AN ADAPTATIVE MODEL FOR A MULTISTAGE STRUCTURED POPULATION UNDER FLUCTUATING ENVIRONMENT

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Abstract. We consider a modified version of a mathematical model describing the dynamics of the European Grapevine Moth, studied by Ainseba, Picart and Thiery. The improvement consists in including adaptation at the larval stage. We establish well-posedness of the model under suitable hypothesis.

1. Introduction. Models with continuous age structure represent a special case of physiologically structured population models. These latter may incorporate other state variables like size, genetic trait and level of maturation of individuals.

Age structured models were first introduced by Sharp and Lotka [25], then Mc Kendrick and Von Foester [21] studied the following linear case

\[
\begin{aligned}
\frac{dn}{dt} + \frac{dn}{da} &= -d(a)n(a,t), \\
n(0, t) &= \int_0^{+\infty} b(a)n(a, t)da, \\
n(a, 0) &= n_0(a),
\end{aligned}
\]

where \(d, b\) and \(n_0\) are continuous functions.

Webb [31] analyzed this class of models using linear semigroup theory.

Later on, Gurtin and Mac Camy [12] turned their attention to the nonlinear case. They considered the case where \(b\) and \(d\) become functions of the total population

\[P(t) = \int_0^{+\infty} n(a, t)da.\]

We refer the reader to Iannelli [13] for mathematical analysis of nonlinear age structured models.

The long term behavior of solutions was studied by Marcati [20], Farkas [10] and Slobodkin [26].

Slobodkin [26] found that age or size alone was not sufficient to characterize the physiological behavior of Daphnia Obtuse population. Then a major change in populations analysis was introduced by Sinko and Streifer [27] with age and size structure. This class of models has been formulated and analyzed by Calsina, Saldana [7], Metz and Diekmann [22], to cite a few.

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The goal of this paper is to present a modified version of a system describing the life cycle of the European Grapevine Moth, introduced in [2], and [1]. In contrast to the works in [2], and [1], where the trait is assumed to be constant, we investigate a model with phenotypical trait and age structure. The trait can vary at the pupae stage when insect changes its morphology, and adapt its strategies to maximize the fitness.

We establish well posedness of the system. Our approach is similar to that used in [16], [15], [5], [6] and [3].

The paper is organized as follows: In the next section, we give the formulation of the model. Section 3 is devoted to existence of solutions and continuous dependence on initial values. In section 4, we discuss the achievements of the paper.

2. Description of the model. We divide the population of insects into five classes: eggs, larvae, pupae, female and male. In the first stage, females lay eggs. Eggs remain until natural death or emerge into larvae. After a few days, larvae die by natural death or evolves into pupae. Individuals stay approximately seven days before encloding into adult. For each category, we consider the chronological age of the individuals that we denotes a. The population changes over time t, and a physiologically trait denoted x.

The following model describes the dynamics of these populations, for \( a \in (0, L_i) \), \( t \geq 0 \), \( x \in \Omega \), and \( i = e, l, f, m, c \).

\[
\begin{align*}
\frac{\partial}{\partial t} u^e(t, a, x) + \frac{\partial}{\partial a} [v^e(E(t), a) u^e(t, a, x)] &= \quad -\mu^e(E(t), a) u^e(t, a, x) - \beta^e(E(t), a) u^e(t, a, x), \\
\frac{\partial}{\partial t} u^l(t, a, x) + \frac{\partial}{\partial a} [v^l(E(t), a) u^l(t, a, x)] &= \quad -\mu^l(P^l(t, x), E(t), a) u^l(t, a, x) - \beta^l(E(t), a) u^l(t, a, x), \\
\frac{\partial}{\partial t} u^c(t, x) &= \int_{\Omega} \int_{0}^{L_i} \beta^i(E(t), a) \gamma(x, y) u^l(t, a, y) \, da \, dy - \mu^c(t) u^c(t, x), \\
\frac{\partial}{\partial t} u^f(t, a, x) + \frac{\partial}{\partial a} [v^f(E(t), a) u^f(t, a, x)] &= \quad -\mu^f(E(t), a) u^f(t, a, x), \\
\frac{\partial}{\partial t} u^m(t, a, x) + \frac{\partial}{\partial a} [v^m(E(t), a) u^m(t, a, x)] &= \quad -\mu^m(E(t), a) u^m(t, a, x).
\end{align*}
\]

The boundary conditions are defined by

\[
\begin{align*}
u^e(E(t), 0) u^e(t, 0, x) &= \int_{0}^{L_f} \beta^f(P^f(t, x), P^m(t, x), E(t), s) u^f(t, s, x) \, ds, \\
v^l(E(t), 0) u^l(t, 0, x) &= \int_{0}^{L_e} \beta^e(E(t), s) u^e(t, s, x) \, ds, \\
v^f(E(t), 0) u^f(t, 0, x) &= \int_{0}^{L_f} \sigma \beta^l(E(t), s) u^l(t, s, x) \, ds, \\
v^m(E(t), 0) u^m(t, 0, x) &= \int_{0}^{L_i} (1 - \sigma) \beta^l(E(t), s) u^l(t, s, x) \, ds,
\end{align*}
\]

where \( \sigma \) denotes the sex ratio and \( t > 0 \).

The system is completed with initial conditions

\[
u^k(0, a, x) = u^k_0(a, x) \text{ in } (0, L^k) \times \Omega, k = e, l, f, m,
\]

(3)
and
\[ u^e (0, x) = u^e_0 (x) \quad \text{a.e in } \Omega. \tag{4} \]

The first equation of (1) gives the evolution of the eggs density. The second equation of (1) describes the temporal variation of the larval density. The third equation of (1) gives the density of individuals at the pupae stage and takes into account the changes and the adaptation of the new generation. The fourth and last equations of (1) describes respectively the evolution of female and male densities.

Here \( L_k \) is the maximum age for the \( k \)-stage. The quantity \( E \) corresponds to the climatic and environmental factors and it is time-dependent. The function \( \mu^k \) is the \( k \)-stage specific per capita mortality function. The total population at time \( t \) and trait \( x \) for the \( k \)-stage is then defined by

\[ P^k (t, x) := \int_0^{L_k} u^k (t, a, x) \, da. \]

Because of the inter-individual competition among larvae for food, we suppose that \( \mu^l \) depends on the total population \( P^l (t, x) \). The function \( \beta^k \) denotes the \( k \)-stage age specific transition function. The function \( v^k \) represents the \( k \)-stage specific per capita growth function which depends on the physiological age. The inflow of the mutant newborns at the pupae stage is given by the integral operator

\[ \int_0^{L_l} \int \Omega \beta^l (E(t), a) \gamma (x, y) u^l (t, a, y) \, dady, \]

where \( \Omega \) is a bounded domain of \( \mathbb{R}^n \). The quantity

\[ \gamma (x, y), \]

represents the probability that a larva with trait \( y \) emerges with a new trait \( x \), see [5], [8], and [9] for similar representation of mutation.

The main idea behind the model is that individuals of the pest population change their behavior, and adapt their dynamic at the pupae stage, where the insect changes morphologically and adapt its abilities and strategy to maximize its fitness, to delay emergence and resist to hostile environment.

**Remark 1.** Note that in the integral giving the newborns (2), we use 0 as a minimal value only for mathematical convenience. It may be replaced by a positive value.

3. **Main result.** We develop a solution determined by the method of characteristics. We formulate the problem as a vector integral equation. Then, we prove the existence and uniqueness of solutions using the Banach Contraction Principle. First, we introduce some notations. For every \( t > 0 \), and \( x \in \Omega \), we define

\[ F^e \left( u^f (t, \cdot, x) \right) = \int_0^{L_f} \beta^f \left( P^f (t, x), P^m (t, x), E(t), s \right) u^f (t, s, x) \, ds, \]
\[ F^\ell (u^\ell (t,.,x)) = \int_0^{L_\ell} \beta^\ell (E(t), s) u^\ell (t, s, x) \, ds, \]
\[ F^f (u^f (t,.,x)) = \int_0^{L_f} \sigma \beta^f (E(t), s) u^f (t, s, x) \, ds, \]
and
\[ F^m (u^m (t,.,x)) = \int_0^{L_m} (1 - \sigma) \beta^m (E(t), s) u^m (t, s, x) \, ds. \]

For \( k = e, l, f, m \), Let
\[
\widetilde{G}^k (t, u^k (t,.,x)) (a) = G^k (u^k (t,.,x)) (a) - \frac{\partial v^k}{\partial a} (E(t), a) u^k (t, a, x)
\]
where
\[
G^e (u^e (t,.,x)) (a) = -\mu^e (E(t), a) u^e (t, a, x) - \beta^e (E(t), a) u^e (t, a, x),
\]
\[
G^l (u^l (t,.,x)) (a) = -\mu^l (P^l (t, x), E(t), a) u^l (t, a, x) - \beta^l (E(t), a) u^l (t, a, x),
\]
\[
G^f (u^f (t,.,x)) (a) = -\mu^f (E(t), a) u^f (t, a, x),
\]
\[
G^m (u^m (t,.,x)) (a) = -\mu^m (E(t), a) u^m (t, a, x).
\]

For every \( t > 0 \), and \( x \in \Omega \), let
\[
H (t, x, u^l (t,.,x)) = \int_\Omega \int_0^{L_l} \beta^l (E(t), a) \gamma (x, y) u^l (t, a, y) \, dady.
\]

### 3.1. Assumptions.
To obtain the main results, we require the following assumptions

(V) For each \( k \in \{e, l, f, m\} \), the function
\[
v^k (E(t), a),
\]
is strictly positive and continuously differentiable with respect to \( a \). In addition, there exists a positive constant \( L_v^k \) such that
\[
\left| \frac{\partial v^k}{\partial a} (E(t), a) \right| \leq L_v^k,
\]
uniformly with respect to \( t \) and \( a \). Let
\[
L_v = \max \{L_v^k, \text{ for } k = e, l, f, m\}.
\]

(B) The birth functions
\[
\beta^l (P^l (t, x), P^m (t, x), E(t), a), \text{ and } \beta^k (E(t), a), k = l, e,
\]
are bounded, nonnegative. The function \( \beta^l \) is Lipschitz continuous with constant \( L_{\beta^l} \) with respect to the variables \( P^l (t, x), P^m (t, x) \).

(M) The mortality function \( \mu^l (P^l (t, x), E(t), a) \) is non-negative and locally bounded. The mortality function \( \mu^k (E(t), a) \), is non-negative and bounded for each \( k = e, f, m \).
The mortality function of larvae stage
\[ \mu^l (P^l(t,x), E(t), a) , \]
is nonnegative, locally bounded, Lipschitz continuous with respect to the first variable.

The Pupae mortality \( \mu_c \) is nonnegative, continuous and bounded.

(K) the kernel \( \gamma(x,y) \) is a probability density function, it satisfies
\[ \gamma(x,y) \geq 0, \]
and is uniformly bounded in \( \Omega \times \Omega \) by \( \|\gamma\|_{\infty} \).

3.2. Existence of solutions. For \( k \in \{e, l, f, m \} \), we define the characteristic curve
\( X^k(t; t_0, a_0) \).

through \( (t_0, a_0) \) by the solution of the differential equation
\[
\left\{ \begin{array}{l}
\frac{da}{dt} = v^k (E(t), a(t)), \\
a(t_0) = a_0.
\end{array} \right.
\]

Assumptions on the function \( v^k \) imply that there exists a unique solution
\( a(t) = X^k(t; t_0, a_0) \).

Let
\[ z_k(t) = X^k(t; 0, 0), \]
be the characteristic curve through the origin. This curve separates the trajectories of individuals that were present at the initial time \( t = 0 \) from the trajectories of those individuals born after the initial time.

We define
\( \tau_k = \tau(t, a) \),
implicitly by the relation
\[ X^k(t; \tau_k, 0) = a. \]

We define
\[ U^k_{(t_0, a_0)}(t, x) = u^k(t, X^k(t; t_0, a_0), x), \]
than
\[ \frac{dU^k_{(t_0, a_0)}(t, x)}{dt} = \tilde{G}^k(t, u^k(t, . , x)) (X^k(t; t_0, a_0)). \]

Integrating this last equation, gives
\[ U^k_{(t_0, a_0)}(t, x) = U^k_{(t_0, a_0)}(t_*, x) + \int_{t_*}^t \tilde{G}^k(s, u^k(s, . , x)) (X^k(s; t_0, a_0)) ds, \]
for some initial time \( t_* \geq 0 \).

We distinguish two cases:

i) If \( a_0 \leq z(t_0) \), then \( \tau_k = \tau(t_0, a_0) > 0 \) and the solution is given by
\[ U^k_{(t_0, a_0)}(t, x) = U^k_{(t_0, a_0)}(\tau_k, x) + \int_{\tau_k}^t \tilde{G}^k(s, u^k(s, . , x)) (X^k(s, \tau_k, 0)) ds. \]
ii) If \( a_0 \geq z(t_0) \), then we take \( \tau_k = 0 \) as the initial time and the solution is given by

\[
U_k(t_0, a_0) (t, x) = U_k(t_0, a_0) (0, x) + \int_0^t \tilde{G}^k(s, u_k(s, . , x)) (X^k(s; t_0, a_0)) ds,
\]
with

\[
U_k(t_0, a_0) (0, x) = u_k(0, X^k(0; t_0, a_0), x) = u_0^k(X^k(0, t_0, a_0), x).
\]

**Definition 2.** A vector \((u^e, u^l, u^f, u^m, u^c)\) is a solution of the system (1) up to time \( T > 0 \), with boundary and initial conditions (2), (3) and (4), if

\[
u^k \in L^\infty ([0, T] \times \Omega, L^1 (0, L^k)) \quad \text{for} \quad k = e, l, f, m \quad \text{and} \quad u^c \in L^\infty ([0, T] \times \Omega),
\]

And

\[
u^e(t, a, x) = \begin{cases}
\frac{e^e(t, \tau e, a, x)}{\nu_0^e(E(t, \tau e, 0))} + \int_{\tau e}^t \tilde{G}^e(s, u^e(s, . , x)) (X^e(s, \tau e, 0)) ds, \quad \text{if} \quad a \leq z_e(t), \\
u_0^e(X^e(0, t, a), x) + \int_0^t \tilde{G}^e(s, u^e(s, . , x)) (X^e(s, t, a)) ds, \quad \text{if} \quad a \geq z_e(t),
\end{cases}
\]

\[
u^l(t, a, x) = \begin{cases}
\frac{e^l(t, \tau l, a, x)}{\nu_0^l(E(t, \tau l, 0))} + \int_{\tau l}^t \tilde{G}^l(s, u^l(s, . , x)) (X^l(s, \tau l, 0)) ds, \quad \text{if} \quad a \leq z_l(t), \\
u_0^l(X^l(0, t, a), x) + \int_0^t \tilde{G}^l(s, u^l(s, . , x)) (X^l(s, t, a)) ds, \quad \text{if} \quad a \geq z_l(t),
\end{cases}
\]

\[
u^f(t, a, x) = \begin{cases}
\frac{e^f(t, \tau f, a, x)}{\nu_0^f(E(t, \tau f, 0))} + \int_{\tau f}^t \tilde{G}^f(s, u^f(s, . , x)) (X^f(s, \tau f, 0)) ds, \quad \text{if} \quad a \leq z_f(t), \\
u_0^f(X^f(0, t, a), x) + \int_0^t \tilde{G}^f(s, u^f(s, . , x)) (X^f(s, t, a)) ds, \quad \text{if} \quad a \geq z_f(t),
\end{cases}
\]

\[
u^m(t, a, x) = \begin{cases}
\frac{e^m(t, \tau m, a, x)}{\nu_0^m(E(t, \tau m, 0))} + \int_{\tau m}^t \tilde{G}^m(s, u^m(s, . , x)) (X^m(s, \tau m, 0)) ds, \quad \text{if} \quad a \leq z_m(t), \\
u_0^m(X^m(0, t, a), x) + \int_0^t \tilde{G}^m(s, u^m(s, . , x)) (X^m(s, t, a)) ds, \quad \text{if} \quad a \geq z_m(t),
\end{cases}
\]

\[
u^c(t, a, x) = - \int_0^t \mu_c(s) ds + \int_0^t \mu_c(r) dr + \int_0^t e^{H(s, x, u^l(s, . , .))} ds.
\]

For \( k = e, l, f, m \), let \( L^1 (0, L^k; R^m) \) be the Banach space of Lebesgue integrable function with the norm

\[
\|u\|_{L^1} := \int_0^L |u(a)| da.
\]
Let $Q = [0, T] \times \Omega$, $E^k = L^\infty (Q, L^1 (0, L^k))$ and $E$ be the Banach space $E = E^e \times E^l \times E^f \times E^m \times L^\infty (Q)$, endowed with the norm $\| u \| = \left( \sup_Q \| u^e \|_{L^1} + \sup_Q \| u^l \|_{L^1} + \sup_Q \| u^f \|_{L^1} + \sup_Q \| u^m \|_{L^1} \right) + \| u^e \|_\infty$.

Let $M > 0$, be chosen such that $0 \leq u_0^k (a, x) \ a.e \ in \ (0, L^k) \times \Omega,$

$$\int_0^{L_k} u_0^k \, da \leq M, \ k = e, l, f, m \ and \ 0 \leq u_0^k (x) \leq M \ a.e \ in \ \Omega.$$  

We denote $u = (u^e, u^l, u^f, u^m, u^e)$ and define the closed subset in $E$ by

$$X = \left\{ u \in E : \sup_Q \int_0^{L_k} (t, a, x) \, da \leq 2M, \ k = e, l, f, m \ and \ \sup_Q u^c (t, x) \leq 2M \right\}.$$  

The space $X$ is complete since it is a closed subset of $E$.

**Remark 3.** For simplicity of notations, except precision, we shall write $L^1$ instead of $L^1 (0, L^k; R^m)$.

We define the map $K : X \to X,$ such that

$$K (u^e, u^l, u^f, u^m, u^e) = (\tilde{u}^e, \tilde{u}^l, \tilde{u}^f, \tilde{u}^m, \tilde{u}^e),$$

where

$$\tilde{u}^e (t, a, x) = \begin{cases} 
\frac{F^e (u^l (\tau_e, \ldots, x))}{\sigma^e (E(\tau_e), 0)} + \int_{\tau_e}^t \widetilde{G}^e (s, u^e (s, \ldots, x)) \left( X^e (s, \sigma_e, 0) \right) \, ds, & \text{if } a \leq z_e (t), \\
u_0^e (X^e (0, t, a), x) + \int_0^t \widetilde{G}^e (s, u^e (s, \ldots, x)) \left( X^e (s, t, a) \right) \, ds, & \text{if } a \geq z_e (t), 
\end{cases}$$

$$\tilde{u}^l (t, a, x) = \begin{cases} 
\frac{F^l (u^l (\tau_l, \ldots, x))}{\sigma^l (E(\tau_l), 0)} + \int_{\tau_l}^t \widetilde{G}^l (s, u^l (s, \ldots, x)) \left( X^l (s, \sigma_l, 0) \right) \, ds, & \text{if } a \leq z_l (t), \\
u_0^l (X^l (0, t, a), x) + \int_0^t \widetilde{G}^l (s, u^l (s, \ldots, x)) \left( X^l (s, t, a) \right) \, ds, & \text{if } a \geq z_l (t), 
\end{cases}$$

$$\tilde{u}^f (t, a, x) = \begin{cases} 
\frac{F^f (u^l (\tau_f, \ldots, x))}{\sigma^f (E(\tau_f), 0)} + \int_{\tau_f}^t \widetilde{G}^f (u^l (s, \ldots, x)) \left( X^f (s, \sigma_f, 0) \right) \, ds, & \text{if } a \leq z_f (t), \\
u_0^f (X^f (0, t, a), x) + \int_0^t \widetilde{G}^f (u^l (s, \ldots, x)) \left( X^f (s, t, a) \right) \, ds, & \text{if } a \geq z_f (t), 
\end{cases}$$
unique solution in
Theorem 4. Under conditions (V), (B), (K) and (M), the system
(2).
Auxiliary results.
\[ \begin{align*}
\text{a) For } k \in \{1, f, m\}, \text{ the function } F^k : L^1 \rightarrow R \text{ is locally lipschitz, in the sense that for any } r > 0 \text{ there exists a positive constant } c_1 > 0 \text{ independent of } k \text{ such that } \\
\left| F^k (\varphi_1 (t, ., x)) - F^k (\varphi_2 (t, ., x)) \right| \leq c_1 \left\| \varphi_1 (t, ., x) - \varphi_2 (t, ., x) \right\|_{L^1} \\
\text{for } \varphi_1 (t, ., x), \varphi_2 (t, ., x) \in L_1, \text{ with } \left\| \varphi_1 (t, ., x) \right\|_{L^1}, \left\| \varphi_2 (t, ., x) \right\|_{L^1} \leq r. \\
\text{For } k = e, \text{ the function } F^e : L^1 \rightarrow R \text{ satisfies } \\
\left| F^e (\varphi_1^e (t, ., x)) - F^e (\varphi_2^e (t, ., x)) \right| \leq c_1 \left( \left\| \varphi_1^e (t, ., x) - \varphi_2^e (t, ., x) \right\|_{L^1} + \\
\left\| \varphi_1^m (t, ., x) - \varphi_2^m (t, ., x) \right\|_{L^1} \right) \\
\text{for } \varphi_1^e (t, ., x), \varphi_2^e (t, ., x) \in L_1, \text{ with } \left\| \varphi_1^e (t, ., x) \right\|_{L^1}, \left\| \varphi_2^e (t, ., x) \right\|_{L^1} \leq r. \\
\text{b) For } k \in \{e, l, f, m\}, \text{ the function } G^k : L^1 (0, L_k) \rightarrow L^1 (0, L_k) \text{ is locally lipschitz, in the sense that for any } r > 0 \text{ there exists a positive constant } c_2 \text{ independent of } k \text{ such that } \\
\left\| G^k (\varphi_1 (t, ., x)) - G^k (\varphi_2 (t, ., x)) \right\|_{L_1} \leq c_2 \left\| \varphi_1 (t, ., x) - \varphi_2 (t, ., x) \right\|_{L^1}, \\
\text{for } \varphi_1 (t, ., x), \varphi_2 (t, ., x) \in L_1, \text{ with } \left\| \varphi_1 (t, ., x) \right\|_{L^1}, \left\| \varphi_2 (t, ., x) \right\|_{L^1} \leq r. \\
\text{c) For every } (t, x) \in R \times \Omega, \text{ the mapping } u^l (t, ., .) \rightarrow H (t, x, u^l (t, ., .)) \text{ is lipschitzian from } L^1 ((0, L^1) \times \Omega) \text{ to } R, \text{ in the sense that there exists a positive constant } c_3 \text{ such that } \\
\left| H (t, x, u^l_1 (t, ., .)) - H (t, x, u^l_2 (t, ., .)) \right| \leq c_3 \left\| u^l_1 (t, ., .) - u^l_2 (t, ., .) \right\|_{L_1 ((0, L^1) \times \Omega)}, \\
\text{for } u^l_1 (t, ., .), u^l_2 (t, ., .) \in L^1 ((0, L^1) \times \Omega). 
\end{align*} \]
Proof. a) For \( k = e \), let \( \varphi^1(t,.,x), \varphi^2(t,.,x) \in L^1(0,L_f) \). We denote here by \( \varphi^j(t,s,x) = \varphi^1(t,s,x) - \varphi^2(t,s,x), j = e, l, m \) then we obtain

\[
\left| F^e \left( \varphi^1(t,.,x) \right) - F^e \left( \varphi^2(t,.,x) \right) \right| \leq \int_0^{L_f} \left| \beta^f \left( P^1_1, P^m_1, E, s \right) \varphi^1(t,s,x) - \beta^f \left( P^l_2, P^m_2, E, s \right) \varphi^2(t,s,x) \right| ds \leq 0
\]

where

\[
c^1_e = 0
\]

The other cases are obtained quite similarly. To finish the proof it suffices to take

\[
c_1 = \max \left( c^k_e \right) \text{ for } k \in \{e, l, f, m\}.
\]

b) The idea of the proof is the same as for the case (a), so we omit it.

c) From the definition of the function \( H \), we have

\[
H(t,x,u^1_1(t,.,.)) - H(t,x,u^1_2(t,.,.)) \leq \int_0^{L_i} \int_0^{L_f} \beta^f(E(t),a) \gamma(x,y) \left( u^1_1(t,a,y) - u^1_2(t,a,y) \right) dady
\]

\[
\leq \left\| \beta^f \right\|_{L_\infty} \left\| \gamma \right\|_{L_\infty} \left\| u^1_1(t,.,.) - u^1_2(t,.,.) \right\|_{L_2}
\]

For the reader’s convenience, we include the following statement.

**Lemma 6.** Let \( a = X^k(t;\tau,\eta) \), then i) the function \( a \) is differentiable with respect to \( \tau \), and

\[
\frac{da}{d\tau} = -v^k(E(\tau),\eta) \exp \left( \int_\tau^t \frac{\partial v^k}{\partial a}(E(\sigma),X^k(\sigma;\tau,\eta)) d\sigma \right),
\]

ii) the function \( a \) is differentiable with respect to \( \eta \), and

\[
\frac{da}{d\eta} = \exp \left( \int_\tau^t \frac{\partial v^k}{\partial a}(E(\sigma),X^k(\sigma;\tau,\eta)) d\sigma \right)
\]

A proof of this Lemma, can be found in [16] and [4] p.114.
3.2.2. A priori estimates. We need some a priori estimates on \((u^c, u^l, \tilde{u}^l, u^m, \tilde{u}^c)\).

Lemma 7. For \(k \in \{c, l, f, m\}\), we have
a) \(\sup_Q \|u^k\|_{L^1} \leq M e^{TL_v} + \{2MC_2 + 2ML_v + e^{TL_v} 4MC_1 \} T\),
b) \(\sup_Q \|u^c(t, x)\| \leq \{M + \text{meas}(\Omega) c_3 2MT\}\).

Proof. a) We give the proof only for Lemma 7.

For \(I_2 + I_4\), we make the change of variables
\[ \eta = X^c(s, t, a) = X^c(s, \tau_4, 0). \]
By assumptions (V) and lemma 5, we have
\[
\int_0^{L_e} |G^e (u^e (s, ., x)) (\eta)| \, d\eta = \|G^e (u^e (s, ., x))\|_{L^1} = \|G^e (u^e (s, ., x)) - G^e (0)\|_{L^1} \\
\leq 2Mc_2,
\]
and
\[
\int_0^{L_e} \left| \frac{\partial u^e}{\partial a} (E(s), \eta) u^e (s, \eta, x) \right| \, d\eta \leq 2ML_v.
\]
Therefore, we obtain that
\[
I_2 + I_4 \leq \{2Mc_2 + 2ML_v\} T.
\]
To estimate \(I_3\), we use the change of variables
\[
\zeta = X^e (0, t, a),
\]
this gives that
\[
I_3 \leq e^{TL_v} \int_0^{L_e} |u^e_0 (\zeta)| \, d\zeta \leq Me^{TL_v}.
\]
Hence
\[
I_1 + I_2 + I_3 + I_4 \leq Me^{TL_v} + \{2Mc_2 + 2ML_v + e^{TL_v} 4Mc_1\} T.
\]
Similarly, we get the other inequalities.

b) since \(\mu_c\) is nonnegative, we have
\[
|\tilde{w}^e (t, x)| \leq |\tilde{w}^e (0, x)| + \int_0^t |H(s, x, u^l (s, ., .))| \, ds \leq \\
|\tilde{w}^e (0, x)| + \int_0^t \left| H(s, x, u^l (s, ., .)) - H(0) \right| \, ds \leq \\
|\tilde{w}^e (0, x)| + \int_0^t c_3 \|u^l (s, ., .)\|_{L^1} \, ds \leq \\
M + c_3 \text{meas}(\Omega) \sup_Q \|u^l\|_{L^1} T \leq M + \text{meas}(\Omega) c_3 2MT.
\]

Let
\[
K^i (u^e, u^l, u^f, u^m, u^c) = \tilde{w}^i \text{ for } i \in \{e, l, f, m, c\}.
\]
To show that \(K\) is a strict contraction, we derive some estimates.
Lemma 8.

\( a) \| K^\varepsilon \left( u_1^v, u_1^f, u_1^m, u_1^\varepsilon \right) - K^\varepsilon \left( u_2^v, u_2^f, u_2^m, u_2^\varepsilon \right) \|_{L^1} \leq e^{TLv} C_1 (sup_Q \| u_1^f - u_2^f \|_{L^1} + sup_Q \| u_1^m - u_2^m \| _{L^1}) + Te^{TLv} (c_2 + L_v) \sup_Q \| u_1^\varepsilon - u_2^\varepsilon \| _{L^1} , \)

\( b) \| K^f \left( u_1^v, u_1^f, u_1^m, u_1^\varepsilon \right) - K^f \left( u_2^v, u_2^f, u_2^m, u_2^\varepsilon \right) \|_{L^1} \leq e^{TLv} C_1 sup_Q \| u_1^f - u_2^f \|_{L^1} + Te^{TLv} (c_2 + L_v) \sup_Q \| u_1^\varepsilon - u_2^\varepsilon \| _{L^1} , \)

\( c) \| K^f \left( u_1^v, u_1^f, u_1^m, u_1^\varepsilon \right) - K^f \left( u_2^v, u_2^f, u_2^m, u_2^\varepsilon \right) \|_{L^1} \leq e^{TLv} C_1 sup_Q \| u_1^f - u_2^f \|_{L^1} + Te^{TLv} (c_2 + L_v) \sup_Q \| u_1^\varepsilon - u_2^\varepsilon \| _{L^1} , \)

\( d) \| K^m \left( u_1^v, u_1^f, u_1^m, u_1^\varepsilon \right) - K^m \left( u_2^v, u_2^f, u_2^m, u_2^\varepsilon \right) \|_{L^1} \leq e^{TLv} C_1 sup_Q \| u_1^f - u_2^f \|_{L^1} + Te^{TLv} (c_2 + L_v) \sup_Q \| u_1^\varepsilon - u_2^\varepsilon \| _{L^1} , \)

\( e) \| K^\varepsilon \left( u_1^v, u_1^f, u_1^m, u_1^\varepsilon \right) - K^\varepsilon \left( u_2^v, u_2^f, u_2^m, u_2^\varepsilon \right) \|_{L^\infty} \leq c_3 T meas(\Omega) \sup_Q \| u_1^\varepsilon - u_2^\varepsilon \| _{L^1} . \)

Proof. We have

\[
\int_0^{L_\varepsilon} \left| K^\varepsilon \left( u_1^v, u_1^f, u_1^m, u_1^\varepsilon \right) - K^\varepsilon \left( u_2^v, u_2^f, u_2^m, u_2^\varepsilon \right) \right| da \\
\leq \int_0^{L_\varepsilon} \left| \left| F^\varepsilon \left( u_1^f (\tau \varepsilon, \cdot) \right) - F^\varepsilon \left( u_2^f (\tau \varepsilon, \cdot) \right) \right| \right|_2 \left| \left| G^\varepsilon \left( s, u_1^v (s, \cdot, x) \right) \left( X^\varepsilon \left( s, \tau \varepsilon, 0 \right) \right) - G^\varepsilon \left( s, u_2^v (s, \cdot, x) \right) \left( X^\varepsilon \left( s, \tau \varepsilon, 0 \right) \right) \right| \right|_2 \right| dsda + \\
\int_0^{L_\varepsilon} \left| \left| G^\varepsilon \left( s, u_1^v (s, \cdot, x) \right) \left( X^\varepsilon \left( s, \tau \varepsilon, 0 \right) \right) - G^\varepsilon \left( s, u_2^v (s, \cdot, x) \right) \left( X^\varepsilon \left( s, \tau \varepsilon, 0 \right) \right) \right| \right|_2 \right| dsda \\
\int_0^{L_\varepsilon} \left| \left| \left| G^\varepsilon \left( s, u_1^v (s, \cdot, x) \right) \left( X^\varepsilon \left( s, \tau \varepsilon, 0 \right) \right) - G^\varepsilon \left( s, u_2^v (s, \cdot, x) \right) \left( X^\varepsilon \left( s, \tau \varepsilon, 0 \right) \right) \right| \right|_2 \right| dsda \\
= P_1 + P_2 + P_3 .
\]

Note that

\[
P_1 \leq e^{TLv} \int_0^t \left| F^\varepsilon \left( u_1^f (\tau \varepsilon, \cdot, x) \right) - F^\varepsilon \left( u_2^f (\tau \varepsilon, \cdot, x) \right) \right| d\tau \varepsilon \leq e^{TLv} C_1 (sup_Q \| u_1^f - u_2^f \|_{L^1} + sup_Q \| u_1^m - u_2^m \| _{L^1}) .
\]

Using the change of variables

\[
\eta = X^\varepsilon \left( s, \tau \varepsilon, 0 \right) = X^\varepsilon \left( s, \tau \varepsilon, 0 \right) ,
\]

we obtain that

\[
P_2 + P_3 \leq \int_0^t \int_0^{\tau \varepsilon (s)} \left| G^\varepsilon \left( s, u_1^v (s, \cdot, x) \right) (\eta) - G^\varepsilon \left( s, u_2^v (s, \cdot, x) \right) (\eta) \right| d\eta da + 
\]
\[
\int_{z_e(t)}^t \int_0^t \left| \tilde{G}^e(s, u_1^e(s,.,x))(\eta) - \tilde{G}^e(s, u_2^e(s,.,x))(\eta) \right| \, d\eta \, da \leq \\
\int_{z_e(t)}^t \int_0^t \left| G^e(s, u_1^e(s,.,x))(\eta) - G^e(s, u_2^e(s,.,x))(\eta) \right| \, d\eta \, da + \\
\int_{z_e(t)}^t \int_0^t \left| \frac{\partial v^e}{\partial a} \left( E(t), \eta \right) (u_1^e(s,\eta,x) - u_2^e(s,\eta,x)) \right| \, d\eta \, da \leq T e^{T L_v} (c_2 + L_v) \sup_Q \| u_1^e - u_2^e \|_{L^1}.
\]

Consequently
\[
P_1 + P_2 + P_3 \leq e^{T L_v} T c_1 (\sup_Q \| u_1^e - u_2^e \|_{L^1}) + \sup_Q \| u_1^m - u_2^m \|_{L^1} T e^{T L_v} (c_2 + L_v) \sup_Q \| u_1^e - u_2^e \|_{L^1}.
\]

The other cases are similar, so we omit the proof.

By Lemma 5, we obtain
\[
\left\| K^c \left( u_1^e, u_1^l, u_1^m, u_1^c \right) - K^c \left( u_2^e, u_2^l, u_2^m, u_2^c \right) \right\|_{L^\infty(0,T) \times \Omega} \leq \text{meas} (\Omega) c_3 T \sup_Q \| u_1^l - u_2^l \|_{L^1}.
\]

\[\Box\]

**Proposition 9.** Under conditions (V), (B), (K) and (M), the operator \( K \) has a unique fixed point in \( X \).

**Proof.** a) For \( T \) small enough, the operator \( K \) maps \( X \) into \( X \).

Indeed, from Lemma 7, it follows that
\[
\sup_Q \| u^k(t,a,x) \|_{L^1} \leq M e^{T L_v} + \left\{ 2M c_2 + 2M L_v + e^{T L_v} 4 M c_1 \right\} T \leq 2M.
\]

for \( T \) small enough.

Similarly, we obtain that
\[
\sup_Q \left| \tilde{u}^c(t,x) \right| \leq \left\{ M + c_3 \text{meas} (\Omega) T 2 M \right\} \leq 2M,
\]

for \( T \) small enough.

b) The operator \( K : X \to X \) is a contraction operator.

It follows from Lemma 8, that
where the solution of the problem (1) depends continuously on the initial conditions.

Let the property of continuous dependence of solutions with respect to initial values.

\[
\begin{align*}
&\sup_{Q} K^c \left( u^c_1, u^c_1, u^c_1, u^m_1, u^c_1 \right) - K^c \left( u^c_2, u^c_2, u^m_2, u^c_2 \right) \\
&\sup_{Q} K^l \left( u^l_1, u^l_1, u^l_1, u^m_1, u^l_1 \right) - K^l \left( u^l_2, u^l_2, u^m_2, u^l_2 \right) \\
&\sup_{Q} K^m \left( u^m_1, u^m_1, u^m_1, u^m_1, u^m_1 \right) - K^m \left( u^m_2, u^m_2, u^m_2, u^m_2 \right) \\
&\left\| K^c \left( u^c_1, u^c_1, u^c_1, u^m_1, u^c_1 \right) - K^c \left( u^c_2, u^c_2, u^m_2, u^c_2 \right) \right\|_{L^\infty} \\
&\leq (e^{TLc} T_{C1} + e^{TLc} (c_2 + L_c)) \sup_{Q} \left\| u^l_1 - u^l_2 \right\|_{L^1} + \\
&(T_{c1} + T_{c1} + e^{TLc} (c_2 + L_c)) \sup_{Q} \left\| u^c_1 - u^c_2 \right\|_{L^1} + \\
&(T_{c1} + T_{c1} + e^{TLc} (c_2 + L_c)) \sup_{Q} \left\| u^m_1 - u^m_2 \right\|_{L^1} \\
&\leq T_{\gamma} \left( \sup_{Q} \left\| u^l_1 - u^l_2 \right\|_{L^1} + \sup_{Q} \left\| u^c_1 - u^c_2 \right\|_{L^1} + \sup_{Q} \left\| u^m_1 - u^m_2 \right\|_{L^1} + \sup_{Q} \left\| u^m_1 - u^m_2 \right\|_{L^1} \right) \\
&= T_{\gamma} \left( \left\| u^l_1, u^l_1, u^l_1, u^m_1, u^l_1 \right\| - \left\| u^l_2, u^l_2, u^l_2, u^m_2, u^l_2 \right\| \right)
\end{align*}
\]

where \( \gamma \) is a positive constant. Note that \( T_{\gamma} \in (0, 1) \) provided that \( T \) is small enough. Hence

\[
\left\| K \left( u^l_1, u^l_1, u^l_1, u^m_1, u^l_1 \right) - K \left( u^l_2, u^l_2, u^m_2, u^l_2 \right) \right\| \leq T_{\gamma} \left\| (u^l_1, u^l_1, u^l_1, u^m_1, u^l_1) - (u^l_2, u^l_2, u^l_2, u^m_2, u^l_2) \right\|.
\]

This proves that \( K \) is a contraction operator on \( X \). \( \square \)

3.3. Continuous dependence on initial conditions. In this section, we examine the property of continuous dependence of solutions with respect to initial values.

Let \( (u^c, u^l, u^m, u^c) \) and \( (\tilde{u}^c, \tilde{u}^l, \tilde{u}^m, \tilde{u}^c) \) be two solutions corresponding respectively to the initial values \( (u^c_0, u^l_0, u^m_0, u^c_0) \) and \( (\tilde{u}^c_0, \tilde{u}^l_0, \tilde{u}^m_0, \tilde{u}^c_0) \).

**Theorem 10.** Let assumptions \((V), (B), (M), \) and \((K)\) hold. If \( T < \frac{1}{\sup_{Q} (\omega \circ \tau_c)} \), then the solution of the problem (1) depends continuously on the initial conditions.

**Proof.** We approach the proof as in contraction property. We have

\[
\left\| u^c - \tilde{u}^c \right\|_{L^1} \leq \int_0^{L_c} \left| u^c - \tilde{u}^c \right| da \leq \int_0^{L_c} E \left( E\left( \tau_c, 0 \right) \right) \left( u^c \left( \tau_c, 0 \right) \right) \left| \frac{\tilde{u}^c \left( \tau_c, 0 \right)}{E \left( \tau_c, 0 \right)} \right| \left| \frac{\tau_c \left( \tau_c, 0 \right)}{E \left( \tau_c, 0 \right)} \right| dsda +
\]

\[
\int_0^{L_c} \int_0^{\tau_c} \left| \tilde{G} \left( u^c \left( \tau_c, 0 \right) \right) \right| \left| X \left( \tau_c, 0 \right) \right| dsda +
\]

\[
\int_0^{L_c} \int_0^{\tau_c} \left| \tilde{G} \left( \tilde{u}^c \left( \tau_c, 0 \right) \right) \right| \left| X \left( \tau_c, 0 \right) \right| dsda +
\]

\[
\int_0^{L_c} \int_0^{\tau_c} \left| \tilde{G} \left( u^c \left( \tau_c, 0 \right) \right) \right| \left| X \left( \tau_c, 0 \right) \right| dsda +
\]

\[
\int_0^{L_c} \int_0^{\tau_c} \left| \tilde{G} \left( \tilde{u}^c \left( \tau_c, 0 \right) \right) \right| \left| X \left( \tau_c, 0 \right) \right| dsda +
\]

\[
\int_{z_{\epsilon}(t)}^{L_v} \left| u_0^\epsilon \left( X^\epsilon (0, t, a), x \right) - \mu_0^\epsilon (X^\epsilon (0, t, a), x) \right| \, da
\]

\[= R_1 + R_2 + R_3 + R_4.\]

Beginning by \( R_1 \), from Lemma 5 and Lemma 7, we obtain that

\[R_1 \leq e^{T L_v} \int_0^t \left| F^\epsilon \left( u_1^f (\tau, x) \right) - F^\epsilon \left( u_2^f (\tau, x) \right) \right| \, d\tau \leq \]

\[e^{T L_v} c_1 \left( \int_0^t \| u_1^f - u_2^f \|_{L^1} + \| u_1^m - u_2^m \|_{L^1} \right) d\tau.\]

For \( R_4 \), we have

\[R_4 \leq e^{T L_v} \int_{z_{\epsilon}(t)}^{L_v} \left| u_0^\epsilon (\eta, x) - \mu_0^\epsilon (\eta, x) \right| \, d\eta = e^{T L_v} \sup_{\Omega} \left\| u_0^\epsilon - \mu_0^\epsilon \right\|_{L^1}.\]

To estimate \( R_2 + R_3 \), we make the change of variables \( \eta = X^\epsilon (s, t, a) = X^\epsilon (s, \tau, a) \). Lemma 5 yields

\[R_2 + R_3 \leq \int_{z_{\epsilon}(t)}^{L_v} \int_0^t \left| \tilde{G}^\epsilon (s, u^\epsilon (s, x)) \left( \eta \right) - \tilde{G}^\epsilon (s, \mu^\epsilon (s, x)) \left( \eta \right) \right| \, d\eta \, ds + \]

\[\int_{z_{\epsilon}(t)}^{L_v} \int_0^t \left| \tilde{G}^\epsilon (s, u^\epsilon (s, x)) \left( \eta \right) - \tilde{G}^\epsilon (s, \mu^\epsilon (s, x)) \left( \eta \right) \right| \, d\eta \, ds \]

\[\leq \int e^{T L_v} (c_2 + L_v) \| u^\epsilon - \mu^\epsilon \|_{L^1} \, ds.\]

We get

\[\| u^\epsilon - \mu^\epsilon \| \leq e^{T L_v} c_1 \left( \int_0^t \| u_1^f - u_2^f \|_{L^1} + \| u_1^m - u_2^m \|_{L^1} \right) d\tau + e^{T L_v} \sup_{\Omega} \left\| u_0^\epsilon - \mu_0^\epsilon \right\|_{L^1} + \]

\[\int_0^t e^{T L_v} (c_2 + L_v) \| u^\epsilon - \mu^\epsilon \|_{L^1} \, ds.\]

Analogously to (a), we obtain the estimates

\[\| u^f - \mu^f \|_{L^1} \leq e^{T L_v} c_1 \int_0^t \| u^\epsilon - \mu^\epsilon \|_{L^1} \, d\tau + e^{T L_v} \sup_{\Omega} \left\| u_0^\epsilon - \mu_0^\epsilon \right\|_{L^1} + \]

\[\int_0^t e^{T L_v} (c_2 + L_v) \| u^f - \mu^f \|_{L^1} \, ds,\]

\[\| u^f - \mu^f \|_{L^1} \leq e^{T L_v} c_1 \int_0^t \| u^f - \mu^f \|_{L^1} \, d\tau + e^{T L_v} \sup_{\Omega} \left\| u_0^f - \mu_0^f \right\|_{L^1} + \]

\[\int_0^t e^{T L_v} (c_2 + L_v) \| u^f - \mu^f \|_{L^1} \, ds.\]
We conclude that
\[ \int_0^t e^{TLv} (c_2 + L_v) \left\| u^f - \tilde{u}^f \right\|_{L^1} ds, \]
and
\[ \left\| u^m - \tilde{u}^m \right\|_{L^1} \leq e^{TLv} c_1 \int_0^t \left\| u^f - \tilde{u}^f \right\|_{L^1} d\tau_m + e^{TLv} \sup_{\Omega} \left\| u^m_0 - \tilde{u}^m_0 \right\|_{L^1} + \]
\[ \int_0^t e^{TLv} (c_2 + L_v) \left\| u^m - \tilde{u}^m \right\|_{L^1} ds. \]

We conclude that
\[ \left\| u^e - \tilde{u}^e \right\|_{L^1} + \left\| u^f - \tilde{u}^f \right\|_{L^1} + \left\| u^f - \tilde{u}^f \right\|_{L^1} + \left\| u^m - \tilde{u}^m \right\|_{L^1} \leq \]
\[ e^{TLv} \left( \sup_{\Omega} \left\| u^e_0 - \tilde{u}^e_0 \right\|_{L^1} + \sup_{\Omega} \left\| u^f_0 - \tilde{u}^f_0 \right\|_{L^1} + \right. \]
\[ + \sup_{\Omega} \left\| u^m_0 - \tilde{u}^m_0 \right\|_{L^1} + \sup_{\Omega} \left\| u^m_0 - \tilde{u}^m_0 \right\|_{L^1} \right) \]
\[ + \int_0^t \left\{ e^{TLv} c_1 \left( \left\| u^f - \tilde{u}^f \right\|_{L^1} + \right. \right. \]
\[ \left. \left. + \left\| u^m - \tilde{u}^m \right\|_{L^1} \right\} \right\} ds + \]
\[ \int_0^t \left\{ e^{TLv} c_1 \left\| u^f - \tilde{u}^f \right\|_{L^1} + e^{TLv} (c_2 + L_v) \left\| u^f - \tilde{u}^f \right\|_{L^1} \right\} ds + \]
\[ \int_0^t \left\{ e^{TLv} c_1 \left\| u^f - \tilde{u}^f \right\|_{L^1} + e^{TLv} (c_2 + L_v) \left\| u^m - \tilde{u}^m \right\|_{L^1} \right\} ds \]
\[ \int_0^t \left\{ e^{TLv} c_1 \left\| u^f - \tilde{u}^f \right\|_{L^1} + e^{TLv} (c_2 + L_v) \left\| u^m - \tilde{u}^m \right\|_{L^1} \right\} ds \]

It follows that
\[ \left\| u^e - \tilde{u}^e \right\|_{L^1} + \left\| u^f - \tilde{u}^f \right\|_{L^1} + \left\| u^f - \tilde{u}^f \right\|_{L^1} + \left\| u^m - \tilde{u}^m \right\|_{L^1} + \left\| u^e - \tilde{u}^e \right\|_{L^\infty} \leq \]
\[ e^{TLv} \left( \sup_{\Omega} \left\| u^e_0 - \tilde{u}^e_0 \right\|_{L^1} + \sup_{\Omega} \left\| u^f_0 - \tilde{u}^f_0 \right\|_{L^1} + \sup_{\Omega} \left\| u^m_0 - \tilde{u}^m_0 \right\|_{L^1} + \right. \]
\[ + \sup_{\Omega} \left\| u^m_0 - \tilde{u}^m_0 \right\|_{L^1} + \sup_{\Omega} \left\| u^m_0 - \tilde{u}^m_0 \right\|_{L^\infty} \right) \]
\[ + \Gamma_1 \int_0^t \left\{ \left\| u^e - \tilde{u}^e \right\|_{L^1} + \left\| u^f - \tilde{u}^f \right\|_{L^1} + \left\| u^f - \tilde{u}^f \right\|_{L^1} + \left\| u^m - \tilde{u}^m \right\|_{L^1} \right. \]
\[ + \left\| u^e - \tilde{u}^e \right\|_{L^\infty} \right\} ds, \]

where \( \Gamma_1 \) is a positive constant depending on \( L_v, T, \) and \( c_i, i = 1, 2, 3. \)
According to Gronwall’s inequality we conclude that
\[
\sup_Q \| u^e - \hat{u}^e \|_{L^1} + \sup_Q \| u^f - \hat{u}^f \|_{L^1} + \sup_Q \| u^m - \hat{u}^m \|_{L^1} \leq e^{\Gamma L_1} \left[ \sup_Q \| u_0^e - \hat{u}_0^e \|_{L^1} + \sup_Q \| u_0^f - \hat{u}_0^f \|_{L^1} + \sup_Q \| u_0^m - \hat{u}_0^m \|_{L^1} \right]^\Gamma \left( e^{(\Gamma_1 + L_1)T} \right).
\]

Now, from lemma 5, we obtain
\[
\| u^e - \hat{u}^e \|_{L^\infty} \leq \| u_0^e - \hat{u}_0^e \|_{L^\infty} + \meas(\Omega) c_3 T \sup_Q \| u^f - \hat{u}^f \|_{L^1}.
\]

This gives that
\[
\sup_Q \| u^e - \hat{u}^e \|_{L^1} + \sup_Q \| u^f - \hat{u}^f \|_{L^1} + \sup_Q \| u^m - \hat{u}^m \|_{L^1} \leq \sup_Q \| u_0^e - \hat{u}_0^e \|_{L^1} + \sup_Q \| u_0^f - \hat{u}_0^f \|_{L^1} + \sup_Q \| u_0^m - \hat{u}_0^m \|_{L^1} e^{(\Gamma_1 + L_1)T} + \meas(\Omega) c_3 T \sup_Q \| u^f - \hat{u}^f \|_{L^1}.
\]

It follows that
\[
\sup_Q \| u^e - \hat{u}^e \|_{L^1} + (1 - \meas(\Omega) c_3 T) \sup_Q \| u^f - \hat{u}^f \|_{L^1} + \sup_Q \| u^m - \hat{u}^m \|_{L^1} \leq \\
\Gamma \left[ \sup_Q \| u_0^e - \hat{u}_0^e \|_{L^1} + \sup_Q \| u_0^f - \hat{u}_0^f \|_{L^1} + \sup_Q \| u_0^m - \hat{u}_0^m \|_{L^1} \right] + \meas(\Omega) c_3 T \sup_Q \| u^f - \hat{u}^f \|_{L^1}.
\]

Here \( \Gamma = \max \left( e^{(\Gamma_1 + L_1)T}, 1 \right) \).

Then, for every positive number \( \varepsilon \), there exists a positive number \( \delta = \delta = \varepsilon \) such that if
\[
\sup_Q \| u^e - \hat{u}^e \|_{L^1} + (1 - c_3 T) \sup_Q \| u^f - \hat{u}^f \|_{L^1} + \sup_Q \| u^m - \hat{u}^m \|_{L^1} + \sup_Q \| u^c - \hat{u}^c \|_{L^\infty} < \delta
\]
then
\[
\sup_Q \| u^e - \hat{u}^e \|_{L^1} + (1 - c_3 T) \sup_Q \| u^f - \hat{u}^f \|_{L^1} + \sup_Q \| u^m - \hat{u}^m \|_{L^1} + \| u^c - \hat{u}^c \|_{L^\infty} < \varepsilon.
\]

This implies that
\[
\| (u^e, u^f, u^m) - (\hat{u}^e, \hat{u}^f, \hat{u}^m, \hat{u}^c) \|_* < \varepsilon.
\]

where
\[
\| (u^e, u^f, u^m, u^c) \|_* = \sup_Q \| u^e \|_{L^1} + (1 - \meas(\Omega)c_3 T) \sup_Q \| u^f \|_{L^1} + \\
\sup_Q \| u^m \|_{L^1} + \sup_Q \| u^c \|_{L^\infty}
\]
is an equivalent norm in $X$. This proves the theorem.

4. **Concluding remarks.** European Grapevine Moth has caused economic damages in southern France and northern Africa where wine and grapes table are produced.

The damages are caused in larval stage where caterpillar may provoke entire loss of vineyard. The caterpillar enter its pupation and stays about seven days before emergence.

Chemical control has been extensively used to reduce the proliferation of the pest population. However, Insecticides can be a factor driving insects to acquire resistance, interpreted by behavioral, physiological change or genetic mutation. To avoid unfavorable conditions and limit the risk of mortality, the insect develop at the pupal stage one or several phenotypes, leading to appropriate timing of emergence. Note, this behavioral change is observed in Malaria Infection where mosquitoes have adapted survival strategies by shifting their biting behavior, see [11].

Other factors can drive the phenology of populations, see for instance, [17], [14], [18], [19], [23], [24] and [28]. Weather variability is another factor which can have large impact on the species. Increasing temperature may affect the rates, fecundity, and larval diet of insects, [30].

In warmer regions, four generations may occur instead of three.

Based on insecticides only, it is much harder to control the pest, and the use of intensive insecticides can produce serious human health consequences, destruction of predator-prey relationships and polluting disasters.

Hence, a need to get better understanding of the mechanisms of insects resistance is crucial.

The advantage of the present study is that it includes in the model endogeneous and exogeneous factors that can explain the observed differences within a cohort.

The validation of the model with experimental data is still in progress.

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