Abstract: The importance of proper tail risk management is a crucial component of the investment process and conditional Value at Risk (CVaR) is often used as a tail risk measure. CVaR is the asymmetric risk measure that controls and manages the downside risk of a portfolio while symmetric risk measures such as variance consider both upside and downside risk. In fact, minimum CVaR portfolio is a promising alternative to traditional mean-variance optimization. However, there are three major challenges in the minimum CVaR portfolio. Firstly, when using CVaR as a risk measure, we need to determine the distribution of asset returns, but it is difficult to actually grasp the distribution; therefore, we need to invest in a situation where the distribution is uncertain. Secondly, the minimum CVaR portfolio is formulated with a single $\beta$ and may output significantly different portfolios depending on the $\beta$. Finally, most portfolio allocation strategies do not account for transaction costs incurred by each rebalancing of the portfolio. In order to improve these challenges, we propose a Regularized Multiple $\beta$ Worst-case CVaR (RM-WCVaR) portfolio. The characteristics of this portfolio are as follows: it makes CVaR robust with worst-case CVaR which is still an asymmetric risk measure, it is stable among multiple $\beta$, and against changes in weights over time. We perform experiments on well-known benchmarks to evaluate the proposed portfolio. RM-WCVaR demonstrates superior performance of having both higher risk-adjusted returns and lower maximum drawdown.

Keywords: RM-WCVaR; tail risk; portfolio optimization

1. Introduction

The problem of finding the optimum portfolio for investors is known as a portfolio optimization problem. The portfolio optimization problem has been an important research theme, both academically and practically as it is a crucial part of managing risk and maximizing returns from a set of investments. The classical portfolio optimization approach is mean-variance optimization (MVO), which mainly concerns the expectation and variability of return (i.e., mean and variance [1]). Although the variance would be the most fundamental risk measure to be minimized, it has a crucial drawback: variance is a symmetric risk measure. Controlling the variance leads to a low deviation from the expected return with regard to both the downside and the upside.

Hence, asymmetric risk measures such as the Value-at-Risk (VaR) measure, which controls and manages the downside risk in terms of percentiles of the loss distribution of portfolio, have been proposed [2].

Instead of considering both the upside and downside of the expected return, the VaR risk measure focuses on only the downside of the expected return as the risk and represents the predicted maximum loss with a specified confidence level $\beta$ (e.g., 99%). VaR became so popular that it was approved as a valid approach for calculating risk charges by bank regulators such as the Basel Accord II [3].

However, the VaR measure, if studied in the framework of coherent risk measures [4], lacks subadditivity, and, therefore, convexity in the case of general loss distributions. This drawback entails both inconsistencies with the well-accepted principle of portfolio
diversification, i.e., diversification reduces risk. The VaR risk measure is non-convex and
not smooth, making it difficult to optimize [5]. To reduce the computational burden of
minimizing VaR, ref. [5] proposed a new mixed integer LP optimization based on the
symmetric property of VaR. Besides, both variance and VaR ignore the magnitude of
extreme or rare losses by their definition. Both risk measures cannot deal with extremely
unlikely, but potentially catastrophic, events i.e., managing the tail risk [6].

The Conditional VaR (CVaR) risk measure responds to the aforementioned drawbacks
of variance and VaR. CVaR is defined as the expected value of the portfolio loss that occurs
beyond a certain probability level \( \beta \). Obviously, CVaR is a more conservative risk measure
than VaR. In [7], it was proven that the CVaR risk measure is a coherent risk measure
that exhibits subadditivity and convexity. Additionally, the minimum CVaR portfolio that
minimizes the CVaR results in a tractable optimization problem [8,9]. For example, when
the portfolio loss is defined as the minus return of the portfolio, and a finite number of
historical observations of returns are used in estimating CVaR, its minimization problem
can be presented as a Linear programming (LP) optimization and can be solved efficiently.
The minimum CVaR portfolio is a promising alternative to MVO for those reasons. In fact,
the effectiveness of CVaR in portfolio construction designs has been demonstrated in a
large number of recently published contributions, including index tracking and enhanced
indexing [10–12].

However, there are three major challenges in the minimum CVaR portfolio. Firstly,
when using CVaR as a risk measure, we need to determine the distribution of asset returns,
but it is difficult to actually grasp the distribution; therefore, we need to invest in a situation
where the distribution is uncertain [13–15]. Secondly, the minimum CVaR portfolio is
formulated with a single \( \beta \) and may output significantly different portfolios depending
on how the \( \beta \) is selected [16]. In the context of MVO, this is called error maximization,
which is the phenomenon that even small changes in the inputs can result in huge changes
in the whole portfolio structure [17]. Thirdly, most portfolio optimization strategies do
not account for transaction costs incurred by each rebalancing of the portfolio [18]. When
buying and selling assets on the markets, commissions and other costs are incurred, such
as globally defined transaction costs that are charged by the brokers or the financial
institutions serving as intermediaries. Most of these transaction costs are incurred for
portfolio turnovers. Transaction costs represent the most important feature to consider
when selecting a real portfolio, given that they diminish net returns and reduce the amount
of capital available for future investments [19].

The objective of this study is to propose a new tail risk-controlling portfolio con-
struction method that addresses the above challenges and to confirm its performance. In
this paper, we propose Regularized Multiple \( \beta \) Worst-case CVaR (RM-WCVaR) Portfolio
Optimization. The characteristics of our portfolio are as follows. It makes CVaR robust
with worst-case CVaR (WCVaR), which is an asymmetric risk measure and used in sit-
uations where the information on the underlying probability distribution is not exactly
known [14,15]. Our portfolio is formulated with the multiple probability levels \( \beta \) of WC-
VaR not to depend on a single \( \beta \) level. Finally, to control transaction costs, we add the
\( L_1 \)-regularization term on the portfolio as stated in [18,20]. However, unlike these studies,
we impose \( L_1 \)-norm penalty on portfolio turnovers rather than portfolio weights.

We also prove that the RM-WCVaR Portfolio Optimization problem is written as an
LP optimization problem such as the single \( \beta \)-CVaR and WCVaR portfolio. We perform ex-
periments on well-known benchmarks to evaluate the proposed portfolio. Compared with
various portfolios, our portfolio demonstrates superior performance of having both higher
risk-adjusted returns and lower maximum drawdown despite the lower turnover rate.

In the following sections, we first review the existing methods in Section 2. We formu-
late the VaR, CVaR, and WCVaR risk measures and the portfolio optimization with them in
Section 3 and then, we propose the RM-WCVaR Portfolio in Section 4 and investigate the
empirical effectiveness of the our portfolio in Section 5. Finally, we conclude in Section 6.
2. Related Work

MVO assumes that investment decisions on getting a diversified portfolio depend on the two inputs: expected returns and the covariances of asset returns. However, as the estimation errors, mainly in expected return parameters, are amplified by optimization and then propagate into the solution of the optimization, extreme portfolio weights and a lack of diversification are commonly observed. This phenomenon has eventually ruined the out-of-sample performance of MVO [17,21]. To date, many efforts have been expended to handle the estimation risk on the parameter uncertainty. In order to reduce estimation error, the conventional regularization models have been applied for the MVO by [18,20]. They demonstrated superior portfolio performances when various types of norm regularities are combined into the optimization problem. Analogously, ref. [22] considered $L_1$ and $L_2$-norms for the mean-CVaR portfolio and [16] considered $L_1$-norms for the multiple CVaR portfolio. Our paper extends this regularization literature to a multiple WCVaR Portfolio.

Robust portfolio optimization is another approach considering the estimation error and has been receiving increased attention [23]. Recently, ref. [24] proposed a robust portfolio optimization approach based on quantile statistics. Robust optimization also has been adopted on the other portfolio optimization problems. In [13], robust portfolio optimization using worst-case VaR was investigated, where only partial information on the distribution was known. In [15], the concept of WC VaR was introduced for the situation where the probability distributions are only partially known, and the properties of WC VaR are studied, such as coherency. Another approach is using a semi-nonparametric distribution, which may asymptotically capture the true density. This approach has been successfully tested for CVaR [25,26]. Our paper extends this robust portfolio optimization literature to a multiple $\beta$ WCVaR portfolio.

Another direction to reduce the estimated error is to construct a risk-based portfolio that does not use expected returns. The minimum variance portfolio [27], risk parity portfolio [28,29], and maximum diversification portfolio [30] have been proposed as representative risk-based portfolio construction methods. The risk-based portfolio has the desirable property that the portfolio and its performance do not change greatly in response to changes to inputs [31,32]. Furthermore, an extension of each of them has been proposed such as minimum VaR and CVaR portfolio [33], risk and complex risk diversification portfolio [34,35] and higher order risk based portfolio [36]. Various empirical analyses and backtests of stock portfolios and asset allocations have shown better performance than mean-variance portfolios and market capitalization-weighted portfolios [37]. Our paper adds a minimum WCVaR-based portfolio to this risk-based portfolio construction literature.

3. Preliminary

In this section, we define VaR, CVaR, and WCVaR. After which, we formulate a minimum WCVaR portfolio optimization problem. Let $r_j$ be the return of stock $j$ ($1 \leq j \leq n$) and $w_j$ be the portfolio weight for stock $j$. We denote $r = (r_1, ..., r_n)^T$ and $w = (w_1, ..., w_n)^T$. Here, $r_j$ is a random variable and follows the continuous joint probability density function $p(r)$. $L(w, r)$ refers to portfolio loss function and throughout this paper, we assume $L(w, r) = -w^T r$. The probability that the loss function is less than $\alpha$ is

$$\Phi(w, \alpha) = \int_{L(w, r) \leq \alpha} p(r)dr$$

(1)

When the portfolio weight $w$ is fixed, $\Phi(w, \alpha)$ which is the function of $\alpha$ is non-decreasing and is continuous from the right, but is generally non-continuous from the left. For simplicity, we assume that $\Phi(w, \alpha)$ is a continuous function with respect to $\alpha$. We can define VaR and CVaR as follows.

**Definition 1.**

$$VaR(w|\beta) := a(w|\beta) = \min(\alpha : \Phi(w, \alpha) \geq \beta)$$

(2)
Definition 2. 
\[ \text{CVaR}(w|\beta) := \phi(w|\beta) = (1 - \beta)^{-1} \int_{L(w,r) \geq a(w|\beta)} L(w,r)p(r)dr \]  

Ref. [4] proposed the coherent risk measure, which characterizes the rationale of risk measure.

Definition 3. The risk measure \( \rho \) that maps random loss \( X \) to a real number and satisfies the below four conditions is called a coherent risk measure. 

**Subadditivity:** for all random losses \( X \) and \( Y \), \( \rho(X + Y) \leq \rho(X) + \rho(Y) \)

**Positive homogeneity:** for positive constant \( a \in \mathbb{R}^+ \), \( \rho(aX) = a\rho(X) \)

**Monotonicity:** if \( X \leq Y \) for each outcome, then \( \rho(X) \leq \rho(Y) \)

**Translation invariance:** for constant \( m \in \mathbb{R} \), \( \rho(X + m) = \rho(X) + m \)

It is well known that CVaR is a coherent risk measure and VaR is not a coherent risk measure as it does not satisfy the Subadditivity [6].

Next, we consider WCVaR. Rather than assuming exact knowledge of the return vector distribution, we presume that the density function \( p(\cdot) \) is only considered to belong to a certain set \( P \) of distributions, i.e., \( p(\cdot) \in P \).

The concept of the WCVaR is introduced in [15] as follows:

Definition 4. 
\[ \text{WCVaR}(w|\beta) := \sup_{p(\cdot) \in P} \text{CVaR}(w|\beta) \]  

Ref. [15] have shown that the WCVaR is a coherent risk measure as well as CVaR.

Hereafter, we assume that the return vector’s distribution is only considered to belong to a set of distributions that includes all mixtures of any predetermined density distributions, i.e.,

\[ P_M = \{ \sum_{i=1}^{l} \lambda_i p^{(i)}; \lambda_i \geq 0, \sum_{i=1}^{l} \lambda_i = 1, i = 1, \ldots, l \} \]  

where \( p^{(i)}(\cdot) \) denotes the \( i \)-th density distribution, and \( l \) denotes the number of the density distributions.

Since it is difficult to handle when the set \( P \) contains an infinite number of \( p^{(i)}(\cdot) \), we consider approximating \( P \) with a convex linear combination of a finite number of \( p^{(i)}(\cdot) \). In this study, the mixture of density distributions \( P_M \) is represented by blocks of divided empirical distributions.

To compute the WCVaR, we define the auxiliary function \( F^{(i)}(w,a|\beta) \) as
\[ F^{(i)}(w,a|\beta) = a + (1 - \beta)^{-1} \int_{R^l} [-w^\top r - a]^+ p^{(i)}(r)dr \]  

where \( i = 1, \ldots, l \) and \([t]^+ := \max(t,0)\). Then, the following lemma holds.

Lemma 1 (Ref. [15]). For an arbitrarily fixed \( w \) and \( \beta \), \( \text{WCVaR}(w|\beta) \) with respect to \( P_M \) is given by
\[ \text{WCVaR}(w|\beta) = \min_{a} \max_{i \in L} F^{(i)}(w,a|\beta) \]  

where \( L = \{1, \ldots, l\} \).
Moreover, denote

\[ F_L(w, \alpha | \beta) = \max_{i \in L} F^{(i)}(w, \alpha | \beta) \quad (8) \]

Minimizing the WCVaR \((w | \beta)\) overall \(w \in X\) is equivalent to minimizing \(F_L(w, \alpha | \beta)\) overall \((w, \alpha) \in X \times R\), in the sense that

\[ \min_{w \in X} \text{WCVaR}(w | \beta) = \min_{(w, \alpha) \in X \times R} F_L(w, \alpha | \beta) \quad (9) \]

From now on, we discuss the computational aspect of minimization of WCVaR. Lemma 1 helps us to translate the minimization problem to a more tractable one. The WCVaR minimization is equivalent to the following problem:

**Problem 1.**

\[
\min_{(w, \alpha, C) \in X \times R \times R} C \\
\text{s.t. } \alpha + (1 - \beta)^{-1} \int_R [-w^\top r - \alpha]^+ p^{(i)}(r)dr \leq C, \ (i = 1, \ldots, l) \quad (10)
\]

We approximate the function \(F^{(i)}(w, \alpha | \beta)\) by sampling a random variable \(r^{(i)}, \ i = 1, \ldots, l\) from the density function \(p^{(i)}(\cdot)\). \(r^{(i)}[q]\) is the \(q\)-th sample with respect to the \(i\)-th density distribution \(p^{(i)}(\cdot)\), and \(N_i \subseteq \{1, \ldots, N\}\) denotes the set of corresponding samples. The auxiliary function \(F^{(i)}(w, \alpha | \beta)\) is approximated as follows.

\[
F^{(i)}(w, \alpha | \beta) \simeq \alpha + (|N_i|(1 - \beta))^{-1} \sum_{q \in N_i} (-w^\top r^{(i)}[q] - \alpha)^+ \quad (12)
\]

Finally, we can formulate the minimum WCVaR portfolio as a linear programming problem, as shown below.

**Problem 2.**

\[
\min_{w, \alpha, t_{iq}} C \\
\text{s.t. } \alpha + \left(\left|N_i\right|(1 - \beta)\right)^{-1} \sum_{q \in N_i} t_{iq} \leq C \quad (i = 1, \ldots, l) \quad (13)
\]

\[
t_{iq} \geq -w^\top r^{(i)}[q] - \alpha \quad (i = 1, \ldots, l, \ q \in N_i) \quad (14)
\]

\[
t_{iq} \geq 0 \quad (i = 1, \ldots, l, \ q \in N_i) \quad (15)
\]

4. Regularized Multiple \(\beta\) WCVaR Portfolio Optimization

In this section, we propose a RM-WCVaR portfolio that takes into account the multiple \(\beta\) WCVaR values and portfolio turnover. Our approach is similar in spirit to that of [38], given that our approach estimates the simultaneously approximating multiple conditional quantiles.

The intuition behind the formulation is to minimize the max margin among multiple \(\beta\) levels of WCVaR (Figure 1). Figure 1 illustrates it where \(\beta_k = 0.97, 0.98, 0.99\) and each \(WC_{\beta_k}\) is given as a solution to Problem 1.

Here, \(WC_{\beta_k}, k = 1, \ldots, K\) is the value of WCVaR obtained by solving Problem 2. Then, we minimized \(C\), considering that \(WC_{\beta_k}\) is a main problem in this paper.
\begin{align}
W_{C,0.97} & \\
W_{C,0.98} & \\
W_{C,0.99} & \\
W_{C,\text{VaR}}(w,0.99) & \\
W_{C,\text{VaR}}(w,0.98) & \\
W_{C,\text{VaR}}(w,0.97) & \\
\end{align}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{The intuition behind the formulation of RM-WCVaR portfolio (Problem 3).}
\end{figure}

\textbf{Problem 3.}
\begin{align}
\min_{(w,C) \in X \times R} & \quad C \\
\text{s.t.} & \quad \text{WCVaR}(w|\beta_k) \leq C + W_{C,\beta_k} \quad (k = 1, \ldots, K)
\end{align}

As in Lemma 1, We define $F^L(w, a_k|\beta_k) = \max_{i \in L} F^i(w, a_k|\beta_k)$. Then, Problem 3 can be written as follows.

\textbf{Problem 4.}
\begin{align}
\min_{(w,C) \in X \times R} & \quad C \\
\text{s.t.} & \quad \min_{a_k} F^L(w, a_k|\beta_k) \leq C + W_{C,\beta_k} \quad (k = 1, \ldots, K)
\end{align}

Thereafter, we consider the following Problem 5.

\textbf{Problem 5.}
\begin{align}
\min_{(w,C,\alpha) \in X \times R \times R^K} & \quad C \\
\text{s.t.} & \quad F^L(w, a_k|\beta_k) \leq C + W_{C,\beta_k} \quad (k = 1, \ldots, K)
\end{align}

where $\alpha = (a_1, \ldots, a_K)^\top$.

According to the Lemma 2, Problem 3 and 5 are equivalent.

\textbf{Lemma 2.}
1. If $(w^*, C^*)$ is the optimal value for Problem 4, then $(w^*, C^*, \alpha^*)$ is the optimal value of Problem 5.
2. If $(w^{**}, C^{**}, \alpha^{**})$ is the optimal value for Problem 5, then $(w^{**}, C^{**})$ is the optimal value for Problem 4.

\textbf{Proof.} Assume that $(w^*, C^*)$ is the optimal value for Problem 4. Given that $(w^*, C^*)$ is a feasible solution of Problem 4, $\min_{a_k} F^L(w^*, a_k|\beta_k) \leq C^* + W_{C,\beta_k}$ holds. We defined $\alpha_k = (a_k^1, \ldots, a_k^K)\top$ as $a_k^k := \arg\min_{a_k \in R} F^L(w^*, a_k|\beta_k)(k = 1, \ldots, K)$. Then, $(w^*, C^*, \alpha^*)$ became
Therefore, (\(\hat{\omega}, \hat{C}\)) is a feasible solution of Problem 4, thereby contradicting that \(C^*\) is the optimal solution of Problem 4. Therefore, \((w^*, C^*, A^*)\) is the optimal solution. Assume that \((w^{**}, C^{**}, A^{**})\) is the optimal value for Problem 5. Then, because \((w^{**}, C^{**}, A^{**})\) is a feasible solution of Problem 5, \(F^L(w^{**}, A^{**}||\beta_k) \leq C^* + WC_{\beta_k}\) holds. \((w^{**}, C^{**})\) is a feasible solution for Problem 4 given that \(\min_{\alpha_i \in R} F^L(\omega, \alpha_k||\beta_k) \leq F^L(\omega^{**}, A^{**}||\beta_k) \leq C^* + WC_{\beta_k}(k = 1, \ldots, K)\) holds. If \((w^{**}, C^{**})\) is not the optimal solution of Problem 5, there exists a feasible solution \((\hat{\omega}, \hat{C})\) satisfying \(\hat{C} < C^*\). We defined \(\hat{\alpha} = (\hat{\alpha}_1, \ldots, \hat{\alpha}_K)^T\) as \(\hat{\alpha}_k := \arg \min F^L_{\beta_k}(\hat{\omega}, \alpha_k)\). Then, \((\hat{\omega}, \hat{C}, \hat{\alpha})\) became a feasible solution of Problem 5, thereby contradicting that \(C^{**}\) is the optimal solution of Problem 5. Therefore, \((w^{**}, C^{**})\) is the optimal solution.

Here, \(r^{(i)}[q]\) is the \(q\)-th sample with respect to the \(i\)-th density distribution \(p^{(i)}(r)\), and \(N_i\) denotes the set of corresponding samples. The function \(F^{(i)}(w, \alpha_k||\beta_k)\) is approximated as follows.

\[
F^{(i)}(w, \alpha_k||\beta_k) \simeq \alpha_k + \frac{1}{|N_i|} \sum_{q \in N_i} [-w^T r^{(i)}[q] - \alpha_k]^+
\]

Finally, we derive the RM-WC\(\alpha\)R Portfolio, where the objectives are minimizing multiple WC\(\alpha\)R values and also controlling the portfolio turnover. Controlling the portfolio turnover is realized through imposing the \(L1\)-regularization term as \(\|w - w^-\|_1 = \sum_{j=1}^n |w_j - w^-_j|\) where \(w^-_j\) denotes the portfolio weight before rebalancing.

Based on the above discussion, the RM-WC\(\alpha\)R Portfolio Optimization problem can be formulated as follows:

**Problem 6.**

\[
\min_{(w, C, \lambda) \in X \times R \times R^k} C + \lambda \|w - w^-\|_1
\]

s.t. \(\hat{F}^{(i)}(w, \alpha_k||\beta_k) \leq C + WC_{\beta_k}\)

\[(i = 1, \ldots, l, \ k = 1, \ldots, K)\]

We can easily prove that Problem 6 is a linear programming problem similar to the usual CV\(\alpha\)R minimization problem.
Theorem 1. The Regularized Multiple $\beta$ WCVaR Portfolio Optimization problem is equivalent to the following linear programming problem.

$$\min_{C, w, \alpha, t, u} C + \sum_{j=1}^{n} u_j$$

s.t. $u_j \geq \lambda \left( w_j - w^- \right)$

$$u_j \geq -\lambda \left( w_j - w^- \right)$$

$$t_{ijk} \geq 0$$

$$t_{ijk} \geq -w^T r_{i}^{q(i)}[q] - \alpha_k$$

$$\alpha_k + \frac{1}{|\mathcal{N}_i|(1 - \beta_k)} \sum_{q \in \mathcal{N}_i} t_{ijk} \leq C + WC_{\beta_k}$$

$$(i = 1, \ldots, l, k = 1, \ldots, K)$$

$$1^T w = 1$$

$$w_j \geq 0 \quad (j = 1, \ldots, n)$$

Proof. Using a standard approach in optimization, we can replace the absolute value term $\lambda \| w - w^- \|_1$ in the objective function with $u \geq \lambda (w - w^-)$ and $u \geq -\lambda (w - w^-)$ in the constraint. Thereafter, the objective and constraints all became linear. □

Algorithm 1 Regularized Multiple $\beta$ WCVaR Portfolio Optimization.

**Input:** $K$ probability levels $\beta_k \in (0, 1) \ (k = 1, \ldots, K)$,
Coefficient of the regularization term $\lambda \in \mathbb{R}_{>0}$,
Number of blocks of a partition $l \in \mathbb{Z}_{>0}$ and
Return matrix $R_i \in \mathbb{R}^{n \times N} \ (t = 1, \ldots, T)$

**Output:** Set of optimal weights $\mathcal{W} = \{w_t \in \mathbb{R}^n\}_{t=1}^T$

1: **for** $t = 1, \ldots, T$ **do**
2: Calculate $WC_{\beta_k}$ via solving Problem 2 $(k = 1, \ldots, K)$.
3: Randomly divide the set $\{1, \ldots, N\}$ into blocks of a partition $\{\mathcal{N}_i\}_{i=1}^l$ s.t. $|\mathcal{N}_i| = \frac{N}{l}$
4: $w^- \leftarrow w_{t-1}$
5: Solve the linear programming introduced in Theorem 1
6: Add to the output set $\mathcal{W}$ the solution $w^*$ as $w_t$
7: **end for**
8: **return** $\mathcal{W}$

5. Experiment

In this section, we report the results of our empirical experiment with well-known benchmarks in finance.

5.1. Dataset

In the experiments, we use well-known academic benchmarks called Fama and French (FF) datasets [39] to ensure the reproducibility of the experiment. This FF dataset is public and is readily available to anyone (https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html). The FF datasets have been recognized as standard datasets and are heavily adopted in finance research because of their extensive coverage of asset classes and very long historical data series. We use FF25, FF48 and FF100 dataset. For
example, the FF25 and FF100 dataset include 25 and 100 portfolios formed based on size and book-to-market ratio, while the FF48 dataset contains monthly returns of 48 portfolios representing different industrial sectors. We use all datasets as monthly data from January 1989 to June 2020.

5.2. Experimental Settings

In our experiment, we use the following portfolio models.

- “EW” stands for equally-weighted (EW) portfolio [40].
- “MV” stands for minimum-variance portfolio. We use the latest 10 years (120 months) to compute for the sample covariance matrix [41].
- “DRP” stands for the doubly regularized minimum-variance portfolio [18]. We use the latest 10 years (120 months) to compute for the sample covariance matrix. We set combinations of two coefficients for regularization terms to \( \lambda_1 = \{0.001, 0.005, 0.01, 0.05\} \) and \( \lambda_2 = \{0.001, 0.005, 0.01, 0.05\} \).
- “EGO” stands for the Kelly growth optimal portfolio with ensemble learning [42]. We set \( n_1 \) (number of resamples) = 50, \( n_2 \) (size of each resample) = 5 \( \tau \), \( \tau \) (number of periods of return data) = 120, \( n_3 \) (number of resampled subsets) = 50, \( n_4 \) (size of each subset) = \( n^{0.7} \), where \( n \) is number of assets (i.e., \( n = 25, 48, 100 \)).
- “RMCVaR” stands for the regularized multiple CVaR portfolio [16]. We set \( K = 5 \) \((k = 1, \ldots, K)\) as five patterns of \( \beta_k = \{0.95, 0.96, 0.97, 0.98, 0.99\} \) to calculate \( C_{\beta_k} \). We also set \( Q \) (number of sampling periods of return data) as 10 years (120 months). For the coefficient of the regularization term, we implemented four patterns of \( \lambda = \{0.001, 0.005, 0.01, 0.05\} \).
- “WCVaR” stands for minimum WCVaR portfolio with \( \beta \) (Problem 2). We implemented five patterns of \( \beta = \{0.95, 0.96, 0.97, 0.98, 0.99\} \). We used the latest 50 years’ \((N = 600 months)\) data and split them randomly into \( l \in \{1, 2, \ldots, 10\} \) divisions.
- “AWCVaR” stands for the average portfolio calculated by the simple average of minimum WCVaR portfolio of different \( \beta = \{0.95, 0.96, 0.97, 0.98, 0.99\} \) at each month.
- “RM-WCVaR” stands for our proposed portfolio. We set \( K = 5 \) \((k = 1, \ldots, K)\) as five patterns of \( \beta_k = \{0.95, 0.96, 0.97, 0.98, 0.99\} \) to calculate \( W_{\beta_k} \). We use the latest 50 years’ \((N = 600 months)\) data and split them randomly into \( l \in \{1, 2, \ldots, 10\} \) divisions. For the coefficient of the regularization term, we implement four patterns of \( \lambda = \{0.001, 0.005, 0.01, 0.05\} \). The RM-WCVaR Portfolio presented in Algorithm 1 is straightforward in terms of implementation.

We use the first-half period, i.e., from January 1989 to December 2004, as the in-sample period in terms of deciding the hyper-parameters of each portfolio. After that, we use the second half-period, i.e., from January 2005 to June 2020, as the out-of-sample period. Each portfolio is updated by sliding one-month-ahead.

5.3. Performance Measures

The following measures widely used in finance to evaluate portfolio strategies [43] are chosen. The portfolio return at time \( t \) is defined as \( R_t = \sum_{j=1}^{n} w_j r_{j,t} \) where \( r_{j,t} \) is the monthly return of \( j \) asset at time \( t \), \( w_{j,t-1} \) is the weight of \( j \) asset in the portfolio at time \( t-1 \), and \( n \) is the total number of assets.

The annualized return (AR), annualized risk as the standard deviation of return (RISK), and risk-adjusted return (R/R) are defined as follows:

\[
AR = \frac{12}{T} \times \sum_{t=1}^{T} R_t, \tag{26}
\]

\[
RISK = \sqrt{\frac{12}{T-1} \times \sum_{t=1}^{T} (R_t - \hat{\mu})^2}, \quad \hat{\mu} = 1/T \times \sum_{t=1}^{T} R_t, \tag{27}
\]

\[
R/R = AR/RISK. \tag{28}
\]
Among them, R/R is the most important measure for a portfolio strategy. We also evaluate the maximum draw-down (MaxDD), which is another widely used risk measure [44] for the portfolio strategy. In particular, MaxDD is the largest drop from a peak defined as

\[
\text{MaxDD} = \min_{t \in [1,T]} \left( 0, \frac{W_t}{\max_{t \in [1,T]} W_t} - 1 \right),
\]

(29)

where \( W_k \) is the cumulative return of the portfolio until time \( k \); that is, \( W_t = \prod_{t'=1}^{t} (1 + R_{t'}) \).

The turnover (TO) indicates the volumes of rebalancing [18]. Since a high TO inevitably generates high explicit and implicit trading costs, the portfolio return tends to be reduced. The TO is a proxy for the transaction costs of the portfolio. The one-way annualized turnover is calculated as an average absolute value of the rebalancing trades over all the trading periods:

\[
\text{TO} = \frac{12}{2(T - 1)} \sum_{t=1}^{T-1} ||w_t - w_{t-1}||_1
\]

(30)

where \( T - 1 \) indicates the total number of the rebalancing periods and \( w_t^{-} = \frac{w_t \otimes (1 + r_t)}{1 + w_t^{-} r_t} \) is the re-normalized portfolio weight vector before rebalance. Here, \( r_t \) is the return vector of the assets at time \( t \), \( w_{t-1} \) is the weight vector at time \( t - 1 \), and the operator \( \otimes \) denotes the Hadamard (element-wise) product.

### 5.4. Experimental Results

Table 1–3 reports the overall performance measures of RM-WCVaR Portfolio, our proposed portfolio, and the 11 compared portfolios introduced in the Experimental Settings section for FF25, FF48, and FF100 dataset. The out-of-sample period is from January 2005 to June 2020. Among the comparisons of the various portfolios, the best performance is highlighted in **bold**.

Table 1 shows that the RM-WCVaR Portfolio outperformed all the compared portfolios in all performance measures. It achieved the highest AR and R/R and the lowest RISK, MaxDD and TO.

| Portfolio | AR[\%] | RISK[\%] | R/R | MaxDD[\%] | TO[\%] |
|-----------|--------|----------|-----|-----------|--------|
| EW        | 8.92   | 18.60    | 0.48| −54.12    | 12.36  |
| MV        | 9.75   | 14.34    | 0.68| −50.69    | 33.10  |
| DRP       | 9.74   | 14.35    | 0.68| −50.72    | 9.35   |
| EGO       | 8.64   | 19.59    | 0.44| −57.26    | 76.52  |
| RMCVaR    | 9.65   | 15.50    | 0.62| −49.59    | 34.96  |
| AWCVaR    | 9.14   | 16.03    | 0.57| −55.82    | 23.11  |
| WCVaR     |        |          |     |          |        |
| 95%       | 9.12   | 16.56    | 0.55| −54.23    | 18.19  |
| 96%       | 9.01   | 15.82    | 0.56| −53.29    | 20.82  |
| 97%       | 9.01   | 15.94    | 0.56| −57.11    | 26.00  |
| 98%       | 9.41   | 16.04    | 0.58| −55.91    | 22.41  |
| 99%       | 9.08   | 16.15    | 0.56| −58.43    | 31.22  |
| RM-WCVaR  | **10.44** | **14.26** | **0.73** | −45.26 | **8.12** |

Performance measures are the annualized return (AR), annualized risk (RISK), annualized return–risk ratio (R/R), maximum drawdown (MaxDD) and turnover rate (TO). The out-of-sample period is from January 2005 to June 2020. Among the comparisons of the various portfolios, the best performance is highlighted in **bold**.

In Table 2, we can see the RM-WCVaR Portfolio had the best AR, R/R, MaxDD and TO. Only the RISK was the best for the DRP portfolio.
Table 2. The out-of-sample performance of each portfolio for FF48 dataset.

| Portfolio | FF48 AR[%] | RISK[%] | R/R | MaxDD[%] | TO[%] |
|-----------|------------|---------|-----|----------|-------|
| EW        | 9.36       | 17.12   | 0.55| -52.90   | 22.03 |
| MV        | 8.86       | 12.77   | 0.69| -43.84   | 28.48 |
| DRP       | 8.78       | 12.20   | 0.72| -38.92   | 17.15 |
| EGO       | 11.11      | 20.61   | 0.54| -57.39   | 79.60 |
| RM-CVaR   | 8.27       | 12.82   | 0.65| -38.28   | 129.31|
| AW-CVaR   | 11.70      | 13.01   | 0.90| -42.60   | 39.44 |
| WCVaR     |             |         |     |          |       |
| 95%       | 11.43      | 13.29   | 0.85| -42.65   | 43.98 |
| 96%       | 10.95      | 13.25   | 0.83| -43.56   | 41.38 |
| 97%       | 11.28      | 13.14   | 0.86| -41.58   | 37.66 |
| 98%       | 12.12      | 13.15   | 0.92| -44.35   | 44.27 |
| 99%       | 12.77      | 13.21   | 0.97| -40.91   | 40.84 |
| RM-WCVaR  | 14.48      | 14.63   | 0.99| -36.66   | 7.87  |

Performance measures are the annualized return (AR), annualized risk (RISK), annualized return–risk ratio (R/R), maximum drawdown (MaxDD) and turnover rate (TO). The out-of-sample period is from January 2005 to June 2020. Among the comparisons of the various portfolios, the best performance is highlighted in bold.

In Table 3, the RM-WCVaR Portfolio had the best AR, R/R, and MaxDD. The TO for the RM-WCVaR portfolio was the second lowest after the EW portfolio.

Table 3. The out-of-sample performance of each portfolio for FF100 dataset.

| Portfolio | FF100 AR[%] | RISK[%] | R/R | MaxDD[%] | TO[%] |
|-----------|------------|---------|-----|----------|-------|
| EW        | 8.86       | 18.87   | 0.47| -54.53   | 16.18 |
| MV        | 9.47       | 14.13   | 0.67| -50.69   | 39.10 |
| DRP       | 9.92       | 14.42   | 0.69| -51.23   | 19.20 |
| EGO       | 8.66       | 20.12   | 0.43| -57.79   | 78.65 |
| RM-CVaR   | 9.87       | 15.42   | 0.64| -49.97   | 35.20 |
| AW-CVaR   | 8.74       | 16.33   | 0.54| -43.02   | 23.27 |
| WCVaR     |             |         |     |          |       |
| 95%       | 7.82       | 16.59   | 0.47| -40.19   | 18.31 |
| 96%       | 9.09       | 16.44   | 0.55| -37.12   | 20.97 |
| 97%       | 9.10       | 16.55   | 0.55| -40.74   | 26.18 |
| 98%       | 9.63       | 16.14   | 0.60| -43.89   | 22.57 |
| 99%       | 8.04       | 16.59   | 0.48| -52.50   | 31.44 |
| RM-WCVaR  | 14.26      | 20.67   | 0.69| -37.10   | 18.00 |

Performance measures are the annualized return (AR), annualized risk (RISK), annualized return–risk ratio (R/R), maximum drawdown (MaxDD) and turnover rate (TO). The out-of-sample period is from January 2005 to June 2020. Among the comparisons of the various portfolios, the best performance is highlighted in bold.

In all datasets, the proposed RM-WCVaR Portfolio achieved the highest AR and R/R, and the lowest MaxDD.

Unsurprisingly, the RM-WCVaR Portfolio was different from ACVaR, which is the simple average of five probability levels’ WCVaR portfolios. RM-WCVaR Portfolio also exceeded the individual $\beta$ levels of WCVaR portfolios in terms of AR, R/R, MaxDD and TO. This is because the RM procedure implies a minimization of the maximum margin among multiple WCVaR levels, which enables more efficient portfolio construction. Analyzing the
relationship between the margin level and performance of RM-WCVaR is an important future task.

Therefore, we can confirm that the RM-WCVaR Portfolio has high R/R and avoids a large drawdown despite the lower turnover rate. Since the TO is the lowest of all compared portfolios, the results do not change when transaction costs are taken into account. We consider the RM-WCVaR portfolio to have had a good R/R because it reduced tail risk, resulting in lower drawdowns and higher returns.

6. Conclusions

Our study makes the following contributions. We propose a Regularized Multiple β WCVaR Portfolio, which solves three challenges in the minimum CVaR portfolio. We prove that the optimization problem reduces to a linear programming problem. We perform experiments on well-known benchmarks in finance to evaluate our proposed portfolio. Our portfolio shows superior performance in terms of having both higher risk-adjusted returns and lower maximum drawdown despite the lower turnover rate.

Directions of promising future work include (1) constructing a more sophisticated mixture distribution by assuming a probability distribution as in [15], rather than a simple empirical distribution in this study, (2) directly using a semi-nonparametric distribution capturing true CVaR as in [25,26] instead of WCVaR, and (3) obtaining a higher R/R by incorporating the expected return into our proposed portfolio.

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References
1. Markowitz, H. Portfolio selection. J. Financ. 1952, 7, 77–91.
2. Morgan, J.; Spencer, M. Riskmetrics Technical Document; Morgan Guaranty Trust Company of New York: New York, NY, USA, 1996.
3. Herring, R.J. The Basel 2 approach to bank operational risk: Regulation on the wrong track. J. Risk Financ. 2002, 4, 42–45. [CrossRef]
4. Artzner, P.; Delbaen, F.; Eber, J.M.; Heath, D. Coherent measures of risk. Math. Financ. 1999, 9, 203–228. [CrossRef]
5. Wang, D.; Chen, Y.; Wang, H.; Huang, M. Formulation of the Non-Parametric Value at Risk Portfolio Selection Problem Considering Symmetry. Symmetry 2020, 12, 1639. [CrossRef]
6. McNeil, A.J.; Frey, R.; Embrechts, P. Quantitative Risk Management: Concepts, Techniques and Tools—Revised Edition; Princeton University Press: Princeton, NJ, USA, 2015.
7. Pflug, G.C. Some remarks on the value-at-risk and the conditional value-at-risk. In Probabilistic Constrained Optimization; Springer: Berlin/Heidelberg, Germany, 2000; pp. 272–281.
8. Rockafellar, R.T.; Uryasev, S. Optimization of conditional value-at-risk. J. Risk 2000, 2, 21–42. [CrossRef]
9. Rockafellar, R.T.; Uryasev, S. Conditional value-at-risk for general loss distributions. J. Bank. Financ. 2002, 26, 1443–1471. [CrossRef]
10. Goel, A.; Sharma, A.; Mehra, A. Index tracking and enhanced indexing using mixed conditional value-at-risk. J. Comput. Appl. Math. 2018, 335, 361–380. [CrossRef]
11. Karmakar, M.; Paul, S. Intraday portfolio risk management using VaR and CVaR: A CGARCH-EVT-Copula approach. Int. J. Forecast. 2019, 35, 699–709. [CrossRef]
12. Guastaroba, G.; Mansini, R.; Ogryczak, W.; Speranza, M.G. Enhanced index tracking with CVaR-based ratio measures. Ann. Oper. Res. 2020, 292, 883–931. [CrossRef]
13. Ghaoui, L.E.; Oks, M.; Oustry, F. Worst-case value-at-risk and robust portfolio optimization: A conic programming approach. Oper. Res. 2003, 51, 543–556. [CrossRef]
14. Čerňáková, J. Worst-case var and cvaR. In Operations Research Proceedings 2005; Springer: Berlin/Heidelberg, Germany, 2006; pp. 817–822.
15. Zhu, S.; Fukushima, M. Worst-case conditional value-at-risk with application to robust portfolio management. Oper. Res. 2009, 57, 1155–1168. [CrossRef]
16. Nakagawa, K.; Noma, S.; Abe, M. RM-CVaR: Regularized Multiple β-CVaR Portfolio. IJCAI 2020, 4562–4568. [CrossRef]
17. Michaud, R.O. The Markowitz optimization enigma: Is ‘optimized’ optimal? Financ. Anal. J. 1989, 45, 31–42. [CrossRef]
18. Shen, W.; Wang, J.; Ma, S. Doubly regularized portfolio with risk minimization. In Proceedings of the Twenty-Eighth AAAI Conference on Artificial Intelligence, Quebec City, QC, Canada, 27–31 July 2014.
19. Mansini, R.; Ogryczak, W.; Speranza, M.G. Portfolio Optimization with Transaction Costs. In Linear and Mixed Integer Programming for Portfolio Optimization; Springer: Berlin/Heidelberg, Germany, 2015; pp. 47–62.
20. DeMiguel, V.; Garlappi, L.; Nogales, F.J.; Uppal, R. A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms. Manag. Sci. 2009, 55, 798–812. [CrossRef]
21. Merton, R.C. On Estimating the Expected Return on the Market: An Exploratory Investigation; Technical report; National Bureau of Economic Research: Cambridge, MA, USA, 1980.
22. Gotoh, J.Y.; Takeda, A. On the role of norm constraints in portfolio selection. Comput. Manag. Sci. 2011, 8, 323. [CrossRef]
23. Tütüncü, R.H.; Koenig, M. Robust asset allocation. Ann. Oper. Res. 2004, 132, 157–187. [CrossRef]
24. Qiu, H.; Han, F.; Liu, H.; Caffo, B. Robust portfolio optimization. Adv. Neural Inf. Process. Syst. 2015, 28, 46–54.
25. Del Brio, E.B.; Mora-Valencia, A.; Perote, J. Expected shortfall assessment in commodity (L) ETF portfolios with semi-parametric specifications. Eur. J. Financ. 2019, 25, 1746–1764. [CrossRef]
26. Molina-Muñoz, E.; Mora-Valencia, A.; Perote, J. Backtesting expected shortfall for world stock index ETFs with extreme value theory and Gram–Charlier mixtures. Int. J. Financ. Econ. 2020. [CrossRef]
27. Clarke, R.; De Silva, H.; Thorley, S. Minimum-variance portfolio composition. J. Portf. Manag. 2011, 37, 31–45. [CrossRef]
28. Qian, E. Risk Parity Portfolios: Efficient Portfolios Through true Diversification. 2005. Available online: https://www.panagora.com/assets/PanAgora-Risk-Parity-Portfolios-Efficient-Portfolios-Through-True-Diversification.pdf (accessed on 12 March 2021).
29. Maillard, S.; Roncalli, T.; Teiletche, J. The properties of equally weighted risk contribution portfolios. J. Portf. Manag. 2010, 36, 60–70. [CrossRef]
30. Choueifaty, Y.; Coignard, Y. Toward maximum diversification. J. Portf. Manag. 2008, 35, 40–51. [CrossRef]
31. Nakagawa, K.; Imamura, M.; Yoshida, K. Risk-based portfolios with large dynamic covariance matrices. Int. J. Financ. Stud. 2018, 6, 52. [CrossRef]
32. Du Plessis, H.; van Rensburg, P. Risk-based portfolio sensitivity to covariance estimation. Invest. Anal. J. 2020, 49, 243–268. [CrossRef]
33. Bodnar, T.; Schmid, W.; Zabolotskyy, T. Minimum VaR and Minimum CVaR optimal portfolios: estimators, confidence regions, and tests. Stat. Risk Model. 2012, 29, 281–314. [CrossRef]
34. Meucci, A. Managing diversification. Risk 2009, 2009, 74–79.
35. Uchiyama, Y.; Kadoya, T.; Nakagawa, K. Complex valued risk diversification. Entropy 2019, 21, 119. [CrossRef]
36. Nakagawa, K.; Uchiyama, Y. GO-GJRSK Model with Application to Higher Order Risk-Based Portfolio. Mathematics 2020, 8, 1990. [CrossRef]
37. Poddig, T.; Unger, A. On the robustness of risk-based asset allocations. Financ. Mark. Portf. Manag. 2012, 26, 369–401. [CrossRef]
38. Zou, H.; Yuan, M. Regularized simultaneous model selection in multiple quantiles regression. Comput. Stat. Data Anal. 2008, 52, 5296–5304. [CrossRef]
39. Fama, E.F.; French, K.R. The cross-section of expected stock returns. J. Financ. 1992, 47, 427–465. [CrossRef]
40. DeMiguel, V.; Garlappi, L.; Uppal, R. Optimal versus naive diversification: How inefficient is the 1/N portfolio strategy? Rev. Financ. Stud. 2007, 22, 1915–1953. [CrossRef]
41. Clarke, R.G.; De Silva, H.; Thorley, S. Minimum-variance portfolios in the US equity market. J. Portf. Manag. 2006, 33, 10–24. [CrossRef]
42. Shen, W.; Wang, B.; Pu, J.; Wang, J. The Kelly Growth Optimal Portfolio with Ensemble Learning. In Proceedings of the AAAI Conference on Artificial Intelligence, Honolulu, HI, USA, January 27–1 February 2019; Volume 33, pp. 1134–1141.
43. Brandt, M.W. Portfolio choice problems. In Handbook of Financial Econometrics: Tools and Techniques; Elsevier: Amsterdam, The Netherlands, 2010; pp. 269–336.
44. Magdon-Ismail, M.; Atiya, A.F. Maximum drawdown. Risk Mag. 2004, 17, 99–102.