Cauchy Noise and Affiliated Stochastic Processes

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Abstract

By departing from the previous attempt (Phys. Rev. E 51, 4114, (1995)) we give a detailed construction of conditional and perturbed Markov processes, under the assumption that the Cauchy law of probability replaces the Gaussian law (appropriate for the Wiener process) as the model of primordial noise. All considered processes are regarded as probabilistic solutions of the so-called Schrödinger interpolation problem, whose validity is thus extended to the jump-type processes and their step process approximants.

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I. Introduction

Probabilistic solutions of the so-called Schrödinger boundary data problem, [1, 2], are known to yield a unique Markovian interpolation between any two strictly positive

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probability densities designed to form the input-output statistics data for a certain
dynamical process, taking place in a finite-time interval. The key problem, if one
attempts to reconstruct the most likely (Markovian) dynamics, is to select the jointly
continuous in space variables positive and contractive semigroup kernel. That issue
was analyzed before in a number of publications, [1]–[8].

In fact, basic stochastic processes of the nonequilibrium statistical physics (Smolu-
chowskii diffusion processes) involve the familiar Feynman-Kac-like kernels as the
building blocks for suitable Markovian transition probability densities, [3, 6, 7]. In
the standard ”free” case (Feynman-Kac potential equal to zero as a necessary con-
dition) the Wiener noise may be recovered.

In the framework of the Schrödinger problem the choice of the integral kernel is
arbitrary, except for the strict positivity (cf. however, [8]) and continuity demand.
It is thus rather natural to ask for the most general stochastic interpolation, that is
admitted under the above premises.

Clearly, the standard Feynman-Kac kernels generated by Laplacians plus suitable
potentials, [9, 10], are very special examples in a surprisingly rich encompassing fam-
ily. First of all, the concept of the Gaussian noise, regarded as a stochastic analogue
of the mechanical ”state of rest” and traditionally linked with a Wiener process,
can be extended to all infinitely divisible probability laws via the Lévy-Khintchine
formula. It expands our framework from continuous diffusion processes to jump or
combined diffusion–jump propagation scenarios, [3], as appropriate mathematical
models of the primordial ”free noise”.

The next natural step in the analysis is to account for typical perturbations
of any given process, according to the pattern of the Feynman-Kac formula hence
in terms of perturbed semigroups, where an appropriate generator (replacing the Laplacian) is additively modified by a suitable potential.

The Feynman-Kac formula is known to extend its validity to the pertinent non-Gaussian measures, [11, 12, 13]. However, to our knowledge, no detailed description of the related Feynman-Kac kernels, with emphasis on their continuity and positivity (those features must be settled, [2, 3, 6], in the context of the above-mentioned Schrödinger interpolation problem), exists in the literature. Quite in contrast with the elaborate analysis that is available with respect to the Wiener measure, [10].

By referring to a physical terminology, let us consider Hamiltonians (semigroup generators) of the form $H = F(\hat{p})$, where $\hat{p} = -i\nabla$ stands for the momentum operator and for $-\infty < k < +\infty$, $F = F(k)$ is a real valued, bounded from below, locally integrable function. Here, $\hbar = c = 1$. We simplify further discussion by considering processes in one spatial dimension. We easily learn that for times $t \geq 0$ there holds

$$\exp(-tH) f(x) = [\exp(-tF(p)) \hat{f}(p)]^{\vee}(x)$$

(1)

where the superscript $\vee$ denotes the inverse Fourier transform and $\hat{f}$ stands for the Fourier transform of $f$.

Let us set $k_t = \frac{1}{\sqrt{2\pi}} [\exp(-tF(p))]^{\vee}$, then the action of $\exp(-tH)$ can be given in terms of a convolution: $\exp(-tH) f = f * k_t$, where $(f * g)(x) := \int_R g(x - z) f(z) dz$.

We are interested in those $F(p)$ which give rise to positivity preserving semigroups: if $F(p)$ satisfies the Lévy-Khintchine formula, then $k_t$ is a positive measure for all $t \geq 0$. Let us concentrate on the integral part of the Lévy-Khintchine formula,
which is responsible for arbitrary stochastic jump features:

\[ F(p) = - \int_{-\infty}^{+\infty} [\exp(iyy) - 1 - \frac{iyy}{1 + y^2}] \nu(dy) \]  

(2)

where \( \nu(dy) \) stands for the so-called Lévy measure.

There are not many explicit examples (analytic formulas for probability densities) for processes governed by (2), except possibly for the so-called stable probability laws. The best known example is the classic Cauchy density. Let us focus our attention on that selected choice for the characteristic exponent \( F(p) \), namely: \( F_0(p) = |p| \) which is the Cauchy process generator. The semigroup generator \( H_0 \) is a pseudodifferential operator. The associated kernel \( k_t \) in view of the "free noise" restriction (no potentials at the moment) is a transition density of the jump-type (Lévy) process, determined by the corresponding Lévy measure \( \nu(dy) = \frac{1}{\pi} \frac{dy}{y^2} \).

It is instructive to notice that a pseudodifferential analog of the Fokker-Planck equation holds true: \( F_0(p) \implies \partial_t \rho(x,t) = -|\nabla|\rho(x,t) \). This evolution rule gives rise to the Cauchy process probability density \( \rho(x,t) = \frac{1}{\pi} \frac{t}{t^2 + x^2} \) and the corresponding space-time homogeneous transition density (e.g. the semigroup kernel in this free propagation case).

As mentioned before, the existence and uniqueness of solutions proof for the Schrödinger problem extends, \( \exists \), to cases governed by infinitely divisible probability laws.

Our principal goal in the present paper is to generalise this observation to encompass the additive perturbations by physically motivated potentials. The construction is based on the Feynman-Kac formula for perturbed semigroups, with strictly positive and jointly continuous kernel functions.
As a byproduct of the discussion we shall give a characterisation of the affiliated Markovian jump-type processes in terms of approximating (convergent) families of more traditional, step processes, that solve a suitable version of the Schrödinger interpolation problem.

The demonstration explicitly pertains to the Cauchy process and its relatives, albeit the techniques and major statements may be extended to a broader class of Lévy processes and their perturbed versions, cf. [5, 11, 12] and [14]-[18] for related mathematical and physical connotations.

II. The Cauchy process and its conditional relatives

We consider Markovian propagation scenarios so remaining within the well established framework, where the input-output statistics data are provided in terms of two strictly positive boundary densities $\rho(x,0)$ and $\rho(x,T)$, $T > 0$. In addition, a bi-variate transition probability density is given in a specific factorized form: $m(x,y) = f(x)k(x,0,y,T)g(y)$, with marginals:

$$\int_R m(x,y)dy = \rho(x,0), \int_R m(x,y)dx = \rho(y,T)$$

(3)

Here, $f(x), g(y)$ are the a priori unknown functions, to come out as strictly positive solutions of the integral system of equations (3), provided that in addition to the density boundary data we have in hands any strictly positive, jointly continuous in space variables function $k(x,0,y,T)$. Additionally, we impose a restriction that $k(x,0,y,T)$ represents a certain strongly continuous dynamical semigroup kernel $k(y,s,x,t), 0 \leq s \leq t < T$, while given at the time interval borders: it secures the
Markov property of the sought for stochastic process.

Under those circumstances, \[ \begin{align*} 
\theta(x, t) &= \int dy k(x, t, y, T) g(y), \quad \theta_s(y, s) = \int dx k(x, 0, y, s) f(x) 
\end{align*} \]

there exists a transition density

\[ p(y, s, x, t) = k(y, s, x, t) \frac{\theta(x, t)}{\theta(y, s)} \]

which implements a Markovian propagation of the probability density

\[ \rho(x, t) = \theta(x, t) \theta_s(x, t) \]

\[ \rho(x, t) = \int p(y, s, x, t) \rho(y, s) dy \]

between the prescribed boundary data.

For a given semigroup which is characterized by its generator (Hamiltonian), the kernel \( k(y, s, x, t) \) and the emerging transition probability density \( p(y, s, x, t) \) are unique in view of the uniqueness of solutions \( f(x), g(y) \). For Markov processes, the knowledge of the transition probability density \( p(y, s, x, t) \) for all intermediate times \( 0 \leq s < t \leq T \) suffices for the derivation of all other relevant characteristics.

At this point, let us make a definite choice of the kernel function, namely that of the Cauchy kernel:

\[ k(y, s, x, t) = \frac{1}{\pi} \frac{t - s}{(t - s)^2 + (x - y)^2}. \]  

We have:

**Theorem 1:**

(a) \( p(y, s, x, t) \) defined by Eqs. (5) and (7) is a Markov transition kernel, that is
(weak limit in below)

\[ \int_R p(y, s, x, t) dx = 1 \]

\[ \lim_{t \downarrow s} p(y, s, x, t) = \delta_y(x) \]

\[ \int_R p(y, t_1, z, t_2) p(z, t_2, x, t_3) dz = p(y, t_1, x, t_3) \]

for all \( 0 \leq t_1 < t_2 < t_3 \leq T \), with \( \delta_y \) standing for the Dirac delta

(b) \( \rho(x, t) \), Eq. (6), is a probability distribution interpolating between \( \rho_0 \) and \( \rho_T \):

\[ \int_R \rho(x, t) dx = 1 \]

\[ \rho(x, 0) = \rho_0(x), \quad \rho(x, T) = \rho_T(x) \]

(c) the process \( X_t \) having \( p(y, s, x, t) \) as the transition kernel is a Markov interpolating process:

\[ \int_R p(y, s, x, t) \rho(y, s) dy = \rho(x, t) \]

for all \( 0 \leq s < t \leq T \).

**Proof:** See e.g. Refs. [5, 6].

Let us notice that the process \( X_t \) is obtained from the Cauchy process \( X_t^C \) by means of a multiplicative transformation of transition function. Clearly, \( \alpha_t^s = \frac{\theta(x, t)}{\theta(s, x)} \) is a multiplicative functional of \( X^C \) such that its average with respect to the Cauchy process reads \( \int \alpha_t^s(\omega) P_x^C(d\omega) = 1 \) for any \( 0 \leq s \leq t \leq T \) and any \( x \in R \), see e.g. [19]. However \( \alpha_t^s \) is not homogeneous and, even worse, not contracting (in fact, not even
bounded). We cannot be a priori sure that the generic sample path properties of the Cauchy process can be attributed to $X_t$ as well. In particular, an approximation of $X_t$ in terms of jump processes with a finite number of jumps in a finite time interval, is by no means obvious and needs a demonstration (to be given in below).

To this end, let us first notice that $\theta^*$ and $\theta$ satisfy the conjugate pseudodifferential equations:

$$\partial_t \theta^* = -|\nabla| \theta^*$$

$$\partial_t \theta = |\nabla| \theta$$

where the operator $|\nabla|$ acts as follows:

$$|\nabla| f(x) = -\frac{1}{\pi} \int_R \left[ f(x+y) - f(x) - \frac{y \nabla f(x)}{1+y^2} \right] \frac{dy}{y^2}.$$  \hspace{1cm} (9)

Let us define a new operator $|\nabla|_\epsilon$ by:

$$|\nabla|_\epsilon f(x) = -\frac{1}{\pi} \int_{|y|>\epsilon} \left[ f(x+y) - f(x) \right] \frac{dy}{y^2}$$  \hspace{1cm} (10)

and, accordingly:

$$\partial_t \theta^*_\epsilon = -|\nabla|_\epsilon \theta^*_\epsilon$$

$$\partial_t \theta^* = |\nabla|_\epsilon \theta^*$$

with $\theta^*_\epsilon(x, 0) = \theta^*(x, 0)$, $\theta^*\epsilon(x, T) = \theta(x, T)$.

Furthermore, let

$$q_\epsilon(x) = \frac{1}{\pi} \chi_{I_\epsilon}(x) \frac{1}{x^2}$$

where $I_\epsilon = [-\epsilon, \epsilon]$ and $\chi_A$ is an indicator function of a set $A$. 

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We have:

**Theorem 2:**

Let us define the Poisson transition kernel corresponding to the measure \( q_\epsilon(x)dx \):

\[
k_\epsilon(x,t) = \left[ \exp\left(-\frac{2t}{\epsilon} \right) \right] \left[ \delta_0(x) + tq_\epsilon(x) + \frac{t^2}{2!}(q_\epsilon * q_\epsilon)(x) + \ldots \right].
\]

Then, functions:

\[
\theta^\epsilon_*(x,t) = \int_R k_\epsilon(x-y,t)\theta_*(y,0)dy
\]

\[
\theta^\epsilon_+(x,t) = \int_R k_\epsilon(x-y,T-t)\theta(y,T)dy
\]

solve the Cauchy problem (11).

**Proof:**

The transition function in the above is called the Poisson transition kernel following the terminology of Ref. [19]. We have \( \theta^\epsilon_*(x,0) = \int_R \delta_0(x-y)\theta_*(y,0)dy = \theta_*(x,0) \) and:

\[
\partial_t \theta^\epsilon_*(x,t) = \int_R [\partial_t k_\epsilon(x-y,t)]\theta_*(y,0)dy
\]

where

\[
\partial_t k_\epsilon(x,t) = -\frac{2}{\pi \epsilon} k_\epsilon(x,t) + \left[ \exp\left(-\frac{2t}{\epsilon} \right) \right] \left[ q_\epsilon(x) + t(q_\epsilon * q_\epsilon)(x) + \ldots \right].
\]

Consequently,

\[
[\partial_t k_\epsilon(.,t)*\theta_*(.,0)](x) = -\frac{2}{\pi \epsilon} \theta^\epsilon_* (x,t) + \left[ \exp\left(-\frac{2t}{\epsilon} \right) \right] q_\epsilon * (\delta_0 + tq_\epsilon + \frac{t^2}{2!}(q_\epsilon * q_\epsilon) + \ldots) * \theta_*(x) = \ldots
\]
\[-\frac{2}{\pi\epsilon} \theta_\epsilon^c(x, t) + [q_\epsilon * \theta_\epsilon^c(., t)](x) = -\frac{2}{\pi\epsilon} \theta_\epsilon^c(x, t) + \int_R q_\epsilon(y) \theta_\epsilon^c(x - y, t)dy.\]

But, there holds
\[
\int_R q_\epsilon(y) \theta_\epsilon^c(x - y, t)dy = \frac{1}{\pi} \int_{|y|>\epsilon} \theta_\epsilon^c((x - y, t) \frac{dy}{y^2} = \frac{1}{\pi} \int_{|y|>\epsilon} \theta_\epsilon^c(x + y, t) \frac{dy}{y^2}
\]

and, in view of the obvious identity
\[
\frac{2}{\pi\epsilon} \theta_\epsilon^c(x, t) = \frac{1}{\pi} \int_{|y|>\epsilon} \theta_\epsilon^c(x, t) \frac{dy}{y^2},
\]

we finally arrive at
\[
\partial_t \theta_\epsilon^c(x, t) = \frac{1}{\pi} \int_{|y|>\epsilon} [\theta_\epsilon^c(x + y, t) - \theta_\epsilon^c(x, t)] \frac{dy}{y^2} = -|\nabla|_\epsilon \theta_\epsilon^c(x, t)
\]

An analogous line of arguments follows with respect to \(\theta^c(x, t)\), which completes the proof.

A random process with a Poisson transition function belongs to the class of, so called, step processes, \[19, 20\], that is jump processes with no accumulation points of jumps in a finite time interval: the number of jumps is finite on each finite time interval. We have:

**Lemma 1:**

The Markov process \(Y_t^\epsilon\) given by the transition function \(k_\epsilon(x, t)\) is a step process with a characteristic function:

\[
\Phi_\epsilon(p, t) = \exp(-t[\hat{q}_\epsilon(0) - \hat{q}_\epsilon(p)])
\]

where \(\hat{q}_\epsilon(p)\) is the Fourier transform of \(q_\epsilon(x)\).
Proof:

We need to evaluate the characteristic function of the transition kernel, that is:

\[
\Phi_\epsilon(p,t) = \exp\left(-\frac{2t}{\pi \epsilon}\right) \cdot \int_{-\infty}^{+\infty} \left[ \exp(-ipx) \right] \left[ \delta_0(x) + tq_\epsilon(x) + \frac{t^2}{2!}(q_\epsilon * q_\epsilon)(x) + \ldots \right] dx =
\]

\[
\exp\left(-\frac{2t}{\pi \epsilon}\right) \cdot \left[ 1 + t\hat{q}_\epsilon(p) + \frac{t^2}{2!}(\hat{q}(p))^2 + \ldots \right] = \exp\left[-\frac{2t}{\pi \epsilon} + t\hat{q}_\epsilon(p)\right]
\]

In view of \(\hat{q}_\epsilon(0) = \frac{2}{\pi \epsilon}\), the Lemma holds true.

As a technical warming up we shall now prove that the Cauchy process is the limit (in distributions) of a one-parameter family of step processes \(Y_\epsilon^t\). We touch here an important issue of limits (convergence) of jump processes, [21, 22, 23] and there are many types of the pertinent convergence. For example, it is known that \(Y_\epsilon^t\) tends to the Cauchy process in probability, [21], while major modern techniques refer to the weak convergence of probability measures, [23]. Also, typical proofs refer only to processes with stationary independent increments, while we cannot respect this limitation in the presence of perturbations.

Lemma 2:

There holds: \(\lim_{\epsilon \to 0} \Phi_\epsilon(p,t) = \psi(p,t)\), where \(\psi(p,t)\) is the Cauchy characteristic function \(\psi(p,t) = \exp(-t|p|)\). Moreover, the limit is uniform for all \(t \in [0, T]\).

Proof:
Let us evaluate $\hat{q}_\epsilon(p)$:

$$
\hat{q}_\epsilon(p) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \exp(ipx) \cdot \hat{q}_\epsilon(x) dx = \frac{1}{\pi} \int_{|x|>\epsilon} \exp(ipx) \cdot \frac{dx}{x^2} = \frac{2}{\pi} \int_{\epsilon}^{\infty} \frac{\cos(px) - 1}{x^2} dx + \frac{2}{\pi \epsilon}
$$

Consequently

$$
\Phi_\epsilon(p,t) = \exp[-\frac{2t}{\pi} \int_{\epsilon}^{\infty} \frac{1 - \cos(px)}{x^2} dx].
$$

In view of

$$
\lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{1 - \cos(px)}{x^2} dx = \frac{|p|\pi}{2},
$$

we arrive at:

$$
\lim_{\epsilon \to 0} \Phi_\epsilon(p,t) = \exp[-\frac{2t}{\pi} \cdot \frac{|p|\pi}{2}] = \exp(-t|p|).
$$

The proof is completed.

Clearly, $|\nabla|_\epsilon$ is a well defined semigroup generator for the step process $Y^\epsilon_t$. Let us recall that sample paths of a step process have only a finite number of jumps in each finite time interval, and between jumps the sample path is constant. \[20\]. The limiting Cauchy process belongs to the category of jump-type processes, where apart from the long jumps-tail (no fixed bound can be imposed on their length) that implies the nonexistence of moments of the probability measure, sample paths of the Cauchy process may have an infinite number of jumps of arbitrarily small size. By general arguments, pertaining to the space $D_E[0,\infty)$ of right continuous functions with left limits (cadlag), both in the finite and infinite time interval the number of jumps is at most countable, \[21\], \[24\]. It is also useful to recall that on a finite time interval there can be at most finitely many points $t \in [0,T]$ at which
the jump size exceeds a given positive number. In view of that, \( \sup_{t \in [0, T]} |Y_t^\varepsilon| < \infty \). Obviously, there is no fixed upper bound for the size of jumps (except for being finite), since a stochastically continuous process with independent increments having, with probability 1, no jumps exceeding a certain constant \( C \), would possess all moments, \([20]\).

Now, we shall pass to a slightly more involved demonstration that a well defined family of Markov processes \( X_t^\varepsilon \) (in fact, step ones) can be constructed, such that the process \( X_t \) of Theorem 1 can be approximated (in the sense of suitable convergence) to an arbitrary degree of accuracy.

Here, we are motivated by a heuristic analysis carried out in our earlier paper, \([3]\). There, we have found that after neglecting "small jumps", the time evolution of the resultant probability density \( \tilde{\rho}_\varepsilon \) may be written as:

\[
\partial_t \tilde{\rho}_\varepsilon (A, t) = \int_R q_\varepsilon(t, x, A) \tilde{\rho}_\varepsilon (x, t) dx + <v >_A (t) \int_{|y| > \varepsilon} \frac{y}{1 + y^2} d\nu (y). \tag{13}
\]

The measure \( d\nu \) is symmetric around the point \{0\}, hence the second term cancels, and we arrive at

\[
\partial_t \tilde{\rho}_\varepsilon (A, t) = \int_R q_\varepsilon(t, x, A) \tilde{\rho}_\varepsilon (x, t) dx \tag{14}
\]

where the so-called jump intensity reads

\[
q_\varepsilon(t, y, A) = \int_{|y| > \varepsilon} \frac{\theta^\varepsilon(y + x, t)}{\theta^\varepsilon(y, t)} [\chi_A(x + y) - \chi_A(y)] d\nu(x) \tag{15}
\]

and \( \theta^\varepsilon(x, t) \) comes out as a solution of the second pseudodifferential equation in the formula (11).
Let us define (cf. Eq. (12))

$$h_\varepsilon(t, y) = \int_{-\infty}^{+\infty} \frac{\theta_\varepsilon(x + y, t)}{\theta_\varepsilon(y, t)} q_\varepsilon(x) dx$$

(16)

and

$$h_\varepsilon(t, y, x) = \frac{\theta_\varepsilon(x, t)}{\theta_\varepsilon(y, t)} q_\varepsilon(x - y).$$

(17)

Then, clearly the jump intensity (14) takes the form

$$q_\varepsilon(t, y, A) = \int_A h_\varepsilon(t, y, x) dx - h_\varepsilon(t, y) \chi_A(y)$$

(18)

With those notations, we have:

**Lemma 3**

If the function $g(y)$ (cf. Eq. (4)) is uniformly bounded, then $h_\varepsilon(t, y, x)$ is a density of a finite measure and $h_\varepsilon(t, y) = \int_R h_\varepsilon(t, y, x) dx$.

**Proof:**

By our assumption, $g(y) \leq M$ for all $y \in R$. Because of $\theta^\varepsilon(x, t) = \int_R k_\varepsilon(T - t, x - y) g(y) dy$, we have a bound

$$\theta^\varepsilon(x, t) \leq M \int_R k_\varepsilon(T - t, x - y) dy = M.$$

Hence

$$\int_{-\infty}^{+\infty} h_\varepsilon(t, y, x) dx = \int_{-\infty}^{+\infty} \frac{\theta^\varepsilon(x, t)}{\theta^\varepsilon(y, t)} q_\varepsilon(x - y) dx = h_\varepsilon(t, y)$$
and
\[ h_\epsilon(t, y) \leq \frac{M - 2}{\theta^\epsilon(y, t)} \epsilon. \]

It is also clear that \( h_\epsilon(t, y, x) \geq 0 \), which completes the proof.

Let us define \( \bar{h}_\epsilon(t, y, A) = -h_\epsilon(t, y)\chi_A(y) + \int_A h_\epsilon(t, y, x)dx \). It is obvious that \( \bar{h}_\epsilon \)

is a charge (that is a real-valued measure with the property \( \bar{h}_\epsilon(t, y, R) = 0 \)).

We shall show that there exists a step process corresponding to the charge \( \bar{h}_\epsilon \).

To this end let us first prove:

**Lemma 4**

For any Borel set \( A \subset R \), the function \( t \rightarrow \int_A h_\epsilon(t, y, x)dx \) is continuous in \( t \), uniformly in \( A \).

**Proof:**

We have the following estimate (cf. Eq. (18) and Lemma 3):

\[
| \int_A h_\epsilon(t, y, x)dx - \int_A h_\epsilon(t_0, y, x)dx | = | \int_A \frac{\theta^\epsilon(y + x, t)}{\theta^\epsilon(y, t)} q_\epsilon(x)dx - \int_A \frac{\theta^\epsilon(y + x, t_0)}{\theta^\epsilon(y, t_0)} q_\epsilon(x)dx | \\
\leq | \int_{A \cap K^c} \frac{\theta^\epsilon(y + x, t)}{\theta^\epsilon(y, t)} q_\epsilon(x)dx - \frac{\theta^\epsilon(y + x, t_0)}{\theta^\epsilon(y, t_0)} q_\epsilon(x)dx | + | \int_{A \cap K} \frac{\theta^\epsilon(y + x, t)}{\theta^\epsilon(y, t)} q_\epsilon(x)dx - \frac{\theta^\epsilon(y + x, t_0)}{\theta^\epsilon(y, t_0)} q_\epsilon(x)dx | + \\
| \int_A \frac{\theta^\epsilon(y + x, t_0)}{\theta^\epsilon(y, t)} - \frac{\theta^\epsilon(y + x, t_0)}{\theta^\epsilon(y, t_0)} q_\epsilon(x)dx | + \]

where \( K \) is a compact set while \( K^c \) is its complement.
Let us denote the summands $A_1, A_2, A_3$ respectively. For the first summand we have

$$A_1 \leq \frac{1}{\theta^c(y, t)} \sup_{x \in \mathbb{R}} (\theta^c(x, t) + \theta^c(x, t_0)) \int_{K^c} q_e(x)dx.$$ 

But:

$$\sup_{x \in \mathbb{R}} \theta^c(x, t) = \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} k^c(T-t, x-y)g(y)dy \leq M \sup_{x \in \mathbb{R}} \int_{\mathbb{R}} k^c(T-t, x-y)dy = M$$

By defining $N(y) = \sup_{t \in [t_0, t_0+1]} \frac{1}{\theta^c(y, t)}$ and adjusting the compact set $K$ so that

$$\int_{K^c} q_e(x)dx \leq \frac{\delta}{3MN(y)},$$

we arrive at $A_1 \leq \frac{\delta}{3}$.

With the second summand, $A_2$, we proceed as follows:

$$A_2 = | \int_{A \cap K} \left[ \frac{\theta^c(x, t)}{\theta^c(y, t)} - \frac{\theta^c(x, t_0)}{\theta^c(y, t_0)} \right] q_e(y-x)dx | \leq N(y) \sup_{x \in K} |\theta^c(x, t) - \theta^c(x, t_0)| \frac{2}{\pi \epsilon}$$

By choosing $t$ so close to $t_0$ that

$$\sup_{x \in K} |\theta^c(x, t) - \theta^c(x, t_0)| \leq \frac{\pi \epsilon}{6N(y)},$$

we get $A_2 \leq \frac{\delta}{3}$.

Analogously with $A_3$:

$$A_3 \leq \left| \frac{1}{\theta^c(y, t)} - \frac{1}{\theta^c(y, t_0)} \right| 2 \sup_{x \in \mathbb{R}} \theta^c(x, t_0) \frac{2}{\pi \epsilon} \leq \frac{4}{\pi \epsilon} MN^2(y) |\theta^c(y, t_0) - \theta^c(y, t)|$$

where by taking $t$ such that $|\theta^c(y, t_0) - \theta^c(y, t)| \leq \frac{\pi \epsilon}{12MN^2(y)}$ we shall get $A_3 \leq \frac{\delta}{3}$. The overall bound is thus $\delta$, and the Lemma is proved.

As a byproduct of the above demonstration, we realise that the function $t \rightarrow h_\epsilon(t, x, A)$ is continuous in $t$ uniformly on compact sets. As a consequence, see e.g. Theorem 4 in chap. 7, sec. 7 of Ref. [20], there exists a stochastically continuous Markov process $X^\epsilon_t$ with continuous from the right sample paths. Moreover, for any $s \in [0, T]$, $y \in \mathbb{R}$ and $A \subset \mathbb{R}$, there holds:

$$\lim_{t \downarrow s} \frac{p_\epsilon(y, s, A, t) - \chi_A(y)}{t - s} = h_\epsilon(s, y, A)$$

(19)
where \( p_\epsilon(y, s, A, t) \) is the transition kernel of the process \( X^\epsilon_t \).

There follows:

**Theorem 3**

The transition probability density of \( X^\epsilon_t \) reads:

\[
p_\epsilon(y, s, x, t) = k_\epsilon(t - s, x - y) \frac{\theta^\epsilon(x, t)}{\theta^\epsilon(y, s)}
\]

and is a solution of the first Kolmogorov equation:

\[
\partial_sp_\epsilon(y, s, x, t) = -\int_R p_\epsilon(z, s, x, t)\bar{h}_\epsilon(s, y, z)dz
\]

**Proof:**

We must demonstrate that Eq. (19) is valid for the just introduced transition density (compare e.g. also Theorem 1), i.e. there holds:

\[
\lim_{t\downarrow s} \frac{1}{t-s}[k_\epsilon(t - s, x - y) \frac{\theta^\epsilon(x, t)}{\theta^\epsilon(y, s)} - \delta_y(x)] = \bar{h}_\epsilon(s, y, x).
\]

To this end, let us notice (adding and subtracting the same summand) that

\[
\bar{h}_\epsilon(s, y, x) = \frac{\theta^\epsilon(x, s)}{\theta^\epsilon(y, s)} \lim_{t\downarrow s} \frac{1}{t-s}[k_\epsilon(t - s, x - y) - \delta_y(x)] + \frac{\delta_y(x)}{\theta^\epsilon(y, s)} \lim_{t\downarrow s} \frac{1}{t-s}[(\theta^\epsilon(x, t) - \delta_y(x)]
\]

To evaluate the second term, let us take a continuous and bounded function \( a(x) \) and consider

\[
\lim_{t\downarrow s}\int_R \frac{\delta_y(x)}{\theta^\epsilon(y, s)} \frac{1}{t-s}[(\theta^\epsilon(x, t) - \delta_y(x)]a(x)dx =
\]
\[ \lim_{t \downarrow s} \frac{a(y)}{\theta^\varepsilon(y, s)} \frac{1}{t - s} [\theta^\varepsilon(y, t) - \theta^\varepsilon(y, s)] = \frac{a(y)}{\theta^\varepsilon(y, s)} \partial_s \theta^\varepsilon(y, s). \]

So, the second term converges weakly to
\[ \frac{\delta_y(x)}{\theta^\varepsilon(y, s)} \partial_s \theta^\varepsilon(y, s). \]

We know that
\[ \partial_s \theta^\varepsilon(y, s) = |\nabla| \theta^\varepsilon(y, s) = - \int_R [\theta^\varepsilon(y + z, s) - \theta^\varepsilon(y, s)]q_\varepsilon(z)dz. \]

Consequently
\[ \frac{\partial_s \theta^\varepsilon(y, s)}{\theta^\varepsilon(y, s)} = - \int_R \frac{\theta^\varepsilon(y + z, s)}{\theta^\varepsilon(y, s)}q_\varepsilon(z)dz + \frac{2}{\pi \varepsilon} = \frac{2}{\pi \varepsilon} - h_\varepsilon(s, y) \]

and thus
\[ \lim_{t \downarrow s} \frac{1}{t - s} [p_\varepsilon(y, s, x, t) - \delta_y(x)] = \]
\[ \frac{\theta^\varepsilon(x, s)}{\theta^\varepsilon(y, s)}q_\varepsilon(x - y) - \frac{2}{\pi \varepsilon} \delta_y(x) + \frac{2}{\pi \varepsilon} \delta_y(x) - h_\varepsilon(s, y)\delta_y(x) = \]
\[ h_\varepsilon(s, y, x) - h_\varepsilon(s, y)\delta_y(x) = \bar{h}_\varepsilon(s, y, x) \]

The first part of our Theorem is proved, and we can pass to its second part.

To check the validity of the Kolmogorov equation, we shall begin from
\[ \partial_s p_\varepsilon(y, s, x, t) = [\partial_s k_\varepsilon(t - s, x - y)] \frac{\partial^\varepsilon(x, t)}{\theta_\varepsilon(y, s)} - p_\varepsilon(y, s, x, t) \frac{\partial_s \theta^\varepsilon(y, s)}{\theta_\varepsilon(y, s)} \]

But:
\[ \partial_s k_\varepsilon(t - s, x - y) = -[q_\varepsilon * k_\varepsilon(t - s, .)](x - y) + k_\varepsilon(x - y) \frac{2}{\pi \varepsilon} \]

and
\[ \frac{\partial_s \theta^\varepsilon(y, s)}{\theta_\varepsilon(y, s)} = \frac{2}{\pi \varepsilon} - h_\varepsilon(s, y) \]
which leads to

\[ \partial_s p(y, s, x, t) = \]

\[ -[q_e * k_e(t - s, .)](x - y) \frac{\theta^e(x, t)}{\theta^e(y, s)} + \frac{2}{\pi \epsilon} p_e(y, s, x, t) - \]

\[ \frac{2}{\pi \epsilon} p_e(y, s, x, t) + p_e(y, s, x, t) h_e(s, y) = \]

\[ -\frac{\theta^e(x, t)}{\theta^e(y, s)} \int_R q_e(x - y - z) k_e(t - s, z) dz + p_e(y, s, x, t) \int_R \theta^e(y, s) q_e(x) dx. \]

On the other hand

\[ -\int_R p_e(z, s, x, t) h(s, y, z) dz = \]

\[ -\int_R k_e(t - s, x - z) \frac{\theta^e(x, t)}{\theta^e(z, s)} \left[ \frac{\theta^e(z, s)}{\theta^e(y, s)} q_e(z - y) - \delta_y(z) h_e(s, y) \right] dz = \]

\[ -\frac{\theta^e(x, t)}{\theta^e(y, s)} \int_R k_e(t - s, x - z) q_e(z - y) dz + p_e(y, s, x, t) h_e(s, y). \]

Since we know that \( h_e(s, y) = \int_R \frac{\theta^e(x + y, s)}{\theta^e(y, s)} q_e(x) dx \), the assertion (e.g. the validity of the first Kolmogorov equation) follows.

**Corollary**

\( X^e_t \) is a step process.

**Proof:**

It suffices to check that \( p_e(y, s, R, t) = 1 \) (cf. Ref. [20]). Since

\[ p_e(y, s, R, t) = \int_R p_e(y, s, x, t) dx = \int_R k_e(t - s, x - y) \frac{\theta^e(x, t)}{\theta^e(y, s)} dx \]
and, by Theorem 2,

$$\int_R k_\epsilon(t - s, x - y)\theta^\epsilon(x, t)dx = \theta^\epsilon(y, s)$$

the Corollary holds true.

All previous considerations can be finally summarized by showing that the family $X^\epsilon_t$ of step processes consistently approximates (converges to) the process $X_t$. Indeed, we have:

**Theorem 4**

The limit:

$$\lim_{\epsilon \downarrow 0} X^\epsilon_t = X_t$$

holds true in distributions and uniformly in $t \in [0, T]$. Moreover, the transition probability density $p_\epsilon$ converges pointwise to $p$ when $\epsilon \downarrow 0$.

**Proof:**

The probability density of the process $X^\epsilon_t$ equals to $\rho_\epsilon(x, t) = \theta^\epsilon_*(x, t)\theta^\epsilon(x, t)$ and that of the process $X_t$ is given by $\rho(x, t) = \theta_*(x, t)\theta(x, t)$. But, $\theta^\epsilon_*(x, t) = \int_R k_\epsilon(t, x - y)f(y)dy$ and $k_\epsilon(t, x - y)$ converges weakly to the Cauchy kernel $k(t, x - y)$, uniformly in $t$. Consequently $\lim_{\epsilon \downarrow 0} \theta^\epsilon_*(x, t) = \theta_*(x, t)$ also uniformly in $t \in [0, T]$. The same holds true for $\theta^\epsilon(x, t)$, and the first assertion follows.
The second statement follows from the fact that $k_\epsilon(t, x)$ tends to the Cauchy kernel $k(t, x)$ (see Lemma 2) when $\epsilon \downarrow 0$.

As stated before, considerations of the present section were mostly a preparation to the study of perturbed problems. However, it is useful to mention that the conditional Cauchy processes are covered by the developed scheme. In fact, we can here adjust to the Cauchy noise an observation previously utilized in the context of the Wiener noise, \[2, 3, 4\]. The pertinent density can be given in the following form:

$$
\rho(x, t) = \frac{k(y_0, t_0, x, t)k(x, t, z_T, T)}{k(y_0, t_0, z_T, T)}
$$  

(20)

with $y_0, z_T \in R$ and $0 < t_0 < t < T$. All previous considerations directly apply to the interpolating process supported by this density. See also for a discussion of Lévy bridges (while specialised to the Cauchy context) in Ref. \[25\].

III. Perturbations of the Cauchy noise

An important conceptual input in probabilistic solutions of the Schrödinger interpolation problem was the clean identification of the rôle played by the Feynman-Kac kernels, specifically by their joint continuity in spatial variables. This technical feature received proper attention in constructions based on the conditional Wiener measure, \[3, 10\], but no analogous results seem to be in existence relative to other conditional measures, even if the pertinent process and its sample paths are deduced from an infinitely divisible probability law (this issue we have analyzed in the previous section). The same obstacle appears in the context of perturbed processes, where the Feynman-Kac formula is known to be valid, \[11, 12, 13\], but the relevant
properties of the Feynman-Kac kernels have not been investigated in the literature. We are motivated by the strategy of Refs. [3, 5], and the techniques developed in the previous section. Let us address the problem analogous to that of Eq. (11), but now in reference to a perturbed semigroup, [11]:

\[ \partial_t \theta_* = -|\nabla|_\theta - V \theta_* \]  \hspace{1cm} (21)

\[ \partial_t \theta = |\nabla| \theta + V \theta \]

where \( V \) is a measurable function such that:

(a) for all \( x \in R \), \( V(x) \geq 0 \),

(b) for each compact set \( K \subset R \) there exists \( C_K \) such that for all \( x \in K \), \( V \) is locally bounded \( V(x) \leq C_K \).

Then \( V \) is locally integrable and for any compact \( K \) we have

\[ \lim_{t \downarrow 0} \sup_{x \in R} E_x^C \{ \int_0^t \chi_K(X_s^C) V(X_s^C) ds \} = 0. \]  \hspace{1cm} (22)

As a consequence, there holds

**Lemma 5**

If \( 1 \leq r \leq p \leq \infty \) and \( t > 0 \), then the operators \( T_t^V \) defined by

\[ (T_t^V f)(x) = E_x^C \{ f(X_t^C) \exp[-\int_0^t V(X_s^C) ds] \} \]

are bounded from \( L^r(R) \) into \( L^p(R) \). Moreover, for each \( r \in [1, \infty] \) and \( f \in L^r(R) \), \( T_t^V f \) is a bounded and continuous function.
Proof:

See e.g. Ref. [11], Proposition III.1.

We shall also use another identity proved by Carmona, [11], namely:

Lemma 6

For any real-valued \( f, g \in L^2(\mathbb{R}) \) there holds

\[
\int_{\mathbb{R}} dx \ f(x) E_x^C \{ g(X_t^C) \exp \left[ - \int_0^t V(X_s^C) ds \right] \} = \int_{\mathbb{R}} dx \ g(x) E_x^C \{ f(X_t^C) \exp \left[ - \int_0^t V(X_s^C) ds \right] \} .
\]

Proof:

Cf. Eq. (III.9) in Ref. [11].

We need to prove that \( T_t^V \) is an integral operator. To this end, a direct transfer of Simon’s arguments, cf. Ref. [26], originally with respect to the Laplace differential operator, i.e. the usage of the Dunford-Pettis theorem (see pp. 450 in [26]) and Lemma 5, gives rise to:

Lemma 7
For any $p \in [1, \infty]$ and $f \in L^p(R)$ there holds

$$(T^V_t f)(x) = \int_R k^V_t(x,y)f(y)dy$$

where $k^V_t(x,y) \geq 0$ almost everywhere and, for $q$ such that $\frac{1}{q} + \frac{1}{p} = 1$, the kernel satisfies

$$\sup_{x \in R}[\int_R [k^V_t(x,y)]^qdy]^{1/q} < \infty$$

**Proof:**

Cf. Theorem A.1.1 and Corollary A.1.2 in Ref. [26].

Notice that by putting $p = 1$ and thus $q = \infty$ we obtain that $k^V_t(x,y) \in L^\infty(R^2)$.

Our ultimate goal is to utilize $k^V_t(x,y)$ in the context of the Schrödinger boundary data and interpolation problem, [2, 6], hence suitable properties of the kernel must be established. For our purposes, the joint continuity and positivity of the kernel is essential.

**Lemma 8**

$k^V_t(x,y)$ is jointly continuous in $(x,y)$.

**Proof:**

We begin from demonstrating that $k^V_t(x,y) = k^V_t(y,x)$ almost everywhere.
By Lemma 6, we have

$$\int \int \mathbb{R}^2 dx \, dy \, f(x) k^V_t(x, y) g(y) = \int \int \mathbb{R}^2 dx \, dy \, g(x) k^V_t(x, y) f(y),$$

hence

$$\int \int \mathbb{R}^2 dx \, dy \, f(x) g(y) [k^V_t(x, y) - k^V_t(y, x)] = 0$$

for all $f, g \in L^2(R) \cap L^1(R)$. The same holds true for all finite combinations $\sum_{i,j} a_{ij} f_i(x) g_j(y)$. Therefore

$$\int \int \mathbb{R}^2 \left[ k^V_t(x, y) - k^V_t(y, x) \right] f(x, y) dx \, dy = 0$$

for all $f(x, y)$ from a dense subset of $L^1(R^2)$. Because $L^\infty(R^2)$ is the dual space to $L^1(R^2)$, we conclude that $k^V_t(x, y) = k^V_t(y, x)$ almost everywhere.

Let us exploit the semigroup property of $k^V_t(x, y)$:

$$k^V_t(x, y) = \int_R k^V_{t/2}(x, w) k^V_{t/2}(w, y) dw.$$ 

For each $y$, $w \rightarrow k^V_{t/2}(w, y) \in L^\infty(R)$ so, by Lemma 5, $k^V_t(x, y)$ is continuous in $x$.

By the symmetry, $k^V_t(x, y)$ is separately continuous in $x$ and $y$.

Let us consider a sequence $(x_n, y_n) \rightarrow (x, y)$. Then:

$$|k^V_t(x_n, y_n) - k^V_t(x_0, y_0)| \leq$$

$$| \int \int \mathbb{R}^2 dw \, dz \, [k^V_{t/3}(x_n, w) - k^V_{t/3}(x_0, w)] [k^V_{t/3}(w, z) - k^V_{t/3}(w, y_n)] +$$

$$| \int \int \mathbb{R}^2 dw \, dz \, k^V_{t/3}(x, w) [k^V_{t/3}(w, z) - k^V_{t/3}(w, y_n)] | +$$

$$| \int_R dw \, [k^V_{t/3}(x, w) - k^V_{t/3}(x_0, w)] [k^V_{2t/3}(w, y_n)] + |k^V_t(x_0, y_n) - k^V_t(x_0, y_0)|.$$ 

Because of

$$||k^V_{2t/3}(\cdot, y_n)||_{L^\infty} < C$$

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for all $y_n$, knowing that $\sup_n k_t^{V_{n/3}}(x_n, w)$ exists and is integrable with respect to $w$, by the Lebesgue dominated convergence theorem the first summand tends to zero. Hence, $k_t^{V}(x, y)$ is jointly continuous in $(x, y)$.

Lemma 9

$k_t^{V}(x, y)$ is strictly positive.

Proof:

Because for the Cauchy process we have, [27] (more general estimates of the growth of random walks and Lévy processes can be found in [28]):

$$E^C_x \{ \sup_{0 \leq s \leq t} |X_s^C| > n \} \leq 3 \sup_{0 \leq s \leq t} E^C_x \{ |X_s^C| > \frac{n}{3} \}$$

and

$$\sup_{0 \leq s \leq t} E^C_x \{ |X_s^C| > \frac{n}{3} \} = E^C_x \{ |X_t^C| > \frac{n}{3} \} = 1 - \frac{2}{\pi} \arctan \left( \frac{n}{3t} \right)$$

there follows:

$$\lim_{n \to \infty} E^C_x \{ \sup_{0 \leq s \leq t} |X_s^C| > n \} = 0$$

This property will be used in below.

Let $0 < \delta < 1$, then:

$$\int_{y-\delta}^{y+\delta} dy k_t^{V}(x, y) = E^C_x \{ \chi_{[y-\delta,y+\delta]}(X_t^C) \exp[- \int_0^t V(X_s^C) ds] \} .$$

By the previously deduced property, for fixed $x$ and $y$, we can choose a compact set
$[-n, n]$ such that

$$E^C_x \{ \Omega^{(0,x)}_{(t, [y-\delta, y+\delta])}(n) \} > \frac{1}{2} \int_{y-\delta}^{y+\delta} k_t(x, y) dy$$

where

$$\Omega^{(0,x)}_{(t, [y-\delta, y+\delta])}(n) = \{ \omega : \omega(0) = x, \omega(t) \in [y-\delta, y+\delta]; s \in [0, t] \Rightarrow \omega(s) \in [-n, n] \}$$

and $k_t(x, y)$ is the Cauchy kernel. Hence

$$\int_{y-\delta}^{y+\delta} dy k_t^V(x, y) \geq \int_{\Omega(n)} \exp\left[-\int_0^t V(X_s^C)ds\right] dP^C_x(\omega) \geq \frac{1}{2} \exp(-c_n t) \cdot \int_{y-\delta}^{y+\delta} k_t(x, y) dy$$

where $c_n = \sup_{x \in [-n, n]} V(x)$.

Because $k_t^V(x, y)$ is continuous and $\delta$ was arbitrary, we get

$$k_t^V(x, y) \geq \frac{1}{2} \exp(-c_n t) \cdot k_t(x, y)$$

The assertion of Lemma 9 is thus valid.

Lemma 8 and 9 provide us with a strictly positive and jointly continuous in space variables kernel, which can be directly exploited for the analysis of the Schrödinger interpolation problem, as exemplified by Eqs. (3)- (6), see also [2, 3, 6]. Indeed, let $\rho_0(x)$ and $\rho_T(x)$ be strictly positive densities. Then, the Markov process $X_t^V$ characterized by the transition probability density:

$$p^V(y, s, x, t) = k_{t-s}^V(x, y) \frac{\theta(x, t)}{\theta(y, s)}$$

and the density of distributions

$$\rho(x, t) = \theta_*(x, t) \theta(x, t)$$

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where:

\[ \theta_s(x, t) = \int_R k_t^V(x, y)f(y)dy \]
\[ \theta_s(y, t) = \int_R k_t^V(x, y)g(x)dx \]

is precisely that interpolating Markov process to which Theorem 1 extends its validity, when the perturbed semigroup kernel replaces the Cauchy kernel.

Clearly, for all \(0 \leq s \leq t \leq T\) we have

\[ \theta_s(x, t) = \int_R k_{t-s}^V(x, y)\theta_s(y, s)dy \]
\[ \theta(y, s) = \int_R k_{t-s}^V(x, y)\theta(x, t)dx \]

and that suffices for the Theorem 1 to hold true in the present case as well.

Following the strategy of the previous section, we shall investigate an issue of approximating the perturbed Cauchy process (set by Lemmas 8, 9 and Theorem 1) by means of step processes.

Let us first invoke the step process \(Y_\epsilon^t\) of Lemma 1. It corresponds to the unperturbed generator \(|\nabla|_\epsilon\). To account for a perturbation and the involved perturbed semigroup, let us consider a multiplicative, homogeneous and contracting functional:

\[ \alpha_t^\epsilon(\omega) = \exp[-\int_s^t V(Y_\epsilon^\tau(\omega))d\tau] \]

of the process \(Y_\epsilon^t\), for times \(0 \leq s \leq t \leq T\).

We recall that the process \(Y_\epsilon^t\) is a step process obtained from the Cauchy process by neglecting ”small jumps” (the \(\epsilon\)-cutoff).

We shall associate with the multiplicative functional (25) the process \(Y_\epsilon^t, V\) and prove that under additional restrictions on the potential \(V\), the pertinent perturbed
process is also a step process.

**Theorem 5**

Let $0 \leq V(x) \leq M$ for all $x \in R$. The transition function:

$$p_{\epsilon,V}(t, x, \Gamma) = E_x^\epsilon \{ \chi_{\Gamma}(Y_t^\epsilon) \exp[- \int_0^t V(Y_s^\epsilon)ds] \}$$

determines the step process $Y_t^{\epsilon,V}$.

**Proof:**

By Theorem 3.8 of Ref. [19], a sufficient condition for the existence of a Markovian step process $Y_t^{\epsilon,V}$ is that its transition function obeys

$$\lim_{t \downarrow 0} p_{\epsilon,V}(t, x, \{x\}) = 1$$

uniformly in $x \in R$.

Let us choose $t_1 > 0$ so that $1 - \delta \leq \exp(-Mt_1)$ is secured. In view of

$$\exp(-Mt) \leq \exp[- \int_0^t V(Y_s^\epsilon(\omega))ds] \leq 1$$

for all $\omega$, we have for all $t < t_1$ the following estimate:

$$(1 - \delta)p_\epsilon(t, x, \Gamma) \leq p_{\epsilon,V}(t, x, \Gamma) \leq p_\epsilon(t, x, \Gamma).$$

On the other hand, there exists $t_2$ such that for all $t < t_2$

$$p_\epsilon(t, x, \{x\}) \geq 1 - \delta$$
is valid for all $x \in R$.
Hence, for all $t < \min(t_1, t_2)$ we get

$$(1 - \delta)^2 \leq p_{\epsilon,V}(t, x, \{x\}) \leq 1$$

Because $\delta$ is arbitrary, after taking $\delta \to 0$, the assertion follows.

From the formula $p_{\epsilon,V}(t, x, \Gamma) \leq p_{\epsilon}(t, x, \Gamma)$ we conclude that the transition function $p_{\epsilon,V}(t, x, \Gamma)$ is absolutely continuous with respect to the Lebesgue measure, and hence posesses a density $k_{\epsilon,V}(t, x, y)$.

A new process $X_{t}^{\epsilon,V}$ can be defined by considering a multiplicative transformation of the process $Y_{t}^{\epsilon,V}$ by means of

$$\alpha_{s}^{t} = \frac{\theta^{\epsilon}(Y_{t}^{\epsilon,V}, t)}{\theta^{\epsilon}(Y_{s}^{\epsilon,V}, s)}$$

where $\theta^{\epsilon}$ is a solution of $\partial_{t}\theta^{\epsilon} = |\nabla|_{\epsilon}\theta^{\epsilon} + V\theta^{\epsilon}$.

The transition probability density of $X_{t}^{\epsilon,V}$ reads

$$p_{\epsilon,V}(s, y, t, x) = k_{\epsilon,V}(t - s, y, x)\frac{\theta^{\epsilon}(x, t)}{\theta^{\epsilon}(y, s)}$$

and by repeating arguments mimicking those of Section II, one can show that the perturbed step process $X_{t}^{\epsilon,V}$ converges in distribution to the perturbed Cauchy process $X_{t}^{V}$, when $\epsilon \to 0$, uniformly in $t \in [0, T]$.

A concise summary of all mathematical arguments of sections II and III, reads:
(a) We have found a solution of the Schrödinger interpolation problem whose kernel function is determined by the Cauchy generator plus a potential.
(b) We have described the pertinent process (and its simpler versions, like the conditional Cauchy process of section II) as a limit of step processes.

(c) The developed techniques can be used to investigate the existence issue (including that of the step process approximation) of more general jump-type processes, in particular those related to the quantum evolution with relativistic Hamiltonians, [5, 29].

Remark:

In the present paper, to simplify calculations and to make formulas more transparent, we have considered processes associated with the Cauchy generator (and thus with the \( \alpha \)-stable symmetric process as a major tool) in space dimension 1. A glance at the construction of solutions of the Schrödinger problem makes clear that the previous limitations are inessential. In fact, we could consider any \( \alpha \in (0, 2) \) -symmetric stable processes on \( \mathbb{R}^n \), for arbitrary \( n \geq 1 \), and secure the strict positivity and joint continuity in space variables of the corresponding transition density. Such properties for \( n \geq 2 \) and for potentials from the Kato class \( K_{n,\alpha} \) were established in the very recent publication, [30], Theorems 3.3 and 3.5.

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