I. INTRODUCTION

Superconductivity, a phenomenon that is typical in condensed matter physics, also both relevant in nuclear and subnuclear physics (see for instance [1, 2]), takes its origin from pairing between fermions. It is typically described assuming a interacting (pairing) Hamiltonian and solving it via the mean-field (MF) approximation [3], which explicitly violates particle number conservation. While this limitation has a small effect for macroscopic systems, it can lead to dramatic deviations when fluctuations are important, i.e. dealing with a fixed small number of particles. This justifies the interest towards exactly-solvable models that avoid any approximation, at the price of assuming specific forms of the study of exactly-solvable models that avoid any approximation, in the present paper we analyze a large family of 2D BCS models with arbitrary $l$-wave $(l \neq 0)$ pairing interaction. We first discuss (Sect. II) the cases that can be exactly solved via Bethe-Ansatz in a finite size system. Later on, we describe a standard MF analysis (Sect. III), and we compare the results from the two different approaches studying the topological properties of their solutions (Sect. IV). In this way, further insight is also achieved about the cases where integrability does not hold, as well as about the role of integrability itself.

The family of superconductive models that we are going to study is described by the Hamiltonians of the form:

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}} - g \sum_{\mathbf{k} \mathbf{k}'} (k_x - ik_y)^l (k'_x + ik'_y)^l c_{\mathbf{k}}^{\dagger} c_{\mathbf{k}'}^{\dagger} c_{-\mathbf{k}} c_{-\mathbf{k}'} .$$

(1)

There $c_{\mathbf{k}}^{\dagger}$ is the creation operators of 2D fermions with momentum $\mathbf{k} = (k_x, k_y)$, and $g$ is the coupling constant, positive for an attractive interaction. Notice that the interaction term creates and annihilates pairs of fermions with opposite momentum. In order to keep the widest generality, at the beginning of our analysis we do not adopt any particular choice for the single particle energy $\epsilon_{\mathbf{k}}$, only assuming it is a function of the modulus $k = |\mathbf{k}|$.

In Eq. (1), we have dropped the spin index $\{\uparrow, \downarrow\}$ in the Fermi operators, so spinless fermions are formally considered. If instead the Cooper pairs are spinful, the symmetry of their spin wavefunctions is univocally determined by the Fermi-Dirac statistics. In fact, when $l$ is even, the Cooper pairs form a spin singlet (antisymmetric), while when $l$ is odd they are in the triplet sector (symmetric and polarized). In both the cases, the structure of the Bethe-Ansatz equations and of the spatial part of the exact Cooper wavefunctions (introduced in Sect. II) in the presence of integrability are the same as in the spinless model described in Eq. (1).

The familiar $s$-wave case corresponds to $l = 0$ and to the singlet sector of the spin wavefunction. This is the sole non symmetry-breaking case under parity and time reversal transformation. The breaking of these symmetries for $l \geq 1$ leads to different kinds of exact solutions, introducing nontrivial topological properties of the paired states (according to the ten-fold way classification for the topological insulators and superconductors, see e. g. [17–

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We analyze a family of two dimensional BCS Hamiltonians with general $l$-wave pairing interactions, classifying the models in this family that are Bethe-Ansatz solvable in the finite-size regime. We show that these solutions are characterized by nontrivial winding numbers, associated to topological phases, in some part of the corresponding phase diagrams. By means of a comparative study, we demonstrate benefits and limitations of the mean-field approximation, which is the standard approach in the limit of large number of particles. The mean-field analysis also allows to extend part of the results beyond integrability, clarifying the peculiarities associable to the integrability itself.

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II. EXACT SOLUTION IN THE INTEGRABLE CASES

A. General setting

In the present Section we address the exact solution of the Hamiltonian in Eq. (1). We find that the precise forms of $\epsilon_k$ and of the Cooper wavefunctions are constrained by requiring the integrability.

The first step to proceed on is to notice that when only a single fermion occupies the level in $k$ or $-k$ (i.e. without its partner), it decouples from the ground-state dynamics, due to the interaction in Eq. (1). So, it is convenient to restrict ourselves to the dynamics of the Cooper pairs, having creation operators $b_k^\dagger = c_k^\dagger c_{-k}^\dagger$ (see e.g. [7]). Accordingly, the Hamiltonian in Eq. (1) takes the form

$$H = \sum_k 2 \epsilon_k b_k^\dagger b_k - g B_k^\dagger B_0. \tag{2}$$

Due to the particular factorized form of the interaction in Eq. (1), $H$ is now quadratic in terms of the new operator $B_k^\dagger = \sum_k z_k b_k^\dagger$ where $z_k = (k_x - ik_y)^\dagger$ are called pairing functions. Clearly, if the $b_k$ operators were truly bosonic, the Hamiltonian would be directly diagonalizable. However, the $b_k$ are instead hard-core bosons obeying the following commutation relations

$$[b_k, b_{k'}^\dagger] = \delta_{kk'} (1 - 2 b_k^\dagger b_k). \tag{3}$$

As a trial wave function for $n$ pairs, we take the following general Ansatz

$$|\Psi_n\rangle = \prod_{\nu=1}^n B_{J_{\nu}} \cdot |0\rangle, \quad B_{J_{\nu}}^\dagger = \sum_k w_k(J_{\nu}) b_k^\dagger. \tag{4}$$

and impose the eigenvalue equation

$$(H - E_n)|\Psi_n\rangle = 0, \tag{5}$$

where the total energy $E_n$ is given by the sum of the pair energies: $E_n = \sum_{\nu=1}^n E_{J_{\nu}}$.

The next two Subsections will be devoted to the solution of Eq. (5) for one single pair and for multi-pairs configurations. Generally, these solutions are obtained analytically using the algebra of the pseudo-bosonic commutation relations to shift $H$ in Eq. (5) through the operators $B_{J_{\nu}}^\dagger$ contained in $|\Psi_n\rangle$, until $H$ acts on the vacuum $|0\rangle$, giving zero. As the detailed calculation is rather cumbersome, it is presented in Appendix A.

B. One pair case

By imposing the eigenvalue equation in Eq. (5) to one pair $|\Psi_1\rangle$ with energy $E_J$, we obtain the condition:

$$w_k(J) = g \frac{z_k}{2\epsilon_k - E_J} \sum_{k'} z_{k'}^* w_{k'}(J). \tag{6}$$

Multiplying both the sides by $z_k^*$ and summing in $k$ (as customary for the gap equation in the BCS theory [21, 22]), unless the "order parameter" $W(J) = \sum_k z_k^* w_k(J)$ is zero, we obtain the Richardson equation for one pair:

$$1 - g \sum_k \frac{|z_k|^2}{2\epsilon_k - E_J} = 0, \tag{7}$$

as well as the expressions for the Ansatz’s coefficients

$$w_k(J) = g W(J) \frac{z_k}{2\epsilon_k - E_J}, \tag{8}$$

proportional to the wavefunction $\frac{z_k}{2\epsilon_k - E_J}$. The proportionality factors $g W(J)$ do not depend on $k$, thus they are irrelevant and can be neglected, as they affect only normalizations and global phases. Consequently, without any loss of generality, we can just assume to remain with the wavefunction

$$w_k(J) = \frac{z_k}{2\epsilon_k - E_J}. \tag{9}$$

Notice that, the spatial wavefunction (9) has the same parity of $l$ under the transformation $k \rightarrow -k$. This fact has a direct consequence on the symmetry of the spin part of wavefunction, as discussed in the introduction. Moreover, if two spins $\{\uparrow, \downarrow\}$ are involved in the Cooper pair, still at fixed $l$, the forms of the Hamiltonian in Eq. (2) and of the commutators in Eq. (3) (as well the consequent ones including the operators $B_J$, see the Appendix A) stay unchanged. Therefore, also the structure of the Bethe-Ansatz equations and of the spatial part of the exact Cooper wavefunctions do not change.

C. Many pairs

Similarly to the one-pair case in the previous Subsection, the Ansatz in Eq. (4) for the $n$ pairs case reads

$$|\Psi_n\rangle = \prod_{\nu=1}^n B_{J_{\nu}} \cdot |0\rangle, \quad B_{J_{\nu}} = \sum_k \frac{z_k}{2\epsilon_k - E_J} b_k^\dagger, \tag{10}$$

where we have assumed the expression in Eq. (9) for the wavefunctions. The solution of Eq. (5), discussed in detail in Appendix A, yields the final equations analogous to Eq. (7). These solutions can be classified in three groups, depending on the form of $z_k$:
1. The pairing function $z_k$ is independent of $k$. A relevant case is obtained fixing $z_k = 1$, there from Eq. (A9) we get the well-known Richardson equations
\[
1 - g \sum_k \frac{1}{2\epsilon_k - E_{J_\nu}} + 2g \sum_{\mu=1(\neq \nu)}^n \frac{1}{E_{J_\mu} - E_{J_\nu}} = 0, \quad (11)
\]
whose solutions give the pair energies $E_{J_\nu}$ [7]. It is important to observe that here we have not imposed any restrictions on $\epsilon_k$, then any dispersion relation (included the flat band $\epsilon_k = 0$) allows integrability in this case.

2. Beside the original s-wave case $z_k = 1$, we can include also the choice $z_k = \exp[i\phi(k)]$, where $\phi(k)$ is a real function of momentum. Like in the previous case, the energy solutions are given by Eq. (11) and again there are no restrictions on $\epsilon_k$. The present choice extends the previous case, allowing for possible phases with nontrivial topology (see Appendix C).

3. The pairing function is $z_k \propto (k_x - ik_y)^l$. Since in this case $|z_k|^2$ depends on $k$ (for $l \neq 0$), we are forced to have $|z_k|^2 \propto \epsilon_k$ in order to guarantee integrability. As a consequence, after the substitution $|z_k|^2 = \alpha \epsilon_k = \alpha k^2$, Eq. (A9) becomes
\[
1 - \tilde{g} \sum_k \frac{\epsilon_k}{2\epsilon_k - E_{J_\nu}} + \tilde{g} \sum_{\mu=1(\neq \nu)}^n \frac{E_{J_\mu}}{E_{J_\mu} - E_{J_\nu}} = 0. \quad (12)
\]
with $\tilde{g} = g \alpha$. For $l = 1$, our result coincides with the p-wave solution found in [9], with a massive-like dispersion $\epsilon_k \propto k^2$. Remarkably, Eq. (12) holds also for the exact solution of the interesting d-wave case, where the relative angular momentum $l = 2$ imposes a quartic dispersion $\epsilon_k \propto k^4$.

In [9, 10] a detailed analysis has been performed on the specific case (3) with $n = l = 1$, both by a MF approach in the thermodynamic limit and comparing its results with the properties of the exact wavefunction from the solution of the Bethe-Ansatz equations. The topological aspects of the obtained phases have been also discussed.

In the following, we generalize the latter analysis to the wider situation where $n, l \geq 1$ and $n$ and $l$ can be different. If $l \neq n$, integrability is broken, so that only a MF approach can be used. If instead $n = l$, a deeper knowledge is achieved by studying again the topological properties of the exact wavefunctions.

### III. MEAN-FIELD ANALYSIS

#### A. General formalism

In this Section we analyze the MF properties of the Hamiltonian in Eq. (1). Following the standard approach to MF superconductivity [21, 22], we find that the MF quadratic Hamiltonian, in the thermodynamic limit and in the grand canonical ensemble, deriving from the one in Eq. (1), is:
\[
H = E_c + \sum_k \left( \xi_k c_k^\dagger c_k + \Delta (k_x + ik_y)^l c_k c_{-k} + H.c. \right), \quad (13)
\]
where $E_c$ is the condensation energy, defined below, $\xi_k = (\epsilon_k - \mu) = (k^2 - \mu)$ is the rescaled dispersion, and $n$ is assumed half-integer. In the chemical potential $\mu$, the Hartree terms are also included, coming from the Wick contractions of the interaction term in the Hamiltonian of Eq. (1). According to the analysis performed in Sect. II, the integrable cases correspond to $n = l$, however, for sake of completeness, here we do not fix $n$ and $l$ equal in this MF treatment.

The Hamiltonian in Eq. (13) describes potentially realistic cases if $n = l = 1$ (when two spins are considered) [22], and if $n = l = 1$ [13–15].

In Eq. (13) we set $\Delta = \sum_k g (k_x^l + ik_y^l) \langle c_{-k} c_k \rangle$, being $\langle c_{-k} c_k \rangle$ the vacuum expectation value, not vanishing in presence of superconductivity; in this way the gap function characterizing the superconductor ground-state becomes $G_\ell(k) = \Delta (k_x^l + ik_y^l)$. The quantity $\langle k_x + ik_y \rangle$ coincides, up to a constant, with the spherical harmonic $Y_\ell^l(k)$ projected in the 2D plane (expected the more stable one in the absence of external strains or pressures, see e.g. [22]).

The condensation energy $E_C$ is given by
\[
E_C = -4 \sum_{k, k' > 0} \frac{\Delta_k \Delta_{k'}}{9kk'} = A \frac{M \Delta^2}{g}, \quad (14)
\]
where the integer $M$ denotes the number of states in the considered region of phase-space. In a general case, the quantity $A$ is explicitly depending on the assumed two body potential of the full Hamiltonian. For the Hamiltonian in Eq. (1), the potential reads in momentum space
\[
gkk' = -g(k_x - ik_y)^l (k'_x + ik'_y)^l, \quad (15)
\]
so that $A = 1$. As we will check in the next following, an important feature of the ground-state free energy $F_{GS}$ is that, when expressed as a sum on the momenta via the gap equation, it does not depend on $A$, then on the precise form of the assumed potential.

The Bogoliubov spectrum corresponding to the Hamiltonian in Eq. (13) is
\[
\lambda_k = \sqrt{\xi_k^2 + \Delta^2 k^2l}, \quad (16)
\]
($k$ denoting again the modulus of $k_x - ik_y$). This spectrum is gapless at $\mu = 0$ and $k = 0$.

The ground-state free energy $F_{GS} = E_{GS} + \mu N$ corresponding to the spectrum in Eq. (16) is
\[
F_{GS} = \sum_{k > 0} (\xi_k - \lambda_k) + \frac{M \Delta^2}{g} + \mu N, \quad (17)
\]
independent on $A$, as anticipated. The Bogoliubov coefficients are:

\[ |u_k|^2 = \frac{1}{2} \left( 1 + \frac{\xi_k}{\sqrt{\xi_k^2 + \Delta^2 k^2}} \right), \quad |v_k|^2 = 1 - |u_k|^2, \]

so that the MF wave function results:

\[ u_k^{(MF)} = \frac{v_k}{u_k} = \frac{\lambda_k - \xi_k}{\Delta (k_x + i k_y)}. \]  

(19)

The equations for $\Delta$ and for $\mu$ are found as follows:

\[ \frac{\partial F_{GS}}{\partial \Delta} = 0 \rightarrow M = \frac{1}{2} \sum_{k>0} k^2 \lambda_k, \]  

(20)

\[ \frac{\partial F_{GS}}{\partial \mu} = 0 \rightarrow N = \sum_{k>0} \left( 1 - \frac{\xi_k}{\lambda_k} \right). \]  

(21)

The last equation can be also written as:

\[ \mu \sum_{k>0} \frac{1}{\lambda_k} = N + \sum_{k>0} \frac{k^{2n}}{\lambda_k} - \frac{M}{2}. \]  

(22)

which, in the case of $n = l$, becomes from Eq. (20):

\[ \mu \sum_{k>0} \frac{1}{\lambda_k} = N + 2 \frac{M}{g} - \frac{M}{2}. \]  

(23)

Using Eq. (20), the ground-state energy is written as:

\[ F_{GS} = \sum_{k>0} \left( \xi_k - \lambda_k + \frac{\Delta^2 k^2}{2 \lambda_k} \right) + \mu N. \]  

(24)

and, exploiting Eq. (22), also as:

\[ F_{GS} = \sum_{k>0} k^{2n} \left( 1 - \frac{2k^{2n} - 2\mu + \Delta^2 k^{2l-n}}{2 \lambda_k} \right). \]  

(25)

If $n = l$, the latter expression shows a duality between different MF solutions, consisting in that two solutions (labelled 1 and 2) are related by the equations $\mu_1 = -\mu_2$ and $\Delta_1^2 - 2\mu_1 = \Delta_2^2 - 2\mu_2$, such that the corresponding free energies coincide: $F_{GS}^{(1)} = F_{GS}^{(2)}$. If $n = l = 1$, this duality is justified by the exact solution of the Richardson equations (11).

Once one considers to work in a lattice, as opposed to the continuum, the above analysis can be extended straightforwardly. Some spin models are indeed quadratic in Fermi operators in momentum space with pair creation [23]. For sufficiently small interaction strength $\propto g$, we expect that superconductivity involves only quasiparticles with momenta within a small range $\delta k \approx g^2$ around the Fermi momentum $k_F$. Here the lattice dispersion, with discretized momenta, can be expanded in powers of $k$, such to end up in a power-law dispersion. At that point, the MF analysis proceeds as described before.

### B. Mean-field phase diagram

Using the derived expressions for the ground-state free energy, for the wave functions of the Bogoliubov excitations and for the self consistency equations, it is interesting to characterize the phase diagram of the Hamiltonian in Eq. (13), as a function of $g$ and of the (average) filling $N/M = x$.

Various transition lines, between different quantum phases, can be identified. A notable transition occurs at $\mu = 0$, where the spectrum in Eq. (16) is gapless at $k = 0$. There the MF wavefunction behaves as:

\[ u_k^{(MF)} \approx \frac{(k_x - i k_y)^l k^{2(n-l)}}{(\sqrt{k_x^2 + k_y^2})^l} \]  

if $\mu < 0$ and $n > l$, \[ (k_x - i k_y)^l k^{2(n-l)} \]  

if $\mu < 0$ and $n < l$, \[ \frac{1}{(k_x + i k_y)^l} \]  

if $\mu > 0$.  

(26)

This transition has a similar nature of the Read-Green one described in the case $n = l = 1$ [9, 10, 24], for this reason in the following the same name will be adopted for it. The condition $\mu = 0$ translates, from Eq. (23), in the relation:

\[ x = \frac{1}{2} \left( 1 - \frac{4}{\Delta^2} \right). \]  

(27)

This line does not depend on the distribution of the momenta, then is topologically protected against every perturbation changing it and possibly breaking the integrability of the Hamiltonian in Eq. (1).

Another notable line, denoted as (generalized) Moore-Read line [9], is found for every $n = l$, parametrized by the relation $\mu = \frac{\Delta^2}{4}$; along this line the condition $F_{GS} = 0$ holds: the same free energy of the vacuum, intended as absence of fermions ($x = 0$), is obtained for the superconductive ground-state. Notice that, in order to obtain this result, the positiveness of $\mu$ is crucial. The condition $\mu = \frac{\Delta^2}{4}$ is fulfilled on the line:

\[ x = \left( 1 - \frac{4}{g} \right), \]  

(28)

a result found exploiting Eq. (21). There the mass gap does not vanish but the ground-state free energy is discontinuous in the thermodynamic limit.

As for the case $n = l = 1$ [9], the duality mentioned in the previous section holds between a point $(g, x_w)$ in the weak pairing regime ($\mu > 0$) and a point $(g, x_s)$ in the strong pairing regime ($\mu < 0$); these points are related each other by the relation

\[ x_w + x_s = \left( 1 - \frac{4}{g} \right) \]  

(29)

still obtained directly from Eqs. (21) and (23). Therefore the Read-Green line is self-dual, while the MR state is dual to the vacuum, where $x = 0$.

The Read-Green and Moore-Read line meet at the point $g = 2$, where the limit $x = 0$ is achieved.
By direct numerical inspection, we have found strong indications that the Moore-Read line does not persist in general out of the integrability [25].

If \( n = l \), the minimum \( E_{\text{GAP}} \) of \( \lambda(k) \), Eq. (16), is

\[
E_{\text{GAP}} = \begin{cases} 
|\mu| & \text{if } \mu < \frac{\Delta^2}{4} \\
\Delta \sqrt{|\mu - \frac{\Delta^2}{4}|} & \text{if } \mu > \frac{\Delta^2}{4} 
\end{cases}.
\]

(30)

The condition \( \mu = \frac{\Delta^2}{4} \) defines a third notable transition line, the so-called Volovik line [9, 10]. Along it a first-order quantum phase transition, reminiscent of the Higgs transition, occurs [13]. The same line is depending on the distribution of the momenta, then it is not topologically protected (and its presence must be verified beyond the MF approach); we find that it reads explicitly as:

\[
x = \frac{1}{2} - \frac{1}{M} \sum_{k>0} \frac{k^2 - \mu}{\Delta_k}.
\]

(31)

The Volovik line remains stable if \( n > l \) (the defining equation, as (31), not being easy writable as a closed formula), while it disappears if \( n < l \), since \( E_{\text{GAP}} \) arises always at \( k \neq 0 \).

IV. TOPOLOGICAL PROPERTIES

In this Section we give a deeper characterization of the MF phase diagram, sketched in the previous Section, studying the topology of the various identified phases. Focusing first on the case \( n = l \), we start by taking the MF Cooper wavefunction \( u_k^{(\text{MF})} \) in Eq. (19) to calculate the topological invariant [10]:

\[
I_{\text{MF}} = \frac{1}{4\pi} \int_{S^2} \text{d}k \ u_k
\]

(32)

where \( S^2 \) is the sphere of radius \( |k| = 1 \) obtained from the plane \( R^2 \) by the inverse of the stereographic projection [26, 27]. We obtain \( I_{\text{MF}} = l \) if \( \mu > 0 \), while \( I_{\text{MF}} = 0 \) if \( \mu < 0 \). As generally expected (see e. g. [19]), \( I_{\text{MF}} \) is sensitive to the vanishing of the energy for the Bogoliubov quasiparticles, occurring at \( \mu = 0 \). Alternatively, the same results can be achieved, still in the MF framework, following a procedure common in the study of topological insulators [26]. In particular, denoting by \( |u_k\rangle \) the positive-energy eigenvector of the quadratic Hamiltonian in Eq. (13), \( I_{\text{MF}} \) is expressed as the integral on the momentum space of the Berry curvature:

\[
I_{\text{MF}} = \frac{1}{4\pi} \int_{S^2} \text{d}k \nabla \times \langle u_k | \nabla | u_k \rangle.
\]

(33)

The equivalence between the two MF calculations for \( I_{\text{MF}} \) stems directly from the fact that \( |u_k\rangle \) is an excited state obtained breaking a Cooper pair.

The content in topology obtained using the MF wavefunction can be probed calculating the same quantity in Eq. (32) by exact wavefunction \( w_k \) of a single Cooper pair, then considering again the limit \( x = 0 \). We implicitly assume that fluctuations beyond MF do not change significantly the MF phase diagram, then the solution of the Bethe- Ansatz equations essentially leads to the same phase diagram, this hypothesis will be not contradicted in the following. The exact wavefunction, derived in Section II, reads, up a unimportant multiplicative constant:

\[
w_k = \frac{(k_x - Ik_y)^l}{2\kappa - E}.
\]

(34)

where \( E \) is the pair energy (complex in general [7]), derived from the solution of the Richardson equations. The integral in Eq. (32) can be recast as follows:

\[
I = l^2 \int_0^\infty \text{d}u \ u^{(3l-1)} - E\bar{E} u^{(l-1)} \left( u^2 + (u^2 - \bar{E})(u^2 - \bar{E})\right)^2,
\]

(35)

with \( k^2 \). The result of Eq. (35) is

\[
\begin{cases} 
I = l \text{ if } E = 0 \\
I = 0 \text{ if } E \neq 0.
\end{cases}
\]

(36)

An alternative derivation of the winding number \( I \) is discussed in the Appendix B; this turns out useful also for the pure phase case in the Appendix C.

Referring to the MF diagram in Fig. 1, the condition \( E = 0 \) is fulfilled in the limit \( x \to 0 \) at the intersection with the Moore-Read line, at \( g = 2 \). This results indicates that \( I = l \) in the region between the Read-Green and the Moore-Read lines, while \( I = 0 \) in the other phases. The same result for \( I \) seems to match the MF phase diagram, while \( I_{\text{MF}} \) fails in the region on the
right of the Moore-Read line, being nonvanishing also there (for this reason, the Moore-Read line is not detected by $J_{MP}$). This mismatch is indeed interesting, since it can indicate a general inability of the topological invariants from the MF wavefunctions to correctly detect some phases of (topological) insulators or superconductors. In our case, the mismatch occurs since the mass gap does not vanish on the MR line. It remains an open question whether the origin of the puzzle is due to integrability of the full model in Eq. (1). However, such interpretation is suggested by the fact that from the MF analysis the MR line seems generally absent for $n \neq l$, where integrability is broken (and no divergencies occur in the spectrum, a situation found instead in the presence of long-range Hamiltonian couplings, see [28] and references therein [29]).

V. DISCUSSION AND CONCLUSIONS

In this paper we have extended the family of the superconductive models for which an exact solution is available. In particular, we have considered a large set of two-dimensional systems with a factorized form for the momentum dependent interaction. Beside the known cases of the $s$-wave pairing, solved by Richardson [4], and $p$-wave pairing, discussed for the first time by Ibañez et al. in [9], we have found that in general $l$-wave pairing is exactly solvable on a finite-size system, provided that the single particle dispersion is proportional to $k^{2n}$, with $n = l$.

We also found that in the integrable cases the topological invariants calculated in the framework of the mean-field approach can not correctly reproduce phase diagram, at variance with the corresponding invariants obtained from the exact (Bethe-Ansatz) solutions for the superconductive states. This discussion has shown the potential inadequacy of the mean-field topological invariants to predict the correct phase diagram of (topological) insulators and superconductors. In our cases, the origin of this problem seems the presence of quantum phase transitions without vanishing of the mass gap, a feature possibly related to integrability. We notice that quite recently a change of topology without mass gap closing, in the presence of large interaction, has been found numerically in [30].

In the non-integrable cases $n \neq l$ (as well as for other perturbed where interactions do not assume the special form of Eq. (1)), the exact wavefunctions analogous to Eq. (9) cannot be derived, as well as the exact solutions from Bethe-Ansatz, therefore only the the mean-field approach can not correctly reproduce phase diagram of (topological) insulators and superconductors. This discussion has shown the potential absence of quantum phase transitions with nonvanishing mass gap.

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In the last term of Eq. (A5), we want to commute the operator $b_k^\dagger b_k$. At this point, it is crucial to use the following manageable

\[ L. Lepori, D. Giuliano, and S. Paganelli, \text{arXiv:1707.05777}. \]

where the commutator in the left side expands as

\[ \sum_{\nu=1}^{n} \left\{ \left( \prod_{\eta=1}^{\nu-1} B_{\eta J_{\nu}}^\dagger \right) \left[ H_{\nu J_{\nu}} - \mathcal{E}_n \prod_{\nu=1}^{n} B_{\nu J_{\nu}}^\dagger \right] \right\}. \quad (A2) \]

Putting Eq. (A4) in Eq. (A2) and using the basic relation $H|0\rangle = 0$, we find:

\[
H|\Psi_n\rangle = \mathcal{E}_n|\Psi_n\rangle + \sum_{\nu=1}^{n} \left[ \left( \sum_{k} (2\epsilon_k - E_{J_{\nu}})w_k(J_{\nu})b_k^\dagger - gB_0^\dagger \sum_{k} z_k^* w_k(J_{\nu}) \right) \left( \prod_{\nu \neq \mu \neq 0} B_{\mu J_{\mu}}^\dagger \right) \right] |0\rangle 
+ \sum_{\nu=1}^{n} \left\{ \left( \prod_{\eta=1}^{\nu-1} B_{\eta J_{\nu}}^\dagger \right) 2gB_0^\dagger \sum_{k} z_k^* w_k(J_{\nu})b_k^\dagger b_k \left( \prod_{\mu=\nu+1}^{n} B_{\mu J_{\mu}}^\dagger \right) \right\} |0\rangle. \quad (A5)
\]

In the last term of Eq. (A5), we want to commute the operator $b_k^\dagger b_k$ to the extreme right, where it annihilates the vacuum $|0\rangle$. To this aim, we write this term as

\[
\sum_{\nu=1}^{n} \left\{ \left( \prod_{\eta=1}^{\nu-1} B_{\eta J_{\nu}}^\dagger \right) \sum_{\mu=\nu+1}^{n} \left\{ \sum_{k} z_k^* w_k(J_{\nu})b_k^\dagger b_k B_{\mu J_{\mu}}^\dagger \right\} \right\} |0\rangle. \quad (A6)
\]

At this point, it is crucial to use the following manageable form for the commutator in Eq. (A6):

\[
\sum_{k} z_k^* w_k(J_{\nu})b_k^\dagger b_k, B_{\nu J_{\nu}}^\dagger = \sum_{k} z_k^* w_k(J_{\nu})w_k(J_{\mu})b_k^\dagger. \quad (A7)
\]

In general, for every $\mu$ and $\nu$, we want to express Eq. (A7) in the form $C_{\mu,\nu}B_{\mu J_{\mu}}^\dagger + D_{\mu,\nu}B_{\nu J_{\nu}}^\dagger$, where $C_{\mu,\nu}$ and $D_{\mu,\nu}$ are some coefficients. For this reason, we impose the condition

\[
\sum_{k} z_k^* w_k(J_{\nu})w_k(J_{\mu})b_k^\dagger = C_{\mu,\nu}B_{\mu J_{\mu}}^\dagger + D_{\mu,\nu}B_{\nu J_{\nu}}^\dagger, \quad (A8)
\]

where we have used the symmetry under the exchange $\nu \leftrightarrow \mu$. Assuming that Eq. (A8) is correct, then we find that the eigenvalue equation Eq. (A5) holds, provided that

\[
1 - g \sum_{k} \frac{|z_k|^2}{2\epsilon_k - E_{J_{\nu}}} + 2g \sum_{\nu=1}^{n} C_{\nu,\mu} = 0, \quad (A9)
\]

where we have used the expression for the wave function $w_k(J_{\nu}) = \frac{\epsilon_k - E_{J_{\nu}}}{2\epsilon_k - E_{J_{\nu}}}$. Eq. (A8) gives

\[
(2\epsilon_k - E_{J_{\nu}})C_{\mu,\nu} + (2\epsilon_k - E_{J_{\nu}})C_{\nu,\mu} = |z_k|^2
\]

with two different kind of solutions:

1. **s-wave**. In this case $|z_k|^2 = 1$ and $C_{\mu,\nu} = C_{\nu,\mu} = (E_{J_{\nu}} - E_{J_{\nu}})^{-1}$. Thus, from (A9) we get the well-known Richardson equation Eq. (11), with no restrictions on $\epsilon_k$. Notice that the condition $|z_k|^2 = 1$ is more general than the s-wave case $z_k = 1$. Using the relations

\[
\begin{align*}
B_{\mu J_{\mu}}^\dagger b_k^\dagger b_k, & = w_k(J)b_k^\dagger, \quad (A3) \\
B_0, B_{\mu J_{\mu}}^\dagger & = z_k w_k(J)(1 - 2b_k^\dagger b_k). \quad (A4)
\end{align*}
\]
2. *l*-wave. In this case, \( z_{k} = (k_x - i k_y)^l \) depends on \( k \) (for \( l \neq 0 \)) and the coefficients are given by
\[
C_{\mu, \nu} = \frac{|z_k|^2}{2E_k} \frac{E_{J_{\nu}}}{E_{J_{\mu}} - E_{J_{\mu}}},
\]
but we must have \( |z_k|^2 \propto \epsilon_k \) for having an \( C_{\mu, \nu} \) independent of \( k \). As a consequence, after the substitution \( |z_k|^2 = \alpha \epsilon_k \), Eq.(A9) becomes Eq. (12).

**Appendix B: Alternative calculation of \( I \)**

In this Appendix we discuss an alternative derivation of the winding number \( I \), also useful for the pure phase case in the Appendix C, that can be performed analyzing directly the map \( \omega_k \) in the case of real \( E \). In order to do that, we separate first Eq. (34) as
\[
\omega_k = (f_-(k) + f_+(k)) e^{i\phi_k l}
\]
with \( f_-(k) = \frac{k^l}{|k|^l} \), \( k < E^{1/2l} \) and \( f_+(k) = \frac{k^l}{|k|^l} \), \( k > E^{1/2l} \).

The part \( f_+(k) e^{i\phi_k l} \) gives a contribution \( I_+ = l \) to \( I \), since \( f_+(k) \) is monotonic in \( k \) and assumes values \( [0, \infty) \), so that \( f_+(k) e^{i\phi_k l} \) covers \( l \) times (because of the phase \( l \phi_k \)) the entire plane \( R^2 \sim S^2 \) (the identification relying again in the stereographic projection).

Assuming now \( E \neq 0 \), we put \( k = 1/p \) in \( f_-(k) \), obtaining \( f_-(p) = -\frac{1}{E} \frac{p^l}{|p|^l} = -f_+(p) \), with \( p > E^{1/2l} \). Apart from the unimportant multiplicative factor \( E^{-1} \), we can write (renaming \( p \equiv k \))
\[
\omega_k = (f_-(k) - f_+(k)) e^{i\phi_k l} = 0
\]
showing that \( I = 0 \) if \( E \neq 0 \). The minus sign in \( f_-(p) \), responsible of the vanishing result for \( I \), is related with the fact that, for \( k \) varying, \( f_+(k) \) and \( f_-(k) \) span the space \( R^2 \sim S^2 \) in opposite sense.

The situation is different if \( E = 0 \): in this case we get only
\[
\omega_k = f_+(k) e^{i\phi_k l}
\]
and \( I = I_+ = l \).

**Appendix C: Pure phase gap**

We can also calculate the topological index \( I \) in the case when \( \Delta(k) = e^{i\phi_k l} \). In this condition, we have shown in Section II that we have integrability, no matter the particular single particle dispersion \( \epsilon_k \); therefore, we assume again \( \xi_0(k) = k^{2l} \). The exact wave function reads in momentum space and up a unimportant multiplicative constant:
\[
\omega_k = \frac{(k_x - i k_y)^l}{k^l (2|k| - E)} \tag{C1}
\]
In this case we obtain:
\[
I = 2l^2 \int_0^{\infty} dk \frac{k^{2l-1} [2k^{2l} - (E + \tilde{E})]}{(1 + (k^{2l} - E)(k^{2l} - \tilde{E}))^2}.
\]
This integral yields \( I = \frac{l}{k^{2l-1}} \), a pretty unexpected result, since in general a integer winding number should be expected. However this result can be explained quite naturally by analyzing directly the map (C1). This map can be expressed as:
\[
\omega_k = \frac{1}{k^{2l} - E} e^{i\phi_k l}.
\]
As for (34), we can write again:
\[
\omega_k = (f_-(k) + f_+(k)) e^{i\phi_k l}
\]
with \( f_-(k) = \frac{1}{k^{2l} - E} \), \( k < E^{1/2l} \), and \( f_+(k) = \frac{1}{k^{2l} - E} \), \( k > E^{1/2l} \). We notice that \( f_+(k) e^{i\phi_k l} \) is homotopic to a constant map \( f_-(k) = c \), since \( f_-(k) = (c, -\frac{1}{E}) \) (the minus sign being re-absorbable in the phase \( \phi_k \)) and not every point of the target stereographic plane \( R^2 \) is covered by \( f_-(k) e^{i\phi_k l} \). Then we can write
\[
\omega_k = (f_-(k) + f_+(k)) e^{i\phi_k l} \sim f_-(k) e^{i\phi_k l} \tag{C5}
\]
(the symbol \( \sim \) means here "continuously deformable to"). Since again \( f_+(k) = [0, \infty) \) and is monotonic, it yields a contribution \( I_+ = l \) to \( I \) for every value of \( E \). However \( f_-(k) \) gives a non vanishing contribution to \( I \), covering a part of sphere with area
\[
I_+ = -\frac{1}{\pi} \int_{k}^{\infty} \frac{2\pi k}{1 + k^2} dk = -\frac{E^2}{E^2 + 1},
\]
where the minus sign appears since \( \lim_{k \to \infty} f_-(k) \to -\infty \). This contribute sums up to \( I_+ \), to give the result (C2):
\[
I = I_+ + I_- = l - l \frac{E^2}{E^2 + 1} = l \frac{E^2}{E^2 + 1}.
\]
In spite of the value of \( I \), the real winding number related to (C1) is \( \tilde{I} = I_+ = l \), since we know that \( f_-(k) \) is homotopic to a constant map, a fact also resulting in the value of \( |I_-| \), smaller than 1.

This result matches the fact that the BCS case and the (C3) case are linked by the transformation in the gap \( \Delta \to \Delta(k) = \Delta e^{i\phi_k l} \). However, this map is continuous but not invertible, wrapping \( l \) times: this is the reason of \( I = l \).

In conclusion, the case (C3) describes a phase with winding number \( I = l \frac{N}{2} \) (\( \frac{N}{2} \) being the number of Cooper pairs in the ground-state). However, the energy of Bogoliubov quasiparticles is the same as in the BCS case, always gapped, then no phase transitions arises and the system is always in a phase with nontrivial topology.