Magnetic field-induced quantum criticality via new asymptotically AdS$_5$ solutions

Eric D’Hoker and Per Kraus

Department of Physics and Astronomy, University of California, Los Angeles, CA 90095, USA

E-mail: dhoker@physics.ucla.edu and pkraus@ucla.edu

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Abstract
Using analytical methods, we derive and extend previously obtained numerical results on the low temperature properties of holographic duals to four-dimensional gauge theories at finite density in a nonzero magnetic field. We find a new asymptotically AdS$_5$ solution representing the system at zero temperature. This solution has vanishing entropy density, and the charge density in the bulk is carried entirely by fluxes. The dimensionless magnetic field to charge density ratio for these solutions is bounded from below, with a quantum critical point appearing at the lower bound. Using matched asymptotic expansions, we extract the low temperature thermodynamics of the system. Above the critical magnetic field, the low temperature entropy density takes a simple form, linear in the temperature, and with a specific heat coefficient diverging at the critical point. At the critical magnetic field, we derive the scaling law $s \sim T^{1/3}$ inferred previously from the numerical analysis. We also compute the full scaling function describing the region near the critical point and identify the dynamical critical exponent: $z = 3$. These solutions are expected to holographically represent boundary theories in which strongly interacting fermions are filling up a Fermi sea. They are fully top–down constructions in which both the bulk and boundary theories have well-known embeddings in the string theory.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction and summary of results
Understanding the properties of interacting fermions at finite density is a central problem in diverse areas of physics, ranging from condensed matter to nuclear and astrophysics. The
dual gravity description arising in the AdS/CFT correspondence may provide a new window on the strongly coupled versions of such systems, and the last few years have seen significant effort in this direction.

The model that has attracted the most attention consists of studying bulk fermions propagating on a Reissner–Nordström black brane background [1–4]. The advantage of this model is that it is very simple, yet the fermion correlators display interesting non-Fermi liquid behavior. Keeping this in mind, in order to highlight some distinctive features of the new solutions that we consider in this work, let us note some limitations and puzzles. To harness the full power of gauge–gravity duality it is desirable to have a model in which the fermions take part in the large $N$ gauge dynamics. However, as emphasized in [5], in the setup of [1–4], the fermions can be thought of as gauge singlets coupled to a large $N$ gauge theory. This is reflected in the fact that the Reissner–Nordström solution does not by itself seem to exhibit any evidence, at least at large $N$, of a Fermi surface when probed by correlation functions of bosonic operators, such as the density–density correlator. Another problematic feature is that the non-Fermi liquid behavior of the fermionic correlators hinges on the existence of a near-horizon AdS$_2$ factor with finite ground state entropy density. This makes the nature of the dual field theory rather obscure, and seemingly quite different from the real world systems with which one is hoping to make contact. On the field theory side, the matter content of CFTs dual to the Reissner–Nordström solution typically includes massless charged bosons. At weak coupling there would thus be no Fermi surface but rather a charged Bose condensate. But what, if anything, this has to do with the Reissner–Nordström solution is quite unclear. Other approaches to studying holographic fermions, with various pros and cons, include [6–9].

Here we argue that a promising holographic setup for studying interacting fermions at finite density can be obtained in the context of the duality between five-dimensional Einstein–Maxwell–Chern–Simons theory and four-dimensional supersymmetric gauge theories. Simply turning on a finite charge density in this system is inadequate, as it leads to the same finite entropy density ground state Reissner–Nordström geometry noted above, along with the same Bose condensation puzzle. However, turning on a magnetic field removes these problems: in the gauge theory the bosonic modes are lifted up in energy above the lightest fermion modes, and in the bulk a smooth zero entropy ground state geometry will be seen to arise. The charge density is carried in the bulk by fluxes, rather than hidden behind a horizon as in the Reissner–Nordström solution, and so the sea of boundary fermions appears to be visible in the leading-order gravity solution. The presence of the magnetic field also provides access to a tunable parameter and, just as in the real world, it is interesting to see how the physical properties change as a function of the field strength.

In previous papers [10–12] we have initiated a detailed study of this system, and a rich structure has emerged. Perhaps most interestingly, precision numerical analysis revealed the existence of a quantum critical point [12]. The results are summarized in figure 1 (taken from [12]). The system is studied as a function of the dimensionless magnetic field to the charge density ratio: $\tilde{B} = B/\rho^{2/3}$. The thermodynamics properties are expressed in terms of the dimensionless temperature $\tilde{T}$ and entropy density $\tilde{s}$. For a sufficiently large magnetic field, we observed in [12] a linear dependence of the low temperature entropy density, $\tilde{s} = c(\tilde{B})\tilde{T}$. As the magnetic field was decreased, or equivalently the charge density increased, the coefficient $c(\tilde{B})$ was found to diverge at a critical value $\tilde{B} = \tilde{B}_c$. At this critical point a new scaling law for the entropy emerged, $\tilde{s} \sim \tilde{T}^{1/3}$. Departures from the critical point with respect to both temperature and magnetic field could be expressed in terms of a single scaling function, as is familiar from the study of both classical and quantum critical phenomena. The numerical results suggested that we identify the critical point as having the dynamical critical exponent $z = 3$ and possessing a relevant operator of scaling dimension 2.
Figure 1. Schematic phase diagram illustrating the various behaviors of the entropy density versus temperature and magnetic field. The region inside the dashed line is controlled by the quantum critical point at $\hat{T} = 0$, $\hat{B} = \hat{B}_c$, and the entropy density can be expressed in terms of a single scaling function $f$ of $(\hat{B} - \hat{B}_c)/T^{2/3}$. We move around inside this region by changing the temperature $\hat{T}$ and the relevant coupling $\hat{B}_c$. The boundary of the region is defined to be where irrelevant operators become important. The yellow region denotes a regime where temperature is the largest energy scale, corresponding to the argument of the scaling function $f$ being small. Outside the yellow region the low temperature behavior of the entropy density, for fixed $\hat{B}$, is either constant or linear in $\hat{T}$, depending on whether the quantum critical point is approached from below or from above $\hat{B}_c$ as $\hat{T} \to 0$.

Several considerations render this behavior especially appealing. In terms of the connection to physical materials, we note that tuning a magnetic field is a common way of locating quantum critical points in heavy fermion compounds [13], and the appearance of the critical point closely parallels what we observe. Namely, away from the critical point one has a linear entropy density versus temperature law due to the presence of a Landau–Fermi liquid, but the coefficient diverges at the critical point, giving way to a non-linear relation. Another helpful aspect is that our behavior is arising in a fully ‘top-down’ setup: the gauge theories are explicitly known, and include as one example $\mathcal{N} = 4$ super-Yang–Mills theory.

On the other hand, since the analysis in [12] was heavily numerical the basic mechanism underlying the observed behavior was not entirely clear. Also, numerical results were obtained only for a particular value of the bulk Chern–Simons coupling, namely that dictated by supersymmetry. To both understand and extend our previous results an analytical treatment is clearly desirable. This is the basis of this paper.

1.1. Results

We have found a new asymptotically AdS$_5$ solution, which is at zero temperature and corresponds to a nonzero magnetic field and charge density in the boundary theory. In a

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As discussed in [12] there is also a close connection with the results described in [14].
convenient set of coordinates the solution takes the form
\[ ds^2 = \frac{dr^2}{L^2} + M dt^2 + 2L dr \, dx_3 + e^{2V} (dx_1^2 + dx_2^2) \]
\[ F = b dx_1 \wedge dx_2 + E dr \wedge dt, \]
(1.1)
Here the magnetic field \( b \) is a constant while the functions \((L, M, V, E)\) depend on \( r \) only. The explicit solution can be written almost entirely analytically: the function \( V \) must be determined numerically, but it obeys an equation with no dependence on free parameters and so is determined once and for all, and then the remaining functions may be solved by quadrature in terms of \( V \). In the near-horizon limit (small \( r \)) the functions take the simple form
\[ L = 2br, \quad M = -\tilde{\alpha}r - \frac{q^2 r^{2k}}{k(k - \frac{1}{2})}, \quad e^{2V} = \frac{b}{\sqrt{3}}, \quad E = qr^{k-1}, \]
(1.2)
where \( k \) is the coefficient of the Chern–Simons term in the bulk action, and \( \tilde{\alpha} \) is a constant. The \((r, t, x_3)\) part of this near-horizon metric is equivalent to the three-dimensional part of the Schrödinger spacetime used in non-relativistic holography [15, 16]. It is also a solution of three-dimensional gravity with a gravitational Chern–Simons term and has been studied in this context recently in [17].

Viewing this solution as the zero temperature limit of a finite temperature solution, we demand that \( M \) be negative near the horizon. For \( k > 1/2 \), this implies that the parameter \( \tilde{\alpha} \) must be non-negative. As we show, this inequality translates into a bound on the dimensionless magnetic field to the charge density ratio measured at the asymptotic AdS_5 boundary. We find \( \hat{B} \geq \hat{B}_c \), where \( \hat{B}_c \) is a \( k \)-dependent number that can be computed in terms of integrals involving the \( V \) function. This explains the appearance of a critical magnetic field in our previous numerical work [12], and a quantum critical point at \( \hat{B} = \hat{B}_c \). From the form of \( M(r) \) in (1.2) it is evident that \( k = 1/2 \) is special, and indeed we find that \( \hat{B}_c \) diverges as \( k \to 1/2 \). The quantum critical point is absent for \( k \leq 1/2 \).

We extract the low temperature thermodynamics by perturbing around the zero temperature solutions. The appropriate technique employs a matched asymptotic expansion, where we match two different perturbations, growing away from the horizon and from the AdS_5 region respectively. This analysis provides formulas for the low temperature entropy density, and gives us the dynamical critical exponent since it involves scaling the near-horizon time and space coordinates as the temperature is scaled.

Above the critical magnetic field this analysis yields the simple result
\[ \hat{s} = \frac{\pi}{6} \left( \frac{\hat{B}_c^3}{\hat{B}_c^3 - \hat{B}_c^5} \right) \hat{T} \quad \hat{T} \to 0, \quad \hat{B} > \hat{B}_c. \]
(1.3)
This result displays all the features that arose in our previous numerical analysis. The linear behavior in temperature is the characteristic of a Landau–Fermi liquid, here presumably arising from the fact that in the field theory we are filling up states in the lowest fermionic Landau level. Recall that the specific heat \( C \) obeys the same behavior, since \( C = T \partial s / \partial T \). In the large \( \hat{B} \) limit the result \( \hat{s} = \pi \hat{T} / 6 \) agrees with what we found for the purely magnetic solutions in [10]. In this limit the near-horizon geometry is BTZ \( \times \mathbb{R}^2 \), and the result corresponds to using the Brown–Henneaux central charge [18] and the Cardy formula. Away from this limit, this approach is not applicable, as the near-horizon solutions are no longer BTZ black holes on account of the nonzero charge density. As we approach \( \hat{B}_c \) from above, we see that the coefficient of the linear term diverges, as was seen numerically in [12]. This is directly analogous to the observation of a divergence in the specific heat coefficient in real materials.

\[ \text{By a coordinate transformation the asymptotic metric may be put in the canonical AdS_5 form.} \]
approaching a field-tuned quantum critical point. A finite temperature version of a magnetic field-induced phase transition was studied holographically in [19], and for zero temperature versions involving probe branes see [20, 21].

Tuning to the quantum critical point corresponds to setting $\tilde{\alpha} = 0$ in (1.2). The details of the matched asymptotic expansion now change, and the corresponding result for the low temperature entropy density is

$$\hat{s} = a\hat{T}^{1/3}, \quad a^3 = \frac{\pi}{576k}\hat{B}_c^3, \quad \hat{B} = \hat{B}_c.$$ (1.4)

This is in accord with the numerical results obtained in [12]. The scaling analysis shows that the dynamical critical exponent is now $z = 3$.\(^3\)

We can perturb around the critical point by including both a finite temperature and a deviation in the magnetic field away from the critical value. This leads to the scaling form for the entropy density

$$\hat{s} = \hat{T}^{1/3} f \left( \frac{\hat{B} - \hat{B}_c}{\hat{T}^{2/3}} \right),$$ (1.5)

where the function $f$ can be found in terms of the solution to a cubic equation

$$f(x)^3 + \frac{xf(x)}{32k}\hat{B}_c^4 = a^3.$$ (1.6)

Given the result $z = 3$, this shows that $\hat{B} - \hat{B}_c$ represents a relevant coupling with scale dimension 2 at the fixed point. Again, we have an agreement with the numerics in [12]. The behavior of other quantities, such as the magnetization, near the critical point may be obtained from (1.5) by applying standard thermodynamic relations.

In other studies of quantum critical points in the AdS/CFT correspondence, nontrivial dynamical critical exponents are associated with near-horizon Lifschitz metrics [22]. Our mechanism is different and is tightly linked with the fact that our solutions are stationary but not static. Entropy density and temperature are measured at the horizon, while time and space in the field theory are measured at the AdS5 boundary. If the natural time and space coordinates at the horizon and at the boundary are nontrivially related, as is the case here, new dynamical scaling laws can ensue. This same mechanism can be expected to play a role in other AdS/CMT applications.

A noteworthy feature of our zero temperature solutions is that the charge density measured at infinity is carried entirely by fluxes in the bulk; some other recent solutions with this property include [23, 25]. This is possible by virtue of the Chern–Simons coupling and is in contrast to the Reissner–Nordström solution, where all the charge is hidden behind the horizon. The fermions in the boundary theory are thus described fully holographically as classical bosonic fields in the bulk, and the charge is also accessible since it is outside the horizon. It will be interesting to study the dynamical response of these holographic fermions by studying perturbations at nonzero frequency and wavelength.

The remainder of this paper is organized as follows. In section 2, we review the ansatz, reduced field equations and first integrals, in the presence of a uniform magnetic field and electric charge density, exhibit the allowed near-horizon solutions and review the physical parameters of the problem. In section 3, we derive our new charged asymptotically AdS5 solutions and derive a formula for the critical magnetic field $\hat{B}_c$ as a function of $k$.

\(^3\) For reasons that will be explained in section 4.3.6 below, this scaling relation holds for all $k \geq 3/4$. For $1/2 < k < 3/4$ a different scaling relation holds, to be examined elsewhere [24].
section 4, we develop a procedure for matching the perturbation theories around the near-horizon and asymptotically AdS$_5$ regions, and use this to derive the low temperature formula for the specific heat coefficient as a function of $\hat{B}$, and the scaling function $f$ near the quantum critical point. In section 5, we present a detailed technical discussion of this matching procedure and solve the associated perturbation theories. In appendix A, we derive the most general near-horizon solutions. In appendix B, we present a detailed discussion of the solution of the more involved equations for the asymptotically AdS$_5$ perturbation theory.

2. Field equations and simple analytic solutions

The action for the five-dimensional Einstein–Maxwell theory, with a negative cosmological constant and Chern–Simons term, is given by

$$S_{EM} = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left( R + F^{MN} F_{MN} - \frac{12}{L^2} \right) + S_{CS} + S_{bndy}.$$  

(2.1)

where the Chern–Simons term is

$$S_{CS} = \frac{k}{12\pi G_5} \int A \wedge F \wedge F.$$  

(2.2)

For the value $k = 2/\sqrt{3}$, the action coincides with the bosonic part of $D = 5$ minimal gauged supergravity. In this paper, however, $k$ will often be kept general, thus allowing for values different from the supersymmetric case. We assume $k \geq 0$ without loss of generality, since a sign reversal of $k$ is equivalent to a parity transformation. Boundary terms in the action are required for finiteness of the action and the existence of a well-posed variational problem [26, 27]; their explicit form may be found in [11]. We henceforth set $L = 1$.

The Bianchi identity is $dF = 0$, while the field equations are given by

$$0 = dF + kF \wedge F$$

$$R_{MN} = 4g_{MN} + \frac{1}{3} F^{PQ} F_{PQ} g_{MN} - 2 F_{MP} F^M_N.$$  

(2.3)

With $k = 2/\sqrt{3}$, the action (2.1) is a consistent truncation known to describe all supersymmetric compactifications of type IIB or M-theory to AdS$_5$ [28–30]. This means that solutions of (2.3) are guaranteed to be solutions of the full 10- or 11-dimensional field equations (although for non-supersymmetric solutions there is no guarantee of stability). It also implies that the solutions we find are holographically dual not just to $\mathcal{N} = 4$ super-Yang–Mills, but to the infinite class of supersymmetric field theories dual to these more general supersymmetric AdS$_5$ compactifications.

2.1. Ansatz and reduced field equations

Our general ansatz for the metric and field strength reads as follows:

$$ds^2 = \frac{dr^2}{L^2 - MN} + M dr^2 + 2L dt d\chi_3 + N d\chi_3^2 + e^{2V} \left(d\chi_1^2 + d\chi_2^2\right)$$

$$F = B d\chi_1 \wedge d\chi_2 + E dr \wedge dt + P d\chi_3 \wedge dr,$$

(2.4)

where $L$, $M$, $N$, $V$, $E$ and $P$ are functions of $r$ only, and $B$ is a constant as a consequence of the Bianchi identity. To identify event horizons, it is useful to write the metric in the form

$$ds^2 = \frac{dr^2}{L^2 - MN} + \frac{L^2 - MN}{N} dr^2 + N \left(d\chi_3 + \frac{L}{N} dr\right)^2 + e^{2V} \left(d\chi_1^2 + d\chi_2^2\right).$$  

(2.5)

Conventions: $R_{\mu\nu} = \partial_\mu \Gamma^\alpha_{\nu\lambda} - \partial_\nu \Gamma^\alpha_{\mu\lambda} + \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\lambda} - \Gamma^\alpha_{\nu\beta} \Gamma^\beta_{\mu\lambda}$, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$, and $R = R^{\alpha}_{\alpha\mu\nu}$.
This ansatz differs in its choice of radial coordinate from that used in our previous papers [10–12]. It was motivated by the previous work on three-dimensional gravity [31, 32], and its advantages will become clear as we proceed.

The metric and field strength are form invariant under the following coordinate transformations.

1. $SL(2, R)$:

$$\begin{pmatrix} t \\ x_3 \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ x_3 \end{pmatrix}, \quad ad - bc = 1. \quad (2.6)$$

2. Scale transformations:

$$r \rightarrow \lambda^2 r, \quad t \rightarrow \frac{1}{\lambda} t, \quad x_3 \rightarrow \frac{1}{\lambda} x_3. \quad (2.7)$$

3. $r$ translations:

$$r \rightarrow r + r_0. \quad (2.8)$$

4. Rescaling of $x_{1,2}$:

$$x_{1,2} \rightarrow \kappa x_{1,2}. \quad (2.9)$$

The reduced field equations are

$$M_1 \quad ((NE + LP)e^{2V})' + 2kB P = 0$$

$$M_2 \quad ((LE + MP)e^{2V} - 2kBE = 0$$

$$E_1 \quad L'' + 2V'L' + 4(V'' + V^2)L - 4PE = 0$$

$$E_2 \quad M'' + 2VM' + 4(V'' + V^2)M + 4E^2 = 0$$

$$E_3 \quad N'' + 2VN' + 4(V'' + V^2)N + 4P^2 = 0$$

$$E_4 \quad 6V''f + 8(V')^2 f + 2VV'f' - g + 4B^2e^{-4V} = 0$$

$$CON \quad (V')^2f + V'f' - 6 + \frac{g}{4} + B^2e^{-4V} + MP^2 + 2LEP + NE^2 = 0,$$

where we have defined the following $SL(2, R)$ invariant bilinears:

$$f = L^2 - MN$$

$$g = (L')^2 - M'N'. \quad (2.11)$$

There is some redundancy in this set of equations; by taking the derivative of the constraint equation one can show that one linear combination of the equations is satisfied identically. Thus, one could omit one of the second-order Einstein equations without loss of information.

2.2. First integrals

Eliminating $E, P$ and $B$ between the Einstein equations, we find the following simple result:

$$(f e^{2V})'' = 24e^{2V}. \quad (2.12)$$

Therefore, we can, in complete generality, solve for $f$ in terms of $V$ as

$$f(r) = 24e^{-2V(r)} \int_{r_1}^{r} dr' \int_{r_1}^{r'} dr'' e^{2V(r')}.$$  

(13)

Introducing the potentials $A$ and $C$ respectively for $E = A'$ and $P = C'$, we may integrate both Maxwell’s equations once:

$$(NE + LP)e^{2V} + 2kBC = 0$$

$$(LE + MP)e^{2V} - 2kBA = 0. \quad (2.14)$$
The integration constants that arise here have been absorbed into the definition of the functions $A$ and $C$. Forming the combinations $M \times E_1 - L \times E_2, N \times E_2 - M \times E_3, L \times E_3 - N \times E_1$, and using (2.14), we find three first integrals of motion:

$$
\begin{align*}
\lambda e^{2V} - 4kBA C &= \lambda_0 \\
\mu e^{2V} + 4kBA^2 &= \mu_0 \\
\nu e^{2V} + 4kB C^2 &= \nu_0
\end{align*}
$$

(2.15)

Here, the combinations $\lambda, \mu, \nu$ are the analogs of angular momentum components for $SL(2, R)$ and, just as $L, M, N$ do, they transform under the vector representation of $SL(2, R)$. They satisfy the following purely kinematic relation:

$$
(f')^2 = 4fg + 4(\lambda^2 - \mu \nu).
$$

(2.16)

Finally, $\lambda_0, \mu_0$ and $\nu_0$ are the constant values of the first integrals.

2.3. Asymptotically AdS$^5$ solutions

In these coordinates, the pure AdS$^5$ solution takes the form

$$
\begin{align*}
\text{d}s^2 &= \frac{\text{d}r^2}{4r^2} + 2r(-\text{d}t^2 + \text{d}x_3^2 + \text{d}x_1^2 + \text{d}x_2^2) \\
F &= 0
\end{align*}
$$

(2.17)

More generally, we will be looking for asymptotically AdS$^5$ solutions, for which the leading behavior at large $r$ is given by

$$
\begin{align*}
\text{d}s^2 &\sim \frac{\text{d}r^2}{4r^2} + c_{L}r \text{d}t^2 + c_{N}r \text{d}x_3^2 + 2c_{L}r \text{d}x_1 \text{d}x_2 + c_{V}r(\text{d}x_1^2 + \text{d}x_2^2) \\
F &\sim b \text{d}x_1 \wedge \text{d}x_2 + \frac{c_{E}}{r^2} \text{d}r \wedge \text{d}t + \frac{c_{P}}{r^2} \text{d}x_3 \wedge \text{d}r
\end{align*}
$$

(2.18)

with $c_{L}^2 - c_{M}c_{N} = 4$, as implied by the field equations. Here, we denote the magnetic field $b$, reserving the use of $B$ for the case where $x_1, x_2$ are scaled to put the metric in the canonical form.

By using the coordinate transformations discussed above, we have some freedom to change the asymptotic parameters. For instance, we can use the $SL(2, R)$ and scale transformations to set $c_{L,M,N}$ to any desired values obeying $c_{L}^2 - c_{M}c_{N} = 4$ and also choose an arbitrary value for $c_{E}$, assuming that it is nonzero to begin with. The freedom to translate $r$ can then be used to set the location of the event horizon, if present, at $r = 0$.

With $b = 0$, a familiar asymptotically AdS$^5$ solution is the electrically charged Reissner–Nordström solution, with a near-horizon AdS$_2 \times R^3$ geometry. In this work we are interested in solutions with $b \neq 0$, corresponding to a gauge theory with a nonzero background magnetic field.

2.4. Near-horizon solutions

With $b \neq 0$, it is possible to find exact solutions where $V$ is constant, so that the metric takes the form $M_3 \times R^2$. Such solutions can, and will, serve as near-horizon geometries for full asymptotically AdS$^5$ solutions. In fact, it is not difficult to find the general solution with constant $V$. As shown in appendix A, for $k \neq 1$ (recall that we have restricted to $k \geq 0$), the
general solution with constant $V$ is given by, up to coordinate transformations\textsuperscript{5},

\[ F = b \, dx_1 \wedge dx_2 + qr_r \eta^k d r \wedge d t \]

\[
\text{d} s^2 = \frac{d r^2}{4b^2 r^2} - \left( \frac{\beta + ar + \frac{q^2}{k (k - \frac{1}{2} \eta)} r^{2\eta k}}{d t^2 + 4 \eta b r d t d x_3 + \frac{b}{\sqrt{3}} (dx_1^2 + dx_2^2)} \right) \tag{2.19}
\]

with $\eta = \pm 1$. This solution appears to be new, although some aspects are familiar. If we set $q = 0$ the solution is identified as $\text{AdS}_3 \times \mathbb{R}^2$ which is the near-horizon part of the magnetic brane solutions found in [10]. The $\text{AdS}_3$ part can equally well be taken to be a nonextremal, rotating BTZ solution, since these can be obtained from $\text{AdS}_3$ by applying the coordinate transformations discussed in section 2.1. Note also that we can use the residual coordinate freedom to set $b, q$ and $\alpha$ to convenient values, such as $b = \sqrt{3}$, $q = 1$ (assuming nonzero $q$) and $\alpha = 0$.

A simple but important fact is that the $\alpha = \beta = q = 0$ solution is invariant under the following coordinate transformations, which are a combination of scale and $\text{SL}(2, \mathbb{R})$ transformations:

\[
\begin{align*}
    r &\rightarrow \lambda r, \\
    t &\rightarrow \frac{1}{\lambda^w} t, \\
    x_3 &\rightarrow \frac{1}{\lambda^{1-w}} x_3
\end{align*}
\]

for any parameter $w$. If we write $t \sim (x_3)^z$ we have

\[
z = \frac{w}{1 - w}. \tag{2.21}
\]

Eventually, $z$ will be identified with a dynamical critical exponent. Of course, nothing we have said so far determines any particular value of $w$ and hence of $z$. Preferred values emerge once we match the near-horizon geometry on to a low temperature asymptotically $\text{AdS}_5$ solution. Doing so, we will find that $w = 1/2$ and $w = 3/4$ are realized, leading to $z = 1$ and $z = 3$ for the values of the dynamical critical exponent in the dual field theory.

For nonzero $q$, the $(r, t, x_3)$ part of the metric coincides with the corresponding three-dimensional part of the ‘Schrödinger’ spacetimes, proposed in [15, 16] in connection with non-relativistic $\text{AdS}/\text{CFT}$\textsuperscript{6}. The nonzero $q$ solutions, but still with $\alpha = \beta = 0$, are invariant under

\[
\begin{align*}
    r &\rightarrow \lambda r, \\
    t &\rightarrow \frac{1}{\lambda^w} t, \\
    x_3 &\rightarrow \frac{1}{\lambda^{1-w}} x_3
\end{align*} \tag{2.22}
\]

Perhaps surprisingly, this scale transformation will end up playing no role in our discussion of scaling near quantum critical points; rather, as noted above, it is the scale transformations (2.20) with $w = 1/2$ and $3/4$ that turn out to be important\textsuperscript{7}. Indeed, note that since the supersymmetric value of $k$ is greater than 1, the dynamical critical exponent corresponding to (2.22) would be negative for the supersymmetric case.

Solutions with the same three-dimensional metric, but with the time and space coordinates swapped, have been studied recently as solutions of topologically massive gravity [17]. By virtue of their asymptotic symmetry algebra, they were found to possess a chiral Virasoro algebra.

Precisely at $k = 1$ a new family of solutions emerges. These are the warped $\text{AdS}_3 \times \mathbb{R}^2$ solutions found in [11], whose three-dimensional part was studied previously in the context of

\textsuperscript{5} The $\eta = 1$ solution obviously breaks down for $k = 1/2$; this value of $k$ is one, among several, ‘special’ values of the Chern–Simons coupling $k$.

\textsuperscript{6} In those solutions there are additional spatial directions appearing in the metric as $r dy_idy_i$, and associated Galilean boost symmetries.

\textsuperscript{7} In fact, for $k \leq 3/4$, the scale transformation (2.22) does determine the dynamical critical exponent [24]. The importance of $k = 3/4$ can be seen from the discussion in section 4.3.6.
topologically massive gravity [33, 34]. These solutions will not play any role in what follows, but we present them here for completeness. The solutions are

\[
d s^2 = \frac{d r^2}{12r(r + r_0)} - 2r(q^2 r + 6r_0) \, d r^2 + 4br \, d r \, d t + d x_1^2 + d x_2^2 + d x_3^2 \quad (2.23)
\]

\[
F = b \, d x_1 \wedge d x_2 + q \, d r \wedge d t
\]

with \(q^2 + 2b^2 = 6\).

2.5. Physical parameters

Given an asymptotically AdS\(_5\) black hole solution we will be interested in computing the magnetic field, charge density, temperature and entropy density. These quantities can be read off from the asymptotic data near the horizon and at infinity.

For the near-horizon behavior it is clear to work with the metric in the form (2.23). The horizon occurs when \(L^2 - MN = 0\), which we assume is satisfied at \(r = 0\). Finite entropy density solutions will have \(N(0) \neq 0\). The value of \(L\) at the horizon is interpreted as a chemical potential conjugate to \(P_3\), the momentum along \(x_3\). This follows from the fact that the existence of a smooth Euclidean section requires that periodic shifts of imaginary time be accompanied by a shift of \(x_3\) proportional to \(L(0)\). We will restrict attention to solutions in which the only thermodynamical potentials are temperature and a chemical potential for the charge density, and hence we demand \(L(0) = 0\). For the solutions we find, \(L\) will vanish linearly in \(r\). It then follows that at a finite entropy horizon we require \(M\) to vanish; \(M'(0)\) will be proportional to the Hawking temperature.

We assume a metric with asymptotic behavior (2.18). To read off physical quantities it is convenient to perform coordinate transformations to put the asymptotic metric in the canonical form. The \(SL(2, R)\) transformation

\[
t \to t - \frac{c_L}{c_M} x_3 \quad (2.24)
\]

removes the \(d x_3 \, d t\) cross term from the metric, i.e. sets \(c_L = 0\), while preserving the condition \(L(0) = 0\). Next, we can perform scale and \(SL(2, R)\) transformations to set \(-c_M = c_N = c_V = 2\).

After applying these transformations, which of course also act on the near-horizon part of the metric, it is straightforward to work out expressions for the physical parameters. We then express the results in terms of the asymptotic coefficients appearing in the general solution, with arbitrary \(c_M, c_N, \ldots\). The relevant formulas are

\[
B = \frac{2b}{c_V}
\]

\[
\rho = 4 \sqrt{2} \, \frac{c_E}{-c_M}
\]

\[
s = \frac{1}{4} e^{2V(0)} \sqrt{N(0)} \sqrt{-c_M} \left( \frac{2}{c_V} \right)
\]

\[
T = \frac{1}{4\pi} \sqrt{\frac{2}{-c_M}} (-M'(0) \sqrt{N(0)}).
\]

Here \(s\) is the entropy density with Newton’s constant scaled out: \(s = G_S S / \text{Vol.}\) The charge density \(\rho\) is determined from the boundary current according to the standard AdS/CFT dictionary; see [11] for details.
These physical parameters are dimensionful and can be rescaled by a coordinate transformation that preserves the asymptotic AdS$_5$ metric. It is thus only dimensionless quantities that are meaningful, and these are defined as

\[ \hat{B} \equiv \frac{B}{\rho^{2/3}} \]
\[ \hat{s} \equiv \frac{s}{B^{3/2}} \]
\[ \hat{T} \equiv \frac{T}{B^{1/2}}. \]  

(2.26)

Here we are defining $\hat{s}$ and $\hat{T}$ in a slightly more convenient manner than in [12], where in the denominator we used the combination $B^3 + \rho^2$; this will clearly have no effect on the scaling properties since we will always have nonzero $B$.

3. Zero temperature asymptotically AdS$_5$ solutions

We now turn to the construction of asymptotically AdS$_5$ solutions representing zero temperature, finite density, matter in a magnetic field. As we will see, such solutions can be found by quadrature in terms of a single universal $V$ function; $V$ is ‘universal’ in the sense that it has no dependence on the magnetic field, charge density or Chern–Simons coupling $k$, and hence can be determined (numerically) once and for all.

3.1. Pure magnetic solutions

We begin by recapitulating the solutions obtained in [10], with a magnetic field but with vanishing electric charge. We can choose coordinates such that these solutions obey

\[ M = N = E = P = 0. \]  

(3.1)

Furthermore, by scaling $x_{1,2}$ we can set

\[ b = \sqrt{3}. \]  

(3.2)

The ansatz is then

\[ ds^2 = \frac{dr^2}{L^2} + 2L \, dt \, dx_3 + e^{2V} (dx^2_1 + dx^2_2) \]
\[ F = \sqrt{3} \, dx^1 \wedge dx^2 \]  

(3.3)

so that $t$ and $x_3$ play the role of lightcone coordinates.

The problem is reduced to solving for $L$ and $V$, which obey

\[ e^1 \quad L'' + 2V'L' + 4(V'' + V'^2)L = 0 \]
\[ e^4 \quad \frac{3}{2} L^2 V'' + \frac{1}{2} e^{-4V} (L^2 e^{4V})' V' - \frac{1}{4} L^2 + 3e^{-4V} = 0 \]
\[ \text{con} \quad e^{-V} (L^2 e^{4V})' V' + \frac{1}{4} L^2 + 3e^{-4V} = 0. \]  

(3.4)

From (2.13) we obtain $L$ in terms of $V$. We choose the boundary conditions $L(0) = L'(0) = 0$, corresponding to a zero temperature horizon at $r = 0$. Then,

\[ L(r)^2 = 24e^{-2V(r)} \int_0^r \, dr' \int_0^{r'} \, dr'' e^{2V(r'')} \]  

(3.5)

and the problem is reduced to finding $V$. 

\[ \text{11} \]
As in [10] we are interested in solutions that interpolate between $\text{AdS}_3 \times \mathbb{R}^2$ at small $r$ and $\text{AdS}_5$ at large $r$. At small $r$ the equations admit a solution of the form

$$V(r) = v_1 r^\sigma + v_2 r^{2\sigma} + v_3 r^{3\sigma} + \cdots \quad (3.6)$$

with $\sigma$ obeying the quadratic equation

$$3\sigma^2 + 3\sigma - 4 = 0. \quad (3.7)$$

We choose the root such that $V$ is finite at $r = 0$, and so take

$$\sigma = \frac{\sqrt{57}}{6} - \frac{1}{2} \approx 0.758. \quad (3.8)$$

Using our freedom to perform a scale transformation on $(r, t, x_3)$ we set $v_1 = 1$, and then find for the first few terms at small $r$:

$$V(r) = r^\sigma - 3 \frac{(2\sigma + 1)(3\sigma - 2)}{3\sigma + 1} r^{2\sigma} + \cdots \quad (3.9)$$

$$L(r) = \sqrt{12} r \left( 1 - 2 \frac{2\sigma - 1}{\sigma + 1} r^\sigma + \cdots \right).$$

Followed out to large $r$ these initial data match on to an asymptotically $\text{AdS}_5$ solution. We have not succeeded in solving for $V$ analytically, but numerical integration is straightforward. The resulting functions are shown in figure 2.

The physical interpretation of this solution was explored in [10]. In the dual field theory it corresponds to massless fermions in a magnetic field. The modes occupying the lowest Landau level give rise at low energies to a $D = (1+1)$-dimensional CFT, which accounts for the existence of the near-horizon $\text{AdS}_3$ factor. In the case of $\mathcal{N} = 4$ SYM, the free field central charge was compared with the Brown–Henneaux central charge, and found to differ by a factor of $\sqrt{3/4}$. Exact agreement is not expected since these solutions are non-supersymmetric.
3.2. Charged solutions

We now generalize to include a nonzero electric charge density. Rather remarkably, we find that the full solution can be solved by quadrature in terms of the same $V$ function as appeared in the pure magnetic solution above. This fact allows us to deduce analytically all the most interesting physical properties of these solutions. These properties include the existence of a critical electric charge density at which the system undergoes a quantum phase transition, and expressions for the low temperature thermodynamics at and away from the critical point.

We describe these solutions in coordinates such that $N = P = 0$. Examining the field equations, we find that $L$ and $V$ obey precisely the same equations as in the pure magnetic case, and therefore we can simply carry over the results from that solution. To complete the solution we need to solve for $E$ and $M$.

3.2.1. Solving for $E$. Recalling that we are scaling $x_{1,2}$ to set $b = \sqrt{3}$, equation M2 now reads

$$(Le^{2V}E)' = \sqrt{12k}E$$

which we integrate as

$$E(r) = \frac{2cVc_E}{L(r)e^{2V(r)}} \exp \left\{ \sqrt{12k} \int_{r}^{\infty} \frac{dr'}{L(r')e^{2V(r')}} \right\}$$

where $c_E$ is an integration constant, while $c_V$ is the same number that appeared in (3.10). It will be convenient to define the following function:

$$\psi(r) \equiv \int_{r}^{\infty} \frac{dr'}{L(r')e^{2V(r')}}$$

in terms of which the gauge potential, defined as $E(r) = A'(r)$ with $A(0) = 0$, is given by,

$$A(r) = \frac{cVc_E}{\sqrt{3}k} e^{\sqrt{12k}\psi(r)}.$$  (3.14)

The asymptotics of $\psi$ are found to be

$$\begin{align*}
    r \to 0 & \quad \psi(r) \sim \frac{\ln r}{\sqrt{12}} + \frac{1}{\sqrt{12k}} \ln \left( \frac{\sqrt{3}e_0}{cVc_E} \right) \\
    r \to \infty & \quad \psi(r) \sim -\frac{1}{2cVr},
\end{align*}$$

where $e_0$ in the last term of the first line arises from the regularized integral

$$e_0 = \frac{cVc_E}{\sqrt{3}} \exp \left\{ \sqrt{12k} \int_{0}^{\infty} dr' \left[ \frac{1}{L(r')e^{2V(r')}} - \frac{1}{\sqrt{12k}(r' + 1)} \right] \right\}.$$  (3.16)

This integral is convergent and produces the following asymptotics of $E(r)$:

$$\begin{align*}
    r \to 0 & \quad E(r) \sim e_0 r^{k-1} \\
    r \to \infty & \quad E(r) \sim \frac{c_E}{r^2}
\end{align*}$$

The asymptotics of $\psi$ guarantee that $A$, as defined in (3.14), automatically vanishes at the horizon, as long as $k > 0$. Note that the ratio $e_0/c_E$ depends only on the properties of the purely magnetic $T = 0$ solution.
3.2.2. Solving for $M$. The remaining equation E2 for $M$ may be integrated by noticing that it is a linear equation in $M$, whose homogeneous part coincides with the equation for $L$. Thus, we know that we can integrate by quadrature. We introduce the function $\phi(r)$ by writing

$$M(r) = L(r)\phi(r)$$  \hspace{1cm} (3.18)

and obtain the following equation for $\phi$:

$$(L^2 e^{2V} \phi')' = -4E^2 L e^{2V}.$$  \hspace{1cm} (3.19)

In terms of $\psi$, we have

$$(L^2 e^{2V} \psi')' = -16c^2 e^{4\sqrt{3}k}\psi.$$  \hspace{1cm} (3.20)

It is straightforward to integrate this equation, and we find

$$L^2 e^{2V} \phi' = \sqrt{12}\beta - \frac{4c^2 e^{4\sqrt{3}k}\psi(r)}{\sqrt{3k}}.$$  \hspace{1cm} (3.21)

where $\beta$ is an integration constant. Integrating once more, we find

$$\phi(r) = -\frac{\alpha}{\sqrt{12}} + \sqrt{12}\beta \int_r^\infty \frac{dr'}{L(r')^2 e^{2V(r')}} - \frac{4c^2 e^{4\sqrt{3}k}\psi(r)}{\sqrt{3k}} \int_r^\infty \frac{dr'}{L(r')^2 e^{2V(r')}}$$  \hspace{1cm} (3.22)

with $\alpha$ as another integration constant.

It is now straightforward to evaluate the asymptotics of $M$, and we find

$$r \to 0 \quad M(r) \sim -\beta - \tilde{\alpha}r - \frac{e^{2\sqrt{3}k}}{k(k - \frac{1}{2})}$$  \hspace{1cm} (3.23)

$$r \to \infty \quad M(r) \sim -\frac{\alpha}{\sqrt{3}},$$

where

$$\alpha - \tilde{\alpha} = 16c^2 e^{2\sqrt{3}k} J(k) \quad J(k) = \frac{1}{2k} \int_0^\infty \frac{e^{4\sqrt{3}k}\psi(r)}{L(r')^2 e^{2V(r')}} dr'.$$  \hspace{1cm} (3.24)

The integral $J(k)$ is convergent for all $k > 1/2$ and positive.

3.3. Interpretation and emergence of the critical magnetic field

We are interested in solutions that represent zero temperature limits of black hole solutions, with a horizon at $r = 0$. We therefore demand that $M$ vanish at $r = 0$, which requires that we set $\beta = 0$.\footnote{In fact, the $\beta \neq 0$ solutions also have an interpretation as the zero temperature limit of black hole solutions, but they necessarily involve a non-zero chemical potential conjugate to momentum along $x_3$. This follows since for nonzero $\beta$ we can always shift $r$ such that $M(0) = 0$, but then $L(0) \neq 0$, which is equivalent to having such a chemical potential. It would be interesting to explore these solutions further (the neutral version is studied in appendix D of [11]), but here we exclude them since we are working at the vanishing $P_3$ chemical potential.} From the expression in (3.22) it is then apparent that

$$-\frac{\alpha}{\sqrt{12}} \leq \frac{M(r)}{L(r)} \leq -\frac{\tilde{\alpha}}{\sqrt{12}}$$  \hspace{1cm} (3.25)

uniformly throughout $0 \leq r < \infty$. Since $L(r)$ is positive, $M(r)$ will be negative definite as long as $\tilde{\alpha}$ is non-negative. Finite temperature black hole solutions will have $M(0) = 0$ with $M'(0) < 0$, the Hawking temperature being proportional to $-M'(0)$. Thus, we need $\tilde{\alpha} \geq 0$ in order for our solution to represent the zero temperature limit of such black holes. In view
Table 1. The critical value of the dimensionless magnetic field to the charge density ratio for selected values of $k$.

| $k$ | $\hat{B}_c$ |
|-----|-------------|
| 0.50001 | 35.4050648722 |
| 0.6 | 1.44700934549 |
| 0.75 | 0.916107730288 |
| 1 | 0.600520361557 |
| $2/\sqrt{3}$ | 0.499424265324 |
| 2 | 0.264993652464 |
| 5 | 0.101365592402 |
| 10 | 0.050219317885 |

of (3.24), we thus require

$$\alpha \geq 16c_V^2 J(k)$$  \hspace{1cm} (3.26)

This translates into a bound on the dimensionless magnetic field $\hat{B}$. From (2.25), (2.26) and (3.23), we have

$$\hat{B} = \left(3 \frac{3}{4}\right)^{1/3} \alpha^{1/3} \frac{c^2 E}{c_V c_E^{1/3}}$$  \hspace{1cm} (3.27)

and so the bound (3.26) translates to $\hat{B} \geq \hat{B}_c$ with the critical field given as a function of the universal number $c_V$, and the Chern–Simons coupling $k$ only:

$$\hat{B}_c = \left(12 J(k) \frac{c_V}{c_E}\right)^{1/3}.$$  \hspace{1cm} (3.28)

Numerical integration yields the values shown in table 1 for $\hat{B}_c$ for various values of $k$.

As can be seen from the integral representation, $\hat{B}_c$ diverges at $k = 1/2$ and goes to zero at large $k$. For the supersymmetric value, $k = 2/\sqrt{3}$, we recover the critical value that arose in the numerical studies conducted in [12]. The interpretation of this critical field in terms of the dual gauge theory is of course an interesting question, on which we will comment in section 6.

Before turning to finite temperature, we pause to note a potentially confusing point regarding these zero temperature solutions. After setting $b = \sqrt{3}$ we presented the solution in terms of two integration constants, $c_E$ and $\alpha$, and showed how they determined the value of $\hat{B}_c$.

It is easy to see however that, up to coordinate transformations, solutions with different values of $(c_E, \alpha)$ are in fact equivalent. In particular, the $SL(2, R)$ transformation $x_3 \to x_3 + ct$ can be used to shift $\alpha$, and a rescaling of $t$ and $x_3$ will rescale $c_E$. Even though these solutions are coordinate equivalent, it is appropriate to treat them as physically distinct when one regards them as zero temperature limits of finite temperature solutions. At finite temperature, and in particular when $N(0) \neq 0$, solutions with different $\hat{B}_c$ are not related by an allowed coordinate transformation: we are not allowed to perform the $SL(2, R)$ transformation $x_3 \to x_3 + ct$ due to the condition that $L$ vanish at the horizon, which in turn is mandated by the absence of a chemical potential for $P_3$. In order to have a continuous zero temperature limit, we need to keep the full family of zero temperature solutions. We will return to this topic in section 4.3.6 where we discuss the approach to criticality in the parameters $(b, q)$. 

15
4. Low temperature thermodynamics

The solutions constructed in the last section carry nonzero charge density and magnetic field, but have vanishing temperature and entropy density. We now want to heat them up and study the low temperature behavior of the entropy density.

4.1. Matched asymptotic expansions

Since analytic solutions at arbitrary $T$ are not available, we need to proceed perturbatively in small $T$. However, straightforward perturbation theory around the zero temperature solutions will not work, as the perturbations diverge at the horizon; at the horizon, the change from zero to finite temperature is not a small perturbation. Instead, we need to employ a matched asymptotic expansion. The basic idea is well illustrated by a simple example.

Below is the metric for finite temperature D3-branes in asymptotically flat space:

$$ds^2 = H^{-\frac{1}{2}} (-f dt^2 + dx^i dx^i) + H^{\frac{1}{2}} f^{-1} dr^2 + H^{1/2} r^2 d\Omega_5^2$$

$$H = 1 + \frac{L^4}{r^4}, \quad f = 1 - \frac{r^4}{L^4}.$$ (4.1)

To make the analogy with our problem, suppose we only knew the solution (4.1) in two limiting cases: (1) $r_s = 0$, which is the zero temperature solution; (2) $r \ll L$, corresponding to omitting the 1 in $H$, which yields the AdS$_5$ Schwarzschild solution times the 5-sphere. Using the perturbation theory, how could we construct the full solution in the regime $r_s \ll L$ (the low temperature regime) and extract the low temperature thermodynamics? Note that from the knowledge of the AdS Schwarzschild solution one can of course work out its entropy density and temperature, but to import this result to the asymptotically flat solution one needs additional information, since one does not know a priori how the AdS Schwarzschild time coordinate matches onto the time coordinate in the asymptotically flat region. They could differ by a scale transformation, which would rescale the temperature; this is a key point for understanding the low temperature behavior of the solutions studied in this paper$^9$.

In the matched asymptotic expansion approach we consider two different perturbation problems. In the first, we perturb around the AdS Schwarzschild solution, requiring smoothness at the horizon, which leads to perturbations that grow with $r$. In the second, we perturb around the zero temperature solution, maintaining asymptotic flatness, which gives perturbations that grow at small $r$. If $r_s \ll L$, there is a parametrically large overlap region $r_s \ll r \ll L$, where both expansions are valid. In this region we can match up the free parameters appearing in the perturbations, and so connect near-horizon data with that at infinity. For the D3-brane example, the perturbed solution in the overlap region is given by

$$ds^2 = ds^2_{AdS_5 \times S^5} - \frac{r_6^6}{2L^6} (-dr^2 + dx^2_i) + \frac{r_4^4}{r^2} dr^2 + \frac{r_2^2}{2L^2} dr^2 + \frac{r_0^2}{r^6} dr^2$$ (4.2)

illustrating the presence of perturbations that grow at small and large $r$.

Our problem is similar, with a few extra complications. First, we have two independent perturbations, corresponding to changing $\hat{T}$ and $\hat{B}$. Second, our zero temperature solutions are not given by a closed form expression. But it is still possible to use this method to determine the low temperature thermodynamics.

$^9$ For the D3-brane example, due to the boost invariance of the zero temperature solution there is actually no ambiguity in determining $s(T)$ in this manner, but since boost invariance is absent for our solutions, let us ignore this fact to maintain a faithful analogy.
4.2. Summary of matching computations

We defer a detailed solution of the perturbations problem to the next section. Here we just summarize the basic results that are needed to determine the thermodynamics. The first part of the problem consists of expanding around a BTZ solution, and the second in expanding around the zero temperature asymptotically AdS solutions constructed above. These expansions are matched in an intermediate region where both solutions approach AdS3 × R2 plus small perturbations. This AdS3 × R2 region is the analog of AdS5 × S5 in the D3-brane example. Linearizing around AdS3 × R2 yields the following result:

\[
E(r) = e_0 r^{k-1} \\
P(r) = p_0 r^{-k-1} \\
L(r) = l_0 + 2br - 4\sqrt{3}\left(\frac{2\sigma - 1}{\sigma + 1}\right)v_\sigma r^{\sigma+1} - 4\sqrt{3}\left(\frac{2\sigma + 3}{\sigma}\right)v_\sigma r^{-\sigma} \\
M(r) = m_0 + m_1 r \\
N(r) = n_0 + n_1 r \\
V(r) = v_0 + v_0 + v_\sigma r^\sigma + v_\sigma r^{-\sigma-1},
\]

where all free integration constants are indicated and \( b = \sqrt{3} \). The problem consists of relating these integration constants to the parameters that appear in the BTZ and asymptotically AdS5 perturbation problems.

4.2.1. Perturbations around BTZ. An exact solution to the field equations is given by BTZ × R2 with the magnetic flux, which we write as

\[
ds^2 = \frac{dr^2}{12r^2 + mnr} - mr dt^2 + 4\sqrt{3}r dt dx_3 + n dx_1^2 + dx_2^2, \\
F = \sqrt{3} dx_1 \wedge dx_2,
\]

i.e. \( b = \sqrt{3} \) and

\[
L = 2\sqrt{3} r, \quad M = -mr, \quad N = n, \quad V = E = P = 0.
\]

In order for this solution to match onto a charged asymptotically AdS5 solution, we need to add a two-parameter family of perturbations. One parameter corresponds to adding electric charge, and the second to inducing a flow toward AdS5. The perturbation parameters are \( q \) and \( v_0 \), given by the values of \( E \) and \( V \) at the horizon10:

\[
E(0) = q, \quad V(0) = v_0.
\]

A smooth perturbation with these boundary conditions can be found analytically in terms of hypergeometric functions. For present purposes we just need results for \( E \) and \( V \). At large \( r \), their asymptotics match onto (4.3) with parameters

\[
e_0 = \frac{E_0 q}{(mn)^{k-1}}, \quad v_\sigma = \frac{V_\sigma}{(mn)^{\sigma}}(v_0 + nq^2 A_q)
\]

and where

\[
E_k = \frac{12^{k-1} \Gamma(2k)}{k \Gamma(k)^2}, \quad V_\sigma = \frac{12^\sigma \Gamma(1 + 2\sigma)}{\Gamma(1 + \sigma)^2}.
\]

Here, the coefficient \( A_q \) depends only on the \( k \), but not on \( q, m \) or \( n \). Its construction will be given in (5.25) of section 5.3.

10 Note that the normalization for the electric charge at the horizon used in [12] was chosen with \( n = 1 \) and differs by a factor of \( \sqrt{n} \) from the normalization used here.
We now match this result to the small $r$ asymptotics of the asymptotically $\text{AdS}_5$ solution. Here we note that $e_0$ and $v_+$ are nonzero already in the unperturbed solution, and so to the leading order we need not concern ourselves with the corrections. For $E$ we already defined $e_0$ to yield the small $r$ asymptotics (see (3.17)), and for $V$ we chose $v_+ = 1$ (see (3.9)). Hence the matching yields

$$\frac{E_k q}{(mn)^{k-1}} = e_0, \quad \frac{V_{\sigma}}{(mn)^{\sigma}} (v_0 + n q^2 A_q) = 1.$$  

(4.9)

4.2.2. Perturbations around asymptotically $\text{AdS}_5$ solution. Now consider perturbations around the zero temperature, asymptotically $\text{AdS}_5$ solution. We demand that these perturbations preserve the conformal boundary metric, meaning that they leave the asymptotic metric constants $c_{L,M,N,V}$ unchanged. With these parameters fixed, and with the magnetic field fixed at $b = \sqrt{3}$, it is apparent from (2.25) that any change in $\hat{B}$ occurs via a change in $c_{E}$. Up to coordinate transformations, there is a two-parameter family of such perturbations, corresponding to changing $\hat{T}$ and $\hat{B}$; we denote the corresponding perturbation parameters as $\epsilon_T$ and $\epsilon_B$. After turning on these parameters, we extract the small $r$ asymptotics and read off the parameters in (4.3). Here we only need the results for $M$ and $N$, which are

$$m_0 = 0, \quad m_1 = - (\bar{\alpha} + \epsilon_T), \quad n_0 = C_T \epsilon_T + \epsilon_B, \quad n_1 = 0.$$  

(4.10)

We also need to know the change in the large $r$ asymptotics of the electric field, to keep track of how the charge density changes:

$$r \to \infty \quad E \sim \frac{c_E}{r^2}, \quad c_E = c_{E0} + CB \epsilon_B.$$  

(4.11)

where we now denote the electric field coefficient appearing in the unperturbed solution by $c_{E0}$. The coefficients $C_T$ and $C_B$ depend on the Chern–Simons coupling $k$, and their values may be extracted from the explicit form of the perturbations around the asymptotically $\text{AdS}_5$ solution. They will be computed analytically in terms of these data in section 5.

Since $M$ and $N$ are already nonzero in the unperturbed BTZ solution we can match to their values, neglecting the higher order corrections. This yields

$$\bar{\alpha} + \epsilon_T = m, \quad C_T \epsilon_T + \epsilon_B = n,$$  

(4.12)

where we recall that $\alpha - \bar{\alpha} = 16 c_V^2 c_{E0}^2 J(k)$.

4.3. Thermodynamics and scaling

We now have all the information we need to read off the low temperature thermodynamics. The results are different depending on whether we are at or above the critical magnetic field, and so we treat these cases separately.

4.3.1. Low temperature thermodynamics for $\hat{B} > \hat{B}_c$. We first consider sitting at fixed $\hat{B} > \hat{B}_c$ and lowering the temperature to zero. To stay at fixed $\hat{B}$ we set $\epsilon_B = 0$, and then (4.12) yields

$$C_T \epsilon_T = n.$$  

(4.13)

To go to zero temperature we need to take $\epsilon_T \to 0$. The condition $\hat{B} > \hat{B}_c$ implies $\bar{\alpha} > 0$, so then (4.12) gives that $m$ should asymptote to a finite value

$$m = \bar{\alpha} = \alpha - 16 c_V^2 c_{E0}^2 J(k).$$  

(4.14)
From (3.27) and (3.28) this yields
\begin{equation}
\frac{\alpha}{m} = \frac{\hat{B}^3}{\hat{B}^3 - \hat{B}_c^3}.
\end{equation}

Now, using \( c_M = -\alpha/\sqrt{3} \) (which follows from (3.23)), \( b = \sqrt{3} \) and the formulas in (2.25), (2.26), it is straightforward to evaluate the following results as \( \epsilon_T \to 0 \):
\begin{equation}
\hat{s} = \frac{1}{24} \sqrt{c_V m \alpha}, \quad \hat{T} = \frac{1}{4\pi} \frac{m \sqrt{c_V n}}{\sqrt{\alpha}}.
\end{equation}

Eliminating \( m \) and \( \alpha \) in favor of \( \hat{T} \) and \( \hat{s} \) automatically cancels the dependence on \( n \) and we find the following expression for the leading low temperature behavior of the entropy,
\begin{equation}
\hat{s} = \frac{\pi}{6} \left( \frac{\hat{B}^3}{\hat{B}^3 - \hat{B}_c^3} \right) \hat{T}
\end{equation}
as announced in (1.3) of the introduction. For \( \hat{B} \) near and larger than \( \hat{B}_c \), this formula may be approximated by
\begin{equation}
\hat{s} \sim \frac{\pi}{18} \left( \frac{\hat{B}_c}{\hat{B} - \hat{B}_c} \right) \hat{T}
\end{equation}
which nicely reproduces the numerical result \( c_3 = 0.045 \) obtained in formula (3.9) of [12] (where \( k = 2/\sqrt{3} \)), after taking into account the change in the normalization of \( \hat{s} \) and \( \hat{T} \) between this paper and [12]. The exact correspondence is
\begin{equation}
c_3 = \frac{\pi \hat{B}_c^2}{18(1 + \hat{B}_c^3)^{1/3}}
\end{equation}
giving approximately \( c_3 = 0.04186 \).

### 4.3.2. Low temperature thermodynamics at \( \hat{B} = \hat{B}_c \)

Next, we examine the low temperature entropy at fixed \( \hat{B} = \hat{B}_c \), which means \( \bar{\alpha} = 0 \) or
\begin{equation}
\alpha = 16c_V^3 c_{E0}^2 J(k) = \frac{4}{3} c_V^3 c_{E0}^2 \hat{B}_c^3.
\end{equation}
We again set \( \epsilon_B = 0 \) in order to hold \( c_{E0} \) fixed, and thus (4.13) holds. From (4.12) we also have \( \epsilon_T = m \), and therefore
\begin{equation}
C_T m = n.
\end{equation}
We then find
\begin{equation}
\hat{s} = a \hat{T}^{1/3}
\end{equation}
with the numerical coefficient given by
\begin{equation}
a = \frac{(2\pi C_T c_V^3 c_{E0}^4 J(k)^2)^{1/3}}{3}.
\end{equation}
Since \( C_T c_{E0}^4 \) is a universal number, which depends only on \( k \) and on the properties of the \( T = 0 \) purely magnetic solution, so does the coefficient \( a \). In particular, it is independent of temperature, magnetic field and charge density.
4.3.3. Scaling function. Turning on both $\epsilon_T$ and $\epsilon_B$ allows us to explore a two-dimensional region around the critical point, corresponding to changing the temperature and magnetic field. Using the matching relations, the change in the magnetic field to first order in $\epsilon_B$ can be expressed as

$$\frac{\hat{B} - \hat{B}_c}{\hat{B}_c} = -\frac{2}{3} \frac{C_B}{c_{E0}} \epsilon_B = \frac{2}{3} \frac{C_B}{c_{E0}} (C_T m - n). \quad (4.24)$$

Combining this with

$$\hat{T} = \frac{1}{4\pi} \frac{m \sqrt{cvn}}{\sqrt{\alpha}} \quad (4.25)$$

lets us write the following cubic equation for $n$

$$n^3 + 3 \frac{c_{E0}}{C_B} \frac{\hat{B} - \hat{B}_c}{\hat{B}_c} n^2 + \left(3 \frac{c_{E0}}{2C_B} \frac{\hat{B} - \hat{B}_c}{\hat{B}_c}\right)^2 n - \left(\frac{4\pi}{c_v} \alpha C_T^2 \frac{c}{cv} \hat{T}^2 \right) = 0. \quad (4.26)$$

Here $\alpha$ takes its value at the critical point, $\alpha = 16 c^2 V c E_0 J(k)$. In (5.52) of section 5, we will obtain the following result for $C_B$

$$C_B = 16 k J(k)^2 c^2 V c^3 E_0 = \frac{4}{3} k c^3 V c^3 E_0 \hat{B}_c^3. \quad (4.27)$$

The entropy density is given by $\hat{s} = \sqrt{c_v n \alpha} / 24$, as in (4.16). Combining these facts, together with the result (4.23), we find that the entropy density can be expressed in the form

$$\hat{s} = \hat{T}^{1/3} f \left( \frac{\hat{B} - \hat{B}_c}{\hat{T}^{2/3}} \right) \quad (4.28)$$

with the scaling function $f$ obeying

$$f(x) \left( f(x)^2 + \frac{x}{32k \hat{B}_c^4} \right) = a^3. \quad (4.29)$$

For $\hat{B} > \hat{B}_c$, and $\hat{T}$ small, we have $x > 0$ and large, so that $f$ may be approximated by

$$f(x) \sim \frac{32k \hat{B}_c^4 a^3}{x} \Rightarrow \frac{\hat{s}}{\hat{T}} \sim \frac{32k \hat{B}_c^4 a^3}{\hat{B} - \hat{B}_c}. \quad (4.30)$$

As we have already obtained an exact expression for the coefficient of $1/ (\hat{B} - \hat{B}_c)$ for the ratio $\hat{s}/\hat{T}$ in (4.18), we readily derive an expression for $a$, and thus for $C_T$ in terms of the other parameters

$$a^3 = \frac{\pi}{576k \hat{B}_c^4}, \quad C_T = \frac{27}{8k c^3 V c^3 E_0 \hat{B}_c^9}. \quad (4.31)$$

We have already established earlier that these values are in perfect agreement with our numerical results of [12] in this regime. For $\hat{B} = \hat{B}_c$, we have $x = 0$, $f(0) = a$ and we recover the critical scaling law of (4.22).

4.3.4. Dynamical critical exponent. The change in power law in the $\hat{s}$ versus $\hat{T}$ relation at the critical point is a reflection of the change in the dynamical critical exponent. This can be made precise as follows. As we lower the temperature to zero, the near-horizon solution is becoming $\text{BTZ} \times \mathbb{R}^2$, with parameters $m$ and $n$. Changing the temperature in this regime corresponds to a scale transformation of the BTZ solution, under which $m$ and $n$ transform. Under a scale transformation (2.20) we have

$$m \rightarrow \lambda^{1-2\nu} m, \quad n \rightarrow \lambda^{2\nu - 2} n. \quad (4.32)$$

Here
For $\hat{B} > \hat{B}_c$, taking the temperature to zero at fixed $\hat{B}$ corresponds to taking $n \to 0$ at fixed $m$. Thus, we have $w = 1/2$, and then from (2.21) this yields $\hat{z} = 1$. On the other hand, at the critical point, $\hat{B} = \hat{B}_c$, we saw that we should take both $m$ and $n$ to zero with $m/n$ fixed. This requires that we take $w = 3/4$ and so $\hat{z} = 3$.

4.3.5. Comparison with the numerical results on scaling of [12]. The value of $\hat{s}/\hat{T}^{1/3}$ at $\hat{B} = \hat{B}_c$ was encoded in the parameter $c_2$ of [12], where $k$ was fixed at the supersymmetric value $k = 2/\sqrt{3}$. The precise relation, taking into account the required conversion of normalizations of $\hat{s}$ and $\hat{T}$ between this paper and [12] is given by

$$c_2 = \frac{a\hat{B}_c^{4/3}}{(1 + \hat{B}_c^2)^{1/2}}.$$  \hspace{1cm} (4.33)

We find the value $c_2 = 0.1265$, which compares favorably with the numerical outcome $c_2 = 0.11$ of [12]. Finally, as $\hat{B} < \hat{B}_c$, we find

$$f(x) \sim \frac{(-x)^{1/2}}{4\sqrt{2}k\hat{B}_c^2}, \quad \hat{s} \sim \frac{\sqrt{\hat{B}_c - \hat{B}}}{4\sqrt{2}k\hat{B}_c^2}.$$  \hspace{1cm} (4.34)

This result may be compared with the coefficient $c_1$ of formula (3.7) in [12]. Taking into account the conversion of conventions, we find

$$c_1 = \frac{1}{4\sqrt{2}k(\hat{B}_c)^{1/2}(1 + \hat{B}_c^2)^{1/2}}$$  \hspace{1cm} (4.35)

which gives $c_1 = 0.221$ as compared with the numerical value $c_1 = 0.172$ of [12].

4.3.6. Approach to zero temperature in the $(b, q)$ plane. The data used to parametrize the solutions studied numerically in [11] and [12] consisted of the charge density at the horizon $q$, as well as the magnetic field strength $b$ in the horizon frame\footnote{The normalization of the parameter $q$ used in [12] differs from the normalization of the parameter $q$ used here by a factor of $\sqrt{n}$, so that $\sqrt{n}\hat{q}_{\text{here}} = q_{[12]}$. In this subsection only, $q$ will denote the parameter of [12].}

(A third parameter, denoted by $C(r_s)$ was associated with the momentum $P_3$ of the solution, and could be fixed arbitrarily.) In this parametrization, the electrically charged, finite temperature, Reissner–Nordström solutions correspond to $b = 0$, while the purely magnetic finite temperature solutions of [10] correspond to $q = 0$, their extremal $T = 0$ limits being reached respectively at the endpoints $(b, q) = (0, \sqrt{6})$ and $(b, q) = (\sqrt{3}, 0)$.

For $\hat{B} \geq \hat{B}_c$ (and for $k \geq 3/4$, as will become clear below), lowering the temperature to zero corresponds in terms of $(b, q)$ to approaching the purely magnetic endpoint $(b, q) = (\sqrt{3}, 0)$. This circumstance further illuminates the remark made in the last paragraph of section 3.3 that the low $T$ thermodynamics, for all values of $\hat{B} \geq \hat{B}_c$, are governed by solutions which are all related to one another by coordinate transformations at $T = 0$. We emphasize that even at this endpoint the full asymptotically AdS$_5$ solution carries a nonzero charge density; what is tending to zero here is the charge at the horizon.

The departure infinitesimally away from the purely magnetic fixed point at $(b, q) = (\sqrt{3}, 0)$ caused by turning on a small $q$ is known analytically from our work here. To see this, note that the value $v_0$ of the field $V$ at the horizon is equivalent, via rescaling of $x_{1,2}$, to a change in $b$, which is given to leading order by $\delta b = \sqrt{3} - b = -2v_0$. With the present definition of $q$, the matching equations obtained in (4.9) read

$$\frac{E_4q/\sqrt{n}}{(mn)^{3/2-1}} = e_0, \quad \frac{V_6}{(mn)^9} (v_0 + q^2 A_9) = 1.$$  \hspace{1cm} (4.36)
The parameter \( e_0 \) is kept fixed, since it is related to the value of \( \hat{B} \), which is being held fixed as we lower the temperature. Rearranging and trading \( v_0 \) for \( \delta b \), we have
\[
q = \frac{e_0}{E_k} m^{k-1} n^{k-\frac{1}{2}}, \quad \delta b = 2A_q q^2 - 2(V_e)^{-1}(mn)^{\sigma}. \tag{4.37}
\]
The parameter \( A_q \), which will be obtained in section 5.3, is negative when the \( z' \)-integral is convergent, namely for \( k < 1 + \sigma/2 \). These relations allow us to examine the curve in the \((b, q)\)-plane along which the \( T = 0 \) limit is obtained while keeping \( \hat{B} \) fixed.

- For \( \hat{B} > \hat{B}_c \), the zero temperature limit is obtained by taking \( n \) to zero while holding \( m \) fixed. It is then appropriate to write
\[
\delta b = 2A_q q^2 - 2V_e^{-1} m \frac{n}{m} \left( \frac{E_k}{e_0} \right)^{\frac{1}{1+\sigma}} q^{\frac{\sigma}{1+\sigma}}. \tag{4.38}
\]
The first term dominates for \( \frac{1}{2} < k < \frac{\sigma+1}{2} \); the second for \( \frac{\sigma+1}{2} < k \). In either case the \( q \to 0 \) limit is smooth, since both powers of \( q \) are positive. Thus, as \( T \to 0 \), the \( \hat{B} > \hat{B}_c \) system flows to the purely magnetic critical point, which was used throughout this paper, and will be discussed elsewhere \[24\].

- For \( \hat{B} = \hat{B}_c \), the zero temperature limit is obtained by taking both \( m \) and \( n \) to zero while holding the ratio \( m/n \) fixed. We write
\[
\delta b = 2A_q q^2 - 2V_e^{-1} \left( \frac{E_k}{e_0} \right)^{\frac{1}{1+\sigma}} \left( \frac{m}{n} \right)^{\frac{\sigma}{1+\sigma}} q^{\frac{\sigma}{1+\sigma}}. \tag{4.39}
\]

For \( \frac{\sigma}{2} + \frac{3}{4} < k \) the second term dominates, and thus determines the flow. For \( \frac{3}{2} < k < \frac{\sigma}{2} + \frac{3}{4} \) the first term dominates.

For \( \frac{3}{4} < k \), both terms admit a smooth \( q \to 0 \) limit since, just as in the case \( \hat{B} > \hat{B}_c \), both powers of \( q \) are positive. Thus, as \( T \to 0 \), the \( \hat{B} = \hat{B}_c \) system again flows to the purely magnetic critical point, a result again confirmed by numerical study.

Finally, for \( \frac{1}{2} < k < \frac{3}{4} \) the power of \( q \) in the second term turns negative, indicating that the system no longer flows toward the purely magnetic critical point as \( T \to 0 \). In fact, numerical study shows that as \( T \to 0 \), a finite limiting value for \( q \) emerges. Thus, the perturbative expansion around the purely magnetic critical point, which was used throughout this paper, can no longer be valid for the \( \hat{B} = \hat{B}_c \) and \( \frac{1}{2} < k < \frac{3}{4} \) system, and the scaling law \( \delta \sim T^{1/4} \) is not expected to hold. The precise nature of the flows for \( k < \frac{3}{4} \) is presently under investigation and will be discussed elsewhere.

The critical curve, defined as the set of limiting values in the \((b, q)\) plane at which \( T = 0 \) may be computed from the above result in the neighborhood of the purely magnetic critical point, where it may be defined as the flow at \( \hat{B} = \hat{B}_c \). We deduce from (4.39) that it is given by \( \delta b \sim -q^2 \) for \( 3/4 < k < 3/4 + \sigma/2 \), and by \( \delta b \sim -q^{\sigma/(4k-3)} \) when \( k > 3/4 + \sigma/2 \), a result found to be in accord with numerical results.

5. Matched asymptotic expansion: detailed analysis

To explore the low temperature thermodynamics in the regime \( B \geq B_c \), we proceed perturbatively in small \( T \), namely \( T \ll \sqrt{B_c} \), as outlined in section 4. For \( r \gg T \), the effects of finite temperature are small and may be treated perturbatively; in this region, we use a perturbative solution around the charged, \( T = 0 \), asymptotically \( \text{AdS}_5 \) solution constructed in section 3. For \( r \ll \sqrt{B_c} \), the finite \( T \) asymptotically \( \text{AdS}_5 \) solution reduces to \( \text{BTZ} \times R^2 \); in this region, we use a perturbative solution around finite temperature \( \text{BTZ} \). In the overlap
region $T \ll r \ll \sqrt{Bc}$ the effects of temperature are small and a perturbed AdS$_3$ solution may be used. It is in this region that both the near-horizon perturbed BTZ solution and the asymptotically AdS$_5$ perturbed solutions are both valid, and where their perturbative solutions may be matched. In this section, we present detailed derivations of the perturbative expansions in each one of these regions, and of their matching. This section is somewhat technical, and so we note that the main results have already been summarized in the preceding section.

5.1. Near-horizon region: perturbed BTZ solution

In the near-horizon region, we expand around the BTZ $\times \mathbb{R}^2$ solution of (4.4) and (4.5). Inspection of the reduced field equations (2.10) instructs us to treat $E$ and $P$ to first order, but the other fields to second order. To organize this perturbation theory we introduce a small parameter $\varepsilon$, so that

$$
E(r) = \varepsilon E_1(r) \\
P(r) = \varepsilon P_1(r) \\
L(r) = 2br + \varepsilon^2 L_1(r) \\
M(r) = -mr + \varepsilon^2 M_1(r) \\
N(r) = n + \varepsilon^2 N_1(r) \\
V(r) = \varepsilon^2 V_1(r),
$$

where $b = \sqrt{3}$, and $m$ and $n$ are the constant parameters of the BTZ solution. The corresponding field equations are

$$
M_1 = 0 = nE_1' + 2brP_1' + 2b(k + 1)P_1 \\
M_2 = 0 = 2brE_1' + 2b(1 - k)E_1 - mrP_1' - mP_1 \\
E_1 = 0 = (L_1' + 8brV_1' - 4bV_1) - 4E_1P_1 \\
E_2 = 0 = (M_1' - 4mrV_1' + 2mV_1)' + 4E_1^2 \\
E_3 = 0 = (N_1' + 4nV_1)' + 4P_1^2 \\
E_4' = 0 = 6((12r^2 + mnr)V_1')' - 96V_1 + 4nE_1^2 + 16brE_1P_1 - 4mrP_1^2 \\
CON = 0 = bL_1' + \frac{m}{4}N_1' + (24r + mn)V_1' - 12V_1 + nE_1^2 + 4brE_1P_1 - mrP_1^2.
$$

We have replaced $E_4$ by $E_4' = E_4 + 4\text{CON}$. The boundary conditions at the horizon are

$$
L_1(0) = M_1(0) = M_1'(0) = N_1(0) = 0.
$$

The first two may be chosen by $SL(2, \mathbb{R})$ and the requirement that the horizon remain at $r = 0$; the latter two follow from the fact that any non-zero values may be absorbed into the parameters $m$, $n$ of the zeroth-order solution. The values of $E_1(0)$ and $V_1(0)$ will be turned on by the perturbation, and $P_1(0)$ will be determined by those, as will be shown below.

5.1.1. Solving Maxwell’s equations. Maxwell’s equations $M_1$ and $M_2$ involve only $E_1$ and $P_1$ and may thus be integrated independently from the remaining equations. Throughout, it will often be convenient to use the rescaled coordinate $z$ defined by

$$
z = \frac{-12r}{mn}.
$$

We shall use the same notation for a function of $r$ and its associated function of $z$. Eliminating $P_1'$ between $M_1$ and $M_2$, it follows that $P_1$ is uniquely determined by $E_1$:

$$
P_1 = \frac{\sqrt{12z}}{km}[(1 - z)E_1' + (k - 1)E_1],
$$

23
while $E_1$ satisfies the hypergeometric differential equation
\[ z(1-z)\partial_z^2 E_1 + [c - (a+b+1)z]\partial_z E_1 - ab E_1 = 0 \] (5.6)
with $a + b = 2$, $ab = 1 - k^2$ and $c = 1$. Retaining the solution which is regular at $r = 0$, we impose a boundary condition $E_1(0) = q$, corresponding to finite charge density at the horizon. The solution is then given by the hypergeometric function
\[ E_1(r) = q F(1 + k, 1 - k; 1; z). \] (5.7)

The homogeneous part is the hypergeometric equation with $a+b = -4/3$ and $c = 1$, following the notation of (5.6). As a result, $a$ and $b$ are given by $a = 1 + \sigma$ and $b = -\sigma$, where $\sigma$ was defined in (3.8). There is a unique solution which is regular at $r = 0$, which we shall abbreviate by
\[ F(z) \equiv F(\sigma + 1, -\sigma; 1; z). \] (5.11)

5.1.2. Solving Einstein’s equations. Einstein’s equation $E_{4'}$ may be solved for $V_1$ in terms of $E_1$ and $P_1$. To do so, it will be useful to introduce the composite function $\Lambda$, defined by
\[ nq^2 \Lambda \equiv \frac{1}{18} (n E_1^2 + 4br E_1 P_1 - mr P_1^2) . \] (5.9)

By inspection, it is clear that the function $\Lambda$ depends only on $z$ and $k$. In terms of these variables, $E_{4'}$ for $V_1$ may be expressed as an inhomogeneous hypergeometric equation
\[ z(1-z)^2 V_1 + (1 - 2z)\partial_z V_1 + \frac{4}{3} V_1 = nq^2 \Lambda(z). \] (5.10)

The homogeneous part is the hypergeometric equation with $a+b = -4/3$ and $c = 1$, following the notation of (5.6). As a result, $a$ and $b$ are given by $a = 1 + \sigma$ and $b = -\sigma$, where $\sigma$ was defined in (3.8). There is a unique solution which is regular at $r = 0$, which we shall abbreviate by
\[ F(z) \equiv F(\sigma + 1, -\sigma; 1; z). \] (5.11)

To construct the inhomogeneous solution, we set $V_1(z) = F(z) v(z)$, after which $\partial_z v$ is found to satisfy a first-order equation
\[ \partial_z (z(1-z)F^2 \partial_z v) = nq^2 F \Lambda. \] (5.12)

This equation may be readily integrated, and the resulting general solution which is regular as $z = r = 0$ is given by
\[ V_1(z) = v_0 F(z) + nq^2 F(z) \int_0^z \frac{1}{z''(1-z'')F(z'')^2} \int_0^{z''} dz''' F(z''') \Lambda(z'''). \] (5.13)

Here, $v_0$ is an arbitrary integration constant. Since $E_1$ and $P_1$ are regular as $z \to 0$, both integrals are convergent, so that $V_1(z)$ is a well-defined function of $z$ with $V_1(0) = v_0$.

Einstein’s equations $E_1$, $E_2$, $E_3$ may be solved in terms of $E_1$, $P_1$, $V_1$ as follows:
\[ L_1(r) = 4\alpha_L r - 8br V_1(r) + 12b \int_0^r dr' V_1(r') + 4 \int_0^r dr' \int_0^{r'} dr'' E_1(r'') \]
\[ M_1(r) = 2v_0 r + 4mr V_1(r) - 6m \int_0^r dr' V_1(r') - 4 \int_0^r dr' \int_0^{r'} dr'' E_1^2(r'') \] (5.14)
\[ N_1(r) = 4\alpha_N r - 4n(V_1(r) - v_0) - 4 \int_0^r dr' \int_0^{r'} dr'' P_1^2(r''). \]

These solutions fulfill all the boundary conditions of (5.3). The two remaining integration constants $\alpha_L$, $\alpha_N$ are related by enforcing the constraint equation $\text{CON}$, and we find
\[ nq^2 = -4b\alpha_L - m\alpha_N. \] (5.15)

In summary, given $m$, $n$ of the unperturbed BTZ solution, the perturbation theory around BTZ is governed entirely by two additional parameters $q$, $v_0$ (the parameter $\alpha_L$ will be determined by further gauge fixing and matching).
5.2. Overlap region: perturbed AdS\(_3\) solution

In the overlap region, the unperturbed geometry is simply AdS\(_3\) × R\(^2\), whose fields are given in (3.3) with \(V = 0\), and corresponds to setting \(m = n = 0\) in the BTZ equations (5.1) and (5.2). All fields are now expanded to first order only, and we find the leading-order solutions:

\[
\begin{align*}
E(r) &= e_0 r^{k-1} \\
P(r) &= p_0 r^{-k-1} \\
L(r) &= l_0 + 2b r - 4\sqrt{3} \left(\frac{2\sigma - 1}{\sigma + 1}\right) v_v r^{\sigma+1} - 4\sqrt{3} \left(\frac{2\sigma + 3}{\sigma}\right) v_- r^{-\sigma} \\
M(r) &= m_0 + m_1 r \\
N(r) &= n_0 + n_1 r \\
V(r) &= v_v r^\sigma + v_- r^{-\sigma - 1}.
\end{align*}
\]

Note that the constraint equation CON fixes the linear term in \(L\). There are nine integration constants, \(e_0, p_0, l_0, m_0, n_0, n_1, v_v, v_-\); their number precisely matches the expected number from four second-order equations, two first-order equations and one constraint.

5.3. Matching expansions in near-horizon and overlap regions

To carry out the matching between these two regions, we need to isolate, in the large \(r/(mn)\) the behavior of the perturbative solution around the BTZ solution, those functional dependences that coincide with those identified in the overlap region in (5.16). These large \(r/(mn)\) asymptotics arise from the asymptotic behavior for large \(z\) of the hypergeometric function, which may be obtained using the inversion formula

\[
F(a; b; c; z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} F(a; 1-c+a; 1-b+a; z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b} F(b; 1-c+b; 1-b+a; z^{-1})
\]

which results in the following dominant asymptotics as \(-z \to \infty\):

\[
F(a; b; c; z) \sim \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{-b}.
\]

Isolating the \(r^{k-1}\) term in \(E_1\) and the \(r^{-k-1}\) term in \(P_1\), we find the following relations:

\[
\begin{align*}
e_0 &= q(mn)^{1-k} E_k \\
p_0 &= q(mn)^{1+k} E_{-k}.
\end{align*}
\]

Isolating the \(r^\sigma\) and \(r^{-1-\sigma}\) terms in \(V_1\) proceeds analogously, paying close attention to some extra subtleties. We begin with the asymptotics as \(r/(mn) \to \infty\) of the corresponding hypergeometric function

\[
F(z) \sim V_\sigma (mn)^{-\sigma} r^\sigma + V_{-1-\sigma} (mn)^{1+\sigma} r^{-1-\sigma} \quad V_\sigma = \frac{(12)^\sigma \Gamma(1+2\sigma)}{\Gamma(1+\sigma)^2}.
\]

The subtlety in evaluating the asymptotics of \(V_1\) resides in the fact that contributions arise due to the inhomogeneous part of the solution in (5.13). To identify these contributions, one needs to investigate the asymptotics of the integrals for large \(-z\). The behavior of \(\Lambda\) is readily deduced from that of \(E_1\) (given in (5.16)), and that of \(P_1\), given by

\[
P_1(z) \sim -\frac{4b q (k-1) E_{k-1}}{mk(mn)^{k-2}} r^{k-2}.
\]
As a result, the \( -z \to \infty \) asymptotics of \( \Lambda \) is given by

\[
\Lambda(z) \sim \Lambda_{\infty}(mn)^{2-2kr}r^{2-2k} \quad \Lambda_{\infty} = \frac{E_0^k}{18(2k-1)}, \tag{5.22}
\]

The asymptotics of the integral over \( z'' \) in (5.13) as \( -z \to \infty \) is proportional to \( (-z')^{2k-1+\sigma} \).

For \( k > 1/2 \), as we have been assuming throughout, this exponent is positive. The leading asymptotics of the \( z' \) integral in (5.13) is then governed by the asymptotics of the integrand, namely \( (-z')^{2k-3-\sigma} \). For \( 2k - 2 < \sigma \), the integral converges as \( z \to \infty \), and the leading asymptotics of \( V_1 \) is given by

\[
V_1(r) \sim A_V V_\sigma(mn)^{-\sigma} r^\sigma + A_V V_{1-\sigma}(mn)^{1+\sigma} r^{-1-\sigma}, \tag{5.23}
\]

where \( V_\sigma \) was defined earlier, and \( A_V \) is given by

\[
A_V = v_0 + \eta q^2 A_q, \tag{5.24}
\]

where \( A_q \) is a quantity that depends only on \( k \). For \( 2k - 2 < \sigma \) (or equivalently \( k < 1 + \sigma/2 \sim 1.38 \), so that this range clearly includes the supersymmetric value of \( k \), we have

\[
A_q = \int_0^\infty \frac{dz'}{z'(1-z') F(z')} \int_0^z \frac{dz''}{z''(1-z'') F(z'') \Lambda(z'')} F(z) \Lambda(z'') A_z 0, \tag{5.26}
\]

where

\[
v_\infty = \frac{3\Lambda_{\infty}}{2(6k^2 - 9k + 1)}. \tag{5.27}
\]

Every increment in the range of \( k \) by 1/2 will require an extra subtraction term.

We are now ready to match the asymptotics of \( V_1 \) between the near-horizon and overlap regions, and we find

\[
v_+ = A_V V_\sigma(mn)^{-\sigma} \quad v_- = A_V V_{1-\sigma}(mn)^{1+\sigma}. \tag{5.28}
\]

The ratio \( v_-/v_+ = (V_{1-\sigma}/V_\sigma)(mn)^{1+2\sigma} \) shows that the term proportional to \( v_- \) corresponds to a higher order correction in the asymptotically AdS\(_5\) region and may be neglected there, so that we effectively have \( v_- \sim 0 \).

It remains to match the expansions of the functions \( L_1, M_1, N_1 \). To leading order, we have the matching

\[
l_0 = 0 \\
m_0 = 0 \quad m_1 = -m \\
n_0 = n \quad n_1 = 0. \tag{5.29}
\]

The large \( r/(mn) \) asymptotics of the functions \( L_1, M_1, N_1 \) may be derived in an analogous fashion and will provide higher order corrections to the leading matching of (5.29). The corresponding results will not, however, be needed for the thermodynamic questions that we are addressing here, and we shall not carry them out.
5.4. Asymptotically AdS\(_5\) region: perturbed \(T = 0\) charged solution

In the asymptotically AdS\(_5\) region, perturbation theory is carried out around the \(T = 0\) charged asymptotically AdS\(_5\) solution, which was derived in section 3. Here, we shall denote the fields of this solution with subscripts 0. The functions \(L_0\) and \(V_0\) obey the equations

\[
0 = L''_0 + 2V'_0L_0 + 4L_0(V''_0 + (V'_0)^2) \\
0 = L''_0(V'_0)^2 + V'_0(L''_0) - 6 + \frac{1}{2}(L'_0)^2 + 3e^{-4V_0}
\]

for both neutral and charged solutions. For the charged solutions, we have

\[
E_0 = A'_0 \quad \text{with} \quad A_0(r) = \frac{e\psi}{kb} e^{2kb\psi(r)} \quad \text{and} \quad
M_0(r) = -\frac{\alpha}{\sqrt{12}} L_0(r) - \frac{4e\psi}{kb} e^{4kb\psi(r)} \int_r^\infty \frac{d\tau'}{L_0(\tau')^2 e^{2V_0(\tau')}}
\]

where the function \(\psi\) is defined by

\[
\psi(\tau) \equiv \int_\infty^\tau \frac{d\tau'}{L_0(\tau') e^{2V_0(\tau')}}
\]

as well as \(N_0 = P_0 = 0\). For given \(V_0\), the first equation of (5.30) is a linear second-order differential equation. Besides its solution \(L_0\), it has a conjugate linearly independent solution, which we shall denote by \(L_c_0\) and normalize by

\[
L_c_0(\tau) \equiv \frac{L_0(\tau)}{\int_\infty^\tau \frac{d\tau'}{L_0(\tau')^2 e^{2V_0(\tau')}}}.
\]

This function obeys the following asymptotics:

\[
r \to 0 \quad L_c_0(\tau) \to -\frac{1}{\sqrt{12}} \\
r \to \infty \quad L_c_0(\tau) \sim -\frac{1}{4e\psi(r)}.
\]

This solution will play a key role in the following.

5.4.1. Perturbation equations. To carry out first-order perturbation theory around this solution, we introduce an expansion parameter \(\epsilon\), and set

\[
P(r) = \epsilon P_1(r) \\
N(r) = \epsilon N_1(r) \\
E(r) = E_0(r) + \epsilon E_1(r) \\
L(r) = L_0(r) + \epsilon L_1(r) \\
M(r) = M_0(r) + \epsilon M_1(r) \\
V(r) = V_0(r) + \epsilon V_1(r) \\
f(r) = f_0(r) + \epsilon f_1(r).
\]

Maxwell’s equations for the perturbation functions are given by

\[
M_1 \quad 0 = (e^{2V_0}(E_0 N_1 + L_0 P_1))' + 2kb P_1 \\
M_2 \quad 0 = (e^{2V_0}(2L_0 E_0 V_1 + L_0 E_1 + E_0 L_1 + M_0 P_1))' - 2kb E_1
\]
5.4.3. The translation and dilation modes. Invariance of equations (2.30) with the superscripts guarantee the existence of two independent perturbative solutions to (5.36) and (5.37). They will be denoted with the superscripts $t$ and $d$, respectively, and are given by

$$
E_1^t = E_0^t, \quad E_1^d = r E_0^t + E_0 / 2
$$

$$
L_1^t = L_0^t, \quad L_1^d = r L_0^t - L_0
$$

$$
M_1^t = M_0^t, \quad M_1^d = r M_0^t - M_0
$$

$$
V_1^t = V_0^t, \quad V_1^d = r V_0^t
$$

$$
f_1^t = f_0^t, \quad f_1^d = r f_0^t - 2 f_0
$$

(5.38)

with $P_1^t = P_1^d = N_1^t = N_1^d = 0$. As expected, the translation mode obeys the desired boundary conditions at $r \to \infty$, but the dilation mode does not, because $V_1^d(r)$ and $r^2 E_1^d(r)$ do not tend to 0 there. Thus, the dilation mode must be absent altogether.

5.4.4. Solving and matching for $N_1$. The function $N_1$ satisfies the same differential equation as $L_0$ does. As a result, it must be a linear combination of $L_0$ and its conjugate $L_0^\ast$. The boundary conditions at $r \to \infty$ forces the function $L_0$ to be absent, so that we have

$$
N_1(r) = \tilde{n}_0 L_0^\ast(r)
$$

(5.39)

with asymptotics given by

$$
r \to 0 \quad N_1(r) \sim -\tilde{n}_0 / \sqrt{12}
$$

$$
r \to \infty \quad N_1(r) \sim -\tilde{n}_0 / 4c_1^* r
$$

(5.40)

Matching the $r \to 0$ asymptotics with those of the overlap region, we find that

$$
n_0 = -\tilde{n}_0 / \sqrt{12}, \quad n_1 = 0
$$

(5.41)

since the function $L_0^\ast$ does not contain a dependence on $r$ which is linear for small $r$. 

Here, the last equation corresponds to the combination $E_4' = E_4 + 4CON$, while the next-to-last equation derives from expanding the $f, V$ equation of (2.12).

5.4.2. Boundary conditions. The $T \neq 0$ perturbations on the $T = 0$ asymptotically AdS$_5$ solutions will be required to have specific boundary conditions at $r = \infty$, namely that the spacetime retain the same asymptotic metric as the $T = 0$ solution. This will require $L_1(r)/r, M_1(r)/r, N_1(r)/r, r^2 E_1(r)$ and $V_1(r)$ to tend to 0 as $r \to \infty$. At small $r$, namely $T \ll r \ll \tilde{b}_1$, we will impose matching conditions with the perturbations around the $T \neq 0$ BTZ solution, as given in the overlap region by (5.16).

5.4.3. The translation and dilation modes. Invariance of equations (2.10) for the $T = 0$ unperturbed asymptotically AdS$_5$ solution under translations and dilations in $r$ guarantee the existence of two independent perturbative solutions to (5.36) and (5.37). They will be denoted with the superscripts $t$ and $d$, respectively, and are given by

$$
E_1^t = E_0^t, \quad E_1^d = r E_0^t + E_0 / 2
$$

$$
L_1^t = L_0^t, \quad L_1^d = r L_0^t - L_0
$$

$$
M_1^t = M_0^t, \quad M_1^d = r M_0^t - M_0
$$

$$
V_1^t = V_0^t, \quad V_1^d = r V_0^t
$$

$$
f_1^t = f_0^t, \quad f_1^d = r f_0^t - 2 f_0
$$

(5.38)

with $P_1^t = P_1^d = N_1^t = N_1^d = 0$. As expected, the translation mode obeys the desired boundary conditions at $r \to \infty$, but the dilation mode does not, because $V_1^d(r)$ and $r^2 E_1^d(r)$ do not tend to 0 there. Thus, the dilation mode must be absent altogether.

5.4.4. Solving and matching for $N_1$. The function $N_1$ satisfies the same differential equation as $L_0$ does. As a result, it must be a linear combination of $L_0$ and its conjugate $L_0^\ast$. The boundary conditions at $r \to \infty$ forces the function $L_0$ to be absent, so that we have

$$
N_1(r) = \tilde{n}_0 L_0^\ast(r)
$$

(5.39)

with asymptotics given by

$$
r \to 0 \quad N_1(r) \sim -\tilde{n}_0 / \sqrt{12}
$$

$$
r \to \infty \quad N_1(r) \sim -\tilde{n}_0 / 4c_1^* r
$$

(5.40)

Matching the $r \to 0$ asymptotics with those of the overlap region, we find that

$$
n_0 = -\tilde{n}_0 / \sqrt{12}, \quad n_1 = 0
$$

(5.41)

since the function $L_0^\ast$ does not contain a dependence on $r$ which is linear for small $r$. 

5.4.2. Boundary conditions. The $T \neq 0$ perturbations on the $T = 0$ asymptotically AdS$_5$ solutions will be required to have specific boundary conditions at $r = \infty$, namely that the spacetime retain the same asymptotic metric as the $T = 0$ solution. This will require $L_1(r)/r, M_1(r)/r, N_1(r)/r, r^2 E_1(r)$ and $V_1(r)$ to tend to 0 as $r \to \infty$. At small $r$, namely $T \ll r \ll \tilde{b}_1$, we will impose matching conditions with the perturbations around the $T \neq 0$ BTZ solution, as given in the overlap region by (5.16).
5.4.5. Solving and matching for $P_1$. The function $P_1 = C'_1$ may be obtained by solving M1 in (5.36). It is instructive, however, to solve for $C_1$ from the first integral for $\lambda$ in (2.15). Linearizing this equation gives

$$\left( N_1 M'_0 - N'_1 M_0 \right) e^{2V_0} = 2\lambda_0 + 8kbA_0 C_1.$$  (5.42)

This equation is easily solved, and we find the general solution

$$C_1 = -\tilde{n}_0 A_0 L_0^2 - \frac{\tilde{n}_0 M_0}{8kbL_0A_0} - \frac{\lambda_0}{4kbA_0}. $$  (5.43)

The relevant asymptotics of $P_1$ as $r \to 0$ is the functional dependence $r^{-1-k}$, resulting from the dependence $r^{-k}$ of $C_1$. The first term above behaves as $r^{k-1}$ which has no overlap with $r^{-1-k}$ for $k > 1/2$. The last two terms do contribute, however, and we find

$$r \to 0 \quad C_1 \sim \frac{-\tilde{n}_0 \alpha - 4b\lambda_0}{48e_0} r^{-k}.$$  (5.44)

$$r \to \infty \quad C_1 \sim \frac{-\tilde{n}_0 \alpha - 4b\lambda_0}{16bcV E_0} \left( 1 + \frac{kb}{cV r} \right).$$

As a result, we find

$$r \to 0 \quad P_1 \sim p_0 r^{-k-1} \quad p_0 = -k \frac{-\tilde{n}_0 \alpha - 4b\lambda_0}{48e_0}$$

$$r \to \infty \quad P_1 \sim \frac{p_\infty}{r^2} \quad p_\infty = -k \frac{-\tilde{n}_0 \alpha - 4b\lambda_0}{16c_0^2 cV E_0}. $$  (5.45)

Comparison with the functional behavior in the overlap region, we find that the prefactors $p_0$ must be the same in both expressions. The value of $p_0$ in the overlap region has already been determined and found to be given by $p_0 = qE_{-\Delta}(mn)^{i\Delta}$. This value provides a higher order correction and vanishes to leading order $p_0 \sim 0$. Combining with the result from (5.45), we must have to this order

$$\lambda_0 = -\tilde{n}_0 \alpha / (4b)$$  (5.46)

so that

$$p_\infty = -\tilde{n}_0 k \frac{\alpha - \tilde{\alpha}}{16c_0^2 cV E_0} = -\tilde{n}_0 kE_0 J(k). $$  (5.47)

The large $r$ asymptotics of $P_1$ will play a key role in determining the constant $C_B$.

5.5. Calculation of the coefficient $C_B$

With the matching information on $N_1$ and $P_1$ in hand so far, it is already possible to compute the coefficient $C_B$ which plays a key role in determining the scaling properties near the quantum critical point. Since $C_B$ governs the response of the charge density $cE$ to a change in the parameter $n$ at the horizon, we need to determine the change in the field $E_1$ caused by turning on the perturbations $N_1$ and $P_1$. We begin by solving for $E_1$.

5.5.1. Solving for $E_1$. The function $E_1$ may be obtained by solving equation M2 of (5.36), in terms of the functions $P_1$, $V_1$, $L_1$. In fact, it is instructive to obtain the corresponding potential $A_1$ from the first integral in the second equation of (2.14). Its perturbative expansion is given by

$$L_0 e^{2V_0} A'_1 = 2kbA_1 = -e^{2V_0}(2L_0 E_0 V_1 + E_0 L_1 + M_0 P_1). $$  (5.48)
Dividing both sides by $A_0 L_0 e^{3V_0}$ and using (3.14), we readily integrate this equation and the general solution is given by

$$A_1(r) = A_0(r) \left[ a_1 - \int_{\infty}^{r} \frac{1}{A_0(r')} \left( 2E_0 V_1 + \frac{E_0}{L_0} L_1 + \frac{M_1}{L_0} P_1 \right) (r') \right],$$

(5.49)

where $a_1$ is an integration constant which may be absorbed into the zeroth-order charge $e_0$ or $c_{E0}$. The matching for small $r$ is carried out between the zeroth-order asymptotically $\text{AdS}_5$ solution $E_0$ and $E(r) = e_0 r^{k-1}$ in the overlap region (5.16), so that $c_{E0}$ and $e_0$ are related by the zeroth-order formula (3.16). The corrections provided by $E_1$ only add higher order effects, which may be ignored for our purposes.

To compute the change in charge density $c_E$, we need to obtain only the leading correction to the $1/r$ asymptotics of $A_1$. Remarkably, to do so, we do not need to know the precise solutions for the functions $L_i$ and $V_1$. The only property of $L_i$ and $V_1$ that we will need here is the fact that they preserve the asymptotic form of the metric, namely as $r \to \infty$, the functions $r V_1$ and $L_i$ must have finite limits.

As a result of the above properties of $V_1$ and $L_i$, the leading asymptotics of the integrand of (5.49) derives entirely from the term in $P_1$ and is of the order $1/r^2$. The integral converges at $\infty$, and the asymptotics there is given by $A_1(r) \sim -\alpha p_\infty / (\sqrt{12} r)$, where $p_\infty$ was defined in (5.45). As a result, the asymptotics of $E_1$ is given by

$$E_1(r) \sim \frac{c E_1}{r^2}, \quad c_{E1} = \frac{\alpha p_\infty}{\sqrt{12}} = 16 n_0 c_{E0}^2 k J(k)^2.$$

(5.50)

Thus, the change in the charge density caused by turning on $n_0$ is indeed non-trivial.

### 5.5.2. Calculating $C_B$ at the critical point.

The coefficient $C_B$ is defined as the change in $c_E$ due to turning on $N$ at the horizon

$$c_E = c_{E0} + C_B \epsilon_B$$

(5.51)

while keeping the temperature fixed at $T = 0$. Thus, we need to know the asymptotics of the field $E_1$ due to turning on $n_0$. This is precisely the effect calculated above, and those results may now be applied. The quantity $c_E - c_{E0}$ may be identified with $c_{E1}$ above, and $\epsilon_B$ with $n \sim n_0$ to leading order, so that we find

$$C_B = \frac{c_{E1}}{n_0} = k J(k) c_{E0} = 16k J(k)^2 c_{E0}^2 c_{E0}^3$$

(5.52)

using the results of (5.50). As was shown in section 4.3.3, once $C_B$ is known, the remaining coefficient $C_I$ may be calculated indirectly by matching the scaling function near the quantum critical point onto the low $T$ behavior of the $\tilde{B} \sim \tilde{B}_c$ branch, which is known exactly. In the remainder of this section, however, we will solve for the perturbations around the $T = 0$ asymptotically $\text{AdS}_5$ solution also for the remaining functions $V_1$, $L_i$ and $M_i$ in order to confirm that the expansion and matching procedure works consistently to leading order also for these functions.

### 5.6. Solving and matching for $V_1$, $L_i$ and $M_i$

The calculation of the perturbations $V_1$ and $L_i$ is considerably more complicated than for $N_i$ and $P_i$, because their equations do not decouple. In appendix B, the perturbations of $V$ and of the composite $f = L^2 - M N$ are obtained from the perturbations of a common field $X$, via the relations $e^{2V} = X''$ and $fe^{2V} = 24X$. The zeroth-order solution corresponds to the
The rhs is subdominant and may be dropped in the matching process, yielding the matching components:

\[ V_1 = X_1^\sigma e^{-2V_0} \]
\[ L_1 = -2V_1 L_0^2 + 24X_1 e^{-2V_0} + M_0 N_1. \] (5.53)

It is shown in appendix B that the field \( X_1 \) obeys a third-order linear differential equation, whose inhomogeneous part is sourced by \( N_1 \) and \( P_1 \). Two linearly independent solutions are known to the homogeneous part of this equation, namely the translation and dilation modes of \( X_1 \), which we denote by \( X_1^t \) and \( X_1^d \), respectively. They are given by

\[ X_1^t = X_0' \quad X_1^d = rX_0' - 2X_0. \] (5.54)

The translation and dilation modes of \( V_1 \) and \( L_1 \) (as well as of the remaining fields) were already given in (5.38). The \( n \) and \( p \) modes are given by

\[ 2V_1^n = (X_1^n)^{\sigma^{-1}} e^{-2V_0} \quad 2L_1^n = -2V_1^n L_0^2 + 24X_1^n e^{-2V_0} \]
\[ 2V_1^p = (X_1^p)^{\sigma^{-1}} e^{-2V_0} \quad 2L_1^p = -2V_1^p L_0^2 + 24X_1^p e^{-2V_0} + M_0 N_1. \] (5.56)

The large \( r \) asymptotics of the \( X_1 \) modes was evaluated in (B.26). The \( t, n \) and \( p \) all respect the asymptotically AdS5 metric and are thus allowed from the large \( r \) boundary perspective, while the mode \( d \) violates those boundary conditions and must be absent, so that \( \zeta_d = 0 \). As \( r \to 0 \), the asymptotic behavior of the remaining modes \( X_1^t, X_1^n, X_1^p \) was evaluated in (B.28), and we derive the following asymptotics for the corresponding \( V_1 \) and \( L_1 \) components:

\[ V_1^t \sim r^{\sigma-1} \quad L_1^t \sim r^{\sigma} \]
\[ V_1^n \sim r^{-\sigma-1} \quad L_1^n \sim r^{-\sigma} \]
\[ V_1^p \sim r^{\sigma} \quad L_1^p \sim r^{\sigma+1}. \] (5.57)

The \( V_1^p \) behavior as \( r \to 0 \) requires it to match on to the \( r^{-1-\sigma} \) branch of the solution (5.23) in the overlap region. As was shown in (5.28), however, the coefficient \( v_+ \) of this branch is suppressed by a factor \((mn)^{1+2\sigma} \sim (mn)^{2\sigma}\) compared to the \( v_- \) branch and thus corresponds to a higher order effect, which may be neglected. Thus, the solution in the overlap region instructs us that the \( V_1^p \) mode must be absent to leading order as well, so that \( \zeta_p = 0 \). It remains to match the mode \( V_1^p \). Combining it with the leading contribution from the zeroth-order asymptotics \( r^\sigma \), we have

\[ v_+ = (v_0 + nq^2 A_q) V_0 (mn)^{-\sigma} = 1 + 2\tilde{\alpha}_0 \sigma J_2(k), \] (5.58)

where the coefficient \( J_2(k) \) is defined in (B.29) of appendix B. Clearly, the second term on the rhs is subdominant and may be dropped in the matching process, yielding the matching condition of (4.9).
5.6.1. Solving and matching for $M_1$. The homogeneous part of the only remaining equation $E_2$ is solved by $L_0$ and $L_0^c$. The general solution is given by
\[
M_1(r) = \tilde{m}_0 L_0^c(r) + \tilde{m}_1 L_0(r) \\
- L_0^c(r) \int_0^r dr' L_0 e^{2V_1}(2V_1'M_0'' + 4M_0(V_1'' + 2V_0'V_1') + 8E_0 E_1)
+ L_0(r) \int_r^\infty dr' L_0 e^{2V_1}(2V_1'M_0'' + 4M_0(V_1'' + 2V_0'V_1') + 8E_0 E_1).
\]
(5.59) 

The lower boundaries of the integrations are of course arbitrary, since their change can always be compensated by changing the integration constants $\tilde{m}_0, \tilde{m}_1$.

As $r \to \infty$, the requirement that the perturbation $V_1$ should maintain the asymptotics $e^{2V} \sim 2V \sqrt{r}$ requires that $V_1$ must vanish at least as fast as $1/r$. (The mode $V_1', V_1''$ and $V_1^p$ all satisfy this requirement, while the dilation mode does not; thus the dilation mode must be absent, and we have $\xi_d = 0$.) Given that $M_0$ and $L_0$ are linear in $r$ as $r \to \infty$, while $L_0^c$ goes like $1/r$, it follows that the leading asymptotics as $r \to \infty$ of the integrands is given by
\[
L_0 e^{2V_1}(2V_1'M_0'' + 4M_0(V_1'' + 2V_0'V_1') + 8E_0 E_1) \sim O(r^0)
\]
\[
L_0^c e^{2V_1}(2V_1'M_0'' + 4M_0(V_1'' + 2V_0'V_1') + 8E_0 E_1) \sim O(r^{-2}).
\]
(5.60) 

Hence, the second integral is convergent; the first integral behaves linearly as $r \to \infty$ so that the contributions of both integral terms behave as constants when $r \to \infty$. Demanding that $c_M \neq 0$ be compensated by the perturbation $\varepsilon_T$ then requires that $\tilde{m}_1 = 0$.

5.7. Calculation of $C_T$ at the critical point

The coefficient $C_T$ was already computed in section 4.3.3 by indirect matching methods. For completeness, here we shall also give its derivation directly from the perturbation theory solutions. The coefficient $C_T$ is defined by the following relations as $r \to 0$:
\[
M(r) \sim -\bar{\alpha}r - \varepsilon_T r
\]
\[
N = C_T \varepsilon_T
\]
\[
(5.61)
\]
while keeping $c_E$ constant. Actually, at the critical solution, we have $\bar{\alpha} = 0$. To determine the coefficient $C_T$, we need both the functions $M_1$ and $E_1$, the latter because we need to make sure that when turning on $N$ at the horizon, we keep $c_E$ unchanged. In fact, from formula (5.50), it is immediately obvious that to keep $c_E$ constant, we need to require
\[
a_1 = -n_0 \frac{C_B}{c_E} C_0
\]
(5.62) 

so that the field $E_1 = A_1'$ relevant to the $\varepsilon_T$ perturbation at constant $c_E$ is given by
\[
A_1(r) = A_0(r) \left[ -n_0 \frac{C_B}{c_E} - \int_\infty^r dr' \frac{1}{A_0(r')} \left( 2E_0 V_1 + \frac{E_0}{L_0} L_1 + \frac{M_0}{L_0} P_1 \right) (r') \right].
\]
(5.63) 

We begin by obtaining the $r \to 0$ asymptotics of $E_1$. Since we have $E_0(r)/A_0(r) \sim k/r$, the last two terms dominate and their integral produces the following leading asymptotics:
\[
A_1(r) \sim \frac{n_0 \bar{\alpha} k}{24 \sqrt{12}(2k - 1)} \frac{A_0(r)}{r} + O(r^0) A_0(r).
\]
(5.64) 

Now, the key ingredient we use here is that at the critical point, $\bar{\alpha} = 0$, so this singular leading asymptotics actually drops out. (Away from the critical point, we would have to compensate for this mode by an explicit subtraction of a translation mode.) As a result, the leading asymptotics of $E_1$ at the critical point is just $E_1(r) \sim r^{k-1}$. We are now in a position to
evaluate the asymptotics of the integrands in (5.59) as $r \to 0$, taking $V_1 = V_1^0 \sim V_0 \sim r^\alpha$ and $E_1 \sim E_0 \sim r^{k-1}$. The terms in $M_0$ and $M_0^0$ scale as $r^\alpha$, while the term in $E_1$ scales as $r^{2k-1}$. As a result, the first integral in (5.59) is convergent as $r \to 0$, but in fact for $k > 1/2$, it vanishes faster than $r$ as $r \to 0$, and thus never contributes to the linear behavior of $M(r)$ which we need to extract following (5.61). Thus, the first integral is irrelevant to the calculation of $C_T$. The second integral in (5.59) is convergent as $r \to 0$ as long as $k > 1/2$. Thus, we extract

$$\frac{1}{C_T} = \frac{\sqrt{T^2}}{n_0} \int_0^\infty \, dr \, L_0^p e^{2V_0}(2V_1^0 M_0^0 + 4M_0 V_1'' + 2V_0 V_1') + 8E_0 E_1),$$

(5.65)

where $V_1 = V_1^0$ and $E_1 = A_1'$ is the field defined in (5.63). All dependence on $n_0$ drops out of these formulas, and the remainder is expressed solely in terms of the functions of the $T = 0$ critical solution times a factor of $c_s^2$.  

6. Discussion

In this paper we have restricted attention to the gravity side of the holographic duality, but it will of course be illuminating to further explore the field theory interpretation of the quantum critical point and its properties. In [12] we noted that the scaling properties of the entropy density agree with those near a quantum critical point described by the Hertz–Millis theory [35–37]. To develop this connection further it will be very useful to study finite frequency/wavelength perturbations around the critical point and to compute the associated dispersion relations. Based on our experience with finite temperature perturbations, it seems likely that this is analytically tractable.

Even at the level of free field theory, there are possible origins for the emergence of a critical magnetic field. In the simplest interpretation, we assume that for $\hat{B} > \hat{B}_c$, the charge density in the field theory is due to fermions occupying the lowest Landau level. This is energetically favorable, as both higher fermionic Landau levels and bosonic levels have higher energy. As the magnetic field is decreased, or the charge density is increased, the maximal $p_3$ (the momentum parallel to the magnetic field vector) carried by states in the lowest Landau level increases, which increases their energy $E = |p_3|$. Eventually, the energy exceeds that of the higher fermionic levels and the lowest bosonic level. By dimensional analysis, this occurs when $\hat{B}^3/\rho^2$ is of order unity, with the precise number depending on the charge spectrum. It is possible that the critical point we see on the gravity side is in some way connected to this, although free field theory clearly seems incapable of explaining the details, such as the $\tilde{s} \sim \tilde{T}^{1/3}$ scaling law.

Another interesting question concerns the effective spacetime dimensionality of the fixed point theory, by which we mean the number of dimensions along which long-range correlations are present. Several features seem to point to a $(1 + 1)$-dimensional interpretation, especially the appearance of near-horizon geometries that factorize as $M_3 \times R$. However, examining more closely our analysis of the low temperature thermodynamics, we see that this is not so clear cut. At nonzero charge density and magnetic field, and for vanishing $P_3$ chemical potential corresponding to the vanishing of $L$ at the horizon, there is no finite temperature solution taking the form of a three-dimensional black hole times $R^2$. Indeed, in appendix A we write down the general factorized solution, and none obeys these conditions, except when $k = 1$. Our low temperature solutions always involve some excitation of $V$, corresponding to a fully five-dimensional solution. This seems to suggest that the degrees of freedom contributing to the low temperature entropy density lie in more than $1 + 1$ dimensions. To better understand this, it will again be very useful to have results for the spectrum of finite frequency/wavelength perturbations around the critical point.
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Appendix A. General solution with constant $V$

In this appendix we derive the most general solution for which $V$ is constant, and $b \neq 0$. By rescaling $x_1$ and $x_2$ we set $V = 0$. The reduced field equations are then

\begin{align}
M_1 \quad & (NE + LP)' + 2kbP = 0 \\
M_2 \quad & (LE + MP)' - 2kbE = 0 \\
E_1 \quad & L'' = 4EP \\
E_2 \quad & M'' = -4E^2 \\
E_3 \quad & N'' = -4P^2 \\
E_4 \quad & (L')^2 = M'N' + 4b^2 \\
CON \quad & MP^2 + 2LEP + NE^2 - 6 + \frac{1}{4}(L')^2 - \frac{1}{4}M'N' + b^2 = 0.
\end{align}

We begin by eliminating the bilinears in $E$ and $P$ between equations $E_1$, $E_2$, $E_3$:

\begin{align}
(L'')^2 - M''N'' &= 0. \tag{A.2}
\end{align}

If $N' \neq 0$, we solve for $M'$ in terms of $L'$ and $N'$ using $E_4$:

\begin{align}
M' &= \frac{(L')^2 - 4b^2}{N'}. \tag{A.3}
\end{align}

Differentiating once and substituting $M''$ into (A.2), we obtain a factorized equation

\begin{align}
(N' L'' - L'N' - 2bN'')(N' L'' - L'N'' + 2bN'') = 0. \tag{A.4}
\end{align}

At least one of these factors must vanish. Without loss of generality, we choose this to be the first factor (the case of the second factor vanishing corresponding to $b \to -b$), so that $N' L'' - (L' + 2b)N'' = 0$. This means that the functions $N'$ and $L' + 2b$ must be proportional to one another. Since we have assumed that $N' \neq 0$, we have in all generality

\begin{align}
L' + 2b &= aN' \tag{A.5}
\end{align}

for some constant $a$. As a result, we have $L'' = aN''$ which implies $P(E + aP) = 0$ by equations $E_1$ and $E_3$. By an $SL(2, R)$ transformation, any solution $E + aP = 0$ is equivalent to a solution $P = 0$. If $N' = 0$, then we must also have $P = 0$. We conclude that all solutions with $V$ constant and $b \neq 0$ are $SL(2, R)$-equivalent to a solution with $P = 0$. We shall now proceed to solve the general case where $V$ is constant and $P = 0$.

A.1. The general solutions for $P = 0$

The constant $V$ and $P = 0$ simplify $E_1$, $E_3$ and $CON$ to the following relations:

\begin{align}
L'' &= N'' = 0 \\
NE^2 &= 6 - 2b^2 \tag{A.6}
\end{align}

the first two of which are solved by

\begin{align}
L(r) &= l_0 + l_1 r \\
N(r) &= n_0 + n_1 r \tag{A.7}
\end{align}

for constant $l_0, l_1, n_0, n_1$. We now have the following three classes of solutions.
(1) When \( n_1 = 0 \) and \( n_0 = 0 \), we have \( b^2 = 3 \), with \( l_1 = \pm 2b \), and the only non-trivial remaining equations are M2 and E2, which are solved by

\[
E(r) = q r^{\pm 2k - 1},
\]

\[
M(r) = m_0 + m_1 r - \frac{2q^2}{k(2k \mp 1)} r^{\pm 2k}
\]

for arbitrary constants \( q, m_0, m_1 \). This solution is equivalent to (2.19).

(2) When \( n_1 = 0 \) and \( n_0 \neq 0 \), we rescale to \( n_0 = 1 \), so that \( E = q \) is a constant obeying \( q^2 + 2b^2 = 6 \), with \( L' = \pm 2b \) from E4, the constraint \( q(k \mp 1) = 0 \) from M2 and

\[
M(r) = m_0 + m_1 r - 2q^2 r^2
\]

for arbitrary constants \( m_0 \) and \( m_1 \). This is equivalent to the \( k = \pm 1 \) solution of (2.23).

(3) When \( n_1 \neq 0 \), we have \( M' \) constant by E4, \( E = 0 \) by E2, \( b^2 = 3 \) by E4 and CON, and

\[
M(r) = m_0 + m_1 r
\]

with \( l_1^2 - m_1 n_1 = 12 \). This solution is \( SL(2, R) \)-equivalent to AdS3, with a standard presentation given by \( l_1 = \sqrt{12} \) and \( m_1 = n_1 = 0 \).

Appendix B. Solving for \( V \) perturbations on asymptotically AdS5

To solve for the perturbations \( V_1 \), given that the perturbations \( N_1 \) and \( P_1 \) are now known, we need to solve a coupled set of equations for \( V_1 \) and \( f_1 \), such as \( fV \) and \( E4' \). Eliminating \( f_1 \) would result in a fourth-order differential equation in \( V_1 \), whose solvability is unclear. Here, instead, we shall take a less direct approach; it will result in a single third-order differential equation for the full composite \( fe^2V \); this equation will then be linearized around the \( f_0, V_0 \) solution; two of its solutions will be known as the corresponding translation and dilation modes, and the third solution may then be constructed by quadrature only.

Our starting point will be the following two independent equations for the full fields

\[
0 = (fe^{2V})'' - 24e^{2V}
\]

\[
0 = f(V')^2 + f'V' + \frac{g}{4} + 3e^{-4V} - 6 + MP^2 + 2LEP + NE^2
\]

where \( f, g \) were defined in (2.11). The first equation coincides with (2.12), while the second is the original CON equation of (2.10). These two equations combined are equivalent to E1, E4 and CON.

B.1. Derivation of a single third-order differential equation

Equation (2.16) allows us to express \( g \) in terms of \( f, f' \) and the combination \( \lambda^2 - \mu \nu \), which in turn may be evaluated with the help of the first integrals of (2.15). Carrying out these substitutions, we may re-express the second equation of (B.1) as follows:

\[
f(V')^2 + f'V' + \frac{(f')^2}{16f} + b^2 e^{-4V} - 6 + MP^2 + 2LEP + NE^2
\]

\[
= \frac{1}{4f} \left[ \lambda_0^2 - \mu_0 \nu_0 + 4kb(2\lambda_0 AC + \mu_0 C^2 + \nu_0 A^2) \right] e^{-4V}.
\]

Next, we solve the first equation in (B.1) by parametrizing the most general solution with the help of a function \( X \), such that

\[
f e^{2V} = 24X
\]

\[
e^{2V} = X''
\]
and eliminate \( f \) and \( V \) in favor of \( X \) in (B.2). This gives the following equation:

\[ -3(XX'''')^2 + 6XX''X''' + (X'X'')^2 + 2XX'' - 4X(X')^3 = R, \tag{B.4} \]

where \( R \) is defined by

\[
\begin{align*}
R &= -\frac{1}{36}(MP^2 + 2LEP + NE^2)fe^{8V} \\
&\quad + \frac{1}{144}[\lambda_0^2 - \mu_0 v_0 + 4kb(2\lambda_0 AC + \mu_0 C^2 + v_0 A^2)]e^{4V}. \tag{B.5}
\end{align*}
\]

In the perturbative expansion of this equation, \( R \) will act as source terms. For this reason, we have refrained from recasting its \( f \) and \( V \) dependence in terms of \( X \).

The unperturbed solution in terms of \( L_0, M_0, V_0, A_0 \) has \( N_0 = C_0 = P_0 = 0 \). Its boundary conditions \( A_0(0) = L_0(0) = M(0) = 0 \) force the values of the first integrals \( \lambda_0 = \mu_0 = v_0 = 0 \), so that \( R_0 = 0 \) and \( X_0 \) obeys

\[ -3(X_0X_0''')^2 + 6X_0X_0'X_0''X_0''' + (X_0'X_0'')^2 + 2X_0X_0'' - 4X_0(X_0')^3 = 0. \tag{B.6} \]

First-order perturbation theory around this solution was organized in (5.35), and we set

\[
\begin{align*}
X &= X_0 + \varepsilon X_1 \\
R &= R_0 + \varepsilon \tilde{R}_0 R_1. \tag{B.7}
\end{align*}
\]

By inspection of (2.15), the constants \( \lambda_0, \mu_0 \) and \( v_0 \) are all at least of order \( \varepsilon \) (since they vanished for the unperturbed solution). Thus, \( R_1 \) may be readily evaluated and we find

\[ R_1 = -\frac{1}{36\tilde{h}_0}(2L_0E_0 P_1 + N_1 E_0^2)f_0 e^{8V_0} + \frac{kbv_0}{36\tilde{h}_0}A_0^2 e^{4V_0}. \tag{B.8} \]

The constant \( v_0 \) is readily deduced using the last equation of (2.15), the expression for \( N_1 \) in (5.39), as well as (5.33), and we find

\[ v_0 = -\tilde{h}_0. \tag{B.9} \]

Finally, we linearize also the left side of (B.4) and organize the result as follows:

\[ a_3 X_0''' + a_2 X_0'' + a_1 X_0' + a_0 X_0 = \tilde{h}_0 R_1, \tag{B.10} \]

where the coefficients are found to be

\[
\begin{align*}
a_3 &= -6X_0^2 X_0''' + 6X_0 X_0' X_0'' \\
a_2 &= 6X_0 X_0'' X_0'' + 2(X_0'')^2 X_0'' + 2X_0 - 12X_0(X_0'')^2 \\
a_1 &= 6X_0 X_0'' X_0'' + 2X_0(X_0'')^2 \\
a_0 &= -6X_0(X_0'')^2 + 6X_0 X_0'' X_0''' + 2X_0'' - 4(X_0'')^3.
\end{align*}
\]

It is this equation for \( X_1 \) that remains to be solved.

### B.2. Constructing the general solution for \( X_1 \)

Equation (B.6) for the unperturbed solution is invariant under translations and dilations in \( r \). As a result, the homogeneous part of (B.10) manifestly admits two solutions: the collective coordinate modes \( X_1' \) and \( X_1'' \) associated respectively with translations and dilations of the unperturbed solution \( X_0 \). We know their form explicitly in terms of \( X_0 \):

\[ X_1' = X_0', \quad X_1'' = r X_0' - 2X_0. \tag{B.12} \]

The fact that \( X_1' \) solves the homogeneous part of (B.10) allows one to reduce the order of the differential equation by one. To do so, we introduce a function \( Y \) which satisfies

\[ X_1 = X_0' Y \quad \Rightarrow \quad b_3 Y''' + b_2 Y'' + b_1 Y' = \tilde{h}_0 R_1, \tag{B.13} \]
where
\[ b_3 = a_1 X_0' \]
\[ b_2 = a_2 X_0' + 3a_1 X_0'' \]
\[ b_1 = a_1 X_0' + 2a_2 X_0'' + 3a_3 X_0''' \]

Since the homogeneous part of equation (B.10) also admits a second solution, namely \( X_1^t \), it follows that the homogeneous part of the equation in \( Y \) must admit the solution
\[ Y_0 = \frac{r X_0' - 2X_0}{X_0'} = r - 2 X_0.' \]

Introducing now a function \( Z \) which satisfies
\[ Y' = Y_0' Z \quad Z'' b_3 Y_0 + Z' (2b_1 Y_0' + b_2 Y_0) = \tilde{n}_0 R_1. \]

The homogeneous part of this equation reduces to
\[ \frac{Z''}{Z_0'} + 2 \frac{Y''}{Y_0} + 3 \frac{X_0''}{X_0'} + \frac{a_2}{a_3} = 0. \]

The quantity \( a_2/a_3 \) may be evaluated in terms of the functions \( L_0 \) and \( V_0 \), and may be reduced to the following simple expression:
\[ \frac{a_2}{a_3} = -4 V_0' - \frac{L_0'}{L_0}. \]

Putting all this together allows us to compute
\[ Z_0' = \frac{L_0 L_0' e^{4V_0}}{(X_0')^4 Y_0^2}. \]

Finally, it remains to find a particular solution to the inhomogeneous equation for \( Z \) of (B.16). To do so, we introduce a function \( Q \) which satisfies
\[ Z' = \tilde{n}_0 Z_0' Q_1 \quad Q_1' = \frac{R_1}{a_3 X_0' Y_0' Z_0'}. \]

Clearly, this equation can be solved by quadratures only.

We are now ready to present the general solution for \( X_1 \), in terms of its three linearly independent homogeneous solutions \( X_1^t \), \( X_1^d \) and the third solution which we shall denote \( X_1^n \), as well as a particular solution \( X_1^p \) sourced by \( R_1 \). They are given explicitly as follows:
\[ X_1^t (r) = X_0^t(r) \]
\[ X_1^d (r) = X_0^d(r) Y_0(r) \]
\[ X_1^n (r) = X_0^n(r) \int_0^r \, dr' Y_0'(r') Z_0(r') \]
\[ X_1^p (r) = X_0^p(r) \int_0^r \, dr' Y_0'(r') \int_0^r \, dr'' Z_0(r'') Q_1(r''), \]

where we have defined \( Q_1 \) to vanish at \( r = 0 \):
\[ Q_1(r) = \int_0^r \, dr' Q_1(r'). \]

The general solution for \( X_1 \) is then given by
\[ X_1 = \xi_t X_1^t + \xi_d X_1^d + \xi_n X_1^n + \tilde{n}_0 X_1^p. \]

The scripts \( t, d, n, p \) refer respectively to translations, dilations, non-local and particular solution, and \( \xi_t, \xi_d \) and \( \xi_n \) are integration constants. The normalizations are such that \( Q_1(0) = 0, Y_0(0) = 0 \) and \( Z_0(\infty) = 0 \).
B.3. Asymptotic behavior of $X_1$ and $V_1$

The effects on the fields of the translation and dilation modes are well known, and have already been spelled out in (5.38). To derive the asymptotics of the $X_1^t$ and $X_1^d$ modes, and their corresponding effects on the other fields, we express the modes in terms of the unperturbed solution and the fields $N_1$ and $P_1$. One finds

$$ X_0' = \frac{1}{12} L_0(L_0' + L_0 V_0')e^{2V_0} \quad Y_0' = \frac{12 - (L_0' + L_0 V_0')^2}{(L_0' + L_0 V_0')^2} $$

and as a result, the equation for $Q_1$ becomes

$$ Q_1' = \frac{12 - (L_0' + L_0 V_0')^2}{3L_0'^2 e^{3V_0}} R_1 $$

$$ R_1 = -\frac{1}{36\sigma_0}(2L_0 E_0 P_1 + N_1 E_0^2)L_0^2 e^{4V_0} - \frac{kb}{36} A_0^2 e^{4V_0}.$$  

The $r \to \infty$ asymptotics of $X_1$ may be evaluated using the asymptotics of $V_0$, $L_0$, and of $N_1$, $P_1$ from (5.40) and (5.45), respectively, and we find

$$ X_1^t \sim \frac{2V_0}{2} r^{2} \quad X_1^t \sim 24 r $$

$$ X_1^d \sim \frac{V_0}{6} r^{3} \quad X_1^d \sim J_1(\sigma)X_1^1 + Q_1(\infty)X_1^1, $$

where $J_1(k)$ is defined by the integral

$$ J_1(k) = \int_{0}^{\infty} dr' Y_0(r') \int_{r'}^{\infty} dr'' Z_0(r'') Q_1(r'') $$

which is convergent for all $k > 1/2$. Clearly, the boundary conditions as $r \to \infty$ require the dilation mode to be absent, so that $\zeta_d = 0$, but allows for the three remaining modes. 

The $r \to 0$ asymptotics of $X_1$ may be evaluated similarly. Actually, we shall be interested only in contributions to $X_1$ that produce functional dependences in $V_1$ of the form $r^\sigma$ and $r^{-1+\sigma}$ in this limit, since these are the only modes that enter in the overlap region of (5.16). Since we have $2V_1 e^{2V_0} = X_1^t$, the corresponding modes in $X_1$ behave as $r^{\sigma+2}$ and $r^{-1+\sigma}$. Using the $r \to 0$ asymptotics of $L_0$, $V_0$, and $N_1$, $P_1$ from (5.40) and (5.45), respectively, we find

$$ X_1^t \sim r \quad X_1^t \sim \frac{6r^{1-\sigma}}{\sigma(2\sigma + 1)(3\sigma - 2)} $$

$$ X_1^d \sim \frac{6\sigma - 4}{\sigma + 1} r^{2+\sigma} \quad X_1^d \sim J_2(k)X_1^d, $$

where $J_2(k)$ is defined by the integral

$$ J_2(k) = \int_{0}^{\infty} dr Z_0'(r) Q_1(r). $$

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