On generalized fractional integral inequalities of Ostrowski type

HÜSEYİN YILDIRIM and SEDA KILINÇ YILDIRIM

Abstract. We obtain new generalizations of Ostrowski inequality by using generalized Riemann–Liouville fractional integrals. Some special cases are also discussed.

1. Introduction

Let \( f : [a, b] \rightarrow \mathbb{R} \) be a function continuous on \([a, b]\) and differentiable in \((a, b)\). If \(|f'(x)| \leq M\) for all \(x \in (a, b)\), then (see [14])

\[
|f(x) - M(f; a, b)| \leq \frac{M}{b-a} \left( \frac{(b-x)^2 + (x-a)^2}{2} \right)
\]

for all \(x \in [a, b]\), where \(M(f; a, b) = \frac{1}{b-a} \int_a^b f(x)dx\).

The inequality (1) is well known in the literature as Ostrowski inequality. Over the years, numerous studies have focused on generalizing this inequality, see, for example, [2–4, 6, 12, 14, 21] and the references cited therein.

Grüss [5] proved the inequality

\[
|M(fg; a, b) - M(f; a, b)M(g; a, b)| \leq \frac{1}{4} (M_1 - m_1) (M_2 - m_2),
\]

where \(f\) and \(g\) are two integrable function on \([a, b]\) satisfying the conditions \(m_1 \leq f(x) \leq M_1\) and \(m_2 \leq g(x) \leq M_2\) for all \(x \in [a, b]\). The constant \(\frac{1}{4}\) is the best possible.
Korkine's identity [11] states that if \( f \) and \( g \) are two integrable functions on \([a, b]\), then
\[
M(fg; a, b) - M(f; a, b)M(g; a, b)
= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(t) - f(s))(g(t) - g(s)) \, ds \, dt.
\] (3)

Many researchers have studied various types of integral inequalities for Riemann–Liouville integrals (see [7–9, 15, 18, 19, 20, 22] and references therein).

**Definition 1** (see [17]). Let \( f \in L^1[a, b] \). The Riemann–Liouville fractional integrals \( J^\alpha_a f(x) \) and \( J^\alpha_b f(x) \) of order \( \alpha \geq 0 \) are defined, respectively, by
\[
J^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a,
\]
and
\[
J^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_b^x (t-x)^{\alpha-1} f(t) \, dt, \quad x < b.
\]
Here \( \Gamma(\alpha) \) is Gamma function and \( J^\alpha_a f(x) = J^\alpha_b f(x) = f(x) \).

Hu [7] obtains the following generalizations for (1) by using (2), (3), and Riemann–Liouville fractional integrals.

**Theorem 1** (see [7]). Let \( f \) be a differentiable function on \([a, b]\) and let \( |f'(x)| \leq M \) for any \( x \in [a, b] \). Then the fractional inequality
\[
\left| \frac{(x-a)^\alpha + (b-x)^\alpha}{\Gamma(\alpha+1)} f(x) - J^\alpha_a f(a) - J^\alpha_b f(b) \right| \leq M \frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{\Gamma(\alpha+2)}
\] (4)
holds for any \( x \in [a, b] \) and \( \alpha \geq 0 \).

**Theorem 2** (see [7]). Let \( f: [a, b] \to R \) be a differentiable mapping and let \( f' \in L^2[a, b] \). If \( f' \) is bounded on \([a, b]\) with \( m \leq f'(x) \leq M \), then we have
\[
\left| \frac{\alpha f(x) + f(a)}{\Gamma(\alpha)(\alpha+1)} (x-a)^{\alpha-1} - \frac{\alpha}{x-a} J^\alpha_a f(a) \right| + \left| \frac{\alpha f(x) + f(b)}{\Gamma(\alpha)(\alpha+1)} (b-x)^{\alpha-1} - \frac{\alpha}{b-x} J^\alpha_b f(b) \right| \leq \sqrt{\frac{1}{2\alpha+1} - \frac{1}{(\alpha+1)^2}} \frac{(x-a)^K_1 + (b-x)^K_2}{\Gamma(\alpha)}
\] (5)
for all \( x \in [a, b] \) and \( \alpha \geq 0 \). Here
\[
K_1^2 = M(f^2; a, x) - M^2(f^2; a, x), \quad K_2^2 = M(f^2; x, b) - M^2(f^2; x, b).
\]
Now we will give definitions of generalized fractional integrals.

**Definition 2** (see [9]). The space \( L_{p,k}[a, b] \) is defined as

\[
L_{p,k}[a, b] = \left\{ f : \| f \|_{p,k} = \left( \int_a^b |f(t)|^p t^k dt \right)^{\frac{1}{p}} < \infty \right\}, \quad 1 \leq p < \infty, \ k \geq 0.
\]

For \( k = 0 \), the space \( L_{p,k}[a, b] \) reduces to the classical space \( L^p[a, b] \).

**Definition 3** (see [1, 9, 11]). Let \( h(x) \) be an increasing positive monotone function on \([a, b]\) such that \( h'(x) \) is continuous on \((a, b)\). The space \( X^p_h(a, b) \) \((1 \leq p < \infty)\) is defined as the set of those real-valued Lebesque measurable functions \( f \) on \([a, b]\) for which

\[
\| f \|_{X^p_h} = \left( \int_a^b |f(t)|^p h'(t) dt \right)^{\frac{1}{p}} < \infty.
\]

In particular, if we take \( h(x) = \frac{x^{k+1}}{k+1} \) \((k \geq 0)\), then the space \( X^p_h(a, b) \) coincides with the space \( L_{p,k}[a, b] \). For \( h(x) = x \), the space \( X^p_h(a, b) \) coincides with the classical space \( L^p[a, b] \).

**Definition 4** (see [1, 9, 11, 23]). Let \( f \in X^p_h(a, b) \). The left and right generalized fractional integrals of function \( f \) of order \( \alpha \geq 0 \) are defined, respectively, by

\[
J_{a^+,h}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (h(x) - h(t))^{\alpha-1} h'(t) f(t) dt, \quad x > a,
\]

and

\[
J_{b^-,h}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (h(t) - h(x))^{\alpha-1} h'(t) f(t) dt, \quad b > x.
\]

Here \( \Gamma(\alpha) \) is Gamma function and \( J_{a^+,h}^0 f(x) = J_{b^-,h}^0 f(x) = f(x) \).

**Remark 1.** Letting \( h(x) = x \) in (6) and (7), we obtain the equalities in Definition 1.

In this paper we will generalize expressions (1), (3), (4), and (5) by using generalized Riemann–Liouville fractional integrals.

**2. Main results**

**Theorem 3.** If \( f, g \in X^p_h(a, b) \), then we have, for \( \alpha \geq 0 \), the identity

\[
J_{a^+,h}^\alpha [f(b)g(b)] - \frac{\Gamma(\alpha + 1)}{(h(b) - h(a))^\alpha} J_{a^+,h}^\alpha [f(b)] J_{a^+,h}^\alpha [g(b)]
\]

\[
= \frac{\alpha}{2} \frac{\Gamma(\alpha)}{(h(b) - h(a))^\alpha} \int_a^b \int_a^b (f(t) - f(s)) (g(t) - g(s)) (h(b) - h(s))^{\alpha-1} (h(b) - h(t))^{\alpha-1} h'(t) h'(s) dsdt.
\]
Proof. By $(f(t) - f(s)) (g(t) - g(s)) = f(t)g(t) - f(t)g(s) - f(s)g(t) + f(s)g(s)$ we have
\[
\int_a^b \int_a^b (f(t) - f(s)) (g(t) - g(s)) \left( (h(b) - h(s))^{\alpha - 1} \right) \times (h(b) - h(t))^{\alpha - 1} h'(t) h'(s) ds dt
\]
\[
= 2 \left[ \int_a^b (h(b) - h(s))^{\alpha - 1} h'(s) ds \int_a^b f(t)g(t) (h(b) - h(t))^{\alpha - 1} h'(t) dt \right]
\]
\[
- 2 \left[ \int_a^b g(s) (h(b) - h(s))^{\alpha - 1} h'(s) ds \int_a^b f(t) (h(b) - h(t))^{\alpha - 1} h'(t) dt \right]
\]
\[
= \frac{2}{\alpha} \left( h(b) - h(a) \right)^\alpha \Gamma(\alpha) J_{\alpha, h}^\alpha [f(b)g(b)] - 2\Gamma^2(\alpha) J_{\alpha, h}^\alpha [f(b)] J_{\alpha, h}^\alpha [g(b)].
\]
This completes the proof. \qed

Remark 2. Taking $\alpha = 1$ in (8), we obtain the identity
\[
M_h(fg; a, b) - M_h(f; a, b) M_h(g; a, b)
\]
\[
= \frac{1}{2 (h(b) - h(a))^2} \int_a^b \int_a^b (f(t) - f(s)) (g(t) - g(s)) h'(t) h'(s) ds dt,
\]
where $M_h(f; a, b) = \frac{1}{h(b) - h(a)} \int_a^b f(t) h'(t) dt$.

Remark 3. For $h(x) = x$ in (9), we obtain the Korkine’s identity (3).

Theorem 4. Let $f$ be a differentiable function on $[a, b]$ and let $|f'(x)| \leq M$ for any $x \in [a, b]$. Then we have, for $\alpha \geq 0$, the generalized fractional inequality
\[
|D(f; a, b)| \leq \frac{1}{\Gamma(\alpha + 2)} M [(h(x) - h(a))^{\alpha + 1} + (h(b) - h(x))^{\alpha + 1}],
\]
where
\[
D(f; a, b) = \frac{1}{\Gamma(\alpha + 1)} [(h(x) - h(a))^\alpha + (h(b) - h(x))^\alpha] f(x)
\]
\[
- J_{\alpha, h}^\alpha [f(a)] - J_{\alpha, h}^\alpha [f(b)] - \left[ J_{\alpha, h}^{\alpha + 1} \left( \frac{f(a)}{h(a)} \right) + J_{\alpha, h}^{\alpha + 1} \left( \frac{f(b)}{h(b)} \right) \right].
\]

Proof. Using integration by parts for fractional integrals in Definition 5, we have
\[
J_{\alpha, h}^{\alpha + 1} f'(a) = \frac{1}{\Gamma(\alpha + 1)} (h(x) - h(a))^\alpha f(x) - J_{\alpha, h}^\alpha [f(a)] - J_{\alpha, h}^{\alpha + 1} \left( \frac{f(a)}{h(a)} \right),
\]
\[
J_{\alpha, h}^{\alpha + 1} f'(b) = - \frac{1}{\Gamma(\alpha + 1)} (h(b) - h(x))^\alpha f(x) + J_{\alpha, h}^\alpha [f(b)] + J_{\alpha, h}^{\alpha + 1} \left( \frac{f(b)}{h(b)} \right).
\]
These equalities show that
\[ J^{\alpha+1}_{x-h} f(a) - J^{\alpha+1}_{x+h} f(b) = D(f; a, b). \]
Therefore, using (6) and (7), by \(|f'(x)| \leq M, x \in [a, b]|), we have
\[
|D(f; a, b)| \leq \frac{1}{\Gamma(\alpha+1)} M \left( \int_a^x (h(t) - h(a))^\alpha h'(t) dt + \int_x^b (h(b) - h(t))^\alpha h'(t) dt \right)
\leq \frac{1}{\Gamma(\alpha+2)} M \left( (h(x) - h(a))^{\alpha+1} + (h(b) - h(x))^{\alpha+1} \right),
\]
which completes the proof. □

Remark 4. If we take \( h(x) = x \) in (10), then we obtain the inequality (4) in Theorem 1.

Remark 5. If \( h(x) = 0 \) and \( \alpha = 1 \), then the inequality (10) reduces to Ostrowski inequality (1).

Theorem 5. Let \( f: [a, b] \to R \) be a differentiable mapping. If \( f' \in X_h^p(a, b) \) is bounded on \([a, b] \) with \( m \leq f'(x) \leq M \), then
\[
|B_1(f; a, b) + B_2(f; a, b)| 
\leq \sqrt{\frac{1}{2\alpha+1} - \frac{1}{(\alpha+1)^2} \frac{1}{\Gamma(\alpha)} \left( (h(x) - h(a))^\alpha K_1 + (h(b) - h(x))^\alpha K_2 \right)} \tag{11}
\]
for all \( x \in [a, b] \) and \( \alpha \geq 0 \), where
\[
B_1(f; a, b) = \frac{1}{\Gamma(\alpha)} (h(x) - h(a))^{\alpha-1} \left[ f(x) - \frac{1}{(\alpha+1)} \int_a^x f'(t) h'(t) dt \right]
- \frac{\alpha}{h(x) - h(a)} \left( J_0^{\alpha-1}_{x-h} f(a) + J_0^{\alpha+1}_{x+h} f(a) \right),
\]
\[
B_2(f; a, b) = \frac{1}{\Gamma(\alpha)} (h(b) - h(x))^{\alpha-1} \left[ f(x) + \frac{1}{(\alpha+1)} \int_x^b f'(t) h'(t) dt \right]
- \frac{\alpha}{h(b) - h(x)} \left( J_0^{\alpha-1}_{x-h} f(b) + J_0^{\alpha+1}_{x+h} f(b) \right),
\]
\[ K_1^2 = M_h(f^2; a, x) - M^2_h(f'; a, x), \quad K_2^2 = M_h(f^2; x, b) - M^2_h(f'; x, b). \]

Proof. From (6) and (7) we have
\[
\frac{1}{\Gamma(\alpha)(h(x) - h(a))} \int_a^x (h(t) - h(a))^\alpha h'(t) f'(t) dt = B_1(f; a, b),
\]
\[
-\frac{1}{\Gamma(\alpha)(h(b) - h(x))} \int_x^b (h(b) - h(t))^\alpha h'(t) f'(t) dt = B_2(f; a, b).
\]
Then
\[ B_1(f; a, b) + B_2(f; a, b) = \frac{1}{\Gamma(\alpha)} (h(x) - h(a)) \int_a^x (h(t) - h(a))^{\alpha} h'(t) f'(t) \, dt \]
\[ - \frac{1}{\Gamma(\alpha)(\alpha + 1)} (h(x) - h(a))^{\alpha - 1} \int_a^x f'(t) h'(t) \, dt \]
\[ - \frac{1}{\Gamma(\alpha)} (h(b) - h(x)) \int_x^b (h(b) - h(t))^{\alpha} h'(t) f'(t) \, dt \]
\[ + \frac{1}{\Gamma(\alpha)(\alpha + 1)} (h(b) - h(x))^{\alpha - 1} \int_x^b f'(t) h'(t) \, dt, \]
and by the identity (9) we get
\[ B_1(f; a, b) + B_2(f; a, b) \]
\[ = \frac{1}{2\Gamma(\alpha)(h(x) - h(a))^2} \int_a^x \int_a^x \left[ (h(t) - h(a))^\alpha \right] \]
\[ - (h(s) - h(a))^{\alpha} \left[ f'(t) - f'(s) \right] h'(s) h'(t) ds dt \tag{12} \]
\[ + \frac{1}{2\Gamma(\alpha)(h(b) - h(x))^2} \int_x^b \int_x^b \left[ (h(b) - h(s))^\alpha \right] \]
\[ - (h(b) - h(t))^{\alpha} \left[ f'(t) - f'(s) \right] h'(s) h'(t) ds dt. \]

Using the Cauchy–Schwarz inequality for double integrals in (12), we obtain that
\[ \left| \int_a^x \int_a^x \left[ (h(t) - h(a))^\alpha - (h(s) - h(a))^{\alpha} \right] \left[ f'(t) - f'(s) \right] h'(s) h'(t) \right| ds dt \]
\[ \leq \left( \int_a^x \int_a^x \left[ (h(t) - h(a))^\alpha - (h(s) - h(a))^{\alpha} \right] h'(s) h'(t) ds dt \right)^{\frac{1}{2}} \]
\[ \times \left( \int_a^x \int_a^x \left[ f'(t) - f'(s) \right]^2 h'(s) h'(t) ds dt \right)^{\frac{1}{2}}. \tag{13} \]

Since
\[ \int_a^x \int_a^x \left[ (h(t) - h(a))^\alpha - (h(s) - h(a))^{\alpha} \right] h'(s) h'(t) ds dt \]
\[ = 2 (h(x) - h(a))^{2\alpha + 2} \left( \frac{1}{2\alpha + 1} - \frac{1}{(\alpha + 1)^2} \right) \]
and
\[ \int_a^x \int_a^x \left[ f'(t) - f'(s) \right]^2 h'(s) h'(t) ds dt \]
\[
= 2 \left( h(x) - h(a) \right)^2 \left[ M_h(f^2; a, x) - M_h^2(f'; a, x) \right],
\]

by (13) we have
\[
\left| \frac{1}{2\Gamma(\alpha)} \int_a^x \int_a^x \left[ (h(t) - h(a))^\alpha - (h(s) - h(a))^\alpha - (h(s) - h(a)) \right] \left[ f'(t) - f'(s) \right] h'(s) h'(t) ds dt \right|
\leq \frac{(h(x) - h(a))^\alpha}{\Gamma(\alpha)} \sqrt{ \frac{1}{2\alpha + 1} - \frac{1}{(\alpha + 1)^2} } \left[ M_h(f^2; a, x) - M_h^2(f'; a, x) \right]^{\frac{1}{2}}.
\]

Similarly we find that
\[
\left| \frac{1}{2\Gamma(\alpha)} \int_a^b \int_a^b \left[ (h(b) - h(x))^\alpha - (h(b) - h(x)) \right] \left[ f'(t) - f'(s) \right] h'(s) h'(t) ds dt \right|
\leq \frac{(h(b) - h(x))^\alpha}{\Gamma(\alpha)} \sqrt{ \frac{1}{2\alpha + 1} - \frac{1}{(\alpha + 1)^2} } \left[ M_h(f^2; x, b) - M_h^2(f'; x, b) \right]^{\frac{1}{2}}.
\]

Using (12), (14), and (15), we obtain the first inequality of (11). Moreover, if \( m \leq f'(x) \leq M \) on \([a, b]\), then by Grüss inequality we get
\[
0 \leq \frac{1}{h(x) - h(a)} \left\| f' \right\|_{X^2_h(a, x)}^2 - (M_h(f'; a, x))^2 \leq \frac{1}{2} (M - m)^2,
\]
\[
0 \leq \frac{1}{h(b) - h(x)} \left\| f' \right\|_{X^2_h(x, b)}^2 - (M_h(f'; x, b))^2 \leq \frac{1}{2} (M - m)^2
\]
which proves the last inequality of (11). \( \square \)

**Remark 6.** If we set \( h(x) = x \) in (11), then we obtain the inequality (5) in Theorem 2.

**Corollary 1.** Under the assumptions of Theorem 5 with \( \alpha = 1 \) the following inequality holds:
\[
2f(x) + \frac{1}{2} \int_x^b f'(t)h'(t)dt - \int_a^x f'(t)h'(t)dt
- \frac{1}{h(x) - h(a)} \left[ J_{x-h}^1 f(a) + J_{x+h}^2 \frac{f(a)}{h(a)} \right] \left[ J_{x+h}^1 f(b) + J_{x-h}^2 \frac{f(b)}{h(b)} \right]
\leq \frac{1}{4\sqrt{3}} (h(b) - h(a))(M - m).
\]
\[
(16)
\]
Remark 7. If we set $h(x) = x$ and $\alpha = 1$ in (16), then we obtain the inequality (2.31) of [23].

3. Concluding remarks

In this study, we presented Ostrowski type generalized inequalities via generalized fractional integrals. It is also shown that the results proved here are a strong generalization of some already published results.

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Department of Mathematics, Faculty of Science and Arts, University of Kahramanmaraş Sütçü İmam, 46000, Kahramanmaraş, Turkey

E-mail address: hyildir@ksu.edu.tr
E-mail address: sedakilincmath@gmail.com