EXTREMAL ERGODIC MEASURES AND
THE FINITENESS PROPERTY OF MATRIX SEMIGROUPS

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Abstract. Let \( S = \{S_1, \ldots, S_K\} \) be a finite set of complex \( d \times d \) matrices and \( \Sigma^+_K \) be the compact space of all one-sided infinite sequences \( i : \mathbb{N} \to \{1, \ldots, K\} \). An ergodic probability \( \mu^* \) of the Markov shift \( \theta : \Sigma^+_K \to \Sigma^+_K; i \mapsto i_{+1} \), is called “extremal” for \( S \) if \( \rho(S) = \lim_{n \to \infty} n^{1/n} \sqrt{\|S_{i_1} \cdots S_{i_n}\|} \) holds for \( \mu^* \)-a.e. \( i \in \Sigma^+_K \), where \( \rho(S) \) denotes the generalized/joint spectral radius of \( S \).

Using the extremal norm and the Kingman subadditive ergodic theorem, it is shown that \( S \) has the spectral finiteness property (i.e. \( \rho(S) = n^{1/n} \sqrt{\rho(S_{i_1} \cdots S_{i_n})} \) for some finite-length word \( (i_1, \ldots, i_n) \) ) if and only if for some extremal measure \( \mu^* \) of \( S \), it has at least one periodic density point \( i \in \Sigma^+_K \).

1. Introduction

We consider an arbitrary finite set of complex, \( d \times d \), matrices \( S = \{S_1, \ldots, S_K\} \), where \( d, K \) both are integers with \( 2 \leq d, K < +\infty \). Let \( \Sigma^+_K \) be the compact topological space of all the one-sided infinite sequences \( i : \mathbb{N} \to \{1, \ldots, K\} \) with the product topology, where \( \mathbb{N} = \{1, 2, \ldots\} \). Recall that the generalized spectral radius of \( S \), first introduced by Daubechies and Lagarias in [11], is defined as

\[
\rho(S) = \lim_{n \to \infty} \sup_{i \in \Sigma^+_K} \rho(S_{i_1} \cdots S_{i_n}) \quad \text{ (} = \sup_{n \geq 1, i \in \Sigma^+_K} n^{1/n} \sqrt{\rho(S_{i_1} \cdots S_{i_n})} \text{)}.
\]

Here \( \rho(A) \) stands for the usual spectral radius for an arbitrary matrix \( A \in \mathbb{C}^{d \times d} \). Another critical characterization of all infinite products of the matrices of \( S \) is the so-called joint spectral radius of \( S \), which appeared initially in Rota and Strang [18], and which is given as

\[
\hat{\rho}(S) = \lim_{n \to \infty} \sup_{i \in \Sigma^+_K} \sqrt{\|S_{i_1} \cdots S_{i_n}\|} \quad \text{ (} = \inf_{n \geq 1, i \in \Sigma^+_K} n^{1/n} \sqrt{\|S_{i_1} \cdots S_{i_n}\|} \text{)}.
\]

where \( \|\cdot\| \) can be any matrix norm satisfying the submultiplicativity (also called the ring) property, i.e., \( \|AB\| \leq \|A\| \cdot \|B\| \) for all \( A, B \in \mathbb{C}^{d \times d} \). According to the Berger-Wang formula [2], it follows that

\[
\rho(S) = \hat{\rho}(S).
\]

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The generalized/joint spectral radius has been the subject of substantial recent research interest in many pure and applied mathematical branches, such as matrix analysis, control theory, wavelets, and so on. If one can find some finite-length word, say \((i_1, \ldots, i_n)\), such that \(\rho(S) = \sqrt[n]{\rho(S_{i_1} \cdots S_{i_n})}\), then \(S\) is said to have the spectral finiteness property. It has been known that the finiteness property does not hold in general. But what key condition implies the finiteness property remains unclear in the theory of joint spectral radius. A brief survey for some recent progress regarding this can be found in Dai and Kozyakin [10, Section 1.2].

Although the finiteness property failed to exist, the idea is still attractive and important due to the fact that algorithms for the computation of the joint spectral radius must be implemented in finite arithmetic. Thus criteria for determining if a given matrix family satisfies the finiteness property are critical for us to develop a decidable algorithm for the joint/generalized spectral radius.

In this paper, we will study the spectral finiteness property of \(S\) via ergodic theory. In [9], Dai et al. showed that there exists an ergodic measure \(\mu_*\) of the one-sided Markov shift transformation, \(\Sigma^+\). Our aim in this paper is to show that \(S\) has the finiteness property if and only if there is an extremal measure \(\mu_*\) that has a periodic density point, as shown by our main theorem given in Section 4.

2. Preliminaries

In this section, we will introduce some preliminary notation and lemmas needed for the Main Theorem in Section 4.

2.1. Ergodic measures. Let \(K \geq 2\) be an arbitrary integer. We equip the finite set \(\{1, \ldots, K\}\) with the usual discrete topology. Let \(\Sigma^+_K\) be the one-sided symbolic space that consists of all the one-sided infinite sequences \(i: \mathbb{N} \to \{1, \ldots, K\}\). From the classical Tychonoff product theorem, \(\Sigma^+_K\) is a compact topological space. For any word \((i^*_1, \ldots, i^*_n)\) \(\in \{1, \ldots, K\}^n\) of finite length \(n \geq 1\), it is well known that the corresponding cylinder set

\[ [i^*_1, \ldots, i^*_n] = \{ i. \in \Sigma^+_K \mid i_1 = i^*_1, \ldots, i_n = i^*_n \} \]

is an open and closed subset of \(\Sigma^+_K\). One can easily check that the set of all the cylinders forms a base for the product topology of \(\Sigma^+_K\). Then, the classical one-sided Markov shift transformation,

\[ \theta: \Sigma^+_K \to \Sigma^+_K, \ i. \mapsto i_{+1}, \]

is continuous. By \(\mathcal{B}\) we denote the Borel \(\sigma\)-field of the compact symbolic space \(\Sigma^+_K\).

A probability measure \(\mu\) on \((\Sigma^+_K, \mathcal{B})\) is called \(\theta\)-invariant if \(\mu(\theta^{-1}(B)) = \mu(B)\) for all \(B \in \mathcal{B}\). A \(\theta\)-invariant measure \(\mu\) is called ergodic if the only members \(B\) of \(\mathcal{B}\) with \(\mu(B \triangle \theta^{-1}(B)) = 0\) satisfy \(\mu(B) = 0\) or \(\mu(B) = 1\). See [17, 20].

For any probability measure \(\mu\) on \((\Sigma^+_K, \mathcal{B})\), a point \(i^*_n \in \Sigma^+_K\) is called a density point of \(\mu\) if for every open neighborhood \(U\) around \(i^*_n\), it follows that \(\mu(U) > 0\). Then, \(i^*_n = (i^*_n)^{+\infty}_{n=1}\) is a density point of \(\mu\) if and only if \(\mu((i^*_1, \ldots, i^*_n)) > 0\) for all \(n \geq 1\). Let us denote by \(\text{supp}(\mu)\) the set of all the density points of \(\mu\) in \(\Sigma^+_K\).
The following lemma is important for our later discussion.

**Lemma 2.1** ([17][20]). Let \( \mu \) be a \( \theta \)-ergodic probability on \( (\Sigma_K^+, \mathcal{B}) \). Then, \( \text{supp}(\mu) \), called the “support of \( \mu \)”, is the minimal \( \theta \)-invariant closed set of \( \mu \)-measure 1.

An infinite sequence \( i_\cdot = (i_n)_{n=1}^{+\infty} \in \Sigma_K^+ \) is said to be periodic of period \( \pi \geq 1 \) if \( i_{n+\pi} = i_n \), namely, \( i_{n+\pi} = i_n \) for all \( n \in \mathbb{N} \). Let \( \delta_i \) be the Dirac probability measure concentrated at the point \( i_\cdot \in \Sigma_K^+ \). Then, for a periodic point \( i_\cdot \) of period \( \pi \),

\[
\mathbb{P}_{i_\cdot} := \pi^{-1} (\delta_{i_1} + \cdots + \delta_{i_{\pi-1}}) \quad (= \pi^{-1} (\delta_{i_{\pi+1}} + \cdots + \delta_{i_{\pi+\pi}}))
\]

is the unique \( \theta \)-ergodic probability measure whose support can be readily seen by

\[
\text{supp}(\mathbb{P}_{i_\cdot}) = \{i_1, \ldots, i_{\pi-1}\} \quad (= \{i_{\pi+1}, \ldots, i_{\pi+\pi}\}).
\]

Thus, in the periodic case, the support is quite simple. However, for a general \( \theta \)-ergodic probability measure \( \mu \), the topological structure of its support may become very complicated and the \( \mu \)-almost sure stability of a linear switched dynamical system induced by \( S \) relies on this structure; for example, see [7][8]. Our results obtained in this paper further show that it is closely related to the finiteness property of a finite set of matrices.

Given two sequences \( \sigma_\cdot = (i_n)_{n=1}^{+\infty} \) and \( \xi_\cdot = (i'_n)_{n=1}^{+\infty} \) belonging to \( \Sigma_K^+ \), \( \xi_\cdot \) is called an \( \omega \)-limit point of \( \sigma_\cdot \) under \( \theta \), provided that there is an infinite increasing subsequence \( \{n_\ell\}_{\ell=1}^{+\infty} \) of \( \mathbb{N} \) satisfying

\[
\xi_\cdot = \lim_{\ell \to +\infty} \theta^{n_\ell}(\sigma_\cdot) \quad (= \lim_{\ell \to +\infty} \sigma_{\cdot + n_\ell}).
\]

Below is a lemma needed in the sequel.

**Lemma 2.2.** Let \( \sigma_\cdot \in \text{supp}(\mu) \) for a \( \theta \)-ergodic probability measure \( \mu \) on \( \Sigma_K^+ \), and let \( \xi_\cdot \in \Sigma_K^+ \) be a periodic sequence. If \( \xi_\cdot \) is an \( \omega \)-limit point of \( \sigma_\cdot \) under \( \theta \), then \( \xi_\cdot \) belongs to \( \text{supp}(\mu) \).

**Proof.** Let us consider the closure of the orbit of \( \theta \) passing through \( \sigma_\cdot \):

\[
\mathcal{O} = \text{cls}\{\sigma_{\cdot + n} \mid n = 0, 1, 2, \ldots\}.
\]

Clearly, \( \xi_\cdot \) belongs to \( \mathcal{O} \). Since by Lemma 2.1 \( \text{supp}(\mu) \) is \( \theta \)-invariant, we can get \( \mathcal{O} \subseteq \text{supp}(\mu) \). This proves Lemma 2.2.\)  \( \square \)

2.2. **Subadditive ergodic theorem.** If \( f : X \to \mathbb{R} \cup \{-\infty\} \) is a function, we simply put \( f^+(x) = \max\{0, f(x)\} \) for any \( x \in X \). We need to employ another classical ergodic theorem stated as follows.

**Kingman’s Subadditive Ergodic Theorem** ([15][20]). Let \( T \) be a measure-preserving transformation of a probability space \( (X, \mathcal{B}, \mu) \) into itself. Let \( \{f_\ell\}_{\ell \geq 1} \) be a sequence of measurable functions

\[
f_\ell : X \to \mathbb{R} \cup \{-\infty\}
\]

satisfying the conditions:

1. \( f_1^+ \in L^1(\mu) \);
2. for each \( \ell, m \geq 1 \), \( f_{\ell+m}(x) \leq f_\ell(x) + f_m(T^\ell(x)) \) for \( \mu \)-a.e. \( x \in X \).
Then, there exists a measurable function
\[ \tilde{f} : X \to \mathbb{R} \cup \{-\infty\} \]
such that
\[ \tilde{f}^+ \in L^1(\mu), \quad \tilde{f} \circ T = \tilde{f} \text{ a.e.}, \quad \lim_{n \to +\infty} \frac{1}{n} f_n = \tilde{f} \text{ a.e.}, \]
and
\[ \lim_{n \to +\infty} \frac{1}{n} \int_X f_n \, d\mu = \inf_{n \geq 1} \frac{1}{n} \int_X f_n \, d\mu = \int_X \tilde{f} \, d\mu. \]

Note 1. If \( \mu \) is \( T \)-ergodic, then \( \tilde{f}(x) \equiv \) constant for \( \mu \)-a.e. \( x \in X \).

Note 2. This result in fact implies the classical Multiplicative Ergodic Theorem.

2.3. A realization of the joint spectral radius via ergodic measures. The foundation of our argument later is the following realization theorem of the joint (or generalized) spectral radius:

**Theorem 2.3** ([9]). Let \( S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d} \) be arbitrary. Then, there is at least one ergodic Borel probability measure \( \mu^* \) of the one-sided Markov shift \( \theta : \Sigma_K^+ \to \Sigma_K^+ \), which is extremal for \( S \).

The extremal measure is useful for the study of dynamical behaviors of \( S \), such as the rotation number of extremal trajectories and extremal switching sequences of \( S \), as is shown in [16, 5].

3. Extremal norms

Let \( S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d} \), where \( d \geq 2 \) is an integer. As a special case of the main result of Rota and Strang [18], we have
\[ \hat{\rho}(S) = \inf_{\|\cdot\| \in \mathcal{N}} \max_{1 \leq k \leq K} \|S_k\|, \]
where \( \mathcal{N} \) denotes the set of all possible matrix norms for \( \mathbb{C}^{d \times d} \) induced by vector norms on \( \mathbb{C}^d \); also see [12] [19] for a short proof. Hence, an important problem is whether or not the above infimum is actually attained by some induced matrix norm \( \|\cdot\| \). For this, a norm \( \|\cdot\| \) on \( \mathbb{C}^d \) satisfying the condition
\[ \hat{\rho}(S) = \max_{1 \leq k \leq K} \|S_k\| \left( = \max_{i \in \Sigma_K^+} \sqrt[n]{\|S_{i_1} \cdots S_{i_n}\|} \forall n \geq 1 \right) \]
is called an extremal norm of \( S \). From Barabanov’s extremal norm theorem [11], one can see that if \( S \) is “irreducible”, i.e., there are no nontrivial, common, and proper subspaces of \( \mathbb{C}^d \) for each member \( S_k \) of \( S \), then there exists an extremal norm for \( S \) on \( \mathbb{C}^d \). Here the irreducibility is crucial for Barabanov’s theorem as shown by a simple counterexample
\[ S = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}, \]
which does not have any extremal norms. However, there always exists a lower-dimensional extremal norm.
Theorem 3.1. For $S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d}$, there exists an invariant linear subspace $E$ of $\mathbb{C}^d$ on which there is a norm $\| \cdot \|$ such that

$$\hat{\rho}(S) = \hat{\rho}(S \mid E) = \max_{1 \leq k \leq K} \|S_k \mid E\|.$$  

Proof. If $\rho(S) = 0$, then we simply let $E = \{0\}$, where 0 is the origin of $\mathbb{C}^d$. So, it may be assumed, without loss of generality, that $\rho(S) > 0$. We denote by $\tilde{S}$ the set $\{S_1/\rho(S), \ldots, S_K/\rho(S)\} \subset \mathbb{C}^{d \times d}$. Clearly, $\rho(\tilde{S}) = 1$.

If the multiplicative semigroup $\tilde{S}^+$, generated by $\tilde{S}$, is bounded in $\mathbb{C}^{d \times d}$, then there exists an extremal norm for $S$ on $\mathbb{C}^d$ (see, e.g., [21]) and we define $E = \mathbb{C}^d$ in this case. Otherwise, suppose that $\tilde{S}^+$ is unbounded. Applying Elsner’s reduction theorem [12] repeatedly, there exists a nonsingular matrix $P \in \mathbb{C}^{d \times d}$ and $2 \leq r \leq d$ such that

$$P^{-1}\tilde{S}_kP = \begin{bmatrix} \tilde{S}_k^{(1)} & 0 & \cdots & 0 \\ \blacklozenge_k & \tilde{S}_k^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & \tilde{S}_k^{(r)} \end{bmatrix},$$

where $\tilde{S}_k^{(j)}$ is $d_j \times d_j$ for $1 \leq j \leq r$, $d_1 + \cdots + d_r = d$ and $\tilde{S}(j) := \{\tilde{S}_k^{(j)}, \ldots, \tilde{S}_K^{(j)}\}$ generates a bounded semigroup $\tilde{S}(j)^+$ in $\mathbb{C}^{d_j \times d_j}$. Let us notice here that

$$\rho(\tilde{S}) = 1 = \max_{1 \leq j \leq r} \rho(\tilde{S}(j)).$$

Thus there exists at least one $j$ such that $\rho(\tilde{S}(j)) = 1$. Now, from [21] Lemma 6.2 one can obtain the desired result. To be self-contained, we provide a detailed proof here.

Without loss of generality, we may assume that $\rho(\tilde{S}(1)) < 1$ and $\rho(\tilde{S}(2)) = 1$; this is because the other cases can always be reduced to this one. We now write

$$F = \left\{ F_k = \begin{bmatrix} \tilde{S}_k^{(1)} \\ \blacklozenge_k \end{bmatrix} : 1 \leq k \leq K \right\}.$$  

Then, one can easily see that $\rho(F) = 1$ and the semigroup $F^+$ generated by $F$ is bounded in $\mathbb{C}^{(d_1+d_2) \times (d_1+d_2)}$. Then it follows that there is an extremal norm $\| \cdot \|$ for $F$ on $\mathbb{C}^{(d_1+d_2) \times (d_1+d_2)}$. This implies that

$$\hat{\rho}(S) = \sup_{1 \leq k \leq K} \|(P^{-1}S_kP) \mid \mathbb{C}^{d_1+d_2}\|.$$  

Finally, let $E = P^{-1}(\mathbb{C}^{d_1+d_2} \times \{0\})$, where 0 is the origin of $\mathbb{C}^{d_1+d_2}$ and define

$$\|x\| = \|P(x)\| = \|xP\| \quad \forall x \in E,$$

where $x \in \mathbb{C}^d$ is viewed as a row vector. Clearly, such an $E$ and $\| \cdot \|$ satisfy the requirement.

This completes the proof of Theorem 3.1. □

Another generalization of Barabanov’s theorem can be found in [6]. In addition, to prove our main theorem, we will need a reduction theorem of the ergodic version.
Let \( \text{Main Theorem.} \) can be stated as follows:  

For any \( \| \cdot \| \) satisfying \( 1 \leq \| \cdot \| \leq 2 \leq r \leq d \) such that 

\[
P^{-1}S_kP = \begin{bmatrix} S_k^{(1)} & 0 & \cdots & 0 \\ * & S_k^{(2)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & S_k^{(r)} \end{bmatrix}, \quad 1 \leq k \leq K,
\]

where 

\[
\tilde{S}(j) = \left\{ \tilde{S}_1^{(j)}, \ldots, \tilde{S}_K^{(j)} \right\} \subset \mathbb{C}^{d_j \times d_j}
\]

is irreducible for every \( 1 \leq j \leq r, d_1 + \cdots + d_r = d \), and that there exists some \( 1 \leq j \leq r \) satisfying that \( \rho(S) = \rho(S(j)) \) and \( \mu_* \) is also extremal for \( \tilde{S}(j) \).

4. Spectral finiteness property

In this section, we will present a sufficient and necessary condition for the spectral finiteness property of a finite set \( S \) of \( d \)-by-\( d \) matrices. The main result of this paper can be stated as follows:

**Main Theorem.** Let \( S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d} \) be an arbitrary finite family of matrices. Then, the following two statements are equivalent to each other.

1. \( S \) has the spectral finiteness property; i.e., there is a word \((i_1^*, \ldots, i_n^*)\) in \( \{1, \ldots, K\}^n \) for some \( n \geq 1 \) such that 
\[
\rho(S) = \sqrt[n]{\rho(S_{i_1^*} \cdots S_{i_n^*})}.
\]

2. There is an extremal \( \theta \)-ergodic probability measure \( \mu_* \) for \( S \) on \( \Sigma_K^+ \), which has a periodic density point \( \xi \in \Sigma_K^+ \).

According to the realization theorem (Theorem 1.3), there always exists at least one extremal \( \theta \)-ergodic probability measure \( \mu_* \) for \( S \), on the symbolic space \( \Sigma_K^+ \).

**Proof.** If \( \rho(S) = 0 \), then \( \rho(S_k) = 0 \) for all \( 1 \leq k \leq K \). So in this case, the statement of the Main Theorem holds trivially. Thus we next assume \( \rho(S) > 0 \).

1. \( \Rightarrow \) (2). Let there exist a word \((i_1^*, \ldots, i_n^*) \in \{1, \ldots, K\}^\pi \) for some \( \pi \geq 1 \) satisfying 
\[
\rho(S) = \sqrt[n]{\rho(S_{i_1^*} \cdots S_{i_n^*})}.
\]

Then we consider the periodic switching sequence 
\[
\xi. = (i_n^*)_{n=1}^{+\infty} \in \Sigma_K^+ \quad \text{with} \quad i_{n+\ell} = i_n^* \quad \forall \ell \geq 0 \quad \text{and} \quad 1 \leq n \leq \pi.
\]

For any \( \ell \geq 0 \), we have \( \xi_{+\ell} = (i_{n+\ell})_{n=1}^{+\infty} \) and further from the classical Gel’fand spectral radius formula it follows that 
\[
\lim_{n \to +\infty} \sqrt[n]{\|S_{i_1+\ell} \cdots S_{i_{n+\ell}}\|_2} = \rho(S),
\]

where \( \| \cdot \|_2 \) denotes the matrix norm induced by the standard Euclidean vector norm on \( \mathbb{C}^d \). Thus, 
\[
\mathbb{P}_{\xi.} = \pi^{-1} \left( \delta_{\xi.} + \cdots + \delta_{\xi. + \pi - 1} \right)
\]
is an extremal \( \theta \)-ergodic probability measure for \( S \).
(2) \Rightarrow (1). First of all, there is no loss of generality in assuming \( \rho(S) = 1 \) and supposing that \( S \) has an extremal \( \theta \)-ergodic probability measure \( \mu_* \) on \( \Sigma_K^+ \), which has a periodic density point \( i'_n = (i'_n)_{n=1}^{+\infty} \) with \( i'_{n+\pi} = i'_n \) for \( n \geq 1 \) and some \( \pi \geq 1 \). By Lemma [3.2] it might be assumed, without loss of generality, that \( S \) is irreducible and then it has an extremal norm \( \| \cdot \| \) on \( \mathbb{C}^d \) such that
\[
1 = \hat{\rho}(S) = \max_{1 \leq k \leq K} \| S_k \|.
\]

From the multiplicative ergodic theorem [13] and the Kingman subadditive ergodic theorem, it follows immediately that
\[
\log \rho(S) = \inf_{n \geq 1} \frac{1}{n} \int_{\Sigma_K^+} \log \| S_{i_1} \cdots S_{i_n} \| \, d\mu_*(i).
\]

Therefore,
\[
1 = \inf_{n \geq 1} \prod_{1 \leq i_1, \ldots, i_n \leq K} \| S_{i_1} \cdots S_{i_n} \|^{\mu_*([i_1, \ldots, i_n])/n},
\]

where \([i_1, \ldots, i_n]\) denotes the cylinder set defined by the word \((i_1, \ldots, i_n)\) as in Section 2.1. Hence, we have
\[
1 \leq \prod_{1 \leq i_1, \ldots, i_n \leq K} \| S_{i_1} \cdots S_{i_n} \|^{\mu_*([i_1, \ldots, i_n])/n} \quad \forall n \geq 1.
\]

Since \( i'_n = (i'_n)_{n=1}^{+\infty} \) is a density point of \( \mu_* \) with \( i'_{n+\pi} = i'_n \) for all \( n \geq 1 \), we can obtain that
\[
1 = \sqrt[n]{\| (S_{i'_1} \cdots S_{i'_n})^n \|} \quad \forall n \geq 1.
\]

This implies from the Gel’fand formula that
\[
1 = \sqrt[n]{\rho(S_{i'_1} \cdots S_{i'_n})} \leq \rho(S),
\]

which shows that \( S \) has the spectral finiteness property.

The proof of the Main Theorem is thus completed. \( \square \)

For any irreducible transition probability matrix \( P = (p_{ij}) \in \mathbb{R}^{K \times K} \) with a stationary distribution \( \mathbf{p} = (p_1, \ldots, p_K) \) (in which all \( p_k > 0 \)), one can define a \( \theta \)-ergodic probability measure \( \mu_{\mathbf{p}, P} \) on \( \Sigma_K^+ \) in the following way:
\[
\mu_{\mathbf{p}, P}([i_1]) = p_{i_1} \quad \text{for } n = 1 \quad \text{and} \quad \mu_{\mathbf{p}, P}([i_1, \ldots, i_n]) = p_{i_n} p_{i_1 i_2} \cdots p_{i_{n-1} i_n} \quad \text{for } n \geq 2,
\]

for all words \((i_1, \ldots, i_n) \in \{1, \ldots, K\}^n\). Such a \( \mu_{\mathbf{p}, P} \) is called a canonical \((\mathbf{p}, P)\)-Markovian probability, which is \( \theta \)-ergodic (cf. [20] Theorem 1.13).

The results of our Main Theorem lead to the following two corollaries.

**Corollary 4.1.** Let \( S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d} \). If \( \mu_{\mathbf{p}, P} \) is extremal for \( S \), then \( S \) has the spectral finiteness property.

**Proof.** According to the classical theory of symbolic dynamics, \( \text{supp}(\mu_{\mathbf{p}, P}) = \Sigma_K^+ \). Combining with the Main Theorem, this completes the proof of Corollary 4.1. \( \square \)

**Corollary 4.2.** Let \( S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d} \) satisfy the periodically switched stability condition:
\[
\rho(S_{i_1} \cdots S_{i_n}) < 1 \quad \forall (i_1, \ldots, i_n) \in \{1, \ldots, K\}^n \text{ and } n \geq 1.
\]

If \( \mu_{\mathbf{p}, P} \) is extremal for \( S \), then \( \rho(S) < 1 \).
Proof. According to Corollary 4.1, it follows that $S$ has the spectral finiteness property, and thus $\rho(S) < 1$ by the periodically switched stability. Therefore, the conclusion follows. □

5. Concluding remarks

In this paper, we have proved that $S = \{S_1, \ldots, S_K\} \subset \mathbb{C}^{d \times d}$ has the spectral finiteness property if and only if there exists a periodic switching sequence in the support of some extremal ergodic measure $\mu_*$ of $S$. This result reveals that the topological structure of an extremal (ergodic) measure is closely related to the finiteness property, and the dynamical stability of the linear switched dynamical system induced by $S$ is characterized by this topological structure.

An explicit important example can further illustrate this. Let us consider

$$S(\alpha) = \left\{ \alpha \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}, \text{ where } \alpha \in \mathbb{R}.$$

From [4, 3, 16, 14], it follows that there is an $\alpha > 0$ such that $S(\alpha)$ does not have the spectral finiteness property. Thus, according to the main theorem given in this paper, each extremal $\theta$-ergodic probability of $S(\alpha)$ does not have any periodic density points.

For the symbolic sequence space $\Sigma_K^+$, the observable measures are the canonical Markovian probabilities $\mu_{p,P}$ as defined in Section 4 and such measures all have periodic density points. Therefore, it is necessary to study the topological structure of those nonobservable measures on $\Sigma_K^+$. This indicates that one has to look at these nonobservable measures on $\Sigma_K^+$ in order to disprove the finiteness property of $S$. In the current literature, the disapproval of the finiteness property essentially makes use of this fact implicitly [4, 3, 16, 14].

References

[1] N. Barabanov, Lyapunov exponents of discrete inclusions I–III, Autom. Remote Control, 49 (1988), 152–157, 283–287, 558–565. MR0940263 (89e:34025); MR0943889 (89e:34026); MR0952665 (89m:34060)
[2] M.A. Berger and Y. Wang, Bounded semigroups of matrices, Linear Algebra Appl., 166 (1992), 21–27. MR1152485 (92m:15012)
[3] V.D. Blondel, J. Theys and A.A. Vladimirov, An elementary counterexample to the finiteness conjecture, SIAM J. Matrix Anal. Appl., 24 (2003), 963–970. MR2003315 (2004g:15010)
[4] T. Bousch and J. Mairesse, Asymptotic height optimization for topical IFS, Tetris heaps and the finiteness conjecture, J. Amer. Math. Soc., 15 (2002), 77–111. MR1862798 (2002j:49008)
[5] X. Dai, Weakly Birkhoff recurrent switching signals, almost sure and partial stability of linear switched dynamical systems, J. Differential Equations, 250 (2010), 3584–3629. MR2773179
[6] X. Dai, Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices, J. Math. Anal. Appl., 379 (2011), 827–833. MR2784362
[7] X. Dai, Y. Huang and M. Xiao, Almost sure stability of discrete-time switched linear systems: A topological point of view, SIAM J. Control Optim., 47 (2008), 2137–2156. MR2421343 (2009g:93134)
[8] X. Dai, Y. Huang and M. Xiao, Periodically switched stability induces exponential stability of discrete-time linear switched systems in the sense of Markovian probabilities, Automatica, 47 (2011), 1512–1519.
[9] X. Dai, Y. Huang and M. Xiao, Realization of joint spectral radius via ergodic theory, Electron. Res. Announc. Math. Sci., 18 (2011), 22–30. MR2817101
[10] X. Dai and V. Kozyakin, Finiteness property of a bounded set of matrices with uniformly sub-peripheral spectrum, Information Processes, 11 (2011), 253–261 (Russian). English version in J. Commun. Technology and Electronics, 56 (2011), 1564–1569.

[11] I. Daubechies and J.C. Lagarias, Sets of matrices all infinite products of which converge, Linear Algebra Appl., 161 (1992), 227–263. Corrigendum/addendum 327 (2001), 69–83. MR1427377 (93f:15006) MR1823340 (2002b:15010)

[12] L. Elsner, The generalized spectral-radius theorem: An analytic-geometric proof, Linear Algebra Appl., 220 (1995), 151–159. MR1334574 (96c:15010)

[13] H. Furstenberg and H. Kesten, Products of random matrices, Ann. Math. Statist., 31 (1960), 457–469. MR0121828 (22:12558)

[14] K.G. Hare, I.D. Morris, N. Sidorov and J. Theys, An explicit counterexample to the Lagarias-Wang finiteness conjecture, Adv. Math., 226 (2011), 4667–4701. MR2775881

[15] J.F.C. Kingman, Subadditive ergodic theory, Ann. Probab., 1 (1973), 883–909. MR0356192 (50:8663)

[16] V.S. Kozyakin, Structure of extremal trajectories of discrete linear systems and the finiteness conjecture, Automat. Remote Control, 68 (2007), 174–209.

[17] V.V. Nemytskii and V.V. Stepanov, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, New Jersey, 1960. MR0121520 (22:12258)

[18] G.-C. Rota and G. Strang, A note on the joint spectral radius, Indag. Math., 22 (1960), 379–381. MR0147922 (26:5434)

[19] M.-H. Shih, J.-W. Wu and C.-T. Pang, Asymptotic stability and generalized Gelfand spectral radius formula, Linear Algebra Appl., 252 (1997), 61–70. MR1428628 (97k:15028)

[20] P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Math., 79, Springer-Verlag, New York, 1982. MR0648108 (84e:28017)

[21] F. Wirth, The generalized spectral radius and extremal norms, Linear Algebra Appl., 342 (2002), 17–40. MR1873424 (2003g:15025)

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