Calculating non-perturbative quantities through the world-line formalism

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Abstract. We present two applications of the world-line formalism to the calculation of non-perturbative quantities in QCD. The first quantity is the free energy of the gluon plasma in the high-temperature limit; the second quantity is the pair-production rate in the chromo-electric field of a flux tube. In the first case, where effects of spatial confinement in the dimensionally-reduced 3D Yang-Mills theory are primarily important, we calculate the free-energy density of a gluon propagating in the stochastic background fields through a suitable parametrization of the area- and the perimeter laws of the Wilson loop, which enters the corresponding one-loop effective action. In this way, we find that the order of the leading correction to the Stefan-Boltzmann free energy changes from $O(\lambda)$ for $N \sim 1$ to $O(\lambda^{3/2})$ for $N \gg 1$, where $\lambda = g^2 N$ is the finite-temperature 't Hooft coupling, and $N$ is the number of colors. In the second case, we find that, in the London limit of the dual superconductor, the Schwinger pair-production rate, $\sim e^{-\text{const} \cdot m^2}$, goes over to $e^{-\text{const} \cdot m}$. Given that the flux-tube field is static, we find such a conversion of the Gaussian distribution into an exponential one, remarkable.

1. Free energy of the gluon plasma in the high-temperature limit
In this Section, we address an important issue regarding the leading correction to the Stefan-Boltzmann law for the free-energy density of the gluon plasma at high temperatures. As we will see, this correction has the order $O(g^2)$ for $N \sim 1$, while this order changes to $O(\lambda^{3/2})$ for $N \gg 1$, where $\lambda = g^2 N$ is the finite-temperature 't Hooft coupling, and $N$ is the number of colors. The corrections to the Stefan-Boltzmann law stem from the spatial confinement of gluons constituting the plasma, as well as from the Polyakov loop. For our analysis, we will use the method developed in Refs. [1, 2]. We start with representing the partition function of the finite-temperature Euclidean Yang-Mills theory in the form

$$Z(T) = \left\langle \int \mathcal{D}a_\mu^a \exp \left[ -\frac{1}{4g^2} \int_0^\beta dx_4 \int_V d^3x (F_{\mu\nu}^a [A])^2 \right] \right\rangle,$$

where $\beta \equiv 1/T$, and $V$ is the three-dimensional volume occupied by the system. In Eq. (1), we have modeled spatial confinement of $a_\mu^a$-gluons by means of the stochastic background fields $B_\mu^a$. For this purpose, the full Yang-Mills field $A_\mu^a$ has been represented as a sum $A_\mu^a = B_\mu^a + a_\mu^a$, and the stochastic field $B_\mu^a$ has been averaged over with some measure $\langle \ldots \rangle$. Clearly, at
finite temperature $T$, both the $a_{\mu}^a$- and the $B_{\mu}^a$-fields obey the periodic boundary conditions $a_{\mu}^a(x, \beta) = a_{\mu}^a(x, 0)$ and $B_{\mu}^a(x, \beta) = B_{\mu}^a(x, 0)$. Integrating over the $a_{\mu}^a$-gluons in the Gaussian approximation, and disregarding for simplicity gluon spin degrees of freedom, one obtains

$$Z(T) = \langle \det [-(D_{\mu}^a[B])] \rangle^{-\frac{1}{2}} \exp \{-(N^2 - 1) \text{Tr} \ln [-(D_{\mu}^a[B])]\}, \quad (2)$$

with the covariant derivative $(D_{\mu}^a[B])_{\nu}^a = \partial_{\mu} f_{\nu}^a + f_{abc} B_{\mu}^b f_{\nu}^c$. Equation (2) includes the color degrees of freedom of $a_{\mu}^a$-gluons, and accounts for their $2(N^2 - 1)$ physical polarizations. In the one-loop approximation for the $a_{\mu}^a$-field, this equation can be simplified further:

$$Z(T) \simeq \exp \{-(N^2 - 1) \langle \text{Tr} \ln [-(D_{\mu}^a[B])]\rangle\}. \quad (3)$$

In Eq. (3), "Tr" includes the trace "tr" over color indices and the functional trace over space-time coordinates.

The free-energy density $F(T)$ is defined through the standard formula

$$\beta VF(T) = -\ln Z(T). \quad (4)$$

Using further for $\ln [-(D_{\mu}^a[B])]$ the proper-time representation, one has

$$F(T) = -(N^2 - 1) \cdot 2 \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s} \int Dz_\mu e^{-\frac{1}{2} \int_0^s d\tau \dot{z}_\mu^2} \langle W[z_\mu] \rangle. \quad (5)$$

The integration in Eq. (5) is performed over trajectories $z_\mu(\tau)$, which obey the periodic boundary conditions: $z_\mu(\tau) = z_\mu(0) + \beta n$ and $z(s) = z(0)$. The vector-function $z_\mu(\tau)$ describes therefore only the shape of the trajectory, while the factor $\beta V$ on the left-hand side of Eq. (4) stems from the integration over positions of the trajectories. Furthermore, the summation over the winding number $n$ yields a factor of 2, which accounts for winding modes with $n < 0$. The zero-temperature part of the free-energy density, corresponding to the zeroth winding mode, has been subtracted [1]. Finally, the Wilson loop that enters Eq. (5), reads $W[z_\mu] = \frac{1}{N^2 - 1} \text{tr} \mathcal{P} \exp (i \oint dz_\mu B_\mu)$, where $B_{\mu} = B_{\mu}^a a_\mu$, and $(a_\mu)^{bc} = -i f^{abc}$ is a generator of the adjoint representation of the group SU($N$).

According to the lattice data [3], the correlation function $\langle g^2 H_i(x) H_k(x') \rangle$ exceeds by an order of magnitude the correlation function $\langle g^2 E_i(x) H_k(x') \rangle$. This fact allows one to approximately factorize $\langle W[z_\mu] \rangle$ as $\langle W[z_\mu] \rangle \simeq \langle W[z] \rangle \prod_{n=-\infty}^{\infty} \langle P^n \rangle$, where $\langle W[z] \rangle = \left\langle \frac{1}{N^2 - 1} \text{tr} \mathcal{P} \exp (i \oint dz_\mu B_\mu) \right\rangle$ is the averaged purely spatial Wilson loop, and $\langle P^n \rangle = \left\langle \frac{1}{N^2 - 1} \text{tr} \mathcal{P} \exp (i n \oint dz_\mu B_\mu) \right\rangle$ is a generalization of the Polyakov loop to the case of $n$ windings. Upon this factorization, the world-line integral over $z_\mu(\tau)$ in Eq. (5) becomes that of a free particle, which yields

$$F(T) = -2(N^2 - 1) \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s} e^{-\frac{\beta^2 s^2}{4}} \langle P^n \rangle \int Dz e^{-\frac{1}{2} \int_0^s d\tau \dot{z}_\mu^2} \langle W[z] \rangle. \quad (6)$$

In order to calculate the world-line integral over $z(\tau)$, we notice that the Wilson-loop average in the adjoint representation can be written as [4]

$$\langle W[z] \rangle = \frac{1}{1 + \frac{1}{N^2}} \left( e^{-\sigma \Sigma} + \frac{1}{N^2} e^{-c g^2 \frac{\sigma^2}{4} T \sigma} \right). \quad (7)$$
Here, $\Sigma$ is the area of the minimal surface bounded by the contour $z(\tau)$, and $c$ is some positive dimensionless constant, which will be determined below. Furthermore, Eq. (7) obeys the normalization condition $\langle W[0] \rangle = 1$. The second exponential on the right-hand side of Eq. (7) represents the perimeter law $e^{-mL}$, where $L = \int_0^s d\tau |z|$ is the length of the contour $z(\tau)$, and the constant $m$ has the dimensionality of mass. Here, we have substituted $L$ by $\sqrt{\Sigma}$, and parametrized $m$ through the soft scale $g^2 NT$ as $m = c \cdot g^2 N^2 T$. The spatial string tension $\sigma$ in the adjoint representation can be expressed in terms of the spatial string tension $\sigma_f$ in the fundamental representation by means of Casimir scaling: $\frac{\sigma}{\sigma_f} = \frac{2N^2}{N^2 - 1}$. This ratio is equal to $9/4$ for $N = 3$, while going to $2$ in the large-$N$ limit. At temperatures $T > T_s$ of interest, where $T_s$ is the temperature of dimensional reduction, one can express $\sigma_f$ in terms of the string tension in the 3D Yang-Mills theory, which was calculated analytically in Ref. [5]. The corresponding expression for $\sigma_f$ reads$^3$ $\sigma_f = \frac{N^2 - 1}{2\pi} (g^2 T)^2$, which yields the following spatial string tension in the adjoint representation: $\sigma = \frac{1}{16\pi} (g^2 N^2 T)^2$.

Hence, the free-energy density (6) can be written in the form $F = F_1 + F_2$, where the term $F_1$ corresponds to the exponential $e^{-\sigma \Sigma}$ from Eq. (7), while the term $F_2$ corresponds to the exponential $e^{-c \sigma^2 \Sigma \sqrt{\Sigma}}$ from the same equation. Clearly, in the large-$N$ limit, $F_1 \gg F_2$ due to the relative factor of $\frac{1}{\sqrt{\Sigma}}$, so that the thermodynamics of the gluon plasma in that limit is fully determined by spatial confinement. Therefore, let us start with calculating the world-line integral $I \equiv \oint \mathcal{D}z \, e^{-\frac{i}{2} \int_0^s d\tau |z|^2 - \sigma \Sigma}$, which enters the term $F_1$. To this end, we implement for the minimal area $\Sigma$ the following ansatz: $\Sigma = \frac{1}{2} \int_0^s d\tau |z \times \dot{z}|$. It corresponds to a parasol-shaped surface made of thin segments. Furthermore, since $\int_0^s d\tau \dot{z} = 0$, the point where the segments merge is the origin. Therefore, the chosen ansatz for $\Sigma$ automatically selects from all cone-shaped surfaces bounded by $z(\tau)$ the one of the minimal area. We use further the approximation $\Sigma \approx \sqrt{\frac{1}{2} s}$, where $f \equiv \frac{1}{2} \int_0^s d\tau (z \times \dot{z})$. In general, $\frac{1}{2} \int_0^s d\tau |z \times \dot{z}|$ can be larger than $\sqrt{\frac{1}{2} s}$. This happens if, in the course of its evolution in spatial directions, the gluon performs backward and/or non-planar motions. Once this happens, the vector product $(z \times \dot{z})$ changes its direction, and the integral $\int_0^s d\tau (z \times \dot{z})$ receives mutually cancelling contributions. This so-called non-backtracking approximation is widely used in order to simplify the parametrizations of minimal surfaces allowing for an analytic calculation of the corresponding world-line integrals [7]. Using this approximation, one can calculate the integral $I$ by representing the exponential $e^{-\sigma \Sigma}$ as $e^{-\sigma \Sigma} = \int_0^{\infty} \frac{d\lambda}{\sqrt{\lambda}} \, e^{-\frac{\lambda |z|^2}{4\sigma}}$, and introducing further an auxiliary space-independent magnetic field $H$ according to the formula

$$e^{-Af^2} = \frac{1}{(4\pi A)^{3/2}} \int d^3H \, e^{-\frac{H^2}{4\sigma}} + iHf, \quad \text{where} \quad A > 0. \quad (8)$$

The world-line integral gets then reduced to the one for a spinless particle of an electric charge $1$ interacting with the constant magnetic field $H$, i.e. to the bosonic Euler-Heisenberg-Schwinger Lagrangian, which has the form $[8]$

$$I = \frac{\sigma}{2\pi^{5/2} \sqrt{s}} \int_0^{\infty} dH \, \frac{H^5 / \sinh(Hs)}{(H^2 + \sigma^2)^2}. \quad (9)$$

Integrating further over $\lambda$, we obtain for the world-line integral at issue:

$$I = \frac{\sigma}{2\pi^{5/2} \sqrt{s}} \int_0^{\infty} dH \, \frac{H^5 / \sinh(Hs)}{(H^2 + \sigma^2)^2}. \quad (9)$$

$^3$ Note that, for $N = 3$, the coefficient $\frac{1}{16\pi} \approx 0.32$ in this formula agrees remarkably well with the value of $0.566^2$, which was used in Ref. [6] for the parametrization of $\sigma_f$ at high temperatures.
In the case of \( N = 3 \), the corresponding free-energy density reads
\[
F_1 \big|_{N=3} = -\frac{18\sigma}{5\pi^3} \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^2} e^{-\frac{\sigma^2}{4s} \langle P^n \rangle} \int_0^\infty dH \frac{H^3}{H^2 + \sigma^2 s^2}.
\]

To perform the perturbative expansion of this expression, we introduce a dimensionless integration variable \( h = H/\sigma \). Furthermore, we notice that, in the high-temperature limit of interest, \( \langle P^n \rangle \simeq \langle P \rangle \), where \( \langle \rangle \)

\[
\langle P \rangle = 1 + \mathcal{O}(g^3).
\]

To find the order of the leading \( g \)-dependent term of the perturbative expansion, we use the approximation \( \sinh(\sigma hs) \simeq \sigma hs \cdot \left( 1 + \frac{(\sigma hs)^2}{6} \right) \), which yields for \( F_1 \big|_{N=3} \) the following expression:
\[
F_1 \big|_{N=3} \simeq -\frac{9\langle P \rangle}{10\pi^2} \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^3} \cdot \frac{e^{-\frac{\sigma^2}{4s} \langle P \rangle}}{\left( 1 + \frac{\sigma^2}{\sqrt{6}} \right)^2}.
\]

Approximating further the sum over winding modes by the first two terms, we obtain
\[
F_1 \big|_{N=3} \simeq -\frac{9\langle P \rangle T^4}{10\pi^2} \left[ 17 - \frac{10}{\sqrt{6}} \sigma \beta^2 + \mathcal{O}((\sigma \beta^2)^2) \right].
\]

Clearly, since \( \sigma = \mathcal{O}(g^4) \), the obtained term \( -\frac{10}{\sqrt{6}} \sigma \beta^2 \) also has the order \( \mathcal{O}(g^4) \). Nevertheless, due to Eq. (10), the order of the leading \( g \)-dependent term of the perturbative expansion of \( F_1 \big|_{N=3} \) is 3, rather than 4.

We proceed now to the calculation of the free-energy density \( F_2 \) for \( N = 3 \), which will allow us to find the value of the constant \( c \) in Eq. (7). The corresponding world-line integral \( \int \mathcal{D}z e^{-\frac{1}{2} \int_0^\infty \lambda d\tau - \frac{g^2 T}{2} \sqrt{\Sigma}} \) can be calculated by using again the approximation \( \Sigma \simeq \sqrt{T^2} \). The fourth root in the so-emerging exponential, \( e^{-g^2 T \sqrt{T^2}} \), can be got rid of by using two identical auxiliary integrations as follows:
\[
e^{-g^2 T \sqrt{T^2}} = \frac{1}{\pi} \int_0^\infty d\lambda \frac{1}{\sqrt{\lambda}} \int_0^\infty d\mu \frac{1}{\sqrt{\mu}} e^{-\lambda - \mu - \frac{g^2 T \sqrt{T^2}}{3\lambda \mu}}.
\]

Introducing now once again the auxiliary magnetic field \( H \) according to the formula (8), we obtain for the exponential at issue the following representation:
\[
e^{-g^2 T \sqrt{T^2}} = \frac{64}{\pi^{5/2}} \frac{1}{(g^2 T)^6} \int_0^\infty d\lambda \lambda^{5/2} e^{-\lambda} \int_0^\infty d\mu e^{-\mu} \int_0^\infty dh \frac{h^3}{\sinh((\xi T)^2)} \int_0^\infty d\lambda \frac{\lambda^{5/2} e^{-\lambda}}{(16\lambda^2 h^2 + 1)^2}.
\]

Performing now the functional \( z \)-integration as in Eq. (9), and integrating further over \( \mu \), which can be done analytically, we obtain the following intermediate expression:
\[
F_2 \big|_{N=3} = -\frac{256T^4}{\pi^{7/2}} \xi^2 \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^2} e^{-\frac{\sigma^2}{4s} \langle P^n \rangle} \int_0^\infty dh \frac{h^3}{\sinh((\xi T)^2)} \int_0^\infty d\lambda \frac{\lambda^{5/2} e^{-\lambda}}{(16\lambda^2 h^2 + 1)^2}.
\]

Here, we have denoted \( \xi \equiv g^2, \ h \equiv H/(\xi T)^2 \), and made \( s \) dimensionless by rescaling it as \( s_{\text{new}} = T^2 s_{\text{old}} \). By using the approximation \( \sinh((\xi T)^2) \simeq (\xi T)^2 |1 + (\xi T)^2/6| \), we have
\[
F_2 \big|_{N=3} \simeq -\frac{256T^4}{\pi^{7/2}} \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^3} e^{-\frac{\sigma^2}{4s} \langle P^n \rangle} \int_0^\infty d\lambda \lambda^{5/2} e^{-\lambda} \int_0^\infty dh \frac{h^2}{(16\lambda^2 h^2 + 1)^2} \cdot \frac{1}{1 + (\xi T)^2/6}.
\]
The $h$-integration in this formula can be performed analytically, which yields

$$F_2 \big|_{N=3} \simeq -\frac{16T^4}{\pi^{5/2}} \sum_{n=1}^{\infty} \int_0^\infty \frac{ds}{s^3} e^{-\frac{s^2}{2}} \int_0^\infty d\lambda \frac{\lambda^{3/2} e^{-\lambda}}{(4\lambda + \xi^2 s^2/\sqrt{6})^2}.$$  

Approximating again the sum over winding modes by the first two terms, we further have

$$\int_0^\infty \frac{ds}{s^3} \left(e^{-\frac{s^2}{2}} + e^{-\frac{s^2}{4}}\right) \int_0^\infty d\lambda \frac{\lambda^{3/2} e^{-\lambda}}{(4\lambda + \xi^2 s^2/\sqrt{6})^2} = \frac{17\sqrt{\pi}}{16} - \frac{27\pi^{3/2}}{128 \cdot 6^{1/4}} \cdot \xi + O(\xi^2).$$

This yields the sought free-energy density

$$F_2 \big|_{N=3} \simeq -\frac{\langle P \rangle T^4}{10\pi^2} \left(17 - \frac{27\pi}{8 \cdot 6^{1/4}} \cdot cg^2\right). \quad (13)$$

Once brought together, equations (11) and (13) yield

$$F \big|_{N=3} \simeq -\frac{\langle P \rangle T^4}{\pi^2} \left[17 - \frac{27\pi}{80 \cdot 6^{1/4}} \cdot cg^2 - \frac{9}{\sqrt{6}} \sigma \beta^2 + O((\sigma \beta^2)^2)\right]. \quad (14)$$

The two leading terms of this expression can be compared with the known perturbative expansion of the free-energy density [10],

$$F_2 \big|_{N=3} = -\frac{8\pi^2 T^4}{45} \left[1 - \frac{15g^2}{16\pi^2} + O(g^4)\right]. \quad (15)$$

Comparing the leading term of Eq. (14), $-\frac{17T^4}{\pi^2} \simeq -1.72T^4$, with the Stefan-Boltzmann expression represented by the leading term of Eq. (15), $-\frac{8\pi^2 T^4}{45} \simeq -1.75T^4$, we conclude that the above-used approximation of the full sum over winding modes by the $(n = 1)$- and the $(n = 2)$-terms is very good. Comparing further with each other the $O(g^2)$-terms of Eqs. (14) and (15), we obtain:

$$c = \frac{80\pi}{27 \cdot 6^{3/4}} \simeq 2.4. \quad (16)$$

By using Eq. (7), we proceed now to arbitrary $N$, which yields

$$F_1 = -\frac{T^4}{8\pi^2} \cdot \frac{N^2 (N^2 - 1)}{N^2 + 1} \left(1 + \frac{\lambda^{3/2}}{8\pi \sqrt{3}}\right) \left(17 - \frac{5\lambda^2}{2\pi \sqrt{6}} + O(\lambda^4)\right)$$

and

$$F_2 = -\frac{T^4}{8\pi^2} \cdot \frac{N^2 - 1}{N^2 + 1} \left(1 + \frac{\lambda^{3/2}}{8\pi \sqrt{3}}\right) \left(17 - \frac{9\pi c}{8 \cdot 6^{1/4}} \cdot \lambda + O(\lambda^2)\right).$$

Here, $\lambda = g^2 N$ is the so-called 't Hooft coupling, which stays finite in the large-$N$ limit, and we have used the leading $\lambda$-dependent expression for the Polyakov loop (cf. Eq. (10)) [11]: $\langle P \rangle \simeq 1 + \frac{\lambda^{3/2}}{8\pi \sqrt{3}}$. Accordingly, we obtain for the full free-energy density $F = F_1 + F_2$:

$$F = -\frac{T^4}{8\pi^2} \cdot \frac{N^2 (N^2 - 1)}{N^2 + 1} \left(1 + \frac{\lambda^{3/2}}{8\pi \sqrt{3}}\right) \left[17 \left(1 + \frac{1}{N^2}\right) - \frac{9\pi c}{8 \cdot 6^{1/4}} \cdot \lambda - \frac{5\lambda^2}{2\pi \sqrt{6}} + O(\lambda^4) + O\left(\frac{\lambda^2}{N^2}\right)\right].$$

In the large-$N$ limit of this expression, the $c$-dependent term, which corresponds to the leading perturbative correction from Eq. (15), gets $\frac{1}{N^2}$-suppressed in comparison with the $O(\lambda^2)$-term,
Accordingly, the electric coupling constant \( g \) between the perimeter- and the area-law exponentials in Eq. (7). The large-\( N \) limit of the free-energy density thus reads

\[
F = -\frac{T^4 N^2}{8\pi^2} \left(1 + \frac{\lambda^{3/2}}{8\pi\sqrt{3}}\right) \left[17 - \frac{5\lambda^2}{2\pi\sqrt{6}} + \mathcal{O}\left(\frac{10}{N^2}\right) + \mathcal{O}\left(\frac{10\lambda}{N^2}\right) + \mathcal{O}(\lambda^4) + \mathcal{O}\left(\frac{\lambda^2}{N^2}\right)\right].
\]

Accordingly, \( -\frac{F}{T} \) could have a maximum corresponding to the most probable configuration of the system, once the relation \( \frac{4\lambda^{3/2}}{8\pi\sqrt{3}} = \frac{5\lambda^2}{2\pi\sqrt{6}} \) would hold, i.e. at \( \lambda = \frac{289}{\sqrt{5}} \). However, since this value of \( \lambda \) is much larger than unity, it lies outside the range of applicability of the \( \lambda \)-expansion, so that such a maximum of \( -\frac{F}{T} \) is not realized. Thus, the main qualitative result of our study is that the leading correction to the Stefan-Boltzmann expression, while being \( \mathcal{O}(\lambda) \) for \( N \sim 1 \), becomes \( \mathcal{O}(\lambda^{3/2}) \) for \( N \gg 1 \), and changes its sign.

2. Pair production in the field of a flux tube

In this Section, we present the calculation of the rate of pair production in the field of a flux tube [12]. Such flux tubes model hadronic strings within the dual-superconductor scenario of confinement [13], and can be viewed as dual Abrikosov-Nielsen-Olesen strings [14]. Here, we are mostly interested in the impact of the dispersion of the (chromo-)

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With the neglection of spin degrees of freedom of the produced quarks, the one-loop effective action has the form

\[ \Gamma[A_i] = NN_f \int_0^\infty \frac{ds}{s} e^{-m^2 s} \int D\mathbf{x}_\perp D\mathbf{x}_\parallel \exp \left[ -\int_0^s d\tau \left( \frac{1}{4} \dot{\mathbf{x}}_\perp^2 + \frac{1}{4} \dot{\mathbf{x}}_\parallel^2 - \frac{ig}{2} E(\mathbf{x}_\perp(\tau)) \varepsilon_{ij} x_i \right) \right]. \]  

(20)

Here, \( x_\parallel = (x_3, x_4) \), the indices \( i \) and \( j \) take the values 3 and 4, and the field of the flux tube reads \( A_i = -\frac{1}{2} \varepsilon_{ij} x_j E(\mathbf{x}_\perp) \). To calculate the world-line integral (20), we impose the condition of largeness of the mass \( m \) of the produced pair in comparison with \( m_V \), i.e. \( m \gg \frac{g}{\sqrt{\pi L}} \). This condition allows us to treat the field of the flux tube as a nearly constant one. Accordingly, the leading small-s approximation is compatible with each other at \( g \gg 1 \).

Owing to the smallness of the pair trajectory, the field \( E(\mathbf{x}_\perp(\tau)) \) can be approximated by its value averaged along the trajectory. Namely, one has

\[ \int_0^s d\tau E(\mathbf{x}_\perp(\tau)) \approx \Sigma_{ij} \varepsilon_{ij} \cdot \frac{1}{8} \int_0^s d\tau E(\mathbf{x}_\perp(\tau)), \]  

(22)

where \( \Sigma_{ij} \equiv \int_0^s d\tau x_i \dot{x}_j \) is the \((i, j)\)-th component of the tensor area associated with the trajectory. Furthermore, the leading small-s approximation corresponds to the classical limit of the world-line integral \( \int D\mathbf{x}_\perp = \int d^2 x_\perp(0) \int_{\mathbf{x}_\perp(0)=\mathbf{x}_\perp(s)} D\mathbf{x}_\perp(\tau) \) in Eq. (20):

\[ \frac{1}{4\pi s} \int d^2 x_\perp \exp \left[ -\frac{ig}{2} E(\mathbf{x}_\perp) \varepsilon_{ij} \Sigma_{ij} \right]. \]  

(23)

Accordingly, the effective action in this limit reads

\[ \Gamma[A_i] \approx \frac{NN_f}{4\pi} \int_0^\infty \frac{ds}{s^2} e^{-m^2 s} \int D\mathbf{x}_\parallel \exp \left( -\frac{1}{4} \int_0^s d\tau \dot{\mathbf{x}}_\parallel^2 \right) \int d^2 x_\perp \exp \left[ -\frac{ig}{2} E(\mathbf{x}_\perp) \varepsilon_{ij} \Sigma_{ij} \right]. \]  

(24)

Neglecting for the moment the dispersion of the field \( E(\mathbf{x}_\perp) \), we have

\[ \int d^2 x_\perp \exp \left[ -\frac{ig}{2} E(\mathbf{x}_\perp) \varepsilon_{ij} \Sigma_{ij} \right] \approx S \exp \left[ -\frac{ig}{2} \langle E(\mathbf{x}_\perp) \rangle \Sigma_{ij} \right], \]  

(25)

where we have again used the notation \( \langle \cdots \rangle \equiv \frac{1}{s} \int d^2 x_\perp (\cdots) \). Therefore, within this approximation, we arrive at the Euler-Heisenberg-Schwinger Lagrangian in the constant field \( A_i = -\frac{1}{2} \varepsilon_{ij} x_j(E) \):

\[ \Gamma[A_i] \approx S \frac{NN_f}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} e^{-m^2 s} \frac{g(E)}{\sin(g(E)s)}. \]  

(26)
we obtain for the effective action (24):

$$w \simeq NN_f \frac{(g(E))^2}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \exp \left( -\pi km^2 \frac{g(E)}{E} \right).$$  

(27)

We can further express the inequality $s < \frac{1}{m^2}$ in terms of the parameters of the dual Abelian Higgs model. To this end, we notice that $w$ can only be non-vanishing provided that at least the first pole from the imaginary part of the Euler-Heisenberg-Schwinger Lagrangian, which corresponds to the $(k = 1)$-term from the sum (27), yields its contribution to $w$. For this reason, $s$ may not be arbitrarily small, but it should be bounded from below as $s > \frac{\pi}{g(V)} = \frac{\pi L}{\pi L}$. The inequality $s < \frac{1}{m^2}$ yields then $m < 2\sqrt{\frac{\pi L}{\pi L}}$. This new constraint is stronger than the above-obtained one, which is expressed by the right inequality (21), since the large coupling $g$ is now absent. Representing the new constraint in the form

$$L < \frac{4}{\pi} \frac{\sigma}{m^2} \simeq 1.27 \frac{\sigma}{m^2},$$  

(28)

one can view it as an upper limit for $L$. Using further the standard value of the string tension, $\sigma = (440 \text{MeV})^2$, and a typical value of the hadronic mass, $m = 200 \text{MeV}$, we get an estimate $L < 6.2$, which leaves a sufficient window for having $L \gg 1$. Approximating then the sum (27) by the $(k = 1)$-term, we obtain

$$w \simeq \frac{2NN_f}{\pi^3} \left( \frac{\sigma}{L} \right)^2 \exp \left( -\frac{\pi m^2 L}{4\sigma} \right).$$  

(29)

We will now calculate $w$ in an alternative way, which allows one to avoid the use of approximation (25) by performing the $d^2x_\perp$-integration of every term in the Taylor expansion of the exponential $\left[ -\frac{ig}{2} E(x_\perp) \varepsilon_{ij} \Sigma_{ij} \right]$. In the London limit, by virtue of the explicit form of $E(x_\perp)$, the corresponding calculation can be done analytically. Namely, by using Eq. (18), we have

$$\int d^2x_\perp \exp \left[ -\frac{ig}{2} E(x_\perp) \varepsilon_{ij} \Sigma_{ij} \right] = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{ig}{2} \varepsilon_{ij} \Sigma_{ij} \right)^n \left( \frac{m_V^2}{g m} \right)^n \int d^2x_\perp (K_0(m_V r))^n.$$

The dominant contribution to the integral on the right-hand side of this expression stems from the distances $r < \frac{1}{m_V}$. By using the leading term in the small-$r$ asymptotic behavior of $K_0(m_V r)$, this contribution can be readily evaluated as $\frac{\pi}{m_V} 2^{2-n} n!$. Owing to the factor of $n!$, it yields the following closed-form expression for the sum over $n$:

$$\frac{4\pi}{m_V^2} \sum_{n=0}^{\infty} \left( -\frac{ig}{4} \frac{m_V^2}{g m} \varepsilon_{ij} \Sigma_{ij} \right)^n = \frac{4\pi}{m_V^2} \frac{1}{1 + \frac{g^2 m^2}{8\pi} \varepsilon_{ij} \Sigma_{ij}} = \frac{4\pi}{m_V^2} \int_0^\infty dt e^{-t} \left( 1 + \frac{g^2 m^2}{8\pi} \varepsilon_{ij} \Sigma_{ij} \right).$$  

(30)

For the $n$-series in Eq. (30) to be convergent, the condition $g^2 m^2 / 8\pi < 1$ should hold. This condition yields the following upper limit for $L$:

$$L < \frac{16}{\pi} \frac{\sigma}{m^2} \simeq 5.09 \frac{\sigma}{m^2}. $$  

(31)

Furthermore, by virtue of the integral representation introduced in the last equality of Eq. (30), we obtain for the effective action (24):

$$\Gamma[A_\perp] \simeq$$
\[ \approx S \frac{NN_f}{\pi} \int_{0}^{\infty} \frac{ds}{s^2} e^{-m^2s} \int D\xi_{\parallel} \int_{0}^{\infty} dt e^{-t\left(1 + \frac{ig^2}{8\pi} \varepsilon_{ij} \Sigma_{ij}\right)} = S \frac{NN_f}{4\pi^2} \int_{0}^{\infty} \frac{ds}{s^2} e^{-m^2s} \int_{0}^{\infty} dt e^{-t} \frac{gE}{\sin(qEs)}. \] (32)

Here, \( \Xi \equiv \frac{tgm^2}{4\pi} \) is a yet another \( x \)-independent electric field, which yielded for the integral \( \int D\xi_{\parallel} \) the corresponding Euler-Heisenberg-Schwinger Lagrangian. The pair-production rate stemming from Eq. (32) has the form

\[ w \approx \frac{NN_f}{2\pi^3} \left(\frac{\sigma}{L}\right)^2 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2} \int_{0}^{\infty} dt e^{-t - \frac{gm^2k^2}{\sigma}} = NN_f \frac{m^3}{\pi^{3/2}} \sqrt{\frac{\sigma}{L}} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}} K_3 \left(2m\sqrt{\frac{\pi L k}{\sigma}}\right). \]

Due to the exponential fall-off of the Macdonald function \( K_3 \) at the large values of its argument, only the terms with \( k \lesssim \frac{\sigma}{4\pi m^2} \) are relevant in the latter sum. Using again the values \( \sigma = (440 \text{ MeV})^2 \) and \( m = 200 \text{ MeV} \), we obtain \( k < \frac{1}{L} < 1 \). Therefore, only the first term from the whole sum can be retained, which yields

\[ w \approx NN_f \frac{m^3}{\pi^{3/2}} \sqrt{\frac{\sigma}{L}} K_3 \left(2m\sqrt{\frac{\pi L}{\sigma}}\right). \] (33)

Furthermore, because of the constraint (31), the argument of the Macdonald function in this formula is smaller than 8. Nevertheless, as long as \( L > \frac{\sigma}{4\pi m^2} \), this argument is still larger than unity, which results into the formula

\[ w \approx NN_f \frac{m^5/2}{2\pi^3/4 L^{3/4}} e^{-2m\sqrt{\frac{\pi L}{\sigma}}}. \] (34)

If \( L \) additionally respects the inequality (28), the obtained expression (34) can be compared with Eq. (29). This comparison leads us to the conclusion that, averaging the exponential \( \exp \left[ -\frac{ig}{2} E(\xi_{\parallel}) \varepsilon_{ij} \Sigma_{ij} \right] \) without recourse to the cumulant expansion, one obtains a change of the Gaussian \( m \)-distribution (29) to the exponential distribution (34). It is remarkable that we have obtained this result for the case of a static field \( E \), namely for the field which is produced by the flux tube in the dual-superconductor model of confinement. A similar conversion of the Gaussian mass-distribution of pairs produced in the electric field into an exponential distribution is known to take place for a time-dependent field \( E(t) \) which falls off with \( t \) as fast as a certain exponential. This is, for example, the case if \( E(t) \propto \frac{1}{\cosh^2(\omega t)} \) with sufficiently large \( \omega \)'s [15]. In our case, the obtained exponential \( m \)-distribution is a consequence of the logarithmic growth of the flux-tube field (18) towards the core of the string, which takes place in the London limit of the dual superconductor. Therefore, the exponential \( m \)-distribution is a specific property of the London limit, which does not hold away from that limit. For instance, in the opposite, so-called Bogomolny, limit of \( m_V = m_H \) [16], \( E(0) \) was found to be finite [17], rather than growing as \( \mathcal{O}(\ln \frac{1}{m_{V/r}}) \). Consequently, the \( m \)-distribution in the Bogomolny limit is the standard Gaussian one.

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