CONTINUOUS SOLUTIONS OF ITERATIVE EQUATIONS OF INFINITE ORDER

Abstract. Given a probability space \((\Omega, A, P)\) and a complete separable metric space \(X\), we consider continuous and bounded solutions \(\varphi: X \to \mathbb{R}\) of the equations \(\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega)\) and \(\varphi(x) = 1 - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega)\), assuming that the given function \(f: X \times \Omega \to X\) is controlled by a random variable \(L: \Omega \to (0, \infty)\) with \(-\infty < \int_{\Omega} \log L(\omega) P(d\omega) < 0\). An application to a refinement type equation is also presented.

Keywords: random-valued vector functions, sequences of iterates, iterative equations, continuous solutions.

Mathematics Subject Classification: Primary 45A05, 39B12; Secondary 39B52, 60B12.

1. INTRODUCTION

Throughout this paper we assume that \((\Omega, A, P)\) is a probability space, \((X, d)\) is a complete separable metric space and \(f: X \times \Omega \to X\) is a random-valued function, i.e., it is measurable with respect to the product \(\sigma\)-algebra \(B(X) \otimes \mathcal{A}\), where \(B(X)\) denotes the \(\sigma\)-algebra of all Borel subsets of \(X\). We consider the equation

\[
\varphi(x) = \int_{\Omega} \varphi(f(x, \omega)) P(d\omega),
\]

which has extensively been studied in various classes of functions (see, e.g., [3, 7, 13]). For more details concerning equation (1.1) and its particular cases, we refer the reader to survey papers [2, part 4] and [1]. Following [11], we also examine the equation of the form

\[
\varphi(x) = 1 - \int_{\Omega} \varphi(f(x, \omega)) P(d\omega).
\]
Numerous papers concern equation (1.1) with $f(x, \omega) = L(\omega)x - M(\omega)$, assuming that $0 < \int_\Omega \log L(\omega)P(d\omega) < \infty$. In the present paper we are interested in the opposite case

$$-\infty < \int_\Omega \log L(\omega)P(d\omega) < 0.$$  \hfill (1.3)

More precisely, we adopt the following hypothesis.

(\textbf{H}) There is a measurable function $L: \Omega \rightarrow (0, \infty)$ such that

$$d(f(x, \omega), f(y, \omega)) \leq L(\omega)d(x, y) \quad \text{for} \quad x, y \in X, \omega \in \Omega$$  \hfill (1.4)

and (1.3) holds.

As an application of the results obtained, we get a corollary on $L^1$-solutions of the equation

$$\Phi(x) = \int_\Omega |\det A(\omega)F'(x)|\Phi(A(\omega)F(x) - C(\omega))P(d\omega).$$  \hfill (1.5)

Equation (1.5) extends both the discrete and the continuous refinement equations which have extensively been studied in connection with their applications (see, e.g., [5, 6, 8, 16]).

The presented results are related to invariance properties of the transfer operator for Markov chains associated with iterated random functions (see, e.g., [9]). In fact, the probability distribution of the limit of the sequences of iterates of a random function satisfies (1.1). Our purpose is to investigate solutions of (1.1), as well as (1.2), in wider classes of functions; e.g., in the class of bounded and continuous functions.

2. MAIN RESULTS

We begin with the following simple lemma.

\textbf{Lemma 2.1.} If (1.3) holds, then the sequence $\left( \prod_{n=1}^N L(\omega_n) \right)$ converges a.s. to zero.

\textit{Proof.} By the Kolmogorov strong law of large numbers,

$$\lim_{N \to \infty} \left( \prod_{n=1}^N L(\omega_n) \right)^{\frac{1}{N}} = \exp \left\{ \int_\Omega \log L(\omega)P(d\omega) \right\} < 1 \quad \text{a.s.}$$

Consequently,

$$\lim_{N \to \infty} \prod_{n=1}^N L(\omega_n) = 0 \quad \text{a.s.}$$

\hfill \square
In the proofs of our results, we will iterate the random-valued function \( f \). The iterates of such a function are defined by (see [4,10])

\[
f^1(x, \omega_1, \omega_2, \ldots) = f(x, \omega_1), \quad f^{n+1}(x, \omega_1, \omega_2, \ldots) = f(f^n(x, \omega_1, \omega_2, \ldots), \omega_{n+1}).
\]

Note that \( f^n \) is a random-valued function on the product probability space \((\Omega^\infty, \mathcal{A}^\infty, P^\infty)\).

We are now in a position to formulate our results. First note that the unique constant solution of (1.2) equals \( \frac{1}{2} \) and we will omit this simple fact in all results of this section.

**Proposition 2.2.** Assume \((H)\) and let \((\sigma_n)\) be a sequence of measure preserving transformations of \((\Omega^\infty, \mathcal{A}^\infty, P^\infty)\) such that

\[
\bigwedge_{\omega \in \Omega^\infty} \left[ \left( \lim_{N \to \infty} \prod_{n=1}^{N} L((\sigma_m(\omega))_n) = 0 \right) \Rightarrow \lim_{N \to \infty} \prod_{n=1}^{N} L((\sigma_N(\omega))_n) = 0 \right].
\]

(2.1)

If \( x_0 \in X \) and if \((f^n(x_0, \cdot) \circ \sigma_n)\) has a subsequence which converges in measure, then every continuous and bounded solution \( \phi : X \to \mathbb{R} \) of (1.1) or (1.2) is constant.

**Proof.** Put

\[
A = \bigcap_{m=1}^{\infty} \sigma_m^{-1} \left( \bigg\{ \omega \in \Omega^\infty : \lim_{N \to \infty} \prod_{n=1}^{N} L(\omega_n) = 0 \bigg\} \right).
\]

From Lemma 2.1 it follows that \( P^\infty(A) = 1 \). By (2.1),

\[
\lim_{N \to \infty} \prod_{n=1}^{N} L((\sigma_N(\omega))_n) = 0 \quad \text{for } \omega \in A.
\]

Using (1.4) and a simple induction, we obtain

\[
d(f^N(x, \sigma_N(\omega)), f^N(y, \sigma_N(\omega))) \leq d(x, y) \prod_{n=1}^{N} L((\sigma_N(\omega))_n)
\]

(2.2)

for \( x, y \in X, \omega \in \Omega^\infty, N \in \mathbb{N} \).

Assume now that \((f^{n_k}(x_0, \cdot) \circ \sigma_{n_k})\) converges in measure. Without loss of generality, we can assume that \((n_k)\) contains even (or odd) numbers only. From (2.2) it follows that for every \( x \in X \) the sequence \((f^{n_k}(x, \cdot) \circ \sigma_{n_k})\) converges in measure and the limit \( \xi \) is independent of \( x \).

Let \( \varphi : X \to \mathbb{R} \) be a continuous and bounded solution of (1.1) or (1.2). In both cases

\[
\varphi(x) = \int_{\Omega^\infty} \varphi(f^{2n}(x, \omega)) P^\infty(d\omega),
\]
whence
\[ \varphi(x) = \int_{\Omega^\infty} \varphi(f^{2n}(x, \sigma_n(\omega))) P^\infty(d\omega) \]
for \( x \in X, n \in \mathbb{N} \). Passing to the limit, we get
\[ \varphi(x) = \int_{\Omega^\infty} \varphi(\xi(\omega)) P^\infty(d\omega) \quad \text{for} \quad x \in X, \]
which shows that \( \varphi \) is constant.

The following result gives some condition on \( f \) under which the sequence \((f^n(x, \cdot) \circ \sigma_n)\) converges a.s. for a special sequence \((\sigma_n)\).

**Theorem 2.3.** Assume (H) and let \( x_0 \in X \). If
\[ \int_{\Omega} \log \max\{d(f(x_0, \omega), x_0), 1\} P(d\omega) < \infty, \quad (2.3) \]
then every continuous and bounded solution \( \varphi : X \to \mathbb{R} \) of (1.1) or (1.2) is constant.

**Proof.** Following [14], define a sequence \((\sigma_n)\) by
\[ \sigma_n(\omega_1, \omega_2, \ldots) = (\omega_n, \ldots, \omega_1, \omega_{n+1}, \ldots). \]
Clearly, \( \sigma_n \) preserves the product measure \( P^\infty \) and (2.1) holds. According to Proposition 2.2, it is enough to show the convergence of \((f^n(x_0, \cdot) \circ \sigma_n)\). Since \( f^n(\cdot, \omega) \)
depends exclusively on the first \( n \) coordinates of \( \omega \in \Omega^\infty \), we see that (2.2) implies
\[ d(f^{N+1}(x_0, \sigma_{N+1}(\omega)), f^N(x_0, \sigma_N(\omega))) \leq \prod_{n=1}^N L(\omega_n) d(f(x_0, \omega_{N+1}), x_0), \]
whence
\[ d(f^{N+N'}(x_0, \sigma_{N+N'}(\omega)), f^N(x_0, \sigma_N(\omega))) \leq \sum_{n=N}^{N+N'-1} \prod_{k=1}^n L(\omega_k) d(f(x_0, \omega_{n+1}), x_0) \]
for \( \omega \in \Omega^\infty, N, N' \in \mathbb{N} \). Consequently, in view of [11, Theorem 2] and (2.3), the series
\[ \sum_{N=1}^{\infty} \prod_{n=1}^N L(\omega_n) d(f(x_0, \omega_{N+1}), x_0) \]
converges almost surely on \( \Omega^\infty \) and the required convergence follows. \( \square \)
Theorem 2.4. If (H) holds, then every bounded and uniformly continuous function \( \varphi : X \to \mathbb{R} \) satisfying
\[
|\varphi(x) - \varphi(y)| \leq \int_{\Omega} |\varphi(f(x,\omega)) - \varphi(f(y,\omega))|P(d\omega) \quad \text{for } x, y \in X \tag{2.4}
\]
is constant.

Proof. Let \( \varphi : X \to (-M,M) \) be a uniformly continuous function such that (2.4) holds.

Fix \( x, y \in X, \varepsilon > 0 \) and let \( \delta \) be a positive real such that \( |\varphi(u) - \varphi(v)| \leq \frac{\varepsilon}{2} \), provided \( d(u,v) \leq \delta \) for \( u, v \in X \).

From (1.4) and Lemma 2.1, we infer
\[
\lim_{N \to \infty} d(f^N(x,\omega), f^N(y,\omega)) = 0.
\]
Hence, for a sufficiently large \( N \in \mathbb{N} \) and for suitably chosen set \( A \in \mathcal{A}^\infty \), there holds
\[
P^\infty(\Omega^\infty \setminus A) \leq \frac{\varepsilon}{4M} \quad \text{and} \quad d(f^N(x,\omega), f^N(y,\omega)) \leq \delta \quad \text{for } \omega \in A.
\]

Finally, by iterating (2.4), we obtain
\[
|\varphi(x) - \varphi(y)| \leq \int_{A} |\varphi(f^N(x,\omega)) - \varphi(f^N(y,\omega))|P^\infty(d\omega) + 2MP^\infty(\Omega^\infty \setminus A) \leq \varepsilon,
\]
which completes the proof.

As a consequence of Theorem 2.4, we obtain a result concerning the uniqueness in the class of uniformly continuous and bounded functions.

Corollary 2.5. If (H) holds, then every bounded and uniformly continuous solution \( \varphi : X \to \mathbb{R} \) of (1.1) or (1.2) is constant.

The next two examples show that neither boundedness nor continuity may be omitted in Theorems 2.3, 2.4 and in Corollary 2.5.

Example 2.6. If
\[
f(0,\omega) = 0 \quad \text{and} \quad f(x,\omega) \neq 0 \quad \text{for } x \neq 0, \omega \in \Omega,
\]
then (2.3) holds with \( x_0 = 0 \) and for every \( \alpha, \beta \in \mathbb{R} \) with \( \alpha \neq \beta \), the function
\[
\varphi = \alpha \chi_{\{0\}} + \beta \chi_{\mathbb{R} \setminus \{0\}} \tag{2.5}
\]
is a bounded and discontinuous solution of (1.1). If \( \alpha + \beta = 1 \), then (2.5) is a solution of (1.2), provided
\[
f(0,\omega) \neq 0 \quad \text{and} \quad f(x,\omega) = 0 \quad \text{for } x \neq 0, \omega \in \Omega.
\]

Example 2.7. Let \( \Omega = \{\omega_1, \omega_2\} \), let \( p_1, p_2 \) be positive reals with \( p_1 + p_2 = 1 \) and let \( L_1 > 0 \) satisfy
\[
L_1^{p_1}(1 - p_1L_1)^{p_2} < p_2^{p_2} \quad \text{and} \quad p_1L_1 < 1.
\]
Put 

\[ L_2 = \frac{1-p_1 L_1}{p_2}, \quad L(\omega_i) = L_i \quad \text{and} \quad f(x, \omega_i) = L(\omega_i) x \]

for \( x \in \mathbb{R}, i = 1, 2 \). Clearly, conditions (1.4) and (1.3) are fulfilled. Equation (1.1) now takes the form

\[ \varphi(x) = p_1 \varphi(L_1 x) + p_2 \varphi(L_2 x). \]

Since \( p_1 L_1 + p_2 L_2 = 1 \), the identity function is a solution of the equation above. It is easy to verify that the function \( x \mapsto x + 1/2 \) satisfies

\[ \varphi(x) = 1 - p_1 \varphi(-L_1 x) - p_2 \varphi(-L_2 x). \]

Denote by \( \mathbb{R}^{n \times m} \) the set of all matrices with \( n \) rows and \( m \) columns, and by \( \| \cdot \| \) the maximum norm in \( \mathbb{R}^n \).

From now on we assume that

\[ f(x, \omega) = A(\omega) F(x) - C(\omega), \tag{2.6} \]

where \( A = [A_{ij}]: \Omega \to \mathbb{R}^{n \times m}, C: \Omega \to \mathbb{R}^n \) are measurable and \( F = [F_i]: \mathbb{R}^n \to \mathbb{R}^m \) is continuous. It is clear that the function given by (2.6) is random-valued (see [12]). Equations (1.1) and (1.2) now take the forms

\[ \varphi(x) = \int_{\Omega} \varphi(A(\omega) F(x) - C(\omega)) P(d\omega) \tag{2.7} \]

and

\[ \varphi(x) = 1 - \int_{\Omega} \varphi(A(\omega) F(x) - C(\omega)) P(d\omega), \tag{2.8} \]

respectively.

The following corollary will be useful in the next section.

**Corollary 2.8.** Let \( F(0) = 0 \),

\[ |F_i(x) - F_i(y)| \leq \|x - y\| \quad \text{for} \ x, y \in X, i = 1, \ldots, m \]

and

\[ -\infty < \int_{\Omega} \log \max_{k=1,\ldots,n} \{|A_{k1}(\omega)| + \cdots + |A_{km}(\omega)|\} P(d\omega) < 0. \]

Then:

(i) Every bounded and uniformly continuous solution \( \varphi: \mathbb{R}^n \to \mathbb{R} \) of (2.7) or (2.8) is constant.

(ii) If

\[ \int_{\Omega} \log \max\{\|C(\omega)\|, 1\} P(d\omega) < \infty, \tag{2.9} \]

then every continuous and bounded solution \( \varphi: \mathbb{R}^n \to \mathbb{R} \) of (2.7) or (2.8) is constant.
Proof. Clearly, (H) holds with
\[ L(\omega) = \max_{k=1,\ldots,n} \{|A_{k1}(\omega)| + \cdots + |A_{km}(\omega)|\} \]
and by (2.9) we obtain (2.3) with \( x_0 = 0 \). Hence the assertions follow from Corollary 2.5 and Theorem 2.3, respectively. \( \square \)

3. AN APPLICATION TO A REFINEMENT TYPE EQUATION

Let \( A: \Omega \to \mathbb{R}^{n \times n} \) and \( C: \Omega \to \mathbb{R}^n \) be measurable, \( \det A(\omega) \neq 0 \) for \( \omega \in \Omega \) and let \( F: \mathbb{R}^n \to \mathbb{R}^n \) be a diffeomorphism. Then for an \( f \) of form (2.6), there holds
\[ l_n \otimes P(f^{-1}(B)) = \int\limits_{\Omega} l_n \left( F^{-1}(A(\omega)^{-1}(B + C(\omega))) \right) P(d\omega) = 0 \]
for \( B \in B(\mathbb{R}^n) \) of zero Lebesgue measure \( l_n \). Consequently, if \( \Phi: \mathbb{R}^n \to \mathbb{R} \) is Lebesgue measurable, then \( \Phi \circ f \) is measurable with respect to the completion of the product \( \sigma \)-algebra \( L_n \otimes A \). Moreover, if the measure \( P \) is complete, then equation (1.5) with unknown \( L^1 \)-function \( \Phi: \mathbb{R}^n \to \mathbb{R} \) makes sense. (We omit details, which may be found in [15] for \( n = 1 \)).

Fix measurable functions \( a_1,\ldots,a_n,c_1,\ldots,c_n: \Omega \to \mathbb{R} \) and diffeomorphisms \( F_1,\ldots,F_n \) from \( \mathbb{R} \) onto itself such that
\[ F_i(0) = 0 \quad \text{and} \quad |F_i(x) - F_i(y)| \leq \|x - y\| \quad \text{for} \quad x,y \in \mathbb{R}, \ i = 1,\ldots,n, \]
and define functions \( A = [A_{ij}]: \Omega \to \mathbb{R}^{n \times n}, F: \mathbb{R}^n \to \mathbb{R}^n \) and \( C: \Omega \to \mathbb{R}^n \) putting
\[ F(x) = (F_1(x_1),\ldots,F_n(x_n)), \quad C = (c_1,\ldots,c_n) \]
and
\[ A_{ij} = 0 \quad \text{if} \quad i \neq j \quad \text{and} \quad A_{ii} = a_i \quad \text{for} \quad i,j = 1,\ldots,n. \]

The following corollary concerns a refinement type equation of form (1.5) with a complete measure \( P \) and the functions \( A,F,C \) defined above.

**Corollary 3.1.** Assume that \( a_1,\ldots,a_n \) are positive (resp. negative), \( F_1,\ldots,F_n \) are increasing (resp. decreasing) and
\[ -\infty < \int\limits_{\Omega} \log \max_{k=1,\ldots,n} |a_k(\omega)| P(d\omega) < 0. \]
Then the trivial function is the only \( L^1 \)-solution \( \Phi: \mathbb{R}^n \to \mathbb{R} \) of (1.5).

**Proof.** Suppose that \( \Phi: \mathbb{R}^n \to \mathbb{R} \) is an \( L^1 \)-solution of (1.5). Define \( \varphi: \mathbb{R}^n \to \mathbb{R} \) by
\[ \varphi(x) = \int_{U_x} \Phi(t) dt, \]
where $U_x = (−∞, x_1) \times \cdots \times (−∞, x_n)$ for $x \in \mathbb{R}^n$. Since

$$U_x = f^{-1}(\cdot, \omega)(U_{f(x, \omega)})$$

and the function “$\mathbb{R}^n \times \Omega \ni (x, \omega) \mapsto | \det A(\omega)F'(x)|\Phi(A(\omega)F(x) − C(\omega))$” is product measurable, it follows that

$$\phi(x) = \int_{\Omega} \left( \int_{U_x} | \det A(\omega)F'(t)|\Phi(A(\omega)F(t) − C(\omega))dt \right) P(d\omega) =$$

$$= \int_{\Omega} \left( \int_{U_{f(x, \omega)}} \Phi(t)dt \right) P(d\omega) = \int_{\Omega} \phi(A(\omega)F(x) − C(\omega))P(d\omega).$$

This means that $\phi$ is a bounded and uniformly continuous solution of (2.7). Moreover, all the assumptions of Corollary 2.8(i) are satisfied. Consequently, $\phi$ is constant and so $\Phi$ equals zero.

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Janusz Morawiec
morawiec@math.us.edu.pl
Silesian University
Institute of Mathematics
Bankowa 14, 40-007 Katowice, Poland

Rafal Kapica
rkapica@math.us.edu.pl
Silesian University
Institute of Mathematics
Bankowa 14, 40-007 Katowice, Poland

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