Spreading of wave packets for neutrino oscillations in vacuum

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Abstract

The effects originated in dispersion with time on spreading of wave packets for the time-integrated two-flavor neutrino oscillation probabilities in vacuum are studied in the context of a field theory treatment. The neutrino flavor states are written as superpositions of neutrino mass eigenstates which are described by localized wave packets. This study is performed for the limit of nearly degenerate masses and considering an expansion of the energy until third order in the momentum. We obtain that the time-integrated neutrino oscillation probabilities are suppressed by a factor $1/L^2$ for the transversal and longitudinal dispersion regimes, where $L$ is the distance between the neutrino source and the detector.
I. INTRODUCTION

Knowing the nature of neutrino fields is an open problem in particle physics \[1,2\]. This problem might be solved, experimentally, establishing if neutrinos are: (i) Majorana fermions; (ii) Dirac fermions. For the first case, neutrino and anti-neutrino are the same particle being described by two-component spinorial fields called Majorana fields \[1,3\]. For the second case, neutrinos and anti-neutrinos are described by four-component spinorial fields called Dirac fields \[1,3\]. However, the theoretical description of neutrino oscillations leads to the same results independently if neutrinos are Majorana or Dirac fermions. This argument was established many years ago using the plane wave formalism of quantum mechanics \[4\]. Moreover, the validity of this argument in the context of a quantum field theory treatment can also been proved using the plane wave formalism \[5\].

The standard neutrino oscillation probabilities using the plane wave formalism have been obtained in the context of several quantum mechanics treatments (for instance, see \[6–11\]). On the other hand, in the context of different quantum field theory treatments, neutrino oscillations in vacuum have been extensively studied describing neutrinos by Dirac fields \[12–24\]. In particular, the effects of the spreading of the wave packets on neutrino oscillation probabilities have been widely investigated for the case of considering an expansion of the energy until second order in the momentum \[12–22\]. For this case, the standard neutrino oscillation probabilities using the wave packet formalism are written in terms of the oscillation and coherence lengths \[12–22\]. Moreover, some aspects of the time effects on spreading of wave packets for neutrino oscillation probabilities have been also studied \[15,18\]. Nevertheless, for the case of considering an expansion of the energy until third order in the momentum, the study of the effects originated in dispersion with time on spreading of wave packets for neutrino oscillation probabilities have not been studied until now.

The main goal of this work is to study the effects originated by dispersion in time on spreading of wave packets for the time-integrated two-flavor neutrino oscillations in vacuum. To do it, we perform an expansion of the energy until third order and we consider the limit of nearly degenerate neutrino masses. The time-integrated two-flavor neutrino oscillation probabilities are calculated in the context of a wave packet extension of the quantum field theory treatment that we previously developed for the case in which neutrinos were considered as Majorana fermions and neutrino mass eigenstates were described by plane waves \[5\]. In the present treatment, the neutrino flavor states are considered as superpositions of neutrino mass eigenstates described by localized wave packets. By methodological reasons, the effects of the spreading of the wave packets are studied for two cases: (i) Considering the expansion of the energy until second order in the momentum that leads to the standard time-integrated neutrino oscillation probabilities \[12–22\]; (ii) considering the expansion of the energy until third order in the momentum that, leads to a suppression of the time-integrated neutrino oscillation probabilities by a factor $1/L^2$ for transversal and longitudinal dispersion regimes, where $L$ is the distance between the neutrino source and the detector. This suppression factor is a new result in the context of neutrino oscillation probabilities and is in agreement with the one obtained by Naumov, whom has recently demonstrated for a theory of wave packets that the integral over time of both the flux and probability densities are proportional to a factor $1/L^2$, considering the energy expanded until third order in the momentum \[25\].

The content of this work has been organized as follows: In section two, considering neutrinos as Majorana fermions, we show how is possible to obtain the standard two-flavor neutrino oscillation probabilities in the context of a quantum field theory treatment, for which the mass eigenstate are described by plane waves; in section three, we extend the plane wave treatment presented in the previous section for the case in which the mass eigenstates are described by localized wave packets; in section four, we study the effects of spreading of wave packets by performing an expansion of the energy until second order in the momentum and we obtain the standard time-integrated neutrino oscillation probabilities; in section five, we study the effects originated in dispersion with time on spreading of wave packets for the time-integrated neutrino oscillation probabilities by performing an expansion of the energy until third order in the momentum; finally in section six we present some conclusions.

II. NEUTRINO OSCILLATION PROBABILITIES USING PLANE WAVES

The standard two-flavor neutrino oscillation probabilities were obtained in the context of a treatment developed in the canonical formalism of quantum field theory for the case in which neutrinos were described as Majorana fermions and neutrino mass eigenstates were described by plane waves \[3\]. In this treatment, the flavor neutrinos were considered as superpositions of mass eigenstates with specific momenta. For the case of the relativistic limit $(L \approx T)$ and after including a normalization constant, the standard plane wave expressions for the neutrino oscillation probabilities \[3\] are obtained

$$P_{\nu_e}^{PW}(L) = 1 - \sin^2(2\theta_{12}) \sin^2 \left( \frac{\Delta m_{12}^2 L}{4E} \right),$$

(1)
\[ P_{\nu_L}^{PW} (L) = \sin^2 [2\beta_{12}] \sin^2 \left[ \frac{\Delta m_{12}^2}{4E} L \right], \]  

where \( \Delta m_{12}^2 \equiv m_2^2 - m_1^2 \), \( E \) is the energy of the neutrino, \( L \) is the distance between the neutrino source and the detector and \( \sin^2 [2\beta_{12}] \) is given by

\[ \sin^2 [2\beta_{12}] = \frac{4\Lambda_L^2}{(1 + \Lambda_L^2)^2}, \]  

with \( \beta_{12} \) representing the mixing angle between the two mass eigenstates in the vacuum. In the last expression \( \Lambda_L \) is

\[ \Lambda_L = \frac{m_{\nu_L} - m_{\bar{\nu}_L} + R_L}{2m_{\nu_L} \nu_{\nu_L}}, \]  

where \( R_L \) is defined by means of

\[ R_L^2 = (m_{\nu_L} - m_{\bar{\nu}_L})^2 + 4m_{\nu_L}^2 \nu_{\nu_L}, \]  

and the parameters \( m_{\nu_L}, m_{\bar{\nu}_L} \) and \( m_{\nu_L} \nu_{\nu_L} \) are related with the masses \( m_1 \) and \( m_2 \) of the neutrino fields \( \nu_1 \) and \( \nu_2 \) (with definite masses) through the following relations

\[ m_1 = \frac{1}{2}(m_{\nu_L} + m_{\bar{\nu}_L} - R_L), \]  

\[ m_2 = \frac{1}{2}(m_{\nu_L} + m_{\bar{\nu}_L} + R_L)e^{-i\alpha_L}, \]  

being \( \alpha_L \) the Majorana complex phase \( \[5\] \).

The expressions \( \[11\] \) and \( \[12\] \) are obtained starting from the oscillation probabilities defined by

\[ P_{\nu_L}^{PW} (L) = \left| \langle 0 | \hat{\nu}_{\nu_L} (x) | \nu_{\nu_L}^{PW} (x_0) \rangle \right|^2, \]  

\[ P_{\nu_{\bar{L}}}^{PW} (L) = \left| \langle 0 | \hat{\nu}_{\bar{L}} (x) | \nu_{\nu_L}^{PW} (x_0) \rangle \right|^2, \]  

in such a way that in the space-time production point \( (x_0) \) the initial left-handed neutrino flavor state \( (eL, \mu L) \) is defined as a superposition of the mass eigenstates \( |\nu_{1L}^{PW} (x_0)\rangle \) and \( |\nu_{2L}^{PW} (x_0)\rangle \)

\[ |\nu_{1L}^{PW} (x_0)\rangle = \sum_{h=\pm 1} \frac{\Lambda_L}{\sqrt{1 + \Lambda_L^2}} |\nu_{1L}^{PW} (x_0)\rangle + \frac{e^{-i\alpha_L}}{\sqrt{1 + \Lambda_L^2}} |\nu_{2L}^{PW} (x_0)\rangle, \]  

where a sum over helicities is taken in the superposition. The mass eigenstates \( |\nu_{1L}^{PW} (x_0)\rangle \) and \( |\nu_{2L}^{PW} (x_0)\rangle \) involved in \( \[11\] \) are obtained using plane waves from the vacuum state \( |0\rangle \) as

\[ |\nu_{aL}^{PW} (x_0)\rangle = A e^{ip_{aL} x} \hat{a}_a^\dagger (p, h) |0\rangle \]  

where \( A \) is a normalization constant, \( \hat{a}_a^\dagger \) is the creation operator of a neutrino of defined mass, \( x_0 \) is the space-time point where this neutrino is created, \( p_a = (E_a, \vec{p}_a) \) is the four-momentum of the mass eigenstates and \( a = 1, 2 \). We have assumed that each mass eigenstate involved in \( \[11\] \) has associate a specific four-momentum.

In the oscillation probabilities \( \[8\] \) and \( \[9\] \), the flavor neutrino field operator \( \hat{\nu}_{\alpha} \) is defined as a superposition of field operators of neutrinos with defined mass \( \hat{\nu}_a \) by means of the expression

\[ \hat{\nu}_{\alpha} (x) = \sum_{a} U_{L, a} \hat{\nu}_a (x), \]  

where \( \alpha = eL, \mu L \) and \( U_L \) is an unitarian rotation matrix given by \( \[5\] \)

\[ U_L = \frac{1}{\sqrt{1 + \Lambda_L^2}} \begin{pmatrix} \Lambda_L e^{-i\alpha_L} & 1 \\ -1 & \Lambda_L e^{i\alpha_L} \end{pmatrix}. \]  

The field operators of neutrinos with defined mass \( \hat{\nu}_a \) involved in \( \[12\] \) are defined as \( \[5\] \)

\[ \hat{\nu}_a (x) = \int \frac{d^3 p}{(2\pi)^{3/2}(2E_a)^{1/2}} \sum_{h=\pm 1} \left[ \sqrt{E_a - h} |\vec{p}_a \rangle \hat{a}_a^\dagger (\vec{p}_a, h) \chi^{h^}\dagger (\vec{p}) e^{-i\vec{p} \cdot x} \right. 

\[ \left. - h \sqrt{E_a + h} |\vec{p}_a \rangle \hat{a}_a^\dagger (\vec{p}_a, h) \chi^{h} (\vec{p}) e^{i\vec{p} \cdot x} \right], \]  

where \( E_a^2 = |\vec{p}_a|^2 + m_a^2 \) is the energy of the neutrino field with defined mass and \( \chi^{h} (\vec{p}) \) are Majorana spinors with helicity eigenvalues \( \pm 1 \).
III. NEUTRINO OSCILLATIONS USING THE WAVE PACKETS FORMALISM

In this section, we will extend the plane wave treatment for neutrino oscillations that we have presented briefly in section two, for the case in which neutrino mass eigenstates are described by localized wave packets. In this treatment, where neutrinos are considered as Majorana fermions, we do not focus on the study of the details of the interaction processes in which neutrinos are produced and detected. Here, on the other hand, it is assumed that wave packets describing mass eigenstates are localized and the coefficients of their superpositions are given by the elements of the unitarian rotation matrix $U_L$ given by (13). The matrix $U_L$ establishes a relationship between the flavor and mass eigenstates bases of neutrino fields in vacuum.

There are different reasons for understand why the description of mass eigenstates using wave packets is most appropriate to study the neutrino oscillations with respect to the description from plane waves [1–3]. Some of these reasons are that the neutrino source and the detector are localized and there exists a spread for the neutrino momentum $p$. Additionally, we have to keep in mind that plane waves localized in some point $x_0$ are in an obvious contradiction with the Heisenberg uncertainty principle. Here, we take into account these reasons when we describe the neutrino mass eigenstates in terms of superpositions of localized wave packets. To do the last, we first consider that in a point $x_0 \equiv x_0 = (t_0, \hat{r}_0)$ is created a left-handed electron neutrino which is described by the following superposition of neutrino mass eigenstates $|\nu^1_W(x_0)\rangle$ and $|\nu^2_W(x_0)\rangle$:

$$
|\nu^a_W(x_0)\rangle = \frac{\Lambda_L}{\sqrt{1 + \Lambda_L^2}} |\nu^1_W(x_0)\rangle + \frac{e^{-i\chi}}{\sqrt{1 + \Lambda_L^2}} |\nu^2_W(x_0)\rangle.
$$

In contrast to the expression (11), now the mass eigenstates involved in (16) are written in term of localized wave packets in the form

$$
|\psi_a(\vec{p}, \langle \vec{p}_a \rangle)\rangle \approx (2\pi)^{-3/2} |\text{Det} \Gamma|^{1/4} \exp \left[ -\frac{1}{4} (\vec{p} - \langle \vec{p}_a \rangle)^2 \Gamma_{kj} (\vec{p} - \langle \vec{p}_a \rangle)^2 \right],
$$

in such a way that $\psi_a = \psi_a(\vec{p}, \langle \vec{p}_a \rangle)$ satisfies the conditions [3]

$$
\frac{\partial \ln \psi_a}{\partial \vec{p}} \bigg|_{\vec{p} = \langle \vec{p}_a \rangle} = 0,
$$

$$
\frac{\partial^2 \ln \psi_a}{\partial \vec{p}^2 \partial \vec{p}^j} \bigg|_{\vec{p} = \langle \vec{p}_a \rangle} = -\frac{1}{2} \Gamma_{kj}
$$

In the probability density function given by (17), we have taken the convention of summation over the Latin repeated index $k$ and $j$. Additionally, we have assumed that the dispersion over the mass eigenstates $|\nu^1_W(x_0)\rangle$ and $|\nu^2_W(x_0)\rangle$ is the same, because these eigenstates are created simultaneously by the same weak production process. This fact is the reason that justifies why the matrix $\Gamma_{kj}$ is identic for both mass eigenstates. Moreover, it is important to note that this matrix is symmetric and the eigenvalues of its inverse $\Gamma^{-1}_{kj}$ are the squares of the widths in the momentum space [3].

The oscillation probabilities between two flavor neutrinos using wave packets are

$$
P^W_{\nu_e}(T, L) = |\langle 0 | \hat{\nu}_e(x) \mid \nu^a_W(x_0)\rangle|^2,
$$

$$
P^W_{\nu_\mu}(T, L) = |\langle 0 \mid \hat{\nu}_{\mu}(x) \mid \nu^a_W(x_0)\rangle|^2,
$$

where $|\nu^a_W(x_0)\rangle$ given by (16) is a superposition of mass eigenstate wave packets $|\nu^1_W(x_0)\rangle$ and $|\nu^2_W(x_0)\rangle$) in the creation point. The expressions (20) and (21) describe, respectively, the probabilities of finding an electron neutrino ($\nu_e$) and a muon neutrino ($\nu_{\mu}$) at a distance $L$ in a time $T$ of the creation point $x_0$. In the calculation of the transition
probabilities \(20\) and \(21\), the relativistic dispersion relation is approximated by means of an expansion around the average momentum of the wave packets \(\{\bar{p}_a\}\) \(16\)

\[
E_a(\bar{p}) \approx \bar{E}_a + \bar{u}_a \cdot (\bar{p} - \langle \bar{p}_a \rangle) + \frac{1}{2} \left( \bar{p} - \langle \bar{p}_a \rangle \right)^k \Omega_{kj} \left( \bar{p} - \langle \bar{p}_a \rangle \right)^j + \cdots ,
\]

\(22\)

where \(16\)

\[
\bar{E}_a = E(\langle \bar{p}_a \rangle) = \sqrt{\langle \bar{p}_a \rangle^2 + m_\nu^2},
\]

\(23\)

\[
v_a^k = \frac{\partial E(\bar{p}_a)}{\partial p_k} \bigg|_{\bar{p} = \langle \bar{p}_a \rangle} = \frac{\langle \bar{p}_a \rangle^k}{\bar{E}_a},
\]

\(24\)

\[
\Omega_{kj}^a = \frac{\partial^2 E(\bar{p}_a)}{\partial p^k \partial p^j} \bigg|_{\bar{p} = \langle \bar{p}_a \rangle} = \frac{1}{\bar{E}_a} \left( \delta_{kj} - v_a^k v_a^j \right).
\]

\(25\)

Additionally, it is possible to write that \(3\)

\[
\sqrt{\frac{E_a(\bar{p}) \pm \hbar \bar{p}}{2E_a(\bar{p})}} \approx \sqrt{\frac{E_a \pm \hbar |\langle \bar{p}_a \rangle|}{2E_a}},
\]

\(26\)

\[
\chi^h(\bar{p}) \approx \chi^h(\langle \bar{p}_a \rangle).
\]

\(27\)

The highest power of the momentum \((\bar{p} - \langle \bar{p}_a \rangle)\) in the expansion of the energy given by \(22\) determines the two different cases which we will studied below: (i) If the highest power is taken until second order, the effects of the spreading of the mass eigenstates wave packets lead to the standard time-integrated neutrino oscillation probabilities obtained using the wave packet formalism \(12\) \(22\); (ii) If the highest power is taken until third order, the effects originated in dispersion with time on the spreading of wave packets for the two-flavor neutrino oscillation probabilities are observed as a factor that suppress the standard time-integrated neutrino oscillation probabilities.

**IV. EXPANSION OF THE ENERGY UNTIL SECOND ORDER IN THE MOMENTUM**

In this section, we will obtain the standard time-integrated neutrino oscillation probabilities using the wave packet formalism. To do it, we expand the energy given by \(22\) up to second order in the power series of \((\bar{p} - \langle \bar{p}_a \rangle)\) \(16\). This fact is justified taking into account that the width of the wave packets is very narrow around the average momentum. For this case, the matrix \(\Gamma_{kj}\) can be diagonalized by means of an orthogonal transformation, \(i.e\), a rotation \(3\) \(16\). Given that the expansion of the energy does not change this rotation, without lost of generality we can take a reference frame where the matrix is diagonal

\[
\Gamma_{kj} = \frac{1}{\sigma_p} \delta_{kj},
\]

\(28\)

with \(\sigma_p\) representing the width of the wave packets in the momentum space. We assume that the width has the same value for each of the dimensions of the momentum space, due to the wave packets are taken as isotropic. Additionally, we define the width of the wave packets in the coordinate space \(\sigma_r\) through the uncertainty relation

\[
\sigma_r \sigma_p = \frac{1}{2}.
\]

\(29\)

If the energy given by \(22\) is substituted in \(20\) and \(21\), keeping up to the second order in the power series of \((\bar{p} - \langle \bar{p}_a \rangle)\), we obtain that the neutrino oscillation probabilities are written as

\[
P_{\nu_e}^{SWP}(T, L) = \frac{1}{\left(2\pi \sigma_p^2 \right)^{1/2}} \frac{1}{\left(1 + \Lambda \right)^2} \left\{ \Lambda^4 \exp[-\lambda_1 \phi_1^S(T)] + \exp[-\lambda_2 \phi_2^S(T)] + \Lambda^2 \Re \exp[-\lambda_3 \phi_3^S(T)] \right\},
\]

\(30\)

\[
P_{\nu_\mu}^{SWP}(T, L) = \frac{1}{\left(2\pi \sigma_p^2 \right)^{1/2}} \frac{1}{\left(1 + \Lambda \right)^2} \left\{ \Lambda^2 \exp[-\lambda_1 \phi_1^S(T)] + \Lambda^2 \exp[-\lambda_2 \phi_2^S(T)] - \Lambda^2 \Re \exp[-\lambda_3 \phi_3^S(T)] \right\},
\]

\(31\)
where $\lambda_1 = \lambda_2 = 1/2\sigma_r^2$, $\lambda_3 = 1/4\sigma_r^2$, $T = t - t_0$, $L = |\vec{r} - \vec{r}_0|$ and the functions in the arguments of the exponentials are given by

$$
\phi_1^S(T) = (L - v_1 T)^2,
$$

(32)

$$
\phi_2^S(T) = (L - v_2 T)^2,
$$

(33)

$$
\phi_3^S(T) = (L - v_1 T)^2 + (L - v_2 T)^2 - i 4\sigma_r^2 (\bar{E}_1 - \bar{E}_2) T + i 4\sigma_r^2 (\bar{p}_1 - \bar{p}_2) L,
$$

(34)

with $v_a = |\vec{v}_a|$. The quantity $\mathcal{S}$ that appears in the oscillation probabilities and is written as

$$
\mathcal{S} = \frac{1}{(E_1 E_2)^{1/2}} \sum_h \sqrt{(\bar{E}_1 - h |\langle \bar{p}_1 \rangle|(\bar{E}_2 - h |\langle \bar{p}_2 \rangle|)).
$$

(35)

Now, we take into account the fact that in the atmospheric and reactor neutrino oscillation experiments it is only possible to measure the distance between the neutrino source and the detector $L$, while the neutrino propagation time $T$ is unknown. However, in the case of the accelerator neutrino experiments (for instance K2K, MINOS, OPERA) it is possible to measure the neutrino propagation time $T$. By this reason, if we focus only on the case of atmospheric and reactor neutrino oscillation experiments, then it is necessary the elimination of the time dependence presents in and . This last can be performed, if we take the average on the time of the expressions and in the following form

$$
P^{SWP}_{\nu_e}(L) = \int P^{SWP}_{\nu_e}(T, L) dT,
$$

(36)

$$
P^{SWP}_{\nu_v}(L) = \int P^{SWP}_{\nu_v}(T, L) dT,
$$

(37)

Time integrations can be performed using both Gaussian integration and the Laplace approximation method. After the time integrations are performed, we obtain from and the following time-integrated neutrino oscillation probabilities

$$
P^{SWP}_{\nu_e}(L) = \frac{1}{(1 + \Lambda^2)^2} \left\{ \frac{\Lambda^4}{v_1} + \frac{1}{v_2} + \Lambda^2 \Xi \left( \frac{2}{v_1^2 + v_2^2} \right)^{1/2} \exp \left[ i f_1^S - f_2^S \right] \right\},
$$

(38)

$$
P^{SWP}_{\nu_v}(L) = \frac{1}{(1 + \Lambda^2)^2} \left\{ \frac{\Lambda^2}{v_1} + \frac{\Lambda^2}{v_2} - \Lambda^2 \Xi \left( \frac{2}{v_1^2 + v_2^2} \right)^{1/2} \exp \left[ i f_1^S - f_2^S \right] \right\},
$$

(39)

where

$$
f_1^S = (\bar{E}_1 - \bar{E}_2) \frac{v_1 + v_2}{v_1^2 + v_2^2} L - (\bar{p}_1 - \bar{p}_2) L,
$$

(40)

$$
f_2^S = \frac{(v_1^2 - v_2^2)^2}{v_1^2 + v_2^2} \frac{L^2}{4\sigma_r^2} + \frac{(\bar{E}_1 - \bar{E}_2)^2}{v_1^2 + v_2^2} \frac{\sigma_r^2}{\sigma_r^2}.
$$

(41)

We have explicitly proved that if the average on the time in the expressions and is performed using Gaussian integration, the results are the same as those obtained using the Laplace approximation method. In both cases, we have obtained the oscillation probabilities given by and . The functional form of the oscillation probabilities and is in agreement with the one previously obtained by Giunti, Kim and Lee. These authors used a quantum mechanics treatment in which flavor neutrinos were described by a superposition of mass eigenstates wave packets.

In order to obtain from and expressions for the oscillation probabilities in the relativistic limit, the following relativistic approximations are used

$$
\bar{E}_a \simeq \bar{E} + \frac{\xi m_a^2}{2E},
$$

(42)

$$
\bar{p}_a \simeq \bar{E} + (1 - \delta) \frac{m_a^2}{2E},
$$

(43)

$$
v_a \simeq 1 - \frac{m_a^2}{2E},
$$

(44)
where $\xi$ is a dimensionless coefficient, typically of order unity, that depends of the neutrino production process and $E$ is the neutrino energy determined by the kinematics of the production process for a massless neutrino. After the relativistic limit is taken, we obtain from (38) and (39) the standard time-integrated neutrino oscillation probabilities

$$P_{\nu_e}^{SWP}(L) = 1 - \frac{1}{2} \sin^2[2\theta_{12}] \left\{ 1 - \exp \left[ i \frac{2\pi}{L_{\text{osc}}} - \left( \frac{L}{L_{\text{coh}}} \right)^2 - 2\pi^2 \xi^2 \left( \frac{\sigma_r}{L_{\text{osc}}} \right)^2 \right] \right\}, \quad (45)$$

$$P_{\nu_\mu}^{SWP}(L) = \frac{1}{2} \sin^2[2\theta_{12}] \left\{ 1 - \exp \left[ i \frac{2\pi}{L_{\text{osc}}} - \left( \frac{L}{L_{\text{coh}}} \right)^2 - 2\pi^2 \xi^2 \left( \frac{\sigma_r}{L_{\text{osc}}} \right)^2 \right] \right\}, \quad (46)$$

where $L_{\text{osc}}$ is the oscillation length and $L_{\text{coh}}$ is the coherence length given by

$$L_{\text{osc}} = \frac{4\pi E}{\Delta m_{12}^2}, \quad (47)$$

$$L_{\text{coh}} = \frac{4\sqrt{2} E^2}{\Delta m_{12}^2} \sigma_r, \quad (48)$$

in agreement with the corresponding lengths very well known in the literature [8, 12–22]. To write the expressions (45) and (46), we have used the definition of $\sin^2[2\theta_{12}]$ in terms of the parameter $\Lambda$ given by (3), where $\theta_{12}$ is the mixing angle in the vacuum between the two mass eigenstates. Specifically, the probability (45) represents the survival probability that an electron neutrino ($\nu_e$) be detected at a distance $L$ in a time $T$ of the creation point $x_0 = (0, r_0)$, where by simplicity $t_0 = 0$. On the other hand, the probability (46) represents the probability of oscillation from an electron neutrino ($\nu_e$) created by the source at point $x_0$ to a muon neutrino ($\nu_\mu$) measured by the detector at a distance $L$ in time $T$. The dependence of the oscillation probability (46) respect to the mixing angle $\theta_{12}$ is nearly equal to one [22]. Additionally, it is easy to show that $|v_1 - v_2| L_{\text{coh}} \approx \Delta m_{12}^2 2E \sim \sigma_r$ and thus the term exp $[-2\pi^2 \xi^2 \left( \frac{\sigma_r}{L_{\text{osc}}} \right)^2]$ is nearly equal to one [22]. In this form, the time-integrated oscillation probabilities (45) and (46) can be written as

$$P_{\nu_e}^{SWP}(L) = 1 - \frac{1}{2} \sin^2[2\theta_{12}] \left\{ 1 - \cos \left[ 2\pi \frac{L}{L_{\text{osc}}} \right] \right\}, \quad (52)$$

showing that the coherence length is much larger than the oscillation length [22]. The maximum number of oscillations can be obtained from these lengths as [8]

$$N_{\text{osc}} = \frac{L_{\text{coh}}}{L_{\text{osc}}} \frac{\sqrt{2}}{\pi} \sigma_r E. \quad (51)$$

Because in the neutrino oscillation experiments one has $L \approx L_{\text{osc}}$, then the term exp $[- \left( \frac{L}{L_{\text{coh}}} \right)^2]$ is nearly equal to one [22]. Additionally, it is easy to show that $|v_1 - v_2| L_{\text{coh}} \approx \Delta m_{12}^2 2E \sim \sigma_r$ and thus the term exp $[-2\pi^2 \xi^2 \left( \frac{\sigma_r}{L_{\text{osc}}} \right)^2]$ is nearly equal to one [22]. In this form, the time-integrated oscillation probabilities (45) and (46) can be written as

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showing that the coherence length is much larger than the oscillation length [22]. The maximum number of oscillations can be obtained from these lengths as [8]

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Because in the neutrino oscillation experiments one has $L \approx L_{\text{osc}}$, then the term exp $[- \left( \frac{L}{L_{\text{coh}}} \right)^2]$ is nearly equal to one [22]. Additionally, it is easy to show that $|v_1 - v_2| L_{\text{coh}} \approx \Delta m_{12}^2 2E \sim \sigma_r$ and thus the term exp $[-2\pi^2 \xi^2 \left( \frac{\sigma_r}{L_{\text{osc}}} \right)^2]$ is nearly equal to one [22]. In this form, the time-integrated oscillation probabilities (45) and (46) can be written as

$$P_{\nu_e}^{SWP}(L) = 1 - \frac{1}{2} \sin^2[2\theta_{12}] \left\{ 1 - \cos \left[ 2\pi \frac{L}{L_{\text{osc}}} \right] \right\}, \quad (52)$$

showing that the coherence length is much larger than the oscillation length [22]. The maximum number of oscillations can be obtained from these lengths as [8]

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showing that the coherence length is much larger than the oscillation length [22]. The maximum number of oscillations can be obtained from these lengths as [8]

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Because in the neutrino oscillation experiments one has $L \approx L_{\text{osc}}$, then the term exp $[- \left( \frac{L}{L_{\text{coh}}} \right)^2]$ is nearly equal to one [22]. Additionally, it is easy to show that $|v_1 - v_2| L_{\text{coh}} \approx \Delta m_{12}^2 2E \sim \sigma_r$ and thus the term exp $[-2\pi^2 \xi^2 \left( \frac{\sigma_r}{L_{\text{osc}}} \right)^2]$ is nearly equal to one [22]. In this form, the time-integrated oscillation probabilities (45) and (46) can be written as

$$P_{\nu_e}^{SWP}(L) = 1 - \frac{1}{2} \sin^2[2\theta_{12}] \left\{ 1 - \cos \left[ 2\pi \frac{L}{L_{\text{osc}}} \right] \right\}, \quad (52)$$

showing that the coherence length is much larger than the oscillation length [22]. The maximum number of oscillations can be obtained from these lengths as [8]

$$N_{\text{osc}} = \frac{L_{\text{coh}}}{L_{\text{osc}}} \frac{\sqrt{2}}{\pi} \sigma_r E. \quad (51)$$
\[ P_{\nu_e}^{SWP}(L) = \frac{1}{2} \sin^2[2\theta_{12}] \left\{ 1 - \cos \left[ 2\pi \frac{L}{L_{osc}} \right] \right\}, \] (53)

that reduce to the standard neutrino oscillation probabilities \(^{11}\) and \(^{2}\) obtained using the plane wave formalism.

V. EXPANSION OF THE ENERGY UNTIL THIRD ORDER IN THE MOMENTUM

In this section, we will study the effects originated in dispersion with time on spreading of wave packets for the time-integrated two-flavor neutrino oscillation probabilities by expanding the energy given by \(^{22}\) up to third order in the momentum \(\langle \vec{p} - \langle \vec{p}_n \rangle \rangle\). For this case, we take a reference frame where the matrix \(\Gamma_{kj}\) is diagonal and identical to \(^{25}\). Substituting the energy given by \(^{22}\) in \(^{20}\) and \(^{21}\), keeping up to the third order in the power series of \(\langle \vec{p} - \langle \vec{p}_n \rangle \rangle\), we obtain that the neutrino oscillation probabilities are written as

\[
P_{\nu_e}^{DWP}(T, L) = \frac{1}{2\pi \sigma_T^2 (1 + \lambda^2)^2} \left\{ \frac{\lambda^4}{g_1^2(T)} \exp[-\lambda_1 \phi_1^D(T)] + \frac{1}{g_2^2(T)} \exp[-\lambda_2 \phi_2^D(T)] + \frac{\lambda^8}{g_3^2(T)} \exp[-\lambda_3 \phi_3^D(T)] \right\}, \] (54)

\[
P_{\nu_e}^{DWP}(T, L) = \frac{1}{2\pi \sigma_T^2 (1 + \lambda^2)^2} \left\{ \frac{\lambda^2}{g_1^2(T)} \exp[-\lambda_1 \phi_1^D(T)] + \frac{\lambda^6}{g_2^2(T)} \exp[-\lambda_2 \phi_2^D(T)] - \frac{\lambda^8}{g_3^2(T)} \exp[-\lambda_3 \phi_3^D(T)] \right\}, \] (55)

with \(\lambda\) given by \(^{26}\), \(\lambda_1 = \lambda_2 = 1/2\sigma_T^2\), \(\lambda_3 = 1/4\sigma_T^2\), and the functions in the arguments of the exponentials are

\[
\phi_1^D(T) = \frac{(L - v_1 T)^2}{1 + \frac{T^2}{T_1^2}}, \quad \phi_2^D(T) = \frac{(L - v_2 T)^2}{1 + \frac{T^2}{T_2^2}}, \quad \phi_3^D(T) = \frac{(L - v_3 T)^2}{1 + \frac{T^2}{T_3^2}},
\] (56-58)

while the functions \(g_1^D(T)\), \(g_2^D(T)\) and \(g_3^D(T)\) are

\[
g_1^D(T) = \left( 1 + \frac{T^2}{T_1^2} \right)^{-1/2}, \quad g_2^D(T) = \left( 1 + \frac{T^2}{T_2^2} \right)^{-1/2}, \quad g_3^D(T) = \left( 1 - \frac{T}{T_3} \right)^{-1/2} \left( 1 + \frac{T}{T_3} \right)^{-1/2},
\] (59-61)

where we have defined the longitudinal dispersion time \(T_{\alpha}^L\) as \(T_{\alpha}^L = 2\bar{E}\lambda_{\alpha}^2/m_{\alpha}^2\), with \(a = 1, 2\), while the transversal dispersion time \(T_T\) has been defined as \(T_T^T = 2\bar{E}\lambda_{\alpha}^2\). The longitudinal dispersion time in neutrino oscillations was initially defined in the context of a quantum mechanics treatment \(^{8}\). This time was posteriorly considered in the context of a quantum field theory treatment of neutrino oscillations \(^{17}\). Most recently, the transversal and longitudinal times that we have defined here were considered in the context of a theory of wave packets in which the energy that appears in the wave packets is expanded until third order in the momentum \(^{25}\).

Because the longitudinal and transversal times are related as \(T_{\alpha}^L = \frac{E_{\alpha}^2}{m_{\alpha}^2} T_T\), we observe that \(T_{\alpha}^L \gg T_T\) and these two very separated times can be used to define three dispersion regimes: (i) The minimum dispersion regime is defined for times \(T < T_T\); (ii) the transversal dispersion regime for \(T_T < T < T_{\alpha}^L\); (iii) the longitudinal dispersion regime for \(T > T_{\alpha}^L\). These three dispersion regimes were equivalently considered previously by using the distance between the neutrino source and the detector \(L\) as the quantity to define these regimes \(^{18}\).
We observe in (54) and (55) the existence of two different longitudinal dispersion times $T_1^L$ and $T_2^L$. For simplicity, we will work in the limit in which the masses are nearly degenerate $m_1 = m_2 = \tilde{m}$. For this limit, it is possible to consider $T_1^L = T_2^L$ and to work with only one longitudinal dispersion time defined by $T^L = 2E^3\sigma^2/\tilde{m}^2$, with $\tilde{m}$ the mass in the degenerate limit. For the nearly degenerate limit, the functions given by the expressions (59), (60) and (61) are written as

$$g_1^D(T) = g_2^D(T) = g_3^D(T) = \left(1 + \frac{T^2}{(T^L)^2}\right)^{1/2},$$

(62)

In the next, we will study for the three dispersion regimes previously defined the effects originated in dispersion with time on the spreading of wave packets for the two-flavor neutrino oscillation probabilities.

A. Spreading in the minimum dispersion regime

In the minimum dispersion regime $T < T^T$ and for the limit of nearly degenerate masses, the oscillation probabilities (54) and (55) can be written as

$$P_{\nu_e}^MDWP(T, L) = \frac{1}{(2\pi\sigma^2)^{1/2}} \left(1 + \frac{1}{\Lambda^2}\right) \left\{ \Lambda^4 \exp[-\lambda_1\phi_1^S(T)] \right. $$

$$+ \exp[-\lambda_2\phi_2^S(T)] + \Lambda^2R \exp[-\lambda_3\phi_3^M(T)] \left\}, \right.$$

(63)

$$P_{\nu_\mu}^MDWP(T, L) = \frac{1}{(2\pi\sigma^2)^{1/2}} \left(1 + \frac{1}{\Lambda^2}\right) \left\{ \Lambda^2 \exp[-\lambda_1\phi_1^S(T)] \right. $$

$$+ \Lambda^2 \exp[-\lambda_2\phi_2^S(T)] - \Lambda^2R \exp[-\lambda_3\phi_3^M(T)] \left\}, \right.$$

(64)

with the functions $\phi_1^S(T)$ and $\phi_2^S(T)$ given by (52) and (53) respectively, and the function $\phi_3^M(T)$ is

$$\phi_3^M(T) = (L - v_1)^2 + (L - v_2)^2 + i\{(v_1^2 - v_2^2)T^2 - 2L(v_1 - v_2)T/T^L \}

- 4\sigma^2(E_1 - E_2)T + i4\sigma^2(\bar{p}_1 - \bar{p}_2)L.$$

(65)

where we have neglected the terms with powers higher than $O(T/T^T)$ and $O(T/T^L)$. Now we focus our attention on the elimination of the time dependence that is present in the neutrino oscillation probabilities (53) and (54). To do it, we take the average on the time of the expressions (58) and (64). With the integration on the time we obtain

$$P_{\nu_e}^MDWP(L) = \frac{1}{(2\pi\sigma^2)^{1/2}} \left(1 + \frac{1}{\Lambda^2}\right) \left\{ \Lambda^4 I_1^M + I_2^M + \Lambda^2R I_3^M \right\},$$

(66)

$$P_{\nu_\mu}^MDWP(L) = \frac{1}{(1 + \Lambda^2)^2} \left(1 + \frac{1}{\Lambda^2}\right) \left\{ \Lambda^2 I_1 + \Lambda^2 I_2 - \Lambda^2R I_3 \right\},$$

(67)

where the integrals on the time for the minimum dispersion regime $I_1^M$, $I_2^M$ and $I_3^M$ are

$$I_1^M = \int \exp[-\lambda_1\phi_1^S(T)]dT,$$

(68)

$$I_2^M = \int \exp[-\lambda_2\phi_2^S(T)]dT,$$

(69)

$$I_3^M = \int \exp[-\lambda_3\phi_3^M(T)]dT.$$  

(70)

(71)

The integrals $I_1^M$ and $I_2^M$ can be performed using both Gaussian integration and the Laplace approximation method, while the integral $I_3^M$ can be only performed using the Laplace approximation method. After the time integrations are performed, we obtain from (66) and (67) the following time-integrated neutrino oscillation probabilities

$$P_{\nu_e}^MDWP(L) = \frac{1}{(1 + \Lambda^2)^2} \left\{ \frac{\Lambda^4}{v_1} + \frac{1}{v_2} + \frac{2}{v_1^2 + v_2^2} \right\}^{1/2} \left[f_1^T + f_2^T\right],$$

(72)
\[ P_{\nu}^{MDWP}(L) = \frac{1}{(1 + \Lambda^2)^2} \left\{ \frac{\Lambda^2}{v_1} + \frac{\Lambda^2}{v_2} - 2\Lambda^2 \Xi \left( \frac{2}{v_1^2 + v_2^2} \right)^{1/2} f_3^T \exp \left[ i f_1^T - f_2^T \right] \right\}, \quad (73) \]

with the functions \( f_1^T \), \( f_2^T \) and \( f_3^T \) given by

\[ f_1^T = (E_1 - E_2) \frac{v_1 + v_2}{v_1^2 + v_2^2} L(1 - h_1^T) - (\bar{p}_1 - \bar{p}_2)L, \quad (74) \]
\[ f_2^T = \frac{(v_1 - v_2)^2}{v_1^2 + v_2^2} \frac{L^2}{4\sigma_1^2}(1 + h_2^T) + \frac{(E_1 - E_2)^2}{v_1^2 + v_2^2} \sigma_1^2 (1 - h_2^T), \quad (75) \]
\[ f_3^T = \left( 1 + \frac{6(v_1^2 - v_2^2)(E_1 - E_2)S}{(v_1^2 + v_2^2)^2} \right)^{-1/2}, \quad (76) \]

where

\[ h_1^T = -\frac{1}{2} (E_1 - E_2)(v_1 - v_2) \frac{[(v_1^2 + v_2^2)^2 + 4v_1v_2]}{(v_1^2 + v_2^2)^2} \frac{S}{S}, \quad (77) \]
\[ h_2^T = -\frac{1}{2} (E_1 - E_2)(v_1 + v_2) \frac{S}{S}, \quad (78) \]
\[ h_3^T = -\frac{5}{2} \frac{(E_1 - E_2)(v_1^2 - v_2^2)}{(v_1^2 + v_2^2)^2} \frac{S}{S}, \quad (79) \]

In the relativistic limit, using the approximations (42), (43) and (44), we obtain from (72) and (73) that the time-integrated neutrino oscillation probabilities for the minimum dispersion regime are

\[ P_{\nu}^{MDWP}(L) = 1 - \frac{1}{2} \sin^2[2\theta_{12}] \left\{ \frac{1}{(1 - a_1)^2} \exp \left[ i 2\pi \frac{L_{osc}}{L_{coh}} - \frac{L_{coh}}{L_{osc}} \right] - 2\pi^2 \xi^2 \left( \frac{\sigma_{(1-a_1)}}{L_{osc}} \right)^2 \right\}, \quad (80) \]
\[ P_{\nu}^{MDWP}(L) = \frac{1}{4} \sin^2[2\theta_{12}] \left\{ \frac{1}{(1 - a_1)^2} \exp \left[ i 2\pi \frac{L_{osc}}{L_{coh}} - \frac{L_{coh}}{L_{osc}} \right] - 2\pi^2 \xi^2 \left( \frac{\sigma_{(1-a_1)}}{L_{osc}} \right)^2 \right\}, \quad (81) \]

where \( L_{osc}' \) and \( L_{coh}' \) are written as

\[ L_{osc}' = \frac{L_{osc}}{1 + a_1}, \quad (82) \]
\[ L_{coh}' = \frac{L_{coh}}{(1 + a_2)^{1/2}}, \quad (83) \]

and

\[ a_1 = \frac{1}{8} \frac{\xi^2(\Delta m_{12}^2)^2m^2}{E^6} = 4\xi^2 \frac{\sigma_1^2}{L_{coh} T_L} \frac{T_T}{T_L}, \quad (84) \]
\[ a_2 = \frac{5}{8} \frac{m^2}{E^2} = 5\xi \frac{T_T}{T_L}, \quad (85) \]
\[ a_3 = \frac{5}{16} \frac{\xi(\Delta m_{12}^2)^2m^2}{E^6} = 10\xi \frac{\sigma_1^2}{L_{coh} T_L} \frac{T_T}{T_L}, \quad (86) \]
\[ a_4 = \frac{3}{4} \frac{\xi(\Delta m_{12}^2)^2m^2}{E^6} = 24\xi \frac{\sigma_1^2}{L_{coh} T_L} \frac{T_T}{T_L}. \quad (87) \]

We can observe that the time-integrated oscillation probabilities (80) and (81) have the same functional form that the oscillation probabilities (45) and (46) obtained considering the expansion of energy until second order in the
momentum, but now the exponential is multiplied by a factor that includes $a_4$. But new, due to the effects originated in dispersion with time on the spreading of the wave packets, the expressions (52) and (53) show respectively some changes of the oscillation length (47) and of the coherence length (48). We observe how the oscillation length (52) and the coherence length (53) are respectively a little smaller than the ones obtained for the case in which the energy is expanded until second order in the momentum (47) and (48). Due to the effects originated in dispersion with time, now the maximum number of oscillations is

$$N'_\text{osc} = \frac{L'_\text{coh}}{L'_{\text{osc}}} = \frac{(1 + a_1)}{(1 + a_2)1/2} N_{\text{osc}},$$

(88)

which implies that it is smaller than the one obtained for the case in which the energy is expanded until second order in the momentum (see the expression (51)). However, the quantities $a_i$, with $i = 1, 2, 3, 4$, are very small, so $L'_{\text{osc}} \simeq L_{\text{osc}}, L'_\text{coh} \simeq L_{\text{coh}}, N'_\text{osc} \simeq N_{\text{osc}}, a_3 \simeq 0$ and $a_4 \simeq 0$. In this way, the time-integrated oscillation probabilities (80) and (81) can reduce to (45) and (46). Thus, for the minimum dispersion regime we find that the effects originated in dispersion with time on the spreading of the wave packets for the time-integrated neutrino oscillation probabilities can be neglected.

B. Spreading in the transversal dispersion regime

In the transversal dispersion regime $T^T < T < T^L$ and for the limit of nearly degenerate masses, the oscillation probabilities (54) and (55) can be written as

$$P^{T, \text{MDWP}}_{\nu_e}(T, L) = \frac{1}{(2\pi\sigma_T^2)^{1/2}} \frac{1}{(1+\lambda T)^2} \left\{ \lambda^4 F^T(T) \exp[-\lambda_1 \phi^S_\nu(T)] + F^T(T) \exp[-\lambda_2 \phi^S_\nu(T)] + \frac{\lambda^2 \pi \sigma_T^2}{2} F^T(T) \exp[-\lambda_3 \phi^S_\nu(T)] \right\},$$

(89)

$$P^{T, \text{MDWP}}_{\nu_\mu}(T, L) = \frac{1}{(2\pi\sigma_T^2)^{1/2}} \frac{1}{(1+\lambda T)^2} \left\{ \lambda^2 F^T(T) \exp[-\lambda_1 \phi^S_\nu(T)] + \frac{\lambda^2 \pi \sigma_T^2}{2} F^T(T) \exp[-\lambda_3 \phi^S_\nu(T)] \right\},$$

(90)

where the function on the time $F^T(T)$ is given by $F^T(T) = \frac{(T_2)}{T_2^2}$). In the expressions (89) and (90), we have neglected the terms with powers higher than $O(T/T^L)$ and the functions $\phi^S_\nu(T)$ and $\phi^S_\nu(T)$ are given by (52) and (53) respectively, while the function $\phi^M_\nu(T)$ is given by (56). The three integrals that appear in the expressions (89) and (90) can be only performed using the Laplace approximation method. After the time integrations are performed, we obtain from (89) and (90) the following time-integrated neutrino oscillation probabilities

$$P^{T, \text{MDWP}}_{\nu_e}(L) = \left(\frac{T_2}{L^2}\right)^2 P^{\text{MDWP}}_{\nu_e}(L) = \left(\frac{T_2}{L^2}\right)^2 P^{\text{SWP}}_{\nu_e}(L),$$

(91)

$$P^{T, \text{MDWP}}_{\nu_\mu}(L) = \left(\frac{T_2}{L^2}\right)^2 P^{\text{MDWP}}_{\nu_\mu}(L) = \left(\frac{T_2}{L^2}\right)^2 P^{\text{SWP}}_{\nu_\mu}(L),$$

(92)

where the standard time-integrated neutrino oscillation probabilities $P^{\text{SWP}}_{\nu_e}(L)$ and $P^{\text{SWP}}_{\nu_\mu}(L)$ are given respectively by (44) and (45), and the transversal dispersion time by $T^T = 2\bar{\sigma}_T^2$. We observe for this case that the standard time-integrated neutrino oscillation probabilities $P^{\text{SWP}}_{\nu_e}(L)$ and $P^{\text{SWP}}_{\nu_\mu}(L)$ are suppressed by a factor $(T_2/L^2)^2$. This result is in agreement with the showed by Naumov [25], whom has recently obtained that the integral over time of both the flux and probability densities are asymptotically proportional to the factor $1/L^2$, when he considered a theory of wave packets in which the energy that appears in the wave packets is expanded until third order in the momentum. This author has demonstrated that the origin of the factor $1/L^2$ for quantum objects is their dispersion with time [25].

C. Spreading in the longitudinal dispersion regime

In the longitudinal dispersion regime $T > T^L$ and for the limit of nearly degenerate masses, the oscillation probabilities (54) and (55) can be written as

$$P^{T, \text{MDWP}}_{\nu_e}(T, L) = \frac{1}{(2\pi\sigma_T^2)^{1/2}} \frac{1}{(1+\lambda T)^2} \left\{ \lambda^4 F^L(T) \exp[-\lambda_1 \phi^D_\nu(T)] + F^L(T) \exp[-\lambda_2 \phi^D_\nu(T)] + \frac{\lambda^2 \pi \sigma_T^2}{2} F^L(T) \exp[-\lambda_3 \phi^D_\nu(T)] \right\},$$

(93)
\[
P_{\nu_{\mu}}^{LDWP}(T, L) = \frac{1}{(2\pi)^{1/2}} \frac{1}{\sqrt{1+\Lambda^2 P}} \left\{ \Lambda^2 F^L(T) \exp[-\lambda_1 \phi_1^D(T)] + \Lambda^2 F^L(T) \exp[-\lambda_2 \phi_2^D(T)] - \Lambda^2 \delta F^L(T) \exp[-\lambda_3 \phi_3^D(T)] \right\},
\]

where the function on the time \( F^L(T) \) is given by \( F^L(T) = \frac{(T^T)^2}{2L^2} \). In the expressions (93) and (94), the functions \( \phi_1^D(T) \), \( \phi_2^D(T) \) and \( \phi_3^D(T) \) are given by \( \frac{1}{16} \), \( \frac{1}{4} \) and \( \frac{1}{2} \), respectively. For this case, also the three integrals that appear in the expressions (93) and (94) can be only performed using the Laplace approximation method. After the time integrations are performed, we obtain from (93) and (94) the following time-integrated neutrino oscillation probabilities

\[
P_{\nu_{\mu}}^{LDWP}(L) = \left( \frac{T^T}{L} \right)^2 \frac{P_{\nu_{\mu}}^{MDWP}(L)}{L^2} = \left( \frac{T^T}{L} \right)^2 P_{\nu_{\mu}}^{SWP}(L),
\]

\[
P_{\nu_{\mu}}^{LDWP}(L) = \left( \frac{T^T}{L} \right)^2 \frac{P_{\nu_{\mu}}^{MDWP}(L)}{L^2} = \left( \frac{T^T}{L} \right)^2 P_{\nu_{\mu}}^{SWP}(L),
\]

where the standard time-integrated neutrino oscillation probabilities \( P_{\nu_{\mu}}^{SWP}(L) \) and \( P_{\nu_{\mu}}^{SWP}(L) \) are given respectively by (15) and (16). We observe also for this case that the standard time-integrated neutrino oscillation probabilities \( P_{\nu_{\mu}}^{SWP}(L) \) and \( P_{\nu_{\mu}}^{SWP}(L) \) are suppressed by a factor \( (T^T)^2/L^2 \).

VI. CONCLUSIONS

We have studied the effects originated in dispersion with time on spreading of wave packets for the time-integrated two-flavor neutrino oscillation probabilities in vacuum. We have calculated the time-integrated two-flavor neutrino oscillation probabilities in the context of a wave packet extension of the quantum field theory treatment that we previously developed for the case in which neutrino mass eigenstates were described by plane waves. In the treatment that we have presented here, neutrino flavor states have been considered as superpositions of neutrino mass eigenstates described by localized wave packets.

By methodological reasons, we have initially studied the effects of the spreading of the wave packets by considering the expansion of the energy until second order in the momentum that leads to the standard time-integrated neutrino oscillation probabilities. After this, we have studied the effects originated by dispersion in time on spreading of wave packets for the time-integrated two-flavor neutrino oscillations by considering the expansion of the energy until third order in the momentum. We have observed that the standard time-integrated neutrino oscillation probabilities are suppressed by a factor \( 1/L^2 \) for the transversal and longitudinal dispersion regimes, where \( L \) is the distance between the neutrino source and the detector. The existence of this kind of suppression for the standard time-integrated neutrino oscillation probabilities might be proved in reactor neutrino oscillation experiments with beams very narrow in time or experiments at short enough distances [24].

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[1] Ch. W. Kim and A. Pevsner, *Neutrinos in physics and astrophysics*. (Harwood Academic Publishers, Switzerland, 1993).
[2] P. B. Pal and R. N. Mohapatra, *Massive neutrinos in physics and astrophysics*. (World Scientific Publishing, 2004).
[3] C. Giunti and Ch. Kim, *Fundamental of neutrinos in physics and astrophysics*. (Oxford University Press, New York, 2007).
[4] S. M. Bilenky, J. Hosek and S. T., *Petcov, Phys. Lett. B* 94, 495 (1980).
[5] Y. F. Pérez and C. J. Quimbay, *J. Mod. Phys.* 3, 803 (2012).
[6] S. M. Bilenky and B. Pontecorvo, *Phys. Rep.* 41, 225 (1978).
[7] B. Kayser, *Phys. Rev. D* 24, 110 (1981).
[8] C. Giunti, C. W. Kim and U. W. Lee, Phys. Rev. D 44, 3635 (1991).
[9] J. Rich, Phys. Rev. D 48, 4318 (1993).
[10] M. Zralec, Acta Phys. Polon. B 29, 3925 (1998).
[11] C. Giunti, Found. Phys. Lett. 14, 213 (2001).
[12] C. Giunti, C. W. Kim, J. A. Lee and U. W. Lee, Phys. Rev. D 48, 4310 (1993).
[13] C. Giunti, C. W. Kim and U. W. Lee, Phys. Lett. B 421, 237 (1998).
[14] C. Giunti and C. W. Kim, Phys. Rev. D 58, 017301 (1998).
[15] M. Beuthe, Phys. Rev. D 66, 013003 (2002).
[16] C. Giunti, JHEP 11, 017 (2002).
[17] C. Giunti, Phys. Scr. 67, 29 (2003).
[18] M. Beuthe, Phys. Rep. 375, 105 (2003).
[19] C. Giunti, Found. Phys. Lett. 17, 103 (2004).
[20] C. Giunti, J. Phys G: Nucl. Part. Phys. 34, R93 (2007).
[21] S. M. Bilenky, F. von Feilitzsch and W. Potzel, J. Phys. G 38, 115002 (2011).
[22] S. M. Bilenky, Phys. Part. Nucl. 42, 515 (2011).
[23] A. E. Bernardini and S. De Leo, Phys. Rev. D 70, 053010 (2004).
[24] A. E. Bernardini and S. De Leo, Phys. Rev. D 71, 076008 (2005).
[25] D. V. Naumov, arXiv:1309.1717 (2013).