THE JOINT DISTRIBUTIONS OF RUNNING MAXIMUM OF A SLEPIAN PROCESSES

PINGJIN DENG

Abstract: Consider the Slepian process \( S \) defined by \( S(t) = B(t + 1) - B(t), t \in [0, 1] \) with \( B(t), t \in \mathbb{R} \) a standard Brownian motion. In this contribution we analyze the joint distribution between the maximum \( m_s = \max_{0 \leq u \leq s} S(u) \) certain and the maximum \( M_t = \max_{0 \leq u \leq t} S(u) \) for \( 0 < s < t \) fixed. Explicit integral expression are obtained for the distribution function of the partial maximum \( m_s \) and the joint distribution function between \( m_s \) and \( M_t \). We also use our results to determine the moments of \( m_s \).

Key words and phrases: Gaussian processes; Slepian processes; running maximum.

1. Introduction

Throughout this paper, we consider the one-dimensional Slepian process defined as the increment of a Brownian motion process, namely

\[ S(t) = B(t + 1) - B(t), \quad t \in [0, 1], \]

where \( B(t) \) is a standard Brownian motion define on probability space \((\Omega, \mathcal{F}, P)\). It can be verified easily that \( S(t), t \in [0, 1] \) is a stationary Gaussian process with covariance function

\[ R_S(s, t) := \mathbb{E}[S(s)S(t)] = 1 - |s - t|, \quad s, t \in [0, 1]. \]

The Slepian processes \( S(t) \) which was first defined by Slepian in [1], has been studied extensively in stochastic processes and statistics. Zakai and Ziv [2] gave an application of Slepian processes to the signal shape problem in radar, while the application of these processes to scan statistics and signal detection problem are presented in Cressie [3] and Bischoff and Gegg [4].

Another important topic in stochastic processes, where Slepian processes have been wiedly discussed is the boundary crossing probability. Based on the Markov-like property (or reciprocal property see e.g., [5]) of \( S \), Slepian [1], Mehr and McFadden [6], and Shepp [7][8] studied the crossing probability of \( S \) conditional on \( S(0) \) with constant boundary. For a more general boundary, Bischoff and Gegg [4] and Deng [9] gave analytic formulas for the crossing probabilities of \( S \) with continuous piecewise linear boundary. For recnet results on boundary crossing probabilities we refer the reader to [10, 11, 12, 13, 14, 15, 16, 17].

For general stochastic processes, both the tail asymptotics of supremum, and the joint survival function of supremum of the process over two intervals has been considered in numerous publications, see e.g., [18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]. The extremal value statistics are also important in application, for example, the statistics of a maximum is a key process in risk management, the relationship between the risk achieved on a sub-time interval and on the whole time interval can always be characteized using the joint distribution of the running maximum processes. However, the formula of this joint distribution is difficult to establish. In the case of Brownian motion, an explicit formula for this joint distribution based on the Fokker-Planck equation is given in [30]. Recently, the joint distribution between two running maximum both for Brownian motion and Brownian bridge process are studied (see [31] and [32], respectively).

For the Slepian processes defined in equation (1), a little is known about the partial running maximum and the correlations of different extremes of Slepian process. This paper is concerned with the maximum statistics of Slepian process \( S \). We obtain an explicit expression for the distribution function of the partial maximum
\[ m_s = \max_{0 \leq u \leq s} S(u) \]. Simple integral expressions are given for this distribution function which allow us to compute the moments generating functions of the running maximum process \( m \). We then investigate the joint distribution function between the running maximum \( m_s \) on a certain time interval \([0, s]\) and \( M_t \) on a longer time interval \([0, t]\), see Figure 1. It is interesting that this kind of probability can change into the computation of boundary non-crossing probability of Slepian process with a non-continuous piecewise linear boundary consisting of two lines in finite time interval. Finally, we compute the moments of \( m_s \) based on its distribution function.

![Figure 1](image_url)

**Figure 1.** A trajectory of Slepian process (blue line) and its running maximum (red line) on time interval \([0, 1]\). The partial maxima achieved on time interval \([0, s]\) and a longer time interval \([0, t]\) are denote by \( m \) and \( M \).

2. Results

In what follows, we let \( m_s = \max_{0 \leq u \leq s} S(u) \), \( M_t = \max_{0 \leq u \leq t} S(u) \), where \( S(u) \) is a Slepian process given in (1). We aim to compute the following two kinds of probability distribution functions (pdfs): the pdf of the partial maximum \( \mathbb{P}(m) \) and \( \mathbb{P}(M) \), the joint distribution of these two running maxima \( \mathbb{P}(m, M) \).

We start by citing the famous Bachelier-Levy formula (see e.g. [33]) which is needed for developing our main results. Concretely, suppose that \( a > 0 \), we have

\[
\mathbb{P}\{ B(t) \leq a + bt, \text{ for all } t \in [0, T]\} = \Phi(b\sqrt{T} + \frac{a}{\sqrt{T}}) - e^{-2ab}\Phi(b\sqrt{T} - \frac{a}{\sqrt{T}})
\]

where \( \Phi \) is the distribution of an \( N(0, 1) \) random variable and the above probability is 0 when \( a \leq 0 \).

**Remarks 2.1.** If \( b > 0, T = \infty \), then the probability in equation (2) is

\[
\mathbb{P}\{ B(t) \leq a + bt, \text{ for all } t \geq 0\} = 1 - e^{-2ab}.
\]

Next we present our first result for the partial maximum \( m_s \).

**Theorem 2.2.** If \( s \in [0, 1] \), then the pdf of the running maximum \( m_s \) of the Slepian process \( S \) is given by

\[
\mathbb{P}(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} \exp\left\{-\frac{x^2}{2}\right\} \Phi\left(\frac{m-x}{2\sqrt{\sigma}} + \frac{m+x}{2}\right) dx
\]

\[
- \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{m^2}{2}\right\} \int_{-\infty}^{m} \Phi\left(-\frac{m-x}{2\sqrt{\sigma}} + \frac{m+x}{2}\right) dx,
\]

where \( \sigma = \frac{s}{2-s} \).

The proof of this theorem based on a fact that conditioned on \( S(0) \), the Slepian process is equivalent in distribution with a Brownian motion, we give a proof in Section 3.
Remarks 2.3. (i) When \( s = 0 \), then the pdf of \( m_0 \) is

\[
P(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} \exp\left\{-\frac{x^2}{2}\right\} dx = \Phi(m),
\]

can also be obtained by the fact that \( m_0 = S(0) \).

(ii) When \( s = 1 \), then from Theorem 2.2, we obtain the pdf of the global maximum \( \max_{0 \leq u \leq 1} S(u) \) which we present as follow is also proved in [9],

\[
P(M) = \mathbb{P}\left\{ \max_{0 \leq u \leq 1} S(u) \leq M \right\} = \Phi^2(M) - M \phi(M) \Phi(M) - \phi^2(M),
\]

where \( \phi \) is the pdf of \( \Phi \); recall \( \Phi \) is the df of an \( N(0,1) \) random variable.

Remarks 2.4. If \( m = 0 \) in Theorem 2.2, the probability that the running maximum process \( m_s \) take non-positive values is

\[
P(0) = \int_{-\infty}^{0} \phi(x)\Phi\left(\frac{x-1}{\sqrt{s}}\right) dx - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \Phi\left(\frac{x+1}{\sqrt{s}}\right) dx
\]

\[
= \frac{1}{2\pi} \arctan \frac{2\sqrt{\pi}}{s - 1} - \frac{\sqrt{\pi}}{(s + 1)\pi},
\]

the case \( s = 1 \) is Remark 3.2 in [9].

Next, we establish the joint distribution function of \( m_s \) and \( M_t \), which is divided into two cases: \( s > 0 \) and \( s = 0 \).

Theorem 2.5. If \( 0 < s \leq t \leq 1 \), then the joint pdf of the running maxima \( m_s \) and \( M_t \) of Slepian process \( S \) is given by

\[
P(m, M) = \int_{-\infty}^{m} \int_{-\infty}^{px+\eta-y} \exp\left\{-\frac{y^2}{2s}\right\} \exp\left\{-\frac{x^2}{2}\right\} \left[ 1 - \exp\left\{-\frac{(m-x)(px+q-y)}{s}\right\} \right]
\]

\[
\times \left\{ \Phi\left(\frac{px+\eta-y}{\delta} + \frac{M+x}{2}\right) - \exp\left\{-\frac{(M+x)(px+q-y)}{2}\right\} \Phi\left(\frac{px+\eta-y}{\delta} - \frac{M+x}{2}\delta\right) \right\} dy dx,
\]

where \( p = \frac{1-s}{2} \), \( q = \frac{s+1}{2} \), \( \eta = \frac{s+1}{2} M \), \( \delta = \sqrt{T-s} \), \( s = \frac{\pi}{2-s} \), \( T = \frac{1}{2-t} \).

The proof of this theorem is presented in Section 3.

Theorem 2.6. If \( s = 0 \), then the joint pdf of the running maxima \( m_0 \) and \( M_t \) of Slepian process \( S \) is given by

\[
P(m, M) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} \exp\left\{-\frac{x^2}{2}\right\} \Phi\left(\frac{M-x}{2\sqrt{T}} + \frac{M+x}{2}\sqrt{T}\right) dx
\]

\[- \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{M^2}{2}\right\} \int_{-\infty}^{M} \Phi\left(\frac{M-x}{2\sqrt{T}} + \frac{M+x}{2}\sqrt{T}\right) dx,
\]

where \( T = \frac{1}{2-t} \).

The proof of this theorem is given in Section 3.

2.1. The moments of the partial maximum. Now we begin to compute the moments of the partial maximum \( m_s \), from Theorem 2.2 and after some computation we obtain the density function \( p(m) \) of \( m_s \), which is presented as following:

\[
p(m) = \frac{2}{1+s} \Phi(\sqrt{3m})\phi(m) + \frac{2\pi}{1+s} m^2 \Phi(\sqrt{3m})\phi(m) + \frac{m}{a} \phi(\sqrt{3m})\phi(m),
\]

where \( a = \frac{1}{2\sqrt{\pi}} \) is a constant. From equation (6) (or equation (3)), we can analysis the features of \( m_s \). In Figure 2, we plot the distribution and density of running maximum \( m_t \).
Given $s$, to compute the moments of $m_s$, the moment generating function of $m_s$ is given by
\[ M(\theta) := E[\exp\{\theta m_s\}] = \int_{-\infty}^{\infty} \exp\{\theta m\} p(m) dm, \]
the formula of the k-th moment $E[m_s^k]$ is then given by the k-th derivative of the moment generating function and setting $\theta = 0$, i.e.
\[ E[m_s^k] = \frac{d^k M(\theta)}{d\theta^k} |_{\theta=0}. \]
Using equation (6), we obtain the following:

**Lemma 2.7.** Suppose that $0 \leq s \leq 1$ is fixed, the moment generating function of $m_s$ is
\[ M(\theta) = \exp\left(\frac{\theta^2}{2}\right) G(\theta), \]
where
\[ G(\theta) = \lambda \int_{-\infty}^{\infty} \Phi(\sqrt{s}m) \phi(m-\theta) dm + \mu \int_{-\infty}^{\infty} m^2 \Phi(\sqrt{s}m) \phi(m-\theta) dm + \gamma \int_{-\infty}^{\infty} m \phi(\sqrt{s}m) \phi(m-\theta) dm, \]
and
\[ \lambda = \frac{2}{1 + \frac{s}{\lambda}}, \quad \mu = \frac{2\sqrt{s}}{1 + \frac{s}{\lambda}}, \quad \gamma = \frac{2\sqrt{s}}{1 + \frac{s}{\lambda}}, \quad \beta = \frac{s}{2 - s}. \]

We present the proof of this lemma in Section 3. Using equation (7), we can compute the moments for all order, and the first two moments are collected as the following corollary

**Corollary 2.8.** Given $0 \leq s \leq 1$, then the first and second order moments are given by
\[ p_1 := E[m_s] = \frac{4\sqrt{s}}{\sqrt{2\pi(1 + \frac{s}{\lambda})}}, \]
\[ p_2 := E[m_s^2] = \frac{2 + 3s}{1 + s}. \]

The proof of this corollary is displayed in section 3. Combining equation (8) and (9), we can obtain the variance function of \( m_s \). In Figure 3, we plot the mean and variance functions of \( m_t \).

\[ \text{Figure 3. The mean and variance functions of running maximum process } m_t. \]

3. Proofs

\textbf{Proof of Theorem 2.2}: Observing that the probability distribution function of the running maximum \( m_s \) of Slepian processes is

\[ P(m) = \mathbb{P} \left\{ m_s = \max_{0 \leq u \leq s} S(u) \leq m \right\} = \mathbb{P} \{ S(u) \leq m, \text{ for all } u \in [0, s] \}. \]

By conditioning on \( S(0) \), we represent the above probability as

\[ P(m) = \int_{-\infty}^{m} \mathbb{P} \{ S(u) \leq m, \text{ for all } u \in [0, s] | S(0) = x \} \varphi(S(0) = x) dx, \]

where \( \varphi(S(0) = x) \) is the density of \( S(0) \), i.e.

\[ \varphi(S(0) = x) = \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\}. \]

From Lemma 2.3 in [9], the process \( Y = \{ Y_t = (S(t) | S(0) = x), t \in [0, 1] \} \) is equivalent in distribution with process \( Z = \{ Z_t = (2 - t)B(\frac{u}{2 - u}) + (1 - t)x, t \in [0, 1] \} \), thus

\[ P(m) = \int_{-\infty}^{m} \mathbb{P} \left\{ (2 - u)B(\frac{u}{2 - u}) + (1 - u)x \leq m, \text{ for all } u \in [0, s] \right\} \varphi(S(0) = x) dx \]

\[ = \int_{-\infty}^{m} \mathbb{P} \left\{ B(u) \leq (\frac{m + x}{2})u + \frac{m - x}{2}, \text{ for all } u \in [0, \frac{s}{2 - s}] \right\} \varphi(S(0) = x) dx. \]

Let \( \overline{s} = s^{2} - s \), then from the famous Bachelier-Levy formula (see equation (2)) we have

\[ \mathbb{P} \left\{ B(u) \leq (\frac{m + x}{2})u + \frac{m - x}{2}, \text{ for all } u \in [0, \overline{s}] \right\} \]

\[ = \Phi(\frac{m - x}{2\sqrt{\overline{s}}} + \frac{m + x}{2\sqrt{\overline{s}}}) - \exp(\frac{-m^2 - x^2}{2}) \Phi(-\frac{m - x}{2\sqrt{\overline{s}}} + \frac{m + x}{2\sqrt{\overline{s}}}), \]

\[ (11) \]
where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$ is the cumulative distribution function of standard normal distribution. Substituting equation (11) and $\varphi(S(0) = x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$ into equation (10), we conclude that

$$
\mathbb{P}(m) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} \exp\{-\frac{x^2}{2}\} \Phi\left(\frac{m-x}{\sqrt{2}\sigma}\right) dx - \frac{1}{\sqrt{2\pi}} \exp\{-\frac{m^2}{2}\} \int_{-\infty}^{m} \Phi\left(\frac{m-x}{\sqrt{2}\sigma}\right) dx,
$$

\square

**Proof of Theorem 2.5:** For $0 \leq s \leq t \leq 1$, $m \leq M$, the joint probability distribution function between the running maxima $m_s$ and $M_t$ of Slepian processes is

$$
\mathbb{P}(m, M) = \mathbb{P}\left\{ m_s = \max_{0 \leq u \leq s} S(u) \leq m, M_t = \max_{0 \leq u \leq t} S(u) \leq M \right\} = \mathbb{P}\{ S(u) \leq m, \text{ for all } u \in [0, s] \text{ and } S(u) \leq M \text{ for all } u \in [0, t] \}.
$$

Using again the fact that the conditional process $Y = \{ Y_t = (S(t) \mid S(0) = x), \ t \in [0, 1] \}$ is equivalent in distribution with process $Z = \{ Z_t = (2-t)B\left(\frac{t}{2-t}\right) + (1-t)x, \ t \in [0, 1] \}$, we obtain

$$
\mathbb{P}(m, M) = \int_{-\infty}^{m} \mathbb{P}\{ S(u) \leq m, \text{ for all } u \in [0, s] \text{ and } S(u) \leq M \text{ for all } u \in [0, t] \} \varphi(S(0) = x) dx
$$

(12)

$$
= \int_{-\infty}^{m} \mathbb{P}\{ B(u) \leq \left(\frac{m+x}{2}\right)u + \frac{m-x}{2}, \text{ for all } u \in [0, \frac{s}{2-s}] \text{ and } B(u) \leq \left(\frac{M+x}{2}\right)u + \frac{M-x}{2}, \text{ for all } u \in [0, \frac{t}{2-t}] \} \varphi(S(0) = x) dx,
$$

where $\varphi(S(0) = x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}$ is the density of $S(0)$. Since $0 \leq s \leq t \leq 1$, $m \leq M$, then

$$
\frac{s}{2-s} \leq \frac{t}{2-t}; \quad \frac{m+x}{2}u + \frac{m-x}{2} \leq \frac{M+x}{2}u + \frac{M-x}{2}, \text{ for all } u \in \left[0, \frac{s}{2-s}\right],
$$

therefore, the last probability in equation (12) is equivalent to

$$
\mathbb{P}(m, M) = \int_{-\infty}^{m} \mathbb{P}\{ B(u) \leq \left(\frac{m+x}{2}\right)u + \frac{m-x}{2}, \text{ for all } u \in [0, \frac{s}{2-s}] \text{ and } B(u) \leq \left(\frac{M+x}{2}\right)u + \frac{M-x}{2}, \text{ for all } u \in \left[\frac{s}{2-s}, \frac{t}{2-t}\right] \} \varphi(S(0) = x) dx.
$$

Letting $a = \frac{m+x}{2}$, $b = \frac{m-x}{2}$, $c = \frac{M+x}{2}$, $d = \frac{M-x}{2}$, $\bar{\pi} = \frac{s}{2-s}$, $T = \frac{s}{2-s}$, we can simplify $\mathbb{P}(m, M)$ with these notations as

(13)

$$
\mathbb{P}(m, M) = \int_{-\infty}^{m} \mathbb{P}\{ B(u) \leq au + b, \text{ for all } u \in [0, \bar{\pi}] \text{ and } B(u) \leq cu + d, \text{ for all } u \in [\bar{\pi}, T] \} \varphi(S(0) = x) dx.
$$

In fact, denote by

$$
l(u) = \begin{cases} 
au + b, & u \in [0, \bar{\pi}] 
\end{cases}
\begin{cases} 
cu + d, & u \in [\bar{\pi}, T]
\end{cases}
$$

then equation (13) can be viewed as the boundary non-crossing probabilities of Slepian process with piecewise linear function $l(u)$, however, Theorem 3.7 in [9] can not be used here, because $l(u)$ is not continuous at $\bar{\pi}$. In order to compute $\mathbb{P}(m, M)$ with equation (13), we need compute the non-crossing probabilities of Brownian motion with non-continuous boundary $l(u)$, i.e.

(14)

$$
\mathbb{P}_B^l := \mathbb{P}\{ B(u) \leq au + b, \text{ for all } u \in [0, \bar{\pi}] \text{ and } B(u) \leq cu + d, \text{ for all } u \in [\bar{\pi}, T] \}.
$$
The trick here for computing $\mathbb{P}_B^{t}$ is using the strong Markovian property of standard Brownian motion $B(u)$ (see e.g. [34]). Concretely, by conditioning on $B(\overline{x})$ in equation (14), we get

$$\mathbb{P}_B^{t} = \int_{-\infty}^{\min(a\overline{x} + b, \overline{x} + d)} \mathbb{P}\{B(u) \leq au + b, \text{ for all } u \in [0, \overline{x}] \text{ and } B(u) \leq cu + d, \text{ for all } u \in [\overline{x}, T] \mid B(\overline{x}) = y\} \varphi(B(\overline{x}) = y) dy$$

$$(15)$$

$$= \int_{-\infty}^{a\overline{x} + b} \mathbb{P}\{B(u) \leq au + b, \text{ for all } u \in [0, \overline{x}] \mid B(\overline{x}) = y\} \times \mathbb{P}\{B(u) \leq cu + d, \text{ for all } u \in [\overline{x}, T] \mid B(\overline{x}) = y\} \varphi(B(\overline{x}) = y) dy,$$

where $\varphi(B(\overline{x}) = y) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y^2}{2}\}$ is the density of $B(\overline{x})$, and the second equality above follows from $a\overline{x} + b \leq c\overline{x} + d$ and the independent property of Brownian motion $B(u)$.

In equation (15), the first factor is

$$\mathbb{P}\{B(u) \leq au + b, \text{ for all } u \in [0, \overline{x}] \mid B(\overline{x}) = y\} = \mathbb{P}\left\{uB\left(\frac{1}{u}\right) \leq a + bu, \text{ for all } u \in \left[\frac{1}{\overline{x}}, \infty\right) \mid \overline{x}B\left(\frac{1}{\overline{x}}\right) = y\right\} = \mathbb{P}\{B(u) \leq a + bu, \text{ for all } u \in \left[\frac{1}{\overline{x}}, \infty\right) \mid \overline{x}B\left(\frac{1}{\overline{x}}\right) = y\}$$

$$= \mathbb{P}\left\{B(u) - B\left(\frac{1}{\overline{x}}\right) \leq a + bu - \frac{b}{\overline{x}}, \text{ for all } u \in \left[\frac{1}{\overline{x}}, \infty\right) \mid B\left(\frac{1}{\overline{x}}\right) = \frac{y}{\overline{x}}\right\}$$

$$= \mathbb{P}\{B(u) - B\left(\frac{1}{\overline{x}}\right) \leq a + bu - \frac{b}{\overline{x}}, \text{ for all } u \in \left[\frac{1}{\overline{x}}, \infty\right)\}$$

$$= \mathbb{P}\left\{B(u) \leq a + bu + \frac{b}{\overline{x}} - \frac{y}{\overline{x}}, \text{ for all } u \in [0, \infty) \right\},$$

the second equality above comes from the fact that $\{uB\left(\frac{1}{u}\right); u \in [\frac{1}{\overline{x}}, \infty)\}$ is equivalent in distribution to $\{B(u); u \in [0, \overline{x}]\}$, and the last two equalities above hold since the process $\{B(u) - B\left(\frac{1}{u}\right); u \in [\frac{1}{\overline{x}}, \infty)\}$ is also a standard Brownian motion, and independent with $B\left(\frac{1}{\overline{x}}\right)$. From the Bachelier-Levy formula with infinity time horizon (see Remarks 2.1) we have

$$\mathbb{P}\left\{B(u) \leq a + bu + \frac{b}{\overline{x}} - \frac{y}{\overline{x}}, \text{ for all } u \in [0, \infty) \right\} = 1 - \exp\left\{-\frac{2b(b + a\overline{x} - y)}{\overline{x}}\right\},$$

hence the probability

$$(16)$$

$$\mathbb{P}\{B(u) \leq au + b, \text{ for all } u \in [0, \overline{x}] \mid B(\overline{x}) = y\} = 1 - \exp\left\{-\frac{2b(b + a\overline{x} - y)}{\overline{x}}\right\}.$$ 

Further note that given $B(\overline{x}) = y$, the process $B(u + \overline{x}) - y$ is again a standard Brownian motion and therefore the second factor in equation (15) is

$$\mathbb{P}\{B(u) \leq cu + d, \text{ for all } u \in [\overline{x}, T] \mid B(\overline{x}) = y\} = \mathbb{P}\{B(u) \leq c(u + \overline{x}) + d - y, \text{ for all } u \in [0, T - \overline{x}]\},$$

by using the Bachelier-Levy formula again we obtain

$$(17)$$

$$\mathbb{P}\{B(u) \leq cu + d, \text{ for all } u \in [\overline{x}, T] \mid B(\overline{x}) = y\} = \Phi\left(\frac{d + c\overline{x} - y}{\sqrt{T - \overline{x}}} + c\sqrt{T - \overline{x}} - \exp\{-2c(d + c\overline{x} - y)\}\right) \Phi\left(\frac{d + c\overline{x} - y}{\sqrt{T - \overline{x}}} - c\sqrt{T - \overline{x}}\right),$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$ is the cumulative distribution function of standard normal distribution.

Letting $p = \frac{1 - \overline{x}}{\overline{x}}$, $q = \frac{\overline{x} + 1}{2} m$, $\eta = \frac{\overline{x} + 1}{2} M$, $\delta = \sqrt{T - \overline{x}}$, $\overline{x} = \frac{\delta^2}{2 - \delta}$, $T = \frac{1}{2 - \delta}$, and substituting equation (16) and
equation (17) into equation (15) we conclude that
\[
P(m, M) = \int_{-\infty}^{m} \int_{-\infty}^{\infty} \frac{1}{2\pi \sqrt{\pi}} \exp\left\{-\frac{y^2}{2}\right\} \exp\left\{-\frac{x^2}{2}\right\} \left\{1 - \exp\left\{-\frac{(m-x)(px+q-y)}{\pi}\right\}\right\} dy dx \\
\times \left\{\Phi\left(\frac{px+\eta-y}{\delta} + \frac{M+x}{2}\right) - \exp\left\{-(M+x)(px+\eta-y)\right\} \Phi\left(\frac{px+\eta-y}{\delta} - \frac{M+x}{2}\right)\right\} dy dx,
\]
completing the proof.

**Proof of Theorem 2.6:** For \( s = 0, \ m \leq M \), the joint probability distribution function between the running maxima \( m_0 \) and \( M \) of Slepian processes is
\[
P(m, M) = \mathbb{P}\left\{m_0 = S(0) \leq m, M = \max_{0 \leq u \leq t} S(u) \leq M\right\} = \mathbb{P}\{S(0) \leq m, \ S(u) \leq M \text{ for all } u \in [0, t]\}.
\]
Conditioning on \( S(0) \) and using the same method as in the proof of Theorem 2.2, we have
\[
P(m, M) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m} \exp\left\{-\frac{x^2}{2}\right\} \Phi\left(\frac{M-x}{2\sqrt{T}} + \frac{M+x}{2}\right) dx \\
- \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{M^2}{2}\right\} \int_{-\infty}^{m} \Phi\left(-\frac{M-x}{2\sqrt{T}} + \frac{M+x}{2}\right) dx,
\]
where \( T = \frac{t}{2^{-t}} \), then the claim follows.

**Proof of Lemma 2.7:** Since
\[
M(\theta) := \mathbb{E}\{\exp\{\theta m\}\},
\]
from equation (6) we obtain
\[
M(\theta) = \int_{-\infty}^{\infty} \exp\{\theta m\} \left\{\frac{2}{1 + \frac{\pi}{8}} \Phi(\sqrt{\pi} \phi) + \frac{2\pi}{1 + \frac{\pi}{8}} m^2 \Phi(\sqrt{\pi} \phi) + \frac{m}{a} \Phi(\sqrt{\pi} \phi)\right\} dm.
\]
where \( a = \frac{4\sqrt{\pi}}{2\sqrt{T}} \). Let \( \lambda = \frac{2}{1 + \frac{\pi}{8}}, \ \mu = \frac{2\pi}{1 + \frac{\pi}{8}}, \ \gamma = \frac{m}{a} \), then we have
\[
M(\theta) = \lambda \int_{-\infty}^{\infty} \exp\{\theta m\} \Phi(\sqrt{\pi} \phi) dm + \mu \int_{-\infty}^{\infty} m^2 \exp\{\theta m\} \Phi(\sqrt{\pi} \phi) dm + \gamma \int_{-\infty}^{\infty} m \exp\{\theta m\} \Phi(\sqrt{\pi} \phi) dm.
\]
Observing that for any \( \theta \in \mathbb{R} \), we have
\[
\exp\{\theta m\} \Phi(\sqrt{\pi} \phi) \phi(m) = \frac{1}{\sqrt{2\pi}} \exp\{\theta m\} \exp\left\{-\frac{m^2}{2}\right\} \Phi(\sqrt{\pi} \phi(m)) = \exp\left\{-\frac{1}{2}(m-\theta)^2\right\} \Phi(\sqrt{\pi} \phi(m)) = \exp\{\theta^2 \phi(\sqrt{\pi} \phi(m) - \theta)\}
\]
(18)
similarly, we have
\[
\exp\{\theta m\} \phi(\sqrt{\pi} \phi) \phi(m) = \frac{1}{\sqrt{2\pi}} \exp\{\theta m\} \exp\left\{-\frac{m^2}{2}\right\} \phi(\sqrt{\pi} \phi(m)) = \exp\{\theta^2 \phi(\sqrt{\pi} \phi(m) - \theta)\}
\]
(19)
Substituting equation (18) and (19) into \( M(\theta) \), and let
\[
G(\theta) = \lambda \int_{-\infty}^{\infty} \Phi(\sqrt{\pi} \phi) dm + \mu \int_{-\infty}^{\infty} m^2 \Phi(\sqrt{\pi} \phi) dm + \gamma \int_{-\infty}^{\infty} m \phi(\sqrt{\pi} \phi) dm,
\]
then the lemma established.

**Proof of Corollary 2.8:** Taking the first derivative of equation (7) and letting \( \theta = 0 \), we have
\[
p_1 = \lambda \int_{-\infty}^{\infty} m \Phi(\sqrt{\pi} \phi) dm + \mu \int_{-\infty}^{\infty} m^3 \Phi(\sqrt{\pi} \phi) dm + \gamma \int_{-\infty}^{\infty} m^2 \phi(\sqrt{\pi} \phi) dm,
\]
where $\lambda = \frac{2}{1+s}$, $\mu = \frac{2s}{1+s}$, $\gamma = \frac{2\sqrt{s}}{1+s}$. It is easily to check that

$$
\int m \phi(\sqrt{s}m) \phi(m) dm = \frac{\sqrt{s}}{2\pi \sqrt{1+s}} \Phi(m\sqrt{1+s}) - \Phi(\sqrt{s}m) \phi(m) + C_1,
$$

$$
\int m^3 \phi(\sqrt{s}m) \phi(m) dm = \frac{2T_1^2 + 3\sqrt{s}}{2\pi (1+s)^\frac{3}{2}} \Phi(m\sqrt{1+s}) - (m^2 + 2) \Phi(\sqrt{s}m) \phi(m) - \frac{\sqrt{s}m}{2\pi (1+s)^\frac{3}{2}} \phi(\sqrt{s}m) + C_2,
$$

where $C_1, C_2$ are constant. Thus, we have

$$
a_1 := \int_{-\infty}^{\infty} m \phi(\sqrt{s}m) \phi(m) dm = \frac{\sqrt{s}}{2\sqrt{2\pi(1+s)}}
$$

$$
a_2 := \int_{-\infty}^{\infty} m^3 \phi(\sqrt{s}m) \phi(m) dm = \frac{2T_1^2 + 3\sqrt{s}}{2\pi (1+s)^\frac{3}{2}}.
$$

Using the integral by part formula, we have

$$
a_3 := \int_{-\infty}^{\infty} m^2 \phi(\sqrt{s}m) \phi(m) dm
$$

$$
= \frac{1}{\sqrt{s}} \{ \int_{-\infty}^{\infty} m^3 \phi(\sqrt{s}m) \phi(m) dm - 2 \int_{-\infty}^{\infty} m \phi(\sqrt{s}m) \phi(m) dm \}
$$

$$
= \frac{1}{\sqrt{s}} (a_2 - 2a_1),
$$

Hence we obtain that

$$
p_1 = \lambda a_1 + \mu a_2 + \gamma a_3 = \frac{4\sqrt{s}}{2\pi \sqrt{1+s}}
$$

Taking second derivative of equation (7) and letting $\theta = 0$, we have

$$
p_2 = \lambda \int_{-\infty}^{\infty} m^2 \phi(\sqrt{s}m) \phi(m) dm + \mu \int_{-\infty}^{\infty} m^4 \phi(\sqrt{s}m) \phi(m) dm + \gamma \int_{-\infty}^{\infty} m^3 \phi(\sqrt{s}m) \phi(m) dm,
$$

by an analogy method we get

$$
\int_{-\infty}^{\infty} m^2 \phi(\sqrt{s}m) \phi(m) dm = \frac{1}{2},
$$

$$
\int_{-\infty}^{\infty} m^4 \phi(\sqrt{s}m) \phi(m) dm = \frac{3}{2},
$$

$$
\int_{-\infty}^{\infty} m^3 \phi(\sqrt{s}m) \phi(m) dm = 0.
$$

Hence we have

$$
p_2 := \mathbb{E}[m_s^2] = \frac{2 + 3\sqrt{s}}{1+s}
$$

establishing the proof. \qed

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Pingjin Deng, School of Finance, Nankai University, 300350, Tianjin, PR China, and Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland

E-mail address: Pingjin.Deng@unil.ch