DYNAMICS OF STOCHASTIC RETARDED BENJAMIN-BONA-MAHONY EQUATIONS ON UNBOUNDED CHANNELS

QIANGHENG ZHANG
School of Mathematics and Statistics
Heze University
Heze 274015, China

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Abstract. This article is devoted to the asymptotic behaviour of solutions for stochastic Benjamin-Bona-Mahony (BBM) equations with distributed delay defined on unbounded channels. We first prove the existence, uniqueness and forward compactness of pullback random attractors (PRAs). We then establish the forward asymptotic autonomy of this PRA. Finally, we study the non-delay stability of this PRA. Due to the loss of usual compact Sobolev embeddings on unbounded domains, the forward uniform tail-estimates and forward flattening of solutions are used to prove the forward asymptotic compactness of solutions.

1. Introduction. The BBM equation was introduced in [1] as a mathematical physical model for propagation of long waves. The dynamics of this equation has been studied by many authors, see, e.g., [7, 14, 22, 25, 27, 31, 32, 36] for the non-delay case and [37, 39] for the delay case. In this paper, we consider the long term behaviour of stochastic delay BBM equations. Although the asymptotic behaviour of solutions for delay PDEs has been extensively investigated, see [4, 2, 5, 12, 15, 19, 34, 38] and the references therein, the stochastic delay BBM equations has not been considered.

In this paper, we study the following non-autonomous BBM equation with additive noise and distributed delay defined on $Q = \mathcal{O} \times \mathbb{R}$ ($\mathcal{O} \subset \mathbb{R}^2$ is a bounded domain):

$$
\begin{aligned}
\frac{du}{dt} - d(Du) - \nu\Delta u dt + \nabla \cdot F(u) dt \\
= (\int_{-\rho}^{0} f(\xi, u(t+\xi)) d\xi + g(t,x)) dt + h(x) dW(t), \quad t > \tau, \ x \in Q,
\end{aligned}
$$

where $\tau \in \mathbb{R}$, $\nu > 0$, $h \in H^1_0(Q)$, $g \in L^2_{loc}(\mathbb{R}, L^2(Q))$, $\rho > 0$ is the delay time of (1) and $\mathbb{W}$ is a two-sided real-valued Wiener process on a probability space $(\Omega, \mathfrak{F}, P)$.

The nonlinear vector function $F$ and delay forcing $f$ will be specified later.

The pullback random attractor $\mathcal{A} = \{A(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ is an important role that describes the long-time behaviour of solutions for stochastic non-autonomous PDEs, see, e.g., [10, 13, 18, 20, 23, 28, 29, 30, 33, 34, 35]. Note that the pullback

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random attractor is a bi-parametric set depending on the time and sample parameters. Therefore, a natural idea is to study the time-dependent property of pullback random attractors (PRAs). Recently, Cui et al. [10] established the finite time stability of PRAs:

\[
\lim_{\tau \to \tau_0} \text{dist}_X (\mathcal{A}(\tau, \omega), \mathcal{A}(\tau_0, \omega)) = 0, \quad \text{for all } \omega \in \Omega,
\]

where \((X, \| \cdot \|_X)\) is a Banach space and \(\text{dist}_X (\cdot, \cdot)\) is the Hausdorff semi-distance. Wang et al. [33] considered the backward long time stability of PRAs:

\[
\lim_{\tau \to -\infty} \text{dist}_X (\mathcal{A}(\tau, \omega), \mathcal{A}(\omega)) = 0, \quad \text{for all } \omega \in \Omega,
\]

where \(\mathcal{A}(\omega)\) is a nonempty compact set. Caraballo et al. [3] investigated the backward asymptotic autonomy of PRAs:

\[
\lim_{\tau \to -\infty} \text{dist}_X (\mathcal{A}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0, \quad \text{for all } \omega \in \Omega,
\]

where \(\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega), \omega \in \Omega\}\) is a autonomous random attractor. In this paper, we study the forward compactness and forward asymptotic autonomy of PRAs for Eq. (1). Inspire by [34], we also prove the non-delay stability of PRAs for Eq. (1).

The first aim of this paper is to prove the existence, uniqueness and forward compactness of PRAs for (1). To this end, we need to establish the forward dissipation of Eq. (1) and the forward pullback asymptotic compactness of solutions. Hence, we will have the following difficulties:

- The solution of Eq. (1) has no higher regularity.
- The non-compactness of Sobolev embeddings on unbounded domains.
- The measurability of forward compact bi-parametric pullback attractors.

Fortunately, we can find valid methods (the forward uniform tail-estimates of solutions, the spectrum decomposition technique and the attractors on two different universes are identical) to resolve the above problems.

The second goal is to study the forward asymptotic autonomy of PRAs:

\[
\lim_{\tau \to +\infty} \text{dist}_X (\mathcal{A}_\rho(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0, \quad \text{for all } \omega \in \Omega,
\]

where \(\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega), \omega \in \Omega\}\) is a autonomous random attractor of the autonomous equation corresponding to (1). For deterministic non-autonomous PDEs, this theme has been considered (see [8, 9, 16, 17, 21]). More recently, Yang et al. [35] considered this subject for stochastic lattice systems. Compared with [35], we do not use the continuity of solutions in the initial value space to prove this theme. On the other hand, the forward asymptotic autonomy of PRAs has not been considered for the deterministic delay PDEs, let alone the stochastic delay PDEs.

Finally, we use the convergence of systems from delay to non-delay and the eventually compactness of PRAs to prove the upper semicontinuity of PRAs as the memory time tends to zero:

\[
\lim_{\rho \to 0} \text{dist}_{X_\rho} (\mathcal{A}_\rho(\tau, \omega), \mathcal{A}_0(\tau, \omega)) := \sup_{a \in \mathcal{A}_\rho(\tau, \omega)} \inf_{b \in \mathcal{A}_0(\tau, \omega)} \| a(\xi) - b \|_X = 0, \quad \text{for all } \tau \in \mathbb{R}, \omega \in \Omega,
\]

where \(X_\rho = C([-\rho, 0]; X)\) and \(\mathcal{A}_\rho = \{\mathcal{A}_\rho(\tau, \omega)\}\) is a pullback random attractor of the non-delay equation corresponding to (1). This theme for stochastic delay PDEs defined on bounded domain has been studied (see [19, 34]). In addition, we assume the delay term is pointwise Lipschitz continuous with respect to the memory time.
instead of global Lipschitz continuous used in the literature. As for as we know, there is only one paper [38] on this topic for stochastic delay PDEs defined on unbounded domain.

In the next section, we establish the existence of a continuous cocycle for Eq. (1). In section 3, we prove the existence, uniqueness and forward compactness of PRAs by the forward uniform estimates, forward uniform tail-estimates and forward flattening of solutions. The forward asymptotic autonomy of PRAs is studied in section 4. The last section is devoted to the non-delay stability of PRAs.

2. Stochastic BBM equation with distributed delay. In this paper, we identify the Wiener process \( W(\cdot, \omega) \) with \( \omega(\cdot) \) on \( (\Omega, \mathcal{F}, P) \), where \( \Omega = \{ \omega \in C([\mathbb{R}, \mathbb{R}]) : \omega(0) = 0 \} \), \( \mathcal{F} \) is the Borel \( \sigma \)-algebra induced by the compact-open topology of \( \Omega \), \( P \) is the Wiener measure. We define a group of measure-preserving transformation \( \{ \theta_t \}_{t \in \mathbb{R}} \) on \( \Omega \) by \( \theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t) \) for \( (\omega, t) \in \Omega \times \mathbb{R} \). Then \( (\Omega, \mathcal{F}, P, \{ \theta_t \}_{t \in \mathbb{R}}) \) is a metric dynamical system. Next, we show that the existence of a continuous non-autonomous random dynamical system for Eq. (1) over \( (\Omega, \mathcal{F}, P, \{ \theta_t \}_{t \in \mathbb{R}}) \).

Let \( z(\theta_t \omega) = - \int_0^t e^s(\theta_t \omega)(s) ds \), then it is a stationary solution of the one-dimensional Ornstein-Uhlenbeck equation: \( dz + dzdt = dW(t) \) such that \( t \rightarrow z(\theta_t \omega) \) is pathwise continuous on a \( \{ \theta_t \}_{t \in \mathbb{R}} \)-invariant subset \( \Omega_0 \subset \Omega \) with \( P(\Omega_0) = 1 \). In addition, we have for all \( \omega \in \Omega_0 \),

\[
\lim_{t \rightarrow \pm \infty} \frac{z(\theta_t \omega)}{t} = 0, \quad \lim_{t \rightarrow \pm \infty} \frac{1}{t} \int_0^t z(\theta_s \omega) ds = 0,
\]

\[
\lim_{t \rightarrow \pm \infty} \frac{1}{t} \int_0^t |z(\theta_s \omega)| ds = E|z| = \frac{1}{\sqrt{\pi}}.
\]

For convenience, we are not distinguish two spaces \( \Omega_0 \) and \( \Omega \) in this paper.

For each \( t \geq \tau \), let \( u_t(\cdot) \) is a delay-shift function defined by \( u_t(\xi) = u(t + \xi) \) with \( \xi \in [ - \rho, 0 ] \). We make a suitable change of variable:

\[
v(t, \tau, \omega, \psi) = u(t, \tau, \psi, \xi) - \tilde{z}(\theta_t \omega) \quad \text{with} \quad \tilde{z}(\theta_t \omega) = (I - \Delta)^{-1} h(\theta_t \omega).
\]

Substituting (4) into (1), we obtain the following random equation:

\[
\frac{\partial(v - \Delta v)}{\partial t} - \nu \Delta v + \nabla \cdot \mathbf{F}(v + \tilde{z}(\theta_t \omega))
\]

\[
= f_0 - \rho f_0(s, v(t + \xi) + \tilde{z}(\theta_t + \xi \omega)) d\xi + g(t, x) + \tilde{z}(\theta_t \omega) + (\nu - 1) \Delta \tilde{z}(\theta_t \omega), \quad v|_{\partial Q} = 0, \quad v(t, \xi, x) = v_\tau(\xi, x) := \phi(\xi, x), \quad \xi \in [ - \rho, 0 ], \quad x \in Q.
\]

Throughout this paper, we assume that \( \rho \in (0, \rho_0) \) for some \( \rho_0 > 0 \). For the purposes of this article, we now make some suitable assumptions.

(F) \( \mathbf{F}(s) = (F_1(s), F_2(s), F_3(s)) \) satisfies

\[
F_k(0) = 0, \quad |F'_k(s)| \leq \alpha_1 + \alpha_2 |s|, \quad s \in \mathbb{R}, \quad k = 1, 2, 3,
\]

where \( \alpha_1, \alpha_2 > 0 \), which implies

\[
|F_k(s)| \leq \alpha_1 |s| + \alpha_2 |s|^2, \quad s \in \mathbb{R}, \quad k = 1, 2, 3.
\]

(D) \( f : [-\rho, 0] \times \mathbb{R} \rightarrow \mathbb{R} \) is a measurable function satisfying \( f(\xi, 0) = 0 \) for all \( \xi \in [-\rho, 0] \) and there exists a positive function \( L_f(\cdot) \) such that

\[
|f(\xi, s_1) - f(\xi, s_2)| \leq L_f(\xi)|s_1 - s_2|, \quad \forall \xi \in [-\rho, 0], s_1, s_2 \in \mathbb{R},
\]

where \( L_f(\cdot) : [-\rho_0, 0] \rightarrow \mathbb{R}^+ \) satisfies \( L_f(\cdot) \in L^2(-\rho_0, 0) \).
Let
\[ H_\rho := C([-\rho, 0] ; H) \] with \( H := L^2(Q) \), \( \| \varphi \|_{H_\rho} = \sup_{\xi \in [-\rho, 0]} \| \varphi(\xi) \|, \forall \varphi \in H_\rho, \rho > 0, \)
\[ V_\rho := C([-\rho, 0] ; V) \] with \( V := H^2_\rho(Q) \), \( \| \varphi \|_{V_\rho} = \sup_{\xi \in [-\rho, 0]} \| \varphi(\xi) \|_V, \forall \varphi \in V_\rho, \rho > 0, \)
where \( \| \cdot \| \) denotes the norm of \( H \). In addition, we use \( (\cdot, \cdot) \) to denote the inner product of \( H \).

Since Eq. (5) is a pathwise random equation, by the standard Galerkin method [24] we have the following lemma:

**Lemma 2.1.** Suppose that \( (F), (D) \) hold, \( h \in V \) and \( g \in L^2_{\text{loc}}(\mathbb{R}, H) \). Then for all \( \tau \in \mathbb{R}, \omega \in \Omega \) and \( \phi \in V_\rho \), Eq. (5) has a unique weak solution
\[ v \in C([\tau - \rho, T]; V), \quad \frac{\partial v}{\partial t} \in L^2(\tau, T; V), \quad \forall T > \tau, \] such that \( v(\tau + \xi, \tau, \omega, \phi) = \phi(\xi) \) for all \( \xi \in [-\rho, 0] \).

We define a mapping \( \Phi_\rho : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times V_\rho \to V_\rho \) by
\[ \Phi_\rho(\tau, \rho, \omega) \phi = v_{t+\tau}(\tau, \theta, \omega, \phi), \quad t \in \mathbb{R}^+, \tau \in \mathbb{R}, \omega \in \Omega, \phi \in V_\rho, \] where \( v \) is a solution of (5). By Lemma 2.1, it is easy to prove that \( \Phi_\rho \) satisfies all conditions of [28, Definition 2.1]. Then \( \Phi_\rho \) is a continuous cocycle generated by (5). Moreover, by (4) we get a continuous cocycle \( \Psi_\rho \) associated with (1), which is equivalent to \( \Phi_\rho \), see [11]. So we discuss \( \Phi_\rho \) in this paper only.

(G1) \( g \in L^2_{\text{loc}}(\mathbb{R}, H) \) is forward tempered:
\[ \sup_{s \geq \tau} \int_{-\infty}^{s} e^{b(r-s)} \| g(r) \|^2 dr < +\infty, \forall \tau \in \mathbb{R}, b > 0. \]  
(G2) \( g \in L^2_{\text{loc}}(\mathbb{R}, H) \) is forward tail-small:
\[ \lim_{k \to +\infty} \sup_{s \geq \tau} \int_{-\infty}^{s} e^{b(r-s)} \int_{|x| \geq k} |g(r, x)|^2 dx dr = 0, \forall \tau \in \mathbb{R}, b > 0. \]

In addition, we will frequently use the following embedding inequality (Agmon inequality):
\[ \| w \|_\infty \leq \beta_0 \| w \|_{H^2(Q)}, \forall w \in H^2(Q), \] where \( \beta_0 > 0 \). By (3) there exists a positive constant \( \tilde{\nu} := \tilde{\nu}(\rho_0) < \min\{ \frac{\rho_0}{2}, \frac{\lambda^2}{4} \} \) such that
\[ \tilde{\gamma} := \frac{\tilde{\nu}}{3} - \beta |z| - 2\sqrt{2}e^{\frac{\xi}{2}} \rho^\frac{1}{2} \| L_f(\cdot) \|_{L^2(\rho, 0)} > 0, \forall \rho \in (0, \rho_0], \] where \( \lambda \) is the Poincaré constant and \( \beta = 4\beta_0 \gamma \| h \|_V \). Consider a random variable:
\[ \gamma(\omega) = \frac{\tilde{\nu}}{3} - \beta |z(\omega)| - 2\sqrt{2}e^{\frac{\xi}{2}} \rho^\frac{1}{2} \| L_f(\cdot) \|_{L^2(\rho, 0)}, \forall \rho \in (0, \rho_0], \] which together with (3) and (13) implies
\[ \lim_{t \to +\infty} \frac{1}{t} \int_{0}^{t} \gamma(\theta_t \omega) ds = \mathbb{E}(\gamma) = \tilde{\gamma} > 0. \]
Let \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) be a family of bounded nonempty subsets of \( V_\rho \). If
\[ \lim_{t \to +\infty} e^{-bt} \| D(s - t, \theta_{-t} \omega) \|_{V_\rho}^2 = 0, \forall \tau \in \mathbb{R}, \omega \in \Omega, b > 0, \]
then $D$ is tempered. If
\[
\lim_{t \to +\infty} e^{-bt} \sup_{s \geq \tau} \|D(s-t, \theta_{-\tau}\omega)\|^2_{V_{\rho}} = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega, b > 0,
\] then $D$ is forward tempered.

Finally, we take two different universes of attraction $D$ and $B$. $D$ is the collection of all tempered families in $V_{\rho}$:
\[
D = \{D = \{D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : D \text{ satisfies (16)}\}.
\]
Then $D$ is inclusion-closed universe. $B$ is composed of all forward tempered families in $V_{\rho}$:
\[
B = \{B = \{B(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} : B \text{ satisfies (17)}\}.
\]
Then $B$ is a forward-closed universe: $B \in B$ whenever $B \in \mathcal{B}$ and $\tilde{B}(\tau, \omega) = \cup_{s \geq \tau} B(s, \omega)$. Note that $\mathcal{B} \subset \mathcal{D}$ and forward-closed implies inclusion-closed.

3. Pullback random attractors.

3.1. Forward uniform estimates of solutions. Let $G(s) = (G_1(s), G_2(s), G_3(s))$, where
\[
G_k(s) = \int_{0}^{s} F_k(r)dr, \quad k = 1, 2, 3, \quad s \in \mathbb{R}.
\] By (7) we find
\[
|G_k(s)| \leq \alpha_1 |s|^2 + \alpha_2 |s|^3, \quad s \in \mathbb{R}, \quad k = 1, 2, 3.
\]
Denote by $c > 0$ an intrinsic constant and by $C(\omega) > 0$ an intrinsic random variable, which are uniform w.r.t. $\rho \in (0, \rho_0]$.

**Lemma 3.1.** Suppose that (F), (D), (G1) hold and $h \in H^1_0(Q)$. Let $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we have the following conclusions:

(i) For each $D \in D$, there exists a $T_d := T_d(D, \tau, \omega) \geq 2\rho_0 + 2$ such that
\[
\sup_{\xi \in [-2\rho-2,0]} \|v(\tau + \xi, t, \theta_{-\tau} \omega, \phi)\|^2_{V} \leq cB_d(\tau, \omega),
\] for all $\phi \in D(\tau-t, \theta_{-\tau} \omega)$, $t \geq T_d$ and $\rho \in (0, \rho_0)$, where
\[
B_d(\tau, \omega) = \int_{-\infty}^{0} e^{\int_{0}^{s} \gamma(\theta_{r} \omega)dr} (1 + \|z(\theta_r \omega)\|^2 + \|g(r + \tau)\|^2) dr < +\infty.
\]
(ii) For each $B \in B$, there is a $T_b = T_b(B, \tau, \omega) \geq 2\rho_0 + 2$ such that
\[
\sup_{s \geq \tau} \sup_{\xi \in [-2\rho-2,0]} \|v(s + \xi, s-t, \theta_{-\tau} \omega, \phi)\|^2_{V} \leq cB_b(\tau, \omega),
\] for all $\phi \in B(s-t, \theta_{-\tau} \omega)$ with $s \geq \tau$, $t \geq T_b$ and $\rho \in (0, \rho_0)$, where
\[
B_b(\tau, \omega) = \sup_{s \geq \tau} B_b(s, \omega) < +\infty.
\] _Proof._ Taking the inner product of (5) with $v(r, s-t, \theta_{-\tau} \omega, \phi)$ in $H$ yields
\[
\frac{1}{2} \frac{d}{dr} \|v(r)\|^2_{V} + \nu \|
abla v(r)\|^2 = -(\nabla \cdot F(u), v) + (g(r, \cdot), v)
\]
\[
+ \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + z(\theta_{r+\xi} \omega))d\xi, v \right) + (z(\theta_{r-\xi} \omega) + (\nu - 1)\Delta z(\theta_{r-\xi} \omega), v).\]

(24)
By the Poincaré inequality we find
\[ \nu \| \nabla v(r) \|^2 \geq \frac{\nu}{2} \| \nabla v(r) \|^2 + \frac{\nu \lambda}{2} \| v(r) \|^2. \]  
(25)
For the first term on the right-hand side of (24), by (7) and (18) we have
\[ - (\nabla \cdot F(u), v) = - \int_Q \nabla \cdot F(u) dx + \int_Q \nabla \tilde{z}(\theta_{r-s}) \nabla \cdot F(u) dx \]
\[ = \int_Q \nabla \cdot G(u) dx - \int_Q \nabla \tilde{z}(\theta_{r-s}) \cdot F(u) dx \leq \int_Q |\nabla \tilde{z}(\theta_{r-s})| |F(v + \tilde{z}(\theta_{r-s}))| dx \]
\[ \leq \alpha_1 \int_Q |\nabla \tilde{z}(\theta_{r-s})| |v| + |\nabla \tilde{z}(\theta_{r-s})| \tilde{z}(\theta_{r-s})| dx \]
\[ + 2 \alpha_2 \int |\nabla \tilde{z}(\theta_{r-s})| |v|^2 dx + 2 \alpha_2 |\nabla \tilde{z}(\theta_{r-s})| \| \tilde{z}(\theta_{r-s}) \|^2 \]
\[ \leq \frac{\nu \lambda}{16} \| v(r) \|^2 + c(\| \nabla \tilde{z}(\theta_{r-s}) \|^2 + \| \tilde{z}(\theta_{r-s}) \|^2) \]
\[ + 2 \alpha_2 \int |\nabla \tilde{z}(\theta_{r-s})| |v|^2 dx + 2 \alpha_2 |\nabla \tilde{z}(\theta_{r-s})| \| \tilde{z}(\theta_{r-s}) \|^2 \]
\[ \leq \frac{\nu \lambda}{16} \| v(r) \|^2 + c(|z(\theta_{r-s})|^2 + |z(\theta_{r-s})|^3) + 2 \alpha_2 \int |\nabla \tilde{z}(\theta_{r-s})| |v|^2 dx, \]  
(26)
where we use \((I - \Delta)^{-1} h \in H^1_0(Q) \cap H^2(Q)\). By (12) we get
\[ 2 \alpha_2 \int |\nabla \tilde{z}(\theta_{r-s})| |v|^2 dx \leq 2 \alpha_2 \| \nabla \tilde{z}(\theta_{r-s}) \|_{L^\infty} \| v(r) \|^2 \]
\[ \leq 2 \alpha_2 \beta_0 \| \nabla \tilde{z}(\theta_{r-s}) \|_{H^2(Q)} \| v(r) \|^2 \leq 2 \alpha_2 \beta_0 \| h \|_V \| z(\theta_{r-s}) \| \| v(r) \|^2. \]  
(27)
For the second term and the last term on the right-hand side of (24), by the Young inequality we obtain
\[ (g(r, \cdot), v) + (\tilde{z}(\theta_{r-s}) + (\nu - 1) \Delta \tilde{z}(\theta_{r-s}), v) \]
\[ \leq \frac{\nu \lambda}{16} \| v(r) \|^2 + c(\| g(r) \|^2 + \| \tilde{z}(\theta_{r-s}) \|^2 + \| \Delta \tilde{z}(\theta_{r-s}) \|^2) \]
\[ \leq \frac{\nu \lambda}{16} \| v(r) \|^2 + c(\| g(r) \|^2 + |z(\theta_{r-s})|^2). \]  
(28)
We infer from (24)-(28) that
\[ \frac{d}{dr} \| v(r) \|_V^2 \leq (\beta |z(\theta_{r-s})| - \nu) \| v(r) \|_V^2 + c(1 + |z(\theta_{r-s})|^4 + \| g(r) \|^2) \]
\[ + 2(\int_{-\rho}^{0} f(\xi, v(r + \xi) + \zeta(\theta_{r-s} + \xi), v) d\xi, v). \]
Multiplying the above inequality by \(e^{\int_{\rho}^{r} \gamma(\theta_{r-s}) dt}\), and then integrating this result over \([s - t, s + \bar{\xi}]\) with \(\bar{\xi} \in [-2 \rho - 2, 0]\) and \(t \geq 2 \rho_0 + 2\) we obtain
\[ e^{\int_{s-t}^{s+\bar{\xi}} \gamma(\theta_{r-s}) dt} \| v(s + \bar{\xi}, s - t, \theta_{r-s}, \phi) \|_V^2 \]
\[ \leq \| \phi(0) \|_V^2 + \int_{s-t}^{s+\bar{\xi}} (\beta |z(\theta_{r-s})| + \gamma(\theta_{r-s}) - \nu) e^{\int_{s-t}^{r} \gamma(\theta_{r-s}) dt} \| v(r) \|_V^2 dr \]
Hence, by We now main treat the delay term of (29). By the Young inequality we have

\[
2 \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))d\xi, v \right) \, dr.
\]

It follows from (8) and \( f(\cdot, 0) = 0 \) that

\[
2 \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))d\xi, v \right) \, dr 
\leq \sqrt{2e^{\hat{\rho} \| L_f(\cdot) \|_{L^{2}(-\rho, 0)}}} \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \| v(r) \|^{2} \, dr
\]

\[
+ \frac{\int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \int_{Q} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))d\xi, v \right)^{2} \, dx \, dr}{\sqrt{2e^{\hat{\rho} \| L_f(\cdot) \|_{L^{2}(-\rho, 0)}}}}.
\]

It follows from (8) and \( f(\cdot, 0) = 0 \) that

\[
\int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))d\xi, v \right) \, dr
\leq 2\| \tilde{\gamma} \|_{L^{2}(-\rho, 0)} \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} (v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))^{2}d\xi \right) \, dr
\]

\[
\leq 2\| \tilde{\gamma} \|_{L^{2}(-\rho, 0)} \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} (v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))^{2}d\xi \right) \, dr
\]

\[
\leq 2\| \tilde{\gamma} \|_{L^{2}(-\rho, 0)} \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} (v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))^{2}d\xi \right) \, dr.
\]

Hence, by \( L_f(\cdot) \in L^{2}(-\rho, 0) \) we have

\[
2 \int_{s-t}^{s+\xi} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))d\xi, v \right) \, dr
\leq 2\| \tilde{\gamma} \|_{L^{2}(-\rho, 0)} \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \| v(r) \|^{2} \, dr
\]

\[
+ \frac{1}{\sqrt{2\| \tilde{\gamma} \|_{L^{2}(-\rho, 0)}}} \int_{s-t}^{s+\xi} e^{\int_{t-r}^{r} \gamma(\theta_{t-s}\omega)dr} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{t+\xi-s}\omega))d\xi, v \right)^{2} \, dx \, dr.
\]

By (14) we know

\[
\beta |z(\theta_{t-s}\omega)| + \gamma(\theta_{t-s}\omega) - \nu + 2\sqrt{2\| \tilde{\gamma} \|_{L^{2}(-\rho, 0)}} < 0.
\]
Substituting (30) and (31) into (29) yields
\[
\|v(s + \xi, s - t, \theta_{-s}, \phi)\|^2_{V^2} \leq c e^{\tilde{p}(p+1)} \|\phi\|^2_{V^2} (e^{f_0^- \gamma(\theta_0)} dr + \int_{-t}^{-t-\rho} e^{f_0^- \gamma(\theta) dr})
\]
\[
+ c e^{\tilde{p}(p+1)} \int_{-t-\rho}^{-t} e^{f_0^- \gamma(\theta) dr} (1 + |z(\theta, \omega)|)^4 + \|g(r + s)\|^2) dr
\]
\[
\leq c \|\phi\|^2_{V^2} (e^{f_0^- \gamma(\theta) dr} + \int_{-t}^{-t-\rho} e^{f_0^- \gamma(\theta) dr}) dr + cB_d(s, \omega),
\]
where we use $\gamma(\theta_{-s}) \leq \frac{\delta}{2}$ in (14) and $B_d(s, \omega)$ is defined by (21). By (2) and (15), there exists a $T := T(\tilde{\gamma}, \omega) \geq 2\rho_0 + 2$ such that
\[
|z(\theta, \omega)| \leq \frac{\tilde{\gamma}}{4} |t|, \quad \left| \int_0^t (\gamma(\theta, \omega) - \gamma) dt \right| \leq \frac{\tilde{\gamma}}{2} |t|, \quad \text{for all } |t| \geq T.
\]
(i) Let $s = \tau$ and $\phi \in D(\tau - t, \theta_{-s})$ in (32). By (16) and (33), there exists a $T_d := T_d(\tau, \omega, D) > T$ such that for all $t \geq T_d$
\[
(e^{f_0^- \gamma(\theta) dr} + \int_{-t}^{-t-\rho} e^{f_0^- \gamma(\theta) dr}) \|\phi\|^2_{V^2} \leq \left(\frac{2}{\tilde{\gamma}} + 1\right)e^{-\frac{\tilde{\gamma}}{t}} \|D(\tau - t, \theta_{-s})\|^2_{V^2} \leq B_d(\tau, \omega),
\]
which implies (20) holds.
(ii) If $\phi \in B(s - t, \theta_{-s})$ with $s \geq \tau$ in (32). By (17) and (33), there exists a $T_b := T_b(\tau, \omega, B) > T$ such that for all $t \geq T_b$
\[
\sup_{s \geq \tau} \|\phi\|^2_{V^2} \left(e^{f_0^- \gamma(\theta) dr} + \int_{-t}^{-t-\rho} e^{f_0^- \gamma(\theta) dr} dr\right) \leq \left(\frac{2}{\tilde{\gamma}} + 1\right)e^{-\frac{\tilde{\gamma}}{t}} \sup_{s \geq \tau} \|B(s - t, \theta_{-s})\|^2_{V^2} \leq B_b(\tau, \omega).
\]
Then we take the supremum of (30) over $s \in [\tau, +\infty)$ yields (22) as desired.

We now show that $B_d(\tau, \omega)$ and $B_b(\tau, \omega)$ are finite. By (33) and the continuity of $z(\theta, \omega)$ we obtain, for all $r \leq 0$,
\[
|z(\theta, \omega)| \leq \frac{\tilde{\gamma}}{4} + C(\omega), \quad e^{f_0^- \gamma(\theta) dr} \leq e^{f_0^- \gamma(\theta) dr + \tilde{\gamma} t} \leq e^{\tilde{\gamma} t + C(\omega)}.
\]
Then, we have
\[
B_d(\tau, \omega) = \sup_{s \geq \tau} \int_0^s e^{f_0^- \gamma(\theta) dr} (1 + |z(\theta, \omega)|)^4 + \|g(r + s)\|^2 dr
\]
\[
\leq C(\omega) \sup_{s \geq \tau} \int_{-\infty}^0 e^{\tilde{\gamma} r} (1 + r^4 + \|g(r + s)\|^2 dr < +\infty,
\]
which implies $B_d(\tau, \omega) < +\infty$.

\[\Box\]

**Lemma 3.2.** Suppose that (F), (D), (G1) hold and $h \in H^1_0(\Omega)$. For each $\beta \in \mathcal{B}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$ such that
\[
\sup_{s \geq \tau} \int_{-\rho}^s \|\frac{\partial}{\partial r} v(r, s - t, \theta_{-s}, \phi)\|^2 dr \leq cB_b(\tau, \omega)^2 + G(\tau, \omega),
\]
for all $\phi \in B(s - t, \theta_{-s})$, $t \geq T_b$ (the same number as in Lemma (3.1)) and $\rho \in (0, \rho_0]$, where
\[
G(\tau, \omega) = \sup_{s \geq \tau} \int_{-\infty}^0 e^{\tilde{\gamma} r} (1 + |z(\theta, \omega)|)^4 + \|g(r + s)\|^2 dr.
\]
Proof. Multiplying (5) by $\frac{\partial}{\partial r} v(r, s - t, \theta_{-s} \omega, \phi)$ yields
\[
\| \frac{\partial v}{\partial r} \|^2 + \| \nabla v \cdot \nabla \frac{\partial v}{\partial r} \|^2 + \nu \int_{Q} \nabla v \cdot \nabla \frac{\partial v}{\partial r} \, dx = \int_{Q} \mathbf{F}(v + \tilde{z}(\theta_{-s} \omega)) \cdot \nabla \frac{\partial v}{\partial r} \, dx
\]
\[
+ \int_{Q} \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{r+\xi} \omega)) \, ds \frac{\partial v}{\partial r} \, dx
\]
\[
+ \int_{Q} \left( g(r, x) + \tilde{z}(\theta_{r-s} \omega) + (\nu - 1) \Delta \tilde{z}(\theta_{r-s} \omega) \right) \frac{\partial v}{\partial r} \, dx.
\]
It follows from (7), (8), $f(\cdot, 0) = 0$ and the Young inequality that
\[
\int_{Q} \mathbf{F}(v + \tilde{z}(\theta_{r-s} \omega)) \cdot \nabla \frac{\partial v}{\partial r} \, dx \leq \int_{Q} \left( a_{1} |v + \tilde{z}(\theta_{r-s} \omega)| + a_{2} |v + \tilde{z}(\theta_{r-s} \omega)|^{2} \right) \| \nabla \frac{\partial v}{\partial r} \| \, dx
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + c(\|v\|^{2} + \|v\|^{4} + \|\tilde{z}(\theta_{r-s} \omega)\|^{2} + \|\tilde{z}(\theta_{r-s} \omega)\|^{4})
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + c(\|v\|^{2} + \|v\|^{4} + \|1 - \Delta\|^{-1} h^{2} |z(\theta_{r-s} \omega)|^{2} + \|1 - \Delta\|^{-1} h^{2} |z(\theta_{r-s} \omega)|^{4})
\]
\[
\int_{Q} \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{r+\xi} \omega)) \, ds \frac{\partial v}{\partial r} \, dx
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + \int_{Q} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{r+\xi} \omega)) \, ds \right)^{2} \, dx
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + \int_{Q} \left( \int_{-\rho}^{0} |L_{\ell}(\xi)| v(r + \xi) + \tilde{z}(\theta_{r+\xi} \omega) \, ds \right)^{2} \, dx
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + 2 \|L_{\ell}(\cdot)\|_{L^{2}(-\rho, 0)}^{2} \int_{Q} \int_{-\rho}^{0} (|v(r + \xi)|^{2} + |\tilde{z}(\theta_{r+\xi} \omega)|^{2}) \, dx \, ds \, d\xi
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + 2 \|L_{\ell}(\cdot)\|_{L^{2}(-\rho, 0)}^{2} \int_{Q} \int_{-\rho}^{0} (|v(r + \xi)|^{2} + \|1 - \Delta\|^{-1} h^{2} |z(\theta_{r+\xi} \omega)|^{2}) \, ds \, d\xi.
\]
and
\[
- \nu \int_{Q} \nabla v \cdot \nabla \frac{\partial v}{\partial r} \, dx + \int_{Q} (g(r, x) + \tilde{z}(\theta_{r-s} \omega) + (\nu - 1) \Delta \tilde{z}(\theta_{r-s} \omega)) \frac{\partial v}{\partial r} \, dx
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + \int_{Q} \left( \frac{\partial v}{\partial r} \right)^{2} + c(\|\nabla v\|^{2} + \|g(r)\|^{2} + \|\tilde{z}(\theta_{r-s} \omega)\|^{2} + \|\Delta \tilde{z}(\theta_{r-s} \omega)\|^{2})
\]
\[
\leq \frac{1}{4} \left( \int_{Q} \frac{\partial v}{\partial r} \right)^{2} + \int_{Q} \left( \frac{\partial v}{\partial r} \right)^{2} + c(\|\nabla v\|^{2} + \|g(r)\|^{2})
\]
\[
+ c(\|1 - \Delta\|^{-1} h^{2} |z(\theta_{r-s} \omega)|^{2} + \|\Delta(1 - \Delta)^{-1} h^{2} |z(\theta_{r-s} \omega)|^{2}).
\]
Hence, by the Sobolev embedding $H^{1}(Q) \hookrightarrow L^{p}(Q)$ with $2 \leq p \leq 6$ we obtain
\[
\| \frac{\partial v}{\partial r} \|_{L^{p}} \leq c(1 + |z(\theta_{r-s} \omega)|^{4} + \|g(r)\|^{2} + \|v(r)\|^{4}) + \int_{-\rho}^{0} \int_{Q} (|v(r + \xi)|^{2} + |z(\theta_{r+\xi} \omega)|^{2}) \, ds \, d\xi.
\]
Integrating (37) over $[s - \rho, s]$ yields
\[
\int_{s-\rho}^{s} \| \frac{\partial v}{\partial r} \|_{L^{p}}^{2} \, dr \leq c \int_{s-\rho}^{s} (1 + |z(\theta_{r-s} \omega)|^{4} + \|g(r)\|^{2} + \|v(r)\|^{4}) \, dr
\]
\[
+ c \int_{s-\rho}^{s} \int_{-\rho}^{0} (|v(r + \xi)|^{2} + |z(\theta_{r+\xi} \omega)|^{2}) \, ds \, d\xi \, dr.
\]
Note that
\[
\int_{s-\rho}^{s} \int_{-\rho}^{0} (\|v(r + \xi)\|^2 + |z(\theta_{r+s}\omega)|^2) d\xi dr
\]
\[
= \int_{-\rho}^{0} \int_{s+\xi-\rho}^{s} (\|v(r)\|^2 + |z(\theta_{r-s}\omega)|^2) d\xi dr
\]
\[
\leq \rho \int_{s-2\rho}^{s} (\|v(r)\|^2 + |z(\theta_{r-s}\omega)|^2) dr.
\]
Then we have
\[
\int_{s-\rho}^{s} \frac{\partial v}{\partial r} dV dr \leq c \int_{s-2\rho}^{s} (1 + |z(\theta_{r-s}\omega)|^4 + \|g(r)\|^2 + \|v(r)\|_V^4) dr. \tag{38}
\]
By (22) we obtain that, for all \( t \geq T_b \)
\[
\sup_{s \geq \tau} \int_{s-\rho}^{s} \|v(r)\|_V^4 dr \leq \rho \sup_{s \leq \xi \in [-\rho, 0]} \|v(s + \xi)\|_V^4 \leq c B^2_b(\tau, \omega).
\]
By (10) and (34) we have
\[
\sup_{s \geq \tau} \int_{s-2\rho}^{s} (1 + |z(\theta_{r-s}\omega)|^4 + \|g(r)\|^2) dr
\]
\[
\leq e^{5\rho} \sup_{s \geq \tau} \int_{-\rho}^{0} e^{\frac{2}{r}(1 + |z(\theta_{r-s}\omega)|^4 + \|g(r+s)\|^2)} dr
\]
\[
\leq e^{5\rho} \sup_{s \geq \tau} \int_{-\infty}^{0} e^{\frac{2}{r}(1 + |z(\theta_{r}\omega)|^4 + \|g(r+s)\|^2)} dr < +\infty.
\]
Hence, we take the supremum of (38) over \( s \in [\tau, +\infty) \) obtain (35) holds.

3.2. Forward uniform tail-estimates of solutions. For each \( k > 0 \), let \( Q_k = \{ x = (x_1, x_2, x_3) \in Q : |x_3| < k \} \) and \( Q_k^c = Q \setminus Q_k \). To overcome the difficulty of Sobolev noncompact embeddings on unbounded channel \( Q \), we need to verify the following result:

**Lemma 3.3.** Suppose that (F), (D), G1, G2 hold and \( h \in \dot{H}^1(Q) \). For each \( \mathcal{B} \in \mathfrak{B} \), \( \tau \in \mathbb{R} \), \( \omega \in \Omega \) and any \( \varepsilon > 0 \), there exist \( K := K(\varepsilon, \tau, \omega) > 0 \) and \( \bar{T}_b := \bar{T}_b(\varepsilon, \mathcal{B}, \tau, \omega) \geq T_b \) such that
\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \|v(s + \tilde{\xi}, s-t, \theta_{-s}\omega, \phi)\|_{\dot{H}^1_0(Q^c_k)}^2 < \varepsilon, \tag{39}
\]
for all \( t \geq \bar{T}_b \), \( \phi \in \mathcal{B}(s-t, \theta_{-t}\omega) \) with \( s \geq \tau \) and \( \rho \in (0, \rho_0) \).

**Proof.** For each \( k > 0 \), let \( \eta_k(x) = \eta(\frac{|x_3|^2}{k}) \), \( \forall x = (x_1, x_2, x_3) \in Q \), where \( \eta(\cdot) : [0, +\infty) \to [0, 1] \) be a smooth function satisfying
\[
\eta(r) = \begin{cases} 
0, & 0 \leq r \leq 1, \\
1, & r \geq 4.
\end{cases} \tag{40}
\]
Multiplying (5) by $\eta_k^2 v(r, s - t, \theta_{t-s} \omega, \phi)$ and after some simple calculations we have

$$\frac{1}{2} \frac{d}{dt} \int_Q \eta_k^2 (|v|^2 + \|\nabla v\|^2) dx + \nu \int_Q \eta_k^2 \|\nabla v\|^2 dx$$

$$= - \int_Q v (\nabla (\frac{\partial v}{\partial r}) \cdot \nabla \eta_k^2) dx - \nu \int_Q v (\nabla v \cdot \nabla \eta_k^2) dx$$

$$+ \int_Q \eta_k^2 \mathbf{F}(v + \tilde{z}(\theta_{t-s} \omega)) \cdot \nabla v dx + \int_Q \mathbf{F}(v + \tilde{z}(\theta_{t-s} \omega)) \cdot \nabla \eta_k^2 dx$$

$$+ \int_Q \eta_k^2 v \int_\rho^0 f(\xi, v(r + \xi) + \tilde{z}(\theta_{r+s} \omega)) d\xi dx$$

$$+ \int_Q \eta_k^2 v (g(r) + \tilde{z}(\theta_{t-s} \omega) + (\nu - 1) \Delta \tilde{z}(\theta_{t-s} \omega)) dx. \quad (41)$$

Note that $\|\nabla \eta_k\|_{\infty} \leq \frac{\xi}{k}$ and thus $\|\nabla \eta_k\|_{\infty} \leq \frac{\xi}{k}$. For the first two term on the right-hand side of (41), by (37) we have

$$- \int_Q v (\nabla (\frac{\partial v}{\partial r}) \cdot \nabla \eta_k^2) dx - \nu \int_Q v (\nabla v \cdot \nabla \eta_k^2) dx \leq \frac{c}{k} (\|\nabla (\frac{\partial v}{\partial r})\|^2 + \|v\|_{\infty}^2)$$

$$\leq \frac{c}{k} (1 + |z(\theta_{t-s} \omega)|^4 + ||g(r)||^2 + \|v(r)\|_{\infty}^2 + \int_{-\rho}^0 (||v(r + \xi)||^2 + |z(\theta_{r+s} \omega)|^2) d\xi). \quad (42)$$

For the third term on the right-hand side of (41), we have

$$\int_Q \eta_k^2 \mathbf{F}(v + \tilde{z}(\theta_{t-s} \omega)) \cdot \nabla v dx$$

$$= \int_Q \eta_k^2 (\mathbf{F}(u) \cdot \nabla u) dx - \int_Q \eta_k^2 \mathbf{F}(v + \tilde{z}(\theta_{t-s} \omega)) \cdot \nabla \tilde{z}(\theta_{t-s} \omega) dx.$$ 

By (18) and (19) we obtain

$$\int_Q \eta_k^2 (\mathbf{F}(u) \cdot \nabla u) dx = \int_Q \eta_k^2 (\nabla \cdot \mathbf{G}(u)) dx = - \int_Q \mathbf{G}(u) \cdot \nabla \eta_k^2 dx$$

$$\leq \frac{c}{k} \int_Q (||v + \tilde{z}(\theta_{t-s} \omega)||^2 + |v + \tilde{z}(\theta_{t-s} \omega)|^3) dx$$

$$\leq \frac{c}{k} (||v||^2 + ||v||_{\infty}^3 + \|\tilde{z}(\theta_{t-s} \omega)||^2 + \|\tilde{z}(\theta_{t-s} \omega)||_{\infty}^3).$$

Similarly to (26) and (27), we have

$$- \int_Q \eta_k^2 \mathbf{F}(v + \tilde{z}(\theta_{t-s} \omega)) \cdot \nabla \tilde{z}(\theta_{t-s} \omega) dx$$

$$\leq \frac{\nu \lambda}{32} \int_Q \eta_k^2 |v|^2 dx + 2a_2 \beta_0 ||h|| \|v\|\tilde{z}(\theta_{t-s} \omega)\| \int_Q \eta_k^2 |v|^2 dx$$

$$+ c \int_Q \eta_k^2 (|\tilde{z}(\theta_{t-s} \omega)|^2 + |\nabla \tilde{z}(\theta_{t-s} \omega)|^2) dx + c \int_Q \eta_k^2 \tilde{z}(\theta_{t-s} \omega)^2 |\nabla \tilde{z}(\theta_{t-s} \omega)| dx.$$
Hence, we have
\[
\int_Q \eta_k^2 \mathbf{F}(v + \tilde{z}(\theta_{r-s}\omega)) \cdot \nabla v dx \leq \frac{c}{k} (\|v\|^2 + \|v\|^3 + \|\tilde{z}(\theta_{r-s}\omega)\|^2 + \|\tilde{\tilde{z}}(\theta_{r-s}\omega)\|^3) \\
+ \frac{\nu \lambda}{32} \int_Q \eta_k^2 |v|^2 dx + c \int_Q \eta_k^2 (|\tilde{z}(\theta_{r-s}\omega)|^2 + \|\nabla \tilde{z}(\theta_{r-s}\omega)\|^2) dx \\
+ 2 \alpha_2 \beta_0 \|h\|_V |z(\theta_{r-s}\omega)| \int_Q \eta_k^2 |v|^2 dx + c \int_Q \eta_k^2 |\tilde{z}(\theta_{r-s}\omega)|^2 |\nabla \tilde{z}(\theta_{r-s}\omega)| dx. \tag{43}
\]

For the forth term on the right-hand side of (41), by (7) we have
\[
For the last term on the right-hand side of (41), we have
\[
\int_Q \eta_k^2 v(g(r) + \tilde{z}(\theta_{r-s}\omega) + (\nu - 1)\Delta \tilde{\tilde{z}}(\theta_{r-s}\omega)) dx \leq \frac{\nu \lambda}{32} \int_Q \eta_k^2 |v|^2 dx + c \int_Q \eta_k^2 (|g(r)|^2 + |\tilde{z}(\theta_{r-s}\omega)|^2 + |\Delta \tilde{\tilde{z}}(\theta_{r-s}\omega)|^2) dx. \tag{45}
\]

Note that \(\eta_k v \in V\) and so by the Poincaré inequality we obtain
\[
\int_Q \eta_k^2 v^2 dx \leq \frac{1}{\lambda} \int_Q |\nabla (\eta_k v)|^2 dx \leq \frac{2}{\lambda} \int_Q |\nabla \eta_k|^2 |v|^2 dx + \int_Q \eta_k^2 |\nabla v|^2 dx \\
\leq \frac{c}{\lambda k^2} \|v\|^2 + \frac{2}{\lambda} \int_Q \eta_k^2 |\nabla v|^2 dx. \tag{46}
\]

It follows from (41)-(46) that
\[
d \int Q \eta_k^2 (|v|^2 + |\nabla v|^2) dx \leq (\beta |z(\theta_{r-s}\omega)| - \tilde{v}) \int Q \eta_k^2 (|v|^2 + |\nabla v|^2) dx \]
\[
+ \frac{c}{k} (1 + |z(\theta_{r-s}\omega)|^4 + \|g(r)\|^4 + \|v(r)\|^4 + \int_{-\rho}^0 (\|v(r + \xi)\|^2 + |z(\theta_{r-s}\omega)|^2) d\xi) \\
+ c \int_Q \eta_k^2 (|g(r)|^2 + |\tilde{z}(\theta_{r-s}\omega)|^2 + |\nabla \tilde{z}(\theta_{r-s}\omega)|^2 + |\Delta \tilde{\tilde{z}}(\theta_{r-s}\omega)|^2) dx \\
+ c \int_Q \eta_k^2 |\tilde{z}(\theta_{r-s}\omega)|^2 |\nabla \tilde{z}(\theta_{r-s}\omega)|^2 + 2 \int_Q \eta_k v \int_{-\rho}^0 f(\xi, v(r + \xi) + \tilde{z}(\theta_{r-s}\omega)) d\xi dx.
\]

Multiplying the above inequality by \(e^{\int_{s-t}^\xi \gamma(\theta_{r-s}\omega) dt}\), and integrating this result over \([s - t, s + \xi] \cap \xi \in [-\sigma, 0]\) we get
\[
e^{\int_{s-t}^\xi \gamma(\theta_{r-s}\omega) dt} \int_Q \eta_k^2 (|v(s + \xi, s - t, \theta_{r-s}\omega, \phi)|^2 + |\nabla v|^2) dx \leq \|\phi\|_V^2 + \int_{s-t}^{s+\xi} (\beta |z(\theta_{r-s}\omega)| + 3\gamma(\theta_{r-s}\omega) - \tilde{v}) e^{\int_{s-t}^\xi \gamma(\theta_{r-s}\omega) dt} \int_Q \eta_k^2 (|v|^2 + |\nabla v|^2) dx dr \\
+ \frac{c}{k} \int_{s-t}^{s+\xi} e^{\int_{s-t}^\xi \gamma(\theta_{r-s}\omega) dt} (1 + |z(\theta_{r-s}\omega)|^4 + \|g(r)\|^4) dr \\
+ \frac{c}{k} \int_{s-t}^{s+\xi} e^{\int_{s-t}^\xi \gamma(\theta_{r-s}\omega) dt} (\|v(r)\|_V^2 + \int_{-\rho}^0 (\|v(r + \xi)\|^2 + |z(\theta_{r-s}\omega)|^2) d\xi) dr.
Similarly to (32) we obtain
\[ \|
\int_{Q_s} \eta_k^2 (|v(s + t, \xi, s - t, \theta - \omega, \phi)|^2 + |\nabla v|^2) dx dr
\leq c(e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} + \int_{-\rho}^{\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} dr) ||\phi||_{V_\rho}^2
+ \frac{c}{K} \int_{s-t}^{s+\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} (1 + |z(\theta \omega)|^4 + \|g(r)\|^2) dr
+ \frac{c}{K} \int_{s-t}^{s+\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} (\|v(r)\|^4 + \int_{-\rho}^{0} (\|v(r + \xi)\|^2 + |z(\theta + \xi \omega)|^2) d\xi) dr
+ c \int_{s-t}^{s+\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} \int_{Q_s} \eta_k^2 |g(r, x, \omega)|^2 dx dr
+ c \int_{s-t}^{s+\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} \int_{Q_s} \eta_k^2 (|\bar{z}(\theta \omega)|^2 + |\nabla \bar{z}(\theta \omega)|^2 + |\Delta \bar{z}(\theta \omega)|^2) dx dr
+ c \int_{s-t}^{s+\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} \int_{Q_s} \eta_k^2 |\bar{z}(\theta \omega)|^2 |\nabla \bar{z}(\theta \omega)| dx dr. \tag{47}
\]

We now discuss the limits of each term on the right-hand side of (47). For the first term, By (17) and (33) we obtain, for all \( t \geq T_b \)
\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} (e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} + \int_{-\rho}^{\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} dr) ||\phi||_{V_\rho}^2
\leq c e^{-\frac{3\gamma}{\tau} t} \sup_{s \geq \tau} ||B(s - t, \theta - \omega)||_{V_\rho}^2 \rightarrow 0 \text{ as } t \rightarrow +\infty. \tag{48} \]

For the second term, by (34) we have
\[ \frac{c}{K} \sup_{s \geq \tau} \int_{s-t}^{s+\xi} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} (1 + |z(\theta \omega)|^4 + \|g(r)\|^2) dr
\leq \frac{c}{K} \sup_{s \geq \tau} \int_{-\infty}^{0} e^{t \int_{-\rho}^{\xi} 3\gamma(\theta \omega) dt} (1 + |z(\theta \omega)|^4 + \|g(r)\|^2) dr.
By the same method as in (32) we obtain that, for all $r \geq s - t$

\[
\|v(r, s - t, \theta_{-s}, \phi)\|_{V^r} \leq c\|\phi\|_{V^r} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} \left( e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} + \int_{-t}^{-s} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} d\xi \right)
+ ce^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} \int_{-t}^{-s} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} (1 + |z(\theta_{-s})|^4 + \|g(\tilde{r} + s)\|^2) d\tilde{r}.
\]

(50)

Note that

\[
\int_{s-t}^{s+t} e^{\int_{s-t}^r \gamma(\theta_{-s})d\xi} d\xi \leq e^{\gamma(\tilde{r})} \int_{s-t}^{s+t} e^{\int_{s-t}^r \gamma(\theta_{-s})d\xi} d\xi \leq e^{\gamma(\tilde{r})} \int_{-t}^{-s} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} (1 + |z(\theta_{-s})|^4 + \|g(\tilde{r} + s)\|^2) d\tilde{r}.
\]

(51)

Then, by (50) we get

\[
\int_{s-t}^{s+t} e^{\int_{s-t}^r \gamma(\theta_{-s})d\xi} d\xi \leq c\|\phi\|_{V^r} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} (1 + |z(\theta_{-s})|^4 + \|g(\tilde{r} + s)\|^2) d\tilde{r}.
\]

By (17) and (33) we obtain, for all $t \geq T_0$

\[
c\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \|\phi\|_{V^r} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} \leq c \int_{-t}^{-s} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} d\xi \|B(s - t, \theta_{-t})\|_{V^r}^2
\leq ce^{-\frac{3\tau}{4}} \sup_{s \geq \tau} \|B(s - t, \theta_{-t})\|_{V^r}^2 < +\infty.
\]

(52)

By (17), (33) and (34) we obtain, for all $t \geq T_0$

\[
c\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \|\phi\|_{V^r} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} \times \left( e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} + \int_{-t}^{-s} e^{\int_{s-t}^0 \gamma(\theta_{-s})d\xi} d\tilde{r} \right) d\tilde{r}
\]

(49)
\[ \leq c \int_{-\infty}^{0} e^{\int_{0}^{t} \gamma(\theta_{\omega}) \, dt} \, dr (e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} + \int_{-\infty}^{0} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \, d\bar{t}) \sup_{s \geq \tau} \| B(s-t, \theta_{-t} \omega) \|^2_{V_{\phi}} \]
\[ \leq C(\omega) \int_{-\infty}^{0} e^{\bar{r}} \, dr \sup_{s \geq \tau} \| B(s-t, \theta_{-t} \omega) \|^2_{V_{\phi}} < +\infty. \] 

By (10) and (34) we have

\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{0}^{r} \gamma(\theta_{\omega}) \, dt} \left( e^{\int_{t}^{r} \gamma(\theta_{\omega}) \, dt} (1 + |z(\theta_{r} \omega)|^4 + \|g(\bar{r} + s)\|^2) \, dr \right) \]
\[ \leq \int_{-\infty}^{0} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \, dr \sup_{s \geq \tau} \int_{-\infty}^{0} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} (1 + |z(\theta_{r} \omega)|^4 + \|g(\bar{r} + s)\|^2) \, d\bar{r} \]
\[ \leq C(\omega) \int_{-\infty}^{0} e^{\bar{r}} \, dr \sup_{s \geq \tau} \int_{-\infty}^{0} e^{\frac{\bar{r}}{2}} (1 + r^4 + \|g(r + s)\|^2) \, dr < +\infty. \] 

Note that

\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{0}^{r} \gamma(\theta_{\omega}) \, dt} |z(\theta_{r-\xi} \omega)|^2 \, dr \leq C(\omega) \int_{-\infty}^{0} e^{\frac{\bar{r}}{2}} |\bar{r}|^2 \, dr < +\infty. \] 

Substituting the estimates (52)-(55) into (51) yields

\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{0}^{r} \gamma(\theta_{\omega}) \, dt} \left( \|v_{\phi}(r + \xi)\|^2 + |z(\theta_{r+\xi} \omega)|^2 \right) d\xi < +\infty. \] 

We now treat the biquadrate term. By (50) we have

\[ \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \|v(r)\|^2_{V_{\phi}} \, dr \leq e^{\rho} \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \|v(r)\|^2_{V_{\phi}} \, dr \]
\[ \leq c \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \left( \phi_{\phi}^q \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \left( e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} (1 + |z(\theta_{r} \omega)|^4 + \|g(\bar{r} + s)\|^2) \, dr \right) \right)^2 \left( e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} + \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \right) \|v_{\phi}(r + \xi)\|^2 \, dr \]
\[ + c \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \left( \phi_{\phi}^q \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \left( e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} (1 + |z(\theta_{r} \omega)|^4 + \|g(\bar{r} + s)\|^2) \, dr \right) \right)^2 \left( e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} + \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \right) \|v_{\phi}(r + \xi)\|^2 \, dr \]
\[ + c \int_{-\infty}^{0} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \left( \phi_{\phi}^q \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \left( e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} (1 + |z(\theta_{r} \omega)|^4 + \|g(\bar{r} + s)\|^2) \, dr \right) \right)^2 \left( e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} + \int_{s-t}^{s+\xi} e^{\int_{0}^{\tau} \gamma(\theta_{\omega}) \, dt} \right) \|v_{\phi}(r + \xi)\|^2 \, dr. \]
Similarly to (53) we have
\[
\left(\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \|\phi\| V_s \int_{-\infty}^{0} e^{\int_{s-t}^{t} \gamma(v_{t}) dt} dr (e^{\int_{-\infty}^{0} \gamma(v_{t}) dt} + \int_{-\infty}^{t} e^{\int_{s-t}^{t} \gamma(v_{t}) dt} dr)^2 \right.
\]
\[
\leq C(\omega) \int_{-\infty}^{0} e^{2\tau} dr \left( e^{2\tau} \sup_{s \geq \tau} \|B(s-t, \theta_{\ell} w)\| V_s \right)^2 < +\infty.
\]

Similarly to (54) we find
\[
\left(\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{-\infty}^{0} e^{\int_{s-t}^{t} \gamma(v_{t}) dt} dr \left( \int_{-\infty}^{0} e^{\int_{s-t}^{t} \gamma(v_{t}) dt} (1 + |z(\theta_{\ell} w)|^4 + \|g(r + s)\|^2) dr \right)^2 \right.
\]
\[
\leq C(\omega) \int_{-\infty}^{0} e^{2\tau} dr \left( \sup_{s \geq \tau} \int_{-\infty}^{0} e^{2\tau} (1 + r^4 + \|g(r + s)\|^2) dr \right)^2 < +\infty.
\]

Then we have
\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{s-x}^{x} \gamma(v_{t}) dt} \|v(r)\| V_s dr < +\infty. \tag{57}
\]

For the third term, by (56) and (57) we obtain
\[
\frac{c}{k} \sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{s-x}^{x} \gamma(v_{t}) dt} dr
\]
\[
\times (\|v(r)\| V_s^4 + \int_{-\rho}^{0} (\|v(r + \xi)\|^2 + |z(\theta_{\ell} w)|^4) d\xi) dr \to 0 \text{ as } k \to +\infty. \tag{58}
\]

For the forth term, by (11), (14) and (34) we obtain
\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{s-x}^{x} \gamma(v_{t}) dt} \int_{Q} \eta^2_{K} |g(r, x)|^2 dr dx
\]
\[
\leq c e^{2\rho} \sup_{s \geq \tau} \int_{-\infty}^{0} e^{\int_{s-t}^{t} \gamma(v_{t}) dt} \int_{Q} |g(r + s, x)|^2 dr dx
\]
\[
\leq C(\omega) \sup_{s \geq \tau} \int_{-\infty}^{0} e^{3\tau} \int_{Q} |g(r + s, x)|^2 dr dx \to 0 \text{ as } k \to +\infty. \tag{59}
\]

For the fifth term, by \((I - \Delta)^{-1}h(\cdot) \in H^1_0(Q) \cap H^2(Q)\) we get
\[
c \sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \int_{s-t}^{s+\xi} e^{\int_{s-x}^{x} \gamma(v_{t}) dt}
\]
\[
\times \int_{Q} \eta^2_{K} (|\varepsilon(\theta_{\ell} w)|^2 + |\nabla \varepsilon(\theta_{\ell} w)|^2 + |\Delta \varepsilon(\theta_{\ell} w)|^2) dr dx
\]
\[
\leq C(\omega) \int_{-\infty}^{0} e^{3\tau} |\varepsilon(\theta_{\ell} w)|^2 dr
\]
\[
\times \int_{Q} \left( |(I - \Delta)^{-1} h|^2 + |\nabla (I - \Delta)^{-1} h|^2 + |\Delta (I - \Delta)^{-1} h|^2 \right) dx \to 0. \tag{60}
\]
as \( k \to +\infty \). For the last term, we have

\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \int_{s-t}^{s+\tilde{\xi}} e^{\int_{s-t}^{r} 3\gamma(\theta_r - s\omega)dr} \int_{Q} \eta_k^2|\tilde{z}(\theta_{r-s}\omega)|^2 |\nabla \tilde{z}(\theta_{r-s}\omega)|dxdr
\]

\[
\leq C(\omega) \int_{-\infty}^{0} e^{\frac{3\gamma}{2}|z(\theta_{r}\omega)|^3 dr} \int_{Q_k^\rho} |(I - \Delta)^{-1} h|^2 |\nabla (I - \Delta)^{-1} h| dx
\]

\[
\leq C(\omega) \int_{-\infty}^{0} e^{\frac{3\gamma}{2}|z(\theta_{r}\omega)|^3 dr} \int_{Q_k^\rho} (|(I - \Delta)^{-1} h|^4 + |\nabla (I - \Delta)^{-1} h|^2) dx \to 0, \quad (61)
\]

as \( k \to +\infty \). Substitute (48), (49) and (58)-(61) into (47) yields

\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \|v(s + \tilde{\xi}, s - t, \theta_{-s\omega}, \phi)\|_{H_k^1(Q_2k)}^2
\]

\[
\leq \sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \int_{Q} \eta_k^2 (|v(s + \tilde{\xi}, s - t, \theta_{-s\omega}, \phi)|^2 + |\nabla v|^2) dx \to 0,
\]

as \( k, t \to +\infty \). So, we obtain (39) as desired. \( \square \)

### 3.3. Forward flattening of solutions in a bounded domain

In this subsection, we will show that the forward flattening of \( \Phi_\rho \) in (9) inside a bounded subdomain of \( Q \), which is necessary to establish the forward \( \mathfrak{B} \)-pullback asymptotic compactness of \( \Phi_\rho \). To this end, we define

\[
\varphi_k(x) = 1 - \eta_k(x) = 1 - \eta_k \left( \frac{|x_3|^2}{k^2} \right) = \left\{ \begin{array}{ll}
1, & |x_3| \leq k, \\
0, & |x_3| \geq 2k,
\end{array} \right.
\]

where \( \eta(\cdot) \) is given in (40). Let \( \bar{v} = \varphi_k v \) for each \( v \in V \), then it is easy to check that \( \bar{v} \in H_0^1(Q_{2k}) \). In addition, the operator \(-\Delta \) has a family of eigenfunctions \( \{e_i\}_{i=1}^\infty \subset H_0^1(Q_{2k}) \) with the corresponding eigenvalues: \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_i \to +\infty \) as \( i \to +\infty \). Note that \( \{e_i\}_{i=1}^\infty \) be the orthonormal basis of \( L^2(Q_{2k}) \). Let \( P_i : L^2(Q_{2k}) \to H_i = \text{span}\{e_1, e_2, \cdots, e_i\} \), then \( P_i \) is an orthogonal projection. Hence, \( \bar{v} \) has the following orthogonal decomposition:

\[
\bar{v} = P_i \bar{v} \oplus (I - P_i) \bar{v} = \bar{v}_{i,1} + \bar{v}_{i,2}, \quad i \in \mathbb{N}.
\]

Multiplying Eq. (5) by \( \varphi_k \) yields

\[
\varphi_k \frac{\partial (v - \Delta v)}{\partial t} - \nu \varphi_k \Delta v + \varphi_k \nabla \cdot F(v + \tilde{z}(\theta t \omega)) = \varphi_k \int_{-\rho}^{0} f(\xi, v(t + \xi) + \tilde{z}(\theta_{t+\xi} \omega)) d\xi + \varphi_k g(t) + \varphi_k \bar{z}(\theta t \omega) + (\nu - 1) \varphi_k \Delta \tilde{z}(\theta t \omega).
\]

Then, by \( \bar{v} = \varphi_k v \) and \( \frac{\partial \bar{v}}{\partial t} = \varphi_k \frac{\partial v}{\partial t} \) we obtain
\[
\frac{\partial (\bar{v} - \Delta \bar{v})}{\partial t} - \nu \Delta \bar{v} + \varphi_k \nabla \cdot \mathbf{F}(v + \hat{z}(\theta_{1,\omega}))
\]
\[
= \varphi_k \int_{-\rho}^{0} f(\xi, v(t + \xi) + \hat{z}(\theta_{1,\xi} \omega)) d\xi + \varphi_k g(t)
\]
\[
+ \varphi_k \hat{z}(\theta_{1,\omega}) + (\nu - 1) \varphi_k \Delta \hat{z}(\theta_{1,\omega}) - \frac{\partial v}{\partial t} \Delta \varphi_k
\]
\[
- 2 \nabla \varphi_k \cdot \nabla \frac{\partial v}{\partial t} - \nu v \Delta \varphi_k - 2 \nu \nabla \varphi_k \cdot \nabla v,
\]

(64)

with

\[
\bar{v}(\tau + \xi, x) = \varphi_k \phi := \tilde{\phi}(\xi, x), \quad x \in Q_{2k}; \quad \bar{v}(t, x) = 0, \quad t \geq \tau, \quad x \in \partial Q_{2k}.
\]

**Lemma 3.4.** Suppose that (F), (D), (G1) hold and \( h \in H^1_0(Q) \). For each \( B \in \mathfrak{B}, \tau \in \mathbb{R}, \omega \in \Omega \) and any \( \varepsilon > 0 \), there exist \( i_0 := i_0(\varepsilon, \tau, \omega) \in \mathbb{N} \) and \( \tilde{T}_b := \tilde{T}_b(\varepsilon, B, \tau, \omega) \geq \tilde{T}_b \) such that

\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \|(I - P_l) \bar{v}(s + \xi, s - t, \theta_{-s, \omega}, \tilde{\phi})\|^2_{H^1_0(Q_{2k})} < \varepsilon,
\]

(65)

for all \( i \geq i_0, \ t \geq \tilde{T}_b, \ \phi \in \mathcal{B}(s - t, \theta_{-s, \omega}) \) with \( s \geq \tau \) and \( \rho \in (0, \rho_0] \).

**Proof.** Taking the inner product of (64) with \( \bar{v}_{i,2}(r, s - t, \theta_{-s, \omega}, \tilde{\phi}) \) in \( L^2(Q_{2k}) \) yields

\[
\frac{1}{2} \frac{d}{dr} \|\bar{v}_{i,2}\|^2 + \nu \|\nabla \bar{v}_{i,2}\|^2 + (\varphi_k \nabla \cdot \mathbf{F}(u), \bar{v}_{i,2})
\]
\[
= (\varphi_k \int_{-\rho}^{0} f(\xi, v(r + \xi) + \hat{z}(\theta_{r+\xi}-s\omega)) d\xi + \varphi_k g(r), \bar{v}_{i,2})
\]
\[
+ (\varphi_k \hat{z}(\theta_{r-s\omega}) + (\nu - 1) \varphi_k \Delta \hat{z}(\theta_{r-s\omega}), \bar{v}_{i,2})
\]
\[
- (\frac{\partial v}{\partial r} \Delta \varphi_k + 2 \nabla \varphi_k \cdot \nabla \frac{\partial v}{\partial r} + \nu v \Delta \varphi_k + 2 \nu \nabla \varphi_k \cdot \nabla v, \bar{v}_{i,2}).
\]

(66)

For the nonlinear term in (66), by (6), \( \|\varphi_k\|_{\infty} \leq 1 \), the Hölder inequality and \( \|\nabla \bar{v}_{i,2}\| \geq \lambda_{i+1}^{\frac{1}{2}}\|\bar{v}_{i,2}\| \) we have

\[
- (\varphi_k \nabla \cdot \mathbf{F}(u), \bar{v}_{i,2}) = - \int_{Q_{2k}} \varphi_k \bar{v}_{i,2} \nabla \cdot \mathbf{F}(v + \hat{z}(\theta_{r-s\omega})) dx
\]
\[
\leq \int_{Q_{2k}} (|\alpha_1 + \alpha_2 (v + \hat{z}(\theta_{r-s\omega}))| |\nabla (v + \hat{z}(\theta_{r-s\omega}))|) dx
\]
\[
\leq c(\|\nabla v\| + \|\nabla \hat{z}(\theta_{r-s\omega})\|) \|\bar{v}_{i,2}\| + c(\|\nabla v\| + \|\nabla \hat{z}(\theta_{r-s\omega})\|)(\|v\| + \|\hat{z}(\theta_{r-s\omega})\|) \|\bar{v}_{i,2}\| + c(\|\nabla \hat{z}(\theta_{r-s\omega})\|)^2 \|\bar{v}_{i,2}\|^{\frac{1}{2}}
\]
\[
\leq c \lambda_{i+1}^{-\frac{1}{2}} (|\|v\| + \|\hat{z}(\theta_{r-s\omega})\|) \|\bar{v}_{i,2}\| + c \lambda_{i+1}^{-\frac{1}{2}} (|\|v\| + \|\hat{z}(\theta_{r-s\omega})\|) \|\bar{v}_{i,2}\| \|\nabla \bar{v}_{i,2}\|^{\frac{1}{2}}
\]
\[
\leq \frac{\nu}{10} \|\bar{v}_{i,2}\|^2 + c \lambda_{i+1}^{-1}(\|\|v\| + \|\hat{z}(\theta_{r-s\omega})\|\|\bar{v}_{i,2}\| + \|\nabla \bar{v}_{i,2}\|)^2
\]
\[
\leq \frac{\nu}{10} \|\bar{v}_{i,2}\|^2 + c \lambda_{i+1}^{-1}(\|\|v\| + \|\hat{z}(\theta_{r-s\omega})\|\|\bar{v}_{i,2}\|)
\]
\[
\leq \frac{\nu}{10} \|\bar{v}_{i,2}\|^2 + c (\lambda_{i+1}^{-1} + \lambda_{i+1}^{-\frac{1}{2}})(\|\|v\| + \|\hat{z}(\theta_{r-s\omega})\|)^2 + c \lambda_{i+1}^{-1},
\]

(67)
where we use the Gagliardo-Nirenberg inequality \((\|\bar{v}_{i,2}\|_3 \leq c\|\nabla \bar{v}_{i,2}\|_2^{\frac{3}{2}}\|\bar{v}_{i,2}\|_2^{\frac{1}{2}})\) in the third line. By (8) we obtain

\[
(\varphi_k \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{r+\xi-s}\omega)))d\xi, \bar{v}_{i,2})
\leq \left( \int_{Q} \left( \int_{-\rho}^{0} f(\xi, v(r + \xi) + \tilde{z}(\theta_{r+\xi-s}\omega))d\xi \right)^2 dx \right)^{\frac{1}{2}} \|\bar{v}_{i,2}\|_2^{\frac{1}{2}}
\leq \lambda_{i+1}^{-\frac{1}{2}} \left( \int_{Q} \left( \int_{-\rho}^{0} L_f^2(\xi)d\xi \int_{-\rho}^{0} (|v(r + \xi)| + |\tilde{z}(\theta_{r+\xi-s}\omega)|)^2 d\xi \right)^2 dx \right)^{\frac{1}{2}} \|\nabla \bar{v}_{i,2}\|_2^{\frac{1}{2}}
\leq \frac{\nu}{10} \|\nabla \bar{v}_{i,2}\|^2 + 2 \lambda_{i+1}^{-1} \|L_f(\xi)d\xi\|^2 \int_{-\rho}^{0} (|v(r + \xi)|^2 + |\tilde{z}(\theta_{r+\xi-s}\omega)|^2) d\xi. \quad (68)
\]

Note that

\[
(\varphi_k g(r), \bar{v}_{i,2}) \leq \lambda_{i+1}^{-\frac{1}{2}} \|g(r)||\|\nabla \bar{v}_{i,2}\| \leq \frac{\nu}{10} \|\nabla \bar{v}_{i,2}\|^2 + c \lambda_{i+1}^{-1} \|g(r)||^2, \quad (69)
\]

\[
(\varphi_k \tilde{z}(\theta_{r-s}\omega) + (\nu - 1)\varphi_k \Delta \tilde{z}(\theta_{r-s}\omega), \bar{v}_{i,2}) \leq \frac{\nu}{10} \|\nabla \bar{v}_{i,2}\|^2 + c \lambda_{i+1}^{-1} \|z(\theta_{r-s}\omega)||^2 \quad (70)
\]

and

\[
- \left( \frac{\partial v}{\partial r} \Delta \varphi_k + 2 \varphi_k \nabla \cdot \nabla v + \nu \Delta \varphi_k + 2\nu \varphi_k \cdot \nabla v, \bar{v}_{i,2} \right)
\leq \frac{\nu}{10} \|\nabla \bar{v}_{i,2}\|^2 + c \lambda_{i+1}^{-1} (\|v||^2_v + \|\frac{\partial v}{\partial r}||^2_v). \quad (71)
\]

Substituting (67)-(71) into (66), by (37) we obtain

\[
\frac{d}{dr} \|\bar{v}_{i,2}(r)||^2_v + \bar{v}_i(r)||^2_v
\leq c(\lambda_{i+1}^{-1} + \lambda_{i+1}^{-\frac{1}{2}})(1 + |z(\theta_{r-s}\omega)||^4 + \|g(r)||^2 + \|v(r)||^2_v)
+ c\lambda_{i+1}^{-1} \int_{-\rho}^{0} (|v(r + \xi)||^2 + |z(\theta_{r+\xi-s}\omega)||^2) d\xi.
\]

Since \(\lambda_{i+1} \to +\infty\) as \(i \to +\infty\), we have \(\lambda_{i+1}^{-1} + \lambda_{i+1}^{-\frac{1}{2}} \to 0\) as \(i \to +\infty\). Hence, for any \(\varepsilon > 0\) there exists a \(i_0 = i_0(\varepsilon) > 0\) such that \(\lambda_{i+1}^{-1} + \lambda_{i+1}^{-\frac{1}{2}} < \varepsilon\) for all \(i \geq i_0\). Then, we have for all \(i \geq i_0\)

\[
\frac{d}{dr} e^{\rho r} \|\bar{v}_{i,2}(r)||^2_v \leq c\varepsilon e^{\rho r} (1 + |z(\theta_{r-s}\omega)||^4 + \|g(r)||^2 + \|v(r)||^2_v)
+ c\varepsilon e^{\rho r} \int_{-\rho}^{0} (|v(r + \xi)||^2 + |z(\theta_{r+\xi-s}\omega)||^2) d\xi.
\]

Integrating the above inequality over \([s - t, s + \xi]\) yields

\[
\|\bar{v}_{i,2}(s + \xi, s - t, \theta_{s-s}\omega, \phi)\|_V^2
\leq e^{\rho s} e^{-\rho t} \|\phi\|_V^2 + c\varepsilon e^{\rho s} \int_{s-t}^{s+\xi} e^{\rho (r-s)} (1 + |z(\theta_{r-s}\omega)||^4 + \|g(r)||^2 + \|v(r)||^2_v) dr
+ c\varepsilon e^{\rho s} \int_{s-t}^{s+\xi} e^{\rho (r-s)} \int_{-\rho}^{0} (|v(r + \xi)||^2 + |z(\theta_{r+\xi-s}\omega)||^2) d\xi dr. \quad (72)
\]
Note that
\[ \int_{s-t}^{s-\xi} e^{\rho(r-s)} \int_{-\rho}^{0} \left( \|v(r+\xi)\|^2 + |z(\theta_{r+\xi-s}\omega)|^2 \right) d\xi dr \leq \rho e^{\rho} \int_{s-t-\rho}^{s-\xi} e^{\rho(r-s)} \left( \|v(r)\|^2 + |z(\theta_{r-s}\omega)|^2 \right) dr \]
\[ \leq c \int_{s-t-\rho}^{s-\xi} e^{\rho(r-s)} |z(\theta_{r-s}\omega)|^2 dr + c\|\phi\|_V^2 \int_{-\rho}^{0} e^{\rho} dr \]
\[ + c \int_{s-t-\rho}^{s-\xi} e^{\rho(r-s)} \left( \|\phi\|^2_{V}, e^{l_0} \gamma(\theta \omega) dt + \int_{-\rho}^{0} e^{l_0} \gamma(\theta \omega) dt d\bar{\Gamma} \right) dr \]
\[ + c \int_{s-t-\rho}^{s-\xi} e^{\rho(r-s)} \left( e^{l_0} \gamma(\theta \omega) dt \right) \int_{-\rho}^{0} e^{l_0} \gamma(\theta \omega) dt (1 + |z(\theta \omega)|^4 + \|g(\bar{r}+s)|^2) d\bar{\Gamma} \right) dr. \]

Since \( r-s \leq 0 \) and \( \gamma(\omega) \leq \frac{\bar{\nu}}{3} \), we have \( e^{\rho(r-s)} e^{l_0} \gamma(\theta \omega) dt \leq e^{\frac{\bar{\nu}}{3}(r-s)} \). Hence, by the same method as in (56) yields
\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \int_{s-t}^{s+\xi} e^{\rho(r-s)} \int_{-\rho}^{0} \left( \|v(r+\xi)\|^2 + |z(\theta_{r+\xi-s}\omega)|^2 \right) d\xi dr < +\infty. \] (73)

Similarly, by the same method as in (57) yields
\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \int_{s-t}^{s+\xi} e^{\rho(r-s)} \|v(r)\|_V^2 dr < +\infty. \] (74)

It is easy to see that
\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \int_{s-t}^{s+\xi} e^{\rho(r-s)} (1 + \|g(r)\|^2 + |z(\theta_{r-s}\omega)|^4) dr < +\infty. \] (75)

By (17) we have
\[ e^{\rho} e^{-\rho t} \sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \|\phi\|^2_{V} \leq e^{\rho} e^{-\rho t} \sup_{s \geq \tau} \|B(s-t,\theta_{-t}\omega)\|^2_{V} \rightarrow 0 \text{ as } t \rightarrow +\infty. \] (76)

Inserting (73)-(76) into (72) yields
\[ \sup_{s \geq \tau} \sup_{\xi \in [-\rho,0]} \|\tilde{V}_{s,t} \|_{V} < \varepsilon, \]
for all \( i \geq i_0, t \geq \bar{T}_b, \phi \in B(s-t,\theta_{-t}\omega) \) with \( s \geq \tau \) and \( \rho \in (0,\rho_0] \). The proof is complete. \( \square \)

3.4. Existence and forward compactness of PRAs.

Lemma 3.5. Suppose that (F), (D), (G1) hold and \( h \in H_0^1(Q) \). For each fixed \( \rho > 0 \), we have
(i) \( \Phi_{\rho} \) in (9) has a \( \mathcal{D} \)-pullback random absorbing set \( K_d = \{ K_d(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{D} \), defined by
\[ K_d(\tau,\omega) = \{ \phi \in V_\rho : \|\phi\|^2_{V_\rho} \leq cB_d(\tau,\omega) \}. \] (77)
(ii) \( \Phi_{\rho} \) in (9) has a \( \mathcal{B} \)-pullback absorbing set \( K_b = \{ K_b(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \in \mathcal{B} \), defined by
\[ K_b(\tau,\omega) = \{ \theta \in V_\rho : \|\theta\|^2_{V_\rho} \leq cB_b(\tau,\omega) \} = \bigcup_{s \geq \tau} K_d(s,\omega). \] (78)
Moreover, $K_b$ is forward-uniformly absorbing: for each $B \in \mathcal{B}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is a $T := T(B, \tau, \omega) > 0$ such that

$$\Phi_{\rho}(t, s-t, \theta_{-t}\omega) B(s-t, \theta_{-t}\omega) \subset K_b(\tau, \omega), \forall s \leq \tau, t \geq T,$$

where $B_d(\tau, \omega)$ and $B_b(\tau, \omega)$ are given in (21) and (23), respectively.

Proof. We first show that $K_d \in \mathcal{D}$ and $K_b \in \mathcal{B}$. Given $b > 0$ and let $\delta_1 = \min\{\frac{2}{b}, \frac{1}{b}\}$. Then, by (15) there exists a $T_1 := T_1(\delta_1, \omega) > 0$ such that

$$\int_0^t |(\gamma(\theta_{-t}\omega) - \bar{\gamma})|\,dt \leq \delta_1 |t|, \text{ for all } |t| \geq T_1.$$

Hence, we obtain that, for all $t \geq T_1$ and $r \leq 0$

$$e^{\int_0^r \gamma(\theta_{-t}\omega)\,dt} = e^{\int_0^{t-r} \gamma(\theta_{-t}\omega)\,dt} = e^{\int_0^t \gamma(\theta_{-r}\omega)\,dt + \bar{\gamma}(r-t)} - e^{\int_0^t \gamma(\theta_{-r}\omega)\,dt} \leq e^{\bar{\gamma}(r)} e^{\bar{\gamma}(r)} r \leq e^{\bar{\gamma}(r)} e^{\bar{\gamma}(r)} r.$$

Note that $K_b$ is decreasing ($K_b(\tau_1, \omega) \subset K_b(\tau_2, \omega)$ if $\tau_1 \geq \tau_2$). Then we have

$$e^{-bt} \sup_{s \geq \tau} \|K_b(s-t, \theta_{-t}\omega)\|^2 \leq e^{-bt} \sup_{s \geq \tau} \|B_d(s-t, \theta_{-t}\omega)\|^2$$

$$\leq ce^{-bt} B_b(\tau, \theta_{-t}\omega) = ce^{-bt} \sup_{s \geq \tau} \int_0^s e^{\int_0^r \gamma(\theta_{-t}\omega)\,dt} (1 + |z(\theta_{-t}\omega)|^4 + \|g(r + s - t)\|^2)\,dr$$

$$\leq ce^{-(b-2\delta_1)t} \sup_{s \geq \tau} \int_{-\infty}^0 e^{2\delta_1 r} (1 + |z(\theta_{-t}\omega)|^4 + \|g(r + s - t)\|^2)\,dr$$

$$\leq ce^{-(b-4\delta_1)t} \sup_{s \geq \tau} \int_{-\infty}^0 e^{2\delta_1 r} (1 + |z(\theta_{-t}\omega)|^4 + \|g(r + s - t)\|^2)\,dr \to 0,$$

as $t \to +\infty$. Therefore, we get $K_b \in \mathcal{B}$. Since $B_b(\tau, \omega) \supset B_d(\tau, \omega)$, we have $K_d \in \mathcal{D}$.

We then prove $K_d$ is a random set. It is obvious to see that $\omega \to B_d(\tau, \omega)$ is measurable since it is the integral of some random variables. Hence, $K_d$ is a random set.

Finally, by (20) and (22), we obtain $K_d$ is a $\mathcal{D}$-pullback absorbing set and $K_b$ is a forward-uniformly absorbing set.

We remark here that the measurability of $K_b$ is unknown, since it is the union of random sets $K_d(s, \cdot)$ over an uncountable index set $s \in [\tau, +\infty)$. However, the corresponding attractor is measurable (see Theorem (3.7)). On the other hand, $K_d$ is a common $\mathcal{D}$-pullback absorbing set and $K_b$ is a common $\mathcal{B}$-pullback absorbing set for all $\rho \in (0, \rho_0]$ due to $c$ is uniform w.r.t. $\rho \in (0, \rho_0]$.

**Lemma 3.6.** Suppose that (F), (D), (G1), (G2) hold and $h \in H_1^1(Q)$. For each fixed $\rho > 0$, $\Phi_{\rho}$ in (9) is forward $\mathcal{B}$-pullback asymptotically compact in $V_\rho$, that is, for each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $B \in \mathcal{B}$, the sequence $\{\Phi_{\rho}(t_n, s_n - t_n, \theta_{-t_n}\omega)\}_{n \in \mathbb{N}}$ is pre-compact in $V_\rho$, whenever $s_n \geq \tau$, $t_n \to +\infty$ and $\phi_n \in B(s_n - t_n, \theta_{-t_n}\omega)$.

**Proof.** Based on the Ascoli-Arzela theorem, we divide the proof into two steps.

**Step 1.** We prove $\{\Phi_{\rho}(t_n, s_n - t_n, \theta_{-t_n}\omega)\}_{n \in \mathbb{N}}$ in $V_\rho$ is equi-continuous from $[-\rho, 0]$ to $V$. Without loss of generality, we assume that $t_n \geq T_\rho$ for all $n \in \mathbb{N}$ due
to \( t_n \to +\infty \), where \( \bar{T}_b \) is given in Lemma 3.4. Let \( \xi_1, \xi_2 \in [-\rho, 0] \) with \( \xi_2 > \xi_1 \), by (9) and (35) in Lemma (3.2) we obtain

\[
\begin{align*}
\| (\Phi_\rho(t_n, s_n - t_n, \theta_{-t_n}\omega) \phi_n)(\xi_1) &- (\Phi_\rho(t_n, s_n - t_n, \theta_{-t_n}\omega) \phi_n)(\xi_2) \|_V \\
& \leq \| v(s_n + \xi_1, s_n - t_n, \theta_{-s_n}\omega, \phi_n) - v(s_n + \xi_2, s_n - t_n, \theta_{-s_n}\omega, \phi_n) \|_V \\
& \leq \int_{s_n + \xi_1}^{s_n + \xi_2} \| \frac{\partial}{\partial r} v(r, s_n - t_n, \theta_{-s_n}\omega, \phi_n) \|_V dr \\
& \leq \left( \int_{s_n + \xi_1}^{s_n + \xi_2} \| \frac{\partial}{\partial r} v(r, s_n - t_n, \theta_{-s_n}\omega, \phi_n) \|_V^2 dr \right)^{\frac{1}{2}} |\sigma_1 - \sigma_2|^{\frac{1}{2}} \\
& \leq \left( \int_{s_n - \rho}^{s_n} \| \frac{\partial}{\partial r} v(r, s_n - t_n, \theta_{-s_n}\omega, \phi_n) \|_V^2 dr \right)^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}} \\
& \leq c(B^2_0(\tau, \omega) + G(\tau, \omega))^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}}.
\end{align*}
\]

Hence, we obtain \( \{ \Phi_\rho(t_n, s_n - t_n, \theta_{-t_n}\omega) \phi_n \}_{n \in \mathbb{N}} \) in \( V_\rho \) is equi-continuous from \([-\rho, 0]\) to \( V \).

**Step 2.** For each fixed \( \xi \in [-\rho, 0] \), we prove (9) in Lemma 3.3, for any \( \varepsilon > 0 \) we have, for all \( t \geq \bar{T}_b \),

\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \| v(s + \xi, s - t, \theta_{-s}\omega, \phi) \|_{H^1_\rho(Q_K)} < \varepsilon,
\]

where \( K \) and \( \bar{T}_b \) are given in Lemma 3.3. Let \( k = K \) in Lemma 3.4, for any \( \varepsilon > 0 \) we have, for all \( t \geq \bar{T}_b \),

\[
\sup_{s \geq \tau} \sup_{\xi \in [-\rho, 0]} \| (I - P_{i_0})(\varphi_K v(s + \xi, s - t, \theta_{-s}\omega, \phi)) \|_{H^1_\rho(Q_{2K})} < \varepsilon,
\]

where \( i_0 \) and \( \bar{T}_b \) are given in Lemma 3.4. By (22) in Lemma (3.1) and \( \bar{T}_b \geq \bar{T}_b \geq \bar{T}_b \) we have \( \{ \varphi_K v : v \in E(\bar{T}_b) \} \) is bounded in \( H^1_\rho(Q_{2K}) \) and so \( \{ P_{i_0}(\varphi_K v) : v \in E(\bar{T}_b) \} \) is bounded in \( H^1_\rho(Q_{2K}) \). Since \( P_{i_0} : L^2(Q_{2K}) \mapsto H_{i_0} = \text{span}\{e_1, e_2, \cdots, e_{i_0}\} \) and \( H_{i_0} \) is a finite dimensional subspace, we obtain

\[
\kappa_{H^1_\rho(Q_{2K})} \{ P_{i_0}(\varphi_K v) : v \in E(\bar{T}_b) \} = 0,
\]

where \( \kappa(\cdot) \) denotes the Kuratowski measure. Based on the property of the Kuratowski measure and (81) we have

\[
\kappa_{H^1_\rho(Q_{2K})} \{ \varphi_K v : v \in E(\bar{T}_b) \} \\
\leq \kappa_{H^1_\rho(Q_{2K})} \{ P_{i_0}(\varphi_K v) : v \in E(\bar{T}_b) \} + \kappa_{H^1_\rho(Q_{2K})} \{ (I - P_{i_0})(\varphi_K v) : v \in E(\bar{T}_b) \} < \varepsilon.
\]

Note that \( \varphi_K v = v \) on \( Q_K \), we have

\[
\kappa_{H^1_\rho(Q_K)}(E(\bar{T}_b)) = \kappa_{H^1_\rho(Q_K)} \{ \varphi_K v : v \in E(\bar{T}_b) \} \leq \kappa_{H^1_\rho(Q_{2K})} \{ \varphi_K v : v \in E(\bar{T}_b) \} < \varepsilon,
\]

which along with (80) imply that

\[
\kappa_{V}(E(\bar{T}_b)) \leq \kappa_{H^1_\rho(Q_K)}(E(\bar{T}_b)) + \kappa_{H^1_\rho(Q_K)}(E(\bar{T}_b)) < 2\varepsilon.
\]
Since \( \{\Phi_p(t_n, s_n - t_n, \theta_{-t_n})\phi_n(\xi)\}_{n \in \mathbb{N}} \subset E(T_h) \), we get
\[
\kappa_V(\{\Phi_p(t_n, s_n - t_n, \theta_{-t_n})\phi_n(\xi)\}_{n \in \mathbb{N}}) < 2\varepsilon,
\]
which together with the property of the Kuratowski measure yields \( \{\Phi_p(t_n, s_n - t_n, \theta_{-t_n})\phi_n(\xi)\}_{n \in \mathbb{N}} \) is pre-compact in \( V_\rho \).

We infer from Step 1 and Step 2 that all conditions of the Ascoli-Arzelà theorem are fulfilled. Then the proof is complete. \( \square \)

**Theorem 3.7.** Suppose that (F), (D), (G1), (G2) hold and \( h \in H^1_0(Q) \). For each fixed \( \rho > 0 \) we have
(i) \( \Phi_p \) in (9) has a unique \( \mathcal{D} \)-pullback random attractor \( \mathcal{A}_d^\rho \in \mathcal{D} \), defined by
\[
\mathcal{A}_d^\rho(\tau, \omega) := \bigcap_{T > 0} \bigcup_{t \geq T} \Phi_p(t, \tau - t, \theta_{-t}\omega)K_d(\tau - t, \theta_{-t}\omega). \quad (82)
\]
(ii) \( \Phi_p \) in (9) has a unique \( \mathcal{B} \)-pullback bi-parametric attractor \( \mathcal{A}_b^\rho \in \mathcal{B} \), defined by
\[
\mathcal{A}_b^\rho(\tau, \omega) := \bigcap_{T > 0} \bigcup_{t \geq T} \Phi_p(t, \tau - t, \theta_{-t}\omega)K_b(\tau - t, \theta_{-t}\omega), \quad (83)
\]
which is forward compact: \( \bigcup_{s \geq \tau} \mathcal{A}_b^\rho(s, \omega) \) is pre-compact on \( V_\rho \) for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \).
(iii) \( \mathcal{A}_d^\rho = \mathcal{A}_b^\rho \). So, \( \mathcal{A}_b^\rho(\tau, \cdot) \) is measurable.

**Proof.** (i) By the same method as in Lemma 3.6 we have \( \Phi_p \) is \( \mathcal{D} \)-pullback asymptotically compact, which combine with (i) of Lemma 3.5 implies all conditions of [28, Lemma 2.21] are satisfied. Hence, we obtain (i) holds.
(ii) Similarly, we have \( \Phi_p \) has a unique pullback bi-parametric attractor \( \mathcal{A}_b^\rho \in \mathcal{B} \),
given by (83). We now show that \( \mathcal{A}_b^\rho(\cdot, \omega) \) is forward compact. Let \( \{u_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \geq \tau} \mathcal{A}_b^\rho(s, \omega) \). Then, for each \( n \in \mathbb{N} \) there is a \( s_n \geq \tau \) such that \( u_n \in \mathcal{A}_b^\rho(s_n, \omega) \). By the invariance of \( \mathcal{A}_b^\rho \), there exists a \( v_n \in \mathcal{A}_b^\rho(s_n - t_n, \theta_{-t_n}\omega) \) such that \( \Phi_p(t_n, s_n - t_n, \theta_{-t_n}\omega)v_n = u_n \). Since \( \Phi_p \) is forward \( \mathcal{B} \)-pullback asymptotically compact, we obtain \( \{v_n\}_{n \in \mathbb{N}} \) has a convergence subsequence and so \( \mathcal{A}_b^\rho(\cdot, \omega) \) is forward compact.
(iii) It follows from (78), (82) and (83) that \( \mathcal{A}_d^\rho \subset \mathcal{A}_b^\rho \). Since \( \mathcal{A}_b^\rho \in \mathcal{B} \subset \mathcal{D} \) and \( \mathcal{A}_d^\rho \) is a \( \mathcal{D} \)-pullback attracting set, by the invariance of \( \mathcal{A}_b^\rho \) we obtain, for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \),
\[
\text{dist}_{V_\rho}(\mathcal{A}_d^\rho(\tau, \omega), \mathcal{A}_b^\rho(\tau, \omega)) = \text{dist}_{V_\rho}(\Phi_p(t, \tau - t, \theta_{-t}\omega)\mathcal{A}_d^\rho(\tau, \omega), \mathcal{A}_b^\rho(\tau, \omega)) \to 0 \quad \text{as} \quad t \to +\infty,
\]
which implies \( \mathcal{A}_b^\rho \subset \mathcal{A}_d^\rho \). Then we have \( \mathcal{A}_d^\rho = \mathcal{A}_b^\rho \) and so \( \mathcal{A}_b^\rho(\tau, \cdot) \) is measurable. \( \square \)

4. Forward asymptotic autonomy of pullback random attractors.

4.1. Convergence of systems from non-autonomy to autonomy. Replacing \( g \in L^2_{loc}(\mathbb{R}, H) \) with \( g_{\infty} \in H \) in (1), we obtain the following autonomous equation:
\[
\begin{align*}
\begin{cases}
\frac{d\tilde{u}}{dt} - d(\Delta \tilde{u}) - \nu \Delta \tilde{u}dt + \nabla \cdot F(\tilde{u})dt \\
= (\int_{-\rho}^{0} f(y, \tilde{u}(t + y))dy + g_{\infty} dt + h(x)dW(t), \ t > \tau, \ x \in Q, \quad (84) \\
\tilde{u}|_{\partial Q} = 0, \ \tilde{u}(0 + \xi, x) = \tilde{u}_0(\xi, x), \xi \in [-\rho, 0].
\end{cases}
\end{align*}
\]
By (4) we get the following random equation:
\[
\begin{aligned}
&\frac{\partial (\varepsilon - \Delta \varepsilon)}{\partial t} - \nu \Delta \varepsilon + \nabla \cdot F(\varepsilon + \tilde{z}(\theta \omega)) \\
&= \int_0^\rho f(\xi, \tilde{v}(\tau + \xi) + \tilde{z}(\theta \xi \omega) + g_\infty + \tilde{z}(\theta \omega) + (\nu - 1) \Delta \tilde{z}(\theta \omega), \\
&\tilde{v}|_{\partial Q} = 0, \quad \tilde{v}(0 + \xi, x) = \tilde{v}_0(\xi, x), \quad \xi \in [-\rho, 0], \quad x \in Q.
\end{aligned}
\tag{85}
\]
By the standard Galerkin method, we can prove Eq. (85) has a unique solution \( \tilde{v} \).

Hence, we obtain a continuous random dynamical system \( \Phi : \mathbb{R}^+ \times \Omega \times V_\rho \to V_\rho \), defined by
\[
\Phi(t, \omega) \tilde{v}_0 = \tilde{v}_t(\cdot, \omega, \tilde{v}_0), \quad t \in \mathbb{R}^+, \quad \omega \in \Omega, \quad \tilde{v}_0 \in V_\rho.
\]
By the same method in subsection 3.1-3.4, we get \( \Phi \) generated by (85) has a \( \mathcal{D}_\infty \)-random attractor \( A_\infty = \{ A_\infty(\omega) : \omega \in \Omega \} \), where \( \mathcal{D}_\infty \) satisfies \( \mathcal{D}_\infty = \{ \mathcal{D}_\infty(\omega) \neq \emptyset : \mathcal{D}_\infty(\omega) \} \) is bounded for all \( \omega \in \Omega \) \( \in \mathcal{D}_\infty \) iff
\[
\lim_{t \to +\infty} e^{-bt} \| \mathcal{D}_\infty(\tilde{\theta}_t \omega) \|^2 = 0, \quad \forall \ b > 0, \ \omega \in \Omega.
\tag{86}
\]
In order to the aim of this subsection, we also need the following assumption:
\[
\text{Lemma 4.1. Suppose that (87) holds and } v_\tau, \tilde{v}_0 \in V_\rho \text{ such that }
\lim_{\tau \to +\infty} \| v_\tau - \tilde{v}_0 \|_{V_\rho} = 0.
\tag{87}
\]
Then, we have
\[
\lim_{\tau \to +\infty} \| \Phi(\tau, \omega)v_\tau - \Phi(\tau, \omega)\tilde{v}_0 \|_{V_\rho} = 0,
\tag{89}
\]
for all \( t \geq 0 \) and \( \omega \in \Omega \).

Proof. For each \( \tau \in \mathbb{R} \), let
\[
V^\tau(r) := v(r + \tau + \theta \omega, v_\tau(r, \omega), \tilde{v}(r, \omega, \tilde{v}_0)), \quad \forall \ r \geq -\rho,
\]
where \( v \) and \( \tilde{v} \) are solutions of (5) and (85), respectively. Subtracting (85) from (5), and then multiplying this result by \( V^\tau(r) \) we obtain
\[
\begin{aligned}
&\frac{d}{dr} \| V^\tau(r) \|^2 + \nu \| \nabla V^\tau(r) \|^2 \\
&\leq - (\nabla \cdot F(v(r + \tau + \tilde{z}(\theta \omega))) - \nabla \cdot F(\tilde{v}(r + \tilde{z}(\theta \omega)), V^\tau(r)) \\
&+ \int_{-\rho}^0 f(\xi, \tilde{v}(r + \tau + \xi + \tilde{z}(\theta \xi \omega))d\xi - \int_{-\rho}^0 f(\xi, \tilde{v}(r + \xi + \tilde{z}(\theta \xi \omega))d\xi \|^2 \\
&+ \| g(r + \tau) - g_\infty \|^2.
\end{aligned}
\tag{90}
\]
For the first term on the right-hand side of (90), by (6) we obtain
\[
- (\nabla \cdot F(v(r + \tau + \tilde{z}(\theta \omega))) - \nabla \cdot F(\tilde{v}(r + \tilde{z}(\theta \omega)), V^\tau(r)) \\
= \int_Q (F(v(r + \tau + \tilde{z}(\theta \omega))) - F(\tilde{v}(r + \tilde{z}(\theta \omega)) \cdot \nabla V^\tau(r))dx \\
\leq \alpha_1 \| V^\tau(r) \| \| \nabla V^\tau(r) \| + \alpha_2 (\| v(r + \tau + \tilde{z}(\theta \omega)) \| + \| \tilde{v}(r + \tilde{z}(\theta \omega)) \|) V^\tau(r) \| \| \nabla V^\tau(r) \| \\
\leq c(1 + \| \tilde{z}(\theta \omega) \| + \| v(r + \tau) \| + \| \tilde{v}(r) \| \| V^\tau(r) \| \| \nabla V^\tau(r) \|.
\]
For the delay term of (90), by (8) we have
\[
\| \int_{-\rho}^{0} f(\xi, v(r+\tau+r) + \tilde{z}(\theta_r+\xi)) d\xi - \int_{-\rho}^{0} f(\xi, v(r+\xi) + \tilde{z}(\theta_r+\xi)) d\xi \|_V^2 \\
\leq \| \int_{-\rho}^{0} |f(\xi, v(r+\tau+\xi) + \tilde{z}(\theta_r+\xi\omega)) d\xi - f(\xi, v(r+\xi) + \tilde{z}(\theta_r+\xi\omega)) d\xi \|_V^2 \\
\leq \iint_{\Omega} \left( \int_{-\rho}^{0} L_f(\xi) V^\tau (r+\xi) d\xi \right)^2 dx \leq \rho \| L_f(\cdot) \|_2^2 (\tau_{\rho,0}) \| V^\tau \|_{V^\rho}^2.
\]
Then we have
\[
\frac{d}{dr} \| V^\tau (r) \|_{V^\rho}^2 \leq c (1 + |z(\theta_r,\omega)| + \| v(r+\tau) \|_V + \| \tilde{v}(r) \|_V) \| V^\tau \|_{V^\rho}^2 + \| g(r+\tau) - g_\infty \|_V^2.
\]
Integrating the above inequality over \([0, t]\) with \(t \in [0, T]\) and \(T > 0\) yields
\[
\| V^\tau (t) \|_{V^\rho}^2 \leq \| v_r - \tilde{v}_0 \|_{V^\rho}^2 + c \int_0^t (1 + |z(\theta_r,\omega)| + \| v(r+\tau) \|_V + \| \tilde{v}(r) \|_V) \| V^\tau \|_{V^\rho}^2 dr + \int_0^t \| g(r+\tau) - g_\infty \|_V^2 dr.
\]
Note that \(\| V^\tau (t) \|_{V^\rho}^2 \leq \| v_r - \tilde{v}_0 \|_{V^\rho}^2\) when \(t \in [-\rho, 0]\). Hence, we obtain
\[
\| V^\tau \|_{V^\rho}^2 \leq \| v_r - \tilde{v}_0 \|_{V^\rho}^2 + c \int_0^T (1 + |z(\theta_r,\omega)| + \| v(r+\tau) \|_V + \| \tilde{v}(r) \|_V) \| V^\tau \|_{V^\rho}^2 dr + \int_0^T \| g(r+\tau) - g_\infty \|_V^2 dr.
\]
Using the Gronwall inequality (see [6, page 167]) for the above inequality yields
\[
\| V^\tau \|_{V^\rho}^2 \leq \left( \| v_r - \tilde{v}_0 \|_{V^\rho}^2 + \int_0^\infty \| g(r+\tau) - g_\infty \|_V^2 dr \right) e^{c \int_0^T (1 + |z(\theta_r,\omega)| + \| v(r+\tau) \|_V + \| \tilde{v}(r) \|_V) dr}.
\]
By the same method as in Lemma 3.1 and \(r \to z(\theta_r, \omega)\) is continuous, we have
\[
\int_0^T (1 + |z(\theta_r,\omega)| + \| v(r+\tau) \|_V + \| \tilde{v}(r) \|_V) dr < +\infty.
\]
Then, by (87) and (88) we get
\[
\lim_{\tau \to +\infty} \| V^\tau \|_{V^\rho}^2 = 0,
\]
which implies (89) holds. \(\square\)

4.2. Upper semi-continuity of attractors from non-autonomous to autonomous. Define a new random set \(K_{fu} = \{ K_{fu}(\omega) : \omega \in \Omega \}\) by \(K_{fu}(\omega) := \bigcup_{\tau \geq 0} K_{h}(s, \omega, \theta_{-\tau})\) for all \(\tau \in \mathbb{R}\). We now show that \(K_{fu} \in \mathcal{D}_\infty\). Since \(K_{h}\) is closed and decreasing, similarly to (79) we have
\[
e^{-bt} \| K_{fu}(\theta_{-\tau}) \|_{V^\rho}^2 = e^{-bt} \| \bigcup_{\tau \geq 0} K_{h}(s, \theta_{-\tau}) \|_{V^\rho}^2 \\
e^{-bt} \| K_{h}(0, \theta_{-\tau}) \|_{V^\rho}^2 \leq c e^{-bt} \sup_{s \geq 0} B_d(s, \theta_{-\tau}).
\]
which contradiction with (92). The proof is complete.

Then we get

$$v_n \in K_{fu}(\theta_0, \omega_0),$$

which along with (89) implies $K_{fu} \in D_{\infty}$.

**Theorem 4.2.** Suppose that (F), (D), (G1), (G2) hold and (87) hold. Then we have

$$\lim_{\tau \to +\infty} \text{dist}_{V_\nu} (A^\nu_0(\tau, \omega), A_\infty(\omega)) = 0, \quad \forall \omega \in \Omega. \quad (91)$$

**Proof.** If (91) is false, then there are $\delta > 0$, $\omega_0 \in \Omega$ and $\tau_n \to +\infty$ such that

$$\text{dist}_{V_\nu} (A^\nu_0(\tau_n, \omega_0), A_\infty(\omega_0)) \geq 4\delta, \quad \forall n \in \mathbb{N}.$$ 

Hence, for each $n \in \mathbb{N}$ there exists a $u_n \in A^\nu_0(\tau_n, \omega_0)$ such that

$$\text{dist}_{V_\nu} (u_n, A_\infty(\omega_0)) \geq 3\delta. \quad (92)$$

Since $A_\infty$ is a $D_{\infty}$-pullback attracting set and $K_{fu} \in D_{\infty}$, there exists $T := T(K_{fu}, \omega_0) > 0$ such that

$$\text{dist}_{V_\nu} (\Phi_{\mu}(T, \theta_{-T}\omega_0)K_{fu}(\theta_{-T}\omega_0), A_\infty(\omega_0)) < \delta. \quad (93)$$

Without loss of generality, we assume that $\tau_n \geq T$ for all $n \in \mathbb{N}$ because of $\tau_n \to +\infty$. By the invariance of $A^\nu_0$, for each $n \in \mathbb{N}$ there exists a $v_n \in A^\nu_0(\tau_n - T, \theta_{-T}\omega_0)$ such that

$$\Phi_{\mu}(T, \tau_n - T, \theta_{-T}\omega_0)v_n = u_n. \quad (94)$$

Since $A^\nu_0$ is forward compact in $V_\nu$ and $\{v_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \geq 0} A^\nu_0(s, \theta_{-T}\omega_0)$, we can find a $v \in \bigcup_{s \geq 0} A^\nu_0(s, \theta_{-T}\omega_0)$ such that

$$\lim_{n \to +\infty} \|v_n - v\|_{V_\nu} = 0, \quad (95)$$

which along with (89) implies

$$\lim_{n \to +\infty} \|\Phi_{\mu}(T, \tau_n - T, \theta_{-T}\omega_0)v_n - \Phi_{\infty}(T, \theta_{-T}\omega_0)v\|_{V_\nu} = 0. \quad (96)$$

Note that $\{u_n\}_{n \in \mathbb{N}} \subset \bigcup_{s \geq 0} A^\nu_0(s, \omega_0)$, then there exists a $u \in \bigcup_{s \geq 0} A^\nu_0(s, \omega_0)$ such that

$$\lim_{n \to +\infty} \|u_n - u\|_{V_\nu} = 0, \quad (97)$$

which along with (94) and (96) implies

$$\Phi_{\infty}(T, \theta_{-T}\omega_0)v = u. \quad (98)$$

It follows from $A^\nu_0 \in D$ is invariant and $K_{fu}$ is a $D$-pullback absorbing set that

$$\bigcup_{s \geq 0} A^\nu_0(s, \omega_0) \subset \bigcup_{s \geq 0} K_{fu}(s, \omega_0) = K_{fu}(\omega_0) \in D_{\infty}.$$ 

Then we get $v \in K_{fu}(\theta_{-T}\omega_0)$. By (93) and (97), there exists $N := N(\delta) \in \mathbb{N}$ such that for all $n \geq N$,

$$\text{dist}_{V_\nu} (u_n, A_\infty(\omega_0)) \leq \|u_n - u\|_{V_\nu} + \text{dist}_{V_\nu} (\Phi_{\infty}(T, \theta_{-T}\omega_0)v, A_\infty(\omega_0)) < 2\delta,$$

which contradiction with (92). The proof is complete. □
5. Non-delay stability of pullback random attractors.

5.1. Convergence of systems from delay to non-delay. Let $\rho = 0$ in Eq. (1), we obtain the following non-delay stochastic equation:

$$
\begin{align*}
\left\{\begin{array}{ll}
du^0 - d(\Delta u^0) - \nu \Delta u^0 dt + \nabla \cdot F(u^0) dt &= g(t,x) dt + h(x) dW(t), \ t > \tau, \\
u^0|_{\partial Q} = 0, \ u^0(\tau, x) &= \psi^0(x), \ \tau \in \mathbb{R}, \ x \in Q.
\end{array}\right.
\end{align*}
$$

(99)

Let $v^0(t, \tau, \omega, \psi^0) = u^0(t, \tau, \omega, \phi^0) - \tilde{z}(\theta_t \omega)$, Eq. (99) be rewritten as the following random equation:

$$
\begin{align*}
\left\{\begin{array}{ll}
\frac{d(v^0 - \Delta v^0)}{dt} - \nu \Delta v^0 + \nabla \cdot F(v^0 + \tilde{z}(\theta_t \omega)) &= g(t,x) + \tilde{z}(\theta_t \omega) + (\nu - 1)\Delta \tilde{z}(\theta_t \omega), \ t > \tau, \\
v^0|_{\partial Q} = 0, \ v^0(\tau, x) &= \phi^0(x), \ \tau \in \mathbb{R}, \ x \in Q.
\end{array}\right.
\end{align*}
$$

(100)

By the same method as in (9), we define a continuous cocycle $\Phi_0 : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times V \rightarrow V$ by

$$
\Phi_0(t, \tau, \omega)\phi^0 = v^0(t + \tau, \tau, \theta_{-\tau} \omega, \phi^0), \ t \in \mathbb{R}^+, \ \tau \in \mathbb{R}, \ \omega \in \Omega, \ \phi^0 \in V,
$$

(101)

where $v^0$ is a solution of (100) with initial value $\phi^0$.

**Lemma 5.1.** Suppose $\phi^\rho \in V_\rho$ and $\phi^0 \in V$ such that

$$
d_{\rho}(\phi^\rho, \phi^0) := \sup_{\xi \in [-\rho, 0]} \|\phi^\rho(\xi) - \phi^0\|_V \rightarrow 0 \ as \ \rho \rightarrow 0.
$$

(102)

Then, we have

$$
\lim_{\rho \rightarrow 0} \sup_{\xi \in [-\rho, 0]} \|\Phi_\rho(t, \tau, \omega)\phi^\rho(\xi) - \Phi_0(t, \tau, \omega, \phi^0)\|_V^2 = 0,
$$

(103)

for all $t \geq 0, \ \tau \in \mathbb{R}$ and $\omega \in \Omega$, where $v^\rho$ and $v^0$ be the solutions of Eq. (5) and Eq. (100) with initial data $\phi^\rho$ and $\phi^0$, respectively.

**Proof.** For each $\rho \in (0, \rho_0)$ and $\xi \in [-\rho, 0]$, we define a function $\tilde{v}_\xi^\rho(\cdot)$ by

$$
\tilde{v}_\xi^\rho(r) = v^\rho(r + \xi, \tau, \theta_{-\tau} \omega, \phi^\rho) - v^0(r, \tau, \theta_{-\tau} \omega, \phi^0), \ \forall \tau \in \mathbb{R}, \ \omega \in \Omega, \ r \geq \tau.
$$

Subtracting (100) from (5), and then multiplying this result by $\tilde{v}_\xi^\rho(r + \tau)$ we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{dr} \|\tilde{v}_\xi^\rho(r + \tau)\|_V^2 + \nu \|\nabla \tilde{v}_\xi^\rho(r + \tau)\|^2 &= - (\nabla \cdot F(v^\rho(r + \tau + \xi) + \tilde{z}(\theta_{r+\xi} \omega)) - \nabla \cdot F(v^0(r + \tau) + \tilde{z}(\theta_{r} \omega), \tilde{v}_\xi^\rho(r + \tau)) \\
+ (\int_{-\rho}^{0} \int_{\mathbb{R}^d} f(\xi, v(r + \tau + \xi + \tilde{\xi}(\theta_{r+\xi} \omega)) - \tilde{z}(\theta_{r+\xi+\tilde{\xi}} \omega)) d\xi, \tilde{v}_\xi^\rho(r + \tau)) \\
+ (\tilde{z}(\theta_{r+\xi} \omega) - \tilde{z}(\theta_{r} \omega), \tilde{v}_\xi^\rho(r + \tau)) + (g(r + \tau + \xi) - g(r + \tau), \tilde{v}_\xi^\rho(r + \tau)) \\
+ (\nu - 1)(\Delta \tilde{z}(\theta_{r+\xi} \omega) - \Delta \tilde{z}(\theta_{r} \omega), \tilde{v}_\xi^\rho(r + \tau)).
\end{align*}
$$

(104)
We now treat each term on the right-hand side of (104). For the nonlinear term, by (6) we have
\[-(\nabla \cdot \mathbf{F}(\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)) - \nabla \cdot \mathbf{F}(\nu^0(r + \tau) + \tilde{z}(\theta_r \omega)), \tilde{v}_c^0(r + \tau))\]
\[\leq \int_Q (\alpha_1 + \alpha_2(|\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)| + |\nu^0(r + \tau) + \tilde{z}(\theta_r \omega)|))\]
\[\times |\tilde{v}_c^0(r + \tau) + \tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|\nabla \tilde{v}_c^0(r + \tau)| dx.\]
Note that
\[\int_Q \alpha_1 |\tilde{v}_c^0(r + \tau) + \tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|\nabla \tilde{v}_c^0(r + \tau)| dx\]
\[= \alpha_1 \int_Q |\tilde{v}_c^0(r + \tau)|\nabla \tilde{v}_c^0(r + \tau)| dx + \alpha_1 \int_Q |\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|\nabla \tilde{v}_c^0(r + \tau)| dx\]
\[\leq \frac{\nu \lambda}{4} |\tilde{v}_c^0(r + \tau)|^2 + c|\nabla \tilde{v}_c^0(r + \tau)|^2 + c|\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|^2,\]
and
\[\int_Q \alpha_2(|\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)| + |\nu^0(r + \tau) + \tilde{z}(\theta_r \omega)|)\]
\[\times |\tilde{v}_c^0(r + \tau) + \tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|\nabla \tilde{v}_c^0(r + \tau)| dx\]
\[\leq \alpha_2(|\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)| + |\nu^0(r + \tau) + \tilde{z}(\theta_r \omega)|)\]
\[\times |\tilde{v}_c^0(r + \tau)|\nabla \tilde{v}_c^0(r + \tau)| dx + \alpha_2(|\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)| + |\nu^0(r + \tau) + \tilde{z}(\theta_r \omega)|)\]
\[\times |\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|\nabla \tilde{v}_c^0(r + \tau)| dx\]
\[\leq \alpha_2(|\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)| \cdot |v^0(r + \tau) + \tilde{z}(\theta_r \omega)| |\nabla \tilde{v}_c^0(r + \tau)|^2 + c|\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|^2 |\nabla \tilde{v}_c^0(r + \tau)|^2\]
\[+ c|\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|^2 (|\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)| |\nabla \tilde{v}_c^0(r + \tau)|^2 + |\nu^0(r + \tau)| |\nabla \tilde{v}_c^0(r + \tau)|^2)\]
\[+ c|\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|^2 (|\tilde{z}(\theta_{r + \xi} \omega)|^2 + |\tilde{z}(\theta_r \omega)|^2 + 1)\].

Then, we have
\[-(\nabla \cdot \mathbf{F}(\nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)) - \nabla \cdot \mathbf{F}(\nu^0(r + \tau) + \tilde{z}(\theta_r \omega)), \tilde{v}_c^0(r + \tau))\]
\[\leq \frac{\nu \lambda}{4} |\tilde{v}_c^0(r + \tau)|^2 + c(|\nu^0(r + \tau + \xi)| + |\nu^0(r + \tau)|) |\nabla \tilde{v}_c^0(r + \tau)|^2\]
\[+ c|\tilde{z}(\theta_{r + \xi} \omega) - \tilde{z}(\theta_r \omega)|^2 (|\tilde{z}(\theta_{r + \xi} \omega)|^2 + |\tilde{z}(\theta_r \omega)|^2 + 1) + \int_0^\rho f(\xi, \nu^0(r + \tau + \xi) + \tilde{z}(\theta_{r + \xi} \omega)) d\xi, \tilde{v}_c^0(r + \tau))\]
By the Young inequality we obtain

\[(\hat{z}(\theta_{r+\xi}\omega) - \hat{z}(\theta_r\omega), \hat{v}_\xi^p(r + \tau)) + (g(r + \tau + \xi) - g(r + \tau), \hat{v}_\xi^p(r + \tau)) \leq \frac{\nu \lambda}{4} \|\hat{v}_\xi^p(r + \tau)\|^2 + c \|g(r + \tau + \xi) - g(r + \tau)\|^2 + c \|\hat{z}(\theta_{r+\xi}\omega) - \hat{z}(\theta_r\omega)\|^2, \quad (107)\]

and

\[(\nu - 1)(\Delta \hat{z}(\theta_{r+\xi}\omega) - \Delta \hat{z}(\theta_r\omega), \hat{v}_\xi^p(r + \tau)) \leq \frac{\nu \lambda}{4} \|\hat{v}_\xi^p(r + \tau)\|^2 + c \|\Delta \hat{z}(\theta_{r+\xi}\omega) - \Delta \hat{z}(\theta_r\omega)\|^2. \quad (108)\]

Inserting the inequalities (105)-(108) into (104), and then integrating this result w.r.t. \(r \in [-\xi, t] \) with \(\xi \in [-\rho, 0], \ t \in [-\xi, T] \) and \(T > \rho_0\) we obtain

\[
\|\hat{v}_\xi^p(t + \tau)\|^2_v \leq \|\hat{v}_\xi^p(-\xi + \tau)\|^2_v + C(\omega) \int_{-\xi}^t \left(\|v^0(r + \tau + \xi)\|_V + \|v^0(r + \tau)\|_V + 1\|\hat{v}_\xi^p(r + \tau)\|_V^2 \right) dr \\
+ C(\omega) \int_{-\xi}^t |z(\theta_{r+\xi}\omega) - z(\theta_r\omega)|^2 \left(\|v^0(r + \tau + \xi)\|_V^2 + \|v^0(r + \tau)\|_V^2 + 1\right) dr \\
+ c \int_{-\xi}^t \int_{-\rho}^0 (\|v^0(r + \tau + \xi + \bar{\xi})\|^2 + \|\hat{z}(\theta_{r+\xi+\bar{\xi}}\omega)\|^2) d\bar{\xi} dr \\
+ c \int_{-\xi}^t \|g(r + \tau + \xi) - g(r + \tau)\|^2 dr,
\]

where we use \(r \rightarrow z(\theta_r\omega)\) is continuous. By the same method as in Lemma 3.1 we have

\[
\sup_{r \in [-\xi, t]} \|v^0(r + \tau + \xi, \theta_{-r}\omega, \phi^0)\|_V^2 \leq \sup_{r \in [\tau, \tau + T]} \|v^0(r, \tau, \theta_{-r}\omega, \phi^0)\|_V^2 \leq C(\omega) \left(1 + \|\phi^0\|_V^2 + \int_{-\rho_0}^T \|g(\tilde{r} + \tau)\|d\tilde{r}\right) \\
\leq C(\omega) \left(1 + d_\rho^2(\phi^0, \phi^0) + \|\phi^0\|_V^2 + \int_{-\rho_0}^T \|g(\tilde{r} + \tau)\|d\tilde{r}\right).
\]

Similarly, we have

\[
\sup_{r \in [-\xi, t]} \|v^0(r + \tau, \theta_{-r}\omega, \phi^0)\|_V^2 \leq \sup_{r \in [\tau, \tau + T]} \|v^0(r, \tau, \theta_{-r}\omega, \phi^0)\|_V^2 \leq C(\omega).
\]

Then we get

\[
\|\hat{v}_\xi^p(t + \tau)\|^2_v \leq \|\hat{v}_\xi^p(-\xi + \tau)\|^2_v + C(\omega)(d_\rho(\phi^0, \phi^0) + 1) \int_{-\xi}^t \|\hat{v}_\xi^p(r + \tau)\|^2_v dr \\
+ C(\omega)(d_\rho^2(\phi^0, \phi^0) + 1) \int_{-\xi}^T |z(\theta_{r+\xi}\omega) - z(\theta_r\omega)|^2 dr
\]
Using the Gronwall inequality for the above inequality we obtain
\[
\|\tilde{v}_\xi(t + \tau)\|^2 \leq C(\omega)(\|\tilde{v}_\xi(-\xi + \tau)\|^2 + J_1(\rho, \xi) + J_2(\rho, \xi) + J_3(\rho, \xi))e^{C(\omega)T(d_\rho(\phi^0, \phi^0) + 1)},
\]
where
\[
\begin{align*}
J_1(\rho, \xi) &:= (d_\rho^2(\phi^0, \phi^0) + 1) \int_{-\xi}^{T} |z(\theta_{r+\xi}\omega) - z(\theta_r\omega)|^2 dr, \\
J_2(\rho, \xi) &:= \int_{-\xi}^{T} \int_{-\rho}^{T+\xi+\tilde{\xi}} (\|v^\rho(r + \tau + \xi + \tilde{\xi})\|^2 + \|\tilde{z}(\theta_{r+\xi}\omega)\|^2) d\tilde{\xi} d\tau, \\
J_3(\rho, \xi) &:= \int_{-\xi}^{T} \|g(r + \tau + \xi) - g(r + \tau)\|^2 dr.
\end{align*}
\]
For the $J_1(\rho, \xi)$, by (102) and $r \to z(\theta_r\omega)$ is continuous we get
\[
J_1(\rho, \xi) \leq (d_\rho^2(\phi^0, \phi^0) + 1) \int_{-\xi}^{T} |z(\theta_{r+\xi}\omega) - z(\theta_r\omega)|^2 dr \to 0, \quad \text{as } \rho \to 0.
\]
For the $J_2(\rho, \xi)$, we have
\[
\begin{align*}
J_2(\rho, \xi) &= \int_{-\xi}^{T} \int_{-\rho}^{T+\xi+\tilde{\xi}} (\|v^\rho(r + \tau + \xi + \tilde{\xi})\|^2 + \|\tilde{z}(\theta_{r+\xi}\omega)\|^2) d\tilde{\xi} d\tau \\
&= \int_{-\rho}^{T+\xi+\tilde{\xi}} \int_{-\xi}^{T} (\|v^\rho(r + \tau + \xi)\|^2 + \|\tilde{z}(\theta_{r+\xi}\omega)\|^2) dr d\tilde{\xi} \\
&\leq \rho \int_{-\rho}^{T} \|v^\rho(r + \tau + \xi + \tilde{\xi})\|^2 dr + c \rho \int_{-\rho}^{T} |z(\theta_r\omega)|^2 dr \\
&\leq C(\omega)\rho \left(1 + d_\rho^2(\phi^0, \phi^0) + \|\phi^0\|^2 V + \int_{-\rho}^{T} \|g(\tilde{\tau} + \tau)\|^2 d\tilde{\tau} \right) + c \rho \int_{-\rho}^{T} |z(\theta_r\omega)|^2 dr \to 0,
\end{align*}
\]
as $\rho \to 0$. For the $J_3(\rho, \xi)$, by $g \in L^2_{\text{loc}}(\mathbb{R}, H)$ and $\xi \in [-\rho, 0]$ we have
\[
J_3(\rho, \xi) := \int_{-\xi}^{T} \|g(r + \tau + \xi) - g(r + \tau)\|^2 dr \\
\leq \int_{-\xi}^{T} \|g(r + \tau + \xi) - g(r + \tau)\|^2 dr \to 0, \quad \text{as } \rho \to 0.
\]
By (102) and the continuity of $\phi^0$ at $\tau$ we obtain
\[
\begin{align*}
\|\tilde{v}_\xi(-\xi + \tau)\|^2_v &= \|\phi^0(\tau, \tau, \theta_{-\xi}\omega, \phi^0) - v^0(-\xi + \tau, \tau, \theta_{-\xi}\omega, \phi^0)\|^2_v \\
&\leq 2\|\phi^0(0) - \phi^0\|^2_v + 2\|\phi^0 - v^0(-\xi + \tau, \tau, \theta_{-\xi}\omega, \phi^0)\|^2_v \\
&\leq 2\tilde{F}(\phi^0, \phi^0) + 2\|\phi^0 - v^0(-\xi + \tau, \tau, \theta_{-\xi}\omega, \phi^0)\|^2_v \to 0 \quad \text{as } \rho \to 0.
\end{align*}
\]
Note that $e^{C(\omega)T(d_\rho(\phi^0, \phi^0) + 1)}$ is finite as $\rho \to 0$. Then we obtain, for all $t \in [-\xi, T]$,
\[
\|\tilde{v}_\xi(t + \tau)\|^2_v \to 0, \quad \text{as } \rho \to 0. \quad (109)
\]
When $t \in [0, -\xi]$, we have $\tau + \xi \leq t + \tau + \xi \leq \tau$. Hence, by (101) and the continuity of $v^0$ at $\tau$ we get
\[
\|v^\rho(t + \tau + \xi) - v^0(t + \tau + \xi)\|_V^2 \leq 2\|v^\rho(t + \tau + \xi) - \phi^0\|_V^2 + 2\|v^0(t + \tau) - \phi^0\|_V^2 \\
\leq 2d^2_p(\phi^\rho, \phi^0) + 2\|v^0(t + \tau) - \phi^0\|_V^2 \to 0, \quad \text{as } \rho \to 0. \tag{110}
\]

By (109) and (110) we obtain
\[
\lim_{\rho \to 0} \sup_{\xi \in [-\rho, 0]} \|v^\rho(t + \tau + \xi, \tau, \theta - \tau \omega, \phi^\rho) - v^0(t + \tau, \tau, \theta - \tau \omega, \phi^0)\|_V^2 = 0, \quad \forall t \in [0, T].
\]

Hence we have (102) as desired. $\square$

5.2. Upper semi-continuity of PRAs as the memory time $\rho \to 0$. By the same method as in subsection 3.1-3.4, we can prove $\Phi_0$ in (101) has a $\mathcal{D}_0$-pullback random attractor $\mathcal{A}^0 = \{\mathcal{A}^0(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where $\mathcal{D}_0$ defined by, for all $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $b > 0$,
\[
\mathcal{D}_0 = \{D_0 = \{D_0(\tau, \omega) \neq \emptyset : D_0(\tau, \omega) \text{ is bounded} : \lim_{t \to +\infty} e^{-bt} ||D_0(\tau - t, \theta - t \omega)||_V^2 = 0\}. \tag{115}
\]

We define a new bi-parametric set $\mathcal{K}_0$ by
\[
\mathcal{K}_0(\tau, \omega) = \{\phi \in V : ||\phi||_V^2 \leq cB_d(\tau, \omega), \tau \in \mathbb{R}, \omega \in \Omega\}. \tag{116}
\]

It is easy to obtain that $\mathcal{K}_0 \subset \mathcal{D}_0$. In addition, by (77) we obtain
\[
\lim_{\rho \to 0} \sup_{\xi \in [-\rho, 0]} ||\mathcal{K}_d(\tau, \omega)||_{V^\rho} = ||\mathcal{K}_0(\tau, \omega)||_V. \tag{111}
\]

**Lemma 5.2.** Suppose that $\mu_n \in \mathcal{A}^{\rho_n}_d(\tau, \omega)$ with $\rho_n \to 0$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, then there exist $\mu \in V$ and an index subsequence of $\{n^*\}$ of $\{n\}$ such that
\[
d_{\rho_n^*}(\mu_n^*, \mu) := \sup_{\xi \in [-\rho_n^*, 0]} ||\mu_n^*(\xi) - \mu||_V \to 0 \text{ as } n^* \to +\infty. \tag{112}
\]

**Proof.** By the invariance of $\mathcal{A}^{\rho_n}_d$, there exists a $\tilde{\mu}_n \in \mathcal{A}^{\rho_n}_d(\tau - t_n, \theta - t_n \omega)$ with $t_n \to +\infty$ such that
\[
\mu_n = \Phi_{\rho_n}(t_n, \tau - t_n, \theta - t_n \omega)\tilde{\mu}_n. \tag{113}
\]

Since $\mathcal{A}^{\rho_n}_d \subset \mathcal{D}$, all uniform estimates (w.r.t. $\rho \in (0, \rho_0]$) in subsection 3.1-3.4 hold. Then, by the same method as in **Step 1** of Lemma (3.6), there exists $\delta > 0$ with $|\xi_1 - \xi_2| \leq \delta$, such that for all $\varepsilon > 0$
\[
\|(\Phi_{\rho_n^*}(t_n^*, \tau - t_n^*, \theta - t_n^* \omega)\tilde{\mu}_n^*)(\xi_1) - (\Phi_{\rho_n^*}(t_n^*, \tau - t_n^*, \theta - t_n^* \omega)\tilde{\mu}_n^*)(\xi_2)\|_V < \varepsilon.
\]

Since $\rho_n^* \to 0$ as $n^* \to +\infty$, without loss of generality we assume that $\rho_n^* < \delta$ for all $n^* \in \mathbb{N}$. Then we have, for all $\xi \in [-\rho_n^*, 0]$,
\[
\|(\Phi_{\rho_n^*}(t_n^*, \tau - t_n^*, \theta - t_n^* \omega)\tilde{\mu}_n^*)(\xi) - (\Phi_{\rho_n^*}(t_n^*, \tau - t_n^*, \theta - t_n^* \omega)\tilde{\mu}_n^*)(0)\|_V < \varepsilon. \tag{114}
\]

We then use the same method as in **Step 2** of Lemma (3.6) show that $(\Phi_{\rho_n}(t_n, \tau - t_n, \theta - t_n \omega)\tilde{\mu}_n)(0)$ is pre-compact in $V$. Hence, there exist $\mu \in V$ and an index subsequence of $\{n^*\}$ of $\{n\}$ such that
\[
\|(\Phi_{\rho_n^*}(t_n^*, \tau - t_n^*, \theta - t_n^* \omega)\tilde{\mu}_n^*)(0) - \mu\|_V \to 0 \text{ as } n \to +\infty. \tag{115}
\]
It follows from (113)-(115) that there exists a $N \in \mathbb{N}$ such that for all $n^* \geq N$ and $\xi \in [-p_n^*, 0]$,
\[
\| \mu^* - \mu \|_V = \|(\Phi_{p_n^*}(t_n^*, \tau - t_n^*, \theta_{-t_n^*}, \omega)\mu_n^*)(\xi) - \mu \|_V
\leq \| (\Phi_{p_n^*}(t_n^*, \tau - t_n^*, \theta_{-t_n^*}, \omega)\mu_n^*)(\xi) - (\Phi_{p_n^*}(t_n^*, \tau - t_n^*, \theta_{-t_n^*}, \omega)\mu_n^*)(0) \|_V
+ \| (\Phi_{p_n^*}(t_n^*, \tau - t_n^*, \theta_{-t_n^*}, \omega)\mu_n^*)(0) - \mu \|_V < 2\varepsilon,
\]
which proves (112) as desired.

It follows from (111), Lemma 5.1 and Lemma 5.2 that all conditions of [34, Theorem 2.1] are satisfied. Hence we obtain the main result of this section:

**Theorem 5.3.** Suppose that (F), (D), (G1), (G2) hold and $h \in H^1_0(Q)$. For each $\tau \in \mathbb{R}$ and $\omega \in \Omega$ we have
\[
dist_{V, \omega}(A^0(\tau), \omega, A^0(\tau, \omega)) \to 0 \text{ as } \rho \to 0.
\]

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E-mail address: zqh.math@126.com