The Riordan–Dirichlet Group

E. V. Burlachenko

1"Mathematical Notes," Steklov Mathematical Institute of Russian Academy of Sciences, Moscow, 119991 Russia

Received June 9, 2016; in final form, December 10, 2018; accepted March 20, 2019

Abstract—Riordan matrices are infinite lower triangular matrices corresponding to certain operators in the space of formal power series. In the paper, we introduce analogous matrices for the space of Dirichlet formal series. It is shown that these matrices form a group, which is analogous to the Riordan group. An analog of the Lagrange inversion formula is given. As an example of the application of these matrices, a method for obtaining identities analogous to those obtained by using Riordan matrices is considered.

DOI: 10.1134/S0001434619090219

Keywords: Riordan matrices, formal Dirichlet series, Lagrange series.

1. INTRODUCTION

The Riordan matrices are infinite lower triangular matrices corresponding to certain operators in the space of formal power series over the field of real or complex numbers. In Sec. 3, we introduce analogous matrices for the space of Dirichlet formal series. In Sec. 2, we specify the aspects of the theory of Riordan matrices which we use when constructing analogous objects. We call these objects the Riordan–Dirichlet matrices. An analogy between them and the Riordan matrices is so full that the content of Sec. 3 repeats the content of Sec. 2 almost verbatim. However, this analogy concerns only the listed aspects. The Riordan matrices find widespread application in diverse areas of mathematics, while the question of practical application of the Riordan–Dirichlet matrices is open. In Sec. 4, we consider the situation in which these matrices prove to be useful for obtaining identities analogous to those obtained by using Riordan matrices. As an example, we obtain an analog of the Abel equations.

2. RIORDAN MATRICES

To the columns of matrices we assign the generating functions of their elements, i.e., formal power series. Thus, the expression \( Aa(x) = b(x) \) means that the column vector multiplied by the matrix \( A \) has generating function \( a(x) = \sum_{n=0}^{\infty} a_n x^n \) and the resulting column vector has generating function \( b(x) = \sum_{n=0}^{\infty} b_n x^n \); we denote the \( n \)th coefficient of the series \( a(x) \) and the \( n \)th row of the matrix \( A \) by \( [x^n]a(x) \) and \( [n, \rightarrow]A \), respectively.

We denote the matrix whose \( n \)th column, \( n = 0, 1, 2, \ldots \), has generating function \( x^n a(x) \) by \( (a(x), 1) \):

\[
(a(x), 1) = \begin{pmatrix}
a_0 & 0 & 0 & 0 & \cdots \\
a_1 & a_0 & 0 & 0 & \cdots \\
a_2 & a_1 & a_0 & 0 & \cdots \\
a_3 & a_2 & a_1 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

E-mail: evgeniy_burlachenko@list.ru
Remark 1. There is an alternative notation for the Riordan matrix defining composition. For example, (1,xa(x)) or (1, a(x)), a₀ = 0. We have chosen the notation (see [2]–[4]) similar to that which will occur below in Sec. 3 when constructing the Riordan–Dirichlet matrices.
In the algebra of formal power series, the powers and the logarithm of a series \( a(x) \), \( a_0 = 1 \), are defined as follows:

\[
a^x(x) = \sum_{n=0}^{\infty} \left( \frac{\varphi}{n} \right) (a(x) - 1)^n, \quad \log a(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (a(x) - 1)^n.
\]

One can also define a power of a series as follows:

\[
a^\varphi(x) = \exp(\varphi \log a(x)) = \sum_{n=0}^{\infty} \frac{\varphi^n}{n!} (\log a(x))^n = \sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n;
\]

here the \( s_n(\varphi)/n! \) are polynomials in \( \varphi \) of degree \( \leq n \), the so-called convolution polynomials \([7]\). The explicit form of these polynomials is as follows:

\[
s_0(\varphi) = 1, \quad \frac{s_n(\varphi)}{n!} = \frac{\varphi}{n} \sum_{m=1}^{n} \frac{b_1^{m_1}}{m_1!} \frac{b_2^{m_2}}{m_2!} \cdots \frac{b_n^{m_n}}{m_n!},
\]

where \( b_i = [x^i] \log a(x) \) and the summation of the coefficient of \( \varphi^m \) is over all monomials \( b_1^{m_1} b_2^{m_2} \cdots b_n^{m_n} \) for which \( \sum_{i=1}^{n} m_i = n \) and \( \sum_{i=1}^{n} m_i = m \).

Let \( D_x \) denote the matrix of the differentiation operator on the space of formal power series:

\[
D_x a(x) = a'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}.
\]

Then

\[
(a(x)b(x))' = a(x)b'(x) + a'(x)b(x), \quad (a^n(x))' = na^{n-1}(x)a'(x),
\]

\[
(a^\varphi(x))' = \varphi a'(x) \sum_{n=1}^{\infty} \left( \frac{\varphi-1}{n-1} \right) (a(x) - 1)^{n-1} = \varphi a^{\varphi-1}(x)a'(x),
\]

\[
(\log a(x))' = a'(x) \sum_{n=1}^{\infty} (-1)^{n-1}(a(x) - 1)^{n-1} = a'(x)a^{-1}(x).
\]

**Theorem 1.** (1) For a formal power series \( a(x) \), \( a_0 = 1 \), and every real \( \beta \), the following identity holds:

\[
a^\varphi(x) = \sum_{n=0}^{\infty} x^n \frac{\varphi}{a^{\varphi+\beta n}(x)} [a^n(1 - x \beta (\log a(x)))'] a^{\varphi+\beta n}(x).
\]

(2) There is a family of series \( (\beta) a(x) \) with the following properties: \( (0) a(x) = a(x) \),

\[
(\beta) a(xa^{-\beta}(x)) = a(x), \quad a(x(\beta) a^{\beta}(x)) = (\beta) a(x),
\]

\[
[x^n] (\beta) a^\varphi(x) = [x^n](1 - x \beta (\log a(x)))' a^{\varphi+\beta n}(x) = \frac{\varphi}{\varphi + \beta n} [x^n] a^{\varphi+\beta n}(x),
\]

\[
[x^n](1 + x \beta (\log(\beta) a(x)))' (\beta) a^\varphi(x) = \frac{\varphi + \beta n}{\varphi} [x^n] (\beta) a^\varphi(x) = [x^n] a^{\varphi+\beta n}(x).
\]

**Remark 2.** Identity (1) is the result of expanding the formal power series

\[
f(x) = (1 - x \beta (\log a(x)))' a^\varphi(x)
\]

in a Lagrange series, and the theorem can be derived from this fact. However, in view of our plans, we shall give an alternative proof, which uses only properties of Riordan matrices.
Proof of Theorem 1. If the matrices \((1, a^{-1}(x)), a_0 = 1, \) and \((1, b(x)), b_0 = 1, \) are mutually inverse, then
\[
(1, a^{-1}(x))b(x) = a(x), \quad (1, b(x))a(x) = b(x).
\]
Since
\[
(x^n b^n(x))' = nx^{n-1}b^{n-1}(x)b(x)(1 + x(\log b(x))'),
\]
it follows that
\[
D_x(1, b(x)) = (b(x)(1 + x(\log b(x))'), b(x))D_x,
\]
and
\[
(1, b(x))a'(x) = \frac{(\log b(x))'}{1 + x(\log b(x))'}.
\]
This yields
\[
(1 + x(\log b(x))', b(x))^{-1} = (1 - x(\log a(x))', a^{-1}(x)).
\]
We set
\[
[x^n]a^m(x) = a_n^{(m)}, \quad [x^n](1 - x(\log a(x))')a^m(x) = c_n^{(m)},
\]
\[
a_m(x) = \sum_{n=0}^{\infty} a_n^{(m+n)}x^n, \quad c_m(x) = \sum_{n=0}^{\infty} c_n^{(m+n)}x^n.
\]
Let us construct the matrix \(A\) whose \(m\)th column has generating function \(x^m a_m(x)\) and the matrix \(C\) whose \(m\)th column has generating function \(x^m c_m(x)\):
\[
A = \begin{pmatrix}
  a_0^{(0)} & 0 & 0 & 0 & \ldots \\
  a_1^{(1)} & a_0^{(1)} & 0 & 0 & \ldots \\
  a_2^{(2)} & a_1^{(2)} & a_0^{(2)} & 0 & \ldots \\
  a_3^{(3)} & a_2^{(3)} & a_1^{(3)} & a_0^{(3)} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad C = \begin{pmatrix}
  c_0^{(0)} & 0 & 0 & 0 & \ldots \\
  c_1^{(1)} & c_0^{(1)} & 0 & 0 & \ldots \\
  c_2^{(2)} & c_1^{(2)} & c_0^{(2)} & 0 & \ldots \\
  c_3^{(3)} & c_2^{(3)} & c_1^{(3)} & c_0^{(3)} & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]
Obviously,
\[
[n, \rightarrow] A = [n, \rightarrow] (a^n(x), 1), \quad [n, \rightarrow] C = [n, \rightarrow] ((1 - x(\log a(x))')a^n(x), 1).
\]
Since
\[
(1 - xa'(x)a^{-1}(x))a^m(x) = a^m(x) - \frac{x}{m} (a^m(x))',
\]
or
\[
[x^n](1 - x(\log a(x))')a^m(x) = \frac{m - n}{m} [x^n]a^m(x),
\]
it follows that
\[
[x^{n+m}] A x^m (1 - x \log a(x))' a^{-m}(x) = [x^{n+m}] C x^m a^{-m}(x)
\]
\[
= [x^n](1 - x(\log a(x))')a^n(x) = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases}
\]
Thus,
\[
A = (1 + x(\log b(x))', b(x)), \quad C = (1, b(x)),
\]
\[
[x^n](1 + x(\log b(x))')b^m(x) = \frac{m + n}{m} [x^n]b^m(x) = [x^n]a^{m+n}(x),
\]
\[
[x^n]b^m(x) = [x^n](1 - x(\log a(x))')a^{m+n}(x) = \frac{m}{m + n} [x^n]a^{m+n}(x).
\]
We set
\[(1, a^{-\beta}(x))^{-1} = (1, \langle x \rangle a^\beta(x)).\]

Then
\[[x^n]_{\langle x \rangle} a^{\beta}(x) = \frac{\beta m}{\beta m + \beta n} [x^n] a^{\beta m + \beta n}(x).\]

Let \(s_n(\varphi)/n!\) be the convolution polynomials of the series \(a(x)\). Then
\[\langle x \rangle a^\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi}{\varphi + \beta n} s_n(\varphi + \beta n) n! x^n.\]

Note that the identity
\[[n, \rightarrow](1 - x(\log a(x))^t)^{-1} = [n, \rightarrow](a^n(x), 1)\]
implies the following formula for the Lagrange series expansion of an arbitrary series \(f(x)\):
\[
\frac{f(x)}{1 - x(\log a(x))^t} = \sum_{n=0}^{\infty} \frac{x^n}{a^n(x)} [x^n] f(x) a^n(x). \quad \square
\]

3. RIORDBAN–DIRICHLET MATRICES

In this section, we construct matrices of certain operators on the space of formal Dirichlet series. To the columns of these matrices we assign the generating functions of their elements, i.e., formal Dirichlet series. Thus, an expression \(Aa(s) = b(s)\) means that the column vector multiplied by the matrix \(A\) has generating function \(a(s) = \sum_{n=1}^{\infty} a_n n^{-s}\), and the resulting column vector has generating function \(b(s) = \sum_{n=1}^{\infty} b_n n^{-s}\). The numbering of the rows and columns of the matrices begins with 1. To single out the coefficients of the series, we introduce the operator \([n^{-s}]: [n^{-s}]a(s) = a_n\).

We denote the matrix whose \(n\)th column has generating function \(n^{-s}a(s)\) by \(\langle a(s), 1 \rangle\):
\[
\langle a(s), 1 \rangle = \begin{pmatrix}
   a_1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
   a_2 & a_1 & 0 & 0 & 0 & 0 & 0 & \cdots \\
   a_3 & 0 & a_1 & 0 & 0 & 0 & 0 & \cdots \\
   a_4 & a_2 & 0 & a_1 & 0 & 0 & 0 & \cdots \\
   a_5 & 0 & 0 & a_1 & 0 & 0 & 0 & \cdots \\
   a_6 & a_3 & a_2 & 0 & a_1 & 0 & 0 & \cdots \\
   a_7 & 0 & 0 & 0 & a_1 & 0 & 0 & \cdots \\
   a_8 & a_4 & 0 & a_2 & 0 & 0 & a_1 & \cdots \\
   \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then
\[\langle a(s), 1 \rangle b(s) = \sum_{n=1}^{\infty} b_n n^{-s} a(s) = \sum_{n=1}^{\infty} n^{-s} \sum_{d|n} b_d a_n/d = a(s)b(s),\]

where the symbol \(d | n\) means that the summation is over all positive integer divisors \(d\) of \(n\). Thus, the matrix \(\langle a(s), 1 \rangle\) corresponds to the operator of multiplication by the series \(a(s)\). The set of all such matrices forms an algebra isomorphic to the algebra of formal Dirichlet series:

\[\langle a(s), 1 \rangle + \langle b(s), 1 \rangle = \langle a(s) + b(s), 1 \rangle,\]
\[\langle a(s), 1 \rangle \langle b(s), 1 \rangle = \langle b(s), 1 \rangle \langle a(s), 1 \rangle = \langle a(s)b(s), 1 \rangle.\]
Remark 3. The matrices \( \langle a(s), 1 \rangle \), denoted by \( A = [a_n] \), were introduced in [8] and [9]. They were regarded in [9] and [10] as matrices of operators on a Hilbert space; in [10], these matrices were called Dirichlet matrices, or \( D \)-matrices.

In the algebra of formal Dirichlet series, the powers and the logarithm of a series \( a(s), a_1 = 1 \), are defined as follows:

\[
a^\varphi(s) = \sum_{n=0}^{\infty} \left( \varphi_n \right) (a(s) - 1)^n, \quad \log a(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (a(s) - 1)^n.
\]

A power of a series can also be defined as follows:

\[
a^\varphi(s) = \exp(\varphi \log a(s)) = \sum_{n=0}^{\infty} \varphi_n (\log a(s))^n = \sum_{n=1}^{\infty} h_n(\varphi)n^{-s},
\]

where the \( h_n(\varphi) \) are polynomials in \( \varphi \) of degree \(<n\), which can be called the convolution polynomials, as in the case of power series. The explicit form of these polynomials is

\[
h_1(\varphi) = 1, \quad h_n(\varphi) = \varphi^n \sum_{m=1}^n \frac{b_2^{n_2} b_3^{n_3} \cdots b_m^{n_m}}{m_2! m_3! \cdots m_n!},
\]

where \( b_i = [i^{-n}] \log(a(s)) \) and the summation of the coefficient of \( \varphi^m \) is over all monomials \( b_2^{n_2} b_3^{n_3} \cdots b_m^{n_m} \) for which \( \prod_{i=2}^n i^{m_i} = n \) and \( \sum_{i=2}^n m_i = m \).

We denote the matrix whose \( n \)th column has generating function \( n^{-s} a_{\ln n}(s) \) by \( \langle 1, a(s) \rangle \):

\[
\langle 1, a(s) \rangle = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & a_1(2) & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & a_1(3) & 0 & 0 & 0 & 0 & \cdots \\
0 & a_2(2) & 0 & a_1(4) & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & a_1(5) & 0 & 0 & \cdots \\
0 & a_3(2) & a_2(3) & 0 & 0 & a_1(6) & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & a_1(7) & 0 & \cdots \\
0 & a_4(2) & 0 & a_2(4) & 0 & 0 & a_1(8) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

where

\[
[n^{-s}]a_{\ln n}(s) = a_n^{(m)}, \quad a_1^{(1)} = 1, \quad a_n^{(1)} = 0.
\]

Then

\[
\langle 1, a(s) \rangle b(s) = \sum_{n=1}^{\infty} b_n n^{-s} a_{\ln n}(s) = \sum_{n=1}^{\infty} n^{-s} \sum_{d|n} b_d a_{n/d}^{(d)} = b(s) \circ a(s),
\]

where the symbol \( \circ \) stands for the operation under consideration, which is analogous to the composition of formal power series. We set

\[
\langle b(s), 1 \rangle \langle 1, a(s) \rangle = \langle b(s), a(s) \rangle.
\]

Theorem 2. The matrices \( \langle b(s), a(s) \rangle \), \( b_1 \neq 0, a_1 \neq 0 \), form a group with respect to matrix product, whose elements are multiplied by the following rule:

\[
\langle b(s), a(s) \rangle \langle f(s), g(s) \rangle = \langle b(s)(f(s) \circ a(s)), a(s)(g(s) \circ a(s)) \rangle.
\]

Proof. Since

\[
\langle 1, a(s) \rangle m^{-s} b(s) = \sum_{n=1}^{\infty} b_n (mn)^{-s} a_{\ln mn}(s) = m^{-s} a_{\ln m}(s) \sum_{n=1}^{\infty} b_n n^{-s} a_{\ln n}(s),
\]

MATHEMATICAL NOTES Vol. 106 No. 4 2019
it follows that
\[ \langle 1, a(s) \rangle \langle b(s), 1 \rangle = \langle b(s) \circ a(s), a(s) \rangle. \]

Hence
\[ \langle 1, a(s) \rangle b(s)c(s) = (b(s) \circ a(s))(c(s) \circ a(s)), \]
\[ \langle 1, a(s) \rangle b^n(s) = (b(s) \circ a(s))^n, \]
and, by the definition of a power of a series,
\[ \langle 1, a(s) \rangle b^\varphi(s) = (b(s) \circ a(s))^\varphi. \]
We have
\[ \langle 1, a(s) \rangle m^{-s}b^{\ln m}(s) = m^{-s}a^{\ln m}(s)(b(x) \circ a(x))^{\ln m}, \]
\[ \langle 1, a(s) \rangle \langle 1, b(s) \rangle = \langle 1, a(s)(b(s) \circ a(s)) \rangle. \]
If
\[ \langle 1, a(s) \rangle^{-1}b^{-1}(s) = f(s), \quad \langle 1, a(s) \rangle^{-1}a^{-1}(s) = g(s), \]
then
\[ \langle b(s), a(s) \rangle^{-1} = \langle f(s), g(s) \rangle. \]

Let us denote the matrix of the differentiation operator on the space of formal Dirichlet series by \( D_s \):
\[ D_s a(s) = a'(s) = \sum_{n=1}^{\infty} \ln \left( \frac{1}{n} \right) a_n n^{-s}. \]
Then
\[ (a(s)b(s))' = a(s)b'(s) + a'(s)b(s), \quad (a^n(s))' = n a^{n-1}(s) a'(s), \]
\[ (a^\varphi(s))' = \varphi a'(s) \sum_{n=1}^{\infty} \left( \frac{\varphi - 1}{n - 1} \right) (a(s) - 1)^{n-1} = \varphi a^\varphi(s) a'(s), \]
\[ (\log a(s))' = a'(s) \sum_{n=1}^{\infty} (-1)^{n-1} (a(x) - 1)^{n-1} = a'(s) a^{-1}(s). \]

**Theorem 3.** (1) The following identity holds for any formal Dirichlet series \( a(s), a_1 = 1, \) and every real \( \beta : \)
\[ a^\varphi(s) = \sum_{n=1}^{\infty} \frac{n^{-s}}{a^{\ln n}(s)} [n^{-s}] (1 + \beta(\log a(s))') a^{\varphi + \beta \ln n}(s). \]
(2) There is a family of series \( (\beta) a(s) \) with the following properties: (0) \( a(s) = a(s), \)
\[ (\beta) a(s) \circ a^{-\beta}(s) = a(s), \quad a(s) \circ (\beta) a^{\beta}(s) = (\beta) a(s), \]
\[ [n^{-s}]_{(\beta)} a^\varphi(s) = [n^{-s}] (1 + \beta(\log a(s))') a^{\varphi + \beta \ln n}(s) = \frac{\varphi}{\varphi + \beta \ln n} [n^{-s}] a^{\varphi + \beta \ln n}(s), \]
\[ [n^{-s}] (1 - \beta(\log_{(\beta)} a(s))')_{(\beta)} a^\varphi(s) = \frac{\varphi + \beta \ln n}{\varphi} [n^{-s}]_{(\beta)} a^\varphi(s) = [n^{-s}] a^{\varphi + \beta \ln n}(s). \]

**Proof.** If matrices \( \langle 1, a^{-1}(s) \rangle, a_1 = 1, \langle 1, b(s) \rangle, b_1 = 1, \) are mutually inverse, then
\[ \langle 1, a^{-1}(s) \rangle b(s) = a(s), \quad \langle 1, b(s) \rangle a(s) = b(s). \]
Since
\[ (n^{-s} b^{\ln n}(s))' = n^{-s} b^{\ln n}(s) \ln \left( \frac{1}{n} \right) (1 - b'(s) b^{-1}(s)), \]
it follows that
\[ D_s(1, b(s)) = (1 - (\log b(s))', b(s))D_s, \quad \langle 1, b(s) \rangle a'(s) = \frac{b'(s)}{1 - (\log b(s))'.} \]

This yields
\[ \langle 1 - (\log b(s))', b(s) \rangle^{-1} = \langle 1 + (\log a(s))', a^{-1}(s) \rangle. \]

We set
\[ [n^{-s}]a^{ln_m}(s) = a^{(m)}_n, \quad [n^{-s}](1 + (\log a(s))')a^{ln_m}(s) = c^{(m)}_n, \]
\[ a_m(s) = \sum_{n=1}^\infty a^{(mn)}_n n^{-s}, \quad c_m(s) = \sum_{n=1}^\infty c^{(mn)}_n n^{-s}. \]

Let us construct the matrix \( A \) whose \( m \)th column has generating function \( m^{-s}a_m(s) \) and the matrix \( C \) whose \( m \)th column has generating function \( m^{-s}c_m(s) \):
\[
\begin{pmatrix}
a^{(1)}_1 & 0 & 0 & 0 & 0 & \cdots \\
a^{(2)}_1 & a^{(2)}_1 & 0 & 0 & 0 & \cdots \\
a^{(3)}_1 & 0 & a^{(3)}_1 & 0 & 0 & \cdots \\
a^{(4)}_1 & a^{(4)}_1 & 0 & a^{(4)}_1 & 0 & \cdots \\
a^{(5)}_1 & 0 & 0 & 0 & a^{(5)}_1 & \cdots \\
a^{(6)}_1 & a^{(6)}_1 & a^{(6)}_2 & 0 & 0 & a^{(6)}_1 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\begin{pmatrix}
c^{(1)}_1 & 0 & 0 & 0 & 0 & \cdots \\
c^{(2)}_1 & c^{(2)}_1 & 0 & 0 & 0 & \cdots \\
c^{(3)}_1 & 0 & c^{(3)}_1 & 0 & 0 & \cdots \\
c^{(4)}_1 & c^{(4)}_1 & 0 & c^{(4)}_1 & 0 & \cdots \\
c^{(5)}_1 & 0 & 0 & 0 & c^{(5)}_1 & \cdots \\
c^{(6)}_1 & c^{(6)}_1 & c^{(6)}_2 & 0 & 0 & c^{(6)}_1 \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Obviously,
\[ [n, \rightarrow] A = [n, \rightarrow] \langle a^{ln_m}(s), 1 \rangle , \]
\[ [n, \rightarrow] C = [n, \rightarrow] \langle 1 + (\log a(s))'a^{ln_m}(s), 1 \rangle. \]

Since
\[ (1 + a'(s)a^{-1}(s))a^{ln_m}(s) = a^{ln_m}(s) + \frac{1}{\ln m} (a^{ln_m}(s))', \]

or
\[ [n^{-s}](1 + (\log a(s))')a^{ln_m}(s) = \frac{\ln(m/n)}{\ln m} [n^{-s}]a^{ln_m}(s), \]

it follows that
\[ [(nm)^{-s}]Am^{-s}(1 + (\log a(s))')a^{-ln_m}(s) = [(nm)^{-s}]Cm^{-s}a^{-ln_m}(s) \]
\[ = [n^{-s}](1 + (\log a(s))')a^{ln_n}(s) = \begin{cases} 1, & n = 1, \\ 0, & n > 1. \end{cases} \]

Thus,
\[ A = (1 - (\log b(s))', b(s)), \quad C = (1, b(s)), \]
\[ [n^{-s}](1 - (\log b(s))')b^{ln_m}(s) = \frac{\ln mn}{\ln m} [n^{-s}]b^{ln_m}(s) = [n^{-s}]a^{ln_mn}(s), \]
\[ [n^{-s}]b^{ln_m}(s) = [n^{-s}](1 + (\log a(s))')a^{ln_mn}(s) = \frac{\ln m}{\ln mn} [n^{-s}]a^{ln_mn}(s). \]

We set
\[ \langle 1, a^{-\beta}(s) \rangle^{-1} = \langle 1, (\beta)a^\beta(s) \rangle. \]
Then
\[ [n^{-s}] \beta a^{\beta \ln n} = \frac{\beta \ln m}{\beta \ln m + \beta \ln n} [n^{-s}] a^{\beta \ln m + \beta \ln n}(s). \]

Let \( h_n(\varphi) \) be the convolution polynomials of the series \( a(s) \). Then
\[ (\beta) a^\varphi(s) = \sum_{n=1}^{\infty} \frac{\varphi}{\varphi + \beta \ln n} h_n(\varphi + \beta \ln n)n^{-s}. \]

Note that the identity
\[ [n, \to] (1 + (\log a(s)))^t, a^{-1}(s))^{-1} = [n, \to] (a^{\ln n}(s), 1) \]
implies a formula similar to that of Lagrange series expansion:
\[ \frac{f(s)}{1 + (\log a(s))^t} = \sum_{n=1}^{\infty} \frac{n^{-s}}{a^{\ln n}(s)} [n^{-s}] f(a^{\ln n}(s)). \]

4. SOME EXAMPLES

In this section, we show how one can use the matrices \( (1, a(s)) \) in the following construction. To every formal power series \( a(x) \), \( a_0 = 1 \), with convolution polynomials \( s_n(\varphi)/n! \), i.e.,
\[ a^\varphi(x) = \sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} x^n, \]
we assign a formal Dirichlet series \( A(s) \) with convolution polynomials \( h_n(\varphi) \) such that
\[ A^\varphi(s) = \prod_{p=2}^{\infty} \left( \sum_{n=0}^{\infty} \frac{s_n(\varphi)}{n!} p^{-ns} \right) = 1 + \sum_{n=2}^{\infty} \frac{s_{m_1}(\varphi)s_{m_2}(\varphi)\cdots s_{m_r}(\varphi)}{m_1!m_2!\cdots m_r!} n^{-s} = \sum_{n=1}^{\infty} h_n(\varphi)n^{-s}, \]
where the product is over all primes and \( n = p_1^{m_1}p_2^{m_2}\cdots p_r^{m_r} \) is the prime factorization of a number \( n \). Under this correspondence, the series \( \log a(x) \) and \( \log A(s) \) are connected by the following conditions: suppose that \([x^m]\log a(x) = b_m\); then \([n^{-s}]\log A(s) = b_m \) if \( n = p^m \), where \( p \) is a prime, and \([n^{-s}]\log A(s) = 0 \) otherwise. Let \( \varepsilon(s) \) denote the Dirichlet series corresponding to the exponential series:
\[ \varepsilon^\varphi(s) = 1 + \sum_{n=2}^{\infty} \frac{\varepsilon^{m_1+m_2+\cdots+m_r}}{m_1!m_2!\cdots m_r!} n^{-s}. \]

Note that \( \log \varepsilon(s) = \sum p^{-s} \), where the summation is over all primes. Let \( 1 \varepsilon(s) \) denote the series related to \( \varepsilon(s) \) by Theorem 3. Then
\[ [n^{-s}] \varphi^\varepsilon(s) = \frac{\varphi^{v(n)}}{f(n)}, \quad [n^{-s}] 1 \varepsilon^\varphi(s) = \frac{\varphi(\varphi + \ln n)^{v(n)-1}}{f(n)}, \]
where
\[ v(1) = 0, \quad v(n) = m_1 + m_2 + \cdots + m_r, \quad f(1) = 1, \quad f(n) = m_1!m_2!\cdots m_r!. \]
The identity \( (1)^{\varphi+\beta}(s) = (1)^{\varphi(s)} (1)^{\beta(s)} \) gives the following analog of the generalized Abel binomial formula:
\[ (\varphi + \beta)(\varphi + \beta + \ln n)^{v(n)-1} = \sum_{d|n} \binom{n}{d}_f \varphi(\varphi + \ln d)^{v(d)-1} \beta(\beta + \ln \left(\frac{n}{d}\right))^{v(n/d)-1}, \]
where
\[ \binom{n}{d}_f = \frac{f(n)}{f(d)f(n/d)}. \]
**Remark 4.** A generalization of the Abel polynomials (see [11, p. 117] and [12, p. 114]) is the polynomials \(A_M(x) = x(x + t_M)^{|M| - 1}\) with generating function
\[
\sum_M A_M(x)\tau_M = e^{xF(\tau)}, \quad F(\tau) = \sum_M t_M^{|M| - 1}\tau_M,
\]
where
\[
t_M = t_{i_1} + t_{i_2} + \cdots + t_{|M|} = \sum_{i \in M} t_i, \quad \tau_M = \tau_{i_1}\tau_{i_2}\cdots\tau_{|M|} = \prod_{i \in M} \tau_i,
\]
t_1, t_2, t_3, \ldots, \tau_1, \tau_2, \tau_3, \ldots \text{ are two infinite series of parameters, } M = \{i_1, i_2, \ldots, i_{|M|}\} \text{ is a finite (possibly empty) subset of the set of positive integers, and } |M| \text{ is the number of elements of } M. \text{ The following formula holds:}
\[
A_M(x + y) = \sum_{I \cup J = M} A_I(x)A_J(y),
\]
where the summation is over all ordered partitions of the set \(M\) into two disjoint subsets \(I\) and \(J\). This, for the substitution \(t_{i_k} = 1, \, |M| = n\), we obtain the generalized Abel binomial formula; for the substitution \(t_{i_k} = \ln \tilde{p}_k\), where \(\tilde{p}_k\) stands for the \(k\)th prime factor in the canonical decomposition of \(n\), i.e.,
\[
n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r} = \tilde{p}_1 \tilde{p}_2 \tilde{p}_3 \cdots \tilde{p}_{v(n)}, \quad |M| = v(n),
\]
we obtain formula (2).

Since
\[
[n^{-s}](1 - (\log(1)\varepsilon(s)))^{(1)}\varepsilon^\varphi(s) = [n^{-s}]\varepsilon^{\varphi + \ln n(s)}
\]
it follows from
\[
(1 - (\log(1)\varepsilon(s)))^{(1)}\varepsilon^{\varphi + \beta}(s) = (1 - (\log(1)\varepsilon(s)))^{(1)}\varepsilon^\varphi(s)(1)\varepsilon^\beta(s)
\]
that
\[
(\varphi + \beta + \ln n)^{v(n)} = \sum_{d | n} \binom{n}{d}_f (\varphi + \ln d)^{v(d)} \beta(\beta + \ln(n/d))^{v(n/d) - 1}.
\]

Since
\[
[n^{-s}]\langle 1, a(s) \rangle b(s) = \sum_{d | n} b_d a_{n/d}, \quad a_{n/d} = \left[\left(\frac{n}{d}\right)^{-s}\right] a^{\ln d}(s),
\]
it follows from
\[
(1)\varepsilon^\varphi(s) = \langle 1, (1)\varepsilon(s) \rangle \varepsilon^\varphi(s), \quad \varepsilon^\varphi(x) = \langle 1, \varepsilon^{-1}(s) \rangle (1)\varepsilon^\varphi(s)
\]
that
\[
\varphi(\varphi + \ln n)^{v(n) - 1} = \sum_{d | n} \binom{n}{d}_f \varphi^{v(d)} \ln d(\ln n)^{v(n/d) - 1},
\]
\[
\varphi^{v(n)} = \sum_{d | n} \binom{n}{d}_f \varphi(\varphi + \ln d)^{v(d) - 1}(\ln(1/d))^{v(n/d)}.
\]

For \(n = p^m\), where \(p\) is a prime, the formulas thus obtained take the form of the Abel identities [3] and [13, pp. 96–103 (Russian transl.)]:
\[
(\varphi + \beta)(\varphi + \beta + ma)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} \varphi(\varphi + ka)^{k-1}\beta(\beta + (m - k)a)^{m-k-1},
\]
Then since the Lagrange series:

\[(\varphi + \beta + ma)^m = \sum_{k=0}^{m} \binom{m}{k} (\varphi + ka)^k \beta (\beta + (m - k)a)^{m-k-1},\]

\[\varphi (\varphi + ma)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} \varphi^k ka (ma)^{m-k-1},\]

\[\varphi^m = \sum_{k=0}^{m} \binom{m}{k} \varphi (\varphi + ka)^{k-1} (-ka)^{m-k}, \quad a = \ln p.\]

Let us generalize this example. Suppose given a series \(A(s)\) such that

\[[n^{-s}] A^\varphi(s) = \frac{u_n(\varphi)}{f(n)},\]

where

\[u_n(\varphi) = s_{m_1}(\varphi) s_{m_2}(\varphi) \cdots s_{m_r}(\varphi), \quad n = p_1^{m_1} p_2^{m_2} \cdots p_r^{m_r}.\]

Then

\[[n^{-s}] (\beta) A^\varphi(s) = \frac{\varphi}{\varphi + \beta \ln n} \frac{u_n(\varphi + \beta \ln n)}{f(n)},\]

\[(\beta) A^\varphi(s) = (1, (\beta) A^\beta(s)) A^\varphi(s), \quad A^\varphi(s) = (1, A^{-\beta}(s)) (\beta) A^\varphi(s),\]

\[\frac{\varphi}{\varphi + \beta \ln n} u_n(\varphi + \beta \ln n) = \sum_{d|n} \binom{n}{d} u_d(\varphi) \frac{\ln d}{\ln n} u_{n/d}(\beta \ln n),\]

\[u_n(\varphi) = \sum_{d|n} \binom{n}{d} \frac{\varphi}{\varphi + \beta \ln d} u_d(\varphi + \beta \ln d) u_{n/d}(\beta \ln (1/d)).\]

Since \(u_p^m(\varphi) = s_m(\varphi)\), it follows that, for \(n = p^m\), the formulas become mutually inverse relations for the Lagrange series:

\[\frac{\varphi}{\varphi + ma} s_m(\varphi + ma) = \sum_{k=0}^{m} \binom{m}{k} s_k(\varphi) \frac{k}{m} s_{m-k}(ma),\]

\[s_m(\varphi) = \sum_{k=0}^{m} \binom{m}{k} \frac{\varphi}{\varphi + ka} s_k(\varphi + ka) s_{m-k}(-ka), \quad a = \beta \ln p.\]

We also give identities for the coefficients \(\binom{n}{d}_f\) which are similar to the identities

\[\sum_{k=0}^{n} \binom{n}{k} = 2^n, \quad \sum_{k=0}^{n} \binom{n}{k} k(-1)^{n-k} = 0, \quad n \neq 1.\]

Since \(\varepsilon(s)\varepsilon(s) = \varepsilon^2(s)\) and \(\varepsilon'(s)\varepsilon^{-1}(s) = (\log \varepsilon(s))'\), it follows that

\[\sum_{d|n} \binom{n}{d}_f = 2^{\nu(n)}, \quad \sum_{d|n} \binom{n}{d}_f \ln d(-1)^{\nu(n/d)} = 0, \quad n \neq p.\]

REFERENCES

1. L. W. Shapiro, S. Getu, W. J. Woan and L. C. Woodson, “The Riordan group,” Discrete Appl. Math. 34 (1–3), 229–339 (1991).
2. R. Sprugnoli, “Riordan arrays and combinatorial sums,” Discrete Math. 132, 267–290 (1994).
3. R. Sprugnoli, “Riordan arrays and Abel–Gould identity,” Discrete Math. 142, 213–233 (1995).
4. D. Merlini, D. G. Rogers, R. Sprugnoli and M. C. Verri, “On some alternative characterizations of Riordan arrays,” Canad. J. Math. 49 (2), 301–320 (1997).
5. W. Wang and T. Wang, “Generalized Riordan arrays,” Discrete Math. 308, 6466–6500 (2008).
6. P. Barry, A Study of Integer Sequences, Riordan Arrays, Pascal-like Arrays and Hankel Transforms
   (University College Cork, 2009).
7. D. E. Knuth, “Convolution polynomials,” Mathematica J. 2 (4), 67–78 (1992).
8. A. Sowa, “Factorizing matrices by Dirichlet multiplication,” Linear Algebra Appl. 438 (5), 2385–2393
   (2013).
9. A. Sowa, “The Dirichlet Ring and Unconditional Bases in $L^2[0,2\pi]$,” Funktsional. Anal. i Prilozhen.
   47 (3), 75–81 (2013) [Funct. Anal. Appl. 47 (3), 227–232 (2013)].
10. A. Sowa, “On the Dirichlet matrix operators in sequence spaces,” Appl. Math. (Warsaw) 44 (2), 185–196
    (2017).
11. S. K. Lando, Lectures on Generating Functions (MTsNMO, Moscow, 2007) [in Russian].
12. S. K. Lando, Introduction to Discrete Mathematics (MTsNMO, Moscow, 2012) [in Russian].
13. J. Riordan, Combinatorial Identities (Robert E. Krieger Publishing Co., Huntington, NY, 1979; Nauka,
    Moscow, 1982).