Scaling dimension of quantum Hall quasiparticles from tunneling current noise measurements

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Determination of properties of quasiparticle excitations is an important task in the experimental investigation of the fractional quantum Hall effect (FQHE). We propose a model-independent method for finding the scaling dimension of FQHE quasiparticles from measurements of the electric current tunneling between two FQHE edges and its noise. In comparison to the commonly used method based on measuring the tunneling current only, the proposed method is less prone to the errors due to non-universal physics of tunnel junctions.

\section*{INTRODUCTION}

The fractional quantum Hall effect (FQHE) has long been known to occur due to electrons forming a strongly correlated topologically ordered state \cite{affleck1988}. While the bulk of this state has a gap, gapless excitations are always present at the FQHE edge. The excitation spectrum and dynamical properties of these edge modes can be encoded in an effective low-energy theory. Such theories, called Chiral Luttinger Liquids (CLL) \cite{affleck1989}, provide a powerful theoretical framework for the description of the fractional quantum Hall effect (FQHE). However, for a given filling factor $\nu$ there may exist several candidate theories predicting the same value of the Hall conductance, but possessing different excitation spectra (e.g., they may differ by whether non-Abelian quasiparticles are present). In such a situation, the foremost task in the investigation of the FQHE state is to discriminate between the candidate theories.

An important characteristic of an edge theory is the spectrum of local quasiparticle excitations. Each quasiparticle is characterized by several quantum numbers, of which two are important for the present paper: the electric charge and the scaling dimension. These quantum numbers can, in principle, be determined in experiments involving tunneling of quasiparticles between two FQHE edges. In this article we discuss weak quasiparticle tunneling through the FQHE bulk in a quantum point contact (QPC). In this case the quasiparticle with the smallest scaling dimension (the most relevant quasiparticle) gives the most important contribution to transport. One can hope to extract the charge and scaling dimension of the particle from transport measurements in such a system. Even such a limited amount of data as the properties of the most relevant quasiparticle can significantly reduce the number of candidate theories. This can be seen, for example, from the theoretical study of Ref. \cite{affleck2000}, relating to $\nu = 5/2$.

It is, in principle, possible to extract the charge and the scaling dimension from the tunneling current measurements only (see the experimental work of Refs. \cite{affleck1986} and references to theory therein). Though, it is well known (see e.g. \cite{affleck1986} and references therein) that the tunneling amplitudes in electrostatically confined QPCs strongly depend on the applied bias voltage in an unknown non-universal way, probably due to charging effects. Thus, the charge and scaling dimension extracted from such measurements are prone to large systematic errors. Even in the simplest FQHE case of $\nu = 1/3$ experimental and theoretical curves agree only qualitatively but not quantitatively (see e.g. Ref. \cite{affleck1986}).\textsuperscript{1}

It has been shown in our previous work \cite{affleck2000} that considering the ratio of the tunneling current noise to the tunneling rate allows one to exclude the unwanted non-universal dependence in the weak tunneling regime in the case of a single quasiparticle type tunneling. The fractional charge of the most relevant quasiparticle for $\nu = 1/3$ was first confirmed \cite{affleck1999} by methods essentially equivalent to the analysis of the noise to tunneling rate ratio. In this paper we focus on the possibility to extract the scaling dimension of the most relevant quasiparticle from such data, paying particular attention to the $\nu = 1/3$ case as the simplest one.

\section{Scaling Dimension from the Noise to Tunneling Rate Ratio}

We consider the following experimental setup (see Fig.\textsuperscript{1}). There are two quantum Hall edges, each supporting the same set of excitation modes. By an excitation mode we mean a channel in which long-lived excitations propagate in one direction with the same velocity. We call the propagation direction of a mode its chirality. The set of excitation modes includes a charge carrying mode or, possibly, several such modes, all having the same chirality, and some (possibly, none) neutral modes (that is the modes that do not carry electric charge). Neutral modes can have different chiralities. The two edges are far apart from each other except for the quantum point

\textsuperscript{1} Moreover, in the case of $\nu = 1$ the experimental curves also deviate from the behaviour one would expect theoretically \cite{affleck1986}. Ref. \cite{affleck1986} explains this by emergence of isles of fractional QHE in the QPC region. However, phenomenologically one can interpret this as the bias voltage dependence of the tunneling amplitudes that determine tunneling between the $\nu = 1$ edges.
contact region where they come close to each other and quasiparticle tunneling processes take place. Yellow rectangles are the Ohmic contacts, which absorb everything that flows into them. Contacts Ground 1 and Ground 2 are grounded. Contact Source S is used to inject electric current $I_s$ into the lower edge, and contact Voltage probe is used to measure the electric current flowing into it and the current noise at zero frequency. All components of the system have absolute temperature $T_0$.

In Ref. [1] we developed a framework that allows one to deal with such experiments. There it was primarily developed for the simple case of an Abelian model of $\nu = 2/3$ edge, however, it can be readily generalized for a wide class of typical Abelian and non-Abelian quantum Hall effect (QHE) edge models.\(^2\) Two remarks are due here. First, the edges support different types of quasiparticles, each characterized by the quasiparticle charge $Q$, scaling dimension $\delta$, and, possibly, some other quantum numbers. Second, only the quasiparticles with the smallest scaling dimension give a significant contribution to the tunneling processes. In the following we label such quasiparticle types by $i = 1, \ldots, n$, with the quasiparticle electric charges being $Q_i$ (in the units of the elementary charge $e$) and their common scaling dimension being $\delta_i = \delta$.

Applying the framework to the experimental setup described above, one can find the tunneling rate $r$ (absolute value of the ratio of the electric current tunneling through QPC and the current $I_s$ injected into the lower edge) and the excess noise at zero frequency $\tilde{S}(\omega = 0)$ (which is the electric current noise measured at Voltage probe less the noise value at $I_s = 0$):

$$r = \frac{4e^2(\pi T_0)^{4\delta - 1}}{h^{4\delta + 1}} \sum_i \theta_i G_i,$$

$$\tilde{S}(0) = \frac{4e^2(\pi T_0)^{4\delta - 1}}{h^{4\delta + 1}} \sum_i \theta_i F_i,$$

$$G_i = \sin 2\pi \delta \int_0^\infty dt Q_i \sin Q_i j_s t \frac{1}{(\sinh t)^{4\delta}},$$

$$F_i = F^{\text{TT}}_i \cos 2\pi \delta - \frac{2}{\pi} F^{\text{0T}}_i \sin 2\pi \delta,$$

$$F^{\text{TT}}_i = Q_i^2 \lim_{\epsilon \to 0} \frac{e^{1-4\delta}}{1-4\delta} + \frac{\infty}{\epsilon} \int dt \cos Q_i j_s t \frac{1}{(\sinh t)^{4\delta}},$$

$$F^{\text{0T}}_i = \int_0^\infty dt Q_i^2 \cos Q_i j_s t \frac{1}{(\sinh t)^{4\delta}},$$

$$j_s = \frac{I_s}{I_0}, \quad I_0 = \nu \frac{e}{h} \pi k_B T_0,$$

where $T_0$ is the system temperature, $e$ is the elementary charge, $h = 2\pi \hbar$ is the Planck constant, $k_B$ is the Boltzmann constant, $\nu$ is the filling factor, and $\theta_i \propto |\eta_i|^2$, where $\eta_i$ is the $i$-th quasiparticle tunneling amplitude. The proportionality coefficient between $\theta_i$ and $|\eta_i|^2$ can be expressed in terms of the propagation velocities of all the edge modes. The formulae (1), (2), (3), (4), (5), (6), (7) are correct for $\delta < 1/2$, for $\delta \geq 1/2$ they should be modified.

In practice, the tunneling amplitudes, and therefore the parameters $\theta_i$, have some unknown non-universal dependence on the current $I_s$ [6][10], which complicates a comparison of experimental data with the theory. However, consideration of the ratio of the excess noise to the tunneling rate $(\text{N/TRR})$

$$X(I_s) = \frac{\tilde{S}(0)}{r} = eI_s \frac{\sum_i \theta_i F_i}{\sum_i \theta_i G_i},$$

allows to exclude the unwanted non-universal dependence in the case of one quasiparticle type dominating tunneling and reduce its influence in the case when several quasiparticles participate in the tunneling processes.

Consider the large-$I_s$ limit of Eq. (8). For $|I_s| \gg I_0$ one gets\(^3\)

$$X(I_s) \big|_{|j_s| > 1} = eI_s \sum_i \theta_i F_i \sum_i \theta_i G_i$$

$$= eI_s \sum_i \theta_i Q_i^{4\delta + 1} + eI_0 \frac{2 - 8\delta}{\pi} + O(|j_s|^{-1}).$$

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\(^2\) See Appendix [A] for a detailed discussion of this issue.

\(^3\) See Appendix [B] for derivation.
This is the key result of the present paper. Unlike Eqs. [3], [5], [6], it can be shown to be valid for any \( \delta > 0 \).

The leading term of the asymptotic behaviour [9] gives the well-known result that in the regime of weak tunneling the gradient of the noise to tunneling rate ratio is equal to the tunneling quasiparticle’s charge. In our case it is some average of the charges in the case of several quasiparticles participating in tunneling. Note the sub-leading term: constant offset contains information about the quasiparticles’ scaling dimension. It is important that all the quasiparticles which significantly contribute to tunneling have the same scaling dimension.

Thus, in principle, by fitting large-\( I_s \) experimental data with a linear function one can find not only the “effective charge” of the tunneling quasiparticles but also their scaling dimension (which is the same for all of the most relevant quasiparticles). However, in practice there are some restrictions to the usability of this approach. They are discussed below.

### 1.1. What experimental conditions are necessary for successful extraction of the scaling dimension?

Now we discuss the possibility to extract the scaling dimension from real experimental data. The result [9] shows that it is, in principle, possible to extract the scaling dimension of the tunneling quasiparticles from experimental data on noise to tunneling rate ratio without knowing fully the specific edge theory. However, there are few practical aspects which should be discussed.

First of all, the parameters \( \theta_i \) related to the quasiparticles’ tunneling amplitudes depend on the current \( I_s \) in a non-universal way. Therefore, Eq. [9] is not useful in the case of several different quasiparticle charges as the gradient of the leading term depends on \( I_s \). From now on we concentrate on the case when all the charges of the quasiparticles contributing to tunneling are equal: \( Q_i = Q \). Examples include the states of Laughlin series, the Moore-Read Pfaffian, Jain’s \( \nu = 2/5 \) state etc. Then, independently of \( \theta_i \),

\[
X(I_s)|_{|j_s|>1} = eQ|I_s| + eI_0 \frac{2 - 8\delta}{\pi} + O(|j_s|^{-1}). \tag{10}
\]

Let us note that in this case it is possible to write a simple analytic expression for the NtTRR [5] (not just the large-\( I_s \) asymptote):

\[
X(I_s)|_{Q_i=Q} = \frac{2eQI_s}{\pi} \text{Im} \left[ \psi \left( 2\delta + \frac{iQj_s}{2} \right) \right], \tag{11}
\]

where the digamma function \( \psi(x) = (\ln \Gamma(x))' \) is the logarithmic derivative of the Euler gamma function \( \Gamma(x) \), and \( \text{Im} [...] \) denotes taking of the imaginary part. Alongside the asymptotic expression [10], the full expression [11] can also be used to extract the scaling dimension \( \delta \) from experimental data.

The second issue is that the dynamics of the system changes near a characteristic energy scale in the FQHE system. Namely, there is a bulk gap \( \Delta \). As the typical energies of the system exceed \( \Delta \) bulk dynamics starts being involved. Thus, one should restrict oneself to

\[
|I_s| \lesssim \nu \frac{e}{\hbar} \pi \Delta. \tag{12}
\]

Deviations from our theory can be expected beyond this threshold.

The third issue is the lower validity bound for the asymptotic expression [10]. Fig. 2 shows the comparison of the NtTRR [11] against its asymptotic behaviour [10] for \( Q = 1/3 \) and \( \delta = 1/6 \). These parameters correspond to the most relevant quasiparticle of the simplest \( \nu = 1/3 \) edge model. As one can see, for \( |I_s| \gtrsim 3I_0 \) the exact NtTRR and its large-\( I_s \) asymptote almost coincide.

To estimate how close the asymptote and the original curve are we have done some fitting. Namely, we took part of a part of the original curve with \( |I_s| \) between \( \alpha I_0 \) and \( 10I_0 \) and fitted it with [10] using \( Q \) and \( \delta \) as fitting parameters. For \( \alpha \geq 3 \) the fitted charge and scaling dimension deviate from their correct values by less than 1% and 11% respectively. This gives an idea of how accurate the estimates of \( Q \) and \( \delta \) obtained from fitting

4 Derivation is given in Appendix [C]

Figure 2: Noise to tunneling rate ratio vs its asymptotic behaviour. (Color online). The green solid curve is the NtTRR [11] for \( Q = 1/3, \delta = 1/6 \). The cyan dashed curve is the large-\( I_s \) asymptote of the NtTRR [10] for the same values of \( Q \) and \( \delta \). The asymptote almost coincides with the original curve for \( |I_s| \gtrsim 3I_0 \).
experimental data with formula (10) can be if there are no other sources of errors.

Thus, one can use the asymptotic expression (10) for $|I_s| \gtrsim \alpha I_0$, where $\alpha$ is on the order of 1. The exact value of the multiplier $\alpha$ depends on the values of $Q$ and $\delta$. Of course, this issue does not arise if one uses the full expression (11).

Note that the greater is $I_0$ the more significant is the term containing the scaling dimension in Eq. (10). At the same time, the less is the interval $\nu^{2/3} \pi \Delta \gtrsim |I_s| \gtrsim \alpha I_0$. Thus, the choice of the system temperature should be a matter of trade-off between these to restrictions in order to allow as good determining of the scaling dimension $\delta$ as possible.

Fourthly. The expressions (10), (11) are valid only when the contribution of less relevant quasiparticles (with greater scaling dimensions) to the tunneling processes can be neglected. Otherwise the corrections due to less relevant quasiparticles can hinder finding the scaling dimension using the large-$I_s$ NtTRR behaviour. Unfortunately, there are no known reliable ways to estimate theoretically how significant these corrections are. However, general theoretical arguments, as well as recent Monte Carlo simulations [15], show that the tunneling amplitude of a quasiparticle with scaling dimension $\delta$ is proportional to $(L/l_B)^{-2\delta}$, where $l_B$ is the magnetic length, and $L$ is the edge length. The typical experimental values of these parameters are $l_B \approx 10$ nm, $L \approx 10$ $\mu$m, suggesting that the contribution of less relevant quasiparticles can usually be neglected. However, to be on the safe side, one can estimate them in practice by comparing experimental data with different possible theoretical answers for NtTRR (the answers including and not including less relevant quasiparticles).

Fifth issue is related to measurement errors. Scaling dimension enters Eq. (10) as a subleading term. Thus, finding the scaling dimension demands a very high quality experimental data with very small statistical errors. The NtTRR errors can be made less significant by using greater values of the tunneling rate. This, however, worsens the accuracy of theoretical result (10) which was derived perturbatively in the limit of small tunneling rate. Therefore, the choice of the strength of tunneling in experimental data should be balanced between worsening the applicability of the theory and improving the quality of data for NtTRR.

The latter observation brings up the sixth issue. The theoretical result (10) was derived perturbatively in the limit of weak tunneling of the quasiparticles. One can reasonably expect that if the tunneling rate is about, e.g., 10% the next perturbative correction to (and the inaccuracy of) the NtTRR should also be about 10%. While such an inaccuracy would bring about an error of the same order to the determined charge $Q$, the effect on the subleading term may be much more significant. This imposes a strong restriction on the value of the tunneling rate as is elaborated in the next section. There we find that for $\nu = 1/3$ and typical experimental parameters one needs the tunneling rate $r \leq 5\%$.

To summarize, the NtTRR (11) and its large-$I_s$ asymptotic behaviour (10) can be used to find the scaling dimension of the most relevant quasiparticle. One should, however, take care to choose the appropriate parametric regime in order to reduce errors.

II. EXACT RESULTS FOR $\nu = 1/3$ AND THE CONDITIONS TO EXTRACT THE SCALING DIMENSION BY PERTURBATIVE FORMULAE

In this subsection we concentrate on the filling factor $\nu = 1/3$. The minimal edge model for this filling factor has only one edge mode represented by the chiral bosonic field and can be constructed in the way described in section IV of Ref. [11]. The electric charge and the scaling dimension of the only most relevant quasiparticle in this model are respectively equal to $Q = 1/3$, $\delta = 1/6$.

This model is believed to give the correct description of the FQHE at $\nu = 1/3$. However, there is surprisingly little experimental evidence directly confirming this belief. While the charge of the most relevant quasiparticle has been confirmed long time ago [12,13], this is not true for its statistics or other properties of the model. Therefore, finding the most relevant quasiparticle’s scaling dimension would be an important check of the validity of the minimal model.

As it was noted in the previous section, finding the scaling dimension from large-$I_s$ asymptotic behaviour of NtTRR has a number of difficulties, one of which is related to the perturbative nature of the theoretical formulae. Fortunately, for the minimal model of the $\nu = 1/3$ edge there is an exact solution of the problem of the most relevant quasiparticle tunneling at QPC which allows for finding the tunneling rate and the tunneling current noise [17–19].

In this subsection we compare the perturbative answer for NtTRR with the exact one in order to find out at what tunneling rates the perturbative result can be applied for finding the scaling dimension. We concentrate on the case of zero temperature of the system ($T_0 = 0$), for which analytic expressions are available. The finite temperature case requires solution of thermodynamic Bethe anzatz equations and is beyond the scope of this work.

The exact answer for the tunneling rate $r = |I_T/I_s|$ at zero temperature is as follows:

$$ r \left( |I_s| > \Xi \delta \right) = \nu \sum_{|n| = 1}^{\infty} A_n(\nu) \frac{|I_s|^{2n(\nu-1)}}{3}, \quad (13) $$

Moreover, recently there has been a report [15], results of which may be interpreted as a signature of presence of additional neutral modes in the $\nu = 1/3$ FQHE. However, in the present work we are not going to discuss this evidence.
\begin{equation}
r (|I_s| < \Xi e^\Delta) = 
1 - \nu^{-1} \sum_{n=1}^{\infty} A_n(\nu^{-1}) \left( \frac{|I_s|}{\Xi} \right)^{2n(\nu^{-1} - 1)},
\end{equation}

\begin{equation}
A_n(x) = (-1)^{n+1} \sqrt{\pi T(nx)} \frac{\Gamma(nx)}{2^{n}n^{n} \Gamma(3/2 + n(x - 1))},
\end{equation}

\begin{equation}
\zeta = \frac{1}{2} \ln (1 - \nu) + \frac{\nu}{2(1 - \nu)} \ln \nu.
\end{equation}

The tunneling amplitude \( \eta \), the parameter \( \theta \propto |\eta|^2 \) in the perturbative formulae (13), (14) and the parameter \( \Xi \) here are related: \( \Xi \propto |\eta|^{1/(1 - \nu)} \). Thus, \( \Xi \) characterizes the tunneling strength. The restrictions on \( |I_s| \) in the formulae (13), (14) represent the radii of convergence of the series.\footnote{In Ref. [18] the definition of \( \zeta \) (which is called \( \Delta \) there) contains a misprint. However, one can check and find that the radius of convergence of the series leads to the definition of \( \zeta \) presented here.}

According to Ref. [18], at zero temperature the excess noise at zero frequency \( S(\omega = 0) \) is connected to the tunneling rate \( r \) via

\begin{equation}
S(\omega = 0, I_s) = \frac{\nu e}{2(1 - \nu)} |I_s| \times \Xi \frac{\partial}{\partial |I_s|} r (I_s).
\end{equation}

The explicit series are

\begin{equation}
S (\omega = 0, |I_s| > \Xi e^\Delta) = \nu^2 e |I_s| \sum_{n=1}^{\infty} n A_n(\nu^{-1}) \left( \frac{|I_s|}{\Xi} \right)^{2n(\nu^{-1} - 1)},
\end{equation}

\begin{equation}
S (\omega = 0, |I_s| < \Xi e^\Delta) = \nu^{-1} e |I_s| \sum_{n=1}^{\infty} n A_n(\nu^{-1}) \left( \frac{|I_s|}{\Xi} \right)^{2n(\nu^{-1} - 1)}.
\end{equation}

It is easy to see expansion in the orders of the tunneling amplitude \( \eta \) in the formulae (13), (18). Taking only the first term in the sums in Eqs. (13), (18) one should recover the lowest order perturbation theory result for the regime of weak tunneling. This is indeed the case.\footnote{There is a small subtlety here. To adapt the perturbative answers (1), (2) for \( T_0 = 0 \) one should take the limit \( T_0 \to 0 \) which coincides with the limit \( |I_s| \to 1 \). Then up to a factor one recovers the expression one can get from taking only the first term in the sums in Eqs. (13), (18). This factor is related to the proportionality factor between \( \Xi \) and \( |\eta|^{1/(1 - \nu)} \).}

Note that while the perturbative \( \Xi \)-term \( X_{\text{pert}} (I_s) = S_{\text{pert}} (\omega = 0, I_s) / r_{\text{pert}} (I_s) \) does not depend on the value of the tunneling amplitude \( \eta \) (or \( \Xi \), which is equivalent), the exact \( \Xi \)-term does.

We now compare the exact answers with the perturbative ones. Fig. 3 shows the comparison of the perturbative and the exact answers for the tunneling rate. For tunneling rates\footnote{We remind that the tunneling rate lies between 0 and 1 by definition.} not exceeding 0.2 the two answers are reasonably close. Note, that knowing the tunneling rate at a certain value of the current \( I_s \) one can find the corresponding value of the tunneling amplitude \( \Xi \).

Fig. 4 shows the comparison of the perturbative and the exact answers for the noise to tunneling rate ratio. Since the temperature \( T_0 = 0 \) the perturbative answer for \( \Xi \)-term is just \( X_{\text{pert}} (I_s) = eQ |I_s| \).

Fig. 5 shows the relative deviation of the lowest order perturbative results for the tunneling rate and \( \Xi \)-term from the exacts ones. The horizontal axis is the exact tunneling rate. The relation between the exact tunneling rate and \( |I_s| / \Xi \) can be seen from Fig. 3. It is interesting to note that for a given tunneling rate the error of the perturbative \( \Xi \)-term is generally greater than the error of the perturbative tunneling rate.

While the comparison made in Figs. 3, 4, 5 gives one an idea of how important the higher order corrections are, the curve representing the exact result in Fig. 3 should be taken with a grain of salt in the experimental context. This is because the tunneling amplitude \( \Xi \) in a real experiment exhibits a non-universal dependence on \( I_s \). It is not untypical that experimentalists work in the regime of constant tunneling rate (see, e.g., [20]). As can be seen from Fig. 5, this regime corresponds to the ratio \( |I_s| / \Xi \) being constant.

Fig. 6 shows the comparison of the perturbative and
Figure 4: Noise to tunneling rate ratio at $\nu = 1/3$. Perturbative answer vs exact answer. (Color online). The red dot-dashed curve is the exact $N_{\text{TRR}}$ plotted using the Eqs. (13)–(19). The green solid curve is the lowest order perturbation theory answer for the $N_{\text{TRR}}$, which can be obtained by taking only the first term in the sums in Eqs. (13), (18). We remind the reader that the system temperature is equal to $T_0 = 0$.

Figure 5: Relative errors of the lowest order perturbative results for the tunneling rate and $N_{\text{TRR}}$ at $\nu = 1/3$ as functions of the exact tunneling rate. (Color online). The black solid curve is the relative deviation of the perturbative result for the tunneling rate from the exact one. The grey dashed curve is the relative deviation of the perturbative result for the $N_{\text{TRR}}$ from the exact one.

Figure 6: Noise to tunneling rate ratio at $\nu = 1/3$. Perturbative answer vs exact answer in the regime of constant tunneling rate. (Color online). The red dot-dashed curve is the exact $N_{\text{TRR}}$ plotted using the Eqs. (13)–(19) for $\Xi = 0.5|I_s|$. The green solid curve is the lowest order perturbation theory answer for the $N_{\text{TRR}}$, which can be obtained by taking only the first term in the sums in Eqs. (13), (18). We remind the reader that the system temperature is equal to $T_0 = 0$.

Figure 7: Dependence of the effective charge $Q^*$ on the tunneling rate for $\nu = 1/3$. (Color online). The black solid curve is the dependence of the effective charge $Q^*$ found from the exact $N_{\text{TRR}}$ on the tunneling rate. The blue dashed line shows the value of the true charge $Q = 1/3$ of the tunneling quasiparticle. We remind the reader that the system temperature is equal to $T_0 = 0$.

the exact answers for the noise to tunneling rate ratio for $|I_s|/\Xi = 2$. Since the temperature $T_0 = 0$, the perturbative answer for the $N_{\text{TRR}}$ is just $X'_{\text{pert}}(I_s) = eQ|I_s|$. The exact answer in the regime $|I_s|/\Xi = \text{const}$ is equal to $X'_{\text{exact}}(I_s) = eQ^*|I_s|$. So the exact answer differs from the perturbative one by the gradient value determined by the "effective charge" $Q^*$. As can be seen from Fig. 7 in the limit of infinitely small tunneling rate the effective charge coincides with the true charge of the tunneling quasiparticle: $Q^* \rightarrow Q = 1/3$. However, at non-zero tunneling rate the charges do not coincide: $Q^* < Q$.

Although, at the moment we are not able to estimate the deviation of the perturbative answer for $N_{\text{TRR}}$ from the exact one at non-zero temperature, the observation that has just been made allows us to formulate some qualitative conditions for the applicability of formula (10). Namely, one can compare the difference between the answers at zero temperature $e(Q - Q^*)|I_s|$ at maximum value of $|I_s|$ which is going to be used with the term $eI_0(2 - 8\delta)/\pi = (2 - 8\delta)k_B T_0 e^2/\hbar$ in Eq. (10), where $T_0$...
is the system temperature.

For example, at $T_0 = 10$ mK for $|I_s|/\Xi = 2$ (which corresponds to the tunneling rate $r \approx 26\%$) at $I_s = 1$ nA for Laughlin quasiparticle ($Q = 1/3$, $\delta = 1/6$) the term containing $\delta$ is about 3 times smaller than the error $\epsilon(Q - Q^*)|I_s|$. Therefore, finding the scaling dimension of the Laughlin quasiparticle with the help of Eq. (10) is not possible under these experimental conditions.

For typical experimental values of $T_0 = 30$ mK and $I_s = 1$ nA the term error does not exceed $(2 - 8\delta)k_B T_0 \nu e^2/h$ for $r \leq 27\%$ and does not exceed $0.1 \times (2 - 8\delta) k_B T_0 \nu e^2/h$ for $r \leq 4\%$. When $\epsilon(Q - Q^*)|I_s|$ is 10 times smaller than the term containing $\delta$, one can hope to find $\delta$ with a reasonably small error. Thus, if the quality of the experimental data at $r \approx 4\%$ is high enough, it should be possible to find $\delta$ reasonably accurately (with the systematic relative error $\approx 10 - 20\%$ due to (a) difference between the exact answer and the perturbative answer and (b) difference between the perturbative answer and its large-$I_s$ asymptotic behaviour) by fitting the experimental data for NtTRR with Eq. (10).

One can eliminate the second source of systematic error by using the full formula (11) instead of the asymptotic expression (10).

Apart from that, the deviation of the effective charge $Q^*$ from the quasiparticle charge $Q$ at higher values of the tunneling rate $r$ gives an opportunity to further check the edge model and the tunneling contact model at $\nu = 1/3$.

CONCLUSIONS

We propose a method for finding the scaling dimension of the most relevant quasiparticle at a QH edge using tunneling current and tunneling current noise measurements. The advantages of the method are (a) reduced sensitivity to the non-universal physics of tunneling contacts (compared to methods based solely on tunneling current measurements), (b) a certain degree of model independence. By comparing our perturbative results with the exact results of Ref. [18] in the case of $\nu = 1/3$ we find that our method should be applied for small enough tunneling rates $r \lesssim 5\%$.

Using the exact solution of Ref. [18] at $\nu = 1/3$ for higher tunneling rates, we find that the effective charge $Q^*$ which can be found from an experiment using standard perturbative formulae deviates from the true charge of the most relevant quasiparticle $Q$. We propose to measure and study this difference in order to check the minimal $\nu = 1/3$ edge model and the tunneling contact model.

Acknowledgements

We would like to thank Oles’ Shtanko, Anna Grivnin and Moty Heiblum for useful discussions.

The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement No 279738 - NEDFOQ.

Appendix A: How general are the answers of section I?

The formulae (1)-(7) for the tunneling rate and the tunneling current noise within the second order perturbation theory in tunneling Hamiltonian were originally obtained in Ref. [11] for the case of the minimal $\nu = 2/3$ edge model under certain phenomenological assumptions. However, these formulae and the calculations leading to them are straightforward to generalize to a much wider class of edge theories.

A general Abelian QH edge theory can be constructed in the way outlined in section IV of Ref. [11]. One typically expects all the modes which carry electric charge to have same chirality $\chi_I$. If a theory contains counterflowing charged modes, in the low-energy limit it can become a theory with a set of charged modes propagating in one direction and a set of neutral modes (possibly, with different directions of propagation) according to the mechanism described in Refs. [21][22].

Under the same assumptions on the interaction between the Ohmic contacts and the edge as were used in Ref. [11], in the case of such theories one can show that the formulae (1)-(2) still hold for tunneling of the quasiparticles with $\delta < 1/2$. The only adjustment which has to be made concerns the number of the parameters $\theta_i$ (according to the number of the most relevant excitations) and their definition. For tunneling of the quasiparticles with $\delta \geq 1/2$ only the formulae (3), (5), (8) should be modified with the terms cancelling divergencies of the integrals at $t \rightarrow 0$ similar to the $\epsilon^{1/4 - \delta}$ term in Eq. (6).

A more general class of QH edge theories is where the charged sector is still described in terms of free bosons like in Abelian theories, while the neutral sector is described in terms of a more complicated model — some conformal field theory (CFT). Perhaps, the most famous example of such a model corresponds to the Moore-Read Pfaffian state. A general scheme for construction of such models is described in Ref. [23]. For more details on CFT see Ref. [24]. For the purposes of the present work it suffices to say that the second order perturbation theory results (1)-(7) hold for this class of models as well as they do for the Abelian ones.

We remind the reader that the phenomenological assumptions regarding the interaction of the Ohmic contacts with the edge are important for the derivation of the formulae (1)-(7). Most importantly, the assumption that the lower edge temperature does not depend on the current $I_s$ is crucial for the results of the present work.

So, the formulae (1)-(7) (up to a modification of the number and the exact expression of the parameters $\theta_i$) are valid for a wide class of typical Abelian and non-Abelian FQHE edge models.
Appendix B: Large-$I_s$ asymptotic behaviour of the noise to tunneling rate ratio

Consider the large-$I_s$ limit of Eq. (3). For $|j_s| \gg 1$ one gets

$$G_i = \frac{j_s}{|j_s|} Q_i^{4\delta} |j_s|^{4\delta-1} \sin 2\pi \delta \times \left( \int_0^\infty \frac{dx}{x} \sin x + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right) = \frac{j_s}{|j_s|} \frac{\pi}{2\Gamma(4\delta)} Q_i^{4\delta} |j_s|^{4\delta-1} \left( 1 + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right), \quad (B1)$$

where $\Gamma(x)$ is the Euler gamma function.

Similarly, for Eqs. (5), (6), (4) in the limit $|j_s| \gg 1$ one gets

$$F^{TT}_i = \frac{\pi}{2\Gamma(4\delta) \cos 2\pi \delta} Q_i^{4\delta+1} |j_s|^{4\delta-1} \times \left( 1 + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right), \quad (B2)$$

$$F^{0T}_i = \frac{\pi(4\delta - 1)}{2\Gamma(4\delta) \sin 2\pi \delta} Q_i^{4\delta} |j_s|^{4\delta-2} \times \left( 1 + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right), \quad (B3)$$

$$F_1 = \frac{\pi}{2\Gamma(4\delta)} Q_i^{4\delta} |j_s|^{4\delta-2} \times \left( \frac{Q_i |j_s|}{\pi} + \frac{2 - \delta \lambda}{2} + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right). \quad (B4)$$

Using Eqs. (1), (2), (8), (B1), (B4) one finally gets the asymptotic expression for the NTRR (5):

$$X(I_s) |_{|j_s| \gg 1} = \frac{\bar{S}(0)}{r} = \epsilon I_s \sum_i \theta_i F_i = \epsilon I_s \sum_i \theta_i \left( Q_i^{4\delta+1} |j_s| + Q_i^{4\delta-2 - \delta \lambda} + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right) = \epsilon I_s \sum_i \theta_i \frac{Q_i^{4\delta}}{|j_s|} \left( 1 + O \left( \frac{\lambda^2}{Q_i^2 j_s^2} \right) \right) = \epsilon |I_s| \sum_i \theta_i Q_i^{4\delta} |j_s| + \epsilon |I_s| \sum_i \theta_i Q_i^{\delta} + \epsilon \bar{\theta} \left( 2 - \delta \lambda \right) + O \left( |j_s|^{-1} \right). \quad (B5)$$

Appendix C: Analytic expressions for the noise to tunneling rate ratio

For the following derivation we need several facts about Euler beta function $B(x, y)$ and Euler gamma function $\Gamma(x)$.

$$\Gamma(x) = \frac{\pi}{\sin \pi x}, \quad (C1)$$

$$\Gamma(x) = \frac{x}{\Gamma(x)}, \quad (C2)$$

$$B(x, y) = \frac{\Gamma(x+y)}{\Gamma(x)} \frac{\Gamma(x)}{\Gamma(x+y)}, \quad (C3)$$

$$2^{4\delta-1} B \left( 1 - 4\delta, \alpha \right) \left( \frac{1}{2} + 2\delta \right) = \int_0^\infty dt \left( \frac{e^{-\alpha t}}{\sinh t} \right)^{4\delta}. \quad (C4)$$

The bars in the second equation denote complex conjugation. The last identity holds for $\delta < 1/4$ and $\Re \{\alpha\} / 2 + 2\delta > 0$, where $\Re[...]$ denotes taking of the real part. However, it can be analytically continued beyond these restrictions.

Using Eqs. (C1), (C4), one can get the following analytic expressions for the functions defined in Eqs. (3), (5):

$$G_i = \frac{Q_i 2^{4\delta-2}}{\Gamma(4\delta)} \left| \Gamma \left( 2 + \frac{iQ_i j_s}{\pi} \right) \right|^2 \sinh \frac{\pi Q_i j_s}{2}, \quad (C5)$$

$$F^{TT}_i = \frac{Q_i 2^{4\delta-2}}{\Gamma(4\delta) \cos 2\pi \delta} \left| \Gamma \left( 2 + \frac{iQ_i j_s}{\pi} \right) \right|^2 \cosh \frac{\pi Q_i j_s}{2}. \quad (C6)$$

For the function defined in Eq. (6), noting that

$$F^{0T}_i = \frac{1}{\sin 2\pi \delta} \frac{\partial}{\partial j_s} G_i, \quad (C7)$$

one gets

$$F^{0T}_i = \frac{Q_i 2^{4\delta-2}}{\Gamma(4\delta) \sin 2\pi \delta} \left| \Gamma \left( 2 + \frac{iQ_i j_s}{\pi} \right) \right|^2 \sinh \frac{\pi Q_i j_s}{2} \times \left( \frac{\pi}{2} \coth \frac{\pi Q_i j_s}{2} - \Im \left[ \psi \left( 2 + \frac{iQ_i j_s}{\pi} \right) \right] \right), \quad (C8)$$

where the digamma function $\psi(x) = (\ln \Gamma(x))'$ is the logarithmic derivative of the Euler gamma function $\Gamma(x)$, and $\Im[...]$ denotes taking of the imaginary part.

Thus, for $F_i$ defined in Eq. (4) we have

$$F_i = \frac{Q_i 2^{4\delta-1}}{\pi \Gamma(4\delta)} \left| \Gamma \left( 2 + \frac{iQ_i j_s}{\pi} \right) \right|^2 \sinh \frac{\pi Q_i j_s}{2} \times \sinh \frac{\pi Q_i j_s}{2} \Im \left[ \psi \left( 2 + \frac{iQ_i j_s}{\pi} \right) \right]. \quad (C9)$$

An interesting relation between this fact and the Ward identity arising due to the conservation of electric charge was noted in Ref. [25].
Using Eqs. [1], [2], [8], one straightforwardly gets the analytic expression for the noise to tunneling rate ratio $X(I_s)$. In the case of coinciding charges of all the quasiparticles participating in tunneling this expression simplifies significantly leading to the result [11].

[1] Xiao-Gang Wen. Topological orders and edge excitations in fractional quantum hall states. *Advances in Physics*, 44(5):405–473, 1995.

[2] Xiao-Gang Wen. Theory of the edge states in fractional quantum Hall effects. *International Journal of Modern Physics B*, 6(10):1711–1762, 1992.

[3] Alexey Boyarsky, Vadim Cheianov, and Jürg Fröhlich. Effective-field theories for the $\nu = 5/2$ quantum Hall edge state. *Physical Review B*, 80(23):233302, 2009.

[4] Iuliana P Radu, JB Miller, CM Marcus, MA Kastner, LN Pfeiffer, and KW West. Quasi-particle properties from tunneling in the $\nu = 5/2$ fractional quantum Hall state. *Science*, 320(5878):899–902, 2008.

[5] Stephan Baer, Clemens Rössler, Thomas Ihn, Klaus Ensslin, Christian Reichl, and Werner Wegscheider. Experimental probe of topological orders and edge excitations in the second Landau level. *arXiv preprint arXiv:1405.0428*, 2014.

[6] A. M. Chang. Chiral Luttinger liquids at the fractional quantum Hall edge. *Reviews of Modern Physics*, 75(4):1449–1505, 2003.

[7] M. Grayson, D. C. Tsui, L. N. Pfeiffer, K. W. West, and A. M. Chang. Continuum of chiral Luttinger liquids at the fractional quantum Hall edge. *Physical Review Letters*, 80(5):1062–1065, 1998.

[8] M Heiblum. Quantum shot noise in edge channels. *Physical Review Letters*, 80(23):233302, 2009.

[9] Stefano Roddaro, Vittorio Pellegrini, Fabio Beltram, Giorgio Biasiol, Lucia Sorba, Roberto Raimondi, and Giovanni Vignale. Nonlinear quasiparticle tunneling between fractional quantum Hall edges. *Physical Review Letters*, 90(4):046805, 2003.

[10] Stefano Roddaro, Vittorio Pellegrini, Fabio Beltram, Loren N Pfeiffer, and Ken W West. Particle-hole symmetric luttinger liquids in a quantum Hall circuit. *Physical Review Letters*, 95(15):156804, 2005.

[11] O. Shtanko, K. Sinnhko, and V. Cheianov. Nonequilibrium noise in transport across a tunneling contact between $\nu = 2/3$ fractional quantum hall edges. *Physical Review B*, 89(12):125104, 2014. PRB.

[12] R De-Picciotto, M Reznikov, M Heiblum, V Umansky, G Bunin, and D Mahalu. Direct observation of a fractional charge. *Nature*, 389(6647):162–164, 1997.

[13] L Saminadayar, DC Glattli, Y Jin, and B Etienne. Observation of the e/3 fractionally charged Laughlin quasiparticle. *Physical Review Letters*, 79(13):2526, 1997.

[14] D. Ferraro, A. Braggio, N. Magnoli, and M. Sassetti. Charge tunneling in fractional edge channels. *Physical Review B*, 82(8):085323, 2010.

[15] S. Huntington and V. Cheianov. Unpublished.

[16] Hiroyuki Inoue, Anna Grivnin, Yuval Ronen, Moty Heiblum, Vladimir Umansky, and Diana Mahalu. Pro-leration of neutral modes in fractional quantum Hall states. *arXiv preprint arXiv:1312.7553*, 2013.

[17] P. Fendley, A. W. W. Ludwig, and H. Saleur. Exact nonequilibrium transport through point contacts in quantum wires and fractional quantum Hall devices. *Physical Review B*, 52(12):8934–8950, 1995.

[18] P Fendley, AWW Ludwig, and H Saleur. Exact nonequilibrium dc shot noise in Luttinger liquids and fractional quantum Hall devices. *Physical Review Letters*, 75(11):2196, 1995.

[19] P Fendley and H Saleur. Nonequilibrium dc noise in a Luttinger liquid with an impurity. *Physical Review B*, 54(15):10845, 1996.

[20] Aveek Bid, Nissim Ofek, Hiroyuki Inoue, Moty Heiblum, C. L. Kane, Vladimir Umansky, and Diana Mahalu. Observation of neutral modes in the fractional quantum Hall regime. *Nature*, 466(7306):585–590, 2010.

[21] C. L. Kane and Matthew P. A. Fisher. Impurity scattering and transport of fractional quantum Hall edge states. *Physical Review B*, 51(19):13449–13466, 1995.

[22] C. L. Kane, Matthew P. A. Fisher, and J. Polchinski. Randomness at the edge: Theory of quantum Hall transport at filling $\nu = 2/3$. *Physical Review Letters*, 72(26):4129–4132, 1994.

[23] Jürg Fröhlich, Bill Pedrini, Christoph Schweigert, and Johannes Walcher. Universality in quantum hall systems: coset construction of incompressible states. *Journal of Statistical Physics*, 103(3-4):527–567, 2001.

[24] P. Di Francesco, P. Mathieu, and D. Senechal. *Conformal Field Theory*. Springer, 1997.

[25] Olaf Smits, JK Slingerland, and Steven H Simon. Nonequilibrium noise in the (non-) Abelian fractional quantum Hall effect. *arXiv preprint arXiv:1401.4581*, 2014.