A DESCRIPTION OF AUTOMORPHISM GROUP OF POWER GRAPHS
OF FINITE GROUPS

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Abstract. The power graph of a group is the graph whose vertex set is the set of nontrivial
elements of group, two elements being adjacent if one is a power of the other. We introduce
some way for find the automorphism groups of some graphs. As an application We describe
the full automorphism group of the power graph of all finite groups. Also we obtain the full
automorphism group of power graph of abelian, homocyclic and nilpotent groups.

1. Introduction.

The directed power graph of a semigroup $S$ was defined by Kelarev and Quinn [12] as
the digraph $\mathcal{P}(S)$ with vertex set $S$, in which there is an arc from $x$ to $y$ if and only if
$x \neq y$ and $y=x^m$ for some positive integer $m$. Motivated by this, Chakrabarty et al. [6]
defined the (undirected) power graph $\mathcal{P}(S)$, in which distinct $x$ and $y$ are joined if one is
a power of the other. The concept of power graphs has been studied extensively by many
authors. For a list of references and the history of this topic, the reader is referred to [2, 5-10].

Let $L$ be a graph. We denote $V(L)$ and $E(L)$ for vertices and edges of $L$, respectively. We
use $a \rightarrow b$ if $a$ is adjacent to $b$. Also for a subgraph $H$ of $L$ and $a \in V(H)$, we denote $H-a$ for
the subgraph generated by $V(H) - \{a\}$. The (open) neighborhood $N(a)$ of vertex $a \in V(L)$ is
the set of vertices are adjacent to $a$. Also the closed neighborhood of $a$, $N[a]$ is $N(a) \cup \{a\}$.
Throughout this paper, all groups and graphs are finite and the following notation is used:
$Aut(G)$ denotes the group of automorphisms of $G$; $\mathbb{Z}_m$ the cyclic group of order $m$; $\mathbb{Z}_m^n$ the
direct product of $n$ copies of $\mathbb{Z}_m$.

In this paper we describe the automorphism group of the power graph of finite group.
Also we obtain automorphism group of the power graph of abelian, and homocyclic groups.

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2. AUTOMORPHISM GROUP OF GRAPHS

In this section we provide some ways for calculating automorphism groups of graphs. Let \( L_1, L_2 \) be two graphs, a function \( f : (V(L_1) \cup E(L_1)) \to (V(L_2) \cup E(L_2)) \) to \( V(L) \) is said an isomorphism if \( f \) is bijective, \( f(V(L_1)) = f(V(L_2)), f(E(L_1)) = f(E(L_2)) \) and, \( x - y \in E(L_1) \) if and only if \( f(x) - f(y) \in E(L_2) \).

We say a subset \( H \) of \( V(L) \) is an \( M\text{EN} \)-subset if it is maximal subset which any two elements of \( H \) have equal closed neighborhood in \( L \). We denote \( H \) by \( \overline{a} \) for any \( a \in H \). We define the weighted graph \( \overline{L} \) as follows.

Let \( V(\overline{L}) = \{ \overline{x} \mid x \in V(L) \} \), \( weight(\overline{x}) = |x| \), and two vertices \( \overline{x}, \overline{y} \) are adjacent if \( x \) and \( y \) are adjacent in \( L \). Also in weighted graph any automorphism preserves the weight of each element.

**Theorem 2.1** ([4] Theorem 2.2). For a graph \( L \) with \(|V(L)| < \infty \),
\[
Aut(L) \cong Aut(\overline{L}) \times \prod_{B \in V(\overline{L})} S_{|B|}.
\]

**Theorem 2.2.** Let \( L = L_1 \cup L_2 \cup \cdots \cup L_t \) be a finite graph and \( L_1 \cong L_2 \cong \cdots \cong L_t \). If \( \varphi(L_i) \in \{ L_1, \cdots, L_t \} \) for all \( i \in \{ 1, \cdots, t \} \) and \( \varphi \in Aut(L) \), then \( Aut(L) \cong Aut(L_1) \times S_t \).

**Proof.** We have \( Aut(L) \) act on \( A = \{ V(L_1), \cdots, V(L_t) \} \) and then there exist group homomorphism \( \psi : Aut(L) \to S_A \) such that \( \ker(\psi) = \{ \varphi \mid \varphi(L_i) = L_i \text{ for all } i \} \). Consequently
\[
\ker(\psi) \cong Aut(L_1) \times \cdots \times Aut(L_t).
\]

Since \( L \) is finite, so there exists a totally partial order \(-\) on \( V(L_1) \). Assume that \( L_1 \cong f_i L_i \) for \( i \geq 2 \) and \( f_1 = \text{Id}_{L_1} \). We consider \( f_i(u) \leq f_i(v) \) if \( u \leq v \) for all \( u, v \in V(L_1) \). Thus we give a totally partial order on each \( V(L_i) \). Let \( H = \{ \varphi \in Aut(L) \mid u \leq v \text{ if and only if } \varphi(u) \leq \varphi(v) \} \).

We see that if \( \varphi \in H \) and \( \varphi(L_i) = L_i \) then \( \varphi \) is identity on \( L_i \). Let \( x \) be the minimum element of \( V(L_i) \) and \( B = \{ f_i(x) \mid 1 \leq i \leq t \} \). Thus each element of \( H \) is induced an bijection function on \( B \). Also for any bijection function \( \sigma \) on \( B \), the function \( \varphi \) by definition \( \varphi(u) = f_j((f_i)^{-1}(u)) \), whence \( u \in V(L_i) \) and \( \sigma(f_i(x)) = f_j(x) \) is an automorphism of \( L \). But \( \ker(\psi) \cap H \) is trivial, consequently \( |Aut(L)| \geq |H||\ker(\psi)| \). On the other hand \( |Aut(L)/\ker(\psi)| \leq |S_t| \), which completes the proof. \( \Box \)

3. THE AUTOMORPHISM GROUP OF POWER GRAPH OF FINITE GROUPS

In this section we describe the automorphism group of finite groups, directed product of some groups and nilpotent groups.
By using Theorem 2.1 we have the following which is same to main result of M. Feng, X. Ma and K. Wang in 2016.

**Theorem 3.1.** For a finite group \( G \),
\[
\text{Aut}(\mathcal{P}(G)) \cong \text{Aut}(\overline{\mathcal{P}(G)}) \times \prod_{x \in V(\mathcal{P}(G))} S_{|x|}.
\]

For the cyclic subgroup \( \langle a \rangle \) of the group \( G \), the subset \( \{a^i| (i, o(a)) = 1 \} \) of \( G \), is denoted by \( \text{gen}(\langle a \rangle) \).

We will use the following.

**Lemma 3.2** ([3], Proposition 2.8). Let \( G \) be a finite group and \( a \in G \). If \( |C_G(a)| \) is not prime power then \( \text{gen}(\langle a \rangle) \) is an MEN-subset.

**Lemma 3.3** ([3], Proposition 2.9). Let \( \langle a \rangle \) be a maximal cyclic subgroup of finite group \( G \). If \( C_G(a) \neq \langle a \rangle \) then \( \text{gen}(\langle b \rangle) \) is an MEN-subset for any \( b \in \langle a \rangle \).

**Lemma 3.4** ([3], Theorem 2.10). Let \( G \) be a finite group. Then \( K \subseteq G \) is an MEN-subset if and only if \( K \) satisfies in one of the following conditions:

1. \( K = \langle a \rangle - \langle a^t \rangle \) where \( o(a) = p^n \), \( 1 < t \leq n \), \( N[a^{p^{t-1}}] = \langle a \rangle \) and \( N[a^t] \neq \langle a \rangle \).
2. \( K = \text{gen}(\langle a \rangle) \) for some \( a \in G \).

Let \( x_M \) is an element of maximum order in \( \tau \). By Lemmas 3.2, 3.3 and 3.4 we have the following.

**Corollary 3.5.** Let \( G \) be a finite group and \( x \in G \). Then \( o(x_M) = 1 + \sum_{\pi \in N(\pi_M), |\pi| \leq |\pi_M|} |\pi| \).

**Theorem 3.6.** Let \( G = H \times K \) and \( (|H|, |K|) = 1 \) then
\[
\text{Aut}(\mathcal{P}(G)) = \text{Aut}(\overline{\mathcal{P}(H)}) \times \text{Aut}(\overline{\mathcal{P}(K)}).
\]

**Proof.** Let \( H, K \) are nontrivial groups, \( (a, b) \in G \) and \( \overline{\varphi} \in \text{Aut}(\overline{\mathcal{P}(G)}) \). Then \( C_G(a, b) = C_H(a) \times C_K(b) \) and, \( |C_G(a, b)| \) is not a prime power. Thus by Lemma 3.2 and Corollary 3.3 \( \varphi \) is preserving the order of each element of \( G \). Therefore \( \varphi(H) = H, \varphi(K) = K \). Assume that \( \varphi(a) = a_1, \varphi(b) = b_1 \), we have \( o(\varphi(a, b)) = o(a, b) \) and \( a_1, b_1 \in \langle \varphi(a, b) \rangle \). But exactly subgroup of order \( o(a) o(b) \) containing \( a_1 \) and \( b_1 \) is \( \langle (a_1, b_1) \rangle \). We deduce that \( \varphi(a, b) = (a_1, b_1) \). Therefore \( \text{Aut}(\overline{\mathcal{P}(G)}) \cong \text{Aut}(\overline{\mathcal{P}(H)}) \times \text{Aut}(\overline{\mathcal{P}(K)}) \) where \( \overline{\mathcal{P}(H)} \) is the set of MEN-set of \( \mathcal{P}(G) \) contained in \( H \). But each element of \( \overline{\mathcal{P}(H)} \) is same to an element of \( \overline{\mathcal{P}(H)} \), or union of some elements of \( \overline{\mathcal{P}(H)} \). By Lemma 3.3 we can assume that \( \overline{\mathcal{P}(H)} = \overline{\mathcal{P}(H)} \cup \cdots \cup \overline{\mathcal{P}(H)} \) where \( o(x_1) < \cdots < o(x_t) = o(x) \) and \( N_H[x_1] = \cdots = N_H[x_t] = N_H[x] \). Since we can consider these points as one point in \( \overline{\mathcal{P}(H)} \), hence the desired result follows.

\[ \square \]
A direct result of above theorem is for nilpotent groups as following.

**Theorem 3.7.** Assume that $G$ be a nilpotent finite group and, $G = P_1 \times \cdots \times P_t$ where $P_i$ is sylow subgroup of $G$. Then

$$\text{Aut}(\mathcal{P}(G)) = (\text{Aut}(\mathcal{P}(P_1)) \times \cdots \times \text{Aut}(\mathcal{P}(P_t))) \rtimes \prod_{B \in \mathcal{V}(\mathcal{P}(G))} S_{|B|}.$$

Now we find the automorphism group of power graph of cyclic group when $n$ is not prime power, which is same to [5].

**Corollary 3.8.** Let $G$ be a cyclic group of order $n$ where $n$ is not prime power. Then $\text{Aut}(\mathcal{P}(G)) \cong \prod_{d|n, d>1} S_{\phi(d)}$, whence $\phi$ is Euler-function.

**Proof.** Since $|C(x)|$ is not prime power for all $x \in G$, by Lemma 3.3 $|\bar{a}| = \phi(o(a))$. So

$$\text{Aut}(\mathcal{P}(G)) = (\text{Aut}(\mathcal{P}(P_1)) \times \cdots \times \text{Aut}(\mathcal{P}(P_t))) \rtimes \prod_{d|n, d>1} S_{\phi(d)}.$$

But any sylow subgroup of $G$ is cyclic, and $|\mathcal{P}_1| = 1$, as desired. \qed

### 4. Abelian Groups

In this section we certainly calculate the automorphism group of power graph of homocyclic and abelian finite groups.

Let $G$ be a finite abelian $p$-group and $x$ a nontrivial element of $G$. The height of $x$, denoted by $\text{height}(x)$, is the largest power $p^n$ of the prime $p$ such that $x \in G^{p^n}$. A non-cyclic group $G$ said a homocyclic group if $G$ be a directed product of some copies of cyclic group of order $p^m$ for some integer $m$.

We begin by a famous theorem in group theory which is played main rule in this section.

**Theorem 4.1.** Let $G$ be a finite abelian group and $a$ be an element of $G$ where $o(a) = \text{exp}(G)$. Then there exist a subgroup $H$ of $G$ such that $G = \langle a \rangle \times H$

**Lemma 4.2.** Let $G$ be a homocyclic group. Then Aut$(G)$, the automorphism group of $G$, acts transitively on the set of elements with equal orders.

**Proof.** Let $G \cong \mathbb{Z}_{p^m}$ and $a, b$ be two elements of order $p^t$. Since $G$ is homocyclic, $\text{height}(a) = \text{height}(b) = p^{m-t}$. So there exist $x, y \in G$ such that $x^{p^m-t} = a$ and $y^{p^m-t} = b$. By Theorem 4.1 there exist subgroups $H_1, H_2$ such that $G = \langle x \rangle \times H_1 = \langle y \rangle \times H_2$. From which $H_1 \cong H_2 \cong \mathbb{Z}_{p^{m-t}}$. Assume that $H_1 \subseteq \varphi H_2$. Now $\psi$ by definition $\psi(x^i h) = y^i \varphi(h)$ where $h \in H_1$ and $0 \leq i \leq p^m$, is an automorphism of $G$ and $\psi(a) = b$, as required. \qed
Theorem 4.3. For $G \cong \mathbb{Z}_{p^n}$, 
\[ \text{Aut}(\mathcal{P}(G)) \cong ((\cdots (S_{k_m} \cdots) \times S_{k_2}) \times S_{k_1}) \times (\prod_{t=1}^{m} S_{(p^t - p^{t-1})}), \]
where $r_t = (p^{tn} - p^{(t-1)n})/p^t - p^{t-1}$, $k_1 = r_1$ and $k_{i+1} = r_{i+1}/r_i$.

Proof. By Theorem 4.1, $\text{Aut}(\mathcal{P}(G)) \cong \text{Aut}(\mathcal{P}(G)) \times \prod_{B \in V(\mathcal{P}(G))} S_{|B|}$. Since $G$ is non-cyclic abelian group, by Lemma 3.2, $|\pi| = p^t - p^{t-1}$, where $o(a) = p^t$.

Set $R_t = \{ \pi | o(x) = p^t \}$ and $r_t = |R_t|$. We know that $G$ has exactly $p^{tn} - p^{(t-1)n}$ elements of order $p^t$, thus $r_t = (p^{tn} - p^{(t-1)n})/(p^t - p^{t-1})$. From which the second part of semi-directed product of theorem has been found.

Now we want to find the first part of that product.

In a $p$-group, two elements $a, b$ are in one connected components of $\mathcal{P}(G)$ if and only if \( \langle a \rangle \cap \langle b \rangle \neq \{ \} \). So $\mathcal{P}(G)$ has exactly $r_1$ connected components. On the other hand by Lemma 4.2, $\text{Aut}(G)$, and so $\text{Aut}(\mathcal{P}(G))$ acts transitively on the set of elements of order $p$. Thus $\text{Aut}(\mathcal{P}(G))$ acts transitively on $R_1$. Consequently all connected components of $\mathcal{P}(G)$ are isomorphic. By Theorem 2.2, $\text{Aut}(\mathcal{P}(G)) = \text{Aut}(K_1) \wr S_{r_1}$ where $K_1$ be a one of connected components of $\mathcal{P}(G)$. But there is only one element, say $\overline{a_1}$, in $V(K_1)$ with properties $|\pi| = p - 1$ and $N[\pi] = V(K_1)$, from which $\text{Aut}(K_1) = \text{Aut}(K_1 - \overline{a_1})$. Now two elements $\overline{a}, \overline{b}$ are in one connected components of $K_1 - \overline{a}$ if and only if $\langle a \rangle \cap \langle b \rangle \neq \langle a_1 \rangle$. Since all connected components of $\mathcal{P}(G)$ are isomorphic and $\text{Aut}(\mathcal{P}(G))$ acts transitively on $R_2$, then $K_1 - \overline{a}$ has exactly $k_2 = r_2/r_1$ isomorphic connected components. It follows that $\text{Aut}(K_1) \cong \text{Aut}(K_2) \wr S_{k_2}$ whence $K_2$ be a connected component of $K_1 - \overline{a}$.

By following this process the proof is completed.

Let $G$ be a finite $p$-group and $\exp(G) = p^a$. Set $\Omega_t(G) = \{ x | x^{p^t} = 1 \}$ and, 
\[ H_t(G) = \{ x \in G | o(x) = p, \text{height}(x) = p^{t-1} \}. \]

Lemma 4.4. Let $G$ be an abelian $p$-group and $x$ is an element of order $p$. Then there is an element $a$ and subgroup $L$ of $G$ such that $G = \langle a \rangle \times L$ and $x \in \langle a \rangle$.

Proof. Since $G$ is abelian, then $G \cong G_1 \times \cdots \times G_k$ where $G_1, \ldots, G_k$ are non-isomorphic homocyclic groups. Assume that $\exp(G_i) = p^{n_1}$ and $p^{n_1} < \cdots < p^{n_k}$. Then $x = (x_1, \ldots, x_k) \in G$ has order $p$ if and only if $\max\{ o(x_i) | i \in \{1, \ldots, k\} \} = p$. Also $\text{height}(x) \in \{ p^{n_1 - 1}, \ldots, p^{n_k - 1} \}$ and, 
\[ x \in H_{n_1}(G) \text{ if and only if } x_1 = x_2 = \cdots = x_{t-1} = 1 \text{ and } o(x_t) = p. \]
Assume that $a = (a_1, \ldots, a_k) \in G$ and $a^{p^{n_t - 1}} = x$.

Therefore $o(a_t) = p^{n_t}$. By Theorem 4.1, there is a subgroup $K$ such that $G_t = \langle a_t \rangle \times K$ and
then \( G=\langle a_t \rangle \times G_1 \times \cdots \times G_{t-1} \times K \times G_{t+1} \times \cdots \times G_k \). So there is a subgroup \( L \) with \( G=\langle a_t \rangle \times L \). But \( a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_k \in L \) and \( o(a)=o(a_t) \), thus \( G=\langle a \rangle \times L \).

\[
\square
\]

**Corollary 4.5.** Let \( G \) be an abelian \( p \)-group. Then \( \text{Aut}(G) \), the automorphism group of \( G \), acts transitively on \( H_t(G) \) when \( H_t \) is a nonempty set.

**Lemma 4.6.** Let \( G \) be an abelian \( p \)-group and \( b \in G \) be a nontrivial element of height \( p' \). Then \( \text{Aut}(\mathcal{P}(N_G(b)) - \{ \tau | x \in \langle b \rangle \}) \), acts transitively on the set of elements of order \( po(b) \) with equal heights in \( N_G(b) - \langle b \rangle \).

**Proof.** Let \( K=\langle N_G(b) \rangle \), \( a^p=b \) and \( o(a)=p'o(b) \). Since \( o(a)=\text{exp}(K) \), there is a subgroup \( L \) such that \( K=L \times \langle a \rangle \). Suppose \( (x,y) \in N_G(b) - \langle b \rangle \) and \( (x,y)^n=(1,b) \). Then \( x^n=1 \) and \( y^n=b \) and consequently \( o(x)|p \). So there are non-isomorphic homocyclic subgroups \( L_1, \ldots, L_m \) such that \( \text{exp}(L_1)< \cdots < \text{exp}(L_{m-1}) < p' \) and, \( L_m=1 \) or \( \text{exp}(L_m)=p' \), and \( L=L_1 \times \cdots \times L_{m-1} \times L_m \).

Set \( M=L_1 \times \cdots \times L_m \). Two elements \( u=(x_1, \ldots, x_m, x) \) and \( v=(y_1, \ldots, y_m, y) \) of order \( po(b) \) in \( N(b) \) have equal heights if and only if \( o(x)=o(y)=po(b) \),

\[
\max \{ o(x_1), \ldots, o(x_m), o(y_1), \ldots, o(y_m) \} | p
\]

and

\[
\min \{ i| x_i \neq 1, 1 \leq i \leq m-1 \}=\min \{ i| y_i \neq 1, 1 \leq i \leq m-1 \}.
\]

we consider two cases.

**Case 1.** \( u, v \in H_{\text{exp}(L_i)} \) for some \( i \leq m \). By the proof of Lemma 4.5, \( \text{Aut}(M) \) has element \( \varphi \) such that \( \varphi(x_1, \ldots, x_m)=(y_1, \ldots, y_m) \) and \( \varphi \) is identity on \( L_m \). Thus \( \psi \) by definition \( \psi(g, a^i)=(\varphi(g), a^i) \) when \( j=y \), is a group automorphism and, \( \varphi(N(b) - \langle b \rangle) = N(b) - \langle b \rangle \), as required.

**Case 2.** Let \( x_1=\cdots=x_{m-1}=y_1=\cdots=y_{m-1}=1 \). Then \( \text{height}(u)=\text{height}(v)=\text{height}(x)=p^{i-1} \) and there exist \( c \in L_m \times \langle a \rangle \) such that \( o(c)=o(a) \), \( u \in \langle c \rangle \). Thus \( M=L \times \langle c \rangle \) and, there is \( \psi \in \text{Aut}(G) \) such that \( \psi(a)=c \). Consequently, \( \psi(x)=u \) completes the proof.

\[
\square
\]

For \( x \in G \), set \( \widehat{x}=N_G(x) - \langle x \rangle \).

**Corollary 4.7.** By the hypothesis of last Lemma,

\[
\text{Aut}(\overline{b})=(\text{Aut}(\overline{(x_1, c)} \bowtie S_{k_1}) \times \cdots \times (\text{Aut}(\overline{(x_{m-1}, c)} \bowtie S_{k_{m-1}})) \times (\text{Aut}(\overline{c}) \bowtie S_{k_m}),
\]
where $x_i \in H_n(\Omega_{i+1}(L))$, $k_i = |H_n(\Omega_{i+1}(L))|$, $k_m = p^r$, $s_i = \exp(L_i)$, $r$ is the number of direct factor of $L_m$ and, $c$ is an element of order $po(b)$ in $\langle a \rangle$.

Proof. Since $\hat{b}$ is not connected and any component has an unique element $\overline{a}$ of order $po(b)$, by Lemmas 2.2, 4.6 the result follows. □

**Corollary 4.8.** Let $G$ be abelian $p$-group then

$$\text{Aut}(\overline{\mathcal{P}(G)}) = \prod_{H_t(G) \neq \phi} (\text{Aut}(\overline{H_t(G)})) \wr S(|H_t(G)|-1)/(p-1)$$

where $\alpha(K)$ is an element of prime order in $K$.

Combining Theorems 3.1, 3.7 and Corollaries 4.7, 4.8 automorphism group of power graph of any abelian groups can be computed.

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