Almost sure local limit theorem for the Dickman distribution

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Abstract We study the asymptotic behavior, and more precisely the second order properties, of the probabilistic model introduced in Hwang and Tsai (Comb Probab Comput 11(4):353–371, 2002) for describing the Dickman distribution. This model appears as an extremal example in the theory of the local and almost sure local limit theorem. We establish a delicate correlation inequality for this system. We apply it to obtain a fine almost sure local limit theorem. In doing so, we also give a corrected proof of the corresponding local limit theorem stated in Hwang and Tsai (Comb Probab Comput 11(4):353–371, 2002).

Keywords General almost sure limit theorem · Almost sure local limit theorem · Dickman function · Dickman distribution · Characteristic function · Correlation inequality · Cumulants · Poisson distribution · Bernoulli distribution

1 Introduction and main results

In this work we study the asymptotic behavior, and more precisely the correlation properties of the probabilistic model introduced in Hwang and Tsai [15] for describing the Dickman function. We prove a rather delicate correlation inequality for this system. We next apply it to establish a fine almost sure local limit theorem. In doing so, we also give a corrected
proof of the corresponding local limit theorem stated in [15]. We first briefly recall some basic facts concerning the Dickman function, which originates from the study by Dickman of the asymptotic distribution of the largest prime factor $P^+(n)$ of a natural integer $n$. He has shown that the limit

$$\lim_{n \to \infty} \frac{1}{n} \# \{ k ; 1 \leq k \leq n : P^+(k) \leq n^{1/u} \} = \varrho(u)$$

(1.1)

exists. This limit, called the Dickman function, is the continuous solution of the differential-difference equation $u \varrho'(u) + \varrho(u - 1) = 0$, $u > 1$ with the initial condition $\varrho(u) = 1$ for $0 \leq u \leq 1$. It is a function of first importance in analytic number theory, and also a key tool in the search of sharp estimates of the distribution of $P^+(n)$. See notably Hildebrand [13] and Tenenbaum [22].

Following [15], let $(Z_k)_{k \geq 1}$ be independent random variables defined by

$$Z_k = \begin{cases} 1 & \text{with probability } 1/k \\ 0 & \text{with probability } 1 - 1/k. \end{cases}$$

(1.2)

Put for all integers $m, n$ with $0 \leq m < n$, $T_m^n = \sum_{k=m+1}^{n} k Z_k$, and set $T_n = T_0^n$. Then

$$\lim_{n \to \infty} P \left\{ n^{-1} T_n < x \right\} = e^{-\gamma} \int_0^x \varrho(v)dv, \quad (x > 0)$$

where $\gamma$ is the Euler–Mascheroni constant. As is well-known $\int_0^\infty \varrho(v)dv = e^\gamma$, see ([22] §III.5.4). The Dickman distribution (denoted throughout by $D$) is the distribution function with density $e^{-\gamma \varrho(x)}$, $x \geq 0$. It is known that $D$ is infinitely divisible, see Hensley [14].

Apart from its obvious link with number theory, this model also appears as a borderline case in the theory of the local and almost sure local limit theorem. In [8–12] and [23], precise and general results of this kind, and sometimes optimal, were recently obtained by the authors. This work is thus also continuing the investigations previously made, by studying this time an extremal case. We recall in effect that an important problem inside this theory concerns the study of the local and almost sure local limit theorem for weighted sums of Bernoulli variables. The “simple” case when the weights are increasing is, to say the least, far from being understood. This case is frequently encountered in random models used in number theory. In addition, the well-known Bernoulli part extraction method used for proving local limit theorems becomes ineffective in this case too. See Giuliano and Weber [11] for more details, see also [10, 12].

We now state our main results.

**Theorem 1.1** (Correlation inequality) Let $\kappa = (\kappa_n)_{n \geq 1}$ be a sequence of integers such that

$$\lim_{n \to \infty} \frac{\kappa_n}{n} = x > 0.$$ 

Let $Y_n = n 1_{\{T_n = \kappa_n\}}$, $n \geq 1$. Then there exists a positive constant $C$, such that for any $n > m \geq 2$,

$$|Cov(Y_m, Y_n)| \leq C \left\{ \frac{n}{n-m} \chi_{m,n}^{(\kappa,x)} + \frac{m}{n-m} + \chi_{2,n}^{(\kappa,x)} + \frac{1}{n} \right\}.$$ 

where

$$\chi_{m,n}^{(\kappa,x)} = \frac{n-m}{\kappa_n - \kappa_m} \cdot \log \frac{n}{\sqrt{n-m}} + \frac{n-m}{\kappa_n - \kappa_m} \cdot g_{m,n} + x \left\{ \frac{n-m}{\kappa_n - \kappa_m} - \frac{1}{x} \right\} + \frac{m+1}{\kappa_n - \kappa_m}.$$
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\[ g_{m,n} = \exp \left( C \left\{ \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m+2}{n-m} \right\} \log^2 \frac{n}{m} \right) - 1 + \frac{1}{\log \frac{n}{m}}. \]

**Theorem 1.2** (Almost sure local limit theorem) Let \( \kappa = (\kappa_n)_{n \geq 1} \) be a strictly increasing sequence such that \( \lim_{n \to \infty} \frac{\kappa_n}{n} = x > 0 \). Then we have, recalling that \( \varrho(x) = 1 \) if \( 0 \leq x \leq 1 \),

\[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} 1 \{T_n = \kappa_n\} = e^{-\gamma} \varrho(x), \quad a.s.
\]

**Corollary 1.3** We have

\[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} 1 \{T_n = n\} = e^{-\gamma}, \quad a.s.
\]

As a consequence, for every \( x \geq 1 \),

\[
\lim_{N \to \infty} \frac{\sum_{n=1}^{N} 1 \{T_n = \lfloor x_n \rfloor\}}{\sum_{n=1}^{N} 1 \{T_n = n\}} = \varrho(x), \quad a.s.
\]

Recall that the characteristic function of the Dickman distribution satisfies

\[ \phi(t) = \exp \left\{ \int_0^1 \frac{e^{itu} - 1}{u} \right\} du. \] (1.3)

See [13] or [15]. Consider also the characteristic function of \( Z_k \) and \( T_m^n \) defined in (1.2) and right after. We have

\[
\phi_{Z_k}(t) = 1 + \frac{e^{it} - 1}{k}
\]

\[
\phi_{T_m^n}(t) = \prod_{k=m+1}^{n} \phi_{Z_k}(tk) = \prod_{k=m+1}^{n} \left( 1 + \frac{e^{itk} - 1}{k} \right).
\]

The following result will be crucial in the proof of Proposition 2.3, which together with the above Correlation inequality will lead to the proof of Theorem 1.2.

**Proposition 1.4** We have

(a) \( \int_{-\infty}^{+\infty} |\phi(t)|^2 \, dt < \infty \),

(b) \( \int_{-\pi n}^{\pi n} \left| \phi_{T_m^n}(u) \right|^2 \, du \to \int_{-\infty}^{\infty} |\phi(u)|^2 \, du \), as \( n \to \infty \).

Another consequence of Proposition 1.4 is the following local limit theorem (for a stronger result with symmetrized Dickman density see Remark 3.1).

**Corollary 1.5** (Local limit theorem) Let \( \kappa = (\kappa_n)_{n \geq 1} \) be a sequence of integers such that \( \lim_{n \to \infty} \frac{\kappa_n}{n} = x > 0 \). Then

\[
\lim_{n \to \infty} n P(T_n = \kappa_n) = e^{-\gamma} \varrho(x).
\]

We will also prove a strong form of the local limit theorem for the Dickman density.
**Theorem 1.6** (Strong local limit theorem)

\[ \sum_{\kappa \in \mathbb{N}} |P(T_n = \kappa) - n^{-1} e^{-\gamma} \varphi(n^{-1} \kappa)| \to 0, \quad n \to \infty. \tag{1.4} \]

The paper is organized as follows. Corollary 1.5 and Theorem 1.2 are respectively proved in Sects. 2 and 6. In Sect. 2, Proposition 2.3 is proved. We also discuss two important estimations (Propositions 2.8 and 2.9) that will be used in the proof of the correlation inequality in Sect. 4. Proposition 1.4 and Theorem 1.6 are proved in Sect. 3. In Sect. 5 we establish a new general form of the almost sure theorem which generalizes a previous result by Mori (see Theorem 1 in [17]).

**Notation** We denote by \( C \) a positive constant possibly depending on some parameters and which may change of value at each occurrence.

## 2 Preliminaries

First note that

\[ D(x) - D(x - 1) = e^{-\gamma} x \varphi(x) = x D'(x), \quad x \geq 1. \tag{2.1} \]

In fact, denoting provisorily \( f(x) = D(x) - D(x - 1) \) and \( e^{-\gamma} x \varphi(x) = g(x), \quad x \geq 1, \) we have, by the very definition of \( D, \) see Sect. 1,

\[ f(1) = \int_0^1 D'(t) \, dt = \int_0^1 e^{-\gamma} \varphi(t) \, dt = e^{-\gamma} = e^{-\gamma} \varphi(1) = g(1); \]

and

\[ f'(x) = \frac{d}{dx} \left( \int_{x-1}^{x} D'(t) \, dt \right) = \frac{d}{dx} \left( \int_{x-1}^{x} e^{-\gamma} \varphi(t) \, dt \right) = e^{-\gamma} \varphi(x) - e^{-\gamma} \varphi(x - 1) \]

\[ = e^{-\gamma} (\varphi(x) + x \varphi'(x)) = \frac{d}{dx} (e^{-\gamma} x \varphi(x)) = g'(x), \quad x > 1. \]

The following result is an extension of the one given in [15] (Proposition 2.1) for the case \( m_n \equiv 0. \)

**Proposition 2.1** Let \((m_n)_{n \geq 1}\) be a sequence of integers such that \( \lim_{n \to \infty} (n - m_n) = +\infty \) and \( m_n = o(n) \). Then, as \( n \to \infty \), the sequence \( \frac{T_{m_n}}{n-m_n} \) converges in distribution to the Dickman law.

**Proof** By the Markov inequality, for every \( \delta > 0 \)

\[ P \left( \sum_{k=1}^{m_n} kZ_k \geq \delta(n - m_n) \right) \leq \frac{E \left[ \sum_{k=1}^{m_n} kZ_k \right]}{\delta(n - m_n)} \leq \frac{m_n}{\delta(n - m_n)} \to 0, \quad n \to \infty. \]

This implies that

\[ \frac{T_n - T_{m_n}}{n-m_n} = \frac{\sum_{k=1}^{m_n} kZ_k}{n-m_n} \to 0 \text{ in probability.} \]

Since \( \frac{T_n}{n} \) converges in distribution to the Dickman law by Proposition 2.1 of [15], the result follows. \( \Box \)
Remark 2.2 Notice that $m_n = o(n)$ is the only case when one can have convergence towards the Dickman distribution: if $m_n$ is not $o(n)$, then there exists a positive number $a \in (0, 1)$ such that, for a suitable subsequence of integers $(k_v)_{v \geq 1}$ with $\lim_{v \to \infty} k_v = \infty$ we have

$$m_{n_{k_v}} > an_{k_v}.$$ 

We write $n$ in place of $n_{k_v}$ for simplicity and we obtain from the above

$$P \left( T_{m_n}^n = 0 \right) = P \left( \sum_{k=m_n+1}^{n} Z_k = 0 \right) = \prod_{k=m_n+1}^{n} \left( 1 - \frac{1}{k} \right) \geq \left( 1 - \frac{1}{an} \right)^{(1-\alpha)n} \to e^{-\frac{1}{\alpha}} > 0$$

This means that in general $P \left( T_{m_n}^n = 0 \right)$ is bounded away from zero. In particular, if $m_n = n\gamma_n$ with $\gamma_n \uparrow 1$, then by using a similar argument one can show that,

$$P \left( T_{m_n}^n = 0 \right) = \prod_{k=m_n+1}^{n} \left( 1 - \frac{1}{k} \right) \geq \left( 1 - \frac{1}{n\gamma_n} \right)^{(1-\gamma_n)n} \to 1.$$ 

The next result will be crucial for the proof of the correlation inequality (Sect. 4).

**Proposition 2.3** Let $\kappa = (\kappa_n)_n$ be an increasing sequence of integers. Then, for $n > m \geq 2$,

$$\left| (\kappa_n - \kappa_m) P \left( T_m^n = \kappa_n - \kappa_m \right) - P \left( (\kappa_n - \kappa_m) - n < T_m^n \leq (\kappa_n - \kappa_m) - (m + 1) \right) \right| \leq C \frac{1 + \log \frac{n}{m}}{\sqrt{n-m}}.$$ 

We need a preliminary easy result.

**Lemma 2.4** Let $T$ be a random variable taking integer values and with characteristic function $\phi_T$. For every integers $\kappa$, $a$ and $b$ with $a < b$ we have the formula

$$P(\kappa - b \leq T \leq \kappa - a) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it\kappa} \left( \sum_{j=a}^{b} e^{itj} \right) \phi_T(t) \, dt.$$ 

**Proof of Lemma 2.4** Using Fourier inversion formula we can write

$$P(\kappa - b \leq T \leq \kappa - a) = \sum_{j=\kappa-b}^{\kappa-a} P(T = j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{j=\kappa-b}^{\kappa-a} e^{-itj} \right) \phi_T(t) \, dt.$$ 

Now for any real $t$,

$$\sum_{j=\kappa-b}^{\kappa-a} e^{-itj} = e^{-it(k-b)} + e^{-it(k-b+1)} + \ldots + e^{-it(k-a)} = e^{-itk} \sum_{j=a}^{b} e^{itj}.$$ 

□

**Proof of Proposition 2.3** First, by Lemma 2.4

$$P \left( (\kappa_n - \kappa_m) - n < T_m^n \leq (\kappa_n - \kappa_m) - (m + 1) \right) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \left( \sum_{k=m+1}^{n} e^{iku} \phi_{T_m}(u) \right) du.$$ 

(2.2)
Next by integrating by parts,
\[
\frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u) \, du \\
= \frac{1}{2\pi i} \left\{ \phi_{T_m}^{(n)}(\pi) e^{-i\pi(\kappa_n - \kappa_m)} - \phi_{T_m}^{(n)}(-\pi) e^{i\pi(\kappa_n - \kappa_m)} \right\} \\
+ i(\kappa_n - \kappa_m) \int_{-\pi}^{\pi} \phi_{T_m}^{(n)}(u) e^{-i(\kappa_n - \kappa_m)u} \, du \\
= \frac{1}{2\pi i} \left\{ \left( \phi_{T_m}^{(n)}(\pi) e^{-i\pi(\kappa_n - \kappa_m)} - \phi_{T_m}^{(n)}(-\pi) e^{i\pi(\kappa_n - \kappa_m)} \right) \\
+ 2\pi i(\kappa_n - \kappa_m) P(T_m = \kappa_n - \kappa_m) \right\} \\
= \frac{1}{\pi} \left( \phi_{T_m}^{(n)}(\pi) e^{-i\pi(\kappa_n - \kappa_m)} \right) + (\kappa_n - \kappa_m) P(T_m = \kappa_n - \kappa_m) \\
= (\kappa_n - \kappa_m) P(T_m = \kappa_n - \kappa_m),
\]
(2.3)

noticing that \( \phi_{T_m}^{(n)}(\pi) \) and \( e^{-i\pi(\kappa_n - \kappa_m)} \) are real (recall that \( \kappa_n \) is an integer). Since

\[
\phi_{T_m}^{(n)}(u) = \phi_{T_m}^{(n)}(u) \sum_{k=m+1}^{n} \frac{k\phi_{Z_k}^{(n)}(ku)}{\phi_{Z_k}(ku)},
\]

subtracting (2.2) from (2.3) we obtain

\[
(\kappa_n - \kappa_m) P(T_m^n = \kappa_n - \kappa_m) - P \left( (\kappa_n - \kappa_m) - n < T_m^n \leq (\kappa_n - \kappa_m) - (m + 1) \right) \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u) \left( \sum_{k=m+1}^{n} \frac{k\phi_{Z_k}^{(n)}(ku)}{\phi_{Z_k}(ku)} \right) \, du \\
- \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \left( \sum_{k=m+1}^{n} e^{iku} \right) \phi_{T_m}^{(n)}(u) \, du \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u) \left( \sum_{k=m+1}^{n} \frac{k\phi_{Z_k}^{(n)}(ku)}{\phi_{Z_k}(ku)} - \sum_{k=m+1}^{n} e^{iku} \right) \, du \\
= \frac{n-m}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u) \frac{\sum_{k=m+1}^{n} \frac{k\phi_{Z_k}^{(n)}(ku)}{\phi_{Z_k}(ku)} - \sum_{k=m+1}^{n} e^{iku}}{n-m} \, du \\
= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u) \gamma_{m,n} \left( \frac{u}{n-m} \right) \, du.
\]

Hence

\[
\left| (\kappa_n - \kappa_m) P(T_m^n = \kappa_n - \kappa_m) - P \left( (\kappa_n - \kappa_m) - n < T_m^n \leq (\kappa_n - \kappa_m) - (m + 1) \right) \right| \\
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u) \gamma_{m,n} \left( \frac{u}{n-m} \right) \, du \\
\leq \frac{1}{2\pi} \left\{ \int_{-\pi}^{\pi} e^{-iu(\kappa_n - \kappa_m)} \cdot \phi_{T_m}^{(n)}(u)^2 \, du \right\}^{\frac{1}{2}} \left\{ \int_{-\pi}^{\pi} \gamma_{m,n} \left( \frac{u}{n-m} \right)^2 \, du \right\}^{\frac{1}{2}}
\]
At the end of this proof we will show that
\[
\sup_{-\pi \leq u \leq \pi} |\gamma_{m,n}(u)| \leq C \frac{1 + \log \frac{n}{m}}{n - m}.
\] (2.5)

Using (2.5) in (2.4), we get
\[
\left\{ \int_{-\pi}^{\pi} \left| \phi_{\frac{T_m}{n}}(u) \right|^2 \, du \right\}^{1/2} \left\{ 2\pi(n - m) \sup_{-\pi \leq u \leq \pi} |\gamma_{m,n}(u)|^2 \right\}^{1/2} \leq C \left\{ \int_{-\pi}^{\pi} \left| \phi_{\frac{T_m}{n}}(u) \right|^2 \, du \right\}^{1/2} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}}.
\]

Since
\[
\frac{T_m}{n - m} = T_n \cdot \frac{n}{n - m} - \frac{\sum_{k=1}^{m} kZ_k}{n - m},
\]
putting \( W = -\frac{\sum_{k=1}^{m} kZ_k}{n - m} \) we can write using independence,
\[
\phi_{\frac{T_m}{n}}(u) = \phi_{\frac{T_n}{n}}^u \left( \frac{n}{n - m} \right) \cdot \phi_W(u).
\]

This implies
\[
\left\{ \int_{-\pi}^{\pi} \left| \phi_{\frac{T_n}{n}}^u \left( \frac{n}{n - m} \right) \right|^2 \, du \right\}^{1/2} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}} \leq \left\{ \int_{-\pi}^{\pi} \left| \phi_{\frac{T_n}{n}}(u) \right|^2 \, du \right\}^{1/2} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}}
\]
\[
= \frac{n - m}{n} \left\{ \int_{-\pi}^{\pi} \left| \phi_{\frac{T_n}{n}}(u) \right|^2 \, du \right\}^{1/2} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}}
\]
\[
\leq \left\{ \int_{-\pi}^{\pi} \left| \phi_{\frac{T_n}{n}}(u) \right|^2 \, du \right\}^{1/2} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}}
\]
\[
\leq \left\{ \int_{-\pi}^{\pi} |\phi_{\frac{T_n}{n}}(u)|^2 \, du \right\}^{1/2} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}}
\]
\[
\leq \left\{ \int_{-\pi}^{\pi} \frac{k}{\sqrt{n - m}} \cdot \frac{1 + \log \frac{n}{m}}{\sqrt{n - m}}
\]

The proof is now achieved by using assertion (b) of Proposition 1.4, which we recall
\[
\int_{-\pi}^{\pi} |\phi(u)|^2 \, du \to \int_{-\infty}^{\infty} |\phi(u)|^2 \, du < \infty.
\]

Here \( \phi \) is as in (1.3).

It remains us to prove (2.5). Write
\[
(n - m)\gamma_{m,n}(u) = \sum_{k=m+1}^{n} \left( k \frac{\phi_{Z_k}(ku)}{\phi_{Z_k}(ku)} - e^{iku} \right)
\]
\[
= \sum_{k=m+1}^{n} \left( \frac{k}{k + e^{iku} - 1} - e^{iku} \right) = \sum_{k=m+1}^{n} e^{iku} \left( k - 1 + e^{iku} - 1 \right)
\]
\[ \sum_{k=m+1}^{n} \frac{e^{iku} (1 - e^{iku})}{k - 1 + e^{iku}} = \sum_{k=m+1}^{n} \frac{e^{iku} (1 - e^{iku})(k - 1 + e^{-iku})}{|k - 1 + e^{iku}|^2}. \]

Hence

\[ (n - m)|\gamma_n(u)| \leq 2 \sum_{k=m+1}^{n} \frac{k}{|k - 1 + e^{iku}|^2} \leq 2 \sum_{k=m+1}^{n} \frac{k}{(k - 2)^2} \leq C \left(1 + \log \frac{n}{m}\right), \]

since

\[ |k - 1 + e^{iku}| \geq |(k - 1) - e^{iku}| = k - 2, \quad k \geq m + 1 \geq 3 \]

and

\[ \sum_{k=m+1}^{n} \frac{k}{(k - 2)^2} = \sum_{k=m-1}^{n-2} \frac{1}{k} + \sum_{k=m-1}^{n-2} \frac{2}{k^2} \leq \sum_{k=m-1}^{n-2} \frac{1}{k} + C \leq \frac{1}{m - 1} + \int_{m-1}^{n-2} \frac{1}{x} \, dx + C \]

\[ \leq C + \log \frac{n - 2}{m - 1} = C + \log \frac{n - 2}{m - 1} \leq C + \log 2 \frac{n}{m} = C + \log \frac{n}{m} \leq C \left(1 + \log \frac{n}{m}\right). \]

Now we can give the

**Proof of Corollary 1.5** It is easy to see that it suffices to show that

\[ \lim_{n \to \infty} n P(T_2^n = \kappa_n) = e^{-\gamma} \rho(x). \]

Proposition (2.3) implies that, for \( n \geq 3 \)

\[ |\kappa_n P(T_2^n = \kappa_n) - P(\kappa_n - n < T_2^n \leq \kappa_n - 3)| \leq C \frac{1 + \log \frac{n}{\sqrt{n - 2}}}{\sqrt{n - 2}}, \]

and relation (2.1), Proposition 2.1 and the continuity of the Dickman distribution give

\[ \lim_{n \to \infty} \frac{n}{\kappa_n} P(\kappa_n - n < T_2^n \leq \kappa_n - 3) = \frac{1}{x} \left(D(x) - D(x - 1)\right) = e^{-\gamma} \rho(x). \]

The result follows.

**Remark** Corollary 1.5 can also be derived from Theorem 2.6 of Arratia et al. [2], which we briefly recall

**Theorem 2.5** Let \((B_n)_{n \geq 1}\) be any sequence such that \(\lim_{m \to \infty} \frac{B_n}{n} = 0.\) If the uniform logarithmic condition holds, then

\[ \lim_{n \to \infty} \max_{0 \leq v \leq B_n} \sup_{s \geq 1} \left| s P(T_v^n = s) - P\left(\frac{s - n}{n} \leq X \leq \frac{s - v}{n}\right)\right| = 0. \]

In our case, the uniform logarithmic condition [2, (1.11–12)], i.e.

\[ |k P(Z_k = 1) - 1| \leq \alpha_k c_k; \quad |k P(Z_k = r)| \leq \alpha_r c_k, \]
with $\lim_{k \to \infty} c_k = 0$ and $\sum_r r \alpha_r < \infty$ is obviously satisfied. Taking $v = 0$, $s = \kappa_n \geq 1$ in Theorem 2.5 gives

$$
|\kappa_n P(T_0^n = \kappa_n) - P\left(\frac{\kappa_n - n}{n} \leq X \leq \frac{\kappa_n}{n}\right)| \leq \max \sup_{0 \leq s \leq B_n, s \geq 1} s P(T_s^n = s) - P\left(\frac{s - n}{n} \leq X \leq \frac{s - v}{n}\right) \to 0, \quad n \to \infty.
$$

It is easy to see that

$$
P\left(\frac{\kappa_n - n}{n} \leq X \leq \frac{\kappa_n}{n}\right) \to P(x - 1 \leq X \leq x) = xe^{-\gamma \rho(x)}.
$$

Hence

$$
nP(T_0^n = \kappa_n) - e^{-\gamma \rho(x)} = \frac{n}{\kappa_n} \left\{ \kappa_n P(T_0^n = \kappa_n) - P\left(\frac{\kappa_n - n}{n} \leq X \leq \frac{\kappa_n}{n}\right) \right\} + \frac{n}{\kappa_n} P\left(\frac{\kappa_n - n}{n} \leq X \leq \frac{\kappa_n}{n}\right) - e^{-\gamma \rho(x)}
$$

$$
\to 0, \quad n \to \infty.
$$

**Remark 2.6** Concerning the proof of Proposition 2.3, notice that

$$
\frac{\phi_{Z_k}^{(t)}}{\phi_{Z_k}(t)} = \psi_{Z_k}^{(t)}, \quad (2.6)
$$

where $\psi_{Z_k}(t) = \log \phi_{Z_k}(t)$, i.e. the second characteristic function of $Z_k$.

Write

$$
\psi_{Z_k}(t) = \sum_{j=1}^{\infty} c_j^{(k)} (it)^j \frac{j!}{j!},
$$

where $(c_j^{(k)})_j$ is the sequence of the cumulants of the $\mathcal{B}(1, \frac{1}{k})$ distribution. Thus

$$
\psi_{Z_k}^{(t)} = \sum_{j=1}^{\infty} i c_j^{(k)} (it)^j \frac{1}{(j-1)!}.
$$

(2.7)

Denote by $\psi_{\Pi_k}(t)$ the second characteristic function of the Poisson law with parameter $\frac{1}{k}$, i.e.

$$
\psi_{\Pi_k}(t) = \frac{e^{it} - 1}{k}, \quad \psi_{\Pi_k}^{(t)} = i \frac{e^{it} - 1}{k} = i \frac{\infty}{k} \sum_{j=1}^{(it)^j} \frac{(j-1)!}{(j-1)!}.
$$

(2.8)

By (2.6) and (2.8),

$$
\gamma_{m,n}(t) = \frac{\sum_{k=m+1}^{n} \frac{\phi_{Z_k}^{(tk)}}{\phi_{Z_k}(tk)} - \sum_{k=m+1}^{n} e^{itk}}{n - m} = \frac{\sum_{k=m+1}^{n} \frac{k}{k} \left(\frac{\phi_{Z_k}^{(tk)}}{\phi_{Z_k}(tk)} - \frac{k}{k} e^{itk}\right)}{n - m}
$$

$$
= \frac{\sum_{k=m+1}^{n} \left(\psi_{Z_k}^{(tk)} - \psi_{\Pi_k}^{(tk)}\right)}{n - m}.
$$
Since by (2.7) and (2.8),

$$\psi'_{Zk}(t) - \psi'_{\Pi k}(t) = \sum_{j=1}^{\infty} \left( i c_j^{(k)} - \frac{i}{k} \right) \frac{(it)^{j-1}}{(j-1)!},$$

we get

$$\gamma_{m,n}(t) = \sum_{k=m+1}^{n} \sum_{j=1}^{\infty} \left( k c_j^{(k)} - \frac{k}{(j-1)!} \right) \frac{(it)^{j-1}}{(j-1)!}.$$

Putting

$$\alpha^{(m,n)}_{j} = \frac{\sum_{k=m+1}^{n} k^{j-1}(k c_j^{(k)} - 1)}{n-m},$$

we obtain the formula

$$\gamma_{m,n}(t) = \sum_{j=1}^{\infty} \frac{(it)^{j-1}}{(j-1)!} \alpha^{(m,n)}_{j}. \qquad (2.9)$$

Let $B(1, x)$ be the Bernoullian law with parameter $x \in (0, 1)$ and $c_n(x)$ the nth cumulant of $B(1, x)$, i.e. the nth coefficient in the development of the logarithm of its characteristic function $\phi(t)$:

$$\log \phi(t) = \log \left( 1 + x(e^{it} - 1) \right) = \sum_{n=1}^{\infty} c_n(x) \frac{(it)^n}{n!}.$$  

The following result expresses $c_n(x)$ in terms of the Stirling numbers of the second kind $S(n, k)$ (i.e. the number of ways of partitioning a set of $n$ elements into $k$ non-empty subsets). We believe that our formula is new (though some analogue calculations appear for instance in Theorem A, p. 280 of [5]).

**Proposition 2.7** For every $n \geq 2$ we have

$$c_n(x) = x(1 - x) \left\{ \sum_{k=1}^{n} (-1)^{k+1} k! S(n - 1, k) x^{k-1} \right\}. \qquad (2.11)$$

This formula is a by-product of the equality

$$\sum_{n=1}^{\infty} \frac{(it)^n}{n!} x(1 - x) \sum_{k=1}^{n} x^{k-1} (-1)^{k+1} k! S(n - 1, k) = \log \left( 1 + x(e^{it} - 1) \right),$$

which in turn follows from the relations

$$k S(n - 1, k) + S(n - 1, k - 1) = S(n, k)$$

and

$$(e^z - 1)^k = k! \sum_{n=k}^{\infty} S(n, k) \frac{z^n}{n!}, \quad z \in \mathbb{C},$$
Almost sure local limit theorem for the Dickman distribution

the second one of which can be found in 24.1.4B, p. 824 of [1]. For a proof see [21], p. 75. Equation (2.11) furnishes an explicit form for the quantity

\[ \frac{c_j(x)}{x} - 1; \]

if \( p = \frac{1}{k} \), this quantity is precisely the expression \( kc_j^{(k)} - 1 \) in the previous calculations [see (2.9)].

We believe that the explicit Formula (2.11) can be used for getting good approximations of \( kc_j^{(k)} - 1 \), and in turn of \( \gamma_{m,n} \) [see (2.10)].

The following result specifies Proposition 1 of [15] quantitatively in terms of the characteristic functions.

**Proposition 2.8** There exists an absolute constant \( C \) such that for all integers \( n > m \geq 2 \) and all real numbers \( t \),

\[
\left| \phi_{\frac{t}{n-m}}(t) - \exp \left\{ \int_0^1 \frac{e^{itu} - 1}{u} \, du \right\} \right| \leq f_{m,n}(t),
\]

where

\[
f_{m,n}(t) = \exp \left\{ Ct^2 \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m+2}{n-m} \right) \right\} - 1.
\]

**Proof** First

\[
\left| \phi_{\frac{t}{n-m}}(t) - \exp \left\{ \int_0^1 \frac{e^{itu} - 1}{u} \, du \right\} \right| - 1
\]

\[
\leq \left| \phi_{\frac{t}{n-m}}(t) \right| \cdot \exp \left\{ \int_0^1 \frac{e^{itu} - 1}{u} \, du - \sum_{k=m+1}^n \log \left( 1 + \frac{e^{it \frac{k}{n-m}} - 1}{k} \right) \right\} - 1
\]

\[
\leq \exp \left\{ \sum_{k=m+1}^n \left[ \int_{\frac{k-1}{n-m}}^{\frac{k}{n-m}} \frac{e^{itu} - 1}{u} \, du - \log \left( 1 + \frac{e^{it \frac{k}{n-m}} - 1}{k} \right) \right] \right\} - 1
\]

\[
\leq \exp \left\{ \sum_{k=m+1}^n \left[ \frac{e^{it \frac{k}{n-m}} - 1}{k} - \log \left( 1 + \frac{e^{it \frac{k}{n-m}} - 1}{k} \right) \right] \right\} - 1
\]

\[
+ \sum_{k=m+1}^n \left[ \int_{\frac{k-1}{n-m}}^{\frac{k}{n-m}} \frac{e^{itu} - 1}{u} \, du - \frac{e^{it \frac{k}{n-m}} - 1}{k} \right] - 1.
\]

We shall prove that

\[
\sum_{k=m+1}^n \left| \log \left( 1 + \frac{e^{it \frac{k}{n-m}} - 1}{k} \right) - \frac{e^{it \frac{k}{n-m}} - 1}{k} \right| \leq Ct^2 \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{1}{n-m} \right)
\]

\( \square \)
These inequalities give

\[
\left| \int_0^1 \frac{e^{itu} - 1}{u} \, du - \frac{e^{it \frac{k}{n-m}} - 1}{k} \right| \leq \frac{(m+1)t^2}{2(n-m)^2}, \quad \forall k \in (m,n). \tag{b}
\]

(a) The inequality

\[
\left| e^{ix} - 1 \right| = 2 \left| \sin \frac{x}{2} \right| \leq 2 \leq |x|
\]

applied to \( x = \frac{tk}{n-m} \) gives

\[
\left| e^{it \frac{k}{n-m}} - 1 \right| \leq \frac{2}{k} \cdot \frac{|t|}{n-m}. \tag{2.12}
\]

From

\[
| \log(1+u) - u | \leq \sum_{j=2}^{\infty} \frac{|u|^j}{j} = |u|^2 \sum_{j=0}^{\infty} \frac{|u|^j}{j+2}, \quad |u| < 1,
\]

applied to \( u = \frac{e^{it \frac{k}{n-m}} - 1}{k} \) (with \( k \geq m+1 \); notice that \( \frac{e^{it \frac{k}{n-m}} - 1}{k} < 1 \) for \( k \geq m+1 \geq 3 \)) we get

\[
\sum_{k=m+1}^{n} \left| \log \left( 1 + \frac{e^{it \frac{k}{n-m}} - 1}{k} \right) - \frac{e^{it \frac{k}{n-m}} - 1}{k} \right| \leq \max_{m+1 \leq l \leq n} \left| \frac{e^{it \frac{k}{n-m}} - 1}{k} \right|^2 \sum_{j=0}^{\infty} \frac{1}{j+2} \sum_{k=m+1}^{n} \frac{\left| e^{it \frac{k}{n-m}} - 1 \right|^j}{k}^2
\]

by (2.12)

\[
\leq C \cdot \frac{t^2}{(n-m)^2} \left( (n-m) + \log \frac{n}{m} + \sum_{j=2}^{\infty} \frac{2^j}{j+2} \int_2^\infty \frac{1}{x^j} \, dx \right)
\]

\[
\leq C \cdot \frac{t^2}{(n-m)^2} \left( (n-m) + \log \frac{n}{m} + 1 \right) \leq Ct^2 \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{1}{n-m} \right).
\]

(b) Putting

\[
\eta_t(x) = \frac{e^{itx} - 1}{x}, \quad x > 0, \quad -\pi \leq t \leq \pi,
\]
we can write

\[
\left| \int_{\frac{k-m}{n-m}}^{\frac{k-1-m}{n-m}} \frac{e^{itu} - 1}{u} - \frac{e^{it\frac{k}{n-m}} - 1}{k} \, du \right| = \left| \int_{\frac{k-m}{n-m}}^{\frac{k-1-m}{n-m}} \frac{e^{itu} - 1}{u} - \frac{e^{it\frac{k}{n-m}} - 1}{(n-m)\frac{k}{n-m}} \, du \right|
\]

\[
= \left| \int_{\frac{k-m}{n-m}}^{\frac{k-1-m}{n-m}} \eta_t(u) \, du - \frac{\eta_t\left(\frac{k}{n-m}\right)}{n-m} \right| = \left| \int_{\frac{k-m}{n-m}}^{\frac{k-1-m}{n-m}} \left\{ \eta_t(u) - \eta_t\left(\frac{k}{n-m}\right) \right\} \, du \right|
\]

\[
\leq \frac{1}{n-m} \sup_{u \in \left[\frac{k-1-m}{n-m}, \frac{k-m}{n-m}\right]} \left| \eta_t(u) - \eta_t\left(\frac{k}{n-m}\right) \right|
\]

\[
\leq \frac{1}{n-m} \cdot \left\{ \sup_{u \in \left[\frac{k-1-m}{n-m}, \frac{k-m}{n-m}\right]} \left| u - \frac{k}{n-m} \right| \right\} \cdot \left\{ \sup_{x \in \mathbb{R}} \left| \eta_t'(x) \right| \right\} = \frac{m+1}{(n-m)^2} \cdot \sup_{x \in \mathbb{R}} \left| \eta_t'(x) \right|
\]

\[
\leq \frac{(m+1)t^2}{2(n-m)^2},
\]

since

\[
\sup_{x \in \mathbb{R}} \left| \eta_t'(x) \right| \leq \frac{t^2}{2},
\]

as we are going to prove. First

\[
\left| \eta_t'(x) \right|^2 = \left| \frac{itxe^{itx} - e^{itx} + 1}{x^2} \right|^2 = \frac{\delta(tx)}{x^4},
\]

where \( \delta(u) = 2(1 - u \sin u - \cos u) + u^2 \). Put now \( H(u) = \frac{u^4}{4} - \delta(u) \). We have

\[
H'(u) = u^3 - 2u(1 - \cos u) = 4u \left( \frac{u^2}{4} - \sin^2 \frac{u}{2} \right) \geq 0, \quad u \geq 0;
\]

hence \( H \) is non-decreasing for \( u \geq 0 \), and from the fact that \( H(0) = 0 \), we deduce that \( H(u) \geq 0 \) for every \( u \geq 0 \), hence also for every \( u \in \mathbb{R} \) since \( H(-u) = H(u) \). In other words \( \delta(u) \leq \frac{u^4}{4} \) and as a consequence

\[
\left| \eta_t'(x) \right|^2 = \frac{\delta(tx)}{x^4} \leq \frac{t^4}{4}.
\]

\( \square \)

The following result specifies Proposition 1 of [15] quantitatively in terms of distribution functions.

**Proposition 2.9** There exists an absolute positive constant \( C \) such that, for all positive integers \( n, m \), with \( n > m \geq 2 \),

\[
\sup_{x \in \mathbb{R}} \left| P \left( \frac{T_n}{n-m} \leq x \right) - D(x) \right| \leq C g_{m,n},
\]

recalling that (see Theorem 1.1)

\[
g_{m,n} = \exp \left( C \left\{ \log \frac{n}{m} + \frac{m+2}{n-m} \log \frac{n}{m} \right\} - 1 \right) + \frac{1}{\log \frac{n}{m}}.
\]
**Proof** In view of Theorem 2 p. 109 of [19], if \( \tau \) is an arbitrary positive number, then for \( b > \frac{1}{2\pi} \)

\[
\sup_{x \in \mathbb{R}} \left| P \left( \frac{T^n_m}{n-m} \leq x \right) - D(x) \right| \\
\leq b \int_{-\tau}^{\tau} \exp \left\{ \int_{0}^{1} \frac{e^{itu} - 1}{u} \, du \right\} - \phi \frac{T^n_m}{n-m} (t) \left| \frac{dt}{|t|} \right| + \frac{r(b)}{\tau} \sup_{x \in \mathbb{R}} |D'(x)|
\]

where \( r(b) \) is a positive constant depending on \( b \) only. Hence

\[
\sup_{x \in \mathbb{R}} \left| P \left( \frac{T^n_m}{n-m} \leq x \right) - D(x) \right| \\
\leq C \left\{ \int_{-\tau}^{\tau} \exp \left\{ \int_{0}^{1} \frac{e^{itu} - 1}{u} \, du \right\} - \phi \frac{T^n_m}{n-m} (t) \left| \frac{dt}{|t|} \right| + \frac{1}{\tau} \sup_{x \in \mathbb{R}} |D'(x)| \right\}
\]

by Proposition 2.8. Now, as for \( A > 0 \) we have

\[
\sup_{0 \leq t \leq \tau} e^{Ax^2} - 1 = \frac{e^{A\tau^2} - 1}{\tau},
\]

by applying this with the choice \( A = C \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m+2}{n-m} \right) \), we obtain

\[
\int_{-\tau}^{\tau} \frac{f_{m,n}(t)}{|t|} \, dt \leq 2 \tau e^{C \left( \log \frac{n}{m} + \frac{m+2}{n-m} \right) \tau^2} - 1 = 2 \left( e^{C \left( \log \frac{n}{m} + \frac{m+2}{n-m} \right) \tau^2} - 1 \right).
\]

Hence, for every \( \tau \),

\[
\sup_{x \in \mathbb{R}} \left| P \left( \frac{T^n_m}{n-m} \leq x \right) - D(x) \right| \leq C \left\{ \exp \left( C \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m+2}{n-m} \right) \tau^2 \right) - 1 + \frac{1}{\tau} \right\}.
\]

Taking \( \tau = \log \frac{n}{m} \) we get

\[
\exp \left( C \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m+2}{n-m} \right) \tau^2 \right) - 1 + \frac{1}{\tau} = \exp \left( C \left( \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m+2}{n-m} \right) \log^2 \frac{n}{m} \right)
\]

\[
= g_{m,n}.
\]

So that using (2.13), we obtain

\[
\sup_{x \in \mathbb{R}} \left| P \left( \frac{T^n_m}{n-m} \leq x \right) - D(x) \right| \leq C g_{m,n},
\]

as claimed. \( \square \)
3 Proofs of Proposition 1.4 and Theorem 1.6

Proof of Proposition 1.4  
(a) By symmetry \((t \mapsto |\phi(t)|^2)\) is an even function, it is sufficient to prove that
\[
\int_0^{+\infty} |\phi(t)|^2 \, dt < +\infty.
\]
Theorem 2 p. 11 of [19] assures that there exist positive constants \(\delta\) and \(C\) such that
\[
|\phi(t)| \leq 1 - Ct^2, \quad |t| < \delta.
\]
This implies that
\[
\int_0^{\delta} |\phi(t)|^2 \, dt \leq \int_0^{\delta} (1 - Ct^2)^2 \, dt = C.
\]
Let us turn to \(\int_0^{+\infty} |\phi(t)|^2 \, dt\). First observe that
\[
|\phi(t)|^2 = \phi(t)\phi(-t) = \exp \left\{ \int_0^1 e^{itu} \frac{1 - \cos tu}{u} \, du \right\} \cdot \exp \left\{ \int_0^1 e^{-itu} \frac{1}{u} \, du \right\} \]
\[
= \exp \left\{ -2 \int_0^1 \frac{1 - \cos tu}{u} \, du \right\}. \quad (3.1)
\]
Now, for every \(\epsilon \in (0, t)\)
\[
\int_0^1 \frac{1 - \cos tu}{u} \, du = \int_0^t \frac{1 - \cos z}{z} \, dz \geq \int_\epsilon^t \frac{1 - \cos z}{z} \, dz = \left[ \frac{z - \sin z}{z^2} \right]_\epsilon^t + \int_\epsilon^t \frac{z - \sin z}{z^2} \, dz
\]
\[
= \log \frac{t}{\epsilon} - \frac{\sin t}{t} + \frac{\sin \epsilon}{\epsilon} - \int_\epsilon^t \frac{\sin z}{z^2} \, dz \geq \log \frac{t}{\epsilon} + C
\]
Hence, taking \(\epsilon = \delta\),
\[
\int_\delta^{+\infty} |\phi(t)|^2 \, dt \leq \int_\delta^{+\infty} \exp \left\{ -2 \left( \log \frac{t}{\delta} + C \right) \right\} \, dt \leq C \int_\delta^{+\infty} \frac{1}{t^2} \, dt = C.
\]
(b) By part (a), Proposition 1.4 will be proved if we show that
\[
\int_{-\infty}^{\infty} \left\{ |\phi_{T_n}^{\pi}(u)|^2 - |\phi(u)|^2 \right\} \, du \to 0, \quad n \to \infty. \quad (3.2)
\]
Recall that, by Proposition 2.1, \(|\phi_{T_n}^{\pi}|\) converges to \(|\phi|\) pointwise and uniformly on every bounded interval. Hence, for any positive \(A\),
\[
\int_{-A}^A \left\{ |\phi_{T_n}^{\pi}(u)|^2 - |\phi(u)|^2 \right\} \, du \to 0, \quad n \to \infty.
\]
Thus we are left with the proof of
\[
\int_{\{A \leq |t| \leq n\pi\}} \left\{ |\phi_{T_n}^{\pi}(u)|^2 - |\phi(u)|^2 \right\} \, du \to 0, \quad n \to \infty. \quad (3.3)
\]
We split the first member of (3.3) as follows: for a fixed \(\epsilon \in (0, 1)\), write
\[
\int_{\{A \leq |t| \leq n\pi\}} = \int_{\{A \leq |t| \leq \epsilon n\pi\}} + \int_{\{\epsilon n\pi \leq |t| \leq n\pi\}} = I_1 + I_2.
\]
We now estimate $I_1$ and $I_2$.

$(I_1)$ Notice that

$$
|\phi_{T_n}(t)|^2 = \prod_{k=1}^n \left|1 + \frac{e^{ikt} - 1}{k}\right|^2 = \prod_{k=1}^n \left\{1 - \frac{1}{k} + \frac{1}{k} \cos kt \right\}^2 + \left(\frac{1}{k} \sin kt\right)^2
$$

$$
= \prod_{k=1}^n \left\{1 + \frac{2(k-1)}{k^2} (\cos kt - 1)\right\} = \exp \left\{\sum_{k=1}^n \log \left[1 + \frac{2(k-1)}{k^2} (\cos kt - 1)\right]\right\}.
$$

(3.4)

Hence

$$
|I_1| \leq \int_{|A \leq |t| \leq \frac{\epsilon \pi \sqrt{n}}{5}} \left|\phi_{T_n}(t)\right|^2 - \exp \left\{\sum_{k=1}^n \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right\} \, dt
$$

$$
+ \int_{|A \leq |t| \leq \frac{\epsilon \pi \sqrt{n}}{5}} \left|\exp \left\{\sum_{k=1}^n \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right\} - \exp \left\{\sum_{k=1}^n \frac{2}{k} (\cos \frac{kt}{n} - 1)\right\}\right| \, dt
$$

$$
+ \int_{|A \leq |t| \leq \frac{\epsilon \pi \sqrt{n}}{5}} \left|\exp \left\{\sum_{k=1}^n \frac{2}{k} (\cos \frac{kt}{n} - 1)\right\} - |\phi(t)|^2\right| \, dt = I_{11} + I_{12} + I_{13}.
$$

We now estimate $I_{11}$, $I_{12}$ and $I_{13}$.

$(I_{11})$ First observe that, by relation (3.4),

$$
I_{11} = \int_{|A \leq |t| \leq \frac{\epsilon \pi \sqrt{n}}{5}} \left|\phi_{T_n}(t)\right|^2 - \exp \left\{\sum_{k=1}^n \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right\} \, dt
$$

$$
= \int_{|A \leq |t| \leq \frac{\epsilon \pi \sqrt{n}}{5}} \left|\exp \left\{\sum_{k=1}^n \log \left[1 + \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right]\right\}
$$

$$
- \exp \left\{\sum_{k=1}^n \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right\}\right| \, dt
$$

$$
\leq \int_{|A \leq |t| \leq \frac{\epsilon \pi \sqrt{n}}{5}} \left|\exp \left\{\sum_{k=1}^n \left[\log \left[1 + \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right] - \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right]\right\}\right| - 1 \, dt,
$$

since

$$
0 \leq \exp \left\{\sum_{k=1}^n \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1)\right\} \leq 1.
$$

Now using the development

$$
\log(1 + z) - z = \sum_{j \geq 2} \frac{(-1)^j}{j} z^j, \ |z| < 1
$$

\[\text{Springer}\]
with \( z = \frac{2(k-1)}{k^2} (\cos \frac{kt}{n} - 1) \) (which, for sufficiently large \( n \), is strictly less than 1 in modulus for every \( k \geq 1 \)) we get

\[
\log \left[ 1 + \frac{2(k-1)}{k^2} \left( \cos \frac{kt}{n} - 1 \right) \right] - \frac{2(k-1)}{k^2} \left( \cos \frac{kt}{n} - 1 \right)
= \sum_{j \geq 2} (-1)^j 2^j (k-1)^j \left( \cos \frac{kt}{n} - 1 \right)^j = \sum_{j \geq 2} 2^j (k-1)^j \left( 1 - \cos \frac{kt}{n} \right)^j \geq 0.
\]

It follows that

\[
\int_{|A| \leq \epsilon \frac{x}{\sqrt{n}}} \left| \exp \left\{ \sum_{k=1}^{n} \log \left[ 1 + \frac{2(k-1)}{k^2} \left( \cos \frac{kt}{n} - 1 \right) \right] - \frac{2(k-1)}{k^2} \left( \cos \frac{kt}{n} - 1 \right) \right\} \right| - 1 \, dt
= \int_{|A| \leq \epsilon \frac{x}{\sqrt{n}}} \left( \exp \left\{ \sum_{k=1}^{n} \sum_{j \geq 2} \frac{2^j (k-1)^j}{jk^2} \left( \frac{kt}{n} \right)^j \left( \cos \frac{kt}{n} - 1 \right)^j \right\} - 1 \right) \, dt.
\]

Using the inequality \( 1 - \cos z \leq z^2 \) the above can be bounded by

\[
\int_{|A| \leq \epsilon \frac{x}{\sqrt{n}}} \left( \exp \left\{ \sum_{k=1}^{n} \sum_{j \geq 2} \frac{2^j (k-1)^j}{jk^2} \left( \frac{kt}{n} \right)^j \right\} - 1 \right) \, dt
= \int_{|A| \leq \epsilon \frac{x}{\sqrt{n}}} \left( \exp \left\{ \sum_{j \geq 2} \frac{1}{j} \left( \frac{\sqrt{2}t}{n} \right)^{2j} \sum_{k=1}^{n} (k-1)^j \right\} - 1 \right) \, dt
\]

\[
\leq \int_{|A| \leq \epsilon \frac{x}{\sqrt{n}}} \left( \exp \left\{ \sum_{j \geq 2} \frac{1}{j} \left( \frac{\sqrt{2}t}{n} \right)^{2j} \left( \frac{n^{j+1}}{j+1} \right) \right\} - 1 \right) \, dt
\]

\[
= \int_{|A| \leq \epsilon \frac{x}{\sqrt{n}}} \left( \exp \left\{ \sum_{j \geq 2} \frac{1}{j} \left( \frac{\sqrt{2}t}{n} \right)^{2j} \left( \frac{n^{j+1}}{j+1} \right) \right\} - 1 \right) \, dt
\]

Now, for \( |t| \leq \epsilon \frac{x}{\sqrt{n}} \) we also have \( |t| \leq \pi \frac{x}{\sqrt{n}} \) (recall that \( \epsilon < 1 \)); hence there exists \( n_0 \) not dependent on \( \epsilon \) such that, for \( n > n_0 \)
\[
\frac{(\sqrt{2}t)^2}{n} \leq \frac{C}{n^\frac{2}{5}} \leq \frac{1}{2}.
\]

Hence, for \(n > n_0\),
\[
\sum_{j \geq 0} \frac{1}{(j + 3)(j + 2)} \cdot \frac{(\sqrt{2}t)^{2j}}{n^j} \leq \sum_{j \geq 0} \frac{1}{(j + 3)(j + 2)} \cdot \frac{1}{2^j} = C,
\]
and we get
\[
\int_{|A| \leq |t| \leq \frac{\epsilon \pi}{5} \sqrt{n}} \left( \exp \left\{ \frac{C t^4}{n} \sum_{j \geq 0} \frac{1}{(j + 3)(j + 2)} \cdot \frac{(\sqrt{2}t)^{2j}}{n^j} \right\} - 1 \right) \, dt
\]
\[
\leq \int_{|A| \leq |t| \leq \frac{\epsilon \pi}{5} \sqrt{n}} \left( \exp \left\{ \frac{C t^4}{n} \right\} - 1 \right) \, dt \leq \left( \exp \left\{ \frac{C (\epsilon \pi \sqrt{n})^4}{n} \right\} - 1 \right) \epsilon \pi \frac{\sqrt{n}}{n}
\]
\[
= C \cdot \epsilon \pi \frac{\sqrt{n}}{n} - 1 < C \epsilon^5.
\]

\((I_{12})\) Here we observe that
\[
0 \leq \exp \left\{ \sum_{k=1}^{n} \frac{2}{k} \left( \cos \frac{kt}{n} - 1 \right) \right\} \leq 1,
\]
hence
\[
I_{12} = \int_{|A| \leq |t| \leq \frac{\epsilon \pi}{5} \sqrt{n}} \left| \exp \left\{ \sum_{k=1}^{n} \frac{2(k - 1)}{k^2} \left( \cos \frac{kt}{n} - 1 \right) \right\} - \exp \left\{ \sum_{k=1}^{n} \frac{2}{k} \left( \cos \frac{kt}{n} - 1 \right) \right\} \right| \, dt
\]
\[
\leq \int_{|A| \leq |t| \leq \frac{\epsilon \pi}{5} \sqrt{n}} \left| \exp \left\{ \sum_{k=1}^{n} \frac{2}{k^2} \left( 1 - \cos \frac{kt}{n} \right) \right\} - 1 \right| \, dt
\]
\[
= \int_{|A| \leq |t| \leq \frac{\epsilon \pi}{5} \sqrt{n}} \left( \exp \left\{ \sum_{k=1}^{n} \frac{2}{k^2} \left( \frac{kt}{n} \right)^2 \right\} - 1 \right) \, dt
\]
\[
\leq \int_{|A| \leq |t| \leq \frac{\epsilon \pi}{5} \sqrt{n}} \left( \exp \left\{ \sum_{k=1}^{n} \frac{2}{k^2} \left( \frac{2(\epsilon \pi \sqrt{n})^2}{n} \right) - 1 \right\} \frac{2(\epsilon \pi \sqrt{n})^2}{n^{3/5}} - \epsilon \pi \frac{\sqrt{n}}{n} \right) \, dt
\]
\[
\leq \left( \epsilon \frac{2(\epsilon \pi \sqrt{n})^2}{n^{3/5}} - 1 \right) \epsilon \pi \frac{\sqrt{n}}{n} = \frac{2(\epsilon \pi)^2}{n^{3/5}} - \frac{1}{n^{3/5}} \epsilon \pi \frac{\sqrt{n}}{n} \to 0, \quad n \to \infty.
\]

\((I_{13})\) Recalling the explicit form of \(|\phi(t)|^2\) given in equation (3.1), we have
\[ I_{13} = \int_{\{|A| \leq \epsilon \pi^{5/6} \sqrt{n}\}} \left| \exp \left\{ \frac{1}{2} \sum_{k=1}^{n} \frac{2}{k} \left( \cos \frac{k}{n} - 1 \right) \right\} - |\phi(t)|^2 \right| \, dt \]

\[ \leq \int_{\{|A| \leq \epsilon \pi^{5/6} \sqrt{n}\}} \left| \exp \left\{ \frac{1}{2} \sum_{k=1}^{n} \frac{2}{k} \left( \cos \frac{k}{n} - 1 \right) \right\} - \exp \left\{ 2 \int_{0}^{1} \frac{\cos tu - 1}{u} \, du \right\} \right| \, dt, \]

and, observing again that

\[ 0 \leq \exp \left\{ \sum_{k=1}^{n} \frac{2}{k} \left( \cos \frac{k}{n} - 1 \right) \right\} \leq 1, \]

we get

\[ I_{13} \leq \int_{\{|A| \leq \epsilon \pi^{5/6} \sqrt{n}\}} \left| \left\{ \sum_{k=1}^{n} \frac{2}{k} \left[ \gamma_t \left( \frac{k}{n} \right) - \gamma_t \left( \frac{u}{n} \right) \right] \right\} - 1 \right| \, dt \]

where we have put

\[ \gamma_t(u) = \frac{\cos tu - 1}{u} \]

and have used the inequality \(|e^x - 1| \leq e|\lambda| - 1\).

It is not difficult to see that

\[ \sup_{u \in \mathbb{R}} |\gamma'_t(u)| = Ct^2; \]

in fact \(\gamma'_t(u) = \eta(tu)t^2\), with

\[ \eta(z) = \frac{1 - z \sin z - \cos z}{z^2}, \]

and proving that \(\sup_{z \in \mathbb{R}} |\eta(z)| = C < +\infty\) is a simple exercise. Thus, by Lagrange’s Theorem,

\[ \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left| \gamma_t \left( \frac{k}{n} \right) - \gamma_t \left( \frac{u}{n} \right) \right| \, du \leq Ct^2 \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left| \frac{k}{n} - u \right| \, du \leq C \frac{t^2}{n^2}. \quad (3.5) \]

Using (3.5) in the last bound for \(I_{13}\) we find

\[ I_{13} \leq \int_{\{|A| \leq \epsilon \pi^{5/6} \sqrt{n}\}} \left( \exp \left\{ \frac{1}{2} \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left| \gamma_t \left( \frac{k}{n} \right) - \gamma_t \left( \frac{u}{n} \right) \right| \, du \right\} - 1 \right) \, dt \]

\[ \leq \int_{\{|A| \leq \epsilon \pi^{5/6} \sqrt{n}\}} \left( \exp \left\{ C \frac{t^2}{n^2} \right\} - 1 \right) \, dt \leq \left( \exp \left\{ C \left( \epsilon \pi^{5/6} \sqrt{n} \right)^2 \right\} - 1 \right) \epsilon \pi^{5/6} \]

\[ = C \frac{e^{\pi^{5/6}} - 1}{\sqrt{n}} \to 0, \quad n \to \infty. \]
(I2) We have
\[ I_2 = \int_{\pi / n \leq |t| \leq n \pi} \left\{ \left| \frac{\phi_{I_n}(u)}{n} \right|^2 - |\phi(u)|^2 \right\} du \]
\[ \leq \int_{\pi / n \leq |t| \leq n \pi} \left| \frac{\phi_{I_n}(u)}{n} \right|^2 du + \int_{|e^\pi \leq |t| \leq n \pi} |\phi(u)|^2 du. \]

The second summand above goes to 0 as \( n \to \infty \) since \( |\phi(u)|^2 \) is integrable on \( \mathbb{R} \) (recall point (a) of this proposition); hence we have to prove that
\[ \int_{|e^\pi \leq |t| \leq n \pi} \left| \frac{\phi_{I_n}(u)}{n} \right|^2 du \to 0, \quad n \to \infty. \]

By relation (3.4), we have, for every \( k_0 \in \mathbb{N} \) and \( n \geq k_0 \)
\[ |\phi_{I_n}(u)|^2 = \exp \left\{ \sum_{k=1}^{k_0-1} \log \left[ 1 + \frac{2(k-1)}{k^2} (\cos kt - 1) \right] \right\} \cdot \exp \left\{ \sum_{k=k_0}^n \log \left[ 1 + \frac{2(k-1)}{k^2} (\cos kt - 1) \right] \right\} \leq \exp \left\{ \sum_{k=k_0}^n \log \left[ 1 + \frac{2(k-1)}{k^2} (\cos kt - 1) \right] \right\}.

Now we use the elementary relation \(|\log(1 - z) + z| \leq |z|^2\), valid for any complex number \( z \) such that \(|z| < \frac{1}{2}\). By applying it with \( z = -\frac{2(k-1)}{k^2} (\cos kt - 1) \) and choosing \( k_0 \) such that \( \left| \frac{2(k-1)}{k^2} (\cos kt - 1) \right| < \frac{1}{2} \) for \( k > k_0 \) and every \( t \), we get
\[ \log \left[ 1 + \frac{2(k-1)}{k^2} (\cos kt - 1) \right] \leq -\frac{2(k-1)}{k^2} (1 - \cos kt) + \frac{4(k-1)^2}{k^4} (1 - \cos kt)^2. \]

Next as obviously \( 1 - \cos x \leq 2 \), we can write
\[ \exp \left\{ \sum_{k=k_0}^n \log \left[ 1 + \frac{2(k-1)}{k^2} (\cos kt - 1) \right] \right\} \]
\[ \leq \exp \left\{ - \sum_{k=k_0}^n \frac{2(k-1)}{k^2} (1 - \cos kt) + \sum_{k=k_0}^n \frac{4(k-1)^2}{k^4} (1 - \cos kt)^2 \right\} \]
\[ \leq \exp \left\{ -2 \sum_{k=k_0}^n \frac{k-1}{k^2} + 2 \sum_{k=k_0}^n \frac{k-1}{k^2} \cos kt + \sum_{k=k_0}^n \frac{16(k-1)^2}{k^4} \right\} \]
\[ \leq \exp \left\{ -2 \sum_{k=k_0}^n \frac{1}{k} + 2 \sum_{k=k_0}^n \frac{\cos kt}{k} + C \right\} \leq \exp \left\{ -2 \int_{k_0}^n \frac{1}{x} \, dx + 2 \sum_{k=k_0}^n \frac{\cos kt}{k} + C \right\} \]
\[ = \frac{C}{n^2} \exp \left\{ 2 \sum_{k=k_0}^n \frac{\cos kt}{k} \right\}. \]
By [24], p. 191 we know that

\[
\sup_{n \geq 1} \left| \sum_{k=1}^{n} \frac{\cos kt}{k} \right| \leq \log \frac{1}{t} + C.
\]

Hence

\[
\sup_{n \geq k_0} \left| \sum_{k=k_0}^{n} \frac{\cos kt}{k} \right| \leq \sup_{n \geq 1} \left| \sum_{k=1}^{n} \frac{\cos kt}{k} \right| + \sum_{k=1}^{k_0-1} \frac{\cos kt}{k} \leq \log \frac{1}{t} + C.
\]

It follows that

\[
\frac{C}{n^2} \exp \left\{ 2 \sum_{k=k_0}^{n} \frac{\cos kt}{k} \right\} \leq \frac{C}{n^2} \exp \left\{ 2 \left( \log \frac{1}{t} + C \right) \right\} = \frac{C}{n^2 t^2},
\]

so that

\[
\int_{\{e^{\pi/5} \leq |t| \leq n \pi\}} \left| \phi_{T_n}(u) \right|^2 \, du = n \int_{e^{\pi/5}}^{\pi} \left| \phi_{T_n}(t) \right|^2 \, dt \leq n \int_{e^{\pi/5}}^{\pi} \frac{C}{n^2 t^2} \, dt \leq \frac{C n^{4/5}}{n} = \frac{C}{n^{1/5}} \to 0, \quad n \to \infty.
\]

Now (3.2) is proved by letting \( \epsilon \to 0 \) in the estimation of \( I_{11} \). The proof is now complete.

\[ \square \]

**Remark 3.1** The relation (3.2) yields a weak form of local limit theorem (see Theorem 1.5). Let \( \hat{T}_n \) be the \( n \)th partial sum

\[
\hat{T}_n = \sum_{k=1}^{n} \hat{Z}_k,
\]

where \( (\hat{Z}_n)_{n \geq 1} \) is an independent copy of \( (Z_n)_{n \geq 1} \). Denote by \( d_s \) the symmetrized Dickman density, which has characteristic function \( \left| \phi \right|^2 \). Then, by the inversion formula, (3.2) and Proposition 1.4,

\[
2 \pi \left\{ n P(T_n - \hat{T}_n = \kappa_n) - d_s(n^{-1} \kappa_n) \right\} = \int_{-n \pi}^{n \pi} e^{-itn^{-1} \kappa_n} \left\{ \phi_{T_n}(t) \right\}^2 - \left| \phi(t) \right|^2 \, dt + \int_{\{|t| > n \pi\}} e^{-itn^{-1} \kappa_n} \left| \phi(t) \right|^2 \, dt \to 0, \quad n \to \infty.
\]

Thus we have

\[
\sup_{\kappa \in \mathbb{Z}} \left\{ n P(T_n - \hat{T}_n = \kappa) - d_s(n^{-1} \kappa) \right\} \to 0, \quad n \to \infty
\]

which is a variant of the classical de Moivre–Gnedenko local limit theorem for symmetrized Dickman density.

We now pass to the proof of Theorem 1.6.

**Proof of Theorem 1.6** From (2.21) on p. 429 in [15] we know that for \( x > 1 \)

\[
e^{-\gamma} \varrho(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t) \, dt.
\]
Set \( g(x) = e^{-\rho x} \). We have for \( x \geq 2 \),

\[
n P(S_n = \kappa) = \frac{n}{2\pi} \int_{-\pi}^{\pi} e^{-it\kappa} \phi_{T_n}(t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-it\frac{\pi}{\kappa}} \phi_{T_n}(t) dt.
\]

Now, by the Cauchy–Schwarz inequality and Proposition 1.4

\[
\frac{1}{2\pi} \left| \int_{-\pi}^{\pi} e^{-it\kappa} \left\{ \phi_{\frac{T_n}{\pi}}(t) - \phi(t) \right\} dt \right| \leq \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} \left| \phi_{\frac{T_n}{\pi}}(t) - \phi(t) \right|^2 dt \right)^{\frac{1}{2}}
\]

\[
= \frac{1}{2\pi} \left( \int_{-\pi}^{\pi} \left| \phi_{\frac{T_n}{\pi}}(t) \right|^2 dt + \left| \phi(t) \right|^2 dt - \int_{-\pi}^{\pi} \phi_{\frac{T_n}{\pi}}(t) \overline{\phi(t)} dt + \phi_{\frac{T_n}{\pi}}(t) \phi(t) dt \right)^{\frac{1}{2}} \rightarrow 0.
\]

Therefore, for \( \kappa \geq 2 \)

\[
\sup_{\kappa \in \mathbb{N}} |n P(T_n = \kappa) - g(n^{-1} \kappa)| \to 0, \quad n \to \infty
\] (3.7)

(note that \( Z_1 \equiv 1 \)).

Now, we assume that \( x \geq 2 \). It is well-known that for \( x > 1 \)

\[
\left( \frac{x + 1}{e} \right)^x \leq \Gamma(x + 1) \quad \text{and} \quad \rho(x) \Gamma(x + 1) < 1,
\]

where \( \Gamma \) is Euler’s gamma function. Thus by the Mean Value Theorem we have for some \( \theta \in (x, y) \) (recall \( \rho \) in non-increasing)

\[
\left| \frac{\rho(x) - \rho(y)}{x - y} \right| = |\rho'(\theta)| = \left| \frac{\rho(\theta - 1)}{\theta} \right| \leq \frac{\rho(x - 1)}{x} \leq \frac{1}{x \Gamma(x)} \leq \frac{e^x}{(1 + x)^x}
\]

Whence for \( 2 \leq x \leq y \) we have for some \( C \)

\[
|\rho(x) - \rho(y)| \leq \frac{C|x - y|}{1 + x^2}.
\] (3.8)

Using this we can adapt the argument of proof of Theorem 4.2.2, p. 124–125 in Ibragimov-Linnik [16]. There is no loss of generality in assuming that \( \kappa \geq 2 \). Let

\[
\Delta_n = \Delta = \sup_{\kappa \geq 2} |n P(T_n = \kappa) - g(n^{-1} \kappa)|.
\]

By (3.7) we have \( \Delta_n \rightarrow 0 \) and therefore

\[
\sum_{\kappa \leq n/\sqrt{\Delta}} \left| P(T_n = \kappa) - \frac{1}{n} g \left( \frac{k}{n} \right) \right| = \sum_{\kappa \leq n/\sqrt{\Delta}} \frac{1}{n} \left| n P(T_n = \kappa) - g \left( \frac{k}{n} \right) \right| \leq \sqrt{\Delta} = o(1).
\] (3.9)
Almost sure local limit theorem for the Dickman distribution

Further by (3.8),

\[
\left| \sum_{\kappa \leq n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) - \int_0^{1/\sqrt{\Delta}} g(x) dx \right| \\
= \left| \sum_{\kappa \leq n/\sqrt{\Delta}} \int_{\kappa/n-1/2n}^{\kappa/n+1/2n} \left( g \left( \frac{\kappa}{n} \right) - g(x) \right) dx \right| + O \left( \frac{1}{n} \right) \\
\leq \frac{C}{2n} \sum_{\kappa \leq n/\sqrt{\Delta}} \int_{\kappa/n-1/2n}^{\kappa/n+1/2n} \frac{1}{1+x^2} dx + O \left( \frac{1}{n} \right) \\
= \frac{C}{2n} \sum_{\kappa \leq n/\sqrt{\Delta}} \arctg \left( \frac{\kappa}{n + 1/2} \right) - \arctg \left( \frac{\kappa}{n - 1/2} \right) + O \left( \frac{1}{n} \right) \\
= O \left( 1/n + \sqrt{\Delta} \right).
\]

By this we have

\[
\sum_{\kappa \leq n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) = 1 + O \left( \sqrt{\Delta} + 1/n \right)
\] (3.10)

because \( \int_0^{1/\sqrt{\Delta}} g(x) dx = 1 + O \left( \sqrt{\Delta} \right) \).

On the other hand (cf. (4.2.15) in [16]) we have by (3.8)

\[
\sum_{\kappa > n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) = \left| \sum_{\kappa > n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) - \int_{n/\sqrt{\Delta}}^{\infty} g(x) dx \right| + O \left( \sqrt{\Delta} \right) \\
= \left| \sum_{\kappa > n/\sqrt{\Delta}} \int_{\kappa/n-1/2n}^{\kappa/n+1/2n} \left( g \left( \frac{\kappa}{n} \right) - g(x) \right) dx \right| + O \left( \sqrt{\Delta} \right) \\
\leq \frac{C}{2n} \sum_{\kappa > n/\sqrt{\Delta}} \int_{\kappa/n-1/2n}^{\kappa/n+1/2n} \frac{1}{1+x^2} dx + O \left( \sqrt{\Delta} \right) \\
= \frac{C}{2n} \sum_{\kappa > n/\sqrt{\Delta}} \arctg \left( \frac{\kappa}{n + 1/2} \right) - \arctg \left( \frac{\kappa}{n - 1/2} \right) + O \left( \sqrt{\Delta} \right) \\
= O \left( 1/n + \sqrt{\Delta} \right).
\]

Thus

\[
\sum_{\kappa > n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) = O \left( 1/n + \sqrt{\Delta} \right).
\] (3.11)

Now, by (3.9) and (3.10)

\[
\sum_{\kappa > n/\sqrt{\Delta}} P(T_n = \kappa) \\
\leq 1 - \sum_{\kappa \leq n/\sqrt{\Delta}} P(T_n = \kappa) - 1 + \sum_{\kappa \leq n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) + 1 - \sum_{\kappa \leq n/\sqrt{\Delta}} \frac{1}{n} g \left( \frac{\kappa}{n} \right) \\
\leq \sqrt{\Delta} + O \left( 1/n + \sqrt{\Delta} \right) = O \left( 1/n + 2\sqrt{\Delta} \right).
\]
Whence
\[
\sum_{\kappa > n/\sqrt{\Delta}} P(T_n = \kappa) = O(1/n + 2\sqrt{\Delta}). \tag{3.12}
\]

Finally, by (3.9), (3.12) and (3.11)
\[
\sum_{\kappa \in \mathbb{N}} |P(T_n = \kappa) - n^{-1} g(n^{-1}\kappa)| \\
\leq \sum_{\kappa \leq n/\sqrt{\Delta}} |P(T_n = \kappa) - n^{-1} g(n^{-1}\kappa)| + \sum_{\kappa > n/\sqrt{\Delta}} |P(T_n = \kappa) - n^{-1} g(n^{-1}\kappa)| \\
\leq \sqrt{\Delta} + \sum_{\kappa > n/\sqrt{\Delta}} P(T_n = \kappa) + \sum_{\kappa > n/\sqrt{\Delta}} n^{-1} g(n^{-1}\kappa) \\
\leq \sqrt{\Delta} + O(1/n + 2\sqrt{\Delta}) + O(1/n + \sqrt{\Delta}) = o(1).
\]

\[\square\]

4 Proof of Theorem 1.1

We begin with observing that
\[
\text{Cov}(Y_m, Y_n) = nm \left\{ P(T_m = \kappa_m, T_n = \kappa_n) - P(T_m = \kappa_m)P(T_n = \kappa_n) \right\} \\
= nm \left\{ P(T_m = \kappa_m, T_m^n = \kappa_n - \kappa_m) - P(T_m = \kappa_m)P(T_n = \kappa_n) \right\} \\
= \{ mP(T_m = \kappa_m) \} \left\{ nP(T_m^n = \kappa_n - \kappa_m) - nP(T_n = \kappa_n) \right\}.
\]

Hence by Theorem 1.5, we have
\[
|\text{Cov}(Y_m, Y_n)| \leq C n P(T_m^n = \kappa_n - \kappa_m) - n P(T_n = \kappa_n) \\
\leq C \left( \left| \frac{n}{n - m} \{ (n - m)P(T_m^n = \kappa_n - \kappa_m) - D'(x) \} \right| + \left| \left( \frac{n}{n - m} - 1 \right) D'(x) \right| \right) \\
+ n P(T_n = \kappa_n) - D'(x) \right) \\
\leq C \left( \left| \frac{n}{n - m} \{ (n - m)P(T_m^n = \kappa_n - \kappa_m) - D'(x) \} \right| + \frac{m}{n - m} + \left| n P(T_n = \kappa_n) - D'(x) \right| \right) \\
= C \left( \frac{n}{n - m} \Gamma + \frac{m}{n - m} + \Delta \right), \tag{4.1}
\]

where we have put for simplicity
\[
\Gamma = |(n - m)P(T_m^n = \kappa_n - \kappa_m) - D'(x)|, \quad \Delta = \left| n P(T_n = \kappa_n) - D'(x) \right|.
\]

The aim is to obtain bounds for \( \Gamma \) and \( \Delta \).

(a) First consider \( \Gamma \). Set
\[
\Phi = P \left( (\kappa_n - \kappa_m) - n \leq T_m^n \leq (\kappa_n - \kappa_m) - (m + 1) \right) \\
\Psi = \left| (\kappa_n - \kappa_m) P(T_n = \kappa_n - \kappa_m) - \Phi \right|.
\]
\[
\Gamma \leq \frac{n-m}{\kappa_n - \kappa_m} \Psi + \left| \frac{n-m}{\kappa_n - \kappa_m} \Phi - D'(x) \right|
\]

\[
= \frac{n-m}{\kappa_n - \kappa_m} \Psi + \left| \frac{n-m}{\kappa_n - \kappa_m} \Phi - \frac{D(x) - D(x-1)}{x} \right|
\]

by (2.1). From Proposition 2.3 we know that

\[
\Psi \leq C \frac{\log \frac{n}{m}}{\sqrt{n-m}}.
\]

Put also

\[
\Lambda = \left| P \left( \frac{T_n}{n-m} \leq \frac{(\kappa_n - \kappa_m) - (m+1)}{n-m} \right) - D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (m+1)}{n-m} \right|
\]

\[
\Theta = \left| P \left( \frac{T_n}{n-m} \leq \frac{(\kappa_n - \kappa_m) - (n+1)}{n-m} \right) - D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (n+1)}{n-m} \right|
\]

\[
\Sigma = \frac{n-m}{\kappa_n - \kappa_m} D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (m+1)}{n-m} - \frac{D(x)}{x}
\]

\[
\Omega = \frac{n-m}{\kappa_n - \kappa_m} D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (n+1)}{n-m} - \frac{D(x-1)}{x}
\]

One easily check that

\[
\left| \frac{n-m}{\kappa_n - \kappa_m} \Phi - \frac{D(x) - D(x-1)}{x} \right| \leq \frac{n-m}{\kappa_n - \kappa_m} (\Lambda + \Theta) + \Sigma + \Omega.
\]

We know from Proposition 2.9 that

\[
\sup_{x \in \mathbb{R}} \left| P \left( \frac{T_n}{n-m} \leq x \right) - D(x) \right| \leq C_{g,m,n}.
\]

Hence

\[
\Lambda + \Theta \leq C_{g,m,n},
\]

Moreover

\[
\Sigma \leq \frac{n-m}{\kappa_n - \kappa_m} D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (m+1)}{n-m} - \frac{D(x)}{x} + \left| \frac{n-m}{\kappa_n - \kappa_m} - \frac{1}{x} \right| D(x)
\]

\[
\leq \frac{n-m}{\kappa_n - \kappa_m} D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (m+1)}{n-m} - \frac{D(x)}{x} + \left| \frac{n-m}{\kappa_n - \kappa_m} - \frac{1}{x} \right|
\]

and, by Lagrange Theorem, there exists \( \xi_n \) such that

\[
\frac{n-m}{\kappa_n - \kappa_m} D - \frac{\left(\frac{\kappa_n - \kappa_m}{n-m} \right) - (m+1)}{n-m} - \frac{D(x)}{x} \leq \frac{(\kappa_n - \kappa_m) - (m+1)}{n-m} - \frac{D'(\xi_n)}{n-m}
\]

\[
\leq \frac{n-m}{\kappa_n - \kappa_m} \frac{(\kappa_n - \kappa_m) - (m+1)}{n-m} - \frac{D'(\xi_n)}{n-m}
\]

\[
\leq \frac{n-m}{\kappa_n - \kappa_m} \frac{(\kappa_n - \kappa_m) - (m+1)}{n-m} - \frac{D'(\xi_n)}{n-m} \leq x \left| \frac{n-m}{\kappa_n - \kappa_m} - \frac{1}{x} \right| + \frac{m+1}{\kappa_n - \kappa_m},
\]

since \( \sup_{x>0} D'(x) = 1 \). For \( \Omega \) we get exactly the same bound as in (4.4).

In conclusion, from (4.2), (4.3) and (4.4) we have obtained

\[
\Gamma = \left| (n-m) P \left( \frac{T_n}{n-m} = \kappa_n - \kappa_m \right) - D'(x) \right| \leq C_{\chi,m,n}^{(\kappa_n)}.
\]
(b) Now consider $\Delta$.

Recall that

$$\Delta = \left| n P(T_n = \kappa_n) - D'(x) \right|.$$ 

Notice that we cannot apply (4.5) directly since we have proved it for $m \geq 2$ only. Nevertheless, with $U = Z_1 + 2Z_2$,

$$\Delta = \sum_{j=0}^{3} P(U = j) \left( n P(T_n^2 = \kappa_n - j) - D'(x) \right) \leq \sup_{0 \leq j \leq 3} \left| n P(T_n^2 = \kappa_n - j) - D'(x) \right|$$

$$= \sup_{0 \leq j \leq 3} \left| \frac{n}{n-2} \left( (n-2) P(T_n^2 = \kappa_n - j) - D'(x) \right) + \frac{2}{n-2} D'(x) \right|$$

$$\leq \frac{n}{n-2} \left( \sup_{0 \leq j \leq 3} |(n-2) P(T_n^2 = \kappa_n - j) - D'(x)| + \frac{2}{n-2} \right)$$

$$\leq C \frac{n}{n-2} \left( \chi_{2,n}^{(\kappa,x)} + \frac{2}{n} \right) \cdot \chi_{2,n}^{(\kappa,x)}.$$  \hspace{1cm} (4.6)

applying (4.5) (with $m = 2$) for the sequence $\kappa^{(j)} = (\kappa_n^{(j)})_n$ defined as $\kappa_n^{(j)} = \kappa_n - j$ and noticing that $\chi_{2,n}^{(\kappa^{(j)},x)} = \chi_{2,n}^{(\kappa,x)}$.

The two relations (4.5) and (4.6), inserted into (4.1), conclude the proof.

5 A general form of the almost sure limit theorem

As we pointed out in the Introduction, the Almost Sure Limit Theorem that we are going to prove in the present section (i.e. Theorem 5.8) is in the spirit of Theorem 1 of T. Mori’s paper [17]; in Sect. 6 it will be applied to the sequence $(Y_n)$ defined in Theorem 1.1: notice that Mori’s result is not directly applicable in this context since it requires that $|Cov(Y_m, Y_n)| \leq h(\frac{m}{n})$ for all $1 \leq m \leq n$ (for a suitable function $h$). For $m = n$ this inequality becomes $Var Y_m \leq h(1) = C$, i.e. the sequence $(Var Y_m)_{m \geq 1}$ must be bounded. Unfortunately this is not true in our setting (see Lemma 6.2).

**Theorem 5.1** Let $(U_n)_{n \geq 1}$ be a sequence of centered random variables. Assume that there exist two numbers $\alpha \geq 0$ and $\sigma > 1$, a non-negative function $f(u, z)$ defined on the set $\{u \geq 1, z \geq \sigma \}$, a non-negative double-indexed sequence $g$ defined on the set $\{(m, n) \in \mathbb{N}^2 : \sigma m \leq n \}$ such that

(i) uniformly in $u > 0$ the functions $u \mapsto f(u, \frac{\nu}{u})$ are ultimately non-increasing (i.e. there exists $m_0$ such that $u \mapsto f(u, \frac{\nu}{u})$ is non-increasing on $(m_0, +\infty)$ and for every $u > 0$);

(ii) the functions

$$z \mapsto \phi(z) = \sup_{u \geq 1} \frac{f(u, z)}{z}, \quad u \mapsto F(u) = \int_{\sigma}^{u} \phi(z) \, dz$$

are defined on $[\sigma, +\infty)$;
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(iv) \[ |\text{Cov}(U_m, U_n)| \leq C \begin{cases} m & \text{for } m = n \\ 1 & \text{for } m < n \leq \sigma m; \end{cases} \]

(v) there exists \( m_1 \) such that, for \( n > m \geq m_1 \)
\[ |\text{Cov}(U_m, U_n)| \leq g(m, n) \frac{1}{m^a} f\left(m, \frac{n}{m}\right). \]

Denote \( V_n = \sum_{k=\sigma^{n-1}+1}^{\sigma^n} \frac{U_k}{k} \).

Then, for every \( n \) and every sufficiently large \( m \)
\[ E \left[ \sum_{i=m+1}^{m+n} V_i \right]^2 \leq C \left( n + \frac{1}{\sigma^{a(m+n)}} \int_{\sigma}^{\sigma^n} F(u)u^{a-1} \, du \right). \]

**Proof** We start with
\[ E \left[ \sum_{i=m+1}^{m+n} V_i \right]^2 = \sum_{i=m+1}^{m+n} E[V_i^2] + 2 \sum_{m+1 \leq i < j \leq m+n} E[V_i V_j]. \] (5.1)

At first,
\[ E[V_i^2] = E \left( \sum_{h=\sigma^{i-1}+1}^{\sigma^i} \frac{U_h}{h} \right) \left( \sum_{k=\sigma^{i-1}+1}^{\sigma^i} \frac{U_k}{k} \right) = \sum_{h=k=\sigma^{i-1}+1}^{\sigma^i} \frac{1}{hk} E[U_h U_k] \]
\[ = \sum_{h=\sigma^{i-1}+1}^{\sigma^i} \frac{1}{h^2} E[U_h^2] + 2 \sum_{\sigma^{i-1}+1 \leq h < k \leq \sigma^i} \frac{1}{hk} E[U_h U_k]. \]
(5.2)

By the first inequality in (iv)
\[ \sum_{h=\sigma^{i-1}+1}^{\sigma^i} \frac{1}{h^2} E[U_h^2] \leq C \sum_{h=\sigma^{i-1}+1}^{\sigma^i} \frac{1}{h} \leq C \log \frac{\sigma^i}{\sigma^{i-1}} = C. \] (5.3)

For the second summand in (5.2), i.e.
\[ \sum_{\sigma^{i-1}+1 \leq h < k \leq \sigma^i} \frac{1}{hk} E[U_h U_k], \]
we notice that
\[ \frac{k}{\sigma} \leq \sigma^{i-1} < h \]
so that, by the second inequality in (iv), we have
\[ \sum_{\sigma^{i-1}+1 \leq h < k \leq \sigma^i} \frac{1}{hk} E[U_h U_k] \leq C \left( \sum_{k=\sigma^{i-1}+1}^{\sigma^i} \frac{1}{k} \right) \left( \sum_{h=\sigma^{i-1}+1}^{\sigma^i} \frac{1}{h} \right) \leq C. \] (5.4)
The above relations (5.3) and (5.4), used in (5.2), give

\[ \sum_{i=m+1}^{m+n} E[V_i^2] \leq C \sum_{i=m+1}^{m+n} 1 = Cn. \]  

(5.5)

Now we consider the second sum in (5.1), i.e.

\[ \sum_{m+1 \leq i < j \leq m+n} E[V_i V_j]. \]

We start with a bound for the summand \( E[V_i V_j] \) when \( j \geq i + 2 \). First, notice that here

\[ E[V_i V_j] = E \left[ \left( \sum_{h=\sigma^{-1}+1}^{\sigma^i} \frac{U_h}{h} \right) \left( \sum_{k=\sigma^{-1}+1}^{\sigma^j} \frac{U_k}{k} \right) \right] \]

(5.6)

and

\[ h \leq \sigma^i \leq \sigma^{i-2} \leq \frac{k}{\sigma}, \]  

(5.7)

hence, by assumption (i),

\[ g(h, k) \leq C; \]

thus the inequality in (v) can be simplified into

\[ |Cov(U_h, U_k)| \leq \frac{1}{h^{\alpha+1}} f \left( h, \frac{k}{h} \right) \]

(we incorporate the constant \( C \) into \( f \) for simplicity). Hence, for \( m \) sufficiently large in order that \( \sigma^{i-1} + 1 > m_0 \), by assumption (ii) we have

\[ E[V_i V_j] = \sum_{h=\sigma^{-1}+1}^{\sigma^i} \sum_{k=\sigma^{-1}+1}^{\sigma^j} \frac{1}{hk} E[U_h U_k] \leq \sum_{h=\sigma^{-1}+1}^{\sigma^i} \frac{1}{h^{\alpha+1}} \int_{\sigma^{-1}}^{\sigma^i} \frac{1}{v} f \left( h, \frac{v}{h} \right) dv \]

\[ \leq \int_{\sigma^{-1}}^{\sigma^i} du \frac{1}{u^{\alpha+1}} \int_{\sigma^{-1}}^{\sigma^i} \frac{1}{v} f \left( u, \frac{v}{u} \right) dv. \]

By means of the change of variable \( v = uz \) in the inner integral, the above becomes

\[ \int_{\sigma^{-1}}^{\sigma^i} dx \frac{1}{u^{\alpha+1}} \int_{\sigma^{-1}}^{\sigma^i} \frac{1}{z} f(u, z) dz \leq \int_{\sigma^{-1}}^{\sigma^i} dx \frac{1}{u^{\alpha+1}} \int_{\sigma^{-1}}^{\sigma^i} \phi(z) dz \]

\[ = \int_{\sigma^{-1}}^{\sigma^i} du \frac{1}{u^{\alpha+1}} \left\{ F \left( \frac{\sigma^i}{u} \right) - F \left( \frac{\sigma^{i-1}}{u} \right) \right\}. \]
Hence
\[
\sum_{m+1 \leq i < j \leq m+n} \sum_{j \geq i+2} E[V_i V_j] \leq \sum_{i=m+1}^{m+n} \sum_{j=i+2}^{m+n} \int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} \left\{ F\left( \frac{\sigma_j}{u} \right) - F\left( \frac{\sigma_{j-1}}{u} \right) \right\}
\]
\[
= \sum_{i=m+1}^{m+n-2} \int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} \sum_{j=i+2}^{m+n} \left\{ F\left( \frac{\sigma_j}{u} \right) - F\left( \frac{\sigma_{j-1}}{u} \right) \right\}
\]
\[
= \sum_{i=m+1}^{m+n-2} \int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} \left\{ F\left( \frac{\sigma^{m+n}}{u} \right) - F\left( \frac{\sigma^{i+1}}{u} \right) \right\}
\]
\[
= \sum_{i=m+1}^{m+n-2} \int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} \left\{ F\left( \frac{\sigma^{m+n}}{u} \right) - \sum_{i=m+1}^{m+n-2} \int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} F\left( \frac{\sigma^{i+1}}{u} \right) \right\}
\]
\[
= \int_{\sigma_m}^{\sigma_{m+n-2}} du \frac{1}{u^{\alpha+1}} F\left( \frac{\sigma^{m+n}}{u} \right) - \sum_{i=m+1}^{m+n-2} \int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} F\left( \frac{\sigma^{i+1}}{u} \right). \tag{5.8}
\]

By the change of variable \( \frac{\sigma^{m+n}}{u} = v \) we get
\[
\int_{\sigma_m}^{\sigma_{m+n-2}} du \frac{1}{u^{\alpha+1}} F\left( \frac{\sigma^{m+n}}{u} \right) = \frac{1}{\sigma^{\alpha(m+n)}} \int_{\sigma^2}^{\sigma^n} F(v)v^{\alpha-1} dv
\]
\[
= \frac{1}{\sigma^{\alpha(m+n)}} \left( \int_\sigma^{\sigma^n} F(v)v^{\alpha-1} dv - \int_\sigma^{\sigma^2} F(v)v^{\alpha-1} dv \right). \tag{5.9}
\]

In a similar way, by the change of variable \( \frac{\sigma^{i+1}}{u} = v \) we get
\[
\int_{\sigma_{i-1}}^{\sigma_i} du \frac{1}{u^{\alpha+1}} F\left( \frac{\sigma^{i+1}}{u} \right) = \frac{1}{\sigma^{\alpha(i+1)}} \int_{\sigma}^{\sigma^2} F(v)v^{\alpha-1} dv. \tag{5.10}
\]

By inserting (5.9) and (5.10) into (5.8), we get
\[
\sum_{m+1 \leq i < j \leq m+n} \sum_{j \geq i+2} E[V_i V_j]
\]
\[
\leq \frac{1}{\sigma^{\alpha(m+n)}} \left( \int_\sigma^{\sigma^n} F(u)u^{\alpha-1} du - \int_\sigma^{\sigma^2} F(u)u^{\alpha-1} du \right)
\]
\[
- \sum_{i=m+1}^{m+n-2} \frac{1}{\sigma^{\alpha(i+1)}} \int_\sigma^{\sigma^2} F(u)u^{\alpha-1} du
\]
\[
= \frac{1}{\sigma^{\alpha(m+n)}} \int_\sigma^{\sigma^n} F(u)u^{\alpha-1} du - \sum_{i=m+1}^{m+n} \frac{1}{\sigma^{\alpha i}} \int_\sigma^{\sigma^2} F(u)u^{\alpha-1} du
\]
\[
= \frac{1}{\sigma^{\alpha(m+n)}} \int_\sigma^{\sigma^n} F(u)u^{\alpha-1} du - C \sum_{i=m+1}^{m+n} \frac{1}{\sigma^{\alpha i}}. \tag{5.11}
\]
And now, by (5.2), (5.3) and (5.4),
\[\sum_{m+1 \leq i < j \leq m+n} E[V_i V_j] = \sum_{m+1 \leq i < j \leq m+n} E[V_i V_j] + \sum_{i=m+1}^{m+n} E[V_i V_{i+1}]\]
\[\leq \sum_{m+1 \leq i < j \leq m+n} E[V_i V_j] + \sum_{i=m+1}^{m+n-1} E[V_i^2]^{1/2} E[V_{i+1}^2]^{1/2}\]
\[\leq \frac{1}{\sigma^{\alpha(m+n)}} \int_{\sigma}^{\sigma^n} F(u) u^{\alpha-1} du - C \sum_{i=m+2}^{m+n-1} \frac{1}{\sigma^{\alpha i}} + C \sum_{i=m+1}^{m+n-1} 1\]
\[\leq C \left( n + \frac{1}{\sigma^{\alpha(m+n)}} \int_{\sigma}^{\sigma^n} F(u) u^{\alpha-1} du \right). \tag{5.12}\]

From (5.1), (5.5) and (5.12) we obtain
\[E \left[ \left( \sum_{i=m+1}^{m+n} V_i \right)^2 \right] \leq C \left( n + \frac{1}{\sigma^{\alpha(m+n)}} \int_{\sigma}^{\sigma^n} F(u) u^{\alpha-1} du \right),\]
as claimed. \(\square\)

Similar techniques prove the following more general result:

**Theorem 5.2** Let \((U_n)_{n \geq 1}\) be a sequence of centered random variables. Let \(N\) be an integer and assume that there exist a number \(\sigma > 1\) and for each \(j = 1, 2, \ldots, N\) numbers \(\alpha_j \geq 0\) a non-negative function \(f_j(u, z)\) defined on the set \(\{u \geq 1, z \geq \sigma\}\), a non-negative double-indexed sequence \(g_j\) defined on the set \(\{(m, n) \in \mathbb{N}^2 : \sigma m \leq n\}\) such that

(i) \(\sup_{n \geq \sigma m} g_j(m, n) = C < +\infty;\)

(ii) uniformly in \(u > 0\) the functions \(u \mapsto f_j(u, \frac{v}{u})\) are ultimately non-increasing (i.e. there exists \(m_0\) such that \(v \mapsto f_j(u, \frac{v}{u})\) are non-increasing on \((m_0, +\infty)\), for each \(j = 1, \ldots, N\) and for every \(u > 0\));

(iii) for each \(j = 1, 2, \ldots, N\) the functions \(z \mapsto \phi_j(z) = \sup_{u \geq 1} \frac{f_j(u, z)}{z}\), \(u \mapsto F_j(u) = \int_{\sigma}^{u} \phi_j(z) dz\)

are defined on \([\sigma, +\infty);\)

(iv) \(|\text{Cov}(U_m, U_n)| \leq C \begin{cases} m & \text{for } m = n \\ 1 & \text{for } m < n \leq \sigma m; \end{cases}\)

(v) there exists \(m_1\) such that, for \(n > m \geq m_1\)

\[|\text{Cov}(U_m, U_n)| \leq \sum_{j=1}^{N} g_j(m, n) \frac{1}{m^{\alpha_j}} f_j \left( m, \frac{n}{m} \right).\]

Denote
\[V_n = \sum_{k=\sigma^{n-1}+1}^{\sigma^n} \frac{U_k}{k}.\]
Then, for every \( n \) and every sufficiently large \( m \)

\[
E \left[ \left( \sum_{i=m+1}^{m+n} V_i \right)^2 \right] \leq C \left( n + \sum_{j=1}^{N} \frac{1}{\sigma_{\alpha j} (m+n)} \int_{\sigma}^{\sigma_n} F_j (u) u^{\alpha j-1} \, du \right).
\]

**Corollary 5.3** In the setting of Theorem 5.1, assume in addition that \( \alpha = 0 \) and there exists \( \beta > 1 \) such that \( F(x) \leq C (\log x)^{\beta-1} \) for every \( x > \sigma \). Then, for every sufficiently large \( m \),

\[
E \left[ \left( \sum_{i=m+1}^{m+n} V_i \right)^2 \right] \leq C \left( (m+n)^{\beta} - m^\beta \right).
\] (5.13)

**Proof** Putting \( \alpha = 0 \) in the claim of Theorem 5.1 we obtain

\[
E \left[ \left( \sum_{i=m+1}^{m+n} V_i \right)^2 \right] \leq C \left( n + \int_{\sigma}^{\sigma_n} F(u) \frac{1}{u} \, du \right)
\]

\[
\leq C \left( n + \int_{\sigma}^{\sigma_n} (\log u)^{\beta-1} \frac{1}{u} \, du \right) = C \left( n + \left[ (\log u)^{\beta}\right]_{\sigma_n} \right)
\]

\[
\leq C \left( n + n^\beta \right) \leq C n^\beta.
\]

On the other hand, the function \( z \mapsto \{(z+n)^{\beta} - z^\beta\} \) being increasing (its derivative is \( \beta(z+n)^{\beta-1} - \beta z^{\beta-1} \geq 0 \), we have

\[
n^\beta \leq (1+n)^{\beta} - 1 \leq (m+n)^{\beta} - m^{\beta}.
\]

\[\square\]

**Corollary 5.4** In the setting of Theorem 5.1, assume in addition that \( \alpha > 0 \) and there exists \( \beta > 1 \) such that \( F(x) \leq C (\log x)^{\beta} \) for every \( x > \sigma \). Then, for every sufficiently large \( m \),

\[
E \left[ \left( \sum_{i=m+1}^{m+n} V_i \right)^2 \right] \leq C \left( (m+n)^{\beta} - m^\beta \right).
\] (5.14)

**Proof** In this case Theorem 5.1 gives

\[
E \left[ \left( \sum_{i=m+1}^{m+n} V_i \right)^2 \right] \leq C \left( n + \frac{1}{\sigma^{\alpha \gamma}} \int_{\sigma}^{\sigma_n} F(u) u^{\alpha-1} \, du \right)
\]

\[
\leq C \left( n + \frac{1}{\sigma^{\alpha n}} \int_{\sigma}^{\sigma_n} (\log u)^{\beta} u^{\alpha-1} \, du \right)
\]

\[
\leq C \left( n + \frac{1}{\sigma^{\alpha n}} (\log(\sigma^n))^{\beta} \int_{\sigma}^{\sigma_n} u^{\alpha-1} \, du \right)
\]

\[
= C \left( n + \frac{(\log(\sigma^n))^{\beta} (\sigma^{\alpha n} - \sigma^\alpha)}{\alpha \sigma^{\alpha n}} \right) \leq C \left( n + n^{\beta} \right) \leq C n^\beta.
\]

The remaining part of the proof is identical to Corollary 5.3. \[\square\]
Corollary 5.5 In the setting of Theorem 5.2, assume that there exists $\beta > 1$ such that $\sum_{j=1}^{N} F_{j}(x) \leq C(\log x)^{\beta}$ for every $x > \sigma$. Then

(i) for every sufficiently large $m$ and for every $n$,

$$E \left[ \left( \sum_{i=m+1}^{m+n} V_{i} \right)^{2} \right] \leq C \left( (m + n)^{\beta} - m^{\beta} \right).$$

(ii) for every $\delta > 0$,

$$\sum_{i=1}^{n} V_{i} = O(n^{\beta/2}(\log n)^{2+\delta}), \quad P - a.s.$$ 

Proof Point (i) follows from Corollaries 5.3 and 5.4. Point (ii) is a consequence of the well known Gaal–Koksma Strong Law of Large Numbers (see [20], p. 134), which we recall. □

Theorem 5.6 Let $(V_{n})_{n \geq 1}$ be a sequence of centered random variables with finite variance. Suppose that there exists a constant $\beta > 0$ such that, for all integers $m \geq 0$, $n \geq 1$,

$$E \left[ \left( \sum_{i=m+1}^{m+n} V_{i} \right)^{2} \right] \leq C \left( (m + n)^{\beta} - m^{\beta} \right),$$

for a suitable constant $C$ independent of $m$ and $n$. Then, for every $\delta > 0$,

$$\sum_{i=1}^{n} V_{i} = O(n^{\beta/2}(\log n)^{2+\delta}), \quad P - a.s.$$ 

Remark 5.7 It is not difficult to see that Theorem 5.6 is in force even if the bound (5.15) holds only for all integers $m \geq h_{0}$, $n > 0$, where $h_{0}$ is an integer strictly greater than 0. A rigorous proof of this statement can be found in the appendix of [9]. From now on, this slight generalization will be tacitly used.

Theorem 5.8 (General ASLT) Let $(\Upsilon_{n})_{n \geq 1}$ be a sequence of non-negative (resp. non-positive) random variables with

$$\lim_{n \to \infty} E[\Upsilon_{n}] = \ell > 0 \quad (\text{resp. } \ell < 0)$$

and such that the sequence $(U_{n})_{n \geq 1}$ defined by $U_{n} = \Upsilon_{n} - E[\Upsilon_{n}]$ verifies the assumptions of Theorem 5.2. Assume that there exists $\beta \in (1, 2)$ such that $\sum_{j=1}^{N} F_{j}(x) \leq C(\log x)^{\beta}$ for every $x > \sigma$. Then

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{\Upsilon_{k}}{k} = \ell, \quad a.s.$$ 

Proof By point (ii) of Corollary 5.5, for every $\delta > 0$ we have

$$\frac{\sum_{i=1}^{n} V_{i}}{n} = \frac{O(n^{\beta/2}(\log n)^{2+\delta})}{n} \to 0. \quad (5.16)$$

Since

$$\sum_{i=1}^{n} V_{i} = \sum_{i=1}^{n} \sum_{k=\sigma_{i+1}^{-1}}^{\sigma_{i}} \frac{U_{k}}{k} = \sum_{k=2}^{\sigma_{n}} \frac{U_{k}}{k} = \sum_{k=2}^{\sigma_{n}} \frac{\Upsilon_{k}}{k} - \sum_{k=2}^{\sigma_{n}} \frac{E[\Upsilon_{k}]}{k}$$

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and

\[
\frac{1}{n \log \sigma} \sum_{k=2}^{\sigma n} \frac{E[\Upsilon_k]}{k} \xrightarrow{n \to \infty} \ell,
\]

the relation (5.16) is equivalent to

\[
\frac{1}{\log n} \sum_{k=1}^{n} \frac{\Upsilon_k}{k} \xrightarrow{n \to \infty} \ell.
\]

By the same argument as in [9], pp. 789–790, this in turn implies that

\[
\frac{1}{\log n} \sum_{k=1}^{m} \frac{\Upsilon_k}{k} \xrightarrow{n \to \infty} \ell,
\]

as claimed. \(\square\)

6 Proof of Theorem 1.2

Let \(x > 0\) and let \(\kappa = (\kappa_n)_{n \geq 1}\) be a strictly increasing sequence of integers with \(\kappa_n \sim x n\), fixed throughout the sequel. Some Lemmas are necessary. For every \(\epsilon \in (0, \frac{1}{2x})\) we set

\[
\sigma = \sigma_\epsilon = \frac{1 + x(1 - \epsilon)}{x(1 + \epsilon)} = 1 + \frac{2}{1 + \epsilon} \left( \frac{1}{2x} - \epsilon \right) > 1. \tag{6.1}
\]

Lemma 6.1 Let \(\epsilon \in (0, \frac{1}{2x})\) be fixed. Then there exists \(m_0 = m_0(\epsilon)\) such that, for \(\sigma m > n > m > m_0\),

\[
P \left( T^n_m = \kappa_n - \kappa_m \right) = 0.
\]

Proof Let

\[
A = \bigcap_{k=m+1}^{n} \{ Z_k = 0 \}.
\]

Then

\[
P \left( T^n_m = \kappa_n - \kappa_m \right) = P \left( \left( T^n_m = \kappa_n - \kappa_m \right) \cap A \right) + P \left( \left( T^n_m = \kappa_n - \kappa_m \right) \cap A^c \right).
\]

(i) Let \(m_0\) be such that, for every \(m > m_0\),

\[
xm(1 - \epsilon) < \kappa_m < xm(1 + \epsilon).
\]

Then, for \(\sigma m > n > m > m_0\),

\[
\kappa_n - \kappa_m < x n (1 + \epsilon) - x m (1 - \epsilon) = m \left\{ x \frac{n}{m} (1 + \epsilon) - x (1 - \epsilon) \right\} \leq m \left\{ x \sigma (1 + \epsilon) - x (1 - \epsilon) \right\} = m. \tag{6.2}
\]
Hence
\[
\{T^n_m = \kappa_n - \kappa_m\} \cap A^c = \{T^n_m = \kappa_n - \kappa_m\} \cap \left( \bigcup_{k=m+1}^{n} \{Z_k = 1\} \right)
\]
\[
= \bigcup_{k=m+1}^{n} \{T^n_m = \kappa_n - \kappa_m, Z_k = 1\} \subseteq \bigcup_{k=m+1}^{n} \{T^n_m = \kappa_n - \kappa_m, T^n_m \geq m + 1\}
\]
\[
= \{T^n_m = \kappa_n - \kappa_m, T^n_m \geq m + 1\} = \emptyset,
\]
by (6.2).

(ii)
\[
\{T^n_m = \kappa_n - \kappa_m\} \cap A \subseteq \{T^n_m = \kappa_n - \kappa_m, T^n_m = 0\} = \emptyset,
\]
hence
\[
\{T^n_m = \kappa_n - \kappa_m\} \cap A \subseteq \{T^n_m = \kappa_n - \kappa_m, T^n_m = 0\} = \emptyset,
\]
since \(\kappa_n - \kappa_m > 0\).

\[\square\]

**Lemma 6.2** Let \(\epsilon \in (0, \frac{1}{2\lambda})\) be fixed. Then there exists \(m_0 = m_0(\epsilon)\) such that, for \(n \geq m > m_0\),
\[
|\text{Cov}(Y_m, Y_n)| \leq C \begin{cases} m & \text{for } m = n \\ 1 & \text{for } m < n \leq \sigma m, \end{cases}
\]
where \(C\) is a positive constant.

**Proof** (a) For \(m = n:\)
\[
\text{Cov}(Y_m, Y_m) = m^2 \left\{ P(T_m = \kappa_m) - P^2(T_m = \kappa_m) \right\} = \{mP(T_m = \kappa_m)\} \{m - P(T_n = \kappa_m)\} \leq Cm,
\]
by the local limit theorem (Corollary 1.5).

(b) For \(m < n \leq \sigma m\): let \(m_0\) be as in Lemma 6.1. Then, for \(\sigma m > n > m > m_0\),
\[
|\text{Cov}(Y_m, Y_n)| = mn \left| P(T_m = \kappa_m, T_n = \kappa_n) - P(T_m = \kappa_m)P(T_n = \kappa_n) \right|
\]
\[
= \{mP(T_m = \kappa_m)\} \left| nP(T_m = \kappa_n - \kappa_m) - nP(T_n = \kappa_n) \right|
\]
\[
= \{mP(T_m = \kappa_m)\} \left| nP(T_n = \kappa_n) \right| \leq C,
\]
by the Local Theorem again and observing that \(P(T^n_m = \kappa_n - \kappa_m) = 0\), by Lemma 6.1.
\[\square\]

**Remark 6.3** Notice that
(i) \(\kappa_n = [xn]\) is strictly increasing if \(x \geq 1\);
(ii) If \(\kappa_n = xn\) we can take \(\epsilon = 0\) and \(m_0 = 1\).

**Lemma 6.4** In the setting of Theorem 5.1, assume that \(f\) has the form
\[
f(u, z) = \psi(uz)
\]
where \(t \mapsto \psi(t)\) is a continuous ultimately non-increasing function, i.e. there exists \(t_0\) such that \(t \mapsto \psi(t)\) is non-increasing for \(t \geq t_0\). Then
\[
F(u) \leq C \begin{cases} 1 & \text{for } u \leq t_0 \\ 1 + \int_{t_0}^{u} \frac{\psi(z)}{z} dz & \text{for } u > t_0. \end{cases}
\]
\[\square\]
Proof It is easy to see that
\[
\sup_{x \geq 1} f(x, z) = \sup_{x \geq 1} \psi(xz) = \sup_{u \geq z} \psi(u) \begin{cases} \leq \max_{u \in [1, t_0]} \psi(u) =: M & \text{for } z \leq t_0 \\ = \psi(z) & \text{for } z > t_0. \end{cases}
\]
Hence
\[
\phi(z) \leq \begin{cases} \frac{M}{\psi(z)} & \text{for } z \leq t_0 \\ \phi(z) & \text{for } z > t_0. \end{cases}
\]
and
\[
F(u) = \int_0^u \phi(z) \, dz = \int_{t_0}^u \phi(z) \, dz + \int_{t_0}^u \phi(z) \, dz \leq C \left\{ \frac{1}{1 + \int_{t_0}^u \psi(z) \, dz} \right\} \quad \text{for } u \leq t_0
\]
\[
\int_{t_0}^u \phi(z) \, dz \leq \int_{t_0}^{u_0} \phi(z) \, dz \leq C \left\{ \frac{1}{1 + \int_{t_0}^{u_0} \psi(z) \, dz} \right\} \quad \text{for } u > t_0.
\]
\]
\]
\[
\begin{array}{ll}
\text{Remark 6.5} & \text{Of course, the preceding lemma has an obvious generalization in the setting of Theorem 5.2.} \\
\end{array}
\]
We are now ready to give the
\[
|\text{Cov}(Y_m, Y_n)| \leq \frac{n}{\kappa_n - \kappa_m} \left\{ \frac{1 + \log \frac{n}{m}}{\sqrt{n-m}} + \exp \left( C \left\{ \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m \log^2 \frac{n}{m}}{n-m} \right\} \right) \right\} - 1 + \frac{1}{\log \frac{n}{m}} \\
+ \frac{|(xn - \kappa_n) - (xm - \kappa_m)|}{n-m} + \frac{m}{n-m} \left\{ \frac{1}{\log n} + \frac{|(xn - \kappa_n) - (2x - \kappa_2)|}{n-2} + \frac{1}{n}. \right\}
\]
\[
(6.4)
\]
In fact (look at the formula in the statement of Theorem 1.1)
\[
\frac{n}{n-m} \kappa_n, x = \frac{n}{\kappa_n - \kappa_m} \left( \frac{\log \frac{n}{m}}{\sqrt{n-m}} + g_{m,n} \right) + x \left( \frac{n-m}{\kappa_n - \kappa_m} \right) \frac{1}{x} \left( \frac{n}{n-m} \right) + \frac{n}{n-m} \frac{m+1}{\kappa_n - \kappa_m},
\]
and
\[
\begin{array}{ll}
(a) \quad \frac{n}{\kappa_n - \kappa_m} \left( \frac{1 + \log \frac{n}{m}}{\sqrt{n-m}} + g_{m,n} \right) \\
= \frac{n}{\kappa_n - \kappa_m} \left[ \frac{1 + \log \frac{n}{m}}{\sqrt{n-m}} + \exp \left( C \left\{ \frac{\log \frac{n}{m}}{(n-m)^2} + \frac{m + 2}{n-m} \log^2 \frac{n}{m} \right\} \right) \right] \\
\leq \frac{n}{\kappa_n - \kappa_m} \left[ \frac{1 + \log \frac{n}{m}}{\sqrt{n-m}} + \exp \left( C \left\{ \frac{\log^3 \frac{n}{m}}{(n-m)^2} + \frac{m \cdot \log^2 \frac{n}{m}}{n-m} \right\} \right) \right] - 1 + \frac{1}{\log \frac{n}{m}} \\
\end{array}
\]
Further, by (a), (b) and (c) above
\[ x_{\delta} \leq \frac{n}{\kappa_n - \kappa_m} \left[ \frac{1 + \log \frac{n}{\delta}}{\sqrt{n - 2}} + \exp \left( C \left\{ \log^3 \frac{n}{\delta} + \frac{2 \cdot \log^2 \frac{n}{\delta}}{n - 2} \right\} \right) - 1 + \frac{1}{\log \frac{n}{\delta}} \right]. \]
for sufficiently large \( n \); recall that we are neglecting multiplicative constants).

Theorem 5.2 will be deduced from Theorem 5.8. It is easy to see that assumptions (i)–(iv) of Theorem 5.2 are in force for each summand in the basic correlation inequality, hence we omit the details. We shall check assumption (v) of Theorem 5.2 for each summand in the basic correlation inequality [in the form (6.4)] and use Corollaries 5.3 or 5.4. More precisely, using the notations of Theorem 5.2 and with \( \sigma \) defined in (6.1):

(1) First summand:
\[
\frac{n}{\kappa_n - \kappa_m} \cdot \frac{1 + \log \frac{n}{\delta}}{\sqrt{n - 2}}.
\]

Fix \( \delta \in (0, \frac{\sigma}{\sigma - 1}) \) and let \( m_1 \) be such that \( 1 - \delta < \frac{k_n}{m} < 1 + \delta \), if \( n > m_1 \). Then for \( n > \sigma m \) and \( m > m_1 \),
\[
\frac{\kappa_n - \kappa_m}{n} = \frac{\kappa_n}{n} - \frac{\kappa_m}{m} \cdot \frac{m}{n} \geq (1 - \delta) - \frac{1 + \delta}{\sigma} > 0.
\]
Hence
\[
\sup_{n > \sigma m} \frac{n}{\kappa_n - \kappa_m} \leq \frac{1}{(1 - \delta) - \frac{1 + \delta}{\sigma}}.
\]

Moreover we have \( \frac{1 + \log \frac{\sigma}{\sigma - 1}}{\sqrt{x - 1}} \), hence
\[
g_1(m, n) = \frac{n}{\kappa_n - \kappa_m}, \quad \alpha_1 = \frac{1}{2}, \quad f_1(u, z) = \frac{1 + \log z}{\sqrt{z - 1}}, \quad \phi_1(z) = \frac{1 + \log z}{z(1 + \log z)}.
\]
last
\[
F_1(u) = \int_{\sigma}^{u} \phi_1(z) \, dz = \int_{\sigma}^{u} \frac{1 + \log z}{z} \, dz \leq C \leq (\log u)^{\beta}, \quad \forall \beta > 1.
\]

(2) Second summand:
\[
\frac{n}{\kappa_n - \kappa_m} \cdot \left\{ \exp \left( C \left\{ \frac{\log^3 \frac{n}{m}}{(n - m)^2} + \frac{m \log^2 \frac{n}{m}}{n - m} \right\} \right) - 1 \right\}.
\]
We have again \( g_2(m, n) = \frac{n}{\kappa_n - \kappa_m} \); moreover
\[
\exp \left( C \left\{ \frac{\log^3 \frac{v}{y - x}}{(y - x)^2} + \frac{x \log^2 \frac{v}{y - x}}{y - x} \right\} \right) - 1 = \exp \left( C \left\{ \frac{\log^3 \frac{v}{y}}{x^2(y - 1)^2} + \frac{x \log^2 \frac{v}{y}}{x(y - 1)} \right\} \right) - 1,
\]
so that
\[ f_2(u, z) = \exp\left( C \left\{ \frac{\log^3 z}{u^2(z-1)^2} + \frac{x \log^2 z}{u(z-1)} \right\} \right) - 1, \]
and \( \alpha_2 = 0 \); further
\[ \phi_2(z) = \sup_{u \geq 1} \frac{f_2(u, z)}{z} = \frac{1}{z} \left\{ \exp\left( C \left\{ \frac{\log^3 z}{(z-1)^2} + \frac{\log^2 z}{z-1} \right\} \right) - 1 \right\}. \]

Put
\[ M = \sup_{z \geq \sigma} \exp\left( C \left\{ \frac{\log^3 z}{(z-1)^2} + \frac{\log^2 z}{z-1} \right\} \right) - 1 \]
and
\[ F_2(u) = \int_{\sigma}^{u} \phi(z) \, dz \leq M \int_{\sigma}^{u} \left\{ \frac{\log^3 z}{(z-1)^2} + \frac{\log^2 z}{z-1} \right\} \frac{1}{z} \, dz \leq C \leq C(\log u)^\beta, \quad \forall \beta > 1. \]

(3) Third summand:
\[ g_3(m, n) = \frac{n}{\kappa_n - \kappa_m} \cdot \frac{1}{\log \frac{n}{m}}, \quad \alpha_3 = 0 \]
we have \( g_3(m, n) = \frac{n}{\kappa_n - \kappa_m} \cdot \alpha_3 = 0 \) and
\[ f_3(u, z) = \frac{1}{\log z}; \quad \phi_3(z) = \frac{1}{z \log z}, \]
hence
\[ F_3(u) = \int_{\sigma}^{u} \phi_3(z) \, dz = \int_{\sigma}^{u} \frac{1}{z \log z} \, dz = \left[ \log \log z \right]_{\sigma}^{u} \leq \log \log u \leq (\log u)^\beta, \quad \forall \beta > 1. \]

(4) Fourth summand:
\[ \frac{n}{\kappa_n - \kappa_m} \cdot \frac{|(x_n - \kappa_n) - (x_m - \kappa_m)|}{n - m} \]
Once more, \( g_4(m, n) = \frac{n}{\kappa_n - \kappa_m} \), \( \alpha_4 = 0 \). Let \( \delta > 0 \) be fixed and \( m_0 \) such that
\[ |\kappa_n - nx| < \delta x_n, \quad n > m_0. \]
Then, for \( n > m > m_0 \),
\[ \frac{|(x_n - \kappa_n) - (x_m - \kappa_m)|}{n - m} < \delta x \frac{n + m}{n - m} = \delta x \frac{\frac{n}{m} + 1}{\frac{n}{m} - 1} \]
and
\[ f_4(u, z) = \delta x \frac{z + 1}{z - 1}; \quad \phi_4(z) = \delta x \frac{z + 1}{z(z - 1)} < \frac{C}{z}. \]
Hence

\[ F_4(u) = \int_\sigma^u \phi(z) \, dz < C \int_\sigma^u \frac{1}{z} \, dz < C \log u \leq C \log^\beta u, \quad \forall \beta > 1. \]

(5) Fifth summand:

\[ \frac{n}{\kappa_n - \kappa_m} \cdot \frac{m}{n-m} = \frac{n}{\kappa_n - \kappa_m} \cdot \frac{1}{\frac{n}{m} - 1}. \]

Once more, \( g_5(m, n) = \frac{n}{\kappa_n - \kappa_m}, \alpha_5 = 0 \) and

\[ f_5(u, z) = \frac{1}{z-1}; \quad \phi_5(z) = \frac{1}{z(z-1)}, \]

and

\[ F(u) = \int_\sigma^u \phi(z) \, dz = \int_\sigma^u \frac{1}{z(z-1)} \, dz \leq C \leq (\log u)^\beta, \quad \forall \beta > 1. \]

(6) Sixth summand:

\[ \frac{m}{n-m} = \frac{1}{\frac{n}{m} - 1}. \]

Here \( g_6(m, n) = 1, \alpha_6 = 0 \) and

\[ f_6(u, z) = \frac{1}{z-1}. \]

The argument is identical to the previous one.

(7) Seventh summand:

\[ \frac{\log n}{\sqrt{n}} + \exp \left( C \left\{ \frac{\log^3 n}{n} + \frac{\log^2 n}{n} \right\} \right) - 1 + \frac{1}{\log n} \leq C \left[ \frac{\log n}{\sqrt{n}} + \exp \left( C \left\{ \frac{\log^2 n}{n} \right\} \right) - 1 + \frac{1}{\log n} \right]. \]

Here \( g_7 \equiv 1, \alpha_7 = 0 \) and

\[ f_7(u, z) = \frac{\log u \sqrt{u}}{u^2} + \exp \left( C \left\{ \frac{\log^2 u \sqrt{u}}{u^2} \right\} \right) - 1 + \frac{1}{\log u \sqrt{u}} = \psi_7(uz), \]

with

\[ \psi_7(t) = \frac{\log t}{\sqrt{t}} + \exp \left( C \left\{ \frac{\log^2 t}{t} \right\} \right) - 1 + \frac{1}{\log t}. \]

We can apply Lemma 6.4, and we find

\[ F_7(u) = C + \int_{t_0}^u \left( \frac{\log t}{t \sqrt{t}} + \frac{1}{t} \left\{ \exp \left( C \left\{ \frac{\log^2 t}{t} \right\} \right) - 1 \right\} + \frac{1}{t \log t} \right) \, dt \leq C \log \log u \leq (\log u)^\beta, \quad \forall \beta > 1, \]

for some suitable \( t_0 > \sigma \).

(8) Eighth summand:

\[ \frac{|(xn - \kappa_n) - (2x - \kappa_2)|}{n-2} \leq C. \]
Here $g_8 \equiv 1, \alpha_8 = 0$ and

$$f_8(u, z) = C = \psi_8(uz),$$

with $\psi_8(t) = C$. We can apply Lemma 6.4, and we find

$$F_8(u) = C + \int_{t_0}^{u} \frac{1}{t} \, dt \leq C \log u \leq (\log u)^\beta, \quad \forall \beta > 1,$$

for some suitable $t_0 > \sigma$.

(9) Ninth summand:

$$\frac{1}{n}$$

The argument is the same as in (7) and (8).

\[ \square \]

### 7 Explicit form of the cumulants of the Bernoulli distribution

In this section we prove the explicit formula announced in Remark 2.6. For every integer $n$ and every integer $k$ with $0 \leq k \leq n$ put

$$a_{k, n} = \sum_{j=0}^{k} (-1)^{j+1}\binom{k}{j} j^n.$$

**Remark 7.1** (i) Notice that $a_{1, n} = 1$ for every $n$. (ii) The Stirling number of second kind $S(n, k)$ has the explicit expression

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} j^n.$$

Hence

$$a_{k, n} = \sum_{j=0}^{k} (-1)^{j+1}\binom{k}{j} j^n = (-1)^{k+1} \sum_{j=0}^{k} (-1)^{j-k}\binom{k}{j} j^n = (-1)^{k+1} \sum_{j=0}^{k} (-1)^{k-j}\binom{k}{j} j^n = (-1)^{k+1} k! S(n, k).$$

(ii) We also recall that $S(n, n) = 1$, which implies that $a_{n, n} = (-1)^{n+1} n!$ by the above relation.

Let $B(1, x)$ be the Bernoullian law with parameter $x \in (0, 1)$. Denote by $c_n(x)$ the $n$-th cumulant of $B(1, x)$, i.e. the $n$-th coefficient in the development of the logarithm of its characteristic function $\phi(t)$:

$$\log \phi(t) = \log \left(1 + x(e^{it} - 1)\right) = \sum_{n=1}^{\infty} c_n(x) \frac{(it)^n}{n!}.$$

**Remark 7.2** (i) It is easily seen that $c_1(x) = x$. (ii) It is well known (see [18] ex. 6 p. 312 for instance) that the sequence of functions $(c_n(x))_n$ verifies the recurrence relation

$$c_{n+1}(x) = x(1-x)c'_n(x)$$

(7.1)
Proposition 7.3 For every \( n \geq 2 \) we have

\[
c_n(x) = x(1 - x) \left\{ \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} \right\}.
\] (7.2)

Proof By (7.1), we must prove that, for every \( n \geq 1 \),

\[
c_n'(x) = \sum_{k=1}^{n} a_{k,n} x^{k-1}.
\] (7.3)

We proceed by induction.

For \( n = 1 \) the statement follows from Remarks 7.1 (i) and 7.2 (i).

Assume that (7.3) holds for the integer \( n - 1 \); hence, by (7.1), we have

\[
c_n(x) = x(1 - x) \left\{ \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} \right\}
\]

and differentiating we get

\[
c_n'(x) = (1 - 2x) \left\{ \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} \right\} + (x - x^2) \left\{ \sum_{k=2}^{n-1} (k - 1)a_{k,n-1} x^{k-2} \right\}
\]

\[
= \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} - \sum_{k=1}^{n-1} 2a_{k,n-1} x^{k-1} + \sum_{k=2}^{n-1} (k - 1)a_{k,n-1} x^{k-1} - \sum_{k=2}^{n-1} (k - 1)a_{k,n-1} x^{k-1}
\]

\[
= \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} - \sum_{k=2}^{n-1} 2a_{k-1,n-1} x^{k-1} + \sum_{k=2}^{n-1} (k - 1)a_{k,n-1} x^{k-1}
\]

\[
- \sum_{k=3}^{n-1} (k - 2)a_{k-1,n-1} x^{k-1}
\]

\[
= a_{1,n-1} + (a_{2,n-1} - 2a_{1,n-1} + a_{2,n-1}) x
\]

\[
+ \sum_{k=3}^{n-1} x^{k-1} (a_{k,n-1} - 2a_{k-1,n-1} + (k - 1)a_{k,n-1} - (k - 2)a_{k-1,n-1})
\]

\[
+ (-2a_{n-1,n-1} - (n - 2)a_{n-1,n-1}) x^{n-1}
\]

\[
= 1 + (2a_{2,n-1} - 2a_{1,n-1}) x + \sum_{k=3}^{n-1} x^{k-1} (ka_{k,n-1} - ka_{k-1,n-1}) + (-n a_{n-1,n-1}) x^{n-1}
\]

\[
= 1 + \sum_{k=2}^{n-1} x^{k-1} (ka_{k,n-1} - ka_{k-1,n-1}) + (-n a_{n-1,n-1}) x^{n-1}
\]

\[
= 1 + \sum_{k=2}^{n-1} x^{k-1} (ka_{k,n-1} - ka_{k-1,n-1}) + (-n)^n x^{n-1}
\]

\[
= a_{1,n} + \sum_{k=2}^{n-1} x^{k-1} (ka_{k,n-1} - ka_{k-1,n-1}) + a_{n,n} x^{n-1} = \sum_{k=1}^{n} a_{k,n} x^{k-1},
\]

since

\[
ka_{k,n-1} - ka_{k-1,n-1} = k \left\{ \sum_{j=0}^{k} (-1)^{j+1} \binom{k}{j} j^{n-1} - \sum_{j=0}^{k-1} (-1)^{j+1} \binom{k-1}{j} j^{n-1} \right\}
\]

\[
= k \left\{ \sum_{j=0}^{k-1} (-1)^{j+1} j^{n-1} \left[ \binom{k}{j} - \binom{k-1}{j} \right] + (-1)^{k+1} k^{n-1} \right\}
\]
Corollary 7.4 The following formula holds

\[
\frac{c_n(x)}{x} - 1 = \sum_{k=2}^{n} \frac{a_{k,n}}{k} x^{k-1}.
\]

Proof Write

\[
\frac{c_n(x)}{x} = (1 - x) \left( \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} \right) = \sum_{k=1}^{n-1} a_{k,n-1} x^{k-1} - \sum_{k=1}^{n-1} a_{k,n-1} x^{k}
\]

\[
= 1 + \sum_{k=2}^{n-1} a_{k,n-1} x^{k-1} - \sum_{k=1}^{n-2} a_{k,n-1} x^{k} = 1 + \sum_{k=1}^{n-2} a_{k+1,n-1} x^{k} - \sum_{k=1}^{n-1} a_{k,n-1} x^{k}
\]

\[
= 1 + \sum_{k=1}^{n-2} (a_{k+1,n-1} - a_{k,n-1}) x^{k} - a_{n-1,n-1} x^{n-1}
\]

\[
= 1 + \sum_{k=1}^{n-2} \frac{a_{k+1,n}}{k+1} x^{k} + (-1)^{n-1} + (n-1)! x^{n-1},
\]

since, from the last calculation above

\[
a_{k+1,n-1} - a_{k,n-1} = \frac{a_{k+1,n}}{k+1}.
\]

Using now Remark 7.1 (ii), we get

\[
\frac{c_n(x)}{x} = 1 + \sum_{k=1}^{n-2} \frac{a_{k+1,n}}{k+1} x^{k} + (-1)^{n+1} (n-1)! x^{n-1}
\]

\[
= 1 + \sum_{k=2}^{n-1} \frac{a_{k,n}}{k} x^{k-1} + \frac{(-1)^{n+1} n!}{n} x^{n-1} = 1 + \sum_{k=2}^{n} \frac{a_{k,n}}{k} x^{k-1}.
\]
Thus we have obtained
\[
\frac{c_n(x)}{x} - 1 = \sum_{k=2}^{n} \frac{a_{k,n}}{k} x^{k-1},
\]
as claimed. \qed

8 Concluding remarks

Corollary 1.3 suggests two simulation procedures for estimating (i) Euler’s constant $\gamma$ and (ii) the values of Dickman’s function $\varrho$. Anyway, we have not investigated the goodness of these methods, nor compared them with the existing ones. For the values of the Dickman function see for instance [13], Corollary 2.3. Simulations of the Dickman distribution appear in [3] and [6].

We also refer to [3] where a probabilistic model (a particular case of the so called perpetuity model introduced in [7]) involving the Dickman distribution, is described and is later developed in [4].

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