Inflation versus Cyclic Predictions for Spectral Tilt

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We present a nearly model-independent estimate that yields the predictions of a class of simple inflationary and ekpyrotic/cyclic models for the spectral tilt of the primordial density inhomogeneities that enables us to compare the two scenarios. Remarkably, we find that the two produce an identical result, \( n_s \approx 0.95 \). For inflation, the same estimate predicts a ratio of tensor to scalar contributions to the low \( l \) multipoles of the microwave background anisotropy of \( T/S \approx 20\% \); the tensor contribution is negligible for ekpyrotic/cyclic models, as shown in earlier papers.

Since these conditions must be maintained for the duration of an epoch spanning many more than \( \mathcal{N} \) e-folds, the simplest possibility is to suppose that \( w \) (and correspondingly \( H \) or \( a \)) change slowly and monotonically during that last \( \mathcal{N} \) e-folds. More precisely, we take “simplest” to mean that (i) \( dw/d\mathcal{N} \) is small, and \( d^2w/d\mathcal{N}^2 \) or \( (dw/d\mathcal{N})^2 \) negligible and (ii) in order for inflation (ekpyrosis) to end, \( H \) during inflation (or \( a \) during ekpyrosis) decays by a factor of order unity over the last \( \mathcal{N} \) e-folds. Tilts or spectral features that differ from those presented here can only be produced by introducing by hand unnecessary rapid variations in \( w \) – unnecessary in the sense that they are not required for either model to give a successful account of the standard cosmology.

Note that our condition on the time-variation of \( w \) does not refer directly to any particular inflaton or cyclic scalar field potential. In fact, it does not assume that either scenario is driven by a scalar field at all. But, one might ask: how does our condition on the equation of state translate into a condition on an inflaton potential? The answer is simple: it means that the potential is characterized by a single dimensionful scale, typically \( H_I \), the Hubble parameter during inflation. For example, for many models the effective potential is well characterized as \( M^4f(\phi/M) \), where \( \phi \) is the inflaton field, \( H_I \approx M^2/M_{Pl} \) where \( M_{Pl} \) is the Planck mass, and \( f(x) \) is a smooth function which, when expanded in \( \phi/M \), has all dimensionless parameters of the same order \( 1 \). In these cases, to produce inflationary models in which there are rapid changes in the equation of state in the last \( \mathcal{N} \) e-folds, sharp features have to be introduced in the inflaton potential: bumps, wiggles, steep waterfalls, etc. But recall that the inflaton field is rolling very slowly throughout inflation, including the last \( \mathcal{N} \) e-folds. Typically, \( \phi \) rolls a short distance, \( \Delta \phi \ll M \), during the last \( \mathcal{N} \) e-folds. Hence, any sharp features must take place over a range \( \delta \phi \ll \Delta \phi \ll M \), or equivalently, by introducing new fields or new mass scales much greater than \( M \) in the inflaton potential. For the purposes of comparing the inflationary and ekpyrotic/cyclic predictions, it makes most sense to consider the class with fewest parameters and simplest uniform behavior of the equation of state, a class which is also well-motivated in both models.
Recently, Gratton et al. \cite{13} analyzed the conditions on the equation of state $w$ required in order for quantum fluctuations in a single scalar field to produce nearly scale-invariant density perturbations, including models which (in the four dimensional effective description) bounce from a contracting to an expanding phase. Their analysis showed that there are only two cases which avoid extreme fine-tuning of initial conditions and/or the effective potential: $w \approx -1$ (inflation) and $w \gg 1$ (the ekpyrotic/cyclic scenario).

Following Gratton et al. \cite{13}, we shall discuss the production of long wavelength perturbations in the gauge invariant Newtonian potential $Φ$, which completely characterizes the density perturbation. Defining $u ≡ aΦ/φ'$ (henceforth, primes denote differentiation with respect to conformal time $τ$), then a Fourier mode of $u$ with wavenumber $k$, $u_k$, obeys the differential equation

$$u_k'' + \left(k^2 - \frac{β(τ)}{τ^2}\right) u_k = 0,$$

with

$$β(τ) \equiv τ^2 H^2 a^2 \left\{\bar{ε} - \frac{1 - \bar{ε}^2}{2} \frac{d ln \bar{ε}}{dN} \right\} + \frac{1 - \bar{ε}^2}{2} \left(\frac{d ln \bar{ε}}{dN}\right)^2 - \frac{(1 - \bar{ε})^2}{2} \frac{d^2 ln \bar{ε}}{dN^2},$$

where $H = a'/a^2$ is the Hubble parameter, and $\bar{ε}$ is related to the equation of state parameter $w$ by

$$\bar{ε} \equiv \frac{3}{2} (1 + w).$$

We have introduced the dimensionless time variable $N$, defined by

$$N \equiv \ln \left(\frac{a_{end}H_{end}}{aH}\right),$$

where the subscript “end” denotes that the quantity is to be evaluated at the end of the inflationary expansion phase or ekpyrotic contraction phase (corresponding to $w \gg 1$). Note that $N$ measures the number of e-folds of modes which exit the horizon before the end of the inflationary or ekpyrotic phase. (N.B. $dN = (\bar{ε} - 1) dτ$ where $N = \ln a$ in Ref. \cite{13}.) Indeed, defining as usual the moment of horizon-crossing as $k_N = aH$ for a given Fourier mode with comoving wavenumber $k_N$, then

$$N = \ln \left(\frac{k_{end}}{k_N}\right),$$

where $k_{end}$ is the last mode to be generated.

For nearly constant $w$ (or constant $\bar{ε}$), the unperturbed equations of motion have the approximate solution

$$a(τ) \sim (-τ)^{1/(ε-1)}, \quad H = \frac{1}{(ε-1)aτ}.$$

Substituting the second of these expressions in $β$, we find

$$β(τ) \approx \frac{1}{(1 - \bar{ε}^2)} \left\{\bar{ε} - \frac{1 - \bar{ε}^2}{2} \frac{d ln \bar{ε}}{dN}\right\},$$

where we have assumed that the higher-order derivative terms $d^2 ln \bar{ε}/dN^2$ and $(d ln \bar{ε}/dN)^2$ are much smaller than $d ln \bar{ε}/dN$.

With the approximation that $β$ is nearly constant for all modes of interest, Eq. \ref{10} can be solved analytically, and the resulting deviation from scale invariance is simply given by the master equation

$$n_s - 1 \approx -2β \approx -\frac{2}{1 - \bar{ε}^2} \left\{\bar{ε} - \frac{1 - \bar{ε}^2}{2} \frac{d ln \bar{ε}}{dN}\right\}.$$

**Inflation.** Inflation is characterized by a period of superluminal expansion during which $w \approx -1$; that is, $\bar{ε} \ll 1$. In this case, Eq. \ref{10} reduces to

$$n_s - 1 \approx -2\bar{ε} + \frac{d ln \bar{ε}}{dN},$$

as derived by Wang et al. \cite{14}.

The next step consists in rewriting the above in terms of $N$ only. For this purpose, we need a relation between $\bar{ε}$ and $N$. During inflation, the Hubble parameter is nearly constant, but the “end” means that $H$ begins to change significantly. So, if we are considering the last $N$ e-folds, then, using Eqs. \ref{11} and the definition of $N$ (see Eq. \ref{10}), it must be that $H$ decays by a factor of order unity over those $N$ e-folds or

$$\frac{H_{end}}{H} = \left(\frac{a}{a_{end}}\right)^\bar{ε} \approx e^{-\bar{ε}N} \approx e^{-1},$$

or

$$\bar{ε} \approx \frac{1}{N}.$$

Assuming that this relation holds approximately for all relevant modes, we may substitute in Eq. \ref{11} and obtain

$$(n_s - 1)_{inf} \approx -2 - \frac{1}{N} = -\frac{3}{N}.\quad \text{(12)}$$

Note that, in this approximation, the two terms on the right hand side of Eq. \ref{12} are both of order $1/N$ or a few percent, the result is in agreement with the tilt predicted by simple inflationary models \cite{15}.

To obtain a numerical estimate of $n_s$, we may derive an approximate value for $N$ from the observational constraint that the amplitude of the density perturbations, $δρ/ρ$, be of order $10^{-5}$. In the simplest inflationary models, $δρ/ρ$ is given by

$$δρ/ρ \approx \left(\frac{T_r}{M_{Pl}}\right)^2 \bar{ε}^{-1/2} \approx \left(\frac{T_r}{M_{Pl}}\right)^2 N^{1/2} \sim 10^{-5}.$$

\begin{equation}
\text{(13)}
\end{equation}
where $T_r$ is the reheat temperature. On the scale of the observable universe today, $N$ has the value (see Eq. (11))

$$N = \ln \left( \frac{a_{end}H_{end}}{a_0H_0} \right) \approx \ln \left( \frac{T_r}{T_0} \right),$$

(14)

where $a_0$, $H_0$ and $T_0$ are respectively the current values of the scale factor, Hubble parameter and (photon) temperature. For simplicity, we have assumed that $H_{end} \sim T_r^2/M_{Pl}$. Combining Eqs. (13) and (14), we obtain the constraint

$$e^N N^{1/4} \approx 10^{-5/3} M_{Pl} \frac{P_l}{T_0},$$

(15)

which implies $N \approx 60$. It follows that $T_r \approx 10^{16}$ GeV.

If we substitute $N \approx 60$ in Eq. (14), we obtain $n_s \approx 0.95$, within a percent or two of what is found for the simplest slow-roll and chaotic potentials [16, 17].

The prediction for the ratio of tensor to scalar contributions to the quadrupole of the CMB for a model with 70% dark energy and 30% matter is, then, [16, 17, 18]

$$N \approx \frac{13.8}{N} \approx 23%,$$

(16)

which is very pleasing because it is in the range which is potentially detectable in the fluctuation spectrum and/or the CMB polarization in the near future [19]. (The WMAP collaboration [20] uses a different convention for $T/S$, defining $(T/S)_{W M A P}$ as the ratio of tensor to scalar amplitude of the primordial spectrum. The conversion factor to our $T/S$ is $(T/S)_{W M A P} \approx 1.16 (T/S)$.)

It is sometimes said that it is easy to construct models where $T/S$ is very small, less than 1%, say. The argument is that the amplitude of tensor fluctuations is proportional to $H^2$, and a modest decrease in the energy scale for inflation reduces the tensor amplitude significantly. However, one must also consider Eq. (15) combined with Eq. (11). From Eq. (14), making $T/S$ less than 1%, for instance, requires $\bar{\epsilon} < 10^{-3}$, which implies $N > 1000$. Since we are interested in $T/S$ at $N \approx 60$, however, the only way to accommodate such a small $\bar{\epsilon}$ at $N \approx 60$ is to have $\bar{\epsilon}$ make a rapid change at some point between $N \approx 60$ and the end of inflation. This is precisely what is done in models which yield a small $T/S$ ratio. (Restated in terms of the inflation potential $V(\phi)$, in order to have $T/S \approx 13.8 \bar{\epsilon} \approx 28 (d \ln V/d \phi)^2 \ll 1$, it must be that $d \ln V/d \phi \ll 0.02$, which is too small if inflation is to end in 60 e-folds unless one introduces a very rapid change in the slope during the last 60 e-folds.)

**Ekpyrotic/cyclic models.**  The ekpyrotic phase is characterized by a period of slow contraction with $w \gg 1$; that is, $\bar{\epsilon} \gg 1$. In this case, Eq. (13) reduces to

$$n_s - 1 \approx -\frac{2}{\bar{\epsilon}} - \frac{d \ln \bar{\epsilon}}{d N}.$$  

(17)

Note further that, for all cosmologies, the scale factor is $a \propto t^{1/\bar{\epsilon}} \propto H^{-1/\bar{\epsilon}}$, where $t$ is proper time. Hence, inflation ($\bar{\epsilon} \ll 1$) has a rapidly varying $H$ nearly constant, whereas the ekpyrotic/cyclic model ($\bar{\epsilon} \gg 1$) has $H$ varying and $a$ nearly constant. This suggests an interesting duality between the inflationary and ekpyrotic/cyclic models that reflects itself in the final results.

If the scale factor $a(\tau)$ is nearly constant during the ekpyrotic (contraction) phase, then the phase ends when $a(\tau)$ begins to change significantly. In particular, the condition that the scale factor $a(\tau)$ decays by a factor of order unity during the last $N$ e-folds reads

$$\frac{a_{end}}{a} = \left( \frac{aH}{a_{end}H_{end}} \right)^{1/(\bar{\epsilon} - 1)} \approx e^{-N/\bar{\epsilon}} \approx e^{-1}$$

(18)

(the analogue of Eq. (10) for inflation), which implies

$$\bar{\epsilon} \approx N$$

(19)

(to be compared with Eq. (11) for inflation). Substituting this expression into Eq. (17), one obtains

$$(n_s - 1)_{ek} \approx -\frac{2}{N} - \frac{1}{N} \approx -\frac{3}{N}. $$

(20)

This is the key relation for the ekpyrotic/cyclic models.

In the inflationary case, we estimated $N$ by using the constraint on the amplitude of density perturbations, $\delta \rho/\rho \sim 10^{-5}$. For ekpyrotic and cyclic models, this constraint involves more parameters and is therefore not sufficient by itself to fix $N$ [12, 21]. To estimate $N$, we rewrite Eq. (4) as

$$N \approx \ln \left( \frac{T_r}{T_0} \right) + \ln \left( \frac{a_{end}H_{end}}{a_r H_r} \right),$$

(21)

where the subscript $r$ denotes the onset of the radiation-dominated phase. In inflation, we have $a_{end} \approx a_r$ and $H_{end} \approx H_r$. In the ekpyrotic/cyclic model, however, the end of ekpyrosis occurs during the contracting phase whereas the onset of radiation-domination is during the expanding phase. To estimate the ratio $a_{end}H_{end}/a_r H_r$, we note that, from approximately the end of ekpyrosis, through the bounce, and up to the onset of radiation-domination, the universe is dominated by scalar field kinetic energy, i.e., $w \approx 1$. From Eqs. (3) and (6), we find $a \approx (-\tau)^{1/2} \sim H^{-1/3}$, and therefore

$$\frac{a_{end}H_{end}}{a_r H_r} \approx \left( \frac{H_{end} M_{Pl}}{T_r} \right)^{2/3}.$$  

(22)

Substituting in Eq. (21), we find

$$e^N = \left( \frac{H_{end}^2 M_{Pl}}{T_r^2 M_{Pl}} \right)^{1/3} \frac{M_{Pl}}{T_0},$$

(23)

which is the analogue of Eq. (15).

The constraints on $H_{end}$ and $T_r$ in cyclic models are analyzed in Ref. [21], and the range of allowed values is presented. Central values are $T_r \approx 10^{5}$ GeV and $H_{end} \approx 10^{5}$
GeV, which, from Eq. (23), implies $N \approx 60$. (By pushing parameters, $N$ can be made to vary 20% or so one way or the other.) Substituting $N = 60$ in the expression for the tilt gives $n_s \approx 0.95$, the same estimate obtained for inflation.

**Conclusions.** Remarkably, our estimates for the typical tilt in the inflationary and ekpyrotic/cyclic models are virtually identical. Both models predict a red spectrum, with spectral slope

$$n_s - 1 \approx -\frac{3}{N}. \quad (24)$$

Furthermore, when adding observational constraints such as the COBE constraint that the amplitude of density fluctuations be of order $10^{-5}$, both models yield $N \approx 60$. This results in an identical prediction for the spectral tilt of $n_s \approx 0.95$. Furthermore, in both models, the time-variation of the equation of state contributes a correction of $O(1)$ that reddens the spectrum. We have seen that this occurs because there is fascinating duality ($\bar{\epsilon} \to 1/\bar{\epsilon}$) between inflationary and ekpyrotic/cyclic conditions. This result was neither planned nor anticipated and suggests a deep connection between the expanding inflationary phase and the contracting ekpyrotic/cyclic phase. The key difference is that inflation also predicts a nearly scale-invariant spectrum of gravitational waves with a detectable amplitude. The predicted ratio of tensor to scalar CMB multipole moments at low $l$ is $T/S \approx 20\%$. The tensor spectrum from cyclic models is strongly blue and exponentially small on cosmic scales $\gtrsim 20$. We thank V. Mukhanov for useful discussions. This work was supported in part by US Department of Energy grants DE-FG02-91ER40671 (PJS) and DE-FG02-92ER40699 (JK), the Columbia University Academic Quality Fund (JK), the Ohrstrom Foundation (JK), and PPARC, UK (NT).

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