Wear of a high pressure layered pipe by a rough rigid bush

K E Kazakov\(^1\) and S P Kurdina\(^2\)
\(^1\)Ishlinsky Institute for Problems in Mechanics of the Russian Academy of Sciences, Moscow, Russia
\(^2\)Bauman Moscow State Technical University, Moscow, Russia

E-mail: kazakov-ke@yandex.ru, kurdinasp@bmstu.ru

Abstract. The article describes the solution to the problem of linear wear of a high pressure elastic pipe with an elastic coating by rigid bush worn on the pipe so that there is no gap between them. It is assumed that both the thickness of the coating in undeformed state and the shape of the inner surface of the bush are variable and depend on the coordinate along the axis of pipe. In the case when indicated characteristics are described by complex rapidly changing functions, standard approaches to solving such a problem turn out to be ineffective. The article describes an approach based on the use of special representations and basic functions that take into account the presence of complex characteristics. In the expression for contact pressures, which is the result of solving the problem and allows us to further evaluate the resource and durability of such a connection, the functions that describe the surfaces shapes are highlighted in explicit form. This allows us to perform exact calculations in cases where the forms are described by complex functions.

1. Introduction
In real constructions, pipes loaded with internal pressure are often used. External conditions determine the use of multilayer pipes, in which different layers serve for different purposes. Such pipes can be fastened with bushings, however, during operation due to vibration or torsion of the bush around the pipe, wear occurs, leading to a thinning of the outer layer. To assess the durability of such a connection, it is necessary to monitor the evolution of the stress-strain state of the pipe and evaluate its wear.

2. Mathematical model of contact-wear problem
Consider elastic pipe with inner radius \(r_{in}\) and the thickness \(h\) in covered by thin elastic layer of variable thickness \(h_{out0}(z)\) and inner radius \(r_{out} = r_{in} + h_{in}\). We put rigid bush on such a tube. Inner radius of the bush is variable, depend on the coordinate \(z\) and less or equal to outer radius of pipe with coating in undeformed state, i.e. \(g(z) \leq r_{out} + h_{out0}(z)\). We assume that thickness \(h_{out0}(z)\) is much smaller than bush length \(2a\), inner radius of main pipe \(r_{in}\), and thickness of main pipe \(h_{in}\). The coating is considered soft compared to the lower layer (its rigidity not exceed the rigidity of lower layer \([1]\)). It is also assumed that there is a smooth contact between layers and between bush outer layer along \(z\) axis. At the time \(t_0\) bush start to vibrate or rotate with angular velocity \(\omega\). Due to friction between the coating and the bush coating begins to wear out. At the same time we begin to load the pipe with internal pressure \(P_{in}(t)\). The scheme for such a problem for the case of rotation is presented in the figure 1. It is obvious that the stress-strain state of the layered pipe and, in particular, the contact
pressures under the bush will change over time. Further we will assume that the contact-wear region is constant and bounded by the boundaries $-a$ and $a$ (coordinates of left and right boundaries of the bush). We will construct solution only for the case when wear is less then coating thickness.

The movement of the outer boundary of the coating under the bush consists of displacement due to load and displacement due to wear.

A number of experiments show that the wear velocity linearly depends on the relative velocity of the contacting surfaces and on the normal load and is inversely proportional to the hardness of the material [2–6]:

$$u_w(z,t) = -\frac{k_w V q(z,t)}{H(z)},$$

where $H(z)$ is coating hardness, $q(z,t)$ is contact pressure, $k_w$ is experimentally determined constant.

The velocity $V$ can be estimated with fretting-wear of surfaces, when the angular velocity is absent, however, the coating wears out due to vibrations of one body relative to another. In the case of rotation $V = \omega (r_{out} + \tilde{h}_{out}) \approx \omega r_{out}$, where $\tilde{h}_{out} = h_{out}(z_0)$, $z_0$ is any point in $[-a,a]$ (note that $h_{out}(z) \ll r_{out}$ for all $z \in [-a,a]$ and hence $\tilde{h}_{out} \leq \max_{z \in [-a,a]} h_{out0}(z) \ll r_{out}$). Then the vertical displacement of outer surface due to wear can be determined by integrating of wear velocity over time from start moment until current time

$$u_w(z,t) = \int_{t_n}^{t} u_w(z,\tau) d\tau = -\frac{k_w V}{H(z)} \int_{t_n}^{t} q(z,\tau) d\tau. \quad (1)$$

Based on the results of [7–13], it can be obtained that the displacement of the outer face of the coating, due to the load on the coating outer boundary and pressure on the pipe inner boundary, is represented by the expression

$$u_q(z,t) = \frac{\Theta P_{in}(t)}{E_{in}} - \frac{1-v_{in}^2}{E_{out}} h_{out}(z,t) q(z,t) - \frac{2(1-v_{in}^2)}{\pi E_{in}} \int_{-a}^{a} k_{cy} \left( \frac{z - \zeta}{r_{in}} \right) g(\zeta,t) d\zeta, \quad (2)$$

In this expression $q(z,t)$ is contact pressure on outer surface of the coating (or normal load on outer surface on the region $[-a,a]$), $E_{in}$ and $E_{out}$ are Young’s moduli of inner pipe and coating, $v_{in}$ and $v_{out}$ are...
Poisson’s ratios of inner pipe and coating, $\Theta$ is known coefficient (we use assumptions $h_{\text{out}}(z) \ll \{r_{\text{out}}, h_{\text{in}}\}$):

$$
\Theta = \frac{a_1}{a_2 E_{\text{out}} / E_{\text{in}} + a_3}, \quad a_1 = \frac{4(1 - \nu_{\text{out}}^2)(1 - \nu_{\text{in}}^2) h_{\text{out}}^2 r_{\text{out}}^2}{(r_{\text{out}}^2 + \bar{h}_{\text{out}})(r_{\text{out}}^2 - r_{\text{in}}^2)},
$$

$$
a_2 = \frac{1 + \nu_{\text{in}}^2}{1 + \nu_{\text{out}}^2} \frac{r_{\text{out}}^2 (r_{\text{out}}^2 - r_{\text{in}}^2)(r_{\text{out}}^2 + \bar{h}_{\text{out}}^2)}{r_{\text{out}}^2 + \bar{h}_{\text{out}}^2}, \quad a_3 = \frac{(1 - 2\nu_{\text{out}}) r_{\text{out}}^2 + 1}{(r_{\text{out}}^2 + \bar{h}_{\text{out}}^2)^2},
$$

$k_{\text{cyl}}(s)$ is known kernel of the cylindrical contact problem

$$
k_{\text{cyl}}(s) = \int_0^\infty \frac{L(u)}{u} \cos(su) du,
$$

where

$$
L(u) = \frac{u u^2 B_1^2(u) - f(1,u) D_1^2(u) - 1}{S(u)}, \quad f(r,u) = \frac{2(1 - \nu_{\text{in}}^2)}{r} + u^2 r, \quad k_r = \frac{r_{\text{out}}}{r_{\text{in}}},
$$

$$
S(u) = \frac{f(1,u)}{k_r} + f(k_r u) + k_r u^4 A_1^2(u) - u^2 f(k_r u) B_1^2(u) - k_r u^2 f(1,u) C_1^2(u) + f(1,u) f(k_r u) D_1^2(u),
$$

$$
A_1(u) = I_0(u) K_0(k_r u) - I_1(k_r u) K_0(u), \quad B_1(u) = I_0(u) K_1(k_r u) + I_1(k_r u) K_0(u),
$$

$$
C_1(u) = I_0(k_r u) K_1(u) + I_1(k_r u) K_0(u), \quad D_1(u) = I_0(u) K_1(k_r u) - I_1(k_r u) K_1(u),
$$

$I_0(u)$, $I_1(u)$, $K_0(u)$, and $K_1(u)$ are Bessel functions. Note that $a_2 << a_3$ and $E_{\text{out}} < E_{\text{in}}$. Hence $\Theta \approx a_1 a_3 \approx 2(1 - \nu_{\text{in}}^2) r_{\text{out}}^2 (r_{\text{out}}^2 - r_{\text{in}}^2)$.

Since we assume that wear is much less than the thickness of the coating, then $u_0(z,t) \ll h_{\text{out}}(z)$. Hence expression for displacement due to load (2) can be simplified

$$
u_0(z,t) = \frac{\Theta P_{\text{in}}(t)}{E_{\text{in}}} \frac{1 - \nu_{\text{out}}^2}{E_{\text{out}}} h_{\text{out}}(z) q(z,t) - \frac{2(1 - \nu_{\text{in}}^2)}{\pi E_{\text{in}}} \int_{-\alpha}^{\alpha} k_{\text{cyl}}(\zeta - \zeta') q(\zeta', t) d\zeta'.
$$

(3)

Note that it is linear equation over $q(z,t)$.

Summing displacements due to wear (1) and forces (3), we get following expression for displacement of outer surface of coating

$$
u(z,t) = \frac{\Theta P_{\text{in}}(t)}{E_{\text{in}}} \frac{1 - \nu_{\text{out}}^2}{E_{\text{out}}} h_{\text{out}}(z) q(z,t) - \frac{k_w V}{H(z)} \int_0^\tau q(z, \tau) d\tau - \frac{2(1 - \nu_{\text{in}}^2)}{\pi E_{\text{in}}} \int_{-\alpha}^{\alpha} k_{\text{cyl}}(\zeta - \zeta') q(\zeta', t) d\zeta'.
$$

(4)

On the other hand, the displacement of the coating outer surface can be represented as follows

$$
u(z,t) = g(z) - [r_{\text{out}} + h_{\text{out}}(z,t)],
$$

(5)

where $h_{\text{out}}(z,t)$ determined by (2).

Comparing expressions (4) and (5), we obtain the linear mixed integral equation of the problem

$$
\frac{1 - \nu_{\text{out}}^2}{E_{\text{out}}} h_{\text{out}}(z) q(z,t) + \frac{2 k_w V}{H(z)} \int_0^\tau q(z, \tau) d\tau + \frac{2(1 - \nu_{\text{in}}^2)}{\pi E_{\text{in}}} \int_{-\alpha}^{\alpha} k_{\text{cyl}}(\zeta - \zeta') q(\zeta', t) d\zeta' = \frac{\Theta P_{\text{in}}(t)}{E_{\text{in}}} - g(z) + r_{\text{out}} + h_{\text{out}}(z).
$$

(6)

We will make assumption that due to technological process of coating production its hardness $H(z)$ is greater where coating thickness $h_{\text{out}}(z)$ is less (in areas with less thickness, the surface is hardened
better). In this case \( H(z) = k_D h_{out0}(z) \). Unfortunately in the case when these values are independent or the dependence between them is another, it is necessary to construct a numerical solution to the problem (6). So final mixed integral equation has a form

\[
\frac{1 - v^2}{E_{out}} h_{out0}(z) q(z, t) + \frac{2k_w V_{out0}(z)}{k_H} \int_{r_0}^{r_1} q(z, \tau) d\tau + \frac{2(1 - v^2)}{\pi E_{in}} \int_{-a}^{a} k_{cy1} \left( \frac{z - \zeta}{r_{in}} \right) q(\zeta, t) d\zeta = \frac{\Theta p_n(t)}{E_{in}} - g(z) + r_{out} + h_{out0}(z). \tag{7}
\]

3. Dimensionless form and solution of the problem

We introduce into (7) new variables and functions by the formulas

\[
z^* = \frac{z}{a}, \quad \zeta^* = \frac{\zeta}{a}, \quad t^* = \frac{t}{\tau_0}, \quad P^*(t^*) = \frac{1}{a} \left[ \frac{\Theta p_n(t)}{E_{in}} + r_{out} \right], \quad g^*(z^*) = \frac{g(z) - h_{out0}(z)}{a},
\]

\[
m^*(z^*) = \frac{E_{in}}{2(1 - v^2)a} \frac{1 - v^2}{E_{out}} h_{out0}(z), \quad q^*(z^*, t^*) = \frac{2(1 - v^2)}{E_{in}} q(z, t),
\]

\[
A^* f(z^*) = \int_{-1}^{1} k_{cy1}(z^*, \zeta^*) f(\zeta^*) d\zeta^*, \quad V^* f(t^*) = \int_{1}^{t} K^*(t^*, \tau^*) f(\tau^*) d\tau^*,
\]

\[
k_{cy1}(z^*, \zeta^*) = \frac{1}{\pi} k_{cyl} \left( \frac{z^* - \zeta^*}{r_{in}} \right), \quad K^*(t^*, \tau^*) = -V^* = -\frac{2k_w V_{out} \tau_0}{k_H (1 - v^2)}.
\]

Then mathematical model for contact-wear problem in dimensionless form

\[
m^*(z^*)(I - V^*)q^*(z^*, t^*) + A^* q^*(z^*, t^*) = P^*(t^*) - g^*(z^*). \tag{8}
\]

Here \( I \) is identity operator. It is necessary to determine the function \( q^*(z^*, t^*) \) from the obtained equation (8). Note that functions \( m^*(z^*) \) and \( g^*(z^*) \) in (8) relate with properties and forms of contacting surfaces; these parameters can be described by a rapidly changing function.

Next, we will construct a solution for the dimensionless equation (8), and for brevity and convenience, we will omit the asterisks in notations of constants, variables, functions, and operators.

In integral equation (8), we make new kernel \( k(z, \zeta) \) and unknown function \( Q(z, t) \)

\[
Q(z, t) = \sqrt{m(z)} q(z, t) + (I + R) \frac{g(z)}{\sqrt{m(z)}}, \quad k(z, \zeta) = \frac{k_{cy1}(z, \zeta)}{\sqrt{m(z)} \sqrt{m(\zeta)}}, \tag{9}
\]

\[
Ff(z) = \int_{-1}^{1} k(z, \zeta) f(\zeta) d\zeta, \quad (I + R) = (I - V)^{-1}, \quad RFf(t) = -V \int_{1}^{t} e^{-V(t-\tau)} f(\tau) d\tau,
\]

where \( R \) is Volterra operator with kernel \(-Ve^{-V(t-\tau)}\) (note that \(-Ve^{-V(t-\tau)}\) is kernel of resolvent for Volterra integral equation with kernel \(-V\), see [14]). Then equation (8) take a form

\[
(I - V)Q(z, t) + FQ(z, t) = \frac{P(t)}{\sqrt{m(z)}} + \frac{c(t) g(z)}{\sqrt{m(z)}}, \quad z \in [-1,1], \quad t \geq 1, \tag{10}
\]

where

\[
Ff(z) = \int_{-1}^{1} k(z, \zeta) f(\zeta) d\zeta, \quad \tilde{g}(z) = \int_{-1}^{1} k_{cy1}(z, \zeta) g(\zeta) d\zeta, \quad c(t) = (I + R) = e^{-V(t-1)}.
\]
We will see the solution of our problem in the class of time-continuous functions in a Hilbert space $L_2[-1,1]$ (e.g., see [15]). To this end we will construct orthonormal basis in $L_2[-1,1]$ from the linearly independent system of functions \( \{1/\sqrt{m(z)}, z/\sqrt{m(z)}, z^2/\sqrt{m(z)} \} \) by the formulas

\[
J_j = \int_{-1}^{1} \frac{z^j}{\sqrt{m(z)}} \, dz, \quad d_{-1} = 1, \quad d_j = \begin{bmatrix} J_0 & \Lambda & J_1 \\ M & O & M \\ J_{i+1} & \Lambda & J_{i+1} \end{bmatrix}, \quad p_0(z) = \frac{1}{\sqrt{J_0}}, \quad p^*_0(z) = \frac{1}{\sqrt{J_0 m(z)}},
\]

\[
p_j(z) = \frac{1}{\sqrt{d_{j-1} d_j}} \begin{bmatrix} J_0 & \Lambda & J_1 \\ M & O & M \\ J_{j+1} & \Lambda & J_{j+1} \end{bmatrix} p^*_j(z) = \frac{p_j(z)}{m(z)}, \quad i, j = 0, 1, 2, K.
\]

Note that $p_j(z) (j = 0, 1, 2, \ldots)$ are polynomials. So we will seek the solution in the form

\[
Q(z,t) = \sum_{k=0}^{\infty} f_k(t) \Phi_k(z), \quad (13)
\]

where $f_k(t)$ is the desired function and the eigenfunctions $\Phi_k(z)$ of the operator $F$ are determined by solving the spectral problem for the operator $F$,

\[
F \Phi_k(z) = \gamma_k \Phi_k(z).
\]

The eigenfunction system \( \{ \Phi_k(z) \}_{k=0,1,2,K} \) is related to the initial system \( \{ p^*_k(z) \}_{k=0,1,2,K} \) of basis functions in $L_2[-1,1]$ as

\[
\Phi_k(z) = \sum_{i=0}^{\infty} \psi_i^{(k)} p^*_i(z).
\]

Therefore, solving the spectral problem is reduced to solving the system of algebraic equations

\[
\sum_{i=0}^{\infty} R_i \psi_i^{(k)} = \gamma_k \psi_i^{(k)}, \quad R_i = \int_{-1}^{1} \int_{-1}^{1} \frac{k_{cyl}(z, \zeta) P_i(z) P_i(\zeta)}{m(z)m(\zeta)} \, dz \, d\zeta, \quad i, l, k = 0, 1, 2, K
\]

We substitute the representation (13) into the integral equation (10) and use (11), (12), (14), and (15) to obtain the following expression for the functions $f_k(t)$,

\[
f_k(t) = (I + W_k) \psi_0^{(k)} \sqrt{J_0} P(t) + g_k \tilde{c}(t), \quad g_k = \sum_{i=0}^{\infty} \psi_i^{(k)} \sum_{i=0}^{\infty} R_i \int_{-1}^{1} \frac{P_i(z) g(z)}{m(z)} \, dz, \quad i, l, k = 0, 1, 2, K.
\]

where $W_k$ are the Volterra operators whose kernels $R_k(t, \tau) = -\frac{V(t-\tau)}{1+\gamma_k}$ are resolvents of the kernels $-V/(1 + \gamma_k)$. Then the functions $f_k(t)$ can be represented as

\[
f_k(t) = \psi_0^{(k)} \sqrt{J_0} \left\{ P(t) - \frac{V}{1+\gamma_k} \int_{-1}^{1} \frac{P(t-\tau)}{1+\gamma_k} \, d\tau \right\} + g_k \gamma_k \left\{ e^{-V(t-\gamma_k)} - \frac{1}{1+\gamma_k} \exp \left[ -\frac{V(t-1)}{1+\gamma_k} \right] \right\}.
\]

As a result, by using (9), (11)–(13), (15), we finally obtain the expression for the contact stresses

\[
q(z,t) = \frac{1}{m(z)} \sum_{k=0}^{\infty} f_k(t) \sum_{m=0}^{\infty} \psi_m^{(k)} p_m(z) - e^{-V(t-\gamma_k)} \frac{g(z)}{m(z)},
\]

where the functions $f_k(t)$ are determined by relation (17), the coefficients $\psi_i^{(k)}$ can be obtained from (16), and $p_d(z)$ can be calculated by the formulas (12).
4. Conclusions and remarks

Contact-wear problem for elastic tube with rough coating and rigid bush with complex inner shape is posed and solved. Analytical representation for contact stresses as a series over special basis is obtained. Complex functions connected with surface shapes and coating hardness is represented by separate factors and terms. It allow one to provide effective numerical calculations using a small number of members of the series. It is important in the case when characteristics of coating and bush described by rapidly changing functions. Another known approaches (using Legendre polynomials, trigonometric functions, etc.) lead us to computational errors.

Obtained representation of contact pressures provide solution only for the case when wear is much less then coating thickness. Another important assumptions described in the article. But this solution can be first step for modeling much more complicated cases of wear.

Acknowledgments

The present work was partially supported by the Ministry of Science and Higher Education within the framework of the Russian State Assignment (contract No. AAAA-A20-120011690132-4) and partially supported by Russian Foundation for Basic Research (projects Nos. 18-01-00770 and 19-51-60001).

References

[1] Alexandrov V M and Mkitaryan S M 1979 Contact Problems for Bodies with Thin Coatings and Interlayers (Moscow: Nauka) [in Russian]

[2] Pronikov A S 1957 Wear and Durability of Machines (Moscow: Mashgiz) [in Russian]

[3] Khrushchev M M and Babichev M A 1970 Abrasive Wear (Moscow: Nauka) [in Russian]

[4] Collins J 1993 Failure of Materials in Mechanical Design: Analysis, Prediction, Prevention (New York: Wiley)

[5] Goryacheva I G and Dobychin M N 1988 Contact Problems in Tribology (Moscow: Mashinostroenie) [in Russian]

[6] Soldatenkov I A 2010 Wear Contact Problem with Applications to Engineering Calculation of Wear (Moscow: Fizmatkнiga) [in Russian]

[7] Manzhurov A V and Chernysh V A 1988 On the interaction of a rigid reinforcing sleeve and inhomogeneous aging high-pressure pipes Mech. Solids 22 (6) 104–10

[8] Arutyunyan N Kh and Manzhurov A V 1999 Contact Problems in the Theory of Creep (Yerevan: Izd-vo Inst. Mekhaniki NAN Armenia) [in Russian]

[9] Manzhurov A V and Kazakov K E 2017 Axisymmetric contact between a rigid punch and a coated foundation with rough surfaces J. Phys.: Conf. Ser. 937 012028 doi: 10.1088/1742-6596/937/1/012028

[10] Manzhurov A V and Kazakov K E 2018 Axisymmetric problem of fretting wear for a foundation with a nonuniform coating and rough punch AIP Conf. Proc. 1959 070023 doi: 10.1063/1.5034698

[11] Manzhurov A V and Kazakov K E 2018 Modeling the contact interaction between a nonuniform foundation and a rough punch Math. Models Comput. Simul. 10 (3) 314–21 doi: 10.1134/S2070048218030109

[12] Kurdina S P and Kazakov K E 2019 Axisymmetric contact problem for a punch and nonuniform foundation with rough surfaces AIP Conf. Proc. 2116 380006 doi: 10.1063/1.5114387

[13] Kazakov K E 2019 Wear of foundation with a nonuniform coating by rough punch IOP Conf. Ser.: Mater Sci. Engng 489 012027 doi: 10.1088/1757-899X/489/1/012027

[14] Polyanin A D and Manzhurov A V 2008 Handbook of Integral Equations, 2nd edition (Boca Raton: Chapman & Hall/ CRC)

[15] Manzhurov A V 2016 A mixed integral equation of mechanics and a generalized projection method of its solution Dokl. Phys. 61 (10) 489–93 doi: 10.1134/S1028335816100025