MINKOWSKI MEASURABILITY OF INFINITE CONFORMAL GRAPH DIRECTED SYSTEMS AND APPLICATION TO APOLLONIAN PACKINGS

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Abstract. We give conditions for the existence of the Minkowski content of limit sets stemming from infinite conformal graph directed systems. As an application we obtain Minkowski measurability of Apollonian gaskets, provide explicit formulae of the Minkowski content, and prove the analytic dependence on the initial circles. Further, we are able to link the fractal Euler characteristic, as well as the Minkowski content, of Apollonian gaskets with the asymptotic behaviour of the circle counting function studied by Kontorovich and Oh. These results lead to a new interpretation and an alternative formula for the Apollonian constant. We estimate a first lower bound for the Apollonian constant, namely 0.055, partially answering an open problem by Oh of 2013. In the higher dimensional setting of collections of disjoint balls, generated e.g. by Kleinian groups of Schottky type, we prove that all fractal curvature measures exist and are constant multiples of each other. Further number theoretical applications connected to the Gauss map and to the Riemann $\zeta$-function illustrate our results.

1. Introduction

Limit sets of infinite conformal graph directed systems (cGDS) form a striking class of geometric objects, which comprises the classes of limit sets of Kleinian groups, Apollonian circle packings, self-conformal and self-similar sets.

Our main motivation for the present article is to characterise the geometric structure of limit sets of infinite cGDS beyond their “fractal dimensions”. The class of Apollonian circle packings illustrates well why this is of particular interest: The Hausdorff, packing and Minkowski dimension of any Apollonian circle packing is the same; its numerical value being approximately 1.30568..., see [Boy82, McM98]. Thus, finer characteristics are required for distinguishing between different circle packings. For this, we study their Minkowski content (which can be viewed as “fractal volume”), their surface area based content (“fractal boundary length”) and their fractal Euler characteristic, as well as the respective localised versions, which are finite Borel measures, called the fractal curvature measures (see Sec. 2.2). We discover that all three characteristics, as well as the respective measures, exist and are constant multiples of one another (see Thm. 4.4, Cor. 4.6, Thm. 4.7 and Rem. 4.8) with the constant being independent of the underlying circle packing. Even more
significant, we show that the fractal curvature measures of collections of disjoint balls in arbitrary dimension formed e.g.
by Kleinian groups of Schottky type are constant multiples of each other with the constant being independent of the collection of balls
(see Sec. 4.3). The fractal Euler characteristic in essence provides an asymptotic on the number $R(\varepsilon)$ of balls of radius
bigger than $\varepsilon$ as $\varepsilon \searrow 0$. In the context of circle packings in $\mathbb{R}^2$ the circle-counting function $R$ has been studied
with the help of Laplace eigenfunctions by Kontorovich and Oh in [KO11] (for further references see Sec. 4.2) and its asymptotic behaviour has been derived. More precisely, it has been shown that $\lim_{\varepsilon \to 0} \varepsilon^D R(\varepsilon) = \pi^{-D/2} \cdot c_A \cdot \mathcal{H}^D(F)$, where $c_A$ is a universal constant which does not depend on the circle packing and its residual set $F$. Here $\mathcal{H}^D(F)$ denotes the $D$-dimensional Hausdorff-measure of $F$ with $D$ denoting the Hausdorff-dimension. To compute (or estimate) $c_A$ is formulated as an open problem in [Oh14b]. Our result here (Thm. 4.7) provides a different representation of the leading asymptotic term of $R$ and in this way gives a new geometric interpretation of the Apollonian constant. Through this new formula, we obtain a first lower bound for the Apollonian constant, namely we show $c_A \geq 0.055$ (Thm. 4.11).

Apollonian circle packings do not fall into any category of sets, for which the Minkowski content or the fractal curvature measures have been shown to exist and determined. The reason, why they can now be treated lies in the key novelty of the present article, namely the extension from finitely to infinitely generated systems. The most important tools in the proof of our main results are some recently obtained renewal theorems for subshifts of finite type over an infinite alphabet developed by the authors in [KK17].

For general limit sets of infinite cGDS, we focus on the Minkowski content. Under certain regularity conditions, we show that the cGDS being non-lattice implies existence of the Minkowski content (see Thm. 3.5) and provide a formula. In the lattice case, we obtain a periodic oscillating function for the volume of the $\varepsilon$-parallel set which we also present. In addition to the Apollonian circle packings, we apply our results for general limit sets of infinite cGDS to number theoretically relevant sets, more precisely to sets stemming from restricted continued fraction digits and from restricted Lüroth digits.

The article is organised as follows. In Sec. 2 we provide the basic definitions which we need to present our main results for infinite cGDS in Sec. 3. The following section, Sec. 4, is devoted to applications of the results from Sec. 3 and to examples. Here, major focus lies on Apollonian circle packings (Sec. 4.1), Apollonian sphere packings in $\mathbb{R}^3$, and collections of disjoint balls in $\mathbb{R}^d$ with $d \geq 4$ formed e.g. by Kleinian groups of Schottky type (Sec. 4.3). Connections of our results to circle-counting results by Kontorovich and Oh, which lead to estimates of the Apollonian constant, are stated in Sec. 4.2. The number theoretical examples are given in Sec. 4.4 and 4.5. In Sec. 5 we present the preliminaries for the proofs which are provided in the final Sec. 6.

2. Basic definitions

2.1. Conformal graph directed systems. Let $(V, E, i, t)$ be a directed multigraph with finite vertex set $V$, countable (finite or infinite) set of directed edges
and functions \( i, t : E \to V \) which determine the initial and terminal vertex of an edge. An \((E \times E)\)-matrix \( A = (A_{e,e'})_{e,e' \in E} \) with entries in \( \{0, 1\} \) which satisfies
\[
A_{e,e'} = 1 \quad \text{if and only if} \quad t(e) = i(e') \quad \text{for edges } e, e' \in E \text{ is called an incidence matrix.}
\]

The set of \emph{infinite admissible words} given by \( A \) is defined to be
\[
E^\infty := E^\infty_A := \{ \omega = (\omega_1, \omega_2, \ldots) \in E^\mathbb{N} \mid A_{\omega_n, \omega_{n+1}} = 1 \quad \forall \ n \in \mathbb{N} \}.
\]

The set of sub-words of length \( n \in \mathbb{N} \) is denoted by \( E^n_A \) and the set of all finite sub-words including the empty word \( \emptyset \) by \( E^* \). The incidence matrix \( A \) is said to be \emph{finitely irreducible} if there exists a finite set \( \Lambda \subset E^* \) such that for all \( i, j \in E \) there is an \( \omega \in \Lambda \) with \( i\omega \in E^\ast \).

\textbf{Definition 2.1 (GDS).} A \emph{graph directed system (GDS)} consists of a directed multi-graph \((V, E, i, t)\) with incidence matrix \( A \), a family \((X_v)_{v \in V}\) of non-empty compact connected subsets of the Euclidean space \((\mathbb{R}^d, |\cdot|)\) and for each edge \( e \in E \) an injective contraction \( \phi_e : X_{t(e)} \to X_{i(e)} \) with Lipschitz constant less than or equal to \( r \) for some \( r \in (0, 1) \). Briefly, the family \( \Phi := (\phi_e : X_{t(e)} \to X_{i(e)})_{e \in E} \) is called a GDS.

A GDS is called \emph{conformal (cGDS)} if

\textbf{(cGDS-1) for every vertex } v \in V, \ X_v \text{ is a compact connected set satisfying } X_v = \overline{U_v}\text{ with } U_v := \text{int}(X_v) \text{ denoting the interior of } X_v,\)

\textbf{(cGDS-2) the open set condition (OSC) is satisfied, in that, for all } e \neq e' \in E \text{ we have } \phi_e(U_{i(e)}) \cap \phi_e(U_{i(e')}) = \emptyset,\)

\textbf{(cGDS-3) for every vertex } v \in V \text{ there exists an open connected set } W_v \supset X_v \text{ such that for every } e \in E \text{ with } t(e) = v \text{ the map } \phi_e \text{ extends to a conformal diffeomorphism from } W_v \text{ into } W_{i(e)}, \text{ whose derivative } \phi_e' \text{ is bounded away from zero on } W_v, \text{ and}\)

\textbf{(cGDS-4) the cone condition holds, that is there exist } j, \ell > 0 \text{ with } j < \pi/2 \text{ such that for every } x \in \bigcup_{v \in V} X_v \text{ there exists an open cone } \text{Con}(x, j, \ell) \subset \text{int}(\bigcup_{v \in V} X_v) \text{ with vertex } x, \text{ central angle of measure } j \text{ and altitude } \ell.\)

A cGDS, whose maps \( \phi_e \) are \emph{similarities} for \( e \in E \), is referred to as \emph{sGDS}.

For a finite word \( \omega \in E^* \) we let \( n(\omega) \) denote its length, where \( n(\emptyset) := 0 \), define \( \phi_{\emptyset} \) to be the identity map on the union \( X := \bigcup_{v \in V} X_v \), and for \( \omega \in E^* \setminus \{\emptyset\} \) set
\[
\phi_{\omega} := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n} : X_{i(\omega(1))} \to X_{i(\omega)}. \]

Here \( \omega_i \) denotes the \( i \)-th letter of the word \( \omega \) for \( i \in \{1, \ldots, n(\omega)\} \), that is \( \omega = (\omega_1, \ldots, \omega_{n(\omega)}) \).

For two finite words \( u = (u_1, \ldots, u_n) \), \( \omega = (\omega_1, \ldots, \omega_m) \in E^* \) with \( t(u_n) = i(\omega_1) \), we let \( u \omega := (u_1, \ldots, u_n, \omega_1, \ldots, \omega_m) \in E^\infty \) denote their concatenation. Likewise, we set \( u \omega := (u_1, \ldots, u_n, \omega_1, \omega_2, \ldots) \) if \( \omega = (\omega_1, \omega_2, \ldots) \in E^\infty \). We write \( \omega | n := (\omega_1, \ldots, \omega_n) \) for the initial sub-word of length \( n \in \mathbb{N} \) of \( \omega \in E^\infty \cup \bigcup_{k>n} E_k^A \).

For \( \omega \in E^\infty \) the sets \((\phi_{\omega | n}(X_{t(\omega_n)}))_{n \in \mathbb{N}}\) form a descending sequence of non-empty compact sets and therefore \( \bigcap_{n \in \mathbb{N}} \phi_{\omega | n}(X_{t(\omega_n)}) \neq \emptyset \). Recall from Definition 2.1 that \( r \in (0, 1) \) denotes a common Lipschitz constant of the functions \( \phi_e, e \in E \). Since \( \text{diam}(\phi_{\omega | n}(X_{t(\omega_n)})) \leq r^n \text{diam}(X_{t(\omega_n)}) \leq r^n \max\{\text{diam}(X_v) \mid v \in V\} \) for every \( n \in \mathbb{N} \), the intersection
\[
\bigcap_{n \in \mathbb{N}} \phi_{\omega | n}(X_{t(\omega_n)})
\]
is a singleton and we denote its only element by \( \pi(\omega) \). The map \( \pi: E^\infty \to \bigcup_{v \in V} X_v \) is called the code map.

**Definition 2.2** (Limit set of a cGDS). The limit set of the cGDS \((\phi_v)_{v \in E}\) is defined to be

\[
F := \pi(E^\infty).
\]

Limit sets of cGDS often have a fractal structure. They include invariant sets of conformal iterated function systems. A conformal iterated function system (cIFS) is a cGDS \( \Psi := (\psi_1, \ldots, \psi_N) \) whose set of vertices \( V \) is a singleton and whose set of edges contains \( N \in \mathbb{N} \setminus \{1\} \) elements. The unique limit set of a cIFS is called the self-conformal set associated with \( \Psi \). In the case that the maps \( \psi_1, \ldots, \psi_N \) are similarities, the limit set is called the self-similar set associated with \( \Psi \) and \( \Psi \) is called an sIFS. A core text concerning cGDS is [MU03].

## 2.2. Dimensions, Minkowski content, fractal curvature measures, and Bowen’s formula for limit sets of cGDS

In order to gain a better understanding of the geometry of a limit set \( F \) of a cGDS we study the volume of its \( \varepsilon \)-neighbourhoods

\[
F_\varepsilon := \left\{ x \in \mathbb{R}^d \mid \inf_{y \in F} |x - y| \leq \varepsilon \right\}
\]

for \( \varepsilon > 0 \), which is a well-used approach in fractal geometry. We measure the volume with the \( d \)-dimensional Lebesgue measure \( \lambda_d \) and study the limiting behaviour of \( \lambda_d(F_\varepsilon) \) as \( \varepsilon \downarrow 0 \). Related to this limiting behaviour is the Minkowski dimension of \( F \)

\[
\dim_M(F) := \lim_{\varepsilon \downarrow 0} d - \ln(\lambda_d(F_\varepsilon))/\ln(\varepsilon).
\]

It is given by Bowen’s Formula, which we present in the following, after introducing some central thermodynamical tools.

The space \( E^\infty \) gives rise to a topological dynamical system with the dynamics given by the (left) shift map \( \sigma \) acting on \( E^\infty \cup E^* \) by

\[
\sigma(\omega) = \begin{cases} 
(\omega_2, \omega_3, \ldots) & : \omega = (\omega_1, \omega_2, \ldots) \in E^\infty \\
(\omega_2, \omega_3, \ldots, \omega_n) & : \omega = (\omega_1, \omega_2, \ldots, \omega_n) \in E_A^n, \ n \geq 2 \\
\emptyset & : \omega \in E_A^\infty \cup E_A^1.
\end{cases}
\]

For \( \omega \in E_A^n \) we define its \( n \)-cylinder to be the set

\[
[\omega] := \{ x \in E^\infty \mid \forall i \in \{1, \ldots, n\} : x_i = \omega_i \}.
\]

We equip \( E^\infty \) with the product topology of the discrete topologies on \( E \) and equip \( E^\infty \subset E^\mathbb{N} \) with the subspace topology. By \( \mathcal{C}(E^\infty) \) we denote the set of continuous real-valued functions on \( E^\infty \) and call elements of \( \mathcal{C}(E^\infty) \) potential functions. The set of bounded continuous functions in \( \mathcal{C}(E^\infty) \) with respect to the supremum-norm \( \| \cdot \|_\infty \) is denoted by \( C_b(E^\infty) \). A potential function \( f \) is called co-homologous to a potential function \( \zeta \) if there exists \( \psi \in \mathcal{C}(E^\infty) \) such that \( f = \zeta + \psi - \psi \circ \sigma \). The potential function \( f \) is called lattice, if it is co-homologous to a potential function \( \zeta \) whose range is contained in a discrete subgroup of \( \mathbb{R} \). Otherwise we say that \( f \) is non-lattice.

A central role in our studies is played by the geometric potential function \( \xi: E^\infty \to \mathbb{R} \) associated with a cGDS \( \Psi \), which is defined by \( \xi(\omega) := -\ln|\phi_{\omega}(\pi(\sigma\omega))| \) for
\( \omega = (\omega_1, \omega_2, \ldots) \in E^\infty \). It lies in \( C(E^\infty) \) but is generally unbounded if \( \Phi \) is infinitely generated. We call \( \Phi \) (non-)lattice if its geometric potential function is (non-)lattice. The topological pressure function of a potential function \( f \in C(E^\infty) \) with respect to the shift map \( \sigma: E^\infty \to E^\infty \) is defined by the well-defined limit

\[
P(f) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\omega \in E^n} \exp \left( \sup_{\tau \in [\omega]} S_n f(\tau) \right),
\]

where

\[
S_n f := \sum_{j=0}^{n-1} f \circ \sigma^j \quad \text{for } n \geq 1 \quad \text{and} \quad S_0 f := 0
\]
denotes the \( n \)-th Birkhoff sum of \( f \).

Bowen’s Formula \cite{MU03} states that the Hausdorff dimension \( \dim_H(F) \) and the Minkowski dimension \( \dim_M(F) \) of the limit set \( F \) coincide and are given by

\[
\dim_H(F) = \dim_M(F) = D := \inf \{ s > 0 : P(-s\xi) \leq 0 \}.
\]

The system \( \Phi \) is called regular if \( P(-D\xi) = 0 \). It is called strongly regular if

\[
\theta := \sup \left\{ s \in \mathbb{R} \mid \sum_{e \in E} \exp \left( \sup_{\xi} s\xi \right) < \infty \right\} > -D.
\]

On the region where \( P(s\xi) \) is finite, the map \( s \mapsto P(s\xi) \) is continuous. Therefore, strong regularity implies regularity.

There exists a big variety of limit sets of cGDS of the same Minkowski and Hausdorff dimension, whence finer tools are needed and we propose to study the Minkowski content

\[
\mathcal{M}(F) := \lim_{\varepsilon \to 0} \varepsilon^{D-d} \lambda_d(F_\varepsilon)
\]

of \( F \). A set \( A \) for which \( \mathcal{M}(A) \) exists, is positive and finite is called Minkowski measurable. A refinement is provided by the local Minkowski content \( \mathcal{M}(F,B) \) relative to a Borel set \( B \subseteq \mathbb{R}^d \) if it exists:

\[
\mathcal{M}(F,B) := \lim_{\varepsilon \to 0} \varepsilon^{D-d} \lambda_d(F_\varepsilon \cap B).
\]

Related to the Minkowski content are the fractal curvatures and fractal curvature measures as introduced in \cite{Win08}. These are defined via a (local) Steiner formula for sets of positive reach. A set \( A \subseteq \mathbb{R}^d \) is said to be of positive reach if there exists \( r > 0 \) such that any point \( x \) in \( A \), has a unique closest neighbour \( \pi_A(x) \) in \( A \). The supremum reach \( \text{reach}(A) \) over all such \( r > 0 \) is called the reach of \( A \), and \( \pi_A: \text{int}A \rightarrow A \) is called the metric projection onto \( A \).

**Theorem 2.3** (Local Steiner formula, \cite{Fed59}). Let \( A \subseteq \mathbb{R}^d \) be a compact set of positive reach. Then there exist uniquely determined signed Borel measures \( C_0(A, \cdot) \ldots, C_d(A,\cdot) \) such that for every \( B \in \mathcal{B}(\mathbb{R}^d) \) and every \( 0 \leq \varepsilon < \text{reach}(A) \) we have

\[
\lambda_d(A_\varepsilon \cap \pi_A^{-1}(B)) = \sum_{k=0}^{d} \varepsilon^{d-k} \kappa_{d-k} C_k(A,B),
\]

where \( \kappa_k \) denotes the \( k \)-dimensional volume of the \( k \)-dimensional unit ball.
The signed measure $C_k(A, \cdot)$ is called the $k$-th curvature measure of $A$. $C_k(A) \equiv C_k(A, \mathbb{R}^d)$ is called the $k$-th (total) curvature of $A$. For any set $A$ of positive reach, $C_d(A) = \lambda_d(A)$, $C_{d-1}(A) = \lambda_{d-1}(\partial A)/2$ and $C_0(A) = \chi(A)$, where $\chi$ denotes the Euler characteristic.

Unfortunately, fractal sets are not of positive reach. However, for such sets one can consider their parallel sets. If these are of positive reach (which will be the case in the applications of the present paper) it is of interest to consider the asymptotic behaviour of the curvature measures as the parallel set becomes smaller. For the following we assume that $F_{\varepsilon^{-1}}$ is a set of positive reach for all sufficiently large $t \in \mathbb{R}$ and denote by $D$ the Minkowski dimension of $F$. Whenever, the weak limit

$$C_k^t(F, \cdot) := \text{w-lim}_{t \to \infty} e^{t(k-D)} C_k(F_{\varepsilon^{-1}}, \cdot)$$

exists, it is called the $k$-th fractal curvature measure of $F$. Its existence implies the existence of the $k$-th (total) fractal curvature of $F$, which is defined through

$$C_k(F) := \lim_{t \to \infty} e^{t(k-D)} C_k(F_{\varepsilon^{-1}}).$$

Notice, for $k = d$ we obtain the Minkowski content, for $k = d - 1$ the surface area based content, and for $k = 0$ the fractal Euler characteristic.

3. Main results

Throughout, we let $\Phi$ denote a strongly regular cGDS with limit set $F$ and use the notation from Sec. 2. A non-empty open set $O := \bigcup_{e \in V} O_e$ with $O_e \subseteq U_e$ and $\phi_e O_{t(e)} \subseteq O_{\omega(e)}$ for each $e \in E$ is called a feasible open set for the cGDS $\Phi$. The condition [cGDS-2] implies existence of at least one feasible open set. For convenience we write $\phi_\omega O := \phi_\omega O_{t(\omega)}$ as well as $\phi_\omega F := \phi_\omega(F \cap X_{t(\omega)})$ for $\omega \in E^*$. Here $i(\omega) := i(\omega_1)$ for $\omega \in E^* \cup E^\infty$ and $t(\omega) := t(\omega_n)$ for a finite word $\omega \in E^*_A$. Moreover, we assume the following projection condition.

**Definition 3.1.** The cGDS $\Phi$ together with the feasible open set $O$ is said to satisfy the projection condition if

$$\phi_e O \subseteq \pi_{\mathcal{F}}^{-1}(\phi_e F) \quad \text{for } e \in E.$$ 

Here, $\pi_{\mathcal{F}}$ denotes the metric projection onto the topological closure $\mathcal{F}$ of $F$.

It is straight-forward to see that the projection condition implies

$$F_\varepsilon \cap \phi_e O = (\phi_e F)_\varepsilon \cap \phi_e O$$

for each $\varepsilon > 0$ and $e \in E$.

**Remark 3.2.** Any cGDS $\Phi$ satisfies the projection condition together with its central open set $O := \bigcup_{e \in V} O_e$, where

$$O_v := \text{int} \left( \bigcap_{\omega \in E^* \cap t(\omega) = v} \phi_\omega^{-1} \pi_{\mathcal{F}}^{-1}(\phi_\omega F) \right),$$

whenever $O \neq \emptyset$. Note that when it is non-empty $O$ indeed is a feasible open set for $\Phi$: For all $\omega \in E^*$ with $t(\omega) = i(e)$ we have $\omega e \in E^*$ and $t(\omega e) = t(e)$ implying

$$\phi_\omega \phi_e O_{t(e)} \subseteq \phi_\omega \phi_e \text{int} \left( \phi_\omega^{-1} \pi_{\mathcal{F}}^{-1}(\phi_\omega F) \right) \subseteq \text{int} \left( \pi_{\mathcal{F}}^{-1}(\phi_\omega F) \right)$$

for each $\omega \in E^* \cap t(\omega) = v$. For $e \in E$, $O_e = \text{int} \left( \bigcap_{\omega \in E^* \cap t(\omega) = v} \phi_\omega^{-1} \pi_{\mathcal{F}}^{-1}(\phi_\omega F) \right)$.
and whence \( \phi_t O_{t(e)} \subset O_{t(e)} \).

The central open set was introduced in [BHR06] for finite self-similar systems and proven to be non-empty for such systems. Thus, in this case, the projection condition is not a restriction on \( \Phi \) or \( F \) but rather ensures a convenient choice for a feasible open set. The above definition provides an extension to infinitely generated cGDS.

We write
\[
\Gamma_v := O_v \setminus \bigcup_{e \in E; t(e) = v} \phi_t(O_{t(e)}), \quad \Gamma := \bigcup_{v \in V} \Gamma_v
\]

and assume non-triviality, that is \( \lambda_d(\Gamma) > 0 \).

**Proposition 3.3.** Non-triviality implies \( \dim_M(F) < d \) and thus \( \lambda_d(F) = 0 \).

In Sec.[6] we prove the above proposition, which is an extension of [MU63] Prop. 4.5.9 and Thm. 4.5.10.

Our main result requires the following conditions

(A) The projection and non-triviality conditions are satisfied.

(B) The incidence matrix is finitely irreducible.

(C) \( \Phi \) is strongly regular.

(D) There exists \( c, \gamma > 0 \) for which \( \lambda_d(F_e \cap \Gamma) \leq c \varepsilon^{d-D+\gamma} \).

**Remark 3.4.** If \( E \) is of finite cardinality then conditions \([C]\) and \([D]\) are always satisfied.

Two functions \( f, g : \mathbb{R} \to \mathbb{R} \) are called asymptotic as \( t \to \infty \), written \( f(t) \sim g(t) \) as \( t \to \infty \), if for all \( \varepsilon > 0 \) there exists \( \bar{t} \in \mathbb{R} \) such that for all \( t \geq \bar{t} \) the value \( f(t) \) lies between \( (1-\varepsilon)g(t) \) and \( (1+\varepsilon)g(t) \). For \( t \in \mathbb{R} \) we define \( \lfloor t \rfloor := \max \{ k \in \mathbb{Z} \mid k \leq t \} \) and \( \{ t \} := t - \lfloor t \rfloor \in [0, 1) \). Note that \( \lfloor t \rfloor \) and \( \{ t \} \) respectively are the integer and the fractional part of \( t \geq 0 \).

**Theorem 3.5.** Suppose that conditions \([A]\) to \([D]\) are met. Let \( \mu_{-D\xi} \) denote the unique \( \sigma \)-invariant Gibbs state for \( -D\xi \) and let \( \nu \) be the \( D \)-conformal measure associated with \( \Phi \) (see Sec. 5.1). For any Borel subset \( B \) of \( O \) the following hold:

(i) If \( \xi \) is non-lattice then
\[
\lambda_d(F_{e^{-1}} \cap B) \sim \frac{e^{-t(d-D)}}{\int \xi d\mu_{-D\xi}} \lim_{m \to \infty} \sum_{u \in E^m \cap \Gamma} e^{-T(D-d)} \lambda_d\left( F_{e^{-T}} \cap \phi_u \Gamma_{t(u)} \right) dT \cdot \nu(B).
\]

Thus, the local Minkowski content \( \mathcal{M}(F, B) \) relative to \( B \) exists.

(ii) If \( \xi \) is lattice with \( \xi = \zeta + \psi \circ \sigma \) and if \( a > 0 \) denotes the maximal real for which \( \zeta(E^\infty) \subset a\mathbb{Z} \) then
\[
\lambda_d(F_{e^{-1}} \cap B) \sim \frac{a \cdot e^{-t(d-D)}}{\int \xi d\mu_{-D\xi}} \lim_{m \to \infty} \sum_{u \in E^m} e^{(D-d)\psi(z_u)} \sum_{t = -\infty}^{\infty} e^{-at(D-d)}
\]
\[
\times \int_{E^\infty} 1_B(\pi y) \cdot e^{-a(D-d)\left\{ \frac{t+\psi(z_u)-\psi(y)}{a} \right\}} A_{u,t}(t, y) d\nu_{-D\xi}(y)
\]
for arbitrary \( x_u \in [u] \), where

\[
A_u,\xi(t,y) := \lambda_d \left( F_{e^{-\frac{1}{a + \xi}}(\frac{t + \log(y)}{u} - \nu(y))}} \right) \cap \phi_u \Gamma_{t(u)}).
\]

Immediate consequences of Thm. 3.5 are presented in the following corollaries, which we state without proofs.

**Corollary 3.6.** If the conditions of Thm. 3.5 are met, then

\[
\lambda_d(F_{e^{-t}} \cap B) \sim \lambda_d(F_{e^{-t}} \cap O) \cdot \nu(B).
\]

In particular, in the non-lattice situation, \( B \mapsto \lim_{t \to \infty} e^{-t(D-d)} \lambda_d(F_{e^{-t}} \cap B) \) is a constant multiple of the \( D \)-conformal measure.

**Corollary 3.7.** Additionally to the conditions of Thm. 3.5, suppose that \( d = 1 \).

Let \( \Gamma = \bigcup_k \Gamma_k \) be the decomposition of \( \Gamma \) into its connected components. If \( \xi \) is non-lattice then

\[
\mathcal{M}(F,B) = \frac{2^{1-D}}{D(1-D)} \int \xi \frac{d\mu_{-D,\xi}}{d\mu_{-D}} \lim_{m \to \infty} \sum_{u \in E_A^m} \sum_k |\phi_u \Gamma_k|^D \cdot \nu(B).
\]

**Corollary 3.8.** Additionally to the conditions of Thm. 3.5, suppose that \( \Phi \) is an sIFS and let \( r_i \) denote the similarity ratio of \( \phi_i \). Then the following hold.

(i) If \( \xi \) is non-lattice then

\[
\mathcal{M}(F,B) = \frac{1}{\int \xi \frac{d\mu_{-D,\xi}}{d\mu_{-D}}} \lim_{m \to \infty} \sum_{u \in E_A^m} r_D \int_{-\infty}^\infty e^{-t(D-d)} \lambda_d \left( F_{e^{-t}} \cap \Gamma_{t(u)} \right) \cdot \nu(B).
\]

(ii) If \( \xi \) is lattice and \( a > 0 \) denotes the maximal real for which \( \xi(E^\infty) \subset a\mathbb{Z} \) then

\[
\lambda_d(F_{e^{-t}} \cap B) \sim \lim_{m \to \infty} \sum_{u \in E_A^m} r_D \sum_{\ell=-\infty}^{\infty} e^{-a\ell(D-d)} e^{-a(D-d)} |\phi_u \Gamma_{t(u)}| \lambda_d \left( F_{e^{-(a+\xi)}} \cap \Gamma_{t(u)} \right) \cdot \nu(B).
\]

**Corollary 3.9.** Additionally to the conditions of Thm. 3.5, suppose that \( \Phi \) is an sIFS and let \( r_i \) denote the similarity ratio of \( \phi_i \). Then the following hold.

(i) If \( \xi \) is non-lattice then

\[
\mathcal{M}(F,B) = \frac{1}{-\sum_i r_i^D \ln r_i} \int_{-\infty}^{\infty} e^{-t(D-d)} \lambda_d \left( F_{e^{-t}} \cap \Gamma \right) \cdot \nu(B).
\]

(ii) If \( \xi \) is lattice and \( a > 0 \) denotes the maximal real for which \( \xi(E^\infty) \subset a\mathbb{Z} \) then

\[
\lambda_d(F_{e^{-t}} \cap B) \sim \frac{a e^{-t(D-d)}}{-\sum_i r_i^D \ln r_i} \sum_{\ell=-\infty}^{\infty} e^{-a\ell(D-d)} e^{-a(D-d)} |\phi_u \Gamma_{t(u)}| \lambda_d \left( F_{e^{-(a+\xi)}} \cap \Gamma \right) \cdot \nu(B).
\]
Remark 3.10. The above theorem and corollaries provide analogues of the respective theorems in the case that the alphabet is finite, given in [Kom11, KK12, DKÖ+13, KK15, Win15, Kom15].

4. Applications and examples

4.1. Apollonian gaskets. The study of circle packings has a long history. A result by Apollonius (ca. 262–190 BC), that is of main importance to us is the following, see [Pol15, Thm. 1.1]. Given three mutually tangent circles $C_1, C_2, C_3$ with disjoint interiors there are precisely two circles $C_0, C_4$ which are tangent to each of the original three (see Fig. 4.1(a)). Now, for each triple of mutually tangent circles from the collection $\{C_0, \ldots, C_4\}$, we can again find two circles which are tangent to each of the circles from the triple. For instance the circles $C_3$ and $C_7$ are tangent to the circles $C_0, C_1, C_2$ (see Fig. 4.1(b)). In this way we obtain a packing of the circle $C_0$ generated by $C_1, C_2, C_3$. We call the limiting object an Apollonian circle packing.

We are interested in geometric properties of circle packings. For this we represent the circle packing inside each of the four curvilinear triangles external to $C_1, C_2, C_3$ and internal to $C_0$ (denoted by $T, T_1, T_2, T_3$ in Fig. 4.2(a)) as a limit set of an infinite cGDS. Without loss of generality we focus on the central curvilinear triangle $T$, which is bounded by arcs of the circles $C_1, C_2, C_3$. (Note that the packing inside of $T_j, j \in \{1, 2, 3\}$, is the image of the packing of $T$ under a Möbius transformation. In Thm. 4.5 for instance, we will see that the Minkowski content of the packing inside of $T_j$ can thus be deduced from the Minkowski content of the packing inside of $T$.) In order to define the contractions of the cGDS we introduce dual- and horocircles: If four circles touch each other mutually, another set of four circles of mutual contact can be found whose points of contact coincide with those of the first four. The new

![Figure 4.1](image-url)

(a) Three mutually tangent circles $C_1$, $C_2$, $C_3$ and the circles $C_0$, $C_4$ which are tangent to each of the original three.

(b) For each triple of mutually tangent circles from $\{C_0, \ldots, C_4\}$ there are exactly two circles which are tangent to the circles from the triple.

Figure 4.1. Families of tangent circles generated by $C_1, C_2, C_3$. 
Figure 4.2. Defining the cGDS for an Apollonian circle packing.
four circles are called dual circles. We denote the dual circles of $C_0, C_1, C_2, C_3$ by $K_0, K_1, K_2, K_3$ (see Fig. 4.2(b)). Moreover, for $j \in \{1, 2, 3\}$ we let $H_j$ denote the circle which tangentially touches $C_0$ and $C_j$ at their touching point and which goes through the touching point of $C_{i_1}$ and $C_{i_2}$, where $i_1, i_2 \in \{1, 2, 3\} \setminus \{j\}$. We call $H_j$ the horocircle associated with $C_j$ (see Fig. 4.2(c)).

For a circle $C$ we let $R_C$ denote the reflection on $C$. If $m$ denotes the centre and $r$ the radius of $C$ then

$$R_C : C \setminus \{m\} \to C \setminus \{m\}, \quad R_C(z) = \frac{r^2(z - m)}{|z - m|^2} + m.$$ 

For obtaining the contractions of the cGDS associated with $\Gamma$ we introduce isometries of $\mathbb{H}^2$ the three dimensional hyperbolic space by

$$f_j := R_{H_j} \circ R_{K_j} : K_0 \to K_0$$

for $j \in \{1, 2, 3\}$. With $m, n$ denoting the centres and $r, s$ denoting the radii of $H_j$ and $K_j$ respectively an explicit representation of $f_j$ is given by

$$f_j(z) = \frac{z(r^2 + m(n - m)) - m^2 + ms^2 - mn(n - m)}{z(n - \bar{m}) + s^2 - n(n - m)}.$$ 

Here, $\bar{n}$ denotes the complex conjugate of $n \in \mathbb{C}$.

We define three compact connected sets $X_v := f_v(K_0)$ in $\partial \mathbb{H}^2 = \mathbb{R}^2$ for $v \in V := \{1, 2, 3\}$ (see Fig. 4.2(d)). For $v, w \in V$ with $v \neq w$ we let $E_{v, w} := \{e_{v, w}^k \mid k \in \mathbb{N}\}$ denote a directed set of edges with $t(e_{v, w}^k) = v$, $i(e_{v, w}^k) = w$ and associated contractions

$$\phi_{e_{v, w}^k} := f_{v}^k|_{X_v} : X_v \to X_w$$

(see Fig. 4.2(e)). Let $E := \bigcup_{v \in V} \bigcup_{w \in V \setminus \{v\}} E_{v, w}$ and define an incidence matrix $A = (A_{e, e'})_{e, e' \in E}$ by $A_{e_{v, w}^k, e_{v', w'}^{k'}} = 1_{\{w' = v\}}$. Then $\Phi := \{\phi_e \mid e \in E\}$ defines a cGDS whose limit set is the Apollonian circle packing generated by $C_1, C_2, C_3$ inside of $\mathcal{T}$.

We now check the assumptions of Sec. 3 in order to apply Thm. 3.5. We set $O_v := \text{int}(f_v(\mathcal{T}))$. Then the set $\Gamma_v$ is a union of countably many circles: $\Gamma_v = \bigcup_{k=1}^{\infty} f_v^k(C_4)$. For our depicted example the sets $X_1, O_2$ and $\Gamma_3$ are shown in Fig. 4.2(f). Moreover, $O = \bigcup O_v$ is a feasible open set for $\Phi$ for which the projection and non-triviality conditions are clearly satisfied giving (A). Setting $\Lambda := \{0, e_{1, 2}^1, e_{1, 3}^1, e_{1, 1}^1, e_{2, 3}^1, e_{3, 1}^1, e_{3, 2}^1\}$ shows finite irreducibility of the incidence matrix and thus (B). To show (C) that is strong regularity, we use the next lemma.

**Lemma 4.1.** [MU98] There is a constant $Q > 1$ such that for all $v \in \{1, 2, 3\}$ we have

$$Q^{-1}k^{-2} \leq \sup_{x \in \mathcal{T} \setminus X_v} |(f_v^k)'(x)| \leq Qk^{-2}.$$ 

**Remark 4.2.** In [MU98] the bounds in the above lemma are stated for $\|\phi_{e_{v, w}^k}\|_\infty$ instead of $\sup_{x \in \mathcal{T} \setminus X_v} |(f_v^k)'(x)|$. However, the discrepancy vanishes in the constant $Q$. 

Since $\xi \geq 0$ and $E$ is of infinite cardinality, the expression $\sum_{e \in E} \exp(\sup(\xi|_e))$ is infinite for $s \geq 0$. To determine $\theta$ from (2.2) it thus suffices to consider $s < 0$. For such $s$

$$\sum_{e \in E} \exp\left(\sup_{v,k}(f^k_v')|_v\right) = \sum_{v=1}^{3} \sum_{k=1}^{\infty} \exp\left(-s \ln\sup_{v,k}(f^k_v)|_v\right) \in \left[3Q^2 \sum_{k=1}^{\infty} k^{2s}, 3Q^{-s} \sum_{k=1}^{\infty} k^{2s}\right]$$

by Lem. 4.1. We deduce $\theta = -1/2$. McMullen [Boy82, McM98] determined the Hausdorff, packing and Minkowski dimensions of the Apollonian gasket to be $D = 1.30568...$. Therefore, $\theta > -D$ and strong regularity follows. To verify the last condition [D] from Sec. 3 we study $\lambda_d(F_c \cap \Gamma_v)$ for each $v \in V$. Let $r$ denote the radius of $C_4$. Then by Lem. 4.1 the radius of $f^k(C_4)$ is bounded from above by $rQk^{-2}$. Thus,

$$\lambda_2(F_c \cap \Gamma_v) \leq \sum_{k=1}^{\left\lfloor \sqrt{\frac{\varepsilon r}{4}} \right\rfloor} \pi(2rQk^{-2}\varepsilon - \varepsilon^2) + \sum_{k=\left\lfloor \sqrt{\frac{\varepsilon r}{4}} \right\rfloor + 1}^{\infty} \pi(rQk^{-2})^2$$

$$\leq C \cdot \varepsilon = C \cdot \varepsilon^{2-D+D-1},$$

with some constant $C > 0$. Therefore, [D] is satisfied with $\gamma = D - 1 > 0$.

The following lemma is probably well known to the experts. Here, we provide a short proof for completeness.

**Lemma 4.3.** The cGDS associated with any Apollonian circle packing is non-lattice.

**Proof.** Let $\Phi$ and $\Psi$ denote cGDS associated with different circle packings and let $\xi$ and $\zeta$ denote their respective geometric potential functions. Then there exists a Möbius transformation $g$ such that $\Psi = g \circ \Phi \circ g^{-1}$ and we have $\zeta = \xi - \ln|g' \circ \pi| + \ln|g' \circ \pi \circ \sigma|$. Since $-\ln|g' \circ \pi| \in C(E^\infty)$ it follows that $\Phi$ is lattice if and only if $\Psi$ is lattice. Thus, it suffices to consider one particular circle packing. We choose the Ford-circles (see Fig. 4.3). Here, the cGDS is given through the maps

$$f_1(z) = \frac{-4}{z + 3 + i} + 1 - i, \quad f_2(z) = \frac{4}{-z + 3 - i} - 1 - i, \quad f_3(z) = \frac{1}{z - i} - i.$$
Suppose that the associated geometric potential function $\xi$ is lattice. Then there exist $\zeta, \psi \in \mathcal{C}(E^\infty)$ with $\xi(E^\infty) \subset a\mathbb{Z}$ for some $a > 0$ and $\xi = \zeta + \psi - \psi \circ \sigma$. This implies

$$S_2 \xi = S_2 \zeta + \psi - \psi \circ \sigma^2$$

and whence $S_2 \xi(x) \in a\mathbb{Z}$ for 2-periodic words $x$. Consider the following points given by two-periodic words

$$\pi(e_{1,1}^1e_{1,2}^1) = 2 - \sqrt{5} - i, \quad \pi(e_{1,1}^1e_{1,2}^1) = 1 - i - \frac{2}{3}\sqrt{3}$$

$$\pi(e_{1,2}^1e_{2,1}^1) = -2 + \sqrt{5} - i, \quad \pi(e_{1,2}^1e_{2,1}^1) = -3 - i + 2\sqrt{3}.$$

We have $f_1'(\pi(e_{1,1}^1e_{1,2}^1)) = f_2'(\pi(e_{1,2}^1e_{1,2}^1)) = 4/(1 + \sqrt{5})^2, f_1'(\pi(e_{1,2}^1e_{1,2}^1)) = 1/3$ and $(f_2')'(\pi(e_{1,2}^1e_{1,2}^1)) = 3/(\sqrt{3} + 2)^2$ and conclude

$$\frac{S_2 \xi(e_{1,1}^1e_{1,2}^1)}{S_2 \xi(e_{1,2}^1e_{2,1}^1)} = \frac{-4 \ln(2) + 4 \ln(1 + \sqrt{5})}{2 \ln(\sqrt{3} + 2)} \notin \mathbb{Q},$$

which is a contradiction. \qed

Let $\mu_{-D\xi}$ denote the unique $\sigma$-invariant Gibbs state of $-D\xi$. Applying Thm. 3.5 yields for any $B \in \mathcal{B}(\mathcal{T})$ 

$$\lambda_2(F_{e^{-T}} \cap B) \sim \frac{e^{-(2-D)T}}{\int \xi \, d\mu_{-D\xi}} \lim_{m \to \infty} \sum_{\omega \in E_\mathbb{A}^m} \int_{-\infty}^{\infty} e^{-T(D-2)} \lambda_2(F_{e^{-T}} \cap \bigcup_{k=1}^{\infty} \phi_\omega f_k^1(\nu(C_4))) \, dT \cdot \nu(B).$$

Using that $\phi_\omega f_k^1(\nu(C_4))$ are circles, the above integral can be evaluated and we obtain the following.

**Theorem 4.4** (Apollonian gasket – Minkowski content). *The local Minkowski content exists and we have for any Borel set $B \subset \mathcal{T}$

$$\mathcal{M}(F, B) = \frac{2}{D(2-D)(D-1)} \int_{E_\mathbb{A}} \pi_2^{-D} \lim_{m \to \infty} \sum_{\omega \in E_\mathbb{A}} \sum_{k=1}^{\infty} |\phi_\omega f_k^1(\nu(C_4))|^D \cdot \nu(B),$$

where $\nu$ denotes the $D$-conformal measure associated with $F$.

**Theorem 4.5** (Apollonian gasket – analytic dependence). *Let $F_0$ denote the symmetric Apollonian gasket with corners on the unit circle and $\nu$ the associated $D$-conformal measure. Then for any other Apollonian gasket $F$ there exists a unique Möbius transformation $g$ such that $F = g(F_0)$ and for the Minkowski content we have for any $B \in \mathcal{B}(\mathcal{T}_0)$

$$\frac{\mathcal{M}(F, g(B))}{\mathcal{M}(F_0, B)} = \int |g'|^D \, d\nu.$$

This in particular proves the analytic dependence of $\mathcal{M}(F)$ on the initial circles.

The proof of Thm. 4.5 is given in Sec. 6.
Corollary 4.6 (Apollonian gasket – surface area based content). The first fractal curvature measure, i.e. the surface area based content, exists and we have for any $B \in \mathcal{B}(T)$

$$C_1^f(F, B) = \pi \frac{2^{-D}}{D(D-1)} \int \xi d\mu_{-D\xi} \lim_{m \to \infty} \sum_{\omega \in E_m^\infty} \sum_{k=1}^{\infty} \left| \phi_{\omega} f_{\ell(\omega)}^k (C_4) \right|^D \cdot \nu(B).$$

The preceding corollary follows from a result in [RW10] together with Thm. 4.4.

Theorem 4.7 (Apollonian gasket – fractal Euler characteristic). The 0-th fractal curvature measure, i.e. the localised fractal Euler characteristic, exists and we have for any $B \in \mathcal{B}(T)$

$$C_0^f(F, B) = -\frac{1}{D} \frac{2^{-D}}{D} \int \xi d\mu_{-D\xi} \lim_{m \to \infty} \sum_{\omega \in E_m^\infty} \sum_{k=1}^{\infty} \left| \phi_{\omega} f_{\ell(\omega)}^k (C_4) \right|^D \cdot \nu(B).$$

The proof of Thm. 4.7 is given in Sec. 6.

Remark 4.8. Combining Thm. 4.4, Cor. 4.6, and Thm. 4.7 we see that

$$C_0^f(F, \cdot) = \frac{1}{\kappa_2} (1 - D) C_1^f(F, \cdot) \quad \text{and} \quad C_1^f(F, \cdot) = \frac{1}{\kappa_1} (2 - D) C_2^f(F, \cdot)$$

with $\kappa_k$ denoting the $k$-dimensional volume of the $k$-dimensional unit ball. Thus, the Minkowski content, the surface area based content, and the fractal Euler characteristic of Apollonian circle packings are all constant multiples of one another with the constant being independent of the underlying circle packing. What is more, this result holds even for the respective measures. That the Minkowski content and the surface area based content are constant multiples of each other is precisely the statement of [RW10] which applies to any bounded set. However, this relation between the fractal Euler characteristic and the Minkowski content is not known for general sets and might be specific to circle packings. This observation shows that the surface area based content and the Euler characteristic do not provide any further geometric information on the structure of the underlying Apollonian circle packing in addition to the Minkowski content.

4.2. Circle counting. The fractal Euler characteristic in essence gives an asymptotic on the number $R(\varepsilon)$ of circles in $T$ of radius bigger than $\varepsilon$ as $\varepsilon \searrow 0$. More precisely,

$$R(\varepsilon) \sim -\varepsilon^{-D} C_0^f(F, T)$$

as $\varepsilon \searrow 0$. The circle counting function $R$ has been studied with the help of Laplace eigenfunctions by Kontorovich and Oh in [KO11] (see also [Oh10, KO11, LO13, Oh14a, Oh14b, OS16, Pol15]) and its asymptotic behaviour has been derived. Our result here provides a different representation of the constant factor of the leading asymptotic term and in this way gives a new geometric interpretation.

Definition and Proposition 4.9 (The Apollonian constant, [Oh14b]). Let $F$ denote a circle packing with associated circle counting function $R$ and let $\mathcal{H}^D$ denote the $D$-dimensional Hausdorff measure.

$$c_A := \frac{\pi^{D/2}}{\mathcal{H}^D(F)} \lim_{\varepsilon \to 0} \varepsilon^D R(\varepsilon)$$
is called the Apollonian constant. It is a universal constant, which is independent of the circle packing. What is more, for any set \( B \in \mathcal{B}(\mathbb{T}) \)
\[
C^I_0(F, B) = -c_A \pi^{-D/2} \mathcal{H}^D(F \cap B).
\]

An immediate consequence of the above definition and Thm. 4.7 is the following.

**Corollary 4.10.** We have that
\[
c_A = \frac{2^{-D} \pi^{D/2}}{D \int \xi d\mu - D\xi} \cdot \lim_{m \to \infty} \sum_{\omega \in E^m} \sum_{k=1}^{\infty} \left| \phi_{\omega f_{t(w)}}^{k} (C_4) \right|^{D},
\]
where each of the two fractions are constants that are independent of the particular circle packing.

Moreover, the above corollary immediately implies that \( \mathcal{H}^D \) is a constant multiple of the \( D \)-conformal measure \( \nu \), reproducing a fact that is well known for the Patterson measure.

**Theorem 4.11.** Let \( F_0 \) denote the symmetric Apollonian gasket with corners on the unit circle as depicted in Fig. 4.2. Further, let \( q: \mathbb{C} \to \mathbb{C} \) be the Möbius transform
\[
q(z) := \frac{(1 + (1 + i) \sqrt{3}) z - 1}{z - 1 + (1 + i) \sqrt{3}},
\]
which maps the real line to the circle \( C_2 \), so that \( C_4 \) is mapped to \( \partial X_1 \). Then
\[
c_A \geq \frac{2^{-D} \pi^{D/2}}{D \int \xi d\mu - D\xi} \cdot \lim_{m \to \infty} \sum_{\omega \in E^m} \sum_{k=1}^{\infty} \frac{\left| \phi_{\omega f_{t(w)}}^{k} (C_4) \right|^{D}}{6 \lim_{m \to \infty} \sum_{\omega \in E^m} \sum_{k=1}^{\infty} \left| \phi_{\omega f_{t(w)}}^{k} q (C_4) \right|^{D}} \geq 0.055.
\]

The proof of Thm. 4.11 is presented in Sec. 4.3.

**4.3. Apollonian sphere packings in \( \mathbb{R}^3 \) and collections of balls in higher dimensions.** In case of Apollonian sphere packings in \( \mathbb{R}^3 \) it is interesting to consider not only the Minkowski content, the surface area based content and the fractal Euler characteristic, but also the other fractal curvatures and curvature measures. In analogy to the 2-dimensional setting, we can construct a cGDS which generates the sphere packing in a region \( \mathcal{T} \) (see [GLM+06]). When considering higher dimensions, i.e. \( \mathbb{R}^d \) with \( d \geq 4 \), we look at collections of disjoint balls formed e.g. by Kleinian groups of Schottky type. These have a representation as a limit set of a cGDS, see [MU03, Ch. 5]. In the following we use the same notation as in Sec. 4.1. We see that \( F_{c \cdot t} \) is a set of positive reach for any \( t > 0 \). Through the local Steiner formula we immediately obtain for \( t e^{-t} < |\phi_{u \Gamma_{t(u)}}| \)
\[
C_d(F_{c \cdot t}, \phi_{u \Gamma_{t(u)}}) = \kappa_d \left( \frac{|\phi_{u \Gamma_{t(u)}}|}{2} \right)^d - \kappa_d \left( \frac{|\phi_{u \Gamma_{t(u)}}|}{2} - e^{-t} \right)^d \quad \text{and}
\]
\[
C_k(F_{c \cdot t}, \phi_{u \Gamma_{t(u)}}) = \frac{\kappa_d}{\kappa_{d-k}} \left( \frac{d}{k} \right) (\frac{1}{-1})^{d-k+1} \left( \frac{|\phi_{u \Gamma_{t(u)}}|}{2} - e^{-t} \right)^k, \quad k \leq d - 1.
\]

We apply the same methods that we used in the previous sections. Note, that in general it is not possible to apply the bounded distortion lemma, when considering curvature measures, since \( C_k \) is not monotonic in the first component for \( k \leq d - 1. \)
However, here we may apply bounded distortion to the expressions on the right hand side of (4.1) and get

\[
C_k(F_{e^{-T}}, O) \sim \frac{e^{-t(k-D)}}{\int \xi d\mu_{e^{-T}}} \lim_{m \to \infty} \sum_{u \in E^m} \int_{-\infty}^{\infty} e^{-T(D-d+k)} C_k(F_{e^{-T}}, \phi_u \Gamma_t(u)) dT.
\]

Evaluating the integrals yields

\[
I_u^k = \frac{\kappa_d \cdot d^k}{\kappa_{d-k} \cdot (d-k)! (-1)^{d-k+1} \left( \frac{|\phi_u \Gamma_t(u)|}{2} \right)^D} \frac{1}{D(D-1) \cdots (D-k)}, \quad k \leq d-1.
\]

Altogether, we obtain for \(k \leq d-1\)

\[
C_k^f(F, O) = \frac{\kappa_{d-k-1}}{\kappa_{d-k}} \cdot \frac{k+1-D}{d-k} \cdot C_{k+1}^f(F, O).
\]

This shows that the fractal curvatures of Apollonian sphere packings and limit sets of Kleinian groups of Schottky type are constant multiples of each other, where the constants only depend on the dimension \(d\) and on the indices of the respective curvatures. In particular, when considering sphere packings in \(\mathbb{R}^3\) we deduce that

\[
C_0^f(F, O) = \frac{1}{4} (1-D) \cdot C_1^f(F, O),
\]

\[
C_1^f(F, O) = \frac{1}{\pi} (2-D) \cdot C_2^f(F, O),
\]

\[
C_2^f(F, O) = \frac{1}{2} (3-D) \cdot C_3^f(F, O).
\]

Also in this section it is possible to localise the curvatures and to obtain the analogues results for the fractal curvature measures.

### 4.4. Restricted continued fraction digits

Let us consider the fractal set given by a restricted continued fraction digit set. For \(\Lambda \subset \mathbb{N}\) we define

\[
F_\Lambda := \{[a_1, a_2, \ldots] \mid \forall n \in \mathbb{N}; a_n \in \Lambda\}.
\]

The Hausdorff dimension of these sets has been extensively studied in [KZ06]. In fact, it has been shown that the Texan Conjecture holds, that is \(\{\dim_H(F_\Lambda) \mid \Lambda \subset \mathbb{N}\} = [0, 1]\).

For these sets with \(\text{card}(\Lambda) \geq 2\) we have to consider the following conformal IFS defined on the unit interval

\[
\Phi := \{\phi_k : x \mapsto 1/(x+k) \mid k \in \Lambda\}.
\]

Set \(D := \dim_H(F_\Lambda)\) and suppose that \(D < 1\) which implies \(\Lambda \neq \mathbb{N}\). Let \(\mu\) denote the equilibrium with respect to the geometric potential \(-\delta_\xi\) and \(h_\mu\) its measure theoretical entropy. For simplicity we will assume that for \(k \in \mathbb{N} \setminus \Lambda\) we have \(k \pm 1 \in \Lambda \cup \{0\}\). This assumption guarantees that the cGDS is strongly regular (cf. [MU99] Example 6.5) and gives rise to the handy formula stated next.
Theorem 4.12. The set $F_\Lambda$ is Minkowski measurable and we have
$$\mathcal{M}(F_\Lambda) = \frac{2^{1-D}}{(1-D)\mu} \lim_{m \to \infty} \sum_{a \in \Lambda} \sum_{|\omega| = m} |\Phi_{\omega}(\{a\})|^D.$$ 

4.5. Restricted Lüroth digits. Fix the decreasing sequence $(t_n)$ in $[0,1]$ given, for some $s > 1$, by
$$t_n := \zeta(s)^{-1} \sum_{k=n}^{\infty} \frac{1}{k^s}, \quad n \in \mathbb{N}$$
defining a Lüroth system, where $\zeta$ denotes the Riemann $\zeta$-function (cf. [KMS12]). With $a_n := 1/(n^s \zeta(s))$ and the same conditions on the set $\Lambda \subset \mathbb{N}$ as in Section 4.4 we consider the linear IFS on the unit-interval given by
$$\Phi := \{\phi_n : x \mapsto -a_n x + t_n \mid n \in \Lambda\}.$$ 
Let $\delta > 0$ be the unique number such that
$$\zeta_\Lambda(\delta s) := \sum_{k \in \Lambda} \frac{1}{k^\delta s} = \zeta(s)^\delta.$$ 
Then the fractal set $L_\Lambda$ of all Lüroth expansions omitting the digits from $\Lambda$ has Hausdorff and Minkowski dimension equal to $\delta$. If the system is non-lattice (depending on the particular choice of $\Lambda$ - see below) then the set is Minkowski measurable and its Minkowski content is given by
$$\mathcal{M}(L_\Lambda) = \frac{2^{1-\delta}(\zeta(s^\delta)/\zeta(s)^\delta - 1)}{\delta(1-\delta) \int \xi d\mu_{-\delta \xi}} = \frac{2^{1-\delta}(\zeta(s^\delta)/\zeta(s)^\delta - 1)}{(1-\delta)(\log \zeta_\Lambda(\delta s) - \delta s(\log \zeta_\Lambda(\delta s)))}$$
For an example of a lattice system which is not Minkowski measurable we fix an integer $\ell \geq 2$ and some $s > 1$ for which $s \log \ell/\log \zeta(s) \in \mathbb{Q}$. Then the fractal set $L_\Lambda$ of all Lüroth expansions allowing only the digits from $\Lambda \subset \{\ell^k \mid k \in \mathbb{N}\}$, card($\Lambda$) $\geq 2$, is not Minkowski measurable since $ks \log \ell - \log(\zeta(s))$ lies in the lattice $(\log(\zeta(s))/q)\mathbb{Z}$ for some $q \in \mathbb{N}$.

5. Preliminaries for the proofs

5.1. Thermodynamic formalism. A Borel probability measure $\mu$ on $E^\omega$ is said to be a Gibbs state for $f \in C(E^\omega)$ if there exists a constant $c > 0$ such that
$$c^{-1} \leq \frac{\mu([\omega|n])}{\exp(S_n f(\omega) - nP(f))} \leq c$$
for every $\omega \in E^\omega$ and $n \in \mathbb{N}$.

Definition 5.1 (Hölder continuity). For $f \in C(E^\omega)$, $\theta \in (0,1)$ and $n \in \mathbb{N}$ define
$$\text{var}_n(f) := \sup \{|f(x) - f(y)| \mid x, y \in E^\omega \text{ and } x_i = y_i \text{ for } i \leq n\},$$
$$\|f\|_\theta := \sup_{n \geq 1} \frac{\text{var}_n(f)}{\theta^n} \quad \text{and} \quad \mathcal{F}_\theta(E^\omega) := \{f \in C(E^\omega) \mid \|f\|_\theta < \infty\}.$$ 
A function $f \in \mathcal{F}_\theta(E^\omega)$ is called $\theta$-Hölder continuous. Since by our definition a $\theta$-Hölder continuous function is not necessarily bounded, we introduce the space...
of bounded Hölder continuous functions and denote it by \( \mathcal{F}_\theta^b(E^\infty) := \mathcal{F}_\theta(E^\infty) \cap \mathcal{C}_b(E^\infty) \).

In order to define the central object of this section, namely the Perron-Frobenius operator of a potential function \( f \), we need to assume that
\[
\sum_{e \in E} \exp(\sup(f|_e)) < \infty.
\]
A function \( f \in \mathcal{F}_\theta(E^\infty) \) which satisfies (5.2) is called summable. Note that by [MU03, Thm. 2.1.5] the pressure \( P(u) \) is finite and \( > -\infty \) for any summable Hölder continuous function \( u : E^\infty \rightarrow \mathbb{R} \).

**Definition 5.2 (Perron-Frobenius operator).** Let \( f \in \mathcal{F}_\theta(E^\infty) \) be summable. The Perron-Frobenius operator \( \mathcal{L}_f : \mathcal{C}_b(E^\infty) \rightarrow \mathcal{C}_b(E^\infty) \) acting on \( \mathcal{C}_b(E^\infty) \) is defined by
\[
\mathcal{L}_f(g)(\omega) = \sum_{e \in E : A_{e \omega_1} = 1} e^{f(e \omega)} g(e \omega) = \sum_{y : \sigma y = \omega} e^{f(y)} g(y).
\]

The conjugate Perron-Frobenius operator \( \mathcal{L}_f^* \) acts on \( \mathcal{C}_b^*(E^\infty) \) via
\[
\mathcal{L}_f^*(\mu)(g) = \mu(\mathcal{L}_f(g)) = \int \mathcal{L}_f(g) \, d\mu,
\]
as shown in [KK17].

The following theorem is a combination of Lem. 2.4.1, Thms. 2.4.3, 2.4.6 and Cor. 2.7.5 from [MU03]. Note that Thms. 2.4.3, 2.4.6 in [MU03] are stated and proved under the hypothesis that the incidence matrix \( A \) is finitely primitive. In [KK17] we provided reasoning that the assumption of finitely irreducible \( A \) in fact suffices.

**Theorem 5.3 ([MU03], Ruelle-Perron-Frobenius theorem for infinite alphabets).** Suppose that \( f \in \mathcal{F}_\theta(E^\infty) \) for some \( \theta \in (0, 1) \) is summable. Then \( \mathcal{L}_f \) preserves the space \( \mathcal{F}_\theta^b(E^\infty) \), \( i.e. \mathcal{L}_f|_{\mathcal{F}_\theta^b(E^\infty)} : \mathcal{F}_\theta^b(E^\infty) \rightarrow \mathcal{F}_\theta^b(E^\infty) \). Moreover, the following hold.

(i) There is a unique Borel probability eigenmeasure \( \nu_f \) of the conjugate Perron-Frobenius operator \( \mathcal{L}_f^* \) and the corresponding eigenvalue is equal to \( e^{P(f)} \). Moreover, \( \nu_f \) is a Gibbs state for \( f \).

(ii) The operator \( \mathcal{L}_f|_{\mathcal{F}_\theta^b(E^\infty)} \) has an eigenfunction \( h_f \) which is bounded from above and which satisfies \( \int h_f \, d\nu_f = 1 \). Further, there exists an \( R > 0 \) such that \( h_f \geq R \) on \( E^\infty \).

(iii) The function \( f \) has a unique ergodic \( \sigma \)-invariant Gibbs state \( \mu_f \).

(iv) There exist constants \( M > 0 \) and \( \gamma \in (0, 1) \) such that for every \( g \in \mathcal{F}_\theta^b(E^\infty) \) and every \( n \in \mathbb{N}_0 \)
\[
\left\| e^{-nP(f)} \mathcal{L}_f^n(g) - \int g \, d\nu_f \cdot h_f \right\|_{\theta} \leq M \gamma^n (\|g\|_\theta + \|g\|_\infty).
\]

Directly from (5.3) we infer the following:
Corollary 5.4. In the setting of Thm.\textsuperscript{5.3} (iv) \(e^{P(u)}\) is a simple isolated eigenvalue of \(L_u|_{F^B_\theta(E^\infty,\mathbb{R})}\). The rest of the spectrum of \(L_u|_{F^B_\theta(E^\infty,\mathbb{R})}\) is contained in a disc centred at zero of radius at most \(\gamma < e^{P(u)}\).

The unique probability measure \(\nu\) supported on \(F\) which satisfies
\[
\nu(\phi_i X \cap \phi_j X) = 0 \quad \text{and} \quad \nu(\phi_i B) = \int_B |\phi_i'|^D \, d\nu
\]
for all distinct \(i,j \in E\) and all Borel sets \(B \subset X_{t(i)}\), is called the \(D\)-conformal measure associated with \(\Phi\). Notice, \(\nu_{-D\xi} = \nu \circ \pi\).

5.2. Renewal theorems. In [Lal89] renewal theorems for counting measures in symbolic dynamics were established, where the underlying symbolic space is based on a finite alphabet. These renewal theorems were extended to more general measures in [Kom11, Kom15]. Moreover, the extended versions from [Kom15] were generalised to the setting of an underlying countably infinite alphabet in [KK17] and we summarise these results in the present section.

Fix \(\theta \in (0, 1)\) and let \(\kappa \in F^B_\theta(E^\infty)\) be non-negative but not identically zero. Further, \(\xi, \eta \in F^B_\theta(E^\infty)\) shall satisfy the following:

\(\text{(E) Regular potential.} \) \(\xi \geq 0\) is not identically zero. There exists a unique \(\delta \in \mathbb{R}\) with \(P(\eta - \delta \xi) = 0\). Further, \(\eta < t^* := \sup \{t \in \mathbb{R} \mid \eta + t\xi\}\) is summable and \(\int (\eta + t\xi) \, d\mu_{\eta - \delta \xi} < \infty\) for all \(t\) in a neighbourhood of \(-\delta\).

For fixed \(x \in E^\infty\) the renewal theorem provides the asymptotic behaviour as \(t \to \infty\) of the renewal function
\[
N(t, x) := \sum_{n=0}^{\infty} \sum_{y: \sigma^n y = x} \kappa(y) f_y(t - S_n \xi(y)) e^{S_n \eta(y)},
\]
where \(f_x: \mathbb{R} \to \mathbb{R}\) for \(x \in E^\infty\), needs to satisfy some regularity conditions (see \(\text{(F)}\) below). We call \(N\) a renewal function since it satisfies an analogue to the classical renewal equation:
\[
N(t, x) = \sum_{y: \sigma y = x} N(t - \xi(y), y) e^{\eta(y)} + \kappa(x) f_x(t).
\]
\(\text{(F) Lebesgue integrability.} \) For any \(x \in E^\infty\) the Lebesgue integral
\[
\int_{-\infty}^{\infty} e^{-\delta t} |f_x(t)| \, dt
\]
exists.

\(\text{(G) Boundedness of} \) \(N\). There exists \(\mathcal{C} > 0\) such that \(e^{-\delta t} N_{abs}(t, x) \leq \mathcal{C}\) for all \(x \in E^\infty\) and \(t \in \mathbb{R}\), where
\[
N_{abs}(t, x) := \sum_{n=0}^{\infty} \sum_{y: \sigma^n y = x} \kappa(y) |f_y(t - S_n \xi(y))| e^{S_n \eta(y)}.
\]

\(\text{(H) Exponential decay of} \) \(N\) on the negative half-axis. There exist \(\mathcal{C} > 0, s > 0\) and \(t_0 \in \mathbb{R}\) such that \(e^{-\delta t} N_{abs}(t, x) \leq \mathcal{C} e^{st}\) for all \(t \leq t_0\).

Theorem 5.5 (Renewal theorem, [KK17]). Assume that \(x \mapsto f_x(t)\) is \(\theta\)-Hölder continuous for every \(t \in \mathbb{R}\) and that Conditions \(\text{(E)}\) to \(\text{(H)}\) hold.
(i) If $\xi$ is non-lattice and $f_x$ is monotonic for every $x \in E^\infty$, then

\[ N(t, x) \sim e^{t \delta} h_{\eta - \delta \xi}(x) \frac{1}{\int \xi \, d\mu_{\eta - \delta \xi}} \int_{E^\infty} \kappa(y) \int_{-\infty}^\infty e^{-T \delta} f_y(T) \, dT \, d\nu_{\eta - \delta \xi}(y) \]

as $t \to \infty$, uniformly for $x \in E^\infty$.

(ii) Assume that $\xi$ is lattice and let $\zeta, \psi \in C(E^\infty)$ satisfy the relation

$\xi - \zeta = \psi - \psi \circ \sigma$,

where $\zeta$ is a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be maximal such that $\zeta(E^\infty) \subseteq a \mathbb{Z}$. Then

\[ N(t, x) \sim e^{t \delta} h_{\eta - \delta \zeta}(x) \tilde{G}_x(t) \]

as $t \to \infty$, uniformly for $x \in E^\infty$, where $\tilde{G}_x$ is periodic with period $a$ and

\[ \tilde{G}_x(t) := \int_{E^\infty} \kappa(y) \sum_{\ell = -\infty}^{\infty} e^{-a \delta} f_y \left( a \ell + a \left\{ \frac{t + \psi(x)}{a} \right\} - \psi(y) \right) \, d\nu_{\eta - \delta \zeta}(y) \times e^{-a \left\{ \frac{t + \psi(x)}{a} \right\} \delta} \int \zeta \, d\mu_{\eta - \delta \zeta}. \]

(iii) We always have

\[ \lim_{t \to \infty} \frac{1}{t} \int_0^t e^{T \delta} N(T, x) \, dT = G \cdot h_{\eta - \delta \xi}(x). \]

Remark 5.6. The monotonicity required in (i) can be replaced by other conditions. For instance we could require that there exists $n \in \mathbb{N}$ for which $S_n \xi$ is bounded away from zero and that the family $(t \mapsto e^{-t \delta} |f_x(t)| \mid x \in E^\infty)$ is equi-directly Riemann integrable (cf. [Kom15]).

6. Proofs

**Lemma 6.1** (Bounded Distortion). [KK17] There exists a sequence $(\rho_n)_{n \in \mathbb{N}}$ with $\rho_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} \rho_n = 1$ such that for all $\omega, u \in E^*$ with $u \omega \in E^*$ and $x, y \in \phi_\omega(X_t(\omega))$ we have that

\[ \rho_n^{-1} \leq \frac{|\phi'_u(x)|}{|\phi'_u(y)|} \leq \rho_n(\omega). \]

**Proof of Prop. 5.5**. This proof is an extension of the proof of [MU03] Prop. 4.5.9]. Non-triviality implies the existence of $\tilde{v} \in V$ for which $\lambda_{\tilde{v}}(\Gamma_{\tilde{v}}) > 0$. Finite irreducibility implies that for each $v \in V$ there exists $\omega_v \in \Lambda$ with $\phi_{\omega_v} \Gamma_{\tilde{v}} \subseteq X_v$. Let $m := \max\{n(\omega) \mid \omega \in \Lambda\}$ and for $v \in V$ set

\[ G_v := O_v \setminus \bigcup_{\omega \in E^{m+1}_A} \phi_\omega O_{\Lambda(\omega)}. \]
Then each $G_v$ has positive $d$-dimensional Lebesgue measure, since
\[
G_v \supseteq O_v \setminus \bigcup_{\omega \in E_A^{n(\omega)+1}} \phi_\omega O_t(\omega) \supseteq \bigcup_{\omega \in E_A^{n(\omega)}, t(\omega)=v} \phi_\omega O_t(\omega) \bigcup_{\omega \in E_A^{n(\omega)+1}} \phi_\omega O_t(\omega)
\]
implies $\lambda_d(G_v) \geq \lambda_d(\phi_\omega \Gamma_\overline{\theta}) > 0$. Recall that $V$ has finite cardinality. Therefore, $K := \min_{v \in V} \lambda_d(G_v)/\lambda_d(O_v)\rho_1^{-2d}$ exists and lies in the open interval $(0, 1)$. By bounded distortion and the transformation formula we have that
\[
\lambda_d(\phi_\omega G_t(\omega)) = \int_{G_t(\omega)} |\phi_\omega'|^d d\lambda_d \geq K \int_{O_t(\omega)} |\phi_\omega'|^d d\lambda_d = K \cdot \lambda_d(\phi_\omega O_t(\omega)).
\]
For $n \in \mathbb{N}_0$ write
\[
W_n := \bigcup_{\omega \in E_A^n} \phi_\omega O_t(\omega).
\]
Then $O_v \cap W_{m+1} = O_v \cap \bigcup_{\omega \in E_A^{m+1}} \phi_\omega O_t(\omega) = O_v \setminus G_v$ and whence
\[
W_{m+1+n} = \bigcup_{\omega \in E_A^n} \phi_\omega (W_{m+1}) = \bigcup_{\omega \in E_A^n} \phi_\omega (O_t(\omega) \setminus G_t(\omega)) = W_n \setminus \bigcup_{\omega \in E_A^n} \phi_\omega G_t(\omega).
\]
This yields
\[
\lambda_d(W_{m+1+n}) = \lambda_d(W_n) - \lambda_d \left( \bigcup_{\omega \in E_A^n} \phi_\omega G_t(\omega) \right) = \lambda_d(W_n) - \sum_{\omega \in E_A^n} \lambda_d \left( \phi_\omega G_t(\omega) \right)
\]
\[
\leq \lambda_d(W_n) - K \sum_{\omega \in E_A^n} \lambda_d \left( \phi_\omega O_t(\omega) \right) = (1 - K) \cdot \lambda_d(W_n).
\]
By the bounded distortion property, $\|\phi_\omega\|_\infty^d \leq \rho_1^d \lambda_d(\phi_\omega O)/\lambda_d(O)$. Thus,
\[
P(-d\xi) = \lim_{n \to \infty} \frac{1}{n(m+1)} \log \sum_{\omega \in E_A^{n(m+1)}} \exp \left( \sup_{\tau \in [\omega]} \left| d S_{n(m+1)} \xi(\tau) \right| \right)
\]
\[
\leq \lim_{n \to \infty} \frac{1}{n(m+1)} \log \rho_1 \lambda_d(W_{n(m+1)}) \lambda_d(O) \leq \frac{1}{m+1} \log(1 - K) < 0.
\]
This implies that $\dim_M(F) < d$. \hfill \Box

**Lemma 6.2 [Kom11 Lem. 4.4].** The set-class
\[
E_F := \{\phi_\omega O \mid \omega \in E^*\} \cup K_F \quad \text{with}
\]
\[
K_F := \{K \in B(\mathbb{R}^d) \mid \exists n \in \mathbb{N} : K \subset \mathbb{R}^d \setminus \bigcup_{\omega \in E_A^n} \phi_\omega O\}
\]
forms an intersection stable generator of $B(\mathbb{R}^d)$. 
Proof of Thm. 3.5. Directly from the definition of $\Gamma_v$ in (3.2) we obtain that $\bigcup O_v$ decomposes in the following way.

$$
\bigcup_{v \in V} O_v = \bigcup_{v \in V} \Gamma_v \cup \bigcup_{\epsilon \in E; i(\epsilon) = v} \phi_{\epsilon}(O_{i(\epsilon)}) = \cdots
$$

$$=
\bigcup_{n=0}^{\infty} \bigcup_{\omega \in E_A^n} \phi_{\omega}(\Gamma_{t(\omega)}) \cup \bigcap_{n=0}^{\infty} \bigcup_{\omega \in E_A^n} \phi_{\omega}(O_{t(\omega)}).
$$

Since $\bigcap_{n \in \mathbb{N}} \bigcup_{\omega \in E^n} \phi_{\omega}(O_{t(\omega)}) \subseteq \bigcap_{n \in \mathbb{N}} \bigcup_{\omega \in E_A^n} \phi_{\omega}(X_t(\omega)) = F$ its Lebesgue measure is zero by Prop. 3.3 and it follows that

$$
(6.1) \quad \lambda_d(F \cap B) = \sum_{n=0}^{m-1} \sum_{\omega \in E_A^n} \lambda_d(F \cap \phi_{\omega}(\Gamma_{t(\omega)}) \cap B)
$$

$$+ \sum_{u \in E_A^n} \sum_{n=0}^{\infty} \sum_{\omega \in E_A^n} \lambda_d(F \cap \phi_{\omega \epsilon}(\Gamma_{t(\omega)}) \cap B)
$$

for any fixed $m \in \mathbb{N}$. We write $e^{-t} = \epsilon$ and use that by the projection condition (3.1) we have that $F \cap \phi_{\omega} \Gamma_{t(\omega)} = \phi_{\omega \epsilon} F \cap \phi_{\omega} \Gamma_{t(\omega)}$ for $\omega \in E^*$. We separately consider the two summands on the right hand side of (6.1) and start with the first one. For $\omega \in E^*$, we choose an arbitrary $y_{\omega} \in E^\infty$ for which $\omega y_{\omega} \in E^\infty$ and apply the Bounded Distortion Lemma:

$$
\lambda_d(F \cap \phi_{\omega} \Gamma_{t(\omega)} \triangleleft \lambda_d(\phi_{\omega} F \cap \phi_{\omega} \Gamma_{t(\omega)})
$$

$$\leq \rho_1^d e^{-d S_n \xi(\omega y_{\omega})} \lambda_d(F \cap \Gamma_{t(\omega)} + \ln \rho_1 \cap \Gamma_{t(\omega)})
$$

$$\leq \rho_1^d e^{-d S_n \xi(\omega y_{\omega})} e^{-t + S_n \xi(\omega y_{\omega}) + \ln \rho_1} (d - D + \gamma)
$$

$$\leq \epsilon \rho_1^2 e^{-t (d - D) S_n \xi(\omega y_{\omega})} e^{-t (d - D + \gamma)}.
$$

Thus,

$$
e^{-t(D-d)} \sum_{n=0}^{m-1} \sum_{\omega \in E_A^n} \lambda_d(F \cap \phi_{\omega} \Gamma_{t(\omega)}) \leq \epsilon \rho_1^2 e^{-t(D-d) S_n \xi(\omega y_{\omega})} e^{-t (d - D + \gamma)} \sum_{n=0}^{m-1} \sum_{\omega \in E_A^n} e^{(\gamma - D) S_n \xi(\omega y_{\omega})}
$$

$$= \epsilon \rho_1^2 e^{-t(D-d)} \sum_{n=0}^{m-1} \sum_{\omega \in E_A^n} \mathcal{L}_{\gamma - D} \xi(y_{\omega})
$$

$$\lim_{t \to \infty} 0
$$

for fixed $m \in \mathbb{N}$. This in particular shows that $\lim_{t \to \infty} \lambda_d(F \cap B) = 0 = \nu(B)$ for $B \in \mathcal{K}_F$. Now, we turn to the second summand on the right hand side of (6.1) and consider the case that $B = \phi_{\omega} O \in \mathcal{E}_F \setminus \mathcal{K}_F$. For each $u \in E_A^n$ with $m \geq n(\kappa), \ldots$
we choose an arbitrary $x_u \in [u]$ and again apply the Bounded Distortion Lemma.

$$\sum_{n=0}^{\infty} \sum_{\omega \in E_\lambda^m} \lambda_d \left( F_{e^{-t}} \cap \phi_{\omega u} (\Gamma_{t(u)}) \right) \cdot 1_{[\kappa]}(\omega x_u)$$

$$= \sum_{n=0}^{\infty} \sum_{\omega \in E_\lambda^m} \lambda_d \left( (\phi_{\omega F})_{e^{-t}} \cap \phi_{\omega u} (\Gamma_{t(u)}) \right) \cdot 1_{[\kappa]}(\omega x_u)$$

$$\leq \sum_{n=0}^{\infty} \sum_{y: \sigma^n y = x_u} \rho_m^d e^{-dS_n \xi(y)} \lambda_d \left( F_{e^{-t}} \cap \phi_{\omega u} (\Gamma_{t(u)}) \right) \cdot 1_{[\kappa]}(y)$$

$$= \rho_m^d \sum_{n=0}^{\infty} \sum_{y: \sigma^n y = x_u} e^{-dS_n \xi(y)} F^n (t - S_n \xi(y) - \ln \rho_m) \cdot 1_{[\kappa]}(y),$$

where

$$f^n(t) := \lambda_d \left( F_{e^{-t}} \cap \phi_{\omega u} (\Gamma_{t(u)}) \right).$$

With $\eta := -d \xi, \kappa \equiv 1, f_y = f^u$ for any $y$ with $\sigma^n y = x_u$ and $N$ as in [5.4] we obtain

$$(6.3) \quad \sum_{u \in E_\lambda^m} \sum_{n=0}^{\infty} \sum_{\omega \in E_\lambda^m} \lambda_d \left( F_{e^{-t}} \cap \phi_{\omega u} (\Gamma_{t(u)}) \right) \leq \rho_m^d \sum_{u \in E_\lambda^m} N(t - \ln \rho_m, x_u)$$

and likewise

$$(6.4) \quad \sum_{u \in E_\lambda^m} \sum_{n=0}^{\infty} \sum_{\omega \in E_\lambda^m} \lambda_d \left( F_{e^{-t}} \cap \phi_{\omega u} (\Gamma_{t(u)}) \right) \geq \rho_m^{-d} \sum_{u \in E_\lambda^m} N(t + \ln \rho_m, x_u).$$

We now want to apply the renewal theorem to the right hand side of the above equations and thus check that its assumptions are satisfied.

Ad [E] By strong regularity $P(-D \xi) = 0$ and as $\xi > 0$ the map $t \mapsto P(t \xi)$ is increasing on $\mathbb{R}$ and strictly increasing on $\{t \in \mathbb{R} \mid P(t \xi) < \infty\} \supseteq (-\infty, \theta] \supseteq (-\infty, -D]$ with $\theta$ as in [2.2]. Thus $\delta = D - d$ is unique with the property $P(- (d + \delta) \xi) = 0$. Moreover, $-\delta < \theta + d = t^*$. On $(-\infty, \theta)$ the function $t \mapsto P(t \xi)$ is differentiable with derivative $\int \xi d\mu_t \xi$. Therefore, $\int (d - t) \xi d\mu_{t-(d-\delta)\xi} = (d - t) \int \xi d\mu_{d-\delta} \xi$ is finite.

Ad [F] Is satisfied by [D].

Ad [H] Choose $t_0 < 0$ so that $F_{e^{-t_0}} \cap \phi_{\omega \Gamma_{t(u)} = \phi_{\omega \Gamma_{t(u)}}}$. Then for $t < t_0$:

$$N(t, x) = \sum_{n=0}^{\infty} \sum_{y: \sigma^n y = x} \lambda_d (\phi_{\omega \Gamma_{t(u)}}) e^{-dS_n \xi(y)}$$

$$= \lambda_d (\phi_{\omega \Gamma_{t(u)}}) \sum_{n=0}^{\infty} \mathcal{L}_n^{\omega} \delta \mathbf{1}(x) := \tilde{C} < \infty$$

by the spectral radius formula and Cor. 5.4. The assertion now follows with $s := -\delta$. 

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A standard trick (e.g. used in [Lal89]) is to consider the function $M(t, x) := e^{-t} N(t, x)/h_{-D\xi}(x)$. By the renewal equation we have

$$M(t, x) = \sum_{y : y = x} M(t - \xi(y), y) e^{-D\xi(y)} \frac{h_{-D\xi}(y)}{h_{-D\xi}(x)} + e^{-t} \lambda_d(F_{e^{-t}} \cap \phi_u \Gamma_{t(u)})/h_{-D\xi}(x).$$

Let $\overline{M}(t, x) := \sup_{r \leq t} M(t, x)$, $\overline{M}(t) := \sup_{x \in \overline{M}} \overline{M}(t, x)$ and recall that $\xi(x) \geq -\ln r > 0$ and $h_{-D\xi} \geq R > 0$ (see Thm. 5.3). Then

$$\overline{M}(t, x) \leq \sum_{y : y = x} \overline{M}(t + \ln r, y) e^{-D\xi(y)} \frac{h_{-D\xi}(y)}{h_{-D\xi}(x)} + \sup_{r \leq t} e^{-t} \lambda_d(F_{e^{-t}} \cap \phi_u \Gamma_{t(u)})/h_{-D\xi}(x) \leq \overline{M}(t + \ln r) + \sup_{r \leq t} e^{-t} \lambda_d(F_{e^{-t}} \cap \phi_u \Gamma_{t(u)})/R.$$

Let $t_0$ be as in the proof of (H). Then $\overline{M}(t_0) \leq \tilde{C} e^{-\delta t_0}/R$ and for $t \leq t_0$

$$\overline{M}(t) \leq \overline{M}(t + \ln r) + e^{-t} \lambda_d(\phi_u \Gamma_{t(u)})/R,$$

which for $n \in \mathbb{N}$ implies

$$\overline{M}(t_0 - n \ln r) \leq \tilde{C} e^{-\delta t_0}/R + e^{-t_0} \lambda_d(\phi_u \Gamma_{t(u)})/(R(1 - \delta)) =: B.$$

Whence $\sup_{t \in R} M(t, x) \leq B$ for all $x$. As $h_{-D\xi}$ is bounded also $e^{-t} N(t, x)$ is bounded.

For applying the Renewal Thm. 5.5 to (6.3) resp. (6.4) we distinguish between the lattice and non-lattice situations.

If $\xi$ is non-lattice the renewal theorem yields

$$\sum_{u \in E^n} N(t \pm \ln \rho_m, x_u) \sim \sum_{u \in E^n} e^{t \delta} \rho_m^+(d + \delta) \xi(x_u) \int_\xi \frac{1}{\int -\infty} e^{-T \delta} f_u(T) dT \cdot \nu_{-(d+\delta)\xi}([\kappa])$$

$$= e^{-t(d - D)} \rho_m^+(D - d) \sum_{u \in E^n} h_{-D\xi}(x_u) \int_\xi \frac{1}{\int -\infty} e^{-T(D - d)} f_u(T) dT \cdot \nu_{-D\xi}([\kappa]).$$

Combining (6.1) - (6.4) with the above we obtain that for $B = \phi_n O$

$$\lim_{m \to \infty} \lim_{t \to \infty} e^{-t(D - d)} \lambda_d(F_{e^{-t}} \cap B)$$

$$= \frac{1}{\int \xi \int \mu_{-D\xi}} \lim_{m \to \infty} \sum_{u \in E^n} h_{-D\xi}(x_u) \int_{-\infty}^{\infty} e^{-T(D - d)} \lambda_d(F_{e^{-T}} \cap \phi_u \Gamma_{t(u)}) dT \cdot \nu(B).$$

Eq. (5.3) implies $h_{-D\xi}(x_u) = \lim_{n \to \infty} L^n_{-D\xi} 1(x_u) = \lim_{n \to \infty} \sum_{u \in E^n} |\phi_u(\pi x_u)|^D$ which gives

$$\lim_{m \to \infty} \lim_{t \to \infty} e^{-t(D - d)} \lambda_d(F_{e^{-t}} \cap B)$$

$$= \frac{1}{\int \xi \int \mu_{-D\xi}} \lim_{m \to \infty} \sum_{u \in E^n} \int_{-\infty}^{\infty} e^{-T(D - d)} \lambda_d(F_{e^{-T}} \cap \phi_u \Gamma_{t(u)}) dT \cdot \nu(B).$$
If \( \xi \) is lattice then

\[
\sum_{u \in E^*_m} N(t \pm \ln \rho_m, x_u) \\
\sim \sum_{u \in E^*_m} e^{\delta} \rho_m^d h_{-d\xi - \delta \zeta}(x_u) e^{-a \delta \left\{ \frac{t \pm \ln \rho_m \pm \psi(x_u)}{\chi} \right\}} \frac{ae^{\delta \psi(x_u)}}{\int_{\zeta} d\mu_{-d\xi - \delta \zeta}} \\
\times \int_{[\sqrt{3}]} \sum_{\ell=\infty} e^{-a \delta} \lambda_d \left( F_{e^{-(a+t \pm \ln \rho_m \pm \psi(y))} \cap \phi_u \Gamma_t(u)} \right) d\nu_{-d\xi - \delta \zeta}(y) \\
=: E_m(t)
\]

Eq. \([5.3]\) implies that \( h_{-d\xi - \delta \zeta}(x_u) = \lim_{n \to \infty} \sum_{\omega \in E^*_m} \omega u \in E^* \left| \phi'_\omega(\pi x_u) \right| e^{-\delta S_n(\omega x_u)} \). Using this and that \( S_n(\xi)(E^\infty) \subset a\mathbb{Z} \) we obtain

\[
E_m(t) \leq \frac{a \rho_m^d}{\int_{\zeta} d\mu_{-d\xi - \delta \zeta}} \lim_{n \to \infty} \sum_{\omega \in E^*_m} \sum_{\omega u \in E^*} \int_{[\sqrt{3}]} \sum_{\ell=\infty} e^{-a \delta} \lambda_d \left( F_{e^{-(a+t \pm \ln \rho_m \pm \psi(y))} \cap \phi_u \Gamma_t(u)} \right) d\nu_{-d\xi - \delta \zeta}(y).
\]

A lower bound can be found analogously, yielding

\[
\lim_{m \to \infty} E_m(t) \\
= e^{\delta} \int_{\zeta} d\mu_{-d\xi - \delta \zeta} \lim_{m \to \infty} \sum_{u \in E^*_m} e^{\delta \psi(x_u)} \sum_{\ell=\infty} e^{-a \delta} \\
\times \int_{[\sqrt{3}]} e^{-a \delta \left\{ \frac{t \pm \psi(y) \pm \phi_u(\psi(y))}{\chi} \right\}} - \delta \psi(y) \lambda_d \left( F_{e^{-a(\ell) \pm \ln \rho_m \pm \psi(y) \pm \phi_u(\psi(y))}} \cap \phi_u \Gamma_t(u)} \right) d\nu_{-d\xi - \delta \zeta}(y).
\]

With \( d\nu_{-d\xi - \delta \zeta} = e^{\delta \psi} d\nu_{-(d+\delta)\xi} \), \( \int \zeta d\mu_{-d\xi - \delta \zeta} = \int \xi d\mu_{-d\xi - \delta \zeta} = \int \xi d\mu_{-(d+\delta)\xi} \) and \( \mu_{-d\xi - \delta \zeta} = \mu_{-(d+\delta)\xi} \) the statement follows. \( \square \)

**Proof of Thm. 4.5** Let \( \Phi := (\phi_e : X_t(e) \to X_t(e)) e \in E \) denote the cGDS associated with \( F_0 \). Then \( \Pi := (\psi_e := \phi_0 g^{-1} : g (X_t(e)) \to g (X_t(e))) e \in E \) is a cGDS with invariant set \( F \). Let \( \xi \) and \( \bar{\xi} \) denote the geometric potential functions associated with \( \Phi \) and \( \Psi \), respectively. We have \( \mu_{-D\bar{\xi}} = \mu_{-D\xi} \) and since \( \mu_{-D\xi} \) is \( \sigma \)-invariant, also

\[
(6.5) \quad \int \bar{\xi} d\mu_{-D\bar{\xi}} = \int \xi d\mu_{-D\xi}
\]
Moreover, for \( n < m \) and an arbitrary \( x \in E^\infty \) with \( ux \in E^\infty \) we have

\[
\sum_{n(\omega) = m} \sum_{k=1}^{\infty} \left| \psi_\omega g f^k_{t(\omega)}(C_4) \right|^D = \sum_{n(\omega) = m} \sum_{k=1}^{\infty} \left| g \phi_\omega f^k_{t(\omega)}(C_4) \right|^D
\]

Applying (5.3) to the above expression for \( m \to \infty \) and using that the lower bound can be found analogously yields

\[
\lim_{m \to \infty} \sum_{n(\omega) = m} \sum_{k=1}^{\infty} \left| \psi_\omega g f^k_{t(\omega)}(C_4) \right|^D = \lim_{n(\omega) = m} \sum_{k=1}^{\infty} \left| \phi_u f^k_{t(u)}(C_4) \right|^D \int |g' \circ \pi|^D \, d\nu_{-D\xi}.
\]

This equality in particular holds for \( g = id \), whence

\[
(6.6) \quad \frac{\lim_{m \to \infty} \sum_{n(\omega) = m} \sum_{k=1}^{\infty} \left| \psi_\omega g f^k_{t(\omega)}(C_4) \right|^D}{\lim_{m \to \infty} \sum_{n(\omega) = m} \sum_{k=1}^{\infty} \left| \phi_u f^k_{t(u)}(C_4) \right|^D} = \int |g' \circ \pi|^D \, d\nu_{-D\xi}.
\]

Proof of Thm. 4.7. With the notation of Sec. 4.2 we have

\[
\chi(F(e^{-t} \cap T)) = 1 - R(e^{-t})
\]

\[
= - \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \sum_{\omega \in E^m_n} 1_{(e^{-t}, \infty)} \left( |\phi_\omega f^k_{t(\omega)}(C_4)|/2 \right)
\]

\[
= - \sum_{k=1}^{m-1} \sum_{n=0}^{\infty} \sum_{\omega \in E^m_n} 1_{(e^{-t}, \infty)} \left( |\phi_\omega f^k_{t(\omega)}(C_4)|/2 \right)
\]

\[
- \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \sum_{\omega \in E^m_n} 1_{(e^{-t}, \infty)} \left( |\phi_\omega \phi_u f^k_{t(u)}(C_4)|/2 \right)
\]

for arbitrary \( m \in \mathbb{N} \). Using the Bounded Distortion Lemma (Lem. 6.1) and Lem. 4.1 we obtain that the first series is \( O(t^m e^{t/2}) \) as \( t \to \infty \) with \( O \) denoting the Big-O
Landau symbol. This can be seen as follows.

\[
\sum_{n=0}^{m-1} \sum_{\omega \in E_n^k} \sum_{k=1}^{\infty} \mathbb{I}_{(e^{-t}, \infty)} \left( |\phi_{\omega} f^k_{i_{\omega}}(C_4)|/2 \right) \\
\leq \sum_{n=0}^{m-1} \sum_{j_1, \ldots, j_n \in \mathbb{N}} \sum_{k=1}^{\infty} 3^n \mathbb{I}_{(e^{-t}, \infty)} \left( Q^{n+1}|C_4|/2 \cdot (j_1 \cdots j_n k)^{-2} \right) \\
\leq \sum_{n=0}^{m-1} 3^n \sum_{j_1, \ldots, j_n = 1}^{\infty} e^{t/2} \sqrt{Q^{n+1}|C_4|/2} \cdot (j_1 \cdots j_n)^{-1} \\
\leq e^{t/2} C_1 \sum_{n=0}^{m-1} (3Q)^n (C_2 t)^n
\]

with some constants \( C_1, C_2 > 0 \) for sufficiently large \( t \). With the second series we proceed as in the proof of Thm. 4.4. Again using the Bounded Distortion Lemma and setting

\[
g_{u,k}(t) := \mathbb{I} \left( \frac{-\ln |\phi_u f^k_{i_{u}}(C_4)|/2}{2}, \infty \right)(t)
\]

we obtain with an arbitrary \( x_u \in [u] \)

\[
\sum_{n=0}^{\infty} \sum_{\omega \in E_n^k} \mathbb{I}_{(e^{-t}, \infty)} \left( |\phi_{\omega} f^k_{i_{\omega}}(C_4)|/2 \right) \leq \sum_{n=0}^{\infty} \sum_{\omega \in E_n^k} g_{u,k}(t - S_n \xi(\omega x_u)) \\
\sim e^{t/2} \frac{h_{-D}(x_u)}{D} \int \frac{\mu_{-D} \rho_M 2^{-D} |\phi_u f^k_{i_{u}}(C_4)|^D}{\xi d\mu_{-D} \xi}
\]

as \( t \to \infty \). The last asymptotic is a consequence of Thm. 5.5. Its prerequisites are easily checked:

**Ad (F)**

\[
\int_{-\infty}^{\infty} e^{-tD} |g_{u,k}(t)| \, dt = \int_{-\infty}^{\infty} e^{-tD} \, dt < \infty
\]

**Ad (G)** This is shown in [Lal89], since in the present setting \( N \) is a counting function.

**Ad (H)** \( N^{abs}(t, x) = 0 \) for \( t \leq -\ln |\phi_u f^k_{i_{u}}(C_4)|/2 \).

Using that the Bounded Distortion Lemma provides a lower estimate, too, and applying the same approximation arguments as in the proof of Thm. 3.5, the statement of Thm. 4.4 follows. \( \square \)

**Proof of Thm. 4.11.** The Möbius transform \( q \) is constructed in such a way that \( \{ f^k_3 q(C_4) \mid k \in \mathbb{N} \} \) gives a cover of \( F_0 \cap X_3 \cap \{ z \in \mathbb{C} \mid \Re(z) \geq 0 \} \). Hence, symmetry
implies
\[
\mathcal{H}^D(F_0) \leq 6 \lim_{m \to \infty} \sum_{\omega \in E_n^A, t(\omega) = 3} \sum_{k=1}^{\infty} |\phi_{\omega} f^k_3 (C_4)|^D
\]
\[
\leq 6\rho_0^D \lim_{m \to \infty} \sum_{\omega \in E_n^A, t(\omega) = 3} \sum_{k=1}^{\infty} |\phi_{\omega} f^k_3 (C_4)|^D
\]
\[
= 2\rho_0^D \lim_{m \to \infty} \sum_{\omega \in E_n^A} \sum_{k=1}^{\infty} |\phi_{\omega} f^k_3 (C_4)|^D.
\]
Together with Cor. [4.10] this proves the first inequality of Thm. [4.11]. For obtaining a numeric bound on \(c_A\) we need to determine the bounded distortion constant \(\rho_0\).

For this define \(\Sigma(n) := \{ (\omega_1, \ldots, \omega_n) \in \{1,2,3\}^n \mid \omega_i \neq \omega_{i+1} \text{ for } i \leq n - 1 \}\). Any allowed concatenation of \(\phi^{k_{n-1}}_{\omega_{n-2}} \circ \cdots \circ f^k_{\omega_1}\) for some \(n \in \mathbb{N}, \omega = (\omega_1, \ldots, \omega_n) \in \Sigma(n)\) and \(k = (k_1, \ldots, k_n) \in \mathbb{N}^n\). Moreover, each \(\psi^k_{\omega} \) is a Möbius transform and we write \(\psi^k_{\omega}(z) = (az + b)/(cz + d)\) with \(a = a(\omega, k), b = b(\omega, k), c = c(\omega, k), d = d(\omega, k) \in \mathbb{C} \) and \(ad - bc \neq 0\). Let \(v = v(\omega, k)\) denote the terminal vertex of \(\psi^k_{\omega}\).

Then
\[
\rho_0 = \sup_{\omega, k} \frac{\max_{z \in X_v} |(\psi^k_{\omega})'(z)|}{\min_{z \in X_v} |(\psi^k_{\omega})'(z)|} = \sup_{\omega, k} \frac{\max_{z \in X_v} |cz + d|^2}{\min_{z \in X_v} |cz + d|^2}.
\]
If \(c = 0\) the quotient is minimal. Hence, we can assume that \(c \neq 0\) and obtain
\[
\rho_0 = \sup_{\omega, k} \frac{\max_{z \in X_v} |z + d/c|^2}{\min_{z \in X_v} |z + d/c|^2} = \sup_{\omega, k} \frac{(|S_v + d/c| + R)^2}{(|S_v + d/c| - R)^2},
\]
where \(S_v\) is the centre of \(X_v\) and \(R = 2\sqrt{3} - 3\) its radius. The last equality holds, since existence of the bounded distortion constant (Lem. [6.1]) implies that \(-d/c\) lies in the exterior of \(X_v\). In the following we show by induction that
\[
\hat{\rho}_n := \inf_{(\omega, k) \in \Sigma(n) \times \mathbb{N}^n} \left| S_{v(\omega, k)} + \frac{d(\omega, k)}{c(\omega, k)} \right| \geq \sqrt{33 - 18\sqrt{3}} =: \hat{\rho} \quad \forall n \in \mathbb{N},
\]
yielding
\[
\rho_0 \leq \left( \frac{\hat{\rho} + R}{\hat{\rho} - R} \right)^2 \leq 4.19225.
\]
Note that \(f_3(z) = ((\sqrt{3} - 1)z + 1)/(-z + \sqrt{3} + 1)\) and that
\[
f_3 = g^{-1} \circ h \circ g \quad \text{with} \quad g(z) = \frac{1}{z - 1}, \quad g^{-1}(z) = 1 + \frac{1}{z} \quad \text{and} \quad h(z) = -\frac{\sqrt{3}}{z},
\]
see [MU98]. Thus, \(h^k(z) = z - k/\sqrt{3}\) and
\[
f_3^k(z) = (\sqrt{3} - k)z + k
\]
\[
- kz + k + \sqrt{3}
\]
Because of symmetry
\[
\hat{\rho}_1 = \inf_{k \in \mathbb{N}} \left| S_1 + \frac{d(3,k)}{c(3,k)} \right| = \inf_{k \in \mathbb{N}} \left| -2 + \sqrt{3} + (2\sqrt{3} - 3)k - \frac{k + \sqrt{3}}{k} \right| = \hat{\rho}.
\]
Now, take an arbitrary concatenation with representation of the form \(\psi^k_{\omega}\) with \((\omega, k) \in \Sigma(n + 1) \times \mathbb{N}^{n+1}\). Without loss of generality assume that \(\omega_{n+1} = 3\) and
that the terminal vertex is 1. By induction hypothesis we know that $|S_1 + \tilde{d}/\tilde{c}| \geq \hat{\rho}$, where $\tilde{c} := c(\omega_1, \ldots, \omega_n), (k_1, \ldots, k_n)$, $\tilde{d} := d((\omega_1, \ldots, \omega_n), (k_1, \ldots, k_n))$, yielding $\tilde{d}/\tilde{c} = re^{i\theta} - S_3$ for some $r \geq \hat{\rho}$ and $\theta \in [0, 2\pi)$. Multiplying the associated matrices of the Möbius maps we see that $c(\omega, k) = (\sqrt{3} - k_{n+1})\tilde{c} - k_{n+1}\tilde{d}$ and $d(\omega, k) = k_{n+1}\tilde{c} + (k_{n+1} + \sqrt{3})\tilde{d}$, whence

$$d(\omega, k) = \frac{k_{n+1}}{\sqrt{3} - k_{n+1}} + \frac{3}{\sqrt{3} - k_{n+1}} - \frac{\rho \cdot S_3}{\sqrt{3} - k_{n+1} - k_{n+1}(re^{i\theta} - S_3)}.$$  

For $k_{n+1} \geq 2$ the angle between $S_1 + A$ and $B(\theta, r)$ is acute, as the scalar product of the two vectors is positive for any $\theta \in [0, 2\pi)$ and $r \geq \hat{\rho}$. (Recall that $S_1 = -2 + \sqrt{3} + (2\sqrt{3} - 3)i$.) Therefore, $|S_1 + A + B(\theta, r)| \geq |S_1 + A|$, and whence

$$\inf_{(\omega, k) \in \Sigma_{(n+1)} \times N^{n+1}, \ k_{n+1} \geq 2} |S_1 + \frac{d(\omega, k)}{c(\omega, k)}| \geq \inf_{m \geq 2} |S_1 + \frac{m}{\sqrt{3} - m}| = \hat{\rho}.$$ 

Finally, if $k_{n+1} = 1$ then (6.7) gives that $d(\omega, k)/c(\omega, k) > \hat{\rho}$ for any $\theta$ and $r$. Thus, $\hat{\rho}_n \geq \hat{\rho}$ for all $n \in \mathbb{N}$.

In [Fuc10] the Lyapunov-exponent $\int \xi d\mu_{-D\xi}$ has been computed, where an approximate value of 0.9149 was obtained. All in all, a lower bound for $c_A$ is

$$c_A \geq \frac{\pi^{D/2}}{D^{2D+1}} \cdot \frac{\hat{\rho}_0^{-D}}{0.915} \geq 0.055.$$ 

\[\square\]

REFERENCES

[BHR06] Christoph Bandt, Nguyen Viet Hung, and Hui Rao. On the open set condition for self-similar fractals. Proc. Amer. Math. Soc., 134(5):1369–1374, 2006.

[Boy82] David W. Boyd. The sequence of radii of the Apollonian packing. Math. Comp., 39(159):249–254, 1982.

[DKÖ13] Ali Deniz, Şahin Koçak, Yunus Özdemir, Andrei Ratiu, and Adem E. Üreyen. On the Minkowski measurability of self-similar fractals in $\mathbb{R}^d$. Turkish J. Math., 37(5):830–846, 2013.

[Fed59] Herbert Federer. Curvature measures. Trans. Am. Math. Soc., 93:418–491, 1959.

[Fuc10] Elena Fuchs. Arithmetic properties of Apollonian circle packings. ProQuest LLC, Ann Arbor, MI, 2010. Thesis (Ph.D.)–Princeton University.

[GLM+06] Ronald L. Graham, Jeffrey C. Lagarias, Colin L. Mallows, Allan R. Wilks, and Catherine H. Yan. Apollonian circle packings: geometry and group theory. III. Higher dimensions. Discrete Comput. Geom., 35(1):37–72, 2006.

[KK12] Marc Kesseböhmer and Sabrina Kombrink. Fractal curvature measures and Minkowski content for self-conformal subsets of the real line. Adv. Math., 230(4-6):2474–2512, 2012.

[KK15] Marc Kesseböhmer and Sabrina Kombrink. Minkowski content and fractal Euler characteristic for conformal graph directed systems. J. Fractal Geom., 2(2):171–227, 2015.

[KK17] Marc Kesseböhmer and Sabrina Kombrink. A complex Ruelle-Perron-Frobenius theorem for infinite alphabets with applications to renewal theory. Discrete Contin. Dyn. Syst., Ser. S, 10(2):335–352, 2017.

[KMS12] Marc Kesseböhmer, Sara Munday, and Bernd O. Stratmann. Strong renewal theorems and Lyapunov spectra for $\alpha$-Farey and $\alpha$-Lüroth systems. Ergodic Theory Dynam. Systems, 32(3):989–1017, 2012.
[KO11] Alex Kontorovich and Hee Oh. Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds. *J. Amer. Math. Soc.*, 24(3):603–648, 2011. With an appendix by Oh and Nimish Shah.

[Ko11] Sabrina Kombrink. *Fractal curvature measures and Minkowski content for limit sets of conformal function systems*. PhD thesis, Universität Bremen, 2011. 

[Kom11] Sabrina Kombrink. Renewal theorems for a class of processes with dependent interarrival times. *preprint arXiv*, page 35, 2015. Preprint.

[KZ06] Marc Kesseböhmer and Sanguo Zhu. Dimension sets for infinite IFSs: the Texan conjecture. *J. Number Theory*, 116(1):230–246, 2006.

[Lal89] S. P. Lalley. Renewal theorems in symbolic dynamics, with applications to geodesic flows, non-Euclidean tessellations and their fractal limits. *Acta Math.*, 163(1-2):1–55, 1989.

[LO13] Min Lee and Hee Oh. Effective circle count for Apollonian packings and closed horospheres. *Geom. Funct. Anal.*, 23(2):580–621, 2013.

[McM98] Curtis T. McMullen. Hausdorff dimension and conformal dynamics. III. Computation of dimension. *Amer. J. Math.*, 120(4):691–721, 1998.

[Mu98] R. Daniel Mauldin and Mariusz Urbański. Dimension and measures for a curvilinear Sierpinski gasket or Apollonian packing. *Adv. Math.*, 136(1):26–38, 1998.

[Mu99] R. Daniel Mauldin and Mariusz Urbański. Conformal iterated function systems with applications to the geometry of continued fractions. *Trans. Amer. Math. Soc.*, 351(12):4995–5025, 1999.

[Mu03] R. Daniel Mauldin and Mariusz Urbański. *Graph directed Markov systems. Geometry and dynamics of limit sets*. Cambridge Tracts in Mathematics 148. Cambridge: Cambridge University Press. xi, 281 p., 2003.

[Oh10] Hee Oh. Dynamics on geometrically finite hyperbolic manifolds with applications to Apollonian circle packings and beyond. In *Proceedings of the International Congress of Mathematicians. Volume III*, pages 1308–1331. Hindustan Book Agency, New Delhi, 2010.

[Oh14a] Hee Oh. Apollonian circle packings: dynamics and number theory. *Jpn. J. Math.*, 9(1):69–97, 2014.

[Oh14b] Hee Oh. Harmonic analysis, ergodic theory and counting for thin groups. In *Thin groups and superstrong approximation*, volume 61 of *Math. Sci. Res. Inst. Publ.*, pages 179–210. Cambridge: Cambridge University Press. xi, 281 p., 2003.

[OS16] Hee Oh and Nimish Shah. Counting visible circles on the sphere and Kleinian groups. In *Geometry, topology, and dynamics in negative curvature*, volume 425 of *London Math. Soc. Lecture Note Ser.*, pages 272–288. Cambridge Univ. Press, Cambridge, 2016.

[Pol15] Mark Pollicott. Apollonian circle packings. In Christoph Bandt, Kenneth Falconer, and Martina Zähle, editors, *Fractal Geometry and Stochastics V*, volume 70 of *Progress in Probability*, pages 121–142. Springer International Publishing, 2015.

[RW10] Jan Rataj and Steffen Winter. On volume and surface area of parallel sets. *Indiana Univ. Math. J.*, 59(5):1661–1685, 2010.

[Win08] Steffen Winter. Curvature measures and fractals. *Dissertationes Math. (Rozprawy Mat.),* 453:66, 2008.

[Win15] Steffen Winter. Minkowski content and fractal curvatures of self-similar tilings and generator formulas for self-similar sets. *Adv. Math.*, 274:285 – 322, 2015.