MULTIDIMENSIONAL COSMOLOGY WITH MULTICOMPONENT PERFECT FLUID AND TODA LATTICES

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Abstract

The integration procedure for multidimensional cosmological models with multi-component perfect fluid in spaces of constant curvature is developed. Reduction of pseudo-Euclidean Toda-like systems to the Euclidean ones is done. Some known solutions are singled out from those obtained. The existence of the wormholes is proved.

PACS numbers: 04.20.J, 04.60.+n, 03.65.Ge

Moscow 1994
1 Introduction

We consider dynamical systems with \( n \geq 2 \) degrees of freedom described by the Lagrangian

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{x}^i \dot{x}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sum_{i=1}^{n} b_i^s \dot{x}^i], \quad m \geq 2.
\]  

(1)

A lot of systems in gravitation [10,11] and as well in multidimensional cosmology [12-34] reduce to the systems with such a Lagrangian.

Without loss of generality it can be assumed that matrix \((\eta_{ij})\) is diagonalized and \(\eta_{ii} = \pm 1\) for \(i = 1, \ldots, m\). Such system is an algebraic generalization of a well-known Toda lattice [1] suggested by Bogoyavlensky [2]. We say that it is an Euclidean Toda-like system, if bilinear form of kinetic energy is positively definite, i.e. \(\eta_{ij} = \delta_{ij}\). Nearly nothing is known about Euclidean Toda-like systems with arbitrary sets of vectors \(b_1, \ldots, b_m\), where \(b_s = (b_1^s, \ldots, b_n^s)\) for \(s = 1, \ldots, m\). But, if they form the set of admissible roots of a simple Lie algebra, then the system is completely integrable and possesses a Lax representation. Remind, that the set of roots \(\alpha_1, \ldots, \alpha_m\) is called admissible [2], provided vectors \(\alpha_r - \alpha_s\) are not roots for all \(r, s = 1, \ldots, m\). Each simple Lie algebra possesses the following set of admissible roots

\[
\omega_1, \ldots, \omega_n, -\Omega
\]

(2)

where \(\omega_1, \ldots, \omega_n\) are simple roots and \(\Omega\) is maximal root [3] (usually \(\Omega = \omega_1 + \ldots + \omega_n\)). Any subset of the set (1.2) is also admissible.

If the maximal root holds in the set (1.2), then generalized periodic Toda lattices arise. The different \(L - A\) pairs for them were found by Bogoyavlensky [2]. There were also presented the Hamiltonians of this systems connected with simple Lie algebras.

The further progress in this field was attained by a number of authors (see, for example, [4-8] and refs. therein). In ref. [6] Adler and van Moerbeke established a criterion of algebraic complete integrability for Euclidean Toda-like systems. (This criterion was formally applied to multidimensional vacuum cosmology with \(n\) Einstein spaces in [30].) The explicit integration of the equations of motion for the generalized open Toda lattices (in this case the maximal root is thrown away) was developed by Olshanetsky and Perelomov [4] and Kostant [5]. (See also [7].)

Here we are interested in the problem of integrability of the Toda-like systems with the indefinite bilinear form of the kinetic energy. Let us call such systems pseudo-Euclidean Toda-like systems. To our knowledge, this problem has not been discussed intensively in the mathematical literature before. The reason, as it seems to us, consists in the following. If one try to connect a pseudo-Euclidean Toda-like system by the known manner with simple Lie algebra it reduces to an Euclidean system for the part of coordinates (see Sect. IV). Nevertheless, integrable pseudo-Euclidean Toda-like systems and search for their solutions in explicit form evoke a special interest, because such systems arise in cosmology. For instance, 4-dimensional vacuum homogeneous cosmological model of Bianchi IX-type is described by the Lagrangian (1.1) with \((\eta_{ij})=\text{diag}(-1, +1, +1)\) [2]. (In [9] it was shown, that this model has a rather rich mathematical structure.)

So, in the present paper we study integrable pseudo-Euclidean Toda-like systems appearing in multidimensional cosmology [12-34]. This direction in the modern theoretical physics
has appeared within the new paradigm based on unified theories and hypothesis of additional space-time dimensions. According to this hypothesis the physical space-time manifold has the topology \( M^4 \times B \), where \( M^4 \) is a 4-dimensional manifold, and B is a so-called internal space (or spaces). Nonobservability of additional dimensions is attained in multidimensional cosmology by spontaneous or dynamical compactification of internal spaces to the Planck scale \( (10^{-33} \text{ cm}) \). Integrable cosmological models are of great interest, because the exact solutions allow to study dynamical properties of the model, in particular compactification of internal spaces, in detail.

In the Sect. II, as in [34], we consider the cosmological model where multidimensional space-time manifold \( M \) is a direct product of the time axis \( R \) and of the \( n \) Einstein spaces \( M_1, \ldots, M_n \). We remind, that any manifold of constant curvature is the Einstein one. It is shown that Einstein equations for the scale factors with a source in the multicomponent perfect fluid form correspond to the Lagrangian (1.1) with \((\eta_{ij})=\text{diag}(-1, +1, \ldots, +1)\). We develop the integration procedure to the case of an orthogonal set of vectors \( b_1, \ldots, b_m \) in Sect. III. Sect. IV is devoted to the reduction of pseudo-Euclidean Toda-like system to the Euclidean one for a part of coordinates. This reduction allows us to obtain the class of the exact solutions for some nonorthogonal sets of the vectors \( b_1, \ldots, b_m \). We present the exact solution in the simplest case, when the reducible pseudo-Euclidean system is connected with the Lie algebra \( A_2 \). Discussion of results is presented in Sect. V. We single out some interesting solutions, in particular, Euclidean wormholes.

We denote by \( n \) the number of Einstein spaces and by \( m \) the number of the matter components. Indexes \( i \) and \( j \) run from 1 to \( n \). Index \( s \) runs from 1 to \( m \).

## 2 The model

We consider a cosmological model describing the evolution of \( n \geq 2 \) Einstein spaces in the presence of \( m \)-component perfect-fluid matter [34]. The metric of the model

\[
g = -\exp[2\gamma(t)]dt \otimes dt + \sum_{i=1}^{n} \exp[2x^i(t)]g^{(i)},
\]

is defined on the manifold

\[
M = R \times M_1 \times \ldots \times M_n,
\]

where the manifold \( M_i \) with the metric \( g^{(i)} \) is an Einstein space of dimension \( N_i \), i.e.

\[
R_{m_i n_i}[g^{(i)}] = \lambda^i_{m_i n_i}, \quad m_i, n_i = 1, \ldots, N_i.
\]

The energy-momentum tensor is taken in the following form

\[
T^M_N = \sum_{\alpha=1}^{m} T^{M(\alpha)}_N,
\]

\[
(T^M_N) = \text{diag}(-\rho^{(\alpha)}(t), p_1^{(\alpha)}(t)\delta_{k_1}, \ldots, p_n^{(\alpha)}(t)\delta_{k_n}),
\]

with the conservation law constraints imposed:

\[
\nabla_M T^{M(\beta)}_N = 0
\]
\( \beta = 1, \ldots, m - 1 \). The Einstein equations
\[
R^M_N - \frac{1}{2} \delta^M_N R = \kappa^2 T^M_N
\]  
(\( \kappa^2 \) is gravitational constant) imply \( \nabla_M T^M_N = 0 \) and consequently \( \nabla_M T^M_N(m) = 0 \).

We suppose that for any \( \alpha \)-th component of matter the pressures in all spaces are proportional to the density
\[
p_i^{(\alpha)}(t) = (1 - h_i^{(\alpha)}) \rho^{(\alpha)}(t),
\]
where
\[
h_i^{(\alpha)} = \text{const}.
\]

The non-zero components of the Ricci-tensor for the metric (2.1) are the following
\[
R_{00} = -\sum_{i=1}^{n} N_i [\ddot{x}^i \dot{x}^i - \dot{\gamma} \dot{x}^i + (\ddot{x}^i)^2],
\]
\[
R_{mni} = g_m^{(i)} [\lambda^i + \exp[2x^i - 2\gamma] (\ddot{x}^i + \dot{x}^i (\sum_{i=1}^{n} N_i \dot{x}^i - \dot{\gamma}))],
\]

\( i = 1, \ldots, n \).

We put
\[
\gamma = \gamma_0 \equiv \sum_{i=1}^{n} N_i x^i
\]
in (1.1) (the harmonic time is used).

The conservation law constraint (2.6) for \( \alpha = 1, \ldots, m \) reads
\[
\rho^{(\alpha)} + \sum_{i=1}^{n} N_i \dot{x}^i (\rho^{(\alpha)} + p_i^{(\alpha)}) = 0.
\]

We impose the conditions of state in the form (2.8), (2.9). Then eq. (2.13) gives
\[
\rho^{(\alpha)}(t) = A^{(\alpha)} \exp[-2\gamma_0 + \sum_{i=1}^{n} u_i^{(\alpha)} x^i].
\]

where \( A^{(\alpha)} = \text{const} \) and
\[
u_i^{(\alpha)} = N_i h_i^{(\alpha)}.
\]

The Einstein eqs. (2.7) may be written in the following manner

\[
\frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j + V = 0,
\]

\[
\lambda^i + \ddot{x}^i \exp[2x^i - 2\gamma_0] = -\kappa^2 \sum_{\alpha=1}^{m} u_i^{(\alpha)} A^{(\alpha)} \exp[2x^i - 2\gamma_0 + \sum_{j=1}^{n} u_j^{(\alpha)} x^j].
\]

Here
\[
G_{ij} = N_i \delta_{ij} - N_i N_j
\]
are the components of the minisuperspace metric,

\[ V = -\frac{1}{2} \sum_{i=1}^{n} \lambda^i N_i \exp[-2x^i + 2\gamma_0] + \kappa^2 \sum_{\alpha=1}^{m} A^{(\alpha)} \exp[\sum_{i=1}^{n} u^{(\alpha)}_i x^i]. \]  

(21)

We denote

\[ u^{i}_{(\alpha)} = \sum_{j=1}^{n} G^{ij} u^{(\alpha)}_j, \]  

(22)

where

\[ G^{ij} = \frac{\delta^{ij}}{N_i} + \frac{1}{2 - D}, \]  

(23)

are the components of the matrix inverse to the one \((G_{ij})\) \[24\].

It is not difficult to verify that eqs. (2.17) are equivalent to the Lagrange-Euler eqs. for the Lagrangian

\[ L = \frac{1}{2} \sum_{i,j=1}^{n} G_{ij} \dot{x}^i \dot{x}^j - V. \]  

(24)

Eq. (2.16) is the zero-energy constraint.

We note, that in the framework of our model the curvature induced terms in the potential (2.19) may be considered as additional components of the perfect fluid. The introduction of the cosmological constant \(\Lambda\) into the model is equivalent also to the addition of a new component with \(u_i = 2N_i\) and \(\kappa^2 A = \Lambda\).

Finally, we present the potential (2.19) modified by introduction of \(\Lambda\)-term in the following form

\[ V = \sum_{k=1}^{n} \left(-\frac{1}{2} \lambda^k N_k\right) \exp[\sum_{i,j=1}^{n} G_{ij} v^{(k)}_i x^j] + \sum_{\alpha=1}^{m} \kappa^2 A^{(\alpha)} \exp[\sum_{i,j=1}^{n} G_{ij} u^{i}_i x^j] + \Lambda \exp[\sum_{i,j=1}^{n} G_{ij} u^{i} x^j], \]  

(25)

where we denote:

\[ v^{i}_{(k)} = \sum_{j=1}^{n} G^{ij} v^{(k)}_j = -2 \frac{\delta^i_k}{N_k}, \quad v^{(k)}_j \equiv 2(N_j - \delta^i_j), \]  

(26)

\[ u^{i} = \sum_{j=1}^{n} G^{ij} u^{j}. \]  

(27)

Let \(\langle \ldots \rangle\) be a symmetrical bilinear form defined on n-dimensional real vector space \(R^n\) with the components \(G_{ij} = \langle e_i, e_j \rangle\) in the canonical basis \(e_1, \ldots, e_n\). (\(e_1 = (1, 0, \ldots, 0)\) etc.) It was shown \[22,24\], that the bilinear form \(\langle \ldots \rangle\) is pseudo-Euclidean one with the signature \((-+, +, +, +)\). Then the Lagrangian (2.22) may be written as:

\[ L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[\langle b^{(\alpha)}_i, x \rangle]. \]  

(28)
\[ x = x^1 e_1 + \ldots + x^n e_n, \quad x \in \mathbb{R}^n \]. Here we denoted by \( m \) the total number of components, including curvature, perfect fluid and the cosmological term. We note, that for \( m = 1 \) the Lagrangian system (2.26) is always integrable. The exact solutions were obtained in [34]. (Some special cases were considered in [31,33].) In the present paper we consider multicomponent case: \( m \geq 2 \).

We say that a vector \( y \in \mathbb{R}^n \) is called time-like, space-like or isotropic, if \( \langle y, y \rangle \) has negative, positive or null values correspondingly. Vectors \( y \) and \( z \) are called orthogonal if \( \langle y, z \rangle = 0 \).

### 3 Exact solutions for orthogonal sets of vectors

Let vectors \( b_1, \ldots, b_m \) satisfy the conditions: 1. They are linear independent; 2. \( \langle b_\alpha, b_\beta \rangle = 0 \) for all \( \alpha \neq \beta \), i.e. the set of vectors is orthogonal.

Then \( m \leq n \). It is not difficult to prove

**Proposition 1.** The set of vectors \( b_1, \ldots, b_n \) may contain at most one isotropic vector.

**Proposition 2.** The set of vectors \( b_1, \ldots, b_n \) may contain at most one time-like vector and, if it holds the other vectors must be space-like.

**Remark 1.** The additional term \( a^{(0)} \exp[\langle b_0, x \rangle] \) with zero-vector \( b_0 = 0 \) does not change the equations of motion, but changes the energy constraint (2.16)

\[
\frac{1}{2} \langle \dot{x}, \dot{x} \rangle + \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[\langle b_\alpha, x \rangle] + a^{(0)} = 0. \tag{29}
\]

It corresponds to the perfect fluid with \( b_i^{(0)} = 0 \) for all \( i = 1, \ldots, n \). Such a perfect fluid is called the stiff or Zeldovich matter [35]. It may be considered also as minimally coupled real scalar field [36]. We take into account this additional component by modification of the energy constraint

\[
\frac{1}{2} \langle \dot{x}, \dot{x} \rangle + \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[\langle b_\alpha, x \rangle] = E_0. \tag{30}
\]

These propositions allow to split the class of exact solutions under consideration into following subclasses:

A. There are one time-like vector and at most \((n-1)\) space-like vectors.
B. There are at most \((n-1)\) space-like vectors.
C. There are one isotropic vector and at most \((n-2)\) space-like vectors (this subclass arises for \( n \geq 3 \)).

To integrate eqs. of motion in all subclasses we consider an orthonormal basis \( e'_1, \ldots, e'_n \).

These vectors are such that

\[
\langle e'_i, e'_j \rangle = \eta_{ij}, \tag{31}
\]

where we denote by \( \eta_{ij} \) the components of the matrix

\[
(\eta_{ij}) = \text{diag}(-1, +1, \ldots, +1). \tag{32}
\]

Let us define coordinates of the vectors in this basis by

\[
x = X^1 e'_1 + \ldots + X^n e'_n. \tag{33}
\]
For these new coordinates we have
\[ X^i = \eta_{ii} < e'_i, x >, \quad x^i = \sum_{k=1}^{n} t_k^i X^k, \]  
(34)

where we denoted by \( t_k^i \) the components of a non-degenerate matrix defined by
\[ e'_k = \sum_{i=1}^{n} t_k^i e_i. \]  
(35)

Components \( t_k^i \) satisfy the relations:
\[ \sum_{k,l=1}^{n} G_{kl} t_k^i t_l^j = \eta_{ij}. \]  
(36)

Let us try to find exact solutions for subclasses A, B and C.
A. Let \( b_1 \) be a time-like vector. Then \( < b_r, b_r > 0 \) for \( r = 2, \ldots, m \) (in this case \( m \leq n \)). We choose the orthonormal basis \( e'_1, \ldots, e'_n \) as
\[ e'_s = b_s/|< b_s, b_s >|^{1/2}, \quad s = 1, \ldots, m. \]  
(37)

Then we have:
\[ < b_s, x > = \eta_{ss} < b_s, b_s > |^{1/2} X^s. \]  
(38)

The Lagrangian (2.26) and the energy constraint (3.2) for the coordinates \( X^1, \ldots, X^n \) have the form
\[ L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\eta_{ss} < b_s, b_s > |^{1/2} X^s], \]  
(39)
\[ E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + \sum_{s=1}^{m} a^{(s)} \exp[\eta_{ss} < b_s, b_s > |^{1/2} X^s]. \]  
(40)

Lagrangian (3.11) leads to the set of eqs.
\[ \ddot{X}^s = -|< b_s, b_s >|^{1/2} a^{(s)} \exp[\eta_{ss} < b_s, b_s > |^{1/2} X^s], \]  
(41)
\[ \ddot{X}^m+1 = \ldots = \ddot{X}^n = 0, \]  
(42)

which is easily integrable. We get
\[ X^s = -\eta_{ss} < b_s, b_s > |^{-1/2} \ln[E_s^2 (t - t_{0s})], \]  
(43)
\[ X^{m+1} = p^{m+1} t + q^{m+1} \ldots, X^n = p^n t + q^n, \]  
(44)

where we denoted
\[ F_s (t - t_{0s}) = \sqrt{|a^{(s)}/E_s|} \cosh[\sqrt{|E_s < b_s, b_s > |/2(t - t_{0s})}], \quad \text{if} \ \eta_{ss} a^s > 0, \ \eta_{ss} E_s > 0, \]  
\[ = \sqrt{|a^{(s)}/E_s|} \sinh[\sqrt{|E_s < b_s, b_s > |/2(t - t_{0s})}], \quad \text{if} \ \eta_{ss} a^s < 0, \ \eta_{ss} E_s < 0, \]  
\[ = \sqrt{|a^{(s)}/E_s|} \sin[\sqrt{|E_s < b_s, b_s > |/2(t - t_{0s})}], \quad \text{if} \ \eta_{ss} a^s < 0, \ \eta_{ss} E_s > 0, \]  
\[ = \sqrt{|< b_s, b_s > a^{(s)}|/2(t - t_{0s})}, \quad \text{if} \ \eta_{ss} a^s < 0, \ E_s = 0. \]  
(45)
By \( t_{0s}, E_{0s} \ (s = 1, \ldots, m) \), \( p^{m+1}, \ldots, p^n, q^{m+1}, \ldots, q^n \) we denoted the integration constants. Some of them are not arbitrary and connected by the relation

\[
E_1 + \ldots + E_m + \frac{1}{2}(p^{m+1})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0. \tag{46}
\]

We have for components \( t^i_k \)

\[
t^i_s = b^i_s/|b_s, b_s|^{1/2}. \tag{47}
\]

It is convenient to present the exact solution in a Kasner-like form. Kasner-like parameters are defined by

\[
\alpha^i = t^i_{m+1}p^{m+1} + \ldots + t^i_np^n, \tag{48}
\]

\[
\beta^i = t^i_{m+1}q^{m+1} + \ldots + t^i_nq^n. \tag{49}
\]

Then for the scale factors of the spaces \( M_i \) (see (3.6)) we get

\[
\exp[x^i] = \prod_{s=1}^m (F^2_s(t - t_{0s}))^{-b^i_s/\langle b_s, b_s \rangle} \exp[\alpha^it + \beta^i]. \tag{50}
\]

Vectors \( \alpha, \beta \in R^n \), are defined by

\[
\alpha = \alpha^1e_1 + \ldots + \alpha^ne_n, \quad \beta = \beta^1e_1 + \ldots + \beta^ne_n \tag{51}
\]

satisfy the relations

\[
\langle \alpha, \alpha \rangle = 2(E_0 - E_1 - \ldots - E_m) \geq 0, \tag{52}
\]

\[
\langle \alpha, b_s \rangle = \langle \beta, b_s \rangle = 0, \quad s = 1, \ldots, m. \tag{53}
\]

We remind that \( \langle \alpha, \beta \rangle = \sum_{i,j=1}^n G_{ij}\alpha^i\beta^j \).

Remark 2. If \( m = n \) then \( \alpha = \beta = 0 \).

Remark 3. The set of constants \( E_0, E_s, t_{0s}, \alpha^i \) and \( \beta^i \) is the final set. Only \( 2n \) constants from them are independent.

Remark 4. The subclass of the solutions may be easily enlarged. It is clear, that the addition of new component inducing a vector collinear to one of \( b_1, \ldots, b_m \) leads to the integrable by quadrature model. Let us take into account the following additional terms in the Lagrangian (2.26)

\[
- \sum_{\alpha=1}^{m(\sigma)} a^{(\sigma\alpha)} \exp[b_{(\sigma\alpha)} < b_\sigma, x >], \tag{54}
\]

where \( b_{(\sigma\alpha)} = \text{const} \neq 0 \) for \( \alpha = 1, \ldots, m(\sigma), 1 \leq \sigma \leq m \). It is not difficult to show, that the modification of the exact solution (3.22) only consists in the replacement of the function \( F_\sigma(t - t_{0\sigma}) \) by one \( F(t - t_{0\sigma}) \), satisfying the quadrature

\[
\int dF/\sqrt{E_\sigma F^2 - a^{(\sigma)} - \sum_{\alpha=1}^{m(\sigma)} a^{(\sigma\alpha)} F^2(1-b_{(\sigma\alpha)}) = \langle b_\sigma, b_\sigma \rangle (t - t_{0\sigma})}. \tag{55}
\]
The additional components with other numbers $\sigma$ may be taken into account by the same manner.

B. We have the set of $m$ space-like vectors $b_1, \ldots, b_m$ ($m \leq n - 1$) and the orthonormal basis defined by

$$e'_s = b_s/\sqrt{\langle b_s, b_s \rangle}, \quad s = 1, \ldots, m. \quad (56)$$

The Lagrangian (2.26) and the energy constraint (3.2) in terms of $X$-coordinates have the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp[\sqrt{\langle b_s, b_s \rangle} X^{s+1}], \quad (57)$$

$$E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + \sum_{s=1}^{m} a^{(s)} \exp[\sqrt{\langle b_s, b_s \rangle} X^{s+1}]. \quad (58)$$

The corresponding eqs. of motion

$$\ddot{X}^1 = \dot{X}^{m+2} = \ldots = \dot{X}^n = 0, \quad \ddot{X}^{s+1} = -\sqrt{\langle b_s, b_s \rangle} a^{(s)} \exp[\sqrt{\langle b_s, b_s \rangle} X^{s+1}] \quad (59)$$

lead to the solution

$$X^1 = p^1 t + q^1, \quad (61)$$

$$X^{s+1} = -\frac{1}{\sqrt{\langle b_s, b_s \rangle}} \ln[F^2_s(t-t_{0s})], \quad (62)$$

$$X^{m+2} = p^{m+2} t + q^{m+2}, \ldots, X^n = p^n t + q^n, \quad (63)$$

where functions $F_s(t-t_{0s})$ are defined by (3.17) (in this case all $\eta_{ss} = 1$). Some of integration constants in (3.33)-(3.35) satisfy the relation

$$E_1 + \ldots + E_m - \frac{1}{2} (p^1)^2 + \frac{1}{2} (p^{m+2})^2 + \ldots + \frac{1}{2} (p^n)^2 = E_0. \quad (64)$$

To present the scale factors in a Kasner-like form we define the parameters:

$$\alpha^i = t^i_1 p^1 + t^i_{m+2} p^{m+2} + \ldots + t^i_n p^n, \quad (65)$$

$$\beta^i = t^i_1 q^1 + t^i_{m+1} q^{m+2} + \ldots + t^i_n q^n. \quad (66)$$

Then from (3.6) we obtain the same formula:

$$\exp[x^i] = \prod_{s=1}^{m} [F^2_s(t-t_{0s})]^{-b^i_s/\langle b_s, b_s \rangle} \exp[\alpha^i t + \beta^i]. \quad (67)$$

The relations (3.8) lead to the following constraints for the Kasner-like parameters $\alpha^i$ and $\beta^i$:

$$\langle \alpha, \alpha \rangle = 2(E_0 - E_1 - \ldots - E_m), \quad (68)$$

$$\langle \alpha, b_s \rangle = \langle \beta, b_s \rangle = 0, \quad s = 1, \ldots, m. \quad (69)$$
Remark 5. If \( m = n - 1 \), then either \( \alpha, \alpha > 0 \) or \( \alpha = 0 \); and \( \beta \) has the same properties.

Remark 6. We may also consider the enlargement of this subclass by the manner described in Remark 4. If we add to the Lagrangian (2.26) the terms (3.26) for the some \( \sigma \leq n \), we should replace the function \( F_\sigma(t - t_0\sigma) \) in eq. (3.39) by the function \( F(t - t_0\sigma) \), satisfying (3.27).

C. Let \( b_1 \) be an isotropic vector. Then \( < b_r, b_r > 0 \) for \( r = 2, \ldots, m \) (in this case \( m \leq n - 1 \)). We choose the orthonormal basis \( e_1', \ldots, e_n' \) by

\[
e_r' = b_r/\sqrt{< b_r, b_r >}, \quad b_1 = e_1' + e_{m+1}'.
\]

Then we get

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - a^{(1)} \exp[-X^1 + X^{m+1}] - \sum_{r=2}^{m} a^{(r)} \exp[\sqrt{< b_r, b_r > X^r}],
\]

\[
E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + a^{(1)} \exp[-X^1 + X^{m+1}] + \sum_{r=2}^{m} a^{(r)} \exp[\sqrt{< b_r, b_r > X^r}],
\]

The corresponding eqs. of motion have the form

\[
\ddot{X}^1 = -a^{(1)} \exp[-X^1 + X^{m+1}],
\]

\[
\ddot{X}^{m+1} = -a^{(1)} \exp[-X^1 + X^{m+1}],
\]

\[
\ddot{X}^r = -\sqrt{< b_r, b_r >} a^{(r)} \exp[\sqrt{< b_r, b_r > X^r}],
\]

\[
\ddot{X}^{m+2} = \ldots = \ddot{X}^{n} = 0.
\]

To integrate (3.45), (3.46) it is useful to consider the eqs. of motion for \( X^+ = X^1 + X^{m+1} \) and \( X^- = -X^1 + X^{m+1} \). Then we get the solution

\[
X^1 = \frac{1}{2} (p^+ - p^-) t + \frac{1}{2} (q^+ - q^-) - 2 \ln[f(t)],
\]

\[
X^{m+1} = \frac{1}{2} (p^+ + p^-) t + \frac{1}{2} (q^+ + q^-) - 2 \ln[f(t)],
\]

\[
X^r = \frac{-1}{\sqrt{< b_r, b_r >}} \ln[F_r^2(t - t_0\sigma)],
\]

\[
X^{m+2} = p^m t + q^m + \ldots, X^n = p^n t + q^n,
\]

Here by \( f(t) \) we denoted the function

\[
f(t) = \exp[\frac{a^{(1)}}{2(p^-)^2} \exp[p^- t + q^-]], \quad p^- \neq 0,
\]

\[
= \exp[\frac{a^{(1)}}{4} \exp[q^- t^2]], \quad p^- = 0.
\]

The integration constants satisfy the relations

\[
\frac{1}{2} p^+ p^- + E_2 + \ldots + E_m + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0, \quad p^- \neq 0,
\]

\[
a^{(1)} \exp[q^-] + E_2 + \ldots + E_m + \frac{1}{2}(p^{m+2})^2 + \ldots + \frac{1}{2}(p^n)^2 = E_0, \quad p^- = 0.
\]
The Kasner-like parameters are defined by
\[ \alpha^i = \frac{1}{2} t_i^1 (p^+ - p^-) + \frac{1}{2} t_{i+1}^1 (p^+ + p^-) + t_{i+2}^m p^{m+2} + \ldots + t_n p^n, \]
\[ \beta^i = \frac{1}{2} t_i^1 (q^+ - q^-) + \frac{1}{2} t_{i+1}^1 (q^+ + q^-) + t_{i+2}^m q^{m+2} + \ldots + t_n q^n. \]

Then from (3.6) we obtain the scale factors in a Kasner-like form:
\[ \exp[x^i] = [f(t)]^{-b_i} \prod_{r=2}^m [F_r^2(t - t_0)]^{-b_r/\langle b_r, b_r \rangle} \exp[\alpha^i t + \beta^i]. \]

The Kasner-like parameters satisfy
\[ \langle \alpha, \alpha \rangle = 2(E_0 - E_2 - \ldots - E_m), \quad \langle \alpha, b_1 \rangle \neq 0 \]
\[\langle \alpha, b_r \rangle = \langle \beta, b_r \rangle = 0, \quad r = 2, \ldots, m. \]

Remark 7. For the parameters \( p^- \) and \( q^- \) we get:
\[ p^- = \langle \alpha, b_1 \rangle, \quad q^- = \langle \beta, b_1 \rangle. \]

Remark 8. If \( m = n - 1 \) and \( \langle \alpha, b_1 \rangle = 0 \), then \( \langle \alpha, \alpha \rangle = 0 \), i.e. \( \alpha = p^+ b_1 \). If \( m < n - 1 \) and \( \langle \alpha, b_1 \rangle = 0 \), then \( \langle \alpha, \alpha \rangle \geq 0 \).

Remark 9. Let us consider the enlargement of this subclass by the addition of the terms (3.26) to the Lagrangian. The modification of the exact solution (3.59) for each \( \sigma = 2, \ldots, m \) is described in the Remark 6. Let us take into account the additional components, induced by isotropic vectors collinear to \( b_1 \). It is not difficult to show that in this case (for \( \sigma = 1 \)) the additional terms (3.26) leads to the following modification of the function \( f(t) \)
\[ f(t) = \exp\left\{ \frac{a^{(1)}}{2(p^-)^2} \exp[p^- t + q^-] + \sum_{\alpha=1}^{m(1)} \frac{a^{(1\alpha)}}{2b^{(1\alpha)}(p^-)^2} \exp[b^{(1\alpha)}(p^- t + q^-)] \right\}, \quad p^- \neq 0, \]
\[ = \exp\left\{ (a^{(1)} \exp[q^-] + \sum_{\alpha=1}^{m(1)} a^{(1\alpha)} \exp[b^{(1\alpha)} q^-]) \frac{t^2}{4} \right\}, \quad p^- = 0. \]

In (3.56) and (3.61) the additional terms appear
\[ \sum_{\alpha=1}^{m(1)} a^{(1\alpha)} \exp[b^{(1\alpha)} q^-]. \]

These are all modifications in this case.
4 Reduction of pseudo-Euclidean Toda-like system to Euclidean one

Now we consider the case, when the set of vectors $b_1, \ldots, b_m$ is not orthogonal. It is easily shown that eqs. of motion of our system with the Lagrangian

$$L = \frac{1}{2} \langle \dot{x}, \dot{x} \rangle - \sum_{\alpha=1}^{m} a^{(\alpha)} \exp[< b_\alpha, x >].$$

(94)

for the new variables

$$p = \dot{x} \in \mathbb{R}^n,$$

(95)

$$l_\alpha = a^{(\alpha)} \exp[< b_\alpha, x >]$$

(96)

have the following form

$$\dot{p} = - \sum_{\alpha=1}^{m} l_\alpha b_\alpha,$$

(97)

$$\dot{l}_\alpha = l_\alpha < b_\alpha, p >.$$

(98)

Note that this representation is valid for non-degenerate bilinear form $<.,.>$ with arbitrary signature.

Let us consider a simple complex Lie algebra $G$. Let $H$ be a Cartan subalgebra, and $h_i, e_{\omega_\gamma}$ be a Weyl-Cartan basis in $G$ [3]. We denote by $h_1, \ldots, h_n$ some basis in $H$ and by $\omega_1, \ldots, \omega_N$ the set of roots ($\omega_\gamma \in H, \gamma = 1, \ldots, N$). If the roots $\omega_1, \ldots, \omega_m$ are admissible, then we have [2]

$$[h, e_{\omega_\alpha}] = (\omega_\alpha, h)e_{\omega_\alpha}, \quad h \in H$$

(99)

$$[e_{\omega_\alpha}, e_{-\omega_\beta}] = \delta_{\alpha\beta}\omega_\alpha, \quad \alpha, \beta = 1, \ldots, m,$$

(100)

where we denote by $(.,.)$ the Killing-Cartan form. Let us define in the algebra $G$ the vectors $(L - A$ pair) [2]

$$L(t) = \sum_{\alpha=1}^{m} f_\alpha(t)e_{-\omega_\alpha} + C\sum_{i=1}^{n} h^i(t)h_i + C^2\sum_{\alpha=1}^{m} e_{\omega_\alpha},$$

(101)

$$A(t) = -\frac{1}{C}\sum_{\alpha=1}^{m} f_\alpha(t)e_{-\omega_\alpha},$$

(102)

where $C$ is arbitrary constant. Using (4.6-4.7), it can be easily checked that eq.

$$\dot{L}(t) = [L(t), A(t)]$$

(103)

is equivalent to the following set of eqs. for the variables $f_\alpha(t), h^i(t)$

$$\dot{h} = -\sum_{\alpha=1}^{m} f_\alpha\omega_\alpha,$$

(104)

$$\dot{f}_\alpha = f_\alpha(\omega_\alpha, h),$$

(105)
where we denoted $h = h^1(t) h_1 + \ldots + h^n(t) h_n$, $h \in H$.

Consider the real linear subspace of dimension $n \ H' \in H$ such that the Killing-Cartan form $(\ldots)$ on $H'$ is a real non-degenerate bilinear form with the signature $(-, +, \ldots, +)$, i.e. $<\ldots>$. It is evident, that the sets of eqs. (4.4-4.5) and (4.11-4.12) are identical, if $h, \omega_1, \ldots, \omega_m \in H'$. Thus, if the set of vectors $b_1, \ldots, b_m \in R^n$ equipped with the bilinear form $<\ldots>$ may be identified with a set of admissible roots $\omega_1, \ldots, \omega_m \in H'$, then pseudo-Euclidean Toda-like system with the Lagrangian (4.1) possesses the Lax representation. If such identification is possible, then the system is called to be connected with the simple complex Lie algebra.

Proposition 3. Let a pseudo-Euclidean Toda-like system is connected with a simple complex Lie algebra. Then it is reducible to an Euclidean Toda-like system for a part of coordinates.

Proof. We get in an arbitrary orthonormal basis $e_1', \ldots, e_n'$

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp \left[ \sum_{i=1}^{n} B^s_i X^i \right], \quad (106)$$

where we denoted

$$B^s_i = \sum_{j=1}^{n} \eta_{ij} B^j_s. \quad (107)$$

We remind, that $b_s = B^1_s e_1' + \ldots + B^n_s e_n'$.

It is known [3] that the Killing-Cartan form defined on the real linear span of roots of a simple (or semi-simple) complex Lie algebra is positively definite. But we have the indefinite bilinear form $<\ldots>$. Then the first components of the vectors $b_1, \ldots, b_m$ must be zero in a suitably chosen orthonormal basis, i.e. $B^1_s = 0$ for $s = 1, \ldots, m$. Then, in this basis Lagrangian (4.1) has the form

$$L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - \sum_{s=1}^{m} a^{(s)} \exp \left[ \sum_{k=2}^{m} B^s_k X^k \right]. \quad (108)$$

Coordinate $X^1$ satisfies the eq.: $\ddot{X}^1 = 0$. Eqs. of motion for $X^2, \ldots, X^n$ are followed from the Euclidean Toda-like Lagrangian

$$L_E = \frac{1}{2} \sum_{k,l=2}^{n} \delta_{kl} \dot{X}^k \dot{X}^l - \sum_{s=1}^{m} a^{(s)} \exp \left[ \sum_{k=2}^{m} B^s_k X^k \right]. \quad (109)$$

Thus, we obtained the reduction of a pseudo-Euclidean Toda-like system to the Euclidean one.

Integrating the eqs. of an Euclidean Toda-like system by known methods [4,5,7], we obtain the class of exact solutions for some nonorthogonal set of vectors $b_1, \ldots, b_m$. Here we consider this procedure for the simplest 2-component case $(n \geq 3)$, when Toda lattice is connected with Lie algebra $A_2$.

Suppose, that the vectors $b_1$ and $b_2$, inducing by two components in the Lagrangian

$$L = \frac{1}{2} \ < \dot{x}, \dot{x}> - a^{(1)} \exp[<b_1, x>] - a^{(2)} \exp[<b_2, x>], \quad (110)$$
satisfy the following conditions

\[ < b_1, b_2 >= -\frac{1}{2} < b_1, b_1 >= -\frac{1}{2} < b_2, b_2 > < 0. \]  (111)

Then, we have two space-like vectors with the same lengths. The angle between them is equal to 120°. We denote

\[ \sqrt{< b_1, b_1 >} = \sqrt{< b_2, b_2 >} = b. \]  (112)

Let us define the orthonormal basis \( e'_1, \ldots, e'_n \) in \( \mathbb{R}^n \) by

\[ b_1 = be'_2, \]  (113)

\[ b_2 = b(-\frac{1}{2}e'_2 + \frac{\sqrt{3}}{2}e'_3). \]  (114)

In this basis the Lagrangian (4.17) and corresponding energy constraint have the form

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j - a^{(1)} \exp[bX^2] - a^{(2)} \exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)],
\]  (115)

\[
E_0 = \frac{1}{2} \sum_{i,j=1}^{n} \eta_{ij} \dot{X}^i \dot{X}^j + a^{(1)} \exp[bX^2] + a^{(2)} \exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)]
\]  (116)

For the coordinates \( X^1, X^4, \ldots, X^n \) we get the following eqs. of motion:

\[ \ddot{X}^1 = \ddot{X}^4 = \ldots = \ddot{X}^n. \]  (117)

Therefore

\[ X^1 = p^1t + q^1, \quad X^4 = p^4t + q^4, \ldots, X^n = p^n t + q^n, \]  (118)

where \( p^1, p^4, \ldots, p^n, q^1, q^4, \ldots, q^n \) are arbitrary integration constants. The eqs. of motion for the coordinates \( X^2 \) and \( X^3 \) follow from the Lagrangian

\[
L_E = \frac{1}{2}((\dot{X}^2)^2 + (\dot{X}^3)^2) - a^{(1)} \exp[bX^2] - a^{(2)} \exp[b(-\frac{1}{2}X^2 + \frac{\sqrt{3}}{2}X^3)].
\]  (119)

Let us introduce new coordinates \( y^1 \) and \( y^2 \) as

\[ y_1 = \frac{b}{2\sqrt{2}}X^2, \quad y_2 = \frac{b}{2\sqrt{2}}X^3. \]  (120)

We obtain the Lagrangian of the open Toda lattice connected with the Lie algebra \( A_2 = SL(3, \mathbb{C}) \)

\[
L_T = \frac{1}{2}((\dot{y}^1)^2 + (\dot{y}^2)^2) - \epsilon g_1^2 \exp[2\sqrt{2}y_1] - \epsilon g_2^2 \exp[-\sqrt{2}y_1 + \sqrt{6}y_2],
\]  (121)

where we denoted

\[
b^2 a^{(1)} / 8 = \epsilon g_1^2, \quad b^2 a^{(2)} / 8 = \epsilon g_2^2,
\]  (122)

\[
\epsilon = \text{sgn}[a^{(1)}] = \text{sgn}[a^{(2)}] = \pm 1.
\]  (123)
To study the open Toda lattice it is useful to add the additional coordinate $y_3$:

$$L_T = \frac{1}{2}((\dot{y}^1)^2 + (\dot{y}^2)^2 + (\dot{y}^3)^2) - \epsilon g_1^2 \exp[2\sqrt{2}y_1] - \epsilon g_2^2 \exp[-\sqrt{2}y_1 + \sqrt{6}y_2],$$

(124)

After the orthogonal linear transformation

$$q_1 = \frac{1}{\sqrt{6}}(\sqrt{3}y_1 + y_2 + \sqrt{2}y_3),$$

$$q_2 = \frac{1}{\sqrt{6}}(-\sqrt{3}y_1 + y_2 + \sqrt{2}y_3),$$

$$q_3 = -2y_2 + \sqrt{2}y_3$$

(125)

(126)

the Lagrangian (4.31) take the well-known form [1,4-8]

$$L_T = \frac{1}{2}(\dot{q}_2^1 + \dot{q}_2^2 + \dot{q}_3^2) - \epsilon g_1^2 \exp[2(q_1 - q_2)] - \epsilon g_2^2 \exp[2(q_2 - q_3)].$$

(127)

In this representation the additional degree of freedom corresponds to the free motion of the center of mass ($\ddot{q}_1 + \ddot{q}_2 + \ddot{q}_3 = 0$). The integrating of the eqs. of motion for this system leads to the result [4,5,7]

$$g_1^2 \exp[2(q_1 - q_2)] = \frac{F_+}{F_-}, \quad g_2^2 \exp[2(q_2 - q_3)] = \frac{F_-}{F_+},$$

(128)

where

$$F_\pm = \frac{4}{9A_1A_2(A_1 + A_2)} \{ A_1 \exp[\pm(A_1 + 2A_2)t \pm B_1] + \epsilon(A_1 + A_2) \exp[\pm(A_1 - A_2)t \mp (B_1 - B_2)] + A_2 \exp[\mp(2A_1 + A_2)t \mp B_2] \}. $$

(129)

The integration constants $B_1, B_2$ are arbitrary and $A_1, A_2$ satisfy the condition: $A_1A_2 > 0$. For the energy of the system with Lagrangian (4.24) we have

$$\frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) + \epsilon g_1^2 \exp[2(q_1 - q_2)] + \epsilon g_2^2 \exp[2(q_2 - q_3)] = \frac{3}{4}(A_1^2 + A_1A_2 + A_2^2).$$

(130)

Doing the inverse linear transformation

$$y_1 = \frac{1}{\sqrt{2}}[q_1 - q_2],$$

$$y_2 = \frac{1}{\sqrt{6}}([q_1 - q_2] + 2[q_2 - q_3]),$$

$$y_3 = \frac{1}{\sqrt{3}}(q_1 + q_2 + q_3),$$

(131)

for the system with Lagrangian (4.22) we get the solution

$$X^2 = \frac{1}{b} \ln[\frac{8}{b^2|a^{(1)}|F_+}],$$

$$X^3 = \frac{\sqrt{3}}{b} \ln[\frac{8}{b^2(|a^{(1)}|a^{(2)})^{1/3}F_+}],$$

(132)

(133)
and the following energy constraint

\[ E_0 = -\frac{1}{2}(p^1)^2 + \frac{1}{2}(p^4)^2 + \ldots + \frac{1}{2}(p^n)^2 + \frac{6}{b^2}(A^2_1 + A_1 A_2 + A^2_2). \]  

(134)

To present the scale factors in the Kasner-like form let us introduce the Kasner-like parameters

\[ \alpha^i = t^i_1 p^1 + t^i_4 p^4 + \ldots + t^i_n p^n, \]  

(135)

\[ \beta^i = t^i_1 q^1 + t^i_4 q^4 + \ldots + t^i_n q^n, \]  

(136)

where components \( t^i_k \) are determined by (3.7). In this case they satisfy the relations

\[ t^2_2 = \frac{1}{b^1}, \quad t^3_3 = \frac{2}{\sqrt{3}}(\frac{1}{b^2} + \frac{1}{2b^1}). \]  

(137)

From (3.6) we obtain the coordinates \( x^i \) and finally present the exact solution in the form

\[ \exp[x^i] = [\tilde{F}_-^{-1}]^{-b^1_i/b_{1,1}} \tilde{F}_-^{-1} + b^2_i/b_{2,2} \exp[\alpha^i t + \beta^i], \]  

(138)

where

\[ \tilde{F}_- = \frac{1}{8} b^2 \{(a^{(1)})^2 |a^{(2)}|\}^{\frac{1}{2}} F_-, \]  

(139)

\[ \tilde{F}_+ = \frac{1}{8} b^2 \{(a^{(2)})^2 |a^{(1)}|\}^{\frac{1}{2}} F_. \]  

(140)

The vectors \( \alpha \) and \( \beta \) defined by (3.23) satisfy the relations

\[ < \alpha, \alpha > = 2(E_0 - \frac{6}{b^2}(A^2_1 + A_1 A_2 + A^2_2)), \]  

(141)

\[ < \alpha, b_r > = < \beta, b_r > = 0, \quad r = 1, 2. \]  

(142)

Remark 10. If \( n = 3 \), then \( < \alpha, \alpha > \leq 0 \) and \( < \beta, \beta > \leq 0 \).

5 Discussion

Let us consider some cosmological models corresponding to the introduced in the Sect. III integrable subclasses of pseudo-Euclidean Toda-like systems. For this purpose in Table I we present values of the bilinear form \( <.,.> \) (see Sect. II) for the vectors

\[ v_i \equiv v^1_{(i)} e_1 + \ldots + v^n_{(i)} e_n, \quad v^j_{(i)} = -\frac{\delta^j_i}{N_i}, \]  

(143)

\[ u_\alpha \equiv u^1_{(\alpha)} e_1 + \ldots + u^n_{(\alpha)} e_n, \quad w^j_{(\alpha)} = h^j_{(\alpha)} + \frac{1}{2 - D} \sum_{i=1}^n N_i h^i_{(\alpha)}, \]  

(144)

\[ u \equiv u^1 e_1 + \ldots + u^n e_n, \quad u^j = \frac{2}{2 - D}, \]  

(145)
induced by curvature, perfect fluid and \( \Lambda \)-term correspondingly.

Within the subclass A we are able to construct the model with one Einstein space of non-zero curvature. Let \((n - 1)\) Einstein spaces are Ricci-flat and one, for instance \( M_1 \), have a non-zero Ricci tensor. Then we put \( b_1 \equiv v_1 \). To get the orthogonality with \( b_1 \) for at most \((n - 1)\) available components of the perfect fluid \((b_{\alpha + 1} \equiv u_\alpha) \) for \( \alpha \leq n - 1 \) we put: \( h_1^{(\alpha)} = 0 \) (see Table I). Then, these components appeared to be in the manifold \( M_1 \) in the Zeldovich matter form (see Remark 1). The model of such a type were integrated in \([32]\). In the same way the model with all Ricci-flat spaces and \( \Lambda \)-term arises. In this case we put \( b_1 = u \). The condition of the orthogonality reads: \( \sum_{\alpha = 1}^{n} h_i^{(\alpha)} N_i = 0 \) for all \( \alpha \leq n - 1 \). Then we get the negative values for the some \( h_i^{(\alpha)} \). It means that for such perfect fluids \( p > \rho \) in some spaces (see (2.8)).

The vectors \( v_i \) and \( u \) induced by curvature and \( \Lambda \)-term correspondingly are time-like, therefore subclasses B and C correspond to the Ricci-flat models without \( \Lambda \)-term for some multicomponent perfect fluid source. These vectors can not be roots of any simple complex Lie algebra. Therefore, the models with more than one non-zero curvature space and the models with curvature and \( \Lambda \)-term are not trivially reducible to the Euclidean Toda lattices. Some possibilities of integration of these models were studied in \([27, 29]\).

In conclusion we discuss the existence of the Euclidean wormholes \([37-40]\) within the class of the obtained exact solutions. We consider the simple model within subclass A with the manifold \( R \times M_1 \times M_2 \), when \( M_1 \) has a nonzero Ricci tensor with \( \lambda_1 > 0 \) (see 2.3) and \( M_2 \) is Ricci-flat. The integrable model arises in the presence of the perfect fluid in the Zeldovich matter form for the space \( M_1 \). It means \( h_1 = 0 \) and the other parameter in the equation of state for \( M_2 \) (see 2.8) may be arbitrary positive constant \( h \). If we demand the positiveness of the mass-energy density for the perfect fluid \((A > 0)\), then from (3.22) we get for the scales factors of the \( M_1 \) and \( M_2 \)

\[
\exp[x^1] = \{F_1^2(t - t_{01})\}^{-\frac{1}{M_1 - 1}}, \quad \{F_2^2(t - t_{02})\}^{-\frac{1}{M_2 - 1}}, \quad (146)
\]

\[
\exp[x^2] = \{F_2^2(t - t_{02})\}^{-\frac{1}{M_2}}, \quad (147)
\]

where

\[
F_1(t - t_{01}) = \sqrt{\frac{1}{2} \lambda_1 N_1 / |E_1| \cosh[\sqrt{2|E_1|/(N_1 - 1)/N_1} (t - t_{01})]}, \quad (148)
\]

\[
F_2(t - t_{01}) = \sqrt{\kappa^2 A / E_2} \cosh[h \sqrt{\frac{1}{2} (N_1 - 1) N_2 |E_2|/(N_1 + N_2 - 1)} (t - t_{02})]. \quad (149)
\]

In this case \( E_1 < 0 \) and \( E_2 > 0 \). The energy constraint (3.24) leads to the condition: \(-E_1 = E_2 \equiv E\).

We may suppose that \( M_1 \) is 3-dimensional sphere \( S^3 \) and \( M_2 \) is \( d \)-dimensional torus \( T^d \). Then formulas (5.4-5.7) present the multidimensional generalization of closed Friedmann model. This model may be relevant in the theory of the Early Universe, because the Zeldovich matter equation of state: \( p = \rho \) is valid on the earlier stage of its evolution \([35]\).

To prove the existence of the Euclidean wormholes we use the transformation \( t \to it \).
Then for the case $t_{01} = t_{02} = 0$ we obtain

$$\exp[x^1] = \left\{ \frac{k^2 A}{E} \cos^2 \left[ \sqrt{\frac{Ed}{d+2}} h t \right] \right\}^{1/(2h)} \left\{ \frac{3 \lambda_1}{2E} \cos^2 \left[ \sqrt{\frac{4E}{3}} t \right] \right\}^{-1/4},$$  

(150)

$$\exp[x^2] = \left\{ \frac{k^2 A}{E} \cos^2 \left[ \sqrt{\frac{Ed}{d+2}} h t \right] \right\}^{-1/(h d)}.$$  

(151)

It is easy to see that when $\frac{d}{d+2} h^2 > \frac{4}{3}$ one has wormhole with respect to the internal space $T^d$. The case $\frac{d}{d+2} h^2 < \frac{4}{3}$ corresponds to the wormhole for the external space $S^3$. Note, that for $h = 2$ and $d = 1$ the wormhole for the internal space is accompanied by the static external space. It is not difficult to show that wormhole with respect to the whole space for this model arises in the presence of the additional component in the form of minimally coupled scalar field.

**Acknowledgments**

The authors are grateful to Prof. D.-E.Liebscher for useful discussions.

This work was supported in part by the Russian Ministry of Science.
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\[
\begin{array}{ccc}
\ v_i & 4 \left( \frac{\delta_{ij}}{N_i} - 1 \right) & -2h^{(\beta)}_i & -4 \\
\ u_{\alpha} & -2h^{(\alpha)}_j & \sum_{i=1}^{n} h^{(\alpha)}_i h^{(\beta)}_i N_i + \frac{2}{2-D} \sum_{i=1}^{n} h^{(\alpha)}_i N_i \\
\ u & -4 & \frac{2}{2-D} \sum_{i=1}^{n} h^{(\beta)}_i N_i & -4 \frac{D-1}{D-2} \\
\end{array}
\]

TABLE I