State monads and their algebras

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Abstract

State monads in cartesian closed categories are those defined by the familiar adjunction between product and exponential. We investigate the structure of their algebras, and show that the exponential functor is monadic provided the base category is sufficiently regular, and the exponent is a non-empty object.

1 Introduction

Let $C$ be a cartesian closed category. Any object $S$ in $C$ gives rise to a monad $T = UF$, where

$$U : X \mapsto X^S$$

and

$$F : X \mapsto S \times X$$

$T$, among other monads, has been extensively studied by researchers in functional programming, in order to model computational effects (see for example [3, 4, 6]), and called a state monad since then. A main source of the present work is [5], which investigates the algebras of state monads, in a significantly different setting however, and provides an explicit equational presentation of these algebras.

This raises the question as whether the original functor $U$ itself is monadic: it turns out that the answer is positive, provided $C$ is sufficiently regular, and $S$ is non-empty (Theorem 2).

1.1 Notations

Throughout this article, $S$ will denote the fixed object of $C$, on which $T$ is built. The natural isomorphism

$$\text{Hom}_C(S \times X, Y) \to \text{Hom}_C(X, Y^S)$$

will be denoted by

$$f \mapsto f^*$$

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and its inverse by
\[ g \mapsto g^* \]
In particular, for each \( X \), the identity
\[ \text{id}_X^S : X^S \to X^S \]
gives rise to
\[ (\text{id}_X^S)^* : S \times X^S \to X \]
which is the evaluation morphism \( \langle s, f \rangle \mapsto f[s] \), and will be denoted by \( \epsilon_X \). It is of course the counit of the adjunction. Finally, for each object \( X \) in \( C \), the
projections \( S \times X \to S \) and \( S \times X \to X \) will be denoted by \( p_X, q_X \) respectively.

Note that \( p_X, q_X \) are natural transformations \( F \to 1 \).

## 2 The case of Sets

We suppose in this section that \( C \) is the category of sets. Let us first recall the concrete meaning of \( (T, \mu, \eta) \) in this case. For each set \( X \), \( TX = (S \times X)^S \), so that the unit \( \eta_X \) associates to each \( x \in X \) the map \( \lambda s \langle c[s], x \rangle \), an element of \( TX \); as for \( \mu_X \), it takes an argument of the form
\[ \lambda s \langle c'[s, s'], f[s, s'] \rangle \]
in \( TTX \), and outputs
\[ \lambda s \langle c'[s, c[s]], f[s, c[s]] \rangle \]
which belongs to \( TX \). In other words, \( \mu_X \) is \( \epsilon_{S \times X}^S \). Recall that \( T \)-algebras are pairs \( \langle X, h \rangle \) where \( X \) and \( h : TX \to X \) are such that the following diagrams commute:
\[
\begin{array}{ccc}
T^2X & \xrightarrow{Th} & TX \\
\downarrow{\mu_X} & & \downarrow{h} \\
TX & \xrightarrow{h} & X
\end{array} \quad \quad \begin{array}{ccc}
X & \xrightarrow{\eta_X} & TX \\
\downarrow{id_X} & & \downarrow{h} \\
X & \xrightarrow{h} & X
\end{array}
\]
which means in this case that
\[
\begin{align*}
h(\lambda s \langle c[s], h(\lambda s \langle c'[s, s'], x[s, s'] \rangle) \rangle) & = h(\lambda s \langle c'[s, c[s]], x[s, c[s]] \rangle) \quad (2) \\
h(\lambda s \langle c[s], x \rangle) & = x \quad (3)
\end{align*}
\]
A morphism of \( T \)-algebras \( \langle X, h \rangle, \langle X', h' \rangle \) is a map \( f : X \to X' \) making the following diagram commutative:
\[
\begin{array}{ccc}
TX & \xrightarrow{Tf} & TX' \\
\downarrow{h} & & \downarrow{h'} \\
X & \xrightarrow{f} & X'
\end{array}
\]
Algebras and morphisms build a category $\mathbf{Alg}^T$, with an obvious forgetful functor $U^T : \langle X, h \rangle \mapsto X$.

As with any adjunction, we may define a comparison functor

$$K : C \to \mathbf{Alg}^T$$

which is given, in this case, by

$$Y \mapsto \langle Y^S, \epsilon^S \rangle$$

Call $U$ monadic if $K$ is an equivalence of categories. Note that the same terminology is often applied to a stronger notion, where $K$ is required to be an isomorphism (see [1] and [2] on this issue).

The key idea is now that, in $\mathbf{Sets}$, $T$-algebras are essentially sets of maps $Y^S$, with their evaluation morphisms. Theorem 1 below is a precise formulation of this remark.

**Theorem 1** If $S \neq \emptyset$, then $U$ is monadic.

**Proof.** Beck’s criterion ([1]) applies, proving the statement. However the situation is better understood by a direct study of the comparison functor.

In order to show that $K$ is an equivalence, we define a functor

$$L : \mathbf{Alg}^T \to \mathbf{Sets}$$

such that $KL$ and $LK$ are naturally isomorphic to the identity functor on $\mathbf{Alg}^T$ and $\mathbf{Sets}$ respectively. Let $\langle X, h \rangle$ be a $T$-algebra, we define the set

$$Y = L \langle X, h \rangle$$

as follows. From the projection

$$q_{S \times X} : S \times (S \times X) \to S \times X$$

we get

$$q^*_S \times X : S \times X \to (S \times X)^S = TX$$

hence

$$\phi = h \circ q^*_S \times X : S \times X \to X$$

Then $Y = L \langle X, h \rangle$ will be the image of $\phi$. Concretely, $Y$ is the subset of $X$ given by

$$Y = \{h(\lambda s' \langle s, x \rangle)) | s \in S, x \in X\}$$

and we get a surjective map $e_X : S \times X \to Y$ making the following diagram commutative:

$$\begin{array}{ccc}
S \times X & \xrightarrow{q^*_S \times X} & TX \\
\downarrow{e_X} & & \downarrow{h} \\
Y & \xrightarrow{m_X} & X
\end{array}$$
where $m_X$ is the inclusion monomorphism (in fact $e_X$ and $m_X$ really depend on the algebra $(X,h)$). Now $L$ has to be defined on morphisms as well: let then $u$ be a morphism of algebras $u: (X,h) \to (X',h')$, $Y = L(X,h)$ and $Y' = L(X',h')$. There is a unique map $Lu: Y \to Y'$ such that the following diagram commutes:

\[
\begin{array}{c}
S \times X \xrightarrow{S \times u} S \times X' \\
\downarrow e_X \quad \quad \downarrow e_{X'} \\
Y \xrightarrow{Lu} Y'
\end{array}
\]

Display $e_X$ as a cokernel

\[
\begin{array}{c}
Z \xrightarrow{a} S \times X \xrightarrow{e_X} Y \\
\downarrow b \quad \quad \downarrow
\end{array}
\]

and fit the above diagram into this larger one:

The inner square commutes because $u$ is a morphism of algebras, the top trapezoid commutes by naturality of $q^*$, and the left and right hand trapezoids commute by definition of $e$, $m$. Hence

\[
m_{X'} \circ e_{X'} \circ (S \times u) \circ a = u \circ m_X \circ e_X \circ a = u \circ m_X \circ e_X \circ b = m_{X'} \circ e_{X'} \circ (S \times u) \circ b
\]

and $m_{X'}$ being a monomorphism,

\[
e_{X'} \circ (S \times u) \circ a = e_{X'} \circ (S \times u) \circ b
\]

so that $e_{X'} \circ (S \times u)$ equalizes the pair $a$, $b$. As $e_X$ is a cokernel, there is a unique $Lu$ making the outer square commutative, as required. As usual, uniqueness of $Lu$ ensures functoriality. Note that $e$ is a natural transformation:

\[
e: FU^T \to L
\]
As a consequence, the adjunction gives a natural transformation from $U^T$ to $UL$, or better, as $U = U^T K$
\[ e^*: U^T \to U_T KL \]
Thus we reduced the existence of a natural isomorphism:
\[ KL \simeq 1 \]
to the following statements:
- for all $\langle X, h \rangle$, $e^*_X$ is a bijection;
- for all $\langle X, h \rangle$, $e^*_X$ is a morphism of algebras.
which will be proved separately in Lemmas 1 and 2 below.

Finally, there is a natural isomorphism:
\[ LK \simeq 1 \]
If $Y$ is a set, $LKY$ is nothing but the subset of $Y^S$ consisting of constant maps from $S$ to $Y$, which is naturally isomorphic to $Y$ because $S \neq \emptyset$.

The two following lemmas will complete our argument. Keeping the previous notations,

**Lemma 1** For each $T$-algebra $\langle X, h \rangle$, the map $e^*_X$ is a bijection.

**Proof.** Let us give an explicit construction of the inverse map $f$. For each $y \in Y^S$, we define $f(y) \in X$ by
\[ f(y) = h(\lambda s \langle s, y[s]\rangle) \]
Now, for each $x \in X$,
\[ f(e_X^*(x)) = h(\lambda s \langle s, h(\lambda s' \langle s, x\rangle)\rangle) \]
\[ = h(\lambda s \langle s, x\rangle) \text{ by \( 2 \)} \]
\[ = x \text{ by \( 3 \)} \]
Thus $f$ is a retraction for $e_X^*$.

On the other hand, let $y \in Y^S$,
\[ e_X^*(f(y)) = \lambda sh(\lambda s' \langle s, f(y)\rangle) \]
\[ = \lambda sh(\lambda s' \langle s, h(\lambda s'' \langle s'', y[s'']\rangle)\rangle) \]
\[ = \lambda sh(\lambda s' \langle s, y[s]\rangle) \text{ by \( 2 \)} \]
but $y[s] = h(\lambda s'' \langle c[s], x[s]\rangle)$ for suitable maps $c$ and $x$, by definition of $Y$, so that the last expression reduces to
\[ \lambda sh(\lambda s' \langle c[s], x[s]\rangle) = \lambda sy[s] = y \]
by \( 2 \) again. Thus $e_X^*(f(y)) = y$, and $f$ is also a section of $e_X^*$. Hence $f = (e_X^*)^{-1}$, as required. $\Box$
Lemma 2  For each $T$-algebra $\langle X, h \rangle$, $e_X^*$ is a morphism of algebras.

Proof. Let us prove the commutativity of the following diagram:

$$
\begin{array}{c}
TX \\
\downarrow h \\
X
\end{array} 
\xrightarrow{T e_X^*} 
\begin{array}{c}
T(Y^S) \\
\downarrow e_Y^* \\
Y^S
\end{array}
$$

Let $u \in TX$. It is a $\lambda s \langle c[s], x[s] \rangle$ for a certain pair of maps $c$, $x$. Hence

$$
e_X^* \circ h(u) = \lambda s' h(\lambda s'' \langle s', h(\lambda s \langle c[s], x[s] \rangle) \rangle) = \lambda s' h(\lambda s'' \langle c[s'], x[s'] \rangle) \quad \text{by (2)}$$

On the other hand

$$
e_Y^* \circ T e_X^*(u) = e_Y^*(\lambda s \langle c[s], \lambda s' h(\lambda s'' \langle s', x[s] \rangle) \rangle) = \lambda s h(\lambda s'' \langle c[s], x[s] \rangle)$$

so that

$$
e_X^* \circ h = e_Y^* \circ T e_X^*$$

3  A general theorem

By carefully examining the proofs in section 2 we see that the equations (2) and (3) are only used in proving Lemmas 1 and 2. Thus large parts of our argument still hold beyond Sets. As we shall see, a key hypothesis for generalizing the previous results is that $C$ has regular epi-mono factorization, that is, each arrow in $C$ factorizes as a composition of a regular epimorphism and a monomorphism. The regularity hypothesis implies that such a factorization is unique, up to isomorphism. Thus from now on, we assume that $C$ has regular epi-mono factorization.

3.1 Construction of $L$

Let us first generalize the construction of

$$L : \text{Alg}^T \rightarrow C$$

If $C$ is any cartesian closed category, and $\langle X, h \rangle$ is a $T$-algebra, the morphism $f_X = h \circ q_{S \times X}$ is still defined:

$$
\begin{array}{c}
S \times X \xrightarrow{q_{S \times X}} TX \\
\downarrow h \\
X
\end{array}
$$
so that all we need to extend the previous definition of \( L \) is the existence of an image object for \( f_X \), which immediately follows from the regular epi-mono factorization property on \( C \).

Thus we get a regular epimorphism \( e_X \) and a monomorphism \( m_X \) such that

\[ f_X = m_X \circ e_X \tag{4} \]

and \( L (X, h) \) can be defined as the object \( Y \) in the commutative diagram

\[
\begin{array}{ccc}
S \times X & \xrightarrow{q_{S \times X}} & TX \\
| & e_X | & | \downarrow h \\
Y & \xrightarrow{m_X} & X
\end{array}
\]

Now the regularity of \( e_X \) suffices to make \( L \) a functor, and \( e \) a natural transformation:

\[ e : FU^T \to L \]

as already shown in the proof of Theorem 1. Whence a natural transformation

\[ e^* : U^T \to U^TKL \]

### 3.2 A retraction

**Proposition 1** For each \( T \)-algebra \( (X, h) \), \( e_X^* : X \to Y^S \) has a retraction.

**Proof.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
S \times X & \xrightarrow{q_{S \times X}} & TX \\
| & e_X | & | \downarrow f_X \\
Y & \xrightarrow{m_X} & X
\end{array}
\]

The left lower triangle gives rise by adjunction to the following commutative triangle:

\[
\begin{array}{ccc}
X & \xrightarrow{f_X^*} & Y^S \\
| e_X^* | & | \downarrow m_X^\mathcal{S} \\
Y^S & \xrightarrow{m_X^\mathcal{S}} & X^S
\end{array}
\]

If \( f_X^* \) has a retraction \( r \), then \( r \circ m_X^\mathcal{S} \) is immediately a retraction for \( e_X^* \). Thus we turn to the construction of such a retraction for \( f_X^* \). The upper right triangle of (5) gives rise again by adjunction to the following commutative triangle:

\[
\begin{array}{ccc}
X & \xrightarrow{(q_{S \times X})^*} & (TX)^S \\
| f_X | & | \downarrow h^S \\
X^S & \xrightarrow{h^S} & X^S
\end{array}
\]
Let us introduce a natural transformation \( \theta \) between \( U \) and \( T \). For each object \( Z \), we first have

\[
\langle p_{Z^S}, \epsilon_Z \rangle : S \times Z^S \to S \times Z
\]

whence

\[
\theta_Z = \langle p_{Z^S}, \epsilon_Z \rangle^* : Z^S \to TZ
\]

as required. The triangle (6) then fits into a larger diagram:

\[
\begin{array}{ccc}
X & (TX)^S & TTX \\
\downarrow f_X & \downarrow \theta_X & \downarrow Th \\
X^S & TX & X \\
\end{array}
\] (7)

where \( r \) is defined by

\[
r = h \circ \theta_X
\]

Now (7) commutes: in fact (6) commutes already, the lower right triangle commutes by definition of \( r \), and the square by naturality of \( \theta \). By defining

\[
\psi = \theta_{TX} \circ (q_{S \times X}^*)^*
\]

we get

\[
r \circ f_X^* = h \circ Th \circ \psi
\]

From the definition of a \( T \)-algebra (1), we know that \( h \circ Th = h \circ \mu_X \). We claim that

\[
\mu_X \circ \psi = \eta_X
\] (8)

which implies

\[
r \circ f_X^* = h \circ Th \circ \psi = h \circ \mu_X \circ \psi = h \circ \eta_X = id_X
\]

by (1) again, and the result reduces to (8), proved in Lemma 3. \( \square \)

**Lemma 3** For each object \( X \), \( \mu_X \circ \theta_{TX} \circ (q_{S \times X}^*)^* = \eta_X \).

**Proof.** The lemma states the commutativity of the following diagram:

\[
\begin{array}{ccc}
(X^S) & (TX)^S & TTX \\
\downarrow \psi & \downarrow \theta_X & \downarrow \mu_X \\
X & TX & X
\end{array}
\] (9)
where the upper left triangle commutes by definition of $\psi$. The first step is to note that, as $\theta_{TX} = \langle p_{TX}^*, \epsilon_{TX} \rangle^*$, $\psi$ is of the form $\phi^*$, where $\phi$ makes the following triangle commute:

\[
\begin{array}{c}
S \times (TX)^S \xrightarrow{\langle p_{TX}^*, \epsilon_{TX} \rangle^*} S \times TX \\
\uparrow \phi \\
S \times X
\end{array}
\] (10)

Let us show that

\[
\phi = \langle p_X, q^*_{S \times X} \rangle
\] (11)

This amounts to check the commutativity in the product diagram:

\[
\begin{array}{c}
S \times X \xrightarrow{p_X} S \times TX \\
\phi \downarrow \quad q^*_{S \times X} \downarrow \\
S \quad TX
\end{array}
\] (12)

The left hand side commutes because of the commutativity of:

\[
\begin{array}{c}
S \times X \xrightarrow{\langle p_{TX}^*, \epsilon_{TX} \rangle^*} S \times (TX)^S \xrightarrow{\langle p_{TX}^*, \epsilon_{TX} \rangle^*} S \times TX \\
\uparrow \quad p_{TX} \quad p_{TX} \\
S \quad S
\end{array}
\] (13)

As for the right hand side, it amounts to the commutativity of:

\[
\begin{array}{c}
S \times X \xrightarrow{\langle p_{TX}^*, \epsilon_{TX} \rangle^*} S \times (TX)^S \\
q^*_{S \times X} \downarrow \quad \epsilon_{TX} \downarrow \\
TX \quad S \times TX
\end{array}
\] (14)

Now, in (14), the commutativity of the right hand side is straightforward, so we are reduced to check commutativity on the left hand side, which in turn results from:

\[
\epsilon_{TX} \circ (S \times (q^*_{S \times X})^*) = (\text{id}_{(TX)^s})_* \circ (S \times (q^*_{S \times X})^*)
\]

\[
= (\text{id}_{(TX)^s})_* \circ (q^*_{S \times X})_*
\]

\[
= ((q^*_{S \times X})^* )_* = q^*_{S \times X}
\]
This achieves the proof of (11).

Back to (9), we now prove that the right lower triangle commutes; we just proved that this triangle is in fact:

\[
\begin{array}{c}
\langle p_X, q_{S \times X}^* \rangle \\
\downarrow \eta_X \\
\end{array}
\]

\[
\begin{array}{c}
TX \\
\downarrow \mu_X \\
\end{array}
\]

As \( \mu_X = (\epsilon_{S \times X})^S \) and \( \eta_X = (\text{id}_{S \times X})^* \), (15) commutes if and only if (16) also commutes:

\[
\begin{array}{c}
S \times TX \\
\downarrow \epsilon_{S \times X} \\
S \times X \\
\end{array}
\]

\[
\begin{array}{c}
\langle p_X, q_{S \times X}^* \rangle \\
\downarrow \text{id}_{S \times X} \\
S \times X \\
\end{array}
\]

Let \( \Delta = \langle \text{id}_S, \text{id}_S \rangle : S \to S \times S \), we may express \( \langle p_X, q_{S \times X}^* \rangle \) as a composition:

\[
\langle p_X, q_{S \times X}^* \rangle = (S \times q_{S \times X}^*) \circ (\Delta \times X) \tag{17}
\]

such that the commutativity of (16) reduces to the commutativity of the following diagram:

\[
\begin{array}{c}
S \times S \times X \\
\downarrow S \times \Delta \\
S \times X \\
\end{array}
\]

\[
\begin{array}{c}
S \times TX \\
\downarrow \epsilon_{S \times X} \\
S \times X \\
\end{array}
\]

\[
\begin{array}{c}
S \times q_{S \times X}^* \\
\end{array}
\]

but the adjunction gives:

\[
\epsilon_{S \times X} \circ (S \times q_{S \times X}^*) = (\text{id}_{TX})_* \circ (S \times q_{S \times X}^*) \\
= (\text{id}_{TX} \circ q_{S \times X}^*)_* \\
= (q_{S \times X})_* \\
= q_{S \times X}
\]

Finally

\[
q_{S \times X} \circ (\Delta \times X) = \text{id}_{S \times X}
\]

and the lemma is proved. \(\square\)

### 3.3 Existence of a section

We now turn to conditions ensuring the existence of a section for \( \epsilon_X^* \). We first prove a technical lemma:
Lemma 4 Suppose \( \langle X, h \rangle \) is a \( T \)-algebra, and the epimorphism \( e_X \) has a section. Then \( e_X^* \) also has a section.

Proof. Let \( \sigma \) be a section of \( e_X \), that is
\[
e_X \circ \sigma = \text{id}_Y
\] and define
\[
\Sigma = h \circ \sigma^S
\]
we claim that \( \Sigma \) is a section of \( e_X^* \). Consider the following diagram:

\[
\begin{array}{ccc}
Y^S & \xrightarrow{\sigma^S} & TX \\
\downarrow{\Sigma} & & \downarrow{e_X^*} \\
X & \xrightarrow{(q_{S \times X})^*} & (TX)^S \\
\end{array}
\]

and note that
\[
m_X^S \circ e_X^* \circ \Sigma = m_X^S \circ e_X^* \circ h \circ \sigma^S
\]
\[
= h^S \circ (q_{S \times X})^* \circ h \circ \sigma^S
\]
\[
= h^S \circ (Th)^S \circ (q_{S \times TX})^* \circ \sigma^S
\]
\[
= h^S \circ \mu_X^S \circ (q_{S \times TX})^* \circ \sigma^S
\]
but using the adjunctions:
\[
\mu_X^S \circ (q_{S \times TX})^* = (\mu_X \circ q_{S \times TX})^*
\]
\[
= (\epsilon_{S \times TX} \circ q_{S \times TX})^*
\]
\[
= (q_{S \times X} \circ \epsilon_{S \times X})^*
\]
\[
= (q_{S \times X})^S \circ \epsilon_{S \times X}
\]
\[
= (q_{S \times X})^S \circ ((\text{id}_{TX}))^*
\]
\[
= (q_{S \times X})^S
\]
so that
\[
m_X^S \circ e_X^* \circ \Sigma = h^S \circ (q_{S \times X})^S \circ \sigma^S
\]
\[
= m_X^S \circ e_X^* \circ \sigma^S
\]
\[
= m_X^S
\]
and finally
\[
e_X^* \circ \Sigma = \text{id}_{Y^S}
\]
because \( m_X^S \) is a monomorphism (right-adjoints preserve monomorphisms).

Now \( e_X \) was defined as a regular epimorphism, which does not imply the existence of a section: think for example at the presheaf category of graphs, where all epimorphisms are regular, many of them without a section. Nevertheless, a very simple condition on \( S \) will be sufficient: recall that a (global) \emph{element} of an object \( Z \) in \( C \) is an arrow \( z : 1 \to Z \), where \( 1 \) is the terminal object, then

**Lemma 5** Suppose \( S \) has at least one element and \( \langle X, h \rangle \) is a \( T \)-algebra. Then \( e_X \) has a section.

**Proof.** Let \( s_0 : 1 \to S \) be an element of \( S \); for each object \( Z \), it gives rise to \( \gamma_Z : Z^S \to Z \) by composition of \( Z^{s_0} \) with the canonical isomorphism \( A^1 \simeq A \). Note that \( \gamma \) becomes a natural transformation and that for each \( Z \)

\[
\gamma_Z \circ q_Z^* = \text{id}_Z \tag{23}
\]

Let us now consider an algebra \( \langle X, h \rangle \), and build the following diagram:

\[
\begin{array}{cccccc}
S \times T X & \xrightarrow{\tau_{S \times T X}} & T T X \\
\downarrow{S \times h} & & \downarrow{TTX} \\
S \times X & \xrightarrow{\gamma_{S \times X}} & T X \\
\downarrow{e_X} & & \downarrow{TX} \\
Y & \xrightarrow{m_X} & X \\
\end{array}
\]

Define

\[
\sigma = \gamma_{S \times X} \circ \eta_X \circ m_X \tag{24}
\]

we claim that \( \sigma \) is a section of \( e \). To see this, we compute

\[
m_X \circ e_X \circ \sigma \circ e_X \tag{25}
\]

First, using naturality of \( \eta \), \( \gamma \) and \( q^* \):

\[
m_X \circ e_X \circ \sigma \circ e_X = h \circ q_{S \times X}^* \circ \gamma_{S \times X} \circ \eta_X \circ h \circ q_{S \times X}^* \\
= h \circ q_{S \times X}^* \circ \gamma_{S \times X} \circ Th \circ \eta_{TX} \circ q_{S \times X}^* \\
= h \circ q_{S \times X}^* \circ (S \circ h) \circ \gamma_{S \times TX} \circ \eta_{TX} \circ q_{S \times X}^* \\
= h \circ Th \circ q_{S \times TX}^* \circ \gamma_{S \times TX} \circ \eta_{TX} \circ q_{S \times X}^*
\]

then because \( h \circ Th = h \circ \mu_X \) and \( \mu_X = \epsilon_T^S \)

\[
m_X \circ e_X \circ \sigma \circ e_X = h \circ \epsilon_T^S \circ q_{S \times TX}^* \circ \gamma_{S \times TX} \circ \eta_{TX} \circ q_{S \times X}^*
\]
and by using naturality of $q_*$ and $\gamma$ again, (25) reduces to:

$$h \circ q_{S \times X}^* \circ \gamma_{S \times X} \circ \epsilon_{TX}^* \circ q_{S \times X}^* \circ \eta_{TX} \circ q_{S \times X}^*$$

Now

$$\epsilon_{TX}^* \circ \eta_X = \mu_X \circ \eta_{TX} = \text{id}_{TX}$$

and

$$\gamma_{S \times X} \circ q_{S \times X}^* = \text{id}_{S \times X}$$

so that finally

$$m_X \circ e_X \circ \sigma \circ e_X = h \circ q_{S \times X}^* = m_X \circ e_X$$

As $m_X$ is a monomorphism, and $e_X$ an epimorphism, this implies

$$e_X \circ \sigma = \text{id}_Y$$

which ends the proof.

From Lemmas 4 and 5 immediately follows

**Proposition 2** If $S$ has at least one element, then for each $T$-algebra $(X, h)$, $e_X^*$ has a section.

### 3.4 $e_X^*$ is a morphism of algebras

We show that Lemma 2 extends to the general setting without additional assumptions:

**Lemma 6** For each $T$-algebra $(X, h)$, $e_X^*$ is a morphism of algebras.

**Proof.** Let us consider the following diagram:

$$
\begin{array}{ccc}
TX & \xrightarrow{T(e_X^*)} & T(Y^S) \\
\downarrow h & & \downarrow \epsilon_X^* \\
X & \xrightarrow{e_X} & Y^S
\end{array}
\xrightarrow{T(m_X^S)}
\begin{array}{ccc}
TX & \xrightarrow{T(e_X^*)} & T(X^S) \\
\downarrow h & & \downarrow \epsilon_X^* \\
X & \xrightarrow{e_X} & X^S
\end{array}
$$

We must show that the left hand side square commutes. First, by using the adjunction, together with (1), (4) and the fact that $q^*$ is natural,

$$m_X^S \circ e_X^* \circ h = (m_X \circ e_X)^* \circ h$$

$$= (m_X \circ e_X \circ (S \times h))^*$$

$$= (h \circ q_{S \times X}^* \circ (S \times h))^*$$

$$= (h \circ \mu_X \circ q_{S \times TX}^*)^*$$

$$= (h \circ q_{S \times TX}^*)^*$$

$$= (h \circ q_{S \times TX}^*)^*$$

$$= (h \circ q_{S \times TX}^*)^*$$

$$= (h \circ q_{S \times TX}^*)^*$$
then the same ingredients plus the facts that $T$ is a functor, and $\epsilon^S$ is natural imply:

$$m^S_X \circ \epsilon^S_Y \circ T(e^*_X) = \epsilon^S_X \circ T(m^S_X) \circ T(e^*_X)$$

$$= \epsilon^S_X \circ T(m^S_X \circ e^*_X)$$

$$= \epsilon^S_X \circ T((h \circ q^s_{X \times X})^*)$$

$$= \epsilon^S_X \circ (S \times (h \circ q^s_{X \times X})^*)^S$$

$$= (\epsilon_X \circ (S \times (h \circ q^s_{X \times X})^*))^S$$

$$= ((h \circ q^s_{X \times X})^S)$$

$$= (h \circ q^s_{X \times X})^S \circ ((id_{TX})^*)$$

$$= (h \circ q^s_{X \times X} \circ (id_{TX})^*)$$

$$= (h \circ q^s_{X \times X} \circ \epsilon_{S \times X})^*$$

whence

$$m^S_X \circ e^*_X \circ h = m^S_X \circ \epsilon^S_Y \circ T(e^*_X)$$

and because $m^S_X$ is a monomorphism,

$$\epsilon^*_X \circ h = \epsilon^S_Y \circ T(e^*_X)$$

which ends the proof. □

### 3.5 Monadicity theorem

As a consequence of propositions 1 and 2, if $S$ has at least one element, then $KL$ is naturally isomorphic to the identity on $\text{Alg}_T$. On the other hand,

**Proposition 3** If $S$ has at least one element, $LK \simeq 1$.

**Proof.** Let $Y$ be an object of $C$. The algebra $KY$ is $\langle Y^S, \epsilon^S_Y \rangle$. By naturality of $q^*$, the following diagram commutes:

$$\begin{array}{ccc}
S \times Y^S & \xrightarrow{q^s_{X \times Y}} & T(Y^S) \\
\epsilon_Y \downarrow & & \downarrow \epsilon^S_Y \\
Y & \xrightarrow{q^*_Y} & Y^S
\end{array}$$

Now, $S$ has at least one element, say

$$s_0 : 1 \to S$$

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which easily provides a retraction for \( q_Y^* \), and a section for \( \epsilon_Y \). In particular, \( q_Y^* \) is a monomorphism and \( \epsilon_Y \) a split — thus regular — epimorphism. Hence there is a unique isomorphism \( \xi_Y \) making the following diagram commutative:

\[
\begin{array}{ccc}
S \times Y^S & \xrightarrow{\epsilon_Y \otimes \id} & LKY \\
\downarrow{\epsilon_Y} & & \downarrow{m_Y} \\
Y & \xleftarrow{\xi_Y} & Y^S
\end{array}
\]

Moreover, \( \xi_Y \) is natural in \( Y \), as shown by standard uniqueness arguments. □

We finally state our main result, an immediate consequence of the above discussion:

**Theorem 2** If \( C \) has regular epi-mono factorization, and \( S \) is an object of \( C \) having at least one element, then \( U : X \mapsto X^S \) is monadic.

### 4 Equations

As we pointed out in the introduction, the present work is strongly related to the analysis of global states given in [5], with a significant difference: in [5], \( C \) is any category with countable products and coproducts, and \( S \) is a countable set of states, so that \( S \times X \) (resp. \( X^S \)) now denotes the coproduct (resp. the product) of \( S \) copies of \( X \), and not as in cartesian closed categories the internal product (resp. exponential) by an object \( S \). Suppose however that \( C \) satisfies both sets of conditions, those in [5] and those of the present paper, as for example will be the case of any presheaf category: then a countable set \( S \) can be embedded as an object of \( C \), coproduct of \( S \) copies of the terminal element, and the results of [5] still make sense in our setting, and the two possible interpretations of the notations \( S \times - \) and \((-)^{S} \) coincide.

In particular, under these additional hypotheses, we may revisit the equational presentation of global state algebras. Let \( \Sigma \) be the signature consisting of a symbol \( l \) of arity \( S \), and for each \( s \in S \), a symbol \( u_s \), of arity 1; let \( E \) be the following set of equations among terms generated by \( \Sigma \):

\[
\begin{align*}
\ u_s[u_t[x]] & = u_t[x] \\
\ u_s[1[(a_t)_t]] & = u_s[a_s] \\
\ 1[(u_s[x])_s] & = x \\
\ 1[1[(a_{st})_t)_s] & = 1[(a_{ss})_s]
\end{align*}
\]

A \( (\Sigma, E) \)-algebra in \( C \) is now a pair \( \langle A, |.|_A \rangle \) where \( A \) is an object of \( C \), and \( |.|_A \) assigns to each symbol in \( \Sigma \) an arrow of \( C \) of appropriate arity:

\[
\begin{align*}
|1|_A : & \ A^S \to A \\
|u_s|_A : & \ A \to A
\end{align*}
\]
in such a way that the equations in $E$ are satisfied.

Examples of $\langle \Sigma, E \rangle$-algebras are objects of the form $B^S$, with the following interpretation of $\Sigma$: $|1|_A$ is defined by

$$
(B^S)^S \xrightarrow{\sim} B^S \times S \xrightarrow{B^S \delta} B^S
$$

where $\delta : S \to S \times S$ is the diagonal map, and $|u_s|_A$ by the composite:

$$
B^S \xrightarrow{B^s} B \xrightarrow{B^!} B^S
$$

where $s : 1 \to S$ picks $s \in S$ and $!: S \to 1$. In Sets, $1$ takes a family of maps $(b_s)_{s \in S}$ and returns the diagonal map $s \mapsto b_s(s)$, whereas $u_s$ takes a map $b$ and returns the constant map $s' \mapsto b(s)$. Now, joining Theorem 2 above and Theorem 1 of [5], we may conclude that those are essentially the only examples of $\langle \Sigma, E \rangle$-algebras, and of course of $T$-algebras.

Remark

Because we deal with global states only, we take here the set of locations $L$ as a singleton, which reduces the seven equations of [5], section 3 to the first four.

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References

[1] J. Beck. Triples, algebras and cohomology. Reprints in Theory and Applications of Categories, 2:1–59, 2003. [http://www.tac.mta.ca/tac/reprints/articles/2/tr2abs.html]

[2] S. Mac Lane. Categories for the Working Mathematician. Springer, 1971.

[3] E. Moggi. Computational lambda-calculus and monads. In Logic in Computer Science, pages 14–23, 1989.

[4] E. Moggi. Notions of computation and monads. Inf. Comput., 1:55–92, 1991.

[5] G. Plotkin and J. Power. Notions of computation determine monads. In FOSSACS 2002, volume 2303/2002 of LNCS. Springer Verlag, 2002.

[6] P. Wadler. Comprehending monads. Math. Struct. Comput. Sci, 2(4):461–493, 1992.