Ramsey degrees of ultrafilters, pseudointersection numbers, and the tools of topological Ramsey spaces

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Abstract
This paper investigates properties of $\sigma$-closed forcings which generate ultrafilters satisfying weak partition relations. The Ramsey degree of an ultrafilter $\mathcal{U}$ for $n$-tuples, denoted $t(\mathcal{U},n)$, is the smallest number $t$ such that given any $l \geq 2$ and coloring $c : [\omega]^n \to l$, there is a member $X \in \mathcal{U}$ such that the restriction of $c$ to $[X]^n$ has no more than $t$ colors. Many well-known $\sigma$-closed forcings are known to generate ultrafilters with finite Ramsey degrees, but finding the precise degrees can sometimes prove elusive or quite involved, at best. In this paper, we utilize methods of topological Ramsey spaces to calculate Ramsey degrees of several classes of ultrafilters generated by $\sigma$-closed forcings. These include a hierarchy of forcings due to Laflamme which generate weakly Ramsey and weaker rapid p-points, forcings of Baumgartner and Taylor and of Blass and generalizations, and the collection of non-p-points generated by the forcings $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$. We provide a general approach to calculating the Ramsey degrees of these ultrafilters, obtaining new results as well as streamlined proofs of previously known results. In the second half of the paper, we calculate pseudointersection and tower numbers for these $\sigma$-closed forcings and their relationships with the classical pseudointersection number $p$.

Keywords Topological Ramsey spaces · Forcing · Ultrafilters · Partition relations · Pseudointersection number · Ellentuck space

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1 Introduction

Ramsey’s Theorem states that given any \( n, l \geq 2 \) and a coloring \( c : [\omega]^n \to l \), there is an infinite subset \( X \subseteq \omega \) such that \( c \) is constant on \( [X]^n \) [38]. A Ramsey ultrafilter is an ultrafilter satisfying instances of Ramsey’s Theorem. Thus, an ultrafilter \( U \) on base set \( \omega \) is Ramsey if and only if given any \( U \in U, n, l \geq 2 \), and coloring \( c : [U]^n \to l \), there is some \( X \in U \) with \( X \subseteq U \) such that \( c \) is constant on \( [X]^n \). This is denoted by

\[
U \rightarrow (U)^n_1. \tag{1}
\]

Ramsey ultrafilters can be constructed using CH, or just \( \text{cov}(\mathcal{M}) = c \). They can also be constructed by forcing with \( \mathcal{P}(\omega)/\text{Fin} \), or equivalently, \( ([\omega]^\omega, \subseteq^*) \). Generalizations of this forcing in many different directions have been used to construct ultrafilters which are not Ramsey but have similar properties in the following sense.

**Definition 1** Let \( U \) be an ultrafilter on a countable base set \( S \). For \( n \geq 2 \), define \( t(U, n) \) to be the least number \( t \), if it exists, such that for each \( l \geq 2 \) and each coloring \( c : [S]^n \to l \), there is a member \( X \in U \) such that the restriction of \( c \) to \( [X]^n \) takes no more than \( t \) colors. When \( t(U, n) \) exists, it is called the Ramsey degree of \( U \) for \( n \)-tuples, and we write

\[
U \rightarrow (U)_1^{t(U,n)}. \]

A weakly Ramsey ultrafilter is one satisfying \( t(U, 2) = 2 \). A plethora of ultrafilters with finite Ramsey degrees have been forced by various \( \sigma \)-closed posets in the literature. The following examples are indicative of the variety of such posets. Laflamme in [29] forced a hierarchy of rapid \( p \)-points above a weakly Ramsey ultrafilter. Baumgartner and Taylor in [3] forced \( k \)-arrow, not \( (k+1) \)-arrow ultrafilters, for each \( k \geq 2 \). These are rapid \( p \)-points satisfying an asymmetric partition relation. In [5], Blass constructed a \( p \)-point which has two Rudin-Keisler incomparable \( p \)-points below it, using a sort of two-dimensional forcing. The forcing \( \mathcal{P}(\omega \times \omega)/\text{Fin}^{\otimes 2} \) was investigated by Szymański and Zhou in [40] and shown to produce an ultrafilter, denoted \( G_2 \), which is not a \( p \)-point but has Ramsey degree \( t(G_2, 2) = 4 \). This ultrafilter was shown to be a weak \( p \)-point in [9] and investigations of its Tukey type are included in that paper. Further extensions of \( \mathcal{P}(\omega)/\text{Fin} \) to finite dimensions \( (\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}) \) for \( 2 \leq k < \omega \) were investigated by Kurilić in [28] and by Dobrinen in [14]. In fact, the natural hierarchy of forcings \( \mathcal{P}(\omega^\alpha)/\text{Fin}^{\otimes \alpha} \), for all countable ordinals \( \alpha \), was shown to be forcing equivalent to certain topological Ramsey spaces in [15]. These and other ultrafilters will be investigated in this paper in two directions: First, we will find a general method for calculating Ramsey degrees of ultrafilters from these classes. Second, we will investigate the pseudointersection and tower numbers of the forcings which generate these ultrafilters.
In recent years, the use of topological Ramsey spaces to investigate forcings generating ultrafilters has provided means for obtaining results which remained elusive when simply using the original forcings. This began in [20], where Dobrinen and Todorcevic constructed a Ramsey space dense inside of the forcing of Laflamme in [29] which produces a weakly Ramsey, not Ramsey ultrafilter, denoted $U_1$, in order to calculate the exact Rudin-Keisler and Tukey structures below this ultrafilter. This idea was extended in [14, 18, 21], and [15], providing new collections of topological Ramsey spaces dense in known forcings, such as those of Baumgartner-Taylor, Blass, Laflamme, and Szymański-Zhou mentioned above, as well as creating new forcings which produce ultrafilters with interesting partition relations. Such Ramsey spaces were used to find exact Rudin-Keisler and Tukey structures below those ultrafilters. An overview of this area can be found in [16].

Once constructed, it turned out that the topological Ramsey space structure of these forcings can be used to investigate how closely these ultrafilters resemble a Ramsey ultrafilter. One such investigation was recently carried out by Dobrinen and Hathaway in [17], where they show that each of the ultrafilters mentioned above have properties similar to that of a Ramsey ultrafilter in the sense of the barren extensions of Henle, Mathias, and Woodin in [26]. In this paper, we investigate properties of the same ultrafilters in [17], utilizing topological Ramsey space techniques to better handle the properties of the forcings.

Background on classical results is provided in Sect. 2 and an overview of topological Ramsey spaces and their associated ultrafilters appears in Sect. 3. While there is not space to reproduce all information on the topological Ramsey spaces addressed in this paper, an overview of their salient properties to aid the reader appear in Sect. 4. These include the spaces mentioned above as well as the Carlson–Simpson infinite dual Ramsey space and the space $\text{FIN}^{[\omega]}_k$ of infinite block sequences of Milliken and related spaces $\text{FIN}^{[\omega]}_k$ of Todorcevic, building on work of Gowers.

The first focus of this paper is on finding exact Ramsey degrees of the aforementioned ultrafilters from the papers [3, 5, 28, 29, 40], and [14], as well as new forcings from [18]. Finding the Ramsey degrees for these ultrafilters based on the original forcings is not always simple. A case in point is Laflamme’s proof that the Ramsey degrees for the weakly Ramsey ultrafilter $U_1$ are $t(U_1, n) = 2^{n-1}$, for $n \geq 2$. His proof can be greatly simplified by utilizing the structure of the topological Ramsey space dense in his forcing. In Sect. 5, we provide a simplified proof of this result, and provide streamlined proofs of other results stated without proof in [29] (see Theorem 46). In Theorem 40, we provide a general approach for determining Ramsey degrees for ultrafilters forced by Ramsey spaces with certain natural properties. Then in Sect. 5.2, we apply this to find Ramsey degrees for the ultrafilters of Laflamme, Baumgartner-Taylor, Blass, and the hypercube spaces from [18].

In Sect. 6 we calculate the Ramsey degrees $t(G_k, 2)$, where $G_k$ is the ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}_{\omega^k}$. These results are new, and the approach using Ramsey spaces dense in these forcing is much more succinct than approaching the Ramsey degrees using the forcings $\mathcal{P}(\omega^k)/\text{Fin}^{\otimes k}_{\omega^k}$.

In Sect. 7, we investigate pseudointersection and tower numbers related to several classes of topological Ramsey spaces. In [31], Malliaris and Shelah solved a
longstanding open problem, proving that \( p = t \). Any \( \sigma \)-closed forcing has naturally defined pseudointersection and tower numbers, and it is of interest to know when equality holds between these cardinal invariants. Each topological Ramsey space has a natural \( \sigma \)-closed quasi-ordering such that its separative quotient is isomorphic to the separative quotient of the Ramsey space with its given quasi-order. Given such a \( \sigma \)-closed order on a topological Ramsey space \( R \), one may define its pseudointersection number \( p_R \) and tower number \( t_R \) (see Definition 53).

In Sect. 7.1, we prove that all topological Ramsey spaces whose members are infinite sequences with a certain amount of independence between the entries of the sequence have \( p_R = t_R = p \). Such spaces include those dense in Laflamme’s forcings \( P_\alpha, 1 \leq \alpha < \omega_1 \), the forcings of Baumgartner-Taylor and of Blass, those in [18]. In contrast, in Sect. 7.2, we show that pseudointersection and tower numbers associated with the forcings \( \mathcal{P}(\omega^\alpha)/\text{Fin}^{\otimes \alpha} \) are all \( \omega_1 \). For \( \alpha \) a finite ordinal, this reproduces results of Kurilić in [28] albeit in what we consider a streamlined fashion. However, for infinite countable ordinals \( \alpha \), these results are new. Finally, in Sect. 7.3 we show that the Carlson-Simpson space forces a Ramsey ultrafilter. This is especially interesting as it is a dual space, quite different from the other spaces considered in this paper; and while it generates a Ramsey ultrafilter, its pseudointersection and tower numbers are \( \omega_1 \), proved by Matet in [32].

It remains open whether the pseudointersection and tower numbers are always equal for topological Ramsey spaces, and whether there are any such spaces for which the pseudointersection or tower numbers can lie strictly between \( \omega_1 \) and \( p \). These and other questions appear in Sect. 8.

### 2 Classical background and definitions

This section contains basic definitions and background for some classical results on ultrafilters and cardinal invariants on \( \omega \).

**Definition 2** 1. For two sets \( X, Y \subseteq \omega \) we say that \( X \) is almost contained in \( Y \), denoted \( X \subseteq^* Y \), if \( X \setminus Y \) is finite.

2. We say that a family of sets \( \mathcal{F} \subseteq [\omega]^{<\omega} \) has the strong finite intersection property (SFIP) if for every finite subfamily \( \mathcal{X} \subseteq [\mathcal{F}]^{<\omega} \), \( \bigcap \mathcal{X} \) is an infinite subset of \( \omega \).

3. Given \( \mathcal{F} \subseteq [\omega]^{<\omega} \), a pseudointersection of the family \( \mathcal{F} \) is a set \( Y \in [\omega]^{<\omega} \) such that for every \( X \in \mathcal{F}, Y \subseteq^* X \).

4. The **pseudointersection number** \( p \) is the smallest cardinality of a family \( \mathcal{F} \subseteq [\omega]^{<\omega} \) which has the SFIP but does not have a pseudointersection.

5. A **tower** is a sequence \( \langle X_\alpha : \alpha < \delta \rangle \) of members of \( [\omega]^{<\omega} \) which is linearly ordered by \( \supseteq^* \) and has no pseudointersection. The **tower number** \( t \) is the smallest cardinality of a tower.

It is well-known that Martin’s Axiom implies \( p = c \); see for instance [2] for a proof. The cardinal invariant \( m(\sigma\text{-centered}) \) is defined to be the minimum cardinal \( \kappa \) for which there exists a \( \sigma \)-centered partial order \( \mathcal{P} \) and a family \( \mathcal{D} \) of \( \kappa \) many dense subsets of \( \mathcal{P} \) which does not admit any \( \mathcal{D} \)-generic filter. Bell proved that \( m(\sigma\text{-centered}) = p \) (see [4]). It is clear from the definitions that \( p \leq t \), and one of the most
important longstanding open problems in cardinal invariants was whether the two are equal. Malliaris and Shelah recently proved that, indeed, $p = t$ (see [30] and [31]).

**Definition 3** Let $\mathcal{U}$ be an ultrafilter on $\omega$.

1. $\mathcal{U}$ is **Ramsey** if for each coloring $c : [\omega]^2 \to 2$, there is a $U \in \mathcal{U}$ such that $U$ is homogeneous for $c$, meaning $|c[U]^2| = 1$.
2. $\mathcal{U}$ is **weakly Ramsey** if for each coloring $c : [\omega]^2 \to 3$, there is a $U \in \mathcal{U}$ such that $|c[U]^2| \leq 2$.

It is well known that an ultrafilter $\mathcal{U}$ on $\omega$ is Ramsey if and only if it is **selective**: For each $\supseteq$-decreasing sequence $\langle U_n \rangle_{n \in \omega}$ of members of $\mathcal{U}$, there is an $X \in \mathcal{U}$ such that for each $n < \omega$, $X \subseteq^* U_n$ and moreover $|X \cap (U_n \setminus U_{n+1})| \leq 1$.

If $\mathcal{U}$ is a Ramsey ultrafilter, then it is routine to show that, in the notation of Definition 1, $t(\mathcal{U}, n) = 1$ for all $n \geq 1$. An ultrafilter $\mathcal{U}$ is weakly Ramsey if and only if $t(\mathcal{U}, 2) \leq 2$. It is clear that any Ramsey ultrafilter is weakly Ramsey. On the other hand, Blass proved in [6] that there are weakly Ramsey ultrafilters which are not Ramsey. Proofs of the following facts can be found in [2].

**Theorem 4** $\langle [\omega]^\omega, \subseteq^* \rangle$ forces a Ramsey ultrafilter.

Next, we introduce the Ellentuck topology on $[\omega]^\omega$. In this section, as in the rest of the paper, we use notation and terminology from [41]. Given $a \in [\omega]^{-\omega}$ and $A \in [\omega]^\omega$, define

$$[a, A] = \{ B \in [\omega]^\omega : a \sqsubset B \subseteq A \},$$

where $a \sqsubset B$ denotes that $a$ is an initial segment of $B$. The **Ellentuck topology** on $[\omega]^\omega$ is the topology generated by basic open sets the sets of the form $[a, A]$. This topology refines the standard metric topology on the Baire space. The space $[\omega]^\omega$ with the Ellentuck topology is called the Ellentuck space.

**Definition 5** A set $\mathcal{X} \subseteq [\omega]^\omega$ is called **Ramsey** if for every non-empty basic open set $[a, A]$ there is $B \in [a, A]$ with $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$.

While the Axiom of Choice implies that there are subsets of $[\omega]^\omega$ which are not Ramsey (see [23]), restricting to definable sets allowed for the development of Ramsey theory of subsets of $[\omega]^\omega$. A highlight of this development was the Galvin-Prikry Theorem [24] that all Borel sets are Ramsey. Silver later proved in [39] that all analytic sets are Ramsey. The pinnacle of this line of work was Ellentuck’s topological characterization of those subsets of $[\omega]^\omega$ which are Ramsey. His work was strongly informed by the above mentioned theorems as well as closely related work of Mathias, which was published in [33].

**Theorem 6** (Ellentuck [22]) Let $\mathcal{X} \subseteq [\omega]^\omega$. Then $\mathcal{X}$ is Ramsey if and only if $\mathcal{X}$ has the Baire Property in the Ellentuck topology.

After this work of Ellentuck, many similar topological spaces, members of which are infinite sequences, were formed and their Ramsey properties were investigated.
similarities were noticed by Carlson in [10], where he coined the term Ramsey space and proved a general theorem from which these other results followed as corollaries. This line of work was expanded by Carlson and Simpson in [12]. In his book [41], Todorcevic set forth a simplified set of axioms which guarantee that a space has Ramsey properties similar to the Ellentuck space. This is the subject of the next section.

3 Background on topological Ramsey spaces and their associated ultrafilters

In this section, we introduce topological Ramsey spaces and the main theorems which provide advantageous techniques. The following definition is taken from [41]. The axioms A.1–A.4 are defined for triples $(R, \leq, r)$ of objects with the following properties, where $R$ is a nonempty set, $\leq$ is a quasi-ordering on $R$, and $r : R \times \omega \rightarrow AR$ is a mapping giving us the sequence $(r_n(\cdot) \leq \cdot(p, n))$ of approximation mappings, where

$$AR = \{r_n(A) : A \in R \text{ and } n < \omega\}$$

is the collection of all finite approximations to members of $R$. For $a \in AR$ and $A, B \in R$,

$$[a, B] = \{A \in R : A \leq B \text{ and } \exists n \in \omega (r_n(A) = a)\}.$$ 

For $a \in AR$, let $|a|$ denote the integer $k$ such that $a = r_k(A)$, for some $A \in R$. For each $n < \omega$, $AR_n = \{r_n(A) : A \in R\}$; then $AR = \bigcup_{n<\omega} AR_n$.

A.1 (a) $r_0(A) = \emptyset$ for all $A \in R$.
(b) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some $n$.
(c) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

If $m < n$, $A \in R$, $a = r_m(A)$ and $b = r_n(A)$, then we will write $a \equiv r_m(b)$.

For $a, b \in AR$, $a \sqsubseteq b$ if and only if $a = r_m(b)$ for some $m \leq |b|$; if $a \sqsubseteq b$ and $a \neq b$, we write $a \sqsubset b$.

A.2 There is a quasi-ordering $\leq_{\text{fin}}$ on $AR$ such that

(a) $\{a \in AR : a \leq_{\text{fin}} b\}$ is finite for all $b \in AR$.
(b) $A \leq B$ if and only if for each $n \in \omega$ there exists $m \in \omega$ such that $r_n(A) \leq_{\text{fin}} r_m(B)$.
(c) For every $a, b, c \in AR$, if $a \sqsubseteq b$ and $b \leq_{\text{fin}} c$ then there exists $d \in AR$ such that $d \sqsubseteq c$ and $a \leq_{\text{fin}} d$.

The notation $\text{depth}_B(a)$ is defined to be the least $n$, if it exists, such that $a \leq_{\text{fin}} r_n(B)$. If such an $n$ does not exist, then write $\text{depth}_B(a) = \infty$. If $\text{depth}_B(a) = n < \infty$, then $[\text{depth}_B(a), B]$ denotes $[r_n(B), B]$.

A.3 (a) If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.
(b) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$. 

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If $n > |a|$, then $r_n[a, A]$ is the collection of all $b \in A R_n$ such that $a \sqsubset b$ and $b = r_m(B)$ for some $B \in [a, A]$.

**Definition** 7 A subset $X$ of $\mathcal{R}$ is Ramsey if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq X$ or $[a, B] \cap X = \emptyset$. $X \subseteq \mathcal{R}$ is Ramsey null if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap X = \emptyset$.

**Definition** 8 A triple $(\mathcal{R}, \leq, r)$ is a topological Ramsey space if every subset of $\mathcal{R}$ with the property of Baire is Ramsey and every meager subset of $\mathcal{R}$ is Ramsey null.

The following result is Theorem 5.4 in [41].

**Theorem 9** (Abstract Ellentuck Theorem) If $(\mathcal{R}, \leq, r)$ is closed (as a subspace of $A R^\omega$) and satisfies axioms A.1, A.2, A.3 and A.4, then every subset of $\mathcal{R}$ with the property of Baire is Ramsey, and every meager subset is Ramsey null; in other words, the triple $(\mathcal{R}, \leq, r)$ forms a topological Ramsey space.

Given $\mathcal{F} \subseteq A R$ and $X \in \mathcal{R}$, let $\mathcal{F} \upharpoonright X$ denote the set of all $s \in \mathcal{F}$ such that $s = r_n(Y)$ for some $n \in \omega$ and $Y \leq X$.

**Definition** 10 A family $\mathcal{F} \subseteq A R$ of finite approximations is
1. Nash-Williams if for all $a, b \in \mathcal{F}$, $a \sqsubseteq b$ implies $a = b$.
2. Ramsey if for every partition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ and every $X \in \mathcal{R}$, there are $Y \leq X$ and $i \in \{0, 1\}$ such that $\mathcal{F}_i \upharpoonright Y = \emptyset$.

The next theorem appears as Theorem 5.17 in [41], and follows from the Abstract Ellentuck Theorem.

**Theorem 11** (Abstract Nash-Williams Theorem) Suppose that $(\mathcal{R}, \leq, r)$ is closed (as a subspace of $A R^\omega$) and satisfies A.1–A.4. Then every Nash-Williams family of finite approximations is Ramsey.

The original Nash-Williams Theorem on $\omega$ states that each Nash-Williams subset $\mathcal{F} \subseteq [\omega]^{<\omega}$ is Ramsey [37].

In [34] Mijares introduced a generalization of the quasi-order $\subseteq^*$ on $[\omega]^\omega$ to topological Ramsey spaces.
Definition 12 For $X, Y \in \mathcal{R}$, write $X \preceq^* Y$ if there exists $a \in \mathcal{AR} \upharpoonright X$ such that $[a, X] \subseteq [a, Y]$. In this case we say that $X$ is an almost reduction of $Y$.

Note that for each $a \in \mathcal{AR} \upharpoonright X$, there exists $Z \in \mathcal{R}$ such that $a \preceq Z$ and $Z \preceq X$, so $\emptyset \neq [a, X] \subseteq [a, Y]$.

Remark 1 For the topological Ramsey spaces considered in this paper, $(\mathcal{R}, \preceq^*)$ is a $\sigma$-closed partial order such that $(\mathcal{R}, \preceq)$ and $(\mathcal{R}, \preceq^*)$ have isomorphic separative quotients. Thus, the two quasi-orders are interchangeable from the viewpoint of forcing. However, in certain instances, even coarser $\sigma$-closed quasi-orders will be used. Even so, these will still have separative quotients which are isomorphic to those of $(\mathcal{R}, \preceq)$ and $(\mathcal{R}, \preceq^*)$.

4 Several classes of topological Ramsey spaces and their associated ultrafilters

Given a topological Ramsey space $(\mathcal{R}, \preceq, r)$, the generic filter forced by $(\mathcal{R}, \preceq)$ induces an ultrafilter as we now show: In all known examples of topological Ramsey spaces, the collection of first approximations, $\mathcal{AR}_1$, is a countable set. If that is not the case for some particular space $\mathcal{R}$, the restriction $\mathcal{AR}_1 \upharpoonright A$ for any member $A$ of $\mathcal{R}$ is countable by Axiom A.2, so one may work below a fixed member of $\mathcal{R}$, if necessary.

Definition 13 Given a generic filter $G \subseteq \mathcal{R}$ for the forcing $(\mathcal{R}, \preceq)$, define

$$\mathcal{U}_R = \{ S \subseteq \mathcal{AR}_1 : S \supseteq \mathcal{AR}_1 \upharpoonright A \text{ for some } A \in G \}.$$  \hfill (2)

Lemma 14 Let $(\mathcal{R}, \preceq, r)$ be a topological Ramsey space, $\preceq^*$ be a $\sigma$-closed quasi-order coarsening $\preceq$, and $G \subseteq \mathcal{R}$ be a generic filter for $(\mathcal{R}, \preceq^*)$. Let $\mathcal{U}_R$ be the filter on base set $\mathcal{AR}_1$ defined in (2). Then $\mathcal{U}_R$ is an ultrafilter on the base set $\mathcal{AR}_1$.

Proof This follows from the Abstract Nash–Williams Theorem and genericity of $G$. \hfill $\Box$

This section introduces some topological Ramsey spaces and their associated ultrafilters whose Ramsey degrees, pseudointersection and tower numbers will be investigated in subsequent sections.

4.1 The topological Ramsey spaces $\mathcal{R}_\alpha$, $1 \leq \alpha < \omega_1$

In [29], Laflamme constructed a forcing, denoted $\mathbb{P}_1$, which generates a weakly Ramsey ultrafilter, denoted $\mathcal{U}_1$, which is not Ramsey. Although Blass had already shown such ultrafilters exist (see [6]), the point of $\mathbb{P}_1$ was to construct a weakly Ramsey ultrafilter with complete combinatorics, analogous to the result that any Ramsey ultrafilter in the model $\mathcal{V}[G]$ obtained by Lévy collapsing a Mahlo cardinal to $\aleph_1$ is $([\omega]^\omega, \preceq^*)$-equivalent to the generic ultrafilter on $\omega$.
generic over $\text{HOD}(\mathbb{R})^{V[G]}$ (see [8] and [33]). One of the advantages of forcing with topological Ramsey spaces is that the associated ultrafilter automatically has complete combinatorics in the presence of large cardinals (see [13] for the result and [16] for an overview of this area). In [20], a topological Ramsey space denoted $\mathcal{R}_1$ was constructed which forms a dense subset of Laflamme’s forcing $\mathbb{P}_1$, hence generating the same weakly Ramsey ultrafilter. The motivation for that construction was to find the exact Tukey structure below $\mathcal{U}_1$ as well as the precise structure of the Rudin-Keisler classes within these Tukey types, which were indeed found in [20]. Here, we reproduce a few definitions and facts relevant to this paper.

**Definition 15** ($(\mathcal{R}_1, \leq, r)$, [20]). Let $T_1$ denote the following infinite tree of height 2.

$$T_1 = \{\langle \rangle\} \cup \{\langle n \rangle : n < \omega\} \cup \bigcup_{n<\omega} \{\langle n, i \rangle : i \leq n\}.$$

$T_1$ can be thought of as an infinite sequence of finite trees of height 2, where the $n$-th subtree of $T_1$ is

$$T_1(n) = \{\langle \rangle, \langle n \rangle, \langle n, i \rangle : i \leq n\}.$$

The members of $\mathcal{R}_1$ are infinite subtrees of $T_1$ which have the same structure as $T_1$. That is, a tree $X \subseteq T_1$ is in $\mathcal{R}_1$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega}$ such that

1. $X \cap T_1(k_n) \cong T_1(n)$ for each $n < \omega$; and
2. whenever $X \cap T_1(j) \neq \emptyset$, then $j = k_n$ for some $n < \omega$.

When this holds, we let $X(n)$ denote $X \cap T_1(k_n)$, and call $X(n)$ the $n$-th subtree of $X$.

For $n < \omega$, $r_n(X)$ denotes $\bigcup_{i<n} X(i)$.

For $X, Y \in \mathcal{R}_1$, define $Y \leq X$ if and only if there is a strictly increasing sequence $(k_n)_{n<\omega}$ such that for each $n$, $Y(n)$ is a subtree of $X(k_n)$. Notice that by the structure of the members of $\mathcal{R}_1$, $Y \leq X$ exactly when $Y \subseteq X$. Given $a, b \in \mathcal{A}\mathcal{R}$, define $b \leq_{\text{fin}} a$ if and only if $b \subseteq a$.

Figures 1 present the first five “blocks” of the maximal member of $\mathcal{R}_1$. 

![Fig. 1 $r_5(T_1)$](image-url)
The members of $\mathcal{R}_1$ are subtrees of $\mathbb{T}_1$ which are isomorphic to $\mathbb{T}_1$. As the first step toward the main theorem of [20], the following was proved.

**Theorem 16** (Dobrinen and Todorčević, [20]) $(\mathcal{R}_1, \leq, r)$ is a topological Ramsey space.

Notice that by the structure of the members of $\mathcal{R}_1$, given $X, Y \in \mathcal{R}_1$, $Y \leq^* X$ (recall Definition 12) holds if and only if there is an $i < \omega$ and a strictly increasing sequence $(k_n)_{n \geq i}$ such that for each $n \geq i$, $Y(n) \subseteq X(k_n)$. Thus, the quasi-order $\leq^*$ turns out to be equivalent to $\subseteq^*$, since $Y \leq^* X$ if and only if $Y \subseteq^* X$. By an ultrafilter $\mathcal{U}_{\mathcal{R}_1}$ associated with the forcing $(\mathcal{R}_1, \leq^*)$ we mean the ultrafilter on base set $\mathcal{A}\mathcal{R}_1$ generated by the sets $\mathcal{A}\mathcal{R}_1 \mid X, X \in G$, where $G$ is some generic filter for $(\mathcal{R}_1, \subseteq^*)$. By the density of this topological Ramsey space in Laflamme’s forcing, this ultrafilter $\mathcal{U}_{\mathcal{R}_1}$ is isomorphic to the ultrafilter $\mathcal{U}_1$ generic for Laflamme’s forcing $\mathbb{P}_1$. Hence, it is weakly Ramsey but not Ramsey.

Continuing in this vein, Laflamme constructed a hierarchy of forcings $\mathbb{P}_\alpha$, $1 \leq \alpha < \omega_1$, in order to produce rapid p-points $\mathcal{U}_\alpha$ satisfying partition relations with decreasing strength as $\alpha$ increases, and such that for $\beta < \alpha$, $\mathcal{U}_\beta$ is Rudin-Keisler below $\mathcal{U}_\alpha$. In [29], Laflamme proved that each $\mathcal{U}_\alpha$ has complete combinatorics, and that below $\mathcal{U}_\alpha$, there is a decreasing chain of length $\alpha + 1$ of Rudin-Keisler types, the least one being that of a Ramsey ultrafilter. This left open, though, whether or not this chain is the only Rudin-Keisler structure below $\mathcal{U}_\alpha$.

Topological Ramsey spaces $\mathcal{R}_\alpha$ were constructed in [21] to produce dense subsets of Laflamme’s forcings $\mathbb{P}_\alpha$, hence generating the same generic ultrafilters. The reader is referred to [21] for the definition of these spaces. The Ramsey space techniques provided valuable methods for proving in [21] that indeed the Rudin-Keisler, and moreover, the Tukey structure below $\mathcal{U}_\alpha$ is exactly a chain of length $\alpha + 1$. Since the definition of the $\mathcal{R}_\alpha$ spaces is long, we do not include it. Here, we reproduce $\mathbb{T}_2$ and $\mathbb{T}_\omega$, with a minor modification not affecting its forcing properties which will make it easier to understand. The reader can then infer the structure of $\mathcal{R}_\alpha$ for each $1 \leq \alpha < \omega_1$. In Sect. 5, we will only work with $\mathcal{R}_k$ for $1 \leq k < \omega$, since the Ramsey degree $t(\mathcal{U}_\omega, 2) = \omega$. However, Sect. 7 will consider pseudointersection and tower numbers of $\mathcal{R}_\alpha$, for all $1 \leq \alpha < \omega_1$.

Members of $\mathcal{R}_2$ are subtrees of $\mathbb{T}_2$ with the same structure than $\mathbb{T}_2$ (see Fig. 2). At limit stages, the tree $\mathbb{T}_\omega$ diagonalizes over the previous $\mathbb{T}_n$, $n < \omega$. Members of $\mathcal{R}_\omega$ are subtrees of $\mathbb{T}_\omega$ with the same structure than $\mathbb{T}_\omega$ (see Fig. 3).

---

Fig. 2 $\mathbb{T}_2$
This subsection introduces topological Ramsey spaces constructed in [18]. The motivation for these spaces was to find dense subsets of some forcings of Blass in [5] and of Baumgartner and Taylor in [3] in order to better study properties of their forced ultrafilters (more details provided below). The construction was seen to easily generalize to any Fraïssé classes with the Ramsey property. We provide the basic ideas of the construction here.

Definition 17 (The space $\mathcal{R}(A)$, [18]) Fix some natural number $J \geq 1$, and for each $j < J$, let $\mathcal{K}_j$ be a Fraïssé class of finite linearly ordered relational structures with the Ramsey property. We say that $A = \langle (A_k, j)_{j < J} : k < \omega \rangle$ is a generating sequence if for each $j < J$, the following hold:

1. For each $k < \omega$, $A_k, j$ is a member of $\mathcal{K}_j$, and $A_0, j$ has universe of cardinality 1.
2. Each $A_k, j$ is a substructure of $A_{k+1}, j$.
3. For each structure $B \in \mathcal{K}_j$, there is a $k$ such that $B$ embeds into $A_k, j$.
4. For each pair $k < m < \omega$, there is an $n > m$ large enough that the following Ramsey property holds:

$$A_n, j \rightarrow (A_m, j)^{A_k, j}.$$ 

Let $A_k$ denote the $J$-tuple of structures $(A_k, j)_{j < J}$. It can be convenient to think of this as the product $\prod_{j < J} A_k, j$ with no additional relations. Let $\overline{A} = \langle (A_k, j) : j < \omega \rangle$. This infinite sequence $\overline{A}$ of $J$-tuples of finite structures is the maximal member of the space $\mathcal{R}(\overline{A})$. We define $B$ to be a member of $\mathcal{R}(\overline{A})$ if and only if $B = \langle (n_k, B_k) : k < \omega \rangle$, where

1. $(n_k)_{k < \omega}$ is some strictly increasing sequence of natural numbers; and
2. for each $k < \omega$, $B_k$ is an $J$-tuple $(B_k, j)_{j < J}$, where each $B_k, j$ is a substructure of $A_{n_k, j}$ isomorphic to $A_k, j$.

We use $B(k)$ to denote $(n_k, B_k)$, the $k$-th block of $B$. The $m$-th approximation of $B$ is $r_m(B) = (B(0), \ldots, B(m-1))$. 

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Define the partial order $\leq$ as follows: For $B = \langle \langle m_k, B_k \rangle : k < \omega \rangle$ and $C = \langle \langle n_k, C_k \rangle : k < \omega \rangle$, define $C \leq B$ if and only if for each $k$ there is an $l_k$ such that $n_k = m_{l_k}$ and for all $j < J$, $C_{k, j}$ is a substructure of $B_{l_k, j}$. The partial order $\leq_{\text{fin}}$ on the collection of finite approximations, $\mathcal{AR}$, is defined as follows: For $b = \langle \langle m_k, B_k \rangle : k < p \rangle$ and $c = \langle \langle n_k, C_k \rangle : k < q \rangle$, where $p, q < \omega$, define $c \leq_{\text{fin}} b$ if and only if there are $C \leq B$ such that $c = r_q(C)$, $b = r_p(B)$. For these spaces, the naturally associated $\sigma$-closed partial order $\leq^*$ from Definition 12 is simply $\subseteq^*$.

**Theorem 18** (Dobrinen, Mijares and Trujillo [18]) Given a generating sequence $\langle \langle A_{k,j} \rangle_{k<\omega} : j < J \rangle$, the triple $\langle \mathcal{R}(\mathbb{A}), \leq^* \rangle$ forms a topological Ramsey space.

Letting $\mathcal{R}$ denote $\mathcal{R}(\mathbb{A})$, given a generic filter $G$ for the forcing $(\mathcal{R}, \leq^*)$, we let $\mathcal{U}_G$ denote the ultrafilter on base set $\mathcal{AR}_1$ generated by the sets $\mathcal{AR}_1 \upharpoonright X, X \in G$. The motivation for these spaces came from studying the Tukey types below ultrafilters constructed in [5] and [3]. The special case where $n = 2$ and both $\mathcal{K}_0$ and $\mathcal{K}_1$ are the classes of finite linear orders produces a Ramsey space which is dense inside the $N$-square forcing of Blass in [5], which he constructed to produce a p-point which has two Rudin-Keisler incomparable selective ultrafilters Rudin-Keisler below it.

Given $n \geq 2$, we shall let $\mathcal{H}_m$ denote the Ramsey space produced when each $\mathcal{K}_j$, $j < n$, is the class of finite linear orders; call this space the $n$-hypercube space. The space $\mathcal{H}_2$ is dense in Blass’ forcing, and hence the ultrafilter $\mathcal{U}_{\mathcal{H}_2}$ is isomorphic to the one constructed by Blass. The collection of spaces $\mathcal{H}_m, n \geq 2$, form a hierarchy of forcings such that each ultrafilter $\mathcal{U}_{\mathcal{H}_m}$ projects to the ultrafilter $\mathcal{U}_{\mathcal{H}_2}$ for $m < n$.

It is shown in [18] that the initial Tukey structure below $\mathcal{U}_{\mathcal{H}_2}$ is isomorphic to the Boolean algebra $\mathcal{P}(n)$. In another direction, the special cases where $J = 1, k \geq 3$ is fixed, and $\mathcal{K}_0$ is the class of all finite ordered $k$-clique-free graphs produces Ramsey spaces which are dense inside partial orders constructed by Baumgartner and Taylor in [3] which produce p-points which have asymmetric partition relations, called $k$-arrow ultrafilters. Results on the initial Rudin-Keisler and Tukey structures of ultrafilters constructed by Ramsey spaces from generating sequences appear in [18], which includes some work of Trujillo in his thesis [42].

### 4.3 High dimensional Ellentuck spaces

The next topological Ramsey spaces we present are the high dimensional Ellentuck spaces. We shall let $\mathcal{E}_1$ denote the Ellentuck space; that is $\langle (\omega)^\omega, \subseteq, r \rangle$, where for $X \in (\omega)^\omega$ and $n < \omega$, $r_n(X) = \{x_i : i < n\}$ where $\{x_i : i < \omega\}$ is the increasing enumeration of $X$. The first new space, $\mathcal{E}_2$, was motivated by a problem left open in [9]: finding the precise structure of the ultrafilters Tukey reducible to the generic ultrafilter forced by $\mathcal{P}(\omega^2)/\text{Fin}^{\otimes 2}$, denoted by $\mathcal{G}_2$.

Here, $\text{Fin}^{\otimes 2}$ is the ideal of all subsets $I \subseteq \omega \times \omega$ for which all but finitely many fibers are finite; that is, for all but finitely many $n$, the set $\{i < \omega : (n, i) \in I\}$ is finite. This ultrafilter $\mathcal{G}_2$ is not a p-point, but still satisfies the partition relation $\mathcal{G}_2 \rightarrow (\mathcal{G}_2)^2_{\omega, \leq 4}$, which is the best partition relation a non-p-point can possess. Similarly to the construction of $\text{Fin}^{\otimes 2}$, ideals $\text{Fin}^{\otimes k+1}$ can be recursively defined to consist of those subsets $X$ of $\omega^{k+1}$ such that for all but finitely many $n$, the collection $\{(j_1, \ldots, j_k) : (n, j_1, \ldots, j_k) \in X\}$
is a member of \( \text{Fin}^{\otimes k} \). Each ideal \( \text{Fin}^{\otimes k} \) is \( \sigma \)-closed under the partial order \( \subseteq \text{Fin}^{\otimes k} \), and the Boolean algebras \( P(\omega^k)/\text{Fin}^{\otimes k} \), \( k \geq 2 \), force ultrafilters, denoted \( G_k \), which form a hierarchy in the sense that \( G_j \) is recovered as the projection of \( G_k \) to its first \( j \) coordinates, for \( j < k \). We let \( G_1 \) denote the ultrafilter forced by \( P(\omega)/\text{Fin} \); this is a Ramsey ultrafilter.

High dimensional Ellentuck spaces \( \mathcal{E}_k \) were constructed by the first author in order to form topological Ramsey spaces which are forcing equivalent to the Boolean algebras \( P(\omega^k)/\text{Fin}^{\otimes k} \). This was utilized to find the exact Rudin-Keisler and Tukey structures below each \( G_k \); these turn out to be exactly chains of length \( k \) (see [14]). Having already proved their forcing equivalence to the Boolean algebras \( P(\omega^k)/\text{Fin}^{\otimes k} \) in [14], we provide here the simpler sequence version of these spaces: These are the “domain spaces” in [15], and this formulation can be found also in [1].

**Definition 19** For \( k \geq 2 \), denote by \( \omega^{k\leq k} \) the collection of all non-decreasing sequences of members of \( \omega \) of length less than or equal to \( k \).

The lexicographic order on \( \omega^{k\leq k} \) is defined as usual; it is presented here to aid the reader.

**Definition 20 (The lexicographic order on \( \omega^{k\leq k} \))** Let \( (s_0, \ldots, s_i), (t_0, \ldots, t_j) \in \omega^{k\leq k} \) be given. We say that \( (s_0, \ldots, s_i) \) is lexicographically below \( (t_0, \ldots, t_j) \), written \( (s_0, \ldots, s_i) \prec_{\text{lex}} (t_0, \ldots, t_j) \), if and only if there is a non-negative integer \( m \) with the following properties:

(i) \( m \leq \min(i, j) \);
(ii) for each \( n < m \), \( s_n = t_n \); and
(iii) either \( s_m < t_m \), or else \( m = i < j \) and \( s_i = t_i \).

Whereas for \( k \geq 2 \), \( \prec_{\text{lex}} \) has order type greater than \( \omega \), the next well-order has order-type \( \omega \) for any \( k \geq 2 \). This will aid in defining the finite approximations of members of \( \mathcal{E}_k \), and is central to seeing how each \( \mathcal{E}_k \) can be obtained as a projection of \( \mathcal{E}_{k+1} \).

**Definition 21 (The well-ordered set \( (\omega^{k\leq k}, \prec) \))** Set the empty sequence \( () \) to be the \( \prec \)-minimum element of \( \omega^{k\leq k} \); so, for all nonempty sequences \( s \) in \( \omega^{k\leq k} \), we have \( () \prec s \). In general, given \( (s_0, \ldots, s_i) \) and \( (t_0, \ldots, t_j) \) in \( \omega^{k\leq k} \) with \( i, j \geq 1 \), define \( (s_0, \ldots, s_i) \prec (t_0, \ldots, t_j) \) if and only if either

1. \( s_i < t_j \), or
2. \( s_i = t_j \) and \( (s_0, \ldots, s_i) \prec_{\text{lex}} (t_0, \ldots, t_j) \).

**Notation** Let \( \omega^{k} \) denote the collection of all non-decreasing sequences of length \( k \) of members of \( \omega \), and notice that the \( \prec \) also well-orders \( \omega^{k} \) in order type \( \omega \). Let \( u_n \) denote the \( n \)-th member of \( (\omega^{k}, \prec) \). For \( s, t \in \omega^{k\leq k} \), we say that \( s \) is a proper initial segment of \( t \) and write \( s \subset t \) if \( s = (s_0, \ldots, s_i), t = (t_0, \ldots, t_j), i < j \), and for all \( m \leq i, s_m = t_m \).
Definition 22 (The spaces \((E_k, \leq, r), 2 \leq k < \omega, [14]\)) An \(E_k\)-tree is a function \(\tilde{X}\) from \(\omega^{k \leq k}\) into \(\omega^{k}\) that preserves the well-order \(<\) and proper initial segments \(\subset\). For \(\tilde{X}\) an \(E_k\)-tree, let \(X\) denote the restriction of \(\tilde{X}\) to \(\omega^{k}\). The space \(E_k\) is defined to be the collection of all \(X\), where \(\tilde{X}\) is an \(E_k\)-tree. We identify \(X\) with its range, which is a subset of \(\omega^{k}\), and usually will write \(X = \{x_0, x_1, \ldots\}\), where \(x_0 = X(u_0) < x_1 = X(u_1) < \cdots\). The partial ordering on \(E_k\) is defined to be simply inclusion; that is, given \(X, Y \in E_k\), \(X \leq Y\) if and only if (the range of) \(X\) is a subset of (the range of) \(Y\). For each \(n < \omega\), the \(n\)-th restriction function \(r_n\) on \(E_k\) is defined by \(r_n(X) = \{x_i : i < n\}\). We set
\[
A E_n^k := \{r_n(X) : X \in E_k\} \quad \text{and} \quad A E^k := \{r_n(X) : n < \omega, X \in E_k\}
\]
to denote the set of all \(n\)-th approximations to members of \(E_k\), and the set of all finite approximations to members of \(E_k\), respectively.

Theorem 23 (Dobrinen [14]) For each \(2 \leq k < \omega\), \((E_k, \leq, r)\) is a topological Ramsey space. Moreover, \((E_k, \subseteq \text{Fin}^{\omega_k})\) is forcing equivalent to \(P(\omega^k)/\text{Fin}^{\omega_k}\).

In the notation of this paper, given a generic filter \(G\) for \((E_k, \subseteq \text{Fin}^{\omega_k})\), let \(U_{E_k}\) denote the ultrafilter on base set \(AE_1^k\) generated by the sets \(AE_1^k \upharpoonright X, X \in G\). Since \((E_k, \subseteq \text{Fin}^{\omega_k})\) and \(P(\omega^k)/\text{Fin}^{\omega_k}\) are forcing equivalent, \(U_{E_k}\) is isomorphic to \(G_k\), defined above.

We now provide some details for the spaces \(E_2\) and \(E_3\) in order to provide the reader with more intuition.

Example 1 (The space \(E_2\)) The members of \(E_2\) look like \(\omega\) many copies of the Ellen-tuck space, with finite approximations obeying the well-ordering \(<\). The well-order \(\langle \omega^{k \leq 2}, <\rangle\) begins as follows:
\[
() < (0) < (0, 0) < (0, 1) < (1) < (1, 1) < (0, 2) < (1, 2) < (2) < (2, 2) < \ldots
\]
The lexicographic order on \(\omega^{k \leq 2}\) has order type equal to the countable ordinal \(\omega^2\). Here, we picture an initial segment of \(\omega^{k \leq 2}\); the set of maximal nodes is precisely \(r_{15}(\omega^{k^2})\) (Fig. 4).
Example 2 (The space $E_3$) The well-order $\langle \omega^{I \leq 3}, \prec \rangle$ begins as follows:

\[
\emptyset \prec (0) \prec (0, 0) \prec (0, 0, 0) \prec (0, 0, 1) \prec (0, 1, 1) \prec (1) \\
\prec (1, 1) \prec (1, 1, 1) \prec (0, 0, 2) \prec (0, 1, 2) \prec (0, 2, 2) \\
\prec (1, 1, 2) \prec (1, 2) \prec (1, 2, 2) \prec (2) \prec (2, 2) \prec (0, 0, 3) < \ldots
\]

The set $\omega^{I \leq 3}$ is a tree of height three with each non-maximal node branching into $\omega$ many nodes. Figure 5 shows the initial structure of $\omega^{I \leq 3}$; the set of maximal nodes forms $r_{20}(\omega^{I \leq 3})$.

4.4 Infinite dimensional Ellentuck spaces

Now we present a class of topological Ramsey spaces which are structurally based on uniform barriers. This subsection is taken from [15] where the first author builds a new class of continuum many infinite dimensional Ellentuck spaces to continue the construction of $\mathcal{P}(\omega^{I \leq k})/\text{Fin}^{I \leq k}$ to the countable transfinite.

For $a, b \in [\omega]^{<\omega}$, we shall use the notation $a \sqsubseteq b$ to denote that $a$ is an initial segment of $b$, and $a \triangleleft b$ to denote that $a$ is a proper initial segment of $b$. This will serve to distinguish the partial ordering of initial segment on $[\omega]^{<\omega}$ from the partial ordering $u \sqsupset v$ of initial segments for $u, v \in \mathcal{AR}$, finite approximations of members of a topological Ramsey space $\mathcal{R}$.

Definition 24 A family $B$ of finite subsets of $\omega$ is a barrier on $\bigcup B$ if

1. $a \subset b$ whenever $a \neq b \in B$; and
2. $\bigcup B$ is infinite and for every infinite $M \subset \bigcup B$ there is an $a \in B$ such that $a \triangleleft M$.

For a barrier $B$ and $n \in \omega$, let $B_n = \{ b \in B : n = \min(b) \}$; and let $B_{[n]} = \{ a \in [\omega]^{<\omega} : \min(a) > n \text{ and } \{ n \} \cup a \in B \}$. For $N$ an infinite subset of $\bigcup B$, $B|N = \{ b \in B : b \subset N \}$.

Definition 25 Let $\alpha < \omega_1$ and $M$ be an infinite subset of $\omega$. A subset $B \subset [\omega]^{<\omega}$ is an $\alpha$-uniform family on $M$ provided that

(a) $\alpha = 0$ implies $B = \emptyset$.
(b) $\alpha = \beta + 1$ implies that $\emptyset \notin B$ and $B_{[n]}$ is $\beta$-uniform on $M \setminus (n + 1)$, for all $n \in M$.
(c) $\alpha > 0$ is a limit ordinal implies that there is an increasing sequence $\{ \alpha_n \}_{n \in M}$ of ordinals covering to $\alpha$ such that $B_{[n]}$ is $\alpha_n$-uniform on $M \setminus (n + 1)$, for all $n \in M$. 
A barrier $B \subset [\omega]^{<\omega}$ which is also a uniform family is called a uniform barrier.

In [15], the first author developed the hierarchy of ideals $\text{Fin}^B$ on $\mathcal{P}(B)$, for $B$ a uniform barrier. Given a uniform barrier $B$ on $\omega$ of lexicographic rank $\alpha$, note that $B_{[n]}$ is a uniform barrier on $\omega \setminus (n + 1)$ of lexicographic rank less than $\alpha$, for any $n < \omega$. Assuming we have defined $\text{Fin}^C$ for each uniform barrier $C$ of lexicographic rank less than $\alpha$, then we have also defined $\text{Fin}^{B_{[n]}}$ for each $n < \omega$ (relativizing to $\omega \setminus (n + 1)$). Let $\text{Fin}^{B_n} = \{X \subset B_n : \{a \setminus \{n\} : a \in X\} \in \text{Fin}^{B_{[n]}}\}$.

**Definition 26** For $B$ a uniform barrier, define $\text{Fin}^B = \{A \subset B : \forall \omega \exists n(A_n \in \text{Fin}^{B_n})\}$.

For each uniform barrier $B$ on $\omega$, $\text{Fin}^B$ is a $\sigma$-closed ideal; hence, $\mathcal{P}(B)/\text{Fin}^B \setminus \{0\}$ is a $\sigma$-closed partial order which can be used to construct an ultrafilter $\mathcal{G}_B$.

Let $\subset \text{Fin}^B$ denote the following partial order on $\mathcal{P}(B)$: For $X, Y \subset B$, $X \subset \text{Fin}^B Y$ if and only if $Y \setminus X \in \text{Fin}^B$. $\mathcal{P}(B)/\text{Fin}^B$ is forcing equivalent to $(\mathcal{P}(B), \subset \text{Fin}^B)$.

We do not include the whole definition of spaces $\mathcal{E}_B$ because it is long, but we include the definition of $\mathbb{W}_B$, which is the top member of the space $\mathcal{E}_B$. For $s \in \omega^{k < \omega}$, let $\text{lh}(s)$ denote the length of $s$, that is, the cardinality of the domain of the sequence $s$. Let $\prec_{\text{lex}}$ denote the lexicographic ordering on $\omega^{k < \omega}$, where we also consider any proper initial segment of a sequence to be lexicographically below that sequence. Given $s, t \in \omega^{k < \omega}$, define $s \prec t$ if and only if either

1. $\max(s) < \max(t)$, or
2. $\max(s) = \max(t)$ and $s \prec_{\text{lex}} t$.

Thus, $\prec$ well-orders $\omega^{k < \omega}$ in order-type $\omega$, with the empty sequence $()$ as the $\prec$-minimum.

Define the function $\sigma : [\omega]^{<\omega} \rightarrow \omega^{k < \omega}$ as follows: Let $\sigma(0) = ()$, the empty sequence; and for $\{a_0, ..., a_n\} \in [\omega]^{<\omega} \setminus \{0\}$, let

$$\sigma(a) = (a_0, a_1 - 1, a_2 - 2, ..., a_n - n).$$

Thus, $\sigma$ is a bijection between the collection of all finite subsets of $\omega$ and the collection of all finite non-decreasing sequences of members of $\omega$.

For $B \subset [\omega]^{<\omega}$, let $\hat{B}$ denote $\{a \in [\omega]^{<\omega} : \exists b \in B(a \leq b)\}$. For sequences $s, t \in \omega^{k < \omega}$, we shall also use the notation $s \leq t$ to denote that $s = t \upharpoonright m$ for some $m$. Write $s \prec t$ if $s \leq t$ and $s \neq t$. For $S \subset \omega^{k < \omega}$, let $\hat{S}$ denote $\{s \in \omega^{k < \omega} : \exists t \in S(s \leq t)\}$.

**Definition 27** Let $B$ be an uniform barrier on $\omega$. Let

$$S_{\hat{B}} = \{\sigma(a) : a \in \hat{B}\} \text{ and } S_B = \{\sigma(a) : a \in B\}.$$

Thus, $S_{\hat{B}}$ and $S_B$ are collections of non-decreasing finite sequences of members of $\omega$. Note that $\prec$ well-orders $S_{\hat{B}}$ and $S_B$ in order-type $\omega$.

**Definition 28** (The top member $\mathbb{W}_B$ of $\mathcal{E}_B$) Let $B$ a uniform barrier on $\omega$. Since $(S_B, \prec)$ has order-type $\omega$, let $v_B : (S_B, \prec) \rightarrow (\omega, <)$ denote this order isomorphism. For $s \in S_B$, let $\text{lh}(s)$ denote the length of $s$ and let

$$\mathbb{W}_B(s) = \{v_B(s \upharpoonright m) : 1 \leq m \leq \text{lh}(s)\}.$$
which is a member of \([\omega]^{{lh(s)}}\). Define
\[
\mathbb{W}_B = \{\mathbb{W}_B(s) : s \in S_B\}.
\]

### 4.5 The spaces \(\text{FIN}^{[\infty]}_k\)

Next, we introduce a collection of topological Ramsey spaces that contain infinite sequences of functions. The space \(\text{FIN}^{[\infty]}_1\), also denoted simply as \(\text{FIN}^{[\infty]}\), is connected with the famous Hindman’s Theorem [27]. Milliken later proved that it forms a topological Ramsey space [36]. The general spaces for \(k \geq 2\) are based on work of Gowers in [25]. The presentation here comes from [41].

**Definition 29** For a positive integer \(k\), define
\[
\text{FIN}_k = \{f : \mathbb{N} \rightarrow \{0, 1, \ldots, k\} : \text{range}(f) \text{ is finite and } k \in \text{range}(f)\}.
\]

We consider \(\text{FIN}_k\) a partial semigroup under the operation of taking the sum of two disjointly supported elements of \(\text{FIN}_k\). For \(f \in \text{FIN}_k\), let \(\text{supp}(f) = \{n : f(n) \neq 0\}\). A block sequence of members of \(\text{FIN}_k\) is a (finite or infinite) sequence \(F = (f_n)\) such that
\[
\max \text{ supp}(f_m) < \min \text{ supp}(f_n) \text{ whenever } m < n.
\]

For \(1 \leq d \leq \infty\), let \(\text{FIN}^{[d]}_k\) denote the collection of all block sequences of length \(d\). The notion of a partial subsemigroup generated by a given block sequence depends on the operation \(T : \text{FIN}_k \rightarrow \text{FIN}_{k-1}\) defined as follows:
\[
T(f)(n) = \max\{f(n) - 1, 0\}.
\]

Given a finite or infinite block sequence \(F = (f_n)\) of elements of \(\text{FIN}_k\) and an integer \(j\) \((1 \leq j \leq k)\), the partial subsemigroup \([F]_j\) of \(\text{FIN}_j\) generated by \(F\) is the collection of members of \(\text{FIN}_j\) of the form
\[
T^{(i_0)}(f_{n_0}) + \ldots + T^{(i_l)}(f_{n_l})
\]
for some finite sequence \(n_0 < \ldots < n_l\) of nonnegative integers and some choice \(i_0, \ldots, i_l \in \{0, 1, \ldots, k\}\). For \(F = (f_n), G = (g_n) \in \text{FIN}^{[\leq \infty]}_k\), set \(F \leq G\) if \(f_n \in [G]_k\) for all \(n\) less than the length of the sequence \(F\). Whenever \(F \leq G\), we say that \(F\) is a block-subsequence of \(G\). The partial ordering \(\leq\) on \(\text{FIN}^{[\infty]}_k\) allows the finitization \(\leq_{\text{fin}}\): For \(F, G \in \text{FIN}^{[< \infty]}_k\),
\[
F \leq_{\text{fin}} G \text{ if and only if } F \leq G \text{ and } (\forall l < \text{length}(G)) F \nleq G \upharpoonright l.
\]

**Theorem 30** [41] For every positive integer \(k\), the triple \((\text{FIN}^{[\infty]}_k, \leq, r)\) is a topological Ramsey space.
The space $\text{FIN}_1^{[\infty]}$, also denoted simply as $\text{FIN}^{[\infty]}$, was proved to be a topological Ramsey space by Milliken in [36]. This was the first space that was built on the basis of a substantially different pigeon hole principle, according to Todorcevic in [41]. Its power over the Ellentuck space was not fully realized until Gower’s successful applications of the “Block Ramsey theory” when treating some problems from Banach space geometry.

The ultrafilter $U_{\text{FIN}^{[\infty]}}$ associated with the space $\text{FIN}^{[\infty]}$ is exactly a stable ordered-union ultrafilter, in the terminology of [7]. Given $f \in \text{FIN}$, let $\min(f)$ denote the minimum of the support of $f$, and let $\max(f)$ denote the maximum of the support of $f$. In [7], Blass showed that the min and max projections of the ultrafilter $U_{\text{FIN}^{[\infty]}}$ are selective ultrafilters which are Rudin-Keisler incomparable. In [19], the analogous result for the Tukey order was shown. More recently, Mildenberger showed in [35] that forcing with $\langle \text{FIN}_k^{[\infty]}, \leq^* \rangle$ produces an ultrafilter with at least $k + 1$-near coherence classes of ultrafilters Rudin-Keisler below it.

### 4.6 The Carlson Simpson dual Ramsey space

Infinite dimensional dual Ramsey theory was developed by Carlson and Simpson in [11], where they establish a combinatorial theorem which is the dual of Ellentuck’s Theorem. The dual form is concerned with colorings of the $k$-element partitions of a fixed infinite set. Now, we will introduce the Carlson-Simpson space, also called the dual Ramsey space.

Using notation in [41], let $\mathcal{E}_\infty$ denote the collection of all equivalence relations $E$ on $\omega$ with infinity many equivalence classes. Each equivalence class $[x]_E$ of $E$ has a minimal representative. Let $p(E)$ denote the set of all minimal representatives of classes of $E$, and let $\{p_n(E) : n < \omega\}$ be the increasing enumeration of $p(E)$. Note that for each $E \in \mathcal{E}_\infty$, $0 \in p(E)$ and hence $p_0(E) = 0$.

For $E, F \in \mathcal{E}_\infty$ we say that $E$ is coarser than $F$ and write $E \leq F$ if every equivalence class of $E$ is the union of some finitely or infinitely many equivalence classes of $F$. The $n$-th approximation of $E \in \mathcal{E}_\infty$ is defined as follows:

$$r_n(E) = E \upharpoonright p_n(E).$$

Thus, $r_n(E)$ is simply the restriction of the equivalence relation $E$ to the finite set $\{0, 1, \ldots, p_n(E) - 1\}$ of integers. Let

$$\mathcal{AE}_\infty = \{r_n(E) : E \in \mathcal{E}_\infty \text{ and } n < \omega\}.$$ 

Given $a \in \mathcal{AE}_\infty$, the length of $a$, denoted $|a|$, is the integer $n$ such that $a = r_n(E)$ for some $E \in \mathcal{E}_\infty$. (Equivalently, $|a|$ is the number of equivalence classes of $a$.) The domain of $a$ is the integer $p_{|a|}(E) = \{0, 1, \ldots, p_{|a|}(E) - 1\}$, where $E$ is some member of $\mathcal{E}_\infty$ such that $a = r_{|a|}(E)$.

**Theorem 31** (Carlson and Simpson, [11]) The space $(\mathcal{E}_\infty, \leq, r)$ is a topological Ramsey space.
The space $E_\infty$ can be identified with the set of all rigid surjections from $\omega$ onto $\omega$. A surjection $f : \omega \to \omega$ is rigid if for each $i \in \omega$, $\min(f^{-1}(i)) < \min(f^{-1}(i+1))$. Given $E \in E_\infty$, define $f_E(0) = 0$. For $n \geq 1$, assuming that for every $i \in n$ we have defined $f_E(i)$, if there is an $i \in n$ such that $n$ and $i$ belong to the same equivalence class of $E$, then let $f_E(n) = f_E(i)$; otherwise, let $f_E(n) = \max\{f_E(i) : i \in n\} + 1$. Conversely, given a rigid surjection $h : \omega \to \omega$, the sets $h^{-1}(i)$ form a partition to $\omega$; we will denote this partition as $E_h \in E_\infty$. Given rigid surjections $g, h$, note that $E_g \leq E_h$ if and only if there exists a rigid surjection $f$ such that $f \circ h = g$ if and only if for each pair $m < n < \omega$, $h(n) = h(m)$ implies $g(n) = g(m)$.

In [32], Matet studied the partial order $(E_\infty, \leq)$ as a lattice, and proved the following about the partial order $\leq^*$, which was given in Definition 12.

**Theorem 32** (Matet [32]) $(E_\infty, \leq^*)$ is a $\sigma$-closed partial order.

### 5 Ramsey degrees for ultrafilters associated to Ramsey spaces

As seen in the previous section, many $\sigma$-closed forcings generating ultrafilters of interest have been shown to contain topological Ramsey spaces as dense subsets. The initial purpose for constructing those new Ramsey spaces was to find the exact Rudin-Keisler and Tukey structures below those ultrafilters. The Abstract Ellentuck Theorem proved to be vital to those investigations, which have been the subject of work in [14, 18, 20, 21], and [15]; the paper [16] provides an overview those results.

In this section we develop a general method, utilizing Theorem 11, to calculate Ramsey degrees for ultrafilters forced by topological Ramsey spaces with certain properties, which we call Independent Sequences of Structures, discussed below. These properties are satisfied by the Ellentuck space, the spaces $R_n, 1 \leq n < \omega$ (see Sect. 4.1), and the spaces generated by Fraïssé classes with the Ramsey property (see Sect. 4.2). Some of these ultrafilters are well-known and some are new, having arisen during investigations discussed in the previous paragraph. This method also provides simple, direct proofs for some known Ramsey degrees, in particular, the ultrafilters $U_n$ of Laflamme in [29] mentioned in Sect. 4.1. Without loss of generality (by restricting below some member of the space if necessary), we will assume all topological Ramsey spaces $R$ contain a strongest member, denoted by $A$.

#### 5.1 A general method for Ramsey degrees for ultrafilters associated to Ramsey spaces composed of independent sequences of structures

The Ramsey spaces associated with the ultrafilters of Baumgartner-Taylor, Blass, and Laflamme mentioned in Sect. 4 all have the following property.

**Definition 33** (Independent Sequences of Structures (ISS)) We say that a topological Ramsey space $(R, \leq, r)$ has Independent Sequences of Structures (ISS) if and only if the following hold:

\[ \text{Springer} \]
• There are relations $R_l, l < L$ for some fixed finite integer $L$, where $R_0$ is a linear order, and the domain of a structure $S$ with these relations is denoted $\text{dom}(S)$.

• The largest member $A$ in $\mathcal{R}$ is a sequence $\langle A(i) : i < \omega \rangle$ such that each $A(i)$ is a finite structure with relations $R_l, l < L$.

• For $i < i'$, the domains of $A(i)$ and $A(i')$ are disjoint, and there are no relations between them, meaning that given any tuple $b \in \bigcup_{i < \omega} \text{dom}(A(i))$ such that $b \cap \text{dom}(A(i))$ is non empty for at least two distinct $i < \omega$ and given $l < L$ such that the length of $b$ is equal to the arity of $R_l$, $\neg R_l(b)$ holds.

• Each member $B \in \mathcal{R}$ can be identified with a sequence $\langle B(i) : i \in \omega \rangle$ where each $B(i)$ is isomorphic to $A(i)$, and moreover, there is a strictly increasing sequence $(k_i)_{i < \omega}$ such that each $B(i)$ is an induced substructure of $A(k_i)$.

• For $B, C \in \mathcal{R}$, $C \subseteq B$ if and only if each $C(n)$ is a substructure of some $B(i_n)$ for some strictly increasing sequence $(i_n)_{n < \omega}$.

• The members of $\mathcal{AR}_m$ are simply initial sequences of length $m$ of members of $\mathcal{R}$: for $B \in \mathcal{R}$, $r_m(B) : = \langle B(i) : i < m \rangle$.

• We require that $\text{dom}(A(0))$ is a singleton; hence the members of $\mathcal{AR}_1$ are singletons.

For each $m < \omega$ and $a, b \in \mathcal{AR}_m$ there exists a unique (because of $R_0$) isomorphism $\varphi_{a,b} : a \rightarrow b$. We will write $\varphi$ to denote $\varphi_{a,b}$, when $a$ and $b$ are obvious. If $X \in \mathcal{R}$ and $s \in [\mathcal{AR}_1 \upharpoonright X]^n$ for some $n \in \omega$, we think of $s$ with the structure inherited by $X$. Thus, if $k_0, \ldots, k_m$ are those indices such that $x \cap X(k_i) \neq \emptyset$ for each $i \leq m$, then $s = \langle s_i : i \leq m \rangle$, where each $s_i$ is the structure on $\text{dom}(s) \cap \text{dom}(X(k_i))$ with the substructure inherited from $X(k_i)$. Since each $X(k_i)$ is a substructure of some $A(k_i)$, each $s_i$ is also the structure on $\text{dom}(s) \cap \text{dom}(A(k_i))$ with the substructure inherited from $A(k_i)$. For $s, t \in [\mathcal{AR}_1]^n$ we say that $s$ and $t$ are isomorphic, and write $s \cong t$, if for some $m, s = \langle s_i : i \leq m \rangle$ and $t = \langle t_i : i \leq m \rangle$, and each $s_i$ is isomorphic to $t_i$. For $t \in [\mathcal{AR}_1]^n$, the isomorphism class of $t$ is the collection of substructures $s \in [\mathcal{AR}_1]^n$, such that $s$ and $t$ are isomorphic.

From now on, for $X \in \mathcal{R}$, we shall simply write $[X]^n$ instead of $[\mathcal{AR}_1 \upharpoonright X]^n$.

**Definition 34** (ISS$^+$) Let $\mathcal{R}$ be a space with the ISS. We say that $\mathcal{R}$ satisfies the ISS$^+$ if additionally, the following hold:

(a) There is some $X \in \mathcal{R}$ such that for any two isomorphic members $u, v \in [X]^n$, there exist $m \in \omega, a, b \in \mathcal{AR}_m, s \in [a]^n$ isomorphic to $u$ and $t \in [b]^n$ isomorphic to $v$ such that $\varphi(s) = t$.

(b) For every $n \geq 2$, there exists an $m \in \omega$ such that for every $X \in \mathcal{R}$ and for every $s \in [\mathcal{AR}_1]^n$, there exists some $t \in [r_m(X)]^n$ such that $s$ and $t$ are isomorphic.

For spaces with the ISS, the $\sigma$-closed partial order $\leq^*$ from Definition 12 is simply $\leq$. Given a generic filter $G$ forced by $(\mathcal{R}, \leq^*)$, the ultrafilter $\mathcal{U}_\mathcal{R}$ on base set $\mathcal{AR}_1$ generated by $G$ was presented in Definition 13. If $a \in \mathcal{AR}$ and $A \in \mathcal{R}$, we will write $[a]^n$ to denote $[\mathcal{AR}_1 \upharpoonright a]^n$.
Definition 35 Given a topological Ramsey space \((\mathcal{R}, \leq, r)\), for \(n \geq 1\), define

\[ t(\mathcal{R}, n) \]

to be the least number \(t\), if it exists, such that for each \(l \geq 2\) and each coloring \(c : [\mathcal{A}\mathcal{R}_1]^n \to l\), there is a member \(X \in \mathcal{R}\) such that the restriction of \(c\) to \([X]^n\) takes no more than \(t\) colors.

Lemma 36 Given a topological Ramsey space \((\mathcal{R}, \leq, r)\), the ultrafilter \(\mathcal{U}_R\) generated by any generic filter \(G\) forced by \((\mathcal{R}, \leq)\) satisfies

\[ t(\mathcal{U}_R, n) = t(\mathcal{R}, n) \]

for each \(n \geq 2\).

This follows immediately by genericity of \(G\). Thus, finding the Ramsey degrees for topological Ramsey spaces is equivalent to finding the Ramsey degrees for their forced ultrafilters.

Definition 37 If \((\mathcal{R}, \leq, r)\) is a topological Ramsey space satisfying the ISS\(^+\), define \(k(\mathcal{R}, n)\) to be the number of isomorphism classes for substructures \(b \in [\mathcal{A}\mathcal{R}_1]^n\) such that \(b\) is a substructure of \(\mathcal{A}(i)\) for some \(i \in \omega\).

Notice that b) of Definition 34 guarantees that, for each \(n\), \(k(\mathcal{R}, n)\) is finite.

Lemma 38 If \((\mathcal{R}, \leq, r)\) is a topological Ramsey space with ISS\(^+\), then for each \(n \geq 1\),

\[ t(\mathcal{R}, n) \leq \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} k(\mathcal{R}, j_i). \]  \hspace{1cm} (4)

Proof Fix \(n \geq 1\) and \(p \geq 2\), and let \(c : [\mathcal{A}\mathcal{R}_1]^n \to p\) be a coloring. Let \(\tilde{k}\) denote the right hand side of the inequality in (4), and define

\[ D = \{ X \in \mathcal{R} : |c''[X]^n| \leq \tilde{k}\}. \]

We will prove that \(D\) is dense in \(\mathcal{R}\). Let \(\{s_k : k < k(\mathcal{R}, n)\}\) be the collection of isomorphism classes for members of \([\mathcal{A}\mathcal{R}_1]^n\). Let \(m\) be the least natural number such that for all \(A \in \mathcal{R}\) and \(k < \tilde{k}\), there is some member of the isomorphism class \(s_k\) contained in \(r_m(A)\). Fix \(a \in \mathcal{A}\mathcal{R}_m\), and linearly order the members of \([a]^n\) as \(\{u_l : l < L\}\), where each \(u_l\) is considered as a sequence of structures and where \(L\) is the number of \(n\)-sized subsets of \(a\). By the ISS\(^+\), for every \(b \in \mathcal{A}\mathcal{R}_m\) there is an isomorphism \(\varphi_{a,b} : a \to b\). Note that \(\{\varphi_{a,b}(u_l) : l < L\}\) is an enumeration for \([b]^n\) preserving the structure, so for each \(l < L\), \(u_l\) is isomorphic to \(\varphi_{a,b}(u_l)\).

Let \(\mathcal{I} = \bigcup_{l \in \mathcal{I}}\). For every \(i \in \mathcal{I}\), define

\[ F_i = \{ b \in \mathcal{A}\mathcal{R}_m : (\forall l \in L)c(\varphi_b(u_l)) = \iota(l) \}. \]
Let $A \in \mathcal{R}$ be given. Since $\mathcal{A}R_m$ is a Nash-Williams family and $\mathcal{A}R_m = \bigcup_{i \in I} \mathcal{F}_i$, by Theorem 11 there are $B \subseteq A$ and $i \in I$ such that $\mathcal{A}R_m \upharpoonright B \subseteq \mathcal{F}_i$. Therefore, for every $b \in \mathcal{A}R_m \upharpoonright B$ and for every $l < L$, $c(\varphi_b(u_l)) = \iota(l)$. Hence $|c''[\mathcal{A}R_m \upharpoonright B]^n| \leq L$. Now suppose that $i < j < L$, $u_t$ and $u_j$ are isomorphic. By a) of ISS+, there exist $b, d \in \mathcal{A}R_m \upharpoonright B$, $s \in [b]^n$ isomorphic to $u_t$, and $t \in [d]^n$ isomorphic to $u_j$ such that $\varphi_{b, d}(s) = t$. Therefore, $c(u_t) = c(u_j)$.

Thus, it remains to count the number of isomorphism classes in $[r_m(\tilde{A})]^n$. Let $t$ be a member of $[B]^n$ and note that for at least one $i \in \omega$, the substructure obtained by intersecting $t$ with $B(i)$ is not empty. Let $q$ be the cardinality of $\{i \in \omega : t \cap B(i) \neq \emptyset\}$. Note that if $q = 1$, then $t$ belongs to one of $k(\mathcal{R}, n)$ different isomorphism classes. If $q \geq 2$, let $\{l_i : i < q\}$ be an increasing enumeration of those $l \in \omega$ such that $t \cap B(l_i) \neq \emptyset$, and let $t_i$ denote the substructure on $t \cap B(l_i)$ inherited from $B(l_i)$. For each $i \in [1, q]$, let $j_i$ denote the cardinality of $\mathcal{A}R_1 \upharpoonright (t \cap B(l_i))$. Note that $n = \sum_{1 \leq i < q} j_i$ and every $j_i < n$, and each $t_i$ belongs to one of $k(\mathcal{R}, j_i)$ isomorphism classes. Hence, $t$ belongs to one of $k(\mathcal{R}, j_1) \times \ldots \times k(\mathcal{R}, j_q)$ many equivalence classes. Letting $q$ range from 2 to its maximum possibility of $n$, there are

$$\sum_{1 < q \leq n} \sum_{l_1 + \ldots + l_q = n} \prod_{1 < i < q} k(\mathcal{R}, j_i)$$

different isomorphism classes, for $n$ sized substructures of $B$ that contain substructures from more than one block. Thus, $|c''[B]^n| \leq \tilde{k}$; hence $B \in D$. Thus, $D$ is a dense subset of $\mathcal{R}$. $\square$

The following Lemma tells us that the right hand side in equation (6) is not just an upper bound but it is the Ramsey degree, which will mean that it is enough to calculate the number of different isomorphism classes of $j$-sized substructures of blocks of $\tilde{A}$ to know the exact Ramsey degree.

**Lemma 39** If $(\mathcal{R}, \leq, r)$ is a topological Ramsey space with ISS+, then

$$t(\mathcal{R}, n) \geq \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} k(\mathcal{R}, j_i).$$

**Proof** As in the previous lemma, let $\tilde{k}$ denote the right hand side of equation (6). In the proof Lemma 38, we showed that there are $\tilde{k}$ isomorphism classes for members of $[\mathcal{A}R_1]^n$. Let $c : [\mathcal{A}R_1]^n \to \tilde{k}$ be a coloring such that for every $s, t \in [\mathcal{A}R_1]^n$, $c(s) = c(t)$ if and only if $s$ and $t$ belong to the same isomorphism class. By b) of ISS+, there exists $m \in \omega$ such that for every $X \in \mathcal{R}$, and every $s \in [\mathcal{A}R_1]^n$ there is some member of $[r_m(X)]^n$ isomorphic to $s$. Then for every $X \in \mathcal{R}$, the set $[X]^n$ contains members of every isomorphism class, and hence, $|c''[X]^n| = \tilde{k}$. $\square$
Theorem 40  Let \((\mathcal{R}, \leq, r)\) be a topological Ramsey space with ISS\(^+\). Then

\[
t(\mathcal{R}, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} k(\mathcal{R}, j_i).
\]  (7)

Proof  This follows from Lemmas 38 and 39. \(\square\)

5.2 Calculations of Ramsey degrees of ultrafilters from spaces with the ISS\(^+\)

In this subsection, we will calculate the exact Ramsey degrees for several classes of ultrafilters which are forced by spaces with the ISS\(^+\). First, we will use Theorem 40 to provide a streamlined calculation of the Ramsey degrees for the weakly Ramsey ultrafilters forced by Laflamme’s forcing \(\mathbb{P}_1\). Indeed, Laflamme calculated these in Theorem 1.10 of [29] via a three-page proof which shows its three-way equivalence with a combinatorial property that \(\mathbb{P}_1\) is naturally seen to possess, and an interesting Ramsey property for analytic subsets of the Baire space in terms of the forcing \(\mathbb{P}_1\) reminiscent of work of Mathias and Blass for Ramsey ultrafilters. The proof we present here is direct and short.

For understanding the following proof, first notice that \(\mathcal{A} \mathcal{R}_1\) consists of all single maximal branches in the tree \(\mathbb{T}_1\), that is, a set of the form \(\{\langle \rangle, (i), (i, j)\}\), where \(i \in \omega\) and \(j \leq i\). Note that given \(n \in \omega\) fixed, every two members of \(\mathcal{A} \mathcal{R}_n\) are isomorphic as subtrees of \(\mathbb{T}_1\). This is because if \(a, b \in \mathcal{A} \mathcal{R}_n\), then there is an isomorphism \(\varphi_{a,b} : a \to b\) which sends each node of the tree \(a\) to the node in the same position of the tree \(b\). Figures 6 and 7 show members, \(a\) and \(b\), of \(\mathcal{A} \mathcal{R}_5\). The isomorphism \(\varphi_{a,b}\) for these finite trees sends \((0, 0)\) to \((20, 15)\), \((1, 0)\) to \((30, 23)\), \((1, 1)\) to \((30, 28)\), \((2, 0)\) to \((50, 48)\), etc.

Letting \(b\) denote the member of \(\mathcal{A} \mathcal{R}_5\) in Fig. 7, note that \(b(0) = \{\langle \rangle, (20), (20, 15)\}\), \(b(1) = \{\langle \rangle, (30), (30, 23), (30, 28)\}\), and so forth.
Definition 41 For every \( n \geq 2 \), let \( S_n = \{ x \in q^n : q \subseteq [1, n], \sum_{i < q} x(i) = n \) and \( \forall i \in q \,(x(i) \neq 0) \} \).

Lemma 42 If \( n \geq 2 \), then \( |S_n| = \sum_{p < n} \binom{n-1}{p} = 2^{n-1} \).

Proof For \( q \subseteq [1, n] \) and \( x \in q^n \) satisfying \( \sum_{i < q} x = n \) and for every \( i \in q \), \( x(i) \neq 0 \), let
\[
\psi(x) = \left\{ x(0) - 1, x(0) + x(1) - 1, ..., \sum_{i < q-1} x(i) - 1 \right\}.
\] (8)

Since every \( x(i) \neq 0 \), \( \psi(x) \) is a subset of \( n - 1 \), so \( \psi(x) \) is a member of \( [n - 1]^{q-1} \).

Notice that since \( \sum_{i < q} x(i) = n \), it follows that \( x(q - 1) = n - \sum_{i < q-1} x(i) \) is determined.

Note that the map \( \psi : S_n \to \mathcal{P}(n - 1) \) is one-to-one. Actually, \( \psi \) is also an onto map. For every \( p < n \) and \( \{m_0, ..., m_{p-1}\} \in [n - 1]^p \), a subset of \( n - 1 \) with an increasing enumeration,
\[
\{m_0, ..., m_{p-1}\} = \psi((m_0 + 1, m_1 - m_0, ..., m_{p-1} - m_{p-2}, n - 1 - m_{p-1}))
\]
with
\[
(m_0 + 1, m_1 - m_0, ..., m_{p-1} - m_{p-2}, n - 1 - m_{p-1}) \in (p+1)n \cap S_n.
\]

Then,
\[
|S_n| = \sum_{p < n} |[n - 1]^p| = \sum_{p < n} \binom{n-1}{p} = 2^{n-1}.
\]

\(\square\)

Corollary 43 Let \( \mathcal{U}_1 \) be the weakly Ramsey ultrafilter forced by Laflamme’s forcing \( \mathbb{P}_1 \), equivalently, by \( (\mathcal{R}_1, \leq^*) \). Then for each \( n \geq 1 \), \( t(\mathcal{U}_1, n) = 2^{n-1} \).

Proof Fix \( n \geq 1 \). First, note that the space \( \mathcal{R}_1 \) satisfies ISS\(^+\). By Theorem 40 we have \( t(\mathcal{U}_1, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + ... + j_q = n} \prod_{1 \leq i \leq q} k(\mathcal{R}_1, j_i) \). Fix \( 1 \leq j \leq n \). Given any \( j \)-sized subsets \( s, t \) of \( \mathcal{A}\mathcal{R}_1 \) such that \( s \subseteq \mathbb{T}_1(i) \) and \( t \subseteq \mathbb{T}_1(m) \) for some \( i, m \), then \( s \) and \( t \) are isomorphic. Then for every \( j \in [1, n] \), \( k(\mathcal{R}_1, j) = 1 \). Therefore
\[
t(\mathcal{U}_1, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + ... + j_q = n} \prod_{1 \leq i \leq q} 1 = |S_n| = 2^{n-1}.
\] (9)

Therefore \( t(\mathcal{U}_1, n) = 2^{n-1} \). \(\square\)

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Since \( t(\mathcal{U}_1, 2) = 2 \), for every \( c : [\mathcal{A} \mathcal{R}_1]^2 \to 3 \) coloring there is an \( X \in \mathcal{U}_1 \) such that \( |c \upharpoonright [X]^2| \leq 2 \). If we indentify \( \mathcal{A} \mathcal{R}_1 \) with \( \omega \) then \( \mathcal{U}_1 \) is a weakly Ramsey ultrafilter in the sense of \( \omega \). By Lemma 3, there is a coloring \( c : [\mathcal{A} \mathcal{R}_1]^2 \to 2 \) such that for every \( X \in \mathcal{R}_1 \), \( |c \upharpoonright [X]^2| = 2 \). Thus, we can see in a simple way that \( \mathcal{U}_1 \) is not a Ramsey ultrafilter.

Next, we will calculate Ramsey degrees for ultrafilters \( \mathcal{U}_k \) forced by Laflamme’s forcings \( \mathbb{P}_k \) from [29], \( k \geq 2 \). As noted in the previous section, the topological Ramsey space \( \mathcal{R}_k \) forces the same ultrafilter as \( \mathbb{P}_k \). The Ramsey degrees for \( \mathcal{U}_k \) are stated in Theorem 2.2 of [29], but a concrete proof does not appear in that paper. Rather, Laflamme points out that the proof entirely similar to, but combinatorially more complicated than that of Theorem 1.10 in [29]. Here, we present a straightforward proof based on the Ramsey structure of \( \mathcal{R}_k \). For the following proof, the set of first approximations \( \{ r_1(A) : A \in \mathcal{R}_k \} \) are identified with the maximal nodes of \( T_k \); this also applies for every member of \( \mathcal{R}_k \). The downward closure of any maximal node in \( T_k \) recovers the tree structure below that node, so it suffices to work with the maximal nodes in \( T_k \).

Recall Definition 37 of \( k(\mathcal{R}, n) \) for a topological Ramsey space \( \mathcal{R} \). The next lemma uses the inductive nature of the construction of \( \mathcal{R}_{k+1} \) from \( \mathcal{R}_k \) to show that each \( k(\mathcal{R}_{k+1}, n) \) can be deduced from the Ramsey degrees of \( \mathcal{U}_k \).

**Lemma 44** For any \( k, n \geq 1 \), we have that \( k(\mathcal{R}_{k+1}, n) = t(\mathcal{U}_k, n) \).

**Proof** Let \( s \in [\mathbb{T}_{k+1}]^n \) be such that \( s \subseteq \mathbb{T}_{k+1}(l) \) for some \( l \in \omega \). Recall our convention that \( s \) is a collection of maximal nodes in \( \mathbb{T}_{k+1} \), so each node in \( s \) is a sequence of length \( k + 2 \). Note that since \( s \) is contained in \( \mathbb{T}_{k+1}(l) \), each node in \( s \) end-extends the sequence \( (l) \). Let \( t \) be the set of sequences resulting by taking out the first member of every sequence in \( s \); thus, letting \( k+1 \omega \) denote the set of sequences of natural numbers of length \( k + 1 \),

\[
  t = \{ x \in (k + 1) \omega : \langle l \rangle \dashv x \in s \}.
\]

Then \( t \) is an \( n \)-sized subset of \( \mathbb{T}_k \). Since there are \( t(\mathcal{U}_k, n) \) isomorphism classes for \( [\mathbb{T}_k]^n \), \( k(\mathcal{R}_{k+1}, n) = t(\mathcal{U}_k, n) \). \( \square \)

**Lemma 45** Let \( k \geq 1 \) be given, and suppose that \( \mathcal{U}_{k+1} \) is an \( (\mathcal{R}_{k+1}, \leq^\ast) \)-generic filter. Then for each \( n \geq 1 \),

\[
  t(\mathcal{U}_{k+1}, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} t(\mathcal{U}_k, j_i).
\]

**Proof** This follows from Theorem 40 and Lemma 44. \( \square \)

**Theorem 46** Given \( k, n \geq 1 \), if \( \mathcal{U}_k \) an ultrafilter forced by Laflamme’s \( \mathbb{P}_k \), or equivalently by \( (\mathcal{R}_k, \leq^\ast) \), then \( t(\mathcal{U}_k, n) = (k + 1)^{n-1} \).

**Proof** The proof is by induction on \( k \) over all \( n \geq 1 \). The case when \( k = 1 \) is done by Corollary 43. Now we assume the conclusion for a fixed \( k \geq 1 \) and prove it for \( k + 1 \).
By the Lemma 45,

\[ t(\mathcal{U}_{k+1}, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} \prod_{1 \leq i \leq q} t(\mathcal{U}_k, j_i). \]  

(12)

By inductive hypothesis \( t(\mathcal{U}_k, j_i) = (k + 1)^{j_i - 1} \). Then

\[ t(\mathcal{U}_{k+1}, n) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} (k + 1)^{j_1 - 1} \ldots (k + 1)^{j_q - 1} = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} (k + 1)^{n - q}. \]  

(13)

By the proof of Lemma 42,

\[ \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = n} (k + 1)^{n - q} = \sum_{1 \leq q \leq n} \binom{n-1}{q-1} (k + 1)^{n - q} = \sum_{0 \leq p \leq n-1} \binom{n-1}{p} (k + 1)^{n-1-p}. \]  

(14)

By Newton’s Theorem, the right hand side of equation (14) equals \((k + 1 + 1)^{n-1}\). Therefore \( t(\mathcal{U}_{k+1}, n) = (k + 2)^{n-1} \).

Next, we calculate the Ramsey degree for pairs for the ultrafilters forced by Blass’ \( N \)-square forcing and more generally, the hypercube Ramsey spaces \( \mathcal{H}^n \).

**Corollary 47** Let \( \mathcal{V}_n \) be an \((\mathcal{H}^n, \leq^*)\)-generic filter. Then

\[ t(\mathcal{V}_n, 2) = 1 + \sum_{i=0}^{n-1} 3^i. \]

In particular, \( t(\mathcal{V}_2, 2) = 5 \), where \( \mathcal{V}_2 \) is the ultrafilter generated by Blass’ \( N \)-square forcing.

**Proof** By Theorem 40, we know that for each \( n \geq 2 \),

\[ t(\mathcal{V}_n, 2) = \sum_{1 \leq q \leq n} \sum_{j_1 + \ldots + j_q = 2} \prod_{1 \leq i \leq q} k(\mathcal{H}^n, j_i) = k(\mathcal{H}^n, 2) + k(\mathcal{H}^n, 1)^2. \]

Note that \( k(\mathcal{H}^n, 1) = 1 \) because all the singletons are isomorphic. Given \( n \geq 2 \), let \( \mathcal{A}_k \) be an \( n \)-hypercube with side length \( k \). We will show that \( k(\mathcal{H}^n, 2) = \sum_{i=0}^{n-1} 3^i \).

Fix \( n = 2 \), and take \( a = (a_0, a_1), b = (b_0, b_1) \in \mathcal{A}_k \) for any large enough \( k \in \omega \). Assume that \( a \) lexicographically below \( b \), according to the lexicographical order on \( \omega \times \omega \). There are 4 non-isomorphic options: Either \( a_0 = b_0 \) and \( a_1 < b_1 \), or else...
$a_0 < b_0$ and any of the three relations $a_1 < a_1, a_1 = b_1,$ or $a_1 > b_1$ holds. Therefore $k(\mathcal{H}^2, 2) = 4 = 3^0 + 3^1$, so $t(\mathcal{U}_2, 2) = 5$.

Now suppose that $k(\mathcal{H}^n, 2) = \sum_{i=0}^{n-1} 3^i$. Given $a = (a_0, \ldots, a_n)$ and $b = (b_0, \ldots, b_n)$ in $A_k$ for any large enough $k \in \omega$, there are the following possibilities. If $a_0 = b_0$, then there are $k(\mathcal{H}^n, 2) = \sum_{i=0}^{n-1} 3^i$ many possible relations between $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. If $a_0 < b_0$, then for each $1 \leq i \leq n$, there are three possible configurations for $a_i$ and $b_i$, namely $a_i < b_i, a_i = b_i$, or $a_i < b_i$. Thus, there are $3^n$ many possible configurations for $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$. Therefore, $k(\mathcal{H}^{n+1}, 2) = \sum_{i=0}^{n} 3^i$.

Finally, we calculate the Ramsey degrees for the $k$-arrow, not $(k+1)$-arrow ultrafilters of Baumgartner and Taylor.

**Corollary 48** For $k \geq 2$, let $\mathcal{W}_k$ be the $k$-arrow, not $(k+1)$-arrow ultrafilter of Baumgartner and Taylor. Then

\[ t(\mathcal{W}_k, 2) = 3. \]

**Proof** Let $A_k$ denote the topological Ramsey space constructed from a generating sequence of finite graphs which have no $k$-cliques. As this space is dense in the Baumgartner-Taylor forcing, the two partial orders generate the same ultrafilters. By Theorem 40, we know that for each $n \geq 2$,

\[ t(A_k, 2) = k(A_k, 2) + k(A_k, 1)^2. \]

As in the previous corollary, $k(A_k, 1) = 1$ because all the singletons are isomorphic. Note that $k(A_k, 2) = 2$, since pairs of singletons in $\mathcal{AR}_1$ in this topological Ramsey space have two isomorphism classes: either a pair has an edge or else it has no edge between them. \( \square \)

### 6 Ramsey degrees for ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$

In this Section we will calculate Ramsey degrees of pairs for ultrafilters forced by $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$, for all $k \geq 2$. In this section, let $G_1$ denote the ultrafilter forced by $\mathcal{P}(\omega)/\text{Fin}$ and note that $G_1$ is a Ramsey ultrafilter. Recall from Sect. 4.3 that $\text{Fin}^\otimes k$ is a $\sigma$-closed ideal on $\omega^k$, and that the Boolean algebras $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$ force ultrafilters $G_k$ such that whenever $1 \leq j < k$, the projection of $G_k$ to the first $j$ coordinates of $\omega^k$ forms an ultrafilter on $\omega^j$ which is generic for $\mathcal{P}(\omega^j)/\text{Fin}^\otimes j$. $G_2$ garners much attention as it is a weak p-point which is not a p-point (see [9]). To ease notation, let $\subseteq^{sk}$ denote $\subseteq^{\text{Fin}^\otimes k}$, and note that $(G_k, \subseteq^{sk})$ is forcing equivalent to $\mathcal{P}(\omega^k)/\text{Fin}^\otimes k$ (see [14]). In this section, we use the high dimensional Ellentuck spaces, $\mathcal{E}_k$, to provide concise proofs of the Ramsey degrees $t(G_k, 2)$ for all $k \geq 2$.

**Definition 49** Let $k \geq 2$ and $s, t, u, v \in \omega^{jk}$. Define the relation $\sim_k$ on pairs of $\omega^{jk}$ by $(s, t) \sim_k (u, v)$ if and only if $s < t, u < v$ and for every $i, j \in k$ and $\rho \in \{=, <\}$, $s_i \rho t_j \iff u_i \rho v_j$. 

\( \square \) Springer
Observe that the set of $X \in \mathcal{E}_k$ satisfying

$$\forall s, t \in X, \forall 1 \leq i < k \ (s_i = t_i \rightarrow s_{i-1} = t_{i-1})$$

(15)

is open dense in $\mathcal{E}_k$. Let $\mathcal{D}_k$ denote the set of $X \in \mathcal{E}_k$ satisfying (15). From now on we will work only with members of $\mathcal{D}_k$. Let $k(\mathcal{E}_k, 2)$ be the number of equivalence classes from $\sim_k$ on pairs of $\mathcal{D}_k$. We will first calculate this number, and then show in Theorem 51 that it actually is the Ramsey degree of $\mathcal{G}_k$ for pairs.

**Lemma 50** For every $k \geq 2$, $k(\mathcal{E}_k, 2) = \sum_{i<k} 3^i = \frac{3^k-1}{2}$.

**Proof** The proof will be by induction on $k \geq 2$. Fix $k = 2$, and fix some $X \in \mathcal{D}_2$. Fix $s, t \in X$ such that $t < s$. Then $s_0 < s_1$, $t_0 < t_1$ and $t_1 < s_1$, where $s = (s_0, s_1)$, $t = (t_0, t_1)$. There are four possibilities for ordering the entries of $s$ and $t$:

(i) $t_0 = s_0$ and $t_1 < s_1$.
(ii) $t_0 < t_1 < s_0 < s_1$.
(iii) $t_0 < s_0 < t_1 < s_1$.
(iv) $s_0 < t_0 < s_1$ and $t_1 < s_1$.

Note that these four options are not isomorphic and for every $X \in \mathcal{D}_2$, $[X]^2$ contains all pairs in all four types. Therefore $k(\mathcal{E}_2, 2) = 4 = 1 + 3$.

Now assume the conclusion holds for some fixed $k \geq 2$; we will prove that conclusion holds for $k + 1$. Fix some $X \in \mathcal{D}_{k+1}$ and take $s, t \in X$ such that $t < s$. Then $s = (s_0, \ldots, s_k)$, $t = (t_0, \ldots, t_k)$ with $s_0 < \ldots < s_k$, $t_0 < \ldots < t_k$ and $t_k < s_k$. There are four options for ordering last two members of sequences $s$ and $t$:

(i) $t_k < s_k-1$. In this case $t_{k-1} < s_k-1$. Then $t \upharpoonright k < s \upharpoonright k$ and the pair $\langle s \upharpoonright k, t \upharpoonright k \rangle$ lies in one of $k(\mathcal{E}_k, 2)$ possible equivalence classes. Therefore, there are $k(\mathcal{E}_k, 2)$ options for ordering members of sequences $s$ and $t$.

(ii) $t_k > s_k-1$ and $t_k - 1 < s_k - 1$. In this case $t \upharpoonright k < s \upharpoonright k$ and the pair $\langle s \upharpoonright k, t \upharpoonright k \rangle$ lies in one of $k(\mathcal{E}_k, 2)$ possible equivalence classes. Therefore, there are $k(\mathcal{E}_k, 2)$ options for ordering members of sequences $s$ and $t$.

(iii) $t_k > s_k - 1$ and $t_k - 1 > s_k$. In this case $s \upharpoonright k < t \upharpoonright k$ and the pair $\langle t \upharpoonright k, s \upharpoonright k \rangle$ lies in one of $k(\mathcal{E}_k, 2)$ possible equivalence classes. Therefore, there are $k(\mathcal{E}_k, 2)$ options for ordering members of sequences $s$ and $t$.

(iv) $t_{k-1} = s_{k-1}$. In this case, $t_i = s_i$ for all $i \leq k - 1$.

Then there are $3k(\mathcal{E}_k, 2) + 1$ equivalence classes for the relation $\sim_{k+1}$. Since $k(\mathcal{E}_k, 2) = \sum_{i<k} 3^i$, then

$$k(\mathcal{E}_{k+1}, 2) = 3 \left( \sum_{i<k} 3^i \right) + 1 = \sum_{i<k+1} 3^i = \frac{3^{k+1} - 1}{2}.$$  

(16)
Theorem 51 For every \( k \in \omega \) such that \( k \geq 2 \), \( t(\mathcal{G}_k, 2) = \sum_{i<k} 3^i = \frac{3^k-1}{2} \).

Proof Let \( c : [\omega^{<k}]^2 \to t(\mathcal{G}_k, 2) \) be such that \( c(p) = c(q) \) if and only if \( p \sim_k q \). For each \( X \in \mathcal{E}_k \), \( [X]^2 \) contains members of every equivalence class. Thus, by Lemma 50, \(|c \upharpoonright [X]^2| = t(\mathcal{G}_k, 2) \geq \sum_{i<k} 3^i \).

Now we want to prove that \( t(\mathcal{G}_k, 2) \leq \sum_{i<k} 3^i \). Let \( r \geq 1 \) be a natural number and \( c : [\omega^{<k}]^2 \to r \) be a coloring. Let

\[
\mathcal{D} = \{Y \in \mathcal{E}_k : |c \upharpoonright [Y]^2| \leq k(\mathcal{E}_k, 2)\}.
\]

We will prove that \( \mathcal{D} \) is a dense subset of \( \mathcal{G}_k \). Let \( X \in \mathcal{G}_k \cap \mathcal{D} \), and let \( m \) be a natural number such that \( r_m(X) \) contains pairs of every equivalence relation of \( \sim_k \).

Fix \( a \in \mathcal{AR}_m \) and fix an order for \( [a]^2 = \{p^a_l : l < L\} \) with the induced substructure, where \( L \) is the number of pairs of sequences that belong to \( a \). For each \( b \in \mathcal{AR}_m \) enumerate pairs of \( b \) as \( [b]^2 = \{p^b_l : l < L\} \) such that for every \( l < L \), \( p^a_l \sim_k p^b_l \). Let \( \mathcal{I} = \langle r \rangle \). For every \( i \in \mathcal{I} \), define

\[
\mathcal{F}_i = \{b \in \mathcal{AR}_m : (\forall l < L) c(p^b_l) = i(l)\}.
\]

Since \( \mathcal{AR}_m \) is a Nash-Williams family and \( \mathcal{AR}_m = \bigcup_{i \in \mathcal{I}} \mathcal{F}_i \), by Theorem 11 there exist \( Y \leq X \) and \( i \in \mathcal{I} \) such that \( \mathcal{AR}_m \upharpoonright Y \subseteq \mathcal{F}_i \). Therefore, for every \( b \in \mathcal{AR}_m \upharpoonright Y \) and for every \( l < L \), \( c(p^b_l) = i(l) \). Hence \( |c''[\mathcal{AR}_m \upharpoonright Y]^2| \leq L \).

Note that if \( i, j \) are such that \( p^a_i \sim_k p^a_j \), then there exist \( A, B \in \mathcal{AR}_m \upharpoonright Y \) and \( l < L \) such that \( p^a_i \sim_k p^A_l \) and \( p^a_j \sim_k p^B_l \). This implies that if \( p^a_i \sim_k p^a_j \) with \( i < j \leq L \), then \( c(p^a_i) = c(p^a_j) \). Since there are \( k(\mathcal{E}_k, 2) \) different equivalence classes, and \( [Y]^2 \) contains pairs of every equivalence class, we obtain that \( |c''[\mathcal{AR} \upharpoonright Y]^2| \leq k(\mathcal{E}_k, 2) \). Therefore \( Y \in \mathcal{D} \); hence, \( \mathcal{D} \) is a dense subset of \( \mathcal{E}_k \). It follows from Lemma 50 \( t(\mathcal{G}_k, 2) \leq \sum_{i<k} 3^i \).

\[\square\]

7 Pseudointersection and tower numbers for topological Ramsey spaces

In this Section we investigate the pseudointersection and tower numbers for several classes of topological Ramsey spaces and their relationships to \( p \) (recall Definition 2). The notion of strong finite intersection property, and hence, also the pseudointersection and tower numbers from Sect. 2 can be extended to any \( \sigma \)-closed partial order, in particular, to topological Ramsey spaces.

Definition 52 We say that \( (\mathcal{R}, \leq_\ast, \leq_\ast) \) is a \( \sigma \)-closed topological Ramsey space if \( (\mathcal{R}, \leq) \) is a topological Ramsey space and \( \leq^\ast \) is a \( \sigma \)-closed order on \( \mathcal{R} \) coarsening \( \leq \) such that \( (\mathcal{R}, \leq) \) and \( (\mathcal{R}, \leq^\ast) \) have isomorphic separative quotients. Let \( \mathcal{F} \) be a subset of \( \mathcal{R} \).

1. We say that \( \mathcal{F} \) has the strong finite intersection property (SFIP) if for every finite subfamily \( \mathcal{G} \subseteq \mathcal{F} \), there exists \( Y \in \mathcal{R} \) such that for each \( X \in \mathcal{F} \), \( Y \leq^\ast X \).
2. \( Y \in \mathcal{R} \) is called a pseudointersection of the family \( \mathcal{F} \) if for every \( X \in \mathcal{F} \), \( Y \leq^\ast X \).
Definition 53 Let \((R, \le, \le^*, r)\) be a \(\sigma\)-closed topological Ramsey space.

1. The pseudointersection number \(p_R\) is the smallest cardinality of a family \(F \subseteq R\) which has the SFIP but does not have a pseudointersection.
2. We say that \(F\) is a tower if it is linearly ordered by \(\ge^*\) and has no pseudointersection.

The tower number \(t_R\) is the smallest cardinality of a tower of \((R, \le^*)\).

Note that for every topological Ramsey space \(R\), \(p_R \le t_R\). A recent groundbreaking result of Malliaris and Shelah shows that \(p = t\) (see [30] and [31]). It is not clear at present whether their work implies that \(p_R = t_R\) for all Ramsey spaces with some \(\sigma\)-closed partial order. For all the spaces considered in this section, we will show that they are indeed equal.

As in previous sections, we will assume that \(AR\) is countable. If this is not the case, we tacitly work on \(AR \upharpoonright X\) for some \(X \in R\), which is countable by axiom A.2.

7.1 Pseudointersection and tower numbers for several classes of topological Ramsey spaces

The following property is satisfied by many topological Ramsey spaces, including several discussed in Sect. 4.

Definition 54 (IEP) We say that a topological Ramsey space \((R, \le, r)\) has the Independent Extension Property (IEP) if the following hold: Each \(X \in R\) is a sequence of the form \(\langle X(n) : n \in \omega \rangle\) such that for every \(n \in \omega\), \(r_n(X) = \langle X(i) : i < n \rangle\), and each \(X(i)\) is a finite set, possibly, but not necessarily, with some relational structures on it. Furthermore, for every \(X \in R\), \(k \in \omega\), and \(s \in AR_k\), there exist \(m \in \omega\) and \(s(k) \subseteq X(m)\) such that \(s \sim s(k) \in AR_{k+1}\) and \(s \subseteq s \sim s(k)\).

Theorem 55 Let \((R, \le, \le^*, r)\) be a \(\sigma\)-closed topological Ramsey space with the IEP, and suppose that \(R\) is closed in \(AR^\omega\). Then \(p \le p_R\).

Proof Recall that by Bell’s result, \(p = m(\sigma\text{-centered})\). Let \(\kappa < m(\sigma\text{-centered})\) be given, and let \(F = \{X_\alpha : \alpha < \kappa\} \subseteq R\) be a family with the SFIP. Define \(P\) to be the collection of all ordered pairs \((s, E)\) such that \(s \in AR\) and \(E \in [\kappa]^<\omega\). Since \(R\) satisfies the IEP, define some shorthand notation as follows: For \(s, t \in AR\) with \(s \subseteq t\), note that \(s = \langle s(i) : i < m \rangle\) and \(t = \langle t(i) : i < n \rangle\) for some \(m \le n\). Then let

\[
t/s = \langle t(i) : m \le i < n \rangle.
\] (19)

Define a partial order \(\le\) on \(P\) as follows: Given \(\langle s, E \rangle, \langle t, F \rangle \in P\), let \(\langle t, F \rangle \le \langle s, E \rangle\) iff \(s \subseteq t\), \(E \subseteq F\), and there exists \(X \in R\) such that for every \(\alpha \in E\), \(X \le X_\alpha\) and \(t/s \subseteq X\), which means that for every \(i \in [\vert s\vert, \vert t\vert]\) there exists \(l\) such that \(t(i) \subseteq X(l)\).

For every \(s \in AR\), define \(P_s = \{\langle s, E \rangle : E \in [\kappa]^<\omega\}\). Note that \(P_s\) is centered. Since \(AR\) is countable, \(P = \bigcup_{s \in AR} P_s\) is a \(\sigma\)-centered partial order. Given \(\alpha < \kappa\) and \(m \in \omega\) let

\[
D_{\alpha,m} = \{\langle s, E \rangle \in P : \alpha \in E \text{ and } \vert s\vert > m\}.
\]
Claim 1 \( D_{\alpha,m} \) is dense.

Proof Fix \( \langle t, F \rangle \in \mathbb{P} \). If \( |t| > m \), then the pair \( \langle t, F \cup \{\alpha\} \rangle \leq \langle t, F \rangle \) is in \( D_{\alpha,m} \). If \( |t| \leq m \), since \( \mathcal{F} \) has the SFIP, there exists \( X \in \mathcal{R} \) such that for every \( \beta \in F \), \( X \leq X_\beta \). Let \( i = |t| \). By the IEP, there is some \( u \in \mathcal{A} \mathcal{R}_{m+1} \) such that \( u \) extends \( t \) into \( X \). That is, there is a strictly increasing sequence \( l_1 < \cdots < l_m \) and subsets \( u(j) \in \mathcal{R}(j) \upharpoonright X(l_j) \), for each \( j \in [i, m] \), such that \( u = t \upharpoonright \langle u(j) : i \leq j \leq m \rangle \) is a member of \( \mathcal{A} \mathcal{R}_{m+1} \). Let \( E = F \cup \{\alpha\} \). By the choice of \( u \), it follows that \( \langle u, E \rangle \leq \langle t, F \rangle \) and \( \langle u, E \rangle \in D_{\alpha,m} \). \( \Box \)

Let \( \mathcal{D} = \{D_{\alpha,m} : \alpha \in \kappa \) and \( m \in \omega \} \) and let \( G \) be a filter on \( \mathbb{P} \) meeting each dense set in \( \mathcal{D} \). Let \( X_G = \bigcup \{s : \exists E \in [\kappa]^{<\omega}(s, E) \in G\} \). Since \( \mathcal{R} \) is closed under \( \mathcal{A} \mathcal{R}^\omega \), \( X_G \in \mathcal{R} \).

Claim 2 \( \forall \alpha \in \kappa, X_G \leq^* X_\alpha \).

Proof Fix \( \alpha < \kappa \) and some \( \langle s, E \rangle \in G \cap D_{\alpha,0} \). We will prove that for every \( m \geq |s| \), \( r_m(X_G) \subseteq X_\alpha \). Let \( m > |s| \) and \( \langle t, F \rangle \in G \cap D_{\alpha,m} \) be given. Since \( D_{\alpha,m} \) is an open dense subset of \( \mathbb{P} \) and \( G \) is a filter, there is a condition \( \langle u, H \rangle \) below both \( \langle s, E \rangle \) and \( \langle t, F \rangle \) such that \( \langle u, H \rangle \in G \cap D_{\alpha,m} \). Note that \( u = r_m(X_G) \). Since \( \langle u, H \rangle \leq \langle s, E \rangle \) and \( \alpha \in E \), there exists \( X \in \mathcal{R} \) such that \( X \leq X_\alpha \) and \( u/s \subseteq X \subseteq X_\alpha \). It follows from \( |u| > m \) that \( r_m(X_G)/s \subseteq X_\alpha \). \( \Box \)

Therefore, \( \kappa < p_R \); and hence, \( m(\sigma\text{-centered}) \leq p_R \). \( \Box \)

This immediately leads to the following corollary for all the topological Ramsey spaces from Sects. 4.1, 4.2, and 4.5. For each of these spaces, the relevant \( \sigma \)-closed partial order \( \leq^* \) from Definition 12 is exactly the mod finite partial order, \( \leq^* \).

Corollary 56 Let \( \mathcal{R} \) be any of the following topological Ramsey spaces, with the \( \sigma \)-closed partial order \( \leq^* \):

1. \( \mathcal{R}_\alpha \), where \( 1 \leq \alpha < \omega_1 \).
2. \( \mathcal{R}(\mathbb{A}) \), where \( \mathbb{A} \) is some generating sequence from a collection of \( \leq \omega \) many Fraïssé classes with the Ramsey property as in [18].
3. \( \mathcal{F} \mathcal{I}N_{k}^{[\infty]} \), where \( k \geq 1 \).

Then \( p \leq p_R \).

Proof All of these spaces satisfy the IEP. \( \Box \)

Now we show that each \( \mathcal{F} \mathcal{I}N_{k}^{[\infty]} \) has tower and pseudointersection numbers equal to \( p \).

Theorem 57 \( \forall k \geq 1, t_{\mathcal{F} \mathcal{I}N_{k}^{[\infty]}} = p_{\mathcal{F} \mathcal{I}N_{k}^{[\infty]}} = p \).

Proof Let \( \mathcal{F} \subseteq [\omega]^{<\omega} \) be a family linearly ordered by \( \supseteq^* \); that is, a tower. For \( A \in [\omega]^{<\omega} \), write \( A = \{a_0, a_1, \ldots, a_n, \ldots\} \) with \( a_n < a_{n+1} \) for each \( n \in \omega \). For \( n \in \omega \), define \( f_n : \omega \to \{0, \ldots, k\} \) to be the function such that \( f_n(i) = k \) for each \( i \in [a_n, a_{n+1}) \), and \( f_n(i) = 0 \) for each \( i \notin [a_n, a_{n+1}) \). Note that the sequence \( F_A = (f_n)_{n \in \omega} \) is a
member of $\text{FIN}_k^{[\infty]}$, and moreover, if $A \neq B$ then $F_A \neq F_B$. Furthermore, $A \subseteq^* B$ implies $F_A \preceq^* F_B$. Let $\mathcal{G} = \{F_A : A \in \mathcal{F}\}$ and note that $\mathcal{G}$ is linearly ordered by $\geq^*$. Suppose that $\mathcal{G}$ has a pseudointersection $H = (h_n)_{n \in \omega} \in \text{FIN}_k^{[\infty]}$. Let $C = \{\min(\text{supp}(h_n)) : n \in \omega\}$. Note that $C$ is a pseudointersection for $\mathcal{F}$. By Lemma 60, $t_{\text{FIN}_k^{[\infty]}} \leq t$. Corollary 56 implies that $p \leq p_{\text{FIN}_k^{[\infty]}}$. Thus, it follows that

$$t_{\text{FIN}_k^{[\infty]}} \leq t = p \leq p_{\text{FIN}_k^{[\infty]}} = p,$$

the middle equality holding by the result of Malliaris and Shelah in [31]. \hfill \square

The next results will have proofs that follow the outline of Theorem 57 but will involve stronger hypotheses in order to apply to the spaces from Sects. 4.1 and 4.2.

**Definition 58 (ISS*)** Let $(\mathcal{R}, \leq, r)$ be a topological Ramsey space satisfying Independent Sequences of Structures. Recall that each finite structure $A_i$ is linearly ordered. We say that $\mathcal{R}$ satisfies the ISS* if for all $k < m$, $A_k$ embeds into $A_m$.

It follows from the ISS* that there are functions $\lambda_k$, $k < \omega$, such that for each $m \geq k$, $\lambda_k(A_m)$ is a substructure of $A_m$ which is isomorphic to $A_k$. Moreover, for each triple $k < m < n$, $\lambda_k(A_n)$ is a substructure of $\lambda_m(A_n)$.

**Lemma 59** Let $\mathcal{F}$ be a family infinite subsets of $\omega$ and $\mathcal{R}$ be a topological Ramsey space with the ISS*. Then for each $B \in [\omega]^{\omega}$ there corresponds a unique $X_B \in \mathcal{R}$ so that given any $B, C \in \mathcal{F}$, the following hold:

1. $B \neq C$ implies $X_B \neq X_C$;
2. $B \subseteq C$ implies $X_B \leq X_C$;
3. $B \subseteq^* C$ implies $X_B \leq^* X_C$; and
4. If $\mathcal{G} = \{X_A : A \in \mathcal{F}\}$ has a pseudointersection, then $\mathcal{F}$ also has a pseudointersection.

**Proof** Given $B \in [\omega]^{\omega}$, let $\{b_0, b_1, b_2, \ldots\}$ be the increasing enumeration of $B$. For each $n \in \omega$, let $X(n) = \lambda_n(A_{b_n})$, and define $X_B = \langle X(n) : n \in \omega \rangle$. Then $X_B \in \mathcal{R}$, since $\mathcal{R}$ satisfies the ISS*. Moreover, notice that whenever $B \neq C$ are in $[\omega]^{\omega}$, then $X_B \neq X_C$, so (1) holds. Suppose that $B, C \in [\omega]^{\omega}$ satisfy $C \subseteq B$. Let $k < \omega$ be such that $c_k \in B$. Then $c_k = b_m$ for some $m \geq k$. By our construction, $X_C(k) = \lambda_k(A_{c_k})$ and $X_B(m) = \lambda_m(A_{b_m})$. Since $c_k = b_m$, $X_C(n)$ is a substructure of $X_B(m)$. From these observations, (2) and (3) of the theorem immediately follow.

Fix a family $\mathcal{F} \subseteq [\omega]^{\omega}$, and let $\mathcal{G} = \{X_B : B \in \mathcal{F}\}$. Assume that there exists some $Y \in \mathcal{R}$ which is a pseudointersection of $\mathcal{G}$. We claim that

$$D = \{m \in \omega : (\exists i \in \omega) Y(i) \text{ is substructure of } A_m\}$$

is a pseudointersection of $\mathcal{F}$. Let $B \in \mathcal{F}$. Since $Y$ is a pseudointersection of $\mathcal{G}$, $Y \preceq^* X_B$. Thus, there exists $p < \omega$ such that for every $i > p$, $Y(i)$ is a substructure of some $X_B(j)$ for some $j < \omega$. It follows that $D \setminus p \subseteq B$; hence, $D \subseteq^* B$. \hfill \square
Lemma 60  Let $(\mathcal{R}, \leq, \leq^*, r)$ be a $\sigma$-closed topological Ramsey space such that for every family $\mathcal{F} \subseteq [\omega]^\omega$ linearly ordered by $\supseteq^*$ there exists a family $\mathcal{G} \subseteq \mathcal{R}$ linearly ordered by $\geq^*$ that satisfies the following:

1. $|\mathcal{G}| = |\mathcal{F}|$
2. If $\mathcal{G}$ has a pseudointersection then $\mathcal{F}$ has also a pseudointersection.

Then $t_\mathcal{R} \leq t$ and $p_\mathcal{R} \leq p$.

Proof  Let $\mathcal{F} \subseteq [\omega]^\omega$ be a family linearly ordered by $\supseteq^*$ such that $|\mathcal{F}| < t_\mathcal{R}$. By hypothesis there is a family $\mathcal{G} \subseteq \mathcal{R}$ linearly ordered by $\geq^*$ that satisfies that $|\mathcal{G}| = |\mathcal{F}|$. So $|\mathcal{G}| = |\mathcal{F}| < t_\mathcal{R}$. Therefore $\mathcal{G}$ has a pseudointersection and by hypothesis 2), $\mathcal{F}$ has also a pseudointersection. Hence $t_\mathcal{R} \leq t$. A similar argument proves that $p_\mathcal{R} \leq p$. □

Theorem 61  Let $\mathcal{R}$ be a topological Ramsey space that satisfies the ISS$^*$. Then $p_\mathcal{R} = t_\mathcal{R} = p$.

Proof  Suppose $\mathcal{R}$ satisfies the ISS$^*$. Then by Lemmas 59 and 60, $t_\mathcal{R} \leq t$. Note that $\mathcal{R}$ satisfies the IEP, since this follows from the ISS$^*$. then Theorem 55 implies that $m(\sigma\text{-centered}) \leq p_\mathcal{R}$. Since $p = m(\sigma\text{-centered})$, we have $p \leq p_\mathcal{R} \leq t_\mathcal{R} \leq t$. The equality follows from the result of Malliaris and Shelah, that $p = t$. □

Corollary 62  1. For all $1 \leq \alpha < \omega_1$, $t_{\mathcal{R}_\alpha} = p_{\mathcal{R}_\alpha} = p$.
2. If $\mathcal{R}$ is a topological Ramsey space generated by a collection of Fraïssé classes with the Ramsey property, then $t_\mathcal{R} = p_\mathcal{R} = p$.

Proof  The topological Ramsey space in the hypothesis satisfy the ISS$^*$, so the corollary follows from Theorem 61. □

In particular, $\forall n \geq 1$, the pseudointersection number and tower number for the $n$-hypercube space $H^n$ all equal $p$, since these are special cases of (2) in Corollary 62.

7.2 Pseudointersection and tower numbers for the forcings $\mathcal{P}(\omega^\alpha / \text{Fin}^\otimes \alpha)$

Next, we look at the pseudointersection and tower numbers for the high dimensional Ellentuck spaces $\mathcal{E}_\alpha$, for $2 \leq \alpha < \omega_1$. Recall that $(\mathcal{E}_\alpha, \subseteq^{*\alpha})$ is forcing equivalent to $\mathcal{P}(\omega^\alpha / \text{Fin}^\otimes \alpha)$. Hence, for the high and infinite dimensional Ellentuck spaces, the partial order $\subseteq^*\alpha$ denotes $\subseteq^{*\alpha}$. We point out that for the spaces $\mathcal{E}_\alpha$, the $\sigma$-closed partial order defined by Mijares (recall Definition 12) is intermediate between $\leq$ and $\subseteq^{*\alpha}$ and hence produces the same separative quotient.

The following theorem is proved in [40].

Theorem 63  (Szymański and Zhou [40]) $t(\text{Fin} \otimes \text{Fin}) = \omega_1$.

Proposition 64  $p_{\mathcal{E}_2} = t_{\mathcal{E}_2} = \omega_1$.

Proof  This is a consequence of the last two theorems. □
In what follows, we show that for each \( \alpha \in [2, \omega_1) \), the pseudointersection and tower numbers of \( E_\alpha \) are equal to \( \omega_1 \). In fact, this is true for each space \( E_B \) in [15], where \( B \) is a uniform barrier of infinite rank. We point out that for \( 2 \leq k < \omega \), the following results were found by Kurilic in [28], though we were unaware of those results at the time that our results were found. It is important to note that the forcings \( P(\mathcal{N}) \) in [28] are different from the forcings \( \mathcal{P}(\alpha)/\text{Fin}^{\omega \alpha} \), so for infinite countable ordinals, the results below are new.

**Notation** For every \( k \geq 2, 1 < l < k, X \in \mathcal{E}_k \) and \( x \in \omega^l \):

1. Let \( \max x \) denote the last member of the finite sequence \( x \).
2. Let \( \pi_1(X) \) denote the set \( \{x_0 : x \in X\} \).
3. Denote by \( \pi_l(X) \) the set \( \{x \upharpoonright l : x \in X\} \).

Note that because of the definition of \( \mathcal{E}_k \) spaces, \( \pi_l(X) \in \mathcal{E}_l \).

**Definition 65** Let \( X \) be a member of \( \mathcal{E}_k \) and \( s \) a finite approximation of \( X \). Write \( \pi_1[X] \) as an increasing sequence \( \{n_0, n_1, ..., n_j, ...\} \). We will say that \( s \) is the \( i \)-th full finite approximation of \( X \) if \( s \) is the \( \sqsubseteq \)-least finite approximation of \( X \) such that there exists \( x \in s \) such that \( \min x = n_i \). We will denote by \( a^k_i(X) \) the \( i \)-th full finite approximation of \( X \).

Note that for every \( i \in \omega \), if \( x \) is the \( \angle \)-least member of \( a^k_i(X)/a^k_{i-1}(X) \) then \( x_0 = n_i \). Also note that for every \( x \in X \) such that \( x_0 = n_i \), there exists \( Y \leq X \) such that \( x \) is the \( \angle \)-least member of \( a^k_i(Y)/a^k_{i-1}(Y) \).

**Lemma 66** For every \( k \in \omega \) and \( X \in \mathcal{E}_k \), there exists a \( X' \in \mathcal{E}_{k+1} \) such that for every \( X, Y \in \mathcal{E}_k \) and \( Z \in \mathcal{E}_{k+1} \):

1. \( \pi_k(X') = X \),
2. \( Y \leq X \) implies \( Y' \leq X' \),
3. \( Y \leq X \) implies \( Y' \leq X' \) and
4. \( Z \leq X \) implies \( \pi_k(Z) \leq X \).

**Proof** The proof will be by induction on \( k \in \omega \). Fix \( k = 2 \). Let \( X \) be a member of \( \mathcal{E}_2 \), write \( X = \{x_0, x_1, ..., x_i, ...\} \), with \( x_i < x_{i+1} \) for every \( i \in \omega \). Write \( \pi_1(X) = \{n_m : m \in \omega\} \) where \( n_m < n_{m+1} \) for every \( m \in \omega \). We want to construct a member of \( \mathcal{E}_3 \) such that \( \pi_2(X') = X \). We will construct such \( X' \) step by step by extending members of full finite approximations. Note that \( a^2_1(X) = \{x_0\} \). In this case we extend \( x_0 \) with \( \check{x}_0 = x_0 \). Now fix \( i > 0 \). Let \( l_i = \sum_{j<i+1} j \), then \( a^2_i(X) = \{x_0, x_1, ..., x_i\} \) and \( a^2_i(X)/a^2_{i-1}(X) = \{x_i, x_{i-1}, ..., x_1\} \). Note that for every \( m \in [0, i+1) \), \( \pi_1(x_i, x_{i-1}) = n_m \). Given \( x \in a^2_i(X) \), there exists \( m \in [0, i+1) \) such that \( \pi_1(x) = n_m \). We extend \( x \) to \( \check{x}_i = x \) (max \( x_{i-j-1}+m \)). Note that for every \( i \in \omega \) and \( j \in [0, l_i) \), \( \pi_2(\check{x}_i) = x_j \). Let \( X' = \{\check{x}_i : i \in \omega, j \in [0, l_i)\} \). Note that by the definition of \( X', X' \in \mathcal{E}_3 \) and \( \pi_2(X') = X \). We will prove that if \( Y \leq X \) then \( Y' \leq X' \). Take \( y \in Y' \) then there are \( j, k \in \omega \) with \( j \leq k \) such that \( y = \check{y} \) (max \( y_k \)). Note that by construction, \( \pi_1(y_j) = \pi_1(y_k) \). Since \( Y \leq X \), there are some \( j', k' \in \omega \) with \( j' < k' \) such that \( x_{j'} = y_j \) and \( x_{k'} = y_k \). There exists an \( i \in \omega \) such that \( x_{k'} \in a^2_i(X)/a^2_{i-1}(X) \). 

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Then \( \tilde{x}_{ij} = x_{ij}^- (\max x_{k'}) = y \) and \( \tilde{x}_{ij} \in X' \). So, \( Y' \leq X' \). By similar arguments, \( Y \leq X \) implies \( Y' \leq X' \). Now suppose that \( Z \leq X \), then \(|X'/Z| < \omega\). Note that if \(|X/\pi_2(Z)| > \omega\), by the construction of \( X' \), \(|X'/Z| > \omega\). Therefore \( \pi_2(Z) \leq X \).

Let \( k > 2 \) be a natural number. Now assume we have the conclusion for every member of \([2, k]\) and we want to prove it for \( k + 1 \). Fix \( X \in \mathcal{E}_{k+1} \) and write \( \pi_1(X) = \{n_0, n_1, ..., n_m, \ldots\} \), with \( n_m < n_{m+1} \) for every \( m \in \omega \). Write \( X(m) \) to denote the subtree of \( X \) such that for every \( x \in X(m) \), \( \pi_1(x) = n_m \). Note that every \( X(m) \) contains a member of \( \mathcal{E}_k \). Now, for every \( m \in \omega \) let \( X(m) \uparrow (0, \ldots, k) \) denote the collection of \( x \uparrow (0, \ldots, k) \) where \( x \) is a member of \( X(m) \). Note that \( X(m) \uparrow (0, \ldots, k) \in \mathcal{E}_k \). By induction hypothesis, for every \( m \in \omega \) we can extend \( X(m) \uparrow (0, \ldots, k) \) to a \( X(m)' \in \mathcal{E}_{k+1} \) such that \( \pi_k(X(m)') = X(m) \uparrow (0, \ldots, k) \). Let \( X' = \bigcup_{m \in \omega} \{(n_m)^- x : x \in X(m)'\} \). Note that \( X' \in \mathcal{E}_{k+2} \). Since for every \( m \in \omega \), \( \pi_k(X(m)') = X(m) \uparrow (0, \ldots, k) \), \( \pi_{k+1}(X') = X \). Take \( X, Y \in \mathcal{E}_{k+1} \) such that \( Y \leq X \). Take \( y \in Y' \) then exists \( n_m \in \omega \) such that \( y = (n_m)^- z \) with \( z \in Y(m)' \). Then \( (z \uparrow (0, k)) \in Y(m) \) and since \( Y \leq X \) there exists an \( l \in \omega \) such that \( Y(m) \subset X(l) \). Hence by hypothesis induction \( Y(m)' \leq X(l)' \) and \( y \in X' \). By similar arguments, \( Y \leq X \) implies \( Y' \leq X' \).

Now suppose that \( Z \leq X \), then \(|X'/Z| < \omega\). Note that if \(|X/\pi_{k+1}(Z)| > \omega\), by the construction of \( X' \), \(|X'/Z| > \omega\). Therefore \( \pi_{k+1}(Z) \leq X \).

**Proposition 67** For every \( k \in \omega \) such that \( k > 1 \), \( t_{\mathcal{E}_{k+1}} \leq t_{\mathcal{E}_k} \).

**Proof** Fix \( k \in \omega \). Let \( \kappa < t_{\mathcal{E}_{k+1}} \) be a cardinal, we will prove that \( \kappa < t_{\mathcal{E}_k} \). Let \( \mathcal{F} \subseteq \mathcal{E}_k \) be a family linearly ordered by \( \leq^* \). By the last Lemma for every \( X \in \mathcal{E}_k \) there exists an \( X' \in \mathcal{E}_{k+1} \) with properties 1, 2 and 3. Let \( \mathcal{G} = \{X' : X \in \mathcal{F}\} \) and note that \( \mathcal{G} \) is linearly ordered by \( \leq^* \). Since \( \kappa < t_{\mathcal{E}_{k+1}} \), \( \mathcal{G} \) has a pseudointersection \( Z \). By 3 of the last Lemma, \( \pi_k(Z) \) is a pseudointersection of \( \mathcal{F} \). Then \( \kappa < t_{\mathcal{E}_k} \). Therefore \( t_{\mathcal{E}_{k+1}} \leq t_{\mathcal{E}_k} \).

**Theorem 68** For every \( k > 2 \), \( t_{\mathcal{E}_k} = p_{\mathcal{E}_k} = \omega_1 \).

**Proof** Let \( k > 2 \) be given. Since \( (\mathcal{E}_k, \leq^*) \) is a \( \sigma \)-closed partial order \( \omega_1 \leq t_{\mathcal{E}_k} \). By the last proposition, \( p_{\mathcal{E}_k} \leq t_{\mathcal{E}_k} \leq t_{\mathcal{E}_2} = \omega_1 \). Therefore \( t_{\mathcal{E}_k} = p_{\mathcal{E}_k} = \omega_1 \).

In fact, by a similar proof, we obtain the following result. Spaces \( \mathcal{E}_B \) where \( B \) is a uniform barrier are defined in [15].

**Theorem 69** Let \( B \) be a uniform barrier of countable rank at least 2. Then \( t_{\mathcal{E}_B} \leq t_{\mathcal{E}_2} \).

**Proof** Similarly to the proof of Lemma 66, in fact, for each \( X \in \mathcal{E}_2 \), there is an \( X' \in \mathcal{E}_B \) such that (1) – (4) of that lemma hold, where \( k = 2 \). Then similarly to Proposition 67, it follows that \( t_{\mathcal{E}_B} \leq t_{\mathcal{E}_2} \).

### 7.3 The Carlson-Simpson Space

We finish this section with a proof that the Carlson-Simpson space also has tower number equal to \( \omega_1 \). Recall from Subsection 7.3 the Carlson-Simpson space \( \mathcal{E}_\infty \), which is not related with high dimensional Ellentuck spaces. The following proposition is a consequence of a Proposition from Carlson that appears in [32], where Carlson prove that there is a family \( \{X_\alpha : \alpha < \omega_1\} \subseteq \mathcal{E}_\infty \) such that there are not \( X \in \mathcal{E}_\infty \) such that for every \( \alpha < \omega_1 \), \( X \leq^* X_\alpha \).
**Proposition 70** \( p_{\mathcal{E}_\infty} = t_{\mathcal{E}_\infty} = \omega_1 \).

This is interesting in light of the following proposition.

**Proposition 71** The partial order \((\mathcal{E}_\infty, \leq^*)\) forces a Ramsey ultrafilter.

**Proof** Let \( G \) be an \((\mathcal{E}_\infty, \leq^*)\)-generic filter, and let \( \mathcal{U} \) be the ultrafilter on \( \omega \) generated by sets \( p(E) \) with \( E \in G \). Suppose we are given a coloring \( c : [\omega]^2 \to 2 \). Define \( D = \{ F \in \mathcal{E}_\infty : |c[p(F)]^2| = 1 \} \). We will prove that \( D \) is a dense subset of \( \mathcal{E}_\infty \).

Let \( E \in \mathcal{E}_\infty \) be given. By Ramsey’s Theorem, there exists \( M \in [p(E)]^\omega \) such that \( M \) is monochromatic. Define a rigid surjection \( h : \omega \to \omega \) recursively as follows. Let \( h(0) = f_E(0) = 0 \). Now, fix \( i \geq 1 \) and suppose we have defined \( h(j) \) for each \( j < i \). There are three cases: If there exists \( j < i \) such that \( f_E(i) = f_E(j) \), then let \( h(i) = h(j) \). Otherwise, for all \( j < i \), \( f_E(i) \neq f_E(j) \). If \( i \in M \), let \( h(i) = \max\{h(j) : j < i\} + 1 \). If \( i \notin M \), let \( h(i) = 0 \). Let \( F = E_h \), the partition of \( \omega \) generated by \( h \). Note that \( F \in \mathcal{E}_\infty \) and \( p(F) = M \). Since \( M \) is monochromatic, \( F \in D \). Therefore \( D \) is a dense subset of \((\mathcal{E}_\infty, \leq^*)\). Since \( G \) is a generic filter, there exists \( H \in D \cap G \). Then \( p(H) \) is monochromatic for the coloring \( c \).

In fact, Navarro Flores has proved recently that every topological Ramsey space forces a Ramsey ultrafilter, answering a question of the first author. This result will appear in a forthcoming paper.

### 8 Open problems

The results in Sects. 5 and 6 may be used to calculate more Ramsey degrees of ultrafilters. Especially of interest are the further degrees \( t(\mathcal{E}_k, n) \) for \( n \geq 3 \), for any \( k \geq 2 \).

As we have seen in the previous section, for several different classes of topological Ramsey spaces, their pseudointersection number is equal to the respective tower number. Also, we know that for some topological Ramsey spaces, \( \mathcal{R}, p_{\mathcal{R}} = t_{\mathcal{R}} = p \) and for others, \( p_{\mathcal{R}} = t_{\mathcal{R}} = \omega_1 \). We conclude this paper with the most pressing open problems in this subject.

**Question 1** Is \( p_{\mathcal{R}} \leq p \) for every topological Ramsey space \( \mathcal{R} \)?

**Question 2** Is there a topological Ramsey space with pseudointersection number different from its tower number?

**Question 3** Is there a topological Ramsey space with pseudointersection number different from both \( \omega_1 \) and \( p \)?

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