On Translation Lengths of Anosov Maps on Curve Graph of Torus

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Abstract

We show that an Anosov map has a geodesic axis on the curve graph of a torus. The direct corollary of our result is the stable translation length of an Anosov map on the curve graph is always a positive integer. As the proof is constructive, we also provide an algorithm to calculate the exact translation length for any given Anosov map. The application of our result is threefold: (a) to determine which word realizes the minimal translation length on the curve graph within a specific class of words, (b) to establish the effective bound on the ratio of translation lengths of an Anosov map on the curve graph to that on Teichmüller space, and (c) to estimate the overall growth of the number of Anosov maps which have a sufficient number of Anosov maps with the same translation length.

1 Introduction

Let $S = S_{g,n}$ be an orientable surface with genus $g$ and $n$ punctures. The mapping class group of $S$, denoted by $	ext{Mod}(S)$, is the group of isotopy classes of orientation-preserving homeomorphisms of $S$. An element of a mapping class group is called a mapping class. The Nielsen–Thurston classification theorem [Thu88] states that every mapping class is either periodic, reducible, or pseudo-Anosov. A mapping class $\psi$ in $	ext{Mod}(S)$ is said to be pseudo-Anosov if there exists a pair of transverse invariant singular measured foliations associated to $\psi$, one of which is stretched by a constant $\lambda$, while the other is contracted by $\lambda^{-1}$. When $S$ is the torus $T^2$, there are foliations without singularities and in such case, $\psi$ is called an Anosov element. It is an easy consequence that Anosov elements can be represented as matrices $M \in \text{SL}(2,\mathbb{Z}) \cong \text{Mod}(T^2)$ with $|\text{tr}(M)| > 2$.

The curve complex $\mathcal{C}(S)$ of $S$, first introduced by Harvey [Har81], is a simplicial complex where vertices are isotopy classes of essential simple closed curves and $(k+1)$-vertices span a $k$-simplex if and only if each $(k + 1)$-tuple of vertices has a set of representative curves with the minimal possible geometric intersection in the given surface. The curve graph is the 1-skeleton
of the curve complex. By giving each edge length 1, $C(S)$ becomes a Gromov-hyperbolic metric space with path metric $d_{C}(\cdot, \cdot)$ \cite{MM99}. Then Mod$(S)$ acts on $C(S)$ by isometry. The stable translation length (also known as asymptotic translation length) of a mapping class $f \in$ Mod$(S)$ on $C(S)$ is defined by

$$l_{c}(f) = \lim_{j \to \infty} \inf \frac{d_{C}(\alpha, f^{j}(\alpha))}{j},$$

where $\alpha$ is a vertex in $C(S)$. Using the triangular inequality, one can show that the $l_{c}(f)$ is independent of the choice of $\alpha$. Masur and Minsky \cite{MM99} showed that $l_{c}(f) > 0$ if and only if $f$ is pseudo-Anosov.

There have been many research works on estimating stable translation length for non-sporadic surfaces, that is, the complexity $\xi(S) = 3g - 4 + n$ is positive. Those works can be found in \cite{FLM08}, \cite{GT11}, \cite{GHKL13}, \cite{Val14}, \cite{AT17}, \cite{KS17}, \cite{BS18} and references therein. Algorithmic approaches to calculating stable translation lengths of pseudo-Anosov maps are available in \cite{Sha12} and \cite{Web15}. A polynomial-time algorithm is established by \cite{BW16}.

As far as we know, there is no literature developing a similar theory for sporadic surfaces. As the complexity of sporadic surfaces is low enough, it is to be expected that exact translation lengths can be calculated, rather than merely estimating stable translation lengths. Among sporadic cases, only $S = S_{0,4}, S_{1,0}, S_{1,1}$ have the Farey graph $F$ as their curve graph; the other ones, namely the spheres with at most 3 punctures have the empty set as their curve graph. Thus, in this paper, we discuss a way to find the exact translation length of an Anosov map on the curve graph of a torus.

As noted before, Mod$(S)$ acts on $C(S)$ by isometry. In particular, when $S$ is a torus, we have an isometry SL$(2, \mathbb{Z})$-action on $F$. In fact, we can embed $F$ into $\mathbb{H}^{2}$, so that SL$(2, \mathbb{Z})$-action on $F$ can be seen as a restriction of the PSL$(2, \mathbb{R})$-action on $\mathbb{H}^{2}$, so-called Möbius transformation. Moreover, an Anosov element in SL$(2, \mathbb{Z})$ and a hyperbolic element in PSL$(2, \mathbb{Z}) \subset$ PSL$(2, \mathbb{R})$ both are characterized by their absolute value of trace being bigger than 2. It follows that one can identify an Anosov element in SL$(2, \mathbb{Z})$ with a hyperbolic element in PSL$(2, \mathbb{Z})$. Hence, in the rest of the paper we will interchangeably say “a hyperbolic element of PSL$(2, \mathbb{Z})$”, and “an Anosov element of SL$(2, \mathbb{Z})$.”

Any hyperbolic element of PSL$(2, \mathbb{Z})$ has the unique invariant geodesic axis in $\mathbb{H}^{2}$. This seems to suggest the existence of an invariant bi-infinite geodesic in $F$ associated with an Anosov map. Indeed, this identification allowed us to prove the following main theorem:

**Theorem 18.** For any Anosov map $f$, there exists a bi-infinite geodesic $P$ in $F$ on which $f$ acts transitively.

We remark here that this is not the case for non-sporadic surfaces; namely, a pseudo-Anosov mapping class may not have a geodesic axis in the curve graph. In particular, when the genus is bigger than 2, this can be easily shown using the fact due to \cite{KS17}: When $S = S_{g}$ with $g \geq 3$, the minimal stable translation length among pseudo-Anosov mapping classes in Mod$(S_{g})$ is bounded above $\frac{1}{g^{2} - 2g - 1}$, which is strictly less than 1 for $g \geq 3$. 

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In particular, the stable translation length of every Anosov map on $\mathcal{F}$ is an \textit{integer}. This can be seen as a slight strengthening of Bowditch’s result [Bow08]—the stable translation length of any pseudo-Anosov map on the curve graph is a \textit{rational number} with a bounded denominator— in a special case where the additional information comes from the concrete description of the curve complex of the torus. Also, we emphasize here that the proof of Theorem [18] is constructive, so we can calculate the exact translation length of any Anosov element.

In section 2, we review some of the standard facts on the Farey graph and continued fractions. In section 3, we introduce a special subgraph, called a \textit{ladder}, of the Farey graph, which plays a key role in this paper. In section 4, we look more closely at the $\text{PSL}(2,\mathbb{Z})$-action on the Farey graph. In section 5, we finally derive our main theorem and provide a concrete way to calculate the exact translation length of an Anosov element on $\mathcal{F}$. In section 6, we provide three applications of our main theorem. Within a specific class of words in $\text{PSL}(2,\mathbb{Z})$, we decide which form of words realizes the minimal translation length on $\mathcal{F}$ (Theorem 23). Inspired from [GHKL13], [AT17] and [BRW17], we establish similar results for the translation length of an Anosov map on the Farey graph, namely the effective bound on the ratio of translation lengths of Anosov map on the Farey graph to that on Teichmüller space (Theorem 28), and the overall growth of the number of Anosov maps which have the same translation lengths (Theorem 29). In Appendix, we provide algorithms for generating and calibrating ladder. Both are crucial for calculating the translation length of an Anosov element.

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2 Farey Graph and Continued Fraction

We review some facts of the Farey graph and continued fractions.

\textbf{Definition.} The \textbf{Farey graph} is a simplicial graph where each vertex is an extended rational number denoted by $\frac{p}{q}$, and a pair of vertices is joined by an edge if and only if these two vertices represent $\frac{p}{q}$ and $\frac{r}{s}$ satisfying $|ps - qr| = 1$.

Denote by $\mathbb{H}^2$ the hyperbolic plane. We can naturally embed Farey graph $\mathcal{F}$ into a compactification of the hyperbolic plane $\overline{\mathbb{H}} = \mathbb{H}^2 \cup \partial \mathbb{H}^2$, where the vertices of $\mathcal{F}$ are in correspondence with \textbf{extended rational points} $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\frac{1}{0} = \infty\} \subset \overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, and the edges are represented by hyperbolic geodesics.
Then $\mathbb{H}^2$ is completely partitioned by the ideal triangles whose sides are the edges of the Farey graph. We call those triangles as **Farey triangles**. In the rest of the paper, we will regard the Farey graph as an embedded graph in $\mathbb{H}$.

**Definition.** A **positive** (negative) **continued fraction** is a continued fraction with positive (negative, respectively) integral coefficients. A **periodic continued fraction** is an infinite continued fraction whose coefficients eventually repeat. A continued fraction is denoted by the following notation:

$$[a_0; a_1, a_2, a_3, \ldots, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}}.$$ 

Also, we introduce the **bar notation** to represent periodic continued fractions:

$$[a; b, x, y, z] = [a; b, x, y, z, \ldots].$$

**Remark.**

1. Every negative continued fraction can be written as a negation of a positive continued fraction:

$$[-b_0; -b_1, -b_2, \ldots, -b_n] = -[b_0; b_1, b_2, \ldots, b_n].$$

2. Every positive (negative) real number has a positive (negative, respectively) continued fraction representation. For example:

$$\frac{5 + \sqrt{17}}{6} = [1; 1, 1, 1, 2, 5, 1, 5, 2], \quad \frac{5 - \sqrt{17}}{6} = [0; 6, 1, 5, 2, 1, 11, 1, 2, 5].$$

3. Lagrange showed that a continued fraction is periodic if and only if it represents a **quadratic irrational**. (See [Ste92], for example.)

We now introduce the **cutting sequence** of a geodesic in $\mathbb{H}^2$. A cutting sequence links the Farey graph with continued fractions. The following definition and Proposition 1 are excerpted from [Ser15].

**Definition** ([Ser15]). The **cutting sequence** of $x \in \mathbb{R}$ is a sequence of $L$’s and $R$’s which are constructed in the following way:

Join $x$ to any point on the imaginary axis $\mathbb{I}$ by a hyperbolic geodesic $\gamma$. This arc in $\mathbb{H}^2$ cuts a succession of Farey triangles, which are ideal triangles, so $\gamma$ cuts exactly two edges of each Farey triangle. Then the two edges meet in a vertex $v$ on the left or right of the oriented geodesic $\gamma$. Label this vertex $v$ with $L$ or $R$ accordingly. (In the exceptional case in which $\gamma$ terminates in a vertex of the triangle, **choose either $L$ or $R$**.) Then the resulting sequence $L^{n_0} R^{n_1} L^{n_2} \ldots$ is defined to be the cutting sequence of $x$.

**Remark.** The cutting sequence is independent of the choice of the initial point of $\gamma$ on $\mathbb{I}$.

The following proposition shows cutting sequences are closely related to continued fractions.
Proposition 1 (Ser15). Suppose \( x > 1 \). Then \( x \) has a cutting sequence as \( L^{n_0}R^{n_1}L^{n_2} \cdots, n_i \in \mathbb{N} \) if and only if \( x = [n_0; n_1, n_2, \cdots] \), written in the positive continued fraction. Likewise, when \( 0 < x < 1 \), \( x \) has a cutting sequence \( R^{n_0}L^{n_1}L^{n_2} \cdots, n_i \in \mathbb{N} \) if and only if \( x = [0; n_1, n_2, \cdots] \).

Therefore, the exponents in the cutting sequence of \( x \) are the coefficients of the continued fraction representation of \( x \).

Since \( x \mapsto -1/x \) is a half-turn in \( \mathbb{H}^2 \) in the disk model, the cutting sequence of \( x \) and that of \(-1/x\) are identical. Using this observation, we can extend Proposition 1 to a situation with \( x < 0 \).

Corollary 2. Suppose \( x < -1 \). Then \( x \) has a cutting sequence \( R^{n_1}L^{n_2} \cdots, n_i \in \mathbb{N} \) if and only if \( x = [-n_1; -n_2, \cdots] \), written in the negative continued fraction. Likewise, when \(-1 < x < 0 \), \( x \) has a cutting sequence \( L^{n_0}R^{n_1}L^{n_2} \cdots, n_i \in \mathbb{N} \), if and only if \( x = [0; -n_0, -n_1, -n_2, \cdots] \), written in the negative continued fraction. In particular, for \( x < 0 \), the coefficients of the positive continued fraction of \(-1/x\) and those of the negative continued fraction of \( x \) are identical up to translation of coefficients.

More generally, we can define a cutting sequence of an oriented bi-infinite geodesic in \( \mathbb{H}^2 \) as a bi-infinite sequence of \( L \)'s and \( R \)'s. To be precise, pick a point \( x \) on a bi-infinite geodesic \( g \), and split \( g \) at \( x \) into two geodesic rays \( g_1 \) and \( g_2 \) with induced orientation. We may assume that \( g_1 \) terminates at \( x \), and \( g_2 \) begins at \( x \). Then the desired bi-infinite sequence is obtained by concatenating two cutting sequences: The \( L-R \) flipped sequence of the cutting sequence of the reverse of \( g_1 \), followed by the cutting sequence of \( g_2 \).

As the beginning point of a geodesic only affects the initial part of a cutting sequence, two oriented geodesics with the same endpoints eventually have the same cutting sequence: The following lemma will not be needed until Theorem 17.

Lemma 3 (Ser15). Let \( \gamma, \gamma' \) be oriented geodesics in \( \mathbb{H}^2 \) with the same endpoint. Then the cutting sequence of \( \gamma \) and \( \gamma' \) eventually coincide.

Proof. Refer to Ser15. \( \square \)

3 Ladder in Farey Graph

3.1 Ladder

In this section, we introduce a special class of subsets of Farey graph, called ladders. A ladder is a useful tool to analyze geodesics in the Farey graph. Later on, we will see that a ladder is geodesically convex, so it captures all the geodesics joining two vertices in \( \mathcal{F} \). Hatcher also introduced a ladder with different terminology('fan') in Hat17. With ladders, he studied symmetries of the Farey graph, which are applied to prove number theoretic results. We focus on the dynamics of \( \text{SL}(2, \mathbb{Z}) \) on the Farey graph. To this end, we establish a ladder which is stabilized by the action of a given Anosov element.
**Definition.** Let $g$ be an oriented bi-infinite geodesic in $\mathbb{H}^2$. Then we define the **ladder** associated with $g$ as the collection of all Farey triangles whose interior intersects with $g$. Moreover, for any real numbers $x, y \in \partial \mathbb{H}^2$, we define the **ladder** associated with $x, y$, denoted by $\mathcal{L}(x, y)$, as the ladder associated with the bi-infinite oriented geodesic joining from $x$ to $y$.

**Remark.** From the above definition, we specified the interior of a Farey triangle and a geodesic $g$ must intersect. Thus, if $x, y$ are vertices in $\mathcal{F}$ which form two endpoints of a single Farey edge, then $\mathcal{L}(x, y) = \emptyset$.

Since a geodesic cannot pass all three sides of a single geodesic triangle at once, we can characterize a ladder as following:

**Fact 4.** A ladder is a consecutive chain of Farey triangles; i.e., a countable union of Farey triangles $\{\Delta_i\}$ such that $\Delta_i \cap \Delta_{i+1}$ is a single edge of the Farey graph and $\Delta_i \cap \Delta_j$ is either an empty set or a single point (which will be called as a pivot point, shortly) if $|i - j| \geq 2$.

From this fact, now we can further define each component of a ladder:

![Figure 1: Components of Ladder](image)

**Definition.** An **endpoint** of a ladder is a degree 2 vertex of a ladder. Connect two endpoints with an oriented geodesic $g$. While recording the cutting sequence of $g$, call every $L$ or $R$-labeled vertex as a **pivot point**. Include two endpoints as pivot points as well. An edge in a ladder is called a **rung** if its interior and $g$ intersect.

The **spine** $K$ of a ladder $\mathcal{L}$ is the path in $\mathcal{L}$ with the following property:

- The beginning point and the terminal point of $K$ are endpoints of $\mathcal{L}$.
- All the vertices of $K$ are exactly all the pivot points in $\mathcal{L}$.
- All the edges of $K$ except for the initial and final one are rungs of $\mathcal{L}$.

For the uniqueness of a spine for each ladder, we define a spine for the following exceptional case as Figure 2. See the remark below for the uniqueness of other ladders.

Then the **side** of a ladder is a connected component of the union of all non-rung edges which do not belong to the spine of the ladder. See Figure 1.
Remark. In fact, if a ladder is other than (1, 1)-type, then its spine is uniquely determined. This is because except for the (1, 1)-type ladder, we can connect each pivot point to another pivot point via the unique rung, except two endpoints. After joining them via rungs, we get a path $P'$ in the ladder except for two endpoints. Now we can attach two endpoints to $P$ in a unique way. See Figure 3 for the illustration of this construction of a spine.

We can specify the type of a ladder associated with a geodesic by counting a sequence of consecutive Farey triangles sharing the same pivot point.

Definition. We say a ladder is of type $(a_1, \cdots, a_n)$ if the ladder has $a_1, \cdots, a_n$ consecutive numbers of Farey triangles with the same pivot points read off in the orientation given to the geodesic. As an example, see Figure 4. In this case, we call $n$ as the length of the ladder.

Immediate from the definitions, the number of Farey triangles sharing the same pivot points should coincide with the cutting sequence associated with $g$.

Proposition 5. Let $g$ be an oriented bi-infinite geodesic in $\mathbb{H}^2$. Then the type of the ladder associated with $g$ is identical to the exponents of the cutting sequence of $g$. 

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3.2 Geodesics in Ladder

For two points \(x, y \in \partial \mathbb{H}\), define \(E(x, y)\) as the set of all edges in \(F\) separating \(x, y\). This definition of \(E(x, y)\) generalizes that of \([\text{Min96}]\), in which \(E(x, y)\) is only defined for \(x, y\) being vertices of the Farey graph. This slightly generalized approach is to handle bi-infinite geodesics.

**Proposition 6.** All edges in \(E(x, y)\) are in one-to-one correspondence with the rungs in \(L(x, y)\).

**Proof.** Denote by \(xy\) the bi-infinite geodesic connecting \(x\) and \(y\) in \(H^2\). Firstly, pick an edge \(e \in E(x, y)\). As \(e\) separates \(H^2\) into 2 components, \(x, y\) fall in different parts of them. Thus the interior of \(e\) must intersect with \(xy\). Therefore, \(e\) must be a rung of \(L(x, y)\).

Conversely, pick any rung \(r\) from \(L(x, y)\). Then there are exactly two Farey triangles incident with \(r\), and they meet \(xy\) by the definition of \(L(x, y)\). Since those two triangles reside in different components separated by \(r\), it follows that \(x, y\) also must be separated by \(r\). Therefore, \(r \in E(x, y)\).

When it comes to considering geodesics in the whole Farey graph, Proposition 6 and the following fact from \([\text{Min96}]\) justify the reason why it suffices to consider only the geodesics in ladders.

**Fact 7** (\([\text{Min96}]\)). Let \(x, y\) be two vertices of the Farey graph. Then for any geodesic path \(P\) joining \(x\) and \(y\) in \(F\), each vertex of \(P\) other than \(x\) and \(y\) must be incident with some edge in \(E(x, y)\).

**Corollary 8** (Ladder is Geodesically Convex). Let \(x, y\) be two vertices of the Farey graph, and \(P\) be a path joining \(x\) and \(y\) in \(F\). Then \(P\) is a geodesic in \(F\) if and only if \(P\) is a geodesic in the ladder \(L(x, y)\). In other words, a ladder is geodesically convex in \(F\).

**Proof.** Suppose \(P\) is a geodesic in \(F\). Then by Fact 7 all the vertices in \(P\) must be incident with edges in \(E(x, y)\), which are rungs of \(L(x, y)\) due to Proposition 6. Thus, every vertex of \(P\) is contained in \(L(x, y)\), so \(P\) is a path in \(L(x, y)\). It is a geodesic in \(L(x, y)\) as well, otherwise there is a shorter path than \(P\) connecting \(x\) and \(y\) in \(L(x, y)\), thus in \(F\), which contradicts to the assumption that \(P\) is a geodesic in \(F\). Thus, \(P\) is a geodesic in \(L(x, y)\).

Conversely, let \(P\) be a geodesic in \(L(x, y)\). Suppose there is a shorter path \(P'\) than \(P\) in \(F\) joining \(x\) and \(y\). Then by the same argument, \(P'\) is contained in \(L\), still shorter than \(P\), a contradiction. Therefore, \(P\) is a geodesic in \(F\).

Here we prove a useful lemma which hints the hierarchy between ladders.

**Lemma 9.** Let \(x, y\) be points in the boundary of \(\mathbb{H}^2\), and \(r, s\) be two vertices of \(L(x, y)\). Then \(L(r, s)\) is contained in \(L(x, y)\).

**Proof.** It suffices to show every rung in \(L(r, s)\) is a rung of \(L(x, y)\), since non-rung edges of a ladder are fully determined with rungs by completing Farey triangles.
Pick a rung $e$ from $\mathcal{L}(r,s)$. According to Proposition 6, it suffices to show that $e$ separates $x, y$. Let $A,B$ be two connected components of $\mathbb{H}^2 \setminus e$. Since $e$ is a rung of $\mathcal{L}(r,s)$, $e$ is disjoint from $r, s$. Thus each $A,B$ contains exactly one of $r, s$. Now suppose to the contrary $x, y$ are not separated by $e$. Then the interior of the bi-infinite geodesic $\overline{xy}$ in $\mathbb{H}^2$ must reside in either $A$ or $B$. Hence, $\mathcal{L}(x,y)$ is contained in either $A$ or $B$, since two edges of the Farey graph cannot intersect in interior points. In particular, both vertices $r,s$ of $\mathcal{L}(x,y)$ should be contained together in either $A$ or $B$, which is a contradiction.

Recall that a **bi-infinite geodesic** in $\mathcal{F}$ is a path in $\mathcal{F}$ whose every finite subpath is also a geodesic in $\mathcal{F}$. Note that a ladder associated with a bi-infinite geodesic in $\mathcal{F}$ is bi-infinite as well.

Likewise, one can define a bi-infinite geodesic in a bi-infinite ladder. It is Lemma 9 that makes this definition allowable.

**Definition.** Let $\mathcal{L}$ be a bi-infinite ladder in $\mathcal{F}$. A bi-infinite path $P$ in $\mathcal{L}$ is called a **bi-infinite geodesic** if for every pair of vertices $s,t$ of $P$, the restriction $P(s,t)$ of $P$ to the subladder $\mathcal{L}(s,t)$ is again a geodesic in $\mathcal{L}$.

**Remark.** Since $\mathcal{L}(s,t)$ is geodesically convex by Corollary 8, the last statement of above definition can be reduced to showing $P(s,t)$ is a geodesic in $\mathcal{L}(s,t)$, instead of $\mathcal{L}$.

With putting more effort on the proof of Corollary 8, we can extend Corollary 8 to general result involving bi-infinite objects.

**Proposition 10.** Let $x,y$ be irrational points in $\partial \mathbb{H}^2$ (i.e., points in $\partial \mathbb{H}^2$ which are not vertices of $\mathcal{F}$), and $P$ be a bi-infinite path joining $x,y$ in the Farey graph. Then $P$ is a geodesic in $\mathcal{F}$ if and only if $P$ is a geodesic in the bi-infinite ladder $\mathcal{L}(x,y)$.

**Proof.** Let $P$ be a bi-infinite path joining $x,y$ in $\mathcal{L}(x,y)$. For any two vertices $s,t$ in $P$, denote by $P(s,t)$ the induced subpath joining $s$ and $t$.

Then by definition $P$ is a bi-infinite geodesic in $\mathcal{L}(x,y)$ if and only if $P(s,t)$ is a geodesic in $\mathcal{L}(s,t)$ for any two vertices $s,t$ in $P$. By Corollary 8, this is equivalent to saying $P(s,t)$ is a geodesic in $\mathcal{F}$ for any two vertices $s,t$ of $P$ and by definition it is exactly the case when $P$ is a bi-infinite geodesic in $\mathcal{F}$. $\square$

### 3.3 Efficient Geodesics in Ladder

Let $g$ be a bi-infinite oriented geodesic in $\mathbb{H}^2$ and $\mathcal{L}$ be its associated ladder. As a consecutive chain of Farey triangles, we give the ladder $\mathcal{L}$ an orientation induced by the orientation of $g$. Then the Farey triangles and pivot points of $\mathcal{L}$ have the induced order: Enumerate them as Figure 5.

Then for each $i$-th pivot point, the $(i+1)$-th one is called the **adjacent pivot point on the other side** and the $(i+2)$-th one is called the **adjacent pivot point on the same side**. Those $(i+1)$-th and $(i+2)$-th pivot points are adjacent pivot points of $i$-th one. Moreover, call the first pivot point as **initial**
and the last one as final, whose immediate predecessor is called semi-final. In other words, the semi-final pivot point has the adjacent final point on the other side.

Figure 5: An orientation of a ladder induces an order of pivot points. The point 6 is the adjacent pivot point on the same side of the pivot point 4, which is the adjacent pivot point on the other side of point 3.

Now we define two key moves across pivot points, which will be a building block for efficient moving.

**Definition.** For each non-terminal pivot point, define the following two moves from a pivot point to the next adjacent pivot point:

1. **Transverse** (‘t’): Move to the adjacent pivot point on the other side.
2. **Pass** (‘p’): Move to the adjacent pivot point on the same side.

Figure 6: The pivot point a has two options to move toward adjacent pivot points: b, c.

We say a path $P$ in a ladder $L$ satisfies the **efficient moving condition** if for each move between adjacent pivot points in $P$,

- Move $t$ whenever the move $p$ passes through more than one edge in $L$ ($n \geq 2$ in Figure 6), or the final point can be reached through the move $t$.
- Move $p$ whenever the move $p$ passes exactly one edge in $L$ ($n = 1$ in Figure 6), and the final point of cannot be reached through the move $t$.

See Figure 7 for more illustrations on the efficient moving condition.

Observe that the efficient moving condition is indeed efficient in a sense of the following proposition: It yields a geodesic.

**Proposition 11.** If a path in a ladder satisfies the efficient moving condition then it is a geodesic in the ladder.
Figure 7: Six paths in a $(2, 3, 4, 1, 1, 1)$-ladder. Two blue points for each figure indicate the beginning points and the final points of paths. Green segments stand for efficient moves, and red ones for non-efficient moves. Only (a) and (e) satisfy the efficient moving condition. (b) commits forbidden moves; the path contains a non-pivot point. (c) fails to satisfy the efficient moving condition at 1-block near the end; it should have moved $p$. Similarly, (d) should have moved $t$ at the beginning; note that, however, it is a geodesic, so some geodesics may not meet the efficient moving condition. (See the remark after the proof of Proposition 11). The problem in (f) is regarding the final point. It fails to move $t$ at the end of the path; even though the next one is a 1-block, it should have moved $t$ since the final point could be reached through the move $t$. 
Proof. Let \( \mathcal{L} \) be a ladder and \( a \) be a pivot point which is neither a final point nor a semi-final point of \( \mathcal{L} \). (Such exceptional cases are obvious; the final point has no option to move. The semi-final point has only one option: \( t \). These options satisfy the efficient moving condition.) We will show that the efficient moving condition forces \( a \) to move while minimizing the length of its locus. Then this eventually implies a path \( \mathcal{P} \) in \( \mathcal{L} \) satisfying the efficient moving condition should be a geodesic.

Recall that \( a \) has only two options to move to adjacent pivot points: pass or transverse. (Otherwise, it backtracks, i.e., it moves to pivot points with the lower order, which hinders the path to be a geodesic.) Denote by \( b, c \) the adjacent pivot points of \( a \) on the same and the other side respectively. Also, denote by \( n \) the number of Farey triangles in \( \triangle abc \). See Figure 6. We break into two cases:

- \( n = 1 \). If \( b \) is the endpoint, then \( p \) is shorter than \( tt \). Thus, \( a \) must choose to move \( p \). If \( b \) is not the endpoint, let \( d \) be the adjacent pivot point of \( b \) on the other side. Then \( a \) must visit \( b \) or \( d \). If \( a \) visits \( b \), then \( p \) is the shortest option for \( a \) to choose. If \( a \) visits \( d \), then \( pt \) is always the shortest option for \( a \) to choose, no matter how many Farey triangles are in \( \triangle bcd \). In any case, \( a \) can choose \( p \) to minimize the length of its locus, which aligns with the efficient moving condition.

- \( n \geq 2 \). Note that \( a \) must visit \( b \) or \( c \). If \( a \) visits \( b \), then there are two possible options: \( tt \) or \( p \). However, since \( p \) leaves at least length 2 trail, \( tt \) is always the shortest path for \( a \) to travel toward \( b \). If \( a \) visits \( c \), then obviously \( t \) is the shortest choice. In any case, \( a \) can choose \( t \) to minimize the length of its locus, which aligns with the efficient moving condition.

All in all, it follows that if \( \mathcal{P} \) satisfies the efficient moving condition, then the starting point of \( \mathcal{P} \) must have the minimum length of the locus by the above arguments. Hence \( \mathcal{P} \) should be a geodesic in \( \mathcal{L} \).

Remark. 1. If the semi-final pivot point \( z \) is contained in \( \mathcal{P} \), the move \( t \) is the only option for \( z \). That is why the efficient moving condition contains the final point exceptions.

2. As in the proof, the alternative path by allowing move \( p \) in 2-block does not generate a longer path, so it can be another geodesic. Thus we want to call a geodesic efficient if satisfies the efficient moving condition.

For bi-infinite geodesics, the efficient moving condition is valid as well, since the efficient moving condition is in some sense a local condition.

**Proposition 12.** If a bi-infinite path in a bi-infinite ladder satisfies the efficient moving condition, then it is a bi-infinite geodesic.

**Proof.** By Proposition 11, every restricted finite subpath of a given path is a geodesic in the finite subladder. Hence by definition, it is a bi-infinite geodesic in the whole bi-infinite ladder. \qed
Now we narrow down our interest to bi-infinite geodesics in periodic ladders.

**Definition.** Let \( \mathcal{L} \) be an infinite (not necessarily bi-infinite) periodic ladder with a repeating pattern \((a_1, \ldots, a_n)\). If the pattern \((a_1, \ldots, a_n)\) is minimal up to cyclic permutation, (i.e., it cannot be decomposed into copies of subpatterns up to cyclic permutation) then we say \( \mathcal{L} \) is of **period** \((a_1, \ldots, a_n)\). In this case, a finite subladder of \( \mathcal{L} \) having the following type up to cyclic permutation:

\[
\begin{align*}
(a_1, \ldots, a_n) & \quad \text{if } n \text{ is even}, \\
(a_1, \ldots, a_n, a_1, \ldots, a_n) & \quad \text{if } n \text{ is odd},
\end{align*}
\]

is called a **prime subladder** of \( \mathcal{L} \).

**Remark.** Observe that the length of any prime subladder is always even.

The following proposition says within a bi-infinite ladder \( \mathcal{L} \), we can establish bi-infinite efficient geodesic whenever \( \mathcal{L} \) is periodic, by concatenating finite efficient geodesics in its prime subladder.

**Proposition 13** (Concatenating Geodesics). Let \( \mathcal{L} \) be a periodic bi-infinite ladder with a prime subladder \( \mathcal{L}' \). Then there exists an efficient geodesic \( P' \) whose endpoints reside on the same side of \( \mathcal{L}' \) such that by concatenating \( \mathcal{L}' \) bi-infinitely, the induced concatenated path \( P \) is a bi-infinite efficient geodesic in \( \mathcal{L} \). More generally, if \( \mathcal{L}'' \) is a finite concatenation of prime subladders then there exists an efficient geodesic \( P'' \) in \( \mathcal{L}'' \) whose induced bi-infinitely concatenated path \( P \) is a bi-infinite efficient geodesic in \( \mathcal{L} \).

**Proof.** Since we want to concatenate \( P' \) by juxtaposing \( \mathcal{L}' \) to make one single connected geodesic in \( \mathcal{L} \), we need to consider maximal efficient geodesics in \( \mathcal{L}' \) whose endpoints lying on the same side of \( \mathcal{L}' \). Thus, there are always two efficient geodesics to consider in \( \mathcal{L}' \), each starts and ends on the same side of \( \mathcal{L}' \).

Note the efficiency of a geodesic is determined by the coefficients of the ladder, except for the move from the semi-final point to the final point of the ladder, which we call **reluctant move**. This move is abnormal, as it is always determined to be \( t \) regardless of the succeeding coefficient of the ladder. In particular, even if the associated coefficient is 1, \( \mathcal{L}' \) fails to move \( p \), which hinders \( P' \) to remain efficient when we embed \( P' \) as a subpath of \( P \). See Figure 8.

![Figure 8](image-url)

**Figure 8:** Illustration of Reluctant move. Within a ladder, there are two maximal efficient geodesics available whose endpoints lying on the same side. In this case, only the right one consists of the reluctant move, which is denoted by the dotted line.
Therefore, the only possible obstruction for $\mathcal{P}$ to be efficient in $\mathcal{L}$ arises from the reluctant moves in $\mathcal{P}'$. Thus, now assume that two efficient geodesics in $\mathcal{L}'$ consist of reluctant moves, and suppose further to the contrary that these two moves indeed cause problems during the bi-concatenation. By problems we mean the reluctant move cannot be extended as an efficient move after the bi-concatenation. In fact, the reluctant moves in two efficient geodesics should be as illustrated in Figure 9. To make each reluctant move problematic, both the first and last coefficients of $\mathcal{L}'$ must be 1.

![Figure 9](image)

Figure 9: Both efficient geodesics have problematic reluctant moves (dotted lines). This forces the first and last coefficient of the ladder to be 1.

To deduce a contradiction from this, we need the following lemma:

**Lemma 14.** Unless all the coefficients of a ladder are 1, the two efficient geodesics intersect at some point $x$ in the ladder, and follow the same pivot points starting from $x$, except for the final point.

**Proof of Lemma 14.** Suppose the two geodesics do not intersect. Then both should not contain $t$-moves because $t$ bisects the ladder so it makes the two geodesics inevitably meet each other. Therefore, both geodesics are of the form $p \cdots p$, in turn, all the coefficients of the ladder become 1. Thus, we just showed the contraposition of the first assertion. Furthermore, note each efficient move at pivot points possibly except for the final point is completely determined by the corresponding coefficients of the ladder. Hence, once the two geodesics meet, they have to visit the exact same pivot points in $\mathcal{L}'$, possibly except for the final point. △

Now we are ready to prove our Proposition 13. We claim that two geodesics in $\mathcal{L}'$ should not intersect. Suppose to the contrary that they intersect at some point in the ladder. Then two geodesics eventually visit the same pivot points. However, as in Figure 9, this forces one of two geodesics to move $tt$ at the final block of $\mathcal{L}'$, whose coefficient is 1, which contradicts to the efficiency condition. (It should have moved $p$ at the final block, instead of $tt$.)

Thus, two geodesics must not meet each other; so their coefficient must be all 1’s, by Lemma 14. However, these consecutive $p$-moves forces both geodesics to avoid the semi-final points of the ladder, which contradicts to the assumption that both geodesics consist of reluctant moves.

Therefore, we showed even if both of the two geodesics contain reluctant moves, at least one of them must not cause a problem during the bi-concatenation. All in all, we conclude at least one of two geodesics can be extended to $\mathcal{P}$ within $\mathcal{P}$. □
4 PSL(2, Z)-action on Farey Graph

Recall that PSL(2, Z) is isomorphic to the group of the orientation preserving isometries of \( \mathbb{H}^2 \). Indeed, there is an isometric action PSL(2, Z) on \( \mathcal{F} \subset \mathbb{H}^2 \), defined as

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{p}{q} = \frac{ap + bq}{cp + dq}
\]
on the vertices of \( \mathcal{F} \), and induced action on the edges. (See for instance [Hat17, Prop 3.1]) It follows that any element \( f \) of PSL(2, Z) maps a Farey triangle to another Farey triangle. More generally, \( f \) sends a ladder to the ladder of the same type. This is because a ladder is a consecutive chain of Farey triangles [Fact 4], and a bi-infinite geodesic \( \gamma \) cuts a Farey triangle \( T \) with the same symbol (L or R) as when \( f(\gamma) \) cuts \( f(T) \). (See the proof of [Ser15, Prop 2.2]) Eventually, for a ladder \( \mathcal{L} \) and its associated geodesic \( \gamma \), the pair \( (\mathcal{L}, \gamma) \) yields the same cutting sequence as \( (f(\mathcal{L}), f(\gamma)) \) does. Thus, we proved the following proposition:

**Proposition 15.** PSL(2, Z) preserves the type of a ladder.

For a hyperbolic isometry \( f \) in PSL(2, Z), denote by \( A_f \) the axis of \( f \) in \( \mathbb{H}^2 \). Recall that \( A_f \) is defined to be the bi-infinite geodesic joining the two fixed points of \( f \), which are represented by conjugate quadratic irrational numbers. It is also a basic fact that \( f \) acts on \( A_f \) by translation.

Now we can show that there exists a canonical ladder associated with a given hyperbolic element in PSL(2, Z).

**Proposition 16.** For each hyperbolic element \( f \), there exists the unique bi-infinite ladder \( \mathcal{L} \) stabilized by \( f \). In this case, each side of \( \mathcal{L} \) is stabilized by \( f \) as well.

**Proof.** Let \( \mathcal{L} \) be the ladder associated with the axis \( A_f \). Such a ladder is unique since \( A_f \) is uniquely determined by \( f \). Suppose for the sake of contradiction, \( A_f \) intersects only finitely many Farey triangles. This is equivalent to saying \( \mathcal{L} \) consists of finitely many Farey triangles. Then the ends of \( A_f \) must be vertices of \( \mathcal{F} \). However, as two fixed points of \( f \) on \( A_f \) are irrational, the ends of \( A_f \) cannot be vertices of \( \mathcal{F} \), which is a contradiction. Hence, \( \mathcal{L} \) must be bi-infinite.

By definition of a ladder, \( \mathcal{L} \) covers \( A_f \). As \( f \) bijectively sends a segment of \( A_f \) to another segment of \( A_f \), any subladder of \( \mathcal{L} \) must be bijectively mapped to another subladder of \( \mathcal{L} \). Thus, it follows that \( f \) stabilizes \( \mathcal{L} \).

Figure 10: \( \triangle pqr, \triangle f(p)f(q)f(r) \): Two triangles with different orientations

To show \( f \) stabilizes each side of \( \mathcal{L} \), it suffices to show \( f \) sends a pivot point of \( \mathcal{L} \) to another pivot point on the same side of \( \mathcal{L} \). Assume there is some
pivot point \( p \) in \( L \) such that \( f(p) \) lies on the other side of \( L \), as in Figure 10. Denote by \( q, r \) the adjacent pivot points of \( p \) on the other side and the same side respectively. Since \( p, q, r \) form a triangle, so do \( f(p), f(q), f(r) \). Then as \( f \) acts on \( L' \) by translation, the position of \( f(p), f(q) \) and \( f(r) \) must be as in Figure 10. However, then \( \triangle pqr \) and \( \triangle f(p)f(q)f(r) \) have different orientations, which is impossible since \( f \) is orientation preserving. Therefore, \( f \) must stabilize each side of \( L \).

Now we can show that the canonical ladder associated with given hyperbolic element must be periodic.

**Theorem 17.** Let \( f \) be a hyperbolic element in \( \text{PSL}(2, \mathbb{Z}) \), and \( L \) be the bi-infinite ladder stabilized by \( f \). Then \( L \) is periodic with the period identical to the cutting sequence of a fixed point of \( f \). More precisely, there exists a ladder \( L' \) constructed by a finite concatenation of a prime subladder of \( L \) such that

\[
\bigcup_{k=-\infty}^{\infty} f^k(L') = L.
\]

**Proof.** Let \( \alpha \) be a quadratic irrational fixed point of \( f \). By Proposition 1 and Corollary 2, \( \alpha \) has a periodic cutting sequence. Orient \( A_f \) to make \( \alpha \) be the terminal point of \( A_f \). By Lemma 3, the cutting sequence of \( A_f \) must be eventually periodic. By Proposition 5, this induces the one-sided infinite periodic subladder in \( L \) corresponding to the periodic part of the cutting sequence of \( A_f \).

![Figure 11: Periodic part of bi-infinite ladder L.](image)

Let \( L'' \) be its prime subladder, with orientation inherited from \( A_f \), and \( p, q \) be its first two pivot points. As \( f(p), f(q) \) are also pivot points of \( L \), Now define \( L' \) to be the finite subladder of \( L \) bounded by two edges \( pq \) and \( f(p)f(q) \). Since \( f \) acts on \( L \) by translation, we have

\[
\bigcup_{k=-\infty}^{\infty} f^k(L') = L,
\]

and thus \( L \) is periodic, with the period identical to the period of \( L' \), which is same as the cutting sequence of \( \alpha \).

Now, suppose \( L' \) cannot be formed by a finite concatenation of the prime subladder \( L'' \) of \( L \). Since both \( L' \) and \( L'' \) bi-infinitely cover \( L \) having the same starting pivot points as \( p \) and \( q \), the length of the period of \( L \) must be a common divisor \( d \) of the two lengths of periods of \( L' \) and \( L'' \). Since we assumed \( L' \) cannot be formed by \( L'' \), \( d \) must be strictly less than the length of \( L'' \). However, from
the minimality of prime subladder, the value $d$ must be odd. This implies the period of $\mathcal{L}'$ is odd, since the length of a prime subladder is always even. Thus $p$ and $f(p)$ must lie on the different side of $\mathcal{L}$, which is absurd by the second statement of Proposition 16.

5 Translation Length on Farey Graph

In this section, we provide our main theorem: there exists a geodesic axis in the Farey graph of a given Anosov mapping class. This theorem allows to define the translation length of Anosov mapping class on the Farey graph, which is our initial objective to calculate. Indeed, after proving the main theorem, we provide a concrete algorithm to calculate the translation length of any Anosov element.

Theorem 18 (Existence of Geodesic Axis). Let $f$ be an Anosov map. Then there exists a bi-infinite geodesic $\mathcal{P}$ in $\mathcal{F}$ on which $f$ acts by translation.

Proof. We identify $f$ as a hyperbolic element of $\text{PSL}(2, \mathbb{Z})$. Then the existence of the invariant bi-infinite geodesic $\mathcal{P}$ of $f$ is a direct consequence of Proposition 13 and Theorem 17. Indeed, there exists a ladder $\mathcal{L}'$ which is a finite concatenation of a prime subladder of the associated bi-infinite ladder $\mathcal{L}$ of $f$ satisfying:

$$\bigcup_{k=-\infty}^{\infty} f^k(\mathcal{L}') = \mathcal{L}.$$ 

Hence, there exists an efficient geodesic $\mathcal{P}'$ in $\mathcal{L}'$ whose bi-infinite concatenation makes the efficient geodesic $\mathcal{P}$ in $\mathcal{L}$. Therefore, $\mathcal{P}$ is invariant under $f$, as

$$\bigcup_{k=-\infty}^{\infty} f^k(\mathcal{P}') = \mathcal{P}.$$ 

Since $f$ acts on $\mathcal{F}$ by translation, so does on $\mathcal{P}$, which was what was wanted.

Remark. Since we have considered only efficient bi-infinite geodesics, there may be another geodesic in $\mathcal{L}'$, whose concatenation makes another $f$-invariant bi-infinite geodesic in $\mathcal{F}$. In particular, the geodesic axis $\mathcal{P}$ in $\mathcal{F}$ of $f$ in may not be unique.

Now we provide a concrete algorithm to calculate the translation length of an Anosov mapping class. To this end, we introduce a special map called ancestor map on vertices of the Farey graph, which can be found in [BHS12]. Let $\text{Vert}(\mathcal{F})$ be the set of vertices of the Farey graph. For each $v \in \text{Vert}(\mathcal{F})$, we can give a lexicographical order on the neighbors of $v$, first order by denominator in increasing order, then by numerator as the same way. Since we restrict the denominators of elements of $\text{Vert}(\mathcal{F})$ to be positive, this ordering on $\text{Vert}(\mathcal{F})$ is a well-ordering. Therefore, the map $\alpha : \text{Vert} \mathcal{F} \to \text{Vert} \mathcal{F}$ associating each vertex to its smallest(with respect to the order we gave) neighbor is well-defined.
We call this map $\alpha$ as **ancestor map**. Note that for any extended rational number $p$, we have $\alpha^n(p) = \infty$ for some non-negative $n$. Choosing such $n$ as the minimum one, we have a path from $p$ to $\infty$ obtained by iterating ancestor map on $p$:

$$p, \alpha(p), \alpha^2(p), \ldots, \alpha^n(p) = \infty.$$ 

We call such a path **ancestor path**, which turns out to be a geodesic in $\mathcal{F}$ connecting $p$ and $\infty$. (See [BHS12])

When two fixed points of a hyperbolic element have opposite signs, then the bi-infinite geodesic connecting these two points and the edge $e = 0, \infty$ must intersect. Thus, in this case, $e$ must be a rung of the associated ladder of $f$. We call such a ladder containing $e = 0, \infty$ a **standard ladder**. In fact, whether a given hyperbolic element has a standard invariant ladder or not can be immediately identified by their signs of entries.

**Proposition 19.** Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{Z})$ be a hyperbolic element. Then the associated ladder of $A$ is standard if and only if $b, c$ have the same sign.

**Proof.** Let $x \in \mathbb{R}$ be one of the fixed points of $A$. Then we have $\frac{ax + b}{cx + d} = x$, so $cx^2 + (d - a)x - b = 0$. Since the associated ladder of $A$ contains $e = 0, \infty$ if and only if the hyperbolic axis of $A$ transverses $e$ if and only if two fixed points have different sign if and only if $\frac{c}{d} < 0$ if and only if $b$ and $c$ have the same sign.

The following proposition suggests a hyperbolic element with a standard invariant ladder is quite standard within its conjugacy class of $\text{PSL}(2, \mathbb{Z})$. This result will be needed in section 6.2.

**Proposition 20.** Any hyperbolic element of $\text{PSL}(2, \mathbb{Z})$ is conjugate to an element with standard invariant ladder.

**Proof.** Let $f$ be a hyperbolic element of $\text{PSL}(2, \mathbb{Z})$ with an invariant ladder $\mathcal{L}$. Pick any rung $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ of $\mathcal{L}$. Then one of the following two matrices

$$\begin{pmatrix} p & r \\ q & s \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} r & p \\ s & q \end{pmatrix}$$

must be in $\text{PSL}(2, \mathbb{Z})$. Denote by $A$ the one in $\text{PSL}(2, \mathbb{Z})$. In fact, $A$ maps the rung $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ of $\mathcal{L}$ to an edge $0, \infty$, so $\mathcal{L}$ is mapped to a standard ladder $\mathcal{L}'$ under $A$. In fact, the very hyperbolic element $AfA^{-1} \in \text{PSL}(2, \mathbb{Z})$ is associated with the standard invariant ladder $\mathcal{L}'$, which was what was wanted.

Aside from standard ladders, we can find a rung of the associated ladder using ancestor map:

**Proposition 21.** Let $f$ be a hyperbolic element in $\text{PSL}(2, \mathbb{Z})$. Let $\delta$ and $\bar{\delta}$ be the two fixed points of $f$ and set $m = \frac{\delta + \bar{\delta}}{2}$. Then there exists $k \geq 0$ such that the edge formed by $\alpha^k(m)$ and $\alpha^{k+1}(m)$ is a rung of the associated bi-infinite ladder of $f$. 

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Proof. Without loss of generality, let $\bar{\delta} < \delta$. Note that $m$ is a rational number, so $m$ represents a vertex in $F$. Also, $m$ and $\infty$ are separated by the bi-infinite geodesic with ends $\delta$ and $\bar{\delta}$. As the ancestor path for $m$ is a path from $m$ to $\infty$, there exists $k \geq 0$ such that $\alpha^k(m) \in (\bar{\delta}, \delta)$ and $\alpha^{k+1}(m) \notin (\bar{\delta}, \delta)$. Then the edge $e$ with ends $\alpha^k(m)$ and $\alpha^{k+1}(m)$ must transverse the geodesic connecting $\delta$ and $\bar{\delta}$. By Proposition 6, the edge $e$ must be a rung of the bi-infinite ladder associated with $f$.

Now, write $p = \alpha^k(m)$. Then denote by $\tilde{L}$ the subladder bounded by two edges $p, \alpha(p)$ and $f(p), f(\alpha(p))$. If the length of $\tilde{L}$ is even, then $p$ must be a pivot point, so $\tilde{L}$ comes out to be a finite concatenation of a prime subladder of $L(\delta, \bar{\delta})$. In turn, we are left to realize the efficient geodesic in $\tilde{L}$ and calculate its length to find the translation length of $f$. However, if the length of $\tilde{L}$ is odd, then $p$ is not a pivot point of $L$. In this case, we need calibrate $\tilde{L}$ to find another subladder with an even length. The calibration process is simple: From an odd-length ladder of type $(a_1, \ldots, a_n)$, generate an even-length ladder $(a_1 + a_n, a_2, \ldots, a_{n-1})$ whose bi-infinite concatenation with $f$ yields $L$ as well. (Refer to Algorithm 2 in Appendix) Thus, the case when $\tilde{L}$ has an odd length is reduced to when $\tilde{L}$ has an even length.

It remains to realize an efficient geodesic in the even length subladder $\tilde{L}$ to find the translation length of $f$. Since the types of the ladder are given, we can use the criteria introduced in Section 3.3 to find the efficient geodesic. To illustrate the aforementioned process, we provide some examples:

**Example 1.** Consider an Anosov map $f$ represented by a matrix in $\text{PSL}(2, \mathbb{Z})$:

\[
\begin{pmatrix}
277 & 60 \\
337 & 73
\end{pmatrix}
\]

The fixed points of $f$ are $\frac{77 \pm \sqrt{25149}}{337}$. Since their signs are different, the ladder associated with $f$ must be standard. In fact, the ladder bounded by two edges $e = 0, \infty$ and $f(e)$ is of the type $(1, 4, 1, 1, 1, 1, 1, 4)$, which can be calibrated into one with $(4, 1, 1, 1, 1, 1, 1, 5)$. (See Figure 12) Thus, the corresponding efficient geodesic is $tpppt$, length 5. Hence, the translation length of $f$ is 5.

![Figure 12: Invariant geodesic for $f$ represented by $\begin{pmatrix} 227 & 60 \\ 337 & 73 \end{pmatrix}$](image)

Now we provide a non-standard ladder example which forces to use Proposition 21 to find a rung of the associated ladder.
Example 2. Consider an Anosov map $f$ represented by a matrix in $\text{PSL}(2, \mathbb{Z})$:

$$
\begin{pmatrix}
65 & -56 \\
101 & -87
\end{pmatrix}
$$

In this case, two fixed points of $f$ are $\frac{76 \pm 2\sqrt{30}}{101}$, so its midpoint $m = \frac{76}{101}$. Since two points have the same sign, we exploit Proposition 21 to find a rung in the associated ladder $\mathcal{L}$. The ancestor path for $\frac{76}{101}$ is:

$$
\frac{76}{101} \rightarrow \frac{3}{4} \rightarrow \frac{1}{1} \rightarrow \frac{1}{0}.
$$

Since two fixed points are approximately $0.644 \cdots$ and $0.860 \cdots$, the edge $e$ connecting $\frac{3}{4}$ and $\frac{1}{2}$ must be a rung of $\mathcal{L}$. Then the ladder bounded by $e = \frac{65-56}{101-87}$ and $f(e)$ is of the type $(1, 5, 3)$, which can be calibrated into one with $(5, 4)$, in which an efficient geodesic has the form $tt$. (see Figure 13) Hence, the translation length of $f$ is 2.

![Figure 13: Invariant geodesic for $f$ represented by \( \begin{pmatrix} 65 & -56 \\ 101 & -87 \end{pmatrix} \)](image)

6 Applications

6.1 Minimal Word

Shin and Strenner [SS16] showed that there is a non-Penner type pseudo-Anosov mapping class $\text{Mod}(S)$ when $S$ is a non-sporadic surface. They also showed this is not the case when $S$ is a sporadic surface. In particular, they showed the following (Lemma A.1 in [SS16]):

**Lemma 22.** Let $f$ be a hyperbolic element in $\mathcal{F}$, and $\gamma$ be the associated bi-infinite geodesic. Let $e_0$ be an edge of $\mathcal{F}$ intersecting $\gamma$. Let $a$ and $b$ be the endpoints of $e_0$ on the left and right hand side of $\gamma$, respectively. Following the orientation of $\gamma$, record the cutting sequence $s_1 \cdots s_n$ starting from the next rung of $e_0$ and ending at $e_n = f(e_0)$. Let $\tau_a$ and $\tau_b$ be the rotations of $\mathcal{F}$ by one tile to the right about the points $a$ and $b$, respectively, and introduce the notation

$$
\tau(s) = \begin{cases}
\tau_a^{-1} & \text{if } s = L \\
\tau_b & \text{if } s = R.
\end{cases}
$$
Then
\[ f = \tau(s_1) \circ \cdots \circ \tau(s_n), \]
where the rotations are applied in right-to-left order.

Thus, we can regard a hyperbolic element of PSL(2,\Z) as a word with letters \( \{\tau^{-1}, \tau_b\} \), where \( a \) and \( b \) are chosen to form an edge \( \overline{ab} \) intersecting \( \gamma \), so \( \overline{ab} \) becomes a rung of the associated ladder to \( f \).

Then the natural question to ask is the following: Using the fixed number of \( \tau_a^{-1} \) and \( \tau_b \), what is the form of words whose translation length is minimal? Our result answers to this question.

**Theorem 23.** Let \( a \) and \( b \) be isotopy classes of essential simple closed curves with intersection number 1 on a torus. Let
\[ S = \{\tau_a, \cdots, \tau_a, \tau_b, \cdots, \tau_b\}, \]
where \( m, n \geq 1 \). Then among the words formed by the elements in \( S \), the word \( \tau_a^{-m} \tau_b^n \) (upto cyclic permutation) has the shortest translation length. Moreover, that shortest length is at most 2.

**Proof.** The latter assertion obvious, since the length of the ladder associated with \( \tau_a^{-m} \tau_b^n \) is 2.

When at least one of \( m \) or \( n \) is 1, there is only one word formed by the elements in \( S \) upto cyclic permutation, so there is nothing to prove.

Now assume both \( m \) and \( n \) are bigger than 1. Then a word \( w \) which is not of the form \( \tau_a^{-m} \tau_b^n \) upto cyclic permutation must have the ladder with length at least 4. However, in a ladder with length 4, the length of the efficient geodesic is at least 2(`pp', if possible, is the shortest one), so \( w \) cannot have the shorter translation length than \( \tau_a^{-m} \tau_b^n \) have. Therefore, we proved \( \tau_a^{-m} \tau_b^n \) produces the smallest translation length on \( F \) among the words formed by \( S \). \( \square \)

### 6.2 Effective Bound of Ratio of Teichm"uller to Curve Graph Translation Length

We can define the translation length on *Teichmüller space* as well. More precisely, for a non-sporadic surface \( S \) and any pseudo-Anosov map \( f \in \text{Mod}(S) \), Bers [Ber78, Theorem 5] showed that there exists the axis \( A_f \) in the Teichmüller space \( T(S) \) of \( S \) which is invariant under the action of \( f \) by translation with the Teichmüller distance \( l_T(f) := \log \lambda \), where \( \lambda > 1 \) is the dilatation of \( f \). The analogous property also holds for when \( S \) is a torus: an Anosov map acts on the Teichmüller space of a torus \( T(T^2) \) with the invariant axis by translation with the hyperbolic distance \( \log \lambda \), where \( \lambda > 1 \) is the leading eigenvalue of \( f \).

For non-sporadic surfaces \( S = S_g \), Gadre, Hironaka, Kent and Leininger [GHKL13] showed that the ratio \( \frac{l_T(f)}{l_C(f)} \) is bounded below by a linear function of \( \log(g) \) by constructing a pseudo-Anosov map \( f \) which realizes the minimum.
Aougab and Taylor [AT17] provided another infinite family of such ratio minimizers. We show here the analogous result on the ratio for a torus also holds.

To begin with, we formalize our observations on the connection between continued fractions and the ladders. For a finite sequence of positive integers $S = (a_1, a_2, \cdots, a_n)$, we consider the following two continued fractions: when $n > 1$, write

$$\frac{p(S)}{q(S)} := [a_1; a_2, \cdots, a_n], \quad \frac{r(S)}{s(S)} := [a_1; a_2, \cdots, a_{n-1}]$$

where $(p(S), q(S))$ and $(r(S), s(S))$ are relatively prime positive integer pairs for $n > 1$. When $n = 1$, define $(p(S), q(S), r(S), s(S)) = (a_1, 1, 1, 0)$ to make it consistent with the case of $n > 1$.

Observe that two fractions $\frac{p(S)}{q(S)}$ and $\frac{r(S)}{s(S)}$ are realized by two adjacent pivot points forming an edge $e$ of a standard ladder of type $S$, which is flanked by $e_0 = 0, \infty$ and $e$. We denote by $L(S)$ this standard ladder constructed from a sequence $S$. Now, a matrix defined as

$$M(S) := \begin{pmatrix} p(S) & r(S) \\ q(S) & s(S) \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$$

sends $e_0$ to $e$, so $M(S)$ has the associated ladder as $L(S)$. Indeed, the function $M$ from the set of finite sequences of positive integers to $\text{GL}(2, \mathbb{Z})$ is a structure-preserving map: The matrix constructed from the concatenation of two positive integer sequences can be expressed by the product of each associated matrix.

**Lemma 24.** Let $(a_1, \cdots, a_m)$ and $(b_1, \cdots, b_n)$ be two sequences of positive integers. Then

$$M(a_1, \cdots, a_m)M(b_1, \cdots, b_n) = M(a_1, \cdots, a_m, b_1, \cdots, b_n).$$

**Proof.** We claim

$$M(a_1, \cdots, a_n) = M(a_1)M(a_2, \cdots, a_n)$$

for every $n$. Write $M(a_2, \cdots, a_n) = \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix}$. Then

$$M(a_1)M(a_2, \cdots, a_n) = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p' & r' \\ q' & s' \end{pmatrix} = \begin{pmatrix} a_1p' + q' & a_1r' + s' \\ p' & r' \end{pmatrix}.$$ 

By definition,

$$\frac{a_1p' + q'}{p'} = a_1 + \frac{1}{p'/q'} = a_1 + [0; a_2, \cdots, a_n] = [a_1; a_2, \cdots, a_n],$$

$$\frac{a_1r' + s'}{r'} = a_1 + \frac{1}{r'/s'} = a_1 + [0; a_2, \cdots, a_{n-1}] = [a_1; a_2, \cdots, a_{n-1}].$$
Hence, we have \((a_1 p' + q' \quad a_1 r' + s') = M(a_1, \ldots, a_n)\), as we have claimed.

Finally, the claim implies Lemma 24. This is because:

\[ M(a_1, \ldots, a_m)M(b_1, \ldots, b_n) = M(a_1) \cdots M(a_m)M(b_1) \cdots M(b_n) = M(a_1, \ldots, a_m, b_1, \ldots, b_n). \]

\[ \square \]

To extract an inequality of translation lengths from that of matrices, we introduce the component-wise comparison among matrices.

**Definition.** For two matrices \(A = (a_{ij})\) and \(B = (b_{ij})\) having the same size, we write \(A \succeq B\) if \(a_{ij} \geq b_{ij}\) for all \(i, j\), that is, every component of \(A\) is greater than or equal to that of \(B\). In particular, we will write it simply \(A \succeq 0\) when every component of \(A\) is nonnegative.

It is easily seen that for equisized square matrices \(A, B, C, D\), and \(A \succeq B \succeq 0\) and \(C \succeq D \succeq 0\), we have \(AC \succeq BD \succeq 0\). Hence by Lemma 24 two positive integer sequences \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) with \(a_i \geq b_i\) for every \(i\) induces the inequality \(M(a_1, \ldots, a_n) \succeq M(b_1, \ldots, b_n)\). The same argument proves \(M(a_1, \ldots, a_n) \geq 0\), as \(M(a_i) \geq 0\) for every \(i\). We summarize the observations in the following lemma.

**Lemma 25.** For two positive integer sequences \((a_1, \ldots, a_n)\) and \((b_1, \ldots, b_n)\) with \(a_i \geq b_i\) for every \(i\), we have

\[ M(a_1, \ldots, a_n) \succeq M(b_1, \ldots, b_n) \geq 0. \]

Shortly, we will see that how big is the **trace** of an Anosov map \(f\) is closely related to the translation length of \(f\). Since we are identifying \(f\) as a hyperbolic element in \(\text{PSL}(2, \mathbb{Z})\), we have to first clarify what it means by the **trace** of an element in \(\text{PSL}(2, \mathbb{Z})\). Indeed, due to the ambiguity of signs in \(\text{PSL}(2, \mathbb{Z})\), the trace \(\text{tr} : A \in \text{PSL}(2, \mathbb{Z}) \mapsto \text{tr} A\) is not well-defined. Thus, we use the absolute value \(|\text{tr} A|\) to define the trace on \(\text{PSL}(2, \mathbb{Z})\), which resolves the ambiguity. We write \(\text{tr} A := |\text{tr} A|\) when no confusion can arise.

Note that \(\text{det} M(a_1) = -1\), so we have \(\text{det} M(a_1, \ldots, a_k) = (-1)^k\) by Lemma 24. Thus whenever a sequence \(S\) has an even length, \(M(S)\) represents an element of \(\text{PSL}(2, \mathbb{Z})\), whose translation length actually decides the lower bound of the trace.

**Proposition 26.** Let \((a_1, \ldots, a_{2n})\) be a sequence of positive integers. Then

\[ \text{tr} M(a_1, \ldots, a_{2n}) \geq \text{tr}(M(2)^l), \tag{1} \]

where \(l\) is the translation length of \(M(a_1, \ldots, a_{2n})\) on \(F\).

**Proof.** We first consider the case \(a_1 = a_2 = \cdots = a_{2n} = 1\). Since \(M(1, \ldots, 1)\) has the invariant geodesic of the form \(p \cdots p\) in \(F\), the translation length of \(M(1, \ldots, 1)\) is \(n\). By Lemma 24
\[
\mathcal{M}(a_1, \cdots, a_{2n}) = \mathcal{M}(1)^{2n} = \left( \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right)^{2n} = \left( \begin{array}{cc} 2 & 1 \\ 1 & 1 \end{array} \right)^n \succeq \left( \begin{array}{cc} 2 & 1 \\ 1 & 0 \end{array} \right)^n = \mathcal{M}(2)^n.
\]

Applying the trace function on both sides, we have the inequality (1).

Now suppose \( a_i > 1 \) for some \( i \). Note that \( \mathcal{M}(a_i, \cdots, a_{2n}, a_1, \cdots, a_{i-1}) \) is conjugate to \( \mathcal{M}(a_1, \cdots, a_{2n}) \) for every \( i \) in \( \text{PSL}(2, \mathbb{Z}) \) by Lemma 24. Since trace and translation length are invariant under conjugation, now we may assume \( a_1 > 1 \).

The rest of the proof will be divided into two steps. First, we will downsize \( \mathcal{M}(a_1, \cdots, a_{2n}) \) by constructing a smaller sequence \( (b_1, \cdots, b_{2n}) \) out of \( (a_1, \cdots, a_{2n}) \) while \( \mathcal{M}(b_1, \cdots, b_{2n}) \) has the same translation length. Then we will show \( \mathcal{M}(b_1, \cdots, b_{2n}) \) satisfies the inequality (1) with \( a_i \)'s replaced by \( b_i \)'s.

**Step 1 (Construction of Downsized Sequence.)** Based on the sequence \( (a_1, \cdots, a_{2n}) \), we construct a sequence \( (b_1, \cdots, b_{2n}) \) as follows:

(i) Set \( b_i = 1 \) whenever \( a_i = 1 \).

(ii) For each odd block of 1’s enclosed by non-1’s in \( \{a_i\} \), set the corresponding element of \( \{b_i\} \) in the position of non-1 followed by the odd 1-block to be 1. More precisely, whenever there exist \( i, k \) such that \( a_i > 1, a_{i+1} = \cdots = a_{i+2k-1} = 1, \) and \( a_{i+2k} > 1 \), then let \( b_{i+2k} = 1 \).

(iii) Set all undetermined \( b_i \) to 2.

Recall that the efficient geodesic in a ladder solely depends on the type of the ladder. Thus, obviously the process (i) has no effect on deciding efficient geodesics in the ladder. For (iii), note that the move \( t \) is done whenever the move \( p \) visits \( \geq 2 \) vertices, so changing a non-1 coefficient \( a_i \geq 2 \) into \( b_i = 2 \) does not change the efficient geodesic as well. Therefore, what is left to show is that the transition of non-1 coefficient \( a_{i+2k} \) into 1 in Process (ii) does not affect deciding the efficient geodesic in the ladder. Suppose \( a_i > 1, a_{i+1} = \cdots = a_{i+2k-1} = 1, \) and \( a_{i+2k} > 1 \). Since \( a_i > 1 \), the efficient geodesic must visit the pivot point corresponding to \( a_i \), followed by \( k \)-consecutive \( p \) moves by which it avoids the pivot points associated with \( a_{i+1}, a_{i+3}, \cdots, a_{i+2k-1} \) until it reaches the point associated with \( a_{i+2k} \). In particular, the part of efficient geodesic decided by \( a_i, \cdots, a_{i+2k} \) is completely irrelevant with the value of \( a_{i+2k} \). See Figure 14. Therefore we may set \( b_{i+2k} = 1 \) without changing the efficient geodesic of \( \mathcal{M}(a_1, \cdots, a_n) \).
All in all, the three processes (i), (ii), (iii) preserve the efficient geodesic in the ladder, so \( M(a_1, \cdots, a_n) \) and \( M(b_1, \cdots, b_n) \) have the same translation length on \( F \), satisfying \( a_i \geq b_i \), by construction. By Lemma 25 we get

\[
\text{tr} M(a_1, \cdots, a_n) \geq \text{tr} M(b_1, \cdots, b_n). \tag{2}
\]

**Step 2 (Show Inequality with \( b_i \)'s.)** We show the similar inequality of (1) for \( b_i \) also holds:

\[
\text{tr} M(b_1, \cdots, b_{2n}) \geq \text{tr}(M(2)^l). \tag{3}
\]

By construction (ii), any odd 1-blocks in \( \{a_i\} \) are all transformed to even 1-blocks in \( \{b_i\} \), except for those not enclosed by non-1 blocks. However, as we have assumed \( a_1 > 1 \), an odd block of 1 in \( b_i \), if exists, can only appear at the end of the sequence. Thus, if the number of 1 in \( b_i \) is \( 2k \), then it forces every 1-block in \( \{b_i\} \) to be even. Therefore the efficient geodesic will contain \( 2n - 2k \) of \( t \)-moves and \( k \) of \( p \)-moves, so the translation length of \( M(b_1, \cdots, b_{2n}) \) must be \( 2n - k \). On the other hand, as 1’s are clustered in even blocks, every \( M(1) \) in \( M(b_1)M(b_2) \cdots M(b_{2n}) \) can be paired and reduced into \( M(2) \), for \( M(1)^2 \geq M(2) \). Then we have

\[
M(b_1)M(b_2) \cdots M(b_{2n}) \geq M(2)^{2n-k},
\]

which establishes the inequality (3).

If the number of 1 in \( b_i \) is \( 2k - 1 \), then it means there is an odd block of 1’s at the end of \( \{b_i\} \), and in particular \( b_{2n} = 1 \). To exploit as many as 1’s to minimize the length of the efficient geodesic, we have to start at \( \infty \), which in turn yields a shorter geodesic with and \( 2n - 2k \) of \( t \)-moves and \( k \) of \( p \)-moves, compared to the geodesic starting from 0, which consists of \( 2n - 2k + 2 \) of \( t \)-moves and \( k - 1 \) of \( p \)-moves. Thus, the translation length of \( M(b_1, \cdots, b_{2n}) \) is \( 2n - k \). Since \( M(2)M(1) \geq M(2) \), we can ignore the first 1 appearing in the odd block of 1’s. For the other 1’s, we can pair them as before, which will give

\[
M(b_1)M(b_2) \cdots M(b_{2n}) \geq M(2)^{2n-k},
\]

again showing the inequality (3).

Finally, combining (2) with (3), we conclude

\[
\text{tr} M(a_1, \cdots, a_{2n}) \geq \text{tr} M(b_1, \cdots, b_{2n}) \geq \text{tr}(M(2)^l).
\]

\[\square\]

Relating an arbitrary Anosov map to one with the form \( M(a_1, \cdots, a_{2n}) \), we can extend Proposition 26 to Anosov elements.

**Corollary 27.** Let \( f \) be an Anosov map, identified with a hyperbolic element in \( \text{PSL}(2, \mathbb{Z}) \). Then

\[
\text{tr}(f) \geq \text{tr}(M(2)^l),
\]

where \( l \) is the translation length of \( f \) on \( F \).
Proof. A slight change in the proof of Proposition 20 actually shows that $f$ is conjugate to one associated with a standard ladder in which both 0 and $\infty$ are pivot points. Since conjugate Anosov maps have the same traces and translation lengths on $F$, we may assume the associated ladder of $f$ has 0, $\infty$ as its rung with 0, $\infty$ being pivot points. As $\det f = 1$, we have $\text{tr } f = \text{tr } (f^{-1})$. Also note that $f$ and $f^{-1}$ have the same translation length on $F$. Since either one of $f$ or $f^{-1}$ will send 0 to a positive rational number, it is possible to write either one as $M(a_1, \cdots, a_{2n})$ for some positive integers $a_1, \cdots, a_{2n}$. Finally, by Proposition 26, we have

$$\text{tr } (f) = \text{tr } (f^{-1}) = \text{tr } M(a_1, \cdots, a_{2n}) \geq \text{tr } (M(2)^l).$$

Finally, we establish our effective bound of the ratio of Teichmüller to curve graph translation lengths.

**Theorem 28.** Let $f$ be an Anosov map. Then the ratio $\frac{l_T(f)}{l_C(f)}$ is

1. bounded below $\log(1 + \sqrt{2}) \simeq 0.8814$, and the minimum is achieved when $f = M(2, 2)^n$ for $n \geq 1$.

2. unbounded above. More precisely, there exists a sequence of Anosov maps $\{f_n\}$ such that $\lim_{n \to \infty} \frac{l_T(f_n)}{l_C(f_n)} = \infty$.

**Proof.** (1) Note that for every positive integer $n$, we have

$$l_T(f^n) = nl_T(f), \quad l_C(f^n) = nl_C(f), \quad \frac{l_T(f^n)}{l_C(f^n)} = \frac{l_T(f)}{l_C(f)}.$$ 

Let $\lambda > 1$ be the dilatation of $f$, which is identical to the largest eigenvalue of $f$. Thus $l_T(f) = \log \lambda$, and $\lambda = \frac{1}{x} + 1$. As both $x \mapsto \log x$ and $x \mapsto x + \frac{1}{x}$ are increasing functions for $x > 1$,

$$\text{tr } (f) \geq \text{tr } (g) \quad \text{if and only if} \quad l_T(f) \geq l_T(g).$$

Now write $l_C(f) = l$. Then $l_C(f^2) = 2l = l_C(M(2)^{2l})$ and $\text{tr } f^2 \geq \text{tr } (M(2)^{2l})$ by Corollary 27. Hence,

$$\frac{l_T(f)}{l_C(f)} = \frac{l_T(f^2)}{l_C(f^2)} \geq \frac{l_T(M(2)^{2l})}{l_C(M(2)^{2l})} = \frac{l_T(M(2)^2)}{l_C(M(2)^2)} = \log(1 + \sqrt{2}).$$

Therefore, we established the lower bound of $\frac{l_T(f)}{l_C(f)}$ as $\log(1 + \sqrt{2})$, which is realized by $M(2, 2)^n = \left(\begin{array}{cc} 5 & 2 \\ 2 & 1 \end{array}\right)^n$ for every positive integer $n$. 

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(2) For the upper bound, we have $f_n = \begin{pmatrix} n+1 & 1 \\ n & 1 \end{pmatrix}$ whose dilatation falls in the interval $(n+1, n+2)$, for the trace is the sum of the dilatation and its reciprocal. Also, the translation length of $f_n$ is 1 because its invariant ladder is of type $(1, n)$. Thus we established a sequence of Anosov maps $\{f_n\}$ such that

$$\frac{l_T(f_n)}{l_C(f_n)} > n + 1,$$

for each $n \geq 1$. Therefore, $\lim_{n \to \infty} \frac{l_T(f_n)}{l_C(f_n)} = \infty$, and so ratio is unbounded above.

### 6.3 Evenly Spread Length Spectrum of Translation Lengths

Motivated from [BRW17, Theorem 9], we now present how typical is the set $m_f = \#\{[f'] \in \text{PSL}(2, \mathbb{Z}) : l_C(f') = l_C(f)\}$. Informally, the result can be phrased as “the spectrum of translation length on $\mathcal{F}$ is evenly spread.”

**Theorem 29.** For any positive integer $k$, we have

$$\lim_{R \to \infty} \frac{\#\{[f] \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R, \ m_f \geq k\}}{\#\{[f] \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R\}} \to 1,$$

where $\text{cl}(\text{PSL}(2, \mathbb{Z}))$ is the set of conjugacy classes in $\text{PSL}(2, \mathbb{Z})$, and $\lambda(f)$ is the dilatation of $f \in \text{PSL}(2, \mathbb{Z})$.

To prove this, we need the following asymptotic result on the number of conjugacy classes of $\text{SL}(2, \mathbb{Z})$. For two functions $f, g : \mathbb{Z}^+ \to \mathbb{R}_{\geq 0}$, denote by $f \in_w g$ if $\lim_{t \to \infty} f(t) / g(t) = \infty$.

**Lemma 30 ([CCC80, Theorem 2]).** Denote by $H(t)$ the number of conjugacy classes in $\text{SL}(2, \mathbb{Z})$ with fixed trace $t$. For any $\theta > 0$, there exists $T(\theta) > 0$ such that whenever $|t| > T(\theta)$, then $H(t) > |t|^{1-\theta}$. In particular, $H \in w(\log)$.

**Proof of Theorem 29.** Define $N(R) = \#\{[f] \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R\}$, the number of conjugacy classes in $\text{PSL}(2, \mathbb{Z})$ whose dilatation is less than $R$. Then by Lemma 30

$$N(R) \geq \#\{[f] \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \text{tr}(f) = [R]\} = \#\{[f] \in \text{cl}(\text{SL}(2, \mathbb{Z})) : \text{tr}(f) = [R]\} \in w(\log R),$$

where we used the fact that $\lambda(f) < R$ if and only if $\text{tr}(f) \leq |R|$. Define $N_i(R) = \#\{[f] \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R, \ l_C(f) = i\}$, which is well-defined since conjugate elements have the same translation length. Then the numerator of ratio in (4) can be rewritten as follows:

$$\#\{[f] \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R, \ m_f \geq k\} = \sum_{i=1}^{\infty} N_i(R) 1_{[k, \infty)}(N_i(R)), \quad (5)$$
where \(1_A\) is the characteristic function on a set \(A\). Note the infinite sum in (5) is, in fact, a finite sum, since the index \(i\) is bounded above \(c \log R := \log(1 + \sqrt{2}) \log R\) by Theorem 28(1). Since \(\sum_{i=1}^{\infty} n_i(R) = N(R) \in w(\log R)\), this suggests that for sufficiently large \(R\), the RHS of (5)

\[
\sum_{i=1}^{[c \log R]} n_i(R) 1_{[k, \infty)}(n_i(R)),
\]

is minimized when all but one \(1_{[k, \infty)}(n_i(R))\) vanish. More precisely, it is minimized when \(n_j(R) = k - 1\) except for one \(j = i\), and \(n_i(R) = N(R) - ([c \log R] - 1)(k - 1)\). Therefore, the ratio in (4):

\[
1 \geq \frac{\# \{f \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R, \ m_f \geq k\}}{\# \{f \in \text{cl}(\text{PSL}(2, \mathbb{Z})) : \lambda(f) < R\}} \geq \frac{N(R) - ([c \log R] - 1)(k - 1)}{N(R)}
\]

\[
= 1 - (k - 1) \frac{[c \log R] - 1}{N(R)} \to 1
\]

as \(R \to \infty\), since \(N(R) \in w(\log R)\). Therefore, we established the limit (4). \(\Box\)
A Algorithms for Ladders

In this appendix, we provide two algorithms for ladders: to generate a ladder bounded by two given edges (Algorithm 1), and to calibrate a ladder with odd length into one with even length (Algorithm 2).

Algorithm 1 Generate a ladder bounded by two edges in the Farey graph.

Require: Two pairs of ExtRationals \{pq, rs\} is distinct from \{xy, zw\}

1: function GENERATE_LADDER(pq, rs, xy, zw)
2: pivotList ← empty list
3: typeList ← empty list
4: prevPivot ← None ▷ stores the previous pivot point
5: \(m \leftarrow \text{FAREYSUM}(pq, rs)\) ▷ tells relative positions of \(xy, zw\) to \(pq, rs\)
6: while True do
7: if \(m\) is in between \((pq, rs)\) then
8: \(tu \leftarrow \text{FAREYSUM}(pq, rs)\)
9: else
10: \(tu \leftarrow \text{FAREYSUBTRACT}(pq, rs)\)
11: end if
12: \((d_1, d_2) \leftarrow \text{FAREYSORT}(pq, rs)\) ▷ \(d_1 > d_2\).
13: if \(m\) is in between \((d_1, r)\) then
14: \((choice, non−choice) \leftarrow d_1, d_2\)
15: else
16: \((choice, non−choice) \leftarrow d_2, d_1\)
17: end if
18: if choice == prevPivot then
19: add the last element of typeList by 1
20: else
21: if prevPivot is None then
22: append nonchoice to the end of pivotList.
23: end if
24: append choice to the end of pivotList
25: append 1 to the end of typeList
26: prev ← choice
27: end if
28: if \{choice, tu\} == \{xy, zw\} then
29: append \(tu\) to the end of pivotList
30: return pivotList, typeList
31: end if
32: end while
33: end function
Algorithm 2 Calibrate an odd ladder into even ladder.

Require: $mn$ must be set as $\text{FareySum}(\text{firstPivot}, \text{secondPivot})$

function CALIBRATEDLADDER($\text{pivotList}, \text{typeList}, mn$)
  if the length of $\text{pivotList}$ is even then
    return $\text{pivotList}, \text{typeList}$  \Comment{no need to calibrate}
  end if

  $\text{intTranslate} \leftarrow$ the first element of $\text{ladderType}$
  remove the first element of $\text{ladderType}$
  $\text{lastElement} \leftarrow$ the last element of $\text{pivotList}$
  remove the last element of $\text{pivotList}$
  $\text{lastPivot} \leftarrow$ the last element of $\text{pivotList}$
  for $i = 0$ to $\text{intTranslate}$ do
    if the $mn$ is in between $\text{lastPivot}$ and $\text{lastElement}$ then
      $\text{lastElement} \leftarrow \text{FareySum}(\text{lastPivot}, \text{lastElement})$
    else
      $\text{lastElement} \leftarrow \text{FareySubtract}(\text{lastPivot}, \text{lastElement})$
    end if
  end for
  remove the first element of $\text{pivotList}$
  append $\text{lastElement}$ to the end of $\text{pivotList}$
  add the last element of $\text{typeList}$ by $\text{intTranslate}$
  return $\text{pivotList}, \text{ladderType}$
end function
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