Density waves in strongly correlated quantum chains

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Abstract. We review exact numerical results for one-dimensional quantum systems with half-filled bands. The topics covered include Peierls transitions in Holstein, Fröhlich, Su-Schrieffer-Heeger, and Heisenberg models with quantum phonons, competing fermion-boson and fermion-fermion interactions, as well as symmetry-protected topological states in fermion and anyon models.

1 Introduction

The properties of quasi-one-dimensional materials such as conjugated polymers, charge-transfer salts, halogen-bridged or organic superconductors are the result of a subtle interplay of charge, spin, and lattice fluctuations, in addition to the unique effects of one-dimensional (1D) correlated quantum systems. This has stimulated intense research efforts on paradigmatic fermion and fermion-boson models [1, 2]. In particular, the question how a quasi-1D material evolves from a metal—either a Tomonaga-Luttinger liquid (TLL) [3, 4] or a Luther-Emery liquid (LEL) [5]—to an insulator has remained one of the most heavily debated issues in solid state physics for decades. Apart from band structure [6, 7] and disorder effects [8], electron-electron and electron-phonon interactions are the driving forces behind the metal-insulator transition in the majority of cases. Coulomb repulsion drives the transition to a Mott insulator (MI) with dominant spin-density-wave (SDW) fluctuations [9], whereas the coupling to the vibrational modes of the crystal triggers the Peierls transition [10] to a long-range ordered charge-density-wave (CDW) or bond-order-wave (BOW) insulator [11]. If more than one type of interaction is relevant, quantum phase transitions (QPTs) between different insulating phases become possible. Quite generally, retarded boson-mediated interactions are significantly more difficult to describe theoretically than the instantaneous Coulomb repulsion.

More recently, QPTs between topologically trivial and nontrivial states have come into the focus of attention [12, 13]. Topological phases possess characteristic zero-energy edge excitations that reflect the topological features of the bulk [14] and may either arise from topological band structures or from interactions [15]. The topological properties are protected by certain symmetries (e.g., inversion, time-reversal or dihedral symmetry [16, 17]). Such symmetry-protected topological (SPT) states have short-range quantum entanglement [18] and may displace more conventional CDW, BOW, or SDW phases. Examples include dimerized Su-Schrieffer-Heeger (SSH) models [19] and the Haldane insulator [20].

While the basic mechanisms underlying metal–insulator and insulator–insulator QPTs are well known, their detailed understanding in microscopic models remains a challenging and active field of research. Convincing evidence for TLL–insulator QPTs has been obtained for the $t$-$V$ model [21], the spinless Holstein and SSH models (Secs. 2.1 and 2.2), as well as the Edwards fermion-boson model (Sec. 2.4). Minimal settings for LEL–insulator QPTs are the spinful Holstein and the extended Hubbard model (Secs. 2.1 and 3.4). Insulator–metal–insulator or direct insulator–insulator QPTs have been explored in the extended Hubbard model (Secs. 2.1 and 3.4). Insulator–metal–insulator or direct insulator–insulator QPTs have been explored in the extended Hubbard model (Secs. 2.1 and 3.4), the Holstein-Hubbard model (Sec. 3.1), the Holstein-SSH model with competing bond and site couplings (Sec. 2.3), and the SSH model with additional Coulomb interaction (Sec. 3.2). Extended Falicov-Kimball models (Sec. 3.3) exhibit QPTs between semimetals or semiconductors and excitonic insulators. Extended Hubbard models with either an additional alternating ferromagnetic spin interaction or a bond dimerization have topologically trivial density-wave (DW) states but also SPT phases (Sec. 4.1). Finally, more exotic bosonic or even anyonic models that can be realized, in particular, with highly tunable cold atoms in optical lattices [22] exhibit superfluid, MI, CDW, and SPT states (Sec. 4.2).

In this contribution, we review the physics of a variety of lattice models for quasi-1D strongly correlated particle systems. Focusing on results from numerically exact methods such as Lanczos exact diagonalization [23, 24], the density matrix renormalization group (DMRG) [25–28], and continuous-time quantum Monte Carlo (QMC) [29–34], we discuss ground-state and spectral properties and relate them to the corresponding 1D low-energy theories [21, 35]. Given the enormous literature, we mostly restrict the scope to half-filled bands for which umklapp scattering can give rise to QPTs. Section 2 is devoted to the effects of fermion-boson coupling, whereas Coulomb...
interaction will be discussed in Sec. 3. In Sec. 4 we review recent work on SPT states. Finally, we conclude in Sec. 5.

2 Density waves from fermion-boson coupling

2.1 Holstein and Fröhlich-type models

Perhaps the simplest example of a quantum system of coupled fermions and bosons is the Hubbard model of charge carriers interacting with lattice vibrations, as described by the Holstein Hamiltonian [36] (\( h = 1 \))

\[
\hat{H}_{\text{Hol}} = -t \sum_{\langle i,j \rangle \sigma} \hat{c}_{i \sigma}^\dagger \hat{c}_{j \sigma} + \omega_0 \sum_i \hat{b}_i^\dagger \hat{b}_i - g \omega_0 \sum_i \left( \hat{b}_i^\dagger + \hat{b}_i \right) \hat{n}_i \quad \hat{n}_i = \hat{c}_i^\dagger \hat{c}_i + \hat{b}_i^\dagger \hat{b}_i.
\]

(1)

It accounts for a single tight-binding electron band emerging from nearest-neighbor hopping, quantum phonons in the harmonic approximation, and a local density-displacement electron-phonon coupling. Here, \( \hat{n}_i = \hat{c}_i^\dagger \hat{c}_i \) and \( \hat{b}_i^\dagger \hat{b}_i \) correspond to spinless fermion \( \hat{n}_i \) and boson \( \hat{b}_i \) operators at site \( i \) of a 1D lattice with \( L \) sites. Similarly, \( \hat{b}_i^\dagger \) (\( \hat{b}_i \)) creates (annihilates) a dispersionless optical (Einstein) phonon of frequency \( \omega_0 \). Half-filling corresponds to \( \langle \hat{n}_i \rangle = 1 \) (\( n = 1/2 \)) for spinful (spinless) fermions.

The physics of the Holstein model is governed by the competition between the itinerancy of the electrons and the tendency of the electron-phonon coupling to "immobilize" them. Importantly, the interaction is retarded in nature, as described by the adiabaticity ratio \( \omega_0/t \). Throughout this article, we use \( t \) as the energy unit. The electron-phonon coupling is parameterized by \( \lambda = \varepsilon_p/2t \) in the adiabatic regime (\( \omega_0/t \ll 1 \)) and by \( g = \varepsilon_p/\omega_0 \) in the anti-adiabatic regime (\( \omega_0/t \gg 1 \)) [37–39]. For the single-particle case, where \( \varepsilon_p \) is the polaron binding energy, the Holstein model has provided important insight into the notoriously difficult problem of polaron formation and self-trapping [36, 40]. The half-filled case considered here provides a framework to investigate the even more intricate problem of the Peierls metal-insulator QPT of spinless [38, 41–48] or spinful fermions [34, 41, 42, 49–59].

The spinless Holstein model is obtained from Eq. (1) by dropping spin sums and indices. Figure 1(a) shows the corresponding ground-state phase diagram from DMRG calculations [27]. At a critical coupling \( g_c(\omega_0) \), a QPT from a TLL to a CDW insulator with long-range \( q = 2K_F = \pi \sigma \) order (alternating occupied and empty sites) and a 2\( K_F \) Peierls lattice distortion [10] takes place. The insulating state can be classified as a traditional band insulator in the adiabatic regime, and as a polaronic superlattice in the anti-adiabatic regime [47, 60]. Numerical evidence for the Kosterlitz-Thouless [61] transition expected from the low-energy TLL description and the mapping to the \( t-V \) model at strong coupling [41] comes from, e.g., XXZ-model physics for large \( \omega_0 \) [47] and a cusp in the fidelity susceptibility [48].

The TLL charge parameter \( K_c \)—determining the decay of correlation functions [21, 35]—from a finite-size scaling of the long-wavelength limit of the charge structure factor \( S_c(q) = \frac{1}{Z} \sum_i c_{i-q} c_i c_{i+q} / c_i c_i c_{i-q} \) according to [21, 56]

\[
K_c = \lim_{L \to \infty} K_c(L) \quad , \quad K_c(L) = \pi \frac{S_c(q_1)}{q_1} , \quad q_1 = \frac{2\pi}{L} \quad \quad (2)
\]

is shown in Fig. 1(b). Contrary to earlier numerical results [38, 60], the TLL turns out to be repulsive (\( K_c < 1 \)) for any \( \omega_0 \) [56]. Accordingly, charge correlations (\( \sim r^{-2K_c} \)) dominate over pairing correlations (\( \sim r^{-2J_c} \)) throughout the TLL phase and show a crossover from weak to strong \( 2K_F \) power-law correlations with increasing coupling [48, 62]. As shown in Fig. 1, \( K_c = 1/2 \) at the critical point, as expected for the umklapp-driven Mott transition in a spinless TLL [21].

Figure 2 shows excitation spectra from QMC simulations [63, 64], namely the single-particle spectral function

\[
A(k, \omega) = \frac{1}{Z} \sum_{m} \left| \langle \psi_m | \hat{c}_k | \psi_n \rangle \right|^2 \left[ e^{-\beta E_m} + e^{-\beta E_n} \right] \times \delta(\omega - (E_n - E_m)) \quad \quad (3)
\]

and the phonon spectral function

\[
B(q, \omega) = \frac{1}{\sqrt{2M\omega_0}} \frac{1}{Z} \sum_{m} \left| \langle \psi_m | \hat{b}_q | \psi_n \rangle \right|^2 e^{-\beta E_m} \times \delta(\omega - (E_n - E_m)) ; \quad (4)
\]

\( E_n \) is the eigenvalue for \( |\psi_n\rangle \), \( Z \) the partition function.

In the adiabatic regime, the single-particle spectrum in the TLL phase [Fig. 2(a)] is gapless but significantly modified by the hybridization of charge and phonon modes [47, 63, 65]. In the CDW phase, it exhibits a Peierls gap and backfolded shadow bands [47, 63, 66, 67] [Fig. 2(b)]. Near the critical point, soliton excitations [68] can be observed [63]. The phonon spectrum [64] reveals the renormalization of the phonon mode due to electron-phonon coupling. In the adiabatic regime, the mode softens at the zone boundary in the TLL phase [Fig. 2(c)], becomes completely soft for \( q = 2K_F \) at the critical point [Fig. 2(d)], and hardens again in the CDW phase [63, 64, 69, 70]. In contrast, for \( \omega_0 \gg t \), the phonon mode hardens in the metallic phase and a central mode appears at \( \lambda_c \) [47]. These findings are consistent with a soft-mode transition for \( \omega_0 \ll t \) and a central-peak transition for \( \omega_0 \gg t \) [47].
A complete picture of the physics of the spinful Holstein model (1) has only emerged recently. Whereas early work [41, 54, 55] suggested the absence of a metallic phase, the existence of the latter has since been confirmed [34, 50, 53, 57–59]; for a detailed review see [62]. In terms of g-ology [21], the attractive umklapp scattering arising from the carbon-carbon bond lengths and their effect on the electronic hopping integral [68] (for related earlier work see [82]). It has a coupling term of the form $\frac{e^{-r/\xi}}{(r^2 + 1)^{3/2}} \hat{b}_{i+r} \hat{b}_{i+r} \hat{n}_{i+1}$ which interpolates between a local Holstein and a long-range Fröhlich-type coupling [81]. As shown in Fig. 4(a), for small to intermediate $\xi$, the same LEI and CDW phases are found, but for large $\xi$ and strong coupling, the nonlocal interaction gives rise to multipoles of droplets and phase separation (PS) [77], as detected from the $q = 0$ divergence of the charge structure factor [Fig. 4(b)] that implies $K_\text{c} = \infty$ and hence a divergent compressibility [21]. The CDW–PS QPT appears to be of first order [77]. Increasing the interaction range at a fixed $\lambda$ drives a CDW–LEL QPT. The concomitant suppression of CDW order gives rise to degenerate pairing and charge correlations in the Fröhlich limit $\xi \to \infty$.

### 2.2 Su-Schrieffer-Heeger model

The SSH model of polyacetylene captures fluctuations of the carbon-carbon bond lengths and their effect on the electronic hopping integral [68] (for related earlier work see [82]). It has a coupling term of the form $\sum_i \mathcal{B}_i (\hat{Q}_{i+1} - \hat{Q}_i)$, where $\hat{Q}_i \sim \hat{b}_i^\dagger + \hat{b}_i$ is the displacement of atom $i$ from its equilibrium position and $\mathcal{B}_i = \sum_\sigma (\hat{c}_i^\dagger \sigma \hat{c}_{i+1} \sigma + \text{H.c.})$. The phonons have an acoustic dispersion $\omega_q = \omega_0 \sin(q/2)$. Reviews in the context of conjugated polymers were given in [83, 84]. Theoretical arguments [85, 86] and exact numerical results [86] suggest that, at half-filling, the SSH model...
is equivalent to the simpler optical SSH model [87]

\[ H_{SSH} = -t \sum_i B_i + \omega_0 \sum_i \hat{b}_i^\dagger \hat{b}_i - g \omega_0 \sum_i \hat{B}_i (\hat{b}_i^\dagger + \hat{b}_i) . \]  

Here, \( \hat{b}_i^\dagger \) and \( \hat{b}_i \) are associated with an optical phonon mode describing fluctuations of the bond lengths. Figure 5 illustrates the quantitative agreement of the two models for the single-particle Green function and the dynamic bond structure factor, which can be attributed to the inherent dominance of \( q = 2k_F = \pi \) order at half-filling [86].

For spinless fermions, the model (7) has a repulsive TLL ground state with dominant BOW correlations below a critical coupling \( \lambda_c \) (for SSH models \( \lambda = g^2 \omega_0/2t \)), and an insulating Peierls ground state with a long-range \( 2k_F \) BOW (alternating weak and strong bonds) for \( \lambda > \lambda_c(\omega_0) \) [55, 85, 86]. The transition from power-law to long-range BOW correlations can be seen in Fig. 5(b) from \( S_b(r) = \langle (\hat{B}_r - \langle \hat{B}_r \rangle)(\hat{B}_0 - \langle \hat{B}_0 \rangle) \rangle \). At the critical point, the correlation functions are consistent with \( K_c = 1/2 \). In the adiabatic regime, the phase transition is again of the soft-mode type [86]. Apart from the interchange of the roles of charge and bond degrees of freedom, the spinless SSH model is in many respects similar to the spinless Holstein model, including spectral and thermodynamic properties [86, 88]. However, subtle differences arise due to the different symmetries of the two models (class BDI of the general classification [89] for the SSH model, class AI with broken particle-hole and chiral symmetry for the Holstein model) [88]. Note that the name SSH model is often used to refer to the mean-field approximation of the true SSH model, i.e., a fermionic Hamiltonian with dimerized hopping but no phonons (see also Sec. 4.1).

In contrast to the spinful Holstein model, the spinful SSH model does not have a metallic phase. Although quantum fluctuations significantly reduce the dimerization compared to the mean-field solution [90], the ground state is an insulating BOW-Peierls state (COS) for any \( \lambda > 0 \) irrespective of \( \omega_0 \) [55, 59, 85–87, 91–93]. Direct numerical evidence for this conclusion is shown in Fig. 5(c). Because Eq. (7) is symmetric under the transformation \( \hat{c}_i \rightarrow (-1)^i \hat{c}_i^\dagger \) that interchanges spin and charge operators, spin and charge correlators are exactly equal. Therefore, the finite-size estimate of \( K_c(L) < 1 \) (a reliable indicator of a spin gap for models with \( SU(2) \) spin symmetry [53]) implies \( K_c = K_s = 0 \) by symmetry and hence an insulating state [86]. The spinful SSH model hence provides an example where Peierls’ theorem [10] holds even for quantum phonons. This property can be traced back to the fact that forward scattering vanishes whereas umklapp scattering is repulsive (rather than attractive, as in the Holstein model) and hence always relevant [59]. Excitation spectra for the spinful SSH model closely resemble those of the spinless model in the ordered phase [86].

2.3 Holstein-Su-Schrieffer-Heeger model

Whereas Holstein and SSH models have been studied intensely, the even more complex problem of competing site and bond couplings—which in principle coexist in most materials—has been addressed by QMC only recently [94] using the Holstein-SSH Hamiltonian

\[
\hat{H} = -t \sum_i \hat{B}_i + \sum_{\alpha} \omega_{0,\alpha} \hat{b}_{i,\alpha}^\dagger \hat{b}_{i,\alpha} - g(\omega_{0, s} \sum_i \tilde{n}_{i, s}(\hat{b}_{i,s}^\dagger + \hat{b}_{i,s}) + g(\omega_{0, b} \sum_i \hat{B}_i (\hat{b}_{i,b}^\dagger + \hat{b}_{i,b}) 
\]

with independent site (\( \alpha = s \)) and bond (\( \alpha = b \)) phonon modes as well as corresponding coupling constants \( \lambda_\alpha \).

Of particular interest is the question if the metallic phase of the Holstein model is stable with respect to the SSH coupling, or if metallic behavior is entirely absent as in the SSH model (7). In terms of g-ology, both couplings produce negative backscattering matrix elements that give rise to a spin gap. On the other hand, the umklapp matrix elements have opposite sign and can therefore compensate, allowing for an extended LEL (C1S0) metallic region. This picture is confirmed by QMC data [94] summarized in the qualitative phase diagram in Fig. 6.

Fig. 4. (a) Phase diagram and (b) charge structure factor in the three phases of the nonlocal electron-phonon model (6) from QMC [77]. Here, \( \omega_0/t = 0.5 \).

Fig. 5. (a) Single-particle Green function and dynamic bond structure factor for the spinful optical SSH model (7) and the original SSH model. (b) Real-space bond correlations of the spinless optical SSH model. The full (dashed) line corresponds to \( 1/x^2 (1/x) \). (c) Finite-size estimates of the TLL parameters \( K_c \) and \( K_s \). All results are from QMC simulations [86].

Fig. 6. Qualitative phase diagram of the Holstein-SSH model. Here, \( (\pi, \tau) \) are associated with an optical phonon mode describing fluctuations of the bond lengths. Figure 5 illustrates the quantitative agreement of the two models for the single-particle Green function and the dynamic bond structure factor, which can be attributed to the inherent dominance of \( q = 2k_F = \pi \) order at half-filling [86].
If the SSH coupling dominates, the system is a BOW insulator just like the SSH model. If the Holstein coupling dominates, a CDW ground state exists. Both states are of type $C_0S_0$. If the couplings are comparable, the competition between the two orders results in a metallic LEL phase. Starting in the CDW phase and increasing $\lambda_b$, the correlation functions in Fig. 7 reveal a suppression (enhancement) of CDW (BOW) order and a QPT to the LEL phase with power-law correlations. At stronger SSH couplings, long-range BOW order emerges. For all parameters, spin correlations remain exponential due to the spin gap. The QPT between the two different Peierls states is found to be continuous, and in the adiabatic regime the site it enters. The fermion-boson coupling in Eq. (9) involves two soft-mode transitions for the site and bond phonon modes, respectively [94]. The single-particle gap is minimal but finite at the QPT [94]. These numerical results contradict earlier approximate results suggesting a first-order BOW–CDW transition [95] or a ferroelectric phase with coexistence of BOW and CDW order [96].

### 2.4 Edwards model

The discussion so far has revealed that the coupling to the lattice can modify the transport properties of low-dimensional systems to the point of insulating behavior. Quantum transport in general takes place in some “background”, which may consist of lattice but also spin or orbital degrees of freedom. For instance, a key problem in the ground state of quantum impurities is the spin–orbital mixing. This is understood in terms of a Peierls-type instability, which is driven by the competition between the hopping and on-site interaction terms. The Peierls instability results in a kink in the density of states (DOS) at the Fermi energy. The presence of fermions in a lattice can lead to the formation of a Peierls gap, which suppresses the hopping between nearest-neighbor sites.

In this model, every hop of a (spinless fermionic) charge carrier along a 1D transport path either creates a (local bosonic) excitation with energy $\omega_0$ in the background at the site it leaves, or annihilates an existing excitation at the site it enters. The fermion–boson coupling in Eq. (9) differs significantly from the Holstein and SSH couplings discussed before. In particular, no static distortion arises in the latter case, excitations of the background are energetically costly ($\omega_0/t > 1$) and the background relaxation rate $\lambda$ for example due to quantum fluctuations. Any particle motion is affected by the background and vice versa. In fact, the Edwards model describes three different regimes: quasi-free, dissipative, and boson-assisted transport [101]. In the latter case, excitations of the background are energetically costly ($\omega_0/t > 1$) and the background relaxation rate is small ($\lambda/t \ll 1$), i.e., the background is “stiff”. Then, for a half-filled band, strong correlations can develop and even drive the system into an insulating state by establishing long-range CDW order [102–104].

Figure 8(a) shows the DMRG phase diagram at half-filling. The critical values were determined from the charge gap and the CDW order parameter [103]. The metallic phase found below a critical boson frequency and above a critical relaxation rate is a repulsive TLL ($K_0 < 1$) [103]. Remarkably, particle motion is possible even for $\lambda = 0$, in lowest order by a vacancy-restoring six-step process where 3 bosons are excited in steps 1–3 and afterwards consumed in steps 4–6 with the particle moving two sites [101]. In contrast to the spinless Holstein model, the CDW state
the Edwards model is a few-boson state \([102]\). As shown in Fig. 8(b), at low and high band filling \(n\) the attractive interaction mediated by the slow phonon becomes strong enough to give rise to first an attractive TLL \((K_c > 1)\) and finally electronic phase separation \([105]\).

The metal–insulator QPT at half-filling is also reflected in the photoemission spectra measured by

\[
A^\pm(k, \omega) = \sum_n |\langle \psi_n^\pm | \hat{c}_k^\dagger \psi_0 \rangle|^2 \delta[\omega \mp (E_n^\pm - E_0)],
\]

where \(\hat{c}_k^\dagger = \hat{c}_k^\dagger \equiv \hat{c}_k\); \(|\psi_0\rangle\) is the ground state for \(N_e\) particles and \(|\psi_n^\pm\rangle\) the \(n\)-th excited state with \(N_e \pm 1\) particles. Figure 9 shows \(A(k, \omega) = A^-(k, \omega) + A^+(k, \omega)\) in the regime where background excitations have a large energy and the bosons strongly affect particle transport. The quasiparticle mass is significantly enhanced and a renormalized band structure appears. However, if \(\lambda\) is sufficiently large, the system remains metallic, as indicated by a finite spectral weight at the Fermi energy \(E_F\) [Fig. 9(a)]. As the possibility of relaxation reduces a gap opens at \(k_F = \pi/2\) [Fig. 9(b)], indicating insulating behavior. We note the internal feedback mechanism: The collective boson excitations originate from the motion of the charge carriers and have to persist long enough to finally inhibit particle transport, thereby completely changing the nature of the many-particle ground state. This complex boson-particle dynamics leads to a new, correlation-induced band structure reminiscent of the (extended) Falicov-Kimball model, with a very narrow valence band and a rather broad conduction band [inset of Fig. 9(b)]. The asymmetric masses can be understood by “dopping” a perfect CDW state: To restore the CDW order a doped hole has to be transferred by the above-mentioned six-step process of order \(O(t^2/\omega_0)\), while a doped particle can move by a two-step process of order \(O(t^2/\omega_0)\) \([102]\). By decreasing \(\omega_0\) at fixed \(\lambda\) the fluctuations overcome the correlations and the system returns to a metallic state. However, the latter differs from the state we started with. In particular, \(A(k, \omega)\) in Fig. 9(c) shows sharp absorption features only near \(k_F\) and “overdamping” at the zone boundaries where the spectrum is dominated by bosonic excitations.

### 2.5 Heisenberg spin-Peierls models

The Peierls (dimerization) instability triggered by the lattice degrees of freedom can be observed not only in quasi-1D itinerant electron systems but also in spin chains with magneto-elastic couplings. Experimentally, such behavior was first seen in the 1970s for organic compounds of the TTF and TCNQ family \([107]\). Interest in the subject revived after the discovery of the first inorganic spin-Peierls compound \(\text{CuGeO}_3\) in 1993 \([108]\), in particular due to the fact that the displaceable spin-Peierls transition in this material does not involve phonon softening. Instead, the Peierls-active optical phonon modes with frequencies \(\omega_{0,1} \approx J\) and \(\omega_{0,2} \approx 2J\) \((J\) being the exchange coupling between neighboring \(\text{Cu}^{2+}\) ions) form well separated spin\(\frac{1}{2}\) chains) harden by about 5% at the transition which therefore occurs at very strong spin-phonon coupling \([109]\). Phonon hardening for experimentally relevant parameters was demonstrated for the magnetoresistive XY model by calculating the dynamic structure factor \([110]\). The physics of \(\text{CuGeO}_3\) reveals that the canonical adiabatic treatment of the lattice \([111,112]\) is inadequate for this material \([113]\). Instead, the application of numerical methods to paradigmatic quantum models yields key information about the nature of the phase transition and the correct models for inorganic spin-Peierls materials.

The simplest model containing all important features of a spin-Peierls system is an antiferromagnetic Heisenberg chain, \(\hat{H}_{\text{Heis}} = J \sum_i \hat{S}_i \cdot \hat{S}_{i+1}\) \((J < 0, \hat{S}_i\) is a spin\(\frac{1}{2}\) operator at site \(i\), coupled to Einstein quantum phonons: \(\hat{H} = \hat{H}_{\text{Heis}} + \hat{H}_{\text{SP}}^{(1,d)} + \omega_0 \sum_i \hat{b}_i^\dagger \hat{b}_i\).

Here we consider two different spin-phonon couplings,

\[
\hat{H}_{\text{SP}}^{1,d} = g \omega_0 \sum_i (\hat{b}_i^\dagger + \hat{b}_i) \hat{S}_i \cdot \hat{S}_{i+1},
\]

and

\[
\hat{H}_{\text{SP}}^{\alpha} = g \sum_i (\hat{b}_{i+1}^\dagger + \hat{b}_{i+1} - \hat{b}_{i}^\dagger - \hat{b}_{i}) \hat{S}_i \cdot \hat{S}_{i+1}.
\]

The local coupling \(\hat{H}_{\text{SP}}^{1,d}\) captures the modification of the spin exchange by a local lattice degree of freedom (modeling, e.g., side-group effects) \([114,115]\). The difference coupling \(\hat{H}_{\text{SP}}^{\alpha}\) describes a linear dependence of the spin exchange on the difference between the phonon amplitudes at sites \(i\) and \(i+1\) \([116]\). A first insight into these models can be gained by integrating out the phonons in the anti-adiabatic limit \(\omega_0 \gg J\) to obtain an effective Heisenberg model with longer-ranged interactions that give rise to frustration. The spin Hamiltonian \(\hat{H} = \hat{H}_{\text{Heis}} + \alpha \hat{S}_1 \cdot \hat{S}_{i+2}\) has a dimerized ground state (alternating long and short bonds) for \(\alpha = \alpha_c = 0.241167\) \([117]\). Accordingly, the spin-phonon coupling must be larger than a nonzero critical value \(g_c(\omega_0)\) for the spin-Peierls instability to occur \([115,116,118]\). This is similar to the Holstein model (1) but in contrast to the static limit \(\omega_0/J = 0\).

Figure 10 shows the phase diagram of the model (11) for either the coupling (12) or (13) from two-block \([116]\).
and four-block [119] DMRG calculations, respectively. The QPT from the the gapless spin-liquid state to the gapped dimerized phase was detected using the well-established criterion of a level crossing between the first singlet and the first triplet excitation; the latter was derived for the frustrated spin chain [116, 120, 121]. For finite systems, the singlet lies above the triplet excitation in the spin liquid, and the two levels become degenerate with the singlet ground state as \( L \to \infty \). In the symmetry-broken gapped phase, the lowest singlet state becomes degenerate with the ground state. The Heisenberg spin-Peierls model with quantum phonons is in the same universality class as the frustrated spin chain [116]. The phonon spectral function has been analyzed in [122]. Related spin-boson models exhibiting TLL and CDW phases have also been investigated in the context of dissipative quantum systems [123].

3 Density waves and Coulomb interaction

3.1 Holstein-Hubbard model

From the 1D Hubbard model [124], it is well established that a local Coulomb repulsion favors a correlated MFI with dominant \( 2k_F \) SDW fluctuations [21]. In contrast to CDW and BOW order, the continuous SU(2) spin symmetry cannot be spontaneously broken [125]. Instead, the SDW correlations are critical (\( \sim 1/r \)) [21, 126]. Of key interest is the interplay or competition of retarded electron-phonon and instantaneous electron-electron interactions that determines if the ground state is a CDW/BOW, SDW, or LEL state. A minimal but rich model capturing this interplay is the Holstein-Hubbard model with Hamiltonian

\[
\hat{H} = \hat{H}_{\text{Hol}} + U \sum_i \hat{n}_i^{\uparrow} \hat{n}_i^{\downarrow}. \tag{14}
\]

The ground-state phase diagram of Eq. (14) was the subject of intense debate. Even after early claims [41] (unfounded [62] but supported by RG calculations [54, 55]) of the absence of metallic behavior in the spinful Holstein model were contradicted by DMRG results [50], numerical work on the Holstein-Hubbard model initially focused on strong couplings where a direct SDW–CDW QPT is observed. Evidence for an intermediate metallic phase—expected from the adiabatic connection to the Holstein model as \( U \to 0 \) —at weaker couplings was obtained with a variety of different methods [34, 53, 57, 58, 127–129]. Very recently, it was shown that the absence of such a phase in RG calculations [54, 55] is due to the neglected momentum dependence of the interaction [59]. A detailed discussion of these contradictory findings was given in [62].

The currently most reliable phase boundaries come from DMRG [57, 58] and QMC [34, 53] calculations. The DMRG critical values shown in Fig. 11 were obtained from a finite-size scaling of the single-particle, charge-, spin-, and neutral gaps defined as

\[
\Delta_{c_1} = E_0(L + 1, \frac{1}{2}) + E_0(L - 1, \frac{1}{2}) - 2E_0(L, 0),
\]
\[
\Delta_{c_2} = E_0(L + 2, 0) + E_0(L - 2, 0) - 2E_0(L, 0),
\]
\[
\Delta_n = E_0(L, 1) - E_0(L, 0),
\]
\[
\Delta_s = E_0(L, 0) - E_0(0, 0).
\]

(15)

The single-particle spectral function in Figs. 12(a) and (b) has a gap at \( E_F \) in both phases, but distinct soliton excitations and backfolded shadow bands only in the CDW phase. Spin-charge separation [21] can be observed for strong interactions [76, 130]. The dynamic charge structure factor

\[
S_c(q, \omega) = \frac{1}{Z} \sum_{mn} \langle \psi_m | \hat{\rho}_q | \psi_n \rangle^2 e^{i\omega \beta} \delta(\omega - (E_m - E_n)).
\]

(16)

with \( \hat{\rho}_q = \sum_n e^{iqr}(\hat{n}_n - n)/\sqrt{L} \) in Figs. 12(c),(d) reveals a \( q = 0 \) charge gap in both insulating phases and the renor-
The competition between electron-phonon and electron-electron interaction has also been studied in the framework of the SSH-UV Hamiltonian

$$\hat{H} = \hat{H}_{\text{SSH}} + U \sum_i \hat{n}_{i\uparrow} \hat{n}_{i\downarrow} + V \sum_i \hat{n}_i \hat{n}_{i+1},$$

which is directly relevant for conjugated polymers [83, 138]. The phase diagram from QMC simulations [87] is shown in Fig. 13. A key difference to the Holstein-Hubbard model is that no metallic phase results from the competing interactions. Instead, for $\omega_0 > 0$ and realistic values $U > 2V$, the ground state is a MI with critical SDW correlations (C0S1) for $\lambda < \lambda_c$, and a BOW Peierls state (C0S0) for $\lambda > \lambda_c$ [42, 86, 87, 139]; in contrast to the continuous suppression of CDW correlations by the Hubbard repulsion in the Holstein-Hubbard model, the amplitude of BOW correlations is enhanced by the Coulomb repulsion in the SSH-UV model [86, 140–143]. Finally, for large $U$, the SSH-UV model is closely related to the spin-Peierls models discussed in Sec. 2.5 [87].
The asymmetric Hubbard model has been used to investigate various many-body effects in (mixed)intermediate-valence rare-earth and transition-metal compounds, including the DW–PS QPT [146], electronic ferroelectricity [147, 148] and (pressure-induced) exciton condensation [149], as well as multiorbital correlation physics in cold atoms [150]. Regarding $\sigma$ as an orbital flavor, the asymmetric Hubbard model is equivalent to the extended Falicov-Kimball model (EFKM) [144, 145, 147, 151].

$$\hat{H}_{\text{EFKM}} = -t_c \sum_{\langle i,j \rangle} \hat{c}_i^\dagger \hat{c}_j + U \sum_i \hat{n}_i \hat{n}_i + \frac{D}{2} \sum_i \left( \hat{c}_i^\dagger \hat{c}_i - \frac{1}{2} \right),$$

(19)

describing two species of spinless fermions, namely, light $c$ (or $d$) electrons and heavy $f$ electrons. A finite $f$-bandwidth allows for $f$-electron coherency, which will take account of a mixed-valence situation as well as of $f$-electron $f$-hole (exciton) bound-state formation and condensation [151, 152]. By contrast, $t_f = 0$ in the original FKM [153], so that the number of $f$-electrons is strictly conserved and no coherence between $f$ and $c$ electrons can arise [154].

The left panel of Fig. 14 shows the DMRG phase diagram at half-filling. Depending on the orbital level splitting there exist staggered orbital ordered (SOO) or band insulator (BI) phases, separated by a critical excitonic insulator (EI) [157]. In the absence of true long-range order in one dimension, the characteristic signatures of an excitonic Bose-Einstein condensate are a power-law decay of the correlator $\langle X_i^+ X_j \rangle$ with $X_i^+ = \hat{c}_i^\dagger f_i$ and a divergence of the excitonic momentum distribution $N(q) = \langle X_i^+ X_q \rangle$ with $X_i^+ = \frac{1}{\sqrt{E}} \sum_k \hat{c}_k^\dagger q \hat{f}_k$ for the lowest-energy state (which has $q = 0$ for the case of a direct gap). The criticality of the EI can also be detected from the von Neumann entropy and the central charge $c^c(L) \simeq 1$. Monitoring the coherence length and binding energy with increasing $U$ yields clear evidence for a BCS–BEC crossover [155]. The addition of an electron–phonon coupling term to the EFKM leads to a competition between an “excitonic” CDW and a “phononic” CDW, while an additional Hund’s coupling promotes an excitonic SDW state [158].

### 3.4 Extended Hubbard model

Another important and intensely investigated purely electronic model is the extended Hubbard model (EHM)

$$\hat{H}_{\text{EHM}} = -t \sum_{\langle i,j \rangle} \hat{c}_i^\dagger \sigma \hat{c}_j + U \sum_i \hat{n}_i \hat{n}_i + V \sum_i \hat{n}_i \hat{n}_{i+1}.$$

(20)

It describes the competition between a local Hubbard repulsion $U$ and a nonlocal (nearest-neighbor) repulsion $V$. The phase diagram at half-filling has been determined by analytical [159–162] and numerical [163–166] methods. While there is agreement that for $U < 2V$ ($U > 2V$) the ground state has long-range (critical) 2kF CDW (SDW) correlations, the criticality of the QPTs and the possibility of an intermediate BOW phase remain under debate. The right panel of Fig. 14 shows the currently perhaps most accurate DMRG phase diagram [156]. The CDW phase is of type COS0, whereas the SDW phase has COS1. Below a critical end point, they are separated by a narrow COS0 phase with long-range BOW order [156, 162, 165]. Exactly on the CDW–BOW critical line $\Delta_c = 0$ but $\Delta_c, \Delta_s > 0$, corresponding to an LEL (C1S0) [156, 159, 164, 165]. The CDW–BOW QPT changes from continuous (XY universality, central charge $c = 1$) to first order at the tricritical point ($U/t, V/t \simeq (5.89, 3.10)$) [156]. The SDW–BOW QPT is characterized by the opening of the spin gap. A detailed discussion of the low-energy theory and correlation functions has been given in [159]. Optical excitation spectra were calculated in [167]. While it does not account for retardation effects, the EHM shares many of the features of the Holstein-Hubbard, Holstein-SSH, and SSH-UV models discussed in Sec. 2. Material-specific EHM such as Hückel-Hubbard-Ohno and Peierls-Hubbard-Ohno models have been studied in detail with the DMRG method [168, 169]. Finally, a TLL to $4k_F$-CDW QPT as a function of the Coulomb interaction range can be observed at quarter-filling [170].

### 4 Density waves and symmetry protection

#### 4.1 Dimerized Extended Hubbard model

We now explore the competition between traditional DW insulators and SPT insulators (SPTIs). A prominent representative of an SPTI is the Haldane insulator (HI) phase of the spin-1 Heisenberg chain [20]. Recently, it has been demonstrated that an SPT state also exists in the EHM with an additional ferromagnetic spin interaction ($J < 0$) on every other bond [171],

$$\hat{H} = \hat{H}_{\text{EHM}} + J \sum_{i=1}^{L/2} \hat{S}_{2i-1} \cdot \hat{S}_{2i}.$$

(21)
Here, $\hat{S}_i = \frac{1}{2} \sum_{\sigma} \hat{c}^\dagger_{i\sigma} \sigma_{\alpha\beta} \hat{c}_{i\sigma}$. Since the EHM behaves like a spin-$\frac{1}{2}$ chain for large $U/V$, the Heisenberg term in Eq. (21) promotes the formation of spin-1 moments from the EAHM takes the form [181]. With Haldane’s generalized Pauli principle [184], the anyon concept becomes important also in 1D systems. A fascinating question is if the HI phase observed, e.g., in the extended anyon-Hubbard model (EAHM) [178, 185] also exists in the extended anyon-Hubbard model (EAHM).

After a fractional Jordan-Wigner transformation of the anyon operators, $\hat{a}_i \rightarrow \hat{b}_i e^{\theta \sum_{\ell=1}^{L} \sigma_{i+l}}$ [186], the Hamiltonian of the EAHM takes the form [181]

$$\hat{H}_{\text{EAHM}} = -t \sum_i \left( \hat{b}^\dagger_i \hat{b}_{i+1} e^{i\theta \hat{n}_i} + e^{-i\theta \hat{n}_i} \hat{b}^\dagger_{i+1} \hat{b}_i \right) + U \sum_i \hat{n}_i (\hat{n}_i - 1)/2 + V \sum_i \hat{n}_i \hat{n}_{i+1},$$

4.2 Extended anyon-Hubbard model

Ultracold atomic gases in optical lattices provide the possibility to study not only fermions or bosons but also anyons. Exchanges of the latter result in a phase factor $e^{i\theta}$ in the many-body wave function. The statistical parameter $\theta$ can take on any value between 0 and $\pi$, so that anyons interpolate between bosons and fermions [182, 183]. With Haldane’s generalized Pauli principle [184], the anyon concept becomes important also in 1D systems. A fascinating question is if the HI phase observed, e.g., in the extended Bose-Hubbard model (EBHM) [178, 185] also exists in the extended anyon-Hubbard model (EAHM).

The phase boundaries of the EAHM (EBHM) with $\theta = \pi/4 (\theta = 0)$ and $n_p = 2$ are shown in the top panel of the phase diagram of the dimerized extended Hubbard model [Eq. (22)] from infinite-size DMRG. The CDW–SPTI QPT is continuous with $c = 1/2$ (first order) below (above) the tricritical Ising point where $c = 7/10$ [175]. Dashed and dotted lines are the CDW–BOW–SDW phase boundaries of the pure EHM (cf. Fig. 14). Inset: phase diagram of the extended Hubbard model with an alternating ferromagnetic Heisenberg interaction [Eq. (21)] from infinite-size DMRG [171, 172, 176].

For a nonzero dimerization $\delta$, the pure EHM both $\Delta_s$ and $\Delta_n$ vanish at the continuous CDW–BOW QPT. For a nonzero dimerization $\delta$, the neutral gap closes whereas $\Delta_\alpha$ remains finite, indicating that the CDW–SPTI QPT belongs to the Ising universality class. At strong coupling, all gaps remain finite across the QPT. The jump in the spin gap $\Delta_n$ indicates a first-order transition. At very large $U$, the low-lying excitations of Eq. (22) are related to the chargeless singlet and triplet excitations of an effective spin-Peierls Hamiltonian.

At criticality, the central charge $c$ can easily be extracted from the entanglement entropy [177, 178]. For periodic boundary conditions, conformal field theory predicts the von Neumann entropy to be $S_L(\ell) = \frac{c}{3} \ln \left( \frac{\ell}{\pi} \right) + s_1$ where $s_1$ is a nonuniversal constant [179]. A finite-size estimate for the central charge is then obtained via [180]

$$c^*(L) = \frac{3}{10} \left( S_L(L/2) - S_L(L/2) \right),$$

taking the doubled unit cell of the SPTI into account. The bottom panel of Fig. 16 gives $c^*(L)$, calculated along the PI-CDW QPT line by varying $U$ and $V$ simultaneously at fixed dimerization. With increasing $U$ there is clear evidence for a crossover from $c^*(L) \approx 1/2$ to $c^*(L) \approx 7/10$, signaling Ising tricriticality. A bosonization-based field-theory analysis of the power-law (exponential) decay of the CDW, SDW, and BOW correlations confirms the universality class of the tricritical Ising model [175].
Fig. 16. Top: DMRG charge ($\Delta n$), spin ($\Delta s$), and neutral ($\Delta c$) gaps for the extended Hubbard model with bond dimerization [Eq. (22)]. Here, $\delta/t = 0.2$, $U/t = 4$ (left) and $U/t = 12$ (right). The SPTI (CDW) phase is marked in gray (white). The spin gap exhibits a jump $\delta s \equiv \Delta s(V^+ - V^-)$ at $V^c/t$. Bottom: Central charge $c^*(L)$ along the CDW–SPTI transition line from DMRG calculations. The data indicate Ising universality ($c = 1/2$) for $U < U_t$ and, most notably, a tricritical Ising point with $c = 7/10$ at $U_t$ (red dotted line) [175]. The inset shows results for the extended Hubbard model with an additional spin-spin interaction [Eq. (21)] [171].

Fig. 17. Most notably, the HI—located between MI and DW insulating phases in the EBHM—survives for any finite fractional phase, i.e., in the anyonic case [181]. Likewise, the superfluid (SF) appears for very weak coupling. The critical values for the MI–HI (HI–DW) QPT (squares) and the HI–DW QPT (circles) were determined from a divergence of the correlation length $\xi$ with increasing DMRG bond-dimension $\chi$; the model becomes critical with central charge $c = 1$ and $c = 1/2$, respectively.

The HI may naively be expected to disappear in the EAHM with $\theta > 0$ which has neither time reversal ($\mathcal{T}$) nor inversion ($\mathcal{I}$) symmetry. However, it has been shown that there exists a nontrivial topological phase protected by the combination of $\hat{R}^z = e^{i\pi} \sum (\delta_i - 1)$ and $\hat{K} = e^{i\theta} \sum \delta_i (\delta_i - 1)^{1/2}$ $\hat{I} \hat{I}$ $\mathcal{T}$ [181]. A nonlocal order parameter $O$ can be constructed that discriminates between states that are symmetric under both $\hat{K}$ and $\hat{R}^z$ and states that are not. The middle panel of Fig. 17 demonstrates that $O$ can be used to distinguish the topologically trivial MI and DW phases ($O = 1$) from the topologically nontrivial HI ($O = -1$).

Valuable information about topological phases is also provided by the entanglement spectrum $\xi_{\alpha}$ [188]. The concept of entanglement is inherent in any DMRG algorithm based on matrix-product states. Dividing the system into two subsystems, $\xi_{\alpha} = -2 \ln \lambda_{\alpha}$ is determined by the singular values $\lambda_{\alpha}$ of the reduced density matrix [171]. The lower panel of Fig. 17 shows the entanglement spectrum of the EBHM with $U/t = 5$. In the HI phase the entanglement spectrum is expected to be at least four-fold degenerate, reflecting the broken $Z_2 \times Z_2$ symmetry. This is clearly seen in the HI phase. By contrast, in the trivial MI and DW phases, the lowest entanglement level is always nondegenerate.

5 Conclusions
The 1D correlated quantum systems reviewed here exhibit a remarkably rich variety of physical properties that can
be studied and understood in particular by powerful numerical methods. For the case of half-filled bands considered, metallic phases are either spinless TLLs or spinful LLLs of repulsive nature, i.e., with dominant CDW or BOW correlations. The insulating phases fall into three categories: (i) long-range ordered with a spontaneously broken Ising symmetry (CDW, BOW), (ii) critical with no symmetry breaking (SDW, EI), (iii) topologically nontrivial with short-range entanglement (HI). While the existence of these phases and the phase transitions can in principle be inferred from the low-energy field theory, the details for a given microscopic model typically require numerical solutions. In particular, mean-field, variational or even bosonization/ RG approaches are in general not sufficient, especially for problems with retarded interactions.

Despite the significant advances reviewed here, 1D correlated quantum systems remain an active, rewarding and efficient, especially for problems with retarded interactions.

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Author contributions

Both authors contributed equally to the preparation of the manuscript and approved it in its final form.

References

1. N. Tsuda, K. Nasu, A. Yanase, and K. Siratori, Electronic Conduction in Oxides (Springer-Verlag, Berlin, 1991).
2. A. R. Bishop and B. I. Swanson, Los Alamos Sci. 21, 133 (1993).
3. S. Tomonaga, Prog. Theor. Phys. 5, 544 (1950).
4. J. M. Luttinger, J. Math. Phys. 4, 1154 (1963).
5. A. Luther and V. J. Emery, Phys. Rev. Lett. 33, 589 (1974).
6. F. Bloch, Z. Phys. 57, 545 (1929).
7. A. H. Wilson, Proc. Roy. Soc. London, Ser. A 133, 458 (1931).
8. P. W. Anderson, Phys. Rev. 109, 1492 (1958).
9. N. F. Mott, Metal-Insulator Transitions (Taylor & Francis, London, 1990).
10. R. Peierls, Quantum theory of solids (Oxford University Press, Oxford, 1955).
11. J.-P. Pouget, C. R. Phys. 17, 332 (2016).
12. J. E. Moore, Nature 464, 194 (2010).
13. M. Z. Hasan and C. L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
14. B. I. Halperin, Phys. Rev. B 25, 2185 (1982).
15. M. Hohenadler and F. F. Assaad, J. Phys.: Condens. Matter 25, 14, 143201 (2013).
16. Z.-C. Gu and X.-G. Wen, Phys. Rev. B 80, 155131 (2009).
17. F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B 85, 075125 (2012).
18. X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013).
19. S. Ryu and Y. Hatsugai, Phys. Rev. Lett. 89, 077002 (2002).
20. F. D. M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).
21. T. Giamarchi, Quantum Physics in One Dimension (Clarendon Press, Oxford, 2004).
22. I. Bloch, J. Dalibard, and W. Zwerger, Rev. Mod. Phys. 80, 885 (2008).
23. J. K. Cullum and R. A. Willoughby, Lanczos Algorithms for Large Symmetric Eigenvalue Computations, volume I & II of Progress in scientific computing (Birkhäuser, Boston, 1985).
24. A. Weiße and H. Fehske, Lect. Notes Phys. 739, 529 (2008).
25. S. R. White, Phys. Rev. Lett. 69, 2863 (1992).
26. U. Schollwöck, Rev. Mod. Phys. 77, 259 (2005).
27. E. Jeckelmann and H. Fehske, Riv. Nuovo Cimento 30, 259 (2007).
28. I. P. McCulloch, arXiv:0804.2509 (2008).
29. A. N. Rubtsov, V. V. Savkin, and A. I. Lichtenstein, Phys. Rev. B 72, 035122 (2005).
30. F. F. Assaad and T. C. Lang, Phys. Rev. B 76, 3, 035116 (2007).
31. E. Gull, A. J. Millis, A. I. Lichtenstein, A. N. Rubtsov, M. Troyer, and P. Werner, Rev. Mod. Phys. 83, 349 (2011).
32. O. Syljuasen and A. W. Sandvik, Phys. Rev. E 66, 046701 (2002).
33. A. Sandvik, R. Singh, and D. Campbell, Phys. Rev. B 56, 14510 (1997).
34. R. T. Clay and R. P. Hardikar, Phys. Rev. Lett. 95, 096401 (2005).
35. J. Voit, Rep. Prog. Phys. 58, 977 (1995).
36. T. Holstein, Ann. Phys. (N.Y.) 8, 325 (1959); 8, 343 (1959).
37. G. Wellein and H. Fehske, Phys. Rev. B 58, 6208 (1998).
38. A. Weiße and H. Fehske, Phys. Rev. B 58, 13526 (1998).
39. M. Capone, W. Stephan, and M. Grilli, Phys. Rev. B 80, 155131 (2009).
40. A. Alvermann, H. Fehske, and S. A. Trugman, Phys. Rev. B 81, 165113 (2010).
41. J. E. Hirsch and E. Fradkin, Phys. Rev. B 27, 4302 (1983).
42. L. G. Caron and C. Bourbonnais, Phys. Rev. B 29, 4230 (1984).
43. H. Zheng, D. Feinberg, and M. Avignon, Phys. Rev. B 39, 9405 (1989).
44. R. H. McKenzie, C. J. Hamer, and D. W. Murray, Phys. Rev. B 53, 9676 (1996).
45. R. J. Bursill, R. H. McKenzie, and C. J. Hamer, Phys. Rev. Lett. 80, 5607 (1998).
46. S. Datta, A. Das, and S. Yarlagadda, Phys. Rev. B 71, 235118 (2005).
47. M. Hohenadler, G. Wellein, A. R. Bishop, A. Alvermann, and H. Fehske, Phys. Rev. B 73, 24, 245120 (2006).
48. M. Weber, F. F. Assaad, and M. Hohenadler, arXiv:1704.07913 (2017).
49. F. Guinea, J. Phys. C: Solid State Phys. 16, 4405 (1983).
50. E. Jackelmann, C. Zhang, and S. R. White, Phys. Rev. B 60, 1, 7950 (1999).
51. A. Weiße, H. Fehske, G. Wellein, and A. R. Bishop, Phys. Rev. B 62, R747 (2000).
52. Y. Takada and A. Chatterjee, Phys. Rev. B 67, 081102 (2003).
53. R. P. Hardikar and R. T. Clay, Phys. Rev. B 75, 245103 (2007).
54. I. P. Bindloss, Phys. Rev. B 71, 205113 (2005).
55. H. Bakrim and C. Bourbonnais, Phys. Rev. B 76, 195115 (2007).
56. S. Ejima and H. Fehske, Europhys. Lett. 87, 27001 (2009).
57. H. Fehske, G. Hager, and E. Jackelmann, Europhys. Lett. 84, 57001 (2008).
58. S. Ejima and H. Fehske, J. Phys.: Conf. Ser. 200, 1, 012031 (2010).
59. H. Bakrim and C. Bourbonnais, Phys. Rev. B 91, 085114 (2015).
60. H. Fehske, M. Holicki, and A. Weiße, in Advances in Solid State Physics 40, edited by B. Kramer, 235–249 (Vieweg, Wiesbaden, 2000).
61. J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).
62. J. Greitemann, S. Hesselmann, S. Wessel, F. F. Assaad, and M. Hohenadler, Phys. Rev. B 92, 245132 (2015).
63. M. Hohenadler, H. Fehske, and F. F. Assaad, Phys. Rev. B 83, 11, 115106 (2011).
64. M. Weber, F. F. Assaad, and M. Hohenadler, Phys. Rev. B 94, 245138 (2016).
65. V. Meden, K. Schönhammer, and O. Gunnarsson, Phys. Rev. B 50, 15, 11179 (1994).
66. J. Voit, L. Perfetti, F. Zwick, H. Berger, G. Margaritondo, G. Grüner, H. Höchst, and M. Grioni, Science 290, 501 (2000).
67. H. Fehske, G. Wellein, G. Hager, A. Weiße, and A. R. Bishop, Phys. Rev. B 69, 165115 (2004).
68. W. P. Su, J. R. Schrieffer, and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979).
69. C. E. Creffield, G. Sanguinetti, and M. Capone, Eur. Phys. J. B 44, 175 (2005).
70. S. Sykora, A. Hübsch, K. W. Becker, G. Wellein, and H. Fehske, Phys. Rev. B 71, 045112 (2005).
71. M. Hohenadler and F. F. Assaad, Phys. Rev. B 87, 075149 (2013).
72. L. Balents and M. P. A. Fisher, Phys. Rev. B 53, 12133 (1996).
73. K.-M. Tam, S.-W. Tsai, and D. K. Campbell, Phys. Rev. B 84, 165123 (2011).
74. J. Voit, Eur. Phys. J. B 5, 505 (1998).
75. C. Zhang, E. Jackelmann, and S. R. White, Phys. Rev. B 60, 20, 14 092 (1999).
76. W. Q. Ning, H. Zhao, C. Q. Wu, and H. Q. Lin, Phys. Rev. Lett. 96, 156402 (2006).
77. M. Hohenadler, F. F. Assaad, and H. Fehske, Phys. Rev. Lett. 109, 116407 (2012).
78. M. Weber, F. F. Assaad, and M. Hohenadler, Phys. Rev. B 91, 235150 (2015).
79. J. Cardy, Scaling and Renormalization in Statistical Physics (Cambridge University Press, Cambridge, England, 1996).
80. J. T. Devreese and A. S. Alexandrov, Rep. Prog. Phys. 72, 6, 066501 (2009).
81. A. S. Alexandrov and P. E. Kornilovitch, Phys. Rev. Lett. 82, 807 (1999).
82. S. Barisic, J. Labbé, and J. Friedel, Phys. Rev. Lett. 25, 919 (1970).
83. D. Baeriswyl, D. Campbell, and S. Mazumdar, in Conjugated Conducting Polymers, edited by H. Kiess, volume 102 of Springer Series in Solid-State Sciences, 7–133 (Springer Berlin Heidelberg, 1992).
84. W. Barford, Electronic and Optical Properties of Conjugated Polymers (Oxford University Press, 2005).
85. E. Fradkin and J. E. Hirsch, Phys. Rev. B 27, 1680 (1983).
86. M. Weber, F. F. Assaad, and M. Hohenadler, Phys. Rev. B 91, 245147 (2015).
87. P. Sengupta, A. W. Sandvik, and D. K. Campbell, Phys. Rev. B 67, 245103 (2003).
88. M. Weber, F. F. Assaad, and M. Hohenadler, Phys. Rev. B 94, 155115 (2016).
89. A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008).
90. M. Nakahara and K. Maki, Phys. Rev. B 25, 7789 (1982).
91. W. Su, Solid State Commun. 42, 7, 497 (1982).
92. G. T. Zimanyi, S. A. Kivelson, and A. Luther, Phys. Rev. Lett. 60, 2089 (1988).
93. W. Barford and R. J. Bursill, Phys. Rev. B 73, 045106 (2006).
94. M. Hohenadler, Phys. Rev. Lett. 117, 206404 (2016).
95. S. Kivelson and D. Hone, Phys. Rev. B 28, 4833 (1983).
96. J. Rossler and D. Gottlieb, J. Phys.: Condens. Matter 2, 16, 3723 (1990).
97. I. G. Bednorz and K. A. Müller, Z. Phys. B 64, 189 (1986).
98. G. H. Jonker and J. H. van Santen, Physica B 16, 337 (1950).
99. M. Berciu, Physics 2, 55 (2009).
100. D. M. Edwads, Physica B 378-380, 133 (2006).
101. A. Alvermann, D. M. Edwads, and H. Fehske, Phys. Rev. Lett. 98, 056602 (2007).
102. G. Wellein, H. Fehske, A. Alvermann, and D. M. Edwards, Phys. Rev. Lett. 101, 136402 (2008).
103. S. Ejima, G. Hager, and H. Fehske, Phys. Rev. Lett. 102, 106404 (2009).
104. S. Ejima and H. Fehske, Phys. Rev. B 80, 155101 (2009).
105. S. Ejima, S. Sykora, K. W. Becker, and H. Fehske, Phys. Rev. B 86, 155149 (2012).
106. H. Fehske, S. Ejima, G. Wellein, and A. R. Bishop, J. Phys.: Conf. Ser. 391, 012152 (2012).
107. J. W. Bray, H. R. Hart, Jr., L. V. Interrante, I. S. Jacobs, J. S. Kasper, G. D. Watkins, S. H. Wee, and J. C. Bonner, Phys. Rev. Lett. 35, 744 (1975).
108. M. Hase, I. Terasaki, and K. Uchinokura, Phys. Rev. Lett. 70, 3651 (1993).
179. P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004).
180. S. Nishimoto, Phys. Rev. B 84, 195108 (2011).
181. F. Lange, S. Ejima, and H. Fehske, Phys. Rev. Lett. 118, 120401 (2017).
182. J. M. Leinaas and J. Myrheim, Nuovo Cimento B 37, 1 (1977).
183. F. Wilczek, Phys. Rev. Lett. 49, 957 (1982).
184. F. D. M. Haldane, Phys. Rev. Lett. 67, 937 (1991).
185. E. G. Dalla Torre, E. Berg, and E. Altman, Phys. Rev. Lett. 97, 260401 (2006).
186. T. Keilmann, S. Lanzmich, I. P. McCulloch, and M. Roncaglia, Nat. Commun. 2, 361 (2011).
187. E. Berg, E. G. Dalla Torre, T. Giamarchi, and E. Altman, Phys. Rev. B 77, 245119 (2008).
188. H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2008).