SCHOENBERG CORRESPONDENCE FOR MULTIFACED INDEPENDENCE

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ABSTRACT. We extend the Schoenberg correspondence for universal independences by Schürmann & Voß to the multivariate setting of Manzel & Schürmann, covering, e.g., Voiculescu’s bifreeness as well as Bożejko & Speicher’s c-free independence. At the same time, we free the proof in the univariate situation from its dependence on Muraki’s classification theorem (which states that there are only five univariate independences: tensor, free, Boolean, monotone, and antimonotone independence). In contrast to the univariate case, the class of multivariate independences considered here is infinite and not yet fully understood. Furthermore, helpful tools such quantum stochastic calculi and Fock space constructions, which are used by Schürmann & Voß to prove Schoenberg correspondence in the univariate case, are not available in general. This problem is overcome by relating convolution exponentials with respect to different independences to each other, which ultimately allows to reduce all Schoenberg correspondences for multivariate independences to the case of convolution semigroups on \(*\)-bialgebras.

1. Introduction

The term Schoenberg correspondence refers to theorems that characterize one-parameter semigroups which are (in some sense) positive by a corresponding condition on their generators. Schoenberg’s original version (in today’s language this is the Schoenberg correspondence for positive definite kernels) appeared in [Scho38]. Schoenberg correspondences play an important role in noncommutative probability. Schoenberg correspondences have been proved by Schürmann for convolution semigroups of sesquilinear forms on \(*\)-coalgebras and for linear functionals on \(*\)-bialgebras [Schü85]. The Schoenberg correspondence for \(*\)-bialgebras is also a cornerstone of the theory of quantum Lévy processes as developed in [Schü93]; it describes generating functionals of Lévy processes on a \(*\)-bialgebra as linear functionals on the \(*\)-bialgebra that are hermitian and whose restriction to the kernel of the counit is positive. Lévy processes on \(*\)-bialgebras have, by definition, increments which are independent with respect to tensor independence. In contrast with classical probability, it is a remarkable feature of noncommutative probability that there are other independences which can be used to define Lévy processes. A particularly nice class of independences are the universal independences; noncommutative random variables are called independent if their noncommutative joint distribution (modelled on the free product of algebras) coincides with the product distribution with respect to a universal product in the sense of Ben Ghorbal & Schürmann [BGS05]. Universal products are defined through axioms which reflect

This work was supported by the German Research Foundation (DFG) grant no. 397960675. The work was partly carried out under the tenure of an ERCIM ‘Alain Bensoussan’ Fellowship Programme at NTNU Trondheim and as a postdoctoral scientific employee at University of Greifswald.
desirable properties of the corresponding independence. As explained in [BGS05], a universal independence yields a convolution for linear functionals on dual semigroups. In [SV14] Schürmann & Voß proved Schoenberg correspondences for universal independences. Instead of concluding the result directly from the axioms of universal products, they use Fock space constructions which have been established for specific universal products; this is possible due to Muraki’s classification theorem [Mur02, Mur13] which states that a (positive) universal independence has to be one of the five well-studied examples: tensor, free, Boolean, monotone, and antimonotone independence.

There are generalizations of universal products in which no classification is available, leading to other classes of Lévy processes. This raises the question whether Schoenberg correspondence also holds in these cases. In this paper we address the multivariate setting of independences associated with $m$-$d$-universal products in the sense of Manzel & Schürmann [MS17] (i.e., universal products for $d$-tuples of $m$-dimensional noncommutative distributions); the most prominent examples are probably Bożejko & Speicher’s $c$-freeness [BLS96] (corresponding to $m = 1, d = 2$) and Voiculescu’s bifreeness [Voï14] (corresponding to $m = 2, d = 1$), but many more such examples have been studied (e.g., (free-)free-Boolean independence by Liu in [Liu19, Liu18] or bimonotone independence of type II by the author in [Ger17] and by Gu, Skoufranis & Hasebe in [GS20]). In [GHU23] the author, Hasebe & Ulrich constructed deformations leading to continuous families of independences for $m = 2, d = 1$. Note that Gu & Skoufranis’ $c$-bifree independence ($m = 2, d = 2$) with its derivates bi-Boolean independence and (with Hasebe) bimonotone independence of type I (both $m = 2, d = 1$) [GS17, GS19, GS20], as well as Lachs’ $r$-$s$-independence ($m = d = 1$) [GL15, Lac15] are not positive in the sense that the product of states (more precisely: restricted states, see Section 5) might not be a state in general. More examples with $m = 2, d = 1$ have been exhibited by the author and Varso [GV23], for some of which positivity is yet open.

In this paper we reinvestigate Schürmann & Voß’ proof of the Schoenberg correspondence for positive universal products [SV14, Section 3], observe that a variation of their approach allows to furnish a proof which does not rely on Muraki’s classification, and simultaneously generalize the proof to the multivariate situation, where the problem was still open (cf. [MS17, Remark 4.5]). On the way we present concise and applicable statements about the Lachs functor, a versatile and useful tool to study of universal products developed by Ben Ghorbal & Schürmann [BGS02, BGS05], Lachs [Lac15], and Manzel & Schürmann [MS17].

The paper is organized as follows. In Section 2 we define and review some properties of coalgebras and dual semigroups. A coalgebra $C$ has a comultiplication $\Delta: C \otimes C$ which allows to define a convolution for linear functionals on $C$ based on the tensor product $\varphi_1 \ast \varphi_2 := (\varphi_1 \otimes \varphi_2) \circ \Delta$. Dual semigroups are defined similar to coalgebras, but a dual semigroup $B$ is an algebra and has its comultiplication $\Lambda$ take values in the free product $B \sqcup B$. We will also need a multivariate version of dual semigroups called $m$-faced dual semigroups to formulate the multivariate Schoenberg correspondences. In Section 3 we define (multivariate) universal products, which are ways to form a product of linear functionals $\varphi_1, \varphi_2$ on algebras $A_1, A_2$ to obtain a linear functional $\varphi_1 \circ \varphi_2$ on the free product $A_1 \sqcup A_2$ of the domains. In particular, a universal product gives rise to a convolution of linear functionals $\varphi_1, \varphi_2: B \to C$ on a dual semigroup $(B, \Lambda)$, defined as $\varphi_1 \ast \varphi_2 := (\varphi_1 \circ \varphi_2) \circ \Lambda$.\]
In Section 4 we discuss a key tool to understand universal products and convolutions, the *Lachs functor* $L$, which allows to express (multivariate) dual semigroup convolutions $\varphi_1 \ast \varphi_2$ as usual coalgebraic convolutions on a bialgebra $L(B)$. We formulate precise statements about $L$ in Theorem 4.1 in the very efficient language of monoidal categories and monoidal functors as laid out for example in [AM10]; for readers unfamiliar with those notions, in Remark 4.4, we try to provide enough intuition on what the Lachs functor does to follow the rest of this paper, also the statements of 4.3 and 4.5, which describe the relationship between coalgebraic and dual semigroup convolutions and convolution exponentials, can be understood without background knowledge in monoidal categories, although their proofs become quite lengthy (cf. [BG105, Theorems 3.4 and 4.6] for the concrete proofs in the univariate case). Finally, Section 5 is devoted to the main result of this paper, the Schoenberg correspondence for multivariate universal products, which characterizes the linear functionals $\psi$ on a $*$-$m$-faced dual semigroup $B$ for which the convolution exponentials $\exp_\otimes(t\psi)$ are (restrictions of) states on the unitization of $B$ for all $t > 0$.

Before we dive into the details, let us vaguely outline what is the idea behind the proof of Schoenberg correspondence in [SV14] and what is the difference in this paper. Consider a structure with an underlying vector space $X$ and

- a notion of positivity, i.e., a subset $X_+ \subset X$
- two *multiplications* $\cdot_1, \cdot_2: X \times X \rightarrow X$ which preserve positivity
- two *exponentials* $\exp_{\cdot_1}, \exp_{\cdot_2}: X \rightarrow X$.

Under good circumstances, one can express one exponential via the other as

$$\exp_{\cdot_2}(x) = \lim_{n \rightarrow \infty} \exp_{\cdot_1}(\tfrac{1}{n} x) \cdot_2 \ldots \cdot_2 \exp_{\cdot_1}(\tfrac{1}{n} x) = \lim_{n \rightarrow \infty} \left(\exp_{\cdot_1}(\tfrac{1}{n} x)\right)^{2^n}.$$ 

Now suppose that there is a subset $G \subset X$ of *generators* such that $\exp_{\cdot_1}(tg)$ is positive for all $g \in G$ and $t > 0$. Then one can conclude from Equation (1) that $\exp_{\cdot_2}(tg)$ is positive for all $g \in G$ and $t > 0$ (as long as $X_+$ is closed). Universal products lead to a lot of possibilities make this idea precise. In [SV14] (inspired by similar results of Schürmann, Skeide & Volkwardt for bialgebras [SSV10]), different comultiplications $\Lambda_1, \Lambda_2: B \rightarrow B \oplus B$ lead to different convolutions and convolution exponentials on the dual space $B'$ such that Equation (1) holds. Therefore, Schoenberg correspondence can be transferred from easier to more complicated dual semigroups; the universal product is fixed beforehand and one has to establish one starting point for each universal product, which is where the dependence on classification theorems stems from. In this paper, the different exponentials correspond to different universal products and we still obtain Equation (1). This allows us in the end to reduce all Schoenberg correspondences in the context of universal products to Schürmann’s Schoenberg correspondence for linear functionals on $*$-bialgebras.

## 2. Coalgebras and dual semigroups

A coalgebra is a vector space $C$ with a coassociative comultiplication $\Delta: C \rightarrow C \otimes C$ (i.e., $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$) and a counit $\delta: C \rightarrow \mathbb{C}$ (i.e., $(\delta \otimes \text{id})\Delta = \text{id} = (\text{id} \otimes \delta)\Delta$); we identify $C \equiv C \otimes \mathbb{C} \equiv \mathbb{C} \otimes C$ and $(C \otimes C) \otimes C \equiv C \otimes (C \otimes C)$ in the natural way. In other words, coalgebras are comonoids in the monoidal category $(\text{Vec}, \otimes, \mathbb{C})$. The dual $C'$ of a coalgebra $C$ is a unital algebra with respect to the convolution product $f \ast g = (f \otimes g) \circ \Delta$ and with unit $\delta$. 
Two key properties make coalgebras much easier to work with than dual semigroups:

- The tensor product and, therefore, coalgebraic convolution are bilinear. (Universal products and convolutions on dual semigroups are not!)
- Every finite dimensional subspace of a coalgebra is contained in a finite dimensional subcoalgebra (this is known as fundamental theorem for coalgebras, for a proof see e.g. [DNR01, Theorem 1.4.7]).

These properties alone allow to establish a good functional calculus on the convolution algebra. For convenience of the reader, we present here the specific instances of this calculus which we will need in the sequel.

**Lemma 2.1** (cf. [SSV10, Section 4]). Let $(C, \Delta, \delta)$ be a coalgebra, $\psi \in C'$.

1. For every $c \in C$, the evaluation of the exponential series
   \[ \exp_C(\psi)(c) = \sum_{n=0}^{\infty} \frac{\psi^n(c)}{n!} \]
   is absolutely convergent in $\mathbb{C}$.

2. Suppose that $R_n, n \in \mathbb{N}$, are linear functionals on $C$ such that for each $c \in C$ there is a constant $K_c \in \mathbb{R}_+$ with $|R_n(c)| \leq \frac{1}{n!} K_c$ for all $n \in \mathbb{N}$. Then, for every $c \in C$,
   \[ \exp_C(\psi)(c) = \lim_{n \to \infty} \left( \delta + \frac{\psi}{n} + R_n \right)^n(c). \]

**Proof.** We sketch the arguments given in [SSV10]. Fix $c \in C$ and let coalg$(c)$ be smallest subcoalgebra of $C$ which contains $c$. Then coalg$(c)$ is finite dimensional, so we can choose an arbitrary norm on coalg$(c)$, and this induces (operator) norms on coalg$(c)'$ and $L(\text{coalg}(c))$; we denote all three norms simply $\| \cdot \|$. Then $L(\text{coalg}(c))$ is a finite-dimensional unital normed algebra. With the convolution operator $T_\psi := (\text{id} \otimes \psi) \circ \Delta | \text{coalg}(c) \in L(\text{coalg}(c))$, for every polynomial $p$ we have $p(\psi) = \delta(p(T_\psi))$. Therefore,
   \[ \sum_{n=k}^\ell \left| \frac{\psi^n(c)}{n!} \right| = \sum_{n=k}^\ell \frac{\| \delta(T_\psi^n) \|}{n!} \leq \| \delta \| \| c \| \sum_{n=k}^\ell \frac{\| T_\psi^n \|^n}{n!} \xrightarrow{k, \ell \to \infty} 0. \]

The second claim is a specialization of [SSV10, Lemma 4.2]. The idea is to prove a corresponding result for Banach algebras, estimate the norm of $T_{R_n} = (\text{id} \otimes R_n) \circ \Delta | \text{coalg}(c)$, and conclude
\[ \left( \delta + \frac{\psi}{n} + R_n \right)^n(c) = \delta \left( \text{id} + \frac{T_\psi}{n} + T_{R_n} \right)^n(c) \xrightarrow{n \to \infty} \delta \exp(T_\psi)(c) = \exp_C(\psi)(c). \]

\[ \square \]

In this article, **algebra** always means an associative, not necessarily unital algebra over the complex numbers $\mathbb{C}$. The category of algebras with algebra homomorphisms is denoted Alg. The free product of algebras is denoted $\biguplus$, indicating that this is the coproduct in the category of algebras. This turns Alg into a monoidal category with the trivial algebra $\{0\}$ as unit object.

An **m-faced algebra** is an algebra $A$ together with subalgebras $A^1, \ldots, A^m$ (the **faces** of $A$) which freely generate $A$, i.e., the canonical homomorphism $A^1 \biguplus \ldots \biguplus A^m \to A$ is an isomorphism; we indicate this by writing $A = A^1 \biguplus \ldots \biguplus A^m$. An
**m-faced algebra homomorphism** $j: A \to B$ is an algebra homomorphism between $m$-faced algebras $A,B$ with $j(A^k) \subset B^k$. The category of $m$-faced algebras with $m$-faced algebra homomorphisms is denoted $\text{Alg}_m$. We consider the free product of $m$-faced algebras again an $m$-faced algebra with faces $(A \sqcup B)^k := A^k \sqcup B^k$. Note that the free product of $m$-faced algebras is the coproduct in the category $\text{Alg}_m$. Therefore, $(\text{Alg}_m, \sqcup, \{0\})$ is a monoidal category for every $m \in \mathbb{N}$.

It should be clear from the context whether an upper index labels a face or indicates a power, but to avoid confusion, we will sometimes explicitly write $A^\otimes d$ instead of $A^d$ for the direct sum $A \oplus \ldots \oplus A$ of $d$ copies when $A$ is an $m$-faced algebra.

**Definition 2.2.** An $m$-faced dual semigroup is an $m$-faced algebra $B$ together with an $m$-faced homomorphism $\Lambda: B \to B \sqcup B$ (the comultiplication) which is coassociative and fulfills the counit property, i.e.,

$$(\Lambda \sqcup \text{id})\Lambda = (\text{id} \sqcup \Lambda)\Lambda \quad \text{and} \quad (\text{id} \sqcup 0)\Lambda = (0 \sqcup \text{id})\Lambda$$

(the counit property makes the unique linear map $0: B \to \{0\}$ a counit for $\Lambda$). In other words, an $m$-faced dual semigroup is a comonoid $(B, \Lambda, 0)$ in the monoidal category $(\text{Alg}_m, \sqcup, \{0\})$. A 1-faced dual semigroup is simply called dual semigroup.

**Example 2.3.** The standard example of a dual semigroup is the tensor algebra $T(V)$ over a vector space with primitive comultiplication $\Lambda(v) = v_1 + v_2 \in T(V) \sqcup T(V)$. If $V = V^1 \oplus \cdots \oplus V^m$, $T(V)$ is $m$-faced with $T(V)^k = T(V^k) \subset T(V)$.

One can also define ($m$-faced) dual semigroups in a framework of unital algebras. We briefly sketch the relation of such unital dual semigroups with the dual semigroups defined in Definition 2.2. Details (for $m = 1$, but $m > 1$ works analogous) can be found in [BGS05, Section 3].

Consider the category $\text{uAlg}$ of unital algebras with unital algebra homomorphisms. For $\text{uAlg}$, the coproduct is given by the unital free product $\sqcup_1$ (i.e., the free product with identification of units) and the initial object is $\mathbb{C}$. Therefore, $(\text{uAlg}, \sqcup_1, \mathbb{C})$ is a monoidal category.

Let $\mathds{1}: \text{Alg} \to \text{uAlg}$ denote the unitization functor; for an algebra $A$, $\mathds{1}A$ is the unital algebra which contains $A$ as an ideal of codimension 1 (even if $A$ had a unit beforehand!) and for an algebra homomorphism $j: A \to B$, $\mathds{1}j: \mathds{1}A \to \mathds{1}B$ is the unique unital extension of $j$. This is a monoidal functor, which means, roughly speaking, that one can identify $\mathds{1}(A \sqcup B) \equiv \mathds{1}A \sqcup \mathds{1}B$. A unital dual semigroup is an unital algebra $B$ with unital algebra homomorphism $\Lambda: B \to B \sqcup_1 B$ and $\lambda: B \to \mathbb{C}$ such that coassociativity $((\Lambda \sqcup \text{id})\Lambda = (\text{id} \sqcup \Lambda)\Lambda)$ and the counit property $((\lambda \sqcup_1 \text{id})\lambda = \text{id} = (\text{id} \sqcup_1 \lambda)\lambda)$ are fulfilled; in short, a unital dual semigroup is a comonoid in $(\text{uAlg}, \sqcup_1, \mathbb{C})$. There is a one-to-one correspondence between dual semigroups and unital dual semigroups (up to isomorphism) given by the following prescriptions:

- $(B, \Lambda, \lambda) \mapsto (B, \Lambda | B, 0)$ with $B := \ker \lambda$; one can check that $\Lambda(B) \subset B \sqcup B$ (identified with a subset of $B \sqcup_1 B$) and the rest follows easily.
- $(A, \Lambda, 0) \mapsto (\mathds{1}A, \mathds{1}\Lambda, \mathds{1}0)$.

This establishes an equivalence of the categories of dual semigroups and unital dual semigroups.

**Remark 2.4.** The unital variant is closer to Voiculescu’s original definition of dual groups in [Voi87] (in the context of pro-C*-algebras), while the nonunital variant
is typically preferred in the context of universal products (cf. [BGS05, MS17]); the reason is that, in general, universal products do not respect identification of units, which complicates the formulas and statements considerably. Both versions are $H_0$-algebras in the sense of Zhang [Zha91].

We also chose to define dual semigroups in the nonunital framework to harmonize better with the definition of universal products in Section 3. However, in certain contexts, e.g., in Section 5 of this paper where we study positivity of linear functionals on dual semigroups, one should think of the nonunital algebras as ideals in their unitization in order to get the right intuition.

3. Multivariate universal products

As opposed to the coalgebraic setting, there is no canonical candidate for a convolution product for functionals on a dual semigroup; linear functionals aren’t morphisms in Alg$\_m$, so we can not simply replace the tensor product in the convolution formula by the free product. In this section we recall the concept of (multivariate) universal products; introduced in [BGS02] and generalized in [MS17], universal products are designed as ideal replacements for the tensor product in the definition of convolution.

**Definition 3.1.** A $m$-faced $d$-valued linear functional, $m$-$d$-functional for short, is a pair $(A, \varphi)$, where $A$ is an $m$-faced algebra and $\varphi: A \rightarrow \mathbb{C}^d$ a linear map. By $\mathcal{F}_{m,d}$ we denote the category of $m$-$d$-functionals, i.e.,  
- an object of $\mathcal{F}_{m,d}$ is an $m$-$d$-functional
  
- a morphism $j: (A_1, \varphi_1) \rightarrow (A_2, \varphi_2)$ is an $m$-faced algebra homomorphism $j: A_1 \rightarrow A_2$ with $\varphi_1 = \varphi_2 \circ j$.

The obvious forgetful functor which associates with each $m$-$d$-functional $(A, \varphi)$ its domain $A$ is denoted $\text{dom}: \mathcal{F}_{m,d} \rightarrow \text{Alg}_{m}$. Usually, we will write $\varphi$ instead of $(A, \varphi)$ and $\text{dom}(\varphi)$ to refer to $A$. Note that $\text{dom}(j) = j$ for every morphism.

By $1\varphi$ we denote the unique unital extension $1\varphi: 1A \rightarrow \mathbb{C}^d$. (If $\varphi$ is also an algebra homomorphism, there is a possible conflict of notation, but it should be quite clear from the context which unital extension is meant.)

Of course, a linear map $\varphi: A \rightarrow \mathbb{C}^d$ is uniquely determined by its components $\varphi_k: A \rightarrow \mathbb{C}, k \in \{1, \ldots, d\}$. We will often make use of the transpose of $\varphi$ defined as $\varphi^{tr}: A^{\odot d} \rightarrow \mathbb{C}, \varphi^{tr}(a_1, \ldots, a_d) := \sum_{k=1}^d \varphi_k(a_k)$.

**Remark 3.2.** The category $\mathcal{F}_{m,d}$ is (trivially) isomorphic to the category AlgP$\_d, m$ of [MS17]. The different choice of notation is mainly made because the reader should not be encouraged to think of all linear functionals on algebras as analogues of probability measures; while this interpretation is quite adequate for certain functionals which encode moments of noncommutative random variables, cumulant functionals are also represented by objects of $\mathcal{F}_{m,d}$ and play an equally important, but very different role.

Next, we present the categorical definition of universal products, the traditional definition can be found in Remark 3.4 right below.

**Definition 3.3.** An $m$-$d$-universal product is a bifunctor $\odot: \mathcal{F}_{m,d} \times \mathcal{F}_{m,d} \rightarrow \mathcal{F}_{m,d}$ such that

(UP1) The bifunctor $\odot$ turns $\mathcal{F}_{m,d}$ into a monoidal category with unit object $0: \{0\} \rightarrow \mathbb{C}^d$. 
We introduce a second functor, \( L \), that acts on the same action on morphisms as \( F \). Then \( L \) is a map \( \mathbb{L} \mathbb{A} \mathbb{G} \mathbb{L} \rightarrow \mathbb{C} \mathbb{D} \mathbb{I} \) which associates with functionals \( \varphi_1, \varphi_2 \) on \( m \)-faced algebras \( A_1, A_2 \), respectively, a functional \( \varphi_1 \circ \varphi_2 \) on \( A_1 \sqcup A_2 \) such that

- \( (\varphi_1 \circ j_1) \odot (\varphi_2 \circ j_2) = (\varphi_1 \odot \varphi_2) \odot (j_1 \sqcup j_2) \) (universality)
- \( (\varphi_1 \odot \varphi_2) \odot \varphi_3 = \varphi_1 \odot (\varphi_2 \odot \varphi_3) \) (associativity)
- \( \varphi \odot 0 = \varphi = 0 \odot \varphi \) (unit condition)

Finally, using universality and the fact that the embeddings \( A_k \rightarrow A_1 \sqcup A_2 \) can be written \( \text{id} \sqcup 0 \): \( A_1 \sqcup \{0\} = A_1 \rightarrow A_1 \sqcup A_2 \), \( 0 \sqcup \text{id} : \{0\} \sqcup A_2 = A_2 \rightarrow A_1 \sqcup A_2 \), the last condition is easily seen to be equivalent to

- \( (\varphi_1 \circ \varphi_2) \mid A_1 = \varphi_i \) (restriction property).

**Remark 3.5.** Working without condition (UP2) is possible. However, as an effect of the universal property of the free product (being the coproduct in the category of \( m \)-faced algebras), this would not be a big generalization. For some details on this, we refer the reader to [GLS16, Section 5.4].

**Definition 3.6.** Let \((B, \Lambda)\) be an \( m \)-faced dual semigroup and \( \odot \) an \( m \)-d-universal product. The **convolution** of \( d \)-valued functionals \( \varphi_1, \varphi_2 : B \rightarrow \mathbb{C}^d \) is defined as

\[
\varphi_1 \star \varphi_2 := (\varphi_1 \odot \varphi_2) \circ \Lambda.
\]

**Remark 3.7.** Note that the convolution depends on the choice of a universal product. Also be aware that universal products and, therefore, the just defined convolution are typically not bilinear. In particular, the \( d \)-valued functionals form a monoid, but not an algebra. It is also not meaningful to define a convolution exponential directly via the exponential series with respect to the operation \( \star \).

**4. Convolution Exponentials and Lachs Functor**

In this section we recall a key technique for dealing with convolution on dual semigroups developed by Ben Ghorbal & Schürmann [BGS02, BGS05], Lachs [Lac15] and Manzel & Schürmann [MS17]. Remark 4.4 below explains the respective contributions and sheds some light on the underlying ideas. The results presented in this section, Theorem 4.1, Corollary 4.3 and Corollary 4.5, are not new, but compress the essence of [Lac15, Section 5.2] and [MS17, Section 5] in a way suitable for application in Section 5 as well as for possible future reference.

Fix \( m, d \in \mathbb{N} \). Let \( u\mathbb{A} \mathbb{G} \) denote the category of unital algebras and \( u\mathbb{F} \) the category of linear functionals on unital algebras (the functionals need not be unital). Given an \( m \)-faced algebra \( A \), we denote by \( \mathcal{L}(A) = S(A^{\oplus d}) \) the symmetric tensor algebra over the direct sum of \( d \) copies of \( A \). Clearly, this defines a functor \( \mathcal{L} : \mathbb{L} \mathbb{A} \mathbb{G} \rightarrow u\mathbb{G} \) with the obvious action on morphisms \( \mathcal{L}(j) = S(j^{\oplus d}) \). Note also that \( \mathcal{L}(\{0\}) \equiv \mathbb{C} \). Recall that the transpose of \( \varphi : A \rightarrow \mathbb{C}^d \) is a map \( \varphi^{tr} : A^{\oplus d} \rightarrow \mathbb{C} \). We introduce a second functor, \( \mathcal{L}_F : \mathcal{F}_{m,d} \rightarrow u\mathbb{F} \) with \( \mathcal{L}_F(\varphi) = S(\varphi^{tr}) \) and the same action on morphisms as \( \mathcal{L} \). Then \( \mathcal{L} \circ \text{dom} = u\text{dom} \circ \mathcal{L}_F \) (where \( u\text{dom} \) is the forgetful functor \( u\mathbb{F} \rightarrow u\mathbb{G} \) that maps each functional to its domain). When there
is no fear of confusion, we omit the subscript and denote both functors by the same symbol $\mathcal{L}$.

Recall that a colax monoidal functor (or cotensor functor) is a functor $F: C_1 \to C_2$ between monoidal categories $(C_k, \otimes, E_k)$ together with a natural transformation $\Delta: F(\cdot \otimes_1 \cdot) \to F(\cdot) \otimes_2 F(\cdot)$ and a morphism $\delta: F(E_1) \to E_2$ which fulfill certain compatibilities \( \text{(coassociativity and counitality), see [AM10, Definition 3.2].} \)

**Theorem 4.1.** Let $\otimes$ be an $m$-$d$-universal product. Then there is a natural transformation $\sigma: \mathcal{L}(\cdot \sqcup \cdot) \to \mathcal{L}(\cdot) \otimes \mathcal{L}(\cdot)$ with the following properties.

- $(\mathcal{L}, \sigma, \text{id}_C): (\text{Alg}_m, \sqcup, \{0\}) \to (u\text{Alg}, \otimes, \mathbb{C})$ is a colax monoidal functor.
- $(\mathcal{L}_F, \sigma, \text{id}_C): (F_{m,d}, \otimes, \{0\}) \to (u\mathcal{F}, \otimes, \mathbb{C})$ is a colax monoidal functor, in particular, for all pairs of $m$-$d$ functionals $\varphi_1, \varphi_2$, it holds that
  \[ S((\varphi_1 \otimes \varphi_2)^{tr}) = (S(\varphi_1^{tr}) \otimes S(\varphi_2^{tr})) \circ \sigma_{A_1,A_2}. \]

**Definition 4.2.** The Lachs functor is the colax monoidal functor $\mathcal{L}_\otimes := (\mathcal{L}_F, \sigma, \text{id}_C)$. Note that the natural transformation $\sigma$, which depends on $\otimes$, is an integral part of the structure. We may simply write $\mathcal{L}$ instead of $\mathcal{L}_\otimes$ if the underlying universal product is understood from the context. In the sequel, usage of the symbol $\mathcal{L}$ always means that an underlying universal product is assumed.

Recall that a bialgebra is a vector space with compatible structures of coalgebra and unital algebra (i.e., the comultiplication and counit are unital algebra homomorphisms). In the categorical language, a bialgebra is a comonoid in $(u\text{Alg}, \otimes, \mathbb{C})$.

**Corollary 4.3.** The Lachs functor induces a functor between the categories of comonoids, i.e., it maps an $m$-faced dual semigroup $(B, \Lambda)$ to a (by construction commutative) bialgebra $\mathcal{L}(B) := (S(B^{\otimes d}), \sigma_{B,B} \circ S(\Lambda^d), S(0))$. Furthermore, it intertwines convolutions, i.e.,

$$\mathcal{L}(\varphi_1 \ast \varphi_2) = S(\varphi_1 \ast \varphi_2)^{tr} = S(\varphi_1^{tr}) * S(\varphi_2^{tr}) = \mathcal{L}(\varphi_1) * \mathcal{L}(\varphi_2).$$

**Proof.** Colax monoidal functors preserve comonoids [AM10, Proposition 3.29]. That $\mathcal{L}$ intertwines convolutions is a direct consequence of Equation (2). \qed

**Remark 4.4.** The construction underlying the Lachs functor dates back to Ben Ghorbal & Schürmann [BGS02, BGS05]. Its functorial properties were exhibited by Lachs [Lac15, Section 5.2]. Manzel & Schürmann generalized the construction to $m$-$d$-universal products in [MS17, Section 5]. Theorem 4.1 is closely related to the “universal coefficient theorem”, which states, roughly speaking, that an $m$-$d$-universal product $\varphi_1 \otimes \varphi_2(a_1 \ldots a_n)$ can always be written as a polynomial in “sub-evaluations” $\varphi_i(a_{k_1} \ldots a_{k_d})$ with coefficients which are independent of the $\varphi_i$ and $a_k$ (cf. [MS17, Theorem 4.2] for the precise statement). We are satisfied with showing one example which hopefully makes this relation apparent for the reader. Let $\otimes_f$ be the free product of linear functionals used in free probability. Then

$$\varphi_1 \otimes_f \varphi_2(a_1 b_1 a_2 b_2) = \varphi_1(a_1 a_2) \varphi_2(b_1) \varphi_2(b_2) + \varphi_1(a_1) \varphi_1(a_2) \varphi_2(b_1 b_2) - \varphi_1(a_1) \varphi_1(a_2) \varphi_2(b_1) \varphi_2(b_2)$$
for all \( \varphi_1, \varphi_2 \) and all \( a_1, a_2 \in \text{dom} \varphi_1, b_1, b_2 \in \text{dom} \varphi_2 \). Therefore, a linear map \( \sigma \) with
\[
\sigma(a_1b_1a_2b_2) = a_1a_2 \otimes (b_1 \cdot b_2) + (a_1 \cdot a_2) \otimes b_1b_2 - (a_1 \cdot a_2) \otimes (b_1 \cdot b_2)
\]
("\( \cdot \)" the multiplication of the symmetric tensor algebra) fulfills Equation (2) when evaluated at \( a_1b_1a_2b_2 \in \text{dom} \varphi_1 \sqcup \text{dom} \varphi_2 \). By the universal coefficient theorem, this scheme works for all functionals \( \varphi_1, \varphi_2 \) and all elements of \( \text{dom} \varphi_1 \sqcup \text{dom} \varphi_2 \) and leads to Theorem 4.1. One can also go the other way round, i.e., the natural transformation of the Lachs functor determines a “polynomial form” of the corresponding universal product via Equation (2) and thus implies the universal coefficient theorem.

The exponential on the dual \( B' \) of a bialgebra \( B \) maps a derivation \( d \) (with respect to the comultiplication \( \delta \), i.e. \( d(ab) = d(a)\delta(b) + \delta(a)d(b) \)) to a character \( \exp(d) \) (a multiplicative linear functional); indeed,
\[
d^m \circ \mu = d^{\otimes m} \circ \Delta^{(m)} \circ \mu = d^{\otimes m} \circ \mu^{\otimes n} \circ \Delta_B \otimes B = (d \circ \mu)^n = (d \otimes \delta + \delta \otimes d)^* \mu^n
\]
and therefore, since \( d \otimes \delta \) and \( \delta \otimes d \) commute under the convolution on \( (B \otimes B)' \),
\[
\exp_s d \circ \mu = \frac{1}{n!} d^{*n} \circ \mu = \exp_s (d \otimes \delta + \delta \otimes d) = (\exp_s d \otimes \delta) * (\delta \otimes \exp_s d) = \exp_s d \otimes \exp_s d.
\]
For a linear functional \( f \), \( S(f) \) is the unique extension of \( f \) to a character on \( S(\text{dom} f) \). We denote by \( D(f) \) the unique extension of \( f \) to an \( S(0) \)-derivation on \( S(\text{dom} f) \), i.e.,
\[
D(f)(a_1 \cdot \ldots \cdot a_n) = \begin{cases} f(a_1) & n = 1, \\ 0 & n \neq 1. \end{cases}
\]
This gives us yet another functor, the differential Lachs functor, \( D: \mathcal{F}_{m,d} \rightarrow \mathfrak{uF} \) with \( D(\psi) = D(\psi^{tr}) \). With this notation, we can comfortably describe convolution exponentials on \( m \)-faced dual semigroups.

**Corollary 4.5.** For an \( m \cdot d \)-universal product \( \otimes \) and an \( m \)-faced dual semigroup \( B \), there exists a unique map \( \exp_\otimes: B' \rightarrow B' \) such that, for all \( \psi \in \mathcal{F}_{m,d} \), \( \text{dom} \psi = B \),
\[
\mathcal{L}(\exp_\otimes(\psi)) = \exp_{\mathcal{L}(B)} D(\psi)
\]
or, equivalently,
\[
\exp_\otimes(\psi) = (\exp_{\mathcal{L}(B)} D(\psi) \restriction B^{\otimes d})^{tr}.
\]

**Proof.** Taking \( \exp_\otimes(\psi) := (\exp_{\mathcal{L}(B)} D(\psi) \restriction B^{\otimes d})^{tr} \) as a definition, we see that
\[
\mathcal{L}(\exp_\otimes(\psi)) = \mathcal{L}( (\exp_{\mathcal{L}(B)} D(\psi) \restriction B^d)^{tr} ) = S(\exp_{\mathcal{L}(B)} D(\psi) \restriction B^d) = \exp_{\mathcal{L}(B)} D(\psi),
\]
where the last equality follows from the fact that \( \exp_{\mathcal{L}(B)} D(\psi) \) is a character and, trivially, an extension of its restriction \( \exp_{\mathcal{L}(B)} D(\psi) \restriction B^{\otimes d} \). Uniqueness follows from restricting to \( B^{\otimes d} \).

Note that even for \( m = d = 1 \), \( \exp_{\mathcal{L}(B)} D(\psi) \restriction B \) will typically not agree with \( \psi \), consequently the character \( \exp_{\mathcal{L}(B)} D(\psi) \) is different from \( S(\psi) \).
5. Positivity

For applications in noncommutative probability, it is desirable that universal products can be formed for distributions of noncommutative random variables. There is a small caveat here because positivity is best expressed in a unital framework – many examples of universal products are, however, not fully compatible with the unital structures. The way out is to only consider unital \( \ast \)-algebras \( \mathcal{A} \) of the form \( \mathcal{A} = \mathbb{1} \mathcal{A} \) for a given \( \ast \)-algebra \( \mathcal{A} \). (Alternatively, what is practically the same, one assumes that \( \mathcal{A} \) is an augmented \( \ast \)-algebra, i.e., a unital algebra with a \( \ast \)-character \( \varepsilon: \mathcal{A} \to \mathbb{C} \); then, with \( A := \text{kern} \, \varepsilon \), there is a canonical isomorphism \( \mathcal{A} \cong \mathbb{1} A \).)

In the following definition, we write \( x \geq 0 \) for a vector \( x \in \mathbb{C}^d \) if all its components are non-negative real numbers.

**Definition 5.1.** A \( d \)-valued linear functional \( \varphi: \mathcal{A} \to \mathbb{C}^d \) on a \( \ast \)-algebra \( \mathcal{A} \) is called

- **restricted state** if its unitization \( \mathbb{1} \varphi \) is positive on the unital \( \ast \)-algebra \( \mathbb{1} \mathcal{A} \), i.e., if
  \[ \mathbb{1} \varphi(a^*a) \geq 0 \text{ for all } a \in \mathbb{1} \mathcal{A}, \]
- **restricted generating functional** if it is hermitian and fulfills
  \[ \varphi(a^*a) \geq 0 \text{ for all } a \in \mathcal{A}. \]

For brevity, we decided to omit "\( d \)-valued" when it comes to (restricted) states and generating functionals, but note that \( \varphi \) is a restricted state if and only if all components of \( \mathbb{1} \varphi \) are states.

**Remark 5.2.**

- Restricted states are automatically hermitian. Indeed, a restricted state \( \varphi: \mathcal{A} \to \mathbb{C} \) induces a positive sesquilinear form on \( \mathbb{1} \mathcal{A} \), \( (a, b)_\varphi := \varphi(a^*b) \), therefore \( \varphi(a^*) = (a, 1)_\varphi = \overline{(1, a)_\varphi} = \varphi(a) \). Note that positivity on \( \mathcal{A} \) is not sufficient for the argument to work.

- A restricted generating functional \( \psi \) extends to a generating functional \( \hat{\psi}: \mathbb{1} \text{ dom} \psi \to \mathbb{C}^d \), i.e., to a \( \psi \) which is hermitian, vanishes at 1, and is positive on the ideal \( \text{ dom} \psi \subset \mathbb{1} \text{ dom} \psi \) (conditionally positive).

The free product of \( \ast \)-algebras carries a natural involution (the unique one extending the involutions of the factors) and will thus be considered a \( \ast \)-algebra.

**Definition 5.3.** An \( m \)-faced \( \ast \)-algebra is an \( m \)-faced algebra with an involution such that the faces are \( \ast \)-subalgebras. A \( \ast \)-\( m \)-faced dual semigroup is an \( m \)-faced dual semigroup with an involution such that the faces are \( \ast \)-subalgebras and the comultiplication is a \( \ast \)-homomorphism (i.e, a comonoid in the category of \( m \)-faced \( \ast \)-algebras).

**Definition 5.4.** An \( m \)-\( d \)-\( \ast \)-distribution is a pair \( (\mathcal{A}, \varphi) \), where \( \mathcal{A} \) is an \( m \)-faced \( \ast \)-algebra and the components of \( \varphi: \mathcal{A} \to \mathbb{C}^d \) are restricted states.

By simply ignoring the \( \ast \)-structure, we can lift a universal product to the category \( \ast \)-\( F_{m,d} \) of \( d \)-valued linear functionals on \( m \)-faced \( \ast \)-algebras, leaving us with the question how much of the inherent positivity structure is respected by a given universal product.

**Definition 5.5.** Let \( \odot \) be an \( m \)-\( d \)-universal product. We say that \( \odot \) is
• positive if the universal product $\varphi_1 \odot \varphi_2$ of $m$-$d$-distributions $\varphi_1, \varphi_2$ is an $m$-$d$-distribution,
• half positive if the universal product powers $\varphi^\odot n$ of an $m$-$d$-distribution $\varphi$ are $m$-$d$-distributions for all $n \in \mathbb{N}$,
• Schoenberg if the convolution exponential $\exp_\odot(\psi)$ is a restricted state for every restricted generating functional $\psi$ on an $*$-$m$-faced dual semigroup.

If $\psi$ is a restricted generating functional on a $*$-algebra, then also $t\psi$ is a restricted generating functional for all $t > 0$; therefore, if $\odot$ is Schoenberg and $\psi$ is a restricted generating functional on a $*$-$m$-faced dual semigroup, then $\exp_\odot t\psi$ are restricted states for all $t > 0$. A standard derivation argument can be used to show that if $\exp_\odot t\psi$ is a restricted state for all $t > 0$, then $\psi$ must be a restricted generating functional: indeed, for $b \in B$, the map $f : \mathbb{R}_+ \to \mathbb{C}, t \mapsto \exp_\odot t\psi(b^*b)$ is differentiable, positive and vanishes at zero, therefore $\psi(b^*b) = \frac{d}{dt} f(t)|_{t=0} \geq 0$. Combining the last two arguments, we obtain the following characterization of the Schoenberg property.

**Observation 5.6.** A universal product is Schoenberg if and only if the Schoenberg correspondence holds:

\[
\psi \text{ restricted generating functional } \iff \exp_\odot t\psi \text{ restricted state for all } t > 0
\]

**Observation 5.7.** Let $(B, \Lambda)$ be a $*$-$m$-faced dual semigroup and $\odot$ an $m$-$d$-universal product. Since $\Lambda$ is a $*$-homomorphism, we have the following implications:

- If $\odot$ is positive, then $\varphi_1 \ast \varphi_2 = (\varphi_1 \odot \varphi_2) \circ \Lambda$ is a restricted state for all restricted states $\varphi_1, \varphi_2 : B \to \mathbb{C}^d$.
- If $\odot$ is half positive, then $\varphi^*n = \varphi^\odot n \circ \Lambda_n$ is a restricted state for every restricted state $\varphi : B \to \mathbb{C}^d$, where $\Lambda_n : B \to B^\otimes n$ denotes the iterated comultiplication.

One might think that the Schoenberg correspondence for dual semigroups follows trivially from Schürmann’s Schoenberg correspondence for $*$-bialgebras [SchI85] by application of the Lachs functor, but it is a bit more complicated than that: the natural transformation $\sigma$ of the Lachs functor (Theorem 4.1) does not fulfill any obvious compatibility with the internal multiplication (in particular, $\sigma \mid B^d \sqcup B^d : B^d \sqcup B^d \to S(B^d) \otimes S(B^d)$ is not a $*$-homomorphism), so there is no direct transfer of positivity properties! Still, it is a key tool in the proof of Schürmann and Voß’ Schoenberg correspondence for positive universal products with $m = d = 1$ and it is the key tool in the proof we will present for the general $m$-$d$-case.

**Remark 5.8.** Roughly speaking, the strategy of Schürmann & Voß [SV14] is to apply Lemma 2.1 (2) in order to reduce the Schoenberg correspondence on a complicated dual semigroup to the Schoenberg correspondence on a dual semigroup with primitive comultiplication; using Muraki’s classification, it is known that Schoenberg correspondence holds in those cases by explicit construction of a Lévy process on a suitable Fock space via quantum stochastic calculus. Our strategy is to apply the same Lemma 2.1 (2) in order to reduce the Schoenberg correspondence for a complicated universal product to the Schoenberg correspondence for the tensor product. This has the huge advantage that the proof is then independent of classification results and specialized quantum stochastic calculi, which are not available in the multivariate setting.
Theorem 5.9. For an $m$-d-universal product $\odot$, the following implications hold:

\[
\text{positive} \implies \text{half positive} \implies \text{Schoenberg}
\]

Proof. The first implication is trivial. For the second implication, let $\odot$ be an arbitrary half positive $m$-d-universal product and $\otimes$ the $m$-d-universal product given by componentwise tensor product (ignoring the faces), i.e.,

\[
(\varphi_1 \otimes \varphi_2)_k = (\varphi_1)_k \otimes (\varphi_2)_k.
\]

Let $B$ be a $*$-$m$-faced dual semigroup and $\psi$ a restricted generating functional. We note that, for $b \in B^{\otimes d}$,

\[
\left| \exp_{\mathcal{L}\otimes(B)} \left( \frac{t}{n} \mathcal{D}(\psi) \right) (b) - \delta(b) - \frac{t}{n} \mathcal{D}(\psi)(b) \right| \leq \frac{1}{n^2} \sum_{k=2}^{\infty} \frac{\mathcal{D}(\psi)^*(b)}{n!} \leq \frac{1}{n^2} K_b
\]

by Lemma 2.1 (1). Therefore, Lemma 2.1 (2) combined with Corollary 4.5, for $\odot$ as well as for $\otimes$, implies

\[
\mathcal{L}(\exp_\odot(\psi))(b) = \exp_{\mathcal{L}_\odot(B)}(\mathcal{D}(\psi))(b)
\]

\[
= \lim_{n \to \infty} \left( \exp_{\mathcal{L}_\otimes(B)} \left( \frac{1}{n} \mathcal{D}(\psi) \right) \ast_{\mathcal{L}_\otimes(B)} \cdots \ast_{\mathcal{L}_\otimes(B)} \exp_{\mathcal{L}_\otimes(B)} \left( \frac{1}{n} \mathcal{D}(\psi) \right) \right) (b)
\]

\[
= \lim_{n \to \infty} \mathcal{L} \left( \exp_\otimes \left( \frac{1}{n} \mathcal{D}(\psi) \right) \ast_{\mathcal{L}_\otimes(B)} \cdots \ast_{\mathcal{L}_\otimes(B)} \mathcal{L} \left( \exp_\otimes \left( \frac{1}{n} \mathcal{D}(\psi) \right) \right) \right) (b)
\]

\[
= \lim_{n \to \infty} \mathcal{L} \left( \exp_\otimes \left( \frac{1}{n} \mathcal{D}(\psi) \right) \ast_{\mathcal{L}_\otimes(B)} \cdots \ast_{\mathcal{L}_\otimes(B)} \mathcal{L} \left( \exp_\otimes \left( \frac{1}{n} \mathcal{D}(\psi) \right) \right) \right) (b).
\]

Therefore, for $b \in B$, we find

\[
\exp_\odot(\psi)(b) = \lim_{n \to \infty} \left( \exp_\otimes \left( \frac{1}{n} \mathcal{D}(\psi) \right) \right)^* (b).
\]

By half positivity, $\exp_\odot(\psi)$ is a restricted state if $\exp_\otimes \left( \frac{1}{n} \mathcal{D}(\psi) \right)$ is a restricted state for all $n$. So we are done if we can show that $\otimes$ is Schoenberg. As the tensor product does not depend on the faces and does not mix the components, it is enough to show that it is Schoenberg for $m = d = 1$. This follows for example from the results of [SV14]. We present an alternative path. Note that the canonical $*$-homomorphism $\hat{\sigma} : B \sqcup B \to \mathbf{1}B \otimes \mathbf{1}B$ with $b_1 \mapsto b \otimes 1, b_2 \mapsto 1 \otimes b$ coincides, as a linear map, with $\sigma_\otimes | B \sqcup B$ from Theorem 4.1. It follows that $\mathbf{1}B \subset \mathcal{S}(B)$ is a $*$-bialgebra with comultiplication $\Delta := \mathbf{1}(\hat{\sigma} \circ A)$ and $\mathbf{1} \exp_\otimes(\psi) = \exp_{1B,\Delta}(\hat{\psi})$, where $\hat{\psi}$ is the extension of $\psi$ to a generating functional on $\mathbf{1}B$ (i.e., $\hat{\psi}(1) = 0$). Therefore, $\exp_\otimes \psi$ is a restricted state by the Schoenberg correspondence for $*$-bialgebras [Schü88]. □

6. Open Problems

We conclude with mentioning two natural questions this result provokes and to which we do not know the answer yet.

(1) Are the non-positive universal products derived from c-bifree-independence half positive or Schoenberg?

(2) If the answer to question 1 is negative, do the reverse implications in Theorem 5.9 hold?
Acknowledgements

I thank Michael Schürmann, Michael Skeide, and Philipp Varšo for very helpful discussions and feedback on earlier drafts of these notes. I thank Reviewer 2 at Communications in Mathematical Physics for pointing at a number of places where giving some additional details now hopefully helps readers with a more probabilistic or analytic background to follow the algebraic and categorical arguments.

This work was supported by the German Research Foundation (DFG) grant no. 397960675. The work was partly carried out under the tenure of an ERCIM ‘Alain Bensoussan’ Fellowship Programme at NTNU Trondheim and as a postdoctoral scientific employee at University of Greifswald.

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