“Systemic nonlocality” from changing constraints on sub-quantum kinematics

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Abstract. In a new approach to explain double-slit interference “from the single particle perspective” via “systemic nonlocality”, we answer the question of how a particle going through one slit can “know” about the state of the other slit. We show that this comes about by changed constraints on assumed classical sub-quantum currents, which we have recently employed [1] to derive probability distributions and Bohm-type trajectories in standard double-slit interference on the basis of a modern, 21st century classical physics. Despite claims in the literature that this scenario is to be described by a dynamical nonlocality that could best be understood in the framework of the Heisenberg picture [2], we show that an explanation can be cast within the framework of the intuitively appealing Schrödinger picture as well. We refer neither to potentials nor to a “quantum force” or some other dynamics, but show that a “systemic nonlocality” may be obtained as a phenomenon that emerges from an assumed sub-quantum kinematics, which is manipulated only by changing its constraints as determined by the changes of the apparatus. Consequences are discussed with respect to the prohibition of superluminal signaling by standard relativity theory.

1. Introduction
In two seminal papers [3, 4], Aharonov et al. more than 40 years ago introduced an approach to explain double-slit interference “from the single particle perspective” via dynamical nonlocality, thereby answering the question of how a particle going through one slit can “know” about the state of the other slit (i.e., being open or closed, for example). Dynamical nonlocality may lead to (causality preserving) changes in probability distributions and is thus distinguished from the kinematic nonlocality implicit in many quantum correlations, which, however, does not cause any changes in probability distributions.

In 2009, Tollaksen et al. [2] proposed some gedanken experiments to test the validity of this approach, and in a recent paper by Spence and Parks first experimental evidence for dynamical nonlocality was presented with the aid of weak measurements [5]. Essentially, it turns out that there is a nonlocal effect of an open or a closed first slit on a particle going through the second slit, thereby shifting the so-called modular momentum of the particle whereas the expectation values of (moments of) its momentum are left unchanged.

In the papers introducing dynamical nonlocality, the basis for a mechanism to explain how a particle at the right slit can “know” about what goes on at the left slit (or vice versa) is given by the nonlocal Heisenberg equations of motion for modular variables like the above-
mentioned modular momentum (to be discussed below). Tollaksen et al. [2] claim that one thus arrived at a fundamental difference between classical and quantum mechanics that was “often missed when the Schrödinger picture is taught and classical intuitions are applied to interference.” In contrast, we want to show here not only that a corresponding phenomenon of a “systemic nonlocality” can be accommodated within the Schrödinger picture, too, but also that a completely “classical” approach is feasible if one understands “classical” in the sense of present-day, “21st century classical physics”, i.e., including recent developments such as diffusion wave fields, superstatistics, or ballistic diffusion, for example. It is clear that both the Heisenberg and the Schrödinger pictures, respectively, have their advantages and disadvantages, for example w.r.t. applications to more complex scenarios, but they also complement each other, thereby highlighting different aspects with different useful insights. In this sense, our intention is not to criticize the use of the Heisenberg picture for the problem in question, but to show the usefulness of the Schrödinger picture as well, thus throwing light on “systemic nonlocality” from a perspective which is more in line with common intuitions.

In [2, 3, 4], the Heisenberg equations of motion are shown to be nonlocal because of the dependence on the potential at two distinct locations (i.e., of the slits). Modeled like the scalar Aharonov-Bohm effect, one finds also here that there are no forces acting on the particle. For example, when a particle reaches a slit and the other slit is suddenly closed, the authors claim that the only effect is a nonlocal change of the particle’s modular momentum due to the associated change of the potential at the location of the other (now closed) slit.

In this paper, we substitute the description via potentials in double slit interference by considering only constraints, or boundary conditions, on an assumed sub-quantum kinematics. Thus, putting a hypothesized sub-quantum medium into a systemic context, we shall attempt to model “systemic nonlocality” not via changes of potential differences, but by changes in sub-quantum kinematics due to their changed constraints. This amounts to a conceptual switch from a formal consideration of the macroscopic potentials of the experimental setup to a more “material” consideration of microscopic, i.e., sub-quantum currents whose modifications upon changing boundary conditions will be shown to co-determine the “nonlocal effects”. (The reason for using the terms such as “systemic nonlocality” or “nonlocal effects” only within quotation marks is given by the circumstance, to be discussed in Section 4, that these descriptions are supposed to only hold for the time resolutions of present-day experiments, but to eventually break down at very small time scales.) Just as in the above-quoted papers on dynamical nonlocality, we shall obtain the usual quantum mechanical results by introducing no potential-related forces whatsoever. Although our approach has many features in common with a Bohmian one, this is where an essential difference must be pointed out. Whereas in Bohmian theory, a nonlocal “quantum force” is made responsible for the genuine quantum effects, in our model no such force exists, since the equivalent of the Bohmian quantum potential is given by contributions from purely kinetic energy terms. As we employ no “quantum force”, therefore, we consider “systemic nonlocality” as a phenomenon that emerges from a sub-quantum kinematics, which is manipulated only by changing its constraints as determined by the changes of the apparatus.

In fact, with our approach we have in a series of papers obtained essential elements of quantum theory [1, 6, 7, 8, 9, 10, 11]. They derive from the assumption that a particle of energy $E = \hbar \omega$ is actually an oscillator of angular frequency $\omega$ phase-locked with the zero-point oscillations of the surrounding environment, the latter of which containing both regular and fluctuating components and being constrained by the boundary conditions of the experimental setup via the buildup and maintenance of standing waves. The particle in this approach is an off-equilibrium steady-state maintained by the throughput of zero-point energy from its vacuum surroundings. This is in close analogy to the bouncing/walking droplets in the experiments of Couder’s group [12, 13, 14, 15], which in many respects can serve as a classical prototype guiding our intuition. We have recently applied our model to the case of interference at a double slit [1],
thereby obtaining the exact quantum mechanical probability distributions on a screen behind the double slit, the average particle trajectories (which because of the averaging are shown to be identical to the Bohmian ones), and the involved probability density currents. One aim of the present paper is to extend the applicability of our model to a scenario that is only apparently dynamical, i.e., to describe “systemic nonlocality” as introduced above.

This paper is structured as follows. In Section 2, we give a short introduction to our sub-quantum model, with explanations of Gaussian dispersion and interference at a double slit, among others. Section 3 presents a review of dynamical nonlocality, its explanations in the recent literature via the Heisenberg picture, and our own explanation via the Schrödinger picture. Finally, in Section 4 we show how a corresponding “systemic nonlocality” can be understood as emerging from a classical sub-quantum kinematics.

2. A short introduction to our sub-quantum approach to quantum mechanics

In older approaches to quantum theory by de Broglie or Bohm, for example, one basically focused on a role of the waves, although sometimes possibly “empty”, as essentially “guiding” the particle through the experimental apparatus. However, the Couder experiments explicitly point at a more complex scenario by an observed partial decoupling of wave and particle propagation. While the particle (or bouncer/walker, respectively) still is guided through interfering and fluctuating waves, the latter constitute a landscape that is not only present in the vicinity of the particle, but throughout the whole apparatus. (In a rare illustration of a similar circumstance in quantum physics, Bohm and Hiley write: “…it is only through the existence of... pools of information which are not expressible solely in terms of relationships of actual particles that the notion of an objective whole can be given meaning.” [16])

We transfer the corresponding insight from the Couder experiments into our modeling of quantum systems and assume that the waves are a space-filling phenomenon involving the whole experimental setup. Thus, one can imagine a partial decoupling of the physics of waves and particles in that the latter still may be “guided” through said landscape, but the former may influence other regions of the landscape by providing specific phase information independently of the propagation of the particle. This is why a remote change in the experimental setup, when mediated to the particle via de- and/or re-construction of standing waves, can potentially amount to a nonlocal effect on a particle via the thus modified guiding landscape.

In [8] we presented a model for the classical explanation of the quantum mechanical dispersion of a free Gaussian wave packet. In accordance with the classical model, we shall now relate it more directly to a “double solution” analogy gleaned from Couder and Fort [15]. For, as is shown, e.g., in [17], one can construct various forms of classical analogies to the quantum mechanical Gaussian dispersion. Originally, the expression of a “double solution” refers to an early idea of de Broglie [18] to model quantum behavior by a two-fold process, i.e., by the movement of a hypothetical point-like singularity solution of the Schrödinger equation, and by the evolution of the usual wavefunction that would provide the empirically confirmed statistical predictions. Recently, Couder and Fort [15] used this ansatz to describe the behaviors of their bouncer- (or walker-) droplets: on an individual level, one observes particles surrounded by circular waves they emit through the phase-coupling with an oscillating bath, which provides, on a statistical level, the emergent outcome in close analogy to quantum mechanical behavior (like, e.g., diffraction or double-slit interference).

In the context of the double solution idea, which is related to correlations on a statistical level between individual uncorrelated particle positions $x$ and momenta $p$, respectively, we consider the free Liouville equation. It provides a phase-space distribution $f(x,p,t)$ that shows the emergence of correlations between $x$ and $p$ from an initially uncorrelated product function of non-spreading (classical) Gaussian position distributions as well as momentum distributions. The motivation for their introduction comes exactly from what one observes in the Couder
experiments. In an idealized scenario, we assume that at each point \( x \) an unbiased emission of momentum fluctuations \( \pi \) in all possible directions takes place, thus mimicking (in a two-dimensional scenario) the circular waves emitted from the “particle as bouncer”. If we compare the typical frequency of the bouncers in the Couder experiments (i.e., roughly \( 10^2 \) Hz) with that of an electron, for example (i.e., roughly \( 10^{20} \) Hz), we see that a continuum ansatz is practically plausible, particularly if we are interested in statistical averages over a long series of experimental runs.

Thus, one can construct said phase-space distribution, with \( \sigma_0 \) being the initial \( x \)-space standard deviation, i.e., \( \sigma_0 = \sigma(t = 0) \), and \( \pi_0 := mu_0 \) the momentum standard deviation, such that

\[
f(x, p, t) = \frac{1}{2\pi\sigma_0 mu_0} \exp \left\{ -\frac{(x - pt/m)^2}{2\sigma_0^2} \right\} \exp \left\{ -\frac{p^2}{2m^2u_0^2} \right\}.
\] (1)

Now, the above-mentioned correlations between \( x \) and \( p \) emerge when one considers the probability density in \( x \)-space. It turns out that

\[
P(x, t) = \int f dp = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ -\frac{x^2}{2\sigma^2} \right\},
\] (2)

with the standard deviation at time \( t \) given by

\[
\sigma^2 = \sigma_0^2 + u_0^2 t^2.
\] (3)

In other words, the distribution (2) with (3) describing a spreading Gaussian is obtained from a continuous set of classical, i.e., non-spreading, Gaussian position distributions of particles whose associated momentum fluctuations also have non-spreading Gaussian distributions. One thus obtains the exact quantum mechanical dispersion formula for a Gaussian, as we have obtained also previously from our classical ansatz. For confirmation with respect to our diffusion model [8, 9], we note that with the usual definition of the “osmotic” velocity field \( u = -D\nabla P \), one obtains with (2), and with bars denoting averages,

\[
\overline{u^2} = D^2 \left( \frac{\nabla P}{P} \right)^2 = \frac{D^2}{\sigma^2}, \quad \text{and thus also} \quad u_0 = \frac{D}{\sigma_0},
\] (4)

so that one can rewrite Eq. (3) in the more familiar form

\[
\sigma^2 = \sigma_0^2 \left( 1 + \frac{D^2t^2}{\sigma_0^4} \right).
\] (5)

Note also that by using the Einstein relation \( D = h/(2m) = h/(4\pi m) \) the norm in (1) thus becomes the invariant expression (reflecting the “exact uncertainty relation” [19])

\[
\frac{1}{2\pi\sigma_0 mu_0} = \frac{1}{2\pi mD} = \frac{2}{h}.
\] (6)

Following from (5), in references [1, 8] we obtained for smoothed-out trajectories (i.e., averaged over a very large number of Brownian motions) a sum over an Ehrenfest-type and a fluctuations term, respectively, for the motion in the \( x \)-direction

\[
x_{\text{tot}}(t) = vt + x(t) = vt + x(0) \frac{\sigma}{\sigma_0} = vt + x(0) \sqrt{1 + \frac{D^2t^2}{\sigma_0^4}}.
\] (7)
Thus one obtains the *average velocity field* of a Gaussian wave packet as

\[ v_{\text{tot}}(t) = v(t) + \frac{dx(t)}{dt} = v(t) + \left[ x_{\text{tot}}(t) - vt \right] \frac{u_0^2 t}{\sigma^2}. \] (8)

Note that Eqs. (7) and (8) are derived solely from statistical physics. Still, they are in full accordance with quantum theory, and in particular with Bohmian trajectories [17]. Note also that one can rewrite Eq. (5) such that it appears like a linear-in-time formula for Brownian motion,

\[ \bar{x}^2 = \bar{x}^2(0) + D_t t, \] (9)

where a time dependent diffusivity

\[ D_t = \frac{u_0^2 t}{4m^2\sigma_0^2} t \] (10)

characterizes Eq. (9) as *ballistic diffusion*. This makes it possible to simulate the dispersion of a Gaussian wave packet on a computer by simply employing coupled map lattices for classical diffusion, with the diffusivity given by Eq. (10). (For detailed discussions, see refs. [8] and [9].)

Moreover, one can easily extend this scheme to more than one slit, like, for example, to explain interference effects at the double slit [1, 11]. For this, we chose similar initial situations as in [17], i.e., electrons (represented by plane waves in the forward \(y\)-direction) from a source passing through soft-edged slits 1 and 2 in a barrier (located along the \(x\)-axis) and recorded at a screen. In our model, we therefore note two Gaussians representing the totality of the effectively “heated-up” so-called *path excitation field* (to be detailed below), one for slit 1 and one for slit 2, whose centers have the distances \(+X\) and \((-X\) from the plane spanned by the source and the center of the barrier along the \(y\)-axis, respectively.

With the total amplitude \(R\) of two coherent waves with (suitably normalized) amplitudes \(R_i = \sqrt{P_i}\), and the local phases \(\varphi_i, i = 1 \text{ or } 2\), one has as usual that

\[ R = R_1 \cos(\omega t + \varphi_1) + R_2 \cos(\omega t + \varphi_2). \] (11)

Introducing an arbitrarily chosen unit vector \(\hat{n}\), one may also define \(\cos(\omega t + \varphi_i(x)) = \hat{n} \cdot \hat{k}_i(x, t)\), such that along with the system’s evolution, the emergent outcome of the time evolution of (11) can be written as

\[ R(x, t) = \hat{n} \cdot \left( R_1 \hat{k}_1(x, t) + R_2 \hat{k}_2(x, t) \right), \] (12)

which we shall use later on. According to classical textbook wisdom, the *averaged total intensity* becomes

\[ P_{\text{tot}} := R^2 = R_1^2 + R_2^2 + 2R_1 R_2 \cos \varphi_{12} = P_1 + P_2 + 2\sqrt{P_1 P_2} \cos \varphi_{12}, \] (13)

where \(\varphi_{12}\) is the relative phase \(\varphi_{12} = \varphi_1 - \varphi_2 = (\mathbf{k}_1 - \mathbf{k}_2) \cdot \mathbf{r}\). Note that \(\varphi_{12}\) enters Eq. (13) only via the cosine function, such that, e.g., even if the total wave numbers (and thus also the total momenta) \(\mathbf{k}_i\) were of vastly different size, the cosine effectively makes Eq. (13) independent of said sizes, but dependent only on an angle modulo \(2\pi\). This will turn out as essential for our discussion further below.

The \(x\)-components of the centroids’ motions from the two alternative slits 1 and 2, respectively, are given by the particle velocity components

\[ v_x = \pm \frac{\hbar}{m} k_x, \] (14)
respectively, such that the relative group velocity of the Gaussians spreading into each other is given by $\Delta v_x = 2v_x$. However, in order to calculate the phase difference $\varphi_{12}$ descriptive of the interference term of the intensity distribution (13), one must take into account the total momenta involved, i.e., one must also include the wave packet dispersion as described in the previous Section. Thus, one obtains with the displacement $\pm x(t) = \mp (X + v_xt)$ in Eq. (8) the total relative velocity of the two Gaussians as

$$\Delta v_{tot,x} = 2 \left[ v_x - (X + v_xt) \frac{u_0^2 t}{\sigma^2} \right].$$

(15)

Therefore, the total phase difference between the two possible paths 1 and 2 (i.e., through either slit) becomes

$$\varphi_{12} = \frac{1}{\hbar} (m \Delta v_{tot,x}) = 2m v_x \frac{x}{\hbar} - (X + v_xt)x \frac{1}{D} \frac{u_0^2 t}{\sigma^2}.$$

(16)

In our earlier papers [6, 7, 8], we have shown that, apart from the ordinary particle current $J(x,t) = P(x,t)v$, we are now dealing with two additional, yet opposing, currents $J_\pm = P(x,t)u$, which are on average orthogonal to $J$ [6, 7, 8, 10], and which are the emergent outcome from the presence of numerous corresponding velocities

$$u_\pm = \mp \frac{\hbar}{2m} \frac{\nabla P}{P}.$$  

(17)

We denote with $u_+$ and $u_-$, respectively, the two opposing tendencies of the diffusion process. Moreover, when we take the averages, we obtain a smoothed-out \textit{average velocity field}

$$\overline{u}(x,t) = \int P u(x,t) \, d^n x,$$

(18)

which is all that we need for our further considerations. Similarly, based on the fact that we have an initial Gaussian distribution of velocity vectors $v(x,t)$, we define an average velocity field $\overline{v}$ of the wave propagation as

$$\overline{v}(x,t) = \int P v(x,t) \, d^n x,$$

(19)

and for the free particle make use of an average orthogonality between the two velocity fields, $u$ and $v$, [6, 7, 8, 9],

$$\overline{v} \cdot \overline{u} = \int P v \cdot u \, d^n x = 0.$$

(20)

In effect, then, the combined presence of both velocity fields $u$ and $v$ can be denoted as a \textit{path excitation field}: via diffusion, the bouncer in its interaction with already existing wave-like excitations of the environment creates an “agitated”, or “heated-up”, thermal landscape, which can also be pictured by interacting wave configurations all along between source and detector of an experimental setup. Recall that our prototype of a walking bouncer, i.e., from the experiments of Couder’s group, is always driven by its interactions with a superposition of waves emitted at the points it visited in the past. Couder et al. denote this superposition of in-phase waves the “path memory” of the bouncer [20]. This implies, however, that the bouncers at the points visited in the present necessarily create new wave configurations which will form the basis of a path memory in the future. In other words, the wave configurations of the past determine the bouncer’s path in the present, whereas its bounces in the present co-determine the wave configurations at any of the possible locations it will visit in the future. Therefore, we call the latter configurations the \textit{path excitation field}, which may also be described as heated-up
thermal field. Ideally, as in the coupling of an oscillator with classical diffusion, non-relativistic diffusion wave fields arise with instantaneous field propagation \[7, 21\], one has elements of the whole setup which may be nonlocally oscillating (“breathing”) in phase. This means that the Gaussian of Eq. (2) can be said to represent an idealized nonlocal path excitation field in that it is a physically existing and effective entity responsible for where the bouncing particle can possibly go. (We shall discuss a less idealized approach in the last Section of this paper.)

Let us now consider a single, classical particle (bouncer) following the propagation of a set of waves of equal amplitude \(R_i\), each representing one of \(i\) possible alternatives according to our principle of path excitation, and focus on the specific role of the velocity fields. To describe the required details, each path \(i\) be occupied by a Gaussian wave packet with a forward momentum \(p_i = \hbar k_i = mv_i\). Moreover, due to the stochastic process of path excitation, the latter has to be represented also by a large number \(N\) of consecutive Brownian shifts, \(p_{u,\alpha} = mu_{\alpha}\). Defining (with indices \(i = 1\) or 2 referring to the two slits, and with + and − referring to the right and the left from the average direction of \(v_i\), respectively)

\[
\vec{v}_{\text{tot}, i} := \vec{v}_i + \vec{u}_{i+} + \vec{u}_{i-},
\]

and with two Gaussian distributions \(P_1 = R_1^2\) and \(P_2 = R_2^2\), one has with (12)

\[
R_{\text{tot}}^2 = (R_1 \vec{v}_{\text{tot}, 1} + R_2 \vec{v}_{\text{tot}, 2})^2,
\]

which after a few calculational steps provides (similarly to [1], but now with slightly different labellings which apply more generally) the total average current

\[
J_{\text{tot}} = P_1 \vec{v}_1 + P_2 \vec{v}_2 + \sqrt{P_1 P_2} (\vec{v}_1 + \vec{v}_2) \cos \varphi_{12} + \sqrt{P_1 P_2} (\vec{u}_1 - \vec{u}_2) \sin \varphi_{12}.
\]

A more detailed account of Eq. (23) and its extension to \(n\) slits is in preparation [Fussy \textit{et al.} (2013)]. Note that Eq. (23), upon the identification of \(\vec{u}_i = -\frac{\hbar}{m} \nabla R_i\) from Eq. (17) and with \(P_i = R_i^2\), turns out to be in perfect agreement with a comparable “Bohmian” derivation \[17, 22\]. In fact, with \(\vec{v}_i = \frac{\nabla S_i}{m}\), one can rewrite (23) as

\[
J_{\text{tot}} = R_1 \frac{\nabla S_1}{m} + R_2 \frac{\nabla S_2}{m} + R_1 R_2 \left( \frac{\nabla S_1}{m} + \frac{\nabla S_2}{m} \right) \cos \varphi_{12} + \frac{\hbar}{m} (R_1 \nabla R_2 - R_2 \nabla R_1) \sin \varphi_{12}.
\]

The formula for the averaged particle trajectories, then, simply results from

\[
\vec{v}_{\text{tot}} = \frac{J_{\text{tot}}}{P_{\text{tot}}^2}.
\]

Although we have obtained the usual quantum mechanical results, we have so far not used the quantum mechanical formalism in any way. However, upon employment of the Madelung transformation for each path \(j\) (\(j = 1\) or 2),

\[
\Psi_j = Re^{iS/\hbar},
\]

and thus \(P_j = R_j^2 = |\Psi_j|^2 = \Psi_j^* \Psi_j\), with the definitions (17) and \(v_j := \nabla S_j/m\), \(\varphi_{12} = (S_1 - S_2)/\hbar\), and recalling the usual trigonometric identities such as \(\cos \varphi = \frac{1}{2} (e^{i\varphi} + e^{-i\varphi})\), etc., one can rewrite the total average current (23) immediately as

\[
J_{\text{tot}} = P_{\text{tot}} \vec{v}_{\text{tot}}
\]

\[
= (\Psi_1 + \Psi_2)^*(\Psi_1 + \Psi_2) \frac{1}{2} \left[ \frac{1}{m} \left( -i\hbar \nabla (\Psi_1 + \Psi_2) \right) \right] + \frac{1}{m} \left( i\hbar \nabla (\Psi_1 + \Psi_2)^* \right)
\]

\[
= -\frac{i\hbar}{2m} [\Psi^* \nabla \Psi - \Psi \nabla \Psi^*] = \frac{1}{m} \text{Re} \{ \Psi^*(-i\hbar \nabla) \Psi \},
\]
where \( P_{\text{tot}} = |\Psi_1 + \Psi_2|^2 =: |\Psi|^2 \). The last two expressions of (27) are the exact well-known formulations of the quantum mechanical probability current, here obtained without any quantum mechanics, but just by a re-formulation of (23). In fact, it is a simple exercise to insert the wave functions (26) into (27) to re-obtain (23).

3. “Systemic” versus dynamical nonlocality in double slit interference

In reference [2], Tollaksen et al. discuss interference at a double slit “from a single particle perspective” and ask the following question: If a particle goes through one slit, how does it “know” whether the second slit is open or closed? We shall here first recapitulate the arguments providing these authors’ answer and later provide our own arguments and answer. Of course, the question is about the phase information and how it affects the particle. We know from quantum mechanics that phases cannot be observed on a local basis and that a common overall phase has no observational meaning. Assuming that two initially non-overlapping Gaussian wave functions, \( \Psi_1 \) and \( \Psi_2 \), describe the probability amplitudes for particles emerging from slits 1 or 2, respectively, which are separated by a distance \( D \), the total wave function for the particle exiting the double slit may be written as

\[
\Psi = e^{i\alpha_1} \Psi_1 + e^{i\alpha_2} \Psi_2,
\]

but since a common overall phase is insignificant, one writes the total wave function as

\[
\Psi_\varphi = \Psi_1 + e^{i\varphi} \Psi_2,
\]

where \( \varphi := \alpha_2 - \alpha_1 \) is the physically significant relative phase. Tollaksen et al. now ask where the relative phase appears in the form of (deterministic) observables that describe interference. When looking at the expectation value (in the one-dimensional case, for simplicity)

\[
\bar{\varphi} = \int \bar{\Psi}_\varphi \Psi_\varphi \, dx = \int (\Psi_1^* + e^{-i\varphi}\Psi_2^*) \, (\Psi_1 + e^{i\varphi}\Psi_2) \, dx
\]

\[
= \int (\Psi_1^* \Psi_1 + \Psi_2^* \Psi_2 + \Psi_2^* \Psi_1 + e^{-2i\varphi} \Psi_1 \Psi_2) \, dx + c.c.,
\]

one sees that it is independent of \( \varphi \) because of the vanishing of the last term due to the assumed non-overlapping of \( \Psi_1 \) and \( \Psi_2 \). Similarly, this also holds for all moments \( \bar{p}^n \), and for all moments \( \bar{p}^n \) as well. In particular, one has for the expectation value of the momentum

\[
\bar{p} = \left[ \int \bar{\Psi}_\varphi \Psi_\varphi \, dx \right]^{-1} \int \bar{\Psi}_\varphi \frac{\partial}{\partial x} \Psi_\varphi \, dx
\]

\[
\equiv \Re \int \left( \Psi_1^* \frac{\partial}{\partial x} \Psi_1 + \Psi_2^* \frac{\partial}{\partial x} \Psi_2 \right) \, dx + \Re \int \Psi_2^* \frac{\partial}{\partial x} e^{i\varphi} \Psi_2 \, dx + c.c.,
\]

where the \( \varphi \)-dependent term vanishes identically, because \( \frac{\partial}{\partial x} \Psi_2 = 0 \) for \( \Psi_2 = 0 \). So, again, where does the relative phase appear? The answer of Tollaksen et al. is given by a “shift operator” that shifts the location of, say \( \Psi_1 \), over the distance \( D \) to its new location coinciding with that of \( \Psi_2 \). The expectation value of the shift operator is thus given by

\[
\bar{e^{-i\varphi}} = \int \left( \Psi_1 e^{-i\varphi} + \Psi_2 e^{-i\varphi} \Psi_2 \right) \, dx + \int \Psi_1^* e^{-i\varphi} \Psi_1 \, dx + \int \Psi_2^* e^{-i\varphi} e^{-i\varphi} \Psi_1 \, dx,
\]

where all but the last term vanish identically, thus providing (with the correct normalization)

\[
\bar{e^{-i\varphi}} = e^{-i\varphi}/2 \text{ and } \bar{e^{-i\varphi}} + \bar{e^{i\varphi}} = \cos \varphi.
\]

(33)
In order to make sense of the shift operator, the authors now shift to the Heisenberg picture, thereby providing with a Hamiltonian $H = \frac{p^2}{2m} + V(x)$ the time evolution of the operator as

$$\frac{d}{dt} e^{-i\mathbf{p} \cdot \mathbf{D} t} = i \hbar \left[ e^{-i\mathbf{p} \cdot \mathbf{D} t}, \frac{p^2}{2m} + V(x) \right] = \frac{-iD}{\hbar} e^{-i\mathbf{p} \cdot \mathbf{D} t} \left\{ \frac{V(x) - V(x + D)}{D} \right\}. \quad (34)$$

With its dependence on the distance $D$ between the two slits, Eq. (34) is claimed to be a description of dynamical nonlocality showing how a particle can “know” about the presence of the other slit. Tollaksen et al. maintain that it is possible to understand this dynamical nonlocality only by employing the Heisenberg picture. However, we shall now show that an equivalent understanding is possible also within the Schrödinger picture, and even more intuitively accessible, too. For, there is one assumption in the foregoing analysis that is not guaranteed to hold in general, i.e., the non-overlapping of the wave functions $\Psi_1$ and $\Psi_2$. On the contrary, we now shall assume that the two Gaussians representing the probability amplitudes for the particle immediately after passing one of the two slits do not have any artificial cut-off, but actually extend across the whole slit system, even with only very small (and practically often negligible) amplitudes in the regions further away from the slit proper. (We shall give arguments for this assumption further below.) In other words, we now ask: what if $\Psi_1$ and $\Psi_2$ do overlap, even if only by a very small amount? To answer this question, we consider the expectation value of the momentum, or the total current $J_{\text{tot}} = \int \mathbf{p} d\mathbf{x}$, respectively, and we obtain from Eq. (31) that the terms $\mathbf{p}_\varphi$ involving the relative phase $\varphi$ stem from the interference terms of $J_{\text{tot}}$, i.e.,

$$P_{\text{tot}} \mathbf{p}_\varphi = R_1 R_2 \left( \frac{\nabla S_1}{m} + \frac{\nabla S_2}{m} \right) \cos \varphi + \frac{\hbar}{m} (R_1 \nabla R_2 - R_2 \nabla R_1) \sin \varphi. \quad (35)$$

First of all one notes upon comparison of Eq. (35) with Eqs. (23)–(25) the exact correspondence of (35) with our classically obtained expression for the interference terms of the emerging current, or of the expression for $\nabla_{\text{tot}}$, respectively. Moreover, although the product $R_1 R_2$ is in fact negligibly small for regions where only a long tail of one Gaussian overlaps with another Gaussian (i.e., such that the non-overlapping assumption would be largely justified), nevertheless the second term in (35) can be very large despite the smallness of $R_1$ or $R_2$. It is this latter part which is responsible for the genuinely quantum-like nature of the average momentum, i.e., for its nonlocal nature. This is similar in the Bohmian picture, but here given a more direct physical meaning in that this last term refers to a difference in diffusive currents as explicitly formulated in the last term of Eq. (23). Because of the mixing of diffusion currents from both channels, we call this decisive term in $J_{\text{tot}}$ the “entangling current”, $J_e$ [23].

For illustration, Figs. 1–6 show our classical computer simulations of interference and the role of the entangling current $J_e$ in different situations. Figs. 1 and 2 show the emerging interference pattern and the average trajectories without, and Figs. 3 and 4 with, an applied extra phase shift at one slit. To bring out the shifting of the interference pattern more clearly, in Figs. 5 and 6 we apply – mainly for didactic reasons – the phase shift at much later times than in Figs. 3 and 4. Thereby, also a decoupling of wave and particle behaviors becomes visible.

Finally, there are substantial arguments against the non-overlapping scenario in Tollaksen et al. Firstly, experiments by Rauch et al. have shown that in interferometry interference does not only happen when the main bulks of the Gaussians overlap, but also when a Gaussian interferes with the off-bulk plane-wave components of the other wave function as well [24]. On a theoretical side, we have repeatedly stressed that the diffusion processes employed in our model can ideally be described by nonlocal diffusion wave fields [21, 25] which thus require small but non-zero amplitudes across the whole experimental setup. Our scenario
**Figure 1.** Classical computer simulation of the interference pattern with equal slit widths: intensity distribution with increasing intensity from white through yellow and orange, with averaged trajectories (red) for two Gaussian slits ($v_{x,1} = v_{x,2} = 0$).

**Figure 2.** The corresponding entangling current density $J_e$, i.e., the last term on the r.h.s. of Eq. (23). As this term is characterized by the difference of the diffusive velocities $u_i$, the entangling current is responsible for the “systemically nonlocal” nature of the process forming the interference pattern.

**Figure 3.** Classical computer simulation as in Fig. 1, but with additional phase shift $\Delta \varphi = 3\pi$ gradually accumulated during the time interval between $t_1$ and $t_2$ at slit 1.

**Figure 4.** The corresponding entangling current density $J_e$. Note the shift of maxima and minima in the emerging pattern, as compared to Fig. 2.
Figure 5. Classical computer simulation as in Fig. 3, but with different times \( t_i \) and with gradually accumulated additional phase \( \Delta \varphi = 5\pi \). This results in the same distributions of \( P \) and \( J_e \) for times \( t > t_2 \) and shows the effect of the shifting of the interference fringes more clearly than Fig. 3. Note the radically different behaviors of the probability density related to wave-like interference on one hand, and that of the average particle trajectories on the other.

Figure 6. Although the currents \( J_e \) dramatically cross the central symmetry line separating the areas of the two slits, the average particle trajectories (Fig. 5) strictly obey the no crossing rule familiar from, but not restricted to, the deBroglie-Bohm interpretation. This is a clear demonstration of the partial decoupling of wave and particle behavior as envisaged in our model.

may also relate to the recently developed models of superstatistics, where a macroscopically observed emergent behavior is the result of a superposition of two or more statistical systems operating at vastly different space-time scales. ([26], and Jizba and Scardigli, this volume) Moreover, and more specifically, we have shown [8, 9] that quantum propagation can be identified with sub-quantum anomalous (i.e., “ballistic”) diffusion which is characterized by infinite mean displacements \( \langle x \rangle = \infty \) despite the finite drifts \( \langle x^2 \rangle = u^2 t^2 \) and \( u < c \). In sum, these arguments speak in favor of using small, but non-zero amplitudes from the Gaussian of the other slit which can interfere with the Gaussian at the particle’s location.

4. Emergence of “systemic nonlocality” from sub-quantum kinematics

Let us now see how we can understand “systemic nonlocality” within our sub-quantum approach. For example, we can ask the following question in our context: what if we start with one slit only, and when the particle might pass it we open the second slit? Let us assume for the time being, and without restriction of generality, that the \( x \)-component of the velocity \( v_x \) is zero. Then, according to (16), upon opening the second slit (with the same \( \sigma \)), we obtain a term proportional to the distance \( 2X \) between the two slits. Thus, there will be a shift in momentum on the particle passing the first slit given by

\[
\frac{\Delta p}{\hbar} = \pm \frac{1}{2} \nabla \varphi_{12} = \pm \frac{1}{2\hbar} \nabla (S_1 - S_2) = \frac{\Delta p_{\text{mod}}}{\hbar},
\]
where one effectively uses the “modular momentum” \( p_{\text{mod}} = p \mod \frac{\hbar}{2X} = p - 2n\pi \frac{\hbar}{2X} \) because an added or subtracted phase difference \( \varphi_{12} = 2n\pi \) does not change anything. In other words, by splitting \( X \) into a component \( X_n \) providing \( \varphi_{12} = 2n\pi \) on the one hand, and the modular remainder \( \Delta X \) on the other hand providing \( \varphi_{12} = p \mod x/\hbar \) because an added or subtracted phase difference \( \varphi_{12} = 2n\pi \) does not change anything. In other words, by splitting \( X \) into a component \( X_n \) providing \( \varphi_{12} = 2n\pi \) on the one hand, and the modular remainder \( \Delta X \) on the other hand providing \( \varphi_{12} = p \mod x/\hbar \), one rewrites (16) with \( X := X_n + \Delta X \) (and with \( v_x = 0 \) for simplicity) as

\[
\varphi_{12} = -(X_n + \Delta X)x \left( \frac{u^2t}{D} \right) = -2n\pi - \Delta X \frac{\hbar}{2m} \frac{t}{\sigma^2 \sigma^2_0}.
\]

Therefore, one can further on substitute \( \Delta p \) by \( \Delta p_{\text{mod}} \), and one obtains a momentum shift (with the sign depending on whether the right or left of the two slits is opened as the second one)

\[
\Delta p_{\text{mod}} = \pm \frac{\hbar}{2} \nabla \varphi_{12} = \pm \frac{\Delta X \hbar^2}{2m \sigma^2 \sigma^2_0} = \pm m\Delta X \frac{D^2 t}{\sigma^2 \sigma^2_0} = \pm m\Delta X \frac{\dot{\sigma}}{\sigma}.
\]

For large times, one has \( \dot{\sigma} \approx \sigma/t \) such that

\[
\Delta p_{\text{mod}} \approx \pm \frac{m\Delta X}{t}.
\]

In a separate paper [23] we show that, with \( \xi(t) = x - vt \) describing the location of a particle in a Gaussian field, the action is given by

\[
S = \int mv_{\text{tot}}(t) \, dx - \int E \, dt = S = mvx + \frac{mu^2_0}{2} \left( \frac{\xi(t)}{\sigma(t)} \right)^2 t - Et = mvx + \frac{mu^2_0}{2} \left( \frac{\xi(0)}{\sigma_0} \right)^2 t - Et,
\]

with \( E \) being the system’s total energy. Thus, the expression \( \nabla \varphi_{12} \) of (38) essentially refers to the gradient of the thermal fluctuation field, with the kinetic temperature \( kT = \frac{mv^2_0}{2} \left[ \frac{\xi(0)}{\sigma(t)} \right]^2 \) of what we have termed the path excitation field.

It follows that, due to the vacuum pressure stemming from the opened second slit, there is an emergent “nonlocal force” which does not derive from a potential but from the impinging of the second slit’s sub-quantum diffusive momenta on the particle at the first slit. Although there is a corresponding shift in the latter’s velocity distribution, i.e.,

\[
\frac{\partial}{\partial t} \Delta p_{\text{mod}} = \pm m\Delta X \frac{1}{\sigma} \left( \sigma \dot{\sigma} - \dot{\sigma}^2 \right) = \pm m\Delta X \left( \frac{\dot{\sigma}}{\sigma} - \left[ \frac{\sigma}{\sigma} \right]^2 \right),
\]

which reduces for large times to a gradually diminishing deceleration, or acceleration, respectively,

\[
\frac{\partial}{\partial t} \Delta p_{\text{mod}} \approx \pm m\Delta X \left[ \frac{\sigma}{\sigma} \right]^2 \approx \pm m \frac{\Delta X}{t^2},
\]

in the vicinity of the slits which interests us here primarily, the expression (41) practically vanishes identically, i.e., there is no additional force on the particle near the slit.

Now that we have found a possible “nonlocal momentum transfer”, the question immediately arises whether this would imply superluminal signaling. To answer this question, we recall that contained in the phase difference (37) is the osmotic velocity \( u \), or the diffusion-related momentum fluctuation \( \delta p = mu \), respectively, for which it holds that it must be unbiased [27], i.e.,

\[
\int P\delta p \, dx = 0.
\]
In other words, both with regard to the directions and with regard to the sizes, the average over all momentum fluctuations and positions of them must vanish identically, i.e., there is a “complete uncertainty” of $\delta p$, just as Tollaksen et al. refer to a “complete uncertainty” of the relative phase [2]. This means that, since an experimenter can neither know where exactly within the Gaussian distribution the particle is located nor what quantity $\delta p = \text{mu}$ would actually be transferred in any single run of the experiment upon opening the second slit, one can only obtain information about the average momentum transfer $\Delta p_{\text{mod}}$ in a sufficiently large statistical sample. As the momentum-shifted particle does not allow to extract a “signal” from its experiencing that shift, i.e., as there is no way to distinguish between an unambiguous state before and an unambiguous state after the shift, one can have a “nonlocal momentum transfer” which could at best imply a “hidden” superluminal signaling.

Although we are therefore dealing with an epistemic indeterminism in this case, we nevertheless have to face the problem that on an ontic level our approach is a deterministic one (i.e., based on an assumed stochastic, and thus essentially deterministic, sub-quantum thermodynamics). As nonlocality can “peacefully co-exist” with relativity only if one also has an underlying indeterminism, we shall in the remainder of this paper discuss the possible consequences for and of our model.

Firstly, the foregoing analysis was restricted to the one-dimensional momentum transfer, i.e., in the $x$–direction normal to the forward particle propagation. However, with the spatial components of Gaussians being independent of each other, the corresponding dispersion and momentum transfer processes must also hold in the other directions, e.g., in the $y$–direction of particle propagation when considering the two-dimensional case.

Generally, the question arises of how the calculated momentum transfer can be nonlocal, while discarding the distances $X_n$ which just provide the ineffective phase differences $\varphi_{12} = 2n\pi$. Here, the analogy to the Couder experiments provides another clue. For there, it is clear that the modes of the standing waves emerging between the walls of the bath container depend on the distance between the walls. In other words, changing that distance means selecting another set of modes. Now, it is possible to transfer this reference to the boundary conditions also to the quantum systems we experiment with in the laboratory. For, the whereabouts of the quanta are always restricted to the space between sources and detectors. Changing the experimental setup, e.g., by introducing a phase shifter, by closing a slit, or the like, therefore always amounts to changing boundary conditions in such a way that old modes of standing waves are substituted by new ones to whose phases the particle/bouncer now locks in.

In principle, this implies essentially only two options with respect to the question of superluminal signaling (with the addition of a third option briefly mentioned at the end of this paper): i) either, one has a strict nonlocality, i.e., with instantaneous changes of boundary conditions across spacelike distances, or ii) the changes take a small, but non-zero time to become effective, thus also implying possible superluminal effects. Option i) is an idealized one that is argued for by using the physics of nonrelativistic oscillator-driven diffusion wave fields as introduced above. For example, in [28] it was shown that for the one-dimensional problem of a pulsating particle in a box, the heat distribution turns out classically to be of the same sinusoidal type as the quantum mechanical probability distribution. Then, a shifting of a wall results in an instantaneous shifting of all the nodes of the distribution just like in the quantum case. Therefore, one has in this idealized classical scenario a nonlocality identical with the quantum one, but originating from the interplay of oscillator-driven diffusion wave fields with macroscopic boundaries of the experimental setup. As mentioned, ontologically this would lead to superluminal signaling, even if it was only a “hidden” one for the observers because of the epistemic indeterminism due to the unknowability of the exact (and, on the sub-quantum level: existing!) initial conditions.

This leaves us with option ii) as the possibly more realistic scenario of an emergent
“nonlocality” that we describe as “systemic”. In other words, with the bouncer always oscillating harmonically with the standing waves of the zero-point field, and with the latter being co-determined by the experimental setup, a sudden change in the corresponding standing wave pattern will “practically instantaneously” (i.e., during times of the order \( t \sim \frac{1}{\omega} \)) affect the particle. In the case of the particle at one slit being affected by the opening of the other slit, therefore, it does not matter how far apart the two slits are: As long as the emerging standing wave configurations result in a “systemic nonlocality” (which is a basic assumption of our model with regard to the physical meaning of the zero-point field), the modular remainder \( \Delta X \) will make itself visible in a shift of the particle’s velocity distribution. Although framed in a “systemically nonlocal” setup, therefore, the latter will just reflect the “local” shift \( \Delta X \) in the vicinity of the particle. As this is accompanied by a complete uncertainty of the relative phase, no signal can in practice be extracted from this process. This constitutes again an explanation of the momentum transfer which excludes superluminal signaling in practice, but not in principle. However, in this scenario the simultaneity (i.e., instantaneousness) of nonlocal effects as described by quantum theory is lost. Still, this is very likely a problem that can be pinned down to the problem of too idealistic applications of concepts of relativity. In other words, the emergent quantum mechanics we envisage may very well have to be accompanied by an emergent relativity as it is discussed, for example, within the paradigm of superstatistics mentioned above. Particularly with regard to a “double special relativity” [26, and Jizba and Scardigli, this volume], classical relativity may turn out as a limiting case of a more general approach, just as ordinary quantum theory may be a limiting case of a “deeper level theory” as envisaged by emergent quantum mechanics.

To conclude, we can now compare the metaphysical assumptions underlying standard quantum mechanics with those of our emergent quantum mechanics [29, 30]. Roughly, quantum theory may be classified via two types of approaches: i) indeterministic and ii) deterministic ones. The first type of approaches is consistent with standard relativity, while the second type is not. More specifically, the first type, which one can call standard quantum mechanics (i.e., including interpretations of its formalism like the Copenhagen or the deBroglie-Bohm versions), is characterized by nonlocal indeterminism, i.e., by a principal indeterminism that guarantees that nonlocality cannot be used for superluminal signaling. The second, deterministic type is characteristic for various forms of emergent quantum mechanics. These may either be characterized by local determinism with only apparent nonlocality (see, e.g., the approach of “stochastic electrodynamics” as in [31, 32]) or by a determinism with “systemic nonlocality”, as it characterizes one variant of our model presented here, for example.

Within our model, we discussed in the present paper two options for possible further elaboration. One option involves the idealized version of nonlocal diffusion wave fields. The latter describe a classical analogue representing nonlocality, which combined with an underlying ontic determinism implies “hidden” superluminal signaling (i.e., despite its epistemic impossibility due to the epistemic indeterminism). The other option is systemically nonlocal-like with only “practical instantaneousness” and thus superluminal signaling, which in combination with emergent relativity still may be constructed as free from causal paradoxes. Finally, as mentioned, a third option can be thought of. It would constitute a hybrid of models via the mixing of deterministic and indeterministic elements, which is suggested as a possibility by the assumed partial decoupling of wave and particle physics in our model. Thus, it is at least conceivable that the deterministic “systemic nonlocality” described in one of our discussed options refers only to the proposed wave-like physics, including that involving the boundary conditions, whereas the particle-like bouncer/walker has indeterministic degrees of freedom of its own. In this case, one would have a “nonlocal momentum transfer” which excludes superluminal signaling both in practice and in principle. Hence, the whole theory would effectively be indeterministic and comply with standard relativity.
References

[1] Grüssing G, Fussy S, Mesa Pascasio J and Schwabl H 2012 Ann. Phys. 327(2) 421–437 (Preprint quant-ph/1106.5994v3)
[2] Tollaksen J, Aharonov Y, Casher A, Kaufherr T and Nussinov S 2010 New J. Phys. 12(1) 013023 (Preprint quant-ph/0910.4227v1)
[3] Aharonov Y, Pendleton H and Petersen A 1969 Int. J. Theor. Phys. 2(3) 213–230
[4] Aharonov Y, Pendleton H and Petersen A 1970 Int. J. Theor. Phys. 3(6) 443–448
[5] Spence S and Parks A 2012 Found. Phys. 42(6) 803–815 (Preprint quant-ph/1010.3289v1)
[6] Grüssing G 2008 Phys. Lett. A 372(25) 4556–4563 (Preprint quant-ph/0711.4954v2)
[7] Grüssing G 2009 Physica A 388 811–823 (Preprint quant-ph/0808.3539v1)
[8] Grüssing G, Fussy S, Mesa Pascasio J and Schwabl H 2010 Physica A 389(21) 4473–4484 (Preprint quant-ph/1004.4596v1)
[9] Grüssing G, Fussy S, Mesa Pascasio J and Schwabl H 2011 J. Phys.: Conf. Ser. 306 012046 (Preprint physics.gen-ph/1005.1058v2)
[10] Grüssing G, Mesa Pascasio J and Schwabl H 2011 Found. Phys. 41(9) 1437–1453 (Preprint quant-ph/0812.3561v4)
[11] Grüssing G, Fussy S, Mesa Pascasio J and Schwabl H 2012 J. Phys.: Conf. Ser. 361 012008 (Preprint quant-ph/1205.3393v1)
[12] Couder Y, Protière S, Fort E and Boudaoud A 2005 Nature 437 208–208
[13] Couder Y and Fort E 2006 Phys. Rev. Lett. 97 154101
[14] Couder Y, Boudaoud A, Protière S and Fort E 2010 Europhys. News 41(1) 5
[15] Couder Y and Fort E 2012 J. Phys.: Conf. Ser. 361(1) 012001
[16] Bohm D and Hiley B J 1993 The undivided universe: An ontological interpretation of quantum theory repr. (London: Routledge) ISBN 0415065887
[17] Holland P R 1993 The Quantum Theory of Motion: An account of the de Broglie-Bohm causal interpretation of quantum mechanics (Cambridge: Cambridge University Press) ISBN 0-521-35404-8
[18] de Broglie L V P R 1960 Non-Linear Wave Mechanics: A Causal Interpretation. (Amsterdam: Elsevier)
[19] Hall M J W and Reginatto M 2002 J. Phys. A: Math. Gen. 35(14) 3289–3303
[20] Fort E, Eddi A, Boudaoud A, Moukhtar J and Couder Y 2010 PNAS 107(41) 17515–17520
[21] Mandelis A, Nicolaides L and Chen Y 2001 Phys. Rev. Lett. 87(2) 020801
[22] Sanz A S and Miret-Artés S 2008 J. Phys. A: Math. Gen. 41(43) 435303 (Preprint quant-ph/0806.2105)
[23] Mesa Pascasio J, Fussy S, Schwabl H and Grüssing G 2013 to be published in Physica A
[24] Rauch H 1993 Phys. Lett. A 173(3) 240–242
[25] Mandelis A 2001 Diffusion-wave fields: Mathematical methods and Green functions (New York, NY: Springer) ISBN 0387951490
[26] Jizba P and Scardigli F 2012 Phys. Rev. D 86(2) 025029 (Preprint hep-th/1105.3930)
[27] Grüssing G 2004 Found. Phys. Lett. 17(4) 343–362 (Preprint quant-ph/0311109v2)
[28] Grüssing G 2010 Entropy 12(9) 1975–2044
[29] Walleczek J 2013 to be published in Found. Phys.
[30] Walleczek J and Grüssing G 2013 to be published in Found. Phys.
[31] de la Peña L, Valdés-Hernández A, Cetto A and Franco H 2011 Phys. Lett. A 375(16) 1720–1723
[32] Cetto A M and de la Peña L 2012 J. Phys.: Conf. Ser. 361(1) 012013