A proof of factorization for deep inelastic neutrino scattering

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Keywords: effective field theory, operator product expansion, parton distribution functions, soft radiation

Abstract

It is proven in this paper that the structure functions for the hadron quantity describing deep inelastic neutrino scattering factor into the product of a short-distance coefficient function, the non-perturbative parton distribution function which encompasses the underlying structure of the target, and the function for soft radiation which does not emerge in the case of electron scattering for which weak radiative corrections are usually practically ignored. This is shown to all orders of perturbative quantum chromodynamics and electroweak theory, and to leading order in the power expansion of the effective field theory used as a tool. It is based on the observation that there is no necessity to go into the partonic level of the physical process, for a generalized version of the operator product expansion affords a framework for the study of inclusive processes, where the momentum carried in by one current operator and out by the other is allowed to go to infinity. It is discovered following this line of argument that the objects entering the factorization theorem need not be SU(2) × U(1) gauge singlets, whether or not we perform the factorization in the symmetric phase. The factorization analysis provides initial conditions for evolution to arbitrary energies that allows for re-summation of large logarithms for loop calculations to the extent of accuracy requested.

1. Introduction

The deep inelastic scattering has served as the classic tool for validating the standard model of particle physics. It was the result of a famous 1968 experiment at SLAC on deep inelastic electron-nucleon scattering that had helped shape the modern theory of strong interactions as quantum chromodynamics with asymptotic freedom as its salient feature [1, 2]. Over the years, both charged leptons and neutrinos have been used as probes to determine the internal structure of the nucleon by measuring its structure functions. Of all scattering on the nucleon, neutrino scattering is unique in that it measures the valence quark distributions through measurement of an additionally parity-violating structure function, and the strange quark distribution through detection of neutrino-induced di-muon production, which provides important constraints that cannot be obtained from either electron or muon scattering experiments [3, 4]. The extension to the deep inelastic scattering of the neutrino on a nucleus plays an essential role in determining exceptional properties of the neutrinos beyond the standard model, embracing their non-zero masses [5].

In this paper, a theoretical framework is developed to derive a factorization theorem that is valid to all orders of perturbation theory for the process of inclusive deep inelastic neutrino scattering. To be explicit, two types of factorization are distinguished. First and foremost the factorization that separates hard and infra-red effects is related to defining the correct effective theory. The properties of soft-collinear effective theory (SCET) [6–9] are used in this paper to provide a simplified proof of factorization and describe the deep inelastic scattering processes within an operator formalism. At the same time, for inclusive processes where there is a flow of large momenta, the operator product expansion [10, 11] presents a systematic method to include power corrections to the factorized cross section. Hence SCET coupled with the operator product expansion showcases an instructive and intriguing example in the deep inelastic neutrino scattering, in which matrix elements of the dominant operators can be written as a convolution of coefficient functions with long-distance contributions defined unambiguously in terms of matrix elements that connect in part to the parton distribution functions for...
the nucleon. The second type is the factorization theorems between soft and collinear degrees of freedom. Colour conservation excludes the existence of the soft factors for colour-singlet operators, while for electroweak non-singlet operators the dependence on soft radiation is obvious; the soft contribution are embedded wholly in the soft Wilson lines in the framework of SCET.

One of the early attempts at a factorization proof for deep inelastic scattering was made in the realm of perturbative quantum chromodynamics [12–14]. It is based on a systematic subtraction procedure that takes care of the overlaps between different leading regions of momentum space. Such a procedure also makes it possible to develop bounds on corrections to leading power factorization theorems. In this paper the attention is focused on putting the proof of factorization in the context of SCET combined with operator product expansion, to leading order in the SCET power expansion. Note in addition that the calculations here are done in the Breit frame, for the current operator product is most conveniently expanded in the Breit frame, where the final state of arbitrary hadrons and the incoming nucleon correspond to hard and collinear degrees of freedom, respectively.

2. Effective field theory

Effective field theories provide a simple and elegant way of organizing physics in processes containing disparate energy scales. SCET as an effective field theory we are interested in describes processes with final state particles having energy much larger than their mass. Part of the power of SCET lies in the great variety of processes that it can be used to describe, but let us keep to the example of the main theme of this paper for illustrating this power for the sake of brevity and clarity. Note that merely a brief review instead of a comprehensive treatment of SCET is given in this section.

Consider the process \( \nu_e + N \rightarrow \mu^- + X \), in which a muon-neutrino of four-momentum \( k \) collides with a nucleon \( N \) of four-momentum \( p \), yielding a muon of four-momentum \( k' \) and a general unobserved hadron state \( X \). The momentum transfer \( q = k - k' \) from the leptons to the hadrons \( X \) is large, and the hard scale \( Q \) of the process is defined by \( Q^2 \equiv q^2 \), where \( Q \gg \Lambda \) with \( \Lambda \) the energy scale characteristic of the nucleons. Those degrees of freedom with momenta larger than \( Q \) are integrated out and contribute to Wilson coefficient functions in the effective theory. Furthermore, in the Breit frame where the momentum transfer and nucleon momentum are in back to back directions along the 3-axis, the all kinds of hadrons created in the final states turn out to be hard degrees of freedom, which can be described using the operator product expansion. The remaining infra-red physics can be described by including all on-shell degrees of freedom whose momenta are set by the scales in the process. The nucleon contains quarks and gluons that travel in a direction parallel to the momentum \( p \) scaling as \( (p^+, p^-, p^z) \sim Q(\lambda^3, 1, \lambda) \), where the parameter \( \lambda \sim \Lambda/Q \ll 1 \). All four components of the collinear gluon fields give order-\( \lambda^0 \) interactions with collinear quarks and are responsible for binding the nucleon. In addition, soft gauge bosons, with all four components of their momenta approximated by \( Q\lambda^2 \), can be emitted by a collinear quark without changing the scaling of its momentum, i.e. taking it off its mass shell.

Unlike other effective field theories, multiple quantum fields are defined for the same species of particle in SCET, for the sake of power counting and momentum scale separation. The quantum fields for collinear quarks and gluons are usually labelled by their light cone direction and the larger part of their momenta. This gives \( (A_1^{\pm \alpha}, A_2^{\pm \alpha}, A_3^{\pm \alpha}) \sim (\lambda^2, 1, \lambda) \) for \( l \)-collinear gluon fields, \( \xi_{k,p} \sim \lambda \) for \( l \)-collinear quark fields, etc. Note that to simplify the power counting, fields are rescaled by powers of \( \lambda \) to make all kinetic terms of the order \( \lambda^0 \). The same logic can be used to attain the power counting for soft quantum fields, yielding \( (A_1^{\pm \alpha}, A_2^{\pm \alpha}, A_3^{\pm \alpha}) \sim (\lambda^2, \lambda^2, \lambda^2) \) for soft gluon fields, \( \xi_{k,p} \sim \lambda^2 \), etc.

3. Preliminaries

The scattering amplitude involves the interaction of a neutrino beam with a nucleon target via a virtual massive gauge boson. The leptonic interactions are calculable using electroweak theory in the tree approximation, and will not be discussed here. The quantity of interest is the interaction of the virtual \( W^\pm \) or \( Z \) boson with the

\[\eta_{\mu
u} = \eta_{\mu
u} = \lambda_{\mu
u} = 1, \eta_{\lambda\eta} = -1.\]

A generic four-vector \( p^\nu \) can be expressed in terms of light-cone vectors as \( p^\nu = -\frac{\mu^2}{4}T^\nu + \frac{\mu^2}{4}T^\nu + p^\mu \), where the fixed four-vectors are \( T^\nu = (0, 0, 1, 1) \), and \( \mu^2 = (0, 0, 1, 1) \), with \( \mu \) running in sequence over the four space-time coordinate labels 1, 2, 3, 0.
nucleon target. To calculate the spin-averaged inclusive cross section, we shall focus on the following hadronic quantity:

\[
m_N \frac{W^{\mu\nu}(q, p)}{p^0} \equiv \frac{1}{2} \sum_{\alpha} \sum_{X} \delta^{\mu}(p_X - p - q) \langle X | J^{\mu}(0) | N \rangle \langle X | J^{\nu}(0) | N \rangle^*,
\]

(1)

where \( J^{\mu} \) is the weak charged current, and the final state \( X \) of arbitrary hadrons is summed over. Lorentz invariance and current conservation dictate that \( W^{\mu\nu}(q, p) \) must take the form

\[
W^{\mu\nu}(q, p) = -\left( \frac{q^{\mu}q^{\nu}}{q^2} - \eta^{\mu\nu} \right) T_1(\omega, Q^2) + \frac{1}{m_N^2} \left( p^\mu + \frac{\omega}{2} q^\mu \right) \left( p^\nu + \frac{\omega}{2} q^\nu \right) T_2(\omega, Q^2)
+ \frac{i}{m_N} \epsilon^{\mu
u\rho\sigma} p_\rho q_\sigma T_3(\omega, Q^2),
\]

(2)

where the two independent scalar functions out of \( q \) and \( p \) are \( Q^2 \equiv q^2 \), and the Bjorken variable \( \omega \equiv -2q \cdot p / q^2 = -2q \cdot p / Q^2 \). Also, equation (1) shows that \( W^{\mu\nu} \equiv W^{0\mu} \), so \( W_r \) are all real, with \( r = 1, 2, 3 \).

Using translation invariance and the completeness of the hadron states \( |X> \), equation (1) gives

\[
m_N \frac{W^{\mu\nu}(q, p)}{p^0} = \frac{1}{2(2\pi)^4} \sum_{\alpha} \int d^4 z e^{-i q z} \langle N | J^{\mu}(z) J^{\nu}(0) | N \rangle.
\]

(3)

The asymptotic behaviour of \( W^{\mu\nu}(q, p) \) as \( q \to \infty \) is therefore related to the singularity of the operator product at \( z \to \infty \).

Feynman diagram calculations of the coefficient functions in the operator product expansion refer directly not to the expansion for matrix elements like those for \( W^{\mu\nu} \), but rather to matrix elements of the two-point Green’s function

\[
m_N \frac{T^{\mu\nu}(q, p)}{p^0} = \frac{1}{2(2\pi)^4} \sum_{\alpha} \int d^4 z e^{-i q z} \langle N | T | J^{\nu}(z) J^{\mu}(0) | N \rangle.
\]

(4)

We can express \( T^{\mu\nu} \) in terms of the structure functions for it as defined by the analogue of equation (2):

\[
T^{\mu\nu}(q, p) = -\left( \frac{q^{\mu}q^{\nu}}{q^2} - \eta^{\mu\nu} \right) T_1(\omega, Q^2) + \frac{1}{m_N^2} \left( p^\mu + \frac{\omega}{2} q^\mu \right) \left( p^\nu + \frac{\omega}{2} q^\nu \right) T_2(\omega, Q^2)
+ \frac{i}{m_N} \epsilon^{\mu
u\rho\sigma} p_\rho q_\sigma T_3(\omega, Q^2).
\]

(5)

The connection between the \( T_r(\omega, Q^2) \) and \( W_r(\omega, Q^2) \), with \( r = 1, 2, 3 \), is provided by the dispersion relations for fixed \( Q^2 \):

\[
\text{Re } T_r(\omega, Q^2) = W_r(\omega, Q^2).
\]

(6)

4. Leading operators in power expansion

The effective field theory makes any symmetries which emerge in the \( Q \to \infty \) limit manifest in the Lagrangian and operators, and allow statements to be made to all orders of perturbation theory. Since degrees of freedom with momenta larger than \( Q \) are integrated out, the Wilson coefficient functions in the effective theory are in general arbitrary functions of the large momenta that are of the order of \( \lambda^3 \). This functional dependence is greatly restricted by a symmetry induced by collinear gauge transformations [8]. Under this symmetry, \( \xi_\lambda \) and \( A_l^{\mu} \) transform within the community of collinear degrees of freedom, while soft fields do not by definition, since otherwise the resultant fields would have a large momentum and be therefore no longer soft. The totality of collinear and soft gauge transformations as distinguished respectively by the scales of their transformation parameters presents a recapitulation of the standard gauge transformations.

For deep inelastic electron scattering within the sector of quantum chromodynamics and electrodynamics, the most general leading-order SCET operators out of collinear fields in the light-cone \( l \) direction are \[ \mathcal{O}_f \equiv (\xi_\lambda W_l) \frac{1}{2} U_f(\hat{P}_\mu, \hat{P}_\nu)(W_l^\dagger \xi_\beta), \]

(7)

\[ \mathcal{O}_f \equiv (\xi_\lambda W_l) \frac{1}{2} U_f(\hat{P}_\mu, \hat{P}_\nu)(W_l^\dagger \xi_\beta), \]

4 Radiative corrections that spoil the factorized form of the cross sections for deep inelastic scattering, for which the momentum transfer \( q^2 \) is sufficiently large, as a product of leptonic and hadronic quantities shall be esteemed to be suppressed by powers of \( 1/q^2 \).

5 It may be helpful to spell out that it is the real instead of the imaginary part of \( T \) that accounts for \( W \) here, which is due to the conventions the author uses. See footnote 3.

6 The Dirac matrices \( \gamma_\mu \) are defined so that \( \{ \gamma_\mu, \gamma_\nu \} = 2\eta_{\mu\nu} \) also \( \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \) and \( \beta = i\gamma_0 \).
\( \mathcal{O}_0 \equiv -\text{Tr} \{ (W_1^g \mathcal{G}_2^a W_2) U_0(\mathcal{P}_\perp, \mathcal{P}_\parallel)(W_1^g \mathcal{G}_2^a W_2) \}. 

(8)

Here the high energy behaviour of the structure functions is encoded here in the Wilson function \( U(\mathcal{P}_\perp, \mathcal{P}_\parallel) \), whose arguments are \( \mathcal{P}_\perp \equiv \bar{D} \cdot \mathcal{P}_\perp = \bar{D} \cdot (\mathcal{P}_1^\perp \pm \mathcal{P}_2^\perp) \) with \( \mathcal{P}_1^\perp \) and \( \mathcal{P}_2^\perp \) operators that pick out the sum of label momenta of the collinear fields to their left and right respectively\(^7\), \( \xi_{\perp} \) are the \( l \)-collinear quark fields defined by \( I_{\xi_{\perp}} = 0 \), where \( f \) labels quark flavours; the gluon operator building blocks are defined as

\[
ig \mathcal{G}_2^a \equiv [i\bar{D}_\mu, iD_\mu^a].
\]

(9)

where \( g \) is the strong coupling constant, and the covariant derivatives are defined by \( \bar{D} \equiv \bar{D}_\mu - ig \bar{A}_\mu \) with \( \bar{P} \equiv \bar{D}_\mu - \mathcal{P}_\mu \) and \( \bar{D}_\mu^a \equiv i\mathcal{P}^\mu - ig A^a_{\mu \perp} \); and in terms of the label operators, the \( l \)-collinear Wilson lines are built out of the gluon fields of order \( \lambda^0 \) as

\[
W_\phi = \exp \left\{ \frac{g}{\bar{P}} \bar{D}_\mu A_\mu \right\} + \text{perms}.
\]

(10)

These operators \( \mathcal{O}_l \) and \( \mathcal{O}_0 \) conserve both collinear and soft SU(3) gauge symmetries, with the understanding that \( W_1^g \xi_{\perp} \) and \( \xi_{\perp} W_2 \) are individually invariants under a collinear gauge transformation. Note that it is because \( \mathcal{P}_\perp \) and its dagger do not commute with collinear fields that the short-distance Wilson coefficients \( U_l \) and \( U_0 \) are conveniently included as part of the operators here.

For \( \nu_{\mu} + N \rightarrow \mu^{-} + X \) or \( \nu_{\mu} + N \rightarrow \nu_{\mu} + X \), extra leading-order operators in the power expansion must be added to the list as in the following most general form

\[
(O_a^l)_m \equiv (\mathcal{F}_a W_j) \frac{L_a}{2} T_{\alpha_{f}} U_{l}(\mathcal{P}_\perp, \mathcal{P}_\parallel)(W_j^\dagger \mathcal{F}_a).
\]

(11)

Here \( \mathcal{F}_a \) are the left-handed quark doublets

\[
\mathcal{F}_a \equiv \left( \begin{array}{c} U_{aL} \\ \sum_j V_{ij} T_{\alpha_{j}} D_{ij} \end{array} \right)
\]

(12)

where \( U_{\alpha} \) and \( D_{\alpha} \) with \( i = 1, 2, 3 \) are three independent quark fields of charge \( 2e/3 \) and \( -e/3 \), respectively, \( L \) denotes the left-handed part of the quark fields, \( V \) is a \( 3 \times 3 \) unitarity matrix known as the Kobayashi-Maskawa matrix [16]; and \( t_{\alpha} \) is the three-vector of isospin matrices

\[
t_{\alpha} = \frac{1}{2} \left\{ \begin{array}{c} (0 \ 1) \\ (1 \ 0) \\ (0 \ i) \\ (i \ 0) \\ (1 \ 0) \\ (0 \ -1) \end{array} \right\},
\]

(13)

with \( a = 1, 2, 3 \). In contrast to \( \mathcal{O}_l \) and \( \mathcal{O}_0 \) which are the usual gauge-singlet quark and gluon operators respectively, \( (O_a^l)_m \) are singlets invariant under the gauge group SU(3) × U(1), and are simultaneously nonsinglet quark operators that transform in the adjoint representation of SU(2)\(_L\) [17–19]. Just like singlet quark operators \( \mathcal{O}_l \) in appearance, collinear Wilson lines are inserted to complete the gauge structure of the operators here, and are dependent on the gauge representation of the collinear quark fields to define the operators. In particular, the Wilson lines denoted by \( W_j \) in equation (11) differentiate from their counterparts in equations (7) and (8) in that they are instead SU(3) × SU(2)\(_L\) × U(1) entities in the representation of the quark fields [20].

All the operators \( \mathcal{O}_l \), \( \mathcal{O}_0 \), and \( (O_a^l)_m \), have dimensionality 2 in the SCET parameter \( \lambda \), with \( \xi_{\perp} \sim \mathcal{F}_a \sim G_\perp^a \sim \lambda^1 \); furthermore, \( \mathcal{O}_l \) and \( (O_a^l)_m \), have dimensionality 3 in mass, and \( \mathcal{O}_0 \), 4, for which the coefficient functions are made dimensionless. The collinear fields as the quintessence of these operators are evaluated at the same residual space-time point, the dependence on which is suppressed in the expressions; the presence of Wilson lines and label momenta, however, make the operators non-local along a particular light-cone direction. These non-local operators sum the infinite set of purely local operators of a given twist.

### 5. Decoupling of soft gauge bosons

In order to prove generalized factorization formulae, it is essential to disentangle the collinear and soft modes. SCET furnishes a systematic treatment of infra-red degrees of freedom with collinear and soft sectors distinguishable. In particular, the interactions between collinear and soft fields can be removed from the collinear Lagrangian of quantum chromodynamics by including the following soft Wilson line at residual space-time point \( x \) as [8]

\[\text{For the spin-averaged matrix elements in this problem, the combination } \mathcal{P}_l = \bar{P}^\perp - \bar{D} \text{ behaves like a total derivative, and as we shall see, by momentum conservation this gives the total large momentum label of the effective theory state which is zero; the dependence on the other linear combination } \mathcal{P}_l = \bar{P} + \bar{D} \text{ is meanwhile displayed explicitly.}\]
\[ S_l(x) = P \exp \left\{ ig \int_{-\infty}^{x} ds \, l \cdot A_i(s) \right\}, \]

which sums the emission of soft radiation from the collinear particles. The collinear quark and gluon fields are then re-defined as

\[ \xi_i \rightarrow S_l \xi_i, \quad A_l^\mu \rightarrow S_l A_l^\mu S_l^\dagger, \]

which imply \( W_l \rightarrow S_l W_l S_l^\dagger \) and \( G_l^{\mu \nu} \rightarrow S_l G_l^{\mu \nu} S_l^\dagger \). Hence in the case of gauge-singlet operators \( O_l \) and \( O_0 \), the \( S_l \)'s cancel trivially because of the identity \( S_l S_l^\dagger = 1 \) and the cyclic property of the trace\(^8\).

In the case of non-singlet operators, however, the factors of soft Wilson lines do not cancel. By the field redefinitions equation (15), \( (O_{\alpha \beta})_l \) switch to

\[
(O_{\alpha \beta})_l = (\mathcal{P}_d W_l)_\alpha \frac{1}{2} (t_d)^\beta_{\lambda} U_l(\mathcal{P}_+, \mathcal{P}_-) (W_l^\dagger \mathcal{P}_d)^\beta
\]

\[ \rightarrow (\mathcal{P}_d W_l S_l^\dagger)_\alpha \frac{1}{2} (t_d)^\beta_{\lambda} U_l(\mathcal{P}_+, \mathcal{P}_-) (S_l W_l^\dagger \mathcal{P}_d)^\beta, \]

where \( \alpha \) and \( \beta \) are gauge indices written down explicitly, with Dirac indices suppressed, and, just as in the situation of the collinear Wilson line, \( S_l \) that replaces \( S_l \) is an \( SU(3) \times SU(2)_L \times U(1) \) soft Wilson line in the representation of the quark fields in electroweak theory.

To separate the hard coefficient functions from the long-distance operators, one introduces trivial convolutions to give

\[ O_f = \int d\eta_1 d\tau_2 U_l(\tau_2, \tau_\neq) \chi_{\rho, \gamma_1} \frac{1}{2} \chi_{\rho, \gamma_2}, \]

\[ O_0 = \int d\eta_1 d\tau_2 U_0(\tau_2, \tau_\neq) \text{Tr} \{ B_{\alpha \beta}^\mu B_{\lambda \mu} \}, \]

and

\[ (O_{\alpha \beta})_l = \int d\eta_1 d\tau_2 U_l(\tau_2, \tau_\neq) \mathcal{H}_{\beta \alpha, \gamma_1} \frac{1}{2} t_d \mathcal{H}_{\beta \alpha, \gamma_2}, \]

where

\[ \chi_{\rho, \tau} \equiv \delta(\tau - \mathcal{P})(W_l^\dagger \xi_\rho), \]

\[ B_{\alpha \beta}^\mu \equiv \delta(\tau - \mathcal{P})(W_l^\dagger G^{\mu \nu} W_l), \]

\[ = \frac{i}{g} \delta(\tau - \mathcal{P})(W_l^\dagger [i\nabla^\mu, D_l, iD_l^\mu] W_l), \]

and

\[ \mathcal{H}_{\alpha \beta, \gamma} \equiv \delta(\tau - \mathcal{P})(W_l^\dagger \mathcal{P}_d). \]

The function \( U(\tau_2, \tau_\neq) \) contains all the short-distance physics, and is determined by matching the full theory onto these operators in the effective theory. It will not be unwelcome to emphasize that the dominant terms in the structure functions \( T_r \) with \( r = 1, 2, 3 \) for \( Q^2 \rightarrow \infty \) with fixed \( \omega \) are contributed by these operators of minimum dimensionality in SCET power expansion.

### 6. Generalized factorization

For the purpose of lending a demonstration of factorization which splits \( W^{\mu \nu} \) into hard, collinear and soft parts, we have to first write nucleon matrix elements of operators in terms of parton distribution functions. The singlet parton distribution functions for quarks and gluons can be defined directly by operators themselves in coordinate space, rather than through their moment equations, respectively as

\[ F_f(x) \equiv \frac{1}{4\pi} \sum_{n} \int dy e^{-2i\mu \cdot p} \langle N| \bar{c}\gamma_\mu |y, -\infty \rangle W^{\mu \nu}(y, -\infty) \bar{W}^{\nu}(y, -\infty) |N\rangle, \]

\[ F_0(x) \equiv \frac{1}{2\pi x} \sum_{n} \int dy e^{-2i\mu \cdot p} \frac{1}{2} \langle N| F_\alpha^\mu(y) W_\alpha(y, -\infty) F_\beta^\nu(y, -\infty) |N\rangle, \]

and for anti-quark distribution functions \( \bar{F}_f(x) = -\bar{F}_f(-x) \), where \( \bar{y}^\mu = \bar{l}^\mu y \); the Fourier transform of \( \xi(y) \) is \( \xi_{p^\mu} F_{p^\mu} \); the gluon field strength, and \( W \) and \( W_{\alpha \beta} \) are path-ordered eikonal lines in the respective fundamental and adjoint representations. A straightforward calculation yields that the operators \( \bar{O}_f \) and \( \bar{O}_0 \) have matrix elements of the following forms \(^{[21]}\)

\(^{8}\) Note that the soft Wilson lines commute with \( \mathcal{P}_d^\mu \) and its dagger since soft gauge bosons only carry momenta of lower order in \( \lambda \).
\[
\frac{1}{2} \sum_{\mathcal{O}} \langle N|\mathcal{O}|N \rangle = (\vec{t} \cdot p) \int_0^1 dx [U_f(2x\vec{t} \cdot p, 0)\mathcal{F}_f(x) - U_f(-2x\vec{t} \cdot p, 0)\mathcal{F}_f(x)],
\] (25)

and
\[
\frac{1}{2} \sum_{\mathcal{O}} \langle N|\mathcal{O}_d|N \rangle = (\vec{t} \cdot p)^2 \int_0^1 dx U_f(2x\vec{t} \cdot p, 0)\mathcal{F}_d(x).
\] (26)

This is all that is needed by quantum chromodynamics alone.

In deep inelastic neutrino scattering where electroweak theory prevails, adjoint parton distribution functions for quarks and anti-quarks are required in addition, which are defined analogously by [22]
\[
(\mathcal{F}^{\mu}_f)_\mu(x) \equiv \frac{1}{4\pi} \sum_{\mathcal{O}_f} \int dy e^{-2i\phi_f(y)} \langle N|\tilde{\mathcal{O}}_f(y)\mathcal{W}(y, -\infty)\mathcal{F}_d(y)\mathcal{W}^\dagger(y, -\infty)|N\rangle,
\] (27)

and \((\tilde{\mathcal{F}}^\mu_f)_\mu(x) = (\mathcal{F}^{\mu}_f)_\mu(-x)\). The calculation is not as straightforward as the one which leads to equation (25), because the soft factors that are absent in QCD factorization theorems arise in electroweak cross sections. The Dirac and gauge indices on the fermion fields in the operators equation (19) must be first disentangled, using the following Fierz relations
\[
-\tilde{\mathcal{O}}_\alpha \gamma_{\mu} \tilde{\mathcal{O}}_\beta = \frac{1}{16} \sum_{\mathcal{O}_f} \int dy e^{-2i\phi_f(y)} \langle N|\mathcal{O}_f(y)\mathcal{W}(y, -\infty)\tilde{\mathcal{O}}_f(y)\mathcal{W}^\dagger(y, -\infty)|N\rangle,
\] (28)

where \(\alpha\) and \(\beta\) denote gauge indices, and \(n\) and \(m\) Dirac indices. Along with the unitarity property of the soft Wilson line \(S^\mu_\alpha S^\nu_\alpha = 1\) and the choice of traceless isospin matrices, the operators \((\mathcal{O}^\rho_{\not{a}})_\nu\), then have matrix elements of the following form
\[
\frac{1}{2} \sum_{\mathcal{O}_f} \langle N|\mathcal{O}_{\not{a}}^\rho|N \rangle = (I \cdot \vec{t}) (\vec{t} \cdot p) S^\mu_\rho \int_0^1 dx U_f(2x\vec{t} \cdot p, 0)(\mathcal{F}_d^\mu)_\mu(x) + U_f(-2x\vec{t} \cdot p, 0)(\tilde{\mathcal{F}}_d^\mu)_\mu(x),
\] (29)

where the soft factors take the form of a vacuum matrix element \(S^\mu_{\not{a}} = \langle \text{VAC} | \text{Tr} \{S_{\not{a}} S^\rho_\mu \} | \text{VAC} \rangle\), provided initial state radiation be not taken account of.

In the Breit frame, the intermediate hadron state has invariant mass \(p^2 \sim Q^2\) as long as \(\omega = 1\) is of the order 1, therefore one can perform operator product expansion and match \(T^\mu_\nu(q, p)\) onto sums of SCET operators:
\[
T^\mu_\nu \rightarrow \frac{1}{2} \sum_{\mathcal{O}_f} \frac{\eta^\mu_\nu}{Q} \langle N|\mathcal{O}_f + \frac{\mathcal{O}_{10}}{Q} + \sum_\mu (\mathcal{O}_{1\mu})_\nu|N\rangle
+ \frac{1}{2} \sum_{\mathcal{O}_f} \frac{(\vec{t}^\mu + \vec{p}^\nu)(\vec{t}^\nu + \vec{p}^\nu)}{Q} \langle N|\mathcal{O}_f + \frac{\mathcal{O}_{10}}{Q} + \sum_\mu (\mathcal{O}_{1\mu})_\nu|N\rangle
+ \frac{i}{2} \sum_{\mathcal{O}_f} \frac{\epsilon^{\nu\rho\sigma} t^\rho p^\sigma}{Q} \langle N|\mathcal{O}_f + \frac{\mathcal{O}_{10}}{Q} + \sum_\mu (\mathcal{O}_{1\mu})_\nu|N\rangle.
\] (30)

Here the subscript \(r = 1, 2, 3\) is used to distinguish any different tensor structures in equation (5) for the quark and gluon operators. The Wilson coefficient factors are functions of the re-normalization scale \(\mu\), as well as the label operators \(\vec{P}\) and \(\vec{P}^\dagger\); consequently, the hard functions are defined by
\[
H_i(x) \equiv U_i(-2Qx, 0, Q, \mu),
\] (31)

where \(r = 1, 2, 3\) labels tensor structures, with operator indices suppressed, and the dependence on \(Q\) and \(\mu\) has been made explicit \(^9\). Substitution of equations (25), (26) and (29) gives then the final results
\[
T_1(\omega, Q^2) = -\omega \int_0^1 dx \{-\omega xH_{1,0}(\omega x)\mathcal{F}_d(x) + \sum_f [H_{1,f}(\omega x)\mathcal{F}_f(x) - H_{1,f}(-\omega x)\mathcal{F}_f(x)] + (I \cdot \vec{t}) \sum_{\not{a}=\not{b}} S_{\not{a}}^\rho \{H_{1,f}(\omega x)(\mathcal{F}_d^\rho)_\rho(x) + H_{1,f}(-\omega x)(\mathcal{F}_d^\rho)_\rho(x)\}\},
\] (32)

\[
T_2(\omega, Q^2) = \frac{4m_N^2}{\omega Q^2} \int_0^1 dx \{-\omega x(H_{1,0} + 4H_{2,0})(\omega x)\mathcal{F}_d(x) + \sum_f [H_{1,f} + 4H_{2,f}](\omega x)(\mathcal{F}_f(x) - (H_{1,f} + 4H_{2,f})(\omega x)(\mathcal{F}_f(x)] + (I \cdot \vec{t}) \sum_{\not{a}=\not{b}} S_{\not{a}}^\rho \{H_{1,f} + 4H_{2,f}(\omega x)(\mathcal{F}_d^\rho)_\rho(x) + (H_{1,f} + 4H_{2,f})(\omega x)(\mathcal{F}_d^\rho)_\rho(x)\}\},
\] (33)

\(^9\) In the Breit frame, \(q^\nu = \frac{2}{3} (\vec{p}^\nu - \vec{p}^\nu), \ p^\nu \simeq \frac{2}{3} \omega^\nu.\)

\(^{10}\) The re-normalization scale \(\mu\) can be chosen above the electroweak scale so that factorization is acquired in the symmetric phase.
and
\[
iT_\beta(\omega, Q^2) = -\frac{4m_\omega^2}{Q^2}\omega \int_0^1 dx \{-\omega x H_{f,0}(\omega x) \bar{\mathcal{F}}_0(x) + \sum_f [H_{f,i}(\omega x) \bar{\mathcal{F}}_i(x) - H_{f,i}(-\omega x) \bar{\mathcal{F}}_i(x)] + (l \cdot \bar{l}) \sum_{a,b} S_{ab}^{\mu}[H_{a,i}(\omega x) \bar{\mathcal{F}}_a^\mu(x) + H_{b,i}(-\omega x) \bar{\mathcal{F}}_b^\mu(x)] \}. \tag{34}
\]

The sum rules for parton distribution functions equations (32), (33) and (34) represent the general factorization theorem for deep inelastic neutrino scattering to all orders of perturbation theory. Now we have the computable hard coefficient functions \(H_i\) weighted by the universal non-perturbative parton distribution functions; the adjoint parton distribution functions are always accompanied by soft factors, which shall not contribute to gauge-singlet functions.

The dispersion relation equation (6) tells us that the hadron form factors \(W\) are determined by the real part of the dimensionless coefficient functions, which can only depend on \(Q\) through \(\ln(\mu^2/Q^2)\), thus confirming Bjorken scaling up to logarithmic corrections. Assuming that there are two generations of quarks treated to be of equal masses with \(m_u = m_d = m_s = m_q \equiv m\), while quarks of other flavours are treated as very heavy and integrated out so that they can be ignored, equation (12) then reduces to two \(SU(2) \times U(1)\) doublets:
\[
\left(\frac{1 + \gamma_5}{2}\right) \begin{pmatrix} d \cos \theta_c + s \sin \theta_c \\ -d \sin \theta_c + s \cos \theta_c \end{pmatrix}, \tag{35}
\]
\[
\left(\frac{1 + \gamma_5}{2}\right) \begin{pmatrix} u \\ c \end{pmatrix}, \tag{36}
\]
where \(\theta_c\) is known as the Cabibbo angle \([23, 24]\). In this assumption, by the tree level matching of the scattering of a neutrino on a quark of flavour \(f\) via the charged current, it is found that only the singlet quark coefficient functions \(U_f\) can be non-zero:
\[
\text{Re } H_{f,1}(z) = g^2 \delta(z - 1)/8m, \quad \text{Re } H_{f,2} = 0, \quad \text{Re } H_{fi}(z) = g^2 m_i^2 \delta(z - 1)/4m Q^2, \tag{37}
\]
where \(g\) is the quark-W boson coupling. The vanishing of \(\text{Re } H_{f,2}\) in the tree approximation reproduces the Callan-Gross relation \(W_1/W_2 = \omega^2 Q^2/4m_i^2\) between \(W_1\) and \(W_2\).

7. Conclusion

What has been demonstrated in this paper is the power of effective field techniques in the context of factorization for deep inelastic neutrino scattering processes. The explicit separation of collinear and soft modes and the implementation of gauge invariance for these modes hence, together with the operator product expansion, greatly simplify the analysis of the problem. The fact that nucleons are not electroweak singlets has played a role of great value, leading up to non-singlet parton distribution functions in the operator formalism. The generalized factorization involves contributions sensitive to the exchange of soft gauge bosons as well.

The electroweak gauge group \(SU(2)_L \times U(1)\) is assumed to be spontaneously broken to a subgroup \(U(1)_{em}\) at some energy well above the natural scale of parton distribution functions of non-perturbative nature. In the symmetry-breaking effective field theory, the massive \(W^\pm\) and \(Z\) bosons could be treated as very heavy and integrated out. In this case, \(W_1\) in equation (11) and \(S_i\) in equation (16) would then have to be replaced by \(SU(3) \times U(1)_{em}\) Wilson lines, i.e. they contain only gluon and photon fields, the actually massless gauge fields. Furthermore, as evaluated between equal one-nucleon states, the adjoint parton distribution functions equation (27) for the off-diagonal matrices \(t_t\) with \(t = 1, 2\) would vanish as required by the conservation of electric charge. The remaining contributions made by the non-singlet operator \(O_t^2\) in equation (29) would reduce to the subtraction of singlet contributions for quarks of charge \(-e/3\) from those of charge \(2e/3\), leaving the soft factor trivial. This might suggest that calculations would have to be done for the unbroken theory where the \(SU(2)_L\) gauge fields be restored in the Wilson lines, for the sake of securing the property of the relevant non-singlet contributions that are sensitive to soft radiation.

As an illustration of the hard scattering factorization formulate to leading order in the power expansion of the effective field theory, the tree level matching of the neutrino scattering via the charged current has been given in the approximate massless limit, leaving much to be desired admittedly. It would be very desirable to see how the coefficient functions, the parton distribution functions and the soft factors as indispensable ingredients in the factorization theorem behave for both charged and neutral current processes, for which the quark flavours can be altered by \(W^\pm\) through first-order weak interactions while by \(Z\) only in loop diagrams, in the real world where the quark masses vary. It would be also interesting to make attempts at factorization theorems that hold beyond the leading twist contribution, which might be probably achieved by generalizing the operators equations (7), (8).
and (11). Given the complication that should arise from more precise computations, the relevant further investigations would be done in separate papers.

Acknowledgments

The author would like to thank Jun Gao, Sven-Olaf Moch and Hua Xing Zhu for helpful discussions. This work is supported in part by the National Natural Science Foundation of China Grant No. 11 647 020, and the Training Programme for Young Teachers from Universities and Colleges in Shanghai Grant No. ZZSDJ15033.

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