BPS Wilson Loop T-dual to Spinning String in $AdS_5 \times S^5$

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Abstract

We use the string sigma model action in $AdS_5 \times S^5$ to reconstruct the open string solution ending on the Wilson loop in $S^3 \times R$ parametrized by a geometric angle in $S^3$ and an angle in flavor space. Under the interchange of the world sheet space and time coordinates and the T-duality transformation with the radial inversion, the static open string configuration associated with the BPS Wilson loop with two equal angle parameters becomes a long open spinning string configuration which is produced by taking the equal limit of two frequencies for the folded spinning closed string with two spins in $AdS_5 \times S^5$. 
## 1 Introduction

The AdS/CFT correspondence \[1\] has more and more revealed the deep relations between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory and the string theory in $AdS_5 \times S^5$. A lot of fascinating results have been accumulated in the computation of the planar observables such as the spectrum, Wilson loops and scattering amplitudes \[2\].

By computing the expectation value of the Wilson loop consisting of a pair of antiparallel lines from the string theory in $AdS_5 \times S^5$ the effective potential between a pair of heavy W bosons has been extracted \[3, 4\]. It has been investigated perturbatively in the gauge theory \[5, 6\] and by the strong coupling expansion in the string theory \[7, 8, 9, 10\] (see also \[11\]).

For the circular 1/2 BPS Wilson loop its expectation value evaluated in the string theory has been reproduced by performing the resummation of ladder diagrams in the $\mathcal{N} = 4$ SYM theory \[6, 12\] and using the localization arguments \[13\]. The lower supersymmetric Wilson loops on a two-sphere $S^2$ embedded into the $R^4$ spacetime have been analyzed by finding the corresponding open string solutions as well as by reducing a purely perturbative calculation in the soluble bosonic 2d Yang-Mills on the sphere \[14, 15, 16, 17\].

In \[18\] the effective potential between a generalized quark antiquark pair has been computed by studying a family of Wilson loops in $S^3 \times R$ which are parametrized by two angle parameters $\phi$ and $\theta$. The quark and antiquark lines are extended along the time direction and are separated by an angle $\pi - \phi$ on $S^3$. The parameter $\theta$ is the relative orientation of the extra coupling to the scalar field for the quark and the antiquark. Through the plane to cylinder transformation, the two lines in $S^3 \times R$ map into two half-lines with a cusp of angle $\pi - \phi$ in $R^4$, and the potential energy of static quark and antiquark is identical with the cusp anomalous dimension. The Wilson loop in $S^3 \times R$ with the Minkowski signature interpolates smoothly between the 1/2 BPS two antipodal lines at $\phi = 0$ and the coincident two antiparallel lines at $\phi = \pi$, while the two Wilson lines with a cusp in $R^4$ between the 1/2 BPS one straight line and the coincident two lines with a cusp of zero angle.

In the weak coupling expansion for the $\mathcal{N} = 4$ SYM theory the effective potential has been computed at one-loop order \[19\] and at two-loop order for $\phi = 0$ \[20\], for $\phi \neq 0$ \[18\]. In the semiclassical expansion for the string theory by using the Nambu-Goto string action, the effective potential has been evaluated at leading order \[19, 14\] and at one-loop order \[18\].

An exact formula for the Bremsstrahlung function of the cusp anomalous dimension for small values of $\phi$ and $\theta$ has been found by relating the cusp anomalous dimension to the localization result of certain 1/8 BPS circular Wilson loops \[21\] (see also \[22\]). The first two terms in the weak coupling and the strong coupling expansions of the exact formula agree with the results of the corresponding effective potential in \[18\]. The three loop term in the weak coupling expansion has been produced by the explicit three loop computation \[23\] and the three loop expansion of the TBA equations \[24, 25\]. There have been further studies about the cusp anomalous dimension associated with the cusped Wilson lines \[26, 27, 28\].

Suggested by a striking similarity between the Bremsstrahlung function \[21\] of the cusp anomalous dimension in the small angle limit and the slope function \[29\] found in the small spin limit of the $AdS_5$ folded string energy, there has been a construction of a possible relation between small (nearly point-like) closed strings in $AdS_5$ and long open strings ending at the boundary which correspond to nearly straight Wilson lines \[30\]. Through the T-duality...
along the boundary directions of Lorentzian AdS in the Poincare coordinates together with
the radial inversion $z \rightarrow 1/z$ and the interchange of space and time coordinates
of the Minkowski world sheet, the world sheet of small closed string is related with the
open string surface ending on wavy line representing small-velocity “quark” trajectory at
the boundary. This open string solution corresponds to the small-wave open string solution
in which ends on a time-like near BPS Wilson loop differing by small fluctuations from a
straight line. Further from the computation of the one-loop fluctuations about the classical
small-wave open string solution, the one-loop correction to the energy radiated by the end-
point of a string has been evaluated to be consistent with the subleading term in the
strong coupling expansion of the Bremsstrahlung function in [21].

Instead of the Nambu-Goto action we will use the string sigma model action in the
global coordinates to reconstruct the open string solution ending at the boundary which
is associated with the two antiparallel Wilson lines in $S^3 \times R$ parametrized by two angle
parameters $\phi$ and $\theta$. We will express the Minkowski open string solution associated with
the BPS Wilson loop specified by $\phi = \pm \theta$ in terms of the Poincare coordinates. We will
first make the flip of world sheet coordinates and secondly perform the T-duality along the
boundary directions with the radial inversion $z \rightarrow 1/z$ to see what kind of string configuration
appears and examine how the BPS condition $\phi = \pm \theta$ is encoded in the transformed string
configuration.

2 The open string solution for the BPS Wilson loop

We consider the classical open string solution in AdS$_5 \times S^5$ ending on the Wilson loop at
the boundary which interpolates smoothly between the 1/2 BPS two antipodal lines and the
coincident two antiparallel lines. The Wilson loop for the $\mathcal{N} = 4$ SYM theory in $S^3 \times R$
is given by

$$W = \frac{1}{N} \text{Tr} \mathcal{P} \exp \left[\oint (iA_\mu \dot{x}^\mu + \Phi^I \Theta^I |\dot{x}|)ds\right]$$

and characterized by two parameters $\phi$ and $\theta$, where the loop is made of two lines separated
by an angle $\pi - \phi$ along the big circle on $S^3$, and $\theta$ specifies the coupling to the scalars $\Phi^I$.

By describing the angle along the big circle by $\varphi$ and the time by $t$ we parametrize the two
lines extending to the future and the past time directions as

$$t = s, \quad \varphi = \frac{\phi}{2}, \quad \Theta^1 = \cos \frac{\theta}{2}, \quad \Theta^2 = \sin \frac{\theta}{2},$$

$$t = -s', \quad \varphi = \pi - \frac{\phi}{2}, \quad \Theta^1 = \cos \frac{\theta}{2}, \quad \Theta^2 = -\sin \frac{\theta}{2}.$$ (2)

The general open string solution with two arbitrary angles $\phi$ and $\theta$ was constructed by using
the Nambu-Goto action [18].

We redereive the open string solution associated with the BPS Wilson loop with $\phi = \pm \theta$
by using the conformal gauge for the string sigma model in the global coordinates. For a
static open string in AdS$_3 \times S^1$ with metric

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\varphi^2 + d\theta^2$$

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we make the following ansatz
\[ t = \tau, \quad \rho = \rho(\sigma), \quad \varphi = \varphi(\sigma), \quad \vartheta = \vartheta(\sigma) \] (4)
with the Minkowski signature in the world sheet. The Virasoro constraint yields
\[ -\cosh^2 \rho + (\partial_\sigma \rho)^2 + \sinh^2 \rho (\partial_\sigma \varphi)^2 + (\partial_\sigma \vartheta)^2 = 0, \] (5)
which reads
\[ \cosh^2 \rho = (\partial_\sigma \varphi)^2 \left[ (\partial_\sigma \rho)^2 + \sinh^2 \rho + (\partial_\sigma \vartheta)^2 \right]. \] (6)
We make a choice of sign as
\[ \partial_\sigma \varphi = \frac{\cosh \rho}{\sqrt{(\partial_\sigma \rho)^2 + \sinh^2 \rho + (\partial_\sigma \vartheta)^2}}. \] (7)
The equation of motion for \( \varphi \)
\[ \partial_\sigma (\sinh^2 \rho \partial_\sigma \varphi) = 0 \] (8)
gives one integral of motion \( p \) as
\[ \sinh^2 \rho \partial_\sigma \varphi = \frac{1}{p}, \] (9)
which is expressed through (7) as
\[ \frac{\sinh^2 \rho \cosh \rho}{\sqrt{(\partial_\sigma \rho)^2 + \sinh^2 \rho + (\partial_\sigma \vartheta)^2}} = \frac{1}{p}. \] (10)
where we choose \( p \) as a positive parameter. The equation of motion for \( \vartheta \)
\[ \partial_\sigma^2 \vartheta = 0 \] (11)
gives a solution \( \vartheta = J \sigma \) and the other integral of motion \( J \) defined by \( \partial_\sigma \vartheta = J \) is described through (7) as
\[ \frac{\partial_\sigma \vartheta \cosh \rho}{\sqrt{(\partial_\sigma \rho)^2 + \sinh^2 \rho + (\partial_\sigma \vartheta)^2}} = J. \] (12)
We combine (12) with (10) to have
\[ \frac{\partial_\sigma \vartheta}{\sinh^2 \rho} = pJ. \] (13)
The remaining equation of motion for \( \rho \) is
\[ \partial_\sigma^2 \rho - \cosh \rho \sinh \rho - \cosh \rho \sinh (\partial_\sigma \varphi)^2 = 0, \] (14)
whose first integral is contained in (5). Indeed differentiating (5) with respect to \( \sigma \) we have
\[ \partial_\sigma \rho (\partial_\sigma^2 \rho - \cosh \rho \sinh \rho) + \partial_\sigma \varphi (\sinh^2 \rho \partial_\sigma^2 \varphi + \cosh \rho \sinh (\partial_\sigma \varphi)^2 + \cosh \rho \partial_\sigma \rho \partial_\sigma \varphi) + \partial_\sigma \vartheta \partial_\sigma^2 \vartheta = 0, \] (15)
which yields (14) through (8) and (11). The expressions such as (1 0), (12) and (13) were presented in ref. [18] for the general $\phi \neq \theta$ case with two arbitrary parameters $J$ and $p$ where $p$ is defined to have opposite sign from ours.

Here we restrict ourselves to the $J = \pm 1/p$ case, that is, the $\phi = \pm \theta$ case. Substitution of (13) into (12) generates a differential equation for $\rho$

$$(\partial_\sigma \rho)^2 = p^2 \cosh^2 \rho \sinh^4 \rho - \sinh^4 \rho - \sinh^2 \rho,$$

which becomes through (9) to be

$$(\partial_\sigma \rho)^2 = \frac{\cosh^2 \rho(p^2 \cosh^2 \rho - (p^2 + 1))}{p^2 \sinh^2 \rho}.$$  \hspace{1cm} (17)

The turning point $\rho_0 (\rho_0 \leq \rho)$ of open string is specified by

$$\cosh^2 \rho_0 = \frac{p^2 + 1}{p^2}.$$  \hspace{1cm} (18)

Thus we have dealt with the string sigma model action to recover the same relevant equations as constructed by using Nambu-Goto string action in ref. [18]. Using the string sigma model for the open string in AdS$_3 \times$S$^3$ with a spin in S$^3$ the first integrals for equations of motion have been presented for the $\theta \neq 0, \phi = 0$ case [24] and for the $\phi \neq \theta$ case [26]. Without solving the relevant equations to obtain the string profile the angles and charges have been directly extracted from them.

We are interested in the explicit expression of solution for (17) specified by a single parameter $p$, which can be read off by putting $J = \pm 1/p$ in the general solution for $\rho$ with the two parameters $J$ and $p$ in ref. [18]. It is convenient to rescale the world sheet coordinates $\tau$ and $\sigma$ in the same way

$$\tau \to \frac{p}{\sqrt{p^2 + 1}} \tau, \quad \sigma \to \frac{p}{\sqrt{p^2 + 1}} \sigma.$$  \hspace{1cm} (19)

By integrating the rescaled equation of (17) under a condition $\rho = \rho_0$ at $\sigma = 0$ we have the following open string solution

$$t = \frac{p}{\sqrt{p^2 + 1}} \tau, \quad \vartheta = \pm \frac{1}{\sqrt{p^2 + 1}} \sigma, \quad \cosh \rho = \frac{\sqrt{p^2 + 1}}{p} \cos \sigma,$$

where the ranges of the world sheet coordinates are

$$-\frac{\pi}{2} \leq \sigma \leq \frac{\pi}{2}, \quad -\infty < \tau < \infty.$$  \hspace{1cm} (20)

At the open string ends $\sigma = \pm \pi/2$, the radial coordinate $\rho$ diverges such that the string touches the boundary of the AdS$_3$ space. The angle $\vartheta$ changes in $-\theta/2 \leq \vartheta \leq \theta/2$,

$$\theta = \pm \frac{\pi}{\sqrt{p^2 + 1}}.$$  \hspace{1cm} (21)
Substitution of the third equation in (20) into the rescaled equation of (9) yields
\[ \partial_\sigma \varphi = \frac{1}{\sqrt{p^2 + 1}} \frac{p^2 \cos^2 \sigma}{p^2 + 1 - p^2 \cos^2 \sigma}. \] (23)

The integration under a condition
\[ \varphi(\sigma = -\frac{\pi}{2}) = \frac{\phi}{2} = \pm \frac{\theta}{2} = \frac{\pi}{2\sqrt{p^2 + 1}} \] (24)
as (2) gives a solution for \(-\pi/2 \leq \sigma \leq 0\)
\[ \varphi = -\frac{1}{\sqrt{p^2 + 1}} \left[ \sigma - \sqrt{p^2 + 1} \tan^{-1} \left( \frac{1}{\sqrt{p^2 + 1}} \tan \left( \frac{\pi}{2} + \frac{\sigma}{2} \right) \right) \right]. \] (25)

For \(0 \leq \sigma \leq \pi/2\) we need to analytically continue the solution. In that region it is given by
\[ \varphi = \pi - \frac{\sigma}{\sqrt{p^2 + 1}} - \tan^{-1} \left( \frac{1}{\sqrt{p^2 + 1}} \tan \left( \frac{\pi}{2} - \sigma \right) \right), \] (26)
which becomes \(\pi - \phi/2\) at \(\sigma = \pi/2\) as (2). At \(p = 0\), the angle \(\phi\) becomes \(\pi\) such that the two antiparallel lines are coincident, while at the infinite \(p\) it vanishes such that the two lines are antipodal.

### 3 T-duality to the BPS Wilson loop

We express the string configuration in terms of the Poincare coordinates. The embedding coordinates \(X_M (M = 0, \cdots, 5)\) for the Lorentzian \(AdS_5\) space are described by the global coordinates \((t, \rho, \psi, \varphi_1, \varphi_2)\)
\[ X_{-1} + iX_0 = \cosh \rho e^{i\mu}, \quad X_1 + iX_2 = \sinh \rho \cos \psi e^{i\varphi_1}, \quad X_3 + iX_4 = \sinh \rho \sinh \varphi_1 e^{i\varphi_2}, \]
\[ X_M X^M = -X_{-1}^2 + X_\mu X^\mu + X_4^2 = -1. \] (27)

These coordinates are related with the Poincare coordinates \((z, x_{\mu})\), \(ds^2 = z^{-2}(dz^2 + dx_\mu dx^\mu)\),
\[ X_\mu = \frac{x_\mu}{z}, \quad X_4 = \frac{-1 + z^2 + x_\mu x^\mu}{2z}, \quad X_{-1} = \frac{1 + z^2 + x_\mu x^\mu}{2z}, \] (28)
which also define the light-like coordinates
\[ X_- = X_{-1} - X_4 = \frac{1}{z}, \quad X_+ = X_{-1} + X_4 = \frac{z^2 + x_\mu x^\mu}{z}. \] (29)

In the \(AdS_3\) space specified by \(\varphi_1 = \varphi, \psi = 0\) or \(X_3 = X_4 = 0\) we use the static string solution (20) together with (27) and (28) to derive
\[ z = \frac{p}{\sqrt{p^2 + 1}} \cos \sigma, \quad x_0 = \tan \frac{p}{\sqrt{p^2 + 1}} \tau \] (30)
\[
(x_1, x_2) = \frac{\sqrt{p^2 \sin^2 \sigma + 1}}{\sqrt{p^2 + 1} \cos \frac{p}{\sqrt{p^2 + 1} \tau}} (\cos \varphi, \sin \varphi). \tag{31}
\]

Indeed an equation \( z^2 + x_\mu x^\mu = 1 \), that is, \( X_4 = 0 \) is satisfied.

In the region \( -\pi/2 \leq \sigma \leq 0 \) we substitute (25) into (31) to obtain
\[
x_1 = \frac{1}{\cos \frac{p}{\sqrt{p^2 + 1} \tau}} \left( -\sin \sigma \cos \frac{\sigma}{\sqrt{p^2 + 1} \tau} + \frac{1}{\sqrt{p^2 + 1}} \cos \sigma \sin \frac{\sigma}{\sqrt{p^2 + 1} \tau} \right),
\]
\[
x_2 = \frac{1}{\cos \frac{p}{\sqrt{p^2 + 1} \tau}} \left( \frac{1}{\sqrt{p^2 + 1}} \cos \sigma \cos \frac{\sigma}{\sqrt{p^2 + 1} \tau} + \sin \sigma \sin \frac{\sigma}{\sqrt{p^2 + 1} \tau} \right). \tag{32}
\]

They are combined to be
\[
x_1 + ix_2 = \frac{1}{\cos \frac{p}{\sqrt{p^2 + 1} \tau}} \left( -\sin \sigma + i \frac{\cos \sigma}{\sqrt{p^2 + 1}} \right) e^{-i\sigma/\sqrt{p^2 + 1}}. \tag{33}
\]

In the region \( 0 \leq \sigma \leq \pi/2 \) using (26) we compute \( x_1 \) and \( x_2 \) to derive the same expression as (32) so that we regard (32) as a solution in the whole interval \( -\pi/2 \leq \sigma \leq \pi/2 \). Thus the string configuration is described by \( z \) and \( x_0 \) in (30) together with the single expression (32).

In the Poincaré coordinates the equations of motion for \( x_\mu \) read
\[
\partial_\tau \left( \frac{\partial_\tau x_\mu}{z^2} \right) - \partial_a \left( \frac{\partial_a x_\mu}{z^2} \right) = 0, \quad \mu = 0, 1, 2. \tag{34}
\]

Plugging the expressions of \( z, x_0 \) in (30) and \( x_1, x_2 \) in (32) into (34) we can confirm that they are satisfied. The equation of motion for \( z \) is given by
\[
\partial_\tau \left( \frac{\partial_\tau z}{z^2} \right) - \partial_a \left( \frac{\partial_a z}{z^2} \right) = \frac{1}{z^3} \left( -\partial_a x_0 \partial^a x_0 + \partial_a x_i \partial^a x_i + \partial_a z \partial^a z \right) \tag{35}
\]
with \( i = 1, 2 \) and \( a = \tau, \sigma \). We confirm that (35) is indeed satisfied by deriving
\[
\partial_a x_i \partial^a x_i = \frac{p^2}{p^2 + 1} \cos^2 \frac{p}{\sqrt{p^2 + 1} \tau} \left[ -\left( \sin^2 \sigma + \frac{\cos^2 \sigma}{p^2 + 1} \right) \sin^2 \frac{p}{\sqrt{p^2 + 1} \tau} \right.
\]
\[
+ \frac{p^2}{p^2 + 1} \cos^2 \sigma \cos^2 \frac{p}{\sqrt{p^2 + 1} \tau} \right] \tag{36}
\]
and showing the RHS of (35) to be
\[
\frac{p^2}{z^3 (p^2 + 1) \cos^4 \frac{p}{\sqrt{p^2 + 1} \tau}} \left( 1 + \frac{p^2}{p^2 + 1} \cos^2 \sigma + \sin^2 \sigma \right) \cos^2 \frac{p}{\sqrt{p^2 + 1} \tau}, \tag{37}
\]
where the \( \sin^2 p\tau/\sqrt{p^2 + 1} \) term vanished.
In [36] a family of closed string solutions on $R \times S^3$ subspace of $AdS_5 \times S^5$ were constructed by using the $\tau \leftrightarrow \sigma$ flip which maps spinning closed string states with large spins to oscillating states with large winding numbers. There was a prescription [30] that starting from the closed string solutions in the bulk of AdS space we perform the T-duality along the boundary directions in the Poincare coordinates of AdS space with the radial inversion and relax the condition of periodicity in $\sigma$ to interchange $\tau$ and $\sigma$ for constructing the open string solutions ending at the boundary which are associated with the Wilson loops.

Here we proceed in the inverse direction. Starting from the open string solution (30) with (33) and $\vartheta = \pm \sigma/\sqrt{p^2 + 1}$ which is associated with the BPS Wilson loop we relax the ranges of the world sheet coordinates and interchange $\tau$ and $\sigma$ to have

$$\tilde{z} = \frac{p}{\sqrt{p^2 + 1}} \cos \tau, \quad \tilde{x}_0 = \tan \frac{p}{\sqrt{p^2 + 1}} \sigma,$$

$$\tilde{x}_1 + i \tilde{x}_2 = \frac{1}{\cos \frac{p}{\sqrt{p^2 + 1}} \sigma} \left( -\sin \tau + i \frac{\cos \tau}{\sqrt{p^2 + 1}} \right) e^{-i\sigma/\sqrt{p^2 + 1}},$$

$$\tilde{\vartheta} = \pm \frac{\tau}{\sqrt{p^2 + 1}}. \quad (38)$$

If we make the T-duality along the boundary directions and the inversion of the radial coordinate for some unknown string configuration $z, x_\mu$ as

$$\partial_\tau x_\mu = - \frac{1}{z^2} \partial_\sigma x_\mu, \quad \partial_\sigma x_\mu = - \frac{1}{z^2} \partial_\tau x_\mu, \quad \tilde{z} = \frac{1}{z}, \quad (39)$$

then we suppose to obtain the previous string configuration $\tilde{z}, \tilde{x}_\mu$ in (38). Rewriting (39) as

$$z = \frac{1}{\tilde{z}} = \frac{\sqrt{p^2 + 1}}{p} \cos \frac{p}{\sqrt{p^2 + 1}} \sigma,$$

$$\partial_\sigma x_\mu = \frac{1}{z^2} \partial_\tau x_\mu, \quad \partial_\tau x_\mu = - \frac{1}{z^2} \partial_\sigma x_\mu \quad (40)$$

we substitute the $\tau \leftrightarrow \sigma$ flip solution (38) into (41) to have the differential equations for the combination $x_1 + ix_2$

$$\partial_\sigma (x_1 + ix_2) = \frac{\cos \frac{p}{\sqrt{p^2 + 1}} \sigma}{\cos \tau} e^{-i\sigma/\sqrt{p^2 + 1}},$$

$$\partial_\tau (x_1 + ix_2) = - \frac{\sin \frac{p}{\sqrt{p^2 + 1}} \sigma}{p \cos^2 \tau} \left( -\sin \tau + i \frac{\cos \tau}{\sqrt{p^2 + 1}} \right) e^{-i\sigma/\sqrt{p^2 + 1}}, \quad (42)$$

which can be solved by

$$x_1 + ix_2 = \frac{\sqrt{p^2 + 1}}{p} \sin \frac{p}{\sqrt{p^2 + 1}} \cos \tau e^{-i\sigma/\sqrt{p^2 + 1}}. \quad (43)$$

The other $\mu = 0$ component is obtained by

$$x_0 = - \frac{\sqrt{p^2 + 1}}{p} \tan \tau. \quad (44)$$
It can be also confirmed that the obtained string configuration expressed by (43), (44) and (40) indeed satisfies the string equations of motion (34) and (35) in the same way as shown previously for the starting static open string solution. Here for convenience we write down two relevant equations corresponding to (36) and (37) respectively

\[ \partial_a x_i \partial^a x_i = -\frac{p^2 + 1}{p^2} \cos^4 \tau \left[ - \sin^2 \frac{p}{\sqrt{p^2 + 1}} \sigma \left( \sin^2 \tau + \frac{\cos^2 \tau}{p^2 + 1} \right) \right. 
+ \left. \frac{p^2}{p^2 + 1} \cos^2 \frac{p}{\sqrt{p^2 + 1}} \sigma \cos^2 \tau \right], \]

(45)

and

\[ \frac{p^2 + 1}{z^3 p^2 \cos^4 \tau} \left( 1 + \frac{p^2}{p^2 + 1} \cos^2 \frac{p}{\sqrt{p^2 + 1}} \sigma + \frac{p^2 - 1}{p^2 + 1} \sin^2 \frac{p}{\sqrt{p^2 + 1}} \sigma \right) \cos^2 \tau. \]

(46)

Under the sign change of \( \tau \) the string solution in \( AdS_3 \times S^1 \) becomes

\[ z = \sqrt{p^2 + 1} \cos \frac{p}{\sqrt{p^2 + 1}} \sigma / \cos \tau, \quad x_0 = \sqrt{p^2 + 1} \tan \tau, \]

\[ x_1 + ix_2 = \sqrt{p^2 + 1} \sin \frac{p}{\sqrt{p^2 + 1}} \sigma e^{i\tau/\sqrt{p^2 + 1}}, \quad \vartheta = +\frac{\tau}{\sqrt{p^2 + 1}} \]

(47)

which, however obeys

\[ z^2 - x_0^2 + x_1^2 + x_2^2 = \frac{p^2 + 1}{p^2}. \]

(48)

In terms of the embedding coordinates the string solution is expressed as

\[ X_0 = \frac{\sin \tau}{\cos \frac{p}{\sqrt{p^2 + 1}} \sigma}, \quad X_1 + iX_2 = \tan \frac{p}{\sqrt{p^2 + 1}} \sigma e^{i\tau/\sqrt{p^2 + 1}}, \]

\[ X_+ = \sqrt{p^2 + 1} \frac{\cos \tau}{\cos \frac{p}{\sqrt{p^2 + 1}} \sigma}, \quad X_- = \frac{p}{\sqrt{p^2 + 1} \cos \frac{p}{\sqrt{p^2 + 1}} \sigma}. \]

(49)

This T-dual string solution has nonvanishing \( X_4 \).

Now we make a particular SO(2,4) transformation in the \((X_-1, X_4)\) plane, that is, a dilatation transformation for (49) as

\[ X_+ \rightarrow \lambda X_+, \quad X_- \rightarrow \frac{1}{\lambda} X_-, \quad X_\mu : \text{invariant} \]

(50)

with \( \lambda = p/\sqrt{p^2 + 1} \) to have

\[ X_+ = X_- = \frac{\cos \tau}{\cos \frac{p}{\sqrt{p^2 + 1}} \sigma}. \]

(51)

which implies \( X_4 = 0 \). Accordingly the string solution in the Poincare coordinates is rescaled as \( z \rightarrow \lambda z, \quad x_\mu \rightarrow \lambda x_\mu \)

\[ z = \frac{\cos \frac{p}{\sqrt{p^2 + 1}} \sigma}{\cos \tau}, \quad x_0 = \tan \tau, \quad x_1 + ix_2 = \frac{\sin \frac{p}{\sqrt{p^2 + 1}} \sigma}{\cos \tau} e^{i\tau/\sqrt{p^2 + 1}}, \]

(52)
which obeys $z^2 + x^2 = 1$ to stay in $AdS_3$.

Further the T-dual string solution in $AdS_3 \times S^1$ is expressed in terms of the global coordinates as

$$t = \tau, \quad \cosh \rho = \frac{1}{\cos \frac{\rho}{\sqrt{p^2 + 1}}}, \quad \varphi = \frac{1}{\sqrt{p^2 + 1}} \tau,$$

$$\vartheta = \mp \frac{1}{\sqrt{p^2 + 1}} \tau. \tag{53}$$

The string reaches the $AdS_3$ boundary at

$$\sigma = \pm \frac{\sqrt{p^2 + 1} \pi}{p \cdot 2}. \tag{54}$$

We see that the dilatation transformation plays an important role to derive a compact and consistent real string solution in the global coordinates.

Here we consider the folded spinning closed string in $AdS_3 \times S^1$,

$$t = \tau, \quad \varphi = \omega \tau, \quad \vartheta = \nu \tau, \tag{55}$$

$$\rho = \rho(\sigma) = \rho(\sigma + 2\pi), \tag{56}$$

where $\omega$ and $\nu$ are two frequencies associated with two spins in $AdS_3$ and in $S^1$ respectively. The equation of motion for $\rho$ is

$$\partial_\sigma^2 \rho = (1 - \omega^2) \sinh \rho \cosh \rho. \tag{57}$$

The Virasoro constraint leads to

$$(\partial_\sigma \rho)^2 = (1 - \nu^2) \cosh^2 \rho - (\omega^2 - \nu^2) \sinh^2 \rho. \tag{58}$$

In the special $\omega = \mp \nu$ case the solution for $\rho$ is given by

$$\cosh \rho = \frac{1}{\cos \sqrt{1 - \nu^2} \sigma}, \tag{59}$$

which satisfies the equation of motion for $\rho$, (57). This string solution is regarded as an effectively long open string configuration which is stretched along the radial direction of $AdS_3$ to the boundary at $\sigma = \pm \pi / 2 \sqrt{1 - \nu^2}$. The specific choice $\nu = \mp 1 / \sqrt{p^2 + 1}$ for (55) and (59) produces the same string solution as the T-dual string configuration (53) in $AdS_3 \times S^1$. Thus the T-dual string solution can be viewed as a special equal limit of two frequencies for the folded spinning closed string in $AdS_3 \times S^1$. It is noted that the BPS condition $\phi = \pm \theta$ in the starting static open string solution becomes the equal condition of two frequencies $\omega = \mp \nu$ in the T-dual spinning string solution.
4 Conclusion

Using the string sigma model in the global coordinates on the Lorentzian $AdS_3 \times S^1$ space we have reconstructed the static open string configuration ending on the BPS Wilson loop [18], which consists of the two antiparallel lines specified by two equal angles $\phi = \pm \theta$. We have expressed the open string solution in terms of the Poincare coordinates and have used the prescription of ref. [30] to make the flip of the Minkowski world sheet coordinates and perform the T-duality transformation along the boundary directions with the radial inversion. By solving the T-duality equation we have derived a compact expression of solution, which stays out of the $AdS_3$ space. We have observed that the dilatation transformation plays an important role to put the T-dual string configuration back into the $AdS_3$ space.

We have demonstrated that when the T-dual string solution described by the Poincare coordinates in $AdS_3 \times S^1$ is expressed in terms of the global coordinates it produces a suggestive expression which turns out to be the spinning open string configuration derived by taking the equal limit of two frequencies for the folded spinning closed string with two spins in $AdS_3 \times S^1$. We have observed that the BPS condition $\phi = \pm \theta$ for the starting BPS Wilson loop corresponds to the equal condition of two frequencies associated with two spins for the T-dual spinning string solution.

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