ILL-POSEDNESS FOR THE 2D VISCOUS SHALLOW WATER EQUATIONS IN THE CRITICAL BESOV SPACES

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Abstract. In this paper, we prove that the 2D viscous shallow water equations is ill-posed in the critical Besov spaces $\dot{B}_{2,1}^{s-1}(\mathbb{R}^2)$ with $p > 4$. Our proof mainly depends on the method introduced by Bourgain-Pavlović in [5] and Chen-Miao-Zhang in [10].

1. Introduction and main result

In the paper, we consider the following 2D viscous shallow water equations:

$$\begin{cases}
\partial_t h + \text{div}(hu) = 0, \\
h(\partial_t u + u \cdot \nabla u) - \nu \nabla \cdot (h \nabla u) + h \nabla h = 0, \\
h(0, x) = h_0, \quad u(0, x) = u_0,
\end{cases} \tag{1.1}$$

where $h(t, x)$ represents the height of the fluid level above the solid bed, $u(t, x) = (u^1(t, x), u^2(t, x))$ is the horizontal velocity field and $\nu > 0$ is the viscous coefficient.

The viscous shallow water equations have been widely studied by mathematicians, cf., [2, 3] and references therein. In [4], by using Lagrangian coordinates and Hölder space estimates, Bui obtained the local existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for (1.1) with initial data in $C^{2+\alpha}$. By using the energy method of Matsumura and Nishida [22], Kloeden [18] and Sundbye [23] showed the global existence and uniqueness of classical solutions to the Cauchy-Dirichlet problem for (1.1). Subsequently, the existence and uniqueness of classical solutions to the Cauchy problem for (1.1) was also proved by Sundbye [24]. By applying the Littlewood-Paley decomposition theory for Sobolev spaces to obtain a losing energy estimate in $H^{s+2}$ for any $s > 0$, Wang and Xu [25] showed that the solution of (1.1) exists locally and uniquely for all initial data $u_0$ and exists globally for small initial data $u_0$ if $h_0 - \bar{h}_0$ is small enough. Lately, Liu and Yin [19, 20, 21] improved the result of [25] in the Sobolev spaces with low regularity and inhomogeneous Besov spaces.

In [9], Chen, Miao and Zhang developed a new method which relies on the smoothing properties of the heat equations and introduced some kind of weighted Besov norms to study the well-posedness of (1.1) for the initial data with the minimal regularity and prove the local well-posedness under the more natural assumption that the initial height is bounded away from zero. They obtained the local well-posedness of the system (1.1) for general initial data in critical Besov spaces with $L^p$ type. Now, let us recall some important progress about the global existence results for small data. Motivated by the ideas of Danchin [11, 12], Chen-Miao-Zhang [7] proved the global well-posedness of the system (1.1) for small initial data in critical Besov spaces with $L^2$ type. Chen-Miao-Zhang [8] and Charve-Danchin [6] obtained the global well-posedness of the system (1.1) in the hybrid Besov spaces, in which the part of high frequency of the initial data lies in the critical Besov spaces with $L^p$ type integrability. Subsequently, Haspot [16] improved the results of [6, 8] by a smart use of the viscous effective flux. For more results of the solutions to (1.1), we refer the reader to see [14, 15, 17].
For the sake of convenience, we take $\bar{h}_0 = 1$ and $\nu = 1$. Substituting $h$ by $1 + h$ in (1.1), we have

$$
\begin{cases}
\partial_t h + \text{div } u + u \cdot \nabla h = h \text{div } u, \\
\partial_t u + u \cdot \nabla u - \Delta u + \nabla h = \nabla (\ln(1 + h)) \cdot \nabla u \\
h(0, x) = h_0, \quad u(0, x) = u_0.
\end{cases}
$$

(1.2)

By [9], the system (1.2) is locally well-posed in the critical Besov space with $1 \leq p < 4$. However, the problem whether the system (1.2) is well-posedness in the critical Besov space with $p > 4$ is unsolved. Recently, Chen-Miao-Zhang [10] proved the ill-posedness of the 3D compressible Navier-Stokes equations in critical Besov spaces with $p > 6$. Motivated by Bourgain-Pavlović [5] and Chen-Miao-Zhang [10], we prove that the system (1.2) is ill-posed in the critical Besov spaces with $p > 4$. Our main ill-posedness result reads as follows.

**Theorem 1.1.** Let $p > 4$. For any $\delta > 0$, there exists initial data satisfying

$$
||h_0||_{\dot{B}^{\frac{2}{p},1}_p} + ||u_0||_{\dot{B}^{\frac{2}{p},1}_p} \leq \delta,
$$

such that a solution $(h, u)$ for the system (1.2) satisfies

$$
||u(t)||_{\dot{B}^{\frac{2}{p},1}_p} \geq \frac{1}{\delta}, \quad \text{for some } 0 < t < \delta.
$$

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in the sequel. In Section 3, we will give the proof of Theorem 1.1.

**Notation.** In the following, since all spaces of functions are over $\mathbb{R}^2$, for simplicity, we drop $\mathbb{R}^2$ in our notations of function spaces if there is no ambiguity. Let $C \geq 1$ and $c \leq 1$ denote constants which can be different at different places. We use $A \lesssim B$ to denote $A \leq CB$.

### 2. Littlewood-Paley Analysis

In this section, we first recall some tools from the Littlewood-Paley theory, the definition of homogeneous Besov spaces and some useful properties. Then, we state some applications in the linear transport equation and the heat equation.

First, let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi \in \mathcal{S}(\mathbb{R}^2)$ supported in $\tilde{C} = \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq \xi \leq \frac{8}{3}\}$ such that

$$
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1 \quad \text{for all } \xi \neq 0.
$$

The frequency localization operator $\hat{\Delta}_j$ and $\hat{S}_j$ are defined by

$$
\hat{\Delta}_j f = \varphi(2^{-j} D) f = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f), \quad \hat{S}_j f = \sum_{k \leq j - 1} \hat{\Delta}_k f \quad \text{for } j \in \mathbb{Z}.
$$

With a suitable choice of $\varphi$, one can easily verify that

$$
\hat{\Delta}_j \hat{\Delta}_k f = 0 \quad \text{if } |j - k| \geq 2, \quad \hat{\Delta}_j (\hat{S}_{k-1} f \hat{\Delta}_k f) = 0 \quad \text{if } |j - k| \geq 5.
$$

Next we recall Bony’s decomposition from [1]:

$$
uv = \hat{T}_u v + \hat{T}_v u + \hat{R}(u, v),
$$

with

$$
\hat{T}_u v = \sum_{j \in \mathbb{Z}} \hat{S}_{j-1} u \hat{\Delta}_j v, \quad \hat{R}(u, v) = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u \hat{\Delta}_j v, \quad \hat{\Delta}_j v = \sum_{|j' - j| \leq 1} \hat{\Delta}_{j'} v.
$$

The following Bernstein lemma will be stated as follows:
Definition 2.2. Let $s \in \mathbb{R}$, $1 \leq p, r \leq \infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} = \{ f \in S'_0(\mathbb{R}^d) : \| f \|_{\dot{B}^s_{p,r}} < +\infty \},$$

where

$$\| f \|_{\dot{B}^s_{p,r}} \triangleq \left\| \left( 2^{ks} \| \Delta_k f \|_{L^p(\mathbb{R}^d)} \right) \right\|_{L^r}.$$  

Definition 2.3. Let $s \in \mathbb{R}$, $1 \leq p, q, r \leq \infty$ and $T \in (0, \infty]$. The Chemin-Lerner type Besov space $\dot{L}^q_T(\dot{B}^s_{p,r})$ is defined as the set of all the distributions $f$ satisfying

$$\| f \|_{\dot{L}^q_T(\dot{B}^s_{p,r})} \triangleq \left\| \left( 2^{ks} \| \Delta_k f \|_{L^q_T(\mathbb{R}^d)} \right) \right\|_{L^r} < +\infty.$$  

By Minkowski’s inequality, it is easy to find that

$$\| f \|_{\dot{L}^q_T(\dot{B}^s_{p,r})} \leq \| f \|_{\dot{L}^q_T(\dot{B}^s_{p,r})} \quad \text{if} \quad q \leq r, \quad \| f \|_{\dot{L}^q_T(\dot{B}^s_{p,r})} \geq \| f \|_{\dot{L}^q_T(\dot{B}^s_{p,r})} \quad \text{if} \quad q \geq r.$$  

Let us present the priori estimates of the linear transport equation

$$\partial_t f + v \cdot \nabla f = g, \quad f(0, x) = f_0,$$  

and the heat equation

$$\partial_t u - \Delta u = G, \quad u(0, x) = u_0,$$  

in homogenous Besov spaces. The following estimates will be frequently used in the sequel.

Lemma 2.4. ([13]) Let $s \in (-2 \min \{ \frac{1}{p}, 1 - \frac{1}{p} \} - 1, 1 + \frac{2}{p}]$ and $1 \leq p \leq \infty$. Let $v$ be a vector field such that $\nabla v \in L^1_T(\dot{B}_{p,1}^s)$. Assume that $f_0 \in \dot{B}_{p,1}^s$, $g \in L^1_T(\dot{B}_{p,1}^s)$ and $f$ is the solution of (2.1). Then there holds for $t \in [0, T]$,

$$\| f \|_{L^p_T(\dot{B}_{p,1}^s)} \leq e^{CV(t)}(\| f_0 \|_{\dot{B}_{p,1}^s} + \int_0^t e^{-CV(\tau)}\| g(\tau) \|_{\dot{B}_{p,1}^s} d\tau),$$

or

$$\| f \|_{L^p_T(\dot{B}_{p,1}^s)} \leq e^{CV(t)}(\| f_0 \|_{\dot{B}_{p,1}^s} + \| g \|_{L^1_T(\dot{B}_{p,1}^s)}),$$

where $V(t) = \int_0^t \| \nabla v \|_{L^p_{B_{p,1}^s}}^2 d\tau$.

Lemma 2.5. ([13]) Let $s \in \mathbb{R}$ and $1 \leq q, q_1, p, r \leq \infty$ with $q_1 \leq q$. Assume that $u_0 \in \dot{B}_{p,r}^s$ and $G \in \dot{L}^{q_1}_T(\dot{B}_{p,r}^{s-2+\frac{2}{q_1}})$. Then (2.2) has a unique solution $u \in \dot{L}^q_T(\dot{B}_{p,r}^{s+\frac{2}{q}})$ satisfying

$$\| u \|_{\dot{L}^q_T(\dot{B}_{p,r}^{s+\frac{2}{q}})} \leq C(\| u_0 \|_{\dot{B}_{p,r}^s} + \| G \|_{\dot{L}^{q_1}_T(\dot{B}_{p,r}^{s+\frac{2}{q_1}})}).$$

Finally, we need the following estimates for the product estimates in the next section.

Lemma 2.6. ([1]) Let $T$, $s > 0$ and $1 \leq \rho \leq \infty$. Then it holds that

$$\| fh \|_{\dot{L}^p_T(\dot{B}_{p,1}^s)} \leq C(\| g \|_{L^\infty_T(\dot{L}^q)} + \| f \|_{L^\infty_T(\dot{L}^q)} \| g \|_{\dot{L}^p_T(\dot{B}_{p,1}^s)}).$$
Lemma 2.7. \cite{1, 10} Let $T, s > 0$ and $1 \leq \rho \leq \infty$. Assume that $F \in W^{[\sigma]+3}(\mathbb{R})$ with $F(0) = 0$. Then for any $f \in L^\infty \cap B^s_{p,1}$, we have
\[
\|F(f)\|_{L^p_s(B^q_{p,1})} \leq C(1 + \|f\|_{L^p_s(L^\infty)})^{[\sigma]+2} \|f\|_{L^p_s(B^q_{p,1})}.
\]

Lemma 2.8. Let $1 \leq \rho, \rho_1, \rho_2 \leq \infty$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $2 \leq p \leq 4 < q < \infty$ with $\frac{2}{p} + \frac{2}{q} > 1$. Then, we have
\[
\|fg\|_{L^p_e(B^{\frac{2}{p}-1}_{p,1})} \lesssim \|f\|_{L^p_e(B^{\frac{2}{p}-1}_{p,1})} \|g\|_{L^p_e(B^{\frac{2}{q}}_{q,1})}.
\]

Proof. Let $\frac{1}{q} + \frac{1}{p} = \frac{1}{r}$. According to Hölder’s inequality and Lemma 2.1, we deduce that
\[
\|\hat{T}_g f\|_{L^p_e(B^{\frac{2}{p}-1}_{p,1})} \leq \sum_{k \in \mathbb{Z}} 2^{k(\frac{2}{p}-1)} \|\hat{S}_{k-1} f\|_{L^p_e(L^{\frac{2}{q}})} \|\hat{\Delta}_k g\|_{L^p_e(L^r)} \\
\leq \sum_{k \in \mathbb{Z}} \sum_{k' \leq k-1} 2^{k(\frac{2}{p}-1)} \|\hat{\Delta}_{k'} f\|_{L^p_e(L^p)} \cdot 2^{\frac{2k}{q}} \|\hat{\Delta}_k g\|_{L^p_e(L^r)} \cdot 2^{k-k'(\frac{2}{p} - \frac{2}{q}-1)} \\
\leq C \|f\|_{L^p_e(B^{\frac{2}{p}-1}_{p,1})} \|g\|_{L^p_e(B^{\frac{2}{q}}_{q,1})}.
\]

This completes the proof of this lemma. \hfill \Box

3. PROOF OF THE MAIN THEOREM

In this section, we will give the details for the proof of the main theorem.
Let $p > 4$. For simplicity, we define the following index, that is
\[
\frac{2}{p^*} + \frac{2}{p} = 1, \quad \frac{4}{q} = \frac{2}{p^*} + \frac{1}{2}, \quad \frac{2}{q^*} + \frac{2}{q} = 1, \quad \frac{4}{r} = \frac{2}{q^*} + \frac{1}{2}.
\]
Then, we have
\[
2 \leq p^* < q < 4 < r < q^* < p, \quad \frac{2}{r} - \frac{2}{p} = \frac{1}{2} \left( \frac{2}{p} - \frac{2}{q} \right), \quad \frac{3}{4} \left( 1 - \frac{4}{p^*} \right) = \frac{2}{q} - \frac{2}{p}, \quad \frac{2}{q} + \frac{2}{r} > 1. \tag{3.1}
\]
Define a smooth function $\phi$ with values in $[0, 1]$ which satisfies
\[
\phi(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq \frac{1}{4}, \\ 0, & \text{if } |\xi| \geq \frac{1}{2}, \end{cases}
\]
Motivated by [5, 10], we introduce the initial data \((h_0, u_0)\) as

\[
h_0 = 0, \quad u_0 = 2^{n(1 - \frac{2}{p} - \varepsilon)}(\phi(x - 2^n \epsilon) + \phi(x + 2^n \epsilon), i\phi(x - 2^n \epsilon) - i\phi(x + 2^n \epsilon)),
\]

where \(\epsilon = (1, 1)\) and \(\varepsilon\) satisfying \(\varepsilon = \frac{19}{30}(\frac{1}{4} - \frac{1}{p}) > 0\). Here, we also have

\[
6\varepsilon < 1 - \frac{4}{p}, \quad 5\varepsilon > \frac{2}{q} - \frac{2}{p} = \frac{3}{4}(1 - \frac{4}{p}).
\]

Notice that

\[
u_0 = 2^{n(1 - \frac{2}{p} - \varepsilon)}(e^{i2^n x \cdot \epsilon} + e^{-i2^n x \cdot \epsilon}, ie^{i2^n x \cdot \epsilon} - i e^{-i2^n x \cdot \epsilon})\phi(x)
\]

\[
= 2^{n(1 - \frac{2}{p} - \varepsilon) + 1}(\cos(2^n x \cdot \epsilon), -\sin(2^n x \cdot \epsilon))\phi(x).
\]

Obviously, the initial data \(u_0\) is a real-valued field. Now we decompose the solution \(u(t)\) into \(u(t) = U_0(t) + U_1(t) + U_2(t)\) satisfying

\[
U_0(t) = e^{t\Delta}u_0, \quad U_1 = -\int_0^t e^{(t-\tau)\Delta}(U_0 \cdot \nabla U_0) \, d\tau,
\]

and

\[
\partial_t U_2 - \Delta U_2 = F_1 + \nabla \ln(1 + h) \cdot \nabla (U_0 + U_1 + U_2) - \nabla h,
\]

where

\[
F_1 = -(U_0 \cdot \nabla U_1 + U_0 \cdot \nabla U_2 + U_1 \cdot \nabla U_0 + U_1 \cdot \nabla U_1
\]

\[
+ U_1 \cdot \nabla U_2 + U_2 \cdot \nabla U_0 + U_2 \cdot \nabla U_1 + U_2 \cdot \nabla U_2).
\]

It is easy to show that for any \(s \geq 0\) and \(q_0 \geq 2\),

\[
||U_0||_{L^\infty_t(B^\frac{2}{q_0}m_0)^n} \leq C ||u_0||_{B^\frac{2}{q_0}m_0} \leq C 2^{n(\frac{2}{q_0} - \frac{2}{p} + s - \varepsilon)}. \tag{3.2}
\]

Now, we establish the lower bound estimate of \(||U_1||_{B^{\frac{2}{p} - 1}_{p,1}}\). Since \(B^{\frac{2}{p} - 1}_{p,1} \hookrightarrow B^{-1}_{\infty,\infty}\), we have

\[
||U_1||_{B^{\frac{2}{p} - 1}_{p,1}} \geq c ||T_{\mathbb{R}^2} \varphi(16\xi) \hat{U}_1(\xi) \, d\xi|| \text{ for some } c > 0 \text{ independent of } n.
\]

Note that

\[
(U_0 \cdot \nabla U_0)^1 = U_0^1 \partial_1 U_0^1 + U_0^2 \partial_2 U_0^1, \quad (U_0 \cdot \nabla U_0)^2 = U_0^1 \partial_1 U_0^2 + U_0^2 \partial_2 U_0^2.
\]

It follows from (3.2) that

\[
||\int_0^t e^{(t-\tau)\Delta}(U_0^1 \partial_1 U_0^1) \, d\tau||_{B^{\frac{2}{p} - 1}_{p,1}} + ||\int_0^t e^{(t-\tau)\Delta}(U_0^2 \partial_2 U_0^2) \, d\tau||_{B^{\frac{2}{p} - 1}_{p,1}} \leq C 2^{-2n}.
\]

Therefore, we can show that

\[
||U_1||_{B^{\frac{2}{p} - 1}_{p,1}} \geq c \Big( \int_{\mathbb{R}^2} \int_0^t e^{-(t-\tau)||\xi||^2} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} U_0^2 \partial_2 e^{\tau \Delta} U_0^1)(\xi) \, d\tau \, d\xi \Big)
\]

\[
+ \int_{\mathbb{R}^2} \int_0^t e^{-(t-\tau)||\xi||^2} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} U_0^1 \partial_1 e^{\tau \Delta} U_0^2)(\xi) \, d\tau \, d\xi \Big) - C 2^{-2n}. \tag{3.3}
\]

Since \(\text{supp } \phi(x - a) * \phi(x - b) \subset B(a + b, 1)\), we see that

\[
\text{supp } \phi(x - 2^n \epsilon) * \phi(x - 2^n \epsilon) \subset B(2^{n+1} \epsilon, 1), \quad \text{supp } \phi(x + 2^n \epsilon) * \phi(x + 2^n \epsilon) \subset B(-2^{n+1} \epsilon, 1).
\]
Then, we can rewrite
\[
2^{2n\left(\frac{3}{p} - 1 + \varepsilon\right)} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} u_0^2 \partial_\tau e^{\tau \Delta} u_0^1) (\xi)
\]
\[
= -\varphi(16\xi) \int_{\mathbb{R}^2} \eta_2 e^{-\tau(|\xi - \eta| + |\eta| ^ 2)} \phi(\xi - \eta - 2^n \vec{e}) \phi(\eta + 2^n \vec{e}) \ d\eta
\]
\[
+ \varphi(16\xi) \int_{\mathbb{R}^2} \eta_2 e^{-\tau(|\xi - \eta| + |\eta| ^ 2)} \phi(\xi - \eta + 2^n \vec{e}) \phi(\eta - 2^n \vec{e}) \ d\eta,
\]
and
\[
2^{2n\left(\frac{3}{p} - 1 + \varepsilon\right)} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} u_0^1 \partial_\tau e^{\tau \Delta} u_0^2) (\xi)
\]
\[
= \varphi(16\xi) \int_{\mathbb{R}^2} \eta_1 e^{-\tau(|\xi - \eta| + |\eta| ^ 2)} \phi(\xi - \eta - 2^n \vec{e}) \phi(\eta + 2^n \vec{e}) \ d\eta
\]
\[
- \varphi(16\xi) \int_{\mathbb{R}^2} \eta_1 e^{-\tau(|\xi - \eta| + |\eta| ^ 2)} \phi(\xi - \eta + 2^n \vec{e}) \phi(\eta - 2^n \vec{e}) \ d\eta.
\]
Note that
\[
\int_0^t e^{-(t-\tau)|\xi|^2} e^{-\tau(|\xi - \eta|^2 + |\eta|^2)} \ d\tau = \frac{e^{-\eta_1^2|\xi|^2} - e^{-t(|\eta|^2 + |\eta - \eta|^2)}}{|\eta|^2 + |\xi - \eta|^2 - |\xi|^2}.
\]
Hence, we can show that
\[
\int_0^t e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} u_0^2 \partial_\tau e^{\tau \Delta} u_0^1) (\xi) \ d\tau
\]
\[
= -2^{2n\left(1 - \frac{2}{p} + \varepsilon\right)} \varphi(16\xi) \int_{\mathbb{R}^2} \frac{e^{-\eta_1^2|\xi|^2} - e^{-t(|\xi - \eta|^2 + |\eta|^2)}}{|\xi - \eta|^2 + |\eta|^2 - |\xi|^2} \left( \eta_2 \phi(\xi - \eta - 2^n \vec{e}) \phi(\eta + 2^n \vec{e})
\]
\[
- \eta_2 \phi(\xi - \eta + 2^n \vec{e}) \phi(\eta - 2^n \vec{e}) \right) \ d\eta
\]
\[
= 2^{2n\left(1 - \frac{2}{p} + \varepsilon\right)} \varphi(16\xi) \int_{\mathbb{R}^2} \frac{e^{-\eta_1^2|\xi|^2} - e^{-t(|\xi - \eta|^2 + 2|\eta|^2)}}{|\xi - \eta|^2 + |\eta|^2 - |\xi|^2} (-2\eta_2 + \xi_2) \phi(\xi - \eta - 2^n \vec{e}) \phi(\eta + 2^n \vec{e}) \ d\eta,
\]
and
\[
\int_0^t e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} u_0^1 \partial_\tau e^{\tau \Delta} u_0^2) (\xi) \ d\tau
\]
\[
= 2^{2n\left(1 - \frac{2}{p} + \varepsilon\right)} \varphi(16\xi) \int_{\mathbb{R}^2} \frac{e^{-\eta_1^2|\xi|^2} - e^{-t(|\xi - \eta|^2 + |\eta|^2)}}{|\xi - \eta|^2 + |\eta|^2 - |\xi|^2} \left( \eta_1 \phi(\xi - \eta - 2^n \vec{e}) \phi(\eta + 2^n \vec{e})
\]
\[
- \eta_1 \phi(\xi - \eta + 2^n \vec{e}) \phi(\eta - 2^n \vec{e}) \right) \ d\eta
\]
\[
= 2^{2n\left(1 - \frac{2}{p} + \varepsilon\right)} \varphi(16\xi) \int_{\mathbb{R}^2} \frac{e^{-\eta_1^2|\xi|^2} - e^{-t(|\xi - \eta|^2 + 2|\eta|^2)}}{|\xi - \eta|^2 + |\eta|^2 - |\xi|^2} (2\eta_1 - \xi_1) \phi(\xi - \eta - 2^n \vec{e}) \phi(\eta + 2^n \vec{e}) \ d\eta.
\]
Making a change of variable, we obtain
\[
\int_0^t e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} u_0^2 \partial_\tau e^{\tau \Delta} u_0^1) (\xi) \ d\tau
\]
\[
= 2^{2n\left(1 - \frac{2}{p} + \varepsilon\right)} \varphi(16\xi) \int_{\mathbb{R}^2} \frac{e^{-\eta_2^2|\xi|^2} - e^{-t(|\xi - \eta - 2^n \vec{e}|^2 + |\eta - 2^n \vec{e}|^2)}}{|\xi - \eta + 2^n \vec{e}|^2 + |\eta - 2^n \vec{e}|^2 - |\xi|^2} (-2\eta_2 + 2^n + \xi_2) \phi(\xi - \eta) \phi(\eta) \ d\eta,
\]
and
\[
\int_0^t e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \mathcal{F}(e^{\tau \Delta} u_0^1 \partial_\tau e^{\tau \Delta} u_0^2) (\xi) \ d\tau
\]
\[
= 2^{2n\left(1 - \frac{2}{p} + \varepsilon\right)} \varphi(16\xi) \int_{\mathbb{R}^2} \frac{e^{-\eta_1^2|\xi|^2} - e^{-t(|\xi - \eta + 2^n \vec{e}|^2 + |\eta - 2^n \vec{e}|^2)}}{|\xi - \eta + 2^n \vec{e}|^2 + |\eta - 2^n \vec{e}|^2 - |\xi|^2} (2\eta_1 - 2^n + \xi_1) \phi(\xi - \eta) \phi(\eta) \ d\eta.
\]
Using the Taylor’s formula, we infer that for $t \leq 2^{-2n}$,
\[
\frac{e^{-t|\xi|^2} - e^{-t\left(|\xi-n+2^{n}|\xi|^2 + |\eta-2^n|\xi|^2 \right)}}{|\xi - \eta + 2^n\xi|^2 + |\eta - 2^n\xi|^2 - |\xi|^2} = te^{-t|\xi|^2}(1 + O(t2^n)),
\]
which along with $t = 2^{-(2+4\epsilon)}$ leads to
\[
\left| \int_{\mathbb{R}^2} \int_0^t e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \mathcal{F}(e^{t\Delta}u_0^3 \partial_\tau e^{t\Delta}u_0^1)(\xi) \, d\tau \, d\xi \right| \geq ct^22^{2n(1-\frac{2}{p} - \epsilon)} \geq c2^{n(1-\frac{4}{p} - 6\epsilon)},
\]
and
\[
\left| \int_{\mathbb{R}^2} \int_0^t e^{-(t-\tau)|\xi|^2} \varphi(16\xi) \mathcal{F}(e^{t\Delta}u_0^3 \partial_\tau e^{t\Delta}u_0^1)(\xi) \, d\tau \, d\xi \right| \geq ct^22^{2n(1-\frac{2}{p} - \epsilon)} \geq c2^{n(1-\frac{4}{p} - 6\epsilon)},
\]
for some $c > 0$ independent of $n$. Combining this results (3.3)-(3.5), we have
\[
||U_1(t)||_{B_{p}^{2}} \geq c2^{n(1-\frac{4}{p} - 6\epsilon)} - C2^{-2n} \geq c2^{n(1-\frac{4}{p} - 6\epsilon)}, \quad t = 2^{-n(2+4\epsilon)}, \quad n \gg 1,
\]
for some $c > 0$ independent of $n$. By Lemma 2.5, we also notice that for $T \leq 2^{2n-4\epsilon}$,
\[
||U_0||_{L^2_t(B_{p}^{2})} \leq T^{\frac{1}{2}} ||U_0||_{B_{p}^{2}} \leq 2^{n(\frac{2}{p} - 3\epsilon)},
\]
\[
||U_0||_{L^2_t(B_{r}^{2})} \leq T ||u_0||_{B_{r}^{2}} \leq 2^{n(\frac{2}{p} - 5\epsilon)},
\]
and
\[
||U_0||_{L^2_t(B_{\frac{r}{2}}^{2}) \cap L^2_t(B_{\frac{r}{4}}^{2})} \leq T^{\frac{1}{2}} ||u_0||_{B_{\frac{r}{4}}^{2}} \leq 2^{n(\frac{2}{p} - 3\epsilon)} \leq 2^{n(\frac{2}{p} - 6\epsilon)}.
\]
Although the norm $||U_1||_{L^2_t(B_{p}^{2})}$ is sufficient large for $T \leq 2^{2n-4\epsilon}$. However, we can deduce that the corresponding norm $||U_1||_{L^2_t(B_{r}^{2}) \cap L^2_t(B_{\frac{r}{4}}^{2})}$ is sufficient small when $T \leq 2^{2n-4\epsilon}$. In fact, it follows from Lemmas 2.5-2.6, Hölder’s inequality and (3.2), (3.7) that
\[
||U_1||_{L^2_t(B_{p}^{2}) \cap L^2_t(B_{r}^{2})} \leq T^{\frac{1}{2}} ||u_1||_{L^2_t(B_{r}^{2}) \cap L^2_t(B_{\frac{r}{4}}^{2})} \leq CT^{\frac{1}{2}} ||U_0 \cdot \nabla U_0||_{L^2_t(B_{r}^{2})} \leq CT \left(||U_0||_{L^2_t(B_{r}^{2})} ||U_0||_{L^2_t(B_{r}^{2})} + ||U_0||_{L^2_t(B_{r}^{2})} ||U_0||_{L^2_t(B_{\frac{r}{4}}^{2})} \right) \leq CT2^{n(\frac{2}{p} - 3\epsilon)} \leq 2^{n(\frac{2}{p} - 6\epsilon)}.
\]
Similar argument as in (3.9), we have for $T \leq 2^{2n-4\epsilon}$,
\[
||U_1||_{L^2_t(B_{r}^{2}) \cap L^2_t(B_{\frac{r}{4}}^{2})} \leq 2^{n(\frac{2}{p} - 6\epsilon)}.
\]
Now, we will show that the corresponding norm of $U_2$ is also small when $T \leq 2^{2n-4\epsilon}$. For simplicity, we denote
\[
X_T = ||U_2||_{L^2_t(B_{r}^{2}) \cap L^2_t(B_{\frac{r}{4}}^{2})}, \quad Y_T = ||h||_{L^2_t(B_{r}^{2})}.
\]
According to Lemma 2.8, (3.1) and (3.7)-(3.10), we have

\[
\|U_0 \cdot \nabla U_1 + U_1 \cdot \nabla U_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \\
\lesssim \|U_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim 2^{\frac{3}{4}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon),
\]

(3.11)

\[
\|U_1 \cdot \nabla U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim \|U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim 2^{2n}(\frac{2}{p} - \frac{2}{p} - 6\epsilon),
\]

(3.12)

\[
\|U_0 \cdot \nabla U_2 + U_2 \cdot \nabla U_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \\
\lesssim \|U_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon)X_T,
\]

(3.13)

\[
\|U_1 \cdot \nabla U_2 + U_2 \cdot \nabla U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim \|U_1\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon)X_T,
\]

(3.14)

and

\[
\|U_2 \cdot \nabla U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim \|U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim X_T^2.
\]

(3.15)

Then, combining this estimates (3.11)-(3.15), we have

\[
\|\mathbf{F}_t\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon) + 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon)X_T + X_T^2.
\]

(3.16)

Using Lemmas 2.7-2.8 and (3.8), (3.10), we obtain

\[
\|\nabla \ln(1 + h) \cdot \nabla (U_0 + U_1 + U_2)\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|\nabla h\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \\
\leq C\|\ln(1 + h)\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|U_0 + U_1 + U_2\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|h\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \\
\leq C(\|h\|_{L_\infty(L_2^2)} + 1)^2\|h\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \lesssim 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon) + X_T + 2^{-2n - 4\epsilon n}Y_T \\
\leq C(1 + Y_T^2)(2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon) + X_T)Y_T.
\]

(3.17)

According to Lemma 2.5 and (3.6)-(3.17), we deduce that

\[
X_T \lesssim 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon) + 2^{\frac{1}{2}}n(\frac{2}{p} - \frac{2}{p} - 6\epsilon)(X_T + Y_T^3) + X_T^2 + Y_T^2 + Y_T^6.
\]

(3.18)

Using the fact

\[
\|\text{div} u + h\text{div} u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \leq C(\|u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} \|u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})}) \\
\leq C(2^{n}(\frac{2}{q} - \frac{2}{p} - 5\epsilon) + X_T)(1 + Y_T),
\]

and applying Lemma 2.4, we infer from (3.7)-(3.10) that

\[
Y_T \leq C \exp\{C(\|u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})}) \}(\|h_0\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})} + \|u\|_{L_2^2(B_{q,1}^{\frac{2}{q}+1})}) \\
\leq C \exp\{C(2^{n}(\frac{2}{q} - \frac{2}{p} - 5\epsilon) + CX_T) (2^{n}(\frac{2}{q} - \frac{2}{p} - 5\epsilon) + X_T)(1 + Y_T).
\]

(3.19)

Denote $T_0 = 2^{-2n - 4\epsilon n}$. Then, using the continuation argument, we conclude from (3.18) and (3.19) that for any $T \leq T_0$,

\[
X_T \leq 2^{n}(\frac{2}{q} - \frac{2}{p} - 5\epsilon), \quad Y_T \leq 2^{n}(\frac{2}{q} - \frac{2}{p} - 5\epsilon), \quad n \gg 1.
\]

(3.20)
Summing up (3.2), (3.6), (3.20), we deduce from (3.1) that
\[ ||u(T_0)||_{B_{p,1}^2}^2 \geq ||U_1(T_0)||_{B_{p,1}^2}^2 - ||U_0(T_0)||_{B_{p,1}^2}^2 - ||U_2(T_0)||_{B_{p,1}^2}^2 \]
\[ \geq c2^{n(1-\frac{4}{p}-6\epsilon)} - C2^{n(\frac{2}{q}-\frac{2}{p}-5\epsilon)} - C2^{-n\epsilon} \geq c2^{n(1-\frac{4}{p}-6\epsilon)}, \quad n \gg 1. \]
This along with the fact that
\[ ||h_0||_{B_{p,1}^2} + ||u_0||_{B_{p,1}^2} \leq 2^{-\epsilon n}, \quad n \gg 1, \]
implies the result of Theorem 1.1.

Acknowledgements. The authors want to thank the referees for their constructive comments and helpful suggestions which greatly improved the presentation of this paper. J. Li is supported by the National Natural Science Foundation of China (Grant No.11801090). W. Zhu is partially supported by the National Natural Science Foundation of China (Grant No.11901092) and Natural Science Foundation of Guangdong Province (No.2017A030310634).

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