THE SPINLESS RELATIVISTIC HULTHÉN PROBLEM

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Abstract

The spinless Salpeter equation can be regarded as the eigenvalue equation of a Hamiltonian that involves the relativistic kinetic energy and therefore is, in general, a nonlocal operator. Accordingly, it is hard to find solutions of this bound-state equation by exclusively analytic means. Nevertheless, a lot of tools enables us to constrain the resulting bound-state spectra rigorously. We illustrate some of these techniques for the example of the Hulthén potential.

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1 Incentive: Semirelativistic Bound-State Equations

The spinless Salpeter equation — encountered in the course of the nonrelativistic reduction of the Bethe–Salpeter formalism [1–3] for the relativistic description of bound states within the domain of quantum field theory — is the eigenvalue equation of a Hamiltonian $H$ which involves, apart from some interaction potential $V(x)$, the relativistically correct expression for the kinetic energy $T(p)$ and, for two bound-state constituents of equal masses $m$, reads\footnote{We present the following discussion in terms of natural units convenient for particle physics: $\hbar = c = 1.$}

$$H \equiv T(p) + V(x) , \quad T(p) \equiv 2 \sqrt{p^2 + m^2} ; \quad (1)$$
as such, it provides a relativistic generalization of the nonrelativistic Schrödinger equation.

We study the spectral features of the Hamiltonian operator (1) for the particular case of the interaction potential $V(x)$ being the short-range Hulthén potential $V_H(r)$, a spherically symmetric central potential $V(x) = V(r)$, depending only on the radial coordinate $r \equiv |x|$ and characterized by just two (positive) parameters, its coupling strength $h$ and its range $b$:

$$V_H(r) = -\frac{h}{\exp(br) - 1} , \quad b > 0 , \quad h \geq 0 . \quad (2)$$

This potential exhibits a Coulomb-like singularity at $r = 0$, as is clear from inspection of its behaviour for small $r$, and enjoys frequent application in several different realms of physics.

In our analysis, we would like to demonstrate how to take a rigorous look at the discrete spectrum of the Hamiltonian $H$ with Hulthén potential (2) \textit{without} having to rely either on assumptions allowing us to derive approximate analytical solutions or on merely numerical approaches. After discussing, in Sec. 2, various techniques for deducing bounds on operator eigenvalues and, in Sec. 3, Hulthén-specific issues, we impose, in Sec. 4, all these tools on $H$.

1.1 Nonrelativistic Reduction to Schrödinger Hulthén Problem

For later use, we recall already here well-known results on the nonrelativistic (NR) Hulthén problem, posed by the Schrödinger Hamiltonian $H_{\text{NR}}$ found as NR limit of the operator (1):

$$H_{\text{NR}} \equiv 2m + \frac{p^2}{m} + V_H(r) \quad (m_1 = m_2 = m) . \quad (3)$$

The generic one- or two-particle Schrödinger Hamiltonian operator with Hulthén potential

$$H' \equiv \frac{p^2}{2\mu} + V_H(r) , \quad \mu > 0 ,$$

where here and only here $\mu$ indicates either the mass $m$ of the single bound particle, $\mu = m$, or the reduced mass of the bound two-particle system, $\mu \equiv m_1 m_2/(m_1 + m_2)$, respectively, possesses a highly welcome property. For any bound states with orbital angular momentum quantum number $\ell = 0$ ("s waves"), the eigenvalues of $H'$ may be given in analytic form [4]:

$$E'_n = -\frac{(2\mu h - n^2 b^2)^2}{8 \mu n^2 b^2} , \quad n = 1, 2, 3, \ldots . \quad (4)$$

The parameters $b$ and $h$ determining the Hulthén potential’s shape and the radial quantum number $n$ are subject to a constraint that limits the number of possible $\ell = 0$ bound states:

$$n^2 b^2 \leq 2\mu h \quad \iff \quad n \leq \frac{\sqrt{2\mu h}}{b} .$$

Clearly, the eigenvalues (4) form the $\ell = 0$ binding energies of the bound-state problem (3).
2 Bounds on Spinless-Salpeter Energy Eigenvalues

Before addressing Hulthén peculiarities, we recall standard means of localizing eigenvalues.

2.1 Upper Energy Bounds

2.1.1 Nonrelativistic Kinematics: Schrödinger Upper Bounds on Eigenvalues

Due to the concavity of the square-root operator of the relativistic free energy as a function of \( p^2 \), the nonrelativistic limit of that operator forms the tangent at their point of tangency \( p^2 = 0 \). The implications for the associated Hamiltonians and their eigenvalues are evident:

\[
H \equiv T(p) + V(x) \leq H_{NR} \equiv 2m + \frac{p^2}{m} + V(x) \quad \implies \quad E_k \leq E_{k, NR} \quad \forall k = 0, 1, 2, \ldots .
\]

2.1.2 Relativistic Kinematics: Variational Upper Bounds on Eigenvalues

For arbitrary self-adjoint Hilbert-space operators \( H \) bounded from below, with eigenvalues \( E_k, k = 0, 1, 2, \ldots , \) ordered by \( E_0 \leq E_1 \leq E_2 \leq \cdots , \) the Rayleigh–Ritz variational method offers a proven instrument to localize the eigenvalues \( E_k \): the \( d \) likewise ordered eigenvalues \( \tilde{E}_k, k = 0, 1, \ldots , d-1, \) of this operator \( H \) restricted to some \( d \)-dimensional trial subspace of the domain of \( H \) are upper bounds to the lowest-lying \( d \) eigenvalues of \( H \) below the onset of its essential spectrum, that is to say, \( E_k \leq \tilde{E}_k \) for all \( k = 0, 1, \ldots , d-1 \). It is straightforward to improve the accuracy [5,6] of these upper bounds by enlarging the chosen trial subspace.

If the basis of this trial subspace is given analytically in both configuration and momentum space, finding expectation values of the Hamiltonian (1) may be considerably facilitated by calculating expectation values of \( T(p) \) in momentum space and expectation values of \( V(x) \) in configuration space; we enforce this feature by an appropriate choice of our basis vectors.

In configuration space, our orthonormal basis functions \( \tilde{\phi}_{k, \ell m}(x) \) are defined in terms of the generalized-Laguerre orthogonal polynomials [7,8], \( L_k^{(\gamma)}(x) \), for the parameter \( \gamma = 2\ell + 2\beta \):

\[
\phi_{k, \ell m}(x) = \sqrt{\frac{(2\mu)^{2\ell+2\beta+1}k!}{\Gamma(2\ell+2\beta+k+1)}} |x|^{\ell+\beta-1} \exp(-\mu |x|) L_k^{(2\ell+2\beta)}(2\mu |x|) \mathcal{Y}_{\ell m}(\Omega_x),
\]

\[
L_k^{(\gamma)}(x) = \sum_{t=0}^{k} \binom{k+\gamma}{k-t} \frac{(-x)^t}{t!}, \quad \ell = 0, 1, 2, \ldots , \quad \mu \in (0, \infty), \quad \beta \in \left(\frac{1}{2}, \infty\right).
\]

By Fourier transformation, our orthonormal basis functions in momentum space, \( \tilde{\phi}_{k, \ell m}(p) \), involve the hypergeometric function \( F(u, v; w; z) \), given in terms of the gamma function [7]:

\[
\tilde{\phi}_{k, \ell m}(p) = \sqrt{\frac{(2\mu)^{2\ell+2\beta+1}k!}{\Gamma(2\ell+2\beta+k+1)}} \frac{(-1)^\ell |p|^\ell}{2^{\ell+1/2} \Gamma\left(\ell + \frac{3}{2}\right)}
\]

\[
\times \sum_{t=0}^{k} \frac{(-1)^t}{t!} \binom{k+2\ell+2\beta}{k-t} \frac{\Gamma(2\ell+\beta+t+2)(2\mu)^t}{(p^2+\mu^2)^{(2\ell+2\beta+t+2)/2}}
\]

\[
\times F\left(\frac{2\ell+\beta+t+2}{2}, -\frac{\beta+t+3}{2}; \ell + \frac{3}{2}; \frac{p^2}{p^2+\mu^2}\right) \mathcal{Y}_{\ell m}(\Omega_p),
\]

\[
F(u, v; w; z) = \frac{\Gamma(u)}{\Gamma(u) \Gamma(v)} \sum_{n=0}^{\infty} \frac{\Gamma(u+n) \Gamma(v+n) z^n}{\Gamma(w+n) n!}.
\]
### 2.2 Lower Energy Bounds

Lower limits to the spinless relativistic Hulthén problem result from the Coulomb potential

\[ V_C(r) = \frac{\kappa}{r}. \]  

In the limit \( b \downarrow 0 \), the Hulthén potential approaches from above the Coulomb-like potential

\[ V(r) = -\frac{h}{br}. \]

Accordingly, the Coulomb potential (7) constitutes a lower bound to the Hulthén potential (2) for sufficiently large Coulomb couplings, more precisely, for any coupling \( \kappa \) that satisfies

\[ \kappa \geq \frac{h}{b}. \]

Precisely the same conclusion follows, from the series expansion of the exponential \( \exp(br) \) in the denominator of \( V_H(r) \), Eq. (2): \( \exp(br) \geq 1+br \). We thus get the operator inequality

\[ V_C(r) \equiv -\frac{\kappa}{r} \leq -\frac{h}{br} \leq -\frac{h}{\exp(br)-1} \equiv V_H(r) \quad \text{for} \quad \frac{h}{b} \leq \kappa. \]

For the semirelativistic Coulomb bound states, in turn, there exist well-known lower limits:

- In a thorough mathematical analysis \([9]\) of the spinless relativistic Coulomb problem, Herbst proved\(^2\) that the Hamiltonian (1) with the Coulomb potential \( V_C \) is essentially self-adjoint for all \( \kappa \leq 1 \), that its Friedrichs extension exists up to its critical coupling

\[ \kappa_c = \frac{4}{\pi} = 1.273239 \ldots, \]

and that, for all \( \kappa < \kappa_c \), the spectrum \( \sigma(H) \) of the operator \( H \) is bounded from below:

\[ \sigma(H) \geq 2m \sqrt{1 - \left( \frac{\kappa}{\kappa_c} \right)^2} = 2m \sqrt{1 - \left( \frac{\pi \kappa}{4} \right)^2}. \]

- Martin and Roy \([10]\) sharpened this lower energy bound for coupling constants \( \kappa \leq 1 \):

\[ \sigma(H) \geq 2m \sqrt{\frac{1 + \sqrt{1 - \kappa^2}}{2}}. \]

### 3 Existence and Number of Hulthén Bound States

#### 3.1 Semirelativistic vs. Nonrelativistic Number of Bound States

Already in Sec. 2.1.1, we pointed out a trivial fact \([11]\): since the nonrelativistic free energy,

\[ T_{NR}(p) \equiv 2m + \frac{p^2}{m}, \]

obviously constitutes an upper bound to the corresponding relativistic kinetic energy \( T(p) \), a fixed spinless-Salpeter energy eigenvalue is never larger than its Schrödinger counterpart:

\[ T(p) \leq T_{NR}(p) \implies H \leq H_{NR} \implies E_k \leq E_{k, NR}, \quad k = 0, 1, 2, \ldots. \]

Hence, we are led to conclude that the total number of bound states of the spinless Salpeter equation, \( N \), will not be less than the number of Schrödinger bound states, \( N_{NR} \):

\[ N \geq N_{NR}. \]

\(^2\)We refrain from explicating here in detail the domains on which the encountered operators are defined.
3.2 Maximum Number of Nonrelativistic Hulthén Bound States

The nonrelativistic Hulthén problem as posed by the Schrödinger operator (3) admits — in contrast to the nonrelativistic Coulomb problem — merely a finite number of bound states. Bargmann [12] proved a simple upper bound to the total number of NR bound states, $N_{NR}$:

$$N_{NR} \lesssim \frac{I(I+1)}{2}, \quad I \equiv m \int_0^\infty dr \, r |V_H(r)| = \frac{\pi^2 m \hbar}{6 b^2}.$$ 

3.3 Critical Parameters of the Semirelativistic Hulthén Problem

For a semirelativistic Hamiltonian (1) with Hulthén potential (2), boundedness from below of this operator requires that the ratio of coupling strength $\hbar$ over range parameter $b$ of this potential must not be larger than a certain critical value of this quotient $\hbar/b$. That is to say, for a given value of $b$, the coupling $\hbar$ must not be larger than its critical value, whereas, for a given value of $\hbar$, the range $b$ has to be greater than its critical value. This fact may be easily demonstrated by application of the Rayleigh–Ritz variational technique (briefly recalled in Sec. 2.1.2) to this semirelativistic Hulthén problem. To follow as far as possible the analytic path, we try the simplest of the set of Laguerre basis states in Eqs. (5) or (6), defined by the choices $k = \ell = m = 0$ for its quantum numbers and $\beta = 1$ for the variational parameter $\beta$:  

$$\phi_{0,00}(x) = \sqrt{\frac{\mu^3}{\pi}} \exp(-\mu |x|), \quad \tilde{\phi}_{0,00}(p) = \sqrt{\frac{8 \mu^5}{\pi}} \frac{1}{(p^2 + \mu^2)^2}.$$ 

For each value of the variational parameter $\mu$, i.e., for all $0 < \mu < \infty$, the expectation value $\langle H \rangle$ of our Hamiltonian $H = T(p) + V_H(r)$ with respect to this trial state provides an upper bound to the ground-state energy. The expectation value $\langle T(p) \rangle$ of the kinetic energy reads

$$\langle T(p) \rangle = \frac{4}{3 \pi (m^2 - \mu^2)^{5/2}} \times \left[ \mu \sqrt{m^2 - \mu^2} (3 m^4 - 4 m^2 \mu^2 + 4 \mu^4) + 3 m^4 (m^2 - 2 \mu^2) \sec^{-1}\left(\frac{m}{\mu}\right) \right];$$

the expectation value $\langle V_H(r) \rangle$ of the Hulthén potential makes use of a polygamma function,  

$$\langle V_H(r) \rangle = \frac{4 \hbar \mu^3}{b^2} \psi^{(2)}\left(1 + \frac{2 \mu}{b}\right),$$

defined [7], at some order $n$, as $(n+1)$-th derivative of the logarithm of the gamma function,  

$$\psi^{(n)}(z) \equiv \frac{d^{n+1}}{dz^{n+1}} \ln \Gamma(z), \quad \Re z > 0.$$ 

We are interested in the limit $\mu \to \infty$. Expanding $\langle H \rangle = \langle T(p) \rangle + \langle V_H(r) \rangle$ for large $\mu$ yields

$$\langle H \rangle = \left(\frac{16}{3 \pi} - \frac{\hbar}{b}\right) \mu + \frac{h}{2} + \frac{1}{\mu} \left(\frac{16 m^2}{3 \pi} - \frac{\hbar b}{8}\right) + O\left(\frac{\log \mu}{\mu^3}\right).$$

For negative coefficients of $\mu$, this expectation value decreases, for rising $\mu$, without bound:

$$\langle H \rangle \underset{\mu \to \infty}{\longrightarrow} -\infty \quad \text{for} \quad \frac{\hbar}{b} > \frac{16}{3 \pi}. $$

Thus, boundedness from below of the operator $T(p) + V_H(r)$ requires the ratio $\hbar/b$ to satisfy

$$\frac{\hbar}{b} \leq \frac{16}{3 \pi} = 1.69765 \ldots .$$

(8)
4 Applications

After all the preparatory considerations in Secs. 2 and 3, it is now rather straightforward to apply the insights gained thereby to the semirelativistic Hulthén Hamiltonian under study. For ease of comparison, we would like to do this exercise, of course, for a choice of numerical values of the bound-state components’ mass $m$ and the Hulthén potential parameters $b$ and $h$ that has been also adopted in, at least, one previous investigation of the present problem. We are aware of merely two publications [13,14] discussing spinless Salpeter equations with either the original [14] form (2) of Hulthén’s potential or a properly generalized [13] variant thereof. Both of these works rely on various simplifying modifications in order to arrive at a Schrödinger-like implicit eigenvalue equation that is assumed to represent some reasonable approximation to the spinless Salpeter equation but allows for obtaining analytic solutions. Unfortunately, only Ref. [14] illustrates its resulting expressions by explicit examples; from Table 1 therein, for use as parameter values in what follows, we read off, in arbitrary units,\(^3\)

\[
m = 1, \quad b = 0.15, \quad h = 0.11. \tag{9}
\]

The reliability of these approximate solutions may be immediately checked by our findings:

- First of all, the parameter values (9) satisfy the inequality (8) imposed by demanding the spinless-Salpeter Hamiltonian with Hulthén potential to be bounded from below:

\[
\frac{h}{b} = 0.73 < \frac{16}{3\pi} = 1.69765 \ldots.
\]

Thus, for the setting (9) one may expect, on good grounds, to find bound states at all.

- According to the inequality limiting the quantum number $n$ of $s$-wave bound states in Sec. 1.1, the choice (9) allows for just two nonrelativistic $\ell = 0$ Hulthén bound states:

\[
n \leq 2 < \frac{\sqrt{m}h}{b} = 2.211 \ldots.
\]

- The Bargmann bound of Sec. 3.2 shows that the nonrelativistic Hulthén problem can accommodate at most 36 bound states since, upon use of the values (9), it returns, for the total number of bound states, $N_{NR} < 36.357 \ldots$, which is, potentially, still rather far from optimum. Improvements of the Bargmann bound exist copiously but usually lead to expressions that are much harder to deal with than Bargmann’s simple result.

- In order to maximize our lower bound to the spectrum of the semirelativistic Hulthén problem resulting from the observation made in Sec. 2.2 that for $\kappa \geq h/b$ the Hulthén potential is bounded from below by the Coulomb potential, we present this bound for the minimum possible value of the Coulomb coupling that still guarantees the desired operator inequality, viz., $\kappa = h/b = 0.73$, for which both Coulomb lower limits apply. This yields, for the ground-state energy eigenvalue $E_0$ and the corresponding binding energy $B_0 \equiv E_0 - 2m$, from the Herbst lower bound [9] $E_0 \geq 1.635$ and $B_0 \geq -0.365$, respectively, and from its Martin–Roy counterpart [10] $E_0 \geq 1.833$ and $B_0 \geq -0.167$, respectively. Surprisingly, one entry in Table 1 of Ref. [14] slightly violates this result.

\(^3\)Repeated inspection of the definition of the Hulthén potential provided by Eq. (6) of Ref. [14] prompts us to take the strange minus sign in the caption of Table 1 of Ref. [14] not too literally; moreover, we do not wonder about the meaning of the parameter $h = 1$, mentioned in the caption of this table but nowhere else.
Table 1 summarizes upper limits on the lowest-lying bound-state levels of the spinless relativistic Hulthén problem found along the lines sketched in Secs. 2.1.1 and 2.1.2 by variational approach or standard numerical solution of the Schrödinger equation [15].

Table 1: Upper limits to the binding energy for the lowest-lying bound states of the spinless Salpeter equation with Hulthén’s potential, for the parameter values of Ref. [14]: the trivial Schrödinger bounds \( E_{NR} \) of Subsec. 2.1.1 and the Laguerre bounds \( E \) of Subsec. 2.1.2. Any bound state is identified by its radial quantum number, \( n_r \), and orbital angular momentum quantum number, \( \ell \). Merely for illustration, we keep the dimension \( d \) of the variational trial space \( D_d \) and both variational parameters \( \mu \) and \( \beta \) fixed to the values \( d = 25, \mu = 1, \beta = 1 \).

| Bound state | Spinless Salpeter equation | Schrödinger equation |
|-------------|---------------------------|---------------------|
| \( n_r \)   | \( \ell \)                | \( E(n_r, \ell) \)   | \( E_{NR}(n_r, \ell) \) |
| 0           | 0                         | -0.10577            | -0.0850694               |
| 1           | 0                         | -0.0022398          | -0.001                  |

In order to approximate the spinless Salpeter equation by an equation easier to treat, assuming the bound-state constituents to be sufficiently heavy both of the two earlier investigations mentioned above [13,14] prefer to expand that cumbersome relativistic kinetic energy \( T(p) \) in our semirelativistic Hamiltonian \( H_{\text{nonrel}} \) nonrelativistically and, by retaining terms up to order \( p^4/m^4 \), to get some pseudo spinless-Salpeter Hamiltonian

\[
H_p \equiv 2m + \frac{p^2}{m} - \frac{p^4}{4m^3} + V(x)
\]

that is obviously unbounded from below [16], so that the term \( p^4/(4m^3) \) can be taken into account only perturbatively. Anyway, we may check whether such nonrelativistic expansion is justifiable at all, by assuming that the lowest bound state emerging from our variational procedure provides a satisfactory description of the ground state, and by inspecting, for this state, the expectation value of the next-to-lowest term in \( T(p) \):

\[
\left\langle \frac{p^2}{m^2} \right\rangle \approx 0.26 ;
\]

thus, the system governed by the parameters (9) can be viewed as not too relativistic.

5 Summary and Concluding Remarks

The spinless Salpeter equation forms the penultimate stage in the nonrelativistic reduction of the homogeneous Bethe–Salpeter equation (more details than those presented above can be found in, e.g., Refs. [17–21]). Perhaps because of the paramount importance of its origin or because of the challenge represented by the nonlocal nature of the Hamiltonian operator controlling the bound states under study, we can witness, from time to time, a considerable increase in interest in this equation of motion, which, in turn, motivated the above analysis.
aiming at the discussion of a couple of rigorous constraints on the spectrum of bound states to be expected if in our spinless Salpeter equation all interactions between the bound-state constituents are subsumed by a Hulthén potential. Needless to say, singular potentials such as that introduced by Hulthén pose obstacles which differ from those to be faced if studying non-singular interactions such as the Woods–Saxon potential [16]. Anyway, a solid starting point for studying semirelativistic systems is the corresponding nonrelativistic case [22,23].

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