A NOTE ON DEGENERATE BELL NUMBERS AND POLYNOMIALS

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Abstract. Recently, several authors have studied the degenerate Bernoulli and Euler polynomials and given some interesting identities of those polynomials. In this paper, we consider the degenerate Bell numbers and polynomials and derive some new identities of those numbers and polynomials associated with special numbers and polynomials. In addition, we investigate some properties of the degenerate Bell polynomials which are derived by using the notion of composita. From our investigation, we give some new relations between the degenerate Bell polynomials and the special polynomials.

1. Introduction

As is well known, the ordinary Bernoulli polynomials are defined by the generating function

\[ \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (\text{see } [2, 3]). \]

When \( x = 0 \), \( B_n = B_n(0) \) are called ordinary Bernoulli numbers. From (1.1), we note that

\[ B_n(x) = \sum_{l=0}^{n} \binom{n}{l} B_l x^{n-l}, \quad (n \geq 0), \quad (\text{see } [2, 3]). \]

In [3], L. Carlitz considered the degenerate Bernoulli polynomials which are given by the generating function

\[ \frac{t}{(1 + \lambda t)^{x} - 1} (1 + \lambda t)^{x} = \sum_{n=0}^{\infty} \beta_n(x \mid \lambda) \frac{t^n}{n!}, \quad (\text{see } [1, 6, 8, 10, 12, 17, 18]). \]

When \( x = 0 \), \( \beta_n(\lambda) = \beta_n(0 \mid \lambda) \) are called the degenerate Bernoulli numbers. These degenerate Bernoulli numbers and polynomials are studied by several authors (see [1, 6, 8, 10, 12, 17, 18]).

For \( n \geq 0 \), the Stirling number of the first kind is defined as

\[ (x)_n = x (x - 1) \cdots (x - n + 1) = \prod_{l=0}^{n-1} (x - l) = \sum_{l=0}^{n} S_1(n, l) x^l, \quad (\text{see } [3, 10]), \]

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and the Stirling number of the second kind is defined as

\[(1.5) \quad x^n = \sum_{l=0}^{n} S_2(n, l) (x)_l, \quad (\text{see [11]}) .\]

It is known that the generating functions of \( S_1(n, l) \) and \( S_2(n, l) \) are given by

\[(1.6) \quad (e^t - 1)^n = n! \sum_{l=n}^{\infty} S_2(l, n) \frac{t^l}{l!}, \]

and

\[(1.7) \quad (\log (1 + t))^n = n! \sum_{l=n}^{\infty} S_1(l, n) \frac{t^l}{l!}, \quad (\text{see [9, 16]}) .\]

The Bell polynomials (also called the exponential polynomial and denoted by \( \phi_n(x) \)) are defined by the generating function

\[(1.8) \quad e^x (e^t - 1) = \sum_{n=0}^{\infty} \frac{\text{Bel}_n(x)}{n!} t^n, \quad (\text{see [4, 7, 13–15]}) .\]

It is not difficult to show that the first few of them are given by

\[
\begin{align*}
\text{Bel}_0(x) &= 1, \\
\text{Bel}_1(x) &= x, \\
\text{Bel}_2(x) &= x^2 + x, \\
\text{Bel}_3(x) &= x^3 + 3x^2 + x, \\
\text{Bel}_4(x) &= x^4 + 6x^3 + 7x^2 + x, \\
\text{Bel}_5(x) &= x^5 + 10x^4 + 25x^3 + 15x^2 + x, \\
\text{Bel}_6(x) &= x^6 + 15x^5 + 65x^4 + 90x^3 + 35x^2 + x, \cdots .
\end{align*}
\]

When \( x = 1, \text{Bel}_n = \text{Bel}_n(1) \) are called the Bell numbers.

From (1.8), we can easily derive the following equation:

\[(1.9) \quad \text{Bel}_n(x+y) = \sum_{l=0}^{n} \binom{n}{l} \text{Bel}_l(x) \text{Bel}_{n-l}(y), \quad (n \geq 0),\]

and

\[(1.10) \quad \frac{1}{e^x} \sum_{k=0}^{\infty} k^n \frac{x^k}{k!} = \sum_{k=0}^{n} S_2(n, k) x^k = \text{Bel}_n(x), \quad (n \in \mathbb{N}).\]

If we set \( x = 1 \), then we obtain Dobiński formula as follows:

\[(1.11) \quad \sum_{k=0}^{n} k^n \frac{1}{k!} = e \sum_{k=1}^{n} S_2(n, k) = e \text{Bel}_n, \quad (n \geq 1),\]

which is equivalent to

\[(1.12) \quad \text{Bel}_n = \frac{1}{e} \sum_{k=0}^{n} k^n \frac{1}{k!}, \quad (\text{see [16]}).\]

Let

\[(1.13) \quad G(t, x) = e^{x(e^t - 1)} = \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{t^n}{n!}\]

By differentiating \( G(t, x) \) with respect to \( t \), we get

\[(1.14) \quad \sum_{n=0}^{\infty} \text{Bel}_n(x) \frac{nt^{n-1}}{n!} = \frac{d}{dt} G(t, x) = xe^x e^{x(e^t - 1)}.\]
\[ = \sum_{n=0}^{\infty} \left( x \sum_{j=0}^{n} \text{Bel}_j (x) \binom{n}{j} \right) \frac{t^n}{n!}. \]

From (1.14), we have

\[ (1.15) \quad \text{Bel}_{n+1} (x) = x^{\sum_{j=0}^{n} \binom{n}{j} \text{Bel}_j (x)}. \]

In particular, for \( x = 1 \), we get

\[ (1.16) \quad \text{Bel}_{n+1} = \sum_{j=0}^{n} \binom{n}{j} \text{Bel}_j. \]

Recently, several authors have studied the degenerate Bernoulli and Euler polynomials and given some interesting identities of those polynomials (see [1, 6, 8, 10, 12, 17, 18]). In this paper, we consider the degenerate Bell numbers and polynomials and derive some new identities of those numbers and polynomials associated with special numbers and polynomials. In addition, we investigate some properties of the degenerate Bell polynomials which are derived by using the notion of composita. From our investigation, we give some new relations between the degenerate Bell polynomials and the special polynomials.

### 2. Degenerate Bell Polynomials and Numbers

Now, we consider the degenerate Bell polynomials which are given by the generating function

\[ (2.1) \quad (1 + \lambda)^{\frac{1}{\lambda^t}} (1 + \lambda t)^{-1} = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda} (x) \frac{t^n}{n!}. \]

When \( x = 1 \), \( \text{Bel}_{n,\lambda} = \text{Bel}_{n,\lambda} (1) \) are called the degenerate Bell numbers.

From (2.1), we note that

\[ (2.2) \quad \lim_{\lambda \to 0} \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda} (x) \frac{t^n}{n!} = \lim_{\lambda \to 0} (1 + \lambda)^{\frac{1}{\lambda^t}} (1 + \lambda t)^{-1} = e^{(e^t - 1)} = \sum_{n=0}^{\infty} \text{Bel}_n (x) \frac{t^n}{n!}. \]

Thus, by (2.2), we get

\[ (2.3) \quad \lim_{\lambda \to 0} \text{Bel}_{n,\lambda} (x) = \text{Bel}_n (x), \quad (n \geq 0). \]

From (2.1), we can derive the following equation:

\[ (2.4) \quad \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda} (x) \frac{t^n}{n!} = (1 + \lambda)^{\frac{1}{\lambda^t}} (1 + \lambda t)^{-1} = \sum_{m=0}^{\infty} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^m \frac{1}{m!} x^m \left( (1 + \lambda t)^{\frac{1}{\lambda^t}} - 1 \right)^m. \]
\[ \sum_{m=0}^{\infty} \left( \log \left( 1 + \lambda \right) \right)^{m} \frac{x^{m}}{m!} \left( e^{\frac{x}{\lambda} \log(1+\lambda t) - 1} \right)^{m} \]

\[ = \sum_{m=0}^{\infty} \left( \log \left( 1 + \lambda \right) \right)^{m} x^{m} \sum_{k=m}^{\infty} S_{2} (k, m) \lambda^{-k} \frac{1}{k!} (\log (1 + \lambda t))^{k} \]

\[ = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \left( \log \left( 1 + \lambda \right) \right)^{m} x^{m} S_{2} (k, m) \lambda^{-k} \sum_{n=k}^{\infty} S_{1} (n, k) \frac{\lambda^{n}}{n!} t^{n} \]

\[ = \sum_{n=0}^{\infty} \sum_{\lambda k=0}^{\infty} \sum_{m=0}^{\infty} \left( \log \left( 1 + \lambda \right) \right)^{m} S_{2} (k, m) \lambda^{-k} x^{m} S_{1} (n, k) \frac{t^{n}}{n!}. \]

By comparing the coefficients on the both sides of (2.4), we obtain the following theorem.

**Theorem 1.** For \( n \geq 0 \), we have

\[ \text{Bel}_{n, \lambda} (x) = \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{\log (1 + \lambda)}{\lambda} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^{m} S_{1} (n, k) S_{2} (k, m) \lambda^{-k} x^{m}. \]

Now, we observe that

\[ (1 + \lambda)^{\frac{1}{\lambda} (1 + \lambda t)^{\frac{1}{\lambda}}} = \sum_{k=0}^{\infty} \frac{(x)^{k} (log (1 + \lambda))^{k}}{k!} (1 + \lambda t)^{\frac{k}{\lambda}} \]

\[ = \sum_{k=0}^{\infty} \frac{(x)^{k} (log (1 + \lambda))^{k}}{k!} \frac{1}{k!} e^{\frac{x}{\lambda} \log(1+\lambda t)} \]

\[ = \sum_{k=0}^{\infty} \frac{(x)^{k} (log (1 + \lambda))^{k}}{k!} \frac{1}{k!} \sum_{l=0}^{\infty} \left( \frac{k}{l} \right)^{l} \frac{(log (1 + \lambda))^{l}}{l!} \]

\[ = \sum_{k=0}^{\infty} \frac{(x)^{k} (log (1 + \lambda))^{k}}{k!} \sum_{l=0}^{\infty} \left( \frac{k}{l} \right)^{l} \sum_{n=l}^{\infty} \frac{S_{1} (n, l) \lambda^{n}}{n!} t^{n} \]

\[ = \sum_{k=0}^{\infty} \sum_{\lambda l=0}^{\infty} \left( \frac{x}{\lambda} \right)^{k} \frac{\log (1 + \lambda)}{\lambda} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} k^{l} \lambda^{n-l} S_{1} (n, l) \frac{t^{n}}{n!} \]

Thus, by (2.1) and (2.5), we get

\[ (1 + \lambda)^{\frac{1}{\lambda}} \sum_{n=0}^{\infty} \text{Bel}_{n, \lambda} (x) \frac{t^{n}}{n!} \]

\[ = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{l=0}^{n} \frac{x^{k}}{k!} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^{k} \frac{k^{l} \lambda^{n-l} S_{1} (n, l) \lambda^{n}}{n!} \right) \frac{t^{n}}{n!} \]

Therefore, by Theorem (1) and (2.6), we obtain the following theorem.

**Theorem 2.** For \( n \geq 0 \), we have

\[ (1 + \lambda)^{\frac{1}{\lambda}} \sum_{k=0}^{n} \sum_{m=0}^{k} \frac{\log (1 + \lambda)}{\lambda} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^{m} S_{1} (n, k) S_{2} (k, m) \lambda^{-k} x^{m} \]
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{n} \frac{x^k}{k!} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^k k^l \lambda^{n-l} S_1(n, l).
\]

Remark:
\[
e^x \sum_{m=0}^{n} S_2(n, m) x^m
\]
\[
= \lim_{\lambda \to 0} (1 + \lambda)^{\frac{1}{\lambda}} \sum_{k=0}^{n} \sum_{m=0}^{\infty} \frac{\log (1 + \lambda)}{\lambda}^m S_1(n, k) S_2(k, m) \lambda^{n-k} x^m
\]
\[
= \lim_{\lambda \to 0} \sum_{k=0}^{\infty} \sum_{l=0}^{n} \frac{x^k}{k!} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^k k^l \lambda^{n-l} S_1(n, l)
\]
\[
= \sum_{k=0}^{\infty} k^n \frac{x^k}{k!}.
\]

When \( x = 1 \), we have
\[
\sum_{k=0}^{\infty} \sum_{l=0}^{n} \frac{\log (1 + \lambda)}{\lambda}^k k^l \lambda^{n-l} S_1(n, l).
\]

Note that
\[
e^n \sum_{k=0}^{n} S_2(n, k)
\]
\[
= \lim_{\lambda \to 0} (1 + \lambda)^{\frac{1}{\lambda}} \sum_{k=0}^{n} \sum_{m=0}^{\infty} \frac{\log (1 + \lambda)}{\lambda}^m S_1(n, k) S_2(k, m) \lambda^{n-k}
\]
\[
= \lim_{\lambda \to 0} \sum_{k=0}^{\infty} \sum_{l=0}^{n} \frac{1}{k!} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^k k^l \lambda^{n-l} S_1(n, l)
\]
\[
= \sum_{k=0}^{\infty} k^n \frac{1}{k!}, \text{ where } n \in \mathbb{N}.
\]

Now, we define the degenerate Stirling numbers of the second kind as follows:
\[
\left( e^{\frac{1}{\lambda} \log(1 + \lambda)} - 1 \right)^n = n! \sum_{m=n}^{\infty} S_2(m, n \mid \lambda) \frac{t^m}{m!}.
\]

Thus, from (2.8), we have
\[
\left( e^{\frac{1}{\lambda} \log(1 + \lambda)} - 1 \right) = e^{\frac{1}{\lambda} \log(1 + \lambda)}\left( e^{\frac{1}{\lambda} \log(1 + \lambda)} - 1 \right)
\]
\[
= \sum_{m=0}^{\infty} \frac{1}{m!} \left( \frac{x}{\lambda} \right)^m \log (1 + \lambda) m! \sum_{k=m}^{\infty} S_2(k, m \mid \lambda) \frac{t^k}{k!}
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} \frac{\log (1 + \lambda)}{\lambda}^m x^m S_2(k, m \mid \lambda) \right) \frac{t^k}{k!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^k \right) x^n S_2(n, k | \lambda) \frac{t^n}{n!}.
\]

By (2.9), we get

\[(2.10) \quad \text{Bel}_{n,\lambda}(x) = \sum_{m=0}^{n} S_2(n, m | \lambda) \left( \frac{\log (1 + \lambda)}{\lambda} \right)^m x^m.
\]

We observe that

\[(2.11) \quad \sum_{k=0}^{n} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^k S_2(n, k | \lambda) x^k
= \sum_{k=0}^{n} \sum_{m=0}^{k} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^m S_1(n, k) S_2(k, m) \lambda^{n-k} x^m
= \sum_{m=0}^{n} \left( \sum_{k=m}^{n} S_1(n, k) S_2(k, m) \lambda^{n-k} \right) \left( \frac{\log (1 + \lambda)}{\lambda} \right)^m x^m.
\]

Thus, by (2.11), we get

\[(2.12) \quad S_2(n, m | \lambda) = \sum_{k=m}^{n} S_1(n, k) S_2(k, m) \lambda^{n-k},
\]

where \(0 \leq m \leq n\).

Therefore, by (2.11) and (2.12), we obtain the following theorem.

**Theorem 3.** For \(n \in \mathbb{N}\), we have

\[
\text{Bel}_{n,\lambda}(x) = \sum_{m=0}^{n} S_2(n, m | \lambda) \left( \frac{\log (1 + \lambda)}{\lambda} \right)^m x^m,
\]

where the degenerate Stirling numbers \(S_2(n, m | \lambda)\) of the second kind have the expression

\[
S_2(n, m | \lambda) = \sum_{k=m}^{n} S_1(n, k) S_2(k, m) \lambda^{n-k},
\]

for \(0 \leq m \leq n\).

Let us define the generating function of the degenerate Bell polynomials as follows:

\[(2.13) \quad G_\lambda(t, x) = (1 + \lambda)^{\frac{t(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{t}} = \sum_{n=0}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!}.
\]

Then, by (2.13), we get

\[(2.14) \quad G_\lambda(t, x + y) = (1 + \lambda)^{\frac{t(y + t)}{t}} \left( 1 + \lambda \right)^{\frac{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{t}} (1 + \lambda)^{\frac{(1 + \lambda t)^{\frac{1}{\lambda}} - 1}{t}} = \left( \sum_{m=0}^{\infty} \text{Bel}_{m,\lambda}(x) \frac{t^m}{m!} \right) \left( \sum_{l=0}^{\infty} \text{Bel}_{l,\lambda}(y) \frac{t^l}{l!} \right).
\]
\[
\sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{m,\lambda}(x) \text{Bel}_{n-m,\lambda}(y) \right) \frac{t^n}{n!}.
\]

Therefore, by (2.14), we obtain the following theorem.

**Theorem 4.** For \( n \geq 0 \), we have
\[
\text{Bel}_{n,\lambda}(x + y) = \sum_{m=0}^{n} \binom{n}{m} \text{Bel}_{m,\lambda}(x) \text{Bel}_{n-m,\lambda}(y).
\]

By differentiating \( G_{\lambda}(t, x) \) with respect to \( t \), we get
\[
\sum_{n=1}^{\infty} \text{Bel}_{n,\lambda}(x) \frac{nt^{n-1}}{n!} = \frac{d}{dt} G_{\lambda}(t, x)
\]
\[
= (1 + \lambda)^\frac{t}{\lambda} \left( (1+\lambda t)^\frac{1}{\lambda} - 1 \right) x \log (1 + \lambda) (1 + \lambda t)^{-1+\frac{1}{\lambda}}
\]
\[
= \left( \sum_{k=0}^{\infty} \frac{\text{Bel}_{k,\lambda}(x) t^k}{k!} \right) \frac{x \log (1 + \lambda)}{\lambda} \sum_{m=0}^{\infty} \left( \frac{1 - \lambda}{\lambda} \right)^m \frac{\lambda^n t^m}{m!}
\]
\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k} \text{Bel}_{k,\lambda}(x) (1 - \lambda | \lambda)^{n-k} \right) \frac{x \log (1 + \lambda)}{\lambda} \frac{t^n}{n!},
\]
where \((x | \lambda)_n = x(x - \lambda) \cdots (x - (n - 1) \lambda)\).

Therefore, by (2.15), we obtain the following theorem.

**Theorem 5.** For \( n \geq 0 \), we have
\[
\text{Bel}_{n+1,\lambda}(x) = x \log (1 + \lambda)^\frac{t}{\lambda} \sum_{k=0}^{n} \binom{n}{k} \text{Bel}_{k,\lambda}(x) (1 - \lambda | \lambda)^{n-k}
\]
where
\[
(x | \lambda)_n = x(x - \lambda) \cdots (x - (n - 1) \lambda).
\]

Note that
\[
\sum_{k=0}^{\infty} \text{Bel}_{k,\lambda}(x) \frac{t^k}{k!}
\]
\[
= (1 + \lambda)^\frac{t}{\lambda} \left( (1+\lambda t)^\frac{1}{\lambda} - 1 \right)
\]
\[
= (1 + \lambda)^{-\frac{t}{\lambda}} (1 + \lambda)^\frac{t}{\lambda}(1+\lambda t)^\frac{1}{\lambda}
\]
\[
= (1 + \lambda)^{-\frac{t}{\lambda}} e^{(1+\lambda t)^\frac{1}{\lambda} \log(1+\lambda)^\frac{1}{\lambda}}
\]
\[
= (1 + \lambda)^{-\frac{t}{\lambda}} \sum_{l=0}^{\infty} \frac{x^l}{l!} \left( \log (1 + \lambda)^\frac{1}{\lambda} \right)^l (1 + \lambda t)^\frac{1}{\lambda}
\]
\[
= (1 + \lambda)^{-\frac{t}{\lambda}} \sum_{l=0}^{\infty} \frac{x^l}{l!} \left( \sum_{k=0}^{\infty} \left( \frac{1}{\lambda} \right) \frac{\lambda^k t^k}{k!} \right)
\]
\[
= (1 + \lambda)^{-\frac{t}{\lambda}} \sum_{k=0}^{\infty} \left( \sum_{l=0}^{\infty} \frac{x^l}{l!} \left( \frac{\log (1 + \lambda)}{\lambda} \right)^l (l | \lambda)_k \right) \frac{t^k}{k!}.
\]

Therefore, by (2.16), we obtain the following theorem.
Theorem 6. For $k \geq 0$, we have

\[
\text{Bel}_{k,\lambda}(x) = (1 + \lambda)^{-\frac{x}{\lambda}} \sum_{l=0}^{\infty} \frac{x^l}{l!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^l (l | \lambda)_k.
\]

For $n \in \mathbb{N}$, we have

\[
\text{Bel}_{n,\lambda}(x) = (1 + \lambda)^{-\frac{x}{\lambda}} \sum_{l=0}^{\infty} \frac{x^l}{l!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^l (l | \lambda)_n.
\]

Therefore, by (2.17), we obtain the following theorem.

Theorem 7. For $n \geq 1$, we have

\[
\text{Bel}_{n,\lambda}(x) = \frac{\log(1 + \lambda)}{\lambda} x \sum_{k=1}^{n} \sum_{j=1}^{k} S_1(n, k) \lambda^{n-k} \binom{k-1}{j-1} \text{Bel}_{j-1} \left( \frac{\log(1 + \lambda)}{\lambda} x \right).
\]
We observe that

\begin{equation}
\sum_{n=0}^\infty \frac{d}{dx} \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} = \frac{d}{dx} \left( (1 + \lambda) \frac{\lambda}{1 + \lambda t} \right) \log(1 + \lambda) \nonumber
\end{equation}

\begin{equation}
\frac{d}{dx} \left\{ e^{\frac{\lambda}{1 + \lambda t} \log(1 + \lambda) - \frac{\lambda}{1 + \lambda t} (1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right\} \nonumber
\end{equation}

\begin{equation}
= \frac{1}{\lambda} \left( (1 + \lambda t)^{\frac{1}{\lambda}} - 1 \right) \log (1 + \lambda) \left( 1 + \lambda \right)^{\frac{1}{\lambda}} (1 + \lambda t)^{\frac{1}{\lambda} - 1} \nonumber
\end{equation}

\begin{equation}
= \frac{1}{\lambda} \left( \log (1 + \lambda) \right) \left( 1 + \lambda \right)^{\frac{1}{\lambda} - 1} - \frac{\log (1 + \lambda)}{\lambda} \left( 1 + \lambda \right)^{\frac{1}{\lambda} - 1} \nonumber
\end{equation}

\begin{equation}
= \left( \sum_{l=0}^\infty \lambda \frac{t^l}{l!} \right) \left( \sum_{m=0}^\infty \text{Bel}_{m,\lambda}(x) \frac{t^m}{m!} \right) \frac{\log (1 + \lambda)}{\lambda} - \frac{\log (1 + \lambda)}{\lambda} \sum_{n=0}^\infty \text{Bel}_{n,\lambda}(x) \nonumber
\end{equation}

\begin{equation}
= \sum_{n=0}^\infty \left( \sum_{m=0}^n \frac{n}{m} \text{Bel}_{m,\lambda}(x) \right) \frac{t^n}{n!} \frac{\log (1 + \lambda)}{\lambda} - \frac{\log (1 + \lambda)}{\lambda} \sum_{n=0}^\infty \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} \nonumber
\end{equation}

By comparing the coefficients on the both sides of (2.18), we obtain the following theorem.

**Theorem 8.** For \( n \geq 1 \), we have

\[ \frac{\lambda}{\log (1 + \lambda)} \frac{d}{dx} \text{Bel}_{n,\lambda}(x) = \sum_{m=0}^{n-1} \binom{n}{m} \text{Bel}_{m,\lambda}(x) (1 | \lambda)^{n-m} \cdot \]

3. Further remarks

In [11], V. Kruchinin and D. Kruchinin introduced the notion of composita in order to study the coefficients of the powers of an ordinary generating function and their properties.

Here we apply their technique to find an explicit expression of the degenerate Bell polynomial \( \text{Bel}_{n,\lambda}(x) \). For this, we first note that

\[ \sum_{n=0}^\infty \text{Bel}_{n,\lambda}(x) \frac{t^n}{n!} = R(F(t)) \],

where \( R(t) = (1 + \lambda)^{\frac{1}{\lambda} t} \), \( F(t) = (1 + \lambda t)^{\frac{1}{\lambda} - 1} \).

We recall from [11] that the composita \( G^\Delta(n, k) \) of the ordinary generating function \( G(t) = \sum_{n=1}^\infty g(n) t^n \) is defined as the \( n \)th coefficient of \( G(t)^k \). So we have

\[ G(t)^k = \sum_{n=k}^\infty G^\Delta(n, k) t^n. \]

Then it was noted in [11] that

\[ G^\Delta(n, k) = \sum_{\lambda_1 + \cdots + \lambda_k = n} g(\lambda_1) g(\lambda_2) \cdots g(\lambda_k), \]

where the sum is over all compositions of the positive integer \( n \) with \( k \) parts.
In order to apply the following Theorem 9, we need to determine the composita $F^\Delta (n, k)$ of $F(t)$ and the coefficients $r(k)$ of $R(t) = \sum_{k=0}^{\infty} r(k) t^k$.

$$\sum_{k=0}^{\infty} r(k) t^k = (1 + \lambda)^{\frac{x}{\lambda}} t^k = e^{x \log(1 + \lambda)} t^k$$

Then we obtain, for $k \geq 0$,

$$r(k) = \frac{1}{k!} \left( \frac{\log(1 + \lambda)}{\lambda} \right)^k x^k. \quad (3.1)$$

Also, for $k \geq 1$,

$$\sum_{n=0}^{\infty} F^\Delta (n, k) t^n = F(t)^k$$

$$= \left( (1 + \lambda t)^{\frac{x}{\lambda}} - 1 \right)^k$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} (1 + \lambda t)^x$$

$$= \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{n=0}^{\infty} \frac{(j | \lambda)_{n} t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \frac{(j | \lambda)_{n}}{n!} t^n. \quad (3.2)$$

Hence, we have, for $k \geq 1$,

$$F^\Delta (n, k) = \frac{1}{n!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (j | \lambda)_{n}$$

$$= \frac{1}{n!} \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} (j | \lambda)_{n}.$$
Now, the main result in this section follows from Theorem 9, (3.1) and (3.2).

**Theorem 10.** For all integers \( n \geq 0 \), the degenerate Bell polynomial \( \text{Bel}_{n,\lambda}(x) \) has the following expression

\[
\text{Bel}_{n,\lambda}(x) = \begin{cases} 
1, & \text{for } n = 0, \\
\sum_{k=1}^{n} \left( \sum_{j=1}^{k} \frac{(-1)^{k-j-1}(j)}{n!j!} \left( \frac{(1+\lambda)\log(1+\lambda)}{\lambda} \right)^k (j|\lambda)_n \right)x^k, & \text{for } n \geq 1.
\end{cases}
\]

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