RATIONAL K-MATRICES FOR FINITE-DIMENSIONAL REPRESENTATIONS OF QUANTUM AFFINE ALGEBRAS

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ABSTRACT. Let \( \mathfrak{g} \) be a complex simple Lie algebra. We prove that every finite-dimensional representation of the (untwisted) quantum affine algebra \( U_qL\mathfrak{g} \) gives rise to a family of spectral K-matrices, namely solutions of Cherednik’s generalized reflection equation, which depends upon the choice of a quantum affine symmetric pair \( U_q\mathfrak{k} \subset U_qL\mathfrak{g} \). Moreover, we prove that every irreducible representation over \( U_qL\mathfrak{g} \) remains generically irreducible under restriction to \( U_q\mathfrak{k} \). From the latter result, we deduce that every obtained K-matrix can be normalized to a matrix-valued rational function in a multiplicative parameter, known in the study of quantum integrability as a trigonometric K-matrix. Finally, we show that our construction recovers many of the known solutions of the standard reflection equation and gives rise to a large class of new solutions.

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1. INTRODUCTION

1.1. Let \( \mathfrak{g} \) be a simple complex Lie algebra and \( U_qL\mathfrak{g} \) the corresponding (untwisted) quantum affine algebra. It is well-known that the irreducible finite-dimensional \( U_qL\mathfrak{g} \)-representations provide a rich source of rational R-matrices, \( i.e. \), matrix-valued rational functions which satisfy the Yang-Baxter equation in a multiplicative parameter. The

2020 Mathematics Subject Classification. 81R50, 16T25, 17B37, 81R12.

Key words and phrases. Quantum affine algebra; Quantum Symmetric Pairs; Reflection Equation.

The first author is partially supported by the Programme FIL-Quota Incentivante of the University of Parma and co-sponsored by Fondazione Cariparma.
goal of the present paper is to show that, relying on the choice of a fixed quantum affine symmetric pair $U_q \mathfrak{g} \subset U_q \mathfrak{g}$, the same irreducible representations give rise to a family of rational K-matrices, i.e., matrix-valued rational functions which satisfy the reflection equation (or boundary Yang-Baxter equation) in a multiplicative parameter.

The strategy we follow hinges on a recent result of the authors [AV20], namely the existence of a universal K-matrix for any quantum Kac-Moody algebra. The latter is a natural operator on category $\mathcal{O}$ integrable representations, which depends upon the choice of a quantum symmetric pair and satisfy a generalized version of the constant (i.e., parameter-independent) reflection equation, featuring a twisting operator $\psi$. Relying on this construction, we proceed in close similarity to the case of the R-matrix. We show that the universal K-matrix, modified through the grading shift on $U_q \mathfrak{g}$, can be specialized to a spectral operator $K_V(z)$ on any finite-dimensional $U_q \mathfrak{g}$-representation. Moreover, we prove that irreducible representations remain irreducible under restriction to the quantum symmetric pair subalgebra, thus yielding the existence of a rational operator $K_V(z)$. Finally, we prove that, for suitable choices of the twisting operator $\psi$, the generalized reflection equation reduces to the standard reflection equation. This last step allows us to recover many of the known solutions, while providing a large class of entirely new solutions.

In the rest of this introduction, we give a brief overview of the main steps of our construction.

1.2. Let $\mathcal{V}$ be a collection of vector spaces, equipped with two kinds of formal operators. The first, known as spectral R-matrices, are denoted by $R_{VW}(z)$ ($V, W \in \mathcal{V}$), act on tensor products, and satisfy the spectral Yang-Baxter equation:

$$R_{UV}(z) \cdot R_{UW}(wz) \cdot R_{VW}(w) = R_{VW}(w) \cdot R_{UW}(wz) \cdot R_{UV}(z). \quad (1.1)$$

The second, known as spectral K-matrices, are denoted $K_V(z)$ ($V \in \mathcal{V}$), act on a single vector space, and satisfy, together with the R-matrices, the spectral reflection equation (or spectral boundary Yang-Baxter equation):

$$R_{WV}(w/z)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{WV}(zw) \cdot K_V(z) \otimes \text{id} = K_V(z) \otimes \text{id} \cdot R_{WV}(zw)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{WV}(w/z). \quad (1.2)$$

where $R_{WV}(z)_{21} := (1 2) \circ R_{WV}(z) \circ (1 2)$ and (1.2) is the flip operator.

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1We will use the adjective spectral instead of parameter-dependent to avoid confusion with the dependence on other parameters, notably $q$; the parameter $z$ is known as spectral parameter in the literature on integrability.
1.3. The spectral reflection equation was introduced by Cherednik in [Che84] with interlaced mathematical and physical motivations.

(1) They are consistency conditions for representations of the Artin-Tits braid group of type $B_n$ (and, forgetting all equations of the form (1.2), representations of the braid group of type $A_{n-1}$) on vector-valued formal series of $n$ variables. Here the group action combines a natural “tensor” action on a direct sum of tensor products of the form $V_1 \otimes \ldots \otimes V_n$ with $V_i \in \mathcal{V}$ and a “function” action, through the corresponding Coxeter group $W(B_n)$ (signed symmetric group, hyperoctahedral group), acting faithfully by permutations and inversions of the variables.

(2) They are conditions for factorized particle-particle and particle-boundary scattering on a half-line, involving $n$ particles with state spaces $V_1, \ldots, V_n \in \mathcal{V}$, as a precursor to integrability.

For 1D quantum integrable models (or 2D models from statistical mechanics) with spatial periodicity, the integrability formalism based on the Yang-Baxter equation in terms of transfer matrices and scattering matrices was already well-established in the early 1980s, see e.g. [Yan68, FST79, ZZ79, Bax82, Gau83, KZ84]. For 1D models with two boundaries, the formalism was developed in [Skl88, Che91, Che92] requiring as additional input solutions of (1.2) and of a similar equation associated to the other boundary (see [Vla15] for the connection between two-boundary transfer matrices and scattering matrices).

1.4. It is natural to consider the case that $\mathcal{V}$ is a suitable category of modules of a (unital associative) algebra $A$. Namely, in this case it is conceivable that the family of equations (1.1) can be obtained by applying suitable representation maps to a single equation in $A \otimes A \otimes A$, or rather its completion with respect to the category $\mathcal{V}$. A similar approach is a priori sensible for (1.2).

Let $q$ be a formal parameter, $\mathbb{F} := \overline{\mathbb{C}(q)}$, and $\mathcal{V}$ the category of finite-dimensional representations\(^2\) of the (untwisted) quantum affine algebra $U_q\mathfrak{g}$ associated to the loop algebra $\mathfrak{L}_g \cong \mathfrak{g} \otimes \mathbb{C}[z, z^{-1}]$ of a finite-dimensional simple Lie algebra $\mathfrak{g}$. The affine quantum group $U_q\mathfrak{g}$, like any quantized Kac-Moody algebra, is a quasitriangular Hopf algebra with respect to its category $\mathcal{O}$; that is, in the completion of the tensorial square of $U_q\mathfrak{g}$ with respect to $\mathcal{O}$ lies an element $R$, called *universal $R$-matrix*, which satisfies the *universal Yang-Baxter equation*

$$R_{12} \cdot R_{13} \cdot R_{23} = R_{23} \cdot R_{13} \cdot R_{12}. \quad (1.3)$$

Drinfeld [Dri87b] used properties of $R$ to show the existence of a power series $R(z)$ taking values in a completion of the tensorial square of $U_q\mathfrak{g}$ with respect to its category

\(^2\)We only consider so-called type 1 representations of quantum groups.
of finite-dimensional modules. This $R(z)$ satisfies a universal version of (1.1), i.e. a spectral version of (1.3). By setting $R_{V,W}(z)$ equal to the grading-shifted action of this power series on $V \otimes W$ for any $V, W \in \mathcal{V}$, the existence of solutions of (1.1) for any $U, V, W \in \mathcal{V}$ readily follows. Moreover, if $V, W \in \mathcal{V}$ are irreducible $U_qL\mathfrak{g}$-modules then it is known that the tensor product $V \otimes W$ is generically irreducible, i.e. the formal Laurent series extension $V \otimes W((z))$ is irreducible as a module of $U_q\mathfrak{g} \otimes F((z))$, and hence $R_{V,W}(z)$, up to a scalar-valued formal power series, depends rationally on $z$, see e.g. [FR92, KS95].

1.5. In this paper we consider an analogous universal approach to the construction of solutions of the spectral reflection equation. It is useful to remark here on a crucial subtlety compared to the spectral Yang-Baxter equation. Namely, one may in first instance consider a reflection equation of the form

$$R_{21} \cdot 1 \otimes K \cdot R \cdot K \otimes 1 = K \otimes 1 \cdot R_{21} \cdot 1 \otimes K \cdot R$$

where $K$ lies in (a suitable completion of) a quantum group $U_q\mathfrak{g}$ of finite type, see [tDHO98, BK19] for bialgebraic and categorical-topological formalisms, as well as explicit constructions. This yields solutions of the constant (i.e. parameter-independent) reflection equation in finite-dimensional modules of $U_q\mathfrak{g}$. However, proposing this as the universal version of (1.2) in $U_qL\mathfrak{g}$ will not do: it is easy to see that applying grading-shifted actions on $V, W \in \mathcal{V}$ instead yields

$$R_{WV}(z/w)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VW}(w/z) \cdot K_V(z) \otimes \text{id} =$$

$$= K_V(z) \otimes \text{id} \cdot R_{WV}(w/z)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VW}(w/z).$$

This equation corresponds instead to a non-faithful action of the Coxeter group $W(\mathcal{B}_n)$ on functions or formal series on $n$ variables. More precisely, if we define $W(\mathcal{B}_n)$ as usual with generators $s_0, s_1, \ldots, s_{n-1}$ and relations

$$s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \quad \text{if } 0 < i < n - 1, \quad s_0s_1s_0s_1 = s_1s_0s_1s_0,$$

$$s_is_j = s_js_i \quad \text{if } j > i + 1, \quad s_i^2 = 1,$$

then here we are referring to the action which trivially extends the permutation action of the symmetric group: each $s_i$ ($0 < i < n$) acts as a simple transposition on the variables and $s_0$ acts trivially.

1.6. It is instead natural to consider a twisted universal reflection equation, namely an equation of the form

$$(R^{\psi\psi})_{21} \cdot 1 \otimes K \cdot R^{\psi} \cdot K \otimes 1 = K \otimes 1 \cdot (R^{\psi})_{21} \cdot 1 \otimes K \cdot R,$$

where $\psi$ is an algebra automorphism of $U_qL\mathfrak{g}$ and $R^{\psi} := (\psi \otimes \text{id})(R)$, $R^{\psi\psi} := (\psi \otimes \psi)(R)$. Such universal equations and the supporting bialgebraic and categorical frameworks were considered in [AV20], with a concrete realization in the context of representation categories $\mathcal{O}$ and $\mathcal{O}^{\text{int}}$ of q-deformed enveloping algebras of symmetrizable Kac-Moody
algebras. In the present paper we restrict to the affine case and develop this formalism further in the context of finite-dimensional representations of $U_q\mathfrak{g}$.

Note that, in analogy with R-matrices, the universal K-matrix has an intertwining property. However, in order to state this we need additional datum in the form of a quantum pseudo-symmetric pair $(U_q\tilde{\mathfrak{g}}, U_q\mathfrak{t})$, see [Kol14, RV21], where $U_q\mathfrak{t}$ is a right coideal subalgebra of $U_q\tilde{\mathfrak{g}}$ mimicking the fixed-point subalgebra of an involution of $U_q\tilde{\mathfrak{g}}$ of the second kind, cf. [KW92, 4.6]. In this paper we will call such algebras $U_q\mathfrak{t}$ (affine) QSP subalgebras. For a given affine QSP subalgebra $U_q\mathfrak{t}$, the element $K$ enjoys a universal intertwining property with respect to $U_q\mathfrak{t}$: it lies in the $\psi$-twisted centralizer of $U_q\mathfrak{t}$ (in fact a modified version of this statement is used to define $K$).

1.7. One of the main results of this paper is that, for all $V \in \mathcal{V}$ and all affine QSP subalgebras $U_q\mathfrak{t}$, there exist twists $\psi$ such that the universal element $K$ evaluates to an $\text{End}(V)$-valued formal Laurent series $K_V(z)$, and the intertwining property and the universal twisted equation (1.4) descend to equations for $K_V(z)$. Namely, if $\pi_{V,z}$ is the grading-shifted representation map for $V$ and $\psi^*V$ is the pullback of $V$ (i.e. the module where the algebra action has been twisted by $\psi$), then we have

$$K_V(z)\pi_{V,z}(b) = \pi_{\psi^*V,1/z}(b)K_V(z) \quad \text{for all } b \in U_q\mathfrak{t},$$

(1.5)

and, for $V, W \in \mathcal{V}$,

$$R_{\psi^*(W)\psi^*(V)}(w/z)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{\psi^*(V)W}(zw) \cdot K_V(z) \otimes \text{id} = K_V(z) \otimes \text{id} \cdot R_{\psi^*(W)V}(zw)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VW}(w/z).$$

(1.6)

Precisely this generalized reflection equation was considered in [Che92] and used to set up a generalized formalism quantum Knizhnik-Zamolodchikov equations; also cf. [FM91] for similar generalized reflection equations. It turns out that, to obtain the spectral inversion associated to the “function action” of the generator $s_0 \in W(\mathcal{B}_n)$ in (1.5) and (1.6), it is essential that $\psi$ deforms an automorphism of $\tilde{\mathfrak{g}}$ of the second kind.

1.8. Secondly, we show that, if $V \in \mathcal{V}$ is an irreducible $U_q\mathfrak{g}$-module then it remains so as a module of the q-deformed standard nilpotent subalgebra, building on results by [CG05, Bow07, HJ12] for the corresponding q-deformed Borel subalgebra. We then promote this to generic QSP irreducibility, i.e. $V \otimes \mathbb{F}((z))$ is irreducible as a $U_q\mathfrak{t} \otimes \mathbb{F}((z))$-module, provided the grading shift is the principal one. Its main consequence is that the solution space of (1.5) is one-dimensional (for any suitable grading shift).

In analogy with R-matrix theory, we deduce that $K_V(z)$ factorizes as a product of a scalar-valued formal Laurent series and an $\text{End}(V)$-valued rational function, which itself satisfies the generalized reflection equation (1.6). This connects with the literature in
quantum integrability on the classification and intertwining properties of trigonometric K-matrices, see e.g. [Del02, DG02, DM03, MLS06, RV16, RV18].

1.9. Outline. In Section 2, we recall the definition of quantum affine algebras and the construction of the rational R-matrix on irreducible finite-dimensional representations. In Section 3, we recall the definition of QSP subalgebras and provide a brief review of the construction of universal K-matrices for quantum Kac-Moody algebras as given in [AV20]. In Section 4, we present the first main result of the paper. Namely, the construction of parameter-dependent K-matrices on finite-dimensional representations is described in Theorem 4.2.1. In Section 5, we prove in Theorem 4.2.1 that this yields the existence of rational K-matrices for irreducible representations. Moreover, we discuss unitarity and normalization under additional assumptions in 5.6-5.7. In Section 6, we prove the second main result of the paper, which Theorems 5.2.1, 4.2.1 rely on. Namely, we prove in Theorem 6.2.1 that every irreducible representation over $U_qLg$ remains generically irreducible under restriction to a large class of subalgebras, including all QSP subalgebras. Finally, in Section 7, we show that our construction recovers many of the known solutions of the standard reflection equation and most importantly gives rise to a large class of entirely new solutions.

1.10. Acknowledgments. The authors would like to thank Martina Balagović, Vijayanthi Chari, Gustav Delius, Sachin Gautam, Tomasz Przeździecki, Catharina Stroppel, Valerio Toledano Laredo, and Curtis Wendlandt for their interest in this work and for useful discussions. The essence of this paper was conceived during the hybrid Mini-Workshop Three Facets of R-Matrices at the Mathematisches Forschungsinstitut Oberwolfach. We are grateful to the organizers for their invitation and to the institute for the wonderful working conditions.

2. Quantum affine algebras

In this section we recall the definition of quantum affine algebras and basic results on their irreducible finite-dimensional representations. Given a lattice $\Lambda$, we shall denote the non-negative part by $\Lambda_+$.

2.1. Untwisted affine Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra defined over $\mathbb{C}$ with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Let $I := \{1, 2, \ldots, \text{rank}(\mathfrak{g})\}$ be the set of vertices of the corresponding Dynkin diagram, $A = (a_{ij})_{i,j \in I}$ the Cartan matrix, $(\cdot, \cdot)$ the normalized invariant bilinear form on $\mathfrak{g}$, $\Pi := \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^*$ a basis of simple roots and $\Pi^\vee := \{h_i \mid i \in I\} \subset \mathfrak{h}$ a basis of simple coroots such that $\alpha_j(h_i) = a_{ij}$ for

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3Regarding the nomenclature, observe that rational functions in the (multiplicative) parameter $z$ can naturally be rewritten as functions depending trigonometrically on an (additive) parameter proportional to $\log(z)$.

4The field can be any algebraically closed field of characteristic zero. For simplicity, we shall work with $\mathbb{C}$ in order to emphasize the analytic properties of R-matrices and K-matrices.
all $i, j \in I$. Let $Q := \mathbb{Z}P \subset \mathfrak{h}^*$ and $Q^\nu := \mathbb{Z}P^\nu \subset \mathfrak{h}$ be the root and coroot lattice, respectively. Let $\Delta_+ \subset Q_+$ be the set of positive roots and $\vartheta = \sum_{i \in I} a_i \alpha_i$ the highest root. Finally, let $P := \{ \lambda \in \mathfrak{h}^* \mid \lambda(Q^\nu) \subset \mathbb{Z} \}$ be the weight lattice.

Let $\tilde{g}$ be the (untwisted) affine Lie algebra associated to $g$ with affine Cartan subalgebra $\tilde{h} \subset \tilde{g}$ [Kac90, Ch. 7]. Set $\tilde{g} := \tilde{g}'$ and $\tilde{h} := \tilde{h}'$. Let $\hat{I} := \{ 0 \} \cup I$ be the set of vertices of the affine Dynkin diagram and $\hat{A} = (a_{ij})_{i,j \in \hat{I}}$ the extended Cartan matrix [Kac90, Table Aff. 1]. We denote by $\hat{Q}^\nu \subset \hat{h}$ and $\hat{Q} \subset \hat{h}^*$ the affine coroot and root lattices, respectively, and set $\hat{h} := \mathbb{C}\hat{Q}^\nu \subset \hat{h}$ and $(\hat{h}^*)' := \mathbb{C}\hat{Q} \subset \hat{h}^*$. Let $\delta \in \hat{Q}^\nu$ and $c \in \hat{Q}^\nu$ be the unique elements such that

$$\{ \lambda \in \hat{Q} \mid \forall i \in \hat{I}, \lambda(h_i) = 0 \} = \mathbb{Z}\delta \quad \text{and} \quad \{ h \in \hat{Q}^\nu \mid \forall i \in \hat{I}, \alpha_i(h) = 0 \} = \mathbb{Z}c.$$

In particular, $\delta = \alpha_0 + \vartheta$, $c$ is central in $\tilde{g}$, and, under the identification $\nu : \hat{h} \rightarrow \hat{h}^*$ induced by the bilinear form, one has $\nu(c) = \delta$. The sets of real and imaginary affine positive roots in $\hat{Q}^\nu$ are described by

$$\hat{\Delta}_{\text{re}}^+ = \Delta_+ + \mathbb{Z}_{\geq 0}\delta \quad \text{and} \quad \hat{\Delta}_{\text{im}}^+ = \mathbb{Z}_{> 0}\delta.$$

We fix $d \in \hat{h}$ such that $\alpha_i(d) = \delta d$ for any $i \in \hat{I}$. Note that $d$ is defined up to a summand proportional to $c$ and we obtain a natural identification $\tilde{h} = h \oplus \mathbb{C}c \oplus C d$. In terms of the extended coroot lattice $\hat{Q}^\nu_\text{ext} := \hat{Q}^\nu \oplus \mathbb{Z}d \subset \hat{h}$ we set $\hat{P} := \{ \lambda \in \hat{h}^* \mid \lambda(\hat{Q}^\nu_\text{ext}) \subset \mathbb{Z} \}$. Then, the quotient lattice\(^5\)

$$\hat{P}_\delta := \hat{P}/(\hat{P} \cap \mathbb{Q}\delta) \simeq \text{hom}_\mathbb{Z}(\hat{Q}^\nu, \mathbb{Z})$$

has a basis given by the images of the fundamental weights in $\hat{P}$.

2.2. Drinfeld-Jimbo presentation of the affine quantum group. Let $q$ be an indeterminate, let $\mathbb{F}$ be the field of rational functions in $q$ with complex coefficients. Consider the algebraic closure $\mathbb{F} = \overline{\mathbb{C}(q)}$. Fix non–negative integers $\{ \epsilon_i \mid i \in \hat{I} \}$ such that the matrix $(\epsilon_i a_{ij})_{i,j \in \hat{I}}$ is symmetric and set $q_i := q^{\epsilon_i}$. The quantum Kac–Moody algebra associated to $\tilde{g}$ is the algebra $U_q \tilde{g}$ over $\mathbb{F}$ with generators $E_i, F_i, i \in \hat{I}$, and $K_h, h \in \hat{Q}^\nu_\text{ext}$ subject to the following defining relations:

$$K_h K_{h'} = K_{h + h'}, \quad K_0 = 1,$$

$$K_h E_i = q^{\alpha_i(h)} E_i K_h, \quad K_h F_i = q^{-\alpha_i(h)} F_i K_h,$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\text{Serre}_{ij}(E_i, E_j) = 0 = \text{Serre}_{ij}(F_i, F_j) \quad (i \neq j).$$

\(^5\) $\hat{P}_\delta$ is also referred to as the classical weight lattice (see e.g., [Kas02]).
for any \( i, j \in \hat{I} \) and \( h, h' \in \hat{Q}^\vee \), where \( K_i^{\pm 1} := K_{\pm \alpha_i, h} \) and the last line contains the usual \( q \)-deformed Serre relations from e.g. [Lus10, 3.1.1 (e)].

We denote by \( U_q \mathfrak{g} \) the subalgebra generated by \( E_i, F_i, i \in I \), and \( K_h, h \in Q^\vee \), which is a quantum Kac-Moody algebra of finite type.

**Remark 2.2.1.** For any \( \mu \in \hat{Q} \), we set \( K_\mu := K_{\nu^{-1}(\mu)} \), where \( \nu^{-1} : \hat{Q} \to \hat{Q}^\vee \) is induced by the identification \( \nu : \hat{h} \to \hat{h}^* \). In particular, \( K_i^{\pm 1} = K_{\pm \alpha_i} \).

On \( U_q \hat{\mathfrak{g}} \) we shall consider the Hopf algebra structure determined by

\[
\Delta(K_h) = K_h \otimes K_h, \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]

for any \( i \in \hat{I} \) and \( h \in \hat{Q}^\vee \). Finally, the Chevalley involution \( \omega : U_q \hat{\mathfrak{g}} \to (U_q \hat{\mathfrak{g}})^{\text{op}} \) is the isomorphism of Hopf algebras defined by

\[
\omega(K_h) = K_{-h}, \quad \omega(E_i) = -F_i, \quad \omega(F_i) = -E_i.
\]

for any \( i \in \hat{I} \) and \( h \in \hat{Q}^\vee \).

We denote by \( U_q \hat{\mathfrak{h}}^+ \) and \( U_q \hat{\mathfrak{n}}^- \) the subalgebras generated by the elements \( \{E_i\}_{i \in \hat{I}} \) and \( \{F_i\}_{i \in \hat{I}} \), respectively. We set \( U_q \hat{\mathfrak{b}}^\pm := U_q \hat{\mathfrak{n}}^\pm U_q \hat{\mathfrak{h}} \), where \( U_q \hat{\mathfrak{h}} \) is the commutative subalgebra generated by \( K_h, h \in \hat{Q}^\vee \). Similarly, we denote by \( U_q \hat{\mathfrak{b}}, U_q \hat{\mathfrak{n}}, U_q \hat{\mathfrak{b}}^\pm \subset U_q \hat{\mathfrak{g}} \) their finite-type counterparts.

Finally, we denote by \( U_q \hat{\mathfrak{g}} \) the subalgebra obtained by replacing the extended coroot lattice \( \hat{Q}^\vee \) with the standard coroot lattice \( \hat{Q}^\vee \); note that it is generated by \( E_i, F_i, K_i^{\pm} \) for all \( i \in \hat{I} \).

Consider the central element \( K_c = K_\delta = K_0 K_{\theta} = K_0 \prod_{i \in I} K_i^{\alpha_i} \). The quantum loop algebra \( U_q L \mathfrak{g} \) is the quotient of \( U_q \hat{\mathfrak{g}} \) by the ideal generated by \( K_c - 1 \). Note that the Hopf algebra structure and the Chevalley involution descend to \( U_q L \mathfrak{g} \).

**2.3. Category \( \mathcal{O} \) and finite–dimensional representations.** In the case of \( U_q \mathfrak{g} \), the irreducible objects in the categories of finite–dimensional (type 1) representations and integrable category \( \mathcal{O} \) representations coincide. However, for quantum affine algebras, the two categories share only trivial representations and have significantly different features, as we briefly describe below.

Let \( V \in \text{Rep}(U_q \hat{\mathfrak{g}}) \). Recall that \( V \) is

1. a \textit{(type 1)} weight representation if \( V = \bigoplus_{\mu \in \hat{P}} V_\mu \), where
   \[
   V_\mu := \{ v \in V \mid \forall h \in \hat{Q}^\vee \text{, } K_h \cdot v = q^{\mu(h)} v \};
   \]
2. an integrable representation if it is a weight representation and the action of the elements \( \{E_i, F_i \mid i \in \hat{I} \} \) is locally nilpotent.
(3) a category $O^\pm$ representation if it is a weight representation and the action of $U_q\widehat{\mathfrak{g}}$ is locally finite\(^6\) (for a choice of sign).

We denote by $O^{\pm,\text{int}}(U_q\widehat{\mathfrak{g}})$ the full subcategory of integrable category $O^\pm$ representations. It is well–known that $O^{\pm,\text{int}}(U_q\widehat{\mathfrak{g}})$ is semisimple and that the nontrivial irreducible representations in $O^{\pm,\text{int}}(U_q\widehat{\mathfrak{g}})$ are infinite–dimensional and classified by non–zero dominant weights, see [Lus10, Thm 6.2.2, Cor. 6.2.3]. Moreover, $O^{\pm,\text{int}}(U_q\widehat{\mathfrak{g}})$ has a natural tensor product and a braiding (cf. 2.5).

Let $V \in \text{Rep}^{\text{fd}}(U_q\widehat{\mathfrak{g}})$. Recall that $V$ is a type 1 representation if the generators $K_i$ ($i \in \hat{I}$) act semisimply on $V$ with eigenvalues in $q^\mathbb{Z}$ and the central element $K_c$ acts as 1 on $V$. Note that $V$ admits a weight decomposition over the quotient lattice $\hat{P}_\delta$.

By [CP95, Prop. 12.2.3], every irreducible representation in $\text{Rep}^{\text{fd}}(U_q\widehat{\mathfrak{g}})$ is twist-equivalent to a type 1 representation. In particular, the problem of studying the finite–dimensional representation theory of $U_q\widehat{\mathfrak{g}}$ essentially reduces to studying that of the quantum loop algebra $U_qL\mathfrak{g}$. This is known to be extremely rich. First of all, $\text{Rep}^{\text{fd}}(U_qL\mathfrak{g})$ is not semisimple; nevertheless its irreducible representations are classified, namely by $\text{rank}(\mathfrak{g})$–tuples of monic polynomials over $\mathbb{F}$, see [CP95, Thm. 12.2.6]. Moreover, although $\text{Rep}^{\text{fd}}(U_qL\mathfrak{g})$ is monoidal, it is not braided in the usual sense (cf. Section 2.6).

2.4. Completions. In the following, we shall consider suitable completions of the algebras $U_q\widehat{\mathfrak{g}}$ and $U_qL\mathfrak{g}$, so as to include certain distinguished operators acting on representations in $O^{\pm,\text{int}}(U_q\widehat{\mathfrak{g}})$ and $\text{Rep}^{\text{fd}}(U_qL\mathfrak{g})$, respectively.

Let $A$ be a bialgebra and $\mathcal{C} \subseteq \text{Rep}(A)$ a monoidal subcategory. We denote by $A^\mathcal{C}$ the algebra of endomorphisms of the forgetful functor $\mathcal{C} \to \text{Vect}$. Recall that an element of $A^\mathcal{C}$ is a collection of linear maps $\{\varepsilon_V\}_{V \in \mathcal{C}}$ such that, for any intertwiner $f : V \to W$, one has $f \circ \varepsilon_V = \varepsilon_W \circ f$. Note that there is a natural algebra map $\iota : A \to A^\mathcal{C}$ and $\mathcal{C}$ is said to separate points if $\iota$ is an embedding. Similarly, for any $n > 0$, we denote by $(A^{\otimes n})^\mathcal{C}$ the algebra of endomorphisms of the $n$-fold forgetful functor $\mathcal{C}^n \to \text{Vect}$, given by $(V_1, \ldots, V_n) \mapsto V_1 \otimes \cdots \otimes V_n$.

The monoidal structure on $\mathcal{C}$ induces on the tower of algebras $(A^{\otimes n})^\mathcal{C}$ ($n \geq 1$) the structure of a cosimplicial algebra (see e.g., [AV20, Sec. 2.10-2.11], [ATL19, Sec. 8.9]). Roughly, this means that $A^\mathcal{C}$ can be thought of as a topological bialgebra, whose structure extends that of $A$ through $\iota$. Similarly, every automorphism of $\mathcal{C}$, naturally extends to $A^\mathcal{C}$ by $\phi(\varepsilon_V) = \varepsilon_{\phi(V)}$.

\(^6\)In the case $O^+$, this coincides with the category $C^n$ from [Lus10, Sec. 3.4.7]. Note that $V$ is not required to be finitely-generated nor to have finite-dimensional weight spaces.
We shall be interested in the algebras \((U_q \mathfrak{g})^{\text{int}}\) and \((U_q L \mathfrak{g})^{\text{fd}}\), obtained as the completion of \(U_q \mathfrak{g}\) and \(U_q L \mathfrak{g}\) with respect to categories \(\mathcal{O}^{\text{int}}(U_q \mathfrak{g})\) and \(\text{Rep}^{\text{fd}}(U_q L \mathfrak{g})\), respectively.

### 2.5. The universal R-matrix

The Hopf algebra \(U_q \mathfrak{g}\) is quasitriangular, i.e., it admits a universal R-matrix \(R \in \hat{U}_q \mathfrak{b}^- \hat{\otimes} \hat{U}_q \mathfrak{b}^+\), where \(\hat{\otimes}\) denotes the completion with respect to the \(\mathbb{Q}\)-grading, satisfying the intertwining equation \(R \Delta (x) = \Delta^\text{op}(x) R\) for any \(x \in U_q \mathfrak{g}\) and the coproduct identities

\[
\Delta \otimes \text{id}(R) = R_{13} R_{23} \quad \text{and} \quad \text{id} \otimes \Delta(R) = R_{13} R_{12}.
\]

In particular, it follows that \(R\) satisfies the Yang–Baxter equation

\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]

The operator \(R\) arises from the Drinfeld double construction of \(U_q \mathfrak{g}\) as the canonical tensor of a Hopf pairing between \(\hat{U}_q \mathfrak{b}^-\) and \(\hat{U}_q \mathfrak{b}^+\) (cf [Dri87b] and [Lus10, Ch. 4]).

More specifically, let \(\{u_i\}, \{v^j\} \subset \mathfrak{h}\) be dual bases and set

\[
\Omega_0 := \sum_i u_i \otimes v^i, \quad \hat{\Omega}_0 := m(c \otimes d + d \otimes c) + \Omega_0,
\]

where \(m = 1, 2, 3\) if \(\mathfrak{g}\) is of type ADE, BCF, or G, respectively. Then the R-matrix of \(U_q \mathfrak{g}\) has the form

\[
R = q^{\hat{\Omega}_0} \cdot \sum_{\mu > 0} \Xi_\mu \in \hat{U}_q \mathfrak{b}^- \hat{\otimes} \hat{U}_q \mathfrak{b}^+,
\]

where \(\Xi_\mu \in \hat{U}_q \mathfrak{h}^- \otimes \hat{U}_q \mathfrak{h}^+\) is the \(\mu\)-component of the canonical tensor\(^7\). Finally, note that \(\omega \otimes \omega(R) = R_{21}\). Hence the Chevalley involution is an isomorphism of the quasitriangular Hopf algebras \(U_q \mathfrak{g}\) and \(U_q \mathfrak{g}^{\text{cop}}\).

**Remark 2.5.1.** The universal R-matrix is properly defined as an element of the completion \((U_q \mathfrak{g} \otimes U_q \mathfrak{g})^{\text{op}}\) with respect to category \(\mathcal{O}^{\pm}\) representations. Indeed, \(R\) does not lie in \(U_q \mathfrak{g} \otimes U_q \mathfrak{g}\), but it does evaluate to a well-defined operator \(R_{VW}\) on any tensor product \(V \otimes W\) of category \(\mathcal{O}^{\pm}\) representations. Note, in particular, that the action of \(q^{\hat{\Omega}_0}\) on the tensor product of weight vectors is given by \(q^{\hat{\Omega}_0} \cdot v \otimes w = q^{(\text{wt}(v), \text{wt}(w))} \cdot v \otimes w\).

Clearly, \(R\) defines a braiding in the category \(\mathcal{O}^{\pm}(U_q \mathfrak{g})\).

\(^7\)Note that the operator \(q^{\hat{\Omega}_0}\) is denoted \(\kappa_{\text{id}}\) in [AV20, BK19]). See also Remark 2.5.1.
2.6. **The spectral R-matrix.** The universal R-matrix of $U_q \widehat{\mathfrak{g}}$ does not immediately act on finite-dimensional representations $U_q L\mathfrak{g}$. The first obstacle is given by the operator $q^{m(c \otimes d + d \otimes c)}$. However, this is easily solved by observing that, since the central element $K_c$ acts by 1 if and only if $c$ acts by zero, that factor can be ignored. The second obstacle is given by the fact that the projection of the operator $\Xi := \sum_{\mu > 0} \Xi_\mu$ does not necessarily converge on finite-dimensional representations over $U_q L\mathfrak{g}$. To this end, set

$$U_q \widehat{\mathfrak{g}}[z, z^{-1}] := U_q \widehat{\mathfrak{g}} \otimes \mathbb{F}[z, z^{-1}]$$

and consider the **homogeneous grading shift automorphism**

$$\Sigma_z : U_q \widehat{\mathfrak{g}}[z, z^{-1}] \to U_q \widehat{\mathfrak{g}}[z, z^{-1}]$$

given by $\Sigma_z(K_h) := K_h$, $\Sigma_z(E_i) := z^{\delta_{ia}} E_i$, and $\Sigma_z(F_i) := z^{-\delta_{ia}} F_i$. Note that, by specializing $z$ in $\mathbb{F}$, we obtain a one-parameter family of automorphism of $U_q \widehat{\mathfrak{g}}$. Then, let

$$\Delta_z, \Delta_z^{\text{op}} : U_q \widehat{\mathfrak{g}}[z, z^{-1}] \to (U_q \widehat{\mathfrak{g}} \otimes U_q \widehat{\mathfrak{g}})[z, z^{-1}]$$

be the **shifted coproducts** defined by

$$\Delta_z(x) := \text{id} \otimes \Sigma_z(\Delta(x)), \quad \Delta_z^{\text{op}}(x) := \text{id} \otimes \Sigma_z(\Delta^{\text{op}}(x)).$$

The grading shift is clearly well-defined on $U_q L\mathfrak{g}$. For any $V \in \text{Rep}^{fd}(U_q L\mathfrak{g})$ with action $\pi_V : U_q L\mathfrak{g} \to \text{End}(V)$, we consider the infinite-dimensional representations

$$V(z) := V \otimes \mathbb{F}(z), \quad V((z)) := V \otimes \mathbb{F}((z)),$$

with action given by $\pi_V(\Sigma_z(x))$. By considering the projection of the formal series $\text{id} \otimes \Sigma_z(R) \in (U_q \widehat{\mathfrak{g}} \otimes U_q \widehat{\mathfrak{g}})[[z]]$ on the quantum loop algebra, one obtains the following theorem due to Drinfeld (cf. [Dri87b], see also [FR92, Her19]).

**Theorem 2.6.1.**

1. The quantum loop algebra $U_q L\mathfrak{g}$ has a universal spectral R-matrix, i.e., a distinguished element $R(z) \in (U_q \widehat{\mathfrak{g}} \otimes U_q \widehat{\mathfrak{g}})[[z]]$ such that $\Sigma_a \otimes \Sigma_b(R(z)) = R(\frac{b}{a} z)$ ($a, b \in \mathbb{F}^\times$) and the following identities are satisfied:

$$R(z) \Delta_z(x) = \Delta_z^{\text{op}}(x) R(z), \quad \text{for all } x \in U_q L\mathfrak{g},$$

$$\Delta_z \otimes \text{id}(R(z)) = R_{13}(zw) R_{23}(w), \quad \text{id} \otimes \Delta_z(R(z)) = R_{13}(z) R_{12}(zw).$$

In particular, the spectral Yang–Baxter equation holds:

$$R_{12}(z) R_{13}(zw) R_{23}(w) = R_{23}(w) R_{13}(zw) R_{12}(z). \quad (2.2)$$

2. For any $V, W \in \text{Rep}^{fd}(U_q L\mathfrak{g})$, the operator

$$R_{VW}(z) := \pi_V \otimes \pi_W(R(z)) \in \text{End}(V \otimes W)[[z]]$$

is well-defined and provides an intertwiner

$$\tilde{R}_{VW}(z) := (1.2) \circ R_{VW}(z) : V((z)) \otimes W \to W \otimes V((z)).$$
2.7. Rational R-matrices. In the case of irreducible representations the operator $R_{VW}(z)$ features the following rationality property (see e.g., [Jim86, Dri87a, FR92, KS95, EFK98]).

**Theorem 2.7.1.** Let $V, W \in \text{Rep}^{fd}(U_q Lg)$ be two irreducible representations. There exists a canonical scalar-valued formal Laurent series $f_{VW}(z) \in \mathbb{F}((z))$ such that

$$R_{VW}(z) := f_{VW}(z)^{-1} R(z) \in \text{End}(V \otimes W)((z))$$

is rational and satisfies the spectral Yang-Baxter equation (2.2) and the unitarity relation

$$R_{VW}(z)^{-1} = (1 \, 2) \circ R_{WV}(z^{-1}) \circ (1 \, 2).$$

In particular, $\tilde{R}_{VW}(z) := (1 \, 2) \circ R_{VW}(z)$ is an intertwiner $V \otimes W(z) \to W(z) \otimes V$.

The proof of the theorem relies on the generic irreducibility of the tensor product $V \otimes W$, i.e., on the irreducibility of the representation $V \otimes W((z))$ over $U_q Lg((z))$ (cf. [KS95, Sec. 4.2] or [Cha02, Thm. 3]). We shall use a similar argument in Section 5 to prove the existence of a rational K-matrix. Finally, note that $f_{VW}(z)$ is uniquely determined by the condition $R(z)(v \otimes w) = v \otimes w$, where $v \in V$ and $w \in W$ are highest weight vectors.

**Example 2.7.2.** Let $g = \mathfrak{sl}(2)$, $V_1 = \mathbb{C}^2$ the fundamental representation, and $V_1(a)$ the corresponding evaluation representation of $U_q \mathfrak{sl}(2)$ at $a \in \mathbb{C}^\times$. In the case of $V_1(a) \otimes V_1(bz)$, the rational function $R_{a,b}(z) := R_{V_1(a) \otimes V_1(bz)}(z)$ is easily computed (see e.g., [CP95, 12.5.7] or [Jim86]). Set $\lambda = b/a$. Then

$$R_{a,b}(z) := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q(1-\lambda z)}{q^2-\lambda z} & \frac{\lambda(z^2-1)}{q^2-\lambda z} & 0 \\ 0 & \frac{1-\lambda z}{q^2-\lambda z} & \frac{q(1-\lambda z)}{q^2-\lambda z} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

Note that, if $\lambda = q^2$, $R_{a,b}(z)$ has a pole at $z = 1$, while, if $\lambda = q^{-2}$, $R_{a,b}(z)$ is not invertible at $z = 1$. It is well–known that $V_1(a) \otimes V_1(b)$ fails to be irreducible precisely when $\lambda = q^{\pm 2}$. □

**Remark 2.7.3.** The R-matrix $\tilde{R}_{VW}(z)$ is in fact a meromorphic intertwiner (in the case $q \in \mathbb{C}$) [FR92, KS95]. More specifically, relying on the crossing symmetry, i.e., the functional relation between $R_{VW}(z)$ and $R_{V^*W}(z)$, one proves that the operator $R(z)$ is analytic near zero and therefore extends meromorphically to $\mathbb{C}$ (cf. [EM02, Appendix]). □

3. Quantum affine symmetric pairs

3.1. Generalized Satake diagrams. Classical and quantum Kac-Moody algebras are defined in terms of combinatorial datum encoded by the Dynkin diagram and the
Cartan matrix. Similarly, classical and quantum symmetric pairs \cite{Let02, Kol14} (the latter are also known as quantum symmetric pair (coideal) subalgebras, Letzer-Kolb coideal subalgebras and \(\iota\) quantum groups, see e.g. \cite{LW21}) arise from a refinement of such datum. In the Kac-Moody setting, this relies on the semidirect product decomposition of the automorphism group of the Kac-Moody algebras and the canonical factorization of specific automorphisms given in \cite{KW92, 4.23 and 4.39} and further developed in \cite{Kol14, App. A} and, somewhat more generally, in \cite{RV21}.

Let \(\text{Aut}(\hat{A})\) be the group of diagram automorphisms of the affine Cartan matrix, i.e., the group of bijections \(\tau : \hat{I} \to \hat{I}\) such that \(a_{ij} = a_{\tau(i)\tau(j)}\). Let \(X \subset \hat{I}\) be a proper subset of indices. Note that the corresponding Cartan matrix \(A_X\) is necessarily of finite type. We denote by \(\text{o}i_X \in \text{Aut}(A_X)\) the opposition involution of \(X\), i.e., the involutive diagram automorphism on \(X\) induced by the action of the longest element \(w_X\) of the Weyl group \(W_X\), cf. \cite{AV20, Sec. 3.11}.

We have the following definition, cf. \cite{RV20, RV21}.

**Definition 3.1.1.** We call a pair \((X, \tau)\) a generalized affine Satake diagram and write \((X, \tau) \in \text{GSat}(\hat{A})\) if \(X \subset \hat{I}\), \(\tau\) is an involutive diagram automorphism which preserves \(X\), and

\[
\begin{align*}
(1) \quad & \tau|_X = o\iota_X, \\
(2) \quad & \text{for any } i \in \hat{I}\setminus X \text{ such that } \tau(i) = i, \text{ the connected component of } X \cup \{i\} \text{ containing } i \text{ is not of type } A_2.
\end{align*}
\]

A classification of generalized Satake diagrams for \(\hat{A}\) is given in \cite[App. A, Tables 5, 6 and 7]{RV21}. Henceforth, we fix \((X, \tau) \in \text{GSat}(\hat{A})\).

### 3.2. Pseudo–involutions

The diagram automorphism \(\tau \in \text{Aut}(\hat{A})\) extends canonically to an automorphism of \(\hat{\mathfrak{g}}\), given on the generators by \(\tau(h_i) := h_{\tau(i)}, \tau(e_i) = e_{\tau(i)},\) and \(\tau(f_i) = f_{\tau(i)}\). We associate to the combinatorial datum \((X, \tau)\) the Lie algebra automorphism \(\theta : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}\), given by

\[
\theta := \text{Ad}(w_X) \circ \omega \circ \tau \tag{3.1}
\]

where \(\omega\) denotes the Chevalley involution on \(\hat{\mathfrak{g}}\). Note that, since \(\alpha_i(\tau(c)) = \alpha_{\tau(i)}(c) = 0\), \(\tau\) preserves \(\mathbb{C}c\). It follows that \(\tau(c) = c\) and \(\theta(c) = -c\). Hence, \(\theta\) descends to an automorphism of \(L\mathfrak{g}\).

**Remarks 3.2.1.**

1. In \cite[Sec. 4.9]{KW92}, Kac and Wang defined a canonical procedure (for arbitrary generalized Cartan matrices) to extend a given diagram automorphism from \(\hat{\mathfrak{h}}\) to \(\tilde{\mathfrak{h}}\). It follows that \(\theta\) canonically extends to an automorphism of \(\tilde{\mathfrak{g}}\). Note that
$\theta$ is an automorphism of $\hat{\frak g}$ of the second kind (see [KW92, 4.6]): the two standard nilpotent subalgebras $n^\pm$ are swapped, up to finite-dimensional subalgebras. Moreover, it is an involution on $\hat{\frak h}$; following [RV21], we will therefore refer to $\theta$ as a pseudo–involution of $\hat{\frak g}$ of the second kind. We note that $\hat{\frak h}^\theta \subseteq \frak h$, see [AV20, Sec. 6.2].

(2) Note that the assignment $(X, \tau) \mapsto \theta$ described in (3.1) can be inverted by setting $X = \{ i \in \hat{I} \mid \theta(h_i) = h_i \}$ and subsequently $\tau = \omega \circ \text{Ad}(w_X)^{-1} \circ \theta$. Relying on this correspondence, henceforth we shall use the subscript $\theta$ also for objects explicitly defined in terms of $(X, \tau)$.

(3) The dual map $\theta : \hat{\frak h} \to \hat{\frak h}$ is denoted by the same symbol. Since $\theta(c) = -c$, one has $\theta(\delta) = -\delta$ and hence $\mathbb{Z}\delta \subseteq \hat{\frak q}^{-\theta}$. Moreover, $\hat{\frak q}^+ = Q^+ \oplus \mathbb{Z}_{\geq 0}\delta$ and therefore $\hat{\frak q}^\theta = Q^\theta \oplus \mathbb{Z}_{\geq 0}\delta$. The restricted rank (also known as the relative rank or real rank) of $\theta$ is the rank of $\hat{\frak q}^{-\theta}$ and is equal to the number of $\tau$-orbits in $\hat{\frak L}_X$ (see e.g. [RV21, Sec. 4] and [AV20, Sec. 8.10]). It follows that $\theta$ has restricted rank one if and only if $\hat{\frak q}^{-\theta} = \mathbb{Z}\delta$. Suppose $\theta$ has restricted rank one. For any $j \in \hat{\frak L}_X$, $\theta$ fixes $\alpha_j - \alpha_{\tau(j)}$ and each $\alpha_i$ ($i \in X$), thus yielding

$$\theta(\alpha_j) = \alpha_j - \frac{2}{a_j|\hat{\frak L}_X|}\delta$$

where we have written $\delta = \sum_{i \in \hat{I}} a_i \alpha_i$ with $a_0 = 1$. $\n$

3.3. Quantum pseudo–involutions. We shall consider a distinguished lift of the pseudo–involution $\theta$ to an algebra automorphism $\theta_\phi$ of $U_q\hat{\frak g}$ and $U_qL\frak g$. This is obtained by choosing a suitable lift for each of the three factors in $\theta$. First, we consider the usual Chevalley involution on $U_q\frak g$ given by (2.1). The diagram automorphism $\tau$ extends canonically to an automorphism of $U_q\hat{\frak g}$ given on the generators by $\tau(E_i) = E_{\tau(i)}$, $\tau(F_i) = F_{\tau(i)}$ and $\tau(K_h) = K_{\tau(h)}$.

The action of the Weyl group operator $w_X \in W_X$ is lifted to $U_q\hat{\frak g}$ as follows. Let $S_X$ be the braid group operator corresponding to $w_X$ acting on any representation in $O^{\pm, \text{int}}(U_q\frak g)$, cf. [Lus10, Sec. 5] and also [AV20, Sec. 5]. More precisely, given a reduced expression $s_{i_1} \cdots s_{i_t}$ of $w_X$ in terms of fundamental reflections, $S_X := S_{i_1} \cdots S_{i_t}$, where, as in [AV20], $S_j = T_{j,1}^\mu$ in the notation from [Lus10, 5.2.1]. It follows from the braid relations that $S_X$ is independent of the reduced expression. We consider the Cartan correction of $S_X$ given by

$$S_\theta := \xi_\theta \cdot S_X .$$

Here $\xi_\theta$ is the Cartan operator defined on any weight vector of weight $\lambda$ as the multiplication by $q^{(\theta(\lambda), \lambda)/2 + (\lambda, \rho_X)}$, where $\rho_X$ is the half–sum of the positive roots in

\begin{align*}
\end{align*}
Let \( \Delta_X \) (cf. [AV20, Sec. 4.9]). By [AV20, Lemma 4.3 (iii)], we obtain an automorphism \( T_\theta : U_q \hat{g} \to U_q \hat{g} \) given by \( T_\theta := \text{Ad} S_\theta \).

The quantum pseudo–involution is the automorphism \( \theta_q : U_q \hat{g} \to U_q \hat{g} \) given by
\[
\theta_q := T_\theta \circ \omega \circ \tau.
\]

Note that, as in the classical case, \( \theta_q \) is independent of the order of the three factors. In [AV20, Lem. 6.10], we derive the following important properties of \( \theta_q \):
\[
\begin{align*}
\theta_q(K_h) &= K_{\theta(h)} \\
\theta_q((U_q \hat{g})_\lambda) &= (U_q \hat{g})_{\theta^*(\lambda)} \\
\theta_q|_{U_q \hat{g}_X} &= \text{id}_{U_q \hat{g}_X},
\end{align*}
\]
where for all \( X \subset \hat{T} \) we have introduced the subalgebra \( U_q \hat{g}_X := \langle E_i, F_i, K_i^{\pm 1} \rangle_{i \in X} \) of \( U_q \hat{g} \), which is a Drinfeld-Jimbo quantum group of finite type.

**Remark 3.3.1.** Further to Remark 3.2.1, note that the Kac-Wang extension of the diagram automorphism \( \tau \) to \( \hat{h} \) does not necessarily preserve the extended coroot lattice \( \hat{Q}_{\text{ext}}^\vee \subset \hat{h} \), and therefore does not automatically extend to an automorphism of \( U_q \hat{g} \). To remedy this, one can modify the lattice itself by replacing the standard derivation \( d \in \hat{h} \) with any \( d_r \in \hat{h} \) such that \( \alpha_i(d_r) = \alpha_{r(i)}(d_r) \), see [Kol14, Sec. 2.6].

For the purposes of this paper it suffices to regard \( \theta_q \) as an automorphism of \( U_q \hat{g} \). Furthermore, \( \theta_q \) descends to an automorphism of the quantum loop algebra \( U_q L \hat{g} \).

### 3.4. QSP subalgebras

By [AV20, Sec. 6.2], \( \hat{h}^\theta \subset \hat{h} \), thus one obtains \( (U_q \hat{h})^{\theta_q} = U_q (\hat{h}^\theta) \subseteq U_q \hat{h} \). To a given pseudo–involution \( \theta \) of the second kind, we associate a family of coideal subalgebras of \( U_q \hat{g} \) intersecting \( U_q \hat{h} \) in \( (U_q \hat{h})^{\theta_q} \). This family is parametrized by two sets, \( \Gamma \subset (\mathbb{F}^*)^\hat{T} \) and \( \Sigma \subset \mathbb{F}^\hat{T} \), introduced in [Let02, Kol14, RV20, RV21].

**Definition 3.4.1.** The QSP subalgebra of \( U_q \hat{g} \) corresponding to \( \theta \) with parameters \( (\gamma, \sigma) \in \Gamma \times \Sigma \) is the subalgebra
\[
U_q^\Gamma := U_q^\Gamma_{\theta, \gamma, \sigma} = \langle U_q \hat{g}_X, (U_q \hat{h})^{\theta_q}, B_i \mid i \in \hat{T} \setminus X \rangle,
\]
where the elements \( B_i := B_i_{\theta, \gamma, \sigma} \in U_q \hat{g} \) are given by
\[
B_i := \begin{cases} 
F_i & \text{if } i \in X, \\
F_i + \gamma_i \cdot \theta_q(F_i) + \sigma_i \cdot K_i^{-1} & \text{if } i \notin X.
\end{cases}
\tag{3.2}
\]

**Remark 3.4.2.** More precisely, \( \Gamma \) and \( \Sigma \) are defined as follows, see [Kol14, Eqns. (5.9) and (5.11)]. First, we choose a subset \( \hat{T}^* \subseteq \hat{T} \setminus X \) by picking a representative for every \( \tau \)-orbit in \( \hat{T} \setminus X \). Then, we define
\[
\hat{T}_{\text{diff}} = \{i \in \hat{T}^* \mid \tau(i) \neq i \text{ and } \exists j \in X \cup \{\tau(i)\} \text{ such that } a_{ij} \neq 0\},
\]
\[ \hat{I}_{ns} = \{ i \in \hat{I}^* \mid \tau(i) = i \text{ and } \forall j \in X \ a_{ij} = 0 \}. \]

Now \( \Gamma \) is the set of tuples \( \gamma \in (\mathbb{F}^x)^\hat{I} \) such that \( \gamma_i = 1 \) if \( i \in X \) and \( \gamma_i = \gamma_{\tau(i)} \) if \( \{ i, \tau(i) \} \cap \hat{I}_{\text{diff}} = \emptyset \). Furthermore, \( \Sigma \) is the set of tuples \( \sigma \in \mathbb{F}^\hat{I} \) such that \( \sigma_i = 0 \) if \( i \in \hat{I} \setminus \hat{I}_{ns} \) and, for all \( (i, j) \in \hat{I}^2 \), \( a_{ij} \in 2\mathbb{Z} \) or \( \sigma_j = 0 \). Note that \( \Gamma \) and \( \Sigma \) do not depend upon the choice of \( \hat{I}^* \). The constraints on the parameter sets \( \Gamma \) and \( \Sigma \) are motivated by Proposition 3.4.3 below.

Note that \( U_q^\mathfrak{g}_X(U_q^\mathfrak{h})^{\theta} \subset U_q^\mathfrak{k} \) is a subbialgebra of \( U_q^\mathfrak{g} \). The following result follows directly from [Kol14, Props. 5.2 and 6.2].

**Proposition 3.4.3.** For any pseudo-involution of the second kind \( \theta \), \( U_q^\mathfrak{k} \) is a right coideal subalgebra of \( U_q^\mathfrak{g} \). Moreover, whenever \( (\gamma, \sigma) \in \Gamma \times \Sigma \), \( U_q^\mathfrak{k} \) has minimal intersection with \( U_q^\mathfrak{h} \):

\[ \Delta(U_q^\mathfrak{k}) = U_q^\mathfrak{k} \otimes U_q^\mathfrak{g}, \quad U_q^\mathfrak{k} \cap U_q^\mathfrak{h} = (U_q^\mathfrak{g})^{\theta}. \]

**Remark 3.4.4.** Following [AV20, Sec. 7.4], we shall regard the tuple \( \gamma \) as a diagonal operator on category \( \mathcal{O}^+ \) representations. Namely, we fix henceforth a group homomorphism \( \gamma : \hat{P} \to \mathbb{F}^x \) such that \( \gamma(\alpha_i) := \gamma_i \) (\( i \in \hat{I} \)). Then, \( \gamma \) acts on any weight vector of weight \( \lambda \) as multiplication by \( \gamma(\lambda) \).

3.5. **The standard K-matrix.** In [AV20, Thm. 8.11-8.12], the authors proved that the QSP subalgebra \( U_q^\mathfrak{k} \) gives rise to a discrete family of K-matrices in \( U_q^\mathfrak{g} \), indexed by affine generalized Satake diagrams. In Section 3.6, we shall provide an even more general construction. We first recall the definition of the standard K-matrix.

**Theorem 3.5.1 ([AV20]).** There exists a unique series \( \Upsilon_\theta := 1 + \sum_{\mu \in \hat{\mathfrak{g}}_\theta^-} \Upsilon_{\theta, \mu} \) with \( \Upsilon_{\theta, \mu} \in U_q^\mathfrak{n}_\mu^\mathfrak{h} \) such that the operator \( K_\theta := \gamma^{-1} \cdot \Upsilon_\theta \in (U_q^\mathfrak{g})^{\mathcal{O}^+, \text{int}} \) satisfies the intertwining identity

\[ K_\theta \cdot b = \theta^{-1}_q(b) \cdot K_\theta \quad (b \in U_q^\mathfrak{k}) \]

and the coproduct identity

\[ \Delta(K_\theta) = R_\theta^{-1} \cdot 1 \otimes K_\theta \cdot R_\theta^{\theta^{-1}} \cdot K_\theta \otimes 1 \]

where \( R \) is the R-matrix of \( U_q^\mathfrak{g} \), \( R_\theta^{\theta^{-1}} := \theta^{-1}_q \otimes \text{id}(R) \), and \( R_\theta := (S_\theta \otimes S_\theta)^{-1} \cdot \Delta(S_\theta) \) is (up to a Cartan correction) the R-matrix of \( U_q^\mathfrak{g}_X \). Moreover, a constant version of the generalized reflection equation holds:

\[ R_2^{\theta^{-1}} \cdot 1 \otimes K_\theta \cdot R^{\theta^{-1}} \cdot K_\theta \otimes 1 = K_\theta \otimes 1 \cdot (R^{\theta^{-1}})_{21} \cdot 1 \otimes K_\theta \cdot R. \]

\(^8\text{Kolb's proofs straightforwardly extend to the case of generalized Satake diagrams.}\)
Remark 3.5.2. We shall refer to the operator $\Upsilon_\theta$ as the quasi K-matrix of $U_q\mathfrak{k}$, although in terms of the quasi K-matrix $X_\theta$ defined in [BW18] and [BK19] as the unique intertwiner between the two bar involutions on $U_q\widehat{\mathfrak{g}}$ and $U_q\mathfrak{k}$ we have $\Upsilon_\theta = \overline{X_\theta}$. Moreover, our construction in [AV20] of $\Upsilon_\theta$ does not rely at all on the QSP bar involution. In fact, as later observed in [Kol21], $\Upsilon_\theta$ can be used to define the bar involution on $U_q\mathfrak{k}$.

3.6. Gauging K-matrices. New solutions of the generalized reflection equations are easily obtained by acting simultaneously on the K-matrix $K_\theta$ and the operator $\theta^{-1}_q$. Consider the following group:

$$G := \left\{ g \in \left( (U_q\widehat{\mathfrak{g}})^{\mathcal{O}^{+,\text{int}}} \right)^\times \mid \text{Ad}(g)(U_q\widehat{\mathfrak{g}}) = U_q\widehat{\mathfrak{g}} \right\}.$$ 

The condition $\varepsilon(g) = 1$ is a non-essential convenient normalization condition. We have the following

Corollary 3.6.1. Let $g \in G$ and set $\psi := \text{Ad}(g) \circ \theta^{-1}_q$ and $R_\psi := (gS_\theta \circ gS_\theta)^{-1}\Delta(gS_\theta)$. The operator

$$K_\psi := g \cdot \gamma^{-1} \cdot \Upsilon_\theta \in (U_q\widehat{\mathfrak{g}})^{\mathcal{O}^{+,\text{int}}}$$

satisfies the intertwining identity

$$K_\psi \cdot b = \psi(b) \cdot K_\psi \quad (b \in U_q\mathfrak{k})$$

and the coproduct identity

$$\Delta(K_\psi) = R^{-1}_\psi \cdot 1 \otimes K_\psi \cdot R_\psi^0 \cdot K_\psi \otimes 1$$

where $R$ is the R-matrix of $U_q\widehat{\mathfrak{g}}$ and $R_\psi := \psi \otimes \text{id}(R)$. Moreover, the generalized reflection equation holds:

$$R_{21}^{\psi \psi} \cdot 1 \otimes K_\psi \cdot R_\psi^0 \cdot K_\psi \otimes 1 = K_\psi \otimes 1 \cdot (R_\psi^0)_{21} \cdot 1 \otimes K_\psi \cdot R.$$  \hspace{1cm} (3.6)

Proof. It is enough to observe that, under the identity $K_\psi = g \cdot K_\theta$, the equations (3.4), (3.5), and (3.6) reduce to their analogues for the standard K-matrix $K_\theta$ in Theorem 3.5.1. □

As before, the evaluation of $K_\theta$ on $V \in \mathcal{O}^{+,\text{int}}(U_q\widehat{\mathfrak{g}})$ yields a QSP intertwiner

$$K_{\psi,V} : V \to \psi^*(V)$$

We shall refer to the automorphism $\psi$ as the twisting operator of the reflection equation. The coproduct identity (3.5) and the reflection equation (3.6) both admit a similar representation theoretic interpretation. Namely, note that, for any $U \in \text{Rep}U_q\widehat{\mathfrak{g}}$ and $W \in \mathcal{O}^{+,\text{int}}(U_q\widehat{\mathfrak{g}})$, the operator $R_{UW} \in \text{End}(U \otimes W)$ is well-defined\(^9\), and therefore so is $R_{\psi^*(V)W}$. Moreover, by construction, one has

$$R_{21}^{\psi \psi} = R_{\psi,21} \cdot R \cdot R_\psi^{-1}.$$ \hspace{1cm} (3.6)

\(^9\)Note however that this is not true for $R_{WU}$.\n
The operator $R_{\psi^* (V) \psi^* (W)}$ is also well-defined and (3.6) holds. As mentioned in the statement of the Theorem, the latter is a constant (i.e., parameter-independent) version of the generalized reflection equation introduced by Cherednik in [Che92, Eq. (4.14)] (see also [FM91]), which encodes the quartic relations in the cylindrical braid group, cf. [AV20, Sec. 2]. Note that (3.6) depends on the choice of $\psi$.

**Remark 3.6.2.** We describe several distinguished examples of $g \in G$ and the corresponding K-matrices and twists.

1. First of all, we may consider diagonal modifications of the K-matrix obtained by setting $g = \beta$, where $\beta$ is any map $P \to \mathbb{F}^\times$.

2. If $g = S_\theta$ then the twisting operator is the involution $\psi = \omega \circ \tau$. We refer to the corresponding K-matrix as the semi-standard K-matrix.

3. Let $(Y, \eta) \in GSAT(\hat{A})$ be a Satake diagram and $\zeta$ the corresponding pseudo-involution. For $g = S^{\zeta^{-1}}_\theta$, one recovers the combinatorial family of universal K-matrices constructed in [AV20]. There are two important special cases:
   - If $\zeta = \theta$, we get the standard K-matrix.
   - The second special case is obtained in analogy with the finite-type K-matrix constructed by Balagović and Kolb in [BK19], which corresponds to the choice $\zeta = \zeta = \text{id}$. The affine analogue of this choice, described in [AV20, Sec. 9], is obtained by taking $Y = \hat{I}\setminus \{0, \tau(0)\}$, $\eta(0) = \tau(0)$, and $\eta|_Y = \text{id}$. Casework shows that $\eta$ thus defined is a diagram automorphism, so $(Y, \eta)$ is an affine Satake diagram.

4. **Spectral K-matrices**

In this section, we present the first main result of the paper. Let $(X, \tau) \in GSAT(\hat{A})$ with pseudo-involution $\theta$, $(\gamma, \sigma) \in \Gamma \times \Sigma$, and $U_q \mathfrak{t} \subset U_q \hat{\mathfrak{g}}$ the corresponding QSP subalgebra. We prove that the universal K-matrices constructed in [AV20] and Corollary 3.6.1 specialize under mild assumptions to spectral operators on finite-dimensional $U_q L \mathfrak{g}$-representations.

4.1. **$\tau$-minimal grading shifts.** We shall need to replace the homogeneous grading shift defined in 2.6, most commonly used in the context of quantum loop algebras,
with a distinguished $\tau$-invariant grading shift. Note that for any group homomorphism $s : \hat{Q} \to \mathbb{Z}$, there is a corresponding grading shift $\Sigma_s^\tau : U_q L\mathfrak{g} \to U_q L\mathfrak{g}[z, z^{-1}]$ defined by

$$
\Sigma_s^\tau(E_i) = z^{s(\alpha_i)}E_i, \quad \Sigma_s^\tau(F_i) = z^{-s(\alpha_i)}F_i, \quad \Sigma_s^\tau(K_h) = K_h
$$

for $i \in \hat{I}$, $h \in \hat{Q}^\vee$. Then, $\Sigma_s^\tau$ is $\tau$-invariant if $s \circ \tau = s$, i.e., $\Sigma_s^\tau \circ \tau = \tau \circ \Sigma_s^\tau$. Note that $s$ is $\tau$-invariant if and only if, as a function on the set of affine simple roots $\{\alpha_i\}_{i \in \hat{I}}$, $s$ is the characteristic function of a union of $\tau$-orbits. In particular, the principal grading shift $\Sigma_s^\tau$ defined by $s(\alpha_i) = 1$ ($i \in \hat{I}$) is always $\tau$-invariant. The $\tau$-minimal grading shift $\Sigma_s^{\tau_{\text{min}}}$ corresponds instead to the characteristic function $s_\tau$ of the $\tau$-orbit of the affine node 0, i.e.,

$$
s_\tau(\alpha_i) = \begin{cases} 1 & \text{if } i \in \{0, \tau(0)\}, \\ 0 & \text{otherwise} \end{cases}
$$

Then, $\Sigma_s^{\tau_{\text{min}}} = \Sigma_s^{\text{hom}}$ if and only if $\tau(0) = 0$. Note that the analogue of Theorem 2.6.1 holds if we replace $R(z)$ by $\text{id} \otimes \Sigma_s^{\tau_{\text{hom}}}(R)$ and $\Delta_z$ by $\text{id} \otimes \Sigma_s^{\tau_{\text{min}}} \circ \Delta$.

From now on we shall use the $\tau$-minimal grading shift $\Sigma_s^{\tau_{\text{min}}}$ and drop the upper index $s_\tau$ unless needed. In particular, we shall denote by $\pi_{V,z}$ the action on $V$ shifted by $\Sigma_s^{\tau_{\text{min}}}$.

4.2. **Spectral K-matrices.** Let $\mathcal{G}_{\theta,\gamma} \subset \mathcal{G}$ be the subset of gauge transformations $\mathfrak{g} \in \mathcal{G}$ of the form $\mathfrak{g} := S^{-1}_Y S^\lambda \cdot \beta$, where

1. $Y \subset \hat{I}$ is any proper subdiagram such that $\chi_i = 0$ for any $i \in Y$;
2. $\beta : \hat{P} \to \mathbb{F}^\times$ is any map such that $\beta(\delta) = \gamma(\delta)$.

About the definition of the set $\mathcal{G}_{\theta,\gamma}$ see also Remark 5.5(2).

We refer to a twisting operator of the form $\psi = \text{Ad}(\mathfrak{g}) \circ \theta_q^{-1}$ with $\mathfrak{g} \in \mathcal{G}_{\theta,\gamma}$, as a QSP-admissible twisting operator. We shall prove the following spectral analogue of Theorem 3.5.1 and Corollary 3.6.1.

**Theorem 4.2.1.** The quantum loop algebra $U_q L\mathfrak{g}$ has a $\mathcal{G}_{\theta,\gamma}$-family of universal spectral K-matrices relative to the QSP subalgebra $U_q \mathfrak{k}$. More precisely, for any $\mathfrak{g} \in \mathcal{G}_{\theta,\gamma}$, set

$$
\psi := \text{Ad}(\mathfrak{g}) \circ \theta_q^{-1} \quad \text{and} \quad R_\psi := (S_\psi \otimes S_\psi)^{-1} \cdot \Delta(S_\psi)
$$

where $S_\psi := \mathfrak{g} \cdot S_\theta$. There exists a canonical Laurent series $K_\psi(z) \in (U_q L\mathfrak{g})^{\text{fd}}((z))$ such that $\Sigma_\mathfrak{g}(K_\psi(z)) = K_\psi(az)$ ($a \in \mathbb{F}^\times$) and the following properties hold.

1. For any $b \in U_q \mathfrak{k}$,

$$
K_\psi(z) \cdot \Sigma_z(b) = \psi(\Sigma_{1/z}(b)) \cdot K_\psi(z). \quad (4.1)
$$
(2) Set \( R(z)^\psi := \psi \otimes \text{id}(R(z)) \). Then,
\[
\Delta_{w/z}(K_\psi(z)) = R_{\psi}^{-1} \cdot 1 \otimes K_\psi(w) \cdot R(zw)^\psi \cdot K_\psi(z) \otimes 1.
\]

Moreover, \( K_\psi(z) \) is a solution of the generalized reflection equation
\[
R(w/z)^{\psi\psi}_{21} \cdot 1 \otimes K_\psi(w) \cdot R(zw)^\psi \cdot K_\psi(z) \otimes 1 =
K_\psi(z) \otimes 1 \cdot R(zw)^{\psi\psi}_{21} \cdot 1 \otimes K_\psi(w) \cdot R(w/z),
\]
where \( R(z)^{\psi\psi}_{21} := \psi \otimes \text{id}(R(z))_{21} \).

Remark 4.2.2.

(1) The identities (4.1) and (4.2) hold in \( (U_qLg)^{\otimes 2}_{\text{fd}}((w/z, z)) \), where \( \text{Frac}((w/z, z)) := \text{Frac}(\mathbb{F}[w/z, z]) \). Following [Che84, Eq. (10)], it may be convenient to use an adapted set of “simple root” coordinates, given by \( u = w/z \) and \( v = z \). Then, (4.1) and (4.2) read
\[
\Delta_u(K_\psi(v)) = R_{\psi}^{-1} \cdot 1 \otimes K_\psi(uv) \cdot R(uv^2)^\psi \cdot K_\psi(v) \otimes 1,
\]
and
\[
R(u)^{\psi\psi}_{21} \cdot 1 \otimes K_\psi(v) \cdot R(uv^2)^\psi \cdot K_\psi(uv) \otimes 1 =
K_\psi(uv) \otimes 1 \cdot R(uv^2)^{\psi\psi}_{21} \cdot 1 \otimes K_\psi(v) \cdot R(u),
\]
in \( (U_qLg)^{\otimes 2}_{\text{fd}}((u, v)) \).

(2) Whenever \( g \) is shift-invariant, i.e., \( \Sigma_z(g) = g \), the K-matrix \( K_\psi(z) \) is a formal series in \( (U_qLg)^{\text{fd}}[[z]] \).

(3) The QSP subalgebra of \( U_qLsl_2 \) for which \( X = \emptyset \) and \( \tau \) is the nontrivial diagram automorphism is also known as the augmented \( q \)-Onsager algebra. In [BT18, Sec. 4.1.2] so-called generic K-operators in a completion of \( \langle K_1 \rangle \otimes \mathbb{F}(z) \) are considered. It would be interesting to relate these to the spectral universal K-matrix \( K_{g_q^{-1}}(z) \).

\( \nabla \)

In analogy with the case of the R-matrix, the spectral K-matrix is obtained by applying the shift operator to the universal K-matrix \( K_\psi \) from Theorem 3.5.1, i.e., \( K_\psi(z) := \Sigma_z(K_\psi) \). The identities (4.1), (4.2), and (4.3) are then recovered from their analogues (3.4), (3.5), and (3.6), respectively, by applying the shift operator \( \Sigma_z \otimes \Sigma_w \). Clearly, since the operator \( K_\psi(z) \) is valued in \( (U_qLg)^{\text{fd}} \), the statements above are to be interpreted as operators on finite-dimensional representations in \( \text{Rep}^{\text{fd}}(U_qLg) \) and it is therefore necessary to prove that \( K_\psi(z) \) gives rise to a well defined element \( K_{\psi,V}(z) \in \text{End}(V)((z)) \) for any \( V \in \text{Rep}^{\text{fd}}(U_qLg) \). The proof is carried out in Sections 4.3-4.4.
4.3. **Descent to finite-dimensional representations.** The first step in the proof of Theorem 4.2.1 amounts to proving that, for any $V \in \text{Rep}^{fd}(U_q\mathfrak{g})$ with representation $\pi_V : U_q\mathfrak{g} \to \text{End}(V)$, we obtain a well-defined operator

$$K_{\psi,V}(z) := \pi_{V,z}(K_\psi) = \pi_V \circ \Sigma_z(K_\psi) \in \text{End}(V)((z))$$

More precisely, we shall prove that each of the operators involved in the definition of the universal K-matrix (3.3) and in the coproduct identity (4.2) descends to one on any finite-dimensional representation over $U_q\mathfrak{g}$. We shall need the following

**Proposition 4.3.1.** The following operators in $(U_q\mathfrak{g})^{O^+,\text{int}}$ descend to $(U_q\mathfrak{g})^{fd}$:

1. the operator $\xi_\theta$, defined on any weight vector $v$ of weight $\lambda$ by

$$\xi_\theta(v) = q^{\langle \theta(\lambda),\lambda \rangle/2 + \langle \lambda, \rho_X \rangle} \cdot v,$$

where $\rho_X$ is the half-sum of the positive roots in $\Delta_X$ (cf. 3.3);

2. for any $i \in \hat{I}$, the braid group operator $S_i$;

3. for any map $\psi : \hat{P} \to \mathbb{R}^\times$ such that $\psi(\delta) = 1$, the diagonal operator defined on any weight vector $v$ of weight $\lambda$ by $\psi(v) := \psi(\lambda) \cdot v$.

Moreover,

4. for any $\Psi \in \bigoplus_{\mu \in \widehat{\mathfrak{Q}_+}} U_q\widehat{\mathfrak{h}}^+\mu$, the shifted operator $\Sigma_z(\Psi)$ descends to $(U_q\mathfrak{g})^{fd}[[z]]$.

**Proof.** (1) In analogy with the operator $q^\hat{\theta}_0$ from 2.5, one checks easily that

$$\xi_\theta = q^{\sum_{i=1}^m \theta(u)^t u_i + m(\theta(c)d + \theta(d)c) - \rho_X},$$

where $m = 1, 2, 3$ if $\mathfrak{g}$ is of type ADE, BCF, or G, respectively. Since $\theta(c) = -c$, the term $m(\theta(c)d + \theta(d)c) = m(\theta(d) - d)c$ acts as 0 and can be ignored. Therefore, as in the case of the operator $q^\hat{\theta}_0$ in 2.6, $\xi_\theta$ descends to an operator in $(U_q\mathfrak{g})^{fd}$. Note that $\xi_\theta(\delta) = 1$.

(2) By restriction, $V$ is a finite-dimensional representation of $U_q\widehat{\mathfrak{h}}_{(i)}$ and therefore integrable. In particular, the action of $S_i$ on $V$ is well-defined.

(3) By definition, a type 1 representation $V$ admits a weight decomposition over the quotient lattice $\hat{P}_\delta$, i.e., $V = \bigoplus_{\lambda \in \hat{P}_\delta} V_\lambda$. Therefore, the operator $\psi$ acts on $V$ if and only if it factors through $\hat{P}_\delta$, i.e., if $\psi(\delta) = 1$.

(4) Set $\Psi = \sum_{\mu \in \widehat{\mathfrak{Q}_+}} \Psi_\mu$, with $\Psi_\mu \in U_q\widehat{\mathfrak{h}}^+\mu$. Let $p : \hat{Q}_+ \to \mathbb{Z}_{\geq 0}$ be the evaluation map given by $p(\lambda) := \lambda(d_\Sigma)$, so that $\Sigma_z(\Psi) = \sum_{n \geq 0} \Psi(n) z^n$ where $\Psi(n) := \sum_{\mu \in \mathfrak{p}^{-1}(n)} \Psi_\mu$. Fix $n \geq 0$. We shall prove that $\Psi(n)$ is a well-defined operator on $V$. Note that, in the case of the principal grading shift (cf. 4.1), $p^{-1}(n)$ is a finite set and the result is
clear. Assume without loss of generality that \( \chi_0 \neq 0 \). Any \( \mu \in p^{-1}(n) \) has the form 
\[ \mu = m\alpha_0 + \lambda, \]
where \( m \leq n \) and \( \lambda \in \mathbb{Q}_+ \). However, the action of \( U_q^n \) is locally finite on \( V \) and the number of occurrences of \( E_0 \) is bounded by \( n \). Therefore, the action of \( \Psi_\mu \) on \( V \) is non-zero only for finitely many \( \mu \in p^{-1}(n) \) and \( \Psi(n) \) is well-defined on \( V \). The same argument applies if 0 is replaced by any other node \( i \in \hat{T} \) such that \( \chi_i \neq 0 \). \( \square \)

By Theorem 3.5.1 and (3.3), we have
\[ K_\psi = g \cdot \gamma^{-1} \cdot \Upsilon_\theta \in (U_q \mathcal{G})^{\gamma, \text{int}} \]
where we recall that \( g \in \mathcal{G}_{\theta, \gamma} \) is a gauge, \( \gamma \) is the parameter operator, and \( \Upsilon_\theta \) is the quasi-K-matrix. Note that, by definition of \( G \), \( g \) is shift-invariant. These assumptions become crucial in the next step.

**Remark 4.3.2.** We did not use yet the fact that the grading shift is \( \tau \)-invariant and \( g \cdot S_\theta \) is shift-invariant. These assumptions become crucial in the next step.

### 4.4 Spectral K-matrices on finite-dimensional representations.

We complete the proof of Theorem 4.2.1 by showing that the relations (4.1), (4.2), and (4.3) hold. Note that, since the grading shift is \( \tau \)-invariant and \( g \in \mathcal{G}_{\theta, \gamma} \) is shift-invariant, we have \( \tau \circ \Sigma_z = \Sigma_z \circ \tau \) and \( \text{Ad}(g \cdot S_\theta) \circ \Sigma_z = \Sigma_z \circ \text{Ad}(g \cdot S_\theta) \). Finally, since \( \omega \circ \Sigma_z = \Sigma_{1/\omega} \circ \omega \) and \( \psi = \text{Ad}(g \cdot S_\theta) \circ \omega \circ \tau \), we have
\[ \psi \circ \Sigma_z = \Sigma_{1/\omega} \circ \psi. \]

Let \( V \in \text{Rep}^{\text{fd}}(U_q \mathcal{L}^G) \). The action on the shifted representation \( \psi^*(V)(1/z) \) is, by definition, given by
\[ \pi_{\psi^*(V),1/z}(x) = \pi_{\psi^*(V)}(\Sigma_{1/z}(x)) = \pi_V(\psi \circ \Sigma_{1/z}(x)). \]

Since \( \pi_V(\psi \circ \Sigma_{1/z}(x)) = \pi_V(\Sigma_z \circ \psi(x)) \), one has \( \psi^*(V)(1/z) = \psi^*(V)(z) \). Therefore, the evaluation of the intertwining identity (3.4) on \( V(z) \) through \( \pi_{V,z} \) yields
\[ K_{\psi^*(V),1/z}(b) = \pi_{\psi^*(V),1/z}(b)K_{\psi^*,V}(z). \]

It follows that \( K_{\psi^*,V}(z) \) is a \( U_q \mathcal{G} \)-intertwiner \( V(z) \to \psi^*(V)(1/z) \), which is equivalent to (4.1).

Set \( K_{\psi^*,V/W}(z,w) := \pi_{V,z} \otimes \pi_{W,w}(\Delta(K_\psi)) \). Then, the evaluation of the coproduct identity (3.5) on \( V(z) \otimes W(w) \) through \( \pi_{V,z} \otimes \pi_{W,w} \) yields
\[ K_{\psi^*,V/W}(z,w) = R_{\psi^*,V/W}^{-1} \cdot \text{id} \otimes K_{\psi^*,V/W}(w) \cdot R_{\psi^*(V)/W}(zw) \cdot K_{\psi^*,V}(z) \otimes \text{id}, \]

By (4.4), this is equivalent to (4.2). Finally, the evaluation of the generalized twisted reflection equation (3.6) on \( V(z) \otimes W(w) \) through \( \pi_{V,z} \otimes \pi_{W,w} \) yields
\[ R_{\psi^*(V)}(w/z)_{21} \cdot \text{id} \otimes K_{\psi^*,W}(w) \cdot R_{\psi^*(V)/W}(zw) \cdot K_{\psi^*,V}(z) \otimes \text{id} = \]
In this section, we prove that, in the case of irreducible representations, the spectral K-matrix constructed in Theorem 4.2.1 gives rise to rational solutions of the generalized reflection equation.

5. Rational K-matrices

5.1. Rational QSP intertwiners. Theorem 4.2.1 leads to the existence of a rational QSP intertwiner, as follows.

Lemma 5.1.1. Let $V \in \text{Rep}^{fd}(U_q L\mathfrak{g})$. There exists a QSP intertwiner

$$K_{\psi,V}(z) : V(z) \rightarrow \psi^*(V)(1/z)$$

in $\text{End}(V)(z)$.

Proof. By Theorems 4.2.1 and 4.4, for any $V \in \text{Rep}^{fd}(U_q L\mathfrak{g})$, the element $K_{\psi}(z)$ provides an intertwiner

$$K_{\psi,V}(z) : V((z)) \rightarrow \psi^*(V)((1/z)) .$$

This is equivalent to the existence of a solution $K_{\psi,V}(z) \in \text{End}(V)((z))$ of a finite system of linear equations, namely

$$K_{\psi,V}(z) \cdot \pi_{V,z}(b) = \pi_{\psi^*(V),1/z}(b) \cdot K_{\psi,V}(z) , \tag{5.1}$$

where $b \in U_q \mathfrak{k}$ runs over any finite set of generators of $U_q \mathfrak{k}$. Since the system is consistent and defined over $\mathbb{F}(z)$, it can be solved in $\text{End}(V)(z)$. □

5.2. Rational K-matrices. We shall now prove that, for irreducible representations, the spectral K-matrix $K_{\psi,V}(z)$ is proportional to $K_{\psi,V}(z) \in \text{End}(V)(z)$, i.e., it is rational up to a non-zero scalar in $\mathbb{F}(z)$.

Theorem 5.2.1. Let $V,W \in \text{Rep}^{fd}(U_q L\mathfrak{g})$ be irreducible representations.

(1) The space of QSP intertwiners $V((z)) \rightarrow \psi^*(V)((1/z))$ is one-dimensional.

(2) There exists a formal Laurent series $g_V(z) \in \mathbb{F}(z)$ and a non-vanishing rational operator $K_{\psi,V}(z) \in \text{End}(V)(z)$ such that

$$K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z) .$$

\[\text{Since the QSP subalgebra } U_q \mathfrak{k} \text{ is fixed throughout this Section, we shall use the expression QSP intertwiner instead of } U_q \mathfrak{k}\text{-intertwiner.}\]

\[\text{The authors are grateful to V. Toledano Laredo for pointing out this argument.}\]
(3) The operators $K_{\psi,V}(z)$ and $K_{\psi,W}(w)$ satisfy the generalized reflection equation in $\text{End}(V \otimes W)(z,w)$

$$R_{\psi^*(W)\psi^*(V)}(z) \cdot 1 \otimes K_{\psi,W}(w) \cdot R_{\psi^*(V)\psi^*(W)}(zw) \cdot K_{\psi,V}(z) \otimes 1 = K_{\psi,V}(z) \otimes 1 \cdot R_{\psi^*(W)\psi^*(V)}(zw) \cdot 1 \otimes K_{\psi,W}(w) \cdot R_{WV}(z,w),$$

where $R_{VW}(z)$ is the rational $R$-matrix (see 2.7), and

$$R_{WV}(z)_{21} := (12) \circ R_{WV}(z) \circ (12).$$

From Remark 4.2.2 (2), we get the following

**Corollary 5.2.2.** Assume $\Sigma(z) = g$. Let $V \in \text{Rep}^{fd}(U_qLg)$ be irreducible. The universal $K$-matrix descends to a formal series operator $K_{\psi,V}(z) \in \text{End}(V)[[z]]$, endowed with a factorization

$$K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z)$$

where $g_V(z) \in \mathbb{F}[[z]]$ and the entries of $K_{\psi,V}(z) \in \text{End}(V)(z)$ are rational functions over $\mathbb{F}$ regular at $z = 0$.

The proof of Theorem 5.2.1 is carried out in Sections 5.3-5.4

5.3. **Proof of Theorem 5.2.1: part (1).** We say that an irreducible $V \in \text{Rep}^{fd}U_qLg$ is **generically QSP irreducible** if $V((z))$ is irreducible as a representation over $U_q\mathfrak{t}((z))$. Such a condition is the natural counterpart of the generic irreducibility of the tensor product $V \otimes W((z))$ in Theorem 2.7.1, which holds for any pair of irreducible representations $V,W$.

The condition of generic irreducibility clearly depends upon the choice of a grading shift. In Corollary 6.2.2, we prove that every irreducible $U_qLg$-representation is generically QSP irreducible with respect to the principal grading shift. It is then clear that, in this case, (1) follows by Schur’s lemma. Namely, let $K_1, K_2 \in \text{End}(V)((z))$ be two solutions of (5.1). Let

$$\mathbb{F}\{z\} := \bigcup_{n>0} \mathbb{F}((z^{1/n}))$$

be the field of Puiseux series over $\mathbb{F}$, i.e. the algebraic closure of $\mathbb{F}((z))$, and set $V\{z\} := V \otimes \mathbb{F}\{z\}$. The composition $K_2^{-1}K_1 : V\{z\} \rightarrow V\{z\}$ is an intertwiner. Therefore, by QSP irreducibility and Schur’s lemma, there exists $g(z) \in \mathbb{F}\{z\}$ such that $K_1 = g(z)K_2$. Clearly, since both operators are defined over $\mathbb{F}((z))$, one has $g(z) \in \mathbb{F}((z))$.

The irreducibility result does not immediately carry over to the case of the $\tau$-minimal grading shift. Instead, we generalize (1) by proving that the above result on the one-dimensionality of the space of QSP intertwiners (5.1) for the principal grading implies
the one-dimensionality of the space of QSP intertwiners for the τ-minimal grading shift.

Let \( \text{pr} : \hat{Q} \to \mathbb{Z} \) be the group homomorphism defined by \( \alpha_i \mapsto 1 \) for all \( i \in \hat{I} \) so that, in the notation of Section 4.1, \( \Sigma^\text{pr}_z \) denotes the principal grading shift. Consider any extension of \( \text{pr}|_Q \) and \( s_\tau|_Q \) to group homomorphisms from \( P \to \hat{Q} \), also denoted \( \text{pr} \) and \( s_\tau \), respectively. Note that the extended \( \text{pr} \) and \( s_\tau \) will in fact take images in \( \frac{1}{m}\mathbb{Z} \) for some positive integer \( m \).

Note that in the following we shall regard \( V \) as a \( P \)-graded vector space. Let \( M_V(z) \) be the linear operator on \( V(z^{1/m}) \subset V(z) \) given by \( M_V(z)v_\lambda = z^{\text{pr}(\lambda)}v_\lambda \) for any weight vector \( v_\lambda \) (\( \lambda \in P \)). Let \( h \) be the Coxeter number of \( g \) and set \( h_\tau \in \frac{1}{2}\mathbb{Z} \) by

\[
h_\tau := \begin{cases} h & \text{if } \tau(0) = 0, \\ \frac{h+1}{2} & \text{if } \tau(0) \neq 0. \end{cases}
\]

Since \( h = \text{ht}(\delta) \), it follows that \( M_V(z) \) intertwines between the principal shifted action and the \( \tau \)-minimal shifted action, i.e., \( \text{Ad}(M_V(z)) \circ \pi^*_\tau_{z^{h_\tau},V} = \pi^*_\text{pr}_{z,V} \). Finally, we observe that the linear map

\[
K^\text{pr}_{\psi,V}(z) \mapsto M_{\psi^*(V)}(z^{-1})^{-1} \cdot K^\text{pr}_{\psi,V} \cdot M_V(z) =: K^*_\psi,V(z)
\]

defines a bijection between the spaces of principal and \( \tau \)-minimal QSP intertwiners. Since we have already shown that the space of solutions \( K_{\zeta,V}(z) : V(z) \to \psi^*(V)(1/z) \) of the linear system

\[
K_{\psi,V}(z) \cdot \pi^*_{V,z}(b) = \pi^*_{\psi^*(V),1/z}(b) \cdot K_{\psi,V}(z) \quad \text{for all } b \in U_q F
\]
is one-dimensional, the result follows.

5.4. **Proof of Theorem 5.2.1: part (2) and (3).** Part (2) follows immediately from part (1) and Lemma 5.1.1. Namely, let \( K_\psi(z) \in \text{End}(V)(z) \) be any rational solution of (5.1), whose existence is guaranteed by Lemma 5.1.1. By (1), there exists \( g_V(z) \in \mathbb{F}(z) \) such that the identity \( K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z) \) holds.

It remains to prove part (3). Let \( R_{VW}(z) = f_{VW}(z)^{-1}R_{VW}(z) \) be the rational R-matrix (cf. Thm. 2.7.1). Then, (2) reduces to prove that

\[
f_{\psi^*(W)\psi^*(V)}(\frac{w}{z}) \cdot g_V(z) \cdot f_{\psi^*(V)}(zw) \cdot g_W(w) = g_V(z) \cdot f_{\psi^*(W)}(zw) \cdot g_W(w) \cdot f_{VW}(\frac{w}{z})
\]

Note that, by [AV20, Thm. 8.10], we have \( R^{\psi}_{\psi} = R_{\psi} R R^{-1}_\psi \). Therefore,

\[
R_{\psi^*(W)\psi^*(V)}(\frac{w}{z})_{21} = R_{\psi} \cdot R_{VW}(\frac{w}{z}) \cdot R^{-1}_\psi = f_{VW}(\frac{w}{z}) R_{\psi} \psi^*(W) \psi^*(V) \psi^*(V) \psi^*(V)_{21}.
\]
i.e., \( f_{\psi(W)\psi(V)}(\tilde{w}) = f_{VW}(\tilde{w}) \). Similarly, replacing \( V \) with \( \psi^*V \), we get
\[
f_{\psi^*(V)W}(zw) = f_{\psi^*(W)\psi^2V}(zw) = f_{\psi^*(W)V}(zw).
\]
Note that the second equality holds since \( \psi^2 \) is a weight zero operator. Therefore it is constant in \( z \) and does not impact the normalization function \( f_{\psi^*(W)V}(zw) \). The result follows.

5.5. Remarks.

(1) Our proof of the generalized reflection equation (5.2) bears some similarity in spirit with the proof of the (untwisted) reflection equation in [DG02, Sec. 3] (also cf. [Del02, Sec. 5.2] and [DM03, Sec. 2.2]) and [RV16, Prop. 6.6] from an assumed solution of the intertwining condition (5.1) in the special case \( \psi^*(V) = V \); note that the existence of such an intertwiner follows from generic QSP irreducibility of \( V \). Both are boundary versions of the approach given in [Jim86] for constructing solutions of the parameter-dependent Yang-Baxter equation. It also relies on the generic QSP irreducibility of \( V \otimes W \) (only the case where \( V \) and \( W \) are evaluation representations is considered). Furthermore, in these approaches an additional property\(^{14}\) of the R-matrix is needed to show that the scalar resulting from the application of Schur’s lemma equals 1.

On the other hand, the universal K-matrix approach has a wider applicability: it requires only the generic QSP irreducibility of the two representations \( V \) and \( W \), not of \( V \otimes W \), and guarantees the existence of a matrix-valued rational function which is at the same time a (twisted) \( U_q\mathfrak{g} \)-intertwiner and a solution of a (generalized) reflection equation.

(2) The definition of the set \( \mathcal{G}_{\theta,\gamma} \) may appear at first quite restrictive. However, for the proofs of Theorems 4.2.1 and 5.2.1 to work, a gauge transformation \( g \in \mathcal{G} \) is required to satisfy the following three conditions:

(G1) \( g \cdot \mathcal{S}_{\theta}^{-1} \) is shift-invariant;

(G2) \( g \cdot \mathcal{S}_{\gamma}^{-1} \) descends to an operator on shifted representations;

(G3) the automorphism \( \psi^2 \), where \( \psi = \text{Ad}(g) \circ \theta_q^{-1} \), is weight preserving.

By observing the action of \( g \) on the Cartan subalgebra at \( q = 1 \), one checks that the conditions (G1), (G2), (G3) essentially determine the set \( \mathcal{G}_{\theta,\gamma} \).

5.6. Unitary K-matrices. Let \( V,W \in \text{Rep}^{fd}(U_qL\mathfrak{g}) \) be irreducible representations. By Theorem 2.7.1, the rational R-matrix \( R_{VW}(z) \) satisfies the unitarity condition \( R_{VW}(z)^{-1} = (12) \circ R_{WV}(z^{-1}) \circ (12) \). The analogue result for rational K-matrices

\(^{14}\)In [DG02], this is the property that \( R_{VW}(z)|_{q=1} \) is proportional to the identity. In [RV16] only the case where \( V = W \) is the vector representation is studied and the additional property (known as \textit{regularity}) is that \( R_{VV}(1) \) is proportional to the elementwise flip of tensor factors in \( V \otimes V \).
is less straightforward. Clearly, the R-matrix argument can be used, under suitable assumptions.

**Proposition 5.6.1.** Suppose $V \in \text{Rep}_{\text{fd}}^{\text{id}} U_q L_\mathfrak{g}$ is irreducible and satisfies the following properties.

(a) $V$ is $\psi$-involutive, i.e., $(\psi^2)^*(V) = V$.

(b) There exist non-zero vectors $v, v' \in V$ and functions $f(z), f'(z) \in \mathbb{F}((z))$ such that $K_{\psi,V}(z)v = f(z)v'$ and $K_{\psi,\psi^*(V)}(z)v' = f'(z)v$.

Then, the following holds.

(1) There exists a choice of rational K-matrices for $V$ and $\psi^*(V)$ such that

$$K_{\psi,V}(z)^{-1} = K_{\psi,\psi^*(V)}(z^{-1})$$

(2) Let $\zeta \in \mathbb{F}^\times$. If $V(\zeta)$ is QSP irreducible, then $K_{\psi,V}(\zeta)$ is well-defined and invertible.

**Proof.** (1) Set $K_{\psi,V}(z) = f(z)^{-1}K_{\psi,V}(z)$ and $K_{\psi,\psi^*(V)}(z) = f'(z)^{-1}K_{\psi,\psi^*(V)}(z)$. It is enough to observe that the composition

$$V(z) \xrightarrow{K_{\psi,V}(z)} \psi^*(V)(z^{-1}) \xrightarrow{K_{\psi,\psi^*(V)}(z^{-1})} (\psi^2)^*(V)(z) = V(z)$$

is a $U_q \mathfrak{sl}$-intertwiner, which is the identity on $v$ and therefore

$$K_{\psi,\psi^*(V)}(z^{-1})K_{\psi,V}(z) = \text{id}_{V(z)}.$$

(2) Let $m \geq 0$ be the order of the pole of $K_{\psi,V}(z)$ at $z = \zeta$. Then, the map

$$\lim_{z \to \zeta} (z - \zeta)^m K_{\psi,V}(z)$$

is a non-zero intertwiner $V(\zeta) \to \psi^*(V)(\zeta^{-1})$ and therefore invertible. However, by unitarity, $K_{\psi,\psi^*(V)}(z^{-1})$ has a zero of order $m$. Therefore, $m = 0$. □

**Remarks 5.6.2.**

(1) The condition (a) above is trivially satisfied in the semi-standard case 3.6 (2), where the twisting operator is given by the involution $\omega \circ \tau$. In this case the condition $(\psi^2)^*(V) = V$ hold for any $V \in \text{Rep}_{\text{id}}^{\text{id}} (U_q L_\mathfrak{g})$. In general, however, some further adjustment is required. For instance, we prove in [AV20, Sec. 9] in the case of $U_q \widehat{\mathfrak{sl}_2}$ the twisting operator can be corrected, so as to become an involution on a given irreducible evaluation representation.

(2) The unitary K-matrices induced by the condition (b) are certainly not canonical. We expect however that a natural normalization does exist. In particular, we show in 5.7 that this problem can be partially solved for restricted rank one QSP subalgebras.
5.7. Abelian K-matrices. Let $V \in \text{Rep}^{fd}(U_q Lg)$ be an irreducible representation. By [CP95, Cor. 12.2.5], $V$ is generated by an $\ell$-highest weight vector (or pseudo-highest-weight vector), i.e., a weight vector $v_{\lambda}$ of weight $\lambda$, which is killed by $F_0$ and $E_i$, with $i \in I$, and is also a simultaneous eigenvector for all imaginary root vectors.

**Proposition 5.7.1.** Assume $\Theta$ has restricted rank one and $\Sigma_z(g) = g$. Let $V \in \text{Rep}^{fd}(U_q Lg)$ be a generically QSP irreducible representation with $\ell$-highest weight vector $v_{\lambda} \in V$. There is a unique $g_V(z) \in \mathbb{F}[z]^{\times}$ such that $K_{\psi,V}(z) = g_V(z) \cdot K_{\psi,V}(z)$ and $K_{\psi,V}(z) \in \text{End}(V)(z)$ is a rational operator regular at $z = 0$ and such that

$$K_{\psi,V}(z)v_{\lambda} = g \cdot v_{\lambda}.$$ 

**Proof.** Since $\Theta$ is of restricted rank 1, the quasi-K-matrix defined in Theorem 3.5.1 satisfies $\Upsilon_\Theta = 1 + \sum_{m \in \mathbb{Z}_{\geq 0}} \Upsilon_{\Theta,m\delta} \in U_q^+\hat{\mathfrak{u}}_m$. Note that, since $\Sigma_z(g) = g$, the quasi-K-matrix $\Upsilon_\Theta$ descends to the only spectral component of $K_{\psi,V}(z)$. In particular, there exists $\tilde{g}_V(z) \in \mathbb{F}[z]^{\times}$ such that

$$\Upsilon_{\Theta,V}(z)v_{\lambda} = \tilde{g}_V(z)v_{\lambda},$$

where $\Upsilon_{\Theta,V}(z) := \pi_{\psi,z}(\Upsilon_\Theta)$. Set $g_V(z) := \gamma^{-1}(\lambda)\tilde{g}_V(z)$. It follows that $K_{\psi}(z) := g_V(z)^{-1} \cdot K_{\psi,V}(z)$ is the unique intertwiner $V((z)) \to \psi^*(V)((z^{-1}))$ such that

$$K_{\psi,V}(z)v_{\lambda} = g \cdot v_{\lambda}.$$ 

Therefore, $K_{\psi,V}(z) \in \text{End}(V)(z)$ is a rational operator. □

**Remark 5.7.2.** Clearly, a similar result can be achieved through normalization with respect to any vector which is a simultaneous eigenvector for the imaginary root vectors. In particular, one can consider any extremal vector (cf. [Kas02, Cha02]). ▽

By combination with Proposition 5.6.1, we get the following

**Corollary 5.7.3.** Suppose that $\Theta$ has restricted rank one, $\Sigma_z(g) = g$. Furthermore, let $V \in \text{Rep}^{fd}(U_q Lg)$ be such that $(\psi^*)^*(V) = V$. Let $v_{\lambda} \in V$ be a normalized $\ell$-highest weight vector such that $v_{\lambda} = v_{\lambda}$. Consider the rational K-matrices normalized as in Proposition 5.7.1 by the conditions

$$K_{\psi,V}(z)v_{\lambda} = g \cdot v_{\lambda} \quad \text{and} \quad K_{\psi,\psi^*(V)}(z)g \cdot v_{\lambda} = v_{\lambda}.$$ 

Then, the unitarity condition $K_{\psi,V}(z)^{-1} = K_{\psi,\psi^*(V)}(z^{-1})$ holds.

**Proof.** It suffices to note that, owing to the definition of $g \in \mathcal{G}_{\Theta,\gamma}$, the vector $g \cdot v_{\lambda} \in \psi^*(V)$ is again a simultaneous eigenvector for the imaginary root vectors. Then, the result follows by Propositions 5.6.1 and 5.7.1. □
6. Generic restricted irreducibility

In this stand-alone section, we consider various restricted irreducibility results involving the untwisted quantum loop algebra $U_qLg$. More precisely, fix a proper subalgebra $A \subset U_qLg$. We are interested in the irreducible representations $V \in \text{Rep}^{fd}(U_qLg)$ whose shifted representation $V(z)$ remains irreducible under restriction to $A(z) := A \otimes \mathbb{F}(z)$. We prove that for a class of subalgebras, which we refer to as modified nilpotent, every irreducible representation over $U_qLg$ remains generically irreducible, with respect to the principal grading shift, under restriction to $A$ (cf. Theorem 6.2.1). As a corollary, we obtain generic QSP irreducibility (cf. Corollary 6.2.2).

6.1. Notations. Let $U_qLg$ be an untwisted quantum loop algebra. We use the shorthand notations $\hat{U}^+$, $\hat{U}^0$ and $\hat{U}^-$ to denote the subalgebras of $U_qLg$ generated by all $E_i$, all $K_i^{\pm 1}$, and all $F_i$, respectively. We will also consider the subalgebra $\hat{U}^{\geq 0} := \hat{U}^+ \cdot \hat{U}^0$. In this section, for any $V \in \text{Rep}^{fd}(U_qLg)$, we denote by $V(z)$ the shifted representation defined by the principal grading shift $\Sigma^p_z$ (cf. Section 4.1). Recall that $\Sigma^p_z$ is the morphism of $U_qLg[z, z^{-1}]$ defined by

$$\Sigma^p_z(K_i^{\pm 1}) = K_i^{\pm 1}, \quad \Sigma^p_z(E_i) = zE_i, \quad \Sigma^p_z(F_i) = z^{-1}F_i \quad \text{for all } i \in \hat{T}.$$  

6.2. Principal irreducibility for modified nilpotent subalgebras. We call a subalgebra $A \subset U_qLg$ a modified nilpotent subalgebra if it is generated by elements of the form $\widetilde{F}_i := F_i + E_i'$ for $i \in \hat{T}$, where $E_i' \in \hat{U}^{\geq 0}$.

Theorem 6.2.1. Let $A$ be a modified nilpotent subalgebra. Let $V \in \text{Rep}^{fd}(U_qLg)$ be irreducible. Then, $V(z)$ remains irreducible under restriction to $A(z)$.

We will give the proof of this theorem in Section 6.4.

Let $\theta$ be a pseudo-involution of $\widehat{g}$ of the second kind with associated generalized Satake diagram $(X, \tau)$. Also let $(\gamma, \sigma) \in \Gamma \times \Sigma$ and consider the case

$$E_i' = \begin{cases} \gamma_i\theta_q(F_i) + \sigma_iK_i^{-1} & \text{if } i \in \hat{T}\backslash X, \\ 0 & \text{if } i \in X. \end{cases}$$  

Since $\theta_q(F_i) \in \hat{U}^+ \cdot K_i^{-1}$ if $i \in \hat{T}\backslash X$, we clearly have $E_i' \in \hat{U}^{\geq 0}$. Hence, $\widetilde{F}_i$ specializes to the generator $B_i$ of $U_q\mathfrak{k}$ defined by (3.2) (recall that we may view $U_q\mathfrak{k}$ as a coideal subalgebra of the quantum loop algebra $U_qLg$). In other words, QSP subalgebras contain modified nilpotent subalgebras of $U_qLg$. Hence Theorem 6.2.1 specializes to the following statement of generic QSP irreducibility.

Corollary 6.2.2. Let $V \in \text{Rep}^{fd}(U_qLg)$ be irreducible. Let $\theta$ be a pseudo-involution of $\widehat{g}$ of the second kind and $(\gamma, \sigma) \in \Gamma \times \Sigma$. Set

$$U_q\mathfrak{k}^- := \langle B_{i, \gamma_i, \sigma_i} \rangle_{i \in \hat{T}} \subset U_q\mathfrak{k}, \quad U_q\mathfrak{k}^-(z) := U_q\mathfrak{k}^- \otimes \mathbb{F}(z).$$  

Then $V(z)$ is irreducible as a representation over $U_q\mathfrak{k}^-(z)$ (and hence over $U_q\mathfrak{k}(z)$).
6.3. Restricted irreducibility of finite-dimensional representations with respect to nilpotent subalgebras. In this subsection we will prove the following stepping stone for Theorem 6.2.1: irreducible finite-dimensional representations of $U_q L\mathfrak{g}$ are also irreducible as $\hat{U}^+$-representations. To our best knowledge, this result is not stated in the literature, although the analogous result for $p U_\epsilon$ is known, see [Bow07, HJ12], and will be the starting point for our proof of the stronger result involving $\hat{U}^+$.

First we recall some basic theory of irreducible finite-dimensional representations of $U_q L\mathfrak{g}$. Let $V$ be an irreducible finite-dimensional representation of $U_q L\mathfrak{g}$. Note that $V$ is also a finite-dimensional representation of the quantum group $U_q \mathfrak{g}$ of finite type (its irreducible summands are necessarily connected by the action of $E_0$ and $F_0$). Hence we obtain a weight decomposition $V = \bigoplus_{\lambda \in P} V_\lambda$ where

$$V_\lambda = \{ v \in V \mid \forall i \in I \ K_i \cdot v = q_i^{\lambda(h_i)} v \}.$$ 

Note that $K_0$ acts on each $V_\lambda$ as multiplication by $q_i^{\lambda(h_0)} = q^{-(i,\vartheta)}$. Consider the unique $\mathbb{Z}$-linear projection $\tau : \hat{Q} \to Q$ such that $\tau_i = \alpha_i (i \in I)$ and $\vartheta = 0$. For all $i \in \hat{I}$ and all $\lambda \in P$, we have

$$E_i \cdot V_\lambda \subseteq V_{\lambda+\tau_i}, \quad F_i \cdot V_\lambda \subseteq V_{\lambda-\tau_i}.$$ 

Finally, the support of $V$ is the finite set

$$\text{Supp}(V) := \{ \lambda \in P \mid V_\lambda \neq \{0\} \}.$$ 

As foreshadowed, we start with the following restricted irreducibility result.

**Proposition 6.3.1** ([HJ12, Prop. 3.5] or [Bow07, Thm. 2.3 (ii) for $\epsilon = (1)^n$]). Every finite-dimensional irreducible representation of $U_q L\mathfrak{g}$ remains irreducible under restriction to $\hat{U}^{\geq 0}$.

Fix an irreducible representation $\pi_V : U_q L\mathfrak{g} \to \text{End}(V)$ where $V$ is finite-dimensional. From Proposition 6.3.1 we deduce some useful technical results, the first of which directly generalizes [CG05, Cor. 2.7].

**Lemma 6.3.2.** Let $\lambda \in \text{Supp}(V)$ and $v_\lambda \in V_\lambda \setminus \{0\}$ be arbitrary. The following statements are satisfied.

1. We have $V = \hat{U}^+ \cdot v_\lambda$.
2. For all $\mu \in \text{Supp}(V)$ there exist $\ell \in \mathbb{Z}_{\geq 0}$ and $i \in \hat{I}^\ell$ with the property that $E_{i_1} \cdots E_{i_\ell} \cdot v_\lambda \in V_\mu \setminus \{0\}$.

**Proof.**
(1) This follows from Proposition 6.3.1, the decomposition $\hat{U}^{\geq 0} = \hat{U}^+ \cdot \hat{U}^0$ and the fact that $\hat{U}^0 \cdot v_\lambda = \mathbb{F}v_\lambda$.

(2) Note part (1) yields the existence of $x \in \hat{U}^+$ such that $x \cdot v_\lambda \in V_\mu \setminus \{0\}$. Each monomial in the $E_i$ ($i \in \hat{I}$) occurring in $x$ must send $v_\lambda$ into a weight space. Hence without loss of generality we may assume that all monomials occurring in $x$ map $v$ to $V_\mu$. Since there must at least be one such monomial which does not annihilate $v_\lambda$, we obtain the result. 

We provide two further definitions. For all $v \in V$ there exists a unique subset $\text{Supp}(v) \subseteq \text{Supp}(V)$ such that

$$v = \sum_{\lambda \in \text{Supp}(v)} v_\lambda,$$

where $v_\lambda \in V_\lambda \setminus \{0\}$.

Note that $|\text{Supp}(v)| = 1$ if and only if $v$ is a weight vector. For all $i \in \hat{I}$ we obtain

$$\text{Supp}(E_i \cdot v) \subseteq \text{Supp}(v) + \alpha_i, \quad \text{Supp}(F_i \cdot v) \subseteq \text{Supp}(v) - \alpha_i,$$

and hence

$$|\text{Supp}(E_i \cdot v)|, |\text{Supp}(F_i \cdot v)| \leq |\text{Supp}(v)|. \quad (6.2)$$

Let $\lambda \in \text{Supp}(V)$ and $v_\lambda \in V_\lambda \setminus \{0\}$. By Lemma 6.3.2 (2), the following definition makes sense:

$$d^+(v_\lambda) := \min\{\ell \in \mathbb{Z}_{\geq 0} \mid \exists i \in \hat{I}^\ell E_{i_1} \cdots E_{i_\ell} \cdot v_\lambda \in V_{\mu_0} \setminus \{0\}\} \in \mathbb{Z}_{\geq 0}.$$ 

More generally, for any $v \in V \setminus \{0\}$ we define

$$d^+(v) := \min\{d^+(v_\lambda) \mid \lambda \in \text{Supp}(v)\} \in \mathbb{Z}_{\geq 0}.$$ 

Note that $d^+(v) = 0$ if and only if $\lambda_0 \in \text{Supp}(v)$. We have the following useful result.

**Lemma 6.3.3.** Let $v \in V \setminus \{0\}$. If $d^+(v) > 0$ then there exists $i \in \hat{I}$ such that $E_i \cdot v \neq 0$ and $d^+(E_i \cdot v) < d^+(v)$.

**Proof.** Set $\ell := d^+(v) \in \mathbb{Z}_{>0}$. The definition of $d^+$ implies that $\lambda_0 \notin \text{Supp}(v)$ and there exists $\lambda \in \text{Supp}(v)$ such that $d^+(v_\lambda) = \ell$. By Lemma 6.3.2 (2) there exists $i \in \hat{I}^\ell$ such that $E_{i_1} \cdots E_{i_\ell} \cdot v_\lambda \in V_{\mu_0} \setminus \{0\}$. Set $i = i_\ell$. Then $d^+(E_i \cdot v_\lambda) = \ell - 1$. The relation (6.1) and the definition of $d^+$ now imply $d^+(E_i \cdot v) < \ell$. 

Now we are ready to state and prove the first main result of this section.

**Theorem 6.3.4.** Every finite-dimensional irreducible representation of $U_qL\mathfrak{g}$ remains irreducible under restriction to $\hat{U}^+$. 

**Proof.** We will show that $\hat{U}^+ \cdot v = V$ for all $v \in V \setminus \{0\}$. Considering Lemma 6.3.2 (1), it remains to show that $\hat{U}^+ \cdot v$ contains a weight vector. We will prove this by induction with respect to $s := |\text{Supp}(v)| \in \mathbb{Z}_{>0}$. This is immediate for $s = 1$. It suffices to prove that the statement is true for a given $s \in \mathbb{Z}_{>1}$, assume that the statement is true for
all $v \in V$ such that $|\text{Supp}(v)| < s$.

Suppose on the contrary that there exists $v \in V$ such that $|\text{Supp}(v)| = s$ and $\hat{U}^+ \cdot v$ does not contain a weight vector; among all those $v$, choose one with a minimal value of $d^+(v)$. We shall deduce a contradiction and hence conclude that such $v$ does not exist. We first derive inequalities using some casework.

(1) Suppose $d^+(v) = 0$. Then $\lambda_0 \in \text{Supp}(v)$. Since $|\text{Supp}(v)| > 1$, we may choose $\mu \in \text{Supp}(v) \setminus \{\lambda_0\}$. By Lemma 6.3.2 (2) there exists $i \in \hat{I}$ with $\ell = d^+(\mu)$ such that $E_{i_1} \cdots E_{i_\ell} \cdot v_\mu \in V_{\lambda_0} \setminus \{0\}$. We must have $\mu + \alpha_{i_1} + \ldots + \alpha_{i_\ell} = \lambda_0$ with $\alpha_{i_1} + \ldots + \alpha_{i_\ell} \neq 0$. From $\text{Supp}(V) \subset \lambda_0 - Q^+$ we obtain $\alpha_{i_1} + \ldots + \alpha_{i_\ell} \in Q^+$. Since $\lambda_0 + \alpha_{i_1} + \ldots + \alpha_{i_\ell} \in \lambda_0 + Q^+$, using $\text{Supp}(V) \subset \lambda_0 - Q^+$ again we obtain $\lambda_0 + \alpha_{i_1} + \ldots + \alpha_{i_\ell} \notin \text{Supp}(V)$. Hence $E_{i_1} \cdots E_{i_\ell} \cdot v_{\lambda_0} = \{0\}$. As a consequence, $|\text{Supp}(E_{i_1} \cdots E_{i_\ell} \cdot v)| < |\text{Supp}(v)|$.

(2) Suppose $d^+(v) > 0$. By Lemma 6.3.3 there exists $i \in \hat{I}$ such that $E_i \cdot v \neq 0$ and $d^+(E_i \cdot v) < d^+(v)$. From the minimality of $v$ and (6.2) we deduce that $|\text{Supp}(E_i \cdot v)| < |\text{Supp}(v)|$.

In either case, we obtain an inequality of the form $|\text{Supp}(E_{i_1} \cdots E_{i_\ell} \cdot v)| < |\text{Supp}(v)|$. But now the inclusion $\hat{U}^+ \cdot (E_{i_1} \cdots E_{i_\ell} \cdot v) \subseteq \hat{U}^+ \cdot v$ furnishes a contradiction with the induction hypothesis. This completes the proof.

By pulling back $\pi_V$ by $\omega$, we immediately obtain the following result, which is more useful for our purposes.

**Corollary 6.3.5.** Every finite-dimensional irreducible representation of $U_q \mathfrak{g}$ remains irreducible under restriction to $\hat{U}^-$. 

### 6.4. Proof of Theorem 6.2.1.

The proof of the main theorem of this section relies on the following basic linear-algebraic result.

**Lemma 6.4.1.** Let $z$ be an indeterminate and $V$ a finite-dimensional $\mathbb{F}$-linear space. Let $\mathcal{M}(z) := \{M_i(z)\}_{i \in \hat{I}}$ be a finite set of endomorphisms of $V(z) := V \otimes \mathbb{F}(z)$ depending polynomially on $z$. We may view $\mathcal{M} := \{M_i(0)\}_{i \in \hat{I}}$ as a set of endomorphisms of $V$. Suppose $V$ has no nontrivial proper $\mathcal{M}$-invariant subspace. Then $V(z)$ has no nontrivial proper $\mathcal{M}(z)$-invariant subspace.

**Proof.** Note that evaluation at 0 defines a linear map $\text{ev}_0$ from $V[z] := V \otimes \mathbb{F}[z] \subset V(z)$ to $V$. Let $S(z)$ be a nontrivial $\mathcal{M}(z)$-invariant subspace of $V(z)$ and abbreviate $S[z] := S(z) \cap V[z]$. As a consequence, for all $s(z) \in S[z]$ and for all $i \in \hat{I}$ we have $M_i(z) \cdot s(z) \in S[z]$.

It follows that $\text{ev}_0(S[z])$ is a nontrivial $\mathcal{M}$-invariant subspace of $V$ and therefore $\text{ev}_0(S[z]) = V$. Extending the scalars to $\mathbb{F}(z)$, we obtain $S(z) = V(z)$, as required. □
We can now complete the proof of Theorem 6.2.1. Since $\tilde{F}_i - F_i \in \hat{U}_{ir}$ for all $i \in \hat{I}$, the action of $z\tilde{F}_i \in U_qLg(z)$ on $V(z)$ is polynomial in $z$. We set $M_i(z)$ equal to this action and note that $M_i(0)$ is equal to the action of $zF_i$, which descends to an action on $V$. Combining Corollary 6.3.5 and Lemma 6.4.1 we deduce that $V(z)$ has no nontrivial proper $M_i$-invariant subspace. Since $z$ is invertible in $F(z)$, we may replace $z\tilde{F}_i$ by $F_i$ and we obtain the desired result.

7. Solutions of the reflection equations

In this section we connect to the existing theory of solutions of the parameter-dependent reflection equation as initiated in the fundamental papers [Che84, Skl88].

7.1. The standard reflection equations. Recall that, by Theorem 2.7.1, for any irreducible $V, W \in \text{Rep}^{\text{fd}} U_q Lg$, there exists a rational $R$-matrix $R_{VW}(z)$. Thus, on the tensor product $V(z) \otimes W(w)$ we consider the (standard) reflection equation

$$R_{WV}(\frac{vw}{w})_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VW}(zw) \cdot K_V(z) \otimes \text{id} =$$

$$= K_V(z) \otimes \text{id} \cdot R_{WV}(zw)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VW}(\frac{vw}{w}) \tag{7.1}$$

for unknown operators $K_V(z)$ and $K_W(w)$. These are typically assumed to be invertible matrix-valued formal Laurent series (or generically invertible matrix-valued meromorphic functions when $q \in \mathbb{C}$). There is a rich literature of work comprehensively cataloguing the solutions of the standard reflection equation, see e.g., [MLS06, RV16, RV18] and references therein.

The interplay between the reflection equation and QSP subalgebras has been deeply studied and explored, see e.g. [Del02, DG02, DM03, BB10, RV16, BT18, VW20]. In this context, it is natural to ask whether the equation (7.1) can be obtained as a special case of the generalized reflection equation (5.2) with respect to a twisting operator $\psi$ which leaves the representations $V$ and $W$ invariant under pullback. In the next section we prove that this is indeed the case.

Remark 7.1.1. There exists a common variant of (7.1), which is related to the dual representations [KS92, KSS93, BCDR95, BCR96, Doi00], i.e.,

$$R_{WV}(\frac{vw}{v})^{tv}_{21} \cdot \text{id}_V \otimes \hat{K}_W(w) \cdot (R_{VW}(zw)^{-1})^{tv} \cdot \hat{K}_V(z) \otimes \text{id}_W =$$

$$= \hat{K}_V(z) \otimes \text{id}_W \cdot (R_{WV}(zw)^{-1})^{tw} \cdot \text{id}_V \otimes \hat{K}_W(w) \cdot R_{VW}(\frac{vw}{v}) \tag{7.2}$$

where $tv$ and $tw$ denote the transposition on the first and second component, respectively. The equation (7.2) is associated to the case where the pullback through the twisting operator $\psi$ gives back the dual representations of $V$ and $W$. However, for brevity, we shall not discuss this case here.
7.2. Solutions through QSP intertwiners. We claim that the rational K-matrices constructed in Theorem 5.2.1 do satisfy the reflection equation (7.1), for a suitable choice of the twisting operator $\psi$ and eventually of the QSP subalgebra itself.

Let $(X, \tau) \in \text{GSat}(\hat{A})$ with pseudo-involution $\theta$ and $(\gamma, \sigma) \in \Gamma \times \Sigma$ such that $\gamma(\delta) = 1$, and $U_q \mathfrak{g} \subset U_q \hat{g}$ the corresponding QSP subalgebra. We consider an auxiliary generalized Satake diagram $(Y_0, \eta_0) \in \text{GSat}(\hat{A})$ of restricted rank one and defined as follows. Set

$$Y_0 := \hat{\Gamma}\{0, \tau(0)\}, \quad \eta_0(0) = \tau(0), \quad \eta_0|_{Y_0} = \hat{o}\eta_0.$$

Let $\zeta_0$ be the corresponding pseudo-involution. Note that, by considering the gauge transformation $g_0 := S_{\omega}^{-1}S_{\theta, \gamma}$, we obtain the QSP admissible twisting operator $\psi_0 = \zeta_0^{-1} \circ \eta_0 \circ \tau$. As we pointed out in Remark 3.6.2 (4), $\eta_0$ and $\tau$ commute and the twisting operator acts as the involution $\eta_0 \circ \tau$ on a large class of representations.

**Theorem 7.2.1.** Let $U_q \mathfrak{g} \subset U_q \mathfrak{g}$ be a restrictable affine QSP subalgebra, i.e., $0 \notin X$ and $\tau(0) = 0$. Let $V, W \in \text{Rep} U_q \mathfrak{g}$ be irreducible representations, which remain irreducible under restriction to $U_q \mathfrak{g}$.

1. The rational K-matrix $K_V(z) := K_{\psi_0, V}(z)$ is a QSP intertwiner

$$K_V(z) : V(z) \to V^{\eta_0 \tau}(1/z),$$

where $V^{\eta_0 \tau} = (\eta_0 \tau)^*(V)$ (and similarly for $K_W(z) := K_{\psi_0, W}(z)$).

2. The K-matrices $K_V(z)$ and $K_W(w)$ satisfy the diagrammatic reflection equation

$$R_{VU}(w^{-1})_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VU}^{\eta_0 \tau}(zw) \cdot K_V(z) \otimes \text{id} =
$$

$$= K_V(z) \otimes \text{id} \cdot R_{VU}^{\eta_0 \tau}(zw)_{21} \cdot \text{id} \otimes K_W(w) \cdot R_{VU}(w).$$

3. Whenever $\tau = \eta_0$, the K-matrices $K_V(z)$ and $K_W(w)$ satisfy the standard reflection equation (7.1).

**Proof.** The result follows immediately from Theorem 5.2.1. Namely, it is enough to observe that, if $V \in \text{Rep} U_q \mathfrak{g}$ is an irreducible representation which remains irreducible under restriction to $U_q \mathfrak{g}$, then the defining data of $V$ as a representation over $U_q \mathfrak{g}$ can be expressed entirely in terms of $U_q \mathfrak{g}$. Moreover, when $U_q \mathfrak{k}$ is a restrictable QSP subalgebra, then $Y_0 = I$ and $\zeta_0$ is the maximal pseudo-involution with respect to the finite Dynkin diagram. Therefore, $\zeta_{0, q}$ is the identity on $U_q \mathfrak{g}$, so that $\psi_0^*(V) = (\eta_0 \tau)^*(V)$. The result follows.

**Remarks 7.2.2.**

1. In the case of $U_q \mathfrak{sl}_N$, every evaluation representation satisfy the conditions of the Theorem.

2. In particular, the result applies to the case of the generalized $q$-Onsager algebras, see [BB10], with respect to any evaluation representation of $U_q \mathfrak{sl}_N$. 

□
The constant (i.e., parameter-independent) version of the diagrammatic reflection equation appeared first in the work of Balagovic-Kolb on the universal K-matrices for finite-type quantum groups [BK19].

The constraint $0 \notin X$ is not strictly necessary. However, when $U_q \mathfrak{k}$ is restrictable, the K-matrices are regular at $z = 0$ and one can show that, as in the case of the R-matrix, $K_V(0)$ reduces to the K-matrix of the underlying finite-type QSP subalgebra.

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