ABUNDANCE OF ARITHMETIC PROGRESSIONS IN QUASI CENTRAL SETS

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Abstract. Furstenberg and Glasner proved that for an arbitrary $k \in \mathbb{N}$, any piecewise syndetic set contains $k$–term arithmetic progression and such collection is also piecewise syndetic in $\mathbb{Z}$. They used algebraic structure of $\beta \mathbb{N}$. The above result was extended for arbitrary semigroups by Bergelson and Hindman, again using the structure of Stone-Cech compactification of general semigroup. However they provided the abundances for various types of large sets. But the abundances in Quasi Central was not provided. In this work we will provide a combinatorial proof of the abundance in Quasi Central sets.

In A subset $S$ of $\mathbb{Z}$ is called syndetic if there exists $r \in \mathbb{N}$ such that $\bigcup_{i=1}^{r}(S-i) = \mathbb{Z}$. Again a subset $S$ of $\mathbb{Z}$ is called thick if it contains arbitrary long intervals in it. Sets which can be expressed as intersection of thick and syndetic sets are called piecewise syndetic. All these notions have natural generalization for arbitrary semigroups.

One of the famous Ramsey theoretic result is so called van der Waerden’s Theorem $[\text{vdw}]$ which states that one cell of any partition $\{C_1, C_2, \ldots, C_r\}$ of $\mathbb{N}$ contains arithmetic progression of arbitrary length. Since arithmetic progressions are invariant under shifts, it follows that every piecewise syndetic set contains arbitrarily long arithmetic progressions.

Furstenberg and E. Glasner in $[\text{FG}]$ algebraically and Beiglboeck in $[\text{Bel}]$ combinatorially proved that if $S$ is a piecewise syndetic subset of $\mathbb{Z}$ and $l \in \mathbb{N}$ then the set of all length $l$ progressions contained in $S$ is also large.

**Theorem 1.** Let $k \in \mathbb{N}$ and assume that $S \subseteq \mathbb{Z}$ is piecewise syndetic. Then $\{(a, d) : a, a+d, \ldots, a+kd \in S\}$ is piecewise syndetic in $\mathbb{Z}^2$.

The above theorem can be proved for the set of Natural Numbers $\mathbb{N}$ in a similar way. In $[\text{BH}, \text{HLS}]$ the above result was studied for various large sets viz. Central, Thick, IP sets etc. for general semigroups. But for the Quasi Central sets it was not studied.

The notion of Quasi Central sets was introduced in $[\text{HMS}]$ which was defined in terms of algebraic structure of $\beta \mathbb{N}$. It has a nice combinatorial property which is given:

**Theorem 2.** $[\text{HMS}, \text{Theorem 3.7}]$ For a countable semigroup $(S, \cdot)$, $A \subseteq S$ is said to be Quasi-central iff there is a decreasing sequence $\langle C_n \rangle_{n=1}^{\infty}$ of subsets of $A$ such that,

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(1) for each \( n \in \mathbb{N} \) and each \( x \in C_n \), there exists \( m \in \mathbb{N} \) with \( C_m \subseteq x^{-1}C_n \) and
(2) \( C_n \) is piecewise syndetic \( \forall n \in \mathbb{N} \).

The importance of Quasi Central sets is it is very close to Central Sets and enjoy a close combinatorial property to those sets. However Every Central sets are Quasi Central but not the converse. For detail study one can look into \[\text{HMS}\].

The following lemma will be essential

\textbf{Theorem 3.} For any quasi-central \( M \subseteq \mathbb{N} \) the collection \( \{(a, b) : \{a, a + b, a + 2b, \ldots, a + lb\} \subset M\} \) is quasi-central in \((\mathbb{N} \times \mathbb{N}, +)\).

\textbf{Proof.} As, \( M \) is quasi-central, theorem 2 guarantees that there exists a decreasing sequence piecewise syndetic subsets of \( \mathbb{N} \), \( \{A_n : n \in \mathbb{N}\} \) such that property 2 is satisfied.

As, all \( A_n \) are piecewise syndetic \( \forall n \in \mathbb{N} \) in the following sequence,

1. \( M \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots \)

The set \( B = \{(a, b) : \{a, a + b, a + 2b, \ldots, a + lb\} \subset M\} \) is piecewise syndetic in \( \mathbb{N} \times \mathbb{N} \) from proposition [1]

And for \( i \in \mathbb{N} \), \( B_i = \{(a, b) \in \mathbb{N} \times \mathbb{N} : \{a, a + b, a + 2b, \ldots, a + lb\} \subset A_i\} \neq \phi \) is piecewise syndetic \( \forall i \in \mathbb{N} \), theorem [1]

Consider,

2. \( B \supseteq B_1 \supseteq B_2 \supseteq \ldots \supseteq B_n \supseteq \ldots \)

Now choose \( n \in \mathbb{N} \) and \( (a, b) \in B_n \), then \( \{a, a + b, a + 2b, \ldots, a + lb\} \subset A_n \).

Now choose by property 2

\( A_N \subseteq \bigcap_{i=0}^{l} \{-(a + ib) + A_n\} \)

Now any \( (a_1, b_1) \in B_N \) implies \( \{a_1, a_1 + b_1, a_1 + 2b_1, \ldots, a_1 + lb_1\} \subseteq A_N \subseteq \bigcap_{i=0}^{l} \{-(a + ib) + A_n\} \)

\( (-a + ib) + A_n \)

And so \( (a_1 + a) + i \cdot (b_1 + b) \in A_n \forall i \in \{0, 1, 2, \ldots, l\} \) and so \( (a_1, b_1) \in -(a, b) + B_n \).

This implies \( B_N \subseteq -(a, b) + B_n \), showing the property 2

This proves the theorem. \( \square \)

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