FORMAL ZETA FUNCTION EXPANSIONS AND THE FREQUENCY OF RAMANUJAN GRAPHS

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Abstract. We show that logarithmic derivative of the Zeta function of any regular graph is given by a power series about infinity whose coefficients are given in terms of the traces of powers of the graph’s Hashimoto matrix.

We then consider the expected value of this power series over random, d-regular graph on n vertices, with d fixed and n tending to infinity. Under rather speculative assumptions, we make a formal calculation that suggests that for fixed d and n large, this expected value should have simple poles of residue $-1/2$ at $\pm (d-1)^{-1/2}$. We shall explain that calculation suggests that for fixed d there is an $f(d) > 1/2$ such that a d-regular graph on n vertices is Ramanujan with probability at least $f(d)$ for n sufficiently large.

Our formal computation has a natural analogue when we consider random covering graphs of degree n over a fixed, regular “base graph.” This again suggests that for n large, a strict majority of random covering graphs are relatively Ramanujan.

We do not regard our formal calculations as providing overwhelming evidence regarding the frequency of Ramanujan graphs. However, these calculations are quite simple, and yield intriguing suggestions which we feel merit further study.

CONTENTS

1. Introduction 1
2. Main Results 3
3. Graph Theoretic Preliminaries 6
3.1. Graphs and Morphisms 6
4. Variants of the Zeta Function 8
5. The Expected Value of $L_G$ 10
6. A Simpler Variant of the $P_t$ and $P_1$ 13
7. Random Graph Covering Maps and Other Models 14
8. Numerical Experiments 16
References 17

1. Introduction

In this paper we shall give a calculation that suggests that there should be many Ramanujan graphs of any fixed degree and any sufficiently large number of vertices. Our calculation is quite speculative, and makes a number of unjustified assumptions. However, our calculations are quite simple and give an intriguing

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suggestion; we therefore find these calculations—at the very least—a curiosity that merits further study.

In more detail, we show that the logarithmic derivative of the Zeta function of a regular graph has a simple power series expansion at infinity. The coefficients of this power series involve traces of successively larger powers of the Hashimoto matrix of a graph. For a standard model of a $d$-regular graph on $n$ vertices, we consider the expected value of this logarithmic derivative for fixed $d$ and large $n$. We make a number of assumptions to make a formal computation which suggests that for fixed $d$, the expected number of real poles near each of $\pm(d - 1)^{-1/2}$ tends to $1/2$ as $n$ tends to infinity. Assuming this $1/2$ is caused entirely by real poles of the Zeta function, then the expected number of positive or negative adjacency eigenvalues of absolute value in the interval $[2(d - 1)^{1/2}, d)$ would be $1/4$ for both cases, positive and negative. Under the likely assumption that for fixed $d$, a random, $d$-regular graph on $n$ vertices will have two or more such eigenvalues with probability bounded away from zero, then for sufficiently large $n$, a strict majority of random graphs are Ramanujan. We emphasize that all these conclusions are quite speculative.

Our computation involves a contour integral of the expected logarithmic derivative near $u = (d - 1)^{-1/2}$, where our formal computation suggests that this expected logarithmic derivative has a pole. We write the contour in a particular way that assumes that the residue at this pole arises entirely from real eigenvalues of the Hashimoto matrix. However it is conceivable that some of the complex Hashimoto eigenvalues near $\pm 2(d - 1)^{1/2}$ also contribute to this pole, in which case (if our other assumptions are correct) the limiting expected number of positive and negative real Zeta function poles may each be less than $1/4$.

We caution the reader that part of our contour passes lies the open ball $|u| < (d - 1)^{-1/2}$, which is one serious issue in the above computation. Indeed, for fixed $d$ and $n$ large, the Zeta function of a random, $d$-regular graph on $n$ vertices has poles throughout the circle $|u| = (d - 1)^{-1/2}$; this follows easily from the fact that a random such graph has a bounded expected number of cycles of any fixed length, and so its adjacency eigenvalue distribution tends to the Kesten-McKay distribution (see [McK81]). Hence, for large $n$ the true expected Zeta function should have a pole distribution throughout $|u| = (d - 1)^{-1/2}$, which makes it highly speculative to work in any part of the region $|u| < (d - 1)^{-1/2}$. At the same time, this makes the residue of $-1/2$ at $\pm(d - 1)^{-1/2}$ in the formal calculation all the more interesting, and is why we may conjecture that the imaginary poles near $\pm(d - 1)^{-1/2}$ may possibly contribute this $-1/2$, if there is any true sense to this residue.

It is interesting that the $-1/2$ comes from a computation on the trace method for regular graphs that was essentially done by Broder and Shamir [BS87], although more justification for our computation comes from the asymptotic expansions of expected trace powers given later improvements of the Broder-Shamir method, namely [Fri91, Fri08, FK14].

As far as we know, this is the first direct application of Zeta functions per se to graph theory. Zeta functions of graphs arose first in the study of p-adic groups [Iha66, Sert03], and developed for general graphs by Sunada, Hashimoto, and Bass (see [Ter11], beginning of Part 2). Ihara’s determinantal formula gave rise to what is now often called the Hashimoto matrix of a graph, which can be used to count strictly non-backtracking closed walks in a graph. Although the underlying graph of the Hashimoto matrix appeared in graph theory in the 1940’s and 1960’s (see
the study of its spectral properties seems largely inspired by the above work on graph Zeta functions. Friedman and Kohler [FK14] note that in the trace method for random, regular graphs, one gets better adjacency eigenvalue bounds if one first gets analogous trace estimates for the Hashimoto matrix. Furthermore, the solution to the Alon Second Eigenvalue Conjecture [Fri08] involved trace methods for the Hashimoto matrix rather than for the adjacency matrix. Hence the Hashimoto matrix—and therefore, by implication, also Zeta functions—have played a vital role in graph theory. However, we know of no previous direct applications of Zeta functions to obtain new theorems or conjectures in graph theory. Our theorems and conjectures—although they involve the Hashimoto matrix in their expansion at infinity—seem to fundamentally involve Zeta functions.

We remark that the expected traces of Hashimoto matrix powers are difficult to study directly. Indeed, these expected traces are complicated by tangles [Fri08, FK14], which are—roughly speaking—low probability events that force a graph to have large, positive real Hashimoto eigenvalues. It is known that such tangles must be removed to prove the Alon conjecture [Fri08] or its relativization [FK14]; furthermore, by modifying trace powers to eliminate the pathological effect of tangles, the asymptotic expansions of expected trace powers become much simpler. For this reason we introduce a second formal power series, whose terms are a variant of the above terms, such that (1) it is probably simpler to understand the terms of this second formal power series, and (2) we believe that the second set of terms contain very similar information to the first.

We can generalize the above discussion to random covering maps of degree $n$ over a fixed, regular “base graph.” Doing so gives the analogous formal computation that suggests that for large $n$ we expect that a majority of random cover maps to be relatively Ramanujan. Again, the $1/4$ we get (or $1/2$ for random, bipartite, regular graphs) comes from the analogue of the Broder-Shamir computation [Fri03] for random covering maps, but is further justified by higher order expansions [LP10, Pud12, FK14].

The rest of this paper is organized as follows. In Section 2 we describe our main theorems and conjectures. In Section 3 we describe our terminology regarding graphs and Zeta functions. In Section 4 we prove our expansion near infinity of the logarithmic derivative of the Zeta function of a graph. In Section 5 we make a formal calculation of the expected above logarithmic derivative and make numerous conjectures. In Section 6 we describe variants of this formal computation which we believe will be easier to study, and yet will contain essentially the same information. In Section 7 we describe other models of random graphs, especially covering maps of degree $n$ over a fixed, regular base graph. In Section 8 we briefly describe our numerical experiments and what previous experiments in the literature have suggested.

2. Main Results

In this section we state our main results, although we will use some terminology to be made precise in later sections. Let us begin with the notion of a random graph that we use. For positive integers $n$ and $d \geq 3$ we consider a random $d$-regular graph on $n$ vertices. It is simplest to think of $d$ as an even integer with a random graph generated by $d/2$ permutations on $\{1, \ldots, n\}$, which we denote $G_{n,d}$, as was used in [BS87, Fri91, Fri08]; our models therefore allow for multiple edges...
and self-loops. We remark that there are similar “algebraic” models for $d$ and $n$ of any parity [Fri08] which we shall describe in Section 7.

We will give a formal calculation that indicates that for fixed $d$ the expected number of adjacency eigenvalues of a graph in $G_{n,d}$ of absolute value in $[2(d - 1)^{1/2}, d]$ is one-half, $1/4$ positive and $1/4$ negative; a more conservative conjecture is that $1/4$ is an upper bound on each side. We also conjecture that the probability that for fixed $d$ and $n \to \infty$, a graph of $G_{n,d}$ has at least two such eigenvalues is bounded from below by a positive constant. These two conjectures—if true—imply that for any fixed $d$, for all $n$ sufficiently large a strict majority of graphs in $G_{n,d}$ are Ramanujan.

After explaining our conjecture, we will comment on generalizations to random covering maps of degree $n$ to a fixed, regular graph. Then we will describe some numerical experiments we made to test our conjecture; although these calculations suggest that our formal computation may be close to the correct answer, our calculations are done on graphs with under one million vertices (and assume that certain software is computing correctly); it may be that one needs more vertices to see the correct trend, and it is commonly believed that there are fewer Ramanujan graphs than our formal calculation suggests; see [MN08].

Let us put our conjectures in a historical context. For a graph, $G$, on $n$ vertices, we let $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ be the $n$ eigenvalues of $A_G$, the adjacency matrix of $G$. In [Alo86], Noga Alon conjectured that for fixed integer $d \geq 3$ and $\epsilon > 0$, as $n$ tends to infinity, the probability that a random $d$-regular graph, $G$, on $n$ has

$$\lambda_2(G) \leq 2(d - 1)^{-1/2} + \epsilon$$

tends to one. Alon’s interest in the above conjecture was that the above condition on $\lambda_2(G)$ implies ([GG81, AM85, Tan84]) that $G$ has a number of interesting isoperimetric or “expansion” properties. Broder and Shamir [BS87] introduced a trace method to study the above question; [BS87, FKS89, Fri91] gave high probability bounds on $\lambda_2(G)$ with $2(d - 1)^{-1/2}$ replaced with a larger constant, and [Fri08] finally settled the original conjecture. We remark that all the aforementioned papers actually give stronger bounds, namely with $\lambda_2(G)$ replaced with

$$\rho(G) \equiv \max_{i \geq 2} |\lambda_i(G)|.$$  

For many applications, it suffices to specify one particular graph, $G \in G_{n,d}$, which satisfies the bound in Alon’s conjecture. Such graphs were given in [LPS88, Mar88, Mor94]; [LPS88] coined the term Ramanujan graph to describe a $d$-regular graph, $G$, satisfying

$$\lambda_i(G) \in \{d, -d\} \cup [-2(d - 1)^{1/2}, 2(d - 1)^{1/2}]$$

for all $i$. This is stronger than Alon’s conjecture in that the $\epsilon$ of Alon becomes zero; it is weaker than what trace methods prove, in that $-d$ is permitted as an eigenvalue, i.e., the graph may be bipartite. Recently [MSS13] have proven the existence of a sequence of bipartite Ramanujan graphs of any degree $d$ for a sequence of $n$’s tending to infinity; [LPS88, Mar88, Mor94] constructed sequences for $d$ such that $d - 1$ is a prime power, but the [LPS88, Mar88, Mor94] are more explicit—constructible in polynomial of log $n$ and $d$—than those of [MSS13]—constructible in polynomial of $n$. 

and $d$. Our results do not suggest any obvious method of constructing Ramanujan graphs.

Our formal calculation for $G_{n,d}$ is based on two results: Ihara Zeta functions of graphs (see [Ter11]), and a trace estimate essentially known since [BS87]. We give a stronger conjecture based on the trace methods and estimates of [Fri91, Fri08, FK14], which give somewhat more justification for the formal calculation we describe.

Let us roughly describe our methods. We will consider the function

$$E_{G \in G_{n,d}} \left[ \frac{\zeta'_G(u)}{\zeta_G(u)} \right],$$

i.e., the expected logarithmic derivative of $\zeta_G(u)$; using the results of [Fri08, FK14] regarding the Alon conjecture, we can write the expected number of eigenvalues equal to or greater than $2(d-1)^{1/2}$ as half of a contour integral of the above logarithmic derivative near $2(d-1)^{-1/2}$ (really minus the logarithmic derivative, since we are counting poles). This contour integral is essentially unchanged if we replace minus this logarithmic derivative by the simpler expression

$$L_G(u) = \sum_{k=0}^{\infty} u^{-1-k} \text{Tr}(H^k_G)(d-1)^{-k},$$

where $H_G$ denotes the Hashimoto matrix of a graph, $G$. We recall [Fri08, FK14] (although essentially from [Fri91]) that we can estimate the expected traces of Hashimoto matrices as

$$\mathbb{E}_{G \in G_{n,d}} \left[ \text{Tr}(H^k_G) \right] = P_0(k) + P_1(k)n^{-1} + \cdots + P_{r-1}(k)n^{1-r} + \text{err}_{n,k,r}$$

for some $C = C(r)$. The leading term, $P_0(k)$, has been essentially known since [BS87] (see [Ter11]) to be

$$P_0(k) = (d-1)^k + (d-1)^{k/2}I_{\text{even}}(k) + O(k) (d-1)^{k/3},$$

where $I_{\text{even}}(k)$ is the indicator function that $k$ is even. If we assume that we may evaluate the expected value of $L_G(u)$ by writing is as a formal sum, term by term, using (1), then the term of $P_0(k)$ gives a term of the expected value of $L_G(u)$ equal to

$$\frac{u}{u^2 - (d-1)},$$

whose residue at $u = \pm(d-1)^{-1/2}$ is 1/2. If this exchange of summation gives the correct asymptotics, i.e., the sum of the terms corresponding to $P_i(k)n^{-i}$ for $i \geq 1$ tends to zero as $n \to \infty$, then as $n \to \infty$ we would have that the expected number of positive real poles and negative real poles is 1/2 each (with the $\pm(d-1)^{-1/2}$ contributing one-half times their expected multiplicity).

It is known that for large $i$, the $P_i(k)$ are problematic due to “tangles” [Fri08], which are certain low probability events in $G_{n,d}$ that force a large second adjacency eigenvalue. Hence we might wish to modify the $P_i(k)$, as done in [Fri08, FK14], by introducing a variant of the Hashimoto trace. This leads us to later introduce the related functions $\tilde{P}_i(k)$, which we believe will be easier to study but contain almost
the same information as the \( P_1(k) \). We shall also explain a generalization of this calculation to models of covering graphs of degree \( n \) over a fixed, regular “base graph.”

For reasons that we explain, this formal calculation takes a number of “leaps of faith” that we cannot justify at this point; on the other hand, it seems like a natural formal calculation to make, and it suggests an intriguing conjecture.

We have made a brief, preliminary experimental investigation of this conjecture with \( d = 4 \) and \( d = 6 \) for positive poles; these experiments are not particularly conclusive: according to \([MN08]\) the true trend may require very large values of \( n \): this is based on the calculations that the “width of concentration” of the second largest adjacency eigenvalue will eventually overtake its mean’s distance to \( 2(d − 1)^{1/2} \). However, if this width of concentration is of the same order of magnitude as its distance to \( 2(d − 1)^{1/2} \) as \( n \to \infty \), then our conjecture does not contradict the other findings of \([MN08]\) (regarding a Tracy-Widom distribution over the width of concentration). Our experiments are made with graphs of only \( n \leq 400,000 \) vertices. For graphs of this size or smaller it does not look like the expected number of real poles has stabilized; however, in all of our experiments this expected number is smaller than \( 1/4 \) for all but very small values of \( n \).

3. Graph Theoretic Preliminaries

In this subsection we give specify our precise definitions for a number of concepts in graph and algebraic graph theory. We note that such definitions vary a bit in the literature. For example, in this paper graphs may have multiple edges and two types of self-loops—half-loops and whole-loops—in the terminology of \([Fri93]\); also see \([ST96, ST00, TS07]\), for example, regarding half-loops.

3.1. Graphs and Morphisms.

**Definition 3.1.** A directed graph (or digraph) is a tuple \( G = (V, E^{\text{dir}}, t, h) \) where \( V \) and \( E^{\text{dir}} \) are sets—the vertex and directed edge sets—and \( t: E^{\text{dir}} \to V \) is the tail map and \( h: E^{\text{dir}} \to V \) is the head map. A directed edge \( e \) is called self-loop if \( t(e) = h(e) \), that is, if its tail is its head. Note that our definition also allows for multiple edges, that is directed edges with identical tails and heads. Unless specifically mentioned, we will only consider directed graphs which have finitely many vertices and directed edges.

A graph, roughly speaking, is a directed graph with an involution that pairs the edges.

**Definition 3.2.** An undirected graph (or simply a graph) is a tuple \( G = (V, E^{\text{dir}}, t, h, \iota) \) where \((V, E^{\text{dir}}, t, h)\) is a directed graph and where \( \iota: E^{\text{dir}} \to E^{\text{dir}} \), called the opposite map or involution of the graph, is an involution on the set of directed edges (that is, \( \iota^2 = \text{id}_{E^{\text{dir}}} \) is the identity) satisfying \( t\iota = h \). The directed graph \( G = (V, E^{\text{dir}}, t, h) \) is called the underlying directed graph of the graph \( G \). If \( e \) is an edge, we often write \( e^{-1} \) for \( \iota(e) \) and call it the opposite edge. A self-loop \( e \) is called a half-loop if \( \iota(e) = e \), and otherwise is called a whole-loop.

The opposite map induces an equivalence relation on the directed edges of the graph, with \( e \in E^{\text{dir}} \) equivalent to \( e\iota \); we call the quotient set, \( E \), the undirected edge of the graph \( G \) (or simply its edge). Given an edge of a graph, an orientation of that edge is the choice of a representative directed edge in the equivalence relation (given by the opposite map).
Notation 3.3. For a graph, $G$, we use the notation $V_G, E_G, E^\text{dir}_G, t_G, h_G, \iota_G$ to denote the vertex set, edge set, directed edge set, tail map, head map, and opposite map of $G$; similarly for directed graphs, $G$.

Definition 3.4. Let $G$ be a directed graph. The adjacency matrix, $A_G$, of $G$ is the square matrix indexed on the vertices, $V_G$, whose $(v_1, v_2)$ entry is the number of directed edges whose tail is the vertex $v_1$ and whose head is the vertex $v_2$. The indegree (respectively outdegree) of a vertex, $v$, of $G$ is the number of edges whose head (respectively tail) is $v$.

The adjacency matrix of an undirected graph, $G$, is simply the adjacency matrix of its underlying directed graph. For an undirected graph, the indegree of any vertex equals its outdegree, and is just called its degree. The degree matrix of $G$ is the diagonal matrix, $D_G$, indexed on $V_G$ whose $(v, v)$ entry is the degree of $v$. We say that $G$ is $d$-regular if $D_G$ is $d$ times the identity matrix, i.e., if each vertex of $G$ has degree $d$.

For any non-negative integer $k$, the number of closed walks of length $k$ is a graph, $G$, is just the trace, $\text{Tr}(A^k_G)$, of the $k$-th power of $A_G$.

Notation 3.5. Given a graph, $G$, the matrix $A_G$ is symmetric, and hence the eigenvalues of $A_G$ are real and can be ordered
\[ \lambda_1(G) \geq \cdots \geq \lambda_n(G), \]
where $n = |V_G|$. We reserve the notation $\lambda_i(G)$ to denote the eigenvalues of $A_G$ ordered as above.

If $G$ is $d$-regular, then $\lambda_1(G) = d$.

Definition 3.6. Let $G$ be a graph. We define the directed line graph or oriented line graph of $G$, denoted $\text{Line}(G)$, to be the directed graph $L = \text{Line}(G) = (V_L, E^\text{dir}_L, t_L, h_L)$ given as follows: its vertex set, $V_L$, is the set $E^\text{dir}_G$ of directed edges of $G$; its set of directed edges is defined by
\[ E^\text{dir}_L = \{(e_1, e_2) \in E^\text{dir}_G \times E^\text{dir}_G \mid h_G(e_1) = t_G(e_2) \text{ and } \iota_G(e_1) \neq e_2 \} \]
that is, $E^\text{dir}_L$ corresponds to the non-backtracking walks of length two in $G$. The tail and head maps are simply defined to be the projections in each component, that is by $t_L(e_1, e_2) = e_1$ and $h_L(e_1, e_2) = e_2$.

The Hashimoto matrix of $G$ is the adjacency matrix of its directed line graph, denoted $H_G$, which is, therefore, a square matrix indexed on $E^\text{dir}_G$. We use the symbol $\mu_1(G)$ to denote the Perron-Frobenius eigenvalue of $H_G$, and use $\mu_2(G), \ldots, \mu_m(G)$, where $m = |E^\text{dir}_G|$, to denote the remaining eigenvalues, in no particular order (all concepts we discuss about the $\mu_i$ for $i \geq 2$ will not depend on their order).

If $G$ is $d$-regular, then $\mu_1(G) = d - 1$.

It is easy to see that for any positive integer $k$, the number of strictly non-backtracking closed walks of length $k$ in a graph, $G$, equals the trace, $\text{Tr}(H^k_G)$, of the $k$ power of $H_G$; of course, the strictly non-backtracking walks begin and end in a vertex, whereas $\text{Tr}(H^k_G)$ most naturally counts walks beginning and ending in an edge; the correspondence between the two notions can be seen by taking a walk of $\text{Line}(G)$, beginning and ending an in a directed edge, $e \in E^\text{dir}_G$, and mapping it to the strictly non-backtracking closed walk in $G$ beginning at, say, the tail of $e$. 
For graphs, $G$, that have half-loops, the Ihara determinantal formula takes the form (see [Fri08, ST96, ST00, TS07]):

\[
\det(\mu I - H_G) = \det(\mu^2 I - \mu A_G + (D_G - I))(\mu - 1)^{\frac{1}{2}|\text{half}_G|}(\mu^2 - 1)^{\frac{1}{2}|\text{pair}_G|},
\]

where $\text{half}_G$ is the set of half-loops of $G$, and $\text{pair}_G$ is the set of undirected edges of $G$ that are not half-loops, i.e., the collection of sets of the form, $\{e_1, e_2\}$ with $\iota e_1 = e_2$ but $e_1 \neq e_2$.

4. Variants of the Zeta Function

For any graph, $G$, recall that $A_G$ denotes its adjacency matrix, and $H_G$ denotes its Hashimoto matrix, i.e., the adjacency matrix of what is commonly called $G$’s oriented line graph, Line($G$). If $\zeta_G(u)$ is the Zeta function of $G$, then we have

\[
\zeta_G(u) = \frac{1}{\det(I - uH_G)},
\]

which, for $d$-regular $G$, we may alternatively write via the Ihara determinantal formula

\[
\det(I - uH_G) = \det(I - uA_G - u^2(d - 1))(1 - u^2)^{-\chi(G)}
\]

(provided $G$ has no half-loops, with a simple modification if $G$ does).

**Definition 4.1.** Let $G$ be a $d$-regular graph. We call an eigenvalue of $H_G$ non-Ramanujan if it is purely real and different from $\pm 1, \pm (d - 1)^{1/2}, \pm (d - 1)$.

We say that an eigenvalue of $A_G$ is non-Ramanujan if it is of absolute value strictly between $2(d - 1)^{1/2}$ and $d$. It is known that the number of non-Ramanujan eigenvalues of $H_G$ is precisely twice the number of eigenvalues of $A_G$. $G$ is called Ramanujan if it has no non-Ramanujan $H_G$ eigenvalues, or, equivalently, no non-Ramanujan $A_G$ eigenvalues. Similarly for positive non-Ramanujan eigenvalues of both $H_G$ and $A_G$, and the same with “positive” replaced with “negative.”

**Notation 4.2.** For any $\epsilon, \delta > 0$, let $C_{\epsilon,\delta}^+$ be the boundary of the rectangle

\[
\{x + iy \in \mathbb{C} \mid |1 - x(d - 1)^{1/2}| \leq \epsilon, |y| \leq \delta\};
\]

define $C_{\epsilon,\delta}^-$ similarly, with $x$ replaced with $-x$.

**Definition 4.3.** We say that a $d$-regular graph, $G$, is $\epsilon$-spectral if $G$’s real Hashimoto eigenvalues lie in set

\[
\{-(d - 1), -1, 1, (d - 1)\} \cup \{x \mid |1 - x(d - 1)^{-1/2}| < \epsilon\}.
\]

We remark that for fixed $d \geq 3$ and $\epsilon > 0$, it is known that a fraction $1 - O(1/n)$ of random $d$-regular graphs on $n$ vertices are $\epsilon$-spectral [Fri08].

For $\epsilon$-spectral $G$ we have that the number of real, positive Hashimoto eigenvalues is given by

\[
\frac{1}{2\pi i} \int_{C_{\epsilon,\delta}^+} \frac{-\zeta'_G(u)}{\zeta_G(u)} \, du
\]

for $\delta > 0$ sufficiently small, where $C_{\epsilon,\delta}^+$ is traversed in the counterclockwise direction; similarly for negative eigenvalues.
Observe that for each \( G \), for large \(|u|\) we have
\[
-\zeta_G(u)/\zeta_G(u) = \sum_{\mu \in \text{Spec}(H_G)} -\mu (1 - u\mu)^{-1},
\]
where \( \text{Spec}(H_G) \) denotes the set of eigenvalues of \( H_G \), counted with multiplicity, and hence
\[
-\zeta_G(u)/\zeta_G(u) = \sum_{\mu \in \text{Spec}(H_G)} \sum_{k=0}^{\infty} u^{-1-k}\mu^{-k}.
\]
By the Ihara determinantal formula, the set of eigenvalues, \( \mu \), of \( H_G \), consists of \( \pm 1 \), with multiplicity \( -\chi(G) \), and, in addition, the eigenvalues that arise as the roots, \( \mu_1, \mu_2 \), from an equation
\[
\mu^2 - \mu\lambda + (d-1) = 0,
\]
where \( \lambda \) ranges over all eigenvalues of \( A_G \); in particular, any pair \( \mu_1, \mu_2 \) as such satisfy \( \mu_1\mu_2 = d - 1 \); hence to sum over \( \mu^{-1} \) over the pairs \( \mu_1, \mu_2 \) as such is the same as summing over \( \mu/(d-1) \) of all such eigenvalues. It easily follows that
\[
\sum_{\mu \in \text{Spec}(H_G)} \mu^{-k} = (1+(-1)^k)n(d-2)/2 + (\text{Tr}(H_G^k) - (1+(-1)^k)n(d-2)/2)(d-1)^{-k}
\]
where \( \text{Tr} \) denotes the trace, and hence
\[
-\zeta_G(u)/\zeta_G(u) = \mathcal{L}_G(u) + e(u),
\]
where
\[
\mathcal{L}_G(u) = \sum_{k=0}^{\infty} u^{-1-k} \text{Tr}(H_G^k)(d-1)^{-k},
\]
and
\[
e(u) = \sum_{k=0}^{\infty} u^{-1-k}(1 - (d-1)^{-k})(1 + (-1)^k)n(d-2)/2
\]
\[
= \frac{n(d-2)}{2u} \sum_{k \geq 0} \left[ u^{-k} + (-u)^{-k} - ((d-1)u)^{-k} - ((d-1)u)^{-k} \right]
\]
\[
= \frac{n(d-2)}{2u} \left[ \frac{1}{1-u} + \frac{1}{1+u} - \frac{1}{1-(d-1)u} - \frac{1}{1+(d-1)u} \right].
\]
It follows that \( e(u) \) is a rational function with poles only at \( u = \pm 1 \) and \( \pm 1/(d-1) \). Furthermore, the \( e(u) \) poles at \( \pm 1 \) have residue \( -\chi(G) \).

**Definition 4.4.** Let \( G \) be a \( d \)-regular graph without half-loops. We define the **essential logarithmic derivative** to be the meromorphic function complex function
\[
\mathcal{L}_G(u) = \sum_{k=0}^{\infty} u^{-1-k} \text{Tr}(H_G^k)(d-1)^{-k}.
\]
We caution the reader that \( \mathcal{L}_G(u) \) is the interesting part of **minus** the usual logarithmic derivative of \( \zeta_G(u) \) (since we are interested in poles, not zeros). Clearly \( \text{Tr}(H_G^k) \) is bounded by the number of non-backtracking walks of length \( k \) in \( G \), i.e., \( |V_G|d(d-1)^{k-1} \), and hence the above expansion for \( \mathcal{L}_G(u) \) converges for all \(|u| > 1\).

We summarize the above discussion in the follow proposition.
Proposition 4.5. Let $G$ be a $d$-regular, $\epsilon$-spectral graph. Then, with $C_{\epsilon,\delta}^+$ as in (4), we have that the number of positive non-Ramanujan Hashimoto eigenvalues is given by
\[ \frac{1}{2\pi i} \int_{C_{\epsilon,\delta}^+} L(u) \, du \]
for $\delta > 0$ sufficiently small; similarly for negative eigenvalues.

We remark that if $G$ is $\epsilon$-spectral for $\epsilon > 0$ small, then $\text{Tr}(H_k^G)$ is the sum of $d^k$ plus $nd - 1$ other eigenvalues, all of which are within the ball $|\mu| \leq (d - 1)^{1/2} + \epsilon'$ for some $\epsilon'$ that tends to zero as $\epsilon$ tends to zero; hence, for such $G$, the expression for $L_G$ in Definition 4.4 has a simple pole of residue 1 at $u = 1$, and the power series at infinity for
\[ L_G(u) - \frac{1}{u - 1} \]
converges for all $|u|^{-1} < (d - 1)^{1/2} + \epsilon'$.

5. The Expected Value of $L_G$

For any even integer, $d$, and integer $n > 0$, we define $G_{n,d}$ to the probability space of $d$-regular random graphs formed by independently choosing $d/2$ permutations, $\pi_1, \ldots, \pi_{d/2}$ uniformly from the set of $n!$ permutations of $\{1, \ldots, n\}$; to each such $\pi_1, \ldots, \pi_{d/2}$ we associate the random graph, $G = G(\{\pi_i\})$, whose vertex set is $V_G = \{1, \ldots, n\}$, and whose edge set, $E_G$, consists of all sets
\[ E_G = \{\{i, \pi_j(i)\} \mid i = 1, \ldots, n, \ j = 1, \ldots, d/2\}. \]
It follows that $G$ may have multiple edges and self-loops.

The following is a corollary of [Fri08, FK14].

Theorem 5.1. For a $d$-regular graph, $G$, define define $N_A^+(G)$ to be the number of positive non-Ramanujan adjacency eigenvalues of $G$ plus the multiplicity of $2(d - 1)^{-1/2}$ (if any) as an eigenvalue of $G$. Then for any even $d \geq 10$ and $\epsilon > 0$ we have that
\[ \lim_{n \to \infty} \lim_{\delta \to 0} \mathbb{E}_{G \in G_{n,d}} \left[ N_A^+(G) - \frac{1}{4\pi i} \int_{C_{\epsilon,\delta}^+} L_G(u) \, du \right] = 0, \]
where we interpret the contour integral as its Cauchy principle for graphs, $G$, whose Zeta function has a pole on $C_{\epsilon,\delta}^+$.

Proof. For $d \geq 10$ we know that the probability that a graph has any such eigenvalues is at most $O(1/n^2)$ (see [Fri08] for $d \geq 12$, and [FK14] for $d = 10$), and hence this expected number is at most $1/n$. \qed

For $d \geq 4$ one might conjecture the same theorem holds, although this does not seem to follow literally from [Fri08, FK14]; however, if one conditions on $G \in G_{n,d}$ not having a $(d, \epsilon')$-tangle (in the sense of [FK14]), which is an order $O(1/n)$ probability event, then we get equality. The problem is that one does not know where most of the eigenvalues lie in graphs that have tangles; we conjecture that graphs with tangles will not give more than an $O(1/n)$ expected number of eigenvalues strictly between $2(d - 1)^{1/2} + \epsilon$ and $d$; hence we conjecture that the above above theorem remains true for all $d \geq 4$.\[ \]
We remark that for any $d$-regular graph, we have that
\[
\lim_{\delta \to 0} \frac{1}{4\pi i} \int_{C_{\epsilon, \delta}^+} L_G(u) \, du = N_{A+, \epsilon}(G),
\]
where $N_{A+, \epsilon}$ counts the number of positive, real Hashimoto eigenvalues, $\mu$, such that
\[
|1 - \mu(d - 1)^{1/2}| \leq \epsilon.
\]
Hence for any $d, n, \epsilon$ we have
\[
\lim_{\delta \to 0} \mathbb{E}_{G \in \mathcal{G}_{n,d}} \left[ \frac{1}{4\pi i} \int_{C_{\epsilon, \delta}^+} L_G(u) \, du \right] = \mathbb{E}_{G \in \mathcal{G}_{n,d}} [N_{A+, \epsilon}(G)].
\]
Now we wish to conjecture a value for (5)
\[
\lim_{\delta \to 0} \mathbb{E}_{G \in \mathcal{G}_{n,d}} \left[ \frac{1}{4\pi i} \int_{C_{\epsilon, \delta}^+} L_G(u) \, du \right].
\]
We now seek to use trace methods to in order to conjecture what the value of (5) will be for fixed $d$ and $n \to \infty$.

It is known that for $d, r$ fixed and $n$ large, we have that
\[
\mathbb{E}_{G \in \mathcal{G}_{n,d}} [\text{Tr}(H_k^G)] = P_0(k) + P_1(k)n^{-1} + \ldots + P_{r-1}(k)n^{1-r} + \text{err}_r(n, k),
\]
where $P_i(k)$ are functions of $k$ alone, and
\[
|\text{err}_r(n, k)| \leq C_{r,k^2}d^{-1}k^{-r}.
\]
Furthermore, $P_0(k)$ is known (see [Fri91, Fri08, FK14], but essentially since [BS87]) to equal
\[
P_0(k) = O(kd) + \sum_{k' \mid k} (d - 1)^{k'},
\]
where $k' \mid k$ means that $k'$ is a positive integer dividing $k$; in particular, we have
\[
P_0(k) = (d - 1)^k + I_{\text{even}}(k)(d - 1)^{k/2} + O(d - 1)^{k/3},
\]
where $I_{\text{even}}(k)$ is the indicator function of $k$ being even. Furthermore, we believe the methods of [Fri91, FK14] will show that for each $i$ we have
\[
\mathcal{P}_i(u) = \sum_{k=0}^{\infty} u^{-1-k} P_i(k)(d - 1)^{-k}
\]
is meromorphic with finitely many poles outside any disc about the origin of radius strictly greater than $1/(d - 1)$; our idea is to use the “mod-S” function approach of [FK14] to show that each $P_i(k)$ is a polyexponential plus an error term (see [FK14]) and to argue for each “type” separately, although we have not written and checked this carefully as of the writing of this article; hence this belief may be regarded as a (plausible) conjecture at present. Let give some conjectures based on the above assumption, and the (more speculative) assumption that we can evaluate the above asymptotic expansion by taking expected values term by term, and the (far more speculative) assumption that the $\mathcal{P}_i(u)$, formed by summing over arbitrarily large $k$, reflect the properties of $\mathcal{G}_{n,d}$ with $n$ fixed.
**Definition 5.2.** For any even \( d \geq 4 \), set
\[
N_i = \frac{1}{2\pi i} \int_{C_{i,\delta}^+} P_i(u) \, du,
\]
where we assume the \( P_i(u) \), given in (8), based on functions \( P_i(k) \) given in (6), are meromorphic functions, at least for \( |u| \) near \( (d-1)^{-1/2} \). We say that \( G \cdot d \) is positively approximable to order \( r \) if for some \( \epsilon > 0 \) we have
\[
\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{C_{r,\delta}^+} E_{G \in G_{n,d}} [L_{G}(u)] \, du = N_0 + N_1 n^{-1} + N_2 n^{-2} + \cdots + N_r n^{-r} + o(n^{-r})
\]
as \( n \) tends to infinity; we similarly define negatively approximable with negative real eigenvalues and \( C_{-\epsilon,\delta}^+ \).

We now state a number of conjectures, which are successively weaker.

**Conjecture 5.3.** For any even \( d \geq 4 \) we have \( G \cdot d \) is

1. positively approximable to any order;
2. positively approximable to order \( r(d) \), where \( r(d) \geq 0 \) and \( r(d) \to \infty \) as \( d \to \infty \);
3. positively approximable to order 0; and
4. positively approximable to order 0 for \( d \) sufficiently large;

and similarly with “positively” replaced with “negatively.”

For the above conjecture, we note that \( G \cdot d \) exhibits \( (d,\epsilon') \)-tangles of order 1 (see [Fri08, FK14]) for \( \epsilon' > 0 \) and \( d \leq 8 \); for such reasons, we believe that there may be a difference between small and large \( d \).

Of course, the intriguing part of this conjecture is the calculation taking (7) to show that
\[
P_0(u) = \sum_{k=1}^{\infty} P_0(k) u^{-k-1} = \frac{u^{-1}}{1 - u^{-1}} + \frac{u^{-1}}{1 - (d - 1)^{-1}u^{-2}} + h(u)
\]
\[
= \frac{1}{u - 1} + \frac{u}{u^2 - (d - 1)} + h(u),
\]
where \( h(u) \) is holomorphic in \( |u| > (d - 1)^{-2/3} \). It follows that
\[
N_0 = \frac{1}{2\pi i} \int_{C_{r,\delta}^+} P_0(u) \, du = 1/2;
\]
similarly with “positive” replaced with “negative.” Hence the above conjecture would imply that the expected number of positive, non-Ramanujan adjacency eigenvalues for a graph in \( G_{n,d} \) with \( d \) fixed and \( n \) large, would tend to 1/4. This establishes a main point of interest.

**Proposition 5.4.** Let \( d \geq 4 \) be an even integer for which \( G \cdot d \) is positively approximable to order 0. Then, as \( n \to \infty \), the limit supremum of the expected of positive, non-Ramanujan Hashimoto eigenvalues of \( G \in G_{n,d} \) is at most 1/2; similarly, the same for non-Ramanujan adjacency eigenvalues is at most 1/4. The same holds with “positive(ly)” replaced everywhere with “negative(ly).”

The reason we involve the limit supremum in the above is that it is conceivable (although quite unlikely in our opinion) that there is a positive expected multiplicity of the eigenvalue \( (d - 1)^{1/2} \) in \( H_{G} \) for \( G \in G_{n,d} \).
We finish this section with a few remarks considering the above conjectures.

In the usual trace methods one estimates the expected value of $A_G^k$ or $H_G^k$ for $k$ of size proportional to $\log n$; furthermore, the contributions to $P_0(k)$ consist of “single loops” (see [Fri08, FK14]), which cannot occur unless $k \leq n$. Hence, the idea of fixing $n$ and formally summing in $k$ cannot be regarded as anything but a formal summation.

We also note that for any positive integer, $m$, $P_0(u)$ has poles at $(d - 1)^{(m-1)/m}\omega_m$, where $\omega_m$ is any $m$-th root of unity; hence this function does not resemble $\zeta_G(u)$ for a fixed $d$-regular graph $G$, whose poles are confined to the reals and the complex circle $|u| = (d - 1)^{-1/2}$; hence if $P_0(u)$ truly reflects some average property of $\zeta_G(u)$ for $G \in \mathcal{G}_{n,d}$ everywhere in $|u| > (d - 1)^{-1}$, then there is some averaging effect that makes $P_0(u)$ different that the typical $\zeta_G(u)$.

6. A Simpler Variant of the $P_i$ and $\mathcal{P}_i$

Part of the problem in dealing with the $P_i$ of (6) and $\mathcal{P}_i$ of (8) is that the $P_i$ can, at least in principle (and we think likely), have roughly $i^i$ real poles between 1 and $(d - 1)^{1/2}$, where the $i^i$ represents roughly the number of types [Fri91, Fri08, FK14] of order $i$ (see also [Pud12] for similar problems).

However, the methods of [FK14] (“mod-$S$” functions, Section 3.5) show that for fixed $d$, and fixed $i$ bounded by a constant times $(d - 1)^{1/3}$, we have that $P_i(u)$ has poles at only $u = \pm 1$ and $u = \pm (d - 1)^{-1/2}$ for $|u| > (d - 1)^{-2/3}$. Hence for fixed $d$ and $i$ sufficiently small, the $P_i(u)$ are much simpler to analyze. However the calculations [Fri91, Fri08, FK14] show something a bit stronger: namely if we consider $P_{i,d}(u)$ and $P_{i,d}(u)$ as depending both on $i$ and $d$, then in fact

$$P_i(k) = (d - 1)^k Q_i(k, d - 1) + \frac{2}{d} R_i(k, d - 1)$$

$$+ \frac{2}{d} S_i(k, d - 1) + O((d - 1)^{k/3})$$

for some constant $C = C(i)$, and where $Q_i(k, d - 1), R_i(k, d - 1), S_i(k, d - 1)$ are polynomial in $k$ (whose degree is bounded by a function of $i$), whose coefficients are rational functions of $d - 1$.

Definition 6.1. The large $d$ polynomials of order $i$ are the functions $Q_i(k, d), R_i(k, d), S_i(k, d)$, determined uniquely in (9). The associated approximate principle term to $Q_i, R_i, S_i$ is the function $\hat{P}_i(k, d) = (d - 1)^k Q_i(k, d - 1) + \hat{P}_{i, even}(d - 1)^{k/2} R_i(k, d - 1) + \hat{P}_{i, odd}(d - 1)^{k/2} S_i(k, d - 1)$, and the associated approximate generating function is

$$\hat{P}_i(u) = \sum_{k=0}^{\infty} u^{-1-k} \hat{P}_i(k)(d - 1)^{-k}.$$ 

It follows that the $\hat{P}_i(u)$ have poles only at $u = 1$ and $u = \pm (d - 1)^{-1/2}$ outside any disc about zero of radius strictly greater than $(d - 1)^{-2/3}$.

The benefit of working with the $\hat{P}_i(k, d)$ and $\hat{P}_i(u)$ is that there are various ways of trying to compute these functions. For example, one can fix $k$ and $n$, and consider what happens as $d \to \infty$. In this case we are studying the expected number of fixed points of strictly reduced words of length $k$ in the alphabet

$$\Pi = \{\pi_1, \pi_1^{-1}, \pi_2, \pi_2^{-1}, \ldots, \pi_d/2, \pi_d/2^{-1}\}$$
with $d$ large. If such a word has exactly one occurrence of $\pi_j, \pi_j^{-1}$ for some $j$, which the overwhelmingly typical, since $k$ is fixed and $d$ is large, then the expected number of fixed point is exactly 1. Hence we are lead to consider those words such that for every $j$, either $\pi_j, \pi_j^{-1}$ does not occur, or it occurs at least twice. The study of such words does not seem easy, although perhaps this can be understood, say with the recent works [Pud12, PP12].

This type of study also seems to resemble more closely the standard (and much more studied) random matrix theory than the $d$-regular spectral graph theory with $d$ fixed; perhaps methods from random matrix theory can be applied here.

7. Random Graph Covering Maps and Other Models

We remark that [Fri08] studies other models of random $d$-regular graphs on $n$ vertices for $d$ and $n$ of arbitrary parity (necessarily having half-edges if $d$ and $n$ are odd). We remark that if $d$ is odd and $n$ is even, we can form a model of a $d$-regular graph based on $d$ perfect matchings, called $I_{n,d}$ in [Fri08]. A curious model of random regular graph is $H_{n,d}$ (for $d$ even), which is a variant of $G_{n,d}$ where random permutations are chosen from the subset of permutations whose cyclic structure consists of a single cycle of length $n$; since such permutations cannot have self-loops (for $n > 1$), for most $d$, the probability of eigenvalues lying in the Alon region is larger for $H_{n,d}$ than for $G_{n,d}$.

For the rest of this section we give a natural extension of our main conjectures to the more general model of a random cover of a graph. $G_{n,d}$ and $I_{n,d}$ are special cases of a “random degree $n$ covering map of a base graph,” where the base graphs are, respectively, a bouquet of $d/2$ whole-loops (requiring $d$ to be even), and a bouquet of $d$ half-loops (where $d$ can be either even or odd). We shall be brief, and refer the reader to [FK14] for details.

**Definition 7.1.** A morphism of directed graphs, $\varphi: G \to H$ is a pair $\varphi = (\varphi_V, \varphi_E)$ for which $\varphi_V: V_G \to V_H$ is a map of vertices and $\varphi_E: E^\text{dir}_G \to E^\text{dir}_H$ is a map of directed edges satisfying $h_H(\varphi_E(e)) = \varphi_V(h_G(e))$ and $t_H(\varphi_E(e)) = \varphi_V(t_G(e))$ for all $e \in E^\text{dir}_G$. We refer to the values of $\varphi_V^{-1}$ as vertex fibres of $\varphi$, and similarly for edge fibres. We often more simply write $\varphi$ instead of $\varphi_V$ or $\varphi_E$.

**Definition 7.2.** A morphism of directed graphs $\nu: H \to G$ is a covering map if it is a local isomorphism, that is for any vertex $w \in V_H$, the edge morphism $\nu_E$ induces a bijection (respectively, injection) between $t_H^{-1}(w)$ and $t_G^{-1}(\nu(w))$ and a bijection (respectively, injection) between $h_H^{-1}(w)$ and $h_G^{-1}(\nu(w))$. We call $G$ the base graph and $H$ a covering graph of $G$.

If $\nu: H \to G$ is a covering map and $G$ is connected, then the degree of $\nu$, denoted $[H: G]$, is the number of preimages of a vertex or edge in $G$ under $\nu$ (which does not depend on the vertex or edge). If $G$ is not connected, we insist that the number of preimages of $\nu$ of a vertex or edge is the same, i.e., the degree is independent of the connected component, and we will write this number as $[H: G]$. In addition, we often refer to $H$, without $\nu$ mentioned explicitly, as a covering graph of $G$.

A morphism of graphs is a covering map if the morphism of the underlying directed graphs is a covering map.

**Definition 7.3.** If $\pi: G \to B$ is a covering map of directed graphs, then an old function (on $V_G$) is a function on $V_G$ arising via pullback from $B$, i.e., a function $f\pi$, where $f$ is a function (usually real or complex valued), i.e., a function on $V_G$. 
(usually real or complex valued) whose value depends only on the \( \pi \) vertex fibres. A \textit{new function} (on \( V_G \)) is a function whose sum on each vertex fibre is zero. The space of all functions (real or complex) on \( V_G \) is a direct sum of the old and new functions, an orthogonal direct sum on the natural inner product on \( V_G \), i.e.,

\[
(f_1, f_2) = \sum_{v \in V_G} f_1(v) f_2(v).
\]

The adjacency matrix, \( A_G \), viewed as an operator, takes old functions to old functions and new functions to new functions. The \textit{new spectrum} of \( A_G \), which we often denote \( \text{Spec}^{\text{new}}_B(A_G) \), is the spectrum of \( A_G \) restricted to the new functions; we similarly define the \textit{old spectrum}.

This discussion holds, of course, equally well if \( \pi : G \to B \) is a covering morphism of graphs, by doing everything over the underlying directed graphs.

We can make similar definitions for the spectrum of the Hashimoto eigenvalues. First, we observe that covering maps induce covering maps on directed line graphs; let us state this formally (the proof is easy).

**Proposition 7.4.** Let \( \pi : G \to B \) be a covering map. Then \( \pi \) induces a covering map \( \pi_{\text{Line}} : \text{Line}(G) \to \text{Line}(B) \).

Since \( \text{Line}(G) \) and \( \text{Line}(B) \) are directed graphs, the above discussion of new and old functions, etc., holds for \( \pi_{\text{Line}} : \text{Line}(G) \to \text{Line}(B) \); e.g., new and old functions are functions on the vertices of \( \text{Line}(G) \), or, equivalently, on \( E_{\text{dir}}^G \).

**Definition 7.5.** Let \( \pi : G \to B \) be a covering map. We define the \textit{new Hashimoto spectrum} of \( G \) with respect to \( B \), denoted \( \text{Spec}^{\text{new}}_B(H_G) \) to be the spectrum of the Hashimoto matrix restricted to the new functions on \( \text{Line}(G) \), and \( \rho^{\text{new}}_B(H_G) \) to be the supremum of the norms of \( \text{Spec}^{\text{new}}_B(H_G) \).

We remark that

\[
\sum_{\mu \in \text{Spec}^{\text{new}}_B(H_G)} \mu^k = \text{Tr}(H^k_G) - \text{Tr}(H^k_B),
\]

and hence the new Hashimoto spectrum is independent of the covering map from \( G \) to \( B \); similarly for the new adjacency spectrum.

To any base graph, \( B \), one can describe various models of random covering maps of degree \( n \). The simplest is to assign to each edge, \( e \in E_B \), a permutation \( \pi(e) \) on \( 1, \ldots, n \) with the stipulation that \( \pi(\iota B e) \) is the inverse permutation of \( \pi(e) \); if \( e = \{e_1, e_2\} \) is not a half-loop, and \( e_1 = e_2 \), then we may assign an arbitrary random permutation to \( \pi(e_1) \) and then set \( \pi(e_2) \) to be the inverse permutation of \( \pi(e_1) \); if \( e \) is a half-loop, then we can assign a random perfect matching to \( \pi(e) \) if \( n \) is even, and otherwise choose \( \pi(e) \) to consist of one fixed point (which is a half-loop in the covering graph) and a random perfect matching on the remaining \( n-1 \) elements. See [FK14] for a more detailed description of this model and other “algebraic” models of a random covering graph of degree \( n \) over a fixed based graph.

For \( d \geq 4 \) even, \( G_{n,d} \) is the above model over the base graph which is a bouquet of \( d/2 \) whole-loops, and \( I_{n,d} \) defined at the beginning of this section is the above model over the base graph consisting of a bouquet of \( d \) half-loops.

Again, one can make similar computations and conjectures as with \( G_{n,d} \). The analogue of \( P_0(k) \) for random covers of degree \( n \) over general base graph, \( B \), is
easily seen to be

\[ P_0(k) = O(kd) + \sum_{k' | k} \text{Tr}(H_B^{k'/2}), \]

(see [FK14], but essentially done in [Fri03], a simple variant of [BS87]). As for regular graphs, the \( \text{Tr}(H_B^{k'/2}) \) is essentially the old spectrum, and the new spectrum term is therefore

\[ I_{\text{even}}(k) \text{Tr}(H_B^{k'/2}) + O(k)(d - 1)^{k/3}. \]

The analogue of \( P_0(u) \) is therefore determined by this

\[ \text{Tr}(H_B^{k'/2}) \]

term for \( k \) even. There are two cases of interest:

1. \( B \) connected and not bipartite, in which case \( d - 1 \) is an eigenvalue of \( H_B \), and all other eigenvalues of \( H_B \) are of absolute value strictly less than \( d - 1 \); then we get the similar formal expansions and conjectures as before; and

2. \( B \) connected and bipartite, in which case \( -(d - 1) \) is also an eigenvalue, and we get a conjectured expectation of \( 1/2 \) for the number of positive non-Ramanujan adjacency eigenvalues. In this case, of course, each positive eigenvalue has a corresponding negative eigenvalue. So, again, we would conjecture that there is a positive probability that a random degree \( n \) cover of \( B \) has two pairs (i.e., four) non-Ramanujan eigenvalues; and, again, these two conjectures imply that for fixed \( d \)-regular \( B \) and sufficiently large \( n \), a strict majority of the degree \( n \) covers of \( B \) are relatively Ramanujan.

8. Numerical Experiments

Here we give some preliminary numerical experiments done to test our conjectures for random 4-regular graphs. As mentioned before, the results of [MN08] indicate that one may need graphs of many more than the \( n = 400,000 \) vertices and fewer that we used in our experiments.

We also mention that our experiments are Bernoulli trials with probability of success that seems to be between .17 and .26. Hence for \( n \geq 100,000 \), where we test no more than 500 random sample graphs, the hundredths digit is not significant. We have tested 10,000 examples only for \( n \leq 10,000 \).

For the model \( G_{n,4} \), of a \( d = 4 \)-regular graph generated by two of the \( n! \) random permutations of \( \{1, \ldots, n\} \), we computed the total number of positive eigenvalues no smaller than \( 2(d - 1)^{1/2} \). We sampled

1. 10,000 random graphs with \( n = 100, n = 1000, \) and \( n = 10,000 \), whose average number of such eigenvalues were 1.2681, 1.2258, and 1.1942, respectively;
2. 500 random graphs with \( n = 100,000 \), with an average of 1.176;
3. 250 random graphs with \( n = 200,000 \), with an average of 1.188;
4. 79 random graphs with \( n = 400,000 \), with an average of 1.177.

Of course, there is always \( \lambda_1(G) = d \), and there is a small chance, roughly \( 1/n + O(1/n^2) \) that \( G \) will be disconnected. So our formal calculations suggest that we should see 1.25 for very large \( n \), or no more than this. However there is no particularly evident convergence of these average number of positive non-Ramanujan eigenvalues at these values of \( n \).
We also tried the model $H_{n,4}$, where the permutations are chosen among the $(n-1)!$ permutations whose cyclic structure is a single cycle of length $n$. Since $H_{n,4}$ graphs are always connected, and generally have less tangles than $G_{n,4}$ [Fri08], we felt this model may give more representative results for the same values of $n$. We sampled

(1) 10,000 random graphs with $n = 100$, $n = 1000$, and $n = 10,000$, whose average number of such eigenvalues were $1.1268$, $1.161$, and $1.1693$, respectively;

(2) 500 random graphs with $n = 100,000$, with an average of $1.192$;

(3) 55 random graphs with $n = 200,000$, with an average of $1.163$;

(4) 87 random graphs with $n = 400,000$, with an average of $1.149$.

Again, we see no evident convergence at this point.

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