Existence and Uniqueness of the Motion by Curvature of regular networks

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Abstract
We prove existence and uniqueness of the motion by curvature of networks in $\mathbb{R}^n$ when the initial datum is of class $W^{2-2/p, p}$, with triple junction where the unit tangent vectors to the concurring curves form angles of 120 degrees. Moreover we investigated the regularization effect due to the parabolic nature of the system. An application of this well-posedness result is a new proof of [25, Theorem 3.18] where the possible behaviours of the solutions at the maximal time of existence are described. Our study is motivated by an open question proposed in [24]: does there exist a unique solution of the motion by curvature of networks with initial datum a regular network of class $C^2$? We give a positive answer.

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1 Introduction
The mean curvature flow of surfaces in $\mathbb{R}^n$, and in Riemannian manifold more in general, is one of the most significant example of geometric evolution equations. This evolution can be understood as the gradient flow of the area functional: a time–dependent surface evolves with normal velocity its mean curvature (at any point and time).
From the 80s the curve shortening flow (mean curvature flow of one–dimensional objects) was widely studied by many authors both for closed curves [11, 12, 13, 17] and for curves with fixed end–points [18, 30, 31]. Also two concurring curves forming an angle or a cusp can be regarded as a single curve with a singular point, which will vanish immediately under the flow [4, 5, 6]. When more than two curves meet at a junction, the description of the motion cannot be reduced to the case of a single curve and the problem present new interesting features. The simplest example of motion by mean curvature of a set which is essentially singular is indeed the motion by curvature of networks: one–dimensional connected sets, finite union of curves that meet at junctions.
Although after the seminal work by Brakke [7] several weak definitions of the motion by curvature of singular surfaces has been proposed, the first attempt to find classical/strong

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solutions to the network flow was by Bronsard and Reitich [8] who provided a well posedness result for initial data of class $C^{2+\alpha}$. The analysis of the long time behavior of the evolving networks was undertaken in [25], completed in [23] for trees composed of three curves and extended to more general cases in [19, 24, 26]. In this paper we restrict to regular networks that present only triple junctions where the unit tangent vectors of the concurring curves form angles of $120$ degrees. The motion by curvature of networks turns out to be a boundary value problem where the evolution of each curve is described by a second order quasilinear PDE (Definition 2.25).

Our main result is the following.

**Theorem 1.1** (Short time existence and uniqueness). Let $p \in (3, \infty)$ and $N_0$ be a regular network in $\mathbb{R}^n$ of class $W_p^{2-2/p}$. Then there exists a unique maximal solution $(N(t))_{t \in [0,T_{\text{max}})}$ to the Motion by Curvature with initial datum $N_0$ in the maximal time interval $[0,T_{\text{max}})$.

We underline that several solutions to the motion by curvature can be obtain by parametrising the same set with different maps, hence uniqueness has to understood in a purely geometric sense, up to reparametrisations. Local existence and uniqueness was proved in [8] for regular network of class $C^{2+\alpha}$ with the sum of the curvature at the junctions equal to zero. When the initial datum is a regular network of class $C^2$ without any restriction on the curvature at the junctions existence (but not uniqueness) has been established in [24, Theorem 6.8]. Theorem 1.1 improves the uniqueness result by Bronsard and Reitich passing from initial data in $C^{2+\alpha}$ to $W_p^{2-2/p}$, giving a fortiori uniqueness for regular networks of class $C^2$ and even of class $H^2$ (take any $p \in (3,6]$). Hence combining Theorem 1.1 together with [24, Theorem 6.8] we are able to show wellposedness of the flow for $C^2$ initial data without imposing second order conditions.

We discuss now in more details Theorem 1.1. The motion by curvature of networks is described by a parabolic system of degenerate PDEs, where only the normal movements of the curves are prescribed. We specify a suitable tangential component of the velocity to turn the problem into a system of non–degenerate second order quasilinear PDEs, the so–called Special Flow (Definition 2.30). Then we linearise the Special Flow around the initial datum and prove existence and uniqueness for the linearised problem ($\S$3.1). Wellposedness of the linear system follows by Solonnikov’s theory [29] provided that the system is parabolic and that the complementary conditions hold. Both properties were already shown in [8], nevertheless we present a new and shorter proof of the second one. Solutions to the special flow are obtained by a contraction argument ($\S$3.2). The solution to the Special Flow induces a solution to the Motion by Curvature of networks. To conclude the uniqueness result it is then enough to prove that two different solutions differ only by a reparametrisation but they are actually the same set ($\S$3.3). Existence of a maximal solution is then standard and thus Theorem 1.1 is proved. We consider as solution space

$$W^1_p \left( (0, T); L_p \left( (0, 1); \mathbb{R}^2 \right) \right) \cap L_p \left( (0, T); W^2_p \left( (0, 1); \mathbb{R}^2 \right) \right)$$

that for $p \in (3, \infty)$ embeds into $BUC \left( [0, T]; C^{1+\alpha} \left( [0, 1] \right) \right)$. This choice allows us to define the boundary conditions pointwise and to use the theory of [29] for the associated linear system. Moreover the above regularity is needed in the contraction estimates because of the quasilinear nature of the equations.

Because of the parabolicity of the problem, it is natural to ask whether the regularity of the evolving network increases during the flow. We give a positive answer to this question in
Section 4 proving that the flow is smooth for all positive times (see Theorem 4.8). The basic idea of the proof is based on the so called parameter trick which is due to Angenent [4]. Although this strategy has been generalized to several context [21, 22, 27], it should be pointed out that our system is not among the above treated cases because of the fully non-linear boundary condition

$$\sum_{i=1}^{3} \frac{\gamma_{ix}}{|\gamma_{x}|} = 0.$$ 

In [14] has been developed a strategy to prove smoothness for positive time of the surface diffusion flow for triple junction clusters with the same non-linear boundary condition. We follow this line and modify the arguments to our setting to prove Theorem 4.8.

Finally a description of the possible different behaviors of the solutions as time tends to the maximal time of existence is desirable. Taking advantage of Theorem 1.1 and Theorem 3.13 we are also able to prove the following:

**Theorem 1.2** (Long time behaviour). Let $p \in (3, 6]$ and $(\mathcal{N}(t))_{t \in [0, T_{\text{max}}]}$ be a maximal solution to the Motion by Curvature in $[0, T_{\text{max}})$ with $T_{\text{max}} \in (0, \infty) \cup \{\infty\}$ to the initial datum $\mathcal{N}_0 = \bigcup_{i=1}^{N} \mathcal{N}_{0i}$. Then

$$T_{\text{max}} = \infty$$

or as $t \to T_{\text{max}}$ at least one of the following happens:

i) the inferior limit of the length of at least one curve of the network $\mathcal{N}(t)$ is zero;

ii) the superior limit of the $L^2$–norm of the curvature of the network is $+\infty$.

This result was first shown for planar networks in [25, Theorem 3.18]. The merit of our proof is that avoids completely energy estimates.

We describe here the structure of the paper. In Section 2 we define the motion by curvature of network and we introduce the solutions space together with useful properties. Section 3 is devoted to the proof of Theorem 1.1. Then in Section 4 we explore the regularisation effect of the flow. We conclude with Section 5 with a description of the behaviour of the solutions at the maximal time of existence.

## 2 Solutions to the Motion by Curvature of networks

### 2.1 Preliminaries on function spaces

This paper is devoted to show well-posedness of a second order evolution equation. One natural solution space is given by

$$W^{1,2}_p \left( (0, T) \times (0, 1); \mathbb{R}^d \right) := W^1_p \left( (0, T); L_p((0, 1); \mathbb{R}^d) \right) \cap L_p \left( (0, T); W^2_p \left( (0, 1); \mathbb{R}^d \right) \right)$$

where $T$ is positive representing the time of existence and $d \in \mathbb{N}$ is any natural number. This space should be understood as the intersection of two Bochner spaces, that are Sobolev spaces defined on a measure space with values in a Banach space. We give a brief summary in the case that the measure space is an interval. A detailed introduction on Bochner spaces can be found in [32].
Definition 2.1. Let $I \subset \mathbb{R}$ be an open interval and $X$ be a Banach space. A function $f : I \rightarrow X$ is called strongly measurable if there exist a family of simple functions $f_n : I \rightarrow X$, $n \in \mathbb{N}$ such that for almost every $x \in I$,
\[
\lim_{n \to \infty} \|f_n(x) - f(x)\| = 0.
\]
Here, a function $g : I \rightarrow X$ is called simple if
\[
g = \sum_{k=1}^{N} a_k \chi_{(b_k, c_k)}
\]
for $N \in \mathbb{N}$, $a_k \in X$, $b_k, c_k \in I$ and $b_k < c_k$ for $k \in \{1, \ldots, N\}$.

If $f : I \rightarrow X$ is strongly measurable, then $\|f\|_X : I \rightarrow \mathbb{R}$ is Lebesgue measurable. This justifies the following definition.

Definition 2.2 ($L^p$-spaces). Let $I \subset \mathbb{R}$ be an open interval and $X$ be a Banach space. For $1 \leq p \leq \infty$, we define
\[
L^p(I; X) := \left\{ f : I \rightarrow X \text{ strongly measurable} : \|f\|_{L^p(I; X)} < \infty \right\},
\]
where $\|f\|_{L^p(I; X)} := \|\|f(\cdot)\|_X\|_{L^p(I; \mathbb{R})}$. Furthermore, we let
\[
L^1_{loc}(I; X) := \left\{ f : I \rightarrow X \text{ strongly measurable} : \text{ for all } K \subset I \text{ compact, } f_{|K} \in L^1(K; X) \right\}.
\]

Definition 2.3. Let $I \subset \mathbb{R}$ be an open interval, $X$ be a Banach space, $f \in L^1_{loc}(I; X)$ and $k \in \mathbb{N}_0$. The $k$-th distributional derivative $\partial^k_x f$ of $f$ is the functional on $C_0^\infty(I; \mathbb{R})$ given by
\[
\langle \phi, \partial^k_x f \rangle := (-1)^{|k|} \int_I f(x) \partial^k_x \phi(x) \, dx.
\]
The distribution $\partial^k_x f$ is called regular if there exists $v \in L^1_{loc}(I; X)$ such that
\[
\langle \phi, \partial^k_x f \rangle := \int_I v(x) \phi(x) \, dx.
\]
In this case we write $\partial^k_x f = v \in L^1_{loc}(I; X)$.

Definition 2.4 (Sobolev spaces). Let $m \in \mathbb{N}$ be an integer, $I \subset \mathbb{R}$ an open interval and $X$ a Banach space. For $1 \leq p \leq \infty$ the Sobolev space of order $m \in \mathbb{N}$ is defined as
\[
W^m_p(I; X) := \left\{ f \in L^p(I; X) : \partial^k_x f \in L^p(I; X) \text{ for all } 1 \leq k \leq m \right\},
\]
where $\partial^k_x f$ is the distributional derivative defined in Definition 2.3.

Theorem 2.5. Let $m \in \mathbb{N}$ be an integer, $1 \leq p \leq \infty$, $I \subset \mathbb{R}$ an open interval and $X$ a Banach space. The space $W^m_p(I; X)$ is a Banach space in the norm
\[
\|f\|_{W^m_p(I; X)} := \left\{ \left( \sum_{0 \leq k \leq m} \|\partial^k_x f\|_{L^p(I; X)}^p \right)^{1/p} \right\}^{1/p}, \quad 1 \leq p < \infty,
\]
\[
\max_{0 \leq k \leq m} \|\partial^k_x f\|_{L^\infty(I; X)}, \quad p = \infty.
\]
Elements in the solution space

\[ W^1_p \left( (0, T); L_p((0, 1); \mathbb{R}^d) \right) \cap L_p \left( (0, T); W^2_p \left( (0, 1); \mathbb{R}^d \right) \right) \]

are thus functions \( f : (0, T) \rightarrow L_p \left( (0, 1); \mathbb{R}^d \right) \) with \( f \in L_p \left( (0, T); L_p \left( (0, 1); \mathbb{R}^d \right) \right) \) possessing one distributional derivative with respect to time \( \partial_t f \in L_p \left( (0, T); L_p \left( (0, 1); \mathbb{R}^d \right) \right) \) in the sense of Definition 2.6. Furthermore, for almost every \( t \in (0, T) \), the function \( f(t) \) lies in \( W^2_p \left( (0, 1); \mathbb{R}^d \right) \) and thus has two spacial derivatives \( \partial_x (f(t)), \partial^2_x (f(t)) \in L_p \left( (0, 1); \mathbb{R}^d \right) \). The functions \( t \mapsto \partial^k_x (f(t)), k \in \{1, 2\} \) lie in \( L_p \left( (0, T); L_p \left( (0, 1); \mathbb{R}^d \right) \right) \). One verifies that the map

\[ J : L_p \left( (0, T); L_p \left( (0, 1); \mathbb{R}^d \right) \right) \rightarrow L_p \left( (0, T) \times (0, 1); \mathbb{R}^d \right), \quad (J f)(t, x) := (f(t))(x) \]

defines an isometric isomorphism. Using this property one shows that the solution space \( W^{1,2}_p \left( (0, T) \times (0, 1); \mathbb{R}^d \right) \) can be identified with functions \( f : (0, T) \times (0, 1) \rightarrow \mathbb{R}^d \) having one distributional derivative with respect to time and two distributional derivatives with respect to space almost everywhere in \( (0, T) \times (0, 1) \) such that these derivatives are again elements of \( L_p \left( (0, T) \times (0, 1); \mathbb{R}^d \right) \).

**Definition 2.6.** Given \( d \in \mathbb{N}, p \in [1, \infty) \) and \( \theta \in (0, 1) \) the Slobodeckiĭ seminorm of an element \( f \in L_p \left( (0, 1); \mathbb{R}^d \right) \) is defined as

\[ [f]_{\theta, p} := \left( \int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^\theta p + 1} \, dx \, dy \right)^{1/p}. \]

Let \( s \in (0, \infty) \) be a non–integer. The Sobolev–Slobodeckiĭ space \( W^s_p \left( (0, 1); \mathbb{R}^d \right) \) is defined by

\[ W^s_p \left( (0, 1); \mathbb{R}^d \right) := \left\{ f \in W^{[s]}_p \left( (0, 1); \mathbb{R}^d \right) : \left[ \partial_x^{|s|} f \right]_{s - |s|, p} < \infty \right\}. \]

The following well-known result characterises the regularity of the initial values.

**Proposition 2.7.** Let \( d \in \mathbb{N}, p \in [1, \infty), T \) be positive. The temporal trace

\[ \text{tr}_{t=0} : W^1_p \left( (0, T); L_p((0, 1); \mathbb{R}^d) \right) \cap L_p \left( (0, T); W^2_p \left( (0, 1); \mathbb{R}^d \right) \right) \rightarrow W^{2-\beta/p}_p \left( (0, 1); \mathbb{R}^d \right) \]

is linear, continuous and surjective, that is, there exists a continuous linear extension operator

\[ \mathcal{E} : W^{2-\beta/p}_p \left( (0, 1); \mathbb{R}^d \right) \rightarrow W^{1,2}_p \left( (0, T) \times (0, 1); \mathbb{R}^d \right) \]

such that \( \| \mathcal{E} \sigma \|_{W^{1,2}_p} \leq \| \sigma \|_{W^{2-\beta/p}(0,1)} \) and \( (\mathcal{E} \sigma)|_{t=0} = \sigma \).

**Proof.** The first part of the statement can be found for example in [9, Theorem 4.5]. The function \( \mathcal{E} \) can be obtained as follows. Using reflection and a cut-off function we may construct a linear and continuous extension operator

\[ E : W^{2-\beta/p}_p \left( (0, 1); \mathbb{R} \right) \rightarrow W^{2-\beta/p}_p \left( \mathbb{R}; \mathbb{R} \right) \]

By [27, Corollary 6.1.12] the initial value problem

\[ \partial_t u(t, x) - \Delta u(t, x) = 0, \quad t \in (0, T_0), x \in \mathbb{R}, \]

\[ u(0) = \psi, \]
admits a unique solution $u := \mathcal{L}^{-1} \psi \in W_p^{1,2}((0, T) \times \mathbb{R}; \mathbb{R})$ which depends on the initial value $\psi \in W_p^{2-2/p}((0, T) \times \mathbb{R}; \mathbb{R})$ in a linear and continuous way. Setting $\mathcal{E} := E \circ L^{-1} \circ E$ where $R : W_p^{1,2}((0, T) \times \mathbb{R}; \mathbb{R}) \to W_p^{1,2}((0, T) \times (0,1); \mathbb{R})$ is the restriction operator, we obtain the desired result. 

Similarly, we can specify the spaces of the boundary values.

**Lemma 2.8.** Let $T$ be positive, $d \in \mathbb{N}$ and $p \in [1, \infty)$. Then the operator

$$W_p^{1,2}((0, T) \times (0,1); \mathbb{R}^d) \to W_p^{1/2-1/2p}((0, T); L_p(\{0\}; \mathbb{R}^d)) \cap L_p((0, T); W_p^{1-1/p}(\{0\}; \mathbb{R}^d)),$$

$$\gamma \mapsto (\gamma_x)_{x=0}$$

is linear and continuous with operator norm 1.

**Proof.** This follows from [29, Theorem 5.1].

In this work we will use the following identification.

**Proposition 2.9.** Let $T$ be positive, $d \in \mathbb{N}$ and $p \in [1, \infty)$. There is an isometric isomorphism

$$W_p^{1/2-1/2p}((0, T); L_p(\{0\}; \mathbb{R}^d)) \cap L_p((0, T); W_p^{1-1/p}(\{0\}; \mathbb{R}^d)) \simeq W_p^{1/2-1/2p}((0, T); \mathbb{R}^d)$$

via the map $f \mapsto (t \mapsto f(t, 0))$.

**Proof.** It is shown in [20, page 406] that integration with respect to the volume element on the 0-dimensional manifold $\{0\}$ is given by integration with respect to the counting measure. That allows us to identify the space

$$L_p(\{0\}; \mathbb{R}^d) := \left\{ f : \{0\} \to \mathbb{R}^d : \int_{\{0\}} |f| \, d\sigma = |f(0)| < \infty \right\}$$

with $\mathbb{R}^d$ via the isometric isomorphism $I : L_p(\{0\}; \mathbb{R}^d) \to \mathbb{R}^d, f \mapsto f(0)$. One easily sees that this operator restricts to $I : W_p^s(\{0\}; \mathbb{R}^d) \to \mathbb{R}^d$ for every $s > 0$. 

We collect some important properties of the boundary data space.

**Proposition 2.10.** Let $I \subset \mathbb{R}$ be a bounded open interval, $p \in [1, \infty)$ and $s \in (0, 1)$ with $s - \frac{1}{p} > 0$. Then $W_p^s(I; \mathbb{R})$ is a Banach algebra where the submultiplicativity of the norm holds up to a constant depending only on $p$ and $s$. More precisely, for $f, g \in W_p^s(I; \mathbb{R})$, the product $fg$ lies in $W_p^s(I; \mathbb{R})$ and satisfies

$$\|fg\|_{W_p^s(I; \mathbb{R})} \leq C(s, p) \left( \|f\|_\infty \|g\|_{W_p^s(I; \mathbb{R})} + \|g\|_\infty \|f\|_{W_p^s(I; \mathbb{R})} \right).$$

Furthermore, given a smooth function $F : \mathbb{R} \to \mathbb{R}$ and a function $f \in W_p^s(I; \mathbb{R})$, the function $t \mapsto F(f(t))$ lies in $W_p^s(I; \mathbb{R})$.

**Proof.** As $W_p^s((0,1); \mathbb{R}) \to C(T; \mathbb{R})$, we obtain for $f, g \in W_p^s(I; \mathbb{R})$ the estimate

$$\|fg\|_{L_p(I; \mathbb{R})} \leq \|f\|_\infty \|g\|_{L_p(I; \mathbb{R})} \leq C(s, p) \|f\|_{W_p^s(I; \mathbb{R})} \|g\|_{W_p^s(I; \mathbb{R})}$$
\[ [fg]_{s,p}^p = \int_I \int_I \frac{|(fg)(x) - (fg)(y)|^p}{|x - y|^{p+1}} \, dx \, dy \]
\[ \leq \int_I \int_I \frac{|g(x)|^p|f(x) - f(y)|^p + |f(y)|^p|g(x) - g(y)|^p}{|x - y|^{p+1}} \, dx \, dy \]
\[ \leq \|g\|_{\infty}^p [f]_{s,p}^p + \|f\|_{\infty}^p \|g\|_{s,p}^p \leq C(s,p) \|f\|_{W^s_p(I;\mathbb{R})} \|g\|_{W^s_p(S;\mathbb{R})}. \]

Let \( F : \mathbb{R} \to \mathbb{R} \) be smooth and \( f \in W^s_p(I;\mathbb{R}) \). As \( f \) lies in \( C(\bar{I};\mathbb{R}) \), there exists \( R > 0 \) such that \( f(\bar{I}) \subset B_R(0) \). Thus we obtain
\[ \|F(f)\|_{L^p(I;\mathbb{R})}^p = \int_I |F(f(x))|^p \, dx \leq \max_{z \in B_R(0)} |F(z)|^p |I|, \]
where \( |I| \) denotes the length of the interval \( I \). Using
\[ |F(f(x)) - F(f(y))| = |\int_0^1 (f'(\xi f(x) + (1 - \xi)f(y)) \, d\xi \, (f(x) - f(y))| \]
\[ \leq \max_{z \in B_R(0)} |F'(z)||f(x) - f(y)| \]
we obtain
\[ [F(f)]_{s,p}^p = \int_I \int_I \frac{|F(f(x)) - F(f(y))|^p}{|x - y|^{p+1}} \, dx \, dy \leq [f]_{s,p}^p \max_{z \in B_R(0)} |F'(z)|^p. \]

To show well-posedness of evolution equations it is important to have embeddings with constants independent of the time interval one is working with. To this end one needs to change the norm on the solution space. In the following, we collect the results that are needed in our specific case.

**Theorem 2.11.** Let \( T \) be positive, \( p \in (3, \infty) \) and \( \alpha \in (0, 1 - 3/p] \). We have continuous embeddings
\[ W^{1,2}_p ((0,T) \times (0,1)) \hookrightarrow \text{BUC} \left( [0,T]; W^{2-2/p}_p ((0,1)) \right) \hookrightarrow \text{BUC} \left( [0,T]; C^{1+\alpha} ([0,1]) \right). \]

**Proof.** The first embedding follows from [3, Chapter III, Theorem 4.10.2], the second is an immediate consequence of the Sobolev Embedding Theorem, see for example [1, Theorem 4.12].

**Proposition 2.12.** Let \( T \) be positive, \( p \in (3, \infty) \) and \( \theta \in \left( \frac{1 + 1/p}{2 - 2/p}, 1 \right) \). Then
\[ W^{1,2}_p ((0,T) \times (0,1)) \hookrightarrow C^{(1-\theta)(1-1/p)} ([0,T]; C^1 ([0,1])) \]
with continuous embedding.
Proof. By [28, Corollary 26], \( W^1_p ((0, T); L_p ((0, 1))) \rightarrow C^{1-\frac{1}{p}} (\{0, T]\); L_p ((0, 1))) \). A direct calculation shows that for all Banach spaces \( X_0, X_1 \) and \( Y \) such that \( X_0 \cap X_1 \subset Y \) and \( \| y \|_Y \leq C \| y \|^{1-\theta}_{X_0} \| y \|^{\theta}_{X_1} \) for all \( y \in X_0 \cap X_1 \), one has the continuous embedding

\[
BUC ([0, T]; X_1) \cap C^\alpha ([0, T]; X_0) \hookrightarrow C^{(1-\theta)\alpha} ([0, T]; Y) .
\]

For all \( \theta \in (0, 1) \) the real interpolation method gives

\[
W_p^{\theta(2-2/p)} ((0, 1)) = \left( L_p ((0, 1)); W_p^{2-2/p} ((0, 1)) \right)_{\theta, p}
\]

and hence using Theorem 2.11 the arguments above imply for all \( \theta \in (0, 1) \),

\[
W_p^{1,2} ((0, T) \times (0, 1)) \hookrightarrow C^{(1-\theta)(1-1/p)} (\{0, T]\); W_p^{\theta(2-2/p)} ((0, 1))) .
\]

The assertion now follows from the Sobolev Embedding Theorem. \( \square \)

Corollary 2.13. Let \( p \in (3, \infty) \). For every \( T > 0 \),

\[
\| g \|_{W_p^{1,2}((0,T)\times(0,1))} := \| g \|_{W_p^{1,2}((0,T)\times(0,1))} + \| g(0) \|_{W_p^{2-2/p}((0,1))}
\]

defines a norm on \( W_p^{1,2} ((0, T) \times (0, 1)) \) that is equivalent to the usual one.

Lemma 2.14. Let \( T \) be positive and \( p \in (3, \infty) \). There exists a linear operator

\[
E : W_p^{1,2} ((0, T) \times (0, 1)) \rightarrow W_p^{1,2} ((0, \infty) \times (0, 1))
\]

such that for all \( g \in W_p^{1,2} ((0, T) \times (0, 1)) \), \( (Eg)_{|0,T} = g \) and

\[
\| Eg \|_{W_p^{1,2}((0,\infty)\times(0,1))} \leq C_p \left( \| g \|_{W_p^{1,2}((0,T)\times(0,1))} + \| g(0) \|_{W_p^{2-2/p}((0,1))} \right) = C_p \| g \|_{W_p^{1,2}((0,T)\times(0,1))}
\]

with a constant \( C_p \) depending only on \( p \).

Proof. In the case that \( g(0) = 0 \), the function \( g \) can be extended to \( (0, \infty) \) by reflecting it with respect to the axis \( t = T \). The general statement can be deduced from this case by solving a linear parabolic equation of fourth order and using results on maximal regularity as given in [27, Proposition 3.4.3]. \( \square \)

Applying \( E \) to every component we obtain an extension operator on \( W_p^{1,4} ((0, T) \times (0, 1); \mathbb{R}^d) \) for \( d \geq 1 \). The following Lemma is an immediate consequence of the Sobolev Embedding Theorem [1, Theorem 4.12].

Lemma 2.15. Let \( p \in (1, \infty) \) and \( \alpha > \frac{1}{p} \). For every positive \( T \),

\[
\| b \|_{W_p^\alpha ((0,T); \mathbb{R})} := \| b \|_{W_p^\alpha ((0,T); \mathbb{R})} + | b(0) |
\]

defines a norm on \( W_p^\alpha ((0,T); \mathbb{R}) \) that is equivalent to the usual one.
Lemma 2.16. Let $T$ be positive, $p \in (1, \infty)$ and $\alpha > \frac{1}{p}$. There exists a linear operator

$$E : W_p^\alpha ((0, T); \mathbb{R}) \to W_p^\alpha ((0, \infty); \mathbb{R})$$

such that for all $b \in W_p^\alpha ((0, T); \mathbb{R})$, $(Eb)_{|0,T} = b$ and

$$\|Eb\|_{W_p^\alpha ((0, \infty); \mathbb{R})} \leq C_p \left( \|b\|_{W_p^\alpha ((0, T); \mathbb{R})} + |b(0)| \right) = C_p \|b\|_{W_p^\alpha ((0, T); \mathbb{R})}$$

with a constant $C_p$ depending only on $p$.

Proof. In the case $b(0) = 0$ the operator obtained by reflecting the function with respect to the axis $t = T$ has the desired properties. The general statement can be deduced from this case using surjectivity of the temporal trace $|t_0 : W_p^\alpha ((0, \infty); \mathbb{R}) \to \mathbb{R}$.

Lemma 2.17. Let $p \in (3, \infty)$ and $T_0$ be positive. There exist constants $C(p)$ and $C(T_0, p)$ such that for all $T \in (0, T_0]$ and all $g \in W_p^{1,2} ((0, T) \times (0, 1))$,

$$\|g\|_{BUC((0,T];C^1((0,1)))} \leq C(p) \|g\|_{BUC((0,T);W_p^{2,2}(0,1))} \leq C(T_0, p) \|g\|_{W_p^{1,2}((0,T)\times(0,1))}.$$

Proof. Let $T \in (0, T_0]$ be arbitrary, $g \in W_p^{1,2} ((0, T) \times (0, 1))$ and $Eg$ the extension according to Lemma 2.14. Then $(Eg)_{|0,T_0}$ lies in $W_p^{1,2} ((0, T_0) \times (0, 1))$ and Theorem 2.11 implies

$$\|g\|_{BUC([0,T],W_p^{2,2}(0,1))} \leq \|(Eg)_{|0,T_0}\|_{BUC([0,T],W_p^{2,2}(0,1))} \leq C(T_0, p) \|(Eg)_{|0,T_0}\|_{W_p^{1,2}((0,T_0)\times(0,1))} \leq C(T_0, p) \|g\|_{W_p^{1,2}((0,T_0)\times(0,1))}.$$

Lemma 2.18. Let $p \in (3, \infty)$, $\theta \in \left( \frac{1 + 1/p}{2 - 2/p}, 1 \right)$ and $T_0$ be positive. There exists a constant $C(T_0, p, \theta)$ such that for all $T \in (0, T_0]$ and all $g \in W_p^{1,2} ((0, T) \times (0, 1))$,

$$\|g\|_{C^{(1-\theta)(1-1/p)}((0,T];C^1([0,1]))} \leq C(T_0, p, \theta) \|g\|_{W_p^{1,2}((0,T)\times(0,1))}.$$

Proof. Similarly to the previous proof it holds for $T \in (0, T_0]$ and $g \in W_p^{1,2} ((0, T) \times (0, 1))$,

$$\|g\|_{C^{(1-\theta)(1-1/p)}((0,T];C^1([0,1]))} \leq \|(Eg)_{|0,T_0}\|_{C^{(1-\theta)(1-1/p)}((0,T_0];C^1([0,1]))} \leq C(T_0, p, \theta) \|(Eg)_{|0,T_0}\|_{W_p^{1,2}((0,T_0)\times(0,1))} \leq C(T_0, p, \theta) \|g\|_{W_p^{1,2}((0,T)\times(0,1))}.$$
2.2 Motion by Curvature of networks

Let $n \in \mathbb{N}, n \geq 2$. Consider a curve $\sigma : [0, 1] \to \mathbb{R}^n$ of class $C^1$. A curve is said to be regular if $|\sigma_x(x)| \neq 0$ for every $x \in [0, 1]$. Let us denote with $s$ the arclength parameter. We remind that $\partial_s = \frac{\partial}{\partial |\sigma_s|}$. A curve is said to be regular if $|\sigma_s(x)| \neq 0$ for every $x \in [0, 1]$. Let us denote with $s$ the arclength parameter. We remind that $\partial_s = \frac{\partial}{\partial |\sigma_s|}$. The curvature vector of a regular $C^2$-curve $\sigma$ is defined by

$$\kappa := \sigma_{ss} = \frac{\sigma_{xx}}{|\sigma_x|^2} - \frac{\langle \sigma_{xx}, \sigma_x \rangle \sigma_x}{|\sigma_x|^4}.$$  

The curvature is nothing but $\kappa = |\tau_s|$. If the curve is planar, namely $\sigma : [0, 1] \to \mathbb{R}^2$, we define its unit normal vector $\nu$ to be the anticlockwise rotation of $\tau$ by $\pi/2$. The oriented curvature of $\sigma$ is the function $k : [0, 1] \to \mathbb{R}$ such that $\tau_s = k \nu$. Although $\tilde{\nu}$ is not always defined, the curvature times the normal to the curve $\kappa \tilde{\nu}$ makes sense even if the curve is regular but not bi-regular. Notice that if the bi-regular curve is in the plane both $\nu$ and $\tilde{\nu}$ are defined and may differ of a sign ($\nu(x) = \tilde{\nu}(x)$ if $k(x) > 0$ and $-\nu(x) = \tilde{\nu}(x)$ if $k(x) < 0$).

Definition 2.19. A network $\mathcal{N}$ is a connected set in $\mathbb{R}^n$ consisting of a finite union of regular curves $\mathcal{N}^i$ that meet only at their end-points in junctions. Each curve $\mathcal{N}^i$ admits a regular $C^1$-parametrisation, that is, a map $\gamma^i : [0, 1] \to \mathbb{R}^n$ of class $C^1$ such that $\gamma^i([0, 1]) = \mathcal{N}^i$.

In the planar case we require the networks to be embedded. Although by definition a network is a set, we will mainly deal with its parametrisations. It is then natural to speak about the regularity of these maps.

Definition 2.20. Let $k > 1$ and $1 \leq p \leq \infty$ with $p > \frac{1}{k-1}$. A network $\mathcal{N}$ is of class $C^k$ (or $W^k_p$, respectively) if it admits a regular parametrisation of class $C^k$ (or $W^k_p$, respectively).

In this paper we restrict to the class of regular network.

Definition 2.21. A network is regular if its curves meet at triple junctions forming equal angles.

Notice that this notion is geometric in the sense that it does not depend on the choice of the parametrisations $\sigma^i$ of the curves of the network $\mathcal{N}$.

We define now the Motion by Curvature of regular networks: a time dependent family of regular networks evolves with normal velocity equal to the curvature vector at any point and any time, namely

$$v^\perp = k.$$  

To be more precise, given a time dependent family of curves $\gamma^i$, we introduce $P := \text{Id} - \gamma_s \otimes \gamma_s$ the projection onto the normal part to $\gamma$. The motion equation reads as

$$P \gamma^i = k.$$  

For sake of presentation we restrict to the Motion by Curvature of a Triod.
**Definition 2.22.** A Triod $\mathcal{T} = \bigcup_{i=1}^{3} \sigma^i([0,1])$ is a network composed of three regular $C^1$-curves $\sigma^i : [0,1] \to \mathbb{R}^n$ that intersect each other only at the triple junction $O := \sigma^1(0) = \sigma^2(0) = \sigma^3(0)$. The other three end-points of the curves $\sigma^i(1)$ with $i \in \{1,2,3\}$ coincide with three distinct points $P^i := \sigma^i(1) \in \mathbb{R}^n$.

![Figure 1: A regular Triod in $\mathbb{R}^2$.](image)

**Definition 2.23** (Geometrically admissible initial Triod). A Triod $\mathcal{T}_0$ is a geometrically admissible initial datum for the motion by curvature if it is regular and each of its curves can be parametrised by a regular curve $\sigma^i \in W^{2-2/p}_p([0,1], \mathbb{R}^n)$ with $p \in (3, \infty)$.

**Remark 2.24.** For $p \in (3, \infty)$ the Sobolev Embedding Theorem implies

$$W^{2-2/p}_p((0,1); \mathbb{R}^n) \hookrightarrow C^{1+\alpha}([0,1]; \mathbb{R}^n)$$

for $\alpha \in (0, 1 - 2/p)$ (see for instance [1, Theorem 4.12]). In particular, any admissible initial network is of class $C^1$ and the angle condition at the boundary is well-defined.

**Definition 2.25.** Let $p \in (3, \infty)$ and $T > 0$. Let $\mathcal{T}_0$ be a geometrically admissible initial Triod with fixed endpoints $P^1, P^2, P^3$. A time dependent family of Triods $(\mathcal{T}(t))$ is a solution to Motion by Curvature in $[0,T]$ with initial datum $\mathcal{T}_0$ if and only if there exists a collection of time dependent parametrisations

$$\gamma^i_n \in W^{1}_p(I_n; L_p((0,1); \mathbb{R}^n)) \cap L_p(I_n; W^{2}_p((0,1); \mathbb{R}^n)),$$

with $n \in \{0, \ldots, N\}$ for some $N \in \mathbb{N}$, $I_n := (a_n, b_n) \subset \mathbb{R}, a_n \leq a_{n+1}, b_n \leq b_{n+1}, a_n < b_n$ and $\bigcup_n (a_n, b_n) = (0,T)$ such that for all $n \in \{0, \ldots, N\}$ and $t \in I_n$, $\gamma_n(t) = (\gamma^1(t), \gamma^2(t), \gamma^3(t))$ is a regular parametrisation of $\mathcal{T}(t)$. Moreover each $\gamma_n$ needs to satisfy the following system:

$$
\begin{align*}
\mathbf{P} \gamma^i(t,x) &= k^i(t,x) & \text{Motion by Curvature}, \\
\gamma^1(t,0) &= \gamma^2(t,0) = \gamma^3(t,0) & \text{concurrency condition}, \\
\sum_{i=1}^{3} \tau^i(t,0) &= 0 & \text{angle condition}, \\
\gamma^i(t,1) &= P^i & \text{fixed end-points},
\end{align*}
$$

(2.1)

for almost every $t \in I_n, x \in (0,1)$ and for $i \in \{1,2,3\}$. Finally we ask that $\gamma_n(0,[0,1]) = \mathcal{T}_0$ whenever $a_n = 0$.

**Remark 2.26.** If we let evolve a time dependent family of networks in $\mathbb{R}^2$ we require the networks to be embedded for all times $t \in [0,T]$. 

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Remark 2.27. In the Motion by Curvature equation only the normal component of the velocity is prescribed. This does not mean that there is not tangential motion. Indeed a non–trivial tangential velocity is needed to allow the motion of the triple junction.

Remark 2.28. We are interested in finding a time–dependent family of networks (\(N_t\)) solution to the Motion by Curvature. Our notion of solution allows the network to be parametrised by different sets of functions in different (but overlapping) time intervals. Namely a solution can be parametrised by \(\gamma = (\gamma^1, \gamma^2, \gamma^3)\) with \(\gamma^i : (a_0, b_0) \times [0, 1] \to \mathbb{R}^n\) and \(\sigma = (\sigma^1, \sigma^2, \sigma^3)\) with \(\sigma^i : (a_1, b_1) \times [0, 1] \to \mathbb{R}^n\) if \(a_0 \leq a_1 < b_0 \leq b_1\) and \(\gamma^i((a_1, b_0) \times [0, 1]) = \sigma^i((a_1, b_0) \times [0, 1])\).

Requiring that the family of networks (\(N(t)\)) is parametrised by one map \(\gamma(t) = (\gamma^1(t), \gamma^2(t), \gamma^3(t))\) in the whole time interval of existence \([0, T]\) as in [24] gives a slightly stronger definition of the Motion by Curvature with respect to 2.25. This difference does not affect the proof of the short time existence result, but in principle using our definition the maximal time interval of existence could be longer.

The first step to solve our Problem (2.25) is to turn system (2.1) into a system of quasilinear non–degenerate parabolic PDEs by choosing a suitable tangential velocity \(T\). We choose \(T\) so that

\[
\gamma^i_t(t, x) = P \gamma^i_t(t, x) + \gamma^2_x(t, x) \tau^i(t, x) \tau^i(t, x) = \kappa^i(t, x) + T^i(t, x) \tau^i(t, x) = \frac{\gamma^i_x(t, x)}{|\gamma^i_x(t, x)|^2}.
\]

Since the expression of the curvature reads as \(\kappa^i(t, x) = \frac{\gamma^i_x(t, x)}{|\gamma^i_x(t, x)|^2} - \left\langle \frac{\gamma^i_x(t, x)}{|\gamma^i_x(t, x)|^2}, \tau^i(t, x) \right\rangle \tau^i(t, x)\)

it is enough to choose \(T^i(t, x) = \left\langle \frac{\gamma^i_x(t, x)}{|\gamma^i_x(t, x)|^2}, \tau^i(t, x) \right\rangle\).

The equation \(\gamma_t = \frac{\gamma_x}{|\gamma_x|^2}\) is called Special Flow.

Definition 2.29 (Admissible initial parametrisation). Let \(p \in (3, \infty)\). An admissible initial parametrisation for a Triod \(\mathbb{T}_0\) is a triple \(\sigma = (\sigma^1, \sigma^2, \sigma^3)\) where \(\bigcup_i \sigma^i([0, 1]) = \mathbb{T}_0\), \(\sigma^i(1) = P^i\), \(\sigma^i(0) = \sigma^2(0) = \sigma^3(0)\) and \(\sum_{i=1}^{3} \frac{\sigma^i(0)}{|\sigma^i(0)|^2} = 0\) with \(\sigma^i\) regular and of class \(W^{2-2/p}_p((0, 1), \mathbb{R}^n)\).

Notice that it follows by the very definition that a geometrically admissible Triod admits an admissible parametrisation.

Definition 2.30 (Solution of the Special Flow). Let \(T > 0\) and \(p \in (3, \infty)\). Consider an admissible initial parametrisation \(\sigma = (\sigma^1, \sigma^2, \sigma^3)\) for a Triod \(\mathbb{T}_0\) in \(\mathbb{R}^n\) with \(\sigma^i(1) = P^i \in \mathbb{R}^n\).

Then we say that \(\gamma = (\gamma^1, \gamma^2, \gamma^3)\) is a solution of the Special Flow in the time interval \([0, T]\) if and only if

\[
\gamma^i \in W^{1, p}_p((0, T); L^p_p((0, 1); \mathbb{R}^n)) \cap L^p_p((0, T); W^{2, p}_p((0, 1); \mathbb{R}^n)),
\]

\(|\gamma^i(t, x)| \neq 0\) for all \((t, x) \in [0, T] \times [0, 1]\) and the following system is satisfied for \(i \in \{1, 2, 3\}\) and for almost every \(x \in (0, 1), t \in (0, T)\):

\[
\left\{\begin{array}{l}
\gamma^i_t(t, x) = \frac{\gamma^i_x(t, x)}{|\gamma^i_x(t, x)|^2} \\
\gamma^1(t, 0) = \gamma^2(t, 0) = \gamma^3(t, 0) \\
\sum_{i=1}^{3} \gamma^i_x(t, 0) = 0 \\
\gamma^i(t, 1) = P^i \\
\gamma^i(0, x) = \sigma^i(x)
\end{array}\right.
\]

Special Flow, concurrency condition, angle condition, fixed end–points, initial data.
Remark 2.31. Both in [8] and in [25] the authors define the Motion by Curvature introducing directly the Special Flow. This is not restrictive to get a short time existence result because a solution of the Special Flow 2.30 induces a solution of the Motion by Curvature 2.25 (see Theorem 3.14 below). However we will see that it is not trivial to get uniqueness of solution to the Motion by Curvature by the uniqueness of solutions of the special flow.

3 Existence and uniqueness

Given $M > 0$ we denote by $B_M$ the open ball in

$$W^1_p \left( (0, T); L^p((0, 1); (\mathbb{R}^n)^3) \right) \cap L_p((0, T); W^2_p((0, 1); (\mathbb{R}^n)^3)) =: E_T$$

with radius $M$ centered in the origin. This section is devoted to the proof of the following:

**Theorem 3.1.** Let $p \in (3, \infty)$ and let $\sigma = (\sigma^1, \sigma^2, \sigma^3)$ be an admissible initial parametrisation. There exists a positive radius $M$ and a positive time $T$ such that the system (2.2) has a unique solution in $E_T \cap B_M$.

3.1 Existence and uniqueness of the linearised Special Flow

We fix an admissible initial parametrisation $\sigma = (\sigma^1, \sigma^2, \sigma^3)$. Linearising the main equation of system (2.2) around the initial datum we obtain:

$$\gamma^i(t, x) - \frac{1}{|\sigma^i_x(x)|^2} \gamma^i_{xx}(t, x) = \left( \frac{1}{|\gamma^1_x(t, x)|^2} - \frac{1}{|\sigma^i_x(x)|^2} \right) \gamma^i_{xx}(t, x) =: f^i(\gamma^i)(t, x). \quad (3.1)$$

The linearisation of the angle condition in $x = 0$ is given by

$$- \sum_{i=1}^3 \left( \frac{\gamma^i_1}{|\sigma^i_x|} - \frac{\sigma^i_1 \langle \gamma^i_x, \sigma^i_x \rangle}{|\sigma^i_x|^3} \right) = \sum_{i=1}^3 \left( \left( \frac{1}{|\gamma^1_x|} - \frac{1}{|\sigma^i_x|^2} \right) \gamma^i_x + \frac{\sigma^i_1 \langle \gamma^i_x, \sigma^i_x \rangle}{|\sigma^i_x|^3} \right) =: b(\gamma)(t), \quad (3.2)$$

where we have omitted the dependence of the left-hand side on $(t, 0)$. The concurrency and the fixed end-points conditions are already linear. We obtain the following linearised system for a general right hand side $(f, b, \eta, \psi)$.

$$\begin{align*}
\gamma^i(t, x) - \frac{1}{|\sigma^i_x(x)|^2} \gamma^i_{xx}(t, x) &= f^i(t, x), \\
\gamma^1(t, 0) &= \gamma^2(t, 0) = \gamma^3(t, 0), \\
- \sum_{i=1}^3 \left( \frac{\gamma^i_1(t, 0)}{|\sigma^i_x(0)|} - \frac{\sigma^i_1(0) \langle \gamma^i_x(t, 0), \sigma^i_x(0) \rangle}{|\sigma^i_x(0)|^3} \right) &= b(t), \\
\gamma^i(1, x) &= \eta^i(t) \quad \text{for } i = 1, 2, 3, \\
\gamma^i(0, x) &= \psi^i(x) \quad \text{for } i = 1, 2, 3.
\end{align*} \quad (3.3)$$

**Definition 3.2 (Linear compatibility conditions).** Let $p \in (3, \infty)$. A function $\psi = (\psi^1, \psi^2, \psi^3)$ of class $W^{2-2/p}_p((0, 1); (\mathbb{R}^n)^3)$ satisfies the linear compatibility conditions for system (3.3) with respect to given functions $b \in W^{1/2-1/2p}_p((0, T); \mathbb{R}^n)$, $\eta \in W^{1/2-1/2p}_p((0, T); (\mathbb{R}^n)^3)$ if for $i, j \in \{1, 2, 3\}$ it holds $\psi^j(0) = \psi^j(0)$, $\psi^1(1) = \eta^1(0)$ and

$$- \sum_{i=1}^3 \left( \frac{\psi^i_1(0)}{|\sigma^i_x(0)|} - \frac{\sigma^i_1(0) \langle \psi^i_x(0), \sigma^i_x(0) \rangle}{|\sigma^i_x(0)|^3} \right) = b(0).$$
Let $p \in (3, \infty)$. We want to show that system (3.3) admits a unique solution $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ in $E_T$. The result follows from the classical theory for parabolic linear systems by Solonnikov [29] provided that the system is parabolic and that the complementary conditions hold (see [29, p. 11]). Both the parabolicity and the complementary (initial and boundary) conditions have been proven in [8]. We remark that the complementary conditions follow from the the Lopatinskii–Shapiro condition (see for instance [10, pages 11–15]).

**Definition 3.3** (Lopatinskii–Shapiro condition). Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ be arbitrary. The Lopatinskii–Shapiro condition for system (3.3) is satisfied at the triple junction if every solution $(\gamma^i)_{i \in \{1, 2, 3\}} \in C^2([0, \infty), (\mathbb{C}^2)^3)$ to

$$
\begin{cases}
\lambda \gamma^i(x) - \frac{1}{|\sigma^i_x(0)|^2} \gamma^i_{xx}(x) = 0 & x \in [0, \infty), i \in \{1, 2, 3\} \\
\gamma^2(0) - \gamma^3(0) = 0 & \gamma^1(0) = 0 & \sum_{i=1}^3 \frac{\gamma^i(x)}{|\sigma^i_x(0)|} = 0 & \lambda \gamma^i(x) \rightarrow 0 & \text{concurrency,}
\end{cases}
$$

which satisfies $\lim_{x \to \infty} |\gamma^i(x)| = 0$ is the trivial solution.

Similarly the Lopatinskii–Shapiro condition for system (3.3) is satisfied at the fixed endpoints if every solution $(\gamma^i)_{i \in \{1, 2, 3\}} \in C^2([0, \infty), (\mathbb{C}^2)^3)$ to

$$
\begin{cases}
\lambda \gamma^i(x) - \frac{1}{|\sigma^i_x(0)|^2} \gamma^i_{xx}(x) = 0 & x \in [0, \infty), i \in \{1, 2, 3\} \\
\gamma^i(0) = 0 & i \in \{1, 2, 3\} \text{ fixed endpoints,}
\end{cases}
$$

which satisfies $\lim_{x \to \infty} |\gamma^i(x)| = 0$ is the trivial solution.

**Lemma 3.4.** The Lopatinskii–Shapiro condition is satisfied.

**Proof.** We first check the condition at the triple junction. Let $\gamma$ be a solution to (3.4) satisfying $\lim_{x \to \infty} |\gamma^i(x)| = 0$. We multiply $\lambda \gamma^i(x) - \frac{1}{|\sigma^i_x(0)|^2} \gamma^i_{xx}(x) = 0$ by $|\sigma^i_x(0)| P^i \sigma^i(x)$ (where with $P^i$ we mean $Id - \sigma^i_x \otimes \sigma^i_x$), then we integrate and sum. Using the two conditions at the boundary we get

$$
0 = \sum_{i=1}^3 \int_0^\infty \lambda |\sigma^i_x(0)| \|P^i \gamma^i(x)\|^2 - \frac{1}{|\sigma^i_x(0)|} \langle \gamma^i_{xx}(x), P^i \sigma^i(x) \rangle \, dx
$$

$$
= \sum_{i=1}^3 \int_0^\infty \lambda |\sigma^i_x(0)| \|P^i \gamma^i(x)\|^2 + \frac{|P^i(\gamma^i_x(x))|^2}{|\sigma^i_x(0)|} \, dx - \sum_{i=1}^3 \frac{1}{|\sigma^i_x(0)|} \langle P^i \gamma^i_x(0), P^i \sigma^i(0) \rangle
$$

$$
= \sum_{i=1}^3 \int_0^\infty \lambda |\sigma^i_x(0)| \|P^i \gamma^i(x)\|^2 + \frac{|P^i(\gamma^i_x(x))|^2}{|\sigma^i_x(0)|} \, dx - \gamma^1(0) \sum_{i=1}^3 P^i \left( \frac{\gamma^i_x(0)}{|\sigma^i_x(0)|} \right)
$$

$$
= \sum_{i=1}^3 \int_0^\infty \lambda |\sigma^i_x(0)| \|P^i \gamma^i(x)\|^2 + \frac{|P^i(\gamma^i_x(x))|^2}{|\sigma^i_x(0)|} \, dx .
$$

As a consequence we get that $P^i(\gamma^i(x)) = 0$ for all $x \in [0, \infty)$ and in particular $P^i(\gamma^1(0)) = 0$ for all $i \in \{1, 2, 3\}$. As the orthogonal complements of $\sigma^i_x(0)$ with $i \in \{1, 2, 3\}$ span all $\mathbb{R}^n$ we conclude that $\gamma^1(0) = 0$ for all $i \in \{1, 2, 3\}$. Repeating the argument and testing the motion
equation by \(|\sigma(0)_i|^3\langle \gamma^i(x), \tau^i(0) \rangle \tau^i(0)\) we can conclude that \(\gamma^i(x) = 0\) for every \(x \in [0, \infty)\).
Indeed we obtain
\[
\sum_{i=1}^{3} \lambda |\sigma^i_x(0)| \int_0^\infty |\langle \gamma^i(x), \tau^i(0) \rangle |^2 \, dx + \sum_{i=1}^{3} \frac{1}{|\sigma^i_x(0)|} \int_0^\infty |\langle \gamma^i_x(x), \tau^i(0) \rangle |^2 \, dx + \sum_{i=1}^{3} \frac{1}{|\sigma^i_x(0)|} \langle \gamma^i(0), \tau^i_0 \rangle \langle \gamma^i_x(0), \tau^i_0 \rangle = 0.
\] (3.5)

This time the boundary condition vanishes since we get \(\gamma^i(0) = 0\) from the previous step.
Again taking the real part of (3.5) we can conclude that \(\langle \gamma^i(x), \tau^i(0) \rangle = 0\) for all \(x \in [0, \infty)\).
Hence \(\gamma^i(x) = 0\) for every \(x \in [0, \infty)\) as desired.
The condition at the fixed endpoints follows in exactly the same way using the boundary condition \(\gamma^i(0) = 0\).

Given \(T > 0\) we introduce the spaces
\[
\mathbb{E}_T := \{ \gamma \in \mathbb{E}, \gamma^i(t, 0) = \gamma^2(t, 0) = \gamma^3(t, 0) \text{ for } i \in \{1, 2, 3\}, t \in [0, T] \};
\]
\[
F_T := \{ (f, b, \eta, \psi) \text{ with } f \in L_p((0, T); L_p((0, 1); (\mathbb{R}^n)^3)), b \in W^p_{1/2, -1/2p}((0, T); \mathbb{R}^n), \eta \in W^{1, -1/2p}((0, T); (\mathbb{R}^n)^3), \sigma \in W^{2, 2-2/p}((0, 1); (\mathbb{R}^n)^3) \text{ such that the conditions } 3.2 \text{ hold} \}.
\]

**Theorem 3.5.** Let \(p \in (3, \infty)\). For every \(T > 0\) system (3.3) has a unique solution \(\gamma \in \mathbb{E}_T\) provided that \(f^i \in L_p((0, T); L_p((0, 1); (\mathbb{R}^n)^3)), b \in W^p_{1/2, -1/2p}((0, T); \mathbb{R}^n), \eta \in W^{1, -1/2p}((0, T); (\mathbb{R}^n)^3)\) and \(\psi \in W^p_{2-2/p}((0, 1); (\mathbb{R}^n)^3)\) fulfils the linear compatibility conditions 3.2 with respect to \(b\) and \(\eta\).
Moreover, there exists a constant \(C = C(T) > 0\) such that the following estimate holds:
\[
\|\gamma\|_{\mathbb{E}_T} \leq C \left( \|f\|_{L_p((0, T); L_p((0, 1)))} + \|b\|_{W^p_{1/2, -1/2p}((0, T))} + \|\eta\|_{W^{1, -1/2p}((0, T))} + \|\psi\|_{W^p_{2-2/p}((0, 1))} \right).
\]

Theorem 3.5 implies in particular that the linear operator \(L_T : \mathbb{E}_T \to \mathbb{F}_T\) defined by
\[
L_T(\gamma) = \begin{pmatrix}
\left(\gamma^i_t - \frac{\gamma^i}{|\gamma^i|^2}\right)_{i \in \{1, 2, 3\}} \\
\sum_{i=1}^{3} \frac{1}{|\sigma^i_x(0)|} \langle \gamma^i_x, \nu^i_0 \rangle \nu^i_0 \\
\gamma|_{t=0} \\
\gamma|_{t=0}
\end{pmatrix}
\]
is a continuous isomorphism.
Corollary 2.13 and Lemma 2.15 imply that for every positive \(T\) the spaces \(\mathbb{E}_T\) and \(\mathbb{F}_T\) endowed with the norms
\[
\|\gamma\|_{\mathbb{E}_T} := \|\gamma\|_{W^1, 2((0, T) \times (0, 1); (\mathbb{R}^n)^3)} = \|\gamma\|_{W^1, 2((0, T) \times (0, 1); (\mathbb{R}^n)^3)} + \|\gamma(0)\|_{W^{2, 2-2/p}((0, 1); (\mathbb{R}^n)^3)}
\]
and
\[
\|f, b, \eta, \psi\|_{\mathbb{F}_T} := \|f\|_{L_p((0, T); L_p((0, 1); (\mathbb{R}^n)^3))} + \|b\|_{W^p_{1/2, -1/2p}((0, T); \mathbb{R}^n)} + \|\eta\|_{W^{1, -1/2p}((0, T); (\mathbb{R}^n)^3)} + \|\psi\|_{W^p_{2-2/p}((0, 1); (\mathbb{R}^n)^3)}
\]
respectively, are Banach spaces. Given a linear operator \( A : \mathbb{F}_T \to \mathbb{E}_T \) we let
\[
\|A\|_{\mathcal{L}(\mathbb{F}_T, \mathbb{E}_T)} := \sup \{ \|A(f, b, \eta, \psi)\|_{\mathbb{E}_T} : (f, b, \eta, \psi) \in \mathbb{F}_T, \|(f, b, \eta, \psi)\|_{\mathbb{F}_T} \leq 1 \}.
\]

**Lemma 3.6.** Let \( p \in (3, \infty) \). For all \( T_0 > 0 \) there exists a constant \( c(T_0, p) \) such that
\[
\sup_{T \in (0, T_0]} \left\| \frac{L^{-1}}{T} \right\|_{\mathcal{L}(\mathbb{F}_T, \mathbb{E}_T)} \leq c(T_0, p).
\]

**Proof.** Let \( T \in (0, T_0] \) be arbitrary, \( (f, b, \eta, \psi) \in \mathbb{F}_T \) and \( E_{T_0} b := (Eb)_{| (0, T_0]} \), \( E_{T_0} \eta := (E\eta)_{| (0, T_0]} \) where \( E \) is the extension operator defined in Lemma 2.16. Extending \( f \) by \( 0 \) to \( E_{T_0} f \in L_p ((0, T_0); L_p ((0, 1])) \) we observe that \( (E_{T_0} f, E_{T_0} b, E_{T_0} \eta, \psi) \) lies in \( \mathbb{F}_{T_0} . \) As \( L_T \) and \( L_{T_0} \) are isomorphisms, there exist unique \( \gamma \in \mathbb{E}_T \) and \( \tilde{\gamma} \in \mathbb{E}_{T_0} \) such that \( L_T \gamma = (f, b, \eta, \psi) \) and \( L_{T_0} \tilde{\gamma} = (E_{T_0} f, E_{T_0} b, E_{T_0} \eta, \psi) \) satisfying
\[
L_T \gamma = (f, b, \eta, \psi) = (E_{T_0} f, E_{T_0} b, E_{T_0} \eta, \psi)_{| (0, T]} = (L_{T_0} \tilde{\gamma})_{| (0, T)} = L_T (\tilde{\gamma}_{| (0, T)}
\]
and thus \( \gamma = \tilde{\gamma}_{| (0, T)} \). Using Theorem 3.5 and the equivalence of norms on \( \mathbb{E}_{T_0} \) this implies
\[
\left\| \frac{L^{-1}}{T} (f, b, \eta, \psi) \right\|_{\mathbb{E}_T} = \left\| \left( \frac{L^{-1}}{T} (E_{T_0} f, E_{T_0} b, E_{T_0} \eta, \psi) \right)_{| (0, T]} \right\|_{\mathbb{E}_{T_0}} \leq \left\| \frac{L^{-1}}{T_0} (E_{T_0} f, E_{T_0} b, E_{T_0} \eta, \psi) \right\|_{\mathbb{E}_{T_0}} \leq c(T_0, p) \left\| (f, b, \eta, \psi) \right\|_{\mathbb{F}_{T_0}} \leq c(T_0, p) \left\| (f, b, \eta, \psi) \right\|_{\mathbb{F}_T}.
\]

\( \Box \)

### 3.2 Existence and uniqueness of the Special Flow

Given an admissible initial parametrisation \( \sigma \) and \( T > 0 \) we consider the complete metric spaces
\[
\mathbb{E}_T^\sigma := \{ \gamma \in \mathbb{E}_T \text{ such that } \gamma_{| t=0} = \sigma \text{ and } \gamma_{| x=1} = \sigma(1) \},
\]
\[
\mathbb{F}_T^\sigma := \{ (f, b) \text{ such that } (f, b, \sigma(1), \sigma) \in \mathbb{F}_T \} \times \{ \sigma(1) \} \times \{ \sigma \}.
\]

Furthermore, given \( M \) positive we introduce the notation
\[
\overline{\mathbb{E}_M} := \left\{ \gamma \in \mathbb{E}_T : \|\gamma\|_{\mathbb{E}_T} \leq M \right\}.
\]

**Lemma 3.7.** Let \( p \in (3, \infty) \), \( T > 0 \) and \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \) be an admissible initial parametrisation. Then the space \( \mathbb{E}_T^\sigma \) is nonempty.

**Proof.** Consider for each \( i \in \{ 1, 2, 3 \} \) the linear boundary value problem
\[
\begin{align*}
\eta^i_{tt}(t, x) - \eta^i_{tx}(t, x) &= 0, \quad \text{in } (0, T) \times (0, 1), \\
\eta^i(t, 0) &= \sigma^i(0), \quad \text{in } (0, T), \\
\eta^i(t, 1) &= \sigma^i(1), \quad \text{in } (0, T), \\
\eta^i(0, x) &= \sigma^i(x), \quad \text{in } (0, 1).
\end{align*}
\]
The linear theory in [29] implies that for every $T > 0$ there exists a unique solution $\eta^i \in W^{1,2}_p((0,T) \times (0,1); \mathbb{R}^n)$. Indeed, similar arguments as in the previous subsection show that the system satisfies the Lopatinskii-Shapiro condition and the right hand side $(\sigma^i(0), \sigma^i(1), \sigma^i(x))$ is compatible. As $\sigma$ parametrises a Triod, we obtain $\sigma^i(0) = \sigma^i(0) = \sigma^3(0)$ which shows $\eta := (\eta^1, \eta^2, \eta^3) \in E_T$ and by construction thus $\eta \in E^c_T$. 

Lemma 3.8. Let $p \in (3, \infty)$ and
\[
\mathbf{c} := \frac{1}{2} \min_{i \in \{1,2,3\}, x \in [0,1]} |\sigma^i_2(x)|.
\]
Given $T_0 > 0$ and $M > 0$ there exists a time $\tilde{T}(c, M) \in (0, T_0]$ such that for all $\gamma \in E^c_T \cap \overline{B_M}$ with $T \in [0, \tilde{T}(c, M)]$ it holds
\[
\inf_{x \in [0,1], t \in [0,T], i \in \{1,2,3\}} \left| \frac{\gamma^i_2(t,x)}{} \right| \geq c.
\]
In particular, the curves $\gamma^i(t)$ are regular for all $t \in [0,T]$.

Proof. Let $p \in (3, \infty)$ and $\theta \in \left(\frac{1+\frac{1}{p}}{2p}, 1\right)$. By Lemma 2.18 there exists a constant $C(T_0, p, \theta)$ such that for all $T \in (0, T_0]$ and all $\gamma \in E^c_T \cap \overline{B_M}$ with $\alpha := (1-\theta)(1-\frac{1}{p})$,
\[
\|\gamma\|_{C^\alpha((0,T]; C^1([0,1]; \mathbb{R}^n^3))} \leq C(T_0, p, \theta) \|\gamma\|_{E_T} \leq C(T_0, p, \theta)M,
\]
which implies in particular for all $t \in [0,T]$,
\[
\|\gamma(t) - \sigma\|_{C^1([0,1]; \mathbb{R}^n^3)} \leq T^\alpha C(T_0, p, \theta)M.
\]
We let $\tilde{T}(c, M)$ be so small that $\tilde{T}(c, M)^\alpha C(T_0, p, \theta)M \leq c$. Then it follows for all $\gamma \in E^c_T$ with $T \in (0, \tilde{T}(c, M)]$,
\[
\inf_{t \in [0,T], x \in [0,1]} \left| \frac{\gamma^i_2(t,x)}{} \right| \geq \inf_{x \in [0,1]} \left| \frac{\sigma^i_2(x)}{} \right| - \sup_{t \in [0,T], x \in [0,1]} \left| \frac{\gamma^i_2(t,x) - \frac{\gamma^i_2(0,x)}{}}{} \right| \geq c.
\]

Let us now define the operator $N_T$ that encodes the non–linearity of our problem. The map $N_T : E^c_T \rightarrow E^c_T$ is given by $\gamma \mapsto (N^1_T, N^2_T, \cdot|_{x=1}, \cdot|_{t=0})(\gamma)$, where the two components $N^1_T, N^2_T$ are defined as
\[
N^1_T : \begin{cases}
E^c_T & \rightarrow L_p((0,T); L_p((0,1); (\mathbb{R}^n^3))) \\
\gamma & \mapsto f(\gamma)
\end{cases}
\]
\[
N^2_T : \begin{cases}
E^c_T & \rightarrow W^{1/2-1/2p}_p((0,T); \mathbb{R}^n) \\
\gamma & \mapsto b(\gamma)
\end{cases}
\]
with $f$ and $b$ defined in (3.1) and (3.2), respectively.

Proposition 3.9. Let $p \in (3, \infty)$ and $M$ be positive. Then for all $T \in (0, \tilde{T}(c, M]$ the map
\[
N_T : E^c_T \cap \overline{B_M} \rightarrow E^c_T, \quad N_T(\gamma) := (N^1_T(\gamma), N^2_T(\gamma), \gamma|_{x=1}, \gamma|_{t=0})
\]
is well defined.
Proof. Let \( T \in (0, \tilde{T}(e, M)] \) and \( \gamma \in \mathbb{E}_T^e \cap \overline{B}_M \). Lemma 3.8 implies

\[
\left\| \left( \frac{1}{|\gamma|^2} - \frac{1}{|\sigma|^2} \right) \gamma^i_{xx} \right\|_{L_p((0,T);L_p((0,1)))} = \int_0^T \int_0^1 \left| \frac{1}{|\gamma|^2} - \frac{1}{|\sigma|^2} \right|^p |\gamma^i_{xx}|^p \, dx \, dt
\]

\[
\leq C \left( \sup_{x \in [0,1], t \in [0,T]} \frac{1}{|\gamma|^2} + \sup_{x \in [0,1]} \frac{1}{|\sigma|^2} \right) \int_0^T \int_0^1 |\gamma^i_{xx}|^p \, dx \, dt
\]

\[
\leq C(e)\|\gamma^i_{xx}\|_{L_p((0,T);L_p((0,1)))} \leq C(e, M) < \infty.
\]

We now show that \( N_T^2(\gamma) \) lies in \( W_p^{1/2-1/2p} ((0,T); \mathbb{R}^n) \). Given \( T \in (0, \tilde{T}(e, M)] \) and \( \gamma \in \mathbb{E}_T^e \cap \overline{B}_M \) the expression \( b(\gamma) \) is given by

\[
t \mapsto b(\gamma)(t) = \sum_{i=1}^3 \left( \frac{1}{|\gamma^i_x(t,0)|} - \frac{1}{|\sigma^i_x(0)|} \right) \gamma^i_x + \frac{\sigma^i_x(0) \langle \sigma^i_x(0), \gamma^i_x(0) \rangle}{|\sigma^i_x|^3}.
\]

Let \( h : \mathbb{R}^n \to \mathbb{R}^n \) be a smooth function such that \( h(p) = \frac{\ell(p)}{|p|} \) for all \( p \in \mathbb{R}^n \setminus B_{\epsilon/2}(0) \). Then one observes that for all \( t \in [0,T] \)

\[
b(\gamma)(t) = \sum_{i=1}^3 h(\gamma^i_x(t)) - (Dh)(\sigma^i_x)\gamma^i_x(t)
\]

where we omitted the evaluation in \( x = 0 \) to ease notation. Each term in the sum can be expressed as

\[
h(\gamma^i_x(t)) - (Dh)(\sigma^i_x)\gamma^i_x(t) = \int_0^1 (Dh)(\xi \gamma^i_x(t) + (1-\xi)\sigma^i_x) \, d\xi \left( \gamma^i_x(t) - \sigma^i_x \right)
\]

\[
- (Dh)(\sigma^i_x) \left( \gamma^i_x(t) - \sigma^i_x \right) + h(\sigma^i_x) + Dh(\sigma^i_x)\sigma^i_x.
\]

All terms that are constant in \( t \) are smooth in \( t \) and by Lemma 2.8 we have

\[
t \mapsto \gamma^i_x(t,0) \in W_p^{1/2-1/2p} ((0,T); \mathbb{R}^n).
\]

As \( W_p^{1/2-1/2p} ((0,T); \mathbb{R}) \) is a Banach algebra according to Lemma 2.10, it only remains to show

\[
t \mapsto \int_0^1 (Dh)(\xi \gamma^i_x(t,0) + (1-\xi)\sigma^i_x(0)) \, d\xi \in W_p^{1/2-1/2p} ((0,T); \mathbb{R}^{n \times n})
\]

which follows from the second assertion in Lemma 2.10. Observe that \( \gamma_{x=1} = \sigma(1) \) and \( \gamma_{t=0} = \sigma \) by definition of \( \mathbb{E}_T^e \). As

\[
N_T^2(\gamma)_{|t=0} = \sum_{i=1}^3 \frac{\sigma^i_x(0)}{|\sigma^i_x(0)|}
\]

the function \( \sigma \) satisfies the linear compatibility conditions 3.2 with respect to \( N_T^2(\gamma) \) and \( \gamma_{x=1} \). This allows us to conclude that

\[
(N_T^1(\gamma), N_T^2(\gamma), \gamma_{x=1}, \gamma_{t=0}) \in \mathbb{E}_T^e.
\]
Corollary 3.10. Let $p \in (3, \infty)$ and $M$ be positive. Then for all $T \in [0, \tilde{T}(c, M)]$ the map
\[ K_T : E_T^\gamma \cap \overline{B_M} \to E_T^\gamma, \quad K_T := L_T^{-1}N_T \]
is well defined.

Proof. Let $T \in (0, \tilde{T}(c, M)]$ and $\gamma \in E_T^\gamma \cap \overline{B_M}$. By the previous proof we have
\[ N_T(\gamma) = (N_T^1(\gamma), N_T^2(\gamma), \gamma|_{x=0}, \gamma|_{t=0}) \in E_T^\gamma \subset E_T \]
and thus in particular
\[ K_T(\gamma) = L_T^{-1}(N_T(\gamma)) \in E_T. \]
To verify that $K_T(\gamma)$ lies in $E_T^\gamma$ we observe that
\[ K_T(\gamma)_{t=0} = (L_T(\gamma))_{t=0} = N_T(\gamma)_{t=0} = \gamma_{t=0} = \sigma, \]
\[ K_T(\gamma)_{| x=1 } = (L_T(\gamma))_{| x=1 } = N_T(\gamma)_{| x=1 } = \gamma_{| x=1 } = \sigma(1). \]

□

Proposition 3.11. Let $p \in (3, \infty)$ and $M$ be positive. There exists a time $T(M, c) \in (0, \min\{ \tilde{T}(M, c), T_0 \})$ such that for all $T \in (0, T(M, c)]$ the map $K_T : E_T^\gamma \cap \overline{B_M} \to E_T^\gamma$ is a contraction.

Proof. Let $T \in (0, \tilde{T}(c, M)]$ and $\gamma, \tilde{\gamma} \in E_T^\gamma \cap \overline{B_M}$ be fixed. We begin by estimating
\[ \| N_T^1(\gamma) - N_T^1(\tilde{\gamma}) \|_{L_p((0,T);L_p((0,1);(\mathbb{R}^n)^3))} = \| f(\gamma) - f(\tilde{\gamma}) \|_{L_p((0,T);L_p((0,1);(\mathbb{R}^n)^3))} . \]
The $i$-th component of $f(\gamma) - f(\tilde{\gamma})$ is given by
\[ \left( \frac{1}{|\gamma_x^i|^2} - \frac{1}{|\tilde{\gamma}_x^i|^2} \right) (\gamma_{xx}^i - \tilde{\gamma}_{xx}^i) + \left( \frac{1}{|\gamma_x^i|^2} - \frac{1}{|\tilde{\gamma}_x^i|^2} \right) \gamma_{ix}^i - \tilde{\gamma}_{ix}^i . \]
Lemma 3.8 implies
\[ \sup_{t \in [0,T], x \in [0,1]} \left| \frac{1}{|\gamma_x^i|^2|\sigma_x^i|^2} + \frac{1}{|\gamma_x^i||\sigma_x^i||\sigma_x^i|^2} \right| \leq C(c) < \infty , \]
and
\[ \sup_{t \in [0,T], x \in [0,1]} \left| \frac{1}{|\gamma_x^i||\gamma_x^i||\gamma_x^i|^2} + \frac{1}{|\gamma_x^i||\gamma_x^i||\gamma_x^i|^2} \right| \leq C(c) < \infty . \]
Hence we obtain
\[ \| f(\gamma)^i - f(\tilde{\gamma})^i \|_{L_p((0,T);L_p((0,1);(\mathbb{R}^n)^3))} \leq C(c) \left( \| (|\sigma_x^i| - |\gamma_x^i|) (\gamma_{xx}^i - \tilde{\gamma}_{xx}^i) \|_{L_p((0,T);L_p((0,1)))} + \| (|\tilde{\gamma}_x^i| - |\gamma_x^i|) \tilde{\gamma}_{xx}^i \|_{L_p((0,T);L_p((0,1)))} \right) \]
\[ \leq C(c) \left( \sup_{t \in [0,T], x \in [0,1]} \| (|\sigma_x^i| - |\gamma_x^i|) (|\gamma_x^i|) \| L_p((0,T);L_p((0,1))) \right) \]
\[ + \sup_{t \in [0,T], x \in [0,1]} \| (|\tilde{\gamma}_x^i| - |\gamma_x^i|) (|\gamma_x^i|) \| L_p((0,T);L_p((0,1))) \right) \]
\[ \leq C(c) \left( \sup_{t \in [0,T], x \in [0,1]} |\sigma_x^i| - |\gamma_x^i| \| \| \gamma_x^i \|_{L_p((0,T);L_p((0,1)))} + \sup_{t \in [0,T], x \in [0,1]} |\tilde{\gamma}_x^i - |\gamma_x^i| \| \| \gamma_x^i \|_{L_p((0,T);L_p((0,1)))} \right) . \]
Let $\theta \in \left(\frac{1+1/p}{2-1/p}, 1\right)$ be fixed and define $\alpha := (1 - \theta)(1 - 1/p)$. Lemma 2.18 implies
\[
\sup_{t \in [0,T], x \in [0,1]} |\sigma_x^\gamma(t, x) - \gamma_x^\gamma(t, x)| = \sup_{t \in [0,T]} \|\gamma_x^\gamma(0) - \gamma_x^\gamma(t)\|_{C([0,1]; \mathbb{R}^n)} \leq \sup_{t \in [0,T]} \|\gamma(t) - \gamma^\gamma(0)\|_{C^1([0,1]; \mathbb{R}^n)} \\
\leq \sup_{t \in [0,T]} t^\alpha \|\gamma(t)\|_{C^\alpha([0,T]; C^1([0,1]; \mathbb{R}^n))} \leq T^\alpha \|\gamma\|_{C^\alpha([0,T]; C^1([0,1]; \mathbb{R}^n))} \\
\leq T^\alpha C(T_0, p, \theta) \|\gamma\|_{E_T} \leq C(M) T^\alpha.
\]
Similarly we obtain
\[
\sup_{t \in [0,T], x \in [0,1]} |\tilde{\gamma}_x^\gamma(t, x) - \gamma_x^\gamma(t, x)| = \sup_{t \in [0,T], x \in [0,1]} |(\tilde{\gamma}_x^\gamma - \gamma_x^\gamma)(t, x) - (\tilde{\gamma}_x^\gamma - \gamma_x^\gamma)(0, x)| \\
\leq \sup_{t \in [0,T]} \|\tilde{\gamma}_x^\gamma - \gamma_x^\gamma\|_{C^\gamma([0,T]; C^1([0,1]; \mathbb{R}^n))} \\
\leq T^\alpha \|\tilde{\gamma}_x^\gamma - \gamma_x^\gamma\|_{C^\alpha([0,T]; C^1([0,1]; \mathbb{R}^n))} \leq CT^\alpha \|\tilde{\gamma} - \gamma\|_{E_T}.
\]
This allows us to conclude
\[
\|f(\gamma) - f(\tilde{\gamma})\|_{L_p((0,T); L_p((0,1); (\mathbb{R}^n)^p))} \leq C(c, M) T^\alpha \|\gamma - \tilde{\gamma}\|_{E_T}.
\]
We proceed by estimating
\[
\|N_T^\gamma(\gamma) - N_T^\gamma(\tilde{\gamma})\|_{W_p^{1/2-1/p}(0,T); \mathbb{R}^2} = \|b(\gamma) - b(\tilde{\gamma})\|_{W_p^{1/2-1/p}(0,T); \mathbb{R}^2}.
\]
Let $T \in \left(0, \tilde{T}(c, M)\right)$ be fixed and $h : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth function such that $h(p) = \frac{p}{|p|}$ on $\mathbb{R}^n \setminus B_{\sigma/2}(0)$. As for all $t \in [0,T]$ and $\eta \in E_T^R \cap \overline{B_M}$,
\[
|\gamma_x(t, 0) - \eta_x|^c,
\]
we may conclude that for all $\gamma, \tilde{\gamma} \in E_T^R \cap \overline{B_M}$, the function
\[
t \mapsto g^\gamma(t) := \int_0^t (Dh)(\xi \gamma_x^\gamma(t, 0) + (1 - \xi) \tilde{\gamma}_x^\gamma(t, 0))d\xi
\]
lies in $W_p^{1/2-1/2p}(0,T; \mathbb{R}^{n \times n})$. To ease notation we let $s := 1/2 - 1/2p$. Observe that $g^\gamma(0) = (Dh)(\sigma_x^\gamma(0))$ and thus
\[
b(\gamma) - b(\tilde{\gamma}) = \sum_{i=1}^3 (g^\gamma(t) - g^\gamma(0)) (\gamma_x^\gamma(t, 0) - \tilde{\gamma}_x^\gamma(t, 0)).
\]
Using the product estimate in Proposition 2.10 we obtain
\[
\|\gamma_x^\gamma(t, 0) - \tilde{\gamma}_x^\gamma(t, 0)\|_{W_p^s(0,T; \mathbb{R}^n)} \leq \|g^\gamma(t) - g^\gamma(0)\|_\infty \|\gamma_x^\gamma - \tilde{\gamma}_x^\gamma\|_{W_p^{1/2-1/2p}(0,T; \mathbb{R}^n)} \\
+ \|g^\gamma(t) - g^\gamma(0)\|_{W_p^s(0,T; \mathbb{R}^{n \times n})} \|\gamma_x^\gamma - \tilde{\gamma}_x^\gamma\|_\infty.
\]
As $s - \frac{1}{p} > 0$ due to $p \in (3, \infty)$ there exists $\beta \in (0, 1)$ such that
\[
W_p^s(0,T; \mathbb{R}^n) \hookrightarrow C^\beta([0,T]; \mathbb{R}^n)
\]
with embedding constant $C(s, p)$. This implies in particular

$$\|g^t - g(0)\|_{\infty} = \sup_{t \in [0, T]} |g^t(t) - g^t(0)| \leq T^\theta \|g^t\|_{C^\theta([0, T]; \mathbb{R}^{n \times n})} \leq T^\theta C(s, p) \|g^t\|_{W^p_t([0, T]; \mathbb{R}^{n \times n})}.$$ 

Reading carefully through the estimates in Proposition 2.10 we observe that

$$\|g^t\|_{W^p_t([0, T]; \mathbb{R}^{n \times n})} \leq C(T_0, M).$$

Furthermore, Proposition 2.18 implies with $\alpha := (1 - \theta) (1 - 1/p)$,

$$\sup_{t \in [0, T]} |\gamma^j(t, 0) - \tilde{\gamma}^j(t, 0)| = \sup_{t \in [0, T]} \left| \left( \frac{\gamma^i}{\gamma^i - \tilde{\gamma}^i} \right)(t, 0) - \left( \frac{\gamma^i}{\gamma^i - \tilde{\gamma}^i} \right)(0, 0) \right|
\leq \sup_{t \in [0, T]} \| \left( \frac{\gamma^i - \tilde{\gamma}^i}{\gamma^i} \right)(t) - \left( \frac{\gamma^i - \tilde{\gamma}^i}{\gamma^i} \right)(t) \|_{C^\alpha([0, 1]; \mathbb{R}^n)}
\leq T^\alpha \| \gamma^i - \tilde{\gamma}^i \|_{C^\alpha([0, T]; C^\alpha([0, 1]; \mathbb{R}^n))} \leq T^\alpha \| \gamma - \tilde{\gamma} \|_{E_T}.$$

This allows us to conclude

$$\|b(\gamma) - b(\tilde{\gamma})\|_{W^{1/2 - 1/2p}_p([0, T]; \mathbb{R}^n)} \leq C(s, p, T_0, M)T^\alpha \| \gamma - \tilde{\gamma} \|_{E_T}.$$

Finally, Lemma 3.6 implies for all $T \in (0, \hat{T}(c, M)],$

$$\|K_T(\gamma) - K_T(\tilde{\gamma})\|_{E_T} = \|L_T^{-1}(N_T(\gamma) - N_T(\tilde{\gamma}))\|_{E_T} \leq c(T_0, p)\|N_T(\gamma) - N_T(\tilde{\gamma})\|_{E_T}
= c(T_0, p) \left( \|f(\gamma) - f(\tilde{\gamma})\|_{L_p([0, T]; L_p((0, 1); (\mathbb{R}^2)^3))} + \|b(\gamma) - b(\tilde{\gamma})\|_{W^{1/2 - 1/2p}_p([0, T]; (\mathbb{R}^2)^3)} \right)
\leq C(T_0, p, M, c)\| \gamma - \tilde{\gamma} \|_{E_T}.$$

This completes the proof.

To conclude the existence of a solution with the Banach Fixed Point Theorem we have to show that there exists a radius $M > 0$ such that $K_T$ is a self-mapping of $E_T \cap \overline{B_M}$.

**Proposition 3.12.** Let $p \in (3, \infty)$. There exists a positive radius $M$ depending on the norm of $\sigma$ in $W^{2-2/p}_p((0, 1))$ and a positive time $\hat{T}(M, c)$ such that for all $T \in (0, \hat{T}(M, c)]$ the map

$$K_T : E_T \cap \overline{B_M} \rightarrow E_T \cap \overline{B_M}$$

is well-defined.

**Proof.** We let $T_0 \equiv 1$ and define

$$M := 2 \max \left\{ \sup_{T \in (0, 1)} \|L_T^{-1}\|_{L_F, E_T}, 1 \right\} \max \left\{ \|E\sigma\|_{E_1}, \|(N_1^1(\mathcal{E}\sigma), N_1^2(\mathcal{E}\sigma), \sigma(1), \sigma)\|_{E_1} \right\},$$

where $\mathcal{E} : W^{2-2/p}_p((0, 1); (\mathbb{R}^n)^3) \rightarrow W^{1, 2}_p((0, 1) \times (0, 1); (\mathbb{R}^n)^3)$ denotes the extension operator defined in 2.7. In particular, $\mathcal{E}\sigma$ lies in $E_T \cap \overline{B_M}$ for all $T \in (0, 1]$. Moreover, for all $T \in (0, 1]$,

$$\|K_T(\mathcal{E}\sigma)\|_{E_T} \leq \sup_{T \in (0, 1]} \|L_T^{-1}\|_{L_{F(T, E_T)}} \| (N_1^1(\mathcal{E}\sigma), N_1^2(\mathcal{E}\sigma), \sigma(1), \sigma) \|_{E_T} \leq M/2.$$
Let \( T(M, c) \) be the time as in Proposition 3.11. Given \( T \in (0,T(M,c)] \) and \( \gamma \in E_T^p \cap \overline{B_M} \) we observe that for some \( \beta \in (0,1) \),
\[
\|K_T(\gamma) - K_T(\mathcal{E}\sigma)\|_{E_T} \leq C(M,c)T^\beta\|\gamma - \mathcal{E}\sigma\|_{E_T} \leq C(M,c)T^\beta 2M .
\]
We choose \( \hat{T}(M,c) \in (0,T(M,c)] \) so small that \( C(M,c)T^\beta 2M \leq M/2 \) for all \( T \in (0,\hat{T}(M,c)] \). Finally, we conclude for all \( T \in (0,\hat{T}(M,c)] \) and \( \gamma \in E_T^p \cap \overline{B_M} \),
\[
\|K_T(\gamma)\|_{E_T} \leq \|K_T(\gamma) - K_T(\mathcal{E}\sigma)\|_{E_T} + \|K_T(\mathcal{E}\sigma)\|_{E_T} \leq M/2 + M/2 = M .
\]

\[\square\]

**Theorem 3.13.** Let \( p \in (3,\infty) \) and \( \sigma \) be an admissible initial parametrisation. There exists a positive time \( \hat{T}(\sigma) \) depending on \( \min_{i \in \{1,2,3\},x \in [0,1]} |\sigma^i_x(x)| \) and \( \|\sigma\|_{W^{3-2/p}(\alpha,1)} \) such that for all \( T \in (0,\hat{T}(\sigma)] \) the system (2.2) has a solution in
\[
E_T = W^1_p((0,T);L_p((0,1);(\mathbb{R}^n)^3)) \cap L_p((0,T);W^2_p((0,1);(\mathbb{R}^n)^3))
\]
which is unique in \( E_T \cap \overline{B_M} \) where
\[
M := 2\max \left\{ \sup_{T \in (0,1)} \left\| L^{-1}_T \right\|_{\mathcal{L}(F_T,E_T)}, 1 \right\} \max \left\{ \left\| \mathcal{E}\sigma \right\|_{E_1}, \left\| (N_1^1(\mathcal{E}\sigma),N_1^2(\mathcal{E}\sigma),\sigma(1),\sigma) \right\|_{E_1} \right\} .
\]

**Proof.** Let \( M \) and \( \hat{T}(M,c) \) be as in Proposition 3.12 and let \( T \in (0,\hat{T}(M,c)] \). The fixed points of the mapping \( K_T \) in \( E_T^p \cap \overline{B_M} \) are precisely the solutions of the system (2.2) in the space \( E_T \cap \overline{B_M} \). As \( K_T \) is a contraction of the complete metric space \( E_T^p \cap \overline{B_M} \), existence and uniqueness of a solution follow from the Contraction Mapping Principle. \[\square\]

### 3.3 Existence and uniqueness of solutions to the Motion by Curvature

Now that we obtained existence and uniqueness of solutions for the Special Flow (2.2) we can come back to our geometric problem.

**Theorem 3.14 (Local Existence).** Let \( T_0 \) be a geometrically admissible initial Triad. Then there exists \( T > 0 \) and there exists a solution of the Motion by Curvature 2.25 in \([0,T]\) which can be described by one parametrisation \( \gamma = (\gamma^1,\gamma^2,\gamma^3) \) in the whole time interval \([0,T]\).

**Proof.** By Definition 2.23, \( T_0 \) admits a parametrisation \( \sigma = (\sigma^1,\sigma^2,\sigma^3) \). Then \( \sigma \) is also an admissible initial parametrisation for the Special Flow. By Theorem 3.1 there exists a unique \( \gamma = (\gamma^1,\gamma^2,\gamma^3) \) solution of the Special Flow (2.2) in \([0,T]\) with \( \gamma^i(0,x) = \sigma^i(x) \). Then \( T = \bigcup_{i=1}^3 \gamma^i([0,T] \times [0,1]) \) is a solution of the Motion by Curvature 2.25. \[\square\]

By using the embeddings
\[
W^{1,2}_p((0,T) \times (0,1)) \hookrightarrow BUC([0,T];C^{1+\alpha}((0,1))) .
\]
shown in Theorem 2.11 and following the arguments in [15, Lemma 5.3] one can show the following Lemma.
Lemma 3.15. Let $p \in (3, \infty)$, $T$ be positive and $f, g \in L_p \left(0, T; W^2_p ((0, 1)) \right) \cap W^1_p (0, T; L_p((0, 1)))$ such that for every $t \in [0, T]$ the map $g(t, \cdot) : [0, 1] \to [0, 1]$ is a $C^1$–diffeomorphism. Then the map $h(t, x) := f(t, g(t, x))$ lies in $L_p \left(0, T; W^2_p ((0, 1)) \right) \cap W^1_p (0, T; L_p((0, 1)))$ and all derivatives can be calculated by the chain rule.

Theorem 3.16 (Local Uniqueness). Let $p \in (3, \infty)$, $T_0$ be a geometrically admissible initial Triod and $\hat{T}(t)$, $\hat{T}(t)$ be two solution of the Motion by Curvature 2.25 with initial datum $T_0$ in $[0, T]$ and $[0, \hat{T}]$, respectively. Then there exists a positive time $\hat{T} \leq \min\{T, \hat{T}\}$ such that $T(t) = \hat{T}(t)$ for all $t \in [0, \hat{T}]$. In other words, the Motion by Curvature 2.25 has a unique solution on small time intervals.

Proof. Let $T_0$ be an admissible initial Triod parametrised by $\varphi \in W^{2-2/p}_p ((0, 1); \mathbb{R}^n)$. Then $\varphi$ is an admissible initial parametrisation for system (2.2), hence by Theorem 3.1 there is a unique solution $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ of system (2.2) in $E_T$ for some positive time $T$ inducing a solution $(T(t))$ of the Motion by Curvature (2.1). Suppose that there is another solution $(\hat{T}(t))$ to the Motion by Curvature in the sense of Definition 2.25 with initial datum $T_0$ in a time interval $[0, \hat{T}]$ for some positive $\hat{T}$. By possibly decreasing the time of existence $\hat{T}$ we may assume that there exists one parametrisation $\hat{\gamma} \in E_{\hat{T}}$ for the evolution $(\hat{T}(t))$ in the whole time interval $[0, \hat{T}]$. We show that there exists a family of time–dependent reparametrisations $\psi^i : [0, \hat{T}] \times [0, 1] \to [0, 1]$ such that the equality

$$
(t, x) \mapsto \hat{\gamma}^i(t, \psi^i(t, x)) = \gamma^i
$$

holds in the space $E_{\hat{T}}$ for some $\hat{T} \leq \min\{T, \hat{T}\}$. To this end, we construct the functions $\psi = (\psi^1, \psi^2, \psi^3)$ in such a way that the functions $(t, x) \mapsto \hat{\gamma}^i(t, \psi^i(t, x))$ are a solution to the Special Flow in $E_{\hat{T}}$. The claim then follows from the uniqueness assertion in Theorem 3.1.

Taking into account the special tangential velocity in (2.2) (formal) differentiation shows that the reparametrisations $\psi^i : [0, \hat{T}] \times [0, 1] \to \mathbb{R}^n$ need to satisfy the following boundary value problem:

$$
\begin{align}
\psi^i(t, x) &= \frac{\psi^i_\times(t, x)}{|\vec{\tau}^i_\times(t, \psi^i(t, x))|^2} \vec{\tau}^i_\times(t, \psi^i(t, x)) \left( \frac{\hat{\gamma}_\times^i(t, \psi^i(t, x)) \cdot \vec{\tau}^i_\times(t, \psi^i(t, x))}{|\hat{\gamma}_\times^i(t, \psi^i(t, x))|^2} \right) + \frac{1}{|\vec{\tau}^i_\times(t, \psi^i(t, x))|} \left( \frac{\hat{\gamma}_\times^i(t, \psi^i(t, x))}{|\hat{\gamma}_\times^i(t, \psi^i(t, x))|^2} \right) \\
\psi^i(t, 0) &= 0 \\
\psi^i(t, 1) &= 1 \\
\hat{\gamma}(0, \psi^i(0, x)) &= \varphi^i(x)
\end{align}
$$

(3.6)

By the implicit function theorem, the initial value lies in $W^{2-2/p}_p ((0, 1); \mathbb{R}^n)$. We observe that the boundary value problem for the family of reparametrisations has a very similar structure as the Special Flow. Thus, analogous arguments as in the proof of Theorem 3.1 allow us to conclude that there exists a solution $(\psi^1, \psi^2, \psi^3)$ to system (3.6) in the space $W^1_p \left(0, \hat{T}; L_p ((0, 1); (\mathbb{R}^n)^3) \right) \cap L_p \left(0, \hat{T}; W^2_p ((0, 1); (\mathbb{R}^n)^3) \right)$ for some $\hat{T} \leq \min\{T, \hat{T}\}$. For every $t \in [0, \hat{T}]$ the function $\psi^i(t) \in W^{2-2/p}_p ((0, 1); \mathbb{R})$ satisfies $\psi^i(t)(0) = 0, \psi^i(t)(1) = 1$ and $\psi^i(t, x) \neq \psi^i(t, x) \neq \psi^i(t, x)$ for all $x \in [0, 1]$ as the system is well posed. This shows that $x \mapsto \psi^i(t, x)$ is a $C^1$–diffeomorphism of the interval $[0, 1]$. Lemma 3.15 implies that the composition $(t, x) \mapsto \psi^i(t, x)$
\( \gamma^i(t, \psi^i(t, x)) \) lies in \( E_{\mathcal{T}} \) and by construction, it is a solution to the Special Flow. We may now argue as in the proof of [Theorem 5.4, our new paper] to obtain that \( \gamma^i \) and \((t, x) \mapsto \gamma^i(t, \psi^i(t, x)) \) coincide in \( E_{\mathcal{T}} \). In particular, the networks \( \mathcal{T}(t) \) and \( \tilde{\mathcal{T}}(t) \) coincide for all \( t \in [0, T] \). \( \square \)

### 3.4 Maximal Solutions

**Definition 3.17** (Maximal solution). Let \( p \in (3, \infty) \) and \( T \in (0, \infty) \cup \{ \infty \} \). Consider a geometrically admissible initial network \( \mathcal{N}_0 \). A time–dependent family of Triods \( (\mathcal{N}_t)_{t \in [0,T)} \) is a **maximal solution** to the Motion by Curvature in \([0, T)\) with initial datum \( \mathcal{N}_0 \) if it is a solution (in the sense of Definition 2.25) in \([0, T)\) for all \( T < T \) and if there does not exist a solution \( \left( \tilde{\mathcal{N}}(\tau) \right) \) to the Motion by Curvature in \([0, \tilde{T})\) with initial datum \( \mathcal{N}_0 \) with \( \tilde{T} > T \) and such that \( \mathcal{N} = \tilde{\mathcal{N}} \) in \([0, T)\). In this case we call \( T \) **maximal time of existence**.

**Lemma 3.18** (Existence of maximal solutions). Let \( p \in (3, \infty) \) and \( \mathcal{N}_0 \) be a geometrically admissible initial network. There exists a maximal solution to the Motion by Curvature with initial datum \( \mathcal{N}_0 \).

**Proof.** Theorem 3.14 gives a solution to the Motion by Curvature on \([0, T]\) with \( T > 0 \). By uniqueness we can define a maximal solution \( (\mathcal{N}(t))_{t \in [0, T_{\max}]} \) as follows. Let

\[
T_{\max} := \sup \left\{ T > 0 : \text{there exists a solution } (\mathcal{N}^T(t))_{t \in [0,T]} \text{ to } (2.2) \text{ with initial datum } \mathcal{N}_0^r \right\}.
\]

By geometric uniqueness of the flow, two solutions \( \mathcal{N}^{T_1} \) and \( \mathcal{N}^{T_2} \) starting in \( \mathcal{N}_0 \) have to coincide on \([0, \min\{T_1, T_2\}]\). Hence we can define a maximal solution \( (\mathcal{N}(t))_{t \in [0, T_{\max}]} \) by setting

\[
\mathcal{N}(t) := \mathcal{N}^T(t) \quad \text{for } t \in [0, T].
\]

\( \square \)

**Lemma 3.19** (Uniqueness of maximal solutions). Let \( p \in (3, \infty) \), \( T_{\max} \in (0, \infty) \cup \{ \infty \} \) and \( (\mathcal{N}(t))_{t \in [0, T_{\max}]} \) be a maximal solution to the Motion by Curvature with initial datum \( \mathcal{N}_0 \) in \([0, T_{\max}]\). Then \( (\mathcal{N}(t)) \) is geometrically unique on finite time intervals.

**Proof.** Suppose that \( (\mathcal{N}(t))_{t \in [0, T_{\max}]} \) and \( \left( \tilde{\mathcal{N}}(t) \right)_{t \in [0, T_{\max}]} \) are two solution of the Motion by Curvature in \([0, T_{\max}]\). Suppose that the set \( \left\{ t \in (0, T_{\max}) \mid \mathcal{N}(t) \neq \tilde{\mathcal{N}}(t) \right\} \) is not empty and let

\[
t^* := \inf \left\{ t \in (0, T_{\max}) \mid \mathcal{N}(t) \neq \tilde{\mathcal{N}}(t) \right\}.
\]

By Theorem 3.16 there exists a positive time \( T \) such that \( \mathcal{N}(t) = \tilde{\mathcal{N}}(t) \) on \([0, T]\). Hence \( t^* \) is strictly positive and \( \mathcal{N}(t^*) = \tilde{\mathcal{N}}(t^*) \) is an admissible initial network for Problem 2.25. Again by Theorem 3.16 there exists a positive time \( T \) such that \( \mathcal{N}(t) = \tilde{\mathcal{N}}(t) \) on \([t^*, t^* + T]\), a contradiction. \( \square \)

**Proof of Theorem 1.1.** The result follows directly by Lemma 3.18 and Lemma 3.19. \( \square \)

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4 Smoothness of the Special Flow

This section is devoted to prove that the solution to the Special Flow is smooth for positive times. Heuristically, this regularisation effect is due to the parabolic nature of the problem. The basic idea of the proof is based on the so called parameter trick which is due to Angenent [4]. This strategy has been generalized to several context [21, 22, 27]. Nevertheless, the Special Flow presents a fully non-linear boundary condition

\[ \sum_{i=1}^{3} \frac{\gamma_i^3}{|\gamma_i^3|} = 0, \]

that is not treated in the above mentioned results. A version of the parameter trick tailored to treat fully non-linear boundary terms is presented in [14]. We follow [14, Section 4] modifying the arguments for the application in our Sobolev setting.

In the following we let \( \sigma \) be a fixed admissible initial parametrisation for the Special Flow.

**Definition 4.1.** Let \( T > 0 \) and \( B : E_T := W^{1,2}_p((0, T) \times (0, 1); (\mathbb{R}^n)^3) \rightarrow W^{1-1/2}_p((0, T); (\mathbb{R}^n)^4) \times W^{1/2-1/2p}_p((0, T); (\mathbb{R}^n)^2) \)
be the linearised boundary operator defined by

\[
B(\gamma) = \begin{pmatrix}
\gamma^1(., 0) - \gamma^2(., 0) \\
\gamma^2(., 0) - \gamma^3(., 0) \\
\gamma^1(., 1) - \gamma^2(., 1) \\
\gamma^2(., 1) - \gamma^3(., 1) \\
\sum_{i=1}^{3} \frac{\gamma_i^3(., 0)}{|\sigma_i^3(0)|} - \frac{\gamma_i^3(., 0)\sigma_i^3(0)}{|\sigma_i^3(0)|^2} \\
\sum_{i=1}^{3} \frac{\gamma_i^3(., 0)}{|\sigma_i^3(0)|} - \frac{\gamma_i^3(., 0)\sigma_i^3(0)}{|\sigma_i^3(0)|^2}
\end{pmatrix}
\]

Moreover we let

\[ X_T := \ker(B). \]

We notice that as \( B \) is continuous, the space \( X_T \) is a closed subspace of \( E_T \) and thus a Banach space. In the following Lemma we will construct a partition of the solution space \( E_T = X_T \oplus Z_T \).

**Lemma 4.2.** Let \( T > 0 \). There exists a closed subspace \( Z_T \) of \( E_T \) such that \( E_T = X_T \oplus Z_T \).

**Proof.** We consider the space

\[ Z_T^1 := \left\{ b \in W^{1-1/2p}_p((0, T); (\mathbb{R}^n)^4) \times W^{1/2-1/2p}_p((0, T); (\mathbb{R}^n)^2) : b_{t=0} = 0 \right\}. \]

Notice that with the choice \( f = 0, b \in Z_T^1, \psi = 0 \) is a suitable right hand side for the linear system (3.3). Hence for every \( b \in Z_T^1 \) there exists a unique solution \( L_T^{-1}(0, b, 0) \in E_T \) and the space \( Z_T^1 := L_T^{-1}\big((0, Z_T^1, 0)\big) \) is a closed subspace of \( E_T \).

We define the space

\[ Z_T^2 := (\mathbb{R}^n)^4 \times (\mathbb{R}^n)^2. \]
Given \( \hat{b} \in \mathbb{Z}^2 \) the elliptic system \( \hat{\eta} = (0, \hat{b}) \) defined by

\[
\begin{align*}
\frac{1}{|\sigma^i(x)|} \eta_{xx}^i(x) & = 0, & x \in (0,1), i \in \{1, 2, 3\}, \\
\eta^1(0) & = \hat{b}^1, \\
\eta^2(0) & = \hat{b}^2, \\
\eta^3(1) & = \hat{b}^3, \\
\eta^2(1) - \eta^3(1) & = \hat{b}^4, \\
- \sum_{i=1}^{3} \left( \frac{\eta^i(0)}{|\sigma^i(0)|} - \frac{\sigma^i_0(0)}{|\sigma^i(0)|} \right) & = \hat{b}^5, \\
- \sum_{i=1}^{3} \left( \frac{\eta^i(1)}{|\sigma^i(1)|} - \frac{\sigma^i_1(1)}{|\sigma^i(1)|} \right) & = \hat{b}^6
\end{align*}
\]

has a unique solution \( \eta \in W^2_p((0,1);(\mathbb{R}^n)^3) \) which we denote by \( \hat{L}^{-1}(\hat{f}, \hat{b}) \). This is guaranteed due to the results in [2] and the fact that the boundary operator fulfills the Lopatinski-Shapiro conditions according to Lemma 3.4. We notice that the space \( \hat{L}^{-1}(\mathbb{Z}^2) \) is a closed subspace of \( W^2_p((0,1);(\mathbb{R}^n)^3) \), again using continuity of the solution operator. Extending every function in \( \hat{L}^{-1}(\mathbb{Z}^2) \) constantly in time, we can view \( \hat{L}^{-1}(\mathbb{Z}^2) \) as a closed subspace of \( E_T \). This space will be denoted by \( \mathbb{Z}_T^2 \). It is straightforward to check that \( \mathbb{Z}_T^1 \cap \mathbb{Z}_T^2 = \{0\} \) which allows us to define \( \mathbb{Z}_T \) as the subspace of \( E_T \) given by

\[ \mathbb{Z}_T := \mathbb{Z}_T^1 \oplus \mathbb{Z}_T^2. \]

Note that \( \mathbb{Z}_T \) is indeed a closed subspace, which one sees as follows. Suppose that

\[ (z_n)_{n \in \mathbb{N}} = (z_{n1} + z_{n2})_{n \in \mathbb{N}} \subset \mathbb{Z}_T \]

is a convergent sequence in \( E_T \). Due to \( E_T \hookrightarrow BUC(0, T; C^{1+\alpha}((0,1);(\mathbb{R}^n)^3)) \) this implies that \( (z_n)_{n \in \mathbb{N}} \) converges in \( C^{1+\alpha}((0,1);(\mathbb{R}^n)^3) \). In particular, this yields the convergence of the boundary data needed for the elliptic system we used to construct \( z_{n1} \). Continuity of the elliptic solution operator, which is guaranteed to the energy estimates in [2], shows now that \( (z_{n1})_{n \in \mathbb{N}} \) converges in \( W^2_p((0,1);(\mathbb{R}^n)^3) \). Due to its constant extension in time we see that \( (z_{n1})_{n \in \mathbb{N}} \) converges in \( E_T \) to a limit \( z^1 \), which is also in \( \mathbb{Z}_T^1 \) due to the closedness of this space. Then, \( (z_{n2})_{n \in \mathbb{N}} = (z_n)_{n \in \mathbb{N}} - (z_{n1})_{n \in \mathbb{N}} \) converges in \( E_T \) as sum of two convergent sequences to \( z^2 \), which is also in \( \mathbb{Z}_T^2 \). Again using closedness. In total, we conclude that \( (z_n)_{n \in \mathbb{N}} \) converges to \( z^1 + z^2 \in \mathbb{Z}_T \), which shows the sought closedness of \( \mathbb{Z}_T \).

It remains to show that \( X_T \cap \mathbb{Z}_T = \{0\} \) and \( E_T = X_T + \mathbb{Z}_T \). To this end let \( \gamma \in X_T \cap \mathbb{Z}_T \). By definition of \( X_T \), \( B(\gamma) = 0 \) which implies in particular \( B(\gamma)_{t=0} = 0 \). As \( \gamma \) lies in \( \mathbb{Z}_T \), there exist \( z_1 \in \mathbb{Z}_T^1 \) and \( z_2 \in \mathbb{Z}_T^2 \) with \( \gamma = z_1 + z_2 \). Now we observe that

\[ 0 = B(z_1 + z_2)_{t=0} = B(z_1)_{t=0} + B(z_2)_{t=0} = B(z_2)_{t=0}. \]

as \( B(z_1) \) lies in \( \mathbb{Z}_T^1 \). As \( (z_2)_{t=0} = z_2 \) by definition of \( \mathbb{Z}_T^2 \), we obtain \( B(z_2) = 0 \) and hence \( z_2 = 0 \) by uniqueness of the elliptic system. This implies \( 0 = B(\gamma) = B(z_1) \) which gives \( z_1 = L_T^{-1}(0, 0, 0) = 0 \).

To prove that \( E_T = X_T + \mathbb{Z}_T \) we let \( \gamma \in E_T \). We define

\[ z_1 := L_T^{-1}(0, B(\gamma) - B(\gamma)_{t=0}, 0) \in \mathbb{Z}_T^1. \]
where we view $B(\gamma)|_{t=0} \in (\mathbb{R}^n)^4 \times (\mathbb{R}^n)^2$ as an element of the space $W^{1-1/2p}_p((0, T); (\mathbb{R}^n)^4) \times W^{1/2-1/2p}_p((0, T); (\mathbb{R}^n)^2)$. We further let

$$z_2 := \tilde{L}^{-1}(0, B(\gamma)|_{t=0}).$$

Now it remains to show that $\gamma - z_1 - z_2$ lies in $X_T$ which is equivalent to $B(\gamma - z_1 - z_2) = 0$ which follows by construction. \hfill \Box

**Definition 4.3.** Given $T > 0$ we define the extension operator

$$\mathcal{E} : W^{2-2p}_p((0, 1); (\mathbb{R}^n)^3) \to W^{1, 2}_p((0, T) \times (0, 1); (\mathbb{R}^n)^3)$$

as follows. Given $\sigma \in W^{2-2p}_p((0, 1); (\mathbb{R}^n)^3)$ we let $\mathcal{E}\sigma$ be the unique solution to the Special Flow.

**Lemma 4.4.** Let $T > 0$. There exists a neighbourhood $U$ of 0 in $X_T$, a function $\rho : U \to Z_T$ and a neighbourhood $V$ of $\mathcal{E}\sigma$ in $\mathcal{E}T$ such that

$$\{\mathcal{E}\sigma + u + \rho(u) : u \in U\} = \left\{ \gamma \in V : \sum_{i=1}^3 \gamma_i^x(t, y) = 0, y \in \{0, 1\} \right\}.$$

Furthermore, it holds that $\rho'(0) \equiv 0$.

**Proof.** We let

$$W^{1-1/2p}_p((0, T); (\mathbb{R}^n)^4) \times W^{1/2-1/2p}_p((0, T); (\mathbb{R}^n)^2)$$

and consider the operator

$$F : X_T \oplus Z_T \to Y_T, \quad (x, z) \mapsto \mathcal{G}(\mathcal{E}\sigma + x + z)$$

where $\mathcal{G}$ denotes the operator

$$\gamma \mapsto \mathcal{G}(\gamma) := \left( \begin{array}{c} (\gamma^1 - \gamma^2)|_{x=0} \\ (\gamma^2 - \gamma^3)|_{x=0} \\ (\gamma^1 - \gamma^2)|_{x=1} \\ (\gamma^2 - \gamma^3)|_{x=1} \\ \sum_{i=1}^3 \frac{\gamma^i}{|y_i|^2}|_{x=0} \\ \sum_{i=1}^3 \frac{2\gamma^i}{|y_i|^2}|_{x=1} \end{array} \right).$$

By definition of $\mathcal{E}\sigma$ we have that $F(0, 0) = 0$. We observe that $\partial_2 F(0, 0) = B$. To apply the implicit function theorem we have to show that

$$B : Z_T \to Y_T$$

is an isomorphism. The map is injective as for $x, y \in Z_T$ with $B(x) = B(y)$ we have $x - y \in \ker B \cap Z_T = 0$. Given $b \in Y_T$ we let $z_1 := L^{-1}_T(0, b - b|_{t=0}) \in Z^1_T$ and $z_2 := \tilde{L}^{-1}(0, b|_{t=0}) \in Z^2_T$ and observe that $z_1 + z_2 \in Z_T$ satisfies

$$B(z_1 + z_2) = B(z_1) + B(z_2) = b - b|_{t=0} + b|_{t=0} = b.$$
The implicit function theorem implies that there exist neighbourhoods $U$ and $W$ of 0 in $X_T$ and $Z_T$, respectively, and a function $\rho : U \to W$ with $\rho(0) = 0$ such that for a neighbourhood $V$ of 0 in $E_T$, it holds

$$\{u + \rho(u) : u \in U\} = \{x + z \in E_T : F(x, z) = 0\} \cap \tilde{V}.$$  

To show that $\rho'(0) = 0$ we let $u \in X_T$ be arbitrary. As $\rho'(0) : X_T \to Z_T$ we obtain $\rho'(0)u \in Z_T$. Hence it is enough to show that $\rho'(0)u$ also lies in $X_T$. To this end we differentiate the identity

$$0 = F(\delta u, \rho(\delta u)) = G(\mathcal{E}\sigma + \delta u + \rho(\delta u))$$

with respect to $\delta$ and obtain

$$0 = \frac{d}{d\delta} G(\mathcal{E}\sigma + \delta u + \rho(\delta u))|_{\delta = 0} = (DG)(\mathcal{E}\sigma)(u + \rho'(0)u) = B(u + \rho'(0)u).$$

This implies $u + \rho'(0)u \in \ker B = X_T$ and thus $\rho'(0)u \in X_T$. \hfill $\square$

**Definition 4.5.** Let $\sigma \in W^{2-2/p}_p((0, 1); (\mathbb{R}^n)^3)$ be an admissible initial parametrisation to the Special Flow. We linearise the system \((2.30)\) around $\mathcal{E}\sigma$. To this we consider the full linearisation which gives

$$\begin{cases}
\gamma_i^j(t, x) - \frac{1}{|E_i\sigma|^2(t,x)} \gamma_{xx}^j(t,x) - 2\frac{\langle E_i\sigma\rangle_2^j(t,x)\langle \gamma_i^j(t,x),\langle E_i\sigma\rangle_2^j(t,x)\rangle}{|E_i\sigma|^2(t,x)} = f^j(t,x), \\
\gamma_1^1(t,0) = \gamma_2^2(t,0) = \gamma_3^3(t,0), \\
\gamma_1^1(t,1) = \gamma_2^2(t,1) = \gamma_3^3(t,1), \\
- \sum_{i=1}^3 \left( \gamma_i^j(t,0) - \frac{\langle E_i\sigma\rangle_2^j(t,0)\langle \gamma_i^j(t,0),\langle E_i\sigma\rangle_2^j(t,0)\rangle}{|E_i\sigma|^2(t,0)} \right) = b(t), \\
- \sum_{i=1}^3 \left( \gamma_i^j(t,1) - \frac{\langle E_i\sigma\rangle_2^j(t,1)\langle \gamma_i^j(t,1),\langle E_i\sigma\rangle_2^j(t,1)\rangle}{|E_i\sigma|^2(t,1)} \right) = b(t), \\
\gamma_i^j(0,x) = \psi^j(x)
\end{cases} \quad \text{for } i = 1, 2, 3.$$

Here $\psi$ is an admissible initial value with respect to the given right hand side $b$ and $\eta$. For $\gamma \in E_T$ we define $A(\mathcal{E}\sigma)\gamma$ by

$$(A(\mathcal{E}\sigma)\gamma)^j := \frac{1}{|E_i\sigma|^2(t,x)} \gamma_{xx}^j(t,x) - 2\frac{\langle E_i\sigma\rangle_2^j(t,x)\langle \gamma_i^j(t,x),\langle E_i\sigma\rangle_2^j(t,x)\rangle}{|E_i\sigma|^2(t,x)}. $$

**Proposition 4.6** (Higher time regularity of solutions to the Special Flow).

Let $T > 0$ and $\tilde{t} \in [0, T]$. For any admissible initial parametrisation $\sigma \in W^{2-2/p}_p((0, 1); (\mathbb{R}^n)^3)$ there are constants $\tilde{C}(\sigma, T), \tilde{\varepsilon}(\sigma, T) > 0$ with the following property. For any admissible initial parametrisation $\tilde{\sigma} \in W^{2-2/p}_p((0, 1); (\mathbb{R}^n)^3)$ with $||\tilde{\sigma} - \sigma|| \leq \tilde{\varepsilon}$ we have the increased time regularity

$$\partial_t \mathcal{E}\sigma \in E_T|_{[\tilde{t}, T]}$$

with the estimate

$$||\partial_t \mathcal{E}\sigma - \partial_t \mathcal{E}\tilde{\sigma}||_{E_T|_{[\tilde{t}, T]}} \leq \frac{\tilde{C}}{\tilde{t}} ||\tilde{\sigma} - \sigma||_{W^{2-2/p}_p(0, 1)}. $$

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Proof. We consider the space \( I := \left\{ \psi \in W^{2-2/p}_p ((0, 1)) : \psi^j(y) = \psi^j(y), \sum_{i=1}^3 \frac{\psi_i^j(y)}{|\sigma_i^j(y)|} - \frac{\sigma_i^j(y) \langle \psi_i^j(y), \sigma_i^j(y) \rangle}{|\sigma_i^j(y)|^3} = 0, y \in \{0, 1\} \right\} \).

We let \( U, V \) and \( \rho \) be as in the previous Lemma and define \( \overline{\rho}(u) := \mathcal{E}\sigma + u + \rho(u) \). For some small \( \varepsilon \in (0, 1) \) we consider the map \( G : (1 - \varepsilon, 1 + \varepsilon) \times I \times X_T \to I \times L_p ((0, T) \times (0, 1); \mathbb{R}^n)^3 \),

\[ (\lambda, \psi, u) \mapsto \left( u_{|t=0} - \psi, \partial_t \overline{\rho}(u) - \frac{\overline{\rho}(u)_{xx}}{|\overline{\rho}(u)_x|^2} \right) \]

Notice that \( G(1, 0, 0) = 0 \). The Fréchet derivative

\[ (\partial^u G)(1, 0, 0) : X_T \to I \times L_p ((0, T) \times (0, 1); \mathbb{R}^n)^3 \]

is given by

\[ (\partial^u G)(1, 0, 0)u = (u_{|t=0}, \partial_t u - A(\mathcal{E}\sigma)u) \]

As \( (\partial^u G)(1, 0, 0) : X_T \to I \times L_p ((0, T) \times (0, 1); \mathbb{R}^n)^3 \) is an isomorphism by Theorem 3.5, the implicit function theorem implies the existence of neighbourhoods \( U \) of \( (1, 0) \) in \( (1 - \varepsilon, 1 + \varepsilon) \times I \) and \( V \) of \( 0 \) in \( X_T \) and a function \( \zeta : U \to V \) with \( \zeta((1, 0)) = 0 \) and

\[ \{ (\lambda, \psi, u) \in U \times V : G(\lambda, \psi, u) = 0 \} = \{ (\lambda, \psi, \zeta(\lambda, \psi) : (\lambda, \psi) \in U \} \]

Consider now the map \( P : E_T \to X_T \) given by \( P(\gamma) := P_{X_T}(\gamma - \mathcal{E}\sigma) \) with \( P_{X_T}(\gamma) = u \) for the unique partition \( \gamma = u + \mathcal{P} \in X_T \oplus Z_T \). Clearly, we have that \( \mathcal{P}(P(\gamma)) = \gamma \) for all \( \gamma \) in the neighbourhood from Lemma 4.4. Now, we consider the time-scaled function

\[ (\mathcal{E}\sigma)_\lambda(t, x) := (\mathcal{E}\sigma)(\lambda t, x). \]

By definition, this satisfies for \( \psi := P(\mathcal{E}\sigma)\big|_{t=0} \)

\[ G(\lambda, \psi, P((\mathcal{E}\sigma)_\lambda)) = 0. \]

By uniqueness we conclude that

\[ P((\mathcal{E}\sigma)_\lambda) = \zeta(\lambda, \psi), \]

and therefore

\[ (\mathcal{E}\sigma)_\lambda = \tilde{\rho}(\zeta(\lambda, \psi)). \]

Now, as both \( \zeta \) and \( \tilde{\rho} \) are smooth, this shows that \( (\mathcal{E}\sigma)_\lambda \) is smooth as function in \( \lambda \) with values in \( E_T \). This implies now

\[ t \partial_t (\mathcal{E}\sigma) = \partial_\lambda (\mathcal{E}\sigma)_\lambda \big|_{\lambda=1} \in E_T, \]

from which we directly conclude (4.1). Finally, we have that

\[ \| \partial_\lambda \tilde{\rho}(\zeta(1, \psi)) - \partial_\lambda \tilde{\rho}(\zeta(1, \sigma)) \|_{E_T} \leq C \int_0^1 \| \partial_\lambda \partial_\lambda \zeta(1, (1-s)\sigma + s\psi)(\psi - \sigma) \|_{L^2} ds \]

\[ \leq C \int_0^1 \| \psi - \sigma \|_{L^2} ds \leq C \| \tilde{\sigma} - \sigma \|_{W^{2-2/p}_p (0, 1)}, \]

Hereby, we used in the first two steps boundedness of derivatives of analytic functions on bounded sets and in the last step continuity of the projection \( P \). Together with (4.3) this yields (4.2) and we are done. \( \square \)
Next, we want to derive higher regularity in space for our solution. But this follows almost immediately from the associated ODE we have at a fixed time.

**Corollary 4.7 (Higher space regularity of solutions to the Special Flow).**

Under the assumptions of Proposition 4.6 we have for almost all $t \in [\bar{t}, T]$ that

$$(\mathcal{E}\sigma)(t) \in W^{3,p/2}((0, 1); (\mathbb{R}^n)^3).$$

In particular, we have for almost all $t \in [\bar{t}, T]$ that there is an $\alpha > 0$ with $(\mathcal{E}\sigma)(t) \in C^{2+\alpha}((0, 1); (\mathbb{R}^n)^3)$.

**Proof.** Considering $\partial_t (\mathcal{E}(\sigma^\gamma)(t)$ as given functions $\tilde{\gamma} \in W^{1,p}((0, 1); (\mathbb{R}^n)^3)$ we see that $\gamma^\gamma(t, \cdot)$ solves

$$-\frac{\gamma_{xx}^\gamma(t, \cdot)}{|\gamma^\gamma_x(t, \cdot)|^2} = \tilde{\gamma},$$

in $W^{1,p}((0, 1); (\mathbb{R}^2)^3)$. As we already know that $\gamma^\gamma(t, \cdot)$ is in $W^{2,p}((0, 1); (\mathbb{R}^n)^3)$ this show that $|\gamma_x^\gamma(t, \cdot)|^2 \in W^{1,p/2}((0, 1); (\mathbb{R}^n)^3)$ and thus we can multiply the above equations to see that

$$-\gamma_{xx}^\gamma(t, \cdot) = \tilde{\gamma} \in W^{1,p/2}((0, 1); (\mathbb{R}^n)^3),$$

with new given inhomogeneities $\tilde{\gamma}$. Note that we used here that with our choice of $p$ the Sobolev space $W^{3,p}$ is indeed a Banach on one dimensional domains. But from the last equation we directly conclude $\gamma_{xx}^\gamma \in W^{3,p/2}((0, 1); (\mathbb{R}^n)^3)$. The second claim is just a direct consequence of the Sobolev embeddings.

With the two previous results we are now able to start a bootstrap procedure.

**Theorem 4.8 (Smoothness of solutions to the special Flow).**

Let $\sigma \in W^{3-2/p}_{p}((0, 1); (\mathbb{R}^n)^3)$ be an admissible parameterisation, $I_{\max} := [0, T_{\max})$ the maximal existence time of the solution $\mathcal{E}\sigma$ of the special flow and $\bar{t} \in (0, T_{\max})$. Then, $\mathcal{E}\sigma$ is smooth on $[0, 1] \times (\bar{t}; T_{\max}].$

**Proof.** Due to Corollary 4.7 we can use $\sigma(t)$ for almost all $t > 0$ as initial data for a regularity result in parabolic Hölder space, cf. [16] for such an result for the Willmore flow. As we checked that the Lopatinskis-Shapiro conditions are still valid in higher codimensions, the analysis works as in the planar case. Additionally, the needed compatibility conditions due to the zero order boundary conditions are guaranteed by the fact that $\partial_t \gamma \in C([\bar{t}, T]; C([0, 1]; (\mathbb{R}^n)^3).$ With this new maximal regularity result, which is the key argument in the proof of Proposition 4.6, we can repeat the whole procedure to derive $C^{3+\alpha,(3+\alpha)/2}$ regularity. This starts now the bootstrapping yielding the desired smoothness result. Note that in every step the needed compatibility conditions guaranteed by the fact that our flow already has the regularity related to these compatibility conditions (see for instance [25, Theorem 3.1]).

## 5 Long time behaviour of the Motion by Curvature

**Proof of Theorem 1.2.** Let $\varepsilon \in (0, T_{\max}/1000)$ be fixed. Suppose that $T_{\max}$ is finite and that the two assertions $i)$ and $ii)$ are not fulfilled. Since the flow is smooth on $[\varepsilon, T]$ for all positive $\varepsilon$ and all $T \in (\varepsilon, T_{\max})$ this means that

$$k^i \in L^\infty((\varepsilon, T_{\max}); L^2((0, 1); \mathbb{R}^n)),$$
and that the lengths $L(N^i)$ of the curves of the network are uniformly bounded away from zero in $[0,T_{\text{max}})$. Moreover thanks to the gradient flow structure of the Motion by Curvature we have $L(N^i) \leq L(N_0)$ for all $t \in [0,T_{\text{max}})$. We observe that it is possible to parametrise each curve $N^i$ of the network with constant velocity equal to the length, that is there exists maps $\gamma^i : [0,T_{\text{max}}) \times [0,1] \to \mathbb{R}^n$ such that $\gamma^i(t,[0,1]) = N^i(t)$ and $|\gamma^i_x(t,x)| = L(N^i(t))$. In particular, we obtain

$$0 < c \leq \sup_{t \in [0,T_{\text{max}}),x \in [0,1]} |\gamma^i_x(t,x)| \leq C < \infty.$$  

With this choice of parametrisation the curvature can be expressed as $k^i = \gamma^i_{xx}/L(N^i)^2$ from which we can infer for all $t \in [0,T_{\text{max}})$,

$$\int_0^1 |\gamma^i_{xx}|^2 dx = (L(N^i))^3 \int_{N^i} |k^i|^2 ds \leq C < \infty.$$  

Let $R > 0$ be such that $N_0 \subset B_R(0)$. Then because of the comparison principle for every $t \in [0,T_{\text{max}})$ it holds $N(t) \subset B_R(0)$. With the above arguments we conclude $\gamma^i \in L^\infty((\varepsilon,T_{\text{max}});W_p^2([0,1];\mathbb{R}^n))$.  

The Sobolev Embedding Theorem implies for all $p \in (3,6]$,

$$\gamma^i \in L^\infty((\varepsilon,T_{\text{max}});W_p^{2-2/p}((0,1);\mathbb{R}^n)),$$

and the norm is bounded by a constant $C$. Hence for all $\delta \in (0,T_{\text{max}})$ the parametrisation $\gamma(T_{\text{max}} - \delta) = (\gamma^1(T_{\text{max}} - \delta), \ldots, \gamma^N(T_{\text{max}} - \delta))$ is an admissible initial parametrisation for Problem 2.30. By Theorem 3.13 there exists a uniform time $T$ of existence depending on $C$ for all initial values $\gamma(T_{\text{max}} - \delta)$. Let $\delta := \frac{T_{\text{max}}}{2}$. Then Theorem 3.13 implies the existence of a solution $\eta = (\eta^1,\ldots,\eta^N)$ with $\eta^i$ regular and $\eta^i \in W_p^1((T_{\text{max}} - \delta,T_{\text{max}} + \delta);L_p((0,1);\mathbb{R}^n)) \cap L_p((T_{\text{max}} - \delta,T_{\text{max}} + \delta);W_p^2((0,1);\mathbb{R}^n))$ to the system (2.2) with $\eta(T_{\text{max}} - \delta) = \gamma(T_{\text{max}} - \delta)$. The two parametrisations $\gamma$ and $\eta$ defined on $(0,T_{\text{max}} - \frac{T_{\text{max}}}{4})$ and $(T_{\text{max}} - \frac{T_{\text{max}}}{4},T_{\text{max}} + \delta)$, respectively, define a solution $\widetilde{T}(t)$ to the Motion by Curvature on the time interval $(0,T_{\text{max}} + \delta)$ with initial datum $N_0$ coinciding with $N$ on $(0,T_{\text{max}})$. This contradicts the maximality of $T_{\text{max}}$.  

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