An Explicit, Coupled-Layer Construction of a High-Rate MSR Code with Low Sub-Packetization Level, Small Field Size and \( d < (n - 1) \)

Birenjith Sasidharan, Myna Vajha, and P. Vijay Kumar
Department of Electrical Communication Engineering, Indian Institute of Science, Bangalore.
Email: \{biren, myna, vijay\}@ece.iisc.ernet.in

Abstract
This paper presents an explicit construction for an \((n = 2qt, k = 2q(t - 1), d = n - (q + 1)), (\alpha = q(2q)^{t-1}, \beta = \frac{\alpha}{q})\) regenerating code over a field \(\mathbb{F}_q\) operating at the Minimum Storage Regeneration (MSR) point. The MSR code can be constructed to have rate \(k/n\) as close to \(1\) as desired, sub-packetization level \(\alpha \leq r\) for \(r = (n - k)\), field size \(Q\) no larger than \(n\) and where all code symbols can be repaired with the same minimum data download. This is the first-known construction of such an MSR code for \(d < (n - 1)\).

I. INTRODUCTION

In an \((n, k, d), (\alpha, \beta)\) regenerating code \([1]\) over the finite field \(\mathbb{F}_q\), a file of size \(B\) over \(\mathbb{F}_q\) is encoded and stored across \(n\) nodes in the network with each node storing \(\alpha\) coded symbols. The parameter \(\alpha\) is termed as the sub-packetization level of the code. A data collector can download the data by connecting to any \(k\) nodes. In the event of node failure, node repair is accomplished by having the replacement node connect to any \(d\) nodes and downloading \(\beta \leq \alpha\) symbols from each node. The quantity \(d\beta\) is termed the repair bandwidth. The focus here is on exact repair, meaning that at the end of the repair process, the contents of the replacement node are identical to that of the failed node.

It is well known that the file size \(B\) must satisfy the upper bound (see \([1]\)):

\[
B \leq \sum_{\ell=1}^{k} \min\{\alpha, (d - \ell + 1)\beta\}. \tag{1}
\]

It follows from this that \(B \leq k\alpha\) and equality is possible only if \(\alpha \leq (d - k + 1)\beta\). A regenerating code is said to be a Minimum Storage Regenerating (MSR) code if \(B = \alpha k\) and \(\alpha = (d - k + 1)\beta\), since the amount \(n\alpha\) of data stored for given file size \(B\) is then the minimum possible.

A. Literature and Our Contribution

The definition of an MSR code requires that all nodes be repairable with the same minimum data download. There are papers however in the literature that refer to a code as being an MSR code even if the data download is a minimum only for the repair of systematic nodes. We will distinguish between the two classes by referring to them as all-node-repair and systematic-repair MSR codes respectively.

Several constructions of MSR codes can now be found in the literature. The product-matrix construction \([2]\), provides MSR codes for any \(2k - 2 \leq d \leq n - 1\). A construction for all-node-repair MSR codes with \(d = n - 1 \geq 2k - 1\) is presented in \([3]\) that builds on the systematic-repair codes constructed in \([4]\). In \([5]\), high-rate MSR codes with parameters \((n, k = n - 2, d = n - 1)\) are constructed using Hadamard designs. In \([6]\), high-rate systematic-repair MSR codes, known as zigzag codes, are constructed for \(d = n - 1\). This was subsequently extended to include the repair of parity nodes as well in \([7]\). In \([8]\), a construction of systematic-repair MSR codes is given, that makes use of permutation matrices. In \([9]\), Cadambe et al. show the existence of high-rate MSR codes for any value of \((n, k, d)\) as \(\alpha\) scales to infinity.
Desirable attributes of an MSR code include an explicit construction, high-rate, low values of sub-packetization level $\alpha$ and small field size. While zigzag codes allow arbitrarily high rates to be achieved, a level of sub-packetization that is exponential in $k$ is required. In a subsequent paper [10], a systematic-repair MSR code having $\alpha = r^{\frac{k}{n}}$ is constructed. In [11], the following lower bound on $\alpha$ is presented:

$$2 \log_2 \alpha \left( \frac{\log \left( \frac{q}{\alpha} \right)}{\alpha} + 1 \right) \geq k. \quad (2)$$

A second lower bound on $\alpha$, $\alpha \geq r^{\frac{k}{n}}$, can be found in [12], that applies to a subclass of MSR codes known as help-by-transfer (also known in the literature as access-optimal) MSR codes. For help-by-transfer MSR codes, the number of symbols transmitted as helper data over the network is equal to the number of symbols accessed at the helper nodes. Prior to this in [13], the authors presented a construction of a systematic-repair MSR code that permits rates in the regime $\frac{2}{3} \leq R \leq 1$, and that has an $\alpha$ that is polynomial in $k$. In [14], a high-rate MSR construction for $d = n - 1$ is presented that has sub-packetization level $r^\frac{k}{n}$ and where all nodes are repaired with minimum data download. The construction provided was however, not explicit, and required large field size. This is extended for general $k \leq d \leq n - 1$ in [15]. In [16], the authors provide a construction for a systematic-repair MSR code for all $k \leq d \leq n - 1$, but these constructions are also non-explicit and require large field size. In [17], explicit help-by-transfer systematic-repair MSR codes are presented with sub-packetization meeting the lower bound $\alpha \geq r^{\frac{k}{n}}$. However, the constructions were limited for $r = 2, 3$. Though suboptimal in terms of repair bandwidth, a vector-MDS code supporting a family of $\alpha = r^p, p \geq 1$ and efficient node-repair is presented in [18].

Most recently, in [19], Ye and Barg present an explicit construction of a high-rate MSR code having rate $k/n$ as close to 1 as desired, sub-packetization level $\alpha = r^{\frac{k}{n}}$ for $r = (n-k)$, field size $Q$ no larger than $n$, $d = (n-1)$ and where all code symbols can be repaired with the same minimum data download. Essentially the same construction was rediscovered, albeit some two months later, by the authors of the present paper in [20]. The construction in [20] builds on the earlier construction in [14]. The authors of [18] observe that the construction in [19] can be extended for $d < n - 1$ using the technique suggested in [15], resulting in a non-explicit construction. In [21], the authors present an MSR code construction for $d < n - 1$ that requires large field size.

In the present paper, we show how the construction in [19] (or [20]) can be modified to handle the case when $d < (n - 1)$ without requiring any expansion in field size. A smaller value of $d$ is appealing in practice because it provides greater flexibility in handling node repair. For instance, it allows one to avoid calling upon nodes that are either slow to respond or else, are otherwise occupied.

II. DESCRIPTION OF THE MSR CODE

A. Code Parameters

Let $q \geq 2, t \geq 2$ be integers. Let $\mathbb{Z}_{2q}$ denote the set of integers modulo $2q$, $[t]$ denote the set set $\{1, 2, \cdots, t\}$ and $[0, 2q - 1]$ denote the set of integers $\{0, 1, \cdots, 2q - 1\}$. We describe below the construction of an $(n, k, d; (\alpha, \beta))$ high-rate MSR code over a finite field $\mathbb{F}_Q$ having parameters

$$\begin{align*}
(n = 2qt, \quad k = 2q(t - 1), \quad d = n - (q + 1)) , \\
(\alpha = q \cdot (2q)^{t-1}, \quad \beta = (2q)^{t-1}) \quad \text{and} \quad Q \leq n .
\end{align*}$$

Hence the code has rate $\frac{(t-1)}{t}$ and field size no larger than that of a scalar MDS code of the same block length. We note that through shortening, we can obtain MSR codes having $(n, k, d) = (n - \Delta_s, k - \Delta_s, d - \Delta_s)$ for $0 \leq \Delta_s \leq k - 1$. Through puncturing, we can obtain MSR codes having $(n, k, d) = (n - \Delta_p, k, d)$ for $0 \leq \Delta_p \leq n - d - 1$. A few example parameters are given in the table below:

| $(q, t), \Delta_s/\Delta_p$ | Parameter set |
|-----------------------------|---------------|
| $(2, 3)$                    | $n \quad k \quad d \quad \alpha$ |
| $(2, 3)$                    | 12 8 9 32 |
| $(2, 3), \Delta_p = 1$     | 11 8 9 32 |
| $(2, 3), \Delta_s = 2$     | 10 6 7 32 |
| $(2, 4)$                    | 16 12 13 128 |
| $(3, 4)$                    | 24 16 20 648 |
B. The Data Cube

The MSR code constructed here can be described in terms of an array of symbols over \( \mathbb{F}_q \) as given below:
\[
\mathcal{A} = \left\{ A(x, y; \tilde{z}) \mid x \in \mathbb{Z}_{2q}, y \in [t], \tilde{z} \in \mathbb{Z}_2^t \right\}.
\]
This array can be depicted as a data cube, see Fig. 1(a) of size \( (2q \times t \times (2q)^t) \). In the figure, the cube appears as a collection of \( (2q)^t \) planes, with each horizontal plane indexed by the parameter \( \tilde{z} \).

From the point of view of the MSR code, the data cube corresponds to the data contained in a total of \( n = 2qt \) nodes, where each node is indexed by the pair of variables:
\[
\{(x, y) \mid x \in \mathbb{Z}_{2q}, y \in [t]\}.
\]
The \((x, y)\)th node stores the \( c_0 = (2q)^t \) symbols
\[
C(x, y) = \{ A(x, y; \tilde{z}) \mid \tilde{z} \in \mathbb{Z}_2^t \}.
\]
(3)

Thus each codeword in the MSR code is made up of the \( n = 2qt \) vector code symbols \( \{C(x, y) \mid x \in \mathbb{Z}_{2q}, y \in [t]\} \), in which each vector has \( (2q)^t \) components indexed by \( \tilde{z} \). It will be explained in Sec. [III-A] how the \( c_0 \) components in a vector are mapped to \( \alpha \) symbols of a node in the MSR code. Let \( \Theta \) be a Vandermonde matrix that forms a parity-check matrix of an \([n, k]-\text{MDS}\) code \( \mathcal{J} \). This can be constructed using field size \( n \). We denote by \( \theta_{(x, y)}^\ell \) the entry of \( \Theta \) at the location \((\ell, (x, y)), \ell \in [0, 2q - 1], (x, y) \in \mathbb{Z}_{2q} \times [t]\). Let \( u \in \mathbb{F}_q \) satisfy \( u \neq 0, u^2 \neq 1 \).

By a slight abuse of notation, we will refer to the symbols \( A(x, y; \tilde{z}) \) as code symbols (as opposed to calling them components of a code symbol) as most of our discussion will involve the symbols \( A(x, y; \tilde{z}) \).

C. Companion Terms, Transformed Code Symbols

Let us define
\[
\tilde{z}(x, y) = \begin{cases} 
(x, z_2, \ldots, z_t), & y = 1, \\
(z_1, \ldots, z_{y-1}, x, z_{y+1}, \ldots, z_t), & 2 \leq y \leq t - 1, \\
(z_1, z_2, \ldots, z_{t-1}, x), & y = t,
\end{cases}
\]
in other words, \( \tilde{z}(x, y) \), is obtained by replacing the \( y \)th component of \( \tilde{z} \) by \( x \). We next set
\[
A^c(x, y; \tilde{z}) = A(\tilde{z}_y, y; \tilde{z}(x, y)),
\]
and regard \( \{A(x, y; \tilde{z}), A^c(x, y; \tilde{z})\} \) as a set of paired elements and \( A^c(x, y; \tilde{z}) \) as the companion of \( A(x, y; \tilde{z}) \). Conversely, \( A(x, y; \tilde{z}) \) is the companion of \( A^c(x, y; \tilde{z}) \). Note however, that if \( z_y = x \), then \( A^c(x, y; \tilde{z}) = A(x, y; \tilde{z}) \) and the element \( A(x, y; \tilde{z}) \) is paired with itself. For \( \tilde{z} \) such that \( z_y \neq x \), we introduce the transformed code symbols \( B(x, y; \tilde{z}), B^c(x, y; \tilde{z}) \):
\[
\begin{bmatrix}
B(x, y; \tilde{z}) \\
B^c(x, y; \tilde{z})
\end{bmatrix} = \begin{bmatrix}
1 & u \\
u & 1
\end{bmatrix}
\begin{bmatrix}
A(x, y; \tilde{z}) \\
A^c(x, y; \tilde{z})
\end{bmatrix}.
\]
where the inverse transformation is given by

\[
\begin{bmatrix}
A(x, y; \bar{z}) \\
A^c(x, y; \bar{z})
\end{bmatrix} = \frac{1}{1 - u^2} \begin{bmatrix}
1 & -u \\
-u & 1
\end{bmatrix} \begin{bmatrix}
B(x, y; \bar{z}) \\
B^c(x, y; \bar{z})
\end{bmatrix}.
\]

If however, \( z_y = x \), we simply define

\[ B(x, y; \bar{z}) = B^c(x, y; \bar{z}) = A(x, y; z) = A^c(x, y; \bar{z}). \]

It can be verified that all 4 elements \( \{ B(x, y; \bar{z}), B^c(x, y; \bar{z}), A(x, y; z), A^c(x, y; \bar{z}) \} \) can be determined from any 2 of them.

![Fig. 2. Illustrating 3 sets of paired symbols \((A(x, y; \bar{z}), A^c(x, y; \bar{z}))\).](image)

D. Parity-Check Equations

The parity-check (p-c) equations required to be satisfied by the symbols \( A(x, y; \bar{z}) \) are of two types: \( B \)-plane p-c equations and nodal p-c equations.

The \( B \)-plane p-c equations are expressed in terms of the transformed code symbols \( B(x, y; \bar{z}) \) and are given by:

\[
\sum_{x \in \mathbb{Z}_{2q}} \sum_{y \in [t]} \theta^\ell_{(x,y)} B(x, y; \bar{z}) = 0, \quad z \in \mathbb{Z}_{2q}, \quad \ell \in [0, 2q - 1].
\] (4)

Thus there are in all, \((2q) \times (2q)^t\) \( B \)-plane p-c equations with 2q equations indexed by the parameter \( \ell \) per plane \( \bar{z} \).

The nodal p-c equations involve only the symbols \( A(x_0, y_0; \bar{z}) \) lying within the same node. For fixed \((x_0, y_0) \in \mathbb{Z}_{2q} \times [t] \), there are a total of \((q \times (2q)^{t-1})\) equations of the form

\[
A(x_0, y_0; \bar{z}) \theta^\ell_{(x_0,y_0)} + \sum_{x'_{y_0} \neq x_0, z'_{y_0} \neq y_0} A(x_0, y_0; z') \theta^\ell_{(x_0,y_0)} = 0,
\] (5)

obtained by varying \( \ell \), over \( 0 \leq \ell \leq (q - 1) \) and varying \( z_i, 1 \leq i \leq t, i \neq y_0 \) over all of \( \mathbb{Z}_{2q} \), with \( z_{y_0} = x_0 \) fixed. These can be alternately be described in terms of their companions as given below:

\[
A(x_0, y_0; \bar{z}) \theta^\ell_{(x_0,y_0)} + \sum_{x \neq x_0} A^c(x, y_0; z) \theta^\ell_{(x,y_0)} = 0,
\] (6)

where the \((q \times (2q)^{t-1})\) equations are obtained this time, by varying \( \ell \), over \( 0 \leq \ell \leq (q - 1) \) and varying \( z \in \mathbb{Z}_{2q}^t \) while maintaining \( z_{y_0} = x_0 \).

III. Parameters of the Proposed MSR Code

In the sections to follow, it will be shown that the code constructed above, yields an MSR code having parameters

\((n = 2qt, k = 2q(t - 1), d = n - q - 1), (\alpha = (2q)^t/2, \beta = (2q)^{t-1})\).
A. The Value of $\alpha$

With respect to the data cube $\{A(x, y; \bar{z}) \mid x \in \mathbb{Z}_{2q}, y \in [t], \bar{z} \in \mathbb{Z}_{2q}\}$, each pair $(x, y)$ identifies a distinct node. At the outset each node appears to contain $(2q)^t$ symbols leading to $\alpha = (2q)^t$. However, these symbols are not linearly independent, since they are subject to the nodal parity-check equations (5). For a given node $(x_0, y_0)$, there are a total of $(2q)^t/2$ parity-check equations corresponding to a parity-check matrix $J$ having a block-diagonal form:

$$
J = \begin{bmatrix}
J_0 & J_0 & \cdots & J_0 \\
\vdots & \vdots & \ddots & \vdots \\
J_0 & J_0 & \cdots & J_0
\end{bmatrix}
$$

Each of the matrices $J_0$ is a Vandermonde matrix, hence $J$ has full rank, which means that each node contains just $(2q)^t/2$ linearly independent symbols. We can thus set $\alpha = (2q)^t/2$.

B. The File Size $B$

While setting $\alpha = \alpha_0/2$, we puncture out half of the code symbols from the datacube. Let $S \subset [n\alpha_0]$ be the set of indices at which symbols are punctured out. Then we need to consider only a subset of parity-check equations in (4) and (5), that have all zeros at locations in $S$. It can be observed that the number of such equations is $\leq (n-k)\alpha$, and therefore the file size $B \geq k\alpha$. In Sec. V we will present a decoding algorithm that corrects failure of any $(n-k)$ nodes. Thus the file size $B = k\alpha$.

IV. Pictorial Representation for Planes that Identifies Erased Nodes

We associate with each plane $\bar{z}$, a $(2q \times t) \{0, 1\}$ incidence matrix $P(\bar{z})$ given by

$$
P(x,y)(\bar{z}) = \begin{cases} 1 & z_y = (x-1) \\ 0 & \text{else.} \end{cases}
$$

Let $E = \{(x_i, y_i) \in \mathbb{Z}_{2q} \times [t] \mid 1 \leq i \leq 2q\}$ denote the location of the $2q$ erased nodes. Given an erasure pattern $\mathcal{E}$ and a plane $\bar{z}$ we associate with a $(2q \times t) \{0, 1\}$ incidence matrix $P(E, \bar{z})$ which is the matrix $P(\bar{z})$ with the entries corresponds to the erased nodes circled. For example, if $\mathcal{E} = \{(0, 2), (1, 2), (2, 2), (2, 4)\}$, with $\bar{z} = [1 2 3 1 0]^t$, we obtain:

$$
P(E, \bar{z}) = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}.
$$

A. Intersection Score of an Erasure Pattern on a Plane

Given a plane $\bar{z} \in \mathbb{Z}_{2q}^t$ and an erasure pattern $\mathcal{E}$, we define the intersection score $\sigma(E, \bar{z})$ to be given by

$$
\sigma(E, \bar{z}) = | \{ y \in [t] \mid (z_y, y) \in \mathcal{E} \} |,
$$

and set $\sigma_{\text{max}}(E) = \max \{ \sigma(E, \bar{z}) \mid \bar{z} \in \mathbb{Z}_{2q}^t \}$. In terms of the matrix $P(E, \bar{z})$, the intersection score equals the number of circled entries that equal 1, and hence $\sigma(E, \bar{z}) = 1$ in the example above.

V. Sequential Decoding Approach to Data Collection

The data collection property requires that we can recover the data in the presence of $(n-k) = 2q$ erasures. Let $\mathcal{E} = \{(x_i, y_i) \mid 1 \leq i \leq 2q\}$ be a fixed erasure pattern. First, we make use of the nodal equations to recover $\alpha$ symbols in each of the $k$ surviving nodes. Then the aim is to recover the erased code symbols, $\{A(x_i, y_i; \bar{z}) \mid 1 \leq i \leq [t], \bar{z} \in \mathbb{Z}_{2q}\}$. We adopt a sequential procedure in which the erased symbols are decoded successively in increasing order of intersection score $s$, $0 \leq s \leq \sigma_{\text{max}}(E)$. The decoding algorithm that relies upon only the $B$-plane p-c equations is same as the one in [19], [20], while we adopt the notion used in [20].
A. Case of Zero Intersection Score

Let $\mathbf{z}$ be a fixed plane having intersection score zero. The $2q$ $B$-plane p-c equations associated to $\mathbf{z}$ are given by

$$\sum_{x \in \mathbb{Z}_{2q}, y \in [t]} \left\{ A(x, y; \mathbf{z}) + uA^c(x, y; \mathbf{z}) \right\} \theta^l_{(x, y)} = 0.$$  

Since $\sigma(\mathcal{E}, \mathbf{z}) = 0$, we have that $(z_y, y) \notin \mathcal{E}$, for any $y \in [t]$. As a result, the companion symbol $A^c(x, y; \mathbf{z})$ which lies in node $(z_y, y)$, is not erased. It follows that for symbols $A(x, y; \mathbf{z})$ with $(x, y) \notin \mathcal{E}$, both $A(x, y; \mathbf{z})$ and $A^c(x, y; \mathbf{z})$ are known. The same argument tells us that for symbols $A(x, y; \mathbf{z})$ with $(x, y) \in \mathcal{E}$, while $A(x, y; \mathbf{z})$ is unknown, $A^c(x, y; \mathbf{z})$ is known. Hence, we can rewrite the parity-check equations associated to plane $\mathbf{z}$ equations in the form:

$$\sum_{(x, y) \in \mathcal{E}} A(x, y; \mathbf{z}) \theta^l_{(x, y)} = \kappa_*,$$

where $\kappa_*$ is generic notion for a known element in the finite field $\mathbb{F}_q$ that can be determined from the non-erased code symbols. We are thus left with a set of $2q$ equations involving $2q$ unknowns and a Vandermonde coefficient matrix, so the symbols $A(x, y; \mathbf{z})$ lying in a place $\mathbf{z}$ having intersection-score zero can in this way, be recovered.

B. Case of Intersection Score $\sigma > 0$

We show how one can inductively recover code symbols corresponding to planes $\mathbf{z}$ having intersection score $\sigma(\mathcal{E}, \mathbf{z}) > 0$, given that symbols in planes $\mathbf{z}'$ with $\sigma(\mathcal{E}, \mathbf{z}') < \sigma$ have already been recovered. We have already carried out recovery of code symbols in planes with intersection score 0, settling the first step of the induction.

Let an erasure pattern $\mathcal{E}$ and a plane $\mathbf{z}$ be fixed. We first partition the $2q$-erasure location set $\mathcal{E}$ into disjoint subsets:

$$\mathcal{E}_{0, \mathbf{z}} = \{(x, y) \in \mathcal{E} \mid x = z_y\},$$

$$\mathcal{E}_{1, \mathbf{z}} = \{(x, y) \in \mathcal{E} \mid (z_y, y) \notin \mathcal{E} \text{ hence } x \neq z_y\},$$

$$\mathcal{E}_{2, \mathbf{z}} = \{(x, y) \in \mathcal{E} \mid (z_y, y) \in \mathcal{E}, x \neq z_y\}.$$

It can be verified that in the case of a symbol $A(x, y; \mathbf{z})$ with $(x, y) \notin \mathcal{E}$, the companion symbol $A^c(x, y; \mathbf{z})$ lies either in an unerased node or else in a plane having lower intersection score and which have hence, already been recovered. For this reason, we can assume that the symbols $B(x, y; \mathbf{z})$ with $(x, y) \notin \mathcal{E}$ are known and the parity-check equations in the inductive decoding process, can once again, be restricted to the erased symbols and their companions, i.e., can be assumed to be of the form:

$$\sum_{(x, y) \in \mathcal{E}} B(x, y; \mathbf{z}) \theta^l_{(x, y)} = \kappa_*.$$

These equations allow us to determine the value of the transformed code symbols $\{B(x, y; \mathbf{z}) \mid (x, y) \in \mathcal{E}\}$.

- In the case of symbols $\{B(x, y; \mathbf{z}) \mid (x, y) \in \mathcal{E}_{0, \mathbf{z}}\}$, we have $A(x, y; \mathbf{z}) = B(x, y; \mathbf{z})$ and thus we have recovered the symbols $A(x, y; \mathbf{z})$ in this instance.
- In the case of the symbols $\{B(x, y; \mathbf{z}) \mid (x, y) \in \mathcal{E}_{1, \mathbf{z}}\}$, we have that the complement $A^c(x, y; \mathbf{z})$ does not belong to an erased node and is hence known. From $B(x, y; \mathbf{z})$ and $A^c(x, y; \mathbf{z})$ one can recover $A(x, y; \mathbf{z})$, and so we are done even in this case.
- This leaves us only with having to recover symbols $\{A(x, y; \mathbf{z}) \mid (x, y) \in \mathcal{E}_{2, \mathbf{z}}\}$. In the case of such symbols, the companion $A^c(x, y; \mathbf{z})$ can be verified to also belong to a plane having the same intersection score and hence we can assume that both $B(x, y; \mathbf{z})$ and $B^c(x, y; \mathbf{z})$ have been determined. From these values, once can determine the value of $A(x, y; \mathbf{z})$.

This concludes the decoding process.

VI. NODE REPAIR

We turn in this section to node repair and assume node $(x_1, y_1)$ to be the failed node. Since there are a total of $d = n - q - 1$ helper nodes, there are a set of $q$ nodes which do not participate in the repair process and which we will term as aloof nodes. Nodes that are not aloof and which do not correspond to the failed node, will be termed as helper nodes.
A. Aligned and Unaligned Nodes

We will declare that two nodes to be *aligned* if their $y$ coordinates are the same. Let $\{(x_i, y_i), 2 \leq i \leq (q + m)\}$ denote the coordinates of the helper nodes aligned with $(x_1, y_1)$. Let us assume that of the $q$ aloof nodes, $(q - m)$ aloof nodes, namely, $\{(x_i, y_i), q + m + 1 \leq i \leq 2q\}$, are aligned with the failed node and $m$ of them, namely, $\{(x_i, y_i), 2q + 1 \leq i \leq 2q + m\}$, are not aligned. We set:

$$
N_{ah} := \{(x_i, y_i), i = 2, \ldots, (q + m)\} \text{ (aligned helper nodes)},
$$

$$
N_{aa} := \{(x_i, y_i), i = q + m + 1, \ldots, 2q\} \text{ (aligned aloof nodes)},
$$

$$
N_{ua} := \{(x_i, y_i), i = 2q + 1, \ldots, 2q + m\} \text{ (unaligned aloof nodes)},
$$

$$
\mathcal{N} = (x_1, y_1) \cup N_{ah} \cup N_{aa} \cup N_{ua}.
$$

Fig. 3. Illustrating the partitioning of $\mathcal{E}$ into aligned ($N_{aa}$) and unaligned aloof nodes ($N_{ua}$) and aligned helper nodes ($N_{ah}$).

B. The Starting Equations

During the repair process, the aloof nodes and the single failed node together behave as though they together constitute a set of $(q + 1)$ erased notes. For this reason, we set

$$
\mathcal{E} = \{(x_1, y_1)\} \cup N_{aa} \cup N_{ua},
$$

and retain the notation $\sigma(\mathcal{E}, z)$ with regard to intersection score.

While each node $(x, y)$ only stores $\alpha$ non-redundant symbols, it nevertheless has access through computation, to all $(2q)^t$ symbols $\{A(x, y; z), z \in \mathbb{Z}_{2q}^t\}$. Therefore the code does not support help-by-transfer repair. But the only computation required at any helper node is decoding of a half-rate RS code. During the repair of node $(x_1, y_1)$, we will only call upon the $\beta = (2q)^{t-1}$ symbols $\{A(x, y; z) \mid z_{y_i} = x_1\}$ from a helper node $(x, y)$.

1) Planes with intersection score 1: Consider first, planes $z$ which are such that $z_{y_i} = x_1$ and $z_{y_i} \neq x_i$ for any aloof node. Such planes have intersection score $\sigma(\mathcal{E}, z) = 1$. The $B$-plane p-c equations in such a plane take on the form:

$$
\sum_{x \in \mathbb{Z}_{2q}, y \in [t]} B(x, y; z) \theta_{(x,y)}^\ell = 0. \quad (8)
$$

It can be verified that for $(x, y) \notin \mathcal{N}$, the symbols $A(x, y; z)$ and $A^c(x, y; z)$ are both available for node repair and from these two values, one can compute $B(x, y; z)$. Hence we can rewrite (8) in the form:

$$
\sum_{(x,y) \in \mathcal{N}} B(x, y; z) \theta_{(x,y)}^\ell = \kappa_\ast. \quad (9)
$$

For brevity in writing we set:

$$
a_i = A(x_i, y_i; z), \quad a_i^c = A^c(x_i, y_i; z),
$$

$$
b_i = B(x_i, y_i; z), \quad b_i^c = B^c(x_i, y_i; z),
$$

$$
\theta_i = \theta(x_i, y_i),
$$

$$
\mathbf{a}_{ah} = [a_2^c, \ldots, a_{q+m}]^T,
$$

$$
\mathbf{b}_{aa} = [b_{q+m+1}, \ldots, b_{2q}]^T,
$$

$$
\mathbf{a}_{ua} = [a_{2q+1}, \ldots, a_{2q+m}]^T,
$$

$$
\mathbf{a}_{ua} = [a_{2q+1}, \ldots, a_{2q+m}]^T,
$$

$$
\mathbf{a}_{ua} = [a_{2q+1}, \ldots, a_{2q+m}]^T.
$$
We have the following situation:

| Node in $\mathcal{N}_{ah}$ | $a_i$ known, $a_i^c$ always unknown |
|-----------------------------|--------------------------------------|
| Node in $\mathcal{N}_{aa}$  | $a_i$ unavailable, $a_i^c$ always unknown |
| Node in $\mathcal{N}_{ua}$  | $a_i$ unavailable, $a_i^c$ can be unknown |

The allows us to rewrite (9) in the form:

$$
\begin{bmatrix}
1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_{2q+m} \\
\vdots & \vdots & \vdots \\
\theta_1^{2q-1} & \cdots & \theta_{2q+m}^{2q-1}
\end{bmatrix}
\begin{bmatrix}
a_1^c \\
a_2^c \\
\vdots \\
a_{2q}^c
\end{bmatrix}
= \kappa_*,
$$

(10)

Apart from these $2q$ plane-parity equations, we also have the $q$ nodal parity-equations associated to node $(x_1, y_1)$:

$$
\begin{bmatrix}
1 & \cdots & 1 \\
\theta_1 & \cdots & \theta_{2q} \\
\vdots & \vdots & \vdots \\
\theta_1^{q-1} & \cdots & \theta_{2q}^{q-1}
\end{bmatrix}
\begin{bmatrix}
a_1^c \\
a_2^c \\
\vdots \\
a_{2q}^c
\end{bmatrix}
= \kappa_*,
$$

(11)

Through row-reduction of the parity-check matrix, we can rewrite (11) in the form:

$$
\begin{bmatrix}
C_1^{(m \times q)} & I_m & \begin{bmatrix} 0 \end{bmatrix}^{(m \times (q-m))} \\
0 & I_{q-m} & \begin{bmatrix} 0 \end{bmatrix}^{((q-m) \times m)}
\end{bmatrix}
\begin{bmatrix}
a_1^c \\
a_2^c \\
\vdots \\
a_{2q}^c
\end{bmatrix}
= \kappa_*,
$$

(12)

By restricting attention only to those equations involving the unknowns $\{a_1^c, ua_2^c, \ldots, a_{q+m}^c\}$ appearing in (10), we obtain:

$$
\begin{bmatrix}
C_1^{(m \times q)} & I_m
\end{bmatrix}
\begin{bmatrix}
a_1^c \\
a_{2q}^c
\end{bmatrix}
= \kappa_*,
$$

(13)

Combining equations (10) and (13), we obtain:

$$
\begin{bmatrix}
V \left( \{\theta_1^{2q-1} & \theta_{2q}^{2q-1}, i \in [2q + m]\} \right)
\end{bmatrix}
\begin{bmatrix}
C_1^{(m \times q)} & I_m & \begin{bmatrix} 0 \end{bmatrix}^{(m \times (q-m))} \\
0 & I_{m} & \begin{bmatrix} 0 \end{bmatrix}^{((q-m) \times m)}
\end{bmatrix}
\begin{bmatrix}
a_1^c \\
a_{2q}^c \\
\vdots \\
a_{2q}^c
\end{bmatrix}
= \kappa_*,
$$

(14)
which can successively be row-reduced to obtain matrices of forms

\[
\begin{pmatrix}
C_1 \\
(m\times q) & I_m \\
(m\times m) & [0] \\
(m\times(q-m)) & [0] \\
C_3 \\
(m\times m)
\end{pmatrix}
\begin{pmatrix}
V \{\theta_i^{2q-1} \mid i \in [2q+m]\}
\end{pmatrix}
\begin{pmatrix}
(2q-m)\times (2q+m)
\end{pmatrix}
\]

\[
\Rightarrow
\begin{pmatrix}
C_1 \\
(m\times q) & I_m \\
(m\times m) & [0] \\
(m\times(q-m)) & [0] \\
C_3 \\
(m\times m)
\end{pmatrix}
\begin{pmatrix}
V \{\theta_i^{2q-1} \mid i \in [2q+m]\}
\end{pmatrix}
\begin{pmatrix}
(2q-m)\times (2q+m)
\end{pmatrix}
\]

\[
\Rightarrow
\begin{pmatrix}
C_1 \\
(m\times q) & I_m \\
(m\times m) & [0] \\
(m\times(q-m)) & [0] \\
C_3 \\
(m\times m)
\end{pmatrix}
\begin{pmatrix}
V \{\theta_i^{2q-1} \mid i \in [2q]\}
\end{pmatrix}
\begin{pmatrix}
(2q-m)\times (2q)
\end{pmatrix}
\]

Thus the equation becomes

\[
\begin{pmatrix}
[0] \\
(m\times q) & [0] \\
(m\times m) & [0] \\
(m\times(q-m)) & [0] \\
C_3 \\
(m\times m)
\end{pmatrix}
\begin{pmatrix}
V \{\theta_i^{2q-1} \mid i \in [2q]\}
\end{pmatrix}
\begin{pmatrix}
(2q)\times (2q)
\end{pmatrix}
\]

\[
= \kappa_a.
\]

(15)

Clearly, the matrix on the left is nonsingular since \(C_3\) is a Cauchy matrix and it follows therefore that we can recover the unknown vector: \([a_1^c, u[a_1^c]_h^T, [b_{aa}]_T, [b_{ua}]_T]^T\). The vector \([a_1^c, [a_1^c]_h]^T\) consists of \((q+m)\) symbols from the same node that participate in the \(q\) nodal \(p\)-c equations involving \(2q\) symbols. Thus we can decode \(2q\) symbols \(\{A(x_1, y_1; z_{(x,y)}) \mid x \in Z_{2q}\}\) belonging to the failed node.

The case of planes having intersection score \(> 1\) can be shown to reduce to the case of planes having intersection score \(1\) using arguments similar to those employed in describing how data collection is carried out.

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