ON THE DPG METHOD FOR SIGNORINI PROBLEMS

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Abstract. We derive and analyze discontinuous Petrov-Galerkin methods with optimal test functions for Signorini-type problems as a prototype of a variational inequality of the first kind. We present different symmetric and non-symmetric formulations where optimal test functions are only used for the PDE part of the problem, not the boundary conditions. For the symmetric case and lowest order approximations, we provide a simple a posteriori error estimate. In a second part, we apply our technique to the singularly perturbed case of reaction dominated diffusion. Numerical results show the performance of our method and, in particular, its robustness in the singularly perturbed case.

1. Introduction

In this paper we develop a framework to solve contact problems by the discontinuous Petrov-Galerkin method with optimal test functions (DPG method). We consider a simplified model problem \((-\Delta u + u = f\) in a bounded domain) with unilateral boundary conditions that resembles a scalar version of the Signorini problem in linear elasticity \([31, 14]\). We prove well-posedness of our formulation and quasi-optimal convergence. We then illustrate how the scheme can be adapted to the singularly perturbed case of reaction-dominated diffusion \((-\varepsilon\Delta u + u = f\)). Specifically, we use the DPG setting from \([22]\) to design a method that controls the field variables \((u, \nabla u, \Delta u)\) in the so-called balanced norm (which corresponds to \(\|u\| + \varepsilon^{1/4}\|\nabla u\| + \varepsilon^{3/4}\|\Delta u\|\) with \(L^2\)-norm \(\|\cdot\|\)), cf. \([28]\). The balanced norm is stronger than the energy norm that stems from the Dirichlet bilinear form of the problem.

There is a long history of numerical analysis for contact problems, and more generally for variational inequalities, cf. the books by Glowinski, Lions, Trémolières \([20]\) and Kikuchi, Oden \([27]\). Some early papers on the (standard and mixed) finite element method for Signorini-type problems are \([21, 2, 33, 3, 29]\). Unilateral boundary conditions usually give rise to limited regularity of the solution and authors have made an effort to establish optimal convergence rates of the finite element method, see, e.g., \([8, 12]\). Recently there has been some interest to extend other numerical schemes to unilateral contact problems, e.g., least squares \([1]\), the local discontinuous Galerkin method \([5]\), and a Nitsche-based finite element method \([8]\).

Our objective is to set up an appropriate framework to apply DPG technology to variational inequalities, in this case Signorini-type problems. At this point we do not intend to be more competitive than previous schemes. In particular, we are not concerned with the reduced regularity of solutions that limit convergence orders. In any case, we show well-posedness of our scheme for the minimum regularity of \(u\) in \(H^1\) and right-hand side function \(f\) in \(L^2\). Our primary focus is to use DPG schemes in such a way that their performance is not reduced by the presence of Signorini boundary conditions. The DPG method aims at ensuring discrete inf-sup stability by the choice of norms and test functions, cf. \([9, 10]\). This is particularly important for singularly perturbed...
problems where one can achieve robustness of error control (the discrete inf-sup constant does not depend on perturbation parameters), see [11, 31, 7, 4, 22].

Recently we have found a setting for the coupling of DPG and Galerkin boundary elements (BEM) to solve transmission problems, see [16]. The principal idea is to take a variational formulation of the interior problem (suitable for the DPG method) and to add transmission conditions as a constraint. This constraint can be given as boundary integral equations in a least-squares or Galerkin form. In this paper, we follow this very strategy. The partial differential equation is put in variational form (without considering any boundary condition) and the Signorini conditions are added as constraint. It turns out that the whole scheme can be written as a variational inequality of the first kind where only the PDE part is tested with optimal test functions, as is DPG strategy. Then, proving coercivity and boundedness of the bilinear form, the Lions-Stampacchia theorem proves well-posedness.

Let us note that there are finite element/boundary element coupling schemes for contact problems, see [6, 30, 18]. However, their coupling variants generalize the setting of contact problems on bounded domains. In contrast, our coupling scheme [16] (for a standard transmission problem) separates the PDE on the bounded domain from the transmission conditions (and exterior problem) in such a way that they can be formulated as a variational PDE plus constraint. As we have mentioned, this is critical for combining a DPG method with transmission conditions as in [16], or with contact conditions as we show here.

The singularly perturbed case is more technical for two reasons. First, the DPG setting itself for the PDE is more complicated (we use a robust formulation with three field variables) and, second, the combination of PDE and Signorini conditions has to take into account the diffusion coefficient as scaling parameter. In that way we apply what we have learnt from the DPG scheme [22] for reaction dominated diffusion, and from the DPG-BEM coupling [15] for this case. To the best of our knowledge, this is the first mathematical analysis of a numerical scheme for a singularly perturbed contact problem.

We will see that there are symmetric and non-symmetric forms to include the Signorini boundary conditions. In the symmetric case we are able to provide a simple a posteriori error estimate that is based on the DPG energy error. Let us also remark that we focus on ultra-weak variational formulations. This has the advantage that both the trace and the flux appear as independent unknowns. This allows for a symmetric formulation of the Signorini conditions, needed for our a posteriori error bound. Other variational formulations can be considered analogously and they give rise to non-symmetric well-posed formulations. In those cases, however, we have no a posteriori error analysis. Let us also mention that our scheme and analysis of the scalar Signorini problem can be extended to variational inequalities of the second kind, e.g., including Coulomb friction and to linear elasticity so that, indeed, the Signorini problem can be solved by our DPG scheme.

In the following we continue this introduction by presenting our model problem, by introducing an abstract formulation as variational inequality of the first kind that is suitable for the DPG scheme, and by giving a final overview of the remainder of this paper.

**Model problem.** Let $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) be a simply connected Lipschitz domain with boundary $\Gamma = \partial \Omega$ and unit normal vector $n$ on $\Gamma$ pointing towards $\mathbb{R}^d \setminus \overline{\Omega}$. For $f \in L^2(\Omega)$ we consider the model problem

\begin{equation}
-c\Delta u + u = f \quad \text{in} \quad \Omega
\end{equation}

subject to the Signorini boundary conditions

\begin{equation}
 u \geq 0, \quad \frac{\partial u}{\partial n} \geq 0, \quad u \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma.
\end{equation}

Initially we will study the case of constant $c = 1$. Later we will illustrate the applicability of our DPG scheme to the singularly perturbed problem with constant $c = \varepsilon$, assuming $\varepsilon$ to be a small
positive number. Of course, the restriction of \( u \) to \( \Gamma \) is understood in the sense of the trace, and its normal derivative on \( \Gamma \) is defined by duality.

It is well known that this Signorini problem admits a unique solution \( u \in K := \{ v \in H^1(\Omega) : v \geq 0 \text{ on } \Gamma \} \), see, e.g., [20] [19]. Moreover, \( u \) can be characterized as the unique solution of the variational inequality of the first kind

\[
a(u, v - u) \geq (f, v - u) \quad \text{for all } v \in K
\]

with

\[
a(u, v) := (\nabla u, \nabla v) + (u, v) \quad \text{for all } u, v \in H^1(\Omega).
\]

In fact, by choosing appropriate test functions \( v \in K \) and integrating by parts, one finds that problem (2) is equivalent to (1a) and

\[
\int_{\Gamma} \frac{\partial u}{\partial n} (v - u) \, d\Gamma \geq 0 \quad \text{for all } v \in K,
\]

see, e.g., [20]. The last relation is useful to establish a DPG setting of the variational inequality problem. Note that (1b) implies that \( \Gamma \) is partitioned into two parts \( \Gamma_D \) and \( \Gamma_N \) with \( u|_{\Gamma_D} = 0 \) and \( \partial_n u|_{\Gamma_N} = 0 \). Hence, \( u \) is the unique solution of (1a) with (mixed) boundary condition \( u|_{\Gamma_D} = 0 \), and \( \partial_n u|_{\Gamma_N} = 0 \). However, \( \Gamma_D \) and \( \Gamma_N \) are unknown in general and solving (1) is equivalent to finding these sets.

**Variational formulation.** Let us give a brief overview of our variational setting for the DPG method. In Section 2 we will consider a non-standard variational form of (1a):

\[
\begin{align*}
\mathbf{u} & \in U : \quad b(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V.
\end{align*}
\]

Here, \( U \) and \( V \) are different Hilbert spaces and \( b(\cdot, \cdot) : U \times V \to \mathbb{R} \) is the bilinear form stemming from, in our case, an ultra-weak formulation. At this point no boundary conditions are included so that there is no unique solution to (4). Denoting by \( (\cdot, \cdot)_V \) the inner product in \( V \), we define the trial-to-test operator \( \Theta_\beta : U \to V \) by

\[
\Theta_\beta := \beta \Theta \quad \text{with} \quad (\Theta \mathbf{u}, \mathbf{v})_V := b(\mathbf{u}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in V.
\]

The parameter \( \beta > 0 \) has to be selected. Using this operator, the discretization of (4) will be based on its equivalent variant with so-called optimal test functions:

\[
\begin{align*}
\mathbf{u} & \in U : \quad b(\mathbf{u}, \Theta_\beta \mathbf{v}) = L(\Theta_\beta \mathbf{v}) \quad \text{for all } \mathbf{v} \in U.
\end{align*}
\]

In our formulations, only one component of \( \mathbf{u} \in U \) corresponds to the original unknown \( u \). Depending on the particular case, we have to define appropriate Dirichlet and Neumann trace operators, \( \gamma_0 \) and \( \gamma_n \), acting on \( U \). Then it is left to add the boundary conditions (1b) in the form \( \gamma_0 \mathbf{u} \geq 0 \), \( \gamma_n \mathbf{u} \geq 0 \), and \( \gamma_0 \mathbf{u} \gamma_n \mathbf{u} = 0 \). This transforms (6) into a variational inequality.

Keeping in mind (3), we define the bilinear form

\[
a^0(\mathbf{u}, \mathbf{v}) := b(\mathbf{u}, \Theta_\beta \mathbf{v}) + (\gamma_n \mathbf{u}, \gamma_0 \mathbf{v})_\Gamma \quad \text{for all } \mathbf{u}, \mathbf{v} \in U,
\]

and consider the following formulation: Find \( \mathbf{u} \in K^0 := \{ \mathbf{v} \in U : \gamma_0 \mathbf{v} \geq 0 \} \) such that

\[
a^0(\mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\Theta_\beta(\mathbf{v} - \mathbf{u})) \quad \text{for all } \mathbf{v} \in K^0.
\]

We will show that this problem is equivalent to (1). In particular, it has a unique solution.

An intrinsic feature of ultra-weak formulations is that all boundary conditions are essential. Therefore, we can derive methods that use different convex sets. From (1b) we infer that

\[
\mathbf{u} \left( \frac{\partial \mathbf{v}}{\partial n} - \frac{\partial \mathbf{u}}{\partial n} \right) \geq 0 \quad \text{on } \Gamma \quad \text{for all } \mathbf{v} \in H^1(\Omega) \text{ with } \frac{\partial \mathbf{v}}{\partial n} \geq 0 \text{ on } \Gamma.
\]
giving rise to the formulation: Find \( u \in K^n := \{ v \in U : \gamma_n v \geq 0 \} \) such that
\[
a^n(u, v - u) := b(u, \Theta_\beta(v - u)) + \langle \gamma_n v - \gamma_n u, \gamma_0 u \rangle_T \geq L(\Theta_\beta(v - u)) \quad \text{for all } v \in K^n.
\]
Note that neither \( a^0(\cdot, \cdot) \) nor \( a^n(\cdot, \cdot) \) is symmetric. However, a combination leads to a formulation with symmetric bilinear form: Find \( u \in K^s := \{ v \in U : \gamma_0 v \geq 0, \gamma_n v \geq 0 \} \) such that
\[
a^s(u, v - u) := \frac{1}{2}(a^0(u, v - u) + a^n(u, v - u)) \geq L(\Theta_\beta(v - u)) \quad \text{for all } v \in K^s.
\]
For this symmetric case we will establish a simple a posteriori error estimator.

Overview. The remainder of this paper is structured as follows. Section 2 deals with the unperturbed model problem [1] (diffusion parameter \( c = 1 \)). After introducing some notation in §2.1, we present a variational inequality that represents (1) and is based on an ultra-weak variational formulation. Theorem 1 then states its well-posedness and equivalence. Afterwards, in §2.2 we present a variational inequality that represents (1) and is based on an ultra-weak variational formulation. Theorem 12 states the well-posedness and equivalence of the variational inequality. §3.3 presents and analyzes the discrete scheme, its well-posedness (Theorem 13), quasi-optimal convergence (Theorem 3). An a posteriori error estimate is derived in §2.4. Some technical results are collected in §2.5 and a proof of Theorem 1 is given at the end of Section 2.

There is a similar structure in Section 3 for the singularly perturbed case (problem (1) with small diffusion \( c = \varepsilon \)). Notation is given in §3.1 and a variational formulation is presented and analyzed in §3.2. There, Theorem 12 states the well-posedness and equivalence of the variational inequality. §3.3 presents and analyzes the discrete scheme, its well-posedness (Theorem 13), quasi-optimality (Theorem 14) and an a posteriori error estimate (Theorem 15). Technical results and a proof of Theorem 12 are presented in §3.4 and §3.5 respectively. Finally, in Section 4 we present several numerical examples that underline our theoretical results for the unperturbed and singularly perturbed cases.

Throughout the paper, \( a \lesssim b \) means that \( a \leq kb \) with a generic constant \( k > 0 \) that is independent of involved parameters, functions and the underlying mesh. Similarly, we use the notation \( a \simeq b \).

2. DPG METHOD

2.1. Notation. For Lipschitz domains \( \omega \subset \mathbb{R}^d \) we use the standard Sobolev spaces \( L^2(\omega), H^1(\omega), \) and \( H(\text{div}, \omega) \). The \( L^2(\omega) \) resp. \( L^2(\partial \omega) \) scalar product is denoted by \( (\cdot, \cdot)_\omega \) resp. \( (\cdot, \cdot)_{\partial \omega} \) with induced norm \( \| \cdot \|_\omega \) resp. \( \| \cdot \|_{\partial \omega} \). Also, we define the trace spaces
\[
H^{1/2}(\partial \omega) := \{ \gamma_\omega u : u \in H^1(\omega) \}
\]
and its dual \( H^{-1/2}(\partial \omega) := (H^{1/2}(\partial \omega))^\prime \), where \( \gamma_\omega \) denotes the trace operator.

Here, duality is understood with respect to \( L^2(\partial \omega) \) as a pivot space, i.e., using the extended \( L^2(\partial \omega) \) inner product \( (\cdot, \cdot)_{\partial \omega} \). The \( L^2(\Omega) \) inner product will be denoted by \( (\cdot, \cdot) \) and the corresponding norm by \( \| \cdot \| \). Let \( \mathcal{T} \) denote a disjoint partition of \( \Omega \) into open Lipschitz sets \( T \in \mathcal{T} \), i.e., \( \bigcup_{T \in \mathcal{T}} T = \Omega \). The set of all boundaries of all elements forms the skeleton \( S := \{ \partial T \mid T \in \mathcal{T} \} \). By \( n_M \) we mean the outer normal vector on \( \partial M \) for a Lipschitz set \( M \). On a partition \( \mathcal{T} \) we use the product spaces
\[
H^1(\mathcal{T}) := \{ w \in L^2(\Omega) : w|_T \in H^1(T) \forall T \in \mathcal{T} \},
\]
\[
H(\text{div}, \mathcal{T}) := \{ q \in (L^2(\Omega))^d : q|_T \in H(\text{div}, \Omega) \forall T \in \mathcal{T} \}.
\]
The symbols \( \nabla_T, \text{div}_T \), resp. \( \Delta_T \) denote the \( \mathcal{T} \)-piecewise gradient, divergence, resp. Laplace operators. On the skeleton \( S \) of \( \mathcal{T} \) we introduce the trace spaces
\[
H^{1/2}(S) := \left\{ \tilde{u} \in \Pi_{T \in \mathcal{T}} H^{1/2}(\partial T) : \exists w \in H^1(\Omega) \text{ such that } \tilde{u}|_{\partial T} = w|_{\partial T} \forall T \in \mathcal{T} \right\},
\]
\[
H^{-1/2}(S) := \left\{ \tilde{\sigma} \in \Pi_{T \in \mathcal{T}} H^{-1/2}(\partial T) : \exists q \in H(\text{div}, \Omega) \text{ such that } \tilde{\sigma}|_{\partial T} = (q \cdot n_T)|_{\partial T} \forall T \in \mathcal{T} \right\}.
\]
These spaces are equipped with norms depending on the problem, see §2.2 and §3.1. For functions \( \hat{u} \in H^{1/2}(S) \), \( \tilde{\sigma} \in H^{-1/2}(S) \) and \( \tau \in H(\text{div}, \mathcal{T}) \), \( v \in H^1(\mathcal{T}) \) we define
\[
\langle \hat{u}, \tau \cdot n \rangle_S := \sum_{T \in \mathcal{T}} \langle \hat{u} |_{\partial T}, \tau \cdot n_T \rangle_{\partial T}, \quad \langle \tilde{\sigma}, v \rangle_S := \sum_{T \in \mathcal{T}} \langle \tilde{\sigma} |_{\partial T}, v \rangle_{\partial T}.
\]

With the latter relations we can also define the restrictions \( \hat{u}|_{\Gamma} \in H^{1/2}(\Gamma) \) and \( \tilde{\sigma}|_{\Gamma} \in H^{-1/2}(\Gamma) \) of functions \( \hat{u} \in H^{1/2}(S) \), \( \tilde{\sigma} \in H^{-1/2}(S) \) onto \( \Gamma \). Let \( w \in H^1(\Omega) \) be such that \( w|_{\partial T} = \hat{u}|_{\partial T} \). Then,
\[
\langle \hat{u}, \tau \cdot n \rangle_S = \sum_{T \in \mathcal{T}} \langle w |_{\partial T}, \tau \cdot n_T \rangle_{\partial T} = \sum_{T \in \mathcal{T}} (\nabla w, \tau)_T + (w, \text{div } \tau)_T = (\nabla w, \tau) + (w, \text{div } \tau)
\]
for all \( \tau \in H(\text{div}, \Omega) \).

Note that \( \hat{u}|_{\Gamma} \) is uniquely determined since the above relation is independent of the choice of the extension \( w \in H^1(\Omega) \) and since \( H^{-1/2}(\Gamma) \) is the (normal) trace space of \( H(\text{div}, \Omega) \). Similarly, we define \( \tilde{\sigma}|_{\Gamma} \in H^{-1/2}(\Gamma) \) through
\[
\langle \tilde{\sigma}, v \rangle_S = \sum_{T \in \mathcal{T}} \langle \sigma \cdot n_T, v |_T \rangle_{\partial T} = \sum_{T \in \mathcal{T}} (\text{div } \sigma, v)_T + (\sigma, \nabla v)_T = (\text{div } \sigma, v) + (\sigma, \nabla v)
\]
for all \( \sigma \in H(\text{div}, \Omega) \) with \( \sigma \cdot n_T |_{\partial T} = \tilde{\sigma} |_{\partial T} \) for all \( T \in \mathcal{T} \).

2.2. Ultra-weak variational formulation. We derive an ultra-weak formulation of (1a) with \( c = 1 \). Following (1a), we define \( \sigma = \nabla u \). Then,
\[-\text{div } \sigma + u = f, \quad \sigma - \nabla u = 0.
\]

We define \( \hat{u} \) and \( \tilde{\sigma} \) such that \( \hat{u}|_{\partial T} = u |_{\partial T} \) and \( \tilde{\sigma}|_{\partial T} = \nabla u \cdot n_T |_{\partial T} \) for all \( T \in \mathcal{T} \). Testing the first-order system with functions \( v \in H^1(\mathcal{T}) \), \( \tau \in H(\text{div}, \mathcal{T}) \), and integrating by parts, we end up with the ultra-weak formulation
\[
(\sigma, \nabla_T v) + (u, v) - \langle \tilde{\sigma}, v \rangle_S = (f, v),
\]
(9b)
\[
(\sigma, v) + (u, \text{div}_T \tau) - \langle \hat{u}, \tau \cdot n \rangle_S = 0.
\]

This gives rise to a bilinear form \( b : U \times V \to \mathbb{R} \) and functional \( L : V \to \mathbb{R} \) defined by
\[
b(u, v) := (\sigma, \nabla_T v) + (u, v) - \langle \tilde{\sigma}, v \rangle_S + (\sigma, \tau) + (u, \text{div}_T \tau) - \langle \hat{u}, \tau \cdot n \rangle_S,
\]
\[
L(v) := (f, v)
\]
for all \( u = (u, \sigma, \hat{u}, \tilde{\sigma}) \in U \), \( v = (v, \tau) \in V \), where
\[
U := L^2(\Omega) \times [L^2(\Omega)]^d \times H^{1/2}(S) \times H^{-1/2}(S),
\]
\[
V := H^1(\mathcal{T}) \times H(\text{div}, \mathcal{T}).
\]

We equip these spaces with the norms
\[
\|u\|_U^2 := \|u\|^2 + \|\sigma\|^2 + \|\hat{u}\|_{1/2,S}^2 + \|\tilde{\sigma}\|_{-1/2,S}^2,
\]
\[
\|v\|_V^2 := \|v\|^2 + \|\nabla_T v\|^2 + \|\tau\|^2 + \|\text{div}_T \tau\|^2,
\]
where the norms for the trace variables are given by the minimum energy extensions to \( H^1(\Omega) \) and \( H(\text{div}, \Omega) \), respectively, i.e.,
\[
\|\hat{u}\|_{1/2,S} := \inf \left\{ (\|w\|^2 + \|\nabla w\|^2)^{1/2} : w \in H^1(\Omega), \hat{u}|_{\partial T} = w |_{\partial T} \quad \forall T \in \mathcal{T} \right\},
\]
\[
\|\tilde{\sigma}\|_{-1/2,S} := \inf \left\{ (\|q\|^2 + \|\text{div } q\|^2)^{1/2} : q \in H(\text{div}, \Omega), \tilde{\sigma}|_{\partial T} = (q \cdot n_T) |_{\partial T} \quad \forall T \in \mathcal{T} \right\}.
\]
In $H^{1/2}(\Gamma)$ we use the norm
\[ \|\hat{v}\|_{H^{1/2}(\Gamma)} := \inf \left\{ (\|w\|^2 + \|\nabla w\|^2)^{1/2} : w \in H^1(\Omega), \hat{v} = w|_{\Gamma} \right\}, \]
and define the norm $\| \cdot \|_{H^{-1/2}(\Gamma)}$ by duality.

The bilinear form $b(\cdot, \cdot)$ induces a linear operator $B : U \to V'$ so that (9) can be written as
\[ u \in U : \quad Bu = L. \]

The operator $B$ has a non-trivial kernel. In order to consider boundary conditions we define trace operators
\[ \gamma_0 : U \to H^{1/2}(\Gamma), \quad \gamma_0(u, \sigma, \hat{u}, \hat{\sigma}) := \hat{u}|_{\Gamma} \quad \text{(Dirichlet trace, see (7))}, \]
\[ \gamma_n : U \to H^{-1/2}(\Gamma), \quad \gamma_n(u, \sigma, \hat{u}, \hat{\sigma}) := \hat{\sigma}|_{\Gamma} \quad \text{(Neumann trace, see (8))}. \]

and define the sets
\[ K^0 := \{ u \in U : \gamma_0 u \geq 0 \}, \quad K^n := \{ u \in U : \gamma_n u \geq 0 \}, \quad K^s := \{ u \in U : \gamma_0 u \geq 0, \gamma_n u \geq 0 \}. \]

The relations “$\geq$” are partial orderings. Following [27, Section 5] we define
\[ \hat{v} \geq 0 \text{ in } H^{1/2}(\Gamma) \iff \exists \{\hat{v}_n\}_{n \in \mathbb{N}} \text{ s.t. } v_n \in \text{Lip}(\Gamma) \text{ with } v_n \geq 0 \text{ and } v_n \rightharpoonup v \text{ in } H^{1/2}(\Gamma), \]
where Lip(\Gamma) denotes all Lipschitz continuous functions $\Gamma \to \mathbb{R}$. On $H^{-1/2}(\Gamma)$, “$\geq$” is understood as duality (see, e.g., [23, Section 1.1.11]),
\[ \hat{\sigma} \geq 0 \text{ in } H^{-1/2}(\Gamma) \iff \langle \hat{\sigma}, \hat{v} \rangle_{\Gamma} \geq 0 \text{ for all } \hat{v} \in H^{1/2}(\Gamma) \text{ with } \hat{v} \geq 0. \]

One can verify that $\{ \hat{v} \in H^{1/2}(\Gamma) : \hat{v} \geq 0 \}$ and $\{ \hat{\sigma} \in H^{-1/2}(\Gamma) : \hat{\sigma} \geq 0 \}$ are closed, convex sets. We will also see that $K^0, K^n, \text{ and } K^s$ are non-empty, closed, convex subsets of $U$ (see Lemma 8 below).

Now let $u \in H^1(\Omega)$ denote the solution of problem (1) and define $u = (u, \sigma, \hat{u}, \hat{\sigma}) \in U$ with components as above. Then $Bu = L$ and, considering inequality (3) as a representation of the boundary conditions (16), one sees that $u \in K^* \text{ for } \ast \in \{0, n, s\}$ and
\[ \langle \gamma_n u, \gamma_0 v - \gamma_0 u \rangle_{\Gamma} \geq 0 \quad \text{for all } v \in K^0, \]
\[ \langle \gamma_n v - \gamma_n u, \gamma_0 u \rangle_{\Gamma} \geq 0 \quad \text{for all } v \in K^n, \]
\[ \frac{1}{2} \langle \gamma_n u, \gamma_0 v - \gamma_0 u \rangle_{\Gamma} + \langle \gamma_n v - \gamma_n u, \gamma_0 u \rangle_{\Gamma} \geq 0 \quad \text{for all } v \in K^s. \]

Recalling the formulation (8) of the problem $Bu = L$, this leads us to defining the bilinear forms $a^* : U \times U \to \mathbb{R}, \ast \in \{0, n, s\}$, and the functional $F : U \to \mathbb{R}$ as follows.
\[ a^0(u, v) := b(u, \Theta_0 v) + \langle \gamma_n u, \gamma_0 v \rangle_{\Gamma}, \]
\[ a^n(u, v) := b(u, \Theta_n v) + \langle \gamma_n v, \gamma_0 u \rangle_{\Gamma}, \]
\[ a^s(u, v) := b(u, \Theta_s v) + \frac{1}{2} \langle \gamma_n u, \gamma_0 v \rangle_{\Gamma} + \langle \gamma_n v, \gamma_0 u \rangle_{\Gamma}, \]
\[ F(v) := L(\Theta v) \]
for $u, v \in U$. Here, $\beta > 0$ is a constant that will be fixed below.

We then obtain the following formulations of the model problem (1) (with $c = 1$) as ultra-weak variational inequalities: For fixed $\ast \in \{0, n, s\}$ find $u \in K^*$ such that
\[ a^*(u, v - u) \geq F(v - u) \quad \text{for all } v \in K^*. \]

These are variational inequalities of the first kind and we use a standard framework for their analysis, cf. [20, 19].

The following is one of our main results.
Theorem 1. Fix \( \star \in \{0, n, s\} \). For all \( \beta \geq 2 \) there holds the following: The bilinear form \( a^\star : U \times U \to \mathbb{R} \) is \( U \)-coercive,

\[
\|u\|_U^2 \leq C_1 a^\star(u, u) \quad \text{for all } u \in U,
\]

and bounded,

\[
|a^\star(u, v)| \leq (C_2^\beta + 1)\|u\|_U\|v\|_U \quad \text{for all } u, v \in U.
\]

The constant \( C_1 > 0 \) depends only on \( \Omega \) and \( C_2 > 0 \) is the continuity constant of \( b : U \times V \to \mathbb{R} \).

In particular, the variational inequality (10) is uniquely solvable and equivalent to problem (1) with \( c = 1 \) in the following sense: If \( u \in H^1(\Omega) \) solves problem (1), then \( u = (u, \sigma, \hat{u}, \hat{\sigma}) \in K^\star \) with \( \sigma := \nabla u, \hat{u}|_{\partial T} := u|_{\partial T}, \hat{\sigma}|_{\partial T} := \nabla u \cdot n_T|_{\partial T} \) for all \( T \in \mathcal{T} \) solves (10). On the other hand, if \( u = (u, \sigma, \hat{u}, \hat{\sigma}) \in K^\star \) solves (10), then \( u \in H^1(\Omega) \) solves (1).

Moreover, the unique solution \( u \in K^\star \) of (10) satisfies

\[
\|b(u, \Theta^\beta w) - F(w)\| \quad \text{for all } w \in U.
\]

This theorem is proved in Section 2.6.

2.3. Discretization and convergence. To discretize our variational inequality (10) we use, in this work, lowest-order piecewise polynomial functions. That is, we replace the space \( U \) by

\[
U_h := P^0(\mathcal{T}) \times [P^0(\mathcal{T})]^d \times S^1(\mathcal{S}) \times P^0(\mathcal{S})
\]

where \( P^0 \) denotes the space of element-wise constants on \( \mathcal{T} \) resp. \( \mathcal{S} \), and \( S^1(\mathcal{S}) \) is the space of globally continuous, element-wise affine functions on \( \mathcal{S} \). Defining the non-empty convex subsets

\[
K^0_h := \{ v_h \in U_h : \gamma_0 v_h \geq 0 \}, \quad K^n_h := \{ v_h \in U_h : \gamma_n v_h \geq 0 \}, \quad K^s_h := \{ v_h \in U_h : \gamma_0 v_h \geq 0, \gamma_n v_h \geq 0 \}
\]

we find that \( K^\star_h \subseteq K^\star \).

The discretized version of (10) then reads: For fixed \( \star \in \{0, n, s\} \) find \( u_h \in K^\star_h \) such that

\[
a^\star(u_h, v_h - u_h) \geq F(v_h - u_h) \quad \text{for all } v_h \in K^\star_h.
\]

Coercivity and boundedness of \( a^\star(\cdot, \cdot) \) hold on the full space \( U \) (Theorem 1). Therefore, the Lions-Stampacchia theorem applies also in the discrete case.

Theorem 2. Under the assumptions of Theorem 1 the discrete variational inequality (12) admits a unique solution \( u_h \in K^\star_h \).

Our scheme converges quasi-optimally in the following sense.

Theorem 3. For \( \beta \geq 2 \) let \( u \in K^\star \), \( u_h \in K^\star_h \) denote the exact solutions of (10), (12). Then there holds

\[
\|u - u_h\|_U^2 \lesssim \begin{cases}
\inf_{v_h \in K^0_h} (\|u - v_h\|_U^2 + \{\gamma_n u, \gamma_0 (v_h - u)\}_\Gamma) & (\star = 0),
\inf_{v_h \in K^n_h} (\|u - v_h\|_U^2 + \{\gamma_n (v_h - u), \gamma_0 u\}_\Gamma) & (\star = n),
\inf_{v_h \in K^s_h} (\|u - v_h\|_U^2 + \frac{1}{2}(\{\gamma_n u, \gamma_0 (v_h - u)\}_\Gamma + \{\gamma_n (v_h - u), \gamma_0 u\}_\Gamma)) & (\star = s).
\end{cases}
\]

The generic constant depends on \( \Omega \) and \( \beta \) but not on \( \mathcal{T} \).

Proof. By Theorem 1 \( a^\star(\cdot, \cdot) \) is \( U \)-coercive and bounded. Therefore we can follow Falk’s lemma [13] to deduce that

\[
\|u - u_h\|_U^2 \leq a^\star(u - u_h, u - v_h) + a^\star(u - u_h, v_h - u_h)
\]

\[
\leq \|u - u_h\|_U(C^\beta_2 + 1)\|u - v_h\|_U + a^\star(u - u_h, v_h - u_h)
\]

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for all $v_h \in K_h^* \subseteq K^*$. We consider only the case $*=0$. The remaining cases are treated in the same manner. To tackle the last term on the right-hand side we use (11) and (12) to see that
\[
a^0(u-u_h, v_h-u_h) = b(u, \Theta_\beta(v_h-u_h)) - a^0(u_h, v_h-u_h) \\
= \langle \gamma_n u, \gamma_0(v_h-u_h) \rangle + F(v_h-u_h) - a^0(u_h, v_h-u_h) \\
\leq \langle \gamma_n u, \gamma_0(v_h-u_h) \rangle.
\]
Note that the exact solution $u$ satisfies $\gamma_n u \geq 0$ and $\gamma_0 u \gamma_n u = 0$. For the discrete one there holds $\gamma_0 u_h \geq 0$. Hence, the last term further simplifies to
\[
\langle \gamma_n u, \gamma_0(v_h-u_h) \rangle \leq \langle \gamma_n u, \gamma_0 v_h \rangle = \langle \gamma_n u, \gamma_0(v_h-u) \rangle.
\]
Altogether, Young’s inequality with some parameter $\delta > 0$ shows that
\[
\|u-u_h\|_U^2 \lesssim \frac{\delta}{2} \|u-u_h\|_U^2 + \frac{\delta-1}{2}(C_\beta^2 + 1)^2 \|u-v_h\|_U^2 + \langle \gamma_n u, \gamma_0(v_h-u) \rangle
\]
for arbitrary $v_h \in K_h^*$. This proves the a priori estimate. \hfill \Box

**Remark 4.** To deduce convergence rates assume, for instance, that $u \in H^3(\Omega)$ is the solution of (1). Set
\[
v_h := (\Pi_h^0 u, \Pi_h^0 \nabla u, I_h u|_{\partial T}, (\Pi_h^0 \nabla u \cdot n_T|_{\partial T})_{T \in T}) \in U,
\]
where $\Pi_h^0$ and $\Pi_h^0$ are the $L^2(\Omega)$-orthogonal projections onto the element-wise constant spaces, $I_h$ is the nodal interpolant, and $\Pi_h^0$ is the (lowest-order) Raviart-Thomas projector. Note that $I_h$ preserves non-negativity (in particular, on the boundary) and the normal trace of $\Pi_h^0$ is the $L^2(\Gamma)$-orthogonal projection $\gamma_0$ and, hence, preserves non-negativity on the boundary as well. Thus, $v_h \in K_h^*$ for $* \in \{0, n, s\}$. We refer to [9] to see that $\|u-v_h\|_U = O(h)$. To control the boundary terms we note that
\[
\|\langle \gamma_n u, \gamma_0 v_h - \gamma_0 u \rangle \|_\Gamma \leq \|\gamma_n u\|_\Gamma \|u-I_h u\|_\Gamma = O(h^2).
\]
Let $\Gamma_1, \ldots, \Gamma_L \subseteq \Gamma$ be such that $\bigcup \Gamma_j = \Gamma$ and $n_{\Omega}|_{\Gamma_j}$ is constant. Using the projection property we obtain
\[
\|\langle \gamma_n v_h - \gamma_n u, \gamma_0 u \rangle \|_\Gamma \leq \sum_{E \in S_\Gamma} \|\langle \gamma_n v_h - \gamma_n u, \gamma_0 u \rangle \|_E = \sum_{E \in S_\Gamma} \|\langle \gamma_n v_h - \gamma_n u, \gamma_0 u - \Pi_h^0 \gamma_0 u \rangle \|_E.
\]

\[
\lesssim \sum_{E \in S_\Gamma} \|E\| \|\gamma_n u\|_{H^1(E)} \|\gamma_0 u\|_{H^1(E)} \lesssim h^2 \left( \sum_{j=1}^L \|\gamma_n u\|_{H^1(\Gamma_j)} \right)^{1/2} \|\gamma_0 u\|_{H^1(\Gamma)}.
\]

Altogether, using $v_h$ in Theorem 3 we infer
\[
\|u-u_h\|_U^2 \lesssim \|u-v_h\|_U^2 + \left\{ \begin{array}{ll}
|\langle \gamma_n u, \gamma_0(v_h-u) \rangle \|_\Gamma & (\ast = 0) \\
|\langle \gamma_n (v_h-u), \gamma_0 u \rangle \|_\Gamma & (\ast = n) \\
\frac{1}{2} |\langle \gamma_n u, \gamma_0(v_h-u) \rangle \|_\Gamma + |\langle \gamma_n (v_h-u), \gamma_0 u \rangle \|_\Gamma & (\ast = s)
\end{array} \right.
\]
\[
= O(h^2).
\]
With less regularity of $u$ the treatment of the boundary terms becomes more technical. We refer to [12] for details.

**Remark 5.** Our analysis also allows for non-conforming discrete cones $K_h^* \not\subseteq K^*$. Then, an additional consistency error shows up in Theorem 3 that is, in the case $\ast = 0$,
\[
\|u-u_h\|_U^2 \lesssim \inf_{v_h \in K_h^0} (\|u-v_h\|_U^2 + \langle \gamma_n u, \gamma_0 v_h - \gamma_0 u \rangle) + \inf_{v \in K^0} \langle \gamma_n u, \gamma_0 v - \gamma_0 u \rangle \|_\Gamma.
\]

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Again, this a priori estimate can be derived by using Falk’s lemma.

2.4. A posteriori error estimate. We derive a simple error estimator in the symmetric case, i.e., $\ast = s$. Throughout this section we assume that $\beta > 0$ is a fixed constant such that $a^s(\cdot, \cdot)$ is coercive (Theorem 1).

Let $S_T := \{\partial T \cap \Gamma : T \in \mathcal{T}\}$ denote the mesh on the boundary which is induced by the volume mesh $\mathcal{T}$. Furthermore let $u_h \in K^s_h$ denote the unique solution of (12). We define for all $T \in \mathcal{T}$ local volume error indicators by

$$
\eta(T)^2 := \beta \|R_T^{-1} \Theta_T(L - B u_h)\|_{V(T)}^2.
$$

Here, $V(T) := H^1(T) \times H(\text{div}, T)$, $\| \cdot \|_{V(T)}$ denotes the canonical norm on $V(T)$, $R_T : V(T) \to (V(T))^\prime$ is the Riesz isomorphism, and $\Theta_T$ is the dual of the canonical embedding $\iota_T : V(T) \to V$.

Moreover, for all $E \in S_T$ we define local boundary indicators by

$$
\eta(E)^2 := \langle \gamma_n u_h, \gamma_0 u_h \rangle_E.
$$

Note that $u_h \in K^s_h$ implies that $\eta(E)^2 \geq 0$. The overall estimator is then given by

$$
\eta^2 := \sum_{T \in \mathcal{T}} \eta(T)^2 + \sum_{E \in S_T} \eta(E)^2.
$$

Theorem 6. For $\beta \geq 2$ let $u \in K^s$ and $u_h \in K^s_h$ be the solutions of (10) and (12), respectively. Then, there holds the reliability estimate

$$
\|u - u_h\|_U \leq C_{\text{rel}} \eta
$$

with a constant $C_{\text{rel}} > 0$ that depends on $\Omega$ but not on $\mathcal{T}$ or $\beta$.

Proof. By the $U$-coercivity of $a^s(\cdot, \cdot)$ (see Theorem 1) we have

$$
\|u - u_h\|_U^2 \leq a^s(u - u_h, u - u_h) = b(u - u_h, \Theta_\beta(u - u_h)) + \langle \gamma_n(u - u_h), \gamma_0(u - u_h) \rangle_{\Gamma}
$$

$$
= \beta \|B(u - u_h)\|_{V'}^2 + \langle \gamma_n(u - u_h), \gamma_0(u - u_h) \rangle_{\Gamma}.
$$

Note that $u \in K^s$ and $u_h \in K^s_h$ implies that $\langle \gamma_n u, \gamma_0 u \rangle_{\Gamma} \geq 0$ and $\langle \gamma_n u_h, \gamma_0 u_h \rangle_{\Gamma} \geq 0$. Together with $B u = L$ and $\gamma_0 u \gamma_n u = 0$ we obtain the estimate

$$
\|u - u_h\|_U^2 \leq \beta \| L - B u_h \|_{V'}^2 + \langle \gamma_n u_h, \gamma_0 u_h \rangle_{\Gamma} = \sum_{T \in \mathcal{T}} \eta(T)^2 + \sum_{E \in S_T} \eta(E)^2,
$$

which finishes the proof. \hfill \Box

2.5. Technical details. We start by proving boundedness of our trace operators which are specifically modified for the space $U$.

Lemma 7. $\gamma_0 : (U, \| \cdot \|_U) \to (H^{1/2}(\Gamma), \| \cdot \|_{H^{1/2}(\Gamma)})$ and $\gamma_n : (U, \| \cdot \|_U) \to (H^{-1/2}(\Gamma), \| \cdot \|_{H^{-1/2}(\Gamma)})$ are bounded with constant 1.

Proof. Boundedness of these operators follows basically by definition of the corresponding norms, see also [16] Lemma 3. Specifically, the definitions of the norms $\| \cdot \|_{H^{1/2}(\Gamma)}$, $\| \cdot \|_{1/2, \mathcal{S}}$, and $\| \cdot \|_U$ prove boundedness of $\gamma_0$. Note that $\| \cdot \|_{H^{-1/2}(\Gamma)} \leq \| \cdot \|_{-1/2, \mathcal{S}}$ and, thus, the definition of $\| \cdot \|_U$ shows boundedness of $\gamma_n$.

By the boundedness of the operators $\gamma_0$ and $\gamma_n$ we immediately establish the following result.

Lemma 8. The sets $K^* (\ast \in \{0, n, s\})$ are non-empty, closed, convex subsets of $U$. 

The following steps are to characterize the kernel of the operator $B$. This kernel is non-trivial since $B$ does not include any boundary condition. Our procedure is similar to the one in [16].

For a function $v \in H^{1/2}(\Gamma)$, we define its quasi-harmonic extension $\widetilde{u} \in H^1(\Omega)$ as the unique solution of

$$(-\Delta \widetilde{u} + \widetilde{u} = 0 \quad \text{in } \Omega, \quad \widetilde{u}|_{\Gamma} = v.)$$

Note that the infimum in the definition of $\|v\|_{H^{1/2}(\Gamma)}$ is attained for the function $\widetilde{u}$, i.e., $\|\widetilde{u}\|^2 + \|\nabla \widetilde{u}\|^2 = \|v\|^2_{H^{1/2}(\Gamma)}$. Then, define the operator $\mathcal{E} : H^{1/2}(\Gamma) \to U$ by

$$\mathcal{E} v := (\widetilde{u}, \mathbf{\sigma}, \widetilde{\tau}), \quad \text{where } \mathbf{\sigma} := \nabla \widetilde{u}, \quad \widetilde{u}|_{\partial T} := \widetilde{u}|_{\partial T}, \quad \widetilde{\tau}|_{\partial T} := \nabla \widetilde{u} \cdot \mathbf{n}_T|_{\partial T} \quad \text{for all } T \in T.$$ 

The range of this operator is the kernel of $B$. We combine important properties of $B$ and $\mathcal{E}$ in the following lemma.

**Lemma 9.** The operators $B : U \to V'$, $\mathcal{E} : H^{1/2}(\Gamma) \to U$ have the following properties:

(i) $B$ and $\mathcal{E}$ are bounded. Specifically there holds

$$\|Bu\|_{V'} \lesssim \|u\|_U \quad \text{for all } u \in U,$$

$$\|v\|_{H^{1/2}(\Gamma)} \leq \|\mathcal{E} v\|_U \leq \sqrt{3}\|v\|_{H^{1/2}(\Gamma)} \quad \text{for all } v \in H^{1/2}(\Gamma).$$

(ii) The kernel of $B$ consists of all quasi-harmonic extensions, i.e., $\ker(B) = \text{ran}(\mathcal{E})$.

(iii) $\mathcal{E}$ is a right-inverse of $\gamma_0$.

(iv) $B : U/\ker(B) \to V'$ is inf-sup stable,

$$\|u - \mathcal{E} \gamma_0 u\|_U \lesssim \|Bu\|_{V'} = \sup_{v \in V'} \frac{(Bu, v)}{\|v\|_V} = b(u, \Theta u)^{1/2} \quad \text{for all } u \in U.$$ 

The involved constant depends only on $\Omega$.

**Proof.** In the case of the Poisson equation these results have been established in [16]. In a recent work [17] we analyzed a time-stepping scheme for the heat equation which naturally leads to the equation $-\Delta u + \delta^{-1} u = f$ where $\delta$ corresponds to a time step $\delta$. Setting $\delta = k_n = 1$ in [17] Lemma 8] we obtain boundedness of the operator $B$ and stability

$$\|u_0\|_U \lesssim \|Bu_0\|_{V'} \quad \text{for all } u_0 \in U \text{ with } \gamma_0 u_0 = 0.$$ 

By definition of $\mathcal{E}$ one sees (iii). Furthermore, integration by parts shows $\text{ran}(\mathcal{E}) \subseteq \ker(B)$. For $u \in U$ we define $u_0 := u - \mathcal{E} \gamma_0 u$ and infer

$$\|u - \mathcal{E} \gamma_0 u\|_U \lesssim \|Bu - \mathcal{E} \gamma_0 u\|_{V'} = \|Bu\|_{V'},$$

which proves (iv) as well as $\ker(B) \subseteq \text{ran}(\mathcal{E})$, hence, (iii).

It remains to show the relations for $\mathcal{E}$ in (i). Let $u = (u, \mathbf{\sigma}, \tilde{\mathbf{\sigma}}) = \mathcal{E} v$ for $v \in H^{1/2}(\Gamma)$. Then,

$$\|\mathcal{E} v\|_{\mathcal{L}}^2 \geq \|u\|^2 + \|\mathbf{\sigma}\|^2 = \|u\|^2 + \|\nabla u\|^2 = \|v\|^2_{H^{1/2}(\Gamma)}.$$ 

On the other hand, using that $\Delta u = u$ by construction of $\mathcal{E} v$, we deduce that

$$\|\mathcal{E} v\|_{\mathcal{L}}^2 = \|u\|^2 + \|\mathbf{\sigma}\|^2 + \|\tilde{\mathbf{\sigma}}\|^2_{1/2, S} + \|\tilde{\mathbf{\sigma}}\|^2_{-1/2, \Gamma} \leq \|u\|^2 + \|\nabla u\|^2 + \|u\|^2 + \|\nabla u\|^2 + \|\Delta u\|^2 = 3\|u\|^2 + 3\|\nabla u\|^2 = 3\|v\|^2_{H^{1/2}(\Gamma)}.$$ 

This concludes the proof. \[\square\]

In Lemma [11] below we give an explicit bound of the control of the Neumann trace for functions of the quotient space $U/\ker(B)$. For its proof we need the following technical result.
Lemma 10. Let \( \hat{\nu} \in H^{1/2}(\Gamma) \). The problem
\[
\begin{align*}
(15a) & \quad \text{div} \tau + v = 0 \quad \text{in } \Omega, \\
(15b) & \quad \tau + \nabla v = 0 \quad \text{in } \Omega, \\
(15c) & \quad v|_{\Gamma} = \hat{\nu}
\end{align*}
\]
admits a unique solution \((v, \tau) \in H^{1}(\Omega) \times H(\text{div}, \Omega)\) with \(\Delta v \in H^1(\Omega)\) and
\[
\|\tau\|^2 + \|\text{div } \tau\|^2 = \|\nabla v\|^2 + \|v\|^2 = \|\hat{\nu}\|^2_{H^{1/2}(\Gamma)}.
\]

Proof. Let \( v \in H^1(\Omega) \) be the unique solution of
\[
-\Delta v + v = 0 \quad \text{in } \Omega, \quad v|_{\Gamma} = \hat{\nu}.
\]
Then, \( \|\nabla v\|^2 + \|v\|^2 = \|\hat{\nu}\|^2_{H^{1/2}(\Gamma)} \) by definition of the latter norm. Define \( \tau := -\Delta v \in L^2(\Omega) \). Since \( \Delta v = v \in H^1(\Omega) \), we have \( \tau \in H(\text{div}, \Omega) \), and \( \text{div } \tau = -\Delta v = -v \) shows 15a. To see unique solvability, let additionally \((v_2, \tau_2) \in H^1(\Omega) \times H(\text{div}, \Omega)\) solve 15. The difference \( w := v - v_2 \) satisfies \( -\Delta w + w = 0 \) in \( \Omega \) with \( w|_{\Gamma} = 0 \). Thus \( w = 0 \) and \( \tau = -\nabla v = -\nabla v_2 = \tau_2 \) as well. \( \square \)

Lemma 11. There holds
\[
\|\gamma_n(u - \mathcal{E}v_0u)\|_{H^{1/2}(\Gamma)} \leq \sqrt{2} \|Bu\|_{V'} \quad \text{for all } u \in U.
\]

Proof. Let \( \hat{\nu} \in H^{1/2}(\Gamma) \) and choose the test function \( v = (v, \tau) \in H^1(\Omega) \times H(\text{div}, \Omega) \subseteq V \) to be the solution of 15. By Lemma 10 there holds \( \|v\|_{V} = \sqrt{2} \|\hat{\nu}\|_{H^{1/2}(\Gamma)} \). Then, by the definition of the bilinear form \( b(\cdot, \cdot) \), and since \( \gamma_0(u - \mathcal{E}v_0u) = 0 \), we have
\[
\langle \gamma_n(u - \mathcal{E}v_0u), \hat{\nu}\rangle_{\Gamma} = |b(u - \mathcal{E}v_0u, v)| = \|b(u, v)\| \leq \|Bu\|_{V'} \|v\|_{V} = \sqrt{2} \|Bu\|_{V'} \|\hat{\nu}\|_{H^{1/2}(\Gamma)},
\]
where we have used that \( \mathcal{E}v_0u \in \ker(B) \). Dividing by \( \|\hat{\nu}\|_{H^{1/2}(\Gamma)} \) and taking the supremum over all \( \hat{\nu} \in H^{1/2}(\Gamma) \setminus \{0\} \), this proves 16. \( \square \)

2.6. Proof of Theorem 1. First, we prove boundedness and coercivity of \( a^\theta(\cdot, \cdot) \). Then, the Lions-Stampacchia theorem, see, e.g., 19, 20, 32, proves unique solvability of 10 provided that \( F : U \to \mathbb{R} \) is a linear functional, which follows by the boundedness of \( \Theta : U \to \mathbb{R} \).

We show the boundedness. Note that \( b : U \times V \to \mathbb{R} \) and \( \Theta : U \to V \) are uniformly bounded,
\[
|b(u, \Theta v)| \leq \beta C_2 \|u\|_{U} \|v\|_{V} \leq \beta C_2^2 \|u\|_{U} \|v\|_{V} \quad \text{for all } u, v \in U.
\]
By duality and boundedness of \( \gamma_0, \gamma_n \) (see Lemma 7), we have
\[
\langle \gamma_n(u, \gamma_0 v) \rangle_{\Gamma} \leq \|\gamma_n u\|_{H^{-1/2}(\Gamma)} \|\gamma_0 v\|_{H^{1/2}(\Gamma)} \leq \|u\|_{U} \|v\|_{V} \quad \text{for all } u, v \in U.
\]
Next, we follow 16 to prove coercivity. Note that \( a^0(u, u) = a^\theta(u, u) = a(u, u) \) for all \( u \in U \). Let \( u \in U \). By the triangle inequality and Lemma 9 we get
\[
\|u\|_{U}^2 \leq \|u - \mathcal{E}v_0u\|_{U}^2 + \|\mathcal{E}v_0u\|_{U}^2 \leq \|Bu\|_{V'}^2 + \|\gamma_0 u\|_{H^{1/2}(\Gamma)}^2.
\]
Let \( \tilde{u} \in H^1(\Omega) \) be the quasi-harmonic extension 14 of \( \gamma_0 u \). The definition of the \( H^{1/2}(\Gamma) \)-norm and integration by parts show that
\[
\|\gamma_0 u\|_{H^{1/2}(\Gamma)}^2 = \|\tilde{u}\|_{H^1(\Omega)}^2 = \|	ilde{u}\|_{L^2(\Omega)}^2 + \|\nabla \tilde{u}\|_{L^2(\Omega)}^2 = \langle \partial_n \tilde{u}, \tilde{u}\rangle_{\Gamma} = \langle \gamma_n \mathcal{E}v_0u, \gamma_0 u \rangle_{\Gamma} + \langle \gamma_n u, \gamma_0 u \rangle_{\Gamma}.
\]
This gives
\[
\|Bu\|_{V'}^2 + \langle \gamma_n \mathcal{E}v_0u, \gamma_0 u \rangle_{\Gamma} = \|Bu\|_{V'}^2 + \langle \gamma_n \mathcal{E}v_0u - \gamma_n u, \gamma_0 u \rangle_{\Gamma} + \langle \gamma_n u, \gamma_0 u \rangle_{\Gamma}.
\]
Using duality, Lemma 11 and Young’s inequality with some parameter \( \delta > 0 \), we obtain
\[
\langle \gamma_n \mathcal{E} \gamma_0 u - \gamma_n u, \gamma_0 u \rangle_\Gamma \leq \| \gamma_n \mathcal{E} \gamma_0 u - \gamma_n u \|_{H^{-1/2}(\Gamma)}\| \gamma_0 u \|_{H^{1/2}(\Gamma)} \leq \sqrt{2}\| Bu \|_{V'}\| \gamma_0 u \|_{H^{1/2}(\Gamma)}
\]
\[ \leq \delta^{-1}\| Bu \|_{V'}^2 + \delta\| \gamma_0 u \|_{H^{1/2}(\Gamma)}^2. \]

For \( \delta = 1 \) we get
\[ \| Bu \|_{V'}^2 + \frac{1}{2}\| \gamma_0 u \|_{H^{1/2}(\Gamma)}^2 \leq 2\| Bu \|_{V'}^2 + \langle \gamma_n u, \gamma_0 u \rangle_\Gamma. \]

Thus,
\[ C_1^{-1}\| u \|_U^2 \leq \beta\| Bu \|_{V'}^2 + \langle \gamma_n u, \gamma_0 u \rangle_\Gamma = a^*(u, u) \]
for all \( \beta \geq 2 \), where the constant \( C_1 > 0 \) depends only on \( \Omega \).

Regarding the equivalence of problems (1) and (10) we know that the unique solution \( u \) of (1) with \( u \) defined as in the assertion satisfies (10) by construction. The other direction follows by existence of a unique solution of (10).

Finally, note that (11) follows by construction. However, in the case \( \star = 0 \) we can infer this identity directly from (10). Let \( u \) denote the solution of (10) with \( \star = 0 \). Set \( v = u \pm (w - \mathcal{E} \gamma_0 w) \) for some arbitrary \( w \in U \). Since \( \gamma_0 (w - \mathcal{E} \gamma_0 w) = 0 \), we infer \( v \in K^0 \), so that we can use it as a test function in (10). This gives
\[ \pm a^0(u, w - \mathcal{E} \gamma_0 w) \geq \pm F(w - \mathcal{E} \gamma_0 w). \]

Hence,
\[ a^0(u, w - \mathcal{E} \gamma_0 w) = F(w - \mathcal{E} \gamma_0 w) \quad \text{for all } w \in U. \]

Note that \( \mathcal{E} \gamma_0 w \in \ker B = \ker \Theta \). This leads to
\[
F(w - \mathcal{E} \gamma_0 w) = L(\Theta_\beta(w - \mathcal{E} \gamma_0 w)) = L(\Theta_\beta w) = F(w), \quad \text{and}
\]
\[ a^0(u, w - \mathcal{E} \gamma_0 w) = b(u, \Theta_\beta w) + \langle \gamma_n u, \gamma_0 (w - \mathcal{E} \gamma_0 w) \rangle_\Gamma = b(u, \Theta_\beta w). \]

This concludes the proof.

3. DPG Method for a Singularly Perturbed Problem

In this section we introduce and analyze a DPG method for the Signorini problem of reaction dominated diffusion, that is, for (1) with small positive constant \( c = \varepsilon \) (0 < \( \varepsilon \) ≤ 1).

We mainly stick to the notation as given in Section 2 but need some additional definitions. Also, we redefine some objects like the bilinear form \( b: U \times V \to \mathbb{R} \), the spaces \( U, V \), and some norms. In fact, our objective of robust control of field variables forces us to carefully scale parts of norms with coefficients depending on the diffusion parameter \( \varepsilon \).

3.1. Notation. For functions \( \hat{u} \in H^{1/2}(S) \), \( \hat{\sigma} \in H^{-1/2}(S) \) we define the skeleton norms
\[
\| \hat{u} \|_{1/2,S} := \inf \left\{ (\| w \|^2 + \varepsilon^{1/2}\| \nabla w \|^2)^{1/2} : w \in H^1(\Omega), \hat{u}|_{\partial T} = w|_{\partial T} \forall T \in \mathcal{T} \right\},
\]
\[
\| \hat{\sigma} \|_{-1/2,S} := \inf \left\{ (\| q \|^2 + \varepsilon\| \text{div } q \|^2)^{1/2} : q \in H(\text{div }, \Omega), \hat{\sigma}|_{\partial T} = (q \cdot n_T)|_{\partial T} \forall T \in \mathcal{T} \right\}.
\]

Moreover, for \( \hat{u} \in H^{1/2}(\Gamma) \), \( \hat{\sigma} \in H^{-1/2}(\Gamma) \) we define the boundary norms
\[
\| \hat{u} \|_{1/2,\Gamma} := \inf \left\{ (\| w \|^2 + \varepsilon^{1/2}\| \nabla w \|^2)^{1/2} : w \in H^1(\Omega), w|_{\Gamma} = \hat{u} \right\},
\]
\[
\| \hat{\sigma} \|_{-1/2,\Gamma} := \inf \left\{ (\| q \|^2 + \varepsilon\| \text{div } q \|^2)^{1/2} : q \in H(\text{div }, \Omega), (q \cdot n_\Omega)|_{\Gamma} = \hat{\sigma} \right\}.
\]
We will need another norm in $H^{1/2}(\Gamma)$ defined by
\[\|\tilde{u}\|_{H^{1/2}(\Gamma)} := \inf \left\{ (\|w\|^2 + \varepsilon \|\nabla w\|^2)^{1/2} : w \in H^1(\Omega), w|\Gamma = \tilde{u} \right\}.\]

Obviously, the latter norm is weaker than the previously defined corresponding boundary norm, $\| \cdot \|_{H^{1/2}(\Gamma)} \leq \| \cdot \|_{1/2,\Gamma}$. The ultra-weak formulation from [22] is based on the spaces $U$ and $V$ defined by
\[\tilde{U} := L^2(\Omega) \times [L^2(\Omega)]^d \times L^2(\Omega) \times H^{1/2}(S) \times H^{1/2}(S) \times H^{-1/2}(S) \times H^{-1/2}(S),\]
\[U := \left\{ (u, \sigma, \rho, \tilde{u}, \tilde{u}, \tilde{\sigma}, \tilde{\sigma}) \in \tilde{U} : \tilde{u}|\Gamma = \tilde{u}|\Gamma \right\},\]
\[V := H^1(T) \times H(\text{div}, T) \times H^1(\Delta, T), \quad \text{where}\]
\[H^1(\Delta, T) := \{ w \in H^1(T) : \Delta w|_T \in L^2(T) \forall T \in T \}.\]

For the analysis we need two different norms in $U$,
\[\|u\|_{U,1}^2 := \|u\|^2 + \|\sigma\|^2 + \varepsilon \|\rho\|^2 + \varepsilon^{3/2}\|\tilde{u}\|^2 + \varepsilon^{3/2}\|\tilde{\sigma}\|^2 + \varepsilon^{5/2}\|\tilde{\sigma}\|^2,\]
\[\|u\|_{U,2}^2 := \|u\|^2 + \|\sigma\|^2 + \varepsilon \|\rho\|^2 + \varepsilon^{1/2}\|\tilde{u}\|^2 + \varepsilon^{1/2}\|\tilde{\sigma}\|^2 + \varepsilon^{1/2}\|\tilde{\sigma}\|^2\]
for $u = (u, \sigma, \rho, \tilde{u}, \tilde{u}, \tilde{\sigma}, \tilde{\sigma}) \in U$. These norms differ in their $\varepsilon$-scalings of the skeleton components so that $\| \cdot \|_{U,1} \leq \| \cdot \|_{U,2}$. In both cases, field components are measured in the so-called balanced norm $(\|u\|^2 + \|\sigma\|^2 + \varepsilon \|\rho\|^2)^{1/2}$ which, for the exact solution, is $(\|u\|^2 + \varepsilon^{1/2}\|\nabla u\|^2 + \varepsilon^{3/2}\|\Delta u\|^2)^{1/2}$, cf. [23, 22].

The test space $V$ is equipped with the norm
\[\|v\|_V^2 := \varepsilon^{-1}\|\mu\|^2 + \|\nabla_T \mu\|^2 + \varepsilon^{-1/2}\|\tau\|^2 + \|\text{div}_T \tau\|^2 + \|v\|^2 + (\varepsilon^{1/2} + \varepsilon)\|\nabla_T v\|^2 + \varepsilon^{3/2}\|\Delta_T v\|^2 \quad \text{for} \ v = (\mu, \tau, v) \in V.\]

This norm is induced by the inner product $(\cdot, \cdot)_V$ on $V$. Note that this norm is equivalent to the one defined in [22]. The only difference is that the term $\varepsilon\|\nabla_T v\|^2$ is not present in [22]. We use this norm here, to get a smaller constant in Lemma [21] below.

3.2. Ultra-weak formulation. The ultra-weak formulation taken from [22] is derived by rewriting problem [1] (with $\varepsilon = \varepsilon$) as a first order system,
\[\rho - \text{div} \sigma = 0, \quad \varepsilon^{-1/4} \sigma - \nabla u = 0, \quad -\varepsilon^{3/4} \rho + u = f.\]

We test the first two equations with $\mu \in H^1(T)$ resp. $\tau \in H(\text{div}, T)$ element-wise, integrate by parts and sum up over all elements. The third equation is tested with $v - \varepsilon^{1/2}\Delta_T v$ for $v \in H^1(\Delta, T)$. Integrating by parts and using the first two equations, we get the formulation
\[(17a) \quad (\rho, \mu) + (\sigma, \nabla_T \mu) - \langle \tilde{\sigma}, \mu \rangle_S = 0,\]
\[(17b) \quad \varepsilon^{-1/4}(\sigma, \tau) + (u, \text{div}_T \tau) - \langle \tilde{u}, \tau \cdot n \rangle_S = 0,\]
\[(17c) \quad \varepsilon^{3/4}(\sigma, \nabla_T v) - \varepsilon^{3/4}(\tilde{\sigma}, v)_S + (u, v) + \varepsilon^{5/4}(\rho, \Delta_T v) + \varepsilon^{1/4}(\sigma, \nabla_T v) - \varepsilon^{1/2}(\tilde{u}, \nabla_T v \cdot n)_S = (f, v - \varepsilon^{1/2}\Delta_T v).\]
The left and right-hand sides give rise to the definition of the bilinear form $b : U \times V \to \mathbb{R}$ and the linear functional $L : V \to \mathbb{R}$

$$b(u, v) := (\rho, \mu) + (\sigma, \nabla \tau u) - \langle \tilde{a}^a, \mu \rangle_S + \varepsilon^{-1/4} \langle \sigma, \tau \rangle + (u, \text{div} \tau) - \langle \tilde{u}^a, \tau \cdot n \rangle_S$$

$$+ \varepsilon^{3/4} \langle \sigma, \nabla v \rangle - \varepsilon^{3/4} \langle \tilde{a}^b, v \rangle_S + (u, v)$$

$$+ \varepsilon^{5/4} \langle \rho, \Delta \tau v \rangle + \varepsilon^{1/4} \langle \sigma, \nabla v \rangle - \varepsilon^{1/2} \langle \tilde{a}^b, \nabla v \cdot n \rangle_S,$$

$$L(v) := (f, v - \varepsilon^{1/2} \Delta \tau v),$$

for all $u = (u, \sigma, \rho, \tilde{a}^a, \tilde{a}^b, \tilde{\sigma}^a, \tilde{\sigma}^b) \in U, v = (\mu, \tau, v) \in V$.

The trial-to-test operator $\Theta_\beta : U \to V$ is defined as before, see (5). Again, the operator $B : U \to V'$ is induced by the bilinear form $b(\cdot, \cdot)$ and (17) can be written as

$$u \in U : \quad Bu = L.$$

The non-trivial kernel of $B$ is related to the trace operators. For the present space $U$ we define them by

$$\gamma_0 : U \to H^{1/2}(\Gamma), \quad \gamma_0 u := \tilde{a}^a|_\Gamma,$$

$$\gamma_n : U \to H^{-1/2}(\Gamma), \quad \gamma_n u := \tilde{a}^a|_\Gamma.$$

For simplicity we only consider the symmetric formulation. The other cases can be derived similarly, see also Section 2. Analogously as in the unperturbed case we introduce the non-empty convex subset

$$K^s = \{ u \in U : \gamma_0 u \geq 0, \gamma_n u \geq 0 \},$$

and define the bilinear form $a^s : U \times U \to \mathbb{R}$ and linear functional $F : U \to \mathbb{R}$ by

$$a^s(u, v) := b(u, \Theta_\beta v) + \frac{1}{2} \varepsilon^{1/4} (\langle \gamma_n u, \gamma_0 v \rangle_\Gamma + \langle \gamma_n v, \gamma_0 u \rangle_\Gamma),$$

$$F(v) := L(\Theta_\beta v)$$

for $u, v \in U$. Here, $\beta > 0$ is a constant to be fixed. Then, our variational inequality reads: Find $u \in K^s$ such that

$$a^s(u, v - u) \geq F(v - u) \quad \text{for all } v \in K^s. \tag{18}$$

In the singularly perturbed case it is convenient to state coercivity and boundedness of the bilinear form $a^s(\cdot, \cdot)$ in the energy-based norm

$$\|u\|^2 := \|Bu\|^2_{V'} + \varepsilon^{-1/2} \|\gamma_0 u\|^2_{H^{1/2}(\Gamma)} \quad (u \in U).$$

Note that $\|B \cdot \|_{V'}$ is the energy norm in standard DPG settings whereas in our case it is a semi-norm.

Corresponding to Theorem 1 we have the following result.

**Theorem 12.** For all $\beta \geq 3$ the bilinear form $a^s : U \times U \to \mathbb{R}$ is coercive,

$$\|u\|^2 \leq C_1 a^s(u, u) \quad \text{for all } u \in U,$$

and bounded,

$$|a^s(u, v)| \leq C_2 \|u\| \|v\| \quad \text{for all } u, v \in U.$$

The constants $C_1, C_2 > 0$ do not depend on $\Omega$, $\mathcal{T}$, or $\varepsilon$. $C_1$ is independent of $\beta$ but $C_2$ is not. Furthermore,

$$\|u\|_{U; 1} \leq \|u\| \leq \|u\|_{U; 2} \quad \text{for all } u \in U \tag{19}$$

with generic constants that are independent of $\mathcal{T}$ and $\varepsilon$.

The variational inequality (18) is uniquely solvable and equivalent to problem (1) (setting $c = \varepsilon$) in the following sense: If $u \in H^1(\Omega)$ solves problem (1), then $u = (u, \sigma, \rho, \tilde{a}^a, \tilde{a}^b, \tilde{\sigma}^a, \tilde{\sigma}^b) \in K^s$ with
Theorem 13. \( \sigma := \varepsilon^{1/4} \nabla u, \quad \tilde{u}^*|_{\partial T} := u|_{\partial T}, \quad \tilde{\sigma}^*|_{\partial T} := \sigma \cdot n|_{\partial T} \) for all \( T \in \mathcal{T} \) \((* \in \{a,b\})\) solves (18). On the other hand, if \( u = (u, \sigma, \rho, \tilde{u}^a, \tilde{\sigma}^a, \tilde{\tau}^a) \in K^a \) solves (18), then \( u \in H^1(\Omega) \) solves (1).

Moreover, the unique solution \( u \in K^a \) of (18) satisfies
\[
b(u, \theta_\beta w) = F(w) \quad \text{for all} \ w \in U.
\]

We prove this result in Section 3.5.

3.3. Discretization, convergence and a posteriori error estimate. We replace \( U \) by the lowest order subspace
\[
U_h := P^0(\mathcal{T}) \times [P^0(\mathcal{T})]^d \times P^0(\mathcal{T}) \times S^1(\mathcal{S}) \times S^1(\mathcal{S}) \times P^0(\mathcal{S}) \times P^0(\mathcal{S})
\]
and \( K^* \) by
\[
K_h^* := \{ v_h \in U_h : \gamma_0 v_h \geq 0, \gamma_n v_h \geq 0 \}.
\]
The discrete version of (18) then reads: Find \( u_h \in K_h^* \) such that
\[
a^*(u_h, v_h - u_h) \geq F(v_h - u_h) \quad \text{for all} \ v_h \in K_h^*.
\]
With the same arguments as in Section 2.3 we can prove unique solvability.

**Theorem 13.** Under the same assumptions as in Theorem 12 the discrete variational inequality (20) admits a unique solution \( u_h \in K_h^* \).

We have the following robust quasi-optimal a priori error estimate. Here, robustness means that the hidden constant does not depend on the positive perturbation parameter \( \varepsilon \) (though, for simplicity we have assumed that \( \varepsilon \leq 1 \)).

**Theorem 14.** For \( \beta \geq 3 \) let \( u \in K^a, u_h \in K_h^* \) denote the exact solutions of (18), (20). Then there holds
\[
\|u - u_h\|_{U,1}^2 \lesssim \inf_{v_h \in K_h^*} \left( \|u - v_h\|_{U,2}^2 + \frac{1}{2} \varepsilon^{1/4} (\gamma_n u, \gamma_0 (v_h - u))_\Gamma + (\gamma_n (v_h - u), \gamma_0 u)_\Gamma \right).
\]
The generic constant does depend on \( \Omega \) and \( \beta \) but not on \( \mathcal{T} \) or \( \varepsilon \).

**Proof.** Following the proof of Theorem 3 we obtain (by replacing \( \| \cdot \|_U \) with \( \| \cdot \|_1 \))
\[
\|u - u_h\|_1^2 \lesssim \inf_{v_h \in K_h^*} \left( \|u - v_h\|_1^2 + \frac{1}{2} \varepsilon^{1/4} (\gamma_n u, \gamma_0 (v_h - u))_\Gamma + (\gamma_n (v_h - u), \gamma_0 u)_\Gamma \right).
\]
Then, (19) proves the error bound. \( \square \)

The derivation of an a posteriori error estimate is analogous to Section 2.4. For a function \( u_h \in K_h^* \) we define local error indicators by
\[
\eta(T)^2 := \beta \| R_T^{-1} \nu_T^*(L - B u_h) \|_{V(T)}^2,
\]
\[
\eta(E)^2 := \varepsilon^{1/4} (\gamma_n u_h, \gamma_0 u_h)_E,
\]
and the overall estimator
\[
\eta^2 := \sum_{T \in \mathcal{T}} \eta(T)^2 + \sum_{E \in \mathcal{S}} \eta(E)^2.
\]
Here, \( V(T) := H^1(T) \times \mathbf{H}(\text{div}, T) \times H^1(\Delta, T) \) is equipped with the norm
\[
\|(\mu, \tau, v)\|_{V(T)} := \varepsilon^{-1/2} \|\mu\|_T^2 + \|\nabla \mu\|_T^2 + \varepsilon^{-1/2} \|\tau\|_T^2 + \|\text{div} \tau\|_T^2 + \|v\|_T^2 + (\varepsilon^{1/2} + \varepsilon) \|\nabla v\|_T^2 + \varepsilon^{3/2} \|\Delta v\|_T^2.
\]
\( R_T : V(T) \to (V(T))' \) denotes the Riesz isomorphism, and \( \nu_T^* \) is the dual of the canonical embedding \( \nu_T : V(T) \to V \).
Analogously to the proof of Theorem 6 in conjunction with (19), we obtain the following a posteriori estimate. Like the a priori estimate from Theorem 14, the a posteriori estimate is robust with respect to $\varepsilon$ ($0 < \varepsilon \leq 1$).

**Theorem 15.** For $\beta \geq 3$ let $u \in K^s$ and $u_h \in K^s_h$ be the solutions of (18) and (20), respectively. Then, there holds the reliability estimate

$$\|u - u_h\| \leq C_{\text{rel}} \eta.$$

The constant $C_{\text{rel}} > 0$ is independent of $\Omega$, $T$, $\beta$ and $\varepsilon$. In particular, with (19) we have

$$\|u - u_h\|_{U,1} \leq C_{\text{rel},U} \eta,$$

where $C_{\text{rel},U} := C_{\text{rel}} C_1$ and $C_1$ depends only on $\Omega$.

### 3.4. Technical details

Analogously as in Lemma 7, we obtain boundedness of the trace operators. In this case we use that, by definition of the norms, $\|\hat{\gamma}^a|_{\Gamma}\|_{1/2,\Gamma} \leq \|\hat{\gamma}^a\|_{1/2,S}$ and $\|\hat{\gamma}^a|_{\Gamma}\|_{-1/2,\Gamma} \leq \|\hat{\gamma}^a\|_{-1/2,S}$.

**Lemma 16.** The operators $\gamma_0 : (U, \| \cdot \|_{U,2}) \to (H^{1/2}(\Gamma), \| \cdot \|_{1/2,\Gamma})$ and $\gamma_n : (U, \| \cdot \|_{U,2}) \to (H^{-1/2}(\Gamma), \| \cdot \|_{-1/2,\Gamma})$ are bounded with constant 1.

We now adapt the definition of the previously employed extension operator $E$ to the current situation.

For a function $\hat{v} \in H^{1/2}(\Gamma)$ we define its quasi-harmonic extension $\tilde{u} \in H^1(\Omega)$ as the unique solution of

$$-\varepsilon \Delta \tilde{u} + \tilde{u} = 0 \quad \text{in} \quad \Omega, \quad \tilde{u}|_{\Gamma} = \hat{v},$$

and define $E : H^{1/2}(\Gamma) \to U$ by

$$E \hat{v} := (\hat{u}, \varepsilon^{1/4} \nabla \hat{u}, \varepsilon^{1/4} \Delta \hat{u}, \hat{u}|_S, \hat{u}|_S, \varepsilon^{1/4} (\nabla \hat{u} \cdot n_T)_{\partial T} |_{T \in T}, \varepsilon^{1/4} (\nabla \hat{u} \cdot n_T)_{\partial T} |_{T \in T}).$$

This operator characterizes the kernel of $B$. Note that, in contrast to $\| \cdot \|_{1/2,\Gamma}$, the norm $\| \cdot \|_{H^{1/2}(\Gamma)}$ (recall the definitions in Section 3.1) is inherited from the energy norm associated to problem (21).

**Lemma 17.** For given $\hat{v} \in H^{1/2}(\Gamma)$ let $\tilde{u} \in H^1(\Omega)$ be the unique solution of (21). Then,

$$\|\hat{v}\|^2_{H^{1/2}(\Gamma)} = \varepsilon \|\nabla \hat{u}\|^2 + \|\hat{u}\|^2 = \varepsilon \langle \partial_n \hat{u}, \hat{v} \rangle_{\Gamma} = \varepsilon^{3/4} \langle \gamma_n E \hat{v}, \hat{v} \rangle_{\Gamma}.$$

**Proof.** The last identity follows by definition of the operator $E$. Using the weak formulation of problem (21) we have

$$\varepsilon \|\nabla \hat{u}\|^2 + \|\hat{u}\|^2 = \varepsilon \langle \partial_n \hat{u}, \hat{v} \rangle_{\Gamma} = \varepsilon \langle \nabla \hat{u}, \nabla w \rangle + \langle \hat{u}, w \rangle$$

$$\leq \left( \varepsilon \|\nabla \hat{u}\|^2 + \|\hat{u}\|^2 \right)^{1/2} \left( \varepsilon \|\nabla w\|^2 + \|w\|^2 \right)^{1/2}$$

for all $w \in H^1(\Omega)$ with $w|_{\Gamma} = \hat{v}$. Thus, $\|\hat{v}\|^2_{H^{1/2}(\Gamma)} = \varepsilon \|\nabla \hat{u}\|^2 + \|\hat{u}\|^2$. \hfill $\square$

Similarly to Lemma 9, there holds the following result in the singularly perturbed case.

**Lemma 18.** The operators $B : U \to V^\prime$, $E : H^{1/2}(\Gamma) \to U$ have the following properties:

(i) $B$ and $E$ are bounded,

$$\|Bu\|_{V^\prime} \lesssim \|u\|_{U,2} \quad \text{for all} \quad u \in U, \quad \|E\hat{v}\|_{U,1} \simeq \varepsilon^{-1/4} \|\hat{v}\|_{H^{1/2}(\Gamma)} \quad \text{for all} \quad \hat{v} \in H^{1/2}(\Gamma).$$

The generic constants are independent of $T$ and $\varepsilon$.

(ii) The kernel of $B$ consists of all quasi-harmonic extensions, i.e., $\ker(B) = \text{ran}(E)$.

(iii) $E$ is a right-inverse of $\gamma_0$. 

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(iv) \( B : U / \ker(B) \rightarrow V' \) is inf-sup stable,

\[
\|u - \mathcal{E} \gamma_0 u \|_{U', 1} \lesssim \|Bu\|_{V'} = \sup_{v \in V} \frac{(Bu, v)}{\|v\|_V} = b(u, \Theta u)^{1/2} \quad \text{for all } u \in U.
\]

The generic constant depends on \( \Omega \) but not on \( \mathcal{T} \) or \( \epsilon \).

**Proof.** First, by [22, Lemma 3] we have \( b(u, v) \lesssim \|u\|_{U', 2}\|v\|_V \) for all \( u \in U, v \in V \). Dividing by \( \|v\|_V \) and taking the supremum proves boundedness of \( B \).

Second, for \( \hat{\nu} \in H^{1/2}(\Gamma) \), let \( u = (u, \sigma, \rho, \hat{\nu}, \hat{\sigma}^a, \hat{\sigma}^b) = \mathcal{E} \hat{\nu} \). By definition of the skeleton norms we have

\[
\|\hat{u}\|^2_{1/2, S} \leq \|u\|^2 + \|\sigma\|^2, \quad \|\hat{\sigma}\|^2_{-1/2, S} \leq \|\sigma\|^2 + \epsilon \|\rho\|^2 \quad (\ast \in \{a, b\}).
\]

In the definition of \( \|\cdot\|_{U', 1} \), these norms are scaled with positive powers of \( \epsilon \). Thus,

\[
\|u\|^2 + \|\sigma\|^2 + \epsilon \|\rho\|^2 \leq \|u\|^2_{U', 1} \leq \|u\|^2 + \|\sigma\|^2 + \epsilon \|\rho\|^2.
\]

Using that \( \rho = \epsilon^{1/4} \Delta u = \epsilon^{1/4} \epsilon^{-1} u = \epsilon^{-3/4} u \) and \( \sigma = \epsilon^{1/4} \nabla u \) by the definition of \( u = \mathcal{E} \hat{\nu}, \) cf. (21), we obtain

\[
\|u\|^2 + \|\sigma\|^2 + \epsilon \|\rho\|^2 \simeq \epsilon^{-1} \|\Delta u\|^2.
\]

The last term on the right-hand side is equal to \( \epsilon^{-1/2} \|\hat{\nu}\|^2_{H^{1/2}(\Gamma)} \) by Lemma 17.

Finally, (ii), (iii) and (iv) are proven in [13, Lemmas 3, 4]. \( \square \)

Similarly to Lemma 11 (for the unperturbed case) we need to control the Neumann traces of elements of the quotient space \( U / \ker(B) \). As we have seen, this has a fundamental relation to the stability of the homogeneous adjoint problem with prescribed Dirichlet boundary condition, cf. Lemma 10. In the singularly perturbed case the situation is a little more technical. Following [22] (see Lemmas 8 and 9 there), we split the stability analysis of the adjoint problem into two parts. These are the following Lemmas 19 and 20. The last lemma of this section (Lemma 21) then states the control of the Neumann traces.

**Lemma 19.** Let \( \hat{\nu} \in H^{1/2}(\Gamma) \) be given. The problem

\[
(22a) \quad \text{div } \lambda + \epsilon^{-1/2} w = 0 \quad \text{in } \Omega,
\]

\[
(22b) \quad \lambda + \nabla w = 0 \quad \text{in } \Omega,
\]

\[
(22c) \quad w|_{\Gamma} = \hat{\nu}
\]

admits a unique solution \( (w, \lambda) \in H^1(\Omega) \times H(\text{div}, \Omega) \) with \( \Delta w \in H^1(\Omega) \), and

\[
\|\lambda\|^2 + \epsilon^{1/2} \|\text{div } \lambda\|^2 = \|\nabla w\|^2 + \epsilon^{-1/2} \|w\|^2 \lesssim \epsilon^{-1} \|\hat{\nu}\|^2_{H^{1/2}(\Gamma)}.
\]

**Proof.** The proof follows the same arguments as given in the proof of Lemma 10. To see the estimate for the norms, we make use of the weak formulation and obtain

\[
\|\nabla w\|^2 + \epsilon^{-1/2} \|w\|^2 \lesssim \|\nabla \tilde{u}\|^2 + \epsilon^{-1/2} \|\tilde{u}\|^2 \lesssim \epsilon^{-1} (\epsilon \|\nabla \tilde{u}\|^2 + \|\tilde{u}\|^2)
\]

for all \( \tilde{u} \in H^1(\Omega) \) with \( \tilde{u}|_{\Gamma} = \hat{\nu} \). Taking the infimum over these functions \( \tilde{u} \) finishes the proof. \( \square \)

**Lemma 20.** Let \( \hat{\nu} \in H^{1/2}(\Gamma) \) be given. The problem

\[
(23a) \quad \text{div } \tau + v = 0 \quad \text{in } \Omega,
\]

\[
(23b) \quad \nabla \mu + (\epsilon^{1/4} + \epsilon^{3/4}) \nabla v + \epsilon^{-1/4} \tau = 0 \quad \text{in } \Omega,
\]

\[
(23c) \quad \epsilon^{5/4} \Delta \tau + \mu = 0 \quad \text{in } \Omega,
\]

\[
(23d) \quad v|_{\Gamma} = 0, \quad \Delta v|_{\Gamma} = -\epsilon^{-5/4} \hat{\nu}
\]
has a unique solution \( \mathbf{v} := (\mu, \tau, v) \in H^1(\Omega) \times H(\text{div}, \Omega) \times H^1(\Delta, \Omega) \). It satisfies \( \mu|_{\Gamma} = \tilde{v} \) and

\[
\|\mathbf{v}\|_V \leq 3/\sqrt{2} \varepsilon^{-1/2} \|\tilde{v}\|_{H^{1/2}(\Gamma)}.
\]

**Proof.** Let \( (w, \lambda) \in H^1(\Omega) \times H(\text{div}, \Omega) \) be the solution of (22) with \( \tilde{w} = \varepsilon^{-1/4} \tilde{v} \). Define \( v \in H^1_0(\Omega) \) to be the unique solution of

\[
-\varepsilon \Delta v + v = w \quad \text{in} \quad \Omega, \quad v|_{\Gamma} = 0.
\]

Note that \( \Delta v = \varepsilon^{-1}(v - w) \in H^1(\Omega) \) and \( \Delta v|_{\Gamma} = -\varepsilon^{-1} w|_{\Gamma} = -\varepsilon^{-5/4} \tilde{v} \). We define \( \mu := -\varepsilon^{5/4} \Delta v \in H^1(\Omega) \) and have that \( \mu|_{\Gamma} = \varepsilon^{1/4} w|_{\Gamma} = \tilde{v} \). In particular, (23c) and (23d) are satisfied.

Now, define \( \tau := \varepsilon^{1/2} \lambda - \varepsilon \nabla v \). Note that \( \lambda \in H(\text{div}, \Omega) \) and \( \nabla v \in H(\text{div}, \Omega) \). Hence, \( \tau \in H(\text{div}, \Omega) \). Together with (22a) and \( -\varepsilon \Delta v + v = w \) we get

\[
\text{div} \tau = \varepsilon^{1/2} \text{div} \lambda - \varepsilon \Delta v = -w - \varepsilon \Delta v = -v,
\]

which is (23a). One also establishes that (23b) holds. In fact, by the definition of \( \tau \) and relation (22b), we find

\[
\nabla \mu + (\varepsilon^{1/4} + \varepsilon^{3/4}) \nabla v + \varepsilon^{-1/4} \tau = \nabla \mu + (\varepsilon^{1/4} + \varepsilon^{3/4}) \nabla v + \varepsilon^{-1/4} (\varepsilon^{1/2} \lambda - \varepsilon \nabla v) = \nabla \mu + \varepsilon^{1/4} \nabla v - \varepsilon^{1/4} \nabla w.
\]

The last term vanishes since \( \mu = -\varepsilon^{5/4} \Delta v = -\varepsilon^{1/4}(v - w) \) by definition of \( \mu \) and \( v \).

Now, testing (23c) with \( \varepsilon^{1/4} \Delta z \), (23b) with \( \varepsilon^{1/4} \nabla z \), and (23a) with \( z \) for \( z \in H^1(\Delta, \Omega) \) with \( z|_{\Gamma} = 0 \), and adding the resulting equations, we obtain, after integrating by parts,

\[
\varepsilon^{3/2}(\Delta v, \Delta z) + (\varepsilon^{1/2} + \varepsilon)(\nabla v, \nabla z) + (v, z) = -\varepsilon^{1/4}(\nabla z \cdot \mathbf{n}_\Omega, \tilde{v})_{\Gamma}.
\]

For any \( \tilde{u} \in H^1(\Omega) \) with \( \tilde{u}|_{\Gamma} = \tilde{v} \) we infer

\[
\varepsilon^{3/2}||\Delta v||^2 + (\varepsilon^{1/2} + \varepsilon)||\nabla v||^2 + ||v||^2 = -\varepsilon^{-1/4}(\nabla v \cdot \mathbf{n}_\Omega, \tilde{v})_{\Gamma} = -\varepsilon^{-1/4}(\nabla v, \tilde{u})_{\Gamma} - \varepsilon^{1/4}(\nabla v, \nabla \tilde{u})
\]

\[
\leq \varepsilon^{3/4}||\Delta v||\|\varepsilon^{-1/2} \tilde{u}\| + \varepsilon^{1/4}||\nabla v||\|\nabla \tilde{u}\|
\]

\[
\leq \left( \varepsilon^{3/2}||\Delta v||^2 + \varepsilon^{1/2}||\nabla v||^2 \right)^{1/2} \varepsilon^{-1/2} \left( \|\tilde{u}\|^2 + \varepsilon \|\nabla \tilde{u}\|^2 \right)^{1/2}.
\]

On the one hand, we conclude

\[
(\varepsilon^{3/2}||\Delta v||^2 + (\varepsilon^{1/2} + \varepsilon)||\nabla v||^2 + ||v||^2)^{1/2} \leq -\varepsilon^{-1/2} \|\tilde{v}\|_{H^{1/2}(\Gamma)}.
\]

On the other hand, using Young’s inequality, we also conclude that

\[
\varepsilon^{3/2}||\Delta v||^2 + (\varepsilon^{1/2} + \varepsilon)||\nabla v||^2 + ||v||^2 \leq \frac{\delta^{-1}}{2} \left( \varepsilon^{3/2}||\Delta v||^2 + \varepsilon^{1/2}||\nabla v||^2 \right) + \frac{\delta}{2} \varepsilon^{-1} \|\tilde{v}\|_{H^{1/2}(\Gamma)}^2.
\]

For \( \delta = \frac{1}{2} \) we get

\[
\varepsilon \|\nabla v\|^2 + ||v||^2 \leq \frac{1}{4} \varepsilon^{-1} \|\tilde{v}\|_{H^{1/2}(\Gamma)}^2.
\]

By (23c) and (23a) we have

\[
\varepsilon^{-1}||\mu||^2 + ||\text{div} \tau||^2 = \varepsilon^{3/2}||\Delta v||^2 + ||v||^2.
\]

It remains to estimate the norms of \( \tau \) and \( \nabla v \). To this end we rewrite the term \( (\lambda, \nabla v) \). Integrating by parts, the condition \( v|_{\Gamma} = 0 \), (22a), and the identity \( w = -\varepsilon \Delta v + v \) show that

\[
(\lambda, \nabla v) = -(\text{div} \lambda, v) = -\varepsilon^{-1/2}(w, v) = -\varepsilon^{-1/2}(\Delta v, v) + \varepsilon^{-1/2}(v, v) = \varepsilon^{-1/2} \left( \varepsilon \|\nabla v\|^2 + ||v||^2 \right).
\]

Recall that \( \varepsilon^{-1/2} \tau = \varepsilon^{1/4} \lambda - \varepsilon^{3/4} \nabla v \). Thus,

\[
\varepsilon^{-1/2}||\tau||^2 = \varepsilon^{1/2}||\lambda||^2 - 2\varepsilon(\lambda, \nabla v) + \varepsilon^{3/2}||\nabla v||^2.
\]

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For the estimation of $\|\nabla \mu\|$ we use (23b) and again $\varepsilon^{1/4} \lambda = \varepsilon^{3/4} \nabla v + \varepsilon^{-1/4} \tau$ to obtain

$$\|\nabla \mu\|^2 = \|\varepsilon^{1/4} \nabla v + \varepsilon^{1/4} \lambda\|^2 = \varepsilon^{1/2} \|\nabla v\|^2 + 2 \varepsilon^{1/2} (\lambda, \nabla v) + \varepsilon^{1/2} \|\lambda\|^2.$$

This gives the estimate

$$\varepsilon^{-1} \|\mu\|^2 + \|\text{div} \tau\|^2 + \varepsilon^{-1/2} \|\tau\|^2 + \|\nabla \mu\|^2$$

$$\leq \varepsilon^{3/2} \|\Delta v\|^2 + \|v\|^2 + 2 \varepsilon^{1/2} \|\lambda\|^2 + (\varepsilon^{1/2} + 2\varepsilon) \|\nabla v\|^2 + 2 \|v\|^2$$

$$\leq \varepsilon^{3/2} \|\Delta v\|^2 + (\varepsilon^{1/2} + \varepsilon) \|\nabla v\|^2 + \|v\|^2 + 2 (\varepsilon \|\nabla v\|^2 + \|v\|^2) + 2 \varepsilon^{1/2} \|\lambda\|^2.$$

Using estimates (25), (26), and Lemma 19 we put everything together to conclude for the overall norm

$$\|(\mu, \tau, v)\|_{\tilde{V}}^2 \leq \frac{9}{2} \varepsilon^{-1} \|\tilde{v}\|_{H^{1/2}(\Gamma)}^2.$$

To see uniqueness of $v$, let $(\mu_2, \tau_2, v_2)$ solve (23). Define $w := v - (\mu_2, \tau_2, v_2)$ and set $w := v - v_2$. Note that $w|_{\Gamma} = 0$ and $\Delta w|_{\Gamma} = 0$. The variational formulation for $w$, which can be obtained in the same way as (24), proves

$$\varepsilon^{3/2} \|\Delta w\|^2 + (\varepsilon^{1/2} + \varepsilon) \|\nabla w\|^2 + \|w\|^2 = 0.$$

Thus, $w = 0$ or equivalently $v = v_2$. Equation (23c) then gives $\mu = -\varepsilon^{5/4} \Delta v = -\varepsilon^{5/4} \Delta v_2 = \mu_2$ and, similarly, (23b) shows $\tau = \tau_2$.

**Lemma 21.** There holds

$$\varepsilon^{1/4} |\langle \gamma_1 (u - \mathcal{E} \gamma_0 u), \tilde{v}\rangle| \leq 3/\sqrt{2} \|B u\|_{V'} \varepsilon^{-1/4} \|\tilde{v}\|_{H^{1/2}(\Gamma)}$$

for all $u \in U$, $\tilde{v} \in H^{1/2}(\Gamma)$.

**Proof.** The idea of the proof is the same as in the proof of Lemma 11 using Lemma 20 instead of Lemma 10. □

3.5. **Proof of Theorem 12.** First, we show boundedness and coercivity of $a^\varepsilon(\cdot, \cdot)$ with respect to the norm $\|\cdot\|$. Then the Lions-Stampacchia theorem, see, e.g., [19, 20, 32], proves unique solvability of (10) provided that $F : U \to \mathbb{R}$ is a linear functional, which follows by the boundedness of $\Theta : U \to \mathbb{R}$.

We start by showing the boundedness. Since $b(\cdot, \Theta \cdot)$ is symmetric and positive semi-definite, the Cauchy-Schwarz inequality proves

$$|b(u, \Theta \beta v)| \leq \beta b(u, \Theta u)^{1/2} b(v, \Theta v)^{1/2} = \beta \|B u\|_{V'} \|B v\|_{V'}$$

for all $u, v \in U$. For the boundary terms, we consider

$$\varepsilon^{1/4} |\langle \gamma_1 u, \gamma_0 v\rangle| \leq \varepsilon^{1/4} |\langle \gamma_1 (u - \mathcal{E} \gamma_0 u), \gamma_0 v\rangle| + \varepsilon^{1/4} |\langle \gamma_1 \mathcal{E} \gamma_0 u, \gamma_0 (\Gamma)\rangle|.$$

The first term on the right-hand side is estimated with Lemma 21. Let $\tilde{u} \in H^1(\Omega)$ be the quasi-harmonic extension of $\gamma_0 u$ and let $\tilde{v} \in H^1(\Omega)$ be the quasi-harmonic extension of $\gamma_0 v$. Then, for the second term we get with integration by parts (cf. Lemma 17)

$$\varepsilon^{1/4} |\langle \gamma_1 \mathcal{E} \gamma_0 u, \gamma_0 v\rangle| = \varepsilon^{-1/2} \left( \varepsilon (\nabla \tilde{u}, \nabla \tilde{v}) + (\tilde{u}, \tilde{v}) \right) \leq \varepsilon^{-1/4} \|\gamma_0 u\|_{H^{1/2}(\Gamma)} \varepsilon^{-1/4} \|\gamma_0 v\|_{H^{1/2}(\Gamma)}.$$

Together, this gives for the boundary term

$$\varepsilon^{1/4} |\langle \gamma_1 u, \gamma_0 v\rangle| \leq 3/\sqrt{2} \|B u\|_{V'} \varepsilon^{-1/4} \|\gamma_0 v\|_{H^{1/2}(\Gamma)} + \varepsilon^{-1/4} \|\gamma_0 u\|_{H^{1/2}(\Gamma)} \varepsilon^{-1/4} \|\gamma_0 v\|_{H^{1/2}(\Gamma)}.$$

The second boundary term $\varepsilon^{1/4} |\langle \gamma_1 v, \gamma_0 u\rangle|$ is treated identically. Altogether, this proves the boundedness of $a^\varepsilon(\cdot, \cdot)$. 

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For the proof of coercivity we use Lemma 17, Lemma 21 and Young’s inequality to find that, for \( \delta > 0 \),
\[
\|u\|^2 = \|Bu\|^2 + \varepsilon^{-1/2}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)} = \|Bu\|^2 + \varepsilon^{1/4}\langle \gamma_n \mathcal{E} \gamma_0 u, \gamma_0 u \rangle_{\Gamma} \\
= \|Bu\|^2 + \varepsilon^{1/4}\langle \gamma_n (\mathcal{E} \gamma_0 u - u), \gamma_0 u \rangle_{\Gamma} + \varepsilon^{1/4}\langle \gamma_n u, \gamma_0 u \rangle_{\Gamma} \\
\leq \|Bu\|^2 + 3/\sqrt{2}\|Bu\|_{V'}\varepsilon^{-1/4}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)} + \varepsilon^{1/4}\langle \gamma_n u, \gamma_0 u \rangle_{\Gamma} \\
\leq (1 + \delta^{-1/4})\|Bu\|^2 + \varepsilon^{1/4}\langle \gamma_n u, \gamma_0 u \rangle_{\Gamma} + \delta^{-1/2}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)}.
\]
We choose \( \delta = \frac{9}{8} \), which implies \( 1 + \delta^{-1/4} = 3 \). Then, subtracting the last term on the right-hand side, we obtain for \( \beta \geq 3 \)
\[
\|Bu\|_{V'}^2 + 7/16\varepsilon^{-1/2}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)} \leq 3\|Bu\|_{V'}^2 + \varepsilon^{1/4}\langle \gamma_n u, \gamma_0 u \rangle_{\Gamma} \leq a^\ast(u, u) \quad \text{for all } u \in U.
\]
Next we show (19). By Lemma 18 and the triangle inequality we obtain
\[
\|u\|_{U,1}^2 \leq \|u - \mathcal{E} \gamma_0 u\|_{U,1}^2 + \|\mathcal{E} \gamma_0 u\|_{U,1}^2 \leq \|Bu\|^2_{V'} + \varepsilon^{-1/2}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)}.
\]
For the proof of the upper bound in (19) let \( \tilde{u} \in H^1(\Omega) \) be a function which attains the minimum in the definition of \( \|\tilde{u}\|_{1/2,\Sigma} \). Then,
\[
\varepsilon^{-1/2}\|\tilde{u}\|_{1/2,\Sigma}^2 = \varepsilon^{-1/2}(\|\tilde{u}\|^2 + \varepsilon^{1/4}\|\nabla \tilde{u}\|^2) \geq \varepsilon^{-1/2}(\|\tilde{u}\|^2 + \varepsilon\|\nabla \tilde{u}\|^2).
\]
Since \( \tilde{u}_{|\Gamma} = \hat{u}_{|\Gamma} = \hat{u}\|_{\Gamma} = \gamma_0 u \), the right-hand side is an upper bound of \( \varepsilon^{-1/2}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)} \). Thus, together with Lemma 18 we get
\[
\|u\|^2_{V'} = \|Bu\|^2_{V'} + \varepsilon^{-1/2}\|\gamma_0 u\|^2_{H^{1/2}(\Gamma)} \leq \|u\|_{U,2}^2 + \varepsilon^{-1/2}\|\tilde{u}\|_{1/2,\Sigma}^2 \leq \|u\|^2_{\tilde{V}'}.
\]
The remainder of the proof follows the same arguments as in the proof of Theorem 1.

4. EXAMPLES

In this section we present various numerical examples in two dimensions \( (d = 2) \). For the first example in §4.2 we take a manufactured solution \( u \in H^2(\Omega) \). The standard finite element method with lowest-order discretization on quasi-uniform meshes converges at a rate \( O(h) \), where \( h \) denotes the diameter of elements in \( T \). We observe the same optimal rate for the DPG methods with \( s \in \{0, n, s\} \), analyzed in Section 2. In §4.3 we consider an L-shaped domain with unknown solution and an expected singularity at the reentrant corner. Indeed, we will see that a uniform method gives suboptimal convergence whereas an adaptive method driven by the estimator \( \eta \) from §2.4 recovers the optimal one. Finally, in §4.4 we use a family of manufactured solutions that exhibit typical boundary layers for the reaction dominated diffusion problem. Our numerical results underline the robustness of the a posteriori error estimate, as stated by Theorem 15.

4.1. General setting. As is usual for DPG methods we replace the infinite dimensional test space \( V \) used in the calculation of optimal test functions \( [5] \) by a finite dimensional subspace \( V_h \), that is, we replace the test function \( \Theta_{\beta,h} u_h \) for \( u_h \in U_h \) by \( \Theta_{\beta,h} u_h \) defined through
\[
(\Theta_{\beta,h} u_h, v_h)_{V'} = \beta b(u_h, v_h) \quad \text{for all } v_h \in V_h.
\]
Here we choose
\[
V_h := \begin{cases} P^2(T) \times [P^2(T)]^2 & \text{for the methods from Section 2} \\
P^2(T) \times [P^2(T)]^2 \times P^4(T) & \text{for the method from Section 3}
\end{cases}
\]
These choices are motivated by [21]. We refer the interested reader to this work for more details. The resulting DPG scheme is called practical DPG method. For the scaling parameter of the test functions we choose \( \beta = 2 \) for the methods from Section 2 and \( \beta = 3 \) for the method from Section 3.
We use the standard basis for the lowest order spaces \( U_h \), that is, the element characteristic functions for \( P^0(\mathcal{T}) \), \( P^0(\mathcal{S}) \), and nodal basis functions (hat-functions) for \( S^1(\mathcal{S}) \). These choices allow for a simple implementation of the inequality constraints in the cones \( K_h^* \).

We solve the discrete variational inequalities \((12), (20)\) with a (Primal-Dual) Active Set Algorithm, see \cite{25,26}. More precisely, we implemented a modification of \cite{26} Algorithm A1 (which deals with obstacle problems) to the present problem (here we consider inequality constraints only for degrees of freedom that are associated to the boundary).

For the problems where the solution is known in analytical form we compute different error quantities depending on the underlying problem from Section 2 or 3:

- **Section 2** Let \( u = (u, \sigma, \tilde{u}, \tilde{\sigma}) \) denote the exact solution of \((10)\) and let \( u_h = (u_h, \sigma_h, \tilde{u}_h, \tilde{\sigma}_h) \) be its approximation. We define

\[
\text{err}(u) := \|u - u_h\|, \quad \text{err}(\sigma) := \|\sigma - \sigma_h\|, \\
\text{err}(\tilde{u}) := (\|u - \tilde{u}_h\|^2 + \|\nabla(u - \tilde{u}_h)\|^2)^{1/2}, \quad \text{err}(\tilde{\sigma}) := (\|\sigma - \tilde{\sigma}_h\|^2 + \|\nabla(\sigma - \tilde{\sigma}_h)\|^2)^{1/2}.
\]

Here, \( \tilde{u}_h \in S^1(\mathcal{T}) \) is the nodal interpolant of \( \hat{u}_h \) at the nodes of \( \mathcal{T} \). Similarly, \( \tilde{\sigma}_h \) is the Raviart-Thomas interpolation of \( \hat{\sigma}_h \). Then, it follows by the definition of the trace norms

\[
\|u - u_h\|_U \leq (\text{err}(u)^2 + \text{err}(\sigma)^2 + \text{err}(\tilde{u})^2 + \text{err}(\tilde{\sigma})^2)^{1/2}.
\]

- **Section 3** Let \( u = (u, \sigma, \rho, \tilde{u}^a, \tilde{\sigma}^a, \tilde{u}^b, \tilde{\sigma}^b) \) denote the exact solution of \((18)\) and let \( u_h = (u_h, \sigma_h, \rho_h, \tilde{u}^{a,h}, \tilde{\sigma}^{a,h}, \tilde{u}^{b,h}, \tilde{\sigma}^{b,h}) \) be its approximation. Define \( \text{err}(u) \) and \( \text{err}(\sigma) \) as above and additionally

\[
\text{err}(\tilde{u}^*) := \left(\|u - \tilde{u}^*_h\|^2 + \varepsilon^{1/2}\|\nabla(u - \tilde{u}^*_h)\|^2\right)^{1/2}, \\
\text{err}(\tilde{\sigma}^*) := \left(\|\sigma - \tilde{\sigma}^*_h\|^2 + \varepsilon\|\nabla(\sigma - \tilde{\sigma}^*_h)\|^2\right)^{1/2}, \\
\text{err}(\rho) := \varepsilon^{1/2}\|\rho - \rho_h\|,
\]

for \( \ast \in \{a, b\} \), and \( \tilde{u}^*_h, \tilde{\sigma}^*_h \) are defined in the same way as above. Our total error estimator is

\[
\text{err}(u) := \left(\text{err}(u)^2 + \text{err}(\sigma)^2 + \text{err}(\rho)^2 + \varepsilon^{3/2}\text{err}(\tilde{u}^a)^2 + \varepsilon\text{err}(\tilde{u}^b)^2 + \varepsilon^{3/2}\text{err}(\tilde{\sigma}^a)^2 + \varepsilon^{5/2}\text{err}(\tilde{\sigma}^b)^2\right)^{1/2}
\]

so that

\[
\|u - u_h\|_{U,1} \leq \text{err}(u).
\]

For examples with singularities and/or boundary or interior layers we use a standard adaptive algorithm that uses \( \eta(\mathcal{T}) \) and \( \eta(E) \) to mark elements by the bulk criterion. For convenience we define \( \eta(\mathcal{T}) \) and \( \eta(\mathcal{S}_T) \) by

\[
\eta(\mathcal{T})^2 := \sum_{T \in \mathcal{T}} \eta(\mathcal{T})^2, \quad \eta(\mathcal{S}_T)^2 := \sum_{E \in \mathcal{S}_T} \eta(E)^2.
\]

### 4.2. Piecewise smooth solution with boundary layer (Section 2)

We consider the domain

\[
\Omega := (-1, 1) \times (0, 1)
\]

and the manufactured solution

\[
u(x, y) := \begin{cases} -16x^2(1-x)y(1-y) & x \geq 0, \\ 2(x+1)^3 - 3(x+1)^2 + 1 & x < 0. \end{cases}
\]

This solution satisfies \( u(x, y) = 0 \) on the part of \( \Gamma = \partial \Omega \) where \( x \geq 0 \), and \( \partial_{n\Omega}u = 0 \) on the part of \( \Gamma \) where \( x < 0 \). We calculate \( f := -\Delta u + u \) and note that \( u \in H^1(\Delta, \Omega) \). Also note that \( u(x, y) \)
is smooth in both regions, \(x > 0\) and \(x < 0\). Our initial mesh consists of 8 congruent triangles. We solve (12) for \(\star \in \{0, n, s\}\) and plot the errors \(\text{err}(u), \text{err}(\sigma), \text{err}(\hat{u}),\) and \(\text{err}(\hat{\sigma})\) for a sequence of uniformly refined triangulations. Moreover, in the case \(\star = s\) we compare these error quantities with the reliable error estimator \(\eta\). The results are given in Figure 1 for \(\star = 0\), \(\star = n\) and Figure 2 for \(\star = s\). We observe optimal convergence rates \(O(h^\alpha) = O((\#T)^{-\alpha/2})\) with \(\alpha = 1\) for the error quantities. This rate is visualized by a triangle. In Figure 2 we see that also the estimator \(\eta(T)\) converges with this rate whereas \(\eta(S_\Gamma)\) has a higher convergence rate of approximately \(\alpha = 2\).

![Figure 1. Error quantities for Example 4.2 with \(\star = 0\) (left) and \(\star = n\) (right).](image)

4.3. **Unknown solution (Section 2).** Let \(\Omega = (-1,1)^2 \setminus [-1,0]^2\) with initial triangulation visualized in Figure 3. We define

\[
    f(x,y) := \begin{cases} 
        -1 & |(x,y)| \leq 0.8, \\
        \frac{1}{2} & \text{else}. 
    \end{cases}
\]

For this right-hand side the solution is not known to us in analytical form. Therefore, we only compute the error estimators. The results are plotted in Figure 4. We observe that uniform refinement leads to a reduced order of convergence \(O((\#T)^{-\alpha/2})\) of approximately \(\alpha = 0.7\), whereas adaptive refinement regains the optimal order \(\alpha = 1\). This is a strong indicator that the unknown solution has a singularity at the reentrant corner which is what one expects. Figure 5 visualizes meshes at different steps of the adaptive loop and supports this observation.

4.4. **Piecewise smooth solution with boundary layer (Section 3).** Let \(\Omega := (-1,1) \times (0,1)\) with manufactured solution

\[
    u(x,y) := \begin{cases} 
        -x^2 \left( e^{-2(1-x)/\sqrt{\varepsilon}} - e^{-1/\sqrt{\varepsilon}} - 1 \right) & x \geq 0, \\
        2(x+1)^3 - 3(x+1)^2 + 1 & x < 0. 
    \end{cases}
\]

We choose \(f = -\varepsilon \Delta u + u\) so that \(u\) satisfies (1) with \(c = \varepsilon\). In particular, \(u\) has a layer of order \(\sqrt{\varepsilon}\) at the boundary for \(x \geq 0\). We solve the variational inequality (20) on a sequence of adaptively refined triangulations for \(\varepsilon \in \{10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}\}\). We start with a coarse initial triangulation that consists of only \(\#T_0 = 8\) congruent triangles.
In Figure 6 we compare the estimator $\eta$ with the total error $\text{err}(u)$. As in the works \[22, 15], we observe that, after boundary layers have been resolved, our method leads to an optimal convergence rate $O((\#T)^{-\alpha/2})$ ($\alpha = 1$). We also observe that the curves representing $\eta(T)$ and $\text{err}(u)$ are quite
close together, even in the pre-asymptotic range, and uniformly in \( \varepsilon \) (for the selected values). This confirms the robustness of the a posteriori estimate by Theorem 15. We also see that the boundary estimator \( \eta(S_T) \) is small in comparison to \( \eta(T) \), and has a higher convergence rate.

Our primary interest was to construct a robust method in the sense that the error in the balanced norm of the field variables \( u, \sigma, \rho \) is controlled uniformly in \( \varepsilon \). The error estimator \( \eta \) does precisely this, as stated by Theorem 15 and seen in Figure 6. To further underline this statement, Figure 7 shows the ratio \( \frac{(\text{err}(u)^2 + \text{err}(\sigma)^2 + \text{err}(\rho)^2)^{1/2}}{\eta(T)} \). We observe that, again after boundary layers have been resolved, this ratio is between 0.5 and 0.55 uniformly with respect to \( \varepsilon \). In fact, this number is close to \( \frac{1}{\sqrt{3}} = 1/\sqrt{3} \approx 0.5774 \). Note that, by the product structure of \( V \), \( \| B u_h - L \|_{V'}^2 = \sum_{T \in T} \| B u_h - L \|_{V'(T)}^2 \), so that \( \eta(T) = \sqrt{3} \| B u_h - L \|_{V'} \), cf. [13]. Our numerical results therefore indicate that the slight overestimation of the error by \( \eta \) is due to the choice of \( \beta \), and is (asymptotically) uniform in \( \varepsilon \).

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Figure 6. Comparison of total error $\text{err}(u)$ and estimators $\eta(T)$, $\eta(S_\Gamma)$ for the problem from Section 4.4.

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Figure 7. Ratio \((\text{err}(u)^2 + \text{err}(\sigma)^2 + \text{err}(\rho)^2)^{1/2} / \eta(T)\) for the problem from Section 4.4.
