THE BEZOUT-CORONA PROBLEM REVISITED:
WIENER SPACE SETTING

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ABSTRACT. The matrix-valued Bezout-corona problem $G(z)X(z) = I_m$, $|z| < 1$, is studied in a Wiener space setting, that is, the given function $G$ is an analytic matrix function on the unit disc whose Taylor coefficients are absolutely summable and the same is required for the solutions $X$. It turns out that all Wiener solutions can be described explicitly in terms of two matrices and a square analytic Wiener function $Y$ satisfying $\text{det} Y(z) \neq 0$ for all $|z| \leq 1$. It is also shown that some of the results hold in the $H^\infty$ setting, but not all. In fact, if $G$ is an $H^\infty$ function, then $Y$ is just an $H^2$ function. Nevertheless, in this case, using the two matrices and the function $Y$, all $H^2$ solutions to the Bezout-corona problem can be described explicitly in a form analogous to the one appearing in the Wiener setting.

1. INTRODUCTION AND MAIN RESULTS

Let $G \in H^\infty_{m \times p}$, that is, $G$ is an $m \times p$ matrix function whose entries are $H^\infty$ functions on the open unit disc $\mathbb{D}$. The $H^\infty$-corona problem asks for a function $X \in H^\infty_{p \times m}$ such that

$$G(z)X(z) = I_m \quad (z \in \mathbb{D}).$$

This problem has its roots in the paper [3] for the case $m = 1$, and in [11] for the case $m > 1$. Since then it has been studied in various contexts for which we refer to the books [13, 16, 17, 18] and the recent papers [9, 10, 20, 21, 22]. See also the introduction of [3] for the role of equation (1.1) in mathematical systems and control theory problems. The problem is also closely related to the Leech problem [15] (see also the comments in [14]) where the identity matrix $I_m$ in the right hand side of (1.1) is replaced by another $H^\infty$ matrix function of appropriate size. In the Leech problem as well as in the corona problem norm constraints on the solution $X$ are often the main issue. When norm constraints are not the main issue one often refers to (1.1) as a Bezout problem in a $H^\infty$ setting.

We view the present paper as an addition to the papers [9] and [10] which deal with the Bezout-corona problem in the setting of stable rational matrix functions. Here we consider equation (1.1) in a Wiener space setting. We assume that $G$ belongs to the Wiener space $W^m_{+ \times p}$ and we look for solutions $X$ which belong to the Wiener space $W^p_{+ \times m}$. In other words, $G \in H^\infty_{m \times p}$ and $X \in H^\infty_{p \times m}$ and both

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have the additional property that their Taylor coefficients at zero are absolutely summable. In this case we refer to (1.1) as the Wiener-Bezout problem. We shall be interested in the description of all Wiener solutions and the least square Wiener solution. The Wiener-Bezout problem includes problem (1.1) for the case when \( G \) is a stable rational matrix function and the solution \( X \) is required to be stable rational matrix function too; see \[9\] and \[10\]. For more information on Wiener spaces we refer the reader to the final paragraph of this introduction.

Assuming \( G \in H^\infty_{m \times p} \), we shall also be interested in solutions \( X \) to (1.1) that belong to \( H^2_{p \times m} \), where \( H^2_{p \times m} \) stands for the linear spaces consisting of all \( p \times m \) matrices with entries in \( H^2 \). In that case we refer to (1.1) as the \( H^2 \)-Bezout problem.

Recall, cf., \[18\] Theorem 3.61 or \[3\] Section 2], that the \( H^\infty \)-corona problem is solvable if and only if \( T_G \) admits a right inverse. Here \( T_G \) is the analytic Toeplitz operator

\[
T_G = \begin{bmatrix}
G_0 & 0 & 0 & \cdots \\
G_1 & G_0 & 0 & \cdots \\
G_2 & G_1 & G_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{bmatrix} : \ell^2_+(\mathbb{C}^p) \rightarrow \ell^2_+(\mathbb{C}^m),
\]

where \( G_0, G_1, G_2, \ldots \) are the Taylor coefficients of \( G \) at zero. Note that \( T_G \) has a right inverse if and only if \( T_G^* T_G \) is strictly positive. Since \( W^\infty_{p \times m} \subseteq H^\infty_{p \times m} \), for the Wiener-Bezout problem to be solvable \( T_G^* T_G \) has to be strictly positive. We shall see that this condition is also sufficient and allows one to give a description of all solutions to the Wiener-Bezout problem in a simpler and more concrete form than for the general \( H^\infty \)-corona problem.

For our first main result we need to introduce two matrices \( \Xi_0 \) and \( \Theta_0 \), and a \( p \times p \) matrix function \( Y \) analytic on \( \mathbb{D} \) as follows. Let \( G \in H^\infty_{m \times p} \), and assume that \( T_G T_G^* \) is strictly positive. Then:

(M1) \( \Xi_0 \) is the \( p \times m \) matrix defined by \( \Xi_0 = E_p^* T_G^*(T_G T_G^*)^{-1} E_m \);

(M2) \( \Theta_0 \) is the \( p \times k \) matrix defined by

\[
(1.2) \quad \Theta_0 \Theta_0^* = I_p - E_p^* T_G^*(T_G T_G^*)^{-1} T_G E_p, \quad \text{Ker} \Theta_0 = \{0\}.
\]

Here for any positive integer \( n \) we write \( E_n \) for the canonical embedding of \( \mathbb{C}^n \) onto the first coordinate space of \( \ell^2_+(\mathbb{C}^n) \), that is,

\[
(1.3) \quad E_n = [I_n \ 0 \ 0 \ 0 \ \cdots ]^\top : \mathbb{C}^n \rightarrow \ell^2_+(\mathbb{C}^n).
\]

Since \( \text{Ker} \Theta_0 = \{0\} \), the integer \( k \) in item (b) is equal to the rank of the matrix \( I_p - E_p^* T_G^*(T_G T_G^*)^{-1} T_G E_p \). We shall see (Lemma 2.1 in the next section) that this rank is equal to \( p - m \), even in the \( H^\infty \) setting. Finally, we define \( Y \) to be the analytic \( p \times p \) matrix function on \( \mathbb{D} \) given by

\[
(1.4) \quad Y(z) = I_p - zE_p^* (I - zS_p^*)^{-1} T_G^*(T_G T_G^*)^{-1} H_G E_p \quad (z \in \mathbb{D}).
\]

Here for any positive integer \( n \) the operator \( S_n \) is the block forward shift on \( \mathbb{C}^n \). Furthermore, \( H_G \) is the Hankel operator defined by \( G \), that is,

\[
H_G = \begin{bmatrix}
G_1 & G_2 & G_3 & \cdots \\
G_2 & G_3 & G_4 & \cdots \\
G_3 & G_4 & G_5 & \cdots \\
\vdots & \vdots & \vdots & \ddots 
\end{bmatrix} : \ell^2_+(\mathbb{C}^p) \rightarrow \ell^2_+(\mathbb{C}^m).
\]
In other words, the Taylor coefficients of $Y_0, Y_1, Y_2, \cdots$ of $Y$ at zero are given by

$$Y_0 = I_p \quad \text{and} \quad \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \end{bmatrix} = -T_G^*(T_G T_G^*)^{-1} \begin{bmatrix} G_1 \\ G_2 \\ \vdots \end{bmatrix}. \quad (1.5)$$

Note that the operator $T_G^*(T_G T_G^*)^{-1}$ appearing in the definitions of the matrices $\Xi$ and $\Theta_0$ and the function $Y$ is the Moore-Penrose right inverse of $T_G$. In a less explicit form the function $Y$ already appears in the papers [9, 10]. The central role of this function is a new aspect of the present paper.

Finally, with the function $Y$ and the two matrices $\Xi$ and $\Theta_0$ we associate the following two functions

$$\Xi(z) = Y(z) \Xi_0 \quad \text{and} \quad \Theta(z) = Y(z) \Theta_0 \quad (z \in \mathbb{D}). \quad (1.6)$$

The next theorem is our main result in the Wiener space setting. It shows that with these three entities $\Xi_0, \Theta_0$ and $Y$ all solutions to the Wiener-Bezout problem can be described explicitly, and that the function $\Xi$ defined by the first identity in (1.6) is the least squares solution.

**Theorem 1.1.** Let $G \in \mathcal{W}_+^{m \times p}$, and assume that $T_G T_G^*$ is strictly positive. Then the matrix function $Y$ defined by (1.4) belongs to the Wiener space $\mathcal{W}_+^{p \times p}$, $\det Y(z) \neq 0$ for each $|z| \leq 1$, and

$$Y(z)^{-1} = I_p + z E_p^* T_G^*(T_G T_G^*)^{-1} H_G(I - z S_p)^{-1} E_p \quad (z \in \mathbb{D}). \quad (1.7)$$

In particular, $Y^{-1}$ is a Wiener function, and hence $Y$ is invertible outer. Furthermore,

(i) $G(z) Y(z) = G_0$ for each $|z| \leq 1$,

(ii) the function $\Theta$ defined by the second identity in (1.6) belongs to $\mathcal{W}_+^{p \times (p-m)}$ (in particular, $k = p - m$), and $\Theta$ is an inner function with $\text{Im} T_\Theta = \text{Ker} T_G$,

(iii) the function $H(z) := (\Theta_0^* \Theta_0)^{-1} \Theta_0^* (I_p - \Xi_0 G_0) Y(z)^{-1}$ belongs to the Wiener space $\mathcal{W}_+^{(p-m) \times p}$, and

$$\det \begin{bmatrix} G(z) \\ H(z) \end{bmatrix} \neq 0 \quad \text{and} \quad \begin{bmatrix} G(z) \\ H(z) \end{bmatrix}^{-1} = Y(z) \begin{bmatrix} \Xi_0 & \Theta_0 \end{bmatrix} \quad (|z| \leq 1). \quad (1.8)$$

Furthermore, for any $V \in \mathcal{W}_+^{(p-m) \times m}$ the function

$$X(z) = Y(z) \Xi_0 + Y(z) \Theta_0 V(z) \quad (|z| \leq 1) \quad (1.9)$$

is a solution to the Wiener-Bezout problem associated with $G$, and all solutions are obtained in this way. Moreover, with $X$ given by (1.9) we have

$$\|X(\cdot)u\|_{H_p^m}^2 = \|Y(\cdot) \Xi_0 u\|_{H_p^m}^2 + \|V(\cdot)u\|_{H_{p-m}^m}^2 \quad (u \in \mathbb{C}^m). \quad (1.10)$$

In particular, the function $\Xi(z) = Y(z) \Xi_0$ is the least squares solution to the Wiener-Bezout problem associated with $G$.

Item (iii) in the above theorem is closely related to Tolkonnikov’s lemma [19] (see also [16] Appendix 3, item 10). In fact, from Tolkonnikov’s lemma it follows that (1.8) holds true with $H$ on the unit circle $T$ being given by

$$H(\zeta) = \Theta^*(\zeta)(I_p - \Xi(z) G(\zeta)) \quad (\zeta \in T). \quad (1.11)$$
At the end of Section 3 (see Remark 3.2) we shall show that the function $H$ defined by the above formula and the function $H$ defined in item (iii) of the above theorem are one and the same function. Specifying (1.8) for $z = 0$ we see that

$$
(1.12) \quad \begin{bmatrix} \Xi_0 & \Theta_0 \end{bmatrix}^{-1} = \begin{bmatrix} G_0 & \Theta_0 \end{bmatrix} \quad \text{with} \quad H_0 = (\Theta_0^*\Theta_0)^{-1}\Theta_0^*(I_p - \Xi_0G_0).
$$

Lemma 2.1 in the next section shows that this inversion formula remains true if $G$ is just an $H^\infty$ function.

Theorem 3.1 which is our second main result, presents a (partial) analogue of Theorem 1.1 in an $H^\infty/H^2$ setting. Let $G \in H^\infty_{m \times p}$, and assume that $T_GT_G$ to be strictly positive. Then the function $Y$ is still well defined on the open unit disc $\mathbb{D}$ and $\det Y(z) \neq 0$ for each $z \in \mathbb{D}$. However, in general, the entries of $Y$ and $Y^{-1}$ are just $H^2$ functions, and formula (1.9) yields $H^2$ solutions rather than $H^\infty$ solutions. Moreover, if the free parameter $V$ in (1.9) is taken from $H^2_{[p-m] \times m}$, then all $H^2$ solutions are obtained by (1.9) and the $H^2$ norm in the identity (1.10) appears in a natural way.

Finally, in Section 2 we shall prove that item (ii) carries over to an $H^\infty$ setting (see Proposition 2.5). The fact that $\Theta$ is inner with $\text{Im} T_\Theta = \text{Ker} T_G$ follows from Lemma 2.1 in [7]. A more direct proof is given at the end of Section 2. The statement that $k = p - m$ is new in the $H^\infty$ setting. For the proof see the final part of Lemma 2.1.

The paper consists of five sections, including the present introduction. In the second section we present a number of auxiliary results which are all valid in the $H^\infty$ setting. Section 3 contains the proof of Theorem 1.1. Section 4 deals with the role of the function $Y$ in the $H^\infty$ case and presents a partial analogue of Theorem 1.1 including the description of all $H^2$ solutions. In the final section we present a few concluding remarks and compute the function $Y$ for the case when $G(z) = [1 + z \quad -z]$.

Notation and terminology. By $\mathcal{W}$ we denote the Wiener space (cf., item (a) in [12, Section XXIX.2]) consisting of all functions on the unit circle that have an absolutely summable Fourier expansion, and $\mathcal{W}^{r \times s}$ stands for the linear space of all $r \times s$ matrix functions of which the entries belong to $\mathcal{W}$. Thus

$$
F \in \mathcal{W}^{r \times s} \iff F(e^{it}) = \sum_{\nu = -\infty}^{\infty} F_\nu e^{i\nu t}, \quad \text{where} \quad \sum_{\nu = -\infty}^{\infty} \|F_\nu\| < \infty.
$$

As usual we refer to $F_\nu$ as the $\nu$-th Fourier coefficient of $F$. We also need the space $\mathcal{W}^{r \times s}_+$ which consists of all $F \in \mathcal{W}^{r \times s}$ that have an analytic extension to the open unit disc $\mathbb{D}$, that is,

$$
F \in \mathcal{W}^{r \times s}_+ \iff F(e^{it}) = \sum_{\nu = 0}^{\infty} F_\nu e^{i\nu t}, \quad \text{where} \quad \sum_{\nu = 0}^{\infty} \|F_\nu\| < \infty.
$$

Each $F \in \mathcal{W}^{r \times s}_+$ is continuous on the unit circle, and therefore each $F \in \mathcal{W}^{r \times s}$ defines a (block) Toeplitz operator $T_F$ mapping $\ell_2^r(\mathbb{C}^s)$ into $\ell_2^s(\mathbb{C}^r)$. With $F \in \mathcal{W}^{r \times s}$, we associate the function $F^* \in \mathcal{W}^{s \times r}$ defined by $F^*(z) = F(1/\bar{z})^*$ for each $z \in \mathbb{T}$. Then $T_{F^*} = T_F^*$. Finally, note that $\mathcal{W}^{r \times s}_+ \subset H^\infty_{r \times s} \subset H^2_{r \times s}$, where $H^\infty_{r \times s}$ and $H^2_{r \times s}$ stand for the linear spaces consisting of all $r \times s$ matrices with entries in $H^\infty$ and $H^2$, respectively.
2. Auxiliary results in an $H^\infty$ setting

Throughout this section let $G \in H^\infty_{m \times p}$ and assume that $T_G T_G^*$ is strictly positive. We shall be dealing with the function $Y$ defined by (1.4) and the matrices $\Xi_0$ and $\Theta_0$ defined by items (M1) and (M2) in the previous section. Note that the function $Y$ and the matrices $\Xi_0$ and $\Theta_0$ are well defined when $G \in H^\infty_{m \times p}$ and $T_G T_G^*$ is strictly positive; it is not required for this that $G$ belongs to a Wiener space.

In this section we shall derive a number of auxiliary results that will be useful in proving Theorem 1.1 in Section 3. These auxiliary results will also allow us to derive the later result we require the following observation about the function $H$ in Lemma 2.1.

Combining the above identities shows

$$
(G_0 \Xi_0 - \Theta_0) \Xi_0^* = (I - \Xi_0 G_0) \Xi_0 = \Xi_0 - \Xi_0 = 0 \quad \text{and} \quad (I - \Xi_0 G_0) \Theta_0 = \Theta_0 - 0 = \Theta_0.
$$

Note that $(\Theta_0^* \Theta)\Theta_0^*$ is a left inverse of $\Theta_0$. Hence

$$
H_0 \Xi_0 = 0 \quad \text{and} \quad H_0 \Theta_0 = I_k.
$$

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$$
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$$

Note that $(\Theta_0^* \Theta)\Theta_0^*$ is a left inverse of $\Theta_0$. Hence

$$
H_0 \Xi_0 = 0 \quad \text{and} \quad H_0 \Theta_0 = I_k.
$$

It follows that $[\Xi_0 \Theta_0]$ is invertible and that its inverse is given by (2.1). In particular, $[\Xi_0 \Theta_0]$ is a square matrix, which implies $p = m + k$. Hence $k = p - m$. Moreover, $G_0 \Theta_0 = 0$ implies $\text{Im} \Theta_0 \subset \text{Ker} G_0$. We have $\text{Ker} \Theta_0 = \{0\}$, so that $\text{rank} \Theta_0 = p - m$. Hence $\text{dim} \text{Im} \Theta_0 = p - m$. On the other hand, we have $\text{Im} G_0 = \mathbb{C}^m$, which implies $\text{dim} \text{Ker} G_0 = p - m$. Therefore $\text{Im} \Theta_0 = \text{Ker} G_0$.

Lemma 2.1 can be seen as the special case of Proposition 2.3 below where $z = 0$. To derive the later result we require the following observation about the function $Y$. 

Proposition 2.2. Let $G \in H_{m \times p}^\infty$ and assume that $T^*_G T^*_G$ is strictly positive. Then the function $Y$ defined by (1.3) is analytic on $\mathbb{D}$, det $Y(z) \neq 0$ for each $z \in \mathbb{D}$, and
\[
Y(z)^{-1} = I_p + zE^*_p T^*_G (T^*_G T^*_G)^{-1} H_G (I - zS_p)^{-1} E^*_p \quad (z \in \mathbb{D}).
\]
In particular, the function $Y(\cdot)^{-1}$ is analytic on $\mathbb{D}$. Moreover, we have
\[
G(z) Y(z) = G_0 \quad (z \in \mathbb{D}).
\]
\[
(2.4)
\]

**Proof.** That fact that $S_p$ has spectral radius equal to 1, yields that $Y$ is analytic on $\mathbb{D}$. Since $S^*_p T^*_G = T^*_G S^*_m$, we can rewrite $Y$ as
\[
Y(z) = I_p - zE^*_p (I - zS_p)^{-1} T^*_G (T^*_G T^*_G)^{-1} H_G E^*_p
\]
\[
= I_p - zE^*_p T^*_G (I - zS^*_m)^{-1} (T^*_G T^*_G)^{-1} H_G E^*_p
\]
\[
= D + zC(I - zA)^{-1} B,
\]
where in the last identity
\[
A = S^*_m, \quad B = (T^*_G T^*_G)^{-1} H_G E^*_p, \quad C = -E^*_p T^*_G, \quad D = I_p.
\]
Note that $H_G E^*_p = S^*_m T^*_G E^*_p$ and
\[
S^*_m T^*_G E^*_p E^*_p T^*_G = S^*_m T^*_G(I - S_p S^*_p) T^*_G = S^*_m T^*_G T^*_G - S^*_m S^*_m T^*_G T^*_G S^*_m
\]
\[
= S^*_m T^*_G T^*_G - T^*_G T^*_G S^*_m.
\]
This yields that $A^x := A - BD^{-1}C$ can be written as
\[
A^x = S^*_m + (T^*_G T^*_G)^{-1} H_G E^*_p E^*_p T^*_G = S^*_m + (T^*_G T^*_G)^{-1} S^*_m T^*_G E^*_p E^*_p T^*_G
\]
\[
= S^*_m + (T^*_G T^*_G)^{-1}(S^*_m T^*_G T^*_G - T^*_G T^*_G S^*_m) = (T^*_G T^*_G)^{-1} S^*_m T^*_G T^*_G.
\]
This implies that $A^x$ is similar to $S^*_m$, and hence has spectral radius equal to 1. Then, by standard state space inversion results, cf., Theorem 2.1 in [1] (with $\lambda = 1/2$), it follows that $Y(z)$ is invertible for each $z \in \mathbb{D}$ with inverse given by
\[
Y(z)^{-1} = D^{-1} - zD^{-1}C(I - zA^x)^{-1}BD^{-1}
\]
\[
= I + zE^*_p T^*_G (I - z(T^*_G T^*_G)^{-1} S^*_m T^*_G T^*_G)^{-1} (T^*_G T^*_G)^{-1} H_G E^*_p
\]
\[
= I + zE^*_p T^*_G (T^*_G T^*_G)^{-1}(I - zS^*_m)^{-1} H_G E^*_p.
\]
Since $S^*_m H_G = H_G S_p$, we have $(I - zS^*_m)^{-1} H_G = H_G (I - zS_p)^{-1}$, and hence (2.3) holds. Note that the spectral radius of $S_p$ is equal to 1, which implies that the function $Y(\cdot)^{-1}$ is analytic on $\mathbb{D}$.

Finally, we prove that (2.4) holds. Let $Y_0, Y_1, Y_2, \ldots$ be the Taylor coefficients of $Y$ at zero. As observed in (1.5), we have $Y_0 = I_p$ and
\[
(2.5)
\]
\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots
\end{bmatrix}
= -T^*_G (T^*_G T^*_G)^{-1}
\begin{bmatrix}
G_1 \\
G_2 \\
\vdots
\end{bmatrix}, \quad \text{and hence} \quad T^*_G
\begin{bmatrix}
Y_1 \\
Y_2 \\
\vdots
\end{bmatrix}
= -
\begin{bmatrix}
G_1 \\
G_2 \\
\vdots
\end{bmatrix}.
\]
The latter identity is equivalent to
\[
G(z) \left( \frac{Y(z) - I_p}{z} \right) = - \frac{G(z) - G_0}{z} \quad (z \in \mathbb{D}).
\]
Multiplying both sides of the above identity by $z$ and adding $G(z)$ on either side yields (2.4). \qed
Proposition 2.3. Let \( G \in H_{m \times p}^{\infty} \) and assume that \( T_G T_G^* \) is strictly positive. Let \( Y \) be the function defined by (1.3), and define the functions \( \Xi \) and \( \Theta \) by (1.6), with \( \Xi_0 \) and \( \Theta_0 \) the matrices in items (M1) and (M2) of the previous section. Consider the matrix function \( H \) defined by

\[
H(z) = H_0 \Theta(z)^{-1}, \quad z \in \mathbb{D}, \quad \text{with} \quad H_0 = (\Theta_0^* \Theta_0)^{-1} (I_p - \Xi_0 G_0).
\]

Then \( H \) is analytic on \( \mathbb{D} \),

\[
\det \begin{bmatrix} G(z) \\ H(z) \end{bmatrix} \neq 0 \quad \text{and} \quad \begin{bmatrix} G(z) \\ H(z) \end{bmatrix}^{-1} = [\Xi(z) \quad \Theta(z)] \quad (z \in \mathbb{D}).
\]

Proof. Since \( Y \) is analytic on \( \mathbb{D} \), clearly \( H \) defined by (2.6) is analytic on \( \mathbb{D} \). Furthermore, using Proposition 2.2 we find that \( G(z) = G_0 Y(z)^{-1}, \quad z \in \mathbb{D} \). Thus

\[
\begin{bmatrix} G(z) \\ H(z) \end{bmatrix} = \begin{bmatrix} G_0 \\ H_0 \end{bmatrix} Y(z)^{-1}, \quad \begin{bmatrix} \Xi(z) & \Theta(z) \end{bmatrix} = Y(z) \begin{bmatrix} \Xi_0 & \Theta_0 \end{bmatrix} \quad (z \in \mathbb{D}).
\]

This shows that our claim reduces to the case \( z = 0 \), which was proved in Lemma 2.1. \( \square \)

We conclude with two auxiliary results, the first is about the function \( \Xi \) and the second about \( \Theta \).

Lemma 2.4. Let \( G \in H_{m \times p}^{\infty} \) and assume that \( T_G T_G^* \) is strictly positive. Then the function \( \Xi \) defined by the first part of (1.6) is also given by

\[
\Xi(z) = E_p^* (I - z S_p^*)^{-1} T_G^* (T_G T_G^*)^{-1} E_m \quad (z \in \mathbb{D})
\]

Proof. Recall that \( \Xi_0 = E_p^* T_G^* (T_G T_G^*)^{-1} E_m \). This yields

\[
H_G E_p \Xi_0 = S_m^* T_G E_p E_p^* T_G^* (T_G T_G^*)^{-1} E_m = S_m^* E_m - S_m S_m^* T_G^* S_m^* (T_G T_G^*)^{-1} E_m = -T_G S_m^* (T_G T_G^*)^{-1} E_m.
\]

With this observation we obtain that the function \( \Xi \) is also given by

\[
\Xi(z) = Y(z) \Xi_0 = \left( I - z E_p^* (I - z S_p^*)^{-1} T_G^* (T_G T_G^*)^{-1} H_G E_p \right) \Xi_0 = \Xi_0 + z E_p^* (I - z S_p^*)^{-1} T_G^* S_m^* (T_G T_G^*)^{-1} E_m = E_p^* \left( I + z (I - z S_p^*)^{-1} S_p^* \right) T_G^* (T_G T_G^*)^{-1} E_m = E_p^* (I - z S_p^*)^{-1} T_G^* (T_G T_G^*)^{-1} E_m.
\]

This proves (2.8). \( \square \)

Proposition 2.5. Let \( G \in H_{m \times p}^{\infty} \) and assume that \( T_G T_G^* \) is strictly positive. The function \( \Theta \) defined in (1.6) belongs to \( H_p^{\infty} \) and is an inner function with \( \text{Im} \Theta_0 = \text{Ker} T_G \).

Proof. Using the definition of \( Y \) in (1.4) and the fact \( H_G E_p = S_m^* T_G E_p \), we see that \( \Theta \) is also given by

\[
\Theta(z) = (I_p - z E_p^* (I - z S_p^*)^{-1} T_G^* (T_G T_G^*)^{-1} S_m^* T_G E_p) \Theta_0 \quad (z \in \mathbb{D}).
\]
By comparing this formula with \[7\) Eq. (2.1)] we conclude that \(\Theta\) coincides (up to multiplication with a constant unitary matrix from the right) with the inner function \(\tilde{\Theta}\) satisfying \(\text{Im} \, T_G^* = \ell_2^s(\mathbb{C}^p) \oplus T_{\tilde{\Theta}}^2(\mathbb{C}^k)\), where \(k\) is the number of columns of the matrix \(\Theta(0)\). The existence of \(\tilde{\Theta}\) is guaranteed by the Beurling-Lax theorem. Since \(\text{Ker} \, T_G = (\text{Im} \, T_G^*)^\perp\), we conclude that \(\text{Ker} \, T_G = \text{Im} \, T_{\Theta}\). Finally, that \(k = p - m\), and thus \(\Theta \in H_p^{\infty}(m - p)\), follows from Lemma 2.1. \(\square\)

Note the proof of Proposition 2.6 relies heavily on \[7\) Lemma 2.1]. We also add something to the observations made in Section 2 of \[7\], namely that \(\Theta\) has size \(p \times (p - m)\), and hence \(\Theta\) is a matrix function of size \(p \times (p - m)\).

Direct proof of Proposition 2.6. Let \(\Theta\) be the analytic matrix function on \(\mathbb{D}\) defined by the second identity in (1.6). We already know (see the final part of Lemma 2.1 that \(\Theta\) has size \(p \times (p - m)\), and hence \(\Theta\) is a matrix function of size \(p \times (p - m)\).

To prove that \(\Theta\) is inner, let \(\Gamma_j\) be \(j\)-th column of the block Toeplitz matrix defined by \(\Theta\). Thus

\[
(2.9) \quad [\Gamma_0 \quad \Gamma_1 \quad \Gamma_2 \quad \ldots] = \begin{bmatrix}
Y_0\Theta_0 & 0 & 0 & \cdots \\
Y_1\Theta_0 & Y_0\Theta_0 & 0 & \cdots \\
Y_2\Theta_0 & Y_1\Theta_0 & Y_0\Theta_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

Note that \(\Gamma_0, \Gamma_1, \Gamma_2, \ldots\) are bounded linear operators from \(\mathbb{C}^{p-m}\) into \(\ell_2^s(\mathbb{C}^p)\). This follows from the first identity in (2.5), the fact that \(T_G^*(T_G T_G^*)^{-1}\) is a bounded operator from \(\ell_2^s(\mathbb{C}^m)\) into \(\ell_2^s(\mathbb{C}^p)\), and the fact that the first column of \(H_G\) is a bounded operator from \(\mathbb{C}^p\) into \(\ell_2^s(\mathbb{C}^m)\). To prove that \(\Theta\) is inner it suffices to show that

\[
(C1) \quad \Gamma_j \text{ is an isometry mapping } \mathbb{C}^{p-m} \text{ into } \ell_2^s(\mathbb{C}^p) \text{ for } j = 0, 1, 2, \ldots;
\]

\[
(C2) \quad \text{Im} \, \Gamma_j \perp \text{Im} \, \Gamma_k \text{ for } k \neq j.
\]

To see this, assume that both conditions are satisfied. Then the operator \(T\) defined by be the infinite block lower triangular matrix on the right hand side of (2.9) is an isometry mapping \(\ell_2^s(\mathbb{C}^{p-m})\) into \(\ell_2^s(\mathbb{C}^p)\). Moreover, \(S_p T = T S_{p-m}\). It follows that \(T\) is a Toeplitz operator, and its defining function \(\Theta(\cdot) = Y(\cdot) \Theta\) belongs to \(H_p^{\infty}(p-m)\); cf., \[12\) Section XXIII.3]. Thus \(T = T_{\Theta}\), and \(\Theta\) is inner because \(T_{\Theta} = T\) is an isometry, \[12\) Section XXVI.3] or \[5\) Proposition 2.6.2.\]

In order to show that conditions (C1) and (C2) are satisfied we need the following two lemmas.

**Lemma 2.6.** Let \(G \in H_m^{\infty}\), and assume that \(T_G T_G^{*}\) is strictly positive. Then

\[
\sum_{i=0}^{\infty} Y_i^{*} Y_{i+j} = \begin{cases}
I_p + E_p H_G^* (T_G T_G^{*})^{-1} H_G E_p & \text{when } j = 0, \\
-G_p E_m (S_m^*)^j (T_G T_G^{*})^{-1} H_G E_p & \text{when } j = 1, 2, \ldots.
\end{cases}
\]
Proof. Note that

\[
\sum_{i=0}^{\infty} Y_i^* Y_{i+j} = Y_0^* Y_j + \begin{bmatrix} Y_1^* & Y_2^* & \cdots \end{bmatrix} \begin{bmatrix} Y_{j+1} \\ Y_{j+2} \\ \vdots \end{bmatrix} 
= Y_0^* Y_j + E_p^* H_G(T_G T_G^*)^{-1} T_G (S_p^*)^j T_G^* (T_G^*)^{-1} H_G E_p 
= Y_0^* Y_j + E_p^* H_G(T_G T_G^*)^{-1} T_G^* (S_p^*)^j (T_G^*)^{-1} H_G E_p 
= Y_0^* Y_j + E_p^* H_G(S_p^*)^j (T_G T_G^*)^{-1} H_G E_p, \quad j = 0, 1, \ldots
\]  

(2.10)

Using $Y_0 = I_p$, we see that with $j = 0$ the identity (2.10) yields the first part of the lemma.

Next assume that $j > 0$. Recall that $H_G E_p = S_m^* E_p$. Taking adjoints in the latter identity and using $Y_0 = I_p$ again, we see that (2.10) can be rewritten as

\[
\sum_{i=0}^{\infty} Y_i^* Y_{i+j} = Y_j + E_p^* T_G S_m^* (S_m^*)^j (T_G T_G^*)^{-1} H_G E_p 
= Y_j + E_p^* T_G S_m^* (S_m^*)^j (T_G T_G^*)^{-1} H_G E_p 
= C_1 - C_2, 
\]

where

\[
C_1 = E_p^* T_G^* (S_m^*)^j (T_G^*)^{-1} H_G E_p 
= E_p^* (S_p^*)^j (T_G T_G^*)^{-1} H_G E_p = -Y_j, 
\]

and

\[
C_2 = E_p^* T_G^* E_m^* (S_m^*)^j (T_G T_G^*)^{-1} H_G E_p 
= G_0 E_m^* (S_m^*)^j (T_G T_G^*)^{-1} H_G E_p. 
\]

Thus

\[
\sum_{i=0}^{\infty} Y_i^* Y_{i+j} = Y_j + C_1 - C_2 = -G_0 E_m^* (S_m^*)^j (T_G T_G^*)^{-1} H_G E_p. 
\]

This proves the second part of the lemma.

\[\square\]

Lemma 2.7. Let $G \in H_{m \times p}^\infty$, and assume that $T_G T_G^*$ is strictly positive. Then

\[
(2.11) \quad \Theta_0 \left( I_p + E_p^* H_G^* (T_G T_G^*)^{-1} H_G E_p \right) \Theta_0 = I_{p-m}. 
\]

Proof. Using the definition of $\Theta_0 \Theta_0^*$ in (1.2) we see that

\[
E_p^* H_G^* (T_G T_G^*)^{-1} H_G E_p \Theta_0 \Theta_0^* = A - B, 
\]

where

\[
A = E_p^* H_G^* (T_G T_G^*)^{-1} H_G E_p, \quad B = E_p^* H_G^* (T_G T_G^*)^{-1} H_G E_p E_p^* T_G^* (T_G T_G^*)^{-1} T_G E_p 
= E_p^* H_G^* (T_G T_G^*)^{-1} S_m T_G E_p E_p^* T_G^* (T_G T_G^*)^{-1} T_G E_p. 
\]
Here we used that $H_G E_p = S^*_m T_G E_p$. Next using $E_p E^*_p = I - S_p S^*_p$ we write $B$ as $B = B_1 - B_2$, where

$$
B_1 = E^*_p H_G (T_G T_G^*)^{-1} S^*_m T_G T_G^* (T_G T_G^*)^{-1} T_G E_p
= E^*_p H_G (T_G T_G^*)^{-1} S^*_m T_G E_p
= E^*_p H_G (T_G T_G^*)^{-1} H_G E_p = A,
$$

and

$$
B_2 = E^*_p H_G (T_G T_G^*)^{-1} S^*_m T_G S_p S^*_p T_G (T_G T_G^*)^{-1} T_G E_p
= E^*_p H_G (T_G T_G^*)^{-1} S^*_m S_p T_G S^*_p (T_G T_G^*)^{-1} T_G E_p
= E^*_p H_G (T_G T_G^*)^{-1} T_G S_p S^*_p (T_G T_G^*)^{-1} T_G E_p
= E^*_p H_G S^*_m (T_G T_G^*)^{-1} T_G E_p
= E^*_p T_G S^*_m (T_G T_G^*)^{-1} T_G E_p.
$$

Next we use $S_m S^*_m = I - E_m E^*_m$ to show that

$$
B_2 = E^*_p T_G (T_G T_G^*)^{-1} T_G E_p - E^*_p T_G E_p - G^*_m (T_G T_G^*)^{-1} T_G E_p
(2.12)
= E^*_p T_G (T_G T_G^*)^{-1} T_G E_p - G^*_m (T_G T_G^*)^{-1} T_G E_p.
$$

Recall (see the final part of Lemma 2.6) that $\Theta_0^* G_0^* = 0$, and hence $\Theta_0^* B_2 = \Theta_0^* E^*_p T_G (T_G T_G^*)^{-1} T_G E_p$. Since $A = B_1$ and $\Theta_0^* \Theta_0^*$ is given by (1.2), we conclude that

$$
\Theta_0 \Theta_0^* \left( I_p + E^*_p H_G (T_G T_G^*)^{-1} H_G E_p \right) \Theta_0 \Theta_0^* = \Theta_0 \Theta_0^* \Theta_0^* \Theta_0^* + \Theta_0 \Theta_0^* E^*_p H_G (T_G T_G^*)^{-1} H_G E_p \Theta_0 \Theta_0^*
= \Theta_0 \Theta_0^* \Theta_0^* \Theta_0^* + \Theta_0 \Theta_0^* (A - B)
= \Theta_0 \Theta_0^* \Theta_0^* \Theta_0^* + \Theta_0 \Theta_0^* (A - B_1 + B_2)
= \Theta_0 \Theta_0^* \Theta_0^* \Theta_0^* + \Theta_0 \Theta_0^* B_2
= \Theta_0 \Theta_0^* \Theta_0^* \Theta_0^* + \Theta_0 \Theta_0^* E^*_p T_G (T_G T_G^*)^{-1} T_G E_p
= \Theta_0 \Theta_0^* \left( I_p - E^*_p T_G (T_G T_G^*)^{-1} T_G E_p + E^*_p T_G (T_G T_G^*)^{-1} T_G E_p \right)
= \Theta_0 \Theta_0^*.
$$

Hence $\Theta_0 \Theta_0^* \left( I_p + E^*_p H_G (T_G T_G^*)^{-1} H_G E_p \right) \Theta_0 \Theta_0^* = \Theta_0 \Theta_0^*$. But then, using that $\Theta_0^*$ is surjective and $\Theta_0$ is injective, we obtain (2.11), and the lemma is proved.

We proceed by showing that (C1) and (C2) are satisfied. Let $\Gamma_0, \Gamma_1, \Gamma_2, \cdots$ be given by (2.9). Using the first part of Lemma 2.6 and formula (2.11) we obtain for each $u \in \mathbb{C}^{p-m}$ that

$$
\| \Gamma_j u \|^2 = \left\langle \Gamma_j \Gamma_j u, u \right\rangle = \left\langle \Theta_0^* \left( \sum_{i=0}^{\infty} Y_i^* Y_i \right) \Theta_0 u, u \right\rangle
= \left\langle \Theta_0 \left( I_p + E^*_p H_G (T_G T_G^*)^{-1} H_G E_p \right) \Theta_0 u, u \right\rangle
= \langle u, u \rangle = \| u \|^2 \quad (j = 0, 1, 2, \cdots).
$$

Thus (C1) holds.
Next, in order to derive (C2), we use the second part of Lemma 2.6 and the fact that $\Theta_0^* G_0^* = 0$. For $j > k$ this yields
\[
\Gamma_j^* \Gamma_k = \Theta_0^* \left( \sum_{i=0}^{\infty} Y_i Y_{i+j-k} \right) \Theta_0 \\
= -\Theta_0^* G_0^* E_m^*(S_m^*)^{-k-1}(T_G T_G^{-1}) H_G E_p \Theta_0 = 0.
\]
It follows that $\text{Im} \Gamma_j \perp \text{Im} \Gamma_k$ for $j > k$. Interchanging the role of $j$ and $k$ then yields (C2).

Finally, we prove $\text{Ker} T_G = \text{Im} T_\Theta$. Recall that $G_0 \Theta_0 = 0$ by the final part of Lemma 2.1. Hence using (2.1) we have
\[
G(z) \Theta(z) = G(z) Y(z) \Theta_0 = G_0 \Theta_0 = 0 \quad (z \in \mathbb{D}).
\]
This implies $T_G T_\Theta = 0$, and thus $\text{Im} T_\Theta \subset \text{Ker} T_G$. To prove the reverse inclusion, take $f = [f_0 \quad f_1 \quad f_2 \quad \cdots]^{\top}$ in $\text{Ker} T_G$, and put $F(z) = E_p (I - z S_m^*)^{-1} f$. Since $G(z) F(z) = 0$ on $\mathbb{D}$, the second part of (2.7) shows that
\[
F(z) = \begin{bmatrix} \Xi(z) & \Theta(z) \end{bmatrix} \begin{bmatrix} G(z) \\ H(z) \end{bmatrix} F(z) \\
\Xi(z) & \Theta(z) \end{bmatrix} \begin{bmatrix} 0 \\ H(z)F(z) \end{bmatrix} = \Theta(z) H(z) F(z) \quad (z \in \mathbb{D}).
\]
It follows that $f = T_\Theta f T_H f$, and thus $f \in \text{Im} T_\Theta$ which proves that $\text{Ker} T_G \subset \text{Im} T_\Theta$, and therefore $\text{Ker} T_G = \text{Im} T_\Theta$. This completes the direct proof of Proposition 2.3.

3. PROOF OF THEOREM 1.1

In this section we prove Theorem 1.1. For that purpose we first derive the following lemma.

**Lemma 3.1.** Let $G \in \mathcal{W}^{m \times p}$, and assume that $T_G T_G^*$ is strictly positive. Then $(T_G T_G^{-1})^{-1}$ maps $\ell_+^1(\mathbb{C}^m)$ into itself.

**Proof.** We split the proof into five parts. In the first part we review a few general facts about Toeplitz and Hankel operators (cf., Sections 2.1–2.3 in [2] and Chapter XXIII in [12]), and we recall an inversion formula from [9].

**PART 1.** Let $F$ belong to the Wiener space $\mathcal{W}^{r \times s}$. Then the Toeplitz operator $T_F$ and the Hankel operator $H_F$ both map $\ell_+^1(\mathbb{C}^r)$ (seen as a linear sub-manifold of $\ell_+^1(\mathbb{C}^s)$) into $\ell_+^1(\mathbb{C}^r)$ (seen as a linear sub-manifold of $\ell_+^1(\mathbb{C}^s)$). Moreover, the induced operators acting between these $\ell_+^1$ spaces are bounded too. Furthermore, $H_F$ is compact as an operator from $\ell_+^1(\mathbb{C}^r)$ into $\ell_+^1(\mathbb{C}^r)$ as well as when viewed as an operator from $\ell_+^1(\mathbb{C}^s)$ into $\ell_+^1(\mathbb{C}^r)$ (cf., [2] Sections 2.1). Finally, if $u$ is a $s \times t$ matrix, then the functions $\varphi$ and $\psi$ given by
\[
\varphi(z) = E_r^*(I - z S_r^*)^{-1} T_F E_s u \quad \text{and} \quad \psi(z) = E_r^*(I - z S_r^*)^{-1} H_F E_s u
\]
belong to the Wiener space $\mathcal{W}^{r \times t}$. Next, we recall some facts from [9] Section 2]. Define $R = GG^*$. See the last paragraph of the introduction for the definition of $G^*$. Note that $R \in \mathcal{W}^{m \times m}$. The fact that $T_G T_G^*$ is strictly positive implies that the matrix $R(z)$ is positive definite.
for each $z \in \mathbb{T}$, and hence the Toeplitz operator $T_R$ acting on $\ell^2_+ (\mathbb{C}^m)$ is invertible. Moreover, see [3, Eq. (2.4)], we have

\begin{equation}
(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G T_R^{-1} H_G)^{-1} H_G T_R^{-1}.
\end{equation}

**PART 2.** Since $R$ belongs to $\mathcal{W}^n \times m$ and $R(z)$ is positive definite for each $z \in \mathbb{T}$, the function $R$ admits a a canonical spectral factorization (see Corollary 2.1 in [4, Section III.2]), that is, $R = R^* R_+$ where $R_+$ belongs to $\mathcal{W}^{m \times m}_+$ and $\det (R(z)) \neq 0$ for each $z$ in the closed unit disc. This implies that $T_R = T_{R_+} T_{R_+}$ and both $T_{R_+}$ and $T_{R_+}^*$ are invertible. In fact, $(T_{R_+})^{-1} = T_{R_+}^{-1}$ and $(T_{R_+}^*)^{-1} = (T_{(R_+)^*})^{-1}$, are both Toeplitz operators. We conclude that $T_R$ is invertible and that its inverse is given by

$$T_R^{-1} = (T_{R_+})^{-1} (T_{(R_+^*)}^{-1}).$$

From the remarks in the first paragraph of the proof it then follows that the operators $(T_{R_+})^{-1}$ and $(T_{(R_+^*)}^{-1})$ map $\ell^1_+ (\mathbb{C}^m)$ into itself and act as bounded linear operators on this space. Hence the same holds true for $T_R^{-1}$. Moreover $T_R^{-1}$, as an operator on $\ell^1_+ (\mathbb{C}^m)$, is again invertible.

**PART 3.** From the final remark in the first paragraph of the first part of the proof we know that the Hankel operator $H_G$ maps $\ell^1_+ (\mathbb{C}^p)$ into $\ell^1_+ (\mathbb{C}^m)$. An analogous result holds true for $H_G^*$. To see this note that $H_G^* = H_{G^*}$, where $G^*$ is the function in $\mathcal{W}^m \times p$ given by:

$$G^*(z) = G^*(z) = G_0^* + z G_0^* + z^2 G_2^* + \cdots \quad (|z| \leq 1).$$

Using the result of Part 1 of the proof we conclude that $I - H_G^* T_R^{-1} H_G$ maps $\ell^1_+ (\mathbb{C}^p)$ into itself and act as bounded linear operator on this space.

**PART 4.** Put $M = I - H_G^* T_R^{-1} H_G$. In this part we show that $M$ is invertible as an operator on $\ell^1_+ (\mathbb{C}^p)$. To do this we use the fact that $H_G$ acts as a compact operator from $\ell^1_+ (\mathbb{C}^p)$ to $\ell^1_+ (\mathbb{C}^m)$. It follows that $M$ as an operator on $\ell^1_+ (\mathbb{C}^p)$ is of the form identity operator plus a compact one. Hence $M$ as an operator on $\ell^1_+ (\mathbb{C}^p)$ is a Fredholm operator of index zero. In order to show that $M$ as an operator on $\ell^1_+ (\mathbb{C}^p)$ is invertible, it then suffices to prove that $M$ on $\ell^1_+ (\mathbb{C}^p)$ is one-to-one. Take $h \in \ell^1_+ (\mathbb{C}^p)$, and assume that $M h = 0$. Since $\ell^2_+ (\mathbb{C}^p)$ is contained in $\ell^1_+ (\mathbb{C}^p)$, it follows that $h \in \ell^2_+ (\mathbb{C}^p)$. But on $\ell^2_+ (\mathbb{C}^p)$ the operator $M$ is invertible. Thus $h = 0$, and $M$ is one-to-one on $\ell^1_+ (\mathbb{C}^p)$. Therefore $I - H_G^* T_R^{-1} H_G$ is invertible as an operator on $\ell^1_+ (\mathbb{C}^p)$.

**PART 5.** The results of the preceding parts of the proof show that the operators appearing in (3.1) all map $\ell^1$ spaces into $\ell^1$ spaces, and hence $(T_G T_G^*)^{-1}$ maps $\ell^1_+ (\mathbb{C}^m)$ into itself.

**Proof of Theorem 1.1.** We split the proof into three parts.

**PART 1.** In this part we show that the function $Y$ defined by (1.3) has the desired properties. First we show that $Y$ belongs to the Wiener space $\mathcal{W}^{p \times p}_+$. To do this note that $T_G = T_{G^*}$, and hence $T_G$ maps $\ell^1_+ (\mathbb{C}^m)$ into $\ell^1_+ (\mathbb{C}^p)$. But then Lemma 3.1 tells us that $T_G (T_G T_G^*)^{-1}$ maps $\ell^1_+ (\mathbb{C}^m)$ into $\ell^1_+ (\mathbb{C}^p)$. Since $G \in \mathcal{W}^{m \times p}_+$, its Taylor coefficients $G_0, G_1, G_2, \ldots$ at zero are absolutely summable in norm, and thus we can use (1.5) to show that the same holds true for the Taylor coefficients at zero of $Y$. Therefore $Y \in \mathcal{W}^{p \times p}_+$. 
Next we show that \( \det Y(z) \neq 0 \) when \(|z| \leq 1\). For \(|z| < 1\) this follows from Proposition 2.2. We shall prove that \( \det Y(z) \neq 0 \) for all \( z \in \mathbb{T} \) by contradiction. Assume that there exists \( \lambda \in \mathbb{T} \) such that \( \det Y(\lambda) = 0 \). Then there exists \( u \neq 0 \) such that \( Y(\lambda)u = 0 \). Since \( G \) and \( Y \) are Wiener functions, \( G \) and \( Y \) extend continuously to \( \mathbb{T} \). Thus the equality in (2.3) from Proposition 2.2 also holds for each \(|z| = 1\). It follows that \( G_0u = G(\lambda)Y(\lambda)u = 0 \). So \( u \in \text{Ker } G_0 \). From the final part of Lemma 2.1 we know that \( \text{Ker } G_0 = \text{Im } \Theta_0 \). But then \( u = \Theta_0v \) for some \( v \in \mathbb{C}^{p-m} \), and \( \Theta(\lambda)v = Y(\lambda)\Theta_0v = Y(\lambda)u = 0 \). We obtain that \( v = I_{p-m}v = \Theta^*(\lambda)\Theta(\lambda)v = 0 \). This implies that \( u = \Theta_0v = 0 \), which contradicts our assumption that \( u \neq 0 \). We conclude that \( \det Y(z) \neq 0 \) for all \(|z| \leq 1\).

By Wiener’s theorem, the fact that \( Y \in \mathcal{W}_+^{p \times p} \) and \( \det Y(z) \neq 0 \) for all \(|z| \leq 1\) implies that \( Y^{-1} \) also belongs to \( \mathcal{W}_+^{p \times p} \). Finally, formula (1.7) follows from (2.3). Thus \( Y \) has all properties mentioned in the first paragraph of Theorem 1.1.

PART 2. In this part we deal with items (i)–(iii). Note that Proposition 2.2 and the final part of Lemma 2.1 show that the statements in items (i) and (ii) in Theorem 1.1 hold true, noting that Proposition 2.2 and the final part of Lemma 2.1 show that \( \text{Ker } G_0 = \text{Im } \Theta_0 \). Hence for \( X \) given by (1.9) with \( Y \) given by (1.9) are solutions to the Wiener-Bezout problem associated with \( H \). Note that \( X \) and \( Y \) are analytic Wiener functions as well. Furthermore, item (iii) follows from Proposition 2.3 and the fact that \( G \), \( H \), and \( Y \) extend to continuous functions on \( \mathbb{T} \). This proves items (i)–(iii) Theorem 1.1.

PART 3. It remains to prove the statements in the final paragraph of Theorem 1.1. Put \( \Xi(z) = Y(z)\Xi_0 \). Clearly \( \Xi \in \mathcal{W}_+^{p \times m} \). Using (2.4) from Proposition 2.2 we have
\[
G(z)\Xi(z) = G(z)Y(z)\Xi_0 = G_0\Xi_0 \quad (z \in \mathbb{D}).
\]
Among other things, equality (2.1) shows that \( G_0\Xi_0 = I_m \). It follows that \( \Xi \) is a solution to the Wiener-Bezout problem (1.1). From the equality (2.1) it also follows that \( G_0\Theta_0 = 0 \). Hence for \( X \) given by (1.9) with \( V \) belonging to \( \mathcal{W}_+^{(p-m) \times m} \) we have
\[
G(z)X(z) = G(z)Y(z)(\Xi_0 + \Theta_0V(z))
= G_0\Xi_0 + G_0\Theta_0V(z) = I_m \quad (z \in \mathbb{D}).
\]
Note that \( X \) given by (1.9) belongs to \( \mathcal{W}_+^{p \times m} \), and thus all \( X \) given by (1.9) are solutions to the Wiener–Bezout problem associated with \( G \).

We proceed by proving (1.10). To do this let \( V \in \mathcal{W}_+^{(p-m) \times m} \), and let \( X \) be given by (1.9). From Lemma 2.1 we know that \( \Xi \) is given by (2.3). This implies that
\[
\text{Im } T_\Xi E_m = \text{Im } T_G(T_GT_G^{-1}E_m) \subset \text{Im } T_G^{-1} = (\text{Im } T_\Theta)^{-1}.
\]
Thus for each \( u \in \mathbb{C}^m \) the vector \( T_\Xi E_m u \) is orthogonal to \( \text{Im } T_\Theta \). Using this orthogonality we have
\[
\|X(\cdot)u\|^2_{H^p_+} = \|TXE_m u\|^2_{L^2_2(\mathbb{C}^p)} = \|T_\Xi E_m u + T_\Theta E_m u\|^2_{L^2_2(\mathbb{C}^p)}
= \|T_\Xi E_m u\|^2 + \|T_\Theta E_m u\|^2_{L^2_2(\mathbb{C}^p)}
= \|T_\Xi E_m u\|^2 + \|T V E_m u\|^2_{L^2_2(\mathbb{C}^p)}
= \|\Xi(\cdot)u\|^2_{H^p_+} + \|V(\cdot)u\|^2_{H^p_+},
\]
which proves (1.10).

Finally, let \( X \in \mathcal{W}_+^{p \times m} \) be a solution to the Wiener-Bezout problem associated with \( G \). Thus \( G(z)X(z) = I_m \) for \( z \in \mathbb{D} \). Define \( V(z) = H(z)X(z) \), \( z \in \mathbb{D} \), where
$H$ is defined in item (iii) of Theorem 1.1. Then $V$ belongs to the Wiener space $W_{+}^{(p-m)\times m}$, and formula (1.8) shows that

$$X(z) = \begin{bmatrix} \Xi(z) & \Theta(z) \end{bmatrix} \begin{bmatrix} G(z) \\ H(z) \end{bmatrix} X(z)$$

$$= \Xi(z)G(z)X(z) + \Theta(z)H(z)X(z) = \Xi(z) + \Theta(z)V(z) \quad (|z| \leq 1).$$

Using the formulas for $\Xi(z)$ and $\Theta(z)$ in (1.6) we see that $X$ admits the representation (1.9).

\begin{remark}
In the Wiener setting the function $H$ defined in item (iii) of Theorem 1.1 and the function $H$ defined by (1.11) are equal. To be more precise, put

$$H(z) = (\Theta_p^0\Theta_0)^{-1}\Theta_p^0(I_p - \Xi_0G_0)Y(z)^{-1} \quad (|z| \leq 1),$$

$$\tilde{H}(\zeta) = \Theta^*(\zeta)(I_p - \Xi(z)G(\zeta)) \quad (|\zeta| = 1).$$

Then $H = \tilde{H}$. To see this, fix $|\zeta| = 1$. According to (2.1) we have

$$H(\zeta)Y(\zeta) \begin{bmatrix} \Theta_0 & \Xi_0 \end{bmatrix} = H_0 \begin{bmatrix} \Theta_0 & \Xi_0 \end{bmatrix} = \begin{bmatrix} I_{p-m} & 0 \end{bmatrix}.$$  

On the other hand, according item (i) in Theorem 1.1 we have $G(\zeta)Y(\zeta) = G_0$. Furthermore, by definition, $\Xi(\zeta) = Y(\zeta)\Xi_0$. It follows that

$$\tilde{H}(\zeta)Y(\zeta) = \Theta^*(\zeta)(Y(\zeta) - \Xi(z)G(\zeta)Y(z))$$

$$= \Theta^*(\zeta)(\zeta) - \Xi(z)G(\zeta) = \Theta^*(\zeta)Y(\zeta)(I_p - \Xi_0G_0).$$

Again using (2.1), we obtain $G_0\Theta_0 = 0$ and $G_0\Xi_0 = I_m$, such that

$$(I - \Xi_0G_0) \begin{bmatrix} \Theta_0 & \Xi_0 \end{bmatrix} = \begin{bmatrix} \Theta_0 & 0 \end{bmatrix}.$$  

This yields

$$\tilde{H}(\zeta)Y(\zeta) \begin{bmatrix} \Theta_0 & \Xi_0 \end{bmatrix} = \Theta^*(\zeta)Y(\zeta) \begin{bmatrix} \Theta_0 & 0 \end{bmatrix} = \Theta^*(\zeta) \begin{bmatrix} \Theta(\zeta) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \Theta(\zeta) & 0 \end{bmatrix} = \begin{bmatrix} I_{p-m} & 0 \end{bmatrix}.$$  

Since $\begin{bmatrix} \Theta_0 & \Xi_0 \end{bmatrix}$ and $Y(\zeta)$ both are invertible, we obtain that $H(\zeta) = \tilde{H}(\zeta)$. But $\zeta$ is an arbitrary point on $T$. Therefore, $H = \tilde{H}$.

\section{Solutions to the $H^2$-Bezout problem}

Let $G \in H_{m\times p}^\infty$ and assume $T_GT_G^*$ is strictly positive. If $G$ is not in $W_{+}^{m\times p}$, then the function $\Xi$ defined in (1.6) will, in general, not be in $W_{+}^{p\times m}$, and hence not a solution to the Wiener-Bezout problem associated with $G$. However, by Propositions 2.2 and 2.3 the function $\Xi$ is still analytic on $\mathbb{D}$ and satisfies $G(z)\Xi(z) = I_m$ for each $z \in \mathbb{D}$. It turns out that $r\Xi$ is in $H_{p\times m}^2$ and hence a solution to the $H^2$-Bezout problem associated with $G$. In fact, extending the description of all solutions to the Wiener-Bezout problem of Theorem 1.1 via (1.9) to one where $V$ is taken from $H_{(p-m)\times m}^2$, all solutions to the $H^2$-Bezout problem are obtained, even if $G \not\in W_{+}^{m\times p}$. The details are given in the following theorem.

\begin{theorem}
Let $G \in H_{m\times p}^\infty$ such that $T_GT_G^*$ is strictly positive. Define the functions $\Xi$ and $\Theta$ by (1.6), with $\Xi_0$ and $\Theta_0$ as in (M1) and (M2). Then $\Xi \in H_{p\times m}^2$.
\end{theorem}
is a bounded operator from $F$ which is clearly bounded as an operator from $F$, contrast, for $F$, Then

\begin{equation}
\text{Proof of Proposition 4.2.}
\end{equation}

Identifying $C$ into $F$, we have

\begin{equation}
\|X(z)u\|_{\ell^2_p} = \|\Xi(z)u\|_{\ell^2_p} + \|V(z)u\|_{\ell^2_p} \quad (u \in C^m).
\end{equation}

In particular, $\Xi$ is the last square solution to the $H^2$-Bezout problem associated with $G$.

Theorem 4.1 gives a variation on the last part of our main result, Theorem 4.1 above, under the weaker assumption $G \in H^{\infty}_{m\times p}$. Variations on the other claims made in Theorem 4.1 e.g., (1.7) and items (i)–(iii), under this weaker assumption were proved in Propositions 2.2, 2.3 and 2.7 above.

We shall first prove the next proposition, which contains the key observation needed in the proof of Theorem 4.1.

**Proposition 4.2.** Let $G \in H^{\infty}_{m\times p}$ such that $T_G T_G^*$ is strictly positive. Then the function $Y$ defined by (1.4) as well as the function $Y(\cdot)^{-1}$ are in $H^2_{p\times p}$. In particular, $\det Y(z) \neq 0$ for almost every $z \in \mathbb{T}$.

In order to prove Proposition 4.2 we require some additional notation. Let $F \in H^2_{r\times s}$. Then $F$ admits a Taylor expansion

\begin{equation}
F(z) = F_0 + zF_1 + z^2F_2 + \cdots \quad (z \in \mathbb{D})
\end{equation}

and induces a bounded operator

\begin{equation}
\Gamma_F = \begin{bmatrix}
F_0 \\
F_1 \\
\vdots
\end{bmatrix} : \mathbb{C}^s \rightarrow \ell^2_r(\mathbb{C}^r) \quad \text{with} \quad \|\Gamma_F u\|_{\ell^2_r(\mathbb{C}^r)} = \|F(\cdot)u\|_{H^2} \quad (u \in \mathbb{C}^s).
\end{equation}

In fact, an analytic $r \times s$ matrix function $F$ as in (4.3) is in $H^2_{r\times s}$ if and only if $\Gamma_F$ above induces a bounded operator from $\mathbb{C}^s$ into $\ell^2_r(\mathbb{C}^r)$. On the other hand, if $K$ is a bounded operator from $\mathbb{C}^s$ into $\ell^2_r(\mathbb{C}^r)$, then $K = \Gamma_F$ for some $F \in H^2_{r\times s}$; in this case $F_n := E^*_rS^{n\ast}_rK$ is the $n$-th Taylor coefficient of $F$.

With $F \in H^2_{r\times s}$ we associate a function $F_s$ defined by

\begin{equation}
F_s(z) = F(\bar{z})^* = F_0^* + zF_1^* + z^2F_2^* + \cdots \quad (|z| < 1).
\end{equation}

Then $F_s \in H^2_{s\times r}$. However, $\Gamma_F$ and $\Gamma_{F_s}$ need not have the same operator norm. In contrast, for $F$ in $H^2_{r\times s}$, we have $F_s \in H^\infty_{s\times r}$ and $\|T_F\| = \|T_{F_s}\| = \|T_{F_s}\| = \|T_{F_s}\|$.

**Proof of Proposition 4.2.** Identifying $\ell^2_+(\mathbb{C}^p)$ with $\mathbb{C}^p \oplus \ell^2_+(\mathbb{C}^p)$, we see that

\begin{equation}
\Gamma_Y = \begin{bmatrix}
I_p \\
-T_G^*(T_GT_G)^{-1}H_GE_p
\end{bmatrix} : \mathbb{C}^p \rightarrow \begin{bmatrix}
\mathbb{C}^p \\
\ell^2_+(\mathbb{C}^p)
\end{bmatrix},
\end{equation}

which is clearly bounded as an operator from $\mathbb{C}^p$ into $\ell^2_+(\mathbb{C}^p)$. Hence $Y \in H^2_{p\times p}$.

Now define $F$ on $\mathbb{D}$ by

\begin{equation}
F(z) = I + zE^*_p(I - zS^*_p)^{-1}H^*_G(T_GT_G)^{-1}T_G E_p \quad (z \in \mathbb{D}).
\end{equation}
Again identifying $\ell^2_+ (C^p)$ with $C^p \oplus \ell^2_+ (C^p)$, this in turn shows that

$$
\Gamma_F = \begin{bmatrix}
I_p & H_G(T_GT_G^*)^{-1} T_G E_p \\
H_G^*(T_GT_G^*)^{-1} T_G E_p & C^p
\end{bmatrix} : C^p \to \ell^2_+ (C^p)
$$

Hence $\Gamma_F$ is bounded, and thus $F$ is in $H^2_{p \times p}$. Moreover, by (2.3), we have

$$
F(z) = I + zE_pT_G(T_GT_G^*)^{-1}H_G(I - zS_p)^{-1}E_p = Y(z)^{-1} \quad (z \in \mathbb{D}).
$$

Since $F \in H^2_{p \times p}$, we obtain that $F \in H^2_{p \times p}$. Hence $Y(\cdot)^{-1}$ is in $H^2_{p \times p}$.

Next we show that all solutions to the $H^2_{p \times p}$ problem associated with $G$ are obtained through (1.1). To do this, we first note that (2.8) implies that $\Gamma_{\Xi} = T_G^*(T_GT_G^*)^{-1}E_m$.

Now let $X \in H^2_{p \times m}$ be a solution to (1.1). Then $\Gamma_X$ is bounded and (1.1) translates to $T_G \Gamma_X = E_m$. We thus obtain that

$$
T_G(T_GT_G^*)^{-1} \Gamma_X = T_G^*(T_GT_G^*)^{-1}E_m = \Gamma_{\Xi}.
$$

Note that $T_G(T_GT_G^*)^{-1}T_G = I - P_{\ker T_G} = I - T_0T_0^*$. Hence

$$
\Gamma_X = T_G(T_GT_G^*)^{-1} \Gamma_X + T_0T_0^* \Gamma_X = \Gamma_{\Xi} + T_0 \Gamma_X,
$$

where $V \in H^2_{(p-m) \times m}$ is determined by $\Gamma_V = T_G^* \Gamma_X$. The above identity implies $X$ is given by (1.1) with $V \in H^2_{(p-m) \times m}$ such that $\Gamma_V = T_G^* \Gamma_X$.

It remains to derive the identity (4.2). But this can be done by using the same argumentation as in the proof of Theorem 1.1 (see (3.2)); we omit the details. \qed

Let $G \in H^\infty_{m \times p}$ be such that $T_GT_G^*$ is strictly positive. Then $Y \in H^2_{p \times p}$ by Proposition 4.3. Now assume $Y \in H^\infty_{p \times p}$. In that case all solutions to the $H^\infty$-corona problem associated with $G$ are given by formula (1.9). More precisely we have the following proposition.

Proposition 4.3. Let $G \in H^\infty_{m \times p}$ be such that $T_GT_G^*$ is strictly positive, and assume that the function $Y$ defined by (1.4) belongs to $Y \in H^\infty_{p \times p}$. Then $Y$ is invertible outer, and all solutions to the $H^\infty$-corona problem are given by (1.9) where the free parameter is any $V \in H^\infty_{(p-m) \times m}$.

Proof. Assume the function $Y$ defined by (1.4) belongs to $Y \in H^\infty_{p \times p}$. Then $\Xi$ defined in (1.9) is in $H^\infty_{p \times p - m}$, and hence $\Xi$ is a solution to the $H^\infty$-corona problem associated with $G$. Since the inner function $\Theta$ is in $H^\infty_{p \times p - m}$, we obtain that $X$ defined by (1.9) is in $H^\infty_{p \times m}$ whenever the parameter $V$ is in $H^\infty_{(p-m) \times m}$. Hence the map $V \mapsto X$ in (1.9) produces solutions to the $H^\infty$-corona problem when restricted to parameters $V \in H^\infty_{(p-m) \times m}$.
Next we show that all solutions to the $H^\infty$-corona problem associated with $G$ are obtained by (1.1) when the parameters $V$ are restricted to $H_{(p-m)\times m}^\infty$. Assume $X \in H_{p \times m}^\infty$ satisfies (1.4). Then $X$ is also a $H^2$-solution of (1.4), and hence $X$ is given by (1.1) for some $V \in H_{(p-m)\times m}^2$. It remains to show that $V \in H_{(p-m)\times m}^\infty$. The latter follows by considering the values of $V$ in $\mathbb{T}$, and noting that for almost every $\zeta \in \mathbb{T}$ we have

$$\|V(\zeta)\| = \|\Theta(\zeta)V(\zeta)\| = \|X(\zeta) - \Xi(\zeta)\| \leq \|X(\zeta)\| + \|\Xi(\zeta)\| \leq \|X\|_\infty + \|\Xi\|_\infty.$$ 

Hence $\|V\|_\infty \leq \|X\|_\infty + \|\Xi\|_\infty < \infty$, and thus $V \in H_{(p-m)\times m}^\infty$.

We conclude with the proof that $Y \in H_{p \times p}^\infty$ implies that the function $Y(\cdot)^{-1}$ also belongs to $H_{p \times p}^\infty$, i.e., that $Y$ is invertible outer. To see that this is the case, recall from Section 3 that the function $H$ defined by (2.0), on the circle is given by

$$H(z) = \Theta(z)^*(I - \Xi(z)G(z)) \quad (\text{a.e. } z \in \mathbb{T}).$$

(Note that this observation does not require $G \in \mathcal{W}_{m \times p}^\infty$.) Since $\Theta$, $\Xi$ and $G$ are all $H^\infty$-functions, it follows that $H$ is essentially bounded on $\mathbb{T}$, and thus $H \in H_{(p-m)\times p}^\infty$. This implies that the function

$$z \mapsto \begin{bmatrix} G(z) \\ H(z) \end{bmatrix} = \begin{bmatrix} G_0 \\ H_0 \end{bmatrix} Y(z)^{-1}$$

is in $H_{p \times p}^\infty$. By the invertibility of $[G_0 \thinspace H_0]$, it follows that $Y^{-1}$ is in $H_{p \times p}^\infty$, as claimed. \qed

Note that the description of the solutions to the $H^\infty$-corona problem associated with $G$ obtained in this way is much simpler than the one obtained in [7, Remark 4.1]. However, while the description (1.1) has a favorable behavior with respect to the $H^2$-norm (see (1.2)), there is no clear connection between the supremum norms of the parameter $V$ and the solution $X$ related through (1.1), making it a less suitable way to describe solutions with an additional bound on the supremum norm.

Furthermore, at this stage it is unknown whether the function $Y$ in (1.4) belongs to $H_{p \times p}^\infty$, or not. Can it happen that $Y$ does not belong to $H_{p \times p}^\infty$, and if so, under what conditions on $G$ does $Y$ belong to $H_{p \times p}^\infty$? Theorem 1.1 yields that $Y \in H_{p \times p}^\infty$ whenever $G \in \mathcal{W}_{m \times p}^\infty$.

5. Concluding remarks

Remark 5.1. Let us assume that the $m \times p$ matrix function $G$ is a constant function, that is, $G(z) = G_0$ for all $z \in \mathbb{D}$. In that case the Bezout-corona equation (1.1) reduces to

$$G_0X(z) = I_m. \quad (5.1)$$

This equation has a solution if and only if the $m \times p$ matrix $G_0$ is right invertible, and in that case a straightforward application of Theorem 1.1 shows that all Wiener class solutions of equation (5.1) are given by

$$X(z) = G_0^*(G_0G_0^*)^{-1} + \tau_0V(z), \quad V \in \mathcal{W}_{+}^{(p-m)\times m}. \quad (5.2)$$

Here $\tau_0$ is an isometry mapping $\mathbb{C}^{p-m}$ onto $\text{Ker} G_0$. This result can also be derived directly by elementary linear algebra, using that $G_0^*(G_0G_0^*)^{-1}$ is the Moore-Penrose
right inverse of \(G_0\). Note the definition of \(\tau_0\) implies that \(\tau_0 \tau_0^*\) is the orthogonal projection of \(\mathbb{C}^p\) onto \(\text{Ker} G_0\), and hence (cf., (1.2)) we have

\[
\tau_0 \tau_0^* = I_p - G_0^*(G_0 G_0^*)^{-1} G_0 \quad \text{and} \quad \text{Ker} \tau_0 = \{0\}.
\]

Finally, in this particular case the function defined by (1.4) is just identically equal to the \(p \times p\) identity matrix.

**Remark 5.2.** Let \(G \in \mathcal{W}_{m \times p}^+\), and assume that \(T_G T_G^*\) is strictly positive. Let \(Y \in \mathcal{W}_{m \times p}^+\) be given by (1.4). Then \(\det (z) \neq 0\) and \(G(z) = G_0 Y(z)^{-1}\) for each \(\lvert z \rvert \leq 1\). Hence equation (1.1) can be rewritten as

\[
G_0 \left( Y(z)^{-1} X(z) \right) = I_m.
\]

But then we can apply the result of the previous remark to show that the set of all Wiener solutions to the Wiener-Bezout problem defined by \(G\) is given by

\[
X(z) = Y(z) \left( G_0^*(G_0 G_0^*)^{-1} + \tau_0 V(z) \right), \quad V \in \mathcal{W}_{(p-m) \times m}^+(\mathbb{R}).
\]

The above representation of the set of all Wiener solutions differs from and is less informative than the one given by (1.9). For instance, in general, the function \(Y(\cdot) \cdot (G_0 G_0^*)^{-1}\) is not the least squares solution.

**Remark 5.3.** If \(G\) is a polynomial, then the function \(Y(\cdot)^{-1}\) is also a polynomial and its degree is less than or equal to the degree of \(G\). This fact is a corollary of formula (1.7) in Theorem 1.1. However, in general, the assumption \(G\) is a polynomial does not imply that \(Y\) is a polynomial. To see this we take \(G(z) = [1 + z - z] + \sum \nu \geq 0 \text{ and } \frac{q(1 - q)}{1 - 2q} = 1.

We compute (5.3). via the formula for the Taylor coefficients of \(Y\) given in (1.5). For this purpose we rewrite the right hand side in the second identity of (1.5) as \(-T_G^* (T_G T_G^*)^{-1} H_G E_2\), and we compute this operator following the approach of [9]. Recall (see [9], Eq. (2.4) or (3.4)) that

\[
(T_G T_G^*)^{-1} = T_R^{-1} + T_R^{-1} H_G (I - H_G T_R^{-1} H_G) -1 H_G^* T_R^{-1},
\]

where \(R = G G^*\). In the present example, where \(G(z) = [1 + z - z] + \sum \nu \geq 0 \text{ and } \frac{q(1 - q)}{1 - 2q} = 1\).

\[
R(z) = \phi(1/z)^* \phi(z), \text{ with } \phi(z) = q^{-1/2} (1 + z) \text{ and } q = \frac{1}{2} (3 - \sqrt{5}.
\]

Furthermore, \(R(z)^{-1} = \psi(z) \psi(1/z)^*\) with

\[
\psi(z) = \sqrt{q} (1 + zq)^{-1} = \sum_{\nu=0}^\infty \sqrt{q} (-q)^\nu z^\nu, \quad (z \in \mathbb{D}).
\]
It follows that \((T_R)^{-1} = T_\psi T_\psi^*\), and
\[
T_\psi = \sqrt{q} \begin{bmatrix} v & S v & S^2 v & \cdots \end{bmatrix} \quad \text{with} \quad v = \begin{bmatrix} 1 & -q & (-q)^2 & (-q)^3 & \cdots \end{bmatrix}^*.
\]
Here \(S\) denotes the forward shift operator on \(\ell^2_+ := \ell^2_+ (\mathbb{C})\). Since \(G(z) = [1 + z & -z]\), we have

\[
T_G = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
1 & -1 & 1 & 0 & 0 & \cdots \\
0 & 0 & 1 & -1 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{bmatrix}, \quad H_G = \begin{bmatrix}
1 & -1 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}.
\]

We then obtain
\[
(T_R)^{-1} H_G = T_\psi T_\psi^* H_G = \sqrt{q} T_\psi H_G = q \begin{bmatrix} v & -v & 0 & 0 & \cdots \end{bmatrix}.
\]

Identifying \(\ell^2_+\) with \(\mathbb{C}^2 \oplus \ell^2_+\), we get
\[
(I - H_G T_R^{-1} H_G)^{-1} = \begin{bmatrix} 1 - q & q & 0 \\
q & 1 - q & 0 \\
0 & 0 & I_{\ell^2_+} \end{bmatrix}^{-1} = \begin{bmatrix} 1/q & -1/\sqrt{q} & 0 \\
-1/\sqrt{q} & 1/q & 0 \\
0 & 0 & I_{\ell^2_+} \end{bmatrix}.
\]

Putting the above computations together yields
\[
(T_G T_G^*)^{-1} = (T_R^{-1} + q^2 \begin{bmatrix} v & -v \end{bmatrix} \begin{bmatrix} 1/q & -1/\sqrt{q} \\
-1/\sqrt{q} & 1/q \end{bmatrix} \begin{bmatrix} v^* \end{bmatrix}) = (T_R^{-1} + v \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} q & -q \sqrt{q} \\
-q \sqrt{q} & q \end{bmatrix} \begin{bmatrix} 1 \\
-1 \end{bmatrix} v^*) = (T_R^{-1} + 2(1 - q) v v^*) = (T_R^{-1} + 2 \sqrt{q} v v^*).
\]

In the third step we used \(q \sqrt{q} = q(1 - q) = 1 - 2q\) which follows from the second and third identity in (4.3). Combining the formula for \((T_G T_G^*)^{-1}\) with the one for \((T_R)^{-1} H_G\) yields
\[
(T_G T_G^*)^{-1} H_G E_2 = (T_R)^{-1} H_G E_2 + 2 \sqrt{q} v v^* H_G E_2 = q \begin{bmatrix} v & -v \end{bmatrix} + 2 \sqrt{q} \begin{bmatrix} v & -v \end{bmatrix} = q(1 + 2/\sqrt{q}) \begin{bmatrix} v & -v \end{bmatrix} = q(1 - 2q)^{-1} \begin{bmatrix} v & -v \end{bmatrix}.
\]

To see that the latter identity holds, note that \(1 - 3q + q^2 = 0\) implies \(3 - q = 1/q\), so that together with \(q \sqrt{q} = 1 - 2q\) we obtain
\[
1 + 2/\sqrt{q} = \frac{\sqrt{q} + 2}{\sqrt{q}} = \frac{(1 - q) + 2}{\sqrt{q}} = 3/\sqrt{q} = \frac{1}{q \sqrt{q}} = (1 - 2q)^{-1}.
\]

Next note that \(T_G^* v = v G(\nu)^*\). By (1.3) we then obtain that for \(\nu = 1, 2, \ldots\) that
\[
Y_\nu = \frac{-q}{1 - 2q} (-q)^{\nu - 1} N = \frac{(-q)^\nu}{1 - 2q} N,
\]
where
\[
N = \begin{bmatrix} G(-q)^* & -G(-q)^* \end{bmatrix} = \begin{bmatrix} \sqrt{q} & -\sqrt{q} \\
q & -q \end{bmatrix} = \begin{bmatrix} 1 - q & -(1 - q) \\
q & -q \end{bmatrix}.
\]
Hence

\[
Y(z) = I_z + \sum_{\nu=1}^{\infty} \frac{(-q)^\nu}{1 - 2q} N z^\nu = I_z + \frac{-q}{(1 - 2q)(1 + zq)} N
\]

\[
= \frac{1}{(1 - 2q)(1 + zq)} \begin{bmatrix}
(1 - 2q)(1 + zq) - q(1 - q)z & q(1 - q)z \\
-q^2z & (1 - 2q)(1 + zq) + zq^2
\end{bmatrix}
\]

\[
= \frac{1}{(1 - 2q)(1 + zq)} \begin{bmatrix}
(1 - 2q) - q^2z & (1 - 2q)z \\
-q^2z & (1 - 2q)(1 + z)
\end{bmatrix}
\]

\[
= \frac{1}{1 + zq} \begin{bmatrix}
1 - (1 - q)z & z \\
-(1 - q)z & 1 + z
\end{bmatrix}.
\]

Here we used \((1 - 2q)(1 + zq) = 1 - 2q + qz - 2q^2z\) and \(q(1 - q) = 1 - 2q\) in the last but one identity, and \(q^2/(1 - 2q) = q(1 - q)^2/(1 - 2q) = 1 - q\) (which follows from the last two identities in (5.1) in the last identity. Hence we obtain \(Y\) is given by (5.3) as claimed.

Using similar calculations as in the previous remark one can prove that for \(G(z) = \begin{bmatrix} 1 + z & -z \end{bmatrix}\) the matrices \(\Xi_0\) and \(\Theta_0\) are given by

\[
\Xi_0 = \frac{q}{1 - 2q} \begin{bmatrix} 1 - q \\ q \end{bmatrix} \quad \text{and} \quad \Theta_0 = \begin{bmatrix} 1 \\ \sqrt{q} \end{bmatrix}
\]

where \(q = \frac{1}{2}(3 - \sqrt{5})\). Note, however, that these formulas for \(\Xi_0\) and \(\Theta_0\) can be derived from earlier results in [9] and [10]. Indeed, \(\Xi_0\) is obtained by taking \(z = 0\) in \(X(z)\) on Page 414, and the formula for \(\Theta_0\) is obtained by taking \(z = 0\) in the formula for \(\Theta(z)\) appearing in the final paragraph of [10].

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