Bicovariant differential calculus on quantum groups from Poisson Lie structures

I.Ya.Aref’eva †
G.E.Arutyunov ‡
and
P.B.Medvedev §

Abstract

The aim of this lecture is to give a pedagogical explanation of the notion of a Poisson Lie structure on the external algebra of a Poisson Lie group which was introduced in our previous papers. Using this notion as a guide we construct quantum external algebras on $SL_q(N)$ with proper (classical) dimension.

1 Introduction

The prime objective of this lecture is a pedagogical explanation of our recent results [1],[2] on the problem of constructing the external algebra on the quantum special linear group $SL_q(N)$.

Roughly speaking one can say that there are two main approaches to develop differential calculi on quantum groups.

The first one uses the Connes approach [3] to the noncommutative geometry and it has been initiated by Woronowicz [4]. In this approach a bicovariant complex of differential forms and the nilpotent operator $d$ of exterior derivative obeying the Leibniz rule play a fundamental role.

Another approach utilizes the Faddeev idea [5] that all objects in the theory of quantum groups should appear naturally as the result of quantization of appropriate Poisson brackets [6]-[11].

Aside from the obvious theoretical meaning this approach reveals also the practical significance. The point is that the Woronowicz bicovariant differential calculus (BDC) is formulated in a rather abstract axiomatic way and to develop its concrete realizations one needs to incorporate some additional information specifying a quantum group [12]. For instance, the explicit description of defining relations of a quantum external algebra appears to be a nontrivial task [13]-[15]. Another complicated problem is to check for these algebras the Diamond Condition [16]-[18]. Concerning $SL_q(N)$ one finds that the bicovariant differential

---

*Invited talk given by G.E.Arutyunov at the XXX Winter School of Theoretical Physics in Karpacz
†Steklov Mathematical Institute, Vavilov 42, GSP-1, 117966, Moscow, Russia; arefeva@qft.mian.su
‡Steklov Mathematical Institute, Vavilov 42, GSP-1, 117966, Moscow, Russia; arut@qft.mian.su
§Institute of Theoretical and Experimental Physics,117259 Moscow, Russia
complex involves an extra element which has no natural classical counterpart in $SL(N)$ [19] and this is the main problem that we are going to address in this lecture.

Generally speaking, there are two possibilities to attack the problem. The first one consists in relaxing the Woronowicz BDC axioms. As it was pointed out a couple of years ago by Faddeev [20] one can relax the Leibniz rule assuming something like

$$d(fh) = (df)h + Lf(dh). \quad (1.1)$$

and in a recent paper [21] this idea was successfully applied to $SL_q(N)$ thereby confirming the quantization procedure for $SL_q(2)$ [4].

Another possibility is just the direct use of the semiclassical strategy and the first step here is to endow the algebra of differential forms on a classical matrix Lie group with a natural Poisson Lie structure. Following Woronowicz we employ the bicovariance condition to be the main characteristic property of this structure. Note that in a quantum case the requirement of bicovariance aside from the obvious geometrical meaning has a physical interpretation [22]-[25]. At the second step, these Poisson Lie structures when being quantized should give as the result algebras of quantum differential forms. The operator $d$ defined on these algebras in pure algebraic terms may not satisfy in general the standard requirements, i.e. the non-zero ”anomaly” $L - 1$ appears. In the semiclassical limit the modified Leibniz rule results in a nondifferential Poisson Lie structure on the external algebra:

$$d\{f, h\} \neq \{df, h\} + \{f, dh\}.$$

The lecture consists of two parts. The first part deals with $GL(N)$ and the second one with $SL(N)$. The first part explains the very notion of a Poisson Lie structure on the external algebra of a Lie group and following [1] we give the complete classification of these structures for $GL(N)$. We separate differential structures to reconstruct all BDC on $GL_q(N)$ possessing the usual classical limit and find the classical counterparts for all the ingredients of BDC, e.g. we give the”commutator” representation for the standard $d$. The second part is based on the paper [2]. Here we obtain all Poisson Lie structures on $SL(N)$ and present their quantization which provides the quantum external algebras on $SL_q(N)$ with the classical dimension. We find that all these structures are nondifferential ones that agrees with the result of [21].

The further development of this issue for the case of other simple Lie groups [26, 27] lies beyond the scope of this lecture.

2 Graded Poisson Lie structures on the algebra of differential forms on $GL(N)$

2.1 Definition

We start with recalling the notion of a Poisson Lie group. The most natural way to describe it is to use the Hopf algebra terminology [3].

Let $\mathcal{A}$ be the function algebra on a Lie group $G$ generated by the matrix elements of a fundamental representation $T = |t^j_i| \equiv |t^j_k| \equiv |t^j_l|$ of $G$. It is well known that $\mathcal{A}$ is a Hopf algebra with the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined on the algebra generators by:

$$\Delta t^j_i = t^k_i \otimes t^l_j. \quad (2.1)$$
A group $G$ is called a Poisson Lie group if it is supplied with a Poisson bracket compatible with $\Delta$:
\[
\Delta\{f, h\}_A = \{\Delta f, \Delta h\}_{A \otimes A}, \quad f, h \in A.
\] (2.2)

It is convenient to describe a Poisson structure in terms of algebra generators $t_i^j$. The main theorem in the theory of Poisson Lie groups states that if $G$ is a connected semisimple Lie group then a Poisson Lie structure is given by the Sklyanin bracket:
\[
\{T_1, T_2\} = [r, T_1 T_2].
\] (2.3)

Here we use the standard tensor notation $T_1 = T \otimes I$ and $T_2 = I \otimes T$. The matrix $r$ coming in (2.3) is a classical $r$-matrix, i.e. it satisfies the Classical Yang Baxter Equation (CYBE) [6].

In this section we will treat the not semisimple General Linear Group ($GL(N)$), however, for the purposes of quantization only the brackets associated with classical $r$-matrices are relevant. Thus, following [8] we employ (2.3) for the Poisson Lie structure on $GL(N)$ with $r$ being a trivial lifting of $r$-matrix for $SL(N)$. It is known [11] that the bracket (2.3) does not depend on the $GL(N)$-invariant symmetric part of $r$-matrix. We take $r = r_+$ satisfying the classical Hecke condition: $Pr_+P + r_+ = 2P$, where $P_{ik}^{sp} = \delta_i^p \delta_k^s$ is the permutation operator.

In the following it will be important that the bracket is degenerate and the function $\det T$ lies in its center: $\{T, \det T\} = 0$. By fixing the value of $\det T$ equal to unity we obtain the Poisson Lie structure on $SL(N)$.

Now our goal is to extend the Poisson Lie structure on $G = GL(N)$ to the algebra of differential forms $M$. (We will treat elements of $A$ as forms of order zero i.e. $A \in M$.) The algebra $M$ has a natural $\mathbb{Z}_2$ grading and therefore on homogeneous elements of $M$ the corresponding Poisson bracket should satisfy the graded Jacobi identity, the graded Leibniz rule, and the graded symmetry property:
\[
\{x, y\} = (-1)^{\deg x \deg y+1}\{y, x\}, \quad \deg \{x, y\} = (\deg x + \deg y) \mod 2.
\] (2.4)

To introduce on $M$ the structure of a $\mathbb{Z}_2$-graded Hopf algebra we need to define the coproduct $\Delta$ on odd generators (one-forms) of $M$. The suitable set of odd generators is provided by the matrix elements $\theta_i^j$ of the right-invariant Maurer-Cartan form $\Theta = dTT^{-1}$. The reason for this choice is the well known “gauge” transformation law of $\Theta$ under the group translations $g \to g_1 g$: $\Theta_g \to \Theta_{g_1 g} = g_1 \Theta_g g_1^{-1} + dg_1 g_1^{-1}$. Hence we define the homomorphism $\Delta: M \to M \otimes M$ on the generators $\theta_i^j$ as:
\[
\Delta \theta_i^j = \theta_i^j \otimes I + t_i^k S(t_p^j) \otimes \theta_k^p.
\] (2.5)

It is also possible to introduce the corresponding counit and the antipod [1]. Therefore, $M$ supplied with such a coproduct is a genuine $\mathbb{Z}_2$-graded Hopf algebra. Now we are in position to give the

**Definition** We say that a Poisson bracket defines on $M$ a graded Poisson Lie structure if it is compatible with $\Delta$:
\[
\Delta\{x, y\}_M = \{\Delta x, \Delta y\}_{M \otimes M}, \quad x, y \in M
\] (2.6)

or, in other words, the coproduct $\Delta$ should be a homomorphism of the Poisson algebra $M$ into $M \otimes M$. In the following we will refer to this bracket as to the $\Delta$-covariant one.
2.2 Explicit formulae for graded brackets

The general expression for the bracket involving generators $\Theta$ and $T$ (bracket of the first order) follows from the grading requirement (2.4):

$$\{\theta^{i}_{j}, t^{l}_{k}\} = C^{j}_{ik} \theta^{m}_{s} + C^{j}_{1i2} t^{j}_{3} + \ldots.$$  (2.7)

The structure coefficients $C$ that enter eq. (2.7) are the functions of even generators $t^{i}_{j}$. Applying $\Delta$ to the both sides of (2.7) and imposing (2.6) we obtain the system of equations on the structure tensors $C$. We omit the calculations of $C$ and quote only the result. All candidates for graded $\Delta$-covariant brackets on $\mathcal{M}$ are of the form

$$\{\Theta_{1}, T_{2}\} = r_{A}^{12} \Theta_{1} T_{2} - \Theta_{1} r_{B}^{12} T_{2} + \alpha_{1} \Theta_{2} T_{2} + \alpha_{2} \text{tr} \Theta T_{2} + \alpha_{3} \text{tr} \Theta P^{12} T_{2} + \alpha_{5} \Theta_{1} T_{2}. \quad (2.8)$$

Here $r_{A} = r^{+} + \alpha_{4} P$ and $r_{B} = r^{+} + \alpha_{6} P$.  

At this stage all the numerical coefficients $\alpha_{1}, \ldots, \alpha_{6}$ coming in (2.8) leave to be undetermined. However, as it was mentioned above, to define the genuine Poisson structure on $\mathcal{M}$ the bracket (2.8) should satisfy the graded Jacobi identity. This fixes some of the parameters $\alpha$. It turns out that possible $\Delta$-covariant brackets are divided into two families each of them is parametrized by two continuous parameters $\alpha$ and $\beta$ and the discrete value of $m = \pm 2$:

1. The first family

$$\{\Theta_{1}, T_{2}\}_{\alpha, \beta}^{\pm} = r_{\pm}^{12} \Theta_{1} T_{2} - \Theta_{1} r_{\mp}^{12} T_{2} + \alpha_{2} \Theta_{2} T_{2} + \alpha_{2} \text{tr} \Theta T_{2} + \alpha_{3} \text{tr} \Theta P^{12} T_{2} + \beta \Theta_{1} T_{2} \quad (2.9)$$

where $\alpha_{2}, \alpha_{3}$ are expressed via $\alpha$ as follows

$$\alpha_{2} = -\alpha^{2}/(m + \alpha N), \quad \alpha_{3} = -\alpha m/(m + \alpha N), \quad m = \pm 2, \alpha \neq -m/N. \quad (2.10)$$

2. The second family

$$\{\Theta_{1}, T_{2}\}_{\alpha, \beta}^{\pm} = r_{\pm}^{12} \Theta_{1} T_{2} - \Theta_{1} r_{\mp}^{12} T_{2} + \alpha \text{tr} \Theta T_{2} + \beta \Theta_{1} T_{2}, \quad (2.11)$$

where $r_{-} = r_{+} - 2P$ provides another solution of the CYBE. We clarify the geometrical meaning of $\alpha$ and $\beta$ later and now following the same steps we define the brackets containing two generators $\Theta$ (brackets of the second order).

We take the bracket in the general form:

$$\{\theta^{j}_{i}, \theta^{l}_{k}\} = (W)^{j}_{ik} + W^{j}_{ik} \theta^{m}_{s} \theta^{p}_{n} + \ldots. \quad (2.12)$$

where $W$ are the unknown tensor functions of even variables $t^{i}_{j}$. According to our general strategy we have to define all $W$-s that guarantee the $\Delta$-covariance of a corresponding bracket. Just at this point the crucial difference between the brackets from the first and from the second family arises. Namely, applying $\Delta$ to the both sides of (2.12) and trying to obtain the system of equation on tensors $W$ one has to use the brackets of the first order. If one uses the brackets from the first family (eq.(2.9)) the corresponding system of equations for $W$ tensors can be solved and the solution is given by

$$\{\Theta_{1}, \Theta_{2}\}_{\alpha}^{\pm} = \alpha (\Theta_{1} \Theta_{1} + \Theta_{2} \Theta_{2}) + r_{\mp}^{12} \Theta_{1} \Theta_{2} + \Theta_{1} \Theta_{2} r_{\mp}^{12} - \Theta_{1} r_{\mp}^{12} \Theta_{2} + \Theta_{2} r_{\mp}^{12} \Theta_{1}. \quad (2.13)$$

Here the parameter $\alpha$ is the same as in (2.9) and the signs $\pm$ are in accordance with (2.11).
As for the second family \((2.11)\), the system of equations for \(W\) tensors has no solutions and therefore there is no \(\Delta\)-covariant Poisson bracket of the second order that prolongs the bracket \((2.11)\).

One has also to check the Jacobi identity for the system of brackets given by \((2.3), (2.9)\) and \((2.13)\):

\[
\sum_{\text{deg}(1)\text{deg}(3)}\{\{\Theta, \Theta\}, T\} = 0 \quad \text{and} \quad \sum\{\{\Theta, \Theta\}, \Theta\} = 0,
\]

We omit the corresponding calculations and note only that these equations reduce to the CYBE for \(r_{\pm}\).

### 2.3 Differential calculus from Poisson Lie structure

In the previous section we have described all Poisson Lie structures on \(\mathcal{M}\) without any references to the operator of exterior derivative \(d\). At the same time, in BDC the operator \(d\) defined on the function algebra on a quantum group and obeying the Leibniz rule is one of the essential elements of the construction. Taking the semiclassical limit of BDC \(([a, b] \to h\{a, b\})\) we argue that it is reasonable to introduce the following

**Definition** A graded Poisson structure on \(\mathcal{M}\) is called a differential one if the operator \(d\) satisfies the Leibniz-like rule:

\[
d\{f, h\} = \{df, h\} + (-1)^{\text{deg}f}\{f, dh\}
\]

In this definition the operator \(d\) is the usual differentiation of \(\mathcal{M}\) which in algebraic terms can be specified on generators in a natural way:

\[
dT = \Theta T \quad d\Theta = \Theta \Theta,
\]

and extended to the whole algebra \(\mathcal{M}\) by using \(d^2 = 0\) and the Leibniz rule.

One can see by the straightforward calculation that the only differential Poisson Lie structures on \(\mathcal{M}\) are those for which \(\alpha = \beta\) since, \(\text{e.g.,}\)

\[
\{dT_1, T_2\}_{\alpha, \beta}^\pm + \{T_1, dT_2\}_{\alpha, \beta}^\pm = d\{T_1, T_2\} + (\alpha - \beta)(\Theta_1 - \Theta_2)T_1T_2.
\]

There is an alternative point of view on the operator \(d\) inspired by the ideology of the noncommutative geometry and realized in BDC. With Poisson Lie structures in hands we can follow this ideology just in the classical (commutative) case. Namely, one can define on the Poisson algebra \(\mathcal{M}_{\alpha, \beta}^\pm\), \(\text{i.e. on } \mathcal{M}\) supplied with the bracket \(\{,\}_{\alpha, \beta}^\pm\), the operator

\[
d_{\alpha, \beta}^\pm \equiv \left\{\frac{1}{\alpha N + m} \text{tr } \Theta, \right\}_{\alpha, \beta}^\pm.
\]

The operator \(d_{\alpha, \beta}^\pm\) satisfies the Leibniz rule because the brackets \(\{,\}_{\alpha, \beta}^\pm\) and \(\{,\}_{\alpha}^\pm\) do. The property \(d_{\alpha, \beta}^2 = 0\) is due to the Jacobi identity and \(\{\text{tr } \Theta, \text{tr } \Theta\}_{\alpha}^\pm = 0\). Hence \(d_{\alpha, \beta}^\pm\) is a good candidate for the operator of exterior derivative on \(\mathcal{M}\). However, one finds:

\[
(d_{\alpha, \beta}^\pm T)^{-1} = \Theta - \frac{\alpha - \beta}{\alpha N + m} \text{tr } \Theta I, \quad d_{\alpha, \beta}^\pm \Theta = \Theta \Theta.
\]

Therefore, we realize again that if and only if \(\alpha = \beta\) then \(d_{\alpha, \beta}^\pm\) can be identified with the usual \(d\). In other words, for graded Poisson Lie algebras with \(\alpha = \beta\) the usual operator of
exterior derivative can be expressed as the internal object of a Poisson structure. In this case the brackets are found to be differential ones as a direct consequence of the Jacobi identity.

It is instructive to note that in all the concrete realizations of BDC the quantum $d$ is defined via the graded commutator with some tr-like differential left-right-invariant one form $\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}$.

2.4 Algebraic and geometric meaning of the variety of brackets

In this section we are going to clarify the meaning of the parameters $\alpha$ and $\beta$ that label the variety of $\Delta$-covariant brackets on $\mathcal{M}$. First of all let us note that there is an arbitrariness in the choice of generators $t_i^j$ and $\theta_i^j$ of $\mathcal{M}$. There exists a nondegenerate change of variables:

$$T \rightarrow \tilde{T} = T(\det T)^s, \quad \Theta \rightarrow \tilde{\Theta} = \Theta + k \text{tr} \Theta, \quad k, s \neq -1/N \quad (2.19)$$

that does not affect the form of the coproduct: $\Delta \tilde{t}_i^j = \tilde{t}_i^k \otimes \tilde{t}_k^j$, $\Delta \tilde{\theta}_i^j = \tilde{\theta}_i^k \otimes I + \tilde{t}_i^k S(\tilde{t}_p^l) \otimes \tilde{\theta}_p^r$.

This covariance of the coproduct reflects itself on the level of brackets. Namely, one can check that the bracket (2.3) is not affected by the transformation (2.19):

$$\{\tilde{T}(T)_1, \tilde{T}(T)_2\} = \{\tilde{T}_1, \tilde{T}_2\}|_{\tilde{T}=T(det T)^s}. \quad (2.20)$$

This is a direct consequence of the fact that $\det T$ is the central element of the Sklyanin bracket. However, as for the brackets of the first and the second orders under the transformation (2.19) they transform as follows

$$\{\tilde{\Theta}(\Theta)_1, \tilde{T}(T)_2\}^{\pm}_{\alpha, \beta} = \{\tilde{\Theta}_1, \tilde{T}_2\}^{\pm}_{\alpha', \beta'}|_{\tilde{\Theta} = \Theta + k \text{tr} \Theta, \tilde{T} = T(\det T)^s}, \quad (2.21)$$

where in both cases $\alpha' = \alpha + k(\alpha N + m)$, $\beta' = \beta + s(\beta N + m)$.

So, we see that under the change of generators given by eq.(2.19) the brackets $\{ , \}^{\pm}_{\alpha, \beta}$ and $\{ , \}^{\pm}_{\alpha'}$ are transformed into the brackets $\{ , \}^{\pm}_{\alpha', \beta'}$ and $\{ , \}^{\pm}_{\alpha'}$ with $\alpha'$ and $\beta'$ given by eq.(2.21). The inverse assertion is also true: for any two pairs of admissible $\alpha, \beta$ and $\alpha', \beta'$ ($\alpha, \beta, \alpha', \beta' \neq -m/N$) there exist such $k$ and $s$ that the transformation (2.19) converts the $\alpha, \beta$ brackets into the $\alpha', \beta'$ ones.

Hence, we have proved that by an appropriate change of variables any bracket on $\mathcal{M}$ can be put into the ”canonical” form with $\alpha = 0 = \beta$:

$$\{\Theta_1, T_2\}^{\pm} = r^{12}_{\pm} \Theta_1 T_2 - \Theta_1 r^{12}_{\pm} T_2 \quad (2.22)$$

$$\{\Theta_1, \Theta_2\}^{\pm} = r^{12}_{\pm} \Theta_1 \Theta_2 + \Theta_1 \Theta_2 r^{12}_{\pm} - \Theta_1 r^{12}_{\pm} \Theta_2 + \Theta_2 r^{12}_{\pm} \Theta_1.$$

From the geometric point of view the first equation in (2.19) reflects the existence of nonequivalent fundamental representations of $GL(N)$. Clearly, the representation $\tilde{T} = T(\det T)^s$ is not equivalent to $T$ for they differ by the value of determinant: $\det \tilde{T} = (\det T)^{sN+1}$. The point $s = -1/N$ is forbidden because the corresponding representation becomes non-exact and can not serve as a coordinate system on $GL(N)$.

The basis in the linear space $\Lambda^1$ of the first order differential forms can be chosen from right(left) invariant forms in the following way. Any representation $T$ of $G$ gives rise to the representation of the Lie algebra $gl(N)$. For a given representation $T$ one can define the
Lie-valued right-invariant Maurer-Cartan form by: \( \Theta = dTT^{-1} \). The matrix elements of \( \Theta \) form a basis in \( \Lambda^1 \).

When one changes a coordinate system on \( G \), i.e. one goes to some other representation \( \hat{T} \), one also changes the matrix \( \Theta \). In the new basis one obtains for \( \hat{\Theta} \):

\[
\hat{\Theta} = d\hat{T}\hat{T}^{-1} = d\ln T + s d\ln T = \Theta + s \text{tr} \Theta I. \tag{2.23}
\]

The expressions for \( \hat{T} \) and \( \hat{\Theta} \) literally coincide with formulae (2.19) for the change of variables in \( M \) with the only difference: two parameters \((k, s)\) in the algebra are replaced by the single parameter \( s \) in the group.

Now we see that not every bracket with arbitrary \( \alpha \) and \( \beta \) can be put in the canonical form (2.22) by a change of a coordinate system on \( G \). This gives a reason to divide all Poisson Lie structures on the external algebra into two sets. The first set consists of the brackets with \( \alpha = \beta \) and the second one with \( \alpha \neq \beta \).

Thus, we proved that there exist two different sets of Poisson Lie structures on the external algebra of \( GL(N) \). The brackets from the first set are labeled by the signs \( \pm \) and by the continuous parameter \( \alpha \), which is redundant as it can be removed by the proper choice of a coordinate system on \( G \). These brackets are the differential ones and \( d \) operator can be expressed as

\[
d = \left\{ \frac{1}{\alpha N \pm 2 \text{tr} \Theta}, \right\}_{\pm}^{\pm}. \tag{2.24}
\]

The brackets from the second set are labeled by the signs \( \pm \) and by a pair of parameters and only one parameter can be set to zero by an appropriate change of coordinates on \( G \). In contrast to the first set these brackets are not differential ones and \( d \) cannot be presented as an internal object of this Poisson Lie structure.

### 2.5 Connection with the bicovariant differential calculus on QG

Now we can relate the graded Poisson Lie structures on \( GL(N) \) with BDC of [14, 15, 18] on \( GL_q(N) \).

BDC on \( GL_q(N) \) is a free associative algebra \( M_\hbar \) \((q = e^\hbar)\) generated by the symbols \( T \) and \( dT \) modulo the quadratic relations which in terms of quantum right-invariant forms \( \Theta = dTT^{-1} \) read

\[
R_\pm T_1 T_2 = T_2 T_1 R_\pm, \tag{2.25}
\]

\[
T_2 \Theta_1 = R_\pm \Theta_1 R_\mp^{-1} T_2 \tag{2.26}
\]

\[
-R_\pm \Theta_1 R_\mp^{-1} \Theta_2 = \Theta_2 R_\pm \Theta_1 R_\mp^{-1}. \tag{2.27}
\]

where \( R_+ = R \) is a quantum \( R \)-matrix and \( R_- = \sigma(R^{-1}) \) (\( \sigma \) is a permutation map) obeys the Hecke condition: \( R_\pm = R_\mp \pm \mu P, \quad \mu = q - 1/q \).

Suppose that these relations represent a quantization of a \( Z_2 \)-graded commutative Poisson algebra \( \mathcal{M} \). Then according to the general quantization principle one can define on \( \mathcal{M} \) the Poisson bracket:

\[
\{ \Theta_1, T_2 \} = -\lim_{\hbar \to 0} \frac{1}{\hbar} [\Theta_1, T_2]^\pm \quad \text{and} \quad \{ \Theta_1, \Theta_2 \} = -\lim_{\hbar \to 0} \frac{1}{\hbar} [\Theta_1, \Theta_2]^\pm \tag{2.28}
\]

Here we use the square brackets \([,]\) for the graded commutator.
Since $R_{\pm}$ have the semiclassical form: $R_{\pm} = 1 + hr_{\pm} + o(h)$, the semiclassical expansion of the multiplication law in $\mathcal{M}_h$ reads

$$T_2 \Theta_1 = \Theta_1 T_2 + h(r_{\pm}^{12} \Theta_1 T_2 - \Theta_1 r_{\mp}^{12} T_2) + O(h^2),$$

$$\Theta_1 \Theta_2 = -\Theta_2 \Theta_1 - h(r_{\pm}^{12} \Theta_1 \Theta_2 + \Theta_1 \Theta_2 r_{\mp}^{12} - \Theta_1 r_{\mp}^{12} \Theta_2 + \Theta_2 r_{\mp}^{12} \Theta_1) + O(h^2).$$

From the last equations we can read off at once the expressions for the graded Poisson brackets:

$$\{\Theta_1, T_2\}_{\pm} = r_{\pm}^{12} \Theta_1 T_2 - \Theta_1 r_{\mp}^{12} T_2$$

$$\{\Theta_1, \Theta_2\}_{\pm} = r_{\pm}^{12} \Theta_1 \Theta_2 + \Theta_1 \Theta_2 r_{\mp}^{12} - \Theta_1 r_{\mp}^{12} \Theta_2 + \Theta_2 r_{\mp}^{12} \Theta_1.$$

which literally coincide with the canonical form (eq.(2.22)) of our brackets. The Diamond Condition [18] reduces in the semiclassical limit to the graded Jacobi identity.

For BDC on $GL_q(N)$ the quantum operator $d$ of exterior derivative can be introduced in the following way [13], [15]. Consider the ”right-left invariant” element $tr_q \Theta$ called the quantum trace of $\Theta$:

$$tr_q \Theta = tr(D \Theta),$$

where $D$ is the numerical matrix $D = diag(1, q^2, \ldots, q^{2(N-1)})$ [8].

Using $tr_q \Theta$ let us define for the ” $\pm$ ” calculi the operator

$$d = \frac{1}{\gamma} [tr_q \Theta, ],$$

where $\gamma = \mu q^{2N-1}$ for the ” + ” and $\gamma = \mu q^{-1}$ for the ” $-$ ” calculi respectively. From eq.(2.27) one has (see [13] for details) the nilpotency condition $d^2 = 0$ and the Maurer-Cartan equation: $d \Theta = \Theta^2$. Moreover, one also gets $dT = dT$. As it should be expected from the above discussion in the semiclassical limit the quantum $d$ reproduces eq.(2.24). In other words, the operator $d$ in BDC is a genuine quantization of its classical counterpart.

To conclude this section we give a comment on quantum versions of the noncanonical brackets (2.9) and (2.13). To obtain these let us note that there exists the ”quantum” transformation group

$$T \rightarrow \tilde{T} = T(\det_q T)^{*}, \quad \Theta \rightarrow \tilde{\Theta} = \Theta + kTr_q \Theta$$

being the deformation of eq.(2.19). Since (2.33) does not affect the coproduct the quantization of any bracket from the family (2.9), (2.13) is obtained by substituting $T(\tilde{T})$ and $\Theta(\tilde{\Theta})$ from eq.(2.33) into the defining relations (2.26), (2.27). The resulting algebras are in agreement with the recent classification of quantum covariant algebras on $GL_q(N)$ [29]. The transformations (2.35) will be a useful tool in dealing with $SL_q(N)$.

## 3 Graded Poisson Lie structures on $SL(N)$ and their quantization

### 3.1 Poisson Lie structures

In the previous part of the lecture our aim was to establish a proper classical structure to be the semiclassical limit of BDC on $GL_q(N)$. To solve this task we have introduced the notion of a Poisson Lie structure on the external algebra on $GL(N)$. Concerning $SL_q(N)$
we have to reverse the setting of the problem. The point is that, as we have mentioned in
the Introduction, the known algebras describing BDC on $SL_q(N)$ involve more generators
as compared to the classical case \cite{19, 29}. Here our goal will be to quantize the algebra of
external forms on $SL(N)$ preserving its classical dimension. Our strategy is the following.
Among the variety of graded brackets produced in the previous section we select the brackets
which admit the $SL(N)$ reduction and present their quantization. Carrying out this program
we will be forced to modify slightly the very definition of a graded Poisson Lie structure.

The group $SL(N)$ stands out by the constraint $\det T = 1$ while the external algebra on
$SL(N)$ is generated by the components $\theta^i_j$ of the traceless Maurer-Cartan form. Thus, the
straightforward $SL(N)$ reduction consists in selecting the graded Poisson Lie structures on
$GL(N)$ that are compatible with constraints
\begin{equation}
\det T = 1 \quad \text{and} \quad \tr \Theta = 0. \tag{3.1}
\end{equation}
However, the closer examination (see eq.(2.17)) reveals that among the graded Poisson Lie
structures on $GL(N)$ given by (2.9) and (2.13) there are no candidates to be consistent with
these constraints.

However, let us put $\tr \Theta = 0$ in the brackets from the second family:
\begin{equation}
\{\Theta_1, T_2\}_\beta^\pm = r^{12}_\pm \Theta_1 T_2 - \Theta_1 r^{12}_\pm T_2 + \beta \Theta_1 T_2. \tag{3.2}
\end{equation}
The classical Hecke condition for $r_+$ guarantees that if $\beta = 0$ then $\det T$ and $\tr \Theta$ are the
central elements of these brackets and, therefore, they can be put equal to desired numerical
values. Thus, the brackets of the first order from the second family are the good object to
admit the $SL(N)$ reduction.

Unfortunately, as we realized from the previous analysis these brackets can not be pro-
longed to the $\Delta$-covariant brackets of the second order. It means that $\Delta$-covariance is too
strong to be satisfied for Special groups.

The hint for relaxing the constraint of $\Delta$-covariance is provided by the Woronowicz
notion of bicovariance. Let us introduce the left and right coactions $\Delta_L$ and $\Delta_R$ being
homomorphisms: $\Delta_L : \mathcal{M} \to \mathcal{A} \otimes \mathcal{M}$, $\Delta_R : \mathcal{M} \to \mathcal{M} \otimes \mathcal{A}$ defined on generators $\theta^i_j$ as:
\begin{align*}
\Delta_R \theta^i_j &= \theta^i_j \otimes I, \quad \Delta_L \theta^i_j = t^k_i S(t^j_p) \otimes \theta^p_k
\end{align*}
and on $t^i_j$ as $\Delta_R = \Delta_L = \Delta$. The space of differential forms of the first order supplied with
these coactions is literally a classical counterpart of the Woronowicz bicovariant bimodule.
Hence we introduce the new

**Definition** We say that a Poisson bracket defines on $\mathcal{M}$ a graded Poisson Lie structure if
the following relations are satisfied:
\begin{align}
\Delta_L \{x, y\}_\mathcal{M} &= \{\Delta_L(x), \Delta_L(y)\}_{\mathcal{A} \otimes \mathcal{M}}, \tag{3.3} \\
\Delta_R \{x, y\}_\mathcal{M} &= \{\Delta_R(x), \Delta_R(y)\}_{\mathcal{M} \otimes \mathcal{A}} \quad x, y \in \mathcal{M} \tag{3.4}
\end{align}
or, in other words, the coproducts $\Delta_{L,R}$ should be homomorphisms of the Poisson algebra $\mathcal{M}$
into $\mathcal{A} \otimes \mathcal{M}$, $\mathcal{M} \otimes \mathcal{A}$ respectively.

Strictly speaking to distinguish between the requirements (3.3), (3.4) and (2.6) we have
to consider the last definition as the definition of bicovariant Poisson Lie structure on $\mathcal{M}$.
However for simplicity we say Poisson Lie structure on $\mathcal{M}$ having in mind requirements (3.3),
(3.4). To avoid misunderstanding we call this bracket the *bicovariant* one in contrast to the
δ-covariant bracket in the definition in sect. 2.1 To justify this definition let us note that any bicovariant bracket of the first order linear in θ-s is also Δ-covariant and vice versa.

Now our goal is to define bicovariant brackets of the second order obeying the $SL(N)$ constraints. Solving the bicovariance conditions (3.3) and (3.4) we get a pair of brackets

$$\{Θ_1, Θ_2\}^\pm = \mp \frac{2}{N} (Θ_1Θ_1 + Θ_2Θ_2) + r^{12}_+Θ_1Θ_2 + Θ_1r^{12}_+Θ_2 + Θ_2r^{12}_+Θ_1. \tag{3.5}$$

One can prove that the systems of brackets (3.2) and (3.5) satisfy the graded Jacobi identity regardless of the sign ($\pm$) combinations, that results in four different graded Poisson Lie structures on the external algebra of $SL(N)$.

It is the place to state a question if these structures are the differential ones. The direct calculation reveals that the answer is no, i.e. the operator $d$ does not satisfy the Leibniz-like rule. This is a strong indication that there is no reason to expect the quantum $d$ to obey the Leibniz rule [2]. It seems quite natural because as we have seen in the case of $GL(N)$ the operator $d$ of exterior derivative is generated by tr Θ. This is a left-right-invariant form that represents the nontrivial element from the first cohomology group $H^1(GL(N))$. However, $H^1(SL(N)) = 0$ and therefore $d$ can not be generated by an internal element of the algebra.

### 3.2 Quantization

We start with the quantization of the bracket of the second order eq.(3.5). The hint is provided by the following remark. Comparing eqs.(3.5) and (2.13) we realize that (3.5) is nothing but the bracket (2.13) with $α = -m/N$. But all the brackets with different values of $α$ are connected by the change of variables eq.(2.21). The only exception is just the point $α = -m/N$. To reach this point from the canonical bracket ($α = 0$) we perform the transformation eq.(2.19) with $k = -1/N$ that gives the traceless $\tilde{Θ}$. On the other hand it follows from eqs.(2.17) and (2.18) that under this value of $α$ tr $\tilde{Θ}$ is a central element of the bracket: $\{tr \tilde{Θ}, Θ\} = 0$. Hence, the brackets of the second order can be derived from the canonical bracket on $GL(N)$ by passing to the traceless Maurer-Cartan form.

The necessary tools to perform the same trick in the quantum case have been just prepared in the previous section. These are: the defining relations eq.(2.27) being the quantum version of the canonical bracket on $GL(N)$ and the quantum trace eq.(2.33).

Let us decompose the form Θ in the following way

$$Θ = \tilde{Θ} + \frac{1}{trD}tr_qΘ. \tag{3.6}$$

Let us stress that eq.(3.6) should not be treated as the usual change $Θ → \tilde{Θ}$ of a basis of external forms on $GL_q(N)$. The constraint tr$q$Θ = 0 makes the generators $\tilde{Θ}^i_j$ linearly dependent and to complete a basis one has to add one more generator tr$q$Θ. However, for $SL(N)$ $\tilde{Θ}^i_j$ do compose the basis. Moreover, substituting the decomposition (3.6) into eq.(2.27) we see that generators Θ form the closed subalgebra. Namely,

$$R_±\tilde{Θ}_1R_+^{-1}\tilde{Θ}_2 + \tilde{Θ}_2R_±\tilde{Θ}_1R_+^{-1} = -\frac{1}{trD} \left(R_±[tr_qΘ, \tilde{Θ}_1]R_+^{-1} + [tr_qΘ, \tilde{Θ}_2]\right). \tag{3.7}$$

where we have used $(tr_qΘ)^2 = 0$ and the Hecke condition for $R_±$. Using the Maurer-Cartan equation for Θ one can easily find the commutator for $tr_qΘ$ and $Ω$:

$$[tr_qΘ, Ω] = \frac{γtrD}{trD - γ}Ω^2. \tag{3.8}$$

\[for \ N = 2 \ these \ brackets \ coincide \ owing \ to \ the \ special \ form \ of \ the \ r-matrix\]
Taking into account the last equation we can write down eq.$(3.7)$ as
\[ R_\pm \tilde{\Theta}_1 R_\mp^{-1} \tilde{\Theta}_2 + \tilde{\Theta}_2 R_\pm \tilde{\Theta}_1 R_\mp^{-1} = k_{q^\pm 1} \left( R_\pm \tilde{\Theta}_2^2 R_\mp^{-1} + \tilde{\Theta}_2^2 \right). \] (3.9)

where $k_q = \frac{q^N}{q^{-N} + [N]_q}$ and $[N]_q$ is the q-number: $[N]_q = (q^N - q^{-N})/\mu$. Now one can easily verify that the formulae (3.9) provide the quantization of the bracket (3.3). This formula was proposed in [29] as commutation relations for Cartan 1-forms on $SL_q(N)$.

The quantization of the brackets of the first order is obviously given by
\[ T_2 \tilde{\Theta}_1 = R_\pm \tilde{\Theta}_1 R_\mp^{-1} T_2. \] (3.10)

The algebras with these relations have two central elements: $tr_q \tilde{\Theta}$ and $det_q T$. Combining eqs. (3.9) and (3.10) we get the quantization of the graded Poisson Lie structure on $SL(N)$.

In the recent paper [21] the algebras (3.9) and (3.10) appeared in a pure quantum treatment.

**ACKNOWLEDGMENT** One of the authors (G.A.) is grateful to the organizers of the XXX Karpacz Winter School of Theoretical Physics for their hospitality and to for the partial financial support of the participation in the XXX Karpacz Winter School. The authors are also grateful to J.Lukierski, P.P.Kulish, M.L.Ge, A.P.Isaev and P.N.Pyatov for interesting discussions. This work is supported in part by RFFR under grant N93-011-147 and ISF under grant M1L-000.

**References**

[1] G.E.Arutyunov and P.B.Medvedev, Quantization of the External Algebra on a Poisson Lie Group, Prep. SMI-11-93 (1993), [hep-th/9311096](https://arxiv.org/abs/hep-th/9311096).

[2] I.Ya.Aref’eva, G.E.Arutyunov and P.B.Medvedev, Poisson Lie structures on the external algebra of $SL(2)$ and their quantization, Prep. SMI-5-94 (1994), [hep-th/9401127](https://arxiv.org/abs/hep-th/9401127) (to be published in J.Math.Phys.).

[3] A.Connes, Publ. Math. IHES, 62 (1986) 41.

[4] S.Woronowicz, Publ.RIMS, Kyoto University 23 (1987) 117; Comm.Math.Phys. 122 (1989) 125;

[5] L.D.Faddeev, Integrable Models in 1+1 Dimensional Quantum Field Theory, Les Houches Lectures, 1982.

[6] V.G.Drinfel’d, Quantum groups, Proc. Int. Congr. Math. Berkley, 1 (1986) 798.

[7] M.A.Semenov-Tian-Shansky, Publ.RIMS Kyoto Univ., 21 (1985) 6, 1237; Funkts.Anal.Prilozh, 17 (1983) 17.

[8] L.D.Faddeev, N.Reshetikhin and L.A.Takhtajan, Alg.Anal. 1 (1988) 129.

[9] A.Yu.Alekseev and L.D.Faddeev, Comm.Math.Phys. 141 (1991) 413.

[10] P.P.Kulish, POMI notes, 205 (1992) 85 (in Russian).

[11] L.A.Takhtajan, Elementary course on quantum groups, PAM# 120, Univ.of Colorado, 1992.
[12] B.Jurco, Lett.Math.Phys., 22 177 (1991).

[13] U.Carow-Watamura, M.Schlieker, S.Watamura and W.Weich, Comm.Math.Phys. 142 605 (1991).

[14] F.Muller-Hoissen, J.Phys. A 25 (1992) 1703.

[15] P.Schupp, P.Watts and B.Zumino, Lett.Math.Phys. 25 (1992) 139.

[16] G.Maltsiniotis, C.R.Acad.Sci.Paris, 331 831 (1990).

[17] Yu.Manin, Bonn Prep. MPI/91-47 (1991), Bonn Prep. MPI/91-60 (1991).

[18] A.Sudbery, York Prep. PRINT-91-0498 (YORK), York Prep. PRINT-92-1 (YORK).

[19] B.Zumino, Differential calculus on quantum spaces and quantum groups, Prep. LBL-33249 (1992).

[20] L.D.Faddeev, Lectures on Int. Workshop "Interplay between Mathematics and Physics", Vienna, (1992), unpublished.

[21] L.D.Faddeev and P.N.Pyatov, The differential calculus on quantum linear groups, Prep. POMI (1994), hep-th/9402070.

[22] I.Ya.Aref’eva and I.V. Volovich, Mod.Phys.Lett., A6 (1991) 893.

[23] A.P.Isaev and Z.Popowicz, Phys.Lett. B281 (1992) 271; B307 (1993) 353.

[24] L.Castellani, Phys.Lett. B292 (1992) 93.

[25] I.Ya.Aref’eva and G.E.Arutyunov, Uniqueness of $U_q(N)$ as a quantum gauge group and representations of its differential algebra, Prep. SMI-4-93, hep-th/9305176.

[26] G.E.Arutyunov and P.B.Medvedev, On Poisson Lie structure on the external algebra of classical Lie groups, Prep. SMI-10-94 (1994), hep-th/9404068.

[27] G.E.Arutyunov, A.P.Isaev and Z.Popovicz, The structure of bicovariant differential calculus on quantum Orthogonal and Symplectic groups, in preparation.

[28] E.K.Sklyanin, Funkts.Anal.Prilozh, 16 (1982) N4, 27; 17 (1983) N4, 34.

[29] A.P.Isaev and P.N.Pyatov, Phys.Lett., A179 (1993) 81; Covariant differential complexes on quantum linear groups, Prep. E2-93-416 JINR, Dubna (1993) HEP-TH/931112.