Regularity results for a model in magnetohydrodynamics with imposed pressure

JULIEN POIRIER,∗ NOUR SELOULA†

Abstract

The magnetohydrodynamics (MHD) problem is most often studied in a framework where Dirichlet type boundary conditions on the velocity field is imposed. In this Note, we study the (MHD) system with pressure boundary condition, together with zero tangential trace for the velocity and the magnetic field. In a three-dimensional bounded possibly multiply connected domain, we first prove the existence of weak solutions in the Hilbert case, and later, the regularity in results for some Stokes and elliptic problems with this type of boundary conditions. Furthermore, under the condition of small data, we obtain the existence and uniqueness of solutions in $W^{1,p}(\Omega)$ for $3/2 < p < 2$ by using a fixed-point technique over a linearized (MHD) problem.

Key words: Magnetohydrodynamic system, Pressure boundary conditions, Navier-Stokes, $L^p$-regularity.

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1 Introduction

Let $\Omega$ be an open bounded set of $\mathbb{R}^3$ of class $C^{1,1}$. In this work, we consider the following incompressible stationary magnetohydrodynamics (MHD) system: find the velocity field $u$, the pressure $P$, the magnetic field $b$ and the constant vector $\alpha = (\alpha_1, \ldots, \alpha_l)$ such that for $1 \leq i \leq l$:

\[
\begin{aligned}
- \nu \Delta u + (\nabla \cdot \mathbf{u}) \times \mathbf{u} + \nabla P - \kappa (\nabla \cdot \mathbf{b}) \times \mathbf{b} &= f \quad \text{and} \quad \nabla \cdot \mathbf{u} = h \quad \text{in} \quad \Omega, \\
\kappa \mu \nabla \cdot \mathbf{b} - \kappa (\nabla \cdot \mathbf{u}) \times \mathbf{b} &= g \quad \text{and} \quad \nabla \cdot \mathbf{b} = 0 \quad \text{in} \quad \Omega, \\
\mathbf{u} \cdot \mathbf{n} &= 0 \quad \text{and} \quad \mathbf{b} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma, \\
P &= P_0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad P = P_0 + \alpha_i \quad \text{on} \quad \Gamma_i, \\
(\mathbf{u} \cdot \mathbf{n})|_{\Gamma_i} &= 0, \quad \text{and} \quad (\mathbf{b} \cdot \mathbf{n}, 1)|_{\Gamma_i} = 0, \quad \forall 1 \leq i \leq I
\end{aligned}
\]

where $\Gamma$ is the boundary of $\Omega$ which is not necessary connected. Here $\Gamma = \bigcup_{i=0}^l \Gamma_i$ where $\Gamma_i$ are the connected components of $\Gamma$ with $\Gamma_0$ the exterior boundary which contains $\Omega$ and all the other boundaries. We denote by $\mathbf{n}$ the unit vector normal to $\Gamma$. The constants $\nu$, $\mu$ and $\kappa$ are constant kinematic, magnetic viscosity and a coupling number respectively. We refer to [11,13] for further discussion of typical values for these parameters. The vector $f$, $g$ and the scalar $h$ and $P_0$ are given. In this work, we assume that $\nu = \mu = \kappa = 1$ for convenience. Using the identity $\mathbf{u} \cdot \nabla \mathbf{u} = (\nabla \cdot \mathbf{u}) \times \mathbf{u} + \frac{1}{2} \nabla |\mathbf{u}|^2$, the classical nonlinear term $\mathbf{u} \cdot \nabla \mathbf{u}$ in the Navier-Stokes equations is replaced by $(\nabla \cdot \mathbf{u}) \times \mathbf{u}$. The pressure $P = p + \frac{1}{2} |\mathbf{u}|^2$ is then the Bernoulli (or dynamic) pressure, where $p$ is the kinematic pressure. The boundary conditions involving the pressure are used in various physical applications.

∗Normandie Univ., UNICAEN, CNRS, Laboratoire de Mathématiques Nicolas Oresme, UMR CNRS 6139, 14000 Caen, France.
E-mail: julien.poirier.prof@gmail.com

†Normandie Univ., UNICAEN, CNRS, Laboratoire de Mathématiques Nicolas Oresme, UMR CNRS 6139, 14000 Caen, France.
E-mail: nour-elhouda.seloula@unicaen.fr
For example, in hydraulic networks, as oil ducts, microfluidic channels or the blood circulatory system. Pressure driven flows occur also in the modeling of the cerebral venous network from three-dimensional angiographic images obtained by magnetic resonance. We note that the MHD system (1.1) has been extensively studied by many authors. We note that most of the contributions are often given where Dirichlet type boundary conditions on the velocity field are imposed. At a continuous level, we can refer, for example to [2,25] for the existence and the regularity of the solutions of (1.1), to [1] for the global solvability of (1.1) under mixed boundary conditions for the magnetic field. For the discretization approaches of (1.1), a few related contributions include mixed finite elements [15,17,23], discontinuous Galerkin finite elements [?] or iterative penalty finite element methods [12] and so on. The boundary condition under the form \( P = P_0 + \alpha_i \) on \( \Gamma_i \), \( i = 1, \ldots, I \) was first introduced in [9,10] for the Stokes and the Navier-Stokes systems in steady hilbertian case. The authors studied the differences \( \alpha_i - \alpha_0 \), \( i = 1 \ldots I \) which represent the unknown pressure drop on inflow and outflow sections \( \Gamma_i \) in a network of pipes. This work is extended to \( L^p \)-theory for \( 1 < p < \infty \) in [7]. In our work, we study the MHD system (1.1) with pressure boundary condition, together with no tangential flow and no tangential magnetic field on the boundary. Up to our knowledge, with these type of boundary conditions, this work is the first one to give a complete \( L^p \)-theory for the MHD system (1.1) not only for large values of \( p \geq 2 \) but also for small values \( 3/2 < p < 2 \) in \( \Omega \subset \mathbb{R}^3 \) domain with a boundary \( \Gamma \) not necessary connected.

The work is organized as follows. We start with presenting the main results of our work in section 2. In section 3, we introduce the necessary notations and some useful results. Section 4 is devoted to the study of the linearized MHD system in Hilbert space. Using Lax-Milgram theorem, we prove the existence and uniqueness of weak solution in \( H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \). Later, we study the \( L^p \)-theory for the linearized MHD system in section 5. In particular, the proof of the regularity \( W^{2,p}(\Omega) \) with \( 1 < p < \frac{6}{5} \) for a non-zero divergence condition is presented in the Appendix (see Section 7). Finally, the nonlinear MHD system is discussed in section 6. The proof of the existence of weak solution in the Hilbertian case is based on the Leray-Schauder fixed point theorem. Then, we prove the regularity of the weak solution in \( W^{1,p}(\Omega) \) with \( p > 2 \), and \( W^{2,p}(\Omega) \) with \( p \geq \frac{6}{5} \). For this, we use the regularity results for the Stokes and some elliptic equations combining them with a bootstrap argument. The existence of a weak solution in \( W^{1,p}(\Omega) \) with \( \frac{3}{2} < p < 2 \) is proved by applying Banach’s fixed-point theorem over the linearized problem.

Some results of this work are announced in [21]

## 2 Main results

In this section, we briefly discuss the main results, for which the following notations are needed:

For \( p \in [1,\infty) \), \( p' \) denotes the conjugate exponent of \( p \), i.e. \( \frac{1}{p'} = 1 - \frac{1}{p} \). We introduce the following space

\[
H^{r,p}(\text{curl},\Omega) := \{ v \in L^r(\Omega); \text{curl} \ v \in L^p(\Omega) \}, \quad \text{with} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{3}
\]  

(2.1)

equipped with the norm

\[
\|v\|_{H^{r,p}(\text{curl},\Omega)} = \|v\|_{L^r(\Omega)} + \|\text{curl} \ v\|_{L^p(\Omega)}.
\]

The closure of \( \mathcal{D}(\Omega) \) in \( H^{r,p}(\text{curl},\Omega) \) is denoted by \( H_0^{r,p}(\text{curl},\Omega) \) with

\[
H_0^{r,p}(\text{curl},\Omega) := \{ v \in H^{r,p}(\text{curl},\Omega); \ v \times n = 0 \ \text{on} \ \Gamma \}.
\]

The dual space of \( H_0^{r,p}(\text{curl},\Omega) \) is denoted by \( [H_0^{r,p}(\text{curl},\Omega)]' \) and its characterization is given in Proposition 3.1. We introduce also the kernel

\[
\mathcal{K}_N^p(\Omega) = \{ v \in L^p(\Omega); \ \text{div} \ v = 0, \ \text{curl} \ v = 0, \ v \times n = 0 \ \text{on} \ \Gamma \},
\]
which is spanned by the functions $\nabla q_i^N \in W^{1,q}(\Omega)$ for any $1 < q < \infty$ [6, Corollary 4.2] and $q_i^N$ is the unique solution of the problem

$$
\begin{cases}
-\Delta q_i^N = 0 & \text{in } \Omega, \
q_i^N|_{\Gamma_0} = 0 & \text{and } q_i^N|_{\Gamma_k} = \text{constant}, 1 \leq k \leq I \\
\langle \partial_n q_i^N, 1 \rangle_{\Gamma_0} = \delta_{ik}, 1 \leq k \leq I, & \text{and } \langle \partial_n q_i^N, 1 \rangle_{\Gamma_k} = -1.
\end{cases}
$$

(2.2)

We will use the symbol $\sigma$ to represent a set of divergence free functions. For example the space $L^p(\Omega)$ is the space of functions in $L^p(\Omega)$ with divergence free. We will denote by $C$ an unspecified positive constant which may depend on $\Omega$ and the dependence on other parameters will be specified if necessary.

The first theorem is concerned with the existence of weak solutions in the case of Hilbert spaces for the following MHD problem:

$$
\begin{align*}
-\Delta u + (\text{curl } u) \times u + \nabla P - (\text{curl } b) \times b &= f \quad \text{and } \text{div } u = h \quad \text{in } \Omega, \\
\text{curl } b - \text{curl } (u \times b) &= g \quad \text{and } \text{div } b = 0 \quad \text{in } \Omega, \\
u \times n &= 0 \quad \text{and } b \times n = 0 \quad \text{on } \Gamma, \\
P &= P_0 \quad \text{on } \Gamma_0 \quad \text{and } P = P_0 + \alpha_i \quad \text{on } \Gamma_i, \\
\langle u \cdot n, 1 \rangle_{\Gamma_i} &= 0 \quad \text{and } \langle b \cdot n, 1 \rangle_{\Gamma_i} = 0, \forall 1 \leq i \leq I.
\end{align*}
$$

(MHD)

The proof is given in Subsection 6.1 (see Theorem 6.1). We note that in the case when $\partial \Omega$ is not connected, to ensure the solvability of problem (MHD), we need to impose the conditions for $u$ and $b$ on the connected components $\Gamma_i$: $\langle u \cdot n, 1 \rangle_{\Gamma_i} = 0$ and $\langle b \cdot n, 1 \rangle_{\Gamma_i} = 0$ for $1 \leq i \leq I$. (See [6] and [7] for an equivalent form of these conditions). Of course, if $\partial \Omega$ is connected, the above conditions are no longer necessary.

**Theorem 2.1.** (Weak solutions of the (MHD) system in $H^1(\Omega)$). Let $f, g \in [H^{0,2}_0(\text{curl}, \Omega)]'$, $h = 0$ and $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ with the compatibility conditions

$$
\begin{align*}
\forall v \in K^2_N(\Omega), \quad & \langle g, v \rangle_{\Omega_{6,2}} = 0, \\
\text{div } g &= 0 \quad \text{in } \Omega,
\end{align*}
$$

(2.3)

(2.4)

where $\langle \cdot, \cdot \rangle_{\Omega_{p,R}}$ denotes the duality product between $[H^{0,p}_0(\text{curl}, \Omega)]'$ and $H^{0,p}_0(\text{curl}, \Omega)$. Then the (MHD) problem has at least one weak solution $(u, b, P, \alpha) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I$ such that

$$
\|u\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)} + \|P\|_{L^2(\Omega)} \leq M,
$$

where $M = C(\|f\|_{[H^{0,2}_0(\text{curl}, \Omega)]'} + \|g\|_{[H^{0,2}_0(\text{curl}, \Omega)]'})^r + \|P_0\|_{H^{-1/2}(\Gamma)}$ and $\alpha = (\alpha_1, \ldots, \alpha_I)$ defined by

$$
\alpha_i = \langle f, \nabla q_i^N \rangle_{\Omega_{6,2}} - \langle P_0, \nabla q_i^N \rangle_{\Gamma} + \int_{\Omega} (\text{curl } b) \times b \cdot \nabla q_i^N \, dx - \int_{\Omega} (\text{curl } u) \times u \cdot \nabla q_i^N \, dx,
$$

(2.5)

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality product between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$.

In addition, suppose that $f$, $g$ and $P_0$ are small in the sense that

$$
C_1 C_2^2 M \leq \frac{2}{3C_p^2},
$$

(2.6)

where $C_p$ is the constant in (3.6) and $C_1$, $C_2$ are the constants defined in (6.15). Then the weak solution $(u, b, P)$ of (MHD) is unique.

The next two theorems are concerned with generalized solutions in $W^{1,p}(\Omega)$ for $p > 2$ and strong solutions in $W^{2,p}(\Omega)$ for $p \geq 6/5$. The existence of weak solution in $W^{1,p}(\Omega)$ for $\frac{2}{7} < p < 2$ is not trivial. We will precise this case later.
Theorem 2.2. (Weak solutions in $W^{1,p}(\Omega)$ with $p > 2$ for the (MHD) system). Let $p > 2$. Suppose that $f, g \in [H_0^\nu, p'](\text{curl}, \Omega)'$, $h = 0$ and $P_0 \in W^{1,\frac{1}{p}}(\Gamma)$ with the compatibility condition (2.4) and

$$\forall \psi \in K_N^p(\Omega), \quad (g, \psi)_{\nu', \nu'} = 0. \quad (2.7)$$

Then the weak solution for the (MHD) system given by Theorem 2.1 satisfies

$$(u, b, P) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,\frac{1}{p}}(\Omega).$$

Moreover, we have the following estimate:

$$\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,\frac{1}{p}}(\Omega)} \leq C(\|f\|_{(H_0^\nu, p')(\text{curl}, \Omega)'} + \|g\|_{(H_0^\nu, p')(\text{curl}, \Omega)'} + \|P_0\|_{W^{1,\frac{1}{p}}(\Gamma)})$$

Theorem 2.3. (Strong solutions in $W^{2,p}(\Omega)$ with $p \geq \frac{2}{3}$ for the (MHD) system). Let us suppose that $\Omega$ is of class $C^{2,1}$ and $p \geq \frac{2}{3}$. Let $f$, $g$ and $P_0$ with the compatibility conditions (2.4) and (2.7) and $f \in L^p(\Omega)$, $g \in L^p(\Omega)$, $h = 0$ and $P_0 \in W^{1,\frac{1}{p}}(\Gamma)$.

Then the weak solution $(u, b, P)$ for the (MHD) system given by Theorem 2.1 belongs to $W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,\frac{1}{p}}(\Omega)$ and satisfies the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,\frac{1}{p}}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1,\frac{1}{p}}(\Gamma)})$$

We refer to Theorem 6.3 and Theorem 6.4 for the proof of the above result, where we use the estimates obtained in the Hilbert case and a bootstrap argument using regularity results of some Stokes and elliptic problems in [6] and [7].

To deal with the regularity of the solutions of the (MHD) system in $W^{1,p}(\Omega)$ with $\frac{2}{3} < p < 2$, we need to study the following linearized MHD system: Find $(u, b, P, c)$ with $c = (c_1, \ldots, c_I)$ such that for $1 \leq i \leq I$:

$$\begin{cases}
- \Delta u + (\text{curl } w) \times u + \nabla P - (\text{curl } b) \times d = f \\
\text{curl } \text{curl } b - \text{curl}(u \times d) = g \\
\quad u \times n = 0 \quad \text{and} \quad b \times n = 0 \quad \text{on } \Gamma,
\end{cases}$$

$$P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \quad \text{on } \Gamma_i,$$

$$(u \cdot n, 1)_{\Gamma_i} = 0 \quad \text{and} \quad (b \cdot n, 1)_{\Gamma_i} = 0. \quad (2.8)$$

The next theorem gives existence of weak and strong solutions for the linearized problem (2.8).

Theorem 2.4. (Existence of weak and strong solutions of the linearized MHD problem). Suppose that $f, g \in [H_0^\nu, p'](\text{curl}, \Omega)'$, $P_0 \in W^{1,\frac{1}{p}}(\Gamma)$, $h \in W^{1,r}(\Omega)$

with the compatibility conditions (2.4) and (2.7).

(1) For any $p \geq 2$, if $\text{curl } w \in L^*(\Omega)$, $d \in W_0^{1,s}(\Omega)$ where $s$ is given by

$$s = \frac{3}{2} \quad \text{if} \quad 2 < p < 3, \quad s > \frac{3}{2} \quad \text{if} \quad p = 3, \quad s = r \quad \text{if} \quad p > 3,$$

then the linearized system (2.8) has a unique solution $(u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,\frac{1}{p}}(\Omega) \times \mathbb{R}^I$ with $c = (c_1, \ldots, c_I)$. Moreover, we have the estimate:

$$\begin{align*}
\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,\frac{1}{p}}(\Omega)} &\leq C(1 + \|\text{curl } w\|_{L^*(\Omega)} + \|d\|_{W^{1,s}(\Omega)})(\|f\|_{(H_0^\nu, p')(\text{curl}, \Omega)'} + \|g\|_{(H_0^\nu, p')(\text{curl}, \Omega)'} + (1 + \|\text{curl } w\|_{L^*(\Omega)} + \|d\|_{W^{1,s}(\Omega)}) \|h\|_{W^{1,r}(\Omega)})
\end{align*}$$
(2). Let \( \frac{3}{2} < p < 2 \). If \( \text{curl} \, w \in L^{3/2}(\Omega) \) and \( d \in W_{0}^{1,3/2}(\Omega) \), then the linearized problem (2.8) has a unique solution \((u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^d\). Moreover, we have the following estimates:

\[
\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} \leq C(1 + \|\text{curl} \, w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)})(\|f\|_{H_{0}^{1/2}(\text{curl},\Omega)})^{r} + \|P_0\|_{W^{1,1/2}(\Gamma)}
\]
\[
+ \|g\|_{H_{0}^{1/2}(\text{curl},\Omega)} + (1 + \|\text{curl} \, w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)})(\|h\|_{W^{1,1/2}(\Omega)})
\]

and

\[
\|P\|_{W^{1,1/2}(\Omega)} \leq C(1 + \|\text{curl} \, w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)}) \times (\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1,1/2}(\Gamma)})
\]
\[
+ \|g\|_{H_{0}^{1/2}(\text{curl},\Omega)} + (1 + \|\text{curl} \, w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)})(\|h\|_{W^{1,1/2}(\Omega)}),
\]

(3). Also for any \( p \in (1, \infty) \), if \( \Omega \) is of class \( C^{2,1} \), \( h = 0 \) in \( \Omega \) and

\[
f \in L^p(\Omega), \; g \in L^p(\Omega), \; \text{curl} \, w \in L^{3/2}(\Omega), \; d \in W_{0}^{1,3/2}(\Omega), \; \text{and} \; P_0 \in W^{1,1/2}(\Gamma)
\]

with the compatibility conditions (2.4) and (2.7), then \((u, b, P, c) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^d\) and satisfies the estimate:

\[
\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C(1 + \|\text{curl} \, w\|_{L^{3/2}(\Omega)})
\]
\[
\times (\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1,1/2}(\Gamma)})
\]

where \( C = C(\Omega, p) \) if \( p \geq 6/5 \) and \( C = C(\Omega, p)(1 + \|\text{curl} \, w\|_{L^{3/2}(\Omega)}) + \|d\|_{W^{1,3/2}(\Omega)} \) if \( 1 < p < 6/5 \).

We refer to Theorem 5.4, Theorem 5.7, Theorem 5.1 and Theorem 5.11 for the proof of the above results. Note that in the above theorem, to prove the existence of weak solutions in \( W^{1,p}(\Omega) \) with \( 3/2 < p < 2 \), we use a duality argument. We also note that we proved more general existence results in Corollary 5.10 where the regularity of the pressure is improved by supposing a data \( P_0 \) less regular.

Finally, the next result shows the existence and uniqueness of weak solutions with \( 3/2 < p < 2 \) for the nonlinear (MHD) problem (see Theorem 6.5). The proof is essentially based on the estimates obtained above for the linearized problem (2.8).

**Theorem 2.5.** (Regularity \( W^{1,p}(\Omega) \) with \( \frac{3}{2} < p < 2 \) for the (MHD) system). Assume that \( \frac{3}{2} < p < 2 \) and \( r \) with \( \frac{1}{r} = \frac{1}{p} + \frac{1}{3} \). Let us consider \( f, g \in [H_{0}^{1/2}(\text{curl},\Omega)], \; P_0 \in W^{1,1/2}(\Gamma) \) and \( h \in W^{1,1}(\Omega) \) with the compatibility conditions (2.4) and (2.7).

(i) There exists a constant \( \delta_1 \) such that, if

\[
\|f\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|g\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|P_0\|_{W^{1,1/2}(\Gamma)} + (\|h\|_{W^{1,1}(\Omega)}) \leq \delta_1
\]

Then, the (MHD) problem has at least one solution \((u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^d\). Moreover, we have the following estimates:

\[
\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} \leq C(1 + \|f\|_{H_{0}^{1/2}(\text{curl},\Omega)}) + \|g\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|P_0\|_{W^{1,1/2}(\Gamma)}
\]
\[
+ (\|h\|_{W^{1,1}(\Omega)}) \tag{2.9}
\]
\[
\|P\|_{W^{1,1/2}(\Omega)} \leq C(1 + C^* \eta)(\|f\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|g\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|P_0\|_{W^{1,1/2}(\Gamma)}
\]
\[
+ (\|h\|_{W^{1,1}(\Omega)}), \tag{2.10}
\]

where \( \delta_1 = (2C^2C^*)^{-1}, \; C_1 = C(1 + C^* \eta)^2 \) with \( C > 0, \; C^* > 0 \) are the constants given in (6.25) and \( \eta \) defined by (6.26). Furthermore, we have for all \( 1 \leq i \leq I \)

\[
\alpha_i = (f, \nabla q_i^N) - \int_{\Omega} (\text{curl} \, w) \times u \cdot \nabla q_i^N \, dx + \int_{\Omega} (\text{curl} \, b) \times d \cdot \nabla q_i^N \, dx + \int_{\Gamma} (h - P_0) \nabla q_i^N \cdot n \, ds
\]

(ii) Moreover, if the data satisfy that

\[
\|f\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|g\|_{H_{0}^{1/2}(\text{curl},\Omega)} + \|P_0\|_{W^{1,1/2}(\Gamma)} + (\|h\|_{W^{1,1}(\Omega)} \leq \delta_2,
\]

for some \( \delta_2 \in [0, \delta_1] \), then the weak solution of (MHD) problem is unique.
3 Notations and preliminary results

Before studying the MHD problem (MHD), we introduce some basic notations and specific functional framework. If we do not state otherwise, $\Omega$ will be considered as an open bounded domain of $\mathbb{R}^3$, which is not necessary connected, of class at least $C^{1,1}$ and sometimes of class $C^{2,1}$. We denote by $\Gamma_i$, $0 \leq i \leq I$, the connected components of $\Gamma$, $\Gamma_0$ being the boundary of the only unbounded connected component of $\mathbb{R}^3 \setminus \Omega$.

The vector fields and matrix fields as well as the corresponding spaces are denoted by bold font. We will use $C$ to denote a generic positive constant which may depend on $\Omega$ and the dependence on other parameters will be specified if necessary. For $1 < p < \infty$, $L^p(\Omega)$ denotes the usual vector-valued $L^p$-space over $\Omega$. As usual, we denote by $W^{m,p}(\Omega)$ the Sobolev space of functions in $L^p(\Omega)$ whose weak derivatives of order less than or equal to $m$ are also in $L^p(\Omega)$. In the case $p = 2$, we shall write $H^m(\Omega)$ instead to $W^{m,2}(\Omega)$. If $p \in [1, \infty)$, $p'$ denotes the conjugate exponent of $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We define the spaces

$$X^p(\Omega) = \{ v \in L^p(\Omega) ; \text{div } v \in L^p(\Omega), \text{curl } v \in L^p(\Omega) \},$$

which is equipped with the norm:

$$\| v \|_{X^p(\Omega)} = \| v \|_{L^p(\Omega)} + \| \text{curl } v \|_{L^p(\Omega)} + \| \text{div } v \|_{L^p(\Omega)}.$$

The subspaces $X^p_X(\Omega)$ and $V^p_X(\Omega)$ are defined by

$$X^p_X(\Omega) = \{ v \in L^p(\Omega) ; \text{div } v \in L^p(\Omega), \text{curl } v \in L^p(\Omega), v \times n = 0 \text{ on } \Gamma \},$$

$$V^p_X(\Omega) = \{ v \in X^p_X(\Omega) ; \text{div } v = 0 \text{ in } \Omega \}.$$

When $p = 2$, we will use the notation $X^2_X(\Omega)$ instead to $X^2(\Omega)$. We denote by $D(\Omega)$ the set of smooth functions (infinitely differentiable) with compact support in $\Omega$. For $p, r \in [1, \infty)$, we introduce the following space

$$H^{r,p}(\text{curl},\Omega) := \{ v \in L^r(\Omega) ; \text{curl } v \in L^p(\Omega) \}, \quad \text{with } \frac{1}{r} = \frac{1}{p} + \frac{1}{3}$$

equipped with the norm

$$\| v \|_{H^{r,p}(\text{curl},\Omega)} = \| v \|_{L^r(\Omega)} + \| \text{curl } v \|_{L^p(\Omega)}.$$

It can be shown that $D(\overline{\Omega})$ is dense in $H^{r,p}(\text{curl},\Omega)$ (cf. [24, Proposition 1.0.2] for the case $r = p$). The closure of $D(\Omega)$ in $H^{r,p}(\text{curl},\Omega)$ is denoted by $H^{r,p}(\text{curl},\Omega)$ with

$$H^{r,p}_0(\text{curl},\Omega) := \{ v \in H^{r,p}(\text{curl},\Omega) ; v \times n = 0 \text{ on } \Gamma \}. $$

$D(\Omega)$ is dense in $H^{r,p}_0(\text{curl},\Omega)$ and its dual space denoted by $[H^{r,p}_0(\text{curl},\Omega)]'$ can be characterized as follows (cf. [7, Lemma 2.4], [7, Lemma 2.5] and [24, Proposition 1.0.6] for the case $r = p$):

**Proposition 3.1.** A distribution $f$ belongs to $[H^{r,p}_0(\text{curl},\Omega)]'$ iff there exists $F \in L^r(\Omega)$ and $\psi \in L^p(\Omega)$ such that $f = F + \text{curl } \psi$. Moreover, we have the estimate :

$$\| f \|_{[H^{r,p}_0(\text{curl},\Omega)]'} \leq \inf_{f = F + \text{curl } \psi} \max(\| F \|_{L^r(\Omega)}, \| \psi \|_{L^p(\Omega)}).$$

Next we introduce the kernel

$$K^p_X(\Omega) = \{ v \in L^p(\Omega) ; \text{div } v = 0, \text{curl } v = 0, v \times n = 0 \text{ on } \Gamma \}.$$ 

Thanks to [6, Corollary 4.2], we know that this kernel is of finite dimension and spanned by the functions $\nabla q^N_i$, $1 \leq i \leq I$, where $q^N_i$ is the unique solution of the problem

$$\begin{cases}
-\Delta q^N_i = 0 & \text{in } \Omega, \\
q^N_i |_{\Gamma_0} = 0 & \text{and } q^N_i |_{\Gamma_0} = \text{constant}, \\n\langle \partial_n q^N_i, 1 \rangle_{\Gamma_k} = \delta_{ik}, & 1 \leq k \leq I, \text{ and } \langle \partial_n q^N_i, 1 \rangle_{\Gamma_k} = -1.
\end{cases}$$

(3.2)
Moreover, the functions $\nabla q_i^N, 1 \leq i \leq I$, belong to $W^{1,q}(\Omega)$ for any $1 < q < \infty$. We will use also the symbol $\sigma$ to represent a set of divergence free functions. In other words, if $X$ is a Banach space, then $X_\sigma = \{v \in X; \text{div} v = 0 \in \Omega\}$.

We recall some useful results that play an important role in the proof of the regularity of solutions in this work. We begin with the following result (see [6, Theorem 3.2.])

**Theorem 3.1.** The space $X^p_0(\Omega)$ is continuously embedded in $W^{1,p}(\Omega)$ and there exists a constant $C$, such that for any $v$ in $X^p_0(\Omega)$:

\[
\|v\|_{W^{1,p}(\Omega)} \leq C(\|v\|_{L^p(\Omega)} + \|\text{div} v\|_{L^p(\Omega)} + \|\text{curl} v\|_{L^p(\Omega)} + \sum_{i=1}^{I} |(v \cdot n, 1)_{\Gamma_i}|). \tag{3.3}
\]

And more generally (see [6, Corollary 5.3])

**Corollary 3.2.** Let $m \in \mathbb{N}^*$ and $\Omega$ of class $C^{m,1}$. Then the space

\[
X^{m,p}(\Omega) = \{v \in L^p(\Omega); \text{div} v \in W^{m-1,p}(\Omega), \text{curl} v \in W^{m-1,p}(\Omega), v \cdot n \in W^{m-1,p}(\Gamma)\}
\]

is continuously embedded in $W^{m,p}(\Omega)$ and we have the following estimate: for any function $v$ in $W^{m,p}(\Omega)$,

\[
\|v\|_{W^{m,p}(\Omega)} \leq C\left(\|v\|_{L^p(\Omega)} + \|\text{curl} v\|_{W^{m-1,p}(\Omega)} + \|\text{div} v\|_{W^{m-1,p}(\Omega)} + \|v \cdot n\|_{W^{m-1,p}(\Gamma)}\right) \tag{3.4}
\]

We also recall the following result (cf. [6, Corollary 3.2]) which gives a Poincaré inequality for every function $v \in W^{1,p}(\Omega)$ with $v \cdot n = 0$ on $\Gamma$.

**Corollary 3.3.** On the space $X^p_0(\Omega)$, the seminorm

\[
v \mapsto \|\text{curl} v\|_{L^p(\Omega)} + \|\text{div} v\|_{L^p(\Omega)} + \sum_{i=1}^{I} |(v \cdot n, 1)_{\Gamma_i}| \tag{3.5}
\]

is equivalent to the norm $\cdot \|v\|_{X^p(\Omega)}$ for any $1 < p < \infty$. In particular, we have the following Poincaré inequality for every function $v \in W^{1,p}(\Omega)$ with $v \cdot n = 0$ on $\Gamma$:

\[
\|v\|_{W^{1,p}(\Omega)} \leq C_p \left(\|\text{div} v\|_{L^p(\Omega)} + \|\text{curl} v\|_{L^p(\Omega)} + \sum_{i=1}^{I} |(v \cdot n, 1)_{\Gamma_i}|\right), \tag{3.6}
\]

where $C_p = C_p(\Omega) > 0$. Moreover, the norm (3.5) is equivalent to the full norm $\|\cdot\|_{W^{1,p}(\Omega)}$ on $X^p_0(\Omega)$.

Let us consider the following Stokes problem:

\[
\begin{cases}
- \Delta u + \nabla P = f & \text{in } \Omega, \\
\text{div } u = h & \text{in } \Omega, \\
u \times n = 0 & \text{on } \Gamma, \\
P = P_0 & \text{on } \Gamma_0 \text{ and } P = P_0 + c_i & \text{on } \Gamma_i, \\
(u \cdot n, 1)_{\Gamma_i} = 0, & 1 \leq i \leq I.
\end{cases}
\tag{S_N}
\]

Then, the following proposition is an extension of that in [6, Theorem 5.7] to the case of non-zero divergence condition ($h \neq 0$). It is concerned with the existence and uniqueness of the weak and strong solutions for the Stokes problem (S_N).

**Proposition 3.2.** We assume that $\Omega$ is of class $C^{2,1}$. Let $f$, $h$ and $P_0$ such that

\[
f \in [H^{r,p'}_0(\text{curl}, \Omega)]', \quad h \in W^{1,r}(\Omega) \quad \text{and} \quad P_0 \in W^{1-1/r,r}(\Gamma),
\]

with $r \leq p$ and $\frac{1}{r} \leq \frac{1}{p'} + \frac{1}{p}$. Then, the problem (S_N) has a unique solution $(u, P) \in W^{1,p}(\Omega) \times W^{1,r}(\Omega)$ and constants $c_1, \ldots, c_I$ satisfying the estimate:

\[
\|u\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leq C\left(\|f\|_{H^{r,p'}_0(\text{curl}, \Omega)'}, \|h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1-1/r,r}(\Gamma)}\right), \tag{3.7}
\]
Proof. To reduce the non-vanishing divergence problem \((S_X)\) to the case where \(\text{div} \, u = 0\) in \(\Omega\), we consider the problem
\[
\Delta \theta = h \quad \text{in} \; \Omega \quad \text{and} \quad \theta = 0 \quad \text{on} \; \Gamma.
\]
Since \(h \in W^{1,r}(\Omega)\), it has a unique solution \(\theta \in W^{3,r}(\Omega) \hookrightarrow W^{2,p}(\Omega)\), with (cf. [14, Theorem 1.8])
\[
\|\theta\|_{W^{2,p}(\Omega)} \leq C \|h\|_{W^{1,r}(\Omega)}.
\] (3.10)

Taking \(w = \nabla \theta\) and defining
\[
\tilde{w} = w - \sum_{i=1}^{I} \langle \omega \cdot n, 1 \rangle_{\Gamma_i} \text{grad} \, q_i^N,
\] (3.11)
we see that \(\tilde{w} \in W^{1,p}(\Omega)\) with \(\text{div} \, \tilde{w} = h\), \(\text{curl} \, \tilde{w} = 0\) in \(\Omega\), \(\tilde{w} \times n = 0\) on \(\Gamma\) and \(\langle \tilde{w} \cdot n, 1 \rangle_{\Gamma_i} = 0\) for any \(1 \leq i \leq I\). Finally, taking \(z = u - \tilde{w}\), we see that the problem \((S_X)\) can be reduced to the following problem for \(z\) and \(P\):
\[
\begin{aligned}
-\Delta z + \nabla P &= f + \Delta \tilde{w} \quad \text{and} \quad \text{div} \, z = 0 \quad \text{in} \; \Omega, \\
z \times n &= 0 \quad \text{on} \; \Gamma, \\
P &= P_0 \quad \text{on} \; \Gamma_0 \quad \text{and} \quad \tilde{P} = P_0 + c_i \quad \text{on} \; \Gamma_i, \\
\langle z \cdot n, 1 \rangle_{\Gamma_i} &= 0, \; 1 \leq i \leq I.
\end{aligned}
\] (3.12)

Since \(w = \nabla \theta\) and \(\Delta (\nabla q_i^N) = 0\), it follows from (3.11) that \(\Delta \tilde{w} = \nabla (\Delta \theta) \in L^r(\Omega) \hookrightarrow [H^r_0, v'(\text{curl}, \Omega)]\) and
\[
\int_{\Omega} \Delta \tilde{w} \cdot \nabla q_i^N \, dx = 0,
\]
we deduce from [6, Theorem 5.7], the existence of a unique solution \((z, P, c) \in W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I\) of (3.12) with \(c = (c_1, \ldots, c_I)\) given by (3.8). Moreover, using (3.10), we have \(\|\Delta \tilde{w}\|_{[H^r_0, v'(\text{curl}, \Omega)]} \leq C \|h\|_{W^{1,r}(\Omega)}\) and then \((z, \tilde{P})\) satisfies the estimate:
\[
\|z\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leq C \left( \|f\|_{[H^r_0, v'(\text{curl}, \Omega)]} + \|h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1-1/r, r}(\Gamma)} \right).
\] (3.13)

As a consequence, \((u, P) = (z + \tilde{w}, \tilde{P}) \in W^{1,p}(\Omega) \times W^{1,r}(\Omega)\) is the unique solution of \((S_X)\) and the estimate (3.7) follows from (3.10) and (3.13).

Now, we suppose that \(f \in L^p(\Omega), h \in W^{1,p}(\Omega)\) and \(P_0 \in W^{1-1/p, p}(\Gamma)\). We know that \((u, P)\) belongs to \(W^{1,p}(\Omega) \times W^{1,r}(\Omega)\). We set \(z = \text{curl} \, u\). Since \(u \times n = 0\) on \(\Gamma\), we have \(z \cdot n = 0\) on \(\Gamma\) and then \(z\) belongs to \(X_X(\Omega)\). By Theorem 3.1, the function \(z\) belongs to \(W^{1,p}(\Omega)\). Then, \(u\) satisfies
\[
u \in L^p(\Omega), \; \text{div} \, u = h \in W^{1,p}(\Omega), \; \text{curl} \, u \in W^{1,p}(\Omega) \quad \text{and} \quad u \times n = 0 \quad \text{on} \; \Gamma.
\]
We deduce from Corollary 3.2 (with \(m = 2\)) that \(u\) belongs to \(W^{2,p}(\Omega)\).

We need also some regularity results for the following elliptic problem
\[
(E_X) \begin{cases}
-\Delta b = g & \text{and} \quad \text{div} \, b = 0 \quad \text{in} \; \Omega, \\
\langle b \cdot n, 1 \rangle_{\Gamma_i} = 0, & \forall 1 \leq i \leq I, \\
\langle b \cdot n, 1 \rangle_{\Gamma_i} = 0, & \forall 1 \leq i \leq I,
\end{cases}
\]
which can be seen as a Stokes problem without pressure. We note that \((E_N)\) is well-posed. Indeed, observe that the condition \(\text{div}\, g = 0\) in \(\Omega\) is necessary to solve \((E_N)\) and then we can verify that it is equivalent to the following problem:

\[
\begin{aligned}
-\Delta b &= g & \text{in } \Omega \\
\text{div} \, b &= 0, \quad \text{and} \quad b \times n = 0 & \text{on } \Gamma, \\
\langle b \cdot n, 1 \rangle_{\Gamma_i} &= 0, \quad \forall 1 \leq i \leq I,
\end{aligned}
\]

(3.14)

where we have replaced the condition \(\text{div} \, b = 0\) in \(\Omega\) by \(\text{div} \, b = 0\) on \(\Gamma\). Next, we know that for any \(b \in W^{1,p}(\Omega)\) such that \(\text{div} \, b \in W^{1,p}(\Omega)\), we have (cf. [3] or [16] for \(b \in W^{2,p}(\Omega)\)):

\[
\text{div} \, b = \text{div} \, b_r + \frac{\partial b}{\partial n} \cdot n - 2Kb \cdot n \quad \text{on } \Gamma,
\]

(3.15)

where \(b_r\) is the tangential component of \(b\), \(K\) denotes the mean curvature of \(\Gamma\) and \(\text{div} \, b_r\) is the surface divergence. Then, using (3.15), the problem (3.14) is equivalent to: find \(b \in W^{1,p}(\Omega)\) such that

\[
\begin{aligned}
-\Delta b &= g & \text{in } \Omega, \\
\frac{\partial b}{\partial n} \cdot n - 2Kb \cdot n &= 0 & \text{on } \Gamma, \\
\langle b \cdot n, 1 \rangle_{\Gamma_i} &= 0, \quad \forall 1 \leq i \leq I,
\end{aligned}
\]

(3.16)

where the condition \(\frac{\partial b}{\partial n} \cdot n - 2Kb \cdot n = 0\) on \(\Gamma\) is a Fourier-Robin type boundary condition.

We begin with the following regularity result for \((E_N)\) which can be found in [6, Corollary 5.4.].

**Theorem 3.3.** Assume that \(\Omega\) is of class \(C^{2,1}\). Let \(g \in L^p(\Omega)\) satisfying the compatibility conditions

\[
\forall v \in K_N^p(\Omega), \quad \int_{\Omega} g \cdot v \, dx = 0,
\]

(3.17)

\[
\text{div} \, g = 0 \quad \text{in } \Omega.
\]

(3.18)

Then the elliptic problem \((E_N)\) has a unique solution \(b \in W^{2,p}(\Omega)\) satisfying the estimate

\[
\|b\|_{W^{2,p}(\Omega)} \leq C_E \|g\|_{L^p(\Omega)}.
\]

(3.19)

We need also the following useful result for \((E_N)\) which gives an improvement of that in [6, Proposition 5.1]. Indeed, we consider the dual space \([H^{\pi,p}_0(\text{curl}, \Omega)]'\) with \(\frac{1}{p} = \frac{1}{p} + \frac{1}{q}\) (c.f. (3.1) and Proposition (3.1)) for data in the right-hand side instead of \([H^{\pi,p}_0(\text{curl}, \Omega)]'\).

**Lemma 3.4.** Let \(\Omega\) of class \(C^{2,1}\). Let \(g \in [H^{\pi,p}_0(\text{curl}, \Omega)]'\) satisfying the compatibility conditions

\[
\forall v \in K_N^p(\Omega), \quad \langle g, v \rangle_{[H^{\pi,p}_0(\text{curl}, \Omega)]' \times H^{\pi,p}_0(\text{curl}, \Omega)} = 0,
\]

(3.20)

\[
\text{div} \, g = 0 \quad \text{in } \Omega.
\]

(3.21)

Then, the elliptic problem \((E_N)\) has a unique solution \(b \in W^{1,p}(\Omega)\) satisfying the estimate:

\[
\|b\|_{W^{1,p}(\Omega)} \leq C \|g\|_{[H^{\pi,p}_0(\text{curl}, \Omega)]'}
\]

(3.22)

**Proof.** Using the characterization of the dual space \([H^{\pi,p}_0(\text{curl}, \Omega)]'\) given in Proposition 3.1, we can write \(g\) as:

\[
g = G + \text{curl} \, \Psi, \quad \text{where} \quad G \in L^r(\Omega) \quad \text{and} \quad \Psi \in L^p(\Omega).
\]

(3.23)
Note that, from (3.2), for any $1 \leq i \leq I$, $(\text{curl} \, \Psi, \nabla q_i^N)_\Omega = 0$, then it follows from (3.20) and (3.23) that $G$ also satisfies the compatibility condition (3.20). Similarly, by (3.23), we have $\nabla G = 0$. Thanks to Theorem 3.3, the following problem:

$$
\begin{cases}
-\Delta b_1 = G & \text{in } \Omega \\
\nabla b_1 = 0 & \text{in } \Omega,
\end{cases}
$$

has a unique solution $b_1 \in W^{2,r}(\Omega)$ satisfying the estimate:

$$
\|b_1\|_{W^{2,r}(\Omega)} \leq C \|G\|_{L^r(\Omega)}. \tag{3.24}
$$

Next, since $\text{curl} \, \Psi \in [H^1_0(\text{curl}, \Omega)]'$ and satisfies the compatibility conditions (3.20), by [6, Proposition 5.1.] the following problem

$$
\begin{cases}
-\Delta b_2 = \text{curl} \, \Psi & \text{in } \Omega \\
\nabla b_2 = 0 & \text{in } \Omega,
\end{cases}
$$

has a unique solution $b_2 \in W^{1,q}(\Omega)$ satisfying the estimate:

$$
\|b_2\|_{W^{1,q}(\Omega)} \leq C \|\text{curl} \, \Psi\|_{L^q(\Omega)}. \tag{3.25}
$$

Since $\frac{1}{q} = \frac{1}{r} + \frac{1}{2}$, $W^{2,r}(\Omega) \hookrightarrow W^{1,q}(\Omega)$. Then, $b = b_1 + b_2$ belongs to $W^{1,q}(\Omega)$ and it is the unique solution of $(\mathcal{E}_\lambda)$. The estimate (3.22) follows from (3.24) and (3.25).

**Remark 3.4.**

(1). We note that the regularity $C^{2,1}$ in Lemma 3.4 can be reduced to $C^{1,1}$. Indeed, we can verify that the Stokes problem $(\mathcal{E}_\lambda)$ is equivalent to the following variational formulation (c.f. [6, Proposition 5.1]): Find $b \in W^{1,p}(\Omega)$ such that for any $a \in V^p_\lambda(\Omega)$:

$$
\int_\Omega \text{curl} \, b \cdot \text{curl} \, a \, dx = (g, a)_{H^1_0(\text{curl}, \Omega)'} \times H^1_0(\text{curl}, \Omega)'. \tag{3.26}
$$

Thanks to [6, Lemma 5.1], if $\Omega$ is of class $C^{1,1}$, the following inf-sup condition holds: there exists a constant $\beta > 0$, such that:

$$
\inf_{a \in V^p_\lambda(\Omega)} \sup_{b \in W^{1,p}(\Omega), b \neq 0} \frac{\int_\Omega \text{curl} \, b \cdot \text{curl} \, a \, dx}{\|b\|_{W^{1,p}(\Omega)} \|a\|_{W^{1,p}(\Omega)}} \geq \beta. \tag{3.27}
$$

So, problem (3.26) has a unique solution $u \in V^p_\lambda(\Omega) \subset W^{1,p}(\Omega)$ since the right-hand sides defines an element of $(V^p_\lambda(\Omega))'$.

(2). In the classical study of the Stokes and Navier-Stokes equations, the pressure $P$ is obtained thanks to a variant of De Rham’s theorem (see [3, Theorem 2.8]). Indeed, let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $f \in W^{-1,p}(\Omega)$, $1 < p < \infty$ satisfying

$$
\forall u \in \mathcal{D}'(\Omega), \quad (f, v)_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0.
$$

Then there exists $P \in L^p(\Omega)$ such that $f = \nabla P$. Unlike the case of Dirichlet boundary condition, the pressure in the $(\text{MHD})$ problem can be found independently of the velocity $u$ and the magnetic field $b$. Indeed, the pressure $P$ is a solution of the problem

$$
\begin{cases}
\Delta P = \text{div} \, f - \text{div}((\text{curl} \, u) \times u) + \text{div}((\text{curl} \, b) \times d) & \text{in } \Omega \\
\n P = P_0 & \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + \alpha_i & \text{on } \Gamma_i.
\end{cases}
$$

So, when we talk about the regularity $W^{1,p}(\Omega)$ or $W^{2,p}(\Omega)$ it concerns $(u, b)$ and we mean that $(u, b, P)$ is the weak or strong solution of the $(\text{MHD})$ problem.
4 The linearized MHD system: $L^2$-theory

In this section we take $w$ and $d$ such that:

$$\text{curl } w \in L^{3/2}(\Omega), \quad d \in L^3(\Omega), \quad \text{div } d = 0 \text{ in } \Omega,$$  \hspace{1cm} (4.1)

and we consider the following linearized MHD system: Find $(u, b, P, c)$ with $c = (c_1, \ldots, c_l)$ such that for $1 \leq i \leq l$:

$$
\begin{cases}
- \Delta u + (\text{curl } w) \times u + \nabla P - (\text{curl } b) \times d = f & \text{and } \text{div } u = h \text{ in } \Omega, \\
\text{curl } \text{curl } b - \text{curl}(u \times d) = g & \text{and } \text{div } b = 0 \text{ in } \Omega, \\
u \times n = 0 & \text{and } b \times n = 0 \text{ on } \Gamma,
\end{cases}
$$  \hspace{1cm} (4.2)

The aim of this section is to show, under minimal regularity assumptions on $f$, $g$, $h$ and $P_0$, the existence and the uniqueness of weak solutions $(u, b, P, c)$ in $H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^l$. Classically, the idea is to write an equivalent variational formulation and use Lax Milgram if the bilinear form involved in the variational formulation is coercive. It is natural to look for a solution $(u, b)$ in $V_N(\Omega) \times V_N(\Omega)$ with

$$V_N(\Omega) := \{ v \in H^1(\Omega); \text{div } v = 0 \text{ in } \Omega, \ v \times n = 0 \text{ on } \Gamma, \ (v \cdot n, 1)_{\Gamma_i} = 0, \forall 1 \leq i \leq l \}. $$

Unlike the case of Dirichlet type boundary conditions, the space $H^{-1}(\Omega)$ is not suitable for source terms in the right hand side to find solutions in $H^1(\Omega)$. Let us analyse the case of $f$, it holds true also for $g$. Since $v \in V_N(\Omega)$, then we can firstly consider the duality pairing $(f, v)_{[H^6_0(\text{curl}, \Omega)]'} \times [H^6_0(\text{curl}, \Omega)]$ in view to write an equivalent variational formulation. Then, we must suppose that $f$ belongs to $[H^6_0(\text{curl}, \Omega)]'$. But, we have $v \in H^1(\Omega) \hookrightarrow L^6(\Omega)$. Then, the previous hypothesis on $f$ can be weakened by considering the space $[H^6_0(\text{curl}, \Omega)]'$ which is a subspace of $H^{-1}(\Omega)$. Indeed, thanks to the characterization given in Proposition 3.1, we have for $r = 6$ and $p = 2$,

$$[H^6_0(\text{curl}, \Omega)]' = \{ F + \text{curl } \psi; F \in L^{6/5}(\Omega), \ \psi \in L^2(\Omega) \}. $$  \hspace{1cm} (4.3)

Then, since $V_N(\Omega) \hookrightarrow [H^6_0(\text{curl}, \Omega)]'$, the previous duality is replaced by

$$(f, v)_{[H^6_0(\text{curl}, \Omega)]'} \times [H^6_0(\text{curl}, \Omega)] = \int_{\Omega} F \cdot v \, dx + \int_{\Omega} \psi \cdot \text{curl } v \, dx.$$ 

In the sequel, we will consider the space $[H^6_0(\text{curl}, \Omega)]'$ for $f$ and $g$ to obtain solutions in $H^1(\Omega)$.

**Proposition 4.1.** Let us suppose $h = 0$. Let $f, g \in [H^6_0(\text{curl}, \Omega)]'$ and $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ with the compatibility conditions

$$\forall v \in K^2_N(\Omega), \quad \langle g, v \rangle_{\Omega, 2} = 0, \quad \text{div } g = 0 \text{ on } \Omega,$$  \hspace{1cm} (4.4)

where $(\cdot, \cdot)_{\Omega, 2}$ denotes the duality product between $[H^6_0(\text{curl}, \Omega)]'$ and $H^6_0(\text{curl}, \Omega)$. Then the following two problems are equivalent:

(i) Find $(u, b, P, c) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^l$ solution of (4.2).

(ii) Find $(u, b) \in V_N(\Omega) \times V_N(\Omega)$ and $c \in \mathbb{R}^l$ such that: for all $(v, \Psi) \in V_N(\Omega) \times V_N(\Omega)$

$$
\int_{\Omega} \text{curl } u \cdot \text{curl } v \, dx + \int_{\Gamma} (\text{curl } w) \times u \cdot v \, dx - \int_{\Omega} (\text{curl } b) \times d \cdot v \, dx + \int_{\Omega} \text{curl } b \cdot \text{curl } \Psi \, dx \\
+ \int_{\Omega} (\text{curl } \Psi) \times d \cdot u \, dx = \langle f, v \rangle_{\Omega, 2} + \langle g, \Psi \rangle_{\Omega, 2} - \langle P_0, v \cdot n \rangle_{\Gamma_i} - \sum_{i=1}^l (P_0 + c_i, v \cdot n)_{\Gamma_i},
$$

(4.6)
and \( c = (c_1, \ldots, c_I) \) satisfying for \( 1 \leq i \leq I \):

\[
    c_i = \langle f, \nabla q_i^N \rangle_{\Omega_{h,2}} - \langle P_0, \nabla q_i^N \cdot n \rangle_{\Gamma} - \int_{\Omega} (\text{curl} \, w) \times u \cdot \nabla q_i^N \, dx + \int_{\Omega} (\text{curl} \, b) \times d \cdot \nabla q_i^N \, dx,
\]

where \( \langle \cdot, \cdot \rangle_{\Gamma} \) denotes the duality product between \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \).

**Proof.** Using the same arguments as in [6, Lemma 5.5], we can prove that \( \mathcal{D}_p(\Omega) \times \mathcal{D}(\Omega) \) is dense in the space

\[
    \mathcal{E}(\Omega) = \{(u, P) \in H^1_p(\Omega) \times L^2(\Omega); \ -\Delta u + \nabla P \in [H^1_0(\text{curl}, \Omega)]'\}.
\]

Moreover, we have the following Green formula: For any \( (u, P) \in \mathcal{E}(\Omega) \) and \( \varphi \in H^1_0(\Omega) \) with \( \varphi \times n = 0 \) on \( \Gamma \):

\[
    \langle -\Delta u + \nabla P, \varphi \rangle_{\Omega_{h,2}} = \int_{\Omega} \text{curl} \, u \cdot \text{curl} \, \varphi \, dx + \langle P, \varphi \cdot n \rangle_{\Gamma}.
\]

Using Green formula (4.8), we deduce that any \( (u, b, P, c) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I \) satisfying (4.2) also solves (4.6). It remains to recover the relation (4.7). Let us take \( v \in H^1_0(\Omega) \) with \( v \times n = 0 \) on \( \Gamma \) and set:

\[
    v_0 = v - \sum_{i=1}^I (v \cdot n, 1)_{\Gamma_i} \nabla q_i^N
\]

Observe that \( v_0 \in H^1_0(\Omega) \), \( v_0 \times n = 0 \) on \( \Gamma \) and due to the properties of \( q_i^N \), we have for all \( 1 \leq i \leq I \), \( \langle v_0 \cdot n, 1 \rangle_{\Gamma_i} = 0 \). Then \( v_0 \) belongs to \( V_N(\Omega) \). Multiplying the first equation on the left of the problem (4.2) with \( v = v_0 + \sum_{i=1}^I (v \cdot n, 1)_{\Gamma_i} \nabla q_i^N \), integrating by parts in \( \Omega \), we obtain

\[
    \int_{\Omega} \text{curl} \, u \cdot \text{curl} \, v_0 \, dx + \int_{\Omega} (\text{curl} \, w) \times u \cdot v_0 \, dx - \int_{\Omega} (\text{curl} \, b) \times d \cdot v_0 \, dx - \langle f, v \rangle_{\Omega} + \langle P_0, v \cdot n \rangle_{\Gamma} = 0
\]

Comparing with the variational formulation (4.6) for the test function \( (v_0, 0) \), we obtain for all \( 1 \leq i \leq I \):

\[
    \sum_{i=1}^I c_i (v \cdot n, 1)_{\Gamma_i} = \sum_{i=1}^I (v \cdot n, 1)_{\Gamma_i} \int_{\Omega} (\text{curl} \, w) \times u \cdot \nabla q_i^N \, dx + \int_{\Omega} (\text{curl} \, b) \times d \cdot \nabla q_i^N \, dx
\]

Finally, taking \( v = \nabla q_i^N \), due to the properties of \( q_i^N \) in (3.2), we obtain the relation (4.7) for all \( 1 \leq j \leq I \).

Conversely, let \( (u, b) \in V_N(\Omega) \times \nabla V_N(\Omega) \) be a solution of (4.6) and \( c = (c_1, \ldots, c_I) \) satisfying (4.7). We want to show it implies (i). We note that \( \mathcal{D}_p(\Omega) \) is not a subspace of \( V_N(\Omega) \), so it is not possible to prove directly that (4.6)-(4.7) implies (i). In particular, we can not apply the De Rham’s lemma to recover the pressure. As a consequence, we need to extend (4.6) for all divergence free functions \( (v, \Psi) \in X_N(\Omega) \times X_N(\Omega) \). For this purpose, let \( (v, \Psi) \in X_N(\Omega) \times X_N(\Omega) \) and we consider the decomposition (4.9) for \( v \) to obtain \( v_0 \in V_N(\Omega) \). Similarly, we set

\[
    \Psi_0 = \Psi - \sum_{i=1}^I (\Psi \cdot n, 1)_{\Gamma_i} \nabla q_i^N,
\]
which implies that \( \Psi_0 \) is a function of \( V_N(\Omega) \). Replacing in (4.6), we obtain:

\[
\int_\Omega \text{curl } u \cdot \text{curl } v \, dx + \int_\Omega (\text{curl } w) \times u \cdot v \, dx - \int_\Omega (\text{curl } b) \times d \cdot v \, dx + \int_\Omega \text{curl } b \cdot \text{curl } \Psi \, dx \\
+ \int_\Omega (\text{curl } \Psi) \times d \cdot u \, dx - (f, v)_{\Omega} - (g, \Psi)_{\Omega} + \langle P_0, v \cdot n \rangle_{\Gamma_0} + \sum_{i=1}^{I} (P_0 + c_i, v \cdot n)_{\Gamma_i}
\]

\[
= \int_\Omega (\text{curl } w) \times u \cdot \nabla q_i^N \, dx + \int_\Omega (\text{curl } b) \times d \cdot \nabla q_i^N \, dx - (f, \nabla q_i^N)_{\Omega} + (P_0, \nabla q_i^N \cdot n)_{\Gamma_i}
\]

where we have used the fact that for all \( 1 \leq i \leq I \): \( \sum_{j=1}^{I} c_j (\nabla q_i^N \cdot n, 1)_{\Gamma_j} = c_i \). Note that the compatibility condition (4.4) implies that \( (g, \nabla q_i^N)_{\Omega} = 0 \) for all \( 1 \leq i \leq I \). Thus, the right hand side of the above relation is equal to zero and then for any \( (\mathbf{v}, \Psi) \in X_N(\Omega) \times X_N(\Omega) \), we have

\[
\int_\Omega \text{curl } u \cdot \text{curl } v \, dx + \int_\Omega (\text{curl } w) \times u \cdot v \, dx - \int_\Omega (\text{curl } b) \times d \cdot v \, dx + \int_\Omega \text{curl } b \cdot \text{curl } \Psi \, dx \\
+ \int_\Omega (\text{curl } \Psi) \times d \cdot u \, dx - (f, v)_{\Omega} - (g, \Psi)_{\Omega} - (P_0, \nabla q_i^N \cdot n)_{\Gamma_i} - \sum_{i=1}^{I} (P_0 + c_i, v \cdot n)_{\Gamma_i} = 0.
\]

(4.11)

That means that problem (4.11) and (4.6) are equivalent. So, in the sequel, we will prove that problem (4.11) implies (i).

Choosing \( (\mathbf{v}, \mathbf{0}) \) with \( \mathbf{v} \in \mathcal{D}_s(\Omega) \) as a test function in (4.11), we have

\[
\langle -\Delta u + (\text{curl } w) \times u - (\text{curl } b) \times d - f, \mathbf{v} \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0
\]

So by De Rham’s theorem, there exists a distribution \( P \in \mathcal{D}'(\Omega) \), defined uniquely up to an additive constant such that

\[
-\Delta u + (\text{curl } w) \times u - (\text{curl } b) \times d - f = -\nabla P \quad \text{in } \Omega.
\]

(4.12)

Since \( u \) and \( b \) belong to \( H^1(\Omega) \hookrightarrow L^6(\Omega) \), the terms \( (\text{curl } w) \times u \) and \( (\text{curl } b) \times d \) belong to \( L^\infty(\Omega) \hookrightarrow H^{-1}(\Omega) \). As \( f \in [H^{\infty}(\Omega)]' \hookrightarrow H^{-1}(\Omega) \), we deduce that \( \nabla P \in H^{-1}(\Omega) \) and then \( P \in L^2(\Omega) \) with a trace in \( H^{-\frac{1}{2}}(\Gamma) \) (we refer to [4]). Next, choosing \( (\mathbf{0}, \Psi) \) with \( \Psi \in \mathcal{D}_s(\Omega) \) in (4.11), we have

\[
\langle \text{curl } \text{curl } b - \text{curl}(u \times d) - g, \Psi \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0
\]

Then, applying [20, Lemma 2.2], we have \( \chi \in L^2(\Omega) \) defined uniquely up to an additive constant such that

\[
\text{curl } \text{curl } b - \text{curl}(u \times d) - g = \nabla \chi \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma
\]

We note that the trace of \( \chi \) is well defined and belongs to \( H^{-1/2}(\Gamma) \). Taking the divergence of the above equation, the function \( \chi \) is solution of the harmonic problem

\[
\Delta \chi = 0 \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma.
\]

So, we deduce that \( \chi = 0 \) in \( \Omega \) which gives the second equation in (4.2). Moreover, by the fact that \( u \) and \( b \) belong to the space \( V_N(\Omega) \), we have \( \text{div } u = \text{div } b = 0 \) in \( \Omega \) and \( u \times n = b \times n = 0 \) on \( \Gamma \).

It remains to show the boundary conditions on the pressure. Multiplying equation (4.12) by \( v \in X_N(\Omega) \), using the decomposition (4.9) and integrating on \( \Omega \), we obtain

\[
\int_\Omega \text{curl } u \cdot \text{curl } v_0 \, dx + \int_\Omega (\text{curl } w) \times u \cdot v_0 \, dx - \int_\Omega (\text{curl } b) \times d \cdot v_0 \, dx - (f, v_0)_{\Omega} + \langle P, v_0 \cdot n \rangle_{\Gamma_i}
\]

\[
= \int_\Omega (\text{curl } w) \times u \cdot \nabla q_i^N \, dx + \int_\Omega (\text{curl } b) \times d \cdot \nabla q_i^N \, dx + (f, \nabla q_i^N)_{\Omega} - \langle P, \nabla q_i^N \cdot n \rangle_{\Gamma_i}
\]
Taking \((\mathbf{v}, 0)\) test function in (4.6), we have:
\[
\int_\Omega \mathbf{curl} \mathbf{u} \cdot \mathbf{curl} v_0 \, dx + \int_\Omega (\mathbf{curl} \mathbf{u}) \times \mathbf{u} \cdot v_0 \, dx - \int_\Omega (\mathbf{curl} \mathbf{b}) \times d \cdot v_0 \, dx - \langle f, v_0 \rangle_\Omega + \langle P_0, v_0 \cdot n \rangle_{\Gamma_0} \\
+ \sum_{i=1}^I (P_0 + c_i, v_0 \cdot n)_{\Gamma_i} = \sum_{i=1}^I (\mathbf{v} \cdot n, 1)_{\Gamma_i} \left[ - \int_\Omega (\mathbf{curl} \mathbf{u}) \times \mathbf{u} \cdot \nabla q_i^N \, dx + \int_\Omega (\mathbf{curl} \mathbf{b}) \times d \cdot \nabla q_i^N \, dx \right] + \langle f, \nabla q_i^N \rangle_{\Omega} - \langle P_0, \nabla q_i^N \cdot n \rangle_{\Gamma_0} - \sum_{j=1}^I (P_0 + c_j, \nabla q_i^N \cdot n)_{\Gamma_j}
\]

Subtracting both equations and using again the decomposition (4.9), we obtain:
\[
\langle P, v_0 \cdot n \rangle_{\Gamma} + \sum_{i=1}^I (\mathbf{v} \cdot n, 1)_{\Gamma_i} \langle P, \nabla q_i^N \cdot n \rangle_{\Gamma_i}
\]
\[
= \langle P_0, v_0 \cdot n \rangle_{\Gamma_0} + \sum_{i=1}^I (P_0 + c_i, v_0 \cdot n)_{\Gamma_i} + \langle P, \mathbf{v} \cdot n, 1 \rangle_{\Gamma_0} \left[ \langle P_0, \nabla q_i^N \cdot n \rangle_{\Gamma_0} + \sum_{j=1}^I (P_0 + c_j, \nabla q_i^N \cdot n)_{\Gamma_j} \right]
\]
Since \((v_0 \cdot n, 1)_{\Gamma_i} = 0, 1 \leq i \leq I\), we have
\[
\langle P, \mathbf{v} \cdot n \rangle_{\Gamma} = \langle P_0, \mathbf{v} \cdot n \rangle_{\Gamma} + \sum_{i=1}^I (c_i, \mathbf{v} \cdot n)_{\Gamma_i} = \langle P_0, \mathbf{v} \cdot n \rangle_{\Gamma_0} + \sum_{i=1}^I (P_0 + c_i, \mathbf{v} \cdot n)_{\Gamma_i}
\]
Next, the argument to deduce that \(P = P_0\) on \(\Gamma_0\) and \(P = P_0 + c_j\) on \(\Gamma_j\) is very similar to that of [7, Proposition 3.7], hence we omit it.

**Remark 4.1.** (i) Note that the compatibility condition (4.4) is necessary. Indeed, if we choose \(\mathbf{v} = 0\) and \(\psi = \nabla q_i^N\) in (4.6), we have \(\langle g, \nabla q_i^N \rangle_{\Omega} = 0, 1 \leq i \leq I\). Observe that since \(\Omega\) is of class \(C^{1,1}\), the functions \(q_i^N\) belong to \(H^2(\Omega)\) and then the vectors \(\nabla q_i^N\) belong to \(H^{0,2}(\mathbf{curl}, \Omega)\). From the characterization (4.3), this condition is actually written as \(\int_\Omega F \cdot \nabla q_i^N \, dx = 0, 1 \leq i \leq I\). In the case where \(\Omega\) is simply connected, the compatibility condition (4.4) is not necessary to solve (4.2) because the kernel \(K_N^2(\Omega) = \{0\}\).

(ii) If \(g\) is the \(\mathbf{curl}\) of an element \(\xi \in L^2(\Omega)\), then \(g\) is still an element of \([H_0^{0,2}(\mathbf{curl}, \Omega)]'\). Moreover, since \(\text{div} g = 0\) in \(\Omega\), it always satisfies the compatibility condition (4.4).

We now prove the solvability of the problem (4.6).

**Theorem 4.2.** Let \(\Omega\) be \(C^{1,1}\) and we suppose \(h = 0\). Let
\[
f, g \in \left[H_0^{0,2}(\mathbf{curl}, \Omega)\right]' \quad \text{and} \quad P_0 \in H^{-\frac{1}{2}}(\Gamma)
\]
with the compatibility conditions (4.4)-(4.5). Then the problem (4.2) has a unique weak solution \((\mathbf{u}, b, P, e) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^I\) which satisfies the estimates:
\[
\|u\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)} \leq C(\|f\|_{H_0^{0,2}(\mathbf{curl}, \Omega)'} + \|g\|_{H_0^{0,2}(\mathbf{curl}, \Omega)'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)})
\]
\[
\|P\|_{L^2(\Omega)} \leq C(1 + \|\mathbf{curl} \mathbf{w}\|_{L^3(\Omega)} + \|d\|_{L^3(\Omega)} + \|f\|_{H_0^{0,2}(\mathbf{curl}, \Omega)'} + \|g\|_{H_0^{0,2}(\mathbf{curl}, \Omega)'} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)})
\]
Moreover, if \(\Omega\) is \(C^{2,1}\), \(f, g \in L^{6/5}(\Omega)\) and \(P_0 \in W^{1/6,6/5}(\Gamma)\), then \((\mathbf{u}, b, P) \in W^{2,\frac{6}{5}}(\Omega) \times W^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega)\) and we have the following estimate:
\[
\|u\|_{W^{2,\frac{6}{5}}(\Omega)} + \|b\|_{W^{2,\frac{6}{5}}(\Omega)} + \|P\|_{W^{1,\frac{6}{5}}(\Omega)} \leq C(1 + \|\mathbf{curl} \mathbf{w}\|_{L^3(\Omega)} + \|d\|_{L^3(\Omega)})
\times (\|f\|_{L^{\frac{6}{5}}(\Omega)} + \|g\|_{L^{\frac{6}{5}}(\Omega)} + \|P_0\|_{W^{1,\frac{6}{5}}(\Gamma)})
\]
(4.15)
Proof. We know, according to Proposition 4.1, that the linearized problem (4.2) is equivalent to (4.6)-(4.7). The existence and uniqueness of weak solution \( (u, b) \in H^1(\Omega) \times H^1(\Omega) \) follow from Lax-Milgram theorem. Let us define the bilinear continuous forms \( a : Z_N(\Omega) \times Z_N(\Omega) \to \mathbb{R} \) and \( a_{\omega,d} : Z_N(\Omega) \times Z_N(\Omega) \to \mathbb{R} \) as follows:

\[
\begin{align*}
  a((u, b), (v, \Psi)) &= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \nabla b \cdot \nabla \Psi \, dx \\
  a_{\omega,d}((u, b), (v, \Psi)) &= \int_\Omega (\nabla w) \times u \cdot v \, dx + \int_\Omega (\nabla \Psi) \times d \cdot u \, dx - \int_\Omega (\nabla b) \times d \cdot v \, dx
\end{align*}
\]

(4.16)

where \( Z_N(\Omega) = V_N(\Omega) \times V_N(\Omega) \) equipped with the product norm

\[
\| (v, \Psi) \|^2_{Z_N(\Omega)} = \| v \|^2_{H^1(\Omega)} + \| \Psi \|^2_{H^1(\Omega)}.
\]

(4.17)

Next, we introduce the linear form \( L : Z_N(\Omega) \to \mathbb{R} \) defined as follows

\[
L(v, \Psi) = \langle f, v \rangle_{\Omega_{\omega,2}} + \langle g, \Psi \rangle_{\Omega_{\omega,2}} - \sum_{i=1}^I (P_i + c_i, v \cdot n)_{\Gamma_i}
\]

So, the variational formulation (4.6) can be rewritten as: for any \((v, \Psi) \in Z_N(\Omega)\)

\[
A((u, b), (v, \Psi)) = a((u, b), (v, \Psi)) + a_{\omega,d}((u, b), (v, \Psi)) = L(v, \Psi)
\]

(4.18)

Since \( (\nabla w) \times u = 0 \), then we have \( a_{\omega,d}((u, b), (u, b)) = 0 \) for all \((u, b) \in Z_N(\Omega)\).

Since \( v \) and \( \Psi \) belong to \( V_N(\Omega) \), we have from \([6, Corollary 3.2.]\) that the application \( v \mapsto \| \nabla v \|_{L^2(\Omega)} \) (respectively \( v \mapsto |\nabla v| \|_{L^2(\Omega)} \)) is a norm on \( V_N(\Omega) \) equivalent to the norm \( \| v \|_{H^1(\Omega)} \) (respectively \( \| v \|_{H^1(\Omega)} \)). As a consequence,

\[
A((u, b), (v, \Psi)) = |a((v, \Psi), (v, \Psi))| = \| \nabla v \|^2_{L^2(\Omega)} + \| \nabla \Psi \|^2_{L^2(\Omega)}
\]

(4.19)

where \( C_P \) is the constant given in (3.6). This shows that the bilinear form \( A(\cdot, \cdot) \) is coercive on \( Z_N(\Omega) \). Moreover, applying Cauchy-Schwarz inequality, we have:

\[
|a((u, b), (v, \Psi))| \leq \| \nabla u \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} + \| \nabla b \|_{L^2(\Omega)} \| \nabla \Psi \|_{L^2(\Omega)} \leq C \| (u, b) \|_{Z_N(\Omega)} \| (v, \Psi) \|_{Z_N(\Omega)}
\]

(4.20)

Now, using Hölder inequality, we have

\[
|a_{\omega,d}((u, b), (v, \Psi))| \leq \| \nabla w \|_{L^2(\Omega)} \| u \|_{L^6(\Omega)} \| v \|_{L^6(\Omega)} + \| \nabla \Psi \|_{L^2(\Omega)} \| u \|_{L^6(\Omega)} \| d \|_{L^6(\Omega)}
\]

Now, using again the equivalence of norms \( \| v \|_{V_N(\Omega)} \) and \( \| \Psi \|_{H^1(\Omega)} \), we obtain for \( C_1 > 0 \) the constant of the embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \)

\[
|a_{\omega,d}((u, b), (v, \Psi))| \leq \left(C_1^2 \| \nabla w \|_{L^2(\Omega)} + C_1 \| d \|_{L^6(\Omega)} \right) \| (u, b) \|_{Z_N(\Omega)} \| (v, \Psi) \|_{Z_N(\Omega)}
\]

(4.21)

From (4.20) and (4.21), we can deduce that the form \( A(\cdot, \cdot) \) is continuous. Using similar arguments, we can verify that the right hand side of (4.18) defines an element in the dual space of \( Z_N(\Omega) \). Thus, by Lax-Milgram lemma, there exists a unique \((u, b) \in Z_N(\Omega)\) satisfying (4.18). So, due to Theorem 3.1, we obtain the existence of a unique weak solution \((u, b) \in H^1(\Omega) \times H^1(\Omega)\). Using (4.19), the variational formulation and trace theorem, we obtain the estimate (4.13). The existence of the pressure follows from De Rham’s theorem. Moreover for the pressure estimate, we can write

\[
\| P \|_{L^2(\Omega)} \leq C \| \nabla P \|_{H^{-1}(\Omega)} \leq C \| f \|_{H^{-1}(\Omega)} + \| \Delta u \|_{H^{-1}(\Omega)} + \| (\nabla w) \times u \|_{H^{-1}(\Omega)} + \| (\nabla b) \times d \|_{H^{-1}(\Omega)}
\]

We know that \( \| f \|_{H^{-1}(\Omega)} \leq C \| f \|_{H^{6/2}(\Omega)} \) and \( \| \Delta u \|_{H^{-1}(\Omega)} \leq C \| u \|_{H^1(\Omega)} \). For the two remaining terms, we proceed as follows

\[
\| (\nabla w) \times u \|_{H^{-1}(\Omega)} \leq C \| (\nabla w) \times u \|_{L^{6/5}(\Omega)} \leq C \| \nabla w \|_{L^{3/2}(\Omega)} \| u \|_{L^6(\Omega)}.
\]
so, we have

\[ \| (\text{curl } u) \times u \|_{H^{-1}(\Omega)} \leq CC_1 \| (\text{curl } u) \|_{L^{3/2}(\Omega)} \| u \|_{H^1(\Omega)}. \]

Proceeding similarly, we get

\[ \| (\text{curl } b) \times d \|_{H^{-1}(\Omega)} \leq C \| d \|_{L^3(\Omega)} \| b \|_{H^1(\Omega)}. \]

Hence, using the above estimates together with the estimate (4.13), we deduce the pressure estimate (4.14).

Now, if \( f, g \in L^6(\Omega) \) and \( P_0 \in W^{1,6} \cap \mathcal{H}^1(\Omega) \), then we already know that \((u, b, P) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega)\) is solution of (4.2). We deduce that \((\text{curl } u) \times u + (\text{curl } b) \times d \) belongs to \( L^{6/5}(\Omega) \). Similarly, we have \( \text{curl}(u \times d) = (d \cdot \nabla)u - (u \cdot \nabla)d \) belongs to \( L^{6/5}(\Omega) \). Observe that \((u, P, c)\) is solution of the following Stokes problem

\[
\begin{aligned}
-\Delta u + \nabla P &= F \quad \text{and} \quad \text{div } u = 0 \quad \text{in } \Omega, \\
u \times n &= 0 \quad \text{on } \Gamma \quad \text{and} \quad P = P_0 \quad \text{on } \Gamma_0, \\
(u \cdot n, 1)_{\Gamma_i} &= 0, \quad \forall 1 \leq i \leq I,
\end{aligned}
\]

with \( F = f - (\text{curl } u) \times u + (\text{curl } b) \times d \) in \( L^{6/5}(\Omega) \). Thanks to the regularity of Stokes problem \((S_N)\) (see Proposition 3.2), \((u, P)\) belongs to \( W^{2,6/5}(\Omega) \times W^{1,6/5}(\Omega) \) with the corresponding estimate. Next, since \( b \) is a solution of the following elliptic problem

\[
\begin{aligned}
\text{curl } b &= G \quad \text{and} \quad \text{div } b = 0 \quad \text{in } \Omega, \\
b \times n &= 0 \quad \text{on } \Gamma, \\
(b \cdot n, 1)_{\Gamma_i} &= 0, \quad \forall 1 \leq i \leq I,
\end{aligned}
\]

with \( G = g + \text{curl}(u \times d) \) in \( L^{6/5}(\Omega) \), we can deduce from Theorem 3.3 that \( b \) belongs to \( W^{2,6/5}(\Omega) \). The estimate (4.15) then follows from the regularity estimates of the above Stokes problem on \((u, P)\) and elliptic problem on \( b \).

\( \square \)

5 The linearized MHD system: \( L^p \)-theory

After the study of weak solutions in the case of Hilbert spaces, we are interested in the study of weak and strong solutions in \( L^p \)-theory for the linearized system (4.2). We begin by studying strong solutions. If \( p \geq 6/5 \), it follows that \( L^p(\Omega) \hookrightarrow L^{6/5}(\Omega) \), \( W^{1-1/p, p}(\Omega) \hookrightarrow W^{1,6/5}(\Omega) \). Then, due to Theorem 4.2, we have \((u, b) \in W^{2,6/5}(\Omega) \times W^{2,6/5}(\Omega)\). In the next subsection we will prove that this solution belongs to \( W^{2,p}(\Omega) \times W^{2,p}(\Omega)\) for any \( p > 6/5 \).

5.1 Strong solution in \( W^{2,p}(\Omega) \) with \( p \geq 6/5 \)

The aim of this section is to give an answer to the question of the existence of a regular solution \((u, b, P) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega)\) for the linearized MHD problem (4.2). When \( p < 3 \), we have the embedding \( W^{2,p}(\Omega) \hookrightarrow W^{1,p'}(\Omega) \) with \( \frac{1}{p'} = \frac{1}{p} - \frac{1}{2} \). Then, supposing \( d \in W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega) \) implies that the term \((\text{curl } b) \times d \) belongs to \( L^p(\Omega) \). If \( p < 3/2 \), \( W^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega) \) with \( \frac{1}{p'} = \frac{1}{p} - \frac{1}{4} \) and then supposing \( \text{curl } u \in L^{3/2}(\Omega) \) implies that the term \((\text{curl } u) \times u \) belongs to \( L^p(\Omega) \). So, we can use the well-known regularity of the Stokes problem \((S_N)\) in order to prove the regularity \( W^{2,p}(\Omega) \times W^{1,p}(\Omega) \) with \( p < 3/2 \) for \((u, P)\) since the right hand side \( f - (\text{curl } u) \times u + (\text{curl } b) \times d \) belongs to \( L^p(\Omega) \). Similarly, the term \( \text{curl}(u \times d) = (d \cdot \nabla)u - (u \cdot \nabla)d \) belongs to \( L^p(\Omega) \) and we can use the regularity of the elliptic problem \((E_N)\) to prove the regularity \( W^{2,p}(\Omega) \) with \( p < 3/2 \) for \( b \). Now, if \( 3/2 \leq p < 3 \), the terms \((\text{curl } b) \times d \) and \((d \cdot \nabla)u \) still belong to \( L^p(\Omega) \) but the situation is different for the terms \((\text{curl } u) \times u \) and \((u \cdot \nabla)d \) if \( \text{curl } u \) and \( \nabla d \) belong only to \( L^{3/2}(\Omega) \). Indeed, we must suppose \( \text{curl } u \in L^s(\Omega) \) and \( \nabla d \in L^s(\Omega) \) with

\[
s = \frac{3}{2} \quad \text{if } p < \frac{3}{2}, \quad s > \frac{3}{2} \quad \text{if } p = \frac{3}{2} \quad \text{and} \quad s = p \quad \text{if } p > \frac{3}{2}.
\]
5.1 Strong solution in $W^{2,p}(\Omega)$ with $p \geq 6/5$

Now, if $p \geq 3$, the problem arises for the terms $(\text{curl} \, b) \times d$ and $(d \cdot \nabla)u$ if we suppose only $d$ in $L^2(\Omega)$. So, we must suppose that $d \in L^p(\Omega)$ with

$$s' = 3 \quad \text{if } p < 3, \quad s' > 3 \quad \text{if } p = 3 \quad \text{and} \quad s' = p \quad \text{if } p > 3.$$  

So, to conserve the assumptions $\text{curl} \, w \in L^{3/2}(\Omega)$ and $d \in W^{1,3/2}(\Omega)$ and prove strong solutions $(u, b, P)$ in $W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega)$, we first assume that $w$ and $d$ are more regular and belong to $D(\Omega)$. We will then prove a priori estimates allowing to remove this latter regularity. We refer to [5, Theorem 2.4] for a similar proof for the Oseen problem. The details are given in the following regularity result in a solenoidal framework.

**Theorem 5.1.** Let $\Omega$ be $C^{2,1}$ and $p \geq 6/5$. Assume that $h = 0$, and let $f, g, w, d$ and $P_0$ satisfying (4.5),

$$f \in L^p(\Omega), \quad g \in L^p(\Omega), \quad \text{curl} \, w \in L^{3/2}(\Omega), \quad d \in W^{\alpha,3/2}(\Omega), \quad \text{and} \quad P_0 \in W^{1-\frac{3}{p},p}(\Gamma)$$

with the compatibility condition

$$\forall v \in K_N^p(\Omega), \quad \int_{\Omega} g \cdot v \, dx = 0. \quad (5.1)$$

Then, the weak solution $(u, b, P)$ of the problem (4.2) given by Theorem 4.2 belongs to $W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega)$ which also satisfies the estimate:

$$\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C(1 + \|\text{curl} \, w\|_{L^2(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)})$$

$$\times \left( \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1-\frac{3}{p},p}(\Gamma)} \right) \quad (5.2)$$

**Proof.** We prove it in two steps:

**First step:** We consider the case of $w \in D(\Omega)$ and $d \in D(\Omega)$. We know that for all $p \geq 6/5$ we have

$$L^p(\Omega) \hookrightarrow L^{6/5}(\Omega) \quad \text{and} \quad W^{1-1/p,p}(\Gamma) \hookrightarrow H^{1/2}(\Gamma).$$

Thanks to Theorem 4.2, there exists a unique solution $(u, b, P, c) \in W^{2,\frac{6}{5}}(\Omega) \times W^{2,\frac{6}{5}}(\Omega) \times W^{1,\frac{6}{5}}(\Omega) \times R^2$ verifying the estimates (4.13)-(4.14).

Since $u \in W^{2,\frac{6}{5}}(\Omega) \hookrightarrow L^6(\Omega)$ and $\text{curl} \, b \in W^{1,\frac{6}{5}}(\Omega) \hookrightarrow L^2(\Omega)$, it follows that $(\text{curl} \, w) \times u \in L^6(\Omega)$ and $(\text{curl} \, b) \times d \in L^2(\Omega)$. Note that $L^2(\Omega) \hookrightarrow L^p(\Omega)$ if $p \leq 2$, then we have three cases:

**Case $\frac{5}{3} < p \leq 2$:** Since $f - (\text{curl} \, w) \times u + (\text{curl} \, b) \times d \in L^p(\Omega)$, thanks to the existence of strong solutions for Stokes equations (see Theorem 3.2), we have that $(u, P) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, we have $g + \text{curl} \, (u \times d) \in L^p(\Omega)$. Thanks to the regularity of elliptic problem (see Theorem 3.3), we have that $b \in W^{2,p}(\Omega)$.

**Case $2 < p \leq 6$:** From the previous case, $(u, b, P) \in H^2(\Omega) \times H^{2/3}(\Omega) \times H^1(\Omega)$. Since

$$H^2(\Omega) \hookrightarrow W^{1,6}(\Omega) \hookrightarrow L^\infty(\Omega),$$

then $(\text{curl} \, w) \times u \in L^\infty(\Omega)$ and $(\text{curl} \, b) \times d \in L^6(\Omega)$. Hence we have that $f - (\text{curl} \, w) \times u + (\text{curl} \, b) \times d \in L^p(\Omega)$. Again, according to Theorem 3.2, it follows that $(u, P) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Moreover, we have that $g + \text{curl} \, (u \times d) \in L^6(\Omega)$. Thanks to Theorem 3.3, we have that $b \in W^{2,p}(\Omega)$.

**Case $p > 6$:** We know that $(u, b, P) \in W^{2,6}(\Omega) \times W^{2,6}(\Omega) \times W^{1,6}(\Omega)$. Since

$$W^{2,6}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$$

then $(\text{curl} \, w) \times u \in L^q(\Omega)$, $(\text{curl} \, b) \times d \in L^q(\Omega)$ and $\text{curl} \, (u \times d) \in L^q(\Omega)$ for any $q \geq 1$. Again, according to the regularity of Stokes and elliptic problems, we have $(u, b, P) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and we have the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C_{p,6} \left( \|f\|_{L^p(\Omega)} + \|\text{curl} \, w\|_{L^p(\Omega)} + \|\text{curl} \, b\|_{L^p(\Omega)} + \|P_0\|_{W^{1-\frac{3}{p},p}(\Gamma)} \right)$$

$$+ \sum_{i=1}^l |\alpha_i| + \|g\|_{L^p(\Omega)} + \|\text{curl} \, (u \times d)\|_{L^p(\Omega)}, \quad (5.3)$$
where \( C_{se} = \max(C_{s}, C_{e}) \) with \( C_{s} \) the constant given in (3.9) and \( C_{e} \) the constant given in (3.19).

To prove the estimate (5.2), we must bound the terms \( \|\text{curl} \, w \times u\|_{L^p(\Omega)}, \|\text{curl} \, b \times d\|_{L^p(\Omega)}, \|\text{curl} \, (u \times d)\|_{L^p(\Omega)} \) and \( \sum_{i=1}^t |c_i| \) in the right hand side of (5.3).

For this, let \( \epsilon > 0 \) and \( \rho_{\epsilon/2} \) the classical mollifier. We consider \( \tilde{y} = \text{curl} \, w \) and \( \tilde{d} \) the extensions by 0 of \( y \) and \( d \) to \( \mathbb{R}^3 \), respectively. We decompose \( \text{curl} \, w \) and \( d \):

\[
\text{curl} \, w = y_1^* + y_2^* \quad \text{where} \quad y_1^* = \text{curl} \, w \ast \rho_{\epsilon/2} \quad \text{and} \quad y_2^* = \text{curl} \, w - y_1^*,
\]

\[
d = d_1^* + d_2^* \quad \text{where} \quad d_1^* = \tilde{d} \ast \rho_{\epsilon/2} \quad \text{and} \quad d_2^* = \tilde{d} - \tilde{d} \ast \rho_{\epsilon/2}.
\]

(i) **Estimate of the term** \( \|\text{curl} \, w \times u\|_{L^p(\Omega)} \). First, we look for the estimate depending on \( y_2^* \). Observe that \( W^{2,p}(\Omega) \hookrightarrow L^m(\Omega) \) with

\[
\frac{1}{m} = \frac{1}{p} - \frac{2}{3} \quad \text{if} \quad p < 3/2, \quad m = \frac{3s}{2s - 3} \quad \in [1, \infty[ \quad \text{if} \quad p = 3/2 \quad \text{and} \quad m = \infty \quad \text{if} \quad p > 3/2.
\]

Using the Hölder inequality and Sobolev embedding, we have

\[
\|y_2^* \times u\|_{L^p(\Omega)} \leq \|y_2^*\|_{L^r(\Omega)} \|u\|_{L^m(\Omega)} \leq C \|y_2^*\|_{L^r(\Omega)} \|u\|_{W^{2,p}(\Omega)}
\]

where \( \frac{1}{r} = \frac{1}{m} + \frac{1}{s} \) and \( s \) the real number defined as:

\[
s = \frac{3}{2} \quad \text{if} \quad p < \frac{3}{2}, \quad s = \frac{3}{2} \quad \text{if} \quad p = \frac{3}{2} \quad \text{and} \quad s = p \quad \text{if} \quad p > \frac{3}{2}.
\]

Moreover, we have

\[
\|y_2^*\|_{L^r(\Omega)} = \|\text{curl} \, w - \text{curl} \, w \ast \rho_{\epsilon/2}\|_{L^r(\Omega)} \leq \epsilon.
\]

Then, it follows that

\[
\|y_2^* \times u\|_{L^p(\Omega)} \leq C\|u\|_{W^{2,p}(\Omega)}.
\]

To get the estimate depending on \( y_1^* \), we consider two steps (similar to [7, Theorem 3.5]):

- **Case** \( \frac{6}{5} \leq p \leq 6 \): there exists \( q \in \left[ \frac{3}{2}, \infty \right) \) such that \( \frac{1}{q} = \frac{1}{q} + \frac{1}{6} \). By Hölder inequality, we have

\[
\|y_1^* \times u\|_{L^p(\Omega)} \leq \|y_1^*\|_{L^q(\Omega)} \|u\|_{L^6(\Omega)}.
\]

Let \( t \in [1, 3] \) such that \( 1 + \frac{1}{q} = \frac{2}{q} + \frac{1}{t} \), we obtain

\[
\|y_1^* \times u\|_{L^p(\Omega)} \leq \|\text{curl} \, w\|_{L^t(\Omega)} \|\rho_{\epsilon}\|_{L^t(\Omega)} \|u\|_{L^6(\Omega)} \leq C_{\epsilon} \|\text{curl} \, w\|_{L^t(\Omega)} \|u\|_{L^6(\Omega)},
\]

where \( C_{\epsilon} \) is the constant absorbing the norm of the mollifier. Since \( H^1(\Omega) \hookrightarrow L^6(\Omega) \), it follows from (4.13) that

\[
\|y_1^* \times u\|_{L^p(\Omega)} \leq C_2 C_{\epsilon} \|\text{curl} \, w\|_{L^t(\Omega)} \left( \|f\|_{H^{0,2}_{\text{curl}}(\Omega)} + \|g\|_{H^{0,2}_{\text{curl}}(\Omega)} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)} \right),
\]

where \( C_2 \) is the constant of the Sobolev embedding \( H^1(\Omega) \hookrightarrow L^6(\Omega) \). Since, \( p \geq \frac{6}{5} \), we deduce that

\[
\|y_1^* \times u\|_{L^p(\Omega)} \leq C_3 C_{\epsilon} \|\text{curl} \, w\|_{L^t(\Omega)} \left( \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1,\frac{3}{2}}(\Gamma)} \right),
\]

where \( C_3 \) is a constant which depends on \( C_2 \), \( L^p(\Omega) \hookrightarrow [H^{0,2}_{\text{curl}}(\Omega)]' \) and \( W^{1,\frac{3}{2}}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma) \).

- **Case** \( p > 6 \): we know that the embedding

\[
W^{2,p}(\Omega) \hookrightarrow W^{1,m}(\Omega),
\]

and
is compact for any \( m \in [1, p^*] \) if \( p < 3 \), for any \( m \in [1, \infty] \) if \( p = 3 \) and for \( m \in [1, \infty] \) if \( p > 3 \).

We choose the exponent \( m \) such that \( 6 < m < +\infty \). So, we have:

\[
W^{2,p}(\Omega) \hookrightarrow W^{1,m}(\Omega) \hookrightarrow L^6(\Omega).
\]

Hence, for any \( \varepsilon' > 0 \), we know that there exists a constant \( C_\varepsilon' \) such that the following interpolation inequality holds:

\[
\| u \|_{W^{1,m}(\Omega)} \leq \varepsilon' \| u \|_{W^{2,p}(\Omega)} + C_\varepsilon' \| u \|_{H^1(\Omega)}
\]  

(5.9)

For \( t > 2 \) such that \( 1 + \frac{1}{p} = \frac{2}{3} + \frac{1}{t} \), we have

\[
\| y'_1 \times u \|_{L^p(\Omega)} \leq C \| y'_1 \|_{L^p(\Omega)} \| u \|_{W^{1,m}(\Omega)}
\]

\[
\leq C \| \text{curl } u \|_{L^\frac{2}{3}(\Omega)} \| \rho_{t/2} \|_{L^t(\mathbb{R}^3)} \| u \|_{W^{1,m}(\Omega)}.
\]

Using (5.9), we obtain

\[
\| y'_1 \times u \|_{L^p(\Omega)} \leq CC_\varepsilon \| \text{curl } u \|_{L^\frac{2}{3}(\Omega)} (\varepsilon' \| u \|_{W^{2,p}(\Omega)} + C_\varepsilon' \| u \|_{H^1(\Omega)})
\]  

(5.10)

Thus, choosing \( \varepsilon' > 0 \) small enough, we can deduce from (5.8) or (5.10) that

\[
\| y'_1 \times u \|_{L^p(\Omega)} \leq CC_\varepsilon \| \text{curl } u \|_{L^\frac{2}{3}(\Omega)} (\varepsilon' \| u \|_{W^{2,p}(\Omega)} + C_\varepsilon' \| u \|_{H^1(\Omega)}).
\]  

(5.11)

(ii) Estimate of the term \( \| (\text{curl } b) \times d \|_{L^p(\Omega)} \): Using the decomposition (5.5), as previously, we have for the part \( d^2_2 \):

\[
\| d^2_2 \|_{L^{s'}(\Omega)} \leq \| d - d \ast \rho_{t/2} \|_{L^{s'}(\Omega)} \leq \varepsilon.
\]  

(5.12)

Recall that \( W^{2,p}(\Omega) \hookrightarrow W^{1,k}(\Omega) \) for \( k = p^* = \frac{3p}{3-p} \) if \( p < 3 \), for any \( k \in [1, \infty] \) if \( p = 3 \) and \( k = \infty \) if \( p > 3 \).

Using the Hölder inequality and (5.12), we have

\[
\| (\text{curl } b) \times d^2_2 \|_{L^p(\Omega)} \leq \| \text{curl } b \|_{L^t(\Omega)} \| d^2_2 \|_{L^{s'}(\Omega)} \leq C_\varepsilon \| b \|_{W^{2,p}(\Omega)},
\]

where \( \frac{1}{p} = \frac{1}{t} + \frac{1}{s'} \) for \( s' \) given by:

\[
s' = 3 \quad \text{if } p < 3, \quad s' > 3 \quad \text{if } p = 3 \quad \text{and} \quad s' = p \quad \text{if } p > 3.
\]  

(5.14)

It remains to prove the estimate depending on \( d^1_1 \). We have three cases:

- **Case** \( p \leq 2 \): Using the Hölder inequality, we have

\[
\| (\text{curl } b) \times d^1_1 \|_{L^p(\Omega)} \leq \| d^1_1 \|_{L^t(\Omega)} \| \text{curl } b \|_{L^2(\Omega)},
\]

where \( \frac{1}{p} = \frac{1}{t} + \frac{1}{2} \). Let \( t \in [1, 3/2] \) such that \( 1 + \frac{1}{t} = \frac{3}{4} + \frac{1}{t} \), we obtain (since \( r \geq 3 \))

\[
\| (\text{curl } b) \times d^1_1 \|_{L^p(\Omega)} \leq \| d \|_{L^3(\Omega)} \| \rho_{t/2} \|_{L^t(\mathbb{R}^3)} \| \text{curl } b \|_{L^2(\Omega)}
\]

\[
\leq C_4 \| d \|_{L^3(\Omega)} \| b \|_{H^1(\Omega)}.
\]  

(5.15)

where \( C_4 \) is a constant which depends on \( L^p(\Omega) \hookrightarrow [H^{1,2}_0(\text{curl } \Omega)]' \) and \( W^{1,\frac{2}{3}}(\Gamma) \hookrightarrow H^{\frac{1}{2}}(\Gamma) \).

- **Case** \( 2 < p < 3 \): Assuming \( 2 < q < p^* \), from the relation

\[
W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow H^1(\Omega)
\]

we have for any \( \varepsilon' > 0 \), there exists a constant \( C_\varepsilon \) such that

\[
\| b \|_{W^{1,q}(\Omega)} \leq \varepsilon' \| b \|_{W^{2,p}(\Omega)} + C_\varepsilon \| b \|_{H^1(\Omega)}
\]  

(5.16)
Let $k$ be defined by $\frac{1}{p'} = \frac{1}{q} + \frac{1}{t}$ and $t \geq 1$ defined by $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{t}$. Thus, since $k > 3$, the following estimate holds:

$$
\|\text{curl } b \times d_i^\ast\|_{L^p(\Omega)} \leq \|d_i^\ast\|_{L^k(\Omega)} \|\text{curl } b\|_{L^q(\Omega)} \leq \|d\|_{L^3(\Omega)} \|\rho_{r/2}\|_{L^r(\Gamma)} \|\text{curl } b\|_{L^3(\Omega)}.
$$

Next using (5.16) yields,

$$
\|\text{curl } b \times d_i^\ast\|_{L^p(\Omega)} \leq C_k \|d\|_{L^3(\Omega)} (\epsilon' \|b\|_{W^{2,p}(\Omega)} + C_{\epsilon'} \|b\|_{H^1(\Omega)}).
$$

\section*{Case $p \geq 3$}

For $\frac{1}{p} = \frac{1}{q} + \frac{1}{s'}$ with $s'$ defined in (5.14), we have

$$
\|\text{curl } b \times d_i^\ast\|_{L^p(\Omega)} \leq \|d_i^\ast\|_{L^{s'}(\Omega)} \|\text{curl } b\|_{L^{s'}(\Omega)}.
$$

Let $t$ be defined by $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{t'}$. Thus, using (5.16) with $q = p_s$, we obtain:

$$
\|\text{curl } b \times d_i^\ast\|_{L^p(\Omega)} \leq C_k \|d\|_{L^3(\Omega)} (\epsilon' \|b\|_{W^{2,p}(\Omega)} + C_{\epsilon'} \|b\|_{H^1(\Omega)}).
$$

Choosing $\epsilon' > 0$ small enough, we deduce from (5.15), (5.17) or (5.18) that

$$
\|\text{curl } b \times d_i^\ast\|_{L^p(\Omega)} \leq C_k \|d\|_{L^3(\Omega)} (\epsilon' \|b\|_{W^{2,p}(\Omega)} + C_{\epsilon'} \|b\|_{H^1(\Omega)}).
$$

\section*{(iii) Estimate of the term $\|\text{curl } (u \times d)\|_{L^p(\Omega)}$}

Note that, since div $u = 0$ and div $d = 0$:

$$
\text{curl } (u \times d) = d \cdot \nabla u - u \cdot \nabla d.
$$

\section*{The term $\|d \cdot \nabla u\|_{L^p(\Omega)}$}

Using the decomposition (5.5) and exactly the same analysis as in (ii) for the term $(\text{curl } b) \times d$ with curl $b$ replaced by $\nabla u$, we obtain the following estimates:

$$
\|d_i^\ast \cdot \nabla u\|_{L^p(\Omega)} \leq \|\nabla u\|_{L^k(\Omega)} \|d_i^\ast\|_{L^{s'}(\Omega)} \leq C_k \|u\|_{W^{2,p}(\Omega)},
$$

where $s'$ is defined in (5.14) and

$$
\|d_i^\ast \cdot \nabla u\|_{L^p(\Omega)} \leq C_k \|d\|_{L^3(\Omega)} (\epsilon' \|u\|_{W^{2,p}(\Omega)} + C_{\epsilon'} \|u\|_{H^1(\Omega)}).
$$

\section*{The term $\|u \cdot \nabla d\|_{L^p(\Omega)}$}

The analysis is similar to the case (i). We consider:

$$
\nabla d = z_1^\ast + z_2^\ast \quad \text{where} \quad z_1^\ast = \nabla d + \rho_{r/2} \quad \text{and} \quad z_2^\ast = \nabla d - \nabla d + \rho_{r/2},
$$

$\nabla d$ is the extension by zero of $\nabla d$ to $\mathbb{R}^3$. Observe that

$$
\|z_2^\ast\|_{L^p(\Omega)} \leq \|\nabla d - \nabla d + \rho_{r/2}\|_{L^p(\Omega)} \leq \epsilon,
$$

with $s$ given in (5.6). Using the above estimates and the same arguments as in the case (i), the influence of $z_2^\ast$ in the bound of $\|u \cdot \nabla d\|_{L^p(\Omega)}$ is given by:

$$
\|u \cdot z_2^\ast\|_{L^p(\Omega)} \leq C \epsilon \|z_2^\ast\|_{L^p(\Omega)} \|u\|_{L^p(\Omega)} \leq C \epsilon \|u\|_{W^{2,p}(\Omega)}.
$$

And for the bound depending on $z_1^\ast$, proceeding in the same way as in the case (i), we derive:

$$
\|z_1^\ast \cdot u\|_{L^p(\Omega)} \leq C \epsilon \|z_1^\ast\|_{L^2(\Omega)} (\epsilon' \|u\|_{W^{2,p}(\Omega)} + C_{\epsilon'} \|u\|_{H^1(\Omega)}).
$$

\section*{(iv) Estimate of the constants $\sum_{i=1}^I |c_i|$}

We note that

$$
|c_i| \leq \left| \int_{\Omega} f \cdot \nabla q_i^N \, dx \right| + \left| \int_{\Omega} (\text{curl } b \times d) \cdot \nabla q_i^N \, dx \right| + \left| \int_{\Omega} (\text{curl } u) \cdot \nabla q_i^N \, dx \right| + \left| \int_{\Gamma} P_0 \nabla q_i^N \cdot n \, ds \right|
$$

\begin{align*}
&\leq \|f\|_{L^2(\Omega)} \|\nabla q_i^N\|_{L^k(\Omega)} + \|\text{curl } b\|_{L^2(\Omega)} \|d\|_{L^3(\Omega)} \|\nabla q_i^N\|_{L^k(\Omega)} \\
&+ \|\text{curl } u\|_{L^2(\Omega)} \|u\|_{L^k(\Omega)} \|\nabla q_i^N\|_{L^k(\Omega)} + \|P_0\|_{H^{-\frac{1}{2}}(\Gamma)} \|\nabla q_i^N \cdot n\|_{H^{rac{1}{2}}(\Gamma)}
\end{align*}
5.2. Weak solution in $W^{1,p}(\Omega)$ with $1 < p < +\infty$

Thanks to [24, Corollary 3.3.1], we know that the functions $\nabla q^\nu$ belong to $W^{1,\eta}(\Omega)$ for any $\eta \geq 2$ where each $q^\nu$ is the unique solution of the problem (3.2).

Now, the estimate (4.13) yields:

$$
\sum_{i=1}^I |c_i| \leq C(1 + \|d\|_{L^3(\Omega)} + \|\text{curl } w\|_{L^3(\Omega)})(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_b\|_{W^{1,\frac{3}{2}}(\Gamma)})
$$

(5.25)

Using the embeddings

$$
W^{1,3/2}(\Omega) \hookrightarrow L^3(\Omega), \quad L^p(\Omega) \hookrightarrow [H^3_0((\text{curl } \Omega))]', \quad W^{1,\frac{3}{2}'}(\Gamma) \hookrightarrow H^{-\frac{1}{2}}(\Gamma),
$$

choosing $\epsilon$ and $\epsilon'$ such that

$$
\epsilon' C_{SE} C_s(\|\text{curl } w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,\frac{3}{2}}(\Omega)}) < \frac{1}{2},
$$

we deduce from (5.7), (5.11), (5.15), (5.19)-(5.21), (5.23)-(5.25), the weak estimate (4.13) and the embedding $W^{1,\frac{3}{2}}(\Omega) \hookrightarrow L^3(\Omega)$ that the estimate (5.2) holds in all cases.

**Second step.** The case of $\text{curl } w \in L^{3/2}(\Omega)$ and $d \in W^{1,3/2}_\sigma(\Omega)$.

Let $w_\lambda \in D(\Omega)$ and $d_\lambda \in D(\Omega)$ such that $\text{curl } w_\lambda \rightarrow \text{curl } w$ in $L^{3/2}(\Omega)$ and $d_\lambda \rightarrow d$ in $W^{1,3/2}(\Omega)$.

Consequently, the following problem:

$$
\begin{cases}
-\Delta u_\lambda + (\text{curl } w_\lambda) \times u_\lambda + \nabla P_\lambda - (\text{curl } b_\lambda) \times d_\lambda = f & \text{and } \nabla u_\lambda = 0 \text{ in } \Omega, \\
\text{curl } b_\lambda - \text{curl}(u_\lambda \times d_\lambda) = g & \text{and } b_\lambda = 0 \text{ in } \Omega, \\
u_\lambda \times n = 0 & \text{and } b_\lambda \times n = 0 \text{ on } \Gamma, \\
P_\lambda = P_0 & \text{on } \Gamma_0 \text{ and } P_\lambda = P_0 + c_i \text{ on } \Gamma_i, \\
(u_\lambda \cdot n, 1)_{\Gamma_i} = 0 & \text{and } (b_\lambda \cdot n, 1)_{\Gamma_i} = 0, \quad 1 \leq i \leq I.
\end{cases}
$$

has a unique solution $(u_\lambda, b_\lambda, P_\lambda, c_\lambda) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^2$ and satisfies:

$$
\begin{align*}
\|u_\lambda\|_{W^{2,p}(\Omega)} + \|b_\lambda\|_{W^{2,p}(\Omega)} + \|P_\lambda\|_{W^{1,p}(\Omega)} & \leq C(1 + \|\text{curl } w_\lambda\|_{L^{3/2}(\Omega)} + \|d_\lambda\|_{W^{1,\frac{3}{2}}(\Omega)}), \\
\times \left( \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_b\|_{W^{1,\frac{3}{2}}(\Gamma)} \right),
\end{align*}
$$

(5.26)

where $C$ is independent of $\lambda$. Finally, these uniform bounds enable us to pass to the limit $\lambda \rightarrow 0$. As a consequence, $(u_\lambda, b_\lambda, P_\lambda, c_\lambda)$ converges to $(u, b, P, c)$ the solution of the linearized MHD problem (4.2) and satisfies the estimate (5.2). □

5.2. Weak solution in $W^{1,p}(\Omega)$ with $1 < p < +\infty$

In this subsection, we study the regularity $W^{1,p}(\Omega)$ of the weak solution for the linearized MHD problem (4.2). We begin with the case $p > 2$. The next theorem will be improved in Corollary 5.10 where we consider a data $P_0$ less regular.

In the following, we denote by $(\cdot, \cdot)_{\Omega, p}$ the duality product between $[H^p_0((\text{curl } \Omega))]'$ and $H^p_0((\text{curl } \Omega))$.

**Theorem 5.2.** (Generalized solution in $W^{1,p}(\Omega)$ with $p > 2$). Suppose that $\Omega$ is of class $C^{1,1}$ and $p > 2$. Assume that $h = 0$, and let $f, g \in [H^p_0((\text{curl } \Omega))]'$ and $P_0 \in W^{1,\frac{3}{2}}(\Gamma)$ with the compatibility condition

$$
\forall \nu \in K^p_N(\Omega), \quad (g, \nu)'_{\Omega, p'} = 0,
$$

(5.27)

$$
\text{div } g = 0 \text{ in } \Omega.
$$

(5.28)

and

$$
\text{curl } w \in L^s(\Omega), \quad d \in W^{1,s}_\sigma(\Omega),
$$

(5.29)

with

$$
s = \frac{3}{2} \text{ if } 2 < p < 3, \quad s > \frac{3}{2} \text{ if } p = 3 \quad \text{and} \quad s = r \text{ if } p > 3.
$$

(5.30)
$$r \geq 1 \quad \text{such that} \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{3}.$$  

(5.31)

Then the linearized MHD problem (4.2) has a unique solution \((u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I\). Moreover, we have the following estimate:

$$\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Omega)} \leq C(1 + \|\text{curl } w\|_{L^p(\Omega)} + \|d\|_{W^{1,r}(\Omega)})$$

\times \left( \|f\|_{H^{3'}(\text{curl}, \Omega)} + \|g\|_{H^{3'}(\text{curl}, \Omega)} + \|P_0\|_{W^{1-1/r, r}(\Gamma)} \right).$$

(5.32)

**Proof. A) Existence:** Applying Proposition 3.2, there exists a unique solution \((u_1, P_1, \alpha^{(1)}) \in W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I\) solution of the following problem:

$$\begin{cases}
-\Delta u_1 + \nabla P_1 = f & \text{in } \Omega, \\
u_1 \times n = 0 & \text{on } \Gamma, \\
P_1 = P_0 & \text{on } \Gamma_0, \\
\langle u_1 \cdot n, 1 \rangle_{\Gamma_i} = 0, & \forall 1 \leq i \leq I,
\end{cases}$$

where \(\alpha_i^{(1)} = \langle f, \nabla q_i^N \rangle_{\Omega, \Gamma_i} = -\int_{\Gamma_i} P_0 \nabla q_i^N \cdot n \, d\sigma\) and satisfying the estimate:

$$\|u_1\|_{W^{1,p}(\Omega)} + \|P_1\|_{W^{1,r}(\Omega)} \leq C \left( \|f\|_{H^{3'}(\text{curl}, \Omega)} + \|P_0\|_{W^{1-1/r, r}(\Gamma)} \right).$$

(5.33)

Next, since \(g\) satisfies the compatibility conditions (5.27)-(5.28), due to Lemma 3.4, the following problem:

$$\begin{cases}
-\Delta b_1 = g & \text{in } \Omega, \\
b_1 \times n = 0 & \text{on } \Gamma, \\
\langle b_1 \cdot n, 1 \rangle_{\Gamma_i} = 0 & \forall 1 \leq i \leq I
\end{cases}$$

has a unique solution \(b_1 \in W^{1,p}(\Omega)\) satisfying the estimate:

$$\|b_1\|_{W^{1,p}(\Omega)} \leq C \|g\|_{H^{3'}(\text{curl}, \Omega)}.$$  

(5.34)

Then, since \(\text{curl } w \in L^s(\Omega)\) and \(u_1 \in W^{1,p}(\Omega)\), we have \((\text{curl } w) \times u_1 \in L^s(\Omega)\). Indeed, if \(p < 3\), then \(W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)\) with \(\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}\) and \(\frac{1}{2} + \frac{1}{p} = \frac{1}{3}\). If \(p = 3\), then there exists \(\varepsilon > 0\) such that \(\frac{1}{2} + \frac{\varepsilon}{p^*} = \frac{5}{6}\). Finally, if \(p > 3\), then \(p^* = \infty\) and \(r = s\). Next, since \(s \geq \frac{3}{2}\), then \(W^{1,s}(\Omega) \hookrightarrow L^3(\Omega)\) so \(d \in L^3(\Omega)\), and by the definition of \(r\) in (5.31), we have \((\text{curl } b_1) \times d \in L^4(\Omega)\). Then \(f_1 = -(\text{curl } w) \times u_1 + (\text{curl } b_1) \times d\) belongs to \(L^s(\Omega)\). Furthermore, we set \(g_1 = \text{curl}(u_1 \times d) = (d \cdot \nabla)u_1 - (u_1 \cdot \nabla)d\). By the same way, using the definitions of \(s\) and \(r\), we can check that \(g_1 \in L^r(\Omega)\). Moreover, \(g_1\) satisfies the compatibility conditions (5.27)-(5.28). Observe that with the values of \(s\) given in (5.30) for \(p > 2\), we have \(r \in [\frac{3}{2}, 3)\) and satisfies

$$s = \frac{3}{2} \quad \text{if} \quad \frac{6}{5} < r < \frac{3}{2}, \quad s > \frac{3}{2} \quad \text{if} \quad r = \frac{3}{2} \quad \text{and} \quad s = r \quad \text{if} \quad r > \frac{3}{2},$$

(5.35)

So, \(s \geq \frac{3}{2}\) and then \(\text{curl } w\) is at least in \(L^3(\Omega)\) and \(d\) is at least in \(W^{1,\frac{3}{2}}(\Omega)\). We deduce from Theorem 5.1 that the following problem:

$$\begin{cases}
-\Delta u_2 + (\text{curl } w) \times u_2 + \nabla P_2 - (\text{curl } b_2) \times d = f_1 & \text{in } \Omega, \\
u_2 \times n = 0 & \text{on } \Gamma, \\
P_2 = 0 & \text{on } \Gamma_0, \\
\langle u_2 \cdot n, 1 \rangle_{\Gamma_i} = 0, & \forall 1 \leq i \leq I,
\end{cases}$$

(5.36)
has a unique solution \((u_2, b_2, P_2, \alpha^{(2)}) \in W^{2,r} (\Omega) \times W^{2,r} (\Omega) \times W^{1,r} (\Omega) \times \mathbb{R}^f\) satisfying the estimate:

\[
\|u_2\|_{W^{2,r} (\Omega)} + \|b_2\|_{W^{2,r} (\Omega)} + \|P_2\|_{W^{1,r} (\Omega)} \\
\leq C (1 + \|\text{curl} \ u\|_{L^2 (\Omega)} + \|d\|_{L^2 (\Omega)}) (\|f_1\|_{L^r (\Omega)} + \|g_1\|_{L^r (\Omega)})
\]

(5.37)

with

\[
\alpha^{(2)} = \langle (\text{curl} (b_1 + b_2)) \times d, \nabla q_1^N \rangle_{\alpha_{1,p,r'}} - \langle (\text{curl} \ u) \times (u_1 + u_2), \nabla q_1^N \rangle_{\alpha_{1,p,r'}}.
\]

(5.38)

Finally, using the embedding \(W^{2,r} (\Omega) \hookrightarrow W^{1,p} (\Omega)\), the solution of the linearized MHD problem (4.2) is given by \((u_1 + u_2, b_1 + b_2, P_1 + P_2, \alpha^{(1)} + \alpha^{(2)}) \in W^{1,p} (\Omega) \times W^{1,p} (\Omega) \times W^{1,r} (\Omega) \times \mathbb{R}^f\).

In particular, the constants \(c_i = \alpha_{i,1}^{(1)} + \alpha_{i,2}^{(2)}\) are given by

\[
c_i = \langle f, \nabla q_1^N \rangle_{\alpha_{1,p,r'}} - \int_\Omega P_0 \nabla q_1^N \cdot n \, d\sigma + \langle (\text{curl} \ b) \times d, \nabla q_1^N \rangle_{\alpha_{1,p,r'}} - \langle (\text{curl} \ u) \times u, \nabla q_1^N \rangle_{\alpha_{1,p,r'}}.
\]

(5.39)

B) Estimates: The terms on \(f_1\) and \(g_1\) in (5.37) can be controlled as:

\[
\|f_1\|_{L^r (\Omega)} \leq C (\|\text{curl} \ u\|_{L^r (\Omega)} \|u_1\|_{W^{1,p} (\Omega)} + \|d\|_{L^r (\Omega)} \|b_1\|_{W^{1,p} (\Omega)}).
\]

(5.40)

\[
\|g_1\|_{L^r (\Omega)} \leq C (\|d\|_{L^r (\Omega)} \|u_1\|_{W^{1,p} (\Omega)} + \|\nabla d\|_{L^r (\Omega)} \|u_1\|_{W^{1,p} (\Omega)}).
\]

(5.41)

Then, using the above estimates and the embeddings \(W^{1,r} (\Omega) \hookrightarrow W^{1,\frac{4}{3}} (\Omega) \hookrightarrow L^3 (\Omega)\) for \(s \geq 3/2\), the estimate (5.37) becomes

\[
\|u_2\|_{W^{2,r} (\Omega)} + \|b_2\|_{W^{2,r} (\Omega)} + \|P_2\|_{W^{1,r} (\Omega)} \leq C (1 + \|\text{curl} \ u\|_{L^r (\Omega)} + \|d\|_{W^{1,r} (\Omega)}) \]

\[
\times (\|\text{curl} \ u\|_{L^r (\Omega)} + \|d\|_{W^{1,r} (\Omega)}) (\|u_1\|_{W^{1,p} (\Omega)} + \|b_1\|_{W^{1,p} (\Omega)}).
\]

(5.42)

Thanks to (5.33), (5.34) and (5.42), the solution \((u, b, P)\) satisfies

\[
\|u\|_{W^{1,p} (\Omega)} + \|b\|_{W^{1,p} (\Omega)} + \|P\|_{W^{1,r} (\Omega)} \leq C (1 + \|\text{curl} \ u\|_{L^r (\Omega)} + \|d\|_{W^{1,r} (\Omega)})^2
\]

\[
\times (\|f\|_{H^{0,p} (\text{curl} \ u, \Omega)}^r + \|g\|_{H^{0,p} (\text{curl} \ u, \Omega)}^r + \|P_0\|_{W^{1-\frac{1}{r}, r} (\Gamma)})
\]

(5.43)

This estimate is not optimal and can be improved. For this, we will consider \((u, b, P) \in W^{1,p} (\Omega) \times W^{1,p} (\Omega) \times W^{1,r} (\Omega)\) the solution of (4.2) obtained in the existence part.

Note that, due to the hypothesis on \(d\) and \(\text{curl} \ u\), the terms \((\text{curl} \ u) \times u\), \((\text{curl} \ b) \times d\) and \((\text{curl} \ u \times d)\) belong to \(L^r (\Omega)\) with \(\frac{1}{r} = \frac{1}{p} + \frac{1}{3}\). Thus, according to the regularity of the Stokes problem (\(S_N\)) (see Proposition 3.2) and the elliptic problem (\(E_N\)) (see Lemma 3.4), we have:

\[
\|u\|_{W^{1,p} (\Omega)} + \|b\|_{W^{1,p} (\Omega)} + \|P\|_{W^{1,r} (\Omega)} \\
\leq C \left( \|f\|_{H^{0, p} (\text{curl} \ u, \Omega)} + \|g\|_{H^{0, p} (\text{curl} \ u, \Omega)} + \|P_0\|_{W^{1-\frac{1}{r}, r} (\Gamma)} + \|\text{curl} \ u\|_{L^r (\Omega)} \right)
\]

(5.44)

We proceed in a way similar to the proof of the Theorem 5.1: we bound the three last terms of (5.44), using the decomposition of \(\text{curl} \ u\), \(d\) and \(\nabla d\) given in (5.4)-(5.5) and (5.22) respectively.

(i) The term \(\|\text{curl} \ u\|_{L^r (\Omega)}\): Using the decomposition (5.4) for \(y = \text{curl} \ u\), we obtain:

\[
\|y_2 \times u\|_{L^r (\Omega)} \leq \|y_2\|_{L^r (\Omega)} \|u\|_{L^r (\Omega)} \leq C \|\text{curl} \ u\|_{W^{1,p} (\Omega)}
\]

(5.45)

where \(W^{1,p} (\Omega) \hookrightarrow L^p (\Omega)\) with \(\frac{1}{p} = \frac{1}{p} - \frac{1}{4}\) is \(p < 3\), for \(p^* = \frac{3p}{2p-3}\) if \(p = 3\) and \(p^* = \infty\) if \(p > 3\). Next, for the term \(y_1 \times u\), let us consider the first case \(p < 3\). We have

\[
\|y_1 \times u\|_{L^r (\Omega)} \leq \|y_1\|_{L^r (\Omega)} \|u\|_{L^m (\Omega)} \leq \|y\|_{L^2 (\Omega)} \|ho_{1/2}\|_{L^k (\Omega)} \|u\|_{L^m (\Omega)}
\]
with \( \frac{1}{p} = \frac{1}{q} + \frac{1}{r} \) and \( 1 + \frac{1}{r} = \frac{2}{p} + \frac{1}{q} \). Choosing \( 6 < m < p^{*} \), the embedding \( W^{1,p}(\Omega) \hookrightarrow L^{m}(\Omega) \) is compact. Following this choice, we have \( t \in \left[ \frac{p}{2}, \frac{mr}{2(p-m)} \right] \) and \( k \in \left[ \frac{m}{2(p-m)} \right] \). Then, for any \( \epsilon' > 0 \), there exists \( C_{\epsilon'} > 0 \) such that

\[
\| u \|_{L^{m}(\Omega)} \leq \epsilon' \| u \|_{W^{1,p}(\Omega)} + C_{\epsilon'} \| u \|_{L^{s}(\Omega)}.
\]

So, we deduce

\[
\| y_{1}^{t} \times u \|_{L^{q}(\Omega)} \leq \epsilon' C_{\epsilon'} \| u \|_{L^{q/2}(\Omega)} \| u \|_{W^{1,p}(\Omega)} + C_{\epsilon'} \| u \|_{L^{3/2}(\Omega)} \| u \|_{H^{1}(\Omega)}
\]

(5.46)

where \( C_{\epsilon} \) is the constant which absorbs the norm of the mollifier and \( C_{1} \) is the constant of the Sobolev embedding \( H^{1}(\Omega) \hookrightarrow L^{6}(\Omega) \). If \( p \geq 3 \), we have

\[
\| y_{1}^{t} \times u \|_{L^{q}(\Omega)} \leq \| y_{1} \|_{L^{q}(\Omega)} \| u \|_{L^{m}(\Omega)} \leq \| y_{1} \|_{L^{q}(\Omega)} \| \rho_{1/2} \|_{L^{3}(\Omega)} \| u \|_{L^{m}(\Omega)}.
\]

(5.47)

where we choose \( m = \infty \) if \( p > 3 \) and \( m \in (1, \infty) \) if \( p = 3 \).

(ii) **The term** \( \| \text{curl} b \times d \|_{L^{q}(\Omega)} \): Using the decomposition (5.5) for \( d \), we have:

\[
\| \text{curl} b \times d_{3} \|_{L^{q}(\Omega)} \leq \| \text{curl} b \|_{L^{2}(\Omega)} \| d_{3} \|_{L^{q/2}(\Omega)} \leq C_{\epsilon} \| b \|_{W^{1,-6/5}(\Omega)}
\]

(5.48)

Next, in order to bound the term \( \| \text{curl} b \times d_{1} \|_{L^{q}(\Omega)} \), we have two cases:

- The case \( 2 < p \leq 6 \): we have

\[
\| \text{curl} b \times d_{1} \|_{L^{q}(\Omega)} \leq \| \text{curl} b \|_{L^{2}(\Omega)} \| d_{1} \|_{L^{q}(\Omega)} \leq \| \text{curl} b \|_{L^{2}(\Omega)} \| d \|_{L^{3}(\Omega)} \| \rho_{1/2} \|_{L^{6}(\Omega)}
\]

(5.49)

with \( \frac{1}{q} = \frac{1}{q} + \frac{1}{r} \) and \( 1 + \frac{1}{r} = \frac{2}{p} + \frac{1}{q} \), so we have to take \( t = \frac{m}{2(p-m)} \) and \( k = \frac{2p}{2p-1} \), which are well-defined.

- The case \( p > 6 \): we have

\[
\| \text{curl} b \times d_{1} \|_{L^{q}(\Omega)} \leq \| \text{curl} b \|_{L^{q}(\Omega)} \| d_{1} \|_{L^{q}(\Omega)} \leq \| \text{curl} b \|_{L^{q}(\Omega)} \| d \|_{L^{3}(\Omega)} \| \rho_{1/2} \|_{L^{6}(\Omega)}
\]

(5.50)

with \( \frac{1}{q} = \frac{1}{q} + \frac{1}{r} \) and \( 1 + \frac{1}{r} = \frac{2}{p} + \frac{1}{q} \). We choose \( 3 < q < 6 \), and then we have that \( t \in [3, p] \) and \( k \in [1, \frac{2p}{2p-1}] \). The interpolation estimate of \( W^{1,q}(\Omega) \) between \( H^{1}(\Omega) \) and \( W^{1,3}(\Omega) \) gives (cf. [19]):

\[
\| \text{curl} b \|_{L^{q}(\Omega)} \leq \| b \|_{W^{1,3}(\Omega)} \| b \|_{H^{1}(\Omega)}^{\frac{1}{2}}
\]

Applying the Young inequality, we obtain for small \( \epsilon' > 0 \):

\[
\| \text{curl} b \|_{L^{q}(\Omega)} \leq \epsilon' \| b \|_{W^{1,6/5}(\Omega)} + C_{\epsilon'} \| b \|_{H^{1}(\Omega)}
\]

Replacing this estimate in (5.50), we obtain:

\[
\| \text{curl} b \times d_{1} \|_{L^{q}(\Omega)} \leq \epsilon' C_{\epsilon'} \| d \|_{L^{3}(\Omega)} \| b \|_{W^{1,6/5}(\Omega)} + C_{\epsilon'} \| d \|_{L^{3}(\Omega)} \| b \|_{H^{1}(\Omega)}
\]

(5.51)

(iii) **The term** \( \| \text{curl} u \times d \|_{L^{q}(\Omega)} \): Since \( \text{div} u = 0 \) and \( \text{div} d = 0 \) in \( \Omega \), thus we rewrite \( \text{curl} u \times d = (d \cdot \nabla) u - (u \cdot \nabla) d \).

- **The term** \( \| (d \cdot \nabla) u \|_{L^{q}(\Omega)} \): following the same proof as for the term \( \| \text{curl} b \times d \|_{L^{q}(\Omega)} \) by replacing \( \text{curl} b \) with \( \nabla u \), we obtain

\[
\| (d_{1} \cdot \nabla) u \|_{L^{q}(\Omega)} \leq C_{\epsilon} \| u \|_{W^{1,6/5}(\Omega)}
\]

(5.52)

and

\[
\| (d_{1} \cdot \nabla) u \|_{L^{q}(\Omega)} \leq C_{\epsilon} \| d \|_{L^{3}(\Omega)} \left( \epsilon' \| u \|_{W^{1,6/5}(\Omega)} + C_{\epsilon'} \| u \|_{H^{1}(\Omega)} \right)
\]

(5.53)

- **The term** \( \| (u \cdot \nabla) d \|_{L^{q}(\Omega)} \): In the same way, we remark that we can control this term as for \( \| \text{curl} u \times d \|_{L^{q}(\Omega)} \) by replacing \( \text{curl} u \) with \( \nabla d \). Applying the decomposition (5.22) for \( \nabla d \), we thus prove that:

\[
\| (d_{1} \cdot \nabla) u \|_{L^{q}(\Omega)} \leq C_{\epsilon} \| u \|_{W^{1,6/5}(\Omega)}
\]

(5.54)

and

\[
\| (d_{1} \cdot \nabla) u \|_{L^{q}(\Omega)} \leq C_{\epsilon} \| \nabla d \|_{L^{3}(\Omega)} \left( \epsilon' \| u \|_{W^{1,6/5}(\Omega)} + C_{\epsilon'} \| u \|_{H^{1}(\Omega)} \right)
\]

(5.55)

Finally, taking the estimates (5.45)-(5.55) together with the embedding \( W^{1,s}(\Omega) \hookrightarrow L^{3}(\Omega) \) for \( s \geq 3/2 \) and (5.44), then choosing \( \epsilon, \epsilon' > 0 \) small enough and using the estimate (4.13), we thus obtain (5.32).
Remark 5.1.
(i) The case $p > 2$ can be analyzed in a similar way to the case $p = 2$ to prove that the space $[H_0^{r - p'}(\text{curl}, \Omega)]'$ with $r = \frac{1}{p} + \frac{1}{3}$ is optimal to obtain the regularity $W^{1,p}(\Omega)$.
(ii) Why do we take $P_0 \in W^{1-1/r,r}(\Gamma)$ instead of $W^{-1/p,p}(\Gamma)$? If we take $P_0 \in L^p(\Omega)$ as in the classic case of Navier-Stokes equations with Dirichlet boundary conditions. But we are not able to solve the Stokes problem $(S_\chi)$ because, in this case, $f = \text{curl} (\text{curl} u) + \nabla P \notin [H_0^{r - p'}(\text{curl}, \Omega)]'$.

We also need to study the case where the divergence is not free for the velocity field. The following problem appears as the dual problem associated to the linearized MHD problem (4.2) in the study of weak solutions for $p < 2$:

$$
\begin{cases}
- \Delta u + (\text{curl } w) \times u + \nabla P - (\text{curl } b) \times d = f \quad \text{and} \quad \text{div } u = 0 \quad \text{in } \Omega, \\
\text{curl} \text{curl } b - \text{curl}(u \times d) + \nabla \chi = g \quad \text{and} \quad \text{div } b = 0 \quad \text{in } \Omega, \\
\quad u \times n = 0, \quad b \times n = 0 \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma, \\
P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \quad \text{on } \Gamma_i,
\end{cases}
\tag{5.56}
$$

Observe that the second equation in (4.2) is replaced by $\text{curl} \text{curl } b - \text{curl}(u \times d) + \nabla \chi = g$ in $\Omega$ with $\chi = 0$ on $\Gamma$. The scalar $\chi$ represents the Lagrange multiplier associated with magnetic divergence constraint. Note that, taking the divergence in the above equation, $\chi$ is a solution of the following Dirichlet problem:

$$
\Delta \chi = \text{div } g \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma.
\tag{5.57}
$$

In particular, if $g$ is divergence-free, we have $\chi = 0$. Nevertheless, the introduction of $\chi$ will be useful to enforce zero divergence condition over the magnetic field. First, we give the following result for the case $h = 0$.

Corollary 5.3. Suppose that $p \geq 2$ and $h = 0$. Let $f, g \in [H_0^{r - p'}(\text{curl}, \Omega)]'$, $P_0 \in W^{1-1/r,r}(\Gamma)$ with the compatibility condition (5.27) and $w, d$ defined with (5.29)-(5.31). Then the problem (5.56) has a unique solution $(u, b, P, \chi, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ where $c = (c_1, \ldots, c_I)$ is given by (5.39). Moreover, we have the following estimates:

$$
\begin{align*}
&\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Gamma)} \leq C(1 + \|\text{curl } w\|_{L^\infty(\Omega)} + \|d\|_{W^{1,r}(\Omega)}) \\
&\times (\|f\|_{H_0^{r - p'}(\text{curl}, \Omega)} + \|g\|_{H_0^{r - p'}(\text{curl}, \Omega)}) + \|P_0\|_{W^{1-1/r,r}(\Gamma)}), \\
&\|\chi\|_{W^{1,r}(\Omega)} \leq C \|g\|_{H_0^{r - p'}(\text{curl}, \Omega)}.
\end{align*}
\tag{5.58}
$$

Proof. As mentioned before, the scalar $\chi$ can be found directly as a solution of the Dirichlet problem (5.57). Since $g \in [H_0^{r - p'}(\text{curl}, \Omega)]'$, $\text{div } g \in W^{-1,r}(\Omega)$ and then $\chi$ belongs to $W^{1,r}(\Omega)$ and satisfies the estimate (5.59). We set $g' = g - \nabla \chi$. It is clear that $g'$ is an element of the dual space $[H_0^{r - p'}(\text{curl}, \Omega)]'$. Moreover, it is clear that $\text{div } g' = 0$ in $\Omega$ and $g'$ satisfies the compatibility conditions (5.27). So, problem (5.56) becomes

$$
\begin{cases}
- \Delta u + (\text{curl } w) \times u + \nabla P - (\text{curl } b) \times d = f \quad \text{and} \quad \text{div } u = 0 \quad \text{in } \Omega, \\
\text{curl} \text{curl } b - \text{curl}(u \times d) = g' \quad \text{and} \quad \text{div } b = 0 \quad \text{in } \Omega, \\
\quad u \times n = 0 \quad \text{and} \quad b \times n = 0 \quad \text{on } \Gamma, \\
P = P_0 \quad \text{on } \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \quad \text{on } \Gamma_i,
\end{cases}
\tag{5.60}
$$

Thanks to Theorem 5.2, problem (5.60) has a unique solution $(u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$
\begin{align*}
&\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Gamma)} \leq C(1 + \|\text{curl } w\|_{L^\infty(\Omega)} + \|d\|_{W^{1,r}(\Omega)}) \\
&\times (\|f\|_{H_0^{r - p'}(\text{curl}, \Omega)} + \|g'\|_{H_0^{r - p'}(\text{curl}, \Omega)}) + \|P_0\|_{W^{1-1/r,r}(\Gamma)}.
\end{align*}
\tag{5.61}
$$

Using (5.59), the previous estimate still holds when $g'$ is replaced by $g$. 

\[\square\]
The next theorem gives a generalization for the case $h \neq 0$.

**Theorem 5.4.** Suppose that $p \geq 2$. Let $f, g \in [H_0^\nu(\text{curl}, \Omega)]'$, $P_0 \in W^{1, r}(\Gamma)$ and $h \in W^{1, r}(\Omega)$ with the compatibility condition (5.27) and $w, d$ defined with (5.29)-(5.31). Then the problem (5.56) has a unique solution $(u, b, P, \chi, c) \in W^{1, p}(\Omega) \times W^{1, p}(\Omega) \times W^{1, r}(\Omega) \times W^{1, r}(\Omega) \times \mathbb{R}^I$. Moreover, we have the following estimate for $(u, b, P)$:

$$
\|u\|_{W^{1, p}(\Omega)} + \|b\|_{W^{1, p}(\Omega)} + \|P\|_{W^{1, r}(\Omega)} \leq C(1 + \|\text{curl} \, w\|_{L^\infty(\Omega)} + \|d\|_{W^{1, r}(\Omega)})^2 \times (\|f\|_{[H_0^\nu(\text{curl}, \Omega)]'} + \|g\|_{[H_0^\nu(\text{curl}, \Omega)]'} + \|h\|_{W^{1, r}(\Omega)} + \|P_0\|_{W^{1, r}(\Gamma)}).
$$

(5.62)

**Proof.** The idea is to lift the data $h$ by using the Stokes problem:

$$
\begin{aligned}
\begin{cases}
-\Delta u_1 + \nabla P_1 = f & \text{and} \quad \text{div} \, u_1 = h \quad \text{in} \ Ω, \\
u_1 \times n = 0 & \text{on} \ Γ, \\
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
P_1 = P_0 & \quad \text{on} \ Γ_0 \quad \text{and} \quad P_1 = P_0 + \alpha^{(1)}_i \quad \text{on} \ Γ_i, \\
(u_1 \cdot n, 1)_{Γ_i} = 0 & \quad \forall 1 \leq i \leq I
\end{aligned}
$$

Thanks to Proposition 3.2, there exists a unique solution $(u_1, P_1, \alpha^{(1)}) \in W^{1, p}(\Omega) \times W^{1, r}(\Omega) \times \mathbb{R}^I$ satisfying the estimate:

$$
\|u_1\|_{W^{1, p}(\Omega)} + \|P_1\|_{W^{1, r}(\Omega)} \leq C \left( \|f\|_{[H_0^\nu(\text{curl}, \Omega)]'} + \|g\|_{[H_0^\nu(\text{curl}, \Omega)]'} + \|h\|_{W^{1, r}(\Omega)} + \|P_0\|_{W^{1, r}(\Gamma)} \right).
$$

(5.63)

where $\alpha^{(1)} = (f, \nabla \eta^N)_{\Omega} + \int_{\Gamma} (h - P_0) \nabla \eta^N \cdot n \, d\sigma$.

Next, since $g$ satisfies the compatibility condition (5.27), due to [6, Theorem 5.2], the following problem:

$$
\begin{aligned}
\begin{cases}
-\Delta b_1 + \nabla \chi = g & \text{and} \quad \text{div} \, b_1 = 0 \quad \text{in} \ Ω, \\
\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\begin{cases}
b_1 \times n = 0 & \text{and} \quad \chi = 0 \quad \text{on} \ Γ, \\
b_1 \cdot n, 1)_{Γ_i} = 0 & \quad \forall 1 \leq i \leq I
\end{cases}
\end{aligned}
$$

has a unique solution $(b_1, \chi) \in W^{1, p}(\Omega) \times W^{1, r}(\Omega)$ satisfying the estimate:

$$
\|b_1\|_{W^{1, p}(\Omega)} + \|\chi\|_{W^{1, r}(\Omega)} \leq C \|g\|_{[H_0^\nu(\text{curl}, \Omega)]'}.
$$

(5.64)

Finally, we consider $(u_2, b_2, P_2, P, \alpha^{(2)}) \in W^{2, r}(\Omega) \times W^{2, r}(\Omega) \times W^{1, r}(\Omega) \times \mathbb{R}^I$ the solution of (5.36) satisfying (5.38) and (5.42). Therefore, $(u_1 + u_2, b_1 + b_2, P_1 + P_2, \chi, \alpha^{(1)} + \alpha^{(2)})$ is the solution of (5.56). Estimate (5.62) follows from (5.63),(5.64) and (5.42). \hfill \Box

Note that the estimate (5.62) is not optimal and will be improved in the next result.

**Proposition 5.5.** Under the assumptions of Theorem 5.4, the problem (5.56) has a unique solution $(u, b, P, \chi, c) \in W^{1, p}(\Omega) \times W^{1, p}(\Omega) \times W^{1, r}(\Omega) \times W^{1, r}(\Omega) \times \mathbb{R}^I$ satisfying (5.59) and the following estimate:

$$
\begin{aligned}
\|u\|_{W^{1, p}(\Omega)} + \|b\|_{W^{1, p}(\Omega)} + \|P\|_{W^{1, r}(\Omega)} & \leq C(1 + \|\text{curl} \, w\|_{L^\infty(\Omega)} + \|d\|_{W^{1, r}(\Omega)}) \left( \|f\|_{[H_0^\nu(\text{curl}, \Omega)]'} + \|g\|_{[H_0^\nu(\text{curl}, \Omega)]'} \right) \\
& \quad + \|P_0\|_{W^{1, r}(\Gamma)} + \|h\|_{W^{1, r}(\Omega)} \left( 1 + \|\text{curl} \, w\|_{L^\infty(\Omega)} + \|d\|_{W^{1, r}(\Omega)} \right)
\end{aligned}
$$

(5.65)

**Proof.** We can reduce the non vanishing divergence problem (5.56) for the velocity to the case where $\text{div} \, u = 0$ in $\Omega$, by solving the following Dirichlet problem:

$$
\Delta \theta = h \quad \text{in} \ Ω \quad \text{and} \quad \theta = 0 \quad \text{on} \ Γ
$$

(5.66)
For \( h \in W^{1,r}(\Omega) \), problem (5.66) has a unique solution \( \theta \in W^{3,r}(\Omega) \hookrightarrow W^{2,p}(\Omega) \) satisfying the following estimate:

\[
\|\theta\|_{W^{2,p}(\Omega)} \leq C \|h\|_{W^{1,r}(\Omega)} \tag{5.67}
\]

Setting \( z = u - \nabla \theta \), then (5.66) becomes: Find \((z, b, P, \chi, c)\) solution of problem:

\[
\begin{align*}
-\Delta z + (\text{curl } w) \times z + \nabla P - (\text{curl } b) \times d &= f + \nabla h - (\text{curl } w) \times \nabla \theta \quad \text{in } \Omega \\
\text{curl } b - c &= f + \text{curl}(\nabla \theta \times d) \quad \text{in } \Omega \\
\text{div } z &= 0, \quad \text{div } b = 0 \quad \text{in } \Omega \\
z \times n &= 0, \quad b \times n = 0 \quad \text{and } \chi = 0 \quad \text{on } \Gamma \\
P &= P_0 \quad \text{on } \Gamma_0, \quad P = P_0 + c_i \quad \text{on } \Gamma_i \\
(z \cdot n, 1)_\Gamma = (b \cdot n, 1)_\Gamma &= 0, \quad \forall 1 \leq i \leq I,
\end{align*}
\]

which is a problem treated in the proof of Corollary 5.3. Since \( \nabla \theta \in W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \), by using the definition of \( s \) in (5.6), we have \( (\text{curl } w) \times \nabla \theta \in L^r(\Omega) \) with \( \frac{1}{r} = \frac{1}{p} - \frac{1}{p^*} \) if \( p < 3 \), \( p^* = \frac{r}{s} \) if \( p = 3 \) and \( p^* = \infty \) if \( p > 3 \). So \( f + \nabla h - (\text{curl } w) \times \nabla \theta \in \left[H^{1,p'}(\Omega)\right]' \). Now, we consider the term \( \text{curl}(\nabla \theta \times d) = d \cdot \nabla \nabla \theta - \nabla \theta \cdot \nabla d \). Since \( d \in W^{1,3}(\Omega) \hookrightarrow L^3(\Omega) \) \((s \geq 3/2)\), then \( d \cdot \nabla \nabla \theta \rightarrow L^3(\Omega) \). Moreover, since \( \nabla d \in L^3(\Omega) \), using the same arguments for the term \((\text{curl } w) \times \nabla \theta\), we deduce that \( \nabla \theta \cdot \nabla d \) belongs to \( L^3(\Omega) \). So, \( g + \text{curl}(\nabla \theta \times d) \in \left[H^{1,p'}(\Omega)\right]' \) and satisfies (5.27). Thanks to Theorem 5.2, there exists a unique solution \((z, b, P, \chi, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^l \) satisfying (5.59) and

\[
\begin{align*}
\|z\|_{W^{1,1}(\Omega)} + \|b\|_{W^{1,1}(\Omega)} + \|P\|_{W^{1,1}(\Omega)} \leq C \left(1 + \|\text{curl } w\|_{L^2(\Omega)} + \|d\|_{W^{1,1}(\Omega)}\right) \\
\times \left(\|f\|_{\left[H^{1,p'}(\Omega)\right]'} + \|g\|_{\left[H^{1,p'}(\Omega)\right]'} + \|P_0\|_{W^{1,1,1}(\Omega)} \right),
\end{align*}
\]

with

\[
c_i = (f, \nabla q_N)_{\Omega, \partial} + \int_{\Gamma} (h - R_0)\nabla q_N \cdot n \, ds - ((\text{curl } w) \times \nabla \theta, \nabla q_N)_{\Omega, \partial'} - ((\text{curl } w) \times z, \nabla q_N)_{\Omega, \partial'} + ((\text{curl } b) \times d, \nabla q_N)_{\Omega, \partial'}.
\]

To bound the terms \(\|\text{curl } w\|_{L^2(\Omega)}\) and \(\|\nabla \theta \times d\|_{L^2(\Omega)}\) in (5.69), we write by using (5.67)

\[
\begin{align*}
\|\text{curl } w\|_{L^2(\Omega)} \parallel \nabla \theta\|_{L^{p^*}(\Omega)} &\leq \|\text{curl } w\|_{L^2(\Omega)} \parallel \nabla \theta\|_{L^{p^*}(\Omega)} \leq \|\text{curl } w\|_{L^2(\Omega)} \parallel \nabla \theta\|_{W^{1,1}(\Omega)} \\
&\leq C \|\text{curl } w\|_{L^2(\Omega)} \parallel b\|_{W^{1,1}(\Omega)}.
\end{align*}
\]

In addition we have

\[
\begin{align*}
\|\text{curl } \nabla \theta \times d\|_{L^2(\Omega)} &\leq \|d \cdot \nabla \nabla \theta\|_{L^2(\Omega)} + \|\nabla d \cdot \nabla \theta\|_{L^2(\Omega)} \\
&\leq \|d\|_{L^2(\Omega)} \parallel \nabla \nabla \theta\|_{L^2(\Omega)} + \|\nabla d\|_{L^2(\Omega)} \parallel \nabla \theta\|_{L^{p^*}(\Omega)} \\
&\leq C \|d\|_{W^{1,1}(\Omega)} \parallel b\|_{W^{1,1}(\Omega)}.
\end{align*}
\]

Now plugging the estimates (5.70) and (5.71) in (5.69) gives

\[
\begin{align*}
\|z\|_{W^{1,1}(\Omega)} + \|b\|_{W^{1,1}(\Omega)} + \|P\|_{W^{1,1}(\Omega)} &\leq C \left(1 + \|\text{curl } w\|_{L^2(\Omega)} + \|d\|_{W^{1,1}(\Omega)} \right) \\
\times \left(\|f\|_{\left[H^{1,p'}(\Omega)\right]'} + \|g\|_{\left[H^{1,p'}(\Omega)\right]'} + \|P_0\|_{W^{1,1,1}(\Omega)} \right),
\end{align*}
\]

Thus, summing the resulting estimate (5.72) along with estimate (5.67), we get the bound for \((u, b, P)\) in (5.65). \(\square\)

We are interested now on the existence of solution for the linearized problem (4.2) in \(W^{1,p}(\Omega)\) with \(p < 2\). Since the problem is linear, we will use a duality argument developed by Lions-Magenes [18]. This way ensures the uniqueness of solutions. For this, we must derive that problem (4.2) has an equivalent variational formulation. We
then need adequate density lemma and Green formulae, adapted to our functional framework, to define rigorously each term. We introduce the space

\[ \mathcal{V}(\Omega) := \{ (v, a, \theta, \tau) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,p\prime}(\Omega) ; \text{div} v \in W^{1,p\prime}(\Omega), \]

\[ v \times n = a \times n = 0 \text{ on } \Gamma, \theta = 0, \text{ on } \Gamma_0 \text{ and } \theta = \text{este on } \Gamma_i, \langle v \cdot n, 1 \rangle_{\Gamma_i} = \langle a \cdot n, 1 \rangle_{\Gamma_i} = 0, \forall 1 \leq i \leq I \}

and we recall that \( \langle \cdot, \cdot \rangle_{\Omega^{\prime},p} \) denotes the duality between \( H_0^{\prime,p}(\text{curl}, \Omega) \) and \( [H_0^{r,p}(\text{curl}, \Omega)]' \) with \( \frac{1}{p'} = \frac{1}{p} - \frac{1}{r} \).

**Lemma 5.6.** We suppose \( \Omega \) of class \( C^{1,1} \). Let \( \frac{3}{2} < p < 2 \). Assume that \( f, g \in [H_0^{r,p}(\text{curl}, \Omega)]' \), \( h = 0 \) and \( P_0 \in W^{1,\frac{1}{2},r}(\Gamma) \) satisfying the compatibility conditions (5.27)-(5.28), together with \( \text{curl} w \in L^{3/2}(\Omega) \) and \( d \in W^{1,3/2}(\Omega) \). Then, the following two problems are equivalent:

1. \( (u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^f \) satisfies the linearized problem (4.2).
2. Find \( (u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^f \) with \( u \times n = 0 \) and \( b \times n = 0 \) on \( \Gamma, \langle u \cdot n, 1 \rangle_{\Gamma_i} = 0 \) and \( \langle b \cdot n, 1 \rangle_{\Gamma_i} = 0 \) for any \( 1 \leq i \leq I \) such that:

For any \( (v, a, \theta, \tau) \in \mathcal{V}(\Omega) \),

\[ (u, -\Delta v - (\text{curl } w) \times v + (\text{curl } a) \times v + \nabla \theta)_{\Omega^{\prime},p} - \int_\Omega P \text{div} v \, dx + (b, \text{curl } a) - \int_\Omega (\text{curl } b) \times v + \nabla \theta)_{\Omega^{\prime},p} \]

\[ + (f, v)_{\Omega^{\prime},p} + (g, a)_{\Omega^{\prime},p} - \int_\Gamma P_0 v \cdot n \, d\sigma, \]

(5.73)

\[ c_i = (f, \nabla q_i^N)_{\Omega^{\prime},p} - \int_\Gamma P_0 \nabla q_i^N \cdot n \, d\sigma + \int_\Omega (\text{curl } b) \times d \cdot \nabla q_i^N \, dx - \int_\Omega (\text{curl } w) \times u \cdot \nabla q_i^N \, dx, \]

(5.74)

**Proof.** (1) \( \Rightarrow \) (2) Let \( (u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^f \) solution of the linearized problem (4.2). Let us take \( (v, a, \theta, \tau) \in \mathcal{V}(\Omega) \). We want to multiply the system (4.2) by \( (v, a, \theta, \tau) \) and integrate by parts. Let us study these terms one by one.

Firstly, we note that the duality pairing \( \langle -\Delta u, v \rangle_{\Omega^{\prime},p} \) is well defined. Indeed, since \( -\Delta u = \text{curl} \text{curl } u \) and \( \text{curl } u \in L^p(\Omega) \), it follows that \( -\Delta u \in [H_0^{r,p}(\text{curl}, \Omega)]' \). Besides, recall that \( v \in W^{1,p}(\Omega) \), so \( \text{curl } v \in L^p(\Omega) \), and since

\[ \frac{1}{r'} = 1 - \frac{1}{r} = 1 - \frac{3 + p}{3p} = \frac{1}{p'} - \frac{1}{3}, \]

(5.75)

then we have \( W^{1,p'}(\Omega) \hookrightarrow L^r(\Omega) \). Hence \( v \in H_0^{r,p}(\text{curl}, \Omega) \). Now, by the density of \( \mathcal{D}(\Omega) \) in \( H_0^{r,p}(\text{curl}, \Omega) \) and \( H_0^{r,p}(\text{curl}, \Omega) \), we have

\[ \langle -\Delta u, v \rangle_{\Omega^{\prime},p} = \int_\Omega \text{curl } u \cdot \text{curl } v \, dx = \langle u, \text{curl } \text{curl } v \rangle_{\Omega^{\prime},p}, \]

(5.76)

The last duality pairing is again well defined: the embedding \( W^{1,p}(\Omega) \hookrightarrow L^r(\Omega) \) implies \( u \in H_0^{r,p}(\text{curl}, \Omega) \), and since \( \text{curl } v \in L^r(\Omega) \) then \( \text{curl } \text{curl } v \in [H_0^{r,p}(\text{curl}, \Omega)]' \). Thus, due to the relation \( \text{curl} \text{curl } v = -\Delta v + \nabla \text{div} v \), we deduce that:

\[ \langle -\Delta u, v \rangle_{\Omega^{\prime},p} = \langle u, -\Delta v + \nabla \text{div} v \rangle_{\Omega^{\prime},p} \]

Observe that since \( v \in \mathcal{V}(\Omega) \), we have \( \text{div} v \in W^{1,p}(\Omega) \), and it follows that \( \nabla \text{div} v \in L^{p\prime}(\Omega) \hookrightarrow [H_0^{r,p}(\text{curl}, \Omega)]' \). Therefore, since \( \text{curl} \text{curl } v \) belongs to \( [H_0^{r,p}(\text{curl}, \Omega)]' \), we deduce that \( \Delta v \) also belongs to \( [H_0^{r,p}(\text{curl}, \Omega)]' \). This proves that the last duality makes sense. Next, since \( \text{div} v = 0 \) on \( \Gamma \) and \( \text{div} u = 0 \) in \( \Omega \), we have

\[ \langle u, \nabla \text{div} v \rangle_{\Omega^{\prime},p} = \int_\Omega u \cdot \nabla \text{div} v \, dx = -\int_\Omega \text{div} u \text{div} v \, dx + \int_\Gamma u \cdot n \, \text{div} v \, d\sigma = 0. \]

We conclude that

\[ \langle -\Delta u, v \rangle_{\Omega^{\prime},p} = \langle u, -\Delta v \rangle_{\Omega^{\prime},p}. \]

We now treat the term \( \langle (\text{curl } w) \times u, v \rangle_{\Omega^{\prime},p} \). Since we have \( \text{curl } w \in L^{\frac{2}{3}}(\Omega) \) and \( u \in W^{1,p}(\Omega) \hookrightarrow L^p(\Omega) \), then, by definition of \( r \), \( (\text{curl } w) \times u \in L^{p\prime}(\Omega) \). Besides, \( v \in W^{1,p}(\Omega) \hookrightarrow L^{p\prime}(\Omega) \). So

\[ \langle (\text{curl } w) \times u, v \rangle_{\Omega^{\prime},p} = \int_\Omega (\text{curl } w) \times u \cdot v \, dx = -\int_\Omega (\text{curl } w) \times v \cdot u \, dx \]
with the integral well defined thanks to (5.75) and
\[
\frac{1}{r} + \frac{1}{(p')^r} = \frac{1}{r} + \frac{1}{p} - \frac{1}{3} = \frac{1}{r} + \frac{1}{r'} = 1.
\]

For the term \((-\text{curl} \, b) \times d, v)_{\Omega r', p'}\), we proceed as for the previous pairing: we have \(\text{curl} \, b \in L^p(\Omega)\) and \(d \in W^{1, \frac{2}{p}}(\Omega) \hookrightarrow L^3(\Omega)\), so \((-\text{curl} \, b) \times d \in L^3(\Omega)\). Therefore:
\[
\langle - (\text{curl} \, b) \times d, v \rangle_{\Omega r', p'} = - \int_{\Omega} (\text{curl} \, b) \times d \cdot v \, dx = - \int_{\Omega} \text{curl} \, b \cdot (d \times v) \, dx
\]
Using again the density of \(D(\Omega)\) in \(H^{p-, p'}_0(\text{curl}, \Omega)\), we obtain
\[
- \int_{\Omega} \text{curl} \, b \cdot (d \times v) \, dx = \int_{\Omega} b \cdot \text{curl}(v \times d) \, dx.
\]
It remains us to treat the pressure term. Since \(\nabla P \in L^r(\Omega)\), we have as for the both previous terms the well defined of the integral:
\[
\langle \nabla P, v \rangle_{\Omega r', p'} = \int_{\Omega} \nabla P \cdot v \, dx
\]
Using the same arguments as in [7, Proposition 3.7], and taking into account the boundary conditions on the pressure \(P\), we have
\[
\langle \nabla P, v \rangle_{\Omega r', p'} = \int_{\Gamma} P \, v \cdot n \, ds - \int_{\Omega} P \, \text{div} v \, dx
\]
\[
= \int_{\Gamma_0} P_0 \, v \cdot n \, ds + \sum_{i=1}^{I} \int_{\Gamma_i} (P_0 + c_i) \, v \cdot n \, ds - \int_{\Omega} P \, \text{div} v \, dx
\]
However, since \((v \cdot n, 1)_{\Gamma_i} = 0\) for all \(1 \leq i \leq I\), we have \(\sum_{i=1}^{I} \int_{\Gamma_i} c_i \, v \cdot n \, ds = 0\). Therefore, we obtain:
\[
\langle \nabla P, v \rangle_{\Omega r', p'} = \int_{\Gamma} P_0 \, v \cdot n \, ds - \int_{\Omega} P \, \text{div} v \, dx
\]
Now, multiplying the equation \(\text{div} \, u = 0\) in \(\Omega\) by \(\theta\), we obtain by the density of \(D(\Omega)\) in \(H^{p-, p'}(\text{div}, \Omega)\)
\[
0 = - \int_{\Omega} \theta \, \text{div} \, u \, dx = \int_{\Omega} u \cdot \nabla \theta \, dx - \int_{\Gamma} \theta \, u \cdot n \, ds,
\]
where we have used the fact that \(u \in W^{1, p}(\Omega) \hookrightarrow L^p(\Omega)\) which implies that \(u \in H^{p-, p}(\text{div}, \Omega)\). Combining the boundary conditions of \(\theta\) on \(\Gamma_i, 0 \leq i \leq I\), with zero fluxs of the velocity \((u \cdot n, 1)_{\Gamma_i} = 0\) for all \(1 \leq i \leq I\), we have: \(\int_{\Gamma} \theta \, u \cdot n \, ds = 0\). Thus, summing the above resulting terms, we obtain:
\[
\langle f, v \rangle_{\Omega r', p'} = \langle u, \text{curl} \, v \rangle_{\Omega r', p} - \int_{\Omega} (\text{curl} \, u) \times v \cdot u \, dx - \int_{\Omega} \text{curl}(d \times v) \cdot b \, dx
\]
\[
- \int_{\Omega} P \, \text{div} v \, dx + \int_{\Gamma_0} P_0 \, v \cdot n \, ds + \int_{\Omega} u \cdot \nabla \theta \, dx.
\]
(5.77)

Now, we treat the terms of the second equation of (4.2). For the term \(\langle \text{curl} \, b, a \rangle_{\Omega r', p'}\), the duality pairing is well defined and following the same reasoning than for (5.70), we have
\[
\langle \text{curl} \, b, a \rangle_{\Omega r', p'} = \int_{\Omega} \text{curl} \, b \cdot \text{curl} \, a \, dx = \langle b, \text{curl} \, a \rangle_{\Omega r', p}
\]
Next, for the term \(\langle u \times d, a \rangle_{\Omega r', p'}\), the duality pairing is again well defined: since \(u \in W^{1, p}(\Omega) \hookrightarrow L^p(\Omega)\) and \(d \in W^{1, \frac{2}{p}}(\Omega) \hookrightarrow L^3(\Omega)\), then \(u \times d \in L^p(\Omega)\) so \(\text{curl} \, (u \times d) \in [H^{r-, p'}_0(\text{curl}, \Omega)]'\). Similarly, by the density of \(D(\Omega)\) in \(H^{r-, p'}_0(\text{curl}, \Omega)\), we have
\[
\langle \text{curl} \, (u \times d), a \rangle_{\Omega r', p'} = \int_{\Omega} \text{curl} \, a \cdot (u \times d) \, dx = \int_{\Omega} u \cdot (d \times \text{curl} \, a) \, dx
\]
Here also the integrals are well defined. Indeed, for the first integral, \(\text{curl} \, a \in L^p(\Omega), u \in L^p(\Omega), d \in W^{1, \frac{2}{p}}(\Omega) \hookrightarrow L^3(\Omega)\) and \(\frac{1}{p} + \frac{1}{p'} + \frac{1}{2} = 1\).
It remains us to multiply the equation $\div b = 0$ in $\Omega$ by $\tau$. By density of $\mathcal{D}(\Omega)$ in $W_0^{1,p'}(\Omega)$, we have the Green formula: for any $\tau \in W_0^{1,p'}(\Omega)$
\[- \int_\Omega (\div b) \tau \, dx = \int_\Omega b \cdot \nabla \tau \, dx\]
In summary, these terms provided by the second equation of (4.2) give:
\[
\langle g, a \rangle_{\Omega^{p',p'}} = \langle b, \curl a \rangle_{\Omega^{p',p'}} - \int_\Omega u \cdot (d \times \curl a) \, dx + \int_\Omega b \cdot \nabla \tau \, dx \tag{5.78}
\]
Finally, adding (5.77) and (5.78), we obtain the variational formulation (5.73).

We now want to determine the constants $c_i$ in (5.74). Let us take $v \in W_0^{1,p}(\Omega)$ with $v \times n = 0$ on $\Gamma$, and set:
\[
v_0 = v - \sum_{i=1}^l \left( \int_{\Gamma_i} v \cdot n \, d\sigma \right) \nabla q_i^N \tag{5.79}
\]
So, $v_0$ belongs to $W_0^{1,p}(\Omega)$ and satisfies $v \times n = 0$ on $\Gamma$, $\langle v_0 \cdot n, 1 \rangle_{\Gamma_i} = 0$, $1 \leq i \leq I$. Multiplying the first equation of (4.2) by $v$ and integrating by parts, we obtain:
\[
\begin{align*}
\int_\Omega \curl u \cdot \curl v \, dx - &\int_\Omega (\curl w) \times v \cdot u \, dx + \int_\Omega \curl v \cdot (v \times d) \, dx \\
+ &\int_{\Gamma_0} P_0 v \cdot n \, d\sigma + \sum_{i=1}^l \int_{\Gamma_i} (P_0 + c_i) v \cdot n \, d\sigma = \int_\Omega f \cdot v \, dx \tag{5.80}
\end{align*}
\]
We now take a test function $(v_0, 0, 0)$ in (5.73). Note that it is possible because of the definition of (5.79):
\[
\begin{align*}
\langle u, -\Delta v_0 \rangle_{\Omega^{p',p'}} - &\int_\Omega u \cdot (\curl w) \times v_0 \, dx - \int_\Omega P \div v_0 \, dx + \int_\Omega b \cdot \curl(v_0 \times d) \, dx \\
= &\langle f, v_0 \rangle_{\Omega^{p',p'}} - \int_\Gamma P_0 v_0 \cdot n \, d\sigma
\end{align*}
\]
By definition of $q_i^N$, we have $\div v_0 = 0$. Besides, $\curl v_0 = \curl v$, so it follows from the same density argument used previously:
\[
\begin{align*}
\int_\Omega \curl u \cdot \curl v \, dx - &\int_\Omega (\curl w) \times v_0 \cdot u \, dx + \int_\Omega b \cdot \curl(v_0 \times d) \, dx = \int_\Omega f \cdot v_0 \, dx - \int_\Gamma P_0 v_0 \cdot n \, d\sigma
\end{align*}
\]
Decomposing $v_0$ with (5.79) in the previous equality, we have:
\[
\begin{align*}
\int_\Omega \curl u \cdot \curl v \, dx - &\int_\Omega (\curl w) \times v_0 \cdot u \, dx + \int_\Omega b \cdot \curl(v_0 \times d) \, dx = \int_\Omega f \cdot v_0 \, dx + \int_\Gamma P_0 v \cdot n \, d\sigma \\
= &\sum_{i=1}^l \left( \int_{\Gamma_i} v \cdot n \, d\sigma \right) \left[ -\int_\Omega (\curl w) \times \nabla q_i^N \cdot u \, dx + \int_\Omega b \cdot \curl(\nabla q_i^N \times d) \, dx - \int_\Omega f \cdot \nabla q_i^N \, dx + \int_\Gamma P_0 \nabla q_i^N \cdot n \, d\sigma \right]
\end{align*}
\]
Injecting (5.80) in this calculus, we thus obtain:
\[
- \sum_{i=1}^l c_i \int_{\Gamma_i} v \cdot n \, d\sigma = \sum_{i=1}^l \left( \int_{\Gamma_i} v \cdot n \, d\sigma \right) \left[ -\int_\Omega (\curl w) \times \nabla q_i^N \cdot u \, dx + \int_\Omega b \cdot \curl(\nabla q_i^N \times d) \, dx \\
- \int_\Omega f \cdot \nabla q_i^N \, dx + \int_\Gamma P_0 \nabla q_i^N \cdot n \, d\sigma \right]
\]
Therefore, taking $v = \nabla q_i^N$ and since, for all $1 \leq i, k \leq I$, $\langle \nabla q_i^N \cdot n, 1 \rangle_{\Gamma_k} = \delta_{i,k}$, we have:
\[
c_i = -\int_\Omega (\curl w) \times u \cdot \nabla q_i^N \, dx - \int_\Omega b \cdot \curl(\nabla q_i^N \times d) \, dx + \int_\Omega f \cdot \nabla q_i^N \, dx - \int_\Gamma P_0 \nabla q_i^N \cdot n \, d\sigma
\]
which gives the relation (5.74).
5.2 Weak solution in $W^{1, p}(\Omega)$ with $1 < p < +\infty$

$(2) \Rightarrow (1)$ Conversely, let $(u, b, P, c) \in W^{1, p}_w(\Omega) \times W^{1, p}_w(\Omega) \times W^{1, r}(\Omega) \times \mathbb{R}^I$ solution of (5.73)-(5.74) with $u \times n = b \times n = 0$ on $\Gamma$, and $(u \cdot n, 1)_{\Gamma_i} = (b \cdot n, 1)_{\Gamma_i} = 0$ for all $1 \leq i \leq I$. We want to prove that $(u, b, P, c)$ satisfies (4.2). Let us take $v \in \mathcal{D}(\Omega)$, $a = 0$, $\theta = \tau = 0$ as test functions in (5.73). We obtain

$$\langle -\Delta u + (\text{curl} w) \times u - (\text{curl} b) \times d - f, v \rangle_{\mathcal{D}'(\Omega) \times \mathcal{D}(\Omega)} = 0, \quad \forall v \in \mathcal{D}(\Omega)$$

So by De Rham’s theorem, there exists $P \in L^p(\Omega)$ such that

$$-\Delta u + (\text{curl} w) \times u - (\text{curl} b) \times d + \nabla P = f$$

So, $(u, b, P)$ satisfies the first equation of (4.2). Let us now take $a \in W^{1, p}_w(\Omega)$ with $a \times n = 0$ on $\Gamma$, $v = 0$, $\theta = \tau = 0$ as test functions in (5.73). We obtain

$$\langle (\text{curl} b - \text{curl}(u \times d) - g, a \rangle_{\Omega^*_p, p}$$

Applying a De Rham Lemma version for functionals acting on vector fields with vanishing tangential components (see [20, Lemma 2.2]), there exists $\chi \in L^2(\Omega)$ defined uniquely up to an additive constant such that:

$$\text{curl} \text{curl} b - \text{curl}(u \times d) + \nabla \chi = g \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma.$$ (5.81)

But taking the divergence in (5.81), $\chi$ is solution of the following Dirichlet problem:

$$\Delta \chi = \text{div} g = 0 \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma.$$

Since $g$ satisfies the compatibility condition (5.28), then $\chi$ is equal to zero and we have:

$$\text{curl} \text{curl} b - \text{curl}(u \times d) = g$$

So $(u, b)$ satisfies the second equation in (4.2).

Next, if we choose $v = a = 0$, $\tau = 0$ and $\theta \in \mathcal{D}(\Omega)$, then we obtain div $u = 0$ in $\Omega$. Similarly, if we choose $v = a = 0$, $\theta = 0$ and $\tau \in \mathcal{D}(\Omega)$, we obtain div $b = 0$ in $\Omega$.

It remains to prove the boundary condition given on the pressure $P$. To this end, we follow a method from [7, Proposition 3.7]. Let us take as test functions $v \in W^{1, p}_w(\Omega)$ with $v \times n = 0$ on $\Gamma$ and div $v \in W^{1, p}_w(\Omega)$, $a = 0$, $\theta \in W^{1, (p')'}(\Omega)$ and $\tau = 0$ in the variational formulation (5.73). Thus, applying Green formulæ as previously, we have:

$$(f, v)_{\Omega^*_p, p} = (u, -\Delta v)_{\Omega^*_p, p} - \int_\Omega (\text{curl} w) \times v \cdot u \, dx + \int_\Omega b \cdot \text{curl}(v \times d) \, dx + \int_\Omega 0 \cdot \theta \cdot u \, dx - \int_\Omega \nabla \theta \cdot u \, dx - \int_\Omega \nabla \theta \cdot u \, dx - \int_\Omega P \, \text{div} v \, dx + \int_\Gamma P \, v \cdot n \, d\sigma$$

We decompose $v$ as in (4.9) and to simplify the presentation, we set $z = \sum_{i=1}^I f_i (\int_{\Gamma_i} v \cdot n \, d\sigma) \nabla q_i^N$. So, $v = v_0 + z$. By definition of $q_i^N$ for $1 \leq i \leq I$, we have $\Delta z = 0$ and div $z = 0$ in $\Omega$. Thus:

$$(f, v_0)_{\Omega^*_p, p} + (f, z)_{\Omega^*_p, p} = (u, -\Delta v_0 - \text{curl} w) \times v_0 + \nabla \theta)_{\Omega^*_p, p} + (b, \text{curl}(v_0 \times d))_{\Omega^*_p, p} - \int_\Omega \nabla \theta \cdot u \, dx$$

Taking $(v_0, 0, 0, 0)$ as a test function in the variational formulation (5.73), we obtain

$$\int_\Gamma P \, v_0 \cdot n \, d\sigma - \int_\Gamma P_0 \, v_0 \cdot n \, d\sigma - \int_\Omega \nabla \theta \cdot u \, dx = 0$$

Note that, since div $u = 0$ in $\Omega$, $\theta = 0$ on $\Gamma_0$, $\theta = \beta$, on $\Gamma_i$ and $(u \cdot n, 1)_{\Gamma_i} = 0$, it follows that $\int_\Omega \nabla \theta \cdot u \, dx = - \int_\Omega \theta \, \text{div} u \, dx + \int_\Gamma \theta u \cdot n \, d\sigma = 0$. Therefore:

$$\int_\Gamma (P - P_0) v_0 \cdot n \, d\sigma = 0$$
Then, we deduce that
\[
\langle f, z \rangle_{\alpha, \beta} + \int_{\Omega} (\text{curl } w) \times z \cdot dx - \int_{\Omega} b \cdot \text{curl}(z \times d) \cdot dx - \int_{\Gamma} P z \cdot n \, d\sigma = 0.
\] (5.84)

Now, using (5.84) and the fact that \[\int_{\Gamma_i} v_0 \cdot n \, d\sigma = 0\] for all \(1 \leq i \leq I\), we have
\[
\int_{\Gamma} P v \cdot n \, d\sigma = \int_{\Gamma} P v_0 \cdot n \, d\sigma + \sum_{i=1}^{I} \left[ \int_{\Gamma_i} \int_{\Gamma_i} v \cdot n \, d\sigma \right] \left( c_i + \int_{\Gamma} P_0 \nabla q_i^N \cdot n \, d\sigma \right)
\]
\[\text{(5.85)}\]

However, we have from (5.73), for all \(1 \leq i \leq I\),
\[
c_i = \langle f, \nabla q_i^N \rangle_{\alpha, \beta} - \int_{\Gamma} P_0 \nabla q_i^N \cdot n \, d\sigma + \int_{\Omega} (\text{curl } b) \times d \cdot \nabla q_i^N \, dx - \int_{\Omega} (\text{curl } w) \times u \cdot \nabla q_i^N \, dx
\]
\[= \langle f, \nabla q_i^N \rangle_{\alpha, \beta} - \int_{\Gamma} P_0 \nabla q_i^N \cdot n \, d\sigma - \int_{\Omega} b \cdot \text{curl}(\nabla q_i^N \times d) \cdot dx - \int_{\Omega} (\text{curl } w) \times u \cdot \nabla q_i^N \, dx.
\]

Therefore, replacing in (5.85), we have:
\[
\int_{\Gamma} P v \cdot n \, d\sigma = \int_{\Gamma} P_0 v_0 \cdot n \, d\sigma + \sum_{i=1}^{I} \left[ \int_{\Gamma_i} \int_{\Gamma_i} v \cdot n \, d\sigma \right] \left( c_i + \int_{\Gamma} P_0 \nabla q_i^N \cdot n \, d\sigma \right)
\]
Moreover, applying directly the decomposition (4.9), we have:
\[
\int_{\Gamma} P_0 v \cdot n \, d\sigma = \int_{\Gamma} P_0 v_0 \cdot n \, d\sigma + \sum_{i=1}^{I} \left[ \int_{\Gamma_i} \int_{\Gamma_i} v \cdot n \, d\sigma \right] c_i = \int_{\Gamma} \left( P_0 + c \right) v \cdot n \, d\sigma,
\]
with \(c = 0\) on \(\Gamma_0\) and \(c = c_i\) on \(\Gamma_i\) for all \(1 \leq i \leq I\). We conclude as in [7, Theorem 3.2] to prove that \(P = P_0\) on \(\Gamma_0\) and \(P = P_0 + c\) on \(\Gamma_i\).

We are now in position to prove the following theorem

**Theorem 5.7.** We suppose \(\Omega\) of classe \(C^{1,1}\). Let \(\frac{3}{2} < p < 2\). Assume that \(f, g \in [H^s_0, \rho'(\text{curl } \Omega)]', P_0 \in W^{1-\frac{1}{p}, \rho}(\Gamma), h \in W^{1, \rho}(\Omega)\) with the compatibility conditions (5.27)-(5.28). Together, with \(\text{curl } w \in L^{3/2}(\Omega)\) and \(d \in W^{1,3/2}(\Omega)\). Then the linearized problem (4.2) has a unique solution \((u, f, b, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,\rho}(\Omega) \times \mathbb{R}I\). Moreover, we have the following estimates:
\[
\|u\|_{W^{1, p}(\Omega)} + \|b\|_{W^{1, p}(\Omega)} \leq C \left( 1 + \|\text{curl } w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)} \right) \left( \|f\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P_0\|_{W^{1-\frac{1}{p}, \rho}(\Gamma)} + \|g\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P\|_{W^{1-\frac{1}{p}, \rho}(\Omega)} \right) \left( \|f\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P_0\|_{W^{1-\frac{1}{p}, \rho}(\Gamma)} + \|g\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P\|_{W^{1-\frac{1}{p}, \rho}(\Omega)} \right).
\] (5.86)

\[
\|P\|_{W^{1-\frac{1}{p}, \rho}(\Omega)} \leq C \left( 1 + \|\text{curl } w\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)} \right)^2 \left( \|f\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P_0\|_{W^{1-\frac{1}{p}, \rho}(\Gamma)} + \|g\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P\|_{W^{1-\frac{1}{p}, \rho}(\Omega)} \right) \left( \|f\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P_0\|_{W^{1-\frac{1}{p}, \rho}(\Gamma)} + \|g\|_{[H^s_0, \rho'(\text{curl } \Omega)]'} + \|P\|_{W^{1-\frac{1}{p}, \rho}(\Omega)} \right).
\] (5.87)

**Proof.** Since \(2 < p' < 3\), thanks to Theorem 5.4, we have for any \((F, G, f, \phi) \in [H^s_0, \rho'(\text{curl } \Omega)]' \times [H^s_0, \rho'(\text{curl } \Omega)]' \times K_0^s(\Omega) \times W^{1,(p')}_0(\Omega)\) that the following problem
\[
- \Delta v - (\text{curl } w) \times v + \nabla \theta + (\text{curl } a) \times d = F \quad \text{and} \quad \text{div } v = 0 \quad \text{in } \Omega,
\]
\[
\text{curl } a + \text{curl}(v \times d) + \nabla \tau = G \quad \text{and} \quad \text{div } a = 0 \quad \text{in } \Omega,
\]
\[
v \times n = 0, \quad a \times n = 0 \quad \text{and} \quad \tau = 0 \quad \text{on } \Gamma, \quad \theta = 0 \quad \text{on } \Gamma_0 \quad \text{and} \quad \theta = \beta_i \quad \text{on } \Gamma_i,
\]
\[
(v \cdot n)_{\Gamma_i} = 0, \quad \text{and} \quad (a \cdot n, 1)_{\Gamma_i} = 0, \quad 1 \leq i \leq I.
\] (5.88)
In order to recover the solution of (5.89), we deduce that the linear mapping \( (F, G, \phi) \rightarrow (f, v)_{\Omega, p'} + (g, a)_{\Omega, p'} - \int_{\Gamma} P_0 v \cdot n \, d\sigma \) defines an element of the dual space of \( H^0_{\text{curl}}(\Omega) \times H^0_{\text{curl}}(\Omega) \times W^{1,p}(\Omega) \). It follows from Riesz' representation theorem that there exists a solution \( (u, b, P) \) in \( H^0_{\text{curl}}(\Omega) \times H^0_{\text{curl}}(\Omega) \times W^{1,-p}(\Omega) \) of the problem

\[
\langle u, F \rangle_{\Omega, p'} + (b, G)_{\Omega, p'} = \langle P, \phi \rangle_{W^{-1,p}(\Omega) \times W^{1,-p'}(\Omega)} = (f, v)_{\Omega, p'} + (g, a)_{\Omega, p'} - \int_{\Gamma} P_0 v \cdot n \, d\sigma
\]

which is the variational formulation (5.73). Moreover, it satisfies the estimate

\[
\|u\|_{H^0_{\text{curl}}(\Omega)} + \|b\|_{H^0_{\text{curl}}(\Omega)} + (1 + \|\text{curl} \, u\|_{L^2(\Omega)})^{-1} \|P\|_{W^{-1,-p}(\Omega)} \leq C(1 + \|\text{curl} \, u\|_{L^2(\Omega)^d} + \|\text{curl} \, v\|_{L^2(\Omega)^d} + \|\text{curl} \, a\|_{L^2(\Omega)^d} + |\Gamma|).
\]

In order to recover the solution of (4.2) through the equivalence result given in Lemma 5.6, it remains to prove that \( u, b \in W^{1,p}(\Omega), P \in W^{1,r}(\Omega) \), that \( \langle u \cdot n, 1 \rangle_{\Gamma_i} = 0, \langle b \cdot n, 1 \rangle_{\Gamma_i} = 0 \) for all \( 1 \leq i \leq I \) and to recover the relation of (5.74). We firstly want to show that \( \int_{\Gamma_i} u \cdot n \, d\sigma = 0 \) and \( \int_{\Gamma_i} b \cdot n \, d\sigma = 0 \). We choose \( (0, 0, \theta, 0) \) with \( \theta \in W^{1,(p')'}(\Omega) \) satisfying \( \theta = 0 \) on \( \Gamma_0 \) and \( \theta = \delta_{ij} \) on \( \Gamma_j \) for all \( 1 \leq j \leq I \) and a fixed \( 1 \leq i \leq I \). Then:

\[
0 = \langle u, \nabla \theta \rangle_{\Omega, p'} = \int_{\Omega} u \cdot \nabla \theta \, dx = \int_{\Gamma_i} \theta u \cdot n \, d\sigma - \int_{\Omega} \text{div} \, u \, \theta \, dx = \int_{\Gamma_i} u \cdot n \, d\sigma
\]

For the condition \( \int_{\Gamma_i} b \cdot n \, d\sigma = 0 \) for all \( 1 \leq i \leq I \), we set \( b = b - \sum_{i=1}^{I} I(b \cdot n, 1)_{\Gamma_i} \nabla q_i^N \). Observe that by the definition of \( q_i^N \), \( \dot{b} \) is also solution of (5.73) and satisfies the condition \( \langle b \cdot n, 1 \rangle_{\Gamma_i} = 0 \).

Next, taking test functions \( (0, 0, \theta, 0) \) and \( (0, 0, \tau, 0) \) with \( \theta \in W^{1,(p')'}(\Omega) \) as above and \( \tau \in \mathcal{D}(\Omega) \), we respectively recover \( \text{div} \, u = 0 \) and \( \text{div} \, b = 0 \) in \( \Omega \). Besides, since \( u, b \in H^0_{\text{curl}}(\Omega) \), we have \( u \) and \( b \) belong to \( X^N_{\text{curl}}(\Omega) \). From Theorem 3.1, we deduce that \( u, b \in W^{1,p}(\Omega) \). Thus, the estimate (5.86) follows from (3.4) and (5.91). Finally, in order to prove that \( P \in W^{1,r}(\Omega) \), we take the test functions \( (v, 0, 0, 0) \) with \( v \in \mathcal{D}(\Omega) \), and we obtain as in the proof of Lemma 5.6 that:

\[
\nabla P = f + \Delta u - (\text{curl} \, w) \times u + (\text{curl} \, b) \times d \quad \text{in} \quad \Omega.
\]

Then taking the divergence, \( P \) is solution of the following problem

\[
\Delta P = \text{div} \, f + \text{div}((\text{curl} \, b) \times d - (\text{curl} \, w) \times u) \quad \text{in} \quad \Omega,
\]

\[
P = P_0 \quad \text{on} \quad \Gamma_0 \quad \text{and} \quad P = P_0 + c_i \quad \text{on} \quad \Gamma_i.
\]
5.2 Weak solution in $W^{1,p}(\Omega)$ with $1 < p < +\infty$

Since $\text{curl } w \in L^p(\Omega)$ and $u \in W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, then $(\text{curl } w) \times u \in L^r(\Omega)$. Besides, $\text{curl } b \in L^p(\Omega)$ and $d \in W^{1,p}(\Omega) \hookrightarrow L^3(\Omega)$. So $(\text{curl } b) \times d \in L^r(\Omega)$. Hence, we obtain that $\Delta P \in W^{-1,r}(\Omega)$. Since $P_0$ belongs to $W^{1-1/r, r}(\Omega)$, we deduce that the solution $P$ of (5.92) belongs to $W^{1-1/r, r}(\Omega)$. Moreover, it satisfies the estimate

$$\|P\|_{W^{1-1/r, r}(\Omega)} \leq \|\text{div } f\|_{W^{-1,r}(\Omega)} + \|\text{div}((\text{curl } b) \times d - (\text{curl } u) \times u)\|_{W^{-1,r}(\Omega)} + \|P_0\|_{W^{1-1/r, r}(\Omega)}$$

Applying the characterization of $[H^p_0(\text{curl}, \Omega)]'$ given in Proposition 3.1, we write $f = F + \text{curl } \Psi$ with $F \in L^r(\Omega)$ and $\Psi \in L^p(\Omega)$. So

$$\|\text{div } f\|_{W^{-1,r}(\Omega)} = \|\text{div } F\|_{W^{-1,r}(\Omega)} = \sup_{\theta \in W^{1,r'}(\Omega)} \frac{|\langle \text{div } F, \theta \rangle|}{\|\theta\|_{W^{1,r'}(\Omega)}} = \sup_{\theta \in W^{1,r'}(\Omega)} \frac{|\langle F, \nabla \theta \rangle|}{\|\theta\|_{W^{1,r'}(\Omega)}} \leq \|F\|_{L^r(\Omega)} ,$$

which implies that

$$\|\text{div } f\|_{W^{-1,r'}(\Omega)} \leq \|f\|_{[H^p_0(\text{curl}, \Omega)]'} \quad (5.93)$$

In the same way, we have:

$$\|\text{div}((\text{curl } b) \times d)\|_{W^{-1,r}(\Omega)} \leq \|\text{curl } b\|_{L^p(\Omega)} \|d\|_{L^r(\Omega)} \leq C_d \|\text{curl } b\|_{W^{1,p}(\Omega)} \|d\|_{W^{1-2,p}(\Omega)} , \quad (5.94)$$

where $C_d$ is the constant related to the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Next,

$$\|\text{div}((\text{curl } u) \times u)\|_{W^{-1,r}(\Omega)} \leq \|\text{curl } u\|_{L^5(\Omega)} \|u\|_{L^r(\Omega)} \leq C \|\text{curl } u\|_{L^5(\Omega)} \|u\|_{W^{1,p}(\Omega)} , \quad (5.95)$$

where we have used the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Using estimates (5.93), (5.94), (5.95) combined with the estimate (5.86), we obtain the estimate (5.87) for the pressure. $\square$

The following result gives the regularity $W^{1,p}(\Omega)$ with $p < 2$ when the divergence of the velocity field $u$ does not vanish.

**Corollary 5.8.** Let $\frac{2}{3} < p < 2$. Assume that $f, g \in [H^p_0(\text{curl}, \Omega)]'$, $P_0 \in W^{1-1/p, p}(\Omega)$, $h \in W^{1,r}(\Omega)$ with the compatibility conditions (5.27)-(5.28), together with $\text{curl } u \in L^{3/2}(\Omega)$ and $d \in W^{1,3/2}_p(\Omega)$ . Then the linearized problem (4.2) has a unique solution $(u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}$. Moreover, we have the following estimates:

$$\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,r}(\Omega)} \leq C(1 + \|\text{curl } u\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)}) \left( \|f\|_{[H^p_0(\text{curl}, \Omega)]'} + \|P_0\|_{W^{1-1/p, p}(\Omega)} + \|g\|_{[H^p_0(\text{curl}, \Omega)]'} + (1 + \|\text{curl } u\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)}) \|h\|_{W^{1,r}(\Omega)} \right) \quad (5.96)$$

and

$$\|P\|_{W^{1,r}(\Omega)} \leq C(1 + \|\text{curl } u\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)})^2 \times \left( \|f\|_{[H^p_0(\text{curl}, \Omega)]'} + \|P_0\|_{W^{1-1/p, p}(\Omega)} + \|g\|_{[H^p_0(\text{curl}, \Omega)]'} + (1 + \|\text{curl } u\|_{L^{3/2}(\Omega)} + \|d\|_{W^{1,3/2}(\Omega)}) \|h\|_{W^{1,r}(\Omega)} \right) . \quad (5.97)$$

**Proof.** We can reduce the non-vanishing divergence problem for the velocity to the case where $\text{div } u = 0$, by solving the Dirichlet problem:

$$\Delta \theta = h \quad \text{ in } \Omega \quad \text{ and } \quad \theta = 0 \text{ on } \Gamma .$$

For $h \in W^{1,r}(\Omega)$, the solution $\theta$ belongs to $W^{3,r}(\Omega)$ and satisfies the estimate

$$\|\theta\|_{W^{3,r}(\Omega)} \leq C \|h\|_{W^{1,r}(\Omega)} . \quad (5.98)$$

Setting $z = u - \nabla \theta$, we obtain that $(z, b, P, c)$ is the solution of the problem treated in the Theorem 5.7 with $f$ and $g$ replaced by $\tilde{f} = f + \nabla h - (\text{curl } u) \times \nabla \theta$ and $\tilde{g} = g + \text{curl}(\nabla \theta \times d)$ respectively.

Indeed, we have $\nabla h \in L^r(\Omega) \hookrightarrow [H^p_0(\text{curl}, \Omega)]'$, and since $\nabla \theta \in W^{2,r}(\Omega) \hookrightarrow L^p(\Omega)$, then $(\text{curl } w) \times \nabla \theta \in L^r(\Omega) \hookrightarrow [H^p_0(\text{curl}, \Omega)]'$, and $\nabla \theta \times d \in L^p(\Omega)$ so $\text{curl}(\nabla \theta \times d) \in [H^p_0(\text{curl}, \Omega)]'$. Therefore, $\tilde{f}, \tilde{g} \in [H^p_0(\text{curl}, \Omega)]'$. Besides,
since we add a curl, then $\tilde{g}$ still satisfies the compatibility conditions (5.27)-(5.28). Thus, applying Theorem 5.7, we have the estimate:

$$\|z\|_{W^{1, p}(\Omega)} + \|b\|_{W^{1, p}(\Omega)} \leq C \left(1 + \|\text{curl } w\|_{L^2(\Omega)} + \|d\|_{W^{1, \frac{2}{3}}(\Omega)}\right) \times \left(\|\tilde{f}\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|\tilde{g}\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|P_0\|_{W^{1, \frac{1}{r}}(\Gamma)}\right)$$

We want to control the terms on $\tilde{f}$ and $\tilde{g}$.

$$\|\tilde{f}\|_{[H^2_0', p'(\text{curl}, \Omega)]'} \leq \|f\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|\nabla h\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|D\text{curl } w\| \times \nabla \theta\|_{[H^2_0', p'(\text{curl}, \Omega)]'}$$

$$\leq \|f\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|\nabla h\|_{L^r(\Omega)} + \|D\text{curl } w\| \times \nabla \theta\|_{L^r(\Omega)}$$

$$\|\tilde{g}\|_{[H^2_0', p'(\text{curl}, \Omega)]'} \leq C \|\text{curl } w\|_{L^2(\Omega)} \|\nabla \theta\|_{W^{1, p}(\Omega)}$$

and

$$\|P\|_{W^{1, r}(\Omega)} \leq C \left(1 + \|\text{curl } w\|_{L^2(\Omega)} + \|d\|_{W^{1, \frac{2}{3}}(\Omega)}\right)^2 \times \left(\|\tilde{f}\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|\tilde{g}\|_{[H^2_0', p'(\text{curl}, \Omega)]'} + \|P_0\|_{W^{1, \frac{1}{r}}(\Gamma)}\right).$$

Using the same arguments as previously, we can obtain the estimate (5.97).

In this subsection, we always take $P_0 \in W^{1, \frac{1}{p'}}(\Gamma)$ to obtain $P \in W^{1, r}(\Omega)$. However, since the pressure is decoupled from the system, we can improve its regularity given in the previous results by choosing a convenient boundary condition. For this, we begin by the following regularity concerning the Stokes problem $(\mathcal{S}_N)$ which is an improvement of [8, Theorem 2.2.6]:

**Theorem 5.9.** Let $\Omega$ be of class $C^{1,1}$. Let us assume $f \in [H^{1, p'}(\text{curl}, \Omega)]'$ and $h \in W^{1, r}(\Omega)$ with $1 \leq r \leq p$ and $\frac{1}{r} \leq \frac{1}{p} + \frac{1}{2}$. Then

1. If $r < 3$ and $P_0 \in W^{-\frac{1}{4}, r}(\Gamma)$, the Stokes problem $(\mathcal{S}_N)$ has a unique solution $(u, P) \in W^{1, p}(\Omega) \times L^r(\Omega)$.
2. If $r \geq 3$ and $P_0 \in W^{-\frac{1}{4}, q}(\Gamma)$ for any finite number $q > 1$, the Stokes problem $(\mathcal{S}_N)$ has a unique solution $(u, P) \in W^{1, p}(\Omega) \times L^q(\Omega)$.

**Proof.** Taking the divergence in the first equation of $(\mathcal{S}_N)$, we have:

$$\begin{cases}
\Delta P = \text{div } f + \Delta h & \text{in } \Omega, \\
P = P_0 & \text{on } \Gamma_0 \text{ and } P = P_0 + c_1 \text{ on } \Gamma_1.
\end{cases}$$

We split this problem in two parts: find $P_1$ such that

$$(P_1) \quad \Delta P_1 = \text{div } f + \Delta h \quad \text{in } \Omega \quad \text{and} \quad P_1 = 0 \text{ on } \Gamma,$$

and find $P_2$ such that

$$(P_2) \quad \Delta P_2 = 0 \quad \text{in } \Omega, \quad P_2 = P_0 \text{ on } \Gamma_0 \quad \text{and} \quad P_2 = P_0 + c_1 \text{ on } \Gamma_1.$$

We note that the regularity of $P_1$ is only dependent of div $f$ and $\Delta h$, and then we choose $P_0$ in order to recover for $P_2$ the same regularity as than for $P_1$. Then, we obtain, we $\Delta h$ of $P$ by adding $P_1$ and $P_2$. Let us analyze problem $(P_1)$. Since $f \in [H^{1, p'}(\text{curl}, \Omega)]'$, there exists $F \in L^r(\Omega)$ and $\Psi \in L^q(\Omega)$ such that $f = F + \text{curl } \Psi$. So div $f = \text{div } F \in W^{-1, r}(\Omega)$. Moreover, we have $\Delta h \in W^{-1, r}(\Omega)$. Then, div $f + \Delta h$ belongs to $\in W^{-1, r}(\Omega)$ which implies that problem $(P_1)$ has a
unique solution $P_1 \in W^{1,r}(\Omega)$. Next, we determine the regularity of $P_2$ with respect to the data $P_0$. We note that $P_0$ must be chosen so that the solution $P_2$ of $(P_2)$ could belong to a class of spaces containing spaces for $P_1$. We distinguish the following cases:

**Case $r < 3$:** If $P_0 \in W^{−1/r'}(\Gamma)$ with $r' = \frac{3}{3−r}$, the solution $P_2$ of problem $(P_2)$ belongs to $L^r(\Omega)$. Since $P_1 \in W^{1,r}(\Omega) \hookrightarrow L^r(\Omega)$, we deduce that $P = P_1 + P_2$ belongs to $L^r(\Omega)$.

**Case $r \geq 3$:** We have $P_1 \in W^{1,r}(\Omega) \hookrightarrow L^q(\Omega)$ for any finite number $q > 1$ if $r = 3$, for $q = \infty$ if $r > 3$. Thus, taking $P_0 \in W^{−\frac{3}{4}q}(\Gamma)$ for any $q > 1$ we have $P_2 \in L^q(\Omega)$ and then $P \in L^q(\Omega)$.

**Remark 5.2.** Observe that, using the above splitting, if $P_0 \in W^{1−1/r,r}(\Gamma)$, we have immediately $P \in W^{1,r}(\Omega)$.

The regularity result given in Theorem 5.9 enables us to improve the pressure in the linearized MHD system (4.2). In particular, we have the following result

**Corollary 5.10.** Let $p > \frac{3}{2}$, $f, g \in [H^{1,6'}(\Omega)]'$, $h \in W^{1,r}(\Omega)$, $P_0 \in W^{−1/r'}(\Gamma)$ satisfying the compatibility condition (5.27)-(5.28). We suppose that

- $\text{curl } w \in L^{\frac{3}{2}}(\Omega)$ and $d \in W^{\frac{1}{2},\frac{3}{2}}(\Omega)$ if $\frac{3}{2} < p < 2$.
- $\text{curl } w \in L^p(\Omega)$ and $d \in W^{1,p}(\Omega)$ if $p \geq 2$, where $s$ is defined in (5.6)

Then, the solution $(u, b, P)$ given in Proposition 5.5 and Theorem 5.7 of the linearized MHD problem (4.2) belongs to $W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times L^r(\Omega)$.

**Proof.** We are going to take advantage of the regularity results for the Stokes problem ($S_N$) given in Theorem 5.9. Then, we can rewrite (4.2) in the following way: Find $(u, P, c)$ such that

$$
\begin{aligned}
\begin{cases}
-\Delta u + \nabla P = \vec{f} & \text{in } \Omega, \\
\text{div } u = h & \text{in } \Omega, \\
u \cdot n = 0 & \text{on } \Gamma, \\
P = P_0 & \text{on } \Gamma_0 \\
P = P_0 + c_1 & \text{on } \Gamma_1,
\end{cases}
\end{aligned}
$$

with $\vec{f} = f - (\text{curl } w) \times u + (\text{curl } b) \times d$ and find $b$ solution of the following elliptic problem

$$
\begin{aligned}
\begin{cases}
\text{curl } b = \vec{g} & \text{in } \Omega, \\
\text{div } b = 0 & \text{in } \Omega, \\
b \cdot n = 0 & \text{on } \Gamma_{\leq 1} \\
(b \cdot n, 1)_{\Gamma_{\leq 1}} = 0, & \forall 1 \leq i \leq I,
\end{cases}
\end{aligned}
$$

with $\vec{g} = g + \text{curl}(u \times d)$. As in the previous proofs, we can easily verify that the assumptions on $f$, $\text{curl } w$ and $d$ imply that the term $\vec{f}$ belongs to $[H^{1,6'}(\Omega)]'$ for both cases $p < 2$ and $p \geq 2$. Thanks to Theorem 5.9, there exists a unique solution $(u, P, c) \in W^{1,p}(\Omega) \times L^r(\Omega) \times \mathbb{R}^d$ for the problem $(S_N)$. Besides, the existence of $b$ is independent of the pressure. Indeed, $\vec{g}$ belongs to $[H^{1,6'}(\Omega)]'$ and satisfies the compatibility conditions (5.27)-(5.28). It follows from Lemma 3.4 that problem $(S_N)$ has a unique solution $b \in W^{1,p}(\Omega)$.

### 5.3 Strong solution in $W^{2,p}(\Omega); 1 < p < 6/5$

The aim of this subsection is to complete the $L^p$–theory for the linearized MHD problem (4.2) by the proof of strong solutions in $W^{2,p}(\Omega)$ with $1 < p < 6/5$. One of the approach that we can use is to consider $w$ and $d$ more regular in a first step and then remove this regularity in a second step. Since the proof with this approach highly mimics that of Theorem 5.1, we put it in the Appendix (see Section 7). We are going to give a shorter and different proof where we take advantage of the regularity $W^{1,p}(\Omega)$ with $1 < p < 2$ for the linearized MHD problem (4.2).

**Theorem 5.11** (Strong solution in $W^{2,p}(\Omega)$ with $1 < p < \frac{6}{5}$). Suppose that $\Omega$ is of class $C^{2,1}$ and $1 < p < \frac{6}{5}$. Assume $h = 0$, and let $f, g \in L^p(\Omega)$, $P_0 \in W^{1−\frac{3}{4}p}(\Gamma)$, $\text{curl } w \in L^{3/2}(\Omega)$ and $d \in W^{1,3/2}(\Omega)$ with the compatibility conditions (4.5)-(5.1).
Then the linearized problem (4.2) has a unique solution \((u, b, P, c) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^l\) satisfying the following estimate:

\[
\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C \left( 1 + \|\text{curl }u\|_{L^2(\Omega)} + \|\mathbf{d}\|_{W^{1,2}(\Omega)} \right)^2 \times \left( \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1,2,p}(\Gamma)} \right),
\]

(5.99)

Proof. Observe that

\[
L^p(\Omega) \hookrightarrow [H^{s,p,\gamma}((\text{curl}, \Omega))]',
\]

\(P_0 \in W^{1-1/p,p}(\Gamma) \hookrightarrow W^{1-1/p,\gamma}(\Gamma),\)

(5.100)

where \(\beta = \frac{1}{p} - \frac{1}{2} = \frac{1}{p} + \frac{1}{2}\). Since \(1 < p < 6/5\), we have \(\frac{1}{p} < p' < 2\). Then applying the regularity \(W^{1,p}(\Omega)\) for the MHD system (4.2) for small values of \(p\) (see Theorem 5.7 with \(h = 0\)), we can deduce the existence of a solution \((u, b, P, c) \in W^{1,p}(\Omega) \times W^{1,p,\gamma}(\Omega) \times W^{1,\gamma}(\Omega) \times \mathbb{R}^l\) satisfying the estimate:

\[
\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} \leq C (1 + \|\text{curl }u\|_{L^2(\Omega)} + \|\mathbf{d}\|_{W^{1,2}(\Omega)}) \left( \|f\|_{L^p(\Omega)} + \|P_0\|_{W^{1,2,p}(\Gamma)} + \|g\|_{L^p(\Omega)} \right),
\]

(5.101)

where we have used the embedding (5.100) and

\[
\|P\|_{W^{1,\gamma}(\Omega)} \leq C (1 + \|\text{curl }u\|_{L^2(\Omega)} + \|\mathbf{d}\|_{W^{1,2}(\Omega)}) \left( \|f\|_{L^p(\Omega)} + \|P_0\|_{W^{1,2,p}(\Gamma)} + \|g\|_{L^p(\Omega)} \right),
\]

Since \(W^{1,p,\gamma}(\Omega) \hookrightarrow L^{p,\gamma}(\Omega)\) with \(\frac{1}{p} = \frac{1}{p} - \frac{1}{2}\), the terms \((\text{curl }u) \times u\) and \((\text{curl }b) \times \mathbf{d}\) belong to \(L^p(\Omega)\). \((u, P, c)\) is then a solution of the problem \((\mathcal{L}_\mathcal{C})\) with \(h = 0\) and a RHS \(f\) in \(L^p(\Omega)\). We deduce from Proposition 3.2 that \((u, P)\) belongs to \(W^{2,p}(\Omega) \times W^{1,p}(\Omega)\) and satisfies the estimate:

\[
\|u\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C (1 + \|\text{curl }u\|_{L^2(\Omega)} + \|\mathbf{d}\|_{W^{1,2}(\Omega)}) \left( \|f\|_{L^p(\Omega)} + \|P_0\|_{W^{1,2,p}(\Gamma)} + \|g\|_{L^p(\Omega)} \right),
\]

(5.102)

Next, \(b\) is a solution of the elliptic problem \((\mathcal{L}_\mathcal{G})\) with a RHS \(g + \text{curl}(u \times d)\) which belongs to \(L^p(\Omega)\). Thanks to Theorem 3.3, the solution \(b\) belongs to \(W^{2,p}(\Omega)\) and satisfies the estimate:

\[
\|b\|_{W^{2,p}(\Omega)} \leq C_E \left( \|g\|_{L^p(\Omega)} + \|\text{curl}(u \times d)\|_{L^p(\Omega)} \right).
\]

(5.103)

Moreover, we have the following bounds:

\[
\|\text{curl }u\|_{L^p(\Omega)} \leq \|\text{curl }u\|_{L^{3/2}(\Omega)} \leq C \|\text{curl }u\|_{L^{2}(\Omega)} \leq C \|\text{curl }u\|_{L^{2}(\Omega)},
\]

(5.104)

and similarly,

\[
\|\text{curl }b\|_{L^p(\Omega)} \leq \|\text{curl }b\|_{L^{2}(\Omega)} \leq C \|\text{curl }b\|_{L^{2}(\Omega)},
\]

(5.105)

Collecting the above bounds together with (5.101) in (5.102)-(5.103) leads to the bound (5.99).

\[\blacksquare\]

Proceeding as in the proof of Corollary 5.8, we can extend the previous result to the non-vanishing divergence case. The proof of the following result can be found in the Appendix (see Section 7).

**Corollary 5.12.** Let \(\Omega\) be of class \(C^{2,1}\) and \(1 < p < \frac{6}{5}\). Assume that \(f, g \in L^p(\Omega), P_0 \in W^{1-1/p,p}(\Gamma), h \in W^{1,p}(\Omega)\), \(\text{curl }u \in L^{3/2}(\Omega)\) and \(d \in W^{1/2,p}(\Omega)\) with the compatibility conditions (4.5)-(4.1). Then the linearized problem (4.2) has a unique solution \((u, b, P, c) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega) \times \mathbb{R}^l\). Moreover, we have the following estimates:

\[
\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C (1 + \|\text{curl }u\|_{L^2(\Omega)} + \|\mathbf{d}\|_{W^{1,2}(\Omega)} \left( \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1,2,p}(\Gamma)} \right) + (1 + \|\text{curl }u\|_{L^2(\Omega)} + \|\mathbf{d}\|_{W^{1,2}(\Omega)}) \|h\|_{W^{1-1/p,p}(\Gamma)}
\]

(5.106)
6 The nonlinear MHD system

In this section, we consider the nonlinear problem and we study the existence of generalized and strong solutions for (MHD).

6.1 Existence and uniqueness: \( L^2 \)-theory

In this subsection, we establish the existence and uniqueness for the weak solution in the Hilbert case for the problem (MHD). The following result is one of the main results. First, we recall that \( \langle \cdot, \cdot \rangle_{\Omega} \) denotes the duality product between \( H_0^{1,p}(\text{curl}, \Omega)' \) and \( H_0^{1,p}(\text{curl}, \Omega) \) and \( \langle \cdot, \cdot \rangle_T \) denotes the duality product between \( H^{-1/2}(\Gamma) \) and \( H^{1/2}(\Gamma) \).

**Theorem 6.1.** (Weak solutions of (MHD) system in \( H^1(\Omega) \)). Let \( \Omega \) be of classe \( C^{1,1} \) and let

\[
\begin{align*}
f, g & \in [H_0^{1,2}(\text{curl}, \Omega)]', \\
h & = 0 \quad \text{and} \quad P_0 \in \dot{H}^{1/2}(\Gamma)
\end{align*}
\]

satisfying the compatibility conditions

\[
\forall v \in K_0^2(\Omega), \quad \langle g, v \rangle_{\Omega} = 0, \quad \text{div} \, g = 0 \quad \text{in} \, \Omega. \tag{6.1}
\]

Then the (MHD) problem has at least one weak solution \((u, b, P, \alpha) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}^2 \) such that

\[
\|u\|_{H^1(\Omega)} + \|b\|_{H^1(\Omega)} + \|P\|_{L^2(\Omega)} \leq M, \tag{6.3}
\]

where \( M = C\|f\|_{[H_0^{1,2}(\text{curl}, \Omega)]'} + \|g\|_{[H_0^{1,2}(\text{curl}, \Omega)]'} + \|P_0\|_{\dot{H}^{-1/2}(\Gamma)} \) with

\[
\alpha_i = \langle f, \nabla q_i^N \rangle_{\Omega} - \langle P_0, \nabla q_i^N \rangle_{\Gamma} + \int_\Omega \langle \text{curl} \, b \times b \rangle \cdot \nabla q_i^N \, dx - \int_\Omega \langle \text{curl} \, u \rangle \times u \cdot \nabla q_i^N \, dx. \tag{6.4}
\]

In addition, suppose that \( f, g \) and \( P_0 \) are small in the sense that

\[
C_1 C_2^2 M \leq \frac{2}{3C^2_T}, \tag{6.5}
\]

where \( C_T \) is the constant in (3.6) and \( C_1, C_2 \) are given in (6.15). Then the weak solution \((u, b, P) \) of (MHD) is unique.

We first recall that the space \( V_N(\Omega) \) denotes

\[
V_N(\Omega) := \{ v \in H^1(\Omega); \, \text{div} \, v = 0 \in \Omega, \, v \times n = 0 \text{ on} \, \Gamma, \langle v \cdot n, 1 \rangle_\Gamma = 0, \, \forall 1 \leq i \leq I \}
\]

and we give the following definition.

**Definition 6.2.** Given \( f, g \in [H_0^{1,2}(\text{curl}, \Omega)]' \) and \( P_0 \in \dot{H}^{1/2}(\Gamma) \) with the compatibility conditions (6.1)-(6.2), \((u, b, \alpha) \in V_N(\Omega) \times V_N(\Omega) \times \mathbb{R} \) is called a weak solution of (MHD) problem if it satisfies: for all \( (v, \Psi) \in V_N(\Omega) \times V_N(\Omega) \),

\[
\begin{align*}
\int_\Omega \langle \text{curl} \, u \rangle \times u & \, dx + \int_\Omega (\text{curl} \, u \times u) \cdot v \, dx - \int_\Omega (\text{curl} \, b \times b) \cdot v \, dx + \int_\Omega \text{curl} \, b \cdot \text{curl} \, \Psi \, dx \\
+ \int_\Omega (\text{curl} \, \Psi \times b) \cdot u \, dx & = \langle f, \Psi \rangle_{\Omega} + \langle g, \Psi \rangle_{\Omega} - \langle P_0, v \cdot n \rangle_{\Gamma} - \sum_{i=1}^I (\langle P_0 + \alpha_i, v \cdot n \rangle_{\Gamma}), \tag{6.6}
\end{align*}
\]

and

\[
\alpha_i = \langle f, \nabla q_i^N \rangle_{\Omega} - \langle P_0, \nabla q_i^N \rangle_{\Gamma} + \int_\Omega \langle \text{curl} \, b \times b \rangle \cdot \nabla q_i^N \, dx - \int_\Omega \langle \text{curl} \, u \rangle \times u \cdot \nabla q_i^N \, dx. \tag{6.7}
\]

To interpret (6.6)-(6.7), it is convenient to remove the constraint of fluxes of the test functions through \( \Gamma_i \). In the following Lemma, we prove that (6.6) can be extended to any test function \((v, \Psi) \in X_N^2(\Omega) \times X_N^2(\Omega)\).

**Lemma 6.1.** Let \( f, g \in [H_0^{1,2}(\text{curl}, \Omega)]' \) and \( P_0 \in \dot{H}^{-1/2}(\Gamma) \) with the compatibility conditions (6.1)-(6.2). Then, the following two statements are equivalent:

1. \((u, b, \alpha) \in V_N(\Omega) \times V_N(\Omega) \times \mathbb{R} \) satisfies (6.6)-(6.7) for any \((v, \Psi) \in V_N(\Omega) \times V_N(\Omega)\).
2. \((u, b, \alpha) \in V_N(\Omega) \times V_N(\Omega) \times \mathbb{R} \) satisfies (6.6)-(6.7) for any \((v, \Psi) \in X_N^2(\Omega) \times X_N^2(\Omega) \).
Proof. Since $V_N(\Omega) \subset X^2_N(\Omega)$, then (ii) implies (i). Conversely, let $(u, b, \alpha_i) \in V_N(\Omega) \times V_N(\Omega) \times \mathbb{R}$ satisfying (6.6)-(6.7) for any $(v, \Psi) \in V_N(\Omega) \times V_N(\Omega)$ and we want to show it implies (ii). The proof is similar to than given for the linearized problem (see Proposition 4.1).

Let $(\tilde{\Psi}, \tilde{v}) \in X^2_N(\Omega) \times X^2_N(\Omega)$. We set

$$
\Psi = \tilde{\Psi} - \sum_{i=1}^I (\tilde{\Psi} \cdot n, 1)_{\Gamma_i} \nabla q_i^N \quad \text{and} \quad v = \tilde{v} - \sum_{i=1}^I (\tilde{\Psi} \cdot n, 1)_{\Gamma_i} \nabla q_i^N.
$$

Then, $(\Psi, v)$ belongs to $V_N(\Omega) \times V_N(\Omega)$. Replacing in (6.6)-(6.7), we obtain

$$
\begin{align*}
\int_\Omega \text{curl} \, b \cdot \text{curl} \, \Psi \, dx &+ \int_\Omega (\text{curl} \, \Psi \times b) \cdot u \, dx - \langle g, \Psi \rangle_{\Omega_\alpha, 2} + \int_\Omega \text{curl} \, u \cdot \text{curl} \, \tilde{v} \, dx \\
&+ \int_\Omega (\text{curl} \, u \times u) \cdot \tilde{v} \, dx - \int_\Omega (\text{curl} \, b \times b) \cdot \tilde{v} \, dx - \langle f, \tilde{v} \rangle_{\Omega_\alpha, 2} + \langle P_0, \tilde{v} \cdot n \rangle_{\Gamma_0} + \sum_{i=1}^I \langle P_0 + \alpha_i, \tilde{v} \cdot n \rangle_{\Gamma_i}
\end{align*}
$$

$$
\begin{align*}
= \sum_{i=1}^I (\tilde{\Psi} \cdot n, 1)_{\Gamma_i} \left( \int_\Omega \text{curl} \, b \cdot \text{curl} (\nabla q_i^N) \, dx + \int_\Omega (\text{curl} \, \nabla q_i^N) \times b \cdot u \, dx - \langle g, \nabla q_i^N \rangle_{\Omega_\alpha, 2} \right) \\
+ \sum_{i=1}^I (\tilde{v} \cdot n, 1)_{\Gamma_i} \left( \int_\Omega \text{curl} \, u \cdot (\text{curl} \, \nabla q_i^N) \, dx - \int_\Omega (\text{curl} \, b \times b) \cdot \nabla q_i^N \, dx + \int_\Omega (\text{curl} \, u \times u) \cdot \nabla q_i^N \, dx \\
- \langle f, \nabla q_i^N \rangle_{\Omega_\alpha, 2} + \langle P_0, \nabla q_i^N \cdot n \rangle_{\Gamma_0} + \sum_{j=1}^I \langle P_0 + \alpha_j, \nabla q_i^N \cdot n \rangle_{\Gamma_j} \right)
\end{align*}
$$

(6.8)

So, we have

$$
\sum_{i=1}^I (\tilde{v} \cdot n, 1)_{\Gamma_i} \left( \int_\Omega (\text{curl} \, b \times \nabla q_i^N) \, dx + \int_\Omega (\text{curl} \, u \times u) \cdot \nabla q_i^N \, dx - \langle f, \nabla q_i^N \rangle_{\Omega_\alpha, 2} \\
+ \langle P_0, \nabla q_i^N \cdot n \rangle_{\Gamma_0} + \sum_{j=1}^I \langle P_0 + \alpha_j, \nabla q_i^N \cdot n \rangle_{\Gamma_j} \right)
$$

Since $q_i^N$ satisfies (3.2), we have in particular that $\sum_{j=1}^I (\alpha_j, \nabla q_i^N \cdot n)_{\Gamma_j} = \alpha_i \delta_{ij}$. Then, we obtain

$$
\sum_{j=1}^I \langle P_0 + \alpha_j, \nabla q_i^N \cdot n \rangle_{\Gamma_j} = \sum_{j=1}^I \langle P_0, \nabla q_i^N \cdot n \rangle_{\Gamma_j} + \alpha_i
$$

(6.9)

Replacing (6.7) in (6.9), we obtain from (6.8) that

$$
\begin{align*}
\int_\Omega \text{curl} \, u \cdot \text{curl} \, \tilde{v} \, dx &+ \int_\Omega (\text{curl} \, u \times u) \cdot \tilde{v} \, dx - \int_\Omega (\text{curl} \, b \times b) \cdot \tilde{v} \, dx + \int_\Omega \text{curl} \, b \cdot \text{curl} \, \tilde{\Psi} \, dx \\
&+ \int_\Omega (\text{curl} \, \tilde{\Psi} \times b) \cdot u \, dx = \langle f, \tilde{v} \rangle_{\Omega_\alpha, 2} + \langle g, \tilde{\Psi} \rangle_{\Omega_\alpha, 2} - \langle P_0, v \cdot n \rangle_{\Gamma_0} - \sum_{i=1}^I \langle P_0 + \alpha_i, \tilde{v} \cdot n \rangle_{\Gamma_i}
\end{align*}
$$

(6.10)

which is (6.6) with test functions $(\tilde{\Psi}, \tilde{v}) \in X^2_N(\Omega) \times X^2_N(\Omega)$. This completes the proof.

$\square$

We can now prove the following result

**Theorem 6.2.** Let $f, g \in [H^0_0(\text{curl}, \Omega)]'$ and $P_0 \in H^{-\frac{1}{2}}(\Gamma)$ with the compatibility conditions (6.1)-(6.2). Then, the following two statements are equivalent:

(i) $(u, b, P, \alpha_i) \in H^1(\Omega) \times H^1(\Omega) \times L^2(\Omega) \times \mathbb{R}$ is a solution of (MHD),

(ii) $(u, b, \alpha_i) \in V_N(\Omega) \times V_N(\Omega) \times \mathbb{R}$ is a weak solution of (MHD), in the sense of Definition (6.2).

**Proof.** The proof that (i) implies (ii) is very similar to that of Proposition 4.1, hence we omit it. Let $(u, b, \alpha_i) \in V_N(\Omega) \times V_N(\Omega) \times \mathbb{R}$ satisfying (6.6)-(6.7) for any $(v, \Psi) \in V_N(\Omega) \times V_N(\Omega)$. Due to Lemma 6.1, we have that $(u, b, \alpha_i) \in$
\( V_N(\Omega) \times V_N(\Omega) \times \mathbb{R} \) satisfies also (6.6)-(6.7) for any \((v, \Psi) \in X_\kappa^3(\Omega) \times X_\kappa^3(\Omega)\). Choosing \(v \in D_p(\Omega)\) and \(\Psi = 0\) as test functions in (6.6), we have

\[
\langle -\Delta u + (\text{curl } u) \times u - (\text{curl } b) \times b - f, v \rangle_{D_p'(\Omega) \times D'(\Omega)} = 0.
\]

So, by De Rham's theorem, there exists an unique \(P \in L^2(\Omega)\) such that

\[
-\Delta u + \nabla P + (\text{curl } u) \times u - (\text{curl } b) \times b - f = 0 \quad \text{in } \Omega. 
\]

(6.11)

Next, choosing \((0, \Psi)\) with \(\Psi \in D_p(\Omega)\) in (6.6), we have

\[
(\text{curl } \text{curl } b - \text{curl } (u \times b) - g, \Psi)_{D_p'(\Omega) \times D'(\Omega)} = 0.
\]

Then, applying [20, Lemma 2.2], we have \(\chi \in L^2(\Omega)\) defined uniquely up to an additive constant such that

\[
\text{curl } \text{curl } b - \text{curl } (u \times b) - g = \nabla \chi \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma
\]

Since \(\chi\) is solution of the following harmonic problem

\[
\Delta \chi = 0 \quad \text{in } \Omega \quad \text{and} \quad \chi = 0 \quad \text{on } \Gamma.
\]

We deduce that \(\chi = 0\) in \(\Omega\) which gives the second equation in (MHD). Moreover, by the fact that \(u\) and \(b\) belong to the space \(V_N(\Omega)\), we have \(\text{div } u = \text{div } b = 0\) in \(\Omega\) and \(u \times n = b \times n = 0\) on \(\Gamma\). The proof of the boundary conditions on the pressure is fairly similar to that given in [7, Proposition 3.7].

**Proof of Theorem 6.1:**

We use the Schauder fixed point Theorem. We make use of the product space \(Z_N(\Omega) = V_N(\Omega) \times V_N(\Omega)\) defined in (4.17). We define the mapping \(G : Z_N(\Omega) \to Z_N(\Omega)\) such that \(G(w, d) = (u, b)\) with \((u, b) \in Z_N(\Omega)\) a solution of the linearized problem (4.18). By Theorem 4.2, for each pair \((w, d) \in Z_N(\Omega)\) the solution \((u, b) \in Z_N(\Omega)\) of problem (4.18) exists, and is unique satisfying the following estimate:

\[
\| (u, b) \|^2_{Z_N(\Omega)} = \| u \|^2_{H^1(\Omega)} + \| b \|^2_{H^1(\Omega)} \leq C( \| f \|_{H_b^0(\Omega)} + \| g \|_{H_b^0(\Omega)} + \| P_b \|_{H^{-1}(\Omega)} ) := M
\]

(6.12)

for some constant \(C > 0\) independent of \(w\) and \(d\).

We define the ball

\[
B \subset \{ (v, \Psi) \in Z_N(\Omega) : \| (v, \Psi) \|^2_{Z_N(\Omega)} \leq r \},
\]

where \(r = M = C( \| f \|_{H_b^0(\Omega)} + \| g \|_{H_b^0(\Omega)} + \| P_b \|_{H^{-1}(\Omega)} )\). By the definition of \(G\) and (6.12), it follows that \(G(B_r) \subset B_r\), and we prove that such a mapping is compact on \(B_r\). For this, let \(\{ (w_k, d_k) \}_{k \geq 1} \) be an arbitrary sequence in \(B_r\). Since \(H^1(\Omega)\) is reflexive, there exists a subsequence still denoted \(\{ (w_k, d_k) \}_{k \geq 1} \) and a pair \((w, d)\) in \(B_r\) such that \((w_k, d_k)\) converges weakly to \((w, d)\) in \(H^1(\Omega) \times H^1(\Omega)\) as \(k \to \infty\). We set \((u_k, b_k) = G(w_k, d_k)\) and \((\tilde{u}, \tilde{b}) = G(w, d)\). We have that \((u_k, b_k)\) converges strongly to \((\tilde{u}, \tilde{b})\) in \(H^1(\Omega) \times H^1(\Omega)\) as \(k \to \infty\). By the compactness of the embedding \(H^1(\Omega) \hookrightarrow L^4(\Omega)\), we have that \(w_k \to w\) strongly in \(L^4(\Omega)\) and \(d_k \to d\) strongly in \(L^4(\Omega)\). By definition of \((u_k, b_k)\) and \((\tilde{u}, \tilde{b})\) are solutions of

\[
a((u_k, b_k), (v, \Psi)) + a_{w_k, d_k}((u_k, b_k), (v, \Psi)) = \mathcal{L}(v, \Psi),
\]

and

\[
a((\tilde{u}, \tilde{b}), (v, \Psi)) + a_{w, d}((\tilde{u}, \tilde{b}), (v, \Psi)) = \mathcal{L}(v, \Psi).
\]

(6.13)

where the forms \(a\) and \(a_{w, d}\) are defined in (4.16). By subtracting the above problems, we obtain

\[
(\text{curl } u_k - \tilde{u}, \text{curl } v) + (\text{curl } b_k - \tilde{b}, \text{curl } \Psi) + ((\text{curl } w_k) \times u_k - (\text{curl } w) \times \tilde{u}, v) + (u_k, (\text{curl } \Psi) \times d_k) - (\tilde{u}, (\text{curl } \Psi) \times d) - ((\text{curl } b_k) \times d_k, v) + ((\text{curl } \tilde{b}) \times d, v) = 0
\]

(6.14)

Since

\[
((\text{curl } w_k) \times u_k - (\text{curl } w) \times \tilde{u}, v) = (\text{curl } w \times (u_k - \tilde{u}), v) + (\text{curl } (w_k - w) \times u_k, v)
\]
and
\begin{align*}
(u_k, (\text{curl } \Psi) \times d_k) - (\tilde{u}, (\text{curl } \Psi) \times d) - ((\text{curl } b_k) \times d_k, v) + ((\text{curl } \tilde{b}) \times d, v) \\
= (u_k - \tilde{u}, (\text{curl } \Psi) \times d) - (u_k, (\text{curl } \Psi) \times d) - (\text{curl}(b_k - \tilde{b}) \times d, v) \\
+ ((\text{curl } b_k) \times d, v) + (u_k, (\text{curl } \Psi) \times d_k) - ((\text{curl } b_k) \times d_k, v).
\end{align*}

Then, replacing in (6.14), we obtain
\begin{align*}
a((u_k - \tilde{u}, b_k - \tilde{b}), (v, \Psi)) + a_{w,d}((u_k - \tilde{u}, b_k - \tilde{b}), (v, \Psi)) \\
= -(\text{curl}(w_k - w) \times u_k, v) + (u_k, \text{curl} (d - d_k)) - (\text{curl } b_k \times (d - d_k), v)
\end{align*}

By Hölder inequality and the fact that \((u_k, b_k)\) belongs to \(B_r\), we have
\begin{align*}
|((\text{curl } w_k - w) \times u_k, v)| &\leq C_2 \|w_k - w\|_{L^4(\Omega)} \|u_k\|_{H^3(\Omega)} \|v\|_{H^1(\Omega)} \\
&\leq C_2 M \|w_k - w\|_{L^4(\Omega)} \|v\|_{H^1(\Omega)},
\end{align*}
\begin{align*}
|((\text{curl } w_k - w) \times u_k, v)| &\leq \|u_k\|_{L^4(\Omega)} \|\text{curl } \Psi\|_{L^2(\Omega)} \|d - d_k\|_{L^4(\Omega)} \\
&\leq C_1 C_2 M \|\text{curl } \Psi\|_{H^1(\Omega)} \|d - d_k\|_{L^4(\Omega)}
\end{align*}
and
\begin{align*}
|((\text{curl } b_k \times (d - d_k), v)| &\leq C \|\text{curl } b_k\|_{L^2(\Omega)} \|d - d_k\|_{L^4(\Omega)} \|v\|_{L^4(\Omega)} \\
&\leq C_1 C_2 M \|d - d_k\|_{L^4(\Omega)} \|v\|_{H^1(\Omega)}
\end{align*}
where \(C_1 > 0\) and \(C_2 > 0\) are such that
\[ \|\text{curl } v\|_{L^4(\Omega)} \leq C_1 \|v\|_{H^1(\Omega)} \quad \text{and} \quad \|v\|_{L^4(\Omega)} \leq C_2 \|v\|_{H^1(\Omega)}. \]

Choosing \((v, \Psi) = (u_k - \tilde{u}, b_k - \tilde{b})\), thanks to the coercivity of the form \(a\) in (4.19) and the fact that \(a_{w,d}((u_k - \tilde{u}, b_k - \tilde{b}), (u_k - \tilde{u}, b_k - \tilde{b})) = 0\), we obtain
\begin{align*}
\frac{2}{C_{\tilde{b}}^2} \left( \|u_k - \tilde{u}\|^2_{H^1(\Omega)} + \|b_k - \tilde{b}\|^2_{H^1(\Omega)} \right) &\leq |a((u_k - \tilde{u}, b_k - \tilde{b}), (u_k - \tilde{u}, b_k - \tilde{b}))| \leq C_2 M \|w_k - w\|_{L^4(\Omega)} \|u_k - \tilde{u}\|_{H^3(\Omega)} + C_1 C_2 M \|d - d_k\|_{L^4(\Omega)} \|u_k - \tilde{u}\|_{H^3(\Omega)} + \|b_k - \tilde{b}\|_{H^1(\Omega)} \\
&\leq \frac{1}{2} \|u_k - \tilde{u}\|^2_{H^1(\Omega)} + \frac{1}{2} \|b_k - \tilde{b}\|^2_{H^1(\Omega)} + C_2^2 M^2 \|w_k - w\|^2_{L^4(\Omega)} + \frac{3}{2} C_2^2 C_2 M^2 \|d_k - d\|^2_{L^4(\Omega)}.
\end{align*}
So, we have
\[ \|u_k - \tilde{u}\|^2_{H^1(\Omega)} + \|b_k - \tilde{b}\|^2_{H^1(\Omega)} \leq C \left( \|w_k - w\|^2_{L^4(\Omega)} + \|d_k - d\|^2_{L^4(\Omega)} \right) \rightarrow 0 \quad \text{as } k \rightarrow 0, \]
where \(C = C_{\tilde{b}}^2 \max(C_2^2 M^2, C_2^2 C_2 M^2)\) is independent of \(k\). Hence, this gives the compactness of \(G\). From Schauder’s theorem we then find that \(G\) has a fixed point \((\tilde{u}, \tilde{b}) = G(\tilde{u}, \tilde{b}) \in B_r\). This fixed point is solution of (6.13). Moreover, \((\tilde{u}, \tilde{b})\) satisfies (6.12).

Now, we establish the uniqueness of the solution of (MHD). For this, let \((u_1, b_1, P_1)\) and \((u_2, b_2, P_2)\) in \(V_N(\Omega) \times V_N(\Omega) \times L^2(\Omega)\) be two solutions of (MHD). We set \(u = u_1 - u_2, b = b_1 - b_2\) et \(P = P_1 - P_2\) and we want to prove that \(u = b = 0\) and \(P = 0\). Choose \(v = u\) and \(\Psi = b\) in (6.6), then \((u, b)\) satisfies
\[ \|\text{curl } u\|_{L^2(\Omega)} + \|\text{curl } b\|_{L^2(\Omega)} + ((\text{curl } u_1) \times u_1 - (\text{curl } u_2) \times u_2, u) \\
- ((\text{curl } b_1) \times b_1 - (\text{curl } b_2) \times b_2, u) + ((\text{curl } b) \times b_1, u_1) - ((\text{curl } b) \times b_2, u_2) = 0 \]
Observe that
\[ ((\text{curl } u_1) \times u_1, u) - ((\text{curl } u_2) \times u_2, u) = ((\text{curl } u) \times u_1, u). \]
and
\[ ((\text{curl } b_1) \times b_1 - (\text{curl } b_2) \times b_2, u) + ((\text{curl } b) \times b_1, u_1) - ((\text{curl } b) \times b_2, u_2) = ((\text{curl } b) \times b, u_2) - ((\text{curl } b_2) \times b, u) \]
which gives
\[
\|\text{curl } u\|^2_{L^2(\Omega)} + \|\text{curl } b\|^2_{L^2(\Omega)} + ((\text{curl } u_1) \times u_1 - (\text{curl } u_2) \times u_2, u)
\]
\[-((\text{curl } b_1) \times b_1 - (\text{curl } b_2) \times b_2, u) + ((\text{curl } b) \times b_1, u_1) - ((\text{curl } b) \times b_2, u_2)
\]
\[= \|\text{curl } u\|^2 + \|\text{curl } b\|^2 + ((\text{curl } u) \times u_1, u)
\]
\[+ ((\text{curl } b) \times b, u_2) - ((\text{curl } b) \times b, u)
\]
Then,
\[\|\text{curl } u\|^2 + \|\text{curl } b\|^2 = ((\text{curl } b_2) \times b, u) - ((\text{curl } u) \times u_1, u) - ((\text{curl } b) \times b, u_2) \quad (6.16)
\]
We want to bound the terms in the RHS of (6.16). We have
\[
|((\text{curl } u) \times u_1, u)| \leq \|\text{curl } u\|_{L^2(\Omega)} \|u_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}
\]
\[\leq C_1 C_2 \|u\|^2_{H^1(\Omega)} \|u_1\|_{L^2(\Omega)}
\]
\[\leq C_1 C_2 \|u_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq C_1 C_2 M \|u\|^2_{H^1(\Omega)},
\]
\[|((\text{curl } b) \times b, u_2)| \leq \|\text{curl } b\|_{L^2(\Omega)} \|b\|_{L^2(\Omega)} \|u_2\|_{L^2(\Omega)}
\]
\[\leq C_1 C_2 \|b\|^2_{H^1(\Omega)} \|u_2\|_{H^1(\Omega)}
\]
\[\leq C_1 C_2 M \|b\|^2_{H^1(\Omega)},
\]
and
\[|((\text{curl } b_2) \times b, u)| \leq \|\text{curl } b_2\|_{L^2(\Omega)} \|b\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}
\]
\[\leq C_1 C_2 \|b_2\|_{H^1(\Omega)} \|b\|_{H^1(\Omega)} \|u\|_{H^1(\Omega)}
\]
\[\leq \frac{1}{2} C_1 C_2 M (\|u\|^2_{H^1(\Omega)} + \|b\|^2_{H^1(\Omega)}).
\]
Using these estimates in (6.16) together with Poincaré’s type inequality (3.6), we obtain
\[
\frac{1}{C_p} (\|u\|^2_{H^1(\Omega)} + \|b\|^2_{H^1(\Omega)}) \leq \|\text{curl } u\|^2 + \|\text{curl } b\|^2 \leq \frac{3}{2} C_1 C_2 M (\|u\|^2_{H^1(\Omega)} + \|b\|^2_{H^1(\Omega)}),
\]
where \(C_p\) is the constant in (3.6). From this relation, we obtain
\[
\left(\frac{1}{C_p} - \frac{3}{2} C_1 C_2 M\right) (\|u\|^2_{H^1(\Omega)} + \|b\|^2_{H^1(\Omega)}) \leq 0.
\]
This, together with condition (6.5), implies that \(u = b = 0\). The construction of the pressure \(P \in L^2(\Omega)\) follows from De Rham’s Theorem (see 6.2).

### 6.2 Weak solution: \(L^p\)-theory, \(p \geq 2\)

In this subsection, we study the regularity of weak solution of system (MHD) in \(L^p\)-theory. We start with the case \(p \geq 2\). The proof is done essentially using the existence of weak solution in the hilbertian case and a bootstrap argument. To take advantage of the regularity of the Stokes problem (\(S_N\)) and the elliptic problem (\(E_N\)), we can rewrite the (MHD) problem in the following way:

\[
\begin{aligned}
- \Delta u + \nabla P &= f - (\text{curl } u) \times u + (\text{curl } b) \times d & \text{in } \Omega, \\
\text{div } u &= h & \text{in } \Omega, \\
\text{div } b &= 0 & \text{in } \Omega, \\
\text{curl } b &= g + \text{curl}(u \times b) & \text{in } \Omega, \\
\text{curl } u &= 0 & \text{in } \Omega, \\
b \times n &= 0 & \text{on } \Gamma, \\
(u \cdot n, 1)_{\Gamma_i} &= 0, & 1 \leq i \leq I, \\
\end{aligned}
\]

and

\[
\begin{aligned}
\text{div } b &= 0 & \text{in } \Omega, \\
b \times n &= 0 & \text{on } \Gamma, \\
(b \cdot n, 1)_{\Gamma_i} &= 0, & \forall 1 \leq i \leq I. \\
\end{aligned}
\]

The following result can be improved in the same way as in Corollary 5.10 by considering a data \(P_0\) less regular.
Theorem 6.3 (Regularity $W^{1,p}(\Omega)$ with $p > 2$). Let $p > 2$ and $r = \frac{2p}{p-2}$. Suppose that $f, g \in [H^{1,p}_0(\text{curl}, \Omega)]'$, $h = 0$, $P_0 \in W^{1-1/r, r}(\Gamma)$ with the compatibility conditions (5.27)-(5.28). Then the weak solution for the (MHD) system given by Theorem 6.1 satisfies

$$(u, b, P) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega).$$

Moreover, we have the following estimate:

$$\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} + \|P\|_{W^{1,r}(\Gamma)} \leq C(\|f\|_{[H^{1,p}_0(\text{curl}, \Omega)]'} + \|g\|_{[H^{1,p}_0(\text{curl}, \Omega)]'} + \|P_0\|_{W^{1-1/r, r}(\Gamma)}).$$

(6.17)

Proof. Since $p > 2$, we have $[H^{1,p}_0(\text{curl}, \Omega)]' \hookrightarrow [H^{6,2}_0(\text{curl}, \Omega)]'$ and $W^{1-1/r, r}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$. Thanks to Theorem 6.1, there exists $(u, b, P, c) \in H^{1}(\Omega) \times H^{1}(\Omega) \times L^2(\Omega) \times \mathbb{R}^2$ solution of (MHD). By using the embedding $H^{1}(\Omega) \hookrightarrow L^6(\Omega)$, it follows that $(\text{curl } u) \times u$ and $(\text{curl } b) \times b$ belong to $L^2(\Omega)$. To apply the regularity of Stokes problem and obtain weak solutions $(u, P) \in W^{1,p}(\Omega) \times W^{1,r}(\Omega)$, we must justify that the RHS $f - (\text{curl } u) \times u + (\text{curl } b) \times b$ belongs to $[H^{1,p}_0(\text{curl}, \Omega)]'$. Similarly, to apply the regularity of the elliptic problem ($\mathcal{E}_N$) and obtain a solution $b \in W^{1,p}(\Omega)$, we also need to justify that the RHS $g + (\text{curl } u \times b)$ satisfies to $[H^{0,p}_0(\text{curl}, \Omega)]'$. As a consequence, we distinguish according to the values of $p$ the following cases:

(i) If $p \leq 3$, then $f - (\text{curl } u) \times u + (\text{curl } b) \times b \in L^{3/2}(\Omega) \hookrightarrow L^{6/5}(\Omega) = L'(\Omega)$. Since $L'(\Omega) \hookrightarrow [H^{1/3}_0(\text{curl}, \Omega)]'$, thanks to the regularity of the Stokes problem ($\mathcal{S}_N$), see Proposition 3.2, we have that $u \in W^{1,p}(\Omega)$ and $P \in W^{1,r}(\Omega)$. Since $(\text{curl } u \times b) = (b \cdot \nabla) u - (u \cdot \nabla) b$, we have with the same argument above that $\text{curl } (u \times b) \in [H^{1/3}_0(\text{curl}, \Omega)]'$. It follows that $g + \text{curl } (u \times b) \in [H^{0,p}_0(\text{curl}, \Omega)]'$. Moreover, $g + \text{curl } (u \times b)$ satisfies the compatibility conditions (5.27)-(5.28). Consequently, thanks to the regularity of the elliptic problem ($\mathcal{E}_N$), we have that $b \in W^{1,p}(\Omega)$.

(ii) If $p > 3$, from the previous case, we have that $(u, b, P) \in W^{1,3}(\Omega) \times W^{1,3}(\Omega) \times W^{1,3/2}(\Omega)$. Therefore, $(u, b) \in L^q(\Omega) \times L^q(\Omega)$, for any $1 < q < \infty$. Then, $\text{curl } (u \times u) \in L^q(\Omega)$ for all $1 \leq m < 3$. In particular, we take $\frac{1}{m} = \frac{1}{q} + \frac{1}{3}$ with $3/2 < m < 3$. So, we have that $\text{curl } (u \times u) \in L^{\frac{3q}{q+3}}(\Omega) \hookrightarrow [H^{1/3}_0(\text{curl}, \Omega)]'$. Using the same arguments, we obtain that $\text{curl } (b \times b) \in L^{\frac{3q}{q+3}}(\Omega) \hookrightarrow [H^{1/3}_0(\text{curl}, \Omega)]'$. Then, the required regularity for $(u, b, P)$ follows by applying the regularity of the Stokes problem ($\mathcal{S}_N$). Further, we have $u \times b \in L^q(\Omega)$ for any $1 < q < \infty$. In particular, taking $t = p$ with $3 < t < \infty$, we have that $u \times b \in L^t(\Omega)$. So, due to the characterization of the space $[H^{1/3}_0(\text{curl}, \Omega)]'$, we have $\text{curl } (u \times b)$ belongs to $[H^{0,p}_0(\text{curl}, \Omega)]'$ and we finish the proof by applying the regularity of the elliptic problem ($\mathcal{E}_N$).

\[\square\]

6.3 Strong solution: $L^p$-theory, $p \geq 6/5$

In this subsection, we study the existence of strong solutions for more regular data. The following theorem gives the regularity $W^{2,p}(\Omega)$ with $p \geq 6/5$.

Theorem 6.4 (Regularity $W^{2,p}(\Omega)$ with $p \geq \frac{6}{5}$). Let us suppose that $\Omega$ is of class $C^{2,1}$ and $p \geq \frac{6}{5}$. Let $f, g$ and $P_0$ satisfy (5.27)-(5.28) and

$$f \in L^p(\Omega), \quad g \in L^p(\Omega), \quad h = 0 \quad \text{and} \quad P_0 \in W^{1-1/p, p}(\Gamma).$$

Then the weak solution $(u, b, P)$ for the (MHD) system given by Theorem 6.1 belongs to $W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and satisfies the following estimate:

$$\|u\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \leq C(\|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)} + \|P_0\|_{W^{1-1/p, p}(\Gamma)}).$$

(6.18)

Proof. To start the proof, the idea is to use the regularity result for weak solutions in $W^{1,p}(\Omega)$ with $p > 2$ given in Theorem 6.3 instead of the weak solutions in the Hilbert case $H^1(\Omega)$. Observe that if $6/5 \leq p \leq 3/2$, we have $2 < p^* < 3$, where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{3}$. Now, denoting $r(p^*) = \frac{2p}{3p - p} - 1$, we obtain that $r(p^*) = p$. Then, we have from the hypothesis of this Theorem that $f \in L^{r(p^*)}(\Omega)$, $g \in L^{r(p^*)}(\Omega)$ and $P_0 \in W^{1-1/r(p^*)+r(p^*)}(\Gamma)$. Since $L^{r(p^*)}(\Omega) \hookrightarrow [H^{0,p}_0(\text{curl}, \Omega)]'$ and $p^* > 2$, we deduce from the regularity result of the (MHD) problem (see Theorem 6.4) that $(u, b, P) \in W^{1,1/p}(\Omega) \times W^{1,1/p}(\Omega)$ and $W^{1-1/p, p}(\Gamma)$. Then, we have the following three cases:

(i) Case $\frac{6}{5} \leq p < \frac{3}{2}$. We have $(\text{curl } u) \times u \in L^q(\Omega)$ with $\frac{1}{q} = \frac{2}{3} - 1$. Since $t > p$, it follows that $(\text{curl } u) \times u$ belongs to $L^t(\Omega)$. The same argument gives that $(\text{curl } b) \times b$ belongs to $L^t(\Omega)$. Consequently, thanks to the existence of strong
solutions for the Stokes problem $(S_N)$ (see Proposition 3.2), we deduce that $(u, P) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega)$. Since $\text{curl}(u \times b)$ belongs also to $L^t(\Omega)$ with $t > p$, we deduce from the regularity result of the elliptic problem $(E_N)$ (see Theorem 3.3) that $b \in W^{2,p}(\Omega)$.

(ii) Case $p = \frac{3}{2}$: We have in this case $p^* = 3$. From above, we know that $(u, b, P) \in W^{1,3}(\Omega) \times W^{1,3}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$. Since $W^{1,3}(\Omega) \hookrightarrow L^q(\Omega)$ for any $1 < q < \infty$, we deduce that $(\text{curl } u) \times u$ and $(\text{curl } b) \times b$ belong to $L^t(\Omega)$ with $t = \frac{3}{2}$. Choosing $q > 3$ gives $t > \frac{3}{2}$. Thanks to the regularity of the Stokes problem $(S_N)$, we have that $(u, P) \in W^{2,\frac{3}{2}}(\Omega) \times W^{1,\frac{3}{2}}(\Omega)$. Using the same arguments, we have $\text{curl}(u \times u) \in L^t(\Omega)$ with $t > \frac{3}{2}$. Then, we apply the regularity of the elliptic problem $(E_N)$ to obtain $b \in W^{2,\frac{3}{2}}(\Omega)$.

(iii) Case $p > \frac{3}{2}$: We know that $(u, b) \in W^{2,\frac{3}{2}}(\Omega) \times W^{2,\frac{3}{2}}(\Omega)$. Then, $(u, b) \in L^q(\Omega) \times L^q(\Omega)$ with $1 < q < \infty$. We deduce that the terms $(\text{curl } u) \times u$ and $(\text{curl } b) \times b$ belong to $L^t(\Omega)$. Thanks to the regularity of $(S_N)$ and $(E_N)$, we deduce that $(u, b, P)$ belongs to $W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{2,p}(\Omega)$.

(b) If $p \geq 3$, from the above result, we have $(u, b, P) \in W^{2,3}(\Omega) \times W^{2,3}(\Omega) \times W^{2,3}(\Omega)$ with $0 < \varepsilon < \frac{3}{4}$. Observe that $(3 - \varepsilon)^* = \frac{3}{(3-\varepsilon)+3} > 3$. This implies that $u \in L^\infty(\Omega)$ and $b \in L^\infty(\Omega)$. Since $\text{curl } u \in W^{1,3}(\Omega) \hookrightarrow L^{(3-\varepsilon)^*}(\Omega)$, it follows that $(\text{curl } u) \times u$ and $(\text{curl } b) \times b$ belong to $L^{(3-\varepsilon)^*}(\Omega) \hookrightarrow L^3(\Omega)$. Again, according the regularity of Stokes problem, we have $(u, P) \in W^{2,3}(\Omega) \times W^{1,3}(\Omega)$. Similarly, $\text{curl}(u \times b) \in L^t(\Omega)$ and the regularity of the elliptic problem $(E_N)$ implies that $b \in W^{1,3}(\Omega)$. Finally, using the embeddings $W^{2,3}(\Omega) \hookrightarrow L^\infty(\Omega)$ and $W^{1,3}(\Omega) \hookrightarrow L^p(\Omega)$ for $1 < q < \infty$, all the terms $(\text{curl } u) \times u$, $(\text{curl } b) \times b$ and $\text{curl}(u \times b)$ belong to $L^q(\Omega)$. To conclude, we apply once again the regularity of Stokes problem $(S_N)$ and elliptic problem $(E_N)$.

6.4 Existence result of the MHD system for $3/2 < p < 2$

The next theorem tells us that it is possible to extend the regularity $W^{1,p}(\Omega)$ of the solution of the nonlinear (MHD) problem for $\frac{3}{2} < p < 2$. For this, we apply Banach’s fixed-point theorem over the linearized problem (4.2).

**Theorem 6.5** (Regularity $W^{1,p}(\Omega)$ with $\frac{3}{2} < p < 2$). Assume that $\frac{3}{2} < p < 2$ and let $r$ be defined by $\frac{1}{r} = \frac{1}{p} + \frac{1}{2}$. Let us consider $f, g \in [H_0^r]'(\text{curl } \Omega)'$, $P_0 \in W^{1,\frac{1}{p^*}-r}(\Gamma)$ and $h \in W^{1,r}(\Omega)$ with the compatibility conditions (5.27)-(5.28).

(i) There exists a constant $\delta_1$ such that, if

$$
\|f\|_{[H_0^r]'(\text{curl } \Omega)'} + \|g\|_{[H_0^r]'(\text{curl } \Omega)'} + \|P_0\|_{W^{1,\frac{1}{p^*}-r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_1
$$

Then, the (MHD) problem has at least a solution $(u, b, P, \alpha) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,r}(\Omega) \times \mathbb{R}^7$. Moreover, we have the following estimates:

$$
\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} \leq C_1(\|f\|_{[H_0^r]'(\text{curl } \Omega)'} + \|g\|_{[H_0^r]'(\text{curl } \Omega)'} + \|P_0\|_{W^{1,\frac{1}{p^*}-r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)})
$$

$$
\|P\|_{W^{1,r}(\Omega)} \leq C_1(1 + C^*\eta)(\|f\|_{[H_0^r]'(\text{curl } \Omega)'} + \|g\|_{[H_0^r]'(\text{curl } \Omega)'} + \|P_0\|_{W^{1,\frac{1}{p^*}-r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)})
$$

where $\delta_1 = (2C^2C^*)^{-1}$, $C_1 = C(1 + C^*\eta)^2$ with $C > 0$, $C^* > 0$ are the constants given in (6.25) and $\eta$ defined by (6.26).

Furthermore, $\alpha = (\alpha_1, \ldots, \alpha_7)$ satisfies

$$
\alpha_1 = (f, \nabla q_0^N) - \int_{\Omega} (\text{curl } u) \times u \cdot \nabla q_0^N \, dx + \int_{\Omega} (\text{curl } b) \times b \cdot \nabla q_0^N \, dx + \int_{\Gamma} (h - P_0) \nabla q_0^N \cdot n \, d\sigma
$$

(ii) Moreover, if the data satisfy that

$$
\|f\|_{[H_0^r]'(\text{curl } \Omega)'} + \|g\|_{[H_0^r]'(\text{curl } \Omega)'} + \|P_0\|_{W^{1,\frac{1}{p^*}-r}(\Gamma)} + \|h\|_{W^{1,r}(\Omega)} \leq \delta_2,
$$

for some $\delta_2 \in [0, \delta_1)$, then the weak solution of (MHD) is unique.

**Proof.**

(i) **Existence:** Let us define the space

$$
Z^p(\Omega) = W^{1,p}(\Omega) \times W^{1,p}(\Omega)
$$

For given \((w, d) \in B_\eta\), define the operator \(T\) by \(T(w, d) = (u, b)\) where \((u, b)\) is the unique solution of the linearized problem (4.2) given by Theorem 5.7 and the neighbourhood \(B_\eta\) is defined by
\[
B_\eta = \{(w, d) \in Z^p(\Omega), \|(w, d)\|_{Z^p(\Omega)} \leq \eta\}, \quad \eta > 0.
\]
Here \(Z^p(\Omega)\) is equipped with the norm
\[
\|(w, d)\|_{Z^p(\Omega)} = \|w\|_{W^{1,p}(\Omega)} + |d|_{W^{1,p}(\Omega)}.
\]
We have to prove that \(T\) is a contraction from \(B_\eta\) to itself. Let \((w_1, d_1), (w_2, d_2) \in B_\eta\). We show that there exists \(\theta \in (0, 1)\) such that:
\[
\|T(w_1, d_1) - T(w_2, d_2)\|_{Z^p(\Omega)} = \|(w_1, b_1) - (w_2, b_2)\|_{Z^p(\Omega)} \leq \theta \|(w_1, d_1) - (w_2, d_2)\|_{Z^p(\Omega)}
\]
(6.21)

Thanks to Corollary 5.8, each \((u_i, b_i, P_i, c_i), i = 1, 2\), belongs to \(W^{1,p}(\Omega) \times W^{1,p}(\Omega) \times W^{1,p}(\Omega)\) and verifies:
\[
\begin{align*}
-\Delta u_i + (\text{curl } w_i) \times u_i + \nabla P_i - (\text{curl } b_i) \times d_i &= f_i \text{ and } \text{div } u_i = h \text{ in } \Omega \\
\text{curl } b_i - \text{curl}(u_i \times d_i) &= g_i \text{ and } \text{div } b_i = 0 \text{ in } \Omega \\
u_i \times n &= 0 \text{ and } b_i \times n = 0 \text{ on } \Gamma \\
P_i &= P_0 \text{ on } \Gamma_0 \text{ and } P_i = P_0 + c_i^{(j)} \text{ on } \Gamma_j \\
(u_i, n, 1)_{\Gamma_j} &= 0 \text{ and } (b_i \cdot n, 1)_{\Gamma_j} = 0, \quad \forall 1 \leq j \leq I
\end{align*}
\]
(6.22)

Together with following estimate for \(i = 1, 2\):
\[
\|(u_i, b_i)\|_{Z^p(\Omega)} \leq C \left(1 + \|\text{curl } w_i\|_{L^2(\Omega)} + |d_i|_{W^{1,2}(\Omega)}\right) \left(\gamma_1 + (1 + \|\text{curl } w_i\|_{L^2(\Omega)} + |d_i|_{W^{1,2}(\Omega)})^{\gamma_2}\right)
\]
(6.23)

where \(\gamma_1 = \|f\|_{H^{p'}_0(\text{curl}, \Omega)} + \|g\|_{H^{p'}_0(\text{curl}, \Omega)} + \|P_0\|_{W^{1,2}(\Omega)}\) and \(\gamma_2 = \|h\|_{W^{1,2}(\Omega)}\).

Next, the differences \((u, b, P, c) = (u_1 - u_2, b_1 - b_2, P_1 - P_2, c_1 - c_2)\) satisfy
\[
\begin{align*}
-\Delta u + (\text{curl } w_1) \times u_1 + \nabla P - (\text{curl } b_1) \times d_1 &= f_2 \text{ and } \text{div } u_1 = 0 \text{ in } \Omega \\
\text{curl } b_1 - \text{curl}(u_1 \times d_1) &= g_2 \text{ and } \text{div } b_1 = 0 \text{ in } \Omega \\
u \times n &= 0 \text{ and } b \times n = 0 \text{ on } \Gamma \\
P &= 0 \text{ on } \Gamma_0 \text{ and } P = c_j \text{ on } \Gamma_j \\
(u \cdot n, 1)_{\Gamma_j} &= 0 \text{ and } (b \cdot n, 1)_{\Gamma_j} = 0, \quad \forall 1 \leq j \leq I
\end{align*}
\]
(6.24)

with \(f_2 = -(\text{curl } w_1) \times u_2 + (\text{curl } b_2) \times d_2\) and \(g_2 = \text{curl}(u_2 \times d_2)\). Observe that \(f_2\) and \(g_2\) belong to \([H^{p'}_0(\text{curl}, \Omega)]'\). Indeed, Since \(u_2, b_2 \in W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)\), \(\text{curl } w \in L^{2}(\Omega)\) and \(d \in W^{1,2}(\Omega)\), then \((\text{curl } w) \times u_2\) and \((\text{curl } b) \times d\) belong to \(L^{p^*}(\Omega) \hookrightarrow [H^{p'}_0(\text{curl}, \Omega)]'\). Besides, \(u_2 \times d \in L^{p^*}(\Omega)\) so \(\text{curl}(u_2 \times d) \in [H^{p'}_0(\text{curl}, \Omega)]'\). Moreover, since \(g_2\) is a \(\text{curl}\) then it satisfies the conditions (5.27)-(5.28). Hence, we apply the Theorem 5.7 and we have the estimate:
\[
\|(u, b)\|_{Z^p(\Omega)} \leq C \left(1 + \|\text{curl } w_1\|_{L^2(\Omega)} + |d_1|_{W^{1,2}(\Omega)}\right) \left(\|f_2\|_{[H^{p'}_0(\text{curl}, \Omega)]'} + \|g_2\|_{[H^{p'}_0(\text{curl}, \Omega)]'}\right)
\]

By the definition of the norm on \([H^{p'}_0(\text{curl}, \Omega)]'\), the Hölder inequality and the embeddings
\[
W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \text{ and } W^{1,p}(\Omega) \hookrightarrow W^{1,2}(\Omega) \hookrightarrow L^2(\Omega),
\]

it follows:
\[
\|f_2\|_{[H^{p'}_0(\text{curl}, \Omega)]'} \leq \|(\text{curl } w) \times u_2\|_{L^{p^*}(\Omega)} + \|(\text{curl } b_2) \times d\|_{L^{p^*}(\Omega)}
\]
\[
\leq \|(\text{curl } w)\|_{L^{2,2}(\Omega)} \|u_2\|_{L^{p^*}(\Omega)} + \|(\text{curl } b_2)\|_{L^{2,2}(\Omega)} \|d\|_{L^{p^*}(\Omega)}
\]
\[
\leq C(C_w) \|w\|_{W^{1,p}(\Omega)} \|u_2\|_{W^{1,p}(\Omega)} + C_d \|d\|_{W^{1,p}(\Omega)} \leq C(C_w) \|u_2\|_{W^{1,p}(\Omega)} \|d\|_{W^{1,p}(\Omega)}
\]

and
\[
\|g_2\|_{[H^{p'}_0(\text{curl}, \Omega)]'} = \|u_2 \times d\|_{L^{p^*}(\Omega)} \leq \|u_2\|_{L^{p^*}(\Omega)} \|d\|_{L^{p^*}(\Omega)} \leq CC_d \|u_2\|_{W^{1,p}(\Omega)} \|d\|_{W^{1,p}(\Omega)}
\]
where $C_w > 0$ and $C_d > 0$ are respectively defined by $\|\text{curl } w\|_{L^2(\Omega)} \leq C_w \|w\|_{W^{1,p}(\Omega)}$ and $\|d\|_{L^2(\Omega)} \leq C_d \|d\|_{W^{1,p}(\Omega)}$.

Therefore, recalling that $(w_1, d_1)$ belongs to $B_\eta$, we have:

$$\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} \leq C(1 + C^*(w_1, d_1))C_w \|w\|_{W^{1,p}(\Omega)} + C_d \|d\|_{W^{1,p}(\Omega)} \|(u_2, b_2)\|_{Z^p(\Omega)}$$

with $C^* = C_w + C_d$. Combining with (6.23), we thus obtain:

$$\|u\|_{W^{1,p}(\Omega)} + \|b\|_{W^{1,p}(\Omega)} \leq C \|u\|_{Z^p(\Omega)} (\gamma_1 + (1 + C^*)\gamma_2)$$

(6.25)

where $C_1 = (1 + C^*)^2$. Hence, if we choose, for instance:

$$\eta = (C^*/2C^*_\eta)^{\frac{1}{2}} - 1$$

and $\gamma = \gamma_1 + \gamma_2 < (2C^*_\eta)^{-1}$

then $C_1\gamma_1 + (1 + C^*)\gamma_2 < \eta$. Therefore, taking $T$ is a contraction and we obtain the unique fixed-point $(u^*, b^*) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega)$ satisfying

$$\|(u^*, b^*)\|_{Z^p(\Omega)} \leq C(1 + C^*)\|(u^*, b^*)\|_{Z^p(\Omega)} (\gamma_1 + (1 + C^*)\gamma_2)$$

Next, since $(u^*, b^*) \in B_\eta$, we obtain

$$\|(u^*, b^*)\|_{Z^p(\Omega)} \leq C(1 + C^*)\|(u^*, b^*)\|_{Z^p(\Omega)}$$

(6.27)

which implies the estimate (6.19):

Now, we want to prove the estimate for the associated pressure. Taking the divergence in the first equation of problem (MHD), we have that $P^*$ is a solution of the following problem:

$$\begin{cases}
\Delta P^* = \text{div } f + \text{div}(\text{curl } b^*) \times b^* - (\text{curl } u^*) \times u^* + \Delta h & \text{in } \Omega, \\
P^* = P_0 & \text{on } \Gamma_0 \quad \text{and} \quad P^* = P_0 + c_i & \text{on } \Gamma_i
\end{cases}$$

with

$$\|P^*\|_{W^{1,r}(\Omega)} \leq \|\text{div } f\|_{W^{1,r}(\Omega)} + \|\text{div}(\text{curl } b^*) \times b^* - (\text{curl } u^*) \times u^*)\|_{W^{1,r}(\Omega)} + \|\Delta h\|_{W^{1,r}(\Omega)} + \|P_0\|_{W^{1,r}(\Omega)}$$

Following the same calculus as in the proof of Theorem 5.7, we obtain

$$\|P^*\|_{W^{1,r}(\Omega)} \leq C(1 + C^*)\gamma (\gamma_1 + (1 + C^*)\gamma_2)$$

which implies (6.20) and the proof of existence is completed.

(ii) Uniqueness:

Let $(u_1, b_1, P_1)$ and $(u_2, b_2, P_2)$ two solutions of the problem (MHD). Then, we set $(u, b, P) = (u_1 - u_2, b_1 - b_2, P_1 - P_2)$ which satisfies the problem:

$$\begin{cases}
-\Delta u + (\text{curl } u_1) \times u + \nabla P - (\text{curl } b) \times b_1 = f_2 & \text{in } \Omega, \\
\text{curl } \text{curl } b - \text{curl}(u \times b_1) = g_2 & \text{in } \Omega, \\
u \times n = 0 \quad \text{and} \quad b \times n = 0 & \text{on } \Gamma, \\
P = 0 & \text{on } \Gamma_0 \quad \text{and} \quad P = a_i^{(1)} - a_i^{(2)} & \text{on } \Gamma_i, \\
(u \cdot n, 1)_{\Gamma_i} = 0 \quad \text{and} \quad (b \cdot n, 1)_{\Gamma_i} = 0 \quad \forall 1 \leq i \leq I
\end{cases}$$

where $f_2, g_2 \in [H^{1/2}_0(\text{curl}, \Omega)]'$ are already given in (6.24) and satisfy the hypothesis of Theorem 5.7. Applying this theorem, we have the estimate

$$\|(u, b)\|_{Z^p(\Omega)} \leq C(1 + \|\text{curl } u_1\|_{L^\infty(\Omega)} + \|b_1\|_{W^{1,4}(\Omega)})C_w \|u\|_{W^{1,p}(\Omega)} + C_d \|b\|_{W^{1,p}(\Omega)}$$

$$\leq C(1 + C^*\|(u_1, b_1)\|_{Z^p(\Omega)} C^*\|(u_2, b_2)\|_{Z^p(\Omega)} \|(u, b)\|_{Z^p(\Omega)}$$

From (6.27), we obtain that:

$$\|(u, b)\|_{Z^p(\Omega)} \leq C(1 + C^*C_1\gamma)C^*C_1\gamma \|(u, b)\|_{Z^p(\Omega)}$$

Thus, for $\gamma$ small enough such that

$$C(1 + C^*C_1\gamma)C^*C_1\gamma < 1$$

we deduce that $u = b = 0$ and then we obtain the uniqueness of the velocity and the magnetic field which implies the uniqueness of the pressure $P$.
7 Appendix

We begin this section by giving another proof of Theorem 5.11.

A second proof of Theorem 5.11: Let $\lambda > 0$, and let us assume $f_\lambda, g_\lambda \in D(\Omega)$ such that $f_\lambda$ and $g_\lambda$ respectively converge to $f$ and $g$ in $L^p(\Omega)$, and $P_0^\lambda \in C^\infty(\Gamma)$ which converges to $P_0$ in $W^{1-\frac{1}{p},p}(\Gamma)$.

We consider the problem: find $(u_\lambda, b_\lambda, c_\lambda^1)$ solution of:

$$
\begin{align}
-\Delta u_\lambda + (\nabla \times w) \times u_\lambda + \nabla P_\lambda - (\nabla \times b_\lambda) \times d &= f_\lambda \quad \text{and} \quad \text{div} u_\lambda = 0 \quad \text{in} \quad \Omega \\
\nabla \times b_\lambda - \nabla \times (u_\lambda \times d) &= g_\lambda \quad \text{and} \quad \text{div} b_\lambda = 0 \quad \text{in} \quad \Omega \\
P_\lambda &= P_0^\lambda, \quad \text{on} \ \Gamma_0, \quad \text{on} \ \Gamma \\
u_\lambda \times n &= 0, \quad b_\lambda \times n = 0 \quad \text{on} \ \Gamma \\
\langle u_\lambda \cdot n, 1 \rangle_{\Gamma_0} = 0, \quad \langle b_\lambda \cdot n, 1 \rangle_{\Gamma_0} = 0, \quad \forall 1 \leq i \leq I
\end{align}
$$

(7.1)

Note that, since $f_\lambda, g_\lambda \in D(\Omega)$, in particular they belong to $L^6(\Omega)$. Thus, applying Theorem 5.1, we have $(u_\lambda, b_\lambda) \in W^{2,6}(\Omega) \times W^2(\Omega) \hookrightarrow H^1(\Omega) \times H^1(\Omega) \rightarrow L^6(\Omega) \times L^6(\Omega)$. Therefore, $(\nabla \times w) \times u_\lambda, (\nabla \times b_\lambda) \times d \quad \text{and} \quad \nabla \times (u_\lambda \times d)$ belong to $L^6(\Omega) \rightarrow L^6(\Omega)$. Hence, using the regularity results of the Stokes and elliptic problems, we obtain that the problem (7.1) has a unique solution $(u_\lambda, b_\lambda, c_\lambda^1) \in W^{2,6}(\Omega) \times W^{2,6}(\Omega) \times W^{1,6}(\Omega)$ which also satisfies the estimate:

$$
\left\| u_\lambda \right\|_{W^{2,6}(\Omega)} + \left\| b_\lambda \right\|_{W^{2,6}(\Omega)} + \left\| P_\lambda \right\|_{W^{1,6}(\Omega)} \\
\lesssim CSE \left( \left\| f_\lambda \right\|_{L^6(\Omega)} + \left\| g_\lambda \right\|_{L^6(\Omega)} + \left\| P_0^\lambda \right\|_{W^{1,6}(\Omega)} + \sum_{i=1}^{I} |c_\lambda^1| + \left\| (\nabla \times u_\lambda) \times u_\lambda \right\|_{L^6(\Omega)} \right) + \left\| (\nabla \times b_\lambda) \times d \right\|_{L^6(\Omega)} + \left\| (\nabla \times (u_\lambda \times d)) \right\|_{L^6(\Omega)}
$$

(7.2)

with $CSE = \max(C_S, C_E)$ where $C_S$ is the constant given in the Proposition 3.2 and $C_E$ the constant given in Theorem 3.3. We now want to bound the right hand side terms $\left\| (\nabla \times u_\lambda) \times u_\lambda \right\|_{L^6(\Omega)}$, $\left\| (\nabla \times b_\lambda) \times d \right\|_{L^6(\Omega)}$, $\left\| (\nabla \times (u_\lambda \times d)) \right\|_{L^6(\Omega)}$ and $\sum_{i=1}^{I} |c_\lambda^1|$ to obtain the estimate (5.99). In this purpose, we decompose $\nabla \times w$ and $d$ as in (5.4)-(5.5).

Let $\epsilon > 0$ and $\rho_{\epsilon/2}$ the classical mollifier. We consider $\tilde{y} = \nabla \times w$ and $\tilde{d}$ the extensions by 0 of $y = \nabla \times w$ and $d$ in $\mathbb{R}^3$ respectively. We take:

$$
\nabla \times w = y_1^i + y_2^i \quad \text{where} \quad y_1^i = \tilde{y} * \rho_{\epsilon/2} \quad \text{and} \quad y_2^i = \nabla \times w - y_1^i
$$

$$
d = d_1^i + d_2^i \quad \text{where} \quad d_1^i = \tilde{d} * \rho_{\epsilon/2} \quad \text{and} \quad d_2^i = d - d_1^i
$$

(7.3)

For each term, we start by bounding the part depending on $d_2^i$ (resp. $y_2^i$), and then we look at $d_1^i$ (resp. $y_1^i$).

(i) Estimate of the term $\left\| (\nabla \times u_\lambda) \times u_\lambda \right\|_{L^6(\Omega)}$

First, following the definition of the mollifier, we classically obtain:

$$
\left\| y_2^i \right\|_{L^6(\Omega)} = \left\| y - \tilde{y} * \rho_{\epsilon/2} \right\|_{L^6(\Omega)} \leq \epsilon
$$

(7.4)

Then, since we have $W^{2,6}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ and the Hölder inequality, we obtain:

$$
\left\| y_2^i \times u_\lambda \right\|_{L^6(\Omega)} \leq \left\| y_2^i \right\|_{L^6(\Omega)} \left\| u_\lambda \right\|_{L^{p^*}(\Omega)} \leq C_1 \left\| y_2^i \right\|_{L^6(\Omega)} \left\| u_\lambda \right\|_{W^{2,6}(\Omega)}
$$

(7.5)

where $C_1$ is the constant related to the previous Sobolev embedding $W^{2,6}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Thus, injecting (7.4) in (7.5), it follows:

$$
\left\| y_2^i \times u_\lambda \right\|_{L^6(\Omega)} \leq C_1 \epsilon \left\| u_\lambda \right\|_{W^{2,6}(\Omega)}
$$

Now, for the term in $y_1^i$, we apply the Hölder inequality to obtain:

$$
\left\| y_1^i \times u_\lambda \right\|_{L^6(\Omega)} \leq \left\| y_1^i \right\|_{L^{m}(\Omega)} \left\| u_\lambda \right\|_{L^{q}(\Omega)} \leq \left\| y_1^i \right\|_{L^6(\Omega)} \left\| \rho_{\epsilon/2} \right\|_{L^1(\Omega)} \left\| u_\lambda \right\|_{L^q(\Omega)}
$$

with $m, q \geq p$ such that $\frac{1}{q} = \frac{1}{m} + \frac{1}{p}$ and $t > 1$ defined by $1 + \frac{1}{m} = \frac{3}{2} + \frac{1}{p}$. Note that this definition imposes $m > \frac{3}{2}$, and we can take in particular $m = 3$, and hence $q = p^*$. Since, following the properties of the mollifier, there exists $C_\epsilon > 0$ such that, for all $t > 1$:

$$
\left\| \rho_{\epsilon/2} \right\|_{L^t(\Omega)} \leq C_\epsilon
$$

(7.6)
so we have
\[
\|y_1^\ell \times u_\lambda\|_{L^p(\Omega)} \leq C_\ast \|y_1\|_{L^2(\Omega)} \|u_\lambda\|_{L^p(\Omega)}
\]

(ii) **Estimate of the term** \(\|\text{curl } b_\lambda\times d\|_{L^p(\Omega)}\):

As previously, we have from the definition of the mollifier:
\[
\|d_1^\ell\|_{W^{1,2}(\Omega)} = \|d - \tilde{d} + \rho_{1/2}\|_{W^{1,2}(\Omega)} \leq \epsilon
\]

(7.7)

Then, combining the Hölder inequality and the Sobolev embedding \(W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)\), we have:
\[
\|\text{curl } b_\lambda\times d_1^\ell\|_{L^p(\Omega)} \leq \|\text{curl } b_\lambda\|_{L^p(\Omega)} \|d_1^\ell\|_{L^4(\Omega)} \leq C_2 \|b_\lambda\|_{W^{1,p}(\Omega)}
\]

where \(C_2\) is the constant related to the Sobolev embedding \(W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)\). We finally recall the Sobolev embedding \(W^{2,p}(\Omega) \hookrightarrow W^{1,p'}(\Omega)\) to obtain:
\[
\|\text{curl } b_\lambda\times d_1^\ell\|_{L^p(\Omega)} \leq C_3 \|b_\lambda\|_{W^{2,p}(\Omega)}
\]

(7.8)

with \(C_3\) the constant related to the Sobolev embedding \(W^{2,p}(\Omega) \hookrightarrow W^{1,p'}(\Omega)\). It remains to bound the term in \(d_1^\ell\).

Applying the Hölder inequality, we have:
\[
\|\text{curl } b_\lambda\times d_1^\ell\|_{L^p(\Omega)} \leq \|\text{curl } b_\lambda\|_{L^q(\Omega)} \|d_1^\ell\|_{L^{q'}(\Omega)} \leq \|\text{curl } b_\lambda\|_{L^q(\Omega)} \|d\|_{L^{q'}(\Omega)} \|\rho_{1/2}\|_{L^{t}(\Omega)}
\]

(7.9)

with \(m, q \geq p\) such that \(\frac{1}{q} = \frac{1}{p} + \frac{1}{r}\) and \(t > 1\) such that \(1 + \frac{m}{q} = 1/2 + 1/t\). Note that these relations require \(m > 3\), then \(\frac{1}{q} < \frac{1}{p} + \frac{1}{t}\) so \(q < p'\). Therefore, we have the Sobolev embeddings:
\[
W^{2,p}(\Omega) \hookrightarrow \text{compact } W^{1,q}(\Omega) \hookrightarrow \text{continuous } L^{p'}(\Omega)
\]

hence there exists \(\eta > 0\) and \(C_\eta > 0\) such that
\[
\|b_\lambda\|_{W^{1,q}(\Omega)} \leq \eta \|b_\lambda\|_{W^{2,p}(\Omega)} + C_\eta \|b_\lambda\|_{L^{p'}(\Omega)}
\]

(7.10)

Injecting (7.10) in (7.9), combining with (7.6) and the Sobolev embedding \(W^{1,2}(\Omega) \hookrightarrow L^4(\Omega)\), we obtain:
\[
\|\text{curl } b_\lambda\times d_1^\ell\|_{L^p(\Omega)} \leq C_4 C_2 \|d\|_{W^{1,2}(\Omega)} (\eta \|b_\lambda\|_{W^{2,p}(\Omega)} + C_\eta \|b_\lambda\|_{L^{p'}(\Omega)})
\]

(iii) **Estimate of the term** \(\|\text{curl } (u_\lambda \times d)\|_{L^p(\Omega)}\):

Since \(\text{div } u_\lambda = 0\) and \(\text{div } d = 0\) in \(\Omega\), then \(\text{curl } (u_\lambda \times d) = (d \cdot \nabla) u_\lambda - (u_\lambda \cdot \nabla) d\). We thus bound these two terms.

- **Estimate of the term** \(\|d \cdot \nabla) u_\lambda\|_{L^p(\Omega)}\):
  The reasoning is exactly the same as for (ii) with \(\text{curl } b_\lambda\) replacing by \(\nabla u_\lambda\). Then we have:
  \[
  \|d_1^\ell \cdot \nabla) u_\lambda\|_{L^p(\Omega)} \leq C_2 C_\ast \epsilon \|u_\lambda\|_{W^{2,p}(\Omega)}
  \]
  and
  \[
  \|d_1^\ell \cdot \nabla) u_\lambda\|_{L^p(\Omega)} \leq C_2 C_\ast \|d\|_{W^{1,2}(\Omega)} (\eta \|b_\lambda\|_{W^{2,p}(\Omega)} + C_\eta \|b_\lambda\|_{L^{p'}(\Omega)})
  \]

- **Estimate of the term** \(\|(u_\lambda \cdot \nabla) d\|_{L^p(\Omega)}\):
  Again, the reasoning is the same as for (i), with \(\nabla d\) instead of \(\text{curl } w\). Then we have:
  \[
  \|u_\lambda \cdot \nabla) d_1^\ell\|_{L^p(\Omega)} \leq C_1 \epsilon \|u_\lambda\|_{W^{2,p}(\Omega)}
  \]
  and
  \[
  \|u_\lambda \cdot \nabla) d_1^\ell\|_{L^p(\Omega)} \leq C_1 \|d\|_{W^{1,2}(\Omega)} \|u_\lambda\|_{L^{p'}(\Omega)}
  \]

(iv) **Estimate of the term** \(\sum_{i=1}^l |c_i^\ell|\):

With a triangle inequality, we have:
\[
|c_i^\ell| \leq |(f_\lambda, \nabla q_i^N) + |(P_\lambda, \nabla q_i^N - n)| + |\int_\Omega (\text{curl } w) \times u_\lambda \cdot \nabla q_i^N dx| + |\int_\Omega (\text{curl } b_\lambda) \times d \cdot \nabla q_i^N dx|
\]

We can’t directly bound \(|\int_\Omega (\text{curl } w) \times u_\lambda \cdot \nabla q_i^N dx|\) and \(|\int_\Omega (\text{curl } b_\lambda) \times d \cdot \nabla q_i^N dx|\) with an Hölder inequality, we must use again the decomposition of \(\text{curl } w\) and \(d\) in (7.3).
Finally, noting that there exists \( C \) where:

\[
\sum_{i=1}^{t} |c_i^t| \leq IC_q \left( \|f_\lambda\|_{L^p(\Omega)} + \|P_\lambda\|_{W^{-1, \frac{1}{p}}(\Gamma)} + C_1 \|u_\lambda\|_{W^{2,p}(\Omega)} + C_3 \|b_\lambda\|_{W^{2,p}(\Omega)} + \eta \|b_\lambda\|_{W^{2,p}(\Omega)} + \|\text{curl} w\|_{L^2(\Omega)} \right) + C_4 \|b_\lambda\|_{L^p(\Omega)} \right).
\]

Then, injecting (i) – (iv) in (7.2), and taking \( \epsilon \) small enough such that:

\[
\epsilon \left( C_2 C_3 + \max(2CC_1, IC_q) \right) < \frac{1}{4}
\]

and \( \eta \) such that

\[
\eta \|d\|_{W^{1, \frac{2}{p}}(\Omega)} CC_2 C_\epsilon < \frac{1}{4}
\]

where \( C > 0 \) denotes the constant \( C = (1 + IC_q) \), we obtain

\[
\|u_\lambda\|_{W^{2,p}(\Omega)} + \|b_\lambda\|_{W^{2,p}(\Omega)} + \|P_\lambda\|_{W^{1,p}(\Omega)} \leq C_{SR} \left( C_3 \|u_\lambda\|_{L^p(\Omega)} + \|g_\lambda\|_{L^p(\Omega)} + \|P_\lambda\|_{W^{-1, \frac{1}{p}}(\Gamma)} \right) + \left( C_4 (1 + 2CC_2 C_\epsilon) \|d\|_{W^{1, \frac{2}{p}}(\Omega)} + C_1 C \|\text{curl} w\|_{L^2(\Omega)} \right) \left( \|u_\lambda\|_{L^p(\Omega)} + \|b_\lambda\|_{L^p(\Omega)} \right)
\]

(7.11)
Applying the estimate (5.86) in (7.11), we finally obtain the estimate:

\[ \|u_\lambda\|_{W^{2,p}(\Omega)} + \|b_\lambda\|_{W^{2,p}(\Omega)} + \|P_\lambda\|_{W^{1,p}(\Omega)} \]
\[ \leq C_{\text{SF}} \left( \|f\|_{L^p(\Omega)} + \|g_\lambda\|_{L^p(\Omega)} + \|\rho_\lambda\|_{W^{1,p}(\Omega)} \right) \left[ C + C_3 C_f \left( C_f (1 + 2C_2 C_0) \|d\|_{W^{1,2}(\Omega)} \right) \right] \]
\[ + C_{\text{C}} \|\mathbf{curl} u\|_{L^2(\Omega)} \left( 1 + \|\mathbf{curl} u\|_{L^2(\Omega)} + \|d\|_{W^{1,2}(\Omega)} \right) \]
\[ \leq C_{\text{SF}} \left( \|f\|_{L^p(\Omega)} + \|g_\lambda\|_{L^p(\Omega)} + \|\rho_\lambda\|_{W^{1,p}(\Omega)} \right) \max \left( C, C_3 C_f C_f (1 + 2C_2 C_1), C_C \right) \]
\[ \times \left( 1 + \|\mathbf{curl} u\|_{L^2(\Omega)} + \|d\|_{W^{1,2}(\Omega)} \right)^2 \]

(7.12)

where \(C_f\) is the constant of the estimate (5.86) and \(C_3\) is defined in (7.8).

To conclude, from the estimate (7.12) we can extract subsequences of \(u_\lambda, b_\lambda\) and \(P_\lambda\), which are still denoted \(u_\lambda, b_\lambda\) and \(P_\lambda\), such that:

\[ u_\lambda \rightharpoonup u \text{ and } b_\lambda \rightharpoonup b \text{ in } W^{2,p}(\Omega), \quad P_\lambda \rightharpoonup P \text{ in } W^{1,p}(\Omega) \]

where \((u, b, P) \in W^{2,p}(\Omega) \times W^{2,p}(\Omega) \times W^{1,p}(\Omega)\) is solution of (4.2) and satisfies the estimate (5.99).

Next, we give here the proof of Corollary 5.12 which is the extension of the previous result to the case of non-zero divergence condition for \(u\).

**Proof of Corollary 5.12:** We proceed with the same reasoning as in the Corollary 5.8. We recover the solution of a linearized problem with a vanishing divergence by considering the Dirichlet problem:

\[ \Delta \theta = h \text{ in } \Omega \text{ and } \theta = 0 \text{ on } \Gamma \]

and setting \(z = u - \nabla \theta\), thus \((z, b, P, c)\) is the solution of (4.2) in the Theorem 5.11 with \(f\) and \(g\) replaced by \(\tilde{f} = f + \nabla h - (\mathbf{curl} u) \times \nabla \theta\) and \(\tilde{g} = g + \mathbf{curl}(\nabla \theta \times \mathbf{d})\).

Indeed, the assumptions of the Theorem 5.11 are satisfied: since \(\nabla \theta \in W^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)\) and \(\mathbf{curl} u \in L^2(\Omega)\), then \((\mathbf{curl} u) \times \nabla \theta \in L^2(\Omega)\). Moreover, we have by hypothesis \(\nabla h \in L^p(\Omega)\) so \(\tilde{f} \in L^p(\Omega)\). In the same way, we have \(\tilde{g} \in L^p(\Omega)\) and since we only add a curl to \(g\), then \(\tilde{g}\) satisfies the conditions (5.27)-(5.28). Hence, we recover the estimate:

\[ \|z\|_{W^{2,p}(\Omega)} + \|b\|_{W^{2,p}(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \]
\[ \leq C_F \left( 1 + \|\mathbf{curl} u\|_{L^2(\Omega)} + \|d\|_{W^{1,2}(\Omega)} \right)^2 \left( \|\tilde{f}\|_{L^p(\Omega)} + \|\tilde{g}\|_{L^p(\Omega)} + \|P\|_{W^{1,p}(\Omega)} \right) \]

(7.13)

We must bound \(\|\tilde{f}\|_{L^p(\Omega)}\) and \(\|\tilde{g}\|_{L^p(\Omega)}\) term by term:

- Applying the Hölder inequality and the Sobolev embedding \(W^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)\), we have:
  \[ \|\tilde{f}\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|\nabla h\|_{L^p(\Omega)} + \|\mathbf{curl} u\|_{L^2(\Omega)} \|
abla \theta\|_{L^{p^{**}}(\Omega)} \]
  \[ \|\tilde{g}\|_{L^p(\Omega)} \leq \|g\|_{L^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} \|d\|_{W^{1,2}(\Omega)} + C_1 \|\mathbf{curl} u\|_{L^2(\Omega)} \|h\|_{W^{1,p}(\Omega)} \]

(7.14)

where \(C_1\) denotes the constant of the Sobolev embedding \(W^{2,p}(\Omega) \hookrightarrow L^{p^{**}}(\Omega)\).

- Note that \(\text{div} \nabla \theta = \Delta \theta = h\), thus we rewrite \(\mathbf{curl}(\nabla \theta \times \mathbf{d}) = -\mathbf{h}d + (d \cdot \nabla) \nabla \theta - (\nabla \theta \cdot \nabla) \mathbf{d}\). Thus, we have:
  \[ \|\tilde{g}\|_{L^p(\Omega)} \leq \|g\|_{L^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} \|d\|_{L^2(\Omega)} + \|h\|_{W^{1,p}(\Omega)} \|d\|_{L^2(\Omega)} \|\theta\|_{L^{p^{**}}(\Omega)} + \|\nabla \theta\|_{L^{p^{**}}(\Omega)} \|\nabla d\|_{L^2(\Omega)} \]
  \[ \leq \|g\|_{L^p(\Omega)} + \|h\|_{W^{1,p}(\Omega)} \|d\|_{L^2(\Omega)} + \|d\|_{L^2(\Omega)} \|\theta\|_{L^{2,p}(\Omega)} + \|\nabla \theta\|_{L^{2,p}(\Omega)} \|\nabla d\|_{L^2(\Omega)} \]

(7.15)

where \(C_2\) and \(C_3\) are the constants respectively related to the Sobolev embeddings \(W^{1,2}(\Omega) \hookrightarrow L^3(\Omega)\) and \(W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)\).
Hence, combining (7.14) and (7.15) with (7.13), it follows that:

\[
\| z \|_{W^{2,p}(\Omega)} + \| b \|_{W^{2,p}(\Omega)} + \| P \|_{W^{1,p}(\Omega)} \\
\leq C_F \left( 1 + \| \text{curl} w \|_{L^2(\Omega)} + \| d \|_{W^{1,2}(\Omega)} \right)^2 \left( \| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} + \| P_b \|_{W^{1-\frac{1}{p},p}(\Gamma)} \right) \\
+ \| b \|_{W^{1,p}(\Omega)} \left( 1 + C_1 \| \text{curl} w \|_{L^2(\Omega)} + (2C_2C_3 + C_1) \| d \|_{W^{1,2}(\Omega)} \right) \\
\leq C_F \left( 1 + \| \text{curl} w \|_{L^2(\Omega)} + \| d \|_{W^{1,2}(\Omega)} \right)^2 \left( \| f \|_{L^p(\Omega)} + \| g \|_{L^p(\Omega)} + \| P_b \|_{W^{1-\frac{1}{p},p}(\Gamma)} \right) \\
+ \max \left( 1, 2C_2C_3 + C_1 \right) \| b \|_{W^{1,p}(\Omega)} \left( 1 + \| \text{curl} w \|_{L^2(\Omega)} + \| d \|_{W^{1,2}(\Omega)} \right)
\]

Finally, we use triangle inequality and we get the estimate (5.106). □

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