Asymptotic infinite-dimensional theory of Banach spaces

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In this paper we study structural properties of infinite dimensional Banach spaces. The classical understanding of such properties was developed in the 50s and 60s; goals of the theory had direct roots in and were natural expansion of problems from the times of Banach. Most of surveys and books of that period directly or indirectly discussed such problems as the existence of unconditional basic sequences, the $c_0$-$\ell_1$-reflexive subspace problem and others. However, it has been realized recently that such a nice and elegant structural theory does not exist. Recent examples (or counter-examples to classical problems) due to Gowers and Maurey [GM] and Gowers [G.2], [G.3] showed much more diversity in the structure of infinite dimensional subspaces of Banach spaces than was expected.

On the other hand, structure of finite dimensional subspaces of Banach spaces and related local properties have been well understood in the last two decades. Many exciting theorems on the behaviour of high dimensional subspaces, finite rank operators, quotient spaces and others were discovered. They have an asymptotic nature: dimension should increase to infinity to reveal regularities behind an increasing diversity of discussed objects (cf. e.g., [MiSch], [P.1], [P.2], [T]).

In this paper infinite dimensional phenomena are investigated by using a similar asymptotic approach. To envisage such phenomena, we discard all informations of a finite dimensional nature and study properties of a space “at infinity”. This naturally motivates a fundamental concept of asymptotic finite-dimensional spaces of X, which will be explained later in this introduction. The main idea behind it is a stabilization at infinity of finite dimensional subspaces which appear everywhere far away. This further leads
to an infinite-dimensional construction resulting in a notion of an *asymptotic version* of \(X\).

Similar stabilization procedures in the form of the notions of spectrum and tilda-spectrum, were studied back at the end of the 60s in [Mi1] and [Mi2]. Originally a complete stabilization procedure was used in [Mi.1] for stabilizing special geometric moduli, so-called \(\beta\)- and \(\delta\)-moduli (see also recent applications of this approach in [MiT]). This was achieved by considering functions \(f_\lambda(x, y) = \|x + \lambda y\|\) on the unit sphere \(S(X)\) of \(X\); in the case when a complete stabilization of these functions on \(S(X)\) was possible, \(X\) was shown to contain an infinite-dimensional \(l_p\) subspace. Different, although close, asymptotic view was taken in [KM] through the notion of stable spaces. Again, it was proved that a stable space \(X\) contains an infinite-dimensional \(l_p\) subspace. In both approaches strong stabilization conditions allowed a complete recovery of some infinite-dimensional subspaces through a construction of what we would call today a “stabilized asymptotic version”.

We would like to mention in this context that for spaces \(l_p\) for \(1 < p < \infty\) there exist Lipschitz functions on the sphere (in fact, equivalent norms) which do not allow a complete stabilization on any infinite-dimensional subspace of \(l_p\). This is a weak form of a recent distortion result by Odell and Schlumprecht [OS.2], which gives a counter-example to a conjecture posed in [Mi.1] and [Mi.2].

An extension of the approach from [Mi.1] was presented in a recent paper [MiT], in which an isomorphic version of the stabilization property was investigated. Then of course it is necessary to consider several variables analogues of the above moduli, on the expense of a clear geometric interpretation. This leads to a definition of upper and lower *envelopes* (see [MiT] and also [L3]), which in the particular cases of so-called bounded distortions, give raise to the definition of asymptotic \(\ell_p\)-spaces. Under the same assumption of bounded distortions, slightly different stabilization procedure was also considered in [Ma]. It would be interesting to find an isomorphic version of stable spaces.

Notions of asymptotic type and cotype and of asymptotic unconditionality were used in [MiSh] to study complementation.

The mentioned above notion of an asymptotic version of a given space \(X\) should be compared with so-called spreading model (we recall the definition in [L3]), which also reflect some properties of a space “at infinity”. However, the spreading model construction involves only subsequences of a given sequence in \(X\); thus improving properties of underlying space too much, while
possibly missing phenomena occurring on block subspaces. In contrast, our asymptotic versions preserve all asymptotic finite-dimensional properties of a space, just choosing its “right” finite-dimensional pieces positioned everywhere, and then putting them together into one infinite-dimensional space.

Let us now describe in rather imprecise terms the intuition of an asymptotic structure of an infinite dimensional Banach space $X$. Such a structure is defined by a family $\mathcal{B}(X)$ of infinite dimensional subspaces of $X$ satisfying a filtration condition which says that for any two subspaces from $\mathcal{B}(X)$ there is a third subspace from $\mathcal{B}(X)$ contained in both of them (see [1.1]); the main example is the family $\mathcal{B}^0(X)$ of all subspaces of finite codimension in $X$. Then, for every $k$, we define the family $\{X\}_k$ of asymptotic $k$-dimensional spaces associated to this asymptotic structure as follows (exact definitions are given in the next section, see [1.3.3]).

Fix $k$ and $\varepsilon > 0$. Consider a “large enough” number $N_1$, a “far enough” subspace $E_1$ of codim $E_1 = N_1$, and an arbitrary vector $x_1 \in S(E_1)$. Next consider a number $N_2 = N_2(x_1)$, depending on $x_1$ and again “large enough”, a “far enough” subspace $E_2 \subset E_1$ of codimension $N_2(x_1)$ and an arbitrary vector $x_2 \in S(E_2)$. In the last $k$th step, we have already chosen normalized vectors $x_1, \ldots, x_{k-1}$ and subspaces $E_{k-1} \subset \cdots \subset E_2 \subset E_1$; we then choose a “far enough” $E_k \subset E_{k-1}$ with codim $E_k = N_k(x_1, \ldots, x_{k-1})$ and an arbitrary vector $x_k \in S(E_k)$. (Note that this description is intentionally somewhat repetitious–since a natural meaning of “far enough” subspaces should imply that their (finite) codimension is automatically “large enough”.)

We call a space $E = \text{span} \{x_1, \ldots, x_k\}$ a permissible subspace (up to $\varepsilon > 0$) and $\{x_i\}_1^k$—a permissible $k$-tuple if for an arbitrary choice of $N_i$ and $E_i$ (with codim $E_i = N_i$) we would be able to choose normalized vectors $\{y_i \in E_i\}$ so that a basic sequence $\{y_i\}_1^k$ is $(1 + \varepsilon)$-equivalent to $\{x_i\}_1^k$.

Now we can also clarify the imprecise notion of “far enough” subspaces $E_i$: by this we mean that an arbitrary choice as above of $x_i \in E_i$ results in a permissible (up to $\varepsilon > 0$) $k$-tuple $\{x_i\}_1^k$ and a permissible (up to $\varepsilon > 0$) subspace $E = \text{span} \{x_i, \ldots, x_k\}$. The existence of such subspaces “far enough” and of associated $N_i$s, will be proved in the next section by some compactness argument.

If $F(k;\varepsilon)$ is the set of all $k$-dimensional $\varepsilon$-permissible subspaces then we put $\{X\}_k = \bigcap_{\varepsilon>0} F(k;\varepsilon)$, and we call every space from $\{X\}_k$ a $k$-dimensional asymptotic space of $X$. Thus, permissible subspaces are $(1 + \varepsilon)$-realizations
of asymptotic spaces.

Finally, a Banach space $Y$ is an asymptotic version of $X$, if $Y$ has a monotone basis $\{y_i\}_1^\infty$ and for every $n$, $\{y_i\}_1^n$ is a basis in an asymptotic space of $X$ i.e., span $[y_i]_1^n \in \{X\}_n$.

Families of asymptotic spaces and asymptotic versions of a given Banach space have interesting properties and reveal a new structure of the original space. For example, in Section 3 it is proved that for a fixed $p$, with $1 \leq p < \infty$, if $X$ is a Banach space such that there exists $C$ such that for every $n$, every space $E \in \{X\}_n$ is $C$-isomorphic to $\ell_p^n$, then every asymptotic version $Y$ of $X$ is isomorphic to $\ell_p$ and the natural basis of $Y$ is equivalent to the natural basis of $\ell_p$. It means that in such a space (called an asymptotic $l_p$-space) all permissible subspaces lie only along its natural $l_p$ basis.

Some properties of families of asymptotic spaces $\{X\}_n$ can be demonstrated through the notion of envelopes. For any sequence with finite support $a \in c_{00}$ the upper envelope is a function $r(a) = \sup \| \sum_i a_i e_i \|$, where the supremum is taken over all natural bases $\{e_i\}$ of asymptotic spaces $E \in \{X\}_n$ and all $n$. Similarly, the lower envelope is a function $g(a) = \inf \| \sum_i a_i e_i \|$, where the infimum is taken over the same set. The functions $r$ and $g$ are always very close to some $l_p$- (and $l_q$)-norms (see 1.9 for an exact statement).

An interesting general property of asymptotic versions is that some of them are, in a sense, stable under iteration. Precisely, we show in Section 2 that for an arbitrary space $X$ there is a special asymptotic version $Y$, called universal, such that its asymptotic structure is the same as for $X$. In particular this implies that not every space $X$, even with an unconditional basis, can be a universal asymptotic version of any Banach space.

In Section 5 we study a complementation problem, and again the asymptotic approach significantly simplifies the picture with respect to “classical” facts.

1 Asymptotic and permissible spaces

We follow [LT.1] for standard notation in the Banach space theory; in particular, fundamental techniques concerning Schauder basis, which will be repeatedly used throughout the paper, can be found in [LT.1] 1.a.

Let $X$ be a Banach space. By $\mathcal{B}(X)$ we denote the family of all subspaces of $X$ of finite-codimension. If $\{u_i\}$ is a basis in $X$, or more generally, a
minimal system in $X$, by $\mathcal{B}^t(X)$ we denote the family of all tail subspaces of $X$, i.e., subspaces of the form $X^n = \text{span}\{u_i\}_{i>n}$, for some $n \in \mathbb{N}$.

By $\mathcal{M}_n$ we denote the space of all $n$-dimensional Banach spaces with normalized bases whose basis constant is smaller than or equal to 2. Given two such spaces $E$, with the basis $\{e_i\}$ and $F$, with the basis $\{f_i\}$, by $d_b(E, F)$ we denote the equivalence constant between the bases, i.e.,

$$d_b(E, F) = \|I : E \to F\| \|I^{-1} : F \to E\|,$$

where $I$ is defined by $Ie_i = f_i$, for $i = 1, \ldots, n$. Then $\log d_b$ is a metric on $\mathcal{M}_n$ which makes it into a compact space.

1.1 An asymptotic structure of $X$ will be defined with respect to a fixed family $\mathcal{B}(X)$ of infinite-dimensional subspaces of a space $X$, which satisfies the filtration condition

For every $X_1, X_2 \in \mathcal{B}(X)$ there exists $X_3 \in \mathcal{B}(X)$ such that $X_3 \subset X_1 \cap X_2$.

By far most important examples of such a family are $\mathcal{B}^0(X)$ and $\mathcal{B}^t(X)$.

1.2 We will work with asymptotic games in which there are two players $S$ and $V$. Rules of moves are the same for all games. Set $X_0 = X$. In the $k$th move, player $S$ chooses a subspace $X_k \in \mathcal{B}(X)$, and then player $V$ chooses a vector $x_k \in S(X_k)$ in such a way that the vectors $x_1, \ldots, x_k$ form a basic sequence with the basis constant smaller than or equal to 2. Further rules will ensure that the games will stop after a finite number of steps.

1.3 Given a space $E \in \mathcal{M}_n$ with a basis $\{e_i\}$, and $\varepsilon > 0$, the vector game associated to $E$ is an asymptotic game in which the vector player $V$ wins if after $n$ moves the vectors $\{x_i\}$ are $(1 + \varepsilon)$-equivalent to $\{e_i\}$. We say that $V$ has a winning strategy for $E$ and $\varepsilon$, if $V$ can win every vector game as above.

1.3.1 Since choosing by $S$ a smaller subspace puts $V$ in a worst position then the filtration property of $\mathcal{B}$ implies that without loss of generality we can assume that, additionally, $X_k \subset X_{k-1}$, for $1 \leq k \leq n$.

Similarly, given $\delta > 0$, by an appropriate choice of subspaces (cf. [LT.1] 1.a.5), $S$ can always ensure that the vectors $\{x_i\}$ have the basis constant less than $1 + \delta$. 

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1.3.2 It follows that $V$ has a winning strategy for a vector game for $E$ and for every $\varepsilon > 0$ if and only if

$$\sup_{X_1} \inf_{x_1 \in S(X_1)} \sup_{X_2} \inf_{x_2 \in S(X_2)} \ldots \sup_{X_n} \inf_{x_n \in S(X_n)} d_b([x_1, \ldots, x_n], E) = 0,$$

with $X_k \in \mathcal{B}(X)$, and $X_k \subset X_{k-1}$ for $1 \leq k \leq n$. (Similar formulas were used to define fundamental notions in [MiT], which were based on some of concepts introduced in [Mi.1].)

1.3.3 Definition A space $E \in \mathcal{M}_n$ with a basis $\{e_i\}$ is called an asymptotic space for $X$ if $V$ has a winning strategy for a vector game in $X$ for $E$ and for every $\varepsilon > 0$. Vectors $\{x_1, \ldots, x_n\}$ in $X$ resulting from a vector game (for some asymptotic space $E$ and for $\varepsilon > 0$) in which $V$ wins, are called a permissible $n$-tuple and the subspace $\text{span } [x_i]$ is called a permissible subspace of $X$.

So a permissible subspace is a $(1 + \varepsilon)$-realization in $X$ of an asymptotic space (for some $\varepsilon > 0$).

The set of all $n$-dimensional asymptotic spaces for $X$ is denoted by $\{X\}_n$. Every $E \in \{X\}_n$ has the natural basis which is monotone (by the last comment in 1.3.1). It is easy to see that the set $\{X\}_n$ is closed in $\mathcal{M}_n$.

1.4 Given set $F \subset \mathcal{M}_n$ and $\varepsilon > 0$, the subspace game is a game in which the subspace player $S$ wins if after $n$ moves, vectors $\{x_i\}$ resulting from the game are $(1 + \varepsilon)$-equivalent to the basis in some space from $F$.

Player $S$ has a winning strategy for a subspace game for $F$ and $\varepsilon$, if $S$ can win every such game. The filtration property clearly implies that $S$ can always choose subspaces satisfying $X_k \subset X_{k-1}$ for $1 \leq k \leq n$, this way only improving his chances to win. Therefore we will always assume that winning strategy for $S$ in a subspace game satisfies this condition.

1.4.1 It follows that player $S$ has a winning strategy for a subspace game for $F$ and for every $\varepsilon > 0$ if and only if

$$\inf_{X_1} \sup_{x_1 \in S(X_1)} \inf_{X_2} \sup_{x_2 \in S(X_2)} \ldots \sup_{x_n \in S(X_n)} \inf_{F \in F} d_b([x_1, \ldots, x_n], F) = 0,$$

with $X_k \in \mathcal{B}(X)$, and $X_k \subset X_{k-1}$ for $1 \leq k \leq n$. 6
1.4.2 To visualize both formulae 1.3.2 and 1.4.1, we can think about a certain tree-like structure of subspaces $X_k$ from the partially ordered by inverse inclusion set $\mathcal{B}(X)$, and arbitrary vectors $x_k \in S(X_k)$, with choices of subsequent subspaces depending on the earlier vectors. Then the vector player $V$ has a winning strategy in a vector game for some space $E \in \mathcal{M}_n$ and $\varepsilon > 0$, if $V$ can find, arbitrarily far along $\mathcal{B}(X)$, vectors $\{x_i\}$ which are $(1 + \varepsilon)$-equivalent to the basis in $E$. The subspace player $S$ has a winning strategy in a subspace game for a subset $\mathcal{F} \subset \mathcal{M}_n \setminus \emptyset$ and $\varepsilon > 0$, if by choosing subspaces $X_k$ far enough along $\mathcal{B}(X)$, $S$ can ensure that the vectors $\{x_k\}$ are $(1 + \varepsilon)$-equivalent to the basis of some space from $\mathcal{F}$.

1.4.3 We will show in 1.5 below that the subspace player $S$ has a winning strategy in a subspace game for $\{X\}_n$ and for every $\varepsilon > 0$. We will then repeatedly use this fact to show that if in an arbitrary asymptotic game $S$ follows his winning strategy for a fixed $\varepsilon > 0$, then a subspace of $X$ resulting in the game is a $(1 + \varepsilon)$-representation of some asymptotic space from $\{X\}_n$. This yields in particular that this subspace is permissible, without actually stating which space from $\{X\}_n$ does it represent.

1.4.4 Let $\mathcal{W}$ be the family of all closed subsets $\mathcal{F}$ of $\mathcal{M}_n$ such that $S$ has a winning strategy in a subspace game for $\mathcal{F}$ and for every $\varepsilon > 0$. Clearly, $\mathcal{M}_n \in \mathcal{W}$ and $\emptyset \notin \mathcal{W}$. Moreover, the filtration property immediately implies that if $\mathcal{F}_i \in \mathcal{W}$, for $i = 1, \ldots, m$, then $\bigcap_i \mathcal{F}_i \in \mathcal{W}$. Let $\tilde{\mathcal{F}} = \bigcap_{\mathcal{F} \in \mathcal{W}} \mathcal{F}$. This is a non-empty closed subset of $\mathcal{M}_n$. We shall show that $\tilde{\mathcal{F}} \in \mathcal{W}$.

This follows from a compactness argument. Let $\mathcal{D}_\delta(F)$ be the open ball in $\mathcal{M}_n$ of radius $\delta$ and center at $F$. Observe that for $\delta > 0$, the set $\mathcal{F}_\delta = \bigcup_{F \in \tilde{\mathcal{F}}} \mathcal{D}_\delta(F)$ contains an intersection of a finite number of sets from $\mathcal{W}$. Indeed, the complement $(\mathcal{F}_\delta)^c$ of $\mathcal{F}_\delta$ is compact, and it is contained in $(\tilde{\mathcal{F}})^c$, which in turn is covered by the union of complements of sets from $\mathcal{W}$. Thus for every $\delta > 0$, $\mathcal{F}_\delta$ contains a set from $\mathcal{W}$, hence $S$ has a winning strategy in a subspace game for $\mathcal{F}_\delta$ and every $\varepsilon > 0$. Since these sets approximate $\tilde{\mathcal{F}}$ arbitrarily close, for an arbitrary fixed $\varepsilon > 0$, $S$ has a winning strategy in a subspace game for $\tilde{\mathcal{F}}$ as well. Thus $\tilde{\mathcal{F}} \in \mathcal{W}$.

1.5 The set of asymptotic spaces $\{X\}_n$ coincides with $\tilde{\mathcal{F}}$. Therefore the subspace player $S$ has a winning strategy in a subspace game for $\{X\}_n$ and
for every $\varepsilon > 0$. In particular, $\{X\}_n$ is non-empty.

First, $\{X\}_n \subset \bar{F}$. Indeed, let $E \in \{X\}_n$ and let $\varepsilon > 0$. Consider an asymptotic game in which each player follows his own strategy; player $S$ follows the winning strategy for a subspace game for $\bar{F}$, and player $V$ follows the winning strategy for a vector game for $E$. Strategy of $V$ implies that vectors $\{x_i\}$ resulting from this game are $(1 + \varepsilon)$-equivalent to the basis in $E$; strategy of $S$ implies that they are also $(1 + \varepsilon)$-equivalent to the basis in some space from $\bar{F}$. Hence $d_b(E, \bar{F}) \leq (1 + \varepsilon)^2$, for every $\varepsilon > 0$. Thus $E \in \bar{F}$, since $\bar{F}$ is closed.

Next, observe that if $E \not\in \{X\}_n$, then for $\varepsilon_0 > 0$ sufficiently small, player $S$ has a strategy in a subspace game such that resulting vectors $\{x_i\}$ satisfy $d_b(\text{span } [x_i], E) \geq 1 + \varepsilon_0$. Thus for every $\varepsilon < \varepsilon_0/2$, $S$ has a winning strategy in a subspace game for $F' = \bar{F} \setminus D_{\varepsilon_0/2}(E)$. Since $F'$ is closed, the minimality of $\bar{F}$ implies in particular that $E \not\in \bar{F}$.

1.6 We look in more detail at the family of all asymptotic spaces of $X$.

Spaces $l_p$ play a special role here. For $1 \leq p < \infty$, the standard unit vector basis in $l_p$ is denoted by $\{e_i\}$. The same notation is used in $c_0$ and in finite-dimensional spaces $l_\infty^n$.

1.6.1 Denote by $\{X\}_n^0$ and by $\{X\}_n^t$ the sets of all $n$-dimensional asymptotic spaces with respect to the families $B^0(X)$ and $B^t(X)$. Clearly, $\{X\}_n^0 \subset \{X\}_n^t$. In general, this inclusion is proper; however the main property of shrinking systems immediately implies that if a fundamental system in $X$ is shrinking, then $\{X\}_n^0 = \{X\}_n^t$.

As an example of a space for which asymptotic structures depend on a family $B(X)$, consider the space $c$ of all convergent scalar sequences. Let $\{u_i\}$ be the natural basis in $c$, that is, $u_1 = (1, 1, 1, \ldots)$ and $u_i = e_{i-1}$ for $i > 1$, and consider $n$-dimensional asymptotic spaces with respect to the family of tail subspaces of $\{u_i\}$. It is obvious that the only such space is $l_\infty^n$ with the standard unit vector basis. This in particular implies that $\{c\}_n^0 = \{l_\infty^n\}$. On the other hand, consider a conditional basis $\{u_i\}$ in $c$ given by $v_i = \sum_{j=1}^{\infty} e_j$ for $i = 1, 2, \ldots$ and consider the set $\{c\}_n^I$ with respect to this basis. Clearly, $l_\infty^n \in \{c\}_n^I$, however it is easy to see that this set is larger: it also contains the space $E$ which is $l_\infty^n$ with the conditional basis $\bar{v}_i = \sum_{j=i}^{n} e_j$ for $i = 1, \ldots, n$. 8
1.6.2 Recall that a bounded non-convergent sequence \( \{z_i\} \) in a Banach space \( X \) is said to generate spreading model, if for every finite sequence of scalars \((a_1, \ldots, a_k)\) the \( k \)-fold limit \( \lim_{n_1 \ldots n_k} \| \sum_{i=1}^{k} a_i z_{n_i} \| \) exists, as \( n_i \to \infty \), for \( i = 1, \ldots, k \), with \( n_1 < \ldots < n_k \).

Then one can define the spreading model \( F \) as a Banach space with the basis \( \{f_i\} \) such that for every finite sequence of scalars \((a_i)\) one has

\[
\| \sum_{i=1}^{k} a_i f_i \| = \lim_{n_1} \ldots \lim_{n_k} \| \sum_{i=1}^{k} a_i z_{n_i} \|.
\]

Clearly, the basis \( \{f_i\} \) is spreading invariant, i.e., for every finite sequence of scalars \((a_i)\) and every \( n_1 < n_2 < \ldots \) one has \( \| \sum a_i f_i \| = \| \sum a_i f_{n_i} \| \). In such a situation, the sequence of differences \( \{f_{2i} - f_{2i-1}\} \) is unconditional (and clearly still spreading invariant).

It is a well-known result by Brunel and Sucheston [BS] and it follows from Ramsey’s theorem that every bounded sequence with no Cauchy sub-sequences contains a subsequence generating spreading model; and then the differences of this subsequence generate an unconditional spreading model (cf. also [MiSch] Section 11). The reader can consult e.g., [BL] on more details on spreading models.

1.6.3 Recall that if a sequence \( \{z_i\} \) generates an unconditional spreading model, then a direct application of Krivine’s theorem [K] says that there exists \( 1 \leq p \leq \infty \) such that for every \( n \in \mathbb{N} \) and every \( \varepsilon > 0 \) there is a finite scalar sequence \( \alpha = (\alpha_1, \ldots, \alpha_m) \) such that any \( n \) successive blocks \( \{x_j\} \) of \( \{z_i\} \) with the same distribution \( \alpha \) and “far enough”, are \((1 + \varepsilon)\)-equivalent to the basis \( \{e_i\} \) in \( l^p \).

Since every Banach space has a sequence generating unconditional spreading model, there exists \( 1 \leq p \leq \infty \) such that \( l^p \in \{X\}_n \) for every \( n \).

1.6.4 Let us briefly discuss a concept of a stabilized asymptotic structure, which appears implicitly or explicitly in many papers already mentioned ([G.1], [Ma], [MiT]) and others ([C], [G.2]). This concept allows for passing to infinite-dimensional subspaces of a given space \( X \) and hence we can assume that \( X \) has a basis.

By \( B_\infty(X) \) denote the set of all infinite-dimensional block subspaces of \( X \). We can then consider a family \( D = \{D_n\} \) of subsets \( D_n \subset M_n \), for
\( n = 1, 2, \ldots \), such that there exists \( Y \in \mathcal{B}_\infty(X) \) such that for every \( n \in \mathbb{N} \) the following stabilization condition holds: for every \( Z \in \mathcal{B}_\infty(Y) \), we have \( D_n = \{ Z \}_n \). It is not difficult to show that the sets \( D_n \) are non-empty. In fact, there exists \( 1 \leq p \leq \infty \) such that \( l^p_n \in D_n \) for every \( n \). Each space \( E \) from \( D_n \) (for \( n \in \mathbb{N} \)) is called a \textit{stabilized asymptotic space} for \( X \), and \( Y \) is called a stabilizing subspace for \( D \).

To show that all the \( D_n \)'s are non-empty and to construct \( Y \), first observe that the compactness of \( M_n \) and Zorn’s lemma show that for a fixed \( n \), given \( \tilde{Y} \in \mathcal{B}_\infty \), there exists \( Y_n \in \mathcal{B}_\infty(\tilde{Y}) \) such that the \( n \)th stabilization condition holds in \( Y_n \). Then the space \( Y \) is a diagonal subspace of \( Y_n \)'s. The last statement about \( l^p_n \)'s follows from the fact that the set of all \( p \)'s such that \( l^p_n \in \{ X \}_n \) for every \( n \) (cf. [1.6.3]), is closed.

1.6.5 An important recent combinatorial theorem by Gowers [G.1] provides further general information on families of stabilized asymptotic spaces. Let \( X \) be a Banach space with a basis. Let \( \Sigma \) be a set of all sequences \( \{ x_1, \ldots, x_n \} \), where \( n \in \mathbb{N} \) and the vectors are successive normalized blocks of the basis. A subset \( \sigma \subset \Sigma \) is called \textit{large} if for every \( Y \in \mathcal{B}_\infty(X) \), there is a sequence \( \{ x_1, \ldots, x_n \} \in \sigma \) with \( x_i \in Y \) for \( i = 1, \ldots, n \).

Given a subspace \( Y \in \mathcal{B}_\infty(X) \), consider a general infinite-dimensional vector game inside \( Y \), which is in essential way less restrictive than the game introduced in [1.3]. Here for the \( k \)th move of the game, the subspace player \( S \) chooses a subspace \( Y_k \in \mathcal{B}_\infty(Y) \) and then the vector player \( V \) chooses a vector \( x_k \in S(Y_k) \). Given a set \( \sigma \subset \Sigma \) and \( \varepsilon > 0 \), player \( V \) wins the game inside \( Y \), if after some number of moves the sequence \( \{ x_1, \ldots, x_n \} \) he has chosen is \((1 + \varepsilon)\)-equivalent to a sequence from \( \sigma \). Note, in comparison with [1.3] and [1.2], that here subspaces chosen by \( S \) may have infinite codimension and the number of moves in the game is not prescribed in advance; in fact, this number even does not have to be finite, if \( V \) does not have a winning strategy.

Gowers’ theorem says that if \( \sigma \subset \Sigma \) is large on \( X \) then for every \( \varepsilon > 0 \) there is a subspace \( Y \in \mathcal{B}_\infty(X) \) inside which \( V \) has a winning strategy for \( \sigma \) and \( \varepsilon \).

Now let \( X \) be an arbitrary space and let \( n \in \mathbb{N} \). If \( E \in \mathcal{M}_n \) and there is an infinite-dimensional subspace \( Z \subset X \) with a basis such that for every \( Y \in \mathcal{B}_\infty(Z) \) and every \( \varepsilon > 0 \), there are \( n \) successive blocks \( \{ v_1, \ldots, v_n \} \) in \( Y \) such that \( d_b(\text{span} [v_1], E) \leq 1 + \varepsilon \), then \( E \) is a stabilized asymptotic space for
\( \text{X and in particular, } E \in \{X\}_n. \)

Indeed, given \( \varepsilon > 0 \), we use Gowers’ theorem for the set \( \sigma \) of all \( n \)-tuples of successive blocks \( \{v_1, \ldots, v_n\} \) as above; this \( \sigma \) is large on \( Z \).

Notice that this argument does not require the full strength of Gowers’ result: the game used above has a fixed length and in this case the theorem is easier. Let us also mention that as an easy corollary to his general result, Gowers obtained the following attractive structure dichotomy for Banach spaces: every infinite-dimensional Banach space either has a subspace with an unconditional basis or has a hereditarily indecomposable subspace \( X_0 \) (i.e., no subspace of \( X_0 \) is a topological direct sum \( Y \oplus Z \) of infinite-dimensional subspaces.)

1.6.6 Recall a recent construction of Odell and Schlumprecht [OS.1], of a Banach space \( Z \) with a basis \( \{z_i\} \) such that for every \( n \in \mathbb{N} \), every \( n \)-dimensional space \( E \) with a monotone basis, every \( Y \in \mathcal{B}_\infty(Z) \) and every \( \varepsilon > 0 \), there are \( n \) successive blocks \( \{v_1, \ldots, v_n\} \) in \( Y \) such that \( d_b(\text{span } \{v_i\}, E) \leq 1 + \varepsilon \). It follows from [1.6.5] that for this space \( Z \), every finite-dimensional space with a monotone basis is a stabilized asymptotic space.

1.7

Definition A Banach space \( X \) is called an asymptotic-\( l_p \) space, for \( 1 \leq p \leq \infty \), if there is a constant \( C \) such that for every \( n \) and every \( E \in \{X\}_n \) we have \( d_b(E, l_p^\infty) \leq C \). The asymptotic structure \( \{X\}_n \) is determined by the family \( \mathcal{B}_0(c) \) of all finite-codimensional spaces of \( X \), i.e., \( \{X\}_n = \{X\}_n^0 \).

An example discussed in [1.6.1] shows that the restriction of the asymptotic structure to \( \{X\}_n^0 \) is essential in general: the space \( c \) is an asymptotic-\( l_\infty \), but some asymptotic spaces relative to the family \( \mathcal{B}_\delta(c) \) of tail spaces of the conditional asis \( \{v_i\} \), are not equivalent (in sense of \( d_b(\cdot, \cdot) \)) to \( l_n^\infty \) with the standard basis.

1.7.1 If \( l_p^2 \) is the only 2-dimensional asymptotic space for a Banach space \( X \), i.e., \( d_b(E, l_p^2) = 1 \), for every \( E \in \{X\}_2 \), then \( X \) contains almost isometric copies of \( l_p \). Indeed, a well-known easy argument shows that the formula from [1.4.1] allows to construct, for every \( \varepsilon > 0 \), a basic sequence \( \{x_i\} \) in \( X \) such that \( \{x_i\} \overset{1+\varepsilon}{\sim} \{e_i\} \) (cf. e.g., [MiT]).
1.7.2 A trivial example of an asymptotic-$l_p$ space not isomorphic to $l_p$, is an $l_p$-direct sum $(\sum \oplus Z_i)_p$ of finite-dimensional spaces.

A class of much more sophisticated examples are $p$-convexified Tsirelson spaces $T(p)$; these spaces are asymptotic-$l_p$ and they do not contain subspaces isomorphic to $l_p$ (cf. e.g., [CS]).

1.7.3 If all $n$-dimensional stabilized asymptotic spaces (cf. 1.6.4) are uniformly equivalent to the unit vector basis in $l^n_p$, for $n \in \mathbb{N}$, and $X$ itself is a stabilizing subspace, then $X$ is called a stabilized asymptotic-$l_p$ space. These spaces were investigated in [MiT] and [Ma] (where they were called just asymptotic-$l_p$ spaces).

1.7.4 From the point of view of Banach space theory it is tempting to consider a seemingly more general concept than asymptotic-$l_p$ spaces, in which the condition that the basis in $E$ is $C$-equivalent to the natural basis in $l^n_p$, is replaced by the condition that $E$ itself is $C$-isomorphic to $l^n_p$ (for $E \in \{X\}_n$). Recall that a Banach space with a basis has uncountably many mutually non-equivalent bases (cf. [LT.1] 1.a.8); for spaces $l_p$, with $1 < p < \infty$, $p \neq 2$, these bases may be chosen to be even unconditional (cf. [LT.1] 2.b.10). It is therefore rather striking that in the asymptotic setting discussed here for $1 \leq p < \infty$, the more general condition of isomorphism of asymptotic spaces to $l^n_p$ already implies the equivalence of the natural bases. This will be proved in Section 3.

1.8 Let us consider again an asymptotic structure with respect to an arbitrary family $\mathcal{B}$ satisfying the filtration condition 1.1. We shall discuss some properties of asymptotic families which show an interplay between different level families.

1.8.1 Let $X^{(1)}$ and $X^{(2)}$ be two $C$-isomorphic Banach spaces. For every $n \in \mathbb{N}$, the Hausdorff distance (in $\mathcal{M}_n$) between $\{X^{(1)}\}_n$ and $\{X^{(2)}\}_n$ is smaller than or equal to $C$. That is, if $1 \leq i \neq j \leq 2$ then for every $E \in \{X^{(i)}\}_n$ there is $F \in \{X^{(j)}\}_n$ such that $d_0(E, F) \leq C$. In particular, if $X^{(1)}$ is an asymptotic-$l_p$ space then so is $X^{(2)}$.

Indeed, let $T : X^{(1)} \to X^{(2)}$ be an isomorphism. Given $E \in \{X^{(i)}\}_n$, the corresponding space $F$ will be spanned by an $n$-tuple resulting from a
subspace game in \( \{X^{(i)}\}_n \); together with this game one considers a vector
game for \( E \) in \( \{X^{(i)}\}_n \), and the moves between the two games are translated
one to another by the operators \( T \) and \( T^{-1} \).

1.8.2 Let \( n_1, \ldots, n_k \) be natural numbers. Let \( E_j \in \{X\}_{n_j} \), for \( j = 1, \ldots, k \).
For every \( N \geq \sum_j n_j \) and any disjoint subsets \( I_j \) of \( \{1, \ldots, N\} \), with \( |I_j| = n_j \)
for \( j = 1, \ldots, k \), there exists an asymptotic space \( F \in \{X\}_N \) with a basis
\( \{f_i\} \) such that \( d_0(\text{span} [f_i]_{i \in I_j}, E_j) = 1 \) for \( j = 1, \ldots, k \).

Indeed, let \( \varepsilon > 0 \). Consider an asymptotic game which ends after \( N \)
moves. Player \( S \) simply follows his winning strategy for a subspace game for
\( \{X\}_N \) and \( \varepsilon \). Strategy for player \( V \) is more complicated. For \( j = 1, \ldots, k \),
write \( I_j = \{i^{(j)}_1, \ldots, i^{(j)}_{n_j}\} \); if \( i \in I_j \) for some \( 1 \leq j \leq k \), say \( i = i^{(j)}_l \) for
\( 1 \leq l \leq n_j \), then \( V \) makes his choice of vector \( x_i \) following the winning
strategy for the \( l \)-th move in a vector game for \( E_j \) and \( \varepsilon \), as if his previous
choices in this game were the vectors \( x_s \), for \( s = i^{(j)}_1, \ldots, i^{(j)}_{n_j} \). If \( i \not\in I_j \) for
any \( j \), \( V \) picks the vector \( x_i \) arbitrarily.

Consider the vectors \( \{x_i\} \) resulting in the game. The strategy of \( S \) implies
that they are \((1 + \varepsilon)\)-equivalent to the basis \( \{f^{(i)}_s\} \) in some asymptotic \( N \)-
dimensional space \( F^{(i)}_\varepsilon \in \{X\}_N \). The strategy of \( V \) implies in turn that
\( d_0(\text{span} [f^{(i)}_s]_{s \in I_j}, E_j) \leq 1 + \varepsilon \), for \( 1 \leq j \leq k \). Then a required space \( F = X_N \)
is any cluster point in \( \mathcal{M}_N \) of the \( F^{(i)}_\varepsilon \)'s, as \( \varepsilon \to 0 \).

1.8.3 Let \( E \in \{X\}_n \), and let \( F \) be a block subspace of \( E \), that is, \( F \) is
spanned by successive blocks of the basis in \( E \). Then \( F \) is an asymptotic
space, \( F \in \{X\}_m \), where \( m = \dim F \). Moreover, given \( n \in \mathbb{N} \) and \( \varepsilon > 0 \),
the subspace player \( S \) has a strategy in an asymptotic game such that after
\( n \) moves, all normalized successive blocks of the \( n \)-tuple resulting from the
game, are permissible, i.e., each of them is \((1 + \varepsilon)\)-equivalent to the basis in
some asymptotic space.

To prove the first statement, let \( \{e_i\} \) be the basis in \( E \) and let \( \{u_k\} \) be
successive blocks of \( \{e_i\} \) spanning \( F \). Let \( i_0 = 1 < i_1 < \ldots < i_m = n + 1 \)
such that \( u_k = \sum_{i=i_{k-1}}^{i_k} a_i e_i \), for \( k = 1, \ldots, m \). Given \( \varepsilon > 0 \), consider a
vector game for \( E \) and \( \varepsilon \) in which choices of player \( S \) follow the pattern
\( X_1, X_1, \ldots, X_1, X_2, \ldots, X_2, X_3, \ldots \), with the change of a subspace being made
only in the \( i_k \)-th moves and subspaces \( X_k \) being arbitrary \((k = 0, \ldots, m - 1) \),
and player \( V \) follows his winning strategy for \( E \). Denote the resulting per-
missible $n$-tuple by $\{x_i\}$, then $\{x_i\} \sim_1 1 + \varepsilon \{e_i\}$. Moreover, the blocks $v_k = \sum_{i=k+1}^{n} a_i x_i$ obviously satisfy $\{v_k\} \sim_1 1 + \varepsilon \{u_k\}$. This describes a winning strategy for $V$ in a vector game for $F$ and $\varepsilon$. Hence $F \in \{X\}_n$.

For the moreover part, it is not difficult to see from a simple perturbation argument, that if $S$ follows his winning strategy for $\{X\}_n$ and $\delta > 0$, then arbitrary successive normalized blocks $\{w_k\}$ of any $n$-tuple $\{x_i\}$ resulting in the game, are $(1 + \delta)(1 + n\delta)$-equivalent to corresponding normalized blocks of the basis in the space from $\{X\}_n$ associated to $\{x_i\}$. Thus $\{w_k\}$ are permissible.

1.9 We conclude this section by introducing the notion of envelopes which is of independent interest.

1.9.1 Recall that $c_{00}$ denotes the space of all scalar sequences eventually zero. The upper and the lower envelopes for $X$ are functions $r(\cdot)$ and $g(\cdot)$, respectively, defined for $a = (a_1, \ldots, a_n, 0, \ldots) \in c_{00}$ by $r(a) = \sup \| \sum a_i e_i \|$ and $g(a) = \inf \| \sum a_i e_i \|$, where the supremum and the infimum are taken over all natural bases $\{e_i\}$ of asymptotic spaces $E \in \{X\}_n$ and all $n$.

The functions $r(\cdot)$ and $g(\cdot)$ are obviously unconditional and subsymmetric. It is easy to see that $r(\cdot)$ is a norm on $c_{00}$ and that $g(\cdot)$ satisfies triangle inequality on disjointly supported vectors. These functions were used in an essential way in [MiT].

1.9.2 Note that the upper envelope is /it sub-homogeneous. By this we mean that for any finite number of successive vectors $b^i \in c_{00}$ such that $r(b^i) \leq 1$ for $i = 1, 2, \ldots$ and for any vector $a = (a_i) \in c_{00}$, we have

$$r(\sum_i a_i b^i) \leq r(a).$$

Similarly, the lower envelope satisfies the /it super-homogeneity condition: if $g(b^i) = 1$ for $i = 1, 2, \ldots$ then

$$g(\sum_i a_i b^i) \geq g(a).$$

The proof of both inequalities uses [1.8.3] and unconditionality of both functions.
1.9.3 It is a general and interesting fact that sub-homogeneous norms or functions satisfying a weaker triangle inequality as \(g(\cdot)\) does, are always close to some \(l_p\)-norm. We formulate the exact statement for our envelope functions.

There exist \(1 \leq p, q \leq \infty\) and \(C, c > 0\) and for every \(\varepsilon > 0\) there exist \(C_\varepsilon, c_\varepsilon > 0\) such that for \(a \in c_{00}\) we have

\[
c_\varepsilon \|a\|_{q + \varepsilon} \leq g(a) \leq C\|a\|_q \quad \text{and} \quad c\|a\|_p \leq r(a) \leq C_\varepsilon\|a\|_{p - \varepsilon}.
\]

We outline a standard argument for the function \(r(\cdot)\). For a positive integer \(n\) set \(\lambda_r(n) = r((1, \ldots, 1, 0, \ldots))\). Then sub-homogeneity of \(r(\cdot)\) discussed in 1.9.2 implies that \(\lambda_r(n m) \leq \lambda_r(n) \lambda_r(m)\). By induction, we get \(\lambda_r(n^k) \leq \lambda_r(n)^k\). Let \(1/p = \inf \ln \lambda_r(n)/\ln n\). Clearly, \(\lambda_r(n) \geq n^{1/p}\) for all \(n\). On the other hand, for every \(\varepsilon > 0\) there exists a constant \(C_\varepsilon\) such that \(\lambda_r(n) \leq C_\varepsilon n^{1/(p-\varepsilon)}\).

By Krivine’s theorem for the space \((c_{00}, r(\cdot))\), this space contains \(l_p^n\)’s uniformly on successive blocks of the natural basis. Using submultiplicativity of \(r\), it easily follows that \(r(a) \geq c\|a\|_p\), for all \(a \in c_{00}\). On the other hand, it can be also easily seen that an upper power type estimate for \(\lambda_r(n)\) implies a similar estimate for \(r\) (with different \(C_\varepsilon\), though).

2 Asymptotic versions

In this section we introduce infinite-dimensional spaces which reflect properties of the whole sequence \(\{\{X\}_n\}\) of families of \(n\)-dimensional asymptotic spaces of a given Banach space \(X\). This will be done by considering an additional structure given naturally by an inclusion on bases of asymptotic spaces.

2.1 A space \(Y\) with a monotone basis \(\{y_i\}\) is called an asymptotic version of \(X\) if for every \(n \in \mathbb{N}\) we have \(\{y_i\}_{i=1}^n \in \{X\}_n\). The set of all asymptotic versions of \(X\) is denoted by \(A(X)\).

A construction of an asymptotic version of a given space \(X\), fully resembles the concept of an injective limit. First observe that if \(\{f_i\}_{i=1}^n \in \{X\}_n\) then the restriction \(\{f_i\}_{i=1}^{n-1}\) is in \(\{X\}_{n-1}\).

Conversely, for every \(\{e_i\}_{i=1}^n \in \{X\}_n\) there is \(\{f_i\}_{i=1}^{n+1} \in \{X\}_{n+1}\) such that \(\{f_i\}_{i=1}^n \stackrel{1}{\sim} \{e_i\}_{i=1}^n\).
Indeed, given \( \varepsilon > 0 \), consider an asymptotic game in \( X \) which ends after \( n + 1 \) moves; in which player \( S \) follows his winning strategy in a subspace game for \( \{X\}_{n+1} \), and player \( V \), in the first \( n \) moves, follows a winning strategy in a vector game for \( \{e_i\}_{i=1}^n \), and in the \((n+1)\)th move picks an arbitrary vector. Denote the resulting \((n+1)\)-tuple by \( \{f_\varepsilon^i\}_{i=1}^{n+1} \). An argument similar to the one used at the end of 1.8.2 shows that any cluster point of the \( \{f_\varepsilon^i\}_{i=1}^{n+1} \)'s in \( M_{n+1} \) belongs to \( \{X\}_{n+1} \) and its restriction is clearly \( \{e_i\}_{i=1}^n \).

We can construct an increasing sequence \( F_1 \subset \ldots \subset F_n \subset F_{n+1} \subset \ldots \) with bases \( \{f_1\} \subset \ldots \subset \{f_i\}_{i=1}^n \subset \{f_i\}_{i=1}^{n+1} \subset \ldots \) such that \( F_n \in \{X\}_n \). Then \( Y = \bigcup_n F_n \) is an asymptotic version of \( X \).

2.2 Let us consider few simple examples of asymptotic versions.

Clearly, a space \( X \) is an asymptotic-\( l_p \) if and only if all asymptotic versions of \( X \) are uniformly equivalent to the standard unit vector basis in \( l_p \).

The space \( Z \) from 1.6.6 has every Banach space \( Y \) with a monotone basis as its asymptotic version.

2.2.1 A space \( X \) has an asymptotic unconditional structure if there exists \( C \) such that for every asymptotic space \( E \in \{X\}_n \) (where \( n = \text{dim} \ E \)) the natural basis \( \{e_i\} \) in \( E \) is \( C \)-unconditional, i.e., unc \( \{x_i\} \leq C \).

Clearly, \( X \) has an asymptotic unconditional structure if and only if there exists \( C \) such that for every asymptotic version \( Y \) of \( X \) the natural basis in \( Y \) is \( C \)-unconditional.

2.3 Let \( Y \in \mathcal{A}(X) \), let \( n \in \mathbb{N} \) and let \( E \in \{Y\}_n \). Fix \( \varepsilon > 0 \). Then the basis in \( E \) is \((1 + \varepsilon)\)-equivalent to \( n \) successive blocks of some initial interval of the basis in \( Y \), say \( \{y_i\}_{i=1}^N \). Since \( \{y_i\}_{i=1}^N \in \{X\}_N \), then 1.8.3 implies that \( E \) is \((1 + \varepsilon)\)-close to some asymptotic space for \( X \). Thus \( \{Y\}_n \subset \{X\}_n \), for every \( n \in \mathbb{N} \).

The following theorem shows that we can construct an asymptotic version of \( X \) which contains all asymptotic spaces of \( X \) in an asymptotic way.

**Theorem** For every Banach space \( X \) there exists an asymptotic version \( Y \in \mathcal{A}(X) \) such that \( \{Y\}_n = \{X\}_n \) for every \( n \in \mathbb{N} \). Moreover, \( Y \) can be constructed in such a way that every asymptotic space of \( X \) is represented (in an asymptotic way) as a permissible span of basic vectors of \( Y \).

Such a space \( Y \) is called a universal asymptotic version for \( X \).
2.4 It follows that not every Banach space can be a universal asymptotic
version of another Banach space. Examples from 2.2 imply that this is a case
of an asymptotic-$l_p$ space not isomorphic to $l_p$ (see [1.7.2]), or of a space with
an asymptotic unconditional basis which is not unconditional ([G.2]).

2.5 The proof of Theorem 2.3 is based on several lemmas.

2.5.1 The first lemma is similar to 1.8.2 and has an analogous proof which
is left for the reader.

**Lemma** Let $n, m \in \mathbb{N}$, let $N \geq n m$. Let $\mathcal{I} = \{I_j\}_{j=1}^m$ be a family of $m$
subsets of $\{1, \ldots, N\}$, such that $|I_j| = n_j \leq n$ for $j = 1, \ldots, m$, and the
following condition is satisfied: for arbitrary two sets $I_k$ and $I_l$ in $\mathcal{I}$, the
intersection $I_k \cap I_l$ is either empty or it is an initial interval of each of them,
i.e., if $I_k = \{t_1, \ldots, t_{m_k}\}$ and $I_l = \{s_1, \ldots, s_{m_l}\}$, and if $t_\mu = s_\nu$ for some
$\mu, \nu \in \mathbb{N}$, then $\mu = \nu$ and $t_1 = s_1, \ldots, t_\mu = s_\mu$. Let $E \in \{X\}_n$ with a basis
$\{e_l\}$. There exists an asymptotic space $F \in \{X\}_N$ with a basis $\{f_i\}$ such that
\[
\{f_i\}_{i \in I_j} \sim \{e_l\}_{l=1}^{n_j},
\]
for $j = 1, \ldots, m$.

2.5.2 We also require infinite-dimensional facts of a similar nature. To
avoid unnecessary repetitions, let us use the convention that if a basis $\{z_i\}$
of a Banach space $Z$ is understood from the context, for a basic sequence
$\{y_i\}$ we shall write $\{y_i\} \sim Z$ instead of $\{y_i\} \sim \{z_i\}$.

The proof of the next lemma follows by combining 1.8.2 and 2.1.

**Lemma** Let $Y_1$ and $Y_2$ be two asymptotic versions of $X$. Let $I_1$ and $I_2$ be
two infinite disjoint subsets of $\mathbb{N}$. There exists an asymptotic version $Y$ of
$X$ with a basis $\{y_i\}$ such that $\{y_i\}_{i \in I_1} \sim Y_1$ and $\{y_i\}_{i \in I_2} \sim Y_2$.

2.5.3 The final lemma is a version of the latter one for infinitely many
spaces. We leave the proof to the reader.

**Lemma** Let $\{Y_j\}$ be a sequence of asymptotic versions of $X$. Let $\{I_j\}$
be a sequence of infinite mutually disjoint subsets of $\mathbb{N}$. There exists an
asymptotic version $Y$ of $X$ with a basis $\{y_i\}$ such that $\{y_i\}_{i \in I_j} \sim Y_j$ for every
$j = 1, 2, \ldots$. 17
2.5.4 Now we are ready for the proof of the theorem.

Proof Fix an arbitrary asymptotic space $E \in \{X\}_n$ with a basis $\{e_i\}$. First we construct an asymptotic version $Y_1 \in A(X)$ such that $E \in \{Y_1\}_n$. Let $K$ be a family of all $m$-tuples of natural numbers, for all $m \leq n$, which are of the form $K = \{p_1, p_1 p_2, \ldots, \prod_{i=1}^m p_i\}$, where $p_1 < p_2 < \ldots < p_m$ are prime numbers.

For an arbitrary $N \in \mathbb{N}$ sufficiently large, let $K_N \subset K$ consists of all $m$-tuples $K \in K$, for all $m \leq n$, for which $\prod_{i=1}^m p_i \leq N$. Observe that family $K_N$ has the property from Lemma 2.5.1. Therefore there exists an asymptotic space $F_N \in \{X\}_N$ with a basis $\{f_i\}$ such that for any $m$-tuple $K \in K_N$ we have

$$\{f_i\}_{i \in K} \sim \{e_i\}_{i=1}^m.\$$

Similarly as in 2.1, we can then construct an increasing sequence of such spaces $\ldots \subset F_N \subset F_{N+1} \subset \ldots$, with bases $\ldots \subset \{f_i\}_i^N \subset \{f_i\}_i^{N+1} \subset \ldots$, each of them having the above structure (because the restriction of $F_{N+1}$ to the first $N$ basis vectors has the same property). This sequence defines an asymptotic version $Y_1$ and it can be checked that $E \in \{Y_1\}_n$.

Given a finite number of asymptotic spaces $\{E_i\}$, we use Lemma 2.5.2 a finite number of times to build an asymptotic version $\tilde{Y}$ such that every $E_i \in \{\tilde{Y}\}_n$.

The end of the argument is obvious: let $\varepsilon_n \downarrow 0$ as $n \to \infty$. For every $n \in \mathbb{N}$, let $T_n$ be a finite $\varepsilon_n$-net in the set $\bigcup_{k\leq n}\{X\}_k$ of all asymptotic spaces of dimension less than or equal to $n$. Let $Y_n$ be an asymptotic version of $X$ which contains all spaces from $T_n$ as asymptotic spaces. Use Lemma 2.5.3 for the spaces $Y_n$ and the sets $I_n = \{(2n + 1)2^t\}_{t=1}^\infty$. Resulting asymptotic version $Y$ has the required property: for every $n \in \mathbb{N}$ and $E \in \{X\}_n$, there is a sequence $\varepsilon_k \to 0$ such that $Y$ has permissible subspaces $F_k$ with the distance $d_b(F_k, E) \leq 1 + \varepsilon_k$. Thus $E \in \{Y\}_n$. Therefore $\{X\}_n \subset \{Y\}_n$; the converse inclusion has been commented on before the statement of the theorem. \qed

3 Uniqueness of the asymptotic-$l_p$ structure

3.1 The following theorem has been already promised in 1.7.4.
Theorem Let $X$ be a Banach space and consider the asymptotic structure on $X$ determined by the family $\mathcal{B}(X)$ of all finite-codimensional subspaces of $X$. Let $1 \leq p < \infty$. Assume that there exists $C$ such that for every $n \in \mathbb{N}$ and every $E \in \{X\}_n$, the Banach–Mazur distance $d(E, l^n_p) \leq C$. Then $X$ is an asymptotic-$l_p$ space.

3.2 Proof Let $\tilde{Y}$ with the basis $\{\tilde{y}_i\}$ be a universal asymptotic version of $X$. If $E \subset \tilde{Y}$ is a finite-dimensional subspace then for every $\varepsilon > 0$, $E$ is $(1 + \varepsilon)$-isomorphic to a subspace of $Y_N = \text{span}\{\tilde{y}_i\}_{i=1}^N$, which in turn is $C$-isomorphic to $l^n_p$. Therefore $\tilde{Y}$ is an $L_p$-space. It is then well-known ([LR]) that $\tilde{Y}$ is isomorphic to a subspace $Y$ of $L_p[0,1]$. Let $\{y_i\}$ denotes the image of $\{\tilde{y}_i\}$ by this isomorphism. The asymptotic structure from $\tilde{Y}$ induces an asymptotic structure on $Y$, which we will now investigate.

Fix an arbitrary $E \in \{Y\}_n$ with the basis $\{e_i\}$. We will show that there exists a permissible $n$-tuple $\{z_i\} \sim \{e_i\}$ which is also equivalent (up to some constant $D''$) to the unit vector basis in $l^n_p$, where $K_p$ depends on $p$ only.

This already completes the proof for $p = 2$, since it is well-known that every unconditional basis in $l^n_2$ is equivalent to the standard unit vector basis. For $p \neq 2$ we proceed separately in cases $1 < p < 2$ and $p > 2$.

3.3 Let $1 \leq p < 2$. Fix $\varepsilon > 0$ and consider a vector game in $Y$ for $E$ and $\varepsilon$. Let the first move of player $S$ be $Y$ itself and let player $V$ choose $z_1 \in S(Y)$. Considering appropriate choices for the second move of the subspace player $S$, we obtain a sequence $\{v_m\} \subset Y$ of second choices for $V$ (with the first choice being always $z_1$); we can also ensure that the $v_m$'s are successive blocks of the Haar basis.

It is now convenient to describe the argument separately for the reflexive case $1 < p < 2$ and for $p = 1$. 19
3.3.1 Let $1 < p < 2$. Passing to a subsequence, we may assume that $\{v_m\}$ generates a spreading model (see [1.6.2]). It is known that the natural basis of a spreading model of any sequence in $L_p$ is symmetric, rather than spreading invariant as in general. This is true in every stable Banach space, and is a direct consequence of the definition of stability [KM] (the reader not familiar with the notion of ultrafilters may also consult [KM] p. 276). On the other hand, $L_p$ is a stable space for $1 \leq p < \infty$ ([KM]). So from the definition of a spreading model, this means that for a given $n$, any $n$-tuple $v_{m_1}, \ldots, v_{m_n}$, with $m_1 < \ldots < m_n$ and $m_1$ large enough, forms a finite almost symmetric basis in its span.

On the other hand, given any weakly null sequence $\{v_m\}$ and $M_1$, by considering a suitable subspace game we can choose an $n$-tuple $v_{m_1}, \ldots, v_{m_n}$ which is 2-equivalent to some asymptotic space and $m_1 \geq M_1$. By the main assumption, the span of $\{v_m\}_{i=1}^n$ is $C$-isomorphic to $l_p^n$.

3.3.2 For $p = 1$ we need to be slightly more careful, because the sequence $\{v_m\}$ is not weak null. Still, a finite analogue of the previous argument works here. (It actually does not require any assumptions on $p$ at all.)

Fix $N$ to be determined later, and note that in the definition of $\{v_m\}$, by using additionally a subspace game for $\{Y\}_N$ in $Y$, we can also ensure that the vectors $\{v_1, \ldots, v_N\}$ form a permissible $N$-tuple. All infinite arguments from 3.3.1 have finite analogues, this follows from a standard compactness argument, using the stability of $L_1$ under ultraproducts ([DK]). This means that given $n$, there is $N$ such that from every almost monotone normalized basic sequence $v_1, \ldots, v_N$, one can extract an almost symmetric subsequence $v_{m_1}, \ldots, v_{m_n}$ of length $n$. Since $\{v_1, \ldots, v_N\}$ is permissible, by [3.3.1], this subsequence can be assumed to be permissible as well. In particular, as in 3.3.1, its span is $C$-isomorphic to $l_1^n$.

3.3.3 We go back to our more general assumption $1 \leq p < 2$. Now we use [JMST] Theorem 1.5, which says that for $1 \leq p \leq \infty$, every $K$-symmetric basis in $l_p^n$ is $D'$-equivalent to the standard unit vector basis $\{e_i\}$ in $l_p^n$, where $D' = D'(K)$ depends on $K$ only. It follows that there is $D = D(C)$ such that $\{v_{m_i}\}_{i=1}^n$ is $D$-equivalent to the basis $\{e_i\}_{i=1}^n$ in $l_p^n$. 

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3.3.4 By Dor’s result [D], which is valid for $1 \leq p < \infty$, for some $\delta = \delta(D) > 0$, there exist disjoint subsets $A_1, \ldots, A_n$ of $[0, 1]$ such that

$$\int_{A_j} |v_{m_j}|^p \geq \delta \quad \text{for} \quad j = 1, \ldots, n.$$  

Since $\int |z_1|^p = 1$, taking $n$ sufficiently large, we get that at least one of the integrals $\int_{A_j} |z_1|^p$ is smaller than $\delta/4$; denote the corresponding set by $A^{(2)}$ and the corresponding vector $v_{m_j}$ by $z_2$.

Passing to a sequence $\{w_m\}$ of possible third choices for $V$ in the vector game, with the first two choices being $z_1, z_2$, and repeating the argument we get a set $A^{(3)}$ and a vector $z_3$ such that

$$\int_{A^{(3)}} |z_3|^p \geq \delta, \quad \int_{A^{(3)}} |z_1|^p < \delta/8, \quad \int_{A^{(3)}} |z_2|^p < \delta/4.$$  

By an obvious induction we get a permissible $n$-tuple $z_1, \ldots, z_n$, $(1+\varepsilon)$-equivalent to $\{e_i\}$ and disjoint sets $B_1, \ldots, B_n$ such that

$$\int_{B_1} |z_1|^p \geq 1 - \delta/2, \quad \int_{B_i} |z_i|^p \geq \delta/2, \quad \text{for} \quad i > 1.$$  

(We put $B_1 = (\bigcup_{i=2}^n A^{(i)})^c$ and $B_2 = A^{(2)} \setminus (\bigcup_{i=3}^n A^{(i)})$, etc., ). Recall that since $\{z_i\}$ are permissible, they are unconditional. Then the above condition implies a lower $l_p$-estimate: for all $\{a_i\}$ we have

$$\| \sum a_i z_i \|^p \sim \int (\sum |a_i z_i|^2)^{p/2} \geq \sum \int_{B_j} |a_j z_j|^p \geq \delta/2 \sum |a_j|^p.$$  

3.3.5 The upper $l_p$-estimates are easy. For $1 < p < 2$ the estimate follows from the type $p$ and from the unconditionality of the basis $\{e_i\}$, obtained in 3.2.1. For $p = 1$, we use the triangle inequality.

Thus $\{z_i\}_{i=1}^n$ is $D_p^n$-equivalent to the unit vector basis in $l_p^n$, as required, where the constant $D_p^n$ depends on $C$ and on $1 \leq p < 2$.

3.4 Let $p > 2$. We use Kadec–Pełczyński approach (cf. [LT.2], 1.c.8).

For $x \in L_p$ and $\delta > 0$, set $\sigma(x, \delta) = \{t \in [0, 1] \mid |x(t)| \geq \delta \|x\|\}$, and let $M(\delta) = \{x \mid \mu(\sigma(x, \delta)) < \delta\}$.

We start with a couple of general remarks which can be proved by standard well-known arguments.
3.4.1 Recall that if a sequence of functions \( \{w_m\} \) is \( K \)-unconditional and it belongs to \( M(\delta) \), for some \( \delta > 0 \), then \( \{w_m\} \) satisfies a lower \( l_2 \) estimate (with a constant depending on \( K \) and \( \delta \)) (cf. e.g., [LT.2], 1.c.10). If \( p > 2 \), combining this with the type 2 of the space \( L_p \) we get that \( \{w_m\} \) is equivalent to the unit vector basis in \( l_2 \).

3.4.2 Consider a sequence \( \{w_m\} \) such that \( w_m \notin M(2^{-m-2}) \) and let \( \eta_m = \sigma(w_m, 2^{-m-2}) \), for \( m = 1, 2, \ldots \). Given \( z_1, \ldots, z_k \) in \( L_p \), there exists \( m_0 \) such that \( \int_{\eta_{m_0}} |z_i|^p < 2^{-k-2} \) for \( i = 1, \ldots, k \).

3.4.3 Recall that \( E \in \{Y\}_n \) was an arbitrary asymptotic space with a basis \( \{e_i\} \) and consider the same games for \( E \) as in 3.3. Let us outline an inductive argument. Let \( 0 \leq k < n \) and assume that \( z_1, \ldots, z_k \) have been already defined as possible choices for the first \( k \) moves of player \( V \) (in a vector game for \( E \)). With these vectors fixed, consider a \( w \)-null sequence \( \{w_m\} \) of possible choices in the \( (k + 1) \)th move for \( V \). Using 3.4.1 and our main isomorphism assumption, we conclude that there is no \( \delta \) such that \( \{w_m\} \subset M(\delta) \). Passing to a subsequence we may therefore assume that \( w_m \notin M(2^{-m-2}) \), for \( m = 1, 2, \ldots \). Let \( m_0 \) be as in 3.4.2, denote \( w_{m_0} \) by \( z_{k+1} \) and set \( \sigma_{k+1} = \eta_{m_0} \).

Proceeding this way we get a permissible \( n \)-tuple \( \{z_i\} \), \((1 + \varepsilon)\)-isomorphic to \( \{e_i\} \), and subsets \( \sigma_i \) of \([0, 1]\), such that for every \( i = 1, \ldots, n \) we have

\[
\int_{\sigma_k} |z_i|^p < 2^{-k-2} \quad \text{for} \quad i < k \leq n.
\]

Obviously, for \( i = 1, \ldots, n \) we have \( \int_{\sigma_i} |z_i|^p < 2^{(-i-2)p} \). Thus, setting \( B_i = \sigma_i \setminus \bigcup_{k>i} \sigma_k \) we get

\[
\int_{B_i} |z_i|^p < 2^{(-i-2)p} + \sum_{k>i} 2^{-k-2} < 2^{-i-1} \quad \text{for} \quad i = 1, \ldots, n.
\]

Since \( \|z_i\| = 1 \) for \( i = 1, \ldots, n \), then \( \{z_i\} \) are equivalent (up to a universal constant) to the unit vector basis in \( l_p^n \). As already indicated at the end of 3.2, this completes the proof of the theorem. \( \square \)
4 Duality for asymptotic-$l_p$ spaces

4.1 A minimal system in a Banach space $X$ is a sequence $\{u_i\}$ such that there exists a sequence $\{u_i^*\}$ in $X^*$ so that $\{u_i, u_i^*\}$ is a biorthogonal system. Systems considered here will be always fundamental and total, in particular, $X = \text{span}\{u_i\}$. Some more information, and in particular classical definitions of shrinking and boundedly complete minimal systems, can be found e.g., in [LT.1], I.f. Let us just recall that a space $X$ is reflexive if and only if every minimal system in $X$ is both shrinking and boundedly complete. The reader who is not familiar with minimal systems may just think about a basis in $X$.

Recall that $\mathcal{B}'(X)$ denotes the family of all tail subspaces.

4.1.1 Let us recall the following known fact ([Mi], also [MiS], Proposition 2.1). In presence of a basis in $X$ this fact is obvious and does not require the shrinking assumption.

**Lemma** Let $(Y, \| \cdot \|_Y)$ be a Banach space with a shrinking minimal system. There exists an equivalent norm $\| \cdot \|$ on $Y$ such that $\|x\|_Y \leq \|x\| \leq 2\|x\|_Y$ for all $x \in Y$ and that for every $\delta > 0$ and every tail subspace $\tilde{Z} \in \mathcal{B}'(Y^*)$ there exists a tail subspace $\tilde{Y} \in \mathcal{B}'(Y)$ such that for every $x \in S(\tilde{Y})$ there is $f \in S(\tilde{Z})$ with $f(x) \geq 1 - \delta$.

4.2 Let $X$ be an asymptotic-$l_p$ space (with respect to the family $\mathcal{B}^0(X)$) and let $\{u_i\}$ be a minimal system in $X$. If $1 < p \leq \infty$, the system is shrinking. If $1 \leq p < \infty$, the system is boundedly complete.

Assume to the contrary that $\{u_i\}$ is not shrinking, i.e., $X^* \neq \text{span}\{u_i^*\}$. Fix $n \in \mathbb{N}$ to be defined later. There exists $x^* \in X^*$ with $\|x^*\| = 1$, for which one can construct a permissible $n$-tuple $\{x_i\}$ in $X$ (for an arbitrary $\varepsilon > 0$), such that $|x^*(x_i)| > \delta$ for $i = 1, \ldots, n$, where $\delta > 0$ is a universal constant. Then

$$C(1 + \varepsilon)n^{1/p} \geq \left\| \sum_{i=1}^n x_i \right\| \geq \left| x^* \left( \sum_{i=1}^n x_i \right) \right| \geq n\delta,$$

and, if $p > 1$, this is a contradiction for $n$ large enough.

Assume that $\{u_i\}$ is not boundedly complete. For every $n \in \mathbb{N}$, there exists a permissible (normalized) $n$-tuple $\{x_i\}$ such that $\sup_n \| \sum_{i=1}^n x_i \| = M < \infty$, where $M$ is a universal constant. On the other hand $\| \sum_{i=1}^n x_i \| \geq (1/C)n^{1/p}$, which is a contradiction, if $p < \infty$. 

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In particular, for $1 < p < \infty$, an asymptotic-$l_p$ space is reflexive. Note however that $l_1$- and $l_\infty$-spaces may be reflexive as well; such examples are given by the Tsirelson space $T(1)$ and its dual $T(1)^*$ (cf. e.g., [CS]).

4.3

Theorem  Let $1 \leq p \leq \infty$ and let $X$ be an asymptotic-$l_p$ space which is reflexive. Then $X^*$ is an asymptotic-$l_{p'}$, where $1/p + 1/p' = 1$ (with the standard convention for $p = 1$ and $p = \infty$).

4.3.1 Let $\{u_i\}$ be a minimal system in $X$. By [1.8.1] we may assume, without loss of generality, that the norm in $X$ satisfies the conclusion of Lemma [4.1.1]. Moreover, the asymptotic structures of $X$ and of $X^*$ are determined by the families $\mathcal{B}^t(X)$ and $\mathcal{B}^t(X^*)$ associated to $\{u_i\}$ and to $\{u_i^*\}$, respectively.

To make the statements below more intuitively clear and to avoid tiresome repetitions, let us recall (cf. [1.3.3 and 1.4.3]) that if $\varepsilon > 0$ is fixed, then an $n$-tuple in $X$ (resp. in $X^*$) is permissible, if it is $(1+\varepsilon)$-equivalent to the natural basis in an asymptotic space from $\{X\}_n$ (resp. $\{X^*\}_n$). In particular an $n$-tuple is permissible if it is obtained as a result of a subspace game in $X$ (resp. in $X^*$), assuming that player $S$ followed his winning strategy for $\{X\}_n$ (resp. $\{X^*\}_n$) and $\varepsilon$.

4.4 An asymptotic lower $l_{p'}$ estimate in $X^*$ is based on the following lemma.

Lemma  Let $Y$ be a Banach space with a shrinking minimal system. Let $\{e_i\} \in \{Y\}_n$ be an asymptotic $n$-tuple and let $\varepsilon > 0$. There exist a permissible $n$-tuple $\{z_i\}$ in $Y$ satisfying $\{z_i\} \overset{1+\varepsilon}{\sim} \{e_i\}$, and a permissible $n$-tuple $\{g_i\} \subset S(Y^*)$ in $Y^*$, such that $g_i(z_i) \geq 1 - \varepsilon$ for $i = 1, \ldots, n$ and $g_i(z_j) = 0$ if $i \neq j$.

The proof of the lemma requires an asymptotic game in $Y$, which combines strategies for two simultaneous games: a winning strategy for $V$ in a vector game in $Y$ and a winning strategy for $S$ in a subspace game in $Y^*$. The latter strategy ensures permissibility in $Y^*$ and it determines choices of subspaces in $Y$ via [4.1.1] (cf. the proof of Lemma [1.5] below). We leave it for the reader.
Now the proof of the lower $l_{p'}$ estimate follows a standard argument. Given an asymptotic $n$-tuple $\{e_i\}$ in $X^*$ and $\varepsilon > 0$, let $\{z_i\}$ in $X^*$ and $\{g_i\}$ in $X$ be as in the lemma. For any scalar $n$-tuple $a = \{a_i\}$, pick $b = \{b_i\}$ with $\|b\|_p = 1$ such that $\sum_i a_ib_i = \|a\|_{p'}$. Then

$$(1 - \varepsilon)(\sum_i |a_i|^{p'})^{1/p'} \leq (\sum_i b_i g_i)(\sum_i a_i z_i) \leq C(1 + \varepsilon)\|\sum_i a_i e_i\|,$$

as required.

4.5 An asymptotic upper $l_{p'}$ estimate in $X^*$ is based on the following reformulation in our context of Theorem 2.2 from [MiS].

**Lemma** Let $Y$ be a Banach space with a shrinking minimal system. Let $\{e_i\} \in \{Y\}_n$ be an asymptotic $n$-tuple, let $\{a_i\}$ be an arbitrary scalar sequence and let $\varepsilon > 0$. There exist a permissible $n$-tuple $\{y_i\}$ in $Y$ satisfying $\{y_i\}_{1+\varepsilon} \sim \{e_i\}$, and a permissible $n$-tuple $\{g_i\}$ in $Y^*$, and a sequence of scalars $\{b_i\}$, such that $g_i(y_j) = 0$ if $i \neq j$ and

$$(\sum_{i=1}^n b_i g_i) \left(\sum_{j=1}^n a_j y_j\right) \geq (1 - \varepsilon) \left(\sum_{i=1}^n b_i g_i\right) \left\|\sum_{j=1}^n a_j y_j\right\|.$$

4.5.1 This result is based on an argument which might be useful in other context; for the reader convenience we outline the proof.

**Proof** We provide a complete argument for $n = 2$, with few comments concerning the general case. Let $Z = Y^*$ and fix $\delta > 0$ to be defined later. Consider a subspace game in $Z$ for $\{Z\}_2$ and $\varepsilon$. We name the players of this game by $S^*$ and $V^*$ respectively. Let $Z_1 \in B'(Z)$ be a tail subspace chosen by $S^*$ in the first move. Let $Y_1 \in B'(Y)$ be a corresponding subspace (for $\delta$), as in Lemma [1.1.1]. Now consider a vector game in $Y$ for $\{e_i\}$ and $\varepsilon$, with $Y_1$ being the first choice of player $S$. Let player $V$ choose $y \in S(Y_1)$. Considering appropriate choices for the second move of the subspace player $S$, we obtain a sequence of successive blocks $y_1 < y_2 < \ldots$ of second choices for $V$ (with the first choice always being $y$). (If $n > 2$, then with a fixed $m \in \mathbb{N}$ let $\{y_{m,i}\} \subset S(Y_1)$ be a sequence of successive blocks, each of which could be picked by $V$ in his third move, in the game in which his first two moves were $y$ and $y_m$. And so on.)
Fix \(m\). Then \(\{y, y_m\} \overset{1+\varepsilon}{\sim} \{e_1, e_2\}\). Let \(w_m = a_1y + a_2y_m\). Let \(f_m \in S(Z_1)\) be a functional norming \(w_m\) up to \(\delta\), as in Lemma \([4.4.1]\). We will show that there is \(\mu \in \mathbb{N}\) such that \(f_\mu\) can be approximated (up to \(3\delta\)) by a functional of a form \(h = b_1g_1 + b_2g_2\), with \(\{g_i\}\) permissible and satisfying the required biorthogonality condition. In particular, \(h\) will norm \(w_\mu\) up to \(4\delta\), which will give the conclusion by setting \(\delta = \varepsilon/4\).

Let \(f\) be a \(w^*\)-cluster point of \(\{f_m\}_m\). Then \(f \in Z_1\). Let \(h_1 \in Z_1\) be finitely supported such that \(\|h_1 - f\| < \delta\). Let \(g_1 = h_1/\|h_1\|\) and consider this \(g_1\) as a choice for the vector player \(V^*\) in the subspace game in \(Z\), so that \(S^*\) chooses \(Z_1\) and \(V^*\) chooses \(g_1\). Let \(Z_2 \in B(Z)\) be a subspace picked by \(S^*\) in his second move. Then \(Z_2\) is the \(k\)th tail subspace, for some \(k \in \mathbb{N}\) and we may assume without loss of generality that \(k > \max(\text{supp}(g_1) \cup \text{supp}(y))\). Let \(Q_k\) denote the canonical projection in \(Z\) onto \(\text{span}\{u_i^*\}_{i \leq k}\), so that in particular \(Q_kh_1 = h_1\). Pick \(\mu \in \mathbb{N}\) such that \(\|Q_kf_\mu - h_1\| \leq \|Q_k(f_\mu - f)\| + \|Q_k(f - h_1)\| < 2\delta\) and that \(\min \text{supp}(y_\mu) > k\). Then \((I - Q_k)f_\mu \in Z_2\) and pick finitely supported \(h_2 \in Z_2\) such that \(\|(I - Q_k)f_\mu - h_2\| < \delta\). Set \(g_2 = h_2/\|h_2\|\). Then \(\{y, y_\mu\}\) is the required permissible couple in \(Y\). Note that \(g_1\) and \(y_\mu\) are disjointly supported, and so are \(y\) and \(g_2\). Thus \(g_1(y_\mu) = g_2(y) = 0\). Also, the functional \(h = h_1 + h_2\) approximates \(f_\mu\) up to \(3\delta\), as promised. Finally, since \(Z_2\) was a second choice of \(S^*\) and \(g_2 \in Z_2\), then \(\{g_1, g_2\}\) is a permissible couple in \(X^*\) and of course, \(h = b_1g_1 + b_2g_2\), for suitable scalars \(b_1, b_2\). (If \(n > 2\), consider \(g_2\) as a second choice for \(V^*\), and let \(Z_3\) be a subspace picked by \(S^*\), which starts after \(g_2\) and \(y_\mu\) and then repeat the argument.) \(\square\)

4.5.2 Again, the proof of an upper \(l^{p'}\)-estimate is completely standard. Given an asymptotic \(n\)-tuple \(\{e_i\}\) in \(X^*\), scalars \(\{a_i\}\) and \(\varepsilon > 0\), apply the lemma for \(Y = X^*\) to get \(\{y_i\}\) in \(X^*\) and \(\{g_i\}\) in \(X\) and scalars \(\{b_i\}\), with the additional normalization \(\|\sum_i b_ig_i\| = 1\). Then \((\sum |b_i|^{p})^{1/p} \leq C\). Thus

\[
(1 - \varepsilon) \left\| \sum_{j=1}^{n} a_jy_j \right\| \leq \left( \sum_{i=1}^{n} b_ig_i \right) \left( \sum_{j=1}^{n} a_jy_j \right) = \sum_{i=1}^{n} a_i b_i \leq C \left( \sum_{i=1}^{n} |a_i|^{p'} \right)^{1/p'},
\]

as required. Combined with \([4.4]\) this concludes the proof of Theorem \([4.3]\). \(\square\)
5 Complemented permissible subspaces

It is well-known and easy to see that every block subspace of \( l_p \) is complemented; the same is true for Tsirelson spaces \( T(p) \), although in this case it is much more difficult to prove (here \( 1 \leq p < \infty \))(cf. [CS]). To get a related complementation property which actually characterizes spaces \( l_p \) or \( c_0 \), one needs to add an unconditionality assumption and to consider all permutations of a given basis ([LT.3], cf. also [LT.1] 2.a.10). In the asymptotic setting the situation is more natural and elegant, and a natural complementation condition fully characterizes asymptotic-\( l_p \) spaces.

5.1 We start by describing few more asymptotic notions. Let \( \mathcal{P} \) be a property of finite-dimensional subspaces of a given Banach space.

**Definition** We say that \( \mathcal{P} \) is satisfied by permissible subspaces of \( X \) far enough, if for every \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), the subspace player \( S \) has a winning strategy in a subspace game for \( \{X\}_n \) and \( \varepsilon \) such that arbitrary \( n \)-tuple \( \{x_i\} \) resulting from the game spans a subspace with property \( \mathcal{P} \). (This subspace is automatically permissible, since the strategy is winning for \( \{X\}_n \).)

We have a similar definition if \( \mathcal{P}_n \) is a property of \( n \)-dimensional subspaces of \( X \), with \( n \in \mathbb{N} \) fixed. Any strategy for \( S \) as above will be called a \( \mathcal{P} \)-strategy.

5.1.1 Recall our intuition of a tree-like structure of subspaces and vectors, as in 1.4.2. Then \( \mathcal{P} \) is satisfied by permissible subspaces of \( X \) far enough, if and only if for an arbitrary \( n \in \mathbb{N} \) and \( \varepsilon > 0 \), by pushing subspaces far enough along \( \mathcal{B}(X) \) player \( S \) can ensure that the subspaces spaned by all resulting \( n \)-tuples are not only permissible but they also have property \( \mathcal{P} \).

5.1.2 Assume that \( \mathcal{P} \) is satisfied by permissible subspaces of \( X \) far enough and let \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). By combining the strategy for the subspace player \( S \) discussed in 1.8.3 with a \( \mathcal{P} \)-strategy, and using filtration condition [1], we obtain a strategy for \( S \) such that arbitrary normalized successive blocks of any \( n \)-tuple \( \{x_i\} \) resulting from the game, are permissible and their span has property \( \mathcal{P} \).
Of course the set of all \( n \) -tuples resulting from the game above represents all spaces from \( \{X\}_n \). In other words, for any \( E \in \{X\}_n \) there is \( \{x_i\} \) as above, \((1 + \varepsilon)\)-equivalent to the basis in \( E \). Indeed, we could appropriately instruct player \( V \) to achieve this \( E \), up to \( 1 + \varepsilon \).

5.1.3 Let \( X \) be a Banach space with a minimal system \( \{u_i\} \). Let \( Y = \text{span} \[ y_i \] \) be a subspace of \( X \). A projection \( P : X \to Y \) is called \( \{u_i\}\)-permissible (or just permissible, if the system \( \{u_i\} \) is understood from the context) if \( P \) can be written as \( P = \sum_i g_i \otimes y_i \), with \( g_i \in X^* \) finitely supported and \( \max \text{supp} (g_i) < \min \text{supp} (g_{i+1}) \), for \( i = 1, 2, \ldots \).

5.2 The duality theorem 4.3 implies (and in fact is equivalent to) a complementation property of asymptotic-\( l_p \) spaces.

**Corollary** Let \( X \) be an asymptotic-\( l_p \) space for some \( 1 < p < \infty \). Then there is \( D \) such that permissible subspaces of \( X \) far enough are \( D \)-complemented by means of permissible projections.

**Proof** Let \( \{u_i\} \) be a minimal system in \( X \), and without loss of generality let us make all the assumptions as in 4.3.1. Let \( n \in \mathbb{N} \) and \( \varepsilon > 0 \). Player \( S \) has a strategy in an asymptotic game in \( X \) such that if \( \{x_i\} \) is a resulting permissible \( n \)-tuple in \( X \), then there exists a permissible \( n \)-tuple \( \{g_i\} \subset S(X^*) \), with \( \max \text{supp} (g_i) < \min \text{supp} (g_{i+1}) \), such that \( g_i(x_i) \geq 1 - \varepsilon \) for \( i = 1, \ldots, n \) and \( g_i(x_j) = 0 \) if \( i \neq j \). Indeed, the strategy for \( S \) is essentially the same as in Lemma 4.4, with an additional requirement for successiveness of the \( g_i \)’s. This property formally implies the existence of a required permissible projection onto \( \text{span} [x_i] \).

Let \( P = \sum_i g_i \otimes x_i \). Clearly, \( P \) is a permissible projection onto \( \text{span} [x_i] \). Fix an arbitrary vector \( x \in X \) and pick scalars \( \{b_i\} \) such that \( \sum_i g_i(x) b_i = (\sum_i |g_i(x)|^p)^{1/p} \) and \( \sum_i |b_i|^p = 1 \). Since \( X \) is asymptotic-\( l_p \) space and, by Theorem 4.3, \( X^* \) is asymptotic-\( l_{p'} \) space (with a constant \( C \)), then

\[
\|Px\| = \| \sum_i g_i(x) x_i \| \leq C (\sum_i |g_i(x)|^p)^{1/p} = C \sum_i g_i(x) b_i \leq C \| \sum_i b_i g_i \| \leq C^2.
\]

Thus \( \|P\| \leq C^2 \).  \( \square \)
5.3 For spaces with basis the converse is true.

**Theorem** Let $X$ be a Banach space with a basis. Assume that there exists a constant $C$ such that permissible subspaces of $X$ far enough are $C$-complemented by means of permissible projections. Then $X$ is an asymptotic-$l_p$ space for some $1 \leq p \leq \infty$.

The asymptotic structure in $X$ may be naturally taken with respect to either family $\mathcal{B}^0(X)$ or $\mathcal{B}^t(X)$. Then the conclusion of the theorem relates to the same structure.

5.4 Before we pass to the proof of the theorem, let us make some comments.

5.4.1 The argument below shows that if the basis in $X$ is unconditional then the assumption that projections are permissible can be dropped.

5.4.2 For arbitrary Banach spaces we have

**Corollary** Let $X$ be a Banach space. Assume that a universal asymptotic space $Y \in \mathcal{A}(X)$ has the property that there exists a constant $C$ such that permissible subspaces of $Y$ far enough are $C$-complemented (in $Y$) by means of permissible projections. Then $X$ is an asymptotic-$l_p$ space for some $1 \leq p \leq \infty$.

This corollary follows immediately by applying Theorem 5.3 to $Y$.

5.5 The argument below is an asymptotic analogue of the original proof as presented e.g. in [LT.1] 2.a.10.

**Proof** Let $n \in \mathbb{N}$ and let $\{v_i\} \in \{X\}_n$ be an asymptotic $n$-tuple. Fix Krivine’s $p \in [1, \infty]$, as in [16.3]. Let $I_1 = \{k(n + 1) + 1 \mid k = 0, \ldots, n-1\}$ and let $I_2 = \{1, \ldots, (n+1)^2\} \setminus I_1$. By [18.2] there exists an asymptotic $(n+1)^2$-tuple $\{f_i\}$ such that $\{f_i\}_{i \in I_1} \sim \{v_i\}$ and $\{f_i\}_{i \in I_2} \sim \{e_k\}$, where $\{e_k\}$ is the unit vector basis in $l_p^{n(n+1)}$.

Now fix $\varepsilon > 0$ and let $\{u_i\}$ be a permissible $(n+1)^2$-tuple of successive blocks of the basis in $X$, $(1+\varepsilon)$-equivalent to $\{f_i\}$ and such that all subspaces spaned by successive blocks of $\{u_i\}$ admit permissible projections of norm $\leq C$. This is possible by the final comment in [5.1.2]. Set $F = \text{span} [u_i]_{i=1}^{n(n+1)}$ and $E = \text{span} [u_i]_{i \in I_1}$ and relabel the basis in $E$ by $\{x_i\}_{i=1}^n$. 

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5.5.1 By the assumption, there exists a projection $Q : F \to E$ with $\|Q\| \leq C$. Since $\text{codim ker } Q = n$, for every $j = 1, \ldots, n$, we can find vectors $e_j \in \ker Q \cap \text{span} [u_{(j-1)(n+1)+1}, \ldots, u_{jn+1}]$ with $\|e_j\| = 1$. Thus we have successive blocks of the basis $x_1, e_1, x_2, \ldots, x_n, e_n$ and we denote their span by $Z$. Of course, $\{e_j\} \overset{1+\varepsilon}{\sim} \{e_i\}$, and it will cause no confusion to write $l_p^n$ for $\text{span} [e_j]$.

By the construction, $Z$ is a permissible $2n$-dimensional subspace of $X$ and $Z = E \oplus l_p^n$; the natural projection $Q$ on the first coordinate has norm $\leq C$ (hence the norm of the projection on the second coordinate is $\leq C + 1$).

5.5.2 Fix $\lambda > 0$ and let

$$G = \text{span} [x_1 + \lambda e_1, x_2 + \lambda e_2, \ldots, x_n + \lambda e_n] \subset Z.$$ 

Since $G$ is a block subspace of $Z$, there is a permissible projection $P : Z \to G$ onto $G$ with $\|P\| \leq C$. The form of $G$ implies that $P$ written in the $C$-direct sum decomposition of $Z$ has a matrix of the form

$$P_Z = \begin{bmatrix} A & (1/\lambda)B \\ \lambda A & B \end{bmatrix}$$

In other words, writing $P = \sum_i z_i^* \otimes (x_i + \lambda e_i)$, as in 1.1.3, we have

$$a_{i,j} = z_i^*(x_j) \quad \text{and} \quad b_{i,j} = \lambda z_i^*(e_j) \quad \text{for } i, j = 1, \ldots, n.$$

5.5.3 Since $P$ is a projection, we have $A + B = I$; that is,

$$a_{i,j} + b_{i,j} = z_i^*(x_j + \lambda e_j) = \delta_{ij} \quad \text{for } i, j = 1, \ldots, n.$$

Since $P$ is permissible, we have $\max \text{ supp } z_i^* < \min \text{ supp } z_{i+1}^*$, for all $i = 1, \ldots, n - 1$. Finally, the form of $P$ implies the norm estimates:

$$\|A : E \to l_p^n\| \leq (1/\lambda)C(C + 1) \quad \text{and} \quad \|B : l_p^n \to E\| \leq \lambda C^2.$$

Since $\text{ supp } z_i^* \cap \text{ supp } (x_i + \lambda e_i) \neq \emptyset$, and $\max \text{ supp } z_{i-1}^* < \min \text{ supp } z_i^*$ and $\max \text{ supp } z_i^* < \min \text{ supp } z_{i+1}^*$, then $\text{ supp } z_i^* \cap \text{ supp } x_j = \emptyset$, if $|i - j| > 1$ and $i = 1, \ldots, n$. In particular, $a_{i,j} = 0$ if $|i - j| > 1$ and $i = 1, \ldots, n$. Similarly, $b_{i,j} = 0$ if $|i - j| > 1$ and $i = 1, \ldots, n$. So for any $\lambda > 0$, the matrices of operators $A$ and $B$ are tri-diagonal.
5.5.4 Let $\lambda = 1/4C^2$. Then, by 5.5.3, $|b_{i,j}| \leq \|B : l^n_p \to E\| \leq 1/4$, for $i, j = 1, \ldots, n$. Since the matrix of $B$ is tri-diagonal, $\|B : l^n_p \to l^n_p\| \leq 3 \max_{i,j} |b_{i,j}| \leq 3/4$. Since $I - A = B$, this implies that $A$ is invertible on $l^n_p$ and

$$\|A^{-1} : l^n_p \to l^n_p\| \leq 4.$$ 

Combining with norm estimates from 5.5.3 we get

$$\|I : E \to l^n_p\| \leq \|A : E \to l^n_p\| \|A^{-1} : l^n_p \to l^n_p\| \leq 16C^3(C + 1).$$

This means that the vectors $\{x_i\}$ satisfy the lower $l_p$-estimate with the constant $C' = 16C^3(C + 1)$.

5.5.5 Let $\lambda = 4C(C + 1)$. By 5.5.3, $|a_{i,j}| \leq 1/4$, for $i, j = 1, \ldots, n$, hence $\|A : l^n_p \to l^n_p\| \leq 3/4$. Thus $\|B^{-1} : l^n_p \to l^n_p\| \leq 4$, and hence

$$\|I : l^n_p \to E\| \leq \|B^{-1} : l^n_p \to l^n_p\| \|B : l^n_p \to E\| \leq 16C^3(C + 1) = C'.$$

It follows that the vectors $\{x_i\}$ satisfy the upper $l_p$-estimate with the constant $C'$. Thus, $\{x_i\} \sim C^2_\| \{e_i\}$. By the construction at the beginning of the proof, the same holds for $\{v_i\} \subset \{X\}_n$, hence $X$ is an asymptotic-$l_p$. \qedsymbol

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