Generalizations of two-stack-sortable permutations

A Dissertation

Presented to

The Faculty of the Graduate School of Arts and Sciences

Brandeis University

Department of Mathematics

Professor Ira Gessel, Advisor

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

by

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September, 2002
Acknowledgment

My deepest gratitude goes to my advisor, Ira Gessel, without whom this work would not be possible; I would like to thank him for his generosity in sharing his insight and time with me, and his constant patience and encouragement. It was and will always be my pleasure to work with him.

I would like to thank Daniel Ruberman, Kiyoshi Igusa, Alan Mayer and Mark Adler for teaching me a lot of mathematics. I would like to thank Susan Parker for helping me to be a better teacher.

I’d also like to thank the Mathematics Department faculty and staff for making my life here more enjoyable.
ABSTRACT

Generalizations of two-stack-sortable permutations

A dissertation presented to the faculty of
the Graduate School of Arts and Sciences of
Brandeis University, Waltham, Massachusetts

by Dapeng Xu

In this thesis, we apply the stack sorting operator to $r$-permutations and construct the functional equation for the generating function of two-stack-sortable $k$-tuple $r$-permutations counted by descents by using a factorization similar to Zeilberger’s. We solve the functional equation and give explicit formulas for the number of two-stack-sortable $r$-permutations.
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CHAPTER 1

Introduction

The operation of stack sorting was first studied by Knuth [13, p. 239]. He described the operation by a railroad cars swapping algorithm. Later West described this operation in terms of a simple card game [27].

The operation uses a stack and can be described as follows. Let $\pi$ be a word with distinct letters in the alphabet $\{1, 2, 3, \ldots\}$. We call $\pi$ the input of the operation. At the $i$th step, we compare the $i$th letter $a$ of the word $\pi$ with the top letter on the stack (if any). If $a$ is smaller or the stack is empty, we put $a$ on the top of the stack. If $a$ is bigger, we move the top letter on the stack to the output and then again compare $a$ with the new top letter on the stack (if any). We repeat this until we put $a$ onto the stack. When the input is empty, we move all the letters on the stack (if any) to the output in order from top to bottom.

West also gave a recursive definition of the stack sorting operation. [27, 26].

Definition 1.0.1. Let $\pi = a_1a_2\cdots a_n$ be a word in the alphabet $\mathbb{P} = \{1, 2, \ldots\}$, having all its letters distinct. If $n = 0$, let $S(\pi)$ be the empty word. Otherwise, let $S(\pi)$ be obtained by permuting the letters of $\pi$ as follows: if $m = \max(a_1, a_2, \ldots, a_n)$ and $\pi = \pi_l m \pi_r$, then

$$S(\pi) = S(\pi_l)S(\pi_r)m.$$ 

It is easy to see that this recursive definition of the stack sorting operation is equivalent to the description in terms of stacks. For a given word $\pi = a_1a_2\cdots a_n$ in
the alphabet \( \mathbb{P} = \{1, 2, \ldots \} \), having all its letters distinct, we can always write it as \( \pi = \pi_l m \pi_r \) where \( m = \max(a_1, a_2, \ldots, a_n) \). According to the description, since \( m \) is bigger than any letter in \( \pi_l \), all the letters in \( \pi_l \) will be moved to the output before we can put \( m \) onto the stack. Also since \( m \) is bigger than any letter in \( \pi_r \), \( m \) will not be removed from the stack until all the letters in \( \pi_r \) go through the stack. That is exactly what the recursive definition says.

The problem of stack sorting was then generalized and researched in a number of ways. Among these, the problems of enumerating \( t \)-stack-sortable permutations interest most people. We say that a permutation \( \pi \) of \( [n] \) is a \( t \)-stack-sortable permutation if \( S^t(\pi) \) is the identity permutation, and that \( \pi \) is exactly \( t \)-stack-sortable if \( \pi \) is \( t \)-stack-sortable but not \((t - 1)\)-stack-sortable. Knuth [15, p. 239] proved that the number of \( 1 \)-stack-sortable permutations of \( [n] \) is the Catalan number \( C_n = \frac{(2n)}{n+1} \). West [26, 27] proved that all permutations of \( [n] \) are \((n - 1)\)-stack-sortable, that \((n - 2)! \) are exactly \((n - 1)\)-stack-sortable, and that \( \frac{7}{2}(n - 2)! + (n - 3)! \) are exactly \((n - 2)\)-stack-sortable.

Characterizations of \( 1 \)-stack-sortable permutations and \( 2 \)-stack-sortable permutations were also given using the following notion of pattern avoidance. Let \( \pi_1 = a_1a_2 \cdots a_n \) be a permutation of \( [n] \) and let \( \pi_2 = b_1b_2 \cdots b_k \) be a permutation of \( [k] \). We say that \( \pi_1 \) contains a type \( \pi_2 \) subsequence if there exists \( 1 \leq i_{b_1} < i_{b_2} < \cdots < i_{b_k} \leq n \) such that \( a_{i_1} < a_{i_2} < \cdots < a_{i_k} \). We say \( \pi_1 \) avoids \( \pi_2 \) if \( \pi_1 \) contains no \( \pi_2 \)-subsequence. For example, a permutation \( \pi \) avoids 231 if it cannot be written as \( \cdots a \cdots b \cdots c \cdots \) so that \( c < a < b \).

Tarjan [24] proved that a permutation is \( 1 \)-stack-sortable if and only if it avoids the pattern 231. West [26, 27] proved that a permutation fails to be \( 2 \)-stack-sortable if it contains a subsequence of type 2341 or a subsequence of type 3241 which is not
part of a subsequence of type 35241 and it is two-stack-sortable if it contains no such subsequence. West also conjectured the number of 2-stack-sortable permutations of \([n]\) to be 
\[
\frac{2(3n)!}{(n + 1)! (2n + 1)!}.
\]

The conjecture was first proved by D. Zeilberger \[28\]. Later, two bijections between two-stack-sortable permutations and non-separable planar graphs were given by S. Dulucq, S. Gire and O. Guibert \[7\]; I. P. Goulden and J. West \[11\]. More contributions to this problem were given by Miklós Bóna, Mireille Bousquet-Mélou, Leopold Travis and others \[2, 4, 5, 6, 16, 25\].

1.1. The generalization

In this thesis, we generalize the ordinary permutations to \(r\)-permutations \[13, 14, 16, 17, 18\] and then enumerate the number of 2-stack-sortable \(r\)-permutations under the generalized stack-sorting operation on \(r\)-permutations.

**Definition 1.1.1.** If a permutation \(a_1a_2\cdots a_{rn}\) of \(\{1^r, 2^r, \ldots, n^r\}\) satisfies the condition that if \(i < j < k\) and \(a_i = a_k\) then \(a_j \leq a_i\), we call it an \(r\)-permutation of \([n]\).

**Definition 1.1.2.** Let \(\pi\) be an \(r\)-permutation of \([n]\). For \(n > 0\), we can write \(\pi\) as \(\pi = \alpha_1n\alpha_2n\cdots n\alpha_{r+1}\). The stack-sorting operation \(S\) on \(r\)-permutations is defined by

\[
S(\pi) = S(\alpha_1)S(\alpha_2)\cdots S(\alpha_{r+1})n.
\]

When \(n = 0\), i.e., when \(\pi\) is an empty permutation, we define \(S(\pi)\) to be \(\emptyset\). Note that \(S(\pi)\) is an ordinary permutation.
Definition 1.1.3. Given an $r$-permutation $\pi$ and a letter $a$ in $\pi$, we call $a$ a type $i$ descent ($i = 1, \ldots, r$) if the $i$th occurrence of $a$ is immediately followed by a smaller letter, and we call $a$ a type 0 descent if the first occurrence of $a$ immediately follows a smaller letter. We denote the set of type $i$ descents of an $r$-permutation $\pi$ by $\pi^{(i)}$ ($i = 0, \ldots, r$).

Remark 1.1.4. Notice that the $r$-permutations are usually defined the other way, i.e., a permutation $a_1a_2\cdots a_n$ of $\{1^r, 2^r, \ldots, n^r\}$ is an $r$-permutation if it satisfies the condition that if $i < j < k$ and $a_i = a_k$, then $a_j \geq a_i$. The reason for the change is that we expect $S(\pi)$ to be an ordinary permutation for an $r$-permutation $\pi$, while the traditional definition of $r$-permutation can not give us that. For example, 123321 is a 2-permutation under the traditional definition, but $S(123321) = 121233$. This behavior is very different from what we study here.

Remark 1.1.5. Note that a type 0 descent is actually an ascent.

Remark 1.1.6. Often, the last letter (or, the last position) of a permutation is considered to be a descent but not here.

Example 1.1.7. When $r = 3$, $n = 6$,

$$\pi = 544453222335611166$$
is a 3-permutation of \(\{1, 2, \ldots, 6\}\). Then

\[
S(\pi) = S(54453222335)S(111)6
= S(444)S(322233)516
= 4S(222)3516
= 423516.
\]

The sets of type \(i\) descents \((i = 0, \ldots, 3)\) of \(\pi\) are \(\pi^{(0)} = \{6\}\), \(\pi^{(1)} = \{3, 5, 6\}\), \(\pi^{(2)} = \{5\}\) and \(\pi^{(3)} = \emptyset\).

The \(r\)-permutations can be represented as \((r + 1)\)-ary decreasing trees. If we have an \(r\)-permutation \(\pi\) of \([n]\) and \(\pi = \alpha_1 n \alpha_2 n \cdots n \alpha_{r+1}\), to get the tree representation of \(\pi\), we set \(n\) to be the root of the tree and recursively set the tree representation of \(\alpha_i\) \((i = 1, 2, \ldots, r+1)\) to be the \(i\)th child of the root. If \(\alpha_i\) is nonempty, then \(\pi\) has an \(i\)th child and \(n\) is a type \(i - 1\) descent. Therefore it is clear that each \(i\)th child corresponds to a type \(i - 1\) descent. For example, the 3-permutation above can be represented as in Figure 1.1.

There are three kinds of traversals of a tree: preorder, inorder, and postorder [23, p. 243; 15, p. 315]. In a preorder traversal of a tree, the root is visited first and then the subtrees rooted at its children are traversed recursively. A postorder traversal recursively traverses the subtrees rooted at the children of the root first, and then visits the root. An inorder traversal, in our case, recursively traverses the subtree rooted at the first child of the root, then visits the root, then recursively traverses the subtree rooted at the second child of the root, then visits the root, and so on. Hence, if we read the tree in Figure 1.1 in inorder, we get back the original \(r\)-permutation \(\pi\). If we read it in postorder, we get an ordinary permutation and it corresponds to \(S(\pi)\).
1. INTRODUCTION

Because any \((r + 1)\)-ary tree on \(n\) vertices has \(n - 1\) children, we have

**Lemma 1.1.8.** Every \(r\)-permutation of \([n]\) has a total of \(n - 1\) descents of types 0, 1, \ldots, \(r\).

1.2. Stack-sortable \(r\)-permutations

Now let us first consider the enumeration of 1-stack-sortable \(r\)-permutations (which we call here stack-sortable \(r\)-permutations) with all types of descents.

We weight any type \(i\) descent by \(x_i\). Therefore the weight of the \(r\)-permutation \(\pi\) in Example 1.1.7 is \(x_0x_1^3x_2\). Because of Lemma 1.1.8, we do not need a parameter to keep track of the number of different letters in an \(r\)-permutation.

For any unlabeled \((r + 1)\)-ary tree, there is one and only one way to label the nodes to make it an \((r + 1)\)-ary decreasing tree such that if we read it in postorder, we
get an identity ordinary permutation. That gives a bijection between stack-sortable $r$-permutations and unlabeled $(r + 1)$-ary trees. Therefore, if we weight an unlabeled $(r + 1)$-ary tree by weighting any $i$th child by $x_{i-1}$, then the total weight of stack-sortable $r$-permutations is the total weight of unlabeled $(r + 1)$-ary trees. Let $A(x)$ be the weight of all stack-sortable $r$-permutations. Then we have

$$A(x) = \prod_{i=0}^{r} (1 + x_i A(x))$$

where $x = (x_0, x_1, \ldots, x_r)$.

To solve this functional equation, we use Lagrange inversion [10, p. 21; 9].

**Lemma 1.2.1. (Multivariable Lagrange Inversion)** Let $f(\lambda) \in \mathbb{R}[[\lambda]]$ and $\phi_1(\lambda), \ldots, \phi_m(\lambda) \in \mathbb{R}[[\lambda]]$ where $\lambda = (\lambda_1, \ldots, \lambda_m)$. Suppose that $w_i = t_i \phi_i(w)$ for $i = 1, \ldots, m$, where $w = (w_1, \ldots, w_m)$. Let $\phi = (\phi_1, \ldots, \phi_m)$, $t = (t_1, \ldots, t_m)$ and $k = (k_1, \ldots, k_m)$. Then

$$f(w(t)) = \sum_k t^k [\lambda^k] \left\{ f(\lambda) \phi^k(\lambda) \left| \delta_{ij} - \frac{\lambda_j}{\phi_i(\lambda)} \frac{\partial \phi_i(\lambda)}{\partial \lambda_j} \right. \right\},$$

where $[\lambda^k] f(\lambda)$ is the coefficient of $\lambda_1^{k_1} \cdots \lambda_m^{k_m}$ of the formal power series $f(\lambda)$.

In our case, we let $w_i = x_i A(x)$ and $\phi_i(\lambda) = \prod_{j=0}^{r} (1 + \lambda_j)$ for $i = 0, \ldots, r$. Then $w_i$ satisfy the condition that $w_i = x_i \phi_i(w)$. Also let $f(\lambda) = \prod_{j=0}^{r} (1 + \lambda_j)$. Then $f(w) = A(x)$. Using multivariable Lagrange inversion, we get

$$A(x) = f(w)$$

$$= \sum_k x^k [\lambda^k] \left\{ \left( \prod_{i=0}^{r} (1 + \lambda_i) \right)^{k_0 + \cdots + k_r + 1} \left| \delta_{ij} - \frac{\lambda_j}{1 + \lambda_j} \right. \right\}. $$

Now we introduce some notation. Let

$$(1.2.1) \quad e_k(x) = \sum_{0 \leq i_1 < i_2 < \cdots < i_k \leq r} x_{i_1} x_{i_2} \cdots x_{i_k}$$
be the $k$th elementary symmetric function of $x = (x_0, x_1, \ldots, x_r)$. Define

\begin{equation}
E(x, u) = E(x_0, x_1, \ldots, x_r; u) = \prod_{i=0}^{r} (1 + x_i u) = \sum_{i=0}^{r} e_i(x) u^i
\end{equation}

\section*{Lemma 1.2.2} Let $M(x) = |\delta_{ij} + (\delta_{ij} - 1)x_j|$, where $0 \leq i, j \leq r$, be the determinant of the $(r + 1) \times (r + 1)$ matrix. Then

\[ M(x) = E(x) \left( 1 - \sum_{i=0}^{r} \frac{x_i}{1 + x_i} \right). \]

**Proof.**

\[
M(x) = \begin{vmatrix}
1 & -x_1 & -x_2 & \cdots & -x_{r-1} & -x_r \\
-x_0 & 1 & -x_2 & \cdots & -x_{r-1} & -x_r \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
-x_0 & -x_1 & -x_2 & \cdots & 1 & -x_r \\
-x_0 & -x_1 & -x_2 & \cdots & -x_{r-1} & 1
\end{vmatrix}.
\]
Subtracting the last row from every other row, we get

\[ M(x) = \begin{pmatrix}
1 + x_0 & 0 & 0 & \cdots & 0 & -(1 + x_r) \\
0 & 1 + x_1 & 0 & \cdots & 0 & -(1 + x_r) \\
0 & 0 & 1 + x_2 & \cdots & 0 & -(1 + x_r) \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 1 + x_{r-1} & -(1 + x_r) \\
-x_0 & -x_1 & -x_2 & \cdots & -x_{r-1} & 1
\end{pmatrix}.
\]

If we let \( x' = (x_1, x_2, \ldots, x_r) \), then by expanding the first column, we get

\[ M(x) = (1 + x_0)
\begin{pmatrix}
1 + x_1 & 0 & \cdots & -(1 + x_r) \\
0 & 1 + x_2 & \cdots & -(1 + x_r) \\
\cdots & \cdots & \cdots & \cdots \\
-x_1 & -x_2 & \cdots & 1
\end{pmatrix}
- (-1)^r x_0
\begin{pmatrix}
0 & 0 & \cdots & 0 & -(1 + x_r) \\
1 + x_1 & 0 & \cdots & 0 & -(1 + x_r) \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & 1 + x_{r-1} & 1
\end{pmatrix}
\]

\[ = (1 + x_0)M(x') - (-1)^r x_0 (-1)^r \prod_{i=1}^{r}(1 + x_i)
\]

\[ = (1 + x_0)M(x') - x_0 \prod_{i=1}^{r}(1 + x_i).
\]

Now we use induction. It is easy to see that the lemma is true when \( r = 0 \). Suppose the lemma is true for \( M(x') \), i.e.,

\[ M(x') = E(x') \left( 1 - \sum_{i=1}^{r} \frac{x_i}{1 + x_i} \right).
\]
Then
\[ M(x) = (1 + x_0)E(x') \left( 1 - \sum_{i=1}^{r} \frac{x_i}{1 + x_i} \right) - x_0 \prod_{i=1}^{r} (1 + x_i) \]
\[ = E(x) \left( 1 - \sum_{i=1}^{r} \frac{x_i}{1 + x_i} \right) - E(x) \frac{x_0}{1 + x_0} \]
\[ = E(x) \left( 1 - \sum_{i=0}^{r} \frac{x_i}{1 + x_i} \right). \]

The lemma is proved. \(\square\)

With Lemma 1.2.2, it follows that
\[
A(x) = \sum_{k} x^k [\lambda^k] \left\{ \left( \prod_{i=0}^{r} (1 + \lambda_i) \right)^{k_0 + \cdots + k_r + 1} \left( \sum_{i=0}^{r} \frac{1}{1 + \lambda_i} - r \right) \right\}
\]
\[ = \sum_{k} x^k \left\{ \sum_{i} \binom{n}{k_i} \cdots (n - 1) \cdots \binom{n}{k_r} - r \prod_{i} \binom{n}{k_i} \right\}
\[ = \sum_{k} x^k \left\{ \left( \sum_{i} \frac{n - k_i}{n} - r \right) \prod_{i} \binom{n}{k_i} \right\}
\[ = \sum_{k} x^k \frac{1}{n} \prod_{i=0}^{r} \binom{n}{k_i} \]

where \(n = 1 + k_0 + \cdots + k_r\). Therefore,

**Theorem 1.2.3.** The number of stack-sortable \(r\)-permutations with \(k_i\) descents of type \(i\) is

\[
\frac{1}{n} \prod_{i=0}^{r} \binom{n}{k_i}
\]

where \(n = 1 + k_0 + \cdots + k_r\).
1.2. STACK-SORTABLE $r$-PERMUTATIONS

**Remark 1.2.4.** This functional equation can also be solved by one variable Lagrange inversion by introducing a new variable, i.e., letting

$$A(z) = z \prod_{i=0}^{r}(1 + x_i A(z)).$$

**Remark 1.2.5.** These numbers also come up in counting noncrossing partitions [21, 20, 8].

**Remark 1.2.6.** When $r = 1$, these numbers are Narayana numbers [22, 12].
CHAPTER 2

Two-stack-sortable \( r \)-permutations

2.1. The functional equations

Now let us consider the case of two-stack-sortable \( r \)-permutations. An \( r \)-permutation \( \pi \) of \([n]\) is two-stack-sortable if \( S^2(\pi) \) is an identity permutation. Let \( \pi = \alpha_1 n\alpha_2 n \cdots n\alpha_{r+1} \) where \( n \) is the largest element in \( \pi \). Then

\[
S(\pi) = S(\alpha_1) S(\alpha_2) \cdots S(\alpha_{r+1}) n
\]

\[
S^2(\pi) = S(S(\alpha_1) S(\alpha_2) \cdots S(\alpha_{r+1})) n
\]

Denote the identity permutation by \( I \). (Here we abuse the notation by letting \( I \) be the identity permutation of \([n]\) for any positive integer \( n \).) One can notice that if \( S^2(\pi) = I \) and if \( m \) is the largest number in \( \alpha_1 \alpha_2 \cdots \alpha_{r+1} \), then \( m \) can occur in only one of the \( \alpha_i \) because if \( m \) appeared in two of the \( \alpha_i \), the \( r \)-permutation condition for \( \pi \) would be violated. Thus \( (\alpha_1, \alpha_2, \ldots, \alpha_{r+1}) \) is an \( (r+1) \)-tuples of \( r \)-permutations satisfying the condition that

\[
S(S(\alpha_1) S(\alpha_2) \cdots S(\alpha_{r+1})) = I.
\]

This suggests that we study the subject defined as follows.

**Definition 2.1.1.** \( (\alpha_1, \alpha_2, \ldots, \alpha_k) \) is a \( k \)-tuple \( r \)-permutation of \([n]\) if \( \alpha_1 \alpha_2 \cdots \alpha_k \) is an \( r \)-permutation of \([n]\) and any letter appears in only one of the \( \alpha_i \). Also we call \( \alpha_i \) the \( i \)th component of the \( k \)-tuple \( r \)-permutation.
We call a $k$-tuple $r$-permutation $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ a two-stack-sortable $k$-tuple $r$-permutation if it satisfies the condition that

$$S(S(\alpha_1)S(\alpha_2) \cdots S(\alpha_k)) = I.$$  

From the definition, we know that each component $\alpha_i$ of a $k$-tuple $r$-permutation is itself an $r$-permutation so that we can interpret a $k$-tuple $r$-permutation of $[n]$ as a forest of $k$ $(r + 1)$-ary decreasing trees. The set of the descents of type $i$ of $k$-tuple $r$-permutations $\pi = (\alpha_1, \ldots, \alpha_k)$ is the union of the sets of descents of type $i$ of $\alpha_j$ ($j = 1, \ldots, k$), i.e.

$$\pi^{(i)} = \bigcup_{j=1}^{k} \alpha_j^{(i)}.$$  

For example, let $\pi = (\alpha_1, \alpha_2)$ where $\alpha_1 = 553111335$ and $\alpha_2 = 4422477667$. Then $(\alpha_1, \alpha_2)$ is a 2-tuple 3-permutation of $\{1, 2, \ldots, 7\}$ and it can be represented as in Figure 2.1. Since

$$S(S(\alpha_1)S(\alpha_2)) = S(1352467)$$  

$$= 1324567,$$

$\pi$ is not two-stack-sortable. The sets of descents of $\pi$ are $\pi^{(0)} = \{7\}$, $\pi^{(1)} = \{3\}$, $\pi^{(2)} = \{4, 5, 7\}$, $\pi^{(3)} = \emptyset$. 

![Forest representation of $\pi$](image-url)
2.1. THE FUNCTIONAL EQUATIONS

Let \( g_n^{(k)} \) be the sum of the weights of two-stack-sortable \( k \)-tuple \( r \)-permutations of \( n \) different letters such that every component is nonempty. We need every component to be nonempty so that we can keep track of every type of descent. Suppose \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) is a two-stack-sortable \( k \)-tuple \( r \)-permutation of \([n]\) in which every component is nonempty and the largest element \( n \) appears in \( \alpha_l \). Let \( \alpha_l = \beta_1 n \beta_2 n \cdots n \beta_{r+1} \).

Then

\[
\begin{align*}
I &= S(S(\alpha_1)S(\alpha_2) \cdots S(\alpha_k)) \\
&= S(S(\alpha_1)S(\alpha_2) \cdots S(\alpha_{l-1})S(\beta_1)S(\beta_2) \cdots S(\beta_{r+1}) n S(\alpha_{l+1}) \cdots S(\alpha_k)) \\
&= S(S(\alpha_1)S(\alpha_2) \cdots S(\alpha_{l-1})S(\beta_1)S(\beta_2) \cdots S(\beta_{r+1})) S(S(\alpha_{l+1}) \cdots S(\alpha_k)) n.
\end{align*}
\]

Therefore, \((\alpha_1, \ldots, \alpha_{l-1}, \beta_1, \ldots, \beta_{r+1})\) and \((\alpha_{l+1}, \ldots, \alpha_k)\) are both two-stack-sortable tuples of \( r \)-permutations.

Let us first consider the case when \( l = 1 \). By (2.1.1), we have

\[
I = S(S(\beta_1)S(\beta_2) \cdots S(\beta_{r+1})) S(S(\alpha_2) \cdots S(\alpha_k)) n.
\]

Suppose there are \( n - i \) letters (including \( n \)) in \( \alpha_1 \). Then the weight of \((\alpha_2, \alpha_3, \ldots, \alpha_k)\) is a term of \( g_i^{(k-1)} \). For \((\beta_1, \ldots, \beta_{r+1})\), if all \( \beta_j \) are nonempty, then the weight of \((\beta_1, \ldots, \beta_{r+1})\) is a term of \( g_{n-1-i}^{(r+1)} \). But by the definition of the descents of \( r \)-permutations, there is a descent of each type in \( \alpha_1 \) for every appearance of \( n \). So the weight of \( \alpha_1 \) is \( x_0 x_1 \cdots x_r \) times the weight of \((\beta_1, \ldots, \beta_{r+1})\). Therefore the sum of the weights of all such \( \alpha_1 \) is \( e_{r+1}(x) g_{n-1-i}^{(r+1)} \). If only one of the \( \beta_j \) is empty, then the weight of \((\beta_1, \ldots, \beta_{r+1})\) is a term of \( g_{n-1-i}^{(r)} \). Again, by the definition of the descents of \( r \)-permutations, there
are \( r - 1 \) descents of different types for \( r - 1 \) appearances of \( n \). So, if \( \beta_{j_0} \) is empty but \( \beta_j \) is not empty for \( j \neq j_0 \), then the weight of \( \alpha_1 \) is \( x_0 x_1 \cdots \hat{x}_{j_0} \cdots x_r \) times the weight of \( (\beta_1, \ldots, \beta_{r+1}) \). Therefore the sum of the weights of all such \( \alpha_1 \) is \( e_r(x)g_{n-1-i}^{(r)} \). In general, if there are \( m \) of the \( \beta_j \) that are nonempty, then the sum of the weights of all such \( \alpha_1 \) is \( e_m(x)g_{n-1-i}^{(m)} \). Therefore, the sum of weights of all the \( k \)-tuple \( r \)-permutation 

\[(\alpha_1, \alpha_2, \ldots, \alpha_k)\] in which \( n \) is in \( \alpha_1 \) is

\[
\sum_{i=0}^{n-1} g_i^{(k-1)} \left[ g_{n-1-i}^{(0)} + g_{n-1-i}^{(1)} e_1(x) + \cdots + g_{n-1-i}^{(r+1)} e_{r+1}(x) \right],
\]

For the same reason, when \( n \) is in \( \alpha_2 \), we have

\[
I = S(S(\alpha_1)S(\beta_1)S(\beta_2) \cdots S(\beta_{r+1})) S(S(\alpha_3) \cdots S(\alpha_k)) n.
\]

Suppose there are \( n - i \) letters in \((\alpha_1, \alpha_2)\). Then the sum of the weights of all the \( k \)-tuple \( r \)-permutation \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) in which \( n \) is in \( \alpha_2 \) is

\[
\sum_{i=0}^{n-1} g_i^{(k-2)} \left[ g_{n-1-i}^{(1)} + g_{n-1-i}^{(2)} e_1(x) + \cdots + g_{n-1-i}^{(r+2)} e_{r+1}(x) \right],
\]

and so on. Thus we have the crucial recursive formula

\[
(2.1.2) \quad g_n^{(k)} = \sum_{i=0}^{n-1} \sum_{l=0}^{k-1} \sum_{j=0}^{r+1} g_{n-i-l}^{(l+j)} e_j(x)
\]

with the initial conditions that \( g_0^{(k)} = \delta_{0k} \) and \( g_n^{(0)} = \delta_{n0} \).

Now let

\[
(2.1.3) \quad f_n^{(k)} = \sum_{j=0}^{r+1} g_n^{(k+j)} e_j(x).
\]

Then

\[
(2.1.4) \quad g_n^{(k)} = f_{n-1}^{(k-1)} + \sum_{i=0}^{n-1} f_{n-1-i}^{(k-2)} g_i^{(1)} + \cdots + \sum_{i=0}^{n-1} f_{n-1-i}^{(0)} g_i^{(k-1)}.
\]
Following the analysis above one can easily see that $f_n^{(k)}$ is the sum of the weights of $(k + 1)$-tuple $r$-permutations $(\alpha_1, \alpha_2, \ldots, \alpha_{k+1})$ of $[n + 1]$ such that the $\alpha_i$ are all nonempty, $S(S(\alpha_1)S(\alpha_2) \cdots S(\alpha_{k+1})) = I$ and the largest letter only appears in $\alpha_{k+1}$.

Also let

\begin{align}
(2.1.5) & \quad f_k = \sum_{n=0}^{\infty} f_n^{(k)}, \quad F = \sum_{k=0}^{\infty} f_k z^k, \\
(2.1.6) & \quad g_k = \sum_{n=0}^{\infty} g_n^{(k)}, \quad G = \sum_{k=0}^{\infty} g_k z^k.
\end{align}

Then it follows that

\begin{align}
(2.1.7) & \quad f_k = g_k + g_{k+1} e_1(x) + \cdots + g_{r+k+1} e_{r+1}(x), \\
(2.1.8) & \quad g_k = f_{k-1} g_0 + f_{k-2} g_1 + \cdots + f_0 g_{k-1},
\end{align}

and from (2.1.8), we get

\begin{equation}
(2.1.9) \quad G = 1 + zFG.
\end{equation}

Notice that if we let $r$ go to infinity, we can interpret $k$-tuple $r$-permutations in terms of forests of decreasing trees and equations (2.1.7) to (2.1.9) are still well defined. Therefore, we have the functional equations as follows.
Theorem 2.1.2. Let $F$ and $G$ be defined as in (2.1.3) and (2.1.4). Then $F$ and $G$ satisfy the functional equations

\[(2.1.10) \quad f_k = \sum_{j=0}^{\infty} g_{k+j} e_j(x) \text{ for } k \geq 0,\]

\[(2.1.11) \quad G = 1 + zFG\]

and $F$ and $G$ are uniquely determined as power series by these functional equations.

Remark 2.1.3. From (2.1.4), we see that $g^{(k)}_n$ are uniquely determined by $f^{(j)}_i$ and $g^{(j)}_i$ where $i < n$. From (2.1.3), we see that $f^{(k)}_n$ are uniquely determined by $g^{(j)}_n$. Therefore, given $g^{(0)}_0 = 1$ and $g^{(k)}_0 = 0$ when $k \geq 1$, all $g^{(k)}_n$ and $f^{(k)}_n$ are uniquely determined by (2.1.3) and (2.1.4). Hence, $F$ and $G$ are uniquely determined by the functional equations in Theorem 2.1.2.

Remark 2.1.4. Notice that $g_1 = f_0$ is the generating function of two-stack-sortable $r$-permutations.

Remark 2.1.5. Notice that in the ring $\mathbb{Q}[[z, \frac{z}{x}, \frac{z}{y}, \frac{z}{x^2}, \ldots]]$,

\[G \cdot E \left( x, \frac{1}{z} \right) = \sum_{k=-\infty}^{\infty} \sum_{j=0}^{\infty} g_{k+j} e_j(x) z^k.\]

Therefore, by (2.1.10), $F - G \cdot E \left( x, \frac{1}{z} \right)$ has only negative powers of $z$. This is the key to solving the functional equations (2.1.10) and (2.1.11).

2.2. The solution to the functional equations

Let $y = (y_0, y_1, y_2, \ldots)$, where the $y_i$ are uniquely determined as formal power series in the $x_i$ by

\[x_j = \frac{y_j(1 + y_j)}{\prod_{i=0}^{\infty}(1 + y_i)^2} = \frac{y_j(1 + y_j)}{E^2(y)}.\]
Let \( t = zE^2(y) \) where \( E(y) \) is defined as in (1.2.2) and (1.2.3). Let \( c(t) = \frac{1 - \sqrt{1 - 4t}}{2t} \) be the generating function of the Catalan numbers. Then

Theorem 2.2.1. The solution to the functional equations in Theorem 2.1.2 is:

\[(2.2.1) \quad G = \frac{c(t)E(y)}{E(y, c(t))}, \]

\[(2.2.2) \quad F = \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right). \]

The proof of the theorem consists of the following lemmas.

Lemma 2.2.2. In the ring \( \mathbb{Q}[[z, \frac{x_0}{z}, \frac{x_1}{z}, \ldots]] \),

\[ E \left( x, \frac{1}{z} \right) = E(y, c(t))E \left( y, \frac{1}{tc(t)} \right) \]

Proof. First, we notice that \( E \left( x, \frac{1}{z} \right) \in \mathbb{Q}[[z, \frac{x_0}{z}, \frac{x_1}{z}, \ldots]] \) and \( E \left( y, \frac{1}{tc(t)} \right) \in \mathbb{Q}[[t, y_0, y_1, \ldots]] \). It is also easy to verify that \( \mathbb{Q}[[z, \frac{x_0}{z}, \frac{x_1}{z}, \ldots]] = \mathbb{Q}[[t, y_0, y_1, \ldots]] \).

Therefore, since \( c(t) \) satisfies \( tc^2(t) - c(t) + 1 = 0 \),

\[(1 + c(t)y_i) \left( 1 + \frac{y_i}{tc(t)} \right) = 1 + c(t)y_i + \frac{y_i}{tc(t)} + \frac{y_i^2}{t} \]

\[= 1 + \frac{y_i + y_i^2}{t} \]

\[= 1 + \frac{x_i}{z}. \]
Hence,

\[ E \left( x, \frac{1}{z} \right) = \prod_{i=0}^{\infty} \left( 1 + \frac{x_i}{z} \right) \]

\[ = \prod_{i=0}^{\infty} \left( 1 + \frac{y_i}{t} \right) \]

\[ = \prod_{i=0}^{\infty} (1 + c(t)y_i) \left( 1 + \frac{y_i}{tc(t)} \right) \]

\[ = E(y, c(t))E \left( y, \frac{1}{tc(t)} \right). \]

\[ \square \]

**Lemma 2.2.3.** Let

\[ \hat{G} = \frac{c(t)E(y)}{E(y, c(t))}, \]

\[ \hat{F} = \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right). \]

Then \( \hat{G} \) and \( \hat{F} \) satisfy

\[ \hat{G} = 1 + z\hat{F}\hat{G}. \]

**Proof.**

\[ 1 - z\hat{F} = 1 - z \cdot \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right) \]

\[ = 1 - \frac{zE^2(y)}{t} \left( 1 - \frac{E(y, c(t))}{c(t)E(y)} \right) \]

\[ = \frac{E(y, c(t))}{c(t)E(y)} \]

\[ = \frac{1}{G} \]

\[ \square \]
Lemma 2.2.4. In the ring $\mathbb{Q}[[z, \frac{x_0}{z}, \frac{x_1}{z}, \frac{x_2}{z}, \ldots]]$, define

$$\hat{F}^- = \hat{G} \cdot E \left( x, \frac{1}{z} \right) - \hat{F}.$$

Then $\hat{F}^-$ has only negative powers of $z$.

**Proof.**

$$\hat{F}^- = \hat{G} \cdot E \left( x, \frac{1}{z} \right) - \hat{F}$$

$$= \frac{c(t)E(y)}{E(y, c(t))} \cdot E \left( x, \frac{1}{z} \right) - \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right)$$

$$= \frac{c(t)E(y)}{E(y, c(t))} \cdot E(y, c(t)) \cdot E \left( y, \frac{1}{tc(t)} \right) - \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right)$$

$$= \frac{E(y)}{t} \left[ tc(t)E \left( y, \frac{1}{tc(t)} \right) + \frac{E(y, c(t))}{c(t)} - E(y) \right]$$

$$= \frac{E(y)}{t} \sum_{n=0}^{\infty} [c(t)^{n-1} + t^{1-n}c(t)^{1-n} - 1] e_n(y)$$

$$= E(y) \sum_{n=0}^{\infty} \left[ \frac{1}{tn} \left[ (tc(t))^{n-1} + c(t)^{1-n} \right] - \frac{1}{t} \right] e_n(y).$$

Since

$$\tag{2.2.3} (tc(t))^{n-1} + c(t)^{1-n} = \left( \frac{1 - \sqrt{1 - 4t}}{2} \right)^{n-1} + \left( \frac{1 + \sqrt{1 - 4t}}{2} \right)^{n-1}$$

$$= \frac{1}{2^{n-2}} \sum_{k \text{ even}} (\sqrt{1 - 4t})^k \binom{n-1}{k}$$

$$= \frac{1}{2^{n-2}} \sum_{k \text{ even}} (1 - 4t)^{k/2} \binom{n-1}{k}$$

is a polynomial of degree less than or equal to $(n - 1)/2$, $\hat{F}^-$ has only negative powers of $t$ and thus has only negative powers of $z$. \qed
Lemma 2.2.5. Let $\hat{G}$ and $\hat{F}$ be defined as above. Define

$$\hat{f}_k = [z^k] \hat{F}, \quad \hat{g}_k = [z^k] \hat{G}.$$  

Then $\hat{g}_k$ and $\hat{f}_k$ satisfy the functional equation that

$$\hat{f}_k = \sum_{j=0}^{\infty} \hat{g}_{k+j} e_j(x) \text{ for } k \geq 0.$$  

Proof. From the definition of $\hat{f}_k$ and $\hat{g}_k$, we know that

$$\hat{F} = \sum_{k=0}^{\infty} \hat{f}_k z^k,$$

$$\hat{G} = \sum_{k=0}^{\infty} \hat{g}_k z^k.$$  

Then

$$[z^k] \left\{ \hat{G} \cdot E\left( x, \frac{1}{z} \right) - \hat{F} \right\} = \sum_{j=0}^{\infty} \hat{g}_{k+j} e_j(x) - \hat{f}_k.$$  

By Lemma 2.2.4, this sum is zero when $k \geq 0$.

The lemma is proved.  

Now Theorem 2.2.1 follows from Lemma 2.2.3 and Lemma 2.2.5.

2.3. The number of two-stack-sortable $r$-permutations with descents

We know that the generating function for two-stack-sortable $r$-permutations is

$$g_1 = f_0$$

$$= [z^0] F$$

$$= [t^0] E(y) \left( E(y) - \frac{E(y, c(t))}{c(t)} \right)$$

$$= [t^0] \left\{ E(y) \sum_{n=0}^{\infty} \frac{1}{t} (1 - c^{n-1}(t)) e_n(y) \right\}$$
\[ [t^0] \left\{ E(y) \sum_{n=0}^{\infty} \sum_{i=0}^{n} \frac{1-n}{i+1} \binom{n+2i}{i} e_n(y) t^i \right\} \]

\[ = E(y) \sum_{n=0}^{\infty} (1-n) e_n(y) \]

where

\[ x_i = \frac{y_i(1+y_i)}{E^2(y)} \]

We can now evaluate the number of two-stack-sortable \( r \)-permutations with the help of multivariable Lagrange inversion.

**Theorem 2.3.1.** The number of two-stack-sortable \( r \)-permutations with \( k_i \) descents of type \( i \) is

\[ \frac{1}{n^2} \prod_{i=0}^{r} \frac{n}{n-k_i} \left( \frac{2n-1-k_i}{k_i} \right), \]

where \( n = 1 + k_0 + k_1 + \cdots + k_r \).

First, we introduce the following identity:

**Lemma 2.3.2.**

\[ \sum_{i=0}^{r+1} (1-i)e_i(y) = E(y) \left( 1 - \sum_{i=0}^{r} \frac{y_i}{1+y_i} \right). \]

**Proof.** Since \( e_i(y) \) is a homogeneous polynomial in \( y_0, y_1, \ldots, y_r \) of degree \( i \),

\[ ie_i(y) = \sum_{j=0}^{r} \frac{y_j \partial e_i(y)}{\partial y_j}. \]

Also notice that

\[ \frac{\partial E(y)}{\partial y_j} = \frac{E(y)}{1+y_j}. \]
Thus

\[
\sum_{i=0}^{r+1} (1 - i)e_i(y) = E(y) - \sum_{i=0}^{r+1} ie_i(y)
\]

\[
= E(y) - \sum_{i=0}^{r+1} \sum_{j=0}^{r} y_j \frac{\partial e_i(y)}{\partial y_j}
\]

\[
= E(y) - \sum_{j=0}^{r} y_j \sum_{i=0}^{r+1} \frac{\partial e_i(y)}{\partial y_j}
\]

\[
= E(y) - \sum_{j=0}^{r} y_j \frac{E(y)}{1 + y_j}
\]

\[
= E(y) \left( 1 - \sum_{j=0}^{r} \frac{y_j}{1 + y_j} \right).
\]

\[\square\]

Therefore,

\[
(2.3.2) \quad g_1 = E(y) \sum_{n=0}^{\infty} (1 - n)e_n(y)
\]

\[
= E^2(y) \left( 1 - \sum_{j=0}^{r} \frac{y_j}{1 + y_j} \right).
\]

If for any \( y = (y_0, y_1, \ldots, y_r) \), we define

\[
A(y, u) = A(y_0, \ldots, y_r; u)
\]

\[
= 1 - \sum_{i=0}^{r} \frac{y_i u}{1 + y_i u},
\]

and let

\[
A(y) = A(y, 1),
\]

then \( g_1 = E^2(y)A(y) \).

The proof of Theorem 2.3.1 now consists of following lemmas.
2.3. THE NUMBER OF TWO-STACK-SORTABLE $r$-PERMUTATIONS WITH DESCENTS

Notice that for any formal power series $f(x) = \sum_k a_k x^k \in \mathbb{Q}[[x_0, x_1, \ldots, x_r]]$ (where $k = (k_0, \ldots, k_r)$ and $x = (x_0, \ldots, x_r)$), if we define operators $D_x, I_x: \mathbb{Q}[[x_0, x_1, \ldots, x_r]] \to \mathbb{Q}[[x_0, x_1, \ldots, x_r]]$ by

\[
D_x(f(x)) = \sum_{i=0}^{r} x_i \frac{\partial f(x)}{\partial x_i},
\]

\[
I_x(f(x)) = f(x),
\]

then $(D_x + I_x)(f(x)) = \sum_k n a_k x^k$, where $n = k_0 + \cdots + k_r + 1$. Therefore, to prove

\[
[x^k] g_1 = \frac{1}{n} \prod_{i=0}^{r} \frac{n}{n - k_i} \binom{2n - 1 - k_i}{k_i},
\]

we only need to prove that

\[
[x^k](D_x + I_x)^2 g_1 = \prod_{i=0}^{r} \frac{n}{n - k_i} \binom{2n - 1 - k_i}{k_i}.
\]

Let $u_i = \sum_{j=0}^{r} x_j \frac{\partial y_i}{\partial x_j}$. Then

**Lemma 2.3.3.**

\[
u_i = \frac{y_i(1 + y_i)}{1 + 2y_i} \frac{1}{A(y, 2)}
\]

**Proof.** Take the logarithm of both sides of $x_i = \frac{y_i(1 + y_i)}{E^2(y)}$. We get

\[
\ln x_i = \ln y_i + \ln(1 + y_i) - 2 \ln E(y).
\]

Then by differentiating both sides of (2.3.3) with respect to $x_j$, we get

\[
\frac{1}{x_i} = \frac{1 + 2y_i}{y_i(1 + y_i)} \frac{\partial y_i}{\partial x_i} - \frac{2}{E(y)} \frac{\partial E(y)}{\partial x_i}, \quad \text{if } j = i,
\]

\[
0 = \frac{1 + 2y_i}{y_i(1 + y_i)} \frac{\partial y_i}{\partial x_j} - \frac{2}{E(y)} \frac{\partial E(y)}{\partial x_j}, \quad \text{if } j \neq i.
\]
Now multiplying $x_j$ by both (2.3.4) and (2.3.3) and summing on $j$, we get

$$1 = \frac{1 + 2y_i}{y_i(1 + y_i)} \sum_{j=0}^{r} x_j \frac{\partial y_i}{\partial x_j} - \frac{2}{E(y)} \sum_{j=0}^{r} x_j \frac{\partial E(y)}{\partial x_j}$$

(2.3.6)

$$= \frac{1 + 2y_i}{y_i(1 + y_i)} u_i - \frac{2}{E(y)} D_x(E(y)).$$

On the other hand,

$$D_x(E(y)) = \sum_{j=0}^{r} x_j \frac{\partial E(y)}{\partial x_j}$$

$$= \sum_{j=0}^{r} x_j \sum_{i=0}^{r} \frac{\partial E(y)}{\partial y_i} \frac{\partial y_i}{\partial x_j}$$

$$= E(y) \frac{1}{1 + y_i} \sum_{j=0}^{r} x_j \frac{\partial y_i}{\partial x_j}$$

(2.3.7)

$$= E(y) \frac{1}{1 + y_i} u_i$$

By solving (2.3.6) and (2.3.7) for $u_i$, we get

$$u_i = \frac{y_i(1 + y_i)}{1 + 2y_i} \frac{1}{A(y, 2)}.$$

The lemma is proved. \[\square\]

Using the lemma above, the proofs of the following lemmas are just a matter of simple computations.

**Lemma 2.3.4.**

$$D_x(E^2(y)) = \frac{E^2(y)}{A(y, 2)} - E^2(y).$$

**Proof.**

$$D_x(E^2(y)) = \sum_{j=0}^{r} x_j \frac{\partial E^2(y)}{\partial x_j}$$
\[= \sum_{j=0}^{r} x_j \sum_{i=0}^{r} \frac{\partial E^2(y)}{\partial y_i} \frac{\partial y_i}{\partial y_j} \]
\[= \sum_{i=0}^{r} \frac{\partial E^2(y)}{\partial y_i} u_i \]
\[= \sum_{i=0}^{r} \frac{2E^2(y)}{1+y_i} \frac{y_i(1+y_i)}{1+2y_i} \frac{1}{A(y,2)} \]
\[= \frac{E^2(y)}{A(y,2)} \sum_{i=0}^{r} \frac{2y_i}{1+2y_i} \]
\[= \frac{E^2(y)}{A(y,2)} - E^2(y). \]

The lemma is proved. \[\square\]

**Lemma 2.3.5.**

\[D_x(A(y)) = 1 - \frac{A(y)}{A(y,2)}.\]

**Proof.**

\[D_x(A(y)) = \sum_{j=0}^{r} x_j \frac{\partial A(y)}{\partial x_j} \]
\[= \sum_{j=0}^{r} x_j \sum_{i=0}^{r} \frac{\partial A(y)}{\partial y_i} \frac{\partial y_i}{\partial x_j} \]
\[= \sum_{i=0}^{r} \frac{\partial A(y)}{\partial y_i} u_i \]
\[= \sum_{i=0}^{r} -\frac{1}{(1+y_i)^2} \frac{y_i(1+y_i)}{1+2y_i} \frac{1}{A(y,2)} \]
\[= \frac{1}{A(y,2)} \sum_{i=0}^{r} \frac{y_i}{(1+y_i)(1+2y_i)} \]
\[= \frac{1}{A(y,2)} \sum_{i=0}^{r} \left( \frac{y_i}{1+y_i} - \frac{2y_i}{1+2y_i} \right) \]
\[= \frac{1}{A(y, 2)} (A(y, 2) - A(y))\]
\[= 1 - \frac{A(y)}{A(y, 2)}.\]
Thus \( y_i = x_i \phi_i(y) \). Using Lemma \[1.2.2\] to evaluate the determinant, we have

\[
[x^k] \frac{E^2(y)}{A(y, 2)} = [y^k] \left\{ \frac{E^2(y)}{A(y, 2)} \phi^k(y) \left| \delta_{ij} - \frac{y_j}{\phi_i(y)} \frac{\partial \phi_i(y)}{\partial y_j} \right| \right\} 
\]

\[
= [y^k] \left\{ \frac{E^2(y)}{A(y, 2)} \phi^k(y) \frac{\delta_{ij} + 2(\delta_{ij} - 1)y_j}{E(y)} \right\} 
\]

\[
= [y^k] \left\{ \frac{E^{2n-1}(y)}{\prod_i (1 + y_i)^{k_i}} E(y, 2) \right\} 
\]

\[
= \prod_i \left[ \binom{2n - 1 - k_i}{k_i} + 2 \binom{2n - 1 - k_i}{k_i - 1} \right] 
\]

\[
= \prod_i \frac{n}{n - k_i} \binom{2n - 1 - k_i}{k_i}. 
\]

Therefore, Theorem \[2.3.1\] is proved.

**Remark 2.3.8.** When \( r = 1 \), we have \( k_0 + k_1 = n - 1 \). Then the number is

\[
\frac{1}{n^2} \frac{n}{n - k_0} \binom{2n - 1 - k_0}{k_0} \frac{n}{n - k_1} \binom{2n - 1 - k_1}{k_1} = \frac{(n + k_1)!}{(k_1 + 1)! (2k_1 + 1)!} \frac{(2n - 1 - k_1)!}{(n - k_1)! (2n - 1 - 2k_1)!},
\]

which was found and proved by Bóna \[3\], Bousquet-Mélou \[6\], and Travis \[25\].

### 2.4. The number of two-stack-sortable \( r \)-permutations

The following theorem is a special case of Theorem \[2.3.1\] when we do not keep track of descents.

**Theorem 2.4.1.** The number of two-stack-sortable \( r \)-permutations of \([n]\) is:

\[
2(r + 1) \frac{(2r + 1)n)!}{n! (2rn + 2)!}.
\]
First proof:

We know that the generating function for two-stack-sortable \( r \)-permutations is given by (2.3.2). Setting \( y_i = y \) for \( i = 0, \ldots, r \), which is equivalent to setting \( x_i = x \) for \( i = 0, \ldots, r \), we get that

\[
x = \frac{y}{(1+y)^{2r+1}}
\]

and

\[
g_1 = (1+y)^{2r+2} \left( 1 - (r+1) \frac{y}{1+y} \right) \\
= (r+1)(1+y)^{2r+1} - r(1+y)^{2r+2} \\
= (1+y)^{2r+1}(1-ry) \\
= \frac{y(1-ry)}{x}.
\]

(2.4.1)

Using Lagrange inversion [10, p. 17; 9], we set \( \phi(y) = (1+y)^{2r+1} \). Then \( y \) satisfies the condition that \( y = x\phi(y) \). Therefore,

\[
[x^k]g_1 = [x^{k+1}]xg_1 \\
= \frac{1}{k+1} \left[ y^k \right] \left\{ \frac{d}{dy} \phi^{k+1}(y) \right\} \\
= \frac{1}{k+1} \left[ y^k \right] \{(1-2ry)(1+y)^{(2r+1)(k+1)}\} \\
= \frac{1}{k+1} \left\{ \binom{(2r+1)(k+1)}{k} - 2r \binom{(2r+1)(k+1)}{k-1} \right\} \\
= 2(r+1) \frac{((2r+1)(k+1))!}{(k+1)! (2r(k+1)+2)!}.
\]

Setting \( n = k+1 \), then the theorem is proved.

Second proof:
From Theorem 2.3.1 we know that the total number of two-stack-sortable permutations is
\[
\frac{1}{n^2} \sum_{k_0+\cdots+k_r=n-1} \prod_{i=0}^r \frac{n}{n-k_i} \binom{2n-1-k_i}{k_i}.
\]
Also we know that
\[
(2.4.2) \quad c^n(t) = \sum_{i=0}^\infty \frac{n}{2i+n} \binom{2i+n}{i} t^i
\]
for any integer \(n\) [13, p. 154]. Therefore,
\[
[t^{k_i}]c^{-2n}(-t) = [t^{k_i}] \sum_{j=0}^\infty \frac{2n}{2n-2j} \binom{2j-2n}{j} (-t)^j
\]
\[
= \frac{2n}{2n-2k_i} (-1)^{k_i} \binom{2k_i-2n}{k_i}
\]
\[
= \frac{n}{n-k_i} \binom{2n-1-k_i}{k_i}.
\]
Hence,
\[
\frac{1}{n^2} \sum_{k_0+\cdots+k_r=n-1} \prod_{i=0}^r \frac{n}{n-k_i} \binom{2n-1-k_i}{k_i}
\]
\[
= \frac{1}{n^2} [t^{n-1}] \left(c^{2n}(-t)\right)^{r+1}
\]
\[
= \frac{1}{n^2} [t^{n-1}] \sum_{j=0}^\infty \frac{2n(r+1)}{2n(r+1)-2j} \binom{2j-2n(r+1)}{j} (-t)^j
\]
\[
= \frac{1}{n^2} \frac{2n(r+1)}{2n(r+1)-2(n-1)} (-1)^{n-1} \binom{2(n-1)-2n(r+1)}{n-1}
\]
\[
= \frac{r+1}{n(nr+1)} \binom{(2r+1)n}{n-1}
\]
\[
= 2(r+1) \frac{(2r+1)n}{n! (2rn+2)!}.
\]
The theorem is proved. \(\square\)
Remark 2.4.2. When \( r = 1 \), this number is

\[
\frac{4 (3n)!}{n! (2n+2)!} = \frac{2(3n)!}{(n+1)! (2n+1)!},
\]

which was conjectured by West [26, 27] and first proved by Zeilberger [28].
CHAPTER 3

Parallel results

3.1. Another approach

In the previous chapter, we counted the number of two-stack-sortable \( r \)-permutations with all types of descents. Here we approach a special case of the problem from another angle. We now count the number of two-stack-sortable \( r \)-permutations with only two basic parameters, the number of different letters and the number of components. We use the indeterminates \( x \) and \( \bar{z} \) to count these two parameters respectively. The difference is, that any component is allowed to be empty.

Therefore, using the same decomposition method as in Chapter 2, if we let \( p^{(k)}_n \) be the weight of all \( k \)-tuple \( r \)-permutations \( (\alpha_1, \ldots, \alpha_k) \) of \( [n] \) where each of \( \alpha_i \) \( (i = 1, \ldots, k) \) is allowed to be empty, and let \( q^{(k)}_n = p^{(r+k+1)}_n \),

\[
\begin{align*}
  p_k &= \sum_{n=0}^{\infty} p^{(k)}_n x^n, \\
  q_k &= \sum_{n=0}^{\infty} q^{(k)}_n x^n,
\end{align*}
\]

then analogous to Theorem [2.1.2], we have that the formal power series \( P \) and \( Q \) satisfy the functional equations

\[
\begin{align*}
  q_k &= p_{r+k+1}, \\
  P &= \frac{1}{1 - \bar{z}} + x\bar{z}PQ,
\end{align*}
\]
Since

\[ Q = \frac{1}{z^{r+1}} \left( P - \sum_{i=0}^{r} p_i z^i \right), \]

we have that

**Theorem 3.1.1.** *The generating function* \( P \) *of two-stack-sortable* \( k \)-tuple \( r \)-permutations *satisfies*

\[ P = \frac{1}{1 - \bar{z}} + \frac{xP}{z^r} \left( P - \sum_{i=0}^{r} p_i z^i \right) \]

and \( P \) is uniquely determined as a power series by this functional equation.

Now, compared to the functional equations (2.1.10) and (2.1.11), this one is structurally simpler. We will also see that (3.1.5) can be easily derived from the functional equations (2.1.10) and (2.1.11).

If we set \( x_i = x \) in functional equations (2.1.10) and (2.1.11), then they become

\[ f_k = \sum_{j=0}^{r+1} \binom{r+1}{j} g_{k+j} x^j \text{ for } k \geq 0, \]

(3.1.6)

\[ G = 1 + zFG. \]

(3.1.7)

Notice that \( p_k \) is the weight of of all \( k \)-tuple \( r \)-permutations where *any component is allowed to be empty* and \( g_k \) is the weight of of all \( k \)-tuple \( r \)-permutations where *no component is allowed to be empty*. Therefore,

\[ p_k = \sum_{i=0}^{k} \binom{k}{i} g_i x^i, \]

(3.1.8)

which is equivalent to

\[ P(x, \bar{z}) = \frac{1}{1 - \bar{z}} G \left( x, \frac{x\bar{z}}{1 - \bar{z}} \right). \]
3.1. ANOTHER APPROACH

Setting $x_i = x$, which is equivalent to setting $y_i = y$, in the solution of functional equations (Theorem 2.2.1) gives the solution to functional equation (3.1.6) and (3.1.7),

\[ G = \frac{c(t)(1 + y)^{r+1}}{(1 + yc(t))^{r+1}}, \]

where $y$ is uniquely determined as formal power series in $x$ by

\[ x = \frac{y}{(1 + y)^{2r+1}} \]

and $t = z(1 + y)^{2(r+1)}$. Therefore,

**Theorem 3.1.2.** The solution to function equation (3.1.3) is:

\[ P = \left( 1 + \frac{t}{y(1 + y)} \right) \frac{c(t)(1 + y)^{r+1}}{(1 + yc(t))^{r+1}}, \]

where $y$ is uniquely determined as a formal power series in $x$ by

\[ x = \frac{y}{(1 + y)^{2r+1}} \]

and $t = y(1 + y) \frac{z}{1 - z}$.

**Corollary 3.1.3.** $p_k$ is a polynomial in $y$ with degree no greater than $2k$.

**Proof.**

\[ x^kg_k = x^k[z^k]G \]

\[ = \left[ \left( \frac{z}{x} \right)^k \right] G \]

\[ = \left[ \left( \frac{z}{x} \right)^k \right] \left\{ \frac{c(t)(1 + y)^{r+1}}{(1 + yc(t))^{r+1}} \right\} \]

\[ = [z^k] \left\{ c(xt)(1 + y)^{r+1} \right\} \]

\[ = [z^k] \left\{ (1 + y)^{r+1} \sum_{i=0}^{\infty} (-1)^i \binom{r+i}{i} y^i (c(xt))^{i+1} \right\} \]
3. PARALLEL RESULTS

\[
\begin{align*}
&=[z^k] \left\{ (1 + y)^{r+1} \sum_{i=0}^{\infty} (-1)^i \binom{r+i}{i} y^i \sum_{j=0}^{\infty} \frac{i+1}{2j+i+1} \binom{2j+i+1}{j} (xt)^j \right\} \\
&=[z^k] \left\{ (1 + y)^{r+1} \sum_{i=0}^{\infty} (-1)^i \binom{r+i}{i} y^i \sum_{j=0}^{\infty} \frac{i+1}{2j+i+1} \binom{2j+i+1}{j} y^j (1+y)^2 z^j \right\} \\
&=(1 + y)^{r+k+1} y^k \sum_{i=0}^{\infty} (-1)^i \binom{r+i}{i} y^i \frac{i+1}{2k+i+1} \binom{2k+i+1}{k} y^j.
\end{align*}
\]

The sum is a hypergeometric series that can be transformed by Euler’s transformation \[1\]. Thus,

\[
x^k g_k = y^k \sum_{i=0}^{k} (-1)^i \frac{(2i+1)r-(k-i)}{(r-k)(k+i+1)} \binom{r-k}{i} \binom{2k}{k-i} y^i.
\]

Therefore, \(x^k g_k\) is a polynomial in \(y\) with degree no greater than \(2k\). Thus \(p_k\) is a polynomial in \(y\) with degree no greater than \(2k\) because \(p_k = \sum_{i=0}^{k} \binom{k}{i} x^i g_i\) (Eq. (3.1.8)).

\[\square\]

In particular,

**Corollary 3.1.4.** \(p_1\) is the generating function of two-stack-sortable \(r\)-permutations, and

\[
p_1 = 1 + y - ry^2.
\]

**Proof.** Since \(p_1 = 1 + x g_1\) (Eq. (3.1.8)) and \(g_1 = \frac{y(1-ry)}{x}\) (Eq. (2.4.1)),

\[
p_1 = 1 + x g_1
\]

\[=1 + y - ry^2.
\]

\[\square\]
3.2. Connection to Zeilberger’s functional equation

Zeilberger first proved West’s conjecture that the number of two-stack-sortable permutations of length $n$ is \( \frac{2(3n)!}{(n+1)! (2n+1)!} \)\(^{[28]} \). He used a factorization similar to ours and the functional equation he got is equivalent to our functional equation (3.1.5) in the case of $r = 1$.

In Zeilberger’s paper\(^{[28]} \), he defined $i(\pi)$ (where $\pi$ is any permutation of \{1, 2, $\ldots$, $n$\}) to be the largest integer $i$ such that the subsequence of the ‘big $i$’: \{n$-i$+1, $\ldots$, n$-1$, n\} are in decreasing order, defined $W^{(i)}$ to be the set of all permutations (of any length) $\pi$ such that $i(\pi) = i$, and let $W^{\geq i}$ to be the set of all permutations $\pi$ such that $i(\pi) \geq i$.

Also he defined $W^{(i)}(x)$ to be the formal power series that equals to the sum of all the weights of elements of $W^{(i)}$, and $W^{\geq i}(x)$ to be the formal power series that equals to the sum of all the weights of elements of $W^{\geq i}$, and he defined

$$
\Phi(x, t) := \sum_{i=0}^{\infty} W^{(i)}(x)t^i ,
$$

$$
\bar{\Phi}(x, t) := \sum_{i=0}^{\infty} W^{\geq i}(x)t^i .
$$

Then, he got

$$
(3.2.1) \quad \Phi(x, t) = \frac{1}{1 - xt} + \frac{xt(\Phi(x, 1) - t\Phi(x, t))(\Phi(x, 1) - \Phi(x, t))}{(1 - t)^2} .
$$

Now noticing that $\Phi(x, 1) = \bar{\Phi}(x, 0)$ and

$$
\bar{\Phi}(x, t) = \frac{\Phi(x, 1) - t\Phi(x, t)}{1 - t} .
$$

Equation (3.2.1) is equivalent to

$$
\Phi(x, t) = \frac{1}{1 - xt} + \frac{(1 + xt\bar{\Phi}(x, t))(\bar{\Phi}(x, t) - \bar{\Phi}(x, 0))}{t} .
$$
Now it is easy to check that \( P(x, \bar{z}) = 1 + \bar{z} \Phi \left( x, \frac{\bar{z}}{x} \right) \). Therefore, the functional equation (3.2.1) is equivalent to (3.1.3) in the case of \( r = 1 \).

The combinatorial connection between the set \( W^\geq i \) and the set of \( k \)-tuple permutations was clearly stated in Zeilberger’s paper [28]. For a typical member \( \pi \) of \( W^\geq i \), if its length is \( n \), then it has the form

\[
\pi = \alpha_0n\alpha_1(n-1)\cdots(n-i+1)\alpha_i,
\]

where \( \alpha_0, \ldots, \alpha_i \) are (possibly empty) permutations of disjoint smaller sets, the union of whose underlying sets is \( \{1, 2, \ldots, n-i\} \). Now, by iterating the definition of the stack sorting operation \( S \),

\[
S(\pi) = S(\alpha_0)S(\alpha_1)\cdots S(\alpha_i)(n-i+1)(n-i+2)\cdots n,
\]

so that,

\[
S^2(\pi) = S(\alpha_0)S(\alpha_1)\cdots S(\alpha_i))(n-i+1)(n-i+2)\cdots n.
\]

It follows that there is a 1-1 correspondence between the elements of \( W^\geq i \) and \((i+1)\)-tuple permutations \( \alpha_0, \alpha_1, \ldots, \alpha_i \), such that \( S(\alpha_0)\cdots S(\alpha_i) = I \), and the underlying sets of the \( \alpha \)'s are disjoint and their union is \( \{1, 2, \ldots, n-i\} \).
CHAPTER 4

Further results

4.1. A generalization of the stack-sorting operation on $r$-permutations

Here we introduce a more general form of the stack-sorting operation on $r$-permutations.

**Definition 4.1.1.** For a given positive number $r$, let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_l)$ satisfy the condition that $0 < \lambda_0 < \lambda_1 < \cdots < \lambda_l = r$. Let $\pi$ be an $r$-permutation of $[n]$. Then we can write $\pi$ as $\pi = \alpha_0 n \alpha_1 n \cdots n \alpha_r$. The generalized stack-sorting operation $S_\lambda$ on $r$-permutations is defined by

$$S_\lambda(\pi) = S_\lambda(\alpha_0) \cdots S_\lambda(\alpha_{\lambda_0}) n S_\lambda(\alpha_{\lambda_0+1}) \cdots S_\lambda(\alpha_{\lambda_1}) n \cdots n S_\lambda(\alpha_{\lambda_{l-1}}+1) \cdots S_\lambda(\alpha_{\lambda_l}) n.$$  

Notice that $S_\lambda(\pi)$ is an $(l+1)$-permutation and $S_\lambda = S$ when $l = 0$, where $S$ is the ordinary stack sorting operation on $r$-permutations (Definition 1.1.2).

It is clear that it is not interesting to consider stack-sortable $r$-permutation under this definition. Also, in order to consider two-stack-sortable $r$-permutations under this definition, we have to consider $S(S_\lambda(\pi))$ instead of $S^2_\lambda(\pi)$.

To enumerate the number of two-stack-sortable $r$-permutations with descents under this definition of stack-sorting operation, we still weight any type $i$ descent by $x_i$. Furthermore, we denote $x^{(0)} = (x_0, x_1, \ldots, x_{\lambda_0})$ and $x^{(i)} = (x_{\lambda_{i-1}+1}, \ldots, x_{\lambda_i})$ for $1 \leq i \leq l$. Also, we use $u$ to keep track of the number of different letters.
Now let \( g_n^{(k)} \) be the sum of the weights of two-stack-sortable \( k \)-tuple \( r \)-permutations \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) such that every \( \alpha_i \) is nonempty and let

\[
\begin{align*}
  f_{n,0}^{(k)} &= \sum_{j=0}^{\lambda_0+1} g_n^{(k+j)} e_j(x^{(0)}), \\
  f_{n,i}^{(k)} &= \sum_{j=0}^{\lambda_i-\lambda_{i-1}} g_n^{(k+j)} e_j(x^{(i)}) \text{ for } 1 \leq i \leq l.
\end{align*}
\]

Also let

\[
\begin{align*}
  f_{k,i} &= \sum_{n=0}^{\infty} f_{n,i}^{(k)} u^n, & F_i &= \sum_{k=0}^{\infty} f_{k,i} z^k, \\
  g_k &= \sum_{n=0}^{\infty} g_n^{(k)} u^n, & G &= \sum_{k=0}^{\infty} g_k z^k.
\end{align*}
\]

Then we have the following functional equations by the same reasoning as in section 2.1:

**Lemma 4.1.2.** Let \( G \) and \( f_{k,i} \) be defined as above, then

\[
\begin{align*}
  (4.1.2) & \quad G = 1 + zuGF_0 \prod_{i=1}^{l} f_{0,i}, \\
  (4.1.3) & \quad f_{k,0} = \sum_{j=0}^{\lambda_0+1} g_{k+j} e_j(x^{(0)}), \\
  (4.1.4) & \quad f_{k,i} = \sum_{j=0}^{\lambda_i-\lambda_{i-1}} g_{k+j} e_j(x^{(i)}) \text{ for } 1 \leq i \leq l.
\end{align*}
\]

If we let \( A = \prod_{i=1}^{l} f_{0,i} \), we can make the substitution \( x = uA \) so that the functional equations (4.1.2) to (4.1.4) become

\[
\begin{align*}
  (4.1.5) & \quad G = 1 + zxGF_0, \\
  (4.1.6) & \quad f_{k,0} = \sum_{j=0}^{\lambda_0+1} g_{k+j} e_j(x^{(0)}).
\end{align*}
\]
4.1. A GENERALIZATION OF THE STACK-SORTING OPERATION ON r-PERMUTATIONS

By Theorem 2.2.1, the solution to these functional equations is:

\[ G = \frac{c(t)E(y)}{E(y, c(t))}, \]

\[ F_0 = \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right), \]

where \( y = (y_0, \ldots, y_\lambda_0) \) is uniquely determined by \( xx_j = \frac{y_i(1 + y_i)}{E^2(y)} \) for \( 0 \leq j \leq \lambda_0 \), and \( t = zx E^2(y) \). From the solution, we can express \( g_i \) in terms of \( x \) and \( y \), i.e., \( g_i = g_i(x, y) \).

Since \( x = uA \), \( g_i \) can be expressed in terms of \( u, A, y \), i.e., \( g_i = g_i(u, A, y) \). By equation (4.1.4),

\[ A = \prod_{i=1}^{l} f_{0,i} \]

\[ \quad = \prod_{i=1}^{l} \sum_{j=0}^{\lambda_i-\lambda_{i-1}} g_j(u, A, y) e_j(x^{(i)}). \]

Now we can solve for \( A \) from the equation above so that \( A \) is expressed in terms of \( u, y, x^{(i)}, 1 \leq i \leq l \), i.e., \( A = A(u; y; x^{(1)}; \ldots; x^{(l)}) \). Therefore, given \( g_i = g_i(u, A, y) \) and \( x_j = \frac{y_i(1 + y_i)}{x E^2(y)} \) for \( 0 \leq j \leq \lambda_0 \), we can solve for \( g_i \) (for example, using Lagrange inversion) so that \( g_i = g_i(u; x^{(0)}; x^{(1)}; \ldots; x^{(l)}) \).

Therefore,

**Theorem 4.1.3.** If \( G \) and \( F_i \) are defined in (4.1.4), then \( G \) and \( F_i \) satisfy the following functional equations:

\[ G = \frac{c(t)E(y)}{E(y, c(t))}, \]

\[ F_0 = \frac{E(y)}{t} \left( E(y) - \frac{E(y, c(t))}{c(t)} \right), \]

where \( y = (y_0, \ldots, y_\lambda_0) \) is uniquely determined by \( xx_i = \frac{y_i(1 + y_i)}{E^2(y)} \) for \( 0 \leq i \leq \lambda_0 \), and \( t = zx E^2(y) \). In particular, the \( g_i \) are algebraic.
**Remark 4.1.4.** It is easy to get $g_0 = 1$. Also, similar to Equation (2.3.2), we can get

$$g_1 = xE(y) \sum_{n=0}^{\infty} (1-n)e_n(y)$$

in this case.

For example, the simplest nontrivial case is when $l = 1$ and $\lambda = (1, 2)$. We have $x^{(0)} = (x_0, x_1)$, $x^{(1)} = (x_2)$, and the functional equations for this case are:

$$G = 1 + zuGF_0 f_{0,1}$$

$$f_{k,0} = g_k + g_{k+1}(x_0 + x_1) + g_{k+2}x_0 x_1$$

$$f_{k,1} = g_k + g_{k+1}x_2$$

Now we make the substitution $x = uf_{0,1}$. Then the functional equations above become

$$G = 1 + zxGF_0$$

$$f_{k,0} = g_k + g_{k+1}(x_0 + x_1) + g_{k+2}x_0 x_1$$

By Theorem 3.1.2, the solution to these functional equations is:

$$G = \frac{c(t)E(y)}{E(y, c(t))},$$

where $y = (y_0, y_1)$, $E(y) = (1+y_0)(1+y_1)$ and $y$ is uniquely determined by $xx_i = \frac{y_i(1+y_i)}{E^2(y)}$ for $i = 0, 1$, and $t = zx E^2(y)$. 
4.1. A GENERALIZATION OF THE STACK-SORTING OPERATION ON $r$-PERMUTATIONS

Also, from Remark 4.1.4, we have

\begin{equation}
 g_0 = 1,

g_1 = xE(y) \sum_{n=0}^{\infty} (1 - n)e_n(y)

= x(1 + y_0)(1 + y_1)(1 - y_0y_1)

= uf_{0,1}(1 + y_0)(1 + y_1)(1 - y_0y_1).
\end{equation}

Therefore,

\[
 f_{0,1} = g_0 + g_1x_2

= 1 + uf_{0,1}(1 + y_0)(1 + y_1)(1 - y_0y_1)x_2.
\]

By solving for $f_{0,1}$, we get

\[
 f_{0,1} = \frac{1}{1 - u(1 + y_0)(1 + y_1)(1 - y_0y_1)x_2}.
\]

So, by (4.1.7),

\[
 g_1 = \frac{u(1 + y_0)(1 + y_1)(1 - y_0y_1)}{1 - u(1 + y_0)(1 + y_1)(1 - y_0y_1)x_2}.
\]

Now, since $xx_i = \frac{y_i(1 + y_i)}{E^2(y)}$ for $i = 0, 1$ and $x = uf_{0,1}$, we have

\[
 ux_i = \frac{y_i(1 + y_i)(1 - u(1 + y_0)(1 + y_1)(1 - y_0y_1)x_2)}{E^2(y)} \text{ for } i = 0, 1.
\]

Since $(y_0, y_1)$ is uniquely determined by the equation above, we can solve for $g_1$ in terms of $u, x_0, x_1$ and $x_2$ by using Lagrange inversion.
4. FURTHER RESULTS

We can also approach the problem without counting descents in the same way as what we did in Section 3.1. If we let $P$ be the generating function of two-stack-sortable $k$-tuple $r$-permutations under $S_\lambda$ for any $k$, with $u$ and $\bar z$ keeping track of the number of different letters and the number of components, then

**Lemma 4.1.5.** $P$ satisfies

$$P = \frac{1}{1 - \bar z} + \frac{uP}{\bar z^\lambda_0} \left( P - \sum_{i=0}^{\lambda_0} p_i \bar z^i \right) \prod_{i=1}^l p_{\lambda_i - \lambda_{i-1}}. \tag{4.1.8}$$

To solve this functional equation, first we make the substitution

$$x = u \prod_{i=1}^l p_{\lambda_i - \lambda_{i-1}}.$$

Then (4.1.8) becomes

$$P = \frac{1}{1 - \bar z} + \frac{xP}{\bar z^\lambda_0} \left( P - \sum_{i=0}^{\lambda_0} p_i \bar z^i \right).$$

From Theorem 3.1.2, we know that

**Theorem 4.1.6.** The solution to the functional equation above is

$$P = \left( 1 + \frac{t}{y(1+y)} \right) \frac{c(t)(1+y)^{\lambda_0+1}}{(1+yc(t))^{\lambda_0+1}},$$

where $y$ is uniquely determined as a formal power series in $x$ by

$$x = \frac{y}{(1+y)^{2\lambda_0+1}},$$

$$t = x(1+y)^{2(\lambda_0+1)} \frac{\bar z}{1 - \bar z},$$

$$= y(1+y) \frac{\bar z}{1 - \bar z}.$$
and

\[ x = u \prod_{i=1}^{l} p_{\lambda_i - \lambda_{i-1}}. \]

Since \( p_k \) is a polynomial in \( y \) (Theorem 3.1.3) and

\[ u = \frac{y}{(1 + y)^{2\lambda_0 + 1} \prod_{i=1}^{l} p_{\lambda_i - \lambda_{i-1}}}, \]

we can solve for \( p_k \) in terms of \( u \).

Again, the simplest nontrivial case is when \( l = 1 \) and \( \lambda = (1, 2) \), and the functional equation for this case is:

\[ P = \frac{1}{1 - \bar{z}} + up_1 \frac{P(P - p_0 - p_1 \bar{z})}{\bar{z}}. \]

Now make the substitution \( x = up_1 \). Then the functional equation above becomes

\[ P = \frac{1}{1 - \bar{z}} + x \frac{P(P - p_0 - p_1 \bar{z})}{\bar{z}}. \]

From Corollary 3.1.4, \( p_1 = 1 + y - y^2 \), where \( y \) is uniquely determined by \( x = \frac{y}{(1 + y)^3} \), or

\[ u = \frac{y}{(1 + y)^3(1 + y - y^2)}. \]

Now we can use Lagrange inversion [10, p. 17; 9] to solve for \( p_1 \).

Since \( y = u(1 + y)^3(1 + y - y^2) \),

\[ [u^n]p_1 = \frac{1}{n} [y^{n-1}] \frac{dp_1}{dy} (1 + y)^{3n}(1 + y - y^2)^n \]

\[ = \frac{1}{n} [y^{n-1}] (1 - 2y)(1 + y)^{3n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} (1 + y)^i y^{2(n-i)} \]

\[ = \frac{1}{n} [y^{n-1}] (1 - 2y) \sum_{i=0}^{n} \sum_{j=0}^{3n+i} (-1)^i \binom{n}{i} \binom{3n+i}{j} y^j y^{2(n-i)} \]

\[ = \sum_{i=0}^{n} (-1)^{n-i} \frac{1}{n} \binom{n}{i} \left[ \left( \binom{3n+i}{2i - 1} - 2 \binom{3n+i}{2i - 2} \right) \right]. \]
4.2. A characterization of $t$-stack-sortable permutations

It is natural to consider counting $k$-stack-sortable permutations for $k > 2$ now. But although people have been trying, little has been found yet, even for three-stack-sortable permutations.

Similar to West’s characterization of two-stack-sortable permutations (which did not lead to an enumeration) [26, 27], we can give a characterization for $t$-stack-sortable permutations.

For a sequence $a_1 a_2 \cdots a_n$ of different letters, define $(a_i, a_j)$ to be an inversion of the sequence if $1 < i < j < n$ and $a_i > a_j$. We also define $\text{rank}(a_i)$ as follows: $\text{rank}(a_i) = m$ if there are exactly $m - 1$ letters among $a_1, a_2, \ldots, a_n$ that are smaller than $a_i$.

**Theorem 4.2.1.** A permutation $\pi$ is $t$-stack-sortable if and only if it does not contain a subsequence $\pi' = a_1 a_2 \cdots a_{t+2}$ which satisfies the following conditions:
(1) $\text{rank}(a_{t+2}) = 1$;
(2) $\text{rank}(a_{t+1}) = t + 2$;
(3) For any $i$ and $j$ such that $1 \leq i < j \leq t$ and $(a_i, a_j)$ is an inversion, there does not exist a subsequence $\pi'' = c_1 c_2 \cdots c_s$ of $\pi$, where $a_i < c_1 < c_2 < \cdots < c_s$ and $s = t + 2 - \text{rank}(a_i)$, such that $\pi''$ appears between $a_i$ and $a_j$ in $\pi$.

To prove this theorem, we need some lemmas from West [26, 27].

**Lemma 4.2.2.** (West) If $\pi$ is a permutation of $[n]$ and $1 \leq a < b \leq n$, and if $a$ precedes $b$ in $\pi$, then $a$ precedes $b$ in $S(\pi)$.
Lemma 4.2.3. (West) If $\pi$ is a permutation of $[n]$ and $1 \leq a < b \leq n$, and if $b$ precedes $a$ in $\pi$, then $b$ precedes $a$ in $S(\pi)$ if there exists $c > b$ such that $b$ precedes $c$ and $c$ precedes $a$ in $\pi$. If there is no such $c$, then $a$ precedes $b$ in $S(\pi)$.

Lemma 4.2.4. (West) If $b$ and $a$ form an inversion in $S(\pi)$, then there exist $c > b$ such that $b$ precedes $c$ and $c$ precedes $a$ in $\pi$.

Now let’s prove Theorem 4.2.1.

Proof. Tarjan and West proved that the theorem is correct when $t = 1$ and $t = 2$ [24, 26, 27]. We now prove the theorem by induction.

Suppose a permutation $\pi$ contains a subsequence $\pi' = a_1a_2 \cdots a_{t+2}$ satisfying the conditions (1) – (3). Let $\pi_1 = S(\pi)$ and suppose that $b_1b_2 \cdots b_{t+2}$ is the subsequence of $\pi_1$ that was $\pi'$ in $\pi$.

Since $\pi'$ satisfies conditions (1) – (2), by Lemma 4.2.2, $a_i$ precedes $a_{i+1}$ in $S(\pi)$ for $1 \leq i \leq t$. By Lemma 4.2.3, $a_i$ precedes $a_{i+2}$ in $S(\pi)$ for $1 \leq i \leq t$. Thus, we get either rank($b_{t+2}$) = $t + 2$, rank($b_{t+1}$) = 1 or rank($b_{t+2}$) = 1, rank($b_{t+1}$) = $t + 2$. Let $\pi'_1 = b_1b_2 \cdots b_{t+1}$ by supposing rank($b_{t+1}$) = 1 (If rank($b_{t+2}$) = 1, we let $\pi'_1 = b_1b_2 \cdots b_{t+2}$). Therefore, $\pi'_1$ satisfies condition (1) with $t$ replaced by $t - 1$.

Let rank($a_{i_0}$) = $t + 1$. Then $a_{i_0} > a_i$ for $1 \leq i \leq t$ and $i \neq i_0$. By Lemma 4.2.2, $a_i$ precedes $a_{i_0}$ in $\pi_1$ for $1 \leq i < i_0$. If $i_0 \neq t$, then $a_{i_0}$ and $a_i$ form an inversion in $\pi'$ for $i_0 < i \leq t$. Since $t + 2 - \text{rank}(a_{i_0}) = 1$, by condition (3), there is no letter bigger than $a_{i_0}$ appears between $a_{i_0}$ and $a_i$ in $\pi$. By Lemma 4.2.3, $a_i$ precedes $a_{i_0}$ in $\pi_1$ for $i_0 < i \leq t$. Therefore, $a_i$ precedes $a_{i_0}$ in $\pi_1$ for $1 \leq i \leq t$ and $i \neq i_0$. So $b_i = a_{i_0}$ and rank($b_i$) = $t + 1$. Hence $\pi'_1$ satisfies condition (2).

Suppose that $\pi'_1$ does not satisfy condition (3); that is, there exist $1 \leq i < j \leq t - 1$ such that $b_i > b_j$, and there exists a subsequence $\pi''_1 = c_1c_2 \cdots c_s$ where $b_i < c_1 < c_2 <
\[ \cdots < c_s \text{ and } s = t + 1 - \text{rank}(b_i), \text{ such that } \pi''_1 \text{ appears between } b_i \text{ and } b_j \text{ in } \pi_1. \text{ Then } c_s \text{ and } b_j \text{ form an inversion in } \pi_1. \text{ By Lemma 4.2.4, there exists some } c > c_s \text{ such that } c_s \text{ precedes } c \text{ and } c \text{ precedes } b_j \text{ in } \pi. \text{ This contradicts the fact that } \pi' \text{ satisfies condition (3).} \]

Now, since \( \pi'_1 \) satisfies condition (1) – (3), by the induction hypothesis, \( \pi_1 \) is not \((t - 1)\)-stack-sortable. So \( \pi \) is not \( t \)-stack-sortable.

Conversely, we can show that if \( \pi \) fails to be \( t \)-stack-sortable, then it contains a subsequence that satisfies the three conditions.

If \( \pi \) is not \( t \)-stack-sortable, then \( \pi_1 = S(\pi) \) is not \((t - 1)\)-stack-sortable. By the induction hypothesis, \( \pi_1 \) contains a subsequence \( \pi'_1 = b_1b_2 \cdots b_{t+1} \) which satisfies the following conditions:

1. \( \text{rank}(b_{t+1}) = 1; \)
2. \( \text{rank}(b_i) = t + 1; \)
3. For any \( i \) and \( j \) such that \( 1 \leq i < j \leq t - 1 \) and \( b_i > b_j \), there does not exist a subsequence \( \pi''_1 = c_1c_2 \cdots c_s \) where \( b_i < c_1 < c_2 < \cdots < c_s \) and \( s = t + 1 - \text{rank}(b_i) \), such that \( \pi''_1 \) appears between \( b_i \) and \( b_j \) in \( \pi_1 \).

Notice that \( b_i \) and \( b_{t+1} \) form an inversion in \( \pi_1 \) for \( 1 \leq i \leq t \). By Lemma 4.2.4, \( b_i \) precedes \( b_{t+1} \) in \( \pi \) for \( 1 \leq i \leq t \). In particular, there exists some \( a > b_i \) such that \( b_i \) precedes \( a \) and \( a \) precedes \( b_{t+1} \) in \( \pi \) for \( 1 \leq i \leq t \). Suppose that \( a_1a_2 \cdots a_{t+1} \) is the subsequence of \( \pi \) that gets transformed to \( \pi'_1 \). Then it is clear that \( a_{t+1} = b_{t+1} \). Let \( \pi' = a_1a_2 \cdots a_ia_{t+1} \). Then \( \pi' \) satisfies conditions (1) and (2).

If \( \pi' \) does not satisfy condition (3); that is, there exist \( 1 \leq i < j \leq t \) such that \( a_i > a_j \), and there exists a subsequence \( \pi'' = c_1c_2 \cdots c_s \) where \( a_i < c_1 < c_2 < \cdots < c_s \) and \( s = t + 2 - \text{rank}(a_i) \), such that \( \pi'' \) appears between \( a_i \) and \( a_j \), then by Lemma
the subsequence $c_1c_2\ldots c_{s-1}$ appears between $a_i$ and $a_j$ in $\pi_1$. This contradicts the fact that $\pi'_1$ satisfies condition (3). Therefore $\pi'$ satisfies condition (3).

The theorem is proved. \hfill \Box

**Lemma 4.2.5.** (Tarjan) A permutation $\pi$ is stack-sortable if and only if $\pi$ contains no subsequence of type 231.

**Proof.** This lemma is the case $t = 1$ of Theorem 4.2.1.

By Theorem 4.2.1, a permutation $\pi$ is stack-sortable if and only if it does not contain a subsequence $\pi' = a_1a_2a_3$, which satisfies the conditions that $\text{rank}(a_3) = 1$ and $\text{rank}(a_2) = 3$, which means that $a_3 < a_1 < a_2$. Therefore, $\pi'$ is a type 231 subsequence. \hfill \Box

**Lemma 4.2.6.** (West) A permutation fails to be two-stack-sortable if it contains a subsequence of type 2341 or a subsequence of type 3241 which is not part of a subsequence of type 35241. If it contains no such subsequence, $\pi$ is two-stack-sortable.

**Proof.** This lemma is the case $t = 2$ of Theorem 4.2.1.

By Theorem 4.2.1, a permutation $\pi$ is two-stack-sortable if and only if it does not contain a subsequence $\pi' = a_1a_2a_3a_4$, which satisfies the conditions that (1) $\text{rank}(a_4) = 1$, (2) $\text{rank}(a_3) = 4$ and, (3) $a_1 < a_2$ or $a_1 > a_2$. If $a_1 < a_2$, then $\pi'$ is a subsequence of type 2341. If $a_1 > a_2$, then $\pi'$ is a subsequence of type 3241. By condition (3), there does not exist a $c > a_1$ appearing between $a_1$ and $a_2$. If there does exist such a $c$, then when $c > a_3$, $a_1ca_2a_3a_4$ is a subsequence of type 35241; when $c < a_3$, $ca_2a_3a_4$ is a subsequence of 3241, which needs to be considered again in the same way. Therefore, these three conditions suggest that $\pi'$ is a either type 2341 or type 3241 subsequence. If it is a type 3241 subsequence, then it is not part of a subsequence of type 35241. \hfill \Box
To enumerate the three-stack-sortable permutations by the same decomposition we used for two-stack-sortable permutations, we consider the following object, called a $\mu$-tuple permutation, where $\mu = (\mu_1, \ldots, \mu_k)$ and $\mu_i$ are positive integers.

**Definition 4.2.7.** $((\alpha_1, \ldots, \alpha_{\mu_1}), (\alpha_{\mu_1+1}, \ldots, \alpha_{\mu_1+\mu_2}), \ldots, (\alpha_{\mu_1+\cdots+\mu_{k-1}+1}, \ldots, \alpha_{\mu_1+\cdots+\mu_k}))$ is a $\mu$-tuple permutation of $[n]$ if $\alpha_1 \cdots \alpha_{\mu_1+\cdots+\mu_k}$ is a permutation of $[n]$. A $\mu$-tuple permutation is three-stack-sortable if

$S(S(S(\alpha_1) \cdots S(\alpha_{\mu_1}))S(S(\alpha_{\mu_1+1}) \cdots S(\alpha_{\mu_1+\mu_2})) \cdots S(S(\alpha_{\mu_1+\cdots+\mu_{k-1}+1}) \cdots S(\alpha_{\mu_1+\cdots+\mu_k}))) = I.$

By the same reasoning as in the case of two-stack-sortable permutations, if $g_\mu$ is the generating function for three-stack-sortable $\mu$-tuple permutations, we get

$$g_\mu = 1 + x \left( \sum_{i=0}^{\mu_1-1} g(\mu_1, \ldots, \mu_k-i+1, i) + \sum_{i=0}^{\mu_{k-1}-1} g(\mu_1, \ldots, \mu_{k-1}-i+1, i) g(\mu_k) + \cdots + \sum_{i=0}^{\mu_1-1} g(\mu_1-i+1) g(\mu_2, \ldots, \mu_k) \right).$$

It seems that this functional equation is hard to solve.

### 4.3. A modification on the functional equations.

If we replace the elementary symmetric functions $e_i(x)$ by complete homogeneous symmetric functions $h_i(x)$, then the functional equations in Theorem 2.1.2 will become:

$$f_k = \sum_{j=0}^{\infty} g_{k+j} h_j(x)$$

$$G = 1 + zFG$$

Combinatorially, in terms of the tree representation, instead of having at most one child of each type, now we can have any number of children of each type. To solve
4.3. A MODIFICATION ON THE FUNCTIONAL EQUATIONS.

this, everything else will be the same except one modification on the substitution

\[ x_i = \frac{y_i(1 + y_i)}{E^2(y)}, \]

which in this case will be

\[ x_i = \frac{y_i(1 - y_i)}{H^2(y)}, \]

where \( H(y) \) is defined analogously to \( E(y) \). Then analogous to the identity in Lemma 2.2.2, we get

\[ H\left(x, \frac{1}{z}\right) = H(y, c(t))H\left(y, \frac{1}{tc(t)}\right) \]

The solutions to these functional equations will be:

\[ G = \frac{c(t)H(y)}{H(y, c(t))}, \]

\[ F = \frac{H(y)}{t} \left( H(y) - \frac{H(y, c(t))}{c(t)} \right). \]

Therefore,

\[ g_1 = f_0 \]

\[ = H(y) \sum_{n=0}^{\infty} (1 - n)h_n(y) \]

Again, we use multivariable Lagrange inversion to get the coefficient of \( x^k \) in \( g_1 \), where \( x = (x_0, \ldots, x_r) \) and \( k = (k_0, \ldots, k_r) \) and it is

\[ \frac{1}{n^2} \prod_{k=0}^{r} \frac{n}{n + k_i} \binom{2n + 2k_i}{k_i}, \]

where \( n = 1 + k_0 + k_1 + \cdots + k_r. \)

Remark 4.3.1. If we let

\[ A(n; k_0, \ldots, k_r) = \frac{1}{n^2} \prod_{k=0}^{r} \frac{n}{n - k_i} \binom{2n - 1 - k_i}{k_i} \]
and

\[ B(n; k_0, \ldots, k_r) = \frac{1}{n^2} \prod_{k=0}^{r} \frac{n^{n + k_i}}{n + k_i} \binom{2n + 2k_i}{k_i}, \]

then

\[ A(-n; k_0, \ldots, k_r) = (-1)^{n-1} B(n; k_0, \ldots, k_r). \]
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