Introduction

As it is usually presented in textbooks, the Standard Model of “fundamental interactions” is, mathematically speaking, a hideous construction. We can summarize its content thus:

1. The interaction fields are gauge fields with a $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry. In other words, there are three systems of gauge bosons $A^{SU(3)}_\nu$, $A^{SU(2)}_\nu$, $A^{U(1)}_\nu$. They contribute terms $(F | F) := \text{tr} \int F_{\mu\nu}^\dagger F^{\mu\nu} d^4x$ to the Action, where $F$ is the gauge field, which is obtained from the gauge potential by the recipe $F = dA + A \wedge A$.

2. The (fermionic) matter fields $\psi$ contribute terms $\int \bar{\psi}_1 D\psi_2 = \int \bar{\psi}_1 \gamma \psi_2 + \int \bar{\psi}_1 A \psi_2$.

3. Unfortunately, in order to give mass to the electroweak gauge bosons, there is the need to add a colorless scalar “matter” field, called the Higgs particle, with dynamics given by $\int D\phi^{\dagger} D\phi + V(\phi)$ where $V(\phi) = -\mu^2 \phi^{\dagger} \phi + \lambda (\phi^{\dagger} \phi)^2$. The “negative mass” $\mu$ is needed for symmetry breakdown to work. The introduction of the Higgs is justified on a technical basis: it preserves unitarity and renormalizability of the quantized theory and ... it works. It also gives mass to the fermions through the seemingly ad hoc and apparently non gauged ...

4. ... Yukawa interaction terms, $\int \bar{\psi}_1 \phi \psi_2$.

5. We summarize thus the several aesthetically unpleasant features of the SM:

1. The Higgs sector is introduced by hand.

2. The link between the parity violating and the symmetry breaking sector remains mysterious.

3. There is no explanation for the observed number of fermionic generations.
4. The choice of gauge groups and hypercharge assignments seems rather arbitrary, although it has the felicitous result that the model, despite being chiral, is anomaly-free.

5. There is an apparent juxtaposition of gauged and non-gauged interaction sectors.

6. There is no explanation for the huge span of fermionic masses.

Noncommutative geometry goes a good bit of the way to solving these questions —except the last.

**A new framework for thinking about the SM**

In noncommutative geometry (NGC) all the complexities and idiosyncrasies of the SM stem from a “pure QCD-like theory” with a unified noncommutative gauge boson $A$ for the $SU(3)_c \times SU(2)_L \times U(1)_Y$ symmetry. Thus the Lagrangian:

$$\mathcal{L}_{\text{NCG}} = -\frac{i}{4} (\mathcal{F} | \mathcal{F}) + \langle \bar{\Psi} | D(A) \Psi \rangle$$

on a noncommutative space, to wit, the product of $M_4$ by the space of the internal degrees of freedom: colour, weak isospin and hypercharge. Here

$$A = A(A^{SU(3)}, A^{SU(2)}, A^{U(1)}, \phi).$$

That is to say, the Higgs is seen as a gauge boson (this helps to explain its quartic kinetic energy and its pointlike coupling to fermions). We still have $\mathcal{F} = dA + A^2$, and therefore

$$\mathcal{F} = \mathcal{F}(F^{SU(3)}, F^{SU(2)}, F^{U(1)}, D\phi, |V|^{1/2}).$$

**The spaces of noncommutative geometry**

The mathematical framework hinges on two related ideas: (1) geometrical properties of spaces of points (e.g., spacetime without chirality) are determined by their c-number functions; (2) other geometrical settings (e.g., spacetime with chirality) can be accommodated by allowing noncommutative algebras of q-number functions; both are thought of as algebras of operators on Hilbert spaces.

Many structures arising in classical geometry are thus replaced by their quantum counterparts. For instances, measure spaces are replaced by von Neumann algebras, topological spaces by $C^*$-algebras, vector bundles by projective modules, Lie groups by smooth groupoids, de Rham homology by cyclic cohomology, and spin manifolds by spectral triples.

Think of functions as forming an algebra $\mathcal{A}$ of multiplication operators on a Hilbert space $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. If $\Gamma$ is the sign operator ($\Gamma = \pm 1$ on $\mathcal{H}^\pm$), then $\delta f = [\Gamma, f]$ is an “infinitesimal” operator. Differential calculus is done with a “spectral triple” consisting of the algebra $\mathcal{A}$, the Hilbert space $\mathcal{H}$ and an odd selfadjoint operator $D$ on $\mathcal{H}$ (e.g., the Dirac operator on the space of spinors.
L^2(S_M). Integration of functions is effected by the Dixmier trace of operators: if \( T \) has eigenvalues \( \mu_n(T) \geq 0 \), then
\[
\int T = \lim_{n \to \infty} \frac{\mu_0(T) + \cdots + \mu_n(T)}{\log n},
\]
where \( \int f = \int f|D|^{-d} \).

Other classical geometrical objects have their quantum counterparts. A complex variable becomes an operator in \( H \), a real variable is a selfadjoint operator, and an infinitesimal is a compact operator. An infinitesimal of order \( k \) is seen to be a compact operator whose singular values \( \mu_n \) are \( O(n^{-k}) \) as \( n \to \infty \).

The differential of real or complex variable is replaced by \( \delta f \equiv [\Gamma, f] = \Gamma f - f \Gamma \); and the integral of a first-order infinitesimal is given by the Dixmier trace.

The spectral triple \( (\mathcal{A}, \mathcal{H}, D) \) determines the geometry completely. For example, here is the formula for computing distances between points (i.e., pure states of \( \mathcal{A} \)) on a conventional Riemannian manifold:
\[
d(p, q) = \sup \{|f(p) - f(q)| : f \in \mathcal{A}, ||[D, f]|| \leq 1\},
\]
where \( D = \partial \) is the usual Dirac operator. Thus, we now have a fully quantum formalism for the classical world, and we notice that distances are better measured by neutrinos than by scalar particles!

The reconstruction of the SM

We need to have more details on the noncommutative differential calculus. One can embed \( \mathcal{A} \) in the “universal differential algebra” \( \Omega^\bullet \mathcal{A} = \bigoplus_{n \geq 0} \Omega^n \mathcal{A} \), generated by symbols \( a_0 da_1 \ldots da_n \) with a formal antiderivation \( d \) satisfying \( d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n, d1 = 0 \) and \( d^2 = 0 \). Having a spectral triple allows us to condense this large algebra to a more useful one. We first represent the whole of \( \Omega^\bullet \mathcal{A} \) on the Hilbert space \( \mathcal{H} \) by taking:
\[
\pi(a_0 da_1 \ldots da_n) := a_0 [D, a_1] \ldots [D, a_n].
\]
The algebra of operators \( \pi(\Omega^\bullet \mathcal{A}) \) is not a differential algebra, in general. This problem is handled by a standard trick: the differential ideal of “junk”\( J := \{ c' + dc'' \in \Omega^\bullet \mathcal{A} : \pi c' = \pi c'' = 0 \} \) is factored out, thereby obtaining a new graded differential algebra of “noncommutative differential forms” by
\[
\Omega^\bullet_D \mathcal{A} := \pi(\Omega^\bullet \mathcal{A})/\pi(J).
\]
The quotient algebra \( \Omega^\bullet_D C^\infty(M; \mathbb{C}) \) for the standard commutative spectral triple is an algebra of operators on \( L^2(S_M) \) isomorphic to the de Rham complex of differential forms. The Connes model is given by
\[
\mathcal{A} := C^\infty(M, \mathbb{R}) \otimes C_F \simeq C^\infty(M, \mathbb{C}) \oplus C^\infty(M, \mathbb{H}) \oplus M_3(C^\infty(M, \mathbb{C})),
\]
\[
\mathcal{H} := L^2(S_M) \otimes (\mathcal{H}_F^+ \oplus \mathcal{H}_F^-), \quad D := (\partial \otimes 1) \oplus (1 \otimes D_F).
\]
The \( D_F \) operator holds information about the Yukawa–Kobayashi–Maskawa couplings. The minimal coupling recipe leads then to the usual fermionic action.
plus the mass terms. The noncommutative gauge potential $A$ and field $F$, on the boson side, are selfadjoint elements respectively of:

$$\Omega^1_{D^A} \simeq \Lambda^1(M, \mathbb{C}) \oplus \Lambda^0(M, \mathbb{H}) \oplus \Lambda^0(M, \mathbb{H}) \oplus \Lambda^1(M, \mathbb{H}) \oplus M_3(\Lambda^1(M, \mathbb{C}))$$

$$\Omega^2_{D^A} \simeq \Lambda^2(M, \mathbb{C}) \oplus \Lambda^0(M, \mathbb{H}) \oplus \Lambda^0(M, \mathbb{H}) \oplus \Lambda^1(M, \mathbb{H}) \oplus \Lambda^1(M, \mathbb{H}) \oplus \Lambda^2(M, \mathbb{H}) \oplus M_3(\Lambda^2(M, \mathbb{C})),$$

from which the Yang–Mills Action and thus the (classical) Lagrangian are obtained by a noncommutative procedure strictly parallel to the usual one. To avoid a $U(3) \times SU(2) \times U(1)$ theory, however, an ingredient is missing. Following Connes we impose the “unimodularity condition”

$$\text{Str}(A + JA.J) = 0,$$

where the supertrace is taken with respect to particle-antiparticle splitting; here $J$ is the conjugation operator that interchanges particles and antiparticles. One gets the reduction to the SM gauge group and the correct hypercharges; this happens now irrespectively of whether neutrinos are massive or not. We have recently shown that the unimodularity condition is strictly equivalent, within the NCG framework, to anomaly cancellation: a first exciting hint at a deeper relationship between quantum physics and NCG than was known before.

**Recapitulation**

The picture that emerges is that of a “doubling” of the space stemming from chirality, with gauge bosons corresponding to the displacements in continuous directions and the Higgs boson corresponding to the exchange of quanta in the discrete direction.

There are 18 free parameters in the SM (leaving aside the vacuum angle $\theta$): the strong coupling constant $\alpha_3$; the electroweak parameters $\alpha_2$, $\sin^2\theta_W$, $m_W$; the Higgs mass; the nine (or twelve, if neutrinos are massive) fermion masses; and four Kobayashi–Maskawa parameters. One has as inputs the fermionic constants only; one can treat $\alpha_2$ as an adjustable parameter. When all computations are done, one obtains the constrained classical SM Lagrangian:

$$\mathcal{L} = -\frac{1}{4} AB_{\mu\nu} B^{\mu\nu} - \frac{1}{4} EF_{\mu\nu} F^{\mu\nu} - \frac{1}{4} CG_{\mu\nu} G^{\mu\nu} + SD_{\mu}\phi D^{\mu}\phi
- L(\|\phi_1\|^2 + \|\phi_2\|^2)^2 + 2L(\|\phi_1\|^2 + \|\phi_2\|^2)^2,$$

where $B, F, G$ denote respectively the $U(1)$, $SU(2)$, $SU(3)$ gauge fields and the coefficients $A, E, C, S, L$ are given in function of four unknown parameters $C_{\ell f}$, $C_{\ell c}$, $C_{q f}$, $C_{q c}$, which play the role of coupling constants in NCG.

The appearance of parameter restrictions is only natural, as all gauge fields now are part of a unique field. As only the ratios among those NCG parameters are important, there would remain only one “prediction”, i.e., the Higgs particle mass. We can be a little more explicit if we take the values which are more natural in the NCG framework: $C_{\ell f} = C_{\ell c}$, $C_{q f} = C_{q c}$. Introduce the parameter
\[ x := (C_{\ell f} - C_{q f})/(C_{\ell f} + C_{q f}) \]

with range \(-1 \leq x \leq 1\). The most natural value is \(x = 0.5\). When one identifies the previous constrained Lagrangian to the usual SM Lagrangian, it yields:

\[ m_W = m_{\text{top}} \sqrt{\frac{3}{N_F} \frac{1 - x}{4 - 2x}}. \]

Then \(m_{\text{top}} \geq \sqrt{3} m_W\). Similarly, \(g_3 = \frac{1}{\sqrt{2}} g_2 \sqrt{(4 - 2x)/(1 - x)}\).

For the Weinberg angle, in the massive neutrino case, one gets \(\sin^2 \theta_W = (12 - 6x)/(32 - 8x)\). Then one obtains the constraint \(\sin^2 \theta_W \leq 0.45\). Finally, for the mass of the Higgs:

\[ m_H = m_{\text{top}} \sqrt{3 - \frac{3}{N_F} \frac{1 - x}{2 - x}} = m_{\text{top}} \sqrt{3 - \frac{6m_W^2}{N_F m_{\text{top}}^2}}; \]

from which we get the relatively tight constraint \(\sqrt{7}/3 m_{\text{top}} \leq m_H \leq \sqrt{3} m_{\text{top}}\).

**Open problems**

One can accommodate the experimental values of the strong coupling constant and the Weinberg angle by choosing \(C_{\ell c} \gg C_{q c}\). Thus, NCG offers no real predictions for the ratio of the coupling constants to the Weinberg angle. Though \(m_W \leq m_{\text{top}}/\sqrt{N_F}\) is a suggestive constraint—it gives at once the right ballpark—there is no true prediction for the mass of the top quark, either. Rather, the experimentally determined top mass helps to fix the more important parameter of the theory, namely \(x\). Once the top mass is pinned down, the model seems to fix the value of the Higgs mass. For instance, if \(m_{\text{top}} = 2.5 m_W \approx 200 \text{ GeV}\), we get \(x = 0.53\), and then \(m_H = 328.3 \text{ GeV}\). Note that for \(x \geq 0.8\), we are outside the perturbative regime in Quantum Field Theory. If there were a compelling reason to adopt Connes’ relations on-shell, the theory would stand or fall by the value of the Higgs mass.

On the other hand, unless and until someone comes out with a quantization procedure specific to NCG that does the trick, there seems to be no such compelling reason. It is only reasonable to apply the standard renormalization procedures of present-day QFT to Connes’ version of the SM Lagrangian. The constraints are not preserved under the renormalization flow, i.e., they do not correspond to a hidden symmetry of the SM. The view that any constraints can be imposed only in a fully renormalization group invariant way is, nevertheless, theoretically untenable.

It is just conceivable that Nature has chosen for us a scale \(\mu_0\) at which to impose Connes’ restrictions. If we choose \(x = 0.5\), the present experimental values for the strong interaction coupling and Weinberg angle are regained on imposing Connes’ relations at the energy scale \(\mu_C \approx 5 \times 10^8 \text{ GeV}\) (in the massive case). This “intermediate unification scale” would mark the limit of validity of the present, phenomenological NCG model, essentially corresponding to an ordinary, but disconnected, manifold; at higher energy scales, the regime of truly
noncommutative geometries would begin. On imposing the mass relations at $\mu_C$, and running the renormalization equations at one loop, we get $m_{\text{top}} \simeq 215$ GeV (within the error bars of the D0 experiment) and $m_H \simeq 235-240$ GeV. The 1-loop approximation is not very accurate; inclusion of quantum corrections at 2nd order would give somewhat higher Higgs masses.

There is also a direct relation between NCG and gravitation: the noncommutative integral $\int D^{-2}$ gives the Einstein–Hilbert action of general relativity. However, there seems to be at present no unambiguous unification strand, within NCG, of gravitation and the subatomic forces.

Some sources

The original groundbreaking paper was [1]. For the “old scheme” of NCG (as presented in the 1992–94 period), and the introduction of the “new scheme”, see [2]. For the mathematics of NCG, see [3] and [4]. The parameter relations were derived in [5]. Renormalization of NCG models, and the rôle of anomalies in NCG schemes, have been explored in [6]. A noncommutative geometry model with massive neutrinos was proposed in [7]. Links between gravitation and NCG have been studied in [8]. For the philosophy of the whole thing, see [9].

References

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