(0,T_2; W^{1,r})} \leq 4CM.
\]
Also, we see from (5.14) that
\[
\left\| \tau \right\|_{\ell^q(0,T_2; L^{\infty})} \left\| \right\|_{\ell^q(0,T_2; L^{\infty})} \left\| \right\|_{\ell^q(0,T_2; L^{\infty})} \leq C_{d,p}(4CM)^{2p}.
\]
Furthermore, by (2.11) in Lemma 2.6 and (5.1) in Theorem B with \( \psi_1 = \psi \) and \( \psi_2 = 0 \), we have
\[
\left\| \Pi_\tau u^\psi \right\|_{\ell^q(0,T_2; H^1)} + \|\Pi_\tau u^\psi\|_{\ell^q(0,T_2; W^{1,r})} \leq C_{d,p} M.
\]
By applying estimates (5.16), (5.17) and (5.18), we get
\[
\left\| \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \left( \frac{N(\tau)}{\tau} Z^\psi_\tau(k\tau) - \frac{N(\tau)}{\tau} \Pi_\tau u^\psi(k\tau) \right) \right\|_{\ell^q(0,T_2; W^{1,r})} \\
\leq C_{d,p} T_2^{\frac{1}{2}} \left\| Z^\psi_\tau - \Pi_\tau u^\psi \right\|_{\ell^q(0,T_2; H^1)} \\
\leq \frac{1}{2} \left\| Z^\psi_\tau - \Pi_\tau u^\psi \right\|_{\ell^q(0,T_2; H^1)}
\]
provided that \( \beta_{d,p} > 0 \) introduced in (5.3) is small enough. By inserting this estimate into (5.15), we obtain
\[
\left\| Z^\psi_\tau - \Pi_\tau u^\psi \right\|_{\ell^q(0,T_2; W^{1,r})} \\
\leq 2 \left\| \tau \sum_{k=0}^{n-1} S_\tau(n\tau - k\tau) \frac{N(\tau)}{\tau} \Pi_\tau u^\psi(k\tau) - i\lambda \int_0^{n\tau} S_\tau(n\tau - s)|u^\psi|^p u^\psi(s)ds \right\|_{\ell^q(0,T_2; W^{1,r})}.
\]
On the other hand, one has
\[
\left\| \sum_{k=0}^{n-1} S_r(n\tau - k\tau) \frac{N(r)}{\tau} \Pi_r u^\psi(k\tau) - i\lambda \int_0^{n\tau} S_r(n\tau - s)|u^\psi|^p u^\psi(s) ds \right\|_{\ell^2(0,T;W^{1,\infty})} \\
\leq C_{d,p} \tau^{1/2} \left(1 + T_2^\epsilon \right) \left(1 + M_3^2\right),
\]
where the constant \( M_3 = M_3(d,p,M,\psi) > 0 \) is the one introduced in (5.2). The proof of (5.19) is a minor modification of the proof in Lemma 3.1. Indeed, by applying Lemma 5.2 and (5.2) in Theorem B into the argument of Lemma 3.1 instead of using (1.6) in Theorem A, we get (5.19).

Since \( T_2 \) is determined by \( d,p \) and \( M \), we have
\[
\left\| Z_r^\psi - \Pi_r u^\psi \right\|_{\ell^2(0,T_2;W^{1,\infty})} \leq \tau^{1/2} C(d,p,M,\psi),
\]
which completes the proof.

**Proof of (5.6).** Take any small \( \epsilon \in (0, M/2) \), and choose \( \psi \in H^2(\mathbb{R}^d) \) such that
\[
\|u(0) - \psi\|_{H^1} \leq \epsilon.
\]
Then we have \( \|\psi\|_{H^1} \leq M \) since \( \|u(0)\|_{H^1} \leq M/2 \). Now we may apply the stability results (5.1) and (5.4) to yield that
\[
\sup_{n\tau \in [0,R]} \|Z_r^\psi(n\tau) - Z_r^\phi(n\tau)\|_{H^1} \leq C_{d,p} \|\phi - \psi\|_{H^1} \leq C_{d,p} \epsilon
\]
and
\[
\sup_{n\tau \in [0,R]} \|u^\psi(n\tau) - u^\phi(n\tau)\|_{H^1} \leq C_{d,p} \|\phi - \psi\|_{H^1} \leq C_{d,p} \epsilon.
\]
By applying (5.5) we have
\[
\sup_{n\tau \in [0,R]} \|Z_r^\psi(n\tau) - u^\psi(n\tau)\|_{H^1} \leq \tau^{1/2} C(d,p,M,\psi).
\]
Combining the above estimates, we find that
\[
\sup_{n\tau \in [0,R]} \|u(n\tau) - Z_r(n\tau)\|_{H^1} \\
\leq \sup_{n\tau \in [0,R]} \left( \|u(n\tau) - u^\psi(n\tau)\|_{H^1} + \|u^\psi(n\tau) - Z_r^\psi(n\tau)\|_{H^1} + \|Z_r^\psi(n\tau) - Z_r(n\tau)\|_{H^1} \right) \\
\leq \tau^{1/2} C(d,p,M,\psi) + 2C_{d,p} \epsilon.
\]
(5.20)

This enables us to yield that
\[
\limsup_{\tau \searrow 0} \sup_{n\tau \in [0,R]} \|u(n\tau) - Z_r(n\tau)\|_{H^1} \leq 2C_{d,p} \epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary, this establishes the desired limit.

### 6. Global \( H^1 \) stability of \( Z_r \) for \( 1 \leq p < p_d \)

In this section, we prove Theorem 1.6 which corresponds to the case of \( 1 \leq p < p_d \). The rough idea of the proof is to apply the local \( H^1 \) stability inductively after dividing the interval \( (0,T) \) into a set of intervals of the same size. In this procedure, the main task is to control the growth of the \( H^1 \) norm of \( Z_r \).
**Proof of Theorem 1.6.** We take an initial data \( \phi \in H^1 \), and consider a time \( T > 0 \) such that \( \sup_{t \leq T} \|u(t)\|_{H^1(\mathbb{R}^d)} < \infty \). Then, from (1.6) in Theorem A, we can find a constant \( M_1 = M_1(d, p, T, \phi) \geq 1 \) such that
\[
\|u\|_{L^\infty(0, T; H^1)} + \|u\|_{L^\nu(0, T; W^{1, \nu})} \leq \frac{M_1}{2}.
\] (6.1)

Let us take \( \beta_{d, p} > 0 \) in (5.3) and choose \( T_2 > 0 \) as
\[
T_2 = \frac{\beta_{d, p}}{2} M_1^{-\frac{4p(p+2)}{4d(p+2)}}.
\] (6.2)

If \( T < 2T_2 \), then the stability follows just by using the local stability result of Proposition 4.1. Thus we may consider only the case \( T > 2T_2 \).

First we consider the case \( \tau \leq T_2/2 \). As in the proof of Theorem 1.4 we choose \( R \in (T_2/2, T_2] \) such that \( R/\tau \in \mathbb{N} \), and to employ an induction argument, we split \([0, T]\) as
\[
[0, T] = \bigcup_{k=0}^{N-1} [kR, (k+1)R) \cup [NR, T]
\] (6.3)
where \( N \in \mathbb{N} \) is chosen so that \( NR \leq T < (N+1)R \). (We remark that \( N \sim T/T_2 \).

For each \( k = 0, 1, \cdots, N - 1 \), take an auxiliary function \( \psi_k \in H^2(\mathbb{R}^d) \) such that
\[
\|\psi_k - u^\phi(kR)\|_{H^1} < \frac{M_1}{2} \left( \frac{M_1}{10C_{d, p}} \right)^{N+1}.
\] (6.4)

Here, the constant \( C_{d, p} \geq 1 \) is the maximum between the one appearing in (5.1) of Theorem B and the one in (5.4) of Proposition 5.1. We remark that we can choose a common function \( \psi_k \) for any value \( \tau \in (0, \bar{\tau}) \) with a small \( \bar{\tau} = \bar{\tau}(\phi, d, p, M_1) \) since \( u \) is continuous in \( H^1 \) and we may choose \( R \in (T_2/2, T) \) for \( \tau \in (0, \bar{\tau}) \).

Combining (6.4) with (6.1), we find that
\[
\|u(kR)\|_{H^1} \leq \frac{M_1}{2} \quad \text{and} \quad \|\psi_k\|_{H^1} \leq M_1 \quad \text{for all} \quad k = 0, \cdots, N - 1.
\] (6.5)

Given (6.5) and the fact that \( \psi_k \in H^2(\mathbb{R}^d) \), we may apply Proposition 5.1 to obtain
\[
\sup_{n \tau \in [0, R]} \|Z^\psi_k(n\tau) - u^\psi_k(n\tau)\|_{H^1} \leq \tau^{1/2} C(d, p, M_1, \psi_k),
\] (6.6)

where \( C(d, p, M_1, \psi_k) \) denotes the constant determined in (5.5). Now we set the following constant
\[
C(d, p, T, \phi) := \left( \frac{10C_{d, p}}{M_1} \right)^{N+1} \max_{k=0, \cdots, N-1} C(d, p, M_1, \psi_k).
\] (6.7)

Here the constant \( C \) depends only on \( (d, p, T, \phi) \) since \( \psi_k \) depends on \( (d, p, T, T_2, M_1, \phi) \) and \( T_2 \) depends on \( (d, p, M_1) \) while \( M_1 \) depends on \( (d, p, T, \phi) \). Then, for \( \tau < \tau_* := \min \left\{ \left( C(d, p, T, \phi) \right)^{-2}, \bar{\tau} \right\} \), we can deduce from (6.6) and (6.7) the following estimate
\[
\sup_{n \tau \in [0, R]} \|Z^\psi_k(n\tau) - u^\psi_k(n\tau)\|_{H^1} \leq \frac{M_1}{10C_{d, p}} M_1^{N+1}.
\] (6.8)

We shall first prove the theorem for \( \tau \in (0, \tau_*) \). To obtain the stability of \( Z^\psi_\tau \) on \([0, T]\), we claim the following estimates.
Claim. For any $k \in \{0, 1, \cdots, N\}$, we have
\[
\max_{n \tau \in I_k} \left\| Z^\phi(n \tau) - u^\phi(n \tau) \right\|_{H^1} \leq (3C_{d,p})^{k+1} \frac{M_1}{(10C_{d,p})^{N+1}} \tag{6.9}
\]
and
\[
\max_{n \tau \in I_k} \left\| Z^\phi(n \tau) \right\|_{H^1} \leq M_1 \tag{6.10}
\]
for all $\tau \in (0, \tau_*)$.

We prove the claim by induction.

**Step 1.** We show that the claim holds for $k = 0$. By the triangle inequality, we have
\[
\max_{n \tau \in [0,R]} \left\| Z^\phi(n \tau) - u^\phi(n \tau) \right\|_{H^1} \leq \max_{n \tau \in [0,R]} \left( \left\| Z^\phi(n \tau) - Z^\psi(n \tau) \right\|_{H^1} + \left\| Z^\psi(n \tau) - u^\psi(n \tau) \right\|_{H^1} + \left\| u^\psi(n \tau) - u^\phi(n \tau) \right\|_{H^1} \right).
\]
Given the upper bound of initial data (6.5), we may apply (5.4) in Proposition 5.1, (6.8), and (5.1) in Theorem B to yield that
\[
\max_{n \tau \in [0,R]} \left\| Z^\phi(n \tau) - u^\phi(n \tau) \right\|_{H^1} \leq C_{d,p} \| \phi - \psi_0 \|_{H^1} + \frac{M_1}{(10C_{d,p})^{N+1}} + C_{d,p} \| \phi_0 - \psi \|_{H^1}
\]
\[
\leq 2C_{d,p} \frac{M_1}{(10C_{d,p})^{N+1}} + \frac{M_1}{(10C_{d,p})^{N+1}}
\]
\[
\leq 3C_{d,p} \frac{M_1}{(10C_{d,p})^{N+1}},
\]
where the last inequality used that $C_{d,p} \geq 1$. By combining this with (6.1), we obtain that
\[
\max_{n \tau \in [0,R]} \left\| Z^\phi(n \tau) \right\|_{H^1} \leq \max_{n \tau \in [0,R]} \| u^\phi(n \tau) \|_{H^1} + \max_{n \tau \in [0,R]} \left\| u^\phi(n \tau) - Z^\phi(n \tau) \right\|_{H^1}
\]
\[
\leq \frac{M_1}{2} + 3C_{d,p} \frac{M_1}{(10C_{d,p})^{N+1}} \leq M_1.
\]
Therefore, (6.10) and (6.9) hold for $k = 0$.

**Step 2.** Suppose that (6.9) and (6.10) hold for some $k \in \{0, 1, \cdots, N - 1\}$. Then we aim to show that (6.9) and (6.10) hold for the $(k+1)$-step. We have
\[
\max_{n \tau \in I_k} \left\| Z^\phi(n \tau) - u^\phi(n \tau) \right\|_{H^1} = \max_{n \tau \in [0,R]} \left\| Z^\phi(n \tau + n_k \tau) - u^\phi(n \tau + n_k \tau) \right\|_{H^1}, \tag{6.11}
\]
where for $k = N$, we assume that the interval $[0,R]$ is regarded as $[0, T - NR]$, by abusing the notation for the simplicity. By the triangle inequality, we estimate the right hand side as
\[
\max_{n \tau \in [0,R]} \left\| Z^\phi(n \tau + n_k \tau) - u^\phi(n \tau + n_k \tau) \right\|_{H^1}
\]
\[
\leq \max_{n \tau \in [0,R]} \left( \left\| Z^\phi(n \tau + n_k \tau) - Z^\psi(n \tau) \right\|_{H^1}
\]
\[
+ \left\| Z^\psi(n \tau) - u^\psi(n \tau) \right\|_{H^1} + \left\| u^\psi(n \tau) - u^\phi(n_k \tau + n \tau) \right\|_{H^1} \right).
\]
\[
(6.12)
\]
Given the estimates of initial data (6.5) and (6.10) with \(k\), we may apply Lemma 5.1 to yield that

\[
\max_{\tau \in [0,T]} \| Z^\phi_r(n\tau + n_k \tau) - Z^\psi_r(n\tau) \|_{H^1},
\]

\[
= \max_{\tau \in [0,T]} \| Z^\psi_r(n\tau) - Z^\psi_r(n\tau) \|_{H^1},
\]

\[
\leq C_{d,p} \| Z^\phi_r(n_k \tau) - \psi_k \|_{H^1},
\]

\[
\leq C_{d,p} \left( \| Z^\phi_r(n_k \tau) - u^\phi(n_k \tau) \|_{H^1} + \| u^\phi(n_k \tau) - \psi_k \|_{H^1} \right).
\]

Next we deduce from Theorem B that

\[
\sup_{\tau \in [0,T]} \| u^\psi_r(n\tau) - u^\phi(n_k \tau + n\tau) \|_{H^1} = \sup_{\tau \in [0,T]} \| u^\psi_r(n\tau) - u^\phi(n_k \tau)(n\tau) \|_{H^1},
\]

\[
\leq C_{d,p} \| \psi_k - u^\phi(n_k \tau) \|_{H^1}.
\]

By inserting the estimates above into (6.12) and by applying (6.8), we arrive at the following estimate

\[
\sup_{\tau \in [0,T]} \| Z^\phi_r(n\tau + n_k \tau) - u^\phi(n_k \tau + n\tau) \|_{H^1}
\]

\[
\leq C_{d,p} \| Z^\phi_r(n_k \tau) - u^\phi(n_k \tau) \|_{H^1} + 2C_{d,p} \left( \| \psi_k - u^\phi(n_k \tau) \|_{H^1} \right) + \frac{M_1}{(10C_{d,p})^{N+1}}
\]

\[
\leq C_{d,p} \| Z^\phi_r(n_k \tau) - u^\phi(n_k \tau) \|_{H^1} + (2C_{d,p} + 1) \frac{M_1}{(10C_{d,p})^{N+1}}
\]

where we used (6.4) for the second inequality. Now we apply the inductive hypothesis (6.9) at the \(k\)-step in the above inequality. Then we get

\[
\sup_{\tau \in [0,T]} \| Z^\phi_r(n\tau + n_k \tau) - u^\phi(n_k \tau + n\tau) \|_{H^1}
\]

\[
\leq C_{d,p} \left( (3C_{d,p})^{k+1} \frac{M_1}{(10C_{d,p})^{N+1}} \right) + (2C_{d,p} + 1) \frac{M_1}{(10C_{d,p})^{N+1}}
\]

\[
\leq (3C_{d,p})^{k+2} \frac{M_1}{(10C_{d,p})^{N+1}},
\]

where \(2C_{d,p} + 1 \leq 2C_{d,p}(3C_{d,p})^{k+1}\) is used in the second inequality. Thus, we obtain the estimate (6.9) for the \((k+1)\)-th step. This, together with (6.1), implies (6.10). Hence the claim is proved, and so we have

\[
\sup_{\tau \in [0,T]} \| Z^\phi_r(n\tau) \|_{H^1} \leq M_1 \quad \text{for all} \quad \tau \in (0, \tau_\#), \tag{6.13}
\]

which proves the theorem for \((q, r) = (\infty, 2)\). Given the estimate (6.13), the stability of \(Z_r\) for general pair follows from the local \(H^1\) stability result of Proposition 4.1 (see also Remark 4.2). The proof is finished for \(\tau \in (0, \tau_\#)\).

Now, it only remains to consider the case \(\tau_\# \leq \tau < 1\). From the definition of \(Z_r\) in (1.2), we can write \(Z_r = (\Pi_r/2)^2 Z_r\). By the Hölder’s inequality in time, (2.11),
(2.12) and (2.13) in Lemma 2.6, we have
\[
\|Z^\phi_{\tau}(n\tau)\|_{L^q([n\tau\in[0,T]:W^{1,r})} \leq \left( \frac{T+1}{\tau} \right)^{\frac{q}{2}} \sup_{n\tau\in[0,T]} \|\Pi_{\tau/2}^2 Z^\phi_{\tau}(n\tau)\|_{L^r}
\]
\[
\leq C \left( \frac{T+1}{\tau} \right)^{\frac{q}{2}} \left( 1 + \tau^{-\frac{1}{2}} \right) \sup_{n\tau\in[0,T]} \|\Pi_{\tau/2} Z^\phi_{\tau}(n\tau)\|_{L^r}
\]
\[
\leq C \left( \frac{T+1}{\tau} \right)^{\frac{q}{2}} \left( 1 + \tau^{-\frac{1}{2}} \right) \sup_{n\tau\in[0,T]} \|Z^\phi_{\tau}(n\tau)\|_{L^2}
\]
\[
\leq C(T+1)^{\frac{q}{2}} \left( \frac{1}{T^\alpha} \right)^{\frac{q}{2}} \sup_{n\tau\in[0,T]} \|Z^\phi_{\tau}(n\tau)\|_{L^2}
\]
where \((q,r)\) is any admissible pair. On the other hand, we know that the \(L^2\) norm of \(Z^\phi_{\tau}\) does not increase as a consequence of the definition (1.2), i.e.
\[
\|Z^\phi_{\tau}(n\tau)\|_{L^2} \leq \|\phi\|_{L^2} \quad \text{for all} \quad n \in \mathbb{N}.
\]
Thus we have
\[
\|Z^\phi_{\tau}(n\tau)\|_{L^q([n\tau\in[0,T]:W^{1,r})} \leq 2C(T+1)^{\frac{q}{2}} \left( \frac{1}{T^\alpha} \right)^{\frac{q}{2}} \|\phi\|_{L^2} \leq C(d,p,T,\phi),
\]
which proves the theorem for \(\tau_* \leq \tau < 1\).

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