The singular support of the Ising model

Jethro van Ekeren * Reimundo Heluani †

Abstract

We propose a new Fermionic quasiparticle sum expression for the character of the Ising model vertex algebra, related to the Jackson–Slater $q$-series identity of Rogers–Ramanujan type and to Nahm sums for the matrix $\left( \begin{array}{c} 8 \\ 3 \end{array} \right)$. We find, as consequences, an explicit monomial basis for the Ising model, and a description of its singular support. We find that the ideal sheaf of the latter, defining it as a subscheme of the arc space of its associated scheme, is finitely generated as a differential ideal.

1 Introduction

1.1. In [1] Li introduced a canonical decreasing filtration $\{F_p V\}$ on an arbitrary vertex algebra $V$. The associated graded $gr_F V$ of $V$ with respect to this filtration carries the natural structure of a $\mathbb{Z}_{\geq 0}$-graded Poisson vertex algebra and its spectrum is called the singular support of $V$. It is known that $gr_F V$ is generated as a differential algebra by $R_V = gr_0^F V$. By differential algebra we mean a commutative algebra with a derivation as defined in [2]. $R_V$ is a Poisson algebra, and in fact is just the Zhu $\mathbb{C}_2$-algebra of $V$. The spectrum $X_V = \text{Spec}(R_V)$ is a Poisson scheme known as the associated scheme of $V$.

The arc algebra $(JR, \partial)$ of a commutative algebra $R$ is the $\mathbb{Z}_{\geq 0}$-graded differential algebra freely generated by $R$. If $R$ is a Poisson algebra, $JR$ acquires the natural structure of a Poisson vertex algebra. One thus has in general a surjection

$$\pi : JR_V \rightarrow gr_F V \quad (1.1.1)$$

of Poisson vertex algebras [3]. We say that a vertex algebra $V$ is classically free if $\pi$ is an isomorphism, reflecting the fact that $gr_F V$, which is to be thought of as the classical limit of $V$, is freely generated as a Poisson vertex algebra.

1.2. There are several classes of examples of classically free vertex algebras. Universal enveloping vertex algebras of (linear and non-linear) Lie conformal algebras are classically free. Thus the universal affine $\mathbb{C}_2$-algebra $k$ with ideal sheaf given by $\ker (\pi)$ is nonzero, and the singular support of the Ising model

$$\text{is classically free if and only if } p = 2.$$ 

Other examples of rational, $C_2$-cofinite, classically free vertex algebras include the simple affine vertex algebra $V_k(\mathfrak{sl}_2)$ at positive integer level $k$ [4]. It is expected that $V_k(\mathfrak{g})$ is classically free for all simple $\mathfrak{g}$ and $k \in \mathbb{Z}_{\geq 0}$. The simple vertex algebra $V_k(\mathfrak{sl}_2)$ at admissible level $k = -2 + \frac{p'}{p'}$ (with $p'_0 \pm 2$ coprime) is known not to be classically free whenever $p'_0 > p > 2$, and is expected to be classically free for the boundary admissible levels $k = -2 + \frac{p'}{p'}$. Arakawa and Linshaw have provided another example of a non classically free vertex algebra in [5].

1.3. If $V$ is not classically free then the differential ideal $\ker(\pi) \subset gr_F V$ is nonzero, and the singular support of $V$ is a closed proper subscheme of the arc space of $X_V$ with ideal sheaf given by $\ker(\pi)$.

*Instituto de Matemática e Estatística (GMA), UFF, Niterói RJ, Brazil.  
jethrovanekeren@gmail.com

†Instituto de Matemática Pura e Aplicada, Rio de Janeiro, RJ, Brazil  
heluani@potuz.net
was asked in \([9]\), for the case of \(V = W_3\), whether ker \(\pi\) is finitely generated as a differential ideal. The question of finite generation of ker \(\pi\) also has an important application in bounding dimensions of chiral homology groups of elliptic curves \([4]\). In this note we study the structure of the singular support of the Virasoro minimal model Vir_{3,4}, also known as the Ising model.

1.4. The singular support of a conformal vertex algebra (i.e., of a VOA) acquires a \(\mathbb{Z}_{\geq 0}\)-grading induced by conformal weight, since the latter grading is compatible with the Li filtration. We write \(\chi_V(q)\) for the graded dimension \(\sum_{n \in \mathbb{Z}_{\geq 0}} \dim(V_n)q^n\) or character of \(V\), and similarly \(\chi_{\text{gr}}F V(q)\) for the graded dimension of \(\text{gr}_F V\). One has on general grounds that \(\chi_{\text{gr}}F V(q) = \chi_V(q)\). By taking account of the canonical grading of \(\text{gr}_F V\) as well, we can introduce a two-variable graded dimension

\[
\chi_{\text{gr}}F V(t, q) = \sum_{m, n \in \mathbb{Z}_{\geq 0}} \dim(\text{gr}_F^n V_m) t^m q^n, \tag{1.4.1}
\]

which we may think of as a refinement of the character.

There are many well known formulas for the character of Vir_{3,4}, for example:

\[
\chi_{\text{Vir}_{3,4}}(q) = \prod_{n=1}^{\infty} \frac{1}{1 - q^n} \sum_{m \in \mathbb{Z}} \left( q^{12m^2+m} - q^{12m^2+7m+1} \right) = \frac{1}{2} \left( \prod_{m=1}^{\infty} \left( 1 + q^{m-1/2} \right) + \prod_{m=1}^{\infty} \left( 1 - q^{m-1/2} \right) \right) \tag{1.4.2}
\]

The first of these expressions is the \((p, p') = (3, 4)\) case of the Feigin-Fuchs character formula for the \((p, p')\) Virasoro minimal model \([6]\) (see Section 2.1 below). It derives from the BGG-type resolution of modules over the Virasoro Lie algebra. The second expression is directly implied by an isomorphism between \(\text{Vir}_{3,4}\) and the even subalgebra \(F_+\) of the neutral free fermion vertex superalgebra \(F\). The third expression is obtained from the second via a classical identity of Euler \([7]\) Corollary 2.2. The equality between the first and the fourth lines is known as the Jackson-Slater identity: it appeared as identity (39) in Slater’s famous list \([9]\), where it was proved by an application of the method of Bailey pairs, and it had appeared earlier in a disguised form in \([10]\).

These \(q\)-series identities have combinatorial interpretations. For instance the fourth expression of \([11, 12]\) is the generating function \(\sum_{n=0}^\infty a_n q^n\) for the \(a_n\) the number of partitions of \(n\) into parts congruent to \(\pm 2, \pm 3, \pm 4\) and \(\pm 5\) modulo 16. An interpretation of the third expression of \((1.4.4)\) was given by Hirshhorn as the generating function for the number of partitions \([\lambda_1, \ldots, \lambda_m]\) of \(n\) satisfying the difference conditions \([11]\)

\[
\lambda_m \geq 2, \quad \lambda_{m-1} - \lambda_m \geq 0, \quad \lambda_{m-2} - \lambda_{m-1} \geq 2, \quad \lambda_{m-3} - \lambda_{m-2} \geq 0, \ldots.
\]

Further combinatorial interpretations have been given by Subbarao \([12]\) and Ribeiro \([13]\) (see also \([14]\)). The following striking formula was conjectured in \([15]\) and proved in \([16]\).

\[
\chi_{\text{Vir}_{3,4}}(q) = \sum_{k=(k_1, k_2, \ldots, k_s) \in \mathbb{Z}_{\geq 0}^s} \frac{q^{k^T C_{E_8}^{-1} k}}{(q)_{k_1} \cdots (q)_{k_s}}, \tag{1.4.3}
\]

Here \(C_{E_8}\) is the Cartan matrix of the simple Lie algebra \(E_8\). In general \((q)_n\) denotes the \(q\)-Pochhammer symbol \((q)_n = (1-q) \cdots (1-q^n)\). Sums, like \((1.4.3)\), of the general form

\[
\sum_{k=(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}^n} \frac{q^{k^T A k + k^T B + C}}{(q)_{k_1} \cdots (q)_{k_n}}, \tag{1.4.4}
\]
where $A$ is symmetric $n \times n$ positive definite matrix over $\mathbb{Q}$, $B \in \mathbb{Q}^n$ and $C \in \mathbb{Q}$, are sometimes referred to as Nahm sums $[17]$. Sums, like the third line of (1.3.2), of the form (1.3.4) but taken over $k \in \mathbb{Z}_{>0}$ satisfying congruence conditions, as well as linear combinations of such sums, are sometimes referred to as (Fermionic) quasiparticle sums in the physics literature. In this article we propose a new Fermionic quasiparticle formula for $\chi_{\text{Vir}_{3,4}}(q)$.

1.5 Conjecture. 

$$
\chi_{\text{Vir}_{3,4}}(q) = \sum_{(k_1,k_2) \in \mathbb{Z}_{>0}^2} \frac{q^{4k_1^2+3k_1k_2+k_2^2}}{(q)_{k_1}(q)_{k_2}} (1 - q^{k_1} + q^{k_1+k_2}). 
$$ (1.5.1)

We have verified this conjecture to order $q^{2000}$ with SageMath $[18]$.

1.6. In this note we prove several structural results about the singular support of the Ising model $\text{Vir}_{3,4}$. These imply that, if Conjecture [13] is true, then $\text{Vir}_{3,4}$ is the first example of a non-classically free vertex algebra for which $\ker \pi$ is finitely generated as a differential ideal.

Let $V = \text{Vir}_{3,4}$ be the Ising model. Its central charge is $c = \frac{1}{2}$. The Zhu $C_2$-algebra of $V$ is $R_V = \mathbb{C}[L]/(L^3)$. The arc algebra $JR_V$ is therefore the quotient of the polynomial algebra 

$$
JC[L_{-2}] = \mathbb{C}[L_{-2},L_{-3},\ldots],
$$

with its $\mathbb{Z}_{>0}$-grading defined by $\deg(L_{-n}) = n$, by the differential ideal $(L^3_{-2})_\partial$. Here the derivation $\partial$ is defined by $\partial(L_{-n}) = (n-1)L_{-n-1}$, and by differential ideal generated by a set $\{f_1,\ldots,f_n\}$ of elements we mean the ideal generated by all derivatives $\partial^k f_i$. In this note we prove the following

Theorem 1. Let $V = \text{Vir}_{3,4}$ be the Ising model vertex algebra. Provided Conjecture [13] holds, we have an isomorphism of Poisson vertex algebras

$$
JR_V/(b)_3 \sim \text{gr}_F V \quad \text{where} \quad b = \frac{1}{6}L_{-3}L_{-2}^2 + L_{-4}L_{-3}L_{-2}.
$$ (1.6.1)

1.7. In fact we find a surjective morphism (1.6.1) for $V = \text{Vir}_{3,p}$ in general, where now $b$ is an explicit element of degree $2p' + 1$. However for $p' \geq 5$ this morphism is not an isomorphism. For $p' = 5$ for instance the morphism fails to be injective in degrees $\geq 19$. See Section 2.

1.8. To explain the origin of (1.5.1) we return to the morphism of Theorem 1 or more precisely to the corresponding bigraded surjection

$$
JC[L_{-2}]/(a,b)_3 \sim \text{gr}_F V,
$$ (1.8.1)

where $a = L_{-3}^2$ and $b$ is as in (1.6.1). The graded algebra $JC[L_{-2}]$ has a basis consisting of monomials

$$
L_\lambda = L_{-\lambda_1}L_{-\lambda_2}\ldots L_{-\lambda_m}
$$

parametrized by partitions $[\lambda_1,\ldots,\lambda_m]$ into parts $\lambda_1 \geq \ldots \geq \lambda_m \geq 2$. To compute the graded dimension of the quotient of $JC[L_{-2}]$ by $I = (a,b)_3$, or equivalently the Hilbert series of $I$, it suffices to choose a monomial ordering in $JC[L_{-2}]$ and compute the Hilbert series of the leading term ideal $\text{LT}(I)$ of $I$. The task is then reduced to enumerating partitions which satisfy certain generalized difference conditions. For instance the partition corresponding to any monomial multiple of $a$ contains $[2,2,2]$ and so this partition is to be excluded. Similarly the leading term (if we adopt the grevlex monomial order) of a multiple of $\partial^{2(2-p')}a$ contains $[p,p,p]$, etc. See Section 2 for details.

Here and below we say that a partition $\lambda = [\lambda_1,\ldots,\lambda_m]$ contains another partition $\mu = [\mu_1,\ldots,\mu_k]$ if the entries of $\mu$, counted with multiplicity, are contained among the entries of $\lambda$, i.e., if $[\mu_1,\ldots,\mu_k] \subseteq [\lambda_1,\ldots,\lambda_m]$ in the sense of multisets. We say that $\lambda$ avoids $\mu$ if $\lambda$ does not contain $\mu$. 

3
Theorem 2. Let $\mathcal{I}(n)$, respectively $\mathcal{P}(n)$, be the set of partitions $\lambda = [\lambda_1, \ldots, \lambda_m]$ of $n$ such that $\lambda_m \geq 2$ and $\lambda$ contains, respectively avoids, the following partitions

$$
\begin{align*}
[p, p, p], & \quad [p + 1, p, p], & \quad [p + 1, p + 1, p], \\
[p + 2, p + 1, p], & \quad [p + 2, p + 2, p], \\
[p + 2, p, p], & \quad [p + 3, p + 3, p, p], & \quad [p + 4, p + 3, p, p], & \quad [p + 4, p + 4, p + 1, p], \\
[p + 4, p + 3, p + 1, p], & \quad [p + 4, p + 4, p + 1, p], & \quad [p + 6, p + 5, p + 3, p + 1, p], & \quad [5, 4, 2, 2], \\
[7, 7, 4, 2, 2], & \quad [9, 6, 4, 2, 2].
\end{align*}
$$

Then

a) if $I = (a, b) \subset JC[L_{-2}]$ as in (1.8.1) then for all $\lambda \in \mathcal{I}(n)$ we have $L_{\lambda} \in LT(I)$, and

b) if we denote by $p(n, m)$ the number of partitions in $\mathcal{P}(n)$ into exactly $m$ parts, then the generating function of these cardinalities is given by the quasiparticle sum

$$
\sum_{m, n \in \mathbb{Z}_{\geq 0}} p(n, m) t^m q^n = \sum_{(k_1, k_2) \in \mathbb{Z}_{\geq 0}^2} t^{2k_1+k_2} q^{4k_1^2+3k_1k_2+k_2^2} (1 - q^{k_1} + q^{k_1+k_2}).
$$

1.9. Parts (a) and (b) of Theorem 2 occupy Sections 4 and 3, respectively. As a corollary of Conjecture 1.5 we deduce the following result on the structure of $\text{gr}_F \text{Vir}_{3,4}$. In particular the bigraded dimension $\chi_{\text{gr}_F \text{Vir}_{3,4}}(t, q)$ is obtained naturally from (1.5.1). We remark that, by contrast, there seems to be no natural way to introduce $t$ into any of the formulas (1.4.2) and (1.4.3) for $\chi_{\text{Vir}_{3,4}}(q)$ in such a way as to obtain $\chi_{\text{gr}_F \text{Vir}_{3,4}}(t, q)$.

Theorem 3. Let $V = \text{Vir}_{3,4}$. If Conjecture 1.5 holds, then

a) for all $n \geq 1$ the set $\{L_{\lambda}\}_{\lambda \in \mathcal{P}(n)}$ is a basis of $V_n$,

b) the kernel of $\pi : JR_V \rightarrow \text{gr}_F V$ is the differential ideal generated by the element $b = \frac{1}{6} L_{-5}L_{-2}^2 + L_{-4}L_{-3}L_{-2}$.

c) the bigraded character of the singular support of $V$ is given by

$$
\chi_{\text{gr}_F V}(t, q) = P(t^{-2}, tq)
$$

where $P(t, q)$ is the generating function (1.8.3).
1.11. The normalized character \( q^{-1/48} \chi_{\text{Vir}_{3,4}}(q) \) is a modular function. In \cite{21} Nahm conjectured (roughly speaking) that for any matrix \( A \) for which the sum \((1.4.4)\) has appropriate asymptotic behaviour there exists some choice of \( B \) and \( C \) for which the sum is modular. (See Section 4.4 below.) The matrix \( A = (\frac{3}{2} \frac{1}{2}) \) which figures in \((1.5.3)\) above, has appeared in Terhoeven’s list of matrices \( A \) for which \((1.4.4)\) displays the correct asymptotics \cite{22}. But until now, despite computer searches \cite{23, 3.B.d)}], no modular Nahm sum associated with the matrix had been found. Our example shows that modularity can be achieved by considering more general quasiparticle sums.

1.12. The same techniques developed in this article can be used for other vertex algebras or even vertex algebra modules to yield new \( q \)-series and partition identities. For example, the Ising model \( \text{Vir}_{3,4} \) has three irreducible modules, \( V_0 = V \), \( V_{1/2} \) and \( V_{1/16} \) where the subscript labels the degree of the unique singular vector. The (unnormalized) characters of \( V_{1/2} \) and \( V_{1/16} \) are given by

\[
\chi_{V_{1/2}}(q) = \frac{1}{2} \left( \prod_{m=1}^{\infty} (1 + q^{m-1/2}) - \prod_{m=1}^{\infty} (1 - q^{m-1/2}) \right) = q^{1/2} \sum_{k \geq 1} \frac{q^{2k^2-2k}}{(q)^{2k-1}},
\]

\[
\chi_{V_{1/16}}(q) = \prod_{m=1}^{\infty} (1 + q^m) = \sum_{k \geq 0} \frac{k(k+1)}{(q)_k}.
\]

In addition to Conjecture 1.5 pertaining to the character of \( V_0 \), we have

1.13 Conjecture. The (unnormalized) characters of \( V_{1/2} \) and \( V_{1/16} \) are given by

\[
\chi_{V_0}(q) = \sum_{k_1, k_2 \geq 0} \frac{q^{2k_1^2 + 3k_2} (1 - q^{4k_1 + 2k_2 + 1})}{(q)_{k_1}(q)_{k_2}},
\]

\[
\chi_{V_{1/2}}(q) = q^{1/2} \sum_{k_1, k_2 \geq 0} \frac{q^{4k_1^2 + 3k_1 k_2 + k_2^2 + 2k_2} (1 - q^{8k_1 + 4k_2 + 6})}{(q)_{k_1}(q)_{k_2}},
\]

\[
\chi_{V_{1/16}}(q) = \sum_{k_1, k_2 \geq 0} \frac{q^{3k_1 k_2 + k_1^2} (q^{k_1 + k_2} + q^{4k_1 + k_2 + 1})}{(q)_{k_1}(q)_{k_2}}.
\]

The first equation is easily seen to be equivalent to Conjecture 1.5; the other two have been verified to order \( q^{2000} \) with SageMath \cite{18}. The project was funded by CNPq grants 409582/2016-0 and 303806/2017-6.

2 Preliminaries and notation

2.1. In working with \( q \)-series identities the following notation is useful. The \( q \)-Pochhammer symbol is \((q)_n = \prod_{j=1}^{n} (1 - q^j)\). We also write \((q)_{\infty}\) for \( \prod_{j=1}^{\infty} (1 - q^j)\). The \( q \)-binomial coefficient is defined to be

\[
\binom{m}{n}_q = \frac{(q)_m}{(q)_n(q)_{m-n}}
\]

for \(0 \leq n \leq m\) and 0 otherwise.

2.2. We denote by \( \mathcal{L} \) the Virasoro Lie algebra, namely the vector space with basis \( \{ L_n \}_{n \in \mathbb{Z}} \cup \{ C \} \) and Lie brackets given by

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12} \delta_{m-n} C, \quad [C, \mathcal{L}] = 0.
\]

We also write \( \mathcal{L}^+ \) for the subalgebra of \( \mathcal{L} \) spanned by \( \{ L_n \}_{n \geq -1} \cup \{ C \} \). The one dimensional representation \( \mathbb{C} \) of \( \mathcal{L} \) of central charge \( c \) is defined by \( C \mapsto c \) and \( L_n \mapsto 0 \). The induced \( \mathcal{L} \)-module

\[
\text{Vir}^c = U(\mathcal{L}) \otimes_{U(\mathcal{L}^+)} \mathbb{C}.
\]
carries the structure of a (conformal) vertex algebra \[24\]. By the PBW theorem the vectors

\[ L^{-n_1}L^{-n_2}\ldots L^{-n_m}|0\rangle, \quad n_1 \geq n_2 \geq \cdots \geq n_m \geq 2, \quad (2.2.2) \]

constitute a basis of \( \text{Vir}^c \). We introduce a \( \mathbb{Z}_{\geq 0} \)-grading on \( \text{Vir}^c \) by assigning the monomial \((2.2.2)\) degree \( \sum_{i=1}^{m} n_i \). The monomials of degree \( n \) are parametrized by partitions of \( n \) into parts of size greater than or equal to 2.

Indeed if \( p, p' \geq 2 \) are coprime integers and

\[ c = c_{p,p'} = 1 - \frac{6(p - p')^2}{pp'}, \]

the vertex algebra \( \text{Vir}^c \) is not simple, with maximal ideal (equivalently maximal \( \mathcal{L} \)-submodule) generated by the homogeneous singular vector \( v_{p,p'} \) of degree \( (p - 1)(p' - 1) \). The simple quotient, denoted \( \text{Vir}_p \), is a rational vertex algebra known as the \((p,p')\) Virasoro minimal model. The case \((p,p') = (3,4)\), of central charge \( c_{3,4} = \frac{1}{2} \), is known as the Ising model. The singular vector \( v_{3,4} \in \text{Vir}^{1/2} \) is given explicitly by

\[ v_{3,4} = L_{-2}^3|0\rangle + \frac{93}{64}L_{-3}^2 - \frac{27}{16}L_{-6}|0\rangle - \frac{33}{8}L_{-4}L_{-2}|0\rangle. \quad (2.2.3) \]

The graded dimension, or character, of \( \text{Vir}_{p,p'} \) is

\[ \chi_{\text{Vir}_{p,p'}}(q) = \frac{1}{(q)_{\infty}} \sum_{m \in \mathbb{Z}} \left( \frac{q^{(2pp'+mp+mp')^2} - q^{(2pp'+mp+mp')^2-(p-p')^2}}{1-q^{pp'}} \right). \quad (2.2.4) \]

2.3. Let \( V \) be a vertex algebra. We now recall the definition of the Li filtration on \( V \) \[1\]. It is the decreasing filtration \( \{F_pV\} \), defined by letting \( F_pV \) be the linear span of vectors of the form

\[ a_{1}^{(n_1-1)} a_{2}^{(n_2-1)} \cdots a_{k}^{(n_k-1)}|0\rangle, \quad a^i \in V, \quad n_i \geq 0, \quad \sum_{j=1}^{k} n_j \geq p. \]

The associated graded \( \text{gr}_F V \) is known as the singular support of \( V \) and is a \( \mathbb{Z}_{\geq 0} \)-graded Poisson vertex algebra. As remarked in \[1\] the degree 0 component \( \text{gr}^0_F V \) coincides with Zhu’s Poisson algebra \( R_V = V/V_{(-2)} \).

If \( V \) is conformal then a choice of homogeneous strong generators \( \{u^i\} \) permits the introduction of an increasing filtration \( \{G_pV\} \), also introduced by Li \[1\], defined by letting \( G_pV \) be the linear span of vectors of the form

\[ u_{1}^{i_1} \cdots u_{N}^{i_N}|0\rangle, \quad n_i \geq 0, \quad \sum_{j=1}^{N} d_{ij} \leq p. \]

(Here \( d_i \) denotes the degree, or conformal weight, of \( u^i \).) Both filtrations are compatible with conformal weight and in fact it was proved by Arakawa that \[3\]

\[ F_pV_n = G_{n-p}V_n. \quad (2.3.1) \]

For \( \text{Vir}^c \), which has \( \omega = L_{-2}|0\rangle \) as strong generating set, \( G_pV \) is essentially the PBW filtration. Hence \( F_pV_{n}^c \) is the linear span of vectors of the form \((2.2.2)\) satisfying \( 2m \leq n-p \). It is clear that \( R_{\text{Vir}^c} \simeq \mathbb{C}[\omega] \) and we have by the PBW theorem

\[ \text{gr}_F \text{Vir}^c \simeq JR_{\text{Vir}^c} \simeq \mathbb{C}[L_{-2}, L_{-3}, \ldots], \]

where \( L_{-2} = \omega \) and the derivation \( \partial \) is given by \( \partial(L_{-n}) = (n-1)L_{-n-1} \).

As explained in the introduction the \( \text{Li} \) filtration on a vertex algebra \( V \) entails a refinement of the character \( \chi_V(q) \) to the two-variable character \( \chi_{\text{Vir}_p} V(t,q) \) defined in \[1,21\]. In this article we work primarily with the PBW filtration on vertex algebras rather than the \( \text{Li} \) filtration, but due to \(2.3.1\) it is easy to convert generating functions from one to the other. Indeed if \( P(t,q) \) denotes the character of the associated graded of \( V = \text{Vir}^c \) or its quotient with respect to the PBW filtration, then \(2.3.1\) translates to \(1.9.1\).
The filtrations on $\text{Vir}_{p,p'}$ coincide with those induced by the quotient map from $\text{Vir}^c$, where $c = c_{p,p'}$. In particular the quotient induces a surjection $R_{\text{Vir}} \twoheadrightarrow R_{\text{Vir}_{p,p'}}$. Let $s = (p - 1)(p' - 1)/2$. It is known that the coefficient of $L_{-2}^s[0]$ in the singular vector $v_{p,p'}$ is nontrivial [11]. All other monomials of the same degree lie in $F_1 \text{Vir}_{p,p'}$ and hence $L_{-2}^s$ vanishes in $R_{\text{Vir}_{p,p'}}$. Indeed, abusing notation, we have $R_{\text{Vir}_{p,p'}} \simeq \mathbb{C}[L_{-2},L_{-3},\ldots]/(L_{-2}^s)$ and consequently

$$JR_{\text{Vir}_{p,p'}} \simeq \mathbb{C}[L_{-2},L_{-3},\ldots]/(L_{-2}^s) \oplus.$$  

As in (1.11) there is a canonical surjection

$$\pi : \mathbb{C}[L_{-2},L_{-3},\ldots]/(L_{-2}^s) \twoheadrightarrow \text{gr}_F \text{Vir}_{p,p'}.$$  

The graded dimension of the arc algebra has been computed in [25]. There it is shown that the ideal $(L_{-2}^s)$ has a Gröbner basis whose leading terms are, translated to the present context, the monomials

$$s_\lambda = \mathbb{C}[L_{-2},L_{-3},\ldots]/(L_{-2}^s),$$  

for which the partition $\lambda = [\lambda_1,\ldots,\lambda_m]$ satisfies $\lambda_m \geq 2$ and the difference condition

$$\lambda_i - \lambda_{i+1} \geq 2, \quad \text{for } 1 \leq i \leq m + 1 - s.$$  

Such partitions are counted by the left hand side of the following Andrews-Gordon identity [26]

$$\sum_{k=(k_1,\ldots,k_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}} \frac{q^{k_1+\cdots+k_n}B^{(s)}}{(q)_{k_1}\cdots(q)_{k_{n-1}}} = \prod_{n \geq 1} \frac{1}{1-q^n}.$$  

Here $G^{(s)}$ is the matrix with entries $G^{(s)}_{ij} = 2 \min(i,j)$ for $1 \leq i, j \leq s$ and $B^{(s)} = (1,2,\ldots,s-1)$. For $p = 2$ the Jacobi triple product identity reduces (2.2.3) to the right hand side of (2.3.2). The $(2,2s+1)$ Virasoro minimal models are thus classically free.

2.4. For the case of the Ising model $(p,p') = (3,4)$ we have the following result.

**Lemma.** The kernel of the surjection $\pi : \mathbb{C}[L_{-2},L_{-3},\ldots]/(L_{-2}^3) \twoheadrightarrow \text{gr}_F \text{Vir}_{3,4}$ is a nontrivial graded differential ideal. Its lowest graded piece is the linear span of the degree 9 vector

$$b = \frac{1}{6}L_{-5}^2L_{-2} + L_{-3}L_{-2}.$$  

**Proof.** It is easy to check that the lowest graded piece of $\ker \pi$ has degree 9 and is 1-dimensional by comparing the graded dimensions of $\text{Vir}_{3,4}$ and the arc algebra. Now let

$$w_{3,4} = L_{-5}L_{-2}L_{-2}[0] + 6L_{-4}L_{-3}L_{-2}[0] \in F_3 \text{Vir}^{1/2}.$$  

Using (2.2.1) and (2.2.3) we prove by direct computation

$$w_{3,4} = \frac{256}{429}L_{-3}v_{3,4} - \frac{64}{429}L_{-1}L_{-2}v_{3,4} - \frac{31}{286}L_{-1}^3v_{3,4} =$$

$$= \frac{27}{8}L_{-6}L_{-3}[0] + \frac{87}{4}L_{-7}L_{-2}[0] + \frac{147}{32}L_{-9}[0] - \frac{45}{16}L_{-5}L_{-4}[0] \in F_5 \text{Vir}^{1/2}.$$  

The Lemma follows applying $\pi$ to both sides of this equation and noting that the image of the LHS equals $\pi(w_{3,4}).$  

In fact by the same technique we may prove in general:

**Lemma.** Let $V = \text{Vir}_{3,p'}$ where $p' \geq 4$. The kernel of the surjection

$$\pi : \mathbb{C}[L_{-2},L_{-3},\ldots]/(L_{-2}^{p'-1}) \twoheadrightarrow \text{gr}_F \text{Vir}_{3,p'}$$  

is a nontrivial graded differential ideal. Its lowest graded piece is the linear span of the degree $2p' + 1$ vector

$$b(p') = \frac{(9 - 2p')}{3(p' - 2)}L_{-5}L_{-2}^{p'-2} + L_{-4}L_{-3}L_{-2}^{p'-3}.$$  

7
2.5 Corollary. Let $V = \text{Vir}_{3, \nu}$. There is a surjective morphism

$$JR_V/I \xrightarrow{\sim} \text{gr}_V,$$

where $I$ denotes the differential ideal $(h^{(\nu)})_0 \subset JR_V$.

As remarked in the introduction, the surjection of Corollary 2.5 is not an isomorphism for $\nu \geq 5$. For the case of the Ising model $V = \text{Vir}_{3,4}$ we study the structure of the ideal $I$ in Sections 8 and 4 below.

2.6 Asymptotics of Nahm sums. Let $A$ be an $n \times n$ positive definite symmetric matrix with rational coefficients, $B = (b_1, \ldots, b_n) \in \mathbb{Q}^n$ and $C \in \mathbb{Q}$. The formal power series

$$f_{A, B, C}(q) = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{q^{2k^2 + k^2 B + C}}{(q)^{k_1} \cdots (q)^{k_r}},$$

(2.6.1)

converges, upon setting $q = e^{2\pi i \tau}$, to a holomorphic function $F(\tau)$ of $\tau \in \mathbb{H}$ the upper half complex plane. The asymptotic behaviour of $F(\tau)$ as $\tau = it \to 0$ along the positive imaginary axis is known to be

$$F(it) \sim e^{2\pi i \alpha}, \quad \text{where} \quad \alpha = \sum_{i} \left( \frac{\pi^2}{6} - L(Q_i) \right).$$

(2.6.2)

Here

$$L(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} + \frac{1}{2} \log(z) \log(1-z),$$

is the Rogers dilogarithm function, and $(Q_1, \ldots, Q_n)$ is the unique solution of the system of equations

$$1 - Q_i = \prod_j Q_j^{A_{ij}}, \quad 0 < Q_i < 1 \quad \text{for} \quad i = 1, \ldots, n.$$ 

(2.6.3)

For example if $A = \left( \frac{3}{2} \frac{3}{2} \right)$ then

$$Q_1 = \frac{1}{2} \left( \sqrt{2\sqrt{2} - 1 + \sqrt{2} - 1} \right) \quad \text{and} \quad Q_2 = \frac{2}{\sqrt{2\sqrt{2} - 1 - \sqrt{2} + 3}}.$$ 

Then $\alpha = \frac{\pi^2}{12}$ can be deduced without difficulty from the functional equations satisfied by $L(z)$.

The normalized character $q^{-c_{\text{eff}}/24} \chi_V(q)$ of a rational $C_2$-cofinite conformal vertex algebra is a modular function on $\Gamma_0(N)$ for some $N$ [23, 29]. Therefore the asymptotic behaviour of $\chi_V$ is always of the form $c^{\frac{\alpha}{2}}$ where $\alpha = \pi^2 g_V/6$ for some $g_V \in \mathbb{Q}$ known as the effective central charge of $V$. For the Ising model we have $g_V = c_V = 1/2$ and the normalized character is expressible as a linear combination of the Weber modular functions.

The condition $\alpha \in \pi^2 \mathbb{Q}$, where $\alpha$ is given by (2.6.2), places a strong restriction on possible fermionic sum representations of characters of vertex algebras and their modules. The matrix $A = G^{(s)}$ appearing in connection with the Andrews-Gordon identity above satisfies the condition and is associated with the vertex algebra $\text{Vir}_{2,2s+1}$. It was observed by Nahm that the rationality condition is closely related to torsion elements in Bloch groups of number fields [21]. Terhoeven produced a list of matrices for which the rationality condition is satisfied [22]. For most of these matrices $A$ there exists a choice of $B$ and $C$ for which (2.6.1) becomes modular, but for other examples such as $(\frac{3}{2} \frac{3}{2})$, no such choices had been found [23]. Our Conjecture 1 shows that modular candidates can be obtained allowing linear combinations of Nahm sums.

3 A partition identity

In this section we prove Theorem 2(b). We recall the set of partitions $\mathcal{P}(n)$ introduced in Theorem 2. We denote by $p(n, m)$ the number of partitions $\lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{P}(n)$ into exactly $m$ parts and we write the generating function

$$P(t, q) = \sum_{m, n} p(n, m) t^m q^n.$$
In this section we prove fermionic sum expressions for \( P(t, q) \). To achieve this we divide \( \mathcal{P}(n) \) into five disjoint subsets, we find recurrence relations among the cardinalities of these sets, these lead to a system of functional equations which we solve in Proposition 3.4 to obtain \( P(t, q) \). Specialization to \( t = 1 \) yields the generating function \( (1.5.1) \) for the number of partitions in \( \mathcal{P}(n) \).

3.1. It is not difficult to verify that \( \mathcal{P}(n) \) decomposes as the disjoint union

\[
\mathcal{P}(n) = \mathcal{A}(n) \cup \mathcal{B}(n) \cup \mathcal{C}(n) \cup \mathcal{D}(n) \cup \mathcal{E}(n),
\]

where, for \( n \geq 0 \), we define \( \mathcal{A}(0) = \{[\cdot]\} \), \( \mathcal{B}(2) = \{[2]\} \), \( \mathcal{D}(4) = \{[2, 2]\} \) and otherwise

\[
\mathcal{A}(n) = \{ \lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{P}(n) \mid 2 < \lambda_m \},
\]

\[
\mathcal{B}(n) = \{ \lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{P}(n) \mid 2 = \lambda_m < \lambda_{m-1} - 1 \},
\]

\[
\mathcal{C}(n) = \{ \lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{P}(n) \mid 2 = \lambda_m = \lambda_{m-1} - 1 \},
\]

\[
\mathcal{D}(n) = \{ \lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{P}(n) \mid 2 = \lambda_m = \lambda_{m-1} < \lambda_{m-2} - 2 \},
\]

\[
\mathcal{E}(n) = \{ \lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{P}(n) \mid 2 = \lambda_m = \lambda_{m-1} = \lambda_{m-2} - 2 \}.
\]

3.2 Lemma. Let \( a(n, m) \) (resp. \( b(n, m), \ldots, e(n, m) \)) denote the number of partitions \( \lambda \in \mathcal{A}(n) \) (resp. \( \mathcal{B}(n), \ldots, \mathcal{E}(n) \)) into exactly \( m \) parts. The following recursive formulas hold

\[
a(n, m) = a(n - m, m) + b(n - m, m) + c(n - m, m) + d(n - m, m),
\]

\[
b(n, m) = a(n - m - 1, m - 1) - d(n - 2m, m - 1),
\]

\[
c(n, m) = b(n - 2m + 1, m - 1) + d(n - 2m - 1),
\]

\[
d(n, m) = b(n - m - 1) - e(n - 2m + 1, m - 1),
\]

\[
e(n, m) = c(n - m, m - 1).
\]

Proof. Let \( \lambda = [\lambda_1, \ldots, \lambda_m] \in \mathcal{A}(n) \) and consider \( \mu = [\lambda_1 - 1, \ldots, \lambda_m - 1] \in \mathcal{P}(n - m) \). Notice that \( \mu \) avoids \([4, 2, 2]\) since \([5, 3, 3]\) is one of the partitions excluded in the definition of \( \mathcal{P} \) (see the third line of (1.8.2)). It follows that \( \mu \in \mathcal{A}(n - m) \cup \mathcal{B}(n - m) \cup \mathcal{C}(n - m) \cup \mathcal{D}(n - m) \). Conversely, if \( \mu \) lies in the latter set then \( \lambda = [\mu_1 + 1, \ldots, \mu_m + 1] \) lies in \( \mathcal{P}(n) \). The only conditions from (1.8.2) that are not immediate are that \( \lambda \) avoids \([5, 3, 3]\) and the four exceptional partitions listed there. But these partitions are indeed avoided by \( \lambda \) because \( \mu \notin \mathcal{E}(n - m) \) and hence avoids \([4, 2, 2]\), and because \( \lambda_m > 2 \). Indeed this demonstrates that \( \lambda \in \mathcal{A}(n) \), and the bijection we have thus established proves the first equation in (3.2.1). The other equations are proved in a similar way.

3.3 Lemma. Consider the formal power series

\[
A(t, q) = \sum_{n,m \geq 0} a(n, m) q^n t^m,
\]

and define similarly \( B(t, q), C(t, q), E(t, q), D(t, q) \). Then they satisfy the following functional equations:

\[
A(t, q) = A(tq, q) + B(tq, q) + C(tq, q) + D(tq, q), \quad A(0, q) = 1,
\]

\[
B(t, q) = t^2 A(tq, q) - t^2 D(tq^2, q), \quad B(0, q) = 0,
\]

\[
C(t, q) = tq B(tq^2, q) + t^2 D(tq^2, q), \quad C(0, q) = 0,
\]

\[
D(t, q) = tq B(tq, q) - tq E(tq^2, q), \quad D(0, q) = 0,
\]

\[
E(t, q) = tq C(tq, q), \quad E(0, q) = 0.
\]

Proof. The functional equations are obtained by direct translation of the recurrence relations (3.2.1) in terms of the power series we have introduced.
3.4 Proposition. The unique solution of the system of functional equations (3.4.1) is

\[ A(t, q) = \sum_{m \geq 0} \frac{t^m q^m}{(q)_m} \sum_{k=0}^{m} k^k q^{(k+1)m+2k^2} \binom{m}{k}, \]

\[ B(t, q) = \sum_{m \geq 1} \frac{t^m q^m}{(q)_m-1} \sum_{k=0}^{m-1} k^k q^{(k+1)m+2k^2} \binom{m-1}{k}, \]

\[ C(t, q) = \sum_{m \geq 2} \frac{t^m q^m}{(q)_{m-2}} \sum_{k=0}^{m-2} k^k q^{(k+1)m+2k^2} \binom{m-2}{k}, \]

\[ D(t, q) = \sum_{m \geq 2} \frac{t^m q^m}{(q)_{m-2}} \sum_{k=0}^{m-2} k^k q^{(k+1)m+2k^2} \binom{m-2}{k}, \]

\[ E(t, q) = \sum_{m \geq 3} \frac{t^m q^m}{(q)_{m-3}} \sum_{k=0}^{m-3} k^k q^{(k+1)m+2k^2} \binom{m-3}{k}. \]

Proof. The uniqueness is automatic, it suffices to check that the expressions given in (3.4.1) satisfy the functional equations and initial conditions (3.3.1). Directly from the definition we have:

\[ A(tq, q) = \sum_{m \geq 0} \frac{t^m q^m}{(q)_m} \sum_{k=0}^{m} k^k q^{(k+1)m+2k^2} \binom{m}{k} q^{m+k}. \] (3.4.2)

Similarly:

\[ B(tq, q) = \sum_{m \geq 1} \frac{t^m q^m}{(q)_m-1} \sum_{k=0}^{m-1} k^k q^{(k+1)m+2k^2} \binom{m-1}{k} q^{2k}. \] (3.4.3)

For \( C(t, q) \) and \( D(tq, q) \) we perform a simple algebraic manipulation:

\[ C(tq, q) = \sum_{m \geq 2} \frac{t^m q^m}{(q)_{m-1}} \sum_{k=0}^{m-2} k^k q^{k+1} q^{(k+1)m+2k^2} \binom{m-2}{k} q^{k+1} \]

\[ = \sum_{m \geq 1} \frac{t^m q^m (m+1)}{(q)_m-1} \sum_{k=0}^{m-1} k^k q^{k+1} q^{(k+1)m+2k^2} \binom{m-1}{k} q^{k+1} \]

\[ = \sum_{m \geq 1} \frac{t^m q^m (m+1)}{(q)_m-1} \sum_{k=1}^{m} k^k q^{k+1} q^{(k+1)m+2k^2} \binom{m-1}{k-1} q^{k+1} \]

\[ = \sum_{m \geq 1} \frac{t^m q^m (m+1)}{(q)_m-1} \sum_{k=0}^{m-1} k^k q^{(k+1)m+2k^2} \binom{m-1}{k} q^{k}, \]

\[ D(tq, q) = \sum_{m \geq 2} \frac{t^m q^m}{(q)_{m-2}} \sum_{k=0}^{m-2} k^k q^{(k+1)m+2k^2} \binom{m-2}{k} q^{-1} \]

\[ = \sum_{m \geq 1} \frac{t^m q^m (m+2)}{(q)_m-1} \sum_{k=0}^{m-1} k^k q^{(k+1)m+2k^2} \binom{m-1}{k} q^{-1} \]

\[ = \sum_{m \geq 1} \frac{t^m q^m (m+2)}{(q)_m-1} \sum_{k=1}^{m} k^k q^{(k-1)(m+4)+2(k-1)^2} \binom{m-1}{k-1} q^{-1} \]

\[ = \sum_{m \geq 1} \frac{t^m q^m (m+2)}{(q)_m-1} \sum_{k=1}^{m} k^k q^{(k+1)m+2k^2} \binom{m-1}{k-1} q^{-1}. \] (3.4.5)
Summing (3.4.3) and (3.4.6) and using
\[
\frac{1}{(q)_{m-1}} \binom{m}{k} = \frac{1}{(q)_{m}} \binom{m}{k} (1 - q^k),
\]
we obtain
\[
C(tq, q) + D(tq, q) = \sum_{m \geq 0} \frac{t^m q^m (m+1)}{(q)_m} \sum_{k=0}^{m} t^k q^{k(m+1)+2k^2} \binom{m}{k} (1 - q^{2k}). 
\tag{3.4.6}
\]
Adding (3.4.2), (3.4.3) and (3.4.6) we obtain the first equation in (3.3.1).

\[
tq^2 A(tq, q) - tq^2 D(tq^2, q) = \sum_{m \geq 0} \frac{tm + 1}{(q)_m} \sum_{k=0}^{m} t^k q^{k(m+1)+2k^2} \binom{m}{k} q^k
\]
proving the second equation in (3.3.1).
proving the third equation in (3.3.1).

$$tqB(tq, q) - D(t, q) = \sum_{m \geq 2} \frac{t^m q^{m^2}}{(q)_{m-2}} \sum_{k=1}^{m-2} t^k q^{k(m+1)+2k^2} \begin{pmatrix} m-2 \atop k \end{pmatrix}_q (1 - q^k)$$

$$= \sum_{m \geq 3} \frac{t^m q^{m^2}}{(q)_{m-2}} \sum_{k=0}^{m-3} t^{k+1} q^{(k+1)(m+1)+2(k+1)^2} \begin{pmatrix} m-2 \atop k+1 \end{pmatrix}_q (1 - q^{k+1})$$

$$= \sum_{m \geq 2} \frac{t^{m+1} q^{m^2+m+2}}{(q)_{m-3}} \sum_{k=0}^{m-3} t^k q^{k(m+5)+2k^2} \begin{pmatrix} m-3 \atop k \end{pmatrix}_q = tqE(tq^2, q),$$

proving the fourth equation in (3.3.1). Finally to prove the fifth equation:

$$tqC(tq, q) = \sum_{m \geq 2} \frac{t^{m+1} q^{m^2+m+2}}{(q)_{m-2}} \sum_{k=0}^{m-2} t^k q^{k(m+4)+2k^2} \begin{pmatrix} m-2 \atop k \end{pmatrix}_q$$

$$= \sum_{m \geq 3} \frac{t^m q^{m^2-m+2}}{(q)_{m-3}} \sum_{k=0}^{m-3} t^k q^{k(m+3)+2k^2} \begin{pmatrix} m-3 \atop k \end{pmatrix}_q = E(t, q).$$

3.5 Remark. Expanding the q-binomial coefficients with (2.1.1) and replacing $k_1 = k$, $k_2 = m - k$. We obtain the following quasi-particle representations for the above power series:

$$A(t, q) = \sum_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \frac{t^{2k_1+k_2} q^{4k_1^2+3k_1 k_2+k_2^2+2k_1+2k_2}}{(q)_{k_1} (q)_{k_2}},$$

$$B(t, q) = tq^2 \sum_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \frac{t^{2k_1+k_2} q^{4k_1^2+3k_1 k_2+k_2^2+5k_1+3k_2}}{(q)_{k_1} (q)_{k_2}},$$

$$C(t, q) = t^2 q^3 \sum_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \frac{t^{2k_1+k_2} q^{4k_1^2+3k_1 k_2+k_2^2+9k_1+4k_2}}{(q)_{k_1} (q)_{k_2}},$$

$$D(t, q) = t^2 q^4 \sum_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \frac{t^{2k_1+k_2} q^{4k_1^2+3k_1 k_2+k_2^2+8k_1+4k_2}}{(q)_{k_1} (q)_{k_2}},$$

$$E(t, q) = t^3 q^8 \sum_{k_1, k_2 \in \mathbb{Z}_{\geq 0}} \frac{t^{2k_1+k_2} q^{4k_1^2+3k_1 k_2+k_2^2+11k_1+5k_2}}{(q)_{k_1} (q)_{k_2}}.$$

Each of these series, specialized at $t = 1$ is a Nahm sum [23] associated to the matrix $\begin{pmatrix} 8 & 3 \\ 3 & 3 \end{pmatrix}$ as explained in 2.3.

Proof of Theorem 3 b). By summing (3.5.1) we obtain (1.8.3). □

4 A PBW basis for the Ising module

In this section we finish the proof of Theorem 2 and prove Theorem 11 as a consequence of Conjecture 1.5. The computations in this section were carried out with the help of SageMath [18]. For an implementation of vertex algebras see also [30]. We start by showing that Theorem 2 and Conjecture 1.5 imply Theorem 11.

Proof of Theorem 4. Let $V = \text{Vir}_{3,4}$. We define the differential ideal

$$I' = \left( \frac{1}{6} L_{-5} L_{-2}^2 + L_{-4} L_{-3} L_{-2} \right) \subset JR_V,$$
and let $A = JR_V/I'$. This is a $\mathbb{Z}_{\geq 0}$ graded differential algebra $A = \oplus_{n \geq 0} A_n$ with graded dimension

$$\chi_A(q) = \sum_{n \geq 0} q^n \dim A_n.$$ 

Since the map (1.6.1) is surjective, we know $\dim A_n \geq \dim V_n$. It remains to show the other inequality. Due to Theorem (2a) we have $\dim A_n \leq |\mathcal{P}(n)|$. Conjecture (1.3) and Theorem (2b) imply that $|\mathcal{P}(n)| = \dim V_n$.

**Proof of Theorem (3 a).** We consider the differential polynomial algebra $\mathbb{C}[L_{-2}, L_{-3}, \ldots]$, with grading defined by $\deg L_{-n} = n$, and the derivation given by $\partial_{L_{-n}} = (n-1)L_{-n-1}$. The grevlex monomial ordering on this ring is defined as follows. For a partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_m]$ in which $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_m \geq 2$ we write

$$L_\lambda = L_{-\lambda_1} \cdots L_{-\lambda_m} \in \mathbb{C}[L_{-2}, L_{-3}, \ldots].$$

Given two partitions $\lambda, \mu$, we say that $\lambda < \mu$ if $\deg L_\lambda < \deg L_\mu$ or if $\deg L_\lambda = \deg L_\mu$ and there exists $i \geq 1$ such that $\lambda_j = \mu_j$ for $0 < j < i$ and $\lambda_i > \mu_i$. Let

$$I = (a, b) \subset \mathbb{C}[L_{-2}, L_{-3}, \ldots],$$

be the preimage of $I$ by the quotient map, where

$$a = L_{-2}^3, \quad b = L_2L_{-3}L_{-4} + \frac{1}{6}L_5L_{-2}^2.$$

It is enough to show that for every $\lambda$ in the set of partitions $\mathcal{I}$ defined by (1.8.2), there exists an element of the ideal $I$ whose leading monomial is $L_\lambda$. Below we exhibit these elements explicitly.

We will find the required elements of $I$ by taking appropriate combinations of derivatives of $a$ and $b$ in such a way as to cancel leading terms. The first step is to write formulas for $\partial^{(n)}a$ and $\partial^{(n)}b$ (here $\partial^{(n)} = \frac{\partial^n}{n!}$), keeping track of the first few leading terms and the polynomial dependence of their coefficients on $n$. Since our partitions (1.8.2) have length up to and including 6, we need to keep track of around 6 leading terms.

We prove by induction

$$\partial^{(3k+9)}a = L_{-5-k}^3 + 6L_{-6-k}L_{-5-k}L_{-4-k} + 3L_{-6-k}L_{-3-k} + 3L_{-7-k}L_{-4-k} + 6L_{-7-k}L_{-5-k}L_{-3-k} + 6L_{-7-k}L_{-6-k}L_{-2-k} + \ldots,$$

(4.0.1a)

$$\frac{1}{3} \partial^{(3k+10)}a = L_{-6-k}L_{-5-k}^2 + L_{-6-k}^2L_{-4-k} + 2L_{-7-k}L_{-5-k}L_{-4-k} + 2L_{-7-k}L_{-6-k}L_{-3-k} + L_{-7-k}L_{-2-k} + L_{-8-k}L_{-4-k} + 2L_{-8-k}L_{-5-k}L_{-3-k} + 2L_{-8-k}L_{-6-k}L_{-2-k} + \ldots,$$

(4.0.1b)

$$\frac{1}{3} \partial^{(3k+11)}a = L_{-6-k}L_{-5-k}^2 + L_{-7-k}L_{-5-k}^2 + 2L_{-7-k}L_{-6-k}L_{-4-k} + L_{-7-k}L_{-3-k} + 2L_{-8-k}L_{-6-k}L_{-4-k} + 2L_{-8-k}L_{-6-k}L_{-3-k} + 2L_{-8-k}L_{-7-k}L_{-2-k} + \ldots,$$

(4.0.1c)

where “…” means terms that are lower in the monomial order. For $0 \leq n \leq 8$ similar expressions for $\partial^{(n)}a$ can be computed by hand. Again we prove by induction

$$\partial^{(3k+6)}b = \frac{1}{6} \left(19k^3 + 150k^2 + 389k + 330\right) L_{-5-k}^3 + (19k^3 + 150k^2 + 391k + 340) L_{-6-k}L_{-5-k}L_{-4-k} + \frac{1}{2} \left(19k^3 + 150k^2 + 395k + 376\right) L_{-6-k}^2L_{-3-k} + \frac{1}{2} \left(19k^3 + 150k^2 + 395k + 344\right) L_{-7-k}L_{-4-k} + (19k^3 + 150k^2 + 397k + 370) L_{-7-k}L_{-5-k}L_{-3-k} + (19k^3 + 150k^2 + 403k + 448) L_{-7-k}L_{-6-k}L_{-2-k} + \ldots,$$

(4.0.2a)
\[ \partial^{(3k+7)}b = \frac{1}{2}(19k^3 + 169k^2 + 496k + 480)L_{-6-k}L_{-5-k}^2 \\
+ \frac{1}{2}(19k^3 + 169k^2 + 498k + 496)L_{-6-k}^2L_{-4-k} \\
+ (19k^3 + 169k^2 + 500k + 496)L_{-7-k}L_{-5-k}L_{-4-k} \\
+ (19k^3 + 169k^2 + 504k + 544)L_{-7-k}L_{-6-k}L_{-3-k} \\
+ \frac{1}{2}(19k^3 + 169k^2 + 512k + 640)L_{-7-k}L_{-2-k}^2 \\
+ \frac{1}{2}(19k^3 + 169k^2 + 508k + 496)L_{-8-k}L_{-5-k}^2 \\
+ (19k^3 + 169k^2 + 528)L_{-8-k}L_{-5-k}L_{-3-k} \\
+ (19k^3 + 169k^2 + 514k + 624)L_{-8-k}L_{-6-k}L_{-2-k} + \ldots, \tag{4.0.2b} \]

\[ \partial^{(3k+8)}b = \frac{1}{2}(19k^3 + 188k^2 + 615k + 666)L_{-6-k}^2L_{-5-k} \\
+ \frac{1}{2}(19k^3 + 188k^2 + 617k + 672)L_{-7-k}L_{-5-k}^2 \\
+ (19k^3 + 188k^2 + 619k + 694)L_{-7-k}L_{-6-k}L_{-4-k} \\
+ \frac{1}{2}(19k^3 + 188k^2 + 625k + 760)L_{-7-k}^2L_{-3-k} \\
+ (19k^3 + 188k^2 + 623k + 690)L_{-8-k}L_{-5-k}L_{-4-k} \\
+ (19k^3 + 188k^2 + 627k + 750)L_{-8-k}L_{-6-k}L_{-3-k} \\
+ (19k^3 + 188k^2 + 635k + 870)L_{-8-k}L_{-7-k}L_{-2-k} + \ldots, \tag{4.0.2c} \]

Examine the expressions for \( \partial^{(n)}a \) in equations (1.0.1) we see that for partitions \( \lambda \) which contain \([p,p,p]\), etc. (the partitions in first line of (1.8.2)), the monomials \( L_{\lambda} \) are leading monomials of elements of \( I \). From (4.0.1b) and (4.0.2a) we see that the polynomial

\[ r_k = \partial^{(3k+1)}b - \frac{1}{6}(19k^3 + 55k^2 + 48k + 12)\partial^{(3k+4)}a, \]

has \( L_{-4-k}^2L_{-2-k} \) as leading monomial. Similarly the polynomials

\[ s_k = \partial^{(3k+2)}b - \frac{1}{6}(19k^3 + 74k^2 + 91k + 36)\partial^{(3k+5)}a, \]

\[ t_k = \partial^{(3k)}b - \frac{1}{6}(19k^3 + 36k^2 + 17k)\partial^{(3k+3)}a, \]

have \( L_{-5-k}L_{-3-k}^2 \) and \( L_{-4-k}L_{-3-k}L_{-2-k} \) respectively as leading monomials. These show that the monomials corresponding to all elements of length three in (1.8.2) are leading monomials of elements of \( I \). We now define

\[ u_0 = 8L_{-5}t_0 - 6L_{-2}t_1, \]

\[ u_{k+1} = (2k + 10)L_{-6-k}t_{k+1} - (2k + 8)L_{-3-k}t_{k+2} - (2k + 10)(3k + 20)L_{-2-k}\partial^{(3k+10)}a, \quad k \geq 0, \]

\[ v_k = \frac{1}{3}(11k^3 + 318k^2 + 3061k + 9426)L_{-2-k}t_{k+2} - (7k^3 + 90k^2 + 349k + 370)L_{-6-k}t_k \]

\[ - (11k^3 + 191k^2 + 1029k + 1745)L_{-3-k}t_{k+1} - 8(k^3 + 19k^2 + 121k + 255)L_{-4-k}t_{k+1} \]

\[ + \frac{1}{3}(35k^4 + 799k^3 + 5075k^2 + 14763k + 13690)L_{-6-k}\partial^{(3k+5)}a \quad k \geq 0 \]

\[ + \frac{8}{3}(k^4 + 26k^3 + 254k^2 + 1102k + 1785)L_{-3-k}\partial^{(3k+4)}a, \]

\[ w_k = (k + 2)L_{-6-k}r_k - (k + 6)L_{-2-k}r_{k+1}, \quad k \geq 0 \]

\[ y_0 = 42L_{-5}r_0 - 84L_{-2}\partial^{(7)}a - 12L_{-2}r_1 + 108L_{-6}t_0, \]

14
The four remaining cases in (1.8.2).

We see that the leading monomials for these elements are respectively

\[ u_k \] is the differential ideal

This shows that the monomials corresponding to all elements in the second group in (1.8.2), are leading monomials of elements in \( I \)). The leading monomial of

is \( L_{-8-k}L_{-7-k}L_{-5-k}L_{-3-k}L_{-2-k} \), the infinite family of length five in (1.8.2). Defining

we see that the leading monomials for these elements are respectively

\[ L_{-5}L_{-4}L_{-3}L_{-2}L_{-1} \]
\[ L_{-7}L_{-6}L_{-5}L_{-4}L_{-3}L_{-2}L_{-1} \]
\[ L_{-9}L_{-8}L_{-7}L_{-6}L_{-5}L_{-4}L_{-3}L_{-2}L_{-1} \]

The four remaining cases in (1.8.2).

4.1 Remark. In fact it can be shown that three of the four exceptional cases in (1.8.2) already lie in the differential ideal \( I \). For example

A similar computation shows that \( L_{-7}L_{-6}L_{-5}L_{-4}L_{2} \) and \( L_{-9}L_{-8}L_{-7}L_{-6}L_{2} \) lie in \( I \). The smallest expressions we found for these generators are too long to fit in these pages.

4.2 Corollary. The set consisting of the infinite family of polynomials \( \partial^{(k)}a, r_k, s_k, t_k, u_k, v_k, y_k, z_k, k \geq 0 \) together with the four exceptional elements \( e_i, i = 1, \ldots, 4 \), forms a Gr"{o}bner basis of the ideal

\[ I = \left( L_{-3}^2, \frac{1}{6}L_{-5}L_{-3}L_{-2} + L_{-4}L_{-3}L_{-2} \right)_{\partial} \subset \mathbb{C}[L_{-2}, L_{-3}, \ldots] \].
5 Conclusion

In this article we conjectured three new \( q \)-series identities. These imply that linear combinations of Nahm sums for a given matrix may be modular invariant, even when no single summand is. As a consequence of one of these identities we described the singular support of the Ising model vertex algebra. We showed that it is, in a differential sense, a hypersurface of the arc space of its associated scheme, that is, it is defined as the zero set of a single equation and all of its derivatives. This is the first known example of a vertex algebra whose singular support is a proper subscheme of the arc space of its associated scheme defined by finitely many equations and their derivatives. We have also found an explicit basis for the Ising model consisting on monomials in the Virasoro Lie algebra generators applied to the vacuum vector.

The methods of this article can be applied to other Virasoro minimal models and their modules to produce new \( q \)-series identities and their combinatorial interpretations.

References

[1] H. Li. Abelianizing vertex algebras. Comm. Math. Phys., 259:391–411, 2005.
[2] J. F. Ritt. Differential algebra. In Colloquium Publications, volume 33. AMS, 1950.
[3] T. Arakawa. A remark on the \( C_2 \)-cofiniteness condition on vertex algebras. Math. Z., 270(1-2):559–575, 2012.
[4] J. van Ekeren and R. Heluani. Chiral homology of elliptic curves and Zhu algebra. arXiv:1804.00017, 2018.
[5] T. Arakawa and A. R. Linshaw. Singular support of a vertex algebra and the arc space of its associated scheme. In Representations and nilpotent orbits of Lie algebraic systems, volume 330 of Progr. Math., pages 1–17. Birkhäuser/Springer, Cham, 2019.
[6] B. L. Feigin and D. B. Fuchs. Verma modules over the Virasoro algebra. In Topology (Leningrad, 1982), volume 1060 of Lecture Notes in Math., pages 230–245. Springer, Berlin, 1984.
[7] G. E. Andrews. The theory of partitions. Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1976. Encyclopedia of Mathematics and its Applications, Vol. 2.
[8] W. N. Bailey. On the simplification of some identities of the Rogers-Ramanujan type. Proc. London Math. Soc. (3), 1:217–221, 1951.
[9] L. J. Slater. Further identities of the rogers-ramanujan type. Proc. London Math. Soc., 54(2):147–167, 1952.
[10] F. Jackson. Examples of a generalization of Euler’s transformation for power series. Messenger of Math., 57:169–187, 1928.
[11] M. D. Hirschhorn. Some partition theorems of the Rogers-Ramanujan type. Journal of Combinatorial Theory Series A, 27(1):33–37, 1979.
[12] M. Subbarao. Some Rorgers-Ramanujan type partition theorems. Pacific Journal of Mathematics, 120(2), 1985.
[13] A. C. Ribeiro. Aspectos combinatorios de identidades do tipo Rogers-Ramanujan. PhD thesis, UNICAMP, 2006.
[14] J. P. O. S. Paulo Mondek Andria C. Ribeiro. New two-line arrays representing partitions. Annals of combinatorics, 15:341–354, 2011.
[15] R. Kedem, T. R. Klassen, B. M. McCoy, and E. Melzer. Fermionic quasi-particle representations for characters of \( (G^{(1)})_1 \times (G^{(1)})_1/(G^{(1)})_2 \). Phys. Lett. B, 304(3-4):263–270, 1993.
S. O. Warnaar and P. A. Pearce. Exceptional structure of the dilute $A_3$ model: $E_8$ and $E_7$ Rogers-Ramanujan identities. *J. Phys. A*, 27(23):L891–L897, 1994.

F. Calegari, S. Garoufalidis, and D. Zagier. Bloch groups, algebraic K-theory, units, and Nahm’s Conjecture. 12 2017.

The Sage Developers. *SageMath, the Sage Mathematics Software System*, 2020. https://www.sagemath.org.

E. Feigin, B. Frenkel. Coinvariants of nilpotent subalgebras of the Virasoro algebra and partition identities. https://arxiv.org/pdf/hep-th/9301039.pdf, 1993.

B. Feigin, M. Jimbo, T. Miwa, E. Mukhin, and Y. Takeyama. A monomial basis for the Virasoro minimal series $M(p, p')$: the case $1 < p'/p < 2$. *Comm. Math. Phys.*, 257(2):395–423, 2005.

W. Nahm. Conformal field theory and torsion elements of the Bloch group. In *Frontiers in number theory, physics, and geometry. II*, pages 67–132. Springer, Berlin, 2007.

M. Terhoeven. Rationale konforme Feldtheorien, der Dilogarithmus und Invarianten von 3-Mannigfaltigkeiten. PhD thesis, Bonn University, 1995.

D. Zagier. The dilogarithm function. In *Frontiers in number theory, physics, and geometry. II*, pages 3–65. Springer, Berlin, 2007.

I. Frenkel and Y. Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke Mathematical Journal*, 66(1):123–168, 1992.

C. Bruschek, H. Mourtada, and J. Schepers. Arc spaces and the Rogers-Ramanujan identities. *Ramanujan J.*, 30(1):9–38, 2013.

G. E. Andrews. *On the general Rogers-Ramanujan theorem*. American Mathematical Society, Providence, R.I., 1974. Memiors of the American Mathematical Society, No. 152.

M. Vlasenko and S. Zwegers. Nahm’s conjecture: asymptotic computations and counterexamples. *Commun. Number Theory Phys.*, 5(3):617–642, 2011.

Y. Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9(1):237–302, 1996.

C. Dong, X. Lin, and S.-H. Ng. Congruence property in conformal field theory. *Algebra Number Theory*, 9(9):2121–2166, 2015.

R. Heluani. *SageMath implementation of vertex algebras*, 2020. https://trac.sagemath.org/ticket/29610.