EXISTENCE OF WEAK SOLUTIONS FOR A SHARP INTERFACE MODEL FOR PHASE SEPARATION ON BIOLOGICAL MEMBRANES

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Abstract. We prove existence of weak solutions of a Mullins-Sekerka equation on a surface that is coupled to diffusion equations in a bulk domain and on the boundary. This model arises as a sharp interface limit of a phase field model to describe the formation of liquid rafts on a cell membrane. The solutions are constructed with the aid of an implicit time discretization and tools from geometric measure theory to pass to the limit.

1. Introduction and main result. We consider a sharp interface model for the evolution of liquid rafts on a cell membrane $\Gamma = \partial B$, where $B$ is a suitable domain. Moreover, in the following $\Gamma = \Gamma^+(t) \cup \Gamma^-(t) \cup \gamma(t)$, where $\Gamma^\pm(t)$ are disjoint domains in $\Gamma$ with common sufficiently smooth boundary $\gamma(t)$. The model yields the system

$$
\begin{align*}
\partial_t u &= \Delta u & \text{in } B \times (0,T], \\
-\nabla u \cdot \nu &= q & \text{on } \Gamma \times (0,T], \\
\Delta_\Gamma \mu &= 0 & \text{on } \Gamma^\pm(t), t \in (0,T], \\
\partial_t v &= \Delta_\Gamma \theta + q & \text{on } \Gamma^\pm(t), t \in (0,T], \\
\theta &= \frac{2}{5} (2v - 1 \mp 1) & \text{on } \Gamma^\pm(t), t \in (0,T], \\
2\mu + \theta &= \kappa_g & \text{on } \gamma(t), t \in (0,T], \\
[f]^+_\gamma &= [\theta]^+_\gamma &= 0 & \text{on } \gamma(t), t \in (0,T], \\
-2\nu &= [\nabla_\Gamma \mu]^+_\gamma \cdot \nu & \text{on } \gamma(t), t \in (0,T], \\
-\nu &= [\nabla_\Gamma \theta]^+_\gamma \cdot \nu & \text{on } \gamma(t), t \in (0,T],
\end{align*}
$$

where $[f]^+_\gamma(x_0, t_0) = \lim_{h \to 0^+} (f(x_0 + \nu_\gamma(x_0, t_0)) - f(x_0 - \nu_\gamma(x_0, t_0)))$ is the jump of a quantity $f$ across the interface $\gamma(t) := \partial \Gamma^+(t)$ and $\nu_\gamma(x_0, t_0) \in T_{x_0} \Gamma$ denotes the unit normal to $\gamma(t_0)$ in $x_0 \in \gamma(t_0)$, pointing inside $\Gamma^+(t_0)$. The geodesic curvature of $\gamma(t)$ in $\Gamma$ is denoted by $k_\gamma(\cdot, t)$ and $\nu(x_0, t_0)$ denotes the normal velocity of $\gamma(t_0)$ in $x_0 \in \gamma(t_0)$ in direction of $\nu_\gamma(x_0, t_0)$. Here $u$ is the concentration of cholesterol in the cell described by $B$, $v$ is the concentration of cholesterol on the cell membrane $\Gamma$, $\Gamma^+(t)$ is the saturated lipid phase and $\Gamma^-(t)$ the unsaturated lipid phase, $\mu$ is

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the chemical potential associated to the phase separation between saturated and unsaturated lipid phases, \( \theta \) is the chemical potential associated to the membrane bound cholesterol, \( \delta > 0 \) is a constant, which controls how strong the preferred binding between saturated lipids and cholesterol is. Finally, \( q = q(u, v, \pm 1) \) in \( \Gamma^\pm(t) \) is a flux describing the absorption and desorption of cholesterol to the boundary given by suitable constitutive assumptions. The model arises as the sharp interface limit of a phase field model, which was derived in Garcke et al. in [5]. The sharp interface limit was shown with the aid of formally matched asymptotics in [5] and on the level of varifold solutions by the authors in [1].

The construction of weak solutions to geometric evolution equations from time-discrete approximations has been widely used to prove existence results for various equations such as the Stefan problem, the Mullins-Sekerka problem or mean curvature flow, cf. e.g. [6, 9, 7]. As always with discretization schemes, solutions are obtained as limits of sequences of approximate solutions. The existence of these limits of course depends on the concrete discretization scheme and the considered function spaces. Furthermore it needs to be clarified in which sense the limits could possibly be weak solutions to the geometric evolution equation.

In this contribution we construct weak solutions to the problem by using a time-discrete scheme that was proposed by Röger in [9] as a generalization of the scheme by Luckhaus and Sturzenhecker in [7]. One of the main difficulties is to identify the weak formulation of the equation for the mean curvature (6). To this end a compactness theorem for integral varifolds due to Schätzle is used, cf. [10], in a similar manner as in [9]. But suitable adaptations to the present coupled system are needed. One of the new difficulties for the analysis of the present system is to control the spatial mean values of \( u \) and \( v \). This can be done provided the coupling term \( q(\varphi, u, v) \) growth at most linearly.

For technical reasons we will consider a situation in which the boundary \( \Gamma \) of \( B \) is flat and compact. More precisely, we assume that

\[ B := \mathbb{R}^2/2\pi\mathbb{Z}^2 \times (0, 1), \tag{10} \]

i.e., we assume periodic boundary conditions in tangential directions. Consequently, we have

\[ \Gamma = \partial B = \mathbb{R}^2/2\pi\mathbb{Z}^2 \times \{0, 1\}. \tag{11} \]

As usual we identify functions \( \mathbb{R}^2/2\pi\mathbb{Z}^2 \) with functions \( f : \mathbb{R}^2 \to \mathbb{R} \) that are \( 2\pi \)-periodic in both coordinate directions. Hence functions on the boundary \( \Gamma \) are given as \( 2\pi \)-periodic functions on \( \mathbb{R}^2 \), while functions in the bulk \( B \) only need to be \( 2\pi \)-periodic with respect to the horizontal coordinate directions.

**Theorem 1.1.** Let \( T, \delta > 0 \) and suppose that \( q : \mathbb{R}^2 \times \{\pm 1\} \to \mathbb{R} \) is continuous and growth at most linearly, i.e. there is some \( \Lambda > 0 \) such that

\[ |q(u, v, \pm 1)| \leq \Lambda(1 + |u| + |v|) \quad \text{for all } u, v \in \mathbb{R}. \]

Then for any \( u_0 \in H^1(B), \chi_0 \in BV(\Gamma; \{0, 1\}) \), \( v_0 \in L^2(\Gamma) \) there exist functions

\[ u \in L^2(0, T; H^1(B)), \quad \chi \in L^\infty_w(0, T; BV(\Gamma; \{0, 1\})), \quad v \in L^2((0, T) \times \Gamma), \]

\[ \theta \in L^2(0, T; H^1(\Gamma)), \quad \mu \in L^2(0, T; H^1(\Gamma)). \]

which are weak solutions to (1)–(9) in the following sense:

\[ -\int_0^T \int_B \nabla u \cdot \nabla \eta - \int_0^T \int_B u \partial_t \eta + \int_B u_0 \eta|_{t=0}^T + \int_\Gamma q(u, v, 2\chi - 1) \eta = 0 \tag{12} \]
holds for all \( \eta \in C^\infty([0, T], H^1(B)) \) with \( \eta(T) = 0 \), we have
\[
- \int_0^T \int_\Gamma \nabla \theta \cdot \nabla \zeta = - \int_0^T \int_\Gamma v \partial_t \zeta + \int_\Gamma v_0 \zeta|_{t=0} - \int_0^T \int_\Gamma q(u, v, 2\chi - 1) \zeta
\]
for all \( \zeta \in C^\infty([0, T]; H^1(\Gamma)) \) with \( \zeta(T) = 0 \) and
\[
\int_0^T \int_\Gamma \nabla \mu \cdot \nabla \xi - \int_0^T \int_\Gamma 2\chi \partial_t \xi = \int_\Gamma 2\chi_0 \xi|_{t=0}
\]
for all \( \xi \in C^\infty([0, T]; H^1(B)) \) with \( \xi(T) = 0 \). Furthermore
\[
\theta = \frac{2}{\delta} (2v - 2\chi).
\]
The essential boundary \( \partial^* \{ \varphi(\cdot, t) = +1 \} \) has for almost all \( t \in (0, T) \) a generalized mean curvature vector \( \bar{H}(t) \in L^2(\{ \nabla \varphi(t) \}) \) for every \( 1 \leq s < \infty \) as defined in Definition \ref{definition:existence} below such that
\[
\bar{H}(\cdot, t) = (\mu + \theta) \frac{\nabla \chi(\cdot, t)}{||\nabla \chi(\cdot, t)||} \quad \mathcal{H}^1 - \text{a.e. for almost all } t \in (0, T).
\]

Here and in the following we use the abbreviations
\[
\int_B f = \int_B f(x) \, dx, \quad \int_\Gamma f = \int_\Gamma f(x) \, dx.
\]
We note that in the theorem \( \chi \) is the characteristic function of \( \Gamma^+ \).

2. Preliminaries. Let \( \Omega = B \) or \( \Omega = \Gamma \). We use the standard notation \( W^m_p(\Omega) \) for the \( L^p \)-Sobolev spaces (with \( 1 \leq p \leq \infty \)) of order \( m \in \mathbb{N}_0 \) and \( H^m(\Omega) = W^m_2(\Omega) \). We note that \( f \in W^m_p(\Omega) \) if and only if \( f \in W^m_{p,\text{loc}}(\mathbb{R}^2 \times A) \) (with \( A = (0, 1) \) if \( \Omega = B \) and \( A = \{0, 1\} \) if \( B = \Gamma \)) if \( f \) is \( 2\pi \)-periodic with respect to \((x_1, x_2) \in \mathbb{R}^2 \) for almost every \( x_3 \in A \).

Let \( M \subseteq \mathbb{R} \). Then \( BV(\Gamma; M) \) denotes the set of all functions \( f : \Gamma \to M \) of bounded variation, cf. e.g. [3]. As for the Sobolev spaces \( f \in BV(\Gamma; M) \) if and only if \( f \in BV_{\text{loc}}(\mathbb{R}^2 \times A) \) and \( f : \mathbb{R}^2 \times A \to M \) is \( 2\pi \)-periodic with respect to \((x_1, x_2) \in \mathbb{R}^2 \) for almost every \( x_3 \in A \). We note that there is a separable Banach space \( X \) such that \( X' = BV(\Omega) \), cf. e.g. [3]. Moreover, we refer to the latter book for an introduction to sets of finite perimeter. Furthermore, if \( X \) is a Banach space, \( 1 \leq p \leq \infty \), and \( T > 0 \), \( f : L^p(0, T; X) \) denotes the space of all strongly measurable \( f : (0, T) \to X \) with \( \|f(\cdot)\|_X \in L^p((0, T)) \) and \( L^\infty_{\text{loc}}((0, T); X') \) denotes the space of all weakly-* measurable \( f : (0, T) \to X' \) with \( \|f(\cdot)\|_{X'} \in L^\infty((0, T)) \).

Then \( L^\infty_{\text{loc}}((0, T); X') \) can be identified with \((L^1(0, T; X'))'\) in a canonical manner and uniformly bounded sets in \( L^\infty_{\text{loc}}((0, T); X') \) are weakly-* compact, cf. [4].

For the following we denote
\[
L^p_{(\text{ad})}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{1}{|\Omega|} \int_\Omega u = m \right\} \quad \text{for } 1 \leq p \leq \infty, m \in \mathbb{R},
\]
\[
H^1_{(0)}(\Omega) = H^1(\Omega) \cap L^2_{(0)}(\Omega), \quad H^{-1}_{(0)}(\Omega) = (H^1_{(0)}(\Omega))',
\]
and \( BV_{(\text{ad})}(\Gamma; M) = BV(\Gamma; M) \cap L^1_{(\text{ad})}(\Gamma) \). We equip \( H^1_{(0)}(\Omega) \) with the norm defined by \( \|u\|_{H^1_{(0)}(\Omega)} = \|\nabla u\|_{L^2(\Omega)} \) for all \( u \in H^1_{(0)}(\Omega) \). Then the Riesz isomorphisms
\[ \mathcal{R} : H^1_{(0)}(B) \to H^{-1}_{(0)}(B) \text{ and } \mathcal{R} : H^1_{(0)}(\Gamma) \to H^{-1}_{(0)}(\Gamma) \] are given by \( -\Delta_N \) and \( -\Delta_\Gamma \), resp., defined by
\[
\langle -\Delta_N u, \varphi \rangle = \int_B \nabla u \cdot \nabla \varphi \quad \text{for all } u, \varphi \in H^1_{(0)}(B),
\]
\[
\langle -\Delta_\Gamma u, \varphi \rangle = \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \quad \text{for all } u, \varphi \in H^1_{(0)}(\Gamma).
\]

Note that \( L^2_{(0)}(\Omega) \to H^1_{(0)}(\Omega) \) in the standard manner.

Furthermore, we need a generalized notion of the mean curvature for the reduced boundary of a set of finite perimeter \( E \subseteq \Omega \), which uses elements from geometric measure theory. For an introduction to it and the definition of an integral \((n-1)\)-varifolds and the weak mean curvature vector for it we refer to [12]. In the following we denote by \( \mathbb{S}^k(p) \) for \( k \leq n \) the Grassmanian, i.e.,
\[
\mathbb{S}^k(p) := \{ S \mid S \text{ is a } k \text{-dimensional subspace of } \mathbb{R}^n \}.
\]
Moreover, we define for \( U \subseteq \mathbb{R}^n \) open
\[
G_k(U) := \{ (p,S) \mid p \in X, S \in \mathbb{S}^k(p) \}.
\]
We identify \( \mathbb{S}^{n-1}(p) \) with \( S_{n-1}(p) \) mod \( \{ e_1, -e_1 \} \) where \( S_{n-1}(p) \) is the unit sphere in \( \mathbb{R}^n \) and \( e_1 \) is the first unit vector. Hence we identify \( \mathbb{S}^{l-1} \) with the set of all unit normal vectors to unoriented \((n-1)\) planes in \( \mathbb{R}^n \).

**Definition 2.1.** Let \( \Omega \subseteq \mathbb{R}^n \) be open and \( E \subseteq \Omega \) be a set of finite perimeter. If there exists an integral \((n-1)\)-varifold \( \vartheta \) on \( \Omega \) such that
\[
\vartheta^* E \subseteq \text{supp}(\vartheta),
\]
\[
\vartheta \text{ has weak mean curvature vector } \vec{H}_{\vartheta},
\]
\[
\vec{H}_{\vartheta}^n \in L^s_{loc}(\vartheta), \quad s > d - 1, \quad s \geq 2
\]
then we call \( \vec{H} := \vec{H}_{\vartheta}|_{\vartheta^* E} \) the generalized mean curvature vector of \( \vartheta^* E \).

In [9] it is shown that under the above condition the definition is well defined and \( \vec{H} \) is a property of \( E \) and it is independent of the choice of \( \vartheta \).

Finally, we need a convergence result for integral \((n-1)\)-varifolds in \( \Omega \subseteq \mathbb{R}^n \) given by the reduced boundary of sets of finite perimeter with mean curvature in \( W^{1,p}(\Omega) \) given by Schätzle in [10, Theorem 1.1]. To this end let \( n \geq 2 \), \( \Omega \subseteq \mathbb{R}^n \) be open and bounded and let \( E_j \subseteq \Omega \), \( j \in \mathbb{N} \), be sets of finite perimeter and let \( V_j \) be the associated integral \((n-1)\)-varifold to it, which is defined by
\[
\langle V_j, \varphi \rangle := \int_{\Omega} \varphi \left( x, \frac{\nabla \chi_{E_j}}{|\nabla \chi_{E_j}|} \right) d|\nabla \chi_{E_j}| \quad \text{for all } \varphi \in C^0_c(G_{n-1}(\Omega)),
\]
and
\[
m_{V_j} := |\nabla \chi_{E_j}|
\]
be its weight. Moreover, assume that \( \sup_{j \in \mathbb{N}} |\nabla \chi_{E_j}|(\Omega) < \infty \) and that there is bounded sequence \((\mu_j)_{j \in \mathbb{N}} \subseteq W^1_p(\Omega) \) with \( \frac{2}{p} < p \) if \( n \geq 3 \) and \( p \geq \frac{4}{3} \) if \( n = 2 \) such that \( V_j \) possesses a mean curvature vectors \( H_j \) with
\[
\vec{H}_j = \mu_j \nu_j \quad \text{on } \vartheta^* E_j, \quad \text{where } \nu_j := \frac{\nabla \chi_{E_j}}{|\nabla \chi_{E_j}|}
\]
for all $j \in \mathbb{N}$, i.e.,
\[
\int_{\Omega} (I - \nu_j \otimes \nu_j) : D\xi \, d|\nabla \chi_{E_j}| = \int_{\Omega} \chi_{E_j} \, \text{div}(\nu_j \xi)
\]
for all $\xi \in C^1_c(\Omega)^n$. Finally, assume that
\[
\mu_j \to_j \mu \quad \text{in } W^1_1(\Omega),
\]
\[
\chi_{E_j} \to_j \chi E \quad \text{in } L^1(\Omega),
\]
\[
V_j \to_j V \quad \text{as varifolds}.
\]

**Theorem 2.2.** Under the above assumptions, $V$ is an integral $(n - 1)$-varifold in $\Omega$ with locally bounded first variation and mean curvature vector
\[
\bar{H}_V \in L^p_{loc}(m_V) \quad \text{for } s = \frac{p}{(n-p)(n-1)},
\]
where $m_V$ is the weight of $V$. $E$ is a set of finite perimeter such that $\partial^* E \subseteq \text{supp} \, V$. Moreover, $\bar{H}_V = \mu \nu E$ $m_V$-almost every, where $\nu := \frac{\nabla \chi E}{|\nabla \chi E|}$.

We refer to [10, Theorem 1.1] for the proof.

3. **Discretization scheme and time discrete solutions.** We follow the same strategy as in Röger [9] and use an implicit time discretization scheme. The scheme relies on the $BV$-formulation of the Gibbs-Thomson law (6) given by Luckhaus and Sturzenhecker [7], i.e. we construct for each time-step $t_h$ functions $\mu_h, \theta_h \in H^1(\Gamma)$ and $\chi_h$ of bounded variation which fulfil
\[
\int_{\Gamma} \text{div}_\Gamma \xi - \frac{\nabla \chi_h}{|\nabla \chi_h|} \cdot D\xi \frac{\nabla \chi_h}{|\nabla \chi_h|} \, d|\nabla \chi_h| = \int_{\Gamma} \text{div}_\Gamma ((2\mu_h + \theta_h) \xi)
\]
for all $\xi \in C^1(\Gamma, \mathbb{R}^2)$. In the remaining equations (1)–(5) and (7)–(9), we substitute the time derivative by its time-discrete approximation $\frac{\chi_{h+1} - \chi_h}{h}$ etc. and formulate the equations in a weak sense. The existence of time discrete solutions is granted by the following proposition.

**Proposition 3.1.** Let $\tilde{\chi} \in BV_{(m_0)}(\Gamma, \{0, 1\})$, $\tilde{v} \in H^1(\Gamma)$ and $\tilde{u} \in H^1(B)$ be given. Then there exist functions
\[
\chi \in BV_{(m_0)}(\Gamma, \{0, 1\}), \quad \mu \in H^1(\Gamma), \quad v \in L^2(\Gamma), \quad \theta \in H^1(\Gamma), \quad u \in H^1(B)
\]
such that
\[
- \int_B \nabla u \cdot \nabla \eta = \int_B \frac{u - \tilde{u}}{h} \eta + \int_{\Gamma} \frac{q(\tilde{u}, \tilde{v}, 2\chi - 1)}{h} \eta \quad \text{for all } \eta \in H^1(B), \quad (17)
\]
such that
\[
\int_{\Gamma} \left[ \nabla \chi \mu \cdot \nabla \chi \zeta + \frac{2}{h} (\chi - \tilde{\chi}) \zeta \right] = 0 \quad \text{for all } \zeta \in H^1(\Gamma), \quad (18)
\]
and such that $\theta := \frac{2}{3} (2v - 2\chi)$ fulfils
\[
- \int_{\Gamma} \nabla \chi \cdot \nabla \theta \zeta = \int_{\Gamma} \frac{v - \tilde{v}}{h} \zeta - \int_{\Gamma} \frac{q(\tilde{u}, \tilde{v}, 2\chi - 1)}{h} \zeta \quad \text{for all } \zeta \in H^1(\Gamma). \quad (19)
\]
Moreover,
\[
\int_{\Gamma} \left( \text{div}_\Gamma \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot D\chi \frac{\nabla \chi}{|\nabla \chi|} \right) \, d|\nabla \chi| = \int_{\Gamma} \chi \text{div}_\Gamma ((2\mu + \theta) \xi) \quad (20)
\]
for all $\xi \in C^\infty(\Gamma; \mathbb{R}^2)$. Furthermore, the estimates
\[
\int_\Gamma d|\nabla \chi| + \frac{h}{2} \|\nabla \mu\|_{L^2(\Gamma)}^2 + \frac{\delta}{8} \|\theta\|_{L^2(\Gamma)}^2
\]
\[
+ \frac{h}{2} \|\nabla \theta\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_B u^2 + \frac{h}{2} \|\nabla u\|_{L^2(B)}^2
\]
\[
\leq \int_\Gamma d|\nabla \tilde{\chi}| + \frac{\delta}{8} \|\tilde{\theta}\|_{L^2(\Gamma)}^2 + C h \|\tilde{\theta}\|_{L^2(\Gamma)}^2 + Ch \|\tilde{u}\|_{H^1(B)}^2 + \frac{1}{2} \int_B \tilde{u}^2
\]
and
\[
\|\mu\|_{H^1(\Gamma)} \leq c(m_0, \Gamma) \left(1 + \int_\Gamma d|\nabla \chi|\right) \left(\int_\Gamma d|\nabla \chi| + \|\nabla \mu\|_{L^2(\Gamma)}\right)
\]  
(21)
hold true with constants independent of $h \in (0, 1)$.

Let us fix some notation before we start with the proof of Proposition 3.1. We denote by $\delta_\Gamma : L^2(\Gamma) \to (H^1_{(0)}(B))^\prime$ the operator defined by
\[
\langle \delta_\Gamma f, u \rangle_{H^{-1}(B), H^1(B)} := \int_\Gamma f \text{tr } u \, d\mathcal{H}^2
\]
for all $f \in L^2(\Gamma)$ and $u \in H^1_{(0)}(B)$.

The proof of Proposition 3.1 now consists of two steps. In the first step we prove for given $\tilde{\chi} \in BV(m_0(\Gamma, \{0, 1\})$, $\tilde{v} \in H^1(\Gamma)$, and $\tilde{u} \in H^1(B)$ the existence of minimizers to the functional
\[
\mathcal{F}_h : BV(m_0(\Gamma, \{0, 1\}) \times L^2(m_1)(\Gamma) \times L^2(m_2)(B) \to \mathbb{R}
\]
defined by
\[
\mathcal{F}_h(\chi, v, u) := \int_\Gamma d|\nabla \chi| + \frac{1}{h} \|\chi - \tilde{\chi}\|_{H^{-1}_{(0)}(\Gamma)}^2
\]
\[
+ \frac{1}{2h} \int_\Gamma (v - 2\chi)^2 + \frac{1}{2h} \|v - \tilde{\varnothing} - h q(\tilde{u}, \tilde{\varnothing}, 2\tilde{\chi} - 1)\|_{H^{-1}_{(0)}(\Gamma)}^2
\]
\[
+ \frac{1}{2} \int_B u^2 + \frac{1}{2h} \|u - \tilde{u} + h \delta_\Gamma q(\tilde{u}, \tilde{\varnothing}, 2\tilde{\chi} - 1)\|_{H^{-1}_{(0)}(B)}^2
\]
where
\[
m_1 = \frac{1}{|\Gamma|} \int_\Gamma (\tilde{\varnothing} + h q(\tilde{u}, \tilde{\varnothing}, 2\tilde{\chi} - 1)), \quad m_2 = \frac{1}{|B|} \int_B \tilde{u} + \frac{1}{|\Gamma|} \int_\Gamma h q(\tilde{u}, \tilde{\varnothing}, 2\tilde{\chi} - 1).
\]
We note that the choice of $m_1$ and $m_2$ ensures that
\[
\frac{1}{|\Gamma|} \int_\Gamma (v - \tilde{\varnothing} - h q(\tilde{u}, \tilde{\varnothing}, 2\tilde{\chi} - 1)) = 0, \quad \frac{1}{|B|} \int_B (u - \tilde{u}) - \frac{1}{|\Gamma|} \int_\Gamma h q(\tilde{u}, \tilde{\varnothing}, 2\tilde{\chi} - 1) = 0.
\]  
(23)

We then prove that these minimizers fulfill the assertion of the proposition.

**Lemma 3.2.** Let $\tilde{\chi} \in BV(m_0(\Gamma, \{0, 1\})$, $\tilde{v} \in H^1(\Gamma)$ and $\tilde{u} \in H^1(B)$ be given and let
\[
\mathcal{F}_h : BV(m_0(\Gamma, \{0, 1\}) \times L^2(m_1)(\Gamma) \times L^2(m_2)(B) \to \mathbb{R}
\]
be defined as in (22). Then there exists a minimizers to $\mathcal{F}_h$.

**Proof.** The existence of minimizers follows from the direct method of the calculus of variations. Indeed, $\mathcal{F}_h \geq 0$ is bounded below. Furthermore, bounded sequences in $BV(\Gamma)$ are precompact in $L^1(\Gamma)$ and the perimeter $\int_\Gamma d|\nabla \chi|$ is lower semi-continuous w.r.t. $L^1$-convergence while bounded sequences in $L^2(m_1)(\Gamma)$ and
$L^2_{(m_2)}(B)$ allow the extraction of weakly converging subsequences in $L^2_{(m_1)}(\Gamma)$ and $L^2_{(m_2)}(B)$ respectively. Thus each sequence $\{ (\chi_k, v_k, u_k) \}_{k \in \mathbb{N}}$ such that $F_h(\chi_k, v_k, u_k)$ is bounded has a subsequence such that $\chi_k$ converges to a function $\chi \in BV_{(m_0)}(\Gamma; \{0,1\})$ w.r.t the $L^1(\Gamma)$-topology and $v_k$ as well as $u_k$ converge weakly to functions $v \in L^2_{(m_1)}(\Gamma)$ and $u \in L^2_{(m_2)}(B)$ respectively w.r.t the corresponding $L^2$-topology. Using the continuity of the embedding $L^2_{(0)}(\Omega) \hookrightarrow H^{-1}_{(0)}(\Omega)$, the $H^{-1}_{(0)}$-norms on $\Gamma$ and $B$ are weakly lower semi-continuous with respect to the (weak) $L^2$-topology. Finally, the convex and continuous functional $(\chi, v) \mapsto \int_\Gamma (2v - 2\chi)^2$ possesses the same property. Hence $F_h$ is (weakly) lower semi-continuous. Moreover, one easily sees that $F_h$ is coercive using that

$$\int_\Gamma (v - \chi)^2 = \int_\Gamma (v^2 - 2v\chi + \chi^2) \geq \int_\Gamma v^2 - \beta \int_\Gamma \chi^2 + \int_\Gamma \chi^2$$

$$= (1 - \frac{1}{\beta})m_0 + (1 - \beta) \int_\Gamma v^2$$

for any $v \in L^2(\Omega)$, $\beta > 0$ and $\chi \in BV_{(m_0)}(\Gamma; \{0,1\})$. This implies the existence of a minimizer of $F_h$.

**Proof of Proposition 3.1.** Let $(\chi, u, v)$ be a minimizer of $F_h$, which is provided by Lemma 3.2. It remains to show that the minimizer fulfills the assertion of the proposition.

The function $\mu_0$ defined by

$$\mu_0 := -(-\Delta_\Gamma)^{-1}\left(\frac{2}{h}(\chi - \bar{\chi})\right)$$

(24)

fulfills

$$\int \left[ \nabla_\Gamma \mu_0 \cdot \nabla_\Gamma \zeta + \frac{2}{h}(\chi - \bar{\chi})\zeta \right] = 0 \text{ for all } \zeta \in H^1(\Gamma).$$

by the definition of $(-\Delta_\Gamma)^{-1}$ and since $\int_\Gamma (\chi - \bar{\chi}) = 0$. Furthermore, we define $\theta := \frac{2}{h}(2v - 2\chi)$. Calculating the first variation of $F_h$ with respect to $v$ in $(\chi, u, v)$, we find

$$\frac{\delta F_h}{\delta v}(\chi, u, v)(\zeta) = \frac{2}{h} \int_\Gamma (2v - 2\chi)\zeta + \frac{1}{h} \int_\Gamma (-\Delta_\Gamma)^{-1}(v - \bar{v} - hq(\bar{u}, \bar{v}, 2\bar{\chi} - 1))\zeta$$

$$= \int_\Gamma \theta\zeta + \frac{1}{h} \int_\Gamma (-\Delta_\Gamma)^{-1}(v - \bar{v} - hq(\bar{u}, \bar{v}, 2\bar{\chi} - 1))\zeta.$$}

for all $\zeta \in L^2_{(0)}(\Gamma)$. Since $(\chi, v, u)$ minimize $F_h$, the first variation $\frac{\delta F_h}{\delta v}$ has to vanish in $(\chi, v, u)$. Thus one finds

$$\theta_0 := \theta - \frac{1}{|\Gamma|} \int_\Gamma \theta = \frac{1}{h} \Delta_\Gamma^{-1}(v - \bar{v} - hq(\bar{u}, \bar{v}, 2\bar{\chi} - 1))$$

and the function $\theta$ solves

$$\Delta_\Gamma \theta = \frac{v - \bar{v}}{h} - q(\bar{u}, \bar{v}, 2\bar{\chi} - 1) \text{ in } H^{-1}_{(0)}(\Gamma).$$

Hence

$$-\int_\Gamma \nabla_\Gamma \theta \cdot \nabla_\Gamma \zeta = \int_\Gamma \frac{v - \bar{v}}{h}\zeta - \int_\Gamma q(\bar{u}, \bar{v}, 2\bar{\chi} - 1)\zeta \quad \text{for all } \zeta \in H^1_{(0)}(\Gamma).$$

Because of (23), the same is true for all $\zeta \in H^1(\Gamma)$. In particular, this implies $\theta \in H^1(\Gamma)$. 

**Existence Sharp Interface Model 7**
Calculating the first variation $\frac{\delta F_h}{\delta u}$ of $F_h$ with respect to $u$, we obtain

$$0 = \frac{\delta F_h}{\delta u}(\chi, u, v)(\eta) = \int_B u\eta + \int_B (-\Delta_N)^{-1}\left(\frac{1}{h} (u - \tilde{u} + h\delta \Gamma q(\tilde{u}, \tilde{v}, 2\tilde{\chi} - 1))\right)\eta$$

for all $\eta \in L_0^2(B)$ and thus

$$u_0 := u - \frac{1}{|\Gamma|} \int_{\Gamma} u = \Delta_N^{-1}\left(\frac{1}{h} (u - \tilde{u} + h\delta \Gamma q(\tilde{u}, \tilde{v}, 2\tilde{\chi} - 1))\right).$$

(26)

This implies $u \in H^1(B)$. Together with (23) we obtain

$$- \int_B \nabla u \cdot \nabla \eta = \int_B \frac{u - \tilde{u}}{h} \eta + \int_B q(\tilde{u}, \tilde{v}, 2\tilde{\chi} - 1)\eta \quad \text{for all } \eta \in H^1(B)$$

which implies that $u$ is a weak solution to

$$\begin{cases}
\Delta u = \frac{1}{h} (u - \tilde{u}) & \text{in } B \\
\nabla u \cdot n = q(\tilde{u}, \tilde{v}, 2\tilde{\chi} - 1) & \text{on } \Gamma.
\end{cases}$$

Given the fact that we minimized $F_h$ over the domain $BV(\{0, 1\}) \times L_0^2(\Gamma) \times L_0^2(B)$ we have to consider the condition $\int_{\Gamma} \chi = m_0$ while calculating the first variation of $F_h$ with respect to $\chi$. In order to do so, we calculate this variation with respect to volume preserving variations before determining the Lagrange multiplier associated with $\int_{\Gamma} \chi = m_0$. To this end, let $(\Phi_s)_{s \in (-\varepsilon, \varepsilon)}$ be a volume preserving smooth family of diffeomorphisms, $\Phi_s : \Gamma \to \Gamma$, such that $\Phi_0 = \text{Id}$ and $\int_{\Gamma} \chi \circ \Phi_s^{-1} = m_0$ for all $s \in (-\varepsilon, \varepsilon)$. We denote by $\xi \in C^\infty(\Gamma, \mathbb{R}^2)$ the associated vector field $\xi = \frac{d}{ds}|_{s=0} \Phi_s$.

Since for all $x \in \Gamma$

$$0 = \frac{d}{ds}|_{s=0} \Phi_s \circ \Phi_s^{-1}(x) = \xi(x) + \frac{d}{ds}|_{s=0} \Phi_s^{-1}(x)$$

we infer

$$\frac{d}{ds}|_{s=0} \Phi_s^{-1} = -\xi.$$

(27)

Moreover, $(\Phi_s)_{s \in (-\varepsilon, \varepsilon)}$ is volume preserving and together with the foregoing observation and [8, Proposition 17.8] we deduce

$$\frac{d}{ds}|_{s=0} \int_{\Gamma} \chi \circ \Phi_s^{-1} = \int_{\Gamma} \chi \text{div}_\Gamma \xi = 0.$$ 

(28)

As $\chi$ and $\chi \circ \Phi_s^{-1}$ are both functions in $BV(\Gamma)$, the perimeter of $\{\chi \circ \Phi_s^{-1} = 1\}$ coincides with $\int_{\Gamma} d|\Gamma(\chi \circ \Phi_s^{-1})|$, see e.g. [3, Theorem 3.36]. Its first variation is

$$\frac{d}{ds}|_{s=0} \int_{\Gamma} d|\Gamma(\chi \circ \Phi_s^{-1})| = \int_{\Gamma} \left(\text{div}_\Gamma \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot D\xi \frac{\nabla \chi}{|\nabla \chi|}\right) d|\nabla \chi|.$$
Since $-\Delta \Gamma : H^1_{(0)}(\Gamma) \to H^{-1}_{(0)}(\Gamma)$ is the Riesz isomorphism, we obtain

$$
\frac{d}{ds}\bigg|_{s=0} \frac{1}{2h} \| \chi \circ \Phi^s_\ast - \tilde{\chi} \|_{H^{-1}_{(0)}(\Gamma)}^2 = \frac{1}{h} \left( \frac{\partial}{\partial s}\right)_{s=0} \left( \chi \circ \Phi^s_\ast \right)^{-1} \left( (-\Delta \Gamma)^{-1} (\chi - \tilde{\chi}) \right)_{H^1_{(0)}, H^1_{(0)}}
$$

Using the definition of $\mu_0$ in (24), (27), and (28) we thus deduce

$$
\frac{d}{ds}\bigg|_{s=0} \frac{1}{h} \| \chi \circ \Phi^s_\ast - \tilde{\chi} \|_{H^{-1}_{(0)}(\Gamma)}^2 = 2 \langle \nabla \Gamma : \xi, \mu_0 \rangle_{H^1_{(0)}, H^1_{(0)}} = -2 \int_\Gamma \chi \text{div}\Gamma (\mu \xi).
$$

Similarly, [8, Proposition 17.8] and the generalized Gauss-Green formula [3, Theorem 3.36] allows us to calculate

$$
\frac{d}{ds}\bigg|_{s=0} \frac{1}{2\delta} \int_\Gamma (2v - 2 (\chi \circ \Phi^s_\ast))^2 = \int_\Gamma \theta \xi \cdot \nabla \Gamma \chi \, d|\nabla \Gamma \chi| = -\int_\Gamma \chi \text{div}\Gamma (\theta \xi).
$$

We thus deduce

$$
0 = \frac{d}{ds}\bigg|_{s=0} F_h(\chi \circ \Phi^s_\ast, u, v) = \int_\Gamma \left( \text{div}\Gamma \xi - \frac{\nabla \Gamma \chi \cdot D\xi \nabla \Gamma \chi}{|\nabla \Gamma \chi|} \right) |\nabla \Gamma \chi| - \int_\Gamma \chi \text{div}\Gamma ((2\mu_0 + \theta)\xi).
$$

Consider now a vectorfield $\hat{\xi} \in C^\infty(\Gamma, \mathbb{R}^2)$ such that

$$
\int_\Gamma \chi \text{div}\Gamma \hat{\xi} \neq 0,
$$

in contrast to $\xi$ above. Let $(\hat{\Phi}_s)_{s \in (-\varepsilon, \varepsilon)}$ be a family of smooth diffeomorphisms with $\hat{\Phi}_0 = \text{Id}$ and

$$
\frac{d}{ds}\bigg|_{s=0} \hat{\Phi}_s = \hat{\xi}.
$$

Together with $\Phi_s$ as above we can define a function $g : (-\varepsilon, \varepsilon)^2 \to \mathbb{R}$ by

$$
g(s, r) := \int_\Gamma \chi \circ (\Phi_s \circ \hat{\Phi}_r)^{-1} - m_0 \quad \text{for all } s, r \in (-\varepsilon, \varepsilon).
$$

Since $g(0, 0) = 0$ and $\partial_r|_{s=r=0} g(s, r) = \int_\Gamma \chi \text{div}\hat{\xi} \neq 0$, the implicit function theorem yields the existence of an $\varepsilon_1 \leq \varepsilon$ and a function $l : (-\varepsilon_1, \varepsilon_1) \to (-\varepsilon, \varepsilon)$ such that

$$
g(s, l(s)) = 0 \quad \text{for all } s \in (-\varepsilon_1, \varepsilon_1).
$$

The family $(\Phi_s \circ \hat{\Phi}_l(s))_{s \in (-\varepsilon_1, \varepsilon_1)}$ constitutes therefore a family of volume preserving smooth diffeomorphisms. Hence

$$
0 = \frac{d}{ds}\bigg|_{s=0} g(s, l(s)) = \frac{d}{ds}\bigg|_{s=0} \int_\Gamma \chi \circ (\Phi_s \circ \hat{\Phi}_l(s))^{-1} = \int_\Gamma (\text{div}\Gamma \xi + l'(0) \text{div}\Gamma \hat{\xi}) \chi.
$$

We deduce

$$
l'(0) = - \left( \int_\Gamma \chi \text{div}\Gamma \hat{\xi} \right)^{-1} \left( \int_\Gamma \chi \text{div}\Gamma \hat{\xi} \right).
$$
At the same time, (29) and the fact that \((\Phi_\varepsilon \circ \hat{\Phi}_t)\) is a family of volume conserving diffeomorphisms with \(\frac{d}{dt} \big|_{t=0} (\Phi_\varepsilon \circ \hat{\Phi}_t) = \left( \xi + t' \right) \hat{\xi} \) imply

\[
0 = \int_\Gamma \left( \text{div}_\Gamma \xi + t'(0) \text{div}_B \hat{\xi} - \frac{\nabla \chi}{|\nabla \chi|} \cdot D (\xi + t'(0) \hat{\xi}) \frac{\nabla \chi}{|\nabla \chi|} \right) d|\nabla \Gamma \chi| - \int_\Gamma \chi \text{div}_\Gamma \left( (2\mu_0 + \theta) \left( \xi + t'(0) \hat{\xi} \right) \right).
\]

If we plug in (30), we obtain

\[
\int_\Gamma \left( \text{div}_\Gamma \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot D \chi \frac{\nabla \chi}{|\nabla \chi|} \right) d|\nabla \Gamma \chi| - \int_\Gamma \chi \text{div}_\Gamma ((2\mu_0 + \theta)\xi) = -t'(0) \left[ \int_\Gamma \left( \text{div}_\Gamma \hat{\xi} - \frac{\nabla \chi}{|\nabla \chi|} \cdot D \hat{\xi} \frac{\nabla \chi}{|\nabla \chi|} \right) d|\nabla \Gamma \chi| - \int_\Gamma \chi \text{div}_\Gamma ((2\mu_0 + \theta)\hat{\xi}) \right]
\]

and with

\[
\lambda := \left( \int_\Gamma \chi \text{div}_\Gamma \hat{\xi} \right)^{-1} \left[ \int_\Gamma \left( \text{div}_\Gamma \hat{\xi} - \frac{\nabla \chi}{|\nabla \chi|} \cdot D \hat{\xi} \frac{\nabla \chi}{|\nabla \chi|} \right) d|\nabla \Gamma \chi| - \int_\Gamma \chi \text{div}_\Gamma ((2\mu_0 + \theta)\hat{\xi}) \right]
\]

and

\[
\mu := \mu_0 + \lambda/2 \]

we end up with

\[
\int_\Gamma \left( \text{div}_\Gamma \xi - \frac{\nabla \chi}{|\nabla \chi|} \cdot D \chi \frac{\nabla \chi}{|\nabla \chi|} \right) d|\nabla \Gamma \chi| = \int_\Gamma \chi \text{div}_\Gamma ((2\mu + \theta)\xi)
\]

for all \(\xi \in C_0^1(\Gamma; \mathbb{R}^2)\). Since clearly \(\mathcal{F}_h(\chi, v, u) \leq \mathcal{F}_h(\tilde{\chi}, \tilde{v}, \tilde{u})\) as \((\chi, v, u)\) minimize \(\mathcal{F}_h\), (25) and (26) allow us to deduce

\[
\mathcal{F}_h(\chi, v, u) = \int_\Gamma d|\nabla \Gamma \chi| + \frac{1}{2h} \|h\nabla \Gamma \mu_0\|_{L^2(\Gamma)}^2 + \frac{\delta}{8} \|\theta\|_{L^2(\Gamma)}^2
\]

\[
+ \frac{1}{2h} \|h\nabla \Gamma \theta\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_B u^2 + \frac{1}{2h} \|h\nabla u\|_{L^2(B)}^2
\]

\[
\leq \int_\Gamma d|\nabla \Gamma \tilde{\chi}| + \frac{1}{2h} \|2\tilde{v} - 2\tilde{x}\|_{L^2(\Gamma)}^2 + \frac{1}{2h} \|h q(\tilde{u}, \tilde{v}, 2\tilde{x} - 1)\|_{H_{00}^{-1}(\Gamma)}^2
\]

\[
+ \frac{1}{2} \int_B \tilde{u}^2 + \frac{1}{2h} \|h\delta q(\tilde{u}, \tilde{v}, 2\tilde{x} - 1)\|_{H_{00}^{-1}(B)}^2
\]

\[
\leq \int_\Gamma d|\nabla \Gamma \tilde{\chi}| + \frac{2}{5} \|	ilde{v} - \tilde{x}\|_{L^2(\Gamma)}^2 + \frac{Ch}{2} \||\tilde{u}\|_{L^2(B)}^2 + \frac{Ch}{2} \||\tilde{u}\|_{L^2(B)}^2 + \frac{1}{2} \int_B \tilde{u}^2
\]

where we have used that \(q\) only growth linearly in \(\tilde{u}\) and \(\tilde{v}\) and that the embedding \(L^2(\Gamma) \hookrightarrow H_{00}^{-1}(\Gamma)\) is continuous.

By the compactness of the trace operator from \(H^1(B)\) to \(L^2(\Gamma)\) we have \(\tilde{u} \equiv u L^2(\Gamma) \leq \varepsilon \|\nabla \tilde{u}\|_{L^2(B)} + C \|\tilde{u}\|_{L^2(B)}\) for every \(\varepsilon > 0\), which implies

\[
\mathcal{F}_h(\chi, v, u) = \int_\Gamma d|\nabla \Gamma \chi| + \frac{h}{2} \|\nabla \Gamma \mu_0\|_{L^2(\Gamma)}^2 + \frac{\delta}{8} \|\theta\|_{L^2(\Gamma)}^2
\]

\[
+ \frac{h}{2} \|\nabla \Gamma \theta\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_B u^2 + \frac{h}{2} \|\nabla u\|_{L^2(B)}^2
\]

\[
\leq \int_\Gamma d|\nabla \Gamma \tilde{\chi}| + \frac{\delta}{8} \|\tilde{\theta}\|_{L^2(\Gamma)}^2 + \frac{Ch}{2} \||\tilde{u}\|_{L^2(\Gamma)}^2 + \frac{h}{4} \|\nabla \tilde{u}\|_{L^2(B)}^2 + \frac{Ch}{2} \||\tilde{u}\|_{L^2(B)}^2 + \frac{1}{2} \int_B \tilde{u}^2
\]

We finish the proof by deducing estimate (21). The proof relies on choosing a particular vector field \(\xi\) in (31). To this end, let \((\rho_\eta)_{\eta>0}\) be a Dirac sequence. We define by

\[
\chi_\eta := \chi \ast \rho_\eta
\]
a family of smooth functions with \(|\chi_\eta - \frac{1}{|\Gamma|} \int_\Gamma \chi_\eta| \leq 1\) and \(|\nabla \chi_\eta| \leq \eta^{-1} C(\Gamma)\). We also have

\[
\frac{1}{|\Gamma|} \int_\Gamma \chi_\eta \leq \frac{m_0}{|\Gamma|}.
\]

Let furthermore \(\Psi : \Gamma \rightarrow \mathbb{R}\) be the solution to

\[
\Delta_\Gamma \Psi = \chi_\eta - \frac{1}{|\Gamma|} \int_\Gamma \chi_\eta \text{ on } \Gamma, \quad \int_\Gamma \Psi = 0.
\]

By standard regularity theory the function \(\Psi\) satisfies

\[
\|\Psi\|_{C^2(\Gamma)} \leq C(\Gamma) \left\|\chi_\eta - \frac{1}{|\Gamma|} \int_\Gamma \chi_\eta\right\|_{C^1(\Gamma)} \leq \frac{1}{\eta} C(\Gamma).
\]

Moreover, assume for a moment that \(\chi\) is smooth. For \(y \in \mathbb{R}^2 \times \{0\}\) such that \(\|y\| \leq 1\) we can then always write

\[
\chi(p - \eta y) - \chi(p) = -\eta \int_0^1 \nabla_\Gamma \chi(p - s\eta y) \cdot y \, ds.
\]

After integrating both sides with respect to \(p\) we obtain

\[
\int_\Gamma |\chi(p - \eta y) - \chi(p)| \, dp \leq \eta \int_0^1 \int_\Gamma |\nabla_\Gamma \chi(p - s\eta y)| \, dp \, ds \leq \eta \int_\Gamma |\nabla_\Gamma \chi|.
\]

Finally, we multiply both sides by \(\rho(y)\) and integrate with respect to \(y\) to find

\[
\int_\Gamma \int_{\mathbb{R}^2} |\chi(p - \eta y) - \chi(p)| \rho(y) \, dy \, ds \leq \eta \int_\Gamma |\nabla_\Gamma \chi|.
\]

Since every \(BV\)-function can be approximated by a sequence of smooth function (see e.g. [3, Theorem 3.9]), we deduce

\[
\|\chi - \chi_\eta\|_{L^1(\Gamma)} \leq C(\Gamma) \eta \left(1 + \int_\Gamma d|\nabla \chi|\right).
\]

Choosing \(\xi = \nabla_\Gamma \Psi\) yields the estimate

\[
\int_\Gamma \chi \nabla_\Gamma \xi = \int_\Gamma \chi \left(\chi_\eta - \frac{1}{|\Gamma|} \int_\Gamma \chi_\eta\right) = \left(1 - \frac{1}{|\Gamma|} \int_\Gamma \chi_\eta\right) m_0 + \int_\Gamma \chi (\chi_\eta - \chi) \geq \left(1 - \frac{m_0}{|\Gamma|}\right) m_0 - C(\Gamma) \eta \left(1 + \int_\Gamma d|\nabla \chi|\right) \geq c(m_0, \Gamma)
\]

if we choose \(\eta = \eta_0(1 + \int_\Gamma d|\nabla_\Gamma \chi|)^{-1}\) for \(\eta_0 = \eta_0(m_0, \Gamma)\) sufficiently small. Plugging these findings in (31) implies

\[
|\lambda| = \frac{\left|\int_\Gamma \left(\nabla \chi \cdot \nabla_\Gamma \xi - \frac{\nabla_\Gamma \xi}{|\nabla_\Gamma \xi|} \cdot D_\chi \frac{\nabla_\Gamma \xi}{|\nabla_\Gamma \xi|}\right) d|\nabla \Gamma \chi| - \int_\Gamma \chi \nabla_\Gamma ((2\mu_0 + \theta) \xi)\right|}{\int_\Gamma \chi \nabla_\Gamma \xi} \leq \frac{\|\Psi\|_{C^2(\Gamma)} \int_\Gamma d|\nabla \Gamma \chi| + 2 \|\mu_0\|_{H^1(\Gamma)} \|\Psi\|_{C^2(\Gamma)} \int_\Gamma \chi \nabla_\Gamma \xi}{\int_\Gamma \chi \nabla_\Gamma \xi} \leq \frac{C(\Gamma) \eta^{-1} \int_\Gamma d|\nabla \Gamma \chi| + C(\Gamma) \eta^{-1} \|\nabla \mu_0\|_{L^2(\Gamma)}}{c(m_0, \Gamma)} \leq C(m_0, \Gamma) \left(1 + \int_\Gamma d|\nabla \Gamma \chi|\right) \left(\int_\Gamma d|\nabla \Gamma \chi| + \|\nabla \mu_0\|_{L^2(\Gamma)}\right),
\]
where we have chosen \( \eta \) exactly as before. In the third step, we used that \( \int_{\Gamma} \mu_0 = 0 \) together with Poincaré’s inequality and (32). Since \( \int_{\Gamma} (2\mu_0 + \lambda) = |\Gamma| \lambda \) the final estimate (21) follows from the last estimate and Poincaré’s inequality.

Proposition 3.1 allows us to construct time discrete solutions

\[(u_h, \chi_h, v_h, \mu_h, \theta_h) : [0, T] \rightarrow H^1(B) \times BV(m_0)(\Gamma, \{0, 1\}) \times L^2(\Gamma) \times H^1(\Gamma) \times H^1(\Gamma)\]

If there are given functions \((u^k_h, \chi^k_h, v^k_h, \mu^k_h, \theta^k_h)\) on some interval \([(k-1)h, kh] \subset [0, T]\) where \(k \in \mathbb{N}_0\) is such that \((kh, (k+1)h]\) is a subset of \([0, T]\), we choose \(\bar{u}(t) = u^k_h(t-h), \bar{\chi}(t) = \chi^k_h(t-h)\) and \(\bar{v}(t) = v^k_h(t-h)\). Proposition 3.1 then yields the existence of functions

\[(u^{k+1}_h, \chi^{k+1}_h, v^{k+1}_h, \mu^{k+1}_h, \theta^{k+1}_h)\]
on \((kh, (k+1)h]\), such that the equations (17) – (20) hold.

Starting with \(k = 0\) and \((u_h, \chi_h, v_h, \mu_h, \theta_h)\) on \((-h, 0]\) given as \(\bar{\mu} \equiv 0\) and by the initial data in the case of \(u_h, \chi_h, v_h, \) and \(\theta_h\), we thus iteratively obtain a time discrete solution \((u_h, \chi_h, v_h, \mu_h, \theta_h)\) on \([0, T]\) by setting

\[ (u_h, \chi_h, v_h, \mu_h, \theta_h)(t) := (u^k_h, \chi^k_h, v^k_h, \mu^k_h, \theta^k_h) \text{ for } t \in ((k-1)h, kh]. \]

**Lemma 3.3.** Let \((u_h, \chi_h, v_h, \mu_h, \theta_h) : [0, T] \rightarrow H^1(B) \times BV(m_0)(\Gamma, \{0, 1\}) \times L^2(\Gamma) \times H^1(\Gamma) \times H^1(\Gamma)\) be the time discrete solutions constructed above. Then the estimates

\[
\begin{align*}
\sup_{t \in [0,T]} \int_{\Gamma} d|\nabla \chi_h(t)| + \|\theta_h\|_{L^\infty(0,T; L^2(\Gamma))} + \|u_h\|_{L^\infty(0,T; L^2(B))} + \|\nabla \mu_h\|_{L^2(\Gamma \times (0,T))} + \|\nabla \theta_h\|_{L^2(\Gamma \times (0,T))} + \|\nabla u_h\|_{L^2(\Gamma \times (0,T))} & \leq C(u_0, v_0, \chi_0) \quad (33) \\
\|\mu_h(t)\|_{H^1(\Gamma)} & \leq C \left( 1 + \|\nabla \mu_h(t)\|_{L^2(\Gamma)} \right) \leq C(T) \quad (34)
\end{align*}
\]

hold true.

**Proof.** According to Proposition 3.1 the functions \((u_h, \chi_h, v_h, \mu_h, \theta_h)\) fulfil

\[
\begin{align*}
\int_{\Gamma} d|\nabla \chi_h(t)| & + \frac{h}{2} \|\nabla \mu_h(t)\|_{L^2(\Gamma)}^2 + \frac{\delta}{8} \|\theta_h(t)\|_{L^2(\Gamma)}^2 \\
& + \frac{h}{2} \|\nabla \theta_h(t)\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_{\partial B} u_h(t)^2 + \frac{h}{4} \|\nabla u_h(t)\|_{L^2(B)}^2 \\
\leq \int_{\Gamma} d|\nabla \chi_h(t-h)| & + \frac{\delta}{8} \|\theta_h(t-h)\|_{L^2(\Gamma)}^2 + C \|v_h(t-h)\|_{L^2(\Gamma)}^2 \\
& + C h \left| u_h(t-h) \right|^2_{L^2(B)} + \frac{1}{2} \int_{\partial B} u_h(t-h)^2.
\end{align*}
\]

Since \(\theta_h(t-h) = \frac{3}{2}(2v_h(t-h) - 2\chi_h(t-h))\), we deduce

\[
\|v_h(t-h)\|_{L^2(\Gamma)}^2 \leq \left| \frac{\delta}{4} \theta_h(t-h) + \chi_h(t-h) \right|^2_{L^2(\Gamma)} \\
\leq 2 \left| \frac{\delta}{4} \theta_h(t-h) \right|^2_{L^2(\Gamma)} + 2 \left| \chi_h(t-h) \right|^2_{L^2(\Gamma)}.
\]

Since \(\chi \in \{0, 1\}\) almost everywhere \(\|\chi_h(t-h)\|_{L^2(\Gamma)} \leq C\) and thus

\[
\|v_h(t-h)\|_{L^2(\Gamma)}^2 \leq \frac{\delta^2}{8} \|\theta_h(t-h)\|_{L^2(\Gamma)}^2 + C.
\]
As a consequence, we obtain
\[
\int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t)| + \frac{h}{2} \|\nabla_{\Gamma} \mu_h(t)\|_{L^2(\Gamma)}^2 + \frac{\delta}{8} \|\theta_h(t)\|_{L^2(\Gamma)}^2 \\
+ \frac{h}{2} \|\nabla_{\Gamma} \theta_h(t)\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_{B} u_h(t)^2 + \frac{h}{4} \|\nabla u_h(t)\|_{L^2(B)}^2 \\
\leq \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t-h)| + \frac{\delta}{8} \|\theta_h(t-h)\|_{L^2(\Gamma)}^2 + \frac{1}{2} \int_{B} u_h(t-h)^2 + C h \|u_h(t-h)\|_{L^2(B)}^2 \\
+ C h \delta^2 \|\theta_h(t-h)\|_{L^2(\Gamma)}^2 + C h. \tag{35}
\]

Summing (35) for \(t_k = kh, k = 0, \ldots, [t/h]\) and keeping in mind that \((u_h, \chi_h, v_h, \mu_h, \theta_h)\) are constant in \(t\) on each subinterval \(((k-1)h, kh)\) yields
\[
\int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t)| + \frac{\delta}{8} \|\theta_h(t)\|_{L^2(\Gamma)}^2 + \|u(t)\|_{L^2(B)}^2 \\
+ \int_{h}^{t+h} \left( \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t)| + \|\nabla_{\Gamma} \mu_h(t)\|_{L^2(\Gamma)}^2 + \|\nabla u_h(t)\|_{L^2(B)}^2 \right) \\
\leq C(T) + \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(0)| + \frac{\delta}{8} \|\theta_h(0)\|_{L^2(\Gamma)}^2 + \|u(0)\|_{L^2(B)}^2 \\
+ C \int_{h}^{t+h} \left( \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t)| + \|\theta_h(t)\|_{L^2(\Gamma)}^2 + \|u_h(t-h)\|_{L^2(B)}^2 \right) \\
\leq C(T) + \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(0)| + \frac{\delta}{4} \|\theta_h(0)\|_{L^2(\Gamma)}^2 + \|u(0)\|_{L^2(B)}^2 \\
+ C \int_{h}^{t} \left( \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(s)| + \|\theta_h(s)\|_{L^2(\Gamma)}^2 + \|u_h(s)\|_{L^2(B)}^2 \right)
\]
and (33) follows from Gronwall’s lemma. To deduce the second estimate observe that
\[
\|\mu_h(t)\|_{H^1(\Gamma)} \leq c(m_0, \Gamma) \left( 1 + \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t)| \right) \left( \int_{\Gamma} d|\nabla_{\Gamma} \chi_h(t)| + \|\nabla_{\Gamma} \mu_h(t)\|_{L^2(\Gamma)} \right).
\]
by Proposition 3.1. Since
\[
\sup_{t \in (0,T)} \int_{\Gamma} |d\nabla_{\Gamma} \chi_h(t)| \leq C
\]
by (33), this implies (34).

\[\square\]

**Lemma 3.4.** Let \((u_h, \chi_h, v_h, \mu_h, \theta_h)\) be as in Lemma 3.3. For any sequence \(h \to 0\) there exists a subsequence \(\{h_k\}_{k \in \mathbb{N}}\) such that
\[
\begin{align*}
    u_k \to u & \text{ in } L^2(0,T; H^1(B)) \text{ and } u_k \to u \text{ in } L^2(0,T; H^s(B)) \text{ for all } \frac{1}{2} < s < 1, \\
    \chi_k \to \chi & \text{ in } L^p(0,T) \times \Gamma \text{ for all } 1 \leq p < \infty, \\
    v_k \to v & \text{ in } L^2(0,T) \times \Gamma, \\
    \theta_k \to \theta & \text{ in } L^2(0,T; H^1(\Gamma)) \text{ and } \theta_k \to \theta \text{ in } L^p(0,T; L^2(\Gamma)) \text{ for all } 1 \leq p < 2, \\
    \mu_k \to \mu & \text{ in } L^2(0,T; H^1(\Gamma)).
\end{align*}
\]
where we used the abbreviations $u_k := u_{h_k}, \ldots$ etc. The limit functions fulfil
\[ u \in L^2(0, T; H^1(B)), \quad \chi \in L^\infty_w(0, T; BV_{(m_0)}(\Gamma; \{0, 1\}), \quad v \in L^2((0, T) \times \Gamma), \]
\[ \theta \in L^2(0, T; H^1(\Gamma)), \quad \mu \in L^2(0, T; H^1(\Gamma)). \]

**Proof.** First observe that \( \{\theta_h\}_{h \in (0, 1)} \) is bounded in \( L^2(0, T; H^1(\Gamma)) \) and that \( \theta_h = \frac{2}{3}(2v_h - \chi_h) \). We infer from Equation (19) and the energy bound (33) that for all \( k \in \mathbb{N} \) such that \( kh < T \)
\[ \|\tau_{kh}v_h - v_h\|_{L^2(0, T-kh; H^{-1}(\Gamma))} \leq khC(T), \]
where \( \tau_s(f) := f(t+s) \) for every \( t \in (0, T-s) \). Because of equation (18) we find similarly
\[ \|\tau_{kh}\chi_h - \chi_h\|_{L^2(0, T-kh; H^{-1}(\Gamma))} \leq khC(T). \] (36)
As such, we immediately deduce for every \( 0 < t_1 < t_2 < T \)
\[ \sup_{h \in (0, 1)} \|\tau_{kh}\theta_h - \theta_h\|_{L^2(t_1, t_2; H^{-1}(\Gamma))} \to 0 \text{ as } h \to 0. \]
Moreover, \( \{\theta_h\}_{h \in (0, 1)} \) is bounded in \( L^2(0, T; H^1(\Gamma)) \) by (33). Simon’s compactness criterion [11, Theorem 6] then implies for a subsequence \( \{h_k\}_{k \in \mathbb{N}} \), \( h_k \to 0 \) as \( k \to \infty \), and \( 1 \leq p < 2 \) the convergence \( \theta_{h_k} \to \theta \) in \( L^p(0, T; L^2(\Gamma)) \).
We also deduce from the energy estimate in Lemma 3.3 that \( \{\chi_h\}_{h \in (0, 1)} \) is bounded in \( L^\infty_w(0, T; BV_{(m_0)}(\Gamma; \{0, 1\})) \) and together with estimate (36) we can again apply Simon’s compactness criterion in order to deduce the relative compactness of \( \{\chi_h\}_{h \in (0, 1)} \) in \( L^1((0, T) \times \Gamma) \). Furthermore, \( \|\chi_h\|_{L^\infty((0, T) \times \Gamma)} = 1 \) and thus \( \{\chi_h\}_{h \in (0, 1)} \) is relatively compact in \( L^p((0, T) \times \Gamma) \) for all \( 1 \leq p < \infty \). Therefore we find a subsequence of \( \{h_k\}_{k \in \mathbb{N}} \) (again denoted by \( h_k \)) such that \( \chi_{h_k} \to \chi \) in \( L^p((0, T) \times \Gamma) \).
Moreover, \( \{u_h\}_{h \in (0, 1)} \) is bounded in \( L^2(0, T; H^1(B)) \) by Lemma 3.3. Equation (37) now implies
\[ \|\tau_{kh}u_h - u_h\|_{L^2(0, T-kh; H^{-1}(B))} \leq khC(T) \]
and as before we find by [11, Theorem 6] for \( 1 \leq p < 2 \) and \( 1/2 < s < 1 \) the strong convergence (up to a subsequence again denoted by \( h_k \)) \( u_{h_k} \to u \) in \( L^p(0, T; H^s(B)) \).

The weak convergences follow directly from the energy estimates in Proposition 3.1 if one considers again subsequences of \( \{h_k\}_{k \in \mathbb{N}} \). \( \square \)

3.1. Equations in the limit and proof of Theorem 1.1. The proof of Theorem 1.1 is split in the following lemmas.

**Lemma 3.5.** The functions \( (u, \chi, v, \theta, \mu, \lambda) \) obtained in Lemma 3.4 fulfil (12)–(15).

**Proof.** According to (17) and the construction of the \( u_h \), the subsequence \( \{u_k\}_{k \in \mathbb{N}} \) fulfils
\[ -\int_B \nabla u_k(t) \cdot \nabla \eta = \int_B \frac{u_k(t) - u_k(t-h_k)}{h_k} \eta + \int_\Gamma q(u_k(t-h_k), v_k(t-h_k), 2\chi_k(t-h_k) - 1) \eta \] (37)
for all \( \eta \in H^1(B), t \in [0,T] \). After choosing \( \eta \in C^\infty([0,T];H^1(B)) \) with \( \eta(T) = 0 \) and integrating in time, a change of variables in the time variable yields

\[
- \int_0^T \int_B \nabla u_k \cdot \nabla \eta = \int_0^T \int_B u_k \eta - \frac{1}{h} \int_0^h \int_B u_0 \eta - \frac{1}{h} \int_0^h \int_B u_0 \eta + \frac{1}{h} \int_0^h \int_B u_0 \eta + \int_0^T \int q(u_k(\cdot - h_k), v_k(\cdot - h_k), 2\chi_k(\cdot - h_k) - 1) \eta \tag{38}
\]

for all \( \eta \in C^\infty([0,T];H^1(B)) \) with \( \eta(T) = 0 \).

From Lemma 3.4 we know \( \theta_k \to \theta \) in \( L^p(0,T;L^2(\Gamma)) \) for all \( 1 \leq p < 2 \) and thus up to a subsequence \( \theta_k(x,t) \to \theta(x,t) \) pointwise almost everywhere in \( \Gamma \times (0,T) \). Because of \( \chi_k \to \chi \) in \( L^p((0,T) \times \Gamma) \) for all \( 1 \leq p < \infty \), we also have \( \chi_k(x,t) \to \chi(x,t) \) pointwise almost everywhere in \( \Gamma \times (0,T) \) for some subsequence. Using \( v_k = \frac{1}{h} \theta_k + \chi_k \) we find (again for a subsequence),

\[ v_k(x,t) \to v(x,t) \text{ pointwise almost everywhere in } \Gamma \times (0,T). \]

Moreover, Lemma 3.4 yields for \( 1 \leq p < 2 \) and \( 1/2 < s < 1 \) the strong convergence \( u_k \to u \) in \( L^p(0,T;H^s(B)) \). Thus we have \( \text{tr} u_k \to \text{tr} u \) in \( L^p(0,T;L^2(\Gamma)) \) by the continuity of the trace operator. For a suitable subsequence, we directly deduce \( u_k(x,t) \to u(x,t) \) pointwise almost everywhere on \( \Gamma \times (0,T) \).

Since \( q \) growth at most linearly in both arguments, we immediately have

\[
\|q(u_k, v_k, 2\chi_k - 1)\|_{L^2((0,T) \times \Gamma)} \leq C(1 + \|u_k\|_{L^2((0,T) \times \Gamma)} + \|v_k\|_{L^2((0,T) \times \Gamma)}) \leq C(1 + \|u_k\|_{L^2(0,T;H^1(B))} + \|v_k\|_{L^2((0,T) \times \Gamma)})
\]

by the trace theorem. Because of

\[
\|v_k\|_{L^2(\Gamma)}^2 \leq \left\| \frac{1}{h} \theta_k + \chi_k \right\|_{L^2(\Gamma)}^2,
\]

Lemma 3.3 and again the Poincaré inequality for \( BV \)-functions in [3, Remark 3.50] imply

\[
\|v_k\|_{L^2(0,T;L^2(\Gamma))} \leq C(T).
\]

Thus \( q(\text{tr}(u_k), v_k, 2\chi_k - 1) \) is bounded in the reflexive space \( L^2(0,T;L^2(\Gamma)) \) and we deduce the existence of a function \( \tilde{q} \in L^2(0,T;L^2(\Gamma)) \) such that

\[ q(\text{tr}(u_k), v_k, \cdot) \to \tilde{q} \text{ in } L^2(0,T;L^2(\Gamma)). \]

Moreover, \( q(\text{tr}(u_k), v_k, 2\chi_k - 1) \) converges pointwise almost everywhere to \( q(\text{tr}(u), v, 2\chi - 1) \) on \( (0,T) \times \Gamma \) thanks to the continuity of \( q \) and the convergence results on \( u_k, v_k \), and \( \chi_k \) above. Since pointwise and weak limit must coincide (if they both exist as in this case), we obtain the weak convergence

\[ q(\text{tr}(u_k), v_k, 2\chi_k - 1) \to q(\text{tr}(u), v, 2\chi - 1) \text{ in } L^2(0,T;L^2(\Gamma)). \tag{39} \]

Hence taking the limit in equation (38) yields

\[
\int_0^T \int_B \nabla u \cdot \nabla \eta = - \int_0^T \int_B u \eta - \int_0^T \int_B u_0 \eta + \int_0^T \int q(u, v, 2\chi - 1) \eta
\]

for all \( \eta \in C^\infty([0,T],H^1(B)) \) with \( \eta(T) = 0 \).
Similarly, (19) implies
\[
\int_0^T \int_{\Gamma} \nabla \theta_k \cdot \nabla \zeta = - \int_0^T \int_{\Gamma} v_k \zeta - \zeta (\cdot + h_k) - \frac{1}{h} \int_0^h \int_{\Gamma} v_0 \zeta
\]
\[
\quad - \int_0^T \int_{\Gamma} q(u_k(\cdot - h_k), v_k(\cdot - h_k), 2\chi_k(\cdot - h_k) - 1) \zeta
\]
for all \( \zeta \in C^\infty([0, T]; H^1(B)) \) with \( \eta(T) = 0 \) and (39) allows us to take the limit in this equation to deduce
\[
- \int_0^T \int_{\Gamma} \nabla \theta \cdot \nabla \zeta = - \int_0^T \int_{\Gamma} v \partial_t \zeta - \int_0^T v_0 \zeta(0, \cdot) - \int_0^T \int_{\Gamma} q(u, v, 2\chi - 1) \zeta
\]
for all \( \zeta \in C^\infty([0, T]; H^1(B)) \) with \( \zeta(T) = 0 \).

Given the convergence results in Lemma 3.4, it is easy to pass to the limit in (18) to derive (14) and to obtain \( \theta = \frac{1}{2}(2\nu - 2\chi) \).

Passing to the limit in (20) is more difficult. We expect to find the Gibbs-Thomson law (6) in the limit, with the interface \( \gamma \) given as the essential boundary \( \partial^* \{ \chi = 1 \} \). The proof therefore splits in two parts: We have to show that \( \partial^* \{ \chi = 1 \} \) has a generalized curvature as in Definition 2.1 and we have to show that this generalized curvature fulfills the Gibbs-Thomson law.

For the time discrete solutions, let \( V^k_t \) be the integral \((n - 1)\)-varifold associated to the interfaces \( \partial^* \{ \chi_k(\cdot, t) = 1 \} \), i.e.,
\[
\langle V^k_t, \varphi \rangle := \int_{\Omega} \varphi(x, \nabla \chi_k(\cdot, t)) \, d|\nabla \chi_k(\cdot, t)| \quad \text{for all } \varphi \in C^0_c(G_{n-1}(\Gamma)),
\]
By (20) its first variation \( \delta V^k_t \) is given as
\[
\delta V^k_t(\xi) := \int_{\Gamma} \left( \text{div} \xi - \frac{\nabla \chi_k(\cdot, t)}{|\nabla \chi_k(\cdot, t)|} \cdot D\xi \frac{\nabla \chi_k(\cdot, t)}{|\nabla \chi_k(\cdot, t)|} \right) \, d|\nabla \chi_k(\cdot, t)|
\]
\[
= \int_{\Gamma} \chi_k(\cdot, t) \text{div} \xi((2\mu_k(\cdot, t) + \theta_k(\cdot, t))\xi)
\]
for all \( \xi \in C^\infty(\Gamma, \mathbb{R}^2) \). The first step is to prove that for almost all \( t \in (0, T) \) the phase boundary \( \partial^* \{ \chi(\cdot, t) = 1 \} \) in the limit has a generalized mean curvature which is related to the first variation of the limit of \( V^k_t \).

**Lemma 3.6.** For almost all \( t \in (0, T) \) and \( 1 \leq s < \infty \), the phase boundary \( \partial^* \{ \chi(\cdot, t) = 1 \} \) has a generalized mean curvature \( \bar{H}(t) \in L^s(\Gamma, d|\nabla \chi(\cdot, t)|)^2 \) which fulfills
\[
\int_{\Gamma} |H(\cdot, t)|^s \, d|\nabla \chi(\cdot, t)| \leq C \liminf_{h \to 0} \|2\mu_h(\cdot, t) + \theta_h(\cdot, t)\|_{H^1(\Gamma)}.
\]
Under the assumption that there exists a subsequence \( h_k \to 0 \) such that
\[
\limsup_{k \in \mathbb{N}} \|2\mu_{h_k}(\cdot, t) + \theta_{h_k}(\cdot, t)\|_{H^1(\Gamma)} < \infty
\]
we obtain furthermore
\[
\delta V^k_t(\xi) \overset{k \to \infty}{\to} - \int_{\Gamma} \bar{H}(t) \cdot \xi \, d|\nabla \chi(\cdot, t)|
\]
for all \( \xi \in C^\infty(\Gamma, \mathbb{R}^2) \).
Proof. We first observe that Fatou’s Lemma and the energy estimate (34) imply that $t \mapsto \liminf_{h \to 0} \|2\mu_h(\cdot, t) + \theta_h(\cdot, t)\|_{H^1(\Gamma)}$ belongs to $L^2(0, T)$. Hence
\[
\liminf_{h \to 0} \|2\mu_h(\cdot, t) + \theta_h(\cdot, t)\|_{H^1(\Gamma)}
\]
is finite for almost every $t \in (0, T)$ and we restrict our arguments in the following to those $t \in (0, T)$.

The proof relies now on the convergence result due to Theorem 2.2. In order to apply this result on $\mathbb{R}^n$ to the varifolds $V_{t,\alpha}^k$ on $\Gamma$, we localize the necessary calculations in the following way.

As a first step, we introduce suitable varifolds on $\mathbb{R}^2$. Let $\Pi : \mathbb{R}^2 \to \mathbb{R}^2/2\pi\mathbb{Z} = \Gamma$ be the quotient map mapping $\mathbb{R}^2$ onto $\Gamma$. Moreover, let $\alpha \in \{0, 1\}^2$ be a multi-index and define
\[
Q_\alpha = (0, 2\pi)^2 - \alpha \pi.
\]
To simplify the notation, we set $r_\alpha = \left(\Pi|_{Q_\alpha}\right)^{-1}$. For all $\zeta \in C_c(G_1(Q_\alpha))$ we define $U_{t,\alpha}^k$ as
\[
U_{t,\alpha}^k(\zeta) = V_{t,\alpha}^k(r_\alpha^* \zeta).
\]
Equation (40) directly yields
\[
\delta U_{t,\alpha}^k(\omega) = \int_{[0,2\pi]^2 - \alpha \pi} r_\alpha^* \chi_k(\cdot, t) \text{div}((r_\alpha^* 2\mu_k(\cdot, t) + r_\alpha^* \theta_k(\cdot, t)) \omega)
\]
for all $\omega \in C^1_c(Q_\alpha, \mathbb{R}^2)$, which means that
\[
\tilde{H}_{U_{t,\alpha}^k} = r_\alpha^* 2\mu_k(\cdot, t) + r_\alpha^* \theta_k(\cdot, t).
\]
The map $\Pi$ is isometric and thus Lemma 3.3 implies that the functions
\[
r_\alpha^* \mu_k(\cdot, t) \text{ and } r_\alpha^* \theta_k(\cdot, t)
\]
are bounded in $H^1(Q_\alpha)$. As a direct consequence, we deduce that these functions are also bounded in each $W^{1,p}(Q_\alpha)$ for $1 < p \leq 2$ because of $|Q_\alpha| = 4\pi^2$ for all $\alpha \in \{0, 1\}^2$. We can thus fix some $p \in (1, 2)$ and together with Lemma 3.4 we obtain (up to a subsequence) the weak convergence
\[
r_\alpha^* 2\mu_k(\cdot, t) + r_\alpha^* \theta_k(\cdot, t) \rightharpoonup r_\alpha^* 2\mu(\cdot, t) + r_\alpha^* \theta(\cdot, t).
\]
By the energy estimate (33), the varifolds $U_{t,\alpha}^k$ are all uniformly bounded in $k$ for fixed $\alpha \in \{0, 1\}^2$ and $t \in (0, T)$. Starting with $\alpha_0 = (0, 0)$ we can hence find a Radon measure $U_{t,\alpha_0}$ and a subsequence $k_j(\alpha_0)$ (depending on $\alpha_0$) such that $U_{t,\alpha_0}^k(\alpha_0) \rightharpoonup U_{t,\alpha_0}$ as varifolds on $Q_{\alpha_0}$ as $j \to \infty$.

Continuing with $\alpha_1 = (1, 0)$ we can find a subsequence $k_j(\alpha_0, \alpha_1)$ of $k_j(\alpha_0)$ and a Radon measure on $G_1(U_{t,\alpha_1})$ such that $U_{t,\alpha_1}^{k_j(\alpha_0)} \rightharpoonup U_{t,\alpha_1}$ as varifolds on $Q_{\alpha_1}$ as $j \to \infty$. By the choice of the subsequence $k_j(\alpha_0)$ and the definition of $U_{t,\alpha_0}$ and $U_{t,\alpha_1}^k$ respectively, the two measures $U_{t,\alpha_0}$ and $U_{t,\alpha_1}$ must coincide on $Q_{\alpha_0} \cap Q_{\alpha_1}$.

Since there are only finitely many elements $\alpha \in \{0, 1\}^2$, we can continue in this way and find a subsequence $k_j$ and a Radon measure for each $\alpha$ such that the requirements of Theorem 2.2 on each $Q_\alpha$ are met and such that the measures $U_{t,\alpha}$ coincide with each other on all overlaps.

Finally, Lemma 3.4 guarantees that
\[
r_\alpha^* \chi_{E_k_j} \rightharpoonup r_\alpha^* \chi_E \text{ in } L^1(Q_\alpha).
\]
As a result, we can apply Theorem 2.2 for each $\alpha$ and deduce that $U_{t,\alpha}$ is an integral varifold fulfilling
\begin{equation}
\partial^* \{ r_\alpha^* \chi_E = 1 \} \cap Q_\alpha \subseteq \text{supp} \ U_{t,\alpha} \tag{41}
\end{equation}
and
\[ \tilde{H}_{U_{t,\alpha}} = (r_\alpha^* 2\mu(\cdot, t) + r_\alpha^* \theta(\cdot, t)) \nu_E \ m_{U_{t,\alpha}} \] — almost everywhere on $\text{supp} \ U_{t,\alpha}$
Moreover, $\tilde{H}_{U_{t,\alpha}}$ is the generalized mean curvature vector of $\partial^* \{ r_\alpha^* \chi_E = 1 \} \cap Q_\alpha$.
Choose $z_0, z_1 \in C^\infty(\mathbb{R}/2\pi \mathbb{Z})$ in such a way that
\[ \text{supp} \ z_0_{|[0,2\pi]} \subset (0,2\pi), \quad \text{supp} \ z_1_{|[-\pi,\pi]} \subset (-\pi,\pi) \]
and
\[ z_0(p_1) + z_1(p_1) = 1 \text{ for all } p_1 \in \mathbb{R}/2\pi \mathbb{Z}. \]
With $\alpha \in \{0,1\}^2$ as above, we introduce cut-off functions $z_\alpha$ on $\Gamma$ by setting
\[ z_\alpha(p) = \prod_{j=1}^2 z_{\alpha_j}(p_j) \text{ for } p = (p_1, p_2) \in \Gamma. \]
Note that $\sum_{\alpha \in \{0,1\}^2} z_\alpha(p) = 1$ for all $p \in \Gamma$.
We define the varifold $V_t$ on $\Gamma$ by
\[ V_t(\psi) = \sum_{\alpha \in \{0,1\}^2} U_{t,\alpha}(\pi^{-1,*}(z_\alpha \psi)) \text{ for all } \psi \in C^0_c(G_1(\Gamma)). \]
One readily calculates
\[ V_t^{kj} \to V_t \text{ as varifolds} \]
and from (41) we deduce
\[ \partial^* \{ \chi = 1 \} \subseteq \text{supp} \ V_t. \]
Given that $\delta V_t^{kj}$ is linear in $\xi$, we find
\[
\begin{align*}
\delta V_t^{kj}(\xi) &= \delta V_t^{kj} \left( \sum_{\alpha \in \{0,1\}^2} z_\alpha \xi \right) \\
&= \sum_{\alpha \in \{0,1\}^2} \int_\Gamma \left( \text{div}_\Gamma(z_\alpha \xi) - \frac{\nabla \chi_{kj}^0(\cdot, \cdot)}{[\nabla \chi_{kj}^0(\cdot, \cdot)]} \cdot D(z_\alpha \xi) \frac{\nabla \chi_{kj}^0(\cdot, \cdot)}{[\nabla \chi_{kj}^0(\cdot, \cdot)]} \right) \, d[\nabla \chi_{kj}^0(\cdot, \cdot)].
\end{align*}
\]
Because the functions $z_\alpha$ have compact support, we use the definition of the varifolds $U_{t,\alpha}$ to see that we can transform each summand to obtain
\[
\begin{align*}
\lim_{j \to \infty} \delta V_t^{kj}(\xi) &= \lim_{j \to \infty} \sum_{\alpha \in \{0,1\}^2} \int_{\Gamma \cap \text{supp} \ z_\alpha} \left( \text{div}_\Gamma(z_\alpha \xi) - \frac{\nabla \chi_{kj}^0(\cdot, \cdot)}{[\nabla \chi_{kj}^0(\cdot, \cdot)]} \cdot D(z_\alpha \xi) \frac{\nabla \chi_{kj}^0(\cdot, \cdot)}{[\nabla \chi_{kj}^0(\cdot, \cdot)]} \right) \, d[\nabla \chi_{kj}^0(\cdot, \cdot)] \\
&= \lim_{j \to \infty} \sum_{\alpha \in \{0,1\}^2} \delta U_{t,\alpha}^{kj}(\pi^{-1,*}(z_\alpha \xi)) \\
&= \sum_{\alpha \in \{0,1\}^2} \int_{\{0,2\pi\}^2 - \alpha \pi} r_\alpha^* \chi(\cdot, t) \, \text{div}(r_\alpha^* 2\mu(\cdot, t) + r_\alpha^* \theta(\cdot, t)) \pi^{-1,*}(z_\alpha \xi) \\
&= \int_\Gamma \chi(\cdot, t) \, \text{div}_\Gamma((2\mu(\cdot, t) + \theta(\cdot, t))\xi).
\end{align*}
\]
Proof of Theorem 1.1. Following Lemma 3.5 and 3.6, it remains to show \( \bar{\bar{H}}(\cdot, t) = \mu + \theta \) in \( L^2(0, T, H^1(\Gamma)) \).

By Lemma 3.6, the operator \( T(t) : C^1_c(\Gamma; \mathbb{R}^2) \rightarrow \mathbb{R} \) defined by

\[
\langle T(t), \xi \rangle := \int_{\Gamma} -H(x, t) \cdot \xi \, d|\nabla \chi(x, t)| \quad \text{for all } \xi \in C^1_c(\Gamma; \mathbb{R}^2)
\]

exists for almost every \( t \in (0, T) \). Let furthermore \( T^h(t) \) be given by

\[
\langle T^h(t), \xi \rangle := \delta V^h(t) \quad \text{for all } \xi \in C^1_c(\Gamma; \mathbb{R}^2).
\]

Proposition 3.1 now yields

\[
\langle T^h(t), \xi \rangle = \int_{\Gamma} \chi_{h} \, \text{div}_T ((2\mu + \theta_h) \xi) \quad \text{for all } \xi \in C^1_c(\Gamma; \mathbb{R}^2)
\]

and by Lemma 3.4 there exist subsequences \( \{h_k\}_{k \in \mathbb{N}} \) such that

\[
\lim_{k \rightarrow \infty} \int_0^T \langle T^{h_k}(t), \xi(\cdot, t) \rangle \, dt = \int_0^T \int_{\Gamma} \chi \, \text{div}_T ((2\mu + \theta) \xi) \, dt,
\]

Next we show that

\[
\lim_{k \rightarrow \infty} \int_0^T \langle T^{h_k}(t), \xi(\cdot, t) \rangle \, dt = \int_0^T \langle T(t), \xi(\cdot, t) \rangle \, dt,
\]

using Lemma 3.6 and following the arguments in [2, Lemma 4.6]. Lemma 3.6 yields the desired convergence pointwise almost everywhere in time under a boundedness assumption on \( \mu_{\bar{h}_k}(t) + \theta_{\bar{h}_k}(t) \). To apply this lemma, we introduce \( T^{h_k}_\alpha(t) : (0, T) \rightarrow C^1_c(\Gamma; \mathbb{R}^2)' \) defined by

\[
\langle T^{h_k}_\alpha(t), \xi(\cdot, t) \rangle = \begin{cases} 
\langle T^{h_k}(t), \xi(\cdot, t) \rangle & \text{if } \|2\mu_{\bar{h}_k}(t) + \theta_{\bar{h}_k}(t)\|_{H^1(\Gamma)} \leq \alpha, \\
\langle T(t), \xi(\cdot, t) \rangle & \text{else}.
\end{cases}
\]

Now Lemma 3.6 allows us to fix any \( \xi \in L^2(0, T; C^1_c(\Gamma; \mathbb{R}^2)) \) and to obtain

\[
\lim_{k \rightarrow \infty} \langle T^{h_k}_\alpha(t), \xi(\cdot, t) \rangle = \langle T(t), \xi(\cdot, t) \rangle
\]

for almost all \( t \in (0, T) \) since we have that either \( \langle T^{h_k}_\alpha(t), \xi(\cdot, t) \rangle = \langle T(t), \xi(\cdot, t) \rangle \) by definition or the boundedness condition in the lemma is fulfilled. Because of

\[
\|T^{h_k}(t), \xi(\cdot, t)\| \leq C \|2\mu_{\bar{h}_k}(t) + \theta_{\bar{h}_k}(t)\|_{H^1(\Gamma)} \|\xi(\cdot, t)\|_{C^1_c(\Gamma; \mathbb{R}^2)}
\]

we can deduce

\[
\|T^{h_k}_\alpha(t), \xi(\cdot, t)\| \leq C \|\xi(t)\|_{C^1_c(\Gamma)} (\alpha + \|T(t)\|_{C^0(\Gamma; \mathbb{R}^2)}) \\
\leq C \|\xi(t)\|_{C^1_c(\Gamma)} (\alpha + \|H(t)\|_{L^\infty(\Gamma, \text{div} \chi(x,t))}),
\]

for all \( \xi \in C^1_c(\Gamma; \mathbb{R}^2) \). This proves that the varifold \( V_t \) has a mean curvature vector \( \bar{\bar{H}}_t \) such that

\[
\bar{\bar{H}}_t = (2\mu(\cdot, t) + \theta(\cdot, t)) \nu_E \quad \text{almost everywhere on supp} \, V_t,
\]

where \( \nu_E = \frac{\nabla \chi}{|\nabla \chi|} \), which is set to be equal to 0 outside of \( \partial^* E \).

\[ \square \]
Lemma 3.6 and the calculations that led to (43). Hence (44) implies

$$\lim_{k \to \infty} \int_0^T \langle T_{\alpha_k}^h(t), \xi(\cdot, t) \rangle \, dt = \int_0^T \langle T(t), \xi(\cdot, t) \rangle \, dt.$$  \hfill (44)

In order to complete the proof, we study all sets

$$Z_{\alpha_k}^h := \left\{ t \in (0, T) \mid \|2\mu_{h_k}(t) + \theta_{h_k}(t)\|_{H^1(\Gamma)} > \alpha \right\},$$

i.e. all the set of all times \(t \in (0, T)\) for which we set \(\langle T_{\alpha_k}^h(t), \xi(\cdot, t) \rangle = \langle T(t), \xi(\cdot, t) \rangle\).

Recall that \(\|2\mu_{h_k} + \theta_{h_k}\|_{L^2(0, T; H^1(\Gamma))}\) is bounded uniformly in \(h_k > 0\) by Lemma 3.3. Thus

$$|Z_{\alpha_k}^h| = \int_{Z_{\alpha_k}^h} 1 \, dt \leq \int_{Z_{\alpha_k}^h} \frac{1}{\alpha^2} \|2\mu_{h_k}(t) + \theta_{h_k}(t)\|^2_{H^1(\Gamma)} \, dt \leq \frac{1}{\alpha^2} \|2\mu_{h_k} + \theta_{h_k}\|_{L^2(0, T; H^1(\Gamma))} \leq \frac{C}{\alpha^2}.$$

Next observe that \(T_{\alpha_k}^h(t) = T_{\alpha}^h(t)\) for all \(t \in (0, T)\) \(\setminus Z_{\alpha_k}^h\) and calculate

$$\left| \int_0^T \langle T_{\alpha_k}^h(t) - T_{\alpha}^h(t), \xi(\cdot, t) \rangle \, dt \right| \leq \int_{Z_{\alpha_k}^h} \left| \langle T_{\alpha_k}^h(t) - T(t), \xi(\cdot, t) \rangle \right| \, dt \leq \left( \int_{Z_{\alpha_k}^h} \|\xi(\cdot, t)\|_{C_0^1(\Gamma, \mathbb{R}^2)} \right)^{1/2} \left( \|T_{\alpha_k}^h\|_{L^2(0, T; C_0^1(\Gamma, \mathbb{R}^2))} + \|T\|_{L^2(0, T; C_0^1(\Gamma, \mathbb{R}^2))} \right)$$

which yields

$$\lim_{\alpha \to \infty} \sup_{k \in \mathbb{N}} \int_0^T \langle T_{\alpha_k}^h(t) - T_{\alpha}^h(t), \xi(\cdot, t) \rangle \, dt = 0 \quad \hfill (45)$$

since \(\|T_{\alpha_k}^h\|_{L^2(0, T; C_0^1(\Gamma, \mathbb{R}^2))}\) and \(\|T\|_{L^2(0, T; C_0^1(\Gamma, \mathbb{R}^2))}\) are bounded uniformly in \(h_k\) by Lemma 3.6 and the calculations that led to (43). Hence (44) implies

$$\int_0^T \int_\Gamma \chi_{\text{div} \Gamma} ((2\mu + \theta) \xi) \, dx \, dt = \lim_{k \to \infty} \int_0^T \langle T_{\alpha_k}^h(t), \xi(\cdot, t) \rangle \, dt$$

$$= \lim_{k \to \infty} \left[ \int_0^T \langle T_{\alpha_k}^h(t), \xi(\cdot, t) \rangle + \langle T_{\alpha}^h(t) - T_{\alpha_k}^h(t), \xi(\cdot, t) \rangle \, dt \right]$$

$$= \int_0^T \langle T(t), \xi(\cdot, t) \rangle \, dt + \lim_{k \to \infty} \left[ \int_0^T \langle T_{\alpha_k}^h(t) - T_{\alpha}^h(t), \xi(\cdot, t) \rangle \, dt \right].$$

Due to (45), the remaining term on the right hand-side vanishes for \(k \to \infty\) since the convergence in (45) is uniform in \(k \in \mathbb{N}\). Therefore we deduce

$$\int_0^T \int_\Gamma \chi_{\text{div} \Gamma} ((2\mu + \theta) \xi) \, dx \, dt = \lim_{k \to \infty} \int_0^T \langle T_{\alpha_k}^h(t), \xi(\cdot, t) \rangle \, dt$$

$$= \int_0^T \langle T(t), \xi(\cdot, t) \rangle \, dt.$$

Furthermore,

$$\int_\Gamma \chi(\cdot, t) \text{div} \Gamma ((2\mu(\cdot, t) + \theta(\cdot, t)) \xi(\cdot, t)) \, dx = \langle T(t), \xi(\cdot, t) \rangle.$$
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holds for almost all $t \in (0, T)$ and all $\zeta \in C^1_c(\Gamma, \mathbb{R}^2)$. Finally, we set $\nu(\cdot, t) = \frac{\nabla \chi(\cdot, t)}{\|\nabla \chi(\cdot, t)\|}$ on $\partial^*\{\chi(\cdot, t) = 1\}$ and the divergence theorem yields

$$\int_{\Gamma} (2\mu(\cdot, t) + \theta(\cdot, t))\nu(\cdot, t) \cdot \xi \, d|\nabla \Gamma \chi(\cdot, t)| = \int_{\Gamma} H(\cdot, t) \cdot \xi \, d|\nabla \Gamma \chi(\cdot, t)|,$$

which concludes the proof. 

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