Criticality of the Higgs mass for the long-range quantum $XY$ chain: Amplitude ratio between the Higgs and paramagnetic gaps

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Abstract

The quantum $XY$ spin chain with the interactions decaying as a power law $1/r^{1+\sigma}$ of the distance between spins $r$ was investigated with the exact diagonalization method. Here, the constituent spin is set to $S = 1$, which enables us to incorporate the biquadratic interactions so as to realize the order-disorder transition with the O(2) symmetry maintained. Thereby, in the ordered phase, we resolved the Higgs mass $m_H$ out of the Goldstone-excitation continuum by specifying Higgs-particle’s quantum numbers to adequate indices. We then turn to the analysis of the critical amplitude ratio $m_H/\Delta$ ($\Delta$: paramagnetic gap in the disordered phase). As the power of the algebraic decay $\sigma$ increases, the amplitude ratio $m_H/\Delta$ gets enhanced gradually in agreement with the $\epsilon(= 4 - D)$-expansion-renormalization-group result; here, we resort to the $\sigma \leftrightarrow D$ relation advocated recently in order to establish a relationship between the renormalization-group result and ours.

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1. Introduction

The spin systems with the long-range interactions have been investigated both theoretically [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16] and experimentally [17, 18, 19, 20, 21, 22]. A main concern is to clarify how the criticality between the ordered and disordered phases is affected by the power of the algebraic decay [7, 8, 9]. Meanwhile, the idea was extended to the case of the quantum spin models [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].
33, 34]. In Fig. 1, we present the criticality chart [25, 30, 31, 32, 33, 34] for the quantum XY spin chain with the interactions decaying as a power law $1/r^{1+\sigma}$ of the distance between spins $r$. For $\sigma \gtrsim 2$, the long-range interaction becomes irrelevant, and the criticality falls into the universality class of the classical two-dimensional ($D = 2$) XY model (with the short-range interactions). On the contrary, for $\sigma < 2/3$, the criticality belongs to the mean-field type, namely, the $D = 4$ universality class. In the intermediate regime $2/3 \leq \sigma \lesssim 2$, the power $\sigma$ interpolates both limiting cases, $D = 2$ and 4, smoothly. Actually, relying on the correspondence between the $\sigma$- and $D$-mediated considerations, the authors of Ref. [13] analyzed the criticality for the $D = 3$ (short-range) Ising model quantitatively. Experimentally, systems with variable-range interactions have been synthesized [18, 19, 20]. In this sense, the $\sigma \leftrightarrow D$ relation [8] is not a mere theoretical concept.

The criticality chart, Fig. 1, resembles to that of the classical counterpart [1, 2, 7]. In this paper, aiming to elucidate the quantum nature of this problem, we devote ourselves to the spectrum, namely, the Higgs (paramagnetic) excitation gap, $m_H(\Delta)$, in the (dis)ordered phase. Particularly, we estimate the critical amplitude ratio

$$m_H/\Delta = \left. \frac{m_H(J)}{\Delta(2J_c - J)} \right|_{J \to J_c^+},$$

with the XY interaction $J$ and the power $\sigma$ ranging within $2/3 < \sigma < 2$. In Fig. 2, we present a schematic drawing for the spectrum in the disordered ($J < J_c$) and ordered ($J > J_c$) phases. The paramagnetic gap opens in the disordered phase, whereas in the ordered phase, the Higgs branch $m_H$ is embedded within the Goldstone-excitation continuum. Meanwhile, the amplitude ratio $m_H/\Delta$ was estimated for the generic values of $D$ by means of the $\epsilon$-expansion method [35]. For the fixed-$D = 3$ systems, a good deal of analyses have been made [36, 37, 38, 39, 40, 41]. As mentioned above, via the $\sigma \leftrightarrow D$ relation [8, 25], one is able to establish a relationship among these $D$- and $\sigma$-mediated considerations.

We employed the exact diagonalization method, which enables us to calculate the energy gaps, $m_H$ and $\Delta$, directly by specifying the quantum numbers to Eqs. (3) and (4), respectively. So far, in the quantum-Monte-Carlo approach [36, 37, 38], the dynamical scalar susceptibility $\chi_s(\omega)$ [42, 43] has been enumerated through the inverse Laplace transformation; see Appendix B of Ref. [37]. Because the disturbance $\chi_s(\omega)$ preserves the O(2) symmetry,
it is less sensitive to the Goldstone excitations \([44]\); hence, a \(m_H\) signal can be captured via \(\chi_s(\omega)\) \([43]\). In the recent experiment for the ultra cold atom \([45]\), essentially the same strategy is undertaken; for such boson system, the (dis)ordered phase should be interpreted as the superfluid (Mott insulator) phase. We stress that the exact diagonalization method admits \(m_H\) and \(\Delta\) without resorting to such elaborated techniques.

To be specific, we present the Hamiltonian for the one-dimensional quantum spin-\(S = 1\) \(XY\) model with the long-range interactions

\[
\mathcal{H} = -\frac{1}{N} \sum_{i \neq j} J_{ij} \left( \frac{J}{2} (S_i^+ S_j^- + S_i^- S_j^+) + \frac{J_4}{4} ((S_i^+)^2 (S_j^-)^2 + (S_i^-)^2 (S_j^+)^2) \right) + D_s \sum_{i=1}^{N}(S_i^z)^2.
\]

(2)

Here, the quantum spin-\(S = 1\) operator \(S_i^{\pm, z}\) is placed at each one-dimensional-lattice point, \(i = 1, 2, \ldots, N\). The summation \(\sum_{i \neq j}\) runs over all possible pairs \(1 \leq i, j \leq N\), and the long-range interaction \(J_{ij}\) decays as a power law, \(J_{ij} = 1/\sin(\pi|i - j|/N)^{1+\sigma}\), with the variable exponent \(\sigma\). Here, we implemented the periodic-boundary condition for the spin chain; what is meant by the expression for \(J_{ij}\) is that the distance between the spins, \(i\) and \(j\), is given by the chord length, \(\sin(\pi|i - j|/N)\), rather than the arc, \(|i - j|\). The \(XY\) interaction \(J_4\) induces the order-disorder transition (see Fig. 2), whereas the biquadratic interaction \(J_4 = 1\) \([46, 47, 48]\) is fixed. The single ion anisotropy \(D_s = 0.1(2 - \sigma)\) is included so as to improve the finite-size behaviors \([49]\); the single ion anisotropy drives the \(XY\) phase to the massive phase \([50, 51, 52]\) in \(D = 2\) dimensions (\(\sigma = 2\)), and we incorporated the diminution factor \((2 - \sigma)\). The normalization (Kac) factor \(\mathcal{N}\) \([53, 54]\) is given by \(\mathcal{N} = N^{-1} \sum_{i \neq j} \sin(\pi|i - j|/N)^{-1-\sigma}\).

The rest of this paper is organized as follows. In the next section, we show the numerical results. A brief account of the simulation algorithm is given as well. In the last section, we address the summary and discussions.

2. Numerical results

In this section, we present the numerical results. We employed the exact diagonalization method for the Hamiltonian (2) with \(N \leq 22\) spins. Before commencing detailed scaling analyses for \(m_H\) and \(\Delta\), we give a brief account of the numerical algorithm. The Hamiltonian has a number of symmetries, \(i.e.,\) good quantum numbers, with which we are able to diagonalize the Hamiltonian within the restricted Hilbert space. The quantum-number
specification scheme for $m_H$ and $\Delta$ is as follows: The numerical diagonalization was performed within the zero-momentum subspace $k = 0$, at which the elementary-excitation gap opens; namely, both the ground- and first-excited-state levels locate. We further specify the subspace with an additional quantum number $S^z_{\text{tot}}$, which corresponds to the total magnetization operator, $\sum_{i=1}^{N} S^z_i$; this quantum number reflects the O(2) symmetry of $\mathcal{H}$. Within the restricted Hilbert space specified by $k = 0$ and $S^z_{\text{tot}}$, the Higgs mass is given by

$$m_H = E_1(0) - E_0(0),$$

with the ground- (first excited-) state energy $E_{0(1)}(S^z_{\text{tot}})$ within the sector $S^z_{\text{tot}}$. The paramagnetic excitation belongs to the $S^z_{\text{tot}} = \pm 1$ sector, and hence, the gap is given by the formula

$$\Delta = E_0(1) - E_0(0).$$

This $\Delta$ branch becomes the Goldstone mode in the adjacent phase (ordered phase). Note that the $m_H$ mode exists far above the collection of the Goldstone excitations (continuum), and such level specification is significant to resolve the former out of the latter.

### 2.1. Scaling analysis of the dynamical critical exponent $z$: A preliminary survey

In this section, based on the scaling theory developed in Ref. [25], we analyze the Higgs mass $m_H$, Eq. (3). As a byproduct, we obtain the dynamical critical exponent $z$, which characterizes the anisotropy between the real-space and imaginary-time directions quantitatively.

In Fig. 3, we present the scaling plot, $(J - J_c)N^{1/\nu - N^z m_H}$, for the fixed $\sigma = 1.2$ and various system sizes, (+) $N = 18$, (×) 20, and (*) 22. The scaling parameters, namely, the critical point $J_c = 0.30785$, the reciprocal correlation-length critical exponent $1/\nu = 0.636$, and the dynamical critical exponent $z = 0.594$, were determined as follows. The critical point $J_c = 0.30785(40)$ was obtained through the least-squares fit for the $1/N - J_c(N)$ data with $N = 16, 18, \ldots, 22$; here, the approximate critical point $J_c(N)$ is defined by

$$\partial_J m_H|_{J = J_c(N)} = 0,$$

for each $N$. Likewise, the dynamical critical exponent $z = 0.594(1)$ was obtained via the least-squares fit for the $(N^+(N+2)/2)^{-1} z(N, N+2)$ data with
The approximate dynamical critical exponent $z(N, N')$ is given by the logarithmic derivative of the finite-size Higgs mass

$$z(N, N') = -\frac{\ln m_H(N)|_{J_c(N)} - \ln m_H(N')|_{J_c(N')}}{\ln N - \ln N'},$$

with a pair of system sizes $(N, N')$. Finally, the reciprocal correlation-length critical exponent $1/\nu = 0.636$ was calculated from the above-mentioned result $z = 0.594$ with the aide of $1/z\nu = 1.07$ taken from Fig. 3a of Ref. [25].

The scaled data in Fig. 3 seem to obey the finite-size scaling satisfactorily. The Higgs mass appears to open in the ordered phase $J > J_c$. The criticality for $m_H$ is analyzed in depth afterwards. Noticeably enough, the $m_H$ branch opens also in the adjacent (disordered) phase $J < J_c$. This massive mode may correspond to the particle-hole-excitation threshold in the boson language [37, 38]; further details as to the disordered phase are not pursued in this paper.

Similar analyses as that of Fig. 3 were carried out for various values of $\sigma$. The $\sigma$-dependent dynamical critical exponent $z$ is presented in Fig. 4. Here, as an error margin, we accepted the deviation between different extrapolation schemes, namely, $N^{-1}z(N, N + 2)$ and $N^{-2}z(N, N + 2)$ approaches, aiming to appreciate possible systematic errors other than the least-squares-fit error. The series of results in Fig. 4 indicate that the dynamical critical exponent $z$ reflects the variation of $\sigma$ sensitively. As a reference, in Fig. 4, we also present an approximate formula (dots) [25]

$$z = \sigma/2,$$

which is validated in the small-$\sigma$ side. Our data appear to obey the formula (7) for a considerably wide range of $\sigma$. However, around the upper and lower critical thresholds, $\sigma = 2$ and $2/3$, respectively, there emerge systematic deviations, which may be attributed to the notorious logarithmic corrections to finite-size scaling [3, 10, 11, 26]. Note that these thresholds correspond to the lower and upper critical dimensions, $D = 2$ and 4, respectively (see Eq. (11)), and inevitably, the finite-size scaling is affected by the marginal operators at these thresholds.

We address a number of remarks. First, the dynamical critical exponent $z$ is peculiar to the quantum criticality. The conventional short-range spin models exhibit $z = 1$, namely, the restoration of the symmetry between the real-space and imaginary-time subspaces. On the contrary, as shown above,
the long-range counterpart does exhibit the anisotropy, \( z \neq 1 \) [25], between these subspaces. In the quantum Monte Carlo simulation, a special care has to be paid so as to remedy this anisotropy in order to manage the finite-size-scaling properly. In the exact diagonalization approach, one is able to concentrate on the real-space sector, because the imaginary-time system size is infinite \( \beta (= 1/T) \to \infty \) a priori; note that one can access the ground state \( T = 0 \) directly. Recently, as for the quantum long-range systems, a variety of approaches such as the exact diagonalization method [53], the variational matrix product state with the quasi-particle ansatz [54], and the density-matrix renormalization group under the open boundary condition [55], have been utilized successfully. Because in the present analysis, the mass gaps right at the zone center \( k = 0 \) are required, we resort to the exact diagonalization method under the periodic boundary condition. Last, in Fig. 3, the minimum point for the Higgs gap seemingly locates slightly out of the critical point \( (J - J_c)N^{1/\nu} \approx 0.1 \). However, in the thermodynamic limit, the deviation from the critical point \( J_c \) vanishes as \( J - J_c \approx 0.1/N^{1/\nu} \to 0 \).

### 2.2. Critical amplitude ratio \( m_H/\Delta \)

Based on the finite-size-scaling analysis examined in the preceding section, we turn to the analysis of the critical amplitude ratio \( m_H/\Delta \). To this end, we investigate the scaling behavior of the paramagnetic gap \( \Delta \), Eq. (4). In Fig. 5, we present the scaling plot, \( (J - J_c)N^{1/\nu} - N^z \Delta \), for \( \sigma = 1.2 \) and various system sizes, (+) \( N = 18 \), (×) 20, and (∗) 22. The scaling parameters, \( J_c = 0.30785 \), \( 1/\nu = 0.636 \), and \( z = 0.594 \), are the same as those of Fig. 3. The scaled data fall into the scaling curve satisfactorily. The paramagnetic gap \( \Delta \) opens in the disordered phase \( J < J_c \); see the schematic diagram, Fig. 2, as well. The paramagnetic mode \( \Delta \) in the disordered phase turns into the Goldstone excitation in the adjacent phase (ordered phase) \( J > J_c \). Actually, in \( J > J_c \), there is shown a closure of the \( \Delta \) branch, which is smaller than the Higgs gap \( m_H \) in Fig. 3. Such Goldstone continuum place an obstacle to observe \( m_H \) clearly. In this paper, we pick up the \( m_H \) mode directly by specifying the quantum numbers to Eq. (3) adequately.

We then consider the amplitude ratio \( m_H/\Delta \). In Fig. 6, we present the scaling plot, \( (J - J_c)N^{1/\nu} - m_H(J)/\Delta(2J_c - J) \), for \( \sigma = 1.2 \) and various system sizes, (+) \( N = 18 \), (×) 20, and (∗) 22. The scaling parameters, \( J_c = 0.30785 \), \( 1/\nu = 0.636 \), and \( 1/\nu = 0.636 \), are the same as those of Fig. 3. Again, the collapse of the scaled data seems to be satisfactory. In the ordered phase \( J > J_c \), the amplitude ratio takes a plateau with the height \( m_H/\Delta \approx 1.8 \). Such a feature
indicates that that the amplitude ratio \( m_H/\Delta \) indeed takes a constant value in proximity to the critical point.

The plateau in Fig. 6 exhibits a shallow bottom, which provides a good indicator as to the amplitude ratio for each \( N \). Aiming to extrapolate \( m_H/\Delta \) systematically to the thermodynamic limit \( N \to \infty \), we carried out the least-squares fit for the plot \( N^{-1}\frac{m_H}{\Delta}(N) \) with \( N = 16, 18, \ldots, 22 \); here the approximate amplitude ratio is given by

\[
\frac{m_H}{\Delta}(N) = \frac{m_H(J)}{\Delta(2J_c - J)} \bigg|_{J = J(N)},
\]

at the bottom of the plateau \( \bar{J}(N) \) satisfying the extremum condition

\[
\frac{\partial (m_H(J)/\Delta(2J_c - J))}{\partial J} \bigg|_{J = J(N)} = 0,
\]

for each \( N \). We arrived at \( m_H/\Delta = 1.980(14) \) via the extrapolation scheme.

We carried out similar analyses as that of Fig. 6 for various values of \( \sigma \). The \( \sigma \)-dependent amplitude ratio \( m_H/\Delta \) is presented in Fig. 7. Here, as an error margin, we accepted the deviation between different extrapolation schemes, namely, \( N^{-1}\frac{m_H}{\Delta}(N) \) and \( N^{-2}\frac{m_H}{\Delta}(N) \) fittings, aiming to appreciate possible systematic errors other than a fitting error. Our data in Fig. 7 indicate that the critical amplitude ratio \( m_H/\Delta \) varies monotonically, as the decay rate \( \sigma \) changes. In other words, the low-energy spectrum reflects the long-range-interaction decay rate \( \sigma \) rather sensitively.

We address a number of remarks. First, the solution for the amplitude ratio \( m_H/\Delta \) does not exist for exceedingly large \( \sigma \gtrsim 1.6 \). That is, the Higgs particle becomes unstable for large \( \sigma \), and the plateau as in Fig. 6 disappears (shrinks) eventually. The end point \( \sigma \approx 1.6 \) corresponds to \( D \approx 2.25 \) according to the \( \sigma \leftrightarrow D \) relation (11); the dimensionality \( D = 2.25 \) is about to reach the lower critical dimension \( D = 2 \). Last, the excitation gaps, \( m_H \) and \( \Delta \), as well as the amplitude ratio \( m_H/\Delta \) exhibit smooth variations throughout the intermediate regime \( 2/3 < \sigma < 2 \). As for the Ising counterpart [53, 54], an anomaly (a dynamical phase transition) was observed around \( \sigma \approx 1.3 \). Correspondingly, the critical amplitude ratio \( m_2/m_1 \) (\( m_i \): \( i \)-th excitation gap) displays a non-monotonic behavior [56]; actually, a minimum locates in the midst of the intermediate regime as to \( m_2/m_1 \).
2.3. Comparison with the preceding results via the $\sigma \leftrightarrow D$ relation [8, 25]

According to the elaborated $\epsilon$-expansion analysis, the critical amplitude ratio is given by

$$m_H/\Delta = \sqrt{2} + \sqrt{2}\epsilon \left(\frac{4\ln 2 + 3\sqrt{3}\pi}{40} - \frac{1}{4}\right), \quad (10)$$

with $D = 4 - \epsilon$. As a reference, in Fig. 7, we present the $\epsilon$-expansion result (dots). Here, we made use of the $\sigma \leftrightarrow D$ relation [8, 25]

$$D = 2/\sigma + 1, \quad (11)$$

in order to establish a relationship between the $\epsilon$-expansion result and ours. The $\sigma \leftrightarrow D$ relation has been pursued extensively, and more refined formulas were proposed. The above expression (11) has an advantage in that it is given in a closed form.

The overall features of our data and the analytic result (10) resemble to each other. Our simulation result suggests that a slight enhancement as to the analytic formula for the large-$\sigma$ regime, $\sigma \gtrsim 1$. Remarkably enough, the above formula (11) up to $O(\epsilon^1)$ may be practically of use to establish a relationship between the power of algebraic decay $\sigma$ and the low-lying excitation spectrum.

The Higgs mass $m_H$ may have an uncertainty, because it has a finite life time; the reciprocal life time corresponds to the intrinsic $m_H$-peak width. According to the $\epsilon$-expansion analysis [35], the scaled peak width $\delta m_H/\Delta$ should also take a universal constant

$$\frac{\delta m_H}{\Delta} = \frac{\sqrt{2}\pi\epsilon}{40}. \quad (12)$$

At $\sigma = 1$ ($D = 3$), this formula yields the scaled peak width $\delta m_H/\Delta = 0.11\ldots$. Taking this intrinsic uncertainty into account, our result and the analytical formula (10) do not display substantial disagreement for the small-$\sigma$ side, $\sigma \lesssim 1$. As mentioned in Sec. 2.2, the series of data $m_H/\Delta$ terminate around $\sigma \approx 1.6$ ($D \approx 2.25$). Such a feature is supported by the analytic formula (12) in the sense that for large $\sigma$ (large $\epsilon$), the Higgs particle becomes obscure nonetheless.

We make a brief overview for the fixed-$D = 3 (= 2 + 1)$-lattice analyses. By means of the quantum Monte Carlo method, the estimates, $m_H/\Delta =
2.1(3) [36, 37] and 3.3(8) [38], were reported, whereas the exact diagonalization method yields 2.1(2) [39]. With the functional-renormalization-group method, the results, \( m_H/\Delta = 2.4 \) [40] and 2.2 [41], were obtained, while the \( \epsilon \)-expansion method [35] yields 1.67. The present simulation admits \( m_H/\Delta = 1.78(11) \) at \( \sigma = 1 \) \((D = 3)\). These results seem to be rather scattered. At present, it is not clear whether the discrepancy should be attributed to the above-mentioned intrinsic uncertainty (12), or a systematic one. Actually, the analyticity for the dynamical scalar susceptibility \( \chi_s(\omega) \) [35] reveals an appreciable discrepancy between the \( m_H \)-pole position and the actual peak position; from the latter, the quantum Monte Carlo results were read off. Clearly, detailed information as to the incoherent part of \( \chi_s(\omega) \) would be desirable so as to resolve the peak position out of the incoherent background. To the best of author’s knowledge, neither the \( m_H \)-peak shape nor the life time has been investigated quantitatively. At present, the analytic result (10) and ours would be “easier to reconcile with the lower of” [35] those preceding results [36, 37, 38, 39, 40, 41].

3. Summary and discussions

The quantum XY spin chain (2) with the algebraically decaying interactions, \( 1/r^{1+\sigma} \), was investigated by means of the exact diagonalization method. This method enables us to identify the spectral gaps, \( m_H \) and \( \Delta \), by specifying the quantum numbers to Eqs. (3) and (4), respectively. As a preliminary survey, we analyzed the finite-size-scaling behavior [25] for \( m_H \), and as a byproduct, we obtained the \( \sigma \)-dependent dynamical critical exponent \( z \). The result \( z \) seems to obey the approximate formula (7), suggesting that the formula may be practically of use for a wide range of \( \sigma \); the discrepancies around both upper and lower critical thresholds, \( \sigma = 2 \) and 2/3, respectively, should be attributed to the notorious corrections to finite-size scaling [3, 10, 11, 26].

We then turn to the analysis of the critical amplitude ratio \( m_H/\Delta \). Our data obey the \( \epsilon \)-expansion result, Eq. (10), for the small-\( \sigma \) side, \( \sigma \lesssim 1 \), whereas the simulation result displays a slight enhancement as to the analytical formula for large \( \sigma \). It has to be mentioned that there should exist an intrinsic uncertainty, Eq. (12), as to \( m_H/\Delta \) due to the finite life time for \( m_H \). Taking this uncertainty into account, the overlap between the analytical formula (10) and the numerical result do not exhibit substantial disagreement in the small-\( \sigma \) side, \( \sigma \lesssim 1 \). For exceedingly large \( \sigma \), the Higgs particle gets unstable, and the amplitude ratio \( m_H/\Delta \) becomes obscure nonetheless. Note
Figure 1: The universality chart \[25, 30, 31, 32, 33, 34\] for the quantum XY spin chain with the interactions decaying as a power law \(1/r^{1+\sigma}\) of the distance between spins \(r\) is presented. For \(\sigma \gtrless 2\), the long-range interaction becomes irrelevant, and the universality class of the order-disorder phase transition falls into that of the two-dimensional (\(D = 2\)) classical XY model, whereas for \(\sigma > 2/3\), the system exhibits the mean-field (\(D = 4\)) criticality. In the intermediate regime \(2/3 \leq \sigma \lesssim 2\), the criticality with fractional effective dimensionality \(2 \leq D \leq 4\) is realized. That is, the power of the algebraic decay \(\sigma\) interpolates smoothly the upper and lower critical dimensions. At these boundaries, there arise logarithmic corrections \[3, 10, 11, 26\] to finite-size scaling.

that both gaps \(m_H\) and \(\Delta\) are experimentally observable \[45\] in proximity to the critical point. It would be intriguing that the amplitude ratio \(m_H/\Delta\) reflects the power of algebraic decay \(\sigma\) as a monotonic function; actually, the correlation-length critical exponent \(\nu\) displays a non-monotonic behavior \[11\] as to the variation of \(\sigma\).

As for the fixed-\(D = 3 (= 2 + 1)\) lattice simulation, the results, \(m_H/\Delta = 2.1(3)\) \[36, 37\] and \(3.3(8)\) \[38\], were reported with the quantum Monte Carlo method, whereas the exact diagonalization method yields \(2.1(2)\) \[39\]. Through the functional-renormalization-group method, the estimates, \(m_H/\Delta = 2.4\) \[40\] and \(2.2\) \[41\], were obtained, while the \(\epsilon\)-expansion method yields \(1.67\). The present simulation admits \(m_H/\Delta = 1.78(11)\) at \(\sigma = 1\) \((D = 3)\). These results seem to be rather scattered. At present, it is not clear whether the disagreement should be attributed to the above-mentioned intrinsic uncertainty or a systematic one. Actually, the analyticity for the dynamical scalar susceptibility \(\chi_s(\omega)\) \[35\] reveals an appreciable discrepancy between the \(m_H\)-pole position and the actual spectral-peak position; from the latter, the quantum Monte Carlo results were read off. Clearly, detailed information as to the incoherent part of \(\chi_s(\omega)\) would be desirable so as to resolve \(m_H\) out of the background. This problem is left for the future study.

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Figure 2: A schematic drawing of the low-lying spectrum for the quantum long-range $XY$ spin model (2) with the $XY$ interaction $J$ is presented. In the ordered phase $J > J_c$, there opens the Higgs gap $m_H$ embedded within the Goldstone-excitation continuum, whereas in the disordered phase $J < J_c$, the paramagnetic gap $\Delta$ appears. The scaling behavior for $m_H$ as well as the critical amplitude ratio $m_H(J)/\Delta(2J_c - J)|_{J \rightarrow J_c^+}$ are the main concern.
Figure 3: The scaling plot, $(J - J_c)N^{1/\nu}N^z m_H$, is presented for $\sigma = 1.2$, and the system sizes, (+) $N = 18$, (x) 20, and (*) 22, with the scaling parameters, $J_c = 0.30785$, $z = 0.594$ and, $1/\nu = 0.636$. The Higgs gap opens in the ordered phase $J > J_c$. This mode seems to be massive in the adjacent phase $J < J_c$ as well, supporting the preceding observations with the quantum Monte Carlo method [37, 38].
Figure 4: The dynamical critical exponent $z$ is presented for the power of algebraic decay $\sigma$. An approximate formula [25], Eq. (7), is presented (dotted) as well. This formula is validated at least for small $\sigma$. Deviations around the lower and upper critical thresholds, $\sigma = 2/3$ and 2, respectively, may be attributed to the notorious logarithmic corrections to finite-size scaling [3, 10, 11, 26].
Figure 5: The scaling plot, $(J - J_c)N^{1/\nu} - N^z\Delta$, is presented for $\sigma = 1.2$, and the system sizes, (+) $N = 18$, (×) 20, and (∗) 22. The scaling parameters, $J_c = 0.30785$, $z = 0.594$ and $1/\nu = 0.636$, are the same as those of Fig. 3. The paramagnetic gap $\Delta$ opens in the disordered phase $J < J_c$. This branch becomes the Goldstone gapless mode in the adjacent phase $J > J_c$, forming the Goldstone continuum in the low-energy spectrum.
Figure 6: The scaling plot, \((J - J_c)N^{1/\nu} - m_H(J)/\Delta(2J_c - J)\), is presented for \(\sigma = 1.2\), and the system sizes, (+) \(N = 18\), (×) 20, and (*) 22. The scaling parameters, \(J_c = 0.30785\), and \(1/\nu = 0.636\), are the same as those of Fig. 3. In the ordered phase \(J > J_c\), there appears a plateau with the height \(m_H/\Delta \approx 1.8\), which admits an estimate for the amplitude ratio; see text for details.
Figure 7: The critical amplitude ratio $m_H/\Delta$, Eq. (1), is presented for various $\sigma$. The series of solutions terminate around $\sigma \approx 1.6$, because the Higgs excitation becomes unstable for exceedingly large $\sigma \gtrsim 1.6$. According to the formula (11), the power $\sigma \approx 1.6$ corresponds to $D \approx 2.25$, which is about to reach the lower critical dimension $D = 2$. As a reference, the $\epsilon(=4-D)$-expansion formula [35], Eq. (12), is shown as a dotted curve.
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