Context-free characterizations of indexed languages

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Abstract

Indexed languages are languages recognized by pushdown automata of level 2 and by indexed grammars. We propose here some new characterizations linking indexed languages to context-free languages: the class of indexed languages is the image of the Dyck language by a nice class of context-free transducers, it is also the class of images, by a projection defined by an FO-formula of nested words labelled by a Dyck word. This last result generalize the logical characterization of context-free languages.

1 Introduction

Higher order languages form an infinite class of languages in which the level 0 is the class regular languages, level 1 is the class of context-free languages, and level 2 is the class of indexed languages. Languages of level \( k \) are recognized by \( k \)-pushdown automata whose storage structure, is roughly speaking, the \( k \)-iteration of the operation \( \text{stack} : \Gamma \mapsto (\Gamma^*) \) creating the set of stacks over a (possibly infinite) alphabet \( \Gamma \). These objects are subjects of many works, non exhaustively: in language theory [15, 16], logic [12, 7] and model checking [9, 13].

We propose in this paper two new characterizations of indexed languages, using combinations of “context-free properties”.

The first one use context-free transducers. It is well know (see for example [11]) every context-free language is the image of the Dyck language (the language of well parentheses words) by a rational transduction. In [6], this result is extended to indexed languages (and also to all the hierarchy of languages recognized by \( k \)-pushdown automata): all indexed language is the image of the Dyck\(_2\) language (roughly speaking, a Dyck language whose parenthesis are words instead of letters) by a rational transduction. This result is not really surprising since the Dyck\(_2\) language is an indexed language and describes the moves of stacks of level 2, in the same way that the Dyck languages describes the moves of stacks of level 1. We prove here (Theorem 3.1) that indexed languages are also the images of the Dyck language by context-free transductions. More precisely, we give a complete characterization of indexed languages since we define a class of context-free transductions mapping exactly the Dyck languages to the class of indexed languages.

The second one use logic on Dyck words endowed with a binary relation. Logical characterization of classes of languages has been initiated by Büchi who proved that the class of regular languages corresponds to the class of languages
definable in monadic second order logic. It has also been showed in [10] that the class of context-free languages corresponds to the class of languages definable in First Order logic extended by an existential quantification over a binary relation. This characterization describes context-free languages using nested words which are pairs \((u, M)\), where \(u\) is a word and \(M\) is a non crossing pairing relation on the positions of \(u\) (see for example [3]). We propose an extension of this last result to indexed languages. We will see as intuitied by Proposition 4.4, that adding a second matching to words might give too much expressiveness power than necessary to express indexed languages. Indeed, the matching relation is used to express the stack moves of a stack-automata, but in a stack-automata of level 2, parts of the stack are duplicated, and so, popped several times. In a general setting, a formula on nested words (even with 2 matching relations) does not express this property since it cannot describe the content of the stack. So, we made the choice to add informations on the content of the stack inside the logical structure considered. We define the class of the Dyck Nested Words which are Dyck words endowed with a matching relation compatible with the word (see Definition 4.3) and prove that indexed languages are the image of the class of the Dyck Nested Words, by a projection definable by a FO formula (Theorem 4.3).

In parallel, we also consider realtime indexed languages which are indexed languages recognized by stack automata of level 2 reading an input letter at each transition step. To our knowledge, it is the first work considering this restriction that we conjecture to be strict. We give a version of the two previous characterizations for realtime indexed languages: one for the characterization by context-free transducers given Theorem 3.1, and one for the logical characterization given Theorem 4.2.

The paper is split as follows: in Section 2, we introduce basic notions on languages, transducers and automata. In Section 3, we prove the characterizations by context-free transducers (Theorem 3.1). Section 4 is devoted to the logical characterizations (Theorems 4.2 and 4.3).

2 Preliminaries

2.1 Words and languages

Let \(\Sigma\) be a finite set, \(\Sigma^*\) denotes the set of words (finite sequences) over \(\Sigma\), and \(\varepsilon\) the empty word. For \(u, v \in \Sigma^*\), the length of \(u\) is denoted \(|u|\) and we write \(v \preceq u\) if \(v\) is a prefix of \(u\), i.e. if there exists \(w \in \Sigma^*\) such that \(u = vw\). If \(\Gamma \subseteq \Sigma\), we denote by \(\pi_\Gamma\) the projection \(\Sigma^* \rightarrow \Gamma^*\) deleting in a word the letters that do not belongs to \(\Gamma\).

A word in \(\hat{\Gamma} = \{\hat{a} \mid a \in A\}\), and adopt the convention that \(\hat{a} = a\) for all \(a\). In all the paper, we suppose that \(\perp\) is a special symbol, which does not belong to \(\Gamma\) and define \(\hat{\Gamma} = \Gamma \cup \hat{\Gamma} \cup \{\perp\}\). Let us then consider the reduction system \(S = \{(aa, \varepsilon), (\hat{a}a, \varepsilon), (\perp, \varepsilon)\}\). A word in \(\hat{\Gamma}^*\) is said to be reduced if it is \(S\)-reduced, i.e., it does not contain occurrences of \(aa, \hat{a}a\) or \(\perp\), for \(a \in \Gamma\). As \(S\) is confluent, each word \(w\) is equivalent (mod \(\rightarrow_S^*\)) to a unique reduced word denoted \(\rho(w)\).

A Dyck word over \(\Gamma\) is a word \(u \in \hat{\Gamma}^*\) such that \(\rho(u) = \varepsilon\) and for every prefix
v \preceq u$: $\rho(v) \in \Gamma^*$. We denote by $\mathcal{D}(\Gamma)$ the set of all Dyck word over $\Gamma$ and $\mathcal{D}^\approx(\Gamma) = \{u \mid \exists v \in \mathcal{D}(\Gamma), u \preceq v\}$.  

2.1.1 Transductions  
Let $\Sigma_I$ and $\Sigma_O$ be two finite alphabets, we consider the free monoid over $\Sigma_I^* \times \Sigma_O^*$ whose product is the product on words, extended to pairs of words: $(u_1, v_1)(u_2, v_2) = (u_1u_2, v_1v_2)$. A subset $\tau$ of $\Sigma_I^* \times \Sigma_O^*$ is called a $(\Sigma_I, \Sigma_O)$-transduction. If $u \in \Sigma_I^*$ and $L \subseteq \Sigma_I^*$, then $\tau(u) = \{v \in \Sigma_O^* \mid (u, v) \in \tau\}$ and $\tau(L) = \{v \in \Sigma_O^* \mid (u, v) \in \tau, u \in L\}$.  

2.2 Stacks  
Iterated nested stacks have been defined in [2] as a generalization of pushdown stacks. In this paper, we are only interested by 1-stacks and 2-stacks. A 1-stack, $A$ or $A^-$, is simply be represented by the word $AabB$. A 2-stack ($\Omega$) is exactly the inverse of the instruction $\text{push}_{1,a}$. This sequence can simply be represented by the word $AabBlb$.  

2.2.1 Instructions over stacks  
Let us now define the pushdown instructions allowed on these stacks. Instructions $\text{push}_{1,a}$ and $\text{pop}_{1,a}$ ($a \in \Gamma_1$) are standard instructions on 1-stacks that respectively add the letter $a$, and remove the letter $a$ on the top of a 1-stack (if the letter $a$ is not on the top of the stack, the instruction $\text{pop}_{1,a}$ is undefined). These two instructions can also be applied to a 2-stack. In this case, the letter is added or removed to the content of the top stack: for example, $\text{push}_{1,b}(\Omega_{ax}) = A[ba]B[ba]$ is a 2-stack whose top is $A[a]$.  

2.2.2 Encoding of stacks  
We show now that a 2-stack $\Omega$ can be viewed as a word over a finite alphabet by encoding it as the shortest sequence of instructions to obtain $\Omega$ from the empty stack (see [7] for a more detailed presentation). Let us consider for example $\Omega = B[aa]A[ba]$, the shortest sequence of instructions to obtain $\Omega$ from an empty stack is $\text{push}_{2,a}$ $\text{push}_{1,a}$ $\text{push}_{1,b}$ $\text{push}_{2,b}$ $\text{pop}_{1,b}$ $\text{push}_{1,a}$. This sequence can simply be represented by the word $AabBla$.  

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Given two disjoint alphabets $\Gamma_1$ and $\Gamma_2$, we define $\mathcal{P}_2(\Gamma_1, \Gamma_2)$ as the set of all words $u \in (\Gamma_2 \cup \Gamma_1 \cup \Gamma_1^*)$ such that $\rho(u) = u$ and $\pi_{\Gamma_1}^{-1}(u) \in \mathcal{D}^\epsilon(\Gamma_1)$.

We set $\mathcal{P}_1(\Gamma) = \Gamma_1^*$, and for $i \in \{1, 2\}$, we define the function $\varphi_i$ mapping 2-stacks to words in $\mathcal{P}_i$.

- $\varphi_1(\$[\omega_1]) = \omega_1^R$. In other words, for a 2-stack $\Omega$, $\varphi_1(\Omega)$ is the mirror image of the content of the top-stack of $\Omega$.

- $\varphi_2(\$[\omega_1]) = \omega_1$, $\varphi_2(A[\omega_1]\Omega) = \rho(\varphi_2(\Omega)A\varphi_1(\Omega)\varphi_1(\omega_1))$. For a 2-stack $\Omega$, $\varphi_2(\Omega)$ is the shortest sequence of instruction to obtain $\Omega$ from an empty stack.

Remark that $\varphi_2$ is a bijective map. The next Lemma falls from the definition of $\varphi_2$.

**Lemma 2.1.** For every $u, v \in \mathcal{P}_2$, and $a \in \Gamma_i$, $1 \leq i \leq 2$,

\[
\begin{align*}
v = \rho(ua) & \iff \varphi_2^{-1}(v) = \text{push}_{i,a}(\varphi_2^{-1}(u)) \\
v = \rho(u\bar{a}) & \iff \varphi_2^{-1}(u) = \text{push}_{i,a}(\varphi_2^{-1}(v)) \iff \varphi_2^{-1}(v) = \overline{\text{push}_{i,a}(\varphi_2^{-1}(u))}
\end{align*}
\]

### 2.3 Pushdown stack automata

We introduce in the next definition for all the pushdown stack machines; we will consider: pushdown 2-stack automata, pushdown 1-stack automata, and pushdown 1-stack transducers. The reader can refers to [14, Chap. 3] for a complete introduction to pushdown automata and pushdown transducers.

**Definition 2.1 (Pushdown automata/transducers).** For $i \in \{1, 2\}$ and $k \geq 1$, a $i$-pushdown transducer ($i$-PDT) of rank $k$ is a structure $(Q, \Sigma_I, \Sigma_O, (\Gamma_j)_{1 \leq j \leq i}, q_0, \Delta, F)$ where:

- $Q$ is a finite set of states, $q_0 \in Q$ is the initial state, $F \subseteq Q$ is the set of final states.
- $\Sigma_I$ is the (finite) input alphabet, $\Sigma_O$ the (finite) output alphabet, $\Gamma_1$ and $\Gamma_i$ are finite stack alphabets and $|\Gamma_1| \leq k$.
- $\Delta \subseteq Q \times (\Sigma_I \cup \{\varepsilon\} \mid \Sigma_O \cup \{\varepsilon\}) \times I((\Gamma_j)_{1 \leq j \leq i}) \times Q$ is the transition relation.

We will also consider $i$-pushdown automata ($i$-PDA) which are pushdown stack transducers without output, so in the transition relation, there are elements in $\Sigma_I \cup \{\varepsilon\}$ instead of elements in $(\Sigma_I \cup \{\varepsilon\} \mid \Sigma_O \cup \{\varepsilon\})$.

The next definitions are given for transducers, to get definitions for automata, it suffices to delete the output. Configurations are tuples $Q \times \Sigma_I^* \times i\text{-Stack}((\Gamma_j)_{1 \leq j \leq i}) \times \Sigma_O^*$. The relation on configurations induced by $A$ is denoted $\vdash_A$ and links pairs $(q, au, \Omega, v) \vdash (q', u, \text{instr}(\Omega), bv)$ for which there is a transition $(q, a \mid b, \text{in}str, q') \in \Delta$.

The set recognized by a transducer $A$ is $\mathcal{L}(A) = \{(u, v) \in \Sigma_I^* \times \Sigma_O^* \mid \exists q \in F \colon (q, u, \varepsilon, \varepsilon) \vdash^* (q', \varepsilon, \varepsilon, v)\}$. So, if $A$ is a transducer, then $\mathcal{L}(A)$ is a $(\Sigma_I, \Sigma_O)$-transduction and if $A$ is an automaton, then $\mathcal{L}(A)$ is a subset of $\Sigma_I^*$.

Let $L \subseteq \Sigma_I^*$. If $L$ is recognized by a $1$-PDA, it is a context-free language. If it is recognized by a $2$-PDA, it is an indexed language. If there is a realtime
\(\hat{\tau}\)-PDA of rank \(k\) recognizing \(L\) (that is, the automaton never read \(\varepsilon\) on the input tape), \(L\) is called realtime of rank \(k\). It is well now that all context-free languages are realtime. To our knowledge, there is no such a result for indexed languages, and we conjecture that there are indexed languages which are not realtime (for example, we haven’t found a realtime automaton for the set the words \(\{a_1 \cdots a_n \# a_1a_2 \# \cdots \# a_1a_2 \cdots a_{n-1}\# a_1a_2 \cdots a_n\}\)). However, we prove (Proposition 4.3) that languages recognized by 2-PDA allowed to read \(\varepsilon\) only on transitions performing instructions of level 2 (we call these languages semi-realtime ) can be recognized by realtime 2-PDA that perform at most one instruction of level 2 per transition.

3 Characterization of Indexed languages by context-free transductions

It is proved in [11] that for every context free language \(L \subset \Sigma^*\), there exists a rational transduction \(\tau\) (that is, \(\tau\) is a rational set) such that \(\tau(\mathcal{D}([0,1]) = L\). In [6], this result is extended to indexed languages for all indexed language \(L\) there is a rational transduction \(\tau\) such that \(\tau(\mathcal{D}_2([0,1]) = L\)), where \(\mathcal{D}_2([0,1])\) is a generalization of the set of Dyck words.

We show here that for all indexed language \(L\) there is a context free transduction \(\tau\) such that \(\tau(\mathcal{D}([0,1])) = L\). We will also give a simple condition on context free \((\hat{\Gamma}, \Sigma)\)-transductions \(\tau\), to ensure that \(\tau(\mathcal{D}(\Gamma))\) is an indexed language.

**Definition 3.1.** A context-free transducer (CFT) is a tuple \(G = (N, \Gamma, \Sigma, S, P)\), where \(N\), \(\Gamma\) and \(\Sigma\) are respectively the sets of nonterminal, input-terminal, and output-terminal symbols, \(S\) is the start symbol (with \(S \in N\)) and \(P\) is a finite set of productions of the form \(X \rightarrow u_0X_1 \cdots u_{n-1}X_{n-1}u_n\), with \(n \geq 0\), \(X, X_1, \ldots, X_n \in N\), and \(u_i \in \Gamma^* \times \Sigma^*\).

The derivation in a context-free transducer is defined as for a context free grammar, but by using the concatenation product over \(\Gamma^* \times \Sigma^*\). So the set derived by \(G\), denoted \(\mathcal{T}(G)\), is a \((\Gamma, \Sigma)\)-transducer.

We define now a subclass of context free transducer generating exactly the indexed languages as image of the Dyck words set.

**Definition 3.2** (Dyck context-free transducer). A Dyck context-free transducer (DCTF) over \((\Sigma, \Gamma)\) is a context-free transducer \(G = (N, \hat{\Gamma}, \Sigma, S_\bot, P)\) such that: all the nonterminal symbols are indexed by \(\hat{\Gamma}\) (and then \(N\) is partitioned in \((N_\alpha)_{\alpha \in \hat{\Gamma}}\)) and all the productions have the form:

1. \(X_\alpha \rightarrow (\beta, a)\Theta_\gamma\Theta_\alpha\) where \(a \in \Sigma \cup \{\varepsilon\}\), \(\alpha, \beta \in \hat{\Gamma}\) and for all \(\gamma \in \hat{\Gamma}\), if \(\gamma \neq \bot\) then \(\Theta_\gamma \in N_\gamma\), else \(\Theta_\gamma \in N_\bot\);
2. \(X_\alpha \rightarrow (\hat{\alpha}, a)\), \(a \in \Sigma \cup \{\varepsilon\}\), \(\alpha \in \Gamma \cup \hat{\Gamma}\).

If in each production \(a \neq \varepsilon\), we say that \(G\) is non-erasing.

Remark that for any transduction \(\tau\) generated by a DCTF, the domain of \(\tau\) is included in the language of words \(u\) such that \(\rho(u) = \varepsilon\)
Example 1. Let \( \Gamma = \{ f, g \} \), the following transducer is generated by a DCTF:
\[
\tau = \{(g f n_1 f n_2 g f n_3 f n_4 g f n_5 f n_6 g a_{n_1+1} a_{n_2+1} b_{n_2+1} b_{n_3+1} e_{n_3+1} e_{n_4+1}) \mid n_1, n_2, n_3 \geq 0 \}
\]

Then \( \tau(\mathcal{D}(\Gamma)) = \{ a^n b^n e^n \mid n \geq 1 \} \).

This part is devoted to the proof of the following theorem:

**Theorem 3.1.** Let \( L \subseteq \Sigma^* \).

1. \( L \) is a (semi-real-time) indexed language of rank \( k \) iff there is a (non-erasing) Dyck context-free transducer \( G \) such that \( \mathcal{T}(G)(\mathcal{D}([1, k])) = L \).

2. If \( L \) is an indexed language, there is CF-transducer \( G \) such that \( \mathcal{T}(G)(\mathcal{D}([0, 1])) = L \).

**Proof of Theorem 3.1. (1 \( \Rightarrow \))**. We want to show that for every semi-realtime indexed language of rank \( k \) \( L \) there is a non-erasing Dyck context-free transducer \( G \) such that \( \mathcal{T}(G)(\mathcal{D}([1, k])) = L \). We start with any 2-PDA recognizing a language \( L : A = (Q, \Sigma, (\Gamma_1, \Gamma_2), q_0, \Delta, F) \), and construct a 1-PDT \( B \) recognizing a \((\Gamma_1, \Sigma)\)-transduction \( \tau \) such that \( \tau(\mathcal{D}(\Gamma_1)) = L \). This transducer mimics the behaviour of \( A \) in the following way: when the stack of \( A \) is \( A_1[\omega_1] A_2[\omega] \ldots [\omega_{n+1}] \), the stack of \( B \) is \( \omega'_1 A_1 \omega'_2 A_2 \ldots \omega'_{n} A_n \omega_{n+1} \) and \( \omega' \) is the encoding of a sequence of instructions given \( \omega \) from \( \omega_{n+1} \). Then, when \( \omega' = \varepsilon \), \( \omega_1 = \omega_2 \) and a push2 instruction can be applied. In parallel, the input word \( u \) of \( B \) is the sequence of instructions of level 1 which have been applied during the computation in \( A \). So when \( u \) is a Dyck word, we are sure that \( B \) has well simulated the copy process. So, \( B \) has the same states than \( A \), its stack alphabet is \( \Gamma = \Gamma_2 \cup \Gamma_1 \cup \Gamma_1 \) and its transition relation \( \Delta' \) is constructed as follows:

- for every \((q, a, \text{push}_{2,A}, q') \in \Delta \), add \((q, \varepsilon \mid a, \text{push}_{A}, q') \) to \( \Delta' \);
- for every \((q, a, \text{push}_{2,A}, q') \in \Delta \), add \((q, \varepsilon \mid a, \text{pop}_{A}, q') \) to \( \Delta' \);
- for every \((q, a, \text{push}_{1,f}, q') \in \Delta \), add \((q, f \mid a, \text{push}_{f}, q') \) and \((q, f \mid a, \text{pop}_{f}, q') \) to \( \Delta' \);
- for every \((q, a, \text{pop}_{1,f}, q') \in \Delta \), add \((q, f \mid a, \text{push}_{f}, q') \) and \((q, f \mid a, \text{pop}_{f}, q') \) to \( \Delta' \);
- for every \((q, a, \text{stay}, q') \in \Delta \), add \((q, \varepsilon \mid a, \text{stay}, q') \) to \( \Delta' \).

Using Lemma 2.1 one can easily verify by induction on the length of computations that for all word \( u \in \Sigma^* \), for all 2-stack \( \Omega \) and all state \( q \):
\[
(q_0, u, \varepsilon) \vdash^*_{A} (q, \varepsilon, \Omega) \text{ iff there exists } v \in \mathcal{D}^{\varepsilon}(\Gamma_1) \text{ and a 1-stack } \omega \text{ such that } (q_0, v, \varepsilon, \varepsilon) \vdash^*_{B} (q, \varepsilon, \omega, u) \text{ and } \rho(\omega) = \varphi_2(\Omega) \text{ and } \rho(v) = \varphi_1(\Omega).\]

Now we apply the standard construction removing the states of a pushdown automaton (see for example Example 3) and get directly a Dyck context-free transducer:
\[
G = (\Sigma, N, S_\bot, P) \text{ with } N_a = \{ X_{p,q_{\alpha}} \}_{p,q \in Q} \text{ for all } \alpha \in \widehat{\emptyset} - \{ \bot \} \text{ and } S_\bot = \{ S_\bot \} \cup \{ X_{p,q_{\bot}} \}_{p,q \in Q}, \text{ and such that } \mathcal{T}(G) = \mathcal{L}(B). \text{ We have then proven that if } L \text{ is an indexed language, there is an alphabet } \Gamma \text{ and a Dyck context-free transducer } G \text{ such that } L = \mathcal{T}(G)(\mathcal{D}(\Gamma)).\]
Let us remark that all production of the form \( X \rightarrow (\perp, \varepsilon)\omega \) can be replaced by \( X \rightarrow \omega \). Then all productions \( X_{\perp} \rightarrow (\perp, \varepsilon) \) and \( X_{\alpha} \rightarrow (\perp, \varepsilon)Y_{\perp}Z_{\alpha} \) can be eliminated by replacing \( X_{\perp} \) by \( \varepsilon \) and \( X_{\alpha} \) by \( Y_{\perp}Z_{\alpha} \), in all productions. It follows that if \( L \) is a semi-realtime indexed language, the transducer \( G \) is non-erasing.

\[
\text{Proof of Theorem 3.1} \quad (1 \iff 2). \text{ Let us prove now that for any Dyck context-free transducer } G, \mathcal{T}(G)((\mathcal{D}(\Gamma))) \text{ is an indexed language. Here we use indexed grammars, that are grammars generating exactly indexed languages (see for example } [II]).
\]

**Definition 3.3.** An indexed grammar is a structure \( \mathcal{I} = (N, I, \Sigma, S, P) \), where \( N, \Sigma \) and \( S \) are defined as for a context-free grammars, \( I \) is a finite set of indices, and \( P \subseteq \mathcal{N} \cup \mathcal{E} \) the set of productions. The rewriting defined by \( \mathcal{I} \) is such that \( \Theta_1A^\omega \Theta_2 \rightarrow \Theta_1uB_1^\omega \cdots B_n^\omega \Theta_2 \) if \( A^\omega \rightarrow uB_1^\omega \cdots B_n^\omega \in P \). The language generated by \( \mathcal{I} \) is \( L(\mathcal{I}) = \{ u \in \Sigma^* \mid S \xrightarrow{\omega} u \} \).

Let \( G = (N, \hat{\Gamma}, \Sigma, S, P) \) be a Dyck context-free transducer \( G \), we construct the indexed grammar \( \mathcal{I} = (N, \Gamma, \Sigma, S, \gamma(P)) \) in the following way:

- for each production \( p = X_\alpha \rightarrow (\beta, u)\Theta_\beta \Theta_\perp \Theta_\alpha \in P \):
  - if \( \beta \notin \Gamma \), then \( \gamma(p) = X_\alpha \rightarrow u\Theta_\beta \rho(\beta) \Theta_\perp \Theta_\alpha \in P' \), else \( \gamma(p) = X_\alpha \rightarrow u\Theta_\beta \Theta_\perp \tilde{\beta} \Theta_\alpha \tilde{\beta} \in P' \), with the convention that \((X_1 \cdots X_n)^z = X_1^z \cdots X_n^z \);
- for every production \( X_\alpha \rightarrow (\alpha, u) \in P' \):
  - if \( \alpha \in \Gamma \), then \( \gamma(p) = X_\alpha^{\omega_\alpha} \rightarrow u \in P' \), else \( \gamma(p) = X_\alpha \rightarrow u \in P' \).

Remark that \( \gamma(p) \) defines a bijection between \( P \) and \( P' \).

The following properties can be checked by induction over the steps of leftmost derivations in \( G \) and \( \mathcal{I} \):

- (a) \( S \xrightarrow{\alpha_1} (\omega, u)X_{\alpha_1} \cdots X_{\alpha_n} \) and \( \omega \in \mathcal{D}^\ast(\Gamma) \) implies \( \rho(\omega) = \rho(\alpha_1 \cdots \alpha_1) \)

- (b) \( S \xrightarrow{\alpha_i} uX_{\alpha_1} \cdots X_{\alpha_i} \cdots X_{\alpha_n} \) implies \( \omega_i = \rho(\alpha_i \cdots \alpha_n), \forall i \in [1, n] \).

Now we prove the following property on leftmost derivations: for all \( (\omega, u) \in \mathcal{D}^\ast(\Gamma) \times \Sigma^* : S \xrightarrow{\alpha_1} (\omega, u)X_{\alpha_1}X_{\alpha_2} \cdots X_{\alpha_k} \) if \( S \xrightarrow{\alpha_1} uX_{\alpha_1} \cdots X_{\alpha_k} \).

Obviously, the induction step holds for all production \( p \) such that \( \rho(p) = X_{\alpha_1} \rightarrow (\beta, u)\Theta_\beta \). So, let us suppose that \( \gamma(p) = X_{\alpha_1}^{\beta} \rightarrow (\beta, u)\Theta_\beta \), with \( \beta \notin \Gamma \). We have to check that \( \gamma(p) \) can be applied if \( \omega \beta \in \mathcal{D}^\ast(\Gamma) \), that is, since \( \omega \in \mathcal{D}^\ast(\Gamma) \) by hypothesis that: \( \omega_1 = \beta \omega_1 (\gamma(p) \text{ can be applied} \) if \( \rho(\omega \beta) \in \Gamma^* \).

This is true since by \( \omega = \rho(\alpha_1 \cdots \alpha_1) \) by (a), and \( \omega_1 = \rho(\alpha_1 \cdots \alpha_1) \) by (b). We end by emphasizing the fact that if \( G \) is non-erasing, the indexed grammar constructed can easily be transformed in a semi-realtime 2-PDA.

Finally, the point 2 of Theorem 3.1 follows from the point 1. It suffices to order the set \( \Gamma = \{ \alpha_1, \ldots, \alpha_k \} \) and replace any productions \( X \rightarrow (\alpha_1, u)w \) of the DCFT by \( X \rightarrow (\perp, \varepsilon)w \), any production \( X \rightarrow (\alpha_1, u)w \) by \( X \rightarrow (\varepsilon, u)w \), and any production \( X \rightarrow (\perp, u)w \) by \( X \rightarrow (\varepsilon, u)w \), we get a new context-free transducer \( G \) satisfying \( L = \mathcal{T}(G')((\mathcal{D}(\{0,1\})) \).
4 Logics for indexed languages

In [10], authors give a logical characterization of context-free languages, using nested words. Nested words are words endowed with a matching relation which is a binary relation linking positions of letters in the word. In this section, we extend this result to indexed languages.

4.1 Logic over nested words

In this part we give some definitions and properties of nested words and define logic projections. More precisely, we will define k-nested words and define FO-projections and MSO-projections as logic projections from k-nested words structures to words structures.

Definition 4.1. A nested word over an alphabet \( \Sigma \) is a structure \((u, M)\) such that \( u \in \Sigma^* \) and \( M \subseteq [1, |u|] \times [1, |u|] \) is a non crossing relation on the positions of the letter of \( u \):

- if \( M(i, j) \) then \( i < j \)
- if \( M(i, j) \) then for all \( k \neq i, j \), \((i, k), (k, i), (j, k), (k, j) \) \( \not\in M \)
- if \( M(i, j) \), \( M(k, l) \) and \( i < k \), then either \( l < j \) or \( j < k \) holds.

Any relation \( M \) over an ordered set satisfying these three properties is called a matching relation.

For \( k \geq 0 \), a k-nested word over an alphabet \( \Sigma \) is a structure \((u, M_0, \ldots, M_k)\) such that \( u \in \Sigma^* \) and \( M_i \subseteq [1, |u|] \times [1, |u|] \) are matching relation, for all \( i \in [1, k] \). We denote by \( NW(\Sigma, k) \) the set of all k-nested words over \( \Sigma \).

We associate to every finite alphabet \( \Sigma \) and every integer \( k \geq 0 \) the relational signature \( \Sigma_{\Sigma,k} = \{<, 2\} \cup \{(R_{a, 1})_{a \in \Sigma}\} \cup \{(M_i, 2)_{1 \leq i \leq k}\} \). When \( k = 0 \), we will write \( Sig_{\Sigma} \) rather than \( \Sigma_{\Sigma,0} \). Clearly, each k-nested word over \( \Sigma \) can be represented by a \( \Sigma_{\Sigma,k} \)-structure. In the following, we will always use the nested word instead of its corresponding relational structure.

Given a formula \( \phi \) over the signature \( \Sigma_{\Sigma,k} \), the set defined by \( \phi \) is \( L(\phi) = \{ u \in NW(\Sigma, k) \mid u \models \phi \} \). In order to define set of words, rather than sets of nested words, we will allow us to quantify over all the matching symbols. So for any \( u \in \Sigma^* \), we write \( u \models_3 \phi \) if there exists \( M_0, \ldots, M_k \) such that \((u, M_0, \ldots, M_k) \models \phi \). The set of words defined by \( \phi \) is then \( L_3(\phi) = \{ u \in NW(\Sigma, k) \mid u \models_3 \phi \} \).

Now we define transformations called MSO-projection and FO-projection associating nested words over an alphabet \( \Gamma \) set of words in \( \Sigma^* \). These transformations will be used to give a concise and uniform presentation of Theorems 4.2 and 4.3.

Definition 4.2 (Logical projection). An MSO-projection (resp. FO-projection) from \( NW(\Gamma, 1) \) toward \( \Sigma^* \) is a formula \( \tau \) in MSO(\( \Sigma_{\Gamma,1} \)) (resp. FO(\( \Sigma_{\Gamma,1} \)) having \(|\Sigma|\) second order free variables indexed by \( \Sigma \).

We denote by \( \tau(\omega) \) the set of all words \( u = (D, <, (R_a)_{a \in \Sigma}) \) such that \( D = \bigcup_{a \in \Sigma} R_a \) and \( \omega; [X_a \leftarrow R_a]_{a \in \Sigma} \models \tau((X_a)_{a \in \Sigma}) \).

If each \( u \in \tau(\omega) \) has the same domain than \( \omega \), \( \tau \) is said to be non-erasing.
Remark that sets \((R_a)_{a \in \Sigma}\) have to be pairwise disjoint. To force a projection to be non-erasing, it suffices to intersect it with the formula \(\forall x, \bigvee_{a \in \Sigma} x \in X_a\).

The following property follows directly from the definition of logical projection:

**Proposition 4.1.** Let \(L \subseteq \Sigma^*\) and \(\Gamma\) be a finite alphabet. The following statements are equivalent:

1. there exists an MSO-projection (resp. FO-projection) \(\tau\) from \(NW(\Gamma, 1)\) toward \(\Sigma^*\) such that \(\tau(NW(\Gamma, 1)) = L\)
2. there exists an MSO-\((\mathcal{S}_{\Gamma \times (\Sigma \cup \{\bot\}), 1})\) (resp. FO) formula \(\phi\) such that \(\pi(\mathcal{L}_\exists(\phi)) = L\) and \(\pi\) is the morphism defined by \(\pi(\alpha, \bot) = \varepsilon\) and \(\pi(\alpha, a) = a\), for all \(a \in \Sigma, \alpha \in \Gamma\).

In addition, there exists a non-erasing \(\tau\) satisfying (1) iff there exists a formula \(\phi\) over the signature \(\mathcal{S}_{\Sigma \times \Gamma, 1}\) satisfying (2).

The first result linking nested words and pushdown automata was given in [10] where authors prove the following theorem:

**Theorem 4.1.** For any set \(L \subseteq \Sigma^*\), the three following properties are equivalent:

1. \(L\) is a context-free language
2. there exists an MSO-\((\mathcal{S}_{\Sigma, 1})\)-formula \(\phi\) such that \(\mathcal{L}_\exists(\phi) = L\)
3. there exists a FO-\((\mathcal{S}_{\Sigma, 1})\)-formula \(\phi\) such that \(\mathcal{L}_\exists(\phi) = L\)

### 4.2 Characterization of indexed languages

A natural way to extend Theorem 4.1 to indexed languages is to use a logic over a class of 2-nested words describing the stack moves of a 2-PDA. We will see later (Proposition 4.4) that FO-logic on 2-nested words is more expressive, even when restricting models to very special 2-nested words. So, we choose to use nested words, endowed with Dyck words, describing the constraints of the stacks.

**Definition 4.3.** Given a finite alphabet \(\Gamma\), a Dyck nested word over \(\Gamma\) is a structure \(\omega = \langle u, M \rangle\) such that:

- \(u \in \mathcal{D}(\Gamma)\)
- \(\langle u, M \rangle\) is a nested word
- for every \((x, y) \in M, \rho(u[x, y]) = \varepsilon\)

We denote by \(DNW(\Gamma)\) the set of all these nested words.

This subsection is devoted to the proof of Theorems 4.2 and 4.3 which are generalizations of Theorem 4.1.

**Theorem 4.2.** For any set \(L \subseteq \Sigma^*\), the three following properties are equivalent:

1. \(L\) is a semi-realtime indexed language of rank \(k\);
2. there exists $\Gamma$ of size $k$, and a non-erasing FO-projection $\tau$ such that $\tau(DNW(\Gamma)) = L$;
3. there exists $\Gamma$ of size $k$, and a non-erasing MSO-projection $\tau$ such that $\tau(DNW(\Gamma)) = L$.

If we consider the more general class of indexed languages, we have:

**Theorem 4.3.** For any set $L \subseteq \Sigma^*$, the three following properties are equivalent:
1. $L$ is an indexed language;
2. there exists a FO-projection $\tau$ such that $\tau(DNW(\{0,1\})) = L$;
3. there exists a MSO-projection $\tau$ such that $\tau(DNW(\{0,1\})) = L$.

**4.2.1 Definability of indexed languages**

We describe now the proof of Theorem 4.2. The proof of Theorem 4.3 is rather similar and use the characterization given in Theorem 3.1(2) and show how to construct an FO-projection from an indexed language.

Let $L \subseteq \Sigma^*$ be a semi-realtime indexed language of rank $k$. From Theorem 3.1 there exists an alphabet $\Gamma$ of size $k$ and a non-erasing Dyck context free transducer $G$ satisfying $T(G)(D(\Gamma)) = L$.

Slightly adapting the construction given in [5] to transform a context free grammar into a context free grammar in Double Greibach Normal Form (that is, right members of productions start and end by a terminal symbol) gives the following Lemma:

**Lemma 4.1.** Let $L \subseteq \Sigma^*$ be a semi-realtime indexed language of rank $k$. There exists an alphabet $\Gamma$ of size $k$ and a context free transducer $G = (\hat{\Gamma}, \Sigma, N, S, P)$ such that $L(G)(D(\Gamma)) = L$ and having the following properties:
1. the set of nonterminal symbols $N$ is partitioned in $(N_\alpha)_{\alpha \in \hat{\Gamma}}$ and $S \in N_\perp$;
2. any production is in one of the following form:
   • $X \to u_1\omega u_2$ with $u_1, u_2 \in (\hat{\Gamma} \times \Sigma)^*$, $\omega \in ((\hat{\Gamma} \times \Sigma) \cup N)^*$ and $|u_1| \geq 1$, $|u_2| \geq 2$;
   • $X \to u$ with $u \in (\hat{\Gamma} \times \Sigma)^*$ and $|u| \geq 2$;
3. if $X \to_G (w, u) \in \hat{\Gamma}^* \times \Sigma^*$ and $X \in N_\alpha$ then there exists $w'$ such that $w = w'\rho(\bar{\alpha})$ and $\rho(w) = \varepsilon$.

Let us briefly explain the proof of Theorem 4.1(1 $\Rightarrow$ 3) given in [10]. Starting from a context-free grammar in Double Greibach Normal Form, each production is associated to a pattern. The grammar is modified in a way such that two nonterminal productions starting from 2 different non terminals $X$ and $Y$ have different patterns. So, a pattern is explicitly related to a nonterminal. Finally, patterns can be described by local properties using edges of an nesting word, so one can easily construct a FO-formula defining the language generated by an context-free grammar.

Here we use the same method by modifying the notion of pattern in a such way that the edges of the matching relation delimit parts of words whose reduction by $\rho$ is $\varepsilon$. 

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Definition 4.4 (Pattern). Let \( p = X \rightarrow u_1X_1u_2X_2 \cdots u_nX_nu_{n+1} \) be a production of the transducer \( G \) of Lemma 4.1 (\( u_i \) are terminal words and \( X_i \) are nonterminals). The pattern of \( p \) is \( \text{pat}(p) = u_1\eta_1u_2\eta_2 \cdots u_n\eta_nX \) where

- \( \eta_i = \) if \( X_i \in N \) else there is \( \alpha \) such that \( X_i \in N_\alpha \) and we set \( \eta_i = |(\alpha, \ast)| \);
- if \( X \in N_\alpha \), \( \alpha \neq \ast \), then by Lemma 4.1 (3) \( u_{n+1} = u(\bar{\alpha}, a) \) and we set \( x = u \); else \( x = u_{n+1} \).

Remark that the pattern of a nonterminal production has the form \( u_1 \cdots u_n \), with \( u_1 \neq \varepsilon \) and \( u_n \neq \varepsilon \) and \( \rho(u_1 \cdots u_n) = \varepsilon \).

Now, the construction of the FO-formula is done exactly as in [10]: a derivation tree in \( G \) is associated to a nested word, and the set \( S \) of nested words corresponding to derivation trees of \( G \) is definable in FOL. In a nested word \((u, M) \in S \) each edge \((x, y) \in M \) corresponds to a nonterminal \( X \) and the surface under the edge \((x, y) \) corresponds to the pattern \( u_1 \cdots u_n \) of a production starting from \( X \). The surface corresponding to this pattern is drawn Figure 1. Symbols \((\alpha, \ast)\) in patterns are interpreted as \((\alpha, a)\) for any \( a \in \Sigma \cup \{\ast\} \).

Since for each pattern \( u_1 \cdots u_n \), \( \rho(u_1 \cdots u_n) = \varepsilon \), a nested word corresponding to a derivation of \( G' \) is a Dyck Nested Word. So, using the construction given in [10] and Proposition 4.1, we get:

Proposition 4.2. For every semi-realtime indexed language of rank \( k \), there exists an alphabet \( \Gamma \) of rank \( k \) and a non-erasing FO-projection \( \tau \) such that \( \tau(DNW(\Gamma)) = L \).

4.2.2 From logic to indexed languages

Now we prove that for an alphabet \( \Gamma \) of size \( k \), and a non-erasing MSO-projection \( \tau \) from \( NW(\bar{\Gamma}) \) to \( \Sigma^* \), \( \tau(DNW(\Gamma)) \) is a semi-realtime indexed language of rank \( k \).

Let us denote \( DNW(\Gamma, \Sigma) \) the class of all nested words \((u, M) \) such that \( u \in (\bar{\Gamma} \times \Sigma)^* \) and \( (\rho(u), M) \in DNW(\Gamma) \). From Proposition 4.1, there exists an MSO formula \( \phi \) such that \( \tau(DNW(\Gamma)) = \pi_2(L_3(\phi) \cap DNW(\Gamma, \Sigma)) \).

From Proposition 5.1, there exists a 1-PDA \( A_1 = (Q, (\bar{\Gamma} \times \Sigma), \Gamma_2, q_0, \Delta_1, F) \) recognizing the language \( L_3(\phi) \). In addition, the behavior of this automaton follows exactly the stacks moves described by edges of nested words in \( L(\phi) \).

More precisely, for all nested word \( w = ((\alpha_1, a_1) \cdots (\alpha_n, a_n), M) \), \( w \in L(\phi) \) iff there is a computation \((q_0, (\alpha_1, a_1) \cdots (\alpha_n, a_n), \varepsilon)) \rightarrow_{A_1} (q_1, (\alpha_2, a_2) \cdots (\alpha_n, a_n), \omega_1)) \rightarrow_{A_1} \cdots (q_n, \varepsilon, \omega_n) \) such that (1):

- there exists \( A \in \Gamma_2 \) such that \( \omega_{i+1} = \text{push}_A(\omega_i) \) iff there exists \( j \) such that \( M(i + 1, j) \),
there exists \( A \in \Gamma_2 \) such that \( \omega_{i+1} = \text{pop}_A(\omega_i) \) iff there exists \( j \) such that \( M(j, i + 1) \).

From this automaton, we construct now a 2-PDA \( \mathcal{A} = (Q, \Sigma, (\Gamma, \Gamma_2), q_0, \Delta, F) \) such that for all \( u = a_1 \cdots a_n \in \Sigma^* \):

\[
(q_0, a_1 \cdots a_n, \varepsilon) \xrightarrow{\text{pda}} (q_1, a_1 \cdots, a_n, \Omega_1) \xrightarrow{\text{pda}} \cdots (q_n, \varepsilon, \Omega_n)
\]

iff there exists \( a_1 \cdots a_n \in \mathcal{D}^e(\Gamma_1) \), and \( \omega_1, \ldots, \omega_n \in \Gamma_2^* \) such that

- \( (q_0, (a_1, a_1) \cdots (a_n, a_n), \varepsilon) \xrightarrow{A_1} (q_1, (a_2, a_2) \cdots (a_n, a_n), \omega_1)) \xrightarrow{A_1} \cdots (q_n, \varepsilon, \omega_n) \),

and

- \( \forall i \in [1, n] \), if \( \Omega_i = A_1[\omega_i^j] \cdots A_f[\omega_i^j] \) then \( \omega_i = A_1 \cdots A_f \) and \( \rho(a_1 \cdots a_i) = \omega_i^j \).

Remark that this property ensures that between a \( \text{push}_{2, A} \) at step \( i \) of the computation, and the corresponding \( \text{push}_{2, A} \) at step \( j \), \( \rho(a_1 \cdots a_j) = \varepsilon \). It follows then from (1) that a such a 2-PDA satisfies \( L(\mathcal{A}) = \tau(DNW(\Gamma)) \).

Let us give the construction of \( \mathcal{A} \) (the construction can be checked by a simple induction on computation steps): for all \( a \in \Sigma, \alpha \in \Gamma_1, p, q \in Q, A \in \Gamma_2 \):

- if \( (q, (a, \bot), \text{stay}, p) \in \Delta_1 \) then \( (q, a, \text{stay}, p) \in \Delta \),
- if \( (q, (a, \bot), \text{push}_{1, A}, p) \in \Delta_1 \) then \( (q, a, \text{push}_{2, A}, p) \in \Delta \),
- if \( (q, (a, \alpha), \text{stay}, p) \in \Delta_1 \) then \( (q, a, \text{push}_{1, A}, p) \in \Delta \),
- if \( (q, (a, \alpha), \text{push}_{1, A}, p) \in \Delta_1 \) then \( (q, a, \text{push}_{2, A}, p) \in \Delta \),
- if \( (q, (a, \alpha), \text{pop}_{1, A}, p) \in \Delta_1 \) then \( (q, a, \text{push}_{2, A}, p) \in \Delta \),
- if \( (q, (a, \text{pop}_{1, A}, p) \in \Delta_1 \) then \( (q, a, \text{push}_{2, A}, p) \in \Delta \),
- if \( (q, (a, \text{pop}_{1, A}, p) \in \Delta_1 \) then \( (q, a, \text{push}_{2, A}, p) \in \Delta \).
4.3 Logic over iterated nested words

Now we show that if we replace the Dyck words of the Dyck nested words by the natural matching relation associated to the Dyck word, the FO-logic over this class of 2-nested words (that we call Iterated nested words) is more powerful to characterize indexed languages.

Definition 4.5. An iterated nested word over an alphabet $\Sigma$ is a 2-nested word $\omega = (u, M_1, M_2)$ over $\Sigma$ such that for all $i, j$, if $M_2(i, j)$ then the number of pop nodes in $[i, j]$ is equal to the number of push nodes in $[i, j]$.

A push node is the leftmost extremity of an edge, and a pop node is the rightmost extremity of an edge.

For instance, the 2-nested word represented Figure 2 is a INW.

Remark that INWs give a good representation of stack moves of a 2-PDA, but do not make appears that the content of the stack of level 1 has been copied. In fact, INWs represent moves of a pushdown automata with 2 stacks such that between the pushing and the popping of an element on the stack 2, the size of stack 1 remained unchanged. It follows that INWs are not the good objects for a logical characterization of indexed languages. This is proven in Proposition 4.4.

Proposition 4.4. There exists a language $L \subseteq \Sigma^*$ which is not an indexed language and for which there exists an FO$(\Sigma, 2)$-formula $\phi$ such that $\text{INW}(\Sigma) \cap L_3(\phi) = L$.

Proof. Let us suppose that $\Sigma = A \cup \{\#\}$. Let $L$ be the set of all words of the form

$u_1 \# u_2 \# \ldots \# u_{n+1} \#$, for $n \geq 0$, and $u_i \in A^*$, such that for all $i \in [1, n]$, if $u_i = a_1 \ldots a_k$, then $u_{i+1} = a_k a_1 \ldots a_{k-1}$.

Using the Shrinking Lemma given in [8] for indexed languages, one can easily prove that $L$ is not an indexed language.

Let us now define a set $S$ of $\text{INW}(\Sigma)$ such that $\{u \mid (u, M_1, M_2) \in S\} = L$, and prove that $S$ is definable in FOL. Let $u = u_1 \# u_2 \# \ldots \# u_{n+1} \# \in L$. For sake of simplicity, we suppose that all factors $u_i$ have an odd length: $|u_i| = 2m$. For each $u_i$, let us consider the unique nested word $(u_i, M_{u_i})$ such that each node in $[1, m]$ is a push node and each node in $[m+1, 2m]$ is a pop node. We define $M_1$ as union of all the $M_{u_i}$ (with the appropriate translation on node numbers).

Now, for each $i \in [1, n]$ let us consider the unique nested word $(u_i \# u_{i+1}, M_{u_i}^\prime)$ such that each node in $[m+1, 2m]$ is a push node and each node in $[2m+2, 3m+1]$ is a pop node. We define $M_2$ as union of all the $M_{u_i}^\prime$ (with the appropriate translation on node numbers). A such a nested word is represented Figure 2.

One can easily construct an $\text{FO}(\Sigma, 2)$-formula $\phi$ defining exactly this set of nested words, since the labeling has to satisfy the following local properties:

1. nodes which are node extremity of a matching are labelled by $\#$

2. If $M_1(x, y) \wedge M_2(y, z) \wedge M_1(z, t)$ then $u(x) = u(t-1)$, $u(y) = u(t-1)$, $u(z) = u(x+1)$ and $u(y-1) = u(z+1)$.

□
Figure 2: A INW: $M_1$ is represented by the edges above the word and $M_2$ is represented by the edges beneath the word.

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A Complements of proofs of Section 3

We give here a complete proof of the following Theorem:

**Theorem 3.1** Let $L \subseteq \Sigma^*$.

1. $L$ is a (semi-realtime) indexed language of rank $k$ iff there is a (non-erasing) Dyck context-free transducer $G$ such that $T(G)(\mathbb{D}([1,k])) = L$.

2. If $L$ is an indexed language, there is context-free transducer $G$ such that $T(G)(\mathbb{D}([0,1])) = L$.

**Proof of Theorem 3.1** (1 $\Rightarrow$). Let $A = (Q, \Sigma, (\Gamma_1, \Gamma_2), q_0, \Delta, F)$ be the 2-PDA such that $L(A) = L$ and $B$ be the 1-PDT built from $A$ according to the construction given in section 3 such that $L(A) = L(B)(\mathbb{D}([1,1]))$. We then apply the standard construction removing the states of a pushdown automaton (see for example [1]) and get directly a Dyck context-free transducer: $G = (\Sigma, N, S_\bot, P)$ with $N_\bot = \{X_{p,q,a}\}_{p,q \in Q}$ for all $a \in \Gamma$ and $N_\bot = \{S_\bot\} \cup \{X_{p,q,a}\}_{p,q \in Q}$, and such that $T(G) = L(B)$. The set $P$ of productions is built as follows: for every $A \in \Gamma_2$, $\alpha \in \Gamma_1 \cup \Gamma_2$, $x \in \Gamma$, $p, r \in Q$:

- if $(q, \bot | a, \text{push}, q') \in \Delta'$, add $X_{q,p,x} \rightarrow (\bot, a)X_{q',r,a}X_{r,p,x}$ to $P$;
- if $(q, \bot | a, \text{pop}, q') \in \Delta'$, add $X_{q,q',a} \rightarrow (\bot, a)$ to $P$;
- if $(q, \alpha | a, \text{push}, q') \in \Delta'$, add $X_{q,p,x} \rightarrow (\alpha, a)X_{q',r,a}X_{r,p,x}$ to $P$;
- if $(q, \alpha | a, \text{pop}, q') \in \Delta'$, add $X_{q,q',a} \rightarrow (\alpha, a)$ to $P$;
- if $(q, \bot | a, \text{stay}, q') \in \Delta'$, add $X_{q,p,x} \rightarrow (\bot, a)X_{q',p,x}$ to $P$.

We also add the following productions for the start symbols and the cases where the stack becomes empty.

- if $(q_0, \alpha | a, \text{push}, x, q) \in \Delta'$ and $x \in \Gamma_1 \cup \Gamma_2$ add $S \rightarrow (\alpha, a)X_{q,p,x}$ and $S \rightarrow (\alpha, a)X_{q,r,z}X_{r,p,\bot}$ to $P$ for all $r \in Q, p \in F$;
- if $(r, \alpha | a, \text{push}, x, q) \in \Delta'$ and $x \in \Gamma_1 \cup \Gamma_2$ add $X_{r,p,\bot} \rightarrow (u, a)X_{q,p,x}$ and $X_{r,p,\bot} \rightarrow (u, a)X_{q,r',x}X_{r',p,\bot}$ to $P$ for all $r' \in Q$. 


We have then proven that if $L$ is an indexed language, there is an alphabet $\Gamma$ and a Dyck context-free transducer $G$ such that $L = T(G)(D(\Gamma))$.

Let us now suppose that $L$ is a semi-realtime indexed languages, we transform $G$ to make it non-erasing: only productions of type $X_\perp \rightarrow (\perp, \varepsilon)$ and $X_\alpha \rightarrow (\perp, \varepsilon)Y_\perp Z_\alpha$ can generate $\varepsilon$ as output. These transitions can be replaced by $X_\perp \rightarrow \varepsilon$ and $X_\alpha \rightarrow Y_\perp Z_\alpha$ without changing the language generated by Dyck words. Production of type $X_\perp \rightarrow \varepsilon$ can be removed by adding a production $Z \rightarrow \omega_1 \omega_2$ for every production $Z \rightarrow \omega_1 X_\perp \omega_2$ of the original transducer. This process may have created productions of the form $X_\alpha \rightarrow Z_\alpha$ which also can be removed by the same kind of transformation.

Now, all productions of the transducer are in one of the following forms:

1. $X_\alpha \rightarrow (\bar{a}, a)$
2. $X_\perp \rightarrow (\bar{\beta}, a)X_\beta$
3. $X_\alpha \rightarrow (\bar{\beta}, a)X_\beta X_\alpha$
4. $X_\alpha \rightarrow (\perp, a)X_\alpha$
5. $X_\alpha \rightarrow Y_\perp Z_\alpha$

for $\alpha, \beta \in \hat{\Gamma}$.

We apply the following construction to get a new transducer $G'$ on the set of nonterminals $N' = N \cup (N \times N)$ (this construction can be applied to any grammar an is not specific to this one):

1. $X \rightarrow (\alpha, a)\omega \in P'$ for all $X \rightarrow (\alpha, a)\omega \in P$
2. $X \rightarrow (\alpha, a)\omega[Y, X] \in P'$ for all $Y \rightarrow (\alpha, a)\omega \in P$
3. $[D, X] \rightarrow Q[C, X] \in P'$ for all $C \rightarrow DQ \in P$
4. $[X, X] \rightarrow \varepsilon$, for all $X$

The following two statements can be shown by induction on the length of derivations: for all pairs $w \in \bar{\Gamma}^* \times \Sigma^*$:

- $[Y, A] \xrightarrow{a} G' w$ iff $A \xrightarrow{a} G Yw$
- $A \xrightarrow{a} G' w$ iff $A \xrightarrow{a} G w$

Remark that if $A \xrightarrow{a} G Yw$, then $Y \in N_\perp$, and then we only have to keep nonterminals $[Y, A]$ satisfy all $Y \in N_\perp$. Let us define the following partition of $N'$ of all $\alpha \in \bar{\Gamma}$, $N'_\alpha = N_\alpha \cup \{[X, Y] \mid Y \in N_\alpha\}$.

Now, we get a transducer in Greibach normal form by replacing each production $[D, X] \rightarrow Q[C, X]$ (of type 3) by all productions $[D, X] \rightarrow \omega[C, X]$ for all production $Q \rightarrow \omega$ (of type 1 or 2). In the same way as below, we also eliminate all production of type 4. Productions of this new transducer are in one of the following forms:

1. derived from production of type 1 and 2

- $X_\alpha \rightarrow (\bar{a}, a)$
- $X_\perp \rightarrow (\bar{\beta}, a)X_\beta$
B Proofs of Section 4.2.1

We give here a detailed proof of Proposition 4.2. We start by proving some properties

Lemma 4.1 Let $L \subseteq \Sigma^*$ be a semi-realtime indexed language of rank $k$. There exists an alphabet $\Gamma$ of size $k$ and a context free transducer $G = (\hat{\Gamma}, \Sigma, N, S, P)$ such that $L(G)(D(\Gamma)) = L$ and having the following properties:

1. the set of nonterminal symbols $N$ is partitioned in $(N_\alpha)_{\alpha \in \hat{\Gamma}}$ and $S \in N_\bot$;
2. any production is in one of the following forms:
   1. $X_\alpha \rightarrow u_1\omega u_2$ with $u_1, u_2 \in (\hat{\Gamma} \times \Sigma)^*$, $\omega \in ((\hat{\Gamma} \times \Sigma) \cup N)^*$ and $|u_1| \geq 1$, $|u_2| \geq 2$;
   2. $X_\alpha \rightarrow u$ with $u \in (\hat{\Gamma} \times \Sigma)^* \cup N^*$ and $|u| \geq 2$;
3. if $X \rightarrow (u, a) \in (\hat{\Gamma} \times \Sigma)^* \cup N^* \cup \{\epsilon\}$ and $X \in N_\alpha$ then there exists $w' \in \alpha$ such that $w = w'\alpha$ and $\rho(w) = \epsilon$.

Proof of Lemma 4.1. Let $L \subseteq \Sigma^*$ be a semi-realtime indexed language of rank $k$. From Theorem 3.1, there exists an alphabet $\Gamma$ of size $k$ and a non-erasing Dyck context free transducer $G = (\hat{\Gamma}, \Sigma, N, S, P)$ satisfying $T(G)(D(\Gamma)) = L$.

From definition, all the nonterminal symbols are indexed by $\Gamma$ (and then $N$ is partitioned in $(N_\alpha)_{\alpha \in \hat{\Gamma}}$) and all the productions are in one of the following forms:

1. $X_\alpha \rightarrow (\gamma, a) \Theta_\gamma$ where $\gamma \in \Sigma \cup \{\epsilon\}$, $\alpha, \beta \in \hat{\Gamma}$ and for all $\gamma \neq \bot$ then $\Theta_\gamma \in N_\gamma$, else $\Theta_\gamma \in N_\bot^*$;
2. $X_\alpha \rightarrow (a, a), a \in \Sigma \cup \{\epsilon\}, \alpha \in \Gamma \cup \hat{\Gamma}$.

Adapting the construction given in [5], we transform $G$ into an equivalent transducer in Greibach Double normal form: $G' = (\hat{\Gamma}, \Sigma, N', S, P')$, $N' = N \cup \{(X,Y)_\alpha \mid X_\alpha, Y_\alpha \in N\}$.

We start by constructing a set $P_1$ of productions by applying the following algorithm:

1. $X \rightarrow (a, a) \in P'$ for all $X \rightarrow (a, a) \in P$
2. $X \rightarrow \omega[Z,T](a, a) \in P'$ for all $X \rightarrow \omega Z \in P$ and $T \rightarrow (a, a)$
3. \([X, Z] \rightarrow \Omega \in P'\) for all \(X \rightarrow \Omega Z \in P\)
4. \([X, Z'] \rightarrow \Omega[Z, X']\Omega' \in P'\) for all \(X \rightarrow \Omega Z, X' \rightarrow \Omega'Z' \in P\)
5. \([X, X] \rightarrow \varepsilon\), for all \(X\)

The following two statements can be shown by induction on the length of derivations: for all pairs \(w \in \hat{\Gamma}^* \times \Sigma^*\):

- \([A, Y] \xrightarrow{*}{G} w\) if \(A \xrightarrow{*}{G} wY\)
- \(A \xrightarrow{*}{G} w\) if \(A \xrightarrow{*}{G} w\)

Remark that if \(A \xrightarrow{*}{G} wY\), then except if \(A \in N_\perp\), then \(A\) and \(Y\) are indexed by the same symbol. Then we only have to keep nonterminals \([A, Y]\), for \(Y \in N_\alpha\) satisfy \(A \in N_\perp \cup N_\alpha\).

So, the productions of the obtained transducer are in one of the following forms:

1. derived from production of type 1 and 2:
   - \(X_{\alpha} \rightarrow (\hat{\alpha}, a)\)
   - \(X_{\alpha} \rightarrow (\hat{\beta}, a)\Theta_\beta\Theta_\perp [Z_{\alpha}, T_{\alpha}](\hat{\alpha}, b)\), for all \(\alpha\)
   - \(X_{\perp} \rightarrow (\hat{\beta}, a)\Theta_\beta\Theta_\perp [Y_{\perp}, T_{\alpha}](\hat{\beta}, b)\)
   - \(X_{\perp} \rightarrow (\hat{\beta}, a)[Y_{\beta}, T_{\beta}](\hat{\beta}, b)\)

2. derived from production of type 3:
   - \([X_{\alpha}, Y_{\alpha}] \rightarrow (\hat{\beta}, a)\Theta_\beta\Theta_\perp\), for all \(\alpha\)
   - \([X_{\perp}, Y_{\beta}] \rightarrow (\hat{\beta}, a)\)

3. derived from production of type 4:
   - \([X_{\alpha}, Z_{\alpha}'] \rightarrow (\hat{\beta}, a)\Theta_\beta\Theta_\perp [Z_{\alpha}, X_{\alpha}'][\hat{\alpha}', \alpha']\Theta_\alpha\Theta_\perp\), for all \(\alpha \in \hat{\Gamma}\)
   - \([X_{\perp}, Z_{\beta}] \rightarrow (\hat{\beta}, a)[Z_{\beta}, X_{\beta}'][\alpha, \alpha']\Theta_\alpha\Theta_\perp\),
   - \([X_{\perp}, Z_{\beta}'] \rightarrow (\hat{\beta}, a)\Theta_\beta\Theta_\perp [Z_{\perp}, X_{\beta}'][\hat{\beta}', a']\)

4. production of type 5: \([X, X] \rightarrow \varepsilon\), for all \(X\)

Let us define the following partitions of \(N'\): of all \(\alpha \neq \perp\), \(N'_\alpha = N_\alpha \cup \{[X, Y] | X \in N_\perp, Y \in N_\alpha\}\) and \(N'_\perp = N_\perp \cup \{[X, Y] | X, Y \in N_\alpha, \alpha \in \hat{\Gamma}\}\).

One can easily check that if \(X \xrightarrow{*}{G} (w, w) \in \hat{\Gamma}^* \times \Sigma^*\) and \(X \in N'_\alpha\) then there exists \(w'\) such that \(w = w'/\rho(\hat{\alpha})\) and \(\rho(w) = \varepsilon\).

Now, we get a transducer in Double Greibach normal by replacing each production \([X, Z] \rightarrow \Omega Y\) (of type 3 and 4) by all productions \([X, Z] \rightarrow \Omega\Omega'\) for all production \(Y \rightarrow \Omega'\) (of type 1 or 2). We also eliminate all productions of type 5 by simple substitutions.

We remind that the notion of pattern is introduced Definition [4.4]. For the proof of Lemma [B.1], we follow exactly the proof of [10] Lemma 2.1.2.
Lemma B.1. Let $L \subseteq \Sigma^*$ be an semi-realtime indexed language of rank $k$. There exists an alphabet $\Gamma$ of size $k$ and a context free transducer $G = (\hat{\Gamma}, \Sigma, N, S, P)$ such that $L(G)(\mathcal{D}(\Gamma)) = L$ and having the following properties:

1. the set of nonterminal symbols $N$ is partitioned in $(N_n)_{n \in \hat{\Gamma}}$ and $S \in N_\bot$;
2. any production is in one of the following form:
   - $X \rightarrow u_1 \omega u_2$ with $u_1, u_2 \in (\hat{\Gamma} \times \Sigma)^*$, $\omega \in ((\hat{\Gamma} \times \Sigma) \cup \Sigma)^*$ and $|u_1| \geq 1$, $|u_2| \geq 2$;
   - $X \rightarrow u$ with $u \in (\hat{\Gamma} \times \Sigma)^*$ and $|u| \geq 2$;
3. if $X \overset{\gamma}{\rightarrow}_G (w, u) \in \hat{\Gamma}^* \times \Sigma^*$ and $X \in N_n$ then there exists $w'$ such that $w = w' \bar{\alpha}$ and $\rho(w) = \varepsilon$.
4. if two nonterminal productions have the same pattern, they have the same left-hand side.

Proof. Using Lemma B.1, we get a transducer satisfying the three first points required by the Lemma. Let us apply the following algorithm to transform our transducer so that it will satisfy the last point:

enumerate all nonterminal symbols $X_1, \ldots, X_r$. Starting with $i = 2$, do the following for every $i$: as long as there is a nonterminal production $p = X_i \rightarrow \Omega$ whose pattern also appears as the pattern of a production with left-hand side $X_j$, $j < i$, replace $p$ by all productions which can be obtained from it by substituting one of the nonterminals in $v$ in all possible ways.

This process will eventually terminate, since the substitutions will either make the production terminal, or it will increase its length. In addition, these transformations preserve conditions 1,2,3. 

We can now achieve the main proof of this part:

Proof of Proposition 4.2. Let $L \subseteq \Sigma^*$ be an semi-realtime indexed language of rank $k$ generated by $G = (\hat{\Gamma}, \Sigma, N, S, P)$ as in Lemma B.1.

We transform $G$ into a context-free grammar the following way:

- if $G$ is non-erasing, $G' = (\hat{\Gamma} \times \Sigma, N, S, P')$.
  For each production $X \rightarrow (\omega_1, u_1)X_1 \cdots (\omega_n, u_n)X_n (\omega_{n+1}, u_{n+1}) \in P$, we have $|\omega_i| = |u_i|$ and we put $X \rightarrow \gamma_1X_1 \cdots \gamma_nX_n \gamma_{n+1} \in P'$ with $\gamma_i = (\alpha_1, a_1) \cdots (\alpha_{\ell_i}, a_{\ell_i})$ and $a_1 \cdots a_{\ell_i} = \omega_i$, $a_1 \cdots a_{\ell_i} = u_i$.
- else $G' = (\hat{\Gamma} \times (\Sigma \cup \{\bot\}), N, S, P')$.
  For each production $X \rightarrow (\omega_1, u_1)X_1 \cdots (\omega_n, u_n)X_n (\omega_{n+1}, u_{n+1}) \in P$, we have $|\omega_i| \geq |u_i|$ and we put $X \rightarrow \gamma_1X_1 \cdots \gamma_nX_n \gamma_{n+1} \in P'$ with $\gamma_i = (\alpha_1, a_1) \cdots (\alpha_{\ell_i}, a_{\ell_i})$ and $a_1 \cdots a_{\ell_i} = \omega_i$, $a_1 \cdots a_{\ell_i} \in u_i \cup \{\bot\}$.

Let us denote $\mathcal{D}(\Gamma, \Sigma)$ the class of all words $u \in (\hat{\Gamma} \times \Sigma \cup \{\bot\})^*$ such that $\pi_1(u) \in \mathcal{D}(\Gamma)$. Obviously, $G'$ satisfies:

$$\pi(\mathcal{L}(G') \cap \mathcal{D}(\Gamma, \Sigma)) = L$$

where $\pi$ is the morphism defined by $\pi(\alpha, \bot) = \varepsilon$ and $\pi(\alpha, a) = a$, for all $a \in \Sigma$, $\alpha \in \Gamma$. 

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Now, consider a derivation tree $T$ of a word $u$ in $G'$. We associate to $T$ a matching relation $M_T$ of $u$ defined the following way: the leftmost and the rightmost child of each internal node $e$ of $T$ are leaves, labeled with terminal symbols corresponding to 2 positions $i, j$, $i < i$ in $u$. If the production applied in the derivation $T$ to the node $e$ has its left-hand side in $N_L$, then $(i, j) \in M_T$, else $(i, j - 1) \in M_T$. Thanks to conditions 2 and 3 of Lemma [B.1] we are sure that $j - 1 > i$, and then $M_T$ is well defined.

We show now that there is a formula $\phi$ over $S_{1,\Gamma \times (\Sigma \cup \{\perp\})}$ (or over $S_{1,\Gamma \times \Sigma}$ if $L$ is semi-realtime) such that

$$(u, M) \models \phi \text{ iff there exists a derivation tree } T \text{ of } u \text{ such that } M = M_T$$

Every edge $(i, j) \in M$ defines a substring $w_i \ldots w_j$ in the following way: we say that an edge $(k, \ell) \in M$ $(i < k < \ell < j)$, lies at the surface of $(i, j)$, if there is no other edge between it and $(i, j)$, that is, if there is no edge $(r, s)$ with $i < r < k < s$. Similarly, a position $k$ $(i \leq k \leq j)$ which is not the endpoint of an edge other than $(i, j)$ lies at the surface of $(i, f)$, if there is no other edge between it and $(i, j)$, that is there is no edge $(r, s)$, with $i < r < k < s$. The string of surface symbols, with surface edges replaced by the symbol | is called the pattern of $(i, j)$ (see Figure 1).

We say that an edge $(i, j)$ corresponds to a production $p$, if their patterns are identical (or if one can replace all symbols $\star$ in the production $p$ to make its pattern identical to those of the edge $(i, j)$). This property of a pair of positions can easily be expressed by a FO-formula $\chi_p(x, y)$. Then, we construct the formula $\chi_X(x, y)$ which is the disjunction of all those $\chi_p(x, y)$ for which $X$ is the left-hand side of the production $p$. For each production $p = u_1x_1 \ldots x_uu_{u+1}$ Then we construct a formula $\chi_p(x, y)$ expressing that $(x, y)$ corresponds to $p$ and that the surfaces arches $(x_1, y_1), \ldots, (x_s, y_s)$ correspond to productions with left-hand side $X_1, \ldots, X_s$ respectively.

Finally, the formula $\phi$ expressed either $u$ is the word $w$, for all production $S \rightarrow w$, or for each edge $(x, y) \in M$, there is a production $p$ such that $\chi_p(x, y)$, and $M(min, max)$ and $\chi_S(min, max)$.

The detail of all these formulas is given in [10].

To conclude, remark that the definition chosen for the patterns of productions implies that if $(u, M) \models \phi$ and $(i, j) \in M$, then $\rho(u(i,j)) = \epsilon$. It follows that

$L_3(\phi) \cap D(\Gamma, \Sigma) = \{u \mid \exists M, (u, M) \in DNW(\Gamma, \Sigma) \text{ and } (u, M) \models \phi\}$. Since $L_3(\phi) = L(G')$, (1) implies that $L = \pi\{u \mid \exists M, (u, M) \in DNW(\Gamma, \Sigma)\}$, that is, from Proposition [1.1] there is a FO-projection $\tau$ such that $\tau(DNW(\Gamma)) = L$.

In addition, if $L$ is semi-realtime, then $\tau$ is non-erasing.

$\Box$