CONTROLLABILITY FOR A STRING WITH ATTACHED MASSES AND RIEZS BASES FOR ASYMMETRIC SPACES

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Abstract. We consider the problem of boundary control for a vibrating string with N interior point masses. We assume the control of Dirichlet, or Neumann, or mixed type is at the left end, and the string is fixed at the right end. Singularities in waves are “smoothed” out to one order as they cross a point mass. We characterize the reachable set for an $L^2$ control. The control problem is reduced to a moment problem, which is then solved using the theory of exponential divided differences in tandem with unique shape and velocity controllability results. The results are sharp with respect to both the regularity of the solution and with respect to time. The eigenfunctions of the associated Sturm–Liouville problem are used to construct Riesz bases for a family of asymmetric spaces that include the sets of reachable positions and velocities.

1. Introduction. There has been much interest in so called “hybrid systems” in which the dynamics of elastic systems and possibly rigid structures are related through some form of coupling. The study of controllability and stabilization of such structures has made in a number of works, see [31] and [24] and references therein, also [26]. An important feature that sometimes arises in such systems is that they are well posed with respect to asymmetric spaces. Examples of these systems include strings with attached masses and many other hybrid systems, such as the so-called cascade systems of the wave equations, where controls are applied directly to the first system and every next system is controlled through the previous one (see [5, 15] and references therein).

In this paper, we consider the controllability of a vibrating string with N attached masses. The controllability of a string with a single attached mass was considered in [28], [20], [21], also [34], and more recently in [6], [18] and [19]. In proving our
controllability results we construct Riesz bases of the associated asymmetric spaces. Our method of constructing Riesz bases should be applicable in other settings, and thus provides a potentially useful tool for studying the control theory and analysis of the associated systems.

We consider the wave equation on the interval $[0, \tilde{\ell}]$ with masses $M_j > 0$ attached at the points $\tau_j$, $j = 1, \ldots, N$, where $0 = \tau_0 < \tau_1 < \ldots < \tau_N < \tau_{N+1} = \tilde{\ell}$. We will assume the string and masses are at rest and at equilibrium at time $t = 0$, when a control is applied at the left end of the string with the right end fixed. In what follows, $y_j(\tau, t)$ will denote the vertical displacement of the string in the interval $(\tau_j, \tau_{j+1})$, and $h_j(t)$ will denote the vertical displacement of the mass at $\tau = \tau_j$.

On each interval $(\tau_j, \tau_{j+1})$, let $\rho_j(\tau)$ represent the density of the string, $\xi_j(\tau)$ the tension, and $q_j(\tau)$ some potential. Small amplitude vibration of this system is modeled by the following system,

$$\rho_j(\tau) \frac{\partial^2 y_j}{\partial \tau^2} - \frac{\partial}{\partial \tau} \left( \xi_j(\tau) \frac{\partial y_j}{\partial \tau} \right) + q_j(\tau) y_j = 0, \quad t > 0, \quad \tau \in (\tau_j, \tau_{j+1}), \quad j \geq 0, \quad (1.1)$$

with the motion of the masses governed by

$$y_{j-1}(\tau_j^+, t) = h_j(t) = y_j(\tau_j^+, t), \quad t > 0, \quad j \geq 1, \quad (1.2)$$

$$M_j h_j''(t) = \xi(\tau_j^+) \frac{\partial y_j}{\partial \tau}(\tau_j^+, t) - \xi(\tau_j^-) \frac{\partial y_{j-1}}{\partial \tau}(\tau_j^-, t), \quad t > 0, \quad j \geq 1. \quad (1.3)$$

Here $y(\tau_j^+, t) := \lim_{\varepsilon \to 0^+} y(\tau_j + \varepsilon, t)$ for fixed $t$, and $y(\tau_j^-, t)$ is defined similarly. The initial conditions are given by

$$y_j(\tau, 0) = \frac{\partial y_j}{\partial t}(\tau, 0) = 0, \quad \tau \in (\tau_j, \tau_{j+1}), \quad j \geq 0, \quad (1.4)$$

$$h_j(0) = h_j'(0) = 0, \quad j \geq 1, \quad (1.5)$$

and boundary conditions are

$$y_N(\tau_{N+1}^-, t) = 0, \quad t > 0, \quad (1.6)$$

$$\alpha_1 y_0(0^+, t) + \alpha_2 \frac{\partial y_0}{\partial \tau}(0^+, t) = f(t), \quad t > 0. \quad (1.7)$$

We will refer to $f(t)$ as the control. We assume throughout that $(\alpha_1, \alpha_2) \neq (0, 0)$. Note that $\alpha_2 = 0$ corresponds to Neumann control, while $\alpha_1 = 0$ corresponds to Neumann control. In what follows, we will refer to all cases $\alpha_2 \neq 0$ as “mixed control”.

Throughout this paper we assume $\rho_j, \xi_j \in C^{j+2}[\tau_j, \tau_{j+1}]$ and $q_j \in C^{\max(0, j-2)}[\tau_j, \tau_{j+1}]$ for each $j$. We also assume $\rho_j, \xi_j, q_j$ are all real valued, and that $\xi_j(\tau) > 0$ and $\rho_j(\tau) > 0$ for all $\tau$ and $j$, so the functions $\xi_j, \rho_j$ are all bounded away from zero. In what follows $H^3(\tau_j, \tau_{j+1})$ refers to the standard Sobolev space, with $H^0 = L^2$.

One of the most important features of this problem is that the attached masses will mollify transmitted waves, so that $y_{j+1}$ will be one Sobolev order more regular than $y_j$. The reason for this fact, first observed in [28], can be outlined as follows in the case $N = 1$ and Dirichlet control. As we shall see in Section 3, see 3.59, 1.3 implies the following second order differential-integral equation of Volterra type:

$$h_1''(t) + c_1 h_1'(t) + c_2 h_1(t) + \int_{s=0}^{\tilde{\ell}} K(s) h_1(t - s) ds = c_3 f'(t) + G(t), \quad (1.8)$$
with $c_j$ constants, $K$ continuous, and $G$ has the same order of Sobolev regularity as $f$. A straightforward argument now proves that $h_1$ is one Sobolev order more regular than $f$. Since $h_1$ can be viewed as a Dirichlet control for the interval $(\tau_1, \tau_2)$, it can be shown that $y_1$ has the same regularity with respect to $(x,t)$ as $h_1$, whereas $y_0$ has the same regularity as $f$. The case of mixed control is different because, by 3.37, the right hand side of the analogue of 1.8 will have the same order of regularity as $f$, so $h_1$ will have two more orders of regularity than $f$.

In this introduction, we will state only the results for mixed control; the analogues for Dirichlet control will be found in later sections.

We will now discuss the well-posedness of our system in the general case. With the mollification effect in mind, in Section 3, we will define a family of Hilbert spaces, labeled $W_i$, with $i = 0, 1$, which are subspaces of $\oplus_{j=0}^N H^{1+\gamma_j}(\tau_j, \tau_{j+1}) \oplus \mathbb{R}^N$. Here the terms in $\mathbb{R}^N$ account for the position of the masses, and the superscript $\mathcal{M}$ refers to mixed control. We will prove the following:

**Proposition 1.** Suppose $\alpha_2 \neq 0$. For any $T > 0$, let $f \in L^2(0, T)$. There exists a unique solution

$$u^f := (y_0, h_1, y_1, h_2, ..., h_N, y_N),$$

to System 1.1-1.7. Furthermore,

$$u^f \in C_i([0, T], W_{i-1}^{\mathcal{M}}), \quad i = 0, 1.$$

Define

$$\ell := \int_0^\ell \sqrt{\frac{p(s)}{\xi(s)}} ds.$$ 

Then it is not hard to see that $\ell$ is the time required for a wave to cross the full length of the string (see, e.g. [35, Sec. 2.1]). We now state our main results.

**Theorem 1.1.** (Mixed control). Consider System 1.1-1.7 with $\alpha_2 \neq 0$. Let $T > 2\ell$. Then for any $(\phi_0, \phi_1) \in W_1^{\mathcal{M}} \times W_0^{\mathcal{M}}$, there exists a control $f \in L^2(0, T)$ such that

$$u^f(\cdot, T) = \phi_0(\cdot), \quad u^f_t(\cdot, T) = \phi_1(\cdot).$$

Furthermore,

$$\|f\|_{L^2(0, T)}^2 \asymp \|\phi_0\|_{W_1^{\mathcal{M}}}^2 + \|\phi_1\|_{W_0^{\mathcal{M}}}^2.$$ 

Here and below, $\|f\|_A \asymp \|g\|_B$ means there exist positive constants $C_1, C_2$ such that $C_1\|f\|_A \leq \|g\|_B \leq C_2\|f\|_A$ for all $f \in A, g \in B$.

In what follows, we will refer to the conclusion of Theorem 1.1 as “full controllability”.

Clearly Theorem 1.1 is sharp with respect to the regularity of the solution. It is also sharp with respect to time, as the following shows.

**Theorem 1.2.** System 1.1-1.7 with mixed control is not approximately controllable for $T \leq 2\ell$, i.e. the reachable set $\{ (u^f(\cdot, T), u^f_t(\cdot, T) : f \in L^2(0, T) \}$ is not dense in $W_1^{\mathcal{M}} \times W_0^{\mathcal{M}}$. If $T = 2\ell$, then its codimension is equal to $N$, and for $T < 2\ell$ its codimension is infinite.

One novelty of this paper and [6] is a new method to prove controllability that combines dynamical and spectral approaches. We now state one of the key results.
from our dynamical approach for mixed control; the analogue for Dirichlet control can be found in Section 4. Define a variable \( x(\tau) = \int_0^\tau \sqrt{\frac{\mu(x)}{\xi(x)}} \, ds \). Define

\[
W_{i}^{T,M} = \left\{ \phi(\tau) \in W_i^M : \phi(\tau) = 0 \text{ for } \tau > x(T), \quad T < \ell, \right. \\
\left. \phi(\tau) \in W_i^M : \frac{d^2\phi}{d\tau^2}(\ell) = 0, \quad j = 0, ..., (N + i) \right\}, \\
T = \ell, \\
T > \ell.
\]

**Theorem 1.3.** Assume \( \alpha_2 \neq 0 \). Let \( T > 0 \).

A) Let \( \phi \in W_1^{T,M} \). Then there exists \( f \in L^2(0,T) \) such that \( u^f(x,T) = \phi(x) \) and

\[
\|u^f(\cdot,T)\|_{W_1^M} \asymp \|f\|_{L^2(0,T)}.
\]

If \( T \leq \ell \), then this \( f \) is unique.

B) Let \( \phi \in W_0^{T,M} \). Then there exists \( f \in L^2(0,T) \) such that \( u^f(x,T) = \phi(x) \) and

\[
\|u^f(\cdot,T)\|_{W_0^M} \asymp \|f\|_{L^2(0,T)}.
\]

If \( T \leq \ell \), then this \( f \) is unique.

In addition to being a vital ingredient in the proof of Theorem 1.1, this result is interesting in its own right. Such results are central to the so called “boundary control method” in inverse problems, (see eg. [13], [17] and references therein). The authors are unaware of the results of Theorem 1.3 for mixed control, or even Neumann control for \( N > 0 \). In what follows, we will refer to Part A as “shape control” and Part B as “velocity control”. To prove Theorem 1.3, we derive a representation of the solution of the wave equation using a solution of the Goursat-type problem. This representation reduces the shape and velocity control problems to Volterra type integral equations of the second kind.

To discuss the spectral approach to our problem, in Section 2, we transform the Sturm–Liouville problem associated to 1.1.1.7 into

\[
-\phi''(x) + q(x)\phi(x) = \lambda^2 \phi(x), \quad x \in (0, \ell) - \{a_j\}^N_1, \tag{1.9}
\]

\[
\phi(0) = 0, \tag{1.10}
\]

\[
\beta_1 \phi(0) + \beta_2 \phi'(0) = 0, \tag{1.11}
\]

\[
\phi(a_j^+) = \phi(a_j^-), \tag{1.12}
\]

\[
\rho_j^+ \phi'(a_j^-) = \rho_j^- \phi'(a_j^+) - (M_j \lambda^2 + \delta_j)\phi(a_j^+), \quad j = 1, \ldots, N, \tag{1.13}
\]

where the explicit formulas for \( q, \rho_j^+, \rho_j^-, \delta_j \) and \( \beta_j \) are all given in that section. We remark that \( \beta_2 \neq 0 \) is equivalent to \( \alpha_2 \neq 0 \). Let \( \{\lambda_n^2 : n \in \mathbb{N}\} \) be the set of eigenvalues for System 1.9.1.13, listed in increasing order. Taking (possibly complex) square roots, we then define the associated frequencies \( \Lambda := \{\lambda_k : k = \pm 1, \pm 2, \ldots\} \). The asymptotics of \( \Lambda \), for both \( \beta_2 = 0 \) and \( \beta_2 \neq 0 \), will be presented in Section 5.

In the spectral approach to proving Theorem 1.1, we express \( u^f \) as a Fourier series, which allows us to rephrase the control problem as a moment problem. Solving this moment problem, which involves the exponential family \( \{e^{it\lambda_k}\} \), is non-trivial because generally there does not exist a uniform gap in the frequencies. To overcome this, we will parametrize the frequencies as

\[
\{\lambda_j^p : p \in \mathbb{N}, \ j = 1, \ldots, N^p]\}.
where $N^{(p)} \leq N + 1$. This parametrization, defined in Section 6, is designed so that for each $p$, the set $\{\lambda_1^p, ..., \lambda_{N^{(p)}}^p\}$ are close together. We then use this parametrization to form Riesz sequences of exponential divided differences, with which we can then solve the moment problem. This will give a spectral characterization of the reachable set, but the general statement of the Theorem 1.1, has not yet been proven. We now describe the key ingredient in finishing the proof. Let $\varphi_j^p$ be a unit norm eigenfunction associated to $\lambda_j^p$, and let $A$ be the self-adjoint operator associated to our Sturm–Liouville problem. We use the associated moment problem in tandem with Theorem 1.3 to prove the following:

**Theorem 1.4.** Suppose $\beta_2 \neq 0$, and assume $A$ is invertible. Then for $i = 0, 1$,

$$\{\sum_{j=k}^{N^p} \frac{\varphi_j^p(0)\varphi_j^p(x)}{(\lambda_j^p)^p} \prod_{i=1}^{k-1}(\lambda_j^p - \lambda_i^p) \mid p \geq 1, k = 1, ..., N^{(p)}\} \text{ forms a Riesz basis of } W_i^M.$$ 

This result will establish a Hilbert space isomorphism between the moment space and the space $W_i^M \times W_0^M$, which will complete the proof of Theorem 1.1. The proofs of these results appear in Section 7. In the case where $A$ is non-invertible, Theorem 1.1 and a version of Theorem 1.4 still hold; the proof of this also indicated in Section 7.

Let $E$ be a positive constant so that $(A + E) > 0$. Then we use $(A + E)$ to define $W_i^M$ for all integers $i$. We will also show in Section 7.2 that Theorem 1.4 holds for any integer $i$. Our final theorem in this section pertains to regular controls for regular spaces. In Section 7.3, we prove

**Theorem 1.5.** Assume $A$ is invertible. Suppose $T > 2\ell$, and assume $\rho_j, \xi_j, q_j \in C^\infty[\tau_j, \tau_{j+1}]$ for each $j$. Let $(\phi_0, \phi_1) \in W_{2i+1}^M \times W_{2i}^M$ for $i \geq 1$. Then there exists $g \in H_0^2(0, T)$ solving

$$u^g(x, T) = \phi_0(x), \quad u^g_T(x, T) = \phi_1(x), \quad (1.14)$$

with

$$||g||_{H^2} \asymp ||\phi_0||_{W_i^M}^2 + ||\phi_1||_{W_{2i}^M}^2.$$ 

We believe Theorem 1.5 might also be provable using results in [25] together with our Theorem 1.1. The results in [25] have the advantage that they can be applied in rather general settings, but our proof follows easily from the framework constructed in this paper.

We now compare this paper to other results in the literature. This paper, along with [6], were inspired by the paper of Hansen and Zuazua, [28]. In that work, the authors consider only Dirichlet control with $N = 1$. Their main result is that for $T > 2\ell$, there is exact controllability for terminal data in $W_0^2 \times W_0^2$. Their method of proof is different from ours, and uses the theory of characteristics for the constant coefficient wave equation, along with an energy estimate, to prove an observability inequality. For these proofs, they assume first $\rho_j, \xi_j$, are constants and $q_j = 0$, and then they give some indication of how their results can be extended to the case of variable coefficients. It is not clear that the methods in their paper can easily be extended to our setting with many masses, and they state that this case is beyond the scope of their paper. The authors also consider the special case where $\rho_j = \xi_j = 1$, $q_j = 0$ and $a_1 = \ell/2$, where they combine a spectral argument with their main result to prove exact Dirichlet controllability in the sharp time of $T = 2\ell$. In this special case, they also use exact controllability to construct a Riesz basis for
the observable space. A purely spectral proof of full Dirichlet controllability in the case \( N = 1, \rho_j = \xi_j = 1, q_j = 0, \) and \( a_1 = \ell/2 \) is given in [21], see also [20].

During the preparation of this manuscript, two manuscripts by Ben Amara and Beldi came to our attention, [18], [19]. In the first of these papers, the authors consider a vibrating string with one attached mass, variable \( \rho_j, \xi_j, \) and \( q_j, \) and Dirichlet control applied at one end with the other end fixed, and they prove exact controllability. Their proof uses a precise analysis of the eigenvalue and eigenfunction asymptotics, and a version of Ingham’s inequality that applies in weakened gap conditions [16]. It is not clear that their methods would apply in the \( N \) mass case.

In the second of their papers, they again consider the case of one attached mass, variable \( \rho_j, \xi_j, \) and \( q_j, \) and Neumann control with the other end fixed, and they prove Theorem 1.1. They also require a gap condition in the frequency spectrum, which will hold only with special cases of \( N = 1. \) An advantage of our method is that we require no such gap condition either for Neumann or for mixed control.

Another advantage of our work is that we prove our results are sharp with respect to time.

We now compare our results with [6]. The current work is an improvement in several ways. First, in [6] we assumed \( \rho_j = \xi_j = 1 \) for all \( j, \) and second we only considered Dirichlet control. Third, in the present paper we prove full controllability in the optimal time \( T = 2\ell \) in the case \( N = 1 \) with Dirichlet control, whereas in all other cases Theorems 1.2 and 7.3 show that full controllability in the optimal time \( T = 2\ell \) is not generally possible. Fourth, Theorem 1.5 is new and has no analogues in [6]. Fifth, in Section 7, we also extend Theorem 1.4, and its Dirichlet analogue, from its current \( i = 0, 1 \) to all positive integers \( i. \) We have applied these extensions in [7] to study the control theory associated to a Schrödinger type equation with strong point singularities. In future work we will use this result to study the control theory for a chain of beams connected by point masses [8]. In addition, the proofs of our results here extend the techniques developed in [6] in several directions, including reduction of System 1.1-1.7 to the case \( \rho = \xi = 1, \) and solving a Goursat type problem with mixed boundary conditions. Another complication that arises in this paper is that the generating function associated to the eigenfrequencies is harder to study in this setting – certain formulas that were proven in [6] using trigonometric identities must be proven in a different way here because the coefficient functions are not continuous across the masses.

The rest of this paper is organized as follows. In Section 2, we transform System 1.1-1.7 to the case \( \rho = \xi = 1, \) and we define a scale of Sobolev type spaces using the associated space operator. In Section 3, we represent the solution to 1.1-1.7, and define the spaces \( W^b \) for \( b = M \) or \( D. \) In Section 4, we discuss our shape and velocity control results. In Section 5, we further study the spectrum of the associated space operator, and state the asymptotics of the frequencies. In Section 6, we study the associated generating functions and construct Riesz bases and Riesz sequences of \( L^2(0,T) \) using exponential divided differences. In Section 7, we prove Theorem 1.1 and state Theorem 7.2, which is the analogue for Dirichlet control. In this section, we also prove Theorems 1.2, 1.4, and 1.5 along with their counterparts for Dirichlet control. Finally, in the appendix, we provide some of the technical proofs. In this section, we also consider some examples where there is a spectral gap in the Neumann-Dirichlet case, and discuss proving exact controllability without the use of exponential divided differences.
2. Transformation of control problem and associated Sturm–Liouville problem. In this section, we will transform the our control problem and its associated Sturm–Liouville problem.

In this section, we will transform the our control problem and its associated Sturm–Liouville problem into a more convenient form. The self-adjoint operator $A$ associated to the transformed Sturm–Liouville problem will then be used to define a scale of Sobolev-type spaces that will be useful in studying our control problem.

We assume for this section that the control $f$ and all the coefficients are sufficiently regular that the solution will be classical. This implies we can use $y(\tau_{j-1}, t) = h_j(t) = y(\tau_j^+, t)$, and allows us to write $h_j(t) = y(\tau_j, t)$. In what follows, when convenient we drop the subscript $j$ and the superscript $f$ for readability. We rewrite 1.1–1.7:

$$
\rho(\tau) \frac{\partial^2 y}{\partial t^2} - (\xi(\tau)y_\tau)\tau + \tilde{q}(\tau)u = 0, \quad t > 0, \quad \tau \in (0, \tilde{\ell}) \setminus \{\tau_j\}_{j=1}^{N}, \quad (2.15)
$$

$$
y(\tau, 0) = y_i(\tau, 0) = 0, \quad \forall \tau,
$$

$$
y(\tau_j^+, t) = y(\tau_j, t) = y(\tau_j^+, t), \quad t > 0, \quad j \geq 1,
$$

$$
M_j y_\ell(\tau_j, t) = \xi(\tau_j^+) y_\ell(\tau_j^+, t) - \xi(\tau_j^-) y_\ell(\tau_j^-, t), \quad t > 0,
$$

$$
y(\tilde{\ell}, t) = 0, \quad t > 0, \quad (2.16)
$$

with control

$$
\alpha_1 y(0, t) + \alpha_2 y_\ell(0, t) = f(t), \quad t > 0, \quad (2.17)
$$

with $(\alpha_1, \alpha_2) \neq (0, 0)$. Here $0 = \tau_0 < \tau_1 < \ldots < \tau_{N+1} = \tilde{\ell}$. In what follows, the set of all equations from 2.15 to 2.17 as System 2.15-2.17.

We begin by considering the associated Sturm–Liouville problem,

$$
-(\xi \phi')' + q \phi = \rho \lambda^2 \phi, \quad x \in (0, \tilde{\ell}) \setminus \{\tau_j\}^N, \quad (2.18)
$$

$$
\phi(\tilde{\ell}) = 0,
$$

$$
\alpha_1 \phi(0) + \alpha_2 \phi'(0) = 0,
$$

$$
\phi(\tau_j^-) = \phi(\tau_j) = \phi(\tau_j^+),
$$

$$
\xi(\tau_j^+) \phi'(\tau_j^+) - \xi(\tau_j^-) \phi'(\tau_j^-) = -\lambda^2 M_j \phi(\tau_j), \quad j = 1, \ldots, N. \quad (2.19)
$$

Thus $\alpha_2 = 0$ corresponds to Dirichlet boundary conditions at $x = 0$ (and also at $x = \tilde{\ell}$), and we will refer this case as $\mathcal{D} - \mathcal{D}$. All cases with $\beta_2 \neq 0$ will be referred to as $\mathcal{M} - \mathcal{D}$. Associated to this problem is the Hilbert space

$$
L^2_{M, \rho}(0, \tilde{\ell}) = \{ v \in L^2(0, \tilde{\ell}) \} \oplus \mathbb{R}^N,
$$

with norm

$$
\|v\|_{L^2_{M, \rho}}^2 = \sum_{j=0}^{N} \int_{\tau_j}^{\tau_{j+1}} |v(\tau)|^2 \rho(\tau) d\tau + \sum_{j=1}^{N} M_j |v(\tau_j)|^2.
$$

We now transform System 2.15-2.17. We set $\tilde{\rho} = \rho/\xi$. Define the function $x$ by

$$
x(\tau) = \int_0^\tau (\tilde{\rho}(s))^{1/2} ds,
$$

and its inverse as $\tau(x)$. Define $a_j = x(\tau_j)$, and $\ell = x(\tilde{\ell})$. For $x \in (a_j, a_{j+1})$, $j = 0, \ldots, N$, define the transformation $J$ by

$$
u(x, t) = (Jy)(x, t) := y(\tau(x), t) (\rho(\tau(x))\xi(\tau(x)))^{1/4} \prod_{l=1}^j \left( \frac{\xi(\tau_l^-)\rho(\tau_l^-)}{\xi(\tau_l^+)\rho(\tau_l^+)} \right)^{1/4},$$

where $\tau_0 = 0$ and $\tau_{N+1} = \tilde{\ell}$.
and for each \( j \),
\[
    u(a_j, t) = (Jy)(a_j, \tau_j) := y(\tau_j, t) \left( \rho(\tau_j^+) \xi(\tau_j^+) \right)^{1/4} \prod_{l=1}^{j} \left( \xi(\tau_l^-) \rho(\tau_l^-) \right)^{1/4}.
\]

Here we use the convention that \( \prod_{i=0}^{0} = 1 \).

An exercise in calculus shows that \( u \) satisfies the following system,

\[
    \begin{align*}
    u_{tt} - u_{xx} + qu &= 0, \quad t > 0, \quad x \in (0, \ell) \setminus \{a_j\}_{j=1}^N, \\
    u(x, 0) &= u_t(x, 0) = 0, \\
    u(a_j^+, t) &= u(a_j, t) = u(a_j^+, t), \quad t > 0, \quad j = 1, \ldots, N, \\
    M_j u_{tt}(a_j, t) &= \rho_j^+ u_x(a_j^+, t) - \rho_j^- u_x(a_j^-, t) + \delta_j u(a_j, t), \quad t > 0, \\
    u(\ell, t) &= 0, \quad t > 0.
    \end{align*}
\]

Here

\[
    \begin{align*}
    \rho_j^+ := & \tilde{\rho}(\tau_j^+) \xi(\tau_j^+), \\
    \delta_j := & \left( \frac{\xi'(\tau_j^-)}{2} + \frac{\tilde{\rho}'(\tau_j^-) \xi(\tau_j^-)}{4\tilde{\rho}(\tau_j^-)} \right) - \left( \frac{\xi'(\tau_j^+)}{2} + \frac{\tilde{\rho}'(\tau_j^+) \xi(\tau_j^+)}{4\tilde{\rho}(\tau_j^+)} \right),
    \end{align*}
\]

and

\[
    q = \left( \tilde{q} + \frac{1}{4} \tilde{\rho}'' - \frac{5}{16} \frac{(\tilde{\rho}')^2}{\tilde{\rho}^3} - \frac{(\xi')^2}{4\xi^2} + \frac{\xi''}{2\xi} \right) \circ \tau.
\]

Thus \( q \in C^j[a_j, a_{j+1}] \) for each \( j \). The transformed control equation can be written as

\[
    f(t) = \left( \alpha_1 - \alpha_2 \frac{\xi'(0)}{2\xi(0)} - \alpha_2 \frac{\tilde{\rho}'(0)}{4\tilde{\rho}(0)} \right) u(0, t) + \alpha_2 \sqrt{\tilde{\rho}(0)} u_x(0, t).
\]

Most of the calculations in this paper will be based on System \( 2.20-2.22 \).

**Remark 1.** In the special case where \( \rho, \xi, \tilde{q} \in C^\infty([0, \ell]) \), we have that \( q \) extends to \( C^\infty([0, \ell]) \), \( \delta_j = 0 \), and \( \rho_j^+ = \rho_j^- \). In this case System \( 2.20-2.21 \) with \( \alpha_2 = 0 \), i.e. Dirichlet control, was studied in [6], and the conclusions of that paper will follow immediately from their counterparts in this one.

The following Sturm–Liouville problem, associated to System \( 2.20-2.22 \), is equivalent to the Sturm–Liouville problem \( 2.18-2.19 \):

\[
    \begin{align*}
    -\phi''(x) + q(x) \phi(x) &= \lambda^2 \phi(x), \quad x \in (0, \ell) \setminus \{a_j\}_1^N, \\
    \phi(\ell) &= 0, \\
    \beta_1 \phi(0) + \beta_2 \phi'(0) &= 0, \\
    \phi(a_j^-) &= \phi(a_j^+), \\
    \rho_j^+ \phi'(a_j^+) &= \rho_j^- \phi'(a_j^-) - (M_j \lambda^2 + \delta_j) \phi(a_j), \quad j = 1, \ldots, N, (2.24)
    \end{align*}
\]

for some \( (\beta_1, \beta_2) \in \mathbb{R}^2 \) satisfying \( \beta_1^2 + \beta_2^2 > 0 \). It is clear from \( 2.22 \) that \( \beta_2 = 0 \) if and only if \( \alpha_2 = 0 \). Thus we will refer to the case \( \beta_2 = 0 \) as \( D = D \), and all other cases will be referred to as \( M = M \).

We now discuss the self-adjoint operators associated to \( 2.23-2.24 \). In what follows, it will be convenient to define

\[
    L := -\frac{d^2}{dx^2} + q(x),
\]

the differential operator acting on distributions living on \( (0, a_1) \cup \ldots \cup (a_N, \ell) \).
We define
$$L^2_M(0, \ell) := L^2(0, \ell) \oplus \mathbb{R}^N,$$
where the norm is defined as
$$||u||^2_M = \sum_{j=0}^{N} \int_{a_j}^{a_{j+1}} |u(x)|^2 \, dx + \sum_{j=1}^{N} M_j |u(a_j)|^2.$$

Denote by $<*, *, >_M$ the associated inner product. It is easy to verify that $||u||^2_M \asymp ||J^{-1}u||^2_{M, \rho}$.

**Case i. $D - D$ boundary conditions.**

We define a quadratic form on $L^2_M$ by
$$Q^D(\phi, \psi) = \sum_{j=0}^{N} \int_{\tau_j}^{\tau_{j+1}} (J^{-1}\phi)'(J^{-1}\psi)'\xi + \tilde{q}(J^{-1}\phi)(J^{-1}\psi) \, d\tau,$$
with domain
$$Q_D = \{ \phi \in L^2_M(0, \ell) : \phi|_{(a_j, a_{j+1})} \in H^1(a_j, a_{j+1}), \phi(a^-_j) = \phi(a^+_j), \phi(a^-_j) = \phi(a^+_j), \phi(0^+) = \phi(\ell^-) = 0 \}.$$  

We remark in passing that the masses do not come into the definition of the quadratic form. Associated with this positive definite, closed quadratic form is the self-adjoint operator $A^D$, with operator domain
$$Dom(A^D) = \{ \phi \in Q_D : A^D(\phi) \in L^2_M(0, \ell) \}.$$  

Then for $\phi \in Dom(A^D)$,
$$A^D \phi(x) = \begin{cases} (L\phi)(x), & x \neq a_j, j = 1, ..., N, \\ \frac{1}{M_j} (\phi(a^-_j) - \phi(a^+_j)), & x = a_j, j = 1, ..., N. \end{cases}$$

**Example.** Set $\rho = H = 1$, $q = 0$, $N = M = 1$, $\ell = 2$ and $a_1 = 1$. Thus $\rho^- = \rho^+ = 1$ and $\delta = 0$. Let $\phi(x) = 1 - |x - 1|$, so $u$ satisfies $D - D$. Then
$$A^D \phi(x) = \begin{cases} 0, & x \neq 1, \\ 2, & x = 1. \end{cases}$$

**Case ii. $M - D$ boundary conditions.**

We define a quadratic form
$$Q^M(\phi, \psi) = \sum_{j=0}^{N} \int_{\tau_j}^{\tau_{j+1}} (J^{-1}\phi)'(J^{-1}\psi)'\xi + \tilde{q}(J^{-1}\phi)(J^{-1}\psi) \, d\tau$$
$$+ \frac{\alpha_1}{\alpha_2} \xi(0) (J^{-1}\phi)(0) (J^{-1}\psi)(0),$$
with domain
$$Q_M = \{ \phi \in L^2_M(0, \ell) : \phi|_{(a_j, a_{j+1})} \in H^1(a_j, a_{j+1}), \phi(a^-_j) = \phi(a^+_j), \phi(a^-_j) = \phi(a^+_j), \phi(0^+) = \phi(\ell^-) = 0 \}.$$  

Associated with this semibounded, closed quadratic form is the self-adjoint operator $A^M$, with operator domain
$$Dom(A^M) = \{ \phi \in Q_M : A^M \phi \in L^2_M(0, \ell) \}.$$  

The action of $A^M$ on functions is the same as in the $D - D$ case.

For both the $D - D$ and $M - D$ cases, one can use standard spectral theory arguments to show that the spectrum is discrete. Let $\{ (\lambda^M_n)^2 \}_{n=1}^{\infty}$ be the set of
eigenvalues of System 2.23-2.24 listed in increasing order, with \( b \) equalling \( \mathcal{M} \) or \( \mathcal{D} \). It follows by standard arguments that the eigenvalues are simple, see [6]. Let \( \{\varphi_n^b\} \) be a basis of normalized eigenfunctions of \( A_b^b \). In what follows, for readability, when there is no possible confusion we will drop \( b \) dependence. By simplicity of the spectrum and the self-adjointness of \( A \) we have that the eigenfunctions are orthonormal with respect to \( <\cdot,\cdot>_{\mathcal{M}} \).

We use the spectral representation to create a scale of Sobolev-type spaces.

**Definition 2.1.** For \( b = \mathcal{M} \) or \( \mathcal{D} \), choose \( E \geq 0 \) such that \((\lambda_n^b)^2 + E > 0 \). Define

\[
\mathcal{H}^{p, b} = \{ u(x) = \sum_{n=1}^{\infty} a_n \varphi_n^b(x) : ||u||_p^2 = \sum_{n=1}^{\infty} |a_n|^2((\lambda_n^b)^2 + E)^p < \infty \}, \quad p \in \mathbb{R}.
\]

Thus \( \mathcal{H}^{p, b} = \text{Dom}((A_b^b + E)^{p/2}) \).

Associated to these spaces are various equations that hold at \( x = a_j \). For instance, for \( \phi \in \mathcal{H}^{p, b} \), \( p \geq 1 \), we have

\[
\phi(a_j^-) = \phi(a_j) = \phi(a_j^+), \quad j = 1, \ldots, N, \tag{2.26}
\]

and for \( \phi \in \mathcal{H}^{p, b} \) with \( p \geq 3 \), we have for \( j = 1, \ldots, N \),

\[
\frac{1}{M_j}(\rho_j^- \phi'(a_j^-) - \rho_j^+ \phi'(a_j^+)) = -\phi''(a_j^-) + q(a_j^-)\phi(a_j) + \delta_j\phi(a_j) = -\phi''(a_j^+) + q(a_j^+)\phi(a_j) + \delta_j\phi(a_j),
\]

as well as 2.26. Clearly these equations hold for \( \phi \) an eigenfunction, and so by basic Fourier theory they hold for \( \phi \in \mathcal{H}^{p, b} \). In what follows, we will refer to such equations as “compatibility conditions”. We have the following list of all compatibility conditions:

**Lemma 2.2.** For \( \phi \in \mathcal{H}^{p, b} \), the following compatibility conditions hold for \( j = 1, \ldots, N \), and \( b = \mathcal{M} \) or \( \mathcal{D} \):

\[
L^n \phi(a_j^-) = L^n \phi(a_j^+), \quad 0 \leq n \leq [p/2] - 1, \tag{2.27}
\]

\[
\phi(a_j^-) = \phi(a_j) = \phi(a_j^+), \quad p \geq 1, \tag{2.28}
\]

and for \( 0 \leq n \leq [p/2] - 2 \)

\[
\frac{1}{M_j}(\rho_j^- (L^n \phi)'(a_j^-) - \rho_j^+ (L^n \phi)'(a_j^+)) = L^{n+1} \phi(a_j^+) + \delta_j L^n \phi(a_j^+) = L^{n+1} \phi(a_j^-) + \delta_j L^n \phi(a_j^-). \tag{2.28}
\]

Also, we have

\[
L^n \phi(\ell) = 0, \quad 0 \leq n \leq [p/2] - 1, \tag{2.27}
\]

and for \( x = 0 \) we have for \( \mathcal{M} - \mathcal{D} \) conditions

\[
\beta_1 (L^n \phi)(0) + 2 \beta_2 (L^n \phi)'(0) = 0, \quad 0 \leq n \leq [p/2] - 1, \tag{2.27}
\]

and for \( \mathcal{D} - \mathcal{D} \) conditions

\[
L^n \phi(0) = 0, \quad 0 \leq n \leq [p/2] - 1. \tag{2.27}
\]

**Remark 2.** In 2.27, we interpret the condition \( 0 \leq n \leq -1 \), which holds for \( p = 0 \), to mean no such compatibility condition holds.

We now prove Lemma 2.2
3. Representation of solution to wave equation. In this section, we will discuss the well-posedness of System 2.21-2.22. It will be convenient to rewrite this system in the form
\[ u_{tt} - u_{xx} + qu = 0, \quad t > 0, \quad x \in (0, \ell) \setminus \{a_j\}_{j=1}^N, \]
(3.29)
\[ u(x,0) = u_t(x,0) = 0, \]
\[ u(a_j^+, t) = h_j(t - a_j), \quad u(a_j^-, t) = 0, \quad t > 0, \quad j = 1, \ldots, N, \]
\[ M_j u''(t - a_j) = \rho_j^+ u_x(a_j^+, t) - \rho_j^- u_x(a_j^-, t) + \delta_j u(a_j, t), \quad t > 0, \]
\[ u(\ell,t) = 0, \quad t > 0, \]
\[ \beta_1 u(0,t) + \beta_2 u(0,t) = f(t), \quad t > 0. \]
(3.30)

The representation of the solution to this system will play a crucial role in proving Proposition 1 and Theorem 1.3.

3.1. Notational preliminaries and solution on half line. For functions of only one variable, \( f^{(j)} \) denotes the \( j \)th derivative. When convenient we will denote \( f^{(1)} = f' \), resp. \( f^{(2)} = f'' \). For partial derivatives of \( u = u^j \), we denote \( u_{xx} \) or \( \frac{\partial^2 u}{\partial x^2} \).

Fix \( T > 0 \). Define \( H^j(a,b) \) to be the set of functions in \( L^2[a,b] \) whose weak derivatives up to order \( j \) are in \( L^2[a,b] \). The corresponding norms will be denoted \( \| \cdot \|_{H^j(a,b)} \).

- \( C^j = \{ f \in C^j(\mathbb{R}) : f(t) = 0 \text{ if } t \leq 0 \} \),
- \( E^j = \{ f \in L^2(\mathbb{R}) : f(t) = 0 \text{ if } t \leq 0 \} \),
- \( H_H^n = \{ f \in H^n(\mathbb{R}) : f(t) = 0 \text{ if } t \leq 0 \} \).

Note that \( f \in H_H^n \) implies \( f^{(j)}(0) = 0 \) for \( j = 0, \ldots, n - 1 \), where \( f^{(j)} \) denotes the \( j \)th derivative. We also define
\[ H_H^{-1} = \{ f \in H^{-1}(\mathbb{R}) : f|_{(-\infty,0)} = 0 \text{ as a distribution } \}. \]

For a Banach space \( B \), we will denote its dual by \( B' \). Finally, when convenient we will denote the solution of System 3.29-3.30 by \( u \) rather than \( u^j \).

Let \( b > 0 \). As a first step to solving System 3.29-3.30, we will find a useful representation of the solution to the following system that models a semi-infinite vibrating string with no masses, with control at \( x = b \).

\[ u_{tt} - u_{xx} + q(x)u = 0, \quad t \in (0, \infty), \quad x \in (b, \infty), \]
(3.31)
\[ \beta_1 u(b,t) + \beta_2 u_x(b,t) = f(t), \quad t > 0, \]
(3.32)
\[ u(x,0) = u_t(x,0) = 0, \quad x > b. \]
(3.33)

As a preliminary step, consider the Goursat-like problem in the domain \( D = \{(x,t) \mid 0 < x < t < \infty\} \):
\[
\begin{align*}
  k_{tt}(x,t) - k_{xx}(x,t) + q(x+b)k(x,t) &= 0, \quad (x,t) \in D, \\
  \beta_1 k(0,t) + \beta_2 k_x(0,t) &= 0, \quad t > 0, \\
  k(x,x) &= -\frac{1}{2} \int_0^x q(\eta + b) d\eta, \quad x \geq 0.
\end{align*}
\]  

We have:

**Proposition 2.** Fix \( b > 0 \).

A) Let \( q \in C[b, \infty) \). System 3.34-3.36 has a unique generalized solution, denoted \( k(b^+; x, t) \), such that \( k(b^+; \ast, \ast) \in C^1(D) \) and the boundary conditions hold in a classical sense.

B) Fix \( n \in \mathbb{N} \). Let \( q \in C^n[b, \infty) \). System 3.34-3.36 has a unique solution, denoted \( k(b^+; x, t) \), such that \( k(b^+; \ast, \ast) \in C^{n+1}(D) \) and 3.35 and 3.36 hold in a classical sense.

The solution to the standard Goursat problem in the case \( \beta_2 = 0 \) is well known, see [13], but we could not find in the literature the system in the general case. The proof for the case \( \beta_2 \neq 0 \) will be given in the appendix, Subsection 9.1.

In what follows, it will be useful to denote the kernel \( k \) above as \( k_M \) in the case \( \beta_2 = 0 \), and \( k_D \) in the \( \beta_2 = 0 \). The following now holds by direct calculation.

**Proposition 3.** A. (Mixed control). Assume \( \beta_2 \neq 0 \), and set \( \beta = \beta_1/\beta_2 \). Let \( k_M \) be as in Proposition 2. Let

\[
g(t) = -\frac{1}{\beta_2} \int_0^t e^{\beta(t-s)} f(s) \, ds.
\]

a) Suppose \( f \in C^1_b \). Then System 3.31,3.32,3.33 has unique solution \( u^f(b^+; x, t) \), with

\[
u^f(b^+; x, t) = g(t - x + b) + \int_{s=x-b}^t k_M(b^+; x - b, s) g(t - s) ds,
\]

\( u^f \in H^2((b, \infty) \times (0, T)) \), 3.31 is satisfied almost everywhere, and the boundary and initial conditions are satisfied in a classical sense.

b) For \( f \in L^2_b \), the function \( u^f(b^+; x, t) \) defined above gives a solution to 3.31 in the distribution sense, 3.32 holds almost everywhere in \( t \), and 3.33 holds for all \( x \). Furthermore, \( u^f \in C([0, T]; H^1_b(b, \infty)) \).

B. (Dirichlet control). Assume \( \beta_1 \neq 0 = \beta_2 \). Let \( k_D \) be as in Proposition 2 with \( q \in C \).

a) Suppose \( f \in C^2_b \). Then System 3.31,3.32,3.33 has unique solution \( u^f(b^+; x, t) \), with

\[
u^f(b^+; x, t) = f(t - x + b) + \int_{s=x-b}^t k_D(b^+; x - b, s) f(t - s) ds,
\]

\( u^f \in H^2((b, \infty) \times (0, T)) \) for each \( T > 0 \), the equation 3.31 is satisfied almost everywhere, and the boundary and initial conditions are satisfied in a classical sense.

b) For \( f \in L^2_b \), the function \( u^f(b^+; x, t) \) defined above gives a solution to 3.31 in the distribution sense, 3.32 holds almost everywhere on \( t \), and 3.33 holds for all \( x \). Furthermore, \( u^f \in C([0, T]; L^2(b, \infty)) \).

**Remark 4.** By part B of Proposition 2, greater regularity for \( q \) implies greater regularity for \( k \). As a consequence, if \( f \in H^2_t \) and \( q \in C^{j-2} \), then \( \beta_2 \neq 0 \) implies that \( x \mapsto u^f(b^+; x, t) \) is \( H^{j+1} \) for each fixed \( t \), while \( \beta_2 = 0 \) implies \( x \mapsto u^f(b^+; x, t) \) is \( H^j \) for each fixed \( t \).
Remark 5. Since $f(t) = 0$ for $t < 0$, 3.38 can be rewritten as
\[
u^f(x^+; x, t) = \begin{cases} g(t - x + b) + \int_{s=x-b}^{t} k_M(b^+; x - b, s) g(t - s) ds, & x - b < t, \\ 0, & x - b \geq t, \end{cases}
\]
and similarly for 3.39.

Remark 6. Proposition 3 has an obvious analogue for a control at $x = b$ generating a wave on the half-line $(-\infty, b)$:
\[
u_{tt} - \nu_{xx} + q(x)\nu = 0, \quad t \in (0, \infty), \quad x \in (-\infty, b),
\]
\[
\beta_1 \nu(b, t) + \beta_2 \nu_x(b, t) = f(t), \quad t > 0,
\]
\[
u(x, 0) = \nu_t(x, 0) = 0, \quad x < b.
\]
Below, we will denote the corresponding solution kernels $k_j(b^-; x, s)$ with $j = M$ or $D$.

3.2. Solution for single attached mass with mixed control. We begin by considering the existence, uniqueness, and regularity of the solution to System 3.29-3.30 in the case of a single mass placed at $x = a$ and mixed boundary control \((M - D), \beta_1 \nu(0, t) + \beta_2 \nu_x(0, t) = f(t), \text{ with } \beta_2 \neq 0\). Thus we rewrite 3.29-3.30 as
\[
u_{tt} - \nu_{xx} + q(x)\nu = 0, \quad x \in (0, \ell) \setminus \{a\}, \quad t \leq 0,
\]
\[
u(a^-, t) = h(t - a) = u(a^+, t), \quad t \geq 0,
\]
\[
\beta_1 \nu(0, t) + \beta_2 \nu_x(0, t) = f(t),
\]
\[
u(\ell, t) = 0.
\]
Here we have expressed the initial conditions as 3.41, and we also assume $t \in (-\infty, \infty)$, which will be convenient in the calculations that follow in this subsection. We will also assume $f(t) = 0$ for $t < 0$. Clearly for $t < a$, the solution is given by 3.38. We now discuss the time interval $t \in [a, \min(2a, \ell))$, so the wave will have had time to interact with the mass but not to reflect off either $x = 0$ or $x = \ell$. Since $h(t - a) = u(a, t)$, we have $h(s) = 0$ for $s \leq 0$. Then we have for $x > a$ by Proposition 3 part B:
\[
u(x, t) = h(t - x) + \int_{s=x-a}^{t-a} k_D(a^+; x - a, s) h(t - a - s) ds.
\]
Define $r(t)$ by
\[
r(t - a) = h(t - a) - g(t - a) - \int_{s=a}^{t} k_M(0^+; a, s) g(t - s) ds,
\]
where $g$ was introduced in 3.37. Then for $x < a$, we have
\[
u(x, t) = g(t - x) + \int_{s=x}^{t} k_M(0^+; x, s) g(t - s) ds
\]
\[+ r(t + x - 2a) + \int_{s=a-x}^{t-a} k_D(a^-; a - x, s) r(t - a - s) ds \leq 3.47
\]
\[
u(x, t) = g(t - x) + \int_{s=x}^{t} k_M(0^+; x, s) g(t - s) ds
\]

+h(t + x - 2a) - g(t + x - 2a) - \int_{s=a}^{t-x-a} k_M(0^+; a, s) g(t + x - a - s) ds \\
+ \int_{s=a-x}^{t-a} k_P(a^-; a - x, s) \\
\left( h(t - s - a) - g(t - s - a) - \int_{r=a}^{t-s} k_M(0^+; a, r) g(t - s - r) dr \right) ds.

(3.48)

Thus \( r \) represents the wave reflected off the mass, while \( h \) represents the transmitted wave. Note that by the definition of \( r \), 3.46 and 3.47, we have that the continuity condition, 3.42, is satisfied. While reading the formula above, the reader is reminded that by construction, \( 0 = f(t) = g(t) = h(t) = r(t) \) for \( t < 0 \).

The condition 3.43 implies by 3.46,3.48:

\[
Mh''(t-a) = -\left( \rho^+ + \rho^- \right) \partial_t h(t-a) + \rho^+ \int_{s=0}^{t-a} \frac{\partial k_P}{\partial x}(a^+; 0, s) h(t-a-s) ds \\
+ 2\rho^- g'(t-a) + 2\rho^- k_M(0^+; a, a) g(t-a) \\
- \rho^- \int_{s=a}^{t} \frac{\partial k_M}{\partial x}(0^+; a, s) g(t-s) ds \\
+ \rho^- \int_{s=a}^{t} \frac{\partial k_M}{\partial s}(0^+; a, s) g(t-s) ds + \delta h(t-a) \\
+ \rho^- \int_{s=0}^{t-a} \frac{\partial k_P}{\partial x}(a^-; 0, s) \\
\left( h(t-a-s) - g(t-a-s) - \int_{r=a}^{t-s} k_M(0^+, a, r) g(t - s - r) dr \right) ds.
\]

(3.49)

**Lemma 3.1.** Let \( T > 0 \).

A) Given \( f \in C^1_+ \), there exists a unique \( h \in C^3_+ \) solving 3.49 for all \( t \leq T \).

B) Define the mapping \( S_M \) by

\[(S_M f)(t) = h(t). \]

(3.50)

Then \( S_M \) is well defined, and extends to a bounded and boundedly invertible linear mapping \( L^2_0 \to H^2_0 \).

C) \( S_M \) extends to a bounded and boundedly invertible linear mapping \( H^j_0 \to H^{j+2}_0 \) for any non-negative integer \( j \).

**Proof.** Study of 3.49 shows that the mapping \( g \mapsto h \), which we denote \( G \), is an isomorphism \( H^j_0 \to H^{j+1}_0 \). The details of the proof, which is a straightforward adaptation of the proof of Lemma 1 in [6], are left to the reader. By 3.37, the mapping \( f \mapsto g \), which we denote \( F \), is an isomorphism \( H^j_0 \to H^{j+1}_0 \). Since \( S_M = GF \), the lemma follows. \( \square \)

Equations 3.46,3.48 provide the solution \( u^j \) until a wave reflects off either \( x = 0 \) at time \( t = 2a \), or off \( x = \ell \) at time \( t = \ell \). After these times, the functions given by these equations will no longer solve the correct boundary conditions. This can be corrected by adding extra terms to 3.46 and 3.48. The interested reader is referred to [6].
3.3. Properties of solution for several attached masses with mixed control. In this subsection, we first define $W_i^{s,M}$ as subsets of $H^{s,M}$ for $i \in \mathbb{N}$. We then prove Proposition 1 for mixed control. We then use powers of the operator $A^{s,M}$ to define $W_i^{s,M}$ for negative integers $i$.

Consider System 3.29-3.30 with mixed boundary conditions, which we can rewrite as:

$$ u_{tt} - u_{xx} + q(x)u = 0, \quad t > 0, \quad x \in (0, \ell) \setminus \{a_j\}_{j=1}^N, \quad (3.51) $$

$$ u(x,0) = u_t(x,0) = 0, \quad x \in (0,\ell) \setminus \{a_j\}_{j=1}^N, \quad (3.52) $$

$$ u(a_j^-, t) = h_j(t-a_j) = u(a_j^+, t), \quad t > 0, \quad j = 1, \ldots, N, \quad (3.53) $$

$$ M_j h_j''(t-a_j) = \rho_j^+ u_x(a_j^+, t) - \rho_j^- u_x(a_j^-, t) + \delta_j u(a_j, t), \quad t > 0, \quad j = 1, \ldots, N, \quad (3.54) $$

$$ \beta u(0,t) + u_x(0,t) = f(t), \quad t > 0, \quad (3.56) $$

$$ u(\ell, t) = 0, \quad t > 0. \quad (3.57) $$

We adopt the notation, for non-negative integer $i$,

$$ W_i = (\oplus_{j=0}^N H^{i+j}(a_j, a_{j+1})) \oplus \mathbb{R}^N. $$

Below, we will denote $u_j^i |_{(a_j, a_{j+1})} = y_j$.

**Proposition 4.** Let $T > 0$. For $f \in L^2(0,T)$, there exists a unique solution $u^f = (y_0, h_1, \ldots, h_N, y_N)$ to the initial boundary value problem 3.51-3.57. Furthermore,

A) For $i = 0, 1$,

$$ y_j \in C^i([0,T]), \quad H^{i+1-i}(a_j, a_{j+1}). \quad (3.58) $$

Furthermore, $h_j \in H^{i+1}(0,T)$ for each $j$.

B) For any $T > 0$, the mapping $f \mapsto (u^f(x,T), u_x^f(x,T))$ is a continuous mapping from $L^2(0,T)$ into

$$ W_1 \times W_0. $$

The proof follows from the representations of $u$, together with regularity results found in Section 3.2, along with the fact, proven in ([6] Section 2.2) that waves reflected off either a mass or an endpoint have the same regularity as incoming waves. The details are left to the reader.

Following [6], we state certain compatibility conditions that solution $u^f$ must satisfy at $\{a_j\}$. These conditions arise naturally from the equations in 3.51-3.57 provided $u^f$ is sufficiently regular. One example of this is $u^f(a_j^-, T) = h_j(T-a_j) = u^f(a_j^+, T)$, which by 3.58 and 3.53 must hold for all $j \geq 1$, and another is $h_j'(T-a_j) = u_x^f(a_j^+, T)$. In addition, the boundary condition at $x = \ell$ imposes further compatibility conditions. These conditions resemble those listed for $H^{s,M}$ in Lemma 1, but also account for the fact that a solution $u^f$ to System 3.51-3.57 is more regular to the right of $x = a_j$ than to the left.

Recall the operator $L = -\frac{d^2}{dx^2} + q$.

**Definition 3.2.** Let $j = 1, \ldots, N$, and let $k$ be any integer. A function $\phi(x)$ satisfies the Condition $C^{k,M}$ at $x = a_j$ if

$$ \frac{d}{dx} L^n \phi(a_j^-) = \frac{d}{dx} L^n \phi(a_j^+) - M_j L^{n+1} \phi(a_j^+) $$

is satisfied for $0 \leq n \leq \lfloor k/2 \rfloor - 2$, and

$$ \phi(a_j) = \phi(a_j^+) \text{ if } k \geq 0, \text{ and } \quad L^n \phi(a_j^-) = L^n \phi(a_j^+) $$
is satisfied for $0 \leq n \leq \lfloor k/2 \rfloor - 1$. A function $\phi(x)$ satisfies the Condition $C_{k,M}$ at $x = \ell$ if $L^nu(\ell, t) = 0$ is satisfied for $0 \leq n \leq \lfloor k/2 \rfloor - 1$, and at $x = 0$ if

$$\beta L^nu(0, t) + \frac{d}{dx}L^nu(0, t) = 0$$

is satisfied for $0 \leq n \leq \lfloor k/2 \rfloor - 1$. Here, if we have $0 \leq n \leq -1$, then the equation is vacuous. Also, for convenience, for $k \leq 0$ we denote the condition $C_{k,M}$ at $x = a_j$ to be a vacuous condition.

A function satisfies Condition $C_{i,M}$ if it satisfies Condition $C_{j-1+i,M}$ at $x = a_j$ for all $j = 1, ..., N + 1$, and Condition $C_{i,M}$ at $x = 0$.

**Definition 3.3.** For integer $i \geq 0$, define the space

$$W^i_M := \{ \phi \in W_i : \text{\phi satisfies Condition } C_{i,M} \}.$$

Then $W^i_M$ are Hilbert spaces with inner product

$$<\phi, \psi>_i = \sum_{j=0}^{N} <\phi, \psi>_{H^{i+j}(a_j, a_{j+1})} + \sum_{j=1}^{N} M_j \phi(a_j) \overline{\psi(a_j)}.$$ 

Here we define the norm:

$$||\phi||_{H^i(a_j, a_{j+1})}^2 = ||d^i\phi||_{L^2(a_j, a_{j+1})}^2 + ||\phi||_{L^2(a_j, a_{j+1})}^2.$$ 

We now sketch the proof of Proposition 1. That the map $(x, t) \mapsto u^j = (y_0, h_1, y_1, ..., h_N, y_N)$ has the announced regularity follows from Proposition 4. That the announced compatibility conditions hold follows from applying $u_{tt} = Lu$ to 3.53,3.55, and to the boundary conditions. For details, the reader is referred to a similar argument in ([6], Section 2.3).

It is easy to see that we have a canonical inclusion of $W^i_M \subset H^i_M$. Furthermore, the set of restrictions of elements of $W^i_M$ to the interval $(a_j, a_{j+1})$ will be the same as the set of restrictions of elements of $H^i_{a_j, a_{j+1}}$ to the same interval. The following result will be used in the proof of Theorem 1.5.

**Proposition 5.** Assume $q \in C^\infty[a_j, a_{j+1}]$ for $j = 0, ..., N$.

Choose constant $E$ so that $(A^M + E) > 0$. Then for $i \geq 2$, $(A^M + E)$ maps $W^i_M$ bijectively onto $W^{i-2}_M$, and this mapping is an isomorphism with respect to the norms $||*||_{W^i_M}$ and $||*||_{W^{i-2}_M}$.

**Proof.** In this proof, we drop the $\mathcal{M}$ superscript for readability. We first show that $L$ maps $W_i$ boundedly into $W_{i-2}$. It is well known that

$$L : H^i(a_j, a_{j+1}) \rightarrow H^{i-2}(a_j, a_{j+1})$$

is a bounded map for each $i \geq 2$. Also, there exists a constant $C > 0$ such that for $j \geq 1$,

$$|A\phi(a_j)| = \frac{1}{M_j} |p^- \phi^+(a_j^+) + p^- \phi^-(a_j^-)|$$

$$\leq C(||\phi||_{H^2(a_j, a_{j+1})} + ||\phi||_{H^3(a_j, a_{j+1})}) \leq C||\phi||_{W_i}.$$ 

Next, observe that by definition $L$ will map a function satisfying Condition $C_i^l$ to a function satisfying $C_{i-2}^l$. This proves that $A$ maps $W_i$ boundedly into $W_{i-2}$, and so the same holds for $(A + E)$. Since $(A + E)$ is an injection from $H^i$ to $H^{i-2}$, the same holds for $(A + E)$ as a mapping from $W_i$ to $W_{i-2}$. We now prove surjectivity. Let $\psi \in W_{i-2} \subset H^{i-2}$. Let $\phi = (A + E)^{-1} \psi \in H^i$. By elliptic regularity, we have
φ restricted to \((a_j, a_{j+1})\) is in \(H^{1+j}(a_j, a_{j+1})\). Since \(L\phi\) satisfies Condition \(C^{i-2}\), it follows immediately that \(\phi\) satisfies \(C_i^*\). Thus we have proved that \((A+E)\) is a bounded bijection from \(W_i\) onto \(W_{i-2}\). Hence by the Open Mapping Theorem, \((A+E)\) is an isomorphism.

**Definition 3.4.** For negative integer \(i\), suppose \(i+2n \geq 0\). Choose constant \(E\) so that \((A^M+E) > 0\). Then we define \(W_i^M = (A^M+E)^n(W_{i+2n})\), with norm given by \(\|v\|_{W_i^M} = \|(A^M+E)^{i/2}v\|_{W_0^M}\) for \(i\) even, and \(\|v\|_{W_i^M} = \|(A^M+E)^{(i-1)/2}v\|_{W_0^M}\) for \(i\) odd.

The proof of the following is left to the reader.

**Proposition 6.**

A) Suppose \(i\) is a negative integer. If \(\phi \in W_i^M\), then \(\phi\) satisfies Condition \(C_i^{i,M}\).

B) For any integer \(i\), \(W_{i+1}^M \subseteq W_i^M\).

3.4. **Properties of solution for several attached masses with Dirichlet control.** In this section, we state the analogues for Dirichlet control of those just stated for mixed control. The proofs of most results are straightforward adaptations of the corresponding ones for the mixed case, and so are omitted. First, for future reference, we state the analogue of Lemma 3.1. Thus for System 3.40-3.45, assume \(\beta_1 = 1, \beta_2 = 0\). Then arguing as in Subsection 3.2, we obtain the equation for \(t < \max(2a, \ell)\),

\[
Mh''(t-a) = -(\rho^+ + \rho^-)h'(t-a) + \rho^+ \int_{s=0}^{t-a} \frac{\partial k_D}{\partial x}(a^+; 0, s)h(t-a-s)ds \\
+ \delta h(t-a) + 2\rho^- f(t-a) \\
+ 2\rho^+ k_D(0^+; a, f(t-a) - \rho^- \int_{s=a}^{t} \frac{\partial k_D}{\partial x}(0^+; a, s)f(t-s)ds \\
+ \rho^- \int_{s=a}^{t} \frac{\partial k_D}{\partial s}(0^+, a, s)f(t-s)ds + \rho^- \int_{s=0}^{t-a} \frac{\partial k_D}{\partial x}(a^-; 0, s) \\
\left(h(t-a-s) - f(t-a-s) - \int_{r=a}^{t-s} k_D(0^+, a, r)f(t-s-r)dr\right)ds.
\]

(3.59)

**Lemma 3.5.** Let \(T > 0\).

A) Given \(f \in C_2^+\), there exists a unique \(h \in C_2^+\) solving 3.59 for all \(t \leq T\).

B) Define the mapping \(S_D\) by

\[
(S_Df)(t) = h(t),
\]

with \(h\) solving 3.59. Then \(S_D\) extends to a bounded and boundedly invertible linear mapping \(H_0^{1}\) \(\rightarrow\) \(H_0^{1+1}\) for any non-negative integer \(l\).

For \(i \geq 0\), we define \(W_i\) as in the previous subsection. Let \(\Theta(0, a_1) := \{u \in H^1(0, a_1) : u(0) = 0\}\). Letting \(X'\) denote the dual space to \(X\), define

\[
W_{-1} = \Theta'(0, a_1) \oplus \left( \oplus_{j=1}^{N} H^{-1}(a_j, a_{j+1})\right).
\]

Then \(W_{-1}\) can be viewed as a subspace of \(H^{-1}(0, \ell)\). To see this, let \((\phi_0, ..., \phi_N) \in W_{-1}\), and let \(\psi \in H^0_0(0, \ell)\). Then the following functional is continuous:

\[
\psi \mapsto \langle \phi_0, \psi|_{(0,a_1)} \rangle + \sum_{j=1}^{N} \int_{a_j}^{a_{j+1}} \psi(x)\phi_j(x)dx,
\]
where $<*,*>$ is the $\Theta'(0,a_1) - \Theta(0,a_1)$ pairing.

We now define condition $C^{i,D}_*$ analogously to the case of mixed control.

**Definition 3.6.** A function $\phi(x)$ satisfies the Condition $C^{i,D}_*$ if it satisfies $C^{i,M}$ at $x = a_j$ when $j = 1, ..., N + 1$, and $L^n\phi(0) = 0$, $0 \leq \lfloor i/2 \rfloor - 1$.

**Definition 3.7.** For integer $i \geq -1$, define the space

$$W^D_i := \{ \phi \in W_i : \phi \text{ satisfies Condition } C^{i,D}_* \}.$$

Then $W_i$ are as Hilbert spaces with inner product

$$<u, v>_i = \sum_{j=0}^{N} <u, v>_{H^{i+1}(a_j, a_{j+1})} + \sum_{j=1}^{N} M_j u(a^+_j) v(a^+_j), \text{ for } i \geq 0,$$

and

$$<u, v>_{-1} = \sum_{j=0}^{N} <u, v>_{H_1^{-1}(a_j, a_{j+1})}.$$

Thus we have the canonical inclusion $W^D_i \subset H^i_D$.

**Remark 7.** Recall that $H^{-1,D}$ can be identified with $H^{-1}(0, \ell)$, and hence $W^D_1$ is a subset of $H^{-1}(0, \ell)$. In particular, the terminal velocity of our string system under Dirichlet control should be viewed as an element of $H^{-1}(0, \ell)$. If $\phi \in W^D_1$, then $\phi$ is well defined in a pointwise sense only for $x \geq a_2$.

**Proposition 7.** (Dirichlet control). Suppose $\beta_2 = 0$. There exists a unique solution

$$u^f := (y_0, h_1, y_1, h_2, ..., h_N, y_N)$$

to System 3.29-3.30. Furthermore,

$$u^f \in C^i([0, T], W^D_i), \text{ } i = 0, 1.$$

The proofs of well-posedness of System 3.29-3.30, with Dirichlet control, and of Proposition 7, are straightforward adaptations of the proofs for mixed control and are left to the reader.

**Proposition 8.** Choose constant $E$ so that $(A^D + E) > 0$. Assume $q \in C^\infty[a_j, a_{j+1}]$ for $j = 0, ..., N$.

A) For $i \geq 2$, $(A^D + E)$ maps $W^D_i$ bijectively onto $W^D_{i-2}$, and this mapping is an isomorphism with respect to the norms $\|*\|_{W^D_i}$ and $\|*\|_{W^D_{i-2}}$.

B) $(A^D + E)W^D_i = W^D_1$.

**Proof.** We prove part B. The proof of part A is left to the reader. In what follows, we will drop the superscript $D$ from $A^D$ and $H^D_i$. For simplicity of exposition, assume $N = 1$. By definition,

$$W_{-1} = \Theta'(0, a_1) \times L^2(a_1, \ell).$$

We have for $\phi, \psi \in H^2$,

$$<A\phi, \psi>_{L^2} = Q^D(\phi, \psi) = \int_{0}^{a_1} \phi' \psi' \, dx + \int_{a_1}^{\ell} \phi' \psi' \, dx.$$

Clearly this identity extends by continuity to $\psi \in H^1(0, \ell)$. Assume $\phi \in W_1$. Then the last equation shows that $A\phi$ is an element of $H^{-1}(0, \ell) = H^{-1}(0, \ell)$. 

Since \( \phi|_{(a_1, \ell)} \in H^2(a_1, \ell) \), we have \( A \phi|_{(a_1, \ell)} \in L^2(a_1, \ell) \). Furthermore the mapping \( \phi \mapsto \int_0^a \phi \psi \) extends to a functional on \( \Theta(0, a_1) \), i.e. an element of \( \Theta'(0, a_1) \). This proves \( (A+E)W_1 \subset \Theta'(0, a_1) \times L^2(a_1, \ell) \). Now suppose \( \phi \in \Theta'(0, a_1) \times L^2(a_1, \ell) \), so \( \phi \in H^{-1} \). Then \( \psi := (A+E)^{-1} \phi \in H^1 \). By elliptic regularity, \( \psi|_{(a_1, \ell)} \in H^2(a_1, \ell) \), and hence \( \psi \in W_1 \), from which we conclude \( \phi = (A+E)\psi \in (A+E)W_1 \).

**Definition 3.8.** For negative integer \( i \), suppose \( i + 2n \geq 0 \). Choose constant \( E \) so that \( (A^D + E) > 0 \). Then we define \( W^D_i = (A^D + E)^n(W^D_{i+2n}) \), with norm given by \( ||v||_{W^D_i} = ||(A^D + E)^{i/2}v||_{W^D_i} \) for \( i \) even, and \( ||v||_{W^D_i} = ||(A^D + E)^{(i-1)/2}v||_{W^D_i} \) for \( i \) odd.

The proof of the following is left to the reader.

**Proposition 9.**
A) Suppose \( i \) is a negative integer. If \( \phi \in W^D_i \), then \( \phi \) satisfies Condition \( C^{*,D}_i \).
B) For any integer \( i \), \( W^D_{i+1} \subset W^D_i \).

4. **Shape and velocity control.** The following result is a first step towards proving Theorem 1.3, while also illustrating some of the essential ideas of the proof of the theorem in a technically simpler setting.

**Lemma 4.1.** Let \( T > 0 \) and assume \( q \in C[0, \infty) \). Consider the system

\[
\begin{align*}
  u_{tt} - u_{xx} + q(x)u &= 0, \ x \in (0, \infty), \ t \in (0, T), \\
  \beta u(0, t) + u_x(0, t) &= f(t), \ t > 0, \\
  u(x, 0) &= u_t(x, 0) = 0, \ x > 0, \ t \leq 0.
\end{align*}
\]  

(4.60)  

(4.61)

A) Given \( \phi \in H^1_+(0, T) \) with \( \phi(0) = 0 \), there exists a unique \( f \in L^2 \) such that \( u^f \) solving System 4.60-4.61 satisfies

\[ u^f(x, T) = \phi(x), \ \forall x. \]

Furthermore,

\[ ||\phi||_{H^1(0, T)} \asymp ||f||_{L^2(0, T)}. \]  

(4.62)

B) Given \( \phi \in L^2(0, T) \), there exists a unique \( f \in L^2 \) such that \( u^f \) solving System 4.60-4.61 satisfies

\[ u^f_t(x, T) = \phi(x), \ \forall x. \]

Furthermore,

\[ ||\phi||_{L^2(0, T)} \asymp ||f||_{L^2(0, T)}. \]  

(4.63)

Proof. We begin with part A. First, by 3.37, the mapping \( F : L^2 \to H^1_+ \) given by \( (F f)(t) = g(t) \) is boundedly invertible. Hence

\[ ||g||_{H^1} \asymp ||f||_{L^2}. \]  

(4.64)

Next, by 3.38, the equation \( u^f(x, T) = \phi(x) \) is equivalent to

\[ \phi(x) = g(T - x) + \int_{s=x}^T k_M(0^+; x, s)g(T - s)ds. \]

This is a Volterra equation of the second kind, hence uniquely solvable in the \( L^2 \) class. Since \( \phi \in H^1_+ \) with \( \phi(0) = 0 \), it follows that \( g(0) = 0 \). Extending \( g(t) \) trivially to \( t < 0 \), we have \( g \in H^1_+ \), and hence \( f = F^{-1}g \in L^2 \). It is easy to verify that

\[ ||\phi||_{H^1} \asymp ||g||_{H^1}, \]

so the proof is concluded by applying 4.64.
To prove part B, note first that by 3.38, the equation \( \phi(x) = u_t(x, T) \) implies
\[
\phi(x) = u_t(x, T) = g'(T - x) + \int_x^T k_M(0^+; x, s)g'(T - s)ds.
\]
This is a Volterra equation of the second kind, and hence there is a unique solution \( g' \in L^2(0, T) \). By 3.37
\[
g'(t) = -f(t) - \beta \int_0^t e^{\beta(t-s)}f(s)ds.
\]
This is also a Volterra equation of the second kind, hence there is a unique solution \( f \in L^2(0, T) \). By properties of Volterra equations, we have
\[
||\phi||_{L^2} \asymp ||g'||_{L^2} \asymp ||f||_{L^2}.
\]
Finally, we extend \( f(t) \) trivially to \( t < 0 \).

The proof of Theorem 1.3 appears in the appendix, Subsection 9.3. We now state the analogous result for Dirichlet control. The proof, similar to the one for the mixed control (also see [6]), is left to the reader. Define
\[
W_i^{T, D} = \begin{cases}
\{ \phi \in W_i^D : \phi(x) = 0 \text{ for } x > T \}, & T < \ell, \\
\{ \phi \in W_i^D : \frac{d^j \phi}{dx^j}(\ell), j = 0, \ldots, (N + i - 1) \}, & T = \ell, \\
W_i^D, & T > \ell.
\end{cases}
\]

**Theorem 4.2.** Assume \( \beta_2 = 0 \) in System 3.29-3.30. Let \( T > 0 \).
A) Let \( \phi \in W_0^{T, D} \). Then there exists \( f \in L^2(0, T) \) such that \( u_f(x, T) = \phi(x) \) and
\[
||u_f(\cdot, T)||_{W_0^D} \asymp ||f||_{L^2(0, T)}.
\]
If \( T \leq \ell \), then this \( f \) is unique.
B) Let \( \phi \in W_{-1}^{T, D} \). Then there exists \( f \in L^2(0, T) \) such that \( u_f(x, T) = \phi(x) \) and
\[
||u_f(\cdot, T)||_{W_{-1}^D} \asymp ||f||_{L^2(0, T)}.
\]
If \( T < \ell \), then this \( f \) is unique.

5. **More on spectral theory of system.** In order to study basis properties of our exponential divided differences, it will be necessary to prove a number of estimates pertaining to the spectrum of our system. Among these results will be the asymptotics of the eigenvalues.

Thus set
\[
\hat{\rho}_j = \frac{\rho_j^-}{\rho_j^+}, \quad M_j = \frac{M_j}{\rho_j^+}, \quad \hat{\delta}_j = \frac{\delta_j}{\rho_j^+}.
\]
Define \( \psi \) as the solution to the following system
\[
\begin{align*}
-\psi''(x) &= \lambda^2 \psi(x), \quad x \in (0, \ell) - \{a_j\}_{1}^{N}, \\
\psi(0) &= 0, \\
\psi'(0) &= \lambda, \\
\psi(a_j^-) &= \psi(a_j^+), \\
\psi'(a_j^-) &= \hat{\rho}_j \psi'(a_j^+) - \left( M_j \delta^2 + \hat{\delta}_j \right) \psi(a_j), \quad j = 1, \ldots, N.
\end{align*}
\]
(5.66)
Thus \( \psi = \psi(x, \lambda) \), but in what follows when convenient we will drop the \( \lambda \) dependence from the notation. Note that \( q = 0 \) here.
To define a self-adjoint operator associated to this system, let \((\beta_1, \beta_2) \in \mathbb{R}^2\) satisfy \((\beta_1, \beta_2) \neq (0, 0)\). Later, we will set
\[
\beta_1 \psi(\ell) + \beta_2 \psi'(\ell) = 0. \tag{5.67}
\]
In the case \(\beta_2 \neq 0\), the transformation \(x \mapsto \ell - x\) shows that the study of the spectrum of 5.65,5.66,5.67 is equivalent to the study of the spectrum under \(M - D\) conditions. We choose to study the current system because it simplifies our presentation.

We solve System 5.65-5.66 (but not yet 5.67) by the following procedure. On the interval \((0, a_1)\) we set
\[
\psi(x) = \sin(\lambda x), \ x \in (0, a_1). \tag{5.68}
\]
We then obtain \(\psi(a_1)\) and \(\psi'(a_1)\) from equations 5.65-5.66. Thus
\[
\psi(x) = \psi(a_1) \cos(\lambda(x - a_1)) + \frac{\psi'(a_1)}{\lambda} \sin(\lambda(x - a_1)), \ x \in (a_1, a_2).
\]
Clearly, we can then iteratively solve for \(\psi\) on \((a_j, a_{j+1})\) for each \(j = 0, ..., N\). The following technical lemma is a generalization of one proven in [6], and holds both for \(D - M\) and \(D - D\) boundary conditions.

**Lemma 5.1.** The functions \(\{\lambda \mapsto \frac{\psi(a_1)}{\lambda}\}_{j=1}^{N+1}\) and \(\{\lambda \mapsto \frac{\psi'(a_1)}{\lambda}\}_{j=1}^{N+1}\) extend to entire functions. Also,

\[
i) \quad \psi(a_j) = \sum_{n=1-j}^{j-1} b_j^n(\lambda)\lambda^n, \tag{5.69}
\]

where for each \(n\), \(b_j^n(\lambda)\) is a sum of products of the form
\[
K \prod_{k=0}^{j-1} T_k(\lambda \ell_k),
\]
where \(K\) are various constants independent of \(\lambda\) and \(T_k(\lambda)\) equals either \(\sin(\lambda)\) or \(\cos(\lambda)\), and at least \((n+1)\) of the \(T_k(\lambda)\) equal \(\sin(\lambda)\). Furthermore
\[
b_{j-1,1}(\lambda) = (-1)^j \prod_{k=1}^{j-1} \hat{M}_k \prod_{k=0}^{j-1} \sin(\lambda \ell_k). \tag{5.70}
\]

Here we use the convention \(\prod_{i=0}^{0} = 1\).

\[
ii) \quad \frac{\psi'(a_1)}{\lambda} = \sum_{n=-j}^{j} \tilde{b}_j^n(\lambda)\lambda^n,
\]

where for each \(n\), the \(\tilde{b}_j^n(\lambda)\) is a sum of products of the form
\[
K \prod_{k=0}^{j} T_k(\lambda \ell_k),
\]
where \(K\) are various constants independent of \(\lambda\) and \(T_k(\lambda)\) equals either \(\sin(\lambda)\) or \(\cos(\lambda)\), and at least \(n\) of the \(T_k(\lambda)\) equal \(\sin(\lambda)\). Furthermore,
\[
\tilde{b}_{j-1,1}(\lambda) = (-1)^j \prod_{k=1}^{j} \hat{M}_k \prod_{k=0}^{j-1} \sin(\lambda \ell_k). \tag{5.71}
\]
Proof. We prove the lemma by induction on \( j \). First, for \( j = 1 \), note that \( a_1 = \ell_0 \), so by 5.65-5.66 we have

\[
\psi(a_1) = \sin(\lambda \ell_0), \quad \frac{\psi'(a_1^+)}{\lambda} = -\lambda \hat{M}_1 \sin(\lambda \ell_0) + \hat{\rho}_1 \cos(\lambda \ell_0) - \frac{\bar{\delta}_1}{\lambda} \sin(\lambda \ell_0),
\]

so the lemma holds in this case. Assume now the lemma holds for some \( j < N + 1 \).

On the interval \([a_j, a_{j+1}]\), we have

\[
\psi(x) = \psi(a_j) \cos(\lambda(x - a_j)) + \frac{\psi'(a_j^+)}{\lambda} \sin(\lambda(x - a_j)). \tag{5.72}
\]

Thus, setting \( x = a_{j+1} \) and applying the inductive hypothesis,

\[
\psi(a_{j+1}) = \psi(a_j) \cos(\lambda \ell_j) + \frac{\psi'(a_j^+)}{\lambda} \sin(\lambda \ell_j) = \cos(\lambda \ell_j) \sum_{n=-j}^{j-1} b_j^n \lambda^n + \sin(\lambda \ell_j) \sum_{n=-j}^{j} \tilde{b}_j^n \lambda^n.
\]

Analyticity of \( \psi(a_{j+1}) \) follows applying the inductive hypotheses to the first equation. From the last equation, applying the inductive hypothesis it is clear that

\[
b_j^{j+1}(\lambda) = \sin(\lambda \ell_j) \tilde{b}_j^j(\lambda) = (-1)^j \left( \prod_{k=1}^{j} \hat{M}_k \right) \left( \prod_{k=0}^{j} \sin(\lambda \ell_k) \right),
\]

proving 5.69-5.70 in this case. Furthermore, by 5.65-5.66 and 5.72,

\[
\frac{\psi'(a_{j+1}^+)}{\lambda} = \hat{\rho}_{j+1} \left( -\psi(a_j) \sin(\lambda \ell_j) + \frac{\psi'(a_j^+)}{\lambda} \cos(\lambda \ell_j) \right) - (\hat{M}_{j+1} + \frac{\bar{\delta}_{j+1}}{\lambda}) \psi(a_{j+1}). \tag{5.73}
\]

Applying the inductive hypothesis again, analyticity and 5.71 can easily be deduced. The proof of the rest of ii follows easily.

In what follows, in the case \( q = 0 \), we will denote the eigenvalue set associated to \( A^D \) or \( A^M \) as \( \{ (\gamma_n)^2 : n \in \mathbb{N} \} \), listed in increasing order, while similarly \( \{ (\lambda_n)^2 : n \in \mathbb{N} \} \) will denote eigenvalues for \( q \neq 0 \). We define \( K = \{ \pm 1, \pm 2, \ldots \} \). We can then use the formula \( \gamma_{-n} = -\gamma_n \), to extend the mapping \( n \mapsto \gamma_n \) from \( \mathbb{N} \) to \( K \), where we assume \( \{ \Re(\gamma_n) \} \) is a non-decreasing sequence. We will refer to

\[
\Gamma := \{ \gamma_n : n \in K \}
\]

as the eigenfrequencies of System 5.65-5.66,5.67 for \( q = 0 \).

In what follows, we define

\[
G(x, \lambda) = \frac{\psi(x, \lambda)}{\lambda}.
\]

We then define the generating function

\[
G(\lambda) := \beta_1 G(\ell, \lambda) + \beta_2 G_x(\ell, \lambda).
\]

We wish to examine some properties of \( G(\lambda) \).
Corollary 1.

a) The function $G(\lambda)$ extends to an entire function of exponential type $\ell$ on the upper and lower half planes.

b) The roots of $G(\lambda)$ are precisely the eigenfrequencies of System 5.65-5.67, including multiplicities.

c) Suppose $\beta_2 \neq 0$ ($\mathcal{D} - \mathcal{M}$ boundary conditions). Let $\lambda = x + iy$, $x, y \in \mathbb{R}$. Then there exists $y_0 > 0$ such that if $|y| > y_0$, then there exist constants $C_0, C_1 > 0$ such that

$$C_0 < \frac{|G(\lambda)|}{1 + |\lambda|^N} < C_1, \forall x.$$  

d) Suppose $\beta_2 = 0$ ($\mathcal{D} - \mathcal{D}$ boundary conditions). Let $\lambda = x + iy$, $x, y \in \mathbb{R}$. Then there exists $y_0 > 0$ such that if $|y| > y_0$, then there exist constants $C_0, C_1 > 0$ such that

$$C_0 < \frac{|G(\lambda)|}{1 + |\lambda|^{N-1}} < C_1, \forall x.$$  

Proof. We begin with part a, considering simultaneously $\mathcal{D} - \mathcal{D}$ and $\mathcal{D} - \mathcal{M}$ conditions. We have by Lemma 5.1,

$$G(\lambda) = \beta_1 \frac{\psi'(a_{N+1})}{\lambda} + \beta_2 \left( -\psi(a_N) \sin(\lambda \ell_N) + \frac{\psi'(a_N)}{\lambda} \cos(\lambda \ell_N) \right),$$

$$= \beta_1 \sum_{-N}^{N} b_n^{N+1} \lambda^{-1} + \beta_2 \left( \sin(\lambda \ell_N) \sum_{-N}^{N-1} b_n^{N} \lambda^n + \cos(\lambda \ell_N) \sum_{-N}^{N} b_n^{N} \lambda^n \right).$$

(5.74)

Thus it is clear from the first equation and Lemma 5.1 that $G(\lambda)$ is entire.

To complete the proof of part a, let $\lambda = re^{i\theta}$. Fix $\theta$ and suppose $r$ is large. For $T(z)$ equalling either $\sin(z)$ or $\cos(z)$,

$$|T(\lambda \ell_j)| \sim e^{r|\sin(\theta)|}.$$  

Thus, using the notation of Lemma 5.1,

$$\prod_{k=0}^{N} |T_k(\lambda \ell_k)| \sim e^{r|\sin(\theta)|}.$$  

Part a) now follows from Lemma 5.1.

To prove part c), we apply 5.74 to get

$$\left| \frac{G(\lambda)}{\lambda^N} - (-1)^N \beta_2 \left( \prod_{k=1}^{N} M_k \right) \left( \prod_{k=0}^{N-1} \sin(\lambda \ell_N) \right) \cos(\lambda \ell_N) \right| = O(\lambda^{-1}), \ |\lambda| \gg 1.$$  

(5.75)

Next, it is easy to show that there exists $y_0 > 0$ such that if $|y| > y_0$,

$$|\cos(\ell_N(x + iy))| \prod_{k=0}^{N-1} |\sin(\ell_k(x + iy))| \asymp 1,$$

this estimate being uniform in $x$. The desired inequalities now follow easily.

The proof of d) is similar to the proof c), with the counterpart of 5.75 being

$$\left| \frac{G(\lambda)}{\lambda^{N-1}} - (-1)^N \left( \prod_{k=1}^{N} M_k \right) \left( \prod_{k=0}^{N} \sin(\lambda \ell_k) \right) \right| = O(\lambda^{-1}), \ |\lambda| \gg 1.$$  

We now prove part b). It is clear from the construction of $G(\lambda)$ that for $\lambda \neq 0$, $G(\lambda) = 0$ if and only if $\lambda$ is an eigenfrequency with $G(x, \lambda)$ the corresponding
eigenfunction. We now consider the case $\lambda = 0$. Recall $G(x, \lambda) = \psi(x)/\lambda$. The proof of Lemma 5.1 shows that $G(a_j, \lambda)$ is continuous (in fact analytic) in $\lambda$, while for $x \in [a_j, a_{j+1}]$ we have by 5.72 that $G(x, \lambda)$ is jointly $C^\infty$ in $x, \lambda$. Thus the limit as $\lambda \to 0$ commutes with derivatives in $x$ for $G(x, \lambda)$, so for each $x \neq a_j$,$$
abla^2 G(x, 0) = \lim_{\lambda \to 0} \frac{1}{\lambda} \nabla^2 G(x, \lambda) = -\frac{\lambda}{\lambda} \psi(x, \lambda) = 0,$$and by 5.65-5.66$$G_x(a_j, 0) = \hat{\rho}_j G_x(a_{j+1}, 0) = \lim_{\lambda \to 0} \left( \frac{\psi(a_j^+, \lambda)}{\lambda} - \hat{\rho}_j \frac{\psi(a_j^-, \lambda)}{\lambda} \right)$$Furthermore, by 5.68 we have $G(x, 0) = x$ on $(0, a_1)$, and hence, $G(x, 0)$ is a non-trivial solution to $$-\phi''(x) = 0, \quad x \in (0, \ell) - \{a_j\}_1^N,$$

$\phi(0) = 0,$

$\phi(a_j) = \phi(a_j^+),$

$\phi'(a_j^+) = \hat{\rho}_j \phi'(a_j^-) - \hat{\delta}_j \phi(a_j), \quad j = 1, \ldots, N.$

It follows that

$$0 = \lim_{\lambda \to 0} G(\lambda) = \beta_1 G(a_{N+1}, 0) + \beta_2 G_x(a_{N+1}, 0)$$

is equivalent to $\lambda = 0$ being an eigenfrequency with eigenfunction $G(x, 0)$.

Finally, we discuss the multiplicity of the zeros of $G(\lambda)$. A discussion of $G(\lambda)$ found in ([29], Ch.10, Item 10.72) shows that the zeros of $G(\lambda)$ are simple, except in the case where 0 is an eigenfrequency, in which case $\lambda = 0$ is a double zero. In that work, where the equation $(\beta_1 \psi(\ell, \lambda^2) + \beta_2 \psi_x(\ell, \lambda^2))/\lambda = 0$ is called the “characteristic equation”, there are no masses. But the argument there can easily be adapted to our setting, after transforming the problem to the setting in 2.18-2.19. Thus, the zeros of $G(\lambda)$, including multiplicities, equal the eigenfrequencies of the Sturm–Liouville problem 5.65-5.67.

**Proposition 10.**

$\mathcal{D} - \mathcal{M}$: $\beta_2 \neq 0$. Let $\Gamma'$ be any subset of $\Gamma$ obtained by deleting $2N$ elements. Then $\Gamma'$ can be reparametrized as

$$\Gamma' = \left( \bigcup_{j=0}^{N-1} \{\gamma_m^{(j)}\}_{m \in \mathbb{Z}} \right) \cup \{\gamma_m^{(N)}\}_{m \in \mathbb{Z}},$$

where for each $j < N$,

$|\gamma_m^{(j)} - \frac{\pi m}{\ell_j}| = O(|m|^{-1}),$

and

$|\gamma_m^{(N)} - \frac{(2m + 1) \pi}{2 \ell_N}| = O(|m| + 1)^{-1})$.
\(D - D\): \(\beta_2 = 0\). Let \(\Gamma'\) be any subset of \(\Gamma\) obtained by deleting \(2N\) elements. Then \(\Gamma'\) can be reparametrized as
\[\Gamma' = \bigcup_{j=0}^{N} \{ \gamma_m^{(j)} \}_{m \in \mathbb{K}},\]

where for each \(j\),
\[|\gamma_m^{(j)} - \frac{\pi m}{\ell_j}| = O(|m|^{-1}).\]

The proof, an application of Rouché’s Theorem to \(G\), resembles the proof of Corollary 2 in [6], and is left to the reader.

We now consider the asymptotics of System 2.23-2.24 with \(q \neq 0\). For this paragraph, we parametrize the sets \(\{ (\lambda_n)^2 : n = 1, 2, \ldots \}\) and \(\{ (\gamma_n)^2 : n = 1, 2, \ldots \}\), both in increasing order. We now define the eigenfrequencies \(\{ \lambda_n : n \in \mathbb{K} \}\) associated to \(q \neq 0\); all of these appear by pairs. If the \((\lambda_n)^2 > 0\), the associated pair will be \(\lambda_n, -\lambda_n\). For \((\lambda_n)^2 < 0\) the associated pair will be the complex conjugate pair \(\lambda_{\pm n} = \pm i \sqrt{|\lambda_n|^2}\), and for \((\lambda_n)^2 = 0\), we have \(0 = \lambda_n = -\lambda_n\). We now compare eigenfrequencies \(\Lambda = \{ \lambda_n : n \in \mathbb{K} \}\) associated to \(q \neq 0\) with those associated to \(q = 0\), namely \(\Gamma = \{ \gamma_n : n \in \mathbb{K} \}\).

**Corollary 2.**

\(M - D\): \(\beta_2 \neq 0\). Let \(\Lambda'\) be any subset of \(\Lambda\) obtained by deleting \(2N\) elements. Then \(\Lambda'\) can be reparametrized as
\[\Lambda' = \bigcup_{j=1}^{N} \{ \lambda_m^{(j)} \}_{m \in \mathbb{K}} \cup \{ \lambda_m^{(0)} \}_{m \in \mathbb{Z}},\]

where for each \(j > 0\),
\[|\lambda_m^{(j)} - \frac{\pi m}{\ell_j}| = O(|m|^{-1}),\]

and
\[|\lambda_m^{(0)} - \frac{(2m + 1) \pi}{2} | \ell_N^{-1} = O((|m| + 1)^{-1}).\]

\(D - D\): \(\beta_2 = 0\). Let \(\Lambda'\) be any subset of \(\Lambda\) obtained by deleting \(2N\) elements. Then \(\Lambda'\) can be reparametrized as
\[\Lambda' = \bigcup_{j=0}^{N} \{ \lambda_m^{(j)} \}_{m \in \mathbb{K}},\]

where for each \(j\),
\[|\lambda_m^{(j)} - \frac{\pi m}{\ell_j}| = O(|m|^{-1}).\]

**Proof.** The following applies both for the \(D - D\) case and the \(D - M\) case. By a standard mini-max argument, see [6], one can show that the frequencies \(\{ \lambda_n \}\) associated to \(q \neq 0\) are asymptotic to \(\{ \gamma_n \}\). More precisely, for any \((\beta_1, \beta_2)\), we have a constant \(C > 0\) such that
\[|\lambda_n - \gamma_n| < \frac{C}{n}, \quad (5.76)\]

The corollary follows. \(\square\)
6. Divided differences. In Section 7.1, we will prove Theorems 1.1 and 7.2 by solving a certain moment problem. This approach is complicated by the fact that, by Corollary 2, the frequencies \( \{\mu_n\} \) generically have no uniform spectral gap. As a consequence, the exponential family \( \{e^{i\lambda_n t}\} \) is not a Riesz sequence (i.e. a Riesz basis in the closure of its linear span) in \( L^2(0, T) \) for any \( T > 0 \). The main purpose of this subsection is to construct the needed Riesz sequence using exponential divided differences. We also prove a result, Proposition 14, which will be used in the proof of Theorems 1.2 and 7.3.

**Definition 6.1.** A countable set of complex numbers \( \{\mu_n\} \) called **uniformly discrete** if \( \inf_{n \neq m} |\mu_n - \mu_m| > 0 \). A set is **relatively uniformly discrete** if it is a union of a finite number of uniformly discrete sets.

**Definition 6.2.** Assume \( \{\mu_j\} \) is a non-repeating sequence. The exponential divided difference (E.D.D.) of order zero for \( \{e^{i\mu_n t}\} \) is \( [e^{i\mu_1 t}, ..., e^{i\mu_n t}] := e^{i\mu_1 t} \). The E.D.D. of order \( n - 1 \) is given by

\[
[e^{i\mu_1 t}, ..., e^{i\mu_n t}] = [e^{i\mu_1 t}, ..., e^{i\mu_{n-1} t}] - [e^{i\mu_1 t}, ..., e^{i\mu_{n-2} t}] \quad \frac{\mu_1 - \mu_n}{\mu_1 - \mu_2}.
\]

One then easily derives the formula

\[
[e^{i\mu_1 t}, ..., e^{i\mu_n t}] = \sum_{k=1}^{n} e^{i\mu_k t} \prod_{j \neq k} (\mu_k - \mu_j).
\]

It is shown in [10] that the functions \( [e^{i\mu_1 t}, ..., e^{i\mu_n t}] \) depend on the parameters \( \mu_j \) continuously, and each is invariant if we change the order of the \( \mu_j \).

We will create E.D.D. from the set \( \Lambda \) and other sets. We cite the following facts from [10]. For any \( z \in \mathbb{C} \), denote by \( D_z(r) \) the disk with center \( z \) and radius \( r \). Let \( G^p(r) \), \( p \in \mathbb{K} \) be the connected components of the union \( \cup_{z \in \Lambda} D_z(r) \). Write \( N(r) \) for the subset of \( \Lambda \) lying in \( G^p \), \( \Lambda^p := \{\lambda_n| \lambda_n \in G^p(r)\} \). By Corollary 2, \( \Lambda \) can be decomposed into a union of \( N + 1 \) uniformly discrete sets, which we label \( \Lambda_j \) with \( j = 0, ..., N \). Let

\[
\delta_j := \inf_{\lambda \in \Lambda_j} |\lambda - \mu|, \quad \delta := \min_{j} \delta_j.
\]

Then for \( r < r_0 := \frac{\delta}{2N + 3} \), the number \( N^p(r) \) of elements of \( \Lambda^p \) is at most \( N + 1 \). For each family of E.D.D. that we construct in this section, we will assume \( r < r_0 \) has been chosen for appropriate \( r_0 \).

It is now convenient to reparametrize \( \Lambda \) according to the clusters \( \{\Lambda^p\} \). Also, we will make the dependence of \( \Lambda \) on the boundary parameters \( (\beta_1, \beta_2) \) explicit:

\[
\Lambda(\beta_1, \beta_2) = \{\lambda^p_l : p \in \mathbb{K}, l = 1, ..., N^p, \lambda^p_l \in G^p(r)\}.
\]

Denote the family of E.D.D. associated to \( \Lambda(\beta_1, \beta_2) \) as follows:

\[
\mathcal{E}(\Lambda(\beta_1, \beta_2)) := \bigcup_{p \in \mathbb{K}} \bigcup_{k=1}^{N^p} \{[e^{i\lambda^p_l t}, e^{i\lambda^p_k t}] \cdots [e^{i\lambda^p_l t}, e^{i\lambda^p_{N^p} t}]\}.
\]

The following result follows from our Corollary 2, and Theorem 3 of [14], (also see [10])

**Proposition 11.** For any \( T > 2\ell \) and any non-zero pair \( (\beta_1, \beta_2) \in \mathbb{R}^2 \), the family \( \mathcal{E}(\Lambda(\beta_1, \beta_2)) \) forms a Riesz sequence in \( L^2(0, T) \).

Proposition 11 will be a major ingredient in our proof of Theorems 1.1 and 7.2.

To discuss the sharpness conditions in time of our controllability results, we require more
results about E.D.D. To this end, we first reparametrize $\Gamma$ according to the clusters in the same way as we reparametrized $\Lambda$ above:

$$
\Gamma(\beta_1, \beta_2) = \{ \gamma^p_l : p \in K, \ l = 1, \ldots, N^p, \ \gamma^p_l \in G^p(r) \}.
$$

By Corollary 2, this reparametrization can be chosen so for large $|p|$,

$$
|\gamma^p_l - \lambda^p_l| = O(|p|^{-1}), \ \forall l = 1, \ldots, N^p.
$$

Remark 8. The reader should distinguish between $\{ \gamma_m^{(j)} \}$ from the previous section, where the frequencies have been partitioned by their association with the subinterval $(a_j, a_{j+1})$, and this section’s $\{ \gamma^p_l \}$, where the frequencies have been partitioned according to the clustering.

We wish to cite a result from \cite{10}. First, we recall some facts about the indicator diagram for an entire function, also see \cite{9, 30}. Let $f$ be an entire function of exponential type. The indicator function is a $2\pi$ periodic function defined by

$$
h_f(\theta) = \limsup_{r \to \infty} \frac{\ln |f(re^{i\theta})|}{r}.
$$

The indicator diagram for $f$ is a convex set $G_f$ such that

$$
h_f(\theta) = \sup_{z \in G_f} \Re(ze^{-i\theta}).
$$

Example. Let $f(z) = e^{i\alpha z} + e^{-i\beta z}$, with $\alpha, \beta > 0$. Then

$$
h_f(\theta) = \begin{cases} 
\beta \sin(\theta), & \theta \in [0, \pi], \\
-\alpha \sin(\theta), & \theta \in [\pi, 2\pi].
\end{cases}
$$

Then one computes $G_f = [-i\alpha, i\beta]$.

Theorem 6.3 (\cite{10}). Let $M = \{ \mu_n \}$ be a relatively uniformly discrete sequence with a uniform bound on the imaginary parts, and denote by $m_{\mu_n}$ the multiplicity of $\mu_n$. Let $\{E(M)\}$ be the associated family of E.D.D., constructed by the procedure given above with a fixed $r < r_0$. Then the family $\{E(M)\}$ forms a Riesz basis in $L^2(0, T)$ if and only if there exists an entire function $F$ of exponential type with indicator diagram of width $T$ with zeros at points $\mu_n$ of multiplicity $m_{\mu_n}$ (the generating function of $E(M)$ on the interval $(0, T)$) such that for some real $h$, $|F(x + ih)|^2$ satisfies the Muckenhoupt condition $(A_2)$:

$$
\sup_{J \in J} \left\{ \frac{1}{|J|} \int_J |F(x + ih)|^2 dx, \frac{1}{|J|} \int_J |F(x + ih)|^{-2} dx \right\} < \infty.
$$

Here $J$ is the set of intervals on the real line.

Definition 6.4. Let $F$ be an entire function of exponential type.

a) $F$ is a Cartwright function if

$$
\int_{-\infty}^{\infty} \frac{\ln |F(x)|, 0)}{1 + x^2} dx < \infty.
$$

b) $F$ is a sine-type function if i) the zeros of $F$ lie in a strip $\{ k : |\Re k | < h \}$ for some $h > 0$, and ii) there exists $y_0 \in \mathbb{R}$ such that $|F(x + iy_0)| \asymp 1$ for $x \in \mathbb{R}$.

We will also use the following statement from \cite{10}:
Proposition 12. Let \( \{\mu_n\} \) be the zeros of a sine-type function with indicator diagram of width \( 2\ell \), \( \delta_n \) a bounded sequence of complex numbers, and \( F \) an entire function of Cartwright class with zero set \( \{\mu_n + \delta_n\} \). If
\[
\lim_{N \to \infty} \sup_n \frac{1}{N} |\Re(\delta_{n+1} + \ldots + \delta_{n+N})| =: d < \frac{1}{4},
\]
then \( F \) has an indicator diagram of width \( 2\ell \) and for any \( h > \sup |\Re(\mu_n + \delta_n)| \), the function \(|F(x + ih)|^2\) satisfies the Muckenhoupt condition \((A_2)\).

Proposition 13.

\( A \) (M–D boundary conditions) Suppose \( \beta_2 \neq 0 \), and let \( \sigma(N) \) be any \( N \) element subset of \( \Gamma(\beta_1, \beta_2) \). Then \( \mathcal{E}(\Gamma(\beta_1, \beta_2) \setminus \sigma(N)) \) forms a Riesz basis of \( L^2(0, 2\ell) \).

\( B \) (D–D boundary conditions) Suppose \( \beta_2 = 0 \), and let \( \sigma(N-1) \) be any \( (N-1) \) element subset of \( \Gamma(\beta_1, \beta_2) \). Then \( \mathcal{E}(\Gamma(\beta_1, \beta_2) \setminus \sigma(N-1)) \) forms a Riesz basis of \( L^2(0, 2\ell) \).

Proof. We prove this for case \( A \); case \( B \) is similar. In what follows, we will drop the \((\beta_1, \beta_2)\) dependence for readability. We apply Theorem 6.3. We showed that the set \( \Gamma \) is relatively uniformly discrete in Corollary 2, and hence the same holds for \( \Gamma \setminus \sigma(N) \). For the generating function for \( \Gamma \setminus \sigma(N) \), we set
\[
F_0(\lambda) = \frac{G(\lambda)}{\prod_{\gamma_1 \in \sigma(N)} (\lambda - \gamma_1^p)}.
\]
By Corollary 1 parts a), b), \( F_0 \) is of exponential type and has zero set \( \Gamma \setminus \sigma(N) \), including multiplicities. Using Lemma 5.1 part i, it is not hard to see that
\[
h_{F_0}(\theta) = \begin{cases} 
\ell \sin(\theta), & \theta \in [0, \pi], \\
-\ell \sin(\theta), & \theta \in [\pi, 2\pi].
\end{cases}
\]
By the example presented above prior to Theorem 6.3, we conclude the indicator diagram is \([-\ell, \ell]\), and its width is \( 2\ell \). Furthermore, by Corollary 1 part c), the relation \(|F_0(x + ih)| \approx 1, x \in \mathbb{R}\), is valid for \(|h| > y_0\). This completes the proof. We remark that we have also proven that \( F_0 \) is a sine-type function.

The next result is key to proving Theorems 1.2 and 7.3.

Proposition 14.

\( A \) (M–D boundary conditions) Suppose \( \beta_2 \neq 0 \), and let \( \sigma'(N) \) be any \( N \) element subset of \( \Lambda(\beta_1, \beta_2) \). Then \( \mathcal{E}(\Lambda(\beta_1, \beta_2) \setminus \sigma'(N)) \) forms a Riesz basis of \( L^2(0, 2\ell) \).

\( B \) (D–D boundary conditions) If \( \beta_2 = 0 \), and let \( \sigma'(N-1) \) be any \( (N-1) \) element subset of \( \Lambda(\beta_1, \beta_2) \). Then \( \mathcal{E}(\Lambda(\beta_1, \beta_2) \setminus \sigma'(N-1)) \) forms a Riesz basis of \( L^2(0, 2\ell) \).

Proof. We prove part \( A \); the proof of \( B \) is similar. We drop \((\beta_1, \beta_2)\) dependence for readability. We again apply Theorem 6.3. That the set \( \Lambda \setminus \sigma'(N) \) is relatively uniformly discrete was established in Corollary 2. We define the generating function
\[
F(\lambda) = \prod_{\lambda_i^p \in \Lambda \setminus \sigma'(N)} \left(1 - \frac{\lambda}{\lambda_i^p}\right).
\]
By direct estimate one can see that \( F \) is entire of exponential type. We now show \( F \) is a Cartwright function. Choose \( P \) so that
\[
|p| > P \text{ implies } \lambda_i^p \in \Lambda \setminus \sigma'(N) \text{ and } (\lambda_i^p)^2 > 0.
\]
The latter inequality implies \( \lambda^p_{l} = -\lambda^p_{l'}, l = 0, \ldots, \mathcal{N}(p) \), so that
\[
F(\lambda) = \prod_{(l,p):p>p} \left(1 - \frac{\lambda^2}{(\lambda^p_{l})^2}\right) \prod_{(l,p):|p| \leq P, \lambda \in \sigma(N)} \left(1 - \frac{\lambda}{\lambda^p_{l}}\right).
\]
Letting \( \lambda = x + iy \),
\[
\max(\ln |F(x)|, 0) \leq \sum_{(l,p):|p| \leq P} \max(\ln |1 - \frac{x}{\lambda^p_{l}}|, 0).
\]
Since this latter sum is finite, \( F \) is a Cartwright function. Letting \( \delta^p_{l} = \lambda^p_{l} - \gamma^p_{l} \), by 5.76 we have \( d = 0 \) in 6.77. Let \( F_0 \) be the sine type function mentioned in the proof of Proposition 13. Applying Proposition 12, it follows that \( F \)'s indicator diagram is of width \( 2\ell \) and \( |F(x + ih)|^2 \) satisfies the Muckenhoupt condition (A2) for some \( h \). The proof is completed by applying Theorem 6.3. \( \Box \)

7. Control and Riesz bases for \( W^p_b \), \( b = D \) or \( \mathcal{M} \).

7.1. Proofs of Theorems 1.1 and 1.2. The argument to finish the proof of Theorem 1.1 is basically the same as the one given in [6], Section 5, but we include in here because elements of this proof are used to prove Theorems 1.2 and 7.3.

We consider System 3.29-3.30 with \( \beta \neq 0 \). In what follows, we denote the frequencies of the system 2.23-2.24 by \( \Lambda = \{\lambda^p_{k} : p \in \mathbb{K}, k = 1, \ldots, \mathcal{N}(p)\} \), with \( \varphi^{|p|}_k \) the corresponding eigenfunctions, orthonormal in \( L^2_{\gamma} \). We will assume for simplicity that the eigenvalues satisfy \( (\lambda^p_{k})^2 > 0 \). If this were not the case in what follows, it would suffice to replace in 7.80 the term \( \sin(\lambda t)/\lambda \) by \( \sin(|\lambda|t)/|\lambda| \) in the case \( \lambda^2 < 0 \), and by \( t \) in the case \( \lambda^2 = 0 \).

Step 1. we express the full control problem as a moment problem.

We present the solution of System 3.29-3.30 in the form of the series
\[
u^f(x,t) = \sum_{p=1}^{\mathcal{N}(p)} \sum_{k=1}^{\mathcal{N}(p)} a^p_k(t) f^p_k(x).
\]
In what follows, we will sometimes drop the superscript \( p \) for readability, and for the same reason write \( u \) for \( u^f \) and \( W \) for \( W^f \). Let \( (u_0, u_1) \in W_1 \times W_0 \), and fix \( T > 2\ell \). We wish to find \( f \) solving
\[
u^f(x,T) = u_0(x), \quad u^f(x,T_*) = u_1(x).
\]
For any \( T > 0 \) and \( f \in L^2(0,T) \), standard calculations using the weak solution formulation for System 3.29-3.30 (see, eg. [9] Ch. 3) give for each \( p \),
\[
\frac{\varphi_k(0)}{\lambda_k} \int_0^t f(\tau) \sin(\lambda_k(t-\tau)) d\tau, \quad a^f_k(t) = \frac{\varphi_k(0)}{\lambda_k} \int_0^t f(\tau) \cos(\lambda_k(t-\tau)) d\tau.
\]
Denote \( s_k(x) = \sin(\lambda_kx) \) and \( c_k(x) = \cos(\lambda_kx) \). We set \( a_k = a_k(T), b_k = b^f_k(T) \), and
\[
\alpha_k = \frac{a_k \lambda_k}{\varphi_k(0)}, \quad \beta_k = \frac{b_k}{\varphi_k(0)}.
\]
Let \( < , >_T \) be the standard complex inner product on \( L^2(0,T) \). Let \( f^T(t) = f(T-t) \), so \( \int_0^T f(t) \sin(\lambda(T-t)) dt =< f^T, \sin(\lambda t) >_T \). Then 7.80 for \( t = T \) can be written as
\[
\alpha_k^p =< f^T, s^p_k >_T, \quad p \in \mathbb{N}, \ k = 1, \ldots, \mathcal{N}(p),
\]
Lemma 7.1. For \( p \in \mathbb{N}, k = 1, ..., \mathcal{N}(\nu) \):

\[
\beta_n^p = \langle f^T, c_n^p \rangle_{\mathcal{H}}, \quad p \in \mathbb{N}, \ k = 1, ..., \mathcal{N}(\nu).
\] (7.83)

Using \( e^{it} = \cos(t) + i \sin(t) \), we get the following equations which hold for all \( p \in \mathbb{N}, k = 1, ..., \mathcal{N}(\nu) \):

\[
i \alpha_k^p + \beta_k^p = \langle f^T, e^{-i\lambda_k^p} \rangle_{\mathcal{H}},
\]

\[
-i \alpha_k^p + \beta_k^p = \langle f^T, e^{i\lambda_k^p} \rangle_{\mathcal{H}}.
\]

Recall \( \lambda_k^{-p} = -\lambda_k^p \) for \( p \geq 1 \). Similarly we set \( \alpha_k^{-p} = -\alpha_k^p \), and \( \beta_k^{-p} = \beta_k^{|p|} \) for all \( p \in \mathbb{K} \). Define \( \gamma_k^p \) by

\[
\gamma_k^p = (-i \alpha_k^p + \beta_k^p), \quad \forall \ p \in \mathbb{K}, \ k = 1, ..., \mathcal{N}(\nu).
\]

Then solving the control problem 7.79 is equivalent to solving for \( f \in L^2(0, T) \) in the following moment problem:

\[
< f^T, e^{i\lambda_k^p} >_{\mathcal{H}} = \gamma_k^p, \ p \in \mathbb{K}, \ k = 1, ..., \mathcal{N}(\nu).
\] (7.84)

We cannot yet solve this moment problem because it is not clear that \( \{\gamma_j\} \in \ell^2 \), and also \( \{e^{i\lambda_k^p}\} \) is not necessarily a Riesz sequence in \( L^2(0, T) \) for any given \( T \). Thus, we will restate 7.84 in terms of E.D.D.

**Step 2. We construct various Riesz sequences.**

Recall

\[
\mathcal{E} = \bigcup_{p \in \mathbb{K}} \{ [e^{i\lambda_1^p}], [e^{i\lambda_2^p}], ..., [e^{i\lambda_n^p}], ..., [e^{i\lambda_{\mathcal{N}(\nu)}^p}] \}.
\]

We define \( \mathcal{S} \) and \( \mathcal{C} \) to be corresponding families of divided differences of \( s_n \) and \( c_n \). Thus,

\[
\mathcal{S} = \bigcup_{p \in \mathbb{N}} \{ [s_1^p(t)], ..., [s_n^p(t)], ..., [s_{\mathcal{N}(\nu)}^p(t)] \},
\]

\[
\mathcal{C} = \bigcup_{p \in \mathbb{N}} \{ [c_1^p(t)], ..., [c_n^p(t)], ..., [c_{\mathcal{N}(\nu)}^p(t)] \}.
\]

For any \( N \geq 1 \), we have that \( \mathcal{E} \) is a Riesz sequence on \( L^2(0, T) \) for \( T > 2\ell \) by Proposition 11. It then follows from ([4], Lemma 5.1) that \( \mathcal{S} \) and \( \mathcal{C} \) form Riesz sequences in \( L^2(0, T) \) for \( T > \ell \).

**Step 3. We rewrite moment problem in terms of EDD.**

We wish to rewrite the moment problems above in terms of our EDD, but first we need to develop some notation. For \( \lambda_k^p \) as above and \( a_1^p, ..., a_n^p \in \mathbb{C} \), we construct divided differences of these numbers iteratively by \( [a_1^p] = a_1^p \), and

\[
[a_1^p, ..., a_n^p] = [a_1^p, ..., a_{n-1}^p] - [a_2^p, ..., a_n^p] \frac{1}{\lambda_k^p - \lambda_{n-1}^p}.
\]

The following result was proven in [6].

**Lemma 7.1.** For \( n = 1, ..., \mathcal{N}(\nu) \),

\[
a_n^p = \sum_{k=1}^{n} [a_1^p, ..., a_k^p] \prod_{j=1}^{k-1} (\lambda_j^p - \lambda_j^p).
\]

Here we use the convention that \( \prod_{j=1}^{n} (\lambda_j^p - \lambda_j^p) = 1 \).

It is easy to see 7.84 is equivalent to

\[
< \gamma_1^p, ..., \gamma_k^p > = < f^T, [e^{i\lambda_1^p}, ..., e^{i\lambda_k^p}] >_{\mathcal{H}}, \quad p \in \mathbb{K}, \ k = 1, ..., \mathcal{N}(\nu).
\] (7.85)
Thus, with an ability, Step 4. Moment problems associated to shape and velocity control.

Since $\mathcal{E}$ is a Riesz sequence on $L^2(0, T_*)$, there exists a solution $f \in L^2(0, T_*)$ to the moment problem with

$$
\|f_T\|_{L^2(0, T_*)}^2 = \|f\|_{L^2(0, T_*)}^2 \simeq \sum_{p,k} \|\gamma_k^p, \ldots, \gamma_k^p\|^2.
$$

(7.86)

Thus, we have solved the moment problem, but it is not yet apparent how the moment problem with $\{\gamma_k^p, \ldots, \gamma_k^p\}$ relates to the spaces $W_k^{pM}$. As a first step of this characterization, observe that for all $p \in \mathbb{K}$, and all $k = 1, \ldots, \mathcal{N}(p)$,

$$
[\beta_1^p, \ldots, \beta_k^p]' + i[\alpha_1^p, \ldots, \alpha_k^p]' = [\gamma_1^p, \ldots, \gamma_k^p]'.
$$

Hence

$$
\sum_{p,k} \|\gamma_k^p, \ldots, \gamma_k^p\|^2 = \sum_{p,k} |\alpha_k^p, \ldots, \alpha_k^p\|^2 + \sum_{p,k} |\beta_k^p, \ldots, \beta_k^p\|^2.
$$

(7.87)

**Step 4. Moment problems associated to shape and velocity control.**

We rewrite (7.82) in the form

$$
[a_1^p, \ldots, a_k^p]' = < f_T, [s_{1,k}^p, \ldots, s_{k,k}^p] >_T, \quad p \in \mathbb{N}, \quad k = 1, \ldots, \mathcal{N}(p),
$$

(7.88)

and (7.83) in the form

$$
[\beta_1^p, \ldots, \beta_k^p]' = < f_T, [c_{1,k}^p, \ldots, c_{k,k}^p] >_T, \quad p \in \mathbb{N}, \quad k = 1, \ldots, \mathcal{N}(p).
$$

(7.89)

For any $p$, let $\mathcal{N} = \mathcal{N}(p)$. By Lemma 7.1, dropping the superscript $p$ for readability,

$$
a_1 \varphi_1 + \ldots + a_N \varphi_N = \sum_{n=1}^N \varphi_n \left( \sum_{k=1}^n [a_1, \ldots, a_k]' \prod_{j=1}^{k-1} (\lambda_n - \lambda_j) \right)
$$

$$
= \sum_{k=1}^N [a_1, \ldots, a_k]' \left( \sum_{j=k}^N \varphi_j \left( \prod_{l=1}^{k-1} (\lambda_j - \lambda_l) \right) \right).
$$

Thus, with $a_k^p = \alpha_k^p \varphi_k^p(0)/\lambda_k^p$, we can rewrite (7.78), with $t = T$, as

$$
u(x, T) = \sum_{p=1}^{\infty} \sum_{k=1}^{\mathcal{N}(p)} a_k^p \varphi_k^p(x)
$$

$$
= \sum_{p=1}^{\infty} \sum_{k=1}^{\mathcal{N}(p)} \alpha_k^p \varphi_k^p(x) \varphi_k^p(0)/\lambda_k^p
$$

$$
= \sum_{p=1}^{\infty} \sum_{k=1}^{\mathcal{N}(p)} [\alpha_1^p, \ldots, a_k^p]' \left( \sum_{j=k}^{\mathcal{N}(p)} \varphi_j^p(x) \varphi_j^p(0) \prod_{l=1}^{k-1} (\lambda_j^p - \lambda_l^p) \right).
$$

(7.90)

**Step 5. Riesz bases for $W_1$ and $W_0$.**

Define

$$
\psi_k^p(x) := \sum_{j=k}^{\mathcal{N}(p)} \varphi_j^p(x) \varphi_j^p(0) \prod_{l=1}^{k-1} (\lambda_j^p - \lambda_l^p).
$$

We will prove that the family $\{\psi_k^p \mid p \geq 1, \ k = 1, \ldots, \mathcal{N}(p)\}$ forms a Riesz basis of $W_1$. This result is not only central to the proof of Theorem 1.1, but is of independent interest. Let $v$ be an arbitrary element of $W_1$. We need to prove the following facts:

(i) $v$ can be presented uniquely as a series with respect to $\psi_k^p$, with quadratically summable coefficients and the series converging in $W_1$,

(ii) the $\ell^2$-norm of the coefficients is equivalent to $W_1$—norm of the function.
We begin proving part i. Suppose $T > \ell$. By Theorem 1.3, part A, there exists $g \in L^2(0, T)$ such that $u^g(x, T) = v(x)$, with
\[
||v||_{W_1} \asymp ||g||_{L^2(0,T)}. \tag{7.91}
\]
We introduce simplifying notations and rewrite 7.88 in the form
\[
\nu_n = <g^T, S_n >_T, \quad n \in \mathbb{N}, \tag{7.92}
\]
and rewrite 7.90 in the form
\[
v(x) = u^g(x, T) = \sum_{n \in \mathbb{N}} \nu_n \psi_n. \tag{7.93}
\]
Since $\varphi_n$ form an orthogonal basis in $L^2_M(0, \ell)$, it follows that $v$ can be uniquely represented as series 7.78, and hence as series 7.93. Since $\{S_n : n \geq 1\}$ forms a Riesz sequence on $L^2(0, T)$ and $g \in L^2(0, T)$, it follows from 7.92 that $\nu_n \in \ell^2$. Below we will prove the series 7.93 converges in $W_1$.

We now prove part ii above. Denote by $S$ the closure of linear span of $\{S_n\}$ in $L^2(0, T)$. Let $g_0$ be the orthogonal projection of $g$ onto $S$. Then clearly $g_0$ satisfies the moment problem 7.92, and hence $u^{g_0}(x, T) = v(x)$ and by Proposition 4
\[
||v||_{W_1} \asymp ||g_0||_{L^2(0,T)}. \tag{7.94}
\]
On the other hand, by 7.91 we have
\[
||g_0||_{L^2(0,T)} \leq ||g||_{L^2(0,T)} \asymp ||v||_{W_1}.
\]
Thus
\[
||g_0||_{L^2(0,T)} \asymp ||v||_{W_1}.
\]
Furthermore, since $\{S_n\}$ form a Riesz basis on $S$, we have
\[
||g_0||^2_{L^2(0,T)} \asymp \sum_{n \in \mathbb{N}} |\nu_n|^2.
\]
Combining, we get
\[
||\sum_{n \in \mathbb{N}} \nu_n \psi_n||_{W_1}^2 = ||v||_{W_1}^2 \asymp \sum_{n \in \mathbb{N}} |\nu_n|^2.
\]
This argument also shows that for any sequence $\{\eta_n : n \in \mathbb{N}\} \in \ell^2$,
\[
||\sum_{n \in \mathbb{N}} \eta_n \psi_n||_{W_1}^2 \asymp \sum_{n \in \mathbb{N}} |\eta_n|^2.
\]
This proves ii and also that series 7.93 converges in $W_1$.

We similarly find a Riesz basis for $W_0$. We have
\[
u_t(x, T) = \sum_{p=1}^{\infty} \sum_{k=1}^{N^p} [\beta_1^p, \ldots, \beta_k^p]'(\sum_{j=k}^{N^p} \varphi_j^p(x) \varphi_j^p(0) \prod_{l=1}^{k-1} (\lambda_j^p - \lambda_l^p)).
\]
Let
\[
\tilde{\psi}_k^p(x) = \sum_{j=k}^{N^p} \varphi_j^p(x) \varphi_j^p(0) \prod_{l=1}^{k-1} (\lambda_j^p - \lambda_l^p).
\]
Arguing as above, one can show $\{\tilde{\psi}_k^p \mid p \geq 1, \quad k = 1, \ldots, N^{(p)}\}$ forms a Riesz basis of $W_0$. The details are left to the reader.

**Step 6. completion of proof of Theorem 1.**

Let $(y_0, y_1) \in W_1 \times W_0$. We wish to find $f \in L^2(0, T_*)$ solving
\[
w^f(x, T_*) = y_0(x), \quad u_t(x, T_*) = y_1(x). \tag{7.94}
\]
We begin by expressing 7.94 as a moment problem. We expand $y_0$ and $y_1$ using the Riesz bases from Step 5:

$$y_0(x) = \sum_{p=1}^{\infty} \sum_{k=1}^{N^{(p)}} y_{0,k}^p \psi_k^p(x), \quad y_1(x) = \sum_{p=1}^{\infty} \sum_{k=1}^{N^{(p)}} y_{1,k}^p \psi_k^p(x).$$

Extend $y_{0,k}^p$, resp. $y_{1,k}^p$ as even, resp. odd functions in $p \in \mathbb{K}$. Set $\hat{\gamma}_k^p = y_{0,k}^p + iy_{1,k}^p \in \ell^2$. Then there exists $f \in L^2(0, T_\gamma)$ solving following moment problem:

$$< f^T, [e^{i\lambda_1^k}, ..., e^{i\lambda_{N}^k}] >_{T_\gamma} = \hat{\gamma}_k^p, \quad p \in \mathbb{K}, \quad k = 1, \ldots, N^{(p)},$$

and, hence, the same $f$ solves 7.94. We have by Step 5 along with 7.90 that

$$\sum_{p=1}^{\infty} \sum_{k=1}^{N^{(p)}} |y_{0,k}^p|^2 \simeq \|y_0\|_{W_1}^2.$$

Similarly, Step 5 implies

$$\sum_{p=1}^{\infty} \sum_{k=1}^{N^{(p)}} |y_{1,k}^p|^2 \simeq \|y_1\|_{W_0}^2.$$

Thus by 7.86 and 7.87, we get

$$\|f\|_{L^2(0, T_\gamma)}^2 \simeq \sum_{p,k} |\gamma_k^p|^2 \simeq \|y_0\|_{W_1}^2 + \|y_1\|_{W_0}^2.$$

The following result is proved in the same way; also see [6]:

**Theorem 7.2.** (Dirichlet control). Consider System 1.1-1.7 with $\alpha_2 = 0$. Let $T > 2\ell$ for $N \geq 2$, and $T > 2\ell$ for $N = 1$. Then for any $(y_0, y_1) \in W_0^{D} \times W_1^{D}$, there exists a control $f \in L^2(0, T)$ such that

$$u^f(\cdot, T) = y_0, \quad \text{and} \quad u^f_1(\cdot, T) = y_1.$$

Furthermore,

$$\|f\|_{L^2(0, T)}^2 \simeq \|y_0\|_{W_0}^2 + \|y_1\|_{W_1}^2.$$

We now prove Theorem 1.2 and its Dirichlet analogue.

**Proof.** Suppose first $T = 2\ell$. We rewrite the moment problem 7.85 with simplifying notation

$$\gamma_{pk} = < f^T, e_{pk} >_{T_\gamma}; \quad p \in \mathbb{K}, \quad k = 1, \ldots, N^{(p)}.$$

Let $\mathcal{P} = \{< f^T, e_{pk} >_{T} : f \in L^2(0, T)\}$. Due to Step 5 in our proof of Theorem 1.1, the reachable set is isomorphic to $\mathcal{P}$. Proposition 14 states that the family $\{e_{pk}\}$ become a Riesz basis in $L^2(0, T)$ if we remove from it $N$ functions, so an arbitrary $N$ functions of this family can be presented in a form of the series with respect to the remaining functions. Therefore, the codimension of the reachable set is $N$.

Suppose next $T < 2\ell$. The density of the set $\{\lambda_n\}$ is equal to $\ell/\pi$ by Corollary 2. Therefore the family $\{e_{pk}\}$ can be made into a Riesz basis in $L^2(0, T)$ if we remove from it an infinite subfamily [14]. This proves the second statement of Theorem 1.2. The proof of the Dirichlet analogue is similar. ☐

The following result is proved in the same way as Theorem 1.2.
Theorem 7.3. System 1.1-1.7 with Dirichlet control is not approximately controllable for $T \leq 2\ell$ if $N \geq 2$ and for $T < 2\ell$ if $N = 1$. If $T = 2\ell$, then the codimension of the reachable set in $W^P_{11} \times W^P_{00}$ is equal to $N - 1$, and for $T < 2\ell$ its codimension is infinite.

7.2. Theorem 1.4 and generalizations.

Theorem 7.4. Suppose $\beta_2 \neq 0$, i.e. $\mathcal{M} - \mathcal{D}$ boundary conditions. Assume $A^M$ is invertible. Then for any integer $i$,

$$\{\sum_{j=k}^{N^p} \frac{\varphi_j^p(0)\varphi_j^p(x)}{(\lambda_j^p)^p} \prod_{l=1}^{k-1}(\lambda_j^p - \lambda_l^p) \mid p \geq 1, k = 1, \ldots, N^p\}$$

forms a Riesz basis of $W^M_i$.

Proof. In the proof of Theorem 1.1 we established this result for $i = 0, 1$. We prove for $i = -1$; the proof for other $i$ is similar. Recall the family $\{\psi_k^p \mid p \geq 1, k = 1, \ldots, N^p\}$ forms a Riesz basis of $W^M_1$, where

$$\psi_k^p(x) = \sum_{j=k}^{N^p} \frac{\varphi_j^p(x)\varphi_j^p(0)}{\lambda_j^p} \prod_{l=1}^{k-1}(\lambda_j^p - \lambda_l^p).$$

Applying the operator $A^M$ to this family, and recalling that by definition $A^M : W^M_1 \rightarrow W^M_{-1}$ is an isomorphism, the result follows.

A similar argument yields the following:

Theorem 7.5.

Suppose $\beta_2 = 0$, i.e. $\mathcal{D} - \mathcal{D}$ boundary conditions. Assume $A^P$ is invertible. Then for any integer $i$,

$$\{\sum_{j=k}^{N^p} \frac{\varphi_j^p(0)\varphi_j^p(x)}{(\lambda_j^p)^{p+1}} \prod_{l=1}^{k-1}(\lambda_j^p - \lambda_l^p) \mid p \geq 1, k = 1, \ldots, N^p\}$$

forms a Riesz basis of $W^P_i$.

7.3. Regular control for regular data. For $T > 0$ and $i \geq 1$, we will consider controls in the Sobolev space $H^2(0, T)$ with norm $\|f\|_{H^2} = \|f(\cdot)|_{L^2(0, T)}$. We first state and prove our result here for the case of Dirichlet control. In what follows, $u^f(x, t)$ will denote the solution to System 3.29-3.30. For this subsection, we will assume the coefficients in 1.1 satisfy $\rho_j, \xi_j, q_j \in C^\infty[\tau_j, \tau_{j+1}]$ for each $j$, and hence

$$q \in C^\infty[a_j, a_{j+1}] \quad j = 0, \ldots, N.$$

Proposition 15. Assume $\beta_2 = 0$. For positive integer $i$, the mapping

$$f \rightarrow (u^f(x, T), u^f_t(x, T))$$

is a continuous mapping from $H^2_0(0, T)$ to $W^P_{2i} \times W^P_{2i-1}$.

Proof. An argument similar to the proof of Proposition 4 shows $f \rightarrow (u^f(x, T), u^f_t(x, T))$ to be a continuous mapping from $H^2_0$ to $W_{2i} \times W_{2i-1}$, so it remains to verify the compatibility conditions. First, we prove the following: for $f \in H^2_0$ with $j \leq 2i$, and $t > 0$,

$$\frac{\partial^j(u^f)}{\partial t^j}(\cdot, t) = f^{(j)}(\cdot, t), \quad \frac{\partial^j(u^f_t)}{\partial t^j}(\cdot, t) = f^{(j+1)}(\cdot, t). \quad (7.95)$$

Here for $j = 2i$ the second equation is understood to hold in the sense of elements of $W^P_{2i}$. We prove the first identity for $i = 1, j = 2$; the proofs in the other
cases are similar. Assume first that \( f \in C^2_\omega \). Then by Proposition 3 and the discussion in Subsection 3.2, \( u^f_{tt}(x,t) \) is jointly continuous in \( x \) and \( t \), and hence \( u^f_{tt}(0,t) = f''(t) \). By the uniqueness of the solution of System 3.29-3.30, it follows that \( u^f_{tt}(x,t) = f''(x,t) \) for all \( x, t \). This proves the result for \( f \in C^2_\omega \). Now suppose \( f \in H^2_\omega \). By the proof of Proposition 4, we have that the mapping \( (t, f) \mapsto u^f_{tt}(x,t) \) is jointly continuous \( \mathbb{R} \times H^2_\omega \rightarrow W^D_\omega \), and also \((t, f'') \mapsto u^{f''}(x,t) \) is jointly continuous \( \mathbb{R} \times L^2(0,T) \rightarrow W^D_\omega \). Thus, approximating \( f \) by functions \( f_n \in C^2_\omega \), the desired equation holds.

We now verify the compatibility conditions at \( x = 0 \). Recall \( \phi(x) \) satisfies \( C^{k,D}_\omega \) if \( L^\omega \phi(0) = 0 \), \( 0 \leq n \leq \lfloor k/2 \rfloor - 1 \). For integer \( j \leq 2i - 2 \), \( 7.95 \) holds pointwise for all \( x \), so for \( n \leq i - 1 \),

\[
L^n u^f(0,T) = \frac{\partial^{2n} u^f}{\partial t^{2n}}(0,T) = u^{f(2n)}(0,T) = f(2n)(T) = 0,
\]

from which it follows that \( u^f \) satisfies \( C^{2i,D}_\omega \) at \( x = 0 \). A similar argument applies to \( u^f_{tt} \); for integer \( n \leq i - 1 \), by 7.95

\[
L^n u^f_{tt}(0,T) = u^{f(2n+1)}_{tt}(0,T) = f^{(2n+1)}(0,T) = 0,
\]

which shows \( u^f_{tt} \) satisfies \( C^{2i-1,D}_\omega \) at \( x = 0 \). Verification of the compatibility conditions at \( x = a_j \) for \( j \geq 1 \) is left to the reader.

We now prove

**Theorem 7.6.** Assume the hypotheses of Theorem 7.2, and \( \rho_i, \xi_j, q_j \in C^{\infty}[\tau_j, \tau_{j+1}] \) for each \( j \). Assume \( A^D \) is invertible. Suppose \( T > 2\ell \) for \( N > 1 \), and \( T \geq 2\ell \) for \( N = 1 \). Let \( (\phi_0, \phi_1) \in W^D_{2i-1} \times W^D_{2i-1} \) for \( i \geq 1 \). Then there exists \( g \in H^2_{2i-1}(0,T) \) solving

\[
u^g(x,T) = \phi_0(x), \quad u^g_0(x,T) = \phi_1(x),
\]

with

\[
|g|_{H^2} \ll ||\phi_0||_{W^D_{2i}}^2 + ||\phi_1||_{W^D_{2i-1}}^2.
\]

**Proof.** For simplicity, we consider \( i = 1 \); the proof for large \( i \) is similar. Let \( (\phi_0, \phi_1) \in W^D_2 \times W^D_2 \). Denote \( A^D \) by \( A \). Thus \( A \phi_0, A \phi_1 \in W^D_2 \times W^D_2 \). Let \( f \in L^2(0,T) \) solve \( u^f(x,t) = \phi_0(x), u^f_0(x,t) = A \phi_1(x) \). Define

\[
h(t) = -\int_t^T f(r)dr, \quad \text{and} \quad g(t) = -\int_{t=0}^T h(s)ds.
\]

Then \( g \in H^2_2 \), and \( g''(t) = f(t) \). Thus, by 7.95, the following equations hold in \( W^D_2 \):

\[
A(u^g(\cdot,T)) = (u^g)_{tt}(\cdot,T)|_{t=T} = (u^{g''})(\cdot,T)|_{t=T} = (u^f)(\cdot,T)|_{t=T} = (u^f)(\cdot,T) = A \phi_0.
\]

Since \( A \) is invertible, we conclude \( u^g(x,T) = \phi_0(x) \). The proof that \( u^g_0(x,T) = \phi_1(x) \) is similar, using the second equation in 7.95. Finally,

\[
||\phi_0||_{W^D_2} + ||\phi_1||_{W^D_2} \ll ||A \phi_0||_{W^D_2}^2 + ||A \phi_1||_{W^D_2}^2 \ll ||f||_{L^2(0,T)}^2 \ll ||g||_{H^2}^2.
\]

The proof of Theorem 1.5 is similar, and is left to the reader.
8. **Conclusion.** In this paper, we have proven exact boundary controllability for a non-homogeneous vibrating string with a finite number of attached masses, with the control being Dirichlet, Neumann, or mixed. We have also shown our results to be optimal both with respect to regularity of the data and with respect to time.

Our method of proof, which is a mixture of dynamical and spectral arguments, should be of independent interest and could have application in other problems. For instance, the construction of Riesz bases for asymmetric spaces used here appears to be new, and has been used to study control problems for Schrödinger-type equations [7]; in future work the techniques developed here will also be used to study chains of vibrating beams connected by point masses. Also, our unique shape and velocity control results can be applied to study inverse problems for vibrating strings with attached masses, see [1].

9. **Appendix.**

9.1. **Proof of proposition 2.**

**Proof.** Since we assume $\beta_2 \neq 0$, we can rewrite the boundary condition at $x = 0$ as $\beta k(0, t) + k_x(0, t) = 0$. Also, for simplicity we set $b = 0$. We transform the system.

First, set $v(x, t) = e^{\beta x} k(x, t)$ and $g(x) = -\frac{1}{2} \int_0^x q(\eta) d\eta$. Then a calculus exercise shows that $v$ solves

$$v_{tt} - v_{xx} + 2\beta v_x + (q - \beta^2)v = 0, \quad (x, t) \in D,$$

$$v_x(0, t) = 0,$$

$$v(x, x) = g(x) e^{\beta x}.$$

Set $p(x) = g(2x)e^{2\beta x}$. Set $\xi = t - x$, $\eta = t + x$. Then set

$$w(\xi, \eta) = v(\eta - \xi, \eta + \xi).$$

Then $w$ satisfies

$$w_{\xi\eta} - \beta w_\xi + \beta w_\eta + (q - \beta^2)w = 0, \quad 0 < \xi < \eta,$$

$$w_\xi(\xi, \eta) - w_\eta(\xi, \eta) = 0,$$

$$w(0, \eta) = p(\eta).$$

Extending $w$ and $q$ by reflection across the line $\xi = \eta$, we get

$$w_{\xi\eta} - \beta w_\xi + \beta w_\eta + (q - \beta^2)w = 0, \quad \eta > 0, \quad \xi > 0,$$

$$w(0, \eta) = p(\eta),$$

$$w(\xi, 0) = p(\xi).$$

Next, we set

$$z(\xi, \eta) = w(\xi, \eta) - p(\xi) - p(\eta).$$

Then $z$ solves for $\eta > 0$, $\xi > 0$,

$$z_{\xi\eta} - \beta z_\xi + \beta z_\eta + (q - \beta^2)z = (\beta^2 - q)[p(\xi) + p(\eta)] + \beta[p'(\eta) - p'(\xi)],$$

$$z(0, \eta) = 0,$$

$$z(\xi, 0) = 0.$$

This initial boundary value problem can be solved using iterative techniques, see ([23], v. 2, Ch. V, S.6.1).

To prove the regularity of $k$, we integrate in $\xi$ and $\eta$ the partial differential equation 9.97 to get an integral equation for $w$, from which the regularity of $w$, and hence $k$, is readily deduced. The details are left to the reader. \qed
9.2. Example with frequency gap. The purpose of this section is to discuss how one could study controllability in a special case without resorting to E.D.D. A more extensive study of these special cases with Neumann control can be found in [19]. This section also gives a purely spectral argument, that applies in some special cases, proving asymmetry of the reachable sets.

Suppose $N = 1$, and assume $\beta_2 \neq 0$ hence $M - D$ boundary conditions for the Sturm–Liouville problem 1.9-1.13. We assume for simplicity of presentation that $\ell = 1$. We denote $\ell_0 = a_1 = a \in (0,1)$. By Corollary 2, the frequency set decomposes in the union of $\{\lambda_n^{(1)}\} \cup \{\lambda_n^{(2)}\}$, with the following asymptotics

$$
\lambda_n^{(1)} = \frac{2n + 1}{\ell_0} + O(1/n),
$$
$$
\lambda_n^{(2)} = \frac{n\pi}{\ell_1} + O(1/n).
$$

An exercise in elementary analysis shows that there exists a uniform gap in the frequencies if and only if

$$
\frac{\ell_1}{\ell_0} = \frac{m}{n}, \text{ with } \gcd(m,n) = 1 \text{ and } [m \text{ not even or } n \text{ not odd}],
$$

which is equivalent to $a = 1 - \frac{p}{r}$ with $p, r \in \mathbb{N}$ with $p$ odd. This observation first appeared in [19]. Because the spectral density is $\frac{1}{\pi}$ in this case, the family $\{e^{i\lambda_n^{(j)}}t\}$ forms a Riesz sequence on $L^2(0,\ell_1)$ for any $\ell_1 > 2\pi$.

Assume further that $a = 1 - \frac{1}{r}$, with $r = 2, 3, \ldots$, and that $q(x) = \delta_j = 0$ and $\rho_j = \rho_j^* = 1$. Let $\phi_n^{(j)}$ be the eigenfunction associated to $\lambda_n^{(j)}$, normalized so that $\langle \phi_n^{(j)}(1) \rangle = 1$. Thus, dropping subscripts and superscripts for the moment, $\phi(x)$ equals for $x < a$,

$$
-\sin(\lambda(1-a)) \cos(\lambda(a-x)) - \sin(\lambda(a-x))(\cos(\lambda(1-a)) - M\lambda \sin(\lambda(1-a))),
$$

and for $x \geq a$,

$$
-\sin(\lambda(1-x)).
$$

Let $\varphi_n^{(j)} = \phi_n^{(j)}/\|\phi_n^{(j)}\|_{L^2_M}$ be the corresponding unit norm eigenfunction.

Then one calculates, using the formula for $\phi(x)$, that

$$
|\varphi_n^{(1)}(0)| = O(1), \quad (9.99)
$$
$$
\|\varphi_n^{(1)}\|_{L^2(a,1)} \asymp 1/n, \quad (9.100)
$$
$$
\|\varphi_n^{(1)}\|_{L^2(0,a)} \asymp 1, \quad (9.101)
$$
$$
|\varphi_n^{(2)}(0)| = O(1/n), \quad (9.102)
$$
$$
\|\varphi_n^{(2)}\|_{L^2(0,1)} \asymp 1. \quad (9.103)
$$

We can rewrite

$$
u^f(x, T) = \sum_{n=1}^{\infty} a_n^1 \varphi_n^{(1)}(x) + \sum_{n=1}^{\infty} a_n^2 \varphi_n^{(2)}(x).
$$

From 7.80,9.102,9.103, we deduce $\{a_n^2\} \in \ell^2_2$, and

$$
\sum_n a_n^2 \varphi_n^{(2)} \in \mathcal{H}^{2,M}.
$$
We remark that the difference between the proof of Theorem 1.1, or to the arguments of [19],
authors how prove that 3.29-3.30. Thus in the coordinate \( x \)
From 7.80, 9.99, 9.101, and 9.100, we get 
\[
\sum_n a_n^1 \varphi_n^{(1)}(t) + \sum_n a_n^2 \varphi_n^{(2)}(t) + \sum_n b_n^1 \varphi_n^{(1)}(t) + \sum_n b_n^2 \varphi_n^{(2)}(t) = 0.
\]
Since \( u = u^f \) will automatically satisfy Condition \( C_1^1 \), it follows that \( u(x, T) \in W_1 \).
A similar argument using the formula for \( (a_n^j)'(t) \) in 7.80 shows that \( u_i(x, T) \in W_0 \).
Define the set \( R \) by
\[
R := \left\{ \left( \sum_n a_n^1 \varphi_n^{(1)}(x) + \sum_n a_n^2 \varphi_n^{(2)}(x), \sum_n b_n^1 \varphi_n^{(1)}(x) + \sum_n b_n^2 \varphi_n^{(2)}(x) \right) : \right. \\
\left. \{a_n^1\} \in \ell_1^2, \{a_n^2\} \in \ell_2^2, \{b_n^1\} \in \ell_0^2, \{b_n^2\} \in \ell_2^2 \right\}.
\]
The argument above gives the following property for the reachable set for time \( T \):
\[
\{(u^f(x, T), u^f_i(x, T)) \} \subset R.
\]
Remark. Let \( \left( \sum_n a_n^1 \varphi_n^{(1)}(x) + \sum_n a_n^2 \varphi_n^{(2)}(x), \sum_n b_n^1 \varphi_n^{(1)}(x) + \sum_n b_n^2 \varphi_n^{(2)}(x) \right) \in R \).
If we could strengthen 9.99, 9.102 to \( |\varphi_n^{(1)}(0)| \approx 1 \) and \( |\varphi_n^{(2)}(0)| \approx 1/n \), then in the notation of 7.81, 7.84 we would have
\[
\{\gamma_n^p\} \in \ell^2.
\]
Hence for \( T > 2\pi \), the moment problem 7.84 would be solvable with \( f \in L^2(0, T) \) satisfying
\[
\|f\|_{L^2(0, T)}^2 \approx \sum_n |\gamma_n^p|^2 = \sum_{n \in \mathbb{N}, j=1,2} \left| \frac{\Delta_n^j a_n^j}{\varphi_n^{(j)}(0)} \right|^2 + \left| \frac{b_n^j}{\varphi_n^{(j)}(0)} \right|^2.
\]
This would prove \( R \) equals the reachable set for time \( T \). It is unclear to the authors how prove that \( \|f\|_{L^2(0, T)}^2 \) is equivalent \( \|u_0\|_{W_1}^2 + \|u_1\|_{W_0}^2 \) without resorting to the proof of Theorem 1.1, or to the arguments of [19].

9.3. Proof of Theorem 1.3. For the proof, we consider the transformed System 3.29-3.30. Thus in the coordinate \( x \), we have
\[
W_1^{T,M} = \left\{ \{\phi(x) \in W_1^M : \phi(x) = 0 \text{ for } x > T \}, \quad T < \ell, \right. \\
\left. \{\phi(x) \in W_1^M : \frac{d\phi}{dx}(t) = 0, \ j = 0, \ldots, (N+i) \}, \quad T = \ell, \right. \\
\left. W_i^M, \quad T > \ell. \right. 
\]
We remark that the difference between \( W_0^{T,M} \) and \( W_0^M \) is that \( \phi \in W_0^M \) implies only \( \phi(t) = 0 \). For Theorem 1.3, we will prove only Part A, leaving part B to the reader. We need to prove the following statements:

I. Suppose \( T \leq \ell \). For any \( \phi \in W_1^{T,M} \), there exists a unique \( f \in L^2(0, T) \) such that \( u^f(x, T) = \phi(x) \). Furthermore,
\[
\|u^f(\cdot, T)\|_{W_1^M} \approx \|f\|_{L^2(0, T)}.
\]

II. Let \( T > \ell \). For any \( \phi \in W_1^M \), there exists a \( f \in L^2(0, T) \) such that \( u^f(x, T) = \phi(x) \) and 9.104 holds.

Fix \( T \leq \ell \) and suppose \( j \) satisfies \( T \in (a_j, a_{j+1}] \). If \( j = 0 \), then the desired result holds by Lemma 4.1. Assume now \( j \geq 1 \). We will prove that for any \( \phi(x) \in W_1^{T,M} \), the equation
\[
u^f(x, T) = \phi(x)
\]
has a unique solution \( f \in L^2(0, T) \). We set \( \Lambda = 2 \min |a_{i+1} - a_i|, \ i = 0, \ldots, N, \) and will solve the equation 9.105 by steps of the length at most \( \Lambda \). This means that we
will move by such steps along the \( x \)-axis from the right to the left starting at the point \( x = T \).

**Step 1.** We solve the equation 9.105 for the set of \( x \in (a_j, T) \) within distance \( \Lambda \) of the the wavefront \( x = T \). On this interval at time \( t = T \), there can be no reflected terms because of unit speed of propagation. We consider separately the two possible cases: (a) \( T - \Lambda \geq a_j \), (b) \( T - \Lambda < a_j \).

**Case a.** Because there are no reflected terms on the interval \( x > T - \Lambda \), arguing as in 3.46 we have that the equation 9.105 reduces to

\[
\phi(x) = h_j(T - x) + \int_{x-a_j}^{T-a_j} k_D(a_j^+; x - a_j, s)h_j(T-a_j-s)ds,
\]

with \( x \in (T - \Lambda, T] \). This is a Volterra equation of the second kind, hence \( \phi \) uniquely determines the function \( h_j(t) \) on the interval \([0, \Lambda]\), and its regularity is the same as the regularity of \( \phi \). Using Lemmas 3.1 and 3.5, we can conclude there exists a unique \( f \in L^2(0, \Lambda) \) such that \( f(t) = (S_{\Lambda}^{-1} S_{\Lambda}^{-1} h_j)(t) \) for \( t \in [0, \Lambda] \).

**Case b.** We use the same argument as in part a over the shorter interval \([a_j, T]\). Thus we first use 9.106 on the interval \( x \in (a_j, T] \) to find \( h_j(t) \) on the interval \([0, T - a_j]\). Then we find \( f(t) = (S_{\Lambda}^{-1} S_{\Lambda}^{-1} h_j)(t) \) on the same time interval.

Define \( c_j \) by \( c_j = \Lambda \) in case a, and \( c_j = T - a_j \) in case b.

**Step 2.** Define \( f_1 \) by

\[
f_1(t) = \begin{cases} f(t), & t < c_1, \\ 0, & t \in [c_1, T]. \end{cases}
\]

Then by construction, \( \phi(x) = u^{f_1}(x, T) \) for \( x < T - c_1 \), and hence \( \phi_j(x) := \phi(x) - u^{f_1}(x, T) \in W_1^{T, M} \) is supported on \([0, T - c_1]\), and hence is in \( W_1^{T-c_1, M} \). If \( T - c_1 > a_j \), then we can repeat the argument in Step 1. If \( T - c_1 \leq a_j \), then the same argument as in Step 1 can be carried out in the interval \((a_j-1, a_j)\). In both cases, we find \( c_2 \) along with the unique \( f_2 \in L^2(0, T) \) supported in \([c_1, c_1 + c_2] \) such that \( u^{f_2}(x, T) = \phi_j(x) \) for \( x \in [T - c_1 - c_2, T - c_1] \).

Thus \( u^{f_1+f_2}(x, T) = u^{f_1}(x, T) + u^{f_2}(x, T) = \phi(x) \) for \( x \in [T - c_1 - c_2, T] \).

**Step 3.** We repeat the arguments of step 1 and step 2 as often as necessary, thus solving for \( f \).

We now prove 9.104. Suppose there exists a sequence \( \{f^n\} \), with \( f^n \in L^2(0, T) \) and \( \{\phi^n = u^{f^n}(x, T)\} \), such that \( \|\phi^n\|_{W_1^T} \to 0 \). We will prove \( \|f^n\|_{L^2(0, T)} \to 0 \).

Consider Step 1. Suppose \( h^n_j(t) \) solves 9.106, i.e.

\[
\phi^n(x) = h^n_j(T - x) + \int_{x-a_j}^{T-a_j} k_D(a_j^+; x - a_j, s)h^n_j(T-a_j-s)ds, \quad T - c_1 < x < T.
\]

By assumption, we have

\[
\|d^m_{x} \phi^n\|_{L^2(T - c_1, T)} \to 0, \quad m = 0, ..., (j + 1).
\]

We apply this with \( m = 0 \) to 9.107, which is a Volterra equation of the second kind. Arguing as in the proof of Lemma 3.5 part B, we deduce \( \|h^n_j\|_{L^2(0, c_1)} \to 0 \). An inductive argument applied to \( x \)-derivatives of 9.107 then easily gives

\[
\|d^m_{T} h^n_j\|_{L^2(0, c_1)} \to 0, \quad m = 0, ..., (j + 1).
\]

It then follows by Lemmas 3.5 and 3.1 that \( \|f^n\|_{L^2(0, c_1)} \to 0 \).
We now use the ideas in Step 2 to finish the proof. Specifically, let

\[
f^n(t) = \begin{cases} f^n(t), & t < c_1, \\ 0, & t \geq c_1. \end{cases}
\]

Then \( \phi_2^n(x) := \phi^n(x) - u^n(x, T) \in W^{1}_1 \) is supported in \([0, T - c_1]\), and \( \|\phi_2^n\|_{W^1_1} \to 0 \). Thus we can repeat the argument above on \( \phi_2^n \). Now iterating this argument as many times as necessary, as in Step 2, we get \( \|f^n\|_{L^2(0, T)} \to 0 \). This proves 9.104. \( \square \)

**Proof of part II.**

Let \( \phi \in W^{1,d}_1 \). Let \( T = \ell + \delta \), where we can assume without loss of generality \( \delta \in (0, \Lambda/4) \). Recall \( \ell_N = \ell - a_N \). Consider the system

\[
\begin{align*}
\ddot{u} - \dddot{u} + g(x)\ddot{u} &= 0, & t \in (0, \ell_N + \delta), & x \in (a_N, \ell), \\
\dot{u}(x, t) &= 0, & t \leq 0, \\
\ddot{u}(a_N, t) &= g(t), \\
\ddot{u}(\ell, t) &= 0.
\end{align*}
\]

By Lemma 2 in [6], there exists \( g(t) \in H^{N+1}_0(0, 2\delta) \) such that \( \ddot{u} = \dddot{u} \) satisfies

\[
\frac{\partial^j \dddot{u}}{\partial x^j}(\ell, \ell_N + \delta) = \phi^{(j)}(\ell), \quad j = 0, \ldots, N.
\]

Since \( g \) is in the range of \( S^{-1}_N S^{N+1}_D \), we can set \( f_g = S^{-1}_M S^{N+1}_D g \in L^2(0, 2\delta) \), so that \( \phi(x) - u^f(x, T) \in W^{1,d}_1 \). Thus we can apply the part I to find \( f \in L^2(0, T) \), supported in \((\delta, T)\), solving \( u^f(x, T) = \phi(x) - u^f(x, T) \). Then \( f + f_g \) is the desired control.

To prove 9.104 in this case, suppose there exists a sequence \( \{\phi^n\} \) such that \( \|\phi^n\|_{W^{1,d}_1} \to 0 \), and suppose we use the procedure from the previous paragraph to find \( \{f^n\} \), with \( f^n \in L^2(0, T) \) such that \( \phi^n = u^{f^n}(x, T) \). We will prove \( \|f^n\|_{L^2(0, T)} \to 0 \).

Since \( \|\phi^n\|_{W^{1,d}_1} \to 0 \), it follows that the restriction of \( \phi^n \) to the interval \((a_N, \ell)\) tends to zero in \( H^{N+1} \) topology, and hence

\[
\sum_{j=0}^N |(\phi^{(j)})(\ell)|^2 \to 0.
\]

By Lemma 2 part i in [6], the associated sequence of functions \( \{g^n\} \) will converge in zero in \( H^{N+1}(0, 2\delta) \). Since \( g^n \) are in the range of \( S^{-1}_M S^{N+1}_D \), we set \( f_g^n := S^{-1}_M S^{N+1}_D g^n \in L^2(0, 2\delta) \). Then

\[
\phi^n(x) - u^{f_g^n}(x, T) = u^{f^n - f_g^n}(x, T) \in W^{1,d}_1.
\]

Since \( f_g^n \) converges to zero in \( L^2(0, 2\delta) \), it follows that

\[
\|\phi^n(x) - u^{f_g^n}(x, T)\|_{W^{1,d}_1} \to 0.
\]

By part I, we conclude \( \|f^n - f_g^n\|_{L^2(0, T)} \to 0 \), so \( \|f^n\|_{L^2(0, T)} \to 0 \). \( \square \)

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