General One-loop Reduction in Generalized Feynman Parametrization Form

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ABSTRACT: For higher loop computation, one of main topics is to find the effective reduction method. Recently there is an alternative reduction method proposed by Chen in [1, 2]. In this paper, using the one-loop scalar integrals with propagators having higher power, we test the power of Chen’s new method. More explicitly, with the improved version of the method we can cancel the dimension shift and terms having unwanted power shifting. Thus the obtained IBP relations are much more simpler and can be solved easily. Using this method we present the explicit examples of bubble, triangle, box and pentagon with one propagators doubled. With these results, we complete our previous computations in [3] with the missed tadpole coefficients and show the potential of Chen’s method for efficient reduction of higher loop integrals.

KEYWORDS: Amplitudes, Reduction.

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1. Introduction

To give more precise theoretical prediction of scatting amplitude of a given process, calculation of high loops integrals becomes important. For these calculations, the PV-reduction method [4] is one of the most used ideas. One way to implement the reduction method is to use Integrating-by-Parts (IBP) relation
As one of the most powerful techniques for loop integrals reduction, IBP gives a large number of recurrence relations, and one could get the reduction of the simpler integrals directly by Gauss elimination. However, as the number and power of propagators become higher and higher, the IBP method becomes hard and inefficient. Finding more efficient reduction methods becomes an important direction.

Unitarity cut method is one alternative reduction method and has been proved to be very useful for one-loop integrals [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19]. For physical one-loop process, the power of propagator is just one, but if the method is a complete method, it should be able to give the reduction of integrals with higher power of propagators. Such a situation is not just a theoretical curiosity. In fact, it appears in the higher loop diagrams as a sub-diagram. Furthermore, although for one-loop integrals the scalar basis is natural, in general the choice of basis can be different, depending on the physical input. For example, for one-loop bubble, the basis with one propagator having power two could be useful as part of UT-basis [20, 21].

In our previous work [3], by combining the trick of differential operators and unitarity cut, we successfully got the analytical reduction result of one-loop integrals with high power propagators and gave the coefficients to all the basis except the tadpoles’ coefficients. Since the tadpole have only one propagator, the unitarity method could not be used to get the tadpole part. To complete our investigation, we want to find the missing tadpole coefficients by some efficient methods.

Except the unitarity cut method, there are other proposals to overcome the difficulty in IBP, by using some tricks and other representations of integrals in recent years, such as Baikov representation [22, 23] and Feynman parametrization representation [24, 25] for loop integrals. In recent years, Chen has proposed a new representation for loop integrals [1, 2]. His method is based on the generalized Feynman parametrization representation, i.e., an extra parameter $x_{n+1}$ has been introduced to combine the $U, F$ in the standard Feynman parametrization representation. Such a generalization will bring some benefits in deriving the IBP recurrence relation, as will shown in this paper.

As a common feature, the IBP recurrence relation derived using the generalized Feynman parametrization representation will naturally have terms in different spacetime dimension. Since we always concern the reduction in a certain dimension $D$, which is usually set to be $4 - 2\epsilon$ for the reason of renormalization, we want to cancel these terms in different dimension. This is usually not an easy work. In [26] Gluza, Kajda and Kosower have shown how to avoid the change of power of propagators in the standard momentum space. Larsen and Zhang have considered the Baikov representation and showed how to eliminate both dimension shifting and the change of power of propagators [27, 28, 29, 30, 31, 32]. These methods require the solution of syzygy equations, which is not easy to figure out in general. In Chen’s second paper [2], he proposed a new technique to simplifying the recurrence relation based on the non-commutative algebra.

Motivated by above discussion and preparing Chen’s method for the high-loop computations, in this paper, we will use the Chen’s method to find the missing tadpole coefficients in our previous work. Furthermore, we will use the idea to remove terms with dimensional shifting in the derived IBP relation to give a simpler reduction method with the analytic results written by the elements of the coefficients matrix.
The plan of the paper is following. In section 2, we have reviewed the Chen’s new method and illustrated with a simple example in the section 2.1. In the example, the integrals in different dimension will naturally emerge. We discussed the physical meaning of the boundary terms, which contributes to the sub-topologies. To cancel the dimension in the parametrization form and simplify the IBP relation, in the section 2.2 we proposed a new trick by adding free auxiliary parameters based on the fact that the $F$ in the integrand is a homogeneous function of $x_1$ with degree $L + 1$. By our trick, we successfully canceled the dimension shift and dropped the terms that we do not concern to a certain extent, and give a simplified IBP relation in which all the integrals are in the certain dimension $D$ and integrals except the target have lower total power of propagators. We gave our analytic result by the determinant of the cofactor of the matrix $\hat{A}$, which is completely determined by the graph. In section 3, combined with our trick, we calculated the triangle $I_3(1, 1, 2)$, box $I_4(1, 1, 1, 2)$, and pentagon $I_5(1, 1, 1, 1, 2)$ in the parametric form proposed by Chen, and gave the analytic result of all the coefficients to the master basis, especially the tadpole parts as the complement of our previous work.

2. Reduction method in parametric form by Chen

In this section, we will introduce a new reduction method proposed by Chen in [1]. The general form of loop integral is given by

$$I[N(l)](k) = \int d^{D}l_1 d^{D}l_2 \cdots d^{D}l_L \frac{N(l)}{D_1^{k_1} D_2^{k_2} \cdots D_n^{k_n}}$$

(2.1)

where for simplicity, we have denoted $l = (l_1, l_2, l_3, \cdots, l_L)$ and $k = (k_1, k_2, k_3, \cdots, k_n)$. Since in this paper, we consider only the scalar’s integrals with $N(l) = 1$, let us label

$$I(L; \lambda_1 + 1, \cdots, \lambda_n + 1) = \int d^{D}l_1 \cdots d^{D}l_L \frac{1}{D_1^{\lambda_1 + 1} \cdots D_n^{\lambda_n + 1}}$$

(2.2)

By the procedure of Feynman parametrization,

$$\sum_i \alpha_i D_i = \sum_{i,j} A_{ij} l_i \cdot l_j + 2 \sum_{i=1}^{L} B_i \cdot l_i + C$$

(2.3)

thus the loop integrals can be done as

$$\int d^{D}l_1 \cdots d^{D}l_L e^{i(\sum \alpha_i D_i)} = e^{i\pi L(1 - \frac{D}{2})/2 \pi^{LD/2} (\text{Det} A)^{-\frac{D}{2}} e^{i(C - \sum A_{ij}^{-1} B_i \cdot B_j)}}$$

(2.4)

Defining $U(\alpha) = \text{Det} A$, and $C - \sum A_{ij}^{-1} B_i \cdot B_j \equiv \frac{V(\alpha)}{U(\alpha)} - \sum m_i^2 \alpha_i^{-1}$, one can see that $U(\alpha)$ is a homogeneous function of $\alpha_i$ with degree $L$, while the $V(\alpha)$ is a homogeneous function of $\alpha_i$ with degree $L + 1$, and the

\footnote{The relation has been verified in many places based on the method in graph theory}
loop integral becomes to

\[
I(L; \lambda_1 + 1, \ldots, \lambda_n + 1) = e^{-\frac{\sum \lambda_i^2}{2} i \pi} \frac{\Pi_{i=1}^n \Gamma(\lambda_i + 1)}{\Pi_{i=1}^n \Gamma(\lambda_i + 1 + 1)} e^{-i \pi L(1 - \frac{D}{2}) / 2 \pi LD / 2} \times \int \prod d \alpha_1 \cdots d \alpha_n U(\alpha)^{-\frac{D}{2}} e^{-i[V(\alpha)/U(\alpha) - \sum m_i^2 \alpha_i]} \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n}
\]  

(2.5)

To derive the parametric form suggested by Chen, we do the following. Using the \( \alpha \)-representation of general propagators,

\[
\frac{1}{(t^2 - m^2)^{\lambda + 1}} = e^{-\frac{\sum \lambda_i^2 + 1}{2} i \pi} \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1)} \int_0^\infty d \alpha e^{i \alpha (t^2 - m^2) \alpha^\lambda}, \quad \text{Im} \{t^2 - m^2\} > 0
\]  

(2.6)

where the ”\( i \epsilon \)” has been neglected, we get

\[
I(L; \lambda_1 + 1, \ldots, \lambda_n + 1) = e^{-\frac{\sum \lambda_i^2 + 1}{2} i \pi} \frac{\Pi_{i=1}^n \Gamma(\lambda_i + 1)}{\Pi_{i=1}^n \Gamma(\lambda_i + 1 + 1)} \int d^D q_1 \cdots d^D q_L \int_0^\infty d \alpha_1 \cdots d \alpha_n e^{i \sum_{i=1}^n \alpha_i D \cdot \alpha_i \lambda_1 \cdots \alpha_n}
\]  

(2.7)

To go further, we change the integral variables as \( \alpha_i = \eta x_i \). Since there are totally \( n \) independent variables, we must put another constraint condition. In general, we could let

\[
\sum_{i \in S(1, 2, 3, \ldots, n)} x_i = 1
\]  

(2.8)

where \( S \) is an arbitrary non-trivial subset of \( \{1, 2, 3, \ldots n\} \). After carrying out the integration over \( \eta \), the second line of eq.(2.5) becomes to

\[
(-i)^{n+\lambda - \frac{DL}{2}} \Gamma(n + \lambda - \frac{DL}{2}) \times \int dx_1 \cdots dx_n \delta(\sum_{j \in S} x_j - 1) \frac{U(x)^{n+\lambda - \frac{DL}{2} + 1}}{[V(x) + U(x) \sum m_i^2 x_i]^{n+\lambda - \frac{DL}{2}}} x_1^{\lambda_1} \cdots x_n^{\lambda_n}
\]  

(2.9)

where

\[
U(x) = \eta^{-L} U(\alpha) = \eta^{-L} U(\eta x_i), \quad V(x) = \eta^{-L-1} V(\alpha) = \eta^{-L-1} V(\eta x), \quad f(x) = -V(x) + U(x) \sum m_i^2 x_i
\]  

\[
\lambda = \sum_{i=1}^{n} \lambda_i, \quad \lambda_u = n + \lambda - \frac{D}{2} (L + 1), \quad \lambda_f = -n - \lambda + \frac{DL}{2}
\]  

(2.10)

Finally we by Mellin transformation\(^2\)

\[
A^{\lambda_1} B^{\lambda_2} = \frac{\Gamma(-\lambda_1 - \lambda_2)}{\Gamma(-\lambda_1) \Gamma(-\lambda_2)} \int_0^\infty dx (A + B x)^{\lambda_1 + \lambda_2} x^{\lambda_1 - 1} x^{\lambda_2 - 1}
\]  

(2.11)

\(^2\)Different from the traditional Feynman parametrization, here we should add a new auxiliary parameter \( x_{n+1} \) to transform the integral into a symmetric form.\(^1\)
Putting all together, we now finally get the parametric form of scalar loop integrals (2.5), we could write the (2.9) as
\[
\frac{(-i)^{n+\lambda} \cdot 2 \Gamma(n + \lambda - \frac{D L}{2}) \Gamma(-\lambda U - \lambda f) \int d x_1 \cdots d x_n \delta(\sum_{j \in S} x_j - 1) \int_0^\infty d x_{n+1}}{\Gamma(-\lambda U) \Gamma(-\lambda f)} \times (U x_{n+1} + f) \lambda_u + \lambda f x^1 \lambda_1 \cdots x^n \lambda_n
\]
\[
\equiv (-i)^{n+\lambda} \cdot 2 \frac{\Gamma(n + \lambda - \frac{D L}{2}) \Gamma(-\lambda U - \lambda f)}{\Gamma(-\lambda U) \Gamma(-\lambda f)} \int d \Pi^{(n+1)} F^\lambda x^1 \lambda_1 \cdots x^n \lambda_n x_{n+1} \lambda_{n+1}
\]
\[
\equiv (-i)^{n+\lambda} \cdot 2 \frac{\Gamma(n + \lambda - \frac{D L}{2}) \Gamma(-\lambda U - \lambda f)}{\Gamma(-\lambda U) \Gamma(-\lambda f)} i_{\lambda_0; \lambda_1; \cdots ; \lambda_n}
\]
(2.12)

where
\[
d \Pi^{(n+1)} = d x_1 \cdots d x_{n+1} \delta(\sum_{j \in S} x_j - 1), \quad F = U x_{n+1} + f, \quad \lambda = \sum_{i=1}^n \lambda_i,
\]
\[
\lambda_0 = \lambda_U + \lambda_f = -\frac{D}{2}, \quad \lambda_{n+1} = -\lambda_U - 1 = -\frac{D}{2}(L + 1) - \lambda - 1 - n
\]
(2.13)

Putting all together, we now finally get the parametric form of scalar loop integrals (2.3),
\[
I(L; \lambda_1 + 1, \cdots , \lambda_n + 1) = (-1)^{n+\lambda} \Gamma(-\lambda_U) \frac{\Gamma(\lambda_1 + 1 \cdots \lambda_n + 1)}{\Gamma(-\lambda U) \Gamma(-\lambda f)} i_{\lambda_0; \lambda_1; \cdots ; \lambda_n}
\]
(2.14)

### 2.1 The IBP identity in parametric represent

The parametric form of (2.14) is the starting point of Chen’s proposal. The IBP relations in this form is given by\(^3\)
\[
\int d \Pi^{(n+1)} \frac{\delta}{\delta x_i} \left\{ F^\lambda x_1 \lambda_1 \cdots x_n \lambda_n x_{n+1} \lambda_{n+1} \right\} + \delta x_i,0 \int d \Pi^{(n)} \left\{ F^\lambda x_1 \lambda_1 \cdots x_n \lambda_n x_{n+1} \lambda_{n+1} \right\} \bigg|_{x_i=0} = 0
\]
(2.15)

where \(i = 1, \ldots, n + 1\) and the \(d \Pi^{(n)}\) in the second term is
\[
d \Pi^{(n)} = d x_1 \cdots d x_i \cdots d x_n d x_{n+1} \delta(\sum_{j \in S} x_j - 1)
\]
(2.16)

The second term in (2.15) contributes to a boundary term which leads to the sub-topologies to the former term.

To illustrate the IBP relation (2.15), we present the reduction of \(I_2(1, 2)\) as an example. The general form of one-loop bubble integrals is given by
\[
I_2(m + 1, n + 1) = \int \frac{d^D l}{(l^2 - m_1^2)^{m+1}(l - l_1)^2 - m_2^{n+1}}
\]
(2.17)

\(^3\)In some sense, the parametric form can be considered as the **generalized Feynman parametrization form**. Thus the IBP relation (2.15) could be called the IBP relation in the generalized Feynman parametrization form.

\(^4\)The IBP relation requires the term in the bracket of the first term to be degree \((-n)\), which can be obtained by multiplying any monomial of degree one. Here in (2.15) we have multiplied \(x_{n+1}\) by our experiences from later examples, but one can make other choices.
and the corresponding parametric form is (in this article we ignore the former factor $\pi^{D/2}$)

$$I_2(m + 1, n + 1) = i(-1)^{m+n+2} \frac{\Gamma(D)}{\Gamma(m+1)\Gamma(n+1)\Gamma(D-2-m-n)} \int d\Pi^{(3)} F^{\lambda_0} x_1^m x_2^n x_3^\lambda$$  \hspace{1cm} (2.18)

where

$$F = (x_1 + x_2)(m_1^2 x_1 + m_2^2 x_2 + x_3) - p_1^2 x_1 x_2$$  \hspace{1cm} (2.19)

and

$$i_{\lambda_0;m,n} = \int d\Pi^{(3)} F^{\lambda_0} x_1^m x_2^n x_3^\lambda$$  \hspace{1cm} (2.20)

with $\lambda_0 = -\frac{D}{2}$ and $\lambda_3 = -3 - m - n - 2\lambda_0$. Using the eq. (2.13), we could get three IBP recurrence relations. Taking $\frac{\partial}{\partial x_1}$ first, the first term in (2.15) gives

$$\lambda_0 i_{\lambda_0 -1;m,n} + 2m_1^2 \lambda_0 i_{\lambda_0 -1;m+1,n} + \Delta\lambda_0 i_{\lambda_0 -1;m,n+1}$$  \hspace{1cm} (2.21)

where $\Delta = m_1^2 + m_2^2 - p_1^2$. The second term gives

$$\delta_{m,0} \int d\Pi^{(2)} (x_3 + m_2^2 x_2)^{\lambda_0} x_2^n \lambda_0 x_3^{2-n-2\lambda_0} = \delta_{m,0} i_{\lambda_0;1,n}$$  \hspace{1cm} (2.22)

Here we need to explain the notation $i_{\lambda_0;1,n}$. From the middle expression of (2.22), we see that it is the parametric form of tadpole $\int \frac{d\Pi}{(x^2-m_2^2)^n}$. To emphasize its origin, i.e., coming from bubble by removing the first propagator, we extend the definition of $i_{\lambda_0;1,...,\lambda_0}$ given in (2.12) by setting $\lambda_1 = -1$ \(^5\). Using the extended notation, we got the first IBP relation

$$\lambda_0 i_{\lambda_0 -1;m,n} + 2m_1^2 \lambda_0 i_{\lambda_0 -1;m+1,n} + \Delta\lambda_0 i_{\lambda_0 -1;m,n+1} + \delta_{m,0} i_{\lambda_0;1,n} = 0$$  \hspace{1cm} (2.23)

When we set $m = n = 0$ in (2.23), it reads

$$\lambda_0 i_{\lambda_0 -1;0,0} + 2m_1^2 \lambda_0 i_{\lambda_0 -1;1,0} + \Delta\lambda_0 i_{\lambda_0 -1;0,1} + i_{\lambda_0;1,0} = 0$$  \hspace{1cm} (2.24)

Similarly, we could take the differential $\frac{\partial}{\partial x_2}$ and get the second IBP relation

$$\lambda_0 i_{\lambda_0 -1;0,0} + \delta_{\lambda_0;i_{\lambda_0 -1;1,0} + 2m_2^2 \lambda_0 i_{\lambda_0 -1;0,1} + i_{\lambda_0;0,-1}} = 0$$  \hspace{1cm} (2.25)

Naively, we should solve $i_{\lambda_0;0,1}$ by $i_{\lambda_0;0,0}$ from (2.24) and (2.23). However, for bubble part, we have $\lambda_0 - 1$ instead of $\lambda_0$. This one could be fixed by rewriting $\lambda_0 \rightarrow \lambda_0 + 1$ since $\lambda_0$ is a free parameter. However, the boundary tadpole part $i_{\lambda_0;0,-1}$ will become $i_{\lambda_0+1;0,-1}$, i.e., having the dimensional shifting, which is a common feature in the parametric IBP relation.

\(^5\)Same notation has also been used in \[2\] (see Eq. (2.5a)).
To deal with it, using the parametric form of tadpoles
\[ i_{\lambda_0; m, -1} = \int d\Pi^{(2)}(1_3 x_3 + m_1^2 x_1^2) \lambda_0 x_1^m x_3^{-2 - m - 2\lambda_0} \]  
(2.26)
and taking the \( \frac{\partial}{\partial x_1} \) and \( \frac{\partial}{\partial x_3} \), we could get two IBP relations
\[ \lambda_0 i_{\lambda_0 - 1; m, -1} + 2m_1^2 \lambda_0 i_{\lambda_0 - 1; m + 1, -1} + m i_{\lambda_0, m - 1, -1} = 0 \]
\[ \lambda_0 i_{\lambda_0 - 1; m + 1, -1} + (-1 - m - 2\lambda_0) i_{\lambda_0, m, -1} = 0 \]  
(2.27)
from which we solve
\[ i_{\lambda_0; 0, -1} = \frac{-\lambda_0}{2m_1^2(2\lambda_0 + 1)} i_{\lambda_0 - 1; 0, -1}, \quad i_{\lambda_0; -1, 0} = \frac{-\lambda_0}{2m_1^2(2\lambda_0 + 1)} i_{\lambda_0 - 1; 0} \]  
(2.28)
Putting (2.28) to (2.24) and (2.25), we can solve the \( i_{\lambda_0; 0, 1} \). After shifting \( \lambda_0 \rightarrow \lambda_0 + 1 \), we finally get
\[ i_{\lambda_0; 0, 1} = \frac{2m_1^2 - \Delta}{\Delta^2 - 4m_1^2 m_2^2} i_{\lambda_0; 0, 0} + \frac{-1}{(2\lambda_0 + 3)(\Delta^2 - 4m_1^2 m_2^2)} i_{\lambda_0; -1, 0} + \frac{\Delta}{2m_2^2(2\lambda_0 + 3)(\Delta^2 - 4m_1^2 m_2^2)} i_{\lambda_0; -1, 0} \]  
(2.29)
Translating back to scalar basis, we get the reduction of \( I_2(1, 2) \) as
\[ I_2(1, 2) = c_{2\rightarrow 2} I_2(1, 1) + c_{2\rightarrow 12} I_2(1, 0) + c_{2\rightarrow 11} I_2(0, 1) \]  
(2.30)
with the coefficients
\[ c_{2\rightarrow 2} = \frac{(D - 3)(\Delta - 2m_1^2)}{\Delta^2 - 4m_1^2 m_2^2}, \quad c_{2\rightarrow 12} = \frac{D - 2}{\Delta^2 - 4m_1^2 m_2^2}, \quad c_{2\rightarrow 11} = \frac{(D - 2)\Delta}{2m_2^2(4m_1^2 m_2^2 - \Delta^2)} \]  
(2.31)
The result is confirmed with the FIRE6\[33, 34\].

### 2.2 Improvement of parametric IBP

As we have seen from the previous subsection, the IBP relation given in (2.13) will contain the integrals with dimension shift, which makes the reduction program a bit troublesome. We would like a recurrence relation without dimension shift. As we reviewed in the introduction there are several references dealt with this or related problems. Based on these work, an improved version of IBP relation has been given in [4] (see Eq.(2.12), (2.13)). All these methods require the solution of syzygy equations, which is not an easy task in general. However, for our one-loop integrals, the function \( F(x) \) is a homogeneous function of \( x_i \) with degree two\(^6\). This good property makes the related syzygy equations simple, which can be solved straightly\(^7\). In this paper, we will develop a direct algorithm to write down IBP relations without the dimension shift and the terms having unwanted higher power of propagators.

\(^6\)Note the \( F(x) \) is a homogeneous function of degree \( L + 1 \) where \( L \) is the number of loops.

\(^7\)In general, this trick could be extended to high loops to avoid the troublesome calculation of syzygy equations.
In the generalized parametric representation, our improved IBP relation is to multiply a degree zero coefficient \( z_i \), for example, \( z_i = x_i^\alpha x_j^\beta x_k^{-\alpha-\beta} \), in (2.15). Since the degree of the new integrand does not change, the IBP identity still holds. Summing them together we get

\[
\sum_{i=1}^{n+1} \int d\Pi^{(n+1)} \frac{\partial}{\partial x_i} \left\{ z_i F^{\lambda_0} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n+1} \right\} + \sum_{i=1}^{n+1} \delta_{\lambda_i,0} \int d\Pi^{(n)} z_i x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} x_{n+1}^{\lambda_{n+1}} |_{x_i=0} = 0 \tag{2.32}
\]

Since the second boundary term involve integrals with sub-topologies, we focus on the first term. Expanding it, we got

\[
\int d\Pi^{(n+1)} \left[ \sum_{i=1}^{n+1} \left( \frac{\partial z_i}{\partial x_i} + \lambda_0 \frac{z_i \partial F}{F} + \lambda_i \frac{z_i}{x_i} \right) + \frac{z_{n+1}}{x_{n+1}} \right] F^{\lambda_0} x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} x_{n+1}^{\lambda_{n+1}} \tag{2.33}
\]

From (2.13), one can see the power \( \lambda_0 \) of \( F \) is related to dimension. To cancel the dimension shift, we need to choose the proper coefficients \( z_i \) so that the \( \sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial x_i} \) is a multiple of the function \( F \), i.e.,

\[
\sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial x_i} + BF = 0 \tag{2.34}
\]

Since coefficients \( z_i \) are not polynomials, (2.34) is not the "normal syzygy equation" and one can not directly use the technique developed for polynomial ring. In [2], Chen developed a method based on the lift and down operators. Here for the one loop integrals, we can solve it directly with some free auxiliary parameters, as we will show shortly. When putting back solutions to the IBP recurrence relation, we could choose these free parameters to cancel both the dimension shift and unwanted terms with higher power of propagators, which leads to a simpler recurrence relation.

Now let us explain the idea in details. Note that in one loop case, the homogeneous function \( F \) is a degree two function of \( x_i \), so we can write

\[
F = \frac{1}{2} A_{ij} x_i x_j \tag{2.35}
\]

where \( A \) is the symmetric matrix\(^9\). Thus we have

\[
f_i \equiv \frac{\partial F}{\partial x_i}, \quad \hat{f} = \hat{A} \hat{x}, \quad \hat{f} \equiv \left[ \begin{array}{c} f_1 \\ f_2 \\ \vdots \\ f_n \\ f_{n+1} \end{array} \right], \quad \hat{x} \equiv \left[ \begin{array}{c} x_1 \\ x_2 \\ \vdots \\ x_n \\ x_{n+1} \end{array} \right]
\]

\(^8\)Note the summation of \( i \) is form 1 to \( n+1 \), where we have included the auxiliary parameter \( x_{n+1} \), which is an apparent different from the tradition Feynman parametrization.

\(^9\)In general it is not necessary to make \( \hat{A} \) be symmetry matrix, and this is just one choice. But for the simplification of the following calculation, since we will later set an antisymmetric matrix \( \hat{K}_A \), it is convenient to make the convention to set \( \hat{A} \) be symmetry matrix.
Solving $\hat{x} = A^{-1} \hat{f}$, we have
\[
F = \frac{1}{2} \hat{x}^T A \hat{x} = \frac{1}{2} \hat{f}^T (A^{-1})^T A A^{-1} \hat{f} = \frac{1}{2} \hat{f}^T (A^{-1})^T \hat{f} \equiv \hat{f}^T \hat{K} \hat{f}, \quad K = \frac{1}{2} A^{-1} \quad (2.36)
\]
where the coefficients’ matrix $\hat{K}$ is a real symmetry matrix. In fact we can do more. Using the trick that
\[
0 = \hat{f}^T \hat{K}_A \hat{f} \quad (2.37)
\]
with any antisymmetric matrix $K_A$, we could add (2.37) to (2.36) to get a more general form
\[
F = \hat{f}^T \hat{K} \hat{f} + \hat{f}^T \hat{K}_A \hat{f} \equiv \hat{f}^T \hat{R} \hat{f} \equiv \hat{f}^T \hat{Q} \hat{x}, \quad \hat{Q} = \frac{1}{2} \hat{I} + \hat{K}_A \hat{A} \quad (2.38)
\]
Noticing that because the arbitrary matrix $\hat{K}_A$ of rank $n+1$, there are $\frac{n(n+1)}{2}$ free independent parameters, $a_1, \cdots, a_{n(n+1)}$ in the matrix $\hat{Q}$ in (2.38).

Now putting (2.38) back to (2.34), we could solve $\hat{z}$ as
\[
\hat{f}^T \hat{z} + B \hat{f}^T \hat{Q} \hat{x} = 0, \quad \Rightarrow \hat{z} = -B \hat{Q} \hat{x} \quad (2.39)
\]
Noticing that since $z$ is degree zero, we should have $B$ homogenous function of degree $-1$. In our article, we choose $B = \frac{1}{x_{n+1}}$. The choice of $\hat{z}$ given by (2.39) will guarantee to remove the dimension shift in the IBP relation. Furthermore, by choosing particular value of these free parameters of $\hat{Q}$, we could cancel some unwanted terms. In the later computations, we will give some examples to illustrate this trick.

3. Reduction of one-loop integrals

As we have mentioned in the introduction, one motivation of the paper is to complete the reduction of scalar basis with general powers. Using the unitarity cut method in [3], we are able to find reduction coefficients of all basis, except the tadpole. In this section, we will use the improved IBP relation (2.32) to find the tadpole coefficients as well as other coefficients.

3.1 The bubble’s case

Let us start from the bubble topology. Although we have done it already in (2.30), here we will redo it using the improved IBP relation (2.32). The parametric form of bubble is given by (2.18), (2.19) and (2.20). Using our label, we have
\[
\hat{f} = \hat{A} \hat{x}, \quad \hat{A} = \begin{bmatrix}
2m_1^2 & \Delta & 1 \\
\Delta & 2m_2^2 & 1 \\
1 & 1 & 0
\end{bmatrix} \quad (3.1)
\]
and

\[ F = \hat{f}^T \hat{K} \hat{f}, \quad \hat{K} = \begin{bmatrix}
\frac{1}{4p_1^2} & -\frac{1}{4p_1^2} & -\frac{m_2^2 + m_3^2 + p_1^2}{4p_1^2} \\
-\frac{1}{4p_1^2} & \frac{1}{4p_1^2} & \frac{m_1^2 - m_2^2 + p_1^2}{4p_1^2} \\
-m_1^2 + m_2^2 + p_1^2 & m_1^2 - m_2^2 + p_1^2 & \frac{\Delta^2 - 4m_1^2m_2^2}{4p_1^2}
\end{bmatrix}\]

(3.2)

Adding the antisymmetric matrix \( K_A \), we have

\[ \hat{K}_A = \begin{bmatrix}
0 & a_1 & a_2 \\
-a_1 & 0 & a_3 \\
-a_2 & -a_3 & 0
\end{bmatrix}, \quad \hat{Q} = \begin{bmatrix}
ant + \frac{1}{2}a_2 + \frac{1}{2}a_1m_1^2 + \frac{1}{2}a_1m_3^2 - 2a_1p_1^2 & a_2 + 2a_1m_2^2 & a_1 \\
a_3 - 2a_1m_1^2 & \frac{1}{2}a_3 - 2a_1m_2^2 - 2a_1m_3^2 + 2a_1p_1^2 & -a_1 \\
-2a_2m_1^2 - a_3(m_1^2 + m_2^2 - p_1^2) - 2a_3m_2^2 - a_2(m_1^2 + m_2^2 - p_1^2) & \frac{1}{2}a_3 - 2a_1m_2^2 - 2a_1m_3^2 + 2a_1p_1^2 & -a_1
\end{bmatrix}\]

(3.3)

### 3.1.1 Deriving the recurrence relation

Taking \( B = \frac{1}{x_2} \) in (2.34), solution (2.39) gives \( z_i = \frac{Q_i x_i}{x_2} \). Expanding the (2.32), we got the IBP recurrence relation

\[ c_{m,n}i\lambda_0;m,n + c_{m+1,n}i\lambda_0;m+1,n + c_{m+1,n-1}i\lambda_0;m+1,n-1 + c_{m,n+1}i\lambda_0;m,n+1 \]

\[ c_{m-1,n+1}i\lambda_0;m-1,n+1 + c_{m,n-1}i\lambda_0;m,n-1 + c_{m-1,n}i\lambda_0;m-1,n + \delta_2 = 0 \]

(3.4)

where the \( \delta_2 \) is the boundary term, which we will compute later. Other coefficients are

\[ c_{m,n} = Q_{11}(1 + m) + Q_{22}(1 + n) + Q_{33}(1 + \lambda_3) + \lambda_0 \]

\[ c_{m+1,n} = Q_{31} = -\lambda_3(a_2A_{11} + a_3A_{21}), \quad c_{m+1,n-1} = Q_{21} = -n(a_1A_{11} - a_3A_{31}) \]

\[ c_{m,n+1} = Q_{32} = -\lambda_3(a_2A_{12} - a_3A_{22}), \quad c_{m-1,n+1} = Q_{12} = m(a_1A_{22} + a_2A_{32}) \]

\[ c_{m,n-1} = Q_{23} = -n(a_1A_{13} - a_3A_{33}), \quad c_{m-1,n} = Q_{13} = m(a_1A_{32} + a_2A_{33}) \]

(3.5)

Since we want to get the reduction of \( I_2(1,2) \), starting from \( m = n = 0 \), we want to eliminate terms with indices \( (m + 1, n) \) and \( (m, n - 1) \), while keeping the term with index \( (m, n + 1) \). Thus we impose \( c_{m+1,n} = 0 \) and \( c_{m+1,n-1} = 0 \), which can be satisfied by choosing the free parameters

\[ a_2 = -\frac{a_1A_{21}}{A_{31}} = -a_1(m_1^2 + m_2^2 - p_1^2), \quad a_3 = \frac{a_1A_{11}}{A_{31}} = 2a_1m_1^2 \]

\[ \hat{Q}_{x} = \begin{bmatrix}
\frac{1}{2} & \frac{a_1}{A_{31}}(A_{22}A_{31} - A_{21}A_{32}) & \frac{a_1}{A_{31}}(A_{23}A_{31} - A_{21}A_{33}) \\
0 & \frac{1}{2} - \frac{a_1}{A_{31}}(A_{12}A_{31} - A_{11}A_{32}) & \frac{a_1}{A_{31}}(A_{11}A_{33} - A_{13}A_{31}) \\
0 & \frac{a_1}{A_{31}}(A_{12}A_{21} - A_{11}A_{22}) & \frac{1}{2} + \frac{a_1}{A_{31}}(A_{13}A_{21} - A_{11}A_{23})
\end{bmatrix} \]

\[ \text{For this example, one can check that we can not add another constraint to fix } a_1. \]
and it left us five terms with non-zero coefficients\textsuperscript{11},

\[
\begin{align*}
c_{m,n+1} &= -\frac{a_1 \lambda_3}{A_{31}}(A_{11}A_{22} - A_{12}A_{21}) = -\frac{a_1 \lambda_3}{A_{31}}|\tilde{A}_{33}| = a_1 \lambda_3((m_1^2 + m_2^2 - p_1^2)^2 - 4m_1^2m_2^2) \\
c_{m-1,n+1} &= -\frac{ma_1}{A_{31}}(A_{21}A_{32} - A_{22}A_{31}) = -\frac{ma_1}{A_{31}}|\tilde{A}_{13}| = -a_1 m(m_1^2 - m_2^2 - p_1^2) \\
c_{m,n-1} &= \frac{ma_1}{A_{31}}(A_{11}A_{33} - A_{13}A_{31}) = \frac{na_1}{A_{31}}|\tilde{A}_{22}| = -a_1 n \\
c_{m-1,n} &= -\frac{ma_1}{A_{31}}(A_{21}A_{33} - A_{23}A_{31}) = -\frac{ma_1}{A_{31}}|\tilde{A}_{12}| = a_1 m \\
c_{m,n} &= \frac{a_1}{A_{31}}((1 + n)(A_{11}A_{32} - A_{12}A_{31}) - (\lambda_3 + 1)(A_{11}A_{23} - A_{13}A_{21})) = \frac{a_1}{A_{31}}(n - \lambda_3)|\tilde{A}_{23}|
\end{align*}
\]

\[(3.7)\]

The boundary $\delta_2$ term: The $\delta_2$ term is given by

\[
\delta_2 = \sum_{i=1}^{3} \delta_{\lambda_i,0} \int d\Pi^{(2)} \left\{ z_i F^{\lambda_0} x_1^{m} x_2^{n} x_3^{\lambda_3 + 1} \right\} |_{x_i=0}
\]

\[(3.8)\]

where the $\lambda_i$ represents the power of $x_i$. It is worth to emphasize that since $z_i$ contains $x_i$, the total power $\lambda_i$ of $x_i$ is not equal to $m, n, \lambda_3$ in general. Expanding it, we get\textsuperscript{12}

\[
\delta_2 = \delta_{\lambda_1,0} \int d\Pi^{(2)} \left( Q_{11} F^{\lambda_0} x_1^{m+1} x_2^{n} x_3^{\lambda_3} + Q_{12} F^{\lambda_0} x_1^{m} x_2^{n+1} x_3^{\lambda_3} + Q_{13} F^{\lambda_0} x_1^{m} x_2^{n} x_3^{\lambda_3 + 1} \right) |_{x_1=0} \\
+ \delta_{\lambda_2,0} \int d\Pi^{(2)} \left( Q_{21} F^{\lambda_0} x_1^{m+1} x_2^{n} x_3^{\lambda_3} + Q_{22} F^{\lambda_0} x_1^{m} x_2^{n+1} x_3^{\lambda_3} + Q_{23} F^{\lambda_0} x_1^{m} x_2^{n} x_3^{\lambda_3 + 1} \right) |_{x_2=0}
\]

\[(3.9)\]

Remembering our extended notation explained under (2.22), we have

\[
\int d\Pi^{(2)} F_{|x_1=0} x_1^{n} = i_{\lambda_0;1,n} \quad \int d\Pi^{(2)} F_{|x_2=0} x_1^{m} = i_{\lambda_0;m,-1}
\]

\[(3.10)\]

and the $\delta_2$ term could be written as

\[
\delta_{2;r} = \delta_{\lambda_1,0} \left( Q_{11;r} i_{\lambda_0;m+1,1} + Q_{12;r} i_{\lambda_0;m,n+1} + Q_{13;r} i_{\lambda_0;m} \right) \\
+ \delta_{\lambda_2,0} \left( Q_{21;r} i_{\lambda_0;m+1,1} + Q_{22;r} i_{\lambda_0;m,n+1} + Q_{23;r} i_{\lambda_0;m} \right) \\
= \delta_{m,-1} Q_{11;r} i_{\lambda_0;1,n} + \delta_{m,0} Q_{12;r} i_{\lambda_0;1,n+1} + \delta_{m,0} Q_{13;r} i_{\lambda_0;1,n} \\
+ \delta_{n,0} Q_{21;r} i_{\lambda_0;m+1,1} + \delta_{n,-1} Q_{22;r} i_{\lambda_0;m,n+1} + \delta_{n,-1} Q_{23;r} i_{\lambda_0;m,1}
\]

\[(3.11)\]

where the subscript $r$ in $\delta_{2;r}$ and $Q_{ij;r}$ means that the $a_2$ and $a_3$ should be replaced by (1.6). Since the $m$ and $n$ could not be $-1$, the first and fifth terms are actually zero.

\textsuperscript{11}where we use the convention $|\tilde{A}_{ij}|$ means the cofactor of matrix element $A_{ij}$

\textsuperscript{12}Since we have kept dimensional regularization $\epsilon$, the $\lambda_3$ can not be zero, thus the corresponding boundary term does not exist.
Now we could use (3.4) and (3.11) to get our result directly. Setting \( m = 0 \) and \( n = 0 \), all other terms in (3.4) are equal to zero, and we are left with\(^{13}\)

\[
c_{0,0}i\lambda_{0,0,0} + c_{0,1}i\lambda_{0,0,1} + \delta_{2,00} = 0
\]

with the coefficients

\[
c_{0,0} = -a_1(D - 3)(m_1^2 - m_2^2 + p_1^2)
\]

\[
c_{0,1} = a_1(D - 3)\left(m_1^4 + m_2^4 + 2m_1^2p_1^2 - 2m_2^2p_1^2 - 2m_1^2m_2^2\right)
\]

\[
\delta_{2,00} = Q_{12:r}i\lambda_{0;1,-1} + Q_{13;r}i\lambda_{0;0,-1} + Q_{21:r}i\lambda_{0;1,-1} + Q_{23:r}i\lambda_{0;0,-1}
\]

where

\[
Q_{21:r} = \frac{-a_1}{A_{31}}(A_{21}A_{32} - A_{22}A_{31}) = \frac{-a_1}{A_{31}}|\bar{A}_{13}|, \quad Q_{23:r} = \frac{-a_1}{A_{31}}(A_{11}A_{33} - A_{13}A_{31}) = \frac{-a_1}{A_{31}}|\bar{A}_{22}|
\]

\[
Q_{12:r} = \frac{-a_1}{A_{31}}(A_{21}A_{32} - A_{22}A_{31}) = \frac{-a_1}{A_{31}}|\bar{A}_{13}|, \quad Q_{13:r} = \frac{-a_1}{A_{31}}(A_{21}A_{33} - A_{23}A_{31}) = \frac{-a_1}{A_{31}}|\bar{A}_{12}|
\]

From it we could directly write down the answer

\[
i\lambda_{0:1,1} = -\frac{c_{0,0}}{c_{0,1}}i\lambda_{0;0,0} - \frac{Q_{21:r}}{c_{0,1}}i\lambda_{0;1,-1} - \frac{Q_{23:r}}{c_{0,1}}i\lambda_{0;0,-1} - \frac{Q_{12:r}}{c_{0,1}}i\lambda_{0;1,-1} - \frac{Q_{13:r}}{c_{0,1}}i\lambda_{0;0,-1}
\]

Translating back to scalar integrals, it is

\[
I_2(2,1) = c_{12-11}I_2(1,1) + c_{12-10}I_2(1,0) + c_{12-20}I_2(2,0) + c_{12-01}I_2(0,1) + c_{12-02}I_2(0,2)
\]

(3.16)

with \( c_{12-20} = 0 \) and

\[
c_{12-11} = \frac{(-3 + D)(m_1^2 - m_2^2 + p_1^2)}{(m_1^4 + (m_2^2 - p_1^2)^2 - 2m_1^2(m_2^2 + p_1^2))}, \quad c_{12-10} = \frac{D - 2}{-2m_1^2(m_2^2 + p_1^2) + m_1^4 + (m_2^2 - p_1^2)^2}
\]

\[
c_{12-01} = \frac{2 - D}{-2m_1^2(m_2^2 + p_1^2) + m_1^4 + (m_2^2 - p_1^2)^2}, \quad c_{12-02} = \frac{-m_1^2 + m_2^2 + p_1^2}{-2m_1^2(m_2^2 + p_1^2) + m_1^4 + (m_2^2 - p_1^2)^2}
\]

(3.17)

Using \( I_2(2,0) = \frac{D-2}{2m_1^2}I_2(1,0)\)\(^{14}\) and \( I_2(0,2) = \frac{D-2}{2m_2^2}I_2(0,1)\) we have our final result of reduction of \( I_2(1,2)\),

\[
I_2(1,2) = c_{2-21}I_2(1,1) + c_{2-20}I_2(1,0) + c_{2-11}I_2(0,1)
\]

(3.18)

\(^{13}\)When setting \( m = n = 0 \), except the boundary term \( \delta_2 \), among other seven terms in (3.4), the coefficients of the second and the third terms have been chosen to be zero. For the other five terms, one can show that \( C_{m-1,n+1}, C_{m,n-1}, C_{m-1,n} \) are zero by using the last line of (3.5). There is another technical point. When \( m = n = 0 \), the seventh term will contain \( i\lambda_{0;1,-1} \), which looks like the one defined in (3.10). But they are, in fact, different. The one appeared in (3.4) with the measure \( d\Pi_2 \) while the one appeared in (3.12) with measure \( d\Pi_2^{(3)} \).

\(^{14}\)The reduction of tadpole with higher power is simple. Noticing that \( I_2(1,0) \propto (m_1^2)^{D-2} \) by dimensional analysis, one can take the derivative over \( m_1^2 \) to get the wanted reduction coefficients.
with the coefficients

\[ c_{2\rightarrow 2} = - \frac{(D - 3) (m_1^2 - m_2^2 + p_1^2)}{-2m_1^2 (m_2^2 + p_1^2) + m_1^4 + (m_2^2 - p_1^2)^2} \]
\[ c_{2\rightarrow 1; 2} = - \frac{D - 2}{-2m_1^2 (m_2^2 + p_1^2) + m_1^4 + (m_2^2 - p_1^2)^2} \]
\[ c_{2\rightarrow 1; 1} = - \frac{(D - 2) (m_1^2 + m_2^2 - p_1^2)}{2m_2^2 \left(-2m_1^2 (m_2^2 + p_1^2) + m_1^4 + (m_2^2 - p_1^2)^2 \right)} \] (3.19)

which is given in (2.30).

### 3.2 The general case of bubbles

Now let us consider the more complicated examples, i.e., the bubble with general higher power of propagators. By the choice (3.6) we got an IBP recurrence relation (3.7) and use it we could reduce the bubbles \( i_{\lambda_0, m, n+1} \) to the simpler bubbles having less total power of propagators and no higher power in \( D_2 \). Similarly, by choosing the different values of \( a_2 \) and \( a_3 \), we could get another IBP recurrence relation to reduce the integral to those having no higher power in \( D_1 \). The choice is

\[ a_2 = - \frac{a_1 A_{22}}{A_{32}}, \quad a_3 = \frac{a_1 A_{12}}{A_{32}} \] (3.20)

and the corresponding IBP recurrence is

\[ c_{m+1, n+i_{\lambda_0, m+1, n}} + c_{m+1, n-1+i_{\lambda_0, m+1, n-1}} + c_{m, n-1+i_{\lambda_0, m-1, n}} + c_{m, n+i_{\lambda_0, m, n}} + \delta_{2, r} = 0 \] (3.21)

with the coefficients

\[ c_{m+1, n} = (|\tilde{A}_{33}|)(D - 3 - m - n), \quad c_{m+1, n-1} = -n|\tilde{A}_{23}| \]
\[ c_{m, n-1} = n|\tilde{A}_{21}|, \quad c_{m-1, n} = -m|\tilde{A}_{11}|, \quad c_{m, n} = |\tilde{A}_{13}|(3 + 2m + n - D) \] (3.22)

and the boundary term

\[ \delta_{2, r'} = -\delta_{m, 0}|\tilde{A}_{11}|i_{\lambda_0, m, n} + \delta_{n, 0} \left( -|\tilde{A}_{32}|i_{\lambda_0, m+1, n} + |\tilde{A}_{21}|i_{\lambda_0, m, n} \right) \] (3.23)

Combining (3.7) and (3.21), we could reduce the general bubbles.

#### 3.2.1 The example: \( I_2(1, 3) \)

In the example \( I_2(1, 3) \), we just need to reduce \( D_2 \) from power 3 to 1. The strategy is to use (3.7) two times. In the first step, by setting \( m = 0 \) and \( n = 1 \) in (3.7) we got

\[ I_2(1, 3) = \frac{|\tilde{A}_{23}|(D - 5)}{2|\tilde{A}_{33}|} I_2(1, 2) + \frac{|\tilde{A}_{22}|(D - 3)}{2|\tilde{A}_{33}|} I_2(1, 1) + \frac{-|\tilde{A}_{12}||(D - 3)}{2|\tilde{A}_{33}|} I_2(0, 2) + \frac{|\tilde{A}_{13}|}{|\tilde{A}_{33}|} I_2(0, 3) \] (3.24)
For the first term in (3.24), setting $m = 0$ and $n = 0$ in (3.7) again we have

$$I_2(1, 2) = \frac{|\tilde{A}_{23}|(D - 3)}{|A_{33}|} I_{2}(1, 1) + \frac{|\tilde{A}_{22}|(D - 2)}{|A_{33}|} I_{2}(1, 0) + \frac{|\tilde{A}_{13}|}{|A_{33}|} I_{2}(0, 2) + \frac{|\tilde{A}_{12}|(D - 2)}{|A_{33}|} I_{2}(0, 1)$$  \ (3.25)

Putting (3.25) into (3.24) and using the reduction of tadpole\(^{15}\) we get

$$I_2(1, 3) = c_{13 \to 11} I_{2}(1, 1) + c_{13 \to 10} I_{2}(1, 0) + c_{13 \to 01} I_{2}(0, 1)$$  \ (3.26)

with the coefficients

$$c_{13 \to 11} = \frac{(|\tilde{A}_{23}| |\tilde{A}_{33}| + |\tilde{A}_{23}|^2 (D - 5))(D - 3)}{2 |A_{33}|^2}$$

$$c_{13 \to 10} = \frac{|\tilde{A}_{22}| |\tilde{A}_{23}| (D - 5)(D - 2)}{2 |A_{33}|^2}$$

$$c_{13 \to 01} = \frac{(D - 2)}{8 |A_{33}|^2 m_2^4} A_{21} (2 A_{32} |\tilde{A}_{23}| (D - 5) m_2^2 + A_{32} A_{33} (D - 4) - A_{33} |\tilde{A}_{23}| (D - 5) m_2^4 - 2 A_{33} |\tilde{A}_{33}| (D - 3) m_2^2)$$

$$- A_{22} A_{31} (2 |\tilde{A}_{23}| (D - 5) m_2^2 + A_{33} (D - 4)) + 2 A_{23} A_{31} m_2^2 (2 |\tilde{A}_{23}| (D - 5) m_2^2 + |\tilde{A}_{33}| (D - 3)).$$  \ (3.27)

The result is confirmed with FIRE6. In this example, we just need to solve 2 equations in reducing bubbles’ topology.

### 3.2.2 The example: \(I_2(3, 5)\)

For this example we need to use (3.21) to lower the power of $D_1$ and (3.7) to lower the power of $D_2$. Setting $m = 1$ and $n = 4$ in (3.21) we can reduce $I_2(3, 5)$ to $I_2(2, 4), I_2(2, 5), I_2(1, 5)$ and $I_2(3, 4)$.

$$I_2(3, 5) = \frac{|\tilde{A}_{11}| (D - 7)}{2 |A_{33}|} I_{2}(2, 1) + \frac{|\tilde{A}_{13}| (D - 9)}{2 |A_{33}|} I_{2}(2, 5) + \frac{|\tilde{A}_{21}| (D - 7)}{2 |A_{33}|} I_{2}(2, 4) + \frac{|\tilde{A}_{23}|}{|A_{33}|} I_{2}(3, 4)$$  \ (3.28)

Then setting $m = 1$ and $n = 3$ in (3.21), we reduce $I_2(3, 4)$ to $I_2(1, 4), I_2(2, 3), I_2(2, 4)$ and $I_2(3, 3)$.

$$I_2(3, 4) = -\frac{|\tilde{A}_{23}|}{|A_{33}|} I_{2}(3, 3) + \frac{|\tilde{A}_{13}| (D - 8)}{2 |A_{33}|} I_{2}(2, 4) + \frac{|\tilde{A}_{21}| (D - 6)}{2 |A_{33}|} I_{2}(2, 3) + \frac{|\tilde{A}_{11}| (D - 6)}{2 |A_{33}|} I_{2}(1, 4)$$  \ (3.29)

With the same idea going down, we just need to solve 14 equation to complete reduce the $I_2(3, 5)$. The analytic expression by these 14 equations have also been confirmed by FIRE6.

\(^{15}\)In general, we could repeat the similar procedure to give the tadpoles’ IBP recurrence relation, and calculate them step by step. Here, for simplicity, we could just use the trick, $I_2(1, 0) \propto (m_2^2)^{-\frac{D-4}{2}}$, and $I_2(0, 1) \propto (m_2^2)^{-\frac{D-2}{2}}$, to directly calculate the $I_2(2, 0) = \frac{\partial}{\partial m_1^2} I_2(1, 0) = \frac{D-4}{2 m_1^2} I_2(1, 0), I_2(0, 2) = \frac{\partial}{\partial m_2^2} I_2(0, 1) = \frac{D-2}{2 m_2^2} I_2(0, 1)$, and $I_2(3, 0) = \frac{1}{2} (\frac{\partial}{\partial m_1^2})^2 I_2(1, 0) = \frac{(D-2)(D-4)}{8 m_1^2} I_2(1, 0), I_2(0, 3) = \frac{1}{2} (\frac{\partial}{\partial m_2^2})^2 I_2(0, 3) = \frac{(D-2)(D-4)}{8 m_2^2} I_2(0, 1)$.
3.3 The triangle’s case

The triangle $I_3(m + 1, n + 1, q + 1)$ is given by

$$I_3(m + 1, n + 1, q + 1) = \int \frac{d\Omega}{(l^2 - m_1^2)^{m+1}(l - p_1)^2 - m_2^2)^{n+1}(l + p_3)^2 - m_3^2)^{q+1}}$$ (3.30)

The parametric form of it is

$$I_3(m + 1, n + 1, q + 1) = i(-1)^{3+m+n+q} \frac{\Gamma(-\lambda_0)}{\Gamma(m+1)\Gamma(n+1)\Gamma(q+1)} i_{\lambda_0,m,n,q}$$ (3.31)

where

$$i_{\lambda_0,m,n,q} = \int d\Pi^{(4)} F^{\lambda_0} x_1^m x_2^n x_3^q x_4^r, \quad \lambda_0 = -\frac{D}{2}, \quad \lambda_4 = -4 - 2\lambda_0 - m - n - q = D - 4 - m - n - q$$ (3.32)

Using the expression (2.10), we have

$$U(x) = x_1 + x_2 + x_3, \quad V(x) = x_1 x_2 p_1^2 + x_1 x_3 p_2^2 + x_2 x_3 p_2^2$$

$$f(x) = -V + U \sum m_i^2 x_i = (x_1 + x_2 + x_3)(x_1 m_1^2 + x_2 m_2^2 + x_3 m_3^2) - x_1 x_2 p_1^2 - x_2 x_3 p_2^2 - x_1 x_3 p_2^2$$

$$F(x) = U(x) + f(x)$$

Thus we can read out matrices

$$\hat{A} = \begin{bmatrix} 2m_1^2 & m_1^2 + m_2^2 - p_1^2 & m_1^2 + m_3^2 - p_2^2 & 1 \\ m_2^2 + m_3^2 - p_1^2 & 2m_2^2 & m_2^2 + m_3^2 - p_2^2 & 1 \\ m_1^2 + m_3^2 - p_2^2 & m_2^2 + m_3^2 - p_2^2 & 2m_3^2 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}, \quad \hat{K}_A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix}$$

$$\hat{Q} = \frac{1}{2} \hat{f} + \hat{K}_A \hat{A}$$ (3.34)

3.3.1 Deriving the recurrence relation

Taking $B = \frac{1}{x_4}$ in (2.39), we got $z_i = \frac{Q_i x_i}{x_4}$. Taking this relation into our IBP identities (2.32), we got

$$\sum_{i=1}^{4} \int d\Pi^{(4)} \left\{ z_i F^{\lambda_0} x_1^m x_2^n x_3^q x_4^{\lambda_4+1} \right\} + \delta_3 = 0$$ (3.35)

where we will deal with the boundary $\delta_3$ term later. After expanding the first term, we got we got

$$c_{m,n,q} + c_{m+1,n,q} i_{\lambda_0;m+1,n,q} + c_{m+1,n,q-1} + c_{m+1,n-q-1} + c_{m+1,n-1,q} i_{\lambda_0;m+1,n-1,q}$$

$$+ c_{m-1,n,q} i_{\lambda_0;m-1,q+1} + c_{m,n,q-1} i_{\lambda_0;m+1,n-1,q} + c_{m,n,q+1} i_{\lambda_0;m+1,n} + c_{m,n,q+1} i_{\lambda_0;m,n+1}$$

$$+ c_{m,n-1,q+1} i_{\lambda_0;m,n-1,q} + c_{m-1,n,q} i_{\lambda_0;m-1,n,q+1} + c_{m-1,n,q+1} i_{\lambda_0;m,n-1,q} + c_{m-1,n,q} i_{\lambda_0;m,n-1,q}$$

$$+ c_{m,n-1,q} i_{\lambda_0;m,n-1,q} + \delta_3 = 0$$ (3.36)
Now, we could choose particular value of our six parameters, $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$ to let the coefficients $c_{m+1,n,q}$, $c_{m+1,n,q-1}$, $c_{m+1,n-1,q}$, $c_{m,n+1,q}$, $c_{m,n+1,q}$, $c_{m,n-1,q}$ be zero. The solution is

\[
\begin{align*}
    a_2 &= -a_1 \frac{A_{21}A_{42} - A_{22}A_{41}}{A_{31}A_{42} - A_{32}A_{41}} = -a_1 \frac{(-m_1^2 + m_2^2 + p_1^2)}{-m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2} \\
    a_3 &= a_1 \frac{(A_{21}A_{32} - A_{22}A_{31})}{A_{31}A_{42} - A_{32}A_{41}} = a_1 \frac{(-m_1^2 + m_2^2 + p_1^2)(m_2^2 + m_3^2 - p_2^2)}{-m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2} - 2a_1 m_2^2 \\
    a_4 &= a_1 \frac{(A_{11}A_{42} - A_{12}A_{41})}{A_{31}A_{42} - A_{32}A_{41}} = a_1 \frac{(-m_1^2 + m_2^2 + p_1^2)}{-m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2} \\
    a_5 &= -a_1 \frac{(A_{11}A_{32} - A_{12}A_{31})}{A_{31}A_{42} - A_{32}A_{41}} = a_1 \frac{(-m_1^2 + m_2^2 + p_1^2)(m_2^2 + m_3^2 - 2(p_1 \cdot p_2) - p_1^2 - p_2^2)}{-m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2} + 2a_1 m_1^2 \\
    a_6 &= a_1 \frac{(A_{11}A_{22} - A_{12}A_{21})}{A_{31}A_{42} - A_{32}A_{41}} = a_1 \frac{(m_1^4 - 2m_1^2(m_2^2 + p_1^2) + (m_2^2 - p_1^2)^2)}{-m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2} 
\end{align*}
\]

Then the matrix $\hat{Q}$ becomes to

\[
\hat{Q}_r = \frac{1}{\Delta_A} \begin{bmatrix}
\frac{1}{2} \Delta_A & 0 & a_1 |\bar{A}_{14}| & a_1 |\bar{A}_{13}| \\
0 & \frac{1}{2} \Delta_A & -a_1 |\bar{A}_{24}| & a_1 |\bar{A}_{23}| \\
0 & 0 & \frac{1}{2} \Delta_A + a_1 |\bar{A}_{34}| & a_1 |\bar{A}_{33}| \\
0 & 0 & -a_1 |\bar{A}_{44}| & \frac{1}{2} \Delta_A - a_1 |\bar{A}_{43}| 
\end{bmatrix}, \quad \Delta_A = Det \begin{bmatrix}
A_{31} & A_{32} \\
A_{41} & A_{42}
\end{bmatrix} = A_{31}A_{42} - A_{32}A_{41}
\]

After this, we have the reduced IBP relation with only the propagator $D_3 = (l + p_3)^2 - m_3^2$ having one increasing power

\[
\begin{align*}
    c_{m,n,q} & \lambda_0; m, n, q + c_{m,n,q+1} \lambda_0; m, n, q+1 + c_{m,n,q+1} \lambda_0; m, n, q+1 + c_{m,n,q+1} \lambda_0; m, n, q+1 \\
    & + c_{m,n,q+1} \lambda_0; m, n, q + c_{m,n,q+1} \lambda_0; m, n, q + c_{m,n,q+1} \lambda_0; m, n, q = 0
\end{align*}
\]
with the coefficients

\[ c_{m,n,q} = \lambda_0 + mQ_{11;r} + nQ_{22;r} + qQ_{33;r} + Q_{11;r} + Q_{22;r} + Q_{33;r} + \lambda_4Q_{44;r} + Q_{44;r} \]

\[ c_{m,n+1,q} = \lambda_4Q_{44;r} = \frac{-a_1\lambda_4}{A_{31}A_{42} - A_{32}A_{41}} \bar{A}_{14} \]

\[ c_{m,1,n+1,q} = mQ_{13;r} = \frac{a_1m}{A_{31}A_{42} - A_{32}A_{41}} \bar{A}_{14} \]

\[ c_{m,n-1,q} = nQ_{24;r} = \frac{-a_1n}{A_{31}A_{42} - A_{32}A_{41}} \bar{A}_{23} \]

\[ \delta_{3;r} = \left( \delta_{m+1,0}Q_{11;r} + \delta_{m,0}Q_{13;r} \right) i_{\lambda_0,1,n,q} + \delta_{m,0}Q_{12;r}i_{\lambda_0,1,n+1,q} + \delta_{m,0}Q_{13;r}i_{\lambda_0,1,n,q+1} \]

\[ + \delta_{q,0}Q_{21;r}i_{\lambda_0,m+1,n-1} + \left( \delta_{n+1,0}Q_{22;r} + \delta_{n,0}Q_{24;r} \right) i_{\lambda_0,m,n-1} \]

\[ + \delta_{q,0}Q_{31;r}i_{\lambda_0,m+1,n} + \delta_{q,0}Q_{32;r}i_{\lambda_0,m,n+1} + \left( \delta_{q+1,0}Q_{33;r} + \delta_{q,0}Q_{34;r} \right) i_{\lambda_0,m,n-1} \]  

(3.40)

The reduction of the boundary \( \delta_3 \) part: Similarly to the bubble’s situation, taking the value of \( z_i \) into the \( \delta_3 \) part, we have the result

\[ \delta_{3;r} \]

(3.41)

the \( i_{\lambda_0,m,n-1} \), \( i_{\lambda_0,m-1,n} \) and \( \delta_{3,0} \) contribute to the sub-topology of triangle, i.e. the bubble\(^{16} \).

3.3.2 The triangle’s example: \( I_3(1,1,2) \)

Now we apply the complete recurrence relation to the example \( I_3(1,1,2) \). Setting \( m = n = q = 0 \) in (3.39), we got our recurrence relation.

\[ c_{0,0,0,0}i_{\lambda_0,0,0,0} + c_{0,0,1}i_{\lambda_0,0,0,1} + \delta_{3,0} = 0 \]  

(3.42)

with the coefficients

\[ c_{0,0,1} = \lambda_4Q_{43;r} = \frac{1}{-m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2} \times \left\{ 2a_1(D - 4) \left( m_1^4p_2 - 2m_2^2((p_1 \cdot p_2) + p_2^2) - m_3^2(p_1 \cdot p_2) + p_2^2 \right) \right. \]

\[ + \left. p_2^2((p_1 \cdot p_2) + p_1^2) + m_3^2(2(p_1 \cdot p_2) + p_1^2 + p_2^2) + m_2^2(2(p_1 \cdot p_2)(2(p_1 \cdot p_2) + p_1^2 + p_2^2) \right) \]

\[ - 2m_3^2((p_1 \cdot p_2) + p_1^2) \]  

\[ + p_1^2(m_3^2 - 2m_2^2(p_1 \cdot p_2) + p_2^2) + p_2^2(2(p_1 \cdot p_2) + p_1^2 + p_2^2) \} \]

\[ c_{0,0,0} = -\frac{D}{2} + Q_{11;r} + Q_{22;r} + Q_{33;r} + (D - 3)Q_{44;r} \]

\[ = -2a_1(D - 4) \left( m_1^2(p_1 \cdot p_2) - m_2^2((p_1 \cdot p_2) + p_2^2) + p_1^2 \left( m_3^2 - (p_1 \cdot p_2) - p_2^2 \right) \right) \]

\[ -m_1^2 + m_2^2 + 2(p_1 \cdot p_2) + p_1^2 \]  

(3.43)

One can see that in (3.42), only two terms of triangle topologies are left: one is the scalar basis and one is the target we want to reduce. Other five terms in (3.39) disappear by the expression in (3.40). Thus there

\(^{16}\)Since the boundary term having only one \( x_i \) = 0, it reduces to the sub-topologies with only one propagator pinched.
is no need to solve mixed IBP relations. The $\delta_3$ term becomes to

$$
\delta_{p:000} = \delta_{p:000} \mid_{m=0,n=q=0} = Q_{14;2}i_{\lambda_0,-1,0,0} + Q_{12;2}i_{\lambda_0,-1,1,0} + Q_{13;2}i_{\lambda_0,-1,0,1} + Q_{21;2}i_{\lambda_0,1,-1,0} + Q_{24;2}i_{\lambda_0,0,-1,0} + Q_{23;2}i_{\lambda_0,0,0,-1}
$$

Translating back to the form of $I$, We have the result

$$
I_3(1,1,2) = c_{3\to111}I_3(1,1,1) + c_{3\to110}I_3(1,1,0) + c_{3\to101}I_3(1,0,1) + c_{3\to011}I_3(0,1,1) + c_{3\to210}I_3(2,1,0) + c_{3\to201}I_3(2,0,1) + c_{3\to120}I_3(1,2,0) + c_{3\to021}I_3(0,2,1) + c_{3\to102}I_3(1,0,2) + c_{3\to012}I_3(0,1,2)
$$

with the coefficients

$$
c_{3\to111} = \frac{c_{0,0,0} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to110} = \frac{-c_{34,2} \Gamma(D-2)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to101} = \frac{-c_{24,2} \Gamma(D-2)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to011} = \frac{-c_{14,2} \Gamma(D-2)}{c_{0,0,1} \Gamma(D-4)}
$$

$$
c_{3\to210} = \frac{c_{31,2} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to201} = \frac{c_{21,2} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to021} = \frac{c_{12,2} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to120} = \frac{c_{32,2} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}
$$

$$
c_{3\to102} = \frac{c_{33,2} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}, \quad c_{3\to012} = \frac{c_{13,2} \Gamma(D-3)}{c_{0,0,1} \Gamma(D-4)}
$$

The last step is to reduce the bubbles with one propagator having power two. This problem has been solved in the previous subsection (see (3.18)). With proper relabeling of external variables for last six terms in (3.45) and collecting all coefficients together, we have we got

$$
I_3(1,1,2) = c_{3\to3}I_3(1,1,1) + c_{3\to2,3}I_3(1,1,0) + c_{3\to2,2}I_3(1,0,1) + c_{3\to2,1}I_3(0,1,1) + c_{3\to1,3}I_3(1,0,0) + c_{3\to1,2,3}I_3(0,1,0) + c_{3\to1,1,2}I_3(0,0,1)
$$

Since the explicit expressions of these coefficients are long, we have given them in the companion Mathematica notebook. The result is confirmed by FIRE6.

### 3.3.3 The general case in triangles

Similarly to the bubbles’ case, by different choices, we could get three IBP recurrence relations, where in each one only one term has one propagator having higher power. For simplicity, let us label the IBP recurrence relation $e_{qi}$ which shifting the propagator $D_i$. Now we could use the $e_{qi}$ with $i = 1, 2, 3$ to calculate the general case of triangles. Let us denote

$$
e_{q1} : (a_1+1^+ + a_1+3^+ + a_1+2^- + a_3^- - a_2^- + a_1^- - a_0) i_{\lambda_0,m,n,q} + \delta_{3,i,e_{q1}} = 0
$$

$$
e_{q2} : (b_2+2^+ + b_2+3^- + b_1^- - 2^+ + b_3^- - b_2^- + b_1^- - b_0) i_{\lambda_0,m,n,q} + \delta_{3,i,e_{q2}} = 0
$$

$$
e_{q3} : (c_3+3^+ + c_2^- + 2^- - c_1^- + 3^+ + c_3^- - c_2^- + c_1^- - c_0) i_{\lambda_0,m,n,q} + \delta_{3,i,e_{q3}} = 0
$$

(3.48)
with all coefficients having the same form as (3.39). Combining them all, we could reduce the general
triangles. For example, for $I_3(2,2,3)$, starting with setting $m = 0$, $n = 1$ and $q = 2$ in eq1, we could reduce
$I_3(2,2,3)$ to $I_3(1,1,3)$, $I_3(1,2,2)$, $I_3(1,2,3)$, $I_3(2,1,3)$ and $I_3(2,2,2)$ and boundary terms, the general
bubbles. Then setting $m = 0$, $n = 0$ and $q = 2$ in eq1, we could reduce $I_3(2,1,3)$ to $I_3(1,1,2)$, $I_3(1,1,3)$,
$I_3(2,1,2)$. After 12 steps, we got the result of the reduction of the triangle’s topology. The boundary
terms involve bubbles and tadpoles, which have been dealt in previous subsections. Finally, we could get
all the coefficients from $I_3(2,2,3)$ to all the scalar basis.

3.4 The box case

The general form of box is given by

$$I_4(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1) = \int \frac{d^D l}{D_1^{n_1+1}D_2^{n_2+1}D_3^{n_3+1}D_4^{n_4+1}}$$

(3.49)

with

$$D_1 = l^2 - m_1^2, \quad D_2 = (l - p_1)^2 - m_2^2, \quad D_3 = (l - p_1 - p_2)^2 - m_3^2, \quad D_4 = (l + p_4)^2 - m_4^2$$

(3.50)

The parametric form of $I_4(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1)$ could be written as

$$I_4(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1) = \frac{i(-1)^{4+n_1+n_2+n_3+n_4}\Gamma(-\lambda_0)}{\Gamma(n_1 + 1)\Gamma(n_2 + 1)\Gamma(n_3 + 1)\Gamma(n_4 + 1)\Gamma(\lambda_5 + 1)} i^{\lambda_0; n_1,n_2,n_3,n_4}$$

(3.51)

where

$$i^{\lambda_0; n_1,n_2,n_3,n_4} = \int d^5\Pi F^{\lambda_0; x_1, x_2, x_3, x_4, x_5} = \int d^5\Pi^{(5)}(U x_5 + f)^{\lambda_0; x_1, x_2, x_3, x_4, x_5}$$

$$d^5\Pi^{(5)} = dx_1 dx_2 dx_3 dx_4 dx_5\delta(\sum x_j - 1), \quad \lambda_0 = -\frac{D}{2}$$

$$\lambda_5 = -5 - n_1 - n_2 - n_3 - n_4 - 2\lambda_0 = D - 5 - n_1 - n_2 - n_3 - n_4$$

(3.52)
and the functions are

\[ U(x) = x_1 + x_2 + x_3 + x_4 \]

\[ V(x) = x_1 x_2 p_1^2 + x_1 x_3 (p_1 + p_2)^2 + x_1 x_4 (p_1 + p_2 + p_3)^2 + x_2 x_3 p_2^2 + x_2 x_4 (p_2 + p_3)^2 + x_3 x_4 p_3^2 \]

\[ f(x) = -V(x) + U(x) \sum m_i^2 x_i \]

\[ = m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4 + (m_1^2 + m_2^2 - p_1^2) x_1 x_2 + \]

\[ + [m_1^2 + m_2^2 - (p_1 + p_2)^2] x_1 x_3 + \]

\[ + [m_1^2 + m_2^2 - (p_1 + p_2 + p_3)^2] x_1 x_4 \]

\[ + (m_2^2 + m_3^2 - p_2^2) x_2 x_3 + [m_2^2 + m_4^2 - (p_2 + p_3)^2] x_2 x_4 + \]

\[ + (m_3^2 + m_4^2 - p_3^2) x_3 x_4 \]

\[ + x_1 x_5 + x_2 x_5 + x_3 x_5 + x_4 x_5 \]

\[ = (x_1 + x_2 + x_3 + x_4) (m_1^2 x_1 + m_2^2 x_2 + m_3^2 x_3 + m_4^2 x_4 + x_5) \]

\[ - x_1 x_2 p_1^2 - x_1 x_3 (p_1 + p_2)^2 - x_1 x_4 (p_1 + p_2 + p_3)^2 - x_2 x_3 p_2^2 - x_2 x_4 (p_2 + p_3)^2 - x_3 x_4 p_3^2 \quad (3.53) \]

Now the matrix are given by

\[
\begin{bmatrix}
2m_1^2 & m_1^2 + m_2^2 - p_1^2 & m_1^2 + m_3^2 - p_1^2 & m_1^2 + m_4^2 - p_1^2 & 1 \\
2m_1^2 + m_2^2 - p_1^2 & 2m_2^2 & m_2^2 + m_3^2 - p_2^2 & m_2^2 + m_4^2 - p_2^2 & 1 \\
m_1^2 + m_2^2 - p_1^2 & m_1^2 + m_3^2 - p_1^2 & 2m_3^2 & m_3^2 + m_4^2 - p_3^2 & 1 \\
m_1^2 + m_2^2 - p_1^2 & m_1^2 + m_3^2 - p_1^2 & m_1^2 + m_4^2 - p_1^2 & 2m_4^2 & 1 \\
1 & 1 & 1 & 1 & 0
\end{bmatrix}, \quad K_A = \begin{bmatrix}
0 & a_1 & a_2 & a_3 & a_4 \\
-1 & 0 & a_5 & a_6 & a_7 \\
a_2 & -a_5 & 0 & a_8 & a_9 \\
a_3 & -a_6 & -a_8 & 0 & a_{10} \\
a_{10} & -a_7 & -a_9 & -a_{10} & 0
\end{bmatrix}
\]

(3.54)

where \( p_{ij} \equiv p_i \cdot p_{i+1} \cdots p_j \).

### 3.4.1 Deriving the recurrence relation

Taking \( B = \frac{1}{x_5} \) in (2.39), we got

- 20 –
\[
\begin{align*}
\{ c_{n1+1,n2,n3,n4} &+ c_{n1+1,n2,n3,n4-1}+4^- + c_{n1+1,n2,n3-1,n4}+1^+3^- + c_{n1+1,n2-1,n3,n4}1^+2^- \\
+c_{n1,n2+1,n3,n4}2^+ + c_{n1,n2+1,n3,n4-1}2^+4^- + c_{n1,n2+1,n3-1,n4}2^+3^- + c_{n1-1,n2+1,n3,n4}2^+1^- \\
+c_{n1,n2+1,n3,n4}3^+ + c_{n1,n2+1,n3,n4-1}3^+4^- + c_{n1,n2-1,n3,n4}3^+2^- + c_{n1-1,n2+1,n3,n4}3^+1^- \\
+c_{n1,n2,n3+1,n4}4^+ + c_{n1,n2,n3+1,n4-1}4^+3^- + c_{n1,n2-1,n3,n4+1}4^+2^- + c_{n1-1,n2,n3,n4+1}4^+1^- \\
+c_{n1,n2,n3,n4}4^- + c_{n1,n2,n3,n4}3^- + c_{n1,n2-1,n3,n4}2^- + c_{n1-1,n2,n3,n4}1^- + c_{n1,n2,n3,n4} \} i_{n1,n2,n3,n4} + \delta_4 = 0 \\
\end{align*}
\]

(3.55)

where

\[
\begin{align*}
j^+ i_{n1\cdots n_j \cdots n_k} &= i_{n1\cdots n_j+1 \cdots n_k}, \quad j^- i_{n1\cdots n_j \cdots n_k} = i_{n1\cdots n_j-1 \cdots n_k} \\
\end{align*}
\]

(3.56)

Similarly, we could choose particular value of the parameters \(a_2\) to \(a_{10}\) with \(a_1\) free to make the coefficients of terms in the first three lines of (3.55) be zero. The analytic solution will be collected in the companion Mathematica notebook, while we could express the solution of the parameters by matrix elements of \(\tilde{A}\).

\[
\begin{align*}
a_2 &= \frac{-a_1}{\Delta_{box}} |\tilde{A}_{13,45}|, \quad a_3 = \frac{a_1}{\Delta_{box}} |\tilde{A}_{14,45}|, \quad a_4 = \frac{-a_1}{\Delta_{box}} |\tilde{A}_{15,45}|, \quad a_5 = \frac{a_1}{\Delta_{box}} |\tilde{A}_{23,45}|, \quad a_6 = \frac{-a_1}{\Delta_{box}} |\tilde{A}_{24,45}| \\
a_7 &= \frac{a_1}{\Delta_{box}} |\tilde{A}_{25,45}|, \quad a_8 = \frac{a_1}{\Delta_{box}} |\tilde{A}_{34,45}|, \quad a_9 = \frac{-a_1}{\Delta_{box}} |\tilde{A}_{35,45}|, \quad a_{10} = \frac{a_1}{\Delta_{box}} |\tilde{A}_{45,45}|, \Delta_{Box} = \begin{vmatrix} A_{31} & A_{32} & A_{33} \\ A_{41} & A_{42} & A_{43} \\ A_{51} & A_{52} & A_{53} \end{vmatrix}
\end{align*}
\]

where \(|\tilde{A}_{i,j,kl}|\) means the determinant of the matrix \(A\) after we removed the \(i, j\)th rows and \(k, l\)th columns. Then the matrix \(\tilde{Q}\) becomes to

\[
\tilde{Q}_r = \frac{1}{\Delta_{Box}} \begin{bmatrix}
\frac{1}{2} \Delta_{Box} & 0 & 0 & -a_1 |\tilde{A}_{15}| & -a_1 |\tilde{A}_{14}| \\
0 & \frac{1}{2} \Delta_{Box} & 0 & a_1 |\tilde{A}_{25}| & a_1 |\tilde{A}_{24}| \\
0 & 0 & \frac{1}{2} \Delta_{Box} & -a_1 |\tilde{A}_{35}| & -a_1 |\tilde{A}_{34}| \\
0 & 0 & 0 & \frac{1}{2} \Delta_{Box} & a_1 |\tilde{A}_{45}| & a_1 |\tilde{A}_{44}| \\
0 & 0 & 0 & 0 & -a_1 |\tilde{A}_{55}| & \frac{1}{2} \Delta_{Box} & a_1 |\tilde{A}_{54}| \\
\end{bmatrix}
\]

After this we got the simplified recurrence relation

\[
\begin{align*}
c_{n1,n2,n3,n4+1}i_{n1,n2,n3,n4+1} &+ c_{n1,n2,n3-1,n4+1}i_{n1,n2,n3-1,n4+1} \\
+c_{n1,n2-1,n3,n4+1}i_{n1,n2-1,n3,n4+1} &+ c_{n1-1,n2,n3,n4}i_{n1-1,n2,n3,n4} \\
+c_{n1,n2,n3-1,n4-1}i_{n1,n2,n3-1,n4-1} &+ c_{n1-1,n2,n3,n4}i_{n1-1,n2,n3-1,n4} \\
+c_{n1,n2-1,n3,n4}i_{n1,n2-1,n3,n4} &+ c_{n1-1,n2,n3,n4}i_{n1-1,n2,n3-1,n4} \\
+c_{n1,n2,n3,n4}i_{n1,n2,n3,n4} &+ \delta_{4,r} = 0
\end{align*}
\]

(3.57)
Now we need to calculate the $\delta_4$ term.

**The Reduction of the boundary $\delta_4$ term:** Similarly to the former case, we could expand the $\delta_4$ term and taking the value of parameters $a_2$ to $a_{10}$ into the $\delta_4$ part. After this, we could get

$$
\delta_{4,r} = \delta_{n_1+1,0} Q_{11} r^{i-1,n_2,n_3,n_4} + \delta_{n_1,0} Q_{12} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_1,0} Q_{13} r^{i-1,n_2,n_3+1,n_4} + \delta_{n_1,0} Q_{14} r^{i-1,n_2,n_3,n_4+1} + \delta_{n_1,0} Q_{15} r^{i-1,n_2,n_3,n_4} + \delta_{n_2,0} Q_{21} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_2,0} Q_{22} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_2,0} Q_{23} r^{i-1,n_2,n_3+1,n_4} + \delta_{n_2,0} Q_{24} r^{i-1,n_2,n_3+1,n_4+1} + \delta_{n_2,0} Q_{25} r^{i-1,n_2,n_3,n_4+1} + \delta_{n_3,0} Q_{31} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_3,0} Q_{32} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_3,0} Q_{33} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_3,0} Q_{34} r^{i-1,n_2+1,n_3,n_4+1} + \delta_{n_3,0} Q_{35} r^{i-1,n_2+1,n_3,n_4+1} + \delta_{n_4,0} Q_{41} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_4,0} Q_{42} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_4,0} Q_{43} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_4,0} Q_{44} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_4,0} Q_{45} r^{i-1,n_2+1,n_3,n_4} + \delta_{n_4,0} Q_{45} r^{i-1,n_2,n_3,n_4+1}
$$

(3.58)

where the subscript "r" means the value of the parameter $Q$ after we set $a_2$ to $a_{10}$.

3.4.2 The example: $I_4(1, 1, 1, 2)$

Now we could use the recurrence relation (3.57) to calculate our example $I_4(1, 1, 1, 2)$. Let $n_1 = n_2 = n_3 = n_4 = 0$, we got (the coefficients of the other terms are all zero)

$$
c_{0,0,0,0} i_{0,0,0,0} + c_{0,0,0,1} i_{0,0,0,1} + \delta_{4:0000} = 0
$$

(3.59)

where $\delta_{4:0000} \equiv \delta_{4,|_{n_1=n_2=n_3=n_4=0}}$. Translating to $I$, we have the result

$$
I_4(1, 1, 1, 2) = c_{4\rightarrow 1111} I_4(1, 1, 1, 1)
$$

$$
+ c_{4\rightarrow 1110} I_4(1, 1, 1, 0) + c_{4\rightarrow 1101} I_4(1, 1, 0, 1) + c_{4\rightarrow 1011} I_4(1, 0, 1, 1) + c_{4\rightarrow 0111} I_4(0, 1, 1, 1)
$$

$$
+ c_{4\rightarrow 2110} I_4(2, 1, 1, 0) + c_{4\rightarrow 2101} I_4(2, 1, 0, 1) + c_{4\rightarrow 2011} I_4(2, 0, 1, 1)
$$

$$
+ c_{4\rightarrow 1210} I_4(1, 2, 1, 0) + c_{4\rightarrow 1201} I_4(1, 2, 0, 1) + c_{4\rightarrow 0211} I_4(0, 2, 1, 1)
$$

$$
+ c_{4\rightarrow 1120} I_4(1, 1, 2, 0) + c_{4\rightarrow 1021} I_4(1, 0, 2, 1) + c_{4\rightarrow 0121} I_4(0, 1, 2, 1)
$$

$$
+ c_{4\rightarrow 1102} I_4(1, 1, 0, 2) + c_{4\rightarrow 1012} I_4(1, 0, 1, 2) + c_{4\rightarrow 0112} I_4(0, 1, 1, 2)
$$

(3.60)
with the coefficients

\[
\begin{align*}
    c_{4 \to 1111} &= \frac{c_{0,0,0,0}}{c_{0,0,0,1}} (D - 5) = \frac{\text{Tr} \hat{Q}_{ij} + (D - 5)Q_{55,r} - \frac{D}{2}}{Q_{54,r}}, \\
    c_{4 \to 0111} &= -\frac{Q_{15,r} \Gamma(D - 3)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 1101} &= -\frac{Q_{35,r} \Gamma(D - 3)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 0211} &= \frac{Q_{12,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 0121} &= \frac{Q_{13,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 2011} &= \frac{Q_{21,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 1021} &= \frac{Q_{23,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 2101} &= \frac{Q_{31,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 1201} &= \frac{Q_{32,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 2110} &= \frac{Q_{41,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 1210} &= \frac{Q_{42,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}, \\
    c_{4 \to 1120} &= \frac{Q_{43,r} \Gamma(D - 4)}{c_{0,0,0,1} \Gamma(D - 5)}
\end{align*}
\]  

(3.61)

Next we need to use the reduction of triangles with one double propagators given in (3.47). Put them into the (3.60), we got the complete reduction of box \( I_4(1,1,1,2) \).

\[
\begin{align*}
    I_4(1,1,1,2) &= c_{4 \to 4} I_4(1,1,1,1) \\
    + c_{4 \to 3;1} I_4(0,1,1,1) + c_{4 \to 3;3} I_4(1,0,1,1) + c_{4 \to 3;3} I_4(1,1,0,1) + c_{4 \to 3;3} I_4(1,1,1,0) \\
    + c_{4 \to 2;12} I_4(0,0,1,1) + c_{4 \to 2;13} I_4(0,1,0,1) + c_{4 \to 2;14} I_4(0,1,1,0) \\
    + c_{4 \to 2;23} I_4(1,0,0,1) + c_{4 \to 2;24} I_4(1,0,1,0) + c_{4 \to 2;34} I_4(1,1,0,0) \\
    + c_{4 \to 1;D_1} I_4(1,0,0,0) + c_{4 \to 1;D_2} I_4(0,1,0,0) + c_{4 \to 1;D_3} I_4(0,0,1,0) + c_{4 \to 1;D_4} I_4(0,0,0,1) \quad \text{(3.62)}
\end{align*}
\]

with the long expressions of these coefficients given in the companion Mathematica notebook. The result is confirmed by FIRE6.

### 3.5 The pentagon’s case

The general form of pentagon is given by

\[
I_5(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1, n_5 + 1) = \int \frac{d^D l}{D_1^{n_1+1}D_2^{n_2+1}D_3^{n_3+1}D_4^{n_4+1}D_5^{n_5+1}} \quad \text{(3.63)}
\]

with

\[
\begin{align*}
    D_1 &= l^2 - m_1^2, & D_2 &= (l - p_1)^2 - m_2^2, & D_3 &= (l - p_1 - p_2)^2 - m_3^2 \\
    D_4 &= (l - p_1 - p_2 - p_3)^2 - m_4^2, & D_5 &= (l + p_3)^2 - m_5^2 \quad \text{(3.64)}
\end{align*}
\]
The parametric form of \( I_5(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1, n_5 + 1) \) could be written as

\[
I_5(n_1 + 1, n_2 + 1, n_3 + 1, n_4 + 1, n_5 + 1) = \frac{i(-1)^{\frac{5}{2} + n_1 + n_2 + n_3 + n_4 + n_5} \Gamma(\lambda_0)}{\sum_{i=1}^5 \Gamma(n_i + 1) \Gamma(\lambda_6 + 1)} i_{\lambda_0, n_1, n_2, n_3, n_4, n_5}
\]

(3.65)

where

\[
i_{\lambda_0, n_1, n_2, n_3, n_4, n_5} = \int d\Pi^{(6)} F^{\lambda_0 x_1 x_2 x_3 x_4 x_5} x_6 \delta(\sum x_j - 1)
\]

\[
d\Pi^{(5)} = dx_1 dx_2 dx_3 dx_4 dx_5 dx_6 \delta(\sum x_j - 1)
\]

\[
\lambda_0 = \frac{D - 6}{2}, \quad \lambda_6 = (D - 6) - n_1 - n_2 - n_3 - n_4 - n_5
\]

(3.66)

and the function

\[
U(x) = x_1 + x_2 + x_3 + x_4 + x_5
\]

\[
V(x) = x_1 x_2 p_1^2 + x_1 x_3 p_1^2 + x_1 x_4 p_1^2 + x_1 x_5 p_1^2
\]

\[
+ x_2 x_3 p_2^2 + x_2 x_4 p_2^2 + x_2 x_5 p_2^2
\]

\[
f(x) = (x_1 + x_2 + x_3 + x_4 + x_5)(m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 + m_5 x_5)
\]

\[
- x_1 x_2 p_1^2 - x_1 x_3 p_1^2 - x_1 x_4 p_1^2 - x_1 x_5 p_1^2 - x_2 x_3 p_2^2 - x_2 x_4 p_2^2 - x_2 x_5 p_2^2
\]

\[
- x_3 x_4 p_3^2 - x_3 x_5 p_3^2
\]

\[
F(x) = (x_1 + x_2 + x_3 + x_4 + x_5)(m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 + m_5 x_5 + x_6)
\]

\[
- x_1 x_2 p_1^2 - x_1 x_3 p_1^2 - x_1 x_4 p_1^2 - x_1 x_5 p_1^2 - x_2 x_3 p_2^2 - x_2 x_4 p_2^2 - x_2 x_5 p_2^2
\]

\[
- x_3 x_4 p_3^2 - x_3 x_5 p_3^2 - x_3 x_6 p_3^2
\]

(3.67)

where \( p_{ij} = p_i + p_{i+1} + \cdots + p_{j-1} + p_j \). Now the matrix are given by

\[
\hat{A} = \begin{bmatrix}
2m_1^2 & m_1^2 + m_2^2 - p_1^2 & m_2^2 + m_3^2 - p_1^2 & m_1^2 + m_4^2 - p_1^2 & m_1^2 + m_5^2 - p_1^2 & 1 \\
m_1^2 + m_2^2 - p_1^2 & 2m_2^2 & m_2^2 + m_3^2 - p_2^2 & m_4^2 + m_5^2 - p_2^2 & m_2^2 + m_5^2 - p_2^2 & 1 \\
m_1^2 + m_3^2 - p_2^2 & m_2^2 + m_3^2 - p_3^2 & 2m_3^2 & m_3^2 + m_4^2 - p_3^2 & m_3^2 + m_5^2 - p_3^2 & 1 \\
m_1^2 + m_4^2 - p_1^2 & m_3^2 + m_4^2 - p_2^2 & m_2^2 + m_4^2 - p_2^2 & 2m_4^2 & m_4^2 + m_5^2 - p_4^2 & 1 \\
m_1^2 + m_5^2 - p_1^2 & m_3^2 + m_5^2 - p_2^2 & m_2^2 + m_5^2 - p_2^2 & m_4^2 + m_5^2 - p_4^2 & 2m_5^2 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}
\]

(3.68)
Taking $B = -\frac{1}{x_0}$ and putting $z_i$ into the IBP identities

\[
\sum_{i=1}^{6} \int \frac{\partial}{\partial x_i} \left\{ z_i F^{\lambda_0 x_1 \ldots x_6} \right\} + \delta_5 = 0
\]  

(3.69)

where the $\delta_5$ is given by

\[
\delta_5 = \sum_{i=1}^{5} \delta_{\lambda_i,0} \int d\Pi(5) \left\{ z_i F^{\lambda_0 x_1 \ldots x_6} \right\} \bigg|_{x_i=0}
\]

(3.70)

### 3.5.1 Deriving the recurrence relation

Similarly to the previous subsections, expanding the IBP relation we get

\[
\left\{ c_{n_1+1,n_2,n_3,n_4,n_5} 1^+ + c_{n_1+1,n_2-1,n_3,n_4,n_5} 1^2 + \ldots + c_{n_1,n_2,n_3-1,n_4,n_5} 1^7 + c_{n_1,n_2,n_3,n_4-1,n_5} 1^4 \right\} i_{n_1,n_2,n_3,n_4,n_5} + \delta_5 = 0
\]

(3.71)

We could choose the particular value of parameter $a_2$ to $a_{15}$ to let the coefficients of the first three line of (3.71) be zero. The solution is

\[
a_2 = -\frac{a_1}{\Delta_{pen}} |\bar{A}_{13,56}|, \quad a_3 = \frac{a_1}{\Delta_{pen}} |\bar{A}_{14,56}|, \quad a_4 = -\frac{a_1}{\Delta_{pen}} |\bar{A}_{15,56}|, \quad a_5 = \frac{a_1}{\Delta_{pen}} |\bar{A}_{16,56}|, \quad a_6 = \frac{a_1}{\Delta_{pen}} |\bar{A}_{23,56}|
\]

\[
a_7 = -\frac{a_1}{\Delta_{pen}} |\bar{A}_{24,56}|, \quad a_8 = \frac{a_1}{\Delta_{pen}} |\bar{A}_{25,56}|, \quad a_9 = -\frac{a_1}{\Delta_{pen}} |\bar{A}_{26,56}|, \quad a_{10} = \frac{a_1}{\Delta_{pen}} |\bar{A}_{34,56}|, \quad a_{11} = -\frac{a_1}{\Delta_{pen}} |\bar{A}_{35,56}|
\]

\[
a_{12} = \frac{a_1}{\Delta_{pen}} |\bar{A}_{36,56}|, \quad a_{13} = \frac{a_1}{\Delta_{pen}} |\bar{A}_{45,56}|, \quad a_{14} = -\frac{a_1}{\Delta_{pen}} |\bar{A}_{46,56}|, \quad a_{15} = \frac{a_1}{\Delta_{pen}} |\bar{A}_{56,56}|
\]

(3.72)

where

\[
\Delta_{pen} =
\begin{vmatrix}
A_{31} & A_{32} & A_{33} & A_{34} \\
A_{41} & A_{42} & A_{43} & A_{44} \\
A_{51} & A_{52} & A_{53} & A_{54} \\
A_{61} & A_{62} & A_{63} & A_{64}
\end{vmatrix}
\]
After this, we got

\[
\begin{align*}
&\left\{ c_{n_1,n_2,n_3,n_4,n_5+1;r} 5^+ + c_{n_1-1,n_2,n_3,n_4,n_5+1;r} 1^{-5^+} + c_{n_1,n_2-1,n_3,n_4,n_5+1;r} 2^{-5^+} + c_{n_1,n_2,n_3-1,n_4,n_5+1;r} 3^{-5^+} \\
+ &c_{n_1,n_2,n_3,n_4-1,n_5+1;r} 4^{-5^+}c_{n_1-1,n_2,n_3,n_4,n_5;r} 1^- + c_{n_1,n_2-1,n_3,n_4,n_5;r} 2^- + c_{n_1,n_2,n_3-1,n_4,n_5;r} 3^- \\
+ &c_{n_1,n_2,n_3,n_4-1,n_5;r} 4^- + c_{n_1,n_2,n_3,n_4-1,n_5;r} 5^- + c_{n_1,n_2,n_3,n_4,n_5;r} \right\} i_{\lambda_0; n_1,n_2,n_3,n_4,n_5} + \delta_{5;r} = 0 \quad (3.73)
\end{align*}
\]

here we have defined

\[
\begin{align*}
i^+ i_{\lambda_0; n_1,n_2,n_3,n_4,n_5} &\equiv i_{\lambda_0; n_1,\cdots n_i+1,\cdots n_5} \\
i^- i_{\lambda_0; n_1,n_2,n_3,n_4,n_5} &\equiv i_{\lambda_0; n_1,\cdots n_i-1,\cdots n_5} \quad (3.74)
\end{align*}
\]

with the coefficients

\[
\begin{align*}
c_{0,0,0,0,1} &= Q_{65;r}\lambda_6, & c_{-1,0,0,0,1} &= n_1 Q_{15;r}, & c_{0,-1,0,0,1} &= n_2 Q_{25;r}, & c_{0,0,-1,0,1} &= n_3 Q_{35;r}, & c_{0,0,0,-1,1} &= n_4 Q_{45;r} \\
c_{-1,0,0,0,0} &= n_1 Q_{16;r}, & c_{0,-1,0,0,0} &= n_2 Q_{26;r}, & c_{0,0,-1,0,0} &= n_3 Q_{36;r}, & c_{0,0,0,-1,0} &= n_4 Q_{46;r}, & c_{0,0,0,0,-1} &= n_5 Q_{56;r} \\
c_{00000;r} &= Tr\hat{Q}_{ij;r} + ((D-6))Q_{66;r} - \frac{D}{2} + n_1 Q_{11;r} + n_2 Q_{22;r} + n_3 Q_{33;r} + n_4 Q_{44;r} + n_5 Q_{55;r} \quad (3.75)
\end{align*}
\]

while the matrix $\hat{Q}$ becomes to

\[
\hat{Q}_r = \frac{1}{\Delta_{pen}} \begin{bmatrix}
\frac{1}{2} \Delta_{pen} & 0 & 0 & 0 & a_1|\bar{A}_{1,6}| & a_1|\bar{A}_{1,5}| \\
0 & \frac{1}{2} \Delta_{pen} & 0 & 0 & -a_1|\bar{A}_{2,6}| & -a_1|\bar{A}_{2,5}| \\
0 & 0 & \frac{1}{2} \Delta_{pen} & 0 & a_1|\bar{A}_{3,6}| & a_1|\bar{A}_{3,5}| \\
0 & 0 & 0 & \frac{1}{2} \Delta_{pen} & -a_1|\bar{A}_{4,6}| & -a_1|\bar{A}_{4,5}| \\
0 & 0 & 0 & 0 & \frac{1}{2} \Delta_{pen} + a_1|\bar{A}_{5,6}| & a_1|\bar{A}_{5,5}| \\
0 & 0 & 0 & 0 & -a_1|\bar{A}_{6,6}| & \frac{1}{2} \Delta_{pen} - a_1|\bar{A}_{6,5}|
\end{bmatrix}
\]
3.6 Reducing the $\delta_5$ term

Similar to the former situation, the $\delta_{6,r}$ term is given by

$$
\delta_{5,r} = Q_{11,r}\delta_{n_1,-1}i_{-1,n_2,n_3,n_4,n_5} + Q_{12,r}\delta_{n_1,0}i_{-1,n_2+1,n_3,n_4,n_5} + Q_{13,r}\delta_{n_1,1}i_{-1,n_2,n_3+1,n_4,n_5}
+ Q_{14,r}\delta_{n_2,0}i_{-1,n_2,n_3,n_4+1,n_5} + Q_{15,r}\delta_{n_2,1}i_{-1,n_2,n_3,n_4,n_5+1} + Q_{16,r}\delta_{n_1,0}i_{-1,n_2,n_3,n_4,n_5}
+ Q_{21,r}\delta_{n_2,0}i_{n_2+1,-1,n_3,n_4,n_5} + Q_{22,r}\delta_{n_2,-1}i_{n_2,-1,n_2,n_3,n_4,n_5} + Q_{23,r}\delta_{n_2,1}i_{n_1,-1,n_3+1,n_4,n_5}
+ Q_{24,r}\delta_{n_2,0}i_{n_1,-1,n_3,n_4+1,n_5} + Q_{25,r}\delta_{n_2,0}i_{n_1,-1,n_3,n_4,n_5+1} + Q_{26,r}\delta_{n_2,0}i_{n_1,-1,n_3,n_4,n_5}
+ Q_{31,r}\delta_{n_3,0}i_{n_1+1,1,n_2,n_3,n_4,n_5} + Q_{32,r}\delta_{n_3,0}i_{n_1,n_2+1,1,n_4,n_5} + Q_{33,r}\delta_{n_3,-1}i_{n_1,n_2,-1,n_4,n_5}
+ Q_{34,r}\delta_{n_3,0}i_{n_1,n_2,-1,n_4+1,n_5} + Q_{35,r}\delta_{n_3,0}i_{n_1,n_2,-1,n_4,n_5+1} + Q_{36,r}\delta_{n_3,0}i_{n_1,n_2,-1,n_4,n_5}
+ Q_{41,r}\delta_{n_4,0}i_{n_1+1,n_2,n_3,-1,n_5} + Q_{42,r}\delta_{n_4,0}i_{n_1,n_2+1,n_3,-1,n_5} + Q_{43,r}\delta_{n_4,0}i_{n_1,n_2,n_3+1,-1,n_5}
+ Q_{44,r}\delta_{n_4,1}i_{n_1,n_2,n_3,-1,n_5} + Q_{45,r}\delta_{n_4,0}i_{n_1,n_2,n_3,-1,n_5+1} + Q_{46,r}\delta_{n_4,0}i_{n_1,n_2,n_3,-1,n_5}
+ Q_{51,r}\delta_{n_5,0}i_{n_1+1,n_2,n_3,n_4,-1} + Q_{52,r}\delta_{n_5,0}i_{n_1,n_2+1,n_3,n_4,-1} + Q_{53,r}\delta_{n_5,0}i_{n_1,n_2,n_3+1,n_4,-1}
+ Q_{54,r}\delta_{n_5,0}i_{n_1,n_2,n_3+1,n_4,-1} + Q_{55,r}\delta_{n_5,-1}i_{n_1,n_2,n_3,n_4,-1} + Q_{56,r}\delta_{n_5,0}i_{n_1,n_2,n_3,n_4,n_5}
$$

(3.76)

3.7 The example: $I_5(1,1,1,1,2)$

Setting $n_1 = n_2 = n_3 = n_4 = n_5 = 0$, we got the IBP recurrence relation (other coefficients are all zero)

$$
c_{0,0,0,0,0,1}i_{\lambda_0;0,0,0,0,0} + c_{0,0,0,0,0,0}i_{\lambda_0;0,0,0,0,0} + \delta_{5,00000} = 0
$$

(3.77)

where $\delta_{5,00000} \equiv \delta_{5,r}|_{n_1=n_2=n_3=n_4=n_5=0}$.

Comparing them with our scalar basis, we have the result

$$
I_5(1,1,1,1,2) = c_{5\rightarrow 5}I_5(1,1,1,1,1) + c_{5\rightarrow 01111}I_4(0,1,1,1,1) + c_{5\rightarrow 10111}I_5(1,0,1,1,1)
+ c_{5\rightarrow 11011}I_5(1,1,0,1,1) + c_{5\rightarrow 11101}I_5(1,1,1,0,1) + c_{5\rightarrow 11110}I_5(1,1,1,1,0)
+ c_{5\rightarrow 20111}I_5(2,0,1,1,1) + c_{5\rightarrow 21011}I_5(2,1,0,1,1) + c_{5\rightarrow 21101}I_5(2,1,1,0,1)
+ c_{5\rightarrow 21110}I_5(2,1,1,1,0) + c_{5\rightarrow 02111}I_5(0,2,1,1,1) + c_{5\rightarrow 12011}I_5(1,2,0,1,1)
+ c_{5\rightarrow 12101}I_5(1,2,1,0,1) + c_{5\rightarrow 12110}I_5(1,2,1,1,0) + c_{5\rightarrow 01211}I_5(0,1,2,1,1)
+ c_{5\rightarrow 10211}I_5(1,0,2,1,1) + c_{5\rightarrow 11201}I_5(1,1,2,0,1) + c_{5\rightarrow 11210}I_5(1,1,2,1,0)
+ c_{5\rightarrow 01211}I_5(0,1,1,2,1) + c_{5\rightarrow 10121}I_5(1,0,1,2,1) + c_{5\rightarrow 11021}I_5(1,1,0,2,1)
+ c_{5\rightarrow 11120}I_5(1,1,1,2,0) + c_{5\rightarrow 01112}I_5(0,1,1,1,2) + c_{5\rightarrow 10112}I_5(1,0,1,1,2)
+ c_{5\rightarrow 11012}I_5(1,1,0,1,2) + c_{5\rightarrow 11102}I_5(1,1,1,0,2)
$$

(3.78)
with the coefficients
\[
c_{5\to 01111} = \frac{(D - 6)c_{0,0,0,0,0}}{c_{0,0,0,0,1}}, \quad c_{5\to 01111} = \frac{(D - 6)(5 - D)Q_{16r}}{c_{0,0,0,0,1}}, \quad c_{5\to 410111} = \frac{(D - 6)(5 - D)Q_{26r}}{c_{0,0,0,0,1}}
\]
\[
c_{5\to 411011} = \frac{(D - 6)(5 - D)Q_{36r}}{c_{0,0,0,0,1}}, \quad c_{5\to 411011} = \frac{(D - 6)(5 - D)Q_{46r}}{c_{0,0,0,0,1}}, \quad c_{5\to 411110} = \frac{(D - 6)(5 - D)Q_{56r}}{c_{0,0,0,0,1}}
\]
\[
c_{5\to 20111} = \frac{(D - 6)Q_{21r}}{c_{0,0,0,0,1}}, \quad c_{5\to 20111} = \frac{(D - 6)Q_{51r}}{c_{0,0,0,0,1}}, \quad c_{5\to 21101} = \frac{(D - 6)Q_{41r}}{c_{0,0,0,0,1}}, \quad c_{5\to 21110} = \frac{(D - 6)Q_{51r}}{c_{0,0,0,0,1}}
\]
\[
c_{5\to 21211} = \frac{(D - 6)Q_{12r}}{c_{0,0,0,0,1}}, \quad c_{5\to 21211} = \frac{(D - 6)Q_{32r}}{c_{0,0,0,0,1}}, \quad c_{5\to 21210} = \frac{(D - 6)Q_{42r}}{c_{0,0,0,0,1}}, \quad c_{5\to 21110} = \frac{(D - 6)Q_{52r}}{c_{0,0,0,0,1}}
\]
\[
c_{5\to 01211} = \frac{(D - 6)Q_{13r}}{c_{0,0,0,0,1}}, \quad c_{5\to 01211} = \frac{(D - 6)Q_{23r}}{c_{0,0,0,0,1}}, \quad c_{5\to 1120} = \frac{(D - 6)Q_{33r}}{c_{0,0,0,0,1}}, \quad c_{5\to 11120} = \frac{(D - 6)Q_{53r}}{c_{0,0,0,0,1}}
\]
\[
c_{5\to 01121} = \frac{(D - 6)Q_{14r}}{c_{0,0,0,0,1}}, \quad c_{5\to 01121} = \frac{(D - 6)Q_{24r}}{c_{0,0,0,0,1}}, \quad c_{5\to 11021} = \frac{(D - 6)Q_{34r}}{c_{0,0,0,0,1}}, \quad c_{5\to 11020} = \frac{(D - 6)Q_{54r}}{c_{0,0,0,0,1}}
\]
\[
c_{5\to 01112} = \frac{(D - 6)Q_{15r}}{c_{0,0,0,0,1}}, \quad c_{5\to 01112} = \frac{(D - 6)Q_{25r}}{c_{0,0,0,0,1}}, \quad c_{5\to 11012} = \frac{(D - 6)Q_{35r}}{c_{0,0,0,0,1}}, \quad c_{5\to 11102} = \frac{(D - 6)Q_{45r}}{c_{0,0,0,0,1}}
\]
(3.79)

The final step is to reduce the coefficients of the general boxes to the scalar basis.

After reduce them to our scalar basis, we got the final answer.

\[
I_5(1, 1, 1, 1, 2)
\]
\[
= c_{5\to 5}I_5(1, 1, 1, 1, 1) + c_{5\to 4,1}I_5(0, 1, 1, 1, 1) + c_{5\to 4,2}I_5(1, 0, 1, 1, 1) + c_{5\to 4,3}I_5(1, 1, 0, 1, 1)
\]
\[
+ c_{5\to 4,4}I_5(1, 1, 0, 1, 0) + c_{5\to 4,5}I_5(1, 1, 1, 1, 0) + c_{5\to 3,12}I_5(0, 0, 1, 1, 1) + c_{5\to 3,13}I_5(0, 1, 0, 1, 1)
\]
\[
+ c_{5\to 3,14}I_5(0, 1, 1, 0, 0) + c_{5\to 3,15}I_5(0, 1, 1, 1, 0) + c_{5\to 3,23}I_5(1, 0, 0, 1, 1) + c_{5\to 3,24}I_5(1, 0, 1, 0, 1)
\]
\[
+ c_{5\to 3,25}I_5(1, 0, 1, 1, 0) + c_{5\to 3,33}I_5(1, 1, 0, 0, 1) + c_{5\to 3,34}I_5(1, 1, 1, 0, 0) + c_{5\to 3,35}I_5(1, 1, 1, 1, 0)
\]
\[
+ c_{5\to 2,1}I_5(1, 1, 1, 0, 0) + c_{5\to 2,1}I_5(1, 0, 1, 0, 0) + c_{5\to 2,2}I_5(1, 0, 0, 1, 0) + c_{5\to 2,2,1}I_5(1, 0, 0, 0, 1)
\]
\[
+ c_{5\to 2,2,3}I_5(0, 1, 1, 0, 0) + c_{5\to 2,2,4}I_5(0, 1, 0, 1, 0) + c_{5\to 2,2,5}I_5(0, 1, 0, 0, 1) + c_{5\to 2,2,6}I_5(0, 0, 1, 0, 0)
\]
\[
+ c_{5\to 2,3}I_5(0, 0, 1, 1, 0) + c_{5\to 2,3,1}I_5(0, 0, 0, 1, 1) + c_{5\to 2,3,2}I_5(0, 0, 0, 1, 0) + c_{5\to 2,3,3}I_5(0, 0, 0, 0, 1)
\]
\[
+ c_{5\to 2,4}I_5(0, 0, 0, 0, 0) + c_{5\to 2,4,1}I_5(0, 0, 0, 0, 0) + c_{5\to 2,4,2}I_5(0, 0, 0, 0, 0) + c_{5\to 2,4,3}I_5(0, 0, 0, 0, 0)
\]
(3.80)

with the coefficients given in the attached Mathematica notebook. Now all coefficients are complete.

4. Analytic result of the coefficients

We give our analytic results in Mathematica notebooks which are put on the publicly available website at [https://github.com/Wanghongbin123/oneloop_parametric](https://github.com/Wanghongbin123/oneloop_parametric).
5. Summary and further discussion

In this paper, we consider the one-loop scalar integrals in the parametric representation given by Chen. However, in the recurrence relation, there are usually some terms that we do not want, as well as some terms with dimensional shifting in general, which makes our calculation not easy and efficient. In Chen’s later paper \[2\], he used a method based on non-commutative algebra to cancel the dimension shift. Different from others methods, in the one loop case, we have used a straight method by solving the linear equation systems to simplify the IBP recurrence relation in the parametric representation. Benefited from the fact that the $F$ is a homogeneous function of $x_i$ with degree two in one-loop’s situation, we could solve the $x_i$ by $\frac{\partial F}{\partial x_i}$ with some free parameters. Then combining all the IBP identities with particular coefficients $z_i$, and then choose particular values of the free parameters, we succeed to cancel the dimension shift and the terms with higher total power. As the complement of the tadpole coefficients in the reduction to our previous paper, we calculated several examples and gave the analytic result of the reduction.

For further research, there are some questions needed to be considered. In the previous calculation, we can see that the coefficients we constructed $z_i$ is not polynomial since it has the denominator with the form $x_{n+1}^\gamma$, so we could not directly use the technic of syzygy. Also, the application of Chen’s method to higher loop is definitely another future direction. For this case the homogeneous function $F(x)$ is of degree $L + 1$, where $L$ is the number of loops. For high loop’s case, we should consider how to construct the coefficients $z_i$ efficiently, and find a relation similar to (2.37) to cancel the terms we do not need. Thirdly, the sub-topologies are totally decided by the boundary term in the parametric representation, and this may lead to some simplification of calculation.

Acknowledgments

I’d like to thank Bo Feng for the inspiring discussion and guidance. This work is supported by Chinese NSF funding under Grant No.11935013.

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