Reconfigurable Decorated PT Nets with Inhibitor Arcs and Transition Priorities

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Abstract. In this paper we deal with additional control structures for decorated PT Nets. The main contribution are inhibitor arcs and priorities. The first ensure that a marking can inhibit the firing of a transition. Inhibitor arcs force that the transition may only fire when the place is empty. An order of transitions restrict the firing, so that a transition may fire only if it has the highest priority of all enabled transitions. This concept is shown to be compatible with reconfigurable Petri nets.

Keywords: reconfigurable Petri nets, decorated Petri nets, category of partially ordered sets, inhibitor arcs, transition priorities

1 Introduction

Motivation for reconfigurable Petri nets, a family of formal modelling techniques (e.g. in [1,2,3,4,5]) is the observation that in increasingly many application areas the underlying system has to be dynamic in a structural sense. Complex coordination and structural adaptation at run-time (e.g. mobile ad-hoc networks, communication spaces, ubiquitous computing) are main features that need to be modelled adequately. The distinction between the net behaviour and the dynamic change of its net structure is the characteristic feature that makes reconfigurable Petri nets so suitable for systems with dynamic structures.

Reconfigurable Petri nets consist of marked Petri nets, i.e. a net with a marking, and a set of rules whose application modifies the net’s structure at runtime. Typical application areas are concerned with the modelling of dynamic structures, for example workflows in a dynamic infrastructure.

As an abstract example of a dynamic system we use a cyclic process that can either be executed or modified. These modifications change the process by inserting additional sequential steps or by forking into parallel steps and they can be reversed too. The net in Fig. 2(a) describes a cyclic process with a distinguished place start that can execute one step and then returns to the start. The modifications are modelled by the rules given in Fig. 1. The colours of the places and transitions indicate the mappings within the rule. Rule sequential_ext_s in Fig. 1(a) models the first possible modification, the insertion of a sequential step after the place start. The left-hand side of the rule is the net L and shows the places that need to be in the context and the transition that is deleted. In the
right hand side of the rule is the net $L$ and shows the added place and transitions as well as the context. For reasons of space we have omitted the intermediate net $K$ that denotes the context explicitly. the rule $\text{sequential\_ext\_s}$ is the first rule that can be applied by matching the place $\text{start}$ in $L$ to the place $\text{start}$ in net $\text{start\_net}$ in Fig. 2(a). The application of a rule via a match from $L$ to the given net leads then to the direct transformation from the given net to the resulting net and is achieved by deleting and adding according to rule.

Reconfigurable Petri nets allow the application of these rules together with the firing of the transitions. Let the application of rule $\text{sequential\_ext\_s}$ be the first step, followed by a firing step. This results in the net in Fig. 2(b). The resulting net has an additional place and an additional transition, denoting the process to have been modified by inserting a sequential step. Moreover, the next step has already been executed denoted by firing the transition in the post-domain of place $\text{start}$. These steps are chosen non-deterministic so the start net in Fig. 2(a) may evolve in ten steps to the net in Fig. 2(c) by firing transitions or applying rules. Due to the application of rule 1(d) we now have a fork and due to the firing of the forking transition we have two token. After another 20 steps it may look like the net in Fig. 3. Note, that the rules 1(b) and 1(c) are inverse to each other as well as the rules 1(d) and 1(e). So, after another 20 steps the net may as well be back to the net in Fig. 2(b), but it cannot reach the start net as there is no inverse rule to rule 1(a).

For the sake of the main focus we have considered merely a small and abstract example. More complex nets and rules can be found in case studies for the applications of reconfigurable Petri nets, see e.g. [6,7,8].
Fig. 2. Start and intermediate nets

Fig. 3. Net after another 20 steps
The paper is organized as follows: First we introduce decorated place/transition nets adding some annotations as names and renewable labels. We motivate changing transition labels and extend the firing of a transition so that the labels may be changed. Nevertheless, this extension is conservative to the firing behavior. Then we define reconfigurable Petri nets based on decorated place/transition nets. In the next section we add inhibitor arcs to decorated PT nets and show, that they are still $M$-adhesive. Section 5 extends the set of transitions with a partial order, describing the priorities between the transitions. We employ the category of partial orders $\text{PoSets}$ and again we obtain an $M$-adhesive category.

2 Reconfigurable Petri Nets

We use the algebraic approach to Petri nets, so a marked place/transition net is given by $N = (P, T, \text{pre}, \text{post}, M)$ with pre- and post-domain functions $\text{pre}, \text{post} : T \to P^\oplus$ and a marking $M \in P^\oplus$, where $P^\oplus$ is the free commutative monoid over the set $P$ of places. To obtain the weight of an arc from a place to a transition $t$ the pre domain function is restricted to that place, i.e. $\text{pre}(t)|_p \in \mathbb{N}$; analogously the weight of an arc from a transition to a place is given by the restriction of the post domain function. For $M_1, M_2 \in P^\oplus$ we have $M_1 \leq M_2$ if $M_1(p) \leq M_2(p)$ for all $p \in P$. A transition $t \in T$ is $M$-enabled for a marking $M \in P^\oplus$ if we have $\text{pre}(t) \leq M$, and in this case the follower marking $M'$ is given by $M' = M \ominus \text{pre}(t) \oplus \text{post}(t)$ and $M[t]M'$ is called firing step. In [9] new features have been added to gain an adequate modelling technique. The extension to capacities and names is quite obvious. More interesting are the transition labels that may change, when the transition is fired. This allows a better coordination of transition firing and rule application, for example can be ensured that a transition has fired (repeatedly) before a transformation may take place. This last extension is conservative with respect to Petri nets as it does not change the net behaviour.

2.1 Decorated Place/Transition Nets

A decorated place/transition net is a marked P/T net $N = (P, T, \text{pre}, \text{post}, M)$ together with names and labels. A capacity is merely a function $\text{cap} : P \to \mathbb{N}^\omega_+$. Based on name spaces $A_P, A_T$ with $\text{pname} : P \to A_P$ and $\text{tname} : T \to A_T$ we have explicit names for places and transitions. Moreover, transitions are equipped with labels that may change when the transition fires. This feature is given by a mapping of transitions to functions. For example the net $N_2$ in Fig. ?? yields the marking $3p_a + p_b + 2p_c$ after firing transitions $t_b$ and $t_d$ in parallel. Furthermore, this parallel firing yields the new transition labels 2 for transition $t_b$ and false for transition $t_d$. So, we compute the follower label $tlb(t_b + t_d)tlb'$, where $tlb, tlb' : T \to W$ are label functions with $tlb'(t_b) = \text{inc}(tlb(t_b)) = \text{inc}(1) = 2$, where the renew function $\text{inc} : \mathbb{N} \to \mathbb{N}$ increases the label by one and $tlb'(t_d) = \text{not}(tlb(t_d)) = \text{not}(true) = \text{false}$. For more details see [9].
Definition 1 (Decorated place/transition net). A decorated place/transition net is a marked place/transition net \( N = (P, T, \text{pre}, \text{post}, M) \) together with

- a capacity as a function \( \text{cap}: P \to \mathbb{N}_0^\omega \)
- name spaces \( A_P, A_T \) with \( \text{pname}: P \to A_P \) and \( \text{tname}: T \to A_T \)
- the function \( \text{tlb}: T \to W \) mapping transitions to transition labels \( W \) and
- the function \( \text{rnw}: T \to \text{END} \) where \( \text{END} \) is a set containing some endomorphisms on \( W \), so that \( \text{rnw}(t): W \to W \) is the function that renews the transition label.

The firing of these nets is the usual for place/transition nets except for changing the transition labels. Moreover, this extension works for parallel firing as well.

Definition 2 (Changing Labels by Parallel Firing). Given a transitions vector \( v = \sum_{t \in T} k_t \cdot t \) then the label is renewed by firing \( \text{tlb}[v] \text{tlb}' \) and for each \( t \in T \) the transition label \( \text{tlb}' : T \to W \) is defined by:

\[
\text{tlb}'(t) = \text{rnw}(t)^{k_t} \circ \text{tlb}(t)
\]

2.2 Transformations of Decorated Nets

For decorated place/transition nets as given above, we obtain with the following notion of morphisms an \( \mathcal{M} \)-adhesive HLR category (see [9]). \( \mathcal{M} \)-adhesive HLR systems can be considered as a unifying framework for graph and Petri net transformations providing enough structure that most notions and results from algebraic graph transformation systems are available, as results on parallelism and concurrency of rules and transformations, results on negative application conditions and constraints, and so on (e.g. in [10,11]).

Net morphisms map places to places and transitions to transitions. They are given as a pair of mappings for the places and the transitions, so that the structure and the decoration is preserved and the marking may be mapped strictly.

Definition 3 (Morphisms between decorated place/transition nets [9]). A net morphism \( f : N_1 \to N_2 \) between two decorated place/transition nets \( N_i = (P_i, T_i, \text{pre}_i, \text{post}_i, M_i, \text{cap}_i, \text{pname}_i, \text{tname}_i, \text{tlb}_i, \text{rnw}_i) \) for \( i \in \{1, 2\} \) is given by \( f = (f_P : P_1 \to P_2, f_T : T_1 \to T_2) \), so that the following equations hold:

1. \( \text{pre}_2 \circ f_T = f_P^{\oplus} \circ \text{pre}_1 \) and \( \text{post}_2 \circ f_T = f_P^{\oplus} \circ \text{post}_1 \)
2. \( \text{cap}_1 = \text{cap}_2 \circ f_P \)
3. \( \text{pname}_1 = \text{pname}_2 \circ f_P \)
4. \( \text{tname}_1 = \text{tname}_2 \circ f_T \) and \( \text{tlb}_1 = \text{tlb}_2 \circ f_T \) and \( \text{rnw}_1 = \text{rnw}_2 \circ f_T \)
5. \( M_1(p) \leq M_2(f_P(p)) \) for all \( p \in P_1 \)

Moreover, the morphism \( f \) is called strict

6. if both \( f_P \) and \( f_T \) are injective and \( M_1(p) = M_2(f_P(p)) \) holds for all \( p \in P_1 \).
A rule in the DPO approach is given by three nets called left hand side \(L\), interface \(K\) and right hand side \(R\), respectively, and a span of two strict net morphisms \(K \rightarrow L\) and \(K \rightarrow R\). Additionally, a match morphism \(m : L \rightarrow N\) is required that identifies the relevant parts of the left hand side in the given net \(N\). Then a transformation step \(N \xrightarrow{(r,m)} M\) via rule \(r\) can be constructed in two steps. Given a rule with a match \(m : L \rightarrow N\) the gluing conditions have to be satisfied in order to apply a rule at a given match. These conditions ensure the result is again a well-defined net. It is a sufficient condition for the existence and uniqueness of the so-called pushout complement which is needed for the first step in a transformation. In this case, we obtain a net \(M\) leading to a direct transformation \(N \xrightarrow{(r,m)} M\) consisting of the following pushouts (1) and (2) in Fig. 4.

\[
\begin{array}{c}
L & \xleftarrow{m} & K & \xrightarrow{r} & R \\
\downarrow & & \downarrow & & \downarrow \\
N & \xleftarrow{D} & D & \xrightarrow{m} & M
\end{array}
\]

Fig. 4. Transformation of a net

Next we show that decorated place/transition nets yield an \(\mathcal{M}\)-adhesive HLR category for \(\mathcal{M}\) being the class of strict morphisms. Hence we obtain all the well-known results, as transformation, local confluence and parallelism, application conditions, amalgamation and so on.

**Lemma 1 (see [9]).** The category \texttt{decoPT} of decorated place/transition nets is an \(\mathcal{M}\)-adhesive HLR category.

This construction as well as a huge amount of notion and results are available since decorated place/transition nets can be proven to be an \(\mathcal{M}\)-adhesive HLR category. Hence we can combine one net together with a set of rules leading to reconfigurable place/transition nets.

**Definition 4 (Reconfigurable Nets).** A reconfigurable decorated place/transition net \(RN = (N,R)\) is given by an decorated \(N\) and a set of rules \(R\).

### 3 Review of \(\mathcal{M}\) adhesive HLR Systems

The theory of HLR systems has been developed as an abstract framework for different types of graph and Petri net transformation systems. Moreover the HLR framework has been applied to algebraic specifications [9], where the interface of an algebraic module specification can be considered as a production of an algebraic specification transformation system [9]. HLR systems are instantiated with various types of graphs, as hypergraphs, attributed and typed graphs, structures, algebraic specifications, various Petri net classes, elementary nets, place/transition nets, Colored Petri nets, or algebraic high-level nets, and more (see [2] and [4]). Adhesive categories have been introduced in [9] and have been combined with HLR categories and systems in [7] leading to the new concept of (weak) adhesive HLR categories and systems. The main reason why adhesive
categories are important for the theory of graph transformation and its generalization to high-level replacement systems is the fact that most of the HLR conditions required in [?] are shown to be already valid in adhesive categories (see [?]). The fundamental construct for (weak) adhesive (HLR) categories and systems are van Kampen (VK) squares.

**Definition 5** (\(M\)-Van Kampen square). A pushout (1) with \(m \in M\) is a \(M\)-van Kampen (VK) square, if for any commutative cube (2) with (1) in the bottom and back faces being pullbacks, the following holds: the top is pushout \(\iff\) the front faces are pullbacks.

\[
\begin{array}{c}
A \xrightarrow{m \in M} B \\
\downarrow \\
C \xrightarrow{n} D
\end{array}
\quad \text{(1)} \\
\begin{array}{c}
A' \xleftarrow{m'} B' \\
\downarrow \\
C' \xleftarrow{n'} D'
\end{array}
\quad \text{(2)}
\]

\(M\)-adhesive HLR systems can be considered as abstract transformation systems in the double pushout approach based on \(M\)-adhesive HLR categories.

**Definition 6** (**\(M\)**-Adhesive HLR Category and PO-PB Compatibility [14]). Given a PO-PB compatible class \(M\) of monomorphisms in \(C\) (see below), then \((C, M)\) is called \(M\)-adhesive HLR-category, if pushouts along \(M\)-morphisms are \(M\)-VK squares (see 5).

A class \(M\) of monomorphisms in \(C\) is called PO-PB compatible, if

1. Pushouts along \(M\)-morphisms exist and \(M\) is stable under pushouts.
2. Pullbacks along \(M\)-morphisms exist and \(M\) is stable under pullbacks.
3. \(M\) contains all identities and is closed under composition.

An \(M\)-adhesive HLR system \(AHS = (C, M, P)\) consists of an adhesive HLR category \((C, M)\) and a set of rules \(P\).

4 **Inhibitor Arcs**

We here introduce generalized inhibitor arcs, that may consider several places to inhibit the transitions firing. So inhibitor arcs are given as a function for transitions to the multiset of places.

**Definition 7** (Generalized inhibitor arcs). Given a decorated place/transition net \(N = (P, T, \text{pre}, \text{post}, M, \text{cap}, \text{pname}, \text{tname}, \text{tlb}, \text{rnw})\) inhibitor arcs are given by \(\text{inh} : T \rightarrow \mathcal{P}(P)\).

A transition is then enabled under a marking \(M_1\) if additionally we have \(M_1(p) = 0\) for all \(p \in \text{inh}(t)\).
Lemma 2. The category decoPTi of decorated place/transition nets with inhibitor arcs is an $\mathcal{M}$-adhesive HLR category with $\mathcal{M}$ being the class of strict, injective net morphisms.

Proof. The proof applies the construction for weak adhesive HLR categories (see Theorem 1 in [12]): Constructing the category decoPTi using comma categories, we use the functor $F : \text{decoPT} \rightarrow \text{Sets}$ yielding the transition set $T$ and the power set functor $P : \text{Sets} \rightarrow \text{Sets}$. The category of decorated place/transition nets is an $\mathcal{M}$-adhesive HLR category (see [9]): Then the comma category $\text{decoPTi} := \text{CommCat}(F, P, \{\text{inh}\})$ yields the category of decorated place/transition nets with inhibitor arcs and is a weak adhesive HLR category as $F$ preserves pushouts and $P$ pullbacks of injective morphisms.

Hence, we have an $\mathcal{M}$-adhesive HLR category, see [14].

5 Transition Priorities

The set of transitions $T$ is equipped with a partial order $\leq$ on the transitions. $t$ is enabled under a marking $M$, if $\text{pre}(t) \geq M$, if $\text{cap}(t) \geq M + \text{post}(t)$ and if all $t'$ being enabled under $M$ we have $t' \leq t$.

We first need to investigate the category PoSets of partially ordered sets. In [15] this category has been examined.

Definition 8 (Category PoSets). The objects are partially orders sets, given by a set $P$ and a partial order $\leq$ over $P$. The morphisms if this category are order-preserving maps, that are maps $f : P_1 \rightarrow P_2$ preserving the order, so $x \leq y$ implies $f(x) \leq f(y)$.

Composition and identity are defined as for sets and are both order-preserving, PoSets is indeed a category [15].

The relation to the category of sets can be given by two functors. The free functor $F : \text{Sets} \rightarrow \text{PoSets}$ is given by $F(M \xrightarrow{f} M') = (M, ID_M) \xrightarrow{f} /\text{nachf}(M', ID_M)$, where $ID_M$ is the identity relation of a set $M$. The forgetful functor $V : \text{PoSets} \rightarrow \text{Sets}$ is defined by $V((P, \leq_P) \xrightarrow{g} (P', \leq_{P'})) = P \xrightarrow{g} P'$.

Lemma 3 (Adjunction to Sets).

Proof. So, we know that $F$ preserves colimits ans $V$ preseves limits.

Lemma 4 (Initial Object and Pushouts in PoSets).

1. The initial object is $(\emptyset, \emptyset)$.
2. Given the span $(P_1, \leq_1) \xrightarrow{f} (P_0, \leq_0) \xrightarrow{g} (P_2, \leq_2)$, then there exists the pushout $(P_1, \leq_1) \xrightarrow{g} (P_3, \leq_3) \xrightarrow{f'} (P_2, \leq_2)$. 
Proof. 1. The initial object is \((\emptyset, \emptyset)\) as there is the empty order preserving mapping to each partially ordered set in \textbf{PoSets}.

2. Given \((P_1, \leq_1) \xleftarrow{f} (P_0, \leq_0) \xrightarrow{g} (P_2, \leq_2)\), then there is in \textbf{Sets} the span \(P_1 \xleftarrow{f} P_0 \xrightarrow{g} P_2\) and its pushout \(P_1 \xrightarrow{\bar{g}} \bar{P}_3 \xleftarrow{h} P_2\), see pushout \((Po)\) in 2 and the relation \(R_3 \subseteq \bar{P}_3 \times P_2\) with

\[(x_3, y_3) \in R_3 \text{ if and only if} \]
\[\exists x_1, y_1 \in P_1 : \bar{g}(x_1) = x_3 \wedge \bar{g}(y_1) = y_3 \wedge x_1 \leq_1 y_1 \quad (1)\]
\[\lor \exists x_2, y_2 \in P_2 : \bar{f}(x_2) = x_3 \wedge \bar{f}(y_2) = y_3 \wedge x_2 \leq_2 y_2\]

Since \(R_3\) is not a partial order\(^1\), we define the relation \(\bar{R}_3\) to be the equivalence closure of all symmetric pairs \(\{(x_3, y_3) \mid (x_3, y_3), (y_3, x_3) \in R_3\} \subseteq R_3\). Then we have the quotient \(P_3 = \bar{P}_3/R_3\) with \(g' := [\_] \circ \bar{g} : P_1 \rightarrow P_3\) and \(f' := [\_] \circ \bar{f} : P_2 \rightarrow P_3\), where \([\_] : \bar{P}_3 \rightarrow \bar{P}_3/R_3 = P_3\) is the natural function mapping each element of \(P_3\) to its equivalence class.

\(\leq_3\) is the transitive closure of

\[
\{(x_3, y_3) \mid x_1 \leq_1 y_1 \text{ for } g'(x_1) = x_3 \text{ and } g'(y_1) = y_3 \}
\]

or

\[
\{x_2 \leq_2 y_2 \text{ for } f'(x_2) = x_3 \text{ and } f'(y_2) = y_3\}
\]

\(\leq_3\) is a partial order, as it is reflexive, antisymmetric and transitive and \(f'\) and \(g'\) are order-preserving maps by construction.

So, in \textbf{PoSets} the category of partially ordered sets \((P_1, \leq_1) \xleftarrow{f} (P_3, \leq_3) \xrightarrow{g'} (P_2, \leq_2)\) is the pushout of \((P_1, \leq_1) \xleftarrow{f} (P_0, \leq_0) \xrightarrow{g} (P_2, \leq_2)\):

\[
\begin{array}{c}
\text{\(P_0\)} \xrightarrow{f} \text{\(P_1\)} \\
\downarrow{g} \quad \downarrow{g''} \\
\text{\(P_2\)} \xrightarrow{f'} \text{\(P_3\)} \xrightarrow{h} \text{\(P_4\)}
\end{array}
\]

\[
\text{(2)}
\]

Obviously \(g' \circ f = f' \circ g\).

For any partially ordered set \((P_4, \leq_4)\) with \(g'' \circ f = f'' \circ g\) we have \(h : P_3 \rightarrow P_4\) in \textbf{Sets} due to the pushout \((Po)\) in Diagram 2. So, we define \(h : P_3 \rightarrow P_4\) with \(h([x]) = \bar{h}(x)\).

\(^1\) Let \(P_0 = \{0, 5\}\) and \(P_1 = \{0, 3, 5\}\) with \(f\) the inclusion and \(P_2 = \{\bullet\}\), then \(3 \leq_1 5\) yields \(([3], [\bullet]) \in R_3\) and \(0 \leq_1 3\) yields \(([\bullet], [3]) \in R_3\), but \([\bullet] = \{0, 5\} \neq \{3\} = [3]\).
To prove that \( h \) is well-defined we show \( h([x_3]) = h([y_3]) \) with \( x_3 \neq y_3 \) but \([y_3] = [x_3]\).

Since \([y_3] = [x_3]\) and \( x_3 \neq y_3 \) there is \((x_3, y_3) \in R_3\) and hence \((x_3, y_3) \in R_3\) and \((y_3, x_3) \in R_3\). Due to the definition of \( R_3 \) there are four cases:

1. \( \exists x_1, y_1 \in P_1 : x_1 \leq_1 y_1 \land \bar{g}(x_1) = x_3 \land \bar{g}(y_1) = y_3 \land \exists x_2, y_2 \in P_2 : x_2 \leq_2 x_1 \land f(x_2) = x_3 \land f(y_2) = y_3 \):
   
   Then we have \( g''(x_1) = \bar{h} \circ \bar{g}(x_1) = \bar{h} \circ \bar{f}(x_2) = f''(x_2) \) and \( g''(y_1) = \bar{h} \circ \bar{g}(y_1) = \bar{h} \circ \bar{f}(y_2) = f''(y_2) \).
   
   This yields \( g''(x_1) \leq_4 g''(y_1) \) and \( g''(y_1) = f''(y_2) \geq_4 f''(x_2) = g''(x_1) \). Since \( \leq_4 \) is a antisymmetric we have \( g''(x_1) = g''(y_1) \).
   
   Hence, we have \( h([x_3]) = \bar{h}(x_3) = \bar{h} \circ \bar{g}(x_1) = g''(x_1) = g''(y_1) = \bar{h} \circ \bar{g}(y_1) = \bar{h}(y_3) = h([y_3]) \).

2. \( \exists x_1, y_1 \in P_1 : y_1 \leq_1 x_1 \land \bar{g}(x_1) = x_3 \land \bar{g}(y_1) = y_3 \land \exists x_2, y_2 \in P_2 : x_2 \leq_2 x_1 \land f(x_2) = x_3 \land f(y_2) = y_3 \) analogously.

3. \( \exists x_1, y_1 \in P_1 : x_1 \leq_1 y_1 \land \bar{g}(x_1) = x_3 \land \bar{g}(y_1) = y_3 \land \exists x_1', y_1' \in P_1 : y_1' \leq_1 x_1' \land \bar{g}(x_1') = x_3 \land \bar{g}(y_1') = y_3' \):
   
   So, we have \( \bar{g}(x_1) = x_3 = \bar{g}(x_1') \) and \( \bar{g}(y_1) = y_3 = \bar{g}(y_1') \). and \( x_1 \leq_1 y_1 \) and \( y_1' \leq_1 x_1' \).
   
   This yields \( g''(x_1) \leq_4 g''(y_1) \) and \( g''(y_1) = g''(y_1') \leq_4 g''(x_1') = g''(x_1) \). Since \( \leq_4 \) is a antisymmetric we have \( g''(x_1) = g''(y_1) \).
   
   Hence, we have \( h([x_3]) = \bar{h}(x_3) = \bar{h} \circ \bar{g}(x_1) = g''(x_1) = g''(y_1) = \bar{h} \circ \bar{g}(y_1) = \bar{h}(y_3) = h([y_3]) \).

4. \( \exists x_1, y_1 \in P_1 : x_1 \leq_1 y_1 \land \bar{g}(x_1) = x_3 \land \bar{g}(y_1) = y_3 \land \exists x_2, y_2 \in P_2 : y_2 \leq_2 x_1 \land f(x_2) = x_3 \land f(y_2) = y_3 \) analogously.

Moreover, \( h \circ g' = h \circ \bar{g} = \bar{h} \circ \bar{g} = g'' \) and \( h \circ f' = h \circ \bar{f} = \bar{h} \circ \bar{f} = f'' \).

Next we introduce the subclass of monomorphisms \( M \). Monomorphisms in \( \text{PoSets} \) are the injective order preserving maps \[15\] and order embeddings - those mappings that satisfy item 1 in Def. 9 - are regular monomorphisms \[15\].

**Definition 9 (Class \( M \)).** The class \( M \) is given by the class of strict order embeddings, that are order preserving mappings \( f : (P, \leq_P) \to (P', \leq_{P'}) \) that additionally satisfy:

1. \( x \leq_P y \) if and only if \( f(x) \leq_{P'} f(y) \) for \( x, y \in P \)
2. for each \( z' \in P' \) with \( f(x) \leq_{P'} z' \leq_{P'} f(y) \) there exists some \( z \in P \) with \( f(z) = z' \) (and hence \( x \leq_P z \leq_P y \)).

Class \( M \) leads to pushouts that are constructed as in the category \( \text{Sets} \), hence the forgetful functor \( V : \text{PoSets} \to \text{Sets} \) preserves pushouts.
Lemma 5 (M-Pushouts in PoSets). Given \((P_1, \leq_1) \xrightarrow{f} (P_0, \leq_0) \xrightarrow{g} (P_2, \leq_2)\) with \(f \in \mathcal{M}\) then there is the pushout \((P_1, \leq_1) \xrightarrow{f} (P_3, \leq_3) \xrightarrow{g} (P_2, \leq_2)\), such that in \textbf{Sets} \(P_1 \xrightarrow{f} P_3 \xleftarrow{g} P_2\) is the pushout of \(P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_2\).

Moreover, \(\mathcal{M}\) is stable under pushouts.

Proof. Obviously, the construction of \(\tilde{R}_3\) in the proof of Lemma 4 yields for \(f \in \mathcal{M}\) that \(\tilde{R}_3 = ID\) the identity relation. Hence, \(P_3 = P_3|\tilde{R}_3 = P_3\).

Moreover, its is \(\mathcal{M}\)-stable:

For \(f \in \mathcal{M}\) in Diagram 2 we know that \(f'\) is injective, as pushouts in \textbf{Sets} preserve monomorphisms, i.e. injective mappings and it is order-preserving by construction.

\(f'\) is an order embedding:

For \(x_2, y_2 \in P_2\) and \(f'(x_2) \leq f'(y_2)\) we have due to the construction of \(\leq_3\) four cases:

1. There are \(x_1, y_1 \in P_1\) with \(x_1 \leq_1 y_1\) so that \(g'(x_1) f'(x_2)\) and \(g'(y_1) = f'(y_2)\). Due to the pushout construction there are \(x_0, y_0 \in P_0\) with \(x_0 \leq_0 y_0\) so that \(f(x_0) = x_1\) and \(g(x_1) = x_2\) and \(f(y_0) = y_1\) and \(g(y_1) = y_2\). Since \(g\) is order preserving, we have \(x_2 \leq_2 y_2\).

2. There is \(x_2 \leq_2 y_2\).

3. There is \(z_3 \in P_3\) with \(f'(x_2) \leq_3 f'(y_3)\), so that there are \(x_1 \leq_1 z_1\) with \(g'(x_1) f'(x_2)\) and \(g'(z_1) = z_3\) and \(z_2 \leq_2 y_2\) and \(f'(z_2) = z_3\).

Due to the pushout construction there are \(x_0, z_0 \in P_0\) with \(x_0 \leq_0 z_0\) so that \(f(x_0) = x_1\) and \(g(x_1) = x_2\) and \(f(z_0) = z_1\) and \(g(z_0) = z_2\). Since \(g\) is order preserving, we have \(x_2 \leq_2 z_2 \leq_2 y_2\).

4. There is \(z_4 \in P_3\) with \(f'(x_2) \leq_3 z_3 \leq_3 f'(y_3)\), so that there are \(z_1 \leq_1 y_1\) with \(g'(y_1) f'(y_2)\) and \(g'(z_1) = z_3\) and \(x_2 \leq_2 z_2\) and \(f'(z_2) = z_3\) analogously.

\(f'\) is a strict order embedding:

Let be \(x_2, y_2 \in P_2\) and \(f'(x_2) \leq_3 z_3 \leq_3 f'(y_2)\) given for \(z_3 \in P_3\). Either \(z_3 \in f'(P_2)\) and hence there is \(f'(z_2) = z_3\) with \(x_2 \leq_2 y_2\) or \(z_3 \notin f'(P_2)\). Then there are \(x_1, y_1, z_1, z'_1 \in P_1\) with \(g'(x_1) = f'(x_2)\) and \(g'(y_1) = f'(y_2)\) and \(g'(z_1) = z_3 = g(z_1)\) and \(x_1 \leq z_1\) and \(z'_1 \leq_1 y_1\). Due to the pushout construction there are \(x_0, y_0 \in P_0\) with \(f(x_0) = x_1\) and \(g(x_1) = x_2\) and \(f(y_0) = y_1\) and \(g(y_1) = y_2\). Since \(f\) is a strict order embedding we have additionally, \(z_0, z'_0\) with \(f(z_0) = z_1\) and \(f(z'_0) = z'_1\) and \(x_0 \leq z_0 \leq z'_0 \leq y_0\). Due to pushout construction \(g(z_0) = g(z'_0)\) and as \(g\) is order preserving we have \(x_2 = g(x_0) \leq_2 g(z_0) \leq_2 g(y_0) = y_2\) with \(f'(g(z_0)) = z_3\).

Next we investigate pullbacks in \textbf{PoSets}.

Lemma 6 (Pullbacks in PoSets). Given \((P_1, \leq_1) \xrightarrow{g} (P_0, \leq_0) \xleftarrow{f} (P_2, \leq_2)\) then there is the pullback \((P_1, \leq_1) \xleftarrow{f} (P_3, \leq_3) \xrightarrow{g} (P_2, \leq_2)\). Moreover, \(\mathcal{M}\) is stable under pullbacks.
Proof. There is the pullback \( P_1 \leftarrow P_3 \xrightarrow{g'} P_2 \) of \( P_1 \xrightarrow{g} P_0 \leftarrow P_2 \) in Sets. \((P_1, \leq_1)\) \( \xleftarrow{f'} (P_3, \leq_3) \xrightarrow{g'} (P_2, \leq_2)\) with \( x_3 \leq y_3 \) if and only if \( f'(x_3) \leq_1 f'(y_3) \) and \( g'(x_3) \leq_3 g'(y_3) \) is pullback in \( \text{PoSets} \). Obviously, \( f' \) and \( g' \) are order-preserving mappings.

\( M \)-morphisms are monomorphisms and hence are preserved by pullbacks.

Theorem 1 (\( \text{PoSets} \) is \( M \)-Adhesive HLR Category.).

Proof.

1. The class \( M \) in \( \text{PoSets} \) is PO-PB compatible, since
   - pushouts along \( M \)-morphisms exist and \( M \) is stable under pushouts,
   - pullbacks along \( M \)-morphisms exist and \( M \) is stable under pullbacks and
   - obviously, \( M \) contains all identities and is closed under composition.

2. In \( \text{PoSets} \) pushouts along \( M \)-morphisms are \( M \)-VK squares: Let be given as : a pushout (1) with \( m \in M \) and some commutative cube (2) with (1) in the bottom and back faces being pullbacks in \( \text{PoSets} \).

\[ A \xrightarrow{m \in M} B \]

\[ C \xrightarrow{n} D \]

\[ A' \xleftarrow{m} B' \]

\[ C' \xrightarrow{n} D' \]

\[ \Rightarrow: \] Let the top be a pushout in \( \text{PoSets} \). Pullbacks preserve \( M \)-morphisms, so \( m' \in M \) and hence the top square is a pushout in \( \text{Sets} \) as well. As \( \text{Sets} \) is adhesive, the front faces are pullbacks in \( \text{Sets} \) as well. Since the construction of pullbacks coincides in \( \text{Sets} \) and \( \text{PoSets} \), the front faces are pullbacks in \( \text{PoSets} \).

\[ \Leftarrow: \] Let the front faces be pullbacks in \( \text{PoSets} \), and hence pullbacks in \( \text{Sets} \). Since \( m \in M \) (1) is pushout in \( \text{Sets} \) as well. So, \( \text{Sets} \) being adhesive, we have the top square being a pushout in \( \text{Sets} \). Moreover, \( M' \in M \) as the back face is a pullback preserving \( M \)-morphisms. So, the top is a pushout along \( M \) is \( \text{PoSets} \).

Hence, by Def. (\( \text{PoSets} \), \( M \)) is an \( M \)-adhesive HLR-category.

Definition 10. The category of place/transition nets with transition priorities \( \text{PTp} \) is given by \( N = (P, (T, \leq_T), \text{pre}, \text{post}, m_0) \) with \( \text{pre}, \text{post} : V(T, \leq_T) \rightarrow P^\oplus \) and morphisms \( f_P, f_T : N_1 \rightarrow N_2 \) where \( f_P \) is a mapping and \( f_T \) is an order-preserving map.

A transition \( t \in T \) is enabled under a marking \( m \), if \( \text{pre}(t) \geq m \) and if for all \( t' \in T \) being enabled under \( m \) we have \( t' \leq_T t \).
Lemma 7 \((\text{PTp}, \mathcal{M})\) is an \(\mathcal{M}\)-adhesive HLR-category\). with \(\mathcal{M}\) the net morphisms where \(f_p\) is strict injective and \(f_T\) is a strict order embedding.

Proof. The proof applies the construction for weak adhesive HLR categories (see Theorem 1 in [12]):
We know that \((\text{Sets}, \mathcal{M})\) with \(\mathcal{M}\) being the injective mappings is an \(\mathcal{M}\)-adhesive HLR category and that \((\_)^\oplus : \text{Sets} \to \text{Sets}\) preserves pullbacks along injective morphisms. As shown above \((\text{PoSets}, \mathcal{M})\) with \(\mathcal{M}\) being the strict order embeddings is an \(\mathcal{M}\)-adhesive HLR category and that \(V : \text{PoSets} \to \text{Sets}\) preserves pushouts along \(\mathcal{M}\)-morphisms. So, the category \(c\text{PTp}\) is isomorphic to the comma category \(\text{ComCat}(V, (\_)^\oplus; I)\) with \(I = 1, 2\), where \(V : \text{PoSets} \to \text{Sets}\) is the forgetful functor from partial ordered sets to sets and \((\_)^\oplus\) is the free commutative monoid functor and hence an \(\mathcal{M}\)-adhesive HLR category.

Lemma 8 \((\text{decoPTip}, \mathcal{M})\) is an \(\mathcal{M}\)-adhesive HLR-category\). with \(\mathcal{M}\) the net morphisms where \(f_p\) is strict injective and \(f_T\) is a strict order embedding.

Proof. Similar to the proof of Lemma 1 in [9] using \(\text{PTp}\) instead of \(\text{PT}\) as the basis.

6 Conclusion

The tool ReCONNet has been developed at the HAW Hamburg in various students projects. Up to now it supports the modelling and simulation of reconfigurable nets. The nets and rules in Figs. 1 and 2 have been edited and computed by ReCONNet. The tool’s most important feature is the ability to create, modify and simulate reconfigurable nets through an intuitive graphic-based user interface (see [17]).

Ongoing work concern the extension of the control structures. This includes the extension of rules with negative application conditions and an explicit representation of an abstract reachability graph based on [9].

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