Anisotropic expansion, dissipative hydrodynamics from kinetic theory

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Abstract

We consider Kasner space-time describing anisotropic three dimensional expansion of the fluid and obtain the dissipative evolution equations for shear stress tensor and energy density from kinetic theory. For this, we use the iterative solution of relativistic Boltzmann equation with relaxation time approximation. We show that our results for second and third order evolution equations reduce to those of one dimensional expansion case under suitable conditions for the anisotropic parameters in Kasner space-time.

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1 Introduction

One of the fundamental question in physics is to understand the properties of matter at extreme density and temperature in the first few microseconds after the big bang. Such a state of matter is known as Quark-Gluon-Plasma (QGP) state where the quarks and the gluons are in deconfined state. A lot of progress has been made in understanding the properties and various aspects of the evolution of the strongly coupled QGP through the heavy-ion collision experiments at RHIC [1, 2] (see also [3]) as well as in LHC (see [4] for a comprehensive review). Hydrodynamics plays an important role after the system undergoes a rapid thermalization and local thermal equilibrium is reached. Since it is difficult to solve the strongly coupled QCD, the qualitative features of the hydrodynamics regime in the evolution of QGP has been extensively studied by using AdS/CFT duality. This has provided an important framework for studying the strongly coupled dynamics in a class of superconformal field theories, in particular, \( \mathcal{N} = 4 \) super Yang-Mills theory and the corresponding gravity dual description in AdS space-time [5, 6]. In the context of heavy ion collisions, AdS/CFT correspondence has led to interesting results like computation of shear viscosity of finite temperature \( \mathcal{N} = 4 \) Supersymmetric Yang-Mills theory [7], viscosity from gravity dual description involving black holes in AdS space [8] etc. As we know, perfect local equilibrium system is described by ideal fluid dynamics. For small departures from equilibrium, the system is described by dissipative fluid dynamics. The nonrelativistic limit of hydrodynamics can be described by Navier-Stokes equations involving the energy density, pressure of the fluid as well as the shear and bulk viscosities. These equations are relevant for describing the fluid at low energy. However, in the context of ultrarelativistic heavy-ion collisions at RHIC and LHC, one needs to use relativistic hydrodynamics to study the evolution and properties of QGP describing fluid at high energy.

The microscopic dynamics describing processes characterized by \( kl_{mfp} \) (\( k \) being the momentum scale and \( l_{mfp} \) is the mean free path of the system) has been studied in great detail using the relativistic hydrodynamics setup. In the framework of gradient expansion in relativistic dissipative hydrodynamics, the zeroth order theory is described by ideal hydrodynamics. First order theory is described by relativistic Navier-Stokes equations. In
first order hydrodynamics, due to the lack of an initial value formulation, signals can be transmitted with arbitrarily high speed thereby violating causality. The system in this case is described by parabolic equations. The first order theory has been extended by Müller [9] and independently by Israel and Stewart(IS) [10] by including the second order gradient terms thereby preserving causality in the resulting relativistic hydrodynamics equations. The set of transport coefficients are extended in the second order hydrodynamics and the resulting equations become hyperbolic. However, the Müller-Israel-Stewart theory does not contain all the corrections to second order in gradient. Subsequently, the authors in refs [11, 12] obtained additional terms in the stress-energy tensor by utilising the conformal symmetries of the theory and the new transport coefficients were explicitly determined. The relaxation time (a transport coefficient in second order viscous hydrodynamics) has also been computed from the analysis of the regularity of the dual geometry [13].

In the second order Israel-Stewart (IS) theory, the equations in relativistic dissipative fluid dynamics have been obtained using the second moment of Boltzmann equation in the kinetic theory description and Grad’s 14-moment expansion [14] for the phase-space distribution function. However, 14-moment approximation used in IS theory does not provide a unique theory as it leads to multiple fluid dynamical equations with similar general structure but different transport coefficients [15, 16]. For one dimensional Bjorken expansion [17], it has been discussed that the second order IS theory has also some unphysical features like reheating of the expanding medium [18], negative pressure [19] for large viscosity and small initial expansion time; large ratio of shear viscosity to entropy density [20] etc. In order to address these problems with IS theory, higher order corrections in the dissipative hydrodynamics were considered to study their effects [21]. Though the resulting evolution equation did not contain all possible terms in the second order dissipative hydrodynamics, the solutions with higher order correction indicated better agreement with the results from kinetic transport theory.

Second-order dissipative equations have been derived from relativistic Boltzmann equation (BE) using gradient expansion of the distribution function thereby including nonlocal effects in the collision term [22]. In the above setup, while deriving the evolution equations, the authors use the definition of dissipative current directly instead of using the second
moment of BE and it is important to note that the evolution equations included all possible second order terms allowed by the symmetry. Subsequently considering the nonlocal collision term, the relativistic third order dissipative evolution equation for the shear stress tensor has been derived from kinetic theory without using Grad’s 14-moment approximation and second moment of BE [23]. In the above formalism, BE has been solved iteratively in the relaxation time approximation (RTA) for the collision term and the nonequilibrium phase space distribution function has been obtained (see also [24] for the discussion in the context of first order and second order evolution equations). The formulation of RTA for the collision term was proposed in a very interesting paper by Anderson and Witting [25] for solving the relativistic BE as a power series in the relaxation time and the transport coefficients for a single component gas were obtained and compared with the relativistic Grad’s moment method. Iterative solution method has also been useful in obtaining higher order corrections to entropy four current [26]. Hydrodynamics gradient expansion in higher orders has been discussed in ref. [27], where, third order corrections in conformal as well as nonconformal hydrodynamics of neutral fluids have been investigated. In the case of nonrelativistic systems, higher order constitutive equations from kinetic theory have been discussed earlier in ref. [28]. There has been progress in hydrodynamical formulations from various other approaches [29, 30, 31]. In this work, we consider a generalization of Bjorken’s one dimensional expansion to three dimensional anisotropic expansion of the relativistic fluid, where the local rest frame (LRF) of the anisotropically expanding fluid is described by time dependent Kasner space-time. For time dependent AdS/CFT correspondence, Kasner space-time has been studied earlier in the context of anisotropic expansion of the RHIC and LHC fireball, where explicit expressions for the hydrodynamic quantities have been obtained in first and second order relativistic viscous hydrodynamics and gravity dual description has been studied [32, 33]. Collisionless Boltzmann equation in Kasner space-time and its relation to anisotropic hydrodynamics has been discussed in ref. [34]. Though Kasner space-time is a curved space-time, Sin et al [32] have shown that under a well controlled approximation, it can be considered as the LRF of the anisotropically expanding fluid on Minkowski space-time. Here, we study the relativistic BE in RTA for the collision term. We extend the results of ref. [23] to three dimensional expansion case.
Using the iterative solution of BE for the nonequilibrium distribution function, we obtain the dissipative evolution equations for the shear stress tensor and energy density to second and third order in gradients from kinetic theory in Kasner space-time. These expressions have not been obtained before. We show that our results reduce to that of one dimensional expansion case under suitable conditions for the Kasner parameters.

The paper is organized as follows: Section 1 contains the introduction and motivation for the present study. Section 2 deals with the basic formalism for solving the relativistic BE iteratively in RTA for the collision term. We discuss the basic set-up in Minkowski space-time. In section 3, we generalize the one dimensional Bjorken expansion to three dimensional anisotropic expansion by considering Kasner space-time. Using the iterative solutions of the Boltzmann equation for the nonequilibrium distribution function, we obtain the second and third order evolution equations for the shear stress tensor and energy density in terms of Kasner parameters. We also show that our evolution equations agree with the one dimensional Bjorken expansion case in the appropriate limit of the Kasner parameters. We summarise and discuss the future perspective in section 4.

2 Kinetic theory and iterative solution of relativistic Boltzmann equation

As we know, the macroscopic state of a system in relativistics fluid dynamics is described by the energy-momentum tensor and the corresponding conservation law plays an important role in the hydrodynamic evolution of a system. The conserved energy-momentum tensor $T^{\mu\nu}$ can be decomposed as,

$$T^{\mu\nu} = \epsilon u^{\mu} u^{\nu} - P \Delta^{\mu\nu} + \pi^{\mu\nu},$$

(2.1)

where, $\epsilon$, $P$ and $\pi^{\mu\nu}$ represent the energy density, pressure and shear stress tensor respectively. The bulk viscosity vanishes for a system of massless particles and the corresponding theory is conformal. We work in the Landau frame. The projection operator $\Delta^{\mu\nu}$ defined on the space orthogonal to the fluid velocity is given by: $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu} u^{\nu}$, where the metric tensor $g^{\mu\nu} = diag(+,-,-,-)$ and $u^{\mu}$ is the fluid 4-velocity. $u^{\mu}$ is an eigen vector
of the energy-momentum tensor \( T^{\mu\nu} u_\nu = \epsilon u^\mu \) and it satisfies the following properties: \( \Delta^{\mu\nu} u_\mu = 0 = \Delta^{\mu\nu} u_\nu; \Delta^{\mu\nu} \Delta^\alpha_\nu = \Delta^{\mu\alpha} \). In the local rest frame of the fluid, the 4-velocity is given by \( u^\mu = (1, 0, 0, 0) \). Projecting the energy-momentum tensor conservation equation in the direction parallel and perpendicular to the fluid 4-velocity results in the following fundamental equations for a relativistic viscous fluid:

\[
\begin{align*}
\dot{\epsilon} + (\epsilon + P) \theta - \pi^{\mu\nu} \sigma_{\mu\nu} &= 0, \\
(\epsilon + P) \dot{u}^\alpha - \nabla^\alpha P + \Delta^\alpha_\mu \partial_\nu \pi^{\mu\nu} &= 0.
\end{align*}
\]  

(2.2)

We use the notations of ref. [23]. Here \( \dot{\epsilon} = u^\mu \partial_\mu \epsilon \) denotes the comoving derivative of the energy density, \( \theta \equiv \nabla_\mu u^\mu \) is the expansion scalar, \( \nabla^\alpha \equiv \Delta^{\mu\alpha} \partial_\mu \) denotes the space-like derivative and \( \sigma^{\mu\nu} \equiv \Delta^{(\mu} u^{\nu)} = \Delta^{\mu\nu} \nabla^\alpha u^\beta \). Symmetrization is defined by \( A(\mu B^\nu) \equiv \frac{1}{2}(A^\mu B^\nu + A^\nu B^\mu) \).

Relativistic hydrodynamics can be derived from kinetic theory where the fundamental equation is Boltzmann equation. Relativistic BE in kinetic theory is given by [35],

\[
p^\mu \partial_\mu f(x, p) = C[f](x, p)
\]  

(2.3)

where \( p^\mu \) is the particle 4-momentum, \( p^\mu = (p^0, p) \) with \( p^0 = \sqrt{p^2 + m^2} \), \( f(x, p) \) is the one particle phase space distribution function and \( C[f](x, p) \) is the collision term. Right hand side of the above equation becomes zero if the collision between particles is neglected. In the kinetic theory, the energy-momentum tensor is expressed in terms of the distribution function \( f(x, p) \) and particle 4-momentum \( p^\mu \):

\[
T^{\mu\nu} = \int dp \ p^\mu p^\nu f(x, p)
\]  

(2.4)

where \( dp \equiv g dp/[(2\pi)^3|p|] \) and \( g \) is the number of internal degrees of freedom and we are considering a system of massless particles.

In the RTA, the collision term is given by [25],

\[
C[f](x, p) = \frac{u \cdot p}{\tau_R} (f(x, p) - f_{eq}(x, p))
\]  

(2.5)
where, \( u \cdot p = u_\mu p^\mu \), \( \tau_R \) is the relaxation time, \( f_{eq}(x,p) \) is the equilibrium distribution function and the deviation \( \delta f \) from the equilibrium value is assumed to be small. We write,

\[
f(x,p) = f_{eq}(x,p) + \delta f(x,p)
\]

(2.6)

The shear stress tensor \( \pi^{\mu\nu} \) in the decomposition of \( T^{\mu\nu} \) can be calculated in terms of \( \delta f \) which is the deviation from the equilibrium distribution function. \( \pi^{\mu\nu} \) can be written as,

\[
\pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int dp \, p^{\alpha} p^{\beta} \delta f,
\]

(2.7)

where,

\[
\Delta^{\mu\nu}_{\alpha\beta} = \frac{1}{2} (\Delta^{\mu}_{\alpha} \Delta^{\nu}_{\beta} + \Delta^{\mu}_{\beta} \Delta^{\nu}_{\alpha} - \frac{2}{3} \Delta^{\mu\nu} \Delta_{\alpha\beta})
\]

(2.8)

is a traceless symmetric projection operator orthogonal to \( u^\mu \) and it satisfies the following properties: \( \Delta^{\mu\nu}_{\alpha\beta} \Delta_{\mu\nu} = 0 = \Delta^{\mu\nu}_{\alpha\beta} \Delta^{\rho\sigma}_{\alpha\beta} \), \( \Delta^{\mu\nu}_{\rho\sigma} \Delta_{\alpha\beta} = \Delta^{\mu\nu}_{\alpha\beta} \).

In order to obtain the expression for the nonequilibrium part \( \delta f \) appearing above in the expression for \( \pi^{\mu\nu} \), one solves the BE iteratively in RTA [24]. In this formalism, one writes the deviation \( \delta f \) in a gradient expansion [36]: \( \delta f = \delta f^{(1)} + \delta f^{(2)} + \delta f^{(3)} + \cdots \), where \( \delta f^{(1)} \) is first order in gradient, \( \delta f^{(2)} \), \( \delta f^{(3)} \) are second and third order in gradients respectively.

Using the expression for the collision functional in the RTA [25],

\[
C[f] = -u^\mu \frac{\delta f}{\tau_R},
\]

(2.9)

the relativistic BE has been solved iteratively [24], where,

\[
\begin{align*}
f_1 &= f_{eq} - \frac{\tau_R}{u^\mu} p^\mu \partial_\mu f_0, \\
\delta f^{(1)} &= -\frac{\tau_R}{u^\mu} p^\mu \partial_\mu f_{eq}, \\
f_2 &= f_{eq} - \frac{\tau_R}{u^\nu} p^\nu \partial_\nu f_1, \\
\delta f^{(2)} &= \frac{\tau_R}{u^\mu} p^\nu \partial_\nu \left( \frac{\tau_R}{u^\nu} \partial_\nu f_{eq} \right).
\end{align*}
\]

(2.10)

(2.11)

(2.12)
The shear stress tensor to first order in gradients can be calculated by using the expression for $\delta f^{(1)}$ given above [eqn.(2.11)] in eqn.(2.7) resulting $\pi^{\mu\nu} = 2\tau_R\beta_\pi \sigma^{\mu\nu}$, $\beta_\pi = \frac{4}{9}P$. Using the above results, one can obtain the expressions for the shear stress tensor and its evolution to higher orders in derivatives.

The above hydrodynamics set up has been discussed in Minkowski space-time. In the next section, we shall discuss the anisotropic expansion of the fluid, in which, Kasner space-time has been considered as the local rest frame of the fluid. It is important to note that though Kasner space-time is a curved space-time, Sin et al [32] have shown that under a well controlled approximation, it can be considered as the LRF of the anisotropically expanding fluid on Minkowski space-time. The classical Boltzmann equation in curved space-time involves the Christoffel symbol, which is given by,

$$
(p^\mu \partial_\mu - \Gamma^\lambda_\mu_\nu p^\mu p^\nu \partial_\lambda)f(p, x) = \mathcal{C}[f]
$$

(2.13)

The stress energy tensor is defined as,

$$
T^{\mu\nu} = \int \sqrt{-g} \frac{d^3p}{p^0} p^\mu p^\nu f(x, p)
$$

(2.14)

where, $g$ is the determinant of the metric tensor $g_{\mu\nu}$. One obtains the hydrodynamic equations by taking the moments w.r.t. to the particle momentum. Baier et al have discussed the set up in curved background in ref.[11]. Hence instead of repeating, We refer to the readers ref. [11] for the discussion in a general curved background. In the present framework, the effect of a general background space-time can be accounted for by replacing the partial derivative $\partial_\mu$ with the covariant derivative $D_\mu$ throughout. For example, we have, $\dot{u}^\alpha = u^\mu D_\mu u^\alpha$ and the covariant derivative of a vector is given by,

$$
D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu_\mu_\rho A^\rho; \quad D_\mu A^\nu = \partial_\mu A^\nu - \Gamma^\rho_\mu_\nu A^\rho
$$

(2.15)

In the next section, we use the covariant derivative for computing various expressions where we consider Kasner space-time as the local rest frame (LRF) of the fluid.
3 Evolution equations for shear stress tensor and Kasner space-time

In order to obtain the evolution equation for the shear stress tensor to higher orders, one needs to compute the comoving derivative of $\delta f$ (which is the deviation from the equilibrium distribution function) expanded in powers of space-time derivatives. In particular, one has,

$$\dot{\pi}^{(\mu\nu)} = \Delta_{\alpha\beta}^{\mu\nu} \int dp \, p^\alpha p^\beta \, \delta \dot{f},$$  \hspace{1cm} (3.16)

where we have used the standard notation $A^{(\mu\nu)} \equiv \Delta_{\alpha\beta}^{\mu\nu} A^{\alpha\beta}$. To second order in gradients, the evolution equation for the shear tensor is given by \cite{24},

$$\dot{\pi}^{(\mu\nu)} + \frac{\pi^{\mu\nu}}{\tau_\pi} = 2 \beta_\pi \sigma^{\mu\nu} + 2 \pi^{(\mu}_\gamma \omega^{\nu)\gamma} - \frac{10}{7} \pi^{(\mu}_\gamma \sigma^{\nu)\gamma} - \frac{4}{3} \pi^{\mu\nu} \theta,$$  \hspace{1cm} (3.17)

where $\omega^{\mu\nu} \equiv \frac{1}{2} (\nabla^\mu u^\nu - \nabla^\nu u^\mu)$ is the fluid vorticity. In the derivation of this equation, one uses the expression for $\delta f^{(1)}$ in the expansion of the particle distribution function $f$ and keeps terms up to quadratic in the gradient expansion. In the above, the Boltzmann relaxation time $\tau_R$ has been replaced by the shear relaxation time $\tau_\pi$ which is a second order transport coefficient and can be expressed as $\tau_\pi = \frac{\eta}{\beta_\pi}$ where $\eta$ is the first order transport coefficient and $\beta_\pi$ is related to the pressure $P$ of the fluid.

Similarly, using the expressions for $\delta f^{(1)}$ (eqn 2.11), $\delta f^{(2)}$ (eqn 2.12), computing their comoving derivatives and keeping terms up to cubic order in derivatives, the third order evolution equation for $\pi^{\mu\nu}$ has been obtained by Jaiswal \cite{23}:

$$\dot{\pi}^{(\mu\nu)} = -\frac{\pi^{\mu\nu}}{\tau_\pi} + 2 \beta_\pi \sigma^{\mu\nu} + 2 \pi^{(\mu}_\gamma \omega^{\nu)\gamma} - \frac{10}{7} \pi^{(\mu}_\gamma \sigma^{\nu)\gamma} - \frac{4}{3} \pi^{\mu\nu} \theta + \frac{25}{7} \pi^{(\mu}_\gamma \omega^{\nu)\gamma} \pi^{\rho\gamma} - \frac{1}{3} \pi^{(\mu}_\gamma \pi^{\nu)\gamma} \theta - \frac{25}{7} \pi^{(\mu}_\gamma \pi^{\nu)\gamma} \sigma_{\rho\gamma} - \frac{25}{7} \pi^{(\mu}_\gamma \pi^{\nu)\gamma} \sigma_{\rho\gamma} - \frac{25}{7} \nabla^{(\mu} \left( \pi^{\nu)\gamma} \tau_\pi \nabla_{\gamma} \pi^{\mu\nu} \right) + \frac{25}{7} \nabla^{(\mu} \left( \tau_\pi \nabla_{\gamma} \pi^{\nu)\gamma} \right)
- \frac{25}{7} \nabla_{\gamma} \left( \tau_\pi \nabla^{(\mu} \pi^{\nu)\gamma} \right) + \frac{12}{7} \nabla_{\gamma} \left( \tau_\pi \nabla^{(\mu} \pi^{\nu)\gamma} \right) - \frac{12}{7} \nabla_{\gamma} \left( \tau_\pi \nabla^{(\mu} \pi^{\nu)\gamma} \right) + \frac{6}{7} \nabla_{\gamma} \left( \tau_\pi \nabla^{(\mu} \pi^{\nu)\gamma} \right)
- \frac{25}{7} \tau_\pi \omega^{(\mu} \pi^{\nu)\gamma} \pi^{\rho\gamma} - \frac{25}{7} \tau_\pi \pi^{(\mu\nu)} \omega^{\rho\gamma} - \frac{10}{63} \tau_\pi \pi^{\mu\nu} \theta^2 + \frac{26}{21} \tau_\pi \pi^{(\mu} \pi^{\nu)\gamma} \theta.$$  \hspace{1cm} (3.18)

Note that the above equation was formulated within the framework of kinetic theory for a system of massless particles which has conformal symmetry. For such a system, dissipation due to bulk viscosity and heat current can be neglected. Using entropy current
and second law of thermodynamics, El et al (ref. [21]) have obtained the evolution equation before. However, in the context of one dimensional Bjorken expansion, the evolution equation obtained there misses out many more terms. Next, we study the evolution equation to second and third order for the shear stress tensor and energy density in the context of three dimensional anisotropic expansion of the conformal fluid. We consider the generalisation of Bjorken’s 1-dimensional expansion to 3-dimensional expansion of the fluid in order to connect to the realistic description of the RHIC and LHC fireball and for this, we consider Kasner space-time as the LRF of the fluid [32, 33, 34]. The metric is [37]:

$$ds^2 = (d\tau)^2 - \tau^{2a}(dx_1)^2 - \tau^{2b}(dx_2)^2 - \tau^{2c}(dx_3)^2$$ (3.19)

Here $x_1, x_2, x_3$ are the comoving coordinates, $\tau$ is the proper time and $a, b, c$ are constants known as Kasner parameters. The Kasner parameters satisfy the conditions,

$$a + b + c = 1, \quad a^2 + b^2 + c^2 = 1$$ (3.20)

The above metric is an exact solution of vacuum Einstein’s equation and it describes a homogeneous and anisotropic expansion of the Universe. The physical quantities are assumed to depend only on proper time $\tau$. The nonzero components of the affine connection for the Kasner metric are given by,

$$\Gamma_{x_1 x_1}^\tau = a\tau^{2a-1}, \quad \Gamma_{x_2 x_2}^\tau = b\tau^{2b-1}, \quad \Gamma_{x_3 x_3}^\tau = c\tau^{2c-1},$$

$$\Gamma_{x_1 x_1}^{\tau} = \frac{a}{\tau}, \quad \Gamma_{x_2 x_2}^{\tau} = \frac{b}{\tau}, \quad \Gamma_{x_3 x_3}^{\tau} = \frac{c}{\tau}$$ (3.21)

The Ricci tensor turns out to be zero upon using Kasner conditions $a + b + c = 1; a^2 + b^2 + c^2 = 1$. Ricci tensor for Kasner space-time is give by,

$$R_{00} = \frac{1}{\tau^2}[(a + b + c) - (a^2 + b^2 + c^2)]$$ (3.22)

$$R_{11} = a[(a + b + c) - 1]\tau^{2a-2}$$

$$R_{22} = b[(a + b + c) - 1]\tau^{2b-2}$$

$$R_{33} = c[(a + b + c) - 1]\tau^{2c-2}$$

where, 1, 2, 3 corresponds to $x_1, x_2, x_3$ coordinates, and 0 corresponds to $\tau$ coordinate. As one can see, the components vanish upon using Kasner conditions. The nonzero compo-
nents of Riemann tensor are given by,

\begin{align*}
R_{0101} &= (1 - a)a\tau^{2a-2}, \quad R_{0202} = (1 - b)b\tau^{2b-2} \\
R_{0303} &= (1 - c)c\tau^{2c-2}, \quad R_{1212} = ab\tau^{2a+2b-2} \\
R_{1313} &= ac\tau^{2a+2c-2}, \quad R_{2323} = bc\tau^{2b+2c-2}
\end{align*}

(3.23)

These expressions become zero for the Bjorken case corresponding to \( a = 1, b = 0, c = 0 \). So the terms involving \( R^{\mu\nu\rho\sigma} \) could contribute to the shear tensor \( \pi^{\mu\nu} \) with a coefficient \( \kappa \) (we refer to eqn. 3.12 in ref. [11]). However, the general expression for the shear stress tensor \( \pi^{\mu\nu} \) when derived from kinetic theory does not contain the \( \kappa \) term which would have involved the Riemann tensor (we refer to eqn. 5.23 of the above reference [11] and subsequent discussion). Boltzmann equation does not contain the term involving \( \kappa \). Hence, there is no inconsistency in the order of the derivative expansion.

We compute various quantities appearing in the second and third order evolution equations for the shear stress tensor. As mentioned in the previous section, here we use the notations involving covariant derivative \( D_\mu \) (as defined in eqn. (2.15)) instead of partial derivative \( \partial_\mu \). We have,

\[ \nabla_\mu u_\nu = \Delta_\mu ^\rho D_\rho u_\nu, \]

\[ \sigma^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \nabla^\alpha u^\beta = \Delta^{\mu\nu}_{\alpha\beta} \Delta^{\alpha\rho} D_\rho u^\beta \]

(3.24)

We obtain the components of the projection operator as,

\[ \Delta^{\mu\nu} \equiv \text{diag}(0, -\frac{1}{\tau^{2a}}, -\frac{1}{\tau^{2b}}, -\frac{1}{\tau^{2c}}) \]

(3.25)

Components of the shear tensor are obtained as,

\[ \sigma^{\tau\tau} = 0, \quad \sigma^{x_1 x_1} = \frac{-2a + b + c}{3} \tau^{-2a-1}, \]

\[ \sigma^{x_2 x_2} = \frac{a - 2b + c}{3} \tau^{-2b-1}, \]

\[ \sigma^{x_3 x_3} = \frac{a + b - 2c}{3} \tau^{-2c-1} \]

(3.26)

where we have used Kasner conditions and,
\[ \theta \equiv \nabla_\mu u^\mu = \Delta_\mu D_\nu u^\mu = \frac{a + b + c}{\tau}. \] (3.27)

Upon using Kasner condition, \( \theta \) becomes \( \frac{1}{\tau} \) which is the same as in one dimensional Bjorken expansion case. The shear stress tensor is diagonal. We assume that it can be characterized by a function \( \pi \) and the Kasner parameters in the following form: \( \pi^{\mu\nu} \equiv \text{diag}(0, -\pi\tau^{-2a}, \frac{\pi}{2}\tau^{-2b}, \frac{\pi}{2}\tau^{-2c}) \). One can check that \( \pi^{\mu\nu} \) is traceless as we are considering a conformal fluid.

The components of \( \dot{\pi}^{(\mu
u)} \) are given by,

\[ \dot{\pi}^{(\tau\tau)} = 0, \quad \dot{\pi}^{(x_1x_1)} = \frac{-1}{\tau^2} \frac{d\pi}{d\tau}, \]
\[ \dot{\pi}^{(x_2x_2)} = \frac{1}{2\tau^2} \frac{d\pi}{d\tau}, \quad \dot{\pi}^{(x_3x_3)} = \frac{1}{2\tau^2} \frac{d\pi}{d\tau}. \] (3.28)

Components of other terms appearing in the second order evolution equation are obtained as,

\[ \pi^{(\gamma\sigma\tau)}_\gamma = 0, \quad \pi^{(x_1\sigma_1)x_1} = \frac{\pi(b + c - 2a)}{6\tau^{2a+1}}, \]
\[ \pi^{(x_2\sigma_2)x_2} = \frac{\pi(a + b - 2c)}{6\tau^{2b+1}}, \]
\[ \pi^{(x_3\sigma_3)x_3} = \frac{\pi(a + c - 2b)}{6\tau^{2c+1}}. \] (3.29)

Substituting the above expressions, the second order evolution equations are given by

\[ \frac{d\epsilon}{d\tau} = -\frac{1}{\tau} (\epsilon + P) - \frac{\pi(b+c-2a)}{2\tau}, \] (3.30)
\[ \frac{d\pi}{d\tau} = -\frac{\pi}{\tau^2} - \frac{2\beta_\pi(-2a+b+c)}{3\tau} - \frac{\pi(38a+23b+23c)}{21\tau}. \] (3.31)

Note that, adding the three equations for the nonzero components of \( \dot{\pi}^{(\mu
u)} \), we get only one independent equation as given above in eqn. (3.31). It is important to note that the Kasner metric in the limit of \( a = 1, b = 0, c = 0 \) reduces to that of the Minkowski metric in Milne coordinates. In this limit, the above second order evolution equations reduce to the evolution equations in the one dimensional Bjorken expansion case [23]:

\[ \frac{d\epsilon}{d\tau} = -\frac{1}{\tau} (\epsilon + P) + \frac{\pi}{\tau}, \] (3.32)
\[ \frac{d\pi}{d\tau} = -\frac{\pi}{\tau^2} + \frac{4\beta_\pi}{3\tau} - \frac{38\pi}{21\tau}. \] (3.33)
Next, we compute the other terms appearing in the third order evolution equation of $\pi^{\mu\nu}$ and give the explicit expressions for the various terms. Components of $\pi^{\gamma}_{\gamma} (\pi^{\tau})_{\gamma} \theta$ are given by,

$$
\pi^{\gamma}_{\gamma} (\pi^{\tau})_{\gamma} \theta = 0, \quad \pi^{(x_1)}_{\gamma} (\pi^{x_1})_{\gamma} \theta = \frac{\pi^2(a + b + c)}{2 \tau^{2a+1}}, \\
\pi^{(x_2)}_{\gamma} (\pi^{x_2})_{\gamma} \theta = \frac{\pi^2(a + b + c)}{4 \tau^{2b+1}}, \\
\pi^{(x_3)}_{\gamma} (\pi^{x_3})_{\gamma} \theta = \frac{\pi^2(a + b + c)}{4 \tau^{2c+1}}
$$

(3.34)

Components of $\pi^{\mu\nu} (\pi^{\rho\gamma})_{\rho\gamma}$ are:

$$
\pi^{x_1 x_1}_{\gamma} (\pi^{\rho\gamma})_{\rho\gamma} = 0, \\
\pi^{x_2 x_2}_{\gamma} (\pi^{\rho\gamma})_{\rho\gamma} = \frac{-\pi^2(2a - b - c)}{2 \tau^{2a+1}}, \\
\pi^{x_3 x_3}_{\gamma} (\pi^{\rho\gamma})_{\rho\gamma} = \frac{\pi^2(2a - b - c)}{4 \tau^{2b+1}}
$$

(3.35)

Components of $\pi^{\rho (\pi^{\mu\nu})}_{\rho \gamma}$ are given by,

$$
\pi^{\rho (\pi^{\tau})}_{\rho \gamma} = 0, \\
\pi^{\rho (x_1)}_{\rho \gamma} (\pi^{x_1})_{\gamma} = \frac{\pi^2(-2a + b + c)}{4 \tau^{2a+1}}, \\
\pi^{\rho (x_2)}_{\rho \gamma} (\pi^{x_2})_{\gamma} = \frac{\pi^2(a - b)}{4 \tau^{2b+1}}, \\
\pi^{\rho (x_3)}_{\rho \gamma} (\pi^{x_3})_{\gamma} = \frac{\pi^2(a - c)}{4 \tau^{2c+1}}
$$

(3.36)

For $\nabla^{(\mu} (\nabla_{\gamma} \pi^{\nu)})_{\gamma}$ we have,

$$
\nabla^{(\tau} (\nabla_{\gamma} \pi^{\tau})_{\gamma}) = 0
$$

$$
\nabla^{(x_1} (\nabla_{\gamma} \pi^{x_1})_{\gamma}) = \frac{\pi}{6 \tau^{2a+2}} (8a^2 + 2b^2 + 2c^2 - 4ab + 2bc - 4ac) \\
\nabla^{(x_2} (\nabla_{\gamma} \pi^{x_2})_{\gamma}) = \frac{\pi}{6 \tau^{2b+2}} (-4a^2 - 4b^2 + 2c^2 + 5ab - bc - ac) \\
\nabla^{(x_3} (\nabla_{\gamma} \pi^{x_3})_{\gamma}) = \frac{\pi}{6 \tau^{2c+2}} (-4a^2 + 2b^2 - 4c^2 - ab - bc + 5ac)
$$

(3.37)

For $\nabla_{\gamma} (\nabla^{(\mu \pi^{\nu})}_{\gamma})$, we get,

$$
\nabla_{\gamma} (\nabla^{(\tau \pi^{\tau})}_{\gamma}) = 0
$$
\[ \nabla_\gamma (\nabla^{x_1} \pi^{x_1}) = \frac{\pi}{6\tau^{2a+2}} (8a^2 + 2b^2 + 2c^2 + 5ab + 2bc + 5ac) \]
\[ \nabla_\gamma (\nabla^{x_2} \pi^{x_2}) = \frac{\pi}{6\tau^{2b+2}} (-4a^2 - 4b^2 + 2c^2 - 4ab - bc - ac) \]
\[ \nabla_\gamma (\nabla^{x_3} \pi^{x_3}) = \frac{\pi}{6\tau^{2c+2}} (-4a^2 + 2b^2 - 4c^2 - ab - bc - 4ac) \] (3.38)

Components of \( \pi^{\mu \nu} \theta^2 \) are given by,
\[ \pi^{\tau \tau} \theta^2 = 0, \pi^{x_1 x_1} \theta^2 = -\frac{\pi}{\tau^{2a}} \left( \frac{a + b + c}{\tau} \right)^2 \]
\[ \pi^{x_2 x_2} \theta^2 = \frac{\pi}{2\tau^{2b}} \left( \frac{a + b + c}{\tau} \right)^2 \]
\[ \pi^{x_3 x_3} \theta^2 = \frac{\pi}{2\tau^{2c}} \left( \frac{a + b + c}{\tau} \right)^2 \] (3.39)

For \( \nabla_\gamma (\tau_\pi \nabla_\pi (\mu \nu)) \), we obtain,
\[ \nabla_\gamma (\tau_\pi \nabla_\pi (\tau \tau)) = 0, \]
\[ \nabla_\gamma (\tau_\pi \nabla_\pi (x_1 x_1)) = \frac{\pi (4a^2 + b^2 + c^2)}{3\tau^{2a+2}}, \]
\[ \nabla_\gamma (\tau_\pi \nabla_\pi (x_2 x_2)) = \frac{\pi (-2a^2 - 2b^2 + c^2)}{3\tau^{2b+2}} \]
\[ \nabla_\gamma (\tau_\pi \nabla_\pi (x_3 x_3)) = \frac{\pi (-2a^2 + b^2 - 2c^2)}{3\tau^{2c+2}} \] (3.40)

All these expressions can be simplified further by using Kasner conditions. Here we have \( \omega^{\mu \nu} = \dot{u}^\mu = \nabla^\mu \tau_\pi = 0 \). The other terms in the evolution equations as given below become zero, namely,
\[ \pi_\gamma^{(\mu \nu)\gamma} = 0 \] (3.41)
\[ \pi_\rho^{(\mu \nu)\gamma} \pi_\rho^{\nu \gamma} = 0 \]
\[ \nabla^{(\mu} \pi_\gamma^{(\nu \gamma \pi_\gamma)) = 0} \]
\[ \nabla_\gamma (\tau_\pi \dot{u}^{\pi} \nabla^{(\mu \nu \gamma)) = 0} \]
\[ \nabla_\gamma (\tau_\pi \dot{u}^{\pi} \nabla^{(\mu \nu \gamma = 0} \]
\[ \tau_\pi \omega^{\rho (\mu \nu \gamma) \nabla^{\pi \rho \gamma} = 0} \]
\[ \tau_\pi \pi_\gamma^{(\mu \nu \gamma} \omega_\rho^{\nu \gamma} = 0} \]
\[ \tau_\pi \pi_\gamma^{(\mu \nu \gamma} \theta = 0 \]
Substituting the expressions for the above nonzero terms, the third order evolution equations for $x_1 x_1$, $x_2 x_2$, $x_3 x_3$ components of the shear stress tensor are obtained as,

\[
\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} - 2\beta_\pi \left( \frac{-2a+b+c}{3\tau} \right) + \frac{\pi(-38a-23b-23c)}{21\tau} + \frac{\pi^2(-803a+34b+34c)}{1470\beta_\pi \tau} \\
+ \frac{\pi^2(106a^2-11b^2-11c^2-13ab-13ac-76bc)}{420\beta_\pi \tau} \quad (3.42)
\]

\[
\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + 4\beta_\pi \left( \frac{a-2b+c}{3\tau} \right) + \frac{\pi(-38a-38b-8c)}{21\tau} + \frac{\pi^2(-803a+199b-131c)}{1470\beta_\pi \tau} \\
+ \frac{\pi^2(53a^2+53b^2-64c^2+25ab-38ac-38bc)}{210\beta_\pi \tau} \quad (3.43)
\]

\[
\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + 4\beta_\pi \left( \frac{a+b-2c}{3\tau} \right) + \frac{\pi(-38a-8b-38c)}{21\tau} + \frac{\pi^2(-803a-131b+199c)}{1470\beta_\pi \tau} \\
+ \frac{\pi^2(53a^2-64b^2+53c^2+25ac-38ab-38bc)}{210\beta_\pi \tau} \quad (3.44)
\]

These equations can be further simplified by using Kasner conditions. One can check that adding the equations for the $x_1 x_1$, $x_2 x_2$ and $x_3 x_3$ components, one gets only one independent equation, which is eq. (3.42). The evolution equation for the energy density $\epsilon$ is obtained as,

\[
\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}(\epsilon + P)\tau - \frac{\pi(b + c - 2a)}{2\tau} \quad (3.45)
\]

In the limit $a = 1, b = 0, c = 0$ for the Kasner parameters, the evolution equation for the energy density reduces to that of the one dimensional Bjorken expansion case [23],

\[
\frac{d\epsilon}{d\tau} = -\frac{1}{\tau}(\epsilon + P - \pi) \quad (3.46)
\]

and the third order evolution equation for the shear stress tensor $\pi^{\mu\nu}$ reduces to that of the equation for the Bjorken’s one dimensional expansion case, namely,

\[
\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + 4\beta_\pi \left( \frac{a^2}{3\tau} \right) - \frac{38\pi}{21\tau} - \frac{72}{245} \frac{\pi^2}{\beta_\pi \tau} \quad (3.47)
\]

By comparing the above equation with the third order evolution equation in the one dimensional expansion case [23],

\[
\frac{d\pi}{d\tau} = -\frac{\pi}{\tau_\pi} + 4\beta_\pi \left( \frac{a^2}{3\tau} \right) - \lambda_\pi \frac{\pi^2}{\beta_\pi \tau} \quad (3.48)
\]
the transport coefficients are given by,

\[ \tau_\pi = \frac{\eta}{\beta_\pi}, \quad \beta_\pi = \frac{4P}{3}, \quad \lambda = \frac{38}{21}, \quad \chi = \frac{72}{245}. \tag{3.49} \]

which matches with the results in the one dimensional expansion case.

Here we would like to point out that one could have introduced three independent fields \( \pi_i(i = 1, 2, 3) \) for the shear stress tensor \( \pi^{\mu\nu} \) instead of a single function \( \pi \), as has been introduced in the above discussion to characterize \( \pi^{\mu\nu} \). We have explicitly checked that by making a general ansatz for the shear stress tensor \( \pi^{\mu\nu} = \text{diag}(0, \pi_1\tau^{-2a}, \pi_2\tau^{-2b}, \pi_3\tau^{-2c}) \) by introducing three independent functions \( \pi_1, \pi_2, \pi_3 \), we get three independent third order evolution equations (also second order) for the components of the shear stress tensor involving \( \pi_1, \pi_2 \) and \( \pi_3 \). These are given by,

\[
\frac{d\pi_1}{d\tau} = -\frac{\pi_1}{\tau_\pi} + 2\beta_\pi \frac{a+2b+c}{3\tau} + \frac{\pi_1(-12a-64b-64c)}{63\tau} + \frac{\pi_2(-10a+20b-10c)}{63\tau} + \frac{\pi_3(-10a-10b+20c)}{63\tau}
-\frac{1}{369}\beta_\pi \frac{\pi_2}{\tau_\pi} \bigg[ \pi_1^2(-386a - 52b - 52c) + \pi_2^2(45a + 155b + 45c) + \pi_3^2(45a + 45b + 155c)
+ \pi_1\pi_2(38a - 76b + 38c) + \pi_1\pi_3(38a + 38b - 76c) \bigg]
+ \frac{\pi_1}{63\tau} \frac{\pi_1}{\tau_\pi} \bigg[ \pi_1(212a^2 - 100b^2 - 100c^2 - 68ab - 68ac
- 200bc) + \pi_2(-156a^2 - 84ab - 48bc) + \pi_3(-156c^2 - 84ac - 48bc) \bigg] \tag{3.50}
\]

\[
\frac{d\pi_2}{d\tau} = -\frac{\pi_2}{\tau_\pi} + 2\beta_\pi \frac{a-2b+c}{3\tau} + \frac{\pi_1(-120a-120b-10c)}{63\tau} + \frac{\pi_2(-64a+124b-64c)}{63\tau} + \frac{\pi_3(-10a+10b+20c)}{63\tau}
-\frac{1}{369}\beta_\pi \frac{\pi_1}{\tau_\pi} \bigg[ \pi_1^2(155a + 45b + 45c) + \pi_2^2(-52a - 386b - 52c) + \pi_3^2(45a + 45b + 155c)
+ \pi_1\pi_2(-76a + 38b + 38c) + \pi_2\pi_3(38a + 38b - 76c) \bigg]
+ \frac{\pi_1}{315\tau} \frac{\pi_1}{\tau_\pi} \bigg[ \pi_1(-78a^2 - 42ab - 24ac) + \pi_2(-50a^2 + 106b^2
- 50c^2 - 34ab - 34bc - 100ac) + \pi_3(-78c^2 - 24ac - 42bc) \bigg] \tag{3.51}
\]

\[
\frac{d\pi_3}{d\tau} = -\frac{\pi_3}{\tau_\pi} + 2\beta_\pi \frac{a+b-2c}{3\tau} + \frac{\pi_1(-120a+10b-10c)}{63\tau} + \frac{\pi_2(-10a+20b-10c)}{63\tau} + \frac{\pi_3(-64a-64b-124c)}{63\tau}
-\frac{1}{369}\beta_\pi \frac{\pi_1}{\tau_\pi} \bigg[ \pi_1^2(155a + 45b + 45c) + \pi_2^2(45a + 155b + 45c) + \pi_3^2(-52a - 52b - 386c)
+ \pi_1\pi_3(-76a + 38b + 38c) + \pi_2\pi_3(38a - 76b + 38c) \bigg]
+ \frac{\pi_1}{315\tau^2} \frac{\pi_1}{\tau_\pi} \bigg[ \pi_1(-78a^2 - 24ab - 42ac) + \pi_2(-78b^2 - 24ab - 42bc) + \pi_3(-50a^2 - 50b^2}
+ 106c^2 - 34ac - 34bc - 100ab) \bigg] \tag{3.52}
\]
Since $\pi^{\mu\nu}$ is traceless, for the Kasner metric $g_{\mu\nu}$, we get the condition

$$\pi_1 + \pi_2 + \pi_3 = 0$$

(3.53)

Using this tracelessness condition, we find that adding the equations for $\pi_2$ and $\pi_3$, we get precisely the equation for $\pi_1$ (where, $\dot{\pi}^{(11)} = \frac{d\pi_1}{d\tau}$). Below we show this explicitly. For making this check, we have also computed all the relevant terms appearing in the third order evolution equations in term of $\pi_1, \pi_2, \pi_3$ (we give these expressions in the appendix).

Adding the equations for $\frac{d\pi_1}{d\tau}$ and $\frac{d\pi_3}{d\tau}$, and using $\pi_2 + \pi_3 = -\pi_1$, we obtain,

$$\frac{d\pi_1}{d\tau} = -\frac{\pi_1}{\tau} + 2\beta \pi (\frac{-2a+b+c}{3\tau}) + \frac{\pi_1(-124a-64b-64c)}{63\tau} + \frac{\pi_2(-10a+20b-10c)}{63\tau} + \frac{\pi_3(-10a-10b+20c)}{63\tau}$$

$$-\frac{1}{\tau^{3\beta_0\tau}}[\pi_1^2(-310a - 90b - 90c) + \pi_2^2(7a + 231b + 7c) + \pi_3^2(7a + 7b + 231c)$$

$$+ \pi_1\pi_2(76a - 38b - 38c) + \pi_1\pi_3(76a - 38b - 38c)$$

$$+ \pi_2\pi_3(-76a + 38b + 38c)] + \frac{\tau_\pi}{315\tau}[\pi_1(156a^2 + 66ab + 66ac) + \pi_2(50a^2 - 28b^2$$

$$+ 50c^2 + 58ab + 76bc + 100ac) + \pi_3(50a^2$$

$$+ 50b^2 - 28c^2 + 76bc + 58ac + 100ab)]$$

(3.54)

This can be simplified to,

$$\frac{d\pi_1}{d\tau} = -\frac{\pi_1}{\tau} + 2\beta \pi (\frac{-2a+b+c}{3\tau}) + \frac{\pi_1(-124a-64b-64c)}{63\tau} + \frac{\pi_2(-10a+20b-10c)}{63\tau} + \frac{\pi_3(-10a-10b+20c)}{63\tau}$$

$$-\frac{1}{\tau^{3\beta_0\tau}}[\pi_1^2(-386a - 52b - 52c) + \pi_2^2(45a + 155b + 45c) + \pi_3^2(45a + 45b + 155c)$$

$$+ \pi_1\pi_2(38a - 76b + 38c) + \pi_1\pi_3(38a + 38b - 76c)]$$

$$+ \frac{\tau_\pi}{630\tau}[\pi_1(212a^2 - 100b^2 - 100c^2 - 68ab - 68ac - 200bc) + \pi_2(-156b^2$$

$$- 84ab - 48bc) + \pi_3(-156c^2 - 84ac - 48bc)] + \frac{(\pi_1+\pi_2+\pi_3)(84a+84b+84c)}{63\tau}$$

$$+ \frac{1}{\tau^{3\beta_0\tau}}[(-38a - 38b - 38c)(\pi_1 + \pi_2 + \pi_3)^2 + 114(a_1 + b_2 + c_3)(\pi_1 + \pi_2 + \pi_3)]$$

$$+ \frac{\tau_\pi}{630\tau}(\pi_1 + \pi_2 + \pi_3)(100a^2 + 100b^2 + 100c^2 + 200ab + 200bc + 200ac)$$

(3.55)

where the last three terms are zero as $\pi_1 + \pi_2 + \pi_3 = 0$. Rest of the terms are precisely the expression for the $\frac{d\pi_1}{d\tau}$ equation. Hence there is no inconsistency and there is only one independent ordinary differential equation (ODE). Hence, for simplicity, we have made the assumption that $\pi_{\mu\nu}$ could be characterized by a single function $\pi$ and the three ODEs involving $x_1x_1, x_2x_2, x_3x_3$ components of the shear tensor are not totally independent, rather
adding the $x_2 x_2$ and $x_3 x_3$ equations, it reduces to the equation for the $x_1 x_1$ component. This we have already discussed with reference to equations (40), (41) and (42) in this section. We have shown above that this is also the case for the general ansatz in terms of $\pi_1, \pi_2$ and $\pi_3$. We have also checked that putting $\pi_1 = -\pi, \pi_2 = \frac{\pi}{2}, \pi_3 = \frac{\pi}{2}$ in the ODEs, we get back the earlier equations in terms of $\pi$, namely, equations (40), (41) and (42) (where we have used the first order relation $\pi = \frac{4\beta_\pi \tau_\pi}{3\tau}$ for rewriting the last term in terms of the coefficient $\frac{x^2}{\beta_\pi \tau}$). All these expressions also reduce to the Bjorken case for $a = 1, b = 0 = c$. Hence the ansatz is consistent. This is also true for the second order evolution equation.

### 4 Summary and discussion

In this work, we have studied relativistic viscous hydrodynamics from kinetic theory to second and third order in gradient expansion. We have considered Kasner space-time as the LRF of the three dimensional anisotropic expansion of a conformal fluid and have obtained the second and third order evolution equations for the shear stress tensor and energy density. We have used the iterative solutions of the Boltzmann equation for the nonequilibrium distribution function in RTA. We have shown that our results for the three dimensional expansion agree with the one dimensional Bjorken expansion case in appropriate limit of the anisotropy parameters. For a general fluid in Kasner space-time, the system may not be conformal, in which case the stress energy tensor will not be traceless. It is a challenging topic to study the evolution of the shear stress tensor for a nonconformal system with nonzero bulk viscosity in a general curved background. Nonconformal fluid models contain a much larger number of transport coefficients. There has been some progress in related aspects from various approaches (see for example, [38, 39, 40]). It will be certainly interesting to explore the dissipative evolution equations in a higher order gradient expansion involving nonconformal fluid in Kasner space-time. One can also work out the higher order corrections to entropy four current in our setup of anisotropic expansion. It will be interesting to relate our results for anisotropic space-time to anisotropic hydrodynamics (see ref. [41] for a review on anisotropic hydrodynamics). It will be worth exploring higher order dissipative hydrodynamics for Gubser flow [42] and relate it to anisotropic hydro-
dynamics in the present scenario. In the context of anisotropic expansion of the fluid, it will also be interesting to study the higher order corrections in relativistic hydrodynamics in extended relaxation time approximation \[43, 44\], where the relaxation time depends on particle energy. We hope to report on these issues in future.

Acknowledgements

We would like to thank Amaresh Jaiswal for helpful discussions.

5 Appendix

We assume that the shear stress tensor is diagonal and is characterized by three functions \(\pi_1\), \(\pi_2\) and \(\pi_3\), namely, \(\pi^{\mu\nu} \equiv \text{diag}(0, \pi_1\tau^{-2a}, \pi_2\tau^{-2b}, \pi_3\tau^{-2c})\). The components of \(\dot{\pi}^{(\mu\nu)}\) are given by,

\[\dot{\pi}^{(\tau\tau)} = 0, \quad \dot{\pi}^{(x_1x_1)} = \frac{1}{\tau^{2a}} \frac{d\pi_1}{d\tau}, \quad \dot{\pi}^{(x_2x_2)} = \frac{1}{\tau^{2b}} \frac{d\pi_2}{d\tau}, \quad \dot{\pi}^{(x_3x_3)} = \frac{1}{\tau^{2c}} \frac{d\pi_3}{d\tau}\] (5.1)

Components of other terms appearing in the third order evolution equation are obtained as,

\[\pi^{(\tau\sigma\tau)} = 0\]

\[\pi^{(x_1\sigma\sigma)x_1}) = \frac{1}{9\tau^{2a+1}}(-2\pi_1(-2a + b + c) + \pi_2(a - 2b + c) + \pi_3(a + b - 2c)\]

\[\pi^{(x_2x_2)} = \frac{1}{9\tau^{2b+1}}(\pi_1(-2a + b + c) - 2\pi_2(a - 2b + c) + \pi_3(a + b - 2c)\]

\[\pi^{(x_3x_3)} = \frac{1}{9\tau^{2c+1}}(\pi_1(-2a + b + c) + \pi_2(a - 2b + c) - 2\pi_3(a + b - 2c)\] (5.2)

Components of \(\pi^{(\tau\tau)}\gamma\theta\) are given by,

\[\pi^{(\tau\tau)} = 0\]

\[\pi^{(x_1x_1)} = \frac{-2\pi_1^2 + \pi_2^2 + \pi_3^2(a + b + c)}{3\tau^{2a+1}}\]
\[ \pi_{\gamma}^{(x_2 \pi x_2)\gamma} \theta = \frac{(\pi_1^2 - 2\pi_2^2 + \pi_3^2)(a + b + c)}{3\tau^{2b+1}} \]
\[ \pi_{\gamma}^{(x_3 \pi x_3)\gamma} \theta = \frac{(\pi_1^2 + \pi_2^2 - 2\pi_3^2)(a + b + c)}{3\tau^{2c+1}} \] (5.3)

Components of \( \pi^{\mu \nu} \pi^{\rho \gamma} \sigma_{\rho \gamma} \) are:
\[ \pi^{\tau \pi} \pi^{\rho \gamma} \sigma_{\rho \gamma} = 0 \]
\[ \pi^{x_1 x_1} \pi^{\rho \gamma} \sigma_{\rho \gamma} = \frac{\pi_1}{3\tau^{2a+1}}[(\pi_1(-2a + b + c) + \pi_2(a - 2b + c) + \pi_3(a + b - 2c)] \]
\[ \pi^{x_2 x_2} \pi^{\rho \gamma} \sigma_{\rho \gamma} = \frac{\pi_2}{3\tau^{2b+1}}[(\pi_1(-2a + b + c) + \pi_2(a - 2b + c) + \pi_3(a + b - 2c)] \]
\[ \pi^{x_3 x_3} \pi^{\rho \gamma} \sigma_{\rho \gamma} = \frac{\pi_3}{3\tau^{2c+1}}[(\pi_1(-2a + b + c) + \pi_2(a - 2b + c) + \pi_3(a + b - 2c)] \] (5.4)

Components of \( \pi^{\rho(\mu \pi \nu)} \pi^{\rho \gamma} \sigma_{\rho \gamma} \) are given by,
\[ \pi^{\rho(\tau \pi \gamma)} \sigma_{\rho \gamma} = 0 \]
\[ \pi^{\rho(x_1 \pi x_1)} \pi^{\rho \gamma} \sigma_{\rho \gamma} = \frac{1}{9\tau^{2a+1}}[(2\pi_1(-2a + b + c) - \pi_2(a - 2b + c) - \pi_3(a + b - 2c)] \]
\[ \pi^{\rho(x_2 \pi x_2)} \pi^{\rho \gamma} \sigma_{\rho \gamma} = \frac{1}{9\tau^{2b+1}}[-\pi_1^2(-2a + b + c) + 2\pi_2^2(a - 2b + c) - \pi_3^2(a + b - 2c)] \]
\[ \pi^{\rho(x_3 \pi x_3)} \pi^{\rho \gamma} \sigma_{\rho \gamma} = \frac{1}{9\tau^{2c+1}}[-\pi_1^2(-2a + b + c) - \pi_2^2(a - 2b + c) + 2\pi_3^2(a + b - 2c)] \] (5.5)

For \( \nabla^{\mu} (\nabla_{\gamma} \pi^{\nu})^{\gamma} \) we have,
\[ \nabla^{(\tau} (\nabla_{\gamma} \pi^{\tau})^{\gamma}) = 0 \]
\[ \nabla^{(x_1} (\nabla_{\gamma} \pi^{x_1})^{\gamma}) = - \frac{1}{3\tau^{2a+2}}(\pi_1(-4a^2 + ab + ac) + \pi_2(2b^2 - 2ab + bc) + \pi_3(2c^2 + bc - 2ac)) \]
\[ \nabla^{(x_2} (\nabla_{\gamma} \pi^{x_2})^{\gamma}) = \frac{1}{3\tau^{2b+2}}(\pi_1(2a^2 - 2ab + ac) + \pi_2(-4b^2 + ab + bc) + \pi_3(2c^2 - 2bc + ac)) \]
\[ \nabla^{(x_3} \left( \nabla_{\gamma} \pi^{x_3)} \right) = \frac{1}{3\tau^{2c+2}}(\pi_1(2a^2 + ab - 2ac) + \pi_2(2b^2 + ab - 2bc) + \pi_3(-4c^2 + bc + ac)) \] (5.6)

For \( \nabla_{\gamma} \left( \nabla^{(\mu \pi^{\nu})} \right) \), we get,

\[ \nabla_{\gamma} \left( \nabla^{(\tau \pi^{\tau})} \right) = 0 \]

\[ \nabla_{\gamma} \left( \nabla^{(x_1 \pi^{x_1})} \right) = \frac{1}{3\tau^{2a+2}}(\pi_1(-4a^2 - 2ab - 2ac) + \pi_2(2b^2 + ab + bc) + \pi_3(2c^2 + bc + ac)) \]

\[ \nabla_{\gamma} \left( \nabla^{(x_2 \pi^{x_2})} \right) = \frac{1}{3\tau^{2b+2}}(\pi_1(2a^2 + ab + ac) + \pi_2(-4b^2 - 2ab - 2bc) + \pi_3(2c^2 + bc + ac)) \]

\[ \nabla_{\gamma} \left( \nabla^{(x_3 \pi^{x_3})} \right) = \frac{1}{3\tau^{2c+2}}(\pi_1(2a^2 + ab + ac) + \pi_2(2b^2 + ab + bc) + \pi_3(-4c^2 - 2bc - 2ac)) \] (5.7)

For \( \nabla_{\gamma} \left( \tau \pi \nabla \pi^{(\mu \nu)} \right) \), we obtain,

\[ \nabla_{\gamma} \left( \tau \pi \nabla \pi^{(\tau \tau)} \right) = 0 \]

\[ \nabla_{\gamma} \left( \tau \pi \nabla \pi^{(x_1 x_1)} \right) = \frac{2\tau_\pi(-2a^2\pi_1 + b^2\pi_2 + c^2\pi_3)}{3\tau^{2a+2}} \]

\[ \nabla_{\gamma} \left( \tau \pi \nabla \pi^{(x_2 x_2)} \right) = \frac{2\tau_\pi(a^2\pi_1 - 2b^2\pi_2 + c^2\pi_3)}{3\tau^{2b+2}} \]

\[ \nabla_{\gamma} \left( \tau \pi \nabla \pi^{(x_3 x_3)} \right) = \frac{2\tau_\pi(a^2\pi_1 + b^2\pi_2 - 2c^2\pi_3)}{3\tau^{2c+2}} \] (5.8)

Components of \( \pi^{\mu \nu} \theta^2 \) are given by,

\[ \pi^{x_1 x_1} \theta^2 = \frac{\pi_1}{\tau^{2a}} \left( \frac{a + b + c}{\tau} \right)^2 \]

\[ \pi^{x_2 x_2} \theta^2 = \frac{\pi_2}{\tau^{2b}} \left( \frac{a + b + c}{\tau} \right)^2 \]

\[ \pi^{x_3 x_3} \theta^2 = \frac{\pi_3}{\tau^{2c}} \left( \frac{a + b + c}{\tau} \right)^2 \] (5.9)

These expressions have been used to obtain the differential equations for the components of \( \pi^{\mu \nu} \) in terms of \( \pi_1, \pi_2, \pi_3 \). All these expressions reduce to our earlier results where we have considered a single function \( \pi \) to characterize the shear stress tensor \( \pi^{\mu \nu} \).
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