NUMERICAL AND ANALYTICAL MODELING OF TRANSIT TIMING VARIATIONS

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ABSTRACT

We develop and apply methods to extract planet masses and eccentricities from observed transit timing variations (TTVs). First, we derive simple analytic expressions for the TTV that include the effects of both first- and second-order resonances. Second, we use $N$-body Markov chain Monte Carlo simulations, as well as the analytic formulae, to measure the masses and eccentricities of 10 planets discovered by Kepler that have not previously been analyzed. Most of the 10 planets have low densities. Using the analytic expressions to partially circumvent degeneracies, we measure small eccentricities of a few percent or less.

Key words: planets and satellites: detection

Supporting material: HDF5 files

1. INTRODUCTION

The Kepler mission has revealed a wide diversity of extrasolar planetary systems. Super-Earth and sub-Neptune planets with radii in the range of $\sim1$–$4\, R_\oplus$ have been shown to be abundant in the Galaxy, even though no such planet exists in our own solar system. Determining the compositions of these abundant planets is important for understanding the planet-formation process. The orbital architectures of many of Kepler’s multiplanet systems are starkly different from our solar system’s as well. Precise measurements of the dynamical states of multiplanet systems offer important clues about their origins and evolution.

Transit timing variations (TTVs) are a powerful tool for measuring masses and eccentricities in multiplanet systems (Agol et al. 2005; Holman 2005). Planets near mean-motion resonances (MMRs) often exhibit large TTV signals, allowing for sensitive measurements of the properties of low-mass planets. However, the conversion of a TTV signal into planet properties is often plagued by a degeneracy between planet masses and eccentricities (Lithwick et al. 2012, hereafter LXW). Nonetheless, $N$-body analyses have provided a number of TTV systems in which planet masses are apparently well constrained and not subject to the predicted degeneracies (e.g., Cochran et al. 2011; Carter et al. 2012; Sanchis-Ojeda et al. 2012; Huber et al. 2013; Lissauer et al. 2013; Masuda et al. 2013; Nesvorný et al. 2013; Jontof-Hutter et al. 2014, 2015; Kipping et al. 2014; Schmitt et al. 2014). Nesvorný & Vokrouhlický (2014) and Deck & Agol (2015a) show that the mean–eccentricity degeneracy can be broken provided that the effects of the planets’ successive conjunctions are seen with sufficient signal to noise in their TTVs.

Characterizing multiplanet systems on the basis of transit timing observations involves fitting noisy data in a high-dimensional parameter space. Bayesian parameter estimation via Markov chain Monte Carlo (MCMC) analysis is well suited to handling such problems and has been applied to the analysis of TTVs previously by numerous authors (e.g., Sanchis-Ojeda et al. 2012; Huber et al. 2013; Masuda et al. 2013; Jontof-Hutter et al. 2015; Schmitt et al. 2014). (Also see Kipping et al. (2014) for an alternative Bayesian approach to dynamical modeling of TTVs.)

In this paper we derive analytic formulae for the TTVs of planets near first- and second-order MMRs. We conduct MCMC analyses using both the analytic model and $N$-body integrations to infer masses and eccentricities of 10 planets in four Kepler multiplanet systems. The analytic model elucidates the degeneracies inherent to inverting TTVs and provides a complimentary approach to parameter inference.

The paper is organized as follows: we summarize the analytic TTV model in Section 2. In Section 3 we detail our methods for inverting TTVs using both $N$-body and analytic methods and apply them to four Kepler multiplanet systems. We summarize our results and conclude in Section 4.

2. ANALYTIC TTV

2.1. The Formulae

In Appendix A we derive the analytic TTV for a pair of planets that lie near (but not in) either a first-order ($J$-$J$-1) MMR or a second-order one ($K$-$K$-2). These formulae should describe the vast majority of TTVs observed by Kepler.

Here we provide a qualitative overview of the formulae because it will help in understanding how well masses and eccentricities can be inferred from observed TTVs (Section 3). We focus first on the case of a planet perturbed by an exterior companion near its $J$-$J$-1 resonance; we then address the other cases of interest, which are almost trivial extensions. The planet’s TTV is $\delta t \equiv O - C$, where $O$ is its observed time of transit and $C$ is the time calculated from its average orbital period, under the assumption of a perfectly periodic orbit. To derive a simple expression for $\delta t$, we expand in powers of the planets’ eccentricities, which is appropriate because Kepler
planets typically have \( e \lesssim 0.2 \) (e.g., see Hadden & Lithwick (2014) and below). However, the most important contributions are not necessarily zeroth order in \( e \). That is because there is a second small parameter that can compensate for a small \( e \): the fractional distance to the nearest first-order MMR:

\[
\Delta = \frac{P' - J - 1}{J} - 1
\]  

(1)

where \( P \) and \( P' \) are the orbital periods of the inner and outer planets. Planets near resonance (\( |\Delta| \ll 1 \)) have particularly large TTVs because the gravitational perturbations add coherently over many orbital periods. Observationally, Kepler pairs with detected TTVs typically have \( |\Delta| \sim 1\%–5\% \).

After expanding in \( e \), we rephrase terms to express the TTV as a sum of three terms that differ in their frequency dependence:

\[
\delta t = \delta t_F + \delta t_C + \delta t_S,
\]

(2)

where \( \delta t_F \) and \( \delta t_S \) are sinusoidal (with different frequencies) and \( \delta t_C \) is the sum of many sinusoidal terms. The three components are given explicitly in Equations (38)–(40) and are described in the following:

1. \( \delta t_F \): The “fundamental” (or alternatively “first harmonic”) has the longest period and typically has the largest amplitude (LXW). Its period is that of the planets’ line of conjunction (the “superperiod”):

\[
P_{\text{super}} = \left| \frac{J}{P'} - \frac{J - 1}{P} \right|^{-1} = \frac{P'}{J|\Delta|},
\]

(3)

and its amplitude is, within order-unity constants\(^6\)

\[
\delta t_F \sim \frac{\mu P}{2\pi|\Delta|} \cdot \max \left\{ 1, \left| \frac{Z}{|\Delta|} \right| \right\},
\]

(4)

where \( \mu \) is the ratio of the outer planet’s mass to the star’s mass, and

\[
Z \equiv \frac{f_{J} z + f_{J'} z'}{\sqrt{f_{J}^2 + f_{J'}^2}} \approx \frac{z' - z}{2}
\]

(5)

is an important variable that consolidates the effect of the planets’ eccentricities on the TTV;\(^7\) in the above definition, \( z \) is the complex eccentricity of the inner planet (\( z \equiv e \cos \omega \)), \( z' \) of the outer, and \( f \) are Laplace coefficients (in the notation of Murray & Dermott 2000). Numerical values for \( f \) are tabulated in the appendix of LXW. The approximate form of Equation (6)—which is independent of \( J \)—is valid to within around 10% fractional error in the coefficients of the \( z \) values (for \( J > 2 \)).

2. \( \delta t_C \): The “chopping” TTV is a sum of many sinusoids that have higher frequencies than the fundamental. These were first derived by Deck & Agol (2015a). The amplitude of each of the terms is

\[
\delta t_C \sim \frac{\mu P}{2\pi} \left( \frac{P}{P - P'} \right)^2,
\]

(7)

within order-unity constants. All first-order and zeroth-order MMRs contribute with roughly this same amplitude—except for the nearby \( J:J-1 \), whose contribution produces \( \delta t_F \). The chopping is independent of eccentricity because there are no resonant denominators (i.e., factors of \( 1/|\Delta| \)), and hence the zeroth-order term in the eccentricity expansion is adequate. Physically, chopping is caused by kicks at conjunctions that can suddenly change the orbit. As a result, the TTV exhibits a strong “chopping” spike at each transit that follows a conjunction (Nesvorný & Vokrouhlický 2014; Deck & Agol 2015a).

3. \( \delta t_S \): The “secondary” (or alternatively “second-harmonic” or “second-order MMR”) term has twice the frequency of \( \delta t_F \). It is caused by proximity to the second-order \( 2J:J-2 \) MMR for a planet pair near the \( J:J-1 \). We derive its effect in the appendix. Its amplitude is, within order-unity constants, a factor of \( |Z| \) smaller than the fundamental:

\[
\delta t_S \sim \frac{\mu P}{2\pi|\Delta|} |Z| \cdot \max \left\{ 1, \left| \frac{Z}{|\Delta|} \right| \right\}.
\]

(8)

Having completed the discussion of an interior planet’s TTV near a first-order resonance, we turn now to the other cases of interest. First, for an exterior planet near a first-order resonance, the discussion above carries through unchanged after replacing mass and period appropriately; \( \mu' \rightarrow \mu \) and \( P \rightarrow P' \). The only other difference is in the order-unity coefficients, which are in any case dropped above. The full formulae that include the order-unity coefficients are given in Appendix A.3. Second, for planets near a second-order \( K:K-2 \) resonance (with \( K \) an odd number), the inner planet’s TTV is \( \delta t = \delta t_C + \delta t_S \); there is no \( \delta t_F \) because it may now be included with the other chopping terms. The secondary TTV is unchanged, with \( 2J \rightarrow K \). See Appendices A.2–A.3 for the full formulae.

2.2. Inferring Planet Parameters

One approach to inferring planet parameters from TTV is with MCMC simulations (Section 3.1). But to understand the MCMC results and to evaluate, for example, the effects of degeneracies and assumed priors on those results, we develop in this section a complementary approach, based on the analytic formulae.

For definiteness, we focus here on a two-planet system near a \( J:J-1 \) MMR. Each planet has, essentially, three unknown parameters: its mass, eccentricity, and longitude of periapse (or equivalently \( \mu \) and complex \( z \)). It also has two additional parameters that are simple to determine accurately and hence we consider “known”: its semimajor axis and the mean longitude at epoch (or equivalently period and the time of a particular transit). For completeness, we note that there are two additional parameters per planet associated with inclinations, but we ignore them here as they usually have a lesser effect on TTVs (see Appendix A.5).

To clarify the parameter inference problem, we rewrite the inner planet’s TTV (Equations (38)–(40)) in a form that
highlights the unknowns (μ' and Z):
\[ \delta T_r = \mu'(A + B Z^s) e^{i \lambda \mu'} + \text{c.c.} \]  
(9)
\[ \delta T_c = \mu' \sum_{j=0}^{2} C_j e^{i \lambda \mu'} + \text{c.c.} \]  
(10)
\[ \delta T_s = \mu'(D Z^s + E Z^s) e^{i \lambda \mu'} + \text{c.c.}, \]  
(11)
where “c.c.” denotes the complex conjugate of the preceding term, and the coefficients A through E are real-valued “known” numbers, that is, determined by the planets’ periods. In addition, \( \lambda = const + \epsilon(2\pi/P) \) is the mean longitude of the outer planet. Note that the period of \( \delta T_e \) is the superperiod (Equation (3)) because \( \lambda' \) is evaluated only when the inner planet transits.

An important feature of these expressions is the fact that they depend on the two planets’ eccentricities only through the combination \( Z \). While that is trivially true for \( \delta T_r \), in Appendix A.4 we show that the same is true for \( \delta T_c \) to a good approximation, due to an apparently coincidental relationship between different Laplace coefficients.\(^8\) Furthermore, the outer planet’s TTV also depends on the same combination \( Z \). As a result, \( Z \) can be determined quite accurately from TTVs. Conversely, even if TTVs are well measured, it is nearly impossible to disentangle the individual planets’ eccentricities. There is an important exception, however, if the pair of planets is close to the 2:1 resonance (or to the 3:1). In these cases, the indirect term leads to a dependence on the individual values of the two planets.\(^9\) The presence of additional planets typically does little to alleviate the degeneracy of inferring individual planet eccentricities.\(^10\)

In addition to the degeneracy between \( z \) and \( z' \) discussed above, there is a second degeneracy: between \( |Z| \) and \( \mu' \). For planet pairs in which only the fundamental TTV is well measured, that degeneracy is evident from Equation (9) since a smaller \( \mu' \) can be compensated for by a larger \( |Z| \) without affecting \( \delta T_r \). Moreover, that degeneracy is not in general removed by observing the outer planet’s \( \delta T_{e'} \) because it depends on the additional unknown \( \mu' \). (For a more detailed discussion, as well as a way to break the degeneracy with a statistical sample of TTVs, see LXW and Hadden & Lithwick (2014)). However, the \( |Z| - \mu' \) degeneracy can be broken if both fundamental and chopping TTVs are observed (Deck & Agol 2015a) or if fundamental and secondary TTVs are observed. We give examples below.

3. FOUR SYSTEMS

We analyze the TTVs in four planetary systems observed by the Kepler telescope, comprising 10 planets. Our analysis is based on the transit times computed by Rowe et al. (2015), which incorporate Kepler long-cadence observations from Quarters 1–17. These four systems exhibit clear TTVs that have not yet been analyzed in detail.

\(^8\) We also show in Appendix A.4 that \( Z \) is nearly independent of \( J \), so if two nearby first-order MMRs both contribute comparable \( b \) values, the eccentricities still enter through the single quantity \( Z \).

\(^9\) Kepler-9 (Holman et al. 2010; Borsato et al. 2014; Dreizler & Ofir 2014), Kepler-18 (Cochran et al. 2011), and Kepler-30 (Sanchis-Ojeda et al. 2012), for which relatively precise individual planet eccentricities measurements are reported, contain planets near the 2:1 MMR.

\(^10\) Three (or more) planets with mutual TTVs in principle yield three distinct \( Z \) values, one for each pairwise interaction, which can be inverted to determine three individual \( z \) values. However, the interactions of the most widely separated pair are typically too weak to constrain their combined \( Z \). Furthermore, the linear transformation from individual \( z \) values to combined \( Z \) values is nearly singular.

3.1. Methods

We employ three complementary methods:

1. N-body MCMC: Our setup is fairly standard and is described in Appendix B.1. We run two N-body MCMC simulations for each system using different priors. Our default priors are logarithmic in masses (\( dP/dM \propto 1/M \)) and uniform in eccentricity (\( dP/de \approx const. \)). Planet densities inferred from TTVs are often surprisingly low (see below and Wu & Lithwick 2013; Hadden & Lithwick 2014; Weiss & Marcy 2014). Therefore we also employ a second set of “high-mass priors” that are uniform in masses (\( dP/dM \propto const. \)) and logarithmic in eccentricity (\( dP/de \propto 1/e \)), where the latter weights more toward lower eccentricity and consequently also toward high masses via Equations (9) and (11).

Our choice of priors is arbitrary, other than that they are somewhat broad. A significant problem with the N-body MCMC method is that the “true” prior is unknown, and very different priors could affect the measured values of the masses and eccentricities.\(^11\) Our solution to this problem is to employ a complementary method based on the analytical formulae, described in the following two points. This can show directly which features in the observed TTV are responsible for the measured values of mass and eccentricity.

2. Analytic MCMC: We run MCMC simulations that model the TTV with analytic formulae (Equations (9)–(11)), rather than with N-body simulations. Details are provided in Appendix B.2. The analytic MCMC results agree well with the N-body ones for the systems considered in this paper (see below). This provides support for the analytic model and, more importantly, shows that the inferred planet parameters can be understood with the help of the analytic model.

3. Analytic Constraint Plot: We use the analytic formulae to infer how each of the TTV components (fundamental, chopping, and secondary) constrains the masses and eccentricities, and thereby show how the overlapping constraints explain the MCMC results. To do so, we first fit for the amplitudes of the sinusoids in Equations (9)–(11). For the inner planet’s TTV, there are five unknowns to be fit for: (a) the complex amplitude of the sinusoid with period equal to the superperiod (Equation (9)), or equivalently the real amplitudes of the sine and cosine components; (b) the real amplitude of the infinite sum of sinusoids in Equation (10) (noting that the phase of this term is known); and (c) the complex amplitude of the secondary TTV. Since the time dependence of each of these terms is known, the fit is done with a simple linear least squares.

Next, setting the complex amplitude inferred from (a) equal to \( \mu'(A + B Z) \), we solve for \( |Z| \) as a function of \( \mu' \). The result is a line in the \( \mu' - |Z| \) plane that is allowed by the fundamental TTV. Accounting for the observational errors turns the line into a band. Similarly, the amplitude of \( \delta T_c \) constrains \( \mu' \), and the complex amplitude of \( \delta T_s \) provides another band of constraint in the \( \mu' - |Z| \) plane.

In the following sections, we describe our results for each of the four systems. Table 1 and Figure 1 summarize the planet masses and densities inferred from the N-body MCMC analysis.

\(^11\) For the reader interested in exploring the sensitivity to alternative priors, we place full posterior samples from our N-body MCMC runs online.
In Table 1, and throughout this paper, measured values refer to the median. The upper and lower error bars demarcate the zone of 68% confidence (“one sigma”) that is bounded by the 84% and 16% percentiles, respectively.

The eccentricity results are in Table 2. We focus on inferring the combined eccentricity $|Z| \approx |z|/\sqrt{2}$ rather than $z$ and $z'$ individually, which are nearly impossible to disentangle from one another. We expect that $|Z|$ is typically a good surrogate for the individual planets’ eccentricities. However, it is conceivable that $z \approx z'$, that the two planets have comparable eccentricities and aligned orbits. If so, the individual eccentricities could be much higher than $|Z|$. Such a situation could arise if damping has acted on the planetary system, removing one of the secular modes but not the other. We do not favor this scenario because it requires a finely tuned damping rate. Nonetheless, it remains a possibility that is difficult to exclude.\footnote{Tides raised on the planets might be dissipative enough to circularize their orbits. The planet Kepler-26 b, for example, circularizes on the timescale $\tau = 2.1 \times Q/k_2$ Myr $\approx 2$ Gyr (Goldreich 1963; Hut 1981), after setting $Q/k_2 \gg 1$, as for Mars and the solid Earth (Lainey 2016), but for this estimate, we have ignored that Kepler-26 b is likely not purely solid and hence might have a higher $Q/k_2$. We have also ignored planet c. In the presence of planet c, the two planets interact secularly. From linear secular theory, their two secular modes damp at the rates $\gamma_2 = 0.4/k_2$ for the aligned mode and $\gamma_2 = 0.6/k_2$ for the antialigned mode (Equation (11) of Wu & Goldreich 2002, to first order in eccentricity). Thus there are three possibilities: (1) both $\gamma_2$ and $\gamma_1$ are longer than the system’s lifetime, in which case tidal circularization is negligible; (2) both are shorter than the lifetime, in which case both planets will be tidally damped; and (3) $\gamma_2 < \gamma_1$ lifetime $< \gamma_1$, in which case only the aligned mode will survive, and the individual $|z|$ and $|Z|$ could be large even though $|Z|$ is very small. Nonetheless, option (3) seems unlikely because it would require fine-tuning to fit the lifetime in the narrow range between $\gamma_1$ and $\gamma_2$. Option (2) also seems unlikely from the fact that this system’s $|Z|$ is nonzero, but an unseen outer planet in this system could alter this conclusion. Turning briefly to the other planets in Table 1, we note that all have longer damping times than Kepler-26 b’s $\gamma_1$ with the possible exception of Kepler-33 c. For Kepler-128, $\gamma_2/k_2 = 1.4$, and for Kepler-307, $\gamma_2/k_2 = 0.5$. Analysis of the Kepler-33 system is more complicated as there are five secular modes, and the properties of the inner planet, Kepler-33 b, are unconstrained (only the TTVs of the outer four are fit in Section 3.5).}

![Figure 1](Figure 1) Top panel: planet mass vs. radius for each planet presented in Section 3. Each splotch shows the 68% joint confidence region in mass and radius, after sampling from the MCMC posteriors, along with stellar parameters and $R_p/R_s$ as described in Table 1. Theoretical mass–radius relationships for planets composed of pure ice and rock from Fortney et al. (2007) are plotted as colored curves. Bottom panel: same as top panel except with density plotted on the vertical axis.
and high-mass priors, along with first- or second-order MMR to the pair’s period ratio is given along with the pair’s distance to resonance, \( \Delta \) (i.e., Equation (1) for the first-order resonances and its obvious extension for the second-order ones: \( \Delta \equiv \frac{e - 2}{e^2} - 1 \)).

### 3.2. Kepler-307 (KOI-1576)

Kepler-307 b and c are a pair of sub-Neptune-sized planets - with radii of \( R_b = 3.0 \pm 0.3 \) \( R_\oplus \) and \( R_c = 2.7 \pm 0.3 \) \( R_\oplus \). The pair were confirmed as planets by Xie (2014) on the basis of their TTVs. The pair’s orbits are near a 5:4 MMR with \( \Delta = 0.005 \). The planets’ TTVs are shown in Figure 2 along with the best-fit \( N \)-body and analytic solutions for the transit times. One can see both the low-frequency fundamental TTV and the high-frequencies from the chopping TTV.

For the \( N \)-body MCMC, an ensemble of 800 walkers was evolved for 250,000 iterations, saving every 800th iteration. A result from the walker autocorrelation lengths (Appendix B.1). The joint posterior distributions of planet masses from analytic and \( N \)-body MCMC are shown in Figure 3. The methods show excellent agreement. Note that the MCMC constrains \( \mu \) (the ratio of planet to star mass), so masses in the figure are in units of

\[
M_{\oplus} \equiv M_{\oplus} \times \frac{M_p}{M_\star},
\]

which differs slightly from an Earth mass. Figure 4 compares \( N \)-body MCMC posteriors computed using our default priors and high-mass priors (see Section 3.1). The inferred planet masses are not strongly affected by the choice of priors.

Figure 5 shows the analytic constraint plots (Section 3.1) for the inner and outer planets. The MCMC result is roughly consistent with where the constraints from the fundamental and chopping components overlap. Hence those two components are primarily responsible for this system’s inferred masses and eccentricities.

Figure 6 illustrates that the combined eccentricity variable \( |\mathcal{Z}| \) is inferred much more accurately than the individual planets’ eccentricities (Section 2.2). The plot shows the posterior distributions of the individual planet eccentricities from the \( N \)-body MCMC, as well as that of \( |\mathcal{Z}| \). The eccentricities of planets b and c are essentially unconstrained by the TTVs. By contrast, the distribution of \( |\mathcal{Z}| \) is sharply peaked around \( |\mathcal{Z}| \sim 0.002 \). The situation illustrated by Figure 6 is typical of the MCMC results for all systems in this paper: only \( |\mathcal{Z}| \) is well constrained, not the individual planets’ eccentricities, which are strongly influenced by the priors.

The light curve of Kepler-307 shows a third (candidate) planet, KOI 1576.03, with a period of 23.34 days and radius of \( \sim 1.2 \) \( R_\oplus \), that we have ignored in our TTV modeling. The period of this candidate planet places it far from any low-order MMRs with the other two planets, and its influence on the

| Planet Pair | Resonance | \( \Delta \) | \( |\mathcal{Z}| \) (default) | \( |\mathcal{Z}| \) (high-mass) |
|-------------|-----------|-------------|-----------------|-----------------|
| Kepler-307 b/c | 5:4 | 0.0050 | 0.0020 -0.0004 | 0.0020 -0.0003 |
| Kepler-128 b/c | 3:2 | 0.0075 | 0.086 -0.033 | 0.015 -0.006 |
| Kepler-26 b/c | 7:5 | 0.0032 | 0.014 -0.001 | 0.010 -0.001 |
| Kepler-33 c/d | 5:3 | -0.0084 | 0.030 -0.013 | 0.016 -0.008 |
| Kepler-33 d/e | 3:2 | -0.0269 | 0.010 -0.004 | 0.009 -0.003 |
| Kepler-33 e/f | 9:7 | 0.0040 | 0.006 -0.002 | 0.005 -0.002 |

Note. Median combined eccentricities from \( N \)-body MCMC with both default and high-mass priors, along with 1\( \sigma \) uncertainties. The nearest first- or second-order MMR to the pair’s period ratio is given along with the pair’s distance to resonance, \( \Delta \) (i.e., Equation (1) for the first-order resonances and its obvious extension for the second-order ones: \( \Delta \equiv \frac{e - 2}{e^2} - 1 \)).
TTVs of Kepler-307 b and c should be negligible, especially given its small size.

### 3.3. Kepler-128 (KOI-274)

Kepler-128 b and c are a pair of super-Earth-sized planets with orbits that place them just wide of the 3:2 MMR ($\Delta = 0.0075$). The pair were confirmed as planets by Xie (2014) on the basis of their TTVs. The TTVs of Kepler-128 b and c are shown in Figure 7. A nonzero secondary component is present in addition to the fundamental TTV and causes the slight “skewness” in the otherwise sinusoidal TTVs.

For the N-body MCMC, an ensemble of 200 walkers was run for 250,000 iterations, saving every 200th iteration, resulting in $\sim$5,300 independent posterior samples, based on analysis of the Markov chains’ autocorrelation lengths. The planet mass constraints derived from MCMC are shown in Figure 8. Figure 9 compares MCMC results using the default and high-mass priors. The marginal mass posterior distributions are quite different for the two cases.

Figure 10 shows the analytic constraint plots (Section 3.1) for the inner and outer planets. The TTVs of both Kepler-128 b and c possess nonzero secondary components in addition to strong fundamental signals. The second-harmonic TTV measurements place an upper limit on $|Z|$, as shown in Figure 10. The
substantial difference between inferred masses for the different priors reflects the fact that the second-harmonic TTVs do not impose any strong upper limit on the masses. The second-harmonic component of planet c’s TTV favors a planet b mass of \( \lesssim 2M_\oplus \) (Figure 10, bottom panel), though the significance of this constraint is modest, and for the high-mass priors it is canceled out by the increased prior probability of smaller eccentricities. The lack of chopping TTVs provides weak 1\( \sigma \) upper bounds of \( M_b < 19.5M_\oplus \) and \( M_c < 10.5M_\oplus \). While the inferred masses and eccentricities of the Kepler-128 system depend strongly on the prior, the inferred ratio of their masses, \( M_b/M_c = 0.8 \pm 0.1 \), is essentially independent of which prior is used. Combining the mass ratio constraint with the transit radius measurements constrains the relative density to \( \rho_b/\rho_c = 0.7 \pm 0.15 \). Both the fundamental TTV phases, as discussed in LXW, and the nonzero second-harmonic TTV amplitudes strongly rule out \( |Z| = 0 \). We find \( |Z| > 0.025 \) (respectively, \( |Z| > 0.007 \)) at 99% confidence using the default (respectively, high-mass) priors.

One can derive an upper limit on planet masses by requiring that the planets have physically plausible bulk densities. The bulk density of Kepler-128 c, the denser of the two planets, provides the most restrictive limit since the planet’s mass ratio is well constrained. A pure iron composition for Kepler-128 c implies a mass of \( \sim 9M_\oplus \) according to the models of Fortney et al. (2007). This limit rules out some of the posterior distribution computed with the high-mass prior. Imposing a maximum mass of \( 9M_\oplus \) on Kepler-128 c requires eccentricities of \( |Z| > 0.008 \) based on planet b’s fundamental TTV amplitude (see Figure 10). If an Earth-like composition is assumed for planet c, then \( M_c \approx 3M_\oplus \) and \( |Z| \approx 0.025 \).

The radii of Kepler-128 b and c in Table 1 are around 20% larger than the radii found by Xie (2014). The transit-fitting

\[ \begin{align*}
\text{Figure 7.} & \text{TTVs of the Kepler-128 system (see Figure 2 for description).}
\end{align*} \]

\[ \begin{align*}
\text{Figure 8.} & \text{MCMC mass posterior for the Kepler-128 system (see Figure 3 for description).}
\end{align*} \]

\[ \begin{align*}
\text{Figure 9.} & \text{Comparison of MCMC priors for the Kepler-128 system (see Figure 4 for description).}
\end{align*} \]

\[ \text{13 The best-fit mass from chopping for planet c is } M_c = -27 \pm 29M_\oplus. \text{ But since the mass must be positive, we adjust this range in a crude way by incorporating the prior that } M_c > 0. \text{ The same is true for } M_b. \]
procedures used by Xie (2014) and Rowe et al. (2015) are quite similar, so the source of this discrepancy is unclear.

3.4. Kepler-26 (KOI-250)

Kepler-26 b and c are a pair of sub-Neptune-sized planets near the second-order 7:5 MMR. The planets were first confirmed by Steffen et al. (2012) on the basis of anticorrelated TTVs. Both planets’ TTVs, shown in Figure 11, show strong \( \delta S \) TTV amplitudes associated with their proximity to the 7:5 MMR, as well as fast frequency chopping.

For the \( N \)-body MCMC, an ensemble of 800 walkers was run for 250,000 iterations, saving every 800th iteration. The MCMC yielded \( \sim 57,000 \) independent posterior samples, based on analysis of the walkers’ autocorrelation lengths. Joint mass constraints for Kepler-26 b and c derived from both the \( N \)-body and analytic MCMCs are plotted in Figure 12. The analytic and \( N \)-body MCMC results show good agreement. Figure 13 shows that the inferred planet masses are marginally smaller assuming the alternate “high-mass” priors (see Section 3.1).

Figure 14 shows the analytic constraints plot for both planets. The combined constraints from the 7:5 MMR \( \delta S \) and chopping signals agree with the MCMC results. The posterior sample computed with high-mass priors is included in Figure 14 as yellow points. This posterior sample is concentrated at lower eccentricity and is almost completely disjointed from the posterior distribution computed with the default priors. This can be explained from the analytic TTV formulae: \( \delta S \) is a quadratic polynomial in \( \mu^2 \), so for any value of \( \mu \) there are different values of \( \mu^2 \) that give the same \( \delta S \) signal. Therefore there are two separate likelihood maxima in the \( \mu-[2] \) plane corresponding to the two roots. The different priors result in posterior distributions concentrated near one of the two minima.

The Kepler-26 system hosts two additional confirmed planets, Kepler-26 d and e. The periods of these two planets, \( P_d = 3.5 \) days and \( P_e = 46.8 \) days, place them far from planets b and c, and they are unlikely to have an appreciable influence on the TTVs of b and c given their sizes, \( R_d \sim 1.1 R_\oplus \) and \( R_e \sim 2.4 R_\oplus \).

3.5. Kepler-33 (KOI-707)

Kepler-33 hosts five planets confirmed by Lissauer et al. (2012) ranging in size from \( \sim 1.7 R_\oplus \) to \( \sim 5 R_\oplus \). We model only the TTVs of the outer four planets, ignoring the innermost planet, Kepler-33 b.\(^{14}\) The outer four planets are arranged in a closely packed configuration near a number of first- and second-order MMRs. Planets c and d lie near the second-order 5:3 MMR (\( \Delta = -0.008 \)). Planets d and e lie near a 3:2 MMR (\( \Delta = -0.027 \)), and the planets e and f are close to the 9:7 MMR (\( \Delta = 0.004 \)) and fall between the 4:3 and 5:4 MMRs (\( \Delta = -0.032 \) and +0.032, respectively). This configuration also places planets d and f somewhat near the 2:1 MMR with \( \Delta = -0.058 \). Figure 15 shows the TTVs of Kepler-33 and the best-fit \( N \)-body and analytic models.

For the \( N \)-body MCMC, an ensemble of 1,000 walkers were evolved for 300,000 iterations, saving every 800th iteration.

\(^{14}\) Kepler-33 b has a period of 5.67 days and a radius of \( 1.7 \pm 0.18 R_\oplus \). The relative distance of Kepler-33 b from any low-order MMRs with the other planets combined with its small size imply its influence on their TTVs should be negligible.
This resulted in ∼36,000 independent posterior samples based on analysis of the walker autocorrelation lengths. The planet mass constraints derived from MCMC for planets d, e, and f are plotted in Figure 16. The mass of the innermost planet, Kepler-33c, is poorly constrained, with the MCMC mainly providing an upper limit (see Figure 17). Figure 17 compares MCMC results using default and high-mass priors. The inferred masses of planets e and f are nearly unaffected by the choice of prior. The inferred masses of planet c and, to a lesser extent, d are sensitive to the assumed prior, indicating that these planets’ masses are not as constrained by the transit time data.

Analytic constraint plots for the Kepler-33 system are shown in Figure 18. The top row shows the masses and $|Z|$ of planets e and f. The MCMC results for planet e and f are explained well by the joint constraints derived from their mutual chopping and 9:7 $\delta t_5$ signals. The MCMC constraints for planet d and e are consistent with the constraint derived from their 3:2 fundamental TTV signals. The masses and $Z$ of planet d and e would be degenerate based solely on the observed $\delta t_5$ signals. However, the mass of planet e is already constrained by interactions with planet f. Since the mass of planet e is constrained, the combined eccentricity, $Z$, of planets d and e can be inferred from the fundamental signal in the TTV of planet d. With $Z$ constrained by the planet d fundamental TTV,
the mass of planet d is in turn constrained by the fundamental TTV signal it induces in planet e.

4. SUMMARY AND DISCUSSION

We have presented an analytic model for the TTVs of multiplanet systems and conducted N-body MCMC simulations to infer planet properties. The analytic constraints show good agreement with N-body fits and provide a clear explanation of the MCMC results. We also explore the degree to which planet masses derived from MCMC are insensitive to the assumed priors.

We summarize the key features of our analytic model:

1. We derive an analytic treatment of the influence of second-order MMRs on TTVs. The effects of second-order MMRs can help to constrain planet masses and eccentricities, both near first-order resonances, as in the case of Kepler-128 (Section 3.3), or planets near a second-order resonance, such as Kepler-26 (Section 3.4).

2. We identify the combined eccentricity, $Z$, as a key parameter in determining the TTV signal. Eccentricities of individual planets will rarely be constrained by TTVs alone. Extracting $|Z|$ from N-body fits provides a useful way to interpret the results.

3. The analytic constraint plots show that a simple linear least-squares fit can be used to derive approximate constraints from TTVs with minimal computational burden.

With the exception of the Kepler-128 system, the planets have low densities, likely less dense than water (Figure 1). These planets are new additions to the growing ranks of low-density sub-Neptune-sized planets that have been characterized via TTV observations. The density uncertainties for Kepler-33 e/f, Kepler-26 b/c, and Kepler-307 b/c, are all dominated by uncertain planet radii (Figure 1).
where they are accompanied by resonant denominators. Our formulae are meant to apply to the bulk of Kepler planets, but they will fail for planets close to a third (or higher) order MMR, or if the planets are librating in resonance.

We start with a detailed derivation of the case of a planet perturbed by an exterior companion, the results of which are in A.2.

A.1. Derivation (External Perturber)

Our notation mostly follows Murray & Dermott (2000). In particular, primed/unprimed variables refer to the outer/inner planet; \( \mathbf{r} \) is the (astrocentric) position vector; \( a, \lambda, e, \varpi \) are the (astrocentric) semimajor axis, mean longitude, eccentricity, and longitude of pericenter, respectively; and \( \alpha = a/a' \). Note, however, that we use \( \mu \) for \( m/M_* \), whereas Murray & Dermott use it for \( Gm \). All longitudes are referred to the observer’s line of sight (LOS).

A.1.1. From Orbital Elements (\( \delta z \) and \( \delta \lambda \)) to TTV (\( \delta t \))

The angular position of a planet relative to the LOS is \( \theta \); it is related to the orbital elements via \( \theta = \lambda + 2e \sin(\theta - \varpi) + O(e^2) \). It will prove convenient to replace the elements \( e \) and \( \varpi \) with the complex eccentricity \( z \) (Ogilvie 2007):

\[
  z \equiv e e^{i \varpi}
\]

implying

\[
  \theta = \lambda + \left( \frac{e^2}{i} e^{i \varpi} + \text{c.c.} \right),
\]

where “c.c.” means the complex conjugate of the preceding term, and we drop the \( O(e^2) \) terms because they are unaccompanied by any resonant denominators. We expand the orbital elements into their unperturbed Keplerian values plus perturbations due to the companion that is \( O(\mu') \):

\[
  a(t) = a_0 + \delta a(t)
\]
\[
  z(t) = z_0 + \delta z(t)
\]
\[
  \lambda(t) = \lambda_0 + \delta \lambda(t),
\]

where \( a_0 \) and \( z_0 \) are constant, and

\[
  \lambda_0 = n(t - T) = \frac{2\pi}{P}(t - T),
\]

expressed in terms of the constants \( n \), \( P \), and \( T \), which are respectively the mean motion, orbital period, and reference time. We write the times of transit as \( t_{\text{trans},0} + \delta t \), where \( \delta t \) is the TTV; that is, it is the \( O(\mu') \) perturbation in the transit time due to the companion. Setting \( \theta = 0 \) in Equation (14) then implies at \( O(\mu') \) (Nesvorný & Morbidelli 2008)

\[
  \delta t = -\frac{P}{2\pi} \left( \delta \lambda + \left( \frac{e^2}{i} \right) + \text{c.c.} \right)
\]

(15)

A.1.2. Equations of Motion

We shall solve perturbatively for \( \delta \lambda \) and \( \delta z \), which then give the TTV via Equation (15). The equations of motion for our preferred variables, \( \{a, z, \lambda\} \), are Hamilton’s equations for the
corresponding canonical variables (Ogilvie 2007):

\[
\frac{dz}{dt} = 2in' \mu' \frac{\partial R}{\partial \alpha} \frac{d\alpha}{dt}, \quad (16)
\]

\[
\frac{d \ln a}{dt} = 2n' \mu' \frac{\partial \alpha}{\partial \lambda} \frac{d \lambda}{dt}, \quad (17)
\]

\[
\frac{d\lambda}{dt} = \frac{n'}{\alpha^{3/2}} \left(1 - \frac{3 \delta a}{2a} - 2n' \mu' \frac{\partial R}{\partial \alpha} \frac{d \alpha}{dt}, \quad (18)
\]

where the bracketed term in \(d\lambda/dt\) comes from the partial derivative of the Keplerian Hamiltonian, expanded to first order in \(\mu'\), and we have dropped terms that are smaller by a factor \(O(e^2)\).\(^{15}\) The disturbing function is

\[
R \equiv -\frac{a'}{|r - r'|} - a' \cdot \frac{r'}{|r'|} = \sum_{j,k} R_{j,k} e^{i(j\lambda' + k\alpha)},
\]

for which the Fourier amplitudes \(R_{j,k}\) are given in Murray & Dermott (2000). For our purposes, the following terms suffice up to \(O(e^2)\):

\[
R_{j,-j} = f_{j1} - \delta j \frac{\alpha}{2}, \quad (20)
\]

\[
R_{j,1-j} = \frac{1}{2} \left( \left(f_{j1}^{*} + \delta j \frac{3}{2} \alpha - \delta j_1 \frac{1}{2} \alpha \right) \zeta^{\#} + (f_{j2}^{1} - \delta j_2 2\alpha) \zeta^{\#*} \right), \quad (21)
\]

\[
R_{j,-2j} = \frac{1}{2} \left( f_{j3}^{*} \zeta^{\#2} + f_{j2}^{*} \zeta^{\#*2} + f_{j3}^{0} - \delta j_3 \frac{27}{8} \alpha \zeta^{\#*2} \right), \quad (22)
\]

where the \(f_{j}^{m}\) are combinations of Laplace coefficients and their derivatives whose explicit form is listed in the appendix of Murray & Dermott (2000).\(^{16}\) Our \(R\) is related to Murray & Dermott’s \(\mathcal{R}\) via \(R = (a'/Gm)\mathcal{R}\).

A.1.3. Solutions for \(\delta z\) and \(\delta \lambda\)

The equations of motion are integrated by (1) replacing \(\lambda\) in the exponential with \(\lambda_{0}\) (after taking the derivative \(\partial R/\partial \lambda\)), which is valid to \(O(\mu')\), and (2) matching Fourier coefficients. The result is

\[
\delta z = \sum_{j,k} \tilde{z}_{j,k} e^{i(j\lambda_{0} + k\alpha)}, \quad \delta \lambda = \sum_{j,k} \lambda_{j,k} e^{i(j\lambda_{0} + k\alpha)},
\]

where

\[
\tilde{z}_{j,k} = 2\mu' \frac{\partial R_{j,k}}{\partial \alpha} \frac{n_{j,k}}{\sqrt{\alpha}}, \quad (23)
\]

\[
\lambda_{j,k} = \mu' \left( -\frac{3k}{\alpha^2} n_{j,k}^2 R_{j,k} - \frac{2}{i} n_{j,k} \sqrt{\alpha} \frac{\partial}{\partial \alpha} R_{j,k} \right), \quad (24)
\]

and we have defined

\[n_{j,k} \equiv \frac{n'}{jn' + kn}. \quad (25)\]

\[\]\(^{15}\)We have dropped a term from the right-hand side of Equation (18). In truth, one should replace \(\partial R/\partial \alpha \rightarrow \partial R/\partial \lambda \rightarrow -\partial R/\partial \alpha\). However, that term does not contribute to the TTV to the order of approximation at which we work. More precisely, its contribution near a \(J:J-1\) MMR is suppressed by the large factor \(n_{j,1-J}\) (see Section A.2).

\[\]\(^{16}\)We omit indirect terms with \(j = -1, 1, 2\) in Equation (22) because they will never appear with small denominators in Equation (34) or (35) below and can therefore be ignored for our purposes.

Note that \(n_{j,k}\) is related to \(\Delta\), the fractional distance to the nearest first-order \(J:J-1\) MMR defined in the body of the paper (Equation (1)), via

\[
n_{j,1-J} = -\frac{1}{J \Delta}, \quad (26)
\]

and hence is large near resonance.

The TTV is obtained by inserting \(\delta z\) and \(\delta \lambda\) into Equation (15) and evaluating at the times of transit, that is, setting \(\lambda_0 = 0\) in the exponent. (By setting \(\lambda_0 = 0\) we ignore order \(e\) corrections to \(\theta\) from Equation (14), which is acceptable because this correction is unaccompanied by a resonant denominator.) We find

\[
\delta t = \frac{P}{2\pi i} \sum_{j>0} e^{i\lambda_{0}} \left( \int_{j,k} \zeta^{*} - \zeta^{*} - i\delta \lambda_{j,k} \right) + \text{c.c.} \quad (27)
\]

To make \(j > 0\), we have rearranged terms, made use of the reality condition \((\lambda_{-j,k} = \lambda_{j,k}^{*})\), and dropped the \(j = 0\) term because it does not contribute to the TTV. Henceforth, \(j\) will be restricted to positive values.

A.2. Explicit TTV Formulae (External Perturber)

We simplify Equation (27) by expanding up to second order in \(e\):

\[
\delta t = \mu' \frac{P}{2\pi i} \sum_{j>0} e^{i\lambda_{0}} \left( t_{j0}^{(1)} + t_{j1}^{(1)} + t_{j2}^{(2)} \right) + \text{c.c.}, \quad (28)
\]

where \(t_{jm}^{(m)}\) is \(m\)th order in eccentricity. The time dependence enters only in the exponent \((\lambda_{0} \propto n't + \text{const.})\), and the \(t_{jm}^{(m)}\) depend on the \(a_0\) and \(z_0\) of the two planets. (Henceforth, we drop the subscript 0). We work out the third \(t_{j3}^{(m)}\) in turn:

1. \(t_{j0}^{(0)}\): The amplitude of an \(m\)th-order MMR \((j:j-m)\) is \(m\)th order in eccentricity, that is, \(R_{j-m,j} = O(e^2\sigma)\), where \(\sigma\) is either planet’s eccentricity (Equations (20)–(22)). Evaluating Equation (27) at zeroth order in \(e\) implies

\[
\mu' t_{j0}^{(0)} = \zeta_{j,1-j} - \zeta_{j,1-j}^{*} - i\delta \lambda_{j,1-j}. \quad (29)
\]

Both zeroth-order and first-order MMRs contribute to this expression: the former through terms in \(\delta \lambda_{j,1-j} \propto R_{j,1-j}\) and \(\partial R_{j,1-j}/\partial \alpha\), and the latter through \(\zeta_{j,1-j} \propto \partial R_{j,1-j}/\partial \alpha^{*}\). Inserting the expressions for \(\zeta\) and \(\lambda\) (Equations (23)–(24)) and then for \(R\) (Equations (20) and (22)) yields

\[
t_{j0}^{(0)} = \frac{2}{\sqrt{\alpha}^{*}} \left( n_{j,1-j} \frac{\partial R_{j,1-j}}{\partial \alpha^{*}} - n_{j,1-j} \left( \frac{\partial R_{j,1-j}}{\partial \alpha} \right)^{*} \right) - \frac{3j}{\alpha^{2}} n_{j,1-j}^{2} R_{j,1-j} + 2n_{j,1-j} \sqrt{\alpha} \frac{\partial}{\partial \alpha} R_{j,1-j} \quad (29)
\]

\[
= \frac{1}{\sqrt{\alpha}^{*}} \left( n_{j,1-j} f_{j2}^{1} + \delta j \frac{3}{2} \alpha - n_{j,1-j} \left( f_{j2}^{1} - \delta j \frac{1}{2} \alpha \right) \right) - \frac{3j}{\alpha^{2}} n_{j,1-j}^{2} - 2n_{j,1-j} \sqrt{\alpha} \frac{\partial}{\partial \alpha} R_{j,1-j} \quad (30)
\]

The quantities entering this expression are all roughly of order unity, with the possible exception of \(n_{j,1-j}\), which is large at \(j = J\) when the planets lie near a \(J:J-1\) MMR.
2. $t^{(1)}_j$: Following the same reasoning as above:

$$\mu' t^{(1)}_j = z_{j,2-j} - z^*_{j,2+j} - i\lambda_{j,1-j}.$$  

For most values of $j$, the $t^{(1)}_j$ are small $O(e)$ corrections to $t^{(0)}_j$. However, for a planet pair near a first-order J:J-1 MMR, the factor $n_{j,1-j}$ is large at $j = J$, and that factor can compensate for the smallness of $e$. Similarly, for a pair near a second-order K:K-2 MMR, the factor $n_{j,2-j}$ is large at $j = K$. We therefore approximate $t^{(1)}_j$ by keeping only terms that are potentially made large by proximity to an MMR:

$$t^{(1)}_j \approx t^{(1)}_{j,F} + t^{(1)}_{j,S}, \tag{31}$$

where

$$t^{(1)}_{j,F} = \frac{3(1-j)^2}{\alpha^2} n_{j,1-j} R_{j,1-j}$$

$$= \frac{3(1-j)^2}{2\alpha^2} n_{j,1-j} \left( f_{j}^{(1)} z^* + f_{j}^{(1)} z^* - 2\alpha j_{j,2} z^* \right)$$

$$t^{(1)}_{j,S} = \frac{2n_{j,2-j} \partial R_{j,2-j}}{\partial e^*}$$

$$= \frac{2n_{j,2-j}}{2\sqrt{\alpha}} \left( 2j_{45} z^* + f_{j}^{(1)} z^* \right), \tag{35}$$

where, at the risk of proliferation of subscripts, the $F$ component is potentially large near a first-order MMR, while the $S$ component is potentially large near a second-order MMR. Note also that we drop a term $\propto n_{j,1-j}$, since it will be much smaller than the $n_{j,1-j}$ in the $F$ component when either is important.

3. $t^{(2)}_j$: Equation (27) implies

$$\mu' t^{(2)}_j = -i\lambda_{j,2-j},$$

where we have ignored the $z$ terms because they can only be large if the planet pair is near a third-order MMR, a possibility we exclude. Again, only terms that are large near MMRs will make a significant contribution to the total TTV. Reasoning as before, we approximate

$$t^{(2)}_j \approx \frac{3(1-j)^2}{\alpha^2} n_{j,2-j} R_{j,2-j}$$

$$= \frac{3(1-j)^2}{2\alpha^2} n_{j,2-j} \left( f_{j}^{(1)} z^* + f_{j}^{(1)} z^* - 2\alpha j_{j,2} z^* \right)$$

$$+ \left( f_{j}^{(1)} - \delta_{j,3} \frac{27\alpha}{8} z^* \right). \tag{37}$$

To summarize, the TTV of a planet with an external perturber is given by Equation (28), with coefficients $t^{(m)}_j$ as listed in this subsection. In order to interpret observed TTVs, it is helpful to decompose the sum in Equation (28) into terms with distinct temporal frequencies, as described in Section 2, and also to drop all subdominant terms at a given frequency. We consider the two cases of relevance separately:

1. Companion near J:J-1 resonance: We decompose the sum as $\delta t = \delta t_F + \delta t_C + \delta t_S$, where the subscripts stand for fundamental, chopping, and secondary (see Equation (2)), where

$$\delta t_F = \mu' \frac{P}{2\pi i} \sum_{j>0} t^{(0)}_j e^{i\lambda_{j,0}} + c.c. \tag{38}$$

$$\delta t_C = \mu' \frac{P}{2\pi i} \sum_{j>0, j\neq 1} t^{(1)}_j e^{i\lambda_{j,0}} + c.c. \tag{39}$$

$$\delta t_S = \mu' \frac{P}{2\pi i} \sum_{j>0} t^{(2)}_j e^{i\lambda_{j,0}} + c.c. \tag{40}$$

At $O(e^2)$ (i.e., terms with superscript $0$), we transfer the $j = J$ term from the sum in Equation (39) to Equation (38) because it has the same frequency and can have comparable amplitude; at $O(e)$, we only include the $j = J$ and $j = 2J$ terms because they are the only ones with near-resonant denominators; and similarly at $O(e^3)$ we only include the $j = 2J$ term. Note that the $\delta t_F$ term has the longest period (given by Equation (3)) because the expressions are evaluated at the transit times of the inner planet ($\lambda_0 = 0$).

2. Companion near K:K-2 resonance, with K odd: We decompose the sum as $\delta t = \delta t_e + \delta t_S$ where

$$\delta t_e = \mu' \frac{P}{2\pi i} \sum_{j>0} (t^{(0)}_j e^{i\lambda_{j,0}} + c.c.) \tag{41}$$

$$\delta t_S = \mu' \frac{P}{2\pi i} (t^{(1)}_{K,S} + t^{(2)}_{K}) e^{i\lambda_{j,0}} + c.c. \tag{42}$$

A.3. Explicit TTV Formulae (Internal Perturber)

Thus far we have considered the case of an external perturber. Here we work through the case of an internal perturber. Since it is largely similar, we skip many of the details. The equations of motion (Equations (16)–(18)) become

$$\frac{dz'}{dt} = 2n' \mu \frac{\partial R}{\partial e^*} \tag{43}$$

$$\frac{d\ln a'}{dt} = 2n' \mu \frac{\partial R}{\partial \lambda'} \tag{44}$$

$$\frac{d\lambda'}{dt} \approx n' \left( 1 - \frac{3}{2} \frac{\partial a'(t)}{a'} \right) + 2n' \mu \left( 1 + \alpha \frac{\partial}{\partial a} \right) R. \tag{45}$$

The disturbing function is the same as before (Equations (19)–(22)), except for the indirect terms: the coefficients of the Kroenecker deltas are to be replaced by

$$R_{1,-1} = \frac{\alpha}{2} \longrightarrow \frac{1}{2\alpha^2} \tag{46}$$

$$R_{2,-1} = 2\alpha \longrightarrow \frac{1}{2\alpha^2} \tag{47}$$

$$R_{3,-1} = \frac{27\alpha}{8} \longrightarrow \frac{3}{8\alpha^2}. \tag{48}$$

The expansion in eccentricity (Equation (28)) becomes

$$\delta t' = \mu' \frac{P}{2\pi i} \sum_{j<0} (t^{(0)}_j + t^{(1)}_j + t^{(2)}_j) e^{i\lambda_{j,0}} + c.c. \tag{49}$$

Note that we choose here the sum to be over negative $j$ values as this allows the $t^{(m)}_j$ to be expressed in terms of the $R_{j,k}$ listed in Equations (20)–(22) (a sum over positive $j$ values would require the complex conjugates, $R_{j,k}^*$).
The coefficients are
\[ t_{j}^{(0)} = n_{1+j} - f_{31}^{(j+1)} - n_{-j} f_{31}^{(1-j)} \]
\[ + \left( 3j^{2} - 2n_{j} \right) \left( 1 + \frac{\partial}{\partial x_{j}} \right) \left( f_{j}^{2} - \delta_{j} \frac{1}{2\alpha x} \right) \]
(50)
\[ t_{j}^{(1)} \approx \left( n_{j+2} - f_{49}^{(j+2)} \right) + 2f_{53}^{(j+2)} z^{*} - \delta_{j} \frac{3}{4\alpha^{2}} z^{*} \]
\[ + \frac{3(j+1)}{2} n_{j+1,j} \times \left( f_{j}^{(j+2)} z^{*} + f_{j}^{(j+1)} z^{*2} - \delta_{j} \frac{1}{2\alpha x} z^{*} \right) \]
(51)
\[ t_{j}^{(2)} \approx \left( \frac{3(j+2)}{2} n_{j+2,j} \right) + \left( f_{j}^{(j+2)} z^{*2} - \delta_{j} \frac{3}{8\alpha^{2}} z^{*2} \right) \]
(52)
Finally, the decomposition into terms with distinct temporal frequencies is essentially the same as Equations (38)–(42), and after the appropriate replacements:
\[ \delta t_{j} = \mu \frac{P_{j}}{2\pi i} (t_{j}^{(0)} + t_{j}^{(1)} + t_{j}^{(2)}) e^{i(1-j)\lambda_{0}} + \text{c.c.}. \]
(53)
\[ \delta t_{j}^{\prime} = \mu \frac{P_{j}^{\prime}}{2\pi i} \sum_{j<0,j-1=j} t_{j}^{(0)} e^{i(j-k)\lambda_{0}} + \text{c.c.}, \]
(54)
\[ \delta t_{j}^{\prime} = \mu \frac{P_{j}}{2\pi i} (t_{-j}^{(1)} + t_{-j}^{(2)}) e^{i(2-k)\lambda_{0}} + \text{c.c.}, \]
(55)
where \( J \) and \( K \) still refer to the nearest \( J \)-\( J \)-1 or \( K \)-\( K \)-2 resonance (for \( J, K > 0 \), and for the case of a first-order MMR, \( K = 2J \)).

A.4. Simplified Dependence on \( Z \)

Here we reparameterize \( \delta t_{r} \) and \( \delta t_{s} \), which have a rather unwieldy dependence on \( z \) and \( z' \), in terms of the single variable \( Z \) introduced in Equation (5) by exploiting some approximate relationships between the \( f \) coefficients appearing in the TTV formulae. We carry out the derivation for a planet with an exterior companion; the derivation for planets with an interior companion is completely analogous, and we merely quote the final result. We assume that the planet is not near a 2:1 or 3:1 MMR since the TTV formulae near these MMRs are complicated by the contribution of indirect terms (see Section 2.2). Using the definition of \( Z \) from Equation (5), the eccentricity-dependent component of the fundamental TTV, \( t_{j}^{(1)} \) (Equation 32), can trivially be rewritten as
\[ t_{j}^{(1)} = \frac{3(1 - J)}{2\alpha x} n_{j-1,j} \sqrt{(f_{j+1,k}^{(2)})^{2} + (f_{j+1,k}^{(2)})^{2}} Z^{*}. \]
(56)

Next we reparameterize \( \delta t_{s} \) in terms of \( Z \). We first consider \( \delta t_{s} \) near a first-order \( J \)-\( J \)-1 MMR; the extension to second-order \( K \)-\( K \)-2 MMRs, described below, is trivial. The first step in simplifying \( \delta t_{s} \) is rewriting \( R_{2J,2-2J} \) as
\[ R_{2J,2-2J} = \frac{1}{2} \left( f_{j}^{(2)} z^{*2} + f_{j}^{(2)} + f_{j}^{(2)} z^{*2} \right) \approx \frac{1}{2} \gamma Z^{*2} \]
(57)
\[ \gamma = \frac{j_{k}^{2}}{2j_{k}^{2} f_{j}^{(2)}} + (f_{j}^{(2)})^{2} \]
(58)

Equations (57) and (58) warrant a few remarks. First, the approximation in Equation (57) expresses an apparently coincidental relationship between Laplace coefficients, namely \( f_{j}^{(2)} / f_{j}^{(2)} \approx f_{j}^{(2)} / f_{j}^{(2)} \approx f_{j}^{(2)} / f_{j}^{(2)} \). Thus, the coefficients of each of the quadratic terms in \( z \) and \( z' \) are equal or nearly equal in the left- and right-hand sides of Equation (57). Equation (57) is extended to second-order \( K \)-\( K \)-2 MMRs by replacing \( 2J \) with \( K \) and defining \( Z \) in terms of \( f_{j}^{(2)} \) and \( f_{j}^{(2)} \) (Equation 5) by taking \( J = [K/2] \), that is, \( K/2 \) rounded up to the nearest whole integer. The approximation matches the values of \( f_{j}^{(2)} \) and \( f_{j}^{(2)} \) with less than 2% fractional error for \( 5 \leq K \leq 11 \) and \( |\Delta| < 0.02 \). When Equation (57) is substituted in (34) and (36), \( t_{s}^{(1)} \) and \( t_{s}^{(2)} \) become
\[ t_{s}^{(1)} = \frac{2n_{k-2-k}^{2} (f_{j}^{(2)})^{2}}{\sqrt{f_{j}^{(2)} + (f_{j}^{(2)})^{2}}} Z^{*} \]
(59)
\[ t_{s}^{(2)} = \frac{3(2 - K)}{2\alpha x} n_{k-2-k}^{2} Z^{*2} \]
(60)
In Equations (38)–(40) we account for the eccentricity-dependent TTV contributions of only the nearest first or second MMRs, which we have parameterized in terms of \( Z \). In fact, to a good approximation, the contributions of all 17 first- and second-order MMRs depend on the planets' complex eccentricities only through the single combination, \( Z \). Additional eccentricity-dependent terms are increasingly important as the planet period ratio approaches unity and successive first- and second-order MMRs become more closely spaced. The TTV formulae can be generalized to incorporate the effects of additional first- and second-order MMRs by adding the appropriate \( t_{j}^{(1)}, t_{j}^{(1)}, \) and \( t_{j}^{(2)} \) terms, defined by Equations (32), (34), and (36), respectively, to the formulae. The additional terms can be expressed in terms of \( Z \) using Equations (56), (59), and (60) by simply replacing \( J \) and \( K \) (though, importantly, not in the definition of \( Z \) ) with the appropriate integer. This is because the ratio of the \( f \) coefficients that determines \( Z \), \( f_{j}^{(2)} / f_{j}^{(2)} \), is nearly independent of the integer \( j \) and is instead primarily determined by the period ratio of the planets (the ratio \( f_{j}^{(2)} / f_{j}^{(2)} \) varies with \( j \) less than 3% for \( 3 \leq j \leq 6 \) when evaluated at a fixed period ratio in the range \( 2/3 < P/P' < 5/6 \)). The combination of \( z \) and \( z' \) that appears in the contribution of a particular MMR to the TTV is determined mainly by the planets’ period ratio and depends only weakly on the particular MMR, allowing Equations (56), (59), and (60) to be used to approximate the TTV contribution of any and all nearby first- and second-order MMRs.

When the definition of \( Z \) and Equation (57) are inserted into (51) and (52), the components comprising the fundamental and secondary TTV of a planet with an interior perturber become
\[ t_{j}^{(1)} = \frac{3}{2} \gamma Z^{*2}. \]
(61)

\footnote{This excludes contributions of the 2:1 and 3:1 MMRs to the TTV because of the associated indirect terms. Planets near any other MMR will be far away from the 3:1 and 2:1 resonances, so the \( O(e) \) and \( O(e^2) \) contributions of these MMRs to the total TTV will be small.}
Table 3

| Nearest Resonance | $t^{(1)}_{-K,S}$ | $t^{(2)}_{-K,S}$ | $t^{(1)}_{-K}$ | $t^{(2)}_{-K}$ | $t^{(1)}_{1-K,S}/Z^a$ | $t^{(1)}_{4-K,S}/Z^a$ | $t^{(2)}_{1-K}/Z^{a^2}$ | $t^{(2)}_{4-K}/Z^{a^2}$ |
|------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 3:2 ($J = 3$)    | $-6.5$          | $-10.4$         | $-2.8 + \frac{0.8}{\Delta}$ | $2.5$           | $0.7$           | $0.3$           | $-1.8 \Delta^{-2}$ | $3.3 \Delta^1$  | $-3.9 \Delta^2$ |
| 7:5 ($K = 7$)    | $-10.7$         | $-13.5$         | $-18.7$         | $10.2$          | $0.8$           | ...             | $3.9 \Delta^{-1}$ | $-4.6 \Delta^2$ | $4.6 \Delta^{-1}$ |
| 4:3 ($J = 4$)    | $-16.0$         | $-17.6$         | $-16.7$         | $-4.2 + \frac{0.8}{\Delta}$ | $5.8$ | $1.9$ | $-1.8 \Delta^{-2}$ | $4.6 \Delta^{-1}$ | $-5.3 \Delta^{-2}$ |
| 9:7 ($K = 9$)    | $-22.6$         | $-22.7$         | $-18.3$         | $-31.4$         | $19.8$          | $4.4$           | ...             | $5.3 \Delta^{-1}$ | $-6.0 \Delta^{-2}$ |
| 5:4 ($J = 5$)    | $-30.6$         | $-28.7$         | $-21.2$         | $-24.6$         | $-5.6 + \frac{0.8}{\Delta}$ | $10.7$ | $-1.8 \Delta^{-2}$ | $6.0 \Delta^{-1}$ | $-6.7 \Delta^{-2}$ |

Note. Numerical values of various components of the analytic TTV formulae appearing in Equations (30), (56), (59), and (60). Each row of the table lists numerical values for components near a particular first- or second-order MMR. The numerical values are computed at the location of exact resonance. Most of the coefficients depend weakly on planet period ratio; a median fractional error of $\sim 8[\Delta]$ is incurred, though in some instances fractional errors can be in excess of $50[\Delta]$.

\[
\begin{align*}
T^{(1)}_{-K,S} & = 2n_{K,2-K} \frac{\gamma f^{(1)}_{33}}{\sqrt{(f^{(2)}_{33})^2 + (f^{(1)}_{33})^2}} Z^a, \\
T^{(2)}_{-K} & = \frac{3K}{2} \eta_{K,2-K} \gamma Z^{a^2},
\end{align*}
\]

(62)

Numerical values for the coefficients appearing in Equations (56), (59), and (60) and Equations (61)–(63) are listed in Table 3.

A.5. Mutual Inclinations

Here we briefly consider the influence of mutual inclinations on the TTV signal. As we shall see, the contribution of inclinations to the TTV is typically smaller than that of eccentricities (for comparable values of $e$ and $I$); hence we assume throughout the remainder of this paper that the planets are coplanar. We define our reference plane so that it contains the line of sight plus an arbitrary orthogonal direction in the plane of the sky. This coordinate system differs from the conventional coordinate system that takes the sky plane as the reference plane. The sky coordinate system is related to the inclination ($I$) and ascending node ($\Omega$) in our coordinate system as $\cos \theta_{sky} = \sin I \sin \Omega$. Any transiting planets have $|\sin I \sin \Omega| \leq R/a$ in our adopted coordinate system.

Inclinations only enter $R_{J,j}$ and $R_{K,j}$ through terms of order $e^2$ and higher so that $\delta_T$ and $\delta_{TT}$ are essentially unchanged for moderate values of mutual inclination. We need only consider the contributions of inclinations to second-order MMRs in our TTV formulae. To leading order, mutual inclinations introduce an additional term that should be added to the disturbing function coefficient $R_{J,j}$ (Equation (22)) and is given by

\[
R^{(\text{inc})}_{J,j} = \frac{1}{2} f^{(2)}_{j} \xi^{a^2},
\]

(64)

$$\xi = \sin(I/2)\exp(i\Omega) - \sin(I'/2)\exp(i\Omega').$$

(65)

Incorporating this term into the TTV formulae near second-order resonances is straightforward, and Equations (60) and (63) for the secondary TTV signals become

\[
T^{(2)}_{K} = \frac{3(2-K)}{2\alpha^2} \eta_{K,2-K} \gamma Z^{a^2} + f^{(2)}_{33} \xi^{a^2},
\]

(66)

\[
T^{(2)}_{-K} = \frac{3K}{2} \eta_{K,2-K} \gamma Z^{a^2} + f^{(2)}_{33} \xi^{a^2}.
\]

(67)

Equations (66) and (67) are compared with $N$-body simulations in Figure 19. The figure plots the secondary TTV amplitudes for two realistic planets near a $3:2$ MMR with $\Delta = 0.007$. Equations (66) and (67) somewhat overpredict the second-harmonic amplitude at large inclinations. We find that accurately reproducing $N$-body results at the higher inclinations requires additional terms in Equation (64) up to sixth order in inclination. The secondary TTV amplitudes predicted by analytic formulae using $R_{K,j}$ expanded to sixth order in inclination are also shown in Figure 19. In sum, we ignore the contribution of mutual inclinations to the secondary TTV because for $|\xi| \sim |Z|$ their contribution to $T_{K}$ will be smaller by a factor of $f^{(2)}_{33}/\gamma < 0.2$ (for $5 \leq K \leq 11$).

A.6. Comparison with Previous Derivations

A number of past studies have derived analytic TTV formulae (e.g., Agol et al. 2005; Nesvorný & Morbidelli 2008; Nesvorný 2009; Lithwick et al. 2012; Deck & Agol 2015a, 2015b; Agol & Deck 2016). Agol et al. (2005), Nesvorný & Vokrouhlický (2014), and Deck & Agol (2015a) all derive formulae for the TTV to zeroth order in eccentricity. Deck & Agol (2015a) express the TTV in terms of Fourier series as we have here; Equations (30) and (50) for the chopping component amplitudes are equivalent to their Equations (11) and (15). Nesvorný (2009) represents the most complete treatment to date, with formulae that compute perturbations to orbital elements to high order in eccentricities and inclinations by using a computer algebra code to include thousands of terms. Our focus here has been on isolating the dominant contributions to the TTV signals of planets near first- and second-order MMRs and providing simple explicit expressions for each component in terms of masses and orbital parameters. These expressions elucidate the role of each component in parameter inference, described in...
Section 2. Agol & Deck (2016) derive the TTV to first order in eccentricity. We have ignored any \( O(e) \) terms that are not potentially accompanied by a small resonant denominator. Deck & Agol (2015b) derive expressions for the second-harmonic/second-order resonance TTV amplitude; their paper was posted to arxiv.org shortly before this one. The second-harmonic/second-order resonance TTV formulae expressed by Equations (66) and (67) are shown in gray. The black points show second-harmonic amplitudes measured in TTVs generated by the \( N \)-body simulation. The red dashed lines show an analytic prediction obtained by including additional terms in the disturbing function coefficient, Equation (64), up to sixth order in inclinations.

**APPENDIX B**

**MCMC METHODS**

**B.1. MCMC with N-Body**

We model each planetary system as point masses orbiting a central star and compute midtransit times via \( N \)-body integration. We use the TTVFast code developed by Deck et al. (2014) to compute transit times. Planets are assumed to have coplanar orbits. We carry out MCMC analyses of each system to infer planet masses and orbits. The MCMC analyses of each multiplanet system are carried out using the EMCEE package’s (Foreman-Mackey et al. 2012) ensemble sampler. The EMCEE package employs the algorithm of Goodman & Weare (2010) to evolve an ensemble of “walkers” in parameter space, with each walker yielding a separate Markov chain of samples from the posterior distribution.

The parameters of the MCMC fits are each planet’s planet-to-star mass ratio, \( \mu_i \), eccentricity vector components \( h_i \equiv e_i \cos(z_i) \) and \( k_i \equiv e_i \sin(z_i) \), initial osculating period, \( P_i \), and initial time of transit, \( T_i \), where \( i = 1, 2, ..., N \) and \( N \) is the number of planets. Errors in the observed transit times are assumed to be independent and Gaussian with standard deviations given by the reported observational uncertainty so that the likelihood of any set of parameters is proportional to \( \exp(-\chi^2/2) \), where \( \chi^2 \) has the standard definition in terms of normalized, squared residuals:

\[
\chi^2 = \sum_{i=1}^{N} \sum_{j} \left( \frac{t_{\text{obs,}i}(j) - t_{N-\text{body,}i}(j)}{\sigma_i(j)} \right)^2,
\]

where the \( t_{\text{obs,}i}(j) \) are the observed transit times, indexed by \( j \), of the \( i \)th planet; \( \sigma_i(j) \) are their reported observational uncertainties; and \( t_{N-\text{body,}i}(j) \) are the transit times computed by \( N \)-body integration. We begin each MCMC ensemble by searching parameter space for a minimum in \( \chi^2 \) with a Levenberg–Marquardt (LM) least-squares minimization algorithm (e.g., Press et al. 1992). Transit time observations that fall more than \( 4\sigma \) away from the initial best fit, measured in terms of the reported uncertainty, are marked as outliers and removed from the data. We find that our MCMC results are largely insensitive to the removal of outliers, having experimented with fitting transit times with outliers included as well as more liberally removing poorly fit transit times. The new transit times are then refit with the LM algorithm, and an ensemble of walkers is initialized in a tight “ball” around the identified minimum. This is done by drawing the walkers’ initial positions from a multivariate Gaussian distribution based on the estimated covariance matrix generated by the LM algorithm.

We estimate the number of independent posterior samples generated by each MCMC run based on the autocorrelation length of each walker’s Markov chain. This is done as follows. First, for each walker in an ensemble, we compute the
autocorrelation functions

\[ \rho_1(\tau) = \frac{\langle X_i(s)X_i(s+\tau) \rangle - \langle X_i(s) \rangle^2}{\langle X_i(s)^2 \rangle - \langle X_i(s) \rangle^2}, \]  

where \( \langle \ldots \rangle \) denotes the average over sample number, \( s \), and the \( X_i \) denote the various model parameters, with \( i \) in the range \( i = 1 \ldots 5N \) for a system of \( N \) planets. We then take the autocorrelation length in each parameter to be the value of \( \tau \) at which \( \rho_1(\tau) \) decreases one \( e \)-folding: \( \rho_1(\tau) < e^{-1} \approx 0.37 \). We assign an autocorrelation length to each walker that is the maximum autocorrelation length, over all the \( 5N \) model parameters, in that walker’s Markov chain. Finally, the number of independent posterior samples generated by an individual walker during an MCMC run is taken to be the total number of samples in the chain divided by the walker’s autocorrelation length. The full posterior samples generated by each MCMC fit are available online.

For each system presented in Section 3, we ran MCMC simulations with two different priors: default and high-mass. Both priors are uniform in all planets’ periods, \( P_i \), and times of initial transit, \( T_i \). Furthermore, we assume that the prior probabilities of each planet’s mass and eccentricity are independent. Therefore the prior probability density for a set of \( M \) planet parameters, \( \theta \), of an \( N \)-planet system can be written as

\[
\text{Prob}(\theta) d\theta = \prod_{i=1}^{N} p(\mu_i)p(h_i, k_i) d\mu_i d\varpi_i dT_i dP_i, \quad (70)
\]

where \( p(\mu_i) \) and \( p(h_i, k_i) \) are the marginal prior probabilities in a planet’s mass and eccentricity components, respectively. The prior probability density, \( p(h_i, k_i) \), for a planet’s eccentricity components can be expressed in terms of the planet’s eccentricity, \( e_i \), and longitude of periapse, \( \varpi_i \), as (Ford 2006)

\[
p(h_i, k_i) d\varpi_i d\varpi_i = p(e_i \cos \varpi_i, e_i \sin \varpi_i) e_i d\varpi_i d\varpi_i, \quad (71)
\]

where the factor of \( e_i \) arises from the Jacobian of the coordinate transformation \((h_i, k_i) \to (e_i, \varpi_i)\). For our default prior we set

\[
p(\mu) \propto \begin{cases} 
\left( \mu + \mu_0 \right)^{-1}; & \mu \geq 0 \\
0; & \text{otherwise}
\end{cases}, \quad (72)
\]

\[
p(h, k) \propto \begin{cases} 
(h^2 + k^2)^{-1/2}; & h^2 + k^2 < 0.9 \\
0; & \text{otherwise}
\end{cases}, \quad (73)
\]

with \( \mu_0 = 3 \times 10^{-7} \). Equation (72) is a logarithmic prior, with the addition of a cutoff at small masses to prevent divergence at \( \mu \to 0 \). The inferred masses of all the planets in Section 3, with the exception of the poorly constrained Kepler-33 c, satisfy \( \mu \gg \mu_0 \). Equation (73) is uniform in eccentricity \((p(h, k) dhdk \propto \text{const} \times dde \) \( \varpi \)), but with an upper cutoff at 0.9 to avoid \( N \)-body integrations that require exceptionally small time steps. In the runs for this paper, the upper cutoff is unimportant because the posterior probabilities are negligible at such high \( e \) values.

For our high-mass prior we choose

\[
p(\mu) \propto \begin{cases} 
\text{const}; & \mu \geq 0 \\
0; & \text{otherwise}
\end{cases}, \quad (74)
\]

\[ p(h, k) \propto \begin{cases} 
(h^2 + k^2)^{-1/2}; & (h^2 + k^2)^{-1/2} < 0.9 \\
0; & \text{otherwise}
\end{cases} \quad (75)\]

with \( e_0 = 10^{-3} \), that is, uniform in mass and logarithmic in eccentricity, but with a cutoff below \( e_0 \). The cutoff at low \( e_0 \) is unimportant for the runs in this paper, aside from preventing divergence at \( e \to 0 \).

B.2. MCMC with Analytic Model

We also carry out full MCMC analyses of each TTV system using the analytic model. In two-planet systems, the TTVs of both planets are fit as a function of the planet-to-star mass ratios and the combined eccentricity, \( \mathcal{Z} \). We only include the pairwise interactions of adjacent planets when fitting the four planets of the Kepler-33 system (Section 3.5), ignoring Kepler-33 d and e’s proximity to a 2:1 MMR. We take the good agreement between the \( N \)-body and analytic MCMC results in Figure 16 as evidence that the 2:1 MMR does not play a significant role in constraining planet parameters. The analytic formulas give TTVs as a function of the planet-to-star mass ratios and the combined complex eccentricity, \( \mathcal{Z} \). This constitutes a significant reduction in the number of required model parameters required for TTV fitting: from the \( 5 \times N \) parameters, where \( N \) is the number of planets, required for a coplanar \( N \)-body fit (see Appendix B.1), to the parameters of the analytic model, one planet–star mass ratio for each planet considered and two components of \( \mathcal{Z} \) for each pairwise interaction considered.

To carry out MCMC fits with the analytic model, each planet’s transit times are first converted to TTVs. Converting transit times to TTVs requires determining a planet’s average period. Average periods are determined by fitting the transit times of planets near first-order MMRs as the sum of a linear trend plus sinusoidal terms with the frequencies expected for the principal and secondary TTV components. If a planet pair is near a second-order MMR, then the planets’ transit times are fit as the sum of a linear trend plus the secondary TTV component. Since the frequencies of the principal and secondary TTV signals depend on the planet periods, we fit the transit times of all planets in a system simultaneously with a nonlinear LM fit. The best-fitting linear trends are subtracted from the observed transit times to yield the TTVs fit by the MCMC.

The likelihood of a set of parameters in the analytic MCMC is computed from their \( \chi^2 \) value as in the \( N \)-body MCMC. The TTV of the inner planet is computed in the analytic MCMC according to Equation (28) by including \( t_j^{(2)} \) only for \( j = K \) where \( K:K \)-2 is the nearest second-order MMR (including \( K = 2J \) near a \( J:1 \) MMR) and including all \( t_j^{(0)} \) and \( t_j^{(1)} \) terms for \( 1 \leq j \leq K \). The term \( t_j^{(2)} \), as well as each \( t_j^{(1)} \), is a function of the variable \( \mathcal{Z} \) and is computed according to the approximations discussed in Appendix A.4. The TTV of the outer planet is computed similarly using Equation (49) with the terms \( t_j^{-K} \) and \( t_j^{(0)} \) and \( t_j^{(1)} \) for \( 1 \leq j \leq 2 - K \) included.

MCMC analyses using the analytic models are carried out using the \texttt{Kombine} MCMC code\textsuperscript{19} (B. Farr & W. M. Farr 2016, in preparation). \texttt{Kombine} is an ensemble sampler that

\texttt{http://home.uchicago.edu/~farr/kombine}
iteratively constructs a kernel-density-estimate-based proposal
distribution to approximate the target posterior distribution.
With Kombine, the proposal distribution is identical for each
Markov chain in the ensemble and is computed to approximate
the underlying posterior distribution, which allows independent
samples to be more rapidly generated than by EMCEE. We find
that the Kombine code fails to converge to a proposal
distribution with a high acceptance fraction when using the
N-body TTV model. Our analytic MCMC uses priors that are
uniform in $\log(\mu_i)$ and $|Z|$.

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