Stepped surfaces and Rauzy fractals induced from automorphisms on the free group of rank 2

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Abstract

For substitution satisfying Pisot, irreducible, unimodular condition, a tiling substitution plays a key role in construction of a stepped surface and Rauzy fractal (see [2]). In this paper we will extend the method to hyperbolic automorphisms on the free group of rank 2 in some class, and obtain set equations of Rauzy fractals by virtue of a tiling substitution. We will also see that the domain exchange transformation on Rauzy fractal is just a two interval exchange transformation.

Keywords: stepped surface, Rauzy fractal, invertible substitution, automorphism on the free group, tiling substitution, Pisot, hyperbolic, interval exchange transformation

0 Introduction

Rauzy fractal [17] has been extensively studied because it plays significant roles in the study of substitutive dynamical system in the case of Pisot, irreducible unimodular substitution (e.g., [2, 14, 16]). Arnoux and Ito [2] gives the way to construct a stepped surface (see Proposition 1.1 and Figure 1) and Rauzy fractal (see Proposition 1.2 and Figure 2) by using a tiling substitution which is sometimes called a dual map, and the set equation of Rauzy fractal related to its self-similality (see Proposition 1.3); and they obtains the domain exchange transformation on Rauzy fractal as the realization of the substitutive dynamical system (see Theorem 1). As the extension to automorphisms on the free group, Arnoux, Berhté, Hilion and Siegel [1] have started the study of the class where cancellation of letters under the iteration of automorphism does not occur; and Berhté and Fernique [4] discussed the action of a tiling substitution for automorphism on the stepped surface.

In this paper we study the natural class of automorphisms related to the companion matrices of quadratic polynomials $x^2 - ax ± 1$ such that

$$A_± = \begin{pmatrix} 0 & ±1 \\ 1 & a \end{pmatrix}, \quad (1)$$

and assume “hyperbolicity” instead of the Pisot condition. The purpose of this paper is to find automorphisms related to the matrices given by (1) which give analogue properties in substitutions case, and to discuss stepped surfaces, Rauzy fractals and dynamical systems on the chosen automorphisms. So we shall extend the way and technique for substitutions
to ones for automorphisms. But we sometimes encounter the problem which is peculiar to automorphisms. For example, take the automorphism defined by

$$
\sigma : \begin{cases} 
1 & \rightarrow 2 \\
2 & \rightarrow 21^{-1}22 
\end{cases},
$$

then cancellation of letters under the iteration of $\sigma$ occurs because

$$
\sigma^2(2) = \sigma(21^{-1}22) = 21^{-1}22 \ 2^{-1}21^{-1}22 \ 21^{-1}22.
$$

Such cancellation never occurs for any substitution and automorphisms discussed in [1]. Main idea to solve this problem is to find a substitution or a “pseud-substitution” $\tau$, which is called an “alternative substitution” in this paper, for each automorphism $\sigma$ satisfying $\sigma = \delta^{-1} \circ \tau \circ \delta$ with some automorphism $\delta$. We will show that many results obtained in substitution case also hold for the chosen automorphisms by using conjugate $\tau$.

In Section 1 recalls results in the case of substitutions of rank 2. So similar results will appear in the case of some automorphisms.

Under the condition of hyperbolicity, there are four cases of the matrix given by (1). In Section 2, we choose automorphisms on the free group of rank 2 for each cases, and find their conjugates which are substitutions or alternative substitutions. These automorphisms are discussed in the following sections.

In Section 3, we show that stepped surfaces related to the chosen automorphisms can be obtain by ones related to their conjugates. By using this fact, we will find appropriate initial elements, so called seeds, for tiling substitutions, and generate the stepped surfaces related to the automorphisms.

In Section 4 is devoted to Rauzy fractals induced from the automorphisms. First we generate Rauzy fractals for the both of the automorphisms and its conjugates by each tiling substitutions with appropriate seeds; and show that Rauzy fractals induced from the automorphisms can be written as a disjoint union of Rauzy fractals related to their conjugates, and thus they are just intervals in Theorem 4. Second we consider measurable dynamical systems with domain exchange transformations on Rauzy fractals, and the structure of its induced transformations in Theorem 5. Finally we see the Rauzy fractals related to the automorphisms are obtained by their fixed points or periodic points in Theorem 6.

1 Results in substitution case

We briefly recall the substitution case. We concentrate substitutions of rank 2 even though some properties are true for any rank. Let $A = \{1, 2\}$ (resp. $\hat{A} = \{1, 2, 1^{-1}, 2^{-1}\}$ ) be an alphabet consisting of two letters (resp. four letters), and $A^*$ (resp. $\hat{A}^*$) the free monoid with the empty word $\epsilon$ generated by $A$ (resp. $\hat{A}$ ). More precisely, a word $W = w_1w_2\cdots w_n \in A^*$ (resp. $\hat{A}^*$) satisfies $w_i \in A$ (resp. $w_i \in \hat{A}$ ) for any $i \in \{1, 2, \ldots, n\}$. We say a word $W = w_1w_2\cdots w_n \in \hat{A}^*$ is reduced if $w_iw_{i+1} \neq \epsilon$ for any $i \in \{1, 2, \ldots, n - 1\}$. A word $ww^{-1}w \in \hat{A}^*$ becomes $w_1w_2$ after cancellation. If two words $W_1, W_2 \in \hat{A}^*$ becomes the same reduced word after cancellation, then we say they are referred to be equivalent, and written as $W_1 \sim W_2$. The free group of rank 2 is defined by $F_2 = \hat{A}^*/\sim$. For simplicity, the concatenation of $k$ copies of some letter
$i \in \mathcal{A}$ (resp. $i^{-1} \in \{1^{-1}, 2^{-1}\}$) is written as $ii \cdots i = i^k$ (resp. $i^{-1}i^{-1} \cdots i^{-1} = i^{-k}$). An endomorphism on $\mathcal{A}^*$ is called a substitution of rank 2 over $\mathcal{A}$ and it is naturally extended to an endomorphism on $F_2$. A substitution is referred to be invertible if it is an automorphism on $F_2$ by the extension.

A canonical homomorphism $f : F_2 \to \mathbb{Z}^2$ is defined by $f(\epsilon) = \mathbf{o}$ and $f(i^{\pm 1}) = \pm \mathbf{e}_i$, $i \in \mathcal{A}$. Then for a matrix $A_\sigma$ defined by $(f(\sigma(1)), f(\sigma(2)))$, so called an incidence matrix associated with an endomorphism $\sigma$ on $F_2$, the following diagram becomes commutative:

$$
\begin{array}{ccc}
F_2 & \overset{\sigma}{\longrightarrow} & F_2 \\
\downarrow f & & \downarrow f \\
\mathbb{Z}^2 & \overset{A_\sigma}{\longrightarrow} & \mathbb{Z}^2
\end{array}
$$

For example, let us consider the substitution $\sigma$ of rank 2 given by

$$
\sigma : \begin{cases}
1 \to 2 \\
2 \to 21
\end{cases}
$$

with the incidence matrix

$$
A_\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.
$$

The substitution is Pisot, irreducible, unimodular, that is, the characteristic polynomial $\Phi_\sigma(x)$ of $A_\sigma$ satisfies the following three conditions:

- (Pisot condition) The maximum root of $\Phi_\sigma(x)$ is Pisot number, that is, the dominant eigenvalue of $A_\sigma$ is greater than one and the other has modulus less than one,

- (Irreducible condition) $\Phi_\sigma(x)$ is irreducible over $\mathbb{Q}$,

- (Unimodular condition) $| \det A_\sigma | = 1$.

A substitution $\sigma$ is referred to be primitive if there exists $n$ such that for any pair $(i, j)$ the letter $i$ occurs in the words $\sigma^n(j)$, in other words, the incidence matrix $A_\sigma$ of $\sigma$ is primitive. In this section we assume a substitution is Pisot, irreducible, unimodular and primitive. By Perron-Frobenius Theorem and the Pisot condition, the incidence matrix $A_\sigma$ of a Pisot irreducible, unimodular and primitive substitution $\sigma$ has a positive column eigenvector $u_\sigma > 0$, a positive lower eigenvector $v_\sigma > 0$ corresponding to the positive eigenvalue $\lambda_\sigma > 1$, and another column eigenvector $u'_\sigma$ corresponding to the other eigenvalue $\lambda'_\sigma$ with $|\lambda'_\sigma| < 1$. It is easy to check the contractive eigenspace $P_\sigma$ of $A_\sigma$ spanned by $u'_\sigma$ is given by $P_\sigma = \{ x \in R^2 \mid \langle x, \cdot v_\sigma \rangle = 0 \}$, where $\langle \cdot, \cdot \rangle$ is an inner product; and the stepped surfaces of $P_\sigma$ are defined by

$$
S_\sigma := \bigcup_{(x, i^*) \in S_\sigma} (x, i^*),
$$

$$
S'_\sigma := \bigcup_{(x, i^*) \in S'_\sigma} (x, i^*),
$$

where

$$
S_\sigma := \{ (x, i^*) \in \mathbb{Z}^2 \times \{1^*, 2^*\} \mid \langle x, i^* v_\sigma \rangle > 0, \langle x - e_i, i^* v_\sigma \rangle \leq 0 \},
$$

$$
S'_\sigma := \{ (x, i^*) \in \mathbb{Z}^2 \times \{1^*, 2^*\} \mid \langle x, i^* v_\sigma \rangle \geq 0, \langle x - e_i, i^* v_\sigma \rangle < 0 \}.\]
Identify \((x, i^*) \in \mathbb{Z}^2 \times \{1^*, 2^*\}\) with the positive oriented unit segment spanned by the fundamental vector \(e_j\) translated by \(x\), where \(\{i, j\} = \{1, 2\}\) (see Figure 5), then these stepped surfaces \(S_\sigma, S'_\sigma\) are discrete approximations of \(P_\sigma\) (see Figure 1). On the other hand, the stepped surface is generated by a tiling substitution. On the free \(\mathbb{Z}\)-module \(\mathcal{G}^*\) defined by

\[
\mathcal{G}^* := \left\{ \sum_{k=1}^{l} n_k(x_k, i_k^*) \mid n_k \in \mathbb{Z}, \ x_k \in \mathbb{Z}^2, \ i_k \in A \text{ for any } k, l < \infty \right\},
\]

an endomorphism \(\sigma^*\), so called a tiling substitution, is given by

\[
\sigma^* (x, i^*) = \sum_{j \in A} \sum_{w(j) = i} (A_{\sigma}^{-1} x + f(S_k^{(j)})), j^*) \text{ for } (x, i^*) \in \mathcal{G}^*,
\]

where \(f\) is a canonical homomorphism from the free monoid \(A^*\) to \(\mathbb{Z}^2\), \(\sigma(j) = w_1^{(j)} w_2^{(j)} \cdots w_{l(j)}^{(j)}\) and \(S_k^{(j)} = w_{k+1}^{(j)} w_{k+2}^{(j)} \cdots w_{l(j)}^{(j)}\). Remark that we usually use the notations \(\mathcal{G}^*_1, E^*_1(\sigma)\) instead of \(\mathcal{G}^*, \sigma^*\) when we consider substitutions of higher rank (cf. [2, 18]). For the substitution \(\sigma\) given by (2), the tiling substitution \(\sigma^*\) is determined as follows:

\[
\sigma^* (x, i^*) = \begin{cases} 
(A_{\sigma}^{-1} x, 2^*) & \text{if } i = 1 \\
(A_{\sigma}^{-1} x - e_1 + e_2, 1^*) & \text{if } i = 2 
\end{cases}
\]

We also identify an element of \(\mathcal{G}^*\) with a union of oriented unit segments with multiplicity. Define the subset of \(\mathcal{G}^*\) which consists of unit segments on the stepped surface without multiplicity as follows:

\[
\mathcal{G}^*_\sigma := \left\{ \sum_{k=1}^{l} (x_k, i_k^*) \mid (x_k, i_k^*) \in S_\sigma, \ l < \infty, \ (x_k, i_k^*) \neq (x_{k'}, i_{k'}^*) \text{ if } k \neq k' \right\},
\]

and \(\mathcal{G}^*_\sigma'\) is also defined in the same way by replacing \(S_\sigma\) with \(S'_\sigma\). By iterating \(\sigma^*\) for the initial elements \(U := (e_1, 1^*) + (e_2, 2^*) \in \mathcal{G}^*_\sigma, \ U' := (o, 1^*) + (o, 2^*) \in \mathcal{G}^*_\sigma'\), the stepped surfaces are obtained.

**Proposition 1.1** ([2]) For a substitution \(\sigma\), we have \(\sigma^* n(U) \in \mathcal{G}^*_\sigma\) (resp. \(\sigma^* n(U') \in \mathcal{G}^*_\sigma'\)) and \(\sigma^* n(U) - \sigma^* n(U') = U - U'\) for any positive integer \(n\).
By using the projection $\pi_\sigma$ from $\mathbb{R}^2$ to $P_\sigma$ along the eigenvector $u_\sigma$, we obtain the quasi-periodic tiling $\mathcal{T}$ on $P_\sigma$ with two prototiles:

$$\mathcal{T} := \bigcup_{(x, i^*) \in S_\sigma} \pi_\sigma(x, i^*).$$

That is the reason why $\sigma^*$ is called a tiling substitution.

We call $\mathcal{U}, \mathcal{U}'$ “seeds” for the tiling substitution $\sigma^*$. The choice of seeds is important when we consider Rauzy fractals and dynamical systems on them generated by a substitution. Recall that we identify an element $\sum_{k=1}^l (x_k, i_k^*) \in G_\sigma^*$ with $\bigcup_{k=1}^l (x_k, i_k^*) \subset S_\sigma$.

**Proposition 1.2** ([2]) There exist the following limit sets in the sense of Hausdorff metric for $i \in A$:

$$X_\sigma := \lim_{n \to \infty} A_\sigma^n \pi_\sigma \sigma^* (U),$$
$$X'^{(i)}_\sigma := \lim_{n \to \infty} A_\sigma^n \pi_\sigma \sigma^* (e_i, i^*),$$
$$X''^{(i)}_\sigma := \lim_{n \to \infty} A_\sigma^n \pi_\sigma \sigma^* (o, i^*).$$

Since the boundaries of the sets $X_\sigma, X'^{(i)}_\sigma, X''^{(i)}_\sigma$ are fractal in the case of substitutions of higher rank, so they are called Rauzy fractals or atomic surfaces.

Figure 2: Rauzy fractals $X'^{(i)}_\sigma, X''^{(i)}_\sigma, i \in A$ related to the substitution $\sigma$ given by (2)

It is well known that these Rauzy fractals are given by a fixed point of a substitution as follows:

$$X'^{(i)}_\sigma = \{-\pi_\sigma f(s_0 s_1 \cdots s_{k-1}) | s_k = i\},$$
$$X''^{(i)}_\sigma = \{-\pi_\sigma f(s_0 s_1 \cdots s_k) | s_k = i\},$$

where the one-sided sequence $s_0 s_1 \cdots$ is a fixed point or a periodic point of a substitution $\sigma$ and $\overline{A}$ means the closure of $A$ (cf. [2, 14]).

By the definition of $\sigma^*$ and $X'^{(i)}_\sigma, X''^{(i)}_\sigma$, we have the proposition:
Proposition 1.3 ([2]) The following set equations hold for \( i \in \mathcal{A} \):

\[
A^{-1}_\sigma X^{(i)}_\sigma = \bigcup_{j \in \mathcal{A}} \bigcup_{w_{k(j)} = i} (-A^{-1}_\sigma \pi_\sigma f(P_{k(j)}^i) + X^{(j)}_\sigma),
\]

\[
A^{-1}_\sigma X^{(i)}_\sigma = \bigcup_{j \in \mathcal{A}} \bigcup_{w_{k(j)} = i} (A^{-1}_\sigma \pi_\sigma f(S_{k(j)}^i) + X^{(j)}_\sigma),
\]

where \( x + S := \{x + y | y \in S\} \) for \( S \subseteq P_\sigma, x \in P_\sigma \).

Moreover, the sets \(-A^{-1}_\sigma \pi_\sigma f(P_{k(j)}^i) + X^{(j)}_\sigma, j \in \mathcal{A} \) such that \( w_{k(j)} = i \) are disjoint in the sense of Lebesgue measure, and the same holds true for the sets \(-A^{-1}_\sigma \pi_\sigma f(S_{k(j)}^i) + X^{(j)}_\sigma \).

\[
\begin{array}{c|c|c}
X^{(1)}_\sigma & X^{(2)}_\sigma \\
\hline
X^{(1)}_\sigma & X^{(2)}_\sigma & -\pi_\sigma e_1 + X^{(2)}_\sigma \\
\hline
A^{-1}_\sigma X^{(2)}_\sigma & A^{-1}_\sigma X^{(1)}_\sigma
\end{array}
\]

Figure 3: The set equations \( A^{-1}_\sigma X^{(1)}_\sigma = -\pi_\sigma e_1 + X^{(2)}_\sigma, A^{-1}_\sigma X^{(2)}_\sigma = X^{(1)}_\sigma \cup X^{(2)}_\sigma \) for the substitution given by (2)

**Definition 1** Let \((X, T, \mu)\) be a measurable dynamical system, \(\sigma\) a substitution over the alphabet \(\mathcal{A}\) such that

\[
\sigma(i) = w^{(i)}_1 w^{(i)}_2 \cdots w^{(i)}_{l(i)},
\]

\(\{X^{(i)} | i \in \mathcal{A}\}\) a measurable partition of \(X\), and \(\{A^{(i)} | i \in \mathcal{A}\}\) a measurable partition of a subset \(A\) of \(X\). We say that the transformation \(T\) has \(\sigma\)-structure with respect to the pair of partitions \(\{X^{(i)}\}, \{A^{(i)}\}\) if the following conditions hold up to set of measure 0:

\[
T^k A^{(i)} \subset X^{(w^{(i)}_{k+1})} \quad \text{for all} \ i \in \mathcal{A}, \ k = 0, 1, \ldots, l^{(i)} - 1.
\]

\[
T^k A^{(i)} \cap A = \emptyset \quad \text{for all} \ i \in \mathcal{A}, \ 0 < k < l^{(i)}.
\]

\[
T^{l(i)} A^{(i)} \subset A \quad \text{for all} \ i \in \mathcal{A}.
\]

\[
X = \bigcup_{i \in \mathcal{A}} \bigcup_{0 \leq k \leq l^{(i)} - 1} T^k A^{(i)} \quad \text{(non-overlapping)}.
\]

**Theorem 1** ([2]) For a Pisot, unimodular, irreducible and primitive substitution, define the map \(T : X_\sigma \to X_\sigma\) by

\[
T(x) := x - \pi_\sigma(e_i) \text{ if } x \in X^{(i)}_\sigma.
\]

The map \(T\), so called a domain exchange transformation, is well-defined; and the measurable dynamical system \((X_\sigma, T, \mu)\) with Lebesgue measure \(\mu\) has \(\sigma\)-structure with respect to the pair of partitions \(\{X^{(i)}_\sigma | i \in \mathcal{A}\}, \{A^{n}_\sigma X^{(i)}_\sigma | i \in \mathcal{A}\}\) (see Figure 4).
For the substitution given by (2), since $A^\sigma X^{(1)} \subset X^{(2)}$, $T(\sigma X^{(1)}) \subset A^\sigma X^{(1)}$ and $A^\sigma X^{(2)} \subset X^{(2)}$, $T(A^\sigma X^{(2)}) \subset X^{(1)}$, $T^2(A^\sigma X^{(2)}) \subset A^\sigma X^{(2)}$, we can check the domain exchange transformation $T$ has $\sigma$-structure, and moreover $\sigma^n$-structure with respect to the partitions \( \{X^{(i)}_\sigma\mid i \in A\} \), \( \{A^\sigma X^{(i)}_\sigma\mid i \in A\} \) for any positive integer $n$. On the other hand, the fixed point $s_0 s_1 \cdots = \lim_{n \to \infty} \sigma^n(2)$ and the origin $o \in A^n X^{(2)}_\sigma$ for any positive integer $n$. Therefore from $\sigma^n$-structure, we have

\[ T^k(o) \in X^{(sk)}_\sigma \text{ for all } k = 0, 1, \cdots. \]

At the end of review of the case of substitutions, recall the following theorem related to the topological property of Rauzy fractals.

**Theorem 2** ([3]) Let a substitution $\sigma$ of rank 2 be Pisot, unimodular, irreducible and primitive. The Rauzy fractals $X_\sigma$, $X^{(i)}_\sigma$, $X'^{(i)}_\sigma$, $i \in A$ are interval if and only if $\sigma$ is invertible. Moreover, $X^{(i)}_\sigma$, $X'^{(i)}_\sigma$ are intervals given by

\[ X^{(i)}_\sigma = \pi_\sigma(e_i, i^*) + h \]

\[ X'^{(i)}_\sigma = \pi_\sigma(o, i^*) + h \]

for some $h \in P_\sigma$.

From the theorem, if $\sigma$ is invertible, then the domain exchange transformation $T$ is just a two interval exchange transformation on the one dimensional torus.

## 2 The choice of automorphisms with incidence matrices of quadratic polynomials

Assume the companion matrix related to a quadratic polynomial $x^2 - ax \mp 1$ which is denoted by $A_{\pm}$:

\[ A_\pm = \begin{pmatrix} 0 & \pm 1 \\ 1 & a \end{pmatrix} \]

is hyperbolic, that is, the dominant eigenvalue $\lambda$ and the other one $\lambda'$ hold $|\lambda| > 1 > |\lambda'|$, then it is easily to check that there are following four cases.
Proposition 2.1 If a matrix $A_\sigma = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$, $a \in \mathbb{Z}$ is hyperbolic, then there are two cases:
(i) $a \geq 3$ and its eigenvalues $\lambda_1, \lambda'_1$ hold $2 \leq a - 1 < \lambda_1 < a$, $0 < \lambda'_1 < 1$,
(ii) $a \leq -3$ and its eigenvalues $\lambda_2, \lambda'_2$ hold $a < \lambda_2 < a + 1 \leq -2$, $-1 < \lambda'_2 < 0$.

If a matrix $A_\tau = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$, $a \in \mathbb{Z}$ is hyperbolic, then there are two cases:
(iii) $a \geq 1$ and its eigenvalues $\lambda_3, \lambda'_3$ hold $1 \leq a < \lambda_3 < a + 1$, $-1 < \lambda'_3 < 0$,
(iv) $a \leq -1$ and its eigenvalues $\lambda_4, \lambda'_4$ hold $a - 1 < \lambda_4 < a \leq -1$, $0 < \lambda'_4 < 1$.

As mentioned in Section 0, the aim of this paper is to find automorphisms related to the matrices $A_\sigma$ and $A_\tau$ with which one can generate a stepped surface and a Rauzy fractal, and discuss a dynamical system on the Rauzy fractal. For this aim, the following automorphisms, which are conjugate to some substitutions or some alternative substitutions, are chosen for each case in Proposition 2.1. We call an endomorphism $\sigma$ on $F_2$ an “alternative” substitution if only two letters $1^-1, 2^-1$ appear in $\sigma(i)$ for all $i \in \mathcal{A}$. If $\sigma$ is an alternative substitution, then $\sigma^2$ becomes a substitution. That is the reason why such an endomorphism on $F_2$ is called an alternative substitution. We say an endomorphism $\sigma$ on $F_2$ is conjugate to an endomorphism $\tau$ if there exists an automorphism $\delta$ such that $\sigma = \delta^{-1} \circ \tau \circ \delta$.

The case (i): A matrix is $A_\sigma$ with $a \geq 3$, and its eigenvalues hold $\lambda_1 > 1$, $0 < \lambda'_1 < 1$. Set the automorphism $\sigma_1$ as

$$
\sigma_1 : \{ 1 \rightarrow 2, 2 \rightarrow 2^{a-2} 1^{-1} 2 22 \}, \quad A_{\sigma_1} = A_- = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}.
$$

The automorphism $\sigma_1$ is conjugate to the invertible substitution $\tau_1$ with the automorphism $\delta_1$ on $F_2$ such that

$$
\tau_1 : \{ 1 \rightarrow 2^{a-3} 12, 2 \rightarrow 2^{a-2} 12 \}, \quad \delta_1 : \{ 1 \rightarrow 21, 2 \rightarrow 2 \} \quad \left( \delta_1^{-1} : \{ 1 \rightarrow 1^{-1} 2, 2 \rightarrow 2 \} \right).
$$

The case (ii): A matrix is $A_\tau$ with $a \leq -3$, and its eigenvalues hold $\lambda_2 < -1$, $-1 < \lambda'_2 < 0$. Set the automorphism $\sigma_2$ as

$$
\sigma_2 : \{ 1 \rightarrow 2, 2 \rightarrow 1^{-1} 2^a \}, \quad A_{\sigma_2} = A_- = \begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}.
$$

The automorphism $\sigma_2$ is conjugate to the alternative substitution $\tau_2$ with the automorphism $\delta_2$ on $F_2$ such that

$$
\tau_2 : \{ 1 \rightarrow 1^{-1} 2^{a+2}, 2 \rightarrow 1^{-1} 2^{a+1} \}, \quad \delta_2 : \{ 1 \rightarrow 2^{-1}, 2 \rightarrow 2 \} \quad \left( \delta_2^{-1} : \{ 1 \rightarrow 21, 2 \rightarrow 2 \} \right).
$$

The case (iii): A matrix is $A_\sigma$ with $a \geq 1$, and its eigenvalues hold $\lambda_3 > 1$, $-1 < \lambda'_3 < 0$. Set the automorphism $\sigma_3$ as

$$
\sigma_3 : \{ 1 \rightarrow 2, 2 \rightarrow 2^a 1 \}, \quad A_{\sigma_3} = A_+ = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}.
$$
In this case, the automorphism $\sigma_3$ is a substitution. Since the property in the case of substitutions is known as we saw in Section 0, so we don’t deal this case in this paper.

The case (iv): A matrix is $A_+$ with $a \leq -1$, and the eigenvalues hold $\lambda_4 < -1$, $0 < \lambda_4' < 1$. Set the automorphism $\sigma_4$ as

$$\sigma_4 : \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 12^a \end{cases}, \quad A_{\sigma_4} = A_+ = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}.$$  

The automorphism $\sigma_4$ is conjugate to the alternative substitution $\tau_4$ with the automorphism $\delta_4$ on $F_2$ such that

$$\tau_4 : \begin{cases} 1 \rightarrow 2^{-1} \\ 2 \rightarrow 1^{-1}2^a \end{cases}, \quad \delta_4 : \begin{cases} 1 \rightarrow 1^{-1} \\ 2 \rightarrow 2 \end{cases} \quad (\delta_4^{-1} = \delta_4).$$

In this paper, we use the following typical examples for each case in figures:

$$\sigma_1 : \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 21^{-1}22 \end{cases}, \quad \tau_1 : \begin{cases} 1 \rightarrow 12 \\ 2 \rightarrow 212 \end{cases}, \quad \delta_1 : \begin{cases} 1 \rightarrow 21^{-1} \\ 2 \rightarrow 2 \end{cases}$$

$$\sigma_2 : \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 1^{-1}2^{-1}1^{-1}2 \end{cases}, \quad \tau_2 : \begin{cases} 1 \rightarrow 1^{-1}2^{-1} \\ 2 \rightarrow 1^{-1}2^{-1}1^{-1}2 \end{cases}, \quad \delta_2 : \begin{cases} 1 \rightarrow 2^{-1} \\ 2 \rightarrow 2 \end{cases}$$

$$\sigma_4 : \begin{cases} 1 \rightarrow 2 \\ 2 \rightarrow 12^{-1} \end{cases}, \quad \tau_4 : \begin{cases} 1 \rightarrow 2^{-1} \\ 2 \rightarrow 1^{-1}2^{-1} \end{cases}, \quad \delta_4 : \begin{cases} 1 \rightarrow 1^{-1} \\ 2 \rightarrow 2 \end{cases}$$

3 Stepped surfaces

In this section, we construct the stepped surface of $P_\tau$, $\sigma = \sigma_1, \sigma_2, \sigma_4$ by using the fact that $\sigma$ is conjugate to some substitution or some alternative substitution. Here, $\tau$ is used for a substitution or an alternative substitution, and $\sigma$ for an endomorphism on the free group $F_2$ of rank 2. First let us consider the stepped surface of $P_\tau$, $\tau = \tau_1, \tau_2, \tau_4$. Notice that from the property of conjugate, the eigenvalues of $A_{\tau_t}$, $t = 1, 2, 4$ are the same as the eigenvalues $\lambda_t, \lambda_t'$ of $A_{\sigma_t}$. The matrices $A_{\tau_1}, -A_{\tau_2}, -A_{\tau_4}$ are primitive, so each incidence matrix $A_{\tau_t}$ of $\tau_t$, $t = 1, 2, 4$ has a positive column eigenvector $u_{\tau_t}$ and a positive low eigenvector $v_{\tau_t}$ corresponding to each eigenvalue $\lambda_1 > 1$, $\lambda_2 < -1$, $\lambda_4 < -1$ by Perron-Frobenius Theorem. When we consider arbitrary substitution or alternative substitution $\tau$, assume that it satisfies the hyperbolic, irreducible, unimodular conditions and $A_{\tau}$ or $-A_{\tau}$ is primitive hereafter. For simplicity, set the low eigenvector of $A_{\tau}$ as $v_{\tau} = (1, \beta)$ with some $\beta > 0$ corresponding to the eigenvalue $\lambda_{\tau}$ with $|\lambda_{\tau}| > 1$. The stepped surfaces $S_t, S_t'$ of the contractive eigenspace of $A_{\tau}$, which is given by $P_\tau = \{x \in R^2 \mid < x, t'v_{\tau} > = 0\}$, are defined analogously as in the case of substitutions as follows:

$$S_\tau := \bigcup_{(x, i^*) \in S_\tau} (x, i^*),$$

$$S'_\tau := \bigcup_{(x, i^*) \in S'_\tau} (x, i^*),$$

where

$$S_\tau := \{ (x, i^*) \in Z^2 \times \{1^*, 2^*\} \mid \langle x, t'v_{\tau} \rangle > 0, \langle x - e_i, t'v_{\tau} \rangle \leq 0 \},$$

$$S'_\tau := \{ (x, i^*) \in Z^2 \times \{1^*, 2^*\} \mid \langle x, t'v_{\tau} \rangle \geq 0, \langle x - e_i, t'v_{\tau} \rangle < 0 \}.$$
We mean by \((x, i^*)\) the positively oriented unit segment translated by \(x\) in \(\mathbb{Z}^2\), that is,
\[
(x, 1^*) := \{x + te_2 \mid 0 \leq t \leq 1\}, \quad (x, 2^*) := \{x + te_1 \mid 0 \leq t \leq 1\}.
\]

\[
\begin{array}{c}
\text{x} \\
(x, 1^*) \quad -\text{x} \\
(x, 2^*) \quad -(-x, 1^*) \\
\end{array}
\]

\[\text{Figure 5: The segments } (x, 1^*), (x, 2^*) \text{ with orientation}\]

Notice that if \(\beta\) is irrational, then
\[
S \setminus S' = \{(e_1, 1^*) \cup (e_2, 2^*)\} \setminus \{(o, 1^*) \cup (o, 2^*)\}.
\]

**Definition 2** For an endomorphism \(\sigma\) on \(F_2\) given by
\[
\sigma(i) = w_1(i)w_2(i) \cdots w_{l(i)}, \quad i \in A,
\]
de define the \(k\)-prefix \(P_k^{(i)}\) and \(k\)-suffix \(S_k^{(i)}\) in \(F_2\) for \(0 \leq k \leq l(i)\) by
\[
P_k^{(i)} := w_1(i)w_2(i) \cdots w_{k-1}(i), \quad S_k^{(i)} := w_{k+1}w_{k+2} \cdots w_{l(i)}.
\]

Sometimes these notations are used for a substitution or an alternative substitution \(\tau\) instead of \(\sigma\). The free \(\mathbb{Z}\)-module \(G^*\) is defined by
\[
G^* := \left\{ \sum_{k=1}^{l} n_k(x_k, i_k^*) \mid n_k \in \mathbb{Z}, \ x_k \in \mathbb{Z}^2, \ i_k \in A \text{ for any } k, \ l < \infty \right\},
\]
whose element is identified with a union of oriented unit segments with their multiplicity. The tiling substitution \(\sigma^*\) for a unimodular endomorphism \(\sigma\) on \(F_2\) such that \(\det(\sigma) = \pm 1\) is defined by
\[
\sigma^*(x, i^*) := \sum_{j \in A} \left\{ \sum_{u_{k}^{(j)} = i} \left( A_{\sigma}^{-1}(x + f(S_{j}^{(k)})), j^* \right) + \sum_{u_{k}^{(j)} = -1} - \left( A_{\sigma}^{-1}(x + f(w_{k+1}^{(j)} S_{j}^{(k)})), j^* \right) \right\}.
\]

**Remark 1** In general, for a unimodular endomorphism \(\sigma\) on the free group \(F_d\) of rank \(d\), a higher dimensional extension \(E_k(\sigma)\) of \(\sigma\) is defined for \(0 \leq k \leq d\), and \(E_k^*(\sigma)\) is determined as its dual map. The tiling substitution \(\sigma^*\) is just \(E_1^*(\sigma)\) (cf. [5, 18]).

Define the subsets of \(G^*\) for a substitution or an alternative substitution \(\tau\) by
\[
G_{\tau}^* := \left\{ \sum_{k=1}^{l} n_k(x_k, i_k^*) \mid n_k \in \{-1, 1\}, \ (x_k, i_k^*) \in S_{\tau}, \ l < \infty \right\}, \quad (7)
\]
and \(G_{\tau'}^*\) is defined by replacing \(S_{\tau}\) with \(S_{\tau}'\) in the formula (7). For an element \(\sum_{k=1}^{l} n_k(x_k, i_k^*) \in G_{\tau}^*\), the condition \(n_k \in \{-1, 1\}\) means that there is no overlap in it, and we identify it with \(\bigcup_{k=1}^{l} (x_k, i_k^*) \subset S_{\tau}\) geometrically. The following two lemmas show that a tiling substitution \(\tau^*\) is well-defined as a map on \(G_{\tau}^*\) (resp. a map from \(G_{\tau}^*\) to \(G_{\tau'}^*\)) for a substitution (resp. an alternative substitution) \(\tau\).
Lemma 1 If \( \tau \) is a substitution or an alternative substitution, then \((x_1, i_1^*), (x_2, i_2^*) \in S_\tau, (x_1, i_1^*) \neq (x_2, i_2^*)\) implies \(\tau^*(x_1, i_1^*) \cap \tau^*(x_2, i_2^*) = \emptyset\), where \(\sum_{k=1}^t n_{k_1}(x_{k_1}, i_{k_1}) \cap \sum_{k=2}^i n_{k_2}(x_{k_2}, i_{k_2}) \neq \emptyset\) means there exist \(k_1 \in \{1, 2, \ldots, l_1\}\) and \(k_2 \in \{1, 2, \ldots, l_2\}\) such that \((x_{k_1}, i_{k_1}) = (x_{k_2}, i_{k_2}^*)\).

Proof. In the case of substitutions, see [2]. We prove it for an alternative substitution. Suppose \((x_1, i_1^*), (x_2, i_2^*) \in S_\tau, (x_1, i_1^*) \neq (x_2, i_2^*)\) and \(\tau^*(x_1, i_1^*) \cap \tau^*(x_2, i_2^*) \neq \emptyset\), then there exists \(j \in A, k_1 \in \{1, 2, \ldots, l(i_1)\}, k_2 \in \{1, 2, \ldots, l(i_2)\}\) such that \(w^{(j)}_{k_1} = i_{k_1}^{-1}, w^{(j)}_{k_2} = i_{k_2}^{-1}\), and

\[
\left(A_{\tau}^{-1}(x_1 + f(w^{(j)}_{k_1} S^{(j)}_{k_1})), j^* \right) = \left(A_{\tau}^{-1}(x_2 + f(w^{(j)}_{k_2} S^{(j)}_{k_2})), j^* \right).
\]

Suppose \(x_1 = x_2\), then \(k_1 = k_2\) because \(\tau\) is an alternative substitution; and \(i_1 = i_2\). It contradicts to \((x_1, i_1^*) \neq (x_2, i_2^*)\). Therefore \(x_1 \neq x_2\), and so \(k_1 \neq k_2\). We can suppose \(k_1 < k_2\) without loss of generality, then

\[
x_1 - e_{i_1} = x_2 + f(S^{(j)}_{k_2-1}) - f(S^{(j)}_{k_1-1}) - e_{i_1} = x_2 - f(w^{(j)}_{k_1+1} \cdots w^{(j)}_{k_2-1})
\]

and

\[
<x_1 - e_{i_1}, t v_\tau > = < x_2 - f(w^{(j)}_{k_1+1} \cdots w^{(j)}_{k_2-1}), t v_\tau > = < x_2, t v_\tau > + < -f(w^{(j)}_{k_1+1} \cdots w^{(j)}_{k_2-1}), t v_\tau > > 0
\]

It contradicts to \((x_1, i_1^*) \in G^*_\tau\).

Lemma 2 If \( \tau \) is a substitution, \((x, i^*) \in G^*_\tau \) (resp. \((x, i^*) \in G^*_{\tau^*}\)) implies \(\tau^*(x, i^*) \in G^*_{\tau^*} \) (resp. \(\tau^*(x, i^*) \in G^*_\tau \)). If \( \tau \) is an alternative substitution, \((x, i^*) \in G^*_\tau \) (resp. \((x, i^*) \in G^*_{\tau^*}\)) implies \(\tau^*(x, i^*) \in G^*_\tau \) (resp. \(\tau^*(x, i^*) \in G^*_{\tau^*}\)).

Proof. In the case of substitutions, see [2]. For an alternative substitution \( \tau \), the tiling substitution is given by \(\tau^*(x, i^*) = \sum_{j \in A} \sum_{w^{(j)}_{i} = i-1} \left(A_{\tau}^{-1}(x + f(S^{(j)}_{k} - e_{i})) \right) \). Suppose \((x, i^*) \in G^*_\tau\). Since the eigenvalue \(\lambda_\tau < 0\) and the eigenvector \(v_\tau >_0\), if \(w^{(j)}_{k} = i^{-1}\), then

\[
< A_{\tau}^{-1}(x + f(S^{(j)}_{k} - e_{i}), t v_\tau > = < x + f(S^{(j)}_{k} - e_{i}), t A_{\tau}^{-1} t v_\tau > = \frac{1}{\lambda_\tau} < x + f(S^{(j)}_{k} - e_{i}), t v_\tau >
\]

and

\[
< A_{\tau}^{-1}(x + f(S^{(j)}_{k} - e_{i}) - e_{j}), t v_\tau > = < x + f(S^{(j)}_{k} - e_{i} - A_{\tau} e_{j}, t A_{\tau}^{-1} t v_\tau > = \frac{1}{\lambda_\tau} < x - f(P^{(j)}_{k}), t v_\tau > < 0,
\]

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and we conclude \((A^{-1}_\tau(x + f(S^{(j)}_k) - e_i), f^*) \in S'_{\tau} \) and \(\tau^*(x, i^*) \in G'_{\tau'}\). \(\square\)

Recall the following lemma and proposition.

**Lemma 3** ([5]) For unimodular endomorphisms \(\sigma, \sigma'\) on \(F_2\), the tiling substitution for their concatenation \(\sigma \circ \sigma'\) is given by

\[
(\sigma \circ \sigma')^* = \sigma'^* \circ \sigma^*.
\]

**Proposition 3.1** ([6]) If a substitution \(\tau\) is invertible, then \(\tau^* n(U), \tau^* n(U')\) and \(\tau^* n(e_i, i^*), \tau^* n(o_i, i^*), \ i \in \mathcal{A}\) are geometrically connected.

Since \((e_1, 1^*), (e_2, 2^*) \in S_{\tau}\) and \((o_1, 1^*), (o_2, 2^*) \in S'_{\tau}\) for a substitution or an alternative substitution \(\tau\), so \(U := (e_1, 1^*) + (e_2, 2^*) \in G^*_{\tau}, U' := (o_1, 1^*) + (o_2, 2^*) \in G'^*_{\tau}\). Even if \(\tau\) is an alternative substitutions, \(\tau^2\) is a substitution and \((\tau^2)^* = (\tau^*)^2\). And it is easy to check the substitutions \(\tau_1, \tau_2, \tau_4\) are invertible. Thus we have the following proposition by Lemma 1, Lemma 2 and Proposition 3.1.

**Proposition 3.2** In the case of (i),

\[
\tau_1^* n(U) \in G^*_{\tau_1}, \ \tau_1^* n(U') \in G'^*_{\tau_1}, \ n \in \mathbb{N},
\]

and \(\tau_1^* n(U), \ \tau_1^* n(U')\) are connected.

In the case of (ii) and (iv),

\[
\tau^* 2n(U) \in G^*_{\tau}, \ \tau^* 2n(U') \in G'^*_{\tau}, \ n \in \mathbb{N},
\]

for \(\tau = \tau_2, \tau_4\), and \(\tau^* 2n(U'), \ \tau^* 2n(U)\) are connected.

**Remark 2** By using the idea of \(C\)-covered property (cf. [13, 7]), we can show that \(\tau_1^* n(U)\) (resp. \(\tau_2^* 2n(U)\)) goes to the stepped surface \(S_{\tau_1}\) (resp. \(S_{\tau_2}\)) geometrically when \(n\) goes to \(\infty\).

The stepped surfaces \(S_{\tau}, S'_{\tau}\) of the line \(P_{\tau}\) for \(\tau = \tau_1, \tau_2, \tau_4\) are generated by using the tiling substitution with the seeds \(U, U'\). From now on we generate the stepped surface of the contractive eigenspace \(P_{\sigma} = \{x \in \mathbb{R}^2 \mid <x, tv_\sigma> = 0\}, \sigma = \sigma_1, \sigma_2, \sigma_4\) related to \(A_{\sigma}\) in each case (i), (ii), (iv). The matrices \(A_{\sigma_1}\) and \(-A_{\sigma_1}\) are positive matrices, so we cannot apply Perron-Frobenius theorem directly for them. In fact, one of the eigenvalues of the incidence matrix \(A_{\sigma_1}\) satisfies \(\lambda_1 > 1\), and its corresponding low eigenvector \(v_{\sigma_1}\) given by \((1, \lambda_1)\) is positive, but one of the eigenvalues of the incidence matrix \(A_{\sigma_1}\) for \(t = 2, 4\) satisfies \(\lambda_t < 1\), and the corresponding low eigenvector \(v_{\sigma_1}\) given by \((-1, -\lambda_t)\) is not positive. So the sets \(S_{\sigma_1}, S'_{\sigma_1}\) related to the stepped surface of the contractive eigenspace \(P_{\sigma}\) are defined as (5), (6), but the sets \(S_{\sigma}, S'_{\sigma}\) are redefined as follows:

\[
S_{\sigma} := \left\{ (x, i^*) \in \mathbb{Z}^2 \times \{1^*, 2^*\} \mid \begin{array}{ll}
<x, tv_\sigma > > 0, <x + e_1, tv_\sigma > \leq 0 & \text{if } i = 1 \\
<x + e_1, tv_\sigma > > 0, <x + e_1 - e_2, tv_\sigma > \leq 0 & \text{if } i = 2
\end{array} \right\}
\]

\[
S'_{\sigma} := \left\{ (x, i^*) \in \mathbb{Z}^2 \times \{1^*, 2^*\} \mid \begin{array}{ll}
<x, tv_\sigma > \geq 0, <x + e_1, tv_\sigma > < 0 & \text{if } i = 1 \\
<x + e_1, tv_\sigma > \geq 0, <x + e_1 - e_2, tv_\sigma > < 0 & \text{if } i = 2
\end{array} \right\}
\]
The subset $G_\sigma^*$, $G_\sigma'$, $\sigma = \sigma_1, \sigma_2, \sigma_4$ of $G^*$ are defined in the same way as (7) by $S_\sigma, S'_\sigma$.

Suppose an automorphism $\sigma$ is conjugate to $\tau$ as $\sigma = \delta^{-1} \circ \tau \circ \delta$ with some automorphism $\delta$ on $F_2$. In general $A_\sigma = A_\delta^{-1} A_\tau A_\delta$, and the contractive eigenspace $P_\sigma$ is given by

\begin{align*}
P_\sigma &= \{ A_\delta^{-1} x \in R^2 \mid x \in P_\tau \} \\
&= A_\delta^{-1} P_\tau.
\end{align*}

By Lemma 3, the tiling substitution $\sigma^*$ of $\sigma$ is

$$\sigma^* = \delta^* \circ \tau^* \circ (\delta^{-1})^*,$$

and moreover,

$$\sigma^* n = \delta^* \circ \tau^* n \circ (\delta^{-1})^*.$$  \hspace{1cm} (8)

The following replacing method will be introduced to understand the relation between the stepped surfaces $S_\tau$ and $S_\sigma$, $t = 1, 2, 4$ by using $\delta^*_\sigma$.

**Replacement Method**

Choose low eigenvectors $v_{\tau_1} = (1, \frac{\lambda_1}{\lambda_1 - 1}), v_{\tau_2} = (1, \frac{\lambda_2}{\lambda_2 + 1}), v_{\tau_4} = (1, -\lambda_4)$ of $A_{\tau}, t = 1, 2, 4$, then we have the following lemmas.

**Lemma 4**

1. If $(x, 1^*) \in G^*_{\tau_1}$, then $(x, 2^*) \in G^*_{\tau_1}$.

2. If $(x, 1^*) \in G^*_{\tau_2}$, then $(x - e_1 + e_2, 2^*) \in G^*_{\tau_2}$.

Proof. We prove the first statement. The second one is proved by the same way. Suppose $(x, 1^*) \in G^*_{\tau_1}$, then $< x, v_{\tau_1} > > 0$ and $< x - e_1, v_{\tau_1} > \leq 0$.

\[
< x - e_2, v_{\tau_1} > = < x - e_1, v_{\tau_1} > + < e_1 - e_2, v_{\tau_1} > = < x - e_1, v_{\tau_1} > + 1 - \frac{\lambda_1}{\lambda_1 - 1} < 0
\]

Therefore $(x, 2^*) \in G^*_{\tau_1}$. \hfill \Box

For a $2 \times 2$ matrix $A$ and $(x, i^*) \in G^*$, $A(x, i^*) := \{ Ax + Ay \mid y \in (x, i^*)) \}$. We get $\delta^*_t(x, i^*), t = 1, 2, 4$, $i \in A$ by the following replacement method.

In the case (i), for $(x, i^*) \in G^*_{\tau_1}$

$$\delta^*_1(x, i^*) = \begin{cases} 
-(A^{-1}_{\delta_1} x + e_1 - e_2, 1^*) & i = 1 \\
(A^{-1}_{\delta_1} x + e_1 - e_2, 1^*) + (A^{-1}_{\delta_1} x, 2^*) & i = 2
\end{cases}$$

Replace $A^{-1}_{\delta_1}(x, 1^*)$ by $-(A^{-1}_{\delta_1} x, 1^*)$, and translate it by $e_1 - e_2$, we get $\delta^*_1(x, 1^*)$. Replace $A^{-1}_{\delta_1}(x, 2^*)$ by $(A^{-1}_{\delta_1} x - e_1 + e_2, 1^*) + (A^{-1}_{\delta_1} x, 2^*)$, and translate it by $e_1 - e_2$, then we get $\delta^*_1(x, 2^*)$. If $(x, 1^*) \in G^*_{\tau_1}$, then $(x, 1^*) + (x, 2^*) \in G^*_{\tau_1}$ by Lemma 4. Therefore the unit segment $\delta^*_1(x, 1^*) = -(A^{-1}_{\delta_1} x + e_1 - e_2, 1^*)$ with negative orientation is always cancelled.
The case (i):

\[
x \xrightarrow{x,1^*} A_{\delta_1}^{-1} x \quad \Rightarrow \quad A_{\delta_1}^{-1} x \quad \Rightarrow \quad A_{\delta_1}^{-1} x + e_1 - e_2
\]

\[
x \xrightarrow{x,2^*} A_{\delta_1}^{-1} x \quad \Rightarrow \quad A_{\delta_1}^{-1} x \quad \Rightarrow \quad A_{\delta_1}^{-1} x + e_1 - e_2
\]

\[
x \xrightarrow{(x,1^*) + (x,2^*)} A_{\delta_1}^{-1} x \quad \Rightarrow \quad A_{\delta_1}^{-1} x \quad \Rightarrow \quad A_{\delta_1}^{-1} x + e_1 - e_2
\]

The stepped surface \( S_{\tau_1} \)

The picture after mapping by \( A_{\delta_1}^{-1} \)

The picture after replacement

The picture after translation by \( e_1 - e_2 \)

Figure 6: Replacement and translation in the case (i)
by $\delta_1^*(x, 2^*)$ (see figure 6).

In the case (ii), for $(x, i^*) \in G^*_{r_2}$

$$
\delta_2^*(x, i^*) = \begin{cases} 
(A_{\delta_2^{-1}}^{-1}x, 1^*) & i = 1 \\
-(A_{\delta_2^{-1}}^{-1}x + e_1, 1^*) + (A_{\delta_2}^{-1}x, 2^*) & i = 2 
\end{cases}
$$

Replace $A_{\delta_2}^{-1}(x, 1^*)$ by $(A_{\delta_2}^{-1}x, 1^*)$, then we get $\delta_2^*(x, 1^*)$. Replace $A_{\delta_2}^{-1}(x, 2^*)$ by $-(A_{\delta_2}^{-1}x + e_1, 1^*) + (A_{\delta_2}^{-1}x, 2^*)$, then we get $\delta_2^*(x, 2^*)$. If $(x, 1^*) \in G^*_{r_2}$, then $(x, 1^*) + (x - e_1 + e_2, 2^*) \in G^*_{r_2}$. Therefore the unit segment $\delta_2^*(x, 1^*) = (A_{\delta_2}^{-1}x, 1^*)$ with positive orientation is always cancelled by $\delta_2^*(x - e_1 + e_2, 2^*)$ (see figure 7).

In the case (iv), for $(x, i^*) \in G^*_{r_4}$

$$
\delta_4^*(x, i^*) = \begin{cases} 
-(A_{\delta_4}^{-1}x + e_1, 1^*) & i = 1 \\
(A_{\delta_4}^{-1}x, 2^*) & i = 2 
\end{cases}
$$

Replace $A_{\delta_4}^{-1}(x, 1^*)$ by $-(A_{\delta_4}^{-1}x, 1^*)$, and translate it by $e_1$, then we get $\delta_4^*(x, 1^*)$. Replace $A_{\delta_4}^{-1}(x, 2^*)$ by $(A_{\delta_4}^{-1}x - e_1, 2^*)$, and translate it by $e_1$, then we get $\delta_4^*(x, 2^*)$ (see figure 8).

The following lemma shows the relation between the stepped surface $S_\sigma$ and $S_{\sigma_t}$, $t = 1, 2, 4$ by using $\delta_1^*$. For $y \in Z^2$ and $\sigma = \sigma_1, \sigma_2, \sigma_3$,

$$
G^*_\sigma + y := \left\{ \sum_{k=1}^l n_k(x_k + y, i_k^*) \mid \sum_{k=1}^l n_k(x_k, i_k^*) \in G^*_\sigma \right\}.
$$

**Lemma 5**

1. $\delta_1^*[(x, 1^*) + (x, 2^*)] \in G^*_\sigma$, if $(x, 1^*) \in G^*_\sigma$.
2. $\delta_2^*[(x, 1^*) + (x - e_1 + e_2, 2^*)] \in G^*_\sigma + e_1$ if $(x, 1^*) \in G^*_\sigma$.
3. $\delta_4^*[(x, 2^*)] \in G^*_\sigma - e_1$ if $(x, i^*) \in G^*_\sigma$.

Proof. In the case where $(x, 1^*) \in G^*_\sigma$, $(x, 1^*) + (x, 2^*) \in G^*_\sigma$ and $\delta_1^*[(x, 1^*) + (x, 2^*)] = (A_{\delta_1}^{-1}x, 2^*)$. From $v_{\sigma_1}A_{\delta_1}^{-1} = (\lambda_1 - 1) v_{\tau_1}$,

$$
< A_{\delta_1}^{-1}x, A_{\delta_1}^{-1}v_{\sigma_1} > = < x, A_{\delta_1} v_{\sigma_1} > = (\lambda_1 - 1) < x, v_{\tau_1} > > 0,
$$

and

$$
< A_{\delta_1}^{-1}x - e_2, v_{\sigma_1} > = < x - e_2, A_{\delta_1}^{-1}v_{\sigma_1} > = (\lambda_1 - 1) < x - e_2, v_{\tau_1} > \leq 0.
$$

Therefore $(A_{\delta_1}^{-1}x, 2^*) \in G^*_\sigma$. 

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The case (ii):

\[
\begin{align*}
(x, 1^*) &\quad \rightarrow \quad A_{\delta_2}^{-1} x \\
A_{\delta_2}^{-1} x &\quad \rightarrow \quad A_{\delta_2}^{-1} x
\end{align*}
\]

\[
\begin{align*}
(x, 2^*) &\quad \rightarrow \quad A_{\delta_2}^{-1} x \\
A_{\delta_2}^{-1} x &\quad \rightarrow \quad A_{\delta_2}^{-1} x \\
(x, 1^*) + (x - e_1 + e_2, 2^*) &\quad \rightarrow \quad A_{\delta_2}^{-1} x \\
A_{\delta_2}^{-1} x &\quad \rightarrow \quad A_{\delta_2}^{-1} x
\end{align*}
\]

The stepped surface \( S_{\tau_2} \)

The picture after mapping by \( A_{\delta_2}^{-1} \)

The picture after replacement

Figure 7: Replacement and translation in the case (ii)
The case (iv):

\[ (x, 1^*) \overset{A_{\delta_4}}{\rightarrow} A_{\delta_4}^{-1}x \overset{\text{replacement}}{\rightarrow} A_{\delta_4}^{-1}x \overset{\text{translation by } e_1}{\rightarrow} A_{\delta_4}^{-1}x + e_1 \]

\[ (x, 2^*) \overset{A_{\delta_4}}{\rightarrow} A_{\delta_4}^{-1}x \overset{\text{replacement}}{\rightarrow} A_{\delta_4}^{-1}x \overset{\text{translation by } e_1}{\rightarrow} A_{\delta_4}^{-1}x \]

The stepped surface \( S_{\tau_4} \)

The picture after mapping by \( A_{\delta_4}^{-1} \)

The picture after replacement

The picture after translation by \( e_1 \)

Figure 8: Replacement and translation in the case (iv)
In the case where \((x, 2^*) \in G_{\tau_1}^*, (x, 1^*) \notin G_{\tau_1}^*, (x, 1^*) \notin G_{\tau_1}^*,\) noticing \(<x - e_1, t^*v_{\tau_1}> > 0\) by \((x, 1^*) \notin G_{\tau_1}^*,\) we have
\[
< A_{\delta_1}^{-1} x + e_1 - e_2, t^*v_{\sigma_1} > = < x - e_1, t^*A_{\delta_1}^{-1} v_{\sigma_1} > \\
= (\lambda_1 - 1) < x - e_1, t^*v_{\tau_1} > > 0, \\
< A_{\delta_1}^{-1} x - e_2, t^*v_{\sigma_1} > = (\lambda_1 - 1) < x - e_2, t^*v_{\tau_1} > \leq 0, \\
< A_{\delta_1}^{-1} x, t^*v_{\sigma_1} > = (\lambda_1 - 1) < x, t^*v_{\tau_1} > > 0.
\]

Therefore \((A_{\delta_1}^{-1} x + e_1 - e_2, 1^*) + (A_{\delta_1}^{-1} x, 2^*) \in G_{\tau_1}^*.\) The first statement is proved, and the others can be proved analogously.

By the replacement method, we have the following lemma.

**Lemma 6** If \(\gamma \in G_{\tau_t}^*, t = 1, 2, 4\) is connected, then \(\delta_t^*(\gamma)\) is also connected.

To generate the stepped surface of \(P_{\sigma_t}, t = 1, 2, 4,\) determine an initial element \(\tilde{U}\) and \(\tilde{U}'\) for \(\sigma_t^*\) as follows:
\[
\tilde{U} := \delta_t^*(U), \tilde{U}' := \delta_t^*(U').
\]

**Theorem 3** For any positive integer \(n,\)

1. \(\sigma_1^* n(\tilde{U}) \in G_{\sigma_1}^*;\)
2. \(\sigma_t^* 2n(\tilde{U}) \in G_{\sigma_t}^* + e_1, t = 2, 4.\)

Moreover, \(\sigma_t^* n(\tilde{U}), t = 1, 2, 4\) are connected.

Proof. From the equality (8),
\[
\sigma_t^* n(\tilde{U}) = \delta_t^* \circ \tau_t^* n(\tilde{U}), t = 1, 2, 4.
\]
By Proposition 3.2, Lemma 5 and Lemma 6, \(\sigma_1^* n(\tilde{U})\) (resp. \(\sigma_t^* 2n(\tilde{U}), t = 2, 4\)) is included in \(G_{\sigma_1}^*\) (resp. \(G_{\sigma_t}^* + e_1\)), and connected. The other cases can be proved analogously.

By Remark 2, \(\sigma_1^* n(\tilde{U})\) (resp. \(\sigma_2^* 2n(\tilde{U})\)) goes to the stepped surface \(S_{\sigma_1}\) (resp. \(S_{\sigma_2}\)) when \(n\) goes to infinity (see Figure 9 and the last pictures of Figure 6, 7, 8).

![Figure 9: The seed \(\tilde{U}\) and \(\sigma_1^* n(\tilde{U})\) in the case (i)](image-url)
4 Rauzy fractals and domain exchange transformations

In this section, we construct Rauzy fractals induced from automorphisms \( \sigma_t, \ t = 1, 2, 4, \) and consider domain exchange transformations.

Define the projection \( \pi_{\tau_t} \) (resp. \( \pi_{\sigma_t} \), \( t = 1, 2, 4 \)) from \( \mathbb{R}^2 \) to the contractive eigenspace \( P_{\tau_t} \) (resp. \( P_{\sigma_t} \)) along a column eigenvector \( \mathbf{u}_{\tau_t} \) (resp. \( \mathbf{u}_{\sigma_t} \)) of \( A_{\tau_t} \) (resp. \( A_{\sigma_t} \)) corresponding to the eigenvalue \( \lambda_t \). First we define Rauzy fractals related to the substitution \( \tau_1 \) and the alternative substitutions \( \tau_2, \tau_4 \) as follows:

\[
X_{\tau} := \lim_{n \to \infty} A_{\tau}^n \pi_{\tau} \tau^* n(\mathcal{U}),
\]

\[
X^{(i)}_{\tau} := \lim_{n \to \infty} A_{\tau}^n \pi_{\tau} \tau^* n(\mathbf{e}_i, \iota^*),
\]

\[
X'^{(i)}_{\tau} := \lim_{n \to \infty} A_{\tau}^n \pi_{\tau} \tau^* n(\mathbf{o}, \iota^*),
\]

\( \tau = \tau_1, \tau_2, \tau_4 \). It is proved that the limit sets exist in the sense of Hausdorff metric by the same way in the case where \( \tau \) is a substitution (cf. [2]).

**Remark 3** Notice that one can replace \( n \) with \( 2n \) in the formulas of definitions of Rauzy fractals above. Thus for alternative substitutions \( \tau = \tau_2, \tau_4, i \in \mathcal{A} \),

\[
X^{(i)}_{\tau} = X'^{(i)}_{\tau}, \ X^{(i)}_{\tau} = X'^{(i)}_{\tau^2}.
\]

Therefore we can also apply Theorem 2 and show that \( X^{(i)}_{\tau}, X'^{(i)}_{\tau} \) are intervals.

**Proposition 4.1** A substitution or an alternative substitution \( \tau \) is written as \( \tau(i) = w_1^{(i)} w_2^{(i)} \cdots w_{p(i)}^{(i)} \) and \( \tau^2(i) = w_1^{(2,i)} w_2^{(2,i)} \cdots w_{p(i)}^{(2,i)} \), \( i \in \mathcal{A} \). We denote by \( P_k^{(i)} \) and \( S_k^{(i)} \) (resp. \( P_k^{(2,i)} \) and \( S_k^{(2,i)} \)) the \( k \)-prefix and the \( k \)-suffix of \( \tau(i) \) (resp. \( \tau^2(i) \)).

In the case of \( (i) \),

\[
A_{\tau_1}^{-1} X^{(i)}_{\tau_1} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} (-A_{\tau_1}^{-1} \pi_{\tau_1} f(P_k^{(j)}) + X^{(j)}_{\tau_1}),
\]

\[
A_{\tau_1}^{-1} X'^{(i)}_{\tau_1} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} (A_{\tau_1}^{-1} \pi_{\tau_1} f(S_k^{(j)}) + X'^{(j)}_{\tau_1}).
\]

In the case of \( (ii) \) and \( (iv) \), for \( \tau = \tau_2, \tau_4 \),

\[
A_{\tau_2}^{-1} X^{(i)}_{\tau_2} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(2,j)} = i} (-A_{\tau_2}^{-1} \pi_{\tau} f(P_k^{(2,j)}) + X^{(j)}_{\tau}),
\]

\[
A_{\tau_2}^{-1} X'^{(i)}_{\tau_2} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(2,j)} = i} (A_{\tau_2}^{-1} \pi_{\tau} f(S_k^{(2,j)}) + X'^{(j)}_{\tau}),
\]

and moreover,

\[
A_{\tau}^{-1} X^{(i)}_{\tau} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} (-A_{\tau}^{-1} \pi_{\tau} f(P_k^{(j)} w_k^{(j)}) + X^{(j)}_{\tau}),
\]

\[
A_{\tau}^{-1} X'^{(i)}_{\tau} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} (A_{\tau}^{-1} \pi_{\tau} f(w_k^{(j)} S_k^{(j)}) + X'^{(j)}_{\tau}).
\]

These unions are pairwise disjoint in the sense of Lebesgue measure.
Proof. For the substitutions $\tau_1, \tau_2^2, \tau_4^2$, these set equations are known (see Proposition 1.3). So we will show the last equations for alternative substitutions $\tau_2, \tau_4$.

$$\begin{align*}
A_{\tau}^{-1}X^{(i)}_{\tau} &= A_{\tau}^{-1} \lim_{n \to \infty} A_{\tau}^{n+1} \pi_{\tau} \tau^{n+1}(e_i, i^*) \\
&= \lim_{n \to \infty} A_{\tau}^{n} \pi_{\tau} \tau^{n} \left( \sum_{j \in A} \sum_{u_k^{(j)} = i-1} -(A_{\tau}^{-1} f(S_k^{(j)}), j^*) \right) \\
&= \lim_{n \to \infty} A_{\tau}^{n} \pi_{\tau} \tau^{n} \left( \sum_{j \in A} \sum_{u_k^{(j)} = i-1} -(e_j - A_{\tau}^{-1} f(P_k^{(j)} w_k^{(j)}), j^*) \right) \\
&= \bigcup_{j \in A} \bigcup_{u_k^{(j)} = i-1} \left( -A_{\tau}^{-1} \pi_{\tau} f(P_k^{(j)} w_k^{(j)}) + X^{(j)}_{\tau} \right),
\end{align*}$$

$\tau = \tau_2, \tau_4$, $i \in A$. The other set equation for $X^{(i)}_{\tau}$ is shown analogously. \hfill $\Box$

Secondly, to construct Rauzy fractals related to automorphism $\sigma = \sigma_1, \sigma_2, \sigma_4$, set seeds $\overline{U}$ and $\overline{U}$ as

$$\begin{align*}
\overline{U} &:= \begin{cases} (e_1, 1^*) + (e_2, 2^*) & \text{if } \sigma = \sigma_2 \\
(o_1, 1^*) + (e_2, 2^*) & \text{if } \sigma = \sigma_1, \sigma_4,
\end{cases} \\
\overline{U} &:= \begin{cases} (o_1, 1^*) + (o_2, 2^*) & \text{if } \sigma = \sigma_2 \\
(e_1, 1^*) + (o_2, 2^*) & \text{if } \sigma = \sigma_1, \sigma_4.
\end{cases}
\end{align*}$$

Rauzy fractals related to $\sigma_t$, $t = 1, 2, 4$ are defined as follows.

**Definition 3** The following limit sets exist in the sense of Lebesgue measure. For $\sigma = \sigma_1, \sigma_2, \sigma_4$,

$$\begin{align*}
X_{\sigma} &= \lim_{n \to \infty} A_{\sigma}^n \pi_{\sigma} \sigma^n(\overline{U}), \\
&= \lim_{n \to \infty} A_{\sigma}^n \pi_{\sigma} \sigma^n(\overline{U}).
\end{align*}$$

In the case of (ii), for $i \in A$,

$$\begin{align*}
X^{(i)}_{\sigma_2} &= \lim_{n \to \infty} A_{\sigma_2}^n \pi_{\sigma_2} \sigma_2^n(e_i, i^*), \\
X^{(i)}_{\sigma_2} &= \lim_{n \to \infty} A_{\sigma_2}^n \pi_{\sigma_2} \sigma_2^n(o, i^*).
\end{align*}$$

In the case of (i) and (iv), for $\sigma = \sigma_1, \sigma_4$,

$$\begin{align*}
X^{(1)}_{\sigma} &= \lim_{n \to \infty} A_{\sigma}^n \pi_{\sigma} \sigma^n(o_1, 1^*), \\
X^{(2)}_{\sigma} &= \lim_{n \to \infty} A_{\sigma}^n \pi_{\sigma} \sigma^n(e_2, 2^*), \\
X^{(1)}_{\sigma} &= \lim_{n \to \infty} A_{\sigma}^n \pi_{\sigma} \sigma^n(e_1, 1^*), \\
X^{(2)}_{\sigma} &= \lim_{n \to \infty} A_{\sigma}^n \pi_{\sigma} \sigma^n(o, 2^*).
\end{align*}$$
For each automorphism $\sigma_t$, $t = 1, 2, 4$, we set
\[
\epsilon_1 := \begin{cases} 
1 & \text{if } \sigma = \sigma_2 \\
-1 & \text{if } \sigma = \sigma_1, \sigma_4 ,
\end{cases}
\]
\[
\epsilon_2 := 1,
\]
then $\delta_t^{-1}(i) \in \{e^1, 2^2\}^*$ for any $t = 1, 2, 4$, $i \in A$. Then we have the theorem which gives the relation between Rauzy fractals $X^{(i)}_t$ and $X^{(i^*)}_t$, $t = 1, 2, 4$.

**Theorem 4** For $t = 1, 2, 4$ and $i \in A$, the following equations hold:
\[
X^{(i^*)}_{\sigma_t} = \bigcup_{j \in A} \bigcup_{w_k(i) = i^*} (\delta \pi f(P_k(j)) + A_{\delta_t}^{-1} X^{(j)}_{\sigma_t}),
\]
\[
X^{(i^*)}_{\sigma_t} = \bigcup_{j \in A} \bigcup_{w_k(i) = i^*} (\pi \sigma f(S_k(j)) + A_{\delta_t}^{-1} X^{(j)}_{\sigma_t}),
\]
where $\delta_t^{-1}(i)$ is written as $\delta_t^{-1}(i) = w_1(i) \cdots w_k(i) \cdots w_{i^*}(i) = P_k(i) w_k(i) S_k(i)$. The unions are disjoint in the sense of Lebesgue measure. Moreover, $X^{(i^*)}_{\sigma_t}, X^{(i^*)}_{\sigma_t^*}, X_{\sigma_t}, X'_{\sigma_t}$ are interval.

Proof. Let us show the first equation for $\sigma = \sigma_2 = \delta_2^{-1} \circ \tau_2 \circ \delta_2$ in the case of (ii). The other cases can be proved analogously. By the definition of $X^{(i)}_{\sigma_t}$,
\[
X^{(i)}_{\sigma_t} = \lim_{n \to \infty} A^n_{\sigma} \pi \sigma f^* n(e_i, i^*)
\]
\[
= \lim_{n \to \infty} A^n_{\sigma} \pi \sigma f^* n (\delta^{-1})^* (e_i, i^*)
\]
\[
= \lim_{n \to \infty} A^n_{\sigma} \pi \sigma f^* n (\sum_{j \in A} \sum_{w_k(j) = i} (A_{\delta} (e_j + f(S_k(j))), j^*)
\]
\[
= \bigcup_{j \in A} \bigcup_{w_k(j) = i} \lim_{n \to \infty} A^n_{\sigma} \pi \sigma f^* n (e_j - A_{\delta} f(P_k(i), j^*).
\]

Put
\[
c_0 := \max_{i \in A} d_H \left( \pi \delta^* (o, i^*), \pi \sigma A_{\delta}^{-1} (o, i^*) \right),
\]
where $d_H$ is the Hausdorff metric. From the property of the Hausdorff metric,
\[
d_H \left( \pi \delta^* \pi^* n (o, i^*), \pi \sigma A_{\delta}^{-1} \pi^* n (o, i^*) \right) \leq c_0,
\]
\[
d_H \left( A^n_{\sigma} \pi \sigma \delta^* \pi^* n (o, i^*), A^n_{\sigma} \pi \sigma A_{\delta}^{-1} \pi^* n (o, i^*) \right) \leq c_0 |\lambda_{\sigma}'| n,
\]
where $\lambda_{\sigma}'$ is the eigenvalue of $A_{\sigma}$ with $|\lambda_{\sigma}'| < 1$. So
\[
\lim_{n \to \infty} d_H \left( A^n_{\sigma} \pi \sigma \delta^* \pi^* n (o, i^*), A^n_{\sigma} \pi \sigma A_{\delta}^{-1} \pi^* n (o, i^*) \right) = 0.
\]
By noticing the equality,
\[
A_{\delta}^{-1} \pi \sigma x = \pi \sigma A_{\delta}^{-1} x, \ x \in R^2
\]
if $\sigma = \delta^{-1} \circ \tau \circ \delta$, we have
\[
\lim_{n \to \infty} A^n_{\sigma} \pi \sigma f^* n (e_i - x, i^*) = \lim_{n \to \infty} A^n_{\sigma} \pi \sigma A_{\delta}^{-1} \pi^* n (e_i - x, i^*)
\]
\[
= \lim_{n \to \infty} A_{\delta}^{-1} A^n_{\sigma} \pi \tau \pi^* n (e_i - x, i^*)
\]
\[
= -\pi \sigma A_{\delta}^{-1} x + A_{\delta}^{-1} X^{(i)}_\tau.
\]
Therefore we have the set equation

\[ X^\sigma_{i} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} ( -\pi_{\sigma}f(P^{(j)}_k) + A^{-1}_\sigma X^\tau_{r} ) . \]

Next we show it is disjoint union and interval through \( A_{\delta}X^\sigma_{i} \). From Theorem 2,

\[
A_{\delta}X^\sigma_{i} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} ( -\pi_{\tau}A_{\delta}f(P^{(j)}_k) + X^\tau_{r} ) \\
= \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} \pi_{\tau}(-A_{\delta}f(P^{(j)}_k) + e_j, j^* + h) \\
= \pi_{\tau}(\delta^{-1})(e_i, i^*) + h
\]

for some \( h \in P_r \). It means we can obtain \( A_{\delta}X^\sigma_{i} \) after projection of \( (\delta^{-1})(e_i, i^*) \) by \( \pi_{\tau} \) and translation by \( h \). From the figure of \( (\delta^{-1})(e_i, i^*) \), the union in the equation are disjoint and \( A_{\delta}X^\sigma_{i} \) is an interval (see Figure 10). The other equation is shown analogously. \( \square \)

**Remark 4** From Figure 10, \( (\delta^{-1})(\mathcal{U}) \notin \mathcal{G}_r^\tau \) in the case of (i), (ii) for \( t = 1, 2 \). Therefore \( \sigma^*(\mathcal{U}) \) is not included in the stepped surface \( S_{\sigma} \). That is the reason why we take the different seeds to construct the stepped surface.

**Definition 4** The domain exchange transformations \( T_{\tau} \) on \( X_{\tau} \) for \( \tau = \tau_1, \tau_2, \tau_4 \) and \( T_{\sigma} \) on \( X_{\sigma} \) for \( \sigma = \sigma_1, \sigma_2, \sigma_4 \) are defined by

\[
T_{\tau} : X_{\tau} \rightarrow X_{\tau} \\
T_{\tau}(x) = x - \pi_{\tau}f(i) \text{ if } x \in X^\tau_{r_{i}}
\]

and

\[
T_{\sigma} : X_{\sigma} \rightarrow X_{\sigma} \\
T_{\sigma}(x) = x - \pi_{\sigma}f(i) \text{ if } x \in X^\sigma_{i}
\]

By the definitions of \( X^\tau_{r_{i}} \), \( \tau = \tau_1, \tau_2, \tau_4 \), and \( X^\sigma_{i} \), \( \sigma = \sigma_1, \sigma_2, \sigma_4 \),

\[
X^\tau_{r_{i}} - \pi_{\tau}f(i) = X^\tau_{r_{i}} - \pi_{\sigma}f(i) = X^\sigma_{i}
\]

therefore the domain exchange transformations are well-defined (see Figure 10).

**Theorem 5** The measurable dynamical system \((X_{\sigma}, T_{\sigma}, \mu)\) with Lebesgue measure \( \mu \) has \( \delta^{-1}\)-structure with respect to the pair of partitions \( \{X^\sigma_{i} \mid i \in \mathcal{A}\}, \{A_{\delta}^{-1}X^\tau_{r_{i}} \mid i \in \mathcal{A}\}\) for \( \sigma = \sigma_1, \tau = \tau_1, \delta = \delta_1, t = 1, 2, 4 \). Moreover, \((X_{\sigma_1}, T_{\sigma_1}, \mu)\) (resp. \((X_{\sigma_2}, T_{\sigma_2}, \mu), t = 2, 4\)) has \( \delta^{-1}_{t}\)-structure (resp. \( \delta_{t}^{-1}\tau_{t}^{-n}\)-structure) with respect to the pair of partitions \( \{X^\sigma_{i} \mid i \in \mathcal{A}\}, \{A_{\delta_1}^{-1}A_{\tau_1}^{-1}X^\tau_{r_{i}} \mid i \in \mathcal{A}\}\) (resp. \( \{X^\sigma_{i} \mid i \in \mathcal{A}\}, \{A_{\delta_1}^{-1}A_{\tau_1}^{-n}X^\tau_{r_{i}} \mid i \in \mathcal{A}\}\)) for any positive integer \( n \) (see Figure 10).

Proof. From the set equation

\[
X^\sigma_{i} = \bigcup_{j \in \mathcal{A}} \bigcup_{w_k^{(j)} = i} ( -\pi_{\sigma}f(P^{(j)}_k) + A^{-1}_\sigma X^\tau_{r_{i}} ) ,
\]

where \( \delta^{-1}(i) \) is written as \( \delta^{-1}(i) = w^{(i)}_1 \cdots w^{(i)}_k \cdots w^{(i)}_l \cdots = P^{(i)}_k w^{(i)}_k \mathcal{S}^{(i)}_k \).
The case (i):

The case (ii):

The case (iv):

Figure 10: \((\delta^{-1}_t)^*(\mathcal{U})\) and domain exchange transformations \(T_{\sigma_t}\) on Rauzy fractals \(X_{\sigma_t}^{(i\epsilon)}\), \(i \in \mathcal{A}, t = 1, 2, 4\)
The induced transformation $T\sigma|_{A_\delta^{-1}X_\tau}(x) = A_\delta^{-1} \circ T_\tau \circ A_\delta(x)$, $x \in X_\sigma$, 

the induced transformation $T\sigma|_{A_\delta^{-1}X_\tau}$ is conjugate to $T_\tau$. Recall that $T_{\tau_1}$ (resp. $T_{\tau_2}$, $t = 2, 4$) has $\tau_1$-structure (resp. $\tau_2^2$-structure) with respect to the pair of partitions $\{X_{\tau_1}^{(i)}| i \in A\}$, $\{A_{\tau_1}^{(i)}X_\tau^{(i)}| i \in A\}$ (resp. $\{X_{\tau_2}^{(i)}| i \in A\}$, $\{A_{\tau_2}^{2n}X_\tau^{(i)}| i \in A\}$) by Theorem 1. Thus the last part is proved.

One-sided sequence $\omega$ is called a fixed point for $\sigma$ if $\sigma(\omega) = \omega$; and $\omega$ is called a periodic point with period $n$ if $\sigma^n(\omega) = \omega$. The substitution $\tau_1$ has a fixed point and the alternative substitutions $\tau_t$, $t = 2, 4$ have periodic points of period 2. We denote the fixed point $\lim_{n \to \infty} \tau_1^n(2)$ by $\omega_{\tau_1}$, and the periodic points $\lim_{n \to \infty} \tau_t^{2n}(2)$, $t = 2, 4$ by $\omega_{\tau_t}$, where these limits exist in the sense of the product topology. Let us define the one-sided sequences

$$\omega_{\tau_1} := \lim_{n \to \infty} \sigma_1^n(2), \quad \omega_{\tau_2} := \lim_{n \to \infty} \sigma_2^n(2), \quad \omega_{\tau_4} := \lim_{n \to \infty} \sigma_4^{2n}(2).$$

These fixed point or periodic points $\omega_{\tau_t}$, $t = 1, 2, 4$ are given by

$$\omega_{\tau_t} = \delta_{\tau_t}^{-1}(\omega_{\tau_t}) \in \{1^\ast, 2^\ast\}^N.$$

The one-sided sequences $\omega_{\tau_1}, \omega_{\tau_2}, t = 1, 2, 4$ are written as

$$\omega_{\tau_t} = s_0s_1 \cdots s_k \cdots,$$

$$\omega_{\tau_t} = t_0t_1 \cdots t_k \cdots.$$

Since $\tau_t^{2n}(e_2, 2^*)$ includes $(e_2, 2^*)$, $t = 1, 2, 4$ for any positive integer $n$, and the origin point $o \in \pi_{\tau_t}(e_2, 2^*)$, so $o \in X_{\tau_1}^{(2)}$. The orbit of the origin point by $T_{\tau_t}$ is described by a fixed point or a periodic point of $\tau_t$ by Theorem 5.

**Corollary 1** For $\sigma = \sigma_1, \sigma_2, \sigma_4$,

$$T_{\sigma}^k(o) \in X_{\sigma}^{(s_k)}, \quad k = 0, 1, \cdots.$$

Finally we will see that Rauzy fractals related to $\sigma_t$, $t = 1, 2, 4$ are also given by the fixed point or the periodic point $\omega_{\tau_t}$ as we saw in Section 0.

For a substitution or an alternative substitution $\tau = \tau_t$, $t = 1, 2, 4$, put

$$Y_{\tau} := \{-\pi_{\tau}f(t_0t_1 \cdots t_k)| k \geq 0\},$$

$$Y_{\tau}^{(i)} := \{-\pi_{\tau}f(t_0t_1 \cdots t_{k-1})| k \geq 0, t_k = i\},$$

$$Y_{\tau}^{(i)} := \{-\pi_{\tau}f(t_0t_1 \cdots t_k)| k \geq 0, t_k = i\}.$$
Then the following equalities hold:

\[ X_\tau = \overline{Y_\tau}, \]
\[ X^{(i)}_\tau = X^{(i)}_{\tau_2} = \overline{Y^{(i)}_\tau}, \]
\[ X^{(i)}_{\tau} = X^{(i)}_{\tau_2} = \overline{Y^{(i)}_\tau}. \]

For automorphisms \( \sigma = \sigma_1, \sigma_2, \sigma_4 \), we have the same result.

**Theorem 6** For \( \sigma = \sigma_1, \sigma_2, \sigma_4 \) and \( i \in A \), put

\[ Y_\sigma := \{-\pi_\sigma f(s_0s_1 \cdots s_k) | k \geq 0\} \]
\[ Y^{(i)}_\sigma := \{-\pi_\sigma f(s_0s_1 \cdots s_{k-1}) | k \geq 0, s_k = i^\epsilon \} \]
\[ Y^{(i)}_{\sigma} := \{-\pi_\sigma f(s_0s_1 \cdots s_k) | k \geq 0, s_k = i^\epsilon \}. \]

Then the following equalities hold:

\[ X_\sigma = Y_\sigma, \quad X^{(i)}_\sigma = Y^{(i)}_\sigma, \quad X^{(i)}_{\sigma} = Y^{(i)}_{\sigma}, \quad i \in A. \]

Proof. By Theorem 4, to prove the equality \( X^{(i)}_{\sigma} = Y^{(i)}_{\sigma} \), it is enough to show that

\[ Y^{(i)}_{\sigma} = \bigcup_{j \in A} \bigcup_{w_{k}^{(i)} = i^\epsilon} (-\pi_\sigma f(P^{(j)}_{k}) + A^{-1}Y^{(j)}_{\tau}), \]

where \( \delta^{-1}(i) = w_{k}^{(i)} = w_{k}^{(i)} \cdots w_{k}^{(i)} = P^{(i)}_{k}w_{k}^{(i)}S^{(i)}_{k}, \quad i \in A \). Take \( -\pi_\sigma f(s_0s_1 \cdots s_{k-1}) \in Y^{(i)}_{\sigma} \) such that \( s_k = i^\epsilon \). There exist \( k_1, k_2 \) such that

\[ s_0s_1 \cdots s_{k-1} = \delta^{-1}(t_0t_1 \cdots t_{k_1-1})P^{(t_{k_1})}_{k_2}, \quad w_{k_2}^{(t_{k_1})} = i^\epsilon. \]

Then

\[ -\pi_\sigma f(s_0s_1 \cdots s_{k-1}) = -\pi_\sigma (A^{-1}_0 f(t_0t_1 \cdots t_{k_1-1}) + f(P^{(t_{k_1})}_{k_2})) \]
\[ = -\pi_\sigma f(P^{(t_{k_1})}_{k_2}) - A^{-1}_0 \pi_\sigma f(t_0t_1 \cdots t_{k_1-1}), \]

and \( -\pi_\sigma f(P^{(t_{k_1})}_{k_2}) \in \bigcup_{j \in A} \bigcup_{w_{k}^{(i)} = i^\epsilon} (-\pi_\sigma f(P^{(j)}_{k}) + A^{-1}Y^{(j)}_{\tau}). \) Therefore we have

\[ Y^{(i)}_{\sigma} \subset \bigcup_{j \in A} \bigcup_{w_{k}^{(i)} = i^\epsilon} (-\pi_\sigma f(P^{(j)}_{k}) + A^{-1}Y^{(j)}_{\tau}). \]

The opposite inclusive relation can be shown easily. \( \square \)

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