Mostow’s Fibration for Canonical Embeddings of Compact Homogeneous CR Manifolds

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Abstract. We define a class of compact homogeneous CR manifolds which are bases of Mostow fibrations having total spaces equal to their canonical complex realizations and Hermitian fibers. This is used to establish isomorphisms between their tangential Cauchy-Riemann cohomology groups and the corresponding Dolbeault cohomology groups of the embeddings.

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1. Introduction and preliminaries

The aim of this paper is to investigate relations between the cohomology groups of the tangential Cauchy Riemann complexes of n-reductive compact homogeneous CR manifolds and the corresponding Dolbeault cohomology groups of their canonical embeddings. The class of n-reductive compact homogeneous CR manifolds was introduced in [1]: its objects are the minimal orbits, in homogeneous spaces of reductive complex groups, of their compact forms.

Results on the cohomology of the tangential CR complexes on general compact CR manifolds of arbitrary codimension were obtained in [14] (see also [8]), under suitable r-pseudoconcavity conditions, involving their scalar Levi forms, that were first introduced in [3, 25]. In this paper we will restrain to the homogeneous case.

The CR structure of a homogeneous CR manifold M0 is efficiently described by considering its CR algebra at any point p0 ∈ M0: it is the pair (κ0, v) consisting of the real Lie algebra κ0 of its transitive group K0 of CR-automorphisms and of the subspace v = dπ−1(Tp0,M0) of the complexification κ of κ0 (see [22]). The formal integrability of the partial complex structure T0,1M0 of M0 is equivalent to the fact that v is a complex Lie subalgebra of κ. The intersection v ∩ ː (conjugation is taken with respect to the real form κ0) is the complexification of the Lie algebra
of the stabilizer of $p_0$ in $K_0$ and the quotient $\mathfrak{v}/(\mathfrak{v} \cap \bar{\mathfrak{v}})$ represents the space $T^{0,1}_{p_0}M_0$ of anti-holomorphic complex tangent vectors at $p_0$.

We call $n$-reductive a homogeneous CR manifolds for which $\mathfrak{v} = (\mathfrak{v} \cap \bar{\mathfrak{v}}) \oplus n(\mathfrak{v})$, i.e. for which $T^{0,1}_{p_0}M_0$ can be identified to the nilradical of $\mathfrak{v}$. It was shown in [11] that the intersection of any pair of Matsuki-dual orbits in a complex flag manifold $M$, with the CR structure inherited from $M$, is an $n$-reductive compact homogeneous CR manifold. Moreover, when $M_0$ is $n$-reductive, $\mathfrak{v}$ is the Lie algebra of a closed complex Lie subgroup $V$ of $K$ that contains the stabilizer of $p_0$ as its maximal compact subgroup, so that $M_0 = K_0/V_0 \hookrightarrow M = K/V$ is a generic CR-embedding. Vice versa, if $M_-$ is a $K$-homogeneous complex algebraic manifold, then a minimal $K_0$-orbit $M_0$ in $M_-$ is an $n$-reductive compact homogeneous CR manifold.

Since $K_0$ is a maximal compact subgroup of a linear algebraic complex group $K$, the quasi-projective manifold $M_-$ can be viewed as a $K_0$-equivariant fiber bundle on the basis $M_0$ (see [24]). We use this Mostow fibration of $M_-$ onto $M_0$ to construct a nonnegative smooth exhaustion $\phi$ of $M_-$, with $\phi^{-1}(0) = M_0$, to relate the Dolbeault cohomology of $M_-$ to the cohomology of the tangential CR-complex on $M_0$. This requires some precision on the structure of the fibers and forces us to introduce a further requirement on the CR algebra $(\mathfrak{k}_0, \mathfrak{v})$, namely to ask that, if $\mathfrak{w}$ is the largest complex subalgebra of $\mathfrak{k}$ with $\mathfrak{v} \subset \mathfrak{w} \subset (\mathfrak{v} + \bar{\mathfrak{v}})$, (see [22] Theorem 5.4), then $n(\mathfrak{w})$ is the nilradical of a parabolic subalgebra of $\mathfrak{k}$. This condition is satisfied in many examples coming from Matsuki duality (cf. [21]) and can always be satisfied by strengthening the CR structure of an $n$-reductive $M_0$.

When we drop this extra assumption, we are still able to construct a continuous exhaustion, which, when $M_0$ is $r$-pseudoconcave, is still strictly $r$-pseudoconcave, allowing us to obtain results on the first $(r - 1)$ tangential Cauchy Riemann and Dolbeault cohomology groups of $M_0$ and $M_-$ (or up to $(r - \text{hd}(\mathcal{F}) - 1)$ if we discuss cohomology with coefficients in a coherent sheaf $\mathcal{F}$).

Earlier versions of some results proved here were discussed in [19] [20].

The paper is organized as follows.

In §2 we discuss some basic facts on $n$-reductive CR manifolds. We skip from basic stuff on CR manifolds and CR algebras, for which we refer, e.g., to [14] [22], and only explain those special features which are necessary for the developments of the next sections.

Cartan and Mostow fibrations are related to the structure of negatively curved Riemannian symmetric space of the set of Hermitian symmetric matrices with determinant one. Hence we found convenient to discuss in §3 as a preliminary, some topics of the geometry of $SL_n(\mathbb{C})/SU(n)$.

In §4 we study decompositions of $K$ with Hermitian fibers.

Example 3.7 shows that a $K_0$-equivariant fibration of $M_-$ with Hermitian fibers, as in [23], is not always possible. In §5 we describe the general structure of the fibers. To this aim, we consider a class of parabolic subalgebras associated to the pair $(\mathfrak{k}_0, \mathfrak{v})$ and find a condition, that we call HNR from horocyclic nilradical, under which we get a Mostow fibration of $M_-$ with Hermitian fibers.

In the final section §6 we apply these results to construct an exhaustion function which permits to relate some cohomology groups of the tangential CR complexes on $M_0$ to the corresponding cohomology groups of the Dolbeault complexes on $M_-$ and analogous results for Čech cohomology with coefficients in a coherent sheaf.
We conclude with the study of an example of a family of intersections of Matsuki-dual orbits and an application of §4 to obtain a pseudoconcavity result for which we do not require the validity of the HNR assumption.

2. Compact homogeneous CR manifolds and ii-reductiveness

In this section we introduce the class of homogeneous CR manifold which is the object of this investigation. We found convenient to recall, in an initial short subsection, the definition of reductive Lie group, as it is not completely standard in the literature.

2.1. Reductive Lie groups. We call reductive a Lie algebra $\kappa$ whose radical is abelian: its commutator subalgebra $[\kappa, \kappa]$ is its semisimple ideal and its radical $\alpha$ equals its center (see [7]).

Reductive $\kappa$’s are characterized by having faithful semisimple representations. An involution $\theta$ on a Lie algebra $\kappa$ yields a direct sum decomposition $\kappa = \kappa_0 \oplus \nu_0$, with $\kappa_0 = \{X \in \kappa \mid \theta(X) = X\}$, $\nu_0 = \{X \in \kappa \mid \theta(X) = -X\}$.

A Lie group $K$ is reductive (see [17]) if its Lie algebra $\kappa$ is reductive and, moreover, there are an involution $\theta$ and an invariant bilinear form $b$ on $\kappa$ such that

(i) $\kappa_0 \perp \nu_b$ for $b$;

(ii) $b < 0$ on $\kappa_0$ and $b > 0$ on $\nu_0$;

(iii) $\kappa_0$ is the Lie algebra of a compact subgroup $K_0$ of $K$ and

\[
K_0 \times \nu_0 \ni (x, X) \mapsto x \cdot \exp(X) \in K
\]

is a diffeomorphism onto;

(iv) every automorphism $\text{Ad}(x)$ of the complexification $\kappa^C$ of $\kappa$, with $x \in K$, is inner, i.e. belongs to the analytic subgroup of the automorphism group of $\kappa^C$ having Lie algebra $\text{ad}(\kappa)$.

Then: $\theta$ is a Cartan involution, $\kappa = \kappa_0 \oplus \nu_0$ and (2.1) are Cartan decompositions, $K_0$ is the associated maximal compact subgroup, $b$ is the invariant bilinear form. The maximal compact subgroup $K_0$ of $K$ intersects all connected component of $K$ (see [17], Proposition 7.19]). In particular, $K$ has finitely many connected components.

2.2. Splittable Lie subalgebras. Let $\kappa$ be a reductive complex Lie algebra, and

\[
\kappa = z \oplus s, \quad \text{with} \quad z = \{X \in \kappa \mid [X, \kappa] = \{0\}\}, \quad s = [\kappa, \kappa]
\]

its decomposition into the direct sum of its center and its semisimple ideal. An element $X$ of $\kappa$ is semisimple if $\text{ad}(X)$ is a semisimple derivation of $\kappa$, and nilpotent if $X \in s$ and $\text{ad}(X)$ is nilpotent.

An equivalent formulation is obtained by considering a faithful matrix representation of $\kappa$ in which the elements of $z$ are diagonal: then semisimple and nilpotent elements correspond to semisimple and nilpotent matrices, respectively.

Each $X \in \kappa$ admits a unique Jordan-Chevalley decomposition

\[
X = X_s + X_n, \quad \text{with} \quad X_s \text{ semisimple, } X_n \text{ nilpotent, and } [X_s, X_n] = 0.
\]

A Lie subalgebra $\nu$ of $\kappa$ is splittable if, for each $X \in \nu$, both $X_s$ and $X_n$ belong to $\nu$.

If $\nu$ is a Lie subalgebra of $\kappa$, the set

\[
\eta_\kappa(\nu) = \{X \in \nu \mid X \text{ is nilpotent}\}
\]
is a nilpotent ideal of \( \mathfrak{v} \), with

\[
\text{rad}_n(\mathfrak{v}) = \text{rad}(\mathfrak{v}) \cap [\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{n}_\kappa(\mathfrak{v}) \subset \text{nil}(\mathfrak{v}),
\]

where \( \text{nil}(\mathfrak{v}) \) is the nilradical, i.e. the maximal nilpotent ideal of \( \mathfrak{v} \), and \( \text{rad}_n(\mathfrak{v}) \) its nilpotent radical, i.e. the intersection of the kernels of all irreducible finite dimensional linear representations of \( \mathfrak{v} \). Note that the nilpotent ideal \( \mathfrak{n}_\kappa(\mathfrak{v}) \), unlike \( \text{nil}(\mathfrak{v}) \) and \( \text{rad}_n(\mathfrak{v}) \), depends on the inclusion \( \mathfrak{v} \subset \kappa \) (cf. [6] §5.3). We recall

**Proposition 2.1** (see [6] §5.4). Every splittable Lie subalgebra \( \mathfrak{v} \) admits a Levi-Chevalley decomposition

\[
\mathfrak{v} = \mathfrak{n}_\kappa(\mathfrak{v}) \oplus \mathfrak{r},
\]

with \( \mathfrak{v} \), reductive and uniquely determined modulo conjugation by elementary automorphisms of \( \mathfrak{v} \), i.e. finite products of automorphisms of the form \( \exp(\text{ad}(X)) \), with \( X \in \mathfrak{v} \) and nilpotent. \( \square \)

### 2.3. Definition of \( \kappa \)-reductive.

Let \( \kappa \) be the complexification of a compact Lie algebra \( \kappa_0 \). Conjugation in \( \kappa \) will be understood with respect to its compact real form \( \kappa_0 \). Note that all Lie subalgebras of a compact Lie algebra are compact and hence reductive.

**Proposition 2.2.** For any complex Lie subalgebra \( \mathfrak{v} \) of \( \kappa \), the intersection \( \mathfrak{v} \cap \bar{\mathfrak{b}} \) is reductive and splittable. In particular, \( \mathfrak{v} \cap \bar{\mathfrak{b}} \cap \mathfrak{n}_\kappa(\mathfrak{v}) = \{0\} \). A splittable \( \mathfrak{v} \) admits a Levi-Chevalley decomposition with a reductive Levi factor containing \( \mathfrak{v} \cap \bar{\mathfrak{b}} \).

**Proof.** We recall that \( \mathfrak{v} \) is splittable if and only if its radical is splittable ([6] Ch.VII, §5, Théorème 2). In this case, \( \mathfrak{v} \) admits a Levi-Chevalley decomposition and all maximal reductive Lie subalgebras of \( \mathfrak{v} \) can be taken as reductive Levi factors. The intersection \( \mathfrak{v} \cap \bar{\mathfrak{b}} \) is reductive, being the complexification of the compact Lie algebra \( \mathfrak{v} \cap \kappa_0 \). Then the reductive Levi factor in the Levi-Chevalley decomposition of \( \mathfrak{v} \) can be taken to contain \( \mathfrak{v} \cap \bar{\mathfrak{b}} \) (see e.g. [26]). \( \square \)

**Notation 2.1.** In the following, for a complex Lie subalgebra \( \mathfrak{v} \) of \( \kappa \), we shall use the notation

\[
\mathfrak{L}_0(\mathfrak{v}) = \mathfrak{v} \cap \kappa_0, \quad \mathfrak{L}(\mathfrak{v}) = \mathfrak{v} \cap \bar{\mathfrak{b}}.
\]

**Definition 2.1.** Let \( \mathbf{K}_0 \) be a compact Lie group with Lie algebra \( \kappa_0 \) and \( M_0 \) a \( \mathbf{K}_0 \)-homogeneous \( CR \) manifold, with isotropy \( \mathbf{V}_0 \) and \( CR \) algebra \( (\kappa_0, \mathfrak{v}) \) at a point \( p_0 \in M_0 \). We say that \( M_0 \), and its \( CR \) algebra \( (\kappa_0, \mathfrak{v}) \), are \( \kappa \)-reductive if

\[
\mathfrak{v} = \mathfrak{n}_\kappa(\mathfrak{v}) \oplus \mathfrak{L}(\mathfrak{v}),
\]

i.e. if \( \mathfrak{L}(\mathfrak{v}) = \mathfrak{v} \cap \bar{\mathfrak{b}} \) is a reductive complement of \( \mathfrak{n}_\kappa(\mathfrak{v}) \) in \( \mathfrak{v} \).

**Remark 2.3.** If \( (\kappa_0, \mathfrak{v}) \) is \( \kappa \)-reductive, then \( \mathfrak{v} \) is splittable. Indeed all elements of \( \mathfrak{n}_\kappa(\mathfrak{v}) \) are nilpotent and all elements of \( \mathfrak{L}(\mathfrak{v}) \) are splittable, because \( \mathfrak{L}(\mathfrak{v}) \) is the complexification of \( \mathfrak{L}_0(\mathfrak{v}) \), which is splittable because consists of semisimple elements. Then \( \mathfrak{v} \) is splittable by [6] Ch.VII, §5, Théorème 1].

All submanifolds which are intersections of dual submanifold in the Matsuki duality, with the \( CR \) structure inherited by the embedding in the ambient flag manifold, are \( \kappa \)-reductive (see [11] §1]). We exhibit here an example of a compact homogeneous \( CR \) manifold \( M_0 \) which is not \( \kappa \)-reductive.
Example 2.4. Let $K_0 = SU(n), n \geq 3$. Fix a complex symmetric nondegenerate $n \times n$ matrix $S$ and consider the subgroup $V = \{ a \in SL(n, \mathbb{C}) \mid a^t S a = S \}$ of $SL(n, \mathbb{C})$, with Lie algebra $\mathfrak{v} = \{ X \in sl(n, \mathbb{C}) \mid X^t S + SX = 0 \}$. Set $V_0 = V \cap K_0$ and $M_0 = K_0 / V_0$. This is a $K_0$-homogeneous $CR$ manifold with $CR$ algebra $(\mathfrak{k}_0, \mathfrak{v})$, where $\mathfrak{k}_0 \simeq su(n), \mathfrak{v} \simeq so(n, \mathbb{C})$. If $S$ and $S^*$ are linearly independent, then $\mathfrak{v}$ is a semisimple Lie subalgebra of $\mathfrak{k}$ distinct from $\mathfrak{v} \cap \mathfrak{b}$.

The $CR$ manifolds of Definition 2.1 have canonical complex realizations:

Theorem 2.5 ([11, Theorem 4.3]). Let $M_0$ be an $\mathfrak{n}$-reductive $K_0$-homogeneous $CR$ manifold, with $CR$ algebra $(\mathfrak{k}_0, \mathfrak{v})$ and isotropy $V_0$ at some point $p_0 \in M_0$. Then there is a closed complex Lie subgroup $V$ of the complexification $K$ of $K_0$ with $K_0 \cap V = V_0$ and $\text{Lie}(V) = \mathfrak{v}$ such that the canonical map

$$M_0 \simeq K_0 / V_0 \longrightarrow M_- = K / V$$

is a generic $CR$ embedding.

Remark 2.6. Vice versa, if $M_- = K / V$ is the homogeneous complex manifold of the complexification $K$ of $K_0$, it is shown in [11, Prop.2.9] that any $K_0$-orbit $M_0$ of minimal dimension in $M_-$. with the $CR$ structure induced by the ambient space, is $\mathfrak{n}$-reductive.

3. Some Remarks on $SL_n(\mathbb{C}) / SU(n)$

Keep the notation of §2. As we explained in the introduction, we need to precise the structure of the fibers of the $K_0$-equivariant Mostow fibration $M_- \rightarrow M_0$.

Mostow fibration ([23, 24]) extends to homogeneous spaces the Cartan decomposition of reductive Lie groups. Both are related to the fact that the positive definite $n \times n$ Hermitian symmetric matrices with determinant one are the points of a Riemannian symmetric space $M_\mathfrak{n}$ with negative sectional curvature. We will discuss some topics on the geometry of $M_\mathfrak{n}$ (see e.g. [11]).

Any compact Lie group $K_0$ has, for some integer $n > 1$, a faithful linear representation in $SU(n)$, which extends to a linear representation $K \hookrightarrow SL_n(\mathbb{C})$. Thus decompositions in $SL_n(\mathbb{C})$ are preliminary to the general case.

The linear group $SL_n(\mathbb{C})$ has the Cartan decomposition

$$SU(n) \times \mathfrak{p}_0(n) \ni (x, X) \longrightarrow x \cdot \exp(X) \in SL_n(\mathbb{C}),$$

where $SU(n) = \{ x \in SL_n(\mathbb{C}) \mid x^t x = I_n \}$ is its maximal compact subgroup consisting of $n \times n$ unitary matrices with determinant one, and $\mathfrak{p}_0(n)$ the subspace of the traceless Hermitian symmetric $n \times n$ matrices in $\mathfrak{s}_0(\mathbb{C})$.

The quotient $M_\mathfrak{n} = SL_n(\mathbb{C}) / SU(n)$ is a symmetric space of the noncompact type and rank $(n-1)$, endowed with a Riemannian metric with negative curvature. We can identify $M_\mathfrak{n}$ with the set $\mathfrak{p}_0(n)$ of positive definite Hermitian symmetric matrices in $SL_n(\mathbb{C})$, which in turn is diffeomorphic to $\mathfrak{p}_0(n)$ via the exponential map. In this way $M_\mathfrak{n}$ can be considered as an open subset of $\mathfrak{p}_0(n)$ and its tangent bundle $T M_\mathfrak{n}$ is naturally diffeomorphic to the subbundle

$$T M_\mathfrak{n} = \{(p, X) \in M_\mathfrak{n} \times \mathfrak{p}(n) \mid p^{-1} X \in \mathfrak{p}_0(n)\}$$

of the trivial bundle $M_\mathfrak{n} \times \mathfrak{p}(n)$, where we set $\mathfrak{p}(n) = \{ X \in \mathbb{C}^{n \times n} \mid X^* = X \}$.

The special linear group $SL_n(\mathbb{C})$ acts on $M_\mathfrak{n}$ as a group of isometries, by

$$SL_n(\mathbb{C}) \times M_\mathfrak{n} \ni (z, p) \longrightarrow z p z^* \in M_\mathfrak{n},$$

where $z p z^*$ is the image of $p$ in $SL_n(\mathbb{C})$. This is seen as a generalization of the classical Mostow fibration $G / K \rightarrow G / P$ of a semisimple Lie group by a Cartan subgroup $K \subset P \subset G$ of $G$.
and $SU(n)$ is the stabilizer of the identity $e = I_n$, that we choose as the base point.

The metric tensor on $\mathcal{M}_n$ is
\[ (X, Y)_p = g_p(X, Y) = \text{trace}(p^{-1} X p^{-1} Y), \quad \forall p \in \mathcal{M}_n, \quad \forall X, Y \in T_p \mathcal{M}_n. \]

The curves \[ \mathbb{R} \ni t \rightarrow z \exp(tX)z^* \in \mathcal{M}_n, \quad \text{for } X \in \mathfrak{p}_0(n), \ z \in SL_n(\mathbb{C}) \] are the complete geodesics in $\mathcal{M}_n$ issued from $p = zz^*$ and
\[ \text{dist}(p_1, p_2) = \left( \sum_{i=1}^n |\log(\lambda_i(p_1^{-1} p_2))|^2 \right)^{1/2}, \]
where $\lambda_i(p_1^{-1} p_2)$ are the eigenvalues of the matrix $p_1^{-1} p_2$, which are real and positive, the Riemannian distance on $\mathcal{M}_n$.

3.1. **Killing and Jacobi Vector Fields.** Since $\mathcal{M}_n$ is a Riemannian symmetric space of $SL_n(\mathbb{C})$, the Lie algebra of its Killing vector fields is isomorphic to $\mathfrak{s}l_n(\mathbb{C})$. The correspondence is
\[ \mathfrak{s}l_n(\mathbb{C}) \ni Z \rightarrow \zeta_Z = \{ p \rightarrow Zp + pZ^* \} \in \mathcal{X}(\mathcal{M}_n). \]

For $H$ in $\mathfrak{p}_0(n)$, the restriction to $[0, 1]$ of the geodesic $t \rightarrow \gamma_H(t) = \exp(tH)$ is the shortest path from $e = \gamma_H(0)$ to $h = \exp(H) = \gamma_H(1)$. We will denote by $\mathcal{J}(H)$ the space of Jacobi vector fields on $\gamma_H$ and by $\mathcal{J}_0(H)$ its subspace consisting of those vanishing at $t = 0$. For each $Z \in \mathfrak{s}l_n(\mathbb{C})$, the restriction of $\zeta_Z$ to $\gamma_H$ is a Jacobi vector field, that we denote by $\theta_Z$:
\[ \mathbb{R} \ni t \rightarrow \theta_Z(t) = Z^* \exp(tH) + \exp(tH)Z \in \mathcal{J}(H). \]

To describe $\mathcal{J}(H)$ it is convenient to consider the commutator of $H$:
\[ \begin{cases} 
C(H) = \{ Z \in \mathfrak{s}l_n(\mathbb{C}) \mid [Z, H] = 0 \} = C_n(H) \oplus C_0(H), & \text{with} \\
C_n(H) = C(H) \cap \mathfrak{su}(n), & C_0(H) = C(H) \cap \mathfrak{p}_0(n).
\end{cases} \]

**Proposition 3.1.** The correspondence $\theta : \mathfrak{s}l_n(\mathbb{C}) \ni Z \rightarrow \theta_Z \in \mathcal{J}(H)$ is a linear map with kernel $C_n(H)$. For each $T \in C_0(H)$, $J(t) = t \cdot \theta_T(t)$ is a Jacobi vector field and
\begin{align*}
\mathcal{J}(H) & = \{ \theta_Z + t \cdot \theta_T \mid Z \in \mathfrak{s}l_n(\mathbb{C}), \ T \in C_0(H) \}, \\
\mathcal{J}_0(H) & = \{ \theta_Y + t \cdot \theta_T \mid Y \in \mathfrak{su}(n), \ T \in C_0(H) \}.
\end{align*}

Fix $Z \in \mathfrak{s}l_n(\mathbb{C})$ and $T \in C_0(H)$. Then
\[ J(t) = \theta_Z(t) + t \cdot \theta_T(t) = Z^* \exp(tH) + \exp(tH)Z + 2t \cdot T \cdot \exp(tH) \]
is the Jacobi vector field on $\gamma_H$ satisfying the initial conditions:
\begin{align*}
J(0) & = Z + Z^*, \\
\frac{d}{dt} J(0) & = \frac{1}{2} [H, Z - Z^*] + 2T,
\end{align*}
and we have
\begin{align*}
J(t) & = \frac{1}{2} \theta_{[H, Z]}(t) + 2T(t), \\
\frac{d^k}{dt^k} J(t) & = 2^{-k} \theta_{ad_H(Z)}(t), \quad \text{for } k \geq 2.
\end{align*}
Proof. If $T \in C_0(H)$, then $0_T$ is parallel and therefore also $t \cdot \theta_T$ is Jacobi on $\gamma_H$. To compute the covariant derivatives of the Jacobi vector field $J(t)$ defined in (3.3), we use the parallel transport $T_{\gamma_H} M_\theta \ni X \to \exp(sH/2)X \exp(sH/2) \in T_{\gamma_H} M_\theta$ along $\gamma_H$. Then

$$\dot{\theta}(t) = \left( \frac{d}{ds} \right)_{s=0} [\exp(-sH/2) [Z \exp((t + s)H) + \exp((t + s)H)Z] \exp(-sH/2)]$$

$$= \frac{1}{2}[Z', H] \exp(tH) + \frac{1}{2} \exp(tH) [H, Z] = \frac{1}{2} \theta(H, Z)(t).$$

By iteration we obtain (3.5) and, in particular, (3.4).

Finally, we need to show that all $J$ in $\gamma(H)$ have the form (3.3). Since $\text{ad}_H$ is semisimple, $s\ell_0(C)$ decomposes into the direct sum of its image and its kernel. Hence $p_0(n) = [H, su(n)] \oplus C_0(H)$, and this yields (3.1) and (3.2). \hfill \Box

For $X \in p_0(n)$, we will denote by $J_X$ the geodesic on $\gamma_H$ with

$$\begin{align*}
J_X(0) &= 0, \\
\dot{J}_X(0) &= X,
\end{align*}$$

while $0_X \in \gamma(H)$ satisfies $0_X(0) = 2X, \dot{0}_X(0) = 0$.

The nonconstant geodesics of a manifold with negative curvature have no conjugate points. Hence the map $\gamma(H) \ni J \to J(t) \in T_{\gamma(H)} M_\theta$ is a linear isomorphism for all $t \neq 0$. Moreover, for every $J \in \gamma(H)$, the real map $\gamma \to ||J(\gamma)||$ is non-negative and convex and therefore a nonzero $J(t) \in \gamma(H)$ vanishes for at most one value of $t \in \mathbb{R}$, corresponding to a minimum of $||J(t)||^2$ and thus to a solution of $(J(t))' = 0$.

**Lemma 3.2.** If $J \in \gamma(H)$ is not parallel along $\gamma_H$ and $(J(0), J(0)) = 0$, then $||J(0)|| < ||J(t)||$ for all $t \neq 0$.

**Lemma 3.3.** The quadratic form

$$||J||_{\gamma}^2 = \int_0^1 (1 - t) \left( ||J(t)||^2 + (J(t), J(t)) \right) dt$$

is positive semidefinite on $\gamma(H)$ and $||J||_{\gamma}^2 = 0$ if and only if $J(t) = 0$ for all $t$. The statement follows because $0_T \in C_0(H)$ is the space of the Jacobi vector fields that are parallel along $\gamma_H$.

**Lemma 3.4.** We have

$$||J(1)||^2 = ||J(0)||^2 + 2(J(0), J(0)) + 2 ||J||_{\gamma}^2, \quad \forall J \in \gamma(H).$$

**Proof.** We apply the integral form of the reminder in the first order Taylor’s expansion to $f(t) = ||J(t)||^2$.

For further reference, we state an easy consequence of Lemma 3.4.
Lemma 3.5. Let \( Z \in \mathfrak{s}_n(\mathbb{C}) \), \( X \in \mathfrak{p}_0(n) \), and \( \text{trace}(X \cdot Z) = 0 \). Then

\[
\|\theta_Z(1) - J_X(1)\|^2 = \|Z + Z^*\|^2 + 2(\mathcal{H}[Z, Z^*]) + \|\theta_Z - J_X\|_H^2.
\]

Proof. We apply (3.8) to \( J = \theta_Z - J_X \).

Then

\[
J(0) = Z + Z^*, \quad \dot{J}(0) = \frac{1}{2}[H, Z - Z^*] - X.
\]

yields

\[
\|\theta_Z(1) - J_X(1)\|^2 = \|J(1)\|^2 = \|Z + Z^*\|^2 + 2(\mathcal{H}[Z, Z^*]) + (J|J)_H
\]

\[
= \|Z + Z^*\|^2 + ([H, Z - Z^*]Z + Z^*) + (J|J)_H
\]

\[
= \|Z + Z^*\|^2 + 2(\mathcal{H}[Z, Z^*]) + (J|J)_H.
\]

Let \( J(t) = \theta_Z(t) + \theta_T(t) \), with \( Z \in \mathfrak{s}_n(\mathbb{C}) \) and \( T \in C_0(H) \). The two commuting Hermitian symmetric matrices \( H \) and \( T \) can be simultaneously diagonalized in an orthonormal basis of \( \mathbb{C}^n \). Let \( \lambda_1, \ldots, \lambda_m \) be the distinct eigenvalues of \( H \), with multiplicities \( n_1, \ldots, n_m \) and choose an orthonormal basis of \( \mathbb{C}^n \) to get matrix representations

\[
H = \begin{pmatrix} \lambda_1 I_{n_1} & & \\ & \lambda_2 I_{n_2} & \\ & & \vdots \\ & & & \lambda_m I_{n_m} \end{pmatrix}, \quad T = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_m \end{pmatrix}.
\]

Let us extend the trace norm of \( \mathfrak{p}_0(n) \) to a norm in \( \mathfrak{s}_n(\mathbb{C}) \), by setting

\[
\|A\| = \sqrt{\text{trace}(AA^*)} \geq 0, \quad \forall A \in \mathfrak{s}_n(\mathbb{C}).
\]

Then

\[
\|J(t)\|^2 = \text{trace}\left( Z^2 + Z^{*2} + 2e^{itH}Ze^{-itH}Z^* + 4t(Z + Z^*)T + 4t^2T^2 \right)
\]

\[
= \text{trace}\left( 2\Re \sum_{i,j=1}^{m} z_{i,j}z_{j,i} + 2\sum_{i,j=1}^{m} z_{i,j}z_{j,i}e^{(\lambda_i - \lambda_j)} + 8t\Re \sum_{i=1}^{m} \tau_i^2 + 4t^2\sum_{i=1}^{m} \tau_i^2 \right)
\]

\[
= 2\sum_{i\neq j} \|z_{i,j}e^{(\lambda_i - \lambda_j)/2} + z_{j,i}e^{(\lambda_i - \lambda_j)/2}\|^2 + 4t^2\sum_{i=1}^{m} \|2\tau_i + z_{i,i}\|^2.
\]

Set \( Z(t) = \exp(itH/2)Z \exp(-itH/2) = (z_{i,j}(t)) \), with \( z_{i,j}(t) = z_{i,j}e^{(\lambda_i - \lambda_j)/2} \in \mathbb{C}^{n \times n} \).

We obtain the expression

\[
\|J(t)\|^2 = \sum_{i\neq j} \|z_{i,j}(t) + z_{j,i}(t)\|^2 + \sum_{i=1}^{m} \|2\tau_i + z_{i,i}\|^2.
\]

If \( J(t) = 0 \), then each summand in (3.11) equals zero. For the terms in the first sum this amounts to the fact that \( [H, Z(t)] = [(\lambda_i - \lambda_j)z_{i,j}(t))]_{i \neq j} \) is Hermitian symmetric. Since \( [H, Z(t)] \) and \( [H, Z] \) are similar, we obtain:

Lemma 3.6. Let \( Z \in \mathfrak{s}_n(\mathbb{C}) \) and \( H \in \mathfrak{p}_0(n) \). A necessary condition in order that there exists \( T \in C_0(H) \) such that the Jacobi vector field \( J(t) = \theta_Z(t) + \theta_T(t) \) on \( \gamma_H \) vanishes at some \( t \in \mathbb{R} \) is that \( [H, Z] \) is semisimple with real eigenvalues.
Example 3.7. We consider the matrices

\[
H = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix} \in \mathfrak{sl}_3(\mathbb{R}), \quad Z = \begin{pmatrix}
0 & a & 0 \\
b & 0 & c \\
0 & d & 0
\end{pmatrix}, \quad Y = \begin{pmatrix}
-\alpha & 0 & \beta \\
0 & 0 & -\beta
\end{pmatrix}.
\]

We impose the conditions that \( Z \) be nilpotent and orthogonal to \( X = [H, Y] \) and that \( \theta_{Z,Y}(1) = 0 \). This translates into the set of equations

\[
\begin{cases}
ab + cd = 0, \\
(\lambda_2 - \lambda_1)(a\alpha + ba) + (\lambda_3 - \lambda_2)(c\beta + d\beta) = 0, \\
\alpha = (ae^{\lambda_1} + \tilde{b}e^{\lambda_2})/(e^{\lambda_2} - e^{\lambda_1}), \\
\beta = (ce^{\lambda_2} + \tilde{d}e^{\lambda_1})/(e^{\lambda_3} - e^{\lambda_2}).
\end{cases}
\]

By using the last two equations we reduce to the system

\[
\begin{pmatrix}
ab + cd = 0, \\
\frac{\lambda_2 - \lambda_1}{e^{\lambda_2} - e^{\lambda_1}}(a^2e^{\lambda_1} + ab(e^{\lambda_1} + e^{\lambda_2}) + |b|^2 e^{\lambda_2}) \\
+ \frac{\lambda_3 - \lambda_2}{e^{\lambda_3} - e^{\lambda_2}}(|c|^2 d^2 - ab(e^{\lambda_2} + e^{-\lambda_1 - \lambda_2}) + |d|^2 e^{\lambda_3}) = 0
\end{pmatrix}
\]

Assuming \( ab \neq 0 \) we obtain from the first equation \( d = -ab/c \) and, as \( \lambda_3 = -\lambda_1 - \lambda_2 \), the system reduces to

\[
(\ast) \quad \begin{pmatrix}
\frac{\lambda_2 - \lambda_1}{e^{\lambda_2} - e^{\lambda_1}}(a^2e^{\lambda_1} + ab(e^{\lambda_1} + e^{\lambda_2}) + |b|^2 e^{\lambda_2}) \\
+ \frac{\lambda_1 + 2\lambda_2}{e^{\lambda_2} - e^{-\lambda_1 - \lambda_2}}(|c|^2 d^2 - ab(e^{\lambda_2} + e^{-\lambda_1 - \lambda_2}) + |d|^2 e^{\lambda_3}) + \frac{|ab|^2}{c^2} e^{\lambda_3 - \lambda_2} = 0
\end{pmatrix}
\]

Let us restrict to the case where \( a, b, c \) are real. For any fixed \( a, b, c \) with \( ab \neq 0 \), the left hand side of (\ast) is positive when \( ab > 0 \) and \( |\lambda_1 + 2\lambda_2| \) is sufficiently small. Let us keep now \( \lambda_1 \) fixed and consider the left hand side of (\ast) as a real valued function \( f(\lambda_2) \) of the parameter \( \lambda_2 \). Then

\[
\lim_{\lambda_2 \to +\infty} \lambda_2^{-1} f(\lambda_2) = |b|^2 + |c|^2 - ab.
\]

If \( ab > 0 \), this is negative for \( |a| \gg 1 \). Then we can choose the parameters to satisfy (\ast). In conclusion: we can find \( H, Z, Y \) with \( H \in \mathfrak{p}_0(3) \), \( Z \in \mathfrak{sl}_3(\mathbb{C}) \) nilpotent, and \( Y \in \mathfrak{su}(3) \) with \( X = [H, Y] \in \mathfrak{p}_0(3) \) trace-orthogonal to \( Z \) such that \( \theta_{Z,Y}(0) \neq 0 \) and \( \theta_{Z,Y}(1) = 0 \).

Jacobi vector fields are used to compute the differential of the exponential map. In fact, for \( H, X \in \mathfrak{p}_0(n) \), the covariant derivative \( \frac{D}{dt} \exp(H + tX)|_{t=0} \) is the value at \( t = 1 \) of the Jacobian vector field \( J_X \in \mathfrak{h}_0(H) \). If \( X = [H, Y] + T \), with \( Y \in \mathfrak{su}(n) \) and \( T \in \mathcal{C}_0(H) \), then

\[
(3.12) \quad \frac{D}{dt} \exp(H + tX)|_{t=0} = J_X(1) = \exp([H, Y]) + T \exp(H).
\]
4. Decompositions with Hermitian fibers

4.1. Decomposition of $\mathbf{SL}_n(\mathbb{C})$. Throughout this section, $V$ is a closed complex Lie subgroup of $\mathbf{SL}_n(\mathbb{C})$, that admits a Levi-Chevalley decomposition $V = V_r \cdot V_n$, with $V_r$ algebraic reductive and $V_n$ unipotent (cf. [31] Ch. I, §6.5). We choose the embedding $V \hookrightarrow \mathbf{SL}_n(\mathbb{C})$ in such a way that $V_0 = V \cap \mathbf{SU}(n)$ is a maximal compact subgroup of $V$ and a real form of $V_r$ and set:

1. $v = \text{Lie}(V), \ v_r = \text{Lie}(V_r), \ v_n = \text{Lie}(V_n), \ v_0 = (v \cap \text{su}(n)) = \text{Lie}(V_0)$,

2. $m_0 = (v + v^*)^+ \cap p_0(n), \ v = v_0 \oplus v', \ \text{with} \ v' = (v \cap p_0(n)) \oplus v_n$.

Remark 4.1. We have $(v + v^*) \cap p_0(n) = [Z + Z^* | Z \in v]$. Indeed, if $Z_1, Z_2 \in v$ and $Z_1 + Z_2 \in v_0(n)$, then $Z = (Z_1 + Z_2)/2 \in v$ and $Z_1 + Z_2^* = Z + Z^*$. Hence the maps

\[ \begin{cases} \exists v' \ni Z \rightarrow (Z + Z^*) \in (v + v^*) \cap p_0(n), \\ \exists v' + m_0 \ni (Z, X) \leftrightarrow (Z^* + X + Z) \in p_0 \end{cases} \]

are $\mathbb{R}$-linear isomorphisms. Often we will write $Z \in v$ as a sum $Z = Z_0 + Z_n$, where it will be understood that $Z_0 \in (v \cap p_0(n))$ and $Z_n \in v_n$.

By (4.3), the Euclidean subspace $\exp(m_0)$ is a natural candidate for the typical fiber $F_0$ of an $\mathbf{SU}(n)$-covariant fibration of $\mathbf{SL}_n(\mathbb{C})/V$. As we will see, this is in fact the case for some important classes of $V$'s.

Being algebraic, $V$ admits the decomposition

\[ V_0 \times v' \ni (u, Z_0 + Z_n) \leftrightarrow u \cdot \exp(Z_0) \cdot \exp(Z_n) \in V, \]

which is a consequence of the Levi-Chevalley decomposition of $V$ and of the polar Cartan decomposition of $V_r$. Set

\[ N = \{ p \in M_0 | p = v^* v, \ \text{for some} \ v \in V \}. \]

Lemma 4.2. The map $v \mapsto v^* v$ defines, by passing to the quotients, an isomorphism

\[ V/V_0 \ni [v] \longrightarrow v^* v \in N. \]

Proof. In fact the right action $v \cdot \xi = v^* \cdot \xi \cdot v$ of $V$ on $N$ is transitive and $V_0$ is the stabilizer of $e = I_n$. \hfill \square

Lemma 4.3. The map

\[ v' \ni (Z_0 + Z_n) \longrightarrow \exp(Z_0^*) \cdot \exp(Z_0) \cdot \exp(Z_n) \in N \]

is a diffeomorphism. In particular, $N$ is diffeomorphic to a Euclidean space.

Proof. In fact, (4.7) is smooth and bijective and its inverse can be computed by using the diffeomorphisms $V/V_0 \ni v' \rightarrow v'$ of (4.4), and (4.5). \hfill \square

Lemma 4.4. We can find a real $r > 0$ such that the map

\[ \lambda : v' \times m_0 \ni (Z_0 + Z_n, H) \longrightarrow \exp(Z_0^*) \exp(Z_0) \exp(H) \exp(Z_0) \exp(Z_n) \in M_0 \]

is a diffeomorphism of $[\|H\| < r]$ onto $\{ p \in M_0 | \text{dist}(p, N) < r \}$.

Proof. By (4.3), $\lambda$ is a local diffeomorphism at all points where it has an injective differential. By using the isometries $p \rightarrow z^* \cdot p \cdot z$ of $M_0$, we may reduce to points $(0, H)$, where, to compute the differential, we can use the Jacobi vector fields $\partial_2$ and $J_2$ on $\gamma_H$, that where defined in (3.1). Indeed, for $(Z, X) \in v' \times m_0$, $d\lambda(0, H)(Z, 0) = \theta_2(1)$ and $d\lambda(0, H)(0, X) = J_2(1)$. Moreover, the maps $v' \ni Z \rightarrow \theta_2(1) \in T_{\exp(\gamma_H)} M_0$
and \( m_0 \ni X \to J_X(1) \in T_{\exp(H)}M_\nu \) both are injective. Thus it suffices to verify that \( \theta_Z(1) \neq J_X(1) \) when \( Z \) and \( X \) are not zero. By Lemma 3.5,

\[
|J_X(1) - \theta_Z(1)|^2 \geq |Z + Z'|^2 + 2(H[Z, Z']), \quad \forall (Z, X) \in v \times m_0.
\]

For \( Z \in v' \), we have \(|Z| = |Z'| \leq |Z + Z'|\). Thus

\[
|H[Z, Z']| \leq |H| \cdot |Z + Z'|^2.
\]

This implies that, for some \( r > 0 \), (4.8) defines a local diffeomorphism, and hence a smooth covering, of \( v' \times \{|H| < r\} \) onto \( \{p \in M_\nu \mid \text{dist}(p, N) < r\} \). This is in fact a global diffeomorphism because both spaces are simply connected. \(\square\)

Set

(4.9)

\[
V' = \{\exp(Z_0)\exp(Z_n) \mid Z_0 + Z_n \in v'\}
\]

and consider the map

(4.10)

\[
u : SU(n) \times m_0 \times V' \ni (u, X, v) \mapsto u \cdot \exp(X) \cdot v \in SL_n(\mathbb{C}).
\]

**Proposition 4.5.** The map (4.10) is onto.

There is a real \( r > 0 \) for which \( \mu \) is a diffeomorphism of \( \{||X|| < r\} \) onto the open manifold \( \{|\zeta| \in SL_n(\mathbb{C}) \mid \text{dist}(\zeta \cdot e, N) < 2r\} \).

**Proof.** The set \( N = \{z^n \mid z \in V\} \) is a properly embedded smooth submanifold of \( M_\nu \). Hence, for each \( p \in M_\nu \), there is a \( \zeta_p \in V \) with

\[
\text{dist}(p, \zeta_p^*-z_p) = \text{dist}(p, N).
\]

The geodesic joining \( \zeta_p^*-z_p \) to \( p \) has the form \([0, 1] \ni t \mapsto \gamma(t) = z_p^* \exp(tH)z_p \) for some \( H \in v_0(n) \), and \( \gamma(0) \) is orthogonal to \( N \) at \( \zeta_p^*-z_p \). The isometry \( q \to z_p^*-1qz_p^{-1} \) maps \( N \) into itself, \( \zeta_p^*-z_p \) to \( e \) and \( \gamma(0) \) to \( H \). Thus \( H \in T_cM_\nu = v_0(n) \) belongs to \( m_0 \).

This shows that, if \( \zeta \in SL_n(\mathbb{C}) \) and \( \zeta_p^*-z_p \) is the nearest point in \( N \) to \( p = \zeta \cdot \zeta_p \), then

\[
p = \zeta \cdot \zeta_p \exp(H)z_p, \quad \text{for some } z_p \in V \text{ and } H \in m_0.
\]

The matrix \( u = \zeta \cdot \zeta_p^{-1} \cdot \exp(-H) \) belongs to \( SU(n) \). Indeed

\[
u u = \exp(-H/2) \cdot [\zeta_p^{-1}]^* \cdot \zeta_p^{-1} \cdot \zeta \cdot \zeta_p^{-1} = \exp(-H/2) \cdot [\zeta_p^{-1}]^* \cdot \zeta_p^{-1} \cdot \exp(H) \cdot z_p \cdot z_p^{-1} \cdot \exp(-H/2) = I_n.
\]

Since \( \zeta = u \cdot \exp(H/2) \cdot z_p \), this proves that (4.10) is onto.

The second part of the statement is then a consequence of Lemma 4.4. \(\square\)

**Corollary 4.6.** The map

(4.11)

\[
SU(n) \times m_0 \ni (x, X) \mapsto \pi(x \cdot \exp(X)) \in SL_n(\mathbb{C})/V,
\]

where \( \pi : SL_n(\mathbb{C}) \to SL_n(\mathbb{C})/V \) is the projection onto the quotient, is onto. By passing to the quotient, it defines a surjective smooth map

(4.12)

\[
SU(n) \times_{V_0} m_0 \to SL_n(\mathbb{C})/V,
\]

where \( SU(n) \times_{V_0} m_0 \) is the quotient of \( SU(n) \times m_0 \) modulo the equivalence relation

\[
(x, X) \sim (x \cdot u, u^*Xu) \quad \text{for } x \in SU(n), X \in m_0 \text{ and } u \in V_0.
\]

\(\square\)
4.2. Decomposition of \(K\). Let \(V\) be a closed subgroup of the complexification \(K\) of a compact Lie group \(K_0\). We can assume that in turn \(K\) is a linear subgroup of \(\text{SL}_n(\mathbb{C})\), with \(K_0 = K \cap \text{SU}(n)\), and \(V_0 = V \cap \text{SU}(n)\) a maximal compact subgroup of \(V\). We obtain:

**Proposition 4.7.** With \(f_0 = m_0\cap X\), we have the commutative diagram with surjective arrows

\[
\begin{array}{ccc}
K_0 \times f_0 & \longrightarrow & K_0 \times V_0 \cap \mathfrak{f}_0 \\
\downarrow & & \downarrow \\
K/V & \longrightarrow & K_0/V.
\end{array}
\]

where the horizontal arrow is the projection onto the quotient, the left one is obtained by restricting (4.11), and the right one by passing to the quotient.

We denoted by \(K_0 \times V_0 \cap \mathfrak{f}_0\) the quotient of the product \(K_0 \times f_0\) by the equivalence relation \((x, X) \sim (x \cdot u, \text{Ad}(u^{-1})(X))\) for \(x \in K_0\), \(X \in f_0\) and \(u \in V_0\). The right arrow maps the equivalence class of \((x, X)\) to \(\pi(x \cdot \exp(X)) \in K/V \subset \text{SL}_n(\mathbb{C})/V\).

**Proof.** It is sufficient to follow the proof of Proposition 4.5 and check that, for \(\zeta \in K\), we obtain \(X \in f_0\) and \(x \in K_0\).

In fact, in this case, \(\zeta^*\zeta = z^* \exp(2X)z \in K \cap P_0(n)\), with \(z \in V\), implies that \(\exp(2X) = z^{-1}\zeta^*\zeta \in \exp(m_0) \cap K = \exp(f_0)\). \(\square\)

We have the analogous of Proposition 4.5.

**Proposition 4.8.** The map

\[
(4.14) \quad K_0 \times f_0 \times V' \ni (u, X, v) \longrightarrow u \cdot \exp(X) \cdot v \in K
\]

is always surjective and there is \(r_0 > 0\) such that, for all \(0 < r \leq r_0\), it is a diffeomorphism of \(||X|| < r\) onto a tubular neighborhood of \(M_0 = K_0/V_0\) in \(M_\ast\).

It is known that the right arrow in (4.13) is the Mostow fibration of \(K/V\) when \(V\) is reductive (see e.g. 23, 29). We give here a simple proof relying on the preparation done in 3.

**Proposition 4.9.** If \(V\) is reductive, then the natural surjective map

\[
(4.15) \quad K_0 \times V_0 \cap \mathfrak{f}_0 \to M_\ast = K/V
\]

is a diffeomorphism.

**Proof.** In this case \(V\), being algebraic and self-adjoint, has the Cartan decomposition \(V = V_0 \times \exp(v')\), with \(v' = v \cap P_0(n)\). By Lemma 3.2, the map

\[
\lambda_\kappa : v' \times f_0 \ni (Z, H) \longrightarrow \exp(Z') \cdot \exp(H) \cdot \exp(Z) \in K \cap P_0(n)
\]

is surjective. Moreover, it is a local diffeomorphism at every point of \(v' \times f_0\). In fact, we can reduce to prove this fact at points \((0, H)\), where the differential at \((Z, X)\) is \(J(1)\) for \(J(i) = \theta_Z + J_X \in \mathfrak{g}(H)\). Then \(||J(1)|| \geq ||J(0)|| = 2||Z|| > 0\) for \(Z \neq 0\), while \(J_X(1) \neq 0\) if \(X \neq 0\). Since \(\kappa \cap P_0(n) = v' \oplus f_0\), this proves that \(d\lambda_\kappa(0, H)\) is a linear isomorphism. Thus, being a connected covering of a simply connected space, \(\lambda_\kappa\) is a global diffeomorphism.

Hence, for every \(\zeta \in K\), there is a unique pair \((Z, H) \in v' \times f_0\) such that

\[
\zeta^* \cdot \zeta = \exp(Z') \cdot \exp(H) \cdot \exp(Z);
\]
then \( u = \zeta \cdot \exp(-Z) \cdot \exp(-\frac{1}{2}H) \in K_0 \) and we obtain the direct product decomposition

\[ (4.16) \quad K = K_0 \cdot \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{n}') , \]

from which the statement follows.

The complex \( K \)-homogeneous \( M_- \) of Proposition 4.9 corresponds to an \( M_- \) which is the Stein complexification of a totally real \( K_0 \)-homogeneous compact \( M_0 \). An \( M_0 \) having a positive CR dimension corresponds to a \( V \) having a nontrivial unipotent radical.

Before investigating cases where, even though \( v_n \neq 0 \), \((4.15)\) is nevertheless a diffeomorphism, we observe that, when we know that decomposition \((4.10)\) is unique, we can extract some extra information from the minimal distance characterization of \( z_0^*z_0^* \) in the proof of Proposition 4.5. For instance, as a corollary of Proposition 4.5, we obtain the following

**Proposition 4.10.** For \( h \in \mathfrak{p}_0(n) \), denote by \( D_\ell(h) \) the minor determinant of the first \( \ell \) rows and columns of \( h \). Set \( D_0(h) = 1 \) and let \( 0 < \lambda_1(h) \leq \cdots \leq \lambda_n(h) \) be the eigenvalues of \( h \). Then

\[ (4.17) \quad \text{dist}(h, e) = \sum_{\ell=1}^{n} |\log(\lambda_\ell(h))|^2 \geq \sum_{\ell=1}^{n} |\log(D_\ell(h)/D_{\ell-1}(h))|^2. \]

If \( h \) is not diagonal, we have strict inequality.

**Proof.** We take \( V \) equal to the group of unipotent upper triangular matrices in \( GL_n(\mathbb{C}) \). The element \( \delta = e^\Delta \in N_h = \{ e^z \mid z \in V \} \), with \( \Delta \in \mathfrak{p}_0 \), at minimal distance from \( e \) satisfies \( \text{trace}([Z + Z^*] \Delta) = 0 \) for all nilpotent upper triangular \( Z \) and hence is diagonal. The unique diagonal \( \delta = z^*h_z \) in \( N_h \) is the one obtained by the Gram-Schmidt orthogonalization procedure. The proof is complete. \( \square \)

The orbit of a point \( p \in \mathcal{M}_n \) by the group of unipotent upper triangular matrices of \( SL_n(\mathbb{C}) \) is an example of a horocycle of maximal dimension in a symmetric space of noncompact type. We will generalize this situation while outlining a class of subgroups \( V \) for which \( F_0 = \exp(\mathfrak{f}_0) \) can be taken as the fiber of the \( K_0 \)-covariant fibration.

Following [32, p.17], we call horocyclic in \( K \) the nilpotent subalgebras which are nilradicals of parabolic subalgebras of \( K \).

**Lemma 4.11.** Let \( q \) be a parabolic subalgebra of \( \mathfrak{s}_n(\mathbb{C}) \), with nilradical \( q_n \). Assume that \( q \cap q^* \) is a reductive Levi factor of \( q \). Let \( H \in q \cap \mathfrak{p}_0(n) \). Then, for \( Z_0 \in q \cap q^* \), \( T \in C_0(\mathfrak{h}) \cap q \) and \( Z_n \in q_n \) the Jacobi vector fields \( J_1 = \theta Z_0 + \theta T \) and \( J_2 = \theta Z_n \) are orthogonal at all points of \( \gamma_H \).

**Proof.** We show, separately, that \( \theta Z_0 \) and \( \theta T \) are both orthogonal to \( \theta Z_n \) at all points of \( \gamma_H \). We have

\[
(\theta_T(t))Z_0(t) = \text{trace}(2Te^{-iH}(e^{iH}Z_n + Z_ne^{iH})) = 2\text{trace}(TZ_n + TZ_n^*) = 0, \\
(\theta_Z_0(t))Z_0(t) = \text{trace}((e^{-iH}Z_0^* + Z_0e^{-iH})(e^{iH}Z_n + Z_n^*e^{iH})) \\
= \text{trace}(Z_0^*(e^{iH}Z_n e^{-iH}) + Z_0^*Z_n^* + Z_0Z_n + (e^{iH}Z_0e^{-iH})Z_n^*) = 0
\]

because \( q \cap q^* \) and \( q_n \) are orthogonal for the trace form of the canonical representation of \( \mathfrak{s}_n(\mathbb{C}) \). Indeed, The expression in the last line is twice the sum of the real parts of the product of \( Z_0 \) and \( Z_n \) and of \( e^{-iH}Z_0^*e^{iH} \in q \cap q^* \) and \( Z_n \). \( \square \)
Proposition 4.12. If \( \nu_n \) is horocyclic in \( \kappa \), then
\[
(4.18) \quad \nu' \times \nu_0 \ni (\nu, H) \mapsto \nu' \exp(H) \nu \in \mathcal{M}(K) = \mathcal{P}_0(n) \cap K 
\]
is a diffeomorphism.

Proof. In fact, we can find a parabolic \( q \) in \( \mathfrak{s\mathfrak{l}}_n(\mathbb{C}) \) such that \( q \cap q^* \) is its reductive Levi factor and \( \nu_n = q_n \cap \kappa \). Then we can reduce to proving the proposition in the case where \( K = \mathbf{S}\mathbf{L}_n(\mathbb{C}) \) and \( \nu_0 = m_0 \). We want to show that (4.18) is a local diffeomorphism. To this aim, with the notation of (3.11) it suffices to prove that, for \( Z \in \nu \) and \( H, X \in m_0 \), we have \( \theta_Z(1) \neq J_X(1) \) when \( Z + X \neq 0 \). We split \( Z \) into the sum \( Z = Z_0 + Z_n \), with \( Z_0 \in \nu \cap \nu_0(n) \) and \( Z_n \in \nu_n \). Then the fact that \( \theta_{Z_0}(1) + J_X(1) \neq 0 \) if \( Z_0 + X \neq 0 \) follows from Lemma 3.2 because of Lemma 4.11. Hence (4.8) is a connected covering of a simply connected manifold and thus a global diffeomorphism.

Proposition 4.12 can be slightly generalized. It was shown in [22, p.251] that there is a unique maximal complex Lie subalgebra \( \omega \) of \( \kappa \) with \( \nu \subseteq \omega \subseteq \nu + \overline{\nu} \). The \( CR \)-algebra \( (\kappa_0, \omega) \) and the corresponding \( K_0 \)-homogeneous \( CR \) manifold \( M_0 \) are called weakly nondegenerate \( K \)-algebra (\( \kappa_0 \), \( \omega \)).

Proposition 4.13. Let \( \omega \) be the largest complex Lie algebra with \( \nu \subseteq \omega \subseteq \nu + \overline{\nu} \). If \( \omega_n = \pi(\omega) \) is horocyclic in \( \kappa \), then (4.18) is a diffeomorphism.

Proof. As above, we reduce the proof to the case where \( K = \mathbf{S}\mathbf{L}_n(\mathbb{C}) \). The proof follows the same pattern of the proof of Proposition 4.12. We denote by \( q \) a parabolic Lie subalgebra of \( \mathfrak{s\mathfrak{l}}_n(\mathbb{C}) \) with \( q_n = w_n \) and use the notation of (3.11). We need to prove that, for \( Z \in \nu' = (\nu \cap \nu_0(n)) \oplus q_n \) and \( X, H \in m_0 \), we have \( \theta_Z(1) + J_X(1) \neq 0 \) if \( Z + X \neq 0 \). To this aim it is convenient to split \( Z \) into a sum \( Z = U + W \), with \( U \in \nu' \cap \omega \cap \overline{\omega} \) and \( W \in q_n \). Let us consider first \( J = \theta_U + J_X \). We note that \( J(0) = X + \frac{1}{2}[X, U - U^*] \) is orthogonal to \( J(0) = U + U^* \). Indeed \( (X|U + U^*) = 0 \) because \( \omega + \overline{\omega} = \nu + \overline{\nu} \) and, since \( [U, U^*] \in \omega \cap \nu_0(n) \),
\[
(U + U^*[H, U - U^*]) = 2\text{trace}(H \cdot [U, U^*]) = 0.
\]
By Lemma 3.2 this implies that \( J(1) \neq 0 \) if \( Z + X \neq 0 \). Finally, we note that \( \theta_W(0) \) and \( \theta_W(0) \) are orthogonal to both \( J(0) \) and \( J(0) \) to conclude, using again Lemma 3.2, that \( J_Z(1) + J_X(1) = J(1) + J_W(1) \neq 0 \) when \( X + Z = (X + U) + W \neq 0 \).

This shows that (4.18), being a connected smooth covering of a simply connected manifold, is a global diffeomorphism.

By using the argument in the proof of Proposition 4.9, we conclude:

Theorem 4.14. Let \( \omega \) be the largest complex Lie algebra with \( \nu \subseteq \omega \subseteq \nu + \overline{\nu} \). If \( \omega_n = \pi(\omega) \) is horocyclic in \( \kappa \), then (4.15) is a global diffeomorphism and therefore we obtain the \( K_0 \)-equivariant Mostow fibration of \( M_- \) over \( M_0 \)
\[
(4.19) \quad K_0 \times \nu_0 \ni_0 \rightarrow M_-
\]
with Hermitian fiber.
We keep the notation of §2.3 and denote by \( w \) the largest Lie subalgebra of \( \kappa \) with 
\[
(4.20) \quad v \subseteq w \subseteq v + \overline{v}.
\]

**Definition 4.1.** We say that \((\kappa_0, v)\) is HNR if \( w_n = n(w) \) is horocyclic.

For further reference, we reformulate the result obtained so far in the following form.

**Theorem 4.15.** If \((\kappa_0, v)\) is HNR, then we have the direct product decomposition
\[
(4.21) \quad K = K_0 \cdot \exp(f_0) \cdot V'. \quad \Box
\]

**Example 4.16.** Minimal orbit of \( SU(2, 2) \) in \( f_{1,2}(C^4) \).

We fix in \( C^4 \) the Hermitian form associated to the matrix
\[
\begin{pmatrix}
I_2 \\
-I_2
\end{pmatrix}
\]
We let the corresponding group \( SU(2, 2) \) operate on the flag manifold \( f_{1,2}(C^4) \), consisting of the pairs \((\ell_1, \ell_2)\) of a line \( \ell_1 \) and a 2-plane \( \ell_2 \) with \( 0 \in \ell_1 \subset \ell_2 \subset C^4 \).

The minimal orbit is
\[
M_0 = \{(\ell_1, \ell_2) \mid \ell_1 \subset \ell_2 = \ell_2^2\},
\]
where the orthogonal is taken with respect to the fixed Hermitian form. It is the total space of a \( \mathbb{CP}^1 \)-bundle over a smooth real manifold and in particular is Levi-flat of \( CR \) dimension 1. With \( K_0 = SL(2) \times U(2) \), \( K = SL(2) \times GL(2) \), the stabilizer
\[
V = \left\{ \begin{pmatrix}
\alpha & 0 \\
0 & \alpha
\end{pmatrix} \right\} \quad a \in ST^1_1(C),
\]
of the base point \( p_0 = \langle (e_1 + e_3), (e_1 + e_2, e_2 + e_4) \rangle \) (here \( T^1_1(C) \) is the group of upper triangular \( 2 \times 2 \) complex matrices with non-vanishing determinant and \( ST^1_1(C) \) its normal subgroup consisting of those having determinant 1) has Lie algebra
\[
v = \left\{ \begin{pmatrix}
\lambda & \alpha \\
0 & -\lambda
\end{pmatrix} \right\} \quad \lambda, \alpha \in \mathbb{C},
\]
Clearly \( v_n \) is not horocyclic. We note that
\[
w = v + \overline{v} = \left\{ \begin{pmatrix}
X & 0 \\
0 & \overline{X}
\end{pmatrix} \right\} X \in sl_2(\mathbb{C}) = v'
\]
is a complex Lie algebra. Thus, although \( V \) is not HNR, nevertheless we have a Mostow fibration with Hermitian fibers by Theorem 4.14.

**Remark 4.17.** Example 3.7 shows that \((4.18)\) is not, in general, a diffeomorphism when \((\kappa_0, v)\) is not HNR.

5. **Mostow fibration in general and the HNR condition**

5.1. **The set \( \Psi_0(v) \).** To better understand the notion introduced in Definition 4.11 and to characterize the fiber of the Mostow fibration of \( M_- \) on \( M_0 \) in general, it is convenient to rehearse some notions that were introduced in §3. We simply assume, at the beginning, that \( \kappa \) is any reductive Lie algebra over \( \mathbb{C} \).
For a Lie subalgebra $\mathfrak{a}$ of $\mathfrak{k}$, let us denote by $\mathfrak{n}(\mathfrak{a})$ the ideal consisting of the ad$_{\mathfrak{a}}$-nilpotent elements of its radical. Starting from any splittable Lie subalgebra $\mathfrak{v}$ of $\mathfrak{k}$ we construct a sequence $\{\mathfrak{v}_0\}$ of Lie subalgebras by setting recursively

\begin{equation}
\begin{cases}
\mathfrak{v}_0 = \mathfrak{v}, \\
\mathfrak{v}_{h+1} = N_{\mathfrak{k}}(\mathfrak{n}(\mathfrak{v}_h)) = \{Z \in \mathfrak{k} | [Z, \mathfrak{n}(\mathfrak{v}_h)] \subset \mathfrak{n}(\mathfrak{v}_h)\}, \quad \forall h \geq 0.
\end{cases}
\end{equation}

Each $\mathfrak{v}_h$, with $h \geq 1$, is the normalizer in $\mathfrak{k}$ of the ideal of ad$_\mathfrak{v}$-nilpotent elements of the radical of $\mathfrak{v}_{h-1}$. It was shown in [1] that $\mathfrak{v}_h \subseteq \mathfrak{v}_{h+1}$ and $\mathfrak{n}(\mathfrak{v}_h) \subseteq \mathfrak{n}(\mathfrak{v}_{h+1})$ for all $h \geq 0$, and that the union $\mathfrak{e} = \bigcup_{h \geq 0} \mathfrak{v}_h$ is a parabolic subalgebra of $\mathfrak{k}$, with $\mathfrak{v} \subset \mathfrak{e}$ and $\mathfrak{n}(\mathfrak{v}) = \mathfrak{v}_0 \subset \mathfrak{n}(\mathfrak{e})$. We call $\mathfrak{e}$ the parabolic regularization of $\mathfrak{v}$. Hence

\begin{equation}
\Psi(\mathfrak{v}) = \{q \in \mathfrak{k} | q \text{ is parabolic in } \mathfrak{k} \text{ and } \mathfrak{v} \subset q, \mathfrak{n}(\mathfrak{v}) \subset \mathfrak{n}(q)\}
\end{equation}

is nonempty. Let us prove a general simple lemma on parabolic Lie subalgebras.

**Lemma 5.1.** If $\mathfrak{q}_1, \mathfrak{q}_2$ are parabolic Lie subalgebras of $\mathfrak{k}$, then the Lie subalgebra $\mathfrak{q} = \mathfrak{q}_1 \cap \mathfrak{q}_2 + \mathfrak{n}(\mathfrak{q}_1)$ is parabolic in $\mathfrak{k}$.

**Proof.** We know (see e.g. [6, Ch.VIII,Prop.10]) that $\mathfrak{q}_1 \cap \mathfrak{q}_2$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. If $\mathcal{R}$ is the corresponding set of roots, then each $\mathfrak{q}_i (i = 1, 2)$ decomposes into a direct sum

$$\mathfrak{q}_i = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}, \alpha(A) \geq 0} \mathfrak{k}_\alpha,$$

where $A_1, A_2 \in \mathfrak{h}_\mathbb{R}$ and, for each $\alpha \in \mathcal{R}$, $\mathfrak{k}_\alpha = \{Z \in \mathfrak{k} | [A, Z] = \alpha(A)Z, \forall A \in \mathfrak{h}_\mathbb{R}\}$ is the root space of $\alpha$.

Take $\epsilon > 0$ so small that $\epsilon \cdot |\alpha(A_2)| < \alpha(A_1)$ if $\alpha(A_1) > 0$. Then

$$\mathfrak{q} = \mathfrak{h} \oplus \sum_{\alpha \in \mathcal{R}, (\alpha(A_1) + \epsilon \alpha(A_2)) > 0} \mathfrak{k}_\alpha,$$

is parabolic. In fact, if $\mathcal{U}(\mathfrak{q}_i)$ are the $\mathfrak{h}$-invariant reductive summands of $\mathfrak{q}_i$ and $\mathfrak{n}(\mathfrak{q}_i)$ the ideals of nilpotent elements of their radicals, we have

$$\mathfrak{q} = (\mathcal{U}(\mathfrak{q}_1) \cap \mathcal{U}(\mathfrak{q}_2)) \oplus (\mathcal{U}(\mathfrak{q}_1) \cap \mathfrak{n}(\mathfrak{q}_2)) \oplus \mathfrak{n}(\mathfrak{q}_1).$$

From now on we assume that $\mathfrak{k}$ is the complexification of its compact real form $\mathfrak{k}_0$. Conjugation in $\mathfrak{k}$ will be understood with respect to $\mathfrak{k}_0$. Using parabolic regularization and Lemma 5.1 we obtain

**Proposition 5.2.** If $(\mathfrak{k}_0, \mathfrak{v})$ is n-reductive, then $\Psi(\mathfrak{v})$ contains a $q$ having a conjugation-invariant reductive Levi subalgebra.

**Proof.** We can take $q = (\epsilon \cap \mathfrak{h}) + \mathfrak{n}(\epsilon)$, for the parabolic regularization $\epsilon$ of $\mathfrak{v}$. □

This shows that, for an n-reductive $(\mathfrak{k}_0, \mathfrak{v})$, the set

\begin{equation}
\Psi_0(\mathfrak{v}) = \{ q \in \Psi(\mathfrak{v}) | q = (q \cap \mathfrak{h}) \oplus \mathfrak{n}(q) \}
\end{equation}

is nonempty. For $q \in \Psi_0(\mathfrak{v})$ we will use $\mathcal{U}(\mathfrak{q}) = q \cap \mathfrak{h}$. The parabolic regularization produces a small $\epsilon$ and a corresponding smaller $(\epsilon \cap \mathfrak{h}) \oplus \mathfrak{n}(\epsilon)$ in $\Psi_0(\mathfrak{v})$. We are however more interested in the maximal elements of $\Psi(\mathfrak{v})$. To explain the meaning of maximality, we prove (cf. [1] Proposition 20)

**Proposition 5.3.** If $(\mathfrak{k}_0, \mathfrak{v})$ is n-reductive and $q$ any maximal element of $\Psi_0(\mathfrak{v})$, then

\begin{equation}
q = \text{Lie}(\mathfrak{n}(\mathfrak{v}) + \mathcal{U}(\mathfrak{q})) \quad \text{and} \quad \mathfrak{n}(q) = \sum_{h} \text{ad}^h(\mathcal{U}(\mathfrak{q}))(\mathfrak{n}(\mathfrak{v})).
\end{equation}
\textbf{Proof.} Let \( q \in \Psi_0(v) \) and denote by \( z \) the center of \( \Psi(q) \). Being invariant under conjugation, it is the complexification of the Lie subalgebra \( 3_0 \) of a maximal torus \( t_0 \) of \( K_0 \). Set \( 3_0 = i3_0 \). Following the construction of Konstant in [18], we consider the set \( Z \) consisting of the nonzero elements \( v \) of the dual \( 3_0^\ast \) for which

\[ \kappa_v = \{ X \in \kappa \mid [Z, X] = v(Z)X, \forall Z \in 3_0 \} \neq \{ 0 \}. \]

This set \( Z \) shares many properties of the root system of a semisimple Lie algebra. With the scalar product defined on \( 3_0 \) by the restriction of the trace form of a faithful linear representation of \( \kappa \) and the corresponding dual scalar product on \( 3_0^\ast \), we have

\begin{enumerate}[(i)]
  \item \( v \in Z \implies -v \in Z \), and \( \overline{v} = \kappa_{-v} \),
  \item \( v_1, v_2, v_1 + v_2 \in Z \implies [\kappa_{v_1}, \kappa_{v_2}] = \kappa_{v_1 + v_2} \),
  \item \( v_1, v_2 \in Z \) and \( (v_1|v_2) > 0 \implies v_1 - v_2 \in Z \),
  \item \( \forall v \in Z, \kappa_v \) is an irreducible \( \mathfrak{L}(q) \)-module,
  \item \( n(q) = \sum_{\nu > 0} \kappa_\nu \), for some lexicographic order in \( Z \),
  \item \( \exists \) a basis \( \{ \mu_1, \ldots, \mu_\ell \} \subset Z \) of positive simple roots of \( 3_0^\ast \).
\end{enumerate}

The Lie subalgebra \( \text{Lie}(n(v) + \mathfrak{L}(q)) \) is contained in \( q \) and is a direct sum

\[ \text{Lie}(n(v) + \mathfrak{L}(q)) = \mathfrak{L}(q) \oplus \sum_{v \in Z} \kappa_v, \]

for a subset \( E \) of \( Z^+ = \{ v > 0 \} \). Assume that there is a positive simple root \( \mu_i \), which does not belong to \( E \). Since \( \mu_i \) is simple, \( q' = q \oplus \kappa_{-\mu_i} \) is still a parabolic Lie subalgebra. Let us show that it is an element of \( \Psi_0(v) \). We have

\[ q' = \mathfrak{L}(q') \oplus n(q'), \quad \mathfrak{L}(q') = \mathfrak{L}(q) \oplus \kappa_{\mu_i} \oplus \kappa_{-\mu_i}, \quad \text{and} \quad n(q') = \sum_{v \in Z \setminus \{ \mu_i \}} \kappa_v. \]

Note that \( \mathfrak{L}(q') = q' \cap q'\prime \). An element \( X \in n(v) \) can be written in a unique way as a sum \( X = \sum_{v \in E} X_v \) with \( X_v \in \kappa_v \). Then \( X \in n(q') \), because \( E \subset Z^+ \setminus \{ \mu_i \} \). This shows that \( n(v) \subset n(q') \), i.e. that \( q' \in \Psi_0(v) \). Thus, if \( q \) is maximal in \( \Psi_0(v) \), then \( \text{Lie}(n(v) + \mathfrak{L}(q)) \) contains all \( \kappa_{\mu_i} \) for \( i = 1, \ldots, \ell \) and thus is equal to \( q \), because \( (ii) \) and the fact that every positive root is a sum of simple positive roots yield that \( \text{Lie}(\sum_{i=1}^\ell \kappa_{\mu_i}) = n(q) \). Finally, it follows from the discussion above that \( n(q) \) is the \( \text{ad}(\mathfrak{L}(q)) \)-module generated by \( n(v) \).

\( \square \)

Analogously, we obtain

\textbf{Proposition 5.4.} If \( q \) is any maximal element of \( \Psi(v) \), then

\[ q = \text{Lie}(n(v) + \mathfrak{L}(q)), \]

for any reductive Levi factor \( \mathfrak{L}(q) \) of \( q \), and \( n(q) \) is the \( \text{ad}(\mathfrak{L}(q)) \)-module generated by \( n(v) \).

\( \square \)

5.2. \textbf{A remark on the HNR condition.} Assume that \( (K_0, v) \) is \( n \)-reductive and let \( Q \) be the parabolic subgroup of \( K \) corresponding to a \( q \) in \( \Psi_0(v) \). Let \( Q_n \) be the unipotent radical of \( Q \) and set \( V' = V \cdot Q_n \). Then \( V' \cap \overline{V} = V \cap \overline{V} \) and therefore the minimal \( K_0 \) orbits in \( M_0 = K/V \) and \( M'_0 = K/V' \) are diffeomorphic as \( K_0 \)-homogeneous manifolds: the \( CR \) algebras \( (K_0, v) \) and \( (K_0, v + q_0) \) define two \( CR \) structures on the same \( M_0 = K_0/V_0 \), the latter being \textit{stronger} than the first. These are the \( CR \) structures inherited from the embeddings \( M_0 \hookrightarrow M_0 \) and \( M_0 \hookrightarrow M'_0 \). Note that \( M'_0 \) is the basis of a complex fiber bundle \( M_0 \to M'_0 \), with Stein fibers.
bi-holomorphic to $\mathbb{C}^k$ for some nonnegative integer $k$ (cf. [1] Thm.30)). The choice of a maximal $q$ in $\Psi_0(v)$ leads to a minimal $v + q_n$, while a minimal $q \in \Psi_0(v)$ to a maximal $v + q_n$, defining, when $(x_0, v)$ is not HNR, a maximal $K_0$-homogeneous CR structure on $M_0$ which is HNR and stronger than the original one.

**Example 5.5.** Minimal orbit of $SU(2, 3)$ in $\mathcal{F}_{1,3}(\mathbb{C}^5)$.

We denote by $\mathcal{F}_{1,3}(\mathbb{C}^5)$ the flag manifold consisting of the pairs $(\ell_1, \ell_3)$ of a line $\ell_1$ and a 3-plane $\ell_3$ of $\mathbb{C}^5$ with $0 \in \ell_1 \subset \ell_3$. We fix the Hermitian symmetric form of signature $(2, 3)$ in $\mathbb{C}^n$, corresponding to the matrix

$$
\begin{pmatrix}
I_2 & \\
& -I_1
\end{pmatrix},
$$

and consider the minimal orbit for the action of the real Lie group $SU(2, 3)$ in $\mathcal{F}_{1,3}(\mathbb{C}^5)$:

$$M_0 = \{(\ell_1, \ell_3) \in \mathcal{F}_{1,3}(\mathbb{C}^5) \mid \ell_1 \subset \ell_1^+ \subset \ell_3\}.$$

Fix on $M_0$ the base point $p_0 = ((e_1 + e_3), (e_1 + e_3, e_2 + e_5)).$ Its stabilizer in $K$ is

$$V = \left\{ \begin{pmatrix}
\lambda_1 & z_1 & 0 \\
0 & \lambda_2 & z_2 \\
0 & 0 & \lambda_3
\end{pmatrix} \middle| \lambda_i, z_i \in \mathbb{C}, \lambda_1 \lambda_2 \lambda_3 = 1 \right\},$$

with Lie algebra

$$v = \left\{ \begin{pmatrix}
\lambda_1 & z_1 & 0 \\
0 & \lambda_2 & z_2 \\
0 & 0 & \lambda_3
\end{pmatrix} \middle| \lambda_i, z_i \in \mathbb{C}, 2\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \right\}.$$

The normalizer of $v_n$ in $K$ is the parabolic

$$q = \left\{ \begin{pmatrix}
\lambda_1 & z_1 & \alpha_1 & 0 \\
0 & \lambda_2 & \alpha_2 & z_2 \\
0 & 0 & \lambda_4 & z_3 \\
0 & 0 & 0 & \lambda_5
\end{pmatrix} \middle| \lambda_i, z_i, \alpha_i \in \mathbb{C}, \sum_{i=1}^5 \lambda_i = 0 \right\},$$

which is also a maximal element in $\Psi_0(v)$ and hence $(s(u(2) \times u(3)), v)$ is not HNR.

The Lie algebra

$$\tilde{v} = v + q_n = \left\{ \begin{pmatrix}
\lambda_1 & z_1 & 0 \\
0 & \lambda_2 & z_2 \\
0 & 0 & \lambda_3
\end{pmatrix} \middle| \lambda_i, z_i \in \mathbb{C}, 2(\lambda_1 + \lambda_2) + \lambda_3 = 0 \right\}$$

is the Lie algebra of the stabilizer $\tilde{V}$ in $K = S(GL_2(\mathbb{C}) \times GL_4(\mathbb{C}))$ of $p_0' = ((e_1 + e_3), (e_1 + e_3, e_4, e_1 + e_3, e_4, e_2 + e_5)).$ This corresponds to the intersection of the $SU(2, 3)$-orbit

$$M'_4 = \{((\ell_1, \ell_2, \ell_4) \in \mathcal{F}_{1,2,4}(\mathbb{C}^5) \mid \ell_1 = \ell_2 \cap \ell_4^+ \}$$

with

$$\dim(\ell_4 \cap \ell_4^+) = 1.$$
with its Matsuki dual $K$-orbit $M'_\infty$.

With $L_2 = \langle e_1, e_2 \rangle$ and $L_3 = \langle e_3, e_4, e_5 \rangle$, we have

$$M'_\infty = \left\{ (\ell_1, \ell_2, \ell_4) \mid \begin{array}{l}
dim(\ell_1 \cap L_2) = 0, \quad \dim(\ell_1 \cap L_3) = 0, \\
\dim(\ell_2 \cap L_2) = 0, \\
\dim(\ell_2 \cap L_3) = 1, \\
\dim(\ell_4 \cap L_2) = 1, \\
\dim(\ell_4 \cap L_3) = 2 \end{array} \right\}. \quad (5.1)$$

This shows that, in this case, the strengthening of the $CR$ structure on $M_0$ corresponds to considering the compact intersection with its Matsuki dual of an intermediate orbit in some complex flag manifold of the same complex semisimple Lie group (in this case of $SL_4(\mathbb{C})$).

**Proposition 5.6.** Assume that $(\kappa_0, \nu)$ is $n$-reductive. Then, if $\nu$ is a complex Lie subalgebra of $\kappa$ with $\nu \subseteq \nu \subseteq \nu \oplus \overline{\nu}$, then also $(\kappa_0, \nu)$ is $n$-reductive.

**Proof.** The reductive Lie group $\kappa$ has an invariant nondegenerate bilinear form $\beta$, which is real and negative definite on $\kappa_0$. We observe that, if the pair $(\kappa_0, \nu)$ is $n$-reductive, then $\nu_n = \nu \cap \nu^\perp$, where $\nu^\perp = \{ Z \in \kappa \mid \beta(Z, Z') = 0, \forall Z' \in \nu \}$, and that $\nu + \overline{\nu}$ has the direct sum decomposition

$$\nu + \overline{\nu} = \nu \oplus \overline{\nu}_n.$$

If $\nu$ is a complex Lie subalgebra with $\nu \subseteq \nu \subseteq \nu + \overline{\nu}$, then $\nu = \nu \oplus (\nu \cap \overline{\nu}_n)$. Since $\beta$ defines a duality pairing between $\nu_n$ and $\overline{\nu}_n$, we obtain the decomposition

$$\nu = (\nu \cap \overline{\nu}) \oplus \nu_n, \quad \text{with} \quad \nu_n = \nu_n \cap (\nu \cap \overline{\nu}_n)^\perp, \quad \nu \cap \overline{\nu} = (\nu \cap \overline{\nu}) \oplus (\nu \cap \overline{\nu}_n) \oplus (\overline{\nu}_n \cap \nu),$$

showing that also $(\kappa_0, \nu)$ is $n$-reductive. \hfill $\square$

**Remark 5.7.** If $(\kappa_0, \nu)$ is $n$-reductive, then $\nu$ is the Lie algebra of an algebraic Lie subgroup $V$ of $K$. This is the content of [1, Thm.26]. In particular, all Lie subalgebras $\nu$ with $\nu \subseteq \nu \subseteq \nu + \overline{\nu}$ are Lie($W$) for an algebraic Lie subgroup $W$ of $K$.

**Example 5.8.** Minimal orbit of $\text{SU}(2,3)$ in $\mathcal{F}_{1,2}(\mathbb{C}^5)$.

We partly use the notation of Example [Example]. Denote by $M_0$ the minimal orbit of $\text{SU}(2,3)$ in the flag $\mathcal{F}_{1,2}(\mathbb{C}^5)$ of nested lines and 2-planes.

$$M_0 = \{(\ell_1, \ell_2) \in \mathcal{F}_{1,2}(\mathbb{C}^5) \mid \ell_2 \subset \ell_2^\perp \}$$

is a $CR$ manifold of type $(3,4)$. It is the total space of a $\mathbb{CP}^1$-bundle on the $CR$ manifold $M'_0$ of isotropic 2-planes in the Grassmannian $G_{1,2}(\mathbb{C}^4)$, which has type $(2,4)$. The stabilizer $V$ of the base point $p_0 = ((e_1 + e_3), (e_1 + e_3, e_2 + e_4))$, has Lie algebra

$$V = \left\{ \begin{bmatrix} \lambda_1 & z_1 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} \lambda_1 & 0 & z_2 \\ 0 & \lambda_2 & z_3 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mid \begin{array}{l}
\lambda_i, z_i \in \mathbb{C} \\
2\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{array} \right\}.$$

The largest $q \in \Psi_0(\nu)$ has

$$q_0 = \left\{ \begin{bmatrix} 0 & z_1 \\ 0 & 0 & z_2 \\ 0 & 0 & z_3 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & z_4 \\ 0 & 0 \end{bmatrix} \right\}.$$
and hence $(s(\mathfrak{su}(2) \times \mathfrak{su}(3)), v)$ is not HNR. We note however that

$$w = \begin{cases} \lambda_1 & \zeta_1 \\ \zeta_2 & \lambda_2 \\ \zeta_1 & \zeta_1 \\ \zeta_2 & \zeta_2 \\ 0 & 0 \\ \lambda_3 & \end{cases} \begin{cases} \lambda_1, \zeta_1, \zeta_l \in \mathbb{C} \\ 2\lambda_1 + 2\lambda_2 + \lambda_3 = 0 \end{cases} \subset v + \mathfrak{f}$$

has a horocyclic $w_n$. The orthogonal $m_0$ of $v + \mathfrak{f}$ in $s(p(2) \times p(3))$ is

$$m_0 = \begin{cases} X \\ -X \\ 0 \end{cases} \mid X \in p(2)$$

and, according to Theorem 4.14 it can be used to describe the typical fiber of the Mostow fibration $M_+ \to M_0$ in this case.

5.3. Decomposition of unipotent Lie groups. A unipotent Lie group is a connected and simply connected Lie group $N$ having a nilpotent Lie algebra $\mathfrak{n}$. Then the exponential map $\exp : \mathfrak{n} \to N$ is a Lie subalgebra, using e.g. [30, Lemma 3.18.5] we conclude that (5.6) is a $\mathfrak{s}$-invariant Lie algebra and hence $(\mathfrak{n}^\circ)$ is also smooth, and $E$ is smooth, because $\exp(I \in \mathfrak{n})$ is a nontrivial normal subgroup of $\mathfrak{n}$. We note however that $\mathfrak{n}$ is not HNR. We conclude however that

$$I \times E \ni (X, x) \mapsto \exp(X) \cdot x \in N$$

is a diffeomorphism onto.

Proof. We argue by recurrence on the sum of the dimension $n$ of $\mathfrak{n}$ and the codimension $k$ of $\mathfrak{g}$ in $\mathfrak{n}$. The statement is indeed trivial when $n = 1$, or $k = 0$. If $k = 1$, then $\mathfrak{g}$ is an ideal in $\mathfrak{n}$ and has a 1-dimensional $S$-invariant complement $I$ in $\mathfrak{n}$. If $I$ is a Lie subalgebra, using e.g. [30, Lemma 3.18.5] we conclude that (5.6) is a diffeomorphism in this case.

Assume now that $k > 1$ and that the statement has already been proved for subalgebras $\mathfrak{g}$ of codimension lesser than $k$ or nilpotent Lie algebras $\mathfrak{n}$ of dimension lesser than $n$. Since $\mathfrak{n}$ is nilpotent, its center $\mathfrak{c}$ has positive dimension and is $S$-invariant. If $\mathfrak{c} \cap \mathfrak{n} = \{0\}$, then $A = \exp(\mathfrak{c} \cap \mathfrak{n})$ is a nontrivial normal subgroup of $N$. Since $\dim(N/A) < n$ and $S$ acts in a completely reducible way on $\mathfrak{n}/(\mathfrak{c} \cap \mathfrak{n})$, by the recursive assumption we can find an S-invariant linear complement I of $\mathfrak{c}$ in $\mathfrak{n}$ such that, for its projection $I'$ in $\mathfrak{n}/(\mathfrak{c} \cap \mathfrak{n})$, the map

$$f' : I' \times (E/A) \ni (X', x') \mapsto \exp(X') \cdot x' \in N/A$$

is a diffeomorphism. This implies that (5.6) is also a diffeomorphism. In fact, if $\zeta \in N$, by the surjectivity of $f'$ there is a pair $(X, x) \in I \times E$ such that $\exp(X) \cdot y = \zeta \cdot a$, for some $a \in A$. This shows that $\zeta = \exp(X) \cdot (y \cdot a^{-1})$ and therefore (5.6) is onto. If $\zeta = \exp(X_1) \cdot (x_1) = \exp(X_2) \cdot (x_2) \cdot a$, with $X_1, X_2 \in I, x_1, x_2 \in E$ and $a \in A$, then $X_1 = X_2 = X$ because the projection $I \to I'$ is a linear isomorphism. Moreover, the correspondence $\zeta \to X$ is $C^\infty$-smooth, because $f'^{-1}$ is smooth. Then $\zeta \to x = \exp(-X) \cdot \zeta \in E$ is also smooth, and $\zeta \to (X, \exp(-X)) \zeta$ yields a smooth inverse of (5.6).
If \( \epsilon \cap \epsilon = \{0\} \), then by the recurrence assumption, we can take an \( S \)-invariant linear complement \( I \) of \( \epsilon \) in \( n \) containing \( \epsilon \) and such that

\[
f' : (I/\epsilon) \times ((E \cdot C)/C) \ni (X', x') \mapsto \exp(X') \cdot x' \in N/C.
\]

is a diffeomorphism. We claim that, with this choice, \((5.6)\) is a diffeomorphism. Indeed, \((E \cdot C)/C \cong E\) and therefore for \( \zeta \in N \) there is a unique \( x \in E \), with \( x = \phi(\zeta) \) for a smooth function \( \phi : N \rightarrow E \), such that, for some \( Z \in \epsilon \) and \( Y \in I \),

\[
\zeta \cdot \exp(Z) = \exp(Y) \cdot x \Rightarrow \zeta = \exp(Y - Z) \cdot x.
\]

The exponential is a diffeomorphism of \( n \) onto \( N \). If we denote by \( \log : N \rightarrow n \) its inverse, we obtain \( X = Y - Z = \log(\zeta \cdot x^{-1}) \in I \) and

\[
N \ni \zeta \mapsto (\log(\zeta \cdot [\phi(\zeta)]^{-1}), \phi(\zeta)) \in I \times E
\]

is a smooth inverse of \((5.6)\). This completes the proof. \( \square \)

With the notation of the previous section, we will apply Proposition \(5.9\) to the case where \( N = Q_n \) and \( n = q_n \), for a minimal \( q \in \Psi(\omega) \), while \( \epsilon = v_n \) and \( S = \text{Ad}(V_0) \). Since \( V_0 \) is compact, its adjoint action on \( q_n \) is completely reducible.

### 5.4. Structure of the typical fiber

The quotient \( K/Q \) of \( K \) by a parabolic subgroup \( Q \) is compact and thus a homogeneous space of its compact form \( K_0 \). Thus

\[
K = K_0 \cdot Q.
\]

Set \( \kappa = \text{Lie}(K) \), \( q = \text{Lie}(Q) \), and choose \( K_0 \) to contain a maximal compact subgroup of \( Q \). Then \( Q \) has a Levi-Chevalley decomposition \( Q = L(Q) \cdot Q_n \), whose reductive factor \( L(Q) \) has Lie algebra \( \mathfrak{u}(q) = q \cap \mathfrak{c} \). The conjugation is taken with respect to the real compact form \( \kappa_0 \) and \( Q_n \) is the unipotent factor of \( Q \), with Lie algebra \( q_n \). We consider the Cartan decomposition \( \kappa = \kappa_0 \oplus \kappa_0^\perp \), with \( \kappa_0 = i \cdot \kappa_0 \). Using the Cartan decomposition of \( L(Q) \), we obtain the direct product decomposition

\[
Q = L(Q) \cdot \exp(n(q)) = L_0(Q) \cdot \exp(v_0 \cap a) \cdot \exp(n(q)).
\]

We keep the notation of the previous sections, with \( \omega \) the maximal complex Lie subalgebra with \( v \subseteq w \subseteq v + \overline{v} \) and take \( q \in \Psi(\omega) \). Then \( \epsilon = v + q_n \) is a Lie subalgebra of \( \kappa \) and the pair \((\kappa_0, \epsilon)\) has the HNR property. Set

\[
\tilde{I}_0 = v_0 \cap (v + q_n)^\perp.
\]

By \((4.21)\), we obtain the direct product decomposition

\[
K = K_0 \cdot \exp(f_0) \cdot \exp(v_n + q_n) \cdot \exp(v \cap v_0).
\]

We use Proposition \(5.9\) to decompose \( \exp(v_n + q_n) \): we can find an \( \text{Ad}(V_0) \)-invariant linear subspace \( I \) of \( (v_n + q_n) \) such that \( v_n + q_n = I \oplus v_n \) and

\[
I \oplus v_n \ni (X, Y) \mapsto \exp(X) \cdot \exp(Y) \in V_n \cdot Q_n = \exp(v_n + q_n)
\]

is a diffeomorphism. We obtained:

**Theorem 5.10.** Let \( \tilde{I}_0 \) and \( I \) be defined by \((5.9)\) and \((5.11)\). Then we have a direct product decomposition

\[
K = K_0 \cdot \exp(f_0) \cdot \exp(I) \cdot V',
\]

where \( V' = \exp(v_n) \cdot \exp(v \cap v_0) \).
Then $F_0 = \exp(\mathfrak{f}_0) \cdot \exp(\mathfrak{I})$, with the adjoint action of $V_0$, is the typical fiber of the Mostow fibration:

$$\mathcal{M}_\kappa = K/V = K_0 \times_{V_0} F_0.$$  

Lemma 5.11. If $N$ is a unipotent subgroup of $K$, then, for every $p \in \mathfrak{p}_0(n)$, the map

$$\mathfrak{n} \ni z \longrightarrow z^*p_0 = \{z^*p_0 \mid z \in \mathfrak{n}\}$$

is a diffeomorphism.

Proof. In fact the stabilizer $\text{Stab}(p)$ of $p$ for the right action

$$K \times \mathfrak{p}_0(\kappa) \ni (z, x) \rightarrow z \cdot x \cdot z \in \mathfrak{p}_0(\kappa)$$

of $K$ on $\mathfrak{p}_0(\kappa)$ is a compact group and hence has trivial intersection with $N$. Thus (5.14) is a diffeomorphism with the image, being the restriction to $N \simeq N/[e_K]$ of the diffeomorphism $K/\text{Stab}(p) \rightarrow \mathfrak{p}_0(\kappa)$.

Corollary 5.12. Fix $q \in \mathfrak{p}_0(\mathfrak{w})$ and let $\mathfrak{f}_0$ and $\mathfrak{l}$ be the corresponding subspaces of $\kappa$ of Theorem 3.19. Then the elements $X \in \mathfrak{f}_0$ and $Z \in \mathfrak{l}$ of the decomposition

$$\zeta = u \cdot \exp(X) \cdot \exp(Z) \cdot v, \quad \text{with} \quad u \in K_0, v \in \exp(\mathfrak{v}_0) \cdot \exp(\mathfrak{v} \cap \mathfrak{p}_0)$$

are obtained in the following way:

(a) $[0, 1] \ni t \rightarrow \exp(2tX)$ is the geodesic in $\mathfrak{p}_0(\kappa)$ joining $e_K$ to the unique point $p_0$ of $\mathcal{N}_{\zeta \cdot \zeta}$ at minimal distance from $e_K$;

(b) $Z$ is the unique element of $\mathfrak{l}$ such that $\exp(Z^*) \cdot p_0 \cdot \exp(Z)$ belongs to $N_{p_0} = \{z^* \cdot p_0 \cdot z \mid z \in \mathfrak{v}\}$.

Proof. Indeed the Mostow fibration of $M_\kappa = K/(\mathfrak{v} \cdot \mathfrak{q}_n)$ can be taken to have a hermitian typical fiber $\exp(\mathfrak{f}_0)$ and correspondingly we obtain a unique decomposition

$$\zeta = u \cdot \exp(X) \cdot \xi \cdot \exp(Y) \quad \text{with} \quad \xi \in \mathfrak{q}_n \quad \text{and} \quad Y \in \mathfrak{v} \cap \mathfrak{p}_0.$$  

The characterization of $X$ coming from the proof of Proposition 4.5 yields (a).

Next we consider $\mathfrak{p}_0 = \mathfrak{q}_n \cdot \exp(2X) \cdot \xi = \mathfrak{q}_n \cdot \xi$. By Lemma 5.11 and the choice of $\mathfrak{l}$ we know that the element $\mathfrak{p}_0$ of $\{z^* \cdot p_0 \cdot z \mid z \in \mathfrak{q}_n\}$ uniquely decomposes as a product $w^* \cdot \exp(Z^*) \cdot p_0 \cdot \exp(Z) \cdot w$ with $w \in \mathfrak{v}$ and $Z \in \mathfrak{l}$. This completes the proof.

6. Application to Dolbeault and CR cohomologies

The cohomology groups of the tangential Cauchy-Riemann complex on real-analytic forms on $M_0$ is the inductive limit of the corresponding Dolbeault cohomology groups of its tubular neighborhoods in $M_\kappa$. We know by [13] that in some degrees these groups coincide with those computed on tangential smooth forms or on currents. We will employ Andreotti-Grauert theory to compare the tangential $CR$ cohomology on $M_0$ with the corresponding global Dolbeault cohomology of $M_\kappa$. To this aim we will use the Mostow fibration $M_\kappa \rightarrow M_0$ to construct a non-negative exhaustion function for $M_\kappa$, vanishing on $M_0$, and having a complex Hessian whose signature reflects the pseudoconvexity/pseudoconcavity of $M_0$. In this way we prove relations of the $CR$ cohomology of $M_0$ with the Dolbeault cohomology of the $K$-orbit $M_\kappa$, similar to what J.A.Wolf did in [27] for the relationship of the open orbits $M_\kappa$ of a real form $G_0$ of a complex semisimple Lie group $G$ in a flag $M$ of $G$ with the structure of their Matsuki duals $M_\kappa = M_0$, which in this case are compact complex manifolds.
6.1. An Exhaustion Function for $M_-$. In [12] H. Grauert noticed that a real-analytic manifold admits a fundamental systems of Stein tubular neighborhoods in any of its complexifications. In fact, a homogeneous analogue of Grauert’s theorem is the fact that the complexification $K$ of a compact Lie group $K_0$ is Stein, and the isomorphism provided by the Cartan decomposition

$$K_0 \times \mathfrak{k}_0 \ni (x, X) \mapsto x \cdot \exp(iX) \in K$$

also yields the exhaustion function

$$K \ni x \cdot \exp(iX) \mapsto \|X\|^2 = -\kappa(X, X) \in \mathbb{R},$$

which is zero on $K_0$, positive on $K \setminus K_0$ and strictly pseudo-convex everywhere. Here and in the following we shall denote by $\kappa$ both the negative definite invariant form of a faithfull unitary representation of $\mathfrak{k}_0$ and its $\mathbb{C}$-bilinear extension to $\kappa$. When $K_0$ is semisimple, the adjoint representation is faithful and we may take as $\kappa$ the Killing form.

We proceed in a similar way to construct an exhaustion function on $M_-$ for the canonical embedding $M_0 \hookrightarrow M_-$ of a $\mathfrak{n}$-reductive $K_0$-homogeneous compact CR manifold $M_0$. We use the notation of the previous sections.

Assume that the pair $(\kappa_0, \nu)$ is $\mathfrak{n}$-reductive and HNR. We already noticed that the last condition is natural if we consider on $M_0$ maximal $K_0$-invariant CR structures. Then, by Corollary [5.12] we have a direct product decomposition

$$K = K_0 \cdot \exp(f_0) \cdot \exp(\nu_0) \cdot \exp(\nu \cap \nu_0)$$

with $\nu_0 = i \cdot \kappa_0$ and $f_0 = (\nu + \overline{\nu})^\perp \cap \nu_0$. Moreover, the exp($f_0$)-term in (6.1) is characterized by

$$\|X\| = \frac{1}{2} \text{dist}(\exp(u)X, N), \quad \text{for} \quad N = \{v^* \cdot v \mid v \in V\} \subset P_0(\kappa).$$

This is indeed a consequence of Corollary [5.12] when $l = \{0\}$.

By passing to the quotient, the map

$$K_0 \times \mathfrak{k}_0 \ni (x, X) \mapsto \|X\|^2 = \kappa(X, X) \in \mathbb{R}.$$ 

defines a smooth exhaustion function (as usual square brackets mean equivalence classes)

$$\Phi : M_- = K_0 \times_{\nu_0} \mathfrak{k}_0 \ni [x, X] \mapsto \|X\|^2 \in \mathbb{R}.$$ 

We have:

**Lemma 6.1.** If $(\kappa_0, \nu)$ is $\mathfrak{n}$-reductive and HNR, then the map $\Phi$ of (6.3) has the properties:

1. $\Phi \in C^\infty(M_-, \mathbb{R})$ and $\Phi \geq 0$ on $M_-;$
2. $\Phi^{-1}(0) = M_0$ and $d\Phi \neq 0$ if $\Phi > 0 ;$
3. $\Phi$ is invariant under the left action of $K_0$ on $M_- :$

$$\Phi(x \cdot p) = \Phi(p), \quad \forall p \in M_-, \forall x \in K_0. \quad \square$$

**Notation 6.1.** The level and sublevel sets of $\Phi$ will be denoted by

$$\Phi_c = \{p \in M_- \mid \Phi(p) = c\} \subset M_- \quad \text{and} \quad \Omega_c = \{p \in M_- \mid \Phi(p) < c\}.$$
6.2. $K_0$-Orbits in $M_\tau$. The level sets $\Phi_c$ are foliated by $K_0$-orbits. Since all points of $M_\tau$ have representatives of the form $x \cdot \exp(X)$ with $x \in K_0$ and $X \in f_0$, then every $K_0$-orbit intersects the fiber $F_0$ over the base point $p_0$ at a point $p_X = [\exp(X)]$, for some $X \in f_0$. An $x \in K_0$ stabilizes $p_X$ if and only if $x \cdot \exp(X)$ is still a representative of $p_X$, and this, by the equivalence relation defining $K_0 \times_{V_0} f_0$, means that $x \in V_0$ and $\Ad(x)(X) = X$. Indeed the equation $x \exp(X)z = \exp(X)$ with $z \in V$ implies, by the uniqueness of the Mostow decomposition, that $z = x^{-1} \in V_0$ and $x \exp(X)x^{-1} = \exp(\Ad(x)(X)) = \exp(X)$, yielding $\Ad(x)(X) = X$.

Thus the $K_0$-orbit

$$\Phi_c = \{x \cdot p_x = [x \cdot \exp(X)] \mid x \in K_0\}$$

in $M_\tau$ through $p_X$ can be identified with the homogeneous space $K_0/V_0$, where

$$V_0 = \{x \in V_0 \mid \Ad(x)(X) = X\},$$

is the stabilizer of $p_X$ in $K_0$. It is a closed Lie subgroup of $K_0$ with Lie algebra

$$\mathfrak{v}_X = \{Y \in \mathfrak{v}_0 \mid [Y, X] = 0\}.$$

**Lemma 6.2.** $M_X$ is a compact $K_0$-homogeneous $CR$-manifold with $CR$-algebra $(\mathfrak{v}_0, \Ad(\exp(X)(0)))$ at $p_X = [\exp(X)]$. \hfill $\Box$

**Remark 6.3.** In general, $M_X$ may not be diffeomorphic to $M_0$. Indeed, $M_0$ is a minimal $K_0$-orbit in $M_\tau$ and $M_X$ is diffeomorphic (and $CR$-diffeomorphic) to $M_0$ if and only if $M_X$ and $M_0$ have the same dimension.

For $X \in f_0$, the left translation $M_\tau \ni p \mapsto \exp(X) \cdot p \in M_\tau$ is a biholomorphism of $M_\tau$ which transforms $M_0$ into a $CR$-diffeomorphic submanifold

$$\tilde{M}_X = \exp(X) \cdot M_0.$$

**Lemma 6.4.** For $X \in f_0$, we have

$$\tilde{M}_X \subset \{\phi \leq \|X\|^2\} = \overline{\Omega_c}.$$

**Proof.** Let $\pi : K \ni \zeta \mapsto [\zeta] \in K/V \simeq M_\tau$ be the canonical projection. Any point of $M_0$ is $\pi(u)$ for some $u \in K_0$ and then the points $p$ of $M_X$ have the form $p = \exp(X)\pi(u) = \pi(\exp(X) \cdot u)$. Set $\zeta = \exp(X) \cdot u$. We know that $\phi(p)$ is the square of the half-distance in $\mathfrak{g}_0(K)$ from the base point $e_K$ to

$$N_{\zeta, \zeta} = \{v^* \cdot \zeta^* \cdot \zeta \cdot v \mid v \in V\}.$$

Since the point $(\zeta^*, \zeta)$ belongs to $N_{\zeta, \zeta}$ and has distance $2\|X\|$ from $e_K$, (in fact $t \mapsto u^* \cdot \exp(2tX) \cdot u$ is the geodesic joining $e_K$ to $(\zeta^*, \zeta)$), it follows that $\phi(p) \leq \|X\|^2$. \hfill $\Box$

We summarize:

**Proposition 6.5.** Let $c > 0$. Then

$$\Phi_c = \bigcup_{X \in f_0} M_X \text{ (disjoint union)}, \quad \tilde{M}_X \subset \{\phi \leq \|X\|^2\}, \quad \forall X \in f_0. \quad \Box$$

In particular, for $c > 0$, we can draw through each point of $\Phi_c$ a translate $\tilde{M}_X$ of $M_0$, which is $CR$-diffeomorphic to $M_0$ and tangent to $\Phi_c$ from inside, i.e. lying in $\overline{\Omega_c}$. This means that the boundary $U_c$ of $\Omega_c$ is at each point less convex than $M_0$. 

6.3. **Application to Dolbeault and CR Cohomologies I.** By Andreotti-Grauert theory (see [4]) we know that for every coherent sheaf \( F \) on an \( r \)-pseudoconcave complex manifold \( X \) we have

\[
\dim(H^j(F, M_0) < \infty, \quad \forall \ j < r - \text{hd}(F),
\]

where \( \text{hd}(F) \) is the homological dimension of \( F \).

We obtain the following:

**Theorem 6.6.** Let \( M_0 \) be a compact \( n \)-reductive homogeneous CR manifold, with \((\kappa_0, \nu)\) HNR and canonical complex embedding \( M_0 \hookrightarrow M_- \).

If \( M_0 \) is an \( r \)-pseudoconvave CR-manifold, then \( M_- \) is an \( r \)-pseudoconcave complex manifold and for every coherent sheaf \( F \) we have

\[
dim(H^j(M_0, F) \approx H^j(M_-, F)) < \infty, \quad \forall \ j < r - \text{hd}(F).
\]

In particular,

\[
dim(H^{p,j}(M_0) \approx H^{p,j}(M_-)) < \infty, \quad \forall \ j < r.
\]

Here we used the notation \( H^{p,j} \) for the \( \bar{\partial} \) and \( \bar{\partial}_{M_0} \)-cohomologies on forms of type \((p, *)\). Because of the validity of the Poincaré lemma in degree \( j \), for \( 0 < j < r \) (see [25]), they coincide with the Čech cohomology with coefficients in the sheaf of germs of \( CR \) or holomorphic \( p \)-forms. Moreover, in this range, the tangential Cauchy-Riemann complexes on currents, \( C^\infty \)-smooth forms and real-analytic forms on \( M_0 \) have isomorphic finite dimensional cohomology groups.

**Proof.** By the HNR assumption, the exhaustion function \( \phi \) in (6.3) is well defined. Then to verify (6.9) we can apply Andreotti-Grauert’s theory, after showing that, for \( c > 0 \), each subdomain \( \Omega_c = \{ \phi < c \} \) is \( r \)-pseudoconcave. To this aim, we prove that the complex Hessian of \( \phi \) admits at least \( r \) negative eigenvalues on the analytic tangent to \( \Phi_c = \partial \Omega_c \). By exploiting the \( K_0 \)-invariance of \( \phi \), we can, without any loss of generality, restrict our consideration to points \( p_0 = [\exp(X)] \in \Phi_c \), with \(|X|^2 = c \in \mathbb{R} \). We may consider \((0,1)\)-vector fields which are tangent to the submanifold \( \tilde{M}_X \), defined in (6.6) and that are also tangent to \( \partial \Omega_c \) at \( p_0 \), because \( \tilde{M}_X \) is tangent to \( \Phi_c \) at \( p_0 \). By Lemma 6.4 \( \tilde{M}_X \) is contained in \( \tilde{\Omega}_c = \{ \phi \leq |X|^2 \} \). Since \( \tilde{M}_X \) is \( CR \)-diffeomorphic to \( M_0 \), it is \( r \)-pseudoconcave. Being \( \tilde{M}_X \subset \tilde{\Omega}_c \), the restriction of the complex Hessian of \( \phi \) to the analytic tangent to \( \tilde{M}_X \) at \( p_0 \) has at least as many negative eigenvalues as the Levi form of \( \tilde{M}_X \) in the codirection \( J\partial \phi([\exp(X)]) \), which, by the assumption, are at least \( r \). This completes the proof.

\[ \square \]

6.4. **Application to Dolbeault and CR cohomologies II.** In this section we want to exploit the amount of pseudo-convexity of the exhaustion function \( \phi \). We keep the assumption that \((\kappa_0, \nu)\) is \( n \)-reductive and HNR and set \( q = \{ \mathbb{Z} \in \kappa \mid \mathbb{Z}, \nu \in \nu_0 \} \) for the maximal parabolic subalgebra \( \mathfrak{p}_0(\nu) \). We recall that \( \nu_0 = \mathfrak{q}_0 \) is the nilradical of \( q \). Let \( Q \) be the parabolic subgroup of \( K \) with \( \text{Lie}(Q) = q \) and \( Q_0 \) its conjugation-invariant reductive factor. Let \( \sigma : K \to M_- = K/V \) be the quotient map. The image of \( Q_0 \) by \( \sigma \) is a \( Q_0 \)-homogeneous complex submanifold \( Q_- \) of \( M_- \).

**Lemma 6.7.** For every \( X \in \mathfrak{i}_0 \), the CR manifold \( \tilde{M}_X \) and the complex manifold \( Q_- \) are transversal at \( p_X \) and their analytic tangent spaces at \( p_X \) are orthogonal for the complex Hessian of \( \phi \).
Proof. The pull-backs of $\tau_{0,1}^p M_X$ and $\tau_{0,1}^p Q_-$ to the base point $p_0$ by the bi-holomorphic map $p \to \exp(X) \cdot p$ are, respectively, $v_r$ and $a_r/(v \cap \mathcal{F})$. This is a consequence of the fact that $X \in a_r$. The statement follows from the fact that $a_r = \bar{a}_r$ and $[\bar{a}_r, v_r] \subset v_n$, $[a_r, \bar{a}_n] \subset \bar{a}_n$.

Theorem 6.8. Let $M_0$ be a compact $\mathfrak{r}$-reductive homogeneous CR manifold of type $(n, k)$, with $(K_0, \nu)$ HNR and canonical complex embedding $M_0 \hookrightarrow M_-$.

If $M_0$ is an $r$-pseudoconvex CR-manifold, then $M_- = n - r$-pseudoconvex complex manifold and for every coherent sheaf $\mathcal{F}$ we have

\[
\dim(H^j(M_0, \mathcal{F}) \simeq H^j(M_-, \mathcal{F})) < \infty, \quad \forall j > n - r.
\]

In particular,

\[
\dim(H^{p,j}(M_0) \simeq H^{p,j}(M_-)) < \infty, \quad \forall j > n - r.
\]

Proof. By [13] Theorem 2.1, under the $r$-pseudoconcavity assumption, the tangential CR cohomology groups on $M_0$ are the inductive limits of the corresponding groups of sheaf and Dolbeault cohomology of the tubular neighborhoods of $M_0$ in $M_-$. While computing the Levi form of $\phi$, it suffices to note that its restriction to $Q_-$ is strictly pseudo-convex, since it is the exhaustion function associated to the canonical CR-embedding $M_0 \cap N_+ \hookrightarrow N_+$. The orbits of the real form $G_0 = SL_{n+1}(\mathbb{C})$ are parametrized by the dimension of the intersection $\ell_p \cap \mathcal{F}_q$ : with $k_0 = \max\{0, p + q - n - 1\}$ we have the orbits

\[
M_+(k) = \{(\ell_p, \ell_q) \in \mathcal{G}_p(C^{n+1}) \times \mathcal{G}_q(C^{n+1}) \mid \dim(\ell_p \cap \mathcal{F}_q) = k\}, \quad k_0 \leq k \leq p.
\]

The complexification $K = SL_{n+1}(\mathbb{C})$ of the compact form $K_0 = SU(n + 1)$ acts on $\mathcal{G}_p(C^{n+1}) \times \mathcal{G}_q(C^{n+1})$ by

\[
a \cdot (\ell_p, \ell_q) = (a(\ell_p), a^{-1}(\ell_q)), \quad \forall a \in SL_{n+1}(\mathbb{C}), \quad \ell_p \in \mathcal{G}_p(C^{n+1}), \quad \ell_q \in \mathcal{G}_q(C^{n+1}).
\]

Consider the polarity $\mathcal{G}_h(C^{n+1}) \ni \ell \mapsto \ell_h^0 \in \mathcal{G}_{n+1-h}(C^{n+1})$ defined by the symmetric bilinear form

\[
b(v, w) = (\ell^0_v) \cdot v = \sum_{i=0}^n v_i w_i.
\]

Then the orbits of $K$ in $\mathcal{G}_p(C^{n+1}) \times \mathcal{G}_q(C^{n+1})$ are parametrized by:

\[
M_-(k) = \{(\ell_p, \ell_q) \in \mathcal{G}_p(C^{n+1}) \times \mathcal{G}_q(C^{n+1}) \mid \dim(\ell_p \cap \mathcal{F}_q) = p - k\}, \quad k_0 \leq k \leq p.
\]
The manifolds $M_+(k)$ and $M_-(k)$ are Matsuki-dual to each other. In fact, since $SU(n + 1)$ preserves Hermitian orthogonality in $\mathbb{C}^{n+1}$ and $\overline{T}_q$ and $\ell_q^0$ are Hermitian orthogonal in $\mathbb{C}^{n+1}$, the pair $(\ell_p, \ell_q)$ belongs to $M_0(k) = M_+(k) \cap M_-(k)$ iff

\[
\ell_p = (\ell_p \cap \overline{T}_q) \oplus (\ell_p \cap \ell_q^0), \quad \text{and either} \dim(\ell_p \cap \overline{T}_q) = k, \text{or dim}(\ell_p \cap \ell_q^0) = p - k.
\]

Set $n_1 = p - k$, $n_2 = k$, $n_3 = n + 1 + k - p - q$, $n_4 = q - k$. Then, taking as base point, with obvious notation, $p_0 = (\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_2}, \mathbb{C}^{n_3} \oplus \mathbb{C}^{n_4})$, the stabilizer of $p_0$ in $K = SL_{n+1}(\mathbb{C})$ has Lie algebra

\[
v = \left\{ \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ 0 & Z_{2,2} & 0 & Z_{2,4} \\ 0 & 0 & Z_{3,3} & Z_{3,4} \\ 0 & 0 & 0 & Z_{4,4} \end{bmatrix} \mid Z_{i,j} \in \mathbb{C}^{n_i \times n_j} \right\} \cap sl_{n+1}(\mathbb{C}).
\]

Indeed, in the block matrix $Z = (Z_{i,j})_{1 \leq i, j \leq 4}$ that have
\[
Z_{3,1} = 0, \quad Z_{3,2} = 0, \quad Z_{4,1} = 0, \quad Z_{4,2} = 0 \quad \text{because} \quad Z(e_1, \ldots, e_p) \subset \langle e_1, \ldots, e_p \rangle.
\]
Moreover, the inclusion $T_Z(\mathbb{C}^{n_2} \oplus \mathbb{C}^{n_3}) \subset \mathbb{C}^{n_2} \oplus \mathbb{C}^{n_3}$ is equivalent to

\[
\begin{bmatrix}
Z_{1,1} & T_{Z_{2,1}} & 0 & 0 \\
Z_{1,2} & T_{Z_{2,2}} & 0 & 0 \\
T_{Z_{1,3}} & T_{Z_{2,3}} & Z_{3,3} & T_{Z_{4,3}} \\
T_{Z_{1,4}} & T_{Z_{2,4}} & Z_{3,4} & T_{Z_{4,4}}
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & X_2 & 0 \\
0 & Y_2 & 0 & 0 \\
0 & 0 & X_4 & 0 \\
0 & 0 & 0 & Y_4
\end{bmatrix}, \quad \forall X_2 \in \mathbb{C}^{n_2}, \quad X_4 \in \mathbb{C}^{n_4},
\]
and this yields $Z_{2,1} = 0, Z_{2,2} = 0, Z_{4,3} = 0$. The compact CR manifold $M_0(k)$ has CR dimension equal to $v = (n_1 n_2 + n_1 n_3 + n_1 n_4 + n_2 n_4 + n_3 n_4)$ and CR-codimension $d = 2n_2 n_3$. The case $k = k_0$, where $n_3 = 0$, is the one where $v$ is parabolic, and $M_0(k_0) = M_-(k_0)$ is a complex flag manifold. In general, $(\kappa_0, v_n)$ is HNR because

\[
v_n = \left\{ \begin{bmatrix} 0 & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ 0 & 0 & 0 & Z_{2,4} \\ 0 & 0 & 0 & Z_{3,4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid Z_{i,j} \in \mathbb{C}^{n_i \times n_j} \right\} \cap sl_{n+1}(\mathbb{C})
\]

is the nilpotent radical of

\[
qu = \left\{ \begin{bmatrix} Z_{1,1} & Z_{1,2} & Z_{1,3} & Z_{1,4} \\ 0 & Z_{2,2} & Z_{2,3} & Z_{2,4} \\ 0 & Z_{3,2} & Z_{3,3} & Z_{3,4} \\ 0 & 0 & Z_{3,4} & Z_{4,4} \end{bmatrix} \mid Z_{i,j} \in \mathbb{C}^{n_i \times n_j} \right\} \cap sl_{n+1}(\mathbb{C})
\]

Then

\[
(6.13) \quad \mathfrak{f}_0 = \mathfrak{m}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{2,3} & 0 \\ 0 & -Z_{2,3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} Z_{2,3} \in \mathbb{C}^{n_2 \times n_3} \cong \mathbb{C}^{n_2 \times n_3}.
\]

The CR algebra $(\kappa_0, v)$ is weakly degenerate when $k < p$ and strictly nondegenerate, according to [22], when $k = p$. The vector valued Levi form is

\[
(Z_{1,2}, Z_{1,3}, Z_{1,4}, Z_{2,4}, Z_{3,4}) \to Z_{1,2}^* Z_{1,3} + Z_{2,4} Z_{3,4}^*
\]
and hence all the nonzero scalar Levi form have a Witt index equal to $\mu = (n_1 + n_4) = (p - k) + (q - k) = p + q - 2k$. The complex manifold $M_-(k)$ has dimension $N = n_1 n_2 + n_1 n_3 + n_1 n_4 + n_2 n_3 + n_2 n_4 + n_3 n_4$ and, according to Theorems [6.6] and [6.8] is $\mu$-pseudoconcave and $(v - \mu)$-pseudoconvex.
6.5. Application to Dolbeault and CR cohomologies III. In this section we extend Theorem 6.6 to the case where we do not assume that \((\kappa_0, \nu)\) is HNR. To this aim we utilize an \(r\)-pseudoconcave exhausting functions which is only continuous (see \[9, 10, 16, 28\]). Namely, we will consider the function
\[(6.14) \quad \phi([\zeta]) = \text{dist}^2(\zeta, \zeta, N), \quad \text{for} \; \zeta \in K,
\]
where \(N = \{v'v \mid v \in V\}\) as in (4.5) and \([\zeta]\) is the element of \(\mathcal{M}_- = \mathcal{K}/\mathcal{V}\) corresponding to \(\zeta \in \mathcal{K}\).

We recall that a continuous function \(\phi\), defined on a complex \(v\)-dimensional manifold \(\mathcal{M}_-\), is said to be weakly \(r\)-pseudoconcave if, for every point \(p \in \mathcal{M}_-\), we can find a coordinate neighborhood \((U, z)\), centered at \(p\), such that, for every \((v - r + 1)\)-dimensional linear subspace \(\ell\) of \(\mathcal{C}'\), for every coordinate ball \(B \subset U\) and \(\psi\) plurisubharmonic on a neighborhood of \(B\), with \(\phi \geq \psi\) on \(\ell \cap \partial B\) we also have \(\phi \geq \psi\) on \(\ell \cap B\).

We say that \(\phi\) is strictly \(r\)-pseudoconcave if, for each \(p \in \mathcal{M}_-\), we can find an open coordinate neighborhood \((U, p)\) centered in \(p\) and an \(\epsilon > 0\) such that \(\phi + \epsilon|z|^2\) is weakly \(r\)-pseudoconcave in \(U\).

By Bungart’s approximation theorem (\[9, \text{Theorem 5.2}\]) strictly \(r\)-pseudoconcave functions can be uniformly approximated on compacts by piece-wise smooth strictly \(r\)-pseudoconcave functions. Thus (see e.g. \[2, \text{Chapter IV}\]) we can still apply the Andreotti-Grauert theory when we have a strictly-\(r\)-pseudoconcave exhaustion function which is only continuous.

Our application relies then on the following lemmas.

**Lemma 6.10.** Let \(\phi\) be a continuous exhaustion function on \(\mathcal{M}_-\) and assume that, for all \(c > 0\) and \(p_0 \in \Phi_c = \{p \in \mathcal{M}_- \mid \phi(p) = c\}\) there is a germ of CR generic \(r\)-pseudoconcave CR submanifold \(\mathcal{M}_0(p_0)\) of \(\mathcal{M}_-\) through \(p_0\) with \(\mathcal{M}_0(p_0) \subset \{\phi_p \leq c\}\). Then \(\phi\) is weakly \(r\)-pseudoconcave.

**Proof.** We argue by contradiction, assuming that, for every coordinate neighborhood \((U, z)\) centered at a point \(p_0 \in \mathcal{M}_-\) we can find a \((v - r + 1)\)-dimensional linear subspace \(\ell\) of \(\mathcal{C}'\) and a plurisubharmonic \(\psi\), defined on a neighborhood of the closure \(\bar{B}\) of a coordinate ball in \(U\), and a point \(p_1 \in \ell \cap \partial B\) where \(\phi(p_1) < \psi(p_1)\), while \(\phi(p) \geq \psi(p)\) for all \(p \in \partial B \cap \ell\). Clearly the same condition is satisfied by any linear \((v - r + 1)\)-plane sufficiently close to \(\ell\), so that we can assume that \(\ell\) intersects \(\mathcal{M}_0(p_1)\) transversally. The intersection \(\mathcal{M}_0(p_1) \cap \ell\) is then a \((1)\)-pseudoconcave CR submanifold of \(\ell\), but the restriction of \(\psi\) to \(\ell \cap \mathcal{M}_0(p_1) \cap B\) contradicts then the maximum principle, since takes at the interior point \(p_1\) a value larger than the supremum of the values taken on the boundary \(\ell \cap \mathcal{M}_0(p_1)\cap \bar{B}\) (see e.g. \[15\]). The contradiction proves that \(\phi\) is weakly \(r\)-pseudoconcave.

**Lemma 6.11.** The exhaustion function \(\phi\) defined by (6.14) is strictly \(r\)-pseudoconcave on \(\mathcal{M}_-\setminus \mathcal{M}_0\).

**Proof.** By Proposition 4.8 there is \(c_0 > 0\) such that \(\phi\) is strictly \(r\)-pseudoconcave when \(0 < \phi(p) \leq c_0\), since, by \[16, \text{Lemma 2.6}\], for a smooth function the notion of strict \(r\)-pseudoconcavity coincides with the requirement about the signature of its complex Hessian.

For \(\zeta \in \mathcal{K}\), we can consider the function \(\phi_\zeta(p) = \phi(\zeta^{-1} \cdot p)\), which is continuous and weakly \(r\)-pseudoconcave on \(\mathcal{M}_- \setminus (\zeta \cdot \mathcal{M}_0)\) and strictly \(r\)-pseudoconcave when it takes positive values smaller than \(c_0^2\). Let \(p_0 \in \mathcal{M}_-\) with \(\phi(p_0) > c_0^2\) and fix a
Theorem 6.12. Let $M_0$ be a compact $r$-reductive homogeneous CR manifold, with canonical complex embedding $M_0 \hookrightarrow M_\ast$.

If $M_0$ is an $r$-pseudoconvave CR-manifold, then $M_\ast$ is an $r$-pseudoconcave complex manifold and for every coherent sheaf $F$ we have

$$\dim (\mathcal{H}^j(M_0, F) \approx \mathcal{H}^j(M_\ast, F)) < \infty, \quad \forall j < r - \text{hd}(F).$$

In particular,

$$\dim (\mathcal{H}^{p,j}(M_0) \approx \mathcal{H}^{p,j}(M_\ast)) < \infty, \quad \forall j < r. \quad \Box$$

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