Reconstruction of the measurable sets in the two dimensional plane by two projections

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Abstract. We discuss the reconstruction of the measurable plane sets from their two projections. Reconstruction of a measurable plane set $F \subset \mathbb{R}^2$ with $\lambda_2(F) < \infty$ from its orthogonal projections is well studied, where $\lambda_i$ is the Lebesgue measure on $\mathbb{R}^i$, $i = 1, 2$. In this paper we first discuss generalization of the known results in the frame of general two projections. The main purpose is to study, for any measurable plane set $F$, whether there are suitable two angles such that $F$ is uniquely reconstructed from the pair of its projections in that two directions. We give an example to show that this problem is negatively solved.

1. Introduction
In this paper, we discuss the reconstruction of the measurable plane sets from their two projections. Let us consider a problem to reconstruct the characteristic function of a measurable plane set $F$ from its line integrals. Since this is a problem in the Radon transform, if we have the all line integrals then the reconstruction is an easy corollary of the results in the Radon transform. Our object, however, is too simple to take its all line integrals. It is better to reconstruct the object with as less data as possible. In this paper, we try the reconstruction with line integrals only in the two directions. Though this problem is well studied for the case where these two projections are orthogonal, few sets can be reconstructed in such a frame. Therefore, in this paper, we take projections in two directions which are not necessarily orthogonal in the reconstruction of the measurable plane sets. Let us formulate our problem.

Definition 1.1. Let $F \subset \mathbb{R}^2$ be a measurable plane set such that $\lambda_2(F) < \infty$, where $\lambda_i$ is the Lebesgue measure on $\mathbb{R}^i$, $i = 1, 2$. Let $f(x, y)$ be the characteristic function of $F$. For $\alpha, \beta$ satisfying $-\pi/2 \leq \alpha + \beta \leq \pi/2$, $-\pi/2 < \alpha - \beta < \pi/2$, we rotate the $x$-axis by the angle $\alpha$ and $y$-axis by the angle $\beta$, which we call the $x(\alpha)$-axis and the $y(\beta)$-axis respectively. Define the coordinates

$$(x, y)(\alpha, \beta) = (x \cos \alpha - y \sin \beta, x \sin \alpha + y \cos \beta), \quad (1)$$

where the coordinates in the right hand side are in the usual frame of the orthogonal coordinates. We call this generalized frame $(\cdot, \cdot)(\alpha, \beta)$ the frame of the $(\alpha, \beta)$ coordinates. For such $\alpha, \beta$, we also define

$${f_1}^{(\alpha, \beta)}(y') := \int_{-\infty}^{\infty} f(-y' \sin \beta + t \cos \alpha, y' \cos \beta + t \sin \alpha) dt,$$

$${f_2}^{(\alpha, \beta)}(x') := \int_{-\infty}^{\infty} f(x' \cos \alpha - t \sin \beta, x' \sin \alpha + t \cos \beta) dt. \quad (2)$$
Figure 1. The $（\alpha, \beta）$ projections

We call these projections $(f_1^{（\alpha, \beta）}, f_2^{（\alpha, \beta）})$ the $(\alpha, \beta)$ projections.

Remark that if $\alpha = \beta \neq 0$, then the projections $(f_1^{（\alpha, \alpha）}, f_2^{（\alpha, \alpha）})$ are orthogonal, but we only call them the $(\alpha, \alpha)$ projections. In this paper, by “the orthogonal projections”, we mean only the $(0, 0)$ projections.

The reconstruction problem of the measurable plane sets from their $(\alpha, \beta)$ projections is as follows.

Problem 1.1. Given two non-negative, integrable functions $P(x)$ and $Q(y)$ having the same $L^1$ norm, find a measurable plane set $F$ with characteristic function $f(x, y)$ such that $P(x) = f_2^{（\alpha, \beta）}(x)$ and $Q(y) = f_1^{（\alpha, \beta）}(y)$ almost everywhere.

As an example of the application of Problem 1.1, consider a homogeneous object in the three dimensional space which contains a hole in its interior. We would like to detect the hole without destructing the object itself. Study this problem on the section by a two dimensional plane and apply the X-ray tomography for the reconstruction. Then the results in Problem 1.1 are applied to solve this problem.

Problem 1.1 is well studied for the orthogonal case where $(\alpha, \beta) = (0, 0)$. It was G.G. Lorentz [6] who first studied Problem 1.1 for $(\alpha, \beta) = (0, 0)$ in 1949, in view of which, let us call this problem the Lorentz X-ray problem. There are a number of studies on this problem for $(\alpha, \beta) = (0, 0)$ ([1, 2, 4, 5, 6, 8] and so on).

In the next section, we review these known results on Problem 1.1 for $(\alpha, \beta) = (0, 0)$. If we study Problem 1.1 only in the frame of the orthogonal coordinates, then we can reconstruct only a few sets. Most sets are non-unique in this frame. Therefore there arises the necessity to study Problem 1.1 in the frame of the general coordinates which are not necessary orthogonal.
In the third section, we discuss this generalization of the frame in the coordinates. In the frame of the generalized coordinates, many non-unique sets in the frame of orthogonal coordinates are made uniquely reconstructed by their two projections in a suitable frame, in view of which, it is natural to pose the following problem.

**Problem 1.2.** For any measurable plane set $F$, are there any pair of angles $\alpha$ and $\beta$ such that $F$ is uniquely reconstructed from the pair of projections $f_1^{(\alpha,\beta)}$ and $f_2^{(\alpha,\beta)}$? It is our main theorem to give an answer to this problem. We are very sorry that the answer to Problem 1.2 is negative. We prove the existence of the non-unique set by any pair of its projections in the fourth section (cf. Theorem 4.1 below), which is our main theorem in this paper. In the last section, we conclude the conclusion and introduce the important open problems left to be solved for further development.

Throughout this paper, unless mentioned otherwise, all discussions are made up to the sets of the measure zero.

### 2. Known results

In this section, we review the known results on Problem 1.1 for $(\alpha, \beta) = (0, 0)$ which are related to our main theorem in this paper. For $(\alpha, \beta) = (0, 0)$, we omit to write “$(0,0)$”, for example, we denote $f_1 := f_1^{(0,0)}$ and so on.

**Definition 2.1.** For non-negative integrable functions $f_1$ and $f_2$ on $\mathbb{R}$, define the projection of $f_1$ by

$$f_{12}(x) := \lambda_1(\{y \mid f_1(y) \geq x\})$$

for $x \geq 0$ and the projection of $f_2$ by

$$f_{21}(y) := \lambda_1(\{x \mid f_2(x) \geq y\})$$

for $y \geq 0$. Similarly, the functions $f_{121}$ and $f_{212}$ are defined respectively as

$$f_{121}(y) := \lambda_1(\{x \mid f_{12}(x) \geq y\}),$$

$$f_{212}(x) := \lambda_1(\{y \mid f_{21}(y) \geq x\}).$$

The answer to Problem 1.1 for the orthogonal case was classified by G.G. Lorentz into the three cases. The following theorem is a reformulation by A. Kuba and A. Volčič [4] of the theorem by G.G. Lorentz [6].

**Theorem 2.1.** (cf. [4] and [6]) Let $f_1(y)$ and $f_2(x)$ be non-negative integrable functions satisfying

$$\int_{-\infty}^{\infty} f_1(y)dy = \int_{-\infty}^{\infty} f_2(x)dx.$$  \hfill (7)

(i) The unique case.

There exists a unique set which has $(f_1, f_2)$ as the orthogonal projections if and only if

$$\int_0^c f_{12}(x)dx = \int_0^c f_{212}(x)dx,$$

for any $c > 0$. \hfill (8)

(ii) The non-unique case.

There exist non-unique sets having $(f_1, f_2)$ as the orthogonal projections if and only if

$$\int_0^c f_{12}(x)dx \geq \int_0^c f_{212}(x)dx,$$

for any $c > 0,$ \hfill (9)

and there is a constant $c > 0$ for which the strict inequality holds.
(iii) **The inconsistent case.**

There exists no set having \( (f_1, f_2) \) as the orthogonal projections if and only if

\[
\int_0^c f_{12}(x)dx < \int_0^c f_{212}(x)dx, \text{ for some } c > 0.
\]  

(10)

In the next section, we generalize this theorem in the frame of the general coordinates (Theorem 3.1 below), which serves as a key lemma to prove our main theorem.

We introduce the following theorems (Theorems 2.2-2.8), since, by their generalizations (Theorems 3.2-3.8), we explain the importance of Problem 1.2. In 1988, A. Kuba and A. Volčič [4] gave a reconstruction formula for the uniquely determined sets.

**Theorem 2.2.** (cf. [4]) If a measurable set \( F \) is uniquely determined by the pair of its orthogonal projections \( (f_1, f_2) \), then

\[
F = \{(x, y) \mid f_2(x) \geq f_{12}(f_1(y))\}.
\]  

(11)

In the same paper [4], they also gave a characterization of the non-uniquely determined sets. For a measurable set \( P \subset \mathbb{R}^2 \), define the parallel translation of \( P \) by

\[
P_{(s, t)} := \{(x, y) \mid (x - s, y - t) \in P\}.
\]  

(12)

A plane set \( F \) is called to have \( (P, P^{12}; P^1, P^2) \) as a switching component if there exist four sets \( P, P^1, P^2, P^{12} \) of the positive measure and two real numbers \( s, t \neq 0 \) such that

\[
P^1 = P_{(s, 0)}, \quad P^2 = P_{(0, t)}, \quad P^{12} = P_{(s, t)}, \quad (P \cup P^{12}) \subseteq F, \quad (P^1 \cup P^2) \cap F = \emptyset.
\]  

(13)

Let a set \( F \) have \( (f_1, f_2) \) as the orthogonal projections and \( (P, P^{12}; P^1, P^2) \) as a switching component. By to switch the switching components \( (P, P^{12}; P^1, P^2) \) in \( F \), we mean the procedure of making another set \( (F \cup P^1 \cup P^2) \setminus (P \cup P^{12}) \) which has the same orthogonal projections as \( F \).

**Theorem 2.3.** (cf. [4]) A measurable plane set having a finite measure is non-uniquely determined by its orthogonal projections if and only if it has a switching component.

They also studied the structure of non-uniquely determined sets (cf. [5]). In 1998, L. Huang and T. Takiguchi [1] proved the following theorem, which proves the \( L^1 \) stability in the reconstruction of the measurable plane sets from the pair of their orthogonal projections in the class of the uniquely determined bounded sets.

**Theorem 2.4.** (cf. [1]) Let \( f_1, f_2, g_1 \) and \( g_2 \) be non-negative, essentially bounded integrable functions. Assume that the pairs of the orthogonal projections, \( (f_1, f_2) \) and \( (g_1, g_2) \), uniquely determine the measurable plane sets \( F \) and \( G \) (with characteristic functions \( f(x, y) \) and \( g(x, y) \)), respectively. Then we have

\[
\|f - g\|_{L^1(\mathbb{R}^2)} \leq C \cdot \max\{\|f_1 - g_1\|_{L^\infty(\mathbb{R})}, \|f_2 - g_2\|_{L^\infty(\mathbb{R})}\},
\]  

(14)

where \( C = C(\|f_1\|_{L^\infty(\mathbb{R})}, \|g_1\|_{L^\infty(\mathbb{R})}, \|f_2\|_{L^\infty(\mathbb{R})}, \|g_2\|_{L^\infty(\mathbb{R})}) \) is a constant.

As an application of this theorem, they also gave an algorithm to construct approximate solutions for the uniquely determined sets from their orthogonal projections possibly containing noise and error. By these results, we can approximately reconstruct the solution from the orthogonal projections possibly containing noise and error if the set to be reconstructed is a priori known to be uniquely determined. In practical applications, however, it hardly happens that the set to be reconstructed is a priori known to be uniquely determined. Although it is possible to judge whether the pair determines a set uniquely, if the projections contain no error, it is impossible to obtain the projections without noise and error. In view of this argument, the best we can hope is to construct an approximate set of the original one from the pair of its orthogonal projections with noise and error. Therefore other problems arise.
Problem 2.1.

(i) Give a sufficient condition for a non-uniquely determined set $F$ from its orthogonal projections, to be approximately reconstructed from the pair of its orthogonal projections possibly containing noise and error.

(ii) Give an algorithm to approximate such non-unique sets.

Since we may assume the existence of the solutions in practical applications, what we have to study is the approximation of the non-unique solutions. A solution to Problem 2.1 was given in [8].

**Theorem 2.5.** (cf. [8]) Assume that a pair of orthogonal projections $(f_1, f_2)$ determines sets non-uniquely. Take a set $F$ whose orthogonal projections are $(f_1, f_2)$. Then any set having $(f_1, f_2)$ as the orthogonal projections is obtained by switching the switching components in $F$ countable times.

**Theorem 2.6.** (cf. [8]) Assume that a non-unique pair $(f_1, f_2)$ by the orthogonal projections, $f_1, f_2 \in L^1 \cap L^\infty(\mathbb{R})$, satisfies

$$\|f_{12} - f_{212}\|_{L^1(\mathbb{R})} < \varepsilon.$$  \hspace{1cm} (15)

Then there exists a uniquely determined set $G$ such that

$$\lambda_2(F \ominus G) < 2\varepsilon \min\{\|f_1\|_{L^\infty(\mathbb{R})}, \|f_2\|_{L^\infty(\mathbb{R})}\},$$  \hspace{1cm} (16)

where $F \ominus G$ is the symmetric difference of $F$ and $G$.

**Theorem 2.7.** (cf. [8]) Assume that two sets $F_1$ and $F_2$ have the same orthogonal projections $(f_1, f_2)$ which satisfy (15). Then there holds

$$\lambda_2(F_1 \ominus F_2) \leq 4\varepsilon \min\{\|f_1\|_{L^\infty(\mathbb{R})}, \|f_2\|_{L^\infty(\mathbb{R})}\}.$$  \hspace{1cm} (17)

By virtue of Theorems 2.6 and 2.7, if the orthogonal projections possibly containing noise and error satisfy (15) then we obtain an approximate solution by approximating the uniquely determined set $G$ in Theorem 2.6. As was mentioned in [8], the construction of $G$ and its orthogonal projections $(g_1, g_2)$ is not difficult. Let

$$g_{212}(x) := \min\{f_{12}(x), f_{212}(x)\},$$  \hspace{1cm} (18)

and

$$\varphi_{f_i}(x) := \lambda_1(\{\xi | f_i(\xi) > f_i(x)\}) + \lambda_1(\{\xi | f_i(\xi) = f_i(x), \xi > x\}),$$  \hspace{1cm} (19)

for $i = 1, 2$. By Propositions 1-3 in [7] (confer also [4]), the function $\varphi_f$ is measure preserving (for the definition of the term “measure preserving”, confer also [7]) and satisfies $f_{212}(\varphi_{f_2}(x)) \equiv f_2(x)$. We define

$$g_2(x) := g_{212}(\varphi_{f_2}(x)).$$  \hspace{1cm} (20)

By the definition, $g_{212}$ is the non-increasing rearrangements of $g_2$ (confer [4, 7], for the definition of the rearrangements). In the same way, we define

$$g_{121}(y) := \min\{f_{21}(y), f_{121}(y)\},$$  \hspace{1cm} (21)

and

$$g_1(y) := g_{121}(\varphi_{f_1}(y)).$$  \hspace{1cm} (22)

By the definition, we obtain $g_{212} \equiv g_2$ a.e., hence the pair of the orthogonal projections $(g_1, g_2)$ determines the set $G$ uniquely. Thus we obtain the unique set $G$ and the pair of its orthogonal projections $(g_1, g_2)$.
For the approximation of the set \(G\), we can apply the algorithm by L. Huang and T. Takiguchi since the set \(G\) is uniquely determined by the pair of its orthogonal projections. Note that no a priori information on the set itself is required in this approximation. Therefore, we obtain the following theorem.

**Theorem 2.8.** (an answer to Problem 2.1) (cf. [8])
In order to approximately reconstruct the non-unique sets whose orthogonal projections satisfy (15), we have only to approximate the set \(G\) defined above. Since \(G\) is uniquely determined by the pair of its orthogonal projections, we can apply the algorithm by Huang-Takiguchi to approximate it. Note that no a priori information on the set itself is required in this approximation.

3. Generalizations in the frame of the generalized coordinates
In the previous section, we reviewed the known results in the frame of the orthogonal coordinates, one of which claims that if the orthogonal projections satisfy (15) then we can reconstruct the parallelogram

\[
F := \{(x, y) \mid y < x < y + 1, \ 0 < y < 1\}. \tag{23}
\]

This set would not satisfy (15) for small \(\varepsilon > 0\). This set is a typical example of the non-uniquely determined ones in the frame of the orthogonal coordinates, however, we can reconstruct it uniquely from its two projections if we rotate the \(y\)-axis by \(-\pi/4\). As well as the example (23), the set

\[
F := \{0 < x < 1, \ 0 < y < 1\} \cup \{1 < x < 2, \ 1 < y < 2\} \tag{24}
\]

is not unique in the orthogonal coordinates, but is made unique in the frame of the \((\pi/4, \pi/4)\) coordinates. For the uniqueness of these two examples in the suitable coordinates, confer Theorem 3.1 below. Besides of these examples, there are a number of the measurable plane sets which are not uniquely reconstructed in the frame of the orthogonal coordinates, but are made unique by a suitable affine transform of the coordinates. By Theorem 2.3, we can claim that the essential character of the non-unique set is that they have the switching components. Therefore, in the examples (23) and (24), we claim that some sets having the switching components in the frame of orthogonal coordinates would not have ones in the frame of the suitable coordinates (cf. Theorem 3.3 below). In this section, we study the reconstruction of the measurable plane sets from their two projections in the frame of such generalized coordinates. We note that A. Kuba [3] discussed the reconstruction of the measurable plane sets from their non-orthogonal two projections, however, our approach is different from his idea. He fixed the \(x\)-axis and studied the rotation in the \(y\)-axis, hence the frames of the type, namely, \((\pi/4, \pi/4)\) coordinates mentioned in the example (24) was not studied by his idea. In order to give generalizations of the theorems mentioned in the previous section, we prepare several definitions.

**Definition 3.1.** For \(f_{1}^{(\alpha, \beta)}, f_{2}^{(\alpha, \beta)}\) defined in Definition 1.1, we define the projections of them by

\[
\begin{align*}
 f_{12}^{(\alpha, \beta)}(x) := & \lambda_{1}(\{y \mid f_{1}^{(\alpha, \beta)}(y) \geq x\}), & f_{21}^{(\alpha, \beta)}(y) := & \lambda_{1}(\{x \mid f_{2}^{(\alpha, \beta)}(x) \geq y\}), \\
 f_{121}^{(\alpha, \beta)}(y) := & \lambda_{1}(\{x \mid f_{12}^{(\alpha, \beta)}(x) \geq y\}), & f_{212}^{(\alpha, \beta)}(x) := & \lambda_{1}(\{y \mid f_{21}^{(\alpha, \beta)}(y) \geq x\}).
\end{align*} \tag{25}
\]

**Definition 3.2.** Let us define the measure in the \((\alpha, \beta)\) coordinates.

\[
\int_{-\infty}^{\infty} f^{(\alpha, \beta)}(\tau) d\tau_{(\alpha, \beta)} := \int_{-\infty}^{\infty} f^{(\alpha, \beta)}(t) \cos(\beta - \alpha) dt. \tag{26}
\]
**Definition 3.3.** For a measurable set \( P \subset \mathbb{R}^2 \), we define the \((\alpha, \beta)\) parallel translation of \( P \) by
\[
P_{(s,t)}^{(\alpha,\beta)} := \{(x,y)(\alpha,\beta)|(x-s,y-t)(\alpha,\beta) \in P\}.
\]

**Definition 3.4.** A measurable plane set \( F \) is defined to have \((P, P_{12}; P^1, P^2)\) as an \((\alpha, \beta)\) switching component if \( P, P^1, P^2, P_{12} \) are of the positive measure and if there exist two real numbers \( s, t \neq 0 \) such that
\[
P^1 = P_{(s,0)}^{(\alpha,\beta)}, \quad P^2 = P_{(0,t)}^{(\alpha,\beta)}, \quad P_{12} = P_{(s,t)}^{(\alpha,\beta)} \quad (P \cup P_{12} \subseteq F, (P^1 \cup P^2) \cap F = \emptyset)
\]
hold. By to switch the \((\alpha, \beta)\) switching components \((P, P_{12}; P^1, P^2)\) in \( F \), we mean the procedure of making another set
\[
\tilde{F} := (F \cup P^1 \cup P^2) \setminus (P \cup P_{12})
\]
which has the same \((\alpha, \beta)\) projections as \( F \).

We claim that all the theorems in the previous section are generalized in the frame of the \((\alpha, \beta)\) coordinates.

**Theorem 3.1.** Let \( f_1^{(\alpha,\beta)}(y) \) and \( f_2^{(\alpha,\beta)}(x) \) be non-negative, integrable functions such that
\[
\int_{-\infty}^{\infty} f_1^{(\alpha,\beta)}(y)dy(\alpha,\beta) = \int_{-\infty}^{\infty} f_2^{(\alpha,\beta)}(x)dx(\alpha,\beta).
\]

(i) The unique case.
There exists a unique set which has \((f_1^{(\alpha,\beta)}), f_2^{(\alpha,\beta)}\) as the \((\alpha, \beta)\) projections if and only if
\[
\int_{0}^{c} f_{12}^{(\alpha,\beta)}(x)dx(\alpha,\beta) = \int_{0}^{c} f_{212}^{(\alpha,\beta)}(x)dx(\alpha,\beta), \text{for any } c > 0.
\]

(ii) The non-unique case.
There exist non-unique sets having \((f_1^{(\alpha,\beta)}), f_2^{(\alpha,\beta)}\) as the \((\alpha, \beta)\) projections if and only if
\[
\int_{0}^{c} f_{12}^{(\alpha,\beta)}(x)dx(\alpha,\beta) \geq \int_{0}^{c} f_{212}^{(\alpha,\beta)}(x)dx(\alpha,\beta), \text{for any } c > 0,
\]
and there is a constant \( c > 0 \) for which the strict inequality holds.

(iii) The inconsistent case.
There exists no set having \((f_1^{(\alpha,\beta)}), f_2^{(\alpha,\beta)}\) as the \((\alpha, \beta)\) projections if and only if
\[
\int_{0}^{c} f_{12}^{(\alpha,\beta)}(x)dx(\alpha,\beta) < \int_{0}^{c} f_{212}^{(\alpha,\beta)}(x)dx(\alpha,\beta), \text{for some } c > 0.
\]

In order to prove this theorem, we modify the proof in [6]. For this modification is very long and contains some important properties in the reconstruction, we divide it into several lemmas.

**Lemma 3.1.** If \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) are the \((\alpha, \beta)\) projections of some measurable plane set \( F \) then (32) holds.

**Proof.** Let us define
\[
\begin{align*}
F_x^{(\alpha,\beta)} & := \{(x,y)(\alpha,\beta) \mid 0 < y < f_{12}^{(\alpha,\beta)}(x)\}, & F_y^{(\alpha,\beta)} & := \{(x,y)(\alpha,\beta) \mid 0 < x < f_{12}^{(\alpha,\beta)}(y)\}, \\
F_{xy}^{(\alpha,\beta)} & := \{(x,y)(\alpha,\beta) \mid 0 < y < f_{212}^{(\alpha,\beta)}(x)\}, & F_{yx}^{(\alpha,\beta)} & := \{(x,y)(\alpha,\beta) \mid 0 < x < f_{212}^{(\alpha,\beta)}(y)\}.
\end{align*}
\]
The set $F_y^{(\alpha, \beta)}$ defined in (34) has $(f_1^{(\alpha, \beta)}, f_2^{(\alpha, \beta)})$ as its $(\alpha, \beta)$ projections.

We first prove that

$$\int_I f_2^{(\alpha, \beta)}(x)dx < \int_0^k f_2^{(\alpha, \beta)}(x)dx \leq \int_0^k f_2^{(\alpha, \beta)}(x)dx \leq \int_0^k f_2^{(\alpha, \beta)}(x)dx$$

for any interval $I \subset \mathbb{R}$, $\lambda_1(I) \leq k$. Let

$$\hat{F} := F \cap \{(x, y)_{(\alpha, \beta)} \mid x \in I\}, \quad \hat{G} := F_y^{(\alpha, \beta)} \cap \{(x, y)_{(\alpha, \beta)} \mid 0 < x < k\}.$$ (36)

We note that what we want to prove is equivalent to $\lambda_2(\hat{F}) \leq \lambda_2(\hat{G})$. Denote by $(\tilde{f}_1^{(\alpha, \beta)}, \tilde{f}_2^{(\alpha, \beta)})$ the $(\alpha, \beta)$ projections of $\hat{F}$ and by $(\hat{f}_1^{(\alpha, \beta)}, \hat{f}_2^{(\alpha, \beta)})$ the $(\alpha, \beta)$ projections of $\hat{G}$. By their definitions, $\hat{f}_1^{(\alpha, \beta)}(y) \leq \tilde{f}_1^{(\alpha, \beta)}(y)$ and $\hat{f}_2^{(\alpha, \beta)}(y) \leq k$, hence

$$\tilde{f}_1^{(\alpha, \beta)}(y) \leq \min(\hat{f}_1^{(\alpha, \beta)}(y), k) = \hat{f}_1^{(\alpha, \beta)}(y).$$ (37)

Therefore, we obtain

$$\lambda_2(\hat{F}) = \int_0^\infty \tilde{f}_1^{(\alpha, \beta)}(y)dy_{(\alpha, \beta)} \leq \int_0^\infty \hat{f}_1^{(\alpha, \beta)}(y)dy_{(\alpha, \beta)} = \lambda_2(\hat{G}),$$ (38)

which proves (35) for any interval $I \subset \mathbb{R}$, $\lambda_1(I) \leq k$. Since the interval $I \subset \mathbb{R}$, $\lambda_1(I) \leq k$ is arbitrary, we also have for the the non-increasing rearrangement $f_2^{(\alpha, \beta)}$ of $f_2^{(\alpha, \beta)}$,

$$\int_0^k f_2^{(\alpha, \beta)}(x)dx \leq \int_0^k f_2^{(\alpha, \beta)}(x)dx \leq \int_0^k f_2^{(\alpha, \beta)}(x)dx$$

for any $k > 0$, which is (32).

\[\square\]

**Lemma 3.2.** For any pair of the functions $(f_1^{(\alpha, \beta)}, f_2^{(\alpha, \beta)})$ satisfying (32), there exists a measurable set $F$ having $(f_1^{(\alpha, \beta)}, f_2^{(\alpha, \beta)})$ as its $(\alpha, \beta)$ projections.

**Proof.** For $x > 0$, $y > 0$, we define

$$D := \{(x, y)_{(\alpha, \beta)} \mid y < f_1^{(\alpha, \beta)}(x), y < f_1^{(\alpha, \beta)}(x)\},$$

$$D^+ := \{(x, y)_{(\alpha, \beta)} \mid f_1^{(\alpha, \beta)}(x) > f_2^{(\alpha, \beta)}(x)\},$$

$$D^- := \{(x, y)_{(\alpha, \beta)} \mid f_2^{(\alpha, \beta)}(x) < f_2^{(\alpha, \beta)}(x)\}.$$ (40)

We first prove that $D^+$ and $D^-$ are represented as

$$D^+ = \sum L^+_i, \quad D^- = \sum L^-_i,$$ (41)

where $i = 1, 2, \cdots, L^+_i$’s and $L^-_i$’s are the *open net lozenges* of the kind

$$\{(x, y)_{(\alpha, \beta)} \mid 2^{-m}l_x < x < 2^{-m}(l_x + 1), \quad 2^{-m}l_y < y < 2^{-m}(l_y + 1)\}.$$ (42)

for $l_x, l_y, m = 0, 1, 2, \cdots, L^+_i = L^-_i$, $L^-_i = L^+_i \cap \{x, y\}$ for some $s > 0$, $t > 0$ and $\lambda_2(L^+_i) \geq \lambda_2(L^-_i) \geq \cdots$. By the definition, $\lambda_2(D^+)$ and $\lambda_2(D^-)$ must be equal. We assume that $\lambda_2(D^+) = \lambda_2(D^-) > 0$, otherwise we have nothing to do for the representation of $D^+$ and $D^-$. 

\[\square\]
Remark that (32) is equivalent to
\[ \lambda_2(D^+ \cap \{(x, y)_{(\alpha, \beta)} \mid 0 < x < k\}) \leq \lambda_2(D^- \cap \{(x, y)_{(\alpha, \beta)} \mid 0 < x < k\}), \]
for any \( k > 0 \). We also note that (32) is represented in another form
\[ \int_0^c f_{211}(y) dy_{(\alpha, \beta)} \geq \int_0^c f_{121}(y) dy_{(\alpha, \beta)}, \text{for any } c > 0, \]
which is equivalent to
\[ \lambda_2(D^- \cap \{(x, y)_{(\alpha, \beta)} \mid 0 < y < k\}) \leq \lambda_2(D^+ \cap \{(x, y)_{(\alpha, \beta)} \mid 0 < y < k\}), \]
for any \( k > 0 \).

In order to obtain the representation of \( D^+ \) and \( D^- \), we first represent \( D^- \) as \( \sum L_i \), where \( L_i \)'s are open net lozenges. By virtue of the assumption \( \lambda_2(D^+) = \lambda_2(D^-) > 0 \) and (32), we can take \( L_0 \) lying in the strip \( \{(x, y)_{(\alpha, \beta)} \mid k^- < y < l^- \} \) such that \( D^+ \cap \{(x, y)_{(\alpha, \beta)} \mid k^- < y < l^- \} = \emptyset \). By (45), there is a \( k^+ < k^- < k \) such that \( \lambda_2(D^+ \cap \{(x, y)_{(\alpha, \beta)} \mid k^- < y < k^- \}) = \lambda_2(L_1) \). For any pair of points \( (x_1, y_1)_{(\alpha, \beta)} \in L_1 \), \( (x_2, y_2)_{(\alpha, \beta)} \in D^+ \cap \{(x, y)_{(\alpha, \beta)} \mid k^- < y < k^- \} \), there are positive numbers \( s \), \( t \) such that \( x_1 = x_2 - s \), \( y_1 = y_2 + t \). Therefore we obtain the representation (41) for \( L_1 \) and \( D^+ \cap \{(x, y)_{(\alpha, \beta)} \mid k^- < y < k^- \} \) by decomposing them into suitable open net lozenges. We apply the same method to \( L_2, L_3, \ldots \) to obtain the representation (41) for \( D^+ \) and \( D^- \).

The set \( F_{xy}^{(\alpha, \beta)} \) defined in (34) has \( (f_{211}^{(\alpha, \beta)}) \) as its \( (\alpha, \beta) \) projections. We inductively construct the set \( F \) having \( (f_{121}^{(\alpha, \beta)}, f_{211}^{(\alpha, \beta)}) \) as its \( (\alpha, \beta) \) projections. Let
\[ L_i := \{(x, y)_{(\alpha, \beta)} \mid k_{x}^{i+} < x < l_{x}^{i+}, k_{y}^{i+} < y < l_{y}^{i+}\}, \]
for \( i = 1, 2, \ldots \).

As the first step, we make a set \( (F_{xy}^{(\alpha, \beta)})_1 \) by either of the following two procedures.
\[
\begin{align*}
(F_{xy}^{(\alpha, \beta)})_1 := (F_{xy}^{(\alpha, \beta)} \cup L_1) \setminus \{(x, y)_{(\alpha, \beta)} \mid k_{x}^{1+} < x < l_{x}^{1+}, k_{y}^{1+} < y < l_{y}^{1+}\}, \\
(F_{xy}^{(\alpha, \beta)})_1 := (F_{xy}^{(\alpha, \beta)} \cup \{(x, y)_{(\alpha, \beta)} \mid k_{x}^{1+} < x < l_{x}^{1+}, k_{y}^{1+} < y < l_{y}^{1+}\}) \setminus L_1^+.
\end{align*}
\]
We note that, for the first step, either of the procedures in (47) is possible and \( (F_{xy}^{(\alpha, \beta)})_1 \) has the same projection \( f_{211}^{(\alpha, \beta)}(x) \) as \( F_{xy}^{(\alpha, \beta)} \).

Let us assume that the \( 1, 2, \ldots, n - 1 \)-st steps are taken and we obtain the sets \( (F_{xy}^{(\alpha, \beta)})_1, \ldots, (F_{xy}^{(\alpha, \beta)})_{n-1} \), all of which have the same projection \( f_{212}^{(\alpha, \beta)}(x) \). Denote by \( ((f_1^{(\alpha, \beta)}), f_{212}^{(\alpha, \beta)}) \) the \( (\alpha, \beta) \) projections of \( (F_{xy}^{(\alpha, \beta)})_j \), \( j = 1, 2, \ldots, n - 1 \). Let \( r_n \) be the length of the sides of \( L_n^+ = L_n^- \). There hold by the definition that
\[ (f_1^{(\alpha, \beta)})_{n-1}(l_{x}^{n+}) \geq f_{212}^{(\alpha, \beta)}(l_{y}^{n+}) + r_n \geq f_{212}^{(\alpha, \beta)}(k_{y}^{n-}) + r_n, \]
\[ (f_1^{(\alpha, \beta)})_{n-1}(l_{x}^{n+}) \leq f_{212}^{(\alpha, \beta)}(l_{y}^{n+}) - r_n. \]
Therefore, the section \( (F_{xy}^{(\alpha, \beta)})_{n-1} \cap \{(x, y)_{(\alpha, \beta)} \mid y = l_{y}^{n+}\} \) contains at least \( [f_{212}^{(\alpha, \beta)}(l_{y}^{n+})/r_n] + 1 \) net-intervals of the length \( r_n \), where \([\cdot]\) is the Gauss bracket. In the same way, the section \( (F_{xy}^{(\alpha, \beta)})_{n-1} \cap \{(x, y)_{(\alpha, \beta)} \mid y = k_{y}^{n-}\} \) is contained in \( [f_{212}^{(\alpha, \beta)}(l_{y}^{n+})/r_n] \) net-intervals of the length \( r_n \). Therefore, there exists \( 0 < k_{x}^{n+} \) such that
\[ \{(x, y)_{(\alpha, \beta)} \mid k_{x}^{n+} < x < l_{x}^{n}, k_{y}^{n+} < y < l_{y}^{n}\} \subset (F_{xy}^{(\alpha, \beta)})_{n-1}, \]
\[ \{(x, y)_{(\alpha, \beta)} \mid k_{x}^{n-} < x < l_{x}^{n}, k_{y}^{n-} < y < l_{y}^{n}\} \subset (F_{xy}^{(\alpha, \beta)})_{n-1} = \emptyset. \]
where \( l^n_x := k^n_x + r^n_x \). Therefore, as the \( n \)-th step, we construct
\[
((F^{(\alpha,\beta)}_{yxy})_n := (F^{(\alpha,\beta)}_{yxy})_{n-1} \cup \{ (x, y)_{(\alpha,\beta)} \mid k^n_x < x < l^n_x, k^n_y < y < l^n_y \} \\
\{ (x, y)_{(\alpha,\beta)} \mid k^n_x < x < l^n_x, k^n_y < y < l^n_y \}).
\] (50)

Note that the set \((F^{(\alpha,\beta)}_{yxy})_n\) also has the same projection \( f^{(\alpha,\beta)}_{1212} \) as \( F^{(\alpha,\beta)}_{yxy} \).

By this inductive procedure, we obtain the set \( \overline{F} \). Denote by \((\overline{f}^{(\alpha,\beta)}_{11}, \overline{f}^{(\alpha,\beta)}_{212})\) the \((\alpha, \beta)\) projections of \( F \).

Let us define the set \( F_x \) by
\[
\overline{F}_x := \{ (x, y)_{(\alpha,\beta)} \mid (\varphi \overline{f}^{(\alpha,\beta)}_{21}(x), y) \in F \},
\] (52)

where \( \varphi \) is defined in (19). \( \overline{F}_x \) has \((\overline{f}^{(\alpha,\beta)}_{11}, \overline{f}^{(\alpha,\beta)}_{21})\) as its \((\alpha, \beta)\) projections. Then we obtain the set \( F \) having \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) as its \((\alpha, \beta)\) projections by the following.

\[
F := \{ (x, y)_{(\alpha,\beta)} \mid (x, \varphi f^{(\alpha,\beta)}_{11}(y)) \in \overline{F}_x \},
\] (53)

which proves the lemma.

\textbf{Lemma 3.3.} If the pair of functions \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) satisfies (32) but not (31) then there exist non-unique sets having \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) as their \((\alpha, \beta)\) projections.

\textit{Proof.} By virtue of the previous lemma, we obtain the proof easily. In the first step in the proof of Lemma 3.2, two ways of the construction of the set \((F^{(\alpha,\beta)}_{yxy})_1\) in (47) are possible. By the following steps, these two \((F^{(\alpha,\beta)}_{yxy})_1\)’s yield different sets having the same \((\alpha, \beta)\) projections. Therefore, we have at least two sets having \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) as their \((\alpha, \beta)\) projections.\]

\textbf{Lemma 3.4.} Assume that a measurable set \( F \) has \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) as its pair of \((\alpha, \beta)\) projections. If for any \( y_1, y_2 \in \mathbb{R} \),
\[
f^{(\alpha,\beta)}_{11}(y_1) \leq f^{(\alpha,\beta)}_{11}(y_2) \iff F^{(\alpha,\beta)}[y_1] \subset F^{(\alpha,\beta)}[y_2],
\] (54)

where for fixed \( y' \in \mathbb{R} \),
\[
F^{(\alpha,\beta)}[y'] := \{ (x, y')_{(\alpha,\beta)} \mid (x, y')_{(\alpha,\beta)} \in F \},
\] (55)

then no set other than \( F \) has \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) as its pair of \((\alpha, \beta)\) projections.

\textit{Proof.} Assume that there exists another set \( F' \) than \( F \) which has \((f^{(\alpha,\beta)}_{11}, f^{(\alpha,\beta)}_{21})\) as its pair of \((\alpha, \beta)\) projections. We can choose \( \varepsilon > 0 \) and a set \( J \subset \mathbb{R} \), \( \lambda_1(J) > 0 \) such that
\[
\lambda_1 (F^{(\alpha,\beta)}[y_0] - F^{(\alpha,\beta)}[y_0]) > \varepsilon, \quad \text{for} \ y_0 \in J.
\] (56)
Take a point of continuity \( y_0 \in J \) of \( f_1^{(α,β)} \) and fix it, since we may assume the existence of such points. Let the sets \( I(y_0), C \) be

\[
I(y_0) := \{ x \in \mathbb{R} \mid (x, y_0)_{(α,β)} \in F \}, \quad C := \{ (x, y)_{(α,β)} \mid x \in I(y_0) \}.
\]

Then

\[
\lambda_2(F \cap C) = \int_{I(y_0)} f_2^{(α,β)}(x)dx_{(α,β)} = \lambda_2(F' \cap C),
\]

therefore

\[
\int_{-∞}^{∞} \lambda_1(C \cap F^{(α,β)}[y])dy_{(α,β)} = \int_{-∞}^{∞} \lambda_1(C \cap F^{(α,β)}[y])dy_{(α,β)}.
\]

If \( f_1^{(α,β)}(y) \leq f_1^{(α,β)}(y_0) \) then by (54) we have \( F^{(α,β)}[y] \subset C \), therefore

\[
\lambda_1(C \cap F^{(α,β)}[y]) = \lambda_1(F^{(α,β)}[y]) = \lambda_1(C \cap F^{(α,β)}[y]) \geq \lambda_1(C \cap F^{(α,β)}[y]).
\]

If \( f_1^{(α,β)}(y) \geq f_1^{(α,β)}(y_0) \) then by (54) we have \( F^{(α,β)}[y] \supset C^{(α,β)}[y_0] \), therefore

\[
\lambda_1(C \cap F^{(α,β)}[y]) = \lambda_1(C^{(α,β)}[y_0]) \geq \lambda_1(C \cap F^{(α,β)}[y]).
\]

Hence \( \lambda_1(C \cap F^{(α,β)}[y]) \geq \lambda_1(C \cap F^{(α,β)}[y]) \), which with (59) yields

\[
\lambda_1(C \cap F^{(α,β)}[y]) = \lambda_1(C \cap F^{(α,β)}[y]).
\]

Let us prove that (56) contradicts (62). For any \( ε > 0 \), there exists \( δ > 0 \) such that

\[
|\lambda_1(F^{(α,β)}[y]) - \lambda_1(F^{(α,β)}[y_0])| < \frac{ε}{2}, \quad \text{for } ∀y \in J, \ |y - y_0| < δ.
\]

Hence we have

\[
\lambda_1(F^{(α,β)}[y] \cap C) > \lambda_1(F^{(α,β)}[y_0]) - \frac{ε}{2}, \quad \text{for } ∀y \in J, \ |y - y_0| < δ,
\]

since if \( f_1^{(α,β)}(y) \geq f_1^{(α,β)}(y_0) \) then by (54), \( F^{(α,β)}[y] \cap C = F^{(α,β)}[y_0] \), otherwise \( F^{(α,β)}[y] \cap C = F^{(α,β)}[y] \). Let the set \( D \) be

\[
D := \{ (x, y)_{(α,β)} \mid (x, y)_{(α,β)} \in F \text{ or } (x, y_0)_{(α,β)} \in F \},
\]

then we have

\[
|\lambda_1(D^{(α,β)}[y] - F^{(α,β)}[y_0])| \leq |\lambda_1(F^{(α,β)}[y] - F^{(α,β)}[y_0])| < \frac{ε}{2},
\]

therefore there holds

\[
\lambda_1(D^{(α,β)}[y]) < \lambda_1(F^{(α,β)}[y_0]) + \frac{ε}{2}, \quad \text{for } ∀y \in J, \ |y - y_0| < δ.
\]

By (56) and the definition of \( D \), we have

\[
\lambda_1(D^{(α,β)}[y] - F^{(α,β)}[y_0]) = \lambda_1(D^{(α,β)}[y] - D \cap F^{(α,β)}[y]) \geq \lambda_1(F^{(α,β)}[y] - F^{(α,β)}[y]) > ε.
\]

Therefore, By (64), (67) and (68) we obtain

\[
\lambda_1(C \cap F^{(α,β)}[y]) \leq \lambda_1(D \cap F^{(α,β)}[y]) < \lambda_1(D^{(α,β)}[y]) - ε < \frac{ε}{2} < \lambda_1(C \cap F^{(α,β)}[y]),
\]

for any \( y \in J, \ |y - y_0| < δ \), which contradicts (62). Hence the lemma is proved.
Lemma 3.5. If a pair of \((\alpha, \beta)\) projections \(f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)}\) satisfies (31) then the set \(F\) having \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) as its \((\alpha, \beta)\) projections satisfies the condition (54).

Proof. Since we have, by the assumption, \(D^+ = D^- = \emptyset\) in (40), the set \(\overline{F}\) constructed in the proof of Lemma 3.2 is represented as \(\overline{F} = F_{xy}^{(\alpha,\beta)} = F_{yx}^{(\alpha,\beta)}\) which has \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) as its \((\alpha, \beta)\) projections. By its definition, \(\overline{F}\) satisfies the condition (54), therefore the set \(\overline{F}_x\) defined in (52) has the same property, so does the set \(F\) defined in (53).

Lemmas 3.1-3.5 prove Theorem 3.1. This theorem takes an important role to prove our main theorem in the next section (cf. the proof of Lemma 4.1 below). In view of Theorem 3.1, the set (23) is uniquely reconstructed in the frame of the \((0, -\pi/4)\) coordinates and so is the set (24) in the frame of the \((\pi/4, \pi/4)\) coordinates.

The following Theorems 3.2-3.8 are generalization of the theorems introduced in the second section. Though they are proved by modifying the proofs of the original theorems (Theorem 2.2-2.8), other proofs are possible, which we shall discuss in our another paper. In this paper, we would not mention the proofs of the following theorems (3.2-3.8), since there are neither our main purpose here nor applied to prove our main theorem. We, however, introduce them in order to explain the importance of Problem 1.2 (cf. also Problem 3.1 below).

Theorem 3.2. If a measurable set \(F\) is uniquely determined by the pair of its \((\alpha, \beta)\) projections \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\), then

\[
F = \{(x, y)_{(\alpha,\beta)}| f_2^{(\alpha,\beta)}(x) \geq f_1^{(\alpha,\beta)}(f_1^{(\alpha,\beta)}(y))\} = \{(x, y)_{(\alpha,\beta)}| f_1^{(\alpha,\beta)}(y) \geq f_2^{(\alpha,\beta)}(f_2^{(\alpha,\beta)}(x))\}.
\]

Theorem 3.3. A measurable plane set having a finite measure is non-uniquely determined by its \((\alpha, \beta)\) projections if and only if it has an \((\alpha, \beta)\) switching component.

Definition 3.5. For a measurable function \(f(x)\) on \(\mathbb{R}\), define

\[
\|f\|_{L^1_{(\alpha,\beta)}(\mathbb{R})} := \int_{\mathbb{R}}|f(x)| \cos(\alpha - \beta) dx,
\]

\[
\|f\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})} := \|f\|_{L^\infty(\mathbb{R})} \cos(\alpha - \beta).
\]

Theorem 3.4. Let \(f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)}, g_1^{(\alpha,\beta)}\) and \(g_2^{(\alpha,\beta)}\) be non-negative, essentially bounded integrable functions. Assume that the pairs of the projections, \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) and \((g_1^{(\alpha,\beta)}, g_2^{(\alpha,\beta)})\), uniquely determine the measurable plane sets \(F\) and \(G\) (with characteristic functions \(f(x, y)\) and \(g(x, y)\)), respectively in the \((\alpha, \beta)\) coordinate. Then we have

\[
\|f - g\|_{L^1(\mathbb{R}^2)} \leq C \cdot \max\{\|f_1^{(\alpha,\beta)} - g_1^{(\alpha,\beta)}\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})}, \|f_2^{(\alpha,\beta)} - g_2^{(\alpha,\beta)}\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})}\},
\]

where

\[
C = C(\|f_1^{(\alpha,\beta)}\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})}, \|g_1^{(\alpha,\beta)}\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})}, \|f_2^{(\alpha,\beta)}\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})}, \|g_2^{(\alpha,\beta)}\|_{L^\infty_{(\alpha,\beta)}(\mathbb{R})}).
\]

Theorem 3.5. Assume that a pair of the \((\alpha, \beta)\) projections \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) determines sets non-uniquely. Take a set \(F\) whose \((\alpha, \beta)\) projections are \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\). Then any set having \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) as the \((\alpha, \beta)\) projections is obtained by switching the \((\alpha, \beta)\) switching components in \(F\) countable times.
Theorem 3.6. Let \( f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)} \in L^1 \cap L^\infty(\mathbb{R}) \). Assume that the non-unique pair \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) of the \((\alpha,\beta)\) projections satisfies
\[
\|f_1^{(\alpha,\beta)} - f_2^{(\alpha,\beta)}\|_{L^1(\mathbb{R})} < \varepsilon. \tag{74}
\]
Then there exists an \((\alpha,\beta)\) uniquely determined set \( G \) such that
\[
\lambda_2(F \oplus G) < 2\varepsilon \min\{\|f_1^{(\alpha,\beta)}\|_{L^\infty(\mathbb{R})}, \|f_2^{(\alpha,\beta)}\|_{L^\infty(\mathbb{R})}\}, \tag{75}
\]
for any set \( F \) having \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) as the \((\alpha,\beta)\) projections.

Theorem 3.7. Assume that two sets \( F_1 \) and \( F_2 \) have the same \((\alpha,\beta)\) projections \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) which satisfy (74). Then there holds
\[
\lambda_2(F_1 \ominus F_2) \leq 4\varepsilon \min\{\|f_1\|_{L^\infty(\mathbb{R})}, \|f_2\|_{L^\infty(\mathbb{R})}\}. \tag{76}
\]

Theorem 3.8. We can approximately reconstruct the measurable plane set \( F \) whose \((\alpha,\beta)\) projections satisfy (74) by approximating the set \( G \) in Theorem 3.6. Since \( G \) is uniquely determined by the pair of its \((\alpha,\beta)\) projections, we can apply an algorithm by Huang-Takiguchi for its approximate reconstruction. In this algorithm, the original set is approximated by a direct sum of parallelograms. Note that no a priori information on the set itself is required in this approximation.

In view of these generalizations, if we find the angles \( \alpha \) and \( \beta \) satisfying the condition (31), then reconstruction is possible by Theorem 3.2. At least, if we find the angles \( \alpha \) and \( \beta \) satisfying the condition (74), in view of Theorems 3.4-3.8, approximate reconstruction is possible for the bounded sets.

It is a very interesting problem to find such angles \( \alpha \) and \( \beta \) for an unknown measurable set by some data of the projections, however, before studying this problem, it is necessary to study the following problem.

Problem 3.1. For any measurable plane set \( F \), do there exist directions \( \alpha \) and \( \beta \) such that (31) or (74) for small \( \varepsilon > 0 \) holds?

If this problem is positively solved, then (approximate) reconstruction of any measurable set by two projection is possible. Unfortunately, the answer to Problem 3.1 is negative. In the next section, we give an example of a measurable set which would satisfy neither (31) nor (74) for small \( \varepsilon > 0 \), for any directions \( \alpha \) and \( \beta \), which is our main purpose in this paper.

4. Main theorem
In this section, we prove that Problem 3.1 is negatively solved. The following is our main theorem in this paper.

Theorem 4.1. For any \( \delta > 0 \), there exists such a bounded measurable set \( F \) having \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) as the \((\alpha,\beta)\) projections that
\[
\|f_1^{(\alpha,\beta)} - f_2^{(\alpha,\beta)}\|_{L^1(\mathbb{R})} \geq \delta \tag{77}
\]
for any \( \alpha, \beta \) satisfying \(-\pi/2 \leq \alpha + \beta \leq \pi/2\), \(-\pi/2 < \alpha - \beta < \pi/2\).

For the preparation of the proof, we introduce a lemma.
Def. 4.1. Let
\[ M(\alpha, \beta) := \lambda_1(\{y \mid f_1^{(\alpha,\beta)}(y) > 0\}) \quad N(\alpha, \beta) := \|f_2^{(\alpha,\beta)}\|_{L^\infty(\mathbb{R})}. \] (78)

Lemma 4.1.

(i) If a pair of the \((\alpha, \beta)\) projections \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\), \(f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)} \in L^1 \cap L^\infty(\mathbb{R})\), determines a set uniquely then
\[ M(\alpha, \beta) = N(\alpha, \beta). \] (79)

(ii) If \(F\) is a bounded measurable plane set having \((f_1^{(\alpha,\beta)}, f_2^{(\alpha,\beta)})\) as the \((\alpha, \beta)\) projections then
\[ M(\alpha, \beta) \geq N(\alpha, \beta). \] (80)

Proof. The first statement is an easy corollary of Theorem 3.1 (i). By their definitions, the functions \(f_1^{(\alpha,\beta)}\) and \(f_2^{(\alpha,\beta)}\) have their values at the origin and there hold that \(\lambda_1(\{y \mid f_1^{(\alpha,\beta)}(y) > 0\}) = f_1^{(\alpha,\beta)}(0)\) and that \(\|f_2^{(\alpha,\beta)}\|_{L^\infty(\mathbb{R})} = f_2^{(\alpha,\beta)}(0)\). By virtue of (31), \(f_1^{(\alpha,\beta)}(0) = f_2^{(\alpha,\beta)}(0)\), which proves the first statement in the lemma.

The proof of the second statement depends on Theorem 3.1 (i) and (ii). The same idea to prove (i) is applied for the proof.

Let us prove our main theorem.

Proof of Theorem 4.1. We prove the theorem by constructing a counterexample. Let \(\varepsilon > 0\) be very small and define the set \(F\) as follows.
\[ F := \{(1, 2) \times \left( (1, 1 + \varepsilon) \cup (2 - \frac{\varepsilon}{2}, 2 + \frac{\varepsilon}{2}) \cup (3 - \varepsilon, 3) \right) \} \]
\[ \cup (1, 1 + \varepsilon) \times \left( 1, 2 + \frac{\varepsilon}{2} \right) \cup (2 - \varepsilon, 2) \times \left( 2 + \frac{\varepsilon}{2}, 3 \right). \] (81)

We claim that the set defined in (81) is the counterexample. We first find the pair \((\alpha, \beta)\) such that \(M(\alpha, \beta)\) is minimal for \(-\pi/2 \leq \alpha + \beta \leq \pi/2\), \(-\pi/2 < \alpha - \beta < \pi/2\).

For the set \(F\) defined in (81), the minimal point of \(M\) is \((\alpha, \beta) = (0, 0)\) and
\[ M(0, 0) = 2. \] (82)

It is easy to verify that for any line \(l\) in the plane
\[ \lambda_1(F \cap l) < 1 + 4\varepsilon, \] (83)
which implies that
\[ N(\alpha, \beta) < 1 + 4\varepsilon \] (84)
for any \(\alpha, \beta\) satisfying \(-\pi/2 \leq \alpha + \beta \leq \pi/2\), \(-\pi/2 < \alpha - \beta < \pi/2\). By virtue of (82), (84), if we take \(0 < \varepsilon < 1/10\), for example, then we obtain
\[ M(\alpha, \beta) - N(\alpha, \beta) > 1/2, \] (85)
for any \(\alpha, \beta\) satisfying \(-\pi/2 \leq \alpha + \beta \leq \pi/2\), \(-\pi/2 < \alpha - \beta < \pi/2\). Lemma 4.1 implies that \(F\) is non-uniquely reconstructed by any pair of the \((\alpha, \beta)\) projections. Therefore there exists such a constant \(\delta_0 > 0\) that
\[ \|f_1^{(\alpha,\beta)} - f_2^{(\alpha,\beta)}\|_{L^\infty(\mathbb{R})} \geq \delta_0 \] (86)
for any \(\alpha, \beta\) satisfying \(-\pi/2 \leq \alpha + \beta \leq \pi/2\), \(-\pi/2 < \alpha - \beta < \pi/2\). The theorem is proved by expanding the set (81) in accordance with the given value \(\delta > 0\).

The following theorem is a straightforward corollary of Theorem 4.1.

Theorem 4.2. There exist measurable plane sets which would not be uniquely reconstructed from any pair of their \((\alpha, \beta)\) projections, where \(-\pi/2 \leq \alpha + \beta \leq \pi/2\), \(-\pi/2 < \alpha - \beta < \pi/2\).
5. Conclusions and open problems
In this section, we mention the concluding remarks and the open problems left to be solved for further development. We first conclude our conclusions.

Conclusion 5.1.
(i) We have proved that the known results on the reconstruction of the measurable plane sets from the orthogonal projections are extended in the frame of the generalized coordinates.
(ii) We have also proved that Problem 3.1 is negatively solved, that is, it is not true that the method of our (approximate) reconstruction would be effective for the all measurable sets.

In view of our conclusions, we pose the following problems to be solved for further development.

Problem 5.1.
(i) Classify the all bounded measurable plane sets into the two classes. One is the class of the measurable plane sets for which there exist such angles \( \alpha \) and \( \beta \) that they are uniquely reconstructed from their \((\alpha, \beta)\) projections. The other one is the class of the measurable plane sets for which no such angles \( \alpha \) and \( \beta \) exist that they are uniquely reconstructed from their \((\alpha, \beta)\) projections.
(ii) Assume that for a measurable set \( F \), there exist such angles \( \alpha \) and \( \beta \) that \( F \) is uniquely reconstructed from \((f_1^{(\alpha, \beta)}, f_2^{(\alpha, \beta)})\). Find such angles \( \alpha \) and \( \beta \) from some data of the projections.
(iii) How much more data of the suitable projections would make the all measurable plane sets reconstructible?

The problem (i) in Problem 5.1 is a natural one. On this problem, the author has the following conjecture.

Conjecture 5.1. For any bounded convex set, there exist such angles \( \alpha \) and \( \beta \) that it is uniquely reconstructed by its \((\alpha, \beta)\) projections.

In this conjecture, the convexity on the set is assumed, which seems to be too strong since there are many non-convex sets which are uniquely reconstructed by their two projections. Anyway, it is important to solve the problem (i) in Problem 5.1 in view of both theoretic interest and practical applications.

The problem (ii) in Problem 5.1 is much more important for applications. For example, the solution of this problem enables us non-destructive reconstruction of the hole in a three dimensional homogeneous object with much less data than the usual X-ray transform.

As for the problem (iii) in Problem 5.1, it is known that arbitrary finite projections would determine not the all measurable sets, which was proved by G.G. Lorentz ([6]). In our problem, we pose that how many suitable projections would determine the all measurable plane sets uniquely, where the directions of the projections may change in accordance with the set, only the number of the projections is fixed. We have proved in this paper that the two projections would not determine the all measurable plane sets. In (iii) in Problem 5.1, we pose that for any measurable plane set whether the suitable three projections or more make it uniquely determined or not.
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