Microscopic theory of superconductivity in twisted double-bilayer graphene

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Small-angle twisted double-bilayer graphene is a correlated moiré superlattice system that has recently been found to exhibit both interaction-induced insulating and superconducting phases with properties different from the related magic-angle twisted bilayer graphene. Here, we develop a microscopic weak-coupling theory for superconductivity in the system starting from a continuum-model description. We study an electron-phonon as well as an entirely electronic pairing mechanism, and discuss the interplay of the two. In each case, the leading superconducting instability transforms under the trivial representation, \( A \), of the point group \( C_3 \) of the system, while the subleading pairing phases belong to the \( E \) channel. We explicitly compute the momentum dependence of the associated order parameters and find that the leading state has no nodal points for electron-phonon pairing but exhibits 6 sign changes on the Fermi surface if the Coulomb interaction dominates. We also discuss the disorder sensitivity of the candidate pairing states and show how triplet superconductivity can be protected against disorder as a consequence of valley-charge conservation.

Introduction.—The zoo of moiré superlattice systems displaying correlated physics has been expanding rapidly [1]. One such heterostructure is twisted double-bilayer graphene (TDBG) [2–5], in which two AB-stacked graphene bilayers are twisted relative to each other. This system stands out not only due to the additional tunability of the band structure via electric fields [6–12], but also because both the gap of the correlated insulating state at half-filling and the critical temperature of the superconducting phase can be enhanced by application of a magnetic field, which hints that its physics is different from that of the well-studied twisted bilayer graphene (TBG) [13–16]. The superconducting phase of TDBG is found to be very fragile [4, 5]: it only emerges [2, 3] on the electron-doped side relative to the insulator at half-filling of the conduction band and quickly weakens away from this optimal doping value.

In a recent publication [17], we performed a systematic classification of the possible pairing instabilities in TDBG, using general constraints based on symmetries and energetics. Here, we complement Ref. 17 by an explicit microscopic calculation using a continuum model for the underlying graphene sheets, inspired by theoretical studies of superconductivity in TBG [18–26]. Despite recent indications pointing towards an electron-phonon based pairing mechanism in TBG [27, 28], the situation is, at present, completely open, at least for TDBG. As a first step towards understanding the pairing glue in TDBG, we analyze both electron-phonon (conventional) and electron-electron (unconventional) interactions-based pairing. The symmetry properties and gap structures of the leading and subleading instabilities, and their doping dependencies are computed and discussed in relation to experiment. We also describe the possibility that electron-phonon coupling provides the main pairing glue, but the dominant interactions that distinguish between singlet and triplet stem from purely electronic processes.

Model and band structure.—The TDBG heterostructure is composed of two individually aligned AB-stacked bilayer graphene (BLG) sheets, twisted relative to each other by a small angle \( \theta \ll 1 \), forming a moiré superlattice. The superlattice structure manifests itself in the coupling between the lower layer of the top BLG and the topmost sheet of the bottom stack, which, in momentum space, hybridizes the eigenstates at a Bloch vector \( \mathbf{k} \) in the moiré Brillouin zone (MBZ) with those related by reciprocal lattice vectors of the moiré superlattice. The resulting electronic band structure of TDBG is well described by a continuum model [10, 29–32] incorporating lattice relaxation effects [33, 34], reviewed in Appendix A.
of the Supplemental Material (SM) [35]. Upon including terms accounting for trigonal warping [36] and particle-hole asymmetry [37], the first conduction and valence bands, around charge neutrality, always acquire a sizable dispersion and unlike TBG [38–40], there no longer exists a sharp “magic angle” at which they are almost perfectly flat. For small \( \theta \), these bands overlap with each other but they can be separated by a gate voltage applied between the top and bottom layers. The 10–15 meV bandwidth of the band hosting superconductivity, shown in green in Fig. 1(a), is still smaller than the interaction scale, revealing the system’s strongly correlated nature. Adding to the distinction from TBG is the fact that the physics of TDBG is dominated by a twofold rotational symmetry \( C_2 \) [9], which formerly protected the Dirac points in TBG. In the following, we focus on this narrow band for a particular twist angle, \( \theta = 1.24^\circ \), at which superconductivity was observed in experimental samples [3]. The associated Fermi surfaces for different band fillings and the density of states (DOS) are sketched in Figs. 1(b) and (c).

**Phonon-mediated superconductivity.** To begin, we consider the option that the attractive interaction between electrons is mediated by the phonon modes of the two-dimensional graphene layers [41]. This is a natural extension of the physics of magic-angle TBG in which the phonon-driven electron-electron attraction, in combination with an enhanced DOS \( \sim 10 \text{ eV}^{-1} \text{ nm}^{-2} \) (nearly ten times larger than \( g(E) \) in Fig. 1), has been predicted to induce intervalley pairing [23–25, 42], yielding a critical temperature \( T_c \sim 1 \text{ K} \) [14]. Given that a similar enhancement of the DOS is lacking in TDBG due to the significantly larger bandwidth, one may question whether such a mechanism still holds in this system.

To this end, we consider, in particular, acoustic phonon modes, prompted by previous works on TBG [24, 42]. While the layer-symmetric modes (all four layers moving together) and the modes where the two BLG move against each other (corresponding to a shift of the moiré lattice) remain gapless, the phonons that shift any of the sheets of the same BLG against each other acquire a mass. Concentrating on the former modes, we assume that their velocity is the same, motivated by the expected weak interlayer phonon coupling [43]. The electron-phonon Hamiltonian [42] can, therefore, be written as a coupling of acoustic phonon modes, \( \alpha_{q,l} \), and the electron density \( \hat{\rho}_{q,l} \) of the lower \( (l=1) \) and upper \( (l=2) \) BLG,

\[
H_{\text{ep}} = \frac{D}{\sqrt{N}\Omega} \sum_{q,l} \sqrt{\frac{\hbar}{2\bar{m}\omega_q}} (-i\hat{q} \cdot \hat{\epsilon}_q) \left( \alpha_{q,l} + \alpha_{q,l}^\dagger \right) \hat{\rho}_{q,l}.
\]

Here, \( \bar{m} \) is the mass density, \( N \) and \( \Omega \) denote the number and area of moiré unit cells, respectively, \( \hat{\epsilon}_q \) is the displacement unit vector, and \( \omega_q = v_{ph} |q| \) the phonon frequency; we take \( v_{ph} = 2 \times 10^6 \text{ cm/s} \) [42]. The deformation potential \( D (\sim 25 \text{ eV}) \) is much larger than the interlayer tunneling parameter (\( \sim 0.1 \text{ eV} \)) [42], which allows for neglecting the effects of the latter. The Debye frequency \( \omega_D (\sim 82.3 \text{ meV} \text{ nm}^2) \) [42, 46], we evaluate the critical temperature \( T_c \) of superconductivity by determining when the largest of the eigenvalues \( \{\lambda_n(T)\} \) of \( \mathcal{M}(T) \) reaches 1; for instance, at three-quarters filling, we obtain \( T_c = 1.90 \text{ K} \).
[see Fig. 2 (b)]. Interestingly, this value is of the same order as the experimental observations and suggests that electron-phonon coupling is strong enough to induce superconductivity in TDBG as well. As seen in Fig. 2 (c), $T_c$ is peaked slightly above half filling of the conduction band; this peak in $T_c$ lines up with the maximum of the DOS in Fig. 1 (c), as expected.

More importantly, beyond these numerical values, the leading instability is found to transform under the trivial representation, $r = A$, while the subleading pairing states transform under $r = E$; the calculated basis functions, $\Delta^\mu_{A,\nu}$, are displayed in Fig. 2 (d). The fact that the dominant basis function respects all symmetries of the system and can be chosen to be real and positive for all momenta in the MBZ is consistent with the general results in Ref. 47. Depending on the sign of the interaction breaking the enhanced spin symmetry, we expect singlet, $\Delta_k = i\sigma_2 \Delta^A_k$, or triplet $\Delta_k = i \sigma_2 \sigma \cdot d \Delta^A_k$, to dominate; here, $\sigma_i$ are Pauli matrices in spin space and the superconducting order parameter couples as $\propto \Delta^i_{k,\sigma,\nu}(\Delta_{k,\sigma,\nu})^i_{k,\sigma,\nu}$ to the low-energy electrons. Finally, we point out that singlet is expected to dominate over triplet if only phonon-mediated interactions are taken into account [47, 48]. If, however, the SU(2)$_+ \times$ SU(2)$_-$-symmetry-breaking interactions are dominated by the Coulomb repulsion, triplet pairing is possible even though the main pairing glue is the SU(2)$_+ \times$ SU(2)$_-$ symmetric electron-phonon-induced interaction: as demonstrated in Appendix D [35], singlet (triplet) will dominate if these symmetry-breaking interactions arise from time-reversal even (odd) particle-hole fluctuations, irrespective of microscopic details.

**Electron-electron interactions and screening.**—The Coulomb repulsion can also be the central driving force of superconductivity [49, 50], as has, e.g., been discussed for monolayer graphene [51–54], artificial graphene [55], and TBG [20, 56]. To explore the scenario of an electronic pairing mechanism in TDBG, we neglect the electron-phonon coupling and focus on the Coulomb interaction. Since the detailed form of the associated interaction $V_{i,j}^\nu s'$ in Eq. (1) at the energy scale of our single-band description—including any potential dependence on the layers $(l, s)$ and $(l', s')$ of the electrons involved—is presently unclear, we assume that it can be described as a pure density-density interaction in the low-energy conduction band: $H_{\text{int}} = N^{-1} \sum_{q} V(q) \rho^i_q \rho^j_{-q}$ with $\rho^i_q = \sum_{k,\sigma,\xi} c_{k,\sigma,\xi}^\dagger c_{k+q,\sigma,\xi}^\dagger$. In momentum space [9],

$$V(q) = \frac{V_0 |g_{\text{el}}|}{2\sqrt{q^2 + \kappa^2}}, \quad V_0 = \frac{e^2}{2 e_0 \epsilon |\Omega| q_{\text{el}}} \approx \frac{e^2 \theta}{4 \pi e_0 \epsilon a},$$

with $e$ the electronic charge and $e_0$ the vacuum permittivity. In Eq. (3), we have already taken into account screening from the gate [57, 58], as described by the screening length $\kappa^{-1} = 2 \times 10^{-8}$ m [9], as well as from the surrounding boron nitride and the higher bands of TDBG, captured by the effective dielectric constant $\epsilon = 5$ [59]. In addition, we consider the crucial momentum and

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**FIG. 2.** Phonon-mediated superconductivity in TDBG. (a) The form factors associated with the projection of the densities on to the lowest band. In the $q = 0$ plane, the threefold rotational symmetry of the lattice is directly visible. (b) The four largest eigenvalues $\lambda$ of the matrix $M$ as a function of temperature $T$; the subleading eigenvalues are doubly degenerate, as marked by $\times 2$. (c) Dependence of the superconducting transition temperature on the chemical potential; dashed lines indicate the five integer fillings of Fig. 1. (d) Basis functions of the leading (left) and subleading (middle and right) superconducting states at 2/3 filling. Note that the extrema of the order parameter trace out the Fermi surface in Fig. 1(b).
doping dependence resulting from the internal screening processes of the electrons in the low-energy bands hosting superconductivity. To describe the latter, we use the random phase approximation (RPA) [60], akin to earlier works on TBG [20, 26, 61–63].

Using the low-energy continuum model and restricting ourselves to the lowest conduction band as before, we compute the Lindhard function [64, 65]

$$\Pi_0(q, \omega = 0) = \frac{2}{N} \sum_{k, \xi} \frac{f_{k, \xi} - f_{k + q, \xi}}{E_{k, \xi} - E_{k + q, \xi}}, \tag{4}$$

where the factor of 2 represents the spin degeneracy, and $f_{k, \xi}$ is the Fermi function evaluated at band energy $E_{k, \xi}$. The resultant effective screened electron-electron interaction is given by $V_{\text{rpa}}(q) = V(q)/(1 - \Pi_0(q) V(q))$, which is plotted in Fig. 3, together with $\chi_0(q) \equiv \Pi_0(q)/\Omega$. The coupling is strongly renormalized as a result of this internal screening and exhibits a significant dependence on electronic filling.

**Coulomb-driven superconductivity.**—The screening of the Coulomb interaction is highly anisotropic, leading to a nontrivial texture of $V_{\text{rpa}}$ in momentum space. If $V_{\text{rpa}}(q) > 0$ $\forall$ $q$ and has a local maximum at $q = 0$, one expects that the kernel $\mathcal{M}(T)$ of the gap equation does not have a positive eigenvalue, signaling the absence of a superconducting instability. This is indeed what we find at hole-doped fillings of the isolated band, with $V_{\text{rpa}}$ illustrated by Figs. 3 (e–f). However, as Fig. 3 (g) conveys, the RPA screening leads to a local minimum at $q = 0$ for 2/3 filling. A superconducting instability now becomes possible as the pairing can gain energy if there is a relative sign change between the order parameters $\Delta_k$ and $\Delta_{k'}$ connected by the momentum $q = k' - k \neq 0$ where $V_{\text{rpa}}(q)$ is maximal. In fact, the deepest local minimum is obtained at approximately $\mu = 0.016$ eV, which roughly coincides with the optimal doping for superconductivity noticed in experiments. Moving away from this filling, on the electron-doped side, the trough quickly disappears [as is visible in Fig. 3 (h)], thereby killing any Coulomb-driven superconductivity in the process.

At the electron concentrations where superconductivity is favored, we find attractive interactions both in the $A$ and $E$ representations with basis functions shown in Fig. 4; as in the case of electron-phonon coupling, the $A$ representation is dominant, but now exhibits sign changes as required due to the repulsive nature of the interactions. It displays 6 nodal points on the Fermi surface [Fig. 4 (a)], which is the minimal nonzero number of nodal points for an order parameter transforming as $A$ of $C_3$. An estimate for the superconducting critical temperature with this electronic mechanism yields $T_c < 0.1$ K, which is significantly lower than the measured value. Since higher-order corrections to RPA can enhance superconductivity [66], this might well be a consequence of our weak-coupling approach and does not rule out an electronic pairing mechanism.

Despite transforming under the same, trivial, representation, the dominant superconducting states in Fig. 2 (d) and Fig. 4 (a) for phonon-mediated and electronic pairing mechanisms, respectively, will have very different thermodynamic properties due to the absence and presence of accidental nodes. Moreover, their behavior in the presence of disorder differs significantly: using the general expression for the reduction, $\delta T_c \sim -(\pi/4) \zeta \tau^{-1}$, of the
superconducting transition temperature with increasing scattering rate $\tau^{-1}$ derived in Ref. [67], we find $\zeta \simeq 0.058$ and $\zeta \simeq 0.498$ for the pairing states in Fig. 2 (d), closed panel, and Fig. 4 (a), respectively, focusing on singlet pairing and nonmagnetic disorder (see also Appendix E [35]). While the former almost enjoys the protection of an “Anderson theorem” [68, 69], the latter is quite sensitive to disorder: its value of $\zeta$ is close to that of a state that transforms under a nontrivial representation ($\zeta = 1/2$). This—or the subtle doping dependence of the electronic pairing mechanism in Fig. 3—could explain why not all experimental groups detect superconductivity [4, 5]. Interestingly, we find that the triplet state with a basis function in Fig. 2 (d), which can be stabilized by a combination of electron-phonon and electron-electron coupling, can also exhibit an Anderson theorem, as long as intervalley scattering is suppressed; the latter is generally expected if the impurity potential is smooth on the atomic scale and only varies on the moiré length scales.

Conclusion and outlook.—In this work, we analyzed the dominant and subleading superconducting instabilities for TDBG using a microscopic continuum model, and discussed both electron-phonon and RPA-screened purely electronic pairing mechanisms, as well as nontrivial aspects of their interplay based on the approximate enhanced spin-rotation symmetry of the system. In particular, the explicit form of the derived superconducting order parameters are expected to be useful for comparison with future experimental investigations of the system such as quasiparticle-interference experiments. On the theoretical side, further first-principles computations of phononic properties [70, 71] of TDBG are needed to obtain a refined description of electron-phonon pairing. In combination with the microscopic theory presented in this study, we hope that this will help to pinpoint the pairing glue and order parameter of superconductivity in TDBG and related moiré superlattice systems.

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Appendix A: Continuum model for TDBG

In this section, we briefly review the continuum model for TDBG introduced by Koshino [1]. Each unrotated bilayer graphene (BLG) sheet has lattice vectors \( \mathbf{a}_1 = a(1,0) \) and \( \mathbf{a}_2 = a(1/2, \sqrt{3}/2) \), with lattice constant \( a \approx 0.246 \text{ nm} \); their reciprocal lattice counterparts are \( \mathbf{b}_1 = (2\pi/a)(1,-1/\sqrt{3}) \) and \( \mathbf{b}_2 = (2\pi/a)(0,2/\sqrt{3}) \). Once a relative rotation is applied between the two BLGs, the lattice vectors differ from the un twisted case, and, for the \( l \)-th BLG, are given by \( \mathbf{a}^{(l)}_{i} = R(\mp \theta/2)\mathbf{a}_i \) (with \( \mp \) for \( l = 1,2 \)), \( R(\theta) \) being the matrix for rotations by angle \( \theta \). Correspondingly, the reciprocal lattice vectors are modified to \( \mathbf{b}^{(l)}_{i} = R(\mp \theta/2)\mathbf{b}_i \). The moiré pattern thus formed is characterized in the limit of small \( \theta \) by the reciprocal lattice vectors \( \mathbf{G}^M = \mathbf{b}^{(1)}_i - \mathbf{b}^{(2)}_i \) (\( l = 1,2 \). In valley \( \xi = \pm 1 \), the Dirac points of graphene are located at \( \mathbf{K}_\xi^{(l)} = -\xi(2\mathbf{b}_1^{(l)} + \mathbf{b}_2^{(l)})/3 \) for the \( l \)-th BLG.

Denoting by \( A_l, B_l \) the sublattice on layer \( l = 1,2,3,4 \) [labeled by the double-index \( (l,s) \) in the main text], the continuum Hamiltonian for TDBG at small twist angles \( \theta \ll 1 \) can be expressed in the Bloch basis of carbon’s \( p_z \) orbitals, \((A_1,B_1,A_2,B_2,A_3,B_3,A_4,B_4)\) as

\[
H_{AB-AB} = \begin{pmatrix}
H_0(k_1) & S(k_1) \\
S(k_1) & H'_0(k_2)
\end{pmatrix}
\begin{pmatrix}
U \\
H_0(k_2)
\end{pmatrix}
+ V,
\]

where \( k_l = R(\pm \theta/2)(\mathbf{k} - \mathbf{K}_\xi^{(l)}) \) with \( \pm \) for \( l = 1,2 \). Using the shorthand \( k_\pm = \xi k_x \pm i k_y \), the other building blocks are

\[
H_0(k) = \begin{pmatrix}
0 & -hv k_- \\
-hv k_+ & d'
\end{pmatrix},
H'_0(k) = \begin{pmatrix}
d' & -hv k_- \\
-hv k_+ & 0
\end{pmatrix},
\]

\[
S(k) = \begin{pmatrix}
\hbar v k_+ & \gamma_1 \\
\hbar v k_- & \hbar v k_+
\end{pmatrix}.
\]

\( H_0 \) and \( H'_0 \) above are the Hamiltonians of monolayer graphene, where the band width \( d \) is \( hv/a = 2.1354 \text{ eV} \) [2, 3]. At so-called dimer sites, the \( A_l \) atoms of the first layer sit atop the \( B_2 \) atoms of the second, and this results in the small additional on-site potential \( d' = 0.050 \text{ eV} \) [4] with respect to nondimer sites. The interlayer coupling of AB-stacked BLG is captured by the matrix \( S \), wherein \( \gamma_1 = 0.4 \text{ eV} \) is the coupling between the abovementioned dimer sites, while the parameters \( v_3 \) and \( v_4 \) are related by \( v_i = (\sqrt{3}/2)\gamma_i a/h \) (\( i = 3,4 \)) [4] to the diagonal hopping elements \( \gamma_3 = 0.32 \text{ eV} \) and \( \gamma_4 = 0.044 \text{ eV} \). In AB-stacked BLG, \( v_3 \) and \( v_4 \) account for trigonal warping of the energy band and electron-hole asymmetry, respectively.

The long-wavelength moiré potential stemming from the angular twist between the two BLGs also induces couplings between the atoms in the second and third layers. This moiré interlayer hopping is effectively described by the matrix \( U \) in Eq. (A1), which is given by [2, 3, 5]

\[
U = \begin{pmatrix}
u & u' \\
-\omega u & \omega v
\end{pmatrix} + e^{i\xi G_1^M \cdot r} \begin{pmatrix}
u & u' \\
-\omega u & \omega v
\end{pmatrix} + e^{i\xi (G_1^M + G_2^M) \cdot r} \begin{pmatrix}
u & u' \\
-\omega u & \omega v
\end{pmatrix}, \quad \omega = e^{2\pi i/3},
\]

where \( u = 0.0797 \text{ eV} \) and \( u' = 0.0975 \text{ eV} \) [3]; crucially, \( u \neq u' \). This difference between the diagonal \( (u) \) and off-diagonal \( (u') \) amplitudes represents the out-of-plane corrugation effect [1]: such lattice relaxation not only expands (shrinks) AB (AA) stacking regions but also enhances the energy gaps separating the lowest-energy and excited bands [3, 6, 7]. In momentum space, the coupling \( U \) hybridizes the eigenstates at a Bloch vector \( \mathbf{k} \) in the moiré Brillouin zone with those at \( \mathbf{q} = \mathbf{k} + \mathbf{G} \), where \( \mathbf{G} = m_1 \mathbf{G}_1^M + m_2 \mathbf{G}_2^M \) for \( m_1, m_2 \in \mathbb{Z} \).

Lastly, another important tuning knob in the experimental setup is the potential difference between the top and bottom graphene layers. This lends another useful degree of controllability to the system, enabling one to change the band separations by adjusting the gate voltage difference. In Eq. (A1), it is parametrized by \( V \), the interlayer asymmetric potential,

\[
V = \text{diag}
\begin{pmatrix}
\frac{3d}{2}, & -\frac{d}{2}, & -\frac{d}{2}, & -\frac{3d}{2}
\end{pmatrix},
\]

where \( \mathbf{1} \) stands for the \( 2 \times 2 \) unit matrix, and \( d \) commutes the electrostatic energy difference between adjacent layers, arising from a constant perpendicular field.

Putting all these ingredients together, we numerically diagonalize the Hamiltonian (A1) in momentum space (with only a limited number of \( q \)-points inside the cut-off circle \( |q - (K_1^{(1)} + K_2^{(2)})/2| < 4|G_1^M| \)) to obtain the low-energy eigenstates and energy bands. Since the inter-valley coupling is negligible at small twist angles, this calculation can be carried out separately for each valley.
Appendix B: Derivation of the form factors

In TDBG, there are four main degrees of freedom for our continuum fields, which are indexed by \( \sigma \) (spin), \( \ell = (l, s) \) (layer), \( \tau \) (sublattice), and \( \xi \) (valley), so the electronic creation operators at (continuum) position \( \mathbf{r} \in \mathbb{R}^2 \) will be denoted by \( a_{\sigma,\ell,\tau,\xi}(\mathbf{r}) \). Although the following framework can be applied to any observable, let us begin with the total electronic density at a two-dimensional position \( \mathbf{r} \), given by

\[
\hat{\rho}(\mathbf{r}) = \sum_{\sigma,\ell,\tau,\xi} \hat{a}_{\sigma,\ell,\tau,\xi}^\dagger(\mathbf{r}) a_{\sigma,\ell,\tau,\xi}(\mathbf{r}) \equiv \sum_{\alpha} \hat{a}_{\alpha}^\dagger(\mathbf{r}) a_{\alpha}(\mathbf{r}),
\]

where \( \alpha \) is a multi-index encompassing all four individual indices. Using the completeness relation

\[
|\alpha, \mathbf{r}\rangle = \sum_n \sum_k \langle \Psi_{k,n} | \Psi_{k,n}^\dagger \rangle |\alpha, \mathbf{r}\rangle,
\]

for the Bloch states \( |\Psi_{k,n}\rangle \)—where the sum involves all momenta of the first moiré Brillouin zone (FBZ) and all bands \( n \) of the system—we can write, at the level of the field operators

\[
a_{\alpha}(\mathbf{r}) = \frac{1}{\sqrt{N}} \sum_n \sum_{k \in \text{FBZ}} \langle \Psi_{k,n} | \Psi_{k,n}^\dagger \rangle \, c_{k,n} \, \hat{c}_{k,n}.
\]

In this equation, we have introduced the vector notation \( \langle \Psi_{k,n} | \Psi_{k,n}^\dagger \rangle \equiv |\alpha, \mathbf{r}\rangle \langle \Psi_{k,n} | \Psi_{k,n}^\dagger \rangle \), and \( c_{k,n} \) are the electronic operators in band \( n \) with momentum \( k \).

When all bands are taken into account, Eq. (B3) is just a change of basis and, hence, exact. However, in the following we will project the theory on the single conduction band (per spin and valley), colored green in Fig. 1 (a), that hosts superconductivity; within the current notation, this subspace is associated with four values of the band index \( n \) that we will, from now on, relabel as \( n \rightarrow (\sigma, \xi) \). The projection amounts to restricting the sum on the right-hand side of Eq. (B2) to only these four values. Plugging this truncated form of Eq. (B3) into Eq. (B1), the projected electronic density operator, \( \rho(\mathbf{r}) \), reads as

\[
\rho(\mathbf{r}) = \frac{1}{N} \sum_{k,k',\sigma,\sigma',\xi,\xi'} \hat{u}_{k,\sigma,\xi}^\dagger(\mathbf{r}) \, \hat{u}_{k',\sigma',\xi'}^\dagger(\mathbf{r}) \, \hat{u}_{k',\sigma',\xi'} \, \hat{u}_{k,\sigma,\xi}.
\]

Recognizing that \( \hat{u}_{k,\sigma,\xi}^\dagger(\mathbf{r}) \, \hat{u}_{k',\sigma',\xi'}(\mathbf{r}) \propto \delta_{\sigma,\sigma'} \) and independent of \( \sigma \), the total density can be split into two components as

\[
\rho(\mathbf{r}) = \rho^{\text{intra}}(\mathbf{r}) + \rho^{\text{inter}}(\mathbf{r}), \quad \text{with}
\]

\[
\rho^{\text{intra}}(\mathbf{r}) = \frac{1}{N} \sum_{k,k',\sigma,\xi} \hat{u}_{k,\sigma,\xi}^\dagger(\mathbf{r}) \, \hat{u}_{k',\sigma,\xi}(\mathbf{r}) \, c_{k,\sigma,\xi}^\dagger \, c_{k',\sigma,\xi},
\]

\[
\rho^{\text{inter}}(\mathbf{r}) = \frac{1}{N} \sum_{k,k',\sigma,\xi} \hat{u}_{k,\sigma,\xi}^\dagger(\mathbf{r}) \, \hat{u}_{k',\sigma,\xi}(\mathbf{r}) \, c_{k,\sigma,\xi} \, c_{k',\sigma,\xi}.
\]

Within the continuum description used in this work, the overlap between the Bloch states for different \( \xi \) and, hence, \( \rho^{\text{inter}} \), vanishes identically. This eventually gives rise to the enhanced SU(2)_+ \times SU(2)_- spin-rotation symmetry mentioned in the main text. In reality, \( \rho^{\text{inter}} \) is nonzero (but small), which breaks this symmetry. However, as has been shown in Ref. 8, the behavior when this symmetry is weakly broken is completely determined by symmetry, which is why we do not have to consider it separately in this work.

Once again turning to Bloch’s theorem, we can write

\[
\hat{u}_{k,\xi}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \, \bar{\Psi}_{k,\xi}(\mathbf{r}) = e^{i\mathbf{k} \cdot \mathbf{r}} \sum_G \bar{u}_{k,\xi}(G) \, e^{iG \cdot \mathbf{r}},
\]

where \( \bar{u}_{k,\xi} \) is lattice periodic, but with the periodicity of the moiré superlattice. The Fourier transform of the density is simply

\[
\rho_q = \int d\mathbf{r} e^{-i\mathbf{q} \cdot \mathbf{r}} \rho(\mathbf{r})
\]

\[
= \frac{1}{N} \sum_{k,k'} \int d\mathbf{r} \sum_{G,G',\sigma,\xi} \left[ \bar{u}_{k,\xi}(G) \bar{u}_{k',\xi}(G') \, c_{k,\sigma,\xi}^\dagger \, c_{k',\sigma,\xi} \right.
\]

\[
\times \exp \left(i(k' - k + G' - G - \mathbf{q}) \cdot \mathbf{r}\right)
\]

\[
= \frac{1}{N} \sum_{k,k',G,\sigma,\xi} c_{k,\sigma,\xi}^\dagger \, c_{k+q-G'G,\sigma,\xi} \sum_G \bar{u}_{k,\xi}(G) \, \bar{u}_{k+q,\xi}(G'),
\]

such that \( k, k + q - G' \in \text{FBZ} \). This condition is satisfied by exactly one \( G' \), therefore allowing us to eliminate the summation over \( G' \) in the last equation above. Note that in the penultimate step, we have made use of the relation \( \bar{u}_{k+G_1G_2}(G_2) = \bar{u}_{k}(G_1 + G_2) \), which we can always enforce; this is equivalent to asserting \( \bar{u}_{k+G}(\mathbf{r}) = \bar{u}_{k}(\mathbf{r}) \forall G \). We finally have

\[
\rho_q = \frac{1}{N} \sum_{k,\sigma,\xi} c_{k,\sigma,\xi}^\dagger \, c_{k+q-G,\sigma,\xi} \, F_{k,\sigma,\xi},
\]

where we have introduced the form factors

\[
F_{k,\sigma,\xi} = \sum_G \bar{u}_{k,\xi}(G) \, \bar{u}_{k+q,\xi}(G),
\]

\[
= \sum_G \bar{u}_{k,\xi}(G) \, \bar{u}_{k+q,\xi}(G) \,(G + G_q),
\]

with \( G_q = q - \text{FBZ}(q) \), i.e., the reciprocal lattice vector which folds the wavevector \( q \) back into the first Brillouin zone. The last form in Eq. (B9) shows most clearly, why the form factors are expected to decay with large \( |q| \).

Finally, let us elucidate how the \( \bar{u}_{k} \) entering the form factors are computed from the continuum model in Appendix A. For notational simplicity, let us recast both
spin and valley blocks of Eq. (A1) in the form of a single Hamiltonian \( H_{AB-AB} = h(-i\nabla) + T(r) \), where \( h \), which is a matrix in \( \alpha \)-space, has no explicit spatial dependence whereas \( T \) encapsulates the interlayer coupling and is moiré-lattice periodic. Appealing to translational symmetry and decomposing the latter in a Fourier series as \( T(r) = \sum_G T_G \exp(iG \cdot r) \), it is straightforward to show that the Schrödinger equation \( H_{AB-AB} |\Psi_{n,k}\rangle = E_{n,k} |\Psi_{n,k}\rangle \) translates to

\[
h(k + G) \tilde{U}_{n,k}(G) + \sum_{G'} T_{G-G'} \tilde{U}_{n,k}(G') = E_{n,k} \tilde{U}_{n,k}(G)
\]

\( \forall k \in \text{FBZ} \). This provides a prescription to read off \( \tilde{U}_k \) from the eigenvectors obtained upon diagonalizing the Hamiltonian \( H_{AB-AB} \) in Eq. (A1).

On a side note, we remark that the density, \( \hat{\rho}_{q,l,s} \), in a particular layer \( l = (l, s) \), rather than the total density, can be projected in a similar fashion, \( \hat{\rho}_{q,l,s} \to \hat{\rho}_{q,l,s} \) as

\[
\hat{\rho}_{q,l,s} = \frac{1}{N} \sum_{k,\sigma,\xi} c_{k,\sigma,\xi}^{\ell} \bar{c}_{\ell,\ell}^{FBZ}(k+q),\sigma,\xi \tilde{C}_{k,q,\ell}^{l,s}(k+q,\sigma,\xi) \quad \text{(C10)}
\]

Using the expression for the densities from Eq. (B10) and the same multi-index notation, \( \ell = (l, s) \) introduced in Appendix A, we find

\[
H_{\text{int}} = \sum_{k,k',q} V_{k-k',q}^{l,l'} c_{k,\sigma,\xi}^{\ell} \bar{c}_{\ell,k',\sigma',\xi}^{l',l} c_{\ell,k',\sigma',\xi}^{FBZ}(k+q),\sigma,\xi \tilde{C}_{k,q,\ell}^{l,s}(k+q,\sigma,\xi) \tilde{C}_{k',q,\ell}^{l,s}(k+q,\sigma,\xi)
\]

with \( k, k' \in \text{FBZ} \). All repeated internal indices (\( \sigma, \sigma', \xi, \xi', \ell, \ell' \)) are summed over, and we adopt this convention for the remainder of the discussion as well. Furthermore, we omit the "FBZ" hereafter and implicitly assume that the momentum arguments of the field operators are folded back into the first Brillouin zone. Retaining only terms in the homogeneous intervalley Cooper channel \( (k = -k', \xi = -\xi') \), \( H_{\text{int}} \) reduces to

\[
H_{\text{int}}^{CC} = - \sum_{k,q} V_{k-k',q}^{l,l'} F_{k,q,\xi}^{l,l} F_{k-k',q,-\xi}^{l,l'} c_{k,\sigma,\xi}^{\ell} \bar{c}_{\ell,k,-\sigma',\xi}^{l,l} c_{\ell,k,-\sigma',\xi}^{FBZ}(k+q),\sigma,\xi \tilde{C}_{k,q,\ell}^{l,s}(k+q,\sigma,\xi) \tilde{C}_{k,-\sigma',\xi}^{l,s}(k+q,\sigma,\xi)
\]

The form factors in this channel can be simplified further: time-reversal symmetry dictates \( F_{k,q,\xi}^{l,l} = (F_{k-q,-\xi}^{l,l})^* \), wherefore

\[
H_{\text{int}}^{CC} = \sum_{k,q} C_{k,q} c_{k,\sigma,\xi}^{\ell} \bar{c}_{\ell,k,-\sigma',\xi}^{l,l} c_{\ell,k,-\sigma',\xi}^{FBZ}(k+q),\sigma,\xi \tilde{C}_{k,q,\ell}^{l,s}(k+q,\sigma,\xi)
\]

An advantage of Eq. (C3)—in deriving which, we have already summed over the valleys—is that it now suffices to compute the wavefunctions for a single valley, say, \( \xi = + \), only. For the electron-phonon coupling discussed in the main text, one finds

\[
C_{k,q} = \delta_{l,l'} \frac{D^2}{m v_{ph}^2} \sum_n \frac{\omega_n^2}{\omega_n^2 + \nu_n} \sum_{l} \sum_{s} F_{k,q,s}^{l,s} \left| F_{k,q,s}^{l,s} \right|^2
\]

whereas for the screened Coulomb interaction, \( V_{\text{RPA}}(q) \),

\[
C_{k,q} = -(V_{\text{RPA}}(q) + V_{\text{RPA}}(-q)) \left| F_{k,q,+} \right|^2
\]

with

\[
F_{k,q,\xi}^{l,s} \equiv \sum_{G} \tilde{U}_{k,\xi}^G(G) P_{l,s}^{G} \tilde{U}_{k+q,\xi}^G(G), \quad \text{(B11)}
\]

where \( P_{l,s}^{G} \) is a square diagonal matrix that projects on to layer \( (l, s) \). By design, it necessarily holds that \( F_{k,q,\xi} = \sum_{s} F_{k,q,\xi}^{l,s} = \sum_{l} F_{k,q,\xi}^{l,s} \).

**Appendix C: Linearized mean-field gap equation**

Having established the procedure for the projection of the density operators to the low-energy subspace formed by the single conduction band (per spin and valley), we are now well-positioned to construct a mean-field theory of superconductivity in TDBG. We start from the density-density interaction given by Eq. (1) of the main text and keep \( V_{l,l'}(q) \) general for now. We also emphasize that \( q \) in Eq. (1) extends over all two-dimensional momenta but \( V_{l,l'}(q) \) need not be periodic in the moiré Brillouin zone.

The mean-field decoupling proceeds by defining, as usual, the expectation value

\[
\langle \hat{\Delta}_{k} \rangle_{\sigma,\sigma'} = \langle c_{k,\sigma,+} c_{-k,\sigma',-} \rangle
\]

whereupon we get (neglecting a constant piece)

\[
H_{\text{int}}^{\text{MF}} = \sum_{k,q} C_{k,q} c_{k,\sigma,\xi}^{\ell} \bar{c}_{\ell,k,-\sigma',\xi}^{l,l} c_{\ell,k,-\sigma',\xi}^{FBZ}(k+q),\sigma,\xi \tilde{C}_{k,q,\ell}^{l,s}(k+q,\sigma,\xi)
\]

where

\[
F_{k,q,\xi}^{l,s} = \sum_{G} \tilde{U}_{k,\xi}^G(G) P_{l,s}^{G} \tilde{U}_{k+q,\xi}^G(G),\quad \text{(B11)}
\]
Compactly expressing the \( q \)-sum as
\[
\left( \Delta_k \right)_{\sigma,\sigma'} \equiv \sum_q C_{k,q} \left( \Delta_{k+q} \right)_{\sigma,\sigma'},
\]  
\( (C8) \)
the mean-field interaction is consolidated into
\[
H_{\text{int}}^{\text{MF}} = \sum_k c_k^{\dagger} \left( \Delta_k \right)_{\sigma,\sigma'} c_{k,-\sigma,-\sigma'} + \text{H.c.}
\]  
\( (C9) \)
In terms of the Nambu spinor \( \psi_{k,\sigma}^T = (c_{k,\sigma,+}, c_{k,\sigma,-}^\dagger)^T \) [9], the full Hamiltonian, including both hopping and pairing, can be knedeed into the Bogoliubov-de Gennes (BdG) [10] form:
\[
H^{\text{MF}} = \sum_k c_k^{\dagger} \xi_k c_k + H_{\text{int}}^{\text{MF}}
\]  
\( (C10) \)
\[
\xi_k = \left[ \begin{array}{cc} E_k & \delta_{\sigma,\sigma'} \\ -\delta_{\sigma,\sigma'} & -E_k \end{array} \right] \Delta_k \psi_{k,\sigma},
\]
\( \psi_{k,\sigma} = \left[ \begin{array}{c} c_{k,\sigma,+} \\ -c_{k,\sigma,-} \end{array} \right] \) taking advantage of the fact that the dispersions in the two valleys are related as \( E_{k,+} = E_{-k,-} \) as a consequence of time-reversal symmetry. All the microscopic details of TDBG are encoded in this mean-field Hamiltonian through the band energies and the form factors.
For self-consistency, Eq. (C8) is required to hold, where the expectation value on the right-hand side [see Eq. (C6)] is calculated within \( H^{\text{MF}} \). To linear order in \( \Delta \), we have
\[
\langle c_{k,\sigma,+} c_{-k,-\sigma'} \rangle = -T \sum_{\omega_n} \left[ \left( \omega_n - E_{\text{BdG}}(k) \right)^{-1} \right]_{\sigma,\sigma'}^{1,2},
\]
the numbers 1, 2 denote the indices in Nambu space while \( \sigma \) and \( \sigma' \) are spin indices. The BdG Green’s function is given by \( G_{\text{BdG}}^{-1} = [\Sigma^{\text{sc}} - i\omega_n] \sim \psi_0 + c^\dagger_0 \Sigma^{\text{sc}} \psi_0 \), where
\[
\Sigma^{\text{sc}} = \left[ \begin{array}{cc} E_{k,+} & 0 \\ 0 & -E_{k,+} \end{array} \right],
\]  
\( (C11) \)
\[
\Delta_k = \left[ \begin{array}{cc} \left( \Delta_k \right)_{\sigma,\sigma'} \\ 0 \end{array} \right].
\]  
\( (C12) \)
Accordingly,
\[
\langle c_{k,\sigma,+} c_{-k,-\sigma'} \rangle = \sum_{\omega_n} \left( \omega_n - E_{k,+} \right) T \left( \omega_n + E_{k,+} \right) \left( \Delta_k \right)_{\sigma,\sigma'}
\]
\[= \frac{1}{2 E_{k,+}} \tanh \left( \frac{E_{k,+}}{2T} \right) \left( \Delta_k \right)_{\sigma,\sigma'}, \]
\( (C13) \)
and Eq. (C8) becomes
\[
\left( \Delta_k \right)_{\sigma,\sigma'} = \sum_{k'} \frac{\nu_{k,k'}}{2 E_{k,+}} \tanh \left( \frac{E_{k,+}}{2T} \right) \left( \Delta_{k'} \right)_{\sigma,\sigma'},
\]  
\( (C14) \)
defining \( \nu_{k,k'} \equiv \xi_{k,k'} \). It is easy to observe that
\[
\nu_{k,k'} = \nu_{k',k},
\]  
\( (C15) \)
\[
\nu_{k,k'} = \nu_{k+G,k'+G}.
\]  
\( (C16) \)
In Eq. (C14), we now see explicitly that singlet and triplet are degenerate as expected due to the presence of the SO(4) \( \cong SU(2) \times SU(2) \) symmetry of independent spin rotations in each valley [11]. This helps us reduce the matrix equation Eq. (C13) to a scalar equation by focusing, say, on singlet pairing where \( \left( \Delta_k \right)_{\sigma,\sigma'} \equiv \Delta_k \left( i\sigma_y \right)_{\sigma,\sigma'} \), resulting in the same eigenvalue equation
\[
\Delta_k = \sum_{k'} M_{k,k'}(T) \Delta_{k'}, \quad k \in \text{FBZ}
\]  
\( (C17) \)
with a kernel in momentum space given by
\[
M_{k,k'} = \nu_{k,k'} \tanh \left( \frac{E_{k,+}}{2T} \right).
\]  
\( (C18) \)
Obviously, due to Eq. (C15), \( \nu \) is Hermitian (it is, in our case, real and symmetric) which guarantees that \( M \) only has real eigenvalues. In general, the sum over \( k' \) in Eq. (C17) involves arbitrarily large momenta. However, due to the decay of the form factors with large momentum transfer \( q = k' - k \) in Eq. (C3), we restrict the sum in Eq. (C14) to \( k \) and \( k' \) in the FBZ.
At sufficiently large \( T \), \( M_{k,k'}(T) \sim 1/T \) and Eq. (C17) does not have a solution. If, below a critical temperature \( T_c \), a solution exists for the order parameter, its structure in momentum space is, of course, interaction-dependent: when \( V_{k'-k} > 0 \) (\(< 0 \), \( i.e. \), the interaction is repulsive (attractive), \( \Delta_k \) and \( \Delta_{k'} \) are favored to have the opposite (same) sign.

**Appendix D: Singlet vs. triplet**

In this appendix, we discuss in more detail the possibility that the dominant \( SU(2)_+ \times SU(2)_- \)-preserving pairing interactions are provided by phonons while those breaking this enhanced spin symmetry—down to the usual total \( SU(2) \) spin symmetry—are dominated by purely electronic physics. Specifically, we show that triplet (singlet) pairing is generically favored if the latter class of interactions are dominated by time-reversal odd (even) particle-hole fluctuations.
To this end, let us assume that, at the energy scales relevant to superconductivity, there is a set of collective electronic modes, \( \phi_j^\alpha, j = 1, 2, \ldots, N_b \), that dominates the part of the pairing mechanism that breaks the \( SU(2)_+ \times SU(2)_- \) symmetry (and, thus, determines whether singlet or triplet will win). As we do not want to make any assumptions about its microscopic form, we take the general form of the coupling to the electrons,
\[
H_{c0} = \sum_{k,q} c_{k+q,\alpha}^\dagger \phi_j^\alpha (k+q,k) c_{k,\beta} \phi_j^\beta,
\]  
\( (D1) \)
that can be momentum dependent and involve arbitrary matrix elements between the different internal degrees of freedom, captured by the multi-indices $\alpha, \beta$ (e.g., spin, valley, different bands, etc.). As we will see below, it is sufficient to specify the behavior of $\phi_q^\dagger$ under time-reversal symmetry: we will focus on all of these modes being time-reversal even, $t_\phi = +$, or odd, $t_\phi = -$, corresponding to

$$\hat{\Theta} \phi_q^\dagger \hat{\Theta} \dagger = t_\phi \phi_{-q}, \ \Theta \lambda(-k, -k') \Theta \dagger = t_\phi \lambda(k, k'),$$

(D2)

where $\hat{\Theta}$ and $\Theta$ are the anti-unitary time-reversal operators in Fock and single-particle space, respectively.

For the remainder of this section, we use the field-integral description and denote the associated field operators of the electrons and collective bosonic modes by the same symbols, $c_{k, \alpha}$ and $\phi_q^\dagger$, employing the combined Matsubara-frequency and momentum notation, $k \equiv (i\omega_n, k)$ and $q \equiv (i\nu_n, q)$; here $\omega_n$ and $\nu_n$ are fermionic and bosonic Matsubara frequencies, respectively. The effective low-energy dynamics of the collective bosonic modes is described by

$$S_\phi = \frac{1}{2} \int \phi_q^\dagger \chi^{-1} (i\nu_n, q) \chi (i\nu_n, -q) \phi_{-q}^\dagger,$$

(D3)

where $\int \cdot \cdots \equiv \int \sum_n \sum_q \cdots$ and $\chi^{-1} (i\nu_n, q)$ is the (full) susceptibility in the corresponding particle-hole channels. It can be shown that [12]

$$\chi (i\nu_n, q) = \chi^T (-i\nu_n, -q),$$

(D4)

$$\chi (i\nu_n, q) = \chi^T (i\nu_n, q).$$

(D5)

By integrating out the bosonic modes $\phi_q^\dagger$, we obtain an effective action $S^f = S^f_0 + S^f_{\text{int}}$ with free contribution $S^f_0 = \int k f_{k\xi} f_{-k\xi} (-i\omega_n + E_{k\xi}) f_{k\xi}$ and electron-electron interactions described by $S^f_{\text{int}}$. The relevant Cooper channel of $S^f_{\text{int}}$ can be written as

$$S^f_{cc} = \int \int \int \int \int \chi^T (i\nu_n, k', k) \chi (i\nu_n, k, k') \chi (i\nu_n, k', k)$

(D8)

where $V_{\sigma \sigma_1} \sigma_2 (k', \xi'; k, \xi) = W_{\sigma \sigma_1} \sigma_2 (k', \xi'; k, \xi) + t_\phi \Delta V_{\sigma \sigma_1} \sigma_2 (k', \xi'; k, \xi)$ is the sum of the SU(2)$_+ \times$ SU(2)$_-$-preserving interactions ($W$) and the additional contribution ($\Delta V$) from the collective bosonic modes. Making use of Eq. (D2) and the fact that time-reversal symmetry implies $\Theta \psi_{k\xi} = s_\sigma \psi_{-k, -\xi}$, where $s_\uparrow = +$, $s_\downarrow = -$ and $\uparrow = \downarrow$, $\downarrow = \uparrow$, the latter can be rewritten as

$$\Delta V_{\sigma \sigma_1} \sigma_2 (k', \xi'; k, \xi) = -\frac{1}{2} \sigma_2 \sigma_3 \lambda^{\xi}_{\sigma_1} \chi (k' - k) \lambda^{\xi}_{\sigma_1, \xi} (k', k),$$

(D9)

introducing the shorthand

$$\lambda^{\xi}_{\sigma_1, \xi} (k', k) = \psi_{k\xi}^\dagger \lambda (k, k') \psi_{k'\xi}.$$

Next, we decouple, as usual, the interaction in Eq. (D8) in the Cooper channel with the help of the Hubbard-Stratonovich fields $\Delta^{\xi}_{\sigma_1, \xi} (k)$ and $\overline{\Delta^{\xi}_{\sigma_1, \xi}} (k)$. To linear order in these fields, the saddle-point equation with respect to

$$\overline{\Delta^{\xi}_{\sigma_1, \xi}} (k) = -\int_{k'} \frac{V_{\sigma \sigma_1} \sigma_2 (k, \xi; k', \xi')}{\omega_{\sigma_1}^\xi + E_{k'\xi}} \Delta^{\xi}_{\sigma_1, \xi'} (k').$$

(D10)

As the system has SU(2) spin-rotation symmetry and

$$S_0 = \int_k c_{k, \alpha}^\dagger (-i\omega_n + h_{\alpha\beta}(k)) c_{k, \beta}.$$

(D6)

the fermion-boson coupling, $S_{c\phi}$, analogous to Eq. (D1), and the contribution $S_{\text{int}}$ describing all other interactions preserving the SU(2)$_+ \times$ SU(2)$_-$ symmetry [such as those in Eq. (C1)].

To study superconductivity, we use the following low-energy description. We diagonalize the free Hamiltonian,

$$h(k) \psi_{k\xi\sigma} = E_{k\xi} \psi_{k\xi\sigma},$$

(D7)

where we have already taken into account that spin-orbit coupling can be neglected (spin $\sigma$ is a good quantum number) and further ignored any intervalley mixing on the level of the noninteracting Hamiltonian $h(k)$. These mixing terms are very small leading to the approximate valley-charge-conservation symmetry, $U(1)_v$, which allows us to focus on intervalley pairing [8]. Breaking $U(1)_v$ weakly (as is the case for the real system) will generate a small admixture of intravalley pairing, which is, however, not of interest to our discussion here. In Eq. (D7) and the following, we concentrate on the eigenstates that give rise to Fermi surfaces and project the electronic fields according to $c_{k\alpha} \rightarrow \sum_{\xi, \xi} \psi_{k\xi\sigma}(\xi) n_{k\xi\sigma}$. 
the enhanced SU(2)− × SU(2)− symmetry is broken, we know that we can discuss spin singlet and triplet separately. Beginning with the former, we write \[ \tilde{\Delta}_k^\xi = \int_{k'} \frac{F_{\xi k, k'}^0 + t_n F_{\xi k, k'} }{(\omega_n^2 + E_{k'}^2)^{1/2} (\omega_n^2 + E_{k'}^2)^{1/2}} \Delta_{k'}^\xi, \] (D11)

where we introduced \[ \tilde{\Delta}_k^\xi = \frac{\Delta_k^\xi}{(\omega_n^2 + E_{k}^2)^{1/2}} \] (to make the kernel Hermitian). \[ F_{\xi k, k'}^0 \] is related to the SU(2)− × SU(2)−-symmetric \( W \), while the contribution of the collective electronic modes, \( \Delta \) in Eq. (D9), is contained in

\[ F_{\xi k, k'} = \frac{1}{2} \sum_{\sigma, \sigma'} \Lambda_{\xi, \sigma, \xi', \sigma'}(k, k') [\chi(k-k')]_{j, j'} \Lambda_{\sigma, \xi', \sigma'}^*(k, k'). \] (D12)

Note that, due to Eq. (D5), \( \chi(q) \) has real eigenvalues, all of which must be positive as required by stability; hence, we find \( F_{\xi k, k'} > 0 \). Furthermore, \( F^* \) is symmetric, \( F_{\xi k, k'} = F_{\xi k', k}^* \), as follows from Eq. (D4). Like in the main text, we view the saddle-point equation as a matrix equation, \( \Delta \lambda(T) = \mathcal{M}(T) \Delta \) with a Hermitian matrix \( \mathcal{M}(T) \): upon decreasing temperature, the largest eigenvalue \( \lambda(T) \) increases and reaches 1 at \( T_c \). Denoting the eigenvalue when setting \( F_{\xi k, k'}^* \rightarrow 0 \) by \( \lambda_0(T) \), and treating \( F_{\xi k, k'} \) as a perturbation yields \( \lambda(T) \sim \lambda_0(T) + t_\phi \delta \lambda_\phi \) at leading order, where

\[ \delta \lambda_\phi(T) = \int_k \int_{k'} \frac{(\tilde{\Delta}_k^\xi)^* F_{\xi k, k'}^0 \tilde{\Delta}_{k'}^\xi}{(\omega_n^2 + E_{k}^2)^{1/2} (\omega_n^2 + E_{k'}^2)^{1/2}}. \] (D13)

Similarly, for triplet, with, say, \( \Delta_{\xi, \sigma, \sigma'}^\xi(k) = \delta_{\sigma, \sigma'} \Delta_{\xi}^\xi \), \( \Delta_{\xi}^\xi = -\tilde{\Delta}_k^\xi \), we find the same equations as above, only with \( F_{\xi k, k'}^0 \) replaced by \( F_{\xi k', k}^0 \).

Most importantly, the eigenvalue will behave as \( \lambda(T) \sim \lambda_0(T) + t_\phi \delta \lambda_\phi(T) \), where \( \delta \lambda_\phi(T) \) is given by Eq. (D13) but with \( F_{\xi k, k'}^0 \rightarrow F_{\xi k', k}^0 \). Since \( \tilde{\Delta}_k^\xi > 0 \) (without loss of generality) if electron-phonon coupling dominates \( F_0 \), as proven in Ref. 12 and can, for example, be seen in Fig. 2(d), we conclude that \( \delta \lambda_\phi(T) > \delta \lambda_\phi(T) \). Therefore, the eigenvalue of singlet (triplet) will be larger and, hence, singlet (triplet) will dominate for \( t_\phi = + \) (\( t_\phi = - \)).

**Appendix E: Disorder sensitivity**

In Ref. [13], a general expression for the disorder-induced suppression, \( \delta T_c = T_c - T_{c,0} \), of the transition temperature \( T_c \) of a superconductor with respect to its clean value \( T_{c,0} \) was derived: for weak scattering and/or low disorder configurations, \( \tau^{-1} \rightarrow 0 \), it holds

\[ \delta T_c \sim -\frac{\pi}{4} \tau^{-1} \zeta. \] (E1)

All nonuniversal details of the system and the impurity potential are encoded in the sensitivity parameter \( \zeta \) that can be written as a Fermi-surface average of the trace \( \text{tr}[C_{k, k'}^0 C_{k, k'}] \) where \( C_{k, k'} \) is the (anti)commutator

\[ C_{k, k'} = \tilde{\Delta}_k \mathbb{T} W_{k, k'} - t_W W_{k, k'} \tilde{\Delta}_{k'} \mathbb{T}. \] (E2)

Here, \( \mathbb{T} \) is the unitary part of the time-reversal operator (in our case \( i\sigma_2 \tau_1 \) with Pauli matrices \( \sigma_j \) and \( \tau_j \) in spin- and valley-space, respectively), \( W_{k, k'} \) is the Fourier transform of the impurity potential (in our low-energy description, a matrix in spin and valley space), and \( \tilde{\Delta}_k \) is the superconducting order parameter (also a matrix in the same space). Furthermore, \( t_W = -1 \) \((t_W = +1)\) for (non)magnetic disorder. With regard to the \( A \) pairing states, we know

\[ \tilde{\Delta}_{k} \mathbb{T} = \Delta_{k}^A \sigma_0 \tau_0 \] (E3)

for singlet pairing and

\[ \tilde{\Delta}_{k} \mathbb{T} = \Delta_{k}^A \sigma \cdot d \tau_3 \] (E4)

in the case of triplet, where \( \Delta_{k}^A \) are the real-valued basis functions sketched in Fig. 2(d) and Fig. 4(a) of the main text. To be concrete, let us focus on spinless, nonmagnetic \((t_W = +1)\) local impurities, \( i.e., W_{k, k'} = \sigma_0 (w_1 \tau_0 + w_2 \tau_1 + w_3 \tau_2), w_j \in \mathbb{R} \), normalized such that \( \sum_j w_j^2 = 1 \); as in Ref. [13], the normalization is chosen so as to ensure that spin-magnetic disorder, \( W_{k, k'} = \sigma_3 (w_1 \tau_0 + w_2 \tau_1 + w_3 \tau_2) \), yields \( \zeta = 1 \).

The sensitivity parameter can be written as \[ \zeta = \left[ \frac{\sum_{k, k'} \mathbb{F}_k \mathbb{T}}{\text{tr}(\Delta_{k}^A \mathbb{T})} \right] \frac{\text{tr}[C_{k, k'}^0 C_{k, k'}]}{4 \sum_{k, k'} \text{tr}[\Delta_{k}^A \Delta_{k}^A]}, \] (E5)

where \( \sum_{k, k'} \mathbb{F}_k \mathbb{T} \) is an average of a function \( F_k \) over the Fermi surfaces of the system. After some algebra, we are led to

\[ \zeta = \frac{\langle |\Delta_{k}^A|^2 \rangle_{FS} - \langle |\Delta_{k}^A|^2 \rangle_{FS}}{2 \langle |\Delta_{k}^A|^2 \rangle_{FS}} \] (E6)

for the singlet state in Eq. (E3). We see that, independent of the ratio of intra- \((w_1)\) to intervalley \((w_2 \text{ and } w_3)\) scattering, the sensitivity of the singlet state is just the variance of the basis function \( \Delta_{k}^A \) on the Fermi surface. For the basis function of the conventional pairing state illustrated in the first panel of Fig. 2(d) of the main text, the variance is small, so \( \zeta \approx 1 \). Contrarily, despite transforming under the trivial representation, the order
parameter in Fig. 4(a) has many sign changes and hence, a small value of $\langle \Delta_A^k \rangle_{FS}$ compared to $|\langle |\Delta_A^k \rangle_{FS}|$. This is why $\zeta$ is close to the maximally possible value $\zeta = 1/2$ in Eq. (E6) that is typically associated only with nontrivial representations, where $\langle \Delta_A^k \rangle_{FS} = 0$ by symmetry.

For completeness, we also discuss the triplet state with the order parameter in Eq. (E4). From Eq. (E5), we find,

$$\zeta = \frac{\langle |\Delta_A^k \rangle_{FS}^2 - (w_1^2 - w_2^2 - w_3^2) \rangle \langle \Delta_A^k \rangle_{FS}^2}{2 \langle |\Delta_A^k \rangle_{FS}^2 \rangle}.$$  \hspace{1cm} (E7)

Consequently, if intravalley scattering dominates, i.e., $w_1 \gg w_{2,3}$, the triplet and singlet states behave identically. Intuitively, this results from the fact that the relative sign change of the triplet order parameter between the two valleys is irrelevant if intervalley scattering is weak. Note that the values of $w_{1,2,3}$ parameterizing the impurity potential are immaterial for the basis function in Fig. 4(a) for which, $|\langle |\Delta_A^k \rangle_{FS}| \ll \langle |\Delta_A^k \rangle_{FS} \rangle$, so $\zeta \approx 0.5$, as in the singlet case. The only difference between singlet and triplet arises for the phonon-mediated state, where $\langle |\Delta_A^k \rangle_{FS} \rangle \approx \sqrt{\langle |\Delta_A^k \rangle_{FS} \rangle^2}$, leading to $\zeta \approx w_2^2 + w_3^2$. Here, the stability of superconductivity is determined by the ratio of intra- to intervalley pairing.

While, in general, impurities can mediate scattering between different valleys, the intervalley scattering amplitudes will be suppressed as long as the impurity potential only varies on the length scales of the moiré lattice, but is smooth on the atomic scale. To see this explicitly, let us start from the tight-binding description of the individual graphene layers. We denote the annihilation operator of an electron on Bravais lattice site $R$, sublattice $\tau$, spin $\sigma$, and layer $\ell$ by $a_{R,\sigma,\ell,\tau}^\dagger$ and consider spinless impurities of the general form

$$H_{\text{imp}} = \sum_R a_{R,\sigma,\ell,\tau}^\dagger \tilde{v}_{\tau\tau'}^{\ell\ell'}(R) a_{R,\sigma,\ell',\tau'}.$$ \hspace{1cm} (E8)

In order to connect to the continuum-model operators, e.g., in Eq. (B1), we first Fourier transform to get

$$H_{\text{imp}} = \sum_{k,q} a_k^{\dagger,\ell\ell'} \tilde{v}_{\tau\tau'}^{\ell\ell'}(q) a_{k+q,\sigma,\ell',\tau'},$$

where $\tilde{v}_{\tau\tau'}^{\ell\ell'}(q) = N^{-1} \sum_{R} v_{\tau\tau'}^{\ell\ell'}(R) e^{iR \cdot q}$ is the Fourier transform of the impurity potential. Clearly, if the potential of the local perturbation is smooth on the atomic length scales, $\tilde{v}_{\tau\tau'}^{\ell\ell'}(q)$ with $q$ connecting the $K$ and $K'$ points (in the original Brillouin zone of graphene) is negligibly small. Transforming back to real space, Eq. (E9) can, thus, be approximately written as

$$H_{\text{imp}} \approx \int dR a_{\sigma,\ell,\tau,\xi}^\dagger \tilde{v}_{\tau\tau'}^{\ell\ell'}(\xi; R) a_{\sigma,\ell',\tau',\xi}(R),$$ \hspace{1cm} (E10)

in the continuum description. Most importantly, it is diagonal in valley space as mentioned in the main text. This property will remain unchanged upon projection onto the bands of the system (see Appendix B).

We finally note that the resulting protection of the triplet state against this type of disorder is not altered by the fact that, in general, the impurity potential will be momentum dependent, $W_{k,k'} = \tau \sigma_0 f_{k,k'}$ with some arbitrary real scalar function $f$: as long as the basis function $\Delta_A^k$ is (almost) momentum independent, the commutator in Eq. (E2) will vanish (approximately), wherefore $\zeta = 0$ ($\zeta \ll 1$).

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