We present an overview of the ways in which D-brane charges are classified in terms of K-theory, emphasizing the natural physical interpretations of a homological classification within a topological setting.

1. Introduction

In this lecture we will discuss one of the most interesting recent developments in mathematical physics, the classification of D-brane charges in string theory using K-theoretic methods [1]–[6]. Our presentation will be in the context of explaining how certain mathematical structures find very natural physical explanations in string theory. In particular, we will focus on two very important features in this subject. The first one is the natural appearance of a homological, as opposed to cohomological, framework in which to describe D-brane charge. This point has been elucidated recently in a variety of different contexts [7]–[11]. While the standard K-theory approach does give very natural insights into the physical basis of D-brane constructions, we shall see that K-homology arises in a far more physically transparent manner. The second point which we will emphasize is that the most basic, physical characteristics of D-branes immediately lead to a definition of their charges in terms of topological, rather than analytical, K-homology. This description thereby provides the correct geometric arena for understanding the physics of D-branes in string theory. For the most part this will be a review of many well-known results which have arisen over the last few years from studies of the relationships between K-theory and D-branes. We will, however, inject some new proposals concerning certain topological K-homology groups.

The organization of this paper is as follows. In the next section we will review the standard physical and geometric arguments leading to the K-theoretic classification of D-brane charges. In section 3 we will then start showing how these constructs can be very naturally reinterpreted in terms of analytic K-homology. In section 4 we present the arguments which imply that D-brane charges are properly classified in terms of topological K-homology. In section 5 we consider the effects of curved backgrounds in string theory.

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and the arguments which imply that an appropriate twisted version of K-theory should classify D-branes in these instances. Here we also propose a new, physically motivated definition of twisted topological K-homology groups, which is also the most natural one from a mathematical perspective.

2. Brane Charges, Superconnections, and K-Theory

In this section we will review some of the standard relationships between K-theory and D-branes, emphasizing the geometrical structures involved. Here we will not give a very precise definition of strings and D-branes, but instead focus on the mathematical entities which describe them.

2.1. Brane-Antibrane Systems

A D-brane may be defined topologically as a relative map

\[(\Sigma, \partial \Sigma) \overset{\chi}{\longrightarrow} (X, M),\]

(2.1)

where \(\Sigma\) is a Riemann surface with boundary called a string worldsheet. We shall work throughout with only “Type II superstrings”, which amounts to assuming that \(\Sigma\) is oriented. The Euclidean spacetime manifold \(X\) is ten-dimensional, oriented and spin, and we assume that \(M \hookrightarrow X\) is a closed, oriented embedded submanifold. For the time-being we will assume that this embedding is topologically trivial, and hence that the D-brane worldvolume \(M\) carries a spin\(^c\) structure \([2, 5]\). These conditions will be relaxed in section 5. Among other things, associated with the D-brane is also a complex vector bundle \(\xi \rightarrow M\) with connection \(\nabla\), called the Chan-Paton bundle. The rank of this bundle is the number of coincident D-branes which wrap the common worldvolume \(M\).

A brane-antibrane system is defined as such a configuration whereby the Chan-Paton bundle carries a \(\mathbb{Z}_2\)-grading

\[\xi = \xi^+ \oplus \xi^-,\]

(2.2)

where the \(\pm\) labels indicate the branes and antibranes, all of which wrap the same worldvolume \(M\). The physical requirement that such a system be invariant under processes involving identical brane-antibrane creation and annihilation is the mathematical statement of stable isomorphism of Chan-Paton bundles. Then, physical quantities which are invariant under deformations of \(\xi\) depend only on the K-theory class \(x = [\xi^+] - [\xi^-] \in K^\ast(M)\). Since \(M\) is spin\(^c\), one can use the Thom isomorphism and the ensuing Atiyah-Bott-Shapiro construction \([2, 12]\) to naturally map \(x\) into a class \(K\ast(M) \to K\ast(X)\) in the K-theory of spacetime (or equivalently in the K-theory \(K\ast(N_X M)\) of the normal bundle to \(M\) in \(X\)). Throughout, when either \(M\) or \(X\) is only locally compact, we shall work in K-theory with compact support, which means physically that we measure D-brane charge with respect to that of the vacuum.

Because of the \(\mathbb{Z}_2\)-grading, the natural geometric object to consider on this system of D-branes is not a connection but rather a superconnection on \(\xi\) \([2, 12]\)

\[A = \begin{pmatrix} \nabla^+ & T \\ \bar{T} & \nabla^- \end{pmatrix},\]

(2.3)
where $\nabla^\pm : \xi^\pm \to \xi^\pm$ are connections on the branes and antibranes, while $T : \xi^+ \to \xi^-$ and $T : \xi^- \to \xi^+$ are called tachyon fields. The brane-antibrane system, with its superconnection (2.3), is depicted schematically in fig. 1.

![Diagram](image)

**Figure 1:** A brane-antibrane system wrapping a common worldvolume $M$. The tachyon field $T$ is associated with open string modes which stretch between the branes and antibranes.

### 2.2. D-Brane Charges

Determining a formula for D-brane charge is the problem of finding the chiral gauge and gravitational anomalies induced on the brane worldvolume $M$ from the restriction of spacetime spinors in $K^1/2_X$ [1, 10, 13]. It amounts to computing the index of the generalized Dirac operator

$$\left( \begin{array}{cc} i \nabla^+ & T \\ T & i \nabla^- \end{array} \right)$$

(2.4)

associated to the superconnection (2.3), by “splitting” the corresponding $\hat{A}$-class in two. Mathematically, this procedure consists of introducing a bilinear pairing on rational cohomology defined by

$$H^*(X, \mathbb{Q}) \otimes H^*(X, \mathbb{Q}) \to \mathbb{Q}, \quad \langle x, y \rangle = (x \cup y)[X],$$

(2.5)

and a homomorphism from K-theory to cohomology given by

$$G : K^*(X) \to H^*(X, \mathbb{Q}), \quad G(\xi) = \text{ch}(\xi) \, \hat{A}^{1/2}(X),$$

(2.6)

where

$$\text{ch}(\xi) = \text{STr} \, e^{-A^2/2\pi} \equiv \text{Tr} \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) e^{-A^2/2\pi}$$

(2.7)

is the Chern character of the graded bundle $\xi$. Then the index theorem gives

$$\langle G(\xi), G(\xi') \rangle = \text{index}(\xi \otimes \xi').$$

(2.8)

Worldvolume integrals of the cohomology class

$$Q = \text{ch}(\phi, \xi) \, \hat{A}^{1/2}(X)$$

(2.9)
then define the D-brane charge \([10, 13, 14]\), where \(\phi\) is the Gysin map on K-theory induced by the worldvolume embedding into \(X\). Note that when the tachyon field is turned off, \(T = \overline{T} = 0\), the Chern character \((2.7)\) is the difference \(\text{ch}(\xi) = \text{ch}(\xi^+) - \text{ch}(\xi^-)\). The corresponding charge \((2.9)\) is then the sum of the charges of the separated branes and antibranes.

In the following we shall also have the occasion to deal with systems of unstable D-branes. They are obtained from brane-antibrane systems by taking the quotient of the Chan-Paton bundle \((2.2)\) by the action of the left-moving, worldsheet fermion parity operator \((-1)^{F_L}\), which amounts to eliminating the \(\mathbb{Z}_2\)-grading and making the identifications

\[
\nabla^+ = \nabla^- \quad \text{and} \quad T = \overline{T}
\]

in the superconnection \((2.3)\). The charge formula is still given by \((2.9)\), but now the supertrace defining the Chern character is modified to

\[
\text{Str} \ e^{-A^2/2\pi} \equiv \text{Tr} \ e^{-A^2/2\pi} \left( \begin{array}{cc} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{array} \right).
\]

**2.3. Tachyon Condensation**

A physical feature of the K-theory classification of D-brane charge is that stable BPS D-branes may be constructed as solitonic decay products in the worldvolumes of brane-antibrane systems \([4, 8, 9]\). For example, let us describe how to build \(k\) D\(_p\)-branes (for odd \(p < 9\)) from \(N = 2^{(9-p)/2} \cdot k\) D\(_9\)-D\(_9\) pairs. The worldvolume \(M\) is \(p + 1\)-dimensional, and for the Chan-Paton bundle \((2.2)\) we take

\[
\xi^\pm = \Delta^\pm \otimes \xi_k,
\]

where \(\Delta^\pm\) are the irreducible, chiral spin bundles over the normal bundle \(N_X M\) of rank \(2^{(9-p)/2}\), and \(\xi_k\) is a trivial complex vector bundle of rank \(k\). It is important to note that, when \(M\) is only spin\(^c\), the usual spinor bundles \(\sigma^\pm\) do not in general exist globally and need to be twisted into globally defined vector bundles over \(M\) \([2, 3, 10]\). For this, let \(\ell \to M\) be a complex line bundle whose first Chern class \(c_1(\ell)\) reduces modulo 2 to the second Stiefel-Whitney class \(w_2(N_X M)\). Then the square root \(\ell^{1/2}\), with \(\ell^{1/2} \otimes \ell^{1/2} = \ell\), also cannot in general be constructed globally, because when there is two-torsion in \(H^2(M, \mathbb{Z})\) then there are different square roots of \(\ell\), and hence more than one spin\(^c\) structure for a given class \(c_1(\ell)\). However, the twisted spinor bundles \(\Delta^\pm = \ell^{1/2} \otimes \sigma^\pm\) do exist as vector bundles over \(M\). This is just the precise meaning of the existence of a spin\(^c\) structure, and it implies the vanishing of global worldsheet anomalies in the case of a topologically trivial B-field background \([15]\). We will return to these issues in section 5.

We choose a coordinate chart on \(X\) in which the connections on the branes and antibranes decompose as

\[
\nabla^\pm = \sum_{m=1}^{p+1} \mathbb{I}_{\Delta^\pm} \otimes \nabla_m \ dx^m + \sum_{i=p+2}^{10} \mathbb{I}_{\Delta^\pm} \otimes \frac{\partial}{\partial x^i} \ dx^i,
\]

where \(m = 1, \ldots, p + 1\) labels directions along the worldvolume \(M\) and \(i = p + 2, \ldots, 10\) labels coordinates in the normal bundle to \(M\) in \(X\). For the tachyon profile we use the
local representation

\[ T_t = \frac{1}{\sqrt{t}} \sum_{i=p+2}^{10} \left[ x^i \gamma_i \otimes \mathbb{I}_{k \times k} - \gamma_i \otimes \Phi^i(x^m) \right], \tag{2.14} \]

where \( t \) is a positive parameter labelling a family of tachyon fields. The first term in (2.14) situates the D-branes at the spacetime location \( x^i = 0 \), where \( \gamma_i \) generate the \( 9-p \)-dimensional Clifford algebra in the Majorana-Weyl representation. It smears the brane charge over a region of size \( \sqrt{t} \) and provides an off-shell interpolation in string theory. The fields \( \Phi \in \Gamma(M, N_\mathbb{C} \otimes \text{End}(\xi_k)) \) extend the maps \( M \hookrightarrow X \) to sections of the normal bundle twisted by the endomorphism bundle \( \text{End}(\xi_k) \). They represent the fluctuations of the relative positions of the \( k \) Dp-branes.

The Chern character (2.7) corresponding to (2.13) and (2.14) is independent of \( t \in \mathbb{R}_+ \) \[12\]. Taking the “on-shell” limit \( t \to 0 \) yields the symmetrized trace formula \[10, 16\]

\[ \text{ch}(\xi) = \text{Tr} \text{Sym} \exp \left( \frac{1}{2\pi} \sum_{m,n=1}^{p+1} [\nabla_n, \nabla_m] \ dx^n \wedge dx^m \right. \]

\[ + \left. \sum_{m=1}^{p+1} [\nabla_m dx^m, \iota_\Phi] + [i_\Phi, i_\Phi] \right), \tag{2.15} \]

where \( \iota_\Phi \) is internal multiplication with \( \Phi \) regarded as a vector field in the transverse space. When substituted into the formula for D-brane charge, the \( \iota_\Phi \) terms in (2.13) yield the Myers terms \[17\] which lead to the “dielectric effect”. The simplest example of this process is a D0-brane blowing up into a spherical D2-brane, or in other words into a D0–D2 bound state.

3. Tachyon Fields and Analytic K-Homology

In this section we shall analyse the topology of the tachyon fields introduced in the previous section in some more detail, and show how they alone lead immediately to the relationship between K-theory and D-branes. After presenting some arguments supporting a homological classification of D-brane charge, we will then show that these same ingredients very naturally give an alternative description in terms of analytic K-homology.

3.1. Tachyons are Fredholm Operators

Let \( \xi \to M \) be a complex Chan-Paton vector bundle over an unstable D-brane world-volume, with structure group \( U(N) \). The tachyon field \( T \) is adjoint-valued, so it is a map \( T : X \to u(N) \) into the Lie algebra of \( U(N) \). The Lie algebra is a contractible space, so \( T \) carries no topology. A map on \( X \to u(N) \) cannot represent an element of \( K^*(X) \), because it does not carry any topological information at all. The solution to this problem \[4, 18\] is to set \( N = \infty \) and use infinitely many unstable D-branes. We then interpret the structure group \( U(\infty) \) as the group \( U(\mathcal{H}) \) of unitary operators on the separable Chan-Paton Hilbert space \( \mathcal{H} = L^2(M, \xi) \) on which the open string zero modes act. Whether or not this makes
sense physically, it can always be done mathematically. From a physical standpoint we may assume that we start with an infinite set of unstable D-branes and reduce it by tachyon condensation to a finite set representing a configuration of finite total energy.

Requiring that the system have finite energy and that the induced D-brane charge be finite is tantamount to assuming that 0 is not an accumulation point of the spectrum of the self-adjoint operator $T$. In other words, $\tilde{T} = T/\sqrt{1 + T^2} \in \mathcal{B}(\mathcal{H})$ is a bounded linear operator on $\mathcal{H}$ such that $\tilde{T}^2 - 1 \in \mathcal{K}(\mathcal{H})$ where $\mathcal{K}(\mathcal{H})$ is the elementary $C^*$-algebra of compact operators on $\mathcal{H}$. These conditions simply mean that the tachyon field is now a map

$$T : X \rightarrow \mathcal{F}(\mathcal{H})$$

into the space $\mathcal{F}(\mathcal{H})$ of Fredholm operators on the Chan-Paton Hilbert space.

Carrying up this argument to brane-antibrane systems, the homotopy classes of maps (3.1) yield an epimorphism

$$\left[ X, \mathcal{F}(\mathcal{H}) \right] \rightarrow K^0(X) \rightarrow 0$$

(3.2)

which is given by taking the index bundle whose fiber over a point $x \in X$ is

$$\text{Index}(T)_x = \ker T(x) \oplus \text{coker} T(x).$$

(3.3)

Then $[\text{Index}(T)]$ is the K-theory class of the D-brane Chan-Paton space. Note that in this case the Fredholm operator $T$ can without loss of generality be taken to be a partial isometry \footnote{More precisely, there is an isomorphism $\left[ X, \mathcal{F}(\mathcal{H}) \right] \rightarrow K^0(X)$. The analogous result for unstable D-branes yields an isomorphism $\left[ X, \mathcal{F}_{\text{skew}}(\mathcal{H}) \right] \rightarrow K^{-1}(X)$, where $\mathcal{F}_{\text{skew}}(\mathcal{H})$ is the space of skew-adjoint Fredholm operators on $\mathcal{H}$ (or, more precisely, its component consisting of operators which are not essentially positive or negative). The analog of (3.3) for this latter map may be found in [4].}, i.e. $T \overline{T} T = T$, which is the basis of the representation of D-branes as noncommutative solitons in string field theory \footnote{[19]}.\footnote{[20]}

3.2. Tachyons are Classifying Maps

A somewhat more direct argument comes from minimizing the tachyon potential $V(T)$. The equations

$$dT = 0, \quad V'(T) = 0$$

(3.4)

are solved by

$$T = \sum_n \lambda_n P_n,$$

(3.5)

where $P_n$ are orthogonal projection operators on $\mathcal{H}$ and $V'(\lambda_n) = 0$. The basic shape of the tachyon potential is such that there are two stationary points \footnote{[19]}. A global minimum $V(0) = 0$ occurs at $\lambda = 0$ representing the closed string vacuum, and an extremum appears at $\lambda = t_*$ with $V(t_*)$ giving the tension of the D-brane in the perturbative open string vacuum. Thus the only non-trivial solution is $T = t_* P_n$, where $P_n$ is a projection operator of rank $n$. 
It follows that slowly-varying tachyonic field configurations on $X$ are given by maps

$$T : X \rightarrow BU(n)$$

(3.6)

into the space $BU(n)$ of rank $n$ projectors on the Chan-Paton Hilbert space. But $BU(n)$ is also the classifying space for complex vector bundles of rank $n$, so that

$$\text{Vect}_n(X) \cong \left[ X : BU(n) \right].$$

(3.7)

Thus homotopy classes of tachyons are directly related to K-theory.

3.3. D-Brane Charge Lives in K-Homology

When the charges of D-branes are defined by the behaviour of Ramond-Ramond fields, which are differential forms on $X$, K-theory is the appropriate contravariant cohomology theory to use [6]. A continuous map $\varphi : X \rightarrow X'$ induces a pull-back on K-theory, $\varphi^* : K^*(X') \rightarrow K^*(X)$. However, D-branes also represent supergravity solutions which are cycles in $X$ corresponding to the worldvolumes of D-branes. They should therefore be properly classified by a homology theory [7]. K-homology carries information of both cycles in $X$ and of gauge bundles over them, and it is naturally implied by Poincaré duality for spin' manifolds $M$. As we shall discuss, D-brane charge is most naturally regarded as an element of K-homology, rather than K-theory, because:

- D-branes a priori carry stable vector bundles, rather than virtual bundles.
- K-homology transforms charges covariantly under maps $\varphi : X \rightarrow X'$ of the space-time, i.e. if $\phi : M \rightarrow X$ is a D-brane, then its push-forward $M \mapsto \varphi_*(M) \subset X'$ gives rise to a map $\varphi_* : K_*(X) \rightarrow K_*(X')$ on K-homology. This makes contact with the covariant, operational definition of D-branes as submanifolds $M$ of the spacetime $X$.
- D-brane charge in the brane-antibrane constructions is determined in a very precise way from the K-theory group of the worldvolume normal bundle.

K-homology is the natural dual classification tool to K-theory. The index map provides a natural bilinear pairing

$$K^*(X) \times K_*(X) \xrightarrow{\text{index}} \mathbb{Z}.\quad (3.8)$$

Furthermore, the K-theory lift of the Poincaré duality relation $H_*(X, \mathbb{Z}) \cong H^{10-*}(X, \mathbb{Z})$ yields

$$K_*(X) \cong K^{10-*}(X). \quad (3.9)$$

The relationship (3.9) has a very natural physical interpretation. Its right-hand side represents the construction of D-branes from non-BPS D9-brane configurations, which classifies D-brane charge defined by Ramond-Ramond fields on $X$. The left-hand side of (3.9) represents the construction of D-branes from non-BPS D-instantons [21] and classifies the D-brane worldvolume embedded into the spacetime manifold $X$. 

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3.4. Tachyonic Configurations Live in Analytic K-Homology

Motivated by the arguments of the previous subsection, let us now turn the analysis of the beginning of this section around somewhat to show that tachyon fields naturally lead to the relationship between D-branes and K-homology. For simplicity and ease of notation, we will assume that the D-brane worldvolume \( M \) is of odd dimension and focus on the tachyon field \( T = \overline{T} \) of a system of unstable branes. The corresponding result for brane-antibrane systems and \( M \) of even dimension is easily obtained by grading all of the structures which follow. The basic observation is that the tachyon field \( T \) (or more precisely its bounded extension \( \tilde{T} \)), along with the worldvolume embedding extension \( \phi \mapsto \Phi \) on the Chan-Paton Hilbert space \( \mathcal{H} \), can be assembled into a “Fredholm module” \((\mathcal{H}, \Phi, T)\) which is specified by the following pieces of data \[19\]:

- \( \mathcal{H} \) is the separable Chan-Paton Hilbert space.
- \( \Phi : C(M) \to \mathcal{B}(\mathcal{H}) \) is a \(*\)-homomorphism. Here we have used Gel’fand duality to replace the manifold \( M \) by the \( C^*\)-algebra \( C(M) \) of continuous complex-valued functions on \( M \). This map is responsible for the representation of D-branes in terms of non-BPS D-instantons.
- \( T \in \mathcal{B}(\mathcal{H}) \) is a self-adjoint operator obeying the finite energy conditions \( T^2 - 1 \in K(\mathcal{H}) \) and \( [T, \Phi] \in K(\mathcal{H}) \).

Then the analytic K-homology \( K_\ast(M) = K^\ast(C(M)) \) is the group of equivalence classes of Fredholm modules with respect to the equivalence relation \( \sim \) generated by the following three relations:

- **Gauge Symmetry**: For any unitary operator \( U \in \mathcal{B}(\mathcal{H}, \mathcal{H}') \),

\[ (\mathcal{H}, \Phi, T) \sim (\mathcal{H}', U \Phi U^{-1}, U T U^{-1}) \]  

- **Annihilation of Virtual Non—BPS D—Instantons by Tachyon Condensation**: Define a degenerate Fredholm module \((\mathcal{H}', \Phi', T')_{\text{deg}}\) as one with a trivial tachyonic field configuration, i.e. \( T'^2 - 1 = [T', \Phi'] = 0 \). Then for any degenerate Fredholm module,

\[ (\mathcal{H}, \Phi, T) \oplus (\mathcal{H}', \Phi', T')_{\text{deg}} \sim (\mathcal{H}, \Phi, T) \]  

for all Fredholm modules \((\mathcal{H}, \Phi, T)\).

- **Continuous Deformations of Tachyons**: Any two Fredholm modules \((\mathcal{H}, \Phi, T)\) and \((\mathcal{H}', \Phi', T')\) are equivalent if there is an operator homotopy \( \mathcal{H} \sim \mathcal{H}', \Phi \sim \Phi' \), and there exists a norm continuous path connecting \( T \) and \( T' \). This condition may be regarded as the statement of D-brane charge conservation in physical processes.

We further require that each equivalence class be unaffected by continuous deformations of \( \mathcal{H} \) and \( \Phi \).
4. D-Branes and Topological K-Homology

In the previous section we arrived at the conclusion that D-brane charge, generated through tachyon condensation on non-BPS brane systems, is most naturally described within the framework of Fredholm modules and analytic K-homology. However, this approach, like other functional analytic approaches to K-homology [7]–[11], obscure the inherent geometry of D-branes. In this section we will accomplish two things. First, we will show that D-branes can be described directly in the language of Fredholm modules, without the need of resorting to their description in terms of tachyon fields on non-BPS configurations. Second, this formulation will immediately lead us into a more geometrical framework for analysing D-brane charges and their properties, which will turn out to be the natural physical arena.

4.1. D-Branes are Fredholm Modules

We will begin by demonstrating that every D-brane provides a Fredholm module over the algebra $C(X)$ of functions on spacetime. Let $\phi : M \rightarrow X$ be a closed, odd dimensional spin$^c$ submanifold of the spacetime $X$, representing the brane worldvolume (Again the result for even dimensional $M$ is easily obtained by introducing appropriate $\mathbb{Z}_2$-gradings). $M$ is equipped with a complex Chan-Paton vector bundle $\xi \rightarrow M$ with connection $\nabla$, and it inherits a metric from $X$. From the spinor representation of the spin$^c$ group, we can define a (twisted) spinor bundle $\Delta \rightarrow M$ and the associated separable Hilbert space $H = L^2(M, \Delta \otimes \xi)$ (4.1)

Let $D / \xi$ be the usual twisted Dirac operator on $\Gamma(M, \Delta \otimes \xi)$ with respect to the chosen connection on the bundle $\Delta \otimes \xi$. Then $D / \xi$ is an unbounded self-adjoint operator on the Hilbert space $L^2(M, \Delta \otimes \xi)$. Let us now define a representation $\Phi : C(X) \rightarrow B(H)$ as pointwise multiplication on $L^2(M, \Delta \otimes \xi)$ by the function

$$\Phi(f) = f \circ \phi, \; \forall f \in C(X).$$

This construction produces an (unbounded) Fredholm module $(H, \Phi, \nabla_\xi)$.

The Fredholm module just built defines an element of K-homology $K_* (X)$ which is independent of the choice of connection on $\Delta \otimes \xi$. Moreover, it can be shown that all classes in $K_* (X)$ can be obtained from this construction by using an appropriate D-brane $(M, \phi, \xi)$ [22]. By putting a suitable equivalence relation on the set of all such triples $(M, \phi, \xi)$, the classes in $K_* (X)$ are in a bijective correspondence with the corresponding equivalence classes $[(M, \phi, \xi)]$. It is precisely this correspondence that now leads us into the geometrical formulation of D-brane charge.

4.2. D-Brane Charge Lives in Topological K-Homology

The fact, discussed in the previous section, that the natural setting in which D-brane charges emerge should be topological motivates the development of a geometric approach
to K-homology. For this, consider the free abelian group generated by certain triples

$$k(X) = \text{Span}_\mathbb{Z} \left\{ (M, \phi, x) \left| \begin{array}{c} M \text{ closed, spin}^c \\ \phi : M \to X \\ x \in K^*(M) \end{array} \right. \right\} ,$$

(4.3)

where in each triple $(M, \phi, x)$ it is understood that the manifold $M$ is equipped with a fixed spin$^c$ structure on its tangent bundle $TM$. Each triple in $k(X)$ will be referred to as a homology K-cycle, because they each naturally give rise to a K-homology class in $X$. The reason for this is very simple. Since $M$ is spin$^c$, we can “cheat” by exploiting the fact that $M$ satisfies Poincaré duality in K-theory. Thus the class $x \in K^*(M)$ has a Poincaré dual in the K-homology $K_*(M)$ of $M$, which under the push-forward $\phi_* : K_*(M) \to K_*(X)$ can be mapped to a class in the K-homology of spacetime $X$. Thus there is a natural homomorphism $k(X) \to K_*(X)$.

This is the geometric description of K-homology, which we shall refer to as topological K-homology to distinguish it from its functional analytic counterpart. It is clear then that any collection of D-branes with worldvolume $\phi : M \to X$ and Chan-Paton bundle $\xi$ naturally defines an element of the topological K-homology $(M, \phi, [\xi]) \in K_*(X)$ of spacetime. The two main, natural physical properties of this description are [22]:

- **Gauge Symmetry Enhancement for Coincident D – Branes**: The disjoint union of two coincident D-brane triples gives

$$\left( M, \phi, [\xi] \right) \coprod \left( M, \phi, [\xi'] \right) \cong \left( M, \phi, [\xi \oplus \xi'] \right) .$$

(4.4)

- **D – Brane Charges**: The Chern character in K-homology may be defined as the rational homology class

$$\text{ch}(M, \phi, x) = \phi_* \left( \text{ch}(x) \cup \widehat{A}(M) \cap [M] \right) .$$

(4.5)

The map $k(X) \to K_*(X)$ is surjective, and its kernel is generated by two relations called “bordism” and “vector bundle modification”. These two equivalence relations required to quotient $k(X)$ have very natural interpretations in stable D-brane physics [4, 14]. The bordism relation is straightforward to describe. It is depicted in fig. 2 and is essentially the requirement of D-brane charge conservation. Namely, two triples are equivalent,

$$(M, \phi, x) \sim (M', \phi', x') ,$$

(4.6)

if there is a K-cycle $(W, \varphi, y)$ which interpolates between the triples in the sense that

$$\left. \partial W, \varphi \right|_{\partial W} \bigg|_y \cong \left( M, \phi, x \right) \coprod (-M', \phi', x') .$$

(4.7)

Here $W$ is a spin$^c$ manifold with boundary, $y = [\xi]$ with $\xi \to W$ a complex vector bundle, $\varphi : W \to X$ is continuous, and $-M'$ denotes the submanifold $M'$ with the reversed spin$^c$ structure. This relation represents “continuous” deformations of the D-brane worldvolume together with the Chan-Paton gauge bundle over it. The other relation is somewhat more involved and is described in the next subsection. It describes a well-known descent relation in D-brane physics which is due to the non-abelian nature of gauge bundles over multiple D-brane configurations.
Figure 2: \textit{D-brane charge conservation in topological K-homology is represented through bordism.}

4.3. \textbf{Vector Bundle Modification is the Dielectric Effect}

Vector bundle modification is the relation that identifies the triples

\[(M, \phi, x) \sim \left(\hat{M}, \phi \circ \rho, [\hat{\xi}] \otimes \rho^*(x)\right),\]  \hspace{1cm} (4.8)

where \(\hat{M} \xrightarrow{\rho} M\) is a sphere bundle over \(M\) whose fiber \(S_p = \rho^{-1}(p), p \in M\) is a sphere of even dimension \(2n\), and \(\hat{\xi} \rightarrow \hat{M}\) is a vector bundle over \(\hat{M}\) such that for all \(p \in M\), \([\hat{\xi}]|_{S_p}\) is the generator of the reduced K-theory group \(\tilde{K}^*(S_p) = \mathbb{Z}\). As we will now demonstrate, the relation (4.8) identifies a spherical D-brane, carrying a non-trivial gauge bundle, with a lower dimensional D-brane. It therefore represents the K-homology manifestation of the dielectric effect [9, 11, 17] that we mentioned in section 2.

Let \(\xi \rightarrow M\) be a spin\(^c\) vector bundle of rank \(2n\), and define

\[\hat{M} = \mathbb{B}(\xi)_+ \bigcup_{S(\xi)} \mathbb{B}(\xi)_-,\]  \hspace{1cm} (4.9)

where \(\mathbb{B}(\xi)_\pm\) are two copies of the unit ball bundle of \(\xi\) whose boundary \(\partial \mathbb{B}(\xi) = S(\xi)\) is the unit sphere bundle of \(\xi\). The copies are glued together using the identity map on \(S(\xi)\), so that (4.9) is a sphere bundle over the original D-brane worldvolume \(M\), with fiber of dimension \(2n\), which is also a spin\(^c\) manifold. The bundles \(\mathbb{B}(\xi)_\pm\) are interpreted as the worldvolumes of D(\(2n+m\))-branes and D(\(2n+m\))-branes, where \(m = \dim(M) - 1\). When glued together, \(\hat{M}\) becomes the worldvolume of a spherical D(\(2n+m\))-brane wrapped on a \(2n\)-dimensional sphere.

Since \(\xi\) has even rank, we can introduce the pull-backs \(\Delta^\pm(\xi)\) of the associated chiral spinor bundles to \(\xi\) using the bundle projection \(\xi \rightarrow M\). We interpret \(\Delta^\pm(\xi)\) as the Chan-Paton vector bundles on the D(\(2n+m\))-branes and D(\(2n+m\))-branes. The graded Chan-Paton bundle on this brane-antibrane system is therefore given by

\[\hat{\xi} = \Delta^+(\xi)\big|_{\mathbb{B}(\xi)_+} \bigcup_{S(\xi)} T|_{S(\xi)} \Delta^-(\xi)\big|_{\mathbb{B}(\xi)_-},\]  \hspace{1cm} (4.10)

where the gluing is done using the transition function \(T\) on \(S(\xi)\) defined by

\[T(p, v) = \sum_{l=1}^{2n} v^l \gamma_l\]  \hspace{1cm} (4.11)
with \( v \) a unit vector in the \( 2n \)-dimensional vector space which is the fiber of \( S(\xi) \) at \( p \in M \), and \( \gamma_l \) are positive chirality \( SO(2n) \) gamma-matrices. We interpret (4.11) as the tachyon field created by open strings stretched between the \( D(2n+m) \)-branes and the \( \overline{D(2n+m)} \)-branes. After tachyon condensation, the tachyonic configuration (4.11) thereby induces \( Dm \)-brane charge. It follows that the configuration \( (\hat{M}, \phi \circ \rho, [\hat{\xi}] \otimes \rho^*(x)) \) should then be physically identified with the \( Dm \)-branes characterized by the K-cycle \((M, \phi, x)\).

5. \( B \)-Fields and Twisted K-Homology

All of our analysis thus far has been implicitly done in flat space. We now consider the generalizations of the previous results to curved string backgrounds. By the string equations of motion, this is tantamount to making a certain supergravity field, called the “\( B \)-field”, non-trivial. We will begin with a purely mathematical explanation of what a \( B \)-field is, and then show that its presence implies that D-brane charge should take values in a certain twisted version of K-theory. The development of twisted K-theory is currently at the very heart of recent activity in this field [2, 9]–[28]. The main result of this section will be a new proposal [29], which we present here without proof, for twisted topological K-homology groups. With this proposal the physical characteristics of twisted K-theory groups become transparent, and it provides the basis for the geometrical characterization of D-branes in curved backgrounds.

5.1. \( B \)-Fields

A \( B \)-field may be defined geometrically as a gerbe with 1-connection [23]. Let us first define precisely what we mean by a gerbe [30] in a form that will be most useful in the following. Recall that integer cohomology groups form a hierarchy that may be used to topologically classify geometrical structures over the spacetime manifold \( X \). For instance, \( H^1(X, \mathbb{Z}) \) classifies functions on \( X \to S^1 \), while \( H^2(X, \mathbb{Z}) \) provides characteristic classes for complex line bundles over \( X \). Gerbes yield geometric realizations for the degree three cohomology classes of \( H^3(X, \mathbb{Z}) \).

For any group \( G \), let \( \underline{G}_X \) denote the corresponding constant sheaf over \( X \) of locally constant \( G \)-valued functions. Then the sheaf cohomology \( H^1(\underline{G}_X) \cong H^1(X, G) \cong [X, BG] \) classifies principal \( G \)-bundles over \( X \). Let us look at some examples of this characterization which will be particularly important for the analysis which follows. Consider the exact sequence

\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C} \xrightarrow{\exp} \mathbb{C}^\times \longrightarrow 0. \tag{5.1}
\]

Since \( \underline{\mathbb{C}}_X \) is a soft sheaf over \( X \), the corresponding long exact sequence in sheaf cohomology yields an isomorphism

\[
H^*(\underline{\mathbb{C}}^\times_X) \xrightarrow{\sim} H^{*+1}(\underline{\mathbb{Z}}_X) \cong H^{*+1}(X, \mathbb{Z}). \tag{5.2}
\]

In particular, \( H^2(X, \mathbb{Z}) \cong H^1(\underline{\mathbb{C}}^\times_X) \cong [X, BU(1)] \), as expected.

Next, if \( \mathcal{H} \) is a separable Hilbert space, then there is an exact sequence

\[
0 \longrightarrow U(1) \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \longrightarrow 0. \tag{5.3}
\]
By Kuiper’s theorem [31], the unitary group $U(\mathcal{H})$ is contractible, and so it yields soft sheaves with no cohomology. Thus again we generate an isomorphism
\[ H^1\left(\text{PU}(\mathcal{H}) \times X\right) \xrightarrow{\sim} H^2(\underline{\mathbb{C}^\times}_X) \cong H^3(X, \mathbb{Z}) \] (5.4)
where we have in addition used (5.3). It follows that the third integer cohomology group
\[ H^3(X, \mathbb{Z}) \cong \left[ X, \text{BPU}(\mathcal{H}) \right] \] (5.5)
classifies principal $\text{PU}(\mathcal{H})$-bundles over the spacetime $X$. For a given such bundle, the corresponding isomorphism class in $H^3(X, \mathbb{Z})$ is known as the Dixmier-Douady class and it represents the gerbe theoretic analog of the first Chern class for line bundles.

As our final example, consider
\[ 0 \rightarrow U(1) \rightarrow \text{Spin}^c(n) \rightarrow SO(n) \rightarrow 0. \] (5.6)
Then the obstruction to lifting any principal $SO(n)$-bundle $SO(n) \hookrightarrow \xi \rightarrow X$ to a spin$^c$-bundle over $X$ is a gerbe
\[ [\xi] \in H^1\left(\text{SO}(n) \times X\right) \xrightarrow{\sim} W_3(\xi) \in H^2(\underline{\mathbb{C}^\times}_X) \cong H^3(X, \mathbb{Z}). \] (5.7)
The obstruction to a spin$^c$ structure on the manifold $X$ is the third Stiefel-Whitney class $W_3(\xi) \in H^3(X, \mathbb{Z})$, which coincides with the Dixmier-Douady invariant of the bundle-lifting gerbe.

Let us now introduce the notion of connections on gerbes. For this, we will use a local formulation in terms of bundle gerbes, which provide a concrete description of a gerbe $H^2(\mathbb{C}^\times_X)$ in the language of Čech-de Rham cohomology. Consider a good cover of the manifold $X$ by open sets $U_\alpha$ with overlaps $U_{\alpha\beta} = U_\alpha \cap U_\beta$ (fig. 3). Instead of defining transition functions on each overlap $U_{\alpha\beta}$, we introduce complex line bundles $\ell_{\alpha\beta} \rightarrow U_{\alpha\beta}$ which satisfy the conditions
\[ \ell_{\alpha\beta} \otimes \ell_{\beta\gamma} = \ell_{\alpha\gamma} \] (5.8)
on triple overlaps $U_\alpha \cap U_\beta \cap U_\gamma$. We define a 0-connection to be a connection $\nabla_{\alpha\beta}$ on each line bundle $\ell_{\alpha\beta} \rightarrow U_{\alpha\beta}$. A 1-connection is then defined to be a 2-form $B_\alpha$ on each $U_\alpha$ such that
\[ B_\alpha - B_\beta = (\nabla_{\alpha\beta})^2 \] (5.9)
on $U_{\alpha\beta}$. The connection must satisfy the local $U(1)$ cocycle conditions
\[ \nabla_{\alpha\beta} + \nabla_{\beta\gamma} + \nabla_{\gamma\alpha} = -i \, d \ln(\zeta_{\alpha\beta\gamma}) \] (5.10)
on triple overlaps, where $\zeta_{\alpha\beta\gamma}$ are $U(1)$-valued 0-forms obeying
\[ \zeta_{\alpha\beta\gamma}^{-1} \zeta_{\beta\gamma\lambda}^{-1} \zeta_{\alpha\gamma\lambda}^{-1} = 1 \] (5.11)
on quadruple overlaps. The de Rham representative $H = dB$ of the curvature of the $B$-field then gives rise to an integer cohomology class $[H] \in H^3(X, \mathbb{Z})$ which coincides
Figure 3: A good covering of spacetime $X$ by open sets $U_\alpha$. 

with the Dixmier-Douady class of the bundle gerbe (Of course $[H]$ vanishes in de Rham cohomology $H^3_{\text{DR}}(X)$). Combining this with the description above in terms of $PU(\mathcal{H})$-bundles, we may alternatively view the $B$-field as a connection $\nabla: \mathcal{A} \to \mathcal{A}$ on the algebra $\mathcal{A} = C(X) \otimes \mathcal{K}(\mathcal{H})$, with $\text{Aut}(\mathcal{K}(\mathcal{H})) = PU(\mathcal{H})$ the group of projective unitary automorphisms of the Hilbert space $\mathcal{H}$. The 0-connections in this setting are derivations on the endomorphism bundle of the given principal $PU(\mathcal{H})$-bundle.

5.2. D-Branes in $B$-Fields

To understand the effect of a non-trivial $B$-field background on the geometry of D-branes, we will describe the Freed-Witten anomaly \cite{15} which arises due to a sign ambiguity in the open string worldsheet functional integral. The result of integrating over worldsheet fermions in $\Gamma(\Sigma, K^{1/2} \otimes \chi^*(TX))$ produces a determinant of the corresponding Dirac operator $\mathcal{D}$ which defines a section of the Pfaffian line bundle $\text{Pfaff}$. The pertinent terms are given in the product

$$\text{Pfaff}(\mathcal{D}) \cdot \text{Hol}_\Sigma(B) \cdot \text{Hol}_{\partial \Sigma}(\nabla).$$

(5.12)

When $\partial \Sigma \neq \emptyset$, the gerbe holonomy is not invariant under gauge transformations \cite{5.9} and must be regarded as a section of a complex line bundle $L_B$ with connection over the free loop space $LM$ of the D-brane worldvolume $M$. The holonomy of the Chan-Paton gauge connection $\nabla$ produces a trivialization of the twisted Pfaffian line bundle $\text{Pfaff} \otimes L_B$ over $LM$. In order for the functional integral to be globally well-defined, we must therefore require that the bundle $\text{Pfaff} \otimes L_B|_{LM}$ be topologically trivial.

To understand what this means, let us consider a family of open string worldsheets $\Sigma_t$, $0 \leq t \leq 1$ ending on $M$, with $\Sigma_0 = \Sigma_1 = \Sigma$. Then, as $t$ varies, the family $\partial \Sigma_t$ sweeps out a 2-cycle $C_2 \subset M$. The holonomy of the fermion determinants around a circle $S^1$ is the sign factor \cite{15}

$$\exp \left( i \pi \left( w_2(M), [C_2] \right) \right),$$

(5.13)

where $w_2(M) \in H^2(M, \mathbb{Z}_2)$ is the second Stiefel-Whitney class which may be related to the third Stiefel-Whitney class introduced in the previous subsection as follows. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{x^2} \mathbb{Z} \xrightarrow{r} \mathbb{Z}_2 \longrightarrow 0,$$

(5.14)
where the map $r$ is reduction modulo 2. This induces a long exact sequence in cohomology

$$
\cdots \longrightarrow \mathbb{H}^2(M, \mathbb{Z}) \xrightarrow{r^*} \mathbb{H}^2(M, \mathbb{Z}_2) \xrightarrow{\beta} \mathbb{H}^3(M, \mathbb{Z}) \longrightarrow \cdots,
$$

(5.15)

where the map $\beta$ is called the Bockstein homomorphism. Then the image of the second Stiefel-Whitney class under the Bockstein map yields the desired class, $W_3(M) = \beta(w_2(M)) \in \mathbb{H}^3(M, \mathbb{Z})$. Note that since the spacetime $X$ is oriented and spin, we further have $W_3(M) = W_3(N_X M)$ [2].

A flat $B$-field has torsion characteristic class $[H]$ which may be computed from the Bockstein map $\beta : \mathbb{H}^2(X, \mathbb{R}/\mathbb{Z}) \rightarrow \mathbb{H}^3(X, \mathbb{Z})$ [15]. The holonomy of this class around an $S^1$ yields the first Chern class of the line bundle $L_B$ on the loop space $LX$. We therefore have

$$
\oint_{\partial \Sigma} \phi^*([H]) = c_1(L_B)
$$

(5.16)

over $LM$. On the other hand, the Chern class of the Pfaffian line bundle over $LM$ may be computed as

$$
c_1(\text{Pfaff}) = \beta \left( \oint_{\partial \Sigma} w_2(M) \right) = \oint_{\partial \Sigma} W_3(M). \tag{5.17}
$$

It follows that $\text{Pfaff} \otimes L_B$ is trivial over $LM$ if

$$
\phi^*([H]) = W_3(M). \tag{5.18}
$$

The main consequence of this condition is that the D-brane worldvolume $\phi : M \rightarrow X$ is not necessarily a spin$^c$ manifold anymore. In particular, it no longer obeys Poincaré duality. Note that in light of the discussion of the previous subsection, the requirement (5.18) is also very natural from a purely mathematical standpoint, as both sides of this equation represent gerbes over $M$. The loss of the spin$^c$ structure requires a modification of the K-theory groups used to classify the D-brane charges.

Because of the condition (5.18), D-brane charge should now live in an appropriate twisted version of K-theory. Just as ordinary cohomology can be twisted by real line bundles (i.e. elements of $\mathbb{H}^2(X, \mathbb{Z}_2)$), so too can K-theory be twisted by gerbes (i.e. elements of $\mathbb{H}^3(X, \mathbb{Z})$). The appropriate receptacle for D-brane charge is then the K-theory of spacetime $X$ twisted by the characteristic class $[H] \in \mathbb{H}^3(X, \mathbb{Z})$ of the $B$-field [2]. A precise definition of this group has been given in [24] based on algebraic K-theory [19]. Instead of using the $C^*$-algebra $C(X)$ of functions on spacetime, one uses the ring of sections $\Gamma(X, \mathcal{E}_{[H]})$, where $\mathcal{E}_{[H]} \rightarrow X$ is the unique locally trivial $PU(H)$-bundle with Dixmier-Douady invariant $[H]$. Thus the twisted K-theory groups are defined as

$$
K^*(X, [H]) = K_*(\Gamma(X, \mathcal{E}_{[H]})) \tag{5.19}
$$

This group coincides with the K-theory of bundle gerbes obtained through the standard Grothendieck construction [27]. When $[H] = 0$, the infinite rank bundle $\mathcal{E}_{[H]=0} = X \times K(H)$ is trivial and $\Gamma(X, \mathcal{E}_{[H]=0}) = C(X) \otimes K(H)$. Then, by Morita equivalence, the
K-theory group \( \langle 5.19 \rangle \) reduces to the usual one \( K^\ast(X, [H] = 0) = K\ast(C(X) \otimes \mathcal{K}(\mathcal{H})) \cong K\ast(X) \). When \([H] \neq 0\) is a torsion class, the definition \( \langle 5.19 \rangle \) reproduces the standard definition in terms of the K-theory of Azumaya algebras \[2, 23\] which leads to the standard brane constructions described in section 2.3. The situations in which \([H] \neq 0\) is not a torsion class require, as in section 3.1, an infinite number of unstable D-branes in processes involving tachyon condensation \[18\].

5.3. D-Brane Charge Lives in Twisted K-Homology

The proposal \( \langle 5.19 \rangle \) for the twisted K-theory groups fits in well with our current understanding of D-branes in curved spaces. However, it does not demonstrate very well the physical significance of the groups \( K^\ast(X, [H]) \). A big quest at present in this field is therefore to provide a clear physical foundation to support the general claim that \( K^\ast(X, [H]) \) is the correct D-brane charge group to use in the presence of topologically non-trivial \( B \)-field backgrounds. This problem has been addressed within different contexts in \[26\].

Following what we did in the previous sections, we can alternatively seek a clearer geometric picture of twisted K-theory. At the functional analytic level, this is relatively straightforward to do using twisted Fredholm modules \[25\]. The definition follows the same pattern as before, except that now the algebra \( C(X) \) is replaced with \( \Gamma(X, \mathcal{E}[H]) \) and one requires a \(*\)-homomorphism

\[
\Phi_{[H]} : \Gamma(X, \mathcal{E}[H]) \longrightarrow \mathcal{B}(\mathcal{H}).
\]

The corresponding equivalence classes generate the analytic K-homology \( K\ast(X, [H]) \) of \( X \) twisted by the characteristic class \([H]\) of the \( B \)-field.

At the geometric level, we propose to interpret \( K\ast(X, [H]) \) as the group spanned by certain triples \[29\]

\[
k(X, [H]) = \text{Span}_\mathbb{Z} \left\{ (M, \phi, x) \mid \begin{array}{l}
M \text{ closed, oriented} \\
\phi : M \longrightarrow X \\
\phi^\ast([H]) = W_3(M) \\
x \in K\ast(M)
\end{array} \right\}.
\]

Note that this definition differs from \( \langle 13 \rangle \) in the weakening of the spin\(^c \) condition, appropriate to a topologically trivial background, to an orientability requirement \((w_1(M) = 0)\) and the incorporation of the anomaly cancellation constraint \( \langle 5.18 \rangle \). It is clear that any D-brane in a topologically non-trivial \( B \)-field defines an element of the abelian group \( \langle 5.21 \rangle \), and also of the twisted K-homology as defined above. The definition \( \langle 5.21 \rangle \) is then the working proposal for twisted topological K-homology. The key point which should make this the correct definition is as follows. The idea is based on the fact that a vector bundle is orientable with respect to complex K-theory if and only if it has a spin\(^c \) structure. This implies that the degree three Stiefel-Whitney class \( W_3 \) acts as an “orientation gerbe” for K-theory, so that for any space, K-theory twisted by \( W_3 \) should give a version of Poincaré duality appropriate to this setting. Then the K-theory class \( x \in K\ast(M) \) has a dual in \( K\ast(M, W_3(M)) \). The condition \( \langle 5.18 \rangle \) then ensures that this will push-forward to a class in the twisted K-homology \( K\ast(X, [H]) \) of spacetime \( X \). The proof that this is really the
case should follow a method analogous to that in the previous section and at the same time this will presumably shed light on some physical properties of D-branes on curved manifolds. In particular, using the recent construction of the Chern character $\text{ch}_{[H]}$ in twisted K-theory, one should now be able to rigorously derive a twisted version of the D-brane charge formula (2.9).

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3There is a slight caveat though. Setting $[H] = 0$ does not quite reduce (5.21) to its untwisted version (5.3), because in the latter case we also require each worldvolume $M$ to carry a specific, fixed choice of spin^c-structure. Presumably this problem can be solved by placing appropriate quotient conditions on (5.21).
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