Heavy Quarkonia and $Bc$ Mesons in the Cornell Potential plus Harmonic Potential in the N-dimensional Radial Schrodinger Equation

M. Abu-Shady

Department of Applied Mathematics, Faculty of Science, Menoufia University, Egypt

Abstract

Heavy Quarkonia and $b\bar{c}$ Mesons masses are calculated within the framework of N-dimensional radial Schrodinger Equation. The Cornell potential is extended by including the harmonic term potential. The energy eigenvalues and the wave functions are calculated in N-dimensional using the Nikiforov-Uvarov Method. The energy eigenvalues in 3-dimensional can be obtained from the N-dimensional form. The mass of spectra of charmonium, bottomonium, and $b\bar{c}$ mesons are calculated. A comparison with other theoretical approaches is discussed. The obtained results are in good agreement with experimental data.

Keywords: Schrodinger equation, Nikiforov-Uvarov method, Meson masses spectra
I. INTRODUCTION

The study of heavy quarkonium systems provides good understating for the quantitative test of quantum chromodynamic (QCD) theory and standard model [1]. There are many techniques that provide us the quantitative and qualitative features about the strong interactions such as lattice QCD, QCD sum rules, and other theoretical methods [2].

The Schrodinger equation (SE) is play an important role in the nuclear and subnuclear physics, in particular, in studying the properties of constituents particles and dynamics of their interactions. The investigation of quarkonium systems is widely studied by using the Schrodinger equation. It is know that the exact solutions of the Schrodinger equation are very difficult when the centrifugal potential is included [3]. Thus, there are some approximation methods that develop for dealing with these problems. In addition, for solving the Schrodinger equation, we should define the potential for the system under study. The methods are widely used for providing the analytic solutions of the Schrodinger equation with using different potentials are the asymptotic iteration method [4] and the Nikiforov-Uvarov Method [5]. In these methods, the authors use an interaction potential such as the Cornell potential as in Refs. [1, 3, 6] or mixed between the Cornell potential and harmonic oscillator potential as in Refs. [2, 7 → 9] or Morse potential [10].

Recently, some authors focus to extend the Schrodinger equation to the higher dimensional which give more details about the systems under study. Moreover, the eigenvalue of spectra and wave function in lower dimensional can be obtained from the N-dimensional form [2].

The aim of this work is study the N-dimensional of radial Schrodinger equation to obtain the eigenvalue of energy and wave function by using the Nikiforov-Uvarov Method. In this work, the Cornell potential plus harmonic term is employed to obtain the good results in comparison with other studies and experimental data.

The paper is organized as follows: In Sec. 2, the Nikiforov-Uvarov Method is briefly explained which is used as the technique for solving the Schrodinger equation. In Sec. 3, The eigenvalue of the energy and the wave function are calculated in the N-dimensional form. In Sec. 4, the results are discussed. In Sec. 5, the summary and conclusion are presented.
II. THEORETICAL DESCRIPTION OF THE NIKIFOROV-UVAROV METHOD

In this section, we briefly give the Nikiforov-Uvarov Method that is used as the technique to solve second-order differential equation as the following form

\[
\Psi''(s) + \frac{\bar{\tau}(s)}{\sigma(s)}\Psi'(s) + \frac{\bar{\sigma}(s)}{\sigma^2(s)}\Psi(s) = 0,
\]

(1)

where \(\sigma(s)\) and \(\bar{\sigma}(s)\) are polynomials of maximum second degree and \(\bar{\tau}(s)\) is a polynomial of maximum first degree. The Nikiforov-Uvarov Method is given in the details in Ref. [5]. The second-order differential equation which takes following form by taking the \(\Psi(s)\) as follows

\[
\Psi(s) = \Phi(s)\chi(s),
\]

(2)

\[
\sigma(s)\chi''(s) + \tau(s)\chi(s) + \lambda\chi(s) = 0.
\]

(3)

Eq. (1) can written as in Ref.[1], where

\[
\sigma(s) = \pi(s)\frac{\Phi(s)}{\Phi'(s)},
\]

(4)

and

\[
\tau(s) = \bar{\tau}(s) + 2\pi(s); \quad \tau'(s) < 0,
\]

(5)

\[
\lambda = \lambda_n = -n\tau'(s) - \frac{n(n-1)}{2}\sigma''(s), \quad n = 0, 1, 2, ...
\]

(6)

\(\chi(s) = \chi_n(s)\) which is polynomial of \(n\) degree which satisfied the hyper geometric function, taking the form

\[
\chi_n(s) = B_n\frac{d^n}{ds^n}(\sigma^n(s)\rho(s)),
\]

(7)

where \(B_n\) is normalized condition and \(\rho(s)\) is a weight function which satisfied the following equation

\[
\frac{d}{ds}\omega(s) = \frac{\tau(s)}{\sigma(s)}\omega(s), \quad \omega(s) = \sigma(s)\rho(s),
\]

(8)

\[
\pi(s) = \frac{\sigma'(s) - \bar{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + K\sigma(s)},
\]

(9)

and

\[
\lambda = K + \pi'(s),
\]

(10)

the \(\pi(s)\) is a polynomial of first degree. The \(k\)-values in the square-root of Eq. (9) is possible to calculate if the polynomials under the square root must be square of polynomials. This is possible if its discriminate is zero.
III. THE SCHRODINGER EQUATION WITH THE CORNALL POTENTIAL PLUS THE HARMONIC POTENTIAL

The SE for two particles interacting via symmetric potential in N-dimensional takes the form as in Ref. [2]

\[
\left[ \frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr} \frac{L(L+N-2)}{r^2} + 2\mu(E - V(r)) \right] \Psi(r) = 0 \tag{11}
\]

where \( L, N, \) and \( \mu \) are the angular momentum quantum number, the dimensionality and the reduced mass for the quarkonium particle. Setting the wave function \( \Psi(r) = R(r) \), the following radial Schrodinger equation is obtained

\[
\left[ \frac{d^2 R(r)}{dr^2} + 2\mu(E - V(r) - \frac{L(L+N-2)}{2\mu r^2}) \right] R(r) = 0. \tag{12}
\]

The Cornell potential plus harmonic term potential is suggested as in Ref. [2]. Thus \( V(r) \) takes the form

\[
V(r) = ar - \frac{b}{r} + cr^2 \tag{13}
\]

where \( a, b, \) and \( c \) are arbitrary constants to be determined later. The potential has distinctive features of strongly interaction, namely, asymptotic freedom and confinement which are represented in the first and the second terms, respectively. The two terms are called as the Cornell potential, the last term is called the harmonic term potential. By substituting Eq. (13) into Eq. (12), we obtain

\[
\left[ \frac{d^2 R(r)}{dr^2} + 2\mu(E - ar + \frac{b}{r} - cr^2 - \frac{L(L+N-2)}{2\mu r^2}) \right] R(r) = 0. \tag{14}
\]

By assuming \( r = \frac{1}{x} \), Eq. (14) takes the following form

\[
\left[ \frac{d^2 R(x)}{dx^2} + \frac{2x}{x^2} \frac{dR(x)}{dx} + \frac{2\mu}{x^4}(E - \frac{a}{x} + bx - cx^2 - \frac{L(L+N-2)}{2\mu}x^2) \right] R(r) = 0, \tag{15}
\]

By following, the technique is mentioned in Sec. 2 as well as in Ref. [1], the eigenvalue of energy of Eq. (14) in N-dimensional is given

\[
E_{nL}^N = \frac{3a}{\delta} + \frac{6c}{\delta^2} - \frac{2\mu(\frac{2a}{\delta^2} + b + \frac{8c}{\delta^4})^2}{[(2n+1) \pm \sqrt{1 + \frac{2\mu a}{\delta^2} + 4((L + \frac{N-2}{2})^2 - \frac{1}{4}) + \frac{24\mu c}{\delta^4})]^2}, \tag{16}
\]

where \( n \) is radial quantum number and \( \delta = \frac{1}{x_0} \), where \( x_0 \) is a characteristic radius of the meson. One obtains the eigenvalue energy in Ref. [1] by setting \( c = 0 \) and \( N = 3 \).
obtains the eigenvalue energy in Ref. [3] by setting $a = 0$ and $N = 3$. By following, the steps in the section 2, The radial of wave function of Eq. (14) takes the following form

$$R_{nL}(r) = C_{nL} r^{-\frac{3a}{\delta^2}} e^{-\sqrt{2\lambda r}} (-r^2 \frac{d}{dr})^n (r^{-2n+2} e^{-\sqrt{2\lambda r}}),$$

(17)

where $C_{nL}$ is the normalization constant that determined by

$$\int_0^\infty |R_{nL}(r)|^2 dr = 0$$

and

$$A = -\mu (E - \frac{3a}{\delta} - \frac{6c}{\delta^2}), \quad B = \mu (\frac{3a}{\delta^3} + \frac{8c}{\delta^3} + b).$$

(18)

We note that the radial wave function in Eq. (17) is independent of N-dimensional. One obtains the wave function in Ref. [3], by setting $a = 0$ and wave function in Ref. [1], by setting $c = 0$.

IV. DISCUSSION OF RESULTS

In this section, we calculate spectra of the heavy quarkonium system such charmonium and bottomonium that have quark and anitquark flavor, the mass of quarkonium is calculated in 3-dimentionnal $(N = 3)$. So we apply the following relation as in Refs. [1, 2]

$$M = 2m + E_{nL}^{N=3},$$

(19)

where $m$ is quarkonium current mass for charmonium or bottomonium. By using Eq. (16), we can write Eq. (19) as follows

$$M = 2m + \frac{3a}{\delta} + \frac{6c}{\delta^2} - \frac{2\mu (\frac{3a}{\delta^3} + b + \frac{8c}{\delta^3})^2}{[(2n+1) \pm \sqrt{1 + \frac{8mu}{\delta^3} + 4L(L+1) + \frac{24\mu e}{\delta^4}}]^2}.$$

(20)

In Table (1), charmonium mass is calculated by using Eq. (20) in comparison with other theoretical methods and the experimental data. In comparison with results in Ref. [2], the authors calculated charmonium mass using asymptotic iteration method (AIM) with employing the same potential that used in Eq. 13. We note that the present results are improved in comparison with results using the AIM and are in good agreement with experimental data. Jamel and Widyon [3] calculated charmonium and bottomonium masses using the Nikiforov-Uvarov method, in which they employed the Coulomb potential plus quadratic potential. We have two advantages in comparison with the results in Ref. [3].

5
Table 1: Mass spectra of charmonium (in GeV) \((m_c = 1.48\ \text{GeV},\ a = 0.255\ \text{GeV}^2, \delta = 0.8\ \text{GeV},\ \text{and}\ c = 0.011\text{GeV}^3)\)

| State | Present work | [2] | [3] | [7] | [1] | [11] | Exp. [12, 13] |
|-------|--------------|-----|-----|-----|-----|-----|--------------|
| 1S    | 1.12         | 3.0689 | 3.078 | 3.096 | 3.096 | 3.078 | 3.068       |
| 2P    | 1.25         | 3.524 | 3.415 | 3.433 | 3.433 | 3.255 | 3.415 | 3.525       |
| 2S    | 1.3          | 3.615 | 4.187 | 3.686 | 3.686 | 3.686 | 3.581 | 3.663       |
| 1D    | 1.3          | 3.757 | 3.752 | 3.767 | 3.770 | 3.504 | 3.749 | 3.770       |
| 2P    | 1.3          | 3.772 | 4.143 | 3.910 | 4.023 | 3.779 | 3.917 | 3.773       |
| 3S    | 0.1          | 3.956 | 5.297 | 3.984 | 4.040 | 4.040 | 4.085 | 4.159       |
| 4S    | 0.1          | 3.921 | 6.407 | 4.150 | 4.358 | 4.269 | 4.589 | 4.421       |

The first, the case of the potential in Ref. [3] is the particular case from the present potential when put \(a = 0\). So we can obtain the eigenvalue of energy and corresponding wave function of Ref. [3] by setting \(a = 0\) and \(N = 3\). The second, the present results are improved in comparison with results in Ref. [3] and are in good agreement with experimental data. In comparison with Ref. [7], the authors used the same potential as in Eq. (13), we have some advantages that the SE is extended to higher dimensional, hence, the energy of quarkonium and wave function is extended to N-dimensional, leading the expression of the energy and wave function are different. Hence, the present results are improved in comparison with Ref. [7] and are in good agreement with experimental data. Also, we add further investigation by calculating \(b\ \bar{c}\) meson mass in the present work as in Table 3. In comparison with Ref. [1], the author used the Cornell potential only. Therefore, the Cornell potential is the particular case at \(c = 0\) in the present potential. The present results are improved in comparison with results in Ref. [1] and are in good agreement with experimental data. In comparison with Ref. [11], we note that the present results are improved and are in good agreement with data.
Table 2: Mass spectra of bottomonium (in GeV) \((m_b = 4.68 \text{ GeV}, \ a = 0.255 \text{ GeV}^2, \delta = 0.8 \text{ GeV}, \text{ and } c = 0.011 \text{GeV}^3)\)

| State | \(b\) | Present work | [2] | [3] | [7] | [1] | [11] | Exp. [12] |
|-------|--------|--------------|-----|-----|-----|-----|-----|----------|
| 1S    | 0.65   | 9.453        | 9.510 | 9.460 | 9.460 | 3.096 | 9.510   | 9.460    |
| 1P    | 0.433  | 9.903        | 9.862 | 9.840 | 9.811 | 3.255 | 9.862   | 9.900    |
| 2S    | 0.5    | 10.025       | 10.627 | 10.023 | 10.023 | 3.686 | 10.038  | 10.023   |
| 1D    | 0.7    | 10.157       | 10.214 | 10.140 | 10.161 | 3.504 | 10.214  | 10.161   |
| 2P    | 0.1    | 10.238       | 10.944 | 10.160 | 10.374 | 3.779 | 10.390  | 10.260   |
| 3S    | 0.01   | 10.295       | 11.726 | 10.280 | 10.355 | 4.040 | 10.566  | 10.355   |
| 4S    | 0.01   | 10.338       | 12.834 | 10.420 | 10.655 | 4.269 | 11.094  | 10.580   |

Table 3: Mass spectra of \(b \bar{c}\) meson (in GeV) \((m_b = 4.823 \text{ GeV}, \ m_c = 1.209 \text{ GeV}, \ a = 0.2 \text{ GeV}^2, \delta = 0.324 \text{ GeV}, \text{ and } c = 0.11 \text{GeV}^3)\)

| State | \(b\) | Present work | [14] | [15] | [16] | Exp. [13] |
|-------|--------|--------------|-----|-----|-----|----------|
| 1S    | 1.79   | 7.276        | 6.349 | 6.264 | 6.270 | 6.277    |
| 1P    | 3.59   | 6.715        | 6.715 | 6.700 | 6.699 | -        |
| 2S    | 6.8    | 6.811        | 6.821 | 6.856 | 6.835 | -        |
| 2P    | 6.72   | 7.016        | 7.102 | 7.108 | 7.091 | -        |
| 3S    | 13.42  | 7.183        | 7.175 | 7.244 | 7.193 | -        |

In Table 2, the bottomonium mass is calculated. We note that all states in the present potential are improved in comparison with other theoretical method that mentioned in the discussion of Table 1. In addition, the present results are in good agreement with experimental data. In Table 3, the \(b \bar{c}\) meson mass is calculated. We find that the 1S state is very good agreement with data. The experimental data of the other states are not available. Hence, the theoretical predictions using the present method and other theoretical are displayed. We note that the present results of the \(b \bar{c}\) meson mass are in good agreement in comparison with Refs. [14, 15, 16]
V. SUMMARY AND CONCLUSION

In the present work, the energy eigenvalues and the wave function are obtained in the N-dimensional form by solving the N-radial Schrodinger equation. The Nikiforov-Uvarov Method is employed as the technique to solve the Schrodinger equation, in which the Cornell potential is extended by including the harmonic oscillator term. The equations of the energy eigenvalue in the 3-dimensional form can obtained from the N-dimensional form and they coincide with the recent works such as in Refs. [1, 3]. A comparison is studied with other theoretical methods, in which the advantages of the present work are displayed. In addition, the obtained results are in good agreement with experimental data. Therefore, we conclude that the present results are improved in comparison with other recent works and are in good agreement with the experimental data. We hope to extend this work for further investigations of other characteristics of quarkonium.

VI. REFERENCES

1. S. M. Kuchin and N. V. Maksimenko, Univ. J. Phys. Appl. 1, 295 (2013).
2. R. Kumar and F. Chand, Commun. Theor. Phys. 59, 528 (2013).
3. A. Al-Jamel and H. Widyan, Appl. Phys. Rese. 4, 94 (2013).
4. H. Ciftci, R. Hall, and N. Saad, J. Phys. A: Math. Gen. 36 11807 (2003).
5. Af. Nikiforov Uvarov, ”Special Functions of Mathematical Physics” Birkhauser, Basel (1988).
6. H. S. Chung and J. Lee, J. Kore. Phys. Soci. 52, 1151 (2008).
7. N. V. Maksimenko and S. M. Kuchin, Russ. Phy. J. 54, 57 (2014).
8. Z. Ghalenovi, A. A. Rajabi, S. Qin and H. Rischke, hep-ph\14034582 (2014).
9. S. M. Ikhdair, quant-ph\11100340 (2011).
10. B. J. Vigo-Aguir and T. E. Simos, Int. J. Quantum Chem. 103, 278 (2005).
11. R. Kumar and F. Chand, Phys. Scr. 85, 055008 (2012).
12. R. M. Barnett et al. [particle Data Group], Phys. Rev. D 54, 1 (1996).

13. J. Beringer et al. [particle Data Group], Phys. Rev. D 86, 1 (2012).

14. A. Kumar Ray and P. C. Vinodkumar, Pramana J. Phys. 66, 958 (2006).

15. E. J. Eichten and C. Quigg, Phys. Rev. D 49, 5845 (1994).

16. D. Ebert, R. N. Faustov, and V. O. Galkin, Phys. Rev. D 67, 014027 (2003).