Superspace calculation of the three-loop dilatation operator of $\mathcal{N} = 4$ SYM theory

Christoph Sieg

Niels Bohr International Academy
Niels Bohr Institute
Blegdamsvej 17
2100 Copenhagen
Denmark

csiegnbi.dk

Abstract. We derive the three-loop dilatation operator of the flavor $SU(2)$ subsector of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in the planar limit by a direct Feynman diagram calculation in $\mathcal{N} = 1$ superspace. The transcendentality three contributions which appear in intermediate steps cancel among each other, leaving a rational result which confirms the predictions from integrability. We derive finiteness conditions that allow us to avoid the explicit evaluation of entire classes of Feynman graphs and also yield constraints on the D-algebra manipulations. Based on these results, we discover universal cancellation mechanisms. As a check for the consistency of our result, we verify the cancellation of all higher-order poles.

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1 Introduction

The hints of integrability found in type II B string theory in $\text{AdS}_5 \times S^5$ and in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory with gauge group $\text{SU}(N)$ in the limit $N \to \infty$ have led to impressive progress in quantitatively testing the AdS/CFT correspondence \cite{1–3}. The correspondence conjectures a duality between these two theories and, in particular, it predicts that, in the limit $N \to \infty$, with the 't Hooft coupling $\lambda = g^2_{\text{YM}}N$ fixed such that the gauge theory becomes planar, the energies of string states should match at any value of $\lambda$ the anomalous dimensions of gauge invariant composite operators with the same quantum numbers.

Tests of this prediction appeared impossible, since, in both theories, the spectra can only be calculated perturbatively in incompatible regimes, i.e., to the first few orders in respective expansions at strong coupling $\lambda \gg 1$ in the string theory and at weak coupling $\lambda \ll 1$ in the gauge theory. Based on the assumption of all-order integrability, this obstacle has been overcome by a unification \cite{4} of the Bethe ansatz of the string theory \cite{5} and of the gauge theory \cite{6, 7}. From these Bethe equations, an integral equation for the so-called cusp anomalous dimension was derived \cite{4, 8}. The found order-by-order solutions at strong and weak coupling \cite{9–12} match with the strong coupling results from string theory \cite{13–17} and with the weak coupling results in the $\mathcal{N} = 4$ SYM theory \cite{18–23} to the known orders. This is an important quantitative test of the AdS/CFT correspondence.

It succeeds, since the cusp anomalous dimension is not affected by corrections due to the finite length of the string states and composite operators that, in general, are not captured by the Bethe ansatz. Based on the assumption of integrability, proposals of how to incorporate these finite size corrections were formulated \cite{24–30} in order to describe the full spectrum. At weak coupling in the $\mathcal{N} = 4$ SYM theory the finite size corrections are the so-called wrapping interactions \cite{6, 31, 32}. In Feynman diagram calculations \cite{33–36}, and from integrability \cite{25, 37}, some leading wrapping corrections were determined, and matching was found. This rules out an earlier conjecture \cite{38} that the finite size effects might be captured by the Hubbard model.

A first sign of integrability in the AdS/CFT correspondence was discovered in a one-loop calculation in the $\mathcal{N} = 4$ SYM theory \cite{39}. The mixing under renormalization among different composite operators in which the scalar fields of the theory appear within a single trace over the gauge group indices was mapped to the integrable Heisenberg spin chain. The spin chain states are thereby identified with the composite operators, and the Hamiltonian acting on these chains is determined by the UV divergences of the underlying Feynman diagrams. The diagonalization of this system, e.g., by means of the Bethe ansatz, yields as eigenvalues the anomalous dimensions. The sums of bare dimensions and the anomalous dimensions are the conformal dimensions, and they are measured by the dilatation operator of the underlying (super-)conformal algebra. The spin chain Hamiltonian is, therefore, also called (the quantum part of) the dilatation operator.

At two loops, the renormalization of composite operators of BMN type \cite{40} was determined by a calculation in component formalism \cite{41}. Thereby, a dilute gas approximation was used, i.e., interactions between the fields that are regarded as impurities in the BMN operators were neglected. Moreover, only the Feynman diagrams that alter the relative positions of the impurities within the gauge trace were explicitly calculated. The
contribution from the remaining most complicated diagrams that leave the flavors unaffected was reconstructed from the condition that the BMN ground state has vanishing anomalous dimension, since it saturates a BPS bound.

Supplementing the aforementioned results with the contributions from the interactions between the impurities, the operator mixing problem in the planar limit is found to be integrable to two-loop order in the flavor $SU(2)$ subsector \[42\]. Each operator in this subsector contains a certain number of elementary fields of two different flavors that are given as two complex combinations of the six real scalar fields. With the assumption of higher-loop integrability, the dilatation operator of the flavor $SU(2)$ subsector was then constructed first to three \[42\] and then to higher loops \[6\]. By considering operator mixing in a bigger subsector, it was then found that the symmetry algebra, together with some assumptions and structural input from the underlying Feynman diagrams, fixes to three-loop order the dilatation operator of this subsector \[43\].

The predictions have been tested by various field theory calculations. The three-loop anomalous dimension of the Konishi operator matches the conjectured eigenvalue from integrability \[20, 44\]. At four loops and beyond, further field theory calculations tested the structure and various eigenvalues of the dilatation operator \[22, 33, 35, 41, 45, 46\]. The respective tests in \[33, 35\] were necessary in order to modify the dilatation operator such that some leading wrapping corrections could be included. In this context, the five-loop result for the Konishi operator \[47, 49\] and six-loop results for twist-three operators \[46\], as obtained from integrability, still have to be tested by direct Feynman diagram calculations.

Albeit the aforementioned tests of the structure and of some eigenvalues at higher loops, a direct field theoretical derivation of the three-loop dilatation operator in the flavor $SU(2)$ subsector is not yet available. In this paper, we perform this calculation in $\mathcal{N} = 1$ superfield formalism. Since our calculation yields the dilatation operator itself, it determines the three-loop planar spectrum of all composite single-trace operators in the flavor $SU(2)$ subsector and goes beyond the existing tests of some eigenvalues. Our result implies three-loop integrability in the flavor $SU(2)$ subsector and it also fixes the coupling dependence in the magnon dispersion relation of the underlying Bethe equations to that order.

Another motivation for this work is to gain insight for similar perturbative calculations at higher orders. We formulate and exploit finiteness conditions for the underlying supergraphs and uncover universal cancellation mechanisms between overall UV divergences of entire classes of graphs. This allows us to reduce the calculational effort significantly, and our findings should be of importance for extending the field theory calculations of the leading wrapping corrections for short operators along the lines of \[32, 35\], also to the next-to-leading order.

The paper is organized as follows:

In Section 2, we introduce our notation and summarize some aspects of operator renormalization in the flavor $SU(2)$ subsector of $\mathcal{N} = 4$ SYM theory.

In Section 3, we reexamine the results from integrability in the flavor $SU(2)$ subsector and the existing tests from field theory to three-loop order. In particular, we argue that our calculation determines to that order the magnon dispersion relation and integrability in this subsector directly from field theory.

In Section 4, we summarize some important implications of the finiteness conditions
for the underlying Feynman diagrams which we derive in Appendix B.

In Section 5, we demonstrate the efficiency of our approach and derive the one- and two-loop results in an instant.

In Section 6, we present the three-loop calculation, classifying the diagrams first according to the range of the interactions, i.e., the number of fields that are involved in the interactions, and then also according to the generated flavor manipulations. By employing the finiteness conditions, we reveal universal cancellations among the overall UV divergences of different Feynman diagrams. Then, we present the final result of our calculation.

In Section 7, we draw our conclusions and comment on the implications our findings should have for calculations of the next-to-leading wrapping corrections along the lines of [32–35].

Several details of the calculation have been delegated to appendices. Based on the D-algebra structure of the Feynman rules listed in Appendix A, the finiteness conditions are derived in Appendix B. They are quite general and also hold, e.g., for the \( \beta \)-deformation [50] of the \( \mathcal{N} = 4 \) SYM theory and for the \( \mathcal{N} = 6 \) Chern-Simons theory [51, 52]. In Appendix C, we determine the expressions of the one- and two-loop subdiagrams that appear in the calculation, and we derive the most complicated cancellation mechanism. In Appendix D, as a consistency check of our result, we explicitly demonstrate the cancellation of higher-order poles in the logarithm of the renormalization constant. Expressions for the relevant integrals and their overall UV divergences are listed in Appendix E.

2 Renormalization in the flavor \( SU(2) \) subsector

In the following, we work in the \( \mathcal{N} = 1 \) superfield formulation of \( \mathcal{N} = 4 \) SYM theory in Fermi-Feynman gauge and use the conventions of [53]. An \( SU(3) \) subgroup of the \( SU(4) \) R-symmetry that we call flavor symmetry is manifest and it transforms the three chiral superfields \( \phi^i = (\phi, \psi, Z) \) into each other.

The composite operators of the aforementioned flavor \( SU(2) \) subsector appear as the lowest components of chiral superfields that themselves are products of chiral superfields, using as building blocks only two of the three different chiral field flavors, e.g., \( \phi \) and \( Z \). The length \( L \) of such an operator is then defined as the number of its constituents \( \phi \) and \( Z \), and we call the appearing \( \phi \) impurities. In order to obtain gauge invariant objects, the color indices of the constituent fields have to be contracted with each other. These contractions form a certain number of cycles that yields the number of gauge traces in the resulting so-called multi-trace operators. Here, we will only consider the planar limit in which it is sufficient to study the mixing of operators involving a single gauge trace only. They are denoted as single-trace operators.

The \( \mathcal{N} = 4 \) SYM theory is finite [54–57], and hence, in terms of \( \mathcal{N} = 1 \) superfields, no infinities are encountered, apart from gauge artefacts [58–61]. This is not the case when quantum corrections are considered for the correlation functions that also involve composite operators. The appearing UV divergences from the loop integrals of the quantum corrections do not cancel and manifest themselves as poles in \( \varepsilon \), where \( \varepsilon \) is the regulator in dimensional reduction [62] in \( D = 4 - 2\varepsilon \) dimensions. These poles have to be absorbed
by the renormalization of the composite operators as
\[ O_{a,\text{ren}} = Z_a^b(\lambda, \varepsilon)O_{b,\text{bare}}, \quad (2.1) \]
where \( Z \) is the matrix-valued renormalization constant that is given as a power series in the 't Hooft coupling constant \( \lambda = g_{YM}^2 N \).

The flavor \( SU(2) \) subsector is closed under renormalization, at least perturbatively [63]. Mixing within this subsector can only occur among composite operators with the same length \( L \) and number of impurities. For appropriately normalized operators, the renormalization constant decomposes as \( Z = 1 + \delta Z \), where \( \delta Z \) can be brought to block-diagonal form. Each block acts within a subset of operators that differ only by permutations of their field content within the gauge trace. The permutations are generated from the nontrivial flavor structure of the chiral and antichiral vertex of the \( \mathcal{N} = 4 \) SYM theory. These vertices and their connections within each Feynman diagram form its chiral structure that acts on the flavors of the interacting fields as a fixed linear combination of products of permutations and of the identity operation. The chiral structure of each diagram is captured by one of the chiral functions that were introduced in [33].

In terms of the permutation \( P_{ij} \) and identity \( 1_{ij} \) that act on the fields at sites \( i \) and \( j \) of a composite operator of length \( L \), they are defined as
\[ \chi(a_1, \ldots, a_n) = \sum_{r=0}^{L-1} \prod_{i=1}^{n} (P - 1)_{a_i+r, a_i+r+1}, \quad (2.2) \]
where \( \chi() \) is the identity. Periodicity with the period \( L \) is understood. The range of the interaction in flavor space, i.e., the number of nearest neighbors that are involved in flavor permutations, is extracted from the argument list \( a_1, \ldots, a_n \) of the chiral functions as
\[ \kappa = \max_{a_1, \ldots, a_n} - \min_{a_1, \ldots, a_n} + 2. \quad (2.3) \]
It must not be confused with the range \( R \) of the Feynman diagram itself, i.e., with the number of fields of the composite operators that are involved in the interaction. In fact, the range \( R \) exceeds \( \kappa \) if flavor-neutral vector fields establish interactions with further fields of the composite operators that are not themselves building up a nontrivial chiral structure.

According to the previous discussion, we can express \( \delta Z \) as a linear combination of chiral functions. The coefficient of each chiral function is determined from the Feynman graphs with the respective chiral structure. It is the negative of the sum of the poles in \( \varepsilon \) that capture the overall UV divergences of the individual graphs. The result immediately determines the renormalization constant in (2.1), and the dilatation operator is then extracted from the latter as
\[ D = \mu \frac{d}{d\mu} \ln Z(\lambda \mu^{2\varepsilon}, \varepsilon) = \lim_{\varepsilon \to 0} \left[ 2\varepsilon \lambda \frac{d}{d\lambda} \ln Z(\lambda, \varepsilon) \right], \quad (2.4) \]
where the second relation holds, since the logarithm cancels all higher-order poles in \( \varepsilon \). This cancellation is an important consistency check for our calculation and it can be found in Appendix [D].
pole of $\delta Z$ and, at a given loop order $K$, multiplies it by a factor $2^K$. This then yields
the dilatation operator (more precisely, the quantum part) as a power series

$$D = \sum_{k \geq 1} g^{2k} D_k , \quad g = \sqrt{\lambda} / 4\pi , \quad (2.5)$$

where we have absorbed the powers of $4\pi$ that appear from the loop integrals into a
rescaled coupling constant $g$.

The expression of the dilatation operator in terms of chiral functions allows for a
general statement when considering the composite operators

$$\text{tr}(Z^L) , \quad \text{tr}(\phi Z^{L-1}) \quad (2.6)$$

that, for each length $L$, are the ground state and, respectively, the first excited state in
the flavor $SU(2)$ subsector. All chiral functions $\chi(n)$ with $n \geq 1$ yield zero when they are
applied to these states. Only the identity $\chi()$ in flavor space yields the length $L$. Since
the operators $\{2.4\}$ are protected and hence are not renormalized, the matrix $\delta Z$, and
thus also $D$, as defined in (2.4), have to vanish when applied to these states and must not
depend explicitly on $\chi()$. We will come back to this statement in Section 4 and relate it
to the preservation of conformal invariance on the quantum level.

3 Three-loop integrability

To three-loop order, rewritten in the basis of chiral functions, the dilatation operator
from integrability is predicted as $[42]$

$$D_1 = -2\chi(1) , \quad D_2 = -2[\chi(1,2) + \chi(2,1)] + 4\chi(1) , \quad D_3 = -4[\chi(1,2,3) + \chi(2,3,1)] + 4i\epsilon_2[\chi(2,1,3) - \chi(1,3,2)] - 4\chi(1,3) + 16[\chi(1,2) + \chi(2,1)] - 24\chi(1) , \quad (3.1)$$

where $\epsilon_2$ remains undetermined and does not enter the spectrum. It is associated with
similarity transformations $[42,64]$.

The above expressions can be applied to an eigenstate of a single magnon with mo-
mentum $p$. This yields the respective coefficients in the weak coupling expansion of the
magnon energy $E(p)$. At each loop order $K$, the coefficient is given as a linear combina-
tion of $1 \leq l \leq K$ individual contributions that are generated as

$$\frac{1}{2}[\chi(1,2,\ldots,l) + \chi(l,\ldots,2,1)] \rightarrow -4\cos(l-1)p \sin^2 \frac{p}{2} \quad (3.2)$$

by the chiral functions that can be associated with the magnon dispersion relation of
the all-order Bethe ansatz formulated in $[6,7]$. The remaining chiral functions $\chi(2,1,3),
\chi(1,3,2),$ and $\chi(1,3)$ of (3.1) only contribute when two magnons are present within their
flavor interaction range $\kappa = 4$ and hence are associated with magnon scattering.

The basis of chiral functions is very convenient, since the coefficients of all chiral
functions of the form (3.2) are directly related to the magnon dispersion relation. For a
single magnon with momentum $p$, the dispersion relation is given by $[6,7]$

$$E(p) = \sqrt{1 + 4h^2(g) \sin^2 \frac{p}{2}} - 1 , \quad (3.3)$$
and it is determined by the underlying symmetry algebra up to an unknown function $h^2(g)$ of the coupling constant $g$. Results for the quantum corrections [66,67] of the giant magnon solution [68] at large $g$ and the field theory results for two-loop anomalous dimensions of the Konishi operator [69,71] and of the BMN operators [41] suggest that $h^2(g) = 4g^2$ is the exact result. This has also been argued using S-duality [72].

Setting $h^2(g) = 4g^2$, the expansion of (3.3) immediately fixes the coefficients of all chiral functions of the form (3.2) not only in (3.1) to three-loop order but in $D_K$ at any loop order $K$. The respective terms in $D_K$ are determined as

$$E(p)|_{g^K} = \left(\frac{-1}{2K-1}\right)^{2K} (4\sin^{2}\frac{p}{g})^{K} \to c_{K,1}\chi(1) + \sum_{i=2}^{K} c_{K,i}[\chi(1,2,\ldots,l) + \chi(l,\ldots,2,1)],$$

$$c_{K,i} = \left(\frac{-1}{2K-1}\right)^{2K} \left(\frac{2(K-1)}{K} \right)^{2(K-1)}.$$

In order to obtain the above expression from the $K$-loop coefficient in the expansion of (3.3), we have kept one factor $\sin^{2}\frac{p}{2}$ and expressed the remaining powers in terms of the cosine of integer multiples of $p$. The respective trigonometric relation can be found, e.g., in [73]. Finally, we have used (3.2) to replace the phase shifts by the respective chiral functions.

Beyond explicit order-by-order evaluations, no guiding principle from which one could determine $h^2(g)$ is presently known. The function $h^2(g)$ might even have a series expansion with coefficients of nonvanishing transcendentality, as found in the case of $\mathcal{N} = 6$ Chern-Simons theory [74,75]. We will, therefore, assume for a moment that $h^2(g)$ has a generic nontrivial expansion at weak coupling as

$$h^2(\lambda) = 4(g^2 + g^4h_2 + g^6h_3 + \ldots).$$

Inserting this expansion into the magnon dispersion relation and using (3.4), it deforms the dilatation operator in (3.1) as

$$D_{1,\text{def}} = -2\chi(1),$$
$$D_{2,\text{def}} = -2[\chi(1,2) + \chi(2,1)] + 2(2 - h_2)\chi(1),$$
$$D_{3,\text{def}} = -4[\chi(1,2,3) + \chi(3,2,1)] + 4i\epsilon_2[\chi(2,1,3) - \chi(1,3,2)] - (4 + s)\chi(1,3) + 4(4 - h_2)[\chi(1,2) + \chi(2,1)] - (12 - 4h_2 + h_3)\chi(1).$$

We have thereby also introduced a deformation $s$ of the only magnon scattering term that affects the spectrum. If $s$ is nonvanishing, $D_{\text{def}}$ no longer commutes with the higher local conserved charges, and integrability is lost.

Note that the all-order resummation of the maximum shuffling terms in [41] assumes that there are no deformations of the form (3.5). The maximum shuffling terms cannot provide any information on the $h_i$, $i \geq 2$, since, at each loop order $K$, the coefficients of the chiral functions (3.2) with $l = K$ do not depend on these $h_i$.

Applying $D_{\text{def}}$ to the Konishi descendant in the flavor $SU(2)$ subsector yields the eigenvalue

$$\gamma = 12g^2 - [48 - 12h_2]g^4 + [336 - 12(8h_2 - h_3 + s)]g^6,$$
and it should match with the anomalous dimension of the Konishi operator to pass
the existing tests from field theory. The one-loop result \[76, 78\] is reproduced, and the
two-loop result \[41, 69–71\] is found for \(h_2 = 0\), as mentioned earlier. The three-loop
eigenvalues for the Konishi operator and for another nonprotected operator obtained
in \[44\] are sufficient to fix \(h_3 = s = 0\) but do not provide direct field theory results
for further operators. Such results would be very desirable, e.g., to test the anomalous
dimensions of twist-two operators of the \(SL(2)\) subsector \[20\] that have been extracted
as highest-transcendentality terms from a full three-loop calculation in QCD \[79, 80\],
exploiting a relation of BFKL and DGLAP evolution in \(N = 4\) SYM theory \[81\]. Our
three-loop calculation of the dilatation operator itself determines all coefficients of \(D_3,\text{def}\)
in \(3.6\) and hence fixes \(h_2, h_3,\) and \(s\) and tests integrability directly from field theory.
In this way, it determines to three-loop order the planar anomalous dimensions of all
single-trace operators of the flavor \(SU(2)\) subsector.

4 Finiteness conditions

Based on power counting and structural properties of the Feynman rules for the \(N = 1\) superfields, in Appendix B we derive finiteness conditions for the diagrams that contribute
to loop corrections of a composite operator in the flavor \(SU(2)\) subsector. Diagrams
in which at least two fields of the composite operators are involved in the interaction
(i.e., they have an interaction range \(R \geq 2\)) and in which all vertices appear in loops
have no overall UV divergence at any loop order. In the flavor \(SU(2)\) subsector, vertices
with vector fields and antichiral vertices can only appear in loops, and each chiral vertex
that appears outside loops leads to a flavor permutation. Diagrams with trivial chiral
function \(\chi()\) hence cannot have vertices outside their loops, and the finiteness conditions
imply that, for \(R \geq 2\), they cannot have an overall UV divergence. In other words, a
contribution of \(\chi()\) to the dilatation operator could only come from a UV divergent chiral
self-energy at the respective order, but it is finite \[54, 61\], and the theory is conformal.
Since all chiral functions but \(\chi()\) vanish when applied to the states \(2.6\), the above property relates the protection of these states to the preservation of conformal invariance at
quantum level.

A further implication of the finiteness condition concerns the D-algebra. The D-
algebra manipulations transform the supergraphs into expressions which are local in the
fermionic coordinates of superspace. The considerations of Appendix B yield restrictions
for the D-algebra manipulations, at the end of which loop integrals with overall UV
divergences are encountered: starting from an initial configuration, where the number of
covariant spinor derivative \(D_\alpha\) inside loops is minimized by convenient choices of their
positions at the chiral composite operator and the chiral vertices, no further \(D_\alpha\) and only a
limited number of \(\bar{D}\) must be transported outside loops by the D-algebra manipulations.
At any loop order \(K\), this restriction leads to complete cancellations among the overall
UV divergences of diagrams with maximum range \(R = K + 1\), if the interaction range in
flavor space \(\kappa\), as defined in \(2.3\), is not maximal; i.e., it obeys \(\kappa < R\) \[34\].

Further details about the finiteness conditions and a discussion that includes also the
case of \(N = 6\) Chern-Simons (CS) theory in \(N = 2\) superspace can be found in Appendix B.
5 One- and two-loop dilatation operator

With the aforementioned finiteness conditions, we can immediately derive the one- and two-loop dilatation operator. No reconstruction of parts of the Feynman diagrams from the BPS condition, as in the original one- and two-loop calculation in component fields in \cite{40} and \cite{41}, is necessary here, but one obtains the full result. At one loop, there is only a single UV divergent Feynman diagram. Its evaluation yields

\begin{align}
\lambda I_1 \chi(1), & \quad Z_1 = -I_1 \chi(1), \\
\end{align}

(5.1)

where, in the diagram, we have omitted all covariant spinor derivatives, and the bold horizontal line at the bottom represents the composite operator (its further noninteracting fields are not drawn). $I_1$ denotes the pole part of the integral $I_1$ given in (E.3). Using (2.4) and casting the result into the form (2.5), we obtain the expression for $D_1$ in (3.1).

The two-loop calculation is reduced to the evaluation of three diagrams, when the finiteness conditions of Appendix B and the finiteness of the two-loop chiral self-energy (C.13) are used. With the one-loop correction of the chiral vertex given in (C.1), the results for the diagrams with overall UV divergences are easily determined as

\begin{align}
\lambda^2 I_2 \chi(1,2), & \quad = -2\lambda^2 I_2 \chi(1), \\
\end{align}

(5.2)

where the equalities hold up to finite terms. Considering also the reflection of the first diagram, the two-loop renormalization constant becomes

\begin{align}
Z_2 = -I_2 [\chi(1,2) + \chi(2,1) - 2\chi(1)], \\
\end{align}

(5.3)

where $I_2 = KR(I_2)$ denotes the overall UV divergence of the integral $I_2$. Thereby, $R$ subtracs the subdivergences, and $K$ extracts the poles in $\varepsilon$. The integral $I_2$ and its overall UV divergence $I_2$ are listed in (E.3). Multiplying the $\frac{1}{\varepsilon}$ pole of (5.3) by 4 yields the result for $D_2$ in (3.1).

6 Three-loop dilatation operator

We organize the diagrams of the three-loop calculation according to their interaction range $R$. This range must not be confused with the range of the flavor interactions $\kappa$ in (2.3) that is restricted as $\kappa \leq R$. At three loops, the maximum range diagrams have $R = 4$. The next-to-maximum range diagrams have $R = 3$, and the diagrams of minimal range have $R = 2$, since, according to the finiteness of the three-loop chiral self-energy [58,\cite{60,61}, the $R = 1$ diagrams are finite. Then, from the finiteness conditions, as summarized in Section 4, we conclude that the simplest chiral function that has to be considered is $\chi(1)$. All equations involving three-loop diagrams are understood to hold up to irrelevant finite contributions.
6.1 Maximum range diagrams

At three loops the maximum number of fields of the composite operators that can interact in a planar diagram is four. The respective diagrams only contain loops that involve the interacting fields of the composite operator; i.e., if the composite operator is removed, the remaining interactions form a tree graph.

There are only four chiral maximum range diagrams. They all give rise to loop integrals with simple poles in $\varepsilon$ and hence contribute to the dilatation operator. They are determined as

\[ \lambda^3 I_3 \chi(1, 2, 3), \quad \lambda^3 I_{3 b} \chi(2, 1, 3), \quad \lambda^3 I_{3 b} \chi(1, 3, 2), \]

where the not-displayed fourth diagram is obtained from the first one by reflection at the vertical axis.

In addition, there are maximum range diagrams which also contain vector fields. They are given by

\[ 0 = 0, \quad 0 = 0, \quad 0 = 0, \]

\[ \lambda^3 I_3 \chi(1, 3), \quad -2\lambda^3 (I_3 + I_{32a}) \chi(1, 3), \quad -2\lambda^3 I_2 \chi(1, 3), \]

where the vanishing of the pole parts, as indicated by the first two equations, is a consequence of the finiteness conditions that are derived in Appendix B. For example, the finiteness of the first diagram follows immediately, since it matches the finiteness condition that all its vertices appear in loops. Because of this condition, we never have to consider graphs of this type and disregard them in the following.

The last diagram in (6.2) only yields higher-order poles in $\varepsilon$, since the interactions occur in disconnected subdiagrams when the composite operator is removed. Although this diagram and its reflection do not contribute to the dilatation operator, we include them for the explicit check of the cancellation of all higher-order poles in Appendix D. In addition to these diagrams, for a composite operator of length $L > 5$, there are similar diagrams which generate higher-order pole terms and have chiral functions $\chi(1, n)$, $n = 4, \ldots, \lfloor \frac{L}{2} \rfloor + 1$. We disregard them, since, to three-loop order, there are no single-pole contributions to the coefficients of these chiral functions, and their cancellation in the logarithm of the renormalization constant is straightforward.
The finiteness conditions of Appendix B restrict the possible D-algebra manipulations which lead to contributions with overall UV divergences. Starting from an initial configuration where a maximum number of covariant derivatives $D_\alpha$ appears at propagators outside loops, no further $D_\alpha$ must be brought outside the loops when performing the D-algebra manipulations. This restriction was found in [34] and it simplified significantly the calculation of the leading wrapping corrections, also in the case of single-impurity operators in the $\beta$-deformed $\mathcal{N} = 4$ SYM theory [82]. The constraint on the D-algebra manipulations implies that, at a given loop order $K$, all diagrams with range $R = K + 1$ that generate flavor interactions of lower range $\kappa < K + 1$ are finite, or their divergences cancel against each other. At three loops, we, therefore, need not consider all remaining range $R = 4$ diagrams containing one of the chiral structures $\chi(1, 2)$, $\chi(2, 1)$, or $\chi(1)$. As an example, we present in (6.3) the cancellations for some particular range $R = 4$ diagrams involving the chiral structure $\chi(1)$ and two vector fields which connect two further chiral fields to it. In the case when both these fields are not direct neighbors, the cancellations are given by

$$+ = 0, \quad + = 0.$$ (6.3)

Our analysis of the range four diagrams is now complete. Including also the reflected diagrams where necessary, the contributions of the range $R = 4$ diagrams to the three-loop renormalization constant is the negative of the sum of all overall UV divergences. We find

$$Z_{3, R=4} = -I_3(\chi(1, 2, 3) + \chi(3, 2, 1)) - I_{\bar{a}b\bar{b}}\chi(2, 1, 3) - I_{\bar{a}b}\chi(1, 3, 2)$$
$$- 2(2I_1I_2 - I_{32t})\chi(1, 3).$$ (6.4)

In order to obtain the above result, we have used the relation $KR(I_1I_2) = -I_1I_2$ for the overall divergence of a product of two integrals.

### 6.2 Next-to-maximum range diagrams

At three loops, the next-to-maximum range diagrams involve $R = 3$ neighboring fields of the composite operator: i.e., two loops contain the three propagators that originate from the composite operator, while one loop also remains in the diagram if the composite operator is removed by cutting the three connecting propagators. Since the one-loop chiral self-energy is identically zero, at least three vertices must be involved in this loop. One either obtains a box formed by only chiral field lines or a loop that obtains at least three vertices and that is built by using chiral and up to two vector fields.

The only chiral diagrams are

$$= \lambda^3I_3\chi(1, 2, 1)$$ (6.5)
and its reflection that comes with the chiral function $\chi(2, 1, 2)$. The appearing chiral functions $\chi(1, 1, 2)$ and $\chi(2, 1, 2)$ can be expressed in terms of a simpler one. Using the definition (2.2) in order to express them in terms of products of permutations, then applying the rules found in [64], we obtain the identities

$$\chi(1, 2, 1) = \chi(2, 1, 2) = \chi(1) . \quad (6.6)$$

However, in favour of a clear identification of the origin of the different contributions, we will keep the original expressions and only make the identification at the very end.

At three loops, the nonvanishing range $R = 3$ diagrams with chiral structure $\chi(1, 2)$ or $\chi(2, 1)$ contain one vector field line which is attached to the chiral lines such that the formed loop involves at least three vertices. The finiteness conditions of Appendix B thereby imply that, in order to obtain a diagram with an overall UV divergence, the chiral vertex which is not part of any loop must not become part of a loop when the vector field interaction is added. Moreover, if the vector field yields a one-loop correction (C.1) of an (anti)chiral vertex which does not lead to the cancellation of a propagator inside a loop, the respective diagram is finite. The remaining diagrams are given by

\[
\begin{align*}
\text{Diagram 1} &= -\lambda^3 I_3 \chi(1, 2) , \\
\text{Diagram 2} &= \lambda^3 I_3 \chi(1, 2) , \\
\text{Diagram 3} &= 0 , \\
\text{Diagram 4} &= \lambda^3 I_3 \chi(1, 2) , \\
\text{Diagram 5} &= -\lambda^3 I_3 \chi(1, 2) .
\end{align*}
\]

and by their reflections. It turns out that, due to the simplicity of the one-loop vertex correction (C.1) and due to the constraints on the D-algebra manipulations, the effect of the vector line is simply to add a triangle to the two-loop integral $I_2$ listed in (E.3). If the vector field interacts with one of the neighboring chiral lines of the composite operator that form the bubble in the lower-right corner of the chiral structure $\chi(1, 2)$, this bubble is removed. Moreover, after D-algebra, there remains maximally one cubic vertex which
is involved with only two lines in the loop integral. There are only two different three-loop integrals with an overall UV divergence that fulfill these restrictions and hence can be obtained after D-algebra. It can either be \( I_3 \) or \( I_{3t} \) if a bubble is, respectively, present or absent. The sign of the individual contributions is determined by the color factor. The relative factor is negative for an odd number of (anti)chiral vertices which appear in the loop that involves the vector propagator and persists when the composite operator is removed. This explains why, in (6.7), there are only five possible results for the individual diagrams. The calculation essentially becomes a simple counting of their multiplicities.

Note that, in (6.7), the contributions which yield \( I_{3t} \) cancel against each other due to a relative sign from the color factors. Since, according to (E.3), this integral has a nonrational simple pole that is proportional to \( \zeta(3) \), a nonvanishing contribution would immediately require that the function \( h^2(g) \) in the magnon dispersion relation (3.3) received a transcendentality three contribution.

The range \( R = 3 \) diagrams with chiral structure \( \chi(1) \) contain up to two flavor-neutral vector connections between \( \chi(1) \) and one of its neighboring field lines. Surprisingly, a partial evaluation of the D-algebra reveals that the overall UV divergences of the diagrams in which the vector fields interact with the additional field line of the composite operator only via cubic vertices cancel among themselves. The precise canceling combinations are

\[
\begin{align*}
\text{Diagram 1} & = 0, \\
\text{Diagram 2} & = 0, \\
\text{Diagram 3} & = 0, \\
\text{Diagram 4} & = 0, \\
\text{Diagram 5} & = 0,
\end{align*}
\]

where the equations hold up to finite parts, which do not enter the result for the dilatation operator. In order to derive the above cancellations, we have used that a loop integral with overall UV divergence can only appear if by the D-algebra manipulations, no spinor derivative \( D_\alpha \) is brought outside the loops \[33\]. This constraint is part of the finiteness conditions derived in Appendix \[3\]. As an example, we derive the last relation of (6.8) in
Appendix C.1. The cancellations in (6.8) are universal and also hold if the chiral structure of the diagram is different from $\chi(1)$, as long as the interactions of the neighboring chiral line with the remaining diagram are not altered. At higher-loop orders, the next-to-maximum range diagrams with two vector field lines involve more than a single chiral vertex. It is then possible that also chiral vertices are involved in loops, and one should try to find cancellation patterns in these cases similar to the ones in (6.8). This was not necessary here, since, at three loops, all next-to-maximum range diagrams have chiral structure $\chi(1)$ with only a single chiral vertex. The diagrams in which this vertex appears in loops are all finite, due to the finiteness conditions.

The cancellations of the overall UV divergences are not complete if two vector field connections with the chiral structure combine into a single quartic vector-matter interaction at the additional chiral field line. The relevant diagrams yield

$$Z_{3,R=3} = -\frac{\lambda^3}{2}I_3\chi(1), \quad \lambda^3 = \frac{\lambda^3}{2}I_3\chi(1), \quad -\frac{\lambda^3}{2}I_3t\chi(1) , \quad (6.9)$$

and we also have to consider their reflections. There is an (accidental) cancellation between the first two contributions in the three-loop graphs which does not seem to hold, in general, if further interactions or other chiral functions are involved at higher loops.

Considering reflected diagrams where necessary, the contribution of all range $R = 3$ diagrams to the renormalization constant is given by

$$Z_{3,R=3} = -\mathcal{I}_3(\chi(1,2,1) + \chi(2,1,2)) - 4\mathcal{I}_3(\chi(1,2) + \chi(2,1)) + \mathcal{I}_3t\chi(1) . \quad (6.10)$$

6.3 Diagrams with nearest neighbor interactions

The remaining diagrams which have to be calculated are corrections of $\chi(1)$ itself, involving no further fields of the composite operator. As mentioned before, the interactions cannot include the chiral vertex in a loop, since the respective graphs have no overall UV divergence, according to the finiteness conditions. Because of the vanishing of the one-loop self-energy, the remaining graphs can either be regarded as two-loop corrections of one of the propagators which connect $\chi(1)$ to the composite operator, or as a two-loop vertex correction of the antichiral vertex within $\chi(1)$. With the chiral self-energy and vertex correction calculated in Appendix C.2, we obtain

$$= 2\lambda^3I_{3\text{t}}\chi(1), \quad = \lambda^3(4I_3 - 3I_{3\text{t}})\chi(1) . \quad (6.11)$$

We also have to consider the reflection of the first diagram. Diagrams of the above type appear likewise in the three-loop contribution to the chiral self-energy. They are shown, respectively, in fig. 1 (f) and (b) of [60], and fig. 7 (f) and (b) of [61], and the expressions are given there in tab. 1. and 2. Apart from a different normalization factor, our results coincide with these expressions. The nearest neighbor interactions contribute to the renormalization constant as

$$Z_{3,R=2} = -(4\mathcal{I}_3 + I_{3\text{t}})\chi(1) . \quad (6.12)$$
6.4 Final result

Since the three-loop chiral self-energy is finite [58, 60, 61], according to the arguments in Section 4, no further graphs have to be considered. The complete three-loop contribution to the renormalization constant is then given by the sum of (6.4), (6.10) and (6.12) and it reads

\[ Z_3 = - \mathcal{I}_3(1, 2, 3) + \mathcal{I}_3(1, 2) - 4(\mathcal{I}_3(1, 2) + \mathcal{I}_3(1, 2)) + 4(\mathcal{I}_3(1, 2) + \mathcal{I}_3(1, 2), 1) + \mathcal{I}_3(1, 2), 1) + \mathcal{I}_3(1, 2), 1) - \mathcal{I}_3(2, 1, 3) - \mathcal{I}_3(2, 1, 3) - \mathcal{I}_3(2, 1, 3). \]  

(6.13)

We stress that the contributions with \( \mathcal{I}_3(1) \) which involve the integral \( \mathcal{I}_3 \) with a non-rational simple pole in (6.10) and (6.12) cancel, as required by the simplest form of the magnon dispersion relation (3.3) with \( h^2(g) = 4g^2 \). As an important consistency check for (6.13), we explicitly demonstrate, in Appendix D, that to three-loop order all higher-order poles cancel in the logarithm of the renormalization constant.

The three-loop dilatation operator is obtained by multiplying by 6 the \( 1/\epsilon \) pole of (6.13) after the expressions of the poles of the integrals given in (E.3) have been inserted. The result reads

\[ D_3 = - 4(\mathcal{I}_3(1, 2, 3) + \mathcal{I}_3(3, 2, 1)) + 2(\mathcal{I}_3(1, 2, 3) - \mathcal{I}_3(1, 2, 3)) - 4(\mathcal{I}_3(1, 2), 1) + \mathcal{I}_3(2, 1, 3) - \mathcal{I}_3(2, 1, 3) - \mathcal{I}_3(2, 1, 3). \]  

(6.14)

It coincides with the prediction from integrability (3.1) if we insert the identities (6.6) that replace the chiral functions \( \chi(1, 2, 1) \) and \( \chi(2, 1, 2) \) each by \( \chi(1) \). The coefficient parameterizing the similarity transformations [42, 64] is fixed to the value

\[ \epsilon_2 = - \frac{i}{2} \]  

(6.15)

in the scheme of Feynman diagrams in \( \mathcal{N} = 1 \) superspace.

7 Conclusions

In this paper we have calculated to three-loop order the dilatation operator in the flavor \( SU(2) \) subsector of \( \mathcal{N} = 4 \) SYM theory and confirmed the predictions from integrability. In particular, we have found cancellations of all contributions of transcendentality three, such that the result only contains rational numbers. Since our calculation is based on Feynman diagrams in \( \mathcal{N} = 1 \) superspace, it is a direct derivation of three-loop integrability and of the planar three-loop spectrum of the flavor \( SU(2) \) subsector from field theory. It also confirms to three-loop order the expectation that there are no corrections to the function \( h^2(g) = 4g^2 \) in the magnon dispersion relation.

From this work, we have also gained insight for future perturbative calculations at higher orders. The finiteness conditions that we have derived in Appendix D predict the finiteness of entire classes of graphs and are quite universal. They also hold, e.g., for the \( \beta \)-deformed \( \mathcal{N} = 4 \) SYM theory and for \( \mathcal{N} = 6 \) Chern-Simons theory in respective formulations in \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) superspace. Here, by making use of the finiteness conditions in the case of \( \mathcal{N} = 4 \) SYM theory, we have found universal cancellations among the overall UV divergences of graphs. They reduced the calculational effort significantly.
Our findings should be of importance for future calculations of wrapping effects for composite operators of length $L$ along the lines of \cite{32,35} beyond the presently known critical loop orders $K = L$. Starting from the $K$-loop dilatation operator, as determined from the underlying integrability, one part of the procedure is the subtraction of the contributions from all diagrams with range $R > L$. At critical order $K = L$, such contributions only stem from maximum range $R = K + 1$ diagrams that also have a chiral structure of maximum range $\kappa = K + 1$. The remaining maximum range diagrams with chiral functions of lower range $\kappa < R$ do not contribute, since their overall UV divergences cancel. In our calculation, we have used this fact at two loops in Section 5 and at three loops in Subsection 6.1. The subtraction of maximum range diagrams from the $K$-loop dilatation operator is performed simply by the omission of all terms with chiral functions of range $\kappa = K + 1$. At the next order $K = L + 1$, the contributions from all maximum and next-to-maximum range diagrams have to be identified and removed from the dilatation operator. At three loops, the next-to-maximum range diagrams show up for the first time and have been evaluated in Subsection 6.2. This analysis gives first hints of how the subtraction procedure has to be extended at the order $K = L + 1$. Because of the identities \cite{6.6}, the contribution of the chiral diagram \cite{6.5} can be captured within a coefficient of a chiral function of lower range. From this example, we conclude that contributions from diagrams of nonmaximal range $R < K + 1$ are encoded within the coefficients of chiral functions of even lower range $\kappa < R$. After making use of relations like \cite{6.6}, the simplified chiral functions no longer contain the information about the number of chiral and antichiral vertices of the underlying diagrams, and, in particular, one cannot recover whether the underlying diagrams are chiral. Relations such as \cite{6.6} only occur between certain chiral functions. They clearly complicate the subtraction procedure at orders $K \geq L + 1$, compared to the one at critical order $K = L$. But, there are also simplifications, as, e.g., the cancellations \cite{6.8} among the overall UV divergences of certain next-to-maximum range diagrams at three loops. They are universal and hold for diagrams with $R = K$ and $\kappa = R - 1$ at any loop order. At three loops, where these diagrams show up for the first time, they can only involve the simplest chiral structure $\chi(1)$ that contains a single chiral vertex. Because of the generalized finiteness conditions, this vertex has to remain outside loop in order not to yield a finite result. At higher loops, the next-to-maximum range diagrams can have more complicated chiral structures with chiral vertices involved in loops, such that the respective finiteness condition is not matched. In these cases, one should try to find universal cancellations similar to the ones in \cite{6.8}. In any case, a complete cancellation of the overall UV divergences of all diagrams with $R = K$ and $\kappa = R - 1$ is excluded, since already at three loops one finds a contribution from the diagrams in \cite{6.9}.

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A Feynman rules in $\mathcal{N} = 1$ superspace

The gauge fixed action $S = S_{\text{gauge}} + S_{\text{FP}} + S_{\text{matter}}$ of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ superspace contains one real vector superfield $V$, three chiral superfields $\phi_i$, $i = 1, 2, 3$ and ghost fields $c$ and $c'$. In the conventions of [53], it reads

$$
S_{\text{gauge}} = \frac{1}{g_{\text{YM}}^2} \left[ \frac{1}{2} \int d^4x d^2\theta \operatorname{tr} (W^\alpha W_\alpha) - \frac{1}{\alpha} \int d^4x d^4\theta \operatorname{tr} ((D^2 V)(\bar{D}^2 V)) \right],
$$

$$
S_{\text{FP}} = \int d^4x d^4\theta \left( (c' + \bar{c}) L_{\text{gymV}}(c + \bar{c} + \coth L_{\text{gymV}}(c - \bar{c})) \right),
$$

$$
S_{\text{matter}} = \int d^4x d^4\theta \left( e^{-g_{\text{YM}}V} \bar{\phi}_i e^{g_{\text{YM}}V} \phi^i \right)
+ i \frac{g_{\text{YM}}}{3!} \int d^4x d^2\theta \epsilon_{ijk} \operatorname{tr} \left( \phi^i \left[ \phi^j, \phi^k \right] \right) + i \frac{g_{\text{YM}}}{3!} \int d^4x d^2\bar{\theta} \epsilon^{ijk} \operatorname{tr} \left( \bar{\phi}_i \left[ \bar{\phi}_j, \bar{\phi}_k \right] \right),
$$

(A.1)

where $W_\alpha = i \bar{D}^2 (e^{-g_{\text{YM}}V} D_\alpha e^{g_{\text{YM}}V})$, $L_{\text{gymV}} X = [V, X]$. The fields decompose as $V = V_a T^a$, $\phi^i = \phi^i_a T^a$, $c = c_a T^a$, and $c' = c'_a T^a$, where the generators $T^a$ satisfy the $SU(N)$ algebra

$$
[T_a, T_b] = i f_{abc} T_c,
$$

(A.2)

and they are normalized as

$$
\operatorname{tr}(T_a T_b) = \delta_{ab}.
$$

(A.3)

We use the Wick rotated Feynman rules, i.e., we have transformed $e^{-iS} \rightarrow e^S$ in the path integral. In supersymmetric Fermi-Feynman gauge, where $\alpha = 1 + \mathcal{O}(g_{\text{YM}}^2)$, the vector, chiral, and ghost propagators are given by[2]

$$
\langle V_a V_b \rangle = \frac{\delta_{ab}}{p^2} \delta^4(\theta_1 - \theta_2),
$$

$$
\langle \phi^i_a \bar{\phi}_j b \rangle = \frac{\delta_{ab}}{p^2} \delta^4(\theta_1 - \theta_2),
$$

$$
\langle c'_a c_b \rangle = -\langle c'_a \bar{c}_b \rangle = \frac{\delta_{ab}}{p^2} \delta^4(\theta_1 - \theta_2).
$$

(A.4)

The cubic gauge vertex is given by

$$
V_{V^3} = \left(-\frac{S^2}{D^2} + D_s^2 + D^2 - D_s^2 + D^2 \bar{D}^2 - D_s^2 \bar{D}^2 \right) \frac{g_{\text{YM}}}{2} \operatorname{tr} \left( T^a [T^b, T^c] \right),
$$

(A.5)

where the color indices are labeled $(a, b, c)$ clockwise. The D-algebra has to be performed for all six permutations of the structure of the covariant derivatives at its legs. The only purpose of the vertices that appear on the right-hand side of the equation is to display this structure. They do not contain any other nontrivial factors.

[2] The corrections from the gauge parameter $\alpha$ do not appear in the diagrams explicitly considered in this paper.
The cubic gauge-matter vertices and gauge-ghost vertices are given by

\[ V_{\phi_1\phi_2\phi_3} = g_{YM} \delta_i^j \text{tr} \left( T^a [T^b, T^c] \right), \]

\[ V_{\phi_1\phi_2} = ig_{YM} \epsilon_{ijk} \text{tr} \left( T^a [T^b, T^c] \right), \]

\[ V_{\bar{\phi}_1\bar{\phi}_2\bar{\phi}_3} = g_{YM} \delta_i^j \text{tr} \left( T^a [T^b, T^c] \right), \]

\[ V_{\bar{\phi}_1\bar{\phi}_2} = ig_{YM} \epsilon_{ijk} \text{tr} \left( T^a [T^b, T^c] \right), \]

\[ V_{\bar{\phi}_1\bar{\phi}_2\phi_3} = g_{YM} \delta_i^j \text{tr} \left( T^a [T^b, T^c] \right), \]

\[ V_{\bar{\phi}_1\bar{\phi}_2} = ig_{YM} \epsilon_{ijk} \text{tr} \left( T^a [T^b, T^c] \right), \]

where the color indices are labeled \((a, b, c)\) clockwise, starting with the leg to the lower-left.

For the three-loop renormalization in the flavor \(SU(2)\) subsector, we only need some of the quartic vertices. They read

\[ V_{V_2\bar{\phi}_i} = \frac{1}{2} g_{YM} \delta_i^j \left( \text{tr} \left( T^a T^b T^c T^d \right) + \text{tr} \left( T^b T^a T^c T^d \right) \right), \]

\[ V_{V_2\phi_i} = \frac{1}{2} g_{YM} \delta_i^j \left( \text{tr} \left( T^a T^b T^c T^d \right) + \text{tr} \left( T^b T^a T^c T^d \right) \right), \]

\[ V_{V\bar{\phi}_i\phi_i} = -g_{YM} \delta_i^j \text{tr} \left( T^a T^b T^c T^d \right), \]

where the color indices are labeled \((a, b, c, d)\) clockwise starting with the leg in the lower left corner.

**B The power of power counting**

In this Appendix, we derive conditions for the finiteness of superfield Feynman diagrams that yield loop corrections for chiral composite operators. Compared to \([34]\), we simplify the derivation and generalize the result, predicting here the finiteness of larger classes of diagrams. The considerations are based on power counting and general arguments and hold for \(N = 4\) SYM theory and, e.g., also for its \(\beta\)-deformation, in terms of \(N = 1\) superfields in Fermi-Feynman gauge. We discuss, in parallel, the case of \(N = 6\) Chern-Simons theory in an \(N = 2\) superfield formalism \([83-85]\) in supersymmetric Landau gauge. The resulting finiteness conditions are the same, but, due to the different D-algebra structure of the propagators, their implications slightly differ from the ones in the SYM case.

In the considered theories, the order in the coupling constant \(k\) can be obtained from
the number \( v_i \) of elementary vertices with \( i \) legs as

\[ k = \sum_{i \geq 3} (i - 2)v_i \, . \]  

(A.1)

A nonamputated Feynman diagram of order \( k \), with \( l \) loops, \( c \) connected pieces, \( e \) external legs, \( p \) propagators of which \( e \) are external, \( v \) elementary vertices, and \( n_D \) and \( n_{\bar{D}} \) spinor derivatives \( D_\alpha \) and \( \bar{D}_\dot{\alpha} \) in \( N = 1 \) superspace in four dimensions or \( D_\alpha \) and \( \bar{D}_\dot{\alpha} \) in \( N = 2 \) superspace in three dimensions, obeys the following relations:

\[ c = v - p + e + l \, , \quad e = k - 2(l - c) \, , \quad n_D + n_{\bar{D}} = \begin{cases} 4v \, , & N = 4 \text{ SYM} \\ k + 4v + 2e_V \, , & N = 6 \text{ CS} \end{cases} \]  

(B.2)

where \( e_V \) is the number of external vector propagators (that contain \( D_\alpha \bar{D}_\dot{\alpha} \) in Landau gauge). The last equation reflects that, in \( N = 4 \) SYM theory, each vertex contributes exactly four spinor derivatives. This holds also for the vertices at higher order, involving increasing numbers of vector fields. In the case of \( N = 6 \) Chern-Simons theory, the last equation is obtained from the relations

\[ n_D = 6v + e \, , \quad n_{\bar{D}} = 2v + p \, , \quad n_D + n_{\bar{D}} = 2 \sum_{i \geq 3} iv_{\phi V_i} + e_v \, , \]  

(B.3)

where the last relation for \( p_V \) follows by using (A.1).

The relations (B.2) have to be modified if a composite operator is part of the Feynman diagram, but they directly hold for the subdiagram, which is obtained after cutting out the composite operator. In the flavor SU(2) subsector of \( N = 4 \) SYM and the flavor SU(2) \( \times \) SU(2) subsector of \( N = 6 \) CS theory, a respective diagram should have even \( e \) and \( e_V = 0 \), and, moreover, half of the external legs should be chiral and antichiral. The number of chiral and antichiral vertices in the subdiagram is then equal. This implies \( n_D = n_{\bar{D}} \).

As a next step, we write down the relations for the full diagram, including an operator composed of \( L \) chiral superfields, of which only \( R = \frac{e}{2} \leq L \) neighboring fields are contracted with the subdiagram. The remaining fields in the composite operator do not interact but become additional external lines. They need not be considered in the power counting for the full diagram; i.e., we can replace the operator by one which has only \( R \) legs. It contains \( 2(R - 1) \) spinor derivatives \( \bar{D} \) to impose the chirality constraint. The following relations then hold

\[ V = v + 1 \, , \quad P = p \, , \quad E = e - R \, , \quad N_D = n_D \, , \quad N_{\bar{D}} = n_D + 2(R - 1) \, , \]  

(B.4)

where the capital variables refer to the full graph with the (shortened) composite operator included.

For the determination of the superficial degree of divergence, the number of propagators which appear in loops is relevant and not the total number of propagators. We
hence first have to amputate the diagram and then also get rid of further propagators which do not appear in any loops. For this purpose, we introduce $v_0$ as the number of chiral vertices that do not appear in any loops and denote by $r_D$ and $r_{\bar{D}}$ the number of the respective spinor derivatives $D$ and $\bar{D}$ that, through the D-algebra manipulations, are brought outside of the loops. We then obtain for the number of propagators and spinor derivatives that appear in loops, the respective relations

$$P_L = p - E - v_0 \ , \quad N_{L_D} = n_D - 2v_0 - r_D \ , \quad N_{L_{\bar{D}}} = n_{\bar{D}} - 2 - r_{\bar{D}} \ , \quad (B.5)$$

where the first equality is only true for a one-particle-irreducible (1PI) connection to the composite operator. In particular, it does not hold for the diagrams that contribute to the self-energy of the external lines. The second equality relies on the fact that, at each antichiral vertex that is involved in only a single loop, one can always place two covariant derivatives $D_{\alpha}$ at the line that is not part of the loop. The last equality follows from the fact that the chiral composite operator contains only $2(R - 1)$ spinor derivatives $\bar{D}$, but the initial configuration of derivatives for the full graph can be chosen such that $2R$ of the $\bar{D}$ do not appear in loops. After D-algebra, the spinor derivatives remaining inside the loops generate the following number of spacetime derivatives, i.e., factors of the loop momenta, in the numerators of the resulting loop integrals:

$$N_{L_{\bar{D}}} = \min(N_{L_D}, N_{L_{\bar{D}}}) - 2L \ . \quad (B.6)$$

Thereby, we have taken into account that two $D$ and two $\bar{D}$ in each loop are not transformed into spacetime derivatives but are required to obtain a nonzero result \[53\]. A $K$-loop integral with $P_L$ (scalar) propagators and $N_{L_{\bar{D}}}$ momenta in the numerators is superficially UV divergent if the following relation holds:

$$DK - 2P_L + N_{L_{\bar{D}}} \geq 0 \ . \quad (B.7)$$

With $e = 2R$, $k = 2K$, $n_D = n_{\bar{D}}$, and the first two relations in (B.2), the above two equations can be combined and rephrased as

$$(D - 4)K - 2v_0 + 2v_0 + n_{\bar{D}} - \max(2v_0 + r_D, 2 + r_{\bar{D}}) \geq 0 \ . \quad (B.8)$$

Using that the value of $n_D$ is one half of the sum given in the last relation in (B.2), we find, with $D = 4$ for the $\mathcal{N} = 4$ SYM theory and $D = 3$ and $e_V = 0$ for the $\mathcal{N} = 6$ CS theory, the following necessary universal condition:

$$2v_0 \geq \max(2v_0 + r_D, 2 + r_{\bar{D}}) \quad (B.9)$$

for obtaining a loop integral with overall UV divergence at the end of the D-algebra manipulations. It constrains the parameters as

$$v_0 \geq 1 \ , \quad r_D = 0 \ , \quad r_{\bar{D}} \leq 2(v_0 - 1) \ . \quad (B.10)$$

This leads to the following finiteness conditions:

*Any Feynman diagram of $\mathcal{N} = 4$ SYM theory in $\mathcal{N} = 1$ superspace in Fermi-Feynman*
gauge or of $\mathcal{N} = 6$ CS theory in $\mathcal{N} = 2$ superspace in Landau gauge that could contribute to loop corrections of a chiral composite operator in the respective flavor $SU(2)$ or $SU(2) \times SU(2)$ subsectors and with an interaction range $R \geq 2$ has no overall UV divergence if at least one of the following criteria is matched:\footnote{\textbf{$R \geq 2$} means that the composite operator is 1PI connected with the rest of the diagram, not including the noninteracting fields of the operator. This excludes diagrams in which one or more fields of the composite operator only involve self-energy corrections.}

1. All of its chiral vertices are part of any loop.

2. One of its spinor derivatives $D$ is brought outside the loops.

3. The number of its spinor derivatives $\bar{D}$ brought outside loops becomes equal or larger than twice the number of chiral vertices that are not part of any loop.

In the flavor $SU(2)$ or $SU(2) \times SU(2)$ subsectors, a chiral vertex that is not part of any loop always generates flavor permutations and thus a nontrivial chiral structure of the diagram. The above finiteness conditions hence imply the following rule:

All diagrams with an interaction range $R \geq 2$ and trivial chiral structure $\chi()$ have no overall UV divergence.

In the case of $\mathcal{N} = 6$ CS theory, the finiteness conditions imply the following statement:

A diagram with interaction range $R \geq 2$ has no overall UV divergence if it contains at least one cubic gauge-matter interaction at which the chiral field line is not part of any loop. In particular, if, in the diagram, exactly one of the chiral vertices appears outside the loops, then it also has no overall UV divergence if the antichiral field of at least one cubic gauge-matter interaction is not part of any loop.

This statement relies on the fact that, in Landau gauge, the vector field propagators carry $D^a \bar{D}_a$. At the designated cubic gauge-matter vertices, at least one of them could only be moved outside the loops, matching at least one criterion of the previously formulated finiteness conditions.

C One- and two-loop subdiagrams

In this Appendix, we derive expressions for the one- and two-loop planar subdiagrams that appear in the three-loop calculation. We sum up all diagrams that contribute to the individual substructures and partially perform the D-algebra manipulations in order to obtain expressions that are local in the fermionic coordinates of superspace. Locality in the fermionic coordinates is displayed by filling out gray the loops of the resulting integrals over the bosonic coordinates. The prefactors of all vertices and propagators that are parts of the loops are considered in the prefactors of the final results. We omit factors of color traces that are identical to the color factors of the respective tree-level diagrams.
C.1 One-loop subdiagrams

The one-loop correction to the chiral vertex is easily evaluated as

\[ \frac{D^2}{p^2} + \cdots = \frac{D^2}{p^2} \cdot \frac{D^2}{p^2} + \cdots = i\lambda g_{YM} \epsilon_{ijk} \quad \text{(C.1)} \]

where the ellipses denote the remaining two diagrams obtained by cyclic permutations of the external legs, and \(\square\) cancels the respective propagator, thereby producing a minus.

The one-loop correction to the cubic gauge-matter vertex is given by

\[ \frac{D^2}{p^2} + \cdots = \frac{D^2}{p^2} \cdot \frac{D^2}{p^2} + \cdots = \frac{D^2}{p^2} \cdot \frac{D^2}{p^2} + \cdots = i\lambda g_{YM} \epsilon_{ijk} \quad \text{(C.2)} \]

where we have omitted the covariant derivatives. In the first diagram, we have to consider the six configurations of the covariant derivatives at the cubic gauge vertex \([A.3]\). Working out the D-algebra for them, we find

\[ \frac{D^2}{p^2} + \cdots = \left( D^\alpha \frac{D^2}{p^2} + \frac{[D_{\dot{\beta}}, D_{\alpha}]}{p^2} \right) \frac{\lambda}{2} g_{YM} \delta^i_j \quad \text{(C.3)} \]

The covariant derivatives and also momenta are read off when leaving the vertices. The above graphical representation is, therefore, translated into the following algebraic expression in the numerator of the respective loop integral:

\[ \frac{\lambda}{2} \left( D^\alpha D^3 D_{\alpha} + \epsilon^{\alpha \beta} [D_{\dot{\beta}}, D_{\alpha}] \right) V(p_1) D^2 \phi(p_2) D^2 \phi'(p_3) \quad \text{(C.4)} \]

where the covariant derivatives act to the right only on the first field that follows them, and we have suppressed the dependence on the fermionic coordinates. In order to correctly apply the procedure of \([53]\) that determines the sign coming from changing the order of the covariant derivatives, one always has to start from an expression with all the indices
in canonical order; i.e., one has to use \( l^{\alpha \beta} \) and \( l^{\bar{\beta} \alpha} \) in the expressions coming, respectively, from the first and the second term of the commutator and then apply the procedure described in [53].

The other two contributions involving only cubic vertices evaluate to

\[
\begin{align*}
\bar{D}^2 D^2 \bar{D}^2 D^2 & = \begin{pmatrix} 0 \end{pmatrix} \lambda g_{\text{YM}} \delta^i_j, \\
\bar{D}^2 D^2 \bar{D}^2 D^2 & = \begin{pmatrix} 0 \end{pmatrix} (-2\lambda) g_{\text{YM}} \delta^i_j.
\end{align*}
\]

The remaining contributions containing a quartic gauge-matter vertex are determined as

\[
\begin{align*}
\bar{D}^2 D^2 \bar{D}^2 D^2 & = \begin{pmatrix} 0 \end{pmatrix} \frac{\lambda}{2} g_{\text{YM}} \delta^i_j, \\
\bar{D}^2 D^2 \bar{D}^2 D^2 & = \begin{pmatrix} 0 \end{pmatrix} \lambda g_{\text{YM}} \delta^i_j,
\end{align*}
\]

where we have inserted \(-\frac{\Box}{p^2} = 1\) in order to obtain triangle integrals. We sum up the above expressions and simplify the numerator. The terms in which the loop momentum \( l \) is contracted with covariant derivatives combine and yield an anticommutator of these derivatives that can be replaced by the momentum \( p_1 = -p_2 - p_3 \) of the vector field they act on. After this step, the dependence on the loop momentum cancels out. The remaining terms simplify further by making use of the identities

\[
p_2^2 - p_3^2 = \frac{1}{2}(p_2 - p_3)_{\alpha \bar{\beta}} \{D_\alpha, \bar{D}_{\bar{\beta}}\}, \quad D^\alpha \bar{D}^2 D_\alpha = p_1^{\alpha \bar{\beta}} D_\alpha \bar{D}_{\bar{\beta}} + 2 D^2 \bar{D}^2,
\]

where \( p_1 \) is the momentum of the field the covariant derivatives act on. The expression for the one-loop correction of the cubic gauge-matter vertex can then be cast into the form

\[
\begin{pmatrix} 0 \end{pmatrix} \lambda g_{\text{YM}} \delta^i_j.
\]

In the following, we will show that the sum of the diagrams in the last equation of (6.8) has no overall UV divergence. The finiteness conditions of Appendix [B] guarantee
that overall UV divergences can only appear if all covariant derivatives $D_\alpha$ remain within the loops of these diagrams. By partial integration, the factor $D^2$ can then be transferred to act on the vector field of the diagrams in (6.8). Here, we evaluate the appearing substructures with such a factor $D^2$ acting on the vector field and show that their sum yields zero.

After D-algebra, we find the following results for the individual contributions that contain the cubic gauge vertex (A.5) or the one-loop correction of the cubic gauge-matter vertex (C.8):

$$
\begin{align*}
\bar{D}^2 D^2 &= \left( \begin{array}{c}
\right) \lambda \frac{1}{2} g_{YM}^2 \epsilon_{ijk} , \\
D^2 \bar{D}^2 D^2 &= \left( \begin{array}{c}
\right) \lambda \frac{1}{2} g_{YM}^2 \epsilon_{ijk} , \\
D^2 \bar{D}^2 D^2 &= \left( \begin{array}{c}
\right) (-i) \lambda \frac{1}{2} g_{YM}^2 \epsilon_{ijk} .
\end{align*}
\right)

\tag{C.9}
$$

In order to obtain the above results, we made use of the relation $D^\alpha \bar{D}^2 D_\alpha = -\Box +$
Furthermore, we consider the following contributions:

\[
\begin{align*}
D^2 \bar{D}^2 + \bar{D}^2 D^2. 
\end{align*}
\]

In the above expression, we have removed and inserted contracted propagators as \(-\frac{a}{p^2} = 1\), where \(p\) is the respective momentum. This does not affect the result and allows us to transform all integrals into box integrals with different numerators. It is then easy to see that the contributions in (C.9) and (C.10) sum up to zero. This demonstrates the cancellation of the overall UV divergences described by the last relation in (6.8).

### C.2 Two-loop subdiagrams

The finite two-loop chiral self-energy and two-loop chiral vertex correction appear as subdiagrams in three-loop diagrams. Here, we derive the results for these subdiagrams. We use that several diagrams are a priori vanishing because the one-loop self-energies of the chiral and vector fields and certain color contractions are zero.

The two-loop chiral self-energy contains the following nonvanishing contributions:

\[
\begin{align*}
S_1 &= \quad \quad = -2\lambda^2 I_{2t}, \\
S_2 &= \quad \quad = \frac{\lambda^2}{2} I_2, \quad S_3 = \quad = \lambda^2 I_2, \\
S_4 &= \quad \quad = \frac{\lambda^2}{2} I_{1t}, \quad S_5 = \quad = \lambda^2 I_{1t}, \\
S_6 &= \quad \quad = \lambda^2 (-I_1^2 - I_{2t}), \\
S_7 &= \quad \quad = -2\lambda^2 I_2, \\
S_8 &= \quad \quad = \frac{\lambda^2}{2} (-I_1^2 + 2I_2 + 2I_{2t}).
\end{align*}
\]

where we have omitted a factor \(p^{2(D-3)}\), and the covariant derivatives \(D^2\) and \(\bar{D}^2\) at the external legs after D-algebra. Expressions for the integrals are given in (E.3) and (E.4). Summing up the above contributions, thereby including also the reflected diagrams where
required, all divergences cancel out, and we find
\[ \Sigma_2 = S_1 + 4S_2 + 4S_3 + 2S_4 + 2S_5 + 2S_6 + 4S_7 + 2S_8 = -2\lambda^2 I_{2t} . \] (C.12)

Restoring the covariant derivatives and the correct proportionality to the external momentum \( p \), the two-loop chiral self-energy can be written as
\[ 2 = -2\lambda^2 p^{2(D-3)} D^2 I_{2t} . \] (C.13)

The gray scaled part of the graph is identified as the integral \( I_{2t} \) given in (E.4).

The two-loop correction of the chiral vertex is given as a sum of the following nonvanishing contributions:
\[ \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \ldots , \] (C.14)
where we have omitted the covariant derivatives. The first and the next two contributions, respectively, contain the one-loop corrections of the chiral vertex in (C.1) and of the cubic gauge-matter vertex in (C.8). The ellipsis denotes the two contributions obtained by cyclic permutations of all displayed diagrams. After D-algebra, the final result can be cast into the form
\[ 2 = \left( \begin{array}{c}
2 \quad \quad \quad \\
-\quad \quad \quad \\
- \quad \quad \quad \\
-2 \quad \quad \quad \\
+ \quad \quad \quad \\
+ \quad \quad \quad \\
+ \quad \quad \quad \\
+ \quad \quad \quad \\
+ \ldots \quad \quad \quad \\
\end{array} \right) \frac{\lambda^2}{2} \epsilon_{ijk} . \] (C.15)

\[ \lambda^2 \epsilon_{ijk} . \] (C.15)

D Cancellation of higher-order poles

In this Appendix, we demonstrate the cancellation of all higher-order poles in the logarithm of the renormalization constant. The cancellations are required in a consistent
where the ellipses denote chiral functions with range $\kappa$ in (2.1) and (2.4), to three-loop order as

$$Z = 1 + \lambda Z_1 + \lambda^2 Z_2 + \lambda^3 Z_3 + \mathcal{O}(\lambda^4) \; .$$  \hspace{1cm} (D.1)

Its logarithm then has the series expansion

$$\ln Z = \lambda Z_1 + \lambda^2 \left( Z_2 - \frac{1}{2} Z_1^2 \right) + \lambda^3 \left( Z_3 - \frac{1}{2} (Z_1 Z_2 + Z_2 Z_1) + \frac{1}{3} Z_1^3 \right) + \mathcal{O}(\lambda^4) \; ,$$  \hspace{1cm} (D.2)

where the one- and two-loop contributions to the renormalization constant were obtained in (5.1) and (5.3) and are given by

$$Z_1 = -I_1 \chi(1) \; , \quad Z_2 = -I_2 (\chi(1, 2) + \chi(2, 1) - 2\chi(1)) + I_1^2 \chi(1, 3) + \ldots \; .$$  \hspace{1cm} (D.3)

In the case of $Z_2$, we have restored one contribution that we neglected in (5.3) because it only contains a quadratic pole in $\varepsilon$ and hence does not contribute to the dilatation operator. Here, this term is required, since its multiplication by the one-loop contribution in (D.2) generates chiral functions that also appear in three-loop diagrams which have simple poles. The respective contributions hence cancel higher-order poles coming from these diagrams. The ellipsis denotes further terms that only involve quadratic poles in $\varepsilon$ and chiral functions with range $\kappa \geq 5$ that cannot appear in the three-loop dilatation operator. The higher-order poles from these contributions cancel separately in a straightforward way, and we have neglected the respective Feynman diagrams from the very beginning.

When (D.3) is inserted into (D.2), one encounters products of chiral functions. They can be expanded in terms of simple chiral functions, thereby taking care of factors of two coming from flavor contractions and minus signs from the color factors. The results read

$$\chi(1)^2 = \chi(1, 2) + \chi(2, 1) + 2\chi(1, 3) - 2\chi(1) + \ldots \; ,$$
$$\chi(1)\chi(1, 2) = \chi(1, 2, 3) - 2\chi(1, 2) + \chi(2, 1, 2) + \chi(1, 3, 2) + \ldots \; ,$$
$$\chi(1)\chi(2, 1) = \chi(1, 3, 2) + \chi(2, 1, 2) - 2\chi(1) + \chi(3, 2, 1) + \ldots \; ,$$
$$\chi(1, 2)\chi(1) = \chi(1, 2, 3) - 2\chi(1, 2) + \chi(2, 1, 2) + \chi(1, 3, 2) + \ldots \; ,$$
$$\chi(2, 1)\chi(1) = \chi(2, 1, 3) + \chi(2, 1, 2) - 2\chi(2, 1) + \chi(3, 2, 1) + \ldots \; ,$$
$$\chi(1)\chi(1, 3) = \chi(2, 1, 3) - 4\chi(1, 3) + \ldots \; ,$$
$$\chi(1, 3)\chi(1) = \chi(1, 3, 2) - 4\chi(1, 3) + \ldots \; ,$$
$$\chi(1)^3 = \chi(1, 2, 3) + \chi(3, 2, 1) + 2(\chi(1, 3, 2) + \chi(2, 1, 3)) - 12\chi(1, 3)$$
$$+ \chi(2, 1, 2) + \chi(1, 2, 1) - 4(\chi(1, 2) - \chi(2, 1)) + 4\chi(1) + \ldots \; ,$$

where the ellipses denote chiral functions with range $\kappa \geq 5$.

The above products are used to reexpress the one- and two-loop renormalization constants in (D.3) and $Z_3$ in (6.13) in a very convenient form

$$Z_1 = -I_1 \chi(1) \; ,$$
$$Z_2 = -I_2 (\chi(1)^2 + (2I_2 + I_1^2)\chi(1, 3) + \ldots \; ,$$
$$Z_3 = -I_3 \chi(1)^3 + 2(-6I_3 - 2I_2 I_1 + I_{32a})\chi(1, 3)$$
$$+ (2I_3 - I_{3bb})\chi(2, 1, 3) + (2I_3 - I_{3bb})\chi(1, 3, 2) + \ldots \; ,$$

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With these expressions, the combination appearing at two loops in (D.2) is then given by

\[ Z_2 - \frac{1}{2} Z_1^2 = -\left( I_2 + \frac{1}{2} I_1^2 \right) (\chi(1)^2 - 2\chi(1, 3)) \, . \]  

(D.6)

Inserting the explicit expressions for the poles listed in (E.3), we find that the quadratic poles cancel in the above combination. Moreover, the linear combination of chiral functions which appears on the right-hand side does not contain \( \chi(1, 3) \) after expanding the product of chiral functions as in (D.4). This is a consequence of the fact that, at two-loop order, all Feynman diagrams which generate \( \chi(1, 3) \) only have double poles.

Inserting the expressions (D.5) into the combination appearing at three loops in (D.2), we find

\[ Z_3 - \frac{1}{2}(Z_1 Z_2 + Z_2 Z_1) + \frac{1}{3} Z_1^3 \]

\[ = -\left( I_3 + I_1 I_2 + \frac{1}{3} I_1^3 \right) \chi(1)^3 - 12 \left( I_1 I_2 + \frac{1}{3} I_1^2 + I_2 I_3 \right) \chi(1, 3) \]

\[ + \frac{1}{2} \left( 4I_3 - 2I_3 - 2I_1 I_2 - 2I_2 I_3 + I_3^3 \right) (\chi(1, 3, 2) + \chi(2, 1, 3)) \]

\[ - \frac{1}{2} (I_3 - I_{12}) (\chi(1, 3, 2) - \chi(2, 1, 3)) \, . \]

(D.7)

With the explicit results for the integrals (E.3), one verifies that the coefficients of \( \chi(1)^3 \), \( \chi(1, 3) \) and of the symmetric combination \( \chi(1, 3, 2) + \chi(2, 1, 3) \) are free of cubic and quadratic poles in \( \varepsilon \) but that the coefficient of the antisymmetric combination is not free of them. This is not an inconsistency. The two chiral functions \( \chi(1, 3, 2) \) and \( \chi(2, 1, 3) \) are different but they yield the same result whenever applied to any state of the flavor \( SU(2) \) subsector. The last contribution, therefore, yields zero whenever we consider the full Feynman diagrams with the composite operator included. In fact, the coefficient of an antisymmetric combination of \( \chi(1, 3, 2) \) and \( \chi(2, 1, 3) \) is given by \( \epsilon_2 \) and it is associated to an ambiguity in fixing a scheme [42, 64]. It does not alter the anomalous dimensions, and, as can be seen from (3.1), integrability makes no prediction for it, while we have found the value given in (6.15).

### E Integrals

In \( D \)-dimensional Euclidean space, the scalar G-function is defined as

\[ G(\alpha, \beta) = \frac{\Gamma(\frac{D}{2} - \alpha) \Gamma(\frac{D}{2} - \beta) \Gamma(\alpha + \beta - \frac{D}{2})}{(4\pi)^{\frac{D}{2}} \Gamma(\alpha) \Gamma(\beta) \Gamma(D - \alpha - \beta)} \, , \]  

(E.1)

and it describes the simple loop integral that involves two propagators of massless fields with respective weights \( \alpha \) and \( \beta \) evaluated with external momentum \( p^2 = 1 \). The G-function with one momentum or, respectively, two momenta in the numerators of the integrals is defined as

\[ G_1(\alpha, \beta) = \frac{1}{2} (-G(\alpha, \beta - 1) + G(\alpha - 1, \beta) + G(\alpha, \beta)) \, , \]

\[ G_2(\alpha, \beta) = \frac{1}{2} (-G(\alpha, \beta - 1) - G(\alpha - 1, \beta) + G(\alpha, \beta)) \, . \]

(E.2)
To three-loop order, we need the following integrals and their overall UV divergences:

\[ I_1 = G(1, 1), \]
\[ I_2 = G(1, 1)G(3 - \frac{D}{2}, 1), \]
\[ I_3 = G(1, 1)G(3 - \frac{D}{2}, 1)G(5 - D, 1), \]
\[ I_{3t} = I_{2t}G(5 - D, 1), \]
\[ I_{3b} = , \]
\[ I_{3bb} = G(1, 1)^2G(3 - \frac{D}{2}, 3 - \frac{D}{2}), \]
\[ I_{32t} = G_1(2, 1)G_1(4 - \frac{D}{2}, 1)G_2(6 - \frac{D}{2}, 1), \]
\[ I_{4} = \frac{1}{2D - 4}G(1, 1)(G(1, 2) + G(3 - \frac{D}{2}, 2)) = \frac{1}{(4\pi)^2} \frac{1}{\varepsilon}, \]
\[ I_4 = \frac{1}{(4\pi)^2} \left( -\frac{1}{2\varepsilon^2} + \frac{1}{2\varepsilon} \right), \]
\[ I_3 = \frac{1}{(4\pi)^6} \left( \frac{1}{6\varepsilon^3} - \frac{1}{2\varepsilon^2} + \frac{2}{3\varepsilon} \right), \]
\[ I_{3t} = \frac{1}{(4\pi)^6} 2\zeta(3), \]
\[ I_{3b} = \frac{1}{(4\pi)^6} \left( \frac{1}{3\varepsilon^3} - \frac{1}{3\varepsilon^2} + \frac{1}{3\varepsilon} \right), \]
\[ I_{3bb} = \frac{1}{(4\pi)^6} \left( \frac{1}{3\varepsilon^3} - \frac{1}{3\varepsilon^2} - \frac{1}{3\varepsilon} \right), \]
\[ I_{32t} = \frac{1}{(4\pi)^6} \left( -\frac{1}{3\varepsilon} \right). \]

(E.3)

where the integral \( I_{3t} \) that appears as substructure in \( I_{3t} \) and in the final expression for the two-loop chiral self-energy (C.13) is finite and given by

\[ I_{2t} = \frac{2}{D - 4}G(1, 1)(G(1, 2) + G(3 - \frac{D}{2}, 2)) = \frac{1}{(4\pi)^4} 6\zeta(3) + O(\varepsilon). \]

(E.4)

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