Finite–Volume Form Factors in Semiclassical Approximation

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Abstract

A semiclassical approach is used to obtain Lorentz covariant expressions for the form factors between the kink states of a quantum field theory with degenerate vacua. Implemented on a cylinder geometry it provides an estimate of the spectral representation of correlation functions in a finite volume. Illustrative examples of the applicability of the method are provided by the Sine-Gordon and the broken $\phi^4$ theories in 1+1 dimensions.
1 Introduction

The problem of understanding quantum field theories (QFT) on a finite volume is of great importance both for theoretical interests and for many realistic applications. In particular, it is crucial to explore the possibility of defining a "form factor representation” for the correlation functions, in analogy to the infinite volume case [1]. There are reasons to expect, in fact, a fast convergent behaviour of these series also for finite volume correlators, as it happens in infinite volume (see, for instance [2]). If this would be indeed the case, accurate estimates of finite volume correlators and other related physical quantities, could be obtained by just using few exact terms of their spectral representations, having consequently a great simplification of the problem. This observation makes clear that it is worth pursuing the research on this topic and looking, in particular, for an efficient scheme of approximation.

It is well known that finite volume quantum field theories are also related to field theories at finite temperature. In 1+1 space-time dimensions, this gives the possibility of viewing a QFT defined on a finite volume of a cylinder geometry in the alternative picture, in which the space variable is infinite and the (euclidean) time variable is instead compactified on a circle of radius $R = 1/T$, where $T$ is the temperature of the system (the so-called Matsubara imaginary time formalism). In the case of integrable theories, promising exact results have been obtained in [3], thanks to the Thermodynamic Bethe Ansatz (TBA) technique, which relates the thermodynamical properties of the system to its exact $S$-matrix [4]. As shown in [3], the finite temperature correlation functions can be expressed in terms of the usual form factors, relative to the infinite volume Hilbert space, dressed with the so-called "filling-fractions”, which encode the thermodynamics of the system. Within this approach, it has been possible to fully understand the finite temperature one-point functions, while a further analysis seems still necessary for the two-point and higher correlation functions.

Coming back to the picture where the time evolution is the usual one while the space variable is compactified, the form factors have to be seen as the matrix elements of local fields between eigenstates of the finite volume hamiltonian. In this case there is still a poor understanding of the correlation functions behaviour, even in the integrable cases. It must be stressed that the reason is not only the present undeveloped and unsatisfactory theory for computing finite volume matrix elements. In fact, contrary to the infinite volume situation, the only knowledge of the form factors is not enough to obtain the correlators in a finite volume since for their computation the energy eigenvalues of the finite volume hamiltonian are also needed. For integrable models, these extra data may be obtained along the
lines discussed in [5], while for non–integrable theories one has necessarily to rely on numerical methods, as the one proposed in [6] for instance.

Leaving apart the above problem of energy eigenvalues and concerning instead about the status of the form factors in a finite volume, there are so far only semiclassical results relative to a conformal theory [7] as well as exact calculations relative to the Ising model [8] (for a Bethe Ansatz approach see, however, [9]). Although these findings are very interesting, the techniques employed in the above papers are however strictly related either to the specific integrable structure of the considered models or to the free nature of the Majorana fermion field of the Ising model. In this paper we are going to use, instead, a semiclassical approach which does not require integrability and it may be then of more general applicability, of course within its range of approximation. As shown below, this approach gives in particular the possibility of checking some of the results of our analysis against the exact known quantities in the case of integrable theories in infinite volume, while it permits to have new predictions on the semiclassical regime of the non–integrable ones. As illustrative examples of this method, we will present its application to two significant models: the integrable QFT of the Sine-Gordon model

$$V(\phi) = \frac{m^2}{\beta^2}(1 - \cos \beta \phi) ,$$

(1.1)

and the non–integrable QFT with a $\phi^4$ interaction in the broken $Z_2$ symmetry phase

$$V(\phi) = \frac{\lambda}{4} \phi^4 - \frac{m^2}{2} \phi^2 + \frac{m^4}{4\lambda} .$$

(1.2)

Both these theories display degenerate minima, that can be chosen as the constant classical solution representing the vacuum state around which one decides to quantize the theory. However, the most interesting classical backgrounds are the kink–type ones, which interpolate between degenerate minima and give rise to a non trivial quantization scheme. Our aim is to define this kind of backgrounds in finite volume and to estimate the relative form factors in an appropriate limit. At the same time, doing so, we will have the possibility of better exploring the properties of the same form factors in the infinite volume case.

2 Classical solutions and form factors

The semiclassical quantization of a field theory defined by a potential $V(\phi)$ is based on the identification of a classical background $\phi_{cl}(x)$ which satisfies the equation of motion

$$\partial_{\mu} \partial^{\mu} \phi_{cl} + V'(\phi_{cl}) = 0 .$$

(2.3)
In the infinite volume, this can be performed with various well established techniques, like the path integral formalism \cite{10} or the solution of the field equations in classical background \cite{11}, usually called the DHN method (for a systematic review, see \cite{12}).

The procedure is particularly simple and interesting if one considers classical field solutions $\phi_{cl}(x)$ in 1+1 dimensions which are static ”kink” configurations interpolating between degenerate minima of the potential, and whose quantization gives rise to a particle-like spectrum. The kink solutions are obtained, firstly, by integrating the first order equation related to (2.3)

$$\frac{1}{2} \left( \frac{\partial \phi_{cl}}{\partial x} \right)^2 = V(\phi_{cl}) + C ,$$ \hspace{1cm} (2.4)

and, secondly, imposing that $\phi_{cl}(x)$ reaches two different minima of the potential as $x \to \pm \infty$. In the infinite volume case, these boundary conditions correspond to fixing the constant of integration $C$ equal to zero. As we will show below, finite volume case can be described, on the contrary, by a non–vanishing value of $C$ which can be directly related to the size of the system. In the following we will be interested to discuss QFT defined on a cylinder geometry where the time variable is infinite while the space one is compactified on a circle of circumference $L$, with twisted boundary conditions, i.e.

$$\phi(x + L) = 2\pi - \phi(x) \quad \text{for Sine-Gordon},$$

$$\phi(x + L) = -\phi(x) \quad \text{for } \phi^4 \text{ theory}$$

A very interesting result, due to Goldstone and Jackiw \cite{13} and well explained in \cite{12}, is that the classical background $\phi_{cl}(x)$ has the quantum meaning of Fourier transform of the form factor of the basic field $\phi(x)$ between kink states\footnote{The classical solution can also be directly seen as the matrix element of $\phi$ between asymptotic states in the space coordinates representation: $<x_2|\phi(0)|x_1>$ = $\delta(x_1 - x_2) \phi_{cl}(x_1)$.}. The technique to derive this result relies on the basic hypothesis that the kink momentum is very small compared to its mass, which is indeed inversely proportional to the coupling constant, considered small in the semiclassical regime. Calling $p_1$ and $p_2$ the momenta of the in and out one-kink states, at the leading order in the coupling constant one obtains

$$<p_2|\phi(0)|p_1> = \int da \ e^{i(p_1 - p_2)a} \phi_{cl}(a) .$$ \hspace{1cm} (2.5)

This important result has, unfortunately, two serious drawbacks: expressing the form factor as a function of the difference of momenta, Lorenz covariance is lost and
moreover, the antisymmetry under the interchange of momenta makes problematic any attempt to go in the crossed channel and obtain the matrix element between the vacuum and a kink–antikink state.

In order to overcome these problems, we need to refine the method proposed in \cite{13}. This can be done by using the rapidity variable $\theta$ of the kink (and considering it as very small), instead of the momentum. For example, in the $\phi^4$ theory \cite{12}, where the kink mass $M$ is of order $1/\lambda$, we will work under the hypothesis that $\theta$ is of order $\lambda$. In this way we get consistently

$$E \equiv M \cosh \theta \simeq M, \quad p \equiv M \sinh \theta \simeq M \theta \ll M.$$  \hfill (2.6)

We can now define the form factor between kink states as the Fourier transform with respect to the Lorentz invariant rapidity difference $\theta \equiv \theta_1 - \theta_2$:

$$< p_1|\phi(0)|p_2> \equiv f(\theta) \equiv M \int da e^{iM\theta a} \hat{f}(a),$$  \hfill (2.7)

with the inverse Fourier transform defined as

$$\hat{f}(a) \equiv \int \frac{d\theta}{2\pi} e^{-iM\theta a} f(\theta).$$  \hfill (2.8)

Following the same procedure used by Goldstone and Jackiw \cite{13}, i.e. starting from the Heisenberg equation of motion for the field $\phi(x,t)$

$$\left(\partial_t^2 - \partial_x^2\right) \phi(x,t) = -V'[\phi(x,t)],$$  \hfill (2.9)

and taking the matrix elements of both sides

$$\left[-(p_1 - p_2)_{\mu}(p_1 - p_2)_{\mu}\right] e^{-i(p_1 - p_2)v_{\mu}} < p_1|\phi(0)|p_2> =$$

$$-e^{-i(p_1 - p_2)v_{\mu}} < p_1|V'[\phi(0)]|p_2>,$$

it is easy to show that, at leading order in $\lambda$, the function $\hat{f}(a)$ obeys the same differential equation satisfied by the kink solution, i.e.

$$\frac{d^2}{da^2} \hat{f}(a) = V'[\hat{f}(a)].$$  \hfill (2.11)

This means that we can take $\hat{f}(a)$ to be equal to $\phi_{cl}(a)$, i.e. $\hat{f}(a) = \phi_{cl}(a)$ and adjusting its boundary conditions by an appropriate choice for the value of the constant $C$ in eq. (2.4).

With the above considerations, it is now possible to express the crossed channel form factor through the variable transformation $\theta \rightarrow i\pi - \theta$:

$$F_2(\theta) \equiv <0|\phi(0)|p_1, \bar{p}_2> = f(i\pi - \theta).$$  \hfill (2.12)
The analysis of this quantity in infinite volume allows us, in particular, to get information about the spectrum of the theory. Its dynamical poles, in fact, located at $\theta^* = i(\pi - u)$ with $0 < u < \pi$, coincide with the poles of the kink–antikink $S$-matrix [1], and the relative bound states masses can be then expressed as

$$m_{(b)} = 2M \sin \frac{u}{2}.$$

(2.13)

It is worth to note that this procedure for extracting the semiclassical bound states masses is remarkably simpler than the standard DHN method of quantizing the corresponding classical backgrounds, because in general these solutions depend also on time and have a much more complicated structure than the kink ones. Moreover, in non–integrable theories these backgrounds could even not exist as exact solutions of the field equations: this happens for example in the $\phi^4$ theory, where the DHN quantization has been performed on some approximate backgrounds [11].

Once the matrix elements (2.12) are known, one can estimate the leading behaviour in $\lambda$ of the spectral function in a regime of the momenta dictated by our assumption of small kink rapidity. In infinite volume the spectral function $\rho(p^2)$ is defined as

$$\langle 0|\phi(x)\phi(0)|0 \rangle \equiv \int \frac{d^2p}{(2\pi)^2} \rho(p^2) e^{ip\cdot x},$$

(2.14)

and has the form factor expansion

$$\rho(p^2) = 2\pi \sum_n \frac{1}{n!} \int d\Omega_1...d\Omega_n \delta(p^0 - E_1... - E_n) \delta(p^1 - p_1... - p_n) |\langle 0|\phi(0)|n \rangle|^2,$$

(2.15)

with $d\Omega \equiv \frac{dp}{2\pi 2E} = \frac{d\theta}{4\pi}$. The delta functions in the above expression make meaningful the use of our form factors, derived in the small $\theta$ approximation, only if we consider a regime in which $p^0 \simeq M$ and $p^1 \ll M$ and, from now on, we will always understand this restriction. The leading $O(1/\lambda)$ contribution to the spectral function, denoted in the following by $\hat{\rho}(p^2)$, is given by the trivial vacuum term plus the kink-antikink contribution:

$$\hat{\rho}(p^2) = 2\pi \delta(p^0)\delta(p^1)|\langle 0|\phi(0)|0 \rangle|^2 + \frac{\pi}{4} \frac{\delta \left(\frac{p^0}{M} - 2 \right)}{M^2} \int \frac{d\theta_1}{2\pi} \left|F_2 \left(2\theta_1 + i\pi - \frac{p^1}{M}\right)\right|^2,$$

(2.16)

\footnote{About the orders in $\lambda$ of the various form factors, we refer to the complete discussion in [12, 13]. Our formalism is slightly different because the $M$ factor in front of (2.7), which is a natural consequence of considering the rapidity as the basic variable, increases by $1/\lambda$ the order of all form factors in the kink sector with respect to [13]; however, since $\theta \simeq O(\lambda)$, in the final expression for the spectral function all orders recombine consistently.}
where the range of integration of the above quantity is of order $p^1/M$ (note that, being $p^1/M \ll 1$, the integral can be roughly estimated by evaluating $|F_2|^2$ at $\theta_1 = 0$: this is what we will do in the next Sections).

The application of this procedure to the finite volume case is straightforward, thanks to the possibility of choosing $\hat{f}(a)$ as a solution of eq. (2.14) with any constant $C$. As explicitly shown by the examples discussed in the next sections, this is equivalent to define a kink solution configuration on a finite volume, with the constant $C$ directly related to the size of the system. We have now to consider the matrix elements of $\phi(0)$ between two eigenstates $|p_{n_1}\rangle$ and $|p_{n_2}\rangle$ of the finite volume hamiltonian. These states can be naturally labelled with the so-called ”quasi-momentum” variable $p_n$, which corresponds to the eigenvalues of the translation operator on the cylinder (multiples of $\pi/L$), and appears in the space dependent part of eq. (2.10) in the case of finite volume. The TBA equations [21], valid for large $L$, are exactly a relation between this variable and the free momentum $p^\infty$ of the infinite volume asymptotic states, through a phase shift $\delta(p^\infty)$ which encodes the information about the interaction:

$$p^\infty_n + \delta(p^\infty_n) = \frac{2n\pi}{L} \equiv p_n .$$  \hspace{1cm} (2.17)

Defining $\theta_n$ as the ”quasi-rapidity” of the kink states by

$$p_n = M(L) \sinh \theta_n \simeq M(L)\theta_n ,$$  \hspace{1cm} (2.18)

we can now write the form factor at a finite volume by replacing the Fourier integral transform with a Fourier series expansion:

$$f(\theta_n) \equiv M(L) \int_{-L/2}^{L/2} da \, e^{iM(L)\theta_n a} \hat{f}(a) ,$$  \hspace{1cm} (2.19)

$$\hat{f}(a) \equiv \frac{1}{LM(L)} \sum_{n=-\infty}^{\infty} e^{-iM(L)\theta_n a} f(\theta_n) ,$$  \hspace{1cm} (2.20)

where

$$M(L)\theta_n \simeq p_{n_1} - p_{n_2} = \frac{(2n_1 - 1)\pi}{L} - \frac{(2n_2 - 1)\pi}{L} \equiv \frac{2n\pi}{L} .$$  \hspace{1cm} (2.21)

Since the energy eigenvalues of the finite volume hamiltonian cannot be related to the quasi-rapidity as in eq. (2.6), in principle we are not allowed to express the crossed channel form factor $F_2(\theta)$ via the change of variable $\theta \rightarrow i\pi - \theta$. However, it is easy to show that in our regime of approximations the deviations of the kink
energy from (2.6) are of higher order in the coupling and can be neglected at this stage. 

On the cylinder, the spectral function can be expressed as a series expansion on the form factors:

\[
\rho(E_k, p_k) = 2\pi \sum_n \frac{1}{n!} \frac{1}{(2L)^n} \sum_{k_1, \ldots, k_n} \frac{1}{E_{k_1} E_{k_2} \ldots E_{k_n}} \delta(E_k - E_{k_1} \ldots - E_{k_n}) \delta(p_k - p_{k_1} \ldots - p_{k_n}) \times \\
\rho(E_k, p_k) = 2\pi \sum_n \frac{1}{n!} (2L)^n \sum_{k_1, \ldots, k_n} \frac{1}{E_{k_1} E_{k_2} \ldots E_{k_n}} \delta(E_k - E_{k_1} \ldots - E_{k_n}) \delta(p_k - p_{k_1} \ldots - p_{k_n}) \times \\
\times |<0|\phi(0)|n>|^2,
\]

(2.22)

where \(p_{k_i}\) are the quasi-momenta of the intermediate states, and \(E_{k_i}\) are the finite volume energy eigenvalues, to be determined by other means (see the comment in Sect. I). In our semiclassical regime, however, this is not necessary: in fact, in order to evaluate the leading contribution \(\hat{\rho}(E_k, p_k)\), we can consistently approximate the kink energies with their classical values (of order \(1/\lambda\)), which can be exactly computed as a function of the volume. We then have

\[
\hat{\rho}(E_k, p_k) = 2\pi \delta(E_k) \delta(p_k)|<0|\phi(0)|0>|^2 + \frac{\pi}{4} \delta \left( \frac{E_k}{M} - 2 \right) \sum_{\theta_{k_1}} |F_2\left(2\theta_{k_1} + i\pi - \frac{p_k}{M}\right)|^2.
\]

(2.23)

As in the infinite volume case, the consistency of the semi-classical approximation selects as the relevant values of the above series those with \(\theta_{k} \simeq 0\) and therefore it can be roughly estimated by simply evaluating \(|F_2|^2\) at \(\theta_{k_1} = 0\).

It is now interesting to apply these general considerations to the analysis of the form factors of the fundamental field \(\phi(x)\) both for an integrable and a non-integrable QFT.

### 3 Sine-Gordon model

#### 3.1 Infinite volume

The Sine-Gordon model, defined in (1.1), is an integrable quantum field theory, for which the infinite volume form factors are exactly known [1]. The (anti)soliton background is given by

\[
\phi_{cl}(x) = \frac{4}{\beta} \text{arctan}(e^{\pm mx})\ .
\]

(3.24)

Its classical energy is \(E_{cl} = 8\frac{m}{\beta}\), and the first quantum corrections are known to be of higher order in \(\beta^2\) [11]. Hence we can consistently approximate the mass \(M_{\infty}\) with this value and assume that the (anti)soliton rapidity will be of order \(\beta^2\). The
semi-classical form factor \( F_2(\theta) \) is explicitly given by

\[
f(\theta) = \frac{4M_\infty}{\beta} \int_{-\infty}^{\infty} da \, e^{i(M_\infty \theta)a} \arctan(e^{ma}) = \frac{2\pi i}{\beta} \frac{1}{\theta \cosh \left[ \frac{4\pi}{\beta^2} \theta \right]} + \frac{2\pi^2}{\beta^3} \delta \left( \frac{\theta}{\beta^2} \right). \tag{3.25}\]

It is an interesting check to see whether our formulation of the semi-classical form factor in terms of the rapidity variable, i.e. the expression \( F_2(\theta) \), matches with the semi-classical limit of the exact one. In doing this check, the only thing to take into account is that, in the definition of the exact form factor of the fundamental field \( \phi(x) \), the asymptotic two-particle state is actually given by the antisymmetric combination of soliton and antisoliton. Since at our level the form factor between antisoliton states is simply

\[
< \bar{p}_1 | \phi(0) | \bar{p}_2 > = \frac{4M_\infty}{\beta} \int_{-\infty}^{\infty} da \, e^{i(M_\infty \theta)a} \arctan(e^{-ma}) = f(-\theta), \tag{3.26}\]

we finally obtain

\[
F_2(\theta) = \frac{4\pi i}{\beta} \frac{1}{(i\pi - \theta) \cosh \left[ \frac{4\pi}{\beta^2} (i\pi - \theta) \right]}, \tag{3.27}\]

which indeed coincides with the exact result in the regime \( i\pi - \theta \simeq O(\beta^2) \). Furthermore, we can also check that the dynamical poles of this quantity, located at

\[
\theta_n = i\pi \left[ 1 - \frac{\beta^2}{8\pi} (2n + 1) \right], \quad -\frac{1}{2} < n < -\frac{1}{2} + \frac{4\pi}{\beta^2}, \tag{3.28}\]

consistently reproduce the odd part of the well-known semiclassical breathers spectrum \( \Pi \)

\[
m_b^{(2n+1)} = 2M_\infty \sin \left[ \frac{\beta^2}{16} (2n + 1) \right] = (2n + 1) m \left[ 1 - \frac{(2n + 1)^2}{3 \times 8^3} \beta^4 + \ldots \right] \tag{3.29}\]

Since in the vacuum sector \(< 0 | \phi | 0 > = 0\), in this model the \( 1/\beta^2 \) leading contribution to the spectral function takes the form:

\[
\hat{\rho}(p^2) = 4\pi^3 \delta \left( \frac{p^2}{M} - 2 \right) \frac{1}{\beta^2 (p^1)^2 \cosh^2 \left[ \frac{2\pi n^1}{m} \right]}, \tag{3.30}\]

### 3.2 Finite volume

In order to consider the effects of a cylindrical geometry, let’s integrate eq. (2.4) with a nonzero constant \( C \). Rescaling for convenience the variables as

\[
\tilde{\phi} = \beta \phi, \quad \tilde{x} = mx, \quad \tilde{C} = \frac{\beta^2}{m^2} C, \tag{3.31}\]
a solution of eq. (2.4), with $-2 < \bar{C} < 0$, can be expressed as

$$\bar{\phi}_{cl}(\bar{x}) = 2 \arccos \left[ \frac{\sqrt{\bar{C} + 2}}{2} \text{sn}(\pm \bar{x}, k^2) \right] , \quad (3.32)$$

where $\text{sn}(\bar{x}, k^2)$ is the Jacobi elliptic function with modulus $k^2 = \frac{\bar{C} + 2}{2}$ (for its properties, see [14]). The plot of this function is drawn in Fig. 1. For a given value of $\bar{C}$, this solution oscillates with a period $4K(k^2)$ between $\bar{\phi}_0$ and $2\pi - \bar{\phi}_0$, where $\bar{\phi}_0$ is defined by the condition $V(\bar{\phi}_0) = -\bar{C}$, and $K(k^2)$ is the complete elliptic integral of the first kind.

The solution (3.32) has been proposed in Ref. [15] as a model of a crystal of solitons and antisolitons in the sine-Gordon theory in infinite volume. In our finite volume case, the solution (3.32) has to be interpreted, instead, as a single (anti)soliton defined on a cylinder of circumference

$$L = \frac{1}{m} 2K \left( \frac{\bar{C} + 2}{2} \right) . \quad (3.33)$$

Within this interpretation, the periodic oscillations of the solution represent the soliton circling around the cylinder. Eq. (3.33) is the explicit relation between the size of the system and the integration constant $\bar{C}$; one can consistently recover the

\[3\] It is worth mentioning the existence of some impressive old papers [16] where the basic ideas discussed in [15] were already present.
infinite volume limit for $C \to 0$: in this limit $L$ goes to infinity and the function $(3.32)$ goes to the standard (anti)soliton solution $(3.24)$.

A strong motivation for the choice of this function as the (anti)soliton background on the cylinder can be obtained by analyzing the so-called ”classical energy per kink” computed in [15]:

$$E_{cl}(L) = \frac{L}{2} \int_{-L/2}^{L/2} \left[ \frac{1}{2} \left( \frac{\partial \phi_{cl}}{\partial x} \right)^2 + \frac{m^2}{\beta^2} (1 - \cos \beta \phi_{cl}) \right] = 8 \frac{m}{\beta^2} \left[ E(k^2) - \frac{1}{2} (1 - k^2) K(k^2) \right],$$

(3.34)

where $E(k^2)$ is the complete elliptic integral of the second kind. In the $L \to \infty$ limit (which corresponds to $k' \to 0$, with $(k')^2 \equiv 1 - k^2 = -\frac{C}{2}$), $E_{cl}(L)$ approaches exponentially the value $M_{\infty}$; the expansions of $E$ and $K$ for small $k'$ [14], compared with

$$e^{-mL} = \frac{1}{16} (k')^2 + \cdots,$$

lead to the following expansion for large $L$ of the classical energy

$$E_{cl}(L) = M_{\infty} - 32 \frac{m}{\beta^2} e^{-mL} + O \left( e^{-2mL} \right).$$

(3.35)

By using the theory of finite volume corrections discussed in [17, 18], the behaviour $(3.35)$ can be put in correspondence with the scattering data of the QFT in infinite volume$^4$:

$$M(L) - M_{\infty} = -\frac{1}{8m_b M_{\infty}^2} g_{kkb}^2 e^{-m_b L} + \cdots,$$

(3.36)

where $g_{kkb}$ is the 3-particle on-shell coupling of kink, antikink and elementary boson, whose mass in infinite volume is $m_b = m$. Using the known expression for the Sine-Gordon kink-breather $S$-matrix [19] and evaluating the limit $\beta \to 0$ of its residue, we exactly get the coefficient reported in $(3.35)$.

We can now write the finite volume form factor $(2.19)$ in terms of the antisoliton background $(3.32)$:

$$f(-\theta_n) = \frac{2M}{\beta} \int_{-L/2}^{L/2} da e^{i M \theta_n a} \text{arccos} \left[ \sqrt{\frac{C + 2}{2}} \text{sn}(ma) \right] =$$

$$= -\frac{2}{\beta \theta_n} e^{i M \theta_n \frac{L}{2} \log(k + ik')} - e^{-i M \theta_n \frac{L}{2} \log(-k + ik')} - \frac{2\pi i}{\beta} \frac{1}{\theta_n} \cosh \left[ \frac{K}{m} M \theta_n \right],$$

(3.37)

$^4$In eq. (3.36), we have only written the term that is relevant in our semiclassical limit, in which the leading contribution to the mass is given by $E_{cl}(L)$.
where \( k' = \sqrt{1 - k^2} \) and \( K'(k^2) = K(k'^2) \). In order to obtain this result one has to use the relation

\[
\arccos [k \text{sn}(ma)] = \frac{1}{i} \log [k \text{sn}(ma) + i \text{dn}(ma)] ,
\]

and, after an integration by parts, finally compare the inverse Fourier transform (2.20) with the expansion (1.3)

\[
\text{cn}(ma) = \frac{2\pi}{k} \frac{1}{mL} \sum_{n=1}^{\infty} \frac{\cos \left[ \frac{L}{m} K'(i\pi - \theta_n) \right]}{\cosh \left[ \frac{(2n-1)\pi a}{L} \right]} .
\]

The form factor (3.37) has the correct IR limit\(^5\), and leads to the following expressions for \( F_2(\theta) \) and for the spectral function:

\[
F_2(\theta_n) = \frac{4\pi i}{\beta(i\pi - \theta_n)} \left\{ \frac{1}{\cosh \left[ \frac{M}{m} K'(i\pi - \theta_n) \right]} + \left( -1 + \frac{2}{\pi} \arctan \frac{k'}{k} \right) \cos \left[ M(i\pi - \theta_n) \frac{L}{2} \right] \right\} ,
\]

\[
\hat{\rho}(E_n, p_n) = 4\pi^3 \delta \left( \frac{E_n}{M} - 2 \right) \frac{1}{\beta^2(p_n)^2} \left\{ \frac{1}{\cosh \left[ \frac{K'}{m} p_n \right]} + \left( -1 + \frac{2}{\pi} \arctan \frac{k'}{k} \right) \cos \left[ p_n \frac{L}{2} \right] \right\}^2 .
\]

Note that the finite volume dependence of both the form factor (3.40) and the spectral function (3.41) is not restricted to the second term only. The \( M(L)K'(k^2) \) factor in the first term carries the main \( L \)-dependence, although it is not manifest but implicitly defined by eq. (3.33).

### 4 \( \phi^4 \) field theory in the broken symmetry phase

The semiclassical analysis performed on the Sine-Gordon model can be repeated very similarly for the quantum field theory defined by the potential (1.2). There is, however, the important conceptual difference that this QFT is non–integrable: as we are going to show, this gives us the possibility of estimating quantities of this quantum field theory which were unknown even in the infinite volume case.

\(^5\)The function \( \frac{e^{-ixL/2}}{x} \) can be shown to tend to \(-i\pi \delta(x)\) in the distributional sense for \( L \to \infty \), and in the same way one can show that \( \frac{\cos(xL/2)}{x^2} \) tends to zero.
4.1 Infinite volume

The standard (anti)kink background is given in this case by

$$\phi_{cl}(x) = (\pm) \frac{m}{\sqrt{\lambda}} \tanh \left( \frac{mx}{\sqrt{2}} \right),$$

(4.42)

with classical energy $M_\infty = \frac{2\sqrt{2}}{3} m^3$.

With our formulation in terms of the rapidity, we can write an unambiguous Lorentz covariant expression for the form factor (2.7)

$$< p_2 | \phi(0) | p_1 > = M_\infty \frac{m}{\sqrt{\lambda}} \int_{-\infty}^{\infty} da e^{iM_\infty \theta a} \tanh \left( \frac{ma}{\sqrt{2}} \right) = \frac{4}{3} i \pi \left( \frac{m}{\sqrt{\lambda}} \right)^3 \frac{1}{\sinh \left( \frac{2\pi}{3} \frac{m^2}{\lambda} \theta \right)}.$$

(4.43)

It is of great interest to analyze in this case the dynamical poles of $F_2(\theta)$ for extracting information about the spectrum of this theory. They are located at

$$\theta_n = i \pi \left[ 1 - \frac{3}{2\pi} \frac{\lambda}{m^2} n \right], \quad 0 < n < \frac{2\pi}{3} \frac{m^2}{\lambda},$$

(4.44)

and the corresponding bound states masses are given by

$$m_{b}^{(n)} = 2M_\infty \sin \left[ \frac{3}{4} \frac{\lambda}{m^2} n \right] = n \sqrt{2} m \left[ 1 - \frac{3}{32} \frac{\lambda^2}{m^4} n^2 + \ldots \right].$$

(4.45)

Note that the leading term is consistently given by multiples of $\sqrt{2}m$, which is the known mass of the elementary boson. Contrary to the Sine-Gordon model, we now have all integer multiples of this mass and not only the odd ones: this is because we are in the broken phase of the theory, where the invariance under $\phi \to -\phi$ is lost. Furthermore, this spectrum exactly coincides with the one derived in [11] by building approximate classical solutions to represent the ”breathers”.

Another important information can be extracted from the residue of $F_2(\theta)$ on the pole corresponding to the lightest bound state $b^{(1)}$. This quantity, indeed, has to be proportional to the one-particle form factor $< 0 | \phi | b^{(1)} >$ through the semiclassical 3-particle on-shell coupling of kink, antikink and elementary boson $g_{kkb}$, a quantity so far unknown in this non integrable theory:

$$\text{Res}_{\theta=\theta_1} F_2(\theta) = i \frac{g_{kkb}}{2\sqrt{2}M_\infty m_{b}^{(1)}} < 0 | \phi | b^{(1)} >.$$

(4.46)

$^6$This has to be contrasted with the tentative covariant extrapolation discussed in [13].
Since the one-particle form factor takes the constant value $1/\sqrt{2}$, at leading order in the coupling we get
\[ g_{kkb} = \frac{32 m^5}{3 \lambda^{3/2}}. \]  
(4.47)

Finally, the $1/\lambda$ leading contribution of the spectral function is given in this case by
\[ \hat{\rho}(p^2) = \frac{2\pi}{\lambda} \delta \left( \frac{p^0}{m} \right) \delta \left( \frac{p^1}{m} \right) + \frac{\pi^3}{2\lambda} \delta \left( \frac{p^0}{M} - 2 \right) \frac{1}{\sinh^2 \left[ \frac{\pi}{\sqrt{2}} \frac{p^1}{m} \right]} . \]  
(4.48)

### 4.2 Finite volume

Our proposal to describe the finite volume (anti)kink is to use the following solution of the differential equation (2.4)
\[ \bar{\phi}_{cl}(\bar{x}) = (\pm) \sqrt{2 - \bar{\phi}_0^2} \text{sn} \left( \frac{\bar{\phi}_0}{\sqrt{2}} \bar{x}, k^2 \right) , \]  
(4.49)

with $k^2 = 2 - \frac{1}{\bar{\phi}_0^2}$, $V(\phi_0) = -C$ and $1 < \bar{\phi}_0 < \sqrt{2}$, where we have rescaled the variables as
\[ \bar{\phi} = \frac{\sqrt{\lambda}}{m} \phi , \quad \bar{x} = m x . \]  
(4.50)

This function oscillates with period
\[ 2\bar{L} = 4 \frac{\sqrt{\bar{x}}}{\bar{\phi}_0} K(k^2) \]

between the two values $-\sqrt{2 - \bar{\phi}_0^2}$ and $\sqrt{2 - \bar{\phi}_0^2}$, and satisfies the antiperiodic boundary condition $\phi(x + L) = -\phi(x)$. Moreover, it goes to the standard (anti)kink solution [4.42] for $\bar{\phi}_0 \to 1$, i.e. when $\bar{L} \to \infty$.

For the "classical energy per kink" in this case we find
\[ E_{cl}(L) = \frac{m^3}{\lambda} \frac{\sqrt{2}}{\bar{\phi}_0} \left( -\frac{1}{6} \bar{\phi}_0^4 K(k^2) + \frac{1}{3} \bar{\phi}_0^2 \left[ 2E(k^2) - K(k^2) \right] + \frac{K(k^2)}{2} \right) , \]  
(4.51)

which for $L \to \infty$ indeed reproduces $M_{\infty}$. Taking into account the $k \to 1$ $(k' \to 0)$ expansions of $E$ and $K$ [41] and noting that
\[ e^{-\sqrt{2}mL} = \frac{1}{256} (k')^4 + \cdots , \]
we derive the following asymptotic expansion of $E_{cl}$ for large $L$:
\[ E_{cl}(L) = M_{\infty} - 8\sqrt{2} \frac{m^3}{\lambda} e^{-\sqrt{2}mL} + O \left( e^{-2\sqrt{2}mL} \right) , \]  
(4.52)

(note that in this theory the mass of the elementary boson is $m_b = \sqrt{2}m$). By using eq. (3.36), it is easy to see that this expansion exactly reproduces the value [4.47].
for the 3-particle coupling $g_{k\bar{k}b}$, previously obtained by looking at the residue of the form factor in infinite volume.

Comparing the inverse Fourier transform [2,20] with the expansion [14]

$$\text{sn}(u) = \frac{\pi}{kK} \sum_{n=1}^{\infty} \frac{\sin \left[ \frac{(2n-1)\pi}{2K} u \right]}{\sinh \left[ \frac{(2n-1)\pi}{2K} K' \right]}, \quad (4.53)$$

we obtain for the form factor in a finite volume the following expression

$$F_2(\theta_n) = M \frac{m}{\sqrt{\lambda}} \frac{\sqrt{2 - \tilde{\phi}_0^2}}{L/2} \int_{-L/2}^{L/2} da e^{i M(i\pi - \theta_n)a} \text{sn} \left( \frac{\tilde{\phi}_0}{\sqrt{2}} ma \right) =$$

$$= i\pi \frac{\sqrt{2} M}{\sqrt{\lambda}} \frac{1}{\sinh \left[ \frac{\sqrt{2} m\tilde{\phi}_0}{m\phi_0} K'M(i\pi - \theta_n) \right]}, \quad (4.54)$$

so that, the $1/\lambda$ leading contribution to the spectral function is given by

$$\hat{\rho}(E_n, p_n) = \frac{2\pi}{\lambda} \delta \left( E_n/m \right) \delta \left( p_n/m \right) + \frac{\pi^3}{2\lambda} \delta \left( E_n - 2 \right) \frac{1}{\sinh^2 \left[ \frac{\sqrt{2} m\tilde{\phi}_0}{m\phi_0} K'p_n \right]} \quad (4.55)$$

Again, as in the Sine-Gordon case, the finite volume dependence of these quantities comes from the factor $M(L)K'(k^2)$, where $M(L)$ is the kink mass given by eq. (4.51).

## 5 Further developments and open problems

Although the form factors provide an important piece of information about the quantum theory, a more complete understanding of the semi–classical behaviour of Sine-Gordon and $\phi^4$ models requires an extension of the DHN procedure to finite volume. This would permit to analyze the energy levels of the kink "particles" and to compute the quantum corrections to their masses. A detailed discussion of these topics will be presented somewhere else [20].

Furthermore, in order to systematically analyze the spectra of the proposed theories on the cylinder, it seems necessary to find the non–trivial periodic classical solutions in the vacuum sector and apply the DHN procedure to them. As a matter of fact, the identification the kink-antikink bound states masses in infinite volume from the dynamical poles of $F_2(\theta)$ is a very powerful tool, whose application to other theories is a topic interesting in itself, but cannot be directly implemented on the cylinder.

Finally, an important open problem which also deserves further investigation is the evaluation of the higher loop corrections to the semiclassical energies (masses),
form factors and Green functions. For the infinite volume case, relevant loop calculations around the non-perturbative kink solutions have been performed for the Sine-Gordon and $\phi^4$ models in the papers [21]. Whether one can extend these loop calculations to the case of finite volume (cylinder geometry), in order to estimate the loop corrections to the semiclassical scaling functions, form factors etc., remains to be seen. This is particularly interesting in the Sine-Gordon case, because it would permit to understand whether also in finite volume the semiclassical results can be related to the exact ones by a simple redefinition of the coupling.

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