Coding Theorem and Strong Converse for Quantum Channels

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Abstract—In this correspondence we present a new proof of Holevo's coding theorem for transmitting classical information through quantum channels, and its strong converse. The technique is largely inspired by Wolfowitz's combinatorial approach using types of sequences. As a by–product of our approach which is independent of previous ones, both in the coding theorem and the converse, we can give a new proof of Holevo’s information bound.

Index Terms—Classical capacity, coding, Holevo bound, quantum channel, strong converse.

I. INTRODUCTION

After the recent achievements in quantum information theory, most notably Schumacher’s quantum data compression [14], and the determination of the quantum channel capacity by Holevo [12] (and independently by Schumacher and Westmoreland [15]), building on ideas of Hausladen et. al. [7], we feel that one should try to convert other and stronger techniques of classical information theory than those used in the cited works to the quantum case. The present work will do this for the method of types, as it is called by Csiszár and Körner [3], and which constitutes a manifestly combinatorial approach to information theory, by rephrasing it in the operator language of quantum theory (section IV). In section V we give a quantitative formulation of the intuition that measurements with high probability of success disturb the measured state only little. These technical results we apply to the coding problem for discrete memoryless quantum channels: we give a new proof of the quantum channel coding theorem (by a maximal code argument, whereas previous proofs adapted the random coding method to quantum states), and prove the strong converse, both in section VI. In section VII we demonstrate how to obtain from these an independent, and completely elementary proof of the Holevo bound [8]. We point out that our technique is also suited to the situations of encoding under linear constraints, and with infinite input alphabet.

It should be mentioned that the strong converse results also in recent independent work of Ogawa and Nagaoka [13], by a different method.

II. PREREQUISITES AND NOTATIONAL CONVENTIONS

We will use the definitions and notation of [16], in particular finite sets will be A, B, M, X, ..., quantum states (density operators) ρ, σ, ..., probability distributions P, Q, ..., and classical–quantum operations V, W, ... (also stochastic matrices), whereas general quantum operations (trace preserving, and completely positive maps) of C∗–algebras X, Y, ... are denoted as (pre-)adjoint maps ϕ∗, ψ∗, ... . Our algebras will be of finite dimension, and apart from commutative ones we will confine ourselves to the C∗–algebra L(H), the algebra of (bounded) linear operators of the complex Hilbert space H, even though everything works equally well for the general case.

The exponential function exp is always understood to basis 2, as well as the logarithm log. The same symbol H denotes the Shannon and the von Neumann entropy (as Shannon’s is the commutative case of von Neumann’s).

We shall need a basic fact about the trace norm || · ||1 of an operator, which is the sum of the absolute values of the eigenvalues.

It is known (and not difficult to prove, using the polar decomposition of α, see [1]) that

$$
\|\alpha\|_1 = \max_{\|B\|_{\infty} \leq 1} |\text{Tr}(\alpha B)|.
$$

If α is selfadjoint we may write it as the difference α+ − α− of its positive and its negative part. Observe that then

$$
\|\alpha\|_1 = \text{Tr}(\alpha+) + \text{Tr}(\alpha-) = \max_{-1 \leq \lambda \leq 1} \text{Tr}(\alpha B).
$$

III. QUANTUM CHANNELS AND CODES

The following definition is from [9]: a (discrete memoryless classical–quantum channel (eq–DMC) is a mapping W from a finite set X into the set of states on the system Y = L(H), taking x ∈ X to Wx (by linear extension we may view this as a quantum operation from (C′X)∗ to Y). For the rest of the correspondence fix H and X, d = dim H and a = |X|.

From [16] we recall: for a probability distribution P on X let PW = ∑x∈XP(x)Wx the average state of the channel W, H(W|P) = ∑x∈X P(x)H(Wx) is the conditional von Neumann entropy, the mutual information between a distribution and the channel is I(P; W) = H(PW) − H(W|P). Finally let C(W) = maxP I(P; W). Note that these notions still make sense for infinite X if only W is required to be measurable (so X has to carry some measurable structure): then also H(W) is measurable, and PW, H(W|P) are expectations over the probability measure P.

An n–block code for a quantum channel W is a pair (f, D), where f is a mapping from a finite set M into Xn, and D is an observable on Y⊗n indexed by M′ ⊆ M, i.e. a partition of ⊗ into positive operators Dm, m ∈ M′. With the convention Wx⊗n = Wx1 ⊗ ... ⊗ Wxn, for a sequence x⊗n = x1 ... xn ∈ Xn the (maximum) error probability of the code is defined as

$$
e(f, D) = \max\{1 - \text{Tr}(W_{f(m)}D_m) : m \in M\}.
$$

We call (f, D) an (n, λ)–code, if e(f, D) ≤ λ. Define N(n, λ) as the maximum size |M| of an (n, λ)–code. The rate of an n–block code is defined as 1/n log |M|. Our main results are summarized in

Theorem 1: For every λ ∈ (0, 1) there exists a constant K(λ, a, d) such that for all eq–DMCs W

$$
|\log N(n, \lambda) - nC(W)| \leq K(\lambda, a, d)\sqrt{n}.
$$
Proof: Combine the code construction Theorem 10 (with a probability distribution P maximizing I(P; W) and A = X^n) and the strong converse Theorem 13.

This theorem justifies the name \textit{capacity} for the quantity C(W), even in the strong sense of Wolfowitz [18].

IV. TYPICAL PROJECTORS AND SHADOWS

Let n a positive integer, and consider sequences x^n = x_1 \ldots x_n \in X^n. For x \in X define the counting function N(x|x^n) = |\{i \in [n] : x_i = x\}|. The type of x^n is the empirical distribution P_{x^n} on X of letters x \in X in x^n: P_{x^n}(x) = \frac{1}{n} N(x|x^n). Obviously the number of types is upper bounded by (n+1)^n; we will refer to this fact as type counting.

Following Wolfowitz [18] we define
\[ T^n_{P,\delta} \{ x^n \in X^n : \forall x \in X \{ N(x|x^n) - nP(x) \leq \delta \sqrt{n} nP(x)(1-P(x)) \}, \]
the set of variance–typical sequences of approximate type P with constant \delta \geq 0. Note that T^n_{P,0} is the set of sequences of type P. Defining K = 2 \log \frac{e}{\epsilon} we have

\[ \left| - \log P_{\otimes n}(x^n) - nH(P) \right| \leq K \delta \sqrt{n}, \]
\[ |T^n_{P,\delta}| \leq \exp (nH(P) + K \delta \sqrt{n}), \]
\[ |T^n_{P,\delta}| \geq \left( 1 - \frac{a}{\delta^2} \right) \exp (nH(P) - K \delta \sqrt{n}). \]

Proof: See [18]. Let us only indicate the proof of the first inequality: T^n_{P,\delta} is the intersection of a events, namely for each x \in X that the mean of the independent Bernoulli variables X_i, with value 1 iff x_i = x has a deviation from its expectation P(x) at most \delta \sqrt{P(x)(1-P(x))}/\sqrt{n}. By Chebyshev’s inequality each of these has probability at least 1 - 1/\delta^2. The rest is in fact contained in Lemma 3 below.

The following definitions are in close analogy to this.

For a state \rho choose a diagonalization \rho = \sum_j \lambda_j \pi_j and observe that the eigenvalue list R is a probability distribution, with H(\rho) = H(R). Thus we may define
\[ \Pi_{P,\delta}^{\otimes n} = \sum_{\rho \in T^n_{P,\delta}} \pi_{j_1} \otimes \cdots \otimes \pi_{j_n}, \]
the variance–typical projector of \rho with constant \delta. It is to be distinguished from the typical projector introduced by Schumacher in [14], which we would rather call \textit{entropy typical}. Observe that \Pi_{P,\delta}^{\otimes n} may depend on the particular diagonalization of \rho. This slight abuse of notation is no harm in the sequel, as we always fix globally diagonalizations of the states in consideration.

Let us say that an operator B, 0 \leq B \leq 1, is an \eta–shadow of the state \rho if Tr (\rho B) \geq \eta. We then have

\textbf{Lemma 3 (Typical projector):} For every state \rho and integral n
\[ \text{Tr} \left( \rho \otimes n \Pi_{P,\delta}^{\otimes n} \right) \geq 1 - \frac{d}{\delta^2}, \]
and with \Pi_{P,\delta} = \Pi_{P,\delta}^{\otimes n},
\[ \Pi_{P,\delta} \exp \left( -nH(\rho) - K \delta \sqrt{n} \right) \leq \Pi_{P,\delta} \exp \left( -nH(\rho) + K \delta \sqrt{n} \right), \]
\[ \text{Tr} \Pi_{P,\delta} \exp \left( nH(\rho) + K \delta \sqrt{n} \right), \]
\[ \text{Tr} \Pi_{P,\delta} \geq \left( 1 - \frac{d}{\delta^2} \right) \exp \left( nH(\rho) + K \delta \sqrt{n} \right). \]
Every \eta–shadow B of \rho^{\otimes n} satisfies
\[ \text{Tr} B \geq \left( \eta - \frac{d}{\delta^2} \right) \exp \left( nH(\rho) - K \delta \sqrt{n} \right). \]

Proof: The first estimate is the Chebyshev inequality, as before: observe that
\[ \text{Tr} \left( \rho \otimes n \Pi_{P,\delta}^{\otimes n} \right) = R^{\otimes n} (T^n_{P,\delta}). \]
The second formula is the key: to prove it let \eta^n = \pi_{j_1} \otimes \cdots \otimes \pi_{j_n}, one of the eigenprojections of \rho^{\otimes n} contributing to \Pi_{P,\delta}^{\otimes n}. Then
\[ \text{Tr} \left( \rho \otimes n \eta^n \right) = R(j_1) \cdots R(j_n) = \prod_j R(j)^n, \]
Taking logs and using the defining relation for the N(j)j^n) we find
\[ \left| \log \text{Tr} \left( \rho \otimes n \eta^n \right) - nH(\rho) \right| = \sum_j \log R(j) \log R(j) - nH(R) \]
\[ = \sum_j \log R(j) (N(jj^n) - nR(j)) \]
\[ \leq -\delta \sqrt{n} \log R(j) \log R(j) \]
\[ \leq \frac{\log e}{\epsilon} \delta \sqrt{n}. \]

The rest follows from the following Lemma 4.

\textbf{Lemma 4 (Shadow bound):} Let 0 \leq \Lambda \leq 1 and \rho a state commuting with \Lambda such that for some \lambda, \mu_1, \mu_2 > 0
\[ \text{Tr} (\rho \Lambda) \geq 1 - \lambda \quad \text{and} \quad \mu_1 \Lambda \leq \sqrt{\Lambda} \rho \sqrt{\Lambda} \leq \mu_2 \Lambda. \]
Then (1 - \lambda)\mu_2^{-1} \leq Tr \Lambda \leq \mu_1^{-1}, and for an \eta–shadow B of \rho one has Tr B \geq (\eta - \mu_2^{-1}).

Proof: The bounds on Tr \Lambda follow by taking traces in the inequalities in \sqrt{\Lambda} \rho \Lambda \sqrt{\Lambda} and using 1 - \lambda \leq Tr (\rho \Lambda) \leq 1. For the \eta–shadow B observe
\[ \mu_2 \text{Tr} B \geq \text{Tr} (\mu_2 \Lambda B) \geq \text{Tr} \left( \sqrt{\Lambda} \rho \sqrt{\Lambda} B \right) \]
\[ = \text{Tr} (\rho B) - \text{Tr} \left( (\rho - \sqrt{\Lambda} \rho \sqrt{\Lambda} B) \right) \]
\[ \geq \eta - \left\| \rho - \sqrt{\Lambda} \rho \sqrt{\Lambda} \right\|_1. \]
Since the trace norm can obviously be estimated by \( \lambda \) we are done.

Fix now diagonalizations \( W_x = \sum_j W(j|x)\pi_{x,j} \) (where \( W(\cdot|x) \) is a stochastic matrix, the double meaning of \( W \) should be no serious ambiguity). Then define the conditional variance–typical projector of \( W \) given \( x^n \) with constant \( \delta \) to be

\[
\Pi_{W,\delta}(x^n) = \bigotimes_{x \in \mathcal{X}} \Pi_{W_x,\delta}^n,
\]

where \( I_x = \{ i \in [n] : x_i = x \} \). With the convention \( W_{x^n} = W_{x_1} \otimes \cdots \otimes W_{x_n} \) we now have

**Lemma 5 (Conditional typical projector):** For all \( x^n \in \mathcal{X}^n \) of type \( P \)

\[
\text{Tr} (W_{x^n} \Pi_{W,\delta}(x^n)) \geq 1 - \frac{ad}{\delta^2},
\]

and with \( \Pi^n = \Pi_{W,\delta}(x^n) \),

\[
\Pi^n \exp \left( -nH(W|P) - Kd\sqrt{a}d\sqrt{n} \right) \leq \Pi^n W_{x^n} \Pi^n \leq \Pi^n \exp \left( -nH(W|P) + Kd\sqrt{a}d\sqrt{n} \right),
\]

\[
\text{Tr} \, \Pi_{W,\delta}(x^n) \leq \exp \left( nH(W|P) + Kd\sqrt{a}d\sqrt{n} \right),
\]

\[
\text{Tr} \, \Pi_{W,\delta}(x^n) \geq \left( 1 - \frac{ad}{\delta^2} \right) \exp \left( nH(W|P) - Kd\sqrt{a}d\sqrt{n} \right).
\]

Every \( \eta \)-shadow \( B \) of \( W_{x^n} \) satisfies

\[
\text{Tr} \, B \geq \left( \eta - \frac{ad}{\delta^2} \right) \exp \left( nH(W|P) - Kd\sqrt{a}d\sqrt{n} \right).
\]

**Proof:** The first estimate follows simply by applying Lemma 3 \( a \) times, the second formula is by piecing together the corresponding formulae from Lemma 3 using \( \sum_{x \in \mathcal{X}} \sqrt{P(x)} \leq \sqrt{a} \). The rest is by the shadow bound Lemma 4.

We need a last result on the behaviour of \( W_{x^n} \) under a typical projector:

**Lemma 6 (Weak law of large numbers):** Let \( x^n \in \mathcal{X}^n \) of type \( P \). Then

\[
\text{Tr} \, (W_{x^n} \Pi_{\Pi_{P,W,\delta,\gamma}}^n) \geq 1 - \frac{ad}{\delta^2}.
\]

**Proof:** Diagonalize \( PW \) \( = \sum_j q_j \pi_j \), and let the quantum operation \( \kappa_* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})_* \) be defined by \( \kappa_* (\sigma) = \sum_j \pi_j \sigma \pi_j \). We claim that

\[
\Pi_{\Pi_{P,W,\delta,\gamma}}^n \geq \Pi_{\kappa_* W,\delta}(x^n).
\]

Indeed let \( \pi^n = \pi_{j_1} \otimes \cdots \otimes \pi_{j_n} \) one of the product states constituting \( \bigotimes_{x \in \mathcal{X}} \Pi_{\kappa_* W,\delta}^n \), i.e. with \( \kappa_* W_x = \sum_j q_{j|x} \pi_j \),

\[
\forall x \in \mathcal{X} \forall j \quad |N(j|x)^{\frac{1}{2}} - q_{j|x}|I_x| \leq \delta \sqrt{|I_x|} \sqrt{q_{j|x}(1 - q_{j|x})}.
\]

Hence (using \( |I_x| = P(x) \))

\[
|N(j|x)^{\frac{1}{2}} - q_{j|x}| \leq \sum_{x \in \mathcal{X}} \left| N(j|x)^{\frac{1}{2}} - q_{j|x} |I_x| \right| \\ \leq \sum_{x \in \mathcal{X}} \delta \sqrt{|I_x|} \sum_{x \in \mathcal{X}} P(x) q_{j|x}(1 - q_{j|x}) \\ \leq \delta \sqrt{|I_x|} \sum_{x \in \mathcal{X}} P(x) q_{j|x}(1 - q_{j|x}),
\]

the last inequality by concavity of the map \( x \mapsto x(1 - x) \), and \( q_j = \sum_{x \in \mathcal{X}} P(x) q_{j|x} \). Hence \( \pi^n \) occurs in the sum for \( \Pi_{\Pi_{P,W,\delta,\gamma}}^n \), and our claim is proved.

Thus we can estimate

\[
\text{Tr} \, (W_{x^n} \Pi_{\Pi_{P,W,\delta,\gamma}}^n) = \text{Tr} \, \left( (\kappa_\sigma \circ \kappa_\gamma W_x^n) \Pi_{P,W,\delta,\gamma}^n \right) \\ \geq \text{Tr} \, \left( (\kappa_\sigma \circ \kappa_\gamma W_x^n) \Pi_{\kappa_* W,\delta}(x^n) \right) \\ \geq 1 - \frac{ad}{\delta^2},
\]

the last line by Lemma 5.

**V. ON GOOD MEASUREMENTS**

We start with a short consideration of fidelity:

Assume in the following that \( \rho \) is a pure state, \( \sigma \) may be mixed. We want to compare the trace norm distance \( D(\rho, \sigma) = \frac{1}{2} \| \rho - \sigma \|_1 \), and the (pure state) fidelity \( F(\rho, \sigma) = \text{Tr}(\rho \sigma) \).

**Lemma 7 (Pure state):** Let \( \rho = |\psi\rangle \langle \psi | \) and \( \sigma = |\phi\rangle \langle \phi | \) pure states. Then

\[1 - F(\rho, \sigma) = D(\rho, \sigma)^2 \leq 1 - |\phi\rangle \langle \psi |.
\]

**Proof:** We may assume \( |\psi\rangle = \alpha |0\rangle + \beta |1\rangle \) and \( |\phi\rangle = \alpha |0\rangle - \beta |1\rangle \), with \( |\alpha|^2 + |\beta|^2 = 1 \). A straightforward calculation shows \( F = |\alpha|^2 - |\beta|^2 \), and \( D = 2 |\alpha| |\beta| \). Now

\[1 - F = 1 - (|\alpha|^2 - |\beta|^2)^2 = (1 + |\alpha|^2 - |\beta|^2)(1 - |\alpha|^2 + |\beta|^2) = 4|\alpha| |\beta|^2 = D^2.
\]

**Lemma 8 (Mixed state):** Let \( \sigma \) an arbitrary mixed state (and \( \rho \) pure as above). Then

\[D \geq 1 - F \geq D^2.
\]

**Proof:** Write \( \sigma = \sum_j q_j \pi_j \) with pure states \( \pi_j \). Then

\[1 - F(\rho, \sigma) = \sum_j q_j (1 - D(\rho, \pi_j))^2 \geq \sum_j q_j D(\rho, \pi_j)^2 \]

Conversely: extend \( \rho \) to the observable \( (\rho, 1 - \rho) \) and consider the quantum operation \( \kappa_* : \sigma \mapsto \rho \sigma + (1 - \rho) \sigma (1 - \rho) \).
Then (with monotonicity of $\| \cdot \|_1$ under quantum operations)
\[ 2D = \|\rho - \sigma\|_1 \geq \|\kappa_+\rho - \kappa_+\sigma\|_1 = \|\rho - \kappa_+\sigma\|_1 \]
(since $\rho = \kappa_+\rho$). Hence with $F = \text{Tr}(\sigma \rho)$
\[ 2D \geq \left\| (1 - F)\rho - \text{Tr}(\sigma(1 - \rho))\pi \right\|_1 \]
\[ = (1 - F) + (1 - F) = 2(1 - F) , \]
for a state $\pi$ supported in $1 - \rho$.

Observe that the inequalities of this lemma still hold if only $\sum_j q_j \leq 1$.

Now we are ready to prove the main object of the present section:

**Lemma 9 (Gentle measurement):** Let $\rho$ be a state, and $X$ a positive operator with $X \leq 1$ and $1 - \text{Tr}(\rho X) \leq \lambda \leq 1$. Then
\[ \left\| \rho - \sqrt{X}\rho\sqrt{X} \right\|_1 \leq \sqrt{8\lambda} . \]

**Proof:** Let $Y = \sqrt{X}$ and write $\rho = \sum_k p_k\pi_k$ with one-dimensional projectors $\pi_k$ and weights $p_k \geq 0$. Now
\[ \left\| \rho - Y\rho Y \right\|_1^2 \leq \left( \sum_k p_k \left\| \pi_k - Y\pi_k Y \right\|_1 \right)^2 \]
\[ \leq \sum_k p_k \left\| \pi_k - Y\pi_k Y \right\|_1^2 \]
\[ \leq 4 \sum_k p_k (1 - \text{Tr}(\pi_k Y\pi_k Y)) \]
\[ \leq 8 \sum_k p_k (1 - \text{Tr}(\pi_k Y)) \]
\[ = 8(1 - \text{Tr}(\rho Y)) \]
\[ \leq 8(1 - \text{Tr}(\rho X)) \leq 8\lambda \]
by triangle inequality, convexity of $x \mapsto x^2$, Lemma 8 $1 - x^2 \leq 2(1 - x)$, and $X \leq Y$.

**VI. CODE BOUNDS**

We can now give a new proof of the quantum channel coding theorem by a maximal code argument (which in the classical case is due to Feinstein [5]), and prove the strong converse.

**Theorem 10 (Code construction):** For $\lambda, \tau \in (0, 1)$ there exist $\delta > 0$ and a constant $K(\lambda, \tau, a, d)$ such that for every cq-DMC $W$, probability distribution $P$ on $X$, $n > 0$, and $A \subset X^n$ with $P^\otimes n(A) \geq \tau$ there is an $(n, \lambda)$-code $(f, D)$ with the properties
\[ \forall m \in M \ f(m) \in A \text{ and } D_m \leq Tr \Pi^W_{W,\delta}(f(m)) , \]
and $|M| \geq \exp(nI(P; W) - K(\lambda, \lambda, \tau, d))$.

**Proof:** Let $A' = A \cap \bigcap^n_{k=1} P^\otimes n(\{P|_{W^{2ad/k}}\})$ (thus $P^\otimes n(A') \geq \tau/2$) and $(f, D)$ a maximal (i.e. non-extendible) $(n, \lambda)$-code with
\[ \forall m \in M \ f(m) \in A' \text{ and } D_m \leq Tr \Pi^W_{W,\delta}(f(m)) , \]
where $\delta = \sqrt{\frac{2ad}{\lambda}}$. In particular (by Lemma 5)
\[ Tr \ D_m \leq \exp(nH(W|P) + (Kd\sqrt{\delta} + Ka\sqrt{\frac{2ad}{\tau}}\log d)\sqrt{n}) . \]

Of course $M$ may be empty. We claim however that $B = \sum_{m \in M} D_m$ is an $\eta$-shadow for all $W_x^n$, $x^n \in A'$, with $\eta = \min\{1 - \lambda, \lambda^2/32\}$.

This is clear for codewords, and for other $x^n$ we could else extend $(f, D)$ with the codeword $x^n$ and corresponding observable operator
\[ D_{x^n} = \sqrt{1 - B} \Pi^W_{W,\delta}(x^n)\sqrt{1 - B} . \]

To see this note first that $D_{x^n} \leq 1 - B$, and $Tr D_m \leq Tr \Pi^W_{W,\delta}(x^n)$. Now apply Lemma 9 to the assumption $Tr (W_{x^n}(1 - B)) \geq 1 - \lambda^2/32$ and obtain
\[ \left\| W_{x^n} - \sqrt{1 - B}W_{x^n}\sqrt{1 - B} \right\|_1 \leq \frac{\lambda}{2} . \]

Hence we can estimate (with $\Pi^n = \Pi^W_{W,\delta}(x^n)$):
\[ Tr (W_{x^n} D_{x^n}) = Tr (W_{x^n}\Pi^n) - \left( (W_{x^n} - \sqrt{1 - B}W_{x^n}\sqrt{1 - B})\Pi^n \right) \geq 1 - \frac{\lambda}{2} - \left\| W_{x^n} - \sqrt{1 - B}W_{x^n}\sqrt{1 - B} \right\|_1 = 1 - \lambda . \]

This proves our claim, and averaging over $P^\otimes n$ we find
\[ Tr ((PW)^\otimes n B) \geq \eta\tau/2 , \]
from which, by Lemma 3 we deduce
\[ \sum_{m \in M} Tr D_m = Tr B \]
\[ \geq \left( \frac{\eta\tau}{2} - \frac{d}{\delta^2} \right) \exp(nH(W) - Kd\delta_0\sqrt{n}) . \]

Choosing $\delta_0 = \sqrt{\frac{2ad}{\tau}}$ the proof is complete.

**Remark 11:** It is interesting to note from the proof that the decoder may be chosen a von Neumann observable (i.e. all its operators are mutually orthogonal projectors). This is because if $(f, D)$ is of this type, then $B$ is a projector, and this means that we may instead of the constructed $D_{x^n} \leq 1 - B$ use the projector $D'_{x^n} = \text{supp} D_{x^n}$; this is still bounded by $1 - B$, only decreases the error probability, and obeys the size condition: $Tr \supp D_{x^n} = \dim \text{im} D_{x^n} \leq \dim \text{im} \Pi^W_{W,\delta}(x^n) = Tr \Pi^W_{W,\delta}(x^n)$.

On the other hand it would be nice if we could decide if the decoder may consist of separable operators. It is clear that a product observable cannot do, as was pointed out by Holevo [10]: otherwise larger capacities could not be reached using block decoding. But it may be that nonlocality as in the recent work of Bennett et. al. [2] is sufficient, and genuine entanglement is not needed (as was proposed in the cited work of Holevo).

**Remark 12:** Our method of proof might seem very abstract. In fact it is not, as the argument in the proof may be understood as a greedy method of extending a given code: start from the empty code, and add codewords after the prescription of the proof, until you are stuck. The theorem then guarantees that the resulting code is rather large.
Theorem 13 (Strong converse): For \( \lambda \in (0,1) \) there exists a constant \( K(\lambda,a,d) \) such that for every cq–DMC \( W \) and \( (n,\lambda) \)-code \((f,D)\)
\[
|M| \leq \exp \left( nC(W) + K(\lambda,a,d)\sqrt{n} \right).
\]

Proof: We will prove even a little more: if additionally all codewords are of the same type \( P \) then
\[
|M| \leq \frac{4}{1 - \lambda} \exp \left( nI(P;W) + 2Kd\sqrt{\alpha\delta}\sqrt{n} \right),
\]
with \( \delta = \frac{\sqrt{2\alpha\delta}}{1 - \lambda} \), which by type counting implies the theorem.

To prove this modify the code as follows: construct new decoding operators
\[
D'_m = \Pi_{P_W,\delta} D_m \Pi_{P_W,\delta}.
\]
Then \((f,D')\) is an \((n,\sqrt{\frac{\lambda}{2}})\)-code because for \( m \in M \), with
\[
I(P;D') = \Pi_{P_W,\delta}(D_m) - \Pi_{P_W,\delta}(W_{f,m} - \Pi^n W_{f,m} \Pi^n) D_m \geq 1 - \lambda - \|W_{f,m} - \Pi^n W_{f,m} \Pi^n\|_1 \\
\geq 1 - \lambda - \frac{\sqrt{\text{Sad}}}{\delta^2} = \frac{1 - \lambda}{2}.
\]

Now by Lemma [5]
\[
\text{Tr} D'_m \geq 1 - \frac{\lambda}{4} \exp \left( nH(W|P) - Kd\sqrt{\alpha}\sqrt{n} \right).
\]
On the other hand (with Lemma [5])
\[
\sum_{m \in M} \text{Tr} D'_m \leq \exp \left( nH(W) + Kd\sqrt{\alpha}\sqrt{n} \right),
\]
and we are done.

VIII. CONCLUSION

We proved the quantum channel coding theorem and its strong converse by methods new to quantum information theory (but which are very close to established methods in classical information theory), and showed how to obtain the Holevo bound as a corollary.

We want to point out that our technique for proving the code bounds yields also the coding theorem and strong converse under linear constraints (see Holevo [11] for definitions and capacity formula): simply because satisfying the linear constraints is a property of whole types, not just individual sequences.

Also we can prove the coding theorem and strong converse in the case of arbitrary (product) signal states in a general (discrete memoryless) quantum–quantum channel (qq–DMC), see [9]: this is a completely positive and unit preserving map \( \varphi : \mathfrak{A}_1 \to \mathfrak{A}_1 \) between finite dimensional \( C^* \)-algebras \( \mathfrak{A}_1, \mathfrak{A}_2 \) (or rather its state map \( \varphi : \mathfrak{S}(\mathfrak{A}_1) \to \mathfrak{S}(\mathfrak{A}_2) \)). This includes the cq–DMC as the special case \( \mathfrak{A}_2 = \mathbb{C}^{2 \times 2} \) and \( \mathfrak{A}_2 = \mathcal{L}(\mathcal{H}) \). An \((n,\lambda)\)-code for this channel is a pair \((F,D)\) with a map \( F : M \to \mathfrak{S}(\mathfrak{A}_1^\otimes n) \) and an observable \( D \) on \( \mathfrak{A}_2^\otimes n \) indexed by \( M' \supset M \), such that the error probability
\[
e(F,D) = \max \left\{ 1 - \text{Tr}(\varphi^n (F(m)) : D_m) : m \in M \right\}
\]
is at most \( \lambda \). We will consider only the case that the \( F(m) \) are product states (such codes we call I–separable, following [15], and the corresponding operational capacity product state capacity \( C^{(1)}(\varphi,\lambda) \), so the channel and all its possible codes are determined by the image of \( \varphi \) in \( \mathfrak{S}(\mathfrak{A}_2) \), a compact convex set. Thus we are back in our original situation, with \( \mathfrak{M} \) now a compact convex set of states in \( \mathcal{L}(\mathcal{H}) \) and \( W = \text{id}_{\mathfrak{M}} \). The capacity we denote \( C(M,\lambda) \), it was for \( \lambda \to 0 \) determined by Schumacher and Westmoreland [14] (an improved argument for their weak converse may be found in [16]).

With the methods presented in this correspondence one can prove

Theorem 14: With the above notations and \( \lambda \in (0,1) \):
\[
C(M,\lambda) = \sup \frac{C(W)}{\text{finite } W \subset M}
\]
Furthermore, the supremum is in fact a maximum, which is assumed by a finite $W \subset W$ of cardinality at most $\dim_C A_2$ and consisting of extremal points of $W$.

The proof of the capacity formula is given in full in [17]. The second part of the statement is from [6].

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