Evaluation problems for the Thompson group and the Brin-Thompson group, and their relation to the word problem

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Abstract

The Thompson group $V$, as well as the Brin-Thompson group $2V$, is finitely generated and can be defined as a monoid acting on bitstrings, respectively pairs of bitstrings. Therefore evaluation problems can be defined for $V$ and $2V$. We show that these evaluation problems reduce to the corresponding word problems, and that in general, these evaluation problems are actually equivalent to the word problems. The long-input version of the evaluation problem is deterministic context-free and reverse deterministic context-free for $V$, and P-complete for $2V$.

1 Introduction

Informally, an evaluation function is of the form

$$E : (p, x) \mapsto E_p(x),$$

where $p$ is a “program” that describes a function $E_p$, $x$ is a data input for $E_p$, and $E_p(x)$ is the corresponding data output. Here we assume that $p$, $x$, and $E_p(x)$ are strings over some (possibly different) alphabets.

The word “input” is ambiguous here, as $E$ has input $(p, x)$, and $E_p$ has input $x$. For clarity we call $x$ the data input.

In this paper, “function” means partial function (unless we explicitly say total function, for a given source set). So $E_p(x)$ can be undefined for some $(p, x)$.

The evaluation decision problem of $E$ is defined as follows.

INPUT: $(p, x, y)$.

QUESTION: $E_p(x) = y$?

When $E_p(x)$ is undefined then $E_p(x) \neq y$; so the decision problem always has a YES/NO answer. From now on we will call the evaluation decision problem simply the evaluation problem.

Evaluation functions show up in many situations, e.g., in relation with universal Turing machines, with interpreters of programming languages, and more generally with exponential objects in a category. But here, the most relevant example here is the following.

Circuits:

The evaluation problem for acyclic boolean circuits, called the circuit-value problem, is defined as follows.

INPUT: $(C, x)$, where $C$ is (an encoding of) an acyclic boolean circuit with just one output wire; and $x$ is a bit-string.

QUESTION: $C(x) = 1$?

The output $C(x)$ can be undefined (if $x$ has the wrong input length for $C$); then the answer is NO. Ladner [17] proved that the circuit-value problem is P-complete; for details about the circuit-value
problem see \[17, 20\]. We will use a more general form of the circuit-value problem, namely with input $(C, x, y)$, and question “$C(x) = y$?” , where $x, y \in \{0, 1\}^*$.  

We can compare the circuit-value problem with the circuit-equivalence problem, where the input consists of two acyclic boolean circuits $C_1, C_2$; the question is whether $C_1$ and $C_2$ have same input-output function. The circuit-equivalence problem is \text{coNP}-complete; this follows from the \text{NP}\-completeness of the satisfiability problem for boolean formulas \[10\]; see \[20\] for details.

It is easy to prove that the circuit-value problem reduces to the circuit-equivalence problem by a many-one log-space reduction; see e.g. Section 7.1 below. But the circuit-equivalence problem does not reduce to the circuit-value problem, unless $P = \text{NP}$.

Languages and complexity:

By an alphabet we mean a finite set, and we only use alphabets that are subsets of a fixed infinite countable set. The set of all finite strings over an alphabet $A$ is denoted by $A^*$; this includes the empty string $\varepsilon$; we denote $A^* \setminus \{\varepsilon\}$ by $A^+$. Strings can be concatenated, which makes $A^*$ the free monoid over $A$. For $m \in \mathbb{N}$, the set of strings in $A^*$ of length $m$ is denoted by $A^m$, and the set of strings of length $\leq m$ is denoted by $A^{\leq m}$; in particular, $A^0 = \{\varepsilon\}$. The length of a string $x$ is denoted by $|x|$. For a set $S$, the cardinal of $S$ is denoted by $|S|$.

We will be interested in the complexity of some evaluation problems, and we will use the well-known complexity classes and reductions below; see e.g. \[10, 20, 12, 14, 11\]; for context-free languages see especially \[13, 16\]. Since all our alphabets are subsets of a fixed infinite countable set, the set of Turing machines is countable, and each complexity “class” below is a countable set.

- **CF** – the context-free languages.
- **coCF** – the co-context-free languages; $\text{coCF} = \{L \subseteq A^* : A$ is an alphabet, $A^* \setminus L \in \text{CF}\}$.
- **DCF** – the deterministic context-free languages.
- **DCF\text{rev}** – the reverse deterministic context-free languages; $\text{DCF}\text{rev} = \{L : L\text{rev} \in \text{DCF}\}$.
- **log-space** computable total functions and languages accepted in logarithmic space.
- **P** – the set of languages accepted by deterministic polynomial-time Turing machines.
- **NP** – the set of languages accepted by nondeterministic polynomial-time Turing machines with existential acceptance.
- **coNP** – the set of languages accepted by nondeterministic polynomial-time Turing machines with universal acceptance; equivalently, $\text{coNP} = \{L \subseteq A^* : A$ is a finite alphabet, $A^* \setminus L \in \text{NP}\}$. The set $\text{coNP}$ has the following useful characterization. For any $L \subseteq A^*$ we have:

  $$L \in \text{coNP} \iff$$

  there exists a two-variable predicate $P_L(.,.) \subseteq A^* \times B^*$ that is decidable in deterministic polynomial-time (where $B$ is an alphabet), and there exists a polynomial $\pi_L(\cdot)$, such that $L = \{x \in A^* : (\forall y \in A^{\leq \pi_L(|x|)}) \ P_L(x, y)\}$

  For $\text{NP}$ a similar characterization applies, but with $\forall$ replaced by $\exists$.

  To define completeness in a complexity class we use various reductions. Let $L_1 \subseteq A^*$ and $L_2 \subseteq B^*$ two languages.

  - A many-one log-space reduction from $L_1 \subseteq A^*$ to $L_2 \subseteq B^*$ is a log-space computable total function $f : A^* \to B^*$ such that $L_1 = f^{-1}(L_2)$; equivalently, for all $x \in A^*$: $x \in L_1$ iff $f(x) \in L_2$.

  - An $N$-ary conjunctive log-space reduction from $L_1 \subseteq A^*$ to $L_2 \subseteq B^*$ (for some $N > 0$) is a log-space computable total function $f : x \in A^* \to (f(x)_1, \ldots, f(x)_N) \in \prod_{i=1}^N B^*$ such that for all $x \in A^*$: $x \in L_1$ iff $f(x)_1 \in L_2$ AND $\ldots$ AND $f(x)_N \in L_2$.  

• A conjunctive log-space reduction of polynomial arity from \( L_1 \subseteq A^* \) to \( L_2 \subseteq B^* \) consists of a polynomial \( \pi() \) and a log-space computable total function \( f : x \in A^* \mapsto f(x) = (f(x)_1, \ldots, f(x)_{\pi(|x|)}) \in \mathcal{X}_{i=1}^{|x|} B^* \) such that for all \( x \in A^* : \ x \in L_1 \iff (\text{for all } i = 1, 2, \ldots, \pi(|x|)): f(x)_i \in L_2. \)

The complexity classes \( \mathcal{P}, \mathcal{NP}, \) and \( \mathcal{coNP} \) are downward closed under these reductions; i.e., if \( L_2 \) is in the class, and \( L_1 \) reduces to \( L_2 \), then \( L_1 \) is in the class.

A language \( L \) is complete in a class \( \mathcal{C} \) for a certain type of reduction iff \( L \in \mathcal{C} \), and every language in \( \mathcal{C} \) reduces to \( L \) for this type of reduction.

Since the Thompson group \( V \) and the Brin-Thompson group \( 2V \) are finitely generated, and are transformation groups (acting on \( \{0, 1\}^\omega \), respectively \( 2 \{0, 1\}^\omega \)), we can also consider evaluation functions and evaluation problems for \( V \) and \( 2V \). For the program input, a string of generators of \( V \) or \( 2V \) is used. For the data input, however, there is a complication: \( V \) and \( 2V \) do not act (as transformation groups) on finite strings. Nevertheless, \( V \) and \( 2V \) can also be defined by partial transformations on \( \{0, 1\}^* \), respectively \( 2 \{0, 1\}^* \), as described below. Hence evaluation problems for \( V \) and \( 2V \) can be defined with a string, respectively a pair of strings, as data input.

Regarding a generating set \( \Gamma \) of the groups \( V \) and \( 2V \) we make the convention that \( \Gamma \) is closed under inverse; i.e., by \( \Gamma^* \) we always mean \((\Gamma^\pm)^*\).

## 2 Evaluation problems for the Thompson group

For the definition of the Thompson group \( V \) we follow [5 Sect. 2.1]; we will repeat some of the definitions, but not all.

For \( x, p \in A^* \) we say that \( p \) is a prefix of \( x \) iff \( (\exists u \in A^*) x = pu \); this is denoted by \( p \leq_{\text{pref}} x \). A prefix code is any subset \( P \subseteq A^* \) such that no element of \( P \) is a prefix of another element of \( P \). A maximal prefix code in \( A^* \) is a prefix code that not a strict subset of any prefix code in \( A^* \). A right ideal is a subset \( R \subseteq A^* \) such that \( RA^* = R \). A right ideal is \( R \) essential in \( A^* \) iff \( R \) has a non-empty intersection with every non-\( \emptyset \) right ideal of \( A^* \). For every right ideal \( R \) there exists a unique prefix code \( P \) such that \( R = PA^* \); and \( R \) is essential iff \( P \) is maximal (see e.g. [1 Lemma 8.1]).

A right ideal morphism of \( A^* \) is a function \( f : A^* \rightarrow A^* \), with domain \( \text{Dom}(f) \) and image set \( \text{Im}(f) \), such that for all \( x \in \text{Dom}(f) \) and all \( w \in A^* : f(xw) = f(x)w \). The unique prefix code that generates the right ideal \( \text{Dom}(f) \) is denoted by \( \text{domC}(f) \), and is called the domain code of \( f \); the unique prefix code that generates the right ideal \( \text{Im}(f) \) is denoted by \( \text{imC}(f) \), and is called the image code. In order to define \( V \) we first introduce the inverse monoid

\[
\mathcal{RI}_A^{\text{fin}} = \{ f : f \text{ is a right ideal morphism of } A^* \text{ such that } f \text{ is injective, and} \\
\text{domC}(f) \text{ and } \text{imC}(f) \text{ are finite maximal prefix codes} \}.
\]

Every \( f \in \mathcal{RI}_A^{\text{fin}} \) has a unique maximum extension to an element of \( \mathcal{RI}_A^{\text{fin}} \) (by [1 Prop. 2.1]).

We define the Higman-Thompson group \( G_{k,1} \) (where \( k = |A| \)) as follows: As a set, \( G_{k,1} \) consists of the right ideal morphisms \( f \in \mathcal{RI}_A^{\text{fin}} \) that are maximum extensions in \( \mathcal{RI}_A^{\text{fin}} \); so \( G_{k,1} \subseteq \mathcal{RI}_A^{\text{fin}} \) (as sets). The multiplication in \( G_{k,1} \) consists of composition, followed by maximum extension. The Thompson group \( V \) is \( G_{2,1} \).

There are other characterizations of \( G_{k,1} \); we give two more, one based on \( \equiv_{\text{end}} \), and one based on a faithful action on \( A^\omega \). For \( f \in \mathcal{RI}_A^{\text{fin}} \), \( p \in \text{domC}(f) \), and \( u \in A^\omega \), we define \( f(pu) = f(p)u \).

(1) The group \( G_{k,1} \) is also a homomorphic image of \( \mathcal{RI}_A^{\text{fin}} \). For \( f_1, f_2 \in \mathcal{RI}_A^{\text{fin}} \) we define the congruence \( \equiv_{\text{end}} \) as follows:

\[
f_1 \equiv_{\text{end}} f_2 \iff f_1 \text{ and } f_2 \text{ agree on } \text{Dom}(f_1) \cap \text{Dom}(f_2).
\]

Then \( G_{k,1} \) is isomorphic to \( \mathcal{RI}_A^{\text{fin}}/\equiv_{\text{end}} \).
(2) The group $G_{k,1}$ is isomorphic to the action monoid of the action of $\mathcal{RI}_A^{\text{fin}}$ on $A^\omega$. Indeed, $f_1 \equiv_{\text{end}} f_2$ iff the actions of $f_1$ and $f_2$ on $A^\omega$ are the same.

In summary, we have three equivalent definitions of $G_{k,1}$:

0 As the subset (not subgroup) of $\mathcal{RI}_A^{\text{fin}}$ consisting of maximally extended right-ideal morphisms, with multiplication consisting of composition, followed by maximum extension. The definition in [9] is a numerical coding of this definition; see [5, Sect. 2.1].

1 As $\mathcal{RI}_A^{\text{fin}}/\equiv_{\text{end}}$.

2 As the action monoid of $\mathcal{RI}_A^{\text{fin}}$ on $A^\omega$.

Every element $f \in \mathcal{RI}_A^{\text{fin}}$ (and in particular, every $f \in G_{k,1}$) is determined by the restriction of $f$ to $\text{domC}(f)$. This restriction $f|_{\text{domC}(f)} : \text{domC}(f) \to \text{imC}(f)$ is a finite bijection, called the table of $f$ (see [5]). When we use tables we do not always assume that $f$ is a maximum extension; but the maximum extension can easily be found from the table.

To define an evaluation function for $V$ we first choose a finite generating set $\Gamma_1$. The evaluation function for $V$ over $\Gamma_1$ is

$$E : (w, x) \in \Gamma_1^* \times \{0,1\}^* \mapsto E_w(x) \in \{0,1\}^*,$$

where, if $w = w_n \ldots w_1$ with $w_n, \ldots, w_1 \in \Gamma_1$, and $E_w$ is the element of $V$ generated by the string $w$. By [1 Prop. 2.1], $E_w$ is the maximum extension of $w_n \circ \ldots \circ w_1(\cdot) \in \mathcal{RI}_A^{\text{fin}}$ to a right-ideal morphism in $\mathcal{RI}_A^{\text{fin}}$, and this maximal extension is unique.

For the Thompson group $V$ it is useful to view the data input as a stack (push-down store); in the Brin-Thompson group, the data input is a pair of stacks. An element of $V$ and $2V$ changes the top bits of the stack(s).

**Definition 2.1** (evaluation problem of $V$ over $\Gamma_1$).

The evaluation problem of $V$ over a finite generating set $\Gamma_1$ is specified as follows.

**INPUT:** $(w, x, y) \in \Gamma_1^* \times \{0,1\}^* \times \{0,1\}^*$.

**QUESTION:** $E_w(x) = y$ ?

In other words, this problem is the set $\{(w, x, y) \in \Gamma_1^* \times \{0,1\}^* \times \{0,1\}^* : E_w(x) = y\}$.

The following gives a connection between the action of $V$ on finite strings and on infinite one. First a general Lemma about the relation between $\{0,1\}^*$ and $\{0,1\}^\omega$.

**Lemma 2.2** The following equivalences hold for every $x, y \in \{0,1\}^*$:

$$x = y \iff x \{0,1\}^* = y \{0,1\}^* \iff x \{0,1\}^\omega = y \{0,1\}^\omega.$$

Similarly,

$$y \preff x \iff x \{0,1\}^* \subseteq y \{0,1\}^* \iff x \{0,1\}^\omega \subseteq y \{0,1\}^\omega.$$

**Proof.** Obviously, $x = y$ implies the other equalities, and $y \preff x$ implies the inclusions. Conversely, if $x \{0,1\}^* \subseteq y \{0,1\}^*$, or $x \{0,1\}^\omega \subseteq y \{0,1\}^\omega$, then $x \in y \{0,1\}^*$, hence $y \preff x$.

Symmetrically, if we have the other inclusions then we also have $x \preff y$, hence, $x = y$. \hfill \Box

**Lemma 2.3** For every element $f \in V$ ($\subseteq \mathcal{RI}_A^{\text{fin}}_{\{0,1\}}$) and every $x, y \in \{0,1\}^*$, the following are equivalent:

1. $f(x) = y$.
2. For all $z \in \{0,1\}^*$: $f(xz) = yz$.
3. There exists a maximal prefix code $P \subseteq \{0,1\}^*$ such that for all $z \in P$: $f(xz) = yz$.
4. For all $u \in \{0,1\}^\omega$: $f(xu) = yu$. 

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**Proof.** The implications (1) ⇒ (2), (1) ⇒ (3), and (1) ⇒ (4) are obvious. And (1) ⇐ (2) is obtained by taking \( z = \varepsilon \).

(1) ⇐ (3): Suppose \( f(xz) = yz \) for all \( z \in P \), where \( P \) is a maximal prefix code. Then \( f(x) = y \) holds in the maximum extension of \( f \) to a right ideal morphism. Since \( f \) is already maximally extended (as \( f \in V \)), we have \( f(x) = y \).

(1) ⇐ (4): Assume \( f(xu) = yu \). Since \( \text{domC}(f) \) is a maximal finite prefix code, every \( u \in \{0,1\}^\omega \) has a prefix \( z_{x,u} \) such that \( f(xz_{x,u}) \) is defined. Let \( v_{x,u} \in \{0,1\}^\omega \) be such that \( u = z_{x,u}v_{x,u} \). So, \( yu = f(xu) = f(xz_{x,u}v_{x,u}) = f(xz_{x,u})v_{x,u} = yz_{x,u}v_{x,u} \). Hence, since \( v_{x,u} \) can be any element of \( \{0,1\}^\omega \), Lemma 2.2 implies that for every \( u \in \{0,1\}^\omega \) : \( f(xz_{x,u}) = yz_{x,u} \). Since \( z_{x,u} \) exists for every \( x \in \text{domC}(f) \) and every \( u \), it follows that \( \{z_{x,u} : x \in \text{domC}(f), \ u \in \{0,1\}^\omega \} \cdot \{0,1\}^* \) is an essential right ideal. Let \( P \) be the maximal prefix code that generates this essential right ideal. Then for all \( p \in P \) : \( f(xp) = yp \). Now (3) holds, which implies (1). \( \Box \)

**Notation:**
For \( x = (x_1, \ldots, x_n) \in n A^* \): \( \text{maxlen}(x) = \max\{|x_i| : i = 1, \ldots, n\} \).
For a finite set \( S \subseteq n A^* \): \( \text{maxlen}(S) = \max\{\text{maxlen}(x) : x \in S\} \).
For a table \( F : P \rightarrow Q \): \( \text{maxlen}(F) = \max\{\text{maxlen}(x) : x \in P \cup Q\} \).
For a set of tables \( S \), \( \text{maxlen}(S) = \max\{\text{maxlen}(F) : F \in S\} \).

**Long versus short data inputs:** When \( V \) is defined by functions on \( \{0,1\}^* \), \( E_w(x) \) is an injective function; but \( E_w(x) \) is not defined for all \( x \in \{0,1\}^* \), except when \( E_w(x) \) is the identity function. Moreover, \( E_w(x) \) can be a strict extension of \( w_n \circ \ldots \circ w_1(x) \).

If \( x \) is long enough then \( E_w(x) = w_n \circ \ldots \circ w_1(x) \); i.e., \( E_w(x) \) is obtained by simply applying the generators \( w_i \in \Gamma_1 \) to the data input, one generator after another. A sufficient (but not necessary) condition for this is given by the following [II Cor. 3.7]:

\[ c_{\Gamma_1} = \text{maxlen}(\Gamma_1), \]

i.e., the length of the longest bit-string in the tables of the generators in \( \Gamma_1 \). Then for every \( w \in \Gamma_1^* \) and \( x \in \{0,1\}^* \):

\[ \text{if } |x| \geq c_{\Gamma_1} |w| \text{ then } w_n \circ \ldots \circ w_1(x) \text{ is defined; } \]

and in that case, \( E_w(x) = w_n \circ \ldots \circ w_1(x) \).

**Definition 2.4 (long versus short data inputs).**
Let \( w \in \Gamma_1^* \) with \( w = w_n \ldots w_1 \) and \( w_n, \ldots, w_1 \in \Gamma_1 \).

We call a data input \( x \in \{0,1\}^* \) a long data input for the word \( w \) iff \( w_n \circ \ldots \circ w_1(x) \) is defined (and then \( E_w(x) = w_n \circ \ldots \circ w_1(x) \)).

A data input \( x \in \text{Dom}(E_w) \) that is not long is called a short data input for the word \( w \).

In summary, for \( w \in \Gamma_1^* \) there are three kinds of data inputs \( x \):

1. data inputs that are *too short*, i.e., \( x \not\in \text{Dom}(E_w) \);
2. data inputs that are *short*, i.e., \( x \in \text{Dom}(E_w) \), but \( w_n \circ \ldots \circ w_1(x) \) is undefined.
3. data inputs that are *long*, i.e., \( x \in \text{Dom}(w_n \circ \ldots \circ w_1(.) \).

For the Thompson group \( V \), one can also consider a *circuit-like generating* set \( \Gamma_1 \cup \tau \), and define the evaluation function, evaluation problem, and domain membership problem of \( V \) over \( \Gamma_1 \cup \tau \). Here, \( \Gamma_1 \) is any finite generating set of \( V \), and \( \tau \) is the set of *bit-transpositions*, i.e., \( \tau = \{\tau_{i,i+1} : i \geq 1\} \), where \( \tau_{i,i+1} \) is the right ideal morphism of \( \{0,1\}^* \) defined by \( \tau_{i,i+1}(ux_i x_{i+1}v) = ux_{i+1} x_i v \) for all \( u \in \{0,1\}^{i-1}, \ v \in \{0,1\}^* \), and \( x_i, x_{i+1} \in \{0,1\} \); \( \tau(z) \) is undefined if \( |z| \leq i \). See [2, 5] for the connection between \( \tau \) and acyclic boolean circuits.
Definition 2.5 (evaluation problem of $V$ over $\Gamma_1 \cup \tau$).

The evaluation problem of $V$ over $\Gamma_1 \cup \tau$ is specified as follows.

**Input:** $(w, x, y) \in (\Gamma_1 \cup \tau)^* \times \{0, 1\}^* \times \{0, 1\}^*$.

**Question:** $E_w(x) = y$ ?

When we consider the evaluation problem and the word problem of $V$ over $\Gamma_1 \cup \tau$, the elements of $\tau$ are encoded over a finite alphabet. We will use the following: $\tau_{j,j+1}$ is encoded by $ab^{j+1}a$. We assume that $\{a, b\}$, $\{0, 1\}$ and $\Gamma_1$ have empty intersection two by two.

Long and short data inputs over $\Gamma_1 \cup \tau$ are defined in the same way as over $\Gamma_1$. For $w \in (\Gamma_1 \cup \tau)^*$, a sufficient condition for $x \in \{0, 1\}^*$ to be a long data input for $w$ is

$$|x| \geq c_{\Gamma_1, w} |w|;$$

where $c_{\Gamma_1, w} = \max\{c_{\Gamma_1}, \maxindex_{\tau}(w)\}$, and $\maxindex_{\tau}(w) = \max\{i \in \mathbb{N}_{>0} : \tau_{i-1,i} \text{ occurs in } w\}$. (i.e., the largest subscript of any element of $\tau$ that occurs in $w$).

Definition 2.6 The evaluation problem for $V$ over $\Gamma_1$ (or over $\Gamma_1 \cup \tau$) for long data inputs is defined as follows.

**Input:** $(w, x, y) \in \Gamma_1^* \times \{0, 1\}^* \times \{0, 1\}^*$ (or $(\Gamma_1 \cup \tau)^* \times \{0, 1\}^* \times \{0, 1\}^*$), where $w = w_n \ldots w_1$, with $w_n, \ldots, w_1 \in \Gamma_1$ (or $\Gamma_1 \cup \tau$).

**Question:** $w_n \circ \ldots \circ w_1(x) = y$ ?

Remark: The question “$E_w(x) = y$?” is equivalent to the question “$w_n \circ \ldots \circ w_1(x) = y$?” iff $x$ is long for $w$. An answer yes to the question “$w_n \circ \ldots \circ w_1(x) = y$?” implies that $x$ is long for $w$. The evaluation problem is not equivalent to the evaluation problem for long data inputs in general.

3 Evaluation problems for the Brin-Thompson group

For the definition of the Brin-Thompson group 2V we follow Sections 2.2 and 2.3 and Def. 2.28 in [5], but we will not repeat everything.

The $n$-fold cartesian products $X_n^\ast A^\ast$ and $X_n^\ast A^\omega$ are denoted by $nA^\ast$, respectively $nA^\omega$. Multiplication in $nA^\ast$ is done coordinatewise, i.e., $nA^\ast$ is the direct product of $n$ copies of the free monoid $A^\ast$. For $u \in nA^\ast$, we denote the coordinates of $u$ by $u_i \in A^\ast$ for $1 \leq i \leq n$; i.e., $u = (u_1, \ldots, u_n)$. The initial factor order on $nA^\ast$ is defined as follows for $u, v \in nA^\ast$: $u \leq_{\text{init}} v$ iff there exists $x \in nA^\ast$ such that $ux = v$. Clearly, $u \leq_{\text{init}} v$ iff $u_i \leq_{\text{pref}} v_i$ for all $i = 1, \ldots, n$.

In $nA^\ast$, similarly to $A^\ast$, we have the concepts of right ideal, essential right ideal, and generating set of a right ideal. An initial factor code is a set $S \subseteq nA^\ast$ such that no element of $S$ is an initial factor of another element of $S$. Every right ideal is generated by a unique initial factor code [5 Lemma 2.7(1)].

An essential initial factor code is, by definition, an initial factor code $S$ such that $S \cap nA^\ast$ is an essential right ideal. It is easy to see that every maximal initial factor code is essential. The converse does not hold. E.g., for $A = \{0, 1\}$, $S = \{(\varepsilon, 0), (0, \varepsilon), (1, 1)\}$ is essential, but not maximal. Indeed, let $P \subseteq \{0, 1\}^\ast$ be any every finite maximal prefix code with $P \neq \{\varepsilon\}$; let $1P = \{1p : p \in P\}$. Then $S \cup (1P \times \{\varepsilon\})$ is an essential initial factor code that has $S$ as a strict subset.

The join $u \vee v$ of $u, v \in nA^\ast$ is defined to be the unique $\leq_{\text{init}}$-minimum common upper bound of $u$ and $v$. Of course, $u \vee v$ does not always exist. The join $u \vee v$ of $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ is defined to be $u \vee v = (\max(u_1, v_1), \max(u_2, v_2), \ldots, \max(u_n, v_n))$.
Moreover, if \( u \lor v \) exists; 
(2) \( u \) and \( v \) have a common upper bound for \( \leq_{\text{init}} \), i.e., \((\exists z)[u \leq_{\text{init}} z \text{ and } v \leq_{\text{init}} z]\); 
(3) for all \( i = 1, \ldots, n \): \( u_i \parallel_{\text{pref}} v_i \) in \( A^* \).

Moreover, if \( u \lor v = \{(u \lor v)_i : i = 1, \ldots, n\} \) exists, then \((u \lor v)_i = u_i\) if \( u_i \leq_{\text{pref}} u_i \), and \((u \lor v)_i = v_i\) if \( u_i \parallel_{\text{pref}} v_i \); see [5, Lemma 2.5].

A set \( S \subseteq nA^* \) is called joinless iff no two elements of \( S \) have a join with respect to \( \leq_{\text{init}} \). Joinless sets will be called joinless codes, since they are a special case of initial factor codes. For \( n \geq 2 \), not every initial factor code is joinless.

A maximal joinless code is a joinless code \( S \) such that for all \( x \in nA^* \), \( S \cup \{x\} \) is not joinless. A right ideal \( R \) is called joinless generated iff the unique initial factor code that generates \( R \) is joinless [5, Def. 2.4 and Lemma 2.7(2)]. Every maximal joinless code is an essential initial factor code [5, Lemma 2.9]; but not every an essential initial factor code is joinless; see the example above, and [5, Remark after Lemma 2.9]. Hence for joinless codes (and in particular for prefix codes of \( A^* \)), essential is equivalent to maximal; but for initial factor codes in general, we saw above that essential is not equivalent to maximal.

We have the following fact: If \( C_1 \) and \( C_2 \) are joinless codes then the right ideal \( C_1 nA^* \cap C_2 nA^* \) is generated by the joinless code \( C_1 \lor C_2 = \{c_1 \lor c_2 : c_1 \in C_1, c_2 \in C_2\} \); moreover, \( C_1 \lor C_2 \) is a maximal joinless code iff \( C_1 \) and \( C_2 \) are both maximal joinless codes [5, Prop. 2.18].

In summary, in \( nA^* \) with \( n \geq 2 \) there are two different generalizations of the concept of prefix code, namely the initial factor codes and the joinless codes. And there are two different generalizations of maximal prefix codes, namely the essential initial factor codes and the maximal joinless codes. Initial factor codes are closely related to right ideals, whereas joinless codes are crucial for defining right ideal morphisms.

Just as for \( A^* \), one defines the concepts of right ideal morphism, domain code, and image code in \( nA^* \). At first we only consider domain and image codes that are joinless. Indeed, if \( P \subseteq nA^* \) is not joinless, the definition of right ideal morphisms on \( P \) can be inconsistent, i.e., the morphisms might not be functions; see Prop. [3.2] below.

Before we get to \( nG_{k,1} \) we define the following monoid:

\[
n\mathcal{R}_{\text{fin}}^n_A = \{ f : f \text{ is a right ideal morphism of } nA^* \text{ such that } f \text{ is injective,} \\
\text{and } \text{dom}(f) \text{ and } \text{im}(f) \text{ are finite, maximal, joinless codes} \}.
\]

Any element of \( n\mathcal{R}_{\text{fin}}^n_A \) is determined by a bijection \( F : P \to Q \) between finite maximal joinless codes \( P, Q \subseteq nA^* \); such a bijection is called a table. Conversely, every table determines an element of \( n\mathcal{R}_{\text{fin}}^n_A \).

Two right ideal morphisms \( f, g \in n\mathcal{R}_{\text{fin}}^n_A \) are called end-equivalent iff \( f \) and \( g \) agree on \( \text{Dom}(f) \cap \text{Dom}(g) \). This will be denoted by \( f \equiv_{\text{end}} g \). By [3, Lemma 2.24]: \( f \equiv_{\text{end}} g \) iff \( f \) and \( g \) have the same action on \( \{0,1\}^\omega \).

**Remarks on maximal extensions of right ideal morphisms in \( nA^* \) when \( n \geq 2 \):**

(1) If one starts out with a table \( F : P \to Q \), where \( F \) is a bijection between finite initial factor codes \( P \) and \( Q \), then the extension of \( F \) to a “right ideal morphism” will not always be a function. E.g., for \( A = \{0,1\} \), the “right ideal morphism” given by the table \( F = \{(0,0)\} \) is not a function, since \( F((0,0)) = F((0,0)) \cdot (\epsilon,0) = (00,\epsilon) \cdot (\epsilon,0) = (00,0) \), and also \( F((0,0)) = F((\epsilon,0)) \cdot (0,\epsilon) = (01,\epsilon) \cdot ((0,0)) = (010,\epsilon) \).

(2) If \( P \) and \( Q \) are joinless codes of equal cardinality, then any bijection \( F : P \to Q \) determines a right ideal morphism \( f : P nA^* \to Q nA^* \) (which is a function); and if \( P \) and \( Q \) are maximal joinless codes then \( f \) belongs to \( n\mathcal{R}_{\text{fin}}^n_A \).
(3) If \( f : PnA^* \to QnA^* \) belongs to \( nRI_A^\text{fin} \), and if \( P \) and \( Q \) are maximal joinless codes, then \( f \) might be extendable to a right ideal morphism whose domain and image codes are essential finite initial factor codes that are not joinless.

Example [5] Lemma 2.27 and Fig. 2:
For 2 \( \{1, 0\}^* \), let 
\[ F = \{(0,0),(0,0),(1,0),(1,0),(0,1),(0,1),(1,10),(1,11),(1,11),(1,10)\}. \]
Then \( F \) determines a right ideal morphism in \( 2RI_2^\text{fin} \) that can be extended to 
\[ F_1 = \{(\varepsilon,0),(\varepsilon,0),(0,1),(0,1),(1,10),(1,11),(1,11),(1,10)\} \in 2RI_2^\text{fin}, \text{ or to} \]
\[ F_2 = \{(0,\varepsilon),(0,\varepsilon),(1,0),(1,0),(1,10),(1,11),(1,11),(1,10)\} \in 2RI_2^\text{fin}. \]
\( F_1 \) and \( F_2 \) have no common extension to an element of \( 2RI_2^\text{fin} \). But both \( F_1 \) and \( F_2 \) can be further extended to 
\[ F_{12} = \{(\varepsilon,0),(\varepsilon,0),(0,\varepsilon),(0,\varepsilon),(1,10),(1,11),(1,11),(1,10)\}. \]
Here, \( F_{12} \notin 2RI_2^\text{fin} \) (its domain and image codes are not joinless); but \( F_{12} \) is nevertheless a welldefined right ideal morphism of 2 \( \{1, 0\}^* \). And \( F_{12} \) is the unique maximum extension of \( F \) to a right ideal morphism of 2 \( \{1, 0\}^* \). [End, Remarks.]

In the above example we see that there are two different kinds of maximal extensions of a right ideal morphism \( f \) in \( nRI_A^\text{fin} \):

1. non-unique maximal extensions of \( f \) to elements of \( nRI_A^\text{fin} \);
2. a unique maximal extension of \( f \) to a right ideal of \( nA^* \) (beyond \( nRI_A^\text{fin} \)).

In general, Prop. 3.2 will give the connection between \( nRI_A^\text{fin} \) and general right ideal morphisms.

In [5] and [6] the definition of \( nRI_A^\text{fin} \) is based on maximal joinless codes, and this is appropriate when \( nV \) and \( nRI_A^\text{fin} \) are considered by themselves. But for the evaluation problem, the action of \( nRI_A^\text{fin} \) and \( nV \) on strings matters, and the fact that right ideal morphisms do not have unique extensions in \( nRI_A^\text{fin} \) is a problem. On the other hand, every essential right ideal morphism has a unique extension to an essential right ideal morphism, generated by an essential initial factor code; so it helps now to consider right ideal morphisms in that setting.

By Prop. 8.4 we can efficiently decide whether a table \( F : P \to Q \), where \( P,Q \subseteq nA^* \) are finite initial factor codes, describes a right ideal morphism that is welldefined (i.e., a function), or that is total, or injective, or surjective.

Lemma 3.1 Let \( f : PnA^* \to QnA^* \) be a welldefined bijective right ideal morphism, described by a bijective table \( F : P \to Q \), where \( P,Q \subseteq nA^* \) are finite essential initial factor codes.

1. Then \( f^{-1} : QnA^* \to PnA^* \) is also a bijective right ideal morphism, with table \( F^{-1} : Q \to P \).
2. If \( S \subseteq PnA^* \) is an (essential) initial factor code then \( f(S) \) is also an (essential) initial factor code.
3. If \( S \subseteq PnA^* \) is a (maximal) joinless code, then \( f(S) \) is also a (maximal) joinless code.

Proof. (1) Since \( f \) is bijective, \( f^{-1} : QnA^* \to PnA^* \) exists as a (bijective) function. Let us prove that it is a right ideal morphism. For any \( qx \in QnA^* \), with \( q \in Q \) and \( x \in nA^* \), we have \( f^{-1}(qx) = pz \in PnA^* \), for some \( p \in P \) and \( z \in nA^* \). Applying \( f \) yields \( qx = f(f^{-1}(qx)) = f(pz) = f(p)z \), where \( f(p) \in Q \).

Since this holds for all \( x \), we have (when \( x = (\varepsilon)^n \)) \( q = f(p)z' \) for some \( z' \). Since \( Q \) is an initial factor code and both \( q \) and \( f(p) \) are in \( Q \), it follows that \( q = f(p) \); hence \( f^{-1}(q) = p \).

Now, going back to \( qx = f(p)z \), we use \( q = f(p) \) and cancelativity of \( nA^* \) to obtain \( x = z \). Thus, \( f^{-1}(qx) = pz = f^{-1}(q)z = f^{-1}(q)x \); so \( f^{-1} \) is a right ideal morphism (given by the table \( F^{-1} \)).
(2) By contraposition, if \( f(S) \) is not an initial factor code then for some \( s_1, s_2 \in S \) we have \( f(s_1) \prec_{\text{init}} f(s_2) = f(s_1) u \) (for some \( u \in nA^* \setminus \{ \varepsilon \}^n \)). Then \( s_2 = f^{-1}(s_2) = f^{-1}(f(s_1) u) = f^{-1}(f(s_1)) \cdot u \) (the latter by (1) above). Hence \( s_1 \prec_{\text{init}} s_2 = s_1 u, \) since \( u \neq (\varepsilon)^n \). This implies that \( S \) is not an initial factor code.

Since \( P \) is essential we have: \( S \) is essential iff for all \( p \in P, w \in nA^* : S \cdot nA^* \cap pw nA^* \neq \emptyset ; \) i.e. for all all \( p \in P, w \in nA^* \) there exist \( s \in S, x, y \in nA^* \) such that \( sx = pwy \). Hence for all \( p \in P, w \in nA^* \) there exist \( f(s) \in f(S) \) and \( x, y \in nA^* \) such that \( f(s)x = f(p)wy \). And \( f(p) \) ranges over all of \( Q \). So for all \( q \in Q, w \in nA^*: f(S) \cap qw nA^* \neq \emptyset \). Hence (since \( Q \) is essential), \( f(S) \) is essential.

(3) By contraposition, if \( f(S) \) is not jointless then \( f(s_1) \cup f(s_2) \) exists in \( f(P nA^*) \), for some \( s_1, s_2 \in S \) with \( s_1 \neq s_2 \). Then \( f(s_1) \cup f(s_2) = f(j) \) for some \( j \in P nA^* \), so \( f(j) = f(s_1) u_1 = f(s_2) u_2 \) for some \( u_1, u_2 \in nA^* \). Hence, applying \( f^{-1} \) yields \( j = s_1 u_1 = s_2 u_2 \), so \( s_1 \cup s_2 \) exists, So \( S \) is not jointless.

By contraposition, if \( f(S) \) is jointless but not maximal jointless then there exists \( f(z) \in Q nA^* \) such that \( f(S) \cup \{ f(z) \} = f(S \cup \{ z \}) \) is jointless. Then, applying \( f^{-1} \) and the result from the paragraph above) shows that \( S \cup \{ z \} \) jointless. Hence, \( S \) is not a maximal jointless code. \( \square \)

**Proposition 3.2 (maximal extensions of a right ideal morphism).**

1. For \( n \geq 2 \) there exist right ideal morphisms in \( nRI_A^\text{fin} \) that can be extended in more than one way to a maximal right ideal morphisms in \( nRI_A^n \).
2. Every right ideal morphism in \( nRI_A^\text{fin} \) has a unique maximum extension to a right ideal morphism of \( nA^* \), with domain code and image code being finite essential initial factor codes.
3. Every right ideal morphism of \( nA^* \), with domain code and image code being finite essential initial factor codes, is an extension of some element of \( nRI_A^\text{fin} \).

**Proof.** (1) This follows from the examples of \( F, F_1, F_2 \) above; see also [5, Lemma 2.27 and Fig. 2].

(2) Existence is trivial, since a jointless code is an initial factor code. To prove uniqueness we first prove two important properties:

[Finite Property] A right ideal morphism \( f \) whose \( \text{domC}(f) \) is finite and essential can only be extended in finitely many ways to a right ideal morphism of \( nA^* \).

Indeed, if \( \text{Dom}(f) \) is a finitely generated essential right ideal, then \( nA^* \setminus \text{Dom}(f) \) is finite. Hence, \( f \) can be defined on \( nA^* \setminus \text{Dom}(f) \) in finitely many ways only.

[Union Property] Let \( f, g \) be a right ideal morphisms of \( nA^* \) such that \( \text{domC}(f) \) and \( \text{imC}(f) \) are finite essential initial factor codes. Let \( f_1 \) and \( f_2 \) be any right ideal morphisms of \( nA^* \) that extend \( f \), such that \( \text{domC}(f_i) \) and \( \text{imC}(f_i) \) are finite essential initial factor codes (for \( i = 1, 2 \)). Then \( f_1 \cup f_2 \) is also a right ideal morphism of \( nA^* \) with finite domain code and finite image code (that are essential initial factor codes), extending \( f \).

Part (2) of the Proposition follows immediately from the Finiteness and Union Properties.

Let us prove the Union Property. Let \( f_{12} = f_1 \cup f_2 \) (the set-theoretic union). So \( \text{Dom}(f_{12}) = \text{Dom}(f_1) \cup \text{Dom}(f_2) \), and this is an essential right ideal (since \( \text{Dom}(f_1) \) and \( \text{Dom}(f_2) \) are essential right ideals). Then for \( x \in \text{Dom}(f_1) \setminus \text{Dom}(f_2) \) we have \( f_{12}(x) = f_1(x) \); and for \( x \in \text{Dom}(f_2) \setminus \text{Dom}(f_1) \) we have \( f_{12}(x) = f_2(x) \). This, and the next Claim, imply that \( f_{12} \) is a function.

**Claim.** For all \( x \in \text{Dom}(f_1) \cap \text{Dom}(f_2) \): \( f_1(x) = f_2(x) \) (\( = f_{12}(x) \)).

**Proof of the Claim:** Let \( x \in \text{Dom}(f_1) \cap \text{Dom}(f_2) \). Since \( f_1 \) and \( f_2 \) are extensions of \( f \), and \( \text{Dom}(f) \) is an essential right ideal, there exists \( \ell \geq 0 \) such that \( x \cdot nA^\ell \subseteq \text{Dom}(f) \). Then for all \( u \in nA^\ell : f_1(x) u = f_1(xu) = f(xu) = f_2(xu) = f_2(x) u, \) hence \( f_1(x) u = f_2(x) u \). Since \( nA^* \) is a cancellative monoid it follows that \( f_1(x) = f_2(x) \). \( \square \)
Now $f_{12}$ is uniquely defined on $\text{Dom}(f_1) \setminus \text{Dom}(f_2)$, $\text{Dom}(f_2) \setminus \text{Dom}(f_1)$, and $\text{Dom}(f_1) \cap \text{Dom}(f_2)$; hence it is a function on all of $\text{Dom}(f_1) \cup \text{Dom}(f_2)$.

It follows also that $f_{12}$ is a right ideal morphism. Indeed, on $\text{Dom}(f_1) = (\text{Dom}(f_1) \setminus \text{Dom}(f_2)) \cup (\text{Dom}(f_1) \cap \text{Dom}(f_2))$ we have $f_{12} = f_1$. And on $\text{Dom}(f_2) = (\text{Dom}(f_2) \setminus \text{Dom}(f_1)) \cup (\text{Dom}(f_1) \cap \text{Dom}(f_2))$ we have $f_{12} = f_2$.

(3) Let $f : P A^* \rightarrow Q A^*$ be a right ideal morphism, where $P, Q \subseteq A^*$ are essential finite initial factor codes, and let $\ell = \text{maxlen}(P)$. We can restrict $f$ to the right ideal $n A^2$, which is generated by the maximal joinless code $n A^2$. By Lemma 3.1, since $n A^2$ is a maximal joinless code, $f(n A^2)$ is a maximal joinless code. □

The Brin-Thompson group $nG_{k,1}$ is defined by $nR_I^{fin} \equiv \equiv_{end}$. Equivalently (by [5] Lemma 2.24): $nG_{k,1}$ is the action monoid of $nR_I^{fin}$ on $nA^\omega$. By the above Remark and Prop. 3.2 $nG_{k,1}$ is also the set of maximally extended right ideal morphisms of $nA^*$, where multiplication is composition followed by maximum extension. When $k = 2$ we obtain $nV$.

The definitions of evaluation functions, evaluation problem, domain membership problem, and long versus short data inputs, can be generalized immediately to $2V$ over a finite generating set $\Gamma_2$.

**Definition 3.3 (evaluation problem of $2V$ over $\Gamma_2$).**

The evaluation problem of $2V$ over a finite generating set $\Gamma_2$ is specified as follows.

**Input:** $(w, x, y) \in \Gamma_2^* \times 2\{0,1\}^* \times 2\{0,1\}^*$.

**Question:** $E_w(x) = y$ ?

Here, $E_w$ is the unique maximum extension of $w_n \circ \ldots \circ w_1(.)$ to a right ideal morphism, where $w = w_n \ldots w_1 \in \Gamma_2$.

For $x = (x_1, x_2) \in 2\{0,1\}^*$ we define $\ell(x) = \max\{|x_1|, |x_2|\}$. Let

$$\lambda_{\Gamma_2} = \text{maxlen}(\Gamma_2) = \max\{\ell(z) : z \in \bigcup_{\gamma \in \Gamma_2} \text{domC}(\gamma) \cup \text{imC}(\gamma)\}.$$ 

By [5] Cor. 3.3 we have for all $f \in 2V$:

if $\ell(x) \geq \lambda_{\Gamma_2}|w|$ then $w_n \circ \ldots \circ w_1(x)$ is defined.

Hence, similarly to Def. 2.4, for a given $w \in \Gamma_2^*$ we call a data input $x \in 2\{0,1\}^*$ a long data input for $w$ iff $w_n \circ \ldots \circ w_1(x)$ is defined. A data input $x \in \text{Dom}(E_w)$ that is not long is called a short data input for $w$.

**Definition 3.4** The evaluation problem for $2V$ over $\Gamma_2$ for long data inputs is defined as follows.

**Input:** $(w, x, y) \in \Gamma_2^* \times 2\{0,1\}^* \times 2\{0,1\}^*$, where $w = w_n \ldots w_1$, with $w_n, \ldots, w_1 \in \Gamma_1$.

**Question:** $w_n \circ \ldots \circ w_1(x) = y$ ?

### 4 Results

We now look at the complexity of the evaluation problem for $V$ and $2V$, and we compare the evaluation problem with the word problem. The evaluation decision problem for $V$ and $2V$, as well as the word problem, depend on $\Gamma_1$, respectively $\Gamma_2$. However, the complexity changes only slightly with the generating set, provided that $\Gamma_1$ and $\Gamma_2$ are finite.

**Proposition 4.1 (evaluation problem of $V$ over $\Gamma_1$ for long data inputs).**

Let $\Gamma_1$ be a finite generating set of the Thompson group $V$. The evaluation problem of $V$ over $\Gamma_1$ for long data inputs is in DCF $\cap$ DCF$^{rev}$. More precisely (assuming $\Gamma_1 \cap \{0,1\} = \emptyset$ and $1 \in \Gamma_1$), the language

$$L_v = \{x^{rev}w : x, y \in \{0,1\}^*, \ w \in \Gamma_1^*, \ w_n \circ \ldots \circ w_1(x) = y\}$$

is deterministic context-free, and its reverse is also deterministic context-free.
**Theorem 4.2** (evaluation problem of \( \Gamma \)).

The evaluation problem of the Brin-Thompson group \( \Gamma \) over \( \text{CF} \) is \( \text{P} \)-complete with respect to log-space many-one reduction.

**Proposition 4.3** (reduction of the word problem to the evaluation problem).

**Theorem 4.2** will be proved in Section 6.

**Relation between the evaluation problem and the word problem**

We will compare the evaluation problems of \( V \) and \( 2V \) with the word problem. The word problem of \( V \) over a finite generating set is in \( \text{coCF} \) (Lehnert and Schweitzer [18]); \( \text{DCF} \cap \text{DCF}^\text{rev} \) and \( \text{DCF} \) are strict subclasses of \( \text{CF} \cap \text{coCF} \) and \( \text{coCF} \). On the other hand, the word problem of \( V \) over a circuit-like generating set \( \Gamma_1 \cup \tau \), and the word problem of \( 2V \) and \( nG_{k,1} \) over a finite generating set, are \( \text{coNP} \)-complete [4 6].

Although the evaluation problem and the word problem of \( V \) and \( 2V \) look similar to the circuit value problem, respectively the circuit equivalence problem, the following proposition shows that there is also a fundamental difference. This is caused by the existence of short (versus long) data inputs.

**Proposition 4.3** (reduction of the word problem to the evaluation problem).

(1) Let \( \Gamma_1 \) be a finite generating set of the Thompson group \( V \). The evaluation problem of \( V \) over \( \Gamma_1 \), or over the circuit-like generating set \( \Gamma_1 \cup \tau \), for the data input and output \( \varepsilon \), is equivalent to the word problem of \( V \) over \( \Gamma_1 \), respectively \( \Gamma_1 \cup \tau \). More precisely, for any \( w \in \Gamma_1^* \) or in \( (\Gamma_1 \cup \tau)^* \),

\[
E_w(\varepsilon) = \varepsilon \quad \text{iff} \quad w = 1 \text{ in } V.
\]

Hence the evaluation problem (in general) of \( V \) over \( \Gamma_1 \) is in \( \text{coCF} \); and the evaluation problem of \( V \) over \( \Gamma_1 \cup \tau \) is \( \text{coNP} \)-complete.

(2) The evaluation problem of the Brin-Thompson group \( 2V \) over a finite generating set \( \Gamma_2 \), for the data input and output \( (\varepsilon, \varepsilon) \), is equivalent to the word problem of \( 2V \). More precisely, for any \( w \in \Gamma_2^* \),

\[
E_w((\varepsilon, \varepsilon)) = (\varepsilon, \varepsilon) \quad \text{iff} \quad w = 1 \text{ in } 2V.
\]
Hence the evaluation problem of $2V$ over $\Gamma_2$ is coNP-complete.

(3) The word problem of $V$ over a finite generating set $\Gamma_1$, or over $\Gamma_1 \cup \tau$, can be reduced to the evaluation problem of $V$ over $\Gamma_1$, respectively $\Gamma_1 \cup \tau$, for data inputs of length $N$, for any fixed $N \leq O(\log |w|)$; the reduction is a polynomial-time conjunctive reduction of polynomial arity.

Similarly, the word problem of $2V$ over a finite generating set $\Gamma_2$ can be reduced to the evaluation problem of $2V$ over $\Gamma_2$ data inputs $(x_1, x_2) \in 2 \{0, 1\}^*$ of length $\max\{|x_1|, |x_2|\} = N$, for any fixed $N \leq O(\log |w|)$; the reduction is a polynomial-time conjunctive reduction of polynomial arity.

Proof. (1), (2): The equivalences are straightforward, since $\{\varepsilon\}$ is a maximal prefix code in $\{0, 1\}^*$, and $\{(\varepsilon, \varepsilon)\}$ is a maximal joinless code in $2 \{0, 1\}^*$.

(3) We observe that $w = 1$ in $V$ iff $w(x) = x$ for all $x \in \{0, 1\}^N$, for any $N \in \mathbb{N}$. Indeed, $\{0, 1\}^N$ is a maximal prefix code. If we pick $N \leq O(\log |w|)$ then the cardinality of $\{0, 1\}^N$ is bounded by a polynomial in $|w|$, and all $x \in \{0, 1\}^N$ can be found in polynomial time.

A similar reasoning applies to $2V$, since $2 \{0, 1\}^N$ is a maximal joinless code. $\square$

**Theorem 4.4 (reduction of the evaluation problem to the word problem).**

(1) The evaluation problem of $V$ over the circuit-like generating set $\Gamma_1 \cup \tau$ is in coNP, and can be reduced to the word problem of $V$ over $\Gamma_1 \cup \tau$ by a many-one log-space reduction.

Similarly, the evaluation problem of $2V$ over a finite generating set $\Gamma_2$ is in coNP, and can be reduced to the word problem of $2V$ over $\Gamma_2$ by a many-one log-space reduction.

(2) The evaluation problem of $V$ over a finite generating set $\Gamma_1$ (and more generally, over a generating set $\Gamma_1 \cup \Delta$), reduces to the word problem of $V$ over $\Gamma_1$ (respectively $\Gamma_1 \cup \Delta$) by an eight-fold conjunctive log-space reduction. Here $\Delta$ is any infinite subset of $V$, coded by a log-space set of strings.

Proof. (1) The evaluation problem is described by the following coNP-formula:

$$E_w(x) = y \quad \text{iff} \quad (\forall z \in \{0, 1\}^*)[|z| = c_{\Gamma_1, w}|w| - |x| \Rightarrow E_w(xz) = yz],$$

where $c_{\Gamma_1, w} = \max\{c_{\Gamma_1}, \text{maxindex}_{\ell}(w)\}$, as seen before.

The length of the string $z$ in the $\forall$-quantifier is polynomially bounded; indeed, $|z| \leq c_{\Gamma_1, w}|w| - |x|$, and $c_{\Gamma_1, w}$ is linearly bounded in terms of the size of $w$.

The predicate $w(xz) = yz$ can be checked in polynomial time. Indeed, by the condition $|xz| = c_{\Gamma_1, w}|w|$, the data input $xz$ is long. Hence $E_w(xz) = w_0 \circ \ldots \circ w_1(xz)$, so $E_w(xz)$ is simply computed (in at most quadratic time) by applying the generators $w_i$ in sequence. Hence the universal formula above is a coNP-formula, so the problem is in coNP.

The word problem of $V$ over $\Gamma_1 \cup \tau$ is coNP-complete, hence the evaluation problem (being in coNP) reduces to the word problem.

For $2V$ over $\Gamma_2$, the same reasoning works; here we use the fact that $\ell(x) = \max\{x_1, x_2\} \leq \lambda_{\Gamma_2}|w|$ implies that $x$ is a long data input.

Proof. (2) This proof is given in Section 7. $\square$

**Open problems:**

- Is the word problem of $M_{2,1}$ over a finite generating set P-complete? (It is in P by [3].)
- Does there exist a finitely presented group, or monoid, whose word problem over a finite generating set is P-complete? (Compare with other P-complete problems for groups in [12], A.8.6 - A.8.17.)
Overview of the reductions between problems

The following graph shows the reductions between the problems considered in this paper. An arrow $A \rightarrow B$ indicates that $A$ reduces to $B$ by many-one log-space reduction or conjunctive log-space reduction. Mutual reduction is indicated by $A \leftrightarrow B$.

Abbreviations:

wp. — the word problem for a given group and generating set  
ev. — the evaluation problem for a given group, generating set, and data input  
$\Gamma_1$ — any finite generating set of $V$  
$\Gamma_2$ — any finite generating set of $2V$  
$\Delta$ — any subset of $V$ that is encoded in binary by a log-space language
5 Complementary prefix codes, partial fixators, commutation tests

This Section introduces tools to be used in Section 6 for proving Theorem 4.2 and in Section 7 for proving Theorem 4.4(2). Let $A$ be a finite alphabet of cardinal $|A| = k \geq 2$.

5.1 Complementary prefix codes

**Definition 5.1** [2 Def. 5.2]. Two prefix codes $P, P' \subseteq A^*$ are complementary prefix codes iff $P \cup P'$ is a maximal prefix code in $A^*$, and $P A^* \cap P'A^* = \emptyset$.

This definition is equivalent to the following: $P \cup P'$ is a maximal prefix code, and $P \cap P' = \emptyset$.

If $P$ is a maximal prefix code then $\emptyset$ is the unique complementary prefix code of $P$; except for this case, the complementary prefix code of $P$ is never unique. E.g., for every $u \in P'$, $(P' \setminus \{u\}) \cup u A$ is also a complementary prefix code of $P$.

**Notation:** For an alphabet $A$ and a set $S \subseteq A^*$, let

$$\text{pref}(S) = \{x \in A^* : (\exists s \in S)[x \leq_{\text{pref}} s]\};$$

i.e., $\text{pref}(S)$ is the set of prefixes of elements of $S$. And

$$\text{Spref}(S) = \{x \in A^* : (\exists s \in S)[x <_{\text{pref}} s]\};$$

i.e., $\text{Spref}(S)$ is the set of strict prefixes of elements of $S$.

**Proposition 5.2 (existence and construction).** For every finite prefix code $P \subseteq A^*$ with $|A| \geq 2$, there exists a finite complementary prefix code $P' \subseteq A^*$. If $P$ is not $\emptyset$ and not a maximal prefix code, then $\maxlen(P') = \maxlen(P)$.

If $P$ is given by a list of strings, then a complementary prefix code $P'$ can be computed from $P$ in log-space.

**Proof.** This is a special case of [7 Lemma 2.27 and Cor. 2.30]. If $P$ is maximal then $P' = \emptyset$, and if $P = \emptyset$ then we can choose $P' = \{\epsilon\}$. If $P$ is not $\emptyset$ and not a maximal prefix code, then a complementary prefix code of $P$ is given by the formula

$$P' = \{xa : x \in \text{Spref}(P), a \in A, xa \notin \text{pref}(P)\}.$$

Claim 1: $P'$ is a prefix code.

Let us consider any $xa, yb \in P'$, where $x, y \in \text{Spref}(P)$ and $a, b \in A$.

Case 1: If $x$ and $y$ are prefixes of a same $p = p_1 \ldots p_k \in P$ then $x \parallel_{\text{pref}} y$; we can assume $x \leq_{\text{pref}} y$.

If $x = y$ and $a \neq b$ then $xa$ and $yb = xb$ differ in position $|x| + 1$, so they are not prefix-comparable. If $x <_{\text{pref}} y$, then $x = p_1 \ldots p_i$, and $y = p_1 \ldots p_ip_{i+1} \ldots p_j$, for some $1 \leq i < j < k$. Since $xa \notin \text{pref}(P)$, $a \neq p_{i+1}$ so $xa$ differs from $yb$ in position $i + 1$; hence $xa$ and $yb$ are not prefix-comparable.

Case 2: If $x, y$ are not in Case 1, and if $x <_{\text{pref}} p, y <_{\text{pref}} q$, for $p, q \in P$, then $x$ and $y$ are not prefix-comparable (otherwise $x, y$ are in Case 1). Then $x$ and $y$ differ in some position $i \leq \min\{|x|, |y|\}$, so $xa$ and $yb$ also differ in that position; hence $xa$ and $yb$ are not prefix-comparable. [End, Claim 1.]

Claim 2: $P \cap P' = \emptyset$.

The formula says that $xa \notin \text{pref}(P)$, hence $xa \notin P$, hence $P \cap P' = \emptyset$.

Claim 3: $P \cup P'$ is maximal.

Let $z \in A^*$ be any string of length $|z| = \maxlen(P)$ such that $z \notin PA^*$. Then exists $s \in \text{Spref}(P)$ such the $s$ is a prefix of $z$ (where $s$ could be $\epsilon$). If $s = z_1 \ldots z_i$ is the longest prefix of $z$ that belongs to $\text{pref}(P)$ then $sz_{i+1} \notin \text{pref}(P)$ with $i = |s|$ (otherwise we would have $z \in PA^*$). Therefore, $sz_{i+1} \in P'$. Thus for every $z \in A^*$ with $|z| = \maxlen(P)$: either $z \in PA^*$, or $z \in P'A^*$. [End, Claim 3.]
Claims 1, 2, 3 imply that \( P' \) is a complement of \( P \).

Thanks to this formula, \( P' \) can be computed from \( P \) in log-space, as follows. First, by comparing all strings in \( P \) one by one, it can be checked that \( P \) is a prefix code. Non-emptiness is trivial to check. (Non)maximality can be checked by the Kraft (in)equality. After that, for every \( p \in P \), every strict prefix \( s \) can be found, hence each string \( sa \) (for \( a \in A \)) is found, after which one can check whether \( sa \) is not a prefix of some element in \( P \). The output of this process is a list of all the elements of \( P' \) in an arbitrary order, with possible repetitions. Since sorting can be done in log-space, \( P' \) can be listed in increasing dictionary order, without repetitions. Recall that the composite of log-space functions is in log-space \( [16, \text{Lemma 13.3}] \). \( \Box \)

A special case is of Prop. \( 5.2 \) is the following.

**Proposition 5.3** For every \( u = u_1 \ldots u_\ell \in A^+ \) with \( u_1, \ldots, u_\ell \in A \) and \( \ell = |u| > 0 \), the following is a complementary prefix code of \( \{u\} \):

\[
\overline{u} = \bigcup_{j=0}^{\ell-1} u_1 \ldots u_j (A \setminus \{u_{j+1}\});
\]

here \( u_1 \ldots u_j = \varepsilon \) when \( j = 0 \). The set \( \overline{u} \) is a prefix code with the following properties:

- \( \text{maxlen}(\overline{u}) = |u| \),
- \( |\overline{u}| = |u|(|A| - 1) \),
- \( \overline{u} \) can be computed (as a list of strings) from \( u \) in time \( O(|u|^2) \), and in log-space. \( \Box \)

If \( u = \varepsilon \) then \( \{u\} \) is a maximal prefix code, so the complementary prefix code is \( \emptyset \).

**Lemma 5.4** (transitivity of \( V \)).

Suppose \( V \) is defined as the set of maximally extended right ideal morphisms in \( RIT_A^{\text{fin}} \).

For any \( u, v \in \{0, 1\}^+ \) there exists \( \psi \in V \) such that \( \psi(u) = v \) and such that the table size of \( \psi \) is \( |\text{domC}(\psi)| = 1 + \max\{|u|, |v|\} \).

**Proof.** Let \( \overline{u} \) and \( \overline{v} \) be complementary prefix codes of \( \{u\} \), respectively \( \{v\} \), as in Prop. \( 5.3 \). So, the cardinalities satisfy \( |\overline{u}| = |u| \) and \( |\overline{v}| = |v| \).

If \( |u| > |v| \), we replace the prefix code \( \overline{u} \) by \( Q_u = (\overline{u} \setminus \{x\}) \cup \{x, 0, 1\} \), for some \( x \in \overline{u} \); this preserves the fact that \( Q_u \) is a complementary prefix code of \( \overline{u} \), but increases its cardinality by 1. We repeat this until \( |Q_u| = |\overline{u}| = |u| \). The case where \( |v| > |u| \) is handled in a similar way. In any case, we obtain complementary prefix codes \( Q_u \) and \( Q_v \) of \( \{u\} \), respectively \( \{v\} \), such that \( |Q_u| = |Q_v| = \max\{|u|, |v|\} \).

Now, \( \psi \) is defined by \( \psi(u) = v \) and \( \psi \) maps \( Q_u \) bijectively onto \( Q_v \), arbitrarily. The table size of \( \psi \) is \( 1 + \max\{|u|, |v|\} \). \( \Box \)

### 5.2 Partial fixators

We generalize the well known concept of fixator to partial injections.

**Definition 5.5** (partial fixator \( [5, \text{Def. 4.13}] \)).

A function \( g \) partially fixes a set \( S \) iff \( g(x) = x \) for every \( x \in S \cap \text{Dom}(g) \cap \text{Im}(g) \). The partial fixator of \( S \) in \( V \) is

\[
p\text{Fix}_V(S) = \{g \in V : (\forall x \in S \cap \text{Dom}(g) \cap \text{Im}(g))(g(x) = x)\}.
\]
This is also called partial pointwise stabilization.

We will only use partial fixators for sets $S$ that are right ideals of $\{0,1\}^*$. If $S = P \{0,1\}^*$ is a right ideal, where $P$ is a prefix code, then $\text{pFix}_V(S)$ is a group \cite[Lemma 4.1]{5}. We abbreviate $\text{pFix}_V(P \{0,1\}^*)$ by $\text{pFix}_V(P)$. In particular, for $z \in \{0,1\}^*$ we abbreviate $\text{pFix}_V(z \{0,1\}^*)$ by $\text{pFix}_V(z)$.

One easily proves that if $P = \emptyset$ then $\text{pFix}_V(P) = V$, and if $P$ is a maximal prefix code then $\text{pFix}_V(P) = \{\text{id}\}$ (the one-element subgroup of $V$).

**Lemma 5.6** Below, $\subseteq$ or $=$ refer to set inclusion or equality of subsets of $V$. For all $u, v \in \{0,1\}^*$:

$$\text{pFix}_V(u) \subseteq \text{pFix}_V(v) \iff v \{0,1\}^* \subseteq u \{0,1\}^* \quad \text{(i.e., } u \leq_{\text{pref}} v).$$

Hence,

$$\text{pFix}_V(u) = \text{pFix}_V(v) \iff u = v.$$

**Proof.** The equality relations immediately follow from the inclusions. Let us prove the inclusions.

[$\subseteq$] If $v = uz \in u \{0,1\}^*$, and if $f$ fixes $u$ then $f$ fixes $uz = v$ (since $f$ is a right ideal and $f(u)$ is defined).

[$\supseteq$] We prove the contrapositive: Suppose $u \not\leq_{\text{pref}} v$.

Case 1: $u$ and $v$ are not prefix-comparable.

Then $\{u,v\}$ is a finite prefix code, so there exists a finite complementary prefix code $Q$.

If $Q = \emptyset$ then $\{u,v\} = \{0,1\}$, since this is the only two-element maximal prefix code in $\{0,1\}^*$.

Suppose $u = 0$ (the case $u = 1$ is similar). Then $f$ with table $\{(0,0), (10,11)(11,10)\}$ satisfies $f \in \text{pFix}_V(u) \setminus \text{pFix}_V(v)$. So, $\text{pFix}_V(u) \not\subseteq \text{pFix}_V(v)$.

If $Q \neq \emptyset$, let $q \in Q$. Then $f$ with table $\{(u,u), (v,q), (q,v)\} \cup \{(z,z) : z \in Q \setminus \{q\}\}$ satisfies $f \in \text{pFix}_V(u) \setminus \text{pFix}_V(v)$. So, $\text{pFix}_V(u) \not\subseteq \text{pFix}_V(v)$.

Case 2: $u$ and $v$ are prefix-comparable.

Then $v \leq_{\text{pref}} u$ (since $u \not\leq_{\text{pref}} v$). So, $u = va$ for some $a \in \{0,1\}^+$. We can assume $a = 0b$ for some $b \in \{0,1\}^*$ (the case $a = 1b$ is similar); so, $u = v0b$. Then $\{u,v10,v11\}$ is a prefix code; let $Q$ be a complementary prefix code. Then $f$ with table $\{(u,u), (v10,v11), (v11,v10)\} \cup \{(z,z) : z \in Q\}$ satisfies $f \in \text{pFix}_V(u) \setminus \text{pFix}_V(v)$. So, $\text{pFix}_V(u) \not\subseteq \text{pFix}_V(v)$. $\square$

The next theorem is a generalization of \cite[Lemma 4.20]{5}.

**Theorem 5.7 (generators of $\text{pFix}_V(P)$).** For any non-empty, non-maximal, finite prefix code $P \subseteq \{0,1\}^*$ we have:

1. $\text{pFix}_V(P)$ is isomorphic to $V$ (hence, $\text{pFix}_V(P)$ is finitely generated).

2. Let $\Gamma_1$ be a finite generating set of $V$; $\Gamma_1$ will be kept fixed. Then, from any $P$ (given as a list of strings), a finite generating set for $\text{pFix}_V(P) \subseteq V$ can be computed in log-space, with each generator of $\text{pFix}_V(P)$ expressed as a word over $\Gamma_1$.

**Proof.** (1) Let $Q = \{q_j : 1 \leq j \leq k\}$ be a finite complementary prefix code of $P$. Let $B = \{b_1, \ldots, b_k\}$ be an alphabet of cardinal $|B| = k = |Q|$, such that $B \cap \{0,1\} = \emptyset$, and let us consider the Higman-Thompson group $G_{2,k}$. Every element of $G_{2,k}$ is given by a bijection between two maximal prefix codes in $B \{0,1\}^*$. For $\psi \in G_{2,k}$, the domain and image codes are of the form $\text{domC}(\psi) = \bigcup_{i=1}^k b_i C_i$, and $\text{imC}(\psi) = \bigcup_{j=1}^k b_j D_j$, where each $C_i$ and $D_j$ is a finite maximal prefix code in $\{0,1\}^*$ (for $i, j = 1, \ldots, k$); and $\psi$ is given by a bijection from $\text{domC}(\psi)$ onto $\text{imC}(\psi)$; so, $|\text{domC}(\psi)| = |\text{imC}(\psi)|$.

To construct an isomorphism from $G_{2,k}$ onto $\text{pFix}_V(P)$ we choose a bijection $f_{B,Q} : B \to Q$. For simplicity, we assume that $Q$ is indexed in increasing dictionary order, and we pick $f_{B,Q}(b_i)$ to be the
\( b \) respondence with \( \Gamma \) determined. As in part complexity of the calculation of \( \Pi \) well-defined. Indeed, since we can also write

\[
\pi(q_i c_i) = q_j d_j \quad \text{iff} \quad \psi(b_i c_i) = b_j d_j.
\]

Then \( \pi(\psi) \) is well-defined; indeed, since \( Q \) is a prefix code, \( q_i c_i \) uniquely determines \( i, b_i \) and \( c_i \); hence \( q_j d_j \) is uniquely determined by \( \psi \) and \( c_i \).

So, \( \text{domC}(\Pi(\psi)) = P \cup \bigcup_{i=1}^{k} q_i C_i \), and \( \text{imC}(\Pi(\psi)) = P \cup \bigcup_{j=1}^{k} q_j D_j \); these are finite maximal prefix codes. Each of \( \bigcup_{i=1}^{k} q_i C_i \) and \( \bigcup_{j=1}^{k} q_j D_j \) is a complementary prefix code of \( P \). It is straightforward to see that \( \Pi \) is invertible and that it is a homomorphism. So, \( G_{2,k} \) is isomorphic to \( \text{pFix}_V(P) \).

Finally, by Higman’s [15 Cor. 2, p. 12], all the free Jónson-Tarski algebras \( V_{2,k} \) are isomorphic to \( V_{2,1} \). And since \( G_{2,k} \) is the group of automorphisms of \( V_{2,k} \), all the groups \( G_{2,k} \) are isomorphic to \( G_{2,1} \) for all \( k \geq 1 \). Hence, \( G_{2,1} \) is isomorphic to \( \text{pFix}_V(P) \).

(2) Recall that \( \Gamma_1 \) is fixed and \( P \) is a variable input; for this input \( P \), our goal is to compute generators of \( \text{pFix}_V(P) \), expressed as words over \( \Gamma_1 \). For this, we first construct an isomorphism \( \Pi_p : V \to \text{pFix}_V(P) \) as in part (1), but we describe this isomorphism more explicitly. (The construction in (1) is thus redundant, but (1) makes (2) easier to understand.)

Let \( B \subseteq \{0, 1\}^* \) be any maximal prefix code such that \( |B| = |Q| \); therefore, now \( B \) is a set of bitstrings, as opposed to the construction in (1), where \( B \) was a new alphabet. Also, \( B \) is a maximal prefix code, whereas \( Q \) is non-maximal. Let \( f_{B,Q} : B \to Q \) be a bijection. For simplicity, we pick \( f_{B,Q} \) so as to preserve the dictionary order of \( \{0, 1\}^* \); then \( f_{B,Q}(b_i) = q_i \) (for \( i = 1, \ldots, k \)), assuming that \( B \) and \( Q \) are both indexed in increasing dictionary order.

For any \( \varphi \in G_{2,1} \) we restrict \( \varphi \) so that \( \text{domC}(\varphi) \cup \text{imC}(\varphi) \subseteq B \{0, 1\}^* \); we still call this restriction \( \varphi \). Then \( \text{domC}(\varphi) = \bigcup_{i=1}^{k} b_i C_i \) and \( \text{imC}(\varphi) = \bigcup_{j=1}^{k} b_j D_j \) for some finite maximal prefix codes \( C_i, D_j \subseteq \{0, 1\}^* \) (for \( i, j = 1, \ldots, k \)). We define an isomorphism from \( G_{2,1} \) onto \( \text{pFix}_V(P) \) by

\[
\Pi_p : \varphi \in G_{2,1} \quad \mapsto \quad \Pi_p(\varphi) = \text{id}_P \cup \pi(\varphi) \in \text{pFix}_V(P),
\]

where \( \text{domC}(\pi(\varphi)) = \bigcup_{i=1}^{k} q_i C_i \) and \( \text{imC}(\pi(\varphi)) = \bigcup_{j=1}^{k} q_j D_j \); and \( \pi(.) \) is such that

\[
\pi(\varphi)(q_i c_i) = q_j d_j \quad \text{iff} \quad \varphi(b_i c_i) = b_j d_j.
\]

We can also write

\[
\Pi_p(\varphi)(\cdot) = \text{id}_P \cup f_{B,Q} \circ \varphi \circ f_{B,Q}^{-1}(\cdot),
\]

where \( f_{B,Q} : B \to Q \) is the bijection that preserves the dictionary order, seen above. Then \( \pi(\varphi) \) is well-defined. Indeed, since \( Q \) is a prefix code, \( q_i c_i \) uniquely determines \( q_i \) and \( c_i \); and \( q_i \) determines \( i \) and \( b_i \). And \( b_i c_i \) determines \( b_j d_j \) (\( = \varphi(b_i c_i) \)), from which \( b_j \) and \( d_j \) hences, \( q_j = \pi(b_j) \), hence \( q_j d_j \) are determined. As in part (1), we see that \( \Pi_p \) is an isomorphism from \( G_{2,1} \) onto \( \text{pFix}_V(P) \).

Let us now find generators for \( \text{pFix}_V(P) \). Since \( \Pi_p(.) \) is an isomorphism from \( G_{2,1} \) onto \( \text{pFix}_V(P) \), it follows immediately that \( \Pi_p(\Gamma_1) \) is a finite generating set of \( \text{pFix}_V(P) \), that is in one-to-one correspondence with \( \Gamma_1 \). In the computation of \( \Pi_p(\Gamma_1) \) from \( P \), only \( P \) is variable (\( \Gamma_1 \) is fixed); so the complexity of the calculation of \( \Pi_p(\gamma) \), for each \( \gamma \in \Gamma_1 \), is measured as a function of \( \sum_{p \in P} |p| \). Since \( \Gamma_1 \) is fixed and finite, we can look at \( \Pi_p(\gamma) \) for one \( \gamma \in \Gamma_1 \) at a time.

The composite of two log-space computable functions is log-space computable [16 Lemma 13.3]. Therefore, in order to compute a word for \( \Pi_p(\gamma) \in \Gamma_1^* \) from \( P \) in log-space, it suffices to show that each one of the following steps is computable in log-space:
• from $P$, find a complementary prefix code $Q$;
• from $Q$, find a maximal prefix code $B \subseteq \{0,1\}^*$ with $|B| = |Q|$;
• from $B$ and $Q$, find $\text{dom}(\Pi_p(\gamma))$ and $\text{im}(\Pi_p(\gamma))$ (by also using $\text{dom}(\gamma)$ and $\text{im}(\gamma)$);
• from $\text{dom}(\Pi_p(\gamma))$ and $\text{im}(\Pi_p(\gamma))$, find the table of $\Pi_p(\gamma)$ (using $\gamma$ and the formula for $\Pi_p(.)$);
• from the table of $\Pi_p(\gamma)$, find a word for $\Pi_p(\gamma) \in \Gamma_1$ such that this word evaluates to $\Pi_p(\gamma)$; this word has length $\leq c(\Pi_p(\gamma) | |\log|\Pi_p(\gamma)|)$ for some constant $c > 0$.

- Finding $Q$: Prop. 5.2 gives a formula for the complementary prefix code,
\[ Q = \{xa : x \in \text{S}_{\text{pref}}(P), a \in \{0,1\}, xa \not\in \text{pref}(P)\}, \]
from which $Q$ can be computed in log space (as a list of strings, sorted in increasing dictionary order).
- Finding $B$: We choose $B = \{0^{k-1}\} \cup 0^{\leq k-2}1$, where $k = |Q|$. The set $B$ (in the form of a list of strings) can be found from $Q$ in log-space.
- Finding the table for $\Pi_p(\gamma)$ from the table of $\gamma$ is done in two steps: First we find the table of $\gamma_B$ (the restriction of $\gamma$ to $B$); second, from this we find the table of $\Pi_p(\gamma)$.

Finding the table of $\gamma_B$: We look at every string $x \in \text{dom}(\gamma)$. If $x$ has a prefix in $B$, then $(x, \gamma(x))$ is already part of the table of $\gamma_B$, so we simply copy it to the output. If $x$ is a strict prefix of any string $b \in B$, i.e., $b = xu$ for some $u \in \{0,1\}^+$, then we include $(b, \gamma(x)u)$ into the table of $\gamma_B$. Recall also that by [1, Lemma 3.3], $\text{dom}(\gamma_B) \subseteq B \cup \text{dom}(\gamma)$, so $\gamma_B$ has table-size $|\text{dom}(\gamma_B)| \leq |B| + |\text{dom}(\gamma)|$. Since $\text{dom}(\gamma)$ and $B$ are given as lists of strings, the above procedure can be carried out in log-space: Only a bounded set of positions in the input needs to be kept track of.

Finding the table of $\Pi_p(\gamma)$ from the table of $\gamma_B$ and $B$: The input is $B$, $\gamma$, $\gamma_B$, and $\pi : B \to Q$ (all given as lists of strings or pairs of strings). By definition, the table of $\Pi_p(\gamma)$ consists of $\{(p,p) : p \in P\} \cup \{(q_i c_i, q_j d_j) : (b_i c_i, b_j d_j) \in \gamma_B\}$.

Outputting $\{(p,p) : p \in P\}$ is easy.

For the second part, we look at each $(b_i c_i, b_j d_j)$ in the table of $\gamma_B$ (for $i = 1, \ldots, k$). From $b_i c_i$ and the prefix code $B$, we find $b_i$; by using the table of $\pi$ we find and output $q_i$; from $b_i c_i$ and the prefix code $B$ we also find and output $c_i$; so we have produced “$(q_i c_i)$.” In a similar, from $b_j d_j$ we find and output “$q_j d_j$”.

All this can be done in log-space, since only a bounded set of positions in the input need to kept track of.

- Finding a word over $\Gamma_1$ for $\Pi_p(\gamma)$: This can be done in log-space by using the construction in [1, Proof of Theorem 3.8(1)]. This word has length $\leq O(|\Pi_p(\gamma) | |\log|\Pi_p(\gamma)|) = O(k \log k)$.

Notation: For any $g \in V$, $\|g\|$ denotes the table size of $g$ (i.e., the number of pairs in the table). The input size for $g$, used for measuring complexity, is $\sum_{x \in \text{dom}(g)} |x| + \sum_{y \in \text{im}(g)} |y|$.

Let us examine the the construction in [1, Proof of Theorem 3.8(1)], where for any $g \in V$, given by its table, a word for $g$ over $\Gamma_1$, of length $O(\|g\| \log \|g\|)$, is computed in space $O(\log(\sum_{x \in \text{dom}(g)} |x| + \sum_{y \in \text{im}(g)} |y|))$. This done in several steps, as follows.

* A maximal prefix code $S_n$ of size $n = \|g\|$ is chosen, such that $S_n \subseteq \{0,1\}^{k-1} \cup \{0,1\}^k$, where $k = \|\log n\|$. See [1, Prop. 3.9 and following].

Let $F$ denote the well-known Thompson group of dictionary order preserving elements of $V$. Now $g$ is factored as $g = \beta_g \pi_g \alpha_g$, where $\beta_g, \alpha_g \in F$, and $\text{dom}(\pi_g) = \text{im}(\pi_g) = S_n$. The three factors are given by their tables; see [1, Prop. 3.9]. This is easily done in log-space.

* The elements $\beta_g, \alpha_g$ are factored over the two well-known generators of $F$; see [1, Prop. 3.10]. This can be implemented in log-space.

* The element $\pi_g$ is factored into $\leq 3n$ string transpositions of the form $(0^{[\log n]} \mid w)$, where $0^{[\log n]} \in S_n$, and $w \in S_n \setminus \{0^k\}$. This factorization is easily carried out in log-space, based on the table of $\pi_g$. 

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Here we define string transpositions \((0^k \mid w)\) as follows. For any \(w \in \{0,1\}^+\), let \(j \in \mathbb{N}\) be the unique number such that \(0^j 1\) is a prefix of \(w\), so \(w = 0^j 1 v\). Now we let

\[(0^k \mid w) = \{(0^k, w), (w, 0^k)\} \cup \{(x, x) : x = 0^i 1, 0 \leq i < k, i \neq j\} \cup \{(x, x) : x = 0^i 1pa, p \in \{0,1\}^*, p \prec \text{pref} v, a \in \{0,1\}, pa \not\prec \text{pref} v\}.

* Every transposition \((0^k \mid w)\) is factored over \(\Gamma_1\) as a word of length \(O(\log n)\). See [1] Lemma 3.11.

The proof of this Lemma leads to four computational steps:

Cases (1.1) and (1.2) can be handled in one step, using log-space.

Cases (1.3) and (1.4) can be handled in one step, using log-space.

Case (2.1) can be handled in one step, using log-space; this leads to case (2.2).

Case (2.2) can be handled in one step, using log-space; this leads to cases (2.3).

Cases (2.3) and (2.4) can be handled jointly, using log-space.

The result is a word over \(\Gamma_1\) whose value in \(G_{2,1}\) is \(\Pi_p(\gamma)\). □

### 5.3 Commutation test

Let \(A\) be a finite alphabet of cardinal \(|A| = k \geq 2\), and let \(G_{k,1}\) be the Higman-Thompson group. The following “commutation test” for membership in \(\text{pFix}_{G_{k,1}}(P)\) was introduced in [2, Cor. 9.15].

**Theorem 5.8 (commutation test for membership in \(\text{pFix}_{G_{k,1}}(P)\)).**

Let \(P, Q \subseteq A^*\) be any finite, non-empty, complementary prefix codes; let \(\Gamma_Q\) be any generating set of \(\text{pFix}_{G_{k,1}}(Q)\). For every \(g \in G_{k,1}\) the following are equivalent:

1. \(g \in \text{pFix}_{G_{k,1}}(P)\),
2. \(g\) commutes with every element of \(\text{pFix}_{G_{k,1}}(Q)\),
3. \(g\) commutes with every generator in \(\Gamma_Q\).

**Proof.** It is straightforward to see that (1) and (2) are equivalent. For the proof that (0) is equivalent to (1), see [2 Cor. 9.15]. □

The commutation test can be effectively used when \(\Gamma_Q\) is finite.

### 6 The evaluation problem of \(V\) over \(\Gamma_1 \cup \tau\), and for \(2V\) over \(\Gamma_2\), for long data inputs

We will prove that these two problems are \(\text{P}\)-complete (Theorem 4.2). Let \(A\) be any finite alphabet with \(|A| = k \geq 2\). As before, let \(\Gamma_1\) and \(\Gamma_2\) be a finite generating set of \(V\), respectively \(2V\).

**Lemma 6.1 (the problems are in \(\text{P}\)).**

The evaluation problem of \(V\) over \(\Gamma_1 \cup \tau\), and the evaluation problem for \(2V\) over \(\Gamma_2\), for long data inputs, belong to \(\text{P}\).

**Proof.** For \(w \in (\Gamma_1 \cup \tau)^*\) and \(x, y \in \{0,1\}^*\), we apply \(w_1\) to \(x\), then \(w_2\) to \(w_1(x)\), \(w_3\) to \(w_2(x)\), etc.; this is straightforward. The length of each \(w_i \circ \ldots \circ w_1(x)\) (for \(i = 1, \ldots, n\)) is at most \(c_{\tau_1, \ldots, \tau_n} \cdot i\).

Here, for any string \(v \in (\Gamma_1 \cup \tau)^*\): \(c_{\tau_1, \ldots, \tau_n} = \max\{c_{\tau_1}, \text{maxindex}_\tau(v)\}\), where \(c_{\tau_1} = \text{maxlen}(\Gamma_1)\), i.e., the length of the longest bit-string in \(\bigcup_{\gamma \in \Gamma_1} \text{domC}(\gamma) \cup \text{imC}(\gamma)\); and \(\text{maxindex}_\tau(v) = \max\{i \in \mathbb{N}^> : \tau_{i-1,i} \text{ occurs in } v\}\) (i.e., the largest subscript of any element of \(\tau\) that occurs in \(v\)).

A very similar reasoning applies to \(2V\) over \(\Gamma_2\). □
Lemma 6.2 (reduction from $V$ over $\Gamma_1 \cup \tau$ to $2V$ over $\Gamma_2$).

The evaluation problem of $V$ over $\Gamma_1 \cup \tau$ can be reduced to the evaluation problem for $2V$ over $\Gamma_2$, by a many-one finite-state reduction. (This holds for all data inputs, long or short.)

Proof. Similarly to [5, Section 4.6], we reduce the evaluation problem $(w, x, y)$ of $V$ over $\Gamma_1 \cup \tau$, to the evaluation problem $(W, (x, \varepsilon), (y, \varepsilon))$ for $2V$ over $\Gamma_2$, where $W$ is obtained from $w$ as follows:

- every $\gamma \in \Gamma_1$ is replaced by $\gamma \times 1$, defined by $\gamma \times 1(u, v) = (\gamma(u), v)$, for all $(u, v) \in 2 \{0, 1\}^*$,
- every $\tau_{i,i+1} \in \tau$ occurring in $w$ is replaced by $\sigma_{i-1} \cdot (\tau_{1,2} \times 1) \cdot \sigma_{i+1}^\tau(.)$.

Then we have $E_w(x) = y$ iff $E_W((x, \varepsilon)) = (y, \varepsilon)$.

Moreover, $W$ can be computed from $w$ in log-space. Outputting $\gamma \times 1$ when $\gamma$ is read can be done by a finite automaton. Outputting $\sigma_{i-1} \cdot (\tau_{1,2} \times 1) \cdot \sigma_{i+1}^\tau(.)$ when $\tau_{i,i+1}$ is read can be done in log-space, assuming that $\tau_{i,i+1}$ is encoded in binary by $ab^{i+1}a$. (See the remark about encoding in Section 1.)

So far we have used the following finite subset $\Gamma_0 = \{\gamma \times 1 : \gamma \in \Gamma_1\} \cup \{\sigma, \tau_{1,2} \times 1\}$ of $2V$ in order to generate all of $\tau$. Next, every element of the finite set $\Gamma_0$ can be replaced by a string over $\Gamma_2$; this can be done by a finite automaton. □

Finally, to prove Theorem 4.2 we need to show that the evaluation problem of $V$ over $\Gamma_1 \cup \tau$ with long data inputs is $P$-hard. We do this by reducing the circuit value problem to this problem, using a many-one log-space reduction. The difficulty is, of course, that we need to simulate arbitrary transformations by bijective functions. For this we adapt the methods in [5] (see also [2]), where the circuit equivalence problem was reduced to the word problem of $V$ over $\Gamma_1 \cup \tau$.

Definition 6.3 [5, Def. 4.10] (simulation): Let $f : \{0, 1\}^m \rightarrow \{0, 1\}^n$ be a total function. An element $\Phi_f \in V$ simulates $f$ iff for all $x \in \{0, 1\}^m$: $\Phi_f(0 \ x) = 0 \ f(x) \ x$.

When $\Phi_f$ is represented by a word $w_f \in (\Gamma_1 \cup \tau)^*$ we also say that $w_f$ simulates $f$.

We want to define a size for every $w \in (\Gamma_1 \cup \tau)^*$, where $\Gamma_1$ is a finite generating set of $V$. First, $\text{size}(\gamma) = 1$ for every $\gamma \in \Gamma_1$, and $\text{size}(\tau_{i,i+1}) = i + 1$ for every $\tau_{i,i+1} \in \tau$. Finally, for $w = w_n \ldots w_1 \in (\Gamma_1 \cup \tau)^*$ with $w_n, \ldots, w_1 \in \Gamma_1 \cup \tau$, the size of $w$ is $\text{size}(w) = \sum_{i=1}^{n} \text{size}(w_i)$.

For a directed acyclic graph, the depth of a vertex $v$, denoted by $\text{depth}(v)$, is the length of a longest directed path ending in $v$. For any $d \in \mathbb{N}$, and any directed acyclic graph $C$, the set of vertices that have depth $d$ is called the $d$th layer. An acyclic circuit (and more generally, a directed acyclic graph) is called strictly layered if for every vertex $v$: all the parents of $v$ have the same depth $\text{depth}(v) - 1$.

For $P$-completeness of the circuit value problem we only need to use acyclic circuits that are strictly layered. Indeed, the proof of $P$-completeness of the circuit-value problem in [20, Thm. 8.1] gives a log-space many-one reduction of the acceptance problem of any one-tape polynomial-time Turing machine $M$ with a data input $x$ to a circuit-value problem $(C, x)$, where $C$ is strictly layered; see also [12, Theorem 6.2.5], where strictly layered circuits are called synchronous. Therefore, [5, Theorem 4.12] can be replaced by the following, which gives a lower complexity.

Theorem 6.4 (existence of a simulation). There is an injective function $C \mapsto w_C$ from the set of strictly layered acyclic boolean circuits to the set of words over $\Gamma_1 \cup \tau$, with the following properties:

1. $w_C$ simulates the input-output function $f_C$ of $C$;
2. the size of $w_C$ satisfies $\|w_C\| < c \ |C|^3$ (for some constant $c > 0$);
3. $w_C$ is computable from $C$ in polynomial time, in terms of $|C|$.

Proof. The proof appears in [5, Theorem 4.12], where we can skip the last step, since $C$ is already strictly layered. □

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Now the circuit value problem for strictly layered circuits (which is \(P\)-complete) can be reduced to the evaluation problem of \(V\) over \(\Gamma_1 \cup \tau\) as follows:

For \((C, x, y)\), where \(C\) is a strictly layered acyclic circuit, and \(x, y \in \{0, 1\}^*\), consider \((w_C, 0x, 0yx) \in (\Gamma_1 \cup \tau)^* \times \{0, 1\}^* \times \{0, 1\}^*\). By Theorem 6.4, we have:
\[
C(x) = y \iff w_C(0x) = 0yx.
\]
The function \((C, x, y) \mapsto (w_C, 0x, 0yx)\) is a many-one polynomial-time reduction, by Theorem 6.4.

The proof of [5, Theorem 4.12] also shows that \(w_C(0x)\) is defined when the generators in \(w_C\) are applied to \(0x\); i.e., \(0x\) is a long data input for \(w_C\).

Finally, let us prove that the reduction \((C, x, y) \mapsto (w_C, 0x, 0yx)\) is actually a many-one log-space reduction (and not just polynomial-time). For this, we review the proof of [5, Theorem 4.12], except that we skip the parts where the circuit is transformed into a strictly layered circuit. For \(1 < j\), let
\[
\sigma_{1,j} = \tau_{j-1} \tau_{j-2} \tau_{j-1} \ldots \tau_{2} \tau_{1} \cdot (i);
\]
in other words, \(\sigma_{1,j}\) is the cyclic shift \(a_1 a_{j-1} \ldots a_2 a_1 \mapsto a_1 a_j a_{j-1} \ldots a_2\).

Let \(C\) be a strictly layered circuit with \(L\) layers, with data input \(x_1 \ldots x_m \in \{0, 1\}^*\), and data output \(y_1 \ldots y_n \in \{0, 1\}^*\). Let \(C_\ell\) be the circuit consisting of layer \(\ell\) (for \(1 \leq \ell \leq L\)). The output of \(C_\ell\) is the bitstring \(Y^\ell\); its input is \(Y^{\ell-1}\). The circuit \(C_\ell\) is simulated by the word \(w_{C_\ell}\) over \(\Gamma_1 \cup \tau\); the function represented by \(w_{C_\ell}\) is \(\Phi_{C_\ell} : 0Y^{\ell-1} \rightarrow 0Y^{Y^{\ell-1}}\).

The word \(w_{C_\ell}\) is constructed by simulating each gate in \(C_\ell\). E.g., if \(w_{C_\ell}\) contains an AND gate in position \(i + 1\) (counting the gates from left to right), with input variables \(x'_{i+1}, x'_{i+2}\) (in \(Y^{\ell-1}\)) and output variable \(y'_{j+1}\) (in \(Y^\ell\)), then
\[
w_{C_\ell} = u_{(C_\ell, > i+1)}(i, j) \sigma_{1,j+2} \tau_{2} \tau_{3} \cdots \tau_{2m} \varphi \cdot \tau_{3} \tau_{5} \cdots \tau_{2m} \varphi \cdot f \cdot u_{(C_\ell, < i+1)}(i, j).
\]
Here \(u_{(C_\ell, > i+1)}(i, j) \in (\Gamma_1 \cup \tau)^*\) simulates the gates to the right of gate number \(i + 1\), and \(u_{(C_\ell, < i+1)}(i, j)\) simulates the AND-gate, and \(\varphi\) simulates the FORK-gate (see [5, following Lemma 4.11]). For the other types of gates (namely OR, NOT, and FORK), the representations by a word over \(\Gamma_1 \cup \tau\) is similar (see [5, proof of Thm. 4.12]).

The entire circuit \(C\) is simulated by the word
\[
w_C = \pi_2 (w_{C_{L-1}} \ldots w_{C_1})^{-1} \pi_1 w_{C_L} w_{C_{L-1}} \ldots w_{C_1},
\]
representing the function
\[
\Phi_C : 0x_1 \ldots x_m \mapsto 0y_1 \ldots y_n x_1 \ldots x_m.
\]
Here, \(\pi_1 = (\sigma_{1,|Z|})^n\), where \(Z = y_1 \ldots y_n Y^{L-1} \ldots Y^{Y^1} x_1 \ldots x_m\), so \(|Z| = 1 + n + m + \sum_{\ell=0}^{L-1} |Y^\ell|\) (i.e., 1 plus the input length, plus the sum of the output lengths of all the layers). And \(\pi_2 = (\sigma_{1,n+m})^m\).

To compute \(w_C\) from \(C\), a Turing machine just needs to keep track of positions inside the circuit \(C\) (namely the layer \(\ell\), and the position within the layer). This can be done in log space.

This is straightforward to see for \(\pi_2\), since from \(C\) one can directly find the number \(m\) of input wires, and the number \(n\) of output wires.

To output \((w_{C_{L-1}} \ldots w_{C_1})^{-1} = (w_{C_1}^{-1} \ldots w_{C_{L-1}}^{-1})\), the Turing machine can compute each \(w_{C_\ell}\) (for \(\ell = L - 1, \ldots, 2, 1\)). The subcircuit \(C_\ell\) is directly obtained from \(C\); and \(w_{C_\ell}\) is the concatenation of the representations of all the gates occurring in \(C_\ell\), from left to right. E.g., for an AND gate at position \(i + 1\) within \(C_\ell\) with output wire at position \(j + 1\), the representation of the gate (seen above) is \(\sigma_{1,j+2} \tau_{3} \tau_{5} \cdots \tau_{2m} \varphi \cdot \tau_{3} \tau_{5} \cdots \tau_{2m} \varphi \cdot f\); this can be computed in log-space, based on \(i\) and \(j\).

To output \(\pi_1\), the Turing machine needs \(|Z|\), which is a number obtained from the size of \(C\), hence in log-space. Then \(\pi_1 = (\sigma_{1,|Z|})^n\) can be directly found.

Finally, \(w_{C_1} w_{C_{L-1}} \ldots w_{C_1}\) is found in a similar way as \((w_{C_{L-1}} \ldots w_{C_1})^{-1}\).

From \(w_C = \pi_2 (w_{C_{L-1}} \ldots w_{C_1})^{-1} \pi_1 w_{C_1} w_{C_{L-1}} \ldots w_{C_1}\), a log-space Turing machine can obtain a word over \(\Gamma_1 \cup \tau\), by eliminating inverses; this mainly involves reordering the word (since the trans-
positions in \( \tau \) are their own inverses). Recall that we assume that \( \Gamma_1^{-1} = \Gamma_1 \). Recall that the composite of log-space computable functions is log-space computable [16] Lemma 13.3. \( \square \)

7 Reduction of evaluation problems to word problems

7.1 Reduction to the monoid word problem

The circuit value problem is easily reduced to the equivalence problem of circuits. A monoid version \( M_{2,1} \) of the Thompson group \( G_{2,1} \) was defined in \([3,4]\). The evaluation problem for the Thompson monoid \( M_{2,1} \) (over a finite generating set \( \Gamma_M \) or a circuit-like generating set \( \Gamma_M \cup \tau \)) can be reduced to the word problem of \( M_{2,1} \) (\( \Gamma_M \), respectively \( \Gamma_M \cup \tau \)), as follows.

For any strings \( u,v \in \{0,1\}^* \) let \( [v \leftarrow u](.) \) denote the element of \( M_{2,1} \) with table \( \{(u,v)\} \). In particular, \([u \leftarrow u] = \mathrm{id}_{u(0,1)} \), i.e. the identity function restricted to \( u \{0,1\}^* \). And \([\varepsilon \leftarrow u](.) \) is the pop \( u \) operation (erasing a prefix \( u \), and undefined on inputs that do not have \( u \) as a prefix); and \([v \leftarrow \varepsilon](.) \) is the push \( v \) operation (prepending a prefix \( v \) to any input string).

Suppose that \( u = u_1 \ldots u_m, v = v_1 \ldots v_n \), where \( u_1, \ldots, u_m, v_1, \ldots, v_n \in \{0,1\} \). To express all functions of the form \([v \leftarrow u](.) \) over a finite generating set, we observe that:

\[
[v \leftarrow u](.) = [v \leftarrow \varepsilon] \cdot [\varepsilon \leftarrow u](.), \\
[\varepsilon \leftarrow u](.) = [\varepsilon \leftarrow u_m] \cdot [\varepsilon \leftarrow u_{m-1}] \cdot \ldots \cdot [\varepsilon \leftarrow u_1](.), \\
[v \leftarrow \varepsilon](.) = [v_1 \leftarrow \varepsilon] \cdot [v_2 \leftarrow \varepsilon] \cdot \ldots \cdot [v_n \leftarrow \varepsilon](.).
\]

Thus all functions of the form \([v \leftarrow u](.) \) are expressed over the set of four generators \( \{[0 \leftarrow \varepsilon], [1 \leftarrow \varepsilon], [\varepsilon \leftarrow 0], [\varepsilon \leftarrow 1]\} \).

For all \( w \in (\Gamma_M \cup \tau)^* \) and \( x,y \in \{0,1\}^* \) we have:

\[
E_w(x) = y \quad \text{iff} \quad E_w \cdot \mathrm{id}_x(.) = [y \leftarrow x](.).
\]

So we have proved:

**Proposition 7.1** The evaluation problem of the monoid \( M_{2,1} \) over \( \Gamma_M \) (or over \( \Gamma_M \cup \tau \)) reduces to the word problem of \( M_{2,1} \) over \( \Gamma_M \) (respectively over \( \Gamma_M \cup \tau \)) by a many-one log-space reduction. \( \square \)

7.2 Reduction of the evaluation problem to the word problem of \( V \) over more general generating sets

We saw already in Theorem \([4,1]\) that the evaluation problem of \( V \) over \( \Gamma_1 \cup \tau \) (and of \( 2V \) over \( \Gamma_2 \)) reduces to the word problem of \( V \) over \( \Gamma_1 \cup \tau \) (respectively \( 2V \) over \( \Gamma_2 \)). More generally, for \( V \) we want to find a reduction from the evaluation problem to the word problem over a finite generating set \( \Gamma_1 \), or over any other generating set containing \( \Gamma_1 \) (Theorem \([4,2]\)).

Below, if \( H \) is a subgroup of \( V \) and \( g \in V \), then \( g \cdot H \) denotes the left coset \( \{g \cdot h : h \in H\} \); and \( H \cdot g \) is the right coset.

**Theorem 7.2** (commutation test for the evaluation problem).

(1) For any \( g \in V \) and \( x,y \in \{0,1\}^* \) we have:

\[
g(x) = y \quad \text{iff} \quad g \cdot \mathrm{pFix}_V(x) = \mathrm{pFix}_V(y) \cdot g.
\]

(2) For any \( x,y \in \{0,1\}^* \), let \( \Gamma_x, \Gamma_y, \Gamma_\mathcal{F}, \Gamma_\mathcal{P} \) be respectively generating sets of \( \mathrm{pFix}_V(x), \mathrm{pFix}_V(y), \mathrm{pFix}_V(\mathcal{F}), \) \( \mathrm{pFix}_V(\mathcal{P}) \). For any \( g \in V \) and \( x,y \in \{0,1\}^* \) we have:

\[
g(x) = y \quad \text{iff} \quad (\forall \alpha \in \Gamma_x) (\forall \beta \in \Gamma_y) ([\delta g \alpha g^{-1} = g \alpha g^{-1} \delta] \quad \text{and} \quad (\forall \gamma \in \Gamma_\mathcal{P}) (\forall \gamma \in \Gamma_\mathcal{F}) [\gamma g^{-1} \beta g = g^{-1} \beta g \gamma]).
\]

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The reduction of the evaluation problem to the word problem uses two commutation tests:

- The first commutation test, for testing membership in a partial fixator (Theorem 5.8): $g \in \text{pFix}_V(P)$ iff $g$ commutes with every element (or every generator) of $\text{pFix}_V(Q)$.
- The second commutation test, for testing an evaluation relation (Theorem 7.2): $g(x) = y$ iff $g \cdot \text{pFix}_V(x) = \text{pFix}_V(y) \cdot g$.

The latter can be reformulated in terms of the generators of the two partial fixators.
8 Appendix

8.1 Evaluation problem for elements of V and 2V given by tables.

We can consider another version of the evaluation problem for V, where every element \( \varphi \) of V are given by a table, instead of a word over a generating set of V. A table is of the form \( \{(u^{(i)}, v^{(i)}): i = 1, \ldots, m\} \), which is a bijection between two finite maximal prefix codes \( \{u^{(i)}: i = 1, \ldots, m\}, \{v^{(i)}: i = 1, \ldots, m\} \subseteq \{0,1\}^* \). The concepts of long and short data inputs also arises for tables, as follows.

A string \( x \in \{0,1\}^* \) is a long data input for the above table iff \( u^{(i)} \leq_{\text{pref}} x \) for some \( i = 1, \ldots, m \). And \( x \) is a short data input for the above table iff \( x = \varepsilon \in \text{Dom}(\varphi) \), but \( x \) is not long.

The evaluation problem of V given by tables is as follows.

**INPUT:** A finite set of pairs \( \{(u^{(i)}, v^{(i)}): i = 1, \ldots, m\} \subseteq \{0,1\}^* \times \{0,1\}^* \), and \( x, y \in \{0,1\}^* \).

**QUESTION 1 (GENERAL PROBLEM):** Is this set of pairs a table for an element of V, and is \( \varphi(x) = y \)?

**QUESTION 2 (FOR LONG DATA INPUTS):** Is this set of pairs a table for an element of V, and is \( x \) a long data input for that table, and is \( \varphi(x) = y \)? Equivalently, the set of pairs a table, and do there exist \( (u_j, v_j) \) in the table and \( x_j, y_j \in \{0,1\}^* \) such that \( x = u_j x_j \) and \( y = v_j y_j \)?

The general evaluation problem with the empty string as data input and output (i.e., \( x = \varepsilon = y \)) is equivalent to the identity problem, where the input is a table, and the question is whether that table represents the identity element of V. This problem is easy to solve, since \( \{(u^{(i)}, v^{(i)}): i = 1, \ldots, m\} \) represents the identity iff \( u^{(i)} = v^{(i)} \) for \( i = 1, \ldots, m \). This also requires checking whether \( \{u^{(i)}: i = 1, \ldots, m\} \) is a maximal prefix code. The general evaluation problem and the evaluation problem with long data inputs are certainly in \( \mathbb{P} \).

Similarly, an element of \( 2V \) could be given by a table that describes a bijection between finite maximal joinless codes. Just as for V, this leads to a general evaluation problem and the evaluation problem with long data inputs. These problems are easily seen to be in \( \mathbb{P} \).

8.2 The evaluation problem of the Thompson monoid \( M_{2,1} \)

The evaluation problem and the concept of long and short data inputs, apply to \( M_{2,1} \) in the same way as for \( G_{2,1} \). We saw in Prop. 7.2 that the evaluation problem for \( M_{2,1} \) reduces to the word problem of \( M_{2,1} \); this holds for finite generating sets \( \Gamma_M \) and for circuit-like generating sets \( \Gamma_M \cup \tau \).

Similarly to Prop. 4.3(1), the word problem reduces to the evaluation problem with data input and output \( \varepsilon \), for finite generating sets \( 
\frac{4}{\text{Prop. 4.3(3)}} \) for \( M_{2,1} \).

For long data inputs, the evaluation problem of \( M_{2,1} \) is in \( \text{DCF} \) over a finite generating set \( \Gamma_M \), and it is \( \mathbb{P} \)-complete over \( \Gamma_M \cup \tau \).

For \( M_{2,1} \) over \( \Gamma_M \cup \tau \) the word problem is \( \text{coNP} \)-complete. For \( M_{2,1} \) over a finite generating set \( \Gamma_M \), the word problem is in \( \mathbb{P} \) (see [4]), but it is still open whether it is \( \mathbb{P} \)-complete.

All these problems can also be considered in the \( 2M_{k,1} \), the monoid generalization of the Brin-Higman-Thompson group (see [7]).

8.3 Essential initial factor codes for defining ideal morphisms

We usually define elements of \( nV \) by tables that use finite maximal joinless codes, because such tables (when bijective) always define elements of \( nV \). On the other hand, for \( n \geq 2 \), tables based on finite essential initial factor codes do not always define elements of \( nV \), as was proved in Prop. 3.2.

Nevertheless, since the unique maximum extension of a right ideal morphism typically requires a table with essential initial factor codes, we also need to consider such tables.
Here we prove that one can decide in log-space whether a table based on initial factor codes defines an element of \( n^V \). The concept of complementary initial factor code is very useful for this, and will be introduced first.

**Definition 8.1** Two finite initial factor codes \( P, Q \subseteq nA^* \) are complementary iff

1. \( P(nA^*) \cap Q(nA^*) = \emptyset \), and
2. \( P \cup Q \) is an essential initial factor code.

If \( P \subseteq nA^* \) is essential then \( \emptyset \) is the unique complementary initial factor code of \( P \). On the other hand, the complementary initial factor codes of \( \emptyset \) are all the essential initial factor codes, e.g., \( \{\varepsilon\}^n \).

**Notation:**

\[ A_\varepsilon = \bigcup_{i=1}^n \{\varepsilon\}^{i-1} \times A \times \{\varepsilon\}^{n-i} \] (this is the minimum generating set of the monoid \( nA^* \)).

For every set \( Q \subseteq nA^* \),

- \( \text{init}(Q) = \{ z \in nA^* : (\exists q \in Q)[z \leq \text{init} q]\} \) (i.e., the set of initial factors of elements of \( Q \));
- \( \text{Sinit}(Q) = \text{init}(Q) \setminus Q \) (i.e., the set of strict initial factors of elements of \( Q \)).

**Proposition 8.2** (complementary initial factor code).

Every finite initial factor code \( P \subseteq nA^* \) has a (usually non-unique) complementary finite initial factor code \( P' \subseteq nA^* \). Moreover, \( P' \) can be chosen so that

- \( |P'| \leq (|A| - 1) \sum_{p \in P} |p| \),
- \( \text{maxlen}(P') = \text{maxlen}(P) \), and
- \( P' \) can be computed from \( P \) in log-space (if \( P \) is given as a finite list of \( n \)-tuples of strings).

**Proof.** We define the set

\[ P^* = \{ \alpha s : s \in \text{Sinit}(P), \alpha \in A_\varepsilon, \{\alpha\} \cup P = \emptyset \} \]

The set \( P^* \) is not necessarily an initial factor code (see the example after this proof), so we let

\[ P' = \text{max}_{\leq \text{init}}(P^*) \],

i.e., \( P' \) consists of the \( \leq \text{init} \)-maximal elements of \( P^* \).

We can compute \( P' \) from \( P^* \) in log-space by eliminating all elements of \( P^* \) that have another element of \( P^* \) as an initial factor. In other words, \( P' \) is the unique maximal initial factor code that is contained in \( P^* \). The formulas for \( P^* \) and \( P' \) immediately imply log-space computability of \( P^* \) and \( P' \), and the formulas for the cardinality \( |P'| \) and for \( \text{maxlen}(P') \).

Let us check that \( P' \) is a complementary initial factor code of \( P \).

The condition \( \{\alpha s\} \cup P = \emptyset \) in the formula for \( P^* \) immediately implies \( P nA^* \cap nA^* = \emptyset \).

Since \( P' \subseteq P^* \), we also have \( P nA^* \cap P' nA^* = \emptyset \).

We prove next that \( P \cup P^* \) is essential. This follows if we show that for every \( x = (x^{(1)}, \ldots, x^{(n)}) \in nA^\ell \) where \( \ell = \text{maxlen}(P) + 1 \): If \( x \not\in P(nA^* \cup P(nA^*) \), then \( x \in P^*(nA^*) \).

Since \( x \not\in P(nA^*) \), and \( |x| = \ell > \text{maxlen}(P) \) for all \( h \in \{1, \ldots, n\} \), it follows that for every \( p \in P \) there exists \( i \in \{1, \ldots, n\} \) such that we have: the longest common prefix of \( x^{(i)} \) and \( p^{(i)} \) is a strict prefix of \( p^{(i)} \) (in \( A^* \)). This prefix is of course also a strict prefix of \( x^{(i)} \), since \( |x^{(i)}| = \ell \). We now consider a maximally long such prefix, i.e., a maximally long string in

\[ \{ r \in A^* : i \in \{1, \ldots, n\}, p \in P, r \leq_{\text{pref}} x^{(i)}, r <_{\text{pref}} p^{(i)} \} \].

Let \( j \in \{1, \ldots, n\} \) and \( q \in P \) be a choice of values in \( \{1, \ldots, n\} \) respectively \( P \) where this maximum is reached. Then a maximally long \( r \) is of the form \( r = x_1^{(j)} \ldots x_m^{(j)} <_{\text{pref}} p^{(j)}, m = |r| < |p^{(j)}| < \ell \),

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where $x^{(j)} = x_1^{(j)} \ldots x_m^{(j)} x_{m+1}^{(j)} \ldots x_{\ell}^{(j)}$. Then $x_1^{(j)} \ldots x_m^{(j)} x_{m+1}^{(j)}$ is a prefix of $x^{(j)}$ that (by maximality of the length $m$) is not a prefix of $p^{(j)}$ for any $p \in P$.

For every $i \in \{1, \ldots, n\}$, let $s^{(i)}$ be the longest common prefix of $x^{(i)}$ and $q^{(i)}$; for $i = j$ we already have $s^{(j)} = x_1^{(j)} \ldots x_m^{(j)}$. Then $s = (s^{(1)}, \ldots, s^{(j-1)}, x_1^{(j)} \ldots x_m^{(j)}, s^{(j+1)}, \ldots, s^{(n)}) <_{\text{init}} q$, hence $s \in \text{Sinit}(P)$. Moreover, $\{s\} \cup P = \emptyset$, where $\{s\} = \{\varepsilon\}^{j-1} \times \{x_m^{(j)}\} \times \{\varepsilon\}^{n-j}$; this holds since $x_1^{(j)} \ldots x_m^{(j)} x_{m+1}^{(j)}$ is not a prefix of $p^{(j)}$ for any $p \in P$. Hence, $s \in P^\#$, and $s <_{\text{init}} x$; thus, $x \in P^\#(nA^*)$.

From the fact that $P \cup P^\#$ is essential it follows that $P \cup P^\#$ is essential, since every element of $P^\#$ has a prefix in $P'$. I.e., $P^\# \subseteq P'(nA^*)$, so $P^\#(nA^*) \subseteq P'(nA^*)$. □

**Example where $P^\#$ is not an initial factor code:**

Let $p = (11,00) \in 2A^*$ with $A = \{0,1\}$ and let $P = \{p\}$. Let $s = (1,0)$ and $t = (1,00)$; so $s, t \in \text{Sinit}(P)$. Let $\alpha = (0,\varepsilon) \in A_\varepsilon$. Now $s\alpha = (10,0)$ and $t\alpha = (10,00)$. Hence $s\alpha \not< p$ and $t\alpha \not< p$ do not exist, so $s\alpha, t\alpha \in P^\#$. But $s\alpha <_{\text{init}} t\alpha$; so $P^\#$ is not an initial factor code in this example.

**Proposition 8.3** It can be decided in log-space whether a finite initial factor code $P \subseteq nA^*$ (given as a finite list of $n$-tuples of strings) is essential.

**Proof.** By Prop. 8.2, a complementary initial factor code $P'$ of $P$ can be computed in log-space. And $P$ is essential iff $P' = \emptyset$. □

**Proposition 8.4** Each one of the following questions can be decided in log-space.

**Input:** A function $F : P \to Q$ from a finite initial factor code $P$ onto a finite initial factor code $Q$ (where $P$ and $Q$ are given as finite lists of $n$-tuples of strings). Let $f : P(nA^*) \to Q(nA^*)$ be the right ideal “morphism” determined by the table $F$.

**Question 1:** Is $f$ a function?

**Question 2:** Is $f$ injective?

**Question 3:** Is $f$ total?

**Question 4:** Is $f$ surjective?

**Question 5:** Does $F$ define an element of the Brin-Higman-Thompson group $nG_{|A|,1}$?

**Proof.** (Q1) For every pair $p, p' \in P$ such that $p \neq p'$ and such that $p \lor p'$ exists, let $p \lor p' = pu = p'v$. If $f(p) u \neq f(p') v$ then $f$ is not a function. If after checking all pairs $p, p'$ as above, no inequality was found, then $f$ is a function.

(Q2) For this we check, as in (Q1), whether the inverse table of $F^{-1} : Q \to P$ is a function.

(Q3) We check whether $P$ is essential, using Prop. 8.3.

(Q4) We check whether $Q$ is essential, using Prop. 8.3.

(Q5) The table $F$ defines an element of $nG_{|A|,1}$ iff $F$ yields a yes answer for all of the above questions. □

**References**

[1] J.C. Birget, “The groups of Richard Thompson and complexity”, *International J. of Algebra and Computation* 14(5, 6) (Dec. 2004) 569-626. (arxiv:math/0204292v2)

[2] J.C. Birget, “Circuits, coNP-completeness, and the groups of Richard Thompson”, *International J. of Algebra and Computation* 16(1) (Feb. 2006) 35-90. (arxiv:math/0310333)
