MODEL-THEORETIC ASPECTS
OF THE GURARIJ OPERATOR SPACE

ISAAC GOLDBRING AND MARTINO LUPINI

Abstract. We show that the theory of the Gurarij operator space is the model-completion of the theory of operator spaces, it has a unique separable 1-exact model, a continuum of separable models, and no prime model. We also establish the corresponding facts for the Gurarij operator system. The proofs involve establishing that the theories of the Fra"{e}ssé limits of the classes of finite-dimensional $M_q$-spaces and $M_q$-systems are separably categorical and have quantifier-elimination. We conclude the paper by showing that no existentially closed operator system can be completely order isomorphic to a C*-algebra.

1. Introduction

The Gurarij Banach space $G$ is a Banach space first constructed by Gurarij in [13]. It has the following universal property: whenever $X \subseteq Y$ are finite-dimensional Banach spaces, $\phi : X \to G$ is a linear isometry, and $\epsilon > 0$, there is an injective linear map $\psi : Y \to G$ extending $\phi$ such that $\|\psi\|\|\psi^{-1}\| < 1 + \epsilon$. The uniqueness of such a space was first proved by Lusky in [20] and later a short proof was given by Kubis and Solecki in [15].

Model-theoretically, $G$ is a relatively nice object. Indeed, Ben Yaacov [1] showed that $G$ is the Fra"{e}ssé limit of the (Fra"{e}ssé) class of finite-dimensional Banach spaces (yielding yet another proof of the uniqueness of $G$). Moreover, Ben Yaacov and Henson [3] showed that $\text{Th}(G)$ is separably categorical and admits quantifier-elimination; since every separable Banach space embeds in $G$, it follows that $\text{Th}(G)$ is the model-completion of the theory of Banach spaces. (On the other hand, it is folklore that $\text{Th}(G)$ is unstable, so perhaps the nice model-theoretic properties of $G$ end here.)

In [21], Oikhberg introduced a noncommutative analog of $G$ which he referred to as (no surprise) a noncommutative Gurarij operator space. Here, “noncommutative” refers to the fact that we are considering operator spaces, the noncommutative analog of Banach spaces. (In Section 2, a primer on operator spaces—amongst other things—will be given.) A Gurarij operator space satisfies the noncommutative analog of the defining property of $G$ mentioned above, where the completely bounded norm replaces the usual norm of linear maps. Approximate uniqueness of a Gurarij operator space was already proved by Oikhberg in [21]. Precise uniqueness was later proved in [17] by realizing the Gurarij operator space (henceforth referred to as $NG$) as the Fra"{e}ssé limit of the class of finite-dimensional 1-exact operator spaces.

In this paper, we establish some of the basic facts about the model theory of $NG$. In analogy with $\text{Th}(G)$, we prove that $\text{Th}(NG)$ has quantifier-elimination and is the model-completion of the theory of operator spaces. However, unlike $\text{Th}(G)$, we prove that $\text{Th}(NG)$ has continuum many separable models and does not even have a prime model. In order to prove the latter result, we prove that $NG$ is the unique 1-exact model
of its theory and then combine this fact with the main result of [12], namely that the
class of 1-exact operator spaces is not an “omitting types class.”

A key tool in our arguments is to consider first the model theory of the spaces \( G_q \) as
introduced in [17]. These spaces are an intermediate generalization of \( G \) to the class of
\( M_q \)-spaces, which is in some sense a reduct of the class of operator spaces. Here, we can
mirror the commutative situation perfectly by proving that \( Th(G_q) \) is separably categori-
cal and has quantifier-elimination. (We offer two proofs of separable categoricity: one
proof proceeds directly and uses a quantitative version of the universal property of \( G_q \)
while the second uses arguments from [4] together with the Ryll-Nardzewski Theorem.)

An operator system analog of the Gurarij space, denoted by \( GS \), was introduced in
[18]. All of our results about \( NSG \) carry over to \( GS \) and we merely indicate what small
changes are needed in the preparatory results.

We conclude the paper by proving that no model of \( Th(GS) \) can be completely order
isomorphic to a C*-algebra. While this fact was proven for \( GS \) itself in [18], our proof
here is somewhat more elementary and covers all models of \( Th(GS) \).

We assume that the reader is familiar with continuous logic as it pertains to operator
algebras (see [10] for a good primer). In Section 2, we describe all of the necessary
background on operator spaces and operator systems.

2. Preliminaries

2.1. Operator spaces and \( M_q \)-spaces. If \( H \) is a Hilbert space, let \( B(H) \) denote the
space of bounded linear operators on \( H \) endowed with the pointwise linear operations
and the operator norm. One can identify \( M_n(B(H)) \) with the space \( B(H^{\otimes n}) \), where
\( H^{\otimes n} \) is the n-fold Hilbertian sum of \( H \) with itself. A (concrete) operator space is a closed
subspace of \( B(H) \). If \( X \) is an operator space, then the inclusion \( M_n(X) \subset M_n(B(H)) \)
induces a norm on \( M_n(X) \) for every \( n \in \mathbb{N} \). If \( X, Y \) are operator spaces, \( \phi : X \to Y \) is a
linear map, and \( n \in \mathbb{N} \), then the \( n \)-th amplification \( \phi^{(n)} : M_n(X) \to M_n(Y) \) is defined by

\[
[x_{ij}] \mapsto [\phi(x_{ij})].
\]

A linear map \( \phi \) is completely bounded if \( \sup_n \|\phi^{(n)}\| < +\infty \), in which case one defines the
completely bounded norm \( \|\phi\|_{cb} := \sup_n \|\phi^{(n)}\| \). We say that \( \phi \) is completely contractive
if \( \phi^{(n)} \) is contractive for every \( n \in \mathbb{N} \) and completely isometric if \( \phi^{(n)} \) is isometric for
every \( n \in \mathbb{N} \).

If \( q \in \mathbb{N}, \alpha, \beta \in M_q \), and \( x \in M_q(X) \) we denote by \( ax \beta \) the element of \( M_q(X) \)
obtained by taking the usual matrix product. The matrix norms on an operator space
satisfy the following relations, known as Ruan’s axioms: for every \( q, k \in \mathbb{N} \) and \( x \in M_q(X) \) we have

\[
\left\| \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right\|_{M_{q+k}(X)} = \|x\|_{M_q(X)},
\]

and for every \( q, n \in \mathbb{N}, \alpha_i, \beta_i \in M_q \) and \( x_i \in M_q(X) \) for \( i = 1, 2, \ldots, n \), we have

\[
\left\| \sum_{i=1}^n \alpha_i x_i \beta_i \right\| \leq \left\| \sum_{i=1}^n \alpha_i \alpha_i^* \right\| \max_{1 \leq i \leq n} \|x_i\| \left\| \sum_{i=1}^n \beta_i^* \beta_i \right\|,
\]

(2.1)

where \( \alpha_i, x_i, \beta_i \) denotes the usual matrix multiplication. Ruan’s theorem [25] asserts
that, conversely, any matricially normed vector space \( X \) with matrix norms satisfying
Ruan’s axioms is linearly completely isometric to a subspace of \( B(H) \); see also [24,
§2.2]. We will regard operator spaces as structures in the language for operator spaces
\( L_{ops} \) introduced in [11, Appendix B]. It is clear that the class of operator spaces, viewed
as \( L_{ops} \)-structures, forms an axiomatizable class by semantic considerations [10, §2.3.2],
Using Ruan’s theorem, concrete axioms for the class of operator spaces are given in [11, Theorem B.3].

A finite-dimensional operator space $X$ is said to be $1$-exact if there are natural numbers $k_n$ and linear maps $\phi_n: X \to M_{k_n}$ such that $\|\phi_n\|_{cb}\|\phi_n^{-1}\|_{cb} \to 1$ as $n \to \infty$. An arbitrary operator space is $1$-exact if all its finite-dimensional subspaces are $1$-exact. It is well known that a C*-algebra is exact if and only if it is $1$-exact when viewed as an operator space. We mention in passing that the $1$-exact operator spaces do not form an axiomatizable class, even amongst the separable ones. Indeed, this follows from two facts: 1) for $n \geq 3$, there is an $n$-dimensional operator space that is not $1$-exact; and 2) for every $n$-dimensional operator space $X$, there are $1$-exact $n$-dimensional operator spaces $X_k$ such that $X \cong \prod_{i=1}^n X_k$ (in other words, the $1$-exact $n$-dimensional operator spaces are weakly dense in the space of all $n$-dimensional operator spaces).

For $q \in \mathbb{N}$, an $M_q$-space is vector space $X$ such that $M_q(X)$ is endowed with a norm satisfying Equation (2.1) for every $n \in \mathbb{N}$, $\alpha_i, \beta_i \in M_q$, and $x_i \in M_q(X)$ for $i = 1, 2, \ldots, n$. Clearly an $M_q$-space is canonically an $M_n$-space for $n \leq q$ via the upper-left corner embedding of $M_n(X)$ into $M_q(X)$. Let $T_{M_q}$ be the reduct of the language of operator spaces where only the sorts for $M_q$-spaces are definable in the language of all $n$-dimensional operator spaces. Furthermore the functor $\text{MIN}$ completely binds, in which case $M_q$-spaces form an axiomatizable class in the language $T_{M_q}$. One can write down explicit axioms using Equation (2.1).

If $\phi: X \to Y$ is a linear map between $M_q$-spaces, then $\phi$ is said to be $q$-bounded if $\phi^{(q)}: M_q(X) \to M_q(Y)$ is bounded. In such a case one sets $\|\phi\| = \|\phi^{(q)}\|$. A linear map $\phi$ is then said to be $q$-contractive if $\phi^{(q)}$ is contractive and $q$-isometric if $\phi^{(q)}$ is isometric.

It is shown in [16, Théorème I.1.9] that any $M_q$-space can be concretely represented as a subspace of $C(K, M_q)$ for some compact Hausdorff space $K$. Here $C(K, M_q)$ is the space of continuous functions from $K$ to $M_q$ endowed with the $M_q$-space structure obtained by canonically identifying $M_q(C(K, M_q))$ with $C(K, M_q \otimes M_q)$, where the latter is endowed with the uniform norm.

An $M_q$-space $X$ admits a canonical operator space structure denoted by $\text{MIN}_q(X)$ [16, I.3]. The corresponding operator norms are defined by

$$\|x\| = \sup_{\phi} \|\phi^{(q)}(x)\|$$

for $n \in \mathbb{N}$ and $x \in M_n(X)$, where $\phi$ ranges over all $q$-contractive linear maps $\phi: X \to M_q$. The $\text{MIN}$ operator space structure on $X$ is characterized by the following property: the identity map $X \to \text{MIN}_q(X)$ is a $q$-isometry, and for any operator space $Y$ and linear map $\phi: Y \to X$, the map $\phi$ is $q$-bounded if and only if $\phi: Y \to \text{MIN}_q(X)$ is completely bounded, in which case $\|\phi: Y \to X\|_q = \|\phi: Y \to \text{MIN}_q(X)\|_q$.

We will call an operator space of the form $\text{MIN}_q(X)$ a $\text{MIN}_q$-space. It is clear that semantically there is really no difference between $M_q$-spaces and $\text{MIN}_q$-spaces. However there is a syntactical difference between these two notions as they correspond to regarding these spaces as structures in two different languages. We will therefore retain the two distinct names to avoid confusion.

It follows from the characterizing property of the functor $\text{MIN}_q$ that $\text{MIN}_q$-spaces are closed under subspaces, isomorphism, and ultraproducts. (For the latter, one needs to observe that the ultraproduct of a family of $q$-bounded maps from $X$ to $M_q$ is again a $q$-bounded map from $X$ to $M_q$. Thus, therefore, $\text{MIN}_q$-spaces form an axiomatizable class in the language of operator spaces. Furthermore the functor $\text{MIN}_q$ is an equivalence of categories from $M_q$-spaces to $\text{MIN}_q$-spaces. It follows from Beth’s definability theorem [10, S 3.4] the that the matrix norms on $M_n(X)$ for $n > q$ are definable in the language of $M_q$-spaces.
2.2. **Operator systems and \( M_q \)-systems.** Suppose that \( X \) is an operator space. An element \( u \in X \) is a **unitary** if there is a linear complete isometry \( \phi : X \rightarrow B(H) \) such that \( \phi(u) \) is the identity operator on \( H \). It is shown in [7] that if \( X \) is a C*-algebra, then this corresponds with the usual notion of unitary. Theorem 2.4 of [7] provides the following abstract characterization of unitaries: \( u \) is a unitary of \( X \) if and only if, for every \( n \in \mathbb{N} \) and \( x \in M_n(X) \), one has that

\[
\left\| \begin{bmatrix} u_n & x \\ x & \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} u_n \\ x \end{bmatrix} \right\|^2 = 1 + \|x\|^2,
\]

where \( u_n \) denotes the diagonal matrix in \( M_n(X) \) with \( u \) in the diagonal entries. A **unital operator space** is an operator space with a distinguished unitary. The abstract characterization of unitaries shows that unital operator spaces form an axiomatizable class in the language \( T_{\text{uosp}} \) obtained by adding to the language of operator spaces a constant symbol for the unit.

If \( X \) is an \( M_q \)-space, then we say that an element \( u \) of \( M_q \) is a unitary if there is a linear \( q \)-isometry \( \phi : X \rightarrow C(K, M_q) \) mapping the distinguished unitary to the function constantly equal to the identity of \( M_q \). Observe that \( u \) is a unitary of \( X \) if and only if it is a unitary of \( \text{MIN}_q(X) \). In fact, if \( u \) is a unitary of \( X \) and \( \phi : X \rightarrow C(K, M_q) \) is a unital \( q \)-isometry, then \( \phi : \text{MIN}_q(X) \rightarrow C(K, M_q) \) is a unital complete isometry. If \( \psi : C(K, M_q) \rightarrow B(H) \) is a unital complete isometry, then \( \phi \circ \psi \) witnesses the fact that \( u \) is a unitary of \( \text{MIN}_q(X) \). Conversely suppose that \( u \) is a unitary of \( \text{MIN}_q(X) \). It follows from the universal property that characterizes the injective envelope of an operator space [6, §4.3] that the injective envelope \( I(\text{MIN}_q(X)) \) is a \( \text{MIN}_q \)-space. Since the C*-envelope \( C^*_e(\text{MIN}_q(X), u) \) of the unital operator space \( \text{MIN}_q(X) \) with unit \( u \) can be realized as a subspace of \( I(\text{MIN}_q(X)) \) by [6, §4.3], it follows that \( C^*_e(\text{MIN}_q(X), u) \) is an \( M_q \)-space. Equivalently \( C^*_e(\text{MIN}_q(X), u) \) is a \( q \)-subhomogeneous C*-algebra [5, IV.1.4.1]. Therefore there is an injective unital *-homomorphism \( \psi : C^*_e(\text{MIN}_q(X), u) \rightarrow \bigoplus_{k \leq q} C(K, M_k) \) for some compact Hausdorff space \( K \); see [5, IV.1.4.3].

Moreover the proof of [7, Theorem 2.4] shows that an element \( u \) of an \( M_q \)-space \( X \) is a unitary if and only if

\[
\left\| \begin{bmatrix} u \quad x \\ x \end{bmatrix} \right\|_{M_{2q}(\text{MIN}_q(X))} = \left\| \begin{bmatrix} u \\ x \end{bmatrix} \right\|_{M_{2q}(\text{MIN}_q(X))} = 1 + \|x\|^2.
\]

A **unital \( M_q \)-space** is an \( M_q \)-space with a distinguished unitary. Let \( T_{uM_q} \) the language of \( M_q \)-spaces with an additional constant symbol for the distinguished unitary. Then the abstract characterization of unitaries in \( M_q \)-spaces provided above together with the fact that the matrix norms on \( \text{MIN}_q(X) \) are definable show that unital \( M_q \)-spaces form an axiomatizable class in the language of unital \( M_q \)-spaces.

An **operator system** is a unital operator space \((X, 1)\) such that there exists a linear complete isometry \( \phi : X \rightarrow B(H) \) with \( \phi(1) = 1 \) and \( \phi[X] \) a self-adjoint subspace of \( B(H) \). By [7, Theorem 3.4], a unital operator space is an operator system if and only if for every \( n \in \mathbb{N} \) and for every \( x \in X \) there is \( y \in Y \) such that \( \|y\| \leq \|x\| \) and

\[
\left\| \begin{bmatrix} n1 & x \\ y & n1 \end{bmatrix} \right\|^2 \leq 1 + n^2.
\]

This shows that operator systems form an axiomatizable class in the language of unital operator spaces.

The representation of \( X \) as a unital self-adjoint subspace of \( X \) induces on \( X \) an involution \( x \mapsto x^* \) and positive cones on \( M_n(X) \) for every \( n \in \mathbb{N} \). If \( X, Y \) are operator systems, then a unital linear map \( \phi : X \rightarrow Y \) is completely positive if and only if it is completely contractive, and in such a case it is necessarily self-adjoint. Therefore by
Beth’s definability theorem again, the involution and the positive cones are definable. Explicitly $x \in M_n(X)$ is positive if and only if
\[
\begin{bmatrix}
1_n & x \\
x & 1_n
\end{bmatrix}
\]
has norm at most 1 [23, Lemma 3.1]. Moreover the adjoint of $x$ is the element $y$ of $X$ that minimizes the left-hand side of Equation 2.2. An alternative axiomatization of operator systems in terms of the unit, the involution, and the positive cones is suggested in [11, Appendix B]. Since in turn the matrix norms are definable from these items, these two axiomatizations are equivalent.

Explicitly Beth’s definability theorem again, the involution and the positive cones are definable. One replaces operator spaces with operator systems, and (complete) contractions with unital (completely) positive maps. An $M_q$-system is a unital $M_q$-space $X$ such there is a unital $q$-isometry $\phi : X \rightarrow C(K,M_q)$ such that the image of $\phi$ is a self-adjoint subspace of $C(K,M_q)$. Equivalently, $X$ is an $M_q$-system if and only if $X$ is a unital $M_q$-space such that $\text{MIN}_q(X)$ is an operator system. The above axiomatizations of operator systems in the language of unital operator spaces and of unital $M_q$-spaces is a Fraïssé class in the sense of [1]. The corresponding Fraïssé limit, which we denote by $G$, is the language of unital $M_q$-spaces show that $\text{MIN}_q$-systems are axiomatizable in the language of unital $M_q$-spaces. Again Beth’s definability theorem shows that the all the matrix norms as well as the positive cones and the involution are definable.

3. The operator spaces $G_q$ and the operator systems $G_q^n$

3.1. The operator spaces $G_q$. It is shown in [17, §3] that the class of finite-dimensional $M_q$-spaces is a Fraïssé class in the sense of [1]. The corresponding Fraïssé limit, which we denoted by $G_q$, is a separable $M_q$-space that is characterized by the following property: whenever $E \subset F$ are finite-dimensional $M_q$-spaces, $f : E \rightarrow G_q$ is a linear $q$-isometry, and $\varepsilon > 0$, then there is a linear extension $g : F \rightarrow G_q$ of $f$ such that $\|g\|_q \|g^{-1}\|_q \leq 1 + \varepsilon$; see [17, Proposition 3.6].

The following amalgamation result is proved in [17, Lemma 3.1]; see also [19, Lemma 2.1].

**Lemma 3.1.** If $X \subset \hat{X}$ and $Y$ are $M_q$-spaces, and $f : X \rightarrow Y$ is a linear injective $q$-contraction such that $\|f^{-1}\|_q \leq 1 + \delta$, then there exists an $M_q$-space $Z$ and $q$-isometric linear maps $i : \hat{X} \rightarrow Z$ and $j : Y \rightarrow Z$ such that $\|i|_X - j \circ f\|_q \leq \delta$.

Arguing as in the proof of [15, Theorem 1.1], where [15, Lemma 2.1] is replaced by Lemma 3.1, shows that $G_q$ has following homogeneity property: whenever $X$ is a finite-dimensional subspace of $G_q$ and $\phi : X \rightarrow G_S$ is a linear map such that $\|\phi\|_q < 1 + \delta$ and $\|\phi^{-1}\|_q < 1 + \delta$, there exists a linear surjective $q$-isometry $\alpha : G_q \rightarrow G_q$ such that $\|\alpha|_X - \phi\|_q < \delta$.

**Proposition 3.2.** $\text{Th}(G_q)$ is separably categorical.

**Proof.** Suppose that $E \subset F$ are finite-dimensional $M_q$-spaces, where $E$ has dimension $k$ and $F$ has dimension $m > k$. Fix also a normalized basis $\bar{a} = (a_1, \ldots, a_m)$ of $F$ such that $(a_1, \ldots, a_k)$ is a basis of $E$. For $1 \leq n \leq m$ we let $X_n$ denote those $n$-tuples $(a_1, \ldots, a_n)$ from $M_q$ such that
\[
\sum_{i=1}^{n} \alpha_i \otimes e_i = 1.
\]
Note that $X_k$ is a compact subset of $M^k_q$, whence definable. We then let $\etaq{a,n}(x_1, \ldots, x_n)$ denote the formula

$$\sup_{(a_1, \ldots, a_n) \in X_n} \max \left\{ \left\| \sum_{i=1}^n a_i \otimes x_i \right\| - 1, 1 - \left\| \sum_{i=1}^n a_i \otimes x_i \right\| \right\}.$$ 

where $a \sim b$ denotes the maximum of $a - b$ and $0$. For the sake of brevity, we write $\etaq{a,n}(\vec{x})$ instead of $\etaq{a,n}(x_1, \ldots, x_n)$; no confusion should arise as the subscript indicates what the free variables are. Furthermore define $\theta(x_1, \ldots, x_k, y_1, \ldots, y_m)$ to be the formula

$$\max \left\{ \etaq{a,n}(y), \sup_{(\beta_1, \ldots, \beta_m) \in X_m} \left\| \sum_{i=1}^k \beta_i \otimes (x_i - y_i) \right\| - 2\etaq{a,k}(\vec{x}) \right\}.$$ 

We now let $\sigmaq{a,k}$ denote the sentence

$$\sup_{x_1, \ldots, x_k} \min_{y_1, \ldots, y_m} \left\{ \frac{1}{2} - \etaq{a,k}(\vec{x}), \inf_{\theta(\vec{x}, \vec{y})} \right\}$$

where $x_1, \ldots, x_k$ and $y_1, \ldots, y_m$ range in the unit ball.

**Claim 1:** $\etaq{a,k} = 0$.

**Proof of Claim 1:** Suppose that $b_1, \ldots, b_k$ are elements in the unit ball of $Gq$ such that $\etaq{a,k}(\vec{b}) < \frac{1}{2}$. Fix $\delta \in (0, \frac{1}{2}]$ such that $\etaq{a,k}(\vec{b}) < \delta$. Define the linear map $f : E \to Gq$ by $f(a_i) = b_i$ for $i \leq k$. Observe that $\|f\|_q < 1 + \delta$ and $\|f^{-1}\|_q < 1 + 2\delta$. Therefore by the above mentioned homogeneity property of $Gq$ there exists a linear $q$-isometry $q : F \to Gq$ such that $\|gq - f\|_q < 2\delta$. Let $c_i = g(a_i)$ for $1 \leq i \leq m$ and observe that $\theta(\vec{b}, \vec{c}) = 0$.

**Claim 2:** If $Z$ is a separable $M_q$-space for which $\sigmaq{a,k} = 0$ for each $k < m$ and $\vec{a}$ as above, then $Z$ is $q$-isometric to $Gq$.

**Proof of Claim 2:** Suppose that $f : E \to Z$ is a linear $q$-isometry, $\dim(E) = k$, $F$ is an $m$-dimensional $M_q$-space containing $E$, and $\varepsilon > 0$ is given. Fix a normalized basis $\vec{a} = (a_1, \ldots, a_m)$ of $F$ for which $a_1, \ldots, a_k$ is a basis of $E$, and $\eta > 0$ small enough. Set $b_i = f(a_i)$ for $i \leq k$. Since $\etaq{a,k}(\vec{b}) = 0$ and $\etaq{a,k} = 0$, there are $c_i \in Z$ for $1 \leq i \leq m$ such that $\thetaq{a,k}(\vec{b}, \vec{c}) \leq \eta$. Therefore the linear map $g : F \to Gq$ defined by $g(a_i) = c_i$ for $1 \leq i \leq m$ is such that $\|g\|_q \leq 1 + \eta$, $\|g^{-1}\|_q \leq 1 + 2\eta$, and $\|gf - f\|_q \leq 1 + \eta$. The “small perturbation argument”—see [8, Lemma 12.3.15] and also [24, §2.13]—allows one to perturb $g$ to a linear map that extends $f$ while only slightly changing the $q$-norms of $g$ and its inverse. Upon choosing $\eta$ small enough, this shows that $Z$ satisfies the approximate homogeneity property that characterizes $Gq$.

We now give an alternate proof of the preceding theorem using the Ryll-Nardzewski Theorem [2, Theorem 12.10].

**Proposition 3.3.** Suppose that $q \in \mathbb{N}$. Then the action of $\text{Aut}(Gq)$ on the unit ball $\text{Ball}(Gq)$ of $Gq$ is approximately oligomorphic.

**Proof.** Observe that the quotient space $\text{Ball}(Gq) // \text{Aut}(Gq)$ is isometric to $[0, 1]$ and hence compact. We need to show that the quotient space $\text{Ball}(Gq)^k // \text{Aut}(Gq)$ is compact for every $k \in \mathbb{N}$. This is essentially shown in [17, Proposition 3.5]. We denote by $[a_1, \ldots, a_k]$ the image of the tuple $(a_1, \ldots, a_k)$ of $\text{Ball}(Gq)^k$ in the quotient $\text{Ball}(Gq)^k // \text{Aut}(Gq)$. Suppose that $[a_1^{(n)}, \ldots, a_k^{(n)}]$ is a sequence in $\text{Ball}(Gq)^k // \text{Aut}(Gq)$. After passing to a subsequence we can assume that, for every $a_1, \ldots, a_k \in M_q$ the sequence

$$\left\| a_1 \otimes a_1^{(n)} + \cdots + a_n \otimes a_k^{(n)} \right\|$$
converges. This implies that the convergence is uniform on the unit ball of $M_q$. Suppose that $a_1, \ldots, a_k \in G_q$ are such that
\[ \| \alpha_1 \otimes a_1 + \cdots + \alpha_n \otimes a_n \| = \lim_n \left\| \alpha_1 \otimes a_1^{(n)} + \cdots + \alpha_n \otimes a_n^{(n)} \right\|. \]
Then [17, Proposition 3.4] shows that $[a_1, \ldots, a_k]$ is the limit of $(\left[ a_1^{(n)}, \ldots, a_k^{(n)} \right])_{n \in \mathbb{N}}$ in $\text{Ball}(G_q^k)$. This shows that every sequence has a convergent subsequence and hence such a space is compact. □

**Corollary 3.4.** $\text{Aut}(G_q)$ is Roelcke precompact for every $q \in \mathbb{N}$

*Proof.* It follows from [4, Theorem 2.4]. □

The Roelcke compactification of a Roelcke precompact group is described model-theoretically in [4, §2.2].

### 3.2. Quantifier-elimination.

Recall from [2, Proposition 13.2] the following test for quantifier-elimination:

**Fact 3.5.** Suppose that, whenever $M, N \models T$, $M_0, N_0$ are finitely generated substructures of $M$ and $N$ respectively, $\Phi : M_0 \to N_0$ is an isomorphism, $\varphi(\vec{x})$ is an $L$-formula, and $\vec{a} \in M_0$, we have
\[ \varphi^M(\vec{a}) = \varphi^N(\Phi(\vec{a})). \]

Then $T$ admits quantifier-elimination.

**Proposition 3.6.** $\text{Th}(G_q)$ has quantifier-elimination.

*Proof.* This follows immediately from the above quantifier-elimination test and the homogeneity and separable categoricity of $G_q$. □

### 3.3. The operator systems $G_q^n$.

It is observed in [18, §4.5] that finite-dimensional $M_q$-systems form a Fraïssé class. The corresponding limit is denoted by $G_q^n$. It is a separable $M_q$-system that is characterized by the following property: whenever $E \subset F$ are finite-dimensional $M_q$-spaces, $f : E \to G_q^n$ is a unital linear $q$-isometry, and $\varepsilon > 0$, then there is a linear extension $g : F \to G_q^n$ of $f$ such that $\|g\|_q \|g^{-1}\|_q \leq 1 + \varepsilon$; see [18, Proposition 4.9].

Arguing as in Subsection 3.1, and replacing Lemma 3.1 with [18, Proposition 4.8], yields the following homogeneity property of $G_q^n$: whenever $E \subset G_q^n$ is a finite-dimensional subsystem and $\phi : E \to G_q^n$ is a unital linear map such that $\max \{ \|\phi\|_q, \|\phi^{-1}\|_q \} < 1 + \delta \leq 2$, there exists a complete order automorphism $\alpha$ of $G_q^n$ such that $\|\alpha|_E - \phi\|_q < 100 \dim(E) \delta^2$. One can then prove similarly as above the following facts.

**Proposition 3.7.** $\text{Th}(G_q^n)$ is separably categorical

**Corollary 3.8.** $\text{Th}(G_q^n)$ has quantifier-elimination.

**Remark.** One can also prove results for $G_q^n$ analogous to Proposition 3.3 and Corollary 3.4. In this case one needs to use results from [18] and in particular [18, Lemma 3.8].
4. The Gurarij operator space $NG$ and the Gurarij system $GS$

4.1. Axioms for the Gurarij operator space.

**Lemma 4.1.** For every $k \in \mathbb{N}$ and $\varepsilon > 0$, there is $\delta = \delta(k, \varepsilon) > 0$ such that if $\phi : M_k \to B(H)$ is a unital linear map satisfying $\|\phi\|_k \leq 1 + \delta$, then $\|\phi\|_{cb} \leq 1 + \varepsilon$.

**Proof.** This follows immediately from [11, Proposition 2.39].

**Lemma 4.2.** For every $k \in \mathbb{N}$ and $\varepsilon > 0$, there is $\delta = \delta(k, \varepsilon) > 0$ such that if $E \subset M_k$ is a subsystem, $X$ is a 1-exact operator system, and $\phi : E \to X$ is a unital linear map such that $\|\phi\|_k \leq 1 + \delta$ and $\|\phi^{-1}\|_k \leq 1 + \delta$, then $\|\phi\|_{cb} \leq 1 + \varepsilon$.

**Proof.** We can assume without loss of generality that $X = GS$ by universality. Observe that by Smith’s lemma [23, Proposition 8.11] $\|\phi^{-1}\|_{cb} = \|\phi^{-1}\|_k \leq 1 + \delta$. The statement then follows from Lemma 4.1, the homogeneity property of $GS$ given by [18, Theorem 4.4], and the “small perturbations argument”.

If $X$ is an operator space, define the Paulsen system $S(X)$ as in [6, §1.3.14]. If $X, Y$ are operator spaces and $\phi : X \to Y$ is a linear map define $\tilde{\phi} : S(X) \to S(Y)$ by

$$\left[ \lambda \begin{array}{c} x \\ y^* \end{array} \right] \mapsto \left[ \lambda \begin{array}{c} \phi(x) \\ \phi(y)^* \end{array} \right].$$

**Lemma 4.3.** If $X, Y$ are operator spaces, $n \in \mathbb{N}$, and $\phi : X \to Y$ is a linear map, then $\|\tilde{\phi}\|_n \leq \|\phi\|_{2n}$.

**Proof.** Suppose that $\varepsilon > 0$ is such that $\|\phi\|_{2n} \leq 1 + \varepsilon$. Then, setting $\psi = \frac{1}{1 + \varepsilon} \phi$, we have that $\tilde{\psi}$ is 2n-positive by [6, Lemma 1.3.15] and hence $n$-contractive by [23, Proposition 3.2]. Therefore if

$$\left[ \begin{array}{c} \alpha x \\ \beta \end{array} \right] \in M_n(S(X))$$

has norm at most 1, then

$$\left\| \tilde{\phi} \begin{array}{c} \alpha x \\ \beta \end{array} \right\| = \left\| \begin{array}{c} \alpha \phi(y)^* \\ \beta \end{array} \right\| = \left\| \begin{array}{c} \alpha \psi(x) \\ \beta \end{array} \right\| - \frac{\varepsilon}{1 + \varepsilon} \left\| \begin{array}{cc} 0 & \phi(x) \\ \phi(y) & 0 \end{array} \right\| \leq 1 + \frac{\varepsilon}{1 + \varepsilon} \max \{\|\phi(x)\|, \|\phi(y)\|\} \leq 1 + \varepsilon \max \{\|x\|, \|y\|\} \leq 1 + \varepsilon \left\| \begin{array}{c} \alpha x \\ \beta \end{array} \right\| \leq 1 + \varepsilon.$$

This shows that $\|\tilde{\phi}\|_n \leq 1 + \varepsilon$.

**Corollary 4.4.** If $X$ is a 1-exact operator space, then $S(X)$ is a 1-exact operator system.

**Corollary 4.5.** If $k \in \mathbb{N}$, $E \subset M_k$, $X$ is a 1-exact operator space, and $\phi : E \to X$ is a linear map, then $\|\phi\|_{cb} = \|\phi\|_{2k}$.

**Proof.** Fix $\delta > 0$ such that $\|\phi\|_{2k} \leq 1 + \delta$. The map $\psi : E \to X \oplus^\infty M_k$ defined by

$$\psi(x) = \left( \frac{1}{1 + \delta} \phi(x), x \right)$$

...
is such that \( \|\psi\|_{2k} = \|\psi^{-1}\|_{2k} = 1 \). Therefore \( \|\tilde{\psi}_k\| = \|\tilde{\psi}^{-1}_{-k}\| \). Hence by Corollary 4.4 and Lemma 4.2 \( \tilde{\psi} \) is a complete contraction. But this implies that \( \psi \) is a complete contraction, \( \|\phi\|_{cb} \leq 1 \), and \( \|\phi\|_{cb} \leq 1 + \delta \).

Recall also that by Smith’s lemma if \( \phi : X \to M_q \) is a linear map then \( \|\phi\|_{cb} = \|\phi\|_q \) [9, Proposition 2.2.2].

**Proposition 4.6.** Suppose that \( Z \) is a separable 1-exact operator space. The following statements are equivalent:

1. \( Z \) is completely isometric to \( NG \);
2. \( Z \) is q-isometric to \( G_q \) for every \( q \in \mathbb{N} \);
3. For every \( q \in \mathbb{N} \) and \( \delta > 0 \) whenever \( E \subset M_q \) is a subspace and \( f : E \to Z \) is a linear map satisfying \( \|f\|_{2q} < 1 + \delta \) and \( \|f^{-1}\|_{2q} < 1 + \delta \), there is a 2q-isometry \( g : M_q \to Z \) such that \( \|g|_E - f\|_{2q} \leq \delta \).

**Proof.** The implication \((1) \Rightarrow (2)\) is observed in [17, Proposition 4.11]. The equivalence of \((2) \) and \((3) \) is a consequence of the characterization of \( G_q \). Finally the implication \((3) \Rightarrow (1)\) is consequence of Corollary 4.5 and the characterization of \( NG \) provided in [17, §4.5].

**Corollary 4.7.** \( NG \) is the unique separable 1-exact model of its theory.

**Proof.** Arguing as in the proof of Corollary 3.2, one can use the equivalence of \((1) \) and \((3) \) in Proposition 4.6 to write down axioms that characterize \( NG \) amongst the separable 1-exact models of its theory. Here is a softer proof: suppose that \( Z \) is a separable 1-exact model of \( \text{Th}(NG) \). By the Keisler-Shelah Theorem, there are ultrafilters \( U \) and \( V \) for which \( Z^U \) is completely isometric to \( NG^V \), whence \( (\text{MIN}_q(Z))^U \) is q-isometric to \( G_q^V \). Consequently, we see that \( \text{MIN}_q(Z) \) is elementarily equivalent to \( G_q \), whence they are q-isometric by Corollary 3.2. Thus, \( Z \) is q-isometric to \( G_q \) for every \( q \), whence, by Proposition 4.6, we have that \( Z \) is completely isometric to \( NG \).

**4.2. Model-completion of the theory of operator spaces.**

**Theorem 4.8.** \( \text{Th}(NG) \) has quantifier-elimination.

**Proof.** If \( \varphi(\vec{x}) \) is a formula in the language of operator spaces, then \( \varphi(\vec{x}) \) is a formula in the language of \( M_q \)-spaces for some \( q \). Since \( NG \) is q-isometric to \( G_q \), the result follows from the fact that \( \text{Th}(G_q) \) has quantifier-elimination.

Suppose that \( T \) is a universal theory and \( T^* \) is a theory with quantifier-elimination. We recall that \( T^* \) is said to be the model-completion of \( T \) if every model of \( T \) embeds in a model of \( T^* \) and vice-versa.

**Corollary 4.9.** \( \text{Th}(NG) \) is the model-completion of the theory of operator spaces.

**Proof.** It suffices to show that any separable operator space embeds into \( NG^U \). This is well-known but we include a proof for completeness. Suppose that \( Z \subset B(H) \) is a separable operator space, where \( H \) is the separable infinite-dimensional Hilbert space. Fix a sequence \( (p_n) \) of projections in \( H \) converging in the strong operator topology to the identity operator such that \( \text{rank}(p_n) = n \). Then the map \( x \mapsto (p_nxp_n)^\ast : Z \to \prod_{n} B(p_nHp_n) \) is a complete isometric embedding. The result follows from the fact that every \( B(p_nHp_n) \cong M_n \) admits a complete order embedding into \( NG \).

**Remark.** By Corollary 3.2 and [17, Proposition 4.11] any two separable models of \( \text{Th}(NG) \) are q-isometric for every \( q \in \mathbb{N} \). However, \( \text{Th}(NG) \) is not separably categorical. In fact, \( \text{Th}(NG) \) has a continuum of pairwise not completely isometric separable models.
To see this, suppose, towards a contradiction, that \( \kappa < \epsilon \) and \((Z_i)_{i<\kappa}\) enumerate all of the separable models of \( \text{Th}(NG) \) up to complete isometry. Let \( Z = \bigoplus_{i<\kappa} Z_i \) be the \( \infty \)-direct sum [6, §1.2.17]. If \( X \) is any separable operator space, then \( X \) embeds into some \( Z_i \) and hence embeds into \( Z \). It follows that \( Z \) is an operator space of density character \( \kappa \) that contains all separable operator spaces. This contradicts the fact that for \( n \geq 3 \) the space of \( n \)-dimensional operator spaces has density character \( \epsilon \) with respect to the completely bounded distance [24, page 20], which is the main result of [14] as formulated in [24], Corollary 21.15 and subsequent remark.

Since the theory of \( NG \) is not separably categorical, and the quotient

\[
\text{Ball}(NG)/\text{Aut}(NG) \cong [0, 1]
\]

is compact, it follows from [4, Theorem 2.4] that \( \text{Aut}(NG) \) is not Roelcke precompact. In particular \( \text{Aut}(NG) \) is not isomorphic as a Polish group to \( \text{Aut}(G_q) \) for \( q \in \mathbb{N} \).

**Problem 4.10.** Are \( \text{Aut}(G_q) \) and \( \text{Aut}(G_{q'}) \) isomorphic for \( q \neq q' \)?

### 4.3. No prime model.

In this subsection, we show that \( \text{Th}(NG) \) does not have a prime model. Besides our work from above, we will need the following result of the first-named author and Thomas Sinclair:

**Fact 4.11 ([12]).** There does not exist a family \( \Gamma_{m,n}(\vec{x}_m) \) of definable predicates in the language of operator spaces (taking only nonnegative values) for which an operator space \( A \) is 1-exact if and only if, for every \( a \in A^{|x_m|} \), we have \( \inf_n \Gamma_{m,n}(a) = 0 \).

**Theorem 4.12.** \( \text{Th}(NG) \) does not have a prime model.

**Proof.** We first observe that if \( \text{Th}(NG) \) had a prime model, then it would have to be \( NG \). Indeed, if \( Z \) is the prime model of \( \text{Th}(NG) \), then \( Z \) embeds (elementarily) into \( NG \), whence \( Z \) is 1-exact and hence completely isometric to \( NG \) by Corollary 4.7.

Suppose, towards a contradiction, that \( NG \) is the prime model of \( \text{Th}(NG) \). For each finite vector \( \vec{x} \) ranging over finite products of unit balls of sorts, let \( (b_n^\vec{x})_n \) denote a countable dense subset of \( NG_{\vec{x}} \). Let \( p_n^\vec{x} := \text{tp}(b_n^\vec{x}) \). Since \( NG \) is the prime model, each \( p_n^\vec{x} \) is isolated, so the predicate \( d(\cdot, p_n^\vec{x}) \) is a definable predicate. Since \( \text{Th}(NG) \) has quantifier elimination, we know that each \( d(\cdot, p_n^\vec{x}) \) is a quantifier-free definable predicate, meaning that it is a limit of quantifier-free formulas.

For an operator space \( E \) and \( a \in E_1^\vec{x} \), let \( \Delta^\vec{x}(a) := \inf_n d(a, p_n^\vec{x}) \). We conclude by showing that \( E \) is a 1-exact operator space if and only if \( \Delta^\vec{x}(a) = 0 \) for all \( a \in E^\vec{x} \), contradicting Fact 4.11.

First suppose that \( \Delta^\vec{x}(a) = 0 \) for all \( a \in E^\vec{x} \); we must show that \( E \) is 1-exact. Fix \( a \in E_1^\vec{x} \); it suffices to show that the operator space generated by \( a \) is 1-exact. Thus, we may assume that \( E \) is generated by \( a \). Let \( M \models \text{Th}(NG) \) contain \( E \). Then \( \text{tp}_M(a) \) is in the metric closure of the isolated types, whence is itself isolated. Since isolated types are realized in all models, there is \( b \in NG_1^\vec{x} \) such that \( \text{tp}_M(a) = \text{tp}_{NG}(b) \). It follows that \( E \) is completely isometric to the operator subspace of \( NG \) generated by \( b \) (say, by embedding \( M \) and \( NG \) elementarily into \( NG^M \) and taking an automorphism of \( NG^M \) sending \( a \) to \( b \)), whence \( E \) is 1-exact.

Conversely, suppose that \( E \) is 1-exact. Fix \( a \in E^\vec{x} \). We must show that \( \Delta^\vec{x}(a) = 0 \). Let \( E_0 \) be the finite-dimensional operator subspace of \( E \) generated by the elements appearing in the various matrices in the elements of \( a \). Since \( E_0 \) completely isometrically embeds in \( NG \), we know that, for any \( \epsilon > 0 \), there is \( b_n^\vec{x} \) such that \( d(a, b_n^\vec{x}) < \epsilon \), whence \( d(a, p_n^\vec{x}) < \epsilon \) and hence \( \Delta^\vec{x}(a) < \epsilon \).

### 4.4. The Gurarij system \( GS \).

Here we state the analogous results for \( GS \). These can be obtained as above, using Subsection 3.3, and the analog of Fact 4.11 for operator systems, which is also proved in [12].
Proposition 4.13. Suppose that $Z$ is a separable 1-exact operator system. The following statements are equivalent:

1. $Z$ is completely order isomorphic to $\mathcal{GS}$;
2. $Z$ is unitally $q$-isometric to $\mathcal{G}_q$ for every $q \in \mathbb{N}$;
3. For every $q \in \mathbb{N}$ and and $\delta > 0$, whenever $E \subset M_q$, and $f : E \to Z$ is a unital linear map such that $\|f\|_q < 1 + \delta$ and $\|f^{-1}\|_q < 1 + \delta$ there is a unital $2q$-isometry $g : M_q \to Z$ such that $\|g|_E - f\|_{2q} < 1 + \delta$.

Corollary 4.14. $\mathcal{GS}$ is the unique separable 1-exact model of its theory.

Corollary 4.15. $\text{Th}(\mathcal{GS})$ has quantifier-elimination and is the model-completion of the theory of operator systems. It has a continuum of separable models and does not admit a prime model.

4.5. A caveat: existentially closed C*-algebras. We recall that the Kirchberg embedding problem (KEP) asks whether every separable C*-algebra embeds into an ultra-power of the Cuntz algebra $\mathcal{O}_2$. In [11], it is proven that the KEP has a positive solution if and only if $\mathcal{O}_2$ is existentially closed in the language of unital C*-algebras.

At first glance, we (mistakenly) thought that the results of the previous subsection could be used to give a negative answer to the KEP. Indeed, suppose that $\mathcal{O}_2$ is existentially closed as a C*-algebra. Then $\mathcal{O}_2$ is also existentially closed as an operator system, whence $\mathcal{O}_2 \equiv \mathcal{GS}$. Since $\mathcal{GS}$ is the unique 1-exact model of its theory, we conclude that $\mathcal{O}_2 \equiv \mathcal{GS}$ as an operator system. However, it is proven in [18] that $\mathcal{GS}$ is not completely order isomorphic to a C*-algebra, yielding a contradiction.

The gap in the above argument is that the statement “$\mathcal{O}_2$ is existentially closed as a C*-algebra” implies the statement “$\mathcal{O}_2$ is existentially closed as an operator system.” In fact, we now show that a (unital) C*-algebra is never existentially closed as an operator system.

Lemma 4.16. Suppose that $\phi : X \to Y$ is a complete order embedding between operator systems. Further suppose that $X$ is existentially closed and $u \in X$ is a unitary. Then $\phi(u)$ is a unitary.

Proof. Suppose that $n \in \mathbb{N}$ and consider the formula $\varphi(u, x)$ defined by

$$\min \left\{ \| [u \otimes I_n, x] \|^2, \| [u \otimes I_n]_x \|^2 \right\} - \|x\|^2.$$

Observe that

$$\left( \inf_{\|x\| \leq 1} \varphi(u, x) \right)^X = 2$$

by [7, Theorem 2.4]. Therefore

$$\left( \inf_{\|x\| = 1} \varphi(\phi(u), x) \right)^Y = 2,$$

whence $\phi(u)$ is a unitary of $Y$. □

A first draft of this paper contained a proof of the next lemma. We thank Thomas Sinclair for pointing out to us that this lemma follows immediately from Pisier’s Linearization Trick (see, for example, [22, Theorem 19]).

Lemma 4.17. Suppose that $\phi : A \to B$ is a ucp map between unital C*-algebras that maps unitaries to unitaries. Then $\phi$ is a $*$-homomorphism.

We thank Thomas Sinclair for providing a proof for the following lemma.
Lemma 4.18. Suppose that $A$ is a unital $C^*$-algebra and $\dim(A) > 1$. Then there is a unital $C^*$-algebra $B$ and a complete order embedding $\phi : A \to B$ that is not a $*$-homomorphism.

Proof. We first remark that $A$ has a nonpure state. Indeed, since the states separate points and every state is a linear combination of pure states, we have that the pure states separate points. Since $\dim(A) > 1$, this implies that there are at least two pure states, whence any proper convex combination of these two pure states is nonpure.

Secondly, we remark that a nonpure state on $A$ is not multiplicative. Indeed, if $\phi$ is a proper convex combination of the distinct pure states $\phi_1$ and $\phi_2$, then taking a unitary $u$ on which $\phi_1$ and $\phi_2$ differ, we have that $\phi(u)$ has modulus strictly smaller than 1.

We are now ready to prove the lemma. Suppose that $A$ is concretely represented as a subalgebra of $B(H)$. Let $\phi$ be a non-pure state. Then the map $x \mapsto (\phi(x) \cdot 1) \oplus x : A \to B(H \oplus H)$ is a complete order embedding that is not a $*$-homomorphism. □

Corollary 4.19. No $C^*$-algebra is existentially closed as an operator system.

Proof. This follows immediately from Lemmas 4.16, 4.17, and 4.18 (noting that all models of $\text{Th}(\mathbb{G}S)$ are infinite-dimensional). □

Remark. As mentioned earlier, it was proven by the second-named author in [18, §4.6] that $\mathbb{G}S$ is not completely order isomorphic to a $C^*$-algebra. Corollary 4.19 provides a new proof of this fact and establishes the same fact for the other models of $\text{Th}(\mathbb{G}S)$.

Remark. Lemma 4.16 remains valid in the operator space category as well (with an identical proof). As a consequence, we see that if $Z$ is an existentially closed operator space, then $Z$ has no unitaries. Indeed, if $Z$ is concretely represented as a subspace of $B(H)$, then the map $x \mapsto x \oplus 0 : Z \to B(H \oplus H)$ is a complete isometric embedding into a $C^*$-algebra whose image contains no unitaries, whence, by Lemma 1, $Z$ cannot contain any unitaries. In particular, we see that $\mathbb{N}G$ contains no unitaries, a fact already observed (implicitly) in [21, Proposition 3.2].

Remark. At the beginning of this subsection, we proved that $\mathcal{O}_2$ cannot be existentially closed as an operator system. We can be a bit more precise about how $\mathcal{O}_2$ fails to be existentially closed as an operator system. Indeed, since $\mathcal{O}_2$ is exact, by universality, there is a complete order embedding $\mathcal{O}_2 \hookrightarrow \mathbb{G}S$. We claim that this embedding is not existential. Indeed, since $\mathbb{G}S$ is existentially closed, if the above embedding were existential, then $\mathcal{O}_2$ would be existentially closed as an operator system, yielding the same contradiction as in the beginning of this subsection. The same argument shows that if $A$ is any separable exact $C^*$-algebra, then the embedding of $A$ into $\mathbb{G}S$ as an operator system is not existential.

Given the above discussion, the following question seems natural:

Question 4.20. Is the class of operator systems unitally completely order isomorphic to a $C^*$-algebra an elementary class?

We now give a condition that would ensure a positive answer to Question 4.20. Suppose that $(X_i : i \in I)$ is a family of operator systems and $\mathcal{U}$ is an ultrafilter on $I$. If $u_i \in X_i$ is a unitary for each $i$, then it is clear that $(u_i)^* \in \prod_{\mathcal{U}} X_i$ is a unitary of $\prod_{\mathcal{U}} X_i$.

Question 4.21. With the preceding notation, if $u$ is a unitary in $\prod_{\mathcal{U}} X_i$, are there unitaries $u_i \in X_i$ for which $u = (u_i)^*$?
We should note that the analog of Question 4.21 for C*-algebras has a positive answer (see [10]).

**Proposition 4.22.** If Question 4.21 has a positive answer, then Question 4.20 has a positive answer.

**Proof.** Clearly the class of operator systems completely order isomorphic to a C*-algebra is closed under isomorphisms and ultraproducts. It suffices to check that it is closed under ultraroots. Towards this end, suppose that $X$ is an operator system for which $X^{\mathcal{U}}$ is a C*-algebra; we need to show that $X$ is a C*-algebra. It suffices to show that $X$ is closed under multiplication. We first show that the product of any two unitaries in $X$ remains in $X$. Suppose that $u, v \in X$ are unitaries. By [7], $uv \in X$ if and only if the matrix $\begin{bmatrix} 1 & u \\ v & x \end{bmatrix}$ is $\sqrt{2}$ times a unitary of $M_2(X)$. However, the aforementioned matrix is $\sqrt{2}$ times a unitary $A$ of $M_2(X^{\mathcal{U}})$; by assumption, $A = (A_n)^*$, where each $A_n$ is a unitary in $M_2(X)$. Since unitaries in an operator space form a closed set, we have the desired conclusion.

In order to finish the proof, it suffices to prove that the linear span of the unitaries in $X$ are dense in $X$. Towards this end, fix $x \in X$ with $\|x\| \leq \frac{1}{2}$. By [5, §II.3.2.16], there are unitaries $u_1, \ldots, u_5 \in X^{\mathcal{U}}$ for which $x = \frac{1}{5}(u_1 + \cdots + u_5)$. By assumption, we may write each $u_i = (u_i^n)^*$, where each $u_i^n$ is a unitary of $X$. It follows that some subsequence of $\left(\frac{1}{5}(u_1^n + \cdots + u_5^n)\right)$ converges to $x$. □

**References**

1. Itai Ben Yaacov, *Fraissé limits of metric structures*, Journal of Symbolic Logic, to appear.
2. Itai Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with applications to algebra and analysis, Vol. 2, London Mathematical Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, p. 315–427.
3. Itai Ben Yaacov and C. Ward Henson, *Generic orbits and type isolation in the Gurarij space*, arXiv:1211.4814 (2012), arXiv: 1211.4814.
4. Itai Ben Yaacov and Todor Tsankov, *Weakly almost periodic functions, model-theoretic stability, and minimality of topological groups*, arXiv:1312.7757 (2013).
5. Bruce Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006.
6. David P. Blecher and Christian Le Merdy, *Operator algebras and their modules—an operator space approach*, London Mathematical Society Monographs. New Series, vol. 30, Oxford University Press, Oxford, 2004.
7. David P. Blecher and Matthew Neal, *Metric characterizations of isometries and of unital operator spaces and systems*, Proceedings of the American Mathematical Society 139 (2011), no. 3, 985–998.
8. Nathanial P. Brown and Narutaka Ozawa, *C*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, 2008.
9. Edward G. Effros and Zhong-Jin Ruan, *Operator spaces*, London Mathematical Society Monographs. New Series, vol. 23, Oxford University Press, 2000.
10. Ilijas Farah, Bradd Hart, Martino Lupini, Leonel Robert, Aaron P. Tikuisis, Alessandro Vignati, and Wilhelm Winter, *Model theory of nuclear C*-algebras*, In preparation.
11. Isaac Goldbring and Thomas Sinclair, *On Kirchberg’s embedding problem*, Journal of Functional Analysis, to appear.
12. Omitting types in operator systems, arXiv:1501.06395 (2015).
13. Vladimir I. Gurari˘ı, Spaces of universal placement, isotropic spaces and a problem of Mazur on rotations of Banach spaces, Siberian Mathematical Journal 7 (1966), 1002–1013.
14. Marius Junge and Gilles Pisier, Bilinear forms on exact operator spaces and $B(H) \otimes B(H)$, Geometric and Functional Analysis 5 (1995), no. 2, 329–363.
15. Wiesław Kubis and Sławomir Solecki, A proof of uniqueness of the Gurari˘ı space, Israel Journal of Mathematics 195 (2013), no. 1, 449–456.
16. Franz Lehner, $M_n$-espaces, sommes d’unitaires et analyse harmonique sur le groupe libre, Ph.D. thesis, Université de Paris 6, 1997.
17. Martino Lupini, Uniqueness, universality, and homogeneity of the noncommutative Gurarij space, arXiv:1410.3345 (2014).
18. A universal nuclear operator system, arXiv:1412.0281 (2014).
19. Operator space and operator system analogs of Kirchberg’s nuclear embedding theorem, arXiv:1502.05966 (2015).
20. Wolfgang Lusky, The Gurarij spaces are unique, Archiv der Mathematik 27 (1976), no. 6, 627–635.
21. Timur Oikhberg, The non-commutative Gurarii space, Archiv der Mathematik 86 (2006), no. 4, 356–364.
22. Narutaka Ozawa, About the Connes embedding conjecture: algebraic approaches, Japanese Journal of Mathematics 8 (2013), no. 1, 147–183.
23. Vern Paulsen, Completely bounded maps and operator algebras, Cambridge Studies in Advanced Mathematics, vol. 78, Cambridge University Press, Cambridge, 2002.
24. Gilles Pisier, Introduction to operator space theory, London Mathematical Society Lecture Note Series, vol. 294, Cambridge University Press, Cambridge, 2003.
25. Zhong-Jin Ruan, Subspaces of C*-algebras, Journal of Functional Analysis 76 (1988), no. 1, 217–230.
26. Blerina Xhabli, The super operator system structures and their applications in quantum entanglement theory, Journal of Functional Analysis 262 (2012), no. 4, 1466–1497.