On the link between mean square-radii and high-order toroidal moments

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Abstract. Multipole expansions of the source play an important role in a broad range of disciplines in modern physics, ranging from the description of exotic states of matter to the design of nanoantennas in photonics. Within the context of the latter, toroidal multipoles, a third group of multipoles complementing the well-known electric and magnetic ones, have been widely investigated since they lead to the formation of non-radiating sources. In the last years, however, the photonics community has brought to light the existence of a fourth type of multipoles that is commonly overlooked. Currently, different groups have provided different mathematical expressions to describe such sources, and they have been coined with different names; on the one hand mean-square radii, and on the other hand, as high order toroidal moments. Despite their clear physical similarity, a formal relation between the two has not yet been established. While explicit formulas for both types have been derived, they are not expressed in the same basis, and therefore it is not possible to draw a clear physical connection between them. In this contribution, we will bridge this gap and rigorously derive the connection between the two representations, taking as an example the cases of the $n\textsuperscript{th}$ order mean square radius of the electric dipole and the $n\textsuperscript{th}$ order electric toroidal dipole. Our results conclusively show that both types of representations are exactly equivalent up to a prefactor.

INTRODUCTION

Multipole decompositions are an essential tool in many areas of modern theoretical physics, ranging from the description of condensed matter systems [1] to the design of antennas [2]. In recent years, they have been actively exploited to understand and efficiently tailor the electromagnetic response of man-made metamaterials and metasurfaces for a broad range of applications in optics [3], and are playing an essential role in the development of the emerging field of all-dielectric nanophotonics [3,4]. Counterintuitively, high-index dielectric particles support multipolar response of both electric and magnetic character [3,5,6]. The multipolar analysis of dielectric nanoantennas sheds light on interesting optical phenomena such as Fano resonances [7], directional scattering [8,9], optical forces [10], or the transverse Kerker effect [11].

The multipole moments of a localized current distribution are commonly classified in terms of electric and magnetic multipoles, associated to oscillating charges and circulating currents, respectively. In general, they can be obtained from a small argument expansion of the vector potential, or equivalently from an expansion of the current in a series of form factors. Through this approach, a third class of multipoles having identical radiation patterns as the conventional ones can be derived, known as toroidal multipoles [12,13]. The interference of the electric dipole with the conventional electric dipole leads to the formation of nonradiating states known as ‘anapoles’ which have spawned intensive research efforts in the photonics community [14]. Notably, the complex current distributions arising within dielectric particles have also been shown to give rise to higher-order [15] and hybrid [16] anapole states of mixed
electric and magnetic character. To understand such an exotic type of nonradiating current distributions, expressions for the so-called higher-order toroidal multipoles were developed [7]. They arise as higher order terms in the expansion of the vector potential, and therefore account for larger retardation effects arising within the source. In parallel, however, there exists an alternative conceptual background to describe these effects, based on a fourth family of multipoles known as the mean-square radii [17,18]. Interestingly, the latter have been proposed as a means to achieve new types of nonradiating current distributions without the need of the conventional electric or magnetic multipoles [18].

Despite their physical similarity, there exists no formal connection between the two electromagnetic descriptions. Mathematical expressions for the high order toroidal moments and the mean-square radii are readily available in the literature [7,17], but are expressed in different basis, and therefore cannot be straightforwardly compared. Here we perform a systematic analysis of the two representations taking as an example the toroidal dipole and demonstrate how both high-order toroidal moments and mean-square radii are in fact equivalent up to a prefactor. Consequently, they can be indifferently used to describe a localized current distribution.

THE NTH ORDER ELECTRIC TOROIDAL DIPOLE

We will derive toroidal moments starting from the expression of what is known as the exact electric dipole, hereby referred to as \( \mathbf{p} \), which groups the contributions to radiation of the elementary electric dipole, the toroidal dipole, and higher order terms, given by [2]:

\[
\mathbf{p} = \frac{1}{\omega} \int d^3 r \left\{ \mathbf{J}(\mathbf{r},t) j_0(\mathbf{r}) + \frac{K^2}{2} \left[ 3(\mathbf{r} \cdot \mathbf{J}(\mathbf{r},t))\mathbf{r} - r^2 \mathbf{J}(\mathbf{r},t) \right] j_2(\mathbf{r}) \right\},
\]

(1)

where \( \mathbf{J}(\mathbf{r},t) \) is the current density of the source, and \( j_0(\mathbf{r}) \) and \( j_2(\mathbf{r}) \) are spherical Bessel functions of the first kind. In the following we will omit the arguments of the current density and assume a time harmonic dependence of all quantities of interest. A simple way to obtain expressions for the \( n \)th order toroidal dipole is to perform a Taylor series of \( j_0(\mathbf{r}) \) and \( j_2(\mathbf{r}) \) in Eq.(1). In the next step we plug in the expansions for \( j_0(\mathbf{r}) \) and \( j_2(\mathbf{r}) \) into Eq.(1) and combine them into one sum. After some calculations one obtains an expression for the \( n \)th order electric toroidal dipole:

\[
\mathbf{T}^{(n)} = -\frac{\alpha^{(n)}}{\omega} \int d^3 r (\mathbf{r})^m \left[ (n+1)\mathbf{J} - n(\mathbf{r} \cdot \mathbf{J}) \frac{\mathbf{r}}{r^2} \right],
\]

(2)

where

\[
\alpha^{(n)} = (-1)^n \frac{6(n+1)}{(2n+3)!}.
\]

(3)

Note that for \( n = 0 \) Eq.(2) yields the well-known electric dipole:

\[
\mathbf{p} = \frac{1}{\omega} \int d^3 r \mathbf{J}.
\]

(4)

We draw attention to the fact that with this notation all prefactors that go with electric toroidal dipole are already included in Eq.(2). A more common way to define toroidal moments is given for example in [7], where a prefactor is extracted from each term, i.e. \( \mathbf{T}^{(n)} \rightarrow \frac{ik}{c} \mathbf{T}^{(n)} \), and similarly for higher order terms:

\[
\mathbf{p} = \mathbf{p} + \frac{ik}{c} \mathbf{T}^{(1)} + \frac{ik^3}{c} \mathbf{T}^{(2)} + \ldots.
\]

(5)
MEAN-SQUARE RADII OF THE TOROIDAL DIPOLE

The expression for the mean-square radii of an arbitrary toroidal moment in the spherical basis can be found in [17], and corresponds to the electromagnetic excitations investigated in [18]:

\[ R_n^{(m)} = -\frac{1}{3c} \sqrt{\frac{4\pi}{2}} \int d^3r r^{2m+2} \left[ \frac{1}{2n+5} Y_{12n}^*(\mathbf{n}) + \frac{\sqrt{2}}{2(n+1)} Y_{10n}^*(\mathbf{n}) \right] \mathbf{J}, \tag{6} \]

where \( m = -1,0,1 \), represents the components of a vector in the spherical basis. Eq. (6) is written in terms of the vector spherical harmonics \( Y_{\ell \pm m}(\mathbf{n}) \), whose definition can also be found in [17]. They are related to the Cartesian components by

\[ R_n^{(m)} = \frac{1}{\sqrt{2}} \left( R_n^{(m)} - R_{n+1}^{(m)} \right), \]
\[ R_n^{(m)} = \frac{i}{\sqrt{2}} \left( R_n^{(m)} + R_{n+1}^{(m)} \right), \]
\[ R_n^{(m)} = -R_{n+1}^{(m)}, \tag{7} \]

In addition, we require the expression for the Cartesian components of each vector spherical harmonic \( Y_{\ell \pm m}(\mathbf{n}) \):

\[ \left( Y_{\ell-1m} \right)_x = -\frac{h_0}{\sqrt{2}} Y_{-1,-m-1} + \frac{h_0}{\sqrt{2}} Y_{-1,m+1}, \]
\[ \left( Y_{\ell-1m} \right)_y = -\frac{ib_0}{\sqrt{2}} Y_{-1,-m-1} - \frac{ib_0}{\sqrt{2}} Y_{-1,m+1}, \]
\[ \left( Y_{\ell-1m} \right)_z = b_0 Y_{-1,m}, \tag{8} \]

and \( Y_{\ell+1m} \):

\[ \left( Y_{\ell+1m} \right)_x = -\frac{c_0}{\sqrt{2}} Y_{1,-m-1} + \frac{c_0}{\sqrt{2}} Y_{1,m+1}, \]
\[ \left( Y_{\ell+1m} \right)_y = -\frac{ic_0}{\sqrt{2}} Y_{1,-m-1} - \frac{ic_0}{\sqrt{2}} Y_{1,m+1}, \]
\[ \left( Y_{\ell+1m} \right)_z = c_0 Y_{1,m}, \tag{9} \]

where the \( h_0 \) and \( c_0 \) coefficients are a function of \( \ell \) and \( m \) [17]. Using Eqs. (8)-(9), one can find the mean-square radii of the toroidal dipole in the spherical basis from Eq. (6) and then, using Eq. (7), convert them into Cartesian. The final result is:

\[ R^{(m)} = \frac{1}{2c} \frac{1}{(n+1)(2n+5)} \int d^3r r^{2m+2} ((n+1)(\mathbf{r} \cdot \mathbf{J})\mathbf{r} - (n+2)r^2) \mathbf{J}. \tag{10} \]

Comparing Eq.(10) with Eq.(2) we found that the \( n^\text{th} \) order mean-square radius and the \( (n+1)^\text{th} \) order electric toroidal dipole have an identical functional form. With Eqs.(2)-(3) and Eq.(10), the relation between them is reduced to:

\[ T^{(n+1)} = i\alpha^{(n)} k^{2n+1} R^{(n)}. \tag{11} \]

Eq.(11) is our central result, demonstrating how the two representations are related.
CONCLUSION

Summarizing, we presented a formula for the $n^{th}$ order electric toroidal dipole (see Eq.(2)), and discussed the mean-square radii of the toroidal dipole. By converting the expressions of the mean-square radii into more intuitive formulas, we have explicitly revealed the connection between the two quantities; Eq.(11) clearly demonstrates that they are equivalent up to a prefactor; i.e. the $n^{th}$ order mean-square radius is nothing more than the $(n+1)^{th}$ order toroidal dipole. We emphasize that the proof presented here can be directly extended to the magnetic toroidal moments and the magnetic mean square radii.

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