Designing Consensus Algorithms for Three-Link Manipulators

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Abstract. We deal with a consensus control problem for a group of three-link manipulators which are networked by digraphs. Assuming that the control inputs of each manipulator are the torques on its links and they are constructed based on weighted difference between its states and those of its neighbor agents, we aim to propose an algorithm on computing the weighting coefficients in the control inputs, so that full consensus is achieved among the manipulators. The control problem is reduced to designing Hurwitz polynomials with complex coefficients. We show that by using Hurwitz polynomials with complex coefficients, a necessary and sufficient condition is obtained for designing the consensus algorithm. Moreover, the discussion is extended to the case of designing convergence rate of consensus. Numerical examples are provided to illustrate the condition and the design algorithms.

1. Introduction

Multi-agent systems are composed of a set of agents (or systems), which are connected by a directed or undirected network. It is usually supposed that each agent can only communicate to its neighboring agents for various information. Cooperative control for such kind of multi-agent systems has been studied extensively, aiming at achieving a collective behavior for the whole system. The desired behavior specifications can be static/dynamic formation achievement, flocking and swarms, wide range coverage, etc. For detailed introduction about various cooperative control problems in multi-agent systems, refer to the survey papers [1, 2], the monographs [3, 4] and the reference papers therein.

It is well known that consensus control is a central problem in cooperative control, which requires an agreement concerning the full/partial states of all agents. Since many cooperative control problems can be reduced to consensus control ones, there have been large quantities of important papers, making great contribution to consensus control of networked agent systems [5, 6, 7]. For networked agents described by second-order integrators, Ref. [8] obtained a necessary and sufficient condition for achieving consensus, and showed that not only real parts but also imaginary parts of the eigenvalues of the Laplacian, denoting the interconnection graph, play important roles in the discussion. Ref. [9] also established a necessary and sufficient
condition for consensus in a set of second-order systems which are controllable canonical. Recently, Ref. [10] proposed a new approach for consensus control in networked third-order agents. In that context, the consensus control was reduced to the problem of designing a Hurwitz polynomial with complex coefficients. Later, the approach in [10] was extended to formation control problems for a set of second-order discrete-time systems with delay of one sample period [11].

The main contribution of the present paper is to extend the discussion and result in [10, 11] to three-link manipulators which are connected by directed interconnection graphs. More precisely, we aim to provide a necessary and sufficient condition for achieving consensus in a group of three-link manipulators. It is well known that multi-link manipulators play important roles in various industries, and the cooperation among manipulators is essential when we expect more intelligent and complex task performed by robots with multi-links. When focusing on consensus among manipulators, one example is that a patient or the patient’s bed has to be moved by several robots, where the manipulators’ angles and angular velocities need to achieve consensus (specified agreement) with each other. Although the main approach is different from the present paper, there have been several papers dealing with various consensus problems for multi-link manipulator systems [12, 13, 14].

The structure of this paper is as follows. In Section 2 we give some preliminaries about graphs, and state two lemmas for the remaining discussion. Section 3 describes the three-link manipulator model under consideration, and formulates the consensus control problem with the control input protocol in each manipulator having two design parameters. It turns out that the consensus problem at hand is reduced to the consensus problem for double integrator systems by introducing proper transformation in control input. In Section 4, the consensus problem is further reduced to designing Hurwitz polynomials with complex coefficients, and a necessary and sufficient condition is then obtained for the parameters design in the control input. A simulation example is also given in Section 4 to show effectiveness of the approach. Section 5 extends the discussion to design the convergence rate of consensus, and also gives an explicit necessary and sufficient condition for the parameters design, together with some detailed remarks on how to solve the condition efficiently.

The notations used in this paper are common in mathematical area. $\mathbb{R}$ denotes the real number, $0_n$ and $I_n$ denote the zero square matrix and the identity matrix with dimension $n$, respectively. For a polynomial with complex or real coefficients, we call it a Hurwitz polynomial if all its zeros are on the open left side of the complex plane.

2. Preliminaries

We review some basic definitions and concepts of graph theory for networked (connected) multi-agents. The interconnection of a set of agents is usually defined by a directed graph (or digraph) $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where the node set $\mathcal{V} = \{1, \ldots, n\}$ denotes the index (or label) of each agent, the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ denotes the interconnections between agents, and $\mathcal{A} = [a_{ik}]_{n \times n}$ is a matrix with nonnegative elements denoting the adjacency of all agents. More precisely, $(i, k) \in \mathcal{E}$ or $i \rightarrow k$ implies that the state/information of the $i$-th agent is available for the $k$-th agent. The element $a_{ik}$ of the matrix $\mathcal{A}$ associated with the edges is positive, i.e., $a_{ik} > 0 \iff (k, i) \in \mathcal{E}$. In addition, we assume that there is no self-loop in the interconnection graph, i.e., $a_{ii} = 0$ for all $i \in \mathcal{V}$. The neighbor set of the $i$-th agent is defined as $\mathcal{N}_i \triangleq \{k \in \mathcal{V} | (k, i) \in \mathcal{E}\}$, which indicates the indexes of the agents from which the $i$-th agent can obtain information.
A directed path in a digraph is a sequence of ordered edges, and a spanning tree of a digraph is defined as a directed path connecting all the nodes of the graph. It is well known that a digraph has a spanning tree if and only if there exists a node which has directed paths to all other nodes.

As in the literature, the Laplacian of a digraph is defined by $L = [l_{ik}]_{n \times n}$, where

$$l_{ik} = \begin{cases} -a_{ik}, & i \neq k \\ n \sum_{j=1, j \neq i}^{n} a_{ij} & i = k. \end{cases}$$

It is seen from the above definition that all row-sums of a Laplacian $L$ are zero. Thus, the Laplacian $L$ of any graph has a zero eigenvalue and a corresponding (right) eigenvector $\mathbf{1}_n = [1 \ 1 \ \cdots \ 1]^T$. Moreover, it is well known that $\text{rank}(L) = n - 1$ and $L$ has only one zero eigenvalue and all other eigenvalues have positive real parts if and only if the graph has a spanning tree. Refer to [15] and other references in graph theory for more detailed properties of graph Laplacians.

The next two lemmas are useful in the discussion of the remaining sections.

**Lemma 1** [16] Let $A$ and $D$ be square matrices of arbitrary dimensions, and consider the determinant of the matrix $W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then,

$$|W| = \begin{cases} |A| \cdot |D - CA^{-1}B| & \text{when } |A| \neq 0 \\ |D| \cdot |A - BD^{-1}C| & \text{when } |D| \neq 0 \\ |AD - CB| & \text{when } AC = CA \\ |DA - BC| & \text{when } BD = DB. \end{cases}$$

**Lemma 2** [16] Suppose that $A_1 \in \mathbb{R}^{n_1 \times n_1}$, $A_2 \in \mathbb{R}^{n_2 \times n_2}$ are square matrices, and $A_3$ is an arbitrary matrix. Then, the following properties concerning Kronecker products are true:

(i) $A_1 \otimes A_3 + A_2 \otimes A_3 = (A_1 + A_2) \otimes A_3$;

(ii) $|A_1 \otimes A_2| = |A_1|^{n_2} |A_2|^{n_1}$.

### 3. Problem Formulation

#### 3.1. System Description

We consider the model of three-link manipulators [17] described in Fig. 1, and the physical parameters are summarized in Table 1. For each link, there is an actuator generating torque.

To write the dynamical equation, we define the Lagrangian $L$, as the difference between the kinetic energy and potential energy of the manipulator. Thus, with the notation $x = [\theta_1 \ \theta_2 \ \theta_3]^T$, we have

$$L(x, \dot{x}) = T(x, \dot{x}) - V(x),$$

where

$$T(x, \dot{x}) = \frac{1}{2} \dot{x}^T M(x) \dot{x}, \quad V(x) = \sum_{k=1}^{3} m_k g h_k(x).$$
Then, using the Lagrange’s equation
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_k} \right) - \frac{\partial L}{\partial \theta_k} = \tau_k, \quad k = 1, 2, 3, \] (3.1)
we obtain the dynamical equation
\[ M(x)\ddot{x} + V(x, \dot{x})\dot{x} + G(x) = u, \] (3.2)
where \( u = [\tau_1 \tau_2 \tau_3]^T \), \( M(x) \) is the manipulator inertia matrix, \( V(x, \dot{x}) \) is the Coriolis matrix for the manipulator (thus \( V(x, \dot{x})\dot{x} \) gives the Coriolis and centrifugal force terms in the dynamical equation), and \( G(x) \) includes gravity terms and other forces acting at the joints. It is known that the matrix \( M(x) \) is always symmetric positive definite.

### 3.2. Consensus Problem

We consider the situation that there are a set of three-link manipulators, whose dynamical equations are given by
\[ M(x_i)\ddot{x}_i + V(x_i, \dot{x}_i)\dot{x}_i + G(x_i) = u_i, \quad i = 1, 2, \ldots, n, \] (3.3)
and the state and the control input vectors of each manipulator are
\[ x_i = \begin{bmatrix} \theta_{1i} & \theta_{2i} & \theta_{3i} \end{bmatrix}^T, \quad u_i = \begin{bmatrix} \tau_{1i} & \tau_{2i} & \tau_{3i} \end{bmatrix}^T. \]

We assume that the manipulators can exchange their state information based on a graph describing the interconnection structure. In other words, each manipulator can only obtain the state information of its neighboring manipulators.

![Figure 2. Four Networked Manipulators](image)

An example of the interconnection graph is given in Figure 2, where four manipulators are numbered as node 1, 2, 3, 4. Then, the information of the whole system is described by the Laplacian
\[ L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \]

which shows that the 1st manipulator obtains state information from the 3rd, and the 4th gets information from the 1st, and so on. In order to achieve consensus among all manipulators, we assume throughout this paper that there is a spanning tree in the interconnection graph, and thus \( \text{rank}(L) = n - 1. \)

Here, we aim to design each manipulator’s control input \( u_i \), based on the interconnection graph, so that the angles and the velocities of all manipulators converge to each other, i.e.,
\[
\lim_{t \to \infty} \| x_i(t) - x_j(t) \| = 0
\]
\[
\lim_{t \to \infty} \| \dot{x}_i(t) - \dot{x}_j(t) \| = 0 \quad (3.4)
\]
hold for all \( i \neq j \). It is noted that the consensus specification, requiring (3.4), is different from stabilization in the sense that converging to an equilibrium point such as the origin is not necessary. In real applications, it is more desirable not to stop the system state but rather to follow each other in a flexible way.

Since the dynamical equation (3.3) is nonlinear, it is generally difficult to analyze and design. Here, we notice that the control input on the right side of the equation is separated from the other state terms, and introduce an auxiliary input \( \bar{u}_i \) as
\[
\bar{u}_i = M_i^{-1}(x_i) (u_i - V(x_i, \dot{x}_i) \dot{x}_i - G(x_i)). \quad (3.5)
\]
Then, the dynamical system (3.3) is transformed to
\[ \ddot{x}_i = \bar{u}_i \] (3.6)
which is a well known double integrator. Then, similar to the literature [8, 10], we design the control input (consensus protocol) \( \bar{u}_i \) as
\[ \bar{u}_i = \sum_{j \in N_i} \left[ \gamma_1 (x_j - x_i) + \gamma_2 (\dot{x}_j - \dot{x}_i) \right] \] (3.7)
where \( \gamma_1 \) and \( \gamma_2 \) are design parameters to be determined.

Combining (3.5) and (3.7), the control input \( u_i \) for each manipulator is
\[ u_i = M(x_i) \sum_{j \in N_i} \left[ \gamma_1 (x_j - x_i) + \gamma_2 (\dot{x}_j - \dot{x}_i) \right] + V(x_i, \dot{x}_i) \dot{x}_i + G(x_i) \] (3.8)
and our control problem is to design \( \gamma_1 \) and \( \gamma_2 \) such that (3.4) is satisfied for all \( i \neq j \).

4. Consensus Design
The closed-loop system composed of the manipulator system (3.3) and the control input (3.8) is equivalent to
\[ \dot{\bar{x}} = (\Theta \otimes I_3) \bar{x} \] (4.1)
where
\[ \bar{x} = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}, \quad x^1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad x^2 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} \]
and
\[ \Theta = \begin{bmatrix} 0_n & I_n \\ -\gamma_1 \mathcal{L} & -\gamma_2 \mathcal{L} \end{bmatrix}. \] (4.2)

According to Lemma 2, we obtain that the characteristic polynomial of (4.1) is
\[ |\lambda I_{6n} - (\Theta \otimes I_3)| = |(\lambda I_{2n} - \Theta) \otimes I_3| = |\lambda I_{2n} - \Theta|^3. \] (4.3)

Then, using similar technique established in the literature such as [4], [8] and [9], we obtain the following fact easily.

**Lemma 3:** Three-link manipulators (3.3) with the control input (3.8) achieve consensus if and only if \( \Theta \) has two zero eigenvalues and all other eigenvalues have negative real parts.

According to the above lemma, our consensus control problem is to choose the parameters \( \gamma_1 \) and \( \gamma_2 \) such that the matrix \( \Theta \) has two zero eigenvalues and all other eigenvalues have negative
real parts. To discuss this requirement, we prove the following lemma to analyze the eigenvalues of $\Theta$.

**Lemma 4:** The characteristic equation of $\Theta$ is

$$|\lambda I_{2n} - \Theta| = \left| \lambda^2 I_n + (\gamma_1 + \gamma_2 \lambda) L \right|. \tag{4.4}$$

Furthermore, if the eigenvalues of $L$ are $\mu_1, \cdots, \mu_n$, then

$$|\lambda I_{2n} - \Theta| = \prod_{i=1}^{n} \left( \lambda^2 + (\gamma_1 + \gamma_2 \lambda) \mu_i \right). \tag{4.5}$$

**Proof:** The characteristic equation of $\Theta$ is computed as

$$|\lambda I_{2n} - \Theta| = \left| \lambda I_n - I_n \right| = \left| \begin{array}{cc} \lambda I_n & -I_n \\ \gamma_1 L & \lambda I_n + \gamma_2 L \end{array} \right|. \tag{4.4}$$

Since $(\lambda I_n)(\gamma_1 L) = (\gamma_1 L)(\lambda I_n)$, we use Lemma 1 to obtain

$$|\lambda I_{2n} - \Theta| = |\lambda I_n(\lambda I_n + \gamma_2 L) + \gamma_1 L| = \left| \lambda^2 I_n + (\gamma_1 + \gamma_2 \lambda) L \right|. \tag{4.5}$$

This completes the proof of (4.4).

Next, let the eigenvalues of $L$ be $\mu_1, \mu_2, \cdots, \mu_n$. Then, the eigenvalues of $-L$ are $-\mu_1, -\mu_2, \cdots, -\mu_n$, which results in

$$|\zeta I_n + L| = \prod_{i=1}^{n} \left( \zeta + \mu_i \right),$$

and thus

$$|\lambda I_{2n} - \Theta| = (\gamma_1 + \gamma_2 \lambda)^n \left| \frac{\lambda^2}{\gamma_1 + \gamma_2 \lambda} I_n + L \right| = (\gamma_1 + \gamma_2 \lambda)^n \prod_{i=1}^{n} \left( \frac{\lambda^2}{\gamma_1 + \gamma_2 \lambda} + \mu_i \right) = \prod_{i=1}^{n} \left( \lambda^2 + (\gamma_1 + \gamma_2 \lambda) \mu_i \right). \tag{4.5}$$

It is observed from (4.5) that $\mu_1 = 0$ corresponds to two zero eigenvalues of $\Theta$. Therefore, according to Lemma 3, our consensus design problem is reduced to seeking $\gamma_1$ and $\gamma_2$ such that the zeros of the polynomial

$$p_\mu(\lambda) = \lambda^2 + (\gamma_1 + \gamma_2 \lambda) \mu, \tag{4.6}$$

has negative real parts for all $\mu = \mu_i, i = 2, \cdots, N$.

Since the eigenvalues $\mu_i$’s are generally complex numbers, the coefficients of $p_\mu(\lambda)$ are also complex. For this reason, we need the following lemma declaring the condition under which a polynomial with complex coefficients is Hurwitz.

**Lemma 5:** [18] The polynomial $p(z)$ with complex coefficients,

$$p(z) = z^2 + \alpha_1 z + \alpha_2, \quad (\alpha_k = p_k + \sqrt{-1} q_k) \tag{4.7}$$
has all its zeros in the open left half-plane if and only if the next two determinants are positive.

\[
\Delta_1 = p_1, \quad \Delta_2 = \begin{vmatrix}
p_1 & 0 & -q_2 \\
p_2 & -q_1 & 0 \\
0 & q_2 & p_1
\end{vmatrix}.
\] (4.8)

From now on, we assume that \(b_i\) and \(c_i\) are the real and imaginary parts of nonzero \(\mu_i\), respectively. Then, we define for \(\mu_i = b_i + \sqrt{-1}c_i\) the polynomial

\[
p_{\mu_i}(\lambda) = \lambda^2 + (\gamma_1 + \gamma_2\lambda)(b_i + \sqrt{-1}c_i) = \lambda^2 + (\gamma_2b_i + \sqrt{-1}\gamma_2c_i)\lambda + (\gamma_1b_i + \sqrt{-1}\gamma_1c_i).
\] (4.9)

Using Lemma 5 for \(p_{\mu_i}(\lambda)\) with \(p_1 = \gamma_2b_i, q_1 = \gamma_2c_i, p_2 = \gamma_1b_i, q_2 = \gamma_1c_i\) in \(p(z)\) of the lemma, we obtain that the polynomial \(p_{\mu_i}(\lambda)\) has all its zeros in the open left half-plane if and only if

\[
\Delta_{1i} = \gamma_2b_i > 0
\] (4.10)

\[
\Delta_{2i} = \begin{vmatrix}
\gamma_2b_i & 0 & -\gamma_1c_i \\
1 & \gamma_1b_i & -\gamma_2c_i \\
0 & \gamma_1c_i & \gamma_2b_i
\end{vmatrix} > 0.
\] (4.11)

Since \(b_i > 0\), the condition (4.10) requires \(\gamma_2 > 0\). Concerning (4.11), we obtain after simple calculation that

\[
\Delta_{2i} = \gamma_1^2\gamma_2^2b_i(b_i^2 + c_i^2) - \gamma_1^2c_i^2 > 0.
\] (4.12)

Then, it is easy to reach \(\gamma_1 > 0\) and

\[
\frac{\gamma_2^2}{\gamma_1} > \frac{c_i^2}{b_i|\mu_i|^2}.
\] (4.13)

Noticing that the above inequality should hold for all \(\mu_i \neq 0\), we obtain the following theorem.

**Theorem 1:** Let the eigenvalues of the Laplacian \(L\) be \(\mu_1 = 0, \mu_2 \neq 0, \cdots, \mu_n \neq 0\). Then, three-link manipulators (3.3) with the control input (3.8) achieve consensus if and only if the design parameters \(\gamma_1\) and \(\gamma_2\) satisfy \(\gamma_1 > 0, \gamma_2 > 0\) and

\[
\frac{\gamma_2^2}{\gamma_1} > \max_{i=2,\cdots,n} \frac{c_i^2}{b_i|\mu_i|^2}.
\] (4.14)

Since there is no limitation on the value of the complex eigenvalues of \(L\), the above theorem is also valid when all \(\mu_i\)'s are real numbers. Substituting \(c_i = 0\) into (4.14), we obtain the following result.

**Corollary 1:** If the Laplacian \(L\) has only real nonzero eigenvalues \(\mu_i > 0, i = 2, \cdots, n\), then consensus is achieved in (3.3) with (3.8) if and only if the design parameters \(\gamma_1\) and \(\gamma_2\) are positive.

**Example 1:** Consider four manipulators which are connected as in Figure 2. Assume that \(m_1 = m_2 = m_3 = 1[kg], l_1 = l_2 = l_3 = 1[m], r_1 = r_2 = r_3 = 0.5[m]\) in the model, and the initial joint angles are

\[
x_1(0) = \begin{bmatrix} 0 & \frac{\pi}{2} & \frac{2\pi}{3} \end{bmatrix}^T,
x_2(0) = \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{2} & \frac{\pi}{6} \end{bmatrix}^T,
x_3(0) = \begin{bmatrix} -\frac{\pi}{2} & -\frac{\pi}{3} & -\frac{\pi}{3} \end{bmatrix}^T,
x_4(0) = \begin{bmatrix} -\frac{\pi}{5} & -\frac{\pi}{6} & -\frac{\pi}{4} \end{bmatrix}^T.
\]
Next, we design the parameters $\gamma_1$ and $\gamma_2$ by using the condition in Theorem 1. Since the eigenvalues of $\mathcal{L}$ are $\{0, 1, \frac{3}{2} \pm \sqrt{-1 \frac{3}{2}}\}$, we only need to deal with $b_3 = \frac{3}{2}$ and $c_3 = \sqrt{\frac{3}{2}}$. Then, the condition (4.14) turns out to be

$$\frac{\gamma_2}{\gamma_1} \geq \frac{c_3^2}{b_3 |\mu_3|^2} = \frac{1}{6},$$

and we choose $\gamma_1 = 8$, $\gamma_2 = 2$ in the control input (3.8).

With the above initial values and control parameters, the trajectories of four manipulators are plotted in Figure 3–8, and the desired consensus has been achieved among the joint angles and the angular velocities of four manipulators.
5. Consensus Rate Design

In this section, we consider the problem of improving convergence of consensus in three-link manipulators. Obviously, this is related to the real parts of the nonzero eigenvalues of Θ. Here, similarly as in stability analysis, we say the consensus is achieved with (convergence) rate $d$ ($d > 0$) if all real parts of the nonzero eigenvalues of Θ are smaller than $-d$. As before, this is equivalent to that the real parts of zeros of $p_i(z)$ for $i \neq 0$ are smaller than $-d$.

To proceed, we first update Lemma 5 for the present design purpose.

**Lemma 6:** The polynomial $p(z)$ with complex coefficients,

$$p(z) = z^2 + \alpha_1 z + \alpha_2, \quad (\alpha_k = p_k + \sqrt{-1} q_k)$$
has all its zeros in the half-plane \( \{ z : \Re(z) < -d \} \) if and only if
\[
\Delta_{1d} = p_1 - 2d > 0
\]
\[
\Delta_{2d} = \begin{vmatrix}
(p_1 - 2d)^2 & p_1 q_1 - q_1 d - q_2 \\
q_1 d - q_2 & p_2 - p_1 d + d^2
\end{vmatrix} > 0.
\] (5.1)

**Proof:** Substituting \( z = \bar{z} - d \) into \( p(z) \), we have
\[
p(z) = p(\bar{z}) = \bar{z}^2 + (\alpha_1 - 2d)\bar{z} + (\alpha_2 - \alpha_1 d + d^2).
\] (5.2)
Since \( \Re(z) < -d \iff \Re(\bar{z}) < 0 \), we obtain the condition by using Lemma 5 that
\[
\Delta_1 = p_1 - 2d > 0
\]
and
\[
\Delta_2 = \begin{vmatrix}
p_1 - 2d & 0 \\
1 & p_2 - p_1 d + d^2
\end{vmatrix} > 0.
\]
Then, we have \( p_1 > 2d \) and
\[
\Delta_2 = (p_1 - 2d)^2(p_2 - p_1 d + d^2) + (q_2 - q_1 d)(p_1 q_1 - q_1 d - q_2) > 0,
\]
which is equivalent to (5.1). This completes the proof.

As in Theorem 1, we substitute \( p_1 = \gamma_2 b_i, p_2 = \gamma_1 b_i, q_1 = \gamma_2 c_i, q_2 = \gamma_1 c_i \) into the condition in Lemma 6 to obtain
\[
\Delta_{1d} = \gamma_2 b_i - 2d > 0
\] (5.3)
\[
\Delta_{2d} = \Delta_{1d}^2(\gamma_1 b_i - \gamma_2 b_i d + d^2) + (\gamma_1 - \gamma_2 d)(\gamma_2^2 b_i - \gamma_2 d - \gamma_1)c_i^2 > 0.
\] (5.4)

The next theorem summarizes the discussion up to now.

**Theorem 2:** Let the eigenvalues of the Laplacian \( L \) be \( \mu_1 = 0, \mu_2 \neq 0, \cdots, \mu_n \neq 0 \). Then, three-link manipulators (3.3) with the control input (3.8) achieve consensus with rate \( d \) if and only if \( \gamma_1 \) and \( \gamma_2 \) satisfy both \( \gamma_2 > \frac{2d}{b_i} \) and (5.4) for all \( i = 2, \cdots, n \).

Although the first condition \( \gamma_2 > \frac{2d}{b_i} \) is easy to satisfy, the condition (5.4) looks complicated. Here, we propose to fix \( \gamma_2 \) and find \( \gamma_1 \) so that (5.4) holds for all \( i = 2, \cdots, n \). To investigate whether this is feasible or not, we define
\[
f_i(\gamma_1) = \Delta_{1d}^2(\gamma_1 b_i - \gamma_2 b_i d + d^2) + (\gamma_1 - \gamma_2 d)(\gamma_2^2 b_i - \gamma_2 d - \gamma_1)c_i^2
\]
\[
= -c_i^2 \gamma_1^2 + b_i(\Delta_{1d}^2 + \gamma_2^2 c_i^2)\gamma_1 - d(\gamma_2 b_i - d)(\Delta_{1d}^2 + \gamma_2^2 c_i^2).
\] (5.5)
(5.6)

According to (5.5), when \( c_i = 0 \), we choose \( \gamma_1 \) satisfying
\[
\gamma_1 > \frac{d(\gamma_2 b_i - d)}{b_i}
\] (5.7)
such that \( f_i(\gamma_1) > 0 \).
Using (5.6), when \(c_i \neq 0\), there exists \(\gamma_1\) satisfying \(f_i(\gamma_1) > 0\) if and only if the discriminant of \(f_i(\gamma_1) = 0\) is positive, i.e.,

\[
b_i^2(\Delta_{1d}^2 + \gamma_2^2c_i^2)^2 - 4dc_i^2(\gamma_2b_i - d)(\Delta_{1d}^2 + \gamma_2^2c_i^2) > 0,
\]

or, equivalently,

\[
b_i^2(\Delta_{1d}^2 + \gamma_2^2c_i^2) > 4dc_i^2(\gamma_2b_i - d).
\]

It is observed that (5.8) always holds for positive \(\gamma_2\) large enough. Thus, in the case of \(c_i \neq 0\), if we choose \(\gamma_2\) large enough satisfying both \(\gamma_2 > \frac{2d}{b_i}\) and (5.8), then we can find \(\gamma_1\) such that \(f_i(\gamma_1) > 0\) (and thus (5.4)) holds. Actually, the procedure of obtaining \(\gamma_1\) can be summarized as follows:

**Step 1** Solve for each \(i\) the second-order linear equation \(f_i(\gamma_1) = 0\) to obtain two solutions \(h_{1i} < h_{2i}\) (and thus \(f_i(\gamma_1) > 0\) on the open interval \((h_{1i}, h_{2i})\)).

**Step 2** Choose \(\gamma_1\) arbitrarily in the set \(\cap_{i=2}^{n}(h_{1i}, h_{2i})\).

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