Numerical cubature from Archimedes’ hat-box theorem

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Dedicated to Krystyna Kuperberg on the occasion of her 60th birthday

Archimedes’ hat-box theorem states that uniform measure on a sphere projects to uniform measure on an interval. This fact can be used to derive Simpson’s rule. We present various constructions of, and lower bounds for, numerical cubature formulas using moment maps as a generalization of Archimedes’ theorem. We realize some well-known cubature formulas on simplices as projections of spherical designs. We combine cubature formulas on simplices and tori to make new formulas on spheres. In particular $S^n$ admits a 7-cubature formula (sometimes a 7-design) with $O(n^4)$ points. We establish a local lower bound on the density of a PI cubature formula on a simplex using the moment map.

Along the way we establish other quadrature and cubature results of independent interest. For each $t$, we construct a lattice trigonometric $(2t+1)$-cubature formula in $n$ dimensions with $O(n^t)$ points. We derive a variant of the Möller lower bound using vector bundles. And we show that Gaussian quadrature is very sharply locally optimal among positive quadrature formulas.

1. INTRODUCTION

Let $\mu$ be a measure on $\mathbb{R}^n$ with finite moments. A cubature formula of degree $t$ for $\mu$ is a set of points $F = \{ \tilde{p}_a \} \subset \mathbb{R}^n$ and a weight function $w_a : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int P(\tilde{x})d\mu = P(F) \overset{\text{def}}{=} \sum_{a=1}^{N} w_a P(\tilde{p}_a)$$

for polynomials $P$ of degree at most $t$. (If $n = 1$, then $F$ is also called a quadrature formula.) The formula $F$ is equal-weight if all $w_a$ are equal; positive if all $w_a$ are positive; and negative if at least one $w_a$ is negative. Let $X$ be the support of $\mu$. The formula $F$ is interior if every point $\tilde{p}_a$ is in the interior of $X$; it is boundary if every $\tilde{p}_a$ is in $X$ and some $\tilde{p}_a$ is in $\partial X$; and otherwise it is exterior. We will mainly consider positive, interior (PI) and positive, boundary (PB) cubature formulas, and we will also assume that $\mu$ is normalized so that total measure is 1. PI formulas are the most useful in numerical analysis [25, Ch. 1]. This application also motivates the main question of cubature formulas, which is to determine how many points are needed for a given formula and a given degree $t$. Equal-weight formulas that are either interior or boundary (EI or EB) are important for other applications, in which context they are also called $t$-designs.

Our starting point is a connection between quadrature on the interval $[-1, 1]$ and cubature on the unit sphere $S^2$, both with uniform measure. By Archimedes’ hat-box theorem [2], the orthogonal projection $\pi$ from $S^2$ to the $z$ coordinate preserves normalized uniform measure. In plainer terms, for any interval $I \subset [a, b]$ or other measurable set, the area of $\pi^{-1}(I)$ is proportional to the length of $I$; see Figure 1. (It is called the hat-box theorem because the surface area of a hemispherical hat equals the area of the side of a cylindrical box containing it.) Therefore if $F$ is a $t$-cubature formula on $S^2$, its projection $\pi(F)$ is a $t$-cubature formula on $[-1, 1]$.

The 2-sphere $S^2$ has 5 especially nice cubature formulas given by the vertices of the Platonic solids. Their cubature properties follow purely from a symmetry argument of Sobolev [25]. Suppose that $G$ is the group of common symmetries of a putative cubature formula $F$ and its measure $\mu$. If $P(\tilde{x})$ is a polynomial and $P_G(\tilde{x})$ is the average of its $G$-orbit, then

$$\int P(\tilde{x})d\mu = \int P(\tilde{x})d\mu \quad P_G(F) = P(F).$$

Therefore it suffices to check $F$ for $G$-invariant polynomials. In particular, if every $G$-invariant polynomial of degree $\leq t$ is constant, then any $G$-orbit is a $t$-design.

By Sobolev’s theorem, the vertices of a regular octahedron form a 3-design on $S^2$. If we project this formula using Archimedes’ theorem, the result is Simpson’s rule. Another projection of the same 6 points yields 2-point Gauss-Legendre quadrature. Figure 2 shows both projections. The 8 vertices of a cube are also a 3-design. One projection is again 2-point Gauss-Legendre quadrature; another is Simpson’s $\frac{3}{2}$ rule. Finally the 12 vertices of a regular icosahedron form a 5-design.
by symmetry. One projection of these 12 points is 4-point Gauss-Lobatto quadrature.

The rest of this article applies toric moment maps, which generalize Archimedes’ theorem to higher dimensions, to the cubature problem. Section 2 shows that several well-known quadrature formulas on the interval and cubature formulas on simplices are projections of higher-dimensional, symmetric formulas. Section 4 combines formulas on tori with formulas on simplices and moment maps to make formulas on tori that are similar to cubature formulas of independent interest. Section 3 establishes new lattice convex polytope. A similar lower bound holds for an arbitrary simplex. One projection of these 12 points is 4-point Gauss-Lobatto quadrature.

Along the way we establish some other quadrature and cubature results that are not derived from moment maps but are of independent interest. Section 5 establishes new lattice cubature formulas on tori that are similar to cubature formulas based on error-correcting codes [17]. In particular it constructs, for each $t$, a trigonometric $(2t+1)$-cubature formula on $[0,2\pi]^n$ of lattice type with $O(n^t)$ points. This improves a construction of Cools, Novak, and Ritter with $O(n^{2t})$ points and negative weights [5], and agrees up to a constant factor with the Stroud-Mysovskikh lower bound [19, 29]. Section 5 presents a refinement of this well-known lower bound in odd degree. It is similar to the Möller bound [18], but applies to some new cases. Section 6 also establishes that Gaussian quadrature is very sharply locally optimal among all positive quadrature formulas (Theorem 6.3). This bound might be previously known since Gaussian quadrature has been widely studied, but the author could not find it in the literature.

2. PROJECTION CONSTRUCTIONS

The immediate higher-dimensional generalization of Archimedes’ theorem replaces the sphere $S^2$ by the complex manifold $\mathbb{C}P^2$. This manifold has a natural metric and a natural real algebraic structure. Concretely, assume that the projective coordinates $(z_0 : z_1 \cdots : z_n)$ of $\mathbb{C}P^n$ are normalized so that

$$|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1.$$ 

Then the coordinates $z_j$ together embed $\mathbb{C}P^2$ into $\mathbb{C}^{(n+1)^2}$ as a real algebraic variety (with $\mathbb{C}^{(n+1)^2}$ interpreted as a $2(n+1)^2$-dimensional real vector space) and a Riemannian manifold. This embedding is familiar in quantum mechanics as the density matrix (or density operator) formalism [20, §2.4]. The induced metric is called the Fubini-Study metric. Since the metric yields a measure on $\mathbb{C}P^n$, and since it is a real algebraic variety, we can consider cubature formulas on it.

There is a projection $\pi : \mathbb{C}P^n \to \Delta_n$ to the $n$-simplex given by

$$\pi(z_0 : z_1 \cdots : z_n) = (|z_0|^2, |z_1|^2, \ldots, |z_n|^2),$$

using normalized coordinates for $\mathbb{C}P^n$ and barycentric coordinates for $\Delta_n$. It is linear and it preserves normalized measure. In more abstract terms, $\pi$ has these properties because $\mathbb{C}P^n$ is a projective toric variety and $\pi$ is its moment map. Archimedes’ theorem is a description of the moment map of $\mathbb{C}P^1 \cong S^2$. Thus, if $F$ is an interior $t$-cubature formula on $\mathbb{C}P^n$, then $\pi(F)$ is a $t$-cubature formula on $\Delta_n$.

Ivanović, Wootters, and Fields [12, 13] defined one interesting family of 2-designs on $\mathbb{C}P^{p-1}$ for $q = p^k$ a prime power. If $p$ is odd, then the 2-design is the orbit of a standard basis vector $e_k$ in the group generated by cyclic permutation and linear operators of the form

$$L(e_k) = \omega^{Tr_p(ak^2 + bk + c)} e_k,$$
where $\omega$ is a $p$th root of unity and $\text{Tr}_p$ is the $\mathbb{F}_p$ trace function on $\mathbb{F}_q$. The construction is more complicated when $p = 2$. In either case, the standard basis projects to the vertices of $\Delta_{q-1}$ and the other $q^2$ vectors project to the center. The result is a standard degree 2 generalization of Simpson’s rule for $\Delta_{q-1}$, shown in Figure 3 when $q = 3$.

Figure 3: A 2-dimensional generalization of Simpson’s rule.

One interesting example is the 240 roots of the $E_8$ root system, one interesting set of orthogonal planes is the 4 eigenplanes of the abelian subgroup of $\text{Aut}(E_8)$ of the form $C_8 \times C_8$. Eric Rains [23] has computed the corresponding 3-cubature formula on $\Delta_3$ using Magma [33]. In barycentric coordinates on $\Delta_3$, its points and weights are the orbits of the two weighted points

$$\bar{p}_1 = \frac{1}{10}(0, 0, 5 - \sqrt{5}, 5 + \sqrt{5}) \quad w_1 = \frac{1}{24}$$

$$\bar{p}_2 = \frac{1}{10}(2, 2, 3 + \sqrt{5}, 3 - \sqrt{5}) \quad w_2 = \frac{5}{24}$$

under the action of the coordinate permutations (34) and (13)(24). In particular, it has 8 points. In conclusion, at least three interesting 3-cubature formulas for $\Delta_3$ arise as projections of $E_8$ root system. The root system model explains the simple rational values of the weights.

The $E_8$ lattice is one of four widely studied and highly symmetric lattices in low dimensions; the other three are the Coxeter-Todd lattice $K_{12}$ in $\mathbb{R}^{12}$, the Barnes-Wall lattice $\Lambda_{16}$ in $\mathbb{R}^{16}$, and the Leech lattice $\Lambda_{24}$ in $\mathbb{R}^{24}$ [24, Ch. 4]. In each case, cases, the set of short vectors has transitive symmetry, and in each case, Sobolev’s theorem establishes its degree as a spherical design.

The 756 short vectors of $K_{12}$ form a 7-design on $S^{11}$. In one of its several presentations as an Eisenstein lattice in $\mathbb{C}^6$ (the “3-base” presentation [24, §7.8]), the short vectors are generated from the two points

$$(1, 1, 1, 1, 1, 1) \quad (1 - \omega, \omega - 1, 0, 0, 0, 0)$$

by freely permuting coordinates, multiplying the coordinates by powers of $\omega$ whose exponents sum to 0, and negating all coordinates. The projection $\tau_2$ sends these points to a 16-point 3-cubature formula on $\Delta_5$ generated from the points

$$\bar{p}_1 = \frac{1}{2}(1, 1, 0, 0, 0, 0) \quad w_1 = \frac{1}{42}$$

$$\bar{p}_2 = \frac{1}{6}(1, 1, 1, 1, 1) \quad w_2 = \frac{27}{42}$$

This projection is analogous to a map $\tau_1 : S^{n-1} \rightarrow \Delta_{n-1}$ defined by Xu [33]:

$$\tau_1(x_1, \ldots, x_n) = (x_1^2, \ldots, x_n^2).$$

The Xu map does not preserve uniform measure. Rather, it takes uniform measure on the sphere to the measure with weight function

$$w_1(\bar{y}) = \frac{2^n \pi^{n/2}}{\pi^{1/n} \sqrt{y_0}y_1 \cdots y_{n-1}}$$

in barycentric coordinates.

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by freely permuting coordinates. This formula was found by Stroud [27, 28].

The 4320 short vectors of $\Lambda_{16}$ form a 7-design on $S^{15}$. In its simplest position (which exhibits its Gaussian lattice structure), the short vectors are generated from the two vectors

$$(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0)$$

$$(2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$$

This projection is analogous to a map $\tau_1 : S^{n-1} \rightarrow \Delta_{n-1}$ defined by Xu [33]:

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The Xu map does not preserve uniform measure. Rather, it takes uniform measure on the sphere to the measure with weight function

$$w_1(\bar{y}) = \frac{2^n \pi^{n/2}}{\pi^{1/n} \sqrt{y_0}y_1 \cdots y_{n-1}}$$

in barycentric coordinates.
by permuting coordinates under the group $GL(4, 2) \ltimes (\mathbb{Z}/2)^4$ of affine automorphisms of $(\mathbb{Z}/2)^4$, together with sign changes that keep the coordinate sums divisible by 4. The projection $r_2$ sends these points to a 51-point 3-cubature formula on $\Delta_7$ generated from the points

\[ \bar{p}_1 = (1, 0, 0, 0, 0, 0, 0) \quad w_1 = \frac{1}{1080} \]
\[ \bar{p}_2 = \frac{1}{2} (1, 1, 0, 0, 0, 0, 0) \quad w_2 = \frac{1}{270} \]
\[ \bar{p}_3 = \frac{1}{4} (1, 1, 1, 0, 0, 0, 0) \quad w_3 = \frac{4}{135} \]
\[ \bar{p}_4 = \frac{1}{8} (1, 1, 1, 1, 1, 1, 1) \quad w_4 = \frac{64}{135}. \]

under the action of the affine group $GL(3, 2) \ltimes (\mathbb{Z}/2)^3$. This is not an optimal PI 3-cubature formula, because the orbit of $\bar{p}_2$ can be eliminated, leaving only 23 points. But it does have a novel property: Instead of full symmetrization, the orbit of $\bar{p}_3$ is in the pattern of the $(8,4,3)$ Steiner system. But this is as good as full symmetrization for 3-cubature, because any monomial of degree 3 involves at most 3 coordinates. The structure of this Barnes-Wall projection led the author to relate cubature to combinatorial $t$-designs and orthogonal arrays [17].

The above position of $\Lambda_{16}$ is compatible with its Gaussian lattice structure. Eric Rains found another interesting position which is compatible with an Eisenstein lattice structure. The corresponding 3-cubature formula on $\Delta_7$ has 50 points. They are generated from

\[ \bar{p}_1 = (1, 0, 0, 0, 0, 0, 0) \quad w_1 = \frac{1}{720} \]
\[ \bar{p}_2 = \frac{1}{4} (1, 1, 1, 0, 0, 0, 0) \quad w_2 = \frac{1}{90} \]
\[ \bar{p}_3 = \frac{1}{3} (1, 1, 0, 0, 1, 0, 0) \quad w_3 = \frac{1}{80} \]
\[ \bar{p}_4 = \frac{1}{12} (4, 0, 0, 4, 1, 1, 3) \quad w_4 = \frac{1}{60} \]
\[ \bar{p}_5 = \frac{1}{12} (4, 0, 4, 0, 1, 1, 1) \quad w_5 = \frac{1}{40} \]
\[ \bar{p}_6 = \frac{1}{12} (3, 1, 3, 1, 1, 1, 1) \quad w_6 = \frac{1}{30} \]
\[ \bar{p}_7 = \frac{1}{12} (3, 1, 1, 1, 1, 3, 1) \quad w_7 = \frac{1}{30}. \]

by the coordinate permutations $(12), (13)(24)(57)(68)$, and $(15)(26)(37)(48)$.

The 196560 short vectors of the Leech lattice form a 11-design on $S^{23}$. The lattice has a space Eisenstein lattice structure which Conway and Sloane call the complex Leech lattice [4, §7.8]. The complex basis that they give leads to a 5-cubature formula on $\Delta_{11}$ generated by the points

\[ \bar{p}_1 = \frac{1}{2} (1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0) \quad w_1 = \frac{1}{10920} \]
\[ \bar{p}_2 = \frac{1}{6} (1, 1, 1, 1, 1, 0, 0, 0, 0, 0, 0) \quad w_2 = \frac{9}{3640} \]
\[ \bar{p}_3 = \frac{1}{18} (7, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \quad w_3 = \frac{27}{1820} \]
\[ \bar{p}_4 = \frac{1}{18} (4, 4, 1, 1, 1, 1, 1, 1, 1, 1, 1) \quad w_4 = \frac{27}{3640}. \]

by the action of the Mathieu group $M_{12}$. In other words, the coordinates of $\bar{p}_2$ are permuted in the pattern of the $(12,6,5)$ Steiner system and the points of the other coordinate are permuted freely. The total is 276 points. Another interesting basis of plane consists of the mutual eigenplanes of the $(\mathbb{Z}/5)^3$
subgroup of the isometry group of the Leech lattice. Eric Rains has computed that the corresponding 5-cubature formula on $\Delta_{11}$ has 498 points, consisting of 22 orbits of the surviving coordinate permutations. However, since none of the Barnes-Wall formulas on $\Delta_t$ are optimal, it is not clear that the smaller of these formulas is either.

3. TORS CONSTRUCTIONS

The constructions in the next section depend on an auxiliary case that generally works out better than cubature on $S^n$, $CP^n$, and $A_n$, namely cubature on algebraic tori. There is a developed theory for a special case of this problem known as trigonometric cubature [4,7]. We will describe a more general class of problems, with one new result for the classic trigonometric cubature problem (Theorem 3.3).

Consider a torus group $T \cong (S^1)^n$ together with a faithful linear action on some real vector space $V \cong \mathbb{R}^n$. Then we can identify $T$ with any faithful orbit $O$ to give it a real algebraic structure. Since $T$ is a compact group, it also comes with Haar measure (i.e., uniform measure). Given both structures, we can then consider cubature formulas for $T$. If a cubature formula $F$ is a $t$-design and forms a subgroup of $T$, then it is called a lattice formula, or an additive $t$-design.

Proposition 3.1. The lattice cubature problem on $T$ is equivalent to a lattice packing problem as follows:

1. The real algebraic structure on $T$ does not depend on the orbit $O$ or the base point chosen on $O$. The ring of polynomials on $T$ is the same as the character ring $R(T)$.

2. Every character $\chi : T \to \mathbb{C}$ is homogeneous as a polynomial on $T$. Its degree defines a norm on $T$, the character group of $T$. The norm is generated by unit steps corresponding to the characters that appear in $V \otimes \mathbb{C}$.

3. The characters that are constant on a subgroup $F \subseteq T$ form a sublattice $\hat{F} \subseteq \hat{T}$. This correspondence is a bijection between finite subgroups and sublattices such that $|F| = \ell_t((\hat{T} : \hat{F}))$.

4. The subgroup $F$ is a $t$-design if and only if $\hat{F}$ has minimum distance $d = t + 1$.

The proof of Proposition 3.1 is lengthy but routine and can be left as an exercise for the reader. It is essentially established in the literature when $T = T(\SO(2n))$ acts on $\mathbb{R}^{2n}$ by separate rotations in $n$ orthogonal planes. This case is equivalent to the (cubic) trigonometric cubature problem, defined as cubature formulas on the $n$-cube $[0, 2\pi]^n$ which are exact for trigonometric polynomials of degree $t$ [7]. All of the arguments generalize without change.

When $T = T(\SO(2n))$, $\hat{T}$ is naturally identified with $\hat{\O}_n$, and its norm is the $\ell_1$ or taxicab norm. Another torus of interest to us is $T = T(\PSU(n + 1))$, the group of diagonal unitary matrices with determinant 1 modulo its center. It acts on $\mathbb{C}^{(n+1)^2}$, interpreted as the space of $(n + 1) \times (n + 1)$ complex matrices, by conjugation. In this case $\hat{T} = A_n$, the root lattice of $\PSU(n + 1)$, and its norm is defined by taking the roots of $A_n$ as unit steps.

Theorem 3.2. Given a real algebraic torus $T$ of dimension $n$, let $K \subseteq \hat{T} \otimes \mathbb{R}$ be the real convex hull of the unit steps in $\hat{T}$. Let $\delta_t(K)$ be the lattice packing density of $K$, and let $\Vol K$ be the volume of $K$ normalized by $\hat{T}$. Then $t \geq 0$ and let $d = t + 1$. Then the best additive $t$-design $F$ on $T$ has at least

$$\frac{d^n(\Vol K)}{2^n\delta_t(K)} \leq |F| \leq \frac{d^n(\Vol K)}{2^n\delta_t(K)}(1 + O(t^{-1}))$$

points.

Theorem 3.2 has been noted independently by several people for trigonometric cubature, but may originally be due to Frolov [11]. In outline, a lattice $\hat{F} \subseteq \hat{T}$ with minimum distance $d$ produces a packing of the dilated body $\frac{d}{n} K$. The packing density $\delta_t(K)$ then yields a lower bound on the index of $\hat{F}$. On the other hand, if $\Lambda$ is the center lattice of the best packing of $K$, then when $t$ is large, $\frac{d}{n} \Lambda$ can be approximated by a sublattice of $\hat{T}$. This establishes the upper bound.

Note also that the best $\Lambda$ has rational coordinates relative to $\hat{T}$ (or they can be made rational if $\Lambda$ is not unique), because $K$ is a rational polytope. Thus there exist special distances $d$ such that the best $F$ has exactly

$$\frac{d^n(\Vol K)}{2^n\delta_t(K)}$$

points. Also if some $d$ achieves exactitude, then so does $kd$ for every $k > 1$.

If $T = T(\SO(2n))$ is the standard cubic $n$-torus, then $K$ is the $n$-cross polytope $C_n^*$. For example, Minkowski established that the lattice packing density of the regular octahedron $C_2^*$ is $\frac{1}{2\pi}$. So there exists an additive $5$-design on $T(\SO(6))$ with 38 points [11, 21].

Since $C_2^*$ is a square, its packing density is 1. Noskov [21] found the best discrete approximation to this packing for every distance $d$ to obtain lattice rules for $T(\SO(4))$. If $d = 2s$, then the best approximation is exact and there is a $(2s - 1)$-design with $2s^2$ points. If $d = 2s + 1$, then the best approximation corresponds to the tiling of $\mathbb{Z}^2$ by the discrete $\ell_1$ ball of radius $s$, or the tiling of the plane $\mathbb{R}^2$ by certain Aztec diamonds, as shown in Figure 5(a). The ball and the corresponding $2s$-design have $s^2 + (s + 1)^2$ points.

Noskov’s designs have a counterpart for $T(\PSU(3))$, where $T(\PSU(3)) = A_2$ is the triangular lattice. If we identify $A_2$ with the Eisenstein integers $\mathbb{Z}[\omega]$, then the highest-density lattice with minimum distance $d$ is the ideal generated by

$$\left[\frac{d}{2}\right] - \omega \left[\frac{d + 1}{2}\right].$$

When $d = 2s$, the dual $(d - 1)$-design has $3s^2$ points and exactly matches the tiling of the plane by regular hexagons. When $d = 2s + 1$, it has $3s^2 + 3s + 1$ points and corresponds to a tiling the plane by the hexagonal polyhex of order $s$ (an “Afghan hexagon”), as shown in Figure 5(b).
Let \( t \geq 0 \). The torus \( T(SO(2n)) \) has a \((2t+1)\)-design with \( O(n^t) \) points. More precisely it has a \( 2t \)-design with \((2n)^t(1+o(1))\) points as \( n \to \infty \) and a \((2t+1)\)-design with twice as many points. The torus \( T(PSU(n+1)) \) has a \( t \)-design with \( n^t(1+o(1)) \) points as \( n \to \infty \).

Remark. Theorem 3.3 can be compared with a prior result by Cools, Novak, and Ritter [5], who obtained \( NI \) formulas with [19, 29] for trigonometric \( 2t \)-cubature with twice as many points. The torus \( T \) has a centrally symmetric algebraic variety. It yields:

\[
|F| \geq \frac{(2n)^t(1-o(1))}{t!}.
\]

The Möller bound applies to trigonometric \((2t+1)\)-cubature in its interpretation as cubature on \( T(SO(2n)) \) because it is a centrally symmetric algebraic variety. It yields:

\[
|F| \geq \frac{2(2n)^t(1-o(1))}{t!}.
\]

Thus for each \( t \), Theorem 3.3 is asymptotically optimal to within a constant factor, even though the lower bounds do not require \( F \) to be positive or interior.

Proof. By Proposition 3.1, our task is to find suitable lattices in \( \mathbb{Z}^n = T(SO(2n)) \) and \( A_n = T(PSU(n+1)) \). Our task is fulfilled by Craig lattices [4, §8.6] in \( A_n \) and skew analogues of Craig lattices in \( \mathbb{Z}^n \). We describe the \( A_n \) case first.

We define the lattice

\[
\Lambda^{(t)}(A_n) = \ker \phi \cap A_n.
\]

Plainly the index of \( \Lambda^{(t)}(A_n) \) is at most \( p^t = n^t(1+o(1)) \). (If \( n \) is large and \( p \approx n \), it is \( p^t \), because any lower power of \( p \) would violate the Stroud-Mysovskikh bound.)

We claim that the distance of \( \Lambda^{(t)}(A_n) \) is \( t+1 \). To show this, we will show that \( \phi \) is injective on the simplex \( \Delta_n^{(t)} \subset \mathbb{Z}_+^{n+1} \) of non-negative vectors with coordinate sum \( t \). A vector \( \vec{x} \in A_n \) with root-step length at most \( t \) can be expressed as the difference of two vectors in \( \Delta_n^{(t)} \); therefore injectivity shows that none of these vectors lie in \( \Lambda^{(t)}(A_n) \).

We can interpret a vector \( \vec{x} \in \Delta_n^{(t)} \) as a multiset of \( S \) over the set \( \{0, \ldots, n\} \) with \( |S| = t \): if

\[
\vec{x} = \sum_{a} m_a \vec{e}_a,
\]

then \( m_a \) is the multiplicity of \( a \in S \). In this interpretation, \( \phi(\vec{x}) \) is the list of power sums

\[
\sum_{a \in S} a^k
\]

for \( 1 \leq k \leq t \). By standard inversion formulas [26], these power sums determine the elementary symmetric functions of the elements of \( S \) when \( p > t \), which are the coefficients of the polynomial

\[
\prod_{a \in S} (x-a).
\]

Thus \( \phi(\vec{x}) \) determines \( S \) as a multiset and the vector \( \vec{x} \), and it is injective on \( \Delta_n^{(t)} \).

For \( \mathbb{Z}^n \) (with the \( \ell_1 \) norm), let \( p > 2n \) be prime. Index the standard basis \( \{\vec{e}_a\} \) of \( \mathbb{Z}^n \) by some subset \( N \subset \mathbb{Z}/p \) such that \( N \) is disjoint from \(-N\). Define \( \phi : \mathbb{Z}^n \to (\mathbb{Z}/p)^t \) by

\[
\phi(\vec{e}_a) = (a, a^3, a^5, \ldots, a^{2^t-1}),
\]
and define
\[ \Lambda^{(1)}(\mathbb{Z}^n) = \ker \phi. \]

Then the index of \( \Lambda^{(1)}(\mathbb{Z}_n) \) is again at most (and usually exactly) \( p^t = n! (1 + o(1)) \). Its distance property can be explained by embedding \( \mathbb{Z}^n \) isometrically into \( A_{2n} \) using the map
\[ \alpha : \bar{e}_a \mapsto \bar{e}_a - \bar{e}_{-a}. \]

Then
\[ \Lambda^{(1)}(\mathbb{Z}^n) = \alpha^{-1}(\Lambda^{(2)}(A_n)). \]

Since \( \Lambda^{(2)}(A_n) \) has distance at least \( 2t + 1 \), so does \( \Lambda^{(1)}(\mathbb{Z}^n) \).

We can boost the distance to \( 2t + 2 \) by passing to its even-sum sublattice.

![Image 6: A Hamming-like “plus” tiling.](Image)

**Remark.** When \( t = 1 \), the number \( p \) in the proof of Theorem 3.3 need not be prime, and the lattices \( \Lambda^{(1)}(\mathbb{Z}_n) \) and \( \Lambda^{(1)}(A_n) \) produce lattice tilings of the ball of \( \ell_1 \)-radius 1 in \( \mathbb{Z}^n \) and the combinatorial simplex \( \Lambda^{(1)} \) in \( A_n \). For example, when \( n = 2 \), they are equivalent to familiar tilings of the plus pentomino (Figure 6) and the triangle trihex. The plus tiling resembles combinatorial tilings coming from Hamming codes [4, §3.2]. More generally, Craig lattices resemble low-distance BCH codes. This resemblance is what led the author to Theorem 3.3.

### 4. FIBRATION CONSTRUCTIONS

The projection construction in Section 3 is instructive, but backwards in a sense: It is harder to make \( t \)-cubature formulas for \( CP^{n-1} \) and \( S^{2n-1} \) than for \( \Delta_{n-1} \) for most values of \( n \) and \( t \). In this section we will use the same projections to lift cubature formulas to spheres and projective spaces from simplices. The construction also requires the definition and constructions of cubature formulas on tori from Section 3.

**Theorem 4.1.** Let \( \alpha : X \to Y \) be one of the three projections \( h, \pi, \) or \( \tau_2 \), and let \( T \) be a generic fiber. Let \( s = 2t + 1 \) when \( X = S^{2n-1} \) and \( s = t \) when \( X = CP^{n-1} \). Given an interior (or boundary) \( t \)-cubature formula \( F \) for \( Y \) and an interior \( s \)-cubature formula \( F_t \) for \( T \), there is a twisted product \( s \)-cubature formula \( F_X = F_T \times F_Y \) for \( X \). It satisfies \( |F_X| = |F_T| \cdot |F_Y| \) and it inherits positivity from its factors. In the boundary case, \( |F_X| \leq |F_T| \cdot |F_Y| \).

Note that in the three cases, \( T \) is isomorphic to \( S^1 \), \( T(SO(2n)) \), and \( T(PSU(n)) \), respectively.

**Proof.** Let \( \sigma_T \) be the discrete measure on \( Y \) corresponding to the cubature formula \( F_Y \), and let \( \sigma_X = \alpha^* \sigma_T \) be the pull-back of \( \sigma_T \) to \( X \). In other words, for each point \( p \) of weight \( w \) in \( F_Y \), \( \sigma_X \) has a term consisting of uniform measure on the torus fiber \( \alpha^{-1}(p) \). Also let \( \mu_X \) and \( \mu_Y \) be normalized uniform measure on \( X \) and \( Y \).

We claim that
\[ \int_X P(\bar{x}) d\mu_X = \int_X P(\bar{x}) d\sigma_X \]
for any polynomial of \( P \) of degree \( s \); in other words \( \mu_X \) and \( \sigma_X \) are \( s \)-cubature equivalent [17]. If we assume the natural group structure on \( T \), then it acts on \( X \) in each of the three cases with \( Y \) as the set of orbits. Then
\[ \int_X P(\bar{x}) d\sigma_X = \int_X P_T(\bar{x}) d\sigma_X \]
\[ = \int_X P_T(\bar{x}) d\mu_X \]
\[ = \int_Y P_T(\bar{y}) d\mu_Y \]
where \( P_T \) is the average of \( P \) with respect to the action of \( T \). The polynomial \( P_T \) then descends to a polynomial \( P_Y \) on \( Y \) of degree \( t \), and
\[ \int_X P_T(\bar{x}) d\mu_X = P(F_Y) \]
\[ \int_Y P_T(\bar{y}) d\mu_Y = P(Y) \]
because \( \alpha \) preserves measure.

The measure \( \sigma_X \) evidently has a twisted product \( s \)-cubature formula \( F_X = F_T \times F_Y \) given by replacing each fiber by a copy of \( F_T \). (A singular fiber corresponding to a boundary point of \( T \) can be replaced by a projection of \( F_T \).) Since \( \mu_X \) and \( \sigma_X \) are \( s \)-cubature equivalent, \( F_X \) is a cubature formula for \( \mu_X \) as well.

**Remark.** The proof of Theorem 4.1 is analogous to Sobolev’s theorem with the finite group \( G \) replaced by the torus \( T \). Indeed the argument works for any compact group.

The simplest case of Theorem 4.1 is the Hopf map \( h \). In this case the theorem says that a \( t \)-cubature formula \( F \) for \( CP^{n-1} \) lifts to a \( (2t+1) \)-cubature formula \( F' \) on \( S^{2n-1} \) with \( (2t+2)|F| \) points. This relation was also observed by König [15].

**Corollary 4.2.** The \( n \)-sphere \( S^n \) has a 7-cubature formula with \( O(n^4) \) points for all \( n \), more precisely \( 4n^4 (1 + o(1)) \) points. The 3-sphere \( S^3 \) has a \( (2s+1) \)-cubature formula with
\[ |F| = \begin{cases} (s+1)(s^2+3) & s \text{ odd} \\ (s+1)(s^2+s+2) & s \text{ even} \end{cases} \]
points.
Proof. The simplex $\Delta_n$ has a 3-cubature formula with $O(n)$ points constructed using Hadamard designs. This can be combined with the 7-design on $T(\text{SO}(2n))$ with $O(n^3)$ points provided by Theorem 4.2, for a total of $O(n^4)$ points. More precisely, the formula on $\Delta_n$ has points at the corners, each of which lifts to $O(n)$ points; a point in the center, which lifts to $O(n^3)$ points; and $2n+o(n)$ points on $|n/2|$-dimensional faces, each of which lift to $2n^3(1+o(1))$ points. Only the last family of points is significant and it comprises $4n^3(1+o(1))$ points.

Noskov's formulas from Section 4.1 include a $(2s+1)$-design on the square torus $T(\text{SO}(4))$ with $2(s+1)^2$ points. When $s$ is odd, this can be combined with the Gauss-Lobatto $s$-quadrature formula on the interval $[s]$ with $s^2$ points. Two of the fibers are circles and can be replaced by $2(s+1)$ points instead of $2(s+1)^2$. The total is then $(s+1)(s^2+3)$ points. When $s$ is even, it can be combined with the Gauss-\text{Radau} $s$-quadrature formula with $s^2$ points. In this case one fiber is a circle.

Remark. The first part of Corollary 4.2 actually yields a 7-design on $S^{n-1}$ with $O(n^6)$ points whenever there is a Hadamard matrix of order $n$. In this case the weights of the 3-cubature formulas on $\Delta_{n-1}$ are $\frac{2}{m(n+1)(n+2)}$ at the corners, $\frac{n}{2(n+1)(n+2)}$ at the faces, and $\frac{d_n}{(n+1)(n+2)}$ at the center. Thus the weights are all commensurable up to a factor of $2n^2$ (note that $n$ is even) and the cubature formula can be interpreted as a 7-design with this multiplicity factor. Moreover, copies of the lattice formulas on the torus fibers can be shifted to make the design multiplicity-free. Better yet, the design need only have $O(n^3)$ points if, for example $n = 4 \cdot 7^k$. In this case the prime $p$ used in the proof of Theorem 4.1 can be replaced by the prime power $7^k$. The number of points on each fiber then compensates for all but a bounded part of the factor of $2n^2$ in the weights.

The previous best construction of 7-designs on $S^{n-1}$ is due to Sidel'nikov [24] and requires $O(2^{(k+1)/2})$ points when $n = 2^k$.

A useful variant of Theorem 4.1 involves the moment map $\tilde{\tau}_2 : \mathbb{R}^{2n} \to \mathbb{R}^n$ defined by the same formula as $\tau_2 : \mathbb{R}^{2n-1} \to \mathbb{R}^n$, namely:

$$\tilde{\tau}_2(x_1, \ldots, x_{2n}) = (x_1^2 + x_2^2, x_3^2 + x_4^2, \ldots, x_{2n-1}^2 + x_{2n}^2).$$

This $\tilde{\tau}_2$ takes uniform measure on the ball $B_n$ to uniform measure on the simplex

$$\Delta_n = \{ x | x_k \geq 0, \sum x_k \leq 1 \}.$$

When $n = 2$, Noskov’s formulas together with some ad hoc cubature formulas for the triangle yield some economical formulas for the 4-ball $B_4$. For example, there is a PB 3-cubature formula on the triangle $x, y > 0, x+y \leq 1$ with points and weights generated from

$$\tilde{p}_1 = (\frac{2}{5}, \frac{2}{5}), \quad w_1 = \frac{25}{48},$$

$$\tilde{p}_2 = (\frac{161 + 17\sqrt{14}}{1344}, 0), \quad w_2 = \frac{16 - 2\sqrt{14}}{25}.$$

by switching the coordinates and negating $\sqrt{14}$. This formula lifts to 1 generic fiber in $B_4$ which can be replaced with 32 points and 4 singular fibers which are circles and can be replaced with 8 points each. The result is a PI 7-cubature formula on $B_4$ with 64 points.

Wandzura and Xiao [31] found competitive PI s-cubature formulas for $s$ up to 30; Figure 7 shows one example. Most of these yield competitive PI $(2s+1)$-cubature formulas on $B_4$ and $S^3$. The formulas could probably be improved further with a search on the triangle that favors nodes on the edges.

The map $\tau_2 : \mathbb{R}^{2n} \to \mathbb{R}^n$ also takes Gaussian measure on $\mathbb{R}^{2n}$ to exponential measure on $\mathbb{R}^n$. For example, there is a PB exponential 4-cubature formula on $\mathbb{R}^7$ with points and weights generated from

$$\tilde{p}_1 \approx (1.50766353, 1.50766353), \quad w_1 \approx 0.354104443,$$

$$\tilde{p}_2 \approx (6.29508677, 1.76717584), \quad w_2 \approx 0.00876905581,$$

$$\tilde{p}_3 \approx (0.285606152, 0), \quad w_3 \approx 0.556110610,$$

$$\tilde{p}_4 \approx (3.27491992, 0), \quad w_4 \approx 0.0722468398$$

by switching the coordinates. It lifts to 3 generic fibers with 50 points each and 4 singular fibers with 10 points each. The result is a positive Gaussian 7-cubature formula on $\mathbb{R}^4$ with 190 points.

5. AN ALGEBRAIC LOWER BOUND

Let $X$ be the Zariski closure of the support of a measure $\mu$ on $\mathbb{R}^d$ and let $A$ be the ring of polynomial functions on $X$. (Recall that the Zariski closure of a set, or closure in the Zariski topology, is the smallest algebraic variety containing it.) In other words, $A$ is the quotient of $\mathbb{R}[\bar{x}]$ by the ideal $l_x$ of polynomials that vanish on $X$. The ring $A$ has a degree filtration coming from the degree filtration of $\mathbb{R}[\bar{x}]$. Stroud [28,29] established an important lower bound on an arbitrary 2r-cubature formula $F$ for $\mu$ (not necessarily positive or interior):

**Theorem 5.1 (Stroud).** If $F$ is a 2r-cubature formula for $\mu$, then

$$|F| \geq \dim A_{\leq r}.$$

Mysovskikh [19] observed that this applies to trigonometric cubature by taking $X = T(\text{SO}(2n))$. (And according to Möller [18], the bound was noted independently in special cases by other authors, e.g., Radon.)

**Proof.** Define a bilinear form

$$b : A_{\leq r} \times A_{\leq r} \to \mathbb{R}$$

by

$$b(P, Q) = \int_X P(\bar{x})Q(\bar{x})d\mu.$$
The form $b$ is positive-definite because the integrand of $b(P,P)$ is non-negative; moreover if the integrand vanishes on $X$, then $P = 0$ as an element of $A$. Therefore $b$ is non-degenerate, and its rank is $\dim A_{\leq t}$. On the other hand, the integrand lies in $A_{\leq 2t}$, so a $2t$-cubature formula $F$ leads to the formula

$$b(P,Q) = \sum w_k P(\tilde{p}_k) Q(\tilde{p}_k).$$

This formula realizes $b$ as a sum of $|F|$ rank 1 forms. Therefore $|F|$ is at least the rank of $b$, as desired. \hfill \square

An interesting scholium of the proof of Theorem 5.1 is that if $F$ is a $2t$-cubature formula, then its points suffice to interpolate polynomials on $X$ of degree $t$.

It is curiously difficult to improve the Stroud bound for odd-degree cubature. However, the inference that lower bounds improve mainly in even degrees is not consistent with the Hopf fibrations

$$h : S^{2n+1} \to \mathbb{C}P^n \quad h : T(S(2n+2)) \to T(\mathbb{PSU}(n+1)).$$

On the one hand, these maps are quadratic and double the degree of cubature in passing from the target to the domain; in particular, they do not preserve odd and even. On the other hand, Sections 2 and 4 together show that cubature in the domain and target are comparably difficult when $n \gg t$.

The Hopf fibration example suggests a generalization of Stroud’s theorem involving actions and degree doubling.

**Theorem 5.2.** Let $\mu$ be measure on $\mathbb{R}^n$, and let $X$ be the Zariski closure of its support. Let $Y \subset \mathbb{R}^k$ be another affine real algebraic variety on which a compact group $G$ acts. Let $A$ and $B$ be the rings of complex-valued polynomials on $X$ and $Y$, and suppose that there is a ring isomorphism

$$A \xrightarrow{\cong} \text{Inv}_G(B)$$

that doubles the filtration degree of $A$. Let $V$ be a unitary representation of $G$ and define the filtered vector space

$$M = \text{Inv}_G(B \otimes V).$$

If $F$ is a $t$-cubature formula for $\mu$, then

$$|F| \geq \frac{\dim M_{\leq t}}{\dim V}.$$

We will always take $Y$ to be the coordinate ring of another algebraic variety $Y$ which is a principal $G$-bundle over $X$, such that the bundle projection $\alpha : Y \to X$ is quadratic. The $A$-module $M$ can then be understood as the space of polynomial sections of a vector bundle $E$ over $X$ with fiber $V$. The sections in $M_{\leq t}$ then behave like polynomials elements of $A$, except that their degrees are half-integers. If $Y = X$ and $G$ is trivial, then $E$ is the trivial line bundle and Theorem 5.2 reduces to Theorem 5.1. The hypotheses of Theorem 5.2 have been chosen so that the proof of Theorem 5.1 generalizes to the case when $E$ is not trivial.

**Proof.** The vector space $M$ (which is naturally an $A$-module) has an $A$-valued Hermitian inner product $a$ induced by the Hermitian inner product on $V$. More precisely, let $\overline{V}$ be the representation conjugate to $V$ and let

$$\overline{M} = \text{Inv}_G(\overline{V} \otimes B)$$

be the corresponding conjugate of $M$. (Note that $A$ and $B$ are both self-conjugate by hypothesis.) Let

$$\mathcal{E} : V \otimes \overline{V} \to \mathbb{C}$$

be the linearization of the standard Hermitian inner product on $V$, and let

$$m : B \otimes B \to B$$

be the linearization of multiplication on $B$. Let $a'$ be the composition

$$B \otimes V \otimes \overline{V} \otimes B \xrightarrow{\mathcal{E} \otimes \mathcal{E} \otimes m} B \otimes B \xrightarrow{a} B.$$

We can restrict the domain to

$$M \otimes \overline{M} = \text{Inv}_G(B \otimes V) \otimes \text{Inv}_G(\overline{V} \otimes B).$$

Since the restricted domain is $G$-invariant, we can then restrict the target to $A$. Let $b$ be this restriction of $a'$. Although given as a linear map on $M \otimes \overline{M}$, it can be reinterpreted as a Hermitian inner product on $M$. In more geometric terms, if $M$ comes from a bundle $E$ over $X$ with fiber $V$, then $a(f,g)$ is the pointwise inner product of two sections $f$ and $g$ of $E$.

Note that $a$ is positive-definite in the sense that

$$a(f,f)(\overline{x}) \geq 0$$

for all $x \in X$, and if $a(f,f) = 0$, then $f = 0 \in M$. The rest of the proof follows that of Theorem 5.1. Define a complex-valued Hermitian inner product $b$ on $M_{\leq t}$ by

$$b(f,g) = \int_X a(f(\overline{x}),g(\overline{x}))d\mu.$$

Then $b$ is also positive-definite, because $a$ is positive-definite and $\mu$ is Zariski-dense in $X$. Thus, $b$ has rank $\dim M_{\leq t}$. A cubature formula $F$ realizes $b$ as a sum of $|F|$ terms of rank at most $\dim V$.

We state three special cases of Theorem 5.2 as corollaries:

**Corollary 5.3.** If $|F|$ is a $(2t+1)$-cubature formula on $\mathbb{C}P^n$, then

$$|F| \leq \binom{n+t}{n} \binom{n+t+1}{n}.$$
Corollary 5.4. If $|F|$ is a $t$-cubature formula on $T(PSU(n + 1))$, then

$$|F| \geq n'(1 - o(1)) \frac{t}{|t/2|!|t/2|!}.$$  

Proof. Let $Y$ be the torus $T(SO(2n + 2))$ and let $G = S^1$ again act by complex rotation in $\mathbb{C}^{n+1}$. In this case $M_{\leq 2t+1}$ is spanned by the space of monomials in $\vec{z}$ and $\vec{\tau}$ of bidegree $(s, t)$ with $s \leq t$ and with the relation

$$z_k \tau_k = 1$$

for all $k$. Its dimension is the number of points in the Minkowski difference

$$\Delta^{(t+1)} - \Delta^{(t)},$$

where $\Delta^{(t)}$ is the discrete simplex defined in the proof of Theorem 3.3. This is very similar to Theorem 5.1 for 2t-cubature, because

$$\dim A_{\leq t} = |\Delta^{(t)}| - |\Delta^{(t)}|.$$  

There is no concise formula for either number, but there is a concise estimate for fixed $t$ in the limit $n \to \infty$. If $E$ is either the trivial bundle when $t$ is even or the bundle $L_1$ (restricted from $\mathbb{C}P^n$) when $t$ is odd, then

$$\dim M_{\leq t} \approx \left(\frac{n + 1}{|t/2|, |t/2|, n + 1 - t}\right) \approx \frac{n'}{|t/2|!|t/2|!}$$

as $t \to \infty$, as desired. $\square$

Remark. When $F$ is a lattice formula, Corollary 5.4 is equivalent to Minkowski’s classic upper bound on the density $F$ as a lattice packing of the discrete simplex $\Delta^{(t)}$. This and the fact that the Hopf fibration is quadratic led the author to Theorem 5.2.

Corollary 5.5. If $\mu$ is a Zariski-dense measure on $S^n$ and $F$ is a $(2t + 1)$-cubature formula for $\mu$, then

$$|F| \geq 2 \left(\frac{n - 1 + t}{t}\right).$$

Proof. The idea is to let $Y = \text{Spin}(n + 1)$ and $G = \text{Spin}(n)$, where Spin$(n)$ is the connected Lie group that double covers SO$(n)$. Then

$$X = \text{Spin}(n + 1)/\text{Spin}(n) = \text{SO}(n + 1)/\text{SO}(n) = S^n.$$

Our choice for the representation $V$ is the spinor representation of Spin$(n)$ when $n$ is odd and a semispinor representation when $n$ is even. The rest of the argument is a review of standard but non-trivial representation theory [14, 50]:

1. The irreducible (unitary) representations of the spin groups Spin$(2n)$ and Spin$(2n + 1)$ are both indexed by vectors

$$\vec{x} = a_1 \vec{\lambda}_1 + a_2 \vec{\lambda}_2 + \ldots + a_n \vec{\lambda}_n$$

with each $a_k$ a non-negative integer. The representation with highest weight $\vec{x}$ can be called $V_{\vec{x}}(\vec{\lambda})$ with $G = \text{Spin}(2n)$ or $G = \text{Spin}(2n + 1)$.

2. The representation $V_C(t \vec{\lambda}_1)$ is realized as the space $A_t$ of homogeneous polynomials of degree $t$ on $S^{2n-1}$ or $S^{2n}$. The representation $V_{\text{Spin}(2n+1)}(\vec{\lambda}_n)$ is the spinor representation of dimension $2^n$. The representations $V_{\text{Spin}(2n)}(\vec{\lambda}_{n-1})$ and $V_{\text{Spin}(2n)}(\vec{\lambda}_n)$ are the semispinor representations of dimension $2^{n-1}$.

3. For any choice of fiber $V = V_{\text{Spin}(n)}(\vec{\lambda})$, the $A$-module $M$ is the induced representation

$$M = M(n, \vec{\lambda}) = \text{Ind}_{\text{Spin}(n)}^{\text{Spin}(2n)} V_{\text{Spin}(n)}(\vec{\lambda}).$$

4. The characters of $V(\vec{\lambda})$ for all $\vec{\lambda}$ generate the natural real algebraic structure on Spin$(2n)$ or Spin$(2n + 1)$ (indeed on any compact Lie group). One degree filtration is defined by letting the characters of every $V(\vec{\lambda})$ have degree 2, except for the spinor or semispinor representations, which have degree 1.

5. The structure of $M(n, \vec{\lambda})$ can be computed by branching formulas for the restriction of an irrept Spin$(n + 1)$ to Spin$(n)$, together with induction-restriction duality. These restriction formulas were computed by Koike [14, Thms. 11.2 & 11.3]. When $\vec{\lambda} = \vec{\lambda}_n$, Koike’s formulas together with the degree filtrations yield

$$M(2n + 1; \vec{\lambda}_n)_{\leq 2t+1} \cong \bigoplus_{s \leq t} V(2n + 1; s \vec{\lambda}_1 + \vec{\lambda}_n)$$

and

$$M(2n; \vec{\lambda}_n)_{\leq 2t+1} \cong \bigoplus_{s \leq t} V(2n; s \vec{\lambda}_1 + \vec{\lambda}_n).$$

6. Finally by the Weyl dimension formula,

$$\dim V(2n + 1; s \vec{\lambda}_1 + \vec{\lambda}_n) = 2^n \left(\frac{2n - 1 + s}{s}\right)$$

and

$$\dim V(2n; s \vec{\lambda}_1 + \vec{\lambda}_n) = \dim V(2n; s \vec{\lambda}_1 + \vec{\lambda}_n)_{\leq 2t+1} = 2^{n-1} \left(\frac{2n - 2 + s}{s}\right).$$

Combining the dimension formulas yields

$$|F| \geq \frac{\dim M(n; \vec{\lambda})_{\leq 2t+1}}{\dim V} = \sum_{s \leq t} 2 \left(\frac{n - 2 + s}{s}\right) = 2 \left(\frac{n - 1 + t}{t}\right),$$

as desired. $\square$

Remark. Corollary 5.5 matches the Möller bound [18] for cubature on $S^n$, but it is more general because the measure $\mu$ need not be centrally symmetric.
6. A LOCAL LOWER BOUND

Our final application of moment maps is to help establish a local lower bound on the density of points of a PI or PB cubature formula on the simplex $\Delta_n$. The bound was originally inspired by PI cubature formulas due to Wandzura and Xiao [31] which were found by simulated annealing. As in the example shown in Figure 7, the points in these formulas accumulate transversely at the edges of the triangle. Another related result is that the limiting density of the points of Gauss-Legendre quadrature (i.e., the zeros of Legendre polynomials) is $1/\pi \sqrt{1-x^2}$ [8]. This density can be interpreted as the linear projection of uniform measure on a circle, which is related to Archimedes’ moment map (Figure 9).

Theorem 6.1 and Corollary 6.2 establish a lower bound on the limiting density of any sequence of PI and PB formulas on $\Delta_n$ that generalizes the limiting density of Legendre zeros. Moreover, if the local density is high in certain regions, in particular near the vertices of $\Delta_n$, then the weights there must be low. By this reasoning, Theorem 6.3 and Corollary 6.4 establish that a $t$-design on $\Delta_n$ requires many more points than an efficient $t$-cubature formula does as $t \to \infty$ (namely $O(t^{2n})$ points versus $O(t^n)$ points). Along the way, Theorem 6.3 establishes that Gaussian quadrature for an arbitrary weight function is very sharply locally optimal among all positive quadrature formulas. Finally Scholium 6.5 generalizes results for uniform measure on $\Delta_n$ to uniform measure on an arbitrary simple convex polytope.

Theorem 6.1. A PI or PB $2t - 1$-cubature formula $F$ on the simplex $\Delta_n$ is an $\varepsilon$-net with respect to the metric $d\hat{s}^2 = \frac{dx_0^2}{2x_0} + \frac{dx_1^2}{2x_1} + \ldots + \frac{dx_n^2}{2x_n}$, in barycentric coordinates, where $\cos 2\varepsilon$ is the highest zero of the Jacobi polynomial $P_{\frac{n-1}{2}}^{(n-1,0)}(x)$.

In the proof and later, we will abbreviate $(n-1,0)$ as “#” in superscripts.

Proof. The idea of the proof is to find, for each $\bar{p} \in \Delta_n$ and each $\varepsilon' > \varepsilon$, a $P$ of degree $2t - 1$ on $\Delta_n$ such that

$$\int_{\Delta_n} P(\bar{x})d\bar{x} > 0,$$
but \( P(\vec{x}) > 0 \) only when \( \vec{x} \in A_n \) is in the \( \epsilon' \)-ball \( B_{\epsilon'}(\vec{p}) \) around \( \vec{p} \). We can call this ball the positive island of \( P(\vec{x}) \); see Figure 8. The existence of such a polynomial \( P \) forces \( F \) to have an evaluation point in \( B_{\epsilon}(\vec{p}) \), for otherwise \( P(F) \leq 0 \).

We first claim that the stated metric is the distance between the Xu map with respect to the Fubini-Study metric on \( CP^n \). To see this it suffices to check the following:

The real locus \( \mathbb{R}P^n \subset CP^n \) is perpendicular to the fibers of \( \pi \) and meets each generic fiber \( 2^n \) times. Indeed, \( \pi \) is a bijection on the orthant \( \mathbb{R}P^n_{\geq 0} \) with non-negative projective coordinates. Moreover, \( \mathbb{R}P^n_{\geq 0} \) is isometric to the orthant \( S^n_{\geq 0} \) of a unit n-sphere, and the restriction of \( \pi \) agrees with the restriction of the Xu map \( \tau_1 \). The metric \( ds^2 \) on \( A_n \) is exactly the push-forward of the standard metric on \( S^n_{\geq 0} \) under \( \tau_1 \). See Figure 9 for an example.

Consider the linear projection \( \alpha : A_n \to [-1,1] \) given by
\[
\alpha(\vec{x}) = 2x_0 - 1.
\]
The map \( \alpha \) sends uniform measure on \( CP^n \) to the measure
\[
\mu(x) = n2^{1-n}(1-x)^{n-1}. \tag{1}
\]
The \( th \) orthogonal polynomial with respect to this measure \( \mu \) is the Jacobi polynomial \( P^n_t = P_t^{(n-1,0)} \). Let \( p_t^\# \) be its highest zero.

Define a polynomial \( Q_\delta : CP^n \to \mathbb{R} \) by
\[
Q_\delta(\vec{x}) = Q_\delta(x) = \frac{P_t^\#(x)^2(x-p_t^\#+\delta)}{(x-p_t^\#)^2}, \tag{2}
\]
where \( x = \alpha(\pi(\vec{z})) \) and \( \delta > 0 \). It has degree \( 2t-1 \) as a polynomial in \( x \), as well as a polynomial on \( CP^n \). Moreover, \( Q_\delta \) vanishes at the zeros of \( P_t^\# \), except at the highest zero, at which its value is positive. Therefore by Gaussian quadrature (1) with respect to the measure \( \mu \),
\[
\int_{CP^n} Q_\delta(\vec{z})d\vec{z} = \int_{-1}^1 Q_\delta(x)d\mu > 0.
\]
At the same time, \( Q \) is non-positive outside of the region
\[
x > p_t^\# - \delta.
\]
This region corresponds to the ball of radius \( \epsilon' \) around \( (1:0:\cdots:0) \), with
\[
2(\cos \epsilon')^2 - 1 = \cos 2\epsilon' = p_t^\# - \delta.
\]
This can be confirmed by comparing with the orthant \( \mathbb{R}P^n_{\geq 0} \) from mentioned previously. Note that \( \epsilon' \to \epsilon \) as \( \delta \to 0 \).

Given \( \vec{q} \in CP^n \), define \( Q_{\epsilon,\vec{q}} \) by rotating \( Q_\delta \) by some isometry of \( CP^n \) that takes \( (1:0:\cdots:0) \) to \( \vec{q} \). Define \( Q_{\epsilon,\vec{q}}^T : A_n \to \mathbb{R} \), where \( \vec{p} = \pi(\vec{q}) \), by averaging \( Q \) over torus fibers:
\[
Q_{\epsilon,\vec{q}}^T(\vec{x}) = \frac{1}{|\pi^{-1}(\vec{x})|} \int_{\pi^{-1}(\vec{x})} Q_{\epsilon,\vec{q}}(\vec{z})d\vec{z}.
\]
Then
\[
\int_{A_n} Q_{\epsilon,\vec{q}}^T(\vec{x})d\vec{x} = \int_{CP^n} Q_{\epsilon,\vec{q}}^T(\vec{x})d\vec{z} > 0,
\]
and \( Q_{\epsilon,\vec{q}}^T \) is non-positive outside of the ball of radius \( \epsilon' \) around \( \vec{p} = \pi(\vec{q}) \) in the induced metric on \( A_n \). Thus, \( Q_{\epsilon,\vec{q}}^T \) has the desired properties.

**Remark.** A somewhat weaker version of Theorem 6.1 holds when \( F \) is positive and exterior, but with real evaluation points. Polynomials similar to \( Q_{\epsilon}^T \) can be constructed directly as products of factors that vanish on quadratic surfaces in \( \mathbb{R}^n \), with only one unsquared factor that vanishes on the boundary of \( B_{\epsilon'}(\vec{p}) \). As it happens, the boundary of \( B_{\epsilon'}(\vec{p}) \) is a quadratic surface. We did not refine this sketched argument into a proof with explicit estimates.

**Corollary 6.2.** Any sequence of PI or PB t-cubature formulas on \( A_n \) has limiting point density \( \Omega(\epsilon^{n} \prod x_k^{-1/2}) \), where \( \vec{x} \in A_n \) is fixed and given in barycentric coordinates, and \( t \to \infty \).

**Proof.** The corollary follows from computing the volume form corresponding to the metric \( ds^2 \) in the statement of Theorem 6.1 and estimating the covering radius \( \epsilon \). The asymptotic behavior of zeros of Jacobi polynomials is given in Abramowitz and Stegun [1, p. 787]. The key step in the estimate is the limit
\[
\lim_{t \to \infty} \frac{P_t^{(a,b)}(\cos \frac{\theta}{t})}{t^{(a+b+1)/2}} = 2^a \theta^{-a} J_a(\theta), \tag{3}
\]
where \( J_a(\theta) \) is the ordinary Bessel function of the first kind. Convergence to the limit is analytic in \( \theta \). Thus
\[
\lim_{t \to \infty} \epsilon \theta^{(a,b)} \theta^{(a+b+1)\epsilon / \theta} = j_{a,b}(\theta).
\]
for every fixed $k$, where $\cos \theta_{i,k}^{(a,b)}/t$ is the $k$th zero of $P_i^{(a,b)}(x)$ and $j_{a,k}$ is the $k$th zero of $j_{a}(x)$. The estimate can be established directly in our geometry by noting that $P_i^{(a,b)}(2|z|^2 - 1)$ is a harmonic function on $\mathbb{C}P^n$. The harmonic equation on $\mathbb{C}P^n$ is then locally approximated by the harmonic (or Helmholtz) equation on $\mathbb{R}^{2n}$, whose radial solutions are derived from Bessel functions.

In our case,

$$\cos 2\epsilon = \cos \frac{\theta^{\text{th}}}{t},$$

for $(2t-1)$ cubature. So

$$\epsilon = \frac{j_{n-1}(1 + o(1))}{2t} = \Theta(t^{-1}),$$

which is also $\Theta(t^{-1})$ for $t$-cubature.

**Theorem 6.3.** Let $\mu$ be an arbitrary normalized measure on $\mathbb{R}$ whose support has at least $2t$ points. Let $p_1, \ldots, p_t$ and $w_1, \ldots, w_t$ be the points and weights of Gaussian $t$-quadrature for the measure $\mu$. Let $F$ be a positive $t$-quadrature formula for $\mu$. Then for each $1 \leq k \leq t$, $F$ has at least one point in the half-open interval $(p_{k-1}, p_k]$, where $p_0 = -\infty$. Moreover, the total weight of all points in $(-\infty, p_1]$ is at most $w_1$, with equality if and only if $F$ is the Gaussian quadrature formula.

**Proof.** Let $\phi_k(x)$ be the $k$th orthonormal polynomial with respect to $\mu$ (with either sign), and let $A_t$ be the leading coefficient of $\phi_k(x)$. If $k = 1$, let

$$P(x) = \frac{\phi_k(x)^2(p_1 + \delta_1 - x)}{(x - p_1)^2}$$

with $\delta_1 > 0$. If $k > 1$, let

$$P(x) = \frac{\phi_k(x)^2(x - p_{k-1} - \delta_0)(p_k + \delta_1 - x)}{(x - p_{k-1})^2(x - p_k)^2}$$

with $\delta_1 \gg \delta_0 > 0$. In both cases,

$$\int_{\mathbb{R}} P(x) d\mu > 0$$

by Gaussian quadrature, while $P$ is only positive on the interval $(p_{k-1}, p_k]$. Therefore $F$ has at least one point in this interval. Since $F$ only has finitely many points, the limit $\delta_1 \to 0$ establishes that $F$ has a point in $(p_{k-1}, p_k]$.

For the second claim, let

$$P(x) = \frac{\phi_k(x)^2}{(x - p_1)^2}.$$ 

Then by Gaussian quadrature,

$$\int_{\mathbb{R}} P(x) d\mu = w_1 P(p_1).$$

Let $q_1, \ldots, q_k$ be the points of $F$ which are at most $p_1$, and let $v_1, \ldots, v_k$ be their weights. Then

$$\int_{\mathbb{R}} P(x) d\mu = P(F) \geq \sum_{j=1}^k w_j P(q_j) \geq P(p_1) \sum_{j=1}^k w_j.$$

The first inequality holds because $P$ is non-negative; the second because $P$ decreases on $(-\infty, p_1]$.

**Corollary 6.4.** The least weight of any positive $t$-cubature formula on $\Delta_n$ (with uniform measure) is $O(t^{-2n})$, uniformly in $t$. Any $t$-design on $\Delta_n$ has $\Omega(t^{2n})$ points.

**Proof.** If $F$ is a $t$-cubature formula on $\Delta_n$, the map $\alpha$ (see equation (1)) sends it to a $t$-quadrature formula $\alpha(F)$ on $[-1, 1]$ with Jacobi-polynomial measure. If $F$ is positive, then the least weight of $\alpha(F)$ is at least that of $F$. On the other hand, Theorem 6.3 establishes that the least weight $\alpha(F)$ is at least the last Christoffel weight $w_t$. The first claim follows by estimating this weight. One of the standard formulas for the general Christoffel weight $w_t$ is

$$w_k = \frac{A_{t+1}||\phi_k(t)||_{\mu}^2}{A_t \phi_k(t) \phi_{k+1}(t)},$$

where $\phi_k(t)$ is the $k$th orthogonal polynomial, $A_t$ is its leading coefficient, and $p_k$ is its $k$th root. In our case, $\phi_k(t) = P_{t,k}, p_k = P_{t,k}$, and $k = t$. We compute:

$$||P_{t,k}||_{\mu}^2 = \frac{n}{2t + n} = \Theta(t^{-1})$$

$$A_t = 2^{-t} \left( n - 1 + 2t \right) = \Theta(2^t).$$

To estimate $P_{t,k}^t(t)$ and $P_{t+1,k}^t(t)$, we again appeal to the limit in equation (3). Differentiating both sides by $\theta$, we obtain

$$\lim_{t \to \infty} \frac{(P_{t,k}^t(t) \cos \theta/t)(\sin \theta/t)}{tP_{t,k}^t(t)} = -a_2 \theta^{-a} a_{t,0}(\alpha).$$

Note that $P_{t,k}^t(t) = (t+n-1) = \Theta(t^{n-1})$. For a fixed value of $\theta$, the various parts of the limit yield

$$(P_{t,k}^t(t) \cos \theta/t) = \Theta(t^{n+1}).$$

By the same token

$$(P_{t,k}^t(t) \cos \theta/t) = \Theta(t^{n+2})$$

when $\theta$ approaches a fixed value of $\theta$, as is the case when $\theta = \theta_{i,t}^t$ is given by

$$p_{i,t}^t = \cos \frac{\theta_{i,t}^t}{t}.$$ 

By a similar calculation,

$$(P_{t+1,k}^t(t) \cos \theta/t) = \Theta(t^{n-2}).$$

The conclusion is that

$$w_t = \Theta(t^{2n}),$$

as desired.

\qed
Scholium 6.5. Let \( K \subset \mathbb{R}^n \) be a convex \( n \)-polytope with \( N \) facets. Let \( F \) be a \( t \)-cubature formula on \( K \) with uniform measure. If \( F \) is PI or PB and if \( K \) is simple, then \( F \) is an \( \varepsilon \)-net with respect to the metric
\[
d s^2 = \frac{dx_1^2}{x_1} + \frac{dx_2^2}{x_2} + \ldots + \frac{dx_N^2}{x_N},
\]
where \( \varepsilon = O(1/t) \). If \( F \) is positive, then its least weight is \( O(t^{-2n}) \). If it is a \( t \)-design, then it has at least \( O(t^{2n}) \) points.

![Diagram of a simplex with facets](image)

Figure 10: A simplex \( L^{-1}(\Delta_n) \) whose facets contain facets of \( K \) that meet at \( \bar{x} \).

Proof. (Sketch) The proof of Theorem 6.1 retains its strength if \( K \subseteq \Delta_n \) and we pass from \( \Delta_n \) to \( K \), provided that the positive island of the polynomial \( Q_{\bar{x},\bar{\beta}} \) lies within \( K \). In this case
\[
\int_K Q_{\bar{x},\bar{\beta}}(\bar{x})d\bar{x} \geq \int_{\Delta_n} Q_{\bar{x},\bar{\beta}}(\bar{x})d\bar{x} > 0.
\]
In order to properly position \( Q_{\bar{x},\bar{\beta}} \) for all \( \bar{\beta} \in K \), we need several embeddings of \( K \) into \( \Delta_n \). For each vertex \( \bar{x} \in K \), choose a linear embedding \( L \) that sends \( \bar{x} \) to some vertex of \( \Delta_n \), and that sends the facets incident to \( \bar{x} \) to facets of \( \Delta_n \). (Equivalently, for each vertex \( \bar{x} \in K \), a simplex \( L^{-1}(\Delta_n) \supseteq K \) whose facets includes all facets of \( K \) that meet at \( \bar{x} \). See Figure 10) Then there exists a finite set of \( L \) such that the positive islands of polynomials of the form \( Q_{\bar{x},\bar{\beta}} \circ L \) cover \( K \). The formula \( F \) must have a point in each island, which establishes that \( F \) is an \( \varepsilon \)-net.

Similarly, the proofs of Theorem 6.3 and Corollary 6.4 retain their strength if uniform measure on \( K \) projects by a map \( \alpha \) to a measure \( \nu \) on \([-1, 1]\) which is dominated by
\[
\mu(x) = 2^{-n}n(1-x)^{n-1}
\]
and which agrees with \( \mu \) in a neighborhood of 1. (Of course \( \mu \) cannot dominate \( \nu \) if \( \nu \) is normalized, so this condition on \( \nu \) must be dropped.) In this case
\[
\int_{\mathbb{R}} P(x)d\nu \geq \int_{\mathbb{R}} P(x)d\mu > 0
\]
for the first half of Theorem 6.3 for the interval \((-\infty, p_1)\), while
\[
\int_{\mathbb{R}} P(x)d\nu \leq \int_{\mathbb{R}} P(x)d\mu
\]
for the second half of Theorem 6.3. A suitable projection \( \alpha \) can be realized by positioning \( K \) in \( \Delta_n \) so that it touches the vertex \( x_0 = 1 \), and then restricting the usual map \( \alpha \) to \( K \).

7. OTHER COMMENTS

In this article we have studied the toric moment map on \( \mathbb{C}P^n \), and on \( \mathbb{C}^n \) restricted to \( S^{2n-1} \) (which can be interpreted as the level surface of an invariant Hamiltonian on \( \mathbb{C}^n \)) as it applies to the cubature problem. Many of the constructions apply equally well to arbitrary toric varieties. To begin with, every complex projective variety \( X \) inherits both a metric and an affine real structure from \( \mathbb{C}P^n \). If \( X \) is toric, it also has a volume-preserving moment map whose image is a centrally symmetric polytope. However, the variety \( X \) rarely has much more symmetry than its moment map image.

The duality between toric cubature (in particular trigonometric cubature) and lattice packings explored in Section 3 suggests a different limit of the cubature problem. Let \( K \subset \mathbb{R}^n \) be a centrally symmetric convex body. For simplicity let \( F = \{\bar{\beta}_a\} \) be a periodic discrete subset of \( \mathbb{R}^n \) with a periodic weight function \( \bar{\beta}_a \leftrightarrow w_a \). Since it is periodic, it has a well-defined Fourier transform \( \hat{F} \). In this setting, \( F \) is a Fourier \( K \)-cubature formula if and only if \( \hat{F} \) is disjoint from the interior of \( K \). The (continuous) Fourier cubature problem is to minimize the density of \( F \) among all \( K \)-cubature formulas or all positive \( K \)-cubature formulas. If \( F \) is a lattice formula with equal weights, then \( \hat{F} \) has the same property and Fourier \( K \)-cubature problem reduces to finding the best lattice packing of \( K \). It would be interesting to find examples of non-lattice formulas that are better than the best lattice formula.

We conjecture that a version of Corollary 5.5 holds, using Theorem 5.2 and the same spinor bundles, for any centrally symmetric subvariety \( X \subset \mathbb{C}^n \). That is, we conjecture Möller’s bound for these varieties, even when the measure \( \mu \) is not centrally symmetric.

Theorem 6.1 shows why some tempting approaches to construct efficient PI or PB formulas on the simplex \( \Delta_n \), even the triangle \( \Delta_2 \), are bound to fail. For example, if the points of a putative cubature formula \( F \) are fixed in advance, the question of whether it admits non-negative weights for \( t \)-cubature reduces to linear programming. But if the points are arranged in some lattice with spacing \( 1/k \), Theorem 6.1 shows that the weights can only be non-negative if \( k = \Omega(t^2) \), so that \( |F| = \Omega(t^{2n}) \).

We believe that the requirement that \( K \) be simple in Scholium 6.5 is not essential. More generally we conjecture that similar results hold if \( K \) is not convex. We also conjecture that the bounds in Theorem 6.1 and Scholium 6.5 are sharp to within a constant factor. The cubature formulas found by Wandzura and Xiao support this conjecture, at least when \( K = \Delta_n \).

The proof of Theorem 6.1 was partly inspired by the linear programming method to bound kissing numbers, \( t \)-designs, and sphere packings [3, 9, 10, 13, 22]. Xu observed that the method for \( t \)-designs also yields bounds on PI \( t \)-cubature [34]. In fact it yields an upper bound on the \( \ell_2 \) norm of the weights of a PI \( t \)-cubature formula, which implies a lower bound on the number of points. We conjecture that linear programming methods could be used to improve the constants in Theorem 6.1.

Krylov [14] established that if \( \{F_i\} \) is a sequence of interior
$t$-cubature formulas for a measure $\mu$, then $\{f(F_t)\}$ converges to $\int f(x)\,d\mu$ for every continuous $f$ if and only if the $\ell_t$ norm of the weights of $F_t$ is bounded as $t \to \infty$. We conjecture then that Theorem 6.1 still holds assuming a bound on the $\ell_t$ norm of the coefficients of $F$ instead of assuming that $F$ is positive.

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[1] Milton Abramowitz and Irene A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, vol. 55. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] Archimedes of Syracuse, *On the sphere and cylinder*, ca. 225BC.

[3] Henry Cohn and Noam Elkies, *New upper bounds on sphere packings. I.*, Ann. of Math. (2) 157 (2003), no. 2, 689–714, arXiv:math.MG/0110009.

[4] John H. Conway and Neil J. A. Sloane, *Sphere packings, lattices and groups*, 3rd ed., Grundlehren der mathematischen Wissenschaften, vol. 290, Springer-Verlag, New York, 1993.

[5] R. Cools, E. Novak, and K. Ritter, *Smolyak's construction of cubature formulas of arbitrary trigonometric degree*, Computing 62 (1999), no. 2, 147–162.

[6] Ronald Cools and James N. Lyness, *Three- and four-dimensional K-optimal lattice rules of moderate trigonometric degree*, Math. Comp. 70 (2001), no. 236, 1549–1567 (electronic).

[7] Ronald Cools and Ian H. Sloan, *Minimal cubature formulae of trigonometric degree*, Math. Comp. 65 (1996), no. 216, 1583–1600.

[8] Jesús S. Dehesa, *Orthogonal polynomials in transport theories*, J. Phys. A 14 (1981), no. 2, 297–302.

[9] P. Delsharte, *Bounds for unrestricted codes, by linear programming*, Philips Res. Rep. 27 (1972), 272–289.

[10] P. Delsharte, J. M. Goethals, and J. J. Seidel, *Spherical codes and designs*, Geometriae Dedicata 6 (1977), no. 3, 363–388.

[11] K. K. Frolov, *The connection of quadrature formulas and sublattices of the lattice of integer vectors*, Dokl. Akad. Nauk SSSR 232 (1977), no. 1, 40–43.

[12] I. D. Ivanović, *Formal state determination*, J. Math. Phys. 24 (1983), no. 5, 1199–1205.

[13] G. A. Kabatjanski and V. I. Levenstein, *Bounds for packings on the sphere and in space*, Problemy Peredači Informacii 14 (1978), no. 1, 3–25.

[14] Kazuhiro Koike, *Representations of spinor groups and the difference characters of SO(2n)*, Adv. Math. 128 (1997), no. 1, 40–81.

[15] Hermann König, *Cubature formulas on spheres*, Advances in multivariate approximation (Witten-Bommerholz, 1998), Math. Res., vol. 107, Wiley-VCH, Berlin, 1999, pp. 201–211.

[16] Vladimir Ivanovich Krylov, *Approximate calculation of integrals*, The Macmillan Co., New York, 1962, Translated by Arthur H. Stroud.

[17] Greg Kuperberg, *Numerical cubature using error-correcting codes*, arXiv:math.NA/0402047.

[18] H. Michael Möller, *Lower bounds for the number of nodes in cubature formulae*, Numerische Integration (Tagung, Math. Forschungsinst., Oberwolfach, 1978), Internat. Ser. Numer. Math., vol. 45, Birkhäuser, Basel, 1979, pp. 221–230.

[19] I. P. Mysovskikh, *Cubature formulas that are exact for trigonometric polynomials*, Dokl. Akad. Nauk SSSR 296 (1987), no. 1, 28–31.

[20] Michael A. Nielsen and Isaac L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge, 2000.

[21] M. V. Noskov, *Formulas for the approximate integration of periodic functions*, Metody Vychisl. (1988), no. 15, 19–22, 178.

[22] A. M. Odlyzko and N. J. A. Sloane, *New bounds on the number of unit spheres that can touch a unit sphere in n dimensions*, J. Combin. Theory Ser. A 26 (1979), no. 2, 210–214.

[23] Eric Rains, 2003, personal communication.

[24] V. M. Sidel’nikov, *Spherical 7-designs in 2n-dimensional Euclidean space*, J. Algebraic Combin. 10 (1999), no. 3, 279–288.

[25] S. L. Sobolev, *Cubature formulas on the sphere which are invariant under transformations of finite rotation groups*, Dokl. Akad. Nauk SSSR 146 (1962), 310–313.

[26] Richard P. Stanley, *Enumerative combinatorics*, vol. 2, Cambridge University Press, 1999.

[27] A. H. Stroud, *Some approximate integration formulas of degree 3 for an n-dimensional simplex*, Numer. Math. 9 (1966), 38–45.

[28] Approximate calculation of multiple integrals, Prentice-Hall Inc., 1971, Prentice-Hall Series in Automatic Computation.

[29] Arthur H. Stroud, *Quadrature methods for functions of more than one variable*, Ann. New York Acad. Sci. 86 (1960), 776–791 (1960).

[30] V. S. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984, Reprint of the 1974 edition.

[31] S. Wandzura and H. Xiao, *Symmetric quadrature rules on a triangle*, Comput. Math. Appl. 45 (2003), no. 12, 1829–1840.

[32] William K. Wootters and Brian D. Fields, *Optimal state-determination by mutually unbiased measurements*, Ann. Physics 191 (1989), no. 2, 363–381.

[33] Yuan Xu, *Orthogonal polynomials and cubature formulae on spheres and on simplices*, Methods Appl. Anal. 5 (1998), no. 2, 169–184.

[34] , *Lower bound for the number of nodes of cubature formulae on the unit ball*, J. Complexity 19 (2003), no. 3, 392–402.

[35] Magma, http://magma.maths.usyd.edu.au/.