Abstract. The loxodromic Eisenstein series is defined for a loxodromic element of cofinite Kleinian groups. It is the analogue of the ordinary Eisenstein series associated to cusps. We study the asymptotic behavior of the loxodromic Eisenstein series for degenerating sequences of three-dimensional hyperbolic manifolds of finite volume. In particular, we prove that if the loxodromic element corresponds to the degenerating geodesic, then the associated loxodromic Eisenstein series converges to the ordinary Eisenstein series associated to the newly developing cusp on the limit manifold.

1. Introduction

1.1. Summary

Let $M$ be a complete, three-dimensional hyperbolic manifold of finite volume with constant negative curvature $-1$. We assume that $M$ is not compact, i.e. $M$ has some cusps. Let $\mathbb{H}^3$ be the three-dimensional hyperbolic space and $\Gamma \subset \text{PSL}(2, \mathbb{C})$ be a cofinite Kleinian group acting on $\mathbb{H}^3$. For $P \in \mathbb{H}^3$, we write $P = z + rj$ with $z \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$. Then the quotient $\Gamma \backslash \mathbb{H}^3$ is a three-dimensional hyperbolic manifold of finite volume with respect to the natural hyperbolic metric induced from $\mathbb{H}^3$. Conversely, for any three-dimensional hyperbolic manifold $M$ of finite volume, we identify the fundamental group of $M$ with a cofinite Kleinian group $\Gamma \subset \text{PSL}(2, \mathbb{C})$. According to this identification, we identify $M$ with $\Gamma \backslash \mathbb{H}^3$.

Let $\xi \in \mathbb{P}^1 \mathbb{C} = \mathbb{C} \cup \{\infty\}$ be a cusp of $M$ and $\Gamma^\prime_\xi$ be the maximal unipotent subgroup of the stabilizer group of $\xi$. Then, for $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with sufficiently large $\text{Re}(s)$, the ordinary Eisenstein series associated to $\xi$ is defined as follows:

$$E_{\xi} (P, s) = E_{\text{par}, \xi} (P, s) := \sum_{M \in \Gamma^\prime_\xi \backslash \Gamma} r (AM P)^s,$$

where $A$ is an element of $\text{PSL}(2, \mathbb{C})$ such that $A\xi = \infty$. In this article, we call $E_{\text{par}, \xi} (P, s)$ the parabolic Eisenstein series in order to distinguish the loxodromic Eisenstein series. The parabolic Eisenstein series $E_{\text{par}, \xi} (P, s)$ converges absolutely and locally uniformly for any $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 2$ and it defines a $\Gamma^\prime$-invariant function where it converges. Furthermore, for Laplace–Beltrami operator $\Delta$, the parabolic Eisenstein series $E_{\text{par}, \xi} (P, s)$

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satisfies the following differential equation:

\[ (-\Delta + s(s - 2)) E_{\text{par}, \gamma}(P, s) = 0. \]

The loxodromic Eisenstein series is the analogue of the parabolic Eisenstein series. It is defined for a loxodromic element of \( \Gamma \). The precise definition of the loxodromic Eisenstein series is as follows. Let \( \gamma \in \Gamma \) be a loxodromic element and \( \Gamma_\gamma \) be its centralizer group in \( \Gamma \). In order to define the loxodromic Eisenstein series, we introduce the change of coordinates \( z := x + iy = e^\rho \cos \varphi (\cos \theta + i \sin \theta), \quad r = e^\rho \sin \varphi \). Under these coordinates, the loxodromic Eisenstein series associated to \( \gamma \) is defined as follows:

\[
E_{\text{lox}, \gamma}(P, s) := \sum_{\eta \in \Gamma_\gamma \setminus \Gamma} \sin \varphi (A\eta P)^s,
\]

where \( s \in \mathbb{C} \) with sufficiently large \( \text{Re}(s) \) and \( A \) is an element in \( \text{PSL}(2, \mathbb{C}) \) such that

\[
A\gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix}
\]

for some \( a(\gamma) \in \mathbb{C} \) with \( |a(\gamma)| > 1 \). The loxodromic Eisenstein series \( E_{\text{lox}, \gamma}(P, s) \) converges absolutely and locally uniformly for any \( P \in \mathbb{H}^3 \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 2 \). It defines a \( \Gamma \)-invariant function where it converges and satisfies the following differential equation:

\[ (-\Delta + s(s - 2)) E_{\text{lox}, \gamma}(P, s) = s^2 E_{\text{lox}, \gamma}(P, s + 2). \]

Furthermore, it is known that it has a meromorphic continuation to all \( s \in \mathbb{C} \) (see [6]).

In this article, we consider the degenerating sequence of three-dimensional hyperbolic manifolds. Since \( M \) is a non-compact, three-dimensional hyperbolic manifold of finite volume, by the cusp closing theorem of W. Thurston [13], we can realize \( M \) as a limit of a non-trivial sequence of finite-volume, three-dimensional hyperbolic manifolds \( \{ M_i = \Gamma_i \setminus \mathbb{H}^3 \}_{i=1}^\infty \). Then, for each newly developing cusp on \( M \), there exists a pinching closed geodesic on \( M_i \) such that the length tends to 0 as \( i \) tends to \( \infty \). In this setting, we investigate the asymptotic behavior of the parabolic and the loxodromic Eisenstein series for degenerating sequences of three-dimensional hyperbolic manifolds and obtain the following results.

**Theorem 1.1.** Let \( \{ M_i = \Gamma_i \setminus \mathbb{H}^3 \}_{i=1}^\infty \) be a degenerating sequence of three-dimensional hyperbolic manifolds of finite volume with the limit manifold \( M_\infty = M = \Gamma \setminus \mathbb{H}^3 \). Then for \( P \in \mathbb{H}^3 \) and any \( s \in \mathbb{C} \) with \( \text{Re}(s) > 2 \) the following assertions hold.

(a) Let \( E_{\text{lox}, M_i, \gamma_i}(P, s) \) be the loxodromic Eisenstein series on \( M_i \) associated to a loxodromic element \( \gamma_i \). If \( \gamma_i \) does not correspond to shrinking closed geodesics, then we have

\[
\lim_{i \to \infty} E_{\text{lox}, M_i, \gamma_i}(P, s) = E_{\text{lox}, M, \gamma}(P, s).
\]

(b) Let \( \xi \in \mathbb{C} \cup \infty \) be a cusp of \( M_i \) for all \( i \) and \( E_{\text{par}, M_i, \xi}(P, s) \) be the parabolic Eisenstein series on \( M_i \) associated to the cusp \( \xi \). Then we have

\[
\lim_{i \to \infty} E_{\text{par}, M_i, \xi}(P, s) = E_{\text{par}, M, \xi}(P, s).
\]
(c) Let \( E_{\text{lox}, M, \gamma_i}(P, s) \) be the loxodromic Eisenstein series on \( M_i \) associated to a loxodromic element \( \gamma_i \). If each \( \gamma_i \) corresponds to the shrinking geodesics \( L_{\gamma_i} \) which results in the new cusp \( \zeta \) on \( M \), then we have

\[
\lim_{i \to \infty} \left( \frac{|P|}{2\pi l_{\gamma_i}} \right)^{s/2} E_{\text{lox}, M, \gamma_i}(P, s) = E_{\text{par}, M, \zeta}(P, s),
\]

where \( |P| \) denotes the Euclidean area determined by the boundary torus of cusp \( \zeta \) and \( l_{\gamma_i} \) is the length of \( L_{\gamma_i} \).

In all three instances, the convergence is absolute and locally uniform away from the developing cusps and any \( s \in \mathbb{C} \) with \( \Re(s) > 2 \).

1.2. Related studies

In our previous article [6], we introduced the loxodromic Eisenstein series as a generalization of the hyperbolic Eisenstein series for Fuchsian groups of the first kind.

The hyperbolic Eisenstein series is defined for hyperbolic fixed points, or equivalently a primitive hyperbolic element of Fuchsian groups of the first kind. It was first introduced by S. S. Kudla and J. J. Millson [9] in 1979 as an analogue of the ordinary Eisenstein series associated to a parabolic fixed point. They established an explicit construction of the harmonic 1-form dual to an oriented closed geodesic on an oriented Riemann surface \( M \) of genus greater than one. Furthermore, they proved the meromorphic continuation to all of \( C \) and gave the location of the possible poles when \( M \) is compact. J. Jorgenson, J. Kramer, and A.-M. v. Pippich [7] established the precise spectral expansion for the hyperbolic Eisenstein series by proving that the hyperbolic Eisenstein series is square integrable. They also proved the meromorphic continuation with the location of the possible poles and their residues from the precise spectral expansion. In our previous article [6], we extended the hyperbolic Eisenstein series for Fuchsian groups of the first kind to the cofinite Kleinian groups and the three-dimensional hyperbolic space and obtained the main results analogous to those of J. Jorgenson, J. Kramer, and A.-M. v. Pippich [7]. T. Falliero [2] studied the asymptotic behavior of the hyperbolic Eisenstein series for a degenerating family of finite-volume hyperbolic Riemann surfaces. They proved that the limit of the hyperbolic Eisenstein series associated to the pinching geodesic for the degenerating family is equal to the parabolic Eisenstein series associated to the newly formed cusp on the limit surface. D. Garbin, J. Jorgenson, and M. Munn [3] proved the same results as T. Falliero [2] using different methods, namely using counting functions and Stieltjes integrals.

Degeneration of three-dimensional hyperbolic manifolds differs from that of hyperbolic Riemann surfaces in several points. However, the methods of D. Garbin, J. Jorgenson, and M. Munn [3] are applicable to the three-dimensional case. In this article, we apply their methods to the loxodromic Eisenstein series and degenerating sequences of three-dimensional hyperbolic manifolds of finite volume and obtain the main results on the asymptotic behavior of loxodromic Eisenstein series for a degenerating sequence of three-dimensional hyperbolic manifolds of finite volume.
2. Preliminaries

In this section, we introduce the basic notation and known results needed later. We introduce the three-dimensional hyperbolic space \( \mathbb{H}^3 \), the group action on \( \mathbb{H}^3 \) of Kleinian groups, three-dimensional hyperbolic manifolds, parabolic and loxodromic Eisenstein series, and so on. We mainly refer to J. Elstrodt, F. Grunewald, and J. Mennicke [1], M. Gromov [5] and M. Taniguchi and K. Matsuzaki [12].

2.1. Three-dimensional hyperbolic space

Let \( \mathbb{H}^3 \) be the three-dimensional hyperbolic space. We use the following coordinates:

\[
\mathbb{H}^3 := \mathbb{C} \times (0, \infty)
\]

\[
= \{(z, r) \mid z \in \mathbb{C}, \ r > 0\}
\]

\[
= \{(x, y, r) \mid x, y \in \mathbb{R}, \ r > 0\}.
\]

For \( P \in \mathbb{H}^3 \), we use the notation

\[
P = (z, r) = (x, y, r) = z + rj,
\]

where

\[
z = x + iy, \quad j = (0, 0, 1).
\]

Then the hyperbolic line element \( d\sigma^2 \) and hyperbolic volume element \( dv \) on \( \mathbb{H}^3 \) are given by

\[
d\sigma^2 := \frac{dx^2 + dy^2 + dr^2}{r^2}, \quad dv := \frac{dx \, dy \, dr}{r^3}.
\]

For any two points \( P, Q \in \mathbb{H}^3 \), we denote the hyperbolic distance between \( P \) and \( Q \) associated to \( d\sigma \) by \( d_{\text{hyp}}(P, Q) \). The hyperbolic Laplace–Beltrami operator is given by

\[
\Delta = r^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) - r \frac{\partial}{\partial r}.
\]

Under the change of coordinates \( x := e^\rho \cos \varphi \cos \theta, \ y := e^\rho \cos \varphi \sin \theta, \ r := e^\rho \sin \varphi \), where \( \rho \in \mathbb{R}, \ 0 < \varphi \leq \pi/2, \ 0 \leq \theta < 2\pi \), the hyperbolic line element and hyperbolic volume element are written as follows:

\[
d\sigma^2 = \frac{d\rho^2 + d\varphi^2 + \cos^2 \varphi \, d\theta^2}{\sin^2 \varphi}, \quad dv = \frac{\cos \varphi}{\sin^3 \varphi} \, d\rho \, d\varphi \, d\theta.
\]

Then the hyperbolic Laplace–Beltrami operator is given by

\[
\Delta = \sin^2 \varphi \left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \varphi^2} + \frac{1}{\cos^2 \varphi} \frac{\partial^2}{\partial \theta^2} \right) - \frac{\sin \varphi}{\cos \varphi} \cdot \frac{\partial}{\partial \varphi}.
\]
2.2. Group action on $\mathbb{H}^3$ and Kleinian groups

The group $\text{PSL}(2, \mathbb{C})$ of complex $2 \times 2$ matrices with determinant one modulo its center $\{\pm I\}$ acts on $\mathbb{H}^3$ naturally. It is defined as follows. For

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$$

and $P \in \mathbb{H}^3$

$$P \mapsto MP := M(P) := (aP + b)(cP + d)^{-1},$$

where the inverse is taken in the skew field of quaternions. More explicitly, we can write this action for $P = z + rJ$ as follows:

$$M(P) = (az + b)(c\bar{z} + \bar{d}) + a\bar{c}r^2 \frac{r}{|cz + d|^2 + |c|^2r^2}.$$ 

Let $\text{Iso}(\mathbb{H}^3)$ be the group of isometries on $\mathbb{H}^3$. We write $\text{Iso}^+(\mathbb{H}^3)$ for the group of orientation-preserving isometries of $\mathbb{H}^3$. Then $\text{Iso}^+(\mathbb{H}^3)$ is isomorphic with $\text{PSL}(2, \mathbb{C})$. A subgroup $\Gamma \subset \text{Iso}(\mathbb{H}^3)$ is called a discontinuous group if and only if for every $P \in \mathbb{H}^3$ and for every sequence $(T_n)_{n \geq 1}$ of distinct elements of $\Gamma$ the sequence $(T_nP)_{n \geq 1}$ has no accumulation point in $\mathbb{H}^3$. In this case $\Gamma$ is also said to act discontinuously on $\mathbb{H}^3$. Discontinuous groups $\Gamma \subset \text{PSL}(2, \mathbb{C})$ can be characterized as groups which are discrete in $\text{PSL}(2, \mathbb{C})$; i.e. a subgroup $\Gamma \subset \text{PSL}(2, \mathbb{C})$ is a discontinuous group if and only if $\Gamma$ is discrete in $\text{PSL}(2, \mathbb{C})$.

**Definition 2.1.** A subgroup $\Gamma \subset \text{PSL}(2, \mathbb{C}) = \text{Iso}^+(\mathbb{H}^3)$ is called a Kleinian group when its action on $\mathbb{H}^3$ is discontinuous.

Each Kleinian group $\Gamma \subset \text{PSL}(2, \mathbb{C})$ has a fundamental domain. An element $\gamma \in \text{SL}(2, \mathbb{C}), \gamma \neq \pm I$, is called

- **parabolic** if $|\text{tr}(\gamma)| = 2$ and $\text{tr}(\gamma) \in \mathbb{R}$,
- **hyperbolic** if $|\text{tr}(\gamma)| > 2$ and $\text{tr}(\gamma) \in \mathbb{R}$,
- **elliptic** if $|\text{tr}(\gamma)| < 2$ and $\text{tr}(\gamma) \in \mathbb{R}$,

and loxodromic in all other cases. An element of $\text{PSL}(2, \mathbb{C})$ is called parabolic, hyperbolic, elliptic, and loxodromic if its preimages in $\text{SL}(2, \mathbb{C})$ have this property. In this article, we define ‘loxodromic’ to be ‘hyperbolic’ or ‘loxodromic’ for simplicity.

Let $\gamma$ be a loxodromic element of $\Gamma$. Then there is an element $A \in \text{PSL}(2, \mathbb{C})$ such that

$$A\gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix} =: D(\gamma).$$

This $D(\gamma)$ is uniquely determined by the condition $|a(\gamma)| > 1$. Note that

$$D(\gamma)(z + rJ) = K(\gamma)z + N(\gamma)rz \quad (z \in \mathbb{C}, \ r > 0),$$

where $K(\gamma) := a(\gamma)^2$ is called the multiplier of $\gamma$ and where $N(\gamma) := |K(\gamma)| = |a(\gamma)^2| > 1$ is called the norm of $\gamma$. When $\gamma$ is a loxodromic element, $\gamma$ has two fixed points in $\mathbb{P}^1 \mathbb{C}$ and no fixed point in $\mathbb{H}^3$.

About the centralizer of $\gamma$ in $\Gamma$, we have the following lemma.
LEMMA 2.2. Let $\gamma \in \Gamma$ be a loxodromic element and $\Gamma_\gamma$ be the centralizer of $\gamma$ in $\Gamma$. Let $(\Gamma_\gamma)_{tor}$ be the torsion subgroup of $\Gamma_\gamma$. Then $(\Gamma_\gamma)_{tor}$ contains only the identity or is the finite cyclic group. Furthermore $\Gamma_\gamma$ is written as the direct product

$$\Gamma_\gamma = (\Gamma_\gamma)_{tor} \times \langle \gamma_0 \rangle,$$

where $\gamma_0$ is a certain element of infinite order in $\Gamma_\gamma$. Then $\gamma_0$ is not uniquely determined by $\gamma$ but the norm $N(\gamma_0)$ is; and $N(\gamma_0)$ is the minimal norm of a loxodromic element from $\Gamma_\gamma$.

Remark 2.3. The elements of $(\Gamma_\gamma)_{tor}$ are hyperbolic rotations around the axis of $\gamma$. $(\Gamma_\gamma)_{tor}$ is generated by a rotation with rotation angle $2\pi/(|\Gamma_\gamma|_{tor})$, where $|(\Gamma_\gamma)_{tor}|$ denotes the order of $(\Gamma_\gamma)_{tor}$.

Proof. See [I, pp. 191–192].

2.3. Hyperbolic 3-manifolds and their degeneration

Let $M$ be a complete, three-dimensional hyperbolic manifold of finite volume with constant negative curvature $-1$. If $\Gamma$ is a torsion-free Kleinian group, then $\Gamma$ acts discontinuously and freely on $\mathbb{H}^3$ so that the quotient $\Gamma \backslash \mathbb{H}^3$ is an orientable hyperbolic 3-manifold. Conversely, the hyperbolic structure of an orientable hyperbolic 3-manifold $M$ can be lifted to the universal cover $\tilde{M}$ which is isometric to $\mathbb{H}^3$. Thus the fundamental group $\pi_1(M)$ can be identified with the covering group $\Gamma$ which is a discrete and torsion-free subgroup of $PSL(2, \mathbb{C})$. Consequently, if $M$ is an orientable hyperbolic 3-manifold, then there exists a torsion-free Kleinian group $\Gamma$ such that $M$ is isometric to $\Gamma \backslash \mathbb{H}^3$.

It is a consequence of the Kazhdan–Margulis theorem that there exists a positive number $\mu$ such that for each orientable hyperbolic manifold $M$ and each $x \in M$ the loops based at $x$ of length $\leq 2\mu$ generate a free Abelian group of rank at most two in $\pi_1(M, x)$. This universal constant $\mu$ is called the Kazhdan–Margulis constant (see [8]). We define the $\varepsilon$-thin part and the $\varepsilon$-thick part as follows.

Definition 2.4. For a complete three-dimensional hyperbolic manifold $M$ and $\varepsilon > 0$, we define $M_{(0, \varepsilon)}$ as the set of $x \in M$ such that there exists a non-contractible loop at $x$ of length $\leq \varepsilon$ and $M_{(\varepsilon, \infty)}$ as $M \backslash M_{(0, \varepsilon)}$.

Then $M_{(0, \varepsilon)}$ and $M_{(\varepsilon, \infty)}$ are called the $\varepsilon$-thin part and the $\varepsilon$-thick part of $M$, respectively. Let $c = c(\varepsilon) > 0$ be the positive real number such that the hyperbolic metric from $(0, 0, c)$ to $(1, 0, c)$ equals $\varepsilon$. Then we define $H_\varepsilon := \{ P \in \mathbb{H}^3 \mid r(P) > c \}$. If $\varepsilon < 2\mu$, then a connected component of the $\varepsilon$-thin part of $M$ can be classified as the following three types (see [12]).

1. Let $T$ be a two-dimensional flat torus and $ds^2$ be its flat metric. Let $C_T$ be the product $T \times \mathbb{R}$ with the metric $e^{-r}ds^2 + dr^2$. Then $C_T$ is a hyperbolic manifold with $\pi_1 = \mathbb{Z} \times \mathbb{Z}$. The connected component of $M_{(0, \varepsilon)}$ with the structure of $C_T$ is called the cusp torus. In this case the connected component of $M_{(0, \varepsilon)}$ is isometric to the quotient space $G_1 \backslash H_\varepsilon$, where $G_1$ is a parabolic Abelian subgroup of rank two generated by $(z \mapsto z + 1, z \mapsto z + \tau)$, $(\text{Im}(\tau) > 0, |\tau| > 1)$.

2. Let $C_Z$ be an infinite cyclic covering of a double infinite cusp. Then $C_Z$ is a hyperbolic manifold with $\pi_1 = \mathbb{Z}$ and has no closed geodesics. The connected component of $M_{(0, \varepsilon)}$ with the structure of $C_Z$ is called the cusp tube. In this case the connected component
of $M_{(0,\varepsilon)}$ is isometric to the quotient space $G_2\backslash H_c$, where $G_2$ is the parabolic cyclic subgroup generated by $z \mapsto z + 1$.

3. Let $\gamma$ be a loxodromic transformation on $\mathbb{H}^3$ and $U$ be the tubular neighborhood of the axis of $\gamma$. We define $T_M$ to be the projection $\langle \gamma \rangle \backslash U$. Then $T_M$ is a hyperbolic manifold with $\pi_1 = \mathbb{Z}$ and has a closed geodesic. The connected component of $M_{(0,\varepsilon)}$ with the structure of $T_M$ is called the Margulis torus. In this case the connected component of $M_{(0,\varepsilon)}$ is isometric to the $\varepsilon$-thin part of $\langle \gamma \rangle \backslash U$.

**Remark 2.5.** Let $M$ be an orientable three-dimensional hyperbolic manifold of finite volume and let $0 < \varepsilon < \frac{1}{2}\mu$, where $\mu$ denotes the Kazhdan–Margulis constant. Then the $\varepsilon$-thin part $M_{(0,\varepsilon)}$ consists of finitely many components and each of these components is isometric to a cusp torus or to a Margulis torus.

We define the convergence of metric spaces as follows.

**Definition 2.6.** Let $X$ and $Y$ be two metric spaces and $f : X \to Y$ be a map from $X$ into $Y$. We define $L(f)$ as

$$L(f) = \sup_{x_1, x_2 \in X} \left| \log \frac{d_X(x_1, x_2)}{d_Y(f(x_1), f(x_2))} \right|,$$

where $d_X$ and $d_Y$ are the distance on $X$ and $Y$, respectively.

**Definition 2.7.** Let $X_i$, $i = 1, 2, \ldots$, be a sequence of metric spaces with base points $x_i \in X_i$. We say that the sequence $(X_i, x_i)$ converges to $(Y, y)$ if, for arbitrary $\varepsilon > 0$ and $r > 0$, there is a natural number $j$ such that for each $i \geq j$ there exists a map $f$ from the $x_i$-centered $r$-ball $B_r(x_i) \subset X_i$ into $Y$ such that

(a) $f(x_i) = y$,

(b) the image $f(B_r(x_i)) \subset Y$ contains the $y$-centered $(r - \varepsilon)$-ball in $Y$, and

(c) $L(f) \leq \varepsilon$.

From the cusp closing theorem of W. Thurston [13], the following assertion holds. Let $M$ be a complete orientable hyperbolic 3-manifold with $\text{Vol}(M) < \infty$ with $p + q$ cusps. Then there is a convergent sequence of hyperbolic three-dimensional manifolds $M_i \to M$ such that each $M_i$ has exactly $p$ cusps and $q$ short geodesics. Especially, $M$ can be written as a limit of compact manifolds.

**Definition 2.8.** A sequence of Kleinian groups $\{\Gamma_i\}$ converges geometrically to a Kleinian group $\Gamma$ if the following conditions hold.

1. If $\gamma \in \Gamma$, then there exists a sequence $\gamma_i \in \Gamma_i$ converging to $\gamma$.

2. If $\gamma \in \text{PSL}(2, \mathbb{C})$ is an accumulation point of a sequence $\{\gamma_i \in \Gamma_i\}$, then $\gamma \in \Gamma$.

If $\{\Gamma_i\}$ is a sequence of torsion-free Kleinian groups converging geometrically to a torsion-free Kleinian group $\Gamma$, then the sequence $\{\Gamma_i \backslash \mathbb{H}^3\}$ of hyperbolic 3-manifolds converges to $\Gamma \backslash \mathbb{H}^3$.

In this article, we use the following setting. Let $M_i \to M$ be the convergent sequence of three-dimensional hyperbolic manifolds with finite volume. Then each corresponding Kleinian group $\Gamma_i$ converges to $\Gamma$ geometrically. Without loss of generality, we may assume that each $M_i$ has exactly one pinching geodesic $L_{\gamma_i}$ which generates the new cusp $\zeta$ on the limit manifold $M = \Gamma \backslash \mathbb{H}^3$. 

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2.4. Counting functions

In this section, we introduce two counting functions, namely the parabolic counting function and the loxodromic counting function. As stated later, the parabolic Eisenstein series and the loxodromic Eisenstein series can be written in the form of the Stieltjes integral by using these counting functions.

Let $\Gamma \subset \text{PSL}(2, \mathbb{C})$ be a Kleinian group and $\zeta \in \mathbb{P}^1 \mathbb{C} = \mathbb{C} \cup \{\infty\}$ be a cusp. We define the stabilizer group of $\zeta$ as

$$
\Gamma_\zeta := \{ M \in \Gamma \mid M \zeta = \zeta \}.
$$

Then we can choose $A \in \text{PSL}(2, \mathbb{C})$ such that $A\zeta = \infty j$. Let $r_0 > 0$ be sufficiently large so $r_0 > r(AP)$ for all $M \in \Gamma$ and let $S_{r_0}$ be the horosphere in $\mathbb{H}^3$ defined by $\{ P \in \mathbb{H}^3 \mid r(P) = r_0 \}$. Here $r_0$ depends on $P \in \mathbb{H}^3$. Then we define the parabolic counting function $N_{\text{par}, \zeta}(T; P, S_{r_0})$ as follows.

**Definition 2.9.** For any positive real number $T > 0$ and $P \in \mathbb{H}^3$, the parabolic counting function $N_{\text{par}, \zeta}(T; P, S_{r_0})$ is defined as

$$
N_{\text{par}, \zeta}(T; P, S_{r_0}) := \#\{ \eta \in \Gamma_\zeta \setminus \Gamma \mid d_{\text{hyp}}(A\eta P, S_{r_0}) < T \}, \tag{1}
$$

where $\#$ denotes the cardinality of the set.

**Lemma 2.10.** Let $r > 0$ be the injective radius at $P \in \mathbb{H}^3$. We assume that $u$ and $T_0$ are positive real numbers satisfying $u > T_0 > r$. Then the following inequality holds:

$$
N_{\text{par}, \zeta}(u; P, S_{r_0}) \leq N_{\text{par}, \zeta}(T_0; P, S_{r_0}) + \frac{\text{vol}_{\text{euc}}(P) \cdot (e^{2u+2r} - e^{2T-2r})}{2\pi r_0^2 \sinh(2r) - 2r}, \tag{2}
$$

where $P$ denotes the boundary torus of cusp $\zeta$ and $\text{vol}_{\text{euc}}(P)$ its Euclidean area.

**Proof.** We can assume $A = I$ without loss of generality. Then the parabolic counting function (1) is written as follows:

$$
N_{\text{par}, \zeta}(T; P, S_{r_0}) = \#\{ \eta \in \Gamma_\zeta \setminus \Gamma \mid d_{\text{hyp}}(\eta P, S_{r_0}) < T \}.
$$

Let $B_P(r) := \{ Q \in \mathbb{H}^3 \mid d_{\text{hyp}}(P, Q) < r \}$ be the $P$-centered hyperbolic ball with hyperbolic radius $r$ and $\mathcal{F}_{\Gamma_\zeta}$ be a fixed fundamental domain of $\Gamma_\zeta$. For $T > 0$, we define $V_{\text{par}}(T)$ as follows:

$$
V_{\text{par}}(T) := \{ P \in \mathcal{F}_{\Gamma_\zeta} \mid r(P) < r_0 \text{ and } d_{\text{hyp}}(P, S_{r_0}) < T \}.
$$

Let $r > 0$ be the injective radius at $P$, i.e. for any $\gamma_1, \gamma_2 \in \Gamma$, we have $B_{\gamma_1 P}(r) \cap B_{\gamma_2 P}(r) = \emptyset$. In addition, for $u$ and $T_0$ satisfying $u > T_0 > r$, the set $\{ \eta_k \} \subset \Gamma_\zeta \setminus \Gamma$ denotes the maximal set such that $\eta P \in V_{\text{par}}(u) \setminus V_{\text{par}}(T_0)$. Then we have

$$
\bigcup_k B_{\eta_k P}(r) \subset V_{\text{par}}(u + r) \setminus V_{\text{par}}(T_0 - r).
$$
Computing the hyperbolic volume of $V_{\text{par}}(T)$ we have

\[
\text{vol}_{\text{hyp}}(V_{\text{par}}(T)) = \int \int_\mathcal{P} \int_{r_0/e^T}^{r_0} \frac{1}{r^3} \, dx \, dy \, dr
\]

\[
= \text{vol}_{\text{euc}}(\mathcal{P}) \cdot \int_{r_0/e^T}^{r_0} \frac{1}{r^3} \, dr
\]

\[
= \text{vol}_{\text{euc}}(\mathcal{P}) \cdot \left[ -\frac{1}{2r^2} \right]_{r_0/e^T}^{r_0}
\]

\[
= \text{vol}_{\text{euc}}(\mathcal{P}) \cdot \frac{e^{2T} - 1}{2r_0^2},
\]

where $\text{vol}_{\text{euc}}(\mathcal{P})$ is the Euclidean area defined by $\{ P \in \mathbb{H}^3 \mid r(P) = r_0 \} \cap \mathcal{F}_\Gamma$. Thus we have the following inequality:

\[
\text{vol}_{\text{hyp}}\left( \bigcup_k B_{n_k P}(r) \right) = \sum_k \text{vol}_{\text{hyp}}(B_{n_k P}(r)) \\
\leq \text{vol}_{\text{hyp}}(V_{\text{par}}(u + r)) - \text{vol}_{\text{hyp}}(V_{\text{par}}(T - r)) \\
= \text{vol}_{\text{euc}}(\mathcal{P}) \cdot \frac{e^{2u+2r} - e^{2T-2r}}{2r_0^2}.
\]

Since the volume of the hyperbolic ball $B_P(r)$ is given by $\pi \{ \sinh(2r) - 2r \}$, we have

\[
\sharp\{ \eta \in \Gamma \backslash \Gamma \mid \eta P \in V_{\text{par}}(u) \setminus V_{\text{par}}(T) \} \cdot \pi \{ \sinh(2r) - 2r \}
\leq \text{vol}_{\text{euc}}(\mathcal{P}) \cdot \frac{e^{2u+2r} - e^{2T-2r}}{2r_0^2}. \tag{3}
\]

Furthermore, from the inclusion of the set, the following equation holds:

\[
\sharp\{ \eta \in \Gamma \backslash \Gamma \mid \eta P \in V_{\text{par}}(u) \setminus V_{\text{par}}(T) \}
= \sharp\{ \eta \in \Gamma \backslash \Gamma \mid \eta P \in V_{\text{par}}(u) \} - \sharp\{ \eta \in \Gamma \backslash \Gamma \mid \eta P \in V_{\text{par}}(T) \}. \tag{4}
\]

From (3) and (4), we have

\[
\sharp\{ \eta \in \Gamma \backslash \Gamma \mid \eta P \in V_{\text{par}}(u) \}
\leq \sharp\{ \eta \in \Gamma \backslash \Gamma \mid \eta P \in V_{\text{par}}(T) \} + \frac{\text{vol}_{\text{euc}}(\mathcal{P}) \cdot (e^{2u+2r} - e^{2T-2r})}{2\pi r_0^2 \{ \sinh(2r) - 2r \}}.
\]

Therefore, the assertion of the lemma holds. \hfill \Box

Next, we define the loxodromic counting function. Let $\gamma \in \Gamma$ be a loxodromic element and $\Gamma_\gamma$ be the centralizer of $\gamma$ in $\Gamma$. Let $L_\gamma$ be the $\gamma$-invariant geodesic in $\mathbb{H}^3$. Then we define the loxodromic counting function $N_{\text{lox},\gamma}(T; P, L_\gamma)$ as follows.

**Definition 2.11.** For any positive real number $T > 0$ and $P \in \mathbb{H}^3$, the loxodromic counting function $N_{\text{lox},\gamma}(T; P, L_\gamma)$ is defined as

\[
N_{\text{lox},\gamma}(T; P, L_\gamma) := \sharp\{ \eta \in \Gamma \backslash \Gamma \mid d_{\text{hyp}}(\eta P, L_\gamma) < T \}, \tag{5}
\]

where $\sharp$ denotes the cardinality of the set.
Lemma 2.12. Let $r > 0$ be the injective radius at $P \in \mathbb{H}^3$. We assume that $u$ and $T_0$ are positive real numbers satisfying $u > T_0 > r$. Then the following inequality holds:

$$N_{\text{lox}}(u; P, L_\gamma) \leq N_{\text{lox}}(T_0; P, L_\gamma) + \frac{\log N(\gamma_0) \cdot ((\cosh(u + r))^2 - (\cosh(T_0 - r))^2)}{\sinh(2r) - 2r},$$

(6)

where $N(\gamma_0)$ denotes the minimal norm determined from the centralizer group $\Gamma_\gamma$.

Proof. Let $\gamma \in \Gamma$ be a loxodromic element. We take $\gamma_0$ as in Lemma 2.2. Let $N(\gamma) = |a(\gamma)|^2$ be the norm of $\gamma$. Then there is an element $A \in \text{PSL}(2, \mathbb{C})$ such that

$$A\gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix}.$$

We can assume $A = I$ without loss of generality. Then the loxodromic counting function (5) is written as follows:

$$N_{\text{lox}}(T; P, L_\gamma) = \#(\eta \in \Gamma_\gamma \setminus \Gamma | d_{\text{hyp}}(\eta P, L_0) < T).$$

Let $B_P(r) := \{ Q \in \mathbb{H}^3 | d_{\text{hyp}}(P, Q) < r \}$ be the $P$-centered hyperbolic ball with hyperbolic radius $r$. Under the coordinates $x = e^\rho \cos \varphi \cos \theta$, $y = e^\rho \cos \varphi \sin \theta$, $r = e^\rho \sin \varphi$, we define $V_{\text{lox}}(T)$ as

$$V_{\text{lox}}(T) := \{ P \in \mathbb{H}^3 | 0 < \rho < \log N(\gamma_0), d_{\text{hyp}}(P, L_0) < T \}.$$

Let $r > 0$ be the injective radius at $P$. For $u > T_0 > r$, the set $\{ \eta_k \}_k \subset \Gamma_\gamma \setminus \Gamma$ denotes the maximal set such that $\eta P \in V_{\text{lox}}(u) \setminus V_{\text{lox}}(T_0)$. Then we have

$$\bigcup_k B_{\eta_k P}(r) \subset V_{\text{lox}}(u + r) \setminus V_{\text{lox}}(T_0 - r).$$

Computing the hyperbolic volume of $V_{\text{lox}}(T)$, we have

$$\text{vol}_{\text{hyp}}(V_{\text{lox}}(T)) = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\log N(\gamma_0)} \frac{\cos \varphi}{\sin^3 \varphi} \, d\rho \, d\varphi \, d\theta$$

$$= 2\pi \log N(\gamma_0) \cdot \int_0^{\pi/2} \frac{\cos \varphi}{\sin^3 \varphi} \, d\varphi \, d\theta$$

$$= 2\pi \log N(\gamma_0) \cdot \left[ -\frac{1}{2 \sin^2 \varphi} \right]_0^{\pi/2}$$

$$= 2\pi \log N(\gamma_0) \cdot \frac{1}{2} \left( -1 + \frac{1}{\sin^2 \varphi} \right)$$

$$= \pi \log N(\gamma_0) \cdot ((\cosh T)^2 - 1).$$
Thus we have the following inequality:

\[
\text{vol}_{\text{hyp}} \left( \bigcup_k B_{n_k P}(r) \right) \\
= \sum_k \text{vol}_{\text{hyp}}(B_{n_k P}(r)) \\
\leq \text{vol}_{\text{hyp}}(V_{\text{lox}}(u + r)) - \text{vol}_{\text{hyp}}(V_{\text{lox}}(T_0 - r)) \\
\leq \pi \log N(\gamma_0) \cdot (\cosh(u + r)^2 - 1) - \pi (\log N(\gamma_0))(\cosh(T_0 - r)^2 - 1) \\
= \pi \log N(\gamma_0) \cdot \{(\cosh(u + r))^2 - (\cosh(T_0 - r))^2\}.
\]

Since the volume of the hyperbolic ball \(B_P(r)\) is given by \(\pi \{\sinh(2r) - 2r\}\), we have

\[
\mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(u) \setminus V_{\text{lox}}(T_0)\} \\
\leq \pi \log N(\gamma_0) \cdot \{(\cosh(u + r))^2 - (\cosh(T_0 - r))^2\}. \quad (7)
\]

Furthermore, from the inclusion of the set, the following equation holds:

\[
\mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(u) \setminus V_{\text{lox}}(T_0)\} \\
= \mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(u)\} - \mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(T_0)\}. \quad (8)
\]

From (7) and (8), we have

\[
\mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(u)\} \\
\leq \mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(T_0)\} \\
+ \frac{\pi \log N(\gamma_0) \cdot \{(\cosh(u + r))^2 - (\cosh(T_0 - r))^2\}}{\pi \{\sinh(2r) - 2r\}} \\
= \mathcal{Z}\{\eta \in \Gamma \setminus \Gamma \mid \eta P \in V_{\text{lox}}(T_0)\} \\
+ \frac{\log N(\gamma_0) \cdot \{(\cosh(u + r))^2 - (\cosh(T_0 - r))^2\}}{\sinh(2r) - 2r}.
\]

Therefore, the assertion of the lemma holds. \(\square\)

The following inequality plays an important role in our analysis.

**Lemma 2.13.** Let \(F\) be a real-valued, smooth, decreasing function defined for \(u > 0\), and let \(g_1\) and \(g_2\) be real-valued, non-decreasing functions defined for \(u \geq a > 0\) and satisfying \(g_1(u) \leq g_2(u)\) for \(u \geq a\). Then the following inequality of Stieltjes integrals holds when both integrals exist:

\[
F(a)g_1(a) + \int_a^\infty F(u) \, dg_1(u) \leq F(a)g_2(a) + \int_a^\infty F(u) \, dg_2(u).
\]

**Proof.** See [3, Section 2.6] or [4, Section 2.7]. \(\square\)
2.5. Parabolic Eisenstein series

In this section, we introduce ordinary Eisenstein series which are constructed by averaging the functions on $\mathbb{H}^3$ defined by $z + rj \mapsto rs$. Since these series are associated to a cusp or equivalently a parabolic element of $\Gamma$, we call them parabolic Eisenstein series in order to distinguish them from the loxodromic Eisenstein series.

Let $\Gamma \subset \text{PSL}(2, \mathbb{C})$ be a Kleinian group and let $\zeta \in \mathbb{P}^1 \mathbb{C} = \mathbb{C} \cup \{\infty\}$ be a cusp. We define the stabilizer group of $\zeta$ as

$$\Gamma_\zeta := \{ M \in \Gamma \mid M \zeta = \zeta \}$$

and its maximal unipotent subgroup as

$$\Gamma_\zeta' := \{ M \in \Gamma \mid M \zeta = \zeta, \ M = I \text{ or } M \text{ is parabolic} \}.$$

Choose $A \in \text{PSL}(2, \mathbb{C})$ such that $A\zeta = \infty$. Then the parabolic Eisenstein series associated to $\zeta$ is defined as follows.

**Definition 2.14.** For any $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with sufficiently large $\text{Re}(s)$, the parabolic Eisenstein series associated to the cusp $\zeta$ is defined as

$$E_{\text{par}, \zeta}(P, s) := \sum_{M \in \Gamma_\zeta' \setminus \Gamma} r(AMP)^s.$$  \hfill (9)

By using the counting function $N_{\text{par}, \zeta}(T; P, S_{r_0})$, we can express the parabolic Eisenstein series as a Stieltjes integral, namely we have

$$E_{\text{par}, \zeta}(P, s) = r_0^s \int_0^\infty e^{-su} dN_{\text{par}, \zeta}(u; P, S_{r_0}).$$  \hfill (10)

**Lemma 2.15.** Let $s \in \mathbb{C}$ be a complex number with $\text{Re}(s) > 2$. Then for any $\varepsilon > 0$ there exists a sufficiently large $T_0 > 0$ such that

$$\left | \int_{T_0}^\infty e^{-su} dN_{\text{par}, \zeta}(u; P, S_{r_0}) \right | < \varepsilon. \hfill (11)$$

**Proof.** For positive real number $u$, we define the functions $F(u)$, $g_1(u)$ and $g_2(u)$ as follows:

$$F(u) := e^{-su},$$
$$g_1(u) := N_{\text{par}, \zeta}(u; P, S_{r_0}),$$
$$g_2(u) := N_{\text{par}, \zeta}(T_0; P, S_{r_0}) + \frac{\text{vol}_{\text{euc}}(P) \cdot (e^{2u+2r} - e^{2T_0-2r})}{2\pi r_0^2 \{ \sinh(2r) - 2r \}},$$

where $r$ denotes the injective radius at $P \in \mathbb{H}^3$ and $\mathcal{P}$ is the boundary torus of the cusp $\zeta$. Then both $g_1$ and $g_2$ are real-valued and non-decreasing functions and $g_1(u) \leq g_2(u)$ for $u \geq T_0 > 0$. Elementary calculations imply that

$$dg_2(u) = \frac{\text{vol}_{\text{euc}}(P) \cdot e^{2u+2r}}{\pi r_0^2 \{ \sinh(2r) - 2r \}} du.$$

By using Lemma 2.13, we have the following estimate:

$$\int_{T_0}^\infty e^{-su} dN_{\text{par}, \zeta}(u; P, S_{r_0})$$

$$\leq \int_{T_0}^\infty e^{-su} dg_2 + e^{-sT_0} \{ g_2(T_0) - g_1(T_0) \}$$
The integral in the first term of (12) converges for \( \text{Re}(s) > 2 \) and we have

\[
(12) = - \frac{\text{vol}_{\text{euc}}(P)}{\pi r_0^2 \{\sinh(2r) - 2r\}} \cdot \frac{e^{(2-s)T_0 + 2r}}{2 - s} + \frac{\text{vol}_{\text{euc}}(P) \cdot e^{(2-s)T_0 + 2r}}{2\pi r_0^2 \{\sinh(2r) - 2r\}}
\]

\[
= \frac{\text{vol}_{\text{euc}}(P) \cdot e^{(2-s)T_0 + 2r}}{2\pi r_0^2 \{\sinh(2r) - 2r\}} \cdot \left( 1 + \frac{2}{s - 2} \right)
\]

\[
= \frac{\text{vol}_{\text{euc}}(P) \cdot e^{(2-s)T_0 + 2r}}{2\pi r_0^2 \{\sinh(2r) - 2r\}} \cdot \frac{s}{s - 2}. \tag{13}
\]

For any \( \varepsilon > 0 \), by choosing

\[
T_0 > \frac{1}{s - 2} \left\{ -\log \varepsilon + \log \left( \frac{\text{vol}_{\text{euc}}(P) \cdot e^{2r}}{2\pi r_0^2 \{\sinh(2r) - 2r\}} \cdot \frac{s}{s - 2} \right) \right\},
\]

we have that the last term of (13) is less than \( \varepsilon \).

The parabolic Eisenstein series (9) converges absolutely and locally uniformly for any \( P \in \mathbb{H}^3 \) and \( s \in \mathbb{C} \) with \( \text{Re}(s) > 2 \). It defines a \( \Gamma \)-invariant function where it converges and satisfies the following differential equation:

\[
(-\Delta + s(s - 2))E_{\text{par}, \zeta}(P, s) = 0. \tag{14}
\]

### 2.6. Loxodromic Eisenstein series

In this section, we define the loxodromic Eisenstein series. Let \( \gamma \in \Gamma \) be a loxodromic element and \( \Gamma_{\gamma} \) the centralizer of \( \gamma \) in \( \Gamma \). For \( P \in \mathbb{H}^3 \), we use the coordinates \( x = e^\rho \cos \varphi \cos \theta, \ y = e^\rho \cos \varphi \sin \theta, \ r = e^\rho \sin \varphi \). Choose \( A \in \text{PSL}(2, \mathbb{C}) \) such that

\[
A \gamma A^{-1} = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1} \end{pmatrix}.
\]

Then the loxodromic Eisenstein series associated to \( \gamma \) is defined as follows.

**Definition 2.16.** For any \( P \in \mathbb{H}^3 \) and \( s \in \mathbb{C} \) with sufficiently large \( \text{Re}(s) \), the loxodromic Eisenstein series associated to \( \gamma \) is defined as

\[
E_{\text{lox}, \gamma}(P, s) := \sum_{\eta \in \Gamma_{\gamma} \backslash \Gamma} \sin \varphi(\eta P)^s. \tag{15}
\]
Let $L_\gamma$ denote the $\gamma$-invariant geodesic in $\mathbb{H}^3$ and $L_0$ the positive $r$-axis. Then $L_\gamma = A^{-1}L_0$. The hyperbolic distance $d_{\text{hyp}}(P, L_0)$ from $P$ to the geodesic line $L_0$ satisfies the following formula:

$$\sin(\varphi(P)) \cosh(d_{\text{hyp}}(P, L_0)) = 1.$$ 

Using this formula, we can rewrite the loxodromic Eisenstein series as follows:

$$E_{\text{lox}, \gamma}(P, s) = \sum_{\eta \in \Gamma \setminus \Gamma_\gamma} \cosh(d_{\text{hyp}}(\eta P, L_\gamma))^{-s}.$$ 

By using the counting function $N_{\text{lox}, \gamma}(T; P, L_\gamma)$, we can express the loxodromic Eisenstein series (15) as a Stieltjes integral, namely we have

$$E_{\text{lox}, \gamma}(P, s) = \int_0^\infty \cosh(u)^{-s} dN_{\text{hyp}, \gamma}(u; P, L_\gamma). \quad (16)$$

**Lemma 2.17.** Let $s \in \mathbb{C}$ be a complex number with $\text{Re}(s) > 2$. Then for any $\varepsilon > 0$ there exists a sufficiently large $T_0 > 0$ such that

$$\left| \int_0^\infty (\cosh(u)^{-s} dN_{\text{lox}, \gamma}(u; P, L_\gamma) \right| < \varepsilon. \quad (17)$$

**Proof.** For positive real number $u > 0$, we define the functions $F(u)$, $g_1(u)$ and $g_2(u)$ as follows:

$$F(u) := (\cosh(u))^{-s},$$

$$g_1(u) := N_{\text{lox}, \gamma}(u; P, L_\gamma),$$

$$g_2(u) := N_{\text{lox}, \gamma}(T_0; P, L_\gamma) + \frac{\log N(\gamma_0) \cdot \{\cosh(u + r)^2 - \cosh(T_0 - r)^2\}}{\sinh(2r) - 2r},$$

where $r$ denotes the injective radius at $P \in \mathbb{H}^3$ and $N(\gamma_0)$ the minimal norm determined from the centralizer group $\Gamma_\gamma$. Then both $g_1$ and $g_2$ are real-valued, non-decreasing functions and $g_1(u) \leq g_2(u)$ for $u \geq T_0 > 0$. Elementary calculations imply that

$$dg_2(u) = \frac{\log N(\gamma_0) \cdot 2 \cosh(u + r) \sinh(u + r)}{\sinh(2r) - 2r} du$$

$$= \frac{\log N(\gamma_0) \cdot \sinh(2(u + r))}{\sinh(2r) - 2r} du.$$ 

By using Lemma 2.13, we have the following estimate:

$$\int_{T_0}^\infty (\cosh(u)^{-s} dN_{\text{lox}, \gamma}(u; P)$$

$$\leq \int_{T_0}^\infty (\cosh(u)^{-s} dg_2 + (\cosh T_0)^{-s} \{g_2(T_0) - g_1(T_0)\})$$

$$= \int_{T_0}^\infty (\cosh(u)^{-s} dg_2$$

$$+ (\cosh T_0)^{-s} \frac{\log N(\gamma_0) \cdot \{(\cosh(T_0 + r)^2 - (\cosh(T_0 - r)^2)\}}{\sinh(2r) - 2r}.$$
For any $\varepsilon > 0$, the integral in the first term of (18) converges for $\text{Re}(s) > 2$ and we have

\[
\frac{2^s \log N(\gamma_0)}{\sinh(2r) - 2r} \cdot \frac{e^{(2-s)T_0+2r}}{2-s} + 2^{s-1} e^{(2-s)T_0+2r} \frac{\log N(\gamma_0)}{\sinh(2r) - 2r}
\]

(18) converges and satisfies the following differential equation:

\[
(-\Delta + s(s - 2)) E_{\text{lox}, \gamma}(P, s) = s^2 E_{\text{lox}, \gamma}(P, s + 2).
\]

(20)

PROPOSITION 2.18. The loxodromic Eisenstein series (15) converges absolutely and locally uniformly for any $P \in \mathbb{H}^3$ and $s \in \mathbb{C}$ with $\text{Re}(s) > 2$. It defines a $\Gamma$-invariant function where it converges and satisfies the following differential equation:

\[
(-\Delta + s(s - 2)) E_{\text{lox}, \gamma}(P, s) = s^2 E_{\text{lox}, \gamma}(P, s + 2).
\]

Proof. See [6].

3. Convergence of Eisenstein series

In this section, we prove Theorem 1.1. We recall the assumptions and notation in Section 2.3. Let $M_i \to M$ be the convergent sequence of three-dimensional hyperbolic manifolds with finite volume. Each corresponding Kleinian group $\Gamma_i$ converges to $\Gamma$ geometrically. We may assume that each $M_i$ has exactly one pinching geodesic $L_{\gamma_i}$ which generates the new cusp $\zeta$ on the limit manifold $M$.
3.1. Convergence of counting functions

Lemma 3.1. Let \( M_i = \Gamma \backslash \mathbb{H}^3 \) be a degenerating sequence of three-dimensional hyperbolic manifolds with the limit manifold \( M = \Gamma \backslash \mathbb{H}^3 \). Then we have the following estimate.

(a) For any cusp \( \zeta \) on \( M_i \), we have
\[
\lim_{i \to \infty} N_{\text{par}, M_i, \zeta}(T; P, S_{r_0}) = N_{\text{par}, M, \zeta}(T; P, S_{r_0}).
\]

(b) If \( \gamma \) does not correspond to a shrinking geodesic, then
\[
\lim_{i \to \infty} N_{\text{lox}, M_i, \gamma}(T; P, L_{\gamma_i}) = N_{\text{lox}, M, \gamma}(T; P, L_{\gamma}).
\]

Proof. We can prove both (a) and (b) by similar argument. Hence we prove (a) only and omit the proof of (b).

For any \( M_i \), let \( M_i, (0, \varepsilon), L_{\gamma_i} \) be the connecting component of the \( \varepsilon \)-thin part of \( M_i \) containing the shrinking geodesic \( L_{\gamma_i} \). Here \( \gamma_i \) are degenerating loxodromic elements forming the new cusp on \( M \). Then \( M_i, (0, \varepsilon), L_{\gamma_i} \) is the Margulis torus and their boundary \( \partial M_i, (0, \varepsilon), L_{\gamma_i} \) is a two-dimensional torus with a flat metric. We choose \( \varepsilon_1 > 0 \) sufficiently small so that the point \( P \) lies in \( M_i \setminus M_i, (0, \varepsilon), L_{\gamma_i} \). Take \( \varepsilon_0 > 0 \) so that \( \varepsilon_1 > \varepsilon_0 > 0 \) and the hyperbolic distance from \( \partial M_i, (0, \varepsilon), L_{\gamma_i} \) to \( \partial M_i, (0, \varepsilon), L_{\gamma_i} \) is greater than \( T \). Then, any geodesic path from \( P \) to the horosphere \( S_{r_0} \) of length bounded by \( T \) is contained in \( M_i \setminus M_i, (0, \varepsilon), L_{\gamma_i} \). Since the hyperbolic metric on \( M_i \) converges away from \( M_i, (0, \varepsilon), L_{\gamma_i} \), the statement of (a) follows. \( \square \)

Lemma 3.2. Let \( L_{\gamma_i} \) be the closed shrinking geodesic in \( M_i \) and \( l_{\gamma_i} \) be its length. Then, for sufficiently large positive real number \( r_0 > 0 \) and \( l_{\gamma_i} \), we define \( g(r_0, l_{\gamma_i}) \) as follows:
\[
g(r_0, l_{\gamma_i}) = \int_{\cot^{-1}\left(\sqrt{|P|}/(2\pi l_{\gamma_i} r_0^2)\right)}^{\pi/2} \frac{d\phi}{\sin \phi},
\]
where \( |P| \) denotes the Euclidean area determined by the boundary torus of the connected component of \( M_i, (0, \varepsilon), L_{\gamma_i} \). Then we have
\[
\lim_{i \to \infty} N_{\text{lox}, M_i, \gamma_i}(T + g(r_0, l_{\gamma_i}); P, L_{\gamma_i}) = N_{\text{par}, M, \zeta}(T; P, S_{r_0}).
\]

Proof. Let \( L_{\gamma_i} \) be the shrinking geodesic loop in \( M_i \) and \( \xi \) be the newly developing cusp. We denote the connected component of the \( \varepsilon \)-thin part of \( M_i \) containing \( L_{\gamma_i} \) by \( M_i, (0, \varepsilon), L_{\gamma_i} \) and the connected component of the \( \varepsilon \)-thin part of \( M \) containing \( \xi \) by \( M_{(0, \varepsilon), \xi} \). From Section 2.3, if \( \varepsilon < 2\mu \), where \( \mu \) is the Kazhdan–Margulis constant, then \( M_i, (0, \varepsilon), L_{\gamma_i} \) and \( M_{(0, \varepsilon), \xi} \) are isometric to the \( \varepsilon \)-thin part of the Margulis torus and the \( \varepsilon \)-thin part of the cusp torus, respectively. We can calculate the volumes of \( M_i, (0, \varepsilon), L_{\gamma_i} \) and \( M_{(0, \varepsilon), \xi} \) as follows:
\[
\text{Vol}(M_i, (0, \varepsilon), L_{\gamma_i}) = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{l_{\gamma_i}} \frac{\cos \phi}{\sin^3 \phi} \rho \ d\phi \ d\rho \ d\theta
\]
\[
= \pi l_{\gamma_i} \left( \frac{1}{\sin^2 \phi} - 1 \right)
\]
\[
= \pi l_{\gamma_i} (\cot^2 \phi),
\]
\[
\text{Vol}(M_{(0, \varepsilon), \xi}) = \int_{r_0}^{\infty} \frac{|P|}{r^3} \ dx \ dy \ dr = \frac{|P|}{2r_0^2},
\]
where the constant $r_0 = r_0(\varepsilon)$ is determined as in Section 2.3, i.e. the hyperbolic metric from $(0, 0, r_0)$ to $(1, 0, r_0)$ equals $\varepsilon$. Furthermore, we may assume that $r_0$ is sufficiently large such that $r_0 > r(AM\mathcal{P})$ for any $M \in \Gamma$. Take $\varphi$ as

$$\varphi = \cot^{-1} \left( \frac{|\mathcal{P}|}{2\pi l_{\mathcal{P}} r_0^2} \right).$$

Then the integral

$$g(r_0, l_{\gamma_i}) = \int_{\cot^{-1}(\sqrt{|\mathcal{P}|}/(2\pi l_{\mathcal{P}} r_0^2))}^{\pi/2} \frac{d\phi}{\sin \phi}$$

describes the geodesic length from the geodesic $L_{\gamma_i}$ to the boundary of the connected component of the $\varepsilon'$-thin part $M_i(0, \varepsilon', L_{\gamma_i})$ such that $\text{Vol}(M_i(0, \varepsilon'), L_{\gamma_i}) = \text{Vol}(M(0, \varepsilon'), \zeta)$. Then we have

$$d_{\text{hyp}}(\eta P, L_{\gamma_i}) = d_{\text{hyp}}(\eta P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) + d_{\text{hyp}}(\partial M_i(0, \varepsilon'), L_{\gamma_i}, L_{\gamma_i})$$

$$= d_{\text{hyp}}(\eta P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) + g(r_0, L_{\gamma_i}).$$

Thus $d_{\text{hyp}}(\eta P, L_{\gamma_i}) < T + g(r_0, L_{\gamma_i})$ if and only if $d_{\text{hyp}}(\eta P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) < T$. Since the hyperbolic metric on $M_i$ converges to the hyperbolic metric on the limit manifold $M$ away from the developing cusp and $M_i(0, \varepsilon'), L_{\gamma_i} \to M(0, \varepsilon'), \zeta$ as $i \to \infty$, we have

$$\lim_{i \to \infty} d_{\text{hyp}}(\eta P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) = d_{\text{hyp}}(\eta P, S_{r_0}).$$

From the above argument, the assertion of the lemma follows.}\hfill\Box

### 3.2. Convergence of Eisenstein series

In this section, we prove Theorem 1.1.

**Proof of Theorem 1.1(a).** For any $T_0 > 0$, the loxodromic Eisenstein series is written as

$$E_{\text{lox}, M_i, \gamma_i}(P, s) = \int_0^{T_0} (\cosh u)^{-s} dN_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i})$$

$$+ \int_{T_0}^\infty (\cosh u)^{-s} dN_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}). \quad (22)$$

From Lemma 2.17, for any $\varepsilon > 0$ there exists a sufficiently large $T_0 > 0$ such that

$$\left| \int_{T_0}^\infty (\cosh u)^{-s} dN_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) \right| < \varepsilon. \quad (23)$$

In addition, we choose $T_0$ to be the continuous point of the counting function $N_{\text{lox}, M, \gamma}(u, P, L_{\gamma})$, i.e. for any $\eta \in \Gamma \setminus \Gamma$ there is no geodesic path from $\eta P$ to $L_{\gamma}$ on the limit manifold $M$ with length equal to $T_0$. From Lemma 3.1(b), for such $T_0$ there exists a sufficiently large natural number $i_0$ such that for any $i > i_0$ the equation $N_{\text{lox}, M_i, \gamma_i}(T_0; P, L_{\gamma_i}) = N_{\text{lox}, M, \gamma}(T_0; P, L_{\gamma})$ holds. Let $\{d_{k, M_i}\}_{k \in [0, T_0]}$ be the set of all lengths on $M_i$ with multiplicity such that for any $\delta > 0$ we have the inequality

$$N_{\text{lox}, M_i, \gamma_i}(d_{k, M_i} - \delta; P, L_{\gamma_i}) < N_{\text{lox}, M_i, \gamma_i}(d_{k, M_i} + \delta; P, L_{\gamma_i}).$$
From Section 2.4, the cardinality of the set \( \{ d_{k, M_i} \}_k \) is finite. Hence the integral of the first term on the right-hand side of (22) can be written as

\[
\int_0^{T_0} (\cosh u)^{-s} d N_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) = \sum_{k=1}^{N} (\cosh d_{k, M_i})^{-s},
\]

where \( N \) is the cardinality of the set \( \{ d_{k, M_i} \}_k \). Thus we can write

\[
\int_0^{T_0} (\cosh u)^{-s} d N_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) - \int_0^{T_0} (\cosh u)^{-s} d N_{\text{lox}, M, \gamma_i}(u; P, L_{\gamma_i}) = \sum_{k=1}^{N} \{(\cosh d_{k, M_i})^{-s} - (\cosh d_{k, M})^{-s}\}.
\]

From Lemma 3.1(a), for any \( \delta > 0 \) there exists an \( i_0' \) such that for any \( i > i_0' \) we have

\[
|d_{k, M_i} - d_{k, M}| < \frac{\delta}{N} \quad \text{for all } k.
\]

Then \( \sum_{k=1}^{N} |d_{k, M_i} - d_{k, M}| < \delta \). Since the function \((\cosh u)^{-s}\) is absolutely continuous on the closed interval \([0, T_0]\), for any \( \varepsilon > 0 \) there exists an \( i_0' \) such that for any \( i > i_0' \) we have

\[
\left| \sum_{k=1}^{N} \{(\cosh d_{k, M_i})^{-s} - (\cosh d_{k, M})^{-s}\} \right| < \varepsilon. \quad (24)
\]

From (22), we can write

\[
|E_{\text{lox}, M_i, \gamma_i}(P, s) - E_{\text{lox}, M, \gamma_i}(P, s)|
\leq \int_0^{T_0} (\cosh u)^{-s} d N_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) - \int_0^{T_0} (\cosh u)^{-s} d N_{\text{lox}, M, \gamma_i}(u; P, L_{\gamma_i}) + \int_{T_0}^{\infty} (\cosh u)^{-s} d N_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) + \int_{T_0}^{\infty} (\cosh u)^{-s} d N_{\text{lox}, M, \gamma_i}(u; P, L_{\gamma_i}).
\]

(25)

Then, from (23) and (24), every term on the right-hand side of (25) can be taken arbitrarily small. Therefore, the proof of part (a) of Theorem 1.1 is complete.

\[ \square \]

\textbf{Proof of Theorem 1.1(b).} As in the proof of part (a), for any \( T_0 > 0 \), we write the parabolic Eisenstein series as

\[
E_{\text{par}, M_i, \zeta}(P, s) = r_0^s \int_0^{T_0} e^{-stu} d N_{\text{par}, M_i, \zeta}(u; P, S_{r_0}) + r_0^s \int_{T_0}^{\infty} e^{-stu} d N_{\text{par}, M_i, \zeta}(u; P, S_{r_0}).
\]

(26)

Then for any \( \varepsilon > 0 \) there exists a sufficiently large \( T_0 > 0 \) such that

\[
\left| r_0^s \int_{T_0}^{\infty} e^{-stu} d N_{\text{par}, M_i, \zeta}(u; P, S_{r_0}) \right| < \varepsilon. \quad (27)
\]

We choose \( T_0 \) to be the continuous point of the counting function \( N_{\text{par}, M_i, \zeta}(T; P, S_{r_0}) \). From Lemma 3.1(a), for such \( T_0 \) there exists a sufficiently large natural number \( i_0 \) such that for
any $i > i_0$ the equation $N_{\text{par}, M_i} (T_0; P, S_{r_0}) = N_{\text{par}, M_i, \xi} (T_0; P, S_{r_0})$ holds. Let $\{d_{k,M_i}\} \subset [0, T_0]$ be the set of lengths on $M_i$ with multiplicity such that for any $\delta > 0$ we have the inequality

$$N_{\text{par}, M_i, \xi} (d_{k,M_i} - \delta; P, S_{r_0}) < N_{\text{par}, M_i, \xi} (d_{k,M_i} + \delta; P, S_{r_0}).$$

Then, as in the proof of part (a), the integral of the first term on the right-hand side of (26) can be written as

$$\int_0^{T_0} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0}) = N \sum_{k=1}^N \exp(-s \cdot d_{k,M_i}),$$

where $N$ is the cardinality of the set $\{d_{k,M_i}\}$. Thus we can write

$$\int_0^{T_0} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0}) - \int_0^{T_0} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0})$$

$$= \sum_{k=1}^N [\exp(-s \cdot d_{k,M_i}) - \exp(-s \cdot d_{k,M_i})].$$

Because of Lemma 3.1(a) and the absolute continuity of the function $\exp(-s \cdot d_{k,M_i})$ on the closed interval $[0, T_0]$, for any $\varepsilon > 0$ there exists $i'_0$ such that for any $i > i'_0$ we have

$$|\sum_{k=1}^N [\exp(-s \cdot d_{i,M_i}) - \exp(-s \cdot d_{k,M_i})| \leq \sum_{k=1}^N |\exp(-s \cdot d_{k,M_i}) - \exp(-s \cdot d_{k,M_i})| < \varepsilon. \quad (28)$$

From (26), we can write

$$|E_{\text{par}, M_i, \xi} (P, s) - E_{\text{par}, M_i, \xi} (P, s)|$$

$$\leq \left| \int_0^{T_0} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0}) - \int_0^{T_0} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0}) \right|$$

$$+ \left| \int_0^{\infty} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0}) \right| + \left| \int_0^{\infty} e^{-su} dN_{\text{par}, M_i, \xi} (u; P, S_{r_0}) \right|. \quad (29)$$

Then, as in the proof of part (a), every term on the right-hand side of (29) can be taken arbitrarily small from (27) and (28). Therefore, the assertion of part (b) of Theorem 1.1 follows.

**Proof of Theorem 1.1(c).** For any $T_0 > 0$, we write the loxodromic Eisenstein series as follows:

$$E_{\text{lox}, M_i, \gamma} (P, s) = \int_0^{T_0 + g(r_0, l_1)} \frac{1}{(\cosh u)^s} dN_{\text{lox}, M_i, \gamma} (u; P, L_{\gamma_1})$$

$$+ \int_{T_0 + g(r_0, l_1)}^{\infty} \frac{1}{(\cosh u)^s} dN_{\text{lox}, M_i, \gamma} (u; P, L_{\gamma_1}). \quad (30)$$

where $g(r_0, l_1)$ is as in Lemma 3.2. Then we have

$$\int_{T_0 + g(r_0, l_1)}^{\infty} \frac{1}{(\cosh u)^s} dN_{\text{lox}, M_i, \gamma} (u; P, L_{\gamma_1})$$

$$= \int_{T_0}^{\infty} (\cosh(u + g(r_0 + l_1)))^{-s} dN_{\text{lox}, M_i, \gamma} (u; P, \partial M_i (0, \varepsilon'), L_{\gamma_1}).$$
From a similar argument to Section 2.4, we derive the following estimates:

\[ N_{\text{lox}, M_i, \gamma_i}(u; P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) \leq N_{\text{lox}, M_i, \gamma_i}(T_0; P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) + (\log N(\gamma_0) + 2r)\{(\cosh(u + r)^2 - \cosh(T_0 - r)^2\}/\sinh(2r) - 2r \]  

and

\[
\int_{T_0}^{\infty} (\cosh u)^{-s} dN_{\text{lox}, M_i, \gamma_i}(u; P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) \\
\leq 2s^{-1}e^{(2-s)T_0+2r(\log N(\gamma_0) + 2r)/\sinh(2r) - 2r} \cdot s/s - 2.
\]  

Therefore, we have

\[
\left| 2^{-s}e^{s g(r_0, l_{\gamma})} \int_{T_0+g(r_0, l_{\gamma})}^{\infty} (\cosh u)^{-s} dN_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) \right| \\
\leq 2^{-1}s^{-1}e^{(2-s)T_0+2g(r_0, l_{\gamma})+2r(\log N(\gamma_0) + 2r)/\sinh(2r) - 2r} \cdot s/s - 2.
\]  

Then the right-hand side of (33) can be taken arbitrarily small for sufficiently large \(T_0\).

Next we consider the integral of the first term on the right-hand side of (30). First, we evaluate the function \(g(r_0, l_{\gamma})\). From the definition (21),

\[
g(r_0, l_{\gamma}) = \log \left( \frac{|P|}{2\pi l_{\gamma} r_0^2} + \sqrt{1 + \frac{|P|}{2\pi l_{\gamma} r_0^2}} \right).
\]

Since \(l_{\gamma} \to 0\) as \(i \to \infty\), we observe \(g(r_0, l_{\gamma}) \to \infty\) as \(i \to \infty\). For fixed \(x > 0\) and \(s \in \mathbb{C}\) with \(\text{Re}(s) > 0\), we have

\[
\lim_{i \to \infty} 2^{-s}e^{sx} \cosh(x + r)^{-s} = e^{-sx}.
\]

Thus we have the following estimate:

\[
\lim_{i \to \infty} 2^{-s}r_0^s e^{s g(r_0, l_{\gamma})} \int_0^{T_0+g(r_0, l_{\gamma})} (\cosh(u))^s dN_{\text{lox}, M_i, \gamma_i}(u; P, L_{\gamma_i}) \\
= \lim_{i \to \infty} 2^{-s}r_0^s e^{s g(r_0, l_{\gamma})} \int_0^{T_0} (\cosh(u + g(r_0, l_{\gamma})))^{-s} dN_{\text{lox}, M_i, \gamma_i}(u; P, \partial M_i(0, \varepsilon'), L_{\gamma_i}) \\
= r_0^s \int_0^{T_0} e^{-su} dN_{\text{par}, M, \xi}(u; P, S_{r_0}).
\]  

To complete the proof of part (c), we give the limit of \(g(r_0, l_{\gamma})\):

\[
2^{-s}r_0^s e^{s g(r_0, l_{\gamma})} = 2^{-s}r_0^s \left( \frac{|P|}{2\pi l_{\gamma} r_0^2} + \sqrt{1 + \frac{|P|}{2\pi l_{\gamma} r_0^2}} \right)^s \\
= \left( \frac{|P|}{2\pi l_{\gamma}} \right)^{s/2} + o\left( \left( \frac{|P|}{2\pi l_{\gamma}} \right)^{s/2} \right).
\]  

as \(l_{\gamma} \to 0\), i.e. \(i \to \infty\). Therefore, from (30) through (35), the assertion of part (c) of Theorem 1.1 follows. \(\square\)
Remark 3.3. We consider the differential equation (20). In the setting of part (c) of Theorem 1.1, the loxodromic Eisenstein series $E_{\text{lox}, \gamma_i}(P, s)$ by multiplying $\left(\frac{|P|}{2\pi l_{\gamma_i}}\right)^{s/2}$ satisfies the following differential equation:

$$\left(-\Delta + s(s-2)\right)\left\{\left(\frac{|P|}{2\pi l_{\gamma_i}}\right)^{s/2} E_{\text{lox}, \gamma_i}(P, s)\right\}$$

$$= \frac{2\pi l_{\gamma_i}s^2}{|P|} \cdot \left\{\left(\frac{|P|}{2\pi l_{\gamma_i}}\right)^{(s+2)/2} E_{\text{lox}, \gamma_i}(P, s+2)\right\}. \quad (36)$$

Since $l_{\gamma_i} \to 0$ as $i \to \infty$, the right-hand side of (36) converges to 0 as $i \to \infty$. Therefore, we can see that the differential equation of the loxodromic Eisenstein series approaches the differential equation of the parabolic Eisenstein series through degeneration.

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