BILINEAR SPACE-TIME ESTIMATES FOR LINEARISED KP-TYPE EQUATIONS ON THE THREE-DIMENSIONAL TORUS WITH APPLICATIONS

AXEL GRÜNROCK

Abstract. A bilinear estimate in terms of Bourgain spaces associated with a linearised Kadomtsev-Petviashvili-type equation on the three-dimensional torus is shown. As a consequence, time localized linear and bilinear space time estimates for this equation are obtained. Applications to the local and global well-posedness of dispersion generalised KP-II equations are discussed. Especially it is proved that the periodic boundary value problem for the original KP-II equation is locally well-posed for data in the anisotropic Sobolev spaces $H^s_x H^\varepsilon_y(T^3)$, if $s \geq \frac{1}{2}$ and $\varepsilon > 0$.

1. Introduction and main results

In a recent paper [7] joint with M. Panthee and J. Silva we investigated local and global well-posedness issues of the Cauchy problem for the dispersion generalised Kadomtsev-Petviashvili-II (KP-II) equation

\[
\begin{cases}
\partial_t u - |D_x|^\alpha \partial_x u + \partial_x^{-1} \Delta y u + u \partial_x u = 0 \\
u(0, x, y) = u_0(x, y)
\end{cases}
\]

on the cylinders $T \times \mathbb{R}$ and $T \times \mathbb{R}^2$, respectively. We considered data $u_0$ satisfying the mean zero condition

\[
\int_0^{2\pi} u_0(x, y) dx = 0
\]

and belonging to the anisotropic Sobolev spaces $H^s_x(T) H^\varepsilon_y(\mathbb{R}^{n-1})$, $n \in \{2, 3\}$. We could prove quite general (with respect to the dispersion parameter $\alpha$) local well-posedness results, to a large extent optimal - up to the endpoint - (with respect to the Sobolev regularity). In two dimensions and for higher dispersion ($\alpha > 3$) in three dimensions, these local results could be combined with the conservation of the $L^2$-norm to obtain global well-posedness.

A key tool to obtain these results were certain bilinear space time estimates for free solutions, similar to Strichartz estimates. A central argument to obtain the space time estimates was the following simple observation. Consider a linearised version of (1) with a more general phase function

\[
\begin{cases}
\partial_t u - i\phi(D_x, D_y) u := \partial_t u - i\phi_0(D_x) u + \partial_x^{-1} \Delta y u = 0 \\
u(0, x, y) = u_0(x, y)
\end{cases}
\]

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where \( \phi_0 \) is arbitrary at the moment, with solution \( u(x, y, t) = e^{it\phi(D_x, D_y)}u_0(x, y) \). Then we can take the partial Fourier transform \( \mathcal{F}_x \) with respect to the first spatial variable \( x \) only to obtain
\[
\mathcal{F}_x e^{it\phi(D_x, D_y)}u_0(k, y) = e^{it\phi_0(k)} e^{it\Delta_y} \mathcal{F}_x u_0(k, y).
\]

Fixing \( k \) we have a solution of the free Schrödinger equation - with rescaled time variable \( s := \frac{t}{\varepsilon} \), and multiplied by a phase factor of size one. Now the whole Schrödinger theory - Strichartz estimates, bilinear refinements thereof, local smoothing and maximal function estimates - is applicable to obtain space time estimates for the linearised KP-type equation \( (4) \).

While in two space dimensions this simple argument has to be supplemented by further estimates depending on \( \phi_0 \), we could obtain (almost) sharp estimates in the three-dimensional \( T \times \mathbb{R}^2 \)-case only by using the "Schrödinger trick" described above. In view of Bourgain’s \( L^4_T \)-estimate for free solutions of the Schrödinger equation with data defined on the two-dimensional torus \( \mathbb{T}^2 \) first part of Prop. 3.6], the question comes up naturally, if our analysis in \( [7] \) concerning \( T \times \mathbb{R}^2 \) can be extended to KP-type equations on \( \mathbb{T}^3 \), and that’s precisely the aim of the present paper.

To state our main results we have to introduce some more notation: We will consider functions \( u, v, \ldots \) of \( (x, y, t) \in T \times \mathbb{T}^2 \times \mathbb{R} \) with Fourier transform \( \hat{u}, \hat{v}, \ldots \), sometimes written as \( \mathcal{F}u, \mathcal{F}v, \ldots \), depending on the dual variables \( (\xi, \tau) := (k, \eta, \tau) \in \mathbb{Z} \times \mathbb{Z}^2 \times \mathbb{R} \). Throughout the paper we assume \( u, v, \ldots \) to fulfill the mean zero condition \( \hat{u}(0, \eta, \tau) = 0 \). For these functions we define the norms
\[
\| u \|_{X_{s, \varepsilon, b}} := \| \| k \|^s \langle \eta \rangle^b \hat{u} \|_{L^2_x, \varepsilon},
\]
where \( \langle x \rangle^2 = 1 + |x|^2 \) and \( \sigma = \tau - \phi(\xi) = \tau - \phi_0(k) + \frac{|k|^2}{2} \). Although some of our arguments do not rely on that, we will always assume \( \phi_0 \) to be odd, in order to have \( \| u \|_{X_{s, \varepsilon, b}} = \| u \|_{X_{s, \varepsilon, b}} \). For \( \varepsilon = 0 \) we abbreviate \( \| u \|_{X_{s, \varepsilon, b}} = \| u \|_{X_{s, b}} \). In these terms our central bilinear space time estimate reads as follows.

**Theorem 1.** Let \( b > \frac{1}{2}, s_{1,2} \geq 0 \) with \( s_1 + s_2 > 1 \) and \( \varepsilon_{0,1,2} \geq 0 \) with \( \varepsilon_0 + \varepsilon_1 + \varepsilon_2 > 0 \). Then the estimate
\[
\| D_y^{-\varepsilon_0}(uv) \|_{L^2_{z,t}} \lesssim \| u \|_{X_{s_1, \varepsilon_1, b}} \| v \|_{X_{s_2, \varepsilon_2, b}}
\]
and its dualized version
\[
\| uv \|_{X_{s_1, \varepsilon_1, -\varepsilon_2, b}} \lesssim \| D_y^{\varepsilon_0} u \|_{L^2_{z,t}} \| v \|_{X_{s_2, \varepsilon_2, b}}
\]
hold true.

Taking \( \varepsilon_0 = 0 \) and \( u = v \) we obtain the linear estimate
\[
\| u \|_{L^4_{z,t}} \lesssim \| u \|_{X_{s, \varepsilon, b}}.
\]
whenever \( s, b > \frac{1}{2} \) and \( \varepsilon > 0 \). The estimate \( (4) \) can be applied to time localised solutions \( e^{it\phi(D_x, D_y)}u_0 \) and \( e^{it\phi(D_x, D_y)}v_0 \) of \( (3) \) to obtain
\[
\| D_y^{-\varepsilon_0}(e^{it\phi(D_x, D_y)}u_0 e^{it\phi(D_x, D_y)}v_0) \|_{L^2_{z,t}[0,1]} \lesssim \| u_0 \|_{H_s^{1, \varepsilon}; H_\tau^{1, \varepsilon}} \| v_0 \|_{H_s^{1, \varepsilon}; H_\tau^{1, \varepsilon}},
\]
provided \( s_{1,2} \) and \( \varepsilon_{0,1,2} \) fulfill the assumptions in Theorem \([11] \). Especially for \( s > \frac{1}{2} \) and \( \varepsilon > 0 \) we have the linear estimate
\[
\| e^{it\phi(D_x, D_y)}u_0 \|_{L^4_{z,t}[0,1]} \lesssim \| u_0 \|_{H_s^{1, \varepsilon}; H_\tau^{1, \varepsilon}}.
\]
Theorem 3. Theorem 2. Let Fourier multiplier $M$ locally well-posed for data \( \phi \) on the right hand side. Here we specialize to the dispersion generalised KP-II equation (1), that is to say, the operator $M$ serves to compensate for the unavoidable loss of the $D_y$ in (1). A careful examination of the proof of Theorem 1 will give the following.

**Theorem 2.** Let \( s, b > \frac{1}{2} \) and \( \epsilon > 0 \). Then
\[
\| M^{-\epsilon}(u, v) \|_{L^1_{x,y} \mu} \lesssim \| u \|_{X_{s,b}} \| v \|_{X_{s,b}}.
\]

The proof of the above theorems will be done in section 2, while section 3 is devoted to the applications. Here we specialize to the dispersion generalised KP-II equation (1), that is to say, the operator $M$ serves to compensate for the unavoidable loss of the $D_y$ in (1). A careful examination of the proof of Theorem 1 will give the following.

**Theorem 3.** Let \( s \geq \frac{1}{2} \) and \( \epsilon > 0 \). Then, for $\alpha = 2$, the Cauchy problem (1) is locally well-posed for data $u_0 \in H_x^sH_y^s(\mathbb{T}^3)$ satisfying the mean zero condition (2).

For high dispersion, i.e. \( \alpha > 3 \), one can allow \( s < 0 \) and \( \epsilon = 0 \). In fact, by the aid of Theorem 2 we can prove:

**Theorem 4.** Let \( 3 < \alpha \leq 4 \) and \( s > \frac{3-\alpha}{2} \). Then the Cauchy problem (1) is locally well-posed for data $u_0 \in H_x^sH_y^s(\mathbb{T}^3)$ satisfying (2). If $s \geq 0$ the corresponding solutions extend globally in time by the conservation of the $L^2_{x,y}$-norm.

More precise statements of the last two theorems will be given in section 3. We conclude this introduction with several remarks commenting on our well-posedness results and their context.

1. Concerning the Cauchy problem for the KP-II equation and its dispersion generalisations on \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \) there is a rich literature, see e.g. [8], [9], [12], [13], [16], [17], [20], [22], this list is by no means exhaustive. For $\alpha = 2$ the theory has even been pushed to the critical space in a recent work of Hadac, Herr, and Koch [10]. On the other hand, for the periodic or semiperiodic problem the theory is much less developed. Besides Bourgain’s seminal paper [2] our only references here are the papers [18], [19] of Saut and Tzvetkov and our own contribution [7] joint with M. Panthee and J. Silva.

2. The results obtained here for the fully periodic case are as good as those in [7] for the $\mathbb{T} \times \mathbb{R}^2$ case and even as those obtained by Hadac [8] for $\mathbb{R}^3$, which are optimal by scaling considerations. We believe this is remarkable since apart from
nonlinear wave and Klein-Gordon equations there are only very few examples in
the literature, where the periodic problem is as well behaved as the corresponding
continuous case. (One example is of course Bourgain’s $L^2_x(T)$ result for the cubic
Schrödinger equation \[3\], but this is half a derivative away from the scaling limit.)
On the other hand there are many examples, such as KdV and mKdV, where at least
the methods applied here lead to (by $\frac{1}{4}$ derivative) weaker results for the periodic
problem. Another example is the KP-II equation itself in two space dimensions,
where in \[7\] we lost $\frac{1}{4}$ derivative when stepping from $\mathbb{R}^2$ to $T \times \mathbb{R}$. Another loss of $\frac{1}{4}$ derivative in the step from $T \times \mathbb{R}$ to $T^2$ is probable.

3. For the semilinear Schrödinger equation

$iu_t + \Delta u = |u|^p u$

on the torus, with $2 < p < 4$ in one, $1 < p < 2$ in two dimensions, one barely
misses the conserved $L^2_x$ norm and thus cannot infer global well-posedness. The
reason behind that is the loss of an $\varepsilon$ derivative in the Strichartz type estimates in
the periodic case. A corresponding derivative loss is apparent in Theorem 1 but
the usually ignored mixed part of the rather comfortable resonance relation of the
dispersion generalised KP-II equation allows (via $M^{-\varepsilon}$) to compensate for this loss,
so that for high dispersion ($3 < \alpha \leq 4$) we can obtain something global. The author
did not expect that, when starting this investigation.

4. We restrict ourselves to the most important (as we believe) values of $\alpha$. Our
arguments work as well for $\alpha \in (2, 3]$ with optimal lower bound for $s$ but possibly
with an $\varepsilon$ loss in the $y$ variable. For $\alpha > 4$ we probably lose optimality.

5. In \[21\] Takaoka and Tzvetkov proved a time localised $L^4-L^2$ Strichartz type
estimate without derivative loss for free solutions of the Schrödinger equation with
data defined on $\mathbb{R} \times T$. Inserting their arguments in our proof of Theorem 1 we
can show a variant thereof with $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0$, if the data live on $T \times \mathbb{R} \times T$.
Consequently our well-posedness results are valid in this case, too.

2. Proof of Theorem 1

The main ingredient in the proof of Bourgain’s Schrödinger estimate

$\|e^{it\Delta}u_0\|_{L^4_x(T^2)} \lesssim \|u_0\|_{H^s_x(T^2)}, \quad (\varepsilon > 0)$

is the well known estimate on the number of representations of an integer $r > 0$
as a sum of two squares: For any $\varepsilon > 0$ there exists $c_\varepsilon$ such that

$\# \{\eta \in \mathbb{Z}^2 : |\eta|^2 = r\} \leq c_{\varepsilon} r^\varepsilon.$

For \[10\], see \[11\] Theorem 338. Our proof of Theorem 1 relies on the following
variant thereof.

Lemma 1. Let $r \in \mathbb{N}$, $\delta \in \mathbb{R}^2$. Then for any $\varepsilon > 0$ there exists $c_{\varepsilon}$, independent of $r$ and $\delta$, such that

$\# \{\eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1\} \leq c_{\varepsilon} r^\varepsilon.$

Proof. In the case where $\delta \in \mathbb{Z}^2$, this follows by translation from \[10\]. So we may assume $\delta \in [0, 1]^2$, and we start by considering the special case $\delta = (\frac{1}{2}, \frac{1}{2})$. Here

$\# \{\eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1\}
= \# \{\eta \in \mathbb{Z}^2 : 4r \leq |2\eta - 2\delta|^2 < 4(r + 1)\}
= \sum_{l=4r+1}^{4r+3} \# \{\eta \in \mathbb{Z}^2 : |2\eta - 2\delta|^2 = l\}.$
But $|2\eta - 2\delta|^2 = 4|\eta|^2 - 4\langle \eta, 2\delta \rangle + 2 \equiv 2 \pmod{4}$, so the only contribution to the above sum comes from $l = 4r + 2$. Thus, by (10), for any $\varepsilon > 0$ there exists $c_\varepsilon$ such that

$$\# \{ \eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1 \} \leq c_\varepsilon (4r + 2)\varepsilon' \leq c_\varepsilon (6r)\varepsilon'. \tag{12}$$

Next we observe that for $\delta \in \{(0, \frac{1}{2}), (\frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2})\}$ we have $|2\eta - 2\delta|^2 \equiv 1 \pmod{4}$, so that the estimate (12) is valid in these cases, too. Iterating the argument, we obtain for $\delta = (\frac{m_1}{2}, \frac{m_2}{2})$ with $m \in \mathbb{N}$ and $0 \leq m_{1,2} \leq 2^m$ the estimate

$$\# \{ \eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1 \} \leq c_\varepsilon (6^{m^*}r)\varepsilon'. \tag{13}$$

Now for an arbitrary $\delta \in [0,1]^2$ we choose $\delta' = (\frac{m_1}{2}, \frac{m_2}{2})$ with $|\delta - \delta'| \sim r^{-\frac{3}{4}}$, so that

$$\{ \eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1 \} \subset \{ \eta \in \mathbb{Z}^2 : r - 1 \leq |\eta - \delta|^2 < r + 2 \}$$

and hence, by (13),

$$\# \{ \eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1 \} \leq 3c_\varepsilon (6^{m^*}r)\varepsilon'.$$

Such a $\delta$ exists for $2^{m^*} \sim r^{\frac{3}{4}}$, estimating roughly, for $6^{m^*} \leq r^{\frac{3}{2}}$. So we have the bound

$$\# \{ \eta \in \mathbb{Z}^2 : r \leq |\eta - \delta|^2 < r + 1 \} \leq 3c_\varepsilon r^{\frac{3}{2}m'}.$$

Choosing $\varepsilon' = \frac{1}{2m'}, c_\varepsilon = 3c_\varepsilon'$, we obtain (11).

\[ \square \]

**Corollary 1.** If $B$ is a disc (or square) of arbitrary position and of radius (side-length) $R$, then for any $\varepsilon > 0$ there exists $c_\varepsilon$ such that

$$\sum_{\eta \in \mathbb{Z}^2 \atop r \leq |\eta - \delta|^2 < r + 1} \chi_B(\eta) \leq c_\varepsilon R^\varepsilon. \tag{14}$$

**Proof.** If $R \geq r^{\frac{1}{2}}$, the estimate (14) follows from Lemma 1. If $R \ll r^{\frac{1}{2}}$, there are at most two lattice points on the intersection of $B$ with the circle of radius $\simeq r^{\frac{1}{2}}$ around $\delta$, by Lemma 4.4 of [4]. \[ \square \]

In the sequel we will use the following projections: For a subset $M \subset \mathbb{Z}^2$ we define $P_M$ by $\mathcal{F}P_M u(k, \eta, \tau) = \chi_M(\eta)\mathcal{F} u(k, \eta, \tau)$. Especially, if $M$ is a ball of radius $2^l$ centered at the origin, we will write $P_l$ instead of $P_M$. Furthermore we have $P_{\Delta l} = P_l - P_{l-1}$, and the $P$-notation will also be used in connection with a sequence $\{Q_{\alpha}^l\}_{\alpha \in \mathbb{Z}^2}$ of squares of sidelenath $2^l$, centered at $2^l\alpha$. Double sized squares with the same centers will be denoted by $Q_{\alpha}^l$.

**Theorem 5.** Let $s > 1$, $b > \frac{1}{2}$ and $\varepsilon > 0$. Then for a disc (or square) $B$ of arbitrary position with radius (side-length) $R$ we have

$$\| (P_B u) v \|_{L^2_{\varepsilon'}(\tau)} \lesssim R^\varepsilon \| u \|_{X_{0, b}} \| v \|_{X_{s, b}}. \tag{15}$$

**Proof.** Choose $f, g$ with $\| f \|_{L^2_s(\tau)} = \| u \|_{X_{0, b}}$ and $\| g \|_{L^2_s(\tau)} = \| v \|_{X_{s, b}}$. Then the left hand side of (15) becomes

$$\| \sum_{\eta \in \mathbb{Z}^2} \sum_{k \in \mathbb{Z}} \chi_B(\eta) f(\xi_1, \tau_1) (\sigma_1)^{-b} g(\xi_2, \tau_2) (\sigma_2)^{-b} \|_{L^2_{\varepsilon'}}. \tag{16}$$

\[ \square \]
Since \( \|uv\|_{L^2_{\tau \xi}} = \|u\|_{L^2_{\tau \xi}} \), which corresponds to \( \|\widehat{u} \widehat{v}\|_{L^2_{\tau \xi}} = \|\widehat{u} \widehat{\tilde{v}}\|_{L^2_{\tau \xi}} \) on Fourier side, where \( \tilde{v}(\xi, \tau) = \overline{v(-\xi, -\tau)} \), and since the phase function \( \phi \) is assumed to be odd, so that \( \|u\|_{X_{\phi, b}} = \|v\|_{X_{\phi, b}} \), we may assume in the estimation on \((10)\), that \( k_1 \) and \( k_2 \) have the same sign, cf. Remark 4.7 in [2]. So it’s sufficient to consider \( 0 < |k_2| \leq |k_1| < |k| \). Now, using Minkowski’s inequality we estimate \((10)\) by

\[
\| \sum_{k_1 \in \mathbb{Z}} |k_2|^{-\alpha} \int d\tau_1 \sum_{\eta_1 \in \mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1, \tau_1) \langle \sigma_1 \rangle^{-b} g(\xi_2, \tau_2) \langle \sigma_2 \rangle^{-b} \|_{L^2_{\tau \xi}} \leq \|k_2|^{-\frac{1}{2}} \int d\tau_1 \sum_{\eta_1 \in \mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1, \tau_1) \langle \sigma_1 \rangle^{-b} g(\xi_2, \tau_2) \langle \sigma_2 \rangle^{-b} \|_{L^2_{\tau \xi}} \|_{L^2_{k_1 k_2}},
\]

where Cauchy-Schwarz was applied to \( \sum_{k_1 \in \mathbb{Z}} \). Thus it is sufficient to show that

\[
(17) \quad \| \int d\tau_1 \sum_{\eta_1 \in \mathbb{Z}^2} \chi_B(\eta_1) f(\xi_1, \tau_1) \langle \sigma_1 \rangle^{-b} g(\xi_2, \tau_2) \langle \sigma_2 \rangle^{-b} \|_{L^2_{\tau \xi}} \lesssim R^c |k_2|^{\frac{1}{2}} \|f(k_1, \cdot, \cdot)\|_{L^2_{\tau \xi}} \|g(k_2, \cdot, \cdot)\|_{L^2_{\tau \xi}}.
\]

By the "Schwarz-method" developed in [14], [15] and by [5, Lemma 4.2], \((17)\) follows from

\[
(18) \quad \sum_{\eta_1 \in \mathbb{Z}^2} \chi_B(\eta_1) \langle \tau - \phi_0(k_1) - \phi_0(k_2) \rangle + \frac{|\eta_1|^2}{k_1} + \frac{|\eta_2|^2}{k_2} \lesssim R^2 |k_2|.
\]

For \( \omega := \eta_1 - \frac{k_1}{k} \eta \), we have \( \frac{|\eta_1|^2}{k_1} + \frac{|\eta_2|^2}{k_2} = \frac{|\eta|^2}{k} + \frac{k}{k_1 k_2} |\omega|^2 \), so that with \( a := \tau - \phi_0(k_1) - \phi_0(k_2) + \frac{1}{k_1 k_2} \eta_2 \), the left hand side of \((18)\) becomes

\[
\sum_{\eta_1 \in \mathbb{Z}^2} \chi_B(\eta_1) (a + \frac{k}{k_1 k_2} \eta_2) - 2b = \sum_{r \geq 0} (a + \frac{k}{k_1 k_2} r)^{-2b} \sum_{r \leq |\eta_1 - \frac{k_1}{k} \eta|^2 < r + 1} \chi_B(\eta_1). \]

By Corollary 1 the inner sum is controlled by \( c \, R^2 \), while

\[
\sum_{r \geq 0} (a + \frac{k}{k_1 k_2} r)^{-2b} \lesssim \frac{|k_1 k_2|}{|k|} \lesssim |k_2|,
\]

which proves \((18)\).

\[ \square \]

**Remark:**

The quantity, which we precisely loose in the application of Lemma 1 is

\[ r^\varepsilon \simeq |\eta_1 - \frac{k_1}{k} \eta|^{2\varepsilon} \lesssim \langle k \eta_1 - k_1 \eta \rangle^{2\varepsilon}, \]

which is the symbol of the Fourier multiplier \( M^{2\varepsilon} \). Rereading carefully the calculation in the previous proof, we see that - instead of \((17)\) - the following estimate holds true as well.

\[
(19) \quad \| \int d\tau_1 \sum_{\eta_1 \in \mathbb{Z}^2} \langle k \eta_1 - k_1 \eta \rangle^{-\varepsilon} f(\xi_1, \tau_1) \langle \sigma_1 \rangle^{-b} g(\xi_2, \tau_2) \langle \sigma_2 \rangle^{-b} \|_{L^2_{\tau \xi}} \lesssim \frac{|k_1 k_2|^{\frac{1}{2}}}{|k|^\varepsilon} \|f(k_1, \cdot, \cdot)\|_{L^2_{\tau \xi}} \|g(k_2, \cdot, \cdot)\|_{L^2_{\tau \xi}}.
\]

(Introducing the \( M^{-\varepsilon} \) we cannot justify the sign assumption on \( k, k_1, \) any more.) Multiplying by \( |k|^\varepsilon \) and summing up over \( k_1 \) using Cauchy-Schwarz we obtain
(20) \[ \|F_xD_y^\pm M^{-\varepsilon}(u,v)\|_{L^\infty_tL^2_y} \lesssim \|u\|_{X_{\frac{1}{2},b}}\|v\|_{X_{\frac{1}{2},b}}, \]

from which (21) follows by a further application of the Cauchy-Schwarz inequality. So Theorem 2 is proved.

Proof of Theorem 1. Since in Corollary 1 the position of the disc is arbitrary, we may replace \( \chi_B(\eta_1) \) by \( \chi_B(\eta_2) \) in the proof of Theorem 3, which gives

(21) \[ \|u(P_Bv)\|_{L^2_y} \lesssim R^\varepsilon\|u\|_{X_{0,b}}\|v\|_{X_{\varepsilon,b}}. \]

Now we have symmetry between \( u \) and \( v \), so that we may interpolate bilinearly to obtain

(22) \[ \|u(P_Bu)v\|_{L^2_y} \lesssim R^\varepsilon\|u\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}} \]

for \( s_{1,2} \geq 0 \) with \( s_1 + s_2 > 1 \). Decomposing dyadically we obtain with \( 0 < \varepsilon' < \varepsilon \)

\[ \|uv\|_{L^2_y} \leq \sum_{l \geq 0} \|P_l(u)v\|_{L^2_y} \lesssim \sum_{l \geq 0} 2^{l\varepsilon'}\|P_lu\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}} \lesssim \sum_{l \geq 0} 2^{l(\varepsilon' - \varepsilon)}\|u\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}}. \]

Exchanging \( u \) and \( v \) again we have shown for \( s_{1,2} \geq 0 \) with \( s_1 + s_2 > 1 \) and \( \varepsilon_{1,2} \geq 0 \) with \( \varepsilon_1 + \varepsilon_2 > 0 \) that

(23) \[ \|uv\|_{L^2_y} \lesssim \|u\|_{X_{s_1,b}}\|v\|_{X_{s_2,b}}, \]

which is the \( \varepsilon_0 = 0 \) part of (4) in Theorem 1. To see the \( \varepsilon_0 > 0 \) part, we decompose

\[ \|D_y^{-\varepsilon_0}(uv)\|_{L^2_y} \leq \sum_{l \geq 0} 2^{-l\varepsilon_0}\|P_l(uv)\|_{L^2_y}, \]

where for fixed \( l \)

\[ \|P_l(uv)\|_{L^2_y}^2 = \sum_{\alpha,\beta \in \mathbb{Z}^2} \langle P_l(uv), P_l(uv) \rangle_{L^2_y}. \]

Now for \( \eta_1 \in Q^l_{\alpha} \), \( |\eta| \leq 2^l \) we have \( \eta_2 = \eta - \eta_1 \in \tilde{Q}^l_{-\alpha} \), so that the latter can be estimated by

\[ \sum_{\alpha,\beta \in \mathbb{Z}^2} \langle (P_{Q^l_{\alpha}}u)(P_{Q^l_{-\alpha}}v), (P_{Q^l_{\beta}}u)(P_{Q^l_{-\beta}}v) \rangle_{L^2_y} \]

\[ \leq \sum_{\alpha,\beta \in \mathbb{Z}^2} \langle (P_{Q^l_{\alpha}}u)(P_{Q^l_{-\alpha}}v), (P_{Q^l_{\beta}}u)(P_{Q^l_{-\beta}}v) \rangle_{L^2_y} \]

\[ \leq \sum_{\alpha,\beta \in \mathbb{Z}^2} \|P_{Q^l_{\alpha}}u\|_{L^2_y}^2\|P_{Q^l_{-\alpha}}v\|_{L^2_y}^2\|P_{Q^l_{\beta}}u\|_{L^2_y}^2\|P_{Q^l_{-\beta}}v\|_{L^2_y}^2. \]

Using (22) and the almost orthogonality of the sequence \( \{P_{Q^l_{\alpha}}v\}_{\alpha \in \mathbb{Z}^2} \) we estimate the latter by

\[ 2^{2l\varepsilon} \sum_{\alpha,\beta \in \mathbb{Z}^2} \|P_{Q^l_{\alpha}}u\|^2_{X_{s_1,b}}\|P_{Q^l_{-\alpha}}v\|^2_{X_{s_2,b}} \lesssim 2^{2l\varepsilon}\|u\|^2_{X_{s_1,b}}\|v\|^2_{X_{s_2,b}}. \]
Choosing $\varepsilon < \varepsilon_0$ the sum over $l$ remains finite and we arrive at
\[
\|D_y^{-\varepsilon_0}(uv)\|_{L^2_y} \lesssim \|u\|_{X_{s,1,b}} \|v\|_{X_{s,2,b}}.
\]
Finally we remark that (4) and (5) are equivalent by duality. $\Box$

3. Applications to KP-II type equations

Here the phase function is specified as $\phi_0(k) = |k|^\alpha k$, $\alpha \geq 2$, so that the mixed weight becomes $\sigma = \tau - |k|^\alpha k + \frac{|y|^2}{k^2}$. To prove the well-posedness results in Theorem 3 and 4 we need some more norms and function spaces, respectively. In both cases we use the spaces $X_{s,\varepsilon,b;\beta}$ with additional weights, introduced in [2] and defined by
\[
\|f\|_{X_{s,\varepsilon,b;\beta}} := \left\| \left( \langle k \rangle^\varepsilon \langle \eta \rangle^\beta 1 + \frac{\langle \sigma \rangle}{\langle k \rangle^{\alpha+1}} \right)^\beta \hat{f} \right\|_{L^2_y}.
\]
We will always have $\beta \geq 0$, so that
\[
\|f\|_{X_{s,b}} \lesssim \|f\|_{X_{s,\varepsilon,b;\beta}}.
\]
Observe that
\[
\|f\|_{X_{s,b}} \sim \|f\|_{X_{s,\varepsilon,b;\beta}}.
\]
if $\langle \sigma \rangle \leq \langle k \rangle^{\alpha+1}$.

The case $\alpha = 2$ corresponding to the original KP-II equation becomes a limiting case in our considerations, where we have to choose the parameter $b = \frac{3}{2}$. Thus we also need the auxiliary norms
\[
\|f\|_{Y_{s,\varepsilon,b;\beta}} := \left\| \left( \langle k \rangle^\varepsilon \langle \eta \rangle^\beta 1 + \frac{\langle \sigma \rangle}{\langle k \rangle^{\alpha+1}} \right)^\beta \hat{f} \right\|_{L^2_y(L^2_y)},
\]
cf. [5], and
\[
\|f\|_{Z_{s,\varepsilon,b;\beta}} := \|f\|_{Y_{s,\varepsilon,b;\beta}} + \|f\|_{X_{s,\varepsilon,-\varepsilon,b;\beta}}.
\]
As before, for $\varepsilon = 0$ we will write $X_{s,b;\beta}$ instead of $X_{s,\varepsilon,b;\beta}$, and if the exponent $\beta$ of the additional weight is zero, we use $X_{s,\varepsilon,b}$ as abbreviation for $X_{s,\varepsilon,b;\beta}$. Similar for the $Y$- and $Z$-norms. In these terms the crucial bilinear estimate leading to Theorem 3 is the following.

**Lemma 2.** Let $\alpha = 2$, $s \geq \frac{1}{2}$ and $\varepsilon > 0$. Then there exists $\gamma > 0$, such that for all $u, v$ supported in $[-T,T] \times T^3$ the estimate
\[
\|\partial_x(uv)\|_{Z_{s,\varepsilon,-\varepsilon/2}} \lesssim T^{\gamma} \|u\|_{X_{s,\varepsilon,-\varepsilon/2}} \|v\|_{X_{s,\varepsilon,-\varepsilon/2}}
\]
holds true.

Correspondingly for Theorem 4 we have

**Lemma 3.** Let $3 < \alpha \leq 4$. Then, for $s > \frac{3-\alpha}{2}$ there exist $b' > -\frac{1}{2}$ and $\beta \in [0, -b']$, such that for all $b > \frac{1}{2}$
\[
\|D_x^{s+1+\varepsilon} M^{-\varepsilon}(u,v)\|_{X_{0,b',\beta}} \lesssim \|u\|_{X_{s,b,\beta}} \|v\|_{X_{s,b,\beta}},
\]
whenever $\varepsilon > 0$ is sufficiently small, and
\[
\|\partial_x(uv)\|_{X_{s,b',\beta}} \lesssim \|u\|_{X_{s,b,\beta}} \|v\|_{X_{s,b,\beta}}.
\]
In the proof of both Lemmas above the resonance relation for the KP-II-type equation with quadratic nonlinearity plays an important role. We have

\begin{equation}
\sigma_1 + \sigma_2 - \sigma = r(k, k_1) + \frac{|k\eta_1 - k_1\eta|^2}{kk_1k_2},
\end{equation}

where

\[|r(k, k_1)| = |k|^\alpha k - |k_1|^\alpha k_1 - |k_2|^\alpha k_2| \sim |k_{\text{max}}|^\alpha |k_{\text{min}}|,\]

see [9]. Both terms on the right of (32) have the same sign, so that

\begin{equation}
\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |k_{\text{min}}||k_{\text{max}}|^\alpha + \frac{|k\eta_1 - k_1\eta|^2}{kk_1k_2}.
\end{equation}

The proof of Lemma 2 is almost the same as that of Lemma 4 in [7], it is repeated here - with minor modifications - for the sake of completeness. We need a variant of Theorem 1 with \( b < \frac{1}{2} \). To obtain this, we first observe that, if \( s_1, s_2 \geq 0 \) with \( s_1 + s_2 > \frac{1}{2}, \varepsilon_{0,1,2} \geq 0 \) with \( \varepsilon_0 + \varepsilon_1 + \varepsilon_2 > 1, 1 \leq p \leq 2 \), and \( b > \frac{1}{1-p} \), then

\begin{equation}
\mathcal{F} D_y^{-\varepsilon_0}(uv) \|_{L_t^p L_x^s} \lesssim \|u\|_{X_{s_1, r, b}} \|v\|_{X_{s_2, r, b}}.
\end{equation}

This follows from Sobolev type embeddings and applications of Young’s inequality. Dualizing the \( p = 2 \) part of (34) we obtain

\begin{equation}
\|uv\|_{X_{s_1, s_2, -b}} \lesssim \|D_y^{\varepsilon_1} u\|_{L_t^2 L_x^s} \|v\|_{X_{s_2, r, b}}.
\end{equation}

Now bilinear interpolation with Theorem 1 gives the following.

**Corollary 2.** Let \( s_1, s_2 \geq 0 \) with \( s_1 + s_2 = 1 \) and \( \varepsilon_{1,2} \geq 0 \) with \( \varepsilon_1 + \varepsilon_2 > 0 \), then there exist \( b < \frac{1}{s} \) and \( p < 2 \) such that

\begin{equation}
\|\mathcal{F}(uv)\|_{L_t^p L_x^s} + \|uv\|_{L_t^p L_x^s} \lesssim \|u\|_{X_{s_1, r, b}} \|v\|_{X_{s_2, r, b}}.
\end{equation}

and

\begin{equation}
\|uv\|_{X_{s_1, -b}} \lesssim \|D_y^{\varepsilon_1} u\|_{L_t^2 L_x^s} \|v\|_{X_{s_2, r, b}}
\end{equation}

hold true.

The purpose of the \( p < 2 \) part in the above Corollary is to deal with the \( Y \)-contribution to the \( Z \)-norm in Lemma 2. Its application will usually follow on an embedding

\[\|\langle \sigma \rangle^{-\frac{1}{2}} \mathcal{F}\|_{L_t^2 L_x^s} \lesssim \|\mathcal{F}\|_{L_t^2 L_x^s},\]

where \( p < 2 \) but arbitrarily closed to 2. Now we’re prepared to establish Lemma 2.

**Proof of Lemma 2.** Without loss of generality we may assume that \( s = \frac{1}{2} \). The proof consists of the following case by case discussion.

**Case a:** \( \langle k \rangle^3 \leq \langle \sigma \rangle \). First we observe that

\begin{equation}
\|\partial_x (uv)\|_{Z_{s, r, \frac{1}{2}}} \lesssim \|D_x^{s+1}(D_y^u u \cdot v)\|_{Z_{0, 0, \frac{1}{2}}} + \|D_x^{s+1}(u \cdot D_y^v v)\|_{Z_{0, 0, \frac{1}{2}}}.
\end{equation}

The first contribution to (38) equals

\begin{align*}
\|\mathcal{F}(D_y^u u \cdot v)\|_{L_t^p} + \|\langle \sigma \rangle^{-\frac{1}{2}} \mathcal{F}(D_y^u u \cdot v)\|_{L_t^p} & \lesssim \|\mathcal{F}(D_y^u u \cdot v)\|_{L_t^p \cap L_t^2 L_x^s} \lesssim \|u\|_{X_{s, r, b}} \|v\|_{X_{s, r, b}}
\end{align*}

by (30), for some \( b < \frac{1}{2} \). Using the fact [4] that under the support assumption on \( u \) the inequality

\begin{equation}
\|u\|_{X_{s, r, b}} \lesssim \mathcal{F}^{-b} \|u\|_{X_{s, r, b}}
\end{equation}

\footnote{for a proof see e. g. Lemma 1.10 in [6]}. 

\( \leq \)
holds, whenever $-\frac{1}{2} < b < \frac{1}{2}$, this can, for some $\gamma > 0$, be further estimated by $T^\gamma \|u\|_{X_{s,\frac{1}{2}+\frac{1}{2}}} \|v\|_{X_{s,\frac{1}{2}+\frac{1}{2}}}$ as desired. The second contribution to (38) can be treated in precisely the same manner.

**Case b:** $\langle k \rangle^3 \geq \langle \sigma \rangle$. Here the additional weight on the left is of size one, so that we have to show

$$\|\partial_x (uv)\|_{Z_{s,\delta}} \lesssim T^\gamma \|u\|_{X_{s,\frac{1}{2}+\frac{1}{2}}} \|v\|_{X_{s,\frac{1}{2}+\frac{1}{2}}}.$$  

**Subcase b.a:** $\sigma$ maximal. Exploiting the resonance relation (33), we see that the contribution from this subcase is bounded by

$$\|FD_x D_y (D_x^{-\frac{1}{2}} u \cdot D_x^{-\frac{1}{2}} v)\|_{L^2_x L^2_y} \lesssim \|FD_x D_y (D_x^{-\frac{1}{2}} u \cdot D_x^{-\frac{1}{2}} v)\|_{L^2_x L^2_y} + \ldots,$$

where $p < 2$. The dots stand for the other possible distributions of derivatives on the two factors, in the same norms, which - by (36) of Corollary 2 - can all be estimated by $c \|u\|_{X_{s,\epsilon,b}} \|v\|_{X_{s,\epsilon,b}}$ for some $b < \frac{1}{2}$. The latter is then further treated as in case a.

**Subcase b.b:** $\sigma_1$ maximal. Here we start with the observation that by Cauchy-Schwarz and (39), for every $b' > -\frac{1}{2}$ there is a $\gamma > 0$ such that

$$\|\partial_x (uv)\|_{Z_{s,\delta}} \lesssim T^\gamma \|D_x^{s+1} (uv)\|_{X_{0,b'}}.$$

With the notation $\Lambda^b = F^{-1}(\sigma)^b F$ we obtain from the resonance relation that

$$\|D_x^{s+1} (uv)\|_{X_{0,b'}} \lesssim \|D_x (D_x^{-\frac{1}{2}} \Lambda^x u \cdot D_x^{-\frac{1}{2}} v)\|_{X_{0,b'}}$$

$$\lesssim \|(D_x^2 D_y \Lambda^x u) (D_x^{-\frac{1}{2}} v)\|_{X_{0,b'}} + \|(D_x^2 \Lambda^x u) (D_x^{-\frac{1}{2}} D_y v)\|_{X_{0,b'}}$$

$$+ \|(D_x^{-\frac{1}{2}} D_y \Lambda^x u) (D_x^2 v)\|_{X_{0,b'}} + \|(D_x^{-\frac{1}{2}} \Lambda^x u) (D_x^2 D_y v)\|_{X_{0,b'}}.$$

Using (37), the first two contributions can be estimated by $c \|u\|_{X_{s,\epsilon,\frac{1}{2}}} \|v\|_{X_{s,\epsilon,b}}$ as desired. The third and fourth term only appear in the frequency range $k \ll |k_1| \sim |k_2|$, where the additional weight in the $\|u\|_{X_{s,\epsilon,\frac{1}{2}}} \|v\|_{X_{s,\epsilon,\frac{1}{2}}}$-norm on the right becomes $\frac{|k_2|}{|k_1|}$, thus shifting a whole derivative from the high frequency factor $v$ to the low frequency factor $u$. So, using (37) again, these contributions can be estimated by

$$c \|u\|_{X_{s,\epsilon,\frac{1}{2}}} \|v\|_{X_{s,\epsilon,b}} \lesssim \|u\|_{X_{s,\epsilon,\frac{1}{2}}} \|v\|_{X_{s,\epsilon,b}}.$$

□

Now we turn to the proof of Lemma 3, where the restrictions to the $b$-parameters can be relaxed slightly, so that the auxiliary $Y^*$- and $Z^*$-norms are not needed. We use again the $\Lambda$-notation, i.e. $\Lambda^b = F^{-1}(\sigma)^b F$.

**Proof of Lemma 3** First we show how (39) implies (40). By the resonance relation (43) we have

$$|k_1 \eta - k_2 \eta|^2 \lesssim |kk_1 k_2 (\langle \sigma \rangle + \langle \sigma_1 \rangle + \langle \sigma_2 \rangle) | \leq |k_1 k_2 \langle \sigma \rangle \langle \sigma_1 \rangle \langle \sigma_2 \rangle|,$$

so that (40) is reduced to

$$\|D_x^{s+1} M^{-\epsilon} (u,v)\|_{X_{0,b'}} \lesssim \|u\|_{X_{s,\frac{1}{2}+\frac{1}{2}}} \|v\|_{X_{s,\frac{1}{2}+\frac{1}{2}}}.$$

Relabelling appropriately and choosing $\varepsilon$ sufficiently small, we see that (40) follows from (39). To prove the latter, we may assume $s \leq 0$. Next we choose $\varepsilon$ small and $b'$ close to $-\frac{1}{2}$ so that

$$s > 2 + (\alpha + 1)b' + 3\varepsilon.$$
and $\beta := \frac{s - b'}{\alpha} \in [0, -b']$. Now the proof consists again of a case by case discussion.

**Case a:** $\langle k \rangle^{n+1} \lesssim \langle \sigma \rangle$. Here it is sufficient to show

$$(41) \quad \|D_x^{s+1+\varepsilon-\alpha\beta-\beta} M^{-\varepsilon}(u, v)\|_{X_{0, \varepsilon'+\beta}} \lesssim \|u\|_{X_{s, b}} \|v\|_{X_{s, b}}$$

**Subcase a.a:** $|k| \ll |k_1| \sim |k_2|$. 

**Subsubcase triple a:** $\langle \sigma \rangle \geq \langle \sigma_{1,2} \rangle$. Here we use the resonance relation (33) to see that the left hand side of (41) is bounded by

$$\|D_x^{s+1+\varepsilon-\alpha\beta+b'} M^{-\varepsilon}(D_x^{(\alpha+1)b} u, D_x^{(\alpha+1)b} v)\|_{L^2_{xyt}} \lesssim \|M^{-\varepsilon}(D_x^{(\alpha+1)b} u, D_x^{(\alpha+1)b} v)\|_{L^2_{xyt}},$$

where we have used the assumption on the frequency sizes in this subcase. Observe that our choice of $\beta$ implies $s + 1 + \varepsilon - \alpha\beta + b' = 1 + \varepsilon + 2b' \geq 0$. Now the bilinear estimate (39) is applied to obtain the upper bound

$$\|D_x^{s+2+3\varepsilon+(\alpha+1)b'} u\|_{X_{0,b}} \|D_x^{s+2+3\varepsilon+(\alpha+1)b'} v\|_{X_{0,b}} \lesssim \|u\|_{X_{s,b}} \|v\|_{X_{s,b}},$$

where in the last step we have used (40).

**Subsubcase a.b:** $\langle \sigma \rangle \geq \langle \sigma_{1,2} \rangle$. Here the resonance relation (33) gives that the left hand side of (41) is bounded by

$$\|D_x^{s+1+\varepsilon-\alpha\beta+b'} M^{-\varepsilon}(D_x^{\alpha} u, D_x^{(\alpha+1)b'} v)\|_{X_{0,-b}} \lesssim \|D_x^{\frac{1}{1-\varepsilon}} M^{-\varepsilon}(D_x^{\alpha} u, D_x^{\frac{1}{1-\varepsilon}+2\varepsilon+(\alpha+1)b'} v)\|_{X_{0,-b}}.$$ 

Now the dual version of estimate (39), that is

$$(42) \quad \|M^{-\varepsilon}(u, v)\|_{X_{\frac{s+1}{1-\varepsilon}, -\frac{1}{1-\varepsilon}}} \lesssim \|u\|_{L^2_{xyt}} \|v\|_{X_{\frac{s+1}{1-\varepsilon}, \frac{1}{1-\varepsilon}}}$$

is applied, which gives, together with the assumption (40), that the latter is bounded by $c\|u\|_{X_{s,b}} \|v\|_{X_{s,b}}$. This completes the discussion of subcase a.a. Concerning subcase a.b, where $|k| \gtrsim |k_{1,2}|$, we solely remark that it can be reduced to the estimation in subsubcase triple a.

**Case b:** $\langle k \rangle^{n+1} \gtrsim \langle \sigma \rangle$. Here the additional weight in the norm on the left of (30) is of size one, so our task is to show

$$(43) \quad \|D_x^{s+1+\varepsilon} M^{-\varepsilon}(u, v)\|_{X_{0,\varepsilon'}} \lesssim \|u\|_{X_{s,b}} \|v\|_{X_{s,b}}.$$

**Subcase b.a:** $\langle \sigma \rangle \geq \langle \sigma_{1,2} \rangle$. Here we may assume by symmetry that $|k_1| \geq |k_2|$. We apply (33) and (30) to see that for $\delta \geq 0$ the left hand side of (42) is controlled by

$$\|M^{-\varepsilon}(D_x^{s+1+\varepsilon+\alpha b'+\delta} u, D_x^{\delta} v)\|_{L^2_{xyt}} \lesssim \|D_x^{\frac{\delta}{2}+2\varepsilon+\alpha b'+\delta} u\|_{X_{s,b}} \|D_x^{\frac{\delta}{2}+\varepsilon-\delta} v\|_{X_{0,b}}.$$ 

The latter is bounded by $c\|u\|_{X_{s,b}} \|v\|_{X_{s,b}}$, provided $\frac{\delta}{2} + 2\varepsilon + \alpha b' + \delta \leq 0$ and $\frac{\delta}{2} + \varepsilon - \delta \leq s$, which can be fulfilled by a proper choice of $\delta \geq 0$, since (40) holds.

**Subcase b.b:** $\langle \sigma_1 \rangle \geq \langle \sigma \rangle, \langle \sigma_{1,2} \rangle$. 

**Subsubcase b.b.a:** \(|k_1| \gtrsim |k_2|\). With \(\delta \geq 0\) as in subcase b.a the contribution here is bounded by

\[
\|D_x^{\frac{d}{4} - \varepsilon} M^{-\varepsilon} (D_x^{s + \frac{d}{4} + 2\varepsilon + \alpha b + \delta} A^n u, D_x^{b'} v)\|_{X_{0,b}} \lesssim \|D_x^{\frac{d}{4} + 2\varepsilon + \alpha b + \delta} u\|_{X_{s,\beta}} \lesssim \|u\|_{X_{s,\beta}} \|v\|_{X_{s,\beta}},
\]

where (43) and the dual version (42) of Theorem 4 were used again. Finally we turn to the

**Subsubcase triple b,** where \(|k_1| \ll |k| \sim |k_2|\). Here the additional weight in \(\|u\|_{X_{s,b,\beta}}\) on the right of (43) behaves like

\[
\|D_x^{s+1+\varepsilon} M^{-\varepsilon} (u, D_x^{-\alpha b} v)\|_{X_{s,\beta}} \lesssim \|u\|_{X_{s-\alpha,\beta}} \|v\|_{X_{s,\beta}}.
\]

Now by (43) the left hand side of (44) can be controlled by

\[
\|D_x^{\frac{d}{4} - \varepsilon} (D_x^{b'} A^n u, D_x^{s + \frac{d}{4} + 2\varepsilon + \alpha (b' - \beta)} v)\|_{X_{0,b}} \lesssim \|D_x^{b'} u\|_{X_{s,\beta}} \|D_x^{2s + 3\varepsilon + (\alpha + 1)b'} v\|_{X_{0,b}}
\]

by (42). Since \(b' = s - \alpha \beta\) the first factor equals \(\|u\|_{X_{s-\alpha,\beta,\beta}}\), while by (40) the second is dominated by \(\|v\|_{X_{s,\beta}}\). This proves (44). \(\square\)

Finally we recall the definition of the Fourier restriction norm spaces from [2]. For a time slab \(I = (-\delta, \delta) \times T^3\) they are given by

\[
X_{s,\varepsilon, b, \beta}^I := \{u|_I : u \in X_{s,\varepsilon, b, \beta}\}
\]

with norm

\[
\|u\|_{X_{s,\varepsilon, b, \beta}^I} := \inf \{\|\tilde{u}\|_{X_{s,\varepsilon, b, \beta}} : \tilde{u} \in X_{s,\varepsilon, b, \beta}, \tilde{u}|_I = u\}.
\]

Now our well-posedness results read as follows.

**Theorem 6** (precise version of Theorem 3). Let \(s \geq \frac{1}{2}\) and \(\varepsilon > 0\). Then for \(u_0 \in H_x^s H_y^\varepsilon(T^3)\) satisfying (2) there exist \(\delta = \delta(\|u_0\|_{H_x^s H_y^\varepsilon}) > 0\) and a unique solution \(u \in X_{s,\varepsilon, b, \beta}^\delta\) of the Cauchy problem (11) with \(\alpha = 2\). This solution is persistent and the mapping \(u_0 \mapsto u, H_x^s H_y^\varepsilon(T^3) \to X_{s,\varepsilon, b, \beta}^\delta\) is locally Lipschitz for any \(\delta_0 \in (0, \delta)\).

**Theorem 7** (precise version of Theorem 4). Let \(3 < \alpha \leq 4, s \geq s' > \frac{3-\alpha}{2}\) and \(\varepsilon \geq 0\). Then for \(u_0 \in H_x^s H_y^\varepsilon(T^3)\) satisfying (2) there exist \(b > \frac{1}{2}, \beta > 0\), \(\delta = \delta(\|u_0\|_{H_x^s H_y^\varepsilon}) > 0\) and a unique solution \(u \in X_{s,\varepsilon, b, \beta} \subset C^0((-\delta, \delta), H_x^s H_y^\varepsilon(T^3))\). This solution depends continuously on the data, and extends globally in time, if \(s \geq 0\) and \(\varepsilon = 0\).

With the estimates from Lemma 2 and 3 at our disposal the proof of these theorems is done by the contraction mapping principle, cf. [2], [3], [14], [15]. The reader is also referred to section 1.3 of [8], where the related arguments are gathered in a general local well-posedness theorem.
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Axel Grünrock: Rheinische Friedrich-Wilhelms-Universität Bonn, Mathematisches Institut, Beringstrasse 1, 53115 Bonn, Germany.
E-mail address: gruenroc@math.uni-bonn.de