Improved bounds on coloring of graphs

Sokol Ndreca\textsuperscript{1}, Aldo Procacci\textsuperscript{2}, Benedetto Scoppola\textsuperscript{3}

\textsuperscript{1}Dep. Estatística-ICEEx, UFMG, CP 702 Belo Horizonte - MG, 30161-970 Brazil
\textsuperscript{2}Dep. Matemática-ICEEx, UFMG, CP 702 Belo Horizonte - MG, 30161-970 Brazil
\textsuperscript{3}Dipartimento di Matematica - Universita Tor Vergata di Roma, 00133 Roma, Italy

emails: sokol@est.ufmg.br; aldo@mat.ufmg.br; scoppola@mat.uniroma2.it

Abstract

Given a graph \( G \) with maximum degree \( \Delta \geq 3 \), we prove that the acyclic edge chromatic number \( \chi'(G) \) of \( G \) is such that \( \chi'(G) \leq [9.62(\Delta - 1)] \). Moreover we prove that: \( \chi'(G) \leq [6.42(\Delta - 1)] \) if \( G \) has girth \( g \geq 5 \); \( \chi'(G) \leq [5.77(\Delta - 1)] \) if \( G \) has girth \( g \geq 7 \); \( \chi'(G) \leq [4.52(\Delta - 1)] \) if \( g \geq 53 \); \( \chi'(G) \leq \Delta + 2 \) if \( g \geq 25.84\Delta \log \Delta(1 + 4.1/\log \Delta) \). We further prove that the acyclic (vertex) chromatic number \( a(G) \) of \( G \) is such that \( a(G) \leq [6.59\Delta^{4/3} + 3.3\Delta] \). We also prove that the star-chromatic number \( \chi_s(G) \) of \( G \) is such that \( \chi_s(G) \leq [4.34\Delta^{3/2} + 1.5\Delta] \). We finally prove that the \( \beta \)-frugal chromatic number \( \chi^\beta(G) \) of \( G \) is such that \( \chi^\beta(G) \leq \max\{k_1(\beta)\Delta, k_2(\beta)\Delta^{1+1/\beta}/(\beta!)^{1/\beta}\} \), where \( k_1(\beta) \) and \( k_2(\beta) \) are decreasing functions of \( \beta \) such that \( k_1(\beta) \in [4,6] \) and \( k_2(\beta) \in [2,5] \). To obtain these results we use an improved version of the Lovász Local Lemma due to Bissacot, Fernández, Procacci and Scoppola \cite{6}.

1 Introduction

Let \( G = (V,E) \) be an undirected graph with vertex set \( V \) and edge set \( E \). Let \( \Delta \) be the maximum degree of \( G \) and \( g \) the girth of \( G \) (i.e. the length of the shortest cycle in \( G \)). A vertex coloring of \( G \) is proper if no two adjacent vertices receive the same color. A proper vertex coloring of \( G \) is acyclic if there are no two-colored cycles in \( G \). A proper vertex coloring of \( G \) is a star coloring if no path of length 3 is bi-chromatic. A proper vertex coloring of \( G \) is \( \beta \)-frugal if any vertex has at most \( \beta \) members of any color class in its neighborhood. Similarly, an edge coloring of \( G \) is said to be proper if no pair of incident edges receive the same color. A proper edge coloring of \( G \) is said to be acyclic if there are no two-colored cycles.

The minimum number of colors required such that a graph \( G \) has at least one proper vertex coloring is called chromatic number of \( G \) and will be denoted by \( c(G) \). The minimum number of colors required such that a graph \( G \) has at least one acyclic proper vertex coloring is called acyclic chromatic number of \( G \) and will be denoted by \( a(G) \). The minimum number of colors required for a graph \( G \) to have at least one star vertex coloring is called the star chromatic number of \( G \) and will be denoted by \( \chi_s(G) \). The minimum number of colors required such that a graph \( G \) has at least one \( \beta \)-frugal proper vertex coloring is called the \( \beta \)-frugal chromatic number of \( G \) and will be denoted by \( \chi^\beta(G) \). The minimum number of colors such that a graph \( G \) has at least one proper edge coloring is called the chromatic edge number and will be denoted by
As far as we know, the best known upper bound for \( a(G) \) in graphs with maximum degree \( \Delta \) has been given in [1] (see there Proposition 2.2), where it is proved that \( a(G) \leq 50\Delta^{4/3} \) for all \( \Delta \geq 1 \). However in [1] authors remarked that the constant 50 is not optimal. The best known upper bound for \( a'(G) \) in a graph with maximum degree \( \Delta \) was obtained in [14] (see there Theorem 2.2) where it is proved that \( a'(G) \leq 16\Delta \) for all \( \Delta \geq 1 \). Recently such bound has been sensibly improved in [16] if one excludes graphs with girth less than 9. Actually it is proved in [16] that, if \( g \geq 9 \) and \( \Delta \geq 4 \), then \( a'(G) \leq 5.91\Delta \) and if \( g \geq 220 \) and \( \Delta \geq 4 \) then \( a'(G) \leq 4.52\Delta \) (see there Theorems 1 and 2). Alon, Sudakov and Zaks have conjectured in [3] that \( a'(G) \leq \Delta + 2 \) and they proved this conjecture for graphs with girth \( g \geq 2000\Delta \log \Delta \) and \( \Delta \geq 3 \) (see there Theorem 4). Also in this case authors did not try to optimize the constant. Recently, the conjecture that \( a'(G) \leq \Delta + 2 \) has been confirmed for some more families of graphs. Namely, complete bipartite graphs [4], outerplanar graphs [17], and graphs with maximum degree four [5].

To our knowledge, the best known upper bound for \( \chi_s(G) \) in graphs of maximum degree \( \Delta \) has been given in [12], where it is proved that \( \chi_s(G) \leq 20\Delta^{3/2} \) for \( \Delta \geq 1 \) (see there Theorem 1.1). Finally, Hind, Molloy and Reed have proved in [13] that \( \chi^\beta(G) \leq \max\{(\beta + 1)\Delta, e^3\Delta^{1+1/\beta}/\beta\} \) for sufficiently large \( \Delta \) (see there Theorem 2). In papers [1], [3], [12], [13], [14], [16] the proofs rely on the Lovász Local Lemma.

The Lovász Local Lemma, which is one of the main tools of the probabilistic method in combinatorics, has been recently related to the cluster expansion of the abstract polymer gas, which in turn is a widely used technique in statistical mechanics. Indeed, during the last decade, the intersection between statistical mechanics and combinatorics has attracted the attention of several researchers and has been increasingly investigated. In particular, the application of cluster expansion methods to coloring problems in graph theory goes back to 2001 with a seminal paper by Sokal [21] relating the anti-ferromagnetic Potts model partition function on a graph \( G \) with the chromatic polynomial on the same graph.

Concerning specifically the Lovász Local Lemma, its surprising and close connection with statistical mechanics was pointed out by Scott and Sokal [20] in 2005. Indeed, in [20], using an old theorem by Shearer [19], the authors showed that the conclusions of the Lovász Local Lemma hold for the dependency graph \( G \) with vertex set \( X \) and probabilities \( \{p_x\}_{x \in X} \) if and only if the independent-set polynomial for \( G \) is non vanishing in the polydisc of radii \( \{p_x\}_{x \in X} \). The relation with statistical mechanics occurs because the independent-set polynomial of \( G \) is, modulo a constant, the partition function of the hard core self repulsive lattice gas on \( G \), so that its logarithm is the pressure of such a gas. From this, Sott and Sokal could conclude that the Lovász Local Lemma is a different way to rephrase the Dobrushin condition [8] for the convergence of the pressure of the hard core lattice gas on \( G \).

In 2007, Fernández and Procacci [10] provided a new criterion for the convergence of the pressure of the lattice gas on a graph \( G \), and showed that this new criterion is always more effective than the Dobrushin’s criterion. Later, the same authors used their criterion in [11] to improve Sokal’s results on zero-free regions of chromatic polynomial of [21].

Very recently, Bissacot et al. [6] used the Fernández-Procacci criterion [10] and the results in [20] to improve the Lovász Local Lemma. This new version of the Lovász Local Lemma has already been used to improve an old result on Latin-transversal [6] and to obtain some new results about colorings of the edges of the complete graph \( K_n \) [7]. Finally, it is worth mentioning that in a very recent paper, Pegden [18] has shown that the new Lemma of [6] also holds in the Moser and Tardos’s algorithmic contest [15].
The present paper thus aims at informing the combinatorics community that many classical bounds obtained via the Lovász Local Lemma can be improved using the new version presented in [6]. In an effort to convince the reader about that, we focus our attention on the six graph coloring problems described above, showing how it is possible to obtain, in a quite straightforward way, improvements on the bounds given in [1], [3], [12], [13], [14] and [16], by simply using the new lemma of [6] in place of the Lovász Local Lemma.

The rest of the paper is organized as follows. In Section 2 we recall the Lovász Local Lemma (Theorem 1), we present the new lemma [6] (Theorem 2) and we state our results on graph colorings (Theorem 3). In Section 3 we prove Theorem 3.

2 The Lovász Local Lemma, the new lemma, and results

We first state the Lovász Local Lemma (LLL) and immediately after the new lemma of [6] with the intent of clarifying how the improvement works.

To state these lemmas we need some definitions. Hereafter, if \( U \) is a finite set, \(|U|\) denotes its cardinality. Let \( X \) be a finite set. Let \( \{A_x\}_{x \in X} \) be a family of events on some probability space, each of which having probability \( \text{Prob}(A_x) = p_x \) to occur. A graph \( H \) with vertex set \( V(H) = X \) is a dependency graph for the family of events \( \{A_x\}_{x \in X} \) if, for each \( x \in X \), \( A_x \) is independent of all the events in the \( \sigma \)-algebra generated by \( \{A_y : y \in X \setminus \Gamma^*_H(x)\} \), where \( \Gamma_H(x) \) denotes the set of vertices of \( H \) adjacent to \( x \) and \( \Gamma^*_H(x) = \Gamma_H(x) \cup \{x\} \).

Denoting \( \bar{A}_x \) the complement event of \( A_x \), let \( \bigcap_{x \in X} \bar{A}_x \) be the event such that none of the events \( \{A_x\}_{x \in X} \) occurs.

**Theorem 1 (Lovász Local Lemma).** Suppose that \( H \) is a dependency graph for the family of events \( \{A_x\}_{x \in X} \) each one with probability \( \text{Prob}(A_x) = p_x \) and there exist \( \{\mu_x\}_{x \in X} \) real numbers in \( [0, +\infty) \) such that, for each \( x \in X \),

\[
p_x \leq \frac{\mu_x}{\varphi_x(\mu)}
\]

with

\[
\varphi_x(\mu) = 1 + \sum_{R \subseteq \Gamma^*_H(x)} \prod_{x \in R} \mu_x
\]

Then

\[
\text{Prob}(\bigcap_{x \in X} \bar{A}_x) > 0
\]

**Remark.** In the literature the LLL is usually written in terms of variables \( r_x = \mu_x/(1 + \mu_x) \in [0, 1) \), so that (2.1) becomes \( p_x \leq r_x \prod_{y \in \Gamma_H(x)} (1 - r_y) \) (see e.g. Lemma 5.1.1 p. 68 in [2]). However, the formulation above, although completely equivalent to the usual one, shows in a clear way the difference and the consequent improvement contained in the following theorem.

**Theorem 2 ([6]).** Suppose that \( H \) is a dependency graph for the family of events \( \{A_x\}_{x \in X} \) each one with probability \( \text{Prob}(A_x) = p_x \) and there exist \( \{\mu_x\}_{x \in X} \) real numbers in \( [0, +\infty) \) such that, for each \( x \in X \),

\[
p_x \leq \frac{\mu_x}{\varphi^*_x(\mu)}
\]

with

\[
\varphi^*_x(\mu) = 1 + \sum_{R \subseteq \Gamma^*_H(x)} \prod_{x \in R} \mu_x
\]
Theorem 3. \[ \text{Prob}(\bigcap_{x \in X} \bar{A}_x) > 0 \] \hspace{1cm} (2.6)

Remark. The only difference between the LLL (as stated in Theorem 1) and Theorem 2 above is that in Theorem 1 the sum of the right hand side of (2.2) is over all the subsets of $\Gamma_H^*(x)$ while in Theorem 2 the same sum is now only over the independent subsets of $\Gamma_H^*(x)$. This yields $\varphi^*(\mu) \leq \varphi(x(\mu))$ so that condition (2.4) in Theorem 2 is always less restrictive than (2.1) in Theorem 1. Moreover, noting that $\varphi(x(\mu)) = (1 + \mu_x) \prod_{y \in \Gamma_H(x)} (1 + \mu_y)$, it is clear that condition (2.1) of Theorem 1 does not depend on the graph structure of $\Gamma_H(x)$ (i.e., on the subgraph of $H$ induced by $\Gamma_H(x)$) but only on its cardinality. For example, condition (2.1) is the same, either if $\Gamma_H(x)$ is an independent set, or $\Gamma_H(x)$ is a clique. In contrast, condition (2.4) in Theorem 2 does depend on the graph structure of $\Gamma_H^*(x)$ (and hence of $\Gamma_H(x)$). Consequently, the improvement brought by Theorem 2 is maximal when the set of vertices $\Gamma_H(x)$ which are neighbors of $x$ form a clique, and it is nearly null when vertices of $\Gamma_H(x)$ form an independent set in the dependency graph (e.g., like in bipartite graphs). In view of this, in the next section we will frequently the following inequality. Let $x$ be a vertex of the dependency graph $H$ for the events $\{A_x\}_{x \in X}$, and suppose that $\Gamma_H^*(x)$ is the union (not necessarily disjoint) of $c_1, \ldots, c_k$ cliques, then, by definition (2.4),

\[ \varphi^*(\mu) \leq 1 + \sum_{s=1}^k \sum_{1 \leq i_1 < \cdots < i_s \leq k} \sum_{x_1 \in c_{i_1}} \cdots \sum_{x_s \in c_{i_s}} \mu_{x_1} \cdots \mu_{x_s} = \prod_{i=1}^k \left[ 1 + \sum_{y \in c_i} \mu_y \right] \] \hspace{1cm} (2.7)

We conclude the section by stating the results contained in the present paper, which can be summarized by the following theorem.

**Theorem 3.** If $G$ is a graph with maximum degree $\Delta \geq 3$ and girth $g$, then

(a) \( a'(G) \leq [9.62(\Delta - 1)] \).

(b) If $g \geq 5$, then $a'(G) \leq [6.42(\Delta - 1)]$. If $g \geq 7$, then $a'(G) \leq [5.77(\Delta - 1)]$. If $g \geq 53$, then $a'(G) \leq [4.52(\Delta - 1)]$.

(c) If $g \geq [25.84\Delta \log \Delta (1 + \frac{1}{4\log \Delta})]$, then $a'(G) \leq \Delta + 2$.

(d) $a(G) \leq [6.59\Delta^{4/3} + 3.3\Delta]$.

(e) $\chi_s(G) \leq [4.34\Delta^{3/2} + 1.5\Delta]$.

(f) For any $\beta \geq 1$, $\chi^\beta(G) \leq [\max\{k_1(\beta)\Delta, k_2(\beta)\Delta^{1+\frac{1}{\beta}}\}]$, where $k_1(\beta)$ and $k_2(\beta)$ are decreasing functions of $\beta$ such that $k_1(\beta) \in [4, 5.27]$ and $k_2(\beta) \in [2, 4.92]$.

Remark. Note that, differently from Theorem 2 in [13], in item (f) it is not required to take $\Delta$ sufficiently large. As for items (a)-(e), to prove item (f) we only need $\Delta \geq 3$. We also stress that we did not attempt to optimize the non leading terms in $\Delta$ in bounds (c)-(e).
3 Proof of Theorem 3

Hereafter $G = (V, E)$ will denote an undirected graph with vertex set $V$, edge set $E$, maximum degree $\Delta \geq 3$ and girth $g$.

3.1 Proof of item (a): acyclic edge chromatic number of $G$

Let $K$ be the set whose elements are the pairs $\{e, e'\} \subset E$ such that $e, e'$ are incident in a common vertex. Let, for $k \geq 2$, $C_{2k}(G)$ be the set of all cycles in $G$ of length $2k$. Finally, let $X = K \cup (\bigcup_{k \geq 2} C_{2k})$. We regard cycles $C_{2k}$ as sets of edges, so that the elements of $X$ are (some of) the subsets of $E$. For each edge $e \in E$, choose a color independently and uniformly among $N$ possible colors such that $N \geq c(\Delta - 1)$ ($c$ is a constant to be determined later). Consider now the following unfavorable events.

I. For $\{e, e'\} \in K$, let $A_{\{e, e'\}}$ be the event that the edges $e$ and $e'$ have the same color.

II. For $c_{2k} \in C_{2k}$ ($k \geq 2$), let $A_{c_{2k}}$ be the event that the cycle $c_{2k}$ is (properly) bichromatic.

If condition (2.4) of Theorem 2 holds, there is a non zero probability that none of the events of type I or II occurs, and hence there exists a proper edge coloring of $G$ with no two-colored cycles. To check condition (2.4) we first observe that, for each $\{e, e'\} \in K$, the probability of the event $A_{\{e, e'\}}$ is

$$\text{Prob}(A_{\{e, e'\}}) = \frac{1}{N}$$

while, for any $k \geq 2$ and $c_{2k} \in C_{2k}$

$$\text{Prob}(A_{c_{2k}}) \leq \frac{1}{N^{2k-2}}$$

Secondly, we have to find a graph with vertex set $X$ which is a dependency graph for the events $\{A_x\}_{x \in X}$. Since we are choosing a color at random for each edge independently, we have clearly that the event $A_{\{e, e'\}}$ is independent of any other event $A_{\{f, f'\}}$ such that $\{e, e'\} \cap \{f, f'\} = \emptyset$ and of all events $A_{c_{2k}}$ with $k \geq 2$ such that $\{e, e'\} \cap c_{2k} = \emptyset$. Analogously, for any $m \geq 2$, the event $A_{c_{2m}}$ is independent of all events $A_{\{f, f'\}}$ with $\{f, f'\} \in K$ and all events $A_{c_{2k}}$ with $k \geq 2$ such that $c_{2m} \cap \{f, f'\} = \emptyset$ and $c_{2m} \cap c_{2k} = \emptyset$. So let $H = (X, F)$ be the graph with vertex set $X$ and edge set $F$ such that the pair $\{x, x'\} \in F$ if and only if $x \cap x' \neq \emptyset$. By construction, $H$ is a dependency graph for the events $\{A_x\}_{x \in X}$. Now observe that

- each edge $e$ is contained in at most $2(\Delta - 1)$ pairs $\{f, f'\} \in K$
- each edge $e$ is contained in at most $(\Delta - 1)^{2k-2}$ cycles $c_{2k} \in C_{2k}$, for any $k \geq 2$

Hence

[a] for each vertex $x = \{e_1, e_2\} \in K$ of $H$, $\Gamma_H^*(x)$ is the union of two sets $\Gamma_1^*(x)$ and $\Gamma_2^*(x)$ such that, for $i = 1, 2$

$$|\Gamma_i^*(x)| \leq 2(\Delta - 1) + \sum_{s \geq 2}(\Delta - 1)^{2s-2}$$

and every element $z \in \Gamma_i^*(x)$ contains $e_i$ ($i = 1, 2$), so that the subgraph of $H$ induced by $\Gamma_i^*(x)$ is a clique.
For \( k \geq 2 \) and for each vertex \( y = c_{2k} = \{e_1, \ldots, e_{2k}\} \in C_{2k} \) of \( H \), \( \Gamma^*_y \) is the union of \( 2k \) sets \( \Gamma^*_{1}(y), \ldots, \Gamma^*_{2k}(y) \) such that, for any \( j = 1, 2, \ldots, 2k \),

\[
|\Gamma^*_j(y)| \leq 2(\Delta - 1) + \sum_{s \geq 2} (\Delta - 1)^{2s-2}
\]

and every element \( z \in \Gamma^*_j(y) \) contains \( e_j \). Hence, for any \( j = 1, 2, \ldots, 2k \), the subgraph of \( H \) induced by \( \Gamma^*_j(y) \) is a clique.

Let us now choose nonnegative numbers \( \{\mu_z\}_{z \in X} \) such that: for any \( x \in K \), \( \mu_x = \mu_1 \); for each \( y \in C_{2k} \), \( \mu_y = \mu_k \). Then, recalling definition (2.5) and inequality (2.7), an easy calculation shows that under conditions [a] and [b] we have

\[
\varphi^*_x(\mu) \leq \left[ 1 + 2(\Delta - 1)\mu_1 + \sum_{s \geq 2} (\Delta - 1)^{2s-2}\mu_s \right]^2
\]

and

\[
\varphi^*_y(\mu) \leq \left[ 1 + 2(\Delta - 1)\mu_1 + \sum_{s \geq 2} (\Delta - 1)^{2s-2}\mu_s \right]^{2k}
\]

and hence conditions (2.4) become

\[
\frac{1}{N} \leq \frac{\mu_1}{\left[ 1 + 2(\Delta - 1)\mu_1 + \sum_{s \geq 2} (\Delta - 1)^{2s-2}\mu_s \right]^2}
\]

\[
\frac{1}{N^{2k-2}} \leq \frac{\mu_k}{\left[ 1 + 2(\Delta - 1)\mu_1 + \sum_{s \geq 2} (\Delta - 1)^{2s-2}\mu_s \right]^{2k}}
\]

Now choosing \( \mu_1 = \mu = \frac{\alpha}{\Delta - 1} \) with \( 0 < \alpha < 1 \) and \( \mu_k = \mu^{2k-2} \), and recalling that \( N \geq c(\Delta - 1) \), the conditions above are satisfied if

\[
\frac{1}{c} \leq \frac{\alpha}{\left( 1 + 2\alpha + \sum_{s \geq 2} \alpha^{2s-2} \right)^2}
\]

\[
\frac{1}{c} \leq \frac{\alpha}{\left( 1 + 2\alpha + \sum_{s \geq 2} \alpha^{2s-2} \right)^{2k/2k-2}}
\]

Since \( k \geq 2 \), the first inequality implies the second. Therefore the condition which guarantees that none of the bad events \( \{A_x\}_{x \in X} \) occurs is

\[
\frac{1}{c} \leq \frac{\alpha}{\left( 1 + 2\alpha + \sum_{s \geq 2} \alpha^{2s-2} \right)^2}
\]

i.e.,

\[
c \geq \alpha^{-1} \left[ 1 + 2\alpha + \frac{\alpha^2}{1 - \alpha^2} \right]^2 \quad (3.1)
\]

The function on the right hand side of (3.1) can be minimized in the interval \( \alpha \in (0, 1) \) and a straightforward calculation gives that (3.1) is satisfied if \( c \geq 9.6130002 \). Hence every graph \( G \) with maximum degree \( \Delta \) such that edges are colored using a number of colors \( N \) greater or equal than \( 9.62(\Delta - 1) \) admits an acyclic proper coloring. \( \square \)
Remark. As observed in [16], using the Lovász Local Lemma one can obtain an upper bound for the edge chromatic number $c'(G)$ of a graph $G$ at best $c'(G) \leq \lceil 4e\Delta \rceil$ and for any $G$ we have clearly that $a'(G) \geq c'(G)$. We leave to the reader to check that one can obtain $c'(G) \leq 4(\Delta - 1)$ using Theorem 2 in place of Theorem 1 and proceeding similarly to the scheme illustrated in the proof of item (a) above.

3.2 Proof of item (b): acyclic edge chromatic number of $G$ when $g \geq 5$

We follow here the strategy described in [16]. Namely we will first consider the following problem.

Let $\eta \geq 2$ be an integer. We want to know the minimum colors needed to find a coloring $C$ of the edges of $G$ such that

1. In any vertex $v$ of $G$ the number of edges incident to $v$ having the same color is at most $\eta$
2. There is no properly bichromatic cycle in $G$
3. There is no monochromatic cycle in $G$

Suppose that we are able to prove that, for some $N \in \mathbb{N}$, we find a coloring $C$ which satisfies 1, 2, 3, using $N$ colors. Then it is also possible to find a coloring $C'$ using $N' = \eta N$ colors which is proper and satisfies 2 (i.e. $C'$ is an acyclic proper coloring). Indeed, just observe that in the coloring $C$ the sets of edges with the same color are forests with maximum degree $\eta$ and one needs $\eta$ colors to proper color a forest with maximum degree $\eta$. So if one recolors each color $c_i$ ($i = 1, 2, \ldots, N$) in the coloring $C$ using $c_i^1, c_i^2, \ldots, c_i^{\eta}$ distinct colors in such a way that monochromatic forests disappear, then one gets a new coloring $C'$ in which $N' = \eta N$ colors are used and by construction $C'$ is proper and satisfies 2.

Now, we use Theorem 2 to show that if $N \geq c(\Delta - 1)$ (where $c$ is a constant to be determined), then the coloring $C$ satisfying properties 1-3 exists and hence, in view of the above argument, if $N' \geq c'(\Delta - 1)$, with $\bar{c} = \eta c$, then there is an acyclic edge coloring $C'$ on $G$ using $N'$ colors.

As we did in the previous subsection, let us choose for each edge $e \in E$ independently a color at random among $N \geq c(\Delta - 1)$ possible colors. Let now $K_\eta$ be the set whose elements are sets of edges $\kappa_\eta = \{e_1, e_2, \ldots, e_{\eta + 1}\} \subset E$ all incident to a common vertex. Let $C_m$ ($m \geq 3$) be the set whose elements are all cycles $c_m$ in $G$ of length $m$. Finally, let $X = \bigcup_{m \geq 3} C_m \cup K_\eta$. We regard cycles as subsets of edges. So again the elements of $X$ are (some of) the subsets of $E$.

We now consider the following unfavorable events

I. For $\kappa_\eta = \{e_1, \ldots, e_{\eta + 1}\} \in K_\eta$, let $A_{\kappa_\eta}$ be the event that all edges $e_1, \ldots, e_{\eta + 1}$ have the same color.

II. For $c_{2k} \in C_{2k}$, let $A_{c_{2k}}$ be the event that the cycle $c_{2k}$ is either properly bichromatic or monochromatic.

III. For $c_{2l+1} \in C_{2l+1}$, let $A_{c_{2l+1}}$ be the event that the cycle $c_{2l+1}$ is monochromatic.

Theorem 2 gives a condition which guarantees that the probability that none of the events of type I or II or III occurs is strictly positive and hence the existence of a coloring $C$ of $G$ with properties 1, 2 and 3 above.

Observe that, for $\kappa_\eta \in K_\eta$, the probability of the event $A_{\kappa_\eta}$ is

$$\text{Prob}(A_{\kappa_\eta}) = \frac{1}{N^\eta}$$
while, for any $k \geq \lfloor g/2 \rfloor$ and $c_{2k} \in C_{2k}$

$$\text{Prob}(A_{c_{2k}}) = \frac{1}{N^{2k-2}}$$

and, for any $l \geq \lfloor g/2 \rfloor$ and $c_{2l+1} \in C_{2l+1}$

$$\text{Prob}(A_{c_{2l+1}}) = \frac{1}{N^{2l}}$$

To prove (3.2) just observe that the total number of ways of coloring an even cycle $c_{2k}$ using $N$ colors is $N^{2k}$ while the number of ways of coloring an even cycle $c_{2k}$ using $N$ colors so that the cycle is either monochromatic or proper bichromatic is $N + N(N - 1) = N^2$, where $N$ is the number of different proper bichromatic ways of coloring the cycle $c_{2k}$ and $N(N - 1)$ is the number of different proper bichromatic ways of coloring the cycle $c_{2k}$. So $\text{Prob}(A_{c_{2k}}) = N^2 / N^{2k} = 1 / N^{2k-2}$.

Now we have to find a graph with vertex set $X$ which is a dependency graph for the events $\{A_x\}_{x \in X}$. Since we are choosing a color at random for each edge independently, we have once again that the event $A_x$ is independent of all other events $A_{x'}$ such that $x \cap x' = \emptyset$. So the graph $H = (X, F)$, with vertex set $X$ and edge set $F$ such that the pair $\{x, x'\} \in F$ if and only if $x \cap x' \neq \emptyset$, is a dependency graph for the events $\{A_x\}_{x \in X}$. Now observe that

- each edge $e$ is contained in at most $2(\Delta - 1)^{l-1} \leq 2(\Delta - 1)^{\eta l}$ distinct sets $\kappa_\eta = \{e_1, \ldots, e_{\eta + 1}\} \in K_\eta$;
- each edge $e$ is contained in at most $(\Delta - 1)^{m-2}$ cycles $c_m \in C_m$ ($m \geq 3$).

Hence we have the following.

[a] For each vertex $x = \kappa_\eta = \{e_1, e_2, \ldots, e_{\eta + 1}\} \in K_\eta$ of $H$, $\Gamma_i^*(x)$ is the union of $\eta + 1$ sets $\Gamma_i^*(x)$ ($i = 1, \ldots, \eta + 1$) such that

$$|\Gamma_i^*(x)| \leq 2\left(\frac{\Delta - 1}{\eta l}\right)^{\eta} + \sum_{k \geq [g/2]} (\Delta - 1)^{2k - 2} + \sum_{l \geq [g/2]} (\Delta - 1)^{2l - 1}$$

and every element of $\Gamma_i^*(x)$ contains $e_i$. Hence the subgraph of $H$ induced by $\Gamma_i^*(x)$ is a clique for $i = 1, \ldots, \eta + 1$.

[b] For $m \geq 3$ and for each vertex $y = c_m = \{e_1, \ldots, e_m\} \in C_m$ of $H$, we have that $\Gamma_i^*(y)$ is the union of $m$ sets $\Gamma_i^*(y)$ such that, for $j = 1, \ldots, m$,

$$|\Gamma_j^*(y)| \leq 2\left(\frac{\Delta - 1}{\eta l}\right)^{\eta} + \sum_{k \geq [g/2]} (\Delta - 1)^{2k - 2} + \sum_{l \geq [g/2]} (\Delta - 1)^{2l - 1}$$

and every element of $\Gamma_j^*(y)$ contains $e_j$ so that the subgraph of $H$ induced by $\Gamma_j^*(y)$ is a clique for all $j = 1, 2, \ldots, m$.

Let us now choose nonnegative numbers $\{\mu_x\}_{x \in X}$ such that: for any $x \in K_\eta$, $\mu_x = \mu_1$; for each $y \in C_m$, $\mu_y = \mu_m$. Then using once again inequality (2.7) one gets, under conditions [a] and [b], that

$$\varphi^{\ast}_x \mu \leq \left[1 + 2\left(\frac{\Delta - 1}{\eta l}\right)^{\eta} \mu_1 + \sum_{k \geq [g/2]} (\Delta - 1)^{2k - 2} \mu_{2k} + \sum_{l \geq [g/2]} (\Delta^{2l - 1} \mu_{2l+1})^{\eta + 1}\right]^{\eta + 1}$$
and

$$\varphi_\eta^\ast(\mu) \leq \left[ 1 + 2 \frac{(\Delta - 1)^\eta}{\eta!} \mu_1 + \sum_{k \geq \lceil \eta/2 \rceil} (\Delta - 1)^{2k-2} \mu_{2k} + \sum_{l \geq \lfloor \eta/2 \rfloor} (\Delta - 1)^{2l-1} \mu_{2l+1} \right]^m$$

and hence condition (2.4) of Theorem 2 becomes

$$\frac{1}{N^\eta} \leq \mu_1 \left[ 1 + 2 \frac{(\Delta - 1)^\eta}{\eta!} \mu_1 + \sum_{k \geq \lceil \eta/2 \rceil} (\Delta - 1)^{2k-2} \mu_{2k} + \sum_{l \geq \lfloor \eta/2 \rfloor} (\Delta - 1)^{2l-1} \mu_{2l+1} \right]^\eta+1$$

$$\frac{1}{N^{2k-2}} \leq \mu_2 \left[ 1 + 2 \frac{(\Delta - 1)^\eta}{\eta!} \mu_1 + \sum_{k \geq \lceil \eta/2 \rceil} (\Delta - 1)^{2k-2} \mu_{2k} + \sum_{l \geq \lfloor \eta/2 \rfloor} (\Delta - 1)^{2l-1} \mu_{2l+1} \right]^{2k}$$

$$\frac{1}{N^{2l}} \leq \mu_{2l+1} \left[ 1 + 2 \frac{(\Delta - 1)^\eta}{\eta!} \mu_1 + \sum_{k \geq \lceil \eta/2 \rceil} (\Delta - 1)^{2k-2} \mu_{2k} + \sum_{l \geq \lfloor \eta/2 \rfloor} (\Delta - 1)^{2l-1} \mu_{2l+1} \right]^{2l+1}$$

Now choose $\mu_1 = \mu_\eta^h$, $\mu_2 = \mu_\eta^{2k-2}$, $\mu_{2l+1} = \mu_\eta^{2l}$ and $\mu = \frac{\alpha}{\Delta - 1}$ with $\alpha \in (0, 1)$. Then, recalling that $N \geq c(\Delta - 1)$, the conditions above are satisfied if

$$\frac{1}{c^\alpha} \leq \left[ 1 + 2 \frac{\alpha^n}{\eta!} + \sum_{k \geq \lceil \eta/2 \rceil} \alpha^{2k-2} + \frac{1}{(\Delta - 1) \sum_{l \geq \lfloor \eta/2 \rfloor} \alpha^{2l}} \right]^{(\eta+1)/\eta}$$

$$\frac{1}{c^\alpha} \leq \left[ 1 + 2 \frac{\alpha^n}{\eta!} + \sum_{k \geq \lceil \eta/2 \rceil} \alpha^{2k-2} + \frac{1}{(\Delta - 1) \sum_{l \geq \lfloor \eta/2 \rfloor} \alpha^{2l}} \right]^{2k/(2k-2)}$$

$$\frac{1}{c^\alpha} \leq \left[ 1 + 2 \frac{\alpha^n}{\eta!} + \sum_{k \geq \lceil \eta/2 \rceil} \alpha^{2k-2} + \frac{1}{(\Delta - 1) \sum_{l \geq \lfloor \eta/2 \rfloor} \alpha^{2l}} \right]^{(2l+1)/2l}$$

If $g \geq 5$, then $k \geq 3$ and $l \geq 2$ and moreover, if $\eta \geq 2$, the three inequalities are satisfied if

$$\frac{1}{c^\alpha} \leq \left[ 1 + 2 \frac{\alpha^n}{\eta!} + \frac{1}{1-\alpha^2} \left( \alpha^{2\lceil \eta/2 \rceil - 2} + \frac{1}{\Delta - 1} \alpha^{2\lfloor \eta/2 \rfloor} \right) \right]^{(\eta+1)/\eta}$$

Hence, recalling that $\bar{c} = \eta c$, observing that $\alpha^{2\lceil \eta/2 \rceil - 2} + \frac{1}{\Delta - 1} \alpha^{2\lfloor \eta/2 \rfloor} \leq \frac{\Delta}{\Delta - 1} \alpha^{2\lceil \eta/2 \rceil - 2}$ for all $\Delta \geq 3$ and all $g \geq 3$, and optimizing with respect to $\alpha \in (0, 1)$, we get

$$\bar{c} \geq \eta \min_{\alpha \in (0, 1)} \alpha' \left[ 1 + 2 \frac{\alpha^n}{\eta!} + \frac{\Delta}{\Delta - 1} \alpha^{2\lceil \eta/2 \rceil - 2} \right]^{(\eta+1)/\eta}$$

(3.3)

If $g \geq 5$, $\eta = 2$, and $\Delta \geq 3$ a rough calculation (bounding $\frac{\Delta}{\Delta - 1}$ by $3/2$ for all $\Delta \geq 3$) gives $\bar{c} \geq 6.42$ and hence $a'(G) \leq [6.42(\Delta - 1)]$; if $g \geq 7$ and $\eta = 2$, we get $\bar{c} \geq 5.77$ and hence $a'(G) \leq [5.77(\Delta - 1)]$. Finally, if $g \geq 53$ and $\eta = 3$, we get $\bar{c} \geq 4.52$ and hence $a'(G) \leq [4.52\Delta]$. Note that $a'(G)/\Delta \leq 4.52$ is obtained in [10] for $g \geq 220$. □

**Remark.** Observe that, for fixed $g, \eta$ the quantity $\bar{c}$ defined in (3.3) slightly decreases as $\Delta \to \infty$ and one can check that $\lim_{\Delta \to \infty} \bar{c}(5, 2, \Delta) \leq 6.159 \ldots$, $\lim_{\Delta \to \infty} \bar{c}(7, 2, \Delta) = 5.654 \ldots$, $\lim_{\Delta \to \infty} \bar{c}(53, 2, \Delta) = 4.511 \ldots$, yielding a slight improvement of $a'(G)$ in all three cases considered as $\Delta \to \infty$. 9
3.3 Proof of item (c): a class of graphs with maximum degree $\Delta$ and acyclic edge chromatic number $\leq \Delta + 2$

We follow [3] using Theorem 2 instead of the Lovász Local Lemma. By Vizing’ Theorem [22], there exists a proper coloring $\mathcal{C}$ of the edges of $G$ with $\Delta + 1$ colors. An even cycle is called properly half-monochromatic with respect to the coloring $\mathcal{C}$, if one of its halves (a set of alternate edges) is monochromatic while the other half is not. Note that a properly half-monochromatic cycle is never properly bichromatic by the coloring $\mathcal{C}$. Observe also that a cycle of odd length can never be properly bichromatic by the coloring $\mathcal{C}$.

Let $K$ be the set whose elements are all pairs of adjacent edges $\{e, e'\}$ of $G$. Let $B_{2m}$ ($m \geq 2$) be the set whose elements are all cycles $b_{2m}$ in $G$ of length $2m$ which are properly bichromatic by the coloring $\mathcal{C}$. Let $H_{2m}$ ($m \geq 2$) be the set whose elements are all cycles $h_{2m}$ in $G$ of length $2m$ which are properly half-monochromatic by the coloring $\mathcal{C}$. Finally, let $X = K \cup (\bigcup_{m \geq 2} B_{2m}) \cup (\bigcup_{m \geq 2} H_{2m})$. Again the elements of $X$ are (some of) the subsets of $E$.

Let us now recolor each edge $e \in E$ using a new color randomly and independently with probability $\frac{c}{\Delta^2}$ (where $c \leq 1$ is a constant to be determined later). Call $\mathcal{C}'$ this new coloring which by construction uses $\Delta + 2$ colors. We need to show that with positive probability the coloring $\mathcal{C}'$ is such that

A. No pair of adjacent edges has the same color

B. There is no properly bichromatic cycle

We use condition (2.4) of Theorem 2. So, once $G$ has been recolored by the coloring $\mathcal{C}'$, consider the following bad events.

I. For each pair of adjacent edges $\{e, e'\} \in K$, let $A_{\{e,e'\}}$ be the event that $e$ and $e'$ have the same color.

II. For each properly bichromatic cycle $b_{2k} \in B_{2k}$ of length $2k$, $k \geq 2$, in $G$ with respect to the coloring $\mathcal{C}$, let $A_{b_{2k}}$ be the event that either no edge is recolored with the new color or one half is recolored and the other half stays unchanged.

III. For each properly half-monochromatic cycle $h_{2m} \in H_{2m}$ of length $2m$, $m \geq 2$, with respect to $\mathcal{C}$, let $A_{h_{2m}}$ be the event that $h_{2m}$ becomes properly bichromatic by recoloring the non monochromatic half of its edges with the new color and leaving the monochromatic part unchanged.

Clearly, if none of the bad events I, II, or III occurs, properties A and B are satisfied. It is straightforward to see that the probabilities of the events I, II, and III are as follows.

For each pair of adjacent edges $\{e, e'\}$

$$\text{Prob}(A_{\{e,e'\}}) = \frac{c^2}{\Delta^2}$$

For $k \geq 2$ and each properly bichromatic cycle $b_{2k}$ of length $2k$,

$$\text{Prob}(A_{b_{2k}}) = \left(1 - \frac{c}{\Delta}\right)^{2k} + 2 \left(1 - \frac{c}{\Delta}\right)^k \left(\frac{c}{\Delta}\right)^k \leq \frac{1}{(1 + \frac{c}{\Delta})^{2k}} \quad (3.4)$$
For \( k \geq 2 \) and each properly half-monochromatic cycle \( h_{2k} \) of length \( 2k \),

\[
\text{Prob}(A_{h_{2k}}) = \frac{c^k}{\Delta^k} \left( 1 - \frac{c}{\Delta} \right)^k
\]

To prove (3.4) let \( w = c/\Delta \). Then (3.4) becomes

\[
(1 - w)^{2k} + 2w^k (1 - w)^k \leq \frac{1}{(1 + w)^{2k}}
\]

i.e., multiplying both side of the inequality by \((1 + w)^{2k}\)

\[
(1 - w^2)^{2k} + 2 (1 - w^2)^k (w+1)^k \leq 1
\]

which is true for all \( k \geq 2 \), if \( w \leq 1/2 \), which is indeed the case since \( w = c/\Delta \leq 1/3 \).

Now, as before, a bad event \( A_x \) of the collection \( \{A_x\}_{x \in X} \) is independent of all other events \( A_{x'} \) such that \( x \cap x' = \emptyset \). So the graph \( H = (X, F) \) with vertex set \( X \) and edge set \( F \) such that the pair \( \{x, x'\} \in F \) if and only if \( x \cap x' \neq \emptyset \) is a dependency graph for the collection of events \( \{A_x\}_{x \in X} \). Moreover, as shown in [3] (see there Lemma 7), in a properly edge-colored graph \( G \)

- each edge \( e \) is contained in at most \( 2\Delta \) pairs \( \{f, f'\} \) of incident edges
- each edge \( e \) is contained in at most \( \Delta \) properly bichromatic cycles of \( G \)
- each edge \( e \) is contained in at most \( 2\Delta^{k-1} \) half-monochromatic cycles of length \( 2k \)

Hence

[a] For each vertex \( x = \{e_1, e_2\} \in K \) of \( H \), \( \Gamma^*_i(x) \) is the union of 2 sets \( \Gamma_i^*(x) \) \((i = 1, 2)\) such that

\[
|\Gamma_i^*(x)| \leq 2\Delta + \Delta + \sum_{k \geq |g/2|} 2\Delta^{k-1}
\]

and every element of \( \Gamma_i^*(x) \) contains \( e_i \). Hence the subgraph of \( H \) induced by \( \Gamma_i^*(x) \) is a clique for \( i = 1, 2 \).

[b] For \( k \geq 2 \) and for each vertex \( y = \{e_1, \ldots, e_{2k}\} \in B_{2k} \cup H_{2k} \) of \( H \), we have that \( \Gamma^*_j(y) \) is the union of \( m \) sets \( \Gamma_j^*(y), \ldots, \Gamma_{2k}^*(y) \) such that, for \( j = 1, \ldots, 2k \),

\[
|\Gamma^*_j(y)| \leq 2\Delta + \Delta + \sum_{k \geq |g/2|} 2\Delta^{k-1}
\]

and every element of \( \Gamma^*_j(y) \) contains \( e_j \) so that the subgraph of \( H \) induced by \( \Gamma^*_j(y) \) is a clique for all \( j = 1, 2, \ldots, 2k \).

Let us now choose nonnegative numbers \( \{\mu_z\}_{z \in X} \) as follows. For any \( x \in K \), put \( \mu_x = \mu_1 \); for each \( y \in B_{2m} \), put \( \mu_y = \mu_2 \); for any \( z \in H_{2m} \) put \( \mu_z = \mu_{2m} \). Then using once again inequality (2.7) one gets, under conditions [a] and [b], that, for any \( x \in K \) and any \( y \in B_{2k} \cup H_{2k} \),

\[
\phi^*_x(\mu) \leq (1 + 2\Delta \mu_1 + \Delta \mu_2 + \sum_{s \geq |g/2|} 2\Delta^{s-1} \mu_{2s})^2
\]
and
\[ \varphi_x^*(\mu) \leq (1 + 2\Delta \mu_1 + \Delta \mu_2 + \sum_{s \geq \lceil g/2 \rceil} 2\Delta s^{-1} \mu_{2s})^{2k} \]

Hence, we have that the condition of Theorem 2 is satisfied if there are positive numbers \( \mu_1, \mu_2, \{\mu_{2k}\}_{k \geq \lceil g/2 \rceil} \) such that the following inequalities are satisfied

\[ \frac{c^2}{\Delta^2} \leq \frac{\mu_1}{(1 + 2\Delta \mu_1 + \Delta \mu_2 + \sum_{s \geq \lceil g/2 \rceil} 2\Delta s^{-1} \mu_{2s})^2} \]
\[ \frac{1}{(1 + \frac{c}{\Delta})^{2k}} \leq \frac{\mu_2}{(1 + 2\Delta \mu_1 + \Delta \mu_2 + \sum_{s \geq \lceil g/2 \rceil} 2\Delta s^{-1} \mu_{2s})^{2k}} \]
\[ \frac{c^k}{\Delta^k} \left(1 - \frac{c}{\Delta}\right)^k \leq \frac{\mu_{2k}}{(1 + 2\Delta \mu_1 + \Delta \mu_2 + \sum_{s \geq \lceil g/2 \rceil} 2\Delta s^{-1} \mu_{2s})^{2k}} \]

Put now \( \mu_1 = \mu_2 = \frac{c^2}{\Delta^2} \) and \( \mu_{2k} = \frac{c^k}{\Delta^k} \). Then the inequalities above become

\[ c \leq \frac{\alpha}{1 + \frac{R_g(\alpha)}{\Delta}} \] (3.5)
\[ \frac{1}{(1 + \frac{c}{\Delta})^k} \leq \frac{\alpha}{\left[1 + \frac{R_g(\alpha)}{\Delta}\right]^k} \] (3.6)
\[ c \left(1 - \frac{c}{\Delta}\right) \leq \frac{\alpha}{\left[1 + \frac{R_g(\alpha)}{\Delta}\right]^2} \] (3.7)

where we have put
\[ R_g(\alpha) = 3\alpha^2 + \frac{2\alpha^{\lceil g/2 \rceil}}{1 - \alpha} \] (3.8)

Now note that (3.6) can be satisfied for all \( k \) greater than some fixed \( k_0 \) only if

\[ c > R_g(\alpha) \] (3.9)

On the other hand, if (3.9) holds, then inequality (3.7) is satisfied if (3.5) is satisfied. Indeed inequality (3.7) can be rewritten as

\[ c \leq \frac{\alpha}{\left(1 + \frac{R_g(\alpha)}{\Delta}\right)^2 \left(1 - \frac{c}{\Delta}\right)} = \frac{\alpha}{\left(1 + \frac{R_g(\alpha)}{\Delta}\right) \left(1 - \frac{\alpha^{\lceil g/2 \rceil}}{1 - \alpha}\right) \left(1 + \frac{R_g(\alpha)}{\Delta}\right)} \]

and \( \left[(1 - \frac{\alpha}{\Delta}) \left(1 + \frac{R_g(\alpha)}{\Delta}\right)\right]^{-1} > 1 \) due to (3.9).

Now let us suppose that \( g \geq 80 \) so that

\[ R_g(\alpha) \leq 3\alpha^2 + \frac{2\alpha^{40}}{1 - \alpha} \]

and let find \( \alpha_0 \) such that

\[ f(\alpha_0) = \frac{\alpha}{\left(1 + \frac{R_{80}(\alpha)}{\Delta}\right)} - R_{80}(\alpha) \]
Finally, let $X$ be the elements of $G$ are four in $u, v$ a pair of non-adjacent vertices among $N_2 \Delta G = \langle \cdot \rangle$ which holds for all integers $k \geq \Delta$ i.e. We follow [1] using Theorem 2 instead of the Lovász Local Lemma. Let $\alpha = 0.0721$, $c_0 - R_{80}(\alpha_0) = 0.07928$, and also using that $\Delta \geq 3$, we get after some easy computations

$$k \log \left\{ \frac{1 + \frac{c_0}{\Delta}}{1 + \frac{R_{80}(\alpha_0)}{\Delta}} \right\} \geq \log \Delta + \log(1/\alpha_0)$$

Observe now that, for $a > b > 0$, $\log \left\{ \frac{1 + a}{1 + b} \right\} = \log \left\{ 1 + \frac{a - b}{1 + b} \right\}$ and, for $w \geq 0$, $\log(1 + w) \geq w(1 - w/2)$. So, recalling that $R_{80}(\alpha_0) = 0.0721$, $c_0 - R_{80}(\alpha_0) = 0.07928$, and also using that $\Delta \geq 3$, we get after some easy computations

$$k \geq 12.92\Delta \log \Delta \left( 1 + \frac{2}{\log \Delta} \right) \left( 1 + \frac{2}{\Delta} \right)$$

which holds for all integers $k \geq \lceil g/2 \rceil$ as soon as $g \geq 25.84\Delta \log \Delta (1 + \frac{2}{\log \Delta})(1 + \frac{2}{\Delta})$ and since $(1 + \frac{2}{\log \Delta})(1 + \frac{2}{\Delta}) \leq 1 + \frac{4.4}{\log \Delta}$ for all $\Delta \geq 3$, Theorem 3 item (c) follows. □

### 3.4 Proof item (d): acyclic chromatic number of $G$

We follow [1] using Theorem 2 instead of the Lovász Local Lemma. Let $C$ be a vertex-coloring of $G = (V, E)$ such that in each vertex the color is chosen at random independently and uniformly among $N \geq c \Delta^{4/3}$ colors ($c$ being a positive constant to be determined later). In the following a pair of non-adjacent vertices $u, v$ of $G$ will be called a special pair if $u$ and $v$ have more than $\Delta^{2/3}$ common neighbors and will be denoted by $\langle u, v \rangle$.

Let $P_4$ be the set whose elements are set of vertices $\{v_0, v_1, v_2, v_3, v_4\}$ forming paths of length four in $G$. Let $C_4$ be the set whose elements are sets of vertices $\{v_1, v_2, v_3, v_4\}$ forming 4-cycles in $G$. Let $S$ be the set whose elements are sets of vertices $\langle v, v' \rangle$ forming special pairs in $G$. Finally, let $X = E \cup P_4 \cup C_4 \cup S$ (here, of course, $E$ is the set of edges of $G$). Observe that now the elements of $X$ are (some of) the subsets of the vertex set $V$ of $G$.

Consider the following unfavorable events.

**I.** For each pair of adjacent vertices $\{u, v\} \in E$ of $G$, let $A_{\{u, v\}}$ be the event that $u$ and $v$ have the same color.

**II.** For each path $P_4 = v_0v_1v_2v_3v_4 \in P_4$ of $G$, let $A_{P_4}$ be the event that vertices $v_0, v_2, v_4$ have the same color and vertices $v_1, v_3$ have the same color.

**III For each induced 4-cycle $C_4 = v_1v_2v_3v_4 \in C_4$ of $G$, in which neither $v_1; v_3$ nor $v_2; v_4$ is a special pair, let $A_{C_4}$ be the event that $v_1; v_3$ have the same color and $v_2; v_4$ have the same color.

**IV For each special pair of vertices $\langle u, v \rangle \in S$ of $G$ let $A_{\{u, v\}}$ be the event that $u$ and $v$ receive the same color.**
Alon, Mc Diarmid and Reed have shown in [1] that if none of the event I, II, III or IV occurs then the graph is properly colored without bichromatic cycles (see in [1], proof of proposition 2.2). We now use Theorem 2 to show that with positive probability none of the events occurs.

We first observe that the probability of an event of type I, II, III and IV respectively are
\[
\text{Prob}(A_{\{u,v\}}) = \frac{1}{N}, \quad \text{Prob}(A_{p_4}) = \frac{1}{N^3}, \quad \text{Prob}(A_{c_4}) = \frac{1}{N^2}, \quad \text{Prob}(A_{\langle u,v \rangle}) = \frac{1}{N}
\]

Secondly, we note that, for \(x, x' \in X\) an event \(A_x\) (where \(x \in X\) can be a pair of adjacent vertices, a path of length four, a cycle of length three or a special pair) is independent of all events \(A_{x'}\) such that \(x \cap x' = \emptyset\). So the graph \(H = (X, F)\) with vertex set \(X\) and edge set \(F\), such that the pair \(\{x, x'\} \in F\) if and only if \(x \cap x' \neq \emptyset\), is a dependency graph for the collection of events \(\{A_x\}_{x \in X}\).

Finally, following [1], (see there the proof of Lemma 2.4) we have that
\begin{itemize}
  \item a vertex \(v \in V\) belongs to at most \(\Delta\) edges of \(G\).
  \item a vertex \(v \in V\) belongs to at most \(\frac{5}{2}\Delta^4\) paths of length 4 in \(G\).
  \item The number of induced 4-cycles in \(G\) containing \(v\) in which no opposite pair of vertices is a special pair is at most \(\frac{1}{2}\Delta^{8/3}\).
  \item The number of special pairs of vertices containing a given vertex \(v\) is at most \(\Delta^{4/3}\).
\end{itemize}

Hence

[a] For each vertex \(x \in E \cup S\) of \(H\) (i.e. either \(x = \{v_1, v_2\}\) or \(x = \langle v_1, v_2 \rangle\)), \(\Gamma^*_H(x)\) is the union of 2 sets \(\Gamma^*_1(x)\) \((i = 1, 2)\) such that
\[
|\Gamma^*_i(x)| \leq \Delta + \frac{5}{2}\Delta^4 + \frac{1}{2}\Delta^{8/3} + \Delta^{4/3}
\]
and every element of \(\Gamma^*_i(x)\) contains \(v_i\). Hence the subgraph of \(H\) induced by \(\Gamma^*_i(x)\) is a clique for \(i = 1, 2\).

[b] For each vertex \(y = \{v_0, v_1, v_2, v_3, v_4\} \in P_4\) of \(H\), we have that \(\Gamma^*_H(y)\) is the union of 5 sets \(\Gamma^*_1(y), \ldots, \Gamma^*_5(y)\) such that, for \(j = 1, \ldots, 5,\)
\[
|\Gamma^*_j(y)| \leq \Delta + \frac{5}{2}\Delta^4 + \frac{1}{2}\Delta^{8/3} + \Delta^{4/3}
\]
and every element of \(\Gamma^*_j(y)\) contains \(v_j\) so that the subgraph of \(H\) induced by \(\Gamma^*_j(y)\) is a clique for all \(j = 1, 2, \ldots, 5\).

[c] For for each vertex \(z = \{v_1, v_2, v_3, v_4\} \in C_4\) of \(H\), we have that \(\Gamma^*_H(z)\) is the union of 4 sets \(\Gamma^*_1(z), \ldots, \Gamma^*_4(z)\) such that, for \(j = 1, \ldots, 4,\)
\[
|\Gamma^*_j(z)| \leq \Delta + \frac{5}{2}\Delta^4 + \frac{1}{2}\Delta^{8/3} + \Delta^{4/3}
\]
and every element of \(\Gamma^*_j(z)\) contains \(v_j\) so that the subgraph of \(H\) induced by \(\Gamma^*_j(z)\) is a clique for all \(j = 1, 2, 3, 4\).
Let us now choose nonnegative numbers \( \{ \mu_u \}_{u \in X} \) as follows. For any \( x \in E \), put \( \mu_x = \mu_1 \); for each \( y \in P_3 \), put \( \mu_y = \mu_2 \); for any \( z \in C_4 \), put \( \mu_z = \mu_3 \); for any \( w \in S \), put \( \mu_z = \mu_4 \). Then using (2.7) one gets, under conditions [a]-[c], that, for any \( x \in E \cup S \), any \( y \in P_4 \), and any \( z \in C_4 \)

\[
\varphi_x^4(\mu) \leq (1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^2
\]

and

\[
\varphi_y^5(\mu) \leq (1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^5
\]

and

\[
\varphi_z^2(\mu) \leq (1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^4
\]

Hence Theorem 2 holds if there are positive numbers \( \mu_1, \mu_2, \mu_3, \) and \( \mu_4 \) such that the following inequalities are simultaneously satisfied:

\[
\begin{align*}
\frac{1}{N} & \leq \frac{\mu_1}{(1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^2} \\
\frac{1}{N^3} & \leq \frac{\mu_2}{(1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^5} \\
\frac{1}{N^2} & \leq \frac{\mu_3}{(1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^4} \\
\frac{1}{N} & \leq \frac{\mu_4}{(1 + \Delta \mu_1 + \frac{5}{2}\Delta^4 \mu_2 + \frac{1}{2}\Delta^8/3 \mu_3 + \Delta^{4/3} \mu_4)^2}
\end{align*}
\]

Taking \( \mu_1 = \mu_4 = \mu, \mu_2 = \mu^3 \) and \( \mu_3 = \mu^2 \) these inequalities are satisfied if, for some \( \mu > 0 \)

\[
\frac{1}{N} \leq \frac{\mu}{(1 + (\Delta + \Delta^{1/3}) \mu + \frac{1}{2}\Delta^8/3 \mu^2 + \frac{5}{2}\Delta^4 \mu^3)^2}
\]

Now choose \( \mu = \alpha/\Delta^{4/3} \). Then, recalling that \( N \geq c\Delta^{4/3} \), inequality above is satisfied if

\[
\frac{1}{c} \leq \frac{\alpha}{(1 + (1 + \Delta^{-1/3}) \alpha + \frac{1}{2}\alpha^2 + \frac{5}{2}\alpha^3)^2}
\]

i.e.

\[
c \geq \frac{1}{\alpha}(1 + \alpha + \frac{1}{2}\alpha^2 + \frac{5}{2}\alpha^3)^2 + \left[ \frac{\alpha}{\Delta^{2/3}} + \frac{2}{\Delta^{1/3}}(1 + \alpha + \frac{1}{2}\alpha^2 + \frac{5}{2}\alpha^3) \right]
\]

(3.10)

and taking \( \alpha = 0.34 \) it is easy to check that the right hand side of (3.10) is less than 6.583 + 3.3/\( \Delta^{1/3} \) for all \( \Delta \geq 3 \). So we get \( c \geq 6.583 + 3.3/\Delta^{1/3} \) and hence \( a(G) \leq \lfloor 6.583\Delta^{4/3} + 3.3\Delta \rfloor \).

\[ \square \]

### 3.5 Proof of item (e): star chromatic number of \( G \)

We follow [12], but we use Theorem 2 in place of the Lovász Local Lemma. Let \( C \) be a vertex-coloring of \( G = (V, E) \) using \( N \geq c\Delta^{3/2} \) colors (\( c \) being a positive constant to be determined later) such that in each vertex the color is chosen at random independently and uniformly among the set of \( N \) colors.
Let $P_3$ be the set whose elements are set of vertices $\{v_1, v_2, v_3, v_4\}$ forming paths of length three in $G$ and let $X = E \cup P_3$. Observe that, as in subsection 3.4, the elements of $X$ are (some of) the subsets of the vertex set $V$ of $G$.

Consider the following unfavorable events.

I. For each pair of adjacent vertices $\{u, v\} \in E$ of $G$, let $A_{\{u,v\}}$ be the event that $u$ and $v$ have the same color.

II. For each path of length three $p_3 = v_1v_2v_3v_4 \in P_3$ in $G$, let $A_{p_3}$ be the event that vertices $v_1, v_3$ have the same color and vertices $v_2$ and $v_4$ have the same color.

Clearly, by definition, if none of the events above occurs then $C$ is a star coloring.

We first observe that the probability of an event of type I, II, respectively are

$$\text{Prob}(A_{\{u,v\}}) = \frac{1}{N}, \quad \text{Prob}(A_{p_3}) = \frac{1}{N^2}$$

Then, as in Subsection 3.4, we observe that, for $x \in X$, the event $A_x$ (where now $x \in X$ can be a pair of adjacent vertices or a path of length four) is independent of all events $A_{x'}$, with $x' \in X$, such that $x \cap x' = \emptyset$. So the graph $H = (X, F)$ with vertex set $X$ and edge set $F$ such that the pair $\{x, x'\} \in F$ if and only if $x \cap x' \neq \emptyset$ is a dependency graph for the collection of events $\{A_x\}_{x \in X}$.

Finally, as observed in [12] (see there Observation 8.1) we have that

- a vertex $v \in V$ belongs to at most $\Delta$ edges of $G$.
- a vertex $v \in V$ belongs to at most $2\Delta(\Delta - 1)^2 \leq 2\Delta^3$ paths of length 3 in $G$.

Hence

[a] For each vertex $x = \{v_1, v_2\} \in E$ of $H$, $\Gamma^*_H(x)$ is the union of 2 sets $\Gamma^*_i(x)$ ($i = 1, 2$) such that

$$|\Gamma^*_i(x)| \leq \Delta + 2\Delta^3$$

and every element of $\Gamma^*_i(x)$ contains $v_i$. Hence the subgraph of $H$ induced by $\Gamma^*_i(x)$ is a clique for $i = 1, 2$.

[b] For for each vertex $y = \{v_1, v_2, v_3, v_4\} \in P_3$ of $H$, we have that $\Gamma^*_H(y)$ is the union of 4 sets $\Gamma^*_1(y), \ldots, \Gamma^*_4(y)$ such that, for $j = 1, \ldots, 4$,

$$|\Gamma^*_j(y)| \leq \Delta + 2\Delta^3$$

and every element of $\Gamma^*_j(y)$ contains $v_j$ so that the subgraph of $H$ induced by $\Gamma^*_j(y)$ is a clique for all $j = 1, 2, \ldots, 4$.

Let us now choose nonnegative numbers $\{\mu_z\}_{z \in X}$ as follows. For any $x \in E$, put $\mu_x = \mu_1$; for each $y \in P_3$, put $\mu_y = \mu_2$. Then, by conditions [a] and [b], using (2.7) we get that, for any $x \in E$, and any $y \in P_3$

$$\varphi^*_x(\mu) \leq (1 + \Delta \mu_1 + 3\Delta^3 \mu_2)^2$$
and

$$\varphi_y^*(\mu) \leq (1 + \Delta \mu_1 + 3 \Delta^3 \mu_2)^4$$

Hence, analogously to the previous sections, Theorem 2 holds if we can find nonnegative numbers \(\mu_1, \mu_2\) such that the following inequalities are simultaneously satisfied:

$$\frac{1}{N} \leq \frac{\mu_1}{(1 + \Delta \mu_1 + 2 \Delta^3 \mu_2)^2}$$

$$\frac{1}{N^2} \leq \frac{\mu_2}{(1 + \Delta \mu_1 + 3 \Delta^3 \mu_2)^4}$$

Now take \(\mu_2 = \mu_1^2\) and \(\mu_1 = \alpha / \Delta^{3/2}\). Then, recalling that \(N \geq c \Delta^{3/2}\), these inequalities are satisfied if, for some \(\alpha > 0\)

$$\frac{1}{c} \leq \frac{\alpha}{(1 + \Delta^{-1/2} \alpha + 2 \alpha^2)^2}$$  \hspace{1cm} (3.11)

Maximizing the right hand side of (3.11) with respect to \(\alpha\) we get that the maximum is reached at

$$\alpha = \alpha_0 = \frac{1}{\sqrt{6} \left( \sqrt{1 + \frac{1}{24\Delta}} + \sqrt{\frac{1}{24\Delta}} \right)}$$

Now observing that \(\alpha_0 \leq 1/\sqrt{6}\) for all \(\Delta \geq 1\), we get that the inequality (3.11) is satisfied for all \(\Delta \geq 3\) as soon as

$$c \geq \sqrt{6} \left( \sqrt{1 + \frac{1}{24\Delta}} + \sqrt{\frac{1}{24\Delta}} \right) \left[ \frac{4}{3} + \frac{1}{\sqrt{6}\Delta} \right]^2$$  \hspace{1cm} (3.12)

It is now easy to check that the left hand side of (3.12) is less than \(\frac{16}{9} \sqrt{6} + \frac{15}{\sqrt{\Delta}}\) for all \(\Delta \geq 3\). So we get \(c \geq \frac{16}{9} \sqrt{6} + \frac{15}{\sqrt{\Delta}}\) and hence \(\chi_s(G) \leq \lceil 4.34 \Delta^{3/2} + 1.5\Delta \rceil\). \(\square\)

### 3.6 Proof of item (f): \(\beta\)-frugal chromatic number

We follow [13], but we use Theorem 2 instead of the Lovász Local Lemma. Let now \(C\) be a vertex-coloring of \(G = (V, E)\) using \(c\) colors (\(c\) being a positive constant to be determined later) such that in each vertex the color is chosen at random independently and uniformly among the set of \(c\) colors.

We may assume \(\beta \geq 2\) since in the case \(\beta = 1\) the 1-frugal chromatic number of \(G\), \(\chi^1(G)\), is just the vertex chromatic number of the graph obtained from \(G\) by adding an edge between any two vertices at distance 2 in \(G\), which has maximum degree at most \(\Delta^2\) and hence, by Vizing \(\chi^1(G) \leq \Delta^2 + 1\). Given \(v \in V\), let \(S^v_\beta\) be the set whose elements are sets of vertices \(\{v_1, \ldots, v_{\beta+1}\}\) such that \(\{v_1, \ldots, v_{\beta+1}\} \subset \Gamma_G(v)\). Let \(S_\beta = \bigcup_{v \in V} S^v_\beta\). Let \(X = E \cup S_\beta\). As in subsections 3.4 and 3.5, the elements of \(X\) are (some of) the subsets of the vertex set \(V\) of \(G\).

Consider the following unfavorable events.

I. For \(\{u, v\} \in E\), let \(A_{\{u,v\}}\) be the event that \(u\) and \(v\) receive the same color.

II. For \(s_\beta = \{v_1, \ldots, v_{\beta+1}\} \in S_\beta\), let \(A_{s_\beta}\) be the event that all vertices in \(s_\beta\) receive the same color.
If none of the events above occurs then, by definition, \( \mathcal{C} \) is a \( \beta \) frugal coloring. We have clearly
\[
\text{Prob}(A_{\{u,v\}}) = \frac{1}{c} \quad \text{and} \quad \text{Prob}(A_{\beta}) = \frac{1}{c^\beta}
\]
As in subsections 3.4 and 3.5, the graph \( H = (X,F) \) with vertex set \( X \) and edge set \( F \) such that the pair \( \{x,x'\} \in F \) if and only \( x \cap x' \neq \emptyset \) is a dependency graph for the events \( \{A_x\}_{x \in X} \). Moreover
- a vertex \( v \in V \) belongs to at most \( \Delta \) edges of \( G \).
- a vertex \( v \in V \) belongs to at most \( \Delta(\frac{\Delta}{\beta}) \leq \Delta^{1+\beta}/\beta! \) sets of type \( \mathcal{h}_\beta = \{v_1,\ldots,v_{\beta+1}\} \).

Hence

[a] For each vertex \( x = \{v_1,v_2\} \in E \) of \( H \), \( \Gamma^*_i(x) \) is the union of two sets \( \Gamma^*_i(x) \) (\( i = 1,2 \)) such that
\[
|\Gamma^*_i(x)| \leq \Delta + \Delta^{1+\beta}/\beta!
\]
and every element of \( \Gamma^*_i(x) \) contains \( v_i \). Hence the subgraph of \( H \) induced by \( \Gamma^*_i(x) \) is a clique for \( i = 1,2 \).

[b] For for each vertex \( y = \{v_1,\ldots,v_{\eta+1}\} \in S_\beta \) of \( H \), we have that \( \Gamma^*_j(y) \) is the union of \( \beta + 1 \) sets \( \Gamma^*_1(y),\ldots,\Gamma^*_j(y) \) such that, for \( j = 1,\ldots,\beta + 1 \),
\[
|\Gamma^*_j(y)| \leq \Delta + \Delta^{1+\beta}/\beta!
\]
and every element of \( \Gamma^*_j(y) \) contains \( v_j \) so that the subgraph of \( H \) induced by \( \Gamma^*_j(y) \) is a clique for all \( j = 1,2,\ldots,\beta + 1 \).

Let us now choose nonnegative numbers \( \{\mu_z\}_{z \in X} \) as follows. For any \( x \in E \), put \( \mu_x = \mu_1 \); for each \( y \in S_\beta \), put \( \mu_y = \mu_2 \). Then, by conditions [a] and [b], using (2.7) we get that, for any \( x \in E \) and any \( y \in S_\beta \)
\[
\varphi_x^*(\mu) \leq (1 + \Delta \mu_1 + \frac{\Delta^{1+\beta}}{\beta!} \mu_2)^2
\]
and
\[
\varphi_y^*(\mu) \leq (1 + \Delta \mu_1 + \frac{\Delta^{1+\beta}}{\beta!} \mu_2)^{\beta+1}
\]
Hence Theorem 2 holds if
\[
\frac{1}{c} \leq \frac{\mu_1}{(1 + \Delta \mu_1 + \frac{\Delta^{1+\beta}}{\beta!} \mu_2)^2} \quad \text{(3.13)}
\]
and
\[
\frac{1}{c^\beta} \leq \frac{\mu_2}{(1 + \Delta \mu_1 + \frac{\Delta^{1+\beta}}{\beta!} \mu_2)^{\beta+1}} \quad \text{(3.14)}
\]
Put \( \mu_1 = \mu \), \( \mu_2 = \beta! \mu^{1+\beta} \) and \( \alpha = \Delta \mu \in (0, +\infty) \), then inequalities (3.13) and (3.14) become
\[
\frac{1}{c} \leq \frac{1}{\Delta (1 + \alpha + \alpha^{1+\beta})^2} \quad \text{(3.15)}
\]

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and

\[
\frac{1}{c} \leq \frac{(\beta!)^{1/\beta}}{\Delta^{1+\frac{1}{\beta}}} \left[ \frac{\alpha}{(1+\alpha+\alpha^{1+\beta})} \right]^{1+\frac{1}{\beta}}
\]

(3.16)

which are satisfied if

\[
c \geq \max \left\{ k_1(\beta) \Delta, \frac{k_2(\beta) \Delta^{1+\frac{1}{\beta}}}{(\beta!)^{1/\beta}} \right\}
\]

(3.17)

where

\[
k_1(\beta) = \min_{\alpha>0} \left( \frac{(1+\alpha+\alpha^{1+\beta})^2}{\alpha} \right)^{1+\frac{1}{\beta}}
\]

\[
k_2(\beta) = \left[ \min_{\alpha>0} \left( \frac{1+\alpha+\alpha^{1+\beta}}{\alpha} \right)^{1+\frac{1}{\beta}} \right]
\]

An easy computation shows that \(k_1(\beta)\) and \(k_2(\beta)\) are both decreasing functions of \(\beta\) and, for \(\beta \geq 2\), we have that \(k_1(\beta) \in [4, 5.27]\) and \(k_2(\beta) \in [2, 4.92]\). Therefore we get that \(c \geq \max\{k_1(\beta) \Delta, k_2(\beta) \Delta^{1+1/\beta}/(\beta!)^{1/\beta}\}\) and hence \(\chi^\beta(G) \leq \lceil \max\{k_1(\beta) \Delta, k_2(\beta) \Delta^{1+1/\beta}/(\beta!)^{1/\beta}\} \rceil\).

\[\blacksquare\]

**Remark.** Note that (3.17) is valid for all \(\Delta \geq 3\). Moreover the upper bound (3.17) of \(\chi^\beta(G)\) is a decreasing function of \(\beta\).

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