SOME PROPERTIES OF THE CLASS $\mathcal{U}$

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Abstract. In this paper we study the class $\mathcal{U}$ of functions that are analytic in the open unit disk $\mathbb{D} = \{z : |z| < 1\}$, normalized such that $f(0) = f'(0) - 1 = 0$ and satisfy
\[ \left| \frac{z}{f(z)} \right|^2 f'(z) - 1 < 1 \quad (z \in \mathbb{D}). \]
For functions in the class $\mathcal{U}$ we give sharp estimate of the second and the third Hankel determinant, its relationship with the class of $\alpha$-convex functions, as well as certain starlike properties.

1. Introduction

Let $\mathcal{A}$ denote the family of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization $f(0) = f'(0) - 1$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ which are starlike and convex in $\mathbb{D}$, respectively, i.e.,
\[ \mathcal{S}^* = \left\{ f \in \mathcal{A} : \Re \left[ \frac{zf'(z)}{f(z)} \right] > 0, \ z \in \mathbb{D} \right\} \]
and
\[ \mathcal{K} = \left\{ f \in \mathcal{A} : \Re \left[ 1 + \frac{zf''(z)}{f'(z)} \right] > 0, \ z \in \mathbb{D} \right\}. \]

Geometrical characterisation of convexity is the usual one, while for the starlikeness we have that $f \in \mathcal{S}^*$, if, and only if, $f(\mathbb{D})$ is a starlike region, i.e.,
\[ z \in f(\mathbb{D}) \quad \Rightarrow \quad tz \in f(\mathbb{D}) \text{ for all } t \in [0, 1]. \]

The linear combination of the expressions involved in the analytical representations of starlikeness and convexity brings us to the classes of $\alpha$-convex functions introduced in 1969 by Mocanu (3) and consisting of functions $f \in \mathcal{A}$ such that
\[ \Re \left\{ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \left[ 1 + \frac{zf''(z)}{f'(z)} \right] \right\} > 0, \quad (z \in \mathbb{D}), \]
where $\frac{f(z)f'(z)}{z} \neq 0$ for $z \in \mathbb{D}$ and $\alpha \in \mathbb{R}$. Those classes he denoted by $\mathcal{M}_\alpha$.

Further, let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying the condition
\[ |U_f(z)| < 1 \quad (z \in \mathbb{D}), \]

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where the operator $U_f$ is defined by

$$U_f(z) := \left[ \frac{z}{f(z)} \right]^2 f'(z) - 1.$$ 

All this classes consist of univalent functions and more details on them can be found in [11, 10].

The class of starlike functions is very large and in the theory of univalent functions it is significant if a class doesn’t entirely lie inside $S^\star$. One such case is the class of functions with bounded turning consisting of functions $f$ from $A$ that satisfy $\Re f'(z) > 0$ for all $z \in \mathbb{D}$. Another example is the class $U$ defined above and first treated in [5] (see also [6, 7, 10]). Namely, the function $-\ln(1 - z)$ is convex, thus starlike, but not in $U$ because $U_f(0.99) = 3.621 \ldots > 1$, while the function $f$ defined by $z f(z) = 1 - \frac{3}{2} z + \frac{1}{2} z^3 = (1 - z)^2 (1 + \frac{i}{2})$ is in $U$ and such that $\frac{z f'(z)}{f(z)} = -\frac{2(z^2 + z + 1)}{z^2 + z - 2} = -\frac{1}{5} + \frac{3i}{5}$ for $z = i$. This rear property is the main reason why the class $U$ attracts huge attention in the past decades.

In this paper we give sharp estimates of the second and the third Hankel determinant over the class $U$ and study its relation with the class of $\alpha$-convex and starlike functions.

2. **Main results**

In the first theorem we give the sharp estimates of the Hankel determinants of second and third order for the class $U$. We first give the definition of the Hankel determinant, whose elements are the coefficients of a function $f \in A$.

**Definition 2.** Let $f \in A$. Then the $q$th Hankel determinant of $f$ is defined for $q \geq 1$, and $n \geq 1$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \ldots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \ldots & a_{n+2q-2} \end{vmatrix}.$$ 

Thus, the second and the third Hankel determinants are, respectively,

$$H_2(2) = a_2 a_4 - a_3^2,$$

$$H_3(1) = a_3 (a_2 a_4 - a_3^2) - a_4 (a_4 - a_2 a_3) + a_5 (a_3 - a_2^2).$$

**Theorem 1.** Let $f \in U$ and $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$. Then we have the sharp estimates:

$$|H_2(2)| \leq 1 \quad \text{and} \quad |H_3(1)| \leq \frac{1}{4}.$$
Given:

Also, from (4) we have

where function \( \omega \) is analytic in \( \mathbb{D} \) with \( \omega(0) = \omega'(0) = 0 \) and \( |\omega(z)| < 1 \) for all \( z \in \mathbb{D} \).

If we put \( \omega_1(z) = \int_0^z \frac{\omega(t)}{t^2} \, dt \), then we easily obtain that \( |\omega_1(z)| \leq |z| < 1 \) and \( |\omega_1'(z)| \leq 1 \) for all \( z \in \mathbb{D} \). If \( \omega_1(z) = c_1z + c_2z^2 + \cdots \), then

\[
\omega_1'(z) = c_1 + 2c_2z + 3c_3z^2 + \cdots \text{ and } |\omega_1'(z)| \leq 1, \quad z \in \mathbb{D},
\]

gives (see relation (13) in the paper of Prokhorov and Szynal [8]):

\[
|c_1| \leq 1, \quad |2c_2| \leq 1 - |c_1|^2 \quad \text{and} \quad 3c_3(1 - |c_1|^2) + 4c_1c_2^2 \leq (1 - |c_1|^2)^2 - 4|c_2|^2.
\]

Also, from (4) we have

\[
f(z) = z + a_2z^2 + (c_1 + a_2^2)z^3 + (c_2 + 2a_2c_1 + a_2^3)z^4 + \cdots.
\]

From the last relation we have

\[
a_3 = c_1 + a_2^2, \quad a_4 = c_2 + 2a_2c_1 + a_2^3, \quad a_5 = c_3 + 2a_2c_2 + c_1^2 + 3a_2^2c_1 + a_2^4.
\]

We may suppose that \( c_1 \geq 0 \), since from (4) we have \( c_1 = a_3 - a_2^2 \) and \( a_3 \) and \( a_2^2 \) have the same turn under rotation. In that sense, from (5) we obtain

\[
0 \leq c_1 \leq 1, \quad |c_2| \leq \frac{1}{2}(1 - c_1^2) \quad \text{and} \quad |c_3| \leq \frac{1}{3}\left(1 - c_1^2 - \frac{4|c_2|^2}{1 + c_1}\right).
\]

If we use (3), (6) and (7), then

\[
|H_2(2)| = |c_2a_2 - c_1^2| \leq |c_2| \cdot |a_2| + c_1^2 \leq \frac{1}{2}(1 - c_1^2) |a_2| + c_1^2
\]

\[
= \frac{1}{2} \cdot |a_2| + \left(1 - \frac{1}{2} \cdot |a_2|\right)c_1^2 \leq 1.
\]

The functions \( k(z) = \frac{z}{(1 - z)^2} \) and \( f_1(z) = \frac{z}{1 - z^2} \) show that the estimate is the best possible.
Similarly, after some calculations we also have
\[
|H_3(1)| = |c_1c_3 - c_2^2| \leq c_1|c_3| + |c_2|^2 \\
\leq \frac{1}{3}c_1 \left( 1 - c_1^2 - \frac{4|c_2|^2}{1 + c_1} \right) + |c_2|^2 \\
= \frac{1}{3} \left( c_1 - c_3^2 + \frac{3 - c_1}{1 + c_1}|c_2|^2 \right) \\
= \frac{1}{3} \left( c_1 - c_3^2 + \frac{3 - c_1}{1 + c_1} \cdot \frac{1}{4} (1 - c_1)^2 \right) \\
= \frac{1}{12} \left( 3 - 2c_1^2 - c_1^4 \right) \leq \frac{3}{12} = \frac{1}{4}.
\]

The function \(f_2(z) = \frac{z}{1 - z^2/2}\) shows that the result is the best possible. □

In the rest of the paper be consider some starlikeness problems for the class \(U\) and its connection with the class of \(\alpha\)-convex functions.

First, let recall the classical results about the relation between the starlike functions and \(\alpha\)-convex functions.

**Theorem 2.**

(a) \(M_\alpha \subseteq S^*\) for every real \(\alpha\) (\([4]\));
(b) for \(0 \leq \frac{\beta}{\alpha} \leq 1\) we have \(M_\alpha \subset M_\beta\) and for \(\alpha > 1\), \(M_\alpha \subset M_1 = K\) (\([9, 4]\)).

As an anologue of the above theorem we have

**Theorem 3.** For the classes \(M_\alpha\) the next results are valid.

(a) \(M_\alpha \subset U\) for \(\alpha \leq -1\);
(b) \(M_\alpha\) is not a subset of \(U\) for any \(0 \leq \alpha \leq 1\).

**Proof.**

(a) Let \(p(z) = U_f(z)\). Then \(p\) is analytic in \(D\) and \(p(0) = p'(0) = 0\). From here we have that \(\left[ \frac{z}{f(z)} \right]^2 f'(z) = p(z) + 1\) and, after some calculations that
\[
2 \frac{zf'(z)}{f(z)} - \left[ 1 + \frac{zf''(z)}{f'(z)} \right] = 1 - \frac{zp'(z)}{p(z) + 1}.
\]

The relation (11) is equivalent to

\[
\text{Re} \left\{ (1 + \alpha) \frac{zf'(z)}{f(z)} - \alpha \left[ 1 - \frac{zp'(z)}{p(z) + 1} \right] \right\} > 0, \ z \in \mathbb{D}.
\]

We want to prove that \(|p(z)| < 1, \ z \in \mathbb{D}\). If not, then according to the Clunie-Jack Lemma (\([2]\)) there exists a \(z_0, |z_0| < 1\), such that \(p(z_0) = e^{i\theta}\)
and \( z_0 p'(z_0) = kp(z_0) = k e^{i\theta}, \ k \geq 2. \) For such \( z_0, \) from (8) we have that

\[
\text{Re} \left\{ (1 + \alpha) \frac{z_0 f'(z_0)}{f(z_0)} - \alpha \left[ 1 - \frac{k e^{i\theta}}{e^{i\theta} + 1} \right] \right\} \\
= (1 + \alpha) \text{Re} \left\{ \frac{z_0 f'(z_0)}{f(z_0)} \right\} + \alpha \frac{k - 2}{2} \leq 0
\]

since \( f \in S^* \) (by Theorem 2) and \( \alpha \leq -1. \) That is the contradiction to (1).

It means that \( |p(z)| = |u_f(z)| < 1, \ z \in \mathbb{D}, \ i.e. \ f \in \mathcal{U}. \)

(b) To prove this part, by using Theorem 2(b), it is enough to find a function \( g \in K \) such that \( g \) not belong to the class \( \mathcal{U}. \) Really, the function \( g(z) = -\ln(1-z) \) is convex but not in \( \mathcal{U}. \)

\[ \square \]

**Open problem.** It remains an open problem to study the relationship between classes \( \mathcal{M}_\alpha \) and \( \mathcal{U} \) when \(-1 < \alpha < 0 \) and \( \alpha > 1. \)

In the next theorem we consider starlikeness of the function

\[
g(z) = \frac{z/f(z) - 1}{-a_2}
\]

where \( f \in \mathcal{U} \) and \( a_2 = \frac{f''(0)}{2} \neq 0, \) i.e., its second coefficient doesn’t vanish.

Namely, we have

**Theorem 4.** Let \( f \in \mathcal{U}. \) Then, for the function \( g \) defined by (9) we have:

(a) \( |g'(z) - 1| < 1 \) for \( |z| < |a_2|/2; \)

(b) \( g \in S^* \) in the disc \( |z| < |a_2|/2 \) and even more

\[
\left| \frac{zg'(z)}{g(z)} - 1 \right| < 1 \quad (|z| < |a_2|/2);
\]

(c) \( g \in \mathcal{U} \) in the disc \( |z| < |a_2|/2 \) if \( 0 < |a_2| \leq 1. \)

The results are best possible.

**Proof.** Let \( f \in \mathcal{U} \) with \( a_2 \neq 0. \) Then, by using (4), we have that

\[
\frac{z}{f(z)} = 1 - a_2 z - z\omega_1(z),
\]

where \( \omega_1 \) is analytic in \( \mathbb{D} \) such that \( |\omega_1(z)| \leq |z| \) and \( |\omega_1'(z)| \leq 1. \) The appropriate function \( g \) from (9) has the form

\[
g(z) = z + \frac{1}{a_2} z\omega_1(z).
\]

From here \( |g'(z) - 1| = \frac{1}{|a_2|} |\omega_1(z) + z\omega_1'(z)| < 1 \) for \( |z| < |a_2|/2. \)

By using previous representation, we obtain

\[
\left| \frac{zg'(z)}{g(z)} - 1 \right| = \left| \frac{z\omega_1'(z)}{a_2 + \omega_1(z)} \right| \leq \frac{|z|}{|a_2| - |z|} < 1
\]

if \( |z| < |a_2|/2. \) It means that the function \( g \) is starlike in the disk \( |z| < |a_2|/2. \)
If we consider function $f_b$ defined by
\begin{equation}
\frac{z}{f_b(z)} = 1 + bz + z^2, \quad 0 < b \leq 2,
\end{equation}
then $f_b \in U$ and
\begin{equation}
g_b(z) = \frac{\frac{z}{b} - 1}{b} = z + \frac{1}{b}z^2.
\end{equation}
For this function we easily have that for $|z| < b/2$:
\[\text{Re} \frac{z g'_b(z)}{g_b(z)} \geq 1 - \frac{2|z|}{1 - \frac{b}{2}|z|} > 0.\]
On the other hand side, since $g'_b(-b/2) = 0$, the function $g_b$ is not univalent in a bigger disc, which implies that our result is best possible.

Also, by using (10) and the next estimation for the function $\omega_1$:
\[|z \omega'_1(z) - \omega_1(z)| \leq \frac{r^2 - |\omega_1(z)|^2}{1 - r^2},\]
(where $|z| = r$ and $|\omega_1(z)| \leq r$), after some calculation we get
\begin{align*}
|U_g(z)| &= \left| \frac{\frac{1}{a_2}(z \omega'_1(z) - \omega_1(z)) - \frac{1}{a_2} \omega_1'(z)}{(1 + \frac{1}{a_2} \omega_1(z))^2} \right| \\
&\leq \frac{|a_2||z \omega'_1(z) - \omega_1(z)| + |\omega_1(z)|^2}{(|a_2| - |\omega_1(z)|)^2} \\
&\leq \frac{|a_2| r^2 - |\omega_1(z)|^2}{1 - r^2} + |\omega_1(z)|^2 \\
&=: \frac{1}{1 - r^2} \varphi(t),
\end{align*}
where we put
\[\varphi(t) = \frac{(1 - r^2 - |a_2|)t^2 + |a_2|r^2}{(|a_2| - t)^2}\]
and $|\omega_1(z)| = t$, $0 \leq t \leq r$. Since
\[\varphi'(t) = \frac{2|a_2|}{(|a_2| - t)^3} ((1 - r^2 - |a_2|)t + r^2) = \frac{2|a_2|}{(|a_2| - t)^3} ((1 - |a_2|)t + (1 - t)r^2) \geq 0,
\]
because $0 < |a_2| \leq 1$ and $0 \leq t < 1$. It means that the function $\varphi$ is an increasing function and that
\[\varphi(t) \leq \varphi(r) = \frac{(1 - r^2)r^2}{(|a_2| - r)^2}.
\]
Finally we have that
\[|U_g(z)| \leq \frac{r^2}{(|a_2| - r)^2} < 1,
\]
since $|z| < |a_2|/2$. That is implies the second statement of the theorem.
As for sharpness, we can also consider the function $f_b$ defined by (10) with $0 < b \leq 1$. For $|z| < \frac{b}{2}$ we have

$$|U_{g_b}(z)| \leq \frac{\frac{1}{b^2}|z|^2}{(1 - \frac{1}{b}|z|)^2} < 1,$$

which implies that $g_b$ belongs to the class $\mathcal{U}$ in the disc $|z| < b/2$. □

We believe that part (b) of the previous theorem is valid for all $0 < |a_2| \leq 2$. In that sense we have the next

**Conjecture 1.** Let $f \in \mathcal{U}$. Then the function $g$ defined by the expression (9) belongs to the class $\mathcal{U}$ in the disc $|z| < |a_2|/2$. The result is the best possible.

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