The necessary and sufficient condition for solvability of a partial integral equation

ESHKABILOV Yu.Kh.
National University of Uzbekistan
e-mail: yusup62@rambler.ru

Abstract
Let \( T_1 : L_2(\Omega^2) \rightarrow L_2(\Omega^2) \) be a partial integral operator with the kernel from \( C(\Omega^3) \) where \( \Omega = [a, b]^{\nu} \). In this paper we investigate solvability of a partial integral equation \( f - \kappa T_1 f = g_0 \) in the space \( L_2(\Omega^2) \) in the case when \( \kappa \) is a characteristic number. We proved the theorem describing the necessary and sufficient condition for solvability of the partial integral equation \( f - \kappa T_1 f = g_0 \).

Key words: partial integral operator, partial integral equation, the Fredholm integral equation, \( L^0 \)-valued internal product.

2000 MSC Subject Classification: 45A05, 45B05, 45C05, 45P05

In the models of solid state physics [1] and also in the lattice field theory [2], there appear so called discrete Schrodinger operators, which are lattice analogues of usual Schrodinger operators in a continuous space. The study of spectra of lattice Hamiltonians (that is discrete Schrodinger operators) is an important matter of mathematical physics. Nevertheless, on studying spectral properties of discrete Schrodinger operators three appear partial integral equations in a Hilbert space of multi-variable functions [1,3]. Therefore, on the investigation of spectra of Hamiltonians considered on a lattice, the study of a solvability problem for partial integral equations in \( L_2 \) is essential (and even interesting from the point of view of functional analysis).

A question on the existence of a solution of partial integral equation (PIE) for functions of two variables was considered in [4-8] and other works. In the work by author [9], the PIE \( f - \kappa T_1 f = g_0 \) was studied in the space \( L_2(\Omega^2) \), where \( \Omega = [a, b]^{\nu} \), for a partial integral operator (PIO) \( T_1 : L_2(\Omega^2) \rightarrow L_2(\Omega^2) \) with the kernel \( k(x, s, y) \) being a three variables continuous function on \( \Omega^3 \). The concept of determinant for the PIE as a continuous function on \( \Omega \) and concepts of regular number, singular number, characteristic number and essential number for a PIE are given. Theorems on solvability of the PIE are proved in the case when \( \kappa \) is the regular and essential number [9]. In this paper we study solvability of the PIE \( f - \kappa T_1 f = g_0 \) when \( \kappa \) is the characteristic number, i.e. the paper continues the work by author [9].

Let \( L^0 = L^0(\Omega) \) be a space of classes of complex measurable functions \( b = b(y) \) on \( \Omega \). We define in the space \( L^0(\Omega) \) the norm of an element \( b \in L^0(\Omega) \) by
the equality \( \|b\| = \sqrt{\int |b(t)|^2 dt} \).

We denote by \( L_{2,0}(\Omega^2) \) the totality of classes of complex measurable functions \( f(x,y) \) on \( \Omega \times \Omega \) satisfying the condition: \( \int |f(x,y)|^2 dx \) exists for almost all \( y \in \Omega \). It is easy to note that \( L_{2,0}(\Omega^2) \) is a linear space over \( \mathbb{C} \) and \( L_2(\Omega^2) \subset L_{2,0}(\Omega^2) \). For each \( b(y) \in L^0 \) and \( f(x,y) \in L_{2,0}(\Omega^2) \), we define the function \( b \circ f \) by the formula \( (b \circ f)(x,y) = b(y)f(x,y) \). Then for any \( b \in L^0 \) we have \( b \circ f \in L_{2,0}(\Omega^2) \), where \( f \in L_{2,0}(\Omega^2) \). For any \( f,g \in L_{2,0}(\Omega^2) \), the integral \( \int f(x,t)g(x,t)dx \) exists at almost all \( t \in \Omega \) and \( \varphi(t) = \int f(x,t)g(x,t)dx \in L^0 \).

Let \( \nabla \) be the Boolean algebra of idempotents in \( L^0 \). A system \( \{ f_1, f_2, \ldots, f_n \} \subset L_{2,0}(\Omega^2) \) is called \( \nabla \)-linearly independent, if for all \( \pi \in \nabla \) and \( b_1(y), b_2(y), \ldots, b_n(y) \in L^0 \) from \( \sum_{k=1}^{n} \pi \circ (b_k \circ f_k) = \theta \) it follows \( \pi \cdot b_1 = \pi \cdot b_2 = \cdots = \pi \cdot b_n = \theta \) [10,11].

Consider the mapping \( \langle \cdot, \cdot \rangle : L_{2,0}(\Omega^2) \times L_{2,0}(\Omega^2) \rightarrow L^0 \) acting by the rule

\[
\langle f, g \rangle = \int_{\Omega} f(s,y)g(s,y)ds, \quad f, g \in L_{2,0}(\Omega^2).
\]

For every \( b \in L^0 \), we have \( \langle b \circ f, g \rangle = b \cdot \langle f, g \rangle \), where \( f, g \in L_{2,0}(\Omega^2) \), i.e. the mapping \( \langle \cdot, \cdot \rangle \) satisfies the condition of \( L^0 \)-valued internal product [11].

In the space \( \mathcal{H} = L_{2,0}(\Omega^2) \), we consider a partial integral operator (PIO) \( S \) defined by

\[
Sf = \int_{\Omega} q(x,s,y)f(s,y)ds, \quad f \in \mathcal{H}
\]

where \( q(x,s,y) \in L_2(\Omega^2) \). The function \( q(x,s,y) \) is called kernel of the PIO \( S_1 \).

A kernel \( q(s,x,y) \) corresponds to the adjoint operator \( S_1^* \), i.e.

\[
S_1^* f = \int_{\Omega} \overline{q(s,x,y)}f(s,y)ds, \quad f \in \mathcal{H}.
\]

Consider a family of operators \( \{ S_\alpha \}_{\alpha \in \Omega} \) in \( L_2(\Omega) \) associated with \( S_1 \) by the following formula

\[
S_\alpha \varphi = \int_{\Omega} q(x,s,\alpha)\varphi(s)ds, \quad \varphi \in L_2(\Omega),
\]

where \( q(x,s,y) \) is the kernel of \( S \).

Further, if the set of integrability of an integral is absent, we mean integrability by the set \( \Omega \).

Now we consider the equation

\[
f - \kappa Sf = g_0, \quad (1)
\]

in the space \( \mathcal{H} \) where \( f \) is an unknown function from \( \mathcal{H} \), \( g_0 \in \mathcal{H} \) is a given function, \( \kappa \in \mathbb{C} \) is a parameter of the equation.
For each \( n \in \mathbb{N} \), we define the measurable function

\[
\Pi^{(n)} = \Pi^{(n)}(x_1, \ldots, x_n, s_1, \ldots, s_n, \alpha)
\]
on \( \Omega^n \times \Omega^n \times \Omega \) by means of an \( n \)-ordinary determinant by the equality

\[
\Pi^{(n)}(x_1, \ldots, x_n, s_1, \ldots, s_n, \alpha) = \begin{vmatrix}
qu(x_1, s_1, \alpha) & \ldots & q(x_1, s_n, \alpha) \\
\vdots & \ddots & \vdots \\
q(x_n, s_1, \alpha) & \ldots & q(x_n, s_n, \alpha)
\end{vmatrix}.
\]

Now, for every \( \varkappa \in \mathbb{C} \) we "formally" define functions \( D_1(y) = D_1(y; \varkappa) \) on \( \Omega \) and \( M_1(x, s, y) = M_1(x, s, y; \varkappa) \) on \( \Omega^3 \) by means of the sum of measurable functional series composed from sequences of measurable functions \( d_n(y) \) on \( \Omega \) and \( q_n(x, s, y) \) on \( \Omega^3 \), respectively, by the following rules

\[
D_1(\alpha) = 1 + \sum_{n \in \mathbb{N}} (-\varkappa)^n \frac{d_n(\alpha)}{n!}, \quad \alpha \in \Omega \quad (a)
\]

and

\[
M_1(x, s, \alpha) = q(x, s, \alpha) \sum_{n \in \mathbb{N}} (-\varkappa)^n \frac{q_n(x, s, \alpha)}{n!}, \quad (x, s, \alpha) \in \Omega^3 \quad (b)
\]

where

\[
d_k(\alpha) = \int \ldots \int \Pi^{(k)}(\xi_1, \ldots, \xi_k, \xi_1, \ldots, \xi_k, \alpha) d\mu(\xi_1) \ldots d\mu(\xi_k),
\]

\[
q_k(x, s, \alpha) = \int \ldots \int \Pi^{(k+1)}(x, \xi_1, \ldots, \xi_k, s, \xi_1, \ldots, \xi_k, \alpha) d\mu(\xi_1) \ldots d\mu(\xi_k).
\]

**Lemma 1.** For each \( \varkappa \in \mathbb{C} \) the functions \( D_1(y) = D_1(y; \varkappa) \) (a) and \( M_1(x, s, y) = M_1(x, s, y; \varkappa) \) (b) are measurable on \( \Omega \) and \( \Omega^3 \), respectively. Moreover, for almost all \( \alpha \in \Omega \), there exists the integral \( \int \int |M_1(x, s, \alpha)|^2 \, dx \, ds \).

**Proof.** It is clear that the operator \( S \) is compact for almost all \( \alpha \in \Omega \). Let \( \varkappa \in \mathbb{C} \) be an arbitrary fixed number. We respectively denote as \( \Delta_\alpha(\varkappa) \) and \( M_\alpha(x, s; \varkappa) \) the determinant and the Fredholm minor of the operator \( I - \varkappa S_\alpha \) for \( \alpha \in \Omega \) where \( I \) is the identity operator in \( L_2(\Omega) \). Let \( \varphi_n(y) \) and \( \psi_n(x, s, y) \) be the partial sums of the functional series (a) and (b), respectively. We have the sequences of measurable functions \( \varphi_n(y) \) on \( \Omega \) and \( \psi_n(x, s, y) \) on \( \Omega^3 \) such that \( \lim_{n \to \infty} \varphi_n(y) = \Delta_\alpha^{(1)}(\varkappa) = D_1(y; \varkappa) \) for almost all \( y \in \Omega \) and \( \lim_{n \to \infty} \psi_n(x, s, y) = M_\alpha^{(1)}(x, s; \varkappa) = M_1(x, s, y; \varkappa) \) for all \( x, s, \alpha \in \Omega \) and for almost all \( y \in \Omega \). Therefore the function \( D_1(y) = D_1(y; \varkappa) \) and the function \( M_1(x, s, y) = M_1(x, s, y; \varkappa) \) are measurable on \( \Omega \) and \( \Omega^3 \), respectively. It is known that if the kernel \( h(x, s) \) of the integral operator \( A \varphi = \int h(x, s) \varphi(s) \, ds, \varphi \in L_2(\Omega) \) is an element of the space \( L_2(\Omega^2) \), then the minor \( M(x, s; \varkappa) \) of the operator \( I - \varkappa A \) is also an element of the space \( L_2(\Omega^2) \). Hence we have

\[
\int \int |M_1(x, s, \alpha)|^2 \, dx \, ds < \infty \quad \text{for almost all} \quad \alpha \in \Omega.
\]
The measurable functions $D_1(y) = D_1(y; \mathcal{X})$ and $M_1(x, s, y) = M_1(x, s, y; \mathcal{X})$ are respectively called the determinant and the minor of the operator $E - \mathcal{X}S$, $\mathcal{X} \in \mathbb{C}$, where $I$ is the identity operator in $L_{2,0}(\Omega^2)$.

**Lemma 2.** Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with the kernel $q \in L_2(\Omega^3)$. If the homogeneous equation $\varphi - \mathcal{X}S_0 \varphi = 0$, $\mathcal{X} \in \mathbb{C}$, has only the trivial solution in $L_2(\Omega)$ for almost all $\mathcal{X} \in \Omega'$, then PIE (1) is solvable in the space $L_{2,0}(\Omega^2)$ for every $g_0 \in L_{2,0}(\Omega^2)$.

**Proof.** Let $x \in \mathbb{C}$, $g_0(x, y)$ be an arbitrary function from the space $L_{2,0}(\Omega^2)$. Let the homogeneous equation $\varphi - \mathcal{X}S_0 \varphi = 0$ has only a trivial solution in the space $L_2(\Omega)$ for almost all $\mathcal{X} \in \Omega'$. Then $D_1(\mathcal{X}) = D_1(\mathcal{X}; \mathcal{X}) \neq 0$ for almost all $\mathcal{X} \in \Omega$ and the equation $\varphi(x) - \mathcal{X}(S_0 \varphi)(x) = h_\mathcal{X}(x)$ has a solution $\varphi_\mathcal{X}(x) \in L_2(\Omega)$ for almost all $\mathcal{X} \in \Omega'$ where $h_\mathcal{X}(x) = g_0(x, \mathcal{X}) \in L_2(\Omega)$. Moreover, the solution $\varphi_\mathcal{X}(x)$ has the form

$$
\varphi_\mathcal{X}(x) = h_\mathcal{X}(x) + \mathcal{X} \int \frac{M_1(x, s, \mathcal{X}; \mathcal{X})}{D_1(\mathcal{X}; \mathcal{X})} h_\mathcal{X}(s) ds.
$$

We have

$$
\int \int \left| \frac{M_1(x, s, \mathcal{X}; \mathcal{X})}{D_1(\mathcal{X}; \mathcal{X})} \right| dxds < \infty \quad \text{for almost all} \quad \mathcal{X} \in \Omega.
$$

It means that we can define a PIO $W = W(\mathcal{X}) : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ with the kernel $\frac{M_1(x, s, \mathcal{X}; \mathcal{X})}{D_1(\mathcal{X}; \mathcal{X})}$. Therefore we have $f_0(x, y) = g_0(x, y) + \mathcal{X}(Wg_0)(x, y) \in L_{2,0}(\Omega^2)$ and $\varphi_\mathcal{X}(x) = f_0(x, \mathcal{X})$ for all $x \in \Omega$ and for almost all $\mathcal{X} \in \Omega$. So the function $f_0(x, y)$ is a solution of the equation (1).

The following two propositions are proved analogously to Propositions 1 and 2 from [].

**Proposition 1.** Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with the kernel $q \in L_2(\Omega^3)$. Then the following two conditions are equivalent:

(i) a number $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $S$;

(ii) a number $\lambda \in \mathbb{C}$ is an eigenvalue of operators $\{S_\mathcal{X}\}_{\mathcal{X} \in \Omega_0}$ where $\Omega_0$ is a subset of $\Omega$ such that $\mu(\Omega_0) > 0$.

**Proposition 2.** If $\lambda \in \mathbb{C}$ is an eigenvalue of the PIO $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ with the kernel $q(x, s, y) \in L_2(\Omega^3)$, then the number $\lambda$ is an eigenvalue of the operator $S^*$.

**Theorem 1.** Let $S : L_{2,0}(\Omega^2) \to L_{2,0}(\Omega^2)$ be a PIO with the kernel $q(x, s, y) \in L_2(\Omega^3)$. Then every eigenvalue of the PIO $S$ corresponds only to a finite number of $\nabla$-linearly independent eigenfunctions.

**Proof.** Let $\lambda \in \mathbb{C}$ be an eigenvalue of the PIO $S$ and

$$
f_1, f_2, \ldots, f_m
$$

(2)
be some $\nabla$-linearly independent eigenfunctions, i.e.
\[
\lambda f_j(x,y) = (S f_j)(x,y), \quad j = 1, 2, \ldots, m. \tag{3}
\]
Since any linear combination of the eigenfunctions (2) of the operator $S$ with coefficients from $L^0$ is also an eigenfunction, we can apply to the functions (2) the process of $L^0$-orthogonalization [12]. Thus, we can assume that the functions (4) are mutually orthogonal and normed in the sense of $M^0$-valued internal products, i.e.
\[
\langle f_i, f_j \rangle = 0, \quad i \neq j \quad \text{and} \quad \langle f_i, f_i \rangle = 1.
\]
Therefore we can rewrite (3) in the following form
\[
\lambda \cdot f_j(x,y) = (S^* f_j)(x,y) = \int q(x,s,y) \cdot f_j(s,y) ds.
\]
From here, it is easy to see that at almost all $x \in \Omega$ the left side of this equality is a $L^0$-valued Fourier coefficient of the function $q(x,s,y)$ as and it is a function of $(s,y)$ with respect to the orthogonal normed system (2). By the Bessel inequality [12], one can write
\[
|\lambda|^2 \sum_{j=1}^m |f_j(x,y)|^2 \leq \int \int |q(s,x,y)|^2 ds \quad \text{for almost all} \quad x \in \Omega.
\]
If we integrate both parts of this inequality by $x$ and $y$, we obtain
\[
m \leq |\lambda|^{-2} \int \int \int |q(s,x,y)|^2 dxdsdy < \infty.
\]
Hence, the number of $\nabla$-linearly independent functions, corresponding to the eigenvalue $\lambda$, is finite.

Let $S$ be a PIO with the kernel $q(x,s,y) \in L^2(\Omega^2)$. A number $\kappa_0 \in \mathbb{C}$ is called a characteristic value of the PIE $f - \kappa_0 S f = g_0$ if the homogeneous equation $f - \kappa_0 S f = 0$ has a non-trivial solution. From here, it is clear that any characteristic value $\kappa_0$ of the PIE $f - \kappa S f = g_0$ is non-zero.

**Corollary 1.** Let $S : L^2_{2,0}(\Omega^2) \to L^2_{2,0}(\Omega^2)$ be a PIO with the kernel $q(x,s,y) \in L^2(\Omega^3)$. Then any characteristic value of the PIE $f - \kappa S f = g_0$ corresponds only to finite number of $\nabla$-linearly independent eigenfunctions.

**Theorem 2.** Let $\kappa$ be a characteristic number of the PIE (1). Then the homogeneous PIE
\[
f - \kappa S f = 0 \tag{4}
\]
and the adjoint homogeneous PIE
\[
f - \kappa S^* f = 0 \tag{4'}
\]
have the same number of $\nabla$-linearly independent solutions.
Proof. Let $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ be $\nabla$-linearly independent solutions of the the homogeneous equations (4) and (4'), respectively. Assume that $m < n$. We can suppose that $f_1, \ldots, f_m$ and $g_1, \ldots, g_n$ are orthonormal systems in the sense of $L^0$-valued internal product.

Define the function

$$p(x, s, y) = q(x, s, y) - \sum_{j=1}^{m} f_j(s, y) g_j(x, y).$$

We have $p(x, s, y) \in L^2(\Omega^3)$ since $f_j, g_k \in L^2(\Omega^2)$. Consider two homogeneous PIE:

$$f - \varpi W f = 0 \quad (5)$$

and

$$f - \varpi W^* f = 0 \quad (5')$$

where $W$ is the PIO with the kernel $p(x, s, y)$.

Let $h(x, y)$ be a solution of the equation (5). Then we have

$$\langle h, g_j \rangle = \langle \varpi W h, g_j \rangle = \langle h, \varpi S^* g_j \rangle - \varpi \langle h, f_j \rangle = \langle h, g_j \rangle - \varpi \langle h, f_j \rangle, \quad j = 1, 2, \ldots, m.$$ 

Hence, by virtue of $\varpi \neq 0$,

$$\langle h, f_j \rangle = 0, \quad j = 1, 2, \ldots, m. \quad (6)$$

Thus, any solution of the equation (5) satisfies the conditions (6). But by virtue of this conditions, one can rewrite the equation (5) in the form $f - \varpi S f = 0$, i.e. any solution of the equation (5) satisfies the equation (4), too. We obtain that a solution $h(x, y)$ of the equation (5) is to be in the form

$$h(x, y) = \sum_{j=1}^{m} \left( b_j \circ f_j \right)(x, y), \quad b_j \in L^0, \quad j = 1, 2, \ldots, m.$$ 

But we have $0 = \langle h, f_k \rangle = \sum_{j=1}^{m} \langle b_j \circ f_j, f_k \rangle = \sum_{j=1}^{m} b_j \langle f_j, f_k \rangle = b_k, k = 1, 2, \ldots, m.$

Thus, we have $h(x, y) = \theta$, i.e. the homogeneous PIE (5) has only the trivial solution. We show that the adjoint equation (5') has non-trivial solutions. If we substitute $g(x, y) = g_k(x, y)$, where $k > m$, in the equation (7') then we obtain $g_k = \varpi^* W^* g_k$. Thus, we obtain the contradiction to Proposition 2: the equation (5) has only the trivial solution, but the adjoint equation (5') has a non-trivial solution. Hence the case $m < n$ is impossible. One can prove similarly that the case $m > n$ is also impossible and we obtain that $m = n.$ \hfill \Box

**Theorem 3.** Let $\varpi_0$ be a characteristic number of the PIE (1). Then:

a) the homogeneous equation $f - \varpi_0 S f = 0$ has a non-trivial solution, moreover the set of all solutions of the homogeneous equation is an infinite dimensional subspace of $\mathcal{H}$;
b) PIE (1) is solvable if and only if a given function \( g_0 \) satisfies the condition
\[
\langle g_0, g \rangle = 0,
\]
where \( g \in \mathcal{H} \) is an arbitrary solution of the adjoint homogeneous equation \( f - \lambda_0 S^* f = 0 \).

**Proof.** The proof of the property a) follows immediately from Propositions 1 and 3. We prove the property b).

i) ("if-part") Let \( \lambda_0 \) be a characteristic number of the PIE (1) and \( f_0 \in \mathcal{H} \) be a solution of the PIE (1) and \( g \in \mathcal{H} \) be an arbitrary solution of the adjoint homogeneous equation \( f - \lambda_0 S^* f = 0 \). Then
\[
\langle f_0, g \rangle = \langle g_0 + \lambda_0 S f_0, g \rangle = \langle g_0, g \rangle + \langle \lambda_0 S f_0, g \rangle = \langle g_0, g \rangle + \langle f_0, \lambda_0 S^* g \rangle = \langle g_0, g \rangle + \langle f_0, g \rangle.
\]
Therefore we have \( \langle g_0, g \rangle = 0 \).

ii) ("only if"-part) Let \( \lambda_0 \) be a characteristic number of the PIE (1). Suppose that \( g_0 \) satisfies the condition (I), i.e. \( \langle g_0, g \rangle = 0 \) for every solution \( g \in \mathcal{H} \) of the equation \( f - \lambda_0 S^* f = 0 \).

Consider the function \( p(x, s, y) \in L_2(\Omega^3) \) given by the equality
\[
p(x, s, y) = q(x, s, y) - \sum_{j=1}^{m} f_j(s, y) g_j(x, y),
\]
where \( f_1, f_2, \ldots, f_m \) and \( g_1, g_2, \ldots, g_m \) are orthonormal systems of the solutions of the equations of (4) and (6'), respectively, in the sense of \( L^0 \)-valued internal product. Then for almost all \( \alpha \in \Omega \) the homogeneous Fredholm equation \( \varphi - \lambda_0 W_\alpha \varphi = 0 \) has in \( L_2(\Omega) \) only the trivial solution [10] where \( W_\alpha \) is an integral operator in \( L_2(\Omega) \) with the kernel \( p(x, s, \alpha) \). Hence by Lemma 2 the PIE \( f - \lambda_0 W f = g_0 \) has the solution \( f_0 \in \mathcal{H} \) of the form
\[
f_0 = g_0(x, y) + \lambda_0 S f_0(x, y) - \lambda_0 \sum_{j=1}^{m} \langle f_0, f_j \rangle \cdot g_j(x, y).
\]
Therefore, we obtain that
\[
\langle f_0, g_k \rangle = \langle g_0, g_k \rangle + \langle \lambda_0 S f_0, g_k \rangle - \lambda_0 \sum_{j=1}^{m} \langle f_0, f_j \rangle \cdot \langle \lambda_0 g_j, g_k \rangle = \langle f_0, \lambda_0 S^* g_k \rangle - \lambda_0 \langle f_0, f_k \rangle = \langle f_0, g_k \rangle - \lambda_0 \langle f_0, f_k \rangle,
\]
i.e. \( \langle f_0, f_k \rangle = 0 \) since \( \lambda_0 \neq 0 \). Thus, the solution \( f_0 \) of the equation \( f - \lambda_0 W f = g_0 \) has the form \( f_0 = g_0 + \lambda_0 S f_0 \) and hence, the function \( f_0 \) is also a solution of the PIE (1) at \( \lambda = \lambda_0 \). \( \square \)
If there exists a number $C$ such that

$$|b(t)| \leq C \text{ for almost all } t \in \Omega,$$

then the PIO $S$ is a bounded operator on the space $L_2(\Omega^2)$, i.e $Sf \in L_2(\Omega^2)$, $\forall f \in L_2(\Omega^2) \subset L_{2,0}(\Omega^2)$ and $\|Sf\|_{L_2(\Omega^2)} \leq C_0\|f\|_{L_2(\Omega^2)}$ for all $f \in L_2(\Omega^2)$ where $C_0$ is a positive number, $b(t) = \int \int |q(x, s, t)|^2 dx ds$.

Let $k(x, s, y) \in C(\Omega^3)$. Then the subspace $L_2(\Omega^2)$ is invariant for the PIO $T_1 : (T_1f)(x, y) = \int k(x, s, y)f(s, y)ds$. Therefore it is possible to study solvability for the PIE

$$f - \kappa T_1f = g_0$$

(7)

and it is uniquely defined by its kernel $k(x, s, y)$, where

$$b(t) = \int \int \int |k(x, s, t)|^2 dx ds.$$

in the space $L_2(\Omega^2)$ where $f$ is an unknown function from $L_2(\Omega^2)$, $g_0 \in L_2(\Omega^2)$ is given (known) functionm and $\kappa \in \mathbb{C}$ is a parameter of the equation.

Let $\chi_{\kappa}$ be a set of characteristic numbers for the PIE (7) (see [9]). Definition of characteristic number [9] and Theorem 3 we obtained imply

**Theorem 4.** Let $\kappa_0 \in \chi_{\kappa}$. Then

a) the homogeneous equation $f - \kappa T_1f = \theta$ has a non-trivial solution, moreover the set of all solutions of the homogeneous equations is an infinite dimensional subspace of $L_2(\Omega^2)$;

b) PIE (7) is solvable if and only if a given function $g_0$ satisfies the condition

$$\int g_0(s, y)g(s, t)ds = 0 \text{ for almost all } t \in \Omega$$

(III)

where $g \in L_2(\Omega^2)$ is an arbitrary solution of the adjoint homogeneous equation $f - \kappa_0 T_1^*f = \theta$.

**References**

[1] Mogilner A.I. Hamiltonians in solid-state physics as multiparticle discrete Schrodinger operators: problems and pesults,- Adv. in Soviet Math., 1991, v.5, p.139-194.

[2] Milnos R.A. Sov. Sci.C. Math.Phys.,1988, v.7. p. 235-280.

[3] S.N. Lakaev, M.E. Muminov Essential and discrete spectra of the three-particle Schrodinger operator on a lattice,- Theor. and math. physics, 2003. V.135, No3.pp.478-503.

[4] Abdus Salam. Fredholm solutions of partial integral equation, – Proc. Cambridge Philos. Soc. **49**, 1952, pp.213-217.
[5] Fenyő S. Beitrag zur Theorie der Linearen Integral Gleichungen, – Publs. mat., 1955, No. 1,2, ss.98-103.

[6] Lichtarnikov L.M. On the spectrum of the one family of linear integral equation with two parameters, – Diff. equations, 1975, Vol.XI, No.6, pp.1108-1117 (in Russian)

[7] Lichtarnikov L.M., Vitova L.Z. On solvability of a linear integral equation with partial integrals, – Ukr. Math. J., 1976, Vol.28, No.1, pp.83-87 (in Russian)

[8] Chulfa E. Fredholm solutions of partial integral equations, – Doklady Akad. Nauk Resp. Uzb., 1997, No.7, pp.9-13

[9] Eshkabilov Yu.Kh. On solvability of a partial integral equation in the space \(L_2(\Omega \times \Omega)\)-MFAT,2007.

[10] Smirnov V.I. Cours of Extra Mathematica, Vol.4, Part I. – Moscow, Nauka, 1974 (in Russian)

[11] Kusraev A.G. Vector duality and its applications. – Novosibirsk, Nauka, 1985 (in Russian)

[12] Ganiev I.G., Kudaybergenov K.K. A finite dimensional module over a ring of measurable functions. – Uzb. Math. Journ., 2004, No.4, p.3-9.

[13] Kusraev A.G. Dominated Operators. – Moscow, Nauka, 2003 (in Russian)

[14] Kudaybergenov K.K. \(\nabla\)-Fredholm operators in Banach-Kantorovich spaces, – MFAT,2006 v.12, No:3, p.234-242.