THE BISHOP-PHELPS-BOLLOBÁS PROPERTY FOR OPERATORS FROM $\mathcal{C}(K)$ TO UNIFORMLY CONVEX SPACES

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Abstract. We show that the pair $(\mathcal{C}(K), X)$ has the Bishop-Phelps-Bollobás property for operators if $K$ is a compact Hausdorff space and $X$ is a uniformly convex space.

1. Introduction

In this paper, we deal with strengthening of the famous Bishop-Phelps theorem. In 1961, Bishop and Phelps [8] showed that the set of all norm attaining functionals on a Banach space $X$ is dense in its dual space $X^*$ which is now called Bishop-Phelps theorem. This theorem has been extended to operators between Banach spaces $X$ and $Y$. In general, the set of norm attaining operators $\mathcal{N}(X, Y)$ is not dense in the space of linear operators $\mathcal{L}(X, Y)$. However, it is true for some pair of Banach spaces $(X, Y)$. One of very well-known examples is the pair of every reflexive Banach space $X$ and every Banach space $Y$, which was shown by Lindenstrauss [24]. After that, this was generalized by Bourgain to Banach space $X$ with Radon-Nikodým property [10], and also there have been many efforts to find other positive examples [12, 13, 15, 17, 19, 26, 27].

Meanwhile, Bollobás sharpened Bishop-Phelps theorem as follows. From now on, the unit ball and the unit sphere of a Banach space $X$ will be denoted by $B_X$ and $S_X$, respectively.

**Theorem 1.1.** ([9]) For an arbitrary $\epsilon > 0$, if $x^* \in S_{X^*}$ satisfies $|1 - x^*(x)| < \frac{\epsilon^2}{4}$ for $x \in B_X$, then there are both $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \epsilon$ and $\|y^* - x^*\| < \epsilon$.

This Bishop-Phelps-Bollobás theorem shows that if a functional almost attains its norm at a point, then it is possible to approximate simultaneously both the functional and the point by norm attaining functionals and their corresponding norm attaining points. Clearly, Bishop-Phelps-Bollobás theorem implies Bishop-Phelps theorem.

Similarly to the case of Bishop-Phelps theorem, Acosta, Aron, García and Maestre [1] started to extend this theorem to bounded linear operators between Banach spaces and introduced the new notion Bishop-Phelps-Bollobás property.

**Definition 1.2.** ([1, Definition 1.1]) Let $X$ and $Y$ be Banach spaces. We say that the pair $(X, Y)$ has the Bishop-Phelps-Bollobás property for operators (BPBp) if, given $\epsilon > 0$, there exists $\eta(\epsilon) > 0$ such that if there exist both $T \in S_{\mathcal{L}(X, Y)}$ and $x_0 \in S_X$ satisfying $\|Tx_0\| > 1 - \eta(\epsilon)$, then there exist both an operator $S \in S_{\mathcal{L}(X, Y)}$ and $u_0 \in S_X$ such that

$$\|Su_0\| = 1, \|x_0 - u_0\| < \epsilon \text{ and } \|T - S\| < \epsilon.$$ 

Acosta et al. showed [1] that the pair $(X, Y)$ has the BPBp for finite dimensional Banach spaces $X$ and $Y$, and that the pair $(\ell^p_\infty, Y)$ has the BPBp for every $n$ if $Y$ is a uniformly convex space. In the same paper, they asked if the pairs $(\ell^p_0, Y)$ and $(\ell^p_\infty, Y)$ have the BPBp for uniformly convex spaces $Y$. The
first author solved the $c_0$ case and proved [20] that $(c_0, Y)$ have the Bishop-Phelps-Bollobás property for all uniformly convex spaces $Y$.

Let $X = L_\infty(\mu)$ or $X = c_0(\Gamma)$ for a set $\Gamma$. Very recently, Lin and authors [23] proved that $(X, Y)$ has the BPBp for every uniformly convex space $Y$. So $(L_\infty(\mu), L_p(\nu))$ has the BPBp for all $1 < p < \infty$ and for all measures $\nu$. They also proved that $(X, Y)$, as a pair of complex spaces, has the BPBp for every uniformly convex space $Y$. In particular, $(L_\infty(\mu), L_1(\nu))$, as a pair of complex spaces, has the BPBp, since $L_1(\nu)$ is uniformly complex convex [18].

On the other hand, there have been several researches about the BPBp for operators into $C(K)$ spaces (or uniform algebras). Even though Schachermayer showed [26] that the set of norm attaining operators is not dense in $\mathcal{L}(L_1[0,1], C[0,1])$, there are some positive results about the BPBp. It is shown [4] that $(X, C(K))$ has the BPBp if $X$ is an Asplund space. This result was extended so that $(X, A)$ has the BPBp if $X$ is Asplund and $A$ is a uniform algebra [11]. The authors also proved [21] that $(X, C(K))$ has the BPBp if $X^*$ admits a uniformly simultaneously continuous retractions. It is also worthwhile to remark that the pair $(C(K), C(L_1))$ of the spaces of real-valued continuous functions has the BPBp for every compact Hausdorff spaces $K$ and $L_1$ [2]. Concerning the results about $L_\infty$ spaces, it is shown [7] that $(L_1(\mu), L_\infty[0,1])$ has the BPBp and this was generalized [14] so that $(L_1(\mu), L_\infty(\nu))$ has the BPBp if $\mu$ is any measure and $\nu$ is a localizable measure. These are the strengthening of the results that the set of norm-attaining operators is dense in $\mathcal{L}(L_1(\mu), L_\infty(\nu))$ [17, 25] for every measure $\mu$ and every localizable measure $\nu$. Finally we remark that if $X$ is uniformly convex, then $(X, Y)$ has the BPBp for every Banach space $Y$ [3, 5, 22].

Throughout this paper, we consider only real Banach spaces. It is the main result of this paper that $(C(K), X)$ has the BPBp for every compact Hausdorff space $K$ and for every uniformly convex space $X$. Recall that Schachermayer showed [26] that every weakly compact operator from $C(K)$ into a Banach space can be approximated by norm attaining weakly compact operators (cf. [6, Theorem 2]). So the set of all norm attaining operators is dense in $\mathcal{L}(C(K), Y)$ for every reflexive space $Y$. Notice that the reflexivity of $Y$ is not sufficient to prove that $(C(K), Y)$ has the BPBp. Indeed, if we take a reflexive strictly convex space $Y_0$ which is not uniformly convex, then $(\ell^1(2), Y_0)$ does not have the BPBp [1, 5]. If we take $K_0$ as the set consisting of only two points, then $C(K_0)$ is isometrically isomorphic to 2-dimensional $\ell^1(2)$ space. Hence $(C(K_0), Y_0)$ does not have the BPBp. However, if $X$ is uniformly convex, then it will be shown that $(C(K), X)$ has the BPBp.

2. Main Result

Given a Banach space $X$, the modulus of convexity $\delta_X(\epsilon)$ of the unit ball $B_X$ is defined by for $0 < \epsilon < 1$,

$$
\delta_X(\epsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in B_X, \|x - y\| \geq \epsilon \right\}.
$$

A Banach space $X$ is said to be uniformly convex if $\delta_X(\epsilon) > 0$ for all $0 < \epsilon < 1$. It is well known that every uniformly convex space is reflexive.

In [20], the following result was shown: Let $1 > \epsilon > 0$ be given and $X$ be a reflexive Banach space and $Y$ be a uniformly convex Banach space with modulus of convexity $\delta_X(\epsilon) > 0$. If $T \in \mathcal{L}(X, Y)$ and $x_1 \in S_X$ satisfy

$$
\|Tx_1\| > 1 - \frac{\epsilon}{2\delta_X(\epsilon)}\frac{\epsilon}{2},
$$

then there exist $S \in \mathcal{L}(X, Y)$ and $x_2 \in S_X$ such that $\|Sx_2\| = 1$, $\|S - T\| < \epsilon$ and $\|Tx_1 - Sx_2\| < \epsilon$.

This says that for a reflexive space $X$ and a uniformly convex space $Y$, the pair $(X, Y)$ has a little weaker property than $\text{BPBp}$. The only difference from the $\text{BPBp}$ and the above is approximating the image of a point if the given operator almost attains its norm. Since the set of all norm attaining operators is dense in $\mathcal{L}(X, Y)$ for every $Y$ if $X$ is reflexive, the following result generalize the result mentioned above [20].
Proposition 2.1. Let $X$ be a Banach space and $Y$ be a uniformly convex space. Suppose that the set of norm attaining operators is dense in $L(X,Y)$. Then, given $0 < \varepsilon < 1$, there exists $\eta(\varepsilon) > 0$ such that if $T \in S_{L(X,Y)}$ and $x_1 \in S_X$ satisfy $\|Tx_1\| > 1 - \eta(\varepsilon)$, then there exist $S \in S_{L(X,Y)}$ and $x_2 \in S_X$ such that $\|Sx_2\| = 1$, $\|S - T\| < \varepsilon$ and $\|Tx_1 - Sx_2\| < \varepsilon$.

Proof. Let $\delta_Y(\cdot)$ be the modulus of convexity of $Y$ and $0 < \varepsilon_1 < \varepsilon$. Choose $\varepsilon_2 > 0$ such that $(1 - \varepsilon_2^2)^2 - 2q_2 - \varepsilon_2^2 > 1 - \delta_Y(\varepsilon_1)$ and $\varepsilon_2^2 + 2q_2 + \varepsilon_1 < \varepsilon$.

We show that $\eta(\varepsilon) = \varepsilon_2^2$ is a suitable number. Assume $\|Tx_1\| > 1 - \varepsilon_2^2$. Choose $y^* \in S_{Y^*}$ such that $y^*Tx_1 = \text{Re} \ y^*Tx_1 > 1 - \varepsilon_2^2$ and define an operator $\tilde{T}_1$ by

$$\tilde{T}_1 x = Tx + \varepsilon_2 y^*(Tx)Tx_1$$

for every $x \in X$.

It is easy to see that $1 - \varepsilon_2 < (1 - \varepsilon_2^2)(1 + \varepsilon_2(1 - \varepsilon_2^2)) \leq \|\tilde{T}_1 x_1\| \leq \|\tilde{T}_1\| \leq 1 + \varepsilon_2$.

Let $T_1 = \tilde{T}_1/\|\tilde{T}_1\|$. Since the set of norm attaining operators is dense in $L(X,Y)$, there exist an operator $S$ and $z \in S_X$ such that $\|T_1 - S\| < \varepsilon_2^2$ and $\|Sz\| = \|S\| = 1$. Since $\|Sz - T_1z\| < \varepsilon_2^2$, we see that $\|T_1z\| > 1 - \varepsilon_2^2$, which means that

$$\|Tz + \varepsilon_2 y^*(Tz)Tx_1\| > (1 - \varepsilon_2^2)\|\tilde{T}_1\| > (1 - \varepsilon_2^2)(1 + \varepsilon_2(1 - \varepsilon_2^2)).$$

Hence, we have $|y^*T(z)| > (1 - \varepsilon_2^2)^2 - 2q_2 - \varepsilon_2^2 > 1 - \delta_Y(\varepsilon_1)$. Choose $\alpha = \pm 1$ satisfying $y^*T(\alpha z) = |y^*T(z)|$ and let $x_2 = \alpha z$.

Then

$$\frac{\|Tx_1 + Tx_2\|}{2} > \frac{y^*Tx_1 + y^*Tx_2}{2} > 1 - \delta_Y(\varepsilon_1).$$

Hence, we see that $\|Tx_1 - Tx_2\| < \varepsilon_2$. Moreover,

$$\|Sx_2 - Tx_1\| \leq \|Sx_2 - T_1x_2\| + \|T_1x_2 - \tilde{T}_1x_2\| + \|\tilde{T}_1x_2 - Tx_2\| + \|Tx_2 - Tx_1\|$$

$$\leq \|T_1 - S\| + \|\tilde{T}_1 - 1\| + \varepsilon_2 + \varepsilon_1$$

$$< \varepsilon_2^2 + \varepsilon_2 + \varepsilon_1 < \varepsilon.$$

This completes the proof. \qed

Now we state the main theorem of this paper.

Theorem 2.2. Let $X$ be a uniformly convex space and $K$ be a compact Hausdorff space. Then the pair $(C(K), X)$ has the BPBp.

Before we present the proof of the main result, we begin with preliminary comments on vector measure and two lemmas. Recall that a vector measure $G : \Sigma \to X$ on a $\sigma$-algebra $\Sigma$ is said to be countably additive if, for every mutually disjoint sequence of $\Sigma$-measurable subsets $\{A_i\}_{i=1}^\infty$, we have

$$G \left( \bigcup_{i=1}^\infty A_i \right) = \sum_{i=1}^\infty G(A_i).$$

For a $\Sigma$-measurable subset $A$, the semi-variation $\|G\|(A)$ of $G$ is defined by

$$\|G\|(A) = \sup \{|x^*G|(A) : x^* \in B_{X^*}\},$$

where $|x^*G|(A)$ is the total variation of the scalar-valued countably additive measure $x^*G$ on $A$. The vector measure $G$ on a Borel $\sigma$-algebra is said to be regular if for each Borel subset $E$ and $\varepsilon > 0$ there exists a compact subset $K$ and an open set $O$ such that $K \subset E \subset O$ and $\|G\|(O \setminus K) < \varepsilon$.

It is well known that if $X$ is reflexive, each operator $T$ in $L(C(K), X)$ has a $X$-valued countably additive representing Borel measure $G$ and the measure is regular (see [16, VI. Theorem 1, 5 and Corollary 14] for a reference). That is, for all $f \in C(K)$ and $x^* \in X^*$, we have

$$Tf = \int_K f \, dG, \quad x^*T(f) = \int_K f \, d^*G \quad \text{and} \quad \|T\| = \|G\|(K).$$
If \( G \) is a countably additive representing measure for an operator \( T \) in \( \mathcal{L}(C(K), X) \), then it is easy to see that for any bounded Borel measurable function \( h : K \to \mathbb{R} \), the mapping \( S \), defined by \( Sf = \int fh \, dG \), is a bounded linear operator and \( \|S\| \leq \|T\| \cdot \|h\|_\infty \), where \( \|h\|_\infty = \sup\{|h(k)| : k \in K\} \).

**Lemma 2.3.** Let \( G \) be a countably additive, Borel regular \( X \)-valued vector measure on a compact Hausdorff space \( K \) with \( \|G\|(K) = 1 \) and let \( 0 < \eta, \gamma < 1 \). Assume that \( f \in S_{C(K)} \) and \( x^* \in S_X^* \) satisfy
\[
\int_K f \, dx^* G > 1 - \eta.
\]
Then, we have
\[
|\lambda G((K \setminus (A^+ \cup A^-))) < \frac{2\eta}{\gamma} + \eta,
\]
where \( A^+_t = \{t \in K \mid f(t) \geq 1 - \gamma\} \) and \( A^-_t = \{t \in K \mid f(t) \leq -1 + \gamma\} \). Moreover, there exist mutually disjoint compact sets \( F^+, F^- \) such that \( x^* G \) is positive on \( F^+ \), negative on \( F^- \) and
\[
\int_{(F^+ \cap A^+_t) \cup (F^- \cap A^-_t)} f \, dx^* G > 1 - \frac{2\eta}{\gamma}.
\]

**Proof.** The Hahn decomposition of \( x^* G \) and the regularity of \( G \) show that there exist mutually disjoint compact sets \( F^+, F^- \) such that \( x^* G \) is positive on \( F^+ \), negative on \( F^- \) and \( \|G\|(K \setminus (F^+ \cup F^-)) < \eta \).

\[
1 - \eta \leq \int_K f \, dx^* G = \int_{F^+} f \, dx^* G + \int_{F^-} f \, dx^* G + \int_{K \setminus (F^+ \cup F^-)} f \, dx^* G
\]
\[
= \int_{F^+ \cap A^+_t} f \, dx^* G + \int_{F^- \cap A^-_t} f \, dx^* G + \int_{F^+ \setminus A^+_t} f \, dx^* G + \int_{F^- \setminus A^-_t} f \, dx^* G + \int_{K \setminus (F^+ \cup F^-)} f \, dx^* G
\]
\[
\leq x^* G(F^+ \cap A^+_t) + (1 - \gamma)x^* G(F^+ \setminus A^+_t) - x^* G(F^+ \cap A^-_t) - (1 - \gamma)x^* G(F^- \setminus A^-_t) + \eta
\]
\[
= x^* G(F^+ - x^* G(F^-) - \gamma(x^* G(F^+ \setminus A^+_t) - x^* G(F^- \setminus A^-_t)) + \eta.
\]

Since \( x^* G(F^+) - x^* G(F^-) = |x^* G((F^+ \cup F^-)) \leq \|G\|(K) = 1 \), we get
\[
|\lambda G((F^+ \setminus A^+_t) \cup (F^- \setminus A^-_t)) = x^* G(F^+ \setminus A^+_t) - x^* G(F^- \setminus A^-_t) \leq \frac{2\eta}{\gamma}.
\]

This shows that
\[
|\lambda G((K \setminus (A^+_t \cup A^-_t))) \leq \|G\|(K \setminus (F^+ \cup F^-)) + |\lambda G((F^+ \cup F^-) \setminus (A^+_t \cup A^-_t))
\]
\[
\leq \|G\|(K \setminus (F^+ \cup F^-)) + |\lambda G((F^+ \setminus A^+_t) \cup (F^- \setminus A^-_t))
\]
\[
< \frac{2\eta}{\gamma} + \eta
\]
and
\[
\int_{(F^+ \cap A^+_t) \cup (F^- \cap A^-_t)} f \, dx^* G = \int_{F^+} f \, dx^* G - \int_{(F^+ \cap A^+_t) \cup (F^- \cap A^-_t)} f \, dx^* G
\]
\[
\geq \int_K f \, dx^* G - \|G\|(K \setminus (F^+ \cup F^-)) - |\lambda G((F^+ \setminus A^+_t) \cup (F^- \setminus A^-_t))
\]
\[
> 1 - 2\eta - \frac{2\eta}{\gamma} > 1 - 4\frac{\eta}{\gamma}.
\]

This completes the proof.

**Lemma 2.4.** Let \( X \) be a uniformly convex space with the modulus of convexity \( \delta_X \) and \( T \in S_{\mathcal{L}(C(K), X)} \) be an operator represented by the countably additive, Borel regular vector measure \( G \). Let \( 0 < \epsilon < 1 \) and \( A \) be a Borel set of \( K \). Suppose that an operator \( S \), defined by \( Sf = \int_A f \, dG \), satisfies \( \|S\| > 1 - \delta_X(\epsilon) \). Then
\[
\|T - S\| \leq \sup_{f \in B_{C(K)}} \left\| \int_{K \setminus A} f \, dG \right\| < \epsilon.
\]
Proof. Choose \( x^* \in S_{X^*}, f_0 \in S_{C(K)} \) such that \( \|Sf_0\| = x^*Sf_0 > 1 - \delta_X(\varepsilon) \). By the regularity of \( G \), we may choose a compact set \( A_1 \subset A \) such that

\[
\int_{A_1} f_0 dx^* G > 1 - \delta_X(\varepsilon).
\]

Fix a closed set \( B \subset K \setminus A \) and \( g \in B_{C(B)} \). Then, choose \( g_+, g_- \in B_{C(K)} \) satisfying

\[
g_+(t) = g_-(t) = f_0(t) \quad \text{for } t \in A_1 \quad \text{and} \quad g_+(t) = -g_-(t) = g(t) \quad \text{for } t \in B.
\]

So, we have

\[
1 - \delta_X(\varepsilon) < \int_{A_1} f_0 dx^* G \leq \left\| \int_{A_1} f_0 dG \right\| = \frac{1}{2} \left\| \int_{A_1 \cup B} g_+ dG + \int_{A_1 \cup B} g_- dG \right\|.
\]

Note that \( \left\| \int_{A_1 \cup B} g_+ dG \right\|, \left\| \int_{A_1 \cup B} g_- dG \right\| \leq 1 \). Thus, from the uniform convexity of \( X \), we get that

\[
\left\| 2 \int_{B} g dG \right\| = \left\| \int_{A_1 \cup B} g_+ dG - \int_{A_1 \cup B} g_- dG \right\| < \varepsilon.
\]

This implies \( \|T - S\| < \varepsilon \) and the proof is done. \( \square \)

Proof of Theorem 2.2. Let \( \delta_X \) be the modulus of convexity for \( B_X \). Fix \( 0 < \varepsilon < \frac{1}{6} \) and let \( \eta \) be the function which appears in Proposition 2.1 for the pair \((C(K), X)\), and let \( \gamma(t) = \min \{ \eta(t), \delta_X(t), \varepsilon \} \) for \( t \in (0, 1) \). Assume that \( T \in S_{L(C(K)), X} \) and \( f_0 \in S_{C(K)} \) satisfy that

\[
\|Tf_0\| > 1 - \frac{\varepsilon}{6} \gamma \left( \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \right).
\]

Let \( G \) be the representing vector measure for \( T \) which is countably additive Borel regular on \( K \). Choose \( x_1^* \in S_{X^*} \) such that \( x_1^*Tf_0 > 1 - \frac{\varepsilon}{6} \gamma \left( \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \right) \). By Lemma 2.3 there exist two mutually disjoint compact sets \( F^+, F^- \) such that \( x^*G \) is positive on \( F^+ \), negative on \( F^- \) and

\[
\int_{(F^+ \cap A_{1/2}) \cup (F^- \cap A_{1/2})} f dx^* G > 1 - \gamma \left( \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \right),
\]

where \( A_{1/2} = \{ t \in K \mid f_0(t) > 1 - \frac{\varepsilon}{2} \} \) and \( A_{1/2} = \{ t \in K \mid f_0(t) < -1 + \frac{\varepsilon}{2} \} \).

Let \( A_1 = F^+ \cap A_{1/2}, A_2 = F^- \cap A_{1/2} \) and \( A = A_1 \cup A_2 \). Then, define \( S_1 \in B_{L(C(K)), X} \) by \( S_1 f = \int_A f dG \) for every \( f \in C(K) \). Then Lemma 2.4 shows that \( \|T - S_1\| < \frac{\varepsilon}{6} \). Choose \( f_1 \in S_{C(K)} \) such that

\[
f_1(t) = 1 \quad \text{for } t \in A_1 \quad \text{and} \quad f_1(t) = -1 \quad \text{for } t \in A_2.
\]

For \( f \in C(K) \), the restriction of \( f \) to \( A \) will be denoted by \( f|_A \). Now consider \( S_1 \) as an operator in \( L(C(A), X) \). Then we have

\[
\|S_1(f_1|_A)\| > 1 - \gamma \left( \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \right),
\]

So Proposition 2.1 shows that there exist \( S_2 \in S_{L(C(A)), X} \) and \( f_2 \in S_{C(A)} \) such that \( \|S_2f_2\| = 1, \|S_2 - \frac{S_1}{\|S_1\|}\| < \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \) and \( \|S_2f_2 - \frac{S_1(f_1|_A)}{\|S_1\|}\| < \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \). Let \( G' \) be the representing vector measure for \( S_2 \) which is countably additive Borel regular on \( A \). Choose \( x_2^* \in S_{X^*} \) so that \( x_2^*S_2f_2 = \|S_2f_2\| = \int_A f_2 dx^* G' = 1 \).

Since

\[
x_2^*S_2(f_1|_A + f_2) \geq 2x_2^*S_2f_2 - \|S_2f_2 - S_2(f_1|_A)\|
\]

\[
\geq 2 - \left\| S_2f_2 - \frac{S_1(f_1|_A)}{\|S_1\|} \right\| - \left\| S_2(f_1|_A) - S_2(f_1|_A) \right\|
\]

\[
> 2 \left( 1 - \frac{\varepsilon}{6} \delta_X \left( \frac{\varepsilon}{6} \right) \right),
\]

then

\[
x_2^*S_2(f_1|_A + f_2) > 1 - \delta_X(\varepsilon).
\]
we get
\[ \int_{A} \frac{f_1 + f_2}{2} dx' G' > 1 - \frac{\epsilon}{6} \delta_X \left( \frac{\epsilon}{6} \right). \]

By applying Lemma 2.3 again, we get a compact subset \( F \) of \( A \) such that
\[ F \subset \{ t \in A : |f_1(t) + f_2(t)| > 2(1 - \epsilon) \} \]
and
\[ \left\| \int_{F} \frac{f_1 + f_2}{2} dG' \right\| > 1 - \delta_X \left( \frac{\epsilon}{6} \right). \]

Let \( B = \{ t \in A : f_1(t)f_2(t) \geq 0 \} \). Then, \( F \subset B \) and
\[ \sup_{f \in B_{C(A)}} \left\| \int_{B} f dG' \right\| > \left\| \int_{F} \frac{f_1 + f_2}{2} dG' \right\| > 1 - \delta_X \left( \frac{\epsilon}{6} \right). \]

By Lemma 2.4, we have
\[ \sup_{f \in B_{C(K)}} \left\| \int_{A \setminus B} f dG' \right\| < \frac{\epsilon}{6}. \]

Define \( S \in \mathcal{L}(C(A), X) \) by, for \( f \in C(A) \),
\[ Sf = \int_{B} f dG' - \int_{A \setminus B} f dG' \]
and let
\[ f_3 = \begin{cases} \frac{|f_2|}{3} & \text{for } t \in A_1, \\ -\frac{|f_2|}{3} & \text{for } t \in A_2. \end{cases} \]

So \( f_3 \in C(A) \) and \( f_3 = f_2\chi_B - f_2\chi_{A \setminus B} \), where \( \chi_S \) is the characteristic function on a set \( S \). Hence we have \( Sf_3 = S_2f_2 \) and \( \|Sf_3\| = \|S\| = 1 \) and \( \|S - S_2\| < \frac{\epsilon}{6} \). On the other hand, we have \( \|2f_3 - f_1\|_A \leq 1 \).

Since \( X \) is uniformly convex and we have \( Sf_3 = \frac{S(f_1) + S(2f_3 - f_1)}{2} \), we get
\[ Sf_3 = \frac{S(f_1) + S(2f_3 - f_1)}{2}. \]

We now consider \( S_1, S_2, S \) as operators in \( \mathcal{L}(C(K), X) \) using the canonical extension. That is, \( S(f) = S(f|_A) \), \( S_i(f) = \frac{S_i(f)}{\|S_i\|} \) for all \( f \in C(K) \) and for \( i = 1, 2 \). Let \( C \) be the compact subset defined by
\[ C = \{ t \in K : |f_1(t) - f_0(t)| \geq \epsilon \}. \]

Note that \( A \) and \( C \) are mutually disjoint. Indeed, if \( t \in A \), then \( |f_0(t) - f_1(t)| \leq \epsilon/2 \). So there is \( \phi \in C(K) \) such that \( 0 \leq \phi \leq 1 \), \( \phi(k) = 1 \) for \( k \in A \) and \( \phi(k) = 0 \) for \( k \in C \). Let \( g = \phi f_1 + (1 - \phi)f_0 \). Then we see that \( \|Sg\| = 1 \),
\[ \|S - T\| \leq \|S - S_2\| + \|S_2 - \frac{S_1}{\|S_1\|}\| + \|\frac{S_1}{\|S_1\|} - S_1\| + \|S_1 - T\| \]
\[ < \frac{\epsilon}{3} + \frac{\epsilon}{6} + \frac{\epsilon}{3} + \frac{\epsilon}{6} = \epsilon \]
and \( \|g - f_0\| = \sup_{k \in K \setminus C} |\phi(k)(f_1(k) - f_0(k))| < \epsilon \). This completes the proof. \[ \square \]

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