A Lazard-like theorem for quasi-coherent sheaves *†

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Abstract

We study filtration of quasi–coherent sheaves. We prove a version of Kaplansky Theorem for quasi–coherent sheaves, by using Drinfeld’s notion of almost projective module and the Hill Lemma. We also show a Lazard-like theorem for flat quasi–coherent sheaves for quasi–compact and semi–separated schemes which satisfy the resolution property.

1 Introduction

The interlacing between quasi–coherent sheaves on an arbitrary scheme $X$ and certain compatible system of modules was outlined in Enochs & Estrada (2005).

This paper is devoted to exploit this relation in case $X$ is quasi–compact and semi–separated, and in particular when $X$ is a closed subscheme of $\mathbb{P}^n(R)$. Namely, recently Drinfeld in Drinfeld (2006) has proposed new notions of infinite dimensional vector bundles by using flat Mittag-Leffler modules, projective modules and almost projective modules. In this paper we focus on the case of almost projective modules. Following Drinfeld, an $R$-module $M$ is almost projective whenever is a direct summand of the coproduct of a projective $R$-module and a finitely generated one. It is then clear that every projective module is in turn almost projective. Then a quasi–coherent sheaf $\mathcal{M}$ on $X$ is locally almost projective if for every open affine subset Spec$R \subseteq X$ the $R$–module of sections $\Gamma(\text{Spec}R,\mathcal{M})$ is almost projective. We will show that every locally almost projective quasi–coherent sheaf on $X$ can be filtered by locally countably generated almost projective quasi–coherent sheaves. The precise formulation of our result is as follows:

**Theorem A.** Let $\mathcal{M}$ be a quasi–coherent sheaf on $X$. Then, there exists a continuous chain of quasi–coherent subsheaves $(\mathcal{M}_\alpha : \alpha < \lambda)$ of $\mathcal{M}$ such that:

- $\mathcal{M} = \bigcup_{\alpha < \lambda} \mathcal{M}_\alpha$.
- $\mathcal{M}_{\alpha+1}/\mathcal{M}_\alpha$ ($\alpha < \lambda$) is locally countably generated almost projective.

In the particular case that $\Gamma(\text{Spec}R,\mathcal{M})$ is in fact projective, we get Drinfeld’s notion of (infinite dimensional) vector bundle (see Drinfeld (2006, Section 2, Definition)) and hence Theorem A specializes to get that $\mathcal{M}_{\alpha+1}/\mathcal{M}_\alpha$ ($\alpha < \lambda$) is a locally countably generated vector bundle. Kaplansky Theorem states that every projective $R$-module $P$ can be written as $P = \bigoplus_{\alpha < \lambda} P_{\alpha}$, with $P_{\alpha}$ countably generated and projective. This is equivalent to saying that every projective module can be filtered by countably generated and projective modules. In the quasi–coherent situation this equivalence is no longer true and in fact it seems unlikely to get such a direct sum decomposition for vector bundles. But our Theorem A precisely states that Kaplansky Theorem still holds in $\mathcal{Qco}(X)$ when replacing direct sum decomposition by filtration. Our proof is

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based on the fact that a Kaplansky-like theorem is also true for almost projective \( R \)-modules (Proposition 2.2).

But perhaps the most interesting application of the techniques provided in this paper is in Theorem B:

**Theorem B.** Let \( X \) be a quasi–compact and semi–separated scheme having enough locally countably generated vector bundles (for instance if \( X \) is noetherian, separated, integral and locally factorial). Let \( \mathcal{F} \) be a flat quasi-coherent sheaf on \( X \). Then \( \mathcal{F} = \lim \rightarrow \mathcal{F}_i \), where \( \mathcal{F}_i \) is locally countably generated and flat with \( \mathcal{V}\text{dim}\mathcal{F}_i \leq 1 \) (where \( \mathcal{V} \) is the class of all vector bundles on \( X \)).

Lazard in [Lazard, 1969, Theorem 1.2] showed that every left \( R \)-module is a directed colimit of finitely generated free modules, so in particular a directed colimit of flat and finitely presented modules. Theorem B shows that every quasi–coherent sheaf is a directed colimit of locally countably generated flats. It is well-known that a finitely related flat module is projective. This seems not to be true for quasi–coherent sheaves, so we replace in Theorem B this condition by saying that the quasi–coherent sheaves \( \mathcal{F}_i \) are locally of finite projective dimension \( \leq 1 \), or, in other words, the dimension of \( \mathcal{F}_i \) with respect to the class of all vector bundles is 1 at most. Finally, in case \( X \subseteq \mathbb{P}^n(R) \) is a closed subscheme (and \( R \) is commutative noetherian, or just commutative if \( n = 1 \)) we point up in Corollary 4.10 that we can find a common upper bound of the projective dimensions of the \( \mathcal{F}_i \)’s.

The paper is structured as follows: in Section 2 we will give all notions and properties of the classes of modules that we will use in the sequel (almost projective and flat Mittag-Leffler modules) as well as, the notion of filtration with respect to a class \( L \) in a Grothendieck category. We also give the statement of the Hill Lemma in this section. Section 3 is devoted to summarize from Enochs & Estrada (2005) the equivalence between \( \mathcal{Qco}(X) \) and certain category of representations by modules of a quiver. Then we will make an explicit construction of this equivalence in case \( X \subseteq \mathbb{P}^n(R) \) is a closed subscheme. Finally Section 4 contains the main results of the paper and, in particular, the proofs of Theorems A and B.

## 2 Preliminaries

Throughout the paper all rings considered are commutative. Let us recall from [Drinfeld, 2006, Section 4] the definition of an almost projective module. As we will see, this notion generalizes the notion of a projective module.

**Definition 2.1.** Let \( R \) be a ring. An **elementary almost projective \( R \)-module** is an \( R \)-module isomorphic to a direct sum of a projective \( R \)-module and a finitely generated one. An **almost projective \( R \)-module** is a direct summand of an elementary almost projective module.

**Proposition 2.2.** Every almost projective module is a direct sum of countably generated almost projective modules.

**Proof.** Let \( T \) be an almost projective \( R \)-module. Then there exists a projective \( R \)-module \( P \) and a finitely generated \( R \)-module \( M \) such that \( T \) is a direct summand of \( P \oplus M \). By Kaplansky’s theorem, we know that \( P \) is a direct sum of countably generated projective \( R \)-modules, say \( P = \bigoplus_{i \in I} P_i \). Then, there exists an \( R \)-module \( K \) such that

\[
\left( \bigoplus_{i \in I} P_i \right) \oplus M = T \oplus K.
\]

Since \( T \) is a direct summand of a direct sum of countably generated modules, \( T \) is again a direct sum of countably generated modules by [Anderson & Fuller, 1992, Theorem 26.1]. Say
T = ∑_{j ∈ J} T_j for some index set J where T_j is a countably generated module for every j ∈ J. Clearly, each T_j is a direct summand of P ⊕ M. This implies that T_j is an almost projective R-module for each j ∈ J and T is a direct sum of countably generated almost projective modules.

**Definition 2.3.** Let R be a ring and M be a right R-module. M is a Mittag-Leffler module if the canonical map \( M ⊗_R \prod_{i ∈ I} M_i \rightarrow \prod_{i ∈ I} M ⊗_R M_i \) is monic for each family of left R-modules \( (M_i| i ∈ I) \).

**Theorem 2.4.** (Raynaud & Gruson (1971)) The following conditions are equivalent:

1. \( M \) is flat Mittag-Leffler R-module.
2. Every finite or countable subset of \( M \) is contained in a countably generated projective submodule \( P ⊆ M \) such that \( M/P \) is flat.

Notice that the second condition of the previous theorem allows to write every flat Mittag-Leffler R-module \( M \) as a direct union of projective and countably generated submodules.

Let \( C \) be a Grothendieck category. A well-ordered direct system of objects of \( C \), \( A = (A_α| α ≤ µ) \), is said to be continuous if \( A_0 = 0 \) and \( A_α = \lim_{β < α} A_β \) for all limit ordinals \( α ≤ µ \). If all morphisms in the system, \( f_{αβ} \), are monomorphisms then the sequence \( A \) is called a continuous chain of modules.

Let \( C' \) be a class of objects of \( C \). An object \( M ∈ C \) is said to be \( C' \)-filtered if there is a continuous chain \( A = (A_α| α ≤ µ) \) of subobjects of \( M \) with \( M = A_µ \) and each of the objects \( A_{α+1}/A_α \) is isomorphic to an object of \( C' \), where \( α < µ \). The chain \( (A_α| α ≤ µ) \) is called a \( C' \)-filtration of \( M \).

The following lemma, known as Hill Lemma, helps us to expand a single filtration of a module \( M \) to a complete lattice of its submodules having some good properties. It is one of the most important tools to prove our main results.

**Lemma 2.5 (Hill Lemma).** (Göbel & Trlifaj, 2006, Theorem 4.2.6) Let R be a ring, \( κ \) an infinite regular cardinal and \( C \) a set of \( < κ \)-presented modules. Let \( M \) be a module with a \( C \)-filtration \( M = (M_α| α ≤ σ) \) for some ordinal \( σ \). Then there is a family \( H \) consisting of submodules of \( M \) such that:

1. \( M ⊆ H \).
2. \( H \) is closed under arbitrary sums and intersections (that is, \( H \) is a complete sublattice of the lattice of submodules of \( M \)).
3. If \( N, P ∈ H \) such that \( N ⊆ P \), then there exists a \( C \)-filtration \( (P_γ| γ ≤ τ) \) of the module \( P/N τ ≤ σ \) such that and for each \( γ < τ \), there is a \( β < σ \) with \( P_γ/P_β \) isomorphic to \( M_{β+1}/M_β \).
4. If \( N ∈ H \) and \( X \) is a subset of \( M \) of cardinality \( < κ \), then there is a \( P ∈ H \) such that \( N \cup X ⊆ P \) and \( P/N \) is \( κ \)-presented.

**3 \( Ωco(X) \) as a Category of Representations**

Let \( X \) be a scheme and \( Ωco(X) \) be the category of quasi-coherent sheaves on \( X \). Following Enochs & Estrada (2003), the aim of this section is to give an equivalent category to \( Ωco(X) \) which will allow us to understand \( Ωco(X) \) in terms of certain compatible systems of modules.
A quiver $Q$ is a directed graph which is given by the pair $(V, E)$, where $E$ denotes the set of all edges of the quiver $Q$ and $V$ is the set of all vertices. A representation $R$ of a quiver $Q$ in the category of commutative rings means that for each vertex $v \in V$ we have a ring $R(v)$ and a ring homomorphism $R(a) : R(v) \rightarrow R(w)$, for each edge $a : v \rightarrow w$.

An $R$-module $M$ is given by an $R(v)$-module $M(v)$, for each vertex $v \in V$, and an $R(v)$-linear morphism $M(a) : M(v) \rightarrow M(w)$ for each edge $a : v \rightarrow w \in E$. Since $R(a)$ is a ring homomorphism for an edge $a : v \rightarrow w$, the $R(v)$-module $M(w)$ can be thought as an $R(v)$-module.

An $R$-module $M$ is quasi-coherent if for each edge $a : v \rightarrow w$, the morphism

$$\text{id}_{R(w)} \otimes_{R(v)} M(a) : R(w) \otimes_{R(v)} M(v) \rightarrow R(w) \otimes_{R(w)} M(w)$$

is an $R(w)$-module isomorphism. For a fixed quiver $Q$ and a fixed representation $R$ of $Q$, the category of quasi-coherent $R$-modules is defined as the full subcategory of the category $R$-Mod containing all quasi-coherent $R$-modules. We will denote it by $R_{Qco}$-Mod. We will say that the representation $R$ is flat if the ring $R(w)$ is a flat $R(v)$-module for each edge $a : v \rightarrow w$. If the representation $R$ is flat, then the category $R_{Qco}$-Mod is a Grothendieck category.

Consider the category of quasi-coherent sheaves on a scheme $(X, \mathcal{O}_X)$, denoted by $\mathcal{Qco}(X)$. By the definition of a scheme, the scheme $X$ has a family $\mathcal{B}$ of affine open subsets which is a base for $X$ such that this family uniquely determines the scheme $(X, \mathcal{O}_X)$ (for example, it is enough to take the family of the affine open subsets covering $X$ and $U \cap V$ for all $U, V$ in this family). And also this family helps to uniquely determine the quasi-coherent $\mathcal{O}_X$-modules. That is, a quasi-coherent $\mathcal{O}_X$-module is determined by giving an $\mathcal{O}_X(U)$-module $M_U$ for each $U$ and a linear map $f_{UV} : M_U \rightarrow M_V$ whenever $V \subseteq U$, $U, V \in \mathcal{B}$, satisfying:

1. $\mathcal{O}_X(V) \otimes_{\mathcal{O}_X(U)} M_U \rightarrow \mathcal{O}_X(V) \otimes_{\mathcal{O}_X(V)} M_V$ is an isomorphism with respect to the morphism $\text{id} \otimes f_{UV}$ for all $V \subseteq U$.

2. If $W \subseteq V \subseteq U$, where $W, V, U \in \mathcal{B}$, then the composition $M_U \rightarrow M_V \rightarrow M_W$ gives $M_U \rightarrow M_W$.

In this way, we are able to construct a quiver $Q = (V, E)$ with respect to the scheme $(X, \mathcal{O}_X)$. Let $\mathcal{B}$ be a base of the scheme $X$ containing affine open subsets such that $\mathcal{O}_X$ is $\mathcal{B}$-sheaf. Now, define a quiver $Q$ having the family $\mathcal{B}$ as the set of vertices, and an edge between two affine open subsets $U, V \in \mathcal{B}$ as the only arrow $U \rightarrow V$ provided that $V \subseteq U$. Fix this quiver. Take the representation $R$ as $R(U) = \mathcal{O}_X(U)$ for each $U \in \mathcal{B}$ and the restriction map $\rho_{UV} : \mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ for the edge $U \rightarrow V$. Then the functor

$$\Phi : \mathcal{Qco}(X) \rightarrow R_{Qco}$-Mod,$$

which was defined by the above argument is well-defined and, in fact, it is an equivalence of categories. Because of this equivalence we will often identify a quasi-coherent sheaf $\mathcal{M}$ with its corresponding quasi-coherent $R$-module $M$ and vice versa.

**Example 3.1.** Let $X = \mathbb{P}^n_R = \text{Proj } R[x_0, \ldots, x_n]$, $n \in \mathbb{N}$. Then take a base containing the affine open sets $D_+(x_i)$ for all $i = 0, \ldots, n$, and all possible intersections. In this case, our base contains basic open subsets of this form

$$D_+(\prod_{i \in V} x_i),$$

where $v \subseteq \{0, 1, \ldots, n\}$. So, the vertices of our quiver are all subsets of $\{0, 1, \ldots, n\}$ and we have only one edge $v \rightarrow w$ for each $v \subseteq w \subseteq \{0, 1, \ldots, n\}$ since $D_+(\prod_{i \in w} x_i) \subseteq D_+(\prod_{i \in v} x_i)$. Then
the structure sheaf takes the following values
\[ \mathcal{O}_{\mathbb{P}^n_R}(D(\prod_{i \in v} x_i)) = R[x_0, \ldots, x_n](\prod_{i \in v} x_i) \]
on each basic open set, and it is isomorphic to the polynomial ring on the ring \( R \) with the variables \( \frac{x_i}{x_j} \) where \( j = 0, \ldots, n \) and \( i \in v \). We will denote this polynomial ring by \( R[v] \). Then the representation \( R \) with respect to this quiver has vertex \( R(v) = R[v] \) and edges \( R[v] \hookrightarrow R[w] \) as long as \( v \subseteq w \).

Finally, an \( R \)-module \( M \) is quasi-coherent if and only if
\[ S^{-1}_{vw}f_{vw} : S^{-1}_{vw}M(v) \longrightarrow S^{-1}_{vw}M(w) = M(w) \]
is an isomorphism as \( R[w] \)-modules for each \( f_{vw} : M(v) \rightarrow M(w) \) where \( S_{vw} \) is the multiplicative group generated by the set \( \{x_j/x_i \mid j \in w \setminus v, \ i \in v \} \cup \{1\} \) and \( v \subset w \).

A closed subscheme \( X \subseteq \mathbb{P}^n_R \) is given by a quasi-coherent sheaf of ideals, i.e. we have an ideal \( J(v) \subseteq R[v] \) for each \( v \) with
\[ R[w] \otimes_{R[v]} J(v) \cong J(w), \]
when \( v \subseteq w \). This means \( J(v) \rightarrow J(w) \) is the localization of \( J(v) \) by the same multiplicative set \( S_{vw} \) as above. Then
\[ \frac{R[v]}{J(v)} \rightarrow \frac{R[w]}{J(w)} \]
is a localization with respect to the set \( \mathcal{S}_{vw} \). To simplify the notation, we will use \( R(v) \) instead of \( \frac{R[v]}{J(v)} \) to represent the representation of rings associated to a closed subscheme \( X \) of \( \mathbb{P}^n_R \).

**Example 3.2.** The construction of the previous example can be extended to quasi–compact and semi–separated schemes. Let \( X \) be a quasi–compact and semi–separated scheme, and let \( \mathcal{U} = \{U_0, \ldots, U_n\} \) be an affine open cover of \( X \). Let us construct a quiver \( Q_X \) whose vertices are the subsets \( v \subseteq \{0, 1, 2, \ldots, n\}, \ v \neq \emptyset, \) and where \( v \) represents the affine open \( \cap_{k \in v} U_k \), and where there is a unique arrow \( v \rightarrow w \) when \( v \subseteq w \), and corresponds to the canonical inclusion \( \cap_{k \in v} U_k \hookrightarrow \cap_{k \in v} U_k \). Then a quasi–coherent sheaf \( \mathcal{M} \) on \( X \) corresponds to a quasi–coherent \( R \)-module \( M \) on \( Q_X \) and vice versa.

## 4 Filtration of quasi–coherent sheaves

It is known that there is a bijection between the class of vector bundles in the sense of classical algebraic geometry and the class of all locally free coherent \( O_X \)-modules of finite rank (see [Hartshorne (1977)]). So, in Sheaf Theory, a vector bundle is a locally free coherent \( O_X \)-module of finite rank. Following [Drinfeld (2006)], we can achieve at least three different generalizations of this definition. The first one is getting just by avoiding the finitely generated assumption.

**Definition 4.1.** ([Drinfeld, 2006], Section 2) Let \( (X, O_X) \) be a scheme. A quasi-coherent \( O_X \)-module \( \mathcal{F} \) is said to be a vector bundle (in the sense of [Drinfeld (2006)]) if \( \mathcal{F}(U) \) is a projective \( O_X(U) \)-module for every affine open subset \( U \) of \( X \).

But then, according to [Drinfeld, 2006, Sections 2 and 4], we get other generalizations of classical vector bundles if in the previous definition we replace “projective” by “flat Mittag-Leffler” or by “almost projective”. These yield to the notions of locally flat Mittag-Leffler and locally almost projective quasi–coherent sheaf.
So, if $R$ is a representation of the structure sheaf of the scheme $(X, \mathcal{O}_X)$, a vector bundle (resp. a locally flat Mittag-Leffler or a locally almost projective) $M$ corresponds to a unique element $M$ in $R_{Q_{\infty}}\text{-Mod}$ such that each $M(u)$ is a projective (resp. flat Mittag–Leffler or almost projective) $R(u)$-module for every vertex $u$.

We are interested whether the converse of this property holds. Namely, if $M$ is a quasi-coherent $R$-module over some quiver $Q = (V, E)$ representing the scheme $(X, \mathcal{O}_X)$, and such that $M(u)$ is projective (resp. flat Mittag–Leffler or almost projective) for each vertex $u \in V$, then $M$ must be a vector bundle (resp. locally flat Mittag–Leffler or locally almost projective) in the above sense. To this end we need the following lemma, which is essentially due to Raynaud Gruson and pointed up by Drinfeld.

**Lemma 4.2.** Let $R \to S$ be a ring map and let $M$ be an $R$-module. If $M$ is a projective (resp. flat Mittag–Leffler or almost projective), then $S \otimes_R M$ is a projective (resp. flat Mittag–Leffler or almost projective) $S$-module. If, in addition, $R \to S$ is faithfully flat then the converse is also true, that is, $S \otimes_R M$ being projective (resp. flat Mittag–Leffler or almost projective) $S$-module implies that $M$ is such.

**Proof.** If $M$ is a projective (resp. flat Mittag–Leffler or almost projective) $R$-module, it is straightforward to check that $S \otimes_R M$ is a projective (resp. flat Mittag–Leffler or almost projective) $S$-module. For the second part see (Drinfeld, 2006, Section 2) and (Drinfeld, 2006, Theorem 4.2(i)).

**Proposition 4.3.** Let $X$ be a scheme. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X$. The following are equivalent:

1. $\mathcal{F}$ is vector bundle (resp. locally flat Mittag–Leffler or locally almost projective) and
2. there exists an affine open covering $X = \bigcup_{i \in I} U_i$ such that the $\mathcal{O}_X(U_i)$-module $\mathcal{F}(U_i)$ is projective (resp. flat Mittag–Leffler or almost projective) for every $i \in I$.

**Proof.** $(1) \Rightarrow (2)$ is immediate.

$(2) \Rightarrow (1)$. Let $U \subseteq X$ be an arbitrary open affine. And write $U = \bigcup_{j=1}^n D(f_j)$ where, for each $j$, $D(f_j)$ is a basic open of some $U_i$, $i \in I$. Let us denote $M = \mathcal{F}(U)$ and $R = \mathcal{O}_X(U)$. Therefore $\mathcal{O}_X(D(f_j)) = R_{f_j}$. Then the hypothesis and Lemma 4.2 applied to the ring map $\mathcal{O}_X(U_i) \to \mathcal{O}_X(D(f_j))$, give that $\mathcal{F}(D(f_j)) = R_{f_j} \otimes_R M = M_{f_j}$ is a projective (resp. flat Mittag–Leffler or almost projective) $R_{f_j}$-module. Let $S = \prod_{j=1}^n R_{f_j}$. Now $R \to S$ is faithfully flat and $S \otimes_R M = \prod_{j=1}^n M_{f_j}$ is a projective (resp. flat Mittag–Leffler and almost projective) $S$-module. Hence we infer, again by Lemma 4.2, that $M$ is projective (resp. flat Mittag–Leffler or almost projective) $R$-module.

From now on we will assume that the scheme $X$ is quasi–compact and semi–separated. Soon it will become clear that the only requirement needed in our results is that $X$ can be covered by at most countably many affine opens. But just for a sake of simplicity the reader may assume that $X \subseteq \mathbb{P}^n_R$ is a closed subscheme. We point out that we will write down explicitly when we need to impose further assumptions on the scheme.

By the previous comments, we will represent each vector bundle (resp. locally flat Mittag–Leffler or locally almost projective) $\mathcal{P}$ on $X$ by a quasi–coherent $R$-module $P$ over some quiver $Q = (V, E)$ representing $X$, such that $P(v)$ is a projective (resp. flat Mittag–Leffler or almost projective) $R(v)$-module for each $v \in V$. Our first aim in this section is to prove Theorem 4.10 (Theorem A in the introduction). As a consequence of it we provide in Corollary 4.17 with a version of Kaplansky Theorem for vector bundles in the Drinfeld’s sense.
The following lemma and proposition, which are modified versions of Enochs & Estrada (2005, Lemma 3.2, Proposition 3.3), have importance in proving our main results.

**Lemma 4.4.** Let $R' \equiv R(v) \to R(w)$ be a part of the ring representation $R$ of $X$ where $w \subseteq v$. Suppose that we have a quasi-coherent $R'$-module $M(v) \xrightarrow{f} M(w)$ and two countable subsets $X(v)$ and $X(w)$ of $M(v)$ and $M(w)$, respectively. Then there exists a quasi-coherent $R'$-submodule $M'(v) \to M'(w)$ of $M(v)$ such that $X(v) \subseteq M'(v) \subseteq M(v)$, $X(w) \subseteq M'(w) \subseteq M(w)$ and $M'(v), M'(w)$ are countably generated modules over $R(v)$ and $R(w)$, respectively.

**Proof.** Let $t \in X(w)$. Then, because of the quasi-coherence, there exists $Y_t = \{y_1, \cdots, y_{k_t}\} \subseteq M(v)$ such that

$$t = \sum_{i=1}^{k_t} r_i f(y_i), \quad r_i \in R(w).$$

Take the submodule $M'(v)$ of $M(v)$ generated by $X(v) \cup Y$ where $Y$ consists of all of $Y_t$ which has been found for each $t \in X(w)$ as above. Since $|X(v) \cup Y| \leq \aleph_0 + \aleph_0 = \aleph_0$, $M'(v)$ is countably generated. Let $M'(w)$ be the $R(w)$-submodule of $M(w)$ generated by $f(M'(v))$. Clearly $M'(w)$ is a countably generated submodule of $M(w)$ containing $X(w)$. Let us see that the submodule

$$M'(v) \xrightarrow{f|_{M'(v)}} M'(w)$$

is quasi-coherent. Since the morphism $\varphi = s \circ (f \otimes_{R(v)} id_{R(w)})$ is an isomorphism (where $s(r_w \otimes m_w) = r_w m_w$, $r_w \in R(w)$ and $m_w \in M(w)$), we only need to show that $\varphi(M'(v) \otimes_{R(v)} R(w))$ is the $R(w)$-module $M'(w) = R(w) f(M'(v))$. Indeed, $\varphi(M'(v) \otimes_{R(v)} R(w))$ is equal to the $R(w)$-module generated by $f(M'(v))$, that is, to the $R(w)$-module $M'(w)$. This implies that $M'(v) \to M'(w)$ is a quasi-coherent $R'$-submodule of $M(v) \to M(w)$. \hfill \Box

**Proposition 4.5.** Let $M$ be a quasi-coherent sheaf on $X$. If $X(v) \subseteq M(v)$ is a countable subset for each $v \subseteq \{0, 1, \ldots, n\}$, then there exists a quasi-coherent submodule $M'$ of $M$ such that $X(v) \subseteq M'(v) \subseteq M(v)$ and $M'(v)$ is a countably generated $R(v)$-module for all $v \subseteq \{0, 1, \ldots, n\}$.

**Proof.** Let $E = \{e_l : 0 \leq l \leq k\}$ be the set of all arrows defining the quiver of $X$ for some natural number $k$. We will construct by induction a family of $R$-submodules $M^{(m)}$ of $M$ satisfying:

1. $X(v) \subseteq M^{(m)}(v) \subseteq M(v)$ is countably generated for each $v$ and $m, l \in \mathbb{N}$.
2. $M^{(m)}(v) \to M^{(m)}(w)$ satisfies the quasi-coherent condition on the edge $l$, whenever $m \equiv l \pmod{(k + 1)}$ for $m \in \mathbb{N}$ and $l \in E$.
3. $M^{(m)} \subseteq M^{(m+1)}$ for all $m \in \mathbb{N}$.

When $m \geq n + 1$, think of $e_m$ as $e_l$, where $m \equiv l \pmod{(k + 1)}$ and $l \in E$. Let us consider the edge $e_0 : v \to w$. By applying Lemma 4.4 to this edge, we obtain $T_0^{(0)}(v) \to T_0^{(0)}(w)$ satisfying the quasi-coherent condition. And say $T_0^{(0)}(u) := X(u)$ for all $u \subseteq \{0, \ldots, n\}$ different from $v$ and $w$. Now from $T_0^{(0)}$, by taking $T_1^{(0)}(u)$ as the $(R(u))$-module generated by the sets $\{T_0^{(0)}(u), f_{u',u}(T_0^{(0)}) \mid f_{u',u} : M(u') \to M(u)\}$ where each morphism $f_{u',u}$ denotes the
morphism $M(a)$ where $a : u' \to u$ $(u, u' \subseteq \{0, \ldots, n\})$, we obtain a locally countably generated $R$-submodule $T_1^{(0)}$. But it is possible that we may have lost the quasi-coherent condition on $e \circ \tau_0 : v \to w$. So, we again apply Lemma 4.4 to obtain $T_2^{(0)}$ such that $T_2^{(0)}(v) \to T_2^{(0)}(w)$ satisfies the quasi-coherent condition. And by the same argument above, we can construct an $R$-submodule $T_3^{(0)}$. Continuing in this way, we obtain the family $\{T_n^{(0)}\}_{n \in \mathbb{N}}$.

Define the first term $M^{(0)}$ as the direct union of this family on $n \in \mathbb{N}$. Now assume we have constructed $M^{(m)}$ for $m \in \mathbb{N}$. Let us define $M^{(m+1)}$. Take the edge $e_{m+1} : v \to w$. We apply Lemma 4.4 to $M^{(m)} : v \to M^{(m)} : w$ to obtain $T_0^{(m+1)} : v \to T_0^{(m+1)} : w$ which satisfies the quasi-coherent condition. Define $T_0^{(m+1)}(u) := M^{(m)}(u)$ for every $u \neq v, w$. From this, we can construct an $R$-submodule $T_1^{(m+1)}$ of $M$ by the same method we did before. Again applying Lemma 4.4 to $T_1^{(m+1)} : v \to T_1^{(m+1)} : w$, we find $T_2^{(m+1)}$ such that $T_2^{(m+1)}(v) \to T_2^{(m+1)}(w)$ is quasi-coherent. By proceeding in the same way, we obtain the family $\{T_n^{(m+1)}\}_{n \in \mathbb{N}}$. So, define $M^{(m+1)} := \bigcup_{n \geq 0} T_n^{(m+1)}$. So we have constructed inductively the desired family $\{M^{(m)}\}_{m \in \mathbb{N}}$.

Finally, if we let $M' : v \to \bigcup_{m \in \mathbb{N}} M^{(m)}(v)$ for all $v \subseteq \{0, 1, 2, \ldots, n\}$, we see that the properties of being an $R$-module and the quasi-coherence condition on each edge are cofinal. So, it follows that $M'$ is a quasi-coherent $R$-submodule of $M$ containing $X(v)$ for all $v \subseteq \{0, 1, 2, \ldots, n\}$. Clearly, $M'$ is locally countably generated, since

$$|M'(v)| = \bigcup_{n \in \mathbb{N}} M^{(m)}(v)$$

for each $v$ and countable union of countable sets is again countable.

For the proof of the next Theorem, we need to fix the following notation: Let $S_v$ be the class of all countably generated almost projective $R(v)$-modules for each $v \subseteq \{0, 1, 2, \ldots, n\}$, $L$ be the class of all locally countably generated almost projective quasi-coherent $R$-modules on $Q_X$ and $C$ be the class of all locally almost projective quasi-coherent $R$-modules. Then the class $L$ contains quasi-coherent $R$-modules $M$ such that $M(v) \in S_v$ for each edge $v \subseteq \{0, 1, 2, \ldots, n\}$.

**Theorem 4.6.** Every locally almost projective quasi-coherent $R$-module is filtered by locally countably generated almost projective quasi-coherent $R$-modules.

**Proof.** Let $T$ be a quasi-coherent $R$-module belonging to the class $C$. By Proposition 2.2 we know that each $T(v)$ has an $S_v$-filtration $M_v$ for all $v \subseteq \{0, 1, 2, \ldots, n\}$. Let $H_v$ be the family associated to $M_v$ by Lemma 2.5 and $\{m_{v, \alpha} | \alpha \leq \tau_v\}$ be an $R(v)$-generating set of the $R(v)$-module $M_v$. Without lost of generality, we can assume that for some ordinal $\tau, \tau = \tau_v$ for all $v$.

We will construct an $L$-filtration $(M_v | \alpha \leq \tau)$ for $T$ by induction on $\alpha$. Let $M_0 = 0$. Assume that $M_\alpha$ is defined for some $\alpha \leq \tau$ such that $M_\alpha(v) \in H_v$ and $m_{v, \beta} \in M(v)$ for all $\beta < \alpha$ and all $v \subseteq \{0, 1, 2, \ldots, n\}$. Set $N_{v, 0} = M_\alpha(v)$. By Lemma 2.5 iv), there is a module $N_{v, 1} \in H_v$ such that $N_{v, 0} \subseteq N_{v, 1}$ and $N_{v, 1}/N_{v, 0}$ is countably generated.

By Proposition 4.5 (with $M$ replaced by $T/M_0$, and $X(v) = N_{v, 1}/M_\alpha(v)$) there is a quasi-coherent $R$-submodule $T_1$ of $T$ such that $M_\alpha \subseteq T_1$ and $T_1/M_\alpha$ is locally countably generated. Then $T_1(v) = N_{v, 1} + \langle T_v \rangle$ for a countably subset $T_v \subseteq T_1(v)$, for each $v$. Again by help of Lemma 2.5 iv), there is a module $N_{v, 2} \in H_v$ such that $T_1(v) = N_{v, 1} + \langle T_v \rangle \subseteq N_{v, 2}$ and $N_{v, 2}/N_{v, 1}$ is countably generated.

Proceeding similarly, we obtain a countable chain $(T_n | n < \aleph_0)$ of quasi-coherent $R$-submodules of $T$, as well as a countable chain $(N_{v, n} | n < \aleph_0)$ of $R(v)$-submodules of $T(v)$, for each $v$. Let $M_{\alpha + 1} = \bigcup_{n < \aleph_0} T_n$. Then $M_{\alpha + 1}$ is a quasi-coherent subsheaf of $T$ satisfying $M_{\alpha + 1}(v) = \bigcup_{n < \aleph_0} M^{(n)}(v)$.
union of countably generated almost projective $R(v)$-modules. Therefore $M_{\alpha+1}/M_\alpha \in \mathcal{L}$.

Assume $M_\beta$ has been defined for all $\beta < \alpha$ where $\alpha$ is a limit ordinal $\leq \tau$. Then we define $M_\alpha := \bigcup_{\beta < \alpha} M_\beta$.

Since $m_{v,\alpha} \in M_{\alpha+1}(v)$ for all $v$ and $\alpha < \tau$, we have $M_\alpha(v) = M(v)$. So $(M_\alpha|_{\alpha \leq \tau})$ is an $\mathcal{L}$-filtration of $M$.

Now, as an application of Theorem 4.6, we can get a version of Kaplansky Theorem for quasi-coherent sheaves on $X$ (cf. Estrada, Asensio Prest & Trlifaj, Corollary 3.12). To do this we just have to restrict $\mathcal{S}_v$ to the class of all countably generated projective $R(v)$-modules for each $v \subseteq \{0, \ldots, n\}$ in the proof of Theorem 4.6 and then using Kaplansky Theorem instead of Proposition 2.2. Notice that in this case, $L$ will be the class of all locally countably generated vector bundles on $X$ and $C$ be the class of all vector bundles.

**Corollary 4.7.** Every vector bundle on $X$ is a filtration of locally countably generated vector bundles.

Now we prove that locally flat Mittag-Leffler quasi-coherent sheaves are direct unions of locally countably generated vector bundles.

**Theorem 4.8.** Every locally flat Mittag-Leffler quasi-coherent sheaf on $X$ is a direct union of locally countably generated vector bundles.

**Proof.** Let $M$ be a locally flat Mittag-Leffler quasi-coherent $R$-module. By Theorem 2.4 each $R(v)$-module $M(v)$ is a union of countably generated projective submodules $v \subseteq \{0, 1, 2, \ldots, n\}$, that is, $M(v) = \bigcup_{i \in I_v} P^i_v$. W.l.o.g., we can assume that $I = I_v$ for each $v \subseteq \{0, 1, 2, \ldots, n\}$.

Let $i \in I$. Set $T^1(v) := P^i_v$. By Proposition 4.5 there is a quasi-coherent $R$-submodule $T^1_i$ of $M$ such that $T^1(v) \subseteq T^1_i(v)$ and $T^1_i$ is locally countably generated. By Theorem 4.6 there is a countably generated projective submodule $T^2(v)$ of $M(v)$ containing $T^1_i(v)$, for each $v$. Again, by Proposition 4.5 there is a quasi-coherent $R$-submodule $T^3_i$ of $M$ such that $T^2_i(v) \subseteq T^3_i(v)$ and $T^3_i$ is locally countably generated. Continuing in the same way, we obtain a countable chain $(T^{n+1}_i)_{n \in \mathbb{N}}$ of locally countably generated quasi-coherent subsheaves of $M$ as well as a countable chain $(T^{2n+1}_i)_{n \in \mathbb{N}}$ of $R(v)$-submodules of $M(v)$ contained in $T^{2n+1}_i(v)$, for each $v$. Let $M_i := \bigcup_{n \in \mathbb{N}} T^{n+1}_i$. Then $M_i$ is a locally countably generated vector bundle since $M(v) = \bigcup_{n \in \mathbb{N}} T^{2n+1}_i(v)$ for each $v$.

Finally, $M = \bigcup_{i \in I} M_i$ since each $M_i(v)$ contains $P^i_v$ for all $i \in I$ and $v \subseteq \{0, 1, 2, \ldots, n\}$.

Now, let $F$ be a flat $R$-module. Then there exists a short exact sequence

$$0 \to M \to \bigoplus_{j} R \to F \to 0.$$ 

Since $F$ is flat, $M$ is a pure submodule of $\bigoplus_{j} R$. Since pure submodules of flat Mittag-Leffler are flat Mittag-Leffler, we follow that $M$ is flat Mittag-Leffler. Therefore $M = \bigcup_{i \in I} M_i$ where $M_i$ is a countably generated projective submodule of $M$ for each $i \in I$. For $J$ a countable subset of $I$, set $M_i,J := M_i$. Then we may find some countable subset $J'$ of $J$ containing $J$ and $M_i,J'$ is submodule of $\bigoplus_{j'} R$. If we denote $R_i,J' := \bigoplus_{j'} R$, we get a commutative diagram

$$0 \to M_i,J' \to R_i,J' \to F_i,J' \to 0.$$ 

$$0 \to M \to \bigoplus_{j} R \to F \to 0.$$
Here, $F_{i,j'}$ is countably generated and flat since $M_{i,j'}$ is pure in $R_{i,j'}$. If we take their direct limits over $I$ and countable subsets $J'$ of $J$, we get $F = \lim_{\to} F_{i,j'}$.

Now we shall prove the main result of our paper. To do so, we will need to assume that our scheme $X$ possesses a family of locally countably generated vector bundles. This is the case whenever $X$ satisfies the resolution property (that is, every coherent sheaf is a quotient of some finite dimensional vector bundle) because, in that situation, every quasi–coherent sheaf on $X$ is the filtered union of coherent subsheafs. So the vector bundles constitute a family of generators of $\mathcal{Qco}(X)$. We find examples of such schemes whenever $X$ is noetherian, separated, integral and locally factorial by a result of Kleiman (see (Hartshorne, 1977, Ex. III.6.8)).

Let us denote by $\mathcal{V}$ the class of all vector bundles on $X$. Given a $M \in \text{R}_{\mathcal{Qco}} \text{-Mod}$ we say that $\mathcal{Vdim} M \leq n$ if there exists an exact sequence in $\text{R}_{\mathcal{Qco}} \text{-Mod}$,

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to M_0 \to 0,$$

such that $P_i \in \mathcal{V}$, for all $i = 0, \ldots, n$.

**Theorem 4.9.** Let $X$ be a scheme having enough locally countably generated vector bundles. Let $F$ be a flat quasi-coherent sheaf on $X$. Then $F = \lim_{\to} F_i$, where $F_i$ is locally countably generated and flat with $\mathcal{Vdim} F_i \leq 1$.

**Proof.** Given a flat quasi-coherent sheaf $F$ we can find a short exact sequence

$$0 \to M \to \bigoplus_{j \in J} P_j \to F \to 0,$$

where $M$ is locally flat Mittag-Leffler. By Theorem 4.8, $M = \bigcup_{i \in I} M_i$, $M_i$ is a locally countably generated vector bundle for each $i \in I$. By the same argument above, we are able to complete commutatively the following diagram

$$
0 \to M_i \xrightarrow{\mathbf{C}} \bigoplus_{j \in J'} P_j \xrightarrow{\mathbf{F}_{i,j'}} \mathbf{F} \to 0
$$

where $J' \subseteq J$ is such that $\bigoplus_{j \in J'} P_j$ is locally countably generated and $\lim \bigoplus_{j \in J'} P_j = \bigoplus_{j \in J} P_j$ and $\mathbf{F}_{i,j'}$ is locally countably generated and flat for each $(i,j')$. Since $\text{R}_{\mathcal{Qco}} \text{-Mod}$ is a Grothendieck category, direct limits are exact, therefore $\lim \mathbf{F}_{i,j'} = \mathbf{F}$. Finally, since both $M_i$ and $\bigoplus_{j \in J} P_j$ are locally countably generated vector bundles, it follows that $\mathcal{Vdim} \mathbf{F}_{i,j'} \leq 1$.

In case $R$ is commutative noetherian and $X \subseteq \mathbb{P}_R^n$ is a closed subscheme, we can replace in Theorem 4.9 the dimension with respect to the class $\mathcal{V}$ by the projective dimension. Recall that, given a quasi-coherent sheaf $M$, we say that projdim $M \leq n$ if Ext$^i(M, -) = 0$ for $i \geq n + 1$. Then $\mathcal{Qco}(X)$ has a family of generators of projective dimension $\leq n$ (see (Enochs Estrada García Rozas 2008, pg. 538)). The generators are provided from the family of $O(k)$, $k \in \mathbb{Z}$, for $\mathbb{P}_R^n$. These give the family $\{i^*(O(k)) : k \in \mathbb{Z}\}$, where $i : X \hookrightarrow \mathbb{P}_R^n(A)$ (see (Hartshorne 1977, p. 120) for notation and terminology) we will let $O(k)$ denote $i^*(O(k))$. Then (Enochs Estrada García Rozas 2008, Corollary 3.10), shows that projdim $O(k) \leq n$ for all $k \in Z$. Now using Serre’s theorem (see for example (Hartshorne 1977, Corollary II.5.18)) and that every quasi-coherent sheaf on $X$ is the filtered union of coherent subsheafs, we get that $\bigoplus_{t \in \mathbb{Z}} O(k)$ is a generator for $\mathcal{Qco}(X)$ of finite projective dimension $\leq n$ by the previous.

In this case Theorem 4.9 specializes as follows:
Corollary 4.10. Let $R$ be a commutative noetherian and $X \subseteq \mathbb{P}^n(R)$ a closed subscheme. Let $F$ be a flat quasi–coherent sheaf on $X$. Then $F = \varprojlim F_i$, where $F_i$ is locally countably generated and flat with $\text{projdim} F_i \leq n + 1$.

Proof. By the previous comments we can replace $\bigoplus_{j \in J} P_j$ by $\bigoplus_{j \in J} \mathcal{O}(k_j)^{m_j}$ in the proof of Theorem 4.9. Then from the proof we get a short exact sequence

$$0 \to M_i \to \bigoplus_{j \in J'} \mathcal{O}(k_j)^{m'_j} \to F_{i,J'} \to 0.$$ 

Now, for any quasi-coherent sheaf $N$, we have an exact sequence

$$\cdots \to \text{Ext}^{l+1}(M_i, N) \to \text{Ext}^{l+2}(F_{i,J'}, N) \to \text{Ext}^{l+2}\left(\bigoplus_{j \in J'} \mathcal{O}(n_j)^{m'_j}, N\right) \to \cdots,$$

for each $l \geq 0$. But, since $\bigoplus_{j \in J'} \mathcal{O}(n_j)^{m'_j}$ and $M_i$ are locally projective, their projective dimensions are $\leq n$ (cf. (Enochs, Estrada García Rozas 2008, Corollary 3.10)). So we get $\text{Ext}^{s+2}(F_{i,J'}, N) = 0$, for each $s \geq n$. That is, $\text{projdim}(F_{i,J'}) \leq n + 1$. \hfill $\square$

Remarks:

(1) According to (Hovey, 2001, Proposition 2.3) a more general version of Corollary 4.10 holds on $\mathcal{Qco}(X)$, for $X$ a noetherian scheme with enough locally frees which is also separated or finite-dimensional.

(2) Theorem 4.9 and Corollary 4.10 are also valid in case $X = \mathbb{P}^1_R$ for $R$ commutative (need not be noetherian). This is because the family $\{\mathcal{O}(k) : k \in \mathbb{Z}\}$ is also a family of generators of finite projective dimension $\leq 2$ in this case (see (Enochs, Estrada García Rozas & Oyonarte 2004a, Proposition 3.4)).

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