HIGH CODIMENSION MEAN CURVATURE FLOW WITH SURGERY

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Abstract. We give a proof of the existence of the mean curvature flow with surgery in high codimension for suitably pinched second fundamental form. As an application we show that pinched high codimension submanifolds are diffeomorphic to $S^n$ or a finite connected sum of $S^{n-1} \times S^1$.

1. Introduction

Let us consider a smooth immersion of a submanifold into Euclidean space $F_0 : \mathcal{M} \to \mathbb{R}^{n+m}$ where $\dim \mathcal{M} \geq 2$, $k \geq 1$. The gradient descent flow of the area is an evolution of the $\mathcal{M}_0 = F(\mathcal{M})$ by mean curvature flow which is a one parameter family of smooth immersions $F : \mathcal{M} \times [0, T) \to \mathbb{R}^{n+m}$ satisfying the following partial differential equation

$$\begin{cases}
\frac{\partial}{\partial t} F(p, t) = \vec{H}(p, t), & p \in \mathcal{M}, t \geq 0, \\
F(\cdot, 0) = F_0,
\end{cases}$$

(1.1)

where $\vec{H}(p, t)$ is the mean curvature vector at the point $F(p, t)$ of the submanifold $\mathcal{M}_t = F_t(\mathcal{M})$. In this paper, we will be interested in the case where $k \geq 2$, which we refer to as high codimension. For closed submanifolds of Euclidean space it is well known that singularities will form in finite time where the curvature becomes unbounded as $t \to T$. Several weak notions of mean curvature flow have been suggested and studied such as [5], [14], [13], [24], [1]. We note that many of these weak flows work best in the hypersurface case ($k = 1$). Here we will consider a different notion of weak solution of mean curvature flow through singularities - mean curvature flow with surgery. This surgery was first introduced by Huisken-Sinestrari [23] in the context of mean curvature flow for two-convex hypersurfaces which in turn was inspired by the procedure of Hamilton [16] for the Ricci flow with surgery deforming metrics on a Riemannian manifold. This Ricci flow with surgery has now been extended to higher dimensions by Brendle [6]). A similar but different Ricci flow with surgery was instrumental in Perelman’s proof of the Geometrization conjecture ([31]). Compared with alternative weak solutions, the mean curvature flow with surgery has the advantage of controlling the topology of the underlying submanifold when passing through singular regions. Furthermore, curvature flows with surgery have been used to classify two convex hypersurfaces in a number of situations, [7], [9] and [8]. Most of the work done on mean curvature flow in higher codimension uses assumptions on the image of the Gauss map. They have either considered graphical submanifolds, [11], [27], [36], [38], submanifolds with additional symplectic or Lagrangian structure [33], [12], [35], [32], [29] or

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using that convex subsets of the Grassmannian are preserved by the mean curvature flow, [34], [37], [39].

The idea of surgery is to stop the flow at a time before the singular time \( T_1 < T \) but which is still close. We then perform the surgery procedure in these high curvature regions. In order to perform the surgery, we required detailed geometric and analytic control of high curvature regions. In particular we need to show the high curvature regions which are nearly singular are either uniformly spherically pinched or contain neck-like regions where the region is close to a standard embedding of cylinder \( \mathbb{S}^{n-1} \times [a, b] \). The surgery is performed on these necks and are removed by surgery. The surgery construction removes these singular regions and replaces them with more regular regions-in the process the maximum curvature is reduced by a certain factor and components diffeomorphic to \( \mathbb{S}^n \) or \( \mathbb{S}^{n-1} \times \mathbb{S}^1 \) are removed. After this the flow is restarted until a new singular time at which the procedure is repeated. We can show the mean curvature flow with surgery depends only on a few parameters that depend on the initial data. Furthermore the mean curvature with surgery terminates after finitely many steps when all remaining components are recognised as diffeomorphic to \( \mathbb{S}^n \) or \( \mathbb{S}^{n-1} \times \mathbb{S}^1 \).

In this paper, the main result is to show for \( n \geq 5 \) with cylindrical pinching, the mean curvature flow with surgery can be constructed with the properties describe above. In this paper, we will extend the singularity analysis of [23], [22] to higher codimension. We note that there are alternative treatments for singularity analysis - see [17], [18] and the resulting mean curvature flow with surgery was developed independently by [7] and [18]. However, these results require a non-collapsedness hypothesis which is a consequence of embeddedness which is not preserved by high codimension mean curvature flow.

However, in high codimension, since we have no direct equivalent of positive mean curvature, we need to consider an alternative condition. Note that on a compact hypersurface if \( H > 0 \), there there exists a \( c > 0 \) such that \( |A|^2 \leq c|H|^2 \) and in fact this bound is preserved by the (co-dimension one) mean curvature flow. In fact such a condition makes sense for all codimensions. This lead Andrews-Baker [3] to consider the following pinching condition on the second fundamental form in high codimension

\[
|A|^2 - c|H|^2 < 0, \quad c < \frac{4}{3n}, \quad |H| > 0
\]

which was shown to be preserved by the mean curvature flow. For \( c = \min\{\frac{4}{3n}, \frac{1}{n-1}\} \), remarkably they were able to prove convergence to a round sphere. We note that the condition \( |A| - \frac{1}{n-1}H^2 < 0, H > 0 \) implies convexity in codimension one. It is an interesting question to consider if \( c \) is optimal. For high dimensions this can be shown to be true, however for low dimensions the pinching condition does not seem to be strong enough to control the normal curvature. For example for surfaces, its seems that the curvature condition should include the normal curvature, see [4].

In this paper, we will study singularity formation in high codimension mean curvature flow and will consider the following curvature pinching

\[
|A|^2 - c_n|H|^2 < 0
\]
where \( c_n := \frac{4}{3n} \). In particular, note that such a solution is said to be quadratically bounded (we note here that this condition implies that \( \mathcal{M} \) satisfies positive isotropic curvature). For future reference, we will say that for \( n \geq 4 \) if

\[
|A|^2 - \frac{1}{n-1}|H|^2 \leq 0
\]

then the second fundamental form is \textit{(quadratically) spherically pinched}. And if

\[
|A|^2 - c_n|H|^2 \leq -\varepsilon|H|^2
\]

for some \( \varepsilon > 0 \), where

\[
c_n := \min \left\{ \frac{4}{3n}, \frac{1}{n-2} \right\}, \quad \text{if } n \geq 5.
\]

then the second fundamental form is \textit{(quadratically) cylindrically pinched}. The goal of this paper is to prove the following theorem:

**Theorem 1.1.** Let \( F_0 : \mathcal{M} \rightarrow \mathbb{R}^{n+m} \) is a smooth immersion of a closed \( n \)-dimensional submanifold with \( n \geq 5 \). Suppose that \( \mathcal{M}_0 = F(\mathcal{M}) \) is cylindrically pinched, that is

\[
|A|^2 - c_n|H|^2 < 0 \quad \text{everywhere on } \mathcal{M}_0.
\]

Then there exists a mean curvature flow with surgery starting from \( \mathcal{M}_0 \) which terminates after finitely many steps.

This is the high codimension analogue of the Huisken-Sinestrari (see also Brendle-Huisken). The proof roughly follows the outline of Husiken-Sinestrari, however with some key differences. One of the keys to Mean curvature flow with surgery is convexity estimates of [21] and [22] and the associated cylindrical estimates [23]. However, as noted before, no such analogue exists for high codimensions. Therefore, we must work differently. In [30], the author showed that it is possible to derive a pointwise gradient estimate directly from the quadratic pinching condition and establish a cylindrical estimate (and a form of the convexity estimates). This gradient estimates allows us to establish a key extension of this estimate for mean curvature flow with surgery, the Neck Detection Lemma, Theorem 5.5. The neck detection lemma is then used to establish Neck Continuation, Theorem 6.3. However, in the case of high codimension, there is an additional complication, when we leave a neck region we are no longer cylindrical - however a result of [28], shows that we are still quantitively close to a hypersurface. This allows us to extend the argument of [23] to high codimension.

The existence of a mean curvature flow with surgery that terminates after finitely many steps has the following topological consequence.

**Corollary 1.2.** Any closed \( n \)-dimensional cylindrically pinched immersed submanifold \( F_0 : \mathcal{M} \rightarrow \mathbb{R}^{n+m} \) with \( n \geq 8 \) is diffeomorphic to either \( S^n \) or a finite connected sum of \( S^{n-1} \times S^1 \).

We note here the quadratic cylindrical pinching implies other natural curvature conditions. For example for codimension one it implies two-convexity. It also implies positive isotropic curvature.

We give an outline of the paper. In section 2 we gather together the basic facts about high codimension submanifolds and mean curvature flow in high codimension. We also
introduce the parameters that will be used to control the flow. We would like to point out that because the gradient estimate shows at the singular regions of high curvature, the singularity is almost codimension one, much of the surgery algorithm in principle follows the method of [23]. However, since the necks are only approximately codimension one and the curvature condition is quadratic pinching and not two convexity, subtle modifications must be made to each step to ensure the algorithm performs mean curvature flow with surgery.

In section 3, we define and investigate cylindrical necks. In particular, we obtain a quantitative description of the neck structure. The notion of a neck was introduced by Hamilton in the paper [16] in the context of (intrinsic) necks in Riemannian manifolds. This notion was then extended to immersed hypersurfaces of Euclidean space by [23]. Here we will need to consider submanifold necks of high codimension. While for high codimension submanifolds the notion of a neck is largely the same, there is the additional subtlety that the normal bundle also needs to be controlled, that is the cylindrical neck is close to immersion of a standard cylinder and submanifold nearly lies on in a subspace of codimension one. In particular, there is a frame of the normal bundle that is nearly flat. Once we have an appropriate notion of high codimension submanifold neck, we introduce the standard surgery that replaces a section of the neck close to a cylinder by (a maximum of) two spherically pinched caps. In section 4, we show the gradient and cylindrical estimates of [30] for smooth mean curvature flow also hold for mean curvature flow with surgery. The gradient estimate allows us to compare different points of the submanifold. The cylindrical estimate shows at points which are not spherical, the submanifold has curvature that is close to the curvature of a standard cylinder. In fact, this can be improved to show in regions of high curvature, there is a part of the submanifold that is close to an embedding of a standard cylinder.

In the final two sections, we adapt the surgery method of Huisken-Sinestrari [23] to high codimension mean curvature flow. Heuristically, the flow runs until it encounters a singular time. We show that before a singular time that unless the submanifold is diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$, we detect neck regions. Such neck detection is the subject of Lemma 5.5. Finally in section 6, we show any neck can be continued until the neck opens up, that is the curvature at the ends is much smaller than in the middle. Hence this shows the surgery can be done in such a way so the curvature is reduced by a fixed multiple. Hence the curvature is uniformly bounded throughout the flow depending only on the initial data. Hence the surgery waiting time is bounded below and hence only finitely many such surgeries can occur and the algorithm terminates after finitely many steps.

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2. Notation and Preliminary Results

In this section, we gather basic facts on the mean curvature flow in high codimension. Let $F : \mathcal{M} \times [0, T) \to \mathbb{R}^{n+m}$ be a solution of the mean curvature flow, (1.1) with closed, smoothly immersed evolving submanifolds $\mathcal{M}_t = F(\mathcal{M}, t)$. It is well known that high codimension mean curvature flow does not preserve embeddedness. We will denote by
$g = \{ g_{ij} \}$ the metric, by $d\mu$ the submanifold measure and by $A = \{ h_{ij} \}$ the vector valued mean curvature and $W = \{ h^i_j \}$ the Weingarten map. The mean curvature will be denoted by $H$ and $|A|$ is the full norm of the second fundamental form.

**Lemma 2.1.** If $M_t$ evolves by mean curvature flow then we have the following associated evolution equations

$$\frac{\partial}{\partial t} g_{ij} = -2H \cdot h_{ij}, \quad \frac{\partial}{\partial t} d\mu = -|H|^2 d\mu.$$ 

We follow to the notation of [3]. A fundamental ingredient in the derivation of the evolution equations is Simons’ identity:

$$\Delta h_{ij} = \nabla_i \nabla_j H + H \cdot h_{ip} h_{pj} - h_{ij} \cdot h_{pq} h_{pq} + 2 h_{jq} \cdot h_{ip} h_{pq} - h_{iq} \cdot h_{qp} h_{pj} - h_{jq} \cdot h_{qp} h_{pi}.$$  

The timelike Codazzi equation combined with Simons’ identity produces the evolution equation for the second fundamental form:

$$\nabla_\partial h_{ij} = \Delta h_{ij} + h_{ij} \cdot h_{pq} h_{pq} + h_{iq} \cdot h_{qp} h_{pj} + h_{jq} \cdot h_{qp} h_{pi} - 2 h_{ip} \cdot h_{jq} h_{pq}.$$  

The evolution equation for the mean curvature vector is found by taking the trace with $g_{ij}$:

$$\nabla_\partial H = \Delta H + H \cdot h_{pq} h_{pq}.$$  

The evolution equations of the norm squared of the second fundamental form and the mean curvature vector are

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2 |\nabla A|^2 + 2 \sum_{\alpha, \beta} \left( \sum_{i,j} h_{i\alpha j} h_{i\beta j} \right)^2 + 2 \sum_{i,j, \alpha, \beta} \left( \sum_p h_{i\alpha p} h_{j\beta p} - h_{j\alpha p} h_{i\beta p} \right)^2.$$  

(2.5)  

$$\frac{\partial}{\partial t} |H|^2 = \Delta |H|^2 - 2 |\nabla^\perp H|^2 + 2 \sum_{i,j} \left( \sum_\alpha H_\alpha h_{ij\alpha} \right)^2.$$  

The last term in (2.4) is the squared length of the normal curvature, which we denote by $|Rm^\perp|^2$. For convenience we label the reaction terms of the above evolution equations by

$$R_1 = \sum_{\alpha, \beta} \left( \sum_{i,j} h_{i\alpha j} h_{i\beta j} \right)^2 + |Rm^\perp|^2$$  

$$R_2 = \sum_{i,j} \left( \sum_\alpha H_\alpha h_{ij\alpha} \right)^2.$$  

2.1. **Preservation of pinching.** We consider the quadratic quantity

$$Q = |A|^2 + a - c|H|^2.$$  

where \( c \) and \( a \) are positive constants. Combining the evolution equations for \(|A|^2\) and \(|H|^2\) yields
(2.7) \[ \partial_t Q = \Delta Q - 2(|\nabla A|^2 - c|\nabla H|^2) + 2R_1 - 2cR_2. \]
We have the following Kato type inequality which is a consequence of the Codazzi equation.

**Lemma 2.2.** For any hypersurface \( M_0 \subset \mathbb{R}^{n+m} \) we have
(2.8) \[ |\nabla A|^2 \geq \frac{3}{n+2}|\nabla H|^2. \]
This is proven in [3] (as in Hamilton [15] and Huisken [20]) and shows that the gradient terms in (2.7) are strictly negative if \( c < \frac{3}{n+2} \). For \( c < \frac{4}{3n} \) we also have \( R_1 - cR_2 < 0 \) (see [3]), so by the maximum principle:

**Lemma 2.3.** Let \( F : M^n \times [0, T) \to \mathbb{R}^{n+m} \) be a solution to the mean curvature flow such that \( M_0 \) satisfies
(2.9) \[ |A|^2 + a \leq c|H|^2 \]
for some \( a > 0 \) and \( c \leq \frac{4}{3n} \). Then this condition is preserved by the mean curvature flow.

In fact we will require a slight modification of this Lemma. For the above reaction terms in the evolution of (2.7), since \( Q \leq 0 \) is preserved by the flow, we have

\[
2R_1 - 2cR_2 \leq 2|A_1|^2Q - 2a|A_1|^2 - \frac{2a}{n} \frac{1}{c - 1/n} |\hat{A}_-|^2 \\
+ \frac{2}{n} \frac{1}{c - 1/n} |A_-|^2Q + \left( 6 - \frac{2}{n(c - 1/n)} \right) |\hat{A}_1|^2 |\hat{A}_-|^2 + \left( 3 - \frac{2}{n(c - 1/n)} \right) |\hat{A}_-|^4
\]

where \( A_1 \) is the second fundamental form in the mean curvature direction and \( \hat{A}_- \) represents (traceless) second fundamental form in the directions orthogonal to the mean curvature. Therefore we have the following Lemma,

**Lemma 2.4.** Let \( F : M^n \times [0, T) \to \mathbb{R}^{n+m} \) be a solution to the mean curvature flow such that \( M_0 \) satisfies
(2.10) \[
Q(x, 0) = |A|^2(x, 0) + a - c|H|^2(x, 0)
\]
for some \( a > 0 \) and \( c \leq \frac{4}{3n} \). Then \( Q(x, t) \leq 0 \) and we have the following evolution inequality

\[
\partial_t Q \leq \Delta Q - 2(|\nabla A|^2 - c|\nabla H|^2) + 2|A_1|^2Q - 2a|A_1|^2 - \frac{2a}{n} \frac{1}{c - 1/n} |\hat{A}_-|^2 \\
+ \frac{2}{n} \frac{1}{c - 1/n} |A_-|^2Q + \left( 6 - \frac{2}{n(c - 1/n)} \right) |\hat{A}_1|^2 |\hat{A}_-|^2 + \left( 3 - \frac{2}{n(c - 1/n)} \right) |\hat{A}_-|^4 \\
\leq 0.
\]

In particular, the reaction terms satisfy \( R_1 - cR_2 \leq 0 \) whenever \( Q \leq 0 \). As a consequence, we see the flow preserves both \( |H| > 0 \) and (2.9).

The following existence theorem holds for the mean curvature flow of \( M_0 \) under the conditions of Theorem 1.1.
**Theorem 2.5.** The mean curvature flow of $\mathcal{M}_0$ exists on a finite maximal time interval $0 \leq t < T < \infty$. Moreover, $\limsup_{t \to T} |A|^2 = \infty$.

The proof that the maximal time of existence is finite follows easily from the evolution equation for the position vector $F$: $\frac{\partial}{\partial t} |F|^2 = \Delta |F|^2 - 2n$. The maximum principle implies $|F(p, t)|^2 \leq R^2 - 2nt$ and thus $T \leq \frac{R^2}{2n}$, where $R = \max \{|F_0(p)| : p \in \Sigma\}$. The proof of the second part of the theorem can be found in [3].

2.2. **Surgery Class.** Here we will introduce a class of submanifolds that is invariant under smooth mean curvature flow and surgery.

**Definition 2.6.** Consider a set of parameters denoted by $R, \alpha_1, \alpha_2, \alpha_3$. Let $C_{n,k}(R, \alpha)$, where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be the class of smooth and closed immersed submanifolds $F : \mathcal{M}^n \to \mathbb{R}^{n+m}$ that satisfy

i) $|A|^2 - \frac{1}{n-2} |H|^2 \leq -\alpha_0 |H|^2$

ii) $|H| \geq \frac{\alpha_1}{R}$

iii) $|\mathcal{M}| \leq \alpha_2 R^n$.

Here $R$ is the scaling parameter chosen so that $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ are scaling invariant in the following way: if $F \in C_{n,k}(R, \alpha)$ then $r_0 F \in C_{n,k}(r_0 R, \alpha)$. Note that $R$ will be chosen so that $|A|^2 \leq \frac{1}{R^2}$ on the initial submanifold $\mathcal{M}_0$.

This class $C_{n,k}(R, \alpha)$ is well adapted to high codimension mean curvature flow.

**Proposition 2.7.** (i) For each $R, \alpha$, the class $C_{n,k}(R, \alpha)$ is invariant under the mean curvature flow.

(ii) For every strictly quadratically pinched, smooth closed submanifold we can choose $R, \alpha$ such that $\mathcal{M} \in C_{n,k}(R, \alpha)$ and $|A|^2 \leq \frac{1}{R^2}$ holds everywhere on $\mathcal{M}$.

**Proof.** For part (i) of the theorem, the preservation of i) from Definition 2.6 follows from the evolution equations in Lemma 2.3. The remaining inequalities ii) and iii) follow from the evolution equations for $|H|$ and $\sqrt{\det g_{ij}}$.

For part (ii) of the theorem, we choose $R$ so that $\sup_{\mathcal{M}} |A|^2 \leq \frac{1}{R^2}$. The existence of $\alpha_0, \alpha_1, \alpha_2$ then follows from compactness. □

2.3. **Mean Curvature Flow with Surgeries.** In this section, we define the mean curvature flow with surgery in high codimension. This surgery procedure will be closely modelled on the procedure in [23] and [16]. The purpose of the mean curvature flow with surgery is to extend the flow past singularities but with control on the underlying topology, geometry and analysis of the submanifold. This distinguishes it from other methods to flow past singularities in high codimension such as [3], [24], [1] or [2].

Surgery can be thought of as reverse connected sum procedure. Here, using the mean curvature flow, we identify neck regions - that is parts of the submanifold that are geometrically close to a standard cylinder $\mathbb{S}^{n-1} \times (a, b) \subset \mathbb{R}^{n+m}$ in a precise quantitative manner. The surgery procedure then replaces part of the neck with two spherical caps, the surgery
is designed to remove regions of high curvature. We note that the surgery may in fact not change the topology of the underlying components. The surgery will be defined with a number of parameters, we will show these can be chose so that the class $\mathcal{C}_{n,k}(R, \alpha)$ is preserved by mean curvature with surgery.

**Definition 2.8** (Mean Curvature Flow with Surgery). The *Mean curvature flow with surgery* is determined by an algorithm that assigns to each initial smooth closed quadratically bounded immersed submanifold $F_0 : M_1 \to \mathbb{R}^{n+m}$ in some surgery class $\mathcal{C}_{n,k}(R, \alpha)$ a sequence of intervals $[0, T_1], [T_1, T_2], [T_2, T_3], \cdots, [T_{N-1}, T_N]$, a sequence of manifolds $M_i, i \leq i \leq N$ and a sequence of smooth mean curvature flows $F^i_t : M_i \to \mathbb{R}^{n+m}, t \in [T_{i-1}, T_i]$ such that the following are true:

1. The initial data submanifold for $F^1_t$ is given by $F_0 : M_1 \to \mathbb{R}^{n+m}$.
2. The initial data submanifold for $F^i_t : M_i \to \mathbb{R}^{n+m}$ on $[T_{i-1}, T_i]$ for $2 \leq i \leq N$ are obtained from $F^{i-1}_{T_{i-1}}$ by the following procedure
   a. Using the standard surgery, a submanifold $\hat{F}^{i-1}_{T_{i-1}} : M_i \to \mathbb{R}^{n+m}$ is obtained from $F^{i-1}_{T_{i-1}} : M_{i-1} \to \mathbb{R}^{n+m}$ by standard surgery by replacing finitely many disjoint necks by two spherical caps.
   b. Finitely many disconnected components that are recognised to be diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

After these two steps are completed, the resulting submanifold becomes the initial data submanifold for $F^i_{T_{i-1}}$ for smooth mean curvature flow on $[T_{i-1}, T_i]$. We say that the mean curvature flow with surgery terminates after finitely many steps at time $T_N$ if either

1. All connected components of $F^N_{T_i}$ are recognised as being diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$.
2. or in step (b) above all components of the proposed initial data $\hat{F}^N_{T_i}$ are recognised as being diffeomorphic to $\mathbb{S}^n$ or $\mathbb{S}^{n-1} \times \mathbb{S}^1$.

A priori, there is no reason for the surgery algorithm to terminate after finitely many steps. In the case of quadratically bounded curvature, we ensure this by carefully choosing the surgery time and scale of the surgery. The scale of surgery means that each surgery removes a fixed portion of the submanifold. In particular, we will show that there are three constants $\omega_1, \omega_2, \omega_3 > 1$ such that the class $\mathcal{C}_{n,k}(R, \alpha)$ is preserved by mean curvature flow with surgery, where $\omega_1, \omega_2, \omega_3$ depend on $\alpha$ so that if we take

$$H_1 \geq \frac{\omega_1}{R}, \quad H_2 = \omega_2 H_1 \quad \& \quad H_3 = \omega_3 H_2$$

we construct the mean curvature flow with surgery so that

- The parameter $H_3$ is the surgery threshold. The surgery times $T_i, 1 \leq i \leq N$ are determined as the first time in $[T_{i-1}, T_i]$ where the norm of the mean curvature $|H|$ has maximum value $H_3$, that is if $t < T_i$ then

$$|H(p, t)| < \max_{q \in M, t \in [T_{i-1}, T_i]} |H(q, \tau)| = H_3.$$
The parameter $H_2$ is the surgery reduction parameter. The maximum mean curvature $H_{\text{max}}^i(T_{i-1})$ of $F_{i-1}^i$ is reduced to $H_2$ by surgery step (ii) for each $1 \leq i \leq N$. Hence, the norm of the mean curvature on $M_t$ will be uniformly bounded by $H_3$ throughout the mean curvature flow with surgery.

The parameter $H_1$ represents the approximate mean curvature of the surgered regions. All surgeries in step 1 of (ii) are performed in regions of the submanifold where the norm of the mean curvature is approximately $H_1$, that is $\frac{H_1^2}{2} \leq |H| \leq 2H_1$.

We note here that the surgery algorithm is completely determined by the choices of surgery parameters in the standard surgery which only depend on $n$ as well as the parameters $H_1, H_2$ and $H_3$ which in turn only depend on the constants $\omega_1, \omega_2, \omega_3$ depending on $\alpha$.

**Theorem 2.9.** For any given initial data submanifold $M_0$ in some $C_{n,k}(R, \alpha)$ there exists a mean curvature flow with surgery that starts from $M_0$ and terminates after a finite number of steps. The submanifolds of the mean curvature flow with surgery satisfy uniform curvature bounds depending on $R, \alpha$ and all time intervals have length bounded from below depending only on $R$ and $\alpha$.

### 3. Necks and Surgery

In this section, we develop the notion of an $(\varepsilon, k, L)$ cylindrical neck region for submanifolds of high codimension. This is based on ideas of Huisken-Sinestraï [23] for hypersurfaces and Hamilton [16] for Ricci Flow (see also Perelman [31]). An $(\varepsilon, k, L)$ neck region in a submanifold of arbitrary codimension $M^n \to \mathbb{R}^{n+m}$ is a region that after appropriate rescaling is close in the $C^{k+2}$ norm to a standard cylinder of length $2L$.

Our construction of surgery is related to Hamilton’s surgery method in [16]. In that paper, Hamilton distinguishes *curvature necks* and *geometric necks*. Curvature necks are regions which have the intrinsic Riemannian curvature close to the curvature of a cylinder whereas geometric necks are regions that have an actual parameterisation that is close to the standard cylinder. Hamilton shows the two notions are essentially equivalent. Furthermore, since we are working in high codimension, we will also need a notion of *almost hypersurface*.

**Definition 3.1** (Extrinsic Curvature Spheres and Almost Hypersurfaces). Let $M^n \subset \mathbb{R}^{n+m}$ be a smooth submanifold of Euclidean space.

(i) We say that the extrinsic curvature is $\varepsilon$-*almost hypersurface at* $p$ if there is a normal vector $\nu$ such that the Weingarten map $W(p) : T_pM^n \oplus N_pM^n \to T_pM^n$ satisfies

$$|W_-(p)| \leq \varepsilon,$$

where $W_\nu$ is the projection of $W$ onto the normal direction $\nu$ and $W_-$ the projection onto the subspace of the normal bundle orthogonal to $\nu$. Furthermore we say that the extrinsic curvature is $\varepsilon$-*spherical at* $p$ if there is a normal vector $\nu$ such that the Weingarten map $W(p) : T_pM^n \oplus N_pM^n \to T_pM^n$

$$|W_\nu(p) - Id| \leq \varepsilon,$$

$$|W_-(p)| \leq \varepsilon.$$
Finally we say that the extrinsic curvature is $\varepsilon$-cylindrical at $p$ if there is a normal vector $\nu$ such that the Weingarten map $W(p) : T_p\mathcal{M}^n \oplus N_p\mathcal{M}^n \to T_p\mathcal{M}^n$

$$|W_\nu(p) - \overline{W}| \leq \varepsilon,$$

$$|W_- (p)| \leq \varepsilon.$$

where $\overline{W}$ is the Weingarten on the tangent space of $S^{n-1} \times \mathbb{R} \to P \subset \mathbb{R}^{n+m}$ where $P$ is a linear subspace of dimension $n + 1$ orthogonal to $\nu$.

(ii) We say that the extrinsic curvature is $(\varepsilon, k)$-parallel at $p$ if

$$|\nabla^l W(p)| \leq \varepsilon \quad \text{for } 1 \leq l \leq k.$$

(iii) We say that extrinsic curvature is $(\varepsilon, k, L)$-almost hypersurface on $\mathcal{M}^n$ if is $\varepsilon$-almost hypersurface and $\varepsilon$-parallel for all $p \in \mathcal{M}^n$. We say that extrinsic curvature is $(\varepsilon, k)$-spherical on $\mathcal{M}^n$ if is $\varepsilon$-spherical and $\varepsilon$-parallel for all $p \in \mathcal{M}^n$. Finally we say that extrinsic curvature is $(\varepsilon, k)$-homothetically almost hypersurface, $(\varepsilon, k)$-homothetically spherical or $(\varepsilon, k, L)$-homothetically cylindrical around $p$ if there exists a scaling constant $\sigma$ such that for $\sigma \mathcal{M}, (iii)$ above holds.

(iv) We say that extrinsic curvature is $(\varepsilon, k)$-homothetically almost hypersurface, $(\varepsilon, k)$-homothetically spherical or $(\varepsilon, k, L)$-homothetically cylindrical around $p$ if there exists a scaling constant $\sigma$ such that for $\sigma \mathcal{M}, (iii)$ above holds.

**Remark 3.2.** Clearly $(\varepsilon, k)$-spherical on $\mathcal{M}^n$ implies $(\varepsilon, k)$-almost hypersurface on $\mathcal{M}^n$.

**Proposition 3.3.** For $\varepsilon > 0 \exists \varepsilon' > 0, \varepsilon' = \varepsilon'(\varepsilon, n)$ such that for all $p \in \mathcal{M}^n$ such that if the extrinsic curvature is $(\varepsilon, k)$-spherical then $\mathcal{M}^n$ is $(\varepsilon', k)$-intrinsic spherical.

**Proof.** This statement is immediate from Gauss’ formula for the curvature

$$R_{ijkl} = \sum_{\alpha=1}^{k} (h_{i\alpha k} h_{j\alpha l} - h_{i\alpha l} h_{j\alpha k})$$

$$= (h_{i\nu k} h_{j\nu l} - h_{i\nu l} h_{j\nu k}) + \sum_{\alpha=1, \alpha \neq \nu}^{k} (h_{i\alpha k} h_{j\alpha l} - h_{i\alpha l} h_{j\alpha k})$$

$\square$

**Proposition 3.4.** For $k \geq 1, \exists \varepsilon(n) > 0 \mid \forall \varepsilon, 0 < \varepsilon \leq \varepsilon(n)$ sufficiently small, and let $F : \Omega \to \mathcal{M} \subset \mathbb{R}^{n+m}$ be an immersion of a submanifold.

i) Suppose that $\mathcal{M}$ is $(\varepsilon, k)$ almost hypersurface inside a ball $B_d(p)$, diameter $d \geq 4$ is a uniformly quadratically spherically pinched submanifold then $\mathcal{M}$ is close in $C^\infty$ to an immersion of a hypersurface, that is there exists an immersion such that

$$f : \Omega \to \mathcal{N} \subset \mathbb{R}^{n+m}, \quad f(\Omega) \subset \mathcal{N} \subset \mathcal{P} \mid \| F - f \|_{C^\infty(\Omega)} \leq \varepsilon(\varepsilon).$$

where $\varepsilon(\varepsilon) \to 0$ as $\varepsilon \to 0$ and $P \simeq \mathbb{R}^{n+1}$ is a linear subspace of dimension $n + 1$.

ii) Suppose that $\mathcal{M}$ is $(\varepsilon, k)$-spherical inside a ball $B_d(p)$, diameter $d \geq 4$ is a uniformly quadratically spherically pinched submanifold then $\mathcal{M}$ is close in $C^\infty$ to (part) the
immersion of a standard $R$-sphere, that is there exists an immersion with some $R > 0$ and $p \in \mathbb{R}^{n+m}$, such that

$$f : \Omega \to \mathbb{R}^{n+m}, \quad f(\Omega) \subset S_R^n(p) \mid \|F - f\|_{C^\infty(\Omega)} \leq \tilde{\varepsilon}(\varepsilon).$$

where $\tilde{\varepsilon}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Proof. We will prove $(ii)$, $(i)$ follows by an exactly analogous argument. We argue by contradiction. Therefore, let us assume there exists a sequence of $\varepsilon_i \to 0$ and immersions $F_i : \Omega \to \mathbb{R}^{n+m}$ such that $F_i$ are $(\varepsilon_i, k)$ spherical but are not close to any immersion of a standard $R$-sphere. In particular, there exists an $\tilde{\varepsilon} > 0$ such that $\|F_i - f\|_{C^k(\Omega)} \geq \tilde{\varepsilon}$ for any immersion, $f$, into a standard $n$-dimensional sphere. However, we have

$$|W_{\nu}^i(p) - Id| \leq \varepsilon_i, \quad |W_i(p)| \leq \varepsilon_i,$$

and

$$|\nabla^l W_i(p)| \leq \varepsilon_i \quad \text{for } 1 \leq l \leq k.$$

In particular we can apply the Langer-Breuning compactness theorem and conclude that there exists an immersion $F \in C^k(\Omega)$ such that $F_i \to F$. Furthermore $|W_{\nu}(p) - Id| = 0, |W_-(p)| = 0$ and $|\nabla^l W(p)| = 0, 0 \leq l \leq k$. But by Schur’s theorem, this shows $F$ is a subset of a standard sphere which is a contradiction. \qed

Remark 3.5. By Theorem 1 of [3], $(ii)$ follows immediately by the exponential convergence of high codimension mean curvature flow to a round point.

3.1. Normal Parameterisations. In [23], developing [16], the notion of a normal parameterisation for maximally extended extrinsic neck for a hypersurface is given. Here we extend this notion to submanifolds of arbitrary codimension. We recall that a topological neck is a diffeomorphism $N : S^{n-1} \times [a, b] \to \mathcal{M}$. For such a neck, we define the average radius $r : [a, b] \to \mathbb{R}$ as follows: Let $\Sigma_z = N(S^{n-1} \times \{z\})$ denote a neck cross section. Then we let $|\Sigma_z|_g$ denote the area of the cross section with respect to the pull back metric, then $r$ is defined as

$$|\Sigma_z|_g = \sigma_{n-1} r(z)^{n-1}$$

where $\sigma_{n-1}$ denotes the area of the standard $(n-1)$ sphere of radius 1. We denote by $\bar{g}$ the standard metric on $S^{n-1} \times [a, b]$.

Here we will determine a neck from its metric structure. The following definition encapsulates this idea.

Definition 3.6 (Geometric Neck). The local diffeomorphism $N : S^{n-1} \times [a, b] \to (\mathcal{M}, g)$ is called an intrinsic $(\varepsilon, k)$- geometric neck if it satisfies the following conditions
(i) The conformal metric $\hat{g} = r(z)^{-2}g$ satisfies the estimates

$$|\hat{g} - \bar{g}|_\bar{g} \leq \varepsilon, \quad |D\hat{g}|_\bar{g} \leq \varepsilon \quad \text{for } i \leq j \leq k,$$

uniformly on $S^{n-1} \times [a, b]$.

(ii) the mean radius function $r : [a, b] \to \mathbb{R}$ satisfies the estimates

$$\left| \frac{d}{dz} \right|^j \left( \frac{d}{dz} \right)^{\log r(z)} \leq \varepsilon$$

for all $1 \leq j \leq k$ everywhere on $[a, b]$.

Remark 3.7. Note that this shows for example that $|g|_\bar{g} \leq Cr^2(z)$.

Definition 3.8 (Normal Neck). A local diffeomorphism $N : S^{n-1} \times [a, b] \to (\mathcal{M}, g)$ is called normal if it satisfies the following conditions

(i) Each cross section $\Sigma_z = N(S^{n-1} \times \{z\}) \subset (\mathcal{M}, g)$ has constant mean curvature.

(ii) The restriction of $N$ to each $S^{n-1} \times \{z\}$ equipped with the standard metric is a harmonic map to $\Sigma_z$ equipped with the metric induced by $g$.

(iii) The volume of any subcylinder with respect to the pull back of $g$ is given by

$$\text{vol}(S^{n-1} \times [v, w], g) = \sigma_{n-1} \int_v^w r(z)^n dz,$$

and

(iv) for any Killing vector field $\bar{V}$ on $S^{n-1} \times \{z\}$ we have

$$\int_{S^{n-1} \times \{z\}} \bar{g}(\bar{V}, U) d\mu = 0$$

where $U$ is the unit normal vector field to $\Sigma_z$ in $(\mathcal{M}, g)$ and $d\mu$ is the measure of the metric $\bar{g}$ on the standard cylinder.

Essentially the idea is that we can detect a geometric neck using a curvature flow (Ricci flow or mean curvature flow) since these are geometric quantities (possibly through the curvature). But to perform the surgery we need a standard parameterisation of the geometric necks. In fact, this is what Hamilton does in [16]. Away from the boundary, for a suitable range of parameters $(\varepsilon, k)$ each geometric $(\varepsilon, k)$-cylindrical neck $N : S^{n-1} \times [a, b] \to (\mathcal{M}, g)$ can be changed by diffeomorphism, to a normal neck $\tilde{N} : S^{n-1} \times [\tilde{a}, \tilde{b}] \to (\mathcal{M}, g)$ which is unique up to isometries of the standard cylinder. This the content of [16, Theorem 2.2].

We now adapt the definition of [16] and [23] to the case of submanifolds with arbitrary codimension. The key difference is that we are required to specify a non-vanishing normal direction.

Definition 3.9 (Submanifold Neck). Let $N : S^{n-1} \times [a, b] \to (\mathcal{M}, g) \subset \mathbb{R}^{n+m}$ be an intrinsic $(\varepsilon, k)$-cylindrical neck in a smooth submanifold $F : \mathcal{M} \hookrightarrow \mathbb{R}^{n+m}$ with induced metric and Weingarten map. We say that $N$ is an $(\varepsilon, k)$ cylindrical submanifold neck if in
addition to the assumptions in Definition 3.6 we have a unit normal direction $\nu \in N\mathcal{M}$ such that
\[ |W_{\nu}(q) - r(z)^{-1}W| \leq \varepsilon r(z)^{-1}, \]
\[ |W_-(q)| \leq \varepsilon r(z)^{-1} \quad \& \quad |\nabla^l W(q)| \leq \varepsilon r(z)^{-1-l}, \quad 1 \leq l \leq k, \]
for all $q \in S^{n-1} \times \{z\}$ and all $z \in [a, b]$.

The key notion here is the combine overlapping necks to form maximal necks.

**Definition 3.10** (Maximal Normal Neck). We call an $(\varepsilon, k)$-cylindrical submanifold neck $N$ a maximal normal $(\varepsilon, k)$-cylindrical submanifold neck if $N$ is normal and if whenever $N^*$ is another neck with $N = N^* \circ G$ for some diffeomorphism then the map $G$ is surjective.

The following theorem is a uniqueness statement about normal and geometric necks.

**Theorem 3.11** ([16] Lemma C2.1 Uniqueness). There exists $(\varepsilon, k)$ so that if $N_1$ and $N_2$ are necks in the same manifold $\mathcal{M}^n$ and are both normal and geometrically $(\varepsilon, k)$ cylindrical, and if there exists a diffeomorphism $F$ of the cylinders such that $N_2 = N_1 F$ then $F$ is an isometry in the standard metrics on the cylinders.

The next theorem tells us that whenever we have a geometric neck, then there exists a (smaller) normal neck. This is crucial for defining mean curvature flow with surgery as the natural curvature estimates from the flow give us criteria to detect geometric necks. However the surgery procedure depends to having a parameterised or normal neck.

**Theorem 3.12** (Existence). For any $\delta > 0$ there exists an $\varepsilon > 0, k$ such that if $N : S^{n-1} \times [a, b] \hookrightarrow \mathcal{M}$ is an $(\varepsilon, k)$-cylindrical submanifold neck with $b - a \geq 3\delta$ then there exists a normal neck $N^*$ and a diffeomorphism $G$ of the domain cylinder of $N^*$ onto a region in the domain cylinder of $N$ containing all points at least $\delta$ from the ends whose image in $\mathcal{M}$ is also in the image of $N_2$ then there exists a normal neck $N$ which is also geometrically $(\varepsilon, k)$ cylindrical and there exist diffeomorphisms $F_1$ and $F_2$ such that $N_1 = N F_1$ and $N_2 = N F_2$ provided $n \geq 3$.

Now we show an extrinsic curvature neck is essentially equivalent to a submanifold neck.

**Theorem 3.14.** For every $(\varepsilon, k, L)$ and $L \geq 10$ there exists $(\varepsilon', k)$ such that if the extrinsic curvature is $(\varepsilon', k', L)$-cylindrical about $p \in \mathcal{M}$ then $p$ lies at the centre of a normal $(\varepsilon, k)$-cylindrical submanifold neck $N : S^{n-1} \times [-L-1, L-1] \hookrightarrow \mathcal{M}$ which is contained in a maximal normal $(\varepsilon, k)$-submanifold neck unless the target submanifold is diffeomorphic to $S^{n-1} \times S^1$. 
3.2. **Standard Surgery.** We now consider $F : \mathcal{M}^n \hookrightarrow \mathbb{R}^{n+m}$ a smooth closed submanifold and let

$$N : \mathbb{S}^{n-1} \times [a, b] \hookrightarrow \mathcal{M}$$

be a maximal normal $(\varepsilon, k)$-submanifold neck, where $(\varepsilon, k)$ lies in the range where Theorems 3.11, Theorem 3.12 and Theorem 3.13 hold. We choose $z_0 \in [a, b]$ with sufficient distance to the ends of the neck $z \in [z_0 - 4\Lambda, z_0 + 4\Lambda] \subset [a, b]$ for some $\Lambda$ to be determined later.

The following definition is taken from [23] with appropriate modifications for the high codimension case. Note that in our case the submanifolds will essentially be codimension one and, intuitively, the submanifold lies mostly in a $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+m}$ subspace of Euclidean space.

**Definition 3.15** (Standard Surgery with Parameters $\tau, B$). Given a maximal normal $(\varepsilon, k)$-submanifold neck and $z_0$ as well as parameters $0 < \tau < 1, B > 10\Lambda$ we now define the standard surgery with parameters $\tau, B$ at the cross section $\Sigma_{z_0} = N(\mathbb{S}^{n-1} \times \{z_0\})$ replacing the cylindrical image of $\mathbb{S}^{n-1} \times [z_0 - 4\Lambda, z_0 + 4\Lambda]$ by smoothly attaching two properly adapted spherical caps. Let us denote by $\tilde{C}_{z_0} : \mathbb{S}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n+m}$ the straight cylinder that best approximates $\mathcal{M}$ at the cross section $\Sigma_{z_0}$

- The radius of $\tilde{C}$ is chosen as the mean radius $r(z_0) = r_0$, a point on its axis if given by the centre of mass of $\Sigma_{z_0}$ with its induced metric.
- its axis is parallel to the average of the unit normal field to $\Sigma_{z_0} \subset (\mathcal{M}, g)$ taken with respect to the induced metric.

The standard surgery with parameters $\tau, B$ is performed as follows

a) The surgery leaves two collars $\mathbb{S}^{n-1} \times [a, z_0 - 3\Lambda]$ and $\mathbb{S}^{n-1} \times [z_0 + 3\Lambda, b]$ unchanged.

b) It replaces the two cylinders $N(\mathbb{S}^{n-1} \times [z_0 - 3\Lambda, z_0])$ and $N(\mathbb{S}^{n-1} \times [z_0, z_0 + 3\Lambda])$ by two $n$-balls attached smoothly to $\Sigma_{z_0 - 3\Lambda}$ and $\Sigma_{z_0 + 3\Lambda}$ respectively. The attachments at $[z_0 - 4\Lambda, z_0]$ and $[z_0, z_0 + 4\Lambda]$ are analogous and we will only describe the procedure for $[z_0 - 4\Lambda, z_0]$. For convenience, let $z_0 = 4\Lambda$ and consider a normal parameterisation $N : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathcal{M}$ in the following.

c) Let $u(z) \equiv r_0 \exp \left( -\frac{B}{z - \Lambda} \right)$ on $[\Lambda, 3\Lambda]$ for $B > 10\Lambda$ in Gaussian normal coordinates to bend the submanifold inwards into a submanifold which is strictly spherically pinched on $\mathbb{S}^{n-1} \times [2\Lambda, 3\Lambda]$ for parameter $0 < \tau < 1$:

\[
\tilde{N}_\tau(\omega, z) := N(\omega, z) - \tau u(z) \nu^+(\omega, z).
\]

(3.1)

d) To blend the resulting submanifold into an axially symmetric one, we choose a fixed smooth transition function $\varphi : [0, 4\Lambda] \to \mathbb{R}^+$ with $\varphi = 1$ on $[0, 2\Lambda], \varphi = 0$ on $[3\Lambda, 4\Lambda]$ with $\varphi' \leq 0$. We denote by $\tilde{C}_{z_0} : \mathbb{S}^{n-1} \times [0, 4\Lambda] \to \mathbb{R}^{n+m}$ the bending of the approximating cylinder defined above, $\tilde{C}_{z_0} = \tilde{C}_{z_0}(\omega, z) - \tau u(z) \nu^+_C(\omega, z)$ we can then interpolate to obtain

$$\tilde{N}(\omega, z) := \varphi(z) \tilde{N}(\omega, z) + (1 - \varphi(z)) \tilde{C}_{z_0}(\omega, z).$$

We note that the function $\varphi$ only depends on $\Lambda$ and that it can be defined in such a way that all its derivatives are smaller if $\Lambda$ is larger. In particular, if we assume $\Lambda \geq 10$, each derivative of $\varphi$ is bounded by some fixed constant.
e) Finally, we suitably change \( u \) on \([3\Lambda, 4\Lambda]\) to a function \( \hat{u} \) to ensure \( \tau \hat{u}(z) \to r(z_0) = r_0 z \) as \( z \) approaches some \( z_1 \in (3\Lambda, 4\Lambda] \) such that \( \tilde{C}_{z_0}([3\Lambda, 4\Lambda]) \) is a smoothly attached axially symmetric and uniformly spherically pinched cap. Since this last deformation on \([3\Lambda, 4\Lambda]\) only concerns the axisymmetric case, it can be made for each pair \( \tau, B \) of parameters in such a way that on the attached strictly spherically pinched cap there is some fixed upper bound for the curvature and each of its derivatives, independent of \( \Lambda \geq 10 \) and the surgery parameters \( \tau, B \).

Next we will need to obtain quantitative estimates on the effect of the bending on the curvature of the submanifold.

3.3. High Codimension Surgery. We consider a neck,
\[
N : S^{n-1} \times [a, b] \to \mathcal{N} \subset \mathbb{R}^{n+m}
\]
with the following Weingarten relations
\[
\frac{\partial^2 N}{\partial x^i \partial x^j} - \Gamma_{ij}^{k} \frac{\partial N}{\partial x^k} = -h_{ij} = -h_{i\alpha}^{\alpha} \nu_{\alpha},
\]
\[
\frac{\partial \nu_{\alpha}}{\partial x^i} = h_{i\alpha}^{\alpha} \nu_{\alpha} + \sum T_{\alpha}^{\beta} \nu_{\beta}
\]
where \( T_{\alpha}^{\beta} \) are the Christoffel symbols of the normal bundle which satisfy \( T_{\alpha}^{\beta} = -T_{\beta}^{\alpha} \) so that \( T_{\alpha}^{\alpha} = 0 \). We assume the neck is in a normal parameterisation and consider a positive scalar function \( u : [a, b] \to \mathbb{R} \) as well as a parameter \( \tau \geq 0 \), such that \( \tau (|u(z)| + |u'(z)|) \leq r(z) \).

We will need here a method that describes the effect of the deformation
\[
(3.2) \quad \tilde{N}(p) = N(p) - \tau u(z) \nu^{+}(p)
\]
where \( \nu^{+} = \frac{H}{\lambda} \) is the principal normal direction. In our situation, we have additional normal directions but we do not need to bend them. As we have shown in \([30]\), the cylindrical regions are close to codimension one and hence most of the curvature lies in the mean curvature direction.

In \([23]\), we have the following calculations that compute the effect of the deformation. This corresponds to the situation of \( m = 1 \), or the hypersurface case.

**Lemma 3.16** ([23, Lemma 3.15]).

(i) \( \tilde{g}^{i}_{ij} = g_{ij} + \tau^{2} \delta^{i}_{j}(u')^{2} + \tau^{2} u^{2} h_{il}^{+} h_{jl}^{+} - 2 \tau u h_{ij}^{+} + \tau^{2} u^{2} \sum T_{i+}^{\beta} T_{j+}^{\gamma} g_{\beta \gamma} \)
(ii) \( \frac{d}{d\tau} \tilde{g}^{i}_{ij} = -2 u h_{ij}^{+} + 2 \tau \delta^{i}_{j}(u')^{2} + 2 \tau u^{2} h_{il}^{+} h_{jl}^{+} + 2 \tau u^{2} \sum T_{i+}^{\beta} T_{j+}^{\gamma} g_{\beta \gamma} \)
(iii) \( \frac{d}{d\tau} \sqrt{\det g^{i}_{ij}} = \sqrt{\det g^{i}_{ij}} (-u h_{ij}^{+} + \tau \delta^{i}_{j}(u')^{2} + \tau u^{2} h_{il}^{+} h_{jl}^{+} + \tau^{2} u^{2} \sum T_{i+}^{\beta} T_{j+}^{\gamma} g_{\beta \gamma}) \)

**Proof.** We see
\[
\tilde{g}^{i}_{ij} = \left\langle \frac{\partial \tilde{N}_{\tau}}{\partial x^i}, \frac{\partial \tilde{N}_{\tau}}{\partial x^j} \right\rangle
\]
and

\[ \frac{\partial \tilde{N}^\tau}{\partial x^i} = \frac{\partial N}{\partial x^i} - \tau u' \delta^1_i \nu^+ - \tau u \frac{\partial \nu^+}{\partial x^i} \]

so that

\[
\tilde{g}^\tau_{ij} = \left\langle \frac{\partial N}{\partial x^i} - \tau u' \delta^1_i \nu^+ - \tau u \frac{\partial \nu^+}{\partial x^i}, \frac{\partial N}{\partial x^j} - \tau u' \delta^1_j \nu^+ - \tau u \frac{\partial \nu^+}{\partial x^j} \right\rangle \\
= g_{ij} - 2\tau \left\langle \frac{\partial N}{\partial x^i}, \frac{\partial \nu^+}{\partial x^j} \right\rangle + \tau^2 (u')^2 \delta^1_i \delta^1_j \\
+ 2\tau^2 u u' \delta^1_i \left\langle \nu^+, \frac{\partial \nu^+}{\partial x^j} \right\rangle + \tau^2 u^2 \left\langle \frac{\partial \nu^+}{\partial x^i}, \frac{\partial \nu^+}{\partial x^j} \right\rangle \\
= g_{ij} - 2\tau \left\langle \frac{\partial N}{\partial x^i}, \frac{\partial \nu^+}{\partial x^j} \right\rangle + \tau^2 (u')^2 \delta^1_i \delta^1_j \\
+ \tau^2 u^2 \left\langle \frac{\partial \nu^+}{\partial x^i}, \frac{\partial \nu^+}{\partial x^j} \right\rangle,
\]

and

\[
\tau^2 u^2 \left\langle \frac{\partial \nu^+}{\partial x^i}, \frac{\partial \nu^+}{\partial x^j} \right\rangle \\
= \tau^2 u^2 \left\langle h^i_l g^k_l \frac{\partial N}{\partial x^k} + \sum T^\beta_{i+} \nu_\beta, h^i_l g^k_l \frac{\partial N}{\partial x^k} + \sum T^\beta_{i+} \nu_\beta \right\rangle \\
= \tau^2 u^2 h^i_l h^j_k g^l_k + \tau^2 u^2 \sum T^\beta_{i+} T^\gamma_{j+} g_{\beta\gamma}.
\]

Therefore

\[ \tilde{g}^\tau_{ij} = g_{ij} + \tau^2 \delta^1_i \delta^1_j (u')^2 + \tau^2 u^2 h^i_i h^j_j - 2\tau u h^i_i + \tau^2 u^2 \sum T^\beta_{i+} T^\gamma_{j+} g_{\beta\gamma}. \]

From this we immediately compute

\[
\frac{d}{d\tau} \tilde{g}^\tau_{ij} = -2u h^j_i + 2\tau \delta^1_i \delta^1_j (u')^2 + 2\tau u^2 h^i_i h^j_j + 2\tau u^2 \sum T^\beta_{i+} T^\gamma_{j+} g_{\beta\gamma}
\]

and

\[
\frac{d}{d\tau} \sqrt{\det \tilde{g}^\tau} = \frac{1}{2} \tilde{g}^\tau_{ij} \frac{d}{d\tau} g^\tau_{ij} \sqrt{\det \tilde{g}^\tau} \\
= \sqrt{\det \tilde{g}^\tau} \tilde{g}^\tau_{ij} \left( -u h_{ij} + \tau \delta^1_i \delta^1_j (u')^2 + \tau u^2 h^i_i h^j_j + \tau^2 u^2 \sum T^\beta_{i+} T^\gamma_{j+} g_{\beta\gamma} \right) .
\]

**Corollary 3.17.** For \( k \geq 1 \) and \( 0 < \varepsilon < \varepsilon_0 \) sufficiently small, there is a fixed constant \( c > 0 \) such that for all deformations \( \tilde{N}_\tau \) given in (3.2) of an \((\varepsilon, k)\)-cylindrical neck in a normal parameterisation we have the estimates

(i) \(| \tilde{g}^\tau_{ij} - (g_{ij} - 2\tau u h^i_i) | \leq c \tau^2 (|u|^2 + |u'|^2) \),

(ii) \(| \tilde{g}^\tau_{ij} - (g_{ij} + 2\tau u h^i_i) | \leq c \tau^2 \tau^{-4} (|u|^2 + |u'|^2) \) and

(iii) \(| \sqrt{\det \tilde{g}^\tau} - \sqrt{\det g(1 - \tau u |H|)} | \leq c \tau^2 \tau^{-2} (|u|^2 + |u'|^2) \).

\( \square \)
Proof. These estimates follow immediately from Lemma 3.16.

In the following, we will use the notation $f = g + \mathcal{O}(h)$ to mean that there exists a constant $C$ such that $|f - g| \leq C|h|$.

**Theorem 3.18.** For $k \geq 1$ and $0 < \varepsilon < \varepsilon_0$ sufficiently small, there is a fixed constant $c > 0$ such that for all deformations $\tilde{N}_\tau$ given in (3.2) of an $(\varepsilon, k)$-cylindrical neck in a normal parameterisation we have the estimates

(i) $\tilde{h}_{ij} = h_{ij} + \tau \delta_i^j \delta^k \frac{\partial^2 N}{\partial x^i \partial x^k} u_i' + \tau u h_{ij}^+ + \tau_0^j \frac{\partial^2 N}{\partial x^i \partial x^k} \frac{\partial N}{\partial x^k}

(ii) $\tilde{h}_{i,j} = h_{i,j} + \tau g^{1j} \delta^i \frac{\partial^2 N}{\partial x^i \partial x^k} u_i' + \tau u h_{i,j}^+ + \tau_0^j \frac{\partial^2 N}{\partial x^i \partial x^k} \frac{\partial N}{\partial x^k}$

(iii) $\tilde{H}^T = H + \tau g^{1i} \frac{\partial^2 N}{\partial x^i \partial x^k} u_i' + \tau u h_{i,j}^+ + \tau_0^j \frac{\partial^2 N}{\partial x^i \partial x^k} \frac{\partial N}{\partial x^k}$

(iv) $|\tilde{A}^\tau|^2 = |A|^2 + 2\tau \delta_i^j \delta^k \frac{\partial^2 N}{\partial x^i \partial x^k} u_i' + 2\tau u h_{ij}^+ + \tau_0^j \frac{\partial^2 N}{\partial x^i \partial x^k} \frac{\partial N}{\partial x^k}$

$+ \mathcal{O}(\varepsilon \tau^2 (|u| + |u'|) + \mathcal{O}(\tau^2 (|u|^2 + |u'|^2 + |u''|^2)),$
Proof. We note that we have the Weingarten relations

\[ -\tilde{h}_{ij} = \frac{\partial^2 \tilde{N}_\tau}{\partial x_i \partial x_j} - \tilde{\Gamma}^\tau_{ij} \frac{\partial \tilde{N}_\tau}{\partial x^k} \]

where \( \tilde{h}_{ij} = \tilde{h}^\tau_{ij} \nu^\tau_\alpha \). Recall that \( \tilde{N}_\tau = N - \tau u \nu^+ \).

Therefore we compute

\[ \frac{\partial \tilde{N}_\tau}{\partial x^j} = \frac{\partial N}{\partial x^j} - \tau \frac{\partial u}{\partial x^j} \nu^+ - \tau \frac{\partial \nu^+}{\partial x^j} \]

\[ = \frac{\partial N}{\partial x^j} - \tau \delta^1_j u' \nu^+ - \tau u h^+_{ij} \frac{\partial N}{\partial x^k} - \tau u \sum T^\alpha_{j+} \nu_\alpha. \]

Next we compute that

\[ \frac{\partial^2 \tilde{N}_\tau}{\partial x^i \partial x^j} = \frac{\partial^2 N}{\partial x^i \partial x^j} - \tau \delta^1_i u' \nu^+ - \tau \delta^1_j u' \frac{\partial \nu^+}{\partial x^i} \]

\[ - \tau \delta^1_i u' h^+_{ij} \frac{\partial N}{\partial x^k} - \tau u \frac{\partial h^+_{ij}}{\partial x^k} - \delta^1_i \sum T^\alpha_{j+} \nu_\alpha - \tau u \sum \frac{\partial T^\alpha_{j+}}{\partial x^i} \nu_\alpha - \tau u \sum T^\alpha_{j+} \frac{\partial \nu_\alpha}{\partial x^i}. \]

Next we note that

\[ \tilde{\Gamma}^\tau_{ij} = \frac{1}{2} \tilde{g}^\tau_{ij} \left( \frac{\partial \tilde{g}^\tau_{ij}}{\partial x^i} + \frac{\partial \tilde{g}^\tau_{ij}}{\partial x^j} - \frac{\partial \tilde{g}^\tau_{ij}}{\partial x^l} \right). \]

Computing we see

\[ \frac{\partial \tilde{g}^\tau_{ij}}{\partial x^i} = \frac{\partial g_{ij}}{\partial x^i} - 2\tau \delta^1_i u' h^+_{ij} + 2\tau \delta^1_i u' \frac{\partial h^+_{ij}}{\partial x^l} + \tau^2 u' u'' \delta^1_i \delta^1_j \]

\[ + 2\tau^2 u' u'' h^+_{ij} h^+_{ip} + \tau^2 u' h^+_{ij} \frac{\partial h^+_{ip}}{\partial x^l} + \tau^2 u^2 \frac{\partial h^+_{ij}}{\partial x^l} \]

\[ + 2\tau^2 u' u'' \sum T^\beta_{i+} T^\gamma_{j+} g_{\beta \gamma} + \tau^2 u^2 \sum \frac{\partial T^\beta_{i+}}{\partial x^l} g_{\beta \gamma} + \tau^2 u^2 \sum T^\beta_{i+} T^\gamma_{j+} \frac{\partial g_{\beta \gamma}}{\partial x^l}, \]

that is we have

\[ \frac{\partial \tilde{g}^\tau_{ij}}{\partial x^i} = \frac{\partial g_{ij}}{\partial x^i} - 2\tau \delta^1_i u' h^+_{ij} + \mathcal{O}(\varepsilon \tau |u|) + \mathcal{O}(\tau^2 (|u|^2 + |u'|^2 + |u''|^2)). \]
Since we have the expansion $|\tilde{g}^{r,ij} - (g^{ij} + 2\tau u h^{+ij})| \leq c\tau^2 r^{-4}(|u|^2 + |u'|^2)$, we see the leading order terms are
\[
g^{-k}_{i} \frac{\partial \tilde{g}^{r}_{ij}}{\partial x^{k}} \simeq g^{-k}_{i} \left( \frac{\partial g^{ij}}{\partial x^{k}} - 2\tau \delta^{1}_{i} u' h^{+}_{ij} \right)
\simeq g^{-k}_{i} \frac{\partial g^{ij}}{\partial x^{k}} - 2\tau g^{1k} u' h^{+}_{ij}.
\]
Hence we find
\[
\tilde{\Gamma}^{r,k}_{ij} \simeq \Gamma^{k}_{ij} - \tau u' (\delta^{1}_{i} h^{+,j}_{j} + \delta^{1}_{j} h^{+i}_{i} - g^{1k} h^{+}_{ij}).
\]
In full, we have the estimate
\[
\tilde{\Gamma}^{r,k}_{ij} = \Gamma^{k}_{ij} - \tau u' (\delta^{1}_{i} h^{+,j}_{j} + \delta^{1}_{j} h^{+i}_{i} - g^{1k} h^{+}_{ij}) + O(\varepsilon\tau |u|) + O(\tau^2 (|u|^2 + |u'|^2 + |u''|^2)).
\]
Since
\[
\frac{\partial \tilde{N}_{r}}{\partial x^{k}} = \frac{\partial N}{\partial x^{k}} - \tau \frac{\partial u}{\partial x^{k}} \nu^{+} - \tau u \frac{\partial \nu^{+}}{\partial x^{k}}
\]
\[
= \frac{\partial N}{\partial x^{k}} - \tau \delta^{1}_{j} u' \nu^{+} - \tau u h^{+}_{i} \frac{\partial N}{\partial x^{k}} - \tau u \sum T^{\alpha}_{j+} \nu_{\alpha}
\]
we get
\[
\tilde{\Gamma}^{r,k}_{ij} \frac{\partial \tilde{N}_{r}}{\partial x^{k}} = \Gamma^{k}_{ij} \frac{\partial N}{\partial x^{k}} - \tau u' (\delta^{1}_{i} h^{+,j}_{j} + \delta^{1}_{j} h^{+i}_{i} - g^{1k} h^{+}_{ij}) \frac{\partial N}{\partial x^{k}}
\]
\[
+ O(\varepsilon\tau |u|) + O(\tau^2 (|u|^2 + |u'|^2 + |u''|^2)).
\]
Therefore we have
\[
\frac{\partial^{2} \tilde{N}_{r}}{\partial x^{i} \partial x^{j}} - \tilde{\Gamma}^{r,k}_{ij} \frac{\partial \tilde{N}_{r}}{\partial x^{k}} = \frac{\partial^{2} N}{\partial x^{i} \partial x^{j}} - \Gamma^{k}_{ij} \frac{\partial N}{\partial x^{k}}
\]
\[
- \tau \delta^{1}_{j} \delta^{1}_{i} u'' \nu^{+} - \tau \delta^{1}_{j} u' \frac{\partial \nu^{+}}{\partial x^{i}}
\]
\[
- \tau \delta^{1}_{i} u' h^{+}_{j} \frac{\partial N}{\partial x^{k}} - \tau u \frac{\partial h^{+}_{j} \frac{\partial N}{\partial x^{k}} - \tau u h^{+}_{j} \frac{\partial N}{\partial x^{i}} - \tau u h^{+}_{k} \frac{\partial N}{\partial x^{j}}}
\]
\[
- \tau u' (\delta^{1}_{i} h^{+}_{j} + \delta^{1}_{j} h^{+i}_{i} - g^{1k} h^{+}_{ij}) \frac{\partial N}{\partial x^{k}}
\]
\[
+ O(\varepsilon\tau |u|) + O(\tau^2 (|u|^2 + |u'|^2 + |u''|^2)).
\]
This then gives us
\[
- \tilde{h}^{r}_{ij} = -h_{ij} - \tau \delta^{1}_{j} \delta^{1}_{i} u'' \nu^{+} - \tau \delta^{1}_{j} u' \frac{\partial \nu^{+}}{\partial x^{i}}
\]
\[
- \tau \delta^{1}_{i} u' h^{+}_{j} \frac{\partial N}{\partial x^{k}} - \tau u \frac{\partial h^{+}_{j} \frac{\partial N}{\partial x^{k}} - \tau u h^{+}_{j} \frac{\partial N}{\partial x^{i}} - \tau u h^{+}_{k} \frac{\partial N}{\partial x^{j}}}
\]
or

$$\tilde{h}_{ij}^* = h_{ij} + \tau \delta^i_j \delta^j_i \nu^+ + \tau u_{h_{ij}^+} \frac{\partial^2 N}{\partial x^i \partial x^j}$$

$$+ \tau \delta^i_j u' \frac{\partial \nu^+}{\partial x^i} + \tau \delta^j_i u' h_{ij}^{+\pm} \frac{\partial N}{\partial x^j} + \tau u \frac{\partial h_{ij}^{+\pm}}{\partial x^j}$$

$$+ \tau u'(\delta^i_j h_{ij}^{+\pm} + \delta^j_i h_{ij}^{+\pm} - g^{1k} h_{ij}^+) \frac{\partial N}{\partial x^k}$$

$$+ O(\varepsilon \tau |u|) + O(\tau^2 (|u|^2 + |u'|^2 + |u''|^2)).$$

Since $|\tilde{g}_{ij} - (g_{ij} + 2\tau u_{h_{ij}^+})| \leq c\tau^2 r^{-4} (|u|^2 + |u'|^2)$, we get

$$-\tilde{h}_{ij}^* = -h_{ij}^* - \tau g^{1j} \delta^i_j \nu^+ - \tau g^{1j} u' \frac{\partial \nu^+}{\partial x^i}$$

$$- \tau \delta^i_j u' h_{ij}^{+\pm} \frac{\partial N}{\partial x^j} - \tau u \frac{\partial h_{ij}^{+\pm}}{\partial x^j}$$

$$+ \tau \delta^1_i u' g^{pj} \sum_{\alpha} T_{p+}^\alpha \nu_\alpha - \tau u g^{pj} \sum_{\alpha} \frac{\partial T_{p+}^\alpha}{\partial x^i} \nu_\alpha - \tau u g^{pj} \sum_{\alpha} \frac{\partial \nu_\alpha}{\partial x^j}$$

$$+ \tau u'(\delta^1_i h_{ij}^{+\pm} + g^{1j} h_{ij}^{+\pm} - g^{1k} h_{ij}^+) \frac{\partial N}{\partial x^k}$$

$$+ O\left(\varepsilon \frac{\tau}{r^2} |u|\right) + O\left(\frac{\tau^2}{r^3} (|u|^2 + |u'|^2 + |u''|^2)\right),$$

or

$$\tilde{h}_{ij}^* = h_{ij}^* + \tau g^{1j} \delta^i_j \nu^+ + \tau u_{h_{ij}^+} \frac{\partial^2 N}{\partial x^i \partial x^j}$$

$$+ \tau g^{1j} u' \frac{\partial \nu^+}{\partial x^i} + \tau \delta^j_i u' h_{ij}^{+\pm} \frac{\partial N}{\partial x^j} + \tau u \frac{\partial h_{ij}^{+\pm}}{\partial x^j}$$

$$+ \tau u'(\delta^1_i h_{ij}^{+\pm} + g^{1j} h_{ij}^{+\pm} - g^{1k} h_{ij}^+) \frac{\partial N}{\partial x^k}$$

$$+ O\left(\varepsilon \frac{\tau}{r^2} (|u| + |u'|)\right) + O\left(\frac{\tau^2}{r^3} (|u|^2 + |u'|^2 + |u''|^2)\right),$$

and

$$\tilde{H}^* = H + \tau g^{11} u'' \nu^+ - \tau u_{h_{ij}^+} \frac{\partial^2 N}{\partial x^i \partial x^j}$$

$$+ \tau g^{1j} u' \frac{\partial \nu^+}{\partial x^i} + \tau u_{h_{ij}^+} \frac{\partial N}{\partial x^j} + \tau u \frac{\partial h_{ij}^{+\pm}}{\partial x^j}.$$
\begin{align*}
+ \tau u'(\delta^1_i h^{+,i}_j + g^{1i} h^{+,k}_j - g^{1k} h^{+,i}_j) \frac{\partial N}{\partial x^k} \\
+ \mathcal{O}\left(\varepsilon \frac{\tau}{r^2}(|u| + |u'|)\right) + \mathcal{O}\left(\frac{\tau^2}{r^3}(|u|^2 + |u'|^2 + |u''|^2)\right).
\end{align*}

This shows the leading order normal terms in the expansion of \(\tilde{h}^{r}_{ij}\) is given by
\[
\tilde{h}^{r}_{ij} \simeq h_{ij} + \tau \delta^1_i \delta^1_j u^{''} + \tau u^{+,k} h_{ik}
\]
and since \(|\tilde{g}^{r,ij} - (g^{ij} + 2\tau u h^{+,ij})| \leq \varepsilon r^{-4}(|u|^2 + |u'|^2),
\]
\[
\tilde{h}^{r}_{ij} \simeq h_{ij} + \tau \delta^1_i \delta^1_j u^{''} + \tau h^{+,k} h_{ij}^{,i} + \tau h^{+,j} h_{ij}^{,j}
\]
and
\[
\tilde{H}^{r} \simeq H + \tau g^{11} u^{''} + \tau u^{+,k} h_{ik}^{,i}.
\]

From this, we see the leading order terms of the expansion of \(|\tilde{A}^r|^2\) is
\[
|\tilde{A}^r|^2 \simeq |A|^2 + 2\tau \delta^1_i \delta^1_j u^{''} h^{+,ij} + 2\tau h^{+,k} h_{ik} h^{,i} j
\]
and \(|\tilde{H}^r|^2\) is
\[
|\tilde{H}^r|^2 \simeq |H|^2 + 2\tau u^{''} g^{11} |H| + 2\tau h^{+,k} h_{ik} h^{,i} j
\]
and with the error terms we get
\[
|\tilde{A}^r|^2 = |A|^2 + 2\tau \delta^1_i \delta^1_j u^{''} h^{+,ij} + 2\tau h^{+,k} h_{ik} h^{,i} j
\]
\[
+ \mathcal{O}\left(\varepsilon \frac{\tau}{r^2}(|u| + |u'|)\right) + \mathcal{O}\left(\frac{\tau^2}{r^3}(|u|^2 + |u'|^2 + |u''|^2)\right),
\]
and \(|\tilde{H}^r|^2\) is
\[
|\tilde{H}^r|^2 = |H|^2 + 2\tau u^{''} g^{11} |H| + 2\tau h^{+,k} h_{ik} h^{,i} j
\]
\[
+ \mathcal{O}\left(\varepsilon \frac{\tau}{r^2}(|u| + |u'|)\right) + \mathcal{O}\left(\frac{\tau^2}{r^3}(|u|^2 + |u'|^2 + |u''|^2)\right).
\]

\[\square\]

In order to blend the neck with a cap, we require the following function, let \(u(z) = r_0 \exp\left(-\frac{B}{z-\Lambda}\right)\) where \(r_0 = r(4\Lambda)\) is the neck scale. The constant \(B\) is to be chosen.

**Lemma 3.19** (cf. [23 Lemma 3.18]).

(i) For \(z \in [\Lambda, 4\Lambda]\), the function \(f(z) = \exp\left(-\frac{B}{z-\Lambda}\right)\) satisfies
\[
f(z) \leq 1, \quad f''(z) \leq \frac{5}{B^2}, \quad f'(z) = \frac{B}{(z-\Lambda)^2} f(z), \quad |f'(z)|^2 \leq \frac{5}{B^2} f(z)
\]
and for $B \geq 12\Lambda$ we have
\[f''(z) \geq \frac{B^2}{2(z - \Lambda)^2} f(z).\]

(ii) Furthermore, $\forall \Lambda, \delta > 0, \exists B$ sufficiently large so that
\[f(z) \leq \delta f''(z), \quad |\tau f'(z)| (1 + |f'(z)|) \leq \delta f''(z), \quad |f''(z)| \leq \delta\]
everywhere on $[\Lambda, 4\Lambda]$.

Therefore combining Lemma 3.19 and Theorem 3.18, we have

**Corollary 3.20.** For any $\Lambda \geq 10$, we can choose $k_0 \geq 1, 0 < \varepsilon_0, 0 < r_0 < 1$ and $B$ sufficiently large such that for all $0 < \varepsilon \leq \varepsilon_0, k \geq k_0$ large enough so that the deformed submanifold $\tilde{N}_{r_0}$ satisfies

(1) $|H'| \geq |H|, \quad |\tilde{A}'|^2 - \frac{1}{n-2} |\tilde{H}'|^2 \leq |A|^2 - \frac{1}{n-2} |H|^2, \quad \sqrt{\det g} \leq \sqrt{\det g}$ on $[\Lambda, 4\Lambda]$,

(2) $|\tilde{A}'|^2 - \frac{1}{n-1} |\tilde{H}'|^2 \leq \frac{1}{2} r_0 v'' \quad \tilde{A}'^2 - \frac{1}{n-2} |\tilde{H}'|^2 \leq |A|^2 - \frac{1}{n-2} |H|^2 - \frac{1}{2} r_0 v''$ on $[2\Lambda, 3\Lambda]$,

(3) $\frac{|A|^2 - \frac{1}{n-2} |H|^2}{|\tilde{H}'|^2} \leq \frac{1}{\det g} \leq \frac{1}{\det g} (1 - \frac{1}{2} r_0 u \tilde{H})$ on $[\Lambda, 4\Lambda]$.

We now show the class of submanifolds $C_{n,k}(R, \alpha)$ is preserved under mean curvature with surgery.

**Theorem 3.21.** For any $\Lambda \geq 10$, we may choose $k_0 \geq 1, 0 < \varepsilon_0, 0 < r_0$ sufficiently small and $B$ sufficiently large so that for all $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ and $R > 0$ the class $C_{n,k}(R, \alpha)$ is invariant under standard surgery with parameters $\tau_0, B$ on a normal $(\varepsilon, k)$-submanifold neck $N : S^{n-1} \times [-4\Lambda, 4\Lambda] \to M$ for all $0 < \varepsilon \leq \varepsilon_0$ and $k \geq k_0$.

**Proof.** The proof of the above estimates essentially follows from Corollary 3.20.\[ \square \]

Here, we will fix the parameters $\Lambda = 10$ and $k_0 = 2$. The parameters $B$ and $\tau_0$ will be fixed so that Theorem 3.21 is valid and so that the algebraic conditions implying the gradient estimate and cylindrical estimate are valid.

Finally, in this section we show surgery is *reverse connected sum* in the following manner.

**Theorem 3.22.** There exists a range of parameters $\Lambda \geq 10, 0 < \varepsilon \leq \varepsilon_0$ and $k \geq k_0$ depending only on $n$, such that the following is true. Suppose that the standard surgery is performed in a normal $(\varepsilon, k)$-submanifold neck $N : S^{n-1} \times [-4\Lambda, 4\Lambda] \to M$ is some connected smooth closed immersed submanifold $F : \hat{M} \to \mathbb{R}^{n+m}$ resulting in a new smooth submanifold $\hat{M}$. If $\hat{M}$ is connected then $\hat{M}$ is diffeomorphic to the manifold obtained from $\hat{M}$ by a standard connected sum with itself. If $\hat{M}$ is disconnected with two components $\hat{M}_1$ and $\hat{M}_2$ then $\hat{M}$ is diffeomorphic to the standard connected sum of $\hat{M}_1$ and $\hat{M}_2$. In particular, if $\hat{M}$ is disconnected and $\hat{M}_2$ is diffeomorphic to $S^n$ then $\hat{M}_1$ is diffeomorphic to $\hat{M}$.
Proof. This follows from the surgery construction: the two open \( n \)-discs attached by the surgery are diffeomorphic to the standard disc and the two collars \([-4\Lambda, 0], [0, 4\Lambda]\) are \((\varepsilon, k)\)-close to the standard cylinder.

4. Gradient Estimates in the Presence of Surgery

In the paper, [30], a key gradient estimate was proved for the smooth mean curvature flow. Here we adapt the estimate to the case of mean curvature flow with surgery.

**Theorem 4.1.** Let \( M_t \in C_{n,k}(R, \alpha) \) be a solution of the mean curvature flow with surgery and normalised initial data. Then there exists a constant \( \gamma_1 = \gamma_1(n, M_0) \) and a constant \( \gamma_2 = \gamma_2(n, M_0) \) such that the flow satisfies the uniform estimate

\[
|\nabla A|^2 \leq \gamma_1 |A|^4 + \gamma_2 R^{-4}
\]

for all \( t \geq R^2/4 \).

**Proof.** We choose here \( \kappa_n = \left(\frac{3}{n+2} - c\right) > 0 \). As \( c = \frac{1}{n-2} - \eta \), \( n \geq 2 \), \( \kappa_n \) is strictly positive. We will consider here the evolution equation for

\[
\frac{|\nabla A|^2}{g^2}
\]

where \( g = \alpha|H|^2 - |A|^2 - \beta \) where \( \alpha \) and \( \beta \) are constants coming from the quadratic bounds on \( |A|^2 \) ensuring that \( g \) is strictly positive. Since \( |A|^2 - c|H|^2 < 0, |H| > 0 \) and \( M_0 \) is compact, there exists an \( \eta(M_0) > 0, C_\eta(M_0) > 0 \) so that

\[
(\varepsilon - \eta)|H|^2 - |A|^2 \geq \frac{C_\eta}{R^2} > 0.
\]

Hence we set

\[
g = c|H|^2 - |A|^2 > \varepsilon|A|^2 > 0
\]

where \( \varepsilon = \frac{n}{n+2} \). From the evolution equations, (2.10), we get

\[
\partial_t g = \Delta g - 2 \left( c|\nabla H|^2 - |\nabla A|^2 \right) + 2 (cR_2 - R_1)
\]

\[
\geq \Delta g - 2 \left( \frac{n+2}{3}c - 1 \right) |\nabla A|^2
\]

\[
\geq \Delta g + 2\kappa_n \frac{n+2}{3} |\nabla A|^2.
\]

The evolution equation for \( |\nabla A|^2 \) is given by

\[
\frac{\partial}{\partial t} |\nabla A|^2 - \Delta |\nabla A|^2 \leq -2|\nabla^2 A|^2 + c_n |A|^2 |\nabla A|^2.
\]

Let \( w, z \) satisfy the evolution equations

\[
\frac{\partial}{\partial t} w = \Delta w + W, \quad \frac{\partial}{\partial t} z = \Delta z + Z
\]
then we find

$$\partial_t \left( \frac{w}{z} \right) = \Delta \left( \frac{w}{z} \right) + \frac{2}{z} \langle \nabla (\frac{w}{z}) , \nabla z \rangle + \frac{W - w Z}{z^2} \nabla \nabla (\frac{w}{z})^2 - \frac{2 w |\nabla z|^2}{z} + \frac{W - w Z}{z^2}.$$ 

Furthermore for any function $g$, we have by Kato's inequality

$$\langle \nabla g , \nabla |\nabla A|^2 \rangle \leq \frac{1}{g} |\nabla g|^2 |\nabla A|^2 + g |\nabla^2 A|^2.$$ 

We then get

$$-\frac{2}{g} |\nabla^2 A|^2 + \frac{2}{g} \langle \nabla g , \nabla (\frac{|\nabla A|^2}{g}) \rangle \leq -\frac{2}{g} |\nabla^2 A|^2 - \frac{2}{g^3} |\nabla g|^2 |\nabla A|^2 + \frac{2}{g^2} \langle \nabla g , \nabla |\nabla A|^2 \rangle \leq 0.$$ 

Then if we let $w = |\nabla A|^2$ and $z = g$ with $W \leq -2 |\nabla^2 A|^2 + c_n |\nabla A|^2 |\nabla A|^2$ and $Z \geq 2 \kappa_n \frac{n+2}{3} |\nabla A|^2$ we get

$$\frac{\partial}{\partial t} \left( \frac{|\nabla A|^2}{g^2} \right) - \Delta \left( \frac{|\nabla A|^2}{g^2} \right) \leq \frac{2}{g} \langle \nabla g , \nabla (\frac{|\nabla A|^2}{g^2}) \rangle + \frac{1}{g} (-2 |\nabla^2 A|^2 + c_n |\nabla A|^2 |\nabla A|^2) \leq c_n |\nabla A|^2 - 2\kappa_n \frac{n+2}{3} \frac{|\nabla A|^4}{g^2}. $$

We repeat the above computation with $w = \frac{|\nabla A|^2}{g}$, $z = g$,

$$W \leq c_n |\nabla A|^2 \frac{|\nabla A|^2}{g} - 2\kappa_n \frac{n+2}{3} \frac{|\nabla A|^4}{g^2}$$

and $Z \geq 0$ to get

$$\frac{\partial}{\partial t} \left( \frac{|\nabla A|^2}{g^2} \right) - \Delta \left( \frac{|\nabla A|^2}{g^2} \right) \leq \frac{2}{g} \langle \nabla g , \nabla \left( \frac{|\nabla A|^2}{g^2} \right) \rangle + \frac{1}{g} \left( c_n |\nabla A|^2 \frac{|\nabla A|^2}{g} - 2\kappa_n \frac{n+2}{3} \frac{|\nabla A|^4}{g^2} \right) \leq c_n |\nabla A|^2 \frac{|\nabla A|^2}{g} - 2\kappa_n \frac{n+2}{3} \frac{|\nabla A|^4}{g^2}. $$

The nonlinearity then is

$$\frac{|\nabla A|^2}{g^2} \left( c_n - \frac{2\kappa_n (n+2)}{3} \frac{|\nabla A|^2}{g} \right) \leq \frac{|\nabla A|^2}{g^2} \left( c_n - \frac{2\kappa_n (n+2)}{3} \frac{|\nabla A|^2}{g} \right).$$

By the maximum principle, there exists a constant (with $\eta$ chosen sufficiently small so that this estimate holds at the initial time) such that

$$\frac{|\nabla A|^2}{g^2} \leq \frac{3c_n}{2\kappa_n (n+2) \eta}. $$
Therefore we see there exists a constant $C = \frac{3c}{2\kappa_n(n+2)\varepsilon} = C(n, \mathcal{M}_0)$ such that

$$\frac{|\nabla A|^2}{g^2} \leq C.$$  

By standard parabolic estimates, at the smooth time $t_0 = \frac{1}{4R^2}$, we get an upper bound of the form

$$|\nabla A|^2 \leq \frac{m_0}{R^4}.$$  

Since $g \geq \frac{c_n}{nR^2}$, we get

$$\frac{|\nabla A|^2}{g^2} \leq \max\left\{\frac{m_0}{C^2}, \frac{3c_n}{2\kappa_n(n+2)\varepsilon}\right\}.$$  

Now in the case of mean curvature flow with surgeries, we have on an exact cylinder

$$|A|^2 - \frac{1}{n-1} |H|^2 = 0.$$  

Therefore we can arrange for reasonable surgery parameters that

$$\frac{1}{n-2} |H|^2 - |A|^2 \geq \frac{1}{2} \left(\frac{1}{n-2} - \frac{1}{n-1}\right) |H|^2 = \frac{1}{(n-1)(n-2)} |H|^2.$$  

Therefore we have

$$g \geq \frac{c_n^2}{(n-1)(n-2)} |H|^2 = c_n^2 |H|^2.$$  

Since on a standard cylinder the second fundamental form is parallel, $|\nabla A|^2 = 0$, and $|H|^2 = \frac{(n-1)^2}{R^2}$, we see that on a $(\varepsilon, k)$-neck with $k \geq 1$, the quantity $|\nabla A|^2$ is uniformly very small with respect to $|H|^4$. Therefore, with our choice of transition function $\varphi$ in $d$ and $e$) of the standard surgery (Definition 3.15), there is a fixed constant $\mu_0$ depending only on $n$ so that for any surgery parameters, we have the uniform estimate $|\nabla A|^2 \leq \mu_0 |H|^4$. Therefore we get

$$\frac{|\nabla A|^2}{g^2} \leq \frac{\mu_0}{c_n^2}$$  

in the surgery region. Iterating this argument for every interval between two surgeries we get

$$\frac{|\nabla A|^2}{g^2} \leq \max\left\{\frac{\mu_0}{c_n^2}, \frac{m_0}{C^2}, \frac{3c_n}{2\kappa_n(n+2)\varepsilon}\right\}.$$  

Therefore we have

$$g^2 \leq |H|^4 + \frac{C(\alpha, n)}{R^4}$$  

and the above estimate implies

$$|\nabla A|^2 \leq c(n) |A|^4 + C(\alpha, n) R^{-4}.$$
We now consider the evolution equation of the term \( |\nabla^2 A|^2 \) for any \( t \geq \frac{1}{4} R^2 \).

**Proof.** We have the following evolution equation
\[
\partial_t |\nabla^2 A|^2 - \Delta |\nabla^2 A|^2 \leq -2|\nabla^3 A|^2 + k_1 |A|^2 |\nabla^2 A|^2 + k_2 |\nabla A|^2 |\nabla^2 A| \\
\leq -2|\nabla^3 A|^2 + \left( k_1 + \frac{k_2}{2} \right) |A|^2 |\nabla^2 A|^2 + \frac{k_2}{2} |\nabla A|^4.
\]

We now consider the evolution equation of the term \( \frac{|\nabla^2 A|^2}{|H|^4} \). Firstly we see
\[
\partial_t |H|^6 \geq \Delta |H|^6 - \alpha(|H|^6)^2 |\nabla H|^2.
\]

Therefore we get
\[
\partial_t \frac{|\nabla^2 A|^2}{|H|^5} - \Delta \frac{|\nabla^2 A|^2}{|H|^5} \leq \frac{1}{|H|^5} \left( -2|\nabla^3 A|^2 + \left( k_1 + \frac{k_2}{2} \right) |A|^2 |\nabla^2 A|^2 + \frac{k_2}{2} |\nabla A|^4 \right) \\
+ \frac{20|H|^3|\nabla^2 A|^2 |\nabla H|^2}{|H|^{10}} + \frac{2}{|H|^{10}} \langle \nabla |H|^5, \nabla |\nabla^2 A|^2 \rangle - \frac{2|\nabla^2 A|^2}{|H|^{15}} |\nabla |H|^5|^2.
\]

We have the terms
\[
\frac{20|H|^3|\nabla^2 A|^2 |\nabla H|^2}{|H|^{10}} - \frac{2|\nabla^2 A|^2}{|H|^{15}} |\nabla |H|^5|^2 \leq \frac{20|\nabla^2 A|^2 |\nabla H|^2}{|H|^7} - \frac{50|\nabla^2 A|^2 |\nabla |H|^2}{|H|^7} \\
\leq \frac{20|\nabla^2 A|^2 |\nabla H|^2}{|H|^7}.
\]

and
\[
\frac{2}{|H|^{10}} \langle \nabla |H|^5, \nabla |\nabla^2 A|^2 \rangle = \frac{10 \langle \nabla |H|, \nabla |\nabla^2 A|^2 \rangle}{|H|^6} \\
\leq \frac{1}{|H|^5} |\nabla^3 A|^2 + \frac{100|\nabla H|^2 |\nabla^2 A|^2}{|H|^7}
\]

Together with the gradient estimate, Theorem 4.1, this gives the following evolution equation
\[
\partial_t \frac{|\nabla^2 A|^2}{|H|^5} - \Delta \frac{|\nabla^2 A|^2}{|H|^5} \leq -\frac{|\nabla^3 A|^2}{|H|^5} + k_3 \frac{|\nabla^2 A|^2}{|H|^4} + \frac{120|\nabla H|^2 |\nabla^2 A|^2}{|H|^7} + \frac{k_2}{2} |\nabla A|^4 \\
\leq -\frac{|\nabla^3 A|^2}{|H|^5} + k_4 \frac{|\nabla^2 A|^2}{|H|^3} + C_1 \frac{|\nabla^2 A|^2}{R^4} + \frac{k_5 |H|^8 + C_2 R^{-8}}{|H|^5}
\]
Similar computations give us
\[
\partial_t |\nabla A|^2 \leq - |\nabla^2 A|^2 + k_6 |H|^8 + C_3 R^{-8},
\]
\[
\partial_t |\nabla A|^2 \leq - |\nabla^2 A|^2 + k_7 |H|^8 + C_4 R^{-8}.
\]
We now set
\[
f = \frac{|\nabla^2 A|^2}{|H|^5} + N \frac{|\nabla A|^2}{|H|^3} + M \frac{|\nabla A|^2}{|H|^7} - \kappa \sqrt{|c| |H|} - |A|^2
\]
We have
\[
\partial_t f - \Delta f \leq k_4 \frac{|\nabla^2 A|^2}{|H|^3} + k_5 |H|^3 + \frac{C_1 |\nabla^2 A|^2}{R^4 |H|^7} + \frac{C_2}{|H|^5 R^8}
\]
\[
- N \frac{|\nabla^2 A|^2}{|H|^3} + Nk_6 |H|^3 + \frac{NC_3}{|H|^5 R^8}
\]
\[
- M \frac{|\nabla^2 A|^2}{|H|^7 R^4} + \frac{k_7 M}{|H|^4 R^4} + \frac{C_4 M}{|H|^4 R^{12}} - \kappa \varepsilon_0 |H|^3.
\]
Therefore we choose
\[
N > k_4, \quad \varepsilon_0 \kappa > Nk_6 + k_5, \quad M > C_1.
\]
Then we find, recalling $|H| \geq \frac{\alpha_1}{R}$,
\[
\partial_t f - \Delta f \leq \frac{C_5}{R^3}
\]
which implies
\[
\max_{M_t} f \leq \max_{M_0} f + C_5 \frac{(t - t_0)}{R^3}.
\]
Since $M_t$ is in $C(R, \alpha)$, we have at time $t = \frac{1}{4} R^2$ the estimate
\[
f \leq \left( \frac{m_1}{\alpha_1^2} + \frac{Nm_0}{\alpha_1^4} + \frac{Mm_0}{\alpha_1^2} - \kappa \alpha_1 \right) R^{-1}.
\]
For surgery modified regions, since on an exact cylinder the second fundamental form is parallel, there exist $\mu_0, \mu_1 > 0$ such that $|\nabla A|^2 \leq \mu_0 |H|^4, |\nabla^2 A|^2 \leq \mu_1 |H|^6$ with constants $\mu_0, \mu_1$ each only depending on the surgery parameters. By choosing $\kappa$ sufficiently large,
\[
\kappa > n(Nk_6 + k_5) + \mu_1 + N\mu_0,
\]
we conclude that on a surgery region, we get
\[
f \leq (\mu_1 + N\mu_0 - \kappa) |H| + M\mu_0 \alpha_1^{-3} R^{-1} \leq M\mu_0 \alpha_1^{-3} R^{-1}.
\]
Given the bound on the maximal time of existence, we have
\[
f \leq C(n, \alpha, \mu_0, \mu_1) R^{-1},
\]
which implies
\[ |\nabla^2 A|^2 \leq c(n, \mu_0, \mu_1)|H|^6 + C(n, \alpha, \mu_0, \mu_1)|H|^5 R^{-1}. \]

Higher order estimates on $|\nabla^m A|$ for $m \leq k_0$ follow by an analogous method (recall $k_0$ is the surgery regularity parameter). Furthermore since we have the evolution equation
\[ |\partial_t A| = |\Delta A + A \ast A| \leq C|\nabla^2 A|^2 + C|A|^3 \leq c_1 A^3 + c_2 R^{-3}. \]

Strictly speaking, this estimate does not hold at surgery times as the flow is not smooth but the flow up to the surgery time and from the surgery time, so taking appropriate limits from below and above, we recover the desired estimates. Hence with this convention understood, we have the following higher order estimates,

**Corollary 4.3.** Let $\mathcal{M}_t \in C(R, \alpha)$ be a solution of the mean curvature flow with surgery and normalised initial data. Then there exists constants $\gamma', \gamma''$ depending only on the dimension, pinching constant and $\alpha$ so that
\[ |\partial_t \nabla^m A|^2 \leq \gamma' A^{4h + 2m + 2} + \gamma'' R^{-4h + 2m + 2} \]
for all $h, m \geq 0$ such that $2h + m \geq k_0$.

As a special case of our estimates we get the following statement. This corollary will be used extensively in the analysis of regions of high curvature. Note that a similar estimate plays an important role in the Ricci flow with surgery, [31, Equation (1.3)].

**Corollary 4.4.** Let $\mathcal{M}_t$ be a solution to the mean curvature flow with surgeries. Then there exists $c^#, H^# > 0$ such that for all $p \in \mathcal{M}$ and $t > 0$ such that
\[ |H(p, t)| \geq H^# \implies |\nabla H(p, t)| \leq c^#|H(p, t)|^2, \quad |\partial_t H(p, t)| \leq c^#|H(p, t)|^3. \]

The following lemma allows us to compare the mean curvature at different points.

**Lemma 4.5.** Let $F : \mathcal{M}^n \to \mathbb{R}^{n+m}$ be an immersed submanifold. Suppose there exists $c^#, H^#$ such that
\[ |\nabla H(p)|^2 \leq c^#|H(p)|^2 \]
for all $p \in \mathcal{M}$ which satisfies the lower bound $|H|(p) \geq H^#$. Then if for some $\gamma > 1$, $p_0 \in \mathcal{M}$ satisfies $|H(p_0)| \geq \gamma H^#$, we have
\[ |H(q)| \geq \frac{|H(p_0)|}{1 + c^#d(p_0, q)|H(p_0)|} \geq \frac{|H(p_0)|}{\gamma}, \quad \forall q \mid d(p_0, q) \leq \frac{\gamma - 1}{c^#} \frac{1}{|H|(p_0)}. \]

**Proof.** This proof follows exactly as in [23] or [30].
5. Neck Detection

We denote by $g(t)$ the metric induced on $\mathcal{M} \subset \mathbb{R}^{n+m}$ at time $t$. Let $\mathcal{B}_{g(t)}(p, r) \subset \mathcal{M}$ be the intrinsic closed ball of radius $r > 0$ about $p \in \mathcal{M}$ with respect to the metric $g(t)$. In addition if $t, \theta$ are given such that $0 \leq t - \theta < t \leq T_0$, we define a (backward) parabolic neighbourhood of $(p, t)$ by

$$\mathcal{P}(p, t, r, \theta) = \{ (q, s) \mid q \in \mathcal{B}_{g(s)}(p, r), s \in [t - \theta, t] \}. \tag{5.1}$$

Furthermore, we also consider normalised (backward) parabolic neighbourhood of $(p, t)$, defined as

$$\hat{\mathcal{P}}(p, t, L, \theta) = \mathcal{P}(p, t, \hat{r}(p, t)L, \hat{r}(p, t)^2\theta).$$

Note the above parabolic neighbourhood is defined only for smooth mean curvature flow. We now extend the above definition for mean curvature with surgery. In order to achieve this, we consider a family of flows $F^i : \mathcal{M}_i \times [T_{i-1}, T_i] \to \mathbb{R}^{n+1}$ where $T_0$ and $0 < T_1 < T_2 < \cdots$ are surgery times. For a flow with surgeries, we define $\mathcal{B}_{g(t)}(p, r)$ as in the smooth case. The neighbourhood $\mathcal{B}_{g(t)}(p, r)$ belongs to the manifold $\mathcal{M}_i$ corresponding to the interval $[T_{i-1}, T_i]$ containing $t$. For surgery times $T_i$, as the flow is not continuous across $T_i$, we will write $g(t-)$ for limits from the left and $g(t+)$ for limits from the right. Furthermore, we will use the convention that $g(t) = g(t-)$, that is our flow will be continuous from the left.

Now, for mean curvature flow with surgeries, since the flow is not continuous across surgery times, the parabolic neighbourhoods $\mathcal{P}(p, t, r, \theta)$ may not be well defined if there are surgeries between $t - \theta$ and $t$.

**Definition 5.1.** Let $F^i : \mathcal{M}_i \times [T_{i-1}, T_i] \to \mathbb{R}^{n+m}, i = 1, 2, \ldots$ be a mean curvature flow with surgery. For $(p, t) \in \mathcal{M}_i \times [T_{i-1}, T_i]$ for some $i$, let $\theta \in (0, t]$ and $r > 0$ we say $\mathcal{B}_{g(t)}(p, r)$ is unchanged by surgery in the interval $[t - \theta, t]$ if there are no points of $\mathcal{B}_{g(t)}(p, r)$ which belong to a region changed by surgery occurring at $s \in (t - \theta, t]$. In this case, the parabolic neighbourhood $\mathcal{P}(p, t, r, \theta)$ is given by (5.1). Alternatively, we write that $\mathcal{P}(p, t, r, \theta)$ does not contain surgeries or is surgery free.

The above definition allows the presence of surgeries on the time interval $(t-\theta, t]$ provided they are performed on parts of the submanifold disjoint from our domain $\mathcal{B}_{g(t)}(p, r)$. In this case $\mathcal{B}_{g(t)}(p, r)$ is a subset of different $\mathcal{M}_i$ before and after the surgery times; the $\mathcal{B}_{g(t)}(p, r)$ are unchanged by surgery and can be considered as a fixed domain in the time interval $[t-\theta, t]$. Also we allow $\mathcal{B}_{g(t)}(p, r)$ to be changed by surgery at the initial time $t-\theta$; in which case at time $t-\theta$ we consider $\mathcal{M}_{t-\theta+}$ after the surgery. Hence the parabolic neighbourhood does not keep track of the surgery at the initial time. It is well known that estimates such as above give local Harnack estimates, that is local control on the size of the curvature in a neighbourhood of a given point.

**Lemma 5.2.** Let $c^#, H#$ be constants of Lemma 4.5. We define $d^# = \frac{1}{8(n-1)^2c^#}$. Then we have the following;
(1) Let \((p, t)\) satisfy the inequality \(|H(p, t)| \geq H^\#\). Then if \(\hat{P}(p, t, r, \theta)\) is surgery free for any given \(r, \theta \in (0, d^\#]\) then
\[
\frac{|H(p, t)|}{2} \leq |H(q, s)| \leq 2|H(p, t)|
\]
for all \((q, s) \in \hat{P}(p, t, r, \theta)\).

(2) Suppose that the regions modified by surgery satisfy \(|H| \geq K\) for some \(K \geq H^\#\) for any time less than \(t\). If \((p, t)\) satisfies \(|H(p, t)| \geq K\) then \(\mathcal{P}(p, t, \frac{1}{8c^\#K}, \frac{1}{8c^\#K^2})\) does not contain surgeries. In particular, \(\hat{P}(p, t, d^\#, d^\#)\) does not contain surgeries and all points \((q, s) \in \hat{P}(p, t, d^\#, d^\#)\) satisfy
\[
\frac{|H(p, t)|}{2} \leq |H(q, s)| \leq 2|H(p, t)|.
\]

As the curvature flow is a one-parameter family of submanifolds, we now introduce time dependent versions of curvature and geometric necks. Therefore, consider for \(s \leq 0\) the function
\[
\rho(r, s) = \sqrt{r^2 - 2(n - 1)s}
\]
where \(\rho(r, s)\) is the radius at time \(s\) of a standard \(n\)-dimensional cylinder which evolves by mean curvature flow and has radius \(r\) at time \(s = 0\). Let us denote that if \(d^\#\) is the constant that appears in Lemma 5.2 then we have
\[
r \leq \rho(r, s) \leq 2r, \quad \forall s \in [-d^\#, 0]
\]
only otherwise we would have a standard cylinder by Lemma 5.2 (2).

**Definition 5.3.** We say a point \((p_0, t_0)\) lies at the center of an \((\varepsilon, k, L, \theta)\)-shrinking neck if after setting \(r_0 = \hat{r}_0(p_0, t_0)\) and \(B_0 = B_{g(t_0)}(p_0, r_0L)\) the following property holds

1. the parabolic neighbourhood \(\hat{P}(p_0, t_0, L, \theta)\) does not contain surgeries;
2. for every \(t \in [t_0, r_0^2\theta +, t_0]\) the region \(B_0\) with respect to the scaled immersion \(\rho(r_0, t-t_0)^{-1}F(\cdot, t)\) is \(\varepsilon\)-cylindrical and \((\varepsilon, k)\)-parallel at every point.

**Remark 5.4.**

1. Since we allow \(t_0 - r_0^2\theta^+\) to be a surgery time, the notation \(t_0 - r_0^2\theta^+\) means the limit from the right.
2. The definition above says at any point of the parabolic neighbourhood \(\mathcal{P}(p_0, t_0, r_0L, r_0^2\theta)\) the Weingarten map of the submanifold and its spatial derivatives up to order \(k\) are \(\varepsilon\)-close to the corresponding ones of the standard shrinking cylinder. Furthermore, using the evolution equations for the Weingarten map, \(\partial_t \nabla^l W\) we can show the derivatives in time are \(\varepsilon\)-close for \(2i + l \leq k\) up to order \(O(\varepsilon)\).

Recall that we will use mean curvature flow with surgery to define a flow beyond the singular time. In order to achieve this, we need to show the submanifold develops necks in
high curvature regions as we approach the singular time. The following is the analogue of Huisken-Sinestrari’s Neck Detection Lemma [23, Lemma 7.4].

**Lemma 5.5 (Neck Detection Lemma).** Let $\mathcal{M}_t, t \in [0, T)$ be a mean curvature flow with surgery in the class $C(R, \alpha)$. The for any $\varepsilon, \theta, L, k \geq k_0 \geq 2$ there exists $\eta_0 > 0, H_0 > 0$ such that the following hold: suppose that

(ND1) \[ |H(p_0, t_0)| \geq H_0, \quad \frac{|A|^2}{|H|^2(p_0, t_0)} \geq \frac{1}{n-1} - \eta_0, \]

(ND2) the neighbourhood $\mathcal{P}(p_0, t_0, L, \theta)$ does not contain surgeries, then

1. the neighbourhood $\mathcal{P}(p_0, t_0, L, \theta)$ is an $(\varepsilon, k_0 - 1, L, \theta)$ shrinking curvature neck and
2. the neighbourhood $\mathcal{P}(p_0, t_0, L - 1, \theta/2)$ is an $(\varepsilon, k, \theta/2)$ shrinking curvature neck.

**Proof.** We proceed by a proof by contradiction. Hence there exists $\varepsilon, L, \theta$ such that the conclusion of the theorem is not true. This implies there exists a sequence of mean curvature flows $\{\mathcal{M}_t^j\}_{j \geq 1}$ and a sequence of times $\{t_j\}_{j \geq 1}$ and points $\{p_j\}_{j \geq 1}, p_j \in \mathcal{M}_t^j$ such that for the space-time points $\{p_j, t_j\}_{j \geq 1}$ at the image points $F_j(p_j, t_j) \in \mathcal{M}_t^j$ then as $t \to T_{\max}, |H(p_j, t_j)| \to \infty$ and

\[ \frac{|A|^2(p_j, t_j)}{|H|^2(p_j, t_j)} \to \frac{1}{n-1} \]

but $(p_j, t_j)$ does not lie at the centre of a $(\varepsilon, k, L - 1, \theta/2)$ neck. Note that $\frac{|A|^2}{|H|^2} < c_n \leq \frac{1}{n-2}$ so the gradient estimates apply.

We perform a parabolic rescaling of $\mathcal{M}_t^k$ in such a way that the norm of the mean curvature at $(p_k, t_k)$ becomes $n - 1$. That is, if $F_k$ is the parameterisation of the original flow $\mathcal{M}_t^k$, we let $\hat{r}_k = \frac{n-1}{|H(p_k, t_k)|}$, and we denote the rescaled flow by $\mathcal{M}_t^k$ and we define its parameterisation as

\[ \tilde{F}_k(p, \tau) = \frac{1}{\hat{r}_k}(F_k(p, \hat{r}_k^2 \tau + t_k) - F_k(p_k, t_k)) \]

For simplicity, we choose for every flow a local co-ordinate system centered at $p_k$. In these co-ordinates we can write $0$ instead of $p_k$. The parabolic neighbourhoods $\mathcal{P}^k(p_k, t_k, \hat{r}_k L, \hat{r}_k^2 \theta)$ in the original flow becomes $\tilde{\mathcal{P}}(0, 0, L, \theta)$. By construction, each rescaled flow satisfies

$\tilde{F}_k(0, 0) = 0, \quad |\tilde{H}_k(0, 0)| = n - 1$.

The gradient estimates give us uniform bounds on $|A|$ and its derivatives up to any order on a neighbourhood of the form $\mathcal{P}(0, 0, d, d)$ for a suitable $d > 0$. This gives us uniform estimates in $C^\infty$ on $\tilde{F}_k$. Hence we can apply Arzela-Ascoli and conclude that there exists a subsequence that converges in $C^\infty$ to some limit flow which we denote by $\tilde{M}_t^\infty$. We now analyse the limit flow $\tilde{M}_t^\infty$. Note that we have

\[ \tilde{A}_k(p, \tau) = \hat{r}_k A_k(p, \hat{r}_k^2 \tau + t_k), \]
so that
\[
\frac{|\bar{A}_k(p, \tau)|^2}{|\bar{H}_k(p, \tau)|^2} = \frac{|A_k(p, \hat{r}_k^2 \tau + t_k)|^2}{|H_k(p, \hat{r}_k^2 \tau + t_k)|^2}
\]
but since \(\hat{r}_k \to 0, t_k \to T_{\text{max}}\) as \(k \to \infty\) this implies
\[
\frac{|\bar{A}(p, \tau)|^2}{|\bar{H}(p, \tau)|^2} = \lim_{k \to \infty} \frac{|\bar{A}_k(p, \tau)|^2}{|\bar{H}_k(p, \tau)|^2} \leq \frac{1}{n-1}
\]
and
\[
\frac{|\bar{A}(0, 0)|^2}{|\bar{H}(0, 0)|^2} = \frac{1}{n-1}.
\]
Hence the flow \(\bar{M}_t^\infty\) has a space-time maximum for \(\frac{|\bar{A}(p, \tau)|^2}{|\bar{H}(p, \tau)|^2}\) at \((0, 0)\). Since the evolution equation for \(\frac{|A|^2}{|H|^2}\) is given by
\[
\partial_t \left( \frac{|A|^2}{|H|^2} \right) - \Delta \left( \frac{|A|^2}{|H|^2} \right) = \frac{2}{|H|^2} \left\langle \nabla |H|^2, \nabla \left( \frac{|A|^2}{|H|^2} \right) \right\rangle - \frac{2}{|H|^2} \left( |\nabla A|^2 - \frac{|A|^2}{|H|^2} |\nabla H|^2 \right)
\]
\[
+ \frac{2}{|H|^2} \left( R_1 - \frac{|A|^2}{|H|^2} R_2 \right).
\]
Now we have
\[
|\nabla H|^2 \leq \frac{3}{n+2} |\nabla A|^2, \quad \frac{|A|^2}{|H|^2} < c_n \leq \frac{1}{n-2}
\]
which gives
\[
- \frac{2}{|H|^2} \left( |\nabla A|^2 - \frac{|A|^2}{|H|^2} |\nabla H|^2 \right) \leq 0.
\]
Furthermore if \(\frac{|A|^2}{|H|^2} = c < c_n \leq \frac{1}{n-2}\) then
\[
R_1 - \frac{|A|^2}{|H|^2} R_2 = R_1 - cR_2 \leq 0.
\]
Hence the strong maximum principle applies to the evolution equation of \(\frac{|A|^2}{|H|^2}\) and shows \(\frac{|A|^2}{|H|^2}\) is constant. The evolution equation then shows that \(|\nabla A|^2 = 0\), that is the second fundamental form is parallel and that \(|A_-|^2 = |\hat{A}_-|^2 = 0\), that is the submanifold is codimension one. Finally this shows locally \(\mathcal{M} = S^{n-k} \times \mathbb{R}^k\), (see [26]). As \(\frac{|A|^2}{|H|^2} < \frac{1}{n-2}\) we can only have \(S^n, S^{n-1} \times \mathbb{R}\) which gives \(\frac{|A|^2}{|H|^2}(x) = \frac{1}{n}, \frac{1}{n-1}\) for every \((p, t)\) which is a contradiction. The proof of (2) is similar. \(\square\)

**Remark 5.6.** We note a variant of the above remark shows about a singularity, if we rescale at the maximum of \(\frac{|A|^2}{|H|^2}\), then a subsequence converges to either a sphere or a cylinder.
We note that the neck detection lemma shows in a region of high curvature, if region is not spherically quadratically pinched then the submanifold is close to a portion of cylinder of the form $\mathbb{S}^{n-1} \times \mathbb{R}$ which lies in a Euclidean subspace $\mathbb{R}^{n+1} \subset \mathbb{R}^{n+m}$. The key to this argument is the gradient estimate, which implies a compactness theorem. In fact, using a theorem of Naff [28], we can show any region which has sufficiently large mean curvature must be close to a submanifold lying in a subspace of codimension one.

The following theorem is due to Naff.

**Theorem 5.7** ([28, Theorem 1.1]). Suppose $n \geq 5$. Suppose $\mathcal{M}_t, t \in [0, T)$ is a smooth, compact $n$-dimensional solution to mean curvature flow in $\mathbb{R}^{n+m}$ such that $|H| > 0$, and $|A| < c_n |H|^2$. Then there are constants $\sigma = \sigma(n, \mathcal{M}_0)$ and $C = C(n, \mathcal{M}_0)$ depending on $n$ and the initial submanifold $\mathcal{M}_0$ such that

$$|A^-|^2 \leq C|H|^{2-\sigma}$$

for all $t \in [0, T)$.

Using this theorem, together with a straightforward modification of the argument of Lemma 5.5, we get

**Lemma 5.8** (Hypersurface Detection Lemma). Let $\mathcal{M}_t, t \in [0, T)$ be a mean curvature flow with surgery in the class $C(R, \alpha)$. Let $\varepsilon, \theta, L, k \geq k_0$ be given where $k_0 \geq 2$. Then there exists a $H'_0 > 0$ with the following properties. Suppose that $|H(p_0, t_0)| \geq H'_0$, then

1. $\hat{P}(p_0, t_0, L, \theta)$ is an $(\varepsilon, k_0, L, \theta)$ almost hypersurface and
2. $\hat{P}(p_0, t_0, L - 2, \theta/2)$ is an $(\varepsilon, k, L, \theta/2)$ almost hypersurface.

We show that as a consequence of the Neck Detection Lemma that the point $(p_0, t_0)$ lies at the centre of a cylindrical immersion and at the centre of a normal submanifold neck. We will investigate here the structure of the submanifold at the final time $t_0$. We define the length of a submanifold neck $\mathbb{S}^{n-1} \times [a, b] \to \mathbb{R}^{n+m}$ equal to $b - a$. Defined in this way, the length of a neck in invariant under scaling. In contrast, we define the distance between points on the neck or manifold to be the distance with respect to the metric $g(t)$. Therefore the distance is not scale-invariant. We also define the term cylindrical submanifold of length $2L$ and $C^{k+2}$-norm less than $\varepsilon$ to denote a region (after appropriate rescaling) is close in norm to some standard immersion of a cylinder $F : \mathbb{S}^{n-1} \times [-L, L] \to \mathbb{R}^{n+1} \subset \mathbb{R}^{n+m}$. Similarly, the length of a cylindrical submanifold is invariant under scaling.

The following Corollary is an immediate consequence of the Neck Detection Lemma

**Corollary 5.9** (cf [23, Corollary 7.7]). Given $\varepsilon, \theta > 0, L \geq 10$ and $k > 0$ integer, then there exists $\eta_0, k_0 > 0$ such that if $(p_0, t_0)$ satisfy the hypotheses (ND1) and (ND2) of the Neck Detection Lemma (Lemma 5.5) then

1. then there is cylindrical submanifold centred at $(p_0, t_0)$ of length $2(L - 2)$ and $C^{k+2}$-norm less than $\varepsilon$,
(2) and there is a normal \((\varepsilon, k, L - 2, \theta)\)-submanifold neck centred at \((p_0, t_0)\).

The following few Lemmas are a consequence of the Neck Detection Lemma and have the same proofs as in \cite{23} with small modifications as we have a quadratic pinching condition and high codimension. We state them for the sake of completeness. Before we state the next lemma, let us make the following observation. Let \(D_p \subset C\) be the set of points of \(C\) having intrinsic distance \(L\) from \(p_0\). Clearly \(B\) cannot be written in the form \(\mathbb{S}^{n-1} \times [a, b]\) for any \(a, b\), however if \(L \geq \frac{n+1}{2}\) (that is sufficiently large so that the ball passes the cut locus) we have

\[
\mathbb{S}^{n-1} \times \{-(L-1), L-1\} \subset B_L \subset \mathbb{S}^{n-1} \times [-L, L].
\]

Hence, if \(L\) is large enough, the ball \(B_L\) is close to the subcylinder of length 2L.

**Lemma 5.10** (cf \cite[Lemma 7.9]{23}). Let \(L \geq 10\) and \(\theta \leq d^\#\) and denote as usual

\[
r_0 = \frac{n - 1}{H(p, t)}, \quad B_0 = B_{g(t_0)}(p_0, r_0L).
\]

Then in the Neck Detection Lemma (Lemma 5.5), \(\eta_0, H_0\) can be chosen so that if \(t \in [t_0 - \theta \tau_0^2, t_0]\), then there exists a \((\varepsilon, k_0 - 1)\)-submanifold neck \(N_t \subset B_0\) centred at \((p_0, t_0)\) which satisfies:

(i) For every cross section of \(N_t\), the mean radius satisfies \(r(\nu) = \rho_0(t_0 - t - \tau_0) (1 + O(\varepsilon))\); 
(ii) The length of \(N_t\) is at least \(L - 2\); 
(iii) There exists a unit vector \(\omega \in \mathbb{R}^{n+m}\) such that \(|\langle \nu^+(p, t), \omega \rangle| \leq \varepsilon\) for any \(p \in N_t\).

To apply the Neck Detection Lemma, we require the hypothesis (NDL2). The following lemma ensures (NDL2) is a consequence of the other assumptions of the Neck Detection Lemma (Lemma 5.5) if the curvature at \((p_0, t_0)\) is fixed factor larger than the curvature in regions previously changed by surgeries.

**Lemma 5.11** (cf \cite[Lemma 7.10]{23}). Consider a flow satisfying the assumptions of the Neck Detection Lemma (Lemma 5.5). Let \(d^\#\) be the value given by Lemma 5.2 and consider \(\varepsilon, k, l, \theta\) with \(\theta \leq d^\#\). Then there exists \(\eta_0, H_0\) so that if \((p_0, t_0)\) satisfies the bounds

\[
|H(p_0, t_0)| \geq \max\{H_0, 5K\}, \quad \frac{|A|^2}{|H|^2}(p_0, t_0) \geq \frac{1}{n - 1} - \eta_0,
\]

where \(K\) is the maximum of the curvature in the regions changed by surgeries at times before \(t_0\). Then \((p_0, t_0)\) satisfies (ND2) and hence the conclusions of the Neck Detection Lemma, Lemma 5.5 are true, that is

(i) \(\hat{\mathcal{P}}(p_0, t_0, L, \theta)\) is an \((\varepsilon, k_0 - 1, L, \theta)\)-shrinker neck and  
(ii) \(\mathcal{P}(p_0, t_0, L - 1, \theta/2)\) is an \((\varepsilon, k, L, \theta/2)\)-shrinker neck.

Furthermore

\[
\mathcal{P} \left( p_0, t_0, \frac{n - 1}{H(p_0, t_0)}L, \left( \frac{n - 1}{K} \right)^2 d^\# \right)
\]

is surgery free. (Note that this set is strictly larger than the neighbourhood of (ND2).
Proof. This proof is a consequence of the Neck Detection Lemma (Lemma 5.5) and hence follows the proof of [23, Lemma 7.10]. □

The hypotheses of the above theorem ensure we can exclude surgeries in the parabolic neighbourhood. If the hypotheses are not satisfied then we may not be able to exclude the presence of surgeries.

Definition 5.12. We say that a parabolic neighbourhood \( \hat{\mathcal{P}}(p_0, t_0, r, \tau) \) is adjacent to a surgery region if it has not been changed by surgeries but there exists a \( p \in \mathcal{M} \) such that \( d_{g(t_0)}(p, p_0) = r \) and which belongs to the boundary of a region changed by surgery at a time \( s \in [t_0 - \tau, t_0] \). We say that a submanifold neck \( \mathcal{N} \subset \mathcal{M} \) is bordered on one side by a disc if one of the two components of \( \partial \mathcal{N} \) is also the boundary of a closed domain \( \mathcal{D} \subset \mathcal{M} \) which is diffeomorphic to a disc and has no interior points in common with \( \mathcal{N} \).

In the next result we assume our flow with surgeries satisfies certain properties, these properties will in fact be consequences of the surgery algorithm defined in the next section:

(s1) There exists a \( K^* > 2H^\# \), where \( H^\# \) is the constant that appears in Corollary 4.4 such that each surgery is performed at \( \Sigma_{z_0} \) which is the cross section of a normal neck with \( r(z_0) = r^* = \frac{n-1}{K^*} \). Hence the surgeries take place in a region where the curvature is approximately \( K^* \).

(s2) The portions of the normal neck have the following two properties

(a) One part belongs to a component that will be discarded after surgery,

(b) In the other part, the portion that is unchanged by surgery has the following structure: on the first cross section (which borders the region changed by surgery) the mean radius satisfies \( r(z) \leq \frac{11}{10}r^* \), the final section \( r(z) \geq 2r^* \) and an intermediate section between the two where \( r^* \leq r(z) \leq 2r^* \).

(s3) Each surgery removes a region of the submanifold with curvature greater than \( 10K^* \). That is, if we consider the surgeries performed at any surgery time, there is a component that is discarded afterwards which contains a point \( p \) with curvature satisfying \( |H(p, t)| \geq 10K^* \). The surgery disconnects this point from the rest of the submanifold.

Remark 5.13.

- Property (s1) implies points modified by surgery satisfy \( \frac{K^*}{2} \leq |H(p, t)| \leq 2K^* \) after surgery. Furthermore, the estimate \( r(z) \leq \frac{11}{10}r^* \) in (s2) is in fact implied by (s1).
- (s2) means that the surgeries are not in fact performed at the end of the necks but are instead performed at a fixed distance so that there is a final unmodified part remaining where the radius doubles. The presence of a long part of the neck is a useful technical device which we will use in the proof of the following Lemma.
- Property (s3) shows surgery reduces the maximum of the mean curvature. However by (s1), the surgery procedure itself does not remove the high curvature regions as the surgery procedure modifies the parts that have curvature approximately \( K^* \). Rather the regions of high curvature become disconnected after the surgery and are
removed because they have known topology. In this sense, surgery is not a purely topological procedure but an analytical method to reduce the curvature.

As we have seen, the hypothesis \((ND2)\) is essential in applying the Neck Detection Lemma. However, we can not always guarantee that we have a surgery free region. In this case, the following Lemma tells us since we have good estimates arising from the surgery procedure, we can topologically recognise the region and still control the maximum of the mean curvature.

\textbf{Lemma 5.14.} Consider a mean curvature flow with surgeries in \(C(r, \alpha)\) that is cylindrically bounded. Furthermore, suppose that \((s1)-(s3)\) above hold. Let \(L, \theta > 0\) so that \(\theta \leq d^\#\) where \(d^\#\) is as in Lemma 5.2 and \(L \geq 20\). There there exists \(\eta_0, H_0\) such that the following hold:

(a) Let \((p_0, t_0)\) satisfy the hypotheses \((ND1)\) and \((ND2)\) of the Neck Detection Lemma, Lemma 5.5,

(b) The parabolic neighbourhood \(\hat{P}(p_0, t_0, L, \theta)\) is adjacent to a surgery region.

Then \((p_0, t_0)\) lies at the centre of a submanifold neck \(\mathcal{N}\) of length at least \(L - 3\), which is bordered on one side by a disc \(\mathcal{D}\). And at time \(t_0\) we have the bounds

\[ |H(q, t_0)| \leq 5K^* \quad q \in \mathcal{N} \cup \mathcal{D}, \]

where \(K^*\) is the constant from \((s1)\).

\textit{Proof.} This proof follows the proof of Proposition 7.12 in [23] which essentially uses the Neck Detection Lemma.

The following theorem and proposition are independent of mean curvature flow and concern the compactness of manifolds (and submanifolds) subject to a curvature inequality.

\textbf{Theorem 5.15 (Bonnet-Myers, Hopf-Rinow).} Let \(\mathcal{M}\) be a complete Riemannian manifold and suppose that \(p \in \mathcal{M}\) such that the sectional curvature satisfies \(K > K_{\min}\) along all geodesics of length \(\frac{\pi}{\sqrt{K_{\min}}}\) from \(p\) or equivalently \(K > K_{\min}\) in a neighbourhood \(d(p, q) \leq \frac{\pi}{\sqrt{K_{\min}}}\).

\textbf{Proposition 5.16 (B.Y Chen [10]).} For \(n \geq 2\) if \(\mathcal{M}^n\) is a submanifold of \(\mathbb{R}^{n+m}\) then at every \(p \in \mathcal{M}^n\),

\[ K_{\min} \geq \frac{1}{2} \left( \frac{1}{n-1} |H(p)|^2 - |A(p)|^2 \right). \]

The following theorem shows either the submanifold is compact or there exists a neck region.

\textbf{Theorem 5.17.} Let \(F : \mathcal{M} \to \mathbb{R}^{n+m}\), be a smooth connected immersed submanifold. Suppose that there exists \(c^\#, H^\# > 0\) such that \(|\nabla H(p)| \leq c^\# |H(p)|^2\) for all \(p \in \mathcal{M}\) where
$|H(p)| \geq H^\#$. Then $\forall \eta_0 > 0$ there exists $\alpha_0 = \alpha_0(c^\#, \eta_0), \gamma_0 = \gamma_0(c^\#, \eta_0)$, if $p \in \mathcal{M}$ satisfies

$$\frac{|A|^2}{|H|^2(p)} < \frac{1}{n - 1} - \eta_0, \quad |H(p)| \geq \gamma_0 H^\#.$$  

then either $\mathcal{M}$ is closed with

$$\frac{|A|^2}{|H|^2} < \frac{1}{n - 1} - \eta_0 \quad \text{everywhere}$$

or there exists a $q \in \mathcal{M}$ such that

(1) $$\frac{|A|^2}{|H|^2(q)} \geq \frac{1}{n - 1} - \eta_0$$

(2) $$d(p, q) \leq \frac{\alpha_0}{|H(p)|}$$

(3) $$|H(q')| \geq \frac{|H(p)|}{\gamma_0}, \quad \forall q' \in \mathcal{M} \mid d(p, q') \leq \frac{\alpha_0}{|H(p)|}$$

in particular $|H(q)| \geq \frac{|H(p)|}{\gamma_0}$.

**Proof.** Given $\alpha_0$, we set $\gamma_0 = 1 + c^\# \alpha_0$. Then for a given $p \in \mathcal{M}$, we let

$$\mathcal{M}_{p, \alpha_0} = \left\{ q \in \mathcal{M} \mid d(p, q) \leq \frac{\alpha_0}{|H(p)|} \right\}.$$  

Then we have that if $|H(p)| \geq \gamma_0 H^\#$, then

$$|H(q)| \geq \frac{|H(p)|}{1 + c^\# d(p, q)|H(p)|} \geq \frac{|H(p)|}{\gamma_0}.$$  

We show if $\alpha_0$ is sufficiently large then this implies $\mathcal{M}$ is compact. Suppose that we have

$$\frac{|A|^2}{|H|^2} < \frac{1}{n - 1} - \eta_0, \quad \forall q \in \mathcal{M}_{q, \alpha_0}.$$  

We have on $\mathcal{M}_{q, \alpha_0}$

$$K_{\min} \geq \frac{1}{2} \left( \frac{1}{n - 1} |H(p)|^2 - |A(p)|^2 \right) \geq \frac{1}{2} \eta_0 |H(p)|^2 \geq \frac{\eta_0}{2} \frac{H^2_{\min}}{H^2} > 0.$$  

Applying the above lemma, if we choose $\alpha_0 = \sqrt{\frac{2\eta_0}{\pi}}$ then we have

$$K_{\min} > 0 \quad \text{in a neighbourhood } d(p, q) \leq \frac{\alpha_0}{|H(p)|} \leq \frac{\pi}{K_{\min}}$$

which shows $\mathcal{M}$ is compact.
The following corollary shows that before the first singular time either a cylindrical neck exists or the flow becomes spherically pinched.

**Corollary 5.18.** Let $M_t$ be a smooth mean curvature flow of a closed submanifold with $|A|^2 \leq c_n|H|^2$. Given neck parameters $\varepsilon, k, L$ there exists $H^*$ depending on the initial data such that $\sup_{p \in M_t_0} |H(p, t_0)| \geq H^*$ then the submanifold at time $t_0$ either contains an $(\varepsilon, k, L)$-cylindrical neck or is spherically pinched, that is

$$|A|^2 - \frac{1}{n-1}|H|^2 < 0.$$

Finally we have the following results on the structure of the submanifold as we leave a neck region. Since in regions of high curvature, we know that we are quantitatively close to a hypersurface over the entire region, the proof follows that of [23, Lemma 7.19]. Let $N$ be an $(\varepsilon, k)$ submanifold neck with axis $\omega$. We define the trajectories

$$\gamma = \frac{\omega^T}{|\omega|^2} \quad \text{for } y \geq 0,$$

$$\gamma(0) = p.$$

**Corollary 5.19 ([23]).** Let $N$ be an $(\varepsilon, k)$ submanifold neck with axis $\omega$ and let $c^\#, H^#$ be as in Lemma [4.3] and let us set $\Theta = 1+(2+\pi)(n-1)c^\#$. Let us define the trajectories $\gamma(p, y)$. Suppose that for some $0 \leq y_1 < y_2 < y_{\max}$ we have $|A|^2(\gamma(p, y)) - \frac{1}{n-1}|H|^2(\gamma(p, y)) < 0$ for all $y \in [y_1, y_2], p \in \Gamma_0$ and that $\omega(0) \cdot \nu^+(p) > 0$ for all $p \in \Sigma_{y_1}$. Suppose that $\Sigma_{y_1}$ has a diameter equal to $\frac{2(n-1)}{K}$ for some $K \geq \Theta H^#$ and that $|H(p)| \geq K$ for all $p \in \Sigma_{y_1}$. Then we have that

$$|H(\gamma(p, y))| \geq \frac{K}{\Theta}, \quad \forall y \in [y_1, y_2], p \in \Sigma_0.$$

6. **The Flow with Surgeries**

This section will be devoted to the proof of the following result :

**Theorem 6.1.** Let $M_0 \in C(R, \alpha)$ be a smooth closed cylindrically bounded submanifold immersed in $\mathbb{R}^n$, that it satisfies (1.15) with $n \geq 5$ and satisfies $|A|^2 \leq R^{-2}$. Then there exists constants $H_1 < H_2 < H_3$ and a mean curvature flow with surgeries starting from $M_0$ with the following properties

- each surgery occurs at a time $T_i$ such that $\max |H(\cdot, T_i-)| = H_3$
- after each surgery, the curvature drops by a finite amount, that is all the components of the manifold satisfy $\max |H(\cdot, T_i+)| \leq H_2$. We note that after surgery there are components that are diffeomorphic to spheres or to $S^{n-1} \times S^1$ with larger curvature but these are discarded afterwards.
- each surgery on a normal submanifold neck starts at a cross section where mean radius satisfies $r(z_0) = \frac{(n-1)}{H_3}$. The mean curvature flow with surgery is terminated after finitely many steps. The constants $H_i$ are arbitrary except that they satisfy $H_1 \geq \omega_1 R^{-1}, H_2 = \omega_2 H_1$ and $H_3 = \omega_3 H_2$ with $\omega_i > 1$ depending only on the parameters $\alpha_i$. 
We note that since we have the gradient estimates and consequently the neck detection lemma, the proof now closely follows that of [23] except that we replace conditions on the first eigenvalue $\lambda_1$.

**Choice of Parameters**

(P0) **Choice of Neck Parameters:** In the previous sections, we have defined a surgery procedure on $(\varepsilon_0, k_0)$-submanifold necks in normal form of length $L$, where $\varepsilon_0$ must be suitably small (but depending only on the dimension), $k_0 \geq 2$ is any integer and $L \geq 10 + 8\Lambda$, where $\Lambda$ is the length parameter in the surgery. We also assume $L \geq 20 + 8\Lambda$ and that $\varepsilon_0$ is small enough so that if $\mathcal{N}$ is a normal $(\varepsilon_0, 1)$-submanifold neck of length $2L$ then the mean curvature at any two points of $\mathcal{N}$ can differ by a factor at most 2.

(P1) We define $c^\#$, $H^\#$ as in Corollary 4.4 and $d^\#$ as Lemma 5.2.

(P2) **First Application of Neck Detection Lemma:** We choose $\eta_0, K_0$ such that if $(p, t_0)$ satisfies

\[(6.1) \quad |H(p, t_0)| \geq K_0, \quad |A|^2(p, t_0) - \frac{1}{n - 1}|H(p, t_0)|^2 \leq -\eta_0|H(p, t_0)|^2,\]

and if $\hat{\mathcal{P}}(p, t_0, L', \theta')$ does not contain surgeries for some $L' \in [L/4, L], \theta' \in [d^\# / 1400, d^\#]$ then $\hat{\mathcal{P}}(p, t_0, L', \theta')$ is a shrinking neck and $(p, t_0)$ lies at the centre of a normal $(\varepsilon_0, k_0)$-submanifold neck of length at least $2L' - 2$. We also require that if $\eta_0, K_0$ are such that if $(p, t_0)$ satisfy (6.1) then and in addition $|H(p, t_0)| \geq 5K$ where $K$ is the maximum of the mean curvature in the regions inserted by the surgeries then the conclusions of Lemma 5.11 apply. Finally we also require that $\eta_0, K_0$ are such that Proposition 5.14 can be applied to the parabolic neighbourhood $\hat{\mathcal{P}}(p, t_0, L', \theta')$ for the values of $\theta', L'$ chosen above.

(P3) **Second Application of Neck Detection Lemma:** Next, we set $\varepsilon_1 = \frac{(n-1)\eta_0}{2}$. We apply Corollary 5.9 to find $\eta_1, K_1$ such that if $(p, t_0)$ satisfies

\[(6.2) \quad |H(p, t_0)| \geq K_1, \quad |A(p, t_0)|^2 - \frac{1}{n - 1}|H(p, t_0)|^2 \leq -\eta_1|H(p, t_0)|^2\]

and the parabolic neighbourhood $\hat{\mathcal{P}}(p, t_0, 10, d^\# / 1400)$ does not contain surgeries, then $(p, t_0)$ lies at the centre of a cylindrical submanifold of length 5 and $C^1$ norm less that $\varepsilon_1$. We will choose $\eta_1, K_1$ such that $K_1 \geq K_0, K_1 \geq H^\#$ and $\eta_1 \leq \eta_0$.

(P4) **Application of the Pinching Theorem 5.17:** Now we choose $\gamma_0$ such that $|H(p, t_0)| > \gamma_0 H^\#$ and $|A(p, t_0)|^2 - \frac{1}{n - 1}|H(p, t_0)|^2 \leq -\eta_1|H(p, t_0)|^2$ then either

- $|A|^2 - \frac{1}{n - 1}|H|^2 \leq \eta_1|H|^2$ everywhere on $\mathcal{M}_{t_0}$ or
- there exists a $q$ such that
  \[|A(q, t_0)|^2 - \frac{1}{n - 1}|H(q, t_0)|^2 \geq -\eta_1|H(q, t_0)|^2\]

and

- $|H(q', t_0)| \geq |H(p, t_0)|/\gamma_0$ for all $q'$ with $d_{g(t_0)}(q', p) \leq d_{g(t_0)}(q, p)$. 

(P5) **Third Application of the Neck Detection Lemma:** Let us set $\theta_2 = (10^4 n^3 \Theta^2 \gamma_0^2)^{-1}$. Then let us choose $K_2, \eta_2$ such that if $|H(p, t_0)| \geq K_2$ and if $|A(p, t_0)|^2 - \frac{1}{n-1} |H(p, t_0)|^2 \geq \eta_2 |H(p, t_0)|^2$ and if $\mathcal{P}(p, t_0, 10, \theta_2)$ does not contain surgeries then $(p, t_0)$ lies on a cylindrical submanifold of length 5 and $C^1$ norm length less than $\varepsilon_1$. We also require that $K_2 \geq K_1$.

(P6) We finally define $H_1$ to be any value such that $H_1 \geq 4 \Theta K_2$ and then define $H_2, H_3$ by

$$H_2 = 10\gamma_0 H_1, \quad H_3 = 10 H_2.$$ 

To fix $H_1$ and hence the other $H_i$, we choose $H_1 = 4 \Theta K_2$, but in principle we can choose $H_i$ to be arbitrarily large.

**Remark 6.2.** An interesting question is what happens if allow $H_i \to \infty$. In the papers, [25] and [19], it was shown that the as the surgery parameters approach infinity, the mean curvature flow with surgery approaches the level set flow.

All parameters introduced above depend only on the parameters $\alpha, R$ describing the initial submanifold. More precisely, the curvature parameters $H_i, K_i, H^\#$ can be written as constants depending only in $\alpha$ multiplied by $R^{-1}$, while the remaining parameters depend only on $\alpha$.

Note that in principle, surgery times may in fact accumulate, that is there is no uniform bound on the time separating successive applications of the surgery procedure. Therefore, we will construct our surgery algorithm so that the following properties are satisfied

(S) Each surgery is performed on a normal $(\varepsilon_0, k_0)$-submanifold neck. The surgeries are performed at times $T_i$ such that $\max |H(\cdot, T_i)| = H_3$. After the surgeries are performed at times $T_i$ and suitable components whose topology is known are removed, we have $\max |H(\cdot, T_i^+)\leq H_2$. In addition, all surgeries satisfy properties (s1) – (s3) with $K^* = H_1$.

Note that property (S) implies the difference between two consecutive surgery times satisfies the uniform lower bound

$$T_{k+1} - T_k \geq \frac{10^2 - 1}{10^2} \frac{1}{2 n H_2^2} > \frac{49}{10^4 n^2 \gamma_0^2 H_2^2}.$$ 

The crucial step in the proof of Theorem 6.1 is the next result which is known as the **Neck Continuation Theorem.** It essentially states that the neck given by the Neck Detection Lemma can be continued until either the submanifold ends in a spherical cap or the curvature decreases by a certain amount. The key point here is that if the curvature is sufficiently large, then we know by some estimates of Naff, that the second fundamental form in the directions normal to the mean curvature vector are close to zero. This in turn shows the submanifold is nearly codimension one with a normal frame that has small torsion.

**Theorem 6.3** (Neck Continuation, cf [23] Theorem 8.2). Suppose that $M_t, t \in [0, t_0]$ is a mean curvature flow with surgeries satisfying property (S) and let $\max_{M_t_0} |H| \geq H_3$. Let
Let $p_0$ be a point where
\begin{equation}
|H(p_0, t_0)| \geq 10H_1, \quad |A|^2(p_0, t_0) - \frac{1}{n-1}|H(p_0, t_0)|^2 \geq -\eta_1|H(p_0, t_0)|^2
\end{equation}
where $\eta_1, H_1$ are surgery parameters defined in (P0)-(P6). Then $(p_0, t_0)$ lies on some $(\varepsilon_0, k_0)$-submanifold neck $N_0$ in normal form (with $(\varepsilon_0, k_0)$ defined in (P0)) which either covers the whole component of $M_{t_0}$ including $p_0$ or has a boundary consisting of two components $\Sigma_1, \Sigma_2$ each of which satisfies one of the following properties

i) the mean radius of $\Sigma$ is $\frac{2(n-1)}{H_1}$ or

ii) the cross section $\Sigma$ is the boundary of a region $D$ diffeomorphic to a disc where the curvature is at least $H_1/\Theta$. The region $D$ lies “after” the cross section $\Sigma$ that is disjoint from $N_0$.

Proof. Let us take $p_0$ such that (6.3) is satisfied. Then from our definitions we have the following bounds
\begin{equation*}
|H(p_0, t_0)| \geq 10K_1 \geq 10K_0, \quad |A|^2(p_0, t_0) - \frac{1}{n-1}|H(p_0, t_0)|^2 \geq -\eta_1|H(p_0, t_0)|^2 \geq -\eta_0|H(p_0, t_0)|^2.
\end{equation*}
Hence at $(p_0, t_0)$ we can apply neck detection at two scales: the “finer” $\varepsilon_1$-level and the “coarser” $\varepsilon_0$-level.

Let us begin our analysis at the “coarser” $\varepsilon_0$ level. Our procedure performs surgeries on neck regions where the curvature is close to $H_1$ so $K = 2H_1$ is a bound from above for the regions modified by surgery. We then apply Lemma 5.11 to ensure the parabolic neighbourhood $\mathcal{P}(p_0, t_0, L, d^\#)$ does not contain surgeries and the point $(p_0, t_0)$ lies at the centre of a normal $(\varepsilon_0, k_0)$-submanifold neck of length at least $2L - 2$. We then consider the maximal neck $N_0$-submanifold neck that contains $p_0$. If $N_0$ is the entire manifold then we are done. If not, then we must follow our neck parameter $z$ in both directions from $p_0$ until we find cross sections of $N_0$ that satisfy either i) or ii) of the theorem.

To this end, let us normalise our neck parameter so that $p_0$ lies in the $z = 0$ cross-section of the neck. In the following, we follow the neck in the increasing $z$ direction. A similar analysis will prove the same conclusion in the decreasing $z$ direction.

If there is a cross section with $r(z) = \frac{2(n-1)}{H_1}$, then we are done. Therefore suppose not, that is there is no such cross section, that is every section satisfies $r(z) < \frac{2(n-1)}{H_1}$, $z \in [0, z_{\text{max}}]$ for $z_{\text{max}}$ is the last value of the cross section of the neck. This implies $|H| > \frac{H_1}{4}$ along the neck. We need to show alternative ii) holds in this case, that is the neck is bordered by a disc.

The strategy of the proof follows [23, Theorem 8.2]. The neck detection lemma ensures the neck $N_0$ can be continued as long as then there exists points satisfying the following

(i) the curvature $|H|$ is large,

(ii) the quantity $\frac{|A|^2}{|H|^2} - \frac{1}{n-1}$ is small and negative,

(iii) a suitable parabolic neighbourhood of the point is surgery free.

Now since the neck $N_0$ must end, one of these three properties must be violated. The first one is ensured by the inequality $|H| > \frac{H_1}{4}$ hence it must one of the other two. If (ii) is
violated, then the neck closes up and ends in a spherically quadratically pinched neck. On the other hand if \((iii)\) is pinched, then we can show the neck is bordered by a cap inserted by a previous surgery.

We define neck continuable points, that is points where we can apply the neck detection lemma. Let \(\Omega\) be the set of neck continuable points where \(p \in \Omega\) if

\[
\begin{align*}
(\Omega_1) & \quad |A|^2(p, t_0) - \frac{1}{n-1}|H|^2(p, t_0) \geq -\eta_0|H|^2(p, t_0) \\
(\Omega_2) & \quad \text{the parabolic neighbourhood,}
\end{align*}
\]

\[
\mathcal{P} \left( p, t_0, \frac{n-1}{|H(p, t_0)|}, \frac{(n-1)^2}{(10H_1)^2}d^\# \right)
\]

does not contain surgeries.

We will show the points of \(\Omega\) satisfy the hypotheses of the neck detection lemma hence the neck can be continued as long as it contains points of \(\Omega\). Hence it follows, since the neck \(N_0\) ends, the last part of \(N_0\) does not contain any points one \(\Omega\).

Let \(p\) be a point that satisfies \((\Omega_1)\) but not \((\Omega_2)\) then by Lemma 5.11 we see \(|H(p, t_0)| < 10H_1\). It follows that \(p_0 \in \Omega\). Let us now follow the neck from \(p_0\) in the direction of increasing \(z\). Such points \(p \in N_0\) satisfy

\[
\frac{(n-1)^2}{(40)^2|H(p, t_0)|^2} \leq \frac{(n-1)^2}{(10H_1)^2}
\]

which implies

\[
\hat{\mathcal{P}}(p, t_0, L, d^#/40^2) \subset \mathcal{P} \left( p, t_0, \frac{n-1}{|H(p, t_0)|}, \frac{(n-1)^2}{10H_1^2}d^\# \right)
\]

Therefore we have from surgery property \((P2)\) that any \(p \in \Omega\) lies at the centre of a normal \((\varepsilon_0, k_0)\)-submanifold neck of length \(2L - 2\). Thus, since the neck ends when \(z = z_{\max}\), there must be an end section of the neck that does not contain points of \(\Omega\), that is there are points \(p\) for which \(z \in (z_{\max} - (L - 1), z_{\max}]\) and \(p \notin \Omega\).

Hence we can consider the “final point” of \(\Omega\), that is let \(z^*\) be the maximal value of \(z\) such that the cross section \(N_0\) with coordinate \(z^*\) contains a point \(p_1 \in \Omega\) while there are no points of \(\Omega\) for \(z \in (z^*, z^* + 10)\).

We now consider two cases

(a) There is at least one point \(p_2\) with \(z \in (z^*, z^* + 10)\) which satisfies \((\Omega_1)\),

(b) All points with \(z \in (z^*, z^* + 10)\) do not satisfy \((\Omega_1)\), that is

\[
|A|^2(p, t_0) - \frac{1}{n-1}|H|^2(p, t_0) \leq -\eta_0|H|^2(p, t_0)|
\]

We start with case \((a)\). This means that there are points that do not satisfy \((\Omega_2)\), that is points that have been modified by surgery. Hence we can apply Proposition 5.14. Hence we need to show the hypotheses of that proposition are satisfied:

Firstly, we see \(p_2\) does not satisfy \((\Omega_2)\), that is the parabolic neighbourhood

\[
\mathcal{P} \left( p_2, t_0, \frac{n-1}{|H(p_2, t_0)|}, \frac{(n-1)^2}{(10H_1)^2}d^\# \right)
\]
has been modified by surgery. Surgery property \((P0)\) shows the curvature varies by at most a factor of two in the surgery modified region in the neck between \(p_1\) and \(p_2\). Therefore, we have

\[
|H(p_2, t_0)| \geq \frac{1}{2}|H(p_1, t_0)|,
\]

\[
d_{g(t_0)}(p_1, p_2) < 2(\pi + 10)\frac{n - 1}{|H(p_2, t_0)|} < \frac{(n - 1)L}{4|H(p_2, t_0)|}.
\]

This implies

\[
\mathcal{P}\left(p_2, r_0, \frac{n - 1}{|H(p_2, t_0)|} \frac{L}{4}, (10H_1)^2d^#\right) \subset \mathcal{P}\left(p_1, t_0, \frac{n - 1}{|H(p_1, t_0)|} L, (n - 1)^2d^#\right).
\]

As \(p_1 \in \Omega\), the neighbourhood at the right hand side does not contain surgeries and hence neither does the left hand side. By continuity, there exists a \(L' \in [L/4, L]\) so that

\[
\mathcal{P}\left(p_2, t_0, \frac{n - 1}{|H(p_2, t_0)|} L', (n - 1)^2 d^#\right)
\]
does not contain surgeries but is adjacent to a surgery on the side of increasing \(z\). We let

\[
\theta' = \frac{|H(p_2, t_0)|^2}{(10H_1)^2} d^#
\]

then the neighbourhood may be written as \(\hat{\mathcal{P}}(p_2, t_0, L', \theta')\). Since \(H_1/4 \leq |H(p_2, t_0)| \leq 10H_1\) we have \(d^#/40^2 \leq \theta' \leq d^#\). Therefore by surgery property \((P2)\), we can apply Proposition \([5.14]\) to conclude that \((p_2, t_0)\) lies in a submanifold neck \(\mathcal{N}\) that is bordered on one side by a disc \(\mathcal{D}\). Furthermore Proposition \([5.14]\) also shows the mean curvature on \(\mathcal{N} \cup \mathcal{D}\) satisfies \(|H| < 10H_1\). Hence the submanifold neck \(\mathcal{N}\) maybe combined with \(\mathcal{N}_0\) to form a unique neck. Finally the side bordered by \(\mathcal{D}\) must be in the direction of increasing \(z\): otherwise \(\mathcal{N}\) would include all the neck \(\mathcal{N}_0\) and this can not happen because \(\mathcal{N}_0\) contains the point \(p_0\) which satisfies \(|H(p_0, t_0)| \geq 10H_1\). This concludes the proof in case \((a)\).

We now consider case \((b)\). We can assume all points in \(\mathcal{N}_0\) with \((z^*, z^* + 10)\) satisfy

\[
|A|^2(p, t_0) - \frac{1}{n - 1}|H|^2(p, t_0) \leq -\eta_0|H|^2(p, t_0)|.
\]

We will show this quadratic spherical pinching condition ensures the neck will start to close up and end in a spherical cap. In order to analyse the end of the neck : let \(\overline{z} \in [0, z^*]\) be the largest value of \(z\) such that the corresponding cross section contains a point \(\overline{q}\) that satisfies

\[
|A|^2(q) - \frac{1}{n - 1}|H|^2(q) > -\eta_1|H|^2(q).
\]

We claim that the parabolic neighbourhood \(\hat{\mathcal{P}}\left(\overline{q}, t_0, 10, \frac{d^#}{1600}\right)\) does not contain surgeries. In fact from our definition, there is a point \(q \in \Omega\) with \(z\) co-ordinate in \([\overline{z} - 10, \overline{z}]\). Then
it is easy to check
\[ \tilde{\mathcal{P}} \left( q, t_0, 10, \frac{d^\#}{1600} \right) \subset \mathcal{P} \left( q, t_0, \frac{n-1}{|H|(q, t_0)} L, \frac{(n-1)^2}{(10H_1)^2} d^\# \right) \]
which does not contain surgeries by definition of \( \Omega \). Hence we know that from (P3) that
these exists a region \( \mathcal{G} \subset \mathcal{N}_0 \) centered at \( \bar{\gamma} \) which is \( C^1 \) close in norm to a standard
embedding of a cylinder with norm less than \( \varepsilon_1 \).

We let \( \omega \) be a unit vector parallel to the axis of our cylinder \( \mathcal{G} \). We assume \( \mathcal{G} \) is parallel
to the \( y \)-axis and that we set \( y = x_{n+1} \) to denote the \((n+1)\) co-ordinate. Let us normalise \( y \)
so that \( F(\bar{\gamma}, t_0) \) lies on the \( \{ y = 0, x_{n+2} = 0, \ldots, x_{n+k} = 0 \} \) subspace. We call the \( \Sigma_0 \)
the intersection of this subspace and \( \mathcal{G} \). For any \( p \in \Sigma_0 \) we consider the curve \( y \mapsto \gamma(y, p) \)
which solves
\[ \frac{d}{dt} \gamma = \frac{\omega^T}{|\omega^T|^2} \text{ for } y \geq 0. \]
We let \( y_{\max} \) denote the supremum of values for which \( \gamma(y, p) \) is defined for all \( p \in \Sigma_0 \). We
set \( \Sigma_y = \{ \gamma(y, p) : p \in \Sigma_0 \} \) for \( 0 \leq y < y_{\max} \). In addition given \( 0 \leq y_1 < y_2 < y_{\max} \), we set
\[ \Sigma(y_1, y_2) = \bigcup \{ \Sigma_y : y_1 \leq y \leq y_2 \}. \]
Let us denote by \( \mathcal{N}'_0 \) the part of \( \mathcal{N}_0 \) corresponding to \( z \in [\bar{\gamma}, z^* + 10] \). The \( z = \bar{\gamma} \) cross
section contains the point \( \bar{\gamma} \) and so it is very close to \( \Sigma_0 \). By definition of \( \bar{\gamma} \) we have
\[ |A|^2 - \frac{1}{n-1} |H|^2 \leq -\eta_0 |H|^2 < 0 \]
and hence the region \( \mathcal{N}'_0 \) is a quadratically spherically pinched region. Furthermore the axis
of \( \mathcal{N}'_0 \) is approximately \( \omega \) everywhere. In addition, the trajectories \( \gamma \) are defined as long as
they remain within \( \mathcal{N}'_0 \). Hence there exists a least value \( y' < y_{\max} \) such that \( \gamma(y', p) \in \partial \mathcal{N}'_0 \)
for some \( p \in \Sigma_0 \). By construction we see \( |\nu^+(p) \cdot \omega| < \varepsilon_1 \) for all \( p \in \Sigma_0 \) since \( \Sigma_0 \) is contained
in the cylindrical submanifold \( \mathcal{G} \). Now on the cylindrical submanifold, we have along the
curves \( \gamma \)
\[ \frac{d}{dy} (\nu^+, \omega) > 0. \]
and in particular we have \( \nu^+(p) \cdot \omega \geq -\varepsilon_1 \) for all \( p \in \Sigma(0, y') \). Now we use the property
that \( |A|^2 - \frac{1}{n-1} |H|^2 \leq -\eta_0 |H|^2 \) on the cross sections corresponding to \( z \in [z^*, z^* + 10] \).
Let us set \( r^* = r(z^*) \) to denote the mean radius of the \( z^* \) section and let \( H^* = \frac{H_1}{2} \). By
assumption \( H^* = \frac{H_1}{2} \), and since the neck is close to \( \omega \) the \( y \) co-ordinate is almost on each
section. Thus the \( y \) co-ordinates on the \( z = z^* \) section and on the \( z = z^* + 10 \) section
differ by approximately \( 1 - r^* \). It follows that at least the points of \( \Sigma(y' - 4r^*, y') \) have
a \( z \) co-ordinate such that \( z \in [z^*, z^* + 10] \). Since \( |H| \) varies slowly on a neck we have
\( |H| \geq H^*/2 \) on \( \Sigma(y' - 5r^*, y') \). Along the curve \( \gamma \) we have of \( y \in [y' - 5r^*, y'] \)
\[ \frac{d}{dy} (\nu^+, \omega) = \sum_{i=1}^n \langle \gamma', e_i \rangle h_i^{k+}(e_j, \omega) \]
\[
+ \sum_{i=1}^{n} \sum_{\beta=2}^{k} T_{i+}^{\beta} \langle \nu_{\beta}, \omega \rangle \langle \gamma', e_i \rangle \\
\geq \eta_0 \frac{H^*}{2}.
\]

Hence for any \( p' \in \Sigma_{y'} \) that is \( p = \gamma(y', p) \) for some \( p \in \Sigma_0 \), we have
\[
\langle \nu^+(p), \omega \rangle = \langle \nu(\gamma(y' - 5r^*, p)), \omega \rangle + \int_{y' - 5r^*}^{y'} \frac{d}{dy} \langle \nu^+, \omega \rangle dy \\
\geq -\epsilon_1 + 5r^* \eta_0 \frac{H^*}{2} \geq -4\epsilon_1.
\]

We now show our neck region ends in a spherically pinched cap region. Roughly speaking, this means that the region that lies past \( y' \) has the structure of a cap. In order to show this, we follow the trajectories of the curves \( \gamma(p, y) \) for \( y > y' \). The region swept out by these curves is generally non longer neck-like as \( y \) grows. However, the curves are well defined until some maximal value \( y_{\text{max}} \). Such a value exists since our submanifold is compact. In order to analyse the post-neck region, we will show for \( y \in [y', y_{\text{max}}) \) the following holds

a) \( |\nu^+ \cdot \omega| < 1 \),

b) \( |A|^2 - \frac{1}{n-1} |H|^2 < -\eta |H|^2 \),

c) \( |H| > \frac{H + 1}{4\Theta} \),

d) \( \nu^+ \cdot \omega > \epsilon_1 \).

Now property a) holds for all \( y \in [y', y_{\text{max}}) \) by construction. Furthermore if any of the inequalities fails, then there is a first \( y' \in (y', y_{\text{max}}) \) such that either b), c) or d) becomes an equality. Let us assume b) fails, that is there exists \( p' \in \Sigma_{y'} \) such that
\[
\left( |A|^2 - \frac{1}{n-1} |H|^2 \right) (p') \leq -\eta |H|^2 (p').
\]

Then from our surgery construction, we have
\[
\theta_2 \frac{(n-1)^2}{|H(p', t_0)|^2} \leq \theta_2 \frac{16(n-1)^2 \Theta^2}{H_1^2} < \frac{16}{10^4 n \gamma_0^2 H_1^4}.
\]

The waiting time estimate for the surgeries then ensures the parabolic neighbourhood \( \hat{P}(p', t_0, 10, \theta_2) \) is surgery free. Therefore by the Neck Detection Lemma, part of the submanifold must be close to a standard embedding of a round cylinder with error in \( C^1 \) norm less than \( \epsilon_1 \). Let \( \tilde{\omega} \) be the axis of this cylinder and \( \tilde{\omega} \neq \omega \) otherwise this would contradict d). In any case let \( v = \omega - \langle \omega, \tilde{\omega} \rangle \tilde{\omega} \). By construction \( v \) is orthogonal to \( \tilde{\omega} \) and \( |v|^2 = 1 - \langle \omega, \tilde{\omega} \rangle^2 \). Therefore, there exists a \( q' \) close to \( p' \) such that \( |v(q') + \frac{v}{|v|}| \leq \epsilon_1 \).

But we also have
\[
4\epsilon_1 < v(q') \cdot \omega = \left( v(q') + \frac{v}{|v|} \right) \cdot \omega - \frac{v}{|v|} \omega.
\]
which is a contradiction.

Next let us assume \( d \) fails. Now the evolution equation of \( \langle \nu^+, \omega \rangle \) is

\[
\frac{d}{dy} \langle \nu^+, \omega \rangle = \sum_{i=1}^{n} \langle \gamma', e_i \rangle \langle \nabla_{e_i} \nu^+, \omega \rangle.
\]

As \( \nabla_{e_i} \nu^+ = \sum_j h^{j,+}_i e_j + \sum_{\beta} T^\beta_{i+} \nu_\beta \), we get

\[
\frac{d}{dy} \langle \nu^+, \omega \rangle = \sum_{i=1}^{n} \langle \gamma', e_i \rangle h^{j,+}_i \langle e_j, \omega \rangle + \sum_{i=1}^{n} \sum_{\beta=2}^{k} T^\beta_{i+} \langle \nu_\beta, \omega \rangle \langle \gamma', e_i \rangle
\]

where \( \gamma' = \frac{\omega^T(\gamma)}{||\omega^T(\gamma)||^2}, \gamma(0) = p \), integrals curves associated to \( \omega^T \). Recall that on an \((\varepsilon, k)\)-submanifold neck there exists a \( C(n) \) such that \( |\Gamma^\beta_{i+}| \leq C(n) \varepsilon \). Note that

\[
\left( |A|^2 - \frac{1}{n-1} |H|^2 \right) \leq -\eta' |H|^2.
\]

implies \( h^{j,+}_i \geq \eta' H_1 \) so that

\[
\sum_{i=1}^{n} \langle \gamma', e_i \rangle h^{j,+}_i \langle e_j, \omega \rangle \geq \eta_1 |H|
\]

and initially on the neck region we have an \((\varepsilon, k)\)-submanifold neck so that there exists a \( C = C(n) \) such that

\[
|\Gamma^\beta_{i+}| \leq c(n) \varepsilon_1.
\]

Therefore, if \( H_1 \) is sufficiently large, we see

\[
\frac{d}{dy} \langle \nu^+, \omega \rangle = \sum_{i=1}^{n} \langle \gamma', e_i \rangle h^{j,+}_i \langle e_j, \omega \rangle + \sum_{i=1}^{n} \sum_{\beta=2}^{k} T^\beta_{i+} \langle \nu_\beta, \omega \rangle \langle \gamma', e_i \rangle
\]

\[
\geq 0.
\]

However, we have left the neck region, so we no longer know that we are a submanifold neck and in particular, that we are close to a hypersurfaces in a linear subspace of \( \mathbb{R}^{n+m} \). But since \( |H| \geq \frac{\varepsilon_1}{20} \), with \( H_1 \) again chosen sufficiently large we get \( |A_-|^2 \leq C \eta(\varepsilon_1) \) where if \( \varepsilon_1 \) is sufficiently small then \( \eta(\varepsilon_1) \) is also small. In particular, we can choose \( \varepsilon_1 \) small so that \( \eta(\varepsilon_1) \) is small so that about \( p^\# \), the submanifold is close to a hypersurface in a
linear subspace $E \simeq \mathbb{R}^{n+1} \subset \mathbb{R}^{n+m}$ together with a normal frame $\nu^+ = \frac{H}{|H|}, \nu_2, \cdots, \nu_k$ with $|T^\beta_{i,+}| \leq \eta_2(\varepsilon_1)$. Therefore we have

$$\frac{d}{dy} \langle \nu^+, \omega \rangle = \sum_{i=1}^{n} \langle \gamma', e_i \rangle h^i_{+,+} \langle e_j, \omega \rangle + \sum_{i=1}^{n} \sum_{\beta=2}^{k} T^\beta_{i,+} \langle \nu^+, \omega \rangle \langle \gamma', e_i \rangle > 0$$

at $y^\#$. Hence $\langle \nu^+, \omega \rangle > \varepsilon_1$ as required.

Finally, let us assume $c)$ fails. note that since $d)$ holds, the diameter of $\Sigma_y$ is less than $8(n-1)/H_1$. Then by $(P3)$ and $(P6)$ of the surgery procedure $H_1 > 4\Theta H^\#$ and we can apply the equivalent of Lemma 7.19 of [23] to show

$$|H(\gamma(p, y^\#))| > \frac{H_1}{4\Theta}.$$

This shows the inequalities $a), b), c)$ and $d)$ are preserved along the trajectories. The claim that $\Sigma_{y_{\text{max}}}$ reduces to a single point which follows as a consequence of standard Morse theory as in [23].

\[\square\]

**Remark 6.4.** • The fact that the region ends in a spherically pinched cap region is related to the possibility of a degenerate neck pinch. Along such a region the curvature remains large, so that we do not perform a surgery to reduce the curvature. Instead the end of the neck closes as a spherical cap. This is exactly what happens for the bowl soliton, the translating self-similar solution that is the model for a degenerate neck pinch.

**Proof of Main Theorem.** We use an iterative argument. Let us consider a flow defined on $[0, t_0]$ which is either smooth or satisfies property $(S)$ of the flow (flow with discard and neck reduction). Let $t_0$ be the next time where the curvature reaches the surgery threshold that is $|H|_{\text{max}}(t_0) = H_3$. Then we we to show at time $t_0$ we can perform a finite number of surgeries so that

(i) $(S)$ is satisfied,

(ii) The mean curvature drops, that is $|H|_{\text{max}} \leq H_2$

or the submanifold is diffeomorphic to $S^n$ or $S^{n-1} \times S$ and is discarded.

To this end, we analyse regions of large curvature. Hence let $p_0 \in \mathbb{M}^{t_0} \mid |H(p_0, t_0)| \geq H_2$. We have two cases

(a),

$$|A|^2(p_0, t_0) - \frac{1}{n-1} |H|^2(p_0, t_0) \leq -\eta_1 |H|^2(p_0, t_0).$$

Hence we may apply Neck Continuation Theorem, Theorem 6.3 and obtain a neck $\mathcal{N}_0 \ni p_0$. Therefore $p_0 \in \mathcal{A}$ and has the following structure of one of the following:

a) Two boundary components and

$$\mathcal{A} \simeq \text{diffeom} S^{n-1} \times [-1, +1]$$
b) one boundary component and
\[ \mathcal{A} \simeq \text{diffeom} \mathbb{B}^n_1 \]
c) No boundary components and is a connected component for which \( p_0 \in \mathcal{A}_0 \) and
\[ \mathcal{A}_0 \simeq \text{diffeo} \mathbb{S}^n, \mathbb{S}^{n-1} \times \mathbb{S}^1. \]

If \( \partial \mathcal{A} \neq \emptyset \) then \( \partial \mathcal{A} \) consists of either one or two cross sections of \( \mathcal{N}_0 \) with mean radius equal to \( \frac{2(n-1)}{H_1} \) and hence has mean curvature approximately \( |H| \simeq \frac{H_0}{2} \).

Let \( \mathcal{A} \) be the neck \( \mathcal{N}_0 \) together with one, or two or no regions given by case (ii) of Neck Continuation Lemma. Then the cross sections have mean radius equal to \( \frac{2(n-1)}{H_1} \) and hence with mean curvature \( |H| \simeq \frac{H_0}{2} \).

(b),
\[ |A|^2(p_0, t_0) - \frac{1}{n-1}|H|^2(p_0, t_0) \geq -\eta_1|H|^2(p_0, t_0). \]

Hence then there exists a \( q_0 \) such that
\[ |A|^2(q_0, t_0) - \frac{1}{n-1}|H|^2(q_0, t_0) \geq -\eta_1|H|^2(q_0, t_0). \]

and
\[ |H|(q, t_0) \geq \frac{|H|(p_0, t_0)}{\gamma_0} \quad \forall d_{t_0}(q, p_0) \leq d_{t_0}(q_0, p_0). \]

In particular, we have
\[ |H|(q_0, t_0) \geq \frac{H_2}{\gamma_0} \geq 10H_1. \]

Therefore with \( q_0 \) then there exists a region \( \mathcal{A} \) with the same properties as above.

**Claim.** \( p_0 \in \mathcal{A} \).

**Proof of claim.** Suppose not, then let \( \gamma \) be a curve such that
\[ \gamma(0) = p_0, \quad \text{and} \quad \gamma(1) = q_0. \]

Then if \( p_0 \notin \mathcal{A} \) there exists a first time \( t' \in [0, 1) \) \( \gamma(t') \in \partial \mathcal{A} \). Now if \( p \in \partial \mathcal{A}, |H|(p, t_0) \simeq \frac{H_0}{2} \) but along the geodesic from \( p_0 \) to \( q_0 \) we have
\[ |H| \geq \frac{|H|(p_0, t_0)}{\gamma_0} \geq 10H_1 \]

which is a contradiction. \( \square \)
We now repeat the process and cover each \( p_0 \) such that \( |H(p_0, t_0)| \geq H_2 \) with regions \( A \).

We now show there are a finite number of such regions and that they are disjoint. Now suppose that \( p' \notin A \mid |H(p', t_0)| \geq H_2 \). We will show the region \( A' \) constructed above is disjoint from \( A \). Recall that \( \partial A \) consists of cross sections with mean radius equal to \( \frac{2(n-1)}{H_1} \). From the neck continuation theorem, if we meet such a cross section we stop there because we have satisfied property ii) of the Theorem. Therefore \( A \) and \( A' \) can only meet at boundary points. Furthermore, any such region has volume bounded below by \((H_2)^{-n}\). Hence there are finitely many such regions. Therefore let \( A, A', A^{(2)}, \ldots, A^{(k)} \) be a finite collection which covers all points of \( M_{t_0} \) such that

\[
|H(p_0, t_0)| \geq H_2.
\]

Surgery : Now we have identified the high curvature regions,

**Step 1:** We discard all \( A^{(i)} \)'s with no boundary components. These are diffeomorphic to \( A^{(i)} \simeq S^n, S^{n-1} \times \mathbb{S}^1 \).

**Step 2:** \( \partial A^{(i)} \) is a cross section of \( N_{0}^{(i)} \) with mean radius \( \frac{2(n-1)}{H_1} \). Now the neck contains \( p_0 \) (or \( q_0 \)) where the curvature is at least \( H_2 \geq 10H_1 \). Therefore there is a first cross-section \( \Sigma^{(i)} \) such that the mean radius is \( \frac{n}{M_1} \), that is half the mean radius of the boundary. We perform the surgery at this cross section \( \Sigma^{(i)} \). If \( \partial A^{(i)} \) has two components, then we perform surgery at both ends. Note that the surgeries on different \( A^{(i)} \) do not interfere because the \( A^{(i)} \)'s only touch at boundary points but the surgeries are performed away from the boundary at points where the mean radius is half the mean radius of the boundary.

In either case, the surgeries (one or two) create a connected component diffeomorphic to \( S^n \) which includes all points of \( A^{(i)} \) with curvature greater than \( H_2 \). Such a component will be discarded hence the maximum of curvature has decreased to below \( H_2 \).

The surgeries performed in this way satisfy property (S) as well as (s1) – (s3) with \( K = H_1 \) and \( r = \frac{n}{M_1} \). After performing the surgeries we restart the flow until we reach the next time \( t_0 \) such that \( |H|_{\text{max}}(t_0) = H_3 \) and repeat the above procedure. Note that the surgery removes a region of volume \( c_n(H_1)^{-n} \) and therefore the surgery halts after a finite number of surgery times.

**Proof of Corollary.** From the proof of the Main Theorem \[\text{1.1}\] after the termination of mean curvature flow with surgery we have finitely many disjoint smooth closed submanifold that are diffeomorphic to \( S^n \) or \( S^{n-1} \times S^1 \) (this includes the submanifolds that were discarded during the flow at surgery times). In the case of cylindrically pinched submanifolds, the smooth mean curvature flow \( F : \mathcal{M} \times [t_0, t_1] \rightarrow \mathbb{R}^{n+m} \) is a smooth immersion of \( \mathcal{M} \times [t_0, t_1] \) into \( \mathbb{R}^{n+m} \) such that all restrictions of \( F : \mathcal{M} \times [a, b] \rightarrow \mathbb{R}^{n+m}, t_0 \leq a < b \leq t_1 \) are isotopic to each other between surgery times. Hence Corollary \[\text{1.2}\] is a direct consequence of Theorem \[3.22\].

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