Integrability and BRST invariance from BF topological theory

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Received 6 April 2023; revised 5 September 2023
Accepted for publication 3 October 2023
Published 12 October 2023

Abstract
We consider the Becchi, Rouet, Stora and Tyutin (BRST) invariant effective action of the non-abelian BF topological theory in two dimensions with gauge group $\text{Sl}(2,\mathbb{R})$. By considering different gauge fixing conditions, the zero-curvature field equation gives rise to several well known integrable equations. We prove that each integrable equation together with the associated ghost field evolution equation, obtained from the BF theory, is a BRST invariant system with an infinite sequence of BRST invariant conserved quantities. We construct explicitly the systems and the BRST transformation laws for the Korteweg-de Vries (KdV) sequence (including the KdV, mKdV and CKdV equations) and Harry Dym integrable equation.

Keywords: integrable systems, gauge field theory, partial differential equations, conservation laws

1. Introduction

Topological field theories were introduced by Schwarz [1], who related the Ray–Singer torsion to the partition function of a quantum field theory, and Witten [2], who introduced the so-called ‘Witten index’ and opened up a wide range of research lines in mathematics and physics.

Topological BF field theories [3–6] have been extensively studied, in particular in relation to gravity theories in several dimensions [7–19]. Two dimensional BF field theories were intensively studied in the context of Poisson sigma models, deformation quantization and non-commutative field theories.
The Becchi, Rouet, Stora and Tyutin formalism (BRST) [20–24] is a fundamental approach in quantum field theory, and as such has been used in the analysis of the BF topological theories, both from the perturbative and non-perturbative point of view [25–33]. The renormalizability of the theory has been proved in [34–40]. BF theories have been also used in the description of topological effects in condensed matter [41–48].

In this work, we study the field equations of the two dimensional BRST formulation of BF theory. These field equations contain in particular the zero curvature equations for the $sl(2, \mathbb{R})$ connection one-form and can be analyzed under several gauge fixing conditions, giving integrable systems. The origin of these integrable equations as solutions of the zero curvature equation is one side of an old conjecture claimed by Ward [49], that all integrable equations should arise from the selfdual field equation for the curvature of a one-form connection arising from a principal bundle with appropriate structure group, which in particular contains the zero curvature equation. The latter has been extensively studied in the literature. A complete analysis of the zero curvature equation was given in [50].

BF theories are topological theories (its action does not depend on a metric) describing the interaction of flat connections with the $p$-form $B$ (in the present case a zero form). Flat connections on oriented two-manifolds have physical applications in several areas. A natural application is in the Aharonov–Bohm effect. In fact, on $\Sigma \equiv \mathbb{R}^2 - \{0\}$, the fundamental group $\Pi_1$ is isomorphic to $\mathbb{Z}$ and related to it there is a flat connection corresponding to a representation of $\Pi_1$ in $U(1)$. Although there is no classical forces on electrical charges moving on $\Sigma$ there are quantum effects due to this non-trivial connection. It has been shown the existence of an isomorphism between the space of flat connections modulo gauge transformations and the representations of the fundamental group $\Pi_1(\Sigma)$ into the structure group $G$ modulo conjugation classes. The proof is via the holonomy on the principle bundle with structure group $G$ and base the manifold $\Sigma$. The explicit solution of the zero curvature equation on a $SL(2, \mathbb{R})$ principal bundle in a partial gauge fixing reduces to integrable equations. In order to work modulo gauge transformations we introduce a BRST formulation of the BF theory on $\Sigma$ and work with the zero curvature condition together with the associated ghost field equations. In this line of thought the main goal would be to obtain explicit solutions of this system of equations formulated on a two-surface with non-trivial fundamental group, since it describes flat connections modulo gauge transformations on the two-surface. In this paper, as a first step, we analyze the integrability properties of the coupled system describing the flat connection together with its associated ghost field. Besides our analysis may be of interest for integrable systems on compact Riemann surfaces [51], also for the formulation of two dimensional gravity, including a dilaton field [52, 53].

The main result of our work is that the field equations of the BRST effective action of the two dimensional BF model, in the above mention gauge fixing procedures, extend the integrable equations to BRST invariant integrable systems where together with the well-known integrable equation there is an evolution equation for the ghost field. The new system has an infinite sequence of conserved quantities, which are BRST invariant. The symmetry is explicitly constructed for the Korteweg-de Vries (KdV) sequence of integrable equations as well as for the Harry Dym integrable equation. An interesting point is that the system describes soliton solutions in the ghost sector of the system. The new integrable system describes the evolution of a real field and its ghost partner, which is an anti-commuting field. In this sense, it is analogous to the case of the supersymmetric extensions of some integrable equations [54–60]. The main difference being that the new system is invariant under a BRST transformation while the latter systems are invariant under a supersymmetric transformation. They have a different analytic structure and different applications. In the first case with interesting applications
on topological problems, in particular related to the moduli space of flat connections on two dimensional manifolds and also to topological gravity.

In section 2 we present the notation used in the work. In section 3 we consider the BF action and gauge symmetry of the theory. In section 4 we present details of the BRST effective BF theory, the BRST charge and associated field equations. In section 5 we discuss the evolution equations for the field and its associated ghost field. In sections 6 and 7 we introduce the new integrable BRST systems extending the KdV, mKdV, CKdV and Harry-Dim integrable equations. In section 8 we give our conclusions.

2. Notation

The generators of the \( \mathfrak{sl}(2, \mathbb{R}) \) algebra will be denoted by

\[
T_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
T_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},
T_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

They satisfy

\[
[T_a, T_b] = f_{ab}^c T_c, a = 0, +, -,
\]

where the structure constants are antisymmetric on the subindices and satisfy

\[
f_{0+}^+ = 1, f_{0-}^- = -1, f_{+}^0 - = 1,
\]

with all other components of \( f_{ab} \), excluding the antisymmetric of the given ones, being zero. Also,

\[
tr(T_a T_b) = \frac{1}{2} \eta_{ab}, \eta_{ab} = \eta_{ba},
\]

where

\[
\eta_{00} = 1, \eta_{0+} = \eta_{0-} = 0, \eta_{++} = 0, \eta_{+-} = 1, \eta_{--} = 0,
\]

and \( f_{abc} = f_{ab}^d \eta_{dc} = \frac{1}{2} \epsilon_{abc} \), where \( \epsilon_{abc} \) is totally antisymmetric with \( \epsilon_{0+-} = 1 \).

The Lie algebra valued connection one-form \( A \equiv A^a_{\mu} dx^\mu \), \( \mu = 0, 1 \), has curvature two-form \( F \equiv dA + A \wedge A = \frac{1}{2} F^a_{\mu \nu} T_a dx^\mu \wedge dx^\nu \), hence

\[
F^a_{\mu \nu} = \partial_\mu A^a_{\nu} - \partial_\nu A^a_{\mu} + f^a_{bc} A^b_{\mu} A^c_{\nu}.
\]

The local coordinates on \( \mathbb{R}^2 \) will be denoted \( x^0 = t, x^1 = x \).

All the fields are functions of \( t \) and \( x \), but we only make explicit the \( x \) dependence.

3. The BF theory

We consider the BF topological theory in \( 1 + 1 \) dimensions valued on the \( \mathfrak{sl}(2, \mathbb{R}) \) algebra.

Its action is given by

\[
S(B, A) \equiv \int_{\Sigma} tr(BF),
\]
where $B$ is a Lie algebra valued zero-form, $F$ is the Lie algebra valued curvature of the connection one-form $A$ and $\Sigma$ is a two-surface.

We consider here local coordinates in $\Sigma$, where one of them can be interpreted as an evolution coordinate, we denote it by $t$.

We have

$$\text{tr} (BF) = B^a F^b \text{tr} (T_a T_b) = \frac{1}{2} g_{ab} B^a F^b = \frac{1}{2} B_a F^a,$$

therefore

$$S (B, A) = \frac{1}{4} \int_{\Sigma} B_a F_{\mu \nu} d x^\mu \wedge d x^\nu = \frac{1}{2} \int_{\Sigma} B_a \left( \partial_x A^a_1 - \partial_t A^a_0 + f^c_{ab} A^b_0 A^c_1 \right) d t \wedge d x.$$

We may introduce its canonical formulation. The canonical conjugate to $A^a_1$, denoted $\Pi_a$, satisfies the constraint

$$\Pi_a = \frac{1}{2} B_a.$$

The canonical Hamiltonian density then can be expressed as

$$\mathcal{H} = - A^a_0 \phi_a,$$

where $\phi_a = \partial_x \Pi_a + f_{abc} A^b_1 \Pi^c$.

We have that $\phi_a = 0$ is a first class constraint, then $\phi_a$ are the generators of the local $sl(2, \mathbb{R})$ algebra:

$$\{ \phi_a (x) , \phi_b (\hat{x}) \}_{PB} = f^c_{ab} \phi_c (x) \delta (x - \hat{x}),$$

where $f^c_{ab}$ are the structure constants of the $sl(2, \mathbb{R})$ algebra, and we do not write explicitly the $t$ dependence of the fields.

The infinitesimal gauge transformation generated by $\phi_a$ is

$$\delta_{\xi} A^a_1 (x) = \{ \langle \xi^c \phi_c \rangle , A^a_1 (x) \}_{PB} = \partial_x \xi^a (x) + f^c_{ab} A^b_0 \xi^c (x)$$

where $\xi^c \equiv \xi^c (t,x)$ are the infinitesimal parameters of the transformation, and $\langle \rangle$ denotes integration on $\mathbb{R}^2$.

The gauge transformation of $\Pi_a$ is

$$\delta_{\xi} \Pi_a (x) = \{ \langle \xi^c \phi_c \rangle , \Pi_a (x) \}_{PB} = - f^c_{ab} \Pi_c (x) \xi^b (x)$$

and of the Lagrange multiplier $A^a_0$,

$$\delta_{\xi} A^a_0 (x) = \partial_t \xi^a (x) + f^a_{bd} A^b_0 (x) \xi^d (x).$$
4. The BRST effective action

The BRST effective action for BF theories has been already discussed in the literature, as we have referred in the introduction, in particular we follow the approach in [61–63].

The BRST charge associated to the first class constraints $\phi_a = 0$ is:

$$\Omega \equiv \left\langle C^a(x) \phi_a(x) - \frac{1}{2} C^a(x) C^b(x) f^d_{ab} \mu_d(x) \right\rangle,$$

where $\langle \rangle$ denotes integration on $x$ for fixed $t$ and $C^a(t,x)$ are the corresponding ghost fields. They are valued on the odd part of a Grassmann algebra and $\mu_a(t,x)$ denote their conjugate momenta.

They satisfy the following Poisson bracket relations

$$\{ C^a(x), \mu_c(\hat{x}) \}_{PB} = \{ \mu_c(\hat{x}), C^a(x) \}_{PB} = \delta_a^c \delta(x - \hat{x}).$$

It follows that

$$\{ \Omega, \Omega \}_{PB} = 0. \quad (1)$$

The BRST transformation of the canonical fields $A^a_1, \Pi_a, C^a, \mu_a$ is given by

$$\delta_{BRST} \equiv \zeta \hat{\delta}$$

where $\zeta$ is the constant BRST parameter valued on the odd part of a Grassmann algebra.

The transformation $\hat{\delta}$ acts as follows

$$\hat{\delta} C^a(x) \equiv \{ \Omega, C^a(x) \}_{PB} = -\frac{1}{2} C^b(x) C^c(x) f^d_{bc} \mu_d(x), \quad (2)$$

$$\hat{\delta} \mu_a(x) \equiv \{ \Omega, \mu_a(x) \}_{PB} = \phi_a(x) - f^d_{ab} C(x) \mu_d(x), \quad (3)$$

$$\hat{\delta} A^a_1(x) \equiv \{ \Omega, A^a_1(x) \}_{PB} = \partial_x C^a(x) + f^c_{ab} A^b_1(x) C^c(x), \quad (4)$$

$$\hat{\delta} \Pi_a(x) \equiv \{ \Omega, \Pi_a(x) \}_{PB} = -f^c_{ab} \Pi_c(x) C^b(x). \quad (5)$$

It follows from (1) that for the canonical fields $f_c(x)$

$$\hat{\delta} \hat{\delta} f_c(x) = 0.$$

In fact,

$$\{ \Omega, \{ \Omega, f_c(x) \}_{PB} \}_{PB} = \{ \{ \Omega, \Omega \}_{PB}, f_c(x) \}_{PB} - \{ \Omega, \{ \Omega, f_c(x) \}_{PB} \}_{PB},$$

hence $\{ \Omega, \{ \Omega, f_c(x) \}_{PB} \}_{PB} = 0$.

We introduce now the BRST transformation for the non-canonical fields, that is, the Lagrange multiplier $A^a_0$ and the antighost field $C^a(x)$. We can also introduce them as canonical fields [64–72], but we consider here a more direct approach [63].
The transformations are
\[ \hat{\delta} \bar{C}_a (x) = D_a (x), \]
\[ \hat{\delta} D_a (x) = 0, \]
\[ \hat{\delta} A^a_0 (x) = \theta^a (x), \]
\[ \hat{\delta} \theta^a (x) = 0, \]
where \( D_a (x) \) and \( \theta^a (x) \) are new auxiliary fields. 
They satisfy: \( \hat{\delta} \hat{\delta} \bar{C}_a (x) = \hat{\delta} \hat{\delta} A^a_0 (x) = \hat{\delta} \hat{\delta} D_a (x) = \hat{\delta} \hat{\delta} \theta^a (x) = 0. \)
Hence all the fields in the effective action belong to the kernel of the operator \( \hat{\delta} \hat{\delta}. \)

The effective action for the BF theory is then given by
\[ S_{\text{eff}} = \int \left[ \sum_i \Pi_i \partial_t A^i_1 + \mu_a \partial_t C^a + \hat{\delta} \left( A^a_0 \mu_a \right) + \hat{\delta} \left( \bar{C}_b \chi^b \right) \right] dt \wedge dx \]
\[ = \int \left[ \sum_i \Pi_i \partial_t A^i_1 + \mu_a \partial_t C^a + \theta^a \mu_a + A^a_0 \delta \mu_a + D_b \chi^b - \bar{C}_b \delta \chi^b \right] dt \wedge dx \]
where \( \chi^a = 0 \) is the gauge fixing condition. 
We notice that
\[ \phi^{\text{eff}}_a = \delta \mu_a = \phi_a - f_{ab}^d \mu_d \]
is the effective constraint which extends the classical constraint \( \phi_a = 0, \) in the effective (quantum) BF action. Its associated Lagrange multiplier is \( A^a_0. \) Besides, \( D_b (t, x) \) is the Lagrange multiplier associated to the gauge fixing condition \( \chi^a = 0. \)

The effective action \( S_{\text{eff}} \) is a functional of the fields \( A^i_1, \Pi_i, C^a, \mu_a, A^a_0, \bar{C}_b, \theta^a, D_a \) and it is BRST invariant:
\[ \hat{\delta} S_{\text{eff}} = 0. \]
In what follows we will consider a gauge fixing condition
\[ \chi^a = 0 \]
which depends only on the canonical fields \( A^i_1 (t, x). \)

The field equations arising from variations of \( S_{\text{eff}} \) are the following:
\[ \phi^{\text{eff}}_a = 0, \]
\[ F^a_{01} = \partial A^a_i - \partial_x A^a_0 + f_{ab}^d A^b_0 A^d_i = 0, \]
\[ \theta^a = \partial C^a + f_{ab}^d A^b_0 C^d, \mu_a = 0 \]
\[ \chi^a = 0, \delta \chi^a = 0, \]
and
\[ \partial_t \Pi_a = f^{i}_{bd} A^b_0 \Pi_d + D_b \frac{\partial \chi^b}{\partial A^i_1} - \bar{C}_b \frac{\partial \left( \delta \chi^b \right)}{\partial A^i_1}. \]
Since the action is BRST invariant, its field equations are also BRST invariant. In particular,
\[ \hat{\delta} F^a_{01} = f_{bd}^a F^b_{01} C^d, \]
hence \( F_{01} = 0 \) implies \( \hat{\delta} F^a_{01} = 0. \)
5. Gauge fixing procedure and integrability

In this and following sections we analyse the gauge fixing procedure and the role of the ghost fields. We consider two-surfaces that can be parametrized by a time coordinate and a space like coordinate which varies from $-\infty$ to $+\infty$. In this context, the emerging real valued functions are supposed to belong to the Schwartz space of functions that rapidly vanish at infinity.

The approach may be generalized to more general surfaces, but we are here interested in analysing the role of ghost fields in the integrability of the BF field equations. The more general formulation will be discussed elsewhere.

The zero curvature equations for the $sl(2,\mathbb{R})$ one-form gauge connection $A$ gives rise to several integrable equations as it has been discussed in [50]. They follow by taking different partial gauge fixing conditions.

We use the following notation, similar to the one in [50] but not equal (the components $A_{a\mu}$ are related by $A_{a\mu} \rightarrow -A_{a\mu}$):

\begin{align*}
A_0^0 &= 2P, \quad A_0^+ = \sqrt{2}u, \quad A_0^- = \sqrt{2}Q, \quad A_1^0 = 2R, \quad A_1^+ = \sqrt{2}S, \quad A_1^- = -\sqrt{2}T.
\end{align*}

The zero curvature equation (8) yields:

\begin{align}
\partial_t R - \partial_x P - uT - QS &= 0, \quad (12) \\
\partial_s S - \partial_x u + 2PS - 2uR &= 0, \quad (13) \\
\partial_t T + \partial_x Q - 2PT - 2QR &= 0. \quad (14)
\end{align}

We impose the partial gauge fixing condition [50]

\begin{align}
R &= 0, \quad S = 1.
\end{align}

It follows from (12)–(14) that

\begin{align}
Q &= -\partial_x P - uT, \quad (15) \\
P &= \frac{1}{2} \partial_x u, \quad (16) \\
\Upsilon &= \partial_t T - \frac{1}{2} \partial_{xx} u - \partial_x (uT) - (\partial_x u) T = 0. \quad (17)
\end{align}

The zero-curvature condition reduces then to equation (17).

Besides, the BF field equation (10) imply

\begin{align}
\chi_0 = A_0^0 &= 0, \quad \chi^+ = A_1^+ - \sqrt{2} = 0, \quad (18) \\
C^0 &= \frac{1}{\sqrt{2}} \partial_x C^+, \quad (19) \\
C^- &= -TC^+ - \frac{1}{2} \partial_x \partial_x C^+. \quad (20)
\end{align}

that is, the ghost fields reduce solely to $C^+(t, x)$.

The equation of motion obtained by variations of $A^a_0$, that is equation (11), determines the Lagrange multipliers $D_a$, while the equation of motion obtained from variations of $A^a_1$ gives rise to the effective constraint given in section 4.

(17) is invariant under the BRST transformations, in fact,
\[ \delta Y = 2Y \partial_x \tilde{C}^+ + (\partial_x Y) \tilde{C}^+, \tilde{C}^+ \equiv \frac{C^+}{\sqrt{2}}, \]

while the BRST transformations of \( u \) and \( T \) are

\[ \begin{align*}
\delta u &= \partial_t \tilde{C}^+ - u \partial_t \tilde{C}^+ + (\partial_x u) \tilde{C}^+ + (\partial_x T) \tilde{C}^+ + 2T \partial_x \tilde{C}^+. \\
\delta T &= \frac{1}{2} \partial_{xx} \tilde{C}^+ + (\partial_x T) \tilde{C}^+ + 2T \partial_x \tilde{C}^+. 
\end{align*} \tag{21} \tag{22} \]

We notice that (21) and (22) are the infinitesimal transformations obtained in [50] for \( u \) and \( T \) in terms of the field \( y(t, x) \) when we replace \( e^{C^+} \rightarrow e \). 

In distinction to the infinitesimal transformations in [50], (21) and (22) are exact transformations under which (17) is invariant.

In [50] they consider also a further gauge restriction \( T = \frac{u}{2} \), where \( \alpha \) and \( s \) are real numbers. (17) reduces then to

\[ \begin{align*}
\partial_t u &= \left( \frac{\alpha + 2}{\alpha} \right) u_{xx} + \frac{s}{2\alpha} u^{1-\alpha} u_{xxx}, \\
\delta T - \delta \left( \frac{u^\alpha}{s} \right) &= 0 
\end{align*} \tag{23} \tag{24} \]

implies, using (21) and (22), the following equation for the ghost \( \tilde{C}^+ \):

\[ \begin{align*}
\partial_t \tilde{C}^+ &= \tilde{C}^+ \partial_x \tilde{C}^+, \\
\delta \tilde{C}^+ &= \frac{s}{2\alpha} u^{1-\alpha} \partial_{xxx} \tilde{C}^+ + \left( \frac{2}{\alpha} + 1 \right) u \partial_x \tilde{C}^+. 
\end{align*} \tag{25} \]

Note that \( s \) is not a genuine parameter, as it can be removed by rescaling.

Equations (23) and (25) become invariant under the following BRST transformation

\[ \begin{align*}
\delta \tilde{C}^+ &= \tilde{C}^+ \partial_x \tilde{C}^+, \\
\delta u &= \frac{s}{2\alpha} u^{1-\alpha} \partial_{xxx} \tilde{C}^+ + \tilde{C}^+ \partial_x u + \frac{2}{\alpha} u \partial_x \tilde{C}^+. 
\end{align*} \tag{26} \tag{27} \]

where (26) arises from (2) and (19), (20) while (27) follows directly from (21), (22) and (24). This is an exact symmetry of the evolution equations (23) and (25).

We notice that the transformation law (26) is only present in a BRST formulation.

Equation (23) for \( \alpha = 1, s = 2 \) becomes the KdV equation and for \( \alpha = -2, s = 1 \) is the Harry Dym integrable equation. In both cases, given a solution \( u(t, x) \) of (23) one can construct an algorithmic procedure to obtain an infinite sequence of solutions of the ghost equations and from them the known infinite sequence of conserved quantities for both integrable systems [50].

In [72] it was shown that for any conserved quantity \( H \) of an evolution equation \( u_t = K(u, u_x, \ldots) \), where dots denote higher order derivative terms with respect to \( x \) up to \( m \) derivatives,

\[ H = \int_{-\infty}^{+\infty} H(u, u_x, \ldots) \, dx \]

the gradient \( \mathcal{M}H \),

\[ \mathcal{M}H \equiv \sum_{r=0}^{m} (-1)^r \partial_t^r \frac{\partial H}{\partial a_i} a_i \equiv \partial_t^r u, \]

\[ \delta Y = 2Y \partial_x \tilde{C}^+ + (\partial_x Y) \tilde{C}^+, \tilde{C}^+ \equiv \frac{C^+}{\sqrt{2}}, \]

\[ \delta u = \partial_t \tilde{C}^+ - u \partial_t \tilde{C}^+ + (\partial_x u) \tilde{C}^+ + (\partial_x T) \tilde{C}^+ + 2T \partial_x \tilde{C}^+. \]

\[ \delta T = \frac{1}{2} \partial_{xx} \tilde{C}^+ + (\partial_x T) \tilde{C}^+ + 2T \partial_x \tilde{C}^+. \]
satisfies the integral equation

\[ \int_{-\infty}^{+\infty} \left( \frac{\partial (\mathcal{H})}{\partial t} \delta u + dK(u, \delta u) \mathcal{H} \right) dx = 0. \] (28)

From equation (28) we obtain an evolution equation for the gradient \( \mathcal{H} \). For the particular values \( \alpha = 1, s = 2 \) this equation is exactly the corresponding evolution equation for the ghost field associated to the KdV equation. This is not the case for generic values of the parameters. In order to generalize this result for any \( \alpha \) and \( s \), it is better to redefine the gauge fixing condition as

\[ u = sT^\beta, \]

and rewrite the system (23) and (25) in terms of \( T \) and \( \tilde{C}^+ \).

We end up with the following system

\[ \partial_t T = \frac{s}{2} \partial_{xxx} (T^\beta) + (2\beta + 1) sT^\beta \partial_x T \] (29)

\[ \partial_t \tilde{C}^+ = \frac{s}{2} \beta T^{\beta-1} \partial_{xxx} \tilde{C}^+ + s(2\beta + 1) T^\beta \partial_x \tilde{C}^+. \] (30)

Note that \( s \) can be removed by rescaling.

If \( H = \int_{-\infty}^{+\infty} \mathcal{H}(u, u_x, \ldots) dx \) is a conserved quantity under the evolution equation (29), for a given value of the parameters \( \beta \) and \( s \), then the gradient \( \mathcal{H} \) satisfies the same evolution equation as the corresponding ghost field. In particular, for \( \beta = 1, s = 2 \) (KdV) and for \( \beta = \frac{1}{2}, s = 1 \) (Harry Dym) there is a sequence of infinite conserved quantities, hence one can obtain from them an infinite set of solutions of equation (30). We notice that it is so, only for these values of the parameters. For generic values of the parameters, it is only guaranteed that for each conserved quantity one can obtain a solution of the ghost equation.

6. The integrable BRST system

In this section we show that the system (29) and (30) for \( \beta = 1, s = 2 \) and for \( \beta = \frac{1}{2}, s = 1 \) are integrable systems with an infinite sequence of conserved quantities which are BRST invariant. That is, the KdV and the Harry Dym equations together with their associated ghost evolution equation have an infinite sequence of BRST invariant conserved quantities. We have considered other values for \( \beta \) and \( s \) but we have not been able to obtain a system with a sequence of infinite conserved local quantities. As far as we know, there is not a systematic argument to show, for generic values of the parameters, whether the resulting system has an infinite sequence of conserved quantities or not. In [30] it was conjectured the integrability for all values of \( \alpha \) in the case without the ghost field. If the conjecture were correct, the integrability property would also be satisfied with the inclusion of the ghost field.

The system (29) and (30) is invariant under the BRST transformations (22) and (26). We notice that the conserved quantities of KdV or Harry Dym equations are not BRST invariant. For example,

\[ H_0 = \int_{-\infty}^{+\infty} T dx \]

is a conserved quantity of the evolution equation (29) for any \( \beta \) and \( s \).
However, it is not BRST invariant under (22):

\[ \hat{\delta}H_0 = \int_{-\infty}^{+\infty} \hat{\delta}T \, dx = \int_{-\infty}^{+\infty} T \partial_c \tilde{C}^+ \, dx \neq 0. \]

Let

\[ H_n = \int_{-\infty}^{+\infty} H_n(T, T_x, \ldots) \, dx \]  \hspace{1cm} (31)

be a conserved quantity of (29).

It then follows the existence of \( J_n \) such that

\[ \partial_t H_n = \partial_x J_n. \]

Consequently,

\[ \hat{\delta} \partial_t H_n = \hat{\delta} \partial_x J_n = \partial_x \hat{\delta} J_n \]

which implies that

\[ \int_{-\infty}^{+\infty} \hat{\delta}H_n \, dx \]  \hspace{1cm} (32)

is a conserved quantity of the system (29) and (30). Moreover, it is invariant under (22) and (26).

In particular, \( H_0 = \int_{-\infty}^{+\infty} T \) has an associated BRST invariant conserved quantity

\[ \tilde{H}_0 = \int_{-\infty}^{+\infty} \tilde{\delta}T \, dx = \int_{-\infty}^{+\infty} T \partial_c \tilde{C}^+ \, dx. \]  \hspace{1cm} (33)

For each conserved quantity of (29) there is a BRST invariant conserved quantity of (29) and (30).

It then follows that for each integrable equation arising from a partial gauge fixing of the one-form connection with zero curvature there is an integrable system, BRST invariant, describing also the evolution of the associated ghost field. The system has an infinite sequence of BRST invariant conserved quantities given by (32).

Another conserved quantity of (29) for any \( \beta \) and \( s \) is

\[ H_1 = \int_{-\infty}^{+\infty} T^{\beta + 1} \, dx. \]  \hspace{1cm} (34)

The associated BRST conserved quantity is

\[ \tilde{H}_1 = \int_{-\infty}^{+\infty} (\beta + 1) T^{\beta} \left( \frac{1}{2} \partial_{xx} \tilde{C}^+ + (\partial_x T) \tilde{C}^+ + 2T \partial_c \tilde{C}^+ \right) \, dx. \]  \hspace{1cm} (35)

In [50] several integrable equations were obtained: KdV, modified KdV, sine-Gordon, Harry Dim, Calogero KdV, nonlinear Schrödinger and KdV sequences. To each of them one can construct, following the BRST approach we have considered, a ghost field evolution equation. The system is BRST invariant and has an infinite sequence of conserved quantities.

For the BRST invariant KdV system (29) and (30) with \( \alpha = 1, s = 2 \), the first few conserved quantities, where we use \( \partial_x g \equiv g_x \), are

\[ H_1 = \int_{-\infty}^{+\infty} \delta u \, dx, H_3 = \int_{-\infty}^{+\infty} \delta \left( \frac{1}{2} u^2 \right) \, dx, H_5 = \int_{-\infty}^{+\infty} \delta \left[ \frac{1}{3} u^3 - \frac{1}{3} (u_x)^2 \right] \, dx \]
which have the explicit expressions
\[ H_1 = \int_{-\infty}^{+\infty} u \tilde{C}_1^+ \, dx, \]
\[ H_3 = \int_{-\infty}^{+\infty} \left( u \tilde{C}_{xxx}^+ + \frac{3}{2} u^2 \tilde{C}_x^+ \right) \, dx, \]
\[ H_5 = \int_{-\infty}^{+\infty} \left( \frac{2}{3} u_x \tilde{C}_{xxx}^+ + \frac{5}{3} u^3 \tilde{C}_x^+ + u^2 \tilde{C}_{xxx}^+ + \frac{1}{3} (u_x)^2 \tilde{C}_x^+ - \frac{4}{3} u u_x \tilde{C}_x^+ \right) \, dx. \]

Since KdV equation has an infinite sequence of conserved quantities, which are not BRST invariant, one can construct an infinite sequence of independent BRST invariant conserved quantities of the KdV and ghost field system.

7. The BRST invariant KdV sequence

We now extend the KdV sequence of integrable equations: KdV, mKdV, CKdV, . . . to BRST invariant systems with an infinite sequence of BRST conserved quantities.

We impose the partial gauge fixing conditions
\[ A_1^+ = \sqrt{2}, \] (36)
\[ A_1^- = 0. \] (37)

Using the same notation for the components of the connection one form \( A_{\mu}^a \) as in section 5, the zero-curvature field equations became
\[ 2 (\partial_x - 2R) (R_t - P_x) \equiv 2 (R_x - R^2) \partial_x - u_{xxx} - 4 (R_x - R^2) u_x - 2 (R_x - R^2) u = 0, \] (38)
where \( P = \frac{1}{2} u_x + u R. \)

If we impose, further on, the gauge fixing condition (Miura transformation)
\[ u = 2 (R_x - R^2). \] (39)

Equation (38) yields the KdV equation for \( u. \)

Moreover, as it is well known,
\[ R_t - P_x \equiv R_t - R_{xxx} + 6R^2 R_x = 0 \] (40)
is the mKdV equation. If \( R \) is a solution of mKdV then \( u \) given in (39) is a solution of the KdV equation.

From \( \hat{\delta} A_1^+ = 0, \hat{\delta} A_1^- = 0, \) using the BRST transformation law in section 3, we obtain
\[ C^0 = (\partial_x + 2R) \tilde{C}^+ \] (41)
\[ (\partial_x - 2R) \tilde{C}^- = 0. \] (42)

Finally,
\[ \hat{\delta} u = 2 \hat{\delta} \left( R_x - R^2 \right) \] (43)
yields the evolution equation for the ghost field \( \tilde{C}^+ \),
\[ \tilde{C}_t^+ = \tilde{C}_{xxx}^+ + 6 (R_x - R^2) \tilde{C}^+. \] (44)
The BRST transformation for $R$ and $\tilde{C}^+$ are

\[
\delta R = \frac{1}{2} \tilde{C}_{\alpha}^+ + \left( R \tilde{C}^+ \right)_{x} + \tilde{C}^{-} = \frac{C^{-}}{\sqrt{2}}
\]

\[
\delta \tilde{C}^+ = \tilde{C}^+ \tilde{C}_{\alpha}^+
\]

(45)

(46)

respectively.

(45) and (46) define a BRST transformation under which the mKdV and associated ghost field given by (44) remain exactly invariant. The system mKdV (44) has an infinite sequence of BRST conserved quantities obtained by the action of the BRST operator $\delta$ to the conserved quantities of the mKdV equation.

Equation (44), with the identification $u = 2(R_x - R^2)$, is the same ghost evolution equation obtained for the KdV equation, that is, (26) with $\alpha = 1, s = 2$.

Also (44) is the same BRST transformation law for the ghost field of the KdV equation.

The next integrable equation in the KdV sequence is the CKdV equation. We consider the partial gauge fixing (36) and (37) together with

\[
R = v + w,
\]

(47)

where $v$ satisfies the mKdV equation, and the gauge fixing condition

\[
u = 2 \left( v_x - v^2 \right).
\]

(48)

We get

\[
R_x - R^2 = v_x - v^2 + \left( w_x - 2vw - w^2 \right),
\]

(49)

hence, if

\[
w_x - 2vw - w^2 = 0,
\]

(50)

then, from (38), $u = 2(v_x - v^2)$, satisfies the KdV equation. Moreover (40) yields

\[
w_t - w_{xxx} + \frac{1}{2} \left( w^3 + 3w_x^2w^{-1} \right) = 0,
\]

(51)

the CKdV equation.

If $w$ satisfies the CKdV equation then $v$ obtained from (50) satisfies the mKdV equation.

The ghost field equations are now given by

\[
\partial_t C^0 = \tilde{C}_x^+ + 2 \left( v + w \right) \tilde{C}^+
\]

\[
[\partial_t - 2 \left( v + w \right)] C^- = 0
\]

and the evolution equation for the ghost field $\tilde{C}^+$ is

\[
\tilde{C}_t^+ = \tilde{C}^+_{xx} + 6 \left( v_x - v^2 \right) \tilde{C}^+.
\]

(52)

The BRST transformation law for $w$ and $\tilde{C}^+$ are

\[
\delta w = \left( w \tilde{C}^+ \right)_x + \tilde{C}_w,
\]

where $\tilde{C}_w = \tilde{C}^- - \tilde{C}_{\alpha}^-$,

\[
[\partial_t - 2v] \tilde{C}_w = 0,
\]

while the BRST transformation of $\tilde{C}^+$ is given by (46).
We notice that (46) and (52) are the same as the ones for the BRST mKdV system and for the BRST KdV system. The other members of the KdV sequence follow from the same gauge fixing conditions (36), (37) and (48) by considering

\[ R = v + w + z + \ldots, \]

using the generalization of (48) and (50) and the evolution equation (40).

All the members of the KdV sequence share the same ghost evolution equation for \( \tilde{c}^+ \) and the same BRST transformation for \( \tilde{c}^+ \).

8. Conclusions

We started from the BF topological theory. Its action does not depend on a given metric. The action was formulated on a topological two-dimensional surface \( \Sigma \). We also introduced a principal bundle on it with structure group \( SL(2, \mathbb{R}) \) and base \( \Sigma \). The BRST action was constructed with the introduction of ghost fields and anti-ghost ones. After imposing three gauge fixing conditions, the independent field equations reduce to an integrable equation, for example the KdV, MKdV, the KdV hierarchy, Harry Dym, sine Gordon, and an equation for the independent ghost field. The complete system, including the ghost field equation, is BRST invariant and has an infinite sequence of conserved quantities. No metric was introduced in the analysis. The interpretation of the local coordinates on \( \Sigma \) may be that one of them is a time coordinate and the other a spatial one or we may consider that both are spatial coordinates. The usual interpretation as evolution equations of the above-mentioned integrable equations requires one of the local coordinates to be a time, however the same equations can be interpreted as equations for a field defined on a two-surface without reference to a time coordinate.

The topological BF theory on the two-surface may be interpreted as a theory of flat connections modulo gauge transformations. That is Lie valued one-form connections on \( \Sigma \) with zero curvature, modulo gauge transformations. We have shown that the field equations of the gauge fixed BF effective action determine a system of integrable equations, formulated on a chart of a two-manifold where no metric has been incorporated. The gauge freedom has been replaced by the presence of ghost and anti-ghost fields. The gauge local symmetry has been replaced by the BRST global symmetry. The solutions of the integrable system, the equations for the field \( u(x,t) \) and the ghost field \( c^+(x,t) \) describe a representative of the equivalence classes of flat connections up to gauge transformations, by considering one element of the class. There is a global symmetry left represented by the infinite sequence of conserved quantities. These conserved quantities, the complete set found in this work, are BRST invariant. The analysis of the moduli space of this integrable system may provide relevant information to the moduli space of flat connections modulo gauge transformations.

Spaces of flat connections have been considered in different contexts: gauge theories, string theory, algebraic geometry, symplectic geometry. The analysis of the space of flat connections on an orientable two-dimensional manifold has relevance in several important topics. To mention some of them: in the representations of the fundamental group into the structure group, \( SL(2, \mathbb{R}) \) in the present work, modulo conjugation, also in the moduli space of holomorphic bundles on Riemann surfaces, a relevant topic in string theory, and in symplectic geometry. In particular, a well-known result is the isomorphism between the space of flat connections modulo gauge transformations and the representations of the fundamental group of \( \Sigma \) in the structure group \( G \) of the principle bundle modulo conjugate classes:

\[ \Pi_1(\Sigma) \to G \] modulo conjugate classes, determined by the holonomy of the flat connection [73].
The moduli space of flat connections has been extensively studied for compact structure groups. The properties of the space of flat connections in two-dimensional manifolds are an archetype for properties of topological spaces arising in gauge theory formulated on manifolds of dimension higher than 2. In the present work we considered $Sl(2, R)$ a connected, non-compact, real Lie group of dimension 3. The search of explicit expressions of flat connections, solitons solutions of the BRST invariant system will allow a specific studied of the singularities of the moduli space of flat connections for the structure group $Sl(2, R)$. This group is very relevant as symmetry of Type IIB string theory. One component of the moduli space of flat connections on principal bundles with $SL(2, R)$ structure group on a genus $g$ compact Riemann surface, as the base manifold, is the Teichmüller space for genus $g$ Riemann surfaces, relevant to describe the moduli space of genus $g$ Riemann surfaces, an important issue in the evaluation of loops diagrams in String and Superstring theories. The analysis of the singularities of the latter space is an open problem relevant to prove the finiteness of Superstring amplitudes. It is also related to topological gravity in two-dimensions as was mention in the introduction. The BRST integrable system we obtained may provide a different approach for the analysis of the above-mentioned moduli space.

In this context, the approach we followed in this paper is to explicitly solve the zero curvature equations, modulo gauge transformations. To do so we introduce the BRST ghost structure for the BF topological theory. The problem of characterizing the space of flat connections on two-surfaces reduces then to the analysis of the solutions of the BRST invariant integrable ghost system we found in this paper. It is interesting that, for the structure group $SL(2, R)$, the moduli space of flat connections modulo gauge transformations is isomorphic to the space of solutions of this extended integrable system. We hope to report advances in this line elsewhere. The group $ SL(2, R) $ is isomorphic to $Spin^+(1, 2)$ and it is then naturally to relate this BF model to Chern–Simon in three-dimensions.

Besides the geometrical relevance of this analysis, the BRST symmetry is a manifest symmetry of the quantum formulation of the BF theory where the ghost and anti-ghost equations play a fundamental role in the proof of the quantum consistency of the theory. It is important to notice that classical and quantum field equations for the BF theory coincide [74], hence all our comments extend to the quantum theory.

The integrable system we obtained in this work was obtained as the stationary points of an action formulated on a two-manifold $M$ where no metric has been incorporated. They are valid of a wide range of two-manifolds, it is a system of partial differential equations on a chart of the two-manifold which should be extended to all the two-manifold. The fundamental group of the two-manifold may be trivial or not, so one interesting question to be analyzed is: What are the soliton solutions of the system in the case of a two-manifold with a non-trivial fundamental group? This is a first step we should follow in order to analyze the space of flat connections on the two-manifold. In the case of a simply connected two-manifold an interesting question is: What is the interpretation of the sequence of infinite conserved quantities in terms of the elevation of the flat connection to the associated principal bundle with its ghost partner?

An extension of the present analysis will be to consider the interaction of the BF-theory with point sources with non-Abelian charges [75]. Explicit solutions of the integrable system introduced in this work will provide an explicit construction of such models. In this case the flat connection interacts with charged point sources and has been related to two-dimensional gravity, relevant as toy model of Quantum Gravity and String theory [52, 53, 76–81].

Finally, if we consider a pseudo-Riemannian manifold, where $t$ is a real time, we may study the topological charges associated to the system, where now the initial and final states correspond to the pair of $(u, C^+)$ fields. This is the correct pair of fields to be considered in the context of a gauge theory.
Data availability statement

No new data were created or analysed in this study.

Conflict of interest

The authors state that there is no conflict of interest.

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