LOWER BOUNDS FOR MOMENTS OF $\zeta'(\rho)$

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Abstract. Assuming the Riemann Hypothesis, we establish lower bounds for moments of the derivative of the Riemann zeta-function averaged over the non-trivial zeros of $\zeta(s)$. Our proof is based upon a recent method of Rudnick and Soundararajan that provides analogous bounds for moments of $L$-functions at the central point, averaged over families.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. In this article we are interested in obtaining lower bounds for moments of the form

$$J_k(T) = \frac{1}{N(T)} \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2k}$$

where $k \in \mathbb{N}$ and the sum runs over the non-trivial (complex) zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. As usual, we let the function

$$N(T) = \sum_{0<\gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T)$$

denote the number of zeros of $\zeta(s)$ up to a height $T$ counted with multiplicity.

Independently, Gonek [3] and Hejhal [5] have conjectured that $J_k(T) \sim (\log T)^{k(k+2)}$ for each $k \in \mathbb{R}$. By modeling the Riemann zeta-function and its derivative using characteristic polynomials of random matrices, Hughes, Keating, and O’Connell [6] have refined this conjecture to state that $J_k(T) \sim C_k (\log T)^{k(k+2)}$ for a precise constant $C_k$ when $k \in \mathbb{C}$ and $\Re k > -3/2$. However, we no longer believe this conjecture to be true for $\Re k < -3/2$. This is since we expect there exist infinitely many zeros $\rho$ such that $|\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon}$ for each $\varepsilon > 0$.

Results of the sort suggested by these conjectures are only known for a few small values of $k$. See, for instance, the results of Gonek [1] for the case $k = 1$ and Ng [8] for the case $k = 2$. Also, Gonek [3] obtained a lower bound in the case $k = -1$. Our main result is to obtain a lower bound for $J_k(T)$ for each $k \in \mathbb{N}$ of the order of magnitude that is suggested by these conjectures.

Theorem 1. Assume the Riemann Hypothesis and let $k \in \mathbb{N}$. Then for sufficiently large $T$ we have

$$\frac{1}{N(T)} \sum_{0<\gamma \leq T} |\zeta'(\rho)|^{2k} \gg_k (\log T)^{k(k+2)}.$$
Under the assumption of the Riemann Hypothesis, Milinovich [7] has recently shown that $J_k(T) \ll_{k, \varepsilon} (\log T)^{k(k+2)+\varepsilon}$ for $k \in \mathbb{N}$ and $\varepsilon > 0$ arbitrary. When combined with Theorem 1, this result lends strong support for the conjecture of Gonek and Hejhal for $k$ a positive integer.

Theorem 1 can be used to exhibit large values of $\zeta'(\rho)$. For example, as an immediate corollary we have the following result.

**Corollary 1.1.** Assume the Riemann Hypothesis and let $\rho = \frac{1}{2} + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then for each $A > 0$ the inequality

$$|\zeta'(\rho)| \geq (\log |\gamma|)^A$$

is satisfied infinitely often.

This result was previously proven by Ng [10] by an application of Soundararajan’s resonance method [13]. The present proof is simpler and provides many more zeros $\rho$ such that (3) is true. On the other hand, the resonance method is capable of detecting much larger values of $\zeta'(\rho)$ assuming a very weak form of the generalized Riemann hypothesis.

Our proof of Theorem 1 relies on combining a method of Rudnick and Soundararajan [11, 12] with a mean-value theorem of Ng (our Lemma 2) and a well-known lemma of Gonek (our Lemma 3). It is likely that our proof can be adapted to prove a lower bound for $J_k(T)$ of the conjectured order of magnitude for all rational $k$ (with $k \geq 1$) in a manner analogous to that suggested in [11].

Let $k \in \mathbb{N}$ and define, for $\xi \geq 1$, the function $A_\xi(s) = \sum_{n \leq \xi} n^{-s}$. Assuming the Riemann Hypothesis, we will estimate

$$\Sigma_1 = \sum_{0 < \gamma \leq T} \zeta'(\rho) A_\xi(\rho)^{k-1} \overline{A_\xi(\rho)}^k \quad \text{and} \quad \Sigma_2 = \sum_{0 < \gamma \leq T} |A_\xi(\rho)|^{2k}$$

where the sums run over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$. Hölder’s inequality implies that

$$\sum_{0 < \gamma \leq T} |\zeta'(\rho)|^{2k} \geq \frac{\left|\Sigma_1\right|^{2k}}{(\Sigma_2)^{k-1}},$$

and so we see that Theorem 1 will follow from the estimates

$$\Sigma_1 \gg T(\log T)^{k^2+2} \quad \text{and} \quad \Sigma_2 \ll T(\log T)^{k^2+1}. \quad (4)$$

It is convenient to express $\Sigma_1$ and $\Sigma_2$ slightly differently. Assuming the Riemann Hypothesis, $1 - \rho = \bar{\rho}$ for any non-trivial zero $\rho$ of $\zeta(s)$. Thus, $\overline{A_\xi(\rho)} = A_\xi(1-\rho)$. This allows us to re-write the sums in (4) as

$$\Sigma_1 = \sum_{0 < \gamma \leq T} \zeta'(\rho) A_\xi(\rho)^{k-1} A_\xi(1-\rho)^k \quad \text{and} \quad \Sigma_2 = \sum_{0 < \gamma \leq T} A_\xi(\rho)^k A_\xi(1-\rho)^k. \quad (5)$$

It is with these representations of $\Sigma_1$ and $\Sigma_2$ that we establish the bounds in (4).
2. Some preliminary estimates

For each real number \( \xi \geq 1 \) and each \( k \in \mathbb{N} \), we define the arithmetic sequence of real numbers \( \tau_k(n; \xi) \) by

\[
\sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)}{n^s} = \left( \sum_{n \leq \xi} \frac{1}{n^s} \right)^k = A_\xi(s)^k. \tag{6}
\]

The function \( \tau_k(n; \xi) \) is a truncated approximation to the arithmetic function \( \tau_k(n) \) (the \( k \)-th iterated divisor function) which is defined by

\[
\zeta^k(s) = \left( \sum_{n=1}^{\infty} \frac{1}{n^s} \right)^k = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s}, \tag{7}
\]

for \( \Re s > 1 \). We require a few estimates for sums involving the functions \( \tau_k(n) \) and \( \tau_k(n; \xi) \) in order to establish the bounds for \( \Sigma_1 \) and \( \Sigma_2 \) in (4).

We use repeatedly that, for \( x \geq 3 \) and \( k, \ell \in \mathbb{N} \),

\[
\sum_{n \leq x} \frac{\tau_k(n)\tau_{\ell}(n)}{n} \asymp_{k, \ell} (\log x)^{k\ell}, \tag{8}
\]

where the implied constants depend on \( k \) and \( \ell \). These bounds are well-known.

From (6) and (7) we notice that \( \tau_k(n; \xi) \) is non-negative and \( \tau_k(n; \xi) \leq \tau_k(n) \) with equality holding when \( n \leq \xi \). In particular, choosing \( k = \ell \) in (8) we find that, for \( \xi \geq 3 \),

\[
(\log \xi)^{k^2} \ll_k \sum_{n \leq \xi} \frac{\tau_k(n)^2}{n} \leq \sum_{n \leq \xi^k} \frac{\tau_k(n; \xi)^2}{n} \leq \sum_{n \leq \xi^k} \frac{\tau_k(n)^2}{n} \ll_k (\log \xi)^{k^2}. \tag{9}
\]

3. A Lower Bound for \( \Sigma_1 \)

In order to establish a lower bound for \( \Sigma_1 \), we require a mean-value estimate for sums of the form

\[
S(X, Y; T) = \sum_{0<\gamma \leq T} \zeta'(\rho)X(\rho)Y(1-\rho)
\]

where

\[
X(s) = \sum_{n \leq N} \frac{x_n}{n^s} \quad \text{and} \quad Y(s) = \sum_{n \leq N} \frac{y_n}{n^s}
\]

are Dirichlet polynomials. For \( X(s) \) and \( Y(s) \) satisfying certain reasonable conditions, a general formula for \( S(X, Y; T) \) has been established by the second author [9]. Before stating the formula, we first introduce some notation. For \( T \) large, we let \( L = \log \frac{T}{\log T} \) and \( N = T^\vartheta \) for some fixed \( \vartheta > 0 \). The functions \( \mu(\cdot) \) and \( \Lambda(\cdot) \) are used to denote the usual arithmetic functions of Möbius and von Mangoldt. Also, we define the arithmetic function \( \Lambda_2(\cdot) \) by \( \Lambda_2(n) = (\mu * \log^2)(n) \) for each \( n \in \mathbb{N} \).
Lemma 2. Let \( x_n \) and \( y_n \) satisfy \(|x_n|, |y_n| \ll \tau_\ell(n)\) for some \( \ell \in \mathbb{N} \) and assume that \( 0 < \vartheta < 1/2 \). Then for any \( A > 0 \), any \( \varepsilon > 0 \), and sufficiently large \( T \) we have

\[
S(X, Y; T) = \frac{T}{2\pi} \sum_{mn \leq N} \frac{x_m y_{mn}}{mn} \left( p_2(\mathcal{L}) - 2p_1(\mathcal{L}) \log n + (\Lambda \ast \log)(n) \right)
- \frac{T}{4\pi} \sum_{mn \leq N} \frac{y_{mn} x_{mn}}{mn} q_2(\mathcal{L} - \log n) + \frac{T}{2\pi} \sum_{a,b \leq N} \frac{r(a; b)}{ab} \sum_{g \leq \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{y_{ag} x_{bg}}{g}
+ O_A(T(\log T)^{-A} + T^{3/4 + \vartheta/2 + \varepsilon})
\]

where \( p_1, p_2, \) and \( q_2 \) are monic polynomials of degrees 1, 2, and 2, respectively, and for \( a, b \in \mathbb{N} \) the function \( r(a; b) \) satisfies the bound

\[
|r(a; b)| \ll A_2(a) + (\log T)A(a) .
\]

Proof. This is a special case of Theorem 1.3 of Ng [9].

Letting \( \xi = T^{1/(4k)} \), we find that the choices \( X(s) = A_\xi(s)^{k-1} \) and \( Y(s) = A_\xi(s)^k \) satisfy the conditions of Lemma 2 with \( \vartheta = 1/4, N = \xi^k, x_n = \tau_{k-1}(n; \xi) \), and \( y_n = \tau_k(n; \xi) \). Consequently, for this choice of \( \xi \),

\[
\Sigma_1 = \frac{T}{2\pi} \sum_{mn \leq \xi^k} \frac{\tau_{k-1}(m; \xi) \tau_k(m; \xi)}{mn} \left( p_2(\mathcal{L}) - 2p_1(\mathcal{L}) \log n + (\Lambda \ast \log)(n) \right)
- \frac{T}{4\pi} \sum_{mn \leq \xi^k} \frac{\tau_k(m; \xi) \tau_{k-1}(mn; \xi)}{mn} q_2(\mathcal{L} - \log n)
+ \frac{T}{2\pi} \sum_{a,b \leq \xi^k} \frac{r(a; b)}{ab} \sum_{g \leq \min\left(\frac{N}{a}, \frac{N}{b}\right)} \frac{\tau_k(ag; \xi) \tau_{k-1}(bg; \xi)}{g} + O(T)
\]

say. To estimate \( \Sigma_1 \), notice that, for \( T \) sufficiently large, \( n \leq \xi^k = T^{1/4} \) implies that

\[
\left( p_2(\mathcal{L}) - 2p_1(\mathcal{L}) \log n + (\Lambda \ast \log)(n) \right) \gg \mathcal{L}^2
\]

and moreover, by [9],

\[
\sum_{mn \leq \xi^k} \frac{\tau_{k-1}(m; \xi) \tau_k(mn; \xi)}{mn} \geq \sum_{n \leq k} \frac{\tau_k(n; \xi)^2}{n} \gg (\log T)^k.
\]

Thus, \( \Sigma_1 \gg T(\log T)^{k^2 + 2} \). Since \( q_2(\mathcal{L} - \log n) \ll \mathcal{L}^2 \), we can bound \( \Sigma_2 \) by using the inequalities \( \tau_k(n; \xi) \leq \tau_k(n) \) and \( \tau_k(mn) \leq \tau_k(m) \tau_k(n) \). In particular, by twice using [9], we find that

\[
\Sigma_2 \ll T \mathcal{L}^2 \sum_{mn \leq \xi^k} \frac{\tau_k(m) \tau_{k-1}(m) \tau_k(n)}{mn} \leq T \mathcal{L}^2 \left( \sum_{m \leq T} \frac{\tau_k(m) \tau_{k-1}(m)}{m} \right) \left( \sum_{n \leq T} \frac{\tau_{k-1}(n)}{n} \right)
\ll T(\log T)^{2 + k(k-1) + k-1} \ll T(\log T)^{k^2 + 1}.
\]
It remains to consider the contribution from $S_{13}$. Again using the inequalities
\[ \tau_k(n;\xi) \leq \tau_k(n) \] and \[ \tau_k(mn) \leq \tau_k(m)\tau_k(n) \] along with (10), it follows that $S_{13}$ is bounded by
\[ \sum_{a,b \leq \xi^k} (\Lambda_2(a) + (\log T)\Lambda(a)) \sum_{g \leq \xi^k} \tau_k(a)\tau_k(g)\tau_{k-1}(b)\tau_{k-1}(g) \]
\[ \ll \sum_{a \leq T} \sum_{b \leq T} \sum_{g \leq T} \tau_k(a)\tau_k(b)\tau_{k-1}(g) \]
\[ \ll (\log T)^{2+(k-1)+k(k-1)} = (\log T)^{k^2+1}. \]
Combining this with our estimates for $S_{11}$ and $S_{12}$, we conclude that $\Sigma_1 \gg T(\log T)^{k^2+2}$.

4. An Upper Bound for $\Sigma_2$

Assuming the Riemann Hypothesis, we interchange the sums in (10) and find that
\[ \Sigma_2 = N(T) \sum_{n \leq \xi^k} \frac{\tau_k(n;\xi)^2}{n} + 2\Re \sum_{m \leq \xi^k} \sum_{m<n \leq \xi^k} \frac{\tau_k(m;\xi)\tau_k(n;\xi)}{n} \sum_{0<\gamma \leq T} \left( \frac{n}{m} \right)^\beta \]
where $N(T)$ denotes the number of non-trivial zeros of $\zeta(s)$ up to a height $T$. Recalling that $\xi = T^{1/(4k)}$ and using (2) and (9), it follows that
\[ N(T) \sum_{n \leq \xi^k} \frac{\tau_k(n;\xi)^2}{n} \ll T(\log T)^{k^2+1}. \]
In order to bound the second sum on the right-hand side of (11), we require the following version of the Landau-Gonek explicit formula.

**Lemma 3.** Let $x, T > 1$ and let $\rho = \beta + i\gamma$ denote a non-trivial zero of $\zeta(s)$. Then
\[ \sum_{0<\gamma \leq T} x^\rho = -\frac{T}{2\pi} \Lambda(x) + O\left(x \log(2xT) \log \log(3x) \right) \]
\[ + O\left( \log x \min\left( T, \frac{x}{\langle x \rangle} \right) \right) + O\left( \log(2T) \min\left( T, \frac{1}{\log x} \right) \right) \]
where $\langle x \rangle$ denotes the distance from $x$ to the closest prime power other than $x$ itself and $\Lambda(x) = \log p$ if $x$ is a positive integral power of a prime $p$ and $\Lambda(x) = 0$ otherwise.

**Proof.** This is a result of Gonek [2, 4].
Applying the lemma, we find that
\[ \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{n} \sum_{0 < \gamma \leq T} \left( \frac{n}{m} \right)^\rho = -\frac{T}{2\pi} \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi) \Lambda \left( \frac{T}{m} \right)}{n} \]
\[ + O \left( \mathcal{L} \log \mathcal{L} \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{m} \right) \]
\[ + O \left( \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi) \log \frac{n}{m}}{m} \right) \]
\[ + O \left( \log T \sum_{m \leq \xi^k} \sum_{m < n \leq \xi^k} \frac{\tau_k(m; \xi) \tau_k(n; \xi)}{n \log \frac{n}{m}} \right) \]
\[ = S_{21} + S_{22} + S_{23} + S_{24}, \]
say. Since we only require an upper bound for \( \Sigma_2 \) (which, by definition, is clearly positive), we can ignore the contribution from \( S_{21} \) because all the non-zero terms in the sum are negative. In what follows, we use \( \varepsilon \) to denote a small positive constant which may be different at each occurrence. To estimate \( S_{22} \), we note that \( \tau_k(n; \xi) \leq \tau_k(n) \ll n^\varepsilon \) which implies \( S_{22} \ll T^{1/4+\varepsilon} \). Turning to \( S_{23} \), we write \( n \) as \( qm + \ell \) with \(-\frac{m}{2} < \ell \leq \frac{m}{2}\) and find that
\[ S_{23} \ll T^\varepsilon \sum_{m \leq \xi^k} \frac{1}{m} \sum_{q \leq \frac{m}{2}} \sum_{\frac{m}{2} < \ell \leq m} \frac{1}{q + \frac{m}{2}} \]
where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). Notice that \( \langle q + \ell \rangle = \frac{|m|}{m} \) if \( q \) is a prime power and \( \ell \neq 0 \), otherwise \( \langle q + \ell \rangle \) is \( \geq \frac{1}{2} \). Hence,
\[ S_{23} \ll T^\varepsilon \left( \sum_{m \leq \xi^k} \frac{1}{m} \sum_{q \leq \frac{m}{2}} \sum_{\frac{m}{2} < \ell \leq m} \frac{1}{q + \frac{m}{2}} \right) \ll T^\varepsilon \left( \sum_{m \leq \xi^k} \sum_{q \leq \frac{m}{2}} 1 \right) \ll T^{1/4+\varepsilon}. \]

It remains to consider \( S_{24} \). For integers \( 1 \leq m < n \leq \xi^k \), let \( n = m + \ell \). Then
\[ \log \frac{n}{m} = -\log \left( 1 - \frac{\ell}{m} \right) > \frac{\ell}{m} \]
Consequently,
\[ S_{24} \ll T^\varepsilon \sum_{m \leq \xi^k} \sum_{1 \leq \ell \leq \xi^k} \frac{1}{(m + \ell) \frac{m}{2}} \ll T^\varepsilon \xi^k = T^{1/4+\varepsilon}. \] (13)
Combining (12) with our estimates for \( S_{22}, S_{23}, \) and \( S_{24} \) we deduce that \( \Sigma_2 \ll T (\log T)^{k^2+1} \) which, when combined with our estimate for \( \Sigma_1 \), completes the proof of Theorem [1].
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References

[1] S. M. Gonek, ‘Mean values of the Riemann zeta-function and its derivatives’, Invent. Math. 75 (1984), 123-141.
[2] S. M. Gonek, “A formula of Landau and mean values of $\zeta(s)$” in Topics in Analytic Number Theory, S. W. Graham and J. D. Vaaler, eds., (Univ. Texas Press, Austin, Tex., 1985), 92-97.
[3] S. M. Gonek, ‘On negative moments of the Riemann zeta-function’, Mathematika 36 (1989), 71-88.
[4] S. M. Gonek, ‘An explicit formula of Landau and its applications to the theory of the zeta function’, Contemp. Math. 143 (1993), 395-413.
[5] D. Hejhal, ‘On the distribution of log $|\zeta'(1/2 + it)|$’, in Number Theory, Trace Formulas, and Discrete Groups, K. E. Aubert, E. Bombieri, and D. M. Goldfeld, eds., Proceedings of the 1987 Selberg Symposium, (Academic Press, 1989), 343-370.
[6] C. P. Hughes, J. P. Keating, and N. O’Connell, ‘Random matrix theory and the derivative of the Riemann zeta-function’, Proc. Roy. Soc. London A 456 (2000), 2611-2627.
[7] M. B. Milinovich, ‘Upper bounds for moments of $\zeta'(\rho)$’, preprint.
[8] N. Ng, ‘The fourth moment of $\zeta'(\rho)$’, Duke Math J. 125 (2004) 243-266.
[9] N. Ng, ‘A discrete mean value of the derivative of the Riemann zeta function’, preprint.
[10] N. Ng, ‘Extreme values of $\zeta'(\rho)$’, preprint.
[11] Z. Rudnick and K. Soundararajan, ‘Lower bounds for moments of L-functions’, Proc. Natl. Sci. Acad. USA 102 (2005), 6837-6838.
[12] Z. Rudnick and K. Soundararajan, ‘Lower bounds for moments of L-functions: symplectic and orthogonal examples’, in Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Friedberg, Bump, Goldfeld, and Hoffstein, eds., (Proc. Symp. Pure Math., vol. 75, Amer. Math. Soc., 2006).
[13] K. Soundararajan, ‘Extreme values of L-functions’, preprint.

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