Chaotic motion of particle around a disformal Kerr black hole in quadratic degenerate higher-order scalar-tensor theories

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Abstract

We have studied the dynamical behaviors of the motion of the timelike particles in the disformal Kerr black hole spacetime with an extra deformation parameter, which is a non-stealth rotating solution in quadratic degenerate higher-order scalar-tensor theories. Our result show that the particle’s motion depends on the sign of the deformation parameter. For the positive deformation parameter, the motion is regular and order. For the negative one, with the change of the deformation parameter, the motion of the particle undergoes a series of transitions betweens chaotic motion and regular motion, and then falls into horizon or escapes to spatial infinity. These mean that the dynamic behavior of timelike particles in the disformal Kerr black hole spacetime becomes richer than that in the usual Kerr black hole case.

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I. INTRODUCTION

Geodesics around black holes have been extensively studied since it could help us to understand the properties of black holes, the geometric structure of spacetimes and the motion of particles. Especially, the timelike geodesics with a back reaction can be also applied to simulate inspiral of black hole binaries with extreme mass ratio due to gravitational wave emission \cite{1, 2, 3, 4, 5}. The negative energy geodesics are always used to estimate the energy absorbed by a particles passing through the ergoregion of a rotating black hole by the Penrose process \cite{6, 7}. Moreover, the null geodesics are also relevant to study the shadow of a black hole, which is observed in the direct imaging of the supermassive black hole M87* by the Event Horizon Telescope \cite{8}.

In general relativity, it is expected that an astrophysical black hole in our Universe should be a Kerr black hole in terms of the well-known no-hair theorem. The geodesic motion of particle in the Kerr black hole spacetime is integrable because the geodesic equation is variable-separable and the number of the integrals of motion in the dynamical system is equal to its degree of freedom \cite{9}. And then the geodesic motion of particle is regular and order in such kind of black hole spacetimes. However, in some alternative theories of gravity, the geometries of black holes could become more complicated so that the geodesic equation is not variable-separable and then chaos may emerge in the geodesic motion of particle.

One of the most fascinating alternative theories of gravity is the so-called degenerate higher-order scalar-tensor (DHOST) theories \cite{10, 11, 12, 13, 14}, which contain the higher order derivative of scalar field and satisfy with a certain set of degeneracy conditions. The degeneracy conditions on Lagrangian ensure that the Ostrogradsky ghost is absent even if there exists higher-order equations of motion in the DHOST theories. Thus, the degeneracy of Lagrangian is crucial for the higher-order theories with only a single scalar degree of freedom \cite{10, 11}. In the general alternative theories of gravity, it is very difficult to obtain exact solutions for black holes because of the more complicated field equations. However, in the DHOST theories, some new black hole solutions are obtained recently \cite{12, 13, 14, 15, 16, 17}. It is found that these solutions can be classified into two types. One of them is the so-called stealth solution whose metric is the same as those in general relativity and the extra scalar field does not emerge in the spacetime metric. The other is named as the non-stealth solution where the parameters of scalar field appear in the metric and then its metric form deviates from that in Einstein’s theory.

Interestingly, the disformal and conformal transformations can take us from some DHOST Ia theory to some other specific DHOST Ia theory \cite{13}. Thus, if making a disformal transformation on a known “seed” metric
In DHOST Ia theory, one could obtain a new solution $g_{\mu\nu}$ in another specific DHOST Ia theory. With such a technique, a disformal rotating black hole solution in quadratic DHOST theories is obtained recently \[18, 19\]. This disformal solution is a non-stealth one and owns three parameters: the mass, the spin and the deformation parameter described the deviation from the Kerr geometry. The scalar field attached to the disformal solution is time-dependent with a constant kinetic density. Although it is a non-stealth solution, the disformed Kerr solution is still asymptotically flat and has a single curvature singularity as in the usual Kerr case. However, the scalar field also makes the disformal spacetime no more Ricci flat. Especially, the presence of the metric function $g_{rt}$ also leads to the lack of circularity \[20, 22\] in the disformal Kerr spacetime \[18, 19\]. These properties of spacetime could modify the geodesic motion of particle and yield some new observational effects differed from those in the Kerr case. The recent study indicates \[22\] that the shadow of a disformal rotating black hole depends heavily on the deformation parameter and some eyebrowlike shadows with the self-similar fractal structures appear as the deformation parameter lies in certain special ranges. Moreover, the deformation parameter in the disformal Kerr black hole is constrained by quasi-periodic oscillations with the observation data of GRO J1655-40 \[24\] and the non-circularity of the spacetime is examined by Sagittarius A* with orbiting pulsars \[25\]. The post-Newtonian (PN) motion of stars orbiting the disformal Kerr black hole has been analyzed with the osculating orbit method \[26\]. The main purpose of this paper is to investigate the chaotic motion of particle around the disformal Kerr black hole in DHOST theories.

The paper is organized as follows. In Sect. II, we introduce briefly a disformal Kerr black hole solution in quadratic DHOST theories and present the geodesic equation of a test timelike particle. In Sect. III, we investigate the chaotic phenomenon in the motion of the scalar particle by techniques including the Poincaré section, Lyapunov exponents, and bifurcation diagram, and then probe the effects of the deformation parameter together with black hole spin parameter on the chaotic motion for a chosen timelike particle. Finally, we end the paper with a summary.

II. GEODESIC EQUATION OF A PARTICLE AROUND A DISFORMAL KERR BLACK HOLE IN QUADRATIC DHOST THEORIES

Let us now to introduce briefly a disformal Kerr black hole solution in quadratic DHOST theories. It belongs to the non-stealth rotating solutions due to an extra deformation parameter. The most general action in quadratic DHOST theory can be expressed as \[18, 19\]

$$S = \int d^4x \sqrt{-g} \left( P(X, \phi) + Q(X, \phi) \Box \phi + F(X, \phi) R + \sum_{i=1}^{5} A_i(X, \phi) L_i \right).$$

(1)
Here $R$ is the usual Ricci scalar and the functions $A_i$, $F$, $Q$ and $P$ depend on the scalar field $\phi$ and its kinetic term $X \equiv \phi_\mu \phi^\mu$ where $\phi_\mu \equiv \nabla_\mu \phi$. $L_i$ are the Lagrangians contained quadratic in second derivatives of the scalar field $\phi$, which are defined by

$$
\begin{align*}
L_1 &\equiv \phi_\mu \phi^\mu, & L_2 &\equiv (\Box \phi)^2, & L_3 &\equiv \phi^\mu \phi^\nu \phi_\mu \phi_\nu, \\
L_4 &\equiv \phi^\mu \phi_\mu \phi^\nu \phi_\nu, & L_5 &\equiv (\phi^\mu \phi_\mu \phi^\nu)^2,
\end{align*}
$$

(2)

where $\phi^\mu \equiv \nabla_\nu \phi_\mu$ denotes the second covariant derivatives of $\phi$. To avoid Ostrogradski instabilities and ensure only an extra scalar degree of freedom besides the usual tensor modes of gravity in quadratic DHOST theory, the functions $F$, $A_i$ must satisfy the so-called degeneracy conditions, but $P$ and $Q$ are totally free. The degeneracy conditions for quadratic DHOST Ia theory is given in Ref. [10]. It is well known that a new solution in DHOST Ia theory can be obtained from a “seed” known solution by performing a disformal transformation. In general, the disformal transformation of the metric can be expressed as

$$
g_{\mu\nu} = A(X, \phi) \tilde{g}_{\mu\nu} - B(X, \phi) \phi_\mu \phi_\nu, \quad (3)
$$

where $g_{\mu\nu}$ is the “disformed” metric and $\tilde{g}_{\mu\nu}$ is the original “seed” one. $A(X, \phi)$ and $B(X, \phi)$ are conformal and disformal factors, respectively. In order to obtain a non-stealth solution, the functions $A$ and $B$ must satisfy the certain conditions so that the two metrics are not degenerate. Starting from the usual Kerr metric in general relativity, and adopting the transformation with $A(X, \phi) = 1$ and $B(X, \phi) = B_0$, one can obtain the disformal Kerr metric in Boyer-Lindquist coordinates [18, 19]

$$
ds^2 = -\frac{\Delta}{\rho} \left( dt - a \sin^2 \theta d\varphi \right)^2 + \frac{\rho}{\Delta} dr^2 + \rho d\theta^2 + \frac{\sin^2 \theta}{\rho} \left( a dt - \left( r^2 + a^2 \right) d\varphi \right)^2 \\
+ \alpha \left( dt + \sqrt{2Mr(r^2 + a^2)} / \Delta \right)^2,
$$

(4)

with

$$
\alpha \equiv -B_0 m^2, \quad \Delta \equiv r^2 - 2Mr + a^2, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta,
$$

(5)

where $M$ and $a$ are the general black hole mass and the spin parameter. $\alpha$ is the deformation parameter of black hole which is related to the rest mass $m$ of the scalar field. Here, the scalar field is taken as only a function of the coordinates $t$ and $r$ with a form [18, 19]

$$
\phi(t, r) = -mt + S_r(r), \quad S_r = -\int \sqrt{\frac{R}{\Delta}} dr, \\
\mathcal{R} = 2Mm^2(r^2 + a^2), \quad \Delta = r^2 + a^2 - 2Mr.
$$

(6)

This form of scalar field can avoid the pathological behavior of the disformal metric at spatial infinite. The choice of $A(X, \phi) = 1$ can avoid a global physically irrelevant constant conformal factor in the metric. Like in
the Kerr black hole case, the disformal Kerr metric \( (4) \) has also an intrinsic ring singularity at \( \rho = 0 \) and the spacetime is asymptotically flat. However, we must note that the disformal Kerr metric \( (4) \) is not Ricci flat, i.e., \( R_{\mu\nu} \neq 0 \), which is different from that in the general Kerr case. Moreover, the presence of the \( drdt \) term yields the lack of circularity in the disformal Kerr spacetime \([18, 19]\) and the spacetime cannot be foliated by 2-dimensional meridional surfaces everywhere orthogonal to the Killing field \( \xi = \partial_t \) and \( \eta = \partial_\phi \) \([20–22]\). This is qualitatively different from that in the usual Kerr spacetime in general relativity. The absence of circularity modifies the structure of the black hole horizons so that the horizons depend on the polar angle \( \theta \) and cannot be given by \( r = \text{const} \) in Boyer-Lindquist coordinates, and then the corresponding surface gravity is no longer a constant \([18, 19]\).

In a curved spacetime, the Lagrangian of a timelike particle moving along the geodesic is

\[
\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \frac{1}{2} \left( g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\phi\phi} \dot{\phi}^2 + 2 g_{t\phi} \dot{t} \dot{\phi} + 2 g_{r\theta} \dot{r} \dot{\theta} \right),
\]

(7)

where the dots denote derivatives with respect to the proper time \( \tau \). Obviously, the metric functions in the disformal Kerr spacetime \( (4) \) are independent of the coordinates \( t \) and \( \phi \). It means that \( t \) and \( \phi \) are cyclic coordinates for the Lagrangian \( (7) \) and then there exist two conserved quantities for a timelike particle in the disformal Kerr spacetime \( (4) \), i.e., the energy \( E \) and the \( z \)-component of the angular momentum \( L \),

\[
E = -p_t = -g_{tt} \dot{t} - g_{r\theta} \dot{r} - g_{t\phi} \dot{\phi}, \quad \quad L = p_\phi = g_{t\phi} \dot{t} + g_{\phi\phi} \dot{\phi}.
\]

(8)

With these two conserved quantities, the geodesic equation for the disformal Kerr black hole spacetime can be written as

\[
\dot{t} = \frac{g_{t\phi} E + g_{t\phi} L + g_{r\theta} g_{\phi\phi} \dot{r}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}}, \quad \quad \dot{\phi} = \frac{g_{t\phi} E + g_{t\phi} L + g_{r\theta} g_{\phi\phi} \dot{r}}{g_{tt} g_{\phi\phi} - g_{t\phi}^2},
\]

(9)

and

\[
\tilde{g}_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 = V_{\text{eff}} (r, \theta; E, L), \quad \quad \tilde{g}_{rr} = \left[ g_{rr} + \frac{g_{t^2} g_{\phi\phi}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} \right],
\]

(10)

with the effective potential

\[
V_{\text{eff}} (r, \theta; E, L) = \frac{E^2 g_{\phi\phi} + 2EL g_{t\phi} + L^2 g_{tt}}{g_{t\phi}^2 - g_{tt} g_{\phi\phi}} - 1.
\]

(11)

Moreover, the timelike particle in the disformal Kerr spacetime \( (4) \) must obey the constraint condition

\[
h = g_{tt} \dot{t}^2 + g_{rr} \dot{r}^2 + g_{\theta\theta} \dot{\theta}^2 + g_{\phi\phi} \dot{\phi}^2 + 2 g_{t\phi} \dot{t} \dot{\phi} + 2 g_{r\theta} \dot{r} \dot{\theta} + 1 = 0.
\]

(12)
It is obvious that in the case with the non-zero deformation parameter $\alpha \neq 0$, the dynamical system is non-integrable because the condition (12) is not variable-separable and then the system admits only two integrals of motion $E$ and $L$. This implies that the motion of the particle could be chaotic in the disformal Kerr black hole spacetime (4). In the next section, we will investigate the effect of the deformation parameter $\alpha$ on geodesic motion of a timelike particle around a disformal Kerr black hole (4).

III. CHAOTIC MOTION OF A TIMELIKE PARTICLE MOVING IN THE DISFORMAL ROTATING KERR BLACK HOLE SPACETIME

In order to probe chaotic behaviors of timelike particles in the spacetime of disformal Kerr black hole (4), we must solve numerically differential equations (9)-(10). Here, we adopt to the corrected fifth-order Runge-Kutta method in which the high precision can be effectively ensured [27–30]. In the spacetime of disformal Kerr black hole (4), the motion of the particle is entirely determined by the black hole background parameters $\{M, a, \alpha\}$, the particle’s parameters $\{E, L\}$ and the corresponding initial conditions of particle $\{r, \theta, \dot{r}, \dot{\theta}\}$. In principle,
particle motion decreases with the decrease of deformation parameters $\alpha$. Especially, as $\alpha$ decreases down to a threshold value $\alpha_c = -0.42195$, the allowed region of the particle motion vanishes and then the motion of particle is prohibited in this case $\alpha \leq \alpha_c$. Moreover, we also present numerically the change of $\alpha_c$ with the black hole spin parameter $a$, and find that $\alpha_c$ increases with the spin parameter. As the spin parameter $a$ tends zero, we find that the $\alpha_c$ approaches to the negative infinity.

![Diagram](image)

**FIG. 2**: The change of Poincaré section ($\theta = \frac{\pi}{2}$) with the deformation parameter $\alpha$ for the motion of the timelike particle in disformal Kerr black hole spacetime (4) with the fixed parameters $a = 0.998$, $E = 0.94$ and $L = 0.8M$.

It is well known that Poincaré section is one of effective tools to identify chaotic motion, which projects trajectories of a continuous dynamical system on a given hypersurface with a pair of conjugate variables in the phase space. The intersection point distribution in Poincaré section [31] reflects the intrinsic dynamical properties of particles’ motions. For example, the periodic motions and the quasi-periodic motions correspond to a finite number of points and a series of close curves in the Poincaré section, respectively. However, the chaotic motions correspond to strange patterns of dispersed points with complex boundaries. In Fig. (2), we show the change of Poincaré section ($\theta = \frac{\pi}{2}$) with the deformation parameter $\alpha$ for the timelike particle’s motion in the disformal Kerr black hole spacetime [41] with the fixed parameters $a = 0.998$, $E = 0.94$ and $L = 0.8M$. It is found that the particle’s motion depends on the sign of the deformation parameter $\alpha$. For the positive $\alpha$, the motion is regular and order, and the region of particle’s motion increases with $\alpha$. For the
negative $\alpha$, with the increase of the absolute value of $\alpha$, the region of particle’s motion first decreases, and then the chaotic motion appears gradually and the corresponding region of particle’s motion increases. With the further increase of $|\alpha|$, the chaotic motion of particle vanishes and the region of particle’s motion decreases. Finally, as $\alpha$ decreases down to a threshold value $\alpha_c = -0.42195$, we find that the region of particle’s motion disappears since the motion of particle is prohibited as shown in the previous analysis.

Moreover, we also present the effect of the spin parameter $a$ on the particle motion for the fixed parameter $\alpha = -0.28$. In Fig. (3), it is shown that the spin parameter $a$ increases the region of chaotic motion for the chosen particle. Here, we select the spin parameter $a \geq 0.9$, the main reason is that as $\alpha = -0.28$ the motion of particle is allowed only in the case of the rapidly rotating black hole.

Let us to detect the chaos in the timelike particle’s motion in the disformal Kerr black hole spacetime by analyzing the Lyapunov exponent of the dynamical system. It is well known that the Lyapunov exponent describes the growth or decline rate of the deviation vector $\Delta X$ between two nearby trajectories, which reflects actually that the motion is highly sensitive to initial conditions or not. In general relativity, the Largest Lyapunov exponent is defined as

$$LE = \lim_{r \to +\infty} \frac{1}{r} \ln \frac{|\Delta X(r)|}{|\Delta X_0|},$$

FIG. 3: The change of Poincaré section ($\theta = \frac{\pi}{2}$) with the spin parameter $a$ for the motion of the timelike particle in disformal Kerr black hole spacetime with the fixed parameters $\alpha = -0.28$, $E = 0.94$ and $L = 0.8M$. 
where the length $|\Delta X(\tau)| \equiv \sqrt{|g_{\mu\nu}\Delta x^\mu \Delta x^\nu|}$ and $\Delta x^\alpha(\tau)$ is the deviation vector between two nearby trajectories at proper time $\tau$. In Fig. (4), we present the Largest Lyapunov exponent with different $\alpha$ for the particle with parameter $E = 0.94, L = 0.8$ and the initial conditions $\{r(0) = 3.5; \dot{r}(0) = 0; \theta(0) = \frac{\pi}{2}\}$. Here we take $a = 0.998$.

with parameter $E = 0.94, L = 0.8$ and the initial conditions $\{r(0) = 3.5; \dot{r}(0) = 0; \theta(0) = \frac{\pi}{2}\}$ in the disformal Kerr black hole spacetime (11) with $a = 0.998$. It is shown that as $\alpha = -0.2$ or $-0.3$, the Largest Lyapunov exponent is negative and then the motion of the particle is order. As $\alpha = -0.28$, the Largest Lyapunov exponent is positive and the corresponding motion is chaotic. In Fig. (5), we also present the Poincaré section for the particle motion in the above three cases. In the case of $\alpha = -0.2$, we find that the phase path of the particle motion is a quasi-periodic Kolmogorov-Arnold-Moser (KAM) tori and the corresponding motion is regular. As $\alpha = -0.3$, there is a chain of islands which is composed of three secondary KAM toris belonging to the same trajectory. However, as $\alpha = -0.28$, we find that the KAM tori is broken and there are many discrete points distributed randomly in the Poincaré section and then corresponding motion is chaotic. The

\[\text{FIG. 4: The Lyapunov exponents with different } \alpha \text{ for the particle with parameter } E = 0.94, L = 0.8 \text{ and the initial conditions } \{r(0) = 3.5; \dot{r}(0) = 0; \theta(0) = \frac{\pi}{2}\}. \text{ Here we take } a = 0.998.\]

\[\text{FIG. 5: The Poincaré surface of section } (\theta = \frac{\pi}{2}) \text{ with different } \alpha \text{ for the particle motion in Fig.4.}\]
dynamical properties of the particle with the chosen initial conditions are consistent with those obtained by analyzing the Lyapunov exponent in Fig.4.

The dependence of dynamical behaviors of system on the black hole parameters can also be visualised in the form of a bifurcation diagram. In Figs. 6 and 7, we plot the bifurcation diagram of the radial coordinate $r(\tau)$ with the deformation parameter $\alpha$ and the spin parameter $a$ for the particle motion with parameter $E = 0.94$, $L = 0.8$ and the initial conditions $\{ r(0) = 3.5; \dot{r}(0) = 0; \theta(0) = \frac{\pi}{2} \}$ in the disformal Kerr black hole spacetime 4. Here, we present the result only in the case with $a > 0.90$ since the we can not find any stable orbit for the above particle with the chosen initial conditions in the slowly rotation black hole case. As
\( \alpha = 0 \), one can find that the radial coordinate \( r(\tau) \) is a periodic function and there is no bifurcation for the dynamical system since the metric (4) reduces to the Kerr one and the corresponding dynamical system of particle is integrable because the condition (12) is variable-separable in this case. For the case with \( \alpha \neq 0 \), it is obvious that there exist periodic, chaotic and escaped solutions which depend on the deformation parameter \( \alpha \) and the spin parameter \( a \). Moreover, we can find that with the change of the parameters \( \alpha \) and \( a \) the motion of particle transforms among single-periodic, multi-periodic and chaotic motions, and the effects of the parameters \( \alpha \) and \( a \) on the motion of the particle are very complex, which are typical features of bifurcation diagram for the usual chaotic dynamical system. Moreover, from Figs. (6) and (7), we find that the range of \( \alpha \) for the existence of the oscillation solution increases with the spin parameter \( a \). With the decrease of \( \alpha \), the range of \( a \) for the existence of the oscillation solution becomes more complicated. These mean that the dynamic behavior of timelike particles in the disformal Kerr black hole spacetime (11) becomes richer than that in the usual Kerr black hole case.

FIG. 7: The bifurcation with the spin parameter \( a \) for the motion of the timelike particle with the parameters with parameter \( E = 0.94, L = 0.8 \) and the initial conditions \{ \( r(0) = 3.5; \dot{r}(0) = 0; \theta(0) = \frac{\pi}{2} \} \) in the disformal Kerr black hole spacetime (4).
IV. SUMMARY

We have studied the dynamical behaviors of the motion of the timelike particles in the disformal Kerr black hole spacetime in quadratic degenerate higher-order scalar-tensor theories. Firstly, we find that the motion of particle is prohibited as the deformation parameter $\alpha$ is less than certain threshold value $\alpha_c$. For the particle with $E = 0.94$ and $L = 0.8M$, threshold value $\alpha_c$ increases with the spin parameter $a$. As the spin parameter $a$ tends zero, we find that the $\alpha_c$ approaches to the negative infinity. Moreover, it is found that the particle’s motion depends on the sign of the deformation parameter $\alpha$. For the positive $\alpha$, the motion is regular and order, and the region of particle’s motion increases with $\alpha$. For the negative $\alpha$, with the increase of the absolute value of $\alpha$, the region of particle’s motion first decreases, and then the chaotic motion appears gradually and the corresponding region of particle’s motion increases. With the further increase of $|\alpha|$, the chaotic motion of particle vanishes and the region of particle’s motion decreases. Finally, as $\alpha$ decreases down to a threshold value $\alpha_c$, the region of particle’s motion disappears and then the motion of particle is prohibited. The presence of chaos in the particle’s motion means that the dynamic behavior of timelike particles in the disformal Kerr black hole spacetime [4] becomes richer than that in the usual Kerr black hole case.

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