Five dimensional formulation of a DSR

Riccardo Junior BUONOCORE
Department of Mathematics, King’s College London, The Strand, London, WC2R 2LS, UK

In this paper we are going to analyze a possible formalization of the DSR as a five dimensional theory. This is not the first attempt to do so (see [6, 7, 9–13]) but we feel that these previous treatments are either too arbitrary in the choice of the new enlarged space or lacks of a satisfactory physical interpretation. In this work we are going to propose an algorithm which fixes the shape of the enlarged space. Afterwards we will focus our attention on the consequences of our formalism proposing a physical interpretation of them.
In the last two decades the interest in the introduction of a new observer independent scale beside the speed of light into a coherent theoretical framework has grown. A turning point has been the demonstration in [1] that the postulates of Special Relativity can be modified in order to accommodate the presence of the Planck length $\ell = \sqrt{\frac{\hbar G}{c^3}}$. This work has then been followed by [2] and [3] where generalizations to other constants such has, respectively, the Plank mass $m = \sqrt{\frac{\hbar G}{c}}$ and the Planck energy $\varepsilon = \sqrt{\frac{\hbar G}{G}}$ have been given. Such proposals go under the name of DSR theories. Despite, from a conceptual point of view, the postulates of Special Relativity can be reformulated in order to accommodate the presence of one of these scales, the study of a theory where both $\hbar$, the reduced Planck constant, and $G$, the Newtonian universal gravitational constant, are different from zero, would clearly require to solve the whole quantum gravity problem. Recently, in [4] and [5], has been proposed a theory, the Relative Locality, which tries to make an intermediate step, suggesting to study a phenomenological limit of the full Quantum Gravity, defined by the limits $\hbar, G \to 0$ but holding fixed their ratio. So pure quantum and gravitational effects are turned off but the possible effects due to the presence of the Planck mass $m$ are kept. The physical role to give to $m$ remains however still unclear.

Recently has been proposed by some authors (see [6, 7]) that a natural way to accommodate the presence of a new observer independent constant should be to formulate the DSR as a five-dimensional theory. As pointed out by the authors this formalism would solve technical and interpretative problems such as the non-linear composition of momenta and the famous soccer-ball problem. However their proposal of the choice and the physical interpretation of the fifth dimension have been given in an arbitrary way. Moreover, as underlined in [8], the addition of one more dimension imposes the enlargement of the isometry group of the theory with the introduction of new transformations which apparently lead to a macroscopic relativity in the concept of mass, which is clearly not observed.

Another five-dimensional formulation, by means of a more formal approach, has been given in [9–13], where the authors derive a precise shape for the fifth coordinate, despite it has an unusual dimension. However, in this proposal too, the interpretation of this fifth dimension and the problem of the variation of the mass remain opened.

Since we believe that it is necessary to understand first, before going to the study of a full quantum gravity theory, the phenomenological limit proposed in [4, 5] and that the introduction of a new observer independent constant should be treated in a five-dimensional setup, the aim of this paper is to give a five-dimensional approach to the DSR which should be both geometrically well-grounded and with reasonable phenomenological consequences. In section II we are going to propose an algorithm which will allow us to derive the Special Relativity (with the intuition of the need of the fourth dimension) from the Galilean Relativity, implementing in the latter the postulate about the role of the speed of light $c$. The reliability of this algorithm is in the fact that we will obtain the right theory, i.e. the Special Relativity. In section III we will apply this procedure to the Special Relativity itself in order to introduce a second fundamental constant. This constant will be the Planck mass $m$ and the role we are going to give to it is to be an upper bound for the square root of the modulus of the 4-momentum of the elementary particles. This assumption, despite it can seem quite arbitrary, is the same which has been found in [14], where it arises in the context of the 3D gravity. Moreover, since elementary particles are considered to be the building blocks of the macroscopic matter, they should not have a rest mass capable to create a black hole. Section IV is dedicated to the introduction of a mass-shell relation over the new five dimensional space. In section V we study the shape of the equation of motion of a freely propagating particle, which in turn will allow us to give an interpretation to the fifth dimension. In sections VI and VII we will analyze systematically the isometry group of our theory giving a physical interpretation to the transformations. Since the analysis done so far will hold only for elementary particles in section VIII we will discuss the soccer-ball problem in our context.

II. FROM GALILEAN RELATIVITY TO EINSTEINIAN RELATIVITY

As anticipated in the introduction, in this section we are going to propose a formal procedure which allows one to deduce the theory of Special Relativity directly by enforcing its postulates into the setup of the Galilean Relativity, giving particular attention to the request that the speed of light in vacuum must be an upper bound to the speed of the other particles. Let us start then with a model of Galilean Relativity defined by the couple $(\mathcal{E}, \delta)$, where $\mathcal{E}$ is the

\footnote{Just like as in the transition from the Galilean Relativity to the Einsteinian Relativity the addition of the observer independent constant $c$ makes time the fourth dimension.}
Euclidean 3-plane equipped with the flat positive-definite metric $\delta$, i.e. $\delta = \text{diag}(1,1,1)$. The only postulate holding here is the well-known Galilean Relativity Principle:

1) The law of physics take the same form in every inertial frame.

Let us then introduce a coordinates system $x^i$, which identifies the points of $\mathcal{E}$ and a parameter $t$ called the universal time. The square modulus of the speed of a particle is defined as

$$v^2 = \frac{dx^i}{dt} \frac{dx_i}{dt}$$

where the Einstein notation is understood and $x^i = x_i$ with $i = 1,2,3$. We now introduce the second postulate imposed by the Special Relativity which can be restated as follows:

2) The speed of light $c$ in vacuum is the same for every inertial observer and it is an upper bound for the speed of the other particles.

Thus for a generic particle with speed $v$ we can write the condition:

$$v^2 \leq c^2,$$

which can be rewritten as

$$\frac{dx^i}{dt} \frac{dx_i}{dt} = c^2 \left(1 - a^2\right),$$

where $a^2 \leq 1$ is a function which will depend on the particle we are considering. With some algebra we can arrange the equation (3) in the following way

$$1 - \frac{dx^i}{d(\text{ct})} \frac{dx_i}{d(\text{ct})} = a^2$$

Now we note that we can always change the parameter with respect to what we are deriving by means of a diffeomorphism provided that $\frac{d(\text{ct})}{ds} > 0$, where we call $s$ the new parameter with $[s] = \text{meters}$. Let us then multiply both sides of equation (4) by $\left(\frac{d(\text{ct})}{ds}\right)^2$ obtaining

$$\frac{d(\text{ct})}{ds} \frac{d(\text{ct})}{ds} - \frac{dx^i}{ds} \frac{dx_i}{ds} = a^2 \left(\frac{d(\text{ct})}{ds}\right)^2.$$ 

Guided by the well-known result, we are led to interpret the quantity $ct$ not as a parameter but as a coordinate in some higher dimensional space, in this case it is 4-dimensional, so that the left hand side of equation (5) seems to be the norm of a vector in this 4-dimensional space. As an intermediate passage we then define $x^0 = ct$:

$$\frac{dx^0}{ds} \frac{dx^0}{ds} - \frac{dx^i}{ds} \frac{dx_i}{ds} = a^2 \left(\frac{dx^0}{ds}\right)^2.$$ 

However the presence of the minus sign between the first and the second term on the left hand side does not allow us to consider it as a true norm. With the intuition in mind that it should be some kind of norm and guided again by the well-known result, we could argue that the spatial coordinates ($x^i$, $i = 1,2,3$) with subscript indexes bring with them a minus sign, while the temporal coordinate ($x^0$) has the same sign in each case. Thus, defining $x^i = -x_i$ and $x^0 = x_0$ we can rewrite the equation (6) as

$$\frac{dx^0}{ds} \frac{dx^0}{ds} = a^2 \left(\frac{dx^0}{ds}\right)^2.$$ 

However the presence of the minus sign between the first and the second term on the left hand side does not allow us to consider it as a true norm. With the intuition in mind that it should be some kind of norm and guided again by the well-known result, we could argue that the spatial coordinates ($x^i$, $i = 1,2,3$) with subscript indexes bring with them a minus sign, while the temporal coordinate ($x^0$) has the same sign in each case. Thus, defining $x^i = -x_i$ and $x^0 = x_0$ we can rewrite the equation (6) as

$$\frac{dx^\alpha}{ds} \frac{dx^\alpha}{ds} = a^2 \left(\frac{dx^0}{ds}\right)^2.$$ 

\[2\] We want to remark that, with this choice, the value of the quantity $\frac{dx^i}{dt} \frac{dx_i}{dt}$ changes from $v^2$ to $-v^2$. The other possibility is $x^0 = -x_0$ and $x^i = x_i$ which would lead also to a change of sign of the right hand side of equation (6).
where $\alpha$ goes from 0 to 3. Since we have shaped the left hand side of equation (7) to be a norm, we are led to infer that the metric of this four dimensional space (now spacetime) is $\eta = \text{diag}(1, -1, -1, -1)$. We are then almost to obtain the Special Relativistic result, however we have still to analyze the right hand side of equation (7). It is well-known from geometry that, if a curve is parametrized by its curvilinear abscissa, its tangent vectors have unitary norm. If we impose then $s$ to be the curvilinear abscissa of the curve $x(s)$ over the spacetime, $a$ must satisfy the condition $a = \frac{ds}{dx}$, so that equation (7) can be written as

$$\frac{dx^\alpha}{ds} \frac{dx_\alpha}{ds} = 1.$$  (8)

In order to be symmetric with respect to the Galilean case we want the parameter over the trajectories of the particles to have the dimension of a time, so we introduce a new parameter $\tau$ defining $s = c\tau$ with $\tau = \text{seconds}$. We underline that in this way we have recovered the usual definition of proper time. Thus we can rewrite equation (8) as

$$\frac{dx^\alpha}{d\tau} \frac{dx_\alpha}{d\tau} = c^2,$$  (9)

which is the Special Relativistic constraint over 4-velocities. Summarizing, the condition (2) imposed over the 3-velocities of physical particles leads us to a theory defined by the couple $(M, \eta)$, where $M$ is a 4-dimensional spacetime (instead of the 3-space we had before) and $\eta$ is the metric over $M$, and the constraint (3) over the 4-velocities of physical particles.

Before closing this section we find useful to stress that, once we have found the semi-riemannian manifold $(M, \eta)$ where the motion takes place, we can easily satisfy the postulate 1) too. In fact we could find the isometry group under which the theory is covariant just by computing the Killing vectors, so that we would be able to satisfy the first postulate by formulating laws which are covariant with respect to this group. However we are not going to do this explicitly since the result is already well-known. Instead from the next section onward we are going to use all this machinery in order to try to address the problem of adding one more universal constant to the Special Relativity.

III. THE EMERGENCE OF A FIFTH DIMENSION

In order to apply again the procedure exposed in the previous section we have to start first by stating the postulate we want to add to the first two:

3) The Planck mass $m$ is equal for every inertial observer and the quantity $mc^2$ is an upper bound for the square root of the modulus of the 4-momentum of the elementary particles.

Following then the path of the previous section, we introduce the condition, analogous to (2),

$$p^\alpha p_\alpha \leq m^2 c^2,$$  (10)

where the momenta are defined by the relation $p_\alpha = m_p \frac{dx_\alpha}{d\tau}$ with $\tau$ the proper time, $m_p$ the mass of the particle with 4-momentum $p$ and $\alpha = 0, 1, 2, 3$. Following equation (1) we rewrite equation (10) as

$$m_p \frac{dx^\alpha}{d\tau} \frac{dx_\alpha}{d\tau} = m^2 c^2 \left(1 - a^2\right),$$  (11)

where $a^2 \leq 1$ is, as before, a function which will depend on the particle under consideration. Let us arrange this equation as

$$1 - \frac{d(x^\alpha)}{d(\frac{m_p}{m} c \tau)} \frac{d(x_\alpha)}{d(\frac{m_p}{m} c \tau)} = a^2.$$  (12)
As we did for the Galilean case, we notice that we can always change the parameter of derivation by means of a
diffeomorphism. We will call then the new parameter \( \Lambda \) with \( [\Lambda] = \text{meters} \), and we impose as before the condition
d\( \frac{m_{\text{p}} c \tau}{d \Lambda} > 0 \). Multiplying both sides of equation (12) by \( \left( \frac{m_{\text{p}} c \tau}{d \Lambda} \right)^2 \), it becomes
\[
\frac{d \left( \frac{m_{\text{p}} c \tau}{d \Lambda} \right)}{d \Lambda} \frac{d \left( \frac{m_{\text{p}} c \tau}{d \Lambda} \right)}{d \Lambda} - \frac{d (x^\alpha)}{d \Lambda} \frac{d (x^\alpha)}{d \Lambda} = a^2 \left( \frac{d \left( \frac{m_{\text{p}} c \tau}{d \Lambda} \right)}{d \Lambda} \right)^2.
\]
(13)

In the spirit of the manipulations of the previous section, we could interpret the quantity \( \frac{m_{\text{p}} c \tau}{d \Lambda} \) no longer as a
parameter but as a coordinate in a higher dimensional space, five dimensional in this case, so that, again, the left
hand side of equation (13) seems to be some sort of norm. Let us call the enlarged space \( B \) and let us define then the
coordinates in this new space in the following way:
\[
\chi^\mu = \begin{cases} 
x^\mu & \mu = 0, 1, 2, 3 
\frac{m_{\text{p}} c \tau}{d \Lambda} & \mu = 4
\end{cases},
\]
(14)
where, clearly, \( [\chi^\mu] = \text{meters} \). Before going on with our analysis we notice that in our context the d
imension of the
fifth coordinate is coherent with the others, so the problem raised in [9–13], which we have recalled in the introduction,
does not apply here.

In light of (14), we rewrite equation (13) as
\[
- \frac{d \chi^\alpha}{d \Lambda} \frac{d \chi^\alpha}{d \Lambda} + \frac{d \chi^4}{d \Lambda} \frac{d \chi^4}{d \Lambda} = a^2 \left( \frac{d \chi^4}{d \Lambda} \right)^2.
\]
(15)

We have now the same problem we had in equation (6): the relative sign between the first and the second term of
the left hand side of equation (15) does not allow us to consider the whole left hand side as a norm. Using the same
trick used in the previous section, we argue that \( \chi^\alpha = - \chi^\alpha \) while \( \chi^4 = \chi^4 \), so equation (15) can now be written as
\[
\frac{d \chi^\mu}{d \Lambda} \frac{d \chi^\mu}{d \Lambda} = a^2 \left( \frac{d \chi^4}{d \Lambda} \right)^2.
\]
(16)

Now the left hand side of equation (16) has the shape of a norm in a 5-dimensional flat semiriemannian manifold
equipped with the metric \( g = \text{diag}(-1, 1, 1, 1, 1) \). If we give again to \( \Lambda \) the role of the curvilinear abscissa over the
curve \( \chi(\Lambda) \subset B \), \( a \) must satisfy the condition \( a = \frac{d \chi^4}{d \Lambda} \) so that equation (16) becomes
\[
\frac{d \chi^\mu}{d \Lambda} \frac{d \chi^\mu}{d \Lambda} = 1.
\]
(17)

Finally, we want to describe the evolution of a particle with respect to a parameter which has the dimension of a
time, so we define \( \Lambda = \frac{m}{m_{\text{p}}} c \lambda \) with \( [\lambda] = \text{seconds} \). A physical interpretation of \( \lambda \) which distinguishes its role from the
one of the \( \tau \) appearing in \( \chi^4 \) will be given in section V. Calling then \( \frac{d \chi^\mu}{d \lambda} \) 5-velocity, its norm computed with respect
to \( \lambda \) reads as
\[
\frac{d \chi^\mu}{d \lambda} \frac{d \chi^\mu}{d \lambda} = \frac{m^2}{m_{\text{p}}^2} \left( \frac{m_{\text{p}} c \tau}{d \Lambda} \right)^2,
\]
(18)
which is then a constraint elementary particles have to satisfy in this five-dimensional framework. Before summarizing
what we have found in this section, once the 5-velocity is defined, it is straightforward to build the 5-momentum as
\[
\Pi_\mu = m_{\text{p}} \frac{d \chi^\mu}{d \lambda}.
\]
(19)

5 We use convention that the index \( \alpha \) runs from 0 to 3 while the index \( \mu \) runs from 0 to 4.
6 We note that this condition corresponds to a change of the signature of the old Minkowskian four dimensional spacetime. This in turn
implies that for example from now on the quantity \( p^\mu p_\mu \) will be negative definite.
which then satisfies the following dispersion relation

$$\Pi^\mu \Pi_\mu = m^2 c^2. \tag{20}$$

We are now going to show briefly that the definition (19) leads to the right Special Relativistic limit. In fact, manipulating first the relation (18) we obtain that

$$\frac{d\tau}{d\lambda} = \frac{1}{\sqrt{1 + \frac{p^\alpha p_\alpha}{m^2 c^2}}} \tag{21}$$

where \( p_\alpha \) is defined just like in Special Relativity as \( m_p \frac{d\chi_\alpha}{d\tau} \). In light of (21) then, we can rewrite the 5-momentum in the following way

$$\Pi_\mu = \left( m_p \frac{d\chi_\alpha}{d\lambda}, mc \frac{d\tau}{d\lambda} \right) = \left( m \frac{d\chi_\alpha}{d\tau}, mc \frac{d\tau}{d\lambda} \right) = \left( \frac{p_\alpha}{\sqrt{1 + \frac{p^\alpha p_\alpha}{m^2 c^2}}}, \frac{mc}{\sqrt{1 + \frac{p^\alpha p_\alpha}{m^2 c^2}}} \right) \tag{22}$$

It is evident that in the limit \( m \to \infty \) the first four components reduce to the usual 4-momentum.\(^7\)

Just like as at the end of the previous section, we end up with a theory defined by a new couple \((B,g)\), where the dimension of the manifold where the motion takes place is grown by one. Relations (18) and (20) are the five dimensional counterparts of the well-known Special Relativistic ones and, as shown by the equation (22) they define quantities which have the right expected limit. The next step is to find a suitable definition of the mass-shell relation for this theory. We remark that obviously in what follows we could have chosen the opposite signature of the metric without changing the results.

**IV. THE MASS-SHELL RELATION**

In Special Relativity the mass-shell relation arise quite naturally from the definitions of 4-velocity and proper time. However in \( B \) there is not such an evident relation between the mass of a particle and its 5-momentum or its 4-momentum. In the context of the Relative Locality, see \([4,5]\), where the momentum space is supposed to be curved, the value of the mass of a particle with 4-momentum \( p \) is defined as the value of the geodesical distance from the origin of the momentum space to the point, over the momentum space itself, identified by the coordinates of \( p \). We want now to take this intuition and move it into the context of this paper.

The dispersion relation (24) is a constraint over the value of the components of the 5-momentum. By a geometrical point of view, it defines a semiriemannian submanifold of \( T^*B \), in particular it is a four dimensional hyperboloid which we will call \( H \). Then, only a four dimensional chart is needed in order to parametrize it and, looking at (22), it is evident that good candidates as coordinates over \( H \) are the components of the 4-momentum \( p_\alpha \). In light of these observations, we argue that the value of the mass of a particle with 4-momentum \( p_\alpha \) is equal to the value of the geodesic distance over \( H \) between the points \((0,0,0,0,mc)\) (written in the embedding space \( T^*B \)), which we will call \( \alpha \) origin of \( H \), and \( p \) itself\(^8\), explicitly

$$D^2(0,p) = m^2 c^2. \tag{23}$$

Thus we are giving to \( H \) partially the same role the curved momentum space has for the Relative Locality\(^9\). A straightforward computation of \( D(0,p) \) (details can be found in appendix A), used in combination with the definition (24), gives the following relation between the 4-momentum and the mass of a particle:

$$p^\alpha p_\alpha = -m^2 c^2 \text{tanh}^2 \left( \frac{m_p}{m} \right), \tag{24}$$

which holds for both massive and massless elementary particles. This is then our proposal of mass-shell relation to enforce on \( B \) which we notice it is just a deformation of the Special Relativistic one and in the limit \( m \to \infty \) it reduces to \( p^\alpha p_\alpha = -m^2 c^2 \). The careful reader could be worried by the fact that the left hand side of equation (24) is no more a scalar in a five dimensional space; we will deal with this problem in section VI.

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\(^7\) We notice that the fifth component diverges just like the zero component of the Special Relativistic 4-momentum in the limit \( c \to \infty \).

\(^8\) Our choice of the point with respect to what we compute the distance appears straightforward analyzing (22), in fact it is obtained just posing \( p_\alpha = 0 \).

\(^9\) The parallelism is not complete since in this context there is no need to introduce any connection which rules the composition of the 4-momenta.
A brief comment upon massless particles

As just stated, relation (24) holds for massless particles too. Despite relation (20) is perfectly well defined, it could seem that equation (18) is not. Actually this is not the case, since equation (18) is simply stating that massless particles have the modulus of their 5-velocity infinite. Moreover, by means of the relation (21), it can be easily seen that this in turn implies that the modulus of their 4-velocity is zero, in perfect agreement with the Special Relativity.

V. THE PROPAGATION OF A FREE PARTICLE

The aim of this section is to give a first phenomenological prediction of the setup developed so far for the propagation of particles, in particular we will focus our attention on the corrections to the speed of a free elementary particle. In order to do this, we have to derive first the shape of the equation of motion of a freely propagating particle over $\mathcal{B}$. To be precise we should first study the group under which the theory should be covariant i.e. which allows to change between inertial frames. However we anticipate that it will be, as expected, the isometry group and thus we will be here satisfied to formulate the law of the propagation of a free particle as a relation involving only 5-vectors.

Let us start observing that the Special Relativistic equations of motion of a free particle, i.e.

$$\frac{d^2 x^\alpha}{d\tau^2} = 0,$$  \(25\)

must hold at least in the limit $m \to \infty$. In light of (25), we infer that the equations of motion of the first four components of the coordinate system should be

$$\frac{d^2 \chi^\alpha}{d\lambda^2} = \left(\frac{d\tau}{d\lambda}\right)^2 \frac{d^2 x^\alpha}{d\tau^2} = 0.$$  \(26\)

It can be argued that the equations $\frac{d^2 x^\alpha}{d\tau^2} = 0$ could not hold exactly in the context we are analyzing, but, in the most general case, on the right hand side terms of order at least $O(m^{-1})$ could appear. However, in the transition from Galilean Relativity to Special Relativity, the equations $\frac{d^2 x^i}{dt^2} = 0$ keep holding exactly, so, arguing that the same feature holds here, we impose that $\frac{d^2 x^\alpha}{d\tau^2} = 0$ on $\mathcal{B}$ too.

For the fifth coordinate we notice that the constraint (18) must always hold, so, combining it with the equation (26), we deduce that also $\frac{d\chi^4}{d\lambda}$ have to be a constant during a free motion i.e.

$$\frac{d^2 \chi^4}{d\lambda^2} = 0.$$  \(27\)

Summarizing, the equations of motion of a freely propagating particle over $\mathcal{B}$ are simply

$$\frac{d^2 \chi^\mu}{d\lambda^2} = 0.$$  \(28\)

It is easy to deduce from (28) the conservation of the 5-momentum, in fact, taking into account the definition (19), equation (28) can be rewritten as

$$\frac{d\Pi_\mu}{d\lambda} = 0.$$  \(29\)

Some kind of link between equation (29) and the well-known conservation of the 4-momentum of the Special Relativity is then expected. We have already established that the first four components of the 5-momentum reduce to the Special Relativistic 4-momentum in the limit $m \to \infty$ (cfr. (22)). It can be easily inferred then that the first four components of the (29) are simply stating a generalization of the conservation of the 4-momentum\(^\text{10}\). For what concerns the fifth

\(^{10}\) In Special Relativity, in the limit $c \to \infty$, the spatial part of the 4-momentum reduces to the Galilean 3-momentum in the same way.
component of the (29), it is instead stating a new conservation law, explicitly
\[
\frac{d\Pi_4}{d\lambda} = \frac{d}{d\lambda} \left( \frac{mc}{\sqrt{1 + \frac{p^\mu p_\mu}{m^2 c^2}}} \right) = 0. \tag{30}
\]
Since the only dynamical quantities between the parenthesis are the \(p_\alpha\)'s, taking into account the relation (24), it follows that the equation of motion (30) is stating that in a free motion the mass of an elementary particle does not change\(^\text{11}\).

Now we have all the instruments to focus our attention on the speed of a freely propagating particle, say, along the positive \(\chi^1\) direction with spatial speed \(v\); then \(\Pi_2 = \Pi_3 = p_2 = p_3 = 0\). The following chain of equalities holds
\[
\frac{v}{c} = \frac{\chi^1}{\chi^0} = \frac{\Pi^1}{\Pi^0} = \frac{\Pi_1}{\Pi_0} = -\frac{p_1}{p_0}, \tag{31}
\]
where the dot means the derivation with respect to \(\lambda\) and the second equality follows from the definition (19). Now we use the mass-shell relation (24) derived in the previous section which can be rewritten, expliciting the contraction over \(\alpha\), as
\[
p_0 = -\sqrt{(p_1)^2 + m^2 c^2 \tanh^2 \left( \frac{m_p}{m} \right)}, \tag{32}
\]
where the presence of the minus sign depends on the signature of the metric. Substituting in (31) the value of \(p_0\) found in (32) we obtain that
\[
\frac{v}{c} = \frac{p_1}{\sqrt{(p_1)^2 + m^2 c^2 \tanh^2 \left( \frac{m_p}{m} \right)}}. \tag{33}
\]
Finally, in order to confront more easily this value with the Special Relativistic one, we develop the denominator in powers of \(m^{-1}\) up to the first non zero correction, obtaining
\[
\frac{v}{c} = \frac{p_1}{\sqrt{(p_1)^2 + m^2 c^2 \tanh^2 \left( \frac{m_p}{m} \right)}} \left[ 1 + \frac{1}{3} \frac{m_p^2 c^2}{m^2} \sqrt{(p_1)^2 + m^2 c^2} \right]. \tag{34}
\]
Evidently the second term in square parenthesis is a correction to the Special Relativistic result which is unfortunately extremely small to be measured. However, from a theoretical point of view, this result protects the setup formulated so far from being a simple non linear reformulation of the Special Relativity. Finally we notice that, since the mass-shell relation (24) holds for both massive and massless particles, the speed of the latter can be computed just posing \(m_p = 0\) in the (33). In this case it reduces exactly to the Special Relativistic result, \(i.e.\) massless particles travels at \(c\). This in turn implies that the context we here propose does not predict any kind of delay, with respect to the Special Relativity, in the detection of massless particles.

**On the difference between \(\tau\) and \(\lambda\)**

Now that we have both the mass-shell relation and the equation of propagation of a free particle we are ready to give an interpretation to the fifth dimension. We recall that its definition is
\[
\chi^4 = \frac{m}{m_p} c \tau. \tag{35}
\]
In Special Relativity, \(c \tau\) is the length of the path travelled by the particle on the spacetime computed from a certain point fixed by the observer. The free equation of motion for \(\chi^4\) is
\[
\frac{d^2 \chi^4}{d\lambda^2} = \frac{mc}{m_p} \frac{d^2 \tau}{d\lambda^2} = 0, \tag{36}
\]
\(^\text{11}\) Despite the empirical results on the neutrino oscillation seems to go against this, we remark that this theory is still classical, since \(\hbar \to 0\), thus purely quantum effects are not taken into account.
then, using the (21) and the mass-shell relation (24), follows that
\[
\frac{d\tau}{d\lambda} = \frac{1}{\sqrt{1 - \tanh^2 \left( \frac{m_p}{m} \right)}} = \text{cost},
\]
(37)
which means that \( \tau \) flows faster than \( \lambda \) by a factor which depends on the mass of the particle we are considering. We now want to take the trace \( \delta \) of a particle over \( B \) and compute the length of its projection over a slice of \( B \), namely a spacetime. Thus consider the following quantity
\[
L[\delta|\mathcal{M}] = \int_a^b d\lambda \sqrt{\frac{dx^\alpha}{d\lambda} \frac{dx^\alpha}{d\lambda}} = \int_a^b d\lambda \sqrt{\frac{dx^\alpha}{d\tau} \frac{dx^\alpha}{d\tau}} = \frac{m}{m_p} c \tanh \left( \frac{m_p}{m} \right) \Delta \lambda \sqrt{1 - \tanh^2 \left( \frac{m_p}{m} \right)},
\]
(38)
where in the third equality we have used the (24) and the (37). Finally, using again the (37), we can write
\[
L[\delta|\mathcal{M}] = \frac{m}{m_p} \tanh \left( \frac{m_p}{m} \right) c \Delta \tau.
\]
(39)
Since in the limit \( m \to \infty \) this expression reduces to \( L[\delta|\mathcal{M}] = c \Delta \tau \), we can infer that the \( c \tau \) appearing in \( \chi^4 \) is the spacetime distance travelled by the observed particle when one neglects the effects due to the mass of the particle itself. It seems natural at this point to give to \( \lambda \) the role of a true proper time, meaning that is the time actually “measured” by the observed particle. A similar result can be found in an observation done in [8], where the author, looking for a purely DSR effect, combines the Compton length of a particle with the fact that, according to General Relativity, the mass of a particle slows its proper time. We finish this section noting that this feature, found using quantum and gravitational arguments, arises naturally in our formalism.

VI. THE ISOMETRY GROUP OF \( B \)

The first postulate of the theory we are here proposing state that the laws of physics must be the same in all inertial reference system. In Special Relativity one deduces that the mutually inertial systems are the ones related by the isometry transformations of the Minkowski spacetime. This deduction is achieved because these transformations are exactly, or a deformation of, the transformations which link mutually inertial frames in classical mechanics. One of the goal of this section is then to demonstrate that the isometry group of \( (B, g) \) plays the same role as the Poincarè group in Special Relativity.

As observed at the end of section [11] once a manifold with its metric are specified, the analysis of its symmetries is straightforward using the Killing vectors which, we recall briefly, are defined by the relation
\[
L_{\xi}g = 0,
\]
(40)
where \( L \) is the Lie derivative and \( \xi = \xi^\mu(\chi) \frac{\partial}{\partial \chi^\mu} \) is a Killing vector field, in this case over \( B \). In coordinates this equation reads as
\[
\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0,
\]
(41)
where the semicolon stands for the covariant derivative. Despite the fifth coordinate \( \chi^5 \) is not strictly a spatial coordinate, geometrically the metric \( g \) is nonetheless flat, so the Killing equation (41) reduces to
\[
\xi_{\mu;\nu} + \xi_{\nu;\mu} = 0,
\]
(42)
where the comma stands for the ordinary derivative. The solution of the (42) is then the well-known expression
\[
\xi^\mu(\chi) = \Lambda^\mu_{\nu} \chi^\nu + d^\mu,
\]
(43)
where \( \Lambda_{\mu\nu} \) is an antisymmetric matrix and \( d^\mu \) is a constant vector with \( [\Lambda_{\mu\nu}] = 1 \) and \( [d^\mu] = \text{meters} \). A first observation is that, since \( b^\mu \) is a vector in a five dimensional space and \( \Lambda \) is a \( 5 \times 5 \) antisymmetric matrix, there will be 15 independent transformations instead of the 10 of the Special Relativity. In order to classify these transformations, we use the fact that the Killing vectors can be thought as infinitesimal transformations, so that we can write
\[
\chi^\mu = \chi^\mu + \sigma \xi^\mu,
\]
(44)
where $\sigma$ is the parameter of the transformation and $\chi'^\mu$ is the value of the coordinate after the transformation. We will then solve the integral curve equation

$$\frac{d\chi'^\mu(\sigma)}{d\sigma} = \xi^\mu$$

(45)

to obtain the finite transformations. Before we begin with their systematic analysis, we give a parametrization of the $\Lambda$ matrix which will turn out to be very useful in the classification, explicitly

$$\chi'^\mu = [g^{\mu\nu} + \sigma (b'^\mu a'^\nu - a'^\mu b'^\nu)] \chi_\nu,$$

(46)

where $a'^\mu$ and $b'^\mu$ are both independent five dimensional vectors of $\mathcal{B}$. Thus the interpretation will rely on the proper choice of such vectors.

### A. Translations

The constant vector $d'^\mu$ clearly induces a translation over $\mathcal{B}$, there is no need then to pass through the infinitesimal transformation and we can just write down the finite result as

$$\chi'^\mu = \chi^\mu + d'^\mu.$$  
(47)

The first four components of $d'^\mu$ clearly produce the usual spacetime translations while for the fifth a little bit more attention is needed. For a generic $d^4$ the fifth component of the (47) reads as

$$\chi'^4 = \chi^4 + d^4.$$  
(48)

Since $\chi^4 = \frac{m}{m_p}ct$, at a first sight it could seem that this fifth translation may induce some kind of change of the mass of the particle we are following. We argue instead that the interpretation is the simplest as possible: according to us, $d^4$ is just a translation of the proper time, i.e. a change of the point on the worldline of the particle from which we are computing the proper time. Explicitly we assume that for a particle with mass $m_p$ the parameter $d^4$ should be interpreted as

$$d^4 = \frac{m}{m_p}c\Delta,$$

(49)

where $\Delta$ is the translation factor over the proper time.

### B. Rotations

Let us now analyze the transformations induced by the matrix $\Lambda_{\mu\nu}$. As said before we just have to focus on the transformations induced by different choices of the vectors $a'^\mu$ and $b'^\mu$. We start with the choice $a'^\mu = (0, 1, 0, 0, 0)$ and $b'^\mu = (0, 0, 1, 0, 0)$. The infinitesimal transformation (46) reads explicitly as

$$\begin{cases}
\chi'^0 = \chi^0 \\
\chi'^1 = \chi^1 - \sigma \chi^2 \\
\chi'^2 = \chi^2 + \sigma \chi^1 \\
\chi'^3 = \chi^3 \\
\chi'^4 = \chi^4
\end{cases}$$

(50)

If we integrate it, it is easy to see that the transformation (50) is a rotation around the $\chi^3$ axis, thus, giving to $\sigma$ the role of an angle and renaming $\sigma = \vartheta$, the finite transformation is

$$\begin{cases}
\chi'^0 = \chi^0 \\
\chi'^1 = \chi^1 \cos \vartheta - \chi^2 \sin \vartheta \\
\chi'^2 = \chi^2 \cos \vartheta + \chi^1 \sin \vartheta \\
\chi'^3 = \chi^3 \\
\chi'^4 = \chi^4
\end{cases}$$

(51)

The choices $a'^\mu = (0, 1, 0, 0, 0)$, $b'^\mu = (0, 0, 0, 1, 0)$ and $a'^\mu = (0, 0, 1, 0, 0)$, $b'^\mu = (0, 0, 0, 1, 0)$, will obviously lead respectively to the spatial rotations around the $\chi^2$ and $\chi^1$ axes.
Finally, rewriting it in terms of the parameters $\beta$ the rapidity and renaming $\sigma$, the integration of the (52) allows to interpret it, at least formally, as a boost along the $\chi^1$ axis. Letting thus $\sigma$ be the rapidity and renaming $\sigma = \psi$, the finite transformation reads as

$$\begin{align*}
\chi^0 &= \chi^0 - \sigma \chi^4 \\
\chi^1 &= \chi^1 - \sigma \chi^0 \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4
\end{align*}$$

(53)

Finally, rewriting it in terms of the parameters $\beta = \sinh \psi$ and $\beta \gamma = \cosh \psi$, with $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$, it becomes

$$\begin{align*}
\chi^0 &= \gamma (\chi^0 - \beta \chi^1) \\
\chi^1 &= \gamma (\chi^1 - \beta \chi^0) \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4
\end{align*}$$

(54)

Since we are not able to find good reasons to give it a different physical meaning with respect to the one it has in Special Relativity, we infer that $\beta = \frac{v}{c}$. The choices $a^{\mu} = (1,0,0,0)$, $b^{\mu} = (0,0,1,0)$ and $a^{\mu} = (1,0,0,0)$, $b^{\mu} = (0,0,0,1)$ lead us to the boosts respectively in the $\chi^2$ and $\chi^3$ direction. Since we have so far recovered, translations, rotations and boosts we can argue that the isometry group of $(\mathbb{E}, g)$ encloses the transformations which link mutually inertial frames. Before moving to the new set of transformations it is worth noting that the mass-shell relation (24) is covariant under the action of the transformations found so far.

### C. Boosts

Our next choice is $a^{\mu} = (1,0,0,0)$ and $b^{\mu} = (0,1,0,0)$ so the infinitesimal transformation (46) becomes

$$\begin{align*}
\chi^0 &= \chi^0 \\
\chi^1 &= \chi^1 - \sigma \chi^0 \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4
\end{align*}$$

(52)

The integration of the (52) allows to interpret it, at least formally, as a boost along the $\chi^1$ axis. Letting thus $\sigma$ be the rapidity and renaming $\sigma = \psi$, the finite transformation reads as

$$\begin{align*}
\chi^0 &= \chi^0 \cosh \psi - \chi^1 \sinh \psi \\
\chi^1 &= \chi^1 \cosh \psi - \chi^0 \sinh \psi \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4
\end{align*}$$

(53)

Finally, rewriting it in terms of the parameters $\beta = \sinh \psi$ and $\beta \gamma = \cosh \psi$, with $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$, it becomes

$$\begin{align*}
\chi^0 &= \gamma (\chi^0 - \beta \chi^1) \\
\chi^1 &= \gamma (\chi^1 - \beta \chi^0) \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4
\end{align*}$$

(54)


### D. Momentum boosts

This subsection is dedicated to the last four transformations left which in literature (see [9–13]) have been referred to as momentum boosts. As we will see by a formal point of view they are not strictly boosts, but we will keep this name. In this subsection we are only going to give their formal definition leaving our physical interpretation of them to the next section. Let us then choose $a^{\mu} = (1,0,0,0)$, $b^{\mu} = (0,0,0,1)$. The infinitesimal transformation (46) becomes then

$$\begin{align*}
\chi^0 &= \chi^0 - \sigma \chi^4 \\
\chi^1 &= \chi^1 \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4 - \sigma \chi^0
\end{align*}$$

(55)

Renaming $\sigma = \psi'$ and integrating, the finite transformation reads as

$$\begin{align*}
\chi^0 &= \chi^0 \cosh \psi' - \chi^4 \sinh \psi' \\
\chi^1 &= \chi^1 \\
\chi^2 &= \chi^2 \\
\chi^3 &= \chi^3 \\
\chi^4 &= \chi^4 \cosh \psi' - \chi^0 \sinh \psi'
\end{align*}$$

(56)
The next choice is \( a^\mu = (0, 1, 0, 0), b^\mu = (0, 0, 0, 1) \) which induces the infinitesimal transformation

\[
\begin{cases}
\chi^0 = \chi^0 \\
\chi^1 = \chi^1 - \sigma \chi^4 \\
\chi^2 = \chi^2 \\
\chi^3 = \chi^3 \\
\chi^4 = \chi^4 + \sigma \chi^1.
\end{cases}
\]  

(57)

Its finite version, renaming \( \sigma = \vartheta' \), reads as

\[
\begin{cases}
\chi^0 = \chi^0 \\
\chi^1 = \chi^1 \cos \vartheta' - \chi^4 \sin \vartheta' \\
\chi^2 = \chi^2 \\
\chi^3 = \chi^3 \\
\chi^4 = \chi^4 \cos \vartheta' + \chi^1 \sin \vartheta'.
\end{cases}
\]  

(58)

We notice that this transformation formally is a rotation. The choices \( a^\mu = (0, 0, 0, 1), b^\mu = (0, 0, 0, 1) \) and \( a^\mu = (0, 0, 1, 0), b^\mu = (0, 0, 0, 1) \) give us the other two possible rotations involving respectively, in place of the \( \chi^1 \) axis, the \( \chi^2 \) and \( \chi^3 \) axes.

VII. PROPOSAL OF INTERPRETATION OF THE MOMENTUM BOOSTS

The physical interpretation of the Lorentz boost can be easily achieved analyzing its infinitesimal version because its spatial part reduces to the well-known Galilean boost. Thus one infers that the Lorentz boost codifies the deformations induced by the change between system of reference with a relative speed. In order to propose a physical interpretation of the momentum boost we then focus our attention on the transformations (55) and (57) which we report here in a synthetic way

\[
\begin{cases}
\chi^0 = \chi^0 - \psi' \chi^4 \\
\chi^4 = \chi^4 + \psi' \chi^0,
\end{cases}
\]  

(59)

\[
\begin{cases}
\chi^1 = \chi^1 - \vartheta' \chi^4 \\
\chi^4 = \chi^4 + \vartheta' \chi^1.
\end{cases}
\]  

(60)

Here the task of the identification of a physical interpretation is more complicated because we do not know yet what is the role of \( \psi' \) and \( \vartheta' \). What we know is that they must be the ratio of a physical quantity, which defines the property that distinguishes between the new and the old system of reference, and a relevant scale of the theory. An obvious choice for the scale is clearly the quantity \( mc \) which is the one we have introduced in order to deform the Special Relativity in section III. Moreover, since the new transformations are exactly four, it seems reasonable to choose the physical quantities at the numerators to be the components of a 4-vector with the dimension of a 4-momentum. At this point it seems evident that these transformations will deal then with the change of 4-momentum of the reference frame. In order to clarify this, we are now going to discuss an explicit example, in a Special Relativistic context, where the change of momentum of the reference frame is implied, creating a link with the (59) and (60). We will then discuss the generalizations.

A. A simple example

Let us consider in a Special Relativistic context a scattering between two elementary particles with masses respectively \( m_p \) and \( m_q \), 4-velocities before the collision \( u \) and \( w \) and after \( u' \) and \( w' \). Let us underline that the two particles before and after the collision are in a free motion. Since we are dealing with a scattering, the masses of the two particles do not change and the process is characterized by the conservation law

\[
p + q = p' + q',
\]

(61)

\footnote{Needless to say that the other two possibilities behave exactly as the (60).}
where $p = m_p u$, $q = m_q w$, $p' = m_p u'$ and $q' = m_q w'$. We now rearrange the (61), expliciting its dependence upon masses and 4-velocities, as

$$u' = u - \frac{m_q}{m_p}(w' - w).$$  \hspace{1cm} (62)

It can be shown (see appendix B) that this relation can be manipulated into

$$x' = x - \frac{m_q}{m_p}(w' - w)\tau,$$  \hspace{1cm} (63)

where $x' = u'\tau$, $x = u\tau$ and $\tau$ is the proper time of the particle with mass $m_p$. As derived in the appendix B the interpretation of this formula is straightforward: the position of the particle with mass $m_p$ after the collision is the one it would have if the collision was not occurred plus a correction given by the second member on the right hand side. Since we are in a Special Relativistic context, relation (63), which has been derived using 4-vectors, holds in any reference frame, in particular it holds in the reference frame attached to the particle with mass $m_q$. Giving then to the particle with mass $m_q$ the role of a massive reference frame, it is evident that the second member on the right hand side of the (63) encloses exactly the change of momentum of the reference frame we were looking for. If we then assume, without loss of generality, that after the collision the massive reference frame travels along the $x^1$ axis (in its coordinatization before the collision), defining

$$\psi' = \frac{m_q(w^0 - w^0)}{mc},$$  \hspace{1cm} (64)

$$\vartheta' = \frac{m_q(w^1 - w^1)}{mc},$$  \hspace{1cm} (65)

and using the definition (14), the transformations (59) and (60) read as

$$\begin{align*}
x^0 &= x^0 - \frac{m_q}{m_p}(w^0 - w^0)\tau \\
\tau' &= \tau - \frac{m_p m_q (w^0 - w^0)}{m^2 c^2} x^0
\end{align*}$$  \hspace{1cm} (66)

$$\begin{align*}
x^1 &= x^1 - \frac{m_q}{m_p}(w^1 - w^1)\tau \\
\tau' &= \tau + \frac{m_p m_q (w^1 - w^1)}{m^2 c^2} x^1
\end{align*}$$  \hspace{1cm} (67)

which in the limit $m \to \infty$ reduce exactly to the (63). According to this result we are led to interpret the momentum boost as the transformation which codifies the deformations in the coordinatization of a reference frame due to the change of 4-momentum of the reference frame itself.

B. Further considerations

The interpretation we gave at the end of the previous subsection gives rise to many questions. The first is that the transformations (59) and (60), as we have derived them, hold in every (Lorentz) boosted system of reference, causing then the values of $\psi'$ and $\vartheta'$, which define the strength of the deformations, to be not uniquely defined. We can easily circumvent this problem stating that their value is the one measured in a system attached to the massive reference frame before the collision and which afterwards keeps on going on the same direction. In order to clarify this we can give an explicit formula: the 4-momentum of the massive reference frame before the collision, measured by a system attached to it, is clearly $q = (m_q c, 0)$. After will be $q' = (m_q \gamma c, m_q \gamma v)$, where $v$ is the spatial speed it will acquire and $\gamma$ is the usual Lorentz factor, thus

$$\psi' = \frac{m_q (\gamma - 1) c}{mc},$$  \hspace{1cm} (68)

$$\vartheta' = \frac{m_q \gamma v}{mc}.  \hspace{1cm} (69)$$

\[^{13}\text{This is true even in our five-dimensional formulation since the Lorentz boost is undeformed.}\]
A second issue is that the (66) and (67) are two different transformations of the isometry group but they have to be applied both in order to give the right Special Relativistic limit. Then it could seem that these two transformations are no more independent. Actually this is not the case. In fact the transformations of the momentum boost are still independent. Moreover, taken separately, they can cause macroscopic changes in the mass of the particles as can be easily seen analyzing the equation (24) and as it was already pointed out in [8]. This problem can be solved noting that the changes of momentum of massive reference frames are not arbitrary because they follow the conservation laws. Thus applied to physical systems, the transformations of the momentum boost are forced by the conservations laws to be taken into account not separately so no macroscopic change of the masses is implied. Furthermore we notice that in the derivation performed in the previous subsection, all the particles are on-shell by construction. We have to underline that however that the transformations (65) and (56) do not commute but, at this stage, this question can be only answered by experiments.

A third problem is that after the collision the massive reference frame has acquired a spatial speed thus we cannot say to be still attached to it and thus we are no longer on a massive reference frame. We then postulate that a momentum boost should be followed by the usual Lorentz boost in the direction and with the speed the massive reference frame has acquired after the collision. We find this consideration particularly interesting since it allows the momentum boost to be considered as a deformation of the Lorentz boost.

Another concern arises about what happen to the coordinatization of the particles which are not involved in the collision. The answer is again in the way we have derived the Special Relativistic limit of the momentum boost. Since a particle which is not involved in the collision does not change its momentum, its contribution in the (61) simply cancels out. Thus the deformations due to the momentum boost occur only for the coordinatization of the particle which exchanges its momentum with the particle reference frame.

Before concluding this subsection we note that it could seems that in some way we have enlarged the class of property which defines two mutually inertial frames. Actually this is note the case: the Galilean definition of mutually inertial frame keeps holding since the massive reference system does change its speed after the collision. The momentum boost adds simply a further specification of the change of its state of motion.

C. Generalizations

So far we have analyzed the application of the momentum boost to a very specific case: the scattering. In this subsection we want to discuss the possible generalizations of its applications recalling that we are always considering the case of elementary particles. A first generalization could be that the two particles before the collision are different from the particles after because for example, in a quantum scenario, an exchange of quantum numbers between the particles have occurred. Despite in a classical setup this could be allowed, we notice that in a quantum perspective the massive reference frame loses its identity after the collision so it is no more identifiable. In fact, the choice of the new massive reference frame after the collision would be totally arbitrary. Thus we feel we can exclude such processes from the possible extension of applicability, in fact despite \(\hbar \to 0\), the strength of the momentum boost is ruled by \(m\) which should enclose to some extent quantum features. We notice that this restriction in turn constrains the other particle (in the binary collision) to keep its nature too.

Finally we make a brief comment on collision of more than two particles. If we consider a collision, for example, of three particles, the conservation law would be

\[ p + q + k = p' + q' + k'. \]

Choosing the particle with momentum \(p\) to be our massive reference frame, if one performs the same manipulations shown in the appendix [B] it can be easily seen that this process would lead to different transformations for the particles with momenta \(q\) and \(k\). Namely the deformation parameters for the particles with momenta \(q\) and \(k\) will be respectively

\[ \sigma_q = \frac{\Delta k + \Delta p}{mc}, \quad \sigma_k = \frac{\Delta q + \Delta p}{mc}, \]

where we have omitted to specify the components of the momenta. This is not acceptable since, for example, a law involving the 5-momenta of different particles would not be covariant because every 5-momentum would change with a different law. Thus a multiparticle collision cannot be treated by the momentum boost. It is worth noting that however more then binary collision are extremely rare.

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14 We recall that the definition is: two frames are mutually inertial if they are standing or travel at a constant speed with respect to each other.
VIII. THE SOCCER-BALL PROBLEM

The soccer-ball problem is an issue which, has been pointed out, arises in theories where another observer invariant scale is tried to be added, has happens in DSR theories (see for example [15, 16]). Briefly, the deformations introduced by the second observer independent scale are usually related to the Planck Energy or (as it is in our case) to the Planck Mass. Since macroscopic bodies have energies and masses far beyond these scales, the effects of the deformations should be easily seen for composite objects. Has already been underlined in [6] and [7] the authors pointed out that this problem can be solved in their five dimensional approach to the DSR since the deformation scale is not the same for elementary particles and composite bodies. Nonetheless a solution has recently been given in the Relative Locality approach too, see for example [17]. In this section we make a simple explicit computation showing that, in our formulation, the soccer-ball problem can be still solved.

Let us then consider the relation (20) which we report here for clarity:

\[ \Pi^\mu \Pi_\mu = m^2 c^2. \]  

(72)

Has we have derived it, this relation should hold only for elementary particles. Let us then consider a body made of two of them with 5-momenta \( \Pi \) and \( \Gamma \) which both satisfy the (72). We parametrize them, following the (22), as

\[
\Pi_\mu = \left( \frac{p_\alpha}{\sqrt{1 + \frac{p^2_{\alpha} m^2}{c^2}}}, \frac{mc}{\sqrt{1 + \frac{p^2_{\alpha} m^2}{c^2}}} \right) = (\gamma'_p p_\alpha, \gamma'_p mc),
\]

(73)

\[
\Gamma_\mu = \left( \frac{q_\alpha}{\sqrt{1 + \frac{q^2_\alpha m^2}{c^2}}}, \frac{mc}{\sqrt{1 + \frac{q^2_\alpha m^2}{c^2}}} \right) = (\gamma'_q q_\alpha, \gamma'_q mc),
\]

(74)

where we have called \( \gamma'_p = \frac{1}{\sqrt{1 + \frac{p^2_{\alpha} m^2}{c^2}}} \). The total 5-momentum of the composite particle will then be \( \Sigma_\mu = \Pi_\mu + \Gamma_\mu \), which norm is

\[
\Sigma^\mu \Sigma_\mu = \Pi^\mu \Pi_\mu + \Gamma^\mu \Gamma_\mu + 2 \Pi^\mu \Gamma_\mu = (2 + \gamma'_p \gamma'_q) m^2 c^2 + \gamma'_p \gamma'_q p^\alpha q_\alpha,
\]

(75)

We finally notice that the quantity \( p^\alpha q_\alpha \) is strictly positive since, in our convention \( p_0, q_0 < 0 \) and \( sgn(p_i) = sgn(q_i) \) because otherwise the composed particle would not exist. It is then evident that the norm of \( \Sigma \), which is made of two particles, is bigger then \( m^2 c^2 \). So we can define the scale of the deformations which affects the composite body as

\[
\Sigma^\mu \Sigma_\mu = m^2 c^2,
\]

(76)

where \( m^2 = (2 + \gamma'_p \gamma'_q) m^2 + \gamma'_p \gamma'_q \frac{p^\alpha q_\alpha}{c^2} \), which roughly grows faster then the number of constituents. We conclude this section noting that the same computation made for bodies composed of more then two particles would lead to even bigger norms, which in the case of macroscopic bodies would be proportional at least to the Avogadro number. Thus according to our interpretation this solves the soccer-ball problem in our framework too.

IX. CONCLUSIONS

In this paper we have shown a procedure to build up a DSR theory once its fundamental postulates are fixed. Clearly, a debate on the way we have chosen such postulates (in particular the last one) is unavoidable. Moreover, such a formal construction with the emergence of a fifth dimension could be just seen as an academic exercise. Nonetheless, to the best of our knowledge, the proposal of DSR here reported is the first which has all together a well-grounded geometrical foundation, a right Special Relativistic limit and a reasonable physical interpretation. The first of them is a natural requirement as long as we want to generalize a theory with deep geometrical foundations such as the Special Relativity, the second is a necessary condition to satisfy for a theory which hopes to be predictive while the third is a test for the coherence of the whole framework.

Despite the emergence of a fifth dimension could seem a radical proposal, the justification for its introduction can be found in the fact that both \( c \) and \( m \) share the same logical role: being observer independent constants. Let us specify this better: section [11] shows a formal procedure, justified by the accuracy of the result, to derive the Special Relativity directly from its postulates, enforcing them in the context of the Galilean Relativity. This procedure give
rise to the intuition of considering the time as an extra dimension to add to the three of the classical mechanics. We argue that, from a formal point of view, the introduction of a second observer independent constant beside the speed of light in vacuum $c$, as required by theories such as DSR, should be treated in the same way. From this perspective the introduction of one more dimension appear to be the natural consequence of having two observer independent constant instead of one.

It is worth noting that the approach here proposed and the Relative Locality, despite they use totally different frameworks, end up both with a relative concept of spacetime. In fact, in the former, a certain spacetime is just a slice of the whole $B$ space, taken at a particular value of the $\chi^4$ axis, while in the latter, different spacetimes are seen as different tangent spaces to the momentum space, which thus has assumed a more fundamental role (cfr. \[4, 5\]).

In concluding this paper, we have however to underline that, at this stage of development, the correction introduce by our theory to the predictions of the Special Relativity (see section $V$) are extremely small ($O(m^2)$), thus a direct test seems to be reachable only far in the future.

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**Appendix A: Geodesic distance over $\mathcal{H}$**

In this appendix we are going to show explicitly how to compute the the geodesical distance from the origin of the hyperboloid $\mathcal{H}$, defined by the relation \[20\], to a generic point upon it. In order to do this, we will use a rescaled version of the $\Pi_\mu$s and not the $p_\alpha$s since, as it will become clear in a while, this choice will make all computations easier. However this choice should not cause any concern since a geodesic distance is a geometric invariant, so the coordinate system used to compute it does not affect the result. We will explicit the dependence over $p$ in the end.

Let us rescale then the 5-momenta defining $\tilde{\Pi}_\mu = \frac{\Pi_\mu}{mc}$ so that the relation \[22\] becomes

$$\tilde{\Pi}_\mu = \left( \frac{p_\alpha/mc}{\sqrt{1 + \frac{p_\alpha^2}{m^2c^2}}}, \frac{1}{\sqrt{1 + \frac{p_\alpha^2}{m^2c^2}}} \right), \quad (A1)$$

with $[\Pi_\mu] = 1$, the dispersion relation \[20\], which defines $\mathcal{H}$, becomes

$$\tilde{\Pi}^\mu\tilde{\Pi}_\mu = 1 \quad (A2)$$

and the origin of the rescaled hyperboloid, which we will call $\tilde{\mathcal{H}}$, becomes $(0, 0, 0, 0, 1)$.

In order to compute the geodesic distance over $\mathcal{H}$ we use the same trick used in \[14\]: we exploit the properties of the embedding space $B$, which is flat, noting that a geodesic over $\mathcal{H}$ can be described by the Lagrangian

$$L = \frac{d\tilde{\Pi}^\mu}{d\rho} \frac{d\tilde{\Pi}_\mu}{d\rho} + \omega(\tilde{\Pi}^\mu\tilde{\Pi}_\mu - 1), \quad (A3)$$

where $\rho$ is the parameter over the geodesic and $\omega$ is a Lagrange multiplier enforcing the constraint on the trajectory of the geodesic $\tilde{\Pi}_\mu(\rho)$. The Lagrange equations obtained from \[A3\] are

$$\ddot{\tilde{\Pi}}^\mu - \omega\dot{\tilde{\Pi}}^\mu = 0, \quad (A4)$$

$$\ddot{\tilde{\Pi}}^\mu\tilde{\Pi}_\mu - 1 = 0 \quad (A5)$$

with $\dot{\tilde{\Pi}}_\mu = \frac{d\tilde{\Pi}}{d\rho}$. In order to solve the equation \[A4\], we have to distinguish the cases $\omega > 0$, $\omega < 0$ and $\omega = 0$. If $\omega > 0$, the solution of \[A4\] is

$$\Pi^\mu(\rho) = A^\mu \cos(\sqrt{\omega}\rho) + B^\mu \sin(\sqrt{\omega}\rho). \quad (A6)$$

Imposing the constraint \[A3\], we obtain that

$$A^\mu A_\mu \cos^2(\sqrt{\omega}\rho) + B^\mu B_\mu \sin^2(\sqrt{\omega}\rho) + 2A^\mu B_\mu \sin(\sqrt{\omega}\rho) \cos(\sqrt{\omega}\rho) = 1, \quad (A7)$$
which implies the conditions

\[ A^\mu A_\mu = B^\mu B_\mu = 1, \quad A^\mu B_\mu = 0. \]  
(A8)

If \( \omega < 0 \), the solution of (A4) is

\[ \Pi^\mu(\rho) = A^\mu e^{-\sqrt{\omega\rho}} + B^\mu e^{\sqrt{\omega\rho}}. \]  
(A9)

Imposing the constraint (A5) we obtain the equation

\[ A^\mu A_\mu e^{-2\sqrt{\omega\rho}} + B^\mu B_\mu e^{2\sqrt{\omega\rho}} + 2A^\mu B_\mu = 1, \]  
(A10)

which implies the conditions

\[ A^\mu A_\mu = B^\mu B_\mu = 0, \quad A^\mu B_\mu = \frac{1}{2}. \]  
(A11)

If \( \omega = 0 \), the solution of (A4) is

\[ \Pi^\mu(\rho) = A^\mu \rho + B^\mu. \]  
(A12)

Imposing the constraint (A5) we obtain the equation

\[ A^\mu A_\mu \rho^2 + B^\mu B_\mu + 2A^\mu B_\mu \rho = 1, \]  
(A13)

which implies the conditions

\[ A^\mu A_\mu = A^\mu B_\mu = 0, \quad B^\mu B_\mu = 1. \]  
(A14)

Now we want to understand which geodesics are timelike, spacelike and lightlike. In fact we are interested in computing the geodesic distance of timelike and lightlike geodesics since we expect the mass-shell relation to be a deformation of the Special Relativistic one. So we need to analyze the sign of the modulus of the tangent vectors to the geodesics we found so far. In order to compute this value, we can use the tangent vectors as seen by the embedding space \( \mathcal{B} \) using the embedding metric \( g \). Thus, once the constraint is taken into account, the modulus of the tangent vectors is simply \( \dot{\tilde{\Pi}}^\mu(\rho)\dot{\tilde{\Pi}}_\mu(\rho) \). If \( \omega > 0 \) we find

\[ \dot{\tilde{\Pi}}^\mu(\rho)\dot{\tilde{\Pi}}_\mu(\rho) = \omega, \]  
(A15)

so the geodesic is spacelike\(^{15}\). If \( \omega < 0 \) we find

\[ \dot{\tilde{\Pi}}^\mu(\rho)\dot{\tilde{\Pi}}_\mu(\rho) = -|\omega|, \]  
(A16)

so the geodesic is timelike. If \( \omega = 0 \) we find

\[ \dot{\tilde{\Pi}}^\mu(\rho)\dot{\tilde{\Pi}}_\mu(\rho) = 0, \]  
(A17)

so the geodesic is lightlike. In light of these results, we discard the geodesic (A6) focusing first on computing the timelike geodesic distance using the relation (A16). Assume we are analyzing the timelike geodesic going out from the origin of \( \tilde{\mathcal{H}} \) (\( \rho^0 = 0 \)) and arriving at a point with coordinates (written in the embedding space) \( \tilde{\Pi}^\mu = \left(\frac{\tilde{p}^\mu}{mc}, \frac{1}{\sqrt{1 + \frac{\tilde{p}^\mu}{mc^2}}} \right) \). Using the definition of length of a curve, holds that

\[ D(0, \tilde{\Pi}) = \int_0^1 d\rho \sqrt{\left| \dot{\tilde{\Pi}}^\mu(\rho)\dot{\tilde{\Pi}}_\mu(\rho) \right|} = |\omega|, \]  
(A18)

\(^{15}\) We recall that \( g = \text{diag}(-1, 1, 1, 1, 1) \).
where we use a parametrization over the geodesic such that $p^0(0) = 0$ and $p^a(1) = \bar{p}^a$. From $[A18]$ and from the observations done at the beginning of this appendix follows that

$$D(0, \bar{\Pi}) = \frac{D(0, \Pi)}{mc} = \frac{D(0, \bar{\Pi})}{mc} = \sqrt{|\omega|}$$  \hfill (A19)

Using now $[A9]$ together with the constraint $[A5]$, we obtain that $\tilde{\Pi}^\mu(1)\tilde{\Pi}_\mu(0) = \cosh \sqrt{|\omega|}$. But we observe that $\tilde{\Pi}^\mu(0) = (0, 0, 0, 0, 1)$ while $\tilde{\Pi}^\mu(1) = \begin{pmatrix} \bar{p}^0/mc \\ \sqrt{1 + \frac{\bar{p}^a\bar{p}_a}{m^2c^2}} \end{pmatrix}$ so

$$\frac{1}{\sqrt{1 + \frac{\bar{p}^a\bar{p}_a}{m^2c^2}}} = \tilde{\Pi}^\mu(1)\tilde{\Pi}_\mu(0) = \cosh \sqrt{|\omega|} = \cosh \left( \frac{D(0, \bar{\Pi})}{mc} \right),$$  \hfill (A20)

where in the last equality we used the $[A10]$. Solving it with respect to $D(0, \bar{\Pi})$, we find that

$$D(0, \bar{\Pi}) = mc \arccosh \left( \frac{1}{\sqrt{1 + \frac{\bar{p}^a\bar{p}_a}{m^2c^2}}} \right).$$  \hfill (A21)

Following the same path for lightlike geodesics we find that

$$D(0, \bar{\Pi}) = 0,$$  \hfill (A22)

and that

$$\bar{p}^a\bar{p}_a = 0.$$  \hfill (A23)

In light of the definition $[23]$ and the equations $[A21]$, $[A22]$ and $[A23]$ we can summarize the results of this section, using a generic value of the momentum $p_\alpha$, with the relation

$$m_p c = mc \arccosh \left( \frac{1}{\sqrt{1 + \frac{p^a p_a}{m^2c^2}}} \right),$$  \hfill (A24)

or its inverse

$$p^a p_a = -m^2c^2 \tanh^2 \left( \frac{m_p}{m} \right),$$  \hfill (A25)

which holds for both timelike and lightlike geodesics.

**Appendix B: Derivation of the relation (63)**

In this appendix we show how to derive formula (63) from (62). Starting from the (62), let us then compute the following integral

$$\int_{\tau}^{\tau'} u'd\tau_u = \int_{\tau}^{\tau'} u \ d\tau_u - \int_{\tau}^{\tau'} m^2_p (u' - v) d\tau_u,$$  \hfill (B1)

where $\tau > 0$ is the proper time of the particle with 4-velocity $u$ and $\tau_u$ is the parameter of integration. Using now the fact that the particles before and after the collision are in a free motion, and that there exists a bijective map between the proper times of the two particles, we notice that $u, u', v$ and $v'$ are all constants with respect to the flow of $\tau$. Moreover, choosing without loss of generality that the proper time of both particles is zero at the collision we have that $u(\tau_u > 0) = v(\tau_u > 0) = u'(\tau_u < 0) = v'(\tau_u < 0) = 0$. Using these conditions the result of the integral (B1) is

$$u' = u - \frac{m^2_p}{m_p} (v' - v)\tau.$$  \hfill (B2)
Finally, choosing the spacetime point of the collision to be the origin of our reference frame, we can interpret the left hand side and the first term of the right hand side to be the positions of the particles with 4-velocity respectively $u'$ and $u$, thus

$$x' = x - \frac{m_q}{m_p} (u' - v) \tau,$$

where we have called $x' = u' \tau$ and $x = u \tau$. It is worth noting that, since $u(r_u > 0) = 0$, $x$ is the position the particle with 4-velocity $u$ would have if the collision had not occurred.

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