Finiteness and Torelli Spaces

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Torelli space $\mathcal{T}_g$ $(g \geq 2)$ is the quotient of Teichmüller space by the Torelli group $T_g$. It is the moduli space of compact, smooth genus $g$ curves $C$ together with a symplectic basis of $H_1(C;\mathbb{Z})$ and is a model of the classifying space of $T_g$. Mess, in his thesis [12], proved that $\mathcal{T}_2$ has the homotopy type of a bouquet of a countable number of circles. Johnson and Millson (cf. [12]) pointed out that a similar argument shows that $H_3(\mathcal{T}_3)$ is of infinite rank. Akita [2] used an indirect argument to prove that $\mathcal{T}_g$ does not have the homotopy type of a finite complex for (almost) all $g \geq 2$. However, the infinite topology of $\mathcal{T}_g$ is not well understood.

The results of Mess and Johnson-Millson are the only ones I know of that explicitly describe some infinite topology of any Torelli space. Moreover, although Johnson [10] proved that $\mathcal{T}_g$ is finitely generated when $g \geq 3$, there is not one $g \geq 3$ for which it is known whether $\mathcal{T}_g$ is finitely presented or not.

The goal of this note is to present a suite of problems designed to probe the infinite topology of Torelli spaces in all genera. These are presented in the fourth section of the paper. The second and third sections present background material needed in the discussion of the problems.

To create a context for these problems, we first review the arguments of Mess and Johnson-Millson. As explained in Section 2, Torelli space in genus 2 is the complement of a countable number of disjoint smooth divisors $D_\alpha$ (i.e., codimension 1 complex subvarieties) in $\mathfrak{h}_2$, the Siegel upper half plane of rank 2. More precisely,

$$\mathcal{T}_2 = \mathfrak{h}_2 - \bigcup_{\phi \in \text{Sp}_2(\mathbb{Z})} \phi(\mathfrak{h}_1 \times \mathfrak{h}_1)$$

where $\text{Sp}_g(R)$ denotes the automorphisms of $R^{2g}$ that fix the standard symplectic inner product.\(^1\) One can now argue, as Mess did [12], that $\mathcal{T}_2$ is homotopy equivalent to a countable wedge of circles. Such an explicit description of $\mathcal{T}_2$ is unlikely in higher genus due to the increasing complexity of the image of the period mapping and the lack of an explicit description of its closure. So it should be fruitful to ponder the source of the infinite rank homology without appeal to this explicit description of $\mathcal{T}_2$. To this end, consider the exact sequence

$$\cdots \rightarrow H_4^b(\mathfrak{h}_2) \rightarrow H_4^c(D) \rightarrow H_3^c(\mathcal{T}_2) \rightarrow H_3^c(\mathfrak{h}_2) \rightarrow \cdots$$

where $D = \cup D_\alpha$ is a locally finite union of smooth divisors indexed by $\text{Sp}_2(\mathbb{Z})$ mod the stabilizer $S_2 \ltimes \text{Sp}_1(\mathbb{Z})$ of $\mathfrak{h}_1 \times \mathfrak{h}_1$.\(^2\) Since $\mathfrak{h}_2$ is a contractible complex manifold of real dimension 6,

$$H^k_c(\mathfrak{h}_2) = H_{6-k}(\mathfrak{h}_2) = 0 \text{ when } k < 6.$$
Thus
\[ H_1(T_2) = H_c^5(T_2) \cong H_c^4(D) \cong \bigoplus_{\alpha \in \text{Sp}_2(Z)/S_2 \times \text{Sp}_1(Z)^2} \mathbb{Z}[D_{\alpha}], \]
a free abelian group of countable rank.

The situation is genus 3 is similar, but slightly more complicated. In this case, the period mapping \( T_3 \to \mathfrak{h}_3 \) is two-to-one, branched along the locus of hyperelliptic curves. The image is the complement \( \mathcal{J}_3 \) of a countable union \( Z \) of submanifolds \( Z_{\alpha} \) of complex codimension 2:
\[ \mathcal{J}_3 = \mathfrak{h}_3 - Z = \mathfrak{h}_3 - \bigcup_{\phi \in \text{Sp}_3(Z)/\text{Sp}_1(Z) \times \text{Sp}_2(Z)} \phi(\mathfrak{h}_1 \times \mathfrak{h}_2); \]
it is the set of framed jacobians of smooth genus 3 curves. By elementary topology,
\[ H_\bullet(\mathcal{J}_3; \mathbb{Q}) \cong H_\bullet(T_3; \mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}, \]
where \( \mathbb{Z}/2\mathbb{Z} \) is the group of automorphisms of the ramified covering \( T_3 \to \mathcal{J}_3 \). If \( H_k(\mathcal{J}_3; \mathbb{Q}) \) is infinite dimensional, then so is \( H_k(T_3; \mathbb{Q}) \). As in the genus 2, we have an exact sequence
\[ 0 = H_8^c(\mathfrak{h}_3) \to H_9^c(Z) \to H_9^c(\mathcal{J}_3) \to H_9^c(\mathfrak{h}_3) = 0 \]
so that
\[ H_3(\mathcal{J}_3) = H_3^c(\mathcal{J}_3) \cong H_3^c(Z) = \bigoplus_{\alpha \in \text{Sp}_3(Z)/\text{Sp}_1(Z) \times \text{Sp}_2(Z)} \mathbb{Z}[Z_{\alpha}]. \]

Three important ingredients in these arguments are:

- \( \mathcal{T}_g \) or a space very closely related to it (e.g., \( \mathcal{J}_g \)) is the complement \( X - Z \) in a manifold \( X \) of a countable union \( Z = \bigcup Z_{\alpha} \) of smooth subvarieties;
- The topology of the manifold \( X \) is very simple — in the cases above, \( X = \mathfrak{h}_g \), which is contractible;
- the “topology at infinity” of \( X \) is very simple — in the cases above the boundary of \( X = \mathfrak{h}_g \) is a sphere;
- the topology of \( X, Z \) and \( \mathcal{T}_g \) are related by a Gysin sequence of compactly supported cohomology, and the relationship between the compactly supported cohomology and ordinary cohomology is entwined with the “topology at infinity” of \( X \).

What is special in genus 2 and 3 is that the closure \( \mathcal{J}_g^c \) of the image of the period mapping \( \mathcal{T}_g \to \mathfrak{h}_g \) is very simple — it is all of \( \mathfrak{h}_g \), a topological ball, which is contractible and has very simple topology at infinity. This fails to be true in genus 4, where \( \mathcal{J}_g^c \) is a singular subvariety of complex codimension 1. Here, even though there is an explicit equation for the \( \mathcal{J}_4^c \) in \( \mathfrak{h}_4 \), we do not understand its topology. The situation only gets worse in higher genus.

1. Preliminaries

Suppose that \( 2g - 2 + n > 0 \). Fix a compact oriented surfaced \( S \) of genus \( g \) and a finite subset \( P \) of \( n \) distinct points in \( S \). The corresponding mapping class group is
\[ \Gamma_{g,n} = \pi_0 \text{Diff}^+(S, P). \]
By a complex curve, or a curve for short, we shall mean a Riemann surface. Denote the Teichmüller space of marked, $n$-pointed, compact genus $g$ curves by $X_{g,n}$. As a set $X_{g,n}$ is

$$\left\{ \text{orientation preserving diffeomorphisms } f : S \to C \text{ to a complex curve} \right\} / \text{isotopies, constant on } P.$$ 

This is a complex manifold of dimension $3g - 3 + n$.

The mapping class group $\Gamma_{g,n}$ acts properly discontinuously on $X_{g,n}$. The quotient $\Gamma_{g,n} \backslash X_{g,n}$ is the moduli space $M_{g,n}$ of $n$-pointed curves of genus $g$.

Set $H_R = H_1(S; R)$. The intersection pairing $H_R \otimes H_R \to R$ is skew symmetric and unimodular. Set

$$\text{Sp}(H_R) = \text{Aut}(H_R, \text{intersection pairing}).$$

The choice of a symplectic basis of $H_R$ gives an isomorphism $\text{Sp}(H_R) \cong \text{Sp}_g(R)$ of $2g \times 2g$ symplectic matrices with entries in $R$. The action of $\Gamma_{g,n}$ on $S$ induces a homomorphism

$$\rho : \Gamma_{g,n} \to \text{Sp}(H_Z)$$

which is well-known to be surjective.

The Torelli group $T_{g,n}$ is defined to be the kernel of $\rho$. It is torsion free. The Torelli space $T_{g,n}$ is defined by

$$T_{g,n} = T_{g,n} \backslash X_{g,n}.$$ 

It is the moduli space of $n$-pointed Riemann surfaces $(C; x_1, \ldots, x_n)$ of genus $g$ together with a symplectic basis of $H_1(C; \mathbb{Z})$. Since $T_{g,n}$ is torsion free, it acts freely on Teichmüller space. Torelli space $T_{g,n}$ is thus a model of the classifying space $BT_{g,n}$. Note that $T_{1,1}$ is trivial and that $X_{1,1} = T_{1,1}$ is just the upper half plane, $\mathbb{H}_1$. Torelli spaces also exist in genus 0 provided $n \geq 3$. In this case, $T_{0,n} = M_{0,n}$.

The Siegel upper half space of rank $g$

$$\mathfrak{h}_g := \text{Sp}_g(\mathbb{R})/U(g) \cong \{ \Omega \in M_g(\mathbb{C}) : \Omega = \Omega^T, \text{Im } \Omega > 0 \}$$

is the symmetric space of $\text{Sp}_g(\mathbb{R})$. It is a complex manifold of dimension $g(g + 1)/2$. It can usefully be regarded as the moduli space of $g$-dimensional principally polarized abelian varieties $(A, \theta)$ together with a symplectic (with respect to the polarization $\theta : H_1(A; \mathbb{Z}) \otimes \mathbb{Z} \to \mathbb{Z}$) basis of $H_1(A; \mathbb{Z})$. The group $\text{Sp}_g(\mathbb{Z})$ acts on the framings, and the quotient $\text{Sp}_g(\mathbb{Z}) \backslash \mathfrak{h}_g$ is the moduli space $A_g$ of principally polarized abelian varieties of dimension $g$.

The decoration $n$ in $\Gamma_{g,n}$, $T_{g,n}$, $T_{g,n}$, etc will be omitted when it is zero.

2. Geography

2.1. The period mapping. A framed genus $g$ curve is a compact Riemann surface of genus $g$ together with a symplectic basis $(a_1, \ldots, a_g, b_1, \ldots, b_g)$ of $H_1(C; \mathbb{Z})$. For each framed curve, there is a unique basis $w_1, \ldots, w_g$ of the holomorphic differentials $H^0(C, \Omega^1_C)$ on $C$ such that

$$\int_{a_j} w_k = \delta_{j,k}.$$ 

The period matrix of $(C; a_1, \ldots, a_g, b_1, \ldots, b_g)$ is the $g \times g$ matrix

$$\Omega = \left( \begin{array}{c} \int_{b_j} w_k \end{array} \right).$$
It is symmetric and has positive definite imaginary part.

As remarked above, $\mathcal{T}_g$ is the moduli space of framed genus $g$ curves. The period mapping

$$\mathcal{T}_g \to \mathfrak{h}_g$$

takes a framed curve to its period matrix. It is holomorphic and descends to the mapping

$$\mathcal{M}_g = \text{Sp}_g(\mathbb{Z}) \backslash \mathcal{T}_g \to \text{Sp}_g(\mathbb{Z}) \backslash \mathfrak{h}_g = \mathcal{A}_g$$

that takes the point $[C]$ of $\mathcal{M}_g$ corresponding to a curve $C$ to the point $[\text{Jac } C]$ of $\mathcal{A}_g$ corresponding to its jacobian.

2.2. The jacobian locus. This is defined to be the image $\mathcal{J}_g$ of the period mapping $\mathcal{T}_g \to \mathfrak{h}_g$. It is a locally closed subvariety of $\mathfrak{h}_g$. It is important to note, however, that it is not closed in $\mathfrak{h}_g$.

To explain this, we need to introduce the locus of reducible principally polarized abelian varieties. This is

$$\mathcal{A}^\text{red}_g = \bigcup_{h=1}^{\lfloor g/2 \rfloor} \text{im}\{\mu_h : \mathcal{A}_h \times \mathcal{A}_{g-h} \to \mathcal{A}_g\}$$

where $\mu_h(A, B) = A \times B$. Set

$$\mathfrak{h}^g_\text{red} = \text{inverse image in } \mathfrak{h}_g \text{ of } \mathcal{A}^\text{red}_g = \bigcup_{h=1}^{\lfloor g/2 \rfloor} \bigcup_{\phi \in \text{Sp}_g(\mathbb{Z}) / \text{Sp}_g(\mathbb{Z})_{h \times \mathfrak{h}_{g-h}}} \phi(\mathfrak{h}_h \times \mathfrak{h}_{g-h})$$

where $\mathfrak{h}_h \times \mathfrak{h}_{g-h}$ is imbedded into $\mathfrak{h}_g$ by taking $(\Omega_1, \Omega_2)$ to the matrix $\Omega_1 \oplus \Omega_2$; the group $\text{Sp}_g(\mathbb{Z})_{h \times \mathfrak{h}_{g-h}}$ denotes the subgroup of $\text{Sp}_g(\mathbb{Z})$ that fixes it set-wise.

For a subset $N$ of $\mathfrak{h}_g$, set $N^\text{red} = N \cap \mathfrak{h}^g_\text{red}$. By [8 Prop. 6],

$$\mathcal{J}_g = \mathcal{J}^c_g - \mathcal{J}^{c,\text{red}}_g$$

from which it follows that

$$\mathcal{J}^{c,\text{red}}_g = \bigcup_{\phi \in \text{Sp}_g(\mathbb{Z})} \bigcup_{h=1}^{\lfloor g/2 \rfloor} \phi(\mathcal{J}^{c,\text{red}}_h \times \mathcal{J}^{c,\text{red}}_{g-h}).$$

2.3. Curves of compact type. A genus $g$ curve $C$ of compact type is a connected, compact, nodal curve satisfying:

(i) the dual graph of $C$ is a tree — this guarantees that the jacobian $\text{Jac } C$ of $C$ is a principally polarized abelian variety;

(ii) the sum of the genera of the components of $C$ is $g$.

These are precisely the stable curves of genus $g$ whose generalized jacobian is compact. The generalized jacobian a genus $g$ curve of compact type is the product of the Jacobians of its components. It is an abelian variety of dimension $g$.

An $n$-pointed nodal curve of genus $g$ is a nodal genus $g$ curve $C$ together with $n$ labeled points in the smooth locus of $C$. An $n$-pointed nodal curve $(C, P)$ of genus

3Note that $\text{im } \mu_h = \text{im } \mu_{g-h}$. This is why we need only those $h$ between 1 and $\lfloor g/2 \rfloor$.

4A nodal curve is a complex analytic curve, all of whose singularities are nodes — that is, of the form $zw = t$. 
g is stable if its automorphism group is finite. This is equivalent to the condition that each connected component of 
\[ C - (C^{\text{sing}} \cup P) \]
has negative Euler characteristic.

Using the deformation theory of stable curves, one can enlarge \( T_{g,n} \) to the moduli space \( T_{0}^{c}g,n \) of framed stable \( n \)-pointed genus \( g \) curves of compact type. This is a complex manifold that contains \( T_{g,n} \) as a dense open subset and on which \( \text{Sp}_{g}(\mathbb{Z}) \) acts (via its action on framings). The quotient \( \text{Sp}_{g}(\mathbb{Z}) \backslash T_{g,n}^{c} \) is the space \( \mathcal{M}_{g,n}^{c} \) of stable \( n \)-pointed, genus \( g \) curves of compact type.

Note that when \( n \geq 3 \), \( T_{0,n}^{c} = \mathcal{M}_{0,n} \), the moduli space of stable \( n \)-pointed curves of genus 0.

**Proposition 1.** If \( 2g - 2 + n > 0 \), then \( T_{g,n}^{c} \) is a smooth complex analytic variety of complex dimension \( 3g - 3 + n \) and
\[ T_{g,n} = T_{g,n}^{c} - Z \]
where \( Z \) is a countable union of transversally intersecting smooth divisors.

The strata of \( Z \) of complex codimension \( k \) are indexed by the \( k \)-simplices of the quotient \( T_{g,n} \backslash K^{\text{sep}}(S, P) \) of the complex of separating curves \( K^{\text{sep}}(S, P) \) of the \( n \)-pointed reference surface \( (S, P) \) by the Torelli group. This correspondence will be made more explicit in the next paragraph.

Since \( T_{g,n} = T_{g,n}^{c} - Z \), where \( Z \) has complex codimension 1, the mapping
\[ \pi_{1}(T_{g,n}, *) \rightarrow \pi_{1}(T_{g,n}^{c}, *) \]
is surjective. Its kernel is generated by the conjugacy classes of small loops about each of the components of \( Z \). But these are precisely the conjugacy classes of Dehn twists on separating simple closed curves (SCCs).

**Proposition 2.** For all \( (g, n) \) satisfying \( 2g - 2 + n > 0 \),
\[ \pi_{1}(T_{g,n}^{c}, *) \cong T_{g,n} / \{ \text{subgroup generated by Dehn twists on separating SCCs} \} \]

As in the previous section, we set \( H_{Z} = H_{1}(S; \mathbb{Z}) \), where \( S \) is the genus \( g \) reference surface. Denote the image of \( u \in \Lambda^{3}H_{Z} \) under the quotient mapping
\[ \Lambda^{3}H_{Z} \rightarrow (\Lambda^{3}H_{Z}) / (\theta \wedge H_{Z}) \]
by \( \overline{\mu} \).

The surjectivity of the “Johnson homomorphism” \( \tau : T_{g,1} \rightarrow \Lambda^{3}H_{Z} \) and Johnson’s result [11] that its kernel is generated by Dehn twists on separating SCCs implies quite directly that:

**Corollary 3.** If \( g \geq 1 \) and \( 2g - 2 + n > 0 \), then
\[ \pi_{1}(T_{g,n}^{c}, *) = H_{1}(T_{g,n}^{c}, \mathbb{Z}) \cong \{ (u_{1}, \ldots, u_{n}) \in (\Lambda^{3}H_{Z})^{n} : \overline{\mu}_{1} = \cdots = \overline{\mu}_{n} \} \]
which is a torsion free abelian group.
2.4. The complex of separating curves, $K^{\text{sep}}(S, P)$. As above $S$ is a compact oriented surface of genus $g$ and $P$ is a subset of cardinality $n$, where $2g - 2 + n > 0$. A simple closed curve $\gamma$ in $S - P$ is separating if $S - (P \cup \gamma)$ is not connected. An SCC is cuspidal if it bounds a disk in $S$.

The simplicial complex $K^{\text{sep}}(S, P)$ has vertices the isotopy classes of separating SCCs $\gamma$ in $S - P$ that are not cuspidal. The isotopy classes of the SCCs $\gamma_0, \ldots, \gamma_k$ of non-cuspidal separating SCCs span a $k$-simplex of $K^{\text{sep}}(S, P)$ if they are disjoint and lie in distinct isotopy classes. When $P$ is empty, we shall abbreviate $K^{\text{sep}}(S, P)$ by $K^{\text{sep}}(S)$.

The correspondence between $T_{g,n}$ orbits of $K^{\text{sep}}(S, P)$ and the strata of $T^{g,\text{red}}_{g,n}$ is given as follows. Given a $k$-simplex $\tilde{\gamma} = (\gamma_0, \ldots, \gamma_k)$ of $K^{\text{sep}}(S, P)$, one can contract each of the SCCs $\gamma_j$. The resulting space $(S/\tilde{\gamma}, P)$ is the topological model of a stable, $n$-pointed, genus $g$ (complex) curve of compact type. Every topological type of stable complex curve of compact type arises in this way.

A marked $n$-pointed, genus $g$ curve of compact type is a homotopy class of homeomorphisms

$$(S/\tilde{\gamma}, P) \rightarrow (C; \{x_1, \ldots, x_n\})$$

to an $n$-pointed, genus $g$ stable curve $(C; \{x_1, \ldots, x_n\})$. For each $\tilde{\gamma} \in K^{\text{sep}}(S, P)$, one can add a connected “rational boundary component” $X_{\gamma}$ to the Teichmüller space $X_{g,n}$ of $(S, P)$ to obtain a topological space. The mapping class group $\Gamma_{g,n}$ acts on this enlarged Teichmüller space $X_{g,c}$ and the quotient is $M_{g}^{c}$ from which it follows that $T_{g,c} = T_{g,n} \backslash X_{g,c}$.

The stratum of $T^{g,\text{red}}_{g,n}$ that has codimension $k$ in $T^{g,c}_{g,n}$ is the locus of stable curves of compact type with precisely $k$ double points. These correspond to the $k - 1$ simplices of $K^{\text{sep}}(S, P)$. It follows that the strata of $T^{g,c}_{g,n}$ correspond to the simplices of $T_{g,n} \backslash K^{\text{sep}}(S, P)$.

Remark 4. Farb and Ivanov [4] have shown that $K^{\text{sep}}(S)$ is connected whenever $g \geq 3$.

2.5. Singularities and dimension. Note that $J_g$ is the quotient of $T_g$ by the involution

$$\sigma : (C; a_1, \ldots, a_g, b_1, \ldots, b_g) \rightarrow (C; -a_1, \ldots, -a_g, -b_1, \ldots, -b_g)$$

Since

$$(C; a_1, \ldots, a_g, b_1, \ldots, b_g) \cong (C; -a_1, \ldots, -a_g, -b_1, \ldots, -b_g)$$

if and only if $C$ is hyperelliptic, this mapping is ramified along the locus $H_g$ of hyperelliptic curves. Since this has complex codimension $g - 2$ in $T_g$, we know that $J_g$ is singular along the locus of hyperelliptic jacobians when $g \geq 4$. It is the quotient of the manifold $T_g$ by $\mathbb{Z}/2$, so $J_g$ is always a $\mathbb{Z}[1/2]$-homology manifold.

The period mapping $T^{g,c}_{g,n} \rightarrow J^{g,c}_{g,n}$ has positive dimensional fibers over $J^{g,c,\text{red}}_{g,n}$ when $g \geq 3$. As a result, $J^{g,c}_{g,n}$ is singular along $J^{g,c,\text{red}}_{g,n}$ when $g \geq 4$, and is not a rational homology manifold.\(^5\)

Since $J^{g,c}_{g,n}$ has dimension $3g - 3$ and $h_g$ dimension $g(g + 1)/2$, $J^{g,c}_{g,n}$ is a proper subvariety of $h_g$ when $g \geq 4$. This and the fact that $J^{g,c}_{g,n}$ is not a rational homology manifold when $g \geq 4$ help explain the difficulty of generalizing Mess’ arguments to any $g \geq 4$.

\(^5\)An explicit description of the links of the singularities of the top stratum of $J^{g,c,\text{red}}_{g,n}$ can be found in [17] Prop. 6.5. There are similar descriptions in higher codimension.
On the positive side, we can say that, since $\mathcal{J}_g^c$ is a closed analytic subvariety of the Stein manifold $h_g$, it is a Stein space. Consequently, by a result of Hamm, we have:

**Proposition 5.** $\mathcal{J}_g^c$ has the homotopy type of a CW-complex of dimension at most $3g - 3$. □

### 3. Homological Tools

This section can be omitted or skimmed on a first reading.

#### 3.1. A Gysin sequence

Since $\mathcal{T}_g$ is obtained from the manifold $\mathcal{T}_g^c$ by removing a countable union of closed subvarieties, we have to be a little more careful than usual when constructing the Gysin sequence.

Suppose that $X$ is a PL manifold of dimension $m$ and that $Y$ is a closed PL subset of $X$. Suppose that $A$ is any coefficient system. For each compact (PL) subset $K$ of $X$, we have the long exact sequence

$$
\cdots \to H^{m-k-1}(Y, Y - (Y \cap K); A) \to H^{m-k}(X, Y - (X - K); A) \to H^{m-k}(Y, Y - (Y \cap K); A) \to \cdots
$$

of the triple $(X, Y \cup (X - K), X - K)$. Taking the direct limit over all such $K$, we obtain the exact sequence

$$
\cdots \to H_{m-k-1}^c(Y; A) \to H_{m-k}^c(X - Y; A) \to H_{m-k}^c(X; A) \to H_{m-k}^c(Y; A) \to \cdots
$$

When $X$ is oriented, we can apply Poincaré duality to obtain the following version of the Gysin sequence:

**Proposition 6.** If $A$ is any coefficient system, $X$ an oriented PL-manifold of dimension $m$ and $Y$ is a closed PL subset of $X$, then there is a long exact sequence

$$
\cdots \to H_{m-k-1}^c(Y; A) \to H_{m-k}^c(X - Y; A) \to H_k(X; A) \to H_{m-k}^c(Y; A) \to \cdots
$$

#### 3.2. A spectral sequence

In practice, we are also faced with the problem of computing $H^*_c(Y)$ in the Gysin sequence above when $Y$ is singular. Suppose that

$$Y = \bigcup_{\alpha \in I} Y_\alpha$$

is a locally finite union of closed PL subspaces of a PL manifold $X$. Set

$$Y_{(\alpha_0, \ldots, \alpha_k)} := Y_{\alpha_0} \cap Y_{\alpha_1} \cap \cdots \cap Y_{\alpha_k}$$

and

$$\mathcal{Y}_k = \prod_{\alpha_0 < \alpha_1 < \cdots < \alpha_k} Y_{(\alpha_0, \ldots, \alpha_k)}.$$

The inclusions

$$d_j : Y_{(\alpha_0, \ldots, \alpha_k)} \hookrightarrow Y_{(\alpha_0, \ldots, \alpha_j, \ldots, \alpha_k)}$$

induce face maps

$$d_j : \mathcal{Y}_k \to \mathcal{Y}_{k-1} \quad j = 0, \ldots, k$$

With these, $\mathcal{Y}_\bullet$ is a strict simplicial space.
Proposition 7. There is a spectral sequence
\[ E_1^{s,t} \cong H_c^s(Y_\bullet; A) \implies H_c^{s+t}(Y; A) \]
whose \( E_1 \) differential
\[ d_1 : H_c^s(Y_\bullet; A) \to H_c^{s+1}(Y_{s+1}; A) \]
is \( \sum_j (-1)^j d_j^s \).

Proof. This follows rather directly from the standard fact that the natural “augmentation”
\[ \epsilon : |Y_\bullet| \to Y \]
from the geometric realization\(^6\) is a homotopy equivalence. Since \( Y \) is a locally finite union of the closed subspaces \( Y_\alpha \), the natural mapping
\[ S_\bullet^c(Y; A) \to S_\bullet^c(Y_\bullet; A) \]
is a quasi-isomorphism. The spectral sequence is that of the double complex \( S_\bullet^c(Y_\bullet; A) \). The quasi-isomorphism implies that the spectral sequence abuts to \( H_\bullet^c(Y; A) \). \( \square \)

Suppose now that each \( Y_\alpha \) is an oriented PL submanifold of dimension 2\( d \) and codimension 2 in \( X \). In addition, suppose that the components of \( Y_\alpha \) of \( Y \) intersect transversally in \( X \), so that each component of \( Y_s \) has dimension \( 2d - 2s \). By duality, the \( E_1 \) term of the spectral sequence can we written as
\[ E_1^{s,t} = H_{2d-2s-t}(Y_s; A) \]
The differential
\[ d_1 : H_{2d-2s-t}(Y_s; A) \to H_{2d-2s-t-2}(Y_{s+1}; A) \]
is the alternating sum of the Gysin mappings \( d_j^s : Y_{s+1} \to Y_s \).

Proposition 8. There is a natural isomorphism \( H_c^{2d}(Y; A) \cong H_0(Y_0; A) \) and an exact sequence
\[ H_c^{2d-2}(Y; A) \to H_2(Y_0; A) \to H_0(Y_1; A) \to H_c^{2d-1}(Y; A) \to H_1(Y_0; A) \to 0 \]

3.3. Cohomology at infinity. For a topological space \( X \) and an \( R \)-module (or local coefficient system of \( R \)-modules) \( A \), set
\[ H_\infty^\bullet(X; A) = \lim_{K \subseteq X \text{ compact}} H^\bullet(X - K; A). \]
We shall call it the cohomology at infinity of \( X \). When \( X \) is a compact manifold with boundary, then there is a natural isomorphism
\[ H_\infty^\bullet(X - \partial X; A) \cong H^\bullet(\partial X; A). \]

Remark 9. I am not sure if this definition appears in the literature, although I would be surprised if it does not. Similar ideas have long appeared in topology, such as in the paper Bestvina and Feighn \([3]\), where they introduce the notion of a space being “\( r \)-connected at infinity.”

\(^6\)This is the quotient of \( \prod_{k \geq 0} Y_k \times \Delta^k \) by the natural equivalence relation generated by \((y, \partial_j \xi) \equiv (d_j y, \xi)\), where \((y, \xi) \in Y_k \times \Delta^{k-1}\), where \( \Delta^d \) denotes the standard \( d \)-simplex, and \( \partial_j : \Delta^{k-1} \to \Delta^k \) is the inclusion of the \( j \)th face.
The direct limit of the long exact sequence of the pairs \((X, X - K)\), where \(K\) is compact, is a long exact sequence

\[
\cdots \to H^{k-1}_\infty(X; A) \to H^k_c(X; A) \to H^k(X; A) \to H^k_\infty(X; A) \to \cdots
\]

When \(X\) is an oriented manifold of dimension \(m\), Poincaré duality gives an isomorphism

\[H_{m-k}(X; A) \to H^k(X; A)\]

Thus, if \(X\) has the homotopy type of a CW-complex of dimension \(d\), then

\[H^k_c(X; A) = 0 \text{ when } k < m - d \quad \text{and} \quad H^k(X; A) = 0 \text{ when } k > d.\]

Plugging these into the long exact sequence (1), we see that

\[H^k_\infty(X; A) \cong H^k(X; A) \text{ when } k < m - d - 1\]

and

\[H^k_\infty(X; A) \cong H^{k+1}_c(X; A) \text{ when } k > d.\]

Moreover, since

\[H^k(X; R) \to \text{Hom}_R \left( H^{m-k}_c(X; R), R \right) \]

is an isomorphism, we have an isomorphism

\[H^k_\infty(X; R) \to \text{Hom}_R \left( H^{m-k-1}_\infty(X; R), R \right) \]

when \(k < m - d - 1\).

One can consider vanishing of \(H_*^\infty(X)\) modulo the Serre class of finitely generated \(R\)-modules. This leads us to the following bizarre result.

**Lemma 10.** Suppose that \(R\) is a PID. If \(H^k_\infty(X; R)\) is a countably generated \(R\)-module when \(a \leq k \leq b\), then \(H^k(X; R)\) is a finitely generated \(R\)-module when \(a \leq k \leq b\).

**Proof.** Since \(X\) has the homotopy type of a countable CW-complex, each \(H^k(X; R)\) is countably generated. Poincaré duality then implies that \(H^k_c(X; R)\) is also countably generated. The Universal Coefficient Theorem implies that \(H^k(X; R)\) surjects onto \(\text{Hom}_R(H^k(X; R), R)\). Consequently, if \(H^k(X; R)\) is countably generated, it is finitely generated. The result now follows from the exact sequence (1). \(\square\)

4. Discussion and Problems

One natural approach to the problem of understanding the topology of \(T_g\) is to view it as the complement of the normal crossings divisor \(T^c_{g, \text{red}}\) in \(T^c_g\). The space \(T^c_g\) can in turn be studied via the period mapping \(T^c_g \to J^c_g\). There are two ways to factorize this. The first is to take the quotient \(Q^c_g := T^c_g / \langle \sigma \rangle\) of \(T^c_g\) by the involution

\[\sigma : [C; a_1, \ldots, a_g, b_1, \ldots, b_g] \to [C; -a_1, \ldots, -a_g, -b_1, \ldots, -b_g].\]

The mapping \(T^c_g \to Q^c_g\) is branched along the hyperelliptic locus \(H^c_g\). The second is to consider the Stein factorization (cf. [3] p. 213) \(T^c_g \to S^c_g \to J^c_g\).
of the period mapping. The important properties of this are that $S^c_g$ is a complex analytic variety, the first mapping has connected fibers, while the second is finite (in the sense of analytic geometry). The two factorizations are related by the diagram

$$
\begin{array}{c}
\mathcal{T}^c_g \\
\downarrow \text{period map} \\
\mathcal{S}^c_g \\
\downarrow \\
\mathcal{Q}_g \\
\downarrow \\
\mathcal{J}^c_g
\end{array}
$$

where all spaces are complex analytic varieties, all mappings are proper, holomorphic and surjective. The horizontal mappings have connected fibers and the vertical mappings are finite and two-to-one except along the hyperelliptic locus.

Perhaps the first natural problem is to understand the topology of $\mathcal{J}^c_g$.

**Problem 1 (Topological Schottky Problem).** Understand the homotopy type of $\mathcal{J}^c_g$ and use it to compute $H^\bullet(\mathcal{J}^c_g)$ and $H^\bullet_{\sigma}(\mathcal{J}^c_g)$.\footnote{When $g = 3$, $\mathcal{Q}_3$ is obtained from $\mathcal{h}_3$ by first blowing up the singular locus of $\mathcal{h}^\text{red}_3$, which is smooth of complex codimension 3, and then blowing up the proper transforms of the components of $\mathcal{h}^\text{red}_3$. The strata of $\mathcal{h}^\text{red}_3$ are described in detail in \cite{[8]}.}

The first interesting case is when $g = 4$, where $\mathcal{J}^c_g$ is a singular divisor in $\mathcal{h}_4$. It would also be interesting and natural to compute the intersection homology of $\mathcal{J}^c_g$.

A knowledge of the topology of $\mathcal{J}_g$, $\mathcal{J}^c_g$ or $\mathcal{Q}^c_g$ should help with the computation of the $\sigma$-invariant homology $H^\bullet_{\sigma}(\mathcal{J}_g)$.\footnote{If one uses intersection homology instead, then duality will still be available. For this reason it is natural to try to compute the intersection homology of $\mathcal{S}^c_g$ and $\mathcal{J}^c_g$.}

When 2 is invertible in the coefficient ring $R$, we can write

$$H^\bullet(\mathcal{T}_g; R) = H^\bullet(\mathcal{T}_g; R)^+ \oplus H^\bullet(\mathcal{T}_g; R)^-$$

where $\sigma$ acts as the identity on $H^\bullet(\mathcal{T}_g; R)^+$ and as $-1$ on $H^\bullet(\mathcal{T}_g; R)^-$.\footnote{When $g = 3$, $\mathcal{Q}^c_3$ is obtained from $\mathcal{h}_3$ by first blowing up the singular locus of $\mathcal{h}^\text{red}_3$, which is smooth of complex codimension 3, and then blowing up the proper transforms of the components of $\mathcal{h}^\text{red}_3$. The strata of $\mathcal{h}^\text{red}_3$ are described in detail in \cite{[8]}.}

**Problem 2.** Determine whether or not $H^\bullet(\mathcal{T}_g; \mathbb{Z}[1/2])^-$ is always a finitely generated $\mathbb{Z}[1/2]$-module. Does the infinite topology of $\mathcal{T}_g$ come from $\mathcal{J}_g$?

To get one’s hands on $H^\bullet_{\sigma}(\mathcal{T}_g)$, it is necessary to better understand the topology of $\mathcal{T}^c_g$ or $\mathcal{S}^c_g$. Since $\mathcal{J}^c_g$ is a Stein space, so is $\mathcal{S}^c_g$. Hamm’s result \cite{[9]} (see also \cite{[5], p. 152}) implies that $\mathcal{S}^c_g$ has the homotopy type of a CW-complex of dimension at most $3g - 3$. Since $\mathcal{S}^c_g$ is not a rational homology manifold when $g \geq 3$, it is probably most useful to study the topology of the manifold $\mathcal{T}^c_g$ as Poincaré duality will then be available.

**Problem 3.** Determine good bounds for the homological dimension (or the CW-dimension) of $\mathcal{T}^c_g$.

The best upper bound that I know of is obtained using Stratified Morse Theory:

**Proposition 11.** If $g \geq 1$ and $2g - 2 + n > 0$, then the dimension of $\mathcal{T}^c_{g,n}$ as a CW-complex satisfies

$$2(g - 2 + n) \leq \dim_{\text{CW}} \mathcal{T}^c_{g,n} \leq \left\lfloor \frac{7g - 8}{2} \right\rfloor + 2n.$$
Proof. Suppose that $g \geq 2$. Since $T_{g,n}^c \to T_g^c$ is proper with fibers of complex dimension $n$, $$\dim_{\text{CW}} T_{g,n}^c = 2n + \dim_{\text{CW}} T_g^c.$$ To establish the lower bound, we need to exhibit a topological $(2g - 4)$-cycle in $T_g^c$ that is non-trivial in $H_\ast(T_g^c)$. Observe that $$Y := (T_{2,1}^c)^2 \times (T_{1,2}^c)^{g-2}$$ is a component of the closure of the locus of chains of $g$ elliptic curves in $T_g^c$. Let $E$ be any elliptic curve. Then $E \subset T_{1,2}$ and $Y$ therefore contains the projective subvariety $E^{g-2}$. The rational homology class of a closed subvariety of a Kähler manifold is always non-trivial (just integrate the appropriate power of the Kähler form over it). Since $T_g^c$ covers $\mathcal{M}_g^c$, which is a Kähler orbifold, $T_g^c$ is a Kähler manifold and the class of $E^{g-2}$ is non-trivial in $H_\ast(T_g^c)$. This establishes the lower bound.

The upper bound is a direct application of Stratified Morse Theory [5]. Since $\mathcal{J}_g^c$ is a Stein space, it is a closed analytic subvariety of $\mathbb{C}^N$ for some $N$. The constant $c$ in [5 Thm. 1.1*, p. 152] is thus 1. Consequently, $\dim_{\text{CW}} T_{g,n}$ is bounded by $$d(g, n) := \sup_{k \geq 0} (2k + f(k))$$ where $f(k)$ is the maximal complex dimension of a subvariety of $T_{g,n}^c$ over which the fiber of $T_{g,n}^c \to \mathcal{J}_g^c$ is $k$ dimensional. Observe that $d(g, n) = d(g) + 2n$, when $g \geq 2$, where $d(g) = d(g, 0)$. Since $T_{1,1}^c \to \mathfrak{h}_1$ is proper with fibers of dimension $n - 1$, we have $d(1, n) = 1 + 2(n - 2) = 2n - 1$.

Since the mapping $T_g \to \mathcal{J}_g$ is finite and since the components of $T_{g,\text{red}}^c \to \mathcal{J}_{g,\text{red}}^c$ are the $\text{Sp}_g(\mathbb{Z})$ orbits of the period mappings $$T_{g,1}^c \times T_{g-h,1}^c \to \mathcal{J}_h^c \times \mathcal{J}_{g-h}^c$$ we have $$d(g) = \max \left\{ 3g - 3, \max \left\{ d(h, 1) + d(g - h, 1) : 1 \leq h \leq g/2 \right\} \right\}.$$ The formula for the upper bound $d(g)$ is now easily proved by induction on $g$. \qed

The upper bound is not sharp when $g \leq 2$; for example, $T_{1,1}^c = \mathfrak{h}_1$ and $T_2^c = \mathfrak{h}_2$, which are contractible. I suspect it is not sharp in higher genus as well. The first interesting case is to determine the CW-dimension of $T_3^c$.

The upper bound on the homological dimension of $T_g^c$ implies that $$H_k(T_g^c) \cong H_{\infty}^{6g - 7 - k}(T_g^c) \text{ when } k < \left\lfloor \frac{5g - 6}{2} \right\rfloor.$$ So the low dimensional homology of $T_g^c$ is related to the topology at infinity of $T_g^c$.

**Problem 4.** Try to understand the “topology at infinity” of $T_g^c$. In particular, try to compute $H_k(T_g^c)$ for $k$ in some range $k \geq d_o$. Alternatively, try to compute the homology $H_\ast(T_g^c)$ in lower degrees.

Note that $T_2^c$ is a manifold with boundary $S^5$. The boundary of $\mathcal{J}_3^c$ is $S^{11}$. The first interesting case is in genus 3.

**Problem 5.** Compute $H_\ast(T_3^c)$. 
The homology of $\mathcal{T}_g^c$ is related to that of $\mathcal{T}_g$ via the Gysin sequence. In order to apply it, one needs to understand the topology of the divisor $\mathcal{T}_g^{c,\text{red}}$. This is built up out of products of lower genus Torelli spaces of compact type. The combinatorics of the divisor is given by the complex $K^{\text{sep}}(S)/T_g$.

**Problem 6.** Compute the $\text{Sp}_g(\mathbb{Z})$-module $H^k_c(\mathcal{T}_g^{c,\text{red}})$ in some range $k \geq k_0$.

We already know that $H^{6g-7}_c(\mathcal{T}_g^{c,\text{red}}) = H_0(B^c_\theta)$ and that there is a surjection $H^{6g-8}_c(\mathcal{T}_g^{c,\text{red}}) \to H_1(B^c_\theta)$, where $B^c_\theta$ denotes the normalization (i.e., disjoint union of the irreducible components) of $\mathcal{T}_g^{c,\text{red}}$. In concrete terms:

$$B^c_\theta = \prod_{h=1}^{[g/2]} \bigoplus_{\phi \in \text{Sp}_g(\mathbb{Z})} \phi(T^c_{h,1} \times T^c_{g-h,1}).$$

Lurking in the background is the folk conjecture $H_k(\mathcal{T}_g)$ is finitely generated when $k < g - 1$.

If true, this places strong conditions on the finiteness of the topology of $\mathcal{T}_g^c$ and $\mathcal{T}_g^{c,\text{red}}$. It is worthwhile to contemplate (for $g \geq 3$) the Gysin sequence:

$$\cdots \to H^{6g-10}_c(\mathcal{T}_g^{c,\text{red}}) \to H_3(\mathcal{T}_g) \to H_3(\mathcal{T}_g^c) \to H^{6g-10}_c(\mathcal{T}_g^c) \to H^{6g-9}_c(\mathcal{T}_g^{c,\text{red}}) \to H_2(\mathcal{T}_g) \to H_2(\mathcal{T}_g^c) \to H^{6g-9}_c(\mathcal{T}_g^c) \to H^{6g-8}_c(\mathcal{T}_g^{c,\text{red}}) \to H_1(\mathcal{T}_g) \to H_1(\mathcal{T}_g^c) \to 0$$

\[ H_0(B^c_\theta) \cong (\Lambda^3 H)/H \]

Here $\tau$ denotes the Johnson homomorphism, realized as the map on $H_1$ induced by the inclusion $\mathcal{T}_g \hookrightarrow \mathcal{T}_g^c$.

Finally, it is interesting to study the topology of the branching locus of $\mathcal{T}_g^c \to \mathcal{J}_g^c$. This is the locus $H^c_\theta$ of hyperelliptic curves of compact type. Using A'Campo’s result $\blacksquare$ that the image of the hyperelliptic mapping class group $^9$ $\Delta_g$ in $\text{Sp}_g(\mathbb{Z})$ contains the level two subgroup $\text{Sp}_g(\mathbb{Z})[2] = \{ A \in \text{Sp}_g(\mathbb{Z})[2] : A \equiv I \mod 2 \}$

$^9$The hyperelliptic mapping class is the centralizer of a hyperelliptic involution in $\Gamma_g$. It is the orbifold fundamental group of the moduli space of smooth hyperelliptic curves of genus $g$. 

of $\text{Sp}_g(\mathbb{Z})$ and the fact that the image $\Delta_g$ in $\text{Sp}_g(\mathbb{F}_2)$ is $S_{2g-2}$, the symmetric group on the Weierstrass points, one can see that $\mathcal{H}_g^c$ has

$$|\text{Sp}_g(\mathbb{F}_2)|/|S_{2g+2}| = \frac{2g^2 \prod_{k=1}^g (2^{2k} - 1)}{(2g + 2)!}$$

components. Each component of $\mathcal{H}_g^c$ is smooth and immerses in $\frak{h}_g$ via the period mapping. The irreducible components of $\mathcal{H}_g$ are disjoint in $\mathcal{J}_g$. In genus 3, $\mathcal{H}_3^c$ has 36 components. Their images are cut out by the 36 even theta nulls $\vartheta_\alpha: \frak{h}_3 \to \mathbb{C}$.

**Conjecture 1.** Each component of $\mathcal{H}_g^c$ is simply connected.

This is trivially true in genus 2, where there is one component which is all of $\frak{h}_2$. If true in genus 3, it implies quite directly the known fact that $T_3$ is generated by $35 = 36 - 1$ bounding pair elements. The number 35 is the rank of $H_1(\mathcal{J}_3, \mathcal{H}_3)$. The inverse images of generators of $\pi_1(\mathcal{J}_3, \mathcal{H}_3)$, once oriented, generate $T_3$.

The conjecture has an equivalent statement in more group theoretic terms. Define the hyperelliptic Torelli group to be the intersection of the hyperelliptic mapping class group and the Torelli group:

$$T\Delta_g := \Delta_g \cap T_g = \ker\{\Delta_g \to \text{Sp}_g(\mathbb{Z})\}.$$ 

It is a subgroup of the Johnson subgroup $K_g := \ker\{\tau : T_g \to \Lambda^3 H/H\}$. Examples of elements in $T\Delta_g$ are Dehn twists on separating simple closed curves that are invariant under the hyperelliptic involution $\sigma$. If $\mathcal{H}_{g,\alpha}$ is a component of $\mathcal{H}_g$, then there is an isomorphism

$$\pi_1(\mathcal{H}_{g,\alpha}, *) \cong T\Delta_g.$$ 

The conjugacy classes of twists on a $\sigma$-invariant separating SCC correspond to loops about components of the divisor $\mathcal{H}_g^{c,\text{red}}$. Van Kampen’s Theorem implies that

$$\pi_1(\mathcal{H}_{g,\alpha}, *) \cong \pi_1(\mathcal{H}_{g,\alpha})/\text{these conjugacy classes} \cong T\Delta_g/\text{these conjugacy classes}.$$ 

The conjecture is thus equivalent to the statement that $T\Delta_g$ is generated by the conjugacy classes of Dehn twists on $\sigma$-invariant separating SCCs.

It is important to understand the topology of the loci of hyperelliptic curves as it is the branch locus of the period mapping and also because it is important in its own right.

**Problem 7.** Investigate the topology of $\mathcal{H}_g$ and $\mathcal{H}_g^c$ and their components. Specifically, compute their homology and the cohomology at infinity of $\mathcal{H}_{g,\alpha}$.

The period mapping immerses each $\mathcal{H}_{g,\alpha}$ in $\frak{h}_g$ as a closed subvariety. Consequently, each $\mathcal{H}_{g,\alpha}$ is a Stein manifold and thus has the homotopy type of a CW-complex of dimension equal to its complex dimension, which is $2g - 1$.

\(^{10}\)As the genus increases, the number of components of $\mathcal{H}_g$ increases exponentially while the minimum number of generators of $T_g$ increases polynomially. The failure of the genus 3 argument presented above in higher genus suggests that $\mathcal{J}_g^c$ is not, in general, simply connected.
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