On the derivation of the Kompaneets equation

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Abstract

The relaxation of a photon bath to thermal equilibrium via Compton scattering with electrons is described in the Kompaneets equation (1956). The equation is mostly known from studies of astrophysical plasmas, for its convergence to the Planck distribution and for possible corrections to that Planck law in the cosmic microwave background, most notably from the Sunyaev-Zeldovich effect. We revisit its derivation emphasizing its structure as a Kramers-Moyal diffusion approximation to the quantum Boltzmann equation or Master equation with stimulated emission. We do not assume that the Planck law is stationary in performing the continuum approximation but we emphasize the necessity of the flux or Møller factor to arrive at a continuity equation. On the other hand, the structure allows more general assumptions than originally envisioned by Kompaneets.

1 Introduction

The interaction between matter and radiation is a standard topic in plasma physics and cosmology. In low-density plasmas, Compton scattering is the dominant mechanism enabling energy exchange [1, 2]. Essentially, this process describes elastic collisions of photons and charged particles (which we assume to consist solely of electrons), preserving photon number. The time evolution of the radiation spectrum subject to Compton scattering on a plasma of nonrelativistic Maxwellian electrons was first obtained by A.S. Kompaneets in 1956 [3, 4]. The Kompaneets equation later found new life in quantitatively describing the Sunyaev-Zeldovich effect, which is a distortion of the cosmic microwave background radiation by Compton scattering of hot electrons during its passage through clusters of galaxies [5, 7, 8].

For a nonrelativistic electron bath at temperature $T$, with $k_B T \ll m_e c^2$ for electron mass $m_e$, the Kompaneets equation

$$\omega^2 \frac{\partial n(\omega, t)}{\partial t} = \frac{n_e \sigma_T c}{m_e c^2} \omega^4 \left\{ k_B T \frac{\partial n(\omega, t)}{\partial \omega} + \hbar \left[ 1 + n(\omega, t) \right] n(\omega, t) \right\}$$

(1)

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1This arose from his theoretical research as part of the Soviet hydrogen bomb program in 1949. When the eponymous relaxation equation turned out to be useless for weapons research, the results were declassified and published by Kompaneets in 1956 [3, 4].
shows how a photon gas under conditions of spatial homogeneity and isotropy relaxes to thermal equilibrium from Compton scattering. It is a continuity equation for the (dimensionless) photon number $n(\omega, t)$ occupying frequency $\omega$ at time $t$. The spectral energy density is proportional to $\hbar (\omega/c)^3 n(\omega, t)$ [9]. The range of frequencies is assumed to have the photon energy $\hbar \omega \sim k_B T \ll m_e c^2$. In (1), $\sigma_T \approx 0.66 b$ is the total Thomson cross section and $n_e$ is the electron density, which together determine the photon mean free path $\ell = (n_e \sigma_T)^{-1}$ and the average time between collisions $\tau = \ell/c$.

The current in frequency space is given by

$$\frac{\partial n}{\partial t}(\omega, t) = \frac{1}{\omega^2} \frac{\partial}{\partial \omega} (\omega^2 j_t(\omega))$$

$$j_t(\omega) = \frac{n_e \sigma_T c}{m_e c^2} \omega^2 \left\{ k_B T \frac{\partial n}{\partial \omega}(\omega, t) + \hbar [1 + n(\omega, t)] n(\omega, t) \right\}$$

(2)

That current (2) vanishes when the photon number $n(\omega, t)$ is given by the Bose-Einstein expression,

$$n_{eq}(\omega) = \frac{1}{\exp(\beta \hbar \omega) - 1}$$

(with photon chemical potential taken zero) at inverse temperature $\beta = 1/k_B T$. The temperature $T$ here refers to the temperature of the electron bath under the assumption that it is in thermal equilibrium, but this can be made more general as shown in Section 6. The relaxation behavior of (1) is obviously an interesting question in itself, but not the subject of the present paper; see e.g. [10, 11] for a numerical code. It suffices to note in addition that depending on the initial number of photons, the limiting photons as $t \uparrow \infty$ will have a Bose-Einstein distribution component ($n_{eq}(\omega)$), as well as a condensate (for $\omega = 0$). That condensate appears when the initial number of photons is larger than the equilibrium number of photons $n_{eq}$ at the given inverse temperature $\beta$. The excess photons cannot disappear, forming a Bose-Einstein condensate at $\omega = 0$, as the Kompaneets equation is photon-number preserving. For this interesting feature we refer to [12, 13] for a more detailed discussion.

Kompaneets himself only derived the exact form of the first term between the brackets in (1), which is purely a diffusive term due to Doppler shift experienced by the photons in the reference frame of the electrons. The (second) drift term, describing stimulated emission and the Compton recoil, were instead derived from the assumption that in thermal equilibrium the stationary distribution ought to be the Planck distribution. The so-called Comptonization, i.e., the redistribution of photon energies scattered by electrons consists of two terms indeed, both suppressed by the same factor of $m_e c^2$, the diffusion because the electron is moving so slowly (depending on $k_B T/m_e c^2$) that the Doppler shift is almost negligible, and the drift is divided by the reduced Compton frequency $\omega_c = m_e c^2/\hbar \approx 7.76 \times 10^{20}$ Hz so that there is barely any recoil.

The derivation of the Kompaneets equation has been called “distinctly non-trivial” [4], and many authors have presented or repeated their best derivation. However, we could not find any reference which carries a full derivation in the same spirit that Kompaneets originally proposed, using the correct relativistic expression of the kinetic (Boltzmann) equation. For example, it is worth making precise when in fact the diffusion
approximation of the Boltzmann equation, the standard way of deriving (1), cannot yield a continuity equation. We found that such features are not fully explored in the existing literature. We believe that a “more didactical” derivation may also be useful to highlight physical limitations and possible extensions. Kompaneets in 1956 is not explicit about the form of the kinetic equation used as starting point. He proposes to carry an expansion up to second order in the photon transferred energy, using the Thomson scattering cross section

\[ \frac{d\sigma^{\text{Th}}}{d\Omega_{\text{rest}}} = \frac{3 \sigma_\text{T}}{16\pi} (1 + \cos^2 \theta_{\text{rest}}) \]

where \( d\Omega_{\text{rest}} \) is the scattering solid angle measured in the electron rest frame. In 1963 Dreicer [14] more carefully describes the Fokker-Planck approximation to the Boltzmann equation for an electron-photon gas. However, he does not give an expression for the time evolution of the photon number. In 1965 Weymann [15] uses Dreicer’s formalism to derive an equation identical to the one by Kompaneets, although not displaying the details of the calculation. Somewhat remarkably, although Dreicer’s and Weymann’s papers are published after Kompaneets’ paper, they do not cite the latter. Nevertheless, those as well as many other references [16, 17, 18, 19] follow Kompaneets’ recipe, also using the Thomson cross section. We highlight that it is not possible to find the Kompaneets equation by that approach while performing the diffusion approximation as proposed by Kompaneets; rather we must use the full relativistic expression with the Klein-Nishina cross section including the Möller velocity factor (see Section 4) in order to correct for the flux of electrons in the kinetic equation (26). In fact, this factor appears naturally while carefully deriving the relativistic Boltzmann equation; see for example [20, 21, 22].

What follows is split in two parts. Sections 2–3 make the first part, where general considerations explain how and why the Planck law is obtained from a Master equation for the bosonic occupation in reciprocal space. By performing a Kramers-Moyal (or diffusion) expansion on the Master equation of a discrete random walk in reciprocal space with suitably chosen transition rates, we show that the Kompaneets equation (and its extensions) can be obtained. That offers a quite different perspective which is especially simple and general. In particular, we believe that this approach is more adapted to carry nonequilibrium extensions on the Kompaneets equation.

The second part starts with Section 4 where we connect with the standard setup for deriving the Kompaneets equation. That centers around the diffusion approximation to the Boltzmann equation with the electron gas as stationary background. It becomes clear then in Section 5 what are the main steps and what is the nature of the approximation. Apart from talking with history and reviewing part of the existing literature on the derivation of the Kompaneets equation, we are motivated by exploring possible extensions. Our motivation for a better understanding of the Kompaneets physics is based on

\footnote{In his original paper [5], Kompaneets mentions in a footnote that his work had already been published as Report No. 336 at the Institute of Chemical Physics, Academy of Sciences, USSR. We exhaustively searched the literature but we could not find this report. Interestingly, Peebles mentions as well that he could not find it either [3].}
where the fascinating hypothesis was formulated of a nonequilibrium effect modifying the Planck spectrum in the cosmic background radiation. We do not quite continue that search in the present paper, but in Section 6 we give generalizations and some possible nonequilibrium extensions of the Kompaneets equation.

There exist many derivations of the Kompaneets equation, witnessing of its fundamental interest in problems of light-matter interaction. Revisiting some of them is useful we believe for clarifying subtleties (and even inconsistencies) that are rarely mentioned (and that a careful reader would stumble upon) even in textbook references. Together with the first part of this paper, a more comprehensive understanding of this famous equation, which is much less known in the statistical mechanics community, pays respect to all these efforts while it may possibly open new avenues to introduce nonequilibrium features in the physics of the Early Universe.

2 Detailed balance with stimulated emission

Pauli was probably the first to dynamically characterize thermal equilibrium for photons in an electron bath, identifying in 1923 the condition of detailed balance in the Master equation describing scattering and recovering the Planck law for its stationary distribution [24]. Transition rates for a jump process arising within a quantum many-particle system are derived from one-particle Green’s functions [25]. Here we take a photonic set up with a symmetrized Fock space taking the tensor product over three-dimensional harmonic oscillators with wave vector \( k \) corresponding to frequency \( \omega \). An elementary transition is the annihilation of a photon with wave vector \( k \) while creating a photon with wave vector \( k' \). When the photons are in weak contact with a thermal bath at inverse temperature \( \beta \), each transition creates a flux in reciprocal space with an expectation given by

\[
j(k \to k') = \alpha(k, k') \, e^{\beta(\omega - \omega')/2} \, |\langle f | a_{k'}^\dag a_k | i \rangle|^2
\]

where \( \alpha(k, k') = \alpha(k', k) \) is symmetric and left unspecified for the moment as a parameter of dynamical activity. We have taken that to lowest order the matrix element will contain a single term annihilating and creating a photon \( a_{k'}^\dag a_k \). Writing \( n(k) \) for the occupation/level at wave vector \( k \), we put

\[
|i\rangle = |\ldots n(k) \ldots n(k') \ldots \rangle
\]

for the initial state. The only non-zero matrix element will between that initial \( |i\rangle \) and the final state

\[
|f\rangle = |\ldots n(k) - 1 \ldots n(k') + 1 \ldots \rangle
\]

Since on each of the Hilbert spaces \( a|n\rangle = \sqrt{n} |n - 1\rangle \) and \( a^\dag |n\rangle = \sqrt{n+1} |n + 1\rangle \), we find that

\[
|\langle f | a_{k'}^\dag a_k | i \rangle|^2 = (1 + n(k')) \, n(k)
\]

Therefore (4) becomes

\[
\begin{align*}
j(k \to k') &= \alpha(k, k') \, w(k, k') \, n(k) \\
w(k, k') &= e^{\beta(\omega - \omega')/2} \, (1 + n(k'))
\end{align*}
\]
In classic texts on Markov processes, (5) makes the sink term into $k'$ from $k$ and one would write a Master equation for the probability of occupying the various wave vectors. The source term is $j(k' \to k)$. Ignoring however correlations between the occupations at different wave vectors, we can write the Master equation directly for the (now expected) occupation numbers

$$\frac{\partial}{\partial t} n_t(k) = \sum_{k'} \alpha(k, k') \left[ e^{\beta(\hbar \omega - \hbar \omega')/2} (1 + n_t(k')) n_t(k) - e^{\beta(\hbar \omega' - \hbar \omega)/2} (1 + n_t(k)) n_t(k') \right] \tag{6}$$

As such, the evolution equation (6) does not need to be photon number-preserving, i.e., it does not follow for free that

$$\sum_k \frac{\partial}{\partial t} n_t(k) = \sum_{k, k'} \alpha(k, k') [w(k', k) n_t(k') - w(k, k') n_t(k)] = 0 \tag{7}$$

unless $\alpha(k, k') = \alpha(k', k)$ is indeed symmetric. Only then, (6) is a continuity equation. Secondly, detailed balance requires that $j(k \to k') - j(k' \to k) = 0$ for all $k, k'$, or (for symmetric $\alpha(k, k')$ always)

$$w(k, k') n(k) = w(k', k) n(k') \quad \text{or} \quad e^{\beta(\hbar \omega - \hbar \omega')/2} (1 + n(k')) n(k) = e^{\beta(\hbar \omega' - \hbar \omega)/2} (1 + n(k)) n(k')$$

$$\implies e^{\beta \hbar \omega} \frac{n(k)}{1 + n(k)} = \text{constant} = e^{\beta \mu} \tag{8}$$

which implies that

$$n(k) = \frac{1}{e^{\beta \hbar \omega} - 1}$$

by assuming that the chemical potential equals zero, $\mu = 0$. Note that we have only used (5) for an environment in thermal equilibrium where the temperature may refer to an electron gas or anything else. The “anything else” would solely show in the prefactor $\alpha(k, k')$ for the kinetics (6). In that sense the evolution given by (6) represents a general Kompaneets equation, before any diffusion approximation.

### 3 Kramers-Moyal expansion

The previous section considers jumps in the space of wave vectors $k$. We associate in the present section an energy $U(k)$ to the system and we expand the Master equation (6) for small energy changes $k \to k'$. That is more generally known as a Kramers-Moyal or diffusion approximation.

Let us start in one dimension, where we consider a lattice mesh $\delta > 0$ for $x \in \delta \mathbb{Z}$. The $x = k_1$ stands for the first component of the (rescaled) wave vector. We imagine a walker hopping on that lattice of wave vectors, to nearest neighbor sites with transition rates

$$w(x, x \pm \delta) = (1 + n(x \pm \delta)) B(x \pm \delta/2) \exp \left\{ \frac{\beta}{2} (U(x \pm \delta) - U(x)) \right\} \exp \left\{ \pm \frac{\beta \delta}{2} f \left( x \pm \frac{\delta}{2} \right) \right\} \tag{9}$$
Here, \( n \) is the instantaneous number of walkers; its presence in the rates represents the stimulated emission. The function \( B > 0 \) is an inhomogeneous activity rate and \( \beta \) is the inverse temperature of a medium enabling the hopping. There is also a driving force \( f \) and a potential \( U \) which are added following the condition of local detailed balance at fixed environment inverse temperature \( \beta = 1/k_B T \). We do not for the moment dwell on the physical meaning of the driving \( f \) and we do not restrict us to photons but to bosonic systems more generally; see also Section 6. The rate (5) is a special case of (9), where \( f \equiv 0 \) and the photon energy \( U = \hbar \omega \). Abusing notation, we also incorporate the symmetric activity \( \alpha(k,k') \) in the rates as the prefactor \( B \).

For fixed \( \delta \) the Master equation as in (6) becomes

\[
\frac{\partial n_t}{\partial t}(x) + j_t(x, x + \delta) - j_t(x - \delta, x) = 0 \quad \text{for (10)}
\]

\[
j_t(x, x + \delta) = n_t(x)w(x, x + \delta) - n_t(x + \delta)w(x, x + \delta)
\]

\[
j_t(x - \delta, x) = n_t(x - \delta)w(x, x - \delta) - n_t(x)w(x, x - \delta)
\]

We expand this last equation to second order in \( \delta \); see Appendix A. The result is

\[
\frac{\partial}{\partial t} n_t = \delta^2 \left\{ (\beta Bg' + \beta B'g)(1 + n)n + (\beta Bg + B')n' + 2\beta Bgnn' + Bn'' \right\}
\]

with \( g(x) = U'(x) - f(x) \). That can be written more explicitly as a continuity equation,

\[
\frac{\partial n_t}{\partial t} = \delta^2 \frac{\partial}{\partial x} \left\{ B(x) \left( \frac{\partial n_t}{\partial x}(x) + \beta g(x)(1 + n_t(x))n_t(x) \right) \right\}
\]

in which we recognize the structural elements of the Kompaneets equation (1).

We can indeed redo that in three dimensions, on \( \Delta Z \times \Delta Z \times \Delta Z \). Taking the same rates in all directions as before with 3-dimensional “force” \( f \), the diffusion approximation now reads

\[
\partial_t n_t = \delta^2 \nabla \cdot j
\]

with in Cartesian coordinates \( (x_\ell, \ell = 1, 2, 3) \) for \( j = \sum_\ell j_\ell \hat{\mathbf{x}}_\ell \),

\[
j_\ell = B \left( \frac{\partial n_t}{\partial x_\ell}(x) + \beta g_\ell(1 + n_t(x)) \right)
\]

for \( g_\ell = \frac{\partial U}{\partial x_\ell} - f_\ell \).

Moving finally to the setup for the Kompaneets equation we enter frequency space by assuming that \( n_t = n(\omega,t) \), \( g = g(\omega) \), \( D = D(\omega) \) with frequency \( \omega = c \sqrt{\sum_\ell x_\ell^2} \) for speed of light \( c \). It means to rewrite (13) in spherical coordinates, with \( \omega \) as radial variable:

\[
\omega^2 \frac{\partial n_t}{\partial t}(\omega,t) = c \delta^2 \frac{\partial}{\partial \omega} \left\{ \omega^2 B(\omega) \left( c \frac{\partial n_t}{\partial \omega}(\omega,t) + \beta g(\omega)(1 + n(\omega,t))n(\omega,t) \right) \right\}
\]

\[\text{Note we are in reciprocal space here so that } \delta \text{ is an inverse length and } f \text{ is measured in multiples of } \hbar c.\]
That is an extended Kompaneets equation, to be compared with (1), where the energy change and the driving combine into 
\[ g(\omega) = c \frac{\partial U}{\partial \omega}(\omega) - f(\omega). \]
Making the choices
\[ c^2 \delta^2 B(\omega) = \frac{k_B T}{m_e c^2} n_e \sigma_T c \omega^2, \quad g(\omega) = \hbar c \]
the above equation (15) becomes exactly the one of Kompaneets (1). Note that 
\[ n_e \sigma_T = \tau^{-1} \]
is the average collision rate, as before. That shows that the full structure of the Kompaneets equation is obtained as the diffusion approximation to a Master equation with stimulated emission, and this holds whenever the limiting activity and drift obey (16). Justifications for the choices (16) come from the physical nature of the process considered in the Kompaneets equation. The photon energy is there \( U(\omega) = \hbar \omega \) and there is no driving \( f \equiv 0 \), making indeed \( g = c \frac{\partial U}{\partial \omega} = \hbar c \). For understanding the first equality in (16), we note that \( c^2 \delta^2 B(\omega) \) appears as the diffusion constant \( D(\omega) \) in (15).

The shift in frequency for a photon undergoing Compton scattering determines that diffusion constant as the conditional average squared shift
\[ D(\omega) = \frac{\langle (\omega' - \omega)^2 \rangle}{2 \tau} \]
(17)
The shift follows from the well-known Compton scattering formula [4]
\[ \omega' - \omega = \frac{c p \cdot (\hat{n}' - \hat{n}) - \hbar \omega(1 - \hat{n} \cdot \hat{n}')}{{\gamma m_e c^2 [1 - p \cdot \hat{n}' / \gamma m_e c + (\hbar \omega / \gamma m_e c^2)(1 - \hat{n} \cdot \hat{n}')] \omega}} \]
(18)
where the electron momentum is \( p \), \( \gamma \) is the relativistic factor and \( \hat{n}' - \hat{n} \) is the scattering vector. In the low-temperature regime where Compton scattering is relevant, under the assumption that the electrons and the photons are of comparable energy much less than \( m_e c^2 \), most of the momentum is carried by the electrons, meaning \( |p| \gg \omega / c \). Hence we can replace the term between the brackets in the denominator by unity, and only retain the first term in the numerator in (18)
\[ \omega' - \omega \approx \frac{p \cdot (\hat{n}' - \hat{n})}{m_e c} \omega \]
(19)
We can assume the square of projection of the scattering vector \( \hat{n}' - \hat{n} \) on the momentum vector \( p \) to average out to a constant of magnitude 1, which we will hereafter ignore. Continuing then the calculation for (17) yields
\[ D \propto \frac{\langle |p|^2 \omega^2 \rangle}{2 m_e^2 c^2 \tau} = \frac{1}{m_e c^2} \frac{\langle |p|^2 \rangle}{2 m_e^2} \frac{\omega^2}{\tau} \times \frac{k_B T}{m_e c^2} \frac{\omega^2}{\tau} \]
(20)
where the temperature \( T \) gives the average kinetic energy of the electron distribution. We thus recover the first equality in (16).

For better understanding and comparison, we add here the heuristics of the opposite situation where we consider an electron in a photon bath (in contact with other matter at temperature \( T \)). Since the electronic density of states goes as the square root of the energy \( E \), we must have an equation for the number \( N(E, t) \) of electrons of the form
\[ \frac{\partial N}{\partial t}(E, t) = \frac{1}{E^{1/2}} \frac{\partial}{\partial E} E^{1/2} j(E) \]
(21)
to be compared with (2). To determine the current \( j_t(E) \) we remember that a crucial property of the Doppler effect, used to compute (17) and to arrive at (16), is that the shift in frequency is linear in the frequency itself, as seen in (19). Here however, because the shift in the electron’s energy equals the same expression, the average squared energy shift is proportional to the square of the momentum instead, hence only linear in the energy. That allows us to write down the electronic version of the Kompaneets equation immediately

\[
E^{1/2} \frac{\partial N}{\partial t}(E, t) = b \frac{\partial}{\partial E} E^{3/2} \left\{ k_B T \frac{\partial N}{\partial E}(E, t) + N(E, t) \right\}
\]  

(22)

for some rate \( b \propto c \sigma_T \frac{U_\gamma}{m_e c^2} \), where \( U_\gamma \) is the energy density of the photon gas. Note finally that we neglect the fermionic nature of the electron, since we presume non-degeneracy of the electron gas (not having thus the fermionic version of stimulated emission). For verification, a derivation of this equation can be found in [27]. For applications to highly dense states of fermionic matter, the Pauli exclusion is significant and departures from (22) are expected. In such a regime however, the long-range Coulomb interactions become relevant and an extra term must be added to the Boltzmann equation (22) anyway.

4 The Boltzmann-Master equation

The Master equation (6) only involves the photon occupation and the electron bath is integrated out and remains present only via the bath temperature. Similarly, the dynamics (9) effectively treats the electron bath via temperature, mobility and possible driving \( f \). To go back one (finer) level of description, we must introduce the integration step. Here we start from a Master equation jointly for electrons and photons. Formally, it has the general structure,

\[
\frac{\partial n}{\partial t}(p, k; t) = \int \frac{d^3 p'}{d^3 k'} \left\{ n(p', k'; t) w(p', k' \rightarrow p, k)(1 + n(k, t)) \right. \\
- n(p, k; t) w(p, k \rightarrow p', k')(1 + n(k', t)) \right\}
\]  

(23)

for the joint occupation \( n(p, k; t) \) at time \( t \) of electron momentum \( p \) and photon wave vector \( k \). The four-momenta participating in the transition are, of course, given by

\[
p^{(r)} = \left( \frac{E^{(r)}}{c}, p^{(r)} \right) \quad \text{and} \quad k^{(r)} = \left( \frac{\hbar \omega^{(r)}}{c}, k^{(r)} \right)
\]

We added the stimulated emission in terms of the photon occupation \( n(k', t) \), but include no such factor for the electrons, assuming them to be nondegenerate (dilute). The transition rates should be obtained from the microscopic process of interaction, here Compton scattering, but we write generally

\[
w(p, k \rightarrow p', k') d^3 p' d^3 k' = \frac{d \sigma}{d \Omega}(p, k) d \Omega c \left( 1 - \frac{v}{c} \cdot \hat{n} \right)
\]  

(24)

\footnote{From here we make a distinction between the vectors \( p \) etc and the four-vectors \( p \).}
for a differential cross section \( \frac{d\sigma}{d\Omega} \), where the equality holds in the sense of distributions. The factor \( c(1 - \mathbf{v}/c \cdot \hat{n}) \) is commonly called the Møller velocity and it is necessary for a consistent microscopic description \([20, 28, 22]\). Neglecting that factor is a common inaccuracy in the literature, e.g. \([27, 29, 30]\). Viewed nonrelativistically it corresponds to the relative velocity between the electrons and the photons, but even with a proper relativistic treatment, where the relative velocity is of course \( c \) in every frame, this exact factor will show in the colliding flux density \([31]\).

We want (23) to correspond to a Boltzmann equation where the total number of electrons and photons is conserved. It suffices here to invoke dynamical reversibility in the form

\[
w(p, k \to p', k') = w(p', k' \to p, k)
\]

which means that the rate function \( w \) is symmetric. Moreover, if we assume that the electron gas evolves on much shorter time-scales than the photons, we can suppose to two distributions are uncorrelated, allowing us to write

\[
n(p, k; t) = \mathcal{N}(p) n(k, t),
\]

where \( \mathcal{N}(p) \) is the momentum distribution of the electrons, which need not necessarily be in thermal equilibrium. We also assume isotropy so that the above leads to the writing of the Boltzmann-type equation:

\[
\frac{\partial n}{\partial t}(\omega) = \int d^3p \, dw \left[ n(\omega') \mathcal{N}(p') (1 + n(\omega)) - n(\omega) \mathcal{N}(p) (1 + n(\omega')) \right]
\]

(26)

where \( dw = d\sigma c(1 - \mathbf{v}/c \cdot \hat{n}) \), \( d\sigma = \frac{d\sigma}{d\Omega} \) stands for the (unspecified for the moment) scattering cross section multiplied with the incoming flux of electrons. The scattering is subject to conservation of energy-momentum corresponding to the collision scheme

\[
(p, k) \rightleftharpoons (p', k')
\]

which means a transition of wave vectors and corresponding frequencies

\[
k \rightleftharpoons k', \quad \omega = c|k| \rightleftharpoons \omega' = c|k'|
\]

(27)

for the photon. The first term on the right-hand side of (26) is the source term where photons with frequency \( \omega \) are created from collisions with electrons with momentum \( p' \), and the second term is the sink term where photons with frequency \( \omega \) collide with electrons of momentum \( p \). In that case, the equation (26) is rewritable indeed as a Master (rate) equation and is really the more precise version of Eq. 1 in \([5]\).

We can integrate out the electron bath to find the rates involving only the photons

\[
w(k' \to k) = \int d^3p \, d^3p' \, w(p', k' \to p, k) \mathcal{N}(p').
\]

(28)

\[\text{To be consistent with literature, we note here that } w(p, k \to p', k') \text{ defined as such is called the non-covariant transition rate. The covariant transition rate } W(p, k \to p', k') \text{ is defined such that } W(p, k \to p', k') = \frac{W(p, k \to p', k')}{EE'/\hbar\omega'}.\]
Invoking the reversibility (25) of the rates, \( w(p', k' \rightarrow p, k) = w(p, k \rightarrow p', k') \), we continue by writing down the inverse rates

\[
w(k \rightarrow k') = \int d^3p d^3p' w(p', k' \rightarrow p, k) \mathcal{N}(p') \frac{\mathcal{N}(p)}{\mathcal{N}(p')}
\]  

To connect this to the setup in Section 2, we have to further include the assumption that the electrons are in thermal equilibrium with inverse temperature \( \beta = 1/k_B T \), which implies that

\[
\mathcal{N}(p) \mathcal{N}(p') = e^{-\beta(E-E')} = e^{-\beta(\hbar \omega' - \hbar \omega)},
\]

where the last equality follows from conservation of energy in the collisions. Inserting (30) into (29) yields the detailed balance relation for the photon transition rates (28),

\[
\frac{w(k' \rightarrow k)}{w(k \rightarrow k')} = e^{-\beta(\hbar \omega - \hbar \omega')}.
\]

5 Diffusion approximation to the Boltzmann Equation

The previous section shows how the kinetic equation which correctly introduces the Compton scattering between the electrons (making the steady thermal bath) and the photons gives rise to a Master equation as treated in Section 2. From there, as shown in Section 3, the Kompaneets equation can be obtained. In the present section, we take the two steps together and directly apply the diffusion approximation to the Boltzmann equation. It is much closer to the published approaches and it gives us the opportunity to clarify a number of points which traditionally are left to the reader. In particular, our framework is more direct than [5, 16, 14, 15] because we do not use (in the derivation itself) that the Planck distribution is the stationary solution of (26). Under certain conditions on the possible cross sections, Escobedo and Mischler [12] rigorously prove the diffusion approximation to the Boltzmann equation in the context of an electron-photon gas. However, they do not explicitly link these conditions to the Thomson/Klein-Nishina cross section nor do they address the same problems we treat here.

We begin with (26) where \( dw = d\Omega \ c \ (1 - \frac{\mathbf{v}}{c} \cdot \mathbf{\hat{n}}) \frac{d\sigma}{d\Omega}(\mathbf{p}, \mathbf{\hat{n}}, \Omega) \) and

\[
\frac{d\sigma}{d\Omega}(\mathbf{p}, \mathbf{\hat{n}}, \Omega) = \frac{3\sigma_T}{16\pi} \frac{1}{\gamma^2 \left( 1 - \frac{p \cdot \mathbf{\hat{n}}'}{\gamma m_e c} + \frac{\hbar \omega}{\gamma m_e c}(1 - \mathbf{\hat{n}} \cdot \mathbf{\hat{n}}') \right)^2} \times \left\{ 1 + \frac{(\frac{\hbar \omega}{\gamma m_e c}(1 - \mathbf{\hat{n}} \cdot \mathbf{\hat{n}}'))^2}{(1 - \mathbf{p} \cdot \mathbf{\hat{n}}'/\gamma m_e c)(1 - \mathbf{p} \cdot \mathbf{\hat{n}}'/\gamma m_e c + \frac{\hbar \omega}{\gamma m_e c}(1 - \mathbf{\hat{n}} \cdot \mathbf{\hat{n}}'))} + \right. \\
+ \left. \left( 1 - \frac{\mathbf{\hat{n}} \cdot \mathbf{\hat{n}}'}{\gamma^2(1 - \mathbf{p} \cdot \mathbf{\hat{n}}'/\gamma m_e c)(1 - \mathbf{p} \cdot \mathbf{\hat{n}}'/\gamma m_e c)} \right)^2 \right\}
\]

as the (correct) Klein-Nishina expression for the cross section. This expression appears in [32], but a derivation is not shown there. Jauch and Rohrlich [33] give a detailed
derivation, but only express the differential cross section in terms of the scattering matrix. As we could not find any reference which contains the exact frame-independent expression for the Klein-Nishina cross section given above we will devote Appendix B to this discussion. However, we also note here that once established the correct (and not so enlightening) expression above, we only need the first few orders of its expansion. As before, we keep in mind the collision scheme
\[ \mathbf{p} + \frac{\hbar \omega}{c} \hat{n} \rightleftharpoons \mathbf{p}' + \frac{\hbar \omega'}{c} \hat{n}' \]
where we now express the wave vector as \( \mathbf{k} = \frac{\hbar \omega}{c} \hat{n} \), defining the scattering angle \( \cos \theta = \hat{n} \cdot \hat{n}' \).

We follow the usual steps, assuming that: (i) the electrons are in thermal equilibrium at temperature \( T \); (ii) the photons are soft, meaning that their energy is very small compared to the rest energy of the electron (\( \hbar \omega \ll m_e c^2 \)), but of same order of the electron bath energy (\( \hbar \omega \sim k_B T \) ); (iii) electrons are nonrelativistic (|\( \mathbf{p} \)| \( \ll m_e c \) or \( k_B T \ll m_e c^2 \)). By combining (ii) and (iii) we conclude also that energy is transferred in small amounts only, permitting the continuum (or diffusion) approximation\(^6\).

Points (ii) and (iii) suggest an expansion in terms of the energy shift, as proposed originally by Kompaneets [5]. For this purpose we define
\[ \Delta = \frac{\hbar(\omega' - \omega)}{k_B T} \]
By looking at the Compton shift \((18)\), we have
\[ \Delta(\omega, \mathbf{p}) = \frac{\hbar \omega}{k_B T \gamma m_e c^2} \left[ \frac{\mathbf{c} \cdot (\hat{n}' - \hat{n}) - \hbar \omega (1 - \hat{n} \cdot \hat{n}')}{1 - \mathbf{p} \cdot \hat{n}' / \gamma m_e c + (\hbar \omega / \gamma m_e c^2)(1 - \hat{n} \cdot \hat{n}')} \right] \]
We refer the reader to [33] for more discussion of expressions \((31)\) and \((32)\).

Let us now go back to \((26)\) to make an expansion up to second order in \( \Delta(\omega, \mathbf{p}) \), mimicking the Kramers-Moyal expansion of Section 3. We make the natural change of variables
\[ \omega \rightarrow x = \frac{\hbar \omega}{k_B T} \]
\[ \omega' \rightarrow x' = \frac{\hbar \omega'}{k_B T} \]
to write the photon numbers up to second order as
\[ n(x', t)(1 + n(x, t)) = n(x, t)(1 + n(x, t)) + (1 + n(x, t)) \frac{\partial n}{\partial x} \Delta + (1 + n(x, t)) \frac{\partial^2 n}{\partial x^2} \frac{\Delta^2}{2} \]
\[ n(x, t)(1 + n(x', t)) = n(x, t)(1 + n(x, t)) + n(x, t) \frac{\partial n}{\partial x} \Delta + n(x, t) \frac{\partial^2 n}{\partial x^2} \frac{\Delta^2}{2} \]
\(^6\)In fact, the transfer of energy depends both on the incoming \( \omega \) and \( \mathbf{p} \) as dictated by the Compton formula and, to the lowest order, it is proportional to the product of both \([19]\).
From (i) above, the electron distribution is Maxwellian at temperature $T$,

$$N(p) d^3 p = N_{eq}(|p|) d^3 p = n_e (2\pi m_e k_B T)^{-3/2} \exp \left(-\frac{p_x^2 + p_y^2 + p_z^2}{2m_e k_B T}\right) d^3 p$$  \hspace{1cm} (35)

where $n_e$ is the electron density. Using conservation of energy

$$E' = E - \Delta k_B T$$

we get

$$N_{eq}(|p'|) = N_{eq}(|p|) \left(1 + \Delta + \frac{\Delta^2}{2}\right)$$  \hspace{1cm} (36)

By taking (33)–(34) and (36) in (26), we obtain

$$\frac{\partial n}{\partial t} = \left[\partial_x n + n(1 + n)\right] I_1 + \left[\frac{1}{2} \partial_{xx} n + (1 + n) \left(\frac{n}{2} + \partial_x n\right)\right] I_2$$  \hspace{1cm} (37)

where, with $\ell = 1, 2,$

$$I_\ell(x) = c \int d^3 p d\Omega \left(1 - \frac{v}{c} \cdot \hat{n}\right) \frac{d\sigma}{d\Omega}(p, \hat{n}, \Omega) N_{eq}(|p|) \Delta^\ell$$  \hspace{1cm} (38)

The computation of the integrals $I_1(x)$ and $I_2(x)$ is presented in Appendices C.1 and C.2 with results

$$I_1(x) = \frac{n_e \sigma_T c k_B T}{m_e c^2} x (4 - x)$$  \hspace{1cm} (39)

$$I_2(x) = \frac{n_e \sigma_T c k_B T}{m_e c^2} 2x^2$$  \hspace{1cm} (40)

After standard manipulations, we end up with the Kompaneets equation

$$\frac{\partial n}{\partial t}(x, t) = \frac{n_e \sigma_T c k_B T}{m_e c^2} \frac{1}{x^2} \partial_x \left\{x^4 (\partial_x n(x, t) + n(x, t)(1 + n(x, t)))\right\}$$  \hspace{1cm} (41)

which is (1).

As already mentioned, the common setup in the literature is to consider the Thomson cross section (3) only, without the Møller velocity factor

$$c \left(1 - \frac{\mathbf{v}}{c} \cdot \hat{n}\right)$$

to compute $I_2(x)$; the value of $I_1(x)$ is then obtained from assuming that (37) must be a continuity equation, with the current vanishing in equilibrium for the Planck distribution. However, when using the Thomson cross section only one finds (see Appendix C.3 for the computation of the integrals in this case)

$$I_1^{Th}(x) = \frac{n_e \sigma_T c k_B T}{m_e c^2} x (1 - x)$$  \hspace{1cm} (42)
instead of (39). To the best of our knowledge that first Kompaneets integral was never computed. As a matter of logic, as we have seen in the previous section, it is not possible with solely the Thomson cross section (which ultimately leads to (42)) to find (41). The Møller velocity factor must be included in the definition of the rates, otherwise consistency in the microscopic description of the collision term is lost. This term naturally appears when carefully deriving (26), see [20], or when going from the covariant description to the non-covariant one. [22]. As mentioned, this term is also needed to yield conservation of the photon number, but what is surprising perhaps, is that none of these effects are seen on \(I_2(x)\), whence making it possible to employ the traditional indirect argument used traditionally (including in Kompaneets’ original paper) to fix the value of \(I_1(x)\).

It is possible in the derivation of the Kompaneets equation to use the Thomson cross section together with the Møller factor (of course) as it satisfies dynamical reversibility as well. Note however that upon writing (3) we are fixing the electron rest frame and, thus, all quantities must be expressed in this frame of reference (including the distribution functions). The diffusion approximation and the integrals must also be performed in this frame. We can avoid this complication by expressing the cross section in a frame-independent manner, just as (31). In that way we have the freedom to work with the Boltzmann equation (26) in the most convenient frame of reference (where the electron distribution is isotropic Maxwellian). We could not find in the literature any frame-independent version of the Thomson cross section, neither a mention of such features which are so important to yield a consistent diffusion approximation.

Traditionally, equation (26) is called the non-covariant Boltzmann equation and while a fully covariant Boltzmann equation does indeed exist, we have chosen to not express it here. However, we invite the reader to check, for example, [22] for the covariant formalism. These two versions are equivalent to each other, provided that we use the rates (24) to connect them. On the other hand, once the link is established, we can regard the Kompaneets equation as the nonrelativistic limit of either Boltzmann equation, which here is concretely realized as the second-order expansion in the energy shift, as expressed in the integrals (39) and (40). In this more kinematic approach, the interpretation of both integrals above is clear as the average shift and shift-squared of the energy, while in covariant approaches the interpretation is not always straightforward. Derivations of the Kompaneets equation which proceed in a covariant manner include [34, 35, 36]. The covariant derivation is simpler in the sense that the main ingredient is the microscopic reversibility of the rates, in the spirit of Sections 2–3. The problems we mention above are taken care of automatically then. For example, using the scattering matrix gives the Møller prefactor and consistency of the different reference frames (since the quantities are always expressed in a Lorentz invariant way). In [37], Challinor and Lasenby man-

\[\text{\footnotesize In fact, any cross section which is obtained from an unitary scattering matrix yields dynamical reversibility; see [22] for a proof.}\]

\[\text{\footnotesize Non-covariance should not be confused with frame-independence. [20]}\] is an example of a non-covariant equation which is written in a frame-independent way.

\[\text{\footnotesize For simplicity, we call any derivation of the Kompaneets equation starting from the covariant Boltzmann equation a covariant approach/derivation. Strictly speaking, the use of the word covariant is not entirely correct because when assuming isotropy to carry the diffusion approximation, one is fixing the reference frame, thus breaking covariance.}\]
aged to identify the moments of the shift (as above) in a covariant approach, and that was repeated later in [35]. Yet, in the integrals, the moments are still multiplied by a Dirac delta, which leaves open the exact analytical computation.

6 Extending the Kompaneets equation

There are various well-known extensions of the Kompaneets equation compared to what was mainly discussed in the previous sections. For example, the condition that the photons are soft, \( \hbar \omega \sim k_B T \) may be relaxed provided that we treat the regime \( \hbar \omega \gg k_B T \) (hard photons) explicitly in the diffusion approximation to the Boltzmann equation. An extra contribution is found then in (1), referred to as the “extended Kompaneets equation to down-Comptonization” [17][19]. The regime of down-Comptonization first appeared in [38] in the so-called Ross-McCrack equation, where radiative transfer of X-ray photons is treated. The equation derived in [38], however, does not yield the Bose-Einstein distribution as stationary solution and should be regarded only as an asymptotic limit of such extended Kompaneets equation.

There are more and different processes in astrophysical plasmas that change the photon occupation number. Beyond Compton scattering, contributions due to (for example) Bremsstrahlung (involving the ions as well) and radiative (or double) Compton scattering may be considered, [5, 39, 4, 16]. Due to the radiative nature of such processes it is not true any longer that the photon number is conserved and we cannot describe them by a Master-Boltzmann equation.

Relativistic extensions to the Kompaneets equation and calculations done in the covariant Boltzmann description include [34, 35, 40, 36, 41, 42, 43]. Barbosa [32] also addresses relativistic extensions, but using a more kinematic approach to the Fokker-Planck approximation.

Pitrou [44] and Buet et al [45] both deal with anisotropic extensions to the Kompaneets equation, where the assumption of isotropy in the distribution functions is no longer required. More general solutions to (1) can be found in [46, 47]. More recently, the diffusion approximation as proposed by Kompaneets is applied to a neutrino gas [48, 49]. For applying the Kompaneets equation in the context of cosmology, the contribution of the cosmic expansion (red shift) must be taken into consideration. In [39, 10, 11, 50], that contribution appears as a convective term in the Boltzmann equation (26). A turbulent Doppler shift (as an integrated Sachs-Wolfe effect for a random gravitational potential field) would give an additional diffusion in frequency but we have not seen that being worked out.

Let us now turn to less standard generalizations. Recall that we have already derived an extended Kompaneets equation in (15) for bosons with energy \( U \) and possibly driven by the “force” \( f \). Here we continue with a modification and two extra constraints where the electron bath need not be in equilibrium. As already pointed out in [32, 51, 52, 53], Kompaneets equation can indeed be recovered for an electron bath which is not in equilibrium, provided that the distribution of the electrons is isotropic, \( i.e. \), that the distribution can be written in terms of the energy. Although some references already considered this fact by using different approaches, including [32, 33], we could not find references which conclude the same while starting from the non-covariant kinetic equation (26). The details of such a derivation are given in Appendix C.4. By assuming:
(i) isotropy of the distribution of the electrons,

\[ \mathcal{N}(\mathbf{p}, t) d^3 \mathbf{p} = \mathcal{N}(|\mathbf{p}|, t) d^3 \mathbf{p} \]  

(43) and (ii) that \( \mathcal{N} \) decays faster than \(|\mathbf{p}|^3\), i.e.,

\[ \lim_{|\mathbf{p}| \to \infty} |\mathbf{p}|^3 \mathcal{N}(|\mathbf{p}|, t) = 0 \]  

(44)

we find Kompaneets equation (41) with an effective temperature

\[ T_{\text{eff}} = \frac{\langle |\mathbf{p}|^2 \rangle}{3 k_B m_e} \]

where

\[ n_e \langle |\mathbf{p}|^2 \rangle = \int d^3 \mathbf{p} |\mathbf{p}|^2 \mathcal{N}(|\mathbf{p}|) \]

Note that it is not so strange to recover the equilibrium (relaxation to) Planck distribution in the diffusion approximation. It only means that nonequilibrium features hide in higher order approximations. As possible scenarios for generating nonequilibrium effects, we may have in mind gravitational sources (still largely unspecifed) or the possible presence of (turbulent) magnetic fields driving the electrons.

7 Conclusions and outlook

Relaxation to quantum equilibrium distributions is a topic of much current interest. In that spirit, the Kompaneets equation for the Comptonization of photons in an electron bath is one of the few examples where a diffusion approximation to the Boltzmann equation can be explicitly done, in such way that we found useful to revisit its derivation.

First, we connected the derivation with statistical mechanical setups, known for example as Kramers-Moyal expansion and clarified a number of points related to the importance of using the physically correct scattering cross sections together with an \textit{ab initio} correct Boltzmann equation (26). We explored and stressed many features which are absent or commonly neglected in literature, such as the appearance of the Møller velocity in the Boltzmann collision term.

Secondly, the re-derivations in the present paper may serve as useful point of departure to understand modifications and nonequilibrium corrections to the Planck law. Indeed, one of the goals of the present paper has been to derive the Kompaneets equation without resorting to the Kompaneets’ approach based on the current vanishing at equilibrium. We found that diffusion in frequency space, with drift determined by exchanges in energy and possibly nonconservative processes combined with stimulated emission, leads to an extension of the Kompaneets equation. Such extensions will be used in future work to motivate specific low-frequency modifications to the Planck law, following the scenario and motivation in [23].

15
A Expansion of the transition rates

The transition rates (9) can be expanded from

\[
\begin{align*}
\exp (x, x \pm \delta) &= (1 + n \pm \delta n') + \frac{\delta^2}{2} n'') \left( B \pm \frac{\delta}{2} B' + \frac{\delta^2}{8} B'' \right) \exp \left\{ -\frac{\beta}{2} \left( \pm \delta (U - f) + \frac{\delta^2}{2} (U'' - f') \right) \right\} \\
\exp (x, x) &= (1 + n) \left( B \pm \frac{\delta}{2} B' + \frac{\delta^2}{8} B'' \right) \exp \left\{ \frac{\beta}{2} \left( \pm \delta (U - f) + \frac{\delta^2}{2} (U'' - f') \right) \right\}
\end{align*}
\]

An expansion up to second order in \( \delta \) yields

\[
\begin{align*}
w(x, x \pm \delta) &= A(x) \pm \frac{\delta}{2} C(x) + \frac{\delta^2}{2} E(x) \\
w(x \pm \delta, x) &= A(x) \pm \frac{\delta}{2} F(x) + \frac{\delta^2}{2} G(x)
\end{align*}
\]

with short-hands

\[
\begin{align*}
A(x) &= B(1 + n) \\
C(x) &= 2Bn' - (\beta Bg - B')(1 + n) \\
E(x) &= \left( \frac{1}{4}(\beta^2 Bg^2 + B'') - \frac{1}{2}(\beta g B' + \beta g'B) \right) (1 + n) - (\beta Bg - B')n' + Bn'' \\
F(x) &= (\beta Bg + B')(1 + n) \\
G(x) &= \left( \frac{1}{4}(\beta^2 Bg^2 + B'') + \frac{1}{2}(\beta g B' + \beta g'B) \right) (1 + n) \\
g(x) &= U''(x) - f(x)
\end{align*}
\]

That gives to leading order in the Master equation

\[
\partial_t n = \delta^2 \{ (G(x) - E(x))n(x) + F(x)\partial_x n(x) + A(x)\partial_{xx} n(x) \}
\]

Substituting the short-hands we get (11).

B Klein-Nishina differential cross section

As before, let us suppose a collision process given by

\((p, k) \leftrightarrow (p', k')\)

with four-momenta

\[
p'^{(\gamma)} = \left( \frac{E^{(\gamma)}}{c}, p'^{(\gamma)} \right) \quad \text{and} \quad k'^{(\gamma)} = \left( \frac{\hbar \omega^{(\gamma)}}{c}, k'^{(\gamma)} \right)
\]

we fix our metric tensor to diag\([- + + +\]) and use the same notation as Jauch and Rohrlich [33]. For convenience, we will use natural units \((\hbar = c = 1)\), which will be restored later in (31). The total Klein-Nishina cross section for Compton scattering is
obtained directly from the trace of the scattering matrix (the scattering amplitude), which is given by

\[
\sigma = \frac{1}{-p' \cdot k' E' \omega' \delta(4)} \int d^3 \mathbf{p}' d\omega' \omega'^2 \delta(4)(p' + k' - p - k) X
\]  

where \( X \) is the scattering amplitude. For a proof of this relation we refer the reader to [33, 52, 53]. By differentiating with respect to the solid angle and using the Fundamental Theorem of Calculus we obtain the differential cross section

\[
\frac{d\sigma}{d\Omega} = \frac{1}{-p' \cdot k' E' \omega' \delta(4)} \int d^3 \mathbf{p}' d\omega' \omega'^2 \delta(4)(p' + k' - p - k) X
\]

this is the frame-independent representation of the differential cross section, where the frame-independent representation of the scattering angle is given by \( \cos \theta = \hat{n} \cdot \hat{n}' \). By evaluating the integrals we get the following expression for the Klein-Nishina differential cross section

\[
\frac{d\sigma}{d\Omega} = \frac{3 \sigma_T}{16 \pi} \left( \frac{\omega'}{\omega} \right)^2 \frac{1}{\gamma^2(1 - \beta \cdot \hat{n})^2} X
\]

where \( \beta = v/c \) (or \( v \) in natural units). The outgoing photon frequency \( \omega' \) is dependent from the ingoing frequency by the Compton shift [18], which can be expressed in the following way

\[
\frac{\omega'}{\omega} = \frac{1 - \beta \cdot \hat{n}}{1 - \beta \cdot \hat{n}' + \frac{\omega}{E}(1 - \hat{n} \cdot \hat{n}')} \tag{50}
\]

The scattering amplitude is given by

\[
X = \frac{p \cdot k}{p \cdot k'} + \frac{p \cdot k'}{p \cdot k} + 2 \left( \frac{m^2}{p \cdot k'} - \frac{m^2}{p \cdot k} \right) + \left( \frac{m^2}{p \cdot k'} - \frac{m^2}{p \cdot k} \right)^2 \tag{51}
\]

Substitute (50) in (49) to yield

\[
\frac{d\sigma}{d\Omega} = \frac{3 \sigma_T}{16 \pi} \left[ \frac{1}{\gamma(1 - \beta \cdot \hat{n}' + \frac{\omega}{E}(1 - \hat{n} \cdot \hat{n}'))} \right]^2 X \tag{52}
\]

In (51), using \( p \cdot k'(\gamma) = -E \omega'(1 - \beta \cdot \hat{n}') \) we have

\[
\frac{p \cdot k}{p \cdot k'} + \frac{p \cdot k'}{p \cdot k} = 1 + \frac{\omega(1 - \hat{n} \cdot \hat{n}')}{E(1 - \beta \cdot \hat{n}')} + \frac{1 - \beta \cdot \hat{n}'}{1 - \beta \cdot \hat{n}' + \frac{\omega}{E}(1 - \hat{n} \cdot \hat{n}')} \quad \text{and} \quad \frac{m^2}{p \cdot k'} - \frac{m^2}{p \cdot k} = -\frac{1 - \hat{n} \cdot \hat{n}'}{\gamma^2(1 - \beta \cdot \hat{n})(1 - \beta \cdot \hat{n}')} \tag{53}
\]

Substituting both expressions into (52) yields

\[
X = 1 + \left[ 1 - \frac{(1 - \hat{n} \cdot \hat{n}')}{\gamma^2(1 - \beta \cdot \hat{n})(1 - \beta \cdot \hat{n}')} \right]^2 + \frac{\omega^2(1 - \hat{n} \cdot \hat{n}')^2}{E^2(1 - \beta \cdot \hat{n}'(1 - \beta \cdot \hat{n}' + \frac{\omega}{E}(1 - \hat{n} \cdot \hat{n}'))) \tag{53}
\]

Putting back (53) into (52) we use

\[
\beta = \frac{v}{E}
\]

to yield the frame-independent representation of the Klein-Nishina cross section (31).
C Computation of integrals

C.1 Integral $I_1$

We compute

$$I_1(x) = c \int d^3p \, d\Omega \left(1 - \frac{v}{c} \cdot \hat{n}\right) \frac{d\sigma}{d\Omega}(p, \hat{n}, \Omega) N_{eq}(|p|) \Delta$$

In what follows we omit the dependencies on the variables for simplicity. The leading order is the second on the electron momenta and the expansion yields

$$I_1(x) = \frac{16\pi}{3\sigma_T} \left(1 - \frac{p}{\gamma m_e c} \cdot \hat{n}\right) \frac{d\sigma}{d\Omega} \Delta = \frac{x p \cdot (\hat{n}' - \hat{n})}{m_e c} (1 + \cos^2 \theta) - \frac{x^2 k_B T}{m_e c^2} (1 - \cos \theta)(1 + \cos^2 \theta)$$

$$+ \frac{x}{(m_e c)^2} \left\{(1 + 2 \cos \theta - \cos^2 \theta)(p \cdot \hat{n})^2 + (3 - 2 \cos \theta + 5 \cos^2 \theta)(p \cdot \hat{n}')^2 - 4(p \cdot \hat{n})(p \cdot \hat{n}')(1 + \cos^2 \theta)\right\}$$

(54)

where we used that $p = \gamma m_e v$.

We first compute the integral over $p$ in Cartesian coordinates, and then over the solid angle. Since the distribution is isotropic, the first part yields zero contribution, i.e.,

$$\int d^3p \, p \cdot (\hat{n}' - \hat{n}) N_{eq}(|p|) = 0 \quad (55)$$

The integral over the momentum in the second parcel is readily done, yielding

$$\int d^3p \, N_{eq}(|p|) = n_e$$

where we used (55). Observe moreover

$$(p \cdot \hat{n})^2 = p_x^2 n_x^2 + p_y^2 n_y^2 + p_z^2 n_z^2 + \text{cross terms in coordinates} \quad \text{(similarly for } \hat{n}'\text{)}$$

$$(p \cdot \hat{n})(p \cdot \hat{n}') = p_x n_x n_x' + p_y n_y n_y' + p_z n_z n_z' + \text{similar to above}$$

Cross terms in the coordinates yield zero contribution for the same reason as (55). Squared terms give a similar contribution, being

$$\int d^3p \, (p \cdot \hat{n})^2 N_{eq}(|p|) = I \cdot \hat{n}^2 = I$$

$$\int d^3p \, (p \cdot \hat{n}')^2 N_{eq}(|p|) = I \cdot \hat{n}'^2 = I$$

$$\int d^3p \, (p \cdot \hat{n})(p \cdot \hat{n}'). N_{eq}(|p|) = I \times \hat{n} \cdot \hat{n}' = I \cos \theta$$

with $I = \int d^3p \, p_x^2 N_{eq}(|p|) = n_e m_e k_B T$ (56)

Using all that in $I_1(x)$ gives

$$I_1(x) = c \frac{3\sigma_T n_e k_B T}{16\pi m_e c^2} \left(-x^2 + 4x\right) \left\{2\pi \int_{-1}^1 d\cos \theta (1 - \cos \theta)(1 + \cos^2 \theta)\right\}$$

18
and since \( \int_{-1}^{1} dy (1 - y)(1 + y^2) = 8/3 \), we find as desired
\[
I_1(x) = \frac{n_e \sigma_T c k_B T}{m_e c^2} x(4 - x)
\]  

(57)

C.2 Integral \( I_2 \)

We compute
\[
I_2(x) = c \int d^3 p d\Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d\sigma}{d\Omega} (\mathbf{p}, \hat{n}, \Omega) N_{eq}(|\mathbf{p}|) \Delta^2
\]

The expansion up to second order in the electron momentum yields
\[
\frac{16\pi}{3\sigma_T} \left( 1 - \frac{\mathbf{p}}{\gamma m_e c} \cdot \hat{n} \right) \frac{d\sigma}{d\Omega} \Delta^2 = \left( \frac{x}{m_e c} \right)^2 (\mathbf{p} \cdot (\hat{n}' - \hat{n}))^2 (1 + \cos^2 \theta)
\]  

(58)

Using the exact same strategy as before, we have
\[
\int d^3 p (\mathbf{p} \cdot (\hat{n}' - \hat{n}))^2 N_{eq}(|\mathbf{p}|) = I \times (\hat{n}' - \hat{n})^2 = 2I(1 - \cos \theta)
\]

with \( I = \int d^3 p p_x^2 N_{eq}(|\mathbf{p}|) = n_e m_e k_B T \)

This yields for \( I_2(x) \)
\[
I_2(x) = \frac{3\sigma_T n_e k_B T}{16\pi} \frac{1}{m_e c^2} 2x^2 \left\{ 2\pi \int_{-1}^{1} d\cos \theta (1 - \cos \theta)(1 + \cos^2 \theta) \right\}
\]
\[
= \frac{n_e \sigma_T c k_B T}{m_e c^2} 2x^2
\]  

as desired.

C.3 The integrals and the Thomson cross section

We demonstrate here that the integrals computed using the Thomson cross section (33) solely (thus neglecting the Møller prefactor) cannot yield the Kompaneets equation.

We calculate
\[
I_{\ell}^{Th}(x) = c \int d^3 \mathbf{p} d\Omega_{rest} \frac{d\sigma^{Th}}{d\Omega_{rest}} N_{eq}(|\mathbf{p}|) \Delta^\ell
\]

with \( \ell = 1, 2 \). The expansion up to second order in the electron momentum yields
\[
\frac{16\pi}{3\sigma_T} \frac{d\sigma^{Th}}{d\Omega_{rest}} N_{eq}(|\mathbf{p}|) \Delta = \frac{x \mathbf{p} \cdot (\hat{n}' - \hat{n})}{m_e c} (1 + \cos^2 \theta) - \frac{x^2 k_B T}{m_e c^2} (1 - \cos \theta)(1 + \cos^2 \theta)
\]
\[
+ \frac{x}{(m_e c)^2} \left\{ (\mathbf{p} \cdot \hat{n})^2 - (\mathbf{p} \cdot \hat{n}) (\mathbf{p} \cdot \hat{n}') \right\} (1 + \cos^2 \theta)
\]  

(60)

\[
\frac{16\pi}{3\sigma_T} \frac{d\sigma^{Th}}{d\Omega_{rest}} N_{eq}(|\mathbf{p}|) \Delta^2 = \left( \frac{x}{m_e c} \right)^2 (\mathbf{p} \cdot (\hat{n}' - \hat{n}))^2 (1 + \cos^2 \theta)
\]  

(61)
where we have dropped the “rest” as subscript in the scattering angle.

We readily observe a change in the last term ($O(|p|^2)$) of expression (60) when comparing to (54), while (61) matches (58). Computing the integrals in the exact same way as before leads to

\[
I_{1a}^{Th}(x) = \frac{n_e \sigma_T c k_B T}{m_e c^2} x (1 - x)
\]

\[
I_{2a}^{Th}(x) = \frac{n_e \sigma_T c k_B T}{m_e c^2} 2x^2
\]

demonstrating (42), as desired. In particular, we observe that $I_{2a}^{Th}(x)$ remains unchanged, i.e., it is insensitive to the use of the Møller prefactor and the frame-independent expression of Klein-Nishina cross section (31). Thus, the change lies on $I_{1a}^{Th}(x)$, where we fail to account for all terms up to $O(|p|^2)$, as the structure of expansion (60) shows.

C.4 Nonequilibrium case

From now on, we denote $|p| = p$. In the same spirit as before we define

\[
x = \frac{\hbar \omega}{\epsilon}, \quad \Delta = \frac{\hbar (\omega' - \omega)}{\epsilon}
\]

where $\epsilon$ is the characteristic energy of the electron bath, $\epsilon = \frac{|p|^2}{2m_e}$. We do the expansion of the photon distribution, which gives the same as (33). The expansion in the electron distribution, however, gives to the leading order

\[
N(p') = N(p) - \frac{m_e \epsilon N'(p)}{p} \Delta + \left( - \frac{(m_e \epsilon)^2 N''(p)}{p^3} + \frac{(m_e \epsilon)^2 N'''(p)}{p^2} \right) \frac{\Delta^2}{2}
\]

with $N'(p) = \frac{\partial N}{\partial p}$ and so on

where we use a first assumption, that the electron distribution is isotropic, $N(p) = N(p)$, with normalization

\[
\int d^3p N(p, t) = n_e
\]

For the equilibrium case we have, of course, $\epsilon = k_B T$ and $N(p, t) = N_{eq}(p)$ as in (35). One checks that in this case

\[
- \frac{m_e \epsilon N'_{eq}(p)}{p} = N_{eq}(p)
\]

\[
\frac{(m_e \epsilon)^2 N''_{eq}(p)}{p^3} - \frac{(m_e \epsilon)^2 N'''_{eq}(p)}{p^2} = N_{eq}(p)
\]

Plugging into (26) reads to the leading order

\[
\partial_t n = n' I_1 (\Delta, N) + n'' I_2 (\Delta^2, N) + n(1 + n) I_3 \left( \Delta, \frac{N'}{p} \right)
\]

\[
+ n'(1 + n) I_4 \left( \Delta^2, \frac{N'}{p} \right) + n(1 + n) I_5 \left( \Delta^2, \frac{N''}{p^2} \right) + I_6 \left( \Delta^2, \frac{N'''}{p^2} \right)
\]

(66)
where

\[
\begin{align*}
I_1 (\Delta, \mathcal{N}) &= c \int d^3 \mathbf{p} d \Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d \sigma}{d \Omega} \Delta \mathcal{N} \\
I_2 (\Delta^2, \mathcal{N}) &= c \int d^3 \mathbf{p} d \Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d \sigma}{d \Omega} \Delta^2 \mathcal{N} \\
I_3 \left( \Delta, \mathcal{N'} \right) &= -c(m_e) \int d^3 \mathbf{p} d \Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d \sigma}{d \Omega} \Delta \mathcal{N'} \\
I_4 \left( \Delta^2, \mathcal{N'} \right) &= -c(m_e) \int d^3 \mathbf{p} d \Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d \sigma}{d \Omega} \Delta^2 \mathcal{N'} \\
I_5 \left( \Delta^2, \mathcal{N''} \right) &= c(m_e)^2 \int d^3 \mathbf{p} d \Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d \sigma}{d \Omega} \Delta^2 \mathcal{N''} \\
I_6 \left( \Delta^2, \mathcal{N''} \right) &= -c(m_e)^2 \int d^3 \mathbf{p} d \Omega \left( 1 - \frac{\mathbf{v}}{c} \cdot \hat{n} \right) \frac{d \sigma}{d \Omega} \Delta^2 \mathcal{N''}
\end{align*}
\]

First and second integral are computed in the exact same way as before and gives

\[
\begin{align*}
I_1 (\Delta, \mathcal{N}) &= \frac{n_e \sigma_T c}{(m_e)^2} \left\langle \frac{p^2}{3} \right\rangle x (4 - x) \\
I_2 (\Delta^2, \mathcal{N}) &= \frac{n_e \sigma_T c}{(m_e)^2} \left\langle \frac{p^2}{3} \right\rangle 2 x^2
\end{align*}
\]

The other integrals are calculated by integration by parts and using the second assumption \( \lim_{p \to \infty} p^3 \mathcal{N}(p) = 0 \).

To compute \( I_4 \) we move to spherical coordinates \( d^3 \mathbf{p} = p^2 dp d\Omega_p \). Using (58) the integral over the electron momentum becomes

\[
\int \frac{(\mathbf{p} \cdot (\hat{n}' - \hat{n}))^2 \mathcal{N}}{p} d^3 \mathbf{p} = \int d\Omega_p |\hat{n}' - \hat{n}|^2 \cos^2 \alpha \int_0^\infty \frac{p^2 \mathcal{N}}{p} p^2 dp
\]

\[
= \int d\Omega_p |\hat{n}' - \hat{n}|^2 \cos^2 \alpha \left\{ p^3 \mathcal{N} \bigg|_0^\infty - 3 \int_0^\infty p^2 \mathcal{N} dp \right\}
\]

\[
= -3 \int_0^\infty d\Omega_p dp |\hat{n}' - \hat{n}|^2 \cos^2 \alpha p^2 \mathcal{N}
\]

\[
= -3 \int d^3 \mathbf{p} (\hat{\mathbf{p}} \cdot (\hat{n}' - \hat{n}))^2 \mathcal{N}
\]

where we used (44), \( \alpha \) for the angle between \( (\hat{n} - \hat{n}') \) and \( \hat{\mathbf{p}} = \mathbf{p}/p \). The last integral is computed in the very same fashion as we did for the equilibrium case. We find

\[
\int d^3 \mathbf{p} (\hat{\mathbf{p}} \cdot (\hat{n}' - \hat{n}))^2 \mathcal{N} = 2 n_e (1 - \cos \theta) \frac{1}{3}
\]

which after integrating the solid angle gives \( I_4 \),

\[
I_4 \left( \Delta^2, \mathcal{N'} \right) = c(m_e) \left( \frac{x}{m_e c} \right)^2 \sigma_T 2 n_e
\]

(69)
Looking at expressions (67), (68) and (69) motivates introducing
\[
\epsilon = k_B T_{\text{eff}}
\]
\[
\langle p^2 \rangle = 3m_e k_B T_{\text{eff}}
\]
Observe that these definitions are compatible with the equilibrium case. We get now
\[
I_1 (\Delta, N) = \frac{n_e \sigma_T c k_B T_{\text{eff}}}{m_e c^2} x (4 - x)
\]
\[
I_2 (\Delta^2, N) = \frac{n_e \sigma_T c k_B T_{\text{eff}}}{m_e c^2} 2x^2
\]
\[
I_4 \left( \Delta^2, \frac{N'}{p} \right) = \frac{n_e \sigma_T c k_B T_{\text{eff}}}{m_e c^2} 2x^2
\]
The other integrals are quite similar, to yield
\[
I_3 \left( \Delta, \frac{N'}{p} \right) = \frac{n_e \sigma_T c k_B T_{\text{eff}}}{m_e c^2} x (4 - x)
\]
\[
I_5 \left( \Delta^2, \frac{N''}{p^2} \right) + I_6 \left( \Delta^2, \frac{N'}{p^2} \right) = \frac{n_e \sigma_T c k_B T_{\text{eff}}}{m_e c^2} 2x^2
\]
We substitute the values back in (66) to find
\[
\omega^2 \frac{\partial n}{\partial t} (\omega, t) = \frac{n_e \sigma_T c}{m_e c^2} \frac{\partial}{\partial \omega} \omega^4 \left\{ k_B T_{\text{eff}} \frac{\partial n}{\partial \omega} (t, \omega) + \hbar [1 + n(t, \omega)] n(t, \omega) \right\}
\]
which is the Kompaneets equation but with an effective (kinetic) temperature
\[
T_{\text{eff}} = \frac{\langle p^2 \rangle}{3m_e k_B}
\]
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