DESCENT OF COHERENT SHEAVES AND COMPLEXES TO GEOMETRIC INVARIANT THEORY QUOTIENTS

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ABSTRACT. Fix a scheme $X$ over a field of characteristic zero that is equipped with an action of a reductive algebraic group $G$. We give necessary and sufficient conditions for a $G$-equivariant coherent sheaf on $X$ or a bounded-above complex of $G$-equivariant coherent sheaves on $X$ to descend to a good quotient $X/G$. This gives a description of the coherent derived category of $X/G$ as an admissible subcategory of the equivariant derived category of $X$.

1. Introduction

Varieties constructed using geometric invariant theory (or GIT) are ubiquitous in algebraic geometry. Many fundamental questions in the geometry of such GIT quotients concern the properties of certain natural vector bundles (or Chern classes of vector bundles) on them—for example, the classes $\kappa_i$ and $\lambda_j$ in the study of the geometry of $M_{g,n}$. It is typical to construct such vector bundles on moduli spaces by descent. That is, one identifies the moduli space as a good quotient (Definition 2.4) $X/G$ of a quasiprojective scheme $X$ by a reductive group $G$ using GIT. Letting $X_{ss} \xrightarrow{\pi} X/G$ denote the quotient map, one identifies a sheaf $M$ on the semistable locus $X_{ss}$ that one expects to have the form $M = \pi^*M$, where $M$ is the desired sheaf on $X/G$. Verifying that such a $M$ exists, however, can be a nontrivial task: the morphism $\pi$ typically does not satisfy hypotheses that would make it possible to apply Grothendieck’s descent machinery.

The following criterion, which may be found in [DN89] (where the authors of that paper attribute it to Kempf), gives a convenient characterization of the vector bundles on $X_{ss}$ that descend to $X/G$.

**Theorem 1.1.** (see [DN89]) Suppose $X$ is a quasiprojective scheme over an algebraically closed field $k$ of characteristic zero, and that $G$ is a reductive algebraic group over $k$ that acts on $X$ with a fixed choice of linearization $H$. Let $E$ be a $G$-vector bundle on $X_{ss}$. Then $E$ descends to $X/G$ if and only if for every closed point $x$ of $X_{ss}$ such that the orbit $G \cdot x$ is closed in $X_{ss}$, the stabilizer of $x$ in $G$ acts trivially on the fiber $E_x$ of $E$ at $x$.

In this paper we extend the descent criterion of Theorem 1.1 in two directions. First, we give a criterion for an arbitrary $G$-equivariant coherent sheaf to descend to a good quotient $X/G$.

**Theorem 1.2.** Suppose that $X$ is a scheme locally of finite type over a field $k$ of characteristic zero and that $G$ is a reductive algebraic group over $k$ that acts on $X$
with good quotient $X \xrightarrow{\pi} X/G$. Let $\mathcal{M}$ denote a $G$-equivariant coherent $\mathcal{O}_X$-module. Then the following are equivalent:

1. $\mathcal{M}$ descends to $X/G$.
2. For every closed point $x \in X$ that lies in a closed $G$-orbit, the $\mathcal{O}_{X,x}$-modules $\mathcal{M} \otimes \mathcal{O}_{X/m_x}$ and $\text{Tor}_1^{\mathcal{O}_X} (\mathcal{M}, \mathcal{O}_{X/m_x})$ are generated by elements invariant under the isotropy subgroup $G_x$.

If $k$ is algebraically closed, these are equivalent also to

3. For every closed point $x \in X$ that lies in a closed $G$-orbit, the $\mathcal{O}_{X,x}$-modules $\mathcal{M} \otimes (\mathcal{O}_{X/m_x})$ and $\text{Tor}_1^{\mathcal{O}_X} (\mathcal{M}, (\mathcal{O}_{X/m_x}))$ are trivial representations of the isotropy group $G_x$.

Generalizing Theorem 1.1 in another direction, we also study the descent of equivariant complexes to $X/G$. Here it makes more sense to consider the complex on $X$ as an object of the (equivariant) derived category,\footnote{The derived category we mean here is the derived category of quasicoherent sheaves. This need not coincide with the “cohomologically quasicoherent” derived category if $X$ is not quasi-compact and separated.} and to ask whether it descends up to quasi-isomorphism, i.e. whether it is in the essential image of the pullback functor from the derived category of $X/G$. We then have the following theorem that describes the difference between the equivariant derived category of $X$—that is, the derived category of the quotient stack $[X/G]$—and the derived category of $X/G$.

**Theorem 1.3.** Suppose $X$ is a scheme locally of finite type over a field $k$ of characteristic zero. Suppose $G$ is a reductive algebraic group over $k$ that acts on $X$ with good quotient $X \xrightarrow{\pi} X/G$. Let $E$ denote a bounded-above $G$-equivariant complex of coherent sheaves on $X$. Then the following are equivalent:

1. $E$ is equivariantly quasi-isomorphic to a complex $E'$ on $X$ that descends to $X/G$.
2. For every closed point $x \in X$ that lies in a closed $G$-orbit, the $\mathcal{O}_X$-modules
   $$H^j (E \otimes \mathcal{O}_{X/m_x})$$
   are generated by elements invariant under the isotropy subgroup $G_x$ for all $j$.

If $k$ is algebraically closed, these are equivalent also to

3. For each closed point $x \in X$ that lies in a closed $G$-orbit, the isotropy representations of $G_x$ on the $\mathcal{O}_X$-modules
   $$H^j (E \otimes \mathcal{O}_{X/m_x})$$
   are trivial for all $j$.

In the case in which $G$ is finite, a similar result appeared in [Ter02].

The author’s original motivation for considering these descent problems was the possibility of applications in “singular symplectic geometry.” More precisely, important examples of singular symplectic moduli stacks come equipped with perfect pairings on their tangent complexes that extend the symplectic structure (in an appropriate sense) on the smooth locus. It is then natural to try to study the singularities of the associated coarse space in terms of the problem of descending
this derived symplectic structure from the stack to the space. We discuss some examples in this context in Section 5.

It seems plausible that the descent results of this paper may be extended to characteristic $p$, provided “reductive” is replaced by “linearly reductive” in appropriate places. It would be nice to replace the hypothesis that $G$ be linearly reductive in characteristic $p$ with a weaker hypothesis that stabilizers of closed points in closed orbits are linearly reductive, but the author does not know how to prove, for example, Lemma 2.12 with such weakened hypotheses.

It is perhaps worth remarking that, for many interesting geometric applications to GIT quotients $X/G$, one needs to compare equivariant sheaves on $X$ and $X^{ss}$, which seems to be extremely difficult in general (see, for example, [Te00]).

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2. Preliminaries

2.1. Reminder about Group Actions.

Convention 2.1. Although it seems common (see, for example, [Ses77]) for connectedness of geometric fibers to be part of the definition of a reductive group scheme, in this paper we allow reductive groups to be disconnected; this causes us no trouble since we work always in characteristic zero.

We remind the reader of a few facts we will need.

Theorem 2.2. (Matsushima) Suppose $G$ is a reductive algebraic group over an algebraically closed field $k$ of characteristic $p \geq 0$, and $H$ is a closed $k$-subgroup of $G$. Then the quotient scheme $G/H$ is affine if and only if (the identity component of) $H$ is reductive.

As a consequence, the stabilizers of closed points are reductive in our sense:

Corollary 2.3. Suppose a reductive group $G$ over $k$ acts on an affine variety $X$ over $k$, and the closed point $x \in X$ has closed $G$-orbit in $X$. Then the isotropy group $G_x$ is reductive.

Recall also the following standard definition of a quotient arising in geometric invariant theory.

Definition 2.4. Let $G$ be an affine algebraic group over $k$ acting on a $k$-scheme $X$. A morphism $\phi : X \to Y$ is called a good quotient if

- $\phi$ is affine and $G$-equivariant,
- $\phi$ is surjective, and $U \subset Y$ is open if and only if $\phi^{-1}(U) \subset X$ is open,
- the natural homomorphism $O_Y \to (\phi_*O_X)^G$ is an isomorphism,
- if $W$ is an invariant closed subset of $X$, then $\phi(W)$ is a closed subset of $Y$; if $W_1$ and $W_2$ are disjoint invariant closed subsets of $X$, then $\phi(W_1) \cap \phi(W_2) = \emptyset$.

Note that any linearized action of a reductive group on a quasi-projective scheme $V$ over a field admits such a quotient of the semistable locus $X = V^{ss}$ (see Theorem 4 of [Ses77]).
2.2. Some Basic Facts. Let 
\[ \pi : X \rightarrow X/G \]
denote the projection morphism. Recall that a quasicoherent sheaf \( \mathcal{M} \) on \( X \) descends to \( X/G \) if there exists a quasicoherent sheaf \( \overline{\mathcal{M}} \) on \( X/G \) and an isomorphism \( \pi^* \overline{\mathcal{M}} \rightarrow \mathcal{M} \). Similarly, a \( G \)-equivariant complex of quasicoherent sheaves \( \mathcal{M} \) on \( X \) is said to descend to \( X/G \) if there is a complex \( \overline{\mathcal{M}} \) on \( X/G \) and a quasiisomorphism \( L\pi^* \overline{\mathcal{M}} \rightarrow \mathcal{M} \).

Recall that one defines the functor \( \pi_G^* \) by \( \pi_G^*(\mathcal{M}) = (\pi_* \mathcal{M})^G \). Since \( \pi \) is affine and \( G \) is reductive, \( \pi_G^* \) is exact.

Lemma 2.5. The functors \((\pi^*, \pi_G^*)\) form an adjoint pair,

\[ \text{qcoh}(X, G) \xrightarrow{\pi_G^*} \text{qcoh}(X/G), \]

where \( \text{qcoh}(X, G) \) denotes the category of \( G \)-equivariant quasicoherent \( \mathcal{O}_X \)-modules. These induce an adjoint pair \((L\pi^*, \pi_G^*)\) of derived functors,

\[ D^-(\text{qcoh}(X, G)) \xrightarrow{\pi_G^*} D^-(\text{qcoh}(X/G)). \]

Furthermore, the functors \( \pi_G^* \pi^* \) and \( \pi_G^* L\pi^* \) are (isomorphic to) the identity functors. We thus obtain:

Corollary 2.6.

1. A \( G \)-equivariant quasicoherent sheaf on \( X \) descends to \( X/G \) if and only if the canonical map

\[ \pi^* \pi_G^* \mathcal{M} \rightarrow \mathcal{M} \]

is an isomorphism.

2. A \( G \)-equivariant quasicoherent complex \( \mathcal{M} \) on \( X \) descends to \( X/G \) if and only if the canonical morphism

\[ L\pi^* \pi_G^* \mathcal{M} \rightarrow \mathcal{M} \]

is a quasi-isomorphism.

Remark 2.7. It is immediate from Corollary 2.6 that the descent criteria of Theorems 1.1 and 1.3 may be checked locally on \( X \). Consequently, we may assume in the proofs that \( X = \text{Spec} \, A \) is affine.

Let \( \mathcal{K} = \ker(\pi_G^*) \) denote the kernel of the functor \( \pi_G^* \): this is the subcategory of objects \( C \) for which \( \pi_G^* C = 0 \). Let \( \mathcal{I} \) denote the essential image of \( L\pi^* \). Standard arguments then give:

Proposition 2.8. The pair \((\mathcal{K}, \mathcal{I})\) forms a semiorthogonal decomposition of \( D^-(\text{qcoh}(X, G)) \).

Thus, Theorem 1.3 may be interpreted as a characterization of the coherent part of the summand \( \mathcal{I} \) in this decomposition.

Example 2.9. Let \( X = \text{Spec} \, \mathbb{C}[z] \), a variety over \( \mathbb{C} \), with the usual \( G_m \)-action. Let \( \mathcal{M} = z^{-1} \mathbb{C}[z] \). Then

\[ \pi^* \pi_G^* \mathcal{M} = \overline{\mathbb{C}[z]} \subset z^{-1} \mathbb{C}[z], \]
and the cone \( C \) on this map is the fiber of \( \mathcal{M} \) at the origin. Thus, this gives an example of a \( C[-1] \in \ker(L\pi^*\pi^*_G) \) and \( \mathcal{F} \in \text{im}(L\pi^*) \) such that \( \text{Hom}(C[-1], \mathcal{F}) \neq 0 \) (in the derived category). So this semiorthogonal decomposition is not an orthogonal decomposition. \( \mathcal{M} \) is also an example of a sheaf whose fiber at every closed point \( x \) in a closed orbit contains no \( G_x \)-invariant elements, but \( \pi^*_G\mathcal{M} \neq 0 \); so \( \mathcal{K} \) cannot be expected to have such a simple fiberwise description as \( \mathcal{I} \).

**Notation 2.10.** We will denote by \( \overline{k} \) a fixed algebraic closure of \( k \), and by \( X_\overline{k}, \mathcal{M}_\overline{k} \) and so on the tensor products with \( \overline{k} \).

**Lemma 2.11.**

1. Suppose that \( \mathcal{M} \) satisfies hypothesis (2) of Theorem 1.6. Then \( \mathcal{M}_\overline{k} \) satisfies hypothesis (3) of Theorem 1.8.

2. Suppose that \( \mathcal{M} \) satisfies hypothesis (2) of Theorem 1.7. Then \( \mathcal{M}_\overline{k} \) satisfies hypothesis (3) of Theorem 1.8.

**Proof.** Let \( x \) denote a closed point of \( X_\overline{k} \) and let \( y \) denote its image in \( X \); this is a closed point of \( X \). There is an inclusion of isotropy groups

\[
(G_\overline{k})_x \subseteq (G_y)_\overline{k};
\]

indeed, the \( G_\overline{k} \)-action on \( X_\overline{k} \) covers the \( G \)-action on \( X \), and so in particular the image of \( (G_\overline{k})_x \) in \( G \) fixes \( y \).

1. Since \( \mathcal{O}_{X_\overline{k}}/m_x = \overline{k} \) is flat over \( \mathcal{O}_X/m_y \), we have: for any \( \mathcal{O}_X \)-module \( N \) and any \( i \geq 0 \),

\[
\text{Tor}^X_i(N_\overline{k}, \mathcal{O}_{X_\overline{k}}/m_x) = \text{Tor}^X_i(N, \mathcal{O}_X/m_y) \otimes \overline{k}.
\]

Combining (2.3) and (2.4), we find that \( \mathcal{M}_\overline{k} \otimes \mathcal{O}_{X_\overline{k}}/m_x \) and \( \text{Tor}^X_i(\mathcal{M}_\overline{k}, \mathcal{O}_{X_\overline{k}}/m_x) \) are generated over \( \overline{k} \) by elements invariant under \( (G_\overline{k})_x \), and thus are in fact trivial representations of this isotropy group.

2. Again, because \( \mathcal{O}_{X_\overline{k}}/m_x = \overline{k} \) is flat over \( \mathcal{O}_X/m_y \), we have

\[
H^i(L\pi^*\mathcal{M} \otimes \mathcal{O}_{X_\overline{k}}/m_x) = H^i(M \otimes \mathcal{O}_{X_\overline{k}/G}/m_y \otimes \mathcal{O}_X/m_x) = H^i(M \otimes \mathcal{O}_{X_\overline{k}/G}/m_y) \otimes \mathcal{O}_X/m_x.
\]

By hypothesis, \( H^i(M \otimes \mathcal{O}_{X_\overline{k}/G}/m_y) \) is generated by \( \mathcal{O}_y \)-invariant elements. Combined with (2.3), this implies that \( H^i(L\pi^*\mathcal{M} \otimes \mathcal{O}_{X_\overline{k}}/m_x) \) is generated by \( \mathcal{O}_X/m_x \)-invariants, as desired.

**Lemma 2.12.** Suppose that \( k \) is an algebraically closed field of characteristic zero and \( A \) is a finitely generated \( k \)-algebra with a rational action of a reductive algebraic \( k \)-group \( G \). Let \( \mathcal{F} \) be a \( G \)-equivariant coherent sheaf on \( \text{Spec} \ A \). Let \( x \in \text{Spec} \ A \) be a closed point the \( G \)-orbit through which is closed in \( \text{Spec} \ A \) and is defined by the ideal \( I \subseteq A \). If \( s \in \mathcal{F} \otimes \mathcal{O}_x/m_x \) is \( G_x \)-invariant, then there is a \( G \)-invariant section \( \tilde{s} \in \mathcal{F}(\text{Spec} \ A) \) such that the image of \( \tilde{s} \) in \( \mathcal{F} \otimes \mathcal{O}_x/m_x \) is the element \( s \).

**Proof.** Restricting \( \mathcal{F} \) to \( \text{Spec} (A/I) \), we obtain a \( G \)-equivariant coherent sheaf on \( G/G_x \), which must therefore be the vector bundle associated to some representation \( V \) of \( G_x \). Now \( s \) determines an element of \( V \), and by hypothesis \( s \) lies in a trivial \( G_x \)-subrepresentation \( W \) of \( V \). The associated bundle \( G \times_{G_x} W \) is trivial, and thus \( s \) extends to a \( G \)-invariant global section of the pullback of \( \mathcal{F}|_{\text{Spec} (A/I)} \). But there is a \( G \)-equivariant surjection \( \mathcal{F} \to \mathcal{F}|_{\text{Spec} (A/I)} \), and because \( G \) is reductive, we may lift this section to a \( G \)-invariant section of \( \mathcal{F} \).
Remark 2.13. The previous lemma easily extends to quasicoherent sheaves, but that extension seems not to be as useful.

Lemma 2.14. Let $k$ be an algebraically closed field of characteristic zero. Suppose $\mathcal{M}$ is a $G$-equivariant quasicoherent sheaf on an affine $G$-scheme $Y$. Then there is a $G$-equivariant locally free (quasicoherent) sheaf $\mathcal{V}$ on $Y$ and a surjective $G$-equivariant homomorphism $\mathcal{V} \to \mathcal{M}$. Moreover, if $\mathcal{M}$ is coherent and the stabilizer $G_x$ acts trivially on the fiber of $\mathcal{M}$ at $x$ for every closed point $x$ in a closed $G$-orbit, then $\mathcal{V}$ may be chosen to be coherent and to descend to $Y//G$.

Proof. For the first statement we may take $\mathcal{V} = \mathcal{O} \otimes_k H^0(\mathcal{M})$. For the second statement, we start with $\mathcal{V}' = \mathcal{O} \otimes_k H^0(\mathcal{M})^G$: the natural map to $\mathcal{M}$ is $G$-equivariant, and Lemma 2.12 implies that it is surjective on fibers at closed points in closed $G$-orbits. Consequently, the cokernel is supported on a $G$-invariant closed subset in the complement of the union of the closed $G$-orbits, implying that its support is empty. Now, since $\mathcal{M}$ is finitely generated, we may replace $H^0(\mathcal{M})^G$ by a finite-dimensional $k$-vector subspace $W$ and take $\mathcal{V} = \mathcal{O} \otimes_k W$. □

3. Proof of Theorem 1.2

Recall that, by Remark 2.4, we may assume that $X$ is affine. We begin with:

Lemma 3.1. A $G$-equivariant quasicoherent $\mathcal{O}_X$-module $M$ descends to $X//G$ if and only if there is a $G$-equivariant presentation

$$P_1 \to P_0 \to M \to 0,$$

where $P_0$ and $P_1$ are $G$-equivariant locally free sheaves that descend to $X//G$.

Proof. The “only if” part is immediate from right-exactness of $\pi^*$ and the equation $1 = \pi^*_G \pi^*$ for the identity functor. Conversely, if $M$ has a presentation as above, then, letting $\overline{M} = \text{coker}(\pi^*_G P_1 \to \pi^*_G P_0)$, (2.1) and right-exactness of $\pi^*$ give $\pi^* \overline{M} = M$, as desired. □

3.1. Necessity of the Criterion. Suppose first that $\mathcal{M}$ descends to $X//G$, we will prove that condition (2) of Theorem 1.2 is satisfied. Let $x \in X$ be a closed point in a closed $G$-orbit, and let $G_x$ denote the isotropy subgroup of $x$.

Choose a presentation

$$P_1 \to P_0 \to \pi^*_G \mathcal{M} \to 0$$

of $\pi^*_G \mathcal{M}$ by locally free coherent sheaves on $X//G$. By Corollary 2.6 and right-exactness of tensor product, $\pi^*(P_1 \to P_0)$ gives a presentation of $\mathcal{M}$ by locally free sheaves on $X$. The fiber of $P_i$ at $\mathcal{O}_X/m_x$ is given by

$$\pi^* P_i \otimes \mathcal{O}_X/m_x = (P_i \otimes \mathcal{O}_{X//G}/m_{\pi(x)}) \otimes \mathcal{O}_X/m_x.$$

These fibers are generated over $\mathcal{O}_X/m_x$ by their subspaces $P_i \otimes \mathcal{O}_{X//G}/m_{\pi(x)}$, which consist of $G_x$-invariants. Moreover, it follows from (3.1) that

$$\mathcal{K} = \text{ker}(\pi^* P_1 \otimes \mathcal{O}_X/m_x \to (\pi^* P_0) \otimes \mathcal{O}_X/m_x)$$

is defined over $\mathcal{O}_{X//G}/m_{\pi(x)}$; consequently, $\mathcal{K}$ is also generated by $G_x$-invariants. But the fiber $\mathcal{M} \otimes \mathcal{O}_X/m_x$ is $G_x$-equivariantly isomorphic to a quotient of the fiber of $\pi^* P_0$, and $\text{Tor}_1(\mathcal{M}, \mathcal{O}_X/m_x)$ is equivariantly isomorphic to a quotient of $\mathcal{K}$. So condition (2) is satisfied.
3.2. **Sufficiency of the Criterion.** Now, suppose that condition (2) holds. We will show that this implies condition (1).

**Case 1.** $k$ algebraically closed. By assumption and Lemma 2.14, there is a surjective $G$-equivariant homomorphism $P_0 \rightarrow \mathcal{M}$ where $P_0$ is a locally free coherent sheaf that descends. Let $\mathcal{K}$ denote its kernel. For every closed point $x \in X$ lying in a closed $G$-orbit, we have an exact sequence

$$\text{Tor}_1(\mathcal{M}, \mathcal{O}_X/m_x) \rightarrow \mathcal{K} \otimes \mathcal{O}_X/m_x \rightarrow P_0 \otimes \mathcal{O}_X/m_x \rightarrow \mathcal{M} \otimes \mathcal{O}_X/m_x \rightarrow 0.$$ 

Because $G_x$ acts trivially on $P_0 \otimes \mathcal{O}_X/m_x$ by construction and on $\text{Tor}_1(\mathcal{M}, \mathcal{O}_X/m_x)$ by assumption, $G_x$ acts trivially on $\mathcal{K} \otimes \mathcal{O}_X/m_x$ for each such $x$. Applying Lemma 2.14 to $\mathcal{K}$, we get a $G$-equivariant homomorphism

$$P_1 \rightarrow P_0,$$

the image of which is $\mathcal{K}$, where $P_1$ is a $G$-equivariant locally free sheaf that descends. Thus, we have a $G$-equivariant presentation $P_1 \rightarrow P_0 \rightarrow \mathcal{M} \rightarrow 0$ in which $P_1$ and $P_0$ descend to $X/G$. By Lemma 3.1, this completes the proof when $k$ is algebraically closed.

**Case 2.** $k$ arbitrary. Now, let $k$ be a field of characteristic 0.

**Lemma 3.2.** If $V$ is a rational $G$-representation defined over $k$, then

$$(V^G)_k = (V_k)^G.$$ 

By Lemma 3.2, we have $(\pi_*^G \mathcal{M})_k = \pi_*^G \mathcal{M}_k$, and consequently

$$(\pi^* \pi_*^G \mathcal{M})_k = \pi_*^G \pi_*^G \mathcal{M}_k.$$ 

Since the canonical map (2.1) for $\mathcal{M}_k$ is an isomorphism by Lemma 2.11 and the $\bar{k}$-case of the proof, (3.2) implies that the pullback to $\bar{k}$ of $\pi^* \pi_*^G \mathcal{M} \rightarrow \mathcal{M}$ is an isomorphism. But now Spec $\bar{k} \rightarrow \text{Spec } k$ is faithfully flat, so $\pi^* \pi_*^G \mathcal{M} \rightarrow \mathcal{M}$ is itself an isomorphism. By Corollary 2.6, this completes the proof.

We note that Condition (2) of Theorem 1.2 cannot be replaced by Condition (3) when $k$ is not algebraically closed:

**Example 3.3.** Let $X = \text{Spec } \mathbb{R}[z, z^{-1}] \cong G_m$, a variety over $\mathbb{R}$ with $G = G_m$ acting in the obvious way. Let $x = (z^2 + 1)$. Then $G_x = \{ \pm 1 \}$. Let $\mathcal{M} = \mathcal{O}_X$. The action of $G_x$ on the fiber $\mathcal{M} \otimes \mathcal{O}_X/m_x \cong \mathbb{C}$ is identified with the action of $\text{Gal}(\mathbb{C}/\mathbb{R})$. In particular, although the fiber is generated by $G_x$-invariants, it does not consist entirely of $G_x$-invariants.

4. **Proof of Theorem 1.3**

4.1. **Preliminaries.**

**Proposition 4.1.** Suppose $E$ is a $G$-equivariant complex of vector bundles on a noetherian affine scheme $Y$ over an algebraically closed field $k$ of characteristic zero. Assume that, for some $n \in \mathbb{Z}$, one is given a $G$-equivariant vector bundle $V$ on $Y$ and a $G$-equivariant homomorphism $V \xrightarrow{f} E_n$ so that

1. $\text{Im}(\phi_n \circ f) = \text{Im}(\phi_n)$ and
(2) the induced homomorphism
\[ \ker(\phi_n \circ f) \to H^n(E) \]
is surjective.

Then there is a \( G \)-equivariant complex \( E' \) of vector bundles and a \( G \)-equivariant quasi-isomorphism \( E' \xrightarrow{\Delta} E \) so that

(1) in degrees \( j > n \) one has \( E'_j = E_j \), and \( q_j : E_j \to E_j \) is the identity map, and

(2) in degree \( n \) the quasi-isomorphism \( q \) restricts to \( V \xrightarrow{f} E_n \).

Proof. We produce \( E' \) recursively, starting from:

\[
\begin{array}{c}
V \\
\downarrow \phi_n \circ f \\
E_{n-1} \\
\downarrow \phi_{n-1} \\
E_n \\
\downarrow \phi_n \\
E_{n+1} \\
\downarrow \phi_{n+1} \\
E_{n+2} \\
\downarrow \\
\vdots
\end{array}
\]

So, suppose we are given a \( G \)-equivariant morphism

\[
\begin{array}{c}
E_j' \\
\downarrow q_j \\
E_{j-1}
\end{array}
\]

which induces

(i) an isomorphism on \( H^i \) for \( i \geq j + 1 \), and

(ii) a surjection on \( H^j \).

Let \( \tilde{E}'_{j-1} = E_{j-1} \times_{E_j} E_j' \). By construction, \( \tilde{E}'_{j-1} \) is equipped with \( G \)-equivariant morphisms

\[
E_{j-1} \xrightarrow{\tilde{q}_{j-1}} \tilde{E}'_{j-1} \xrightarrow{\tilde{\phi}'_{j-1}} E_j'
\]

for which \( \phi_{j-1} \circ \tilde{q}_{j-1} = q_j \circ \tilde{\phi}'_{j-1} \). Furthermore, the induced morphism on cohomology in degree \( j \) is an isomorphism: by assumption the sheaf \( \ker(\phi'_j) \) surjects onto \( H^j(E) \), and so it is enough to check that the image of \( E'_{j-1} \) in \( E'_j \) is the kernel of the map \( \ker(\phi'_j) \to H^j(E) \). A section \( e \) of \( \ker(\phi'_j) \) goes to zero in \( H^j(E) \), however, exactly if there is some \( e' \) in \( E_{j-1} \) for which \( \tilde{\phi}_{j-1}(e') = q_j(e) \), and then the section \( (e', e) \) lies in \( \tilde{E}'_{j-1} \) and maps to \( e \).

In addition, it is easy to see that the kernel of the map

\[
\tilde{E}'_{j-1} \to E'_j
\]
surjects onto \( \ker(\phi_{j-1}) \) (if \( e \) is a section of the kernel, then \( e, 0 \) lies in the kernel of \( (4.1) \)) and hence onto \( H^{j-1}(E) \). Thus, to complete the proof it will be enough to produce a \( G \)-equivariant vector bundle \( E'_{j-1} \) and a \( G \)-equivariant surjective morphism of coherent sheaves \( E'_{j-1} \to \tilde{E}'_{j-1} \). But the existence of such a vector bundle and morphism is guaranteed by Lemma 2.14. It follows that our extension of \( E' \) now satisfies (i) and (ii) with \( j \) replaced by \( j - 1 \), and continuing our recursion gives the desired complex. \( \square \)
Lemma 4.2. Suppose that

\[(4.2) \quad E' \to E \to E'' \to \]

is an exact triangle of $G$-equivariant complexes. Suppose that the canonical map \[(2.2) \quad \] is

1. an isomorphism on $H^i$ for $E'$ for all $i$, and
2. an isomorphism on $H^i$ for $E''$ in degrees $i \geq n$ and a surjection in degree $i = n - 1$.

Then the canonical map \[(2.2) \quad \] is

1. an isomorphism on $H^i$ for $E$ in degrees $i \geq n$ and
2. a surjection on $H^{n-1}$ for $E$.

Proof. This is a restatement of the Five Lemma for \[(4.2) \quad \].

Lemma 4.3. Given an exact triangle of the form \[(4.2) \quad \], if two of $E$, $E'$, and $E''$ satisfy condition (2) of Theorem 1.3, then so does the third.

Proof. This is immediate from the long exact cohomology sequence.

Lemma 4.4. Suppose that $M$ is a $G$-equivariant coherent sheaf on $X$, and that $V$ is a $G$-equivariant vector bundle on $X$ that descends to $X/G$. Suppose that there is a $G$-equivariant surjective map $V \to M$. Then the natural map \[(2.1) \quad \] for $M$ is surjective.

Proof. We have a surjective map

$$\pi^* \pi_G^* V = V \to M$$

that factors through $\pi^* \pi_G^* M$.

Corollary 4.5. Suppose that $M$ is a coherent complex concentrated in degrees $(-\infty, m]$ that satisfies condition (2) of Theorem 1.3. Then the canonical map \[(2.2) \quad \] is surjective on $H^m$.

Proof. By hypothesis and right-exactness of tensor product, $H^m(M)$ satisfies condition (2) of Theorem 1.3 as well. Hence, applying Lemma 2.14 to $H^m(M)$, there are a $G$-equivariant vector bundle $V$ that descends to $X/G$ and an equivariant surjection $V \to H^m(M)$. Lemma 4.4 then implies that $\pi^* \pi_G^* H^m(M) \to H^m(M)$ is surjective.

We now use the exact triangle

$$\tau_{\leq m-1} M \to M \to H^m(M) \to \ldots$$

Applying $L\pi^* \pi_G^*$ to it and taking the long exact cohomology sequence, we get an exact sequence

$$\ldots \to H^m(L\pi^* \pi_G^* M) \to \pi^* \pi_G^* H^m(M) \to H^{m-1}(L\pi^* \pi_G^* \tau_{\leq m-1} M) \to \ldots$$

But the last group above is zero by degree considerations, so the map $H^m(L\pi^* \pi_G^* M) \to \pi^* \pi_G^* H^m(M)$ is surjective. Combining this with the conclusion of the previous paragraph yields the corollary.
4.2. **Necessity of the Criterion.** Suppose that \( E \) is a \( G \)-equivariant bounded abelian complex of vector bundles on \( X \) that descends to \( X/G \).

Let \( x \in X \) be a closed point in a closed \( G \)-orbit. Then

\[
E^L \otimes \mathcal{O}_X/m_x = (L\pi^*\pi_G^*E)^L \otimes \mathcal{O}_X/m_x = \left(\pi_x^G E \otimes \mathcal{O}_{X/G}/m_{\pi(x)}\right)^L \otimes \mathcal{O}_X/m_x.
\]

Now \( \mathcal{O}_{X/G}/m_{\pi(x)} \to \mathcal{O}_X/m_x \) is flat, so

\[
H^i(E^L \otimes \mathcal{O}_X/m_x) = H^i(\pi_x^G E \otimes \mathcal{O}_{X/G}/m_{\pi(x)}) \otimes \mathcal{O}_X/m_x.
\]

The right-hand side is generated over \( \mathcal{O}_X/m_x \) by the \( G_x \)-invariants \( H^i(\pi_x^G E \otimes \mathcal{O}_{X/G}/m_{\pi(x)}) \).

4.3. **Sufficiency of the Criterion.**

**Case 1.** \( k \) algebraically closed. We prove the sufficiency of Condition (2) by induction. Consider the following statement:

\( P(k) \): Suppose that \( E \) is a \( G \)-equivariant complex that satisfies Condition (2) of Theorem 1.3 and that \( E \) is concentrated in degrees \((-\infty, m]\). Then \((2.2)\) induces an isomorphism on \( H^i \) for \( i \geq m - k \) and a surjection on \( H^{m-k-1} \).

To complete the proof of the theorem, it suffices to prove \( P(k) \) for all \( k \). Moreover, \( P(-2) \) holds trivially.

Suppose, by way of inductive hypothesis, that \( P(k) \) holds (that is, for all \( E \)). Let

\[
E : 
\cdots \to E_n \xrightarrow{\phi_1} E_{n+1} \to \cdots \to E_{-2} \xrightarrow{\phi_2} E_{-1} \xrightarrow{\phi_1} E_0 \to 0
\]

be a \( G \)-equivariant complex of vector bundles on \( X \). By Corollary 4.5 and Lemma 2.11 there is a \( G \)-equivariant vector bundle \( E_0' \) that descends to \( X/G \). Suppose, by way of inductive hypothesis, that \( E_0' \) is concentrated in degrees \((-\infty, 0]\) and \( E_0' \) descends. We get an exact triangle

\[
E_0' \to E' \to \sigma_{\leq -1} E' \xrightarrow{[1]}
\]

for which \( E_0' \) descends to \( X/G \) and \( \sigma_{\leq -1} E' \) is concentrated in degrees \((-\infty, -1] \).

By Lemma 4.3 all three terms in \((4.3)\) satisfy condition (2) of the theorem.

We wish to apply Lemma 4.2 to \((4.3)\) and \((4.3)\). Since \( E_0' \) descends, condition (1) of Lemma 4.2 is satisfied. The inductive hypothesis tells us that condition (2) of Lemma 4.2 applied to \( \sigma_{\leq -1} E' \) holds for \( n = -k - 1 \). Thus, Lemma 4.2 implies that \( P(k+1) \) holds for \( E' \). Consequently, \( P(k+1) \) holds for \( E \). Since this conclusion holds for arbitrary \( E \), this completes the inductive step, and thus the proof of the theorem when \( k \) is algebraically closed.

**Case 2.** \( k \) arbitrary. Consider the morphism \((2.2)\) Tensoring with \( \bar{k} \) and using \((3.2)\), we obtain the morphism

\[
(L\pi^*\pi_G^*E)_{\bar{k}} = L\pi^*\pi_G^*E_{\bar{k}} \to E_{\bar{k}}.
\]

By Lemma 4.11 and the algebraically closed case, this is a quasi-isomorphism. But \( \text{Spec} \bar{k} \to \text{Spec} k \) is faithfully flat, so \((2.2)\) is also a quasi-isomorphism for \( E \). By Corollary 2.6 this completes the proof. \( \square \)
5. Some Examples

In this section, we discuss two examples that have motivated the author's interest in the descent questions considered here.

5.1. Cotangent Complex in the Smooth Setting. Let \( X \) be a smooth complex variety with an action of a reductive group \( G \) and good quotient \( \pi : X \to X/G \). The equivariant cotangent complex of \( X \) is the complex

\[
\mathbb{L} : \Omega_X^1 \to \mathcal{O}_X \otimes \mathfrak{g}^*
\]

in which, on fibers, the map is given by the dual of the infinitesimal action of \( \mathfrak{g} = \text{Lie}(G) \) on \( X \). This complex is the pullback of the cotangent complex of the stack quotient \( [X/G] \) along the projection \( X \to [X/G] \), hence it measures "singularities in the equivariant geometry of \( X \)."

Suppose that \( \mathbb{L} \) descends to \( X/G \) on a neighborhood of the image \( \pi(x) \) of a closed point \( x \in X \) lying in a closed \( G \)-orbit. Then the stabilizer \( G_x \) must act trivially on the cohomologies \( H^i(\mathbb{L}_x) \) of the complex \( \Omega^1_{X,x} \to \mathfrak{g}^* \), where \( \Omega^1_{X,x} \) denotes the cotangent space to \( X \) at \( x \). Now \( H^1(\mathbb{L}_x) = \mathfrak{g}^*_x \) is the dual of the Lie algebra of \( G_x \) (with the coadjoint action of \( G_x \)) and \( H^0(\mathbb{L}_x) = (T_{X,x}/\mathfrak{g})^\vee = N^\vee_{G,x/X}(x) \) is the conormal space to the orbit \( G \cdot x \) at \( x \). If \( G_x \) acts trivially on \( N^\vee_{G,x/X}(x) \), it follows from Luna’s Étale Slice Theorem that there then exists a smooth slice \( S \) through \( x \) for the \( G \)-action on \( X \) on which \( G_x \) acts trivially. Moreover, since \( X \) is étale-locally isomorphic to \( G \times_{G_x} S \) in a neighborhood of \( x \), we may conclude that \( X \cong (G/G_x) \times S \) étale-locally and \( G \)-equivariantly near \( x \). In particular, \( x \) lies in the open stratum \( U \) of \( X \) on which the orbit type is constant, i.e. every \( p \in U \) has stabilizer conjugate to \( G_x \). Summarizing:

**Proposition 5.1.** Suppose \( X \) is a smooth complex variety with an action of a reductive group \( G \) with good quotient \( \pi : X \to X/G \). Let \( U \subset X \) denote the open stratum described above. Then \( \pi(U) \) is the largest open subset of the good quotient \( X/G \) to which the equivariant cotangent complex \( \mathbb{L} \) descends.

5.2. Cotangent Complex in the Symplectic Setting. As we mentioned in the introduction, Theorem 4.3 was originally motivated by the following picture coming from (algebraic) symplectic geometry. Let \( M \) be a smooth, affine complex variety with an action of a reductive group \( G \). Let \( \mu : T^*M \to \mathfrak{g}^* \) be a moment map for the induced action on the cotangent bundle. Suppose, for simplicity, that \( \mu \) makes \( N = \mu^{-1}(0) \) a complete intersection, so that the quotient stack \( \mathcal{X} = [N/G] \) is a complete intersection and its cotangent complex \( \mathbb{L}_\mathcal{X} \) is concentrated in \([-1,1]\). Moreover, \( \mathbb{L}_\mathcal{X} \) comes equipped with a map \( \omega : \mathbb{L}_\mathcal{X} \to \mathbb{L}^\vee_X \) that is antisymmetric and nondegenerate (that is, it pairs \( H^0 \) nondegenerately with itself and \( H^1 \) nondegenerately with \( H^{-1} \)—here antisymmetry is taken in the graded sense and the nondegeneracy is a statement about the pairing on cohomology of fibers). The quotient stack \( [N/G] \) equipped with this structure—an antisymmetric, closed, and nondegenerate pairing on its cotangent complex—is what should be properly considered a (local complete intersection) symplectic stack.\(^2\)

\(^2\)If a stack \( M \) is not lci, then the cotangent complex will lie in degrees \((-\infty,1]\) but not in \([-1,1]\). In that case, it is reasonable to hope that \( \mathbb{L}_\mathcal{X} \) will admit a nondegenerate pairing for a suitable derived enhancement \( \mathcal{X} \) of \( \mathcal{X} \).
It is natural to wonder whether one can extract information about the singularities of the quotient space $X = N//G$ from $L_X$: for example, the dimensions of cohomologies of $L_X$ stratify $X$ (and hence $N$) by dimensions of stabilizers, which induces a stratification of $X$ as well. In this light, it would be optimal if the pullback of $L_X$ to $N$, i.e. the $G$-equivariant cotangent complex of $N$, descended to $X$. To check this, one can use the following description of the cotangent complex. First, the cotangent complex of $N$ is given by the pullback map on 1-forms,

$$\mathcal{O}_N \otimes g \xrightarrow{\mu} (\Omega^1_{T^{*}M})|_N.$$  

The cotangent complex of the stack $[N/G]$ is then obtained by descending the $G$-equivariant complex on $N$,

$$\tilde{T}^* : \mathcal{O}_N \otimes g \xrightarrow{\mu^*} (\Omega^1_{T^{*}M})|_N \xrightarrow{da^*} \mathcal{O}_N \otimes g^*,$$

where $da^*$ is the dual of the infinitesimal action map $da : \mathcal{O}_N \otimes g \rightarrow T_{T^{*}M}$.

Unfortunately, examples indicate that this $G$-equivariant perfect complex on $N$ probably rarely (or maybe never) descends to $N//G$; but this failure of descent seems to be an interesting phenomenon worthy of further consideration. The closures of minimal nilpotent orbits in semisimple groups provide an interesting class of examples. In type $A$, such orbit closures are symplectic reductions of $T^*\mathbb{C}^{n+1}$ by the $\mathbb{C}^*$-action induced from scaling on $\mathbb{C}^{n+1}$. The quotient is a subvariety of $\mathfrak{sl}_{n+1}$, namely the set of nilpotent matrices of rank less than or equal to 1. Indeed, the moment map

$$\mu : T^*\mathbb{C}^{n+1} = \mathbb{C}^{n+1} \times (\mathbb{C}^{n+1})^* \rightarrow \mathbb{C}$$

is $\mu(i,j) = j(i)$, which is equivalent to $\text{trace}(ij) = 0$; the map $\mu^{-1}(0) \rightarrow \mathfrak{sl}_{n+1}$ takes $(i,j) \mapsto ij$. The orbit closure $\overline{\mathcal{O}} = \mu^{-1}(0)/\mathbb{C}^*$ has an isolated singularity at the origin in $\mathfrak{sl}_{n+1}$, and it is easy to see using Theorem [Fu03] that the equivariant cotangent complex of $\mu^{-1}(0)$ does not descend to $\overline{\mathcal{O}}$ at the singularity. However, $\overline{\mathcal{O}}$ has a symplectic resolution of singularities by $T^*\mathbb{P}^n$ [Fu03], realized as the quotient of the subset

$$\mu^{-1}(0)^* = \mu^{-1}(0) \cap \left((\mathbb{C}^{n+1} \setminus \{0\}) \times (\mathbb{C}^{n+1})^*\right)$$

by the action of $\mathbb{C}^*$. Every point of $\mu^{-1}(0)^*$ has reductive stabilizer—in fact, the action is free—and so the restriction of the equivariant cotangent complex naturally descends to $T^*\mathbb{P}^n = \mu^{-1}(0)^*/\mathbb{C}^*$.

This is, of course, a rather trivial example, but it is intriguing to compare it to cases in which a symplectic resolution is known not to exist: for example, minimal nilpotent orbit closures in type $C$ [Fu03]. In this case, the orbit closure is a quotient of $\mathbb{C}^{2n}$ by the action of $\mathbb{Z}/2\mathbb{Z}$, and again the cotangent complex—in this case, just the cotangent bundle—does not descend. However, in this example there is no clear method to obtain a quotient to which the cotangent complex does descend: the largest open set on which this does happen, which again consists of free orbits, does not map surjectively to $\mathbb{C}^{2n}//(\mathbb{Z}/2\mathbb{Z})$, and so does not provide a symplectic resolution of the minimal orbit closure.

It would be interesting to know whether one can explain the existence of the symplectic resolution in terms of the behavior of the cotangent complex. It is conjectured (cf. [Fu03]) that every birational contraction of a smooth symplectic variety is locally modelled on a symplectic resolution of some nilpotent orbit closure. Thus, one may hope that understanding the behavior of the cotangent complex for nilpotent orbit closures may shed light on existence of symplectic resolutions.
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