On Calderón’s conjecture

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1. Introduction

This paper is a successor of [4]. In that paper we considered bilinear operators of the form

\[ H_\alpha(f_1, f_2)(x) := \text{p.v.} \int f_1(x - t)f_2(x + \alpha t) \frac{dt}{t}, \]

which are originally defined for \( f_1, f_2 \) in the Schwartz class \( S(\mathbb{R}) \). The natural question is whether estimates of the form

\[ \|H_\alpha(f_1, f_2)\|_p \leq C_{\alpha, p_1, p_2} \|f_1\|_{p_1} \|f_2\|_{p_2} \]

with constants \( C_{\alpha, p_1, p_2} \) depending only on \( \alpha, p_1, p_2 \) and \( p := \frac{p_1 p_2}{p_1 + p_2} \) hold. The first result of this type is proved in [4], and the purpose of the current paper is to extend the range of exponents \( p_1 \) and \( p_2 \) for which (2) is known. In particular, the case \( p_1 = 2, p_2 = \infty \) is solved to the affirmative. This was originally considered to be the most natural case and is known as Calderón’s conjecture [3].

We prove the following theorem:

**Theorem 1.** Let \( \alpha \in \mathbb{R} \setminus \{0, -1\} \) and

\[ \begin{align*}
1 < p_1, p_2 & \leq \infty, \\
\frac{2}{3} < p & := \frac{p_1 p_2}{p_1 + p_2} < \infty.
\end{align*} \]

Then there is a constant \( C_{\alpha, p_1, p_2} \) such that estimate (2) holds for all \( f_1, f_2 \in S(\mathbb{R}) \).

If \( \alpha = 0, -1, \infty \), then we obtain the bilinear operators

\[ H(f_1) \cdot f_2, \, H(f_1 \cdot f_2), \, f_1 \cdot H(f_2), \]

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the last one by replacing $t$ with $t/\alpha$ and taking a weak limit as $\alpha$ tends to infinity. Here $H$ is the ordinary linear Hilbert transform, and $\cdot$ is pointwise multiplication. The $L^p$-bounds of these operators are easy to determine and quite different from those in the theorem. This suggests that the behaviour of the constant $C_{\alpha,p_1,p_2}$ is subtle near the exceptional values of $\alpha$. It would be of interest to know that the constant is independent of $\alpha$ for some choices of $p_1$ and $p_2$.

We do not know that the condition $2/3 < p$ is necessary in the theorem. But it is necessary for our proof. An easy counterexample shows that the unconditionality in inequality (6) already requires $2/3 \leq p$. The cases of $(p_1, p_2)$ being equal to $(1, \infty)$, $(\infty, 1)$, or $(\infty, \infty)$ have to be excluded from the theorem, since the ordinary Hilbert transform is not bounded on $L^1$ or $L^\infty$.

We assume the reader as somewhat familiar with the results and techniques of [4]. The differences between the current paper and [4] manifest themselves in the overall organization and the extension of the counting function estimates to functions in $L^q$ with $q < 2$.

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2. Preliminary remarks on the exponents

Call a pair $(p_1, p_2)$ good, if for all $\alpha \in \mathbb{R} \setminus \{0, -1\}$ there is a constant $C_{\alpha,p_1,p_2}$ such that estimate (2) holds for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. In this section we discuss interpolation and duality arguments. These, together with the known results from [4], show that instead of Theorem 1 it suffices to prove:

**Proposition 1.** If $1 < p_1, p_2 < 2$ and $2/3 < \frac{p_1 p_2}{p_1 + p_2}$, then $(p_1, p_2)$ is good.

In [4] the following is proved:

**Proposition 2.** If $2 < p_1, p_2 < \infty$ and $1 < \frac{p_1 p_2}{p_1 + p_2} < 2$, then $(p_1, p_2)$ is good.

Strictly speaking, this proposition is proved in [4] only in the case $\alpha = 1$, but this restriction is inessential. The necessary modifications to obtain the full result appear in the current paper in Section 3. Therefore we take Proposition 2 for granted.

The next lemma follows by complex interpolation as in [1]. The authors are grateful to E. Stein for pointing out this reference to them.

**Lemma 1.** Let $1 < p_1, p_2, q_1, q_2 \leq \infty$ and assume that $(p_1, p_2)$ and $(q_1, q_2)$ are good. Then

$$
\left( \frac{\theta}{p_1} \frac{1 - \theta}{q_1} + \frac{\theta}{p_2} \frac{1 - \theta}{q_2} \right)
$$

is good for all $0 < \theta < 1$. 

Next we need a duality lemma.

**Lemma 2.** Let \(1 < p_1, p_2 < \infty\) such that \(\frac{p_1 p_2}{p_1 + p_2} \geq 1\). If \((p_1, p_2)\) is good, then so are the pairs

\[
\left( p_1, \left( \frac{p_1 p_2}{p_1 + p_2} \right)' \right) \quad \text{and} \quad \left( \left( \frac{p_1 p_2}{p_1 + p_2} \right)', p_2 \right).
\]

Here \(p'\) denotes as usual the dual exponent of \(p\). To prove the lemma, fix \(\alpha \in \mathbb{R} \setminus \{0, -1\}\) and \(f_1 \in \mathcal{S}(\mathbb{R})\) and consider the linear operator \(H_\alpha(f_1, \omega)\). The formal adjoint of this operator with respect to the natural bilinear pairing is

\[
\text{sgn}(1 + \alpha) H_{-\frac{\alpha}{1+\alpha}}(f_1, \omega),
\]

as the following lines show:

\[
\int \left( \text{p.v.} \int f_1(x - t)f_2(x + \alpha t) \frac{1}{t} dt \right) f_3(x) \, dx
\]

\[
= \text{p.v.} \int \int f_1(x - \alpha t - t)f_2(x)f_3(x - \alpha t) \, dx \frac{1}{t} dt
\]

\[
= \text{sgn}(1 + \alpha) \int \left( \text{p.v.} \int f_1(x - t)f_3(x - \frac{\alpha}{1+\alpha} t) \frac{1}{t} dt \right) f_2(x) \, dx.
\]

Similarly, we observe that for fixed \(f_2\) the formal adjoint of \(H_\alpha(\omega, f_2)\) is \(-H_{-1-\alpha}(\omega, f_2)\). This proves Lemma 2 by duality.

Now we are ready to prove estimate (2) in the remaining cases, i.e., for those pairs \((p_1, p_2)\) for which one of \(p_1, p_2\) is smaller or equal to two, and the other one is greater or equal to two. In this case the constraint on \(p\) is automatically satisfied. By symmetry it suffices to do this for \(p_1 \in [1, 2]\) and \(p_2 \in [2, \infty]\). First observe that the pairs \((3, 3)\) and \((3/2, 3/2)\) are good by the above propositions. Then the pairs \((2, 2)\) and \((2, \infty)\) are good by interpolation and duality. Let \(P\) be the set of all \(p_1 \in [1, 2]\) such that the pair \((p_1, p_2)\) is good for all \(p_2 \in [2, \infty]\). The previous observations show that \(2 \in P\). Define \(p := \inf P\) and assume \(p > 1\). Pick a small \(\varepsilon > 0\) and a \(p_1 \in P\) with \(p_1 < p + \varepsilon\). If \(\varepsilon\) is small enough, we can interpolate the good pairs \((p_1, \varepsilon^{-1})\) and \((1 + \varepsilon, 2 - \varepsilon)\) to obtain a good pair of the form \((q_\varepsilon, q_\varepsilon')\). Since \(\lim_{\varepsilon \to 0} q_\varepsilon = \frac{3p - 2}{2p - 1} < p\) we have \(q_\varepsilon < p\) provided \(\varepsilon\) is small enough. By duality we see that the pair \((q, \infty)\) is good, and by Proposition 1 there is a \(p_2 < 2\) such that \((q, p_2)\) is good. By interpolation \(q \in P\) follows. This is a contradiction to \(p = \inf P\); therefore the assumption \(p > 1\) is false and we have \(\inf P = 1\). Again by interpolation we observe \(P = [1, 2]\), which finishes the prove of estimate (2) for the remaining exponents.
3. Time-frequency decomposition of $H_\alpha$

In this section we write the bilinear operators $H_\alpha$ approximately as finite sums over rank one operators, each rank one operator being well localized in time and frequency. We mostly follow the corresponding section in [4], adopting the basic notation and definitions from there such as that of a phase plane representation.

In contrast to [4] we work out how the decomposition and the constants depend on $\alpha$, and we add an additional assumption (iv) in Proposition 3 which is necessary to prove $L^p$-estimates for $p < 2$. The reader should think of the functions $\theta_{\xi,s}$ in this assumption as being exponentials $\theta_{\xi,s}(x) = e^{i\eta x}$ for certain frequencies $\eta_s = \eta_1(\xi)$.

**Proposition 3.** Assume we are given exponents $1 < p_1, p_2 < 2$ such that $\frac{p_1 p_2}{p_1 + p_2} > \frac{2}{3}$, and we are given a constant $C_m$ for each integer $m \geq 0$. Then there is a constant $C$ depending on these data such that the following holds:

Let $S$ be a finite set, $\phi_1, \phi_2, \phi_3 : S \to \mathcal{S}(\mathbb{R})$ be injective maps, and $I, \omega_1, \omega_2, \omega_3 : S \mapsto J$ be maps such that $I(S)$ is a grid, $J_\omega := \omega_1(S) \cup \omega_2(S) \cup \omega_3(S)$ is a grid, and the following properties (i)--(iv) hold for all $i \in \{1, 2, 3\}$:

(i) The map $\rho_i : \phi_i(S) \to \mathcal{R}$, $\phi_i(s) \mapsto I(s) \times \omega_i(s)$ is a phase plane representation with constants $C_m$.

(ii) $\omega_i(s) \cap \omega_j(s) = \emptyset$ for all $s \in S$ and $j \in \{1, 2, 3\}$ with $i \neq j$.

(iii) If $\omega_i(s) \subset J$ and $\omega_i(s) \neq J$ for some $s \in S$, $J \in J_\omega$, then $\omega_j(s) \subset J$ for all $j \in \{1, 2, 3\}$.

(iv) To each $\xi \in \mathbb{R}$ there is associated a measurable function $\theta_{\xi,s} : \mathbb{R} \to \{z \in \mathbb{C} : |z| = 1\}$ such that for all $s \in S$, $J \in J_\omega$, and $J \in I(S)$ the following holds: If $\xi \in \omega_j(s)$, $|J| \leq |I(s)|$, then

\begin{equation}
\inf_{\lambda \in \mathbb{C}} \|\phi_i(s) - \lambda \theta_{\xi,i}\|_{L^\infty(J)} \leq C_0 |J| |I(s)|^{-\frac{2}{3}} \left(1 + \frac{|c(J)| - c(I(s))}{|I(s)|}\right)^{-2}.
\end{equation}

For all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ and all maps $\varepsilon : S \to [-1, 1]$, we then have:

\begin{equation}
\left\| \sum_{s \in S} \varepsilon(s) |I(s)|^{-\frac{2}{3}} \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \phi_3(s) \right\|_{\frac{p_1 p_2}{p_1 + p_2}} \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}.
\end{equation}

In the rest of this section we prove Proposition 2 under the assumption that Proposition 3 above is true. Let $1 < p_1, p_2 < 2$ with $\frac{2}{3} < p := \frac{p_1 p_2}{p_1 + p_2}$ and $\alpha \in \mathbb{R} \setminus \{0, -1\}$.
Let $L$ be the smallest integer larger than
\[
2^{10} \max \left\{ |\alpha|, \frac{1}{|\alpha|}, \frac{1}{|1 + \alpha|} \right\}.
\]
The dependence on $\alpha$ will enter into our estimate via a polynomial dependence on $L$.

Define $\varepsilon := L^{-3}$. Pick a function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\psi}$ is supported in $[L^3 - 1, L^3 + 1]$ and
\[
\sum_{k \in \mathbb{Z}} \hat{\psi}(2^k \xi) = 1 \text{ for all } \xi > 0.
\]

Define
\[
\psi_k(x) := 2^{-\frac{\varepsilon k}{2}} \hat{\psi}(2^{-\varepsilon k} x)
\]
and
\[
\tilde{H}_\alpha(f_1, f_2)(x) := \sum_{k \in \mathbb{Z}} 2^{-\frac{\varepsilon k}{2}} \int_{\mathbb{R}} f_1(x - t)f_2(x + \alpha t)\psi_k(t) \, dt.
\]

It suffices to prove boundedness of $\tilde{H}_\alpha$. Pick a $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\varphi}$ is supported in $[-1, 1]$ and
\[
\sum_{n,l \in \mathbb{Z}} \left\langle f, \varphi_{k,n,\frac{l}{2}} \right\rangle \varphi_{k,n,\frac{l}{2}} = f
\]
for all Schwartz functions $f$, where
\[
\varphi_{k,n,l}(x) := 2^{-\frac{\varepsilon k}{2}} \varphi(2^{-\varepsilon k} x - n) e^{2\pi i 2^{-\varepsilon k} x l}.
\]

We apply this formula three times in (7) to obtain:
\[
\tilde{H}_\alpha(f_1, f_2)(x) = \sum_{k,n_1,n_2,n_3,l_1,l_2,l_3} C_{k,n_1,n_2,n_3,l_1,l_2,l_3} H_{k,n_1,n_2,n_3,l_1,l_2,l_3}(f_1, f_2)(x)
\]
with
\[
H_{k,n_1,n_2,n_3,l_1,l_2,l_3}(f_1, f_2)(x) := 2^{-\frac{\varepsilon k}{2}} \left\langle f_1, \varphi_{k,n_1,\frac{l_1}{2}} \right\rangle \left\langle f_2, \varphi_{k,n_2,\frac{l_2}{2}} \right\rangle \varphi_{k,n_3,\frac{l_3}{2}}(x)
\]
and
\[
C_{k,n_1,n_2,n_3,l_1,l_2,l_3} := \int \int \varphi_{k,n_1,\frac{l_1}{2}}(x-t)\varphi_{k,n_2,\frac{l_2}{2}}(x+\alpha t)\varphi_{k,n_3,\frac{l_3}{2}}(x)\psi_k(t) \, dt \, dx.
\]

The proof of the following lemma is a straightforward calculation as in [4].

**Lemma 3.** There is a constant $C$ depending on $\phi$ and $\psi$ such that
\[
|C_{k,n_1,n_2,n_3,l_1,l_2,l_3}| \leq C \left( 1 + \frac{1}{L} \text{diam}\{n_1, n_2, n_3\} \right)^{-100}.
\]
Moreover,

\[ C_{k,n_1,n_2,n_3,l_1,l_2,l_3} = 0, \]

unless

\[
(12) \quad l_1 \in \left[ \left( -\frac{\alpha}{1 + \alpha} l_3 + \frac{2}{1 + \alpha} L^3 \right) - L, \left( -\frac{\alpha}{1 + \alpha} l_3 + \frac{2}{1 + \alpha} L^3 \right) + L \right]
\]

and

\[
(13) \quad l_2 \in \left[ \left( -\frac{1}{1 + \alpha} l_3 - \frac{2}{1 + \alpha} L^3 \right) - L, \left( -\frac{1}{1 + \alpha} l_3 - \frac{2}{1 + \alpha} L^3 \right) + L \right].
\]

Now we can reduce Proposition 2 to the following lemma:

**Lemma 4.** There is a constant \( C \) depending on \( p_1, p_2, \varphi, \) and \( \psi \) such that the following holds:

Let \( \nu > 0 \) be an integer and let \( S \) be a finite subset of \( \mathbb{Z}^3 \) such that for \((k,n,l),(k',n',l') \in S \) the following three properties are satisfied:

\[
(14) \quad \text{If } k \neq k', \quad \text{then } |k - k'| > L^{10},
\]

\[
(15) \quad \text{if } n \neq n', \quad \text{then } |n - n'| > L^{10} \nu,
\]

\[
(16) \quad \text{if } l \neq l', \quad \text{then } |l - l'| > L^{10}.
\]

Let \( \nu_1, \nu_2 \) be integers with \( 1 + \max \{|\nu_1|, |\nu_2|\} = \nu \) and let \( \lambda_1, \lambda_2 : \mathbb{Z} \to \mathbb{Z} \) be functions such that \( l_1 := \lambda_1(l_3) \) satisfies (12) and \( l_2 := \lambda_2(l_3) \) satisfies (13) for all \( l_3 \in \mathbb{Z} \). Then we have for all \( f_1, f_2 \in \mathcal{S}(\mathbb{R}) \) and all maps \( \varepsilon : S \to [-1,1] \):

\[
(17) \quad \left\| \sum_{(k,n,l) \in S} \varepsilon(k,n,l) H_{k,n+\nu_1+n+\nu_2,n,l_1,l_2,l}(f_1,f_2) \right\|_p \leq CL^{30} \nu^{10} \|f_1\|_{p_1} \|f_2\|_{p_2}.
\]

Before proving the lemma we show how it implies boundedness of \( \tilde{H}_\alpha \) and therefore proves Proposition 2. First observe that the lemma also holds without the finiteness condition on \( S \). We can also remove conditions (14), (15), and (16) on \( S \) at the cost of some additional powers of \( L \) and \( \nu \), so that the conclusion of the lemma without these hypotheses is

\[
(18) \quad \left\| \sum_{(k,n,l) \in S} \varepsilon(k,n,l) H_{k,n+\nu_1+n+\nu_2,n,l_1,l_2,l}(f_1,f_2) \right\|_p \leq CL^{100} \nu^{20} \|f_1\|_{p_1} \|f_2\|_{p_2}.
\]

Here we have used the quasi triangle inequality for \( L^p \) which is uniform for \( p > \frac{2}{3} \).
Observe that (18) and (11) imply

\[
\left\| \sum_{(k,n,l) \in S} C_{k,n+\nu_1,n+\nu_2,n,l_3} H_{k,n+\nu_1,n+\nu_2,n,l_3} (f_1, f_2) \right\|_p \\
\leq CL^{200} \nu^{-50} \|f_1\|_p \|f_2\|_p.
\]

Conditions (12) and (13) give a bound on the number of values the functions \( \lambda_1 \) and \( \lambda_2 \) can take at a fixed \( l_3 \) so that the coefficient \( C_{k,n+\nu_1,n+\nu_2,n,l_3} \) does not vanish. Moreover there are of the order \( \nu \) pairs \( \nu_1, \nu_2 \) such that \( 1 + \max\{|\nu_1|, |\nu_2|\} = \nu \). Hence,

\[
\left\| \sum_{(k,n,l) \in S, n_1, n_2, l_1, l_2 \in \mathbb{Z}} C_{k,n_1,n_2,n_1,l_1,l_2} H_{k,n_1,n_2,n_1,l_1,l_2} (f_1, f_2) \right\|_p \\
\leq CL^{300} \nu^{-20} \|f_1\|_p \|f_2\|_p.
\]

Summing over all \( \nu \) gives boundedness of \( \tilde{H}_\alpha \).

It remains to prove Lemma 4. Clearly we intend to do this by applying Proposition 3. Fix data \( S, \nu, \nu_1, \nu_2, \lambda_1, \lambda_2 \) as in Lemma 4. Define functions \( \phi_i : S \mapsto \mathcal{S}(\mathbb{R}) \) as follows:

\[
\phi_1(k,n,l) := L^{-10} \nu^{-2} \varphi_{k,n+\nu_1, \lambda_1}, \\
\phi_2(k,n,l) := L^{-10} \nu^{-2} \varphi_{k,n+\nu_2, \lambda_2}, \\
\phi_3(k,n,l) := L^{-10} \nu^{-2} \varphi_{k,n, \lambda_2}.
\]

If \( E \) is a subset of \( \mathbb{R} \) and \( x \neq 0 \) a real number we use the notation \( x \cdot E := \{ xy \in \mathbb{R} : y \in E \} \). This is not to be confused with the previously defined \( xI \) for positive \( x \) and intervals \( I \). Pick three maps \( \omega_1, \omega_2, \omega_3 : S \mapsto \mathcal{J} \) such that the following properties (20)–(25) are satisfied for all \( s = (k,n,l) \in S \):

\[
\frac{1+\alpha}{\alpha} \cdot \text{supp} (\hat{\phi}_1(s)) \subset \omega_1(s),
\]

\[
-(1+\alpha) \cdot \text{supp} (\hat{\phi}_2(s)) \subset \omega_2(s),
\]

\[
\text{supp} (\hat{\phi}_3(s)) \subset \omega_3(s),
\]

\[
2^{-\varepsilon(k+1)}L \leq |\omega_i(s)| \leq 2^{-\varepsilon k}L \text{ for } i = 1, 2, 3,
\]

\[
\mathcal{J}_\omega := \omega_1(S) \cup \omega_2(S) \cup \omega_3(S) \text{ is a grid,}
\]

and, for all \( i, j \in \{1, 2, 3\} \),
If \( \omega_1(s) \subset J \) and \( \omega_1(s) \neq J \) for some \( J \in J_\omega \), then \( \omega_2(s) \subset J \).

The existence of such a triple of maps is proved as in [4].

Next pick a map \( I : S \to J \) which satisfies the following three properties (26)–(28) for all \( s = (k,n,l) \in S \):

(26) \[ |c(I(s)) - 2^\varepsilon k n| \leq 2^{\varepsilon k} \nu, \]

(27) \[ 2^{4^2 \varepsilon k} \nu \leq |I(s)| \leq 2^{\varepsilon} 2^{4^2 \varepsilon k} \nu, \]

(28) \( I(S) \) is a grid.

The existence of such a map is again proved as in [4].

Now Lemma 4 follows immediately from the fact that the data \( S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \) and \( \omega_3 \) satisfy the hypotheses of Proposition 3. The verification of these hypotheses is as in [4] except for hypothesis (iv).

We prove hypothesis (iv) for \( i = 1 \), the other cases being similar. Define for \( \xi \in \mathbb{R} \):

\[ \theta_{\xi,1}(x) := e^{-2\pi i \frac{\alpha}{\alpha + 1} \xi}. \]

Pick \( s = (k,n,l) \in S \). Obviously,

\[ \nu^{-2} \varphi_{k,n+\nu_1,0}(x) \leq C|I(s)|^{-\frac{1}{2}} \left( 1 + \frac{|x - c(I(s))|}{|I(s)|} \right)^{-2} \]

and

\[ \nu^{-2} (\varphi_{k,n+\nu_1,0})'(x) \leq C|I(s)|^{-\frac{3}{2}} \left( 1 + \frac{|x - c(I(s))|}{|I(s)|} \right)^{-2}. \]

Now let \( \xi \in \omega_3(s) \). By choice of \( \theta_{\xi,1} \) we see that the function

\[ \varphi_{k,n+\nu_1,\lambda_1(s)}^{-1} \theta_{\xi,1} \]

arises from \( \varphi_{k,n+\nu_1,0} \) by modulating with a frequency which is contained in \( L^{10}[|I(s)|^{-1}, |I(s)|^{-1}] \). Therefore,

\[ (\phi_1(s)\theta_{\xi,1}^{-1})'(x) \leq C|I(s)|^{-\frac{3}{2}} \left( 1 + \frac{|x - c(I(s))|}{|I(s)|} \right)^{-2}. \]

Now let \( J \in I(S) \) with \( |J| \leq |I(s)| \). Then we have

\[ \inf_{\lambda} \|\phi_1(s)\theta_{\xi,1}^{-1} - \lambda\|_{L^\infty(J)} \leq |J| \left\| (\phi_1(s)\theta_{\xi,1}^{-1})' \right\|_{L^\infty(J)} \]

\[ \leq C|J||I(s)|^{-\frac{3}{2}} \left( 1 + \frac{|c(J) - c(I(s))|}{|I(s)|} \right)^{-2}. \]

This proves hypothesis (iv), and therefore finishes the reduction of Proposition 2 to Proposition 3.
4. Reduction to a symmetric statement

The following proposition is a variant of Proposition 3 which is symmetric in the indices 1, 2, and 3.

**Proposition 4.** Let $1 < p_1, p_2, p_3 < 2$ be exponents with

$$1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 2$$

and let $C_m > 0$ for $m \geq 0$. Then there are constants $C, \lambda_0 > 0$ such that the following holds: Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \omega_3$ be as in Proposition 3, let $f_i, \ i = 1, 2, 3$ be Schwartz functions with $\|f_i\|_{p_i} = 1$, and define

$$E := \{ x \in \mathbb{R} : \max_i (M_{p_i}(M f_i)(x)) \geq \lambda_0 \}.$$  

Then we have

$$\sum_{s \in S : I(s) \notin E} |I(s)|^{-\frac{1}{2}} |\langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \langle f_3, \phi_3(s) \rangle| \leq C.$$  

We now prove that Proposition 3 follows from Proposition 4. Let $1 < p_1, p_2 < 2$ and assume

$$p := \frac{p_1 p_2}{p_1 + p_2} > \frac{2}{3}.$$  

Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \omega_3, \varepsilon$ be as in the proposition and define for each $S' \subset S$

$$H_{S'}(f_1, f_2) = \sum_{s \in S'} \varepsilon(s) |I|^{-\frac{1}{2}} \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \phi_3(s).$$  

By Marcinkiewicz interpolation ([2]), it suffices to prove a corresponding weak-type estimate instead of (6). By linearity and scaling invariance it suffices to prove that there is a constant $C$ such that for $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$ we have

$$|\{ x \in \mathbb{R} : |H_S(f_1, f_2)(x)| \geq 2 \}| \leq C.$$  

Pick an exponent $p_3$ such that the triple $p_1, p_2, p_3$ satisfies the conditions of Proposition 4, and let $\lambda_0$ be as in this proposition. Let $f_1$ and $f_2$ be Schwartz functions with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$.

Define

$$E_0 := \{ x : \max \{ M_{p_1}(M f_1)(x), M_{p_2}(M f_2)(x) \} \geq \lambda_0 \}.$$  

and

$$E_{in} := \left\{ x \in \mathbb{R} : \left| H_{s \in S : I(s) \in E_0} f_1, f_2 \right| \geq 1 \right\},$$

$$E_{out} := \left\{ x \in \mathbb{R} : \left| H_{s \in S : I(s) \notin E_0} f_1, f_2 \right| \geq 1 \right\}.$$
It suffices to bound the measures of $E_{\text{in}}$ and $E_{\text{out}}$ by constants. We first estimate that of $E_{\text{out}}$ using Proposition 4. Let $\delta > 0$ be a small number and let $\theta : [0, \infty) \to [0, 1]$ be a smooth function which vanishes on the interval $[0, 1-\delta]$ and is constant equal to 1 on $[1, \infty)$. Extend this function to the complex plane by defining in polar coordinates $\theta(re^{i\phi}) := \theta(r)e^{-i\phi}$. Assume that $\delta$ is chosen sufficiently small to give

$$|E_{\text{out}}|^{1 \over p_3} \leq \left\| \theta \left( H_{\{s \in S : I(s) \not\subset E_{\text{in}}\}}(f_1, f_2) \right) \right\|_{p_3} \leq 2 |E_{\text{out}}|^{1 \over p_3}.$$

Define

$$f_3 := \frac{\theta \left( H_{\{s \in S : I(s) \not\subset E_{\text{in}}\}}(f_1, f_2) \right)}{\left\| \theta \left( H_{\{s \in S : I(s) \not\subset E_{\text{in}}\}}(f_1, f_2) \right) \right\|_{p_3}}.$$

We can assume that $|E_{\text{out}}| > \lambda_0^{-p_3}$, because otherwise nothing is to prove. This assumption implies $\|M_{p_3}(Mf_3)\|_\infty < \lambda_0$. By applying Proposition 4, we obtain:

$$|E_{\text{out}}|^{1 - \frac{1}{p_3}} \leq 2 \left| \int H_{\{s \in S : I(s) \not\subset E_{\text{in}}\}}(f_1, f_2)(x)f_3(x) \, dx \right| \leq C.$$

Therefore $|E_{\text{out}}|$ is bounded by a constant.

It remains to estimate the measure of the set $E_{\text{in}}$, which is an elementary calculation. We need the following lemma:

**Lemma 5.** Let $J$ be an interval and define

$$S_J := \{s \in S : I(s) = J\}.$$

Then for all $m > 0$ there is a $C_m$ such that for all $A > 1$ and $f_1, f_2 \in S(\mathbb{R})$ we have:

$$\|H_{S_J}(f_1, f_2)\|_{L^1(\mathbb{R}^c)} \leq C_m |J| A^{-m} \left( \inf_{x \in J} M_{p_1} f_1(x) \right) \left( \inf_{x \in J} M_{p_2} f_2(x) \right).$$

We prove the lemma for $|J| = 1$, which suffices by homogeneity. For $m \geq 0$ define the weight

$$w_m(x) := (1 + \text{dist}(x, J))^m.$$

Then for $1 \leq r < 2$ we obtain the estimates

$$\sum_{s \in S_J} \alpha_s \phi_1(s) \|_{L^{r'}(\omega_m)} \leq C_m \left\| (\alpha_s)_{s \in S_J} \right\|_{L^r(S_J)}$$

and

$$\sum_{s \in S_J} \left\| (f, \phi_1(s))_{s \in S_J} \right\|_{L^{r'}(\omega_m)} \leq C_m \|f\|_{L^r(\omega_m^{-1})},$$

which follow easily by interpolation ([6]) from the trivial weighted estimate at $r = 1$ and the nonweighted estimate at $r = 2$. 

Now define \( r \) by
\[
\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2},
\]
in particular we have \( 1 < r < 2 \). By writing \( H_{S_J}(f_1, f_2) = (H_{S_J}(f_1, f_2)w_{i0})w_{j0} \) and applying Hölder we have for large \( m \):
\[
\|H_{S_J}(f_1, f_2)\|_{L^1((A_J)c)} \leq C MA^{-M} \|H_{S_J}(f_1, f_2)\|_{L^r(w_m)}.
\]
Here \( M \) depends on \( m \) and \( r \) and can be made arbitrarily large by picking \( m \) accordingly. By estimates (29) and (30) we can estimate the previously displayed expression further by
\[
\leq C MA^{-M} \left\| \left\{ (f_1, \phi_1(s)) \langle f_2, \phi_2(s) \rangle \right\}_{s \in S_J} \right\|_{L^p(S_J)}
\leq C MA^{-M} \left\| \left\{ (f_1, \phi_1(s)) \right\}_{s \in S_J} \right\|_{L^{p_1}(S)} \left\| \left\{ (f_2, \phi_2(s)) \right\}_{s \in S_J} \right\|_{L^{p_2}(S_J)}
\leq C MA^{-M} \|f_1\|_{L^{p_1}(w_{i0})} \|f_2\|_{L^{p_2}(w_{i0})}
\leq C MA^{-M} \left( \inf_{x \in J} M_{p_1} f_1(x) \right) \left( \inf_{x \in J} M_{p_2} f_2(x) \right).
\]
This finishes the proof of Lemma 5.

We return to the proof of the set \( E_{in} \). Define
\[
E' := E_0 \cup \bigcup_{J \in I(S) : J \subseteq E} 4J.
\]
Since \( |E'| \leq 5|E_0| \leq C \), it suffices to prove
\[
\|H_{\{s \in S : I(s) \subseteq E_0\}}(f_1, f_2)\|_{L^1(E')} \leq C.
\]
Fix \( k > 1 \) and define
\[
\mathcal{I}_k := \{ J \in I(S) : J \subseteq E_0, 2^kJ \subseteq E', 2^{k+1}J \not\subseteq E' \}.
\]
Let \( J \in \mathcal{I}_k \). Then for \( i = 1, 2 \) we have:
\[
\inf_{x \in J} M_{p_i} f_i(x) \leq 2^{k+1} \inf_{x \in 2^{k+1}J} M_{p_i} f_i(x) \leq 2^{k+1},
\]
since outside the set \( E' \) the maximal function is bounded by 1. Hence, by the previous lemma,
\[
\|H_{S_J}(f_1, f_2)\|_{L^1((E')c)} \leq C_m |J| 2^{-km}.
\]
Since \( I(S) \) is a grid, it is easy to see that the intervals in \( \mathcal{I}_k \) are pairwise disjoint; hence we have
\[
\|H_{\{s \in S : I(s) \in \mathcal{I}_k\}}(f_1, f_2)\|_{L^1((E')c)} \leq C_m |E_0| 2^{-km}.
\]
By summing over all \( k > 1 \) we prove (31). This finishes the estimate of the set \( |E_{in}| \) and therefore the reduction of Proposition 3 to Proposition 4.
5. The combinatorics on the set $S$

We prove Proposition 4. Let $1 < p_1, p_2, p_3 < 2$ be exponents with

$$1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 2.$$ 

Let $\eta > 0$ be the largest number such that $\frac{1}{\eta}$ is an integer and

$$\eta \leq 2^{-100} \left( 2 - \sum_i \frac{1}{p_i} \right) \min_j \left( 1 - \frac{1}{p_j} \right).$$

Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2,$ and $\omega_3$ be as in Propositions 3 and 4. Let $f_i, i = 1, 2, 3$ be Schwartz functions with $\|f_i\|_{p_i} = 1$. Without loss of generality we can assume that for all $s \in S$,

$$I(s) \nsubseteq \left\{ x \in \mathbb{R} : \max_i (M_{p_i}(Mf_i)(x)) \geq \lambda_0 \right\},$$

where $\lambda_0$ is a constant which we will specify later.

Define a partial order $\ll$ on the set of rectangles by

$$J_1 \times J_2 \ll J'_1 \times J'_2, \quad \text{if} \quad J_1 \subset J'_1 \quad \text{and} \quad J'_2 \subset J_2.$$ 

A subset $T \subset S$ is called a tree of type $i$, if the set $\rho_i(T)$ has exactly one maximal element with respect to $\ll$. This maximal element is called the base of the tree $T$ and is denoted by $s_T$. Define $J_T := I(s_T)$.

Define $S_{-1} := S$. Let $k \geq 0$ be an integer and assume by recursion that we have already defined $S_{k-1}$. Define

$$S_k := S_{k-1} \setminus \bigcup_{i,j=1}^3 \left( \bigcup_{l=0}^{\infty} T_{k,i,j,l} \right),$$

where the sets $T_{k,i,j,l}$ are defined as follows. Let $k \geq 0$ and $i, j \in \{1, 2, 3\}$ be fixed. Let $l \geq 0$ be an integer and assume by recursion that we have already defined $T_{k,i,j,\lambda}$ for all integers $\lambda$ with $0 \leq \lambda < l$. If one of the sets $T_{k,i,j,\lambda}$ with $\lambda < l$ is empty, then define $T_{k,i,j,l} := \emptyset$. Otherwise let $\mathcal{F}$ denote the set of all trees $T$ of type $i$ which satisfy the following conditions (34)–(36):

$$T \subset S_{k-1} \setminus \bigcup_{\lambda < l} T_{k,i,j,\lambda},$$

(35) if $i = j$, then $|\langle f_j, \phi_j(s) \rangle| \geq 2^{-\eta k^2} \left| I(s) \right|^{1/2}$ for all $s \in T$,

(36) if $i \neq j$, then $\left( \sum_{s \in T} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{\|I(s)\|^2} \right)^{1/2} \geq 2^{4k} 2^{-\eta k^2} |J_T|.$

If $\mathcal{F}$ is empty, then we define $T_{k,i,j,l} := \emptyset$. Otherwise define $\mathcal{F}_{\text{max}}$ to be the set of all $T_{\text{max}} \in \mathcal{F}$ which satisfy:

$$\text{if } T \in \mathcal{F}, \quad T_{\text{max}} \subset T, \quad \text{then } T = T_{\text{max}}.$$
Choose $T_{k, i, j, l} \in \mathcal{F}_{\text{max}}$ such that for all $T \in \mathcal{F}_{\text{max}}$,

\begin{align}
\text{(38)} & \quad \text{if } i < j, \quad \text{then } \omega_i(s_{T_{k, i, j, l}}) \not< \omega_i(s_T), \\
\text{(39)} & \quad \text{if } i > j, \quad \text{then } \omega_i(s_T) \not< \omega(s_{T_{k, i, j, l}}).
\end{align}

Here $[a, b] \not< [a', b']$ means $b > a'$. Observe that $T_{k, i, j, l}$ actually satisfies (38) and (39) for all $T \in \mathcal{F}$. This finishes the definition of the sets $T_{k, i, j, l}$ and $S_k$.

Since $S$ is finite, $T_{k, i, j, l} = \emptyset$ for sufficiently large $l$. In particular, each $s \in S_k$ satisfies

\begin{equation}
\text{(40)} \quad |\langle f_i, \phi_i(s) \rangle| \leq 2^{-\eta k} 2^{-\frac{k}{p_j'}} |I(s)|^{\frac{1}{2}}
\end{equation}

for all $i$, since the set $\{s\}$ is a tree of type $i$ which by construction of $S_k$ does not satisfy (35) for $j = i$. Similarly for $j \neq i$ each tree $T \subset S_k$ of type $i$ satisfies

\begin{equation}
\text{(41)} \quad \left\| \left( \sum_{s \in T} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \leq 2^{4} 2^{-\frac{k}{p_j'}} |J_T|.
\end{equation}

Moreover, (40) implies that the intersection of all $S_k$ contains only elements $s$ with $\prod_j \langle f_j, \phi_j(s) \rangle = 0$.

Let $k \leq \eta^{-2}$ and assume $T_{k, i, j, l}$ is a tree. Observe that (35) and (36) together with Lemma 6 in Section 7 provide a lower bound on the maximal function $M_{p_j}(Mf_j)(x)$ for $x \in J_{T_{k, i, j, l}}$. This lower bound depends only on $\eta$, $p_j$, and the constants $C_m$ of the phase plane representation. Therefore if we choose the constant $\lambda_0$ in (32) small enough depending on $\eta$, $p_j$, and $C_m$, it then is clear that $T_{k, i, j, l} = \emptyset$ for $k \leq \eta^{-2}$.

Now we have

\[
\sum_{s \in S} |I(s)|^{-\frac{1}{2}} \prod_j \langle f_j, \phi_j(s) \rangle \leq \sum_{k>\eta^{-2}} \sum_{i, j} \sum_{l=0}^{\infty} \left( \sup_{s \in T_{k, i, j, l}} |I(s)|^{-\frac{1}{2}} |\langle f_j, \phi_j(s) \rangle| \right)
\times \prod_{\kappa \neq i} \left( \sum_{s \in T_{k, \kappa, l}} |\langle f_k, \phi_k(s) \rangle|^2 \right)^{\frac{1}{2}}.
\]

Using (40), (41) and Lemma 7 of Section 7 we can bound this by

\[
\leq C \sum_{k>\eta^{-2}} 2^{-\sum_j \frac{k}{p_j'}} \sum_{i, j} \sum_{l=0}^{\infty} |J_{T_{k, i, j, l}}|.
\]

Now we apply the estimate

\begin{equation}
\text{(42)} \quad \sum_{l=0}^{\infty} |J_{T_{k, i, j, l}}| \leq C 2^{10m p_j' k} 2^k,
\end{equation}

...
for each \( k > \eta^{-2}, \tau_j \), which is proved in Sections 6 and 8. This bounds the previously displayed expression by

\[
\leq C \sum_{k > \eta^{-2}} 2^{-\eta p_j/k} 2^{10\eta p_j/k} 2^{k}.
\]

This is less than a constant since

\[
\sum_j \frac{1}{p_j'} \geq 1 + 10\eta \max_j p_j'
\]

by the choice of \( \eta \). This finishes the proof of Proposition 4 up to the proof of estimate (42) and Lemmata 6 and 7.

### 6. Counting the trees for \( \tau = j \)

We prove estimate (42) in the case \( \tau = j \). Thus fix \( k > \eta^{-2}, \tau, \tau \) with \( \tau = j \). Let \( \mathcal{F} \) denote the set of all trees \( T_{k,\tau,\tau,\tau} \). Observe that for \( T, T' \in \mathcal{F}, T \neq T' \) we have, by (37), that \( T \cup T' \) is not a tree; therefore

\[
\rho_{\tau}(s_T) \cap \rho_{\tau}(s_{T'}) = \emptyset.
\]

Define \( b := 2^{-\eta k} 2^{-\frac{1}{p_j'}} \). Then by (35) for all \( T \in \mathcal{F} \)

\[
|\langle f_\tau, \phi_{\tau}(s_T) \rangle| \geq b |J_T|^\frac{1}{2}
\]

Finally recall that for all \( s \in \mathcal{S} \):

\[
I(s) \not\subset \{ x : M_{\rho_{\tau}}(M f_\tau)(x) \geq \lambda_0 \}.
\]

Our proof goes in the following four steps:

**Step 1.** Define the counting function

\[
N_{\mathcal{F}}(x) := \sum_{T \in \mathcal{F}} 1_{J_T}(x).
\]

We have to estimate the \( L^1 \)-norm of the counting function. Since the counting function is integer-valued, it suffices to show a weak-type \( 1 + \varepsilon \) estimate for small \( \varepsilon \). More precisely it suffices to show for all integers \( \lambda \geq 1 \) and sufficiently small \( \delta, \varepsilon > 0 \), \( \delta = \delta(\eta, p_i), \varepsilon = \varepsilon(\eta, p_i) \):

\[
|\{ x \in \mathbb{R} : N_{\mathcal{F}}(x) \geq \lambda \}| \leq b^{-p_j'} \delta \lambda^{-1-\varepsilon}.
\]

Fix such a \( \lambda \). As in [4] there is a subset \( \mathcal{F}' \subset \mathcal{F} \) such that, if we define \( N_{\mathcal{F}'} \), analogously to \( N_{\mathcal{F}} \),

\[
\{ x \in \mathbb{R} : N_{\mathcal{F}'}(x) \geq \lambda \} = \{ x \in \mathbb{R} : N_{\mathcal{F}}(x) \geq \lambda \}
\]

and \( \| N_{\mathcal{F}'} \|_{\infty} \leq \lambda \). This is due to the grid structure of \( I(S) \).
Step 2. Let \( A > 1 \) be a number whose value will be specified later. We can write
\[
\mathcal{F}' = \left( \bigcup_{m=1}^{A^{10}} \mathcal{F}_m \right) \cup \mathcal{F}''
\]
such that if \( T, T' \in \mathcal{F}_m \) for some \( m \) and \( T \neq T' \), then
\[
(A_T \times \omega(s_T)) \cap (A_{T'} \times \omega(s_{T'})) = \emptyset,
\]
and
\[
\sum_{T \in \mathcal{F}''} |J_T| \leq C e^{-A} \sum_{T \in \mathcal{F}_1} |J_T|.
\]
For a proof of this fact see the proof of the separation lemma in [4].

Step 3. Let \( 1 \leq m \leq A^{10} \). The following lines hold for all sufficiently small \( \delta, \varepsilon > 0 \). The arguments may require \( \delta, \varepsilon \) to change from line to line. For a tempered distribution \( f, x \in \mathbb{R} \), and \( T \in \mathcal{F}_m \) define
\[
Bf(x)(T) := \frac{\langle f, \phi_s(t_T) \rangle}{|J_T|^\frac{1}{2}} 1_{J_T}(x).
\]
Let \( L^2(\mathbb{R}, l^2(\mathcal{F})) \) be the Banach space of square-integrable functions on \( \mathbb{R} \) with values in \( l^2(\mathcal{F}) \), and analogously for other exponents. Then we have the following estimate by Lemma 4.3 in [4]
\[
\|Bf\|_{L^2(\mathbb{R}, l^2(\mathcal{F}_m))} = \left( \sum_{T \in \mathcal{F}_m} \left| \langle f_s, \phi_s(t_T) \rangle \right|^2 \right)^{\frac{1}{2}} \leq C(1 + A^{-\frac{1}{2}} \lambda) \|f\|_2.
\]
We also trivially have
\[
\|Bf\|_{L^{1+\delta}(\mathbb{R}, l^\infty(\mathcal{F}_m))} = \left( \int \left( \sup_{T \in \mathcal{F}_m, x \in J_T} \left| \frac{\langle f_s, \phi_s(s_T) \rangle}{|J_T|^\frac{1}{2}} \right|^{1+\delta} \right) dx \right)^{\frac{1}{1+\delta}} \leq C\|Mf\|_{1+\delta} \leq C\|f\|_{1+\delta}.
\]
By interpolation we have for \( 1 < p < 2 \):
\[
\|Bf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_m))} \leq C(1 + A^{-\frac{1}{2}} \lambda) \|f\|_p.
\]
Let \( J \in I(S) \), and let \( \mathcal{F}_{m,J} \) be the set of \( T \in \mathcal{F}_m \) such that \( J_T \subset J \). By a localization argument, as in [4], we see that
\[
\|Bf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_{m,J}))} \leq C \lambda^\varepsilon (1 + A^{-\frac{1}{2}} \lambda) |J|^{\frac{1}{p}} \inf_{x \in J} M_p(Mf)(x).
\]
In the following, $g^*$ denotes the sharp maximal function of $g$ with respect to the given grid, as in [4]. We define $N_{F_m,J}$ in analogy to (46) to be the counting function of the trees $T \in F_m$ for which $I_T \subset J$. We apply the previous estimate for $f_i$ and use (44) to obtain
\[
\left( N_{F_m,J}^{\frac{1}{p} + \delta} \right)^2(x) \leq \sup_{J : x \in J} \left( \frac{1}{|J|} \int_J N_{F_m,J}(x) |f_i|^{p} \, dx \right)^{\frac{1}{p}} \\
\leq b^{-\delta} \sup_{J : x \in J} \left( \frac{1}{|J|} \left| \sum_{T \in F_{m,J}} \frac{|\langle f_i, \phi_T(s) \rangle|}{|J_T|^{1/2}} \right|^{p} \right)^{\frac{1}{p}} \\
\leq b^{-\delta} C \left( \lambda^{\varepsilon} (1 + A^{-\frac{\varepsilon}{2}} \lambda) \right)^{p}.
\]
Using (45) we can sharpen this argument in the case $p = p_i$ to
\[
\left( N_{F_m,J}^{\frac{1}{p_i} + \delta} \right)^2(x) \leq C b^{-\delta} \left( \lambda^{\varepsilon} (1 + A^{-\frac{\varepsilon}{2}} \lambda) \right)^{p}.
\]
Taking the $\frac{p_i + \delta}{p}$-th norm on both sides and raising to the $\frac{p_i + \delta}{p}$-th power gives
\[
\| N_{F_m,J} \|_{\frac{p_i + \delta}{p}} \leq C b^{-\delta} \left( \lambda^{\varepsilon} (1 + A^{-\frac{\varepsilon}{2}} \lambda) \right)^{p_i + \delta}.
\]
Step 4. We split the counting function $N_{F'}$ according to (47) and use the weak-type estimate following from (49) on the first part and estimate (48) together with (49) and the fact that the counting function is integer-valued on the second part. This gives
\[
\{ x \in \mathbb{R} : N_{F'}(x) \geq A^{10} \lambda \} \leq C A^{10} \lambda^{-\frac{p_i + \delta}{p_i + \delta}} b^{-\delta} \left( \lambda^{\varepsilon} (1 + A^{-\frac{\varepsilon}{2}} \lambda) \right)^{p_i + \delta} + e^{-A} C b^{-\delta} \left( \lambda^{\varepsilon} (1 + A^{-\frac{\varepsilon}{2}} \lambda) \right)^{p_i + \delta}.
\]
Choosing $A$ of the order $\lambda^{\varepsilon}$ and $\varepsilon \ll \delta$ gives
\[
\{ x \in \mathbb{R} : N_{F'}(x) \geq \lambda \} \leq C \lambda^{-1-\varepsilon} b^{-\delta}.
\]
According to Step 1 this finishes the proof of estimate (42) in the case $i = j$.

7. Estimates on a single tree

This section collects some standard facts from Calderón-Zygmund theory, adapted to the setup of trees.

Lemma 6. Fix $k, i, j, l$ such that $T := T_{k,i,j,l}$ is a tree, assume $i \neq j$, and let $1 < p \leq 2$. We then have
\[
\left\| \left( \sum_{s \in T} \frac{|\langle f, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_p \leq C \|f\|_p.
\]
For each interval $J \in I(S)$ define $T_J := \{ s \in T : I(s) \subset J \}$. Then we obtain

\begin{equation}
\left\| \left( \sum_{s \in T_J} \frac{|f, \phi_J(s)|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_p \leq C \inf_{x \in J} M_p(Mf)(x).
\end{equation}

For each $s \in T$, let $h_s$ be a measurable function supported in $I(s)$ with $\|h_I(x)\|_\infty = |I(s)|^{-\frac{1}{2}}$, $\|h\|_2 = 1$, and $\langle h_s, h_{s'} \rangle = 0$ for $s \neq s'$. Then for all maps $\varepsilon : T \to \{-1, 1\}$, we have

\begin{equation}
\left\| \sum_{s \in T} \varepsilon(s) \langle f, \phi_J(s) \rangle h_s \right\|_p \leq C \|f\|_p.
\end{equation}

First we prove estimate (52). The estimate is true in the case $p = 2$, as is proved in [4]. By interpolation it suffices to prove the weak-type estimate

\begin{equation}
\left\{ x \in \mathbb{R} : \sum_{s \in T} \varepsilon(s) \langle f, \phi_J(s) \rangle h_s(x) \geq C\lambda \right\} \leq C \frac{\|f\|_1}{\lambda}.
\end{equation}

Let $f \in L^1(\mathbb{R})$. We write $f$ as the sum of a good function $g$ and a bad function $b$ as follows. Let $\{I_n\}_n$ be the set of maximal intervals of the grid $I(S)$ for which $\int_{I_n} |f(x)| \, dx \geq \lambda |I_n|.

Let $\xi \in \omega_i(s_T)$, and pick a function $\theta_{\xi, s}$ as in hypothesis (iv) of Proposition 3. For each of the intervals $I_n$, define

\[ b_n(x) := 1_{I_n}(x) (f(x) - \lambda_n \theta_{\xi, s}(x)), \]

where $\lambda_n$ is chosen such that $b_n$ is orthogonal to $\theta_{\xi, s}$. Obviously $\lambda_n$ is bounded by $C \|f\|_{L^1(I_n)}$. Define $b := \sum b_n$ and $g := f - b$. It suffices to prove estimate (53) for the good and bad function separately. The estimate for the good function follows immediately from estimate (52) for $p = 2$. For the bad function we proceed as follows. Since the set

\[ E := \bigcup_n 2I_n \]

is bounded in measure by $C\lambda^{-1}$, it suffices to prove the strong-type estimate

\begin{equation}
\left\| \sum_n \left( \sum_{s \in T} \varepsilon(s) \langle b_n, \phi_J(s) \rangle h_s \right) \right\|_{L^1(E)} \leq C \|f\|_1.
\end{equation}

We estimate each summand separately. Obviously, we have

\[ \left\| \sum_{s \in T} \varepsilon(s) \langle b_n, \phi_J(s) \rangle h_s \right\|_{L^1(E)} \leq \sum_{s \in T : I(s) \not\subset 2I_n} |I(s)|^{\frac{1}{2}} \langle b_n, \phi_J(s) \rangle. \]
For each integer $k$ let $T_k$ be the set of those $s \in T$, for which $|I(s)| \leq 2^k|I_n| < 2|I(s)|$ and $I(s) \nsubseteq 2I_n$. For $k < 2$ we use the estimate

$$\sum_{s \in T_k} |I(s)|^\frac{1}{2} |\langle b_n, \phi_j(s) \rangle| \leq C\|b_n\|_1 \sum_{s \in T_k} \left(1 + \frac{|c(I(s)) - c(I_n)|}{|I(s)|}\right)^{-2}$$

$$\leq C\|b_n\|_1 \int_{(2I_n)^c} \sum_{s \in T_k} \frac{1}{2^k|I_n|} \left(1 + \frac{x - c(I_n)}{2^k|I_n|}\right)^{-2} 1_{I(s)}(x) \, dx$$

$$\leq C\|b_n\|_1 2^k.$$

For the last inequality we have seen that the intervals $I(s)$ with $s \in T_k$ are pairwise disjoint.

For $k > 2$ we use the orthogonality of $b_n$ and $\theta_{\xi,\lambda}$ as well as hypothesis (iv) of Proposition 3 to obtain

$$\sum_{s \in T_k} |I(s)|^\frac{1}{2} |\langle b_n, \phi_j(s) \rangle| \leq C\|b_n\|_1 \sum_{s \in T_k} \inf_{\lambda} \|\phi_j(s) - \lambda\theta_{\xi,\lambda}\|_{L^\infty(I_n)}$$

$$\leq C\|b_n\|_1 \sum_{s \in T_k} \left(1 + \frac{|c(I(s)) - c(I_n)|}{|I(s)|}\right)^{-2} \frac{|I_n|}{|I(s)|}$$

$$\leq C\|b_n\|_1 2^{-k}.$$

The last inequality follows by a similar argument as in the case $k \leq 2$. Summing (55) and (56) over $k$ and $n$ gives (54) and finishes the proof of (52).

We prove estimate (50). Observe that (52) is not void, since functions $h_s$ clearly exist. Therefore we can average (52) over all choices of $\varepsilon$ to obtain:

$$2^{-|T|} \sum_{\varepsilon} \left\|\sum_{s \in T} \varepsilon(s) \langle f, \phi_j(s) \rangle h_s\right\|_p^p = \int_{\mathbb{R}} 2^{-n} \sum_{\varepsilon} \left(\sum_{s \in T} \varepsilon(s) \langle f, \phi_j(s) \rangle h_s(x)\right)^p \, dx$$

$$\leq C\|f\|_p^p.$$

Now Khinchine’s inequality gives

$$\int_{\mathbb{R}} \left(2^{-n} \sum_{\varepsilon} \left(\sum_{s \in T} \varepsilon(s) \langle f, \phi_j(s) \rangle h_s(x)\right)^2\right)^\frac{p}{2} \, dx \leq C\|f\|_p^p,$$

which immediately implies estimate (50).

To prove (51) fix a $J$ and write $f = f1_{2J} + f1_{(2J)^c}$. It suffices to prove the estimate separately for both summands. For the first summand we simply
apply (50). For the second summand we write
\[
\left( \sum_{s \in T_J} \left| \left\langle f_{2 Jo}^c, \phi_J(s) \right\rangle \right|^2 |I(s)|^{-1} \right)^{\frac{1}{2}} \leq C \sum_{x \in I(s)} \frac{Mf(x)|I(s)||J|^{-1}}{\sum_{s \in T_J: x \in I(s)}} \leq CMf(x)1_J(x).
\]

The last inequality follows by summing a geometric series. This proves the estimate for the second summand and finishes the proof of Lemma 6.

**Lemma 7.** Fix \( k \geq \eta^{-2}, i, j, l \) such that \( T := T_{k,i,j,l} \) is a tree and assume \( i \neq j \). Then we have
\[
\left( \sum_{s \in T} \left| \left\langle f_j, \phi_j(s) \right\rangle \right|^2 \right)^{\frac{1}{2}} \leq C \left\| \left( \sum_{s \in T} \frac{\left| \left\langle f_j, \phi_j(s) \right\rangle \right|^2}{|I(s)|} 1_{I(s)}(x) \right)^{\frac{1}{2}} \right\|_{L^1(\mathbb{R})} |J_T|^{-\frac{1}{2}}.
\]

**Proof.** For each \( J \in I(S) \),
\[
\frac{1}{|J|} \int_J \left( \sum_{s \in T: I(s) \subset J} \left| \left\langle f_j, \phi_j(s) \right\rangle \right|^2 \left| \frac{1}{I(s)} 1_{I(s)}(x) \right|^{\frac{1}{2}} \right) dx \leq C 2^{-\frac{1}{v_j}},
\]
since the set \( \{ s \in T : I(s) \subset J \} \) is a union of trees \( \{T_n\}_n \) which satisfy (41) for \( k - 1 \) and
\[
\sum_n |J_{T_n}| \leq |J_T|.
\]
Define for \( x \in \mathbb{R} \) and \( s \in T \):
\[
F(x)(s) := \sum_{s \in T} \frac{\left| \left\langle f_j, \phi_j(s) \right\rangle \right|^2}{|I(s)|} 1_{I(s)}(x).
\]
Since \( F \) is supported on \( J_T \), we have
\[
\|F\|_{L^2(\mathbb{R}, I^2(T))} \leq |J_T|^{\frac{1}{2}} \|F\|_{BMO(\mathbb{R}, I^2(T))}.
\]
Here BMO is understood with respect to the grid \( I(S) \) as in [4]. We prove Lemma 7 by estimating this BMO-norm with (58) and (36).

**8. Counting the trees for \( i \neq j \)**

We prove estimate (42) in the case \( i \neq j \). Thus fix \( k \geq \eta^{-2}, i, j \) with \( i \neq j \). Let \( \mathcal{F} \) denote the set of all trees \( T_{k,i,j,l} \).
As in [4] we define for \( T \in \mathcal{F} \):

\[
\begin{align*}
T^{\min} &= \{ s \in T : \rho_1(s) \text{ is minimal in } \rho(T) \}, \\
T^{\text{fat}} &= \{ s \in T : 2^{\frac{4}{k+2}} |I(s)| \geq |J_T| \}, \\
T^{\partial} &= \{ s \in T : I(s) \cap (1 - 2^{-k})J_T = \emptyset \}, \\
T^{\partial \max} &= \{ s \in T^{\partial} : \rho(s) \text{ is maximal in } \rho(T^{\partial}) \}, \\
T^{\text{nice}} &= T \setminus (T^{\min} \cup T^{\text{fat}} \cup T^{\partial}).
\end{align*}
\]

Define \( b := 2^{-\frac{4}{k+2}} \). By similar arguments as in [4] we have the estimate

\[
(59) \quad \text{if } i \neq j, \quad \left\| \left( \sum_{s \in T^{\text{nice}}} \frac{|\langle f, \phi_j(s) \rangle|^2}{|I(s)|} \right)^{\frac{1}{2}} \right\|_1 \geq b|J_T|.
\]

Define the counting function

\[
N_T(x) := \sum_{T \in \mathcal{F}} 1_{J_T}(x).
\]

As in Section 6 it suffices to show

\[
(60) \quad |\{ x \in \mathbb{R} : N_T(x) \geq \lambda \}| \leq b^{-\frac{4}{k+2}} \lambda^{-1-\epsilon}
\]

for all integers \( \lambda \geq 1 \) and small \( \epsilon, \delta > 0 \). In addition, we can assume that \( \|N_T\|_{\infty} \leq \lambda \).

Let \( y \in \mathbb{R}, T \in \mathcal{F}, x \in J_T \), and \( s \in T \). For \( f \in \mathcal{S}(\mathbb{R}) \) define

\[
Sf(y)(T)(x)(s) := \frac{\langle f, \phi_j(s) \rangle}{|I(s)|^{\frac{1}{2}}} 1_{I(s)}(x) 1_{J_T}(y).
\]

Consider \( J_T \) as a measure space with Lebesgue measure normalized to 1. Then the operator is bounded from \( L^2 \) to \( L^2(\mathbb{R}, L^2(\mathcal{F}, L^2(J_T, l^2(T)))) \), as we see below. We have used a sloppy notation for the second Banach space: The range space \( L^2(J_T, l^2(T)) \) depends on the variable \( T \in \mathcal{F} \). To make this space independent of \( T \), we take the direct sum of these Banach spaces as \( T \) varies over \( \mathcal{F} \), and we let \( Sf(y)(T) \) be nonzero only on the component corresponding to \( T \). This is how we interpret the above notation. To see the claimed estimate we calculate:

\[
\begin{align*}
\int \sum_{T \in \mathcal{F}} \frac{1}{|J_T|} \left\| \sum_{s \in T} \frac{|\langle f, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)}(x) 1_{J_T}(y) \right\| dx dy &= \sum_{s \in \bigcup_{T \in \mathcal{F}} T} |\langle f, \phi_j(s) \rangle|^2 \\
&\leq C(1 + \lambda A^{-\frac{1}{2}}) \|f\|_2^2,
\end{align*}
\]

the last inequality being taken from [4]. The operator is also bounded from \( L^{1+2\delta} \) into \( L^{1+2\delta}(\mathbb{R}, l^\infty(\mathcal{F}, L^{1+\delta}(J_T, l^2(T)))) \).
since by Lemma 6:

\[
\int \left( \sup_{T \in \mathcal{F}} \left( \frac{1}{|J_T|} \int \left( \sum_{s \in T} \left( \frac{|\langle f, \phi_j(s) \rangle|}{|I(s)|^{\frac{1}{2}}} \right)^2 \right)^{\frac{1+\delta}{2}} \right) \right) dx dy \\
\leq \int \left( \sup_{T \in \mathcal{F}: y \in J_T} \frac{1}{|J_T|} \left\| \sum_{s \in T} \left( \frac{|\langle f, \phi_j(s) \rangle|}{|I(s)|^{\frac{1}{2}}} \right)^2 \right\|_1^{\frac{1}{1+\delta}} \right) dy \\
\leq C \int (M_{1+\delta}(Mf)(y))^{1+2\delta} dy \\
\leq C \|f\|_1^{1+2\delta}.
\]

By complex interpolation and the fact that $L^q(J_T) \subset L^1(J_T)$ for $q \geq 1$ we obtain that $S$ maps $L^p$ into $L^p(\mathbb{R}, L^{p+\delta}(\mathcal{F}, L^1(J_T, l^2(T))))$ with norm less than $C(1 + \lambda A^{-\frac{1}{2}})$.

Let $J \in I(S)$ and define $\mathcal{F}_J$ to be the set of $T \in \mathcal{F}$ such that $J_T \subset J$. Then we can localize as before to get

\[
\|Sf\|_{L^p(\mathbb{R}, L^{p+\delta}(\mathcal{F}_J, L^1(J_T, l^2(T))))} \leq C \lambda^\varepsilon (1 + \lambda A^{-\frac{1}{2}}) |J|^{\frac{1}{p}} \inf_{x \in J} M^p(Mf)(x).
\]

Using the estimate (59) on nice trees gives, for $f = f_j$ and $p = p_j$,

\[
\left( N_{\mathcal{F}}^{\frac{p_j}{p_j+\delta}} \right)^2 (x) \leq \sup_{J, x \in J} \left( \frac{1}{|J|} \int J_{\mathcal{F}_J}(x)^{\frac{p_j}{p_j+\delta}} dx \right) \\
\leq b^{-p_j} \sup_{J, x \in J} \left( \frac{1}{|J|} \|Sf_j\|_{L^p(\mathbb{R}, L^{p+\delta}(\mathcal{F}_J, L^1(J_T, l^2(T))))}^{p_j} \right) \\
\leq b^{-p_j} C \lambda^\varepsilon (1 + A^{-\frac{1}{2}} \lambda)^{p_j} (M_{p_j}(Mf_j)(x))^{p_j}.
\]

Again we can sharpen this argument to obtain

\[
\left( N_{\mathcal{F}}^{\frac{p_j}{p_j+\delta}} \right)^2 (x) \leq C b^{-p_j} \lambda^\varepsilon (1 + A^{-\frac{1}{2}} \lambda)^{p_j} \max\{M_{p_j}(Mf_j)(x)^{p_j}, \lambda_0\}.
\]

Taking the $\frac{p_j+\delta}{p_j}$-norm on both sides proves estimate (60) and therefore also (42).

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